A local characterization for static charged black holes

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Abstract

We obtain a purely local characterization that singles out the Majumdar–Papapetrou class, the near-horizon Bertotti–Robinson geometry and the Reissner–Nordström exterior solution, together with its plane and hyperbolic counterparts, among the static electrovacuum spacetimes. These five classes are found to form the whole set of static Einstein–Maxwell fields without sources and conformally flat space of orbits, that is, the conformastat electrovacuum spacetimes. The main part of the proof consists in showing that a functional relationship between the gravitational and electromagnetic potentials must always exist. The classification procedure also provides an improved characterization of Majumdar–Papapetrou, by only requiring a conformally flat space of orbits with a vanishing Ricci scalar of the usual conveniently rescaled 3-metric. A simple global consideration allows us to state that the asymptotically flat subset of the Majumdar–Papapetrou class and the Reissner–Nordström exterior solution are the only asymptotically flat conformastat electrovacuum spacetimes.

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1. Introduction

The (standard) Majumdar–Papapetrou and Reissner–Nordström metrics are known to describe, under rather general conditions, the exterior geometries of the static charged black holes, as shown by the recent uniqueness theorems (see [1] and references therein). The aim of this work is to provide an essentially local characterization for the Majumdar–Papapetrou class and the Reissner–Nordström exterior solutions. Local characterizations are important, not only for being essential ingredients for the improvement of the global characterizations of black holes provided by the uniqueness theorems but also for a better understanding of the solutions and its potential use in stability problems. We first find a purely local uniqueness result that
characterizes Majumdar–Papapetrou, the near-horizon geometry and Reissner–Nordström, together with its plane and hyperbolic counterparts, among the (strict) static electrovacuum spacetimes. Global considerations can then be used to restrict the set conveniently.

The known characterizations of the different families of black hole metrics vary from the (purely) local, the ‘essentially’ local and those of global nature in quite a gradual manner. Whether or not some global property is preferable to some stronger local constraints is not clear a priori (see e.g. the discussion about some possible Kerr characterizations included in [2]). Nevertheless, it is always convenient to try to minimize the number of constraints present in any given local characterization. Take the different characterizations we have for the Schwarzschild metric. Although Birkhoff’s theorem constitutes a nice and purely local characterization, it seems of no use in the uniqueness theorems. Another purely local characterization which involves only the Weyl tensor and the metric itself is given in [3]. A more convenient characterization ingredient appears to be the conformal flatness of the hypersurfaces of constant static time (conformastat), since that constitutes a crucial step in the uniqueness theorems as they stand now. This is, in fact, not purely local, since this characterizes Schwarzschild among the static and asymptotically flat vacuum spacetimes. Indeed, conformastat vacuum spacetimes comprise three [4] (see also [5]) out of the seven families that constitute the whole set of degenerate (type D) static vacuum spacetimes [6, 7]. These three families correspond to the Schwarzschild solution together with its plane and hyperbolic counterparts (class A in table 18.2 in [7]). The Schwarzschild solution can be singled out by requiring asymptotic flatness. This is a simple global consideration, and in this respect one may think of this as being an essentially local characterization.

A natural step to follow is the generalization of the above to static charged black holes. Indeed, global arguments in the uniqueness theorems establish, again, conformal flatness of the hypersurfaces of constant static time [8]. It is important to note, however, that the same global arguments also imply that the gradients of the gravitational and the electromagnetic potentials are aligned or, in other words, that the potentials are functionally related. It is these two facts, together with a ‘non-degeneracy’ restriction and asymptotic flatness, that lead eventually to spherical symmetry and thus to the standard uniqueness results for the non-extreme Reissner–Nordström black hole (see e.g. [8]). One question we address in this paper is up to which extent the alignment and the ‘non-degeneracy’ properties can be relaxed in a local characterization. We believe this may be of use on the improvement of the recent uniqueness theorems of (multi) black holes (see [1] and references therein), since we provide an essentially local uniqueness result for the static charged black hole solutions, binding together the Majumdar–Papapetrou to the exterior Reissner–Nordström solution.

The characterization we present here comes by finding the complete solution of the Einstein–Maxwell field equations without sources for static spacetimes with a conformally flat space of orbits. The only extra assumption made is that the electromagnetic field inherits the symmetry, so that it is also stationary. We call such solutions conformastat electrovacuum spacetimes. Note that we take the definition in [7] as standard, following the original terminology by Synge [9]: conformastationary are those stationary spacetimes with a conformally flat space of orbits and the conformastat comprise the static subset.

Conformal flatness corresponds to the vanishing of the Cotton tensor associated with the induced metric on the space of orbits. A general study of conformastationary spacetimes would then follow an analogous path to the characterization of the Kerr and Kerr–Newman families of black holes among the stationary solutions. In the Kerr case the crucial local property is the vanishing of the complex Simon tensor [10], which generalizes the Cotton tensor on the space of orbits. The characterization of the Kerr metric in [10] comes as a result of the equivalence of the multipole structure of Kerr with that of an asymptotically flat end
with vanishing Simon tensor. The first objection to this characterization is, precisely, that the isometry with Kerr is only established in some neighbourhood of infinity, and hence the extension of this isometry to the whole (strict) stationary region cannot be ensured yet. This motivates, in fact, the search for improved local characterizations, since the problem of the extension of the isometries to whole (strict) stationary regions may be fixed by exploiting the local characterizations to their full extent. Indeed Perjés found [13] that the most general metric with a vanishing Simon tensor depends only on a few parameters, and thus showed that the asymptotically flatness condition in the characterization of Kerr is only necessary in order to fix the value of some constants. In this paper we thus follow an analogous aim, since we exhaust the implications of the vanishing of the Cotton tensor in the static electrovacuum problem.

The second drawback the characterization of Kerr in [10] faces is that, by construction, it is not valid within the ergosphere. To address this problem Mars [2, 11] managed to improve that characterization and include the ergosphere by constructing the so-called Mars–Simon tensor, this time relative to the spacetime. The Kerr characterization in [11] (see also theorem 1 in [2]) is essentially local, since the vanishing of the Mars–Simon tensor produces a family of vacuum solutions depending on two complex constants, only to be fixed by some simple global consideration. On the other hand, in [2], Mars provided a characterization with a much weaker local condition, using more effectively the asymptotic flatness. The work in [11] has been extended recently by Wong in [12], by providing a couple of extended characterizations for the Kerr–Newman family, the first being purely local.

The main assumptions inherent to the spacetime characterizations of the Kerr–Newman family have two crucial direct implications. The first is the degeneration of the Weyl tensor (type D), and the second is the existence of a functional relationship of the gravitational and electromagnetic potentials in the static case. None of these restrictions are taken as assumptions in the present work. Not imposing any restriction on the Petrov type is important in the static case, as otherwise the Majumdar–Papapetrou class would not be taken into consideration. On the other hand, the key result in this paper that leads to the complete solution of the conformastat electrovacuum problem is precisely that the alignment of the gradients of the potentials is necessary. In this sense, in the static case the results found here completely generalize those in [12]. Furthermore, these results suggest that the known local Kerr–Newman characterizations may be improved by relaxing some of the requirements involved.

The vanishing of the Cotton tensor in the stationary vacuum problem was dealt with in a series of three papers by Lukász et al in [14] and Perjés in [15, 16] (see also [17]). They found the whole set of conformastationary vacuum spacetimes. In a first paper [14] the solutions possessing a functional relationship between the real and imaginary parts of the Ernst potential \( E \) were found to consist of three bi-parametric families of solutions generated from the three conformastat vacuum solutions (class A) by the Ehlers transformation. In [15], using the purposely defined ‘Ernst coordinates’, Perjés found that solutions with functionally independent real and imaginary parts of \( E \) necessarily admit a spacelike isometry\(^4\) to conclude in [16] (see also [17]) that this set of solutions is empty. Therefore, all conformastationary vacuum spacetimes belong to the three families presented in [14], which can be thought of as the NUT-type extensions of Schwarzschild and its plane and hyperbolic counterparts.

The plan of this paper is analogous. We begin in sections 2 and 3 by showing how the conformastat electrovacuum problem and the conformastationary vacuum problem can

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\(^3\) Let us note that they refer to conformastationary spacetimes simply as ‘conformastat’.

\(^4\) The authors talk of an ‘axial’ symmetry, but no global property is involved in the result at this point.
be treated within a common framework by using a suitable notation. The motivation is to use the previous works [15–17] as a guide, and additionally, to recover those results. In section 4 we prove the key result: the conformastat electrovacuum spacetimes necessarily contain a functional relationship between the gravitational and electromagnetic potentials. Regarding the use of the procedures in [15, 16] two points must be stressed. First, the ‘common’ proof needs a different approach at many stages, since the variables involved in the general case are not necessarily complex, and thus the positivity of some products cannot be used. Second, the final stages in the proof differ from those in [16] and fix some errors found in [17]. Anyway, to ease the comparison with these works we have kept the same notation whenever possible.

In the second part, section 5 is devoted to complete the study of conformastat electrovacuum spacetimes by classifying and exploiting the necessary functional relationship between the gravitational and electromagnetic potentials. We find that all conformastat electrovacuum spacetimes either belong to the Majumdar–Papapetrou class or correspond to either the Bertotti–Robinson solution or the exterior Reissner–Nordström solution together with its plane and hyperbolic counterparts. Furthermore, the procedure used for the classification provides an improved characterization of the Majumdar–Papapetrou class. This is known to be the class of static electrovacuum spacetimes such that the usual rescaled induced metric in the space of orbits is flat. Here we find that one only needs to ask that metric to be conformally flat and with a vanishing Ricci scalar.

The main result constitutes then a completely local characterization of the static and charged (multi) black hole solutions, plus the ‘non-standard’ Majumdar–Papapetrou solutions, the near-horizon geometry (Bertotti–Robinson) and the plane and hyperbolic counterparts of the exterior Reissner–Nordström. A simple global consideration can be used now to single out the black hole solutions. The essentially local characterization is thus that the conformastat electrovacuum asymptotically flat spacetimes are isometric either to the asymptotically flat subset of the Majumdar–Papapetrou class or the Reissner–Nordström static exterior.

Sign conventions differ from those in [14–16] and follow those in [7]: the metric has signature \((-, +, +, +)\) and the Riemann tensor is defined so that \(2\nabla[\mu \nabla_\rho]w^\lambda = R^\lambda{}_{\gamma a b}w^\gamma\). Greek indices refer to the spacetime and Latin indices to the three-dimensional space of orbits. Units are chosen so that \(G = c = 1\).

2. Conformastationary spacetimes

A stationary spacetime \((\mathcal{M}, g_{\mu\nu})\) is locally defined by the existence of a timelike Killing vector field \(\xi^\mu\), whose space of orbits invariantly determines a differentiable three-dimensional Riemannian manifold \(\Sigma_3\). Local coordinates \([t, x^a]\) exist for which \(\xi^\mu = \partial_t\) and such that the line element can be cast as [7]

\[
\begin{align*}
\mathrm{d}s^2 &= -e^{2U}(\mathrm{d}t + A_a\,\mathrm{d}x^a)^2 + e^{-2U}\hat{h}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b,
\end{align*}
\]

where \(U, A_a\), and \(\hat{h}_{ab}\) do not depend on \(t\). Applying the usual projection formalism [7, 18] we will think of \(U\) as a function on \(\Sigma_3\), \(A_a\) as a 1-form belonging to \(T^*\Sigma_3\) and \(\hat{h}_{ab}\) as a metric on \(\Sigma_3\). Once these three objects are given, the local geometry of the stationary spacetime \((\mathcal{M}, g_{\mu\nu})\) is fully specified by using (1). Let us, from now on, endow \(\Sigma_3\) with the metric \(\hat{h}_{ab}\) and use the first latin indices \(a, b, \ldots\) for objects defined on \((\Sigma_3, \hat{h}_{ab})\). A conformastationary spacetime is a stationary spacetime whose space of orbits \((\Sigma_3, \hat{h}_{ab})\) is conformally flat [7]. Thence, in a conformastationary spacetime there exist coordinates \([x, y, z]\) in which \(\hat{h}_{ab}\,\mathrm{dx}^a\,\mathrm{dx}^b = e^{\phi(x, y, z)}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\). The intrinsic characterization of a conformally flat 3-space is the vanishing of the Cotton tensor \(C_{abc}\) [7, 19], or equivalently, the
York tensor density [20], defined as \( Y^e_a = \hat{\eta}^{bce} C_{abc} \), where \( \hat{\eta}_{abc} \) denotes the volume form of \((\Sigma_3, \hat{h}_{ab})\), which satisfies \( Y^e_a = Y^a_e \) and \( Y^a_a = 0 \). More explicitly, \( \Sigma_3 \) is conformally flat if and only if

\[
Y^e_a = \hat{\eta}^{bce} (2 \hat{\nabla}_c \hat{R}_{ba} - \frac{1}{2} \hat{h}_{ab} \hat{\nabla}_c \hat{R}) = 0, \tag{2}
\]

where \( \hat{R}_{ab} \) and \( \hat{\nabla} \) denote the Ricci tensor and covariant derivative relative to \( \hat{h}_{ab} \).

Conformastat spacetimes are those conformastationary spacetimes which are, in fact, static. In this context, a static spacetime is thus characterized by \( A_a = 0 \).

### 2.1. Electrovacuum field equations

Let us first fix one basic assumption and some notations. First, we will restrict ourselves to Maxwell fields \( F_{\mu\nu} \) in \((M, g_{\mu\nu})\) which inherit the stationary symmetry, i.e. for which \( L_\xi F = 0 \). The Einstein–Maxwell equations outside the sources imply (locally, at least) the existence of two complex scalars, \( \Phi(x^a) \) the electromagnetic potential and \( E(x^a) \) the Ernst potential. These two potentials in \((\Sigma_3, \hat{h}_{ab})\) satisfy the so-called Ernst–Maxwell equations

\[
\hat{\nabla}^a H_a + \frac{1}{2} \bar{G} \cdot H - \frac{1}{2} G \cdot H = 0, \tag{3}
\]

\[
\hat{\nabla}^a G_a - \bar{H} \cdot H - (G - \bar{G}) \cdot G = 0, \tag{4}
\]

where \( H_a \equiv (\text{Re} \, \Phi + \Phi \bar{\Phi})^{-1/2} \Phi, \) and \( G_a \equiv 1/2(\text{Re} \, \Phi + \Phi \bar{\Phi})^{-1}(\bar{\Phi} \cdot \Phi_2 + 2 \Phi \Phi_2) \) and the dot denotes the scalar product. It will be convenient for later to note two identities that \( H_a \) and \( G_a \) satisfy: \( dH = H \wedge \text{Re} G \) and \( dG = G \wedge \bar{G} + \bar{G} \wedge H \). These relations are, in fact, the integrability conditions for the two potentials.

The rest of the Einstein–Maxwell equations without sources reduce to the following problem for \( \hat{h}_{ab} \):

\[
\hat{R}_{ab} = G_a \bar{G}_b + \bar{G}_a G_b - (H_a \bar{H}_b + \bar{H}_a H_b). \tag{5}
\]

Once \( \hat{h}_{ab} \) is known, the geometry and electromagnetic field are recovered from the complex potentials. The metric function \( U \) and the 1-form \( A_a \) are determined by the relations

\[
e^{2U} = \text{Re} \, \Phi + \Phi \bar{\Phi}, \quad (dA)_{ab} = 2 e^{-4U} \hat{\eta}_{abc} \text{Im} \, G^c, \]

taking into account that the freedom in the determination of \( A_a \) corresponds to a transformation of the time coordinate of the form \( t \to t + \chi (x^a) \) [7]. The electromagnetic field, conveniently described by the self-dual 2-form \( F_{\mu\nu} \)

\[
\mathcal{F} = F + i \ast F, \]

where \( \ast \) stands for the Hodge dual in \((M, g_{\mu\nu})\), i.e. \( \ast F_{\mu\nu} = \frac{1}{2} F^{\mu\nu} \eta_{\mu\nu\rho\sigma} \), is thus recovered by

\[
\mathcal{F} = -e^{-U} [H \wedge \xi + i \ast (H \wedge \xi)],
\]

where the 1-form \( H_\mu \) in \((M, g_{\mu\nu})\) is given, in coordinates adapted to the Killing (1), by \( H_\mu = (0, H_\mu) \). Note that \( \xi_\mu = -e^{2U}(1, A_\mu) \). The real and imaginary parts of \( H_\mu \) correspond to the electric and magnetic fields with respect to the observer defined by \( u \equiv e^{-U} \xi \); this is \( H_\mu = E_\mu + i B_\mu = F_{\mu\nu} u^\nu \). For completeness, let us note that the intrinsic definition of \( G_\mu \) in \((M, g_{\mu\nu})\) is given by \( G_\mu \equiv u^\nu (\nabla_\nu u_\mu + i \ast (\nabla_\nu u_\mu) \) and its real and imaginary parts \( G_\mu = a_\mu + i 2 w_\mu \) correspond to the acceleration and twist vectors of the congruence \( u \).
2.2. Vacuum and electro-magnetostatic cases

The stationary vacuum case is characterized by \( \Phi = 0 \), so that \( H_a = 0 \) and hence (5) specializes to

\[
\hat{R}_{ab} = G_a \overline{G}_b + \overline{G}_a G_b
\]

and the Ernst–Maxwell equations reduce to

\[
\hat{\nabla}^a G_a - (G - \overline{G}) \cdot G = 0.
\]

The integrability condition that \( G_a \) satisfies reads simply \( dG = G \wedge \overline{G} \).

The static case is characterized by \( G_a - \overline{G}_a(= 2 \text{Im} G_a) = 0 \). A well-known fact is that the conditions for \( H_a \) and \( G_a \) and the field equations yield \( dG = 0 \) (in fact \( G_a = U_a \)) and \( \overline{G}_a = e^{-2i\theta} H_a \) for some constant \( \theta \) (see e.g. [21]). Let us now define the vector \( X_a \equiv e^{-i\theta} H_a \), which is real by construction and related to the electric and magnetic static fields by \( E_a = \cos \theta X_a \) and \( B_a = \sin \theta X_a \). Instead of working with the complex \( /Phi1 \) let us consider the real potential \( /Psi1 = e^{-i\theta /Phi1} \), so that \( X_a = e^{i\theta} U,a \). We are thus left with two real vectors: \( G_a \) and \( X_a \).

The stationary vacuum and the static electrovacuum cases are known to have an analogous structure, although they are inequivalent (see e.g. chapter 34 in [7]). The analogy has been used previously in the literature in a more or less implicit manner (see e.g. [22]). The fact that the two problems are inequivalent comes most notably from the signature of the potential spaces, which differ in the two cases. Despite this, one can make the analogy explicit and useful in the present study, incorporating that change of signature by making use of a hyperbolic-complex or motor number construction, based on the real Clifford algebra \( C_{1,0}(R) \) (see e.g. [23] and references therein), for the static electrovacuum problem. We call \( j \) the hyperbolic imaginary unit, which satisfies \( j^2 = 1 \), and denote the conjugate operation by \( \tilde{j} = -j \). Note that \( C_{0,1}(R) \) is isomorphic to the field of complex numbers, in which \( i \) is the elliptic imaginary unit.

We are now ready to define

\[
\Sigma_a = \frac{1}{2} (G_a + jX_a),
\]

in terms of which the Einstein–Maxwell and Ernst–Maxwell equations read

\[
\hat{R}_{ab} = 4(\Sigma_a \tilde{\Sigma}_b + \tilde{\Sigma}_a \Sigma_b),
\]

\[
\hat{\nabla}^a \Sigma_a - (\Sigma - \tilde{\Sigma}) \cdot \Sigma = 0,
\]

and the identities for \( G_a \) and \( X_a \) reduce to \( d\Sigma = \Sigma \wedge \tilde{\Sigma} \).

3. A common framework

For the sake of completeness and to allow us to use the techniques and some results of previous works on conformastationary vacuum spacetimes [14, 15], we set up a common and more general problem using a common notation.

Let us denote by \( \iota \) any of both the complex \( i \) and the hypercomplex \( j \), so that \( \iota^2 = \pm 1 \) accordingly, and the general conjugation by \( \tilde{\iota} \), so that \( \tilde{\iota} = -\iota \) stands for either \( \tilde{\iota} = -i \) or \( \tilde{j} = -j \). Any object of the form \( F = f + ig \) will be called a composed object, and \( \Re(F) \) and \( \Im(F) \) will denote its real and imaginary parts.

Consider now a composed vector field \( \mathcal{Y}^a \) and a real metric \( \hat{R}_{ab} \) which satisfy the system of equations

\[
\hat{R}_{ab} = N(\mathcal{Y}_a \tilde{\mathcal{Y}}_b + \tilde{\mathcal{Y}}_a \mathcal{Y}_b),
\]
\[ \nabla^a \gamma_a - (\gamma - \tilde{\gamma}) \cdot \gamma = 0, \]  
(9)  
\[ d\gamma = \gamma \wedge \tilde{\gamma}. \]  
(10)

One could regard this problem at the level of the potentials, but for our purposes it suffices to set up the problem for the vectors and therewith include the integrability conditions as equations. The vacuum case is recovered by taking \( N = 1 \), \( \gamma_a = G_{\alpha} \), a complex 1-form and the conjugate being the complex conjugate. The static case corresponds to \( N = 4 \), \( \gamma_a = \Sigma_{\alpha} \), a j -1-form and the conjugate being the j -conjugate. Note that in both cases the right-hand side of the equation for \( \tilde{R}_{ab} \) is, as it should, a real quantity, whereas equations (9) and (10) yield two real equations each.

3.1. Conformastationarity

Conformastationarity follows by the vanishing of the York tensor density of \( \tilde{\gamma}_{ab} \), that is, by applying equation (2) to the Ricci tensor as expressed in (8). Before writing down the explicit expressions, let us introduce a very convenient vector (see [14])

\[ L \equiv \star(\gamma \wedge \tilde{\gamma}), \]

where \( \star \) denotes the Hodge dual in \( (\Sigma, \tilde{\gamma}_{ab}) \), i.e. \( L_a = \gamma^b \tilde{\gamma}^c \tilde{\eta}_{bac} \). By construction we have \( \bar{L}_a = -L_a \) and \( L \cdot \gamma = L \cdot \tilde{\gamma} = 0 \). Note also that \( L = \star d\gamma \). Let us stress the fact that since \( \bar{L}_a = -L_a \), \( L_a \) is imaginary and thus \( \bar{L}d \gamma \geq 0 \). Introducing (8) into (2) one obtains the real equation

\[ \frac{1}{2N} Y_a^a = (\gamma_a - \tilde{\gamma}_a)L^a + \eta^{bce}(\check{\gamma}_b \nabla_c \gamma_a + \gamma_b \nabla_c \tilde{\gamma}_a) - \frac{1}{2} \tilde{\gamma}_{ab} \tilde{\eta}^{bce} \nabla_c (\gamma \cdot \tilde{\gamma}) = 0. \]  
(11)

Since \( Y_a^a = 0 \) this equation contains at most five independent components. We will exploit the consequences of those equations later.

Two very different situations arise in the study of the system of equations composed of (9), (10), (8) and (11), for \( Y_a \) and \( \tilde{\gamma}_{ab} \) : the class of solutions for which \( L_a \neq 0 \) and those for which \( L_a = 0 \). Nevertheless, before entering into the study of these two cases one has to consider the case \( \gamma \cdot \gamma = 0 \). In the static case \( \gamma_a = \Sigma_a \) is j -composed and \( \Sigma \cdot \Sigma = 0 \) implies, in particular, \( G \cdot G + X \cdot X = 0 \), which clearly leaves us only with the trivial case \( G_a = X_a = 0 \). However, in the vacuum case \( \gamma_a = G_a \) is complex and one can have, in principle, fields for which \( G \cdot G = 0 \). The study of these null fields was performed in [14], where it was proven that no null conformastationary vacuum spacetimes exist apart from the trivial case of flat spacetime. In the following we will therefore take \( \gamma \cdot \gamma \neq 0 \) without loss of generality.

4. The class \( L_a \neq 0 \)

In this section we prove that the class \( L_a \neq 0 \) is empty in two steps. We first show that if \( L_a \neq 0 \) there must be an additional isometry, and then that the existence of that isometry implies the non-existence of solutions with \( L_a \neq 0 \).

Let us take the basis \( \{ L_a, \gamma_a, \tilde{\gamma}_a \} \). (Note that the associated basis for the real tangent vector space is composed of \( iL_a, \gamma_a + \tilde{\gamma}_a \) and \( i(\gamma_a - \tilde{\gamma}_a) \).) The metric \( \tilde{\gamma}_{ab} \) expressed in this basis reads

\[ \tilde{\gamma}_{ab} = \frac{1}{L \cdot L} \{ L_a L_b + (\gamma \cdot \gamma) \tilde{\gamma}_a \tilde{\gamma}_b + (\gamma \cdot \tilde{\gamma}) \gamma_a \tilde{\gamma}_b - (\gamma \cdot \tilde{\gamma})(\gamma_a \tilde{\gamma}_b + \tilde{\gamma}_a \gamma_b) \}. \]  
(12)

where \( L \cdot L = (\gamma \cdot \gamma)(\tilde{\gamma} \cdot \tilde{\gamma}) - (\gamma \cdot \tilde{\gamma})^2 \). Since \( L_a \neq 0 \) we have \( \bar{L}d \gamma \geq 0 \).

Although we have kept the notation as close as possible to that used in [14], the vector \( L \) defined here differs by a multiplicative \( i \).
Using the obvious notation $Y_{ab}^b$ by $Y_a^b$, etc, the tracefree property of $Y_a^c$ translates onto

$$Y_L^L = -(\tilde{Y} \cdot \tilde{Y}) Y_{\tilde{Y}}^\tilde{Y} - (Y \cdot \tilde{Y}) Y_{\tilde{Y}}^\tilde{Y} + (Y \cdot \tilde{Y})(Y_{\tilde{Y}}^{\tilde{Y}} + Y_{\tilde{Y}}^{\tilde{Y}}).$$

Together with the use of the conjugate operation, this allows us to keep all the information contained in $Y_{ab}^b$ in only three components: $Y_{\tilde{Y}}^{\tilde{Y}}$, $Y_{\tilde{Y}}^{\tilde{Y}}$ and $Y_L^L$. (Note that $Y_{\tilde{Y}}^{\tilde{Y}} = Y_{\tilde{Y}}^{\tilde{Y}}$.)

The corresponding three equations in (11), from where the five real independent equations eventually follow, read

$$L^c \nabla_c (\tilde{Y} \cdot Y) = 0,$$
$$L \wedge dL = 0,$$
$$L^c L^d \nabla_c Y_a - \frac{1}{2} \eta_{bce} Y_e L_b \nabla_c (\tilde{Y} \cdot \tilde{Y}) = 0.$$

The interpretation of equations (13) and (14) is straightforward. Equation (14) states that $L_a$ is hypersurface orthogonal, i.e. integrable. Equation (13) implies that the product $\tilde{Y} \cdot Y$ is constant along $L_a$. In the static case this translates onto the fact that the two scalars $G^2 + X^2$ and $G \cdot X$ are constant along the direction orthogonal to the planes spanned by $G_a$ and $X_a$.

4.1. The additional isometry

In this subsection (together with appendix A) we prove that the above equations (13), (14) and (15), together with the Ricci equations (8) and the integrability condition (10), imply the existence of a further isometry along $L_a$.

Since $L_a$ is integrable (14) and imaginary, there exist two real functions $\chi(x^a)$ and $\phi(x^a)$ such that

$$L = \iota \chi d \phi.$$

The function $\phi$ cannot be constant precisely because $L_a \neq 0$, and we can also take $\chi > 0$ without loss of generality. The integrability equations (10) imply, in turn, the existence of two further real functions, encoded in the composed potential $\sigma(x^a)$ so that

$$Y = \frac{1}{\sigma + \sigma} d \sigma.$$

The main idea is to use the three potentials $\sigma$, $\bar{\sigma}$ and $\phi$, as coordinates. In the vacuum (complex) case [15] these particularize to the so-called Ernst coordinates. The independence of $\phi$ and $\sigma$ is ensured by the orthogonality of $L_a$ and $Y_a$. Let us label this coordinate system as

$$x^1 = \sigma, \quad x^2 = \bar{\sigma}, \quad x^3 = \phi.$$

The real coordinates and manifold-related quantities can always be recovered by the obvious linear transformations to the coordinates $\sigma + \bar{\sigma}$ and $\iota(\sigma - \bar{\sigma})$. There exists a freedom in choosing $\phi$, since $L_a$ is invariant under the transformation

$$\phi \rightarrow f(\phi), \quad \chi \rightarrow \chi \left( \frac{df}{d\phi} \right)^{-1},$$

for any smooth function $f$ with a non-vanishing derivative. This freedom will be only used in the last step of the proof (see appendix A).

A simple inspection shows the relationship of $\sigma$ with the original potentials. In the static case one has $\sigma = \frac{i}{2}(\phi^{\prime \prime} + j \psi)$, whereas in the vacuum case one recovers the usual Ernst complex potential in vacuum $\sigma(= E) = e^{2U} + i \Omega$. 

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The form of the metric $\hat{h}_{ab}$ in these coordinates follows directly from (12) together with (16) and (17). With the help of a shorter notation for the products

$$α ≡ (σ + \bar{σ})^2(Y \cdot \bar{Y}),$$
$$β ≡ (σ + \bar{σ})^2(Y \cdot \bar{Y}),$$

(where note that $α$ is composed, we denote by $γ$ its conjugate and $β = \bar{β}$ is real) together with the auxiliary real functions $ρ$, which essentially substitutes $χ$, and $D$ defined by

$$D ≡ αγ − β^2 = (σ + \bar{σ})^2(L \cdot \bar{L}),$$
$$ρ^2 ≡ ιχ^2L \cdot \bar{L} = ιχ^2D − 1(σ + \bar{σ})^4χ^2 > 0,$$

(19)

the line element reads

$$\hat{h}_{ab} dx^a dx^b = \frac{1}{D}[\gamma(dx^1)^2 − 2β dx^1 dx^2 + α(dx^2)^2 + ρ^2D(dx^3)^2].$$

(20)

Since we are dealing with $Y \cdot \bar{Y} ≠ 0$, $α$ cannot vanish, and in the complex case one thus readily has that $αγ > 0$ because $αγ = α\bar{α}$. But in the $j$-composed case this is not ensured a priori. Nevertheless, the real function $D$ satisfies $ι^2D > 0$ by construction, which in the $j$-composed case translates onto $D > 0$ and therefore $αγ > 0$ necessarily. To sum up, in any case we have $αγ > 0$.

Let us also remark that $\det \hat{h} = ρ^2D^{-1}$ in these composed coordinates. On the other hand, given (16) and (17) together with the definition of $L_α$, the volume element is fixed by $\hat{η}_{123} = ρD^{-1}\sqrt{ι^2D}$. We will take the metric to be determined by the four real unknown functions encoded in $α$, $β$, and $ρ$. Without loss of generality we take $ρ > 0$.

It only remains to write equations (9), (8) plus (13) and (15) in this coordinate system. Since

$$\bar{L} = ι \sqrt{ι^2D} \rho(σ + \bar{σ})^2\partial_3,$$

equation (13) holds iff $α$ and $γ$ are functions of $σ$ and $\bar{σ}$ only. With this information at hand, equation (9) translates onto

$$(α\partial_1 + β\partial_2) \ln ρ \sqrt{ι^2D} + \partial_1α + \partial_2β = \frac{2α}{σ + \bar{σ}},$$

(21)

while (15) reads

$$2(α\partial_1 + β\partial_2) ln ρ + \partial_2β − \frac{2β}{σ + \bar{σ}} = 0.$$  

(22)

The components of the equation for the Ricci tensor (8) yield the four independent composed equations

$$\hat{R}_{11} = \hat{R}_{13} = \hat{R}_{33} = 0, \quad \hat{R}_{12} = \frac{N}{(σ + \bar{σ})^2},$$

(23)

which encode the six real equations due to the fact that $\hat{R}_{22} = \hat{R}_{11}$, $\hat{R}_{23} = \hat{R}_{13}$. Note that $\hat{R}_{12}$ and $\hat{R}_{33}$ are real.

The first consequence that the integrability conditions of the system (21)–(23) provide is as follows.

**Proposition 1.** Any solution of the system of equations (21), (22) and (23) for $ι^2D > 0$ necessarily has $β_3(\equiv \partial_3β) = 0$.

To ease the reading the proof is left to appendix A.
The only remaining function in the line element (20) which may still depend on $x^3$ is $\rho$. But this cannot be the case due to (22). Assuming that a solution to (22) exists, integration of $\gamma(22) - \beta \tilde{\gamma}(22)$ yields

$$\ln \rho = \int_{0}^{\sigma} \frac{1}{2D} \left( \beta \beta_1 - \gamma \beta_2 + 2\beta \frac{\gamma - \beta}{x^1 + \tilde{\sigma}} \right) dx_1 + \ln \rho_0.$$ 

The integral does not depend on $x^3$, but the arbitrary term $\rho_0$ depends on $x^3$, in principle. However, this term can be eliminated by using the remaining freedom in choosing the coordinates (18), given by a transformation $x^3' = x^3(x^3)$.

This completes the proof of the existence of an additional spacelike isometry whenever $\tilde{\gamma} \wedge \gamma \neq 0$ and $\gamma \cdot \gamma \neq 0$.

### 4.2. The class $L_a \neq 0$ is empty

In appendix B we prove the following result.

**Proposition 2.** There is no solution of the system of equations (21), (22) and (23) for functions depending on $\sigma$ and $\tilde{\sigma}$ only, with $\iota^2 D > 0$, for $N \neq 16, -2, 8/5$.

Since we are interested only in the cases $N = 1$ and $N = 4$ we do not investigate further the compatibility of (21), (22) and (23) for the special cases $N = 16, -2, 8/5$.

This proposition thus states that the class $L_a \neq 0$ with and additional isometry is empty. Combined with proposition 1 this finally implies that the full class $L_a \neq 0$ is empty.

We are thus only left with $L_a = 0$ necessarily. This means that $\gamma_a$ and $\tilde{\gamma}_a$ are parallel, and $d\gamma = 0$ by (10). Therefore, $\gamma_a$ is a gradient of some composed potential whose real and imaginary parts are functionally dependent.

In particular, on the one hand we have thus recovered the result found in the series of papers [14–16] (see also [17]).

**Theorem 1.** Conformastationary vacuum spacetimes are always characterized by a functional relation between the potentials $U$ and $\Omega$.

On the other hand, in the stationary electrovacuum case we have thus proven

**Theorem 2.** Conformastat electrovacuum spacetimes are always characterized by a functional relation between the potentials $U$ and $\Phi$.

### 5. The complete solution of the conformastat electrovacuum problem

In the conformastationary vacuum case the complete solution is thus given by those spacetimes for which $U = U(\Omega)$. This was studied in [14]. The solution consists of three explicit bi-parametric families of line elements, as described in the introduction. We refer to [14] for the explicit form of the line elements.

In the following we focus on the static case. From the above results we know that we only have to look for solutions for which

$$U = U(\Psi).$$

This is a well-known ansatz used to find electro(mageto)static solutions as described in [7], section 18.6.3. Our work consists in finding all the conformastat solutions among this class.
The divergence equation (9) firstly fixes the functional relationship to be\(^7\) (see e.g. [7])
\[
e^{2U} = 1 - 2\psi + \psi^2
\]
for an arbitrary constant \(c\), which can be rewritten in parametric form in terms of an auxiliary function \(V\) as
\[
c^2 = 1 : \quad \psi = c - 1/V, \quad e^{2U} = V^{-2}, \quad (24)
\]
\[
c^2 > 1 : \quad \psi = c - \sqrt{c^2 - 1} \coth V, \quad e^{2U} = (c^2 - 1) \sinh^{-2} V, \quad (25)
\]
\[
c^2 < 1 : \quad \psi = c - \sqrt{1 - c^2} \cot V, \quad e^{2U} = (1 - c^2) \sin^{-2} V, \quad (26)
\]
and secondly implies
\[
\hat{\nabla}^2 V = 0
\]
in all cases. The Ricci equations (8) reduce now to
\[
c^2 = 1 : \quad \hat{R}_{ab} = 0, \quad (27)
\]
\[
c^2 > 1 : \quad \hat{R}_{ab} = 2V_{,a}V_{,b}, \quad (28)
\]
\[
c^2 < 1 : \quad \hat{R}_{ab} = -2V_{,a}V_{,b}. \quad (29)
\]
The remaining equation that \(\hat{h}_{ab}\) and \(V_a\) have to satisfy corresponds to the conformal flatness of \(\hat{h}_{ab}\) and is encoded in (11).

Let us stress on the fact that either case \(c^2 > 1\) or \(c^2 < 1\) constitutes a more general problem for \(\hat{h}_{ab}\) than the problem for the conformally flat 3-metric that one encounters in the black hole (global) uniqueness theorems (see e.g. [8]). In the uniqueness theorems for charged black holes one establishes from global considerations (using the positive mass theorem) not only the conformal flatness of the 3-metric and that the potentials are functionally related but also that the conformal factor depends only on the potential. Since the conformal factor is not fixed \(a \text{ priori}\) in this study, we cannot use the usual results found in the uniqueness theorems. Instead we follow the procedure used by Das in [4] for obtaining the static vacuum solutions.

5.1. Case \(c^2 = 1\)

Equation (27) does not involve \(V_a\) and simply implies that \(\hat{h}_{ab}\) must be flat, which in turn renders (11) to be automatically satisfied. This is the well-known Majumdar–Papapetrou class of solutions [7]. Given any solution \(V\) of the Laplace equation \(\hat{\nabla}^2 V = 0\) in flat 3-space, the metric of the corresponding member of the Majumdar–Papapetrou class is found by using (24), and thus reads
\[
d\tau^2 = -\frac{1}{V^2}dt^2 + V^2(dx^2 + dy^2 + dz^2),
\]
while the electromagnetic potential \(\Phi = e^{i\phi}\psi\), after a trivial shift, is given by
\[
\Phi = -e^{i\phi} \frac{1}{V}.
\]

\(^7\) The relationship one obtains is in fact \(e^{2U} = b + a\psi + \psi^2\) for arbitrary constants \(a\) and \(b\). The constant \(b\) can be rescaled by using the freedom \(\psi \rightarrow \psi + \text{const. (if } b \leq 0\) or a rescaling of the \(t\) coordinate (if \(b > 0\)).
5.2. Case \( c^2 > 1 \)

In this case we are looking for solutions \( \{ \hat{h}_{ab}, V \} \) with \( V_a \equiv V_{,a} \neq 0 \) of the system

\[
\hat{R}_{ab} = 2V_aV_b, \tag{30}
\]

\[
\hat{\nabla}^2 V = 0, \tag{31}
\]

\[
4V_b [\hat{\nabla}_c]V_a - \hat{h}_{ab}[\hat{\nabla}_c](V \cdot V) = 0, \tag{32}
\]

where the latter stands for (2).

Because of \( \hat{\nabla}^2 V = 0 \) and \( V_a \neq 0 \), local coordinates \( \{ x, y^A \} \) with \( A = 2, 3 \) can be chosen so that \( V = x \), and also such that \( \{ y^A \} \) span the surfaces \( S_2 \) orthogonal to \( V_a \). In these coordinates adapted to \( V_a \) equation (32) implies the following form of the metric:

\[
\hat{h}_{ab} \, dx^a dx^b = W^2(x) \, dx^2 + W(x)/\Omega_1 AB \, dy^A dy^B, \tag{33}
\]

where \( W(x) \) is an arbitrary positive \( C^3 \) function and \( \Omega_1 AB \) is a Riemannian \( C^3 \) metric on \( S_2 \), depending only on \( \{ y^A \} \). The imposition of (30) leads to an equation for \( W(x) \) whose solution reads

\[
W = (A e^x + B e^{-x})^{-2} \tag{34}
\]

with constants \( A \) and \( B \).

It only remains to see that the surfaces \( (S_2, \Omega_{AB}) \) are of constant curvature. Let us consider the unit normal to \( S_2 \), \( n_a = W V_a \), and two vectors tangent to \( S_2 \), \( e^a \), \( n_a e^b \hat{h}_{ab} = 0 \). The second fundamental form of \( S_2 \) in \( \Sigma_3 \) thus reads

\[
K_{AB} = e_A^a e_B^b \nabla_a n_b = W'/2W)\Omega_{AB}. \tag{35}
\]

On the other hand, taking into account the identity between the Riemann and the Ricci tensors in a three-dimensional space, equation (30) is used to obtain the following expression of the Riemann tensor of \( \hat{h}_{ab} \) projected on \( S_2 \):

\[
\hat{R}_{abcd} e^a e^b e^c e^d = \Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}. \tag{36}
\]

This expression is then introduced into the Gauss equation in order to obtain the Riemann tensor for \( W \Omega_{AB} \) on \( S_2 \):

\[
(W \Omega)_{ABCD} = \hat{R}_{abcd} e^a e^b e^c e^d + K_{AC} K_{BD} - K_{AD} K_{BC}

= \left( \frac{W^2}{4W^2 - 1} \right) (\Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC})

= -4ABW(\Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}). \tag{37}
\]

The Riemann tensor for \( \Omega_{AB} \) on \( S_2 \) thus reads

\[
(\Omega)_{ABCD} = W^{-1}(W \Omega)_{ABCD} = -4AB(\Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}). \tag{38}
\]

Therefore, \( (S_2, \Omega_{AB}) \) is a surface of constant curvature \(-4AB\). In principle, three different possibilities arise: (i) \( AB = 0 \), (ii) \( AB < 0 \) and (iii) \( AB > 0 \).

5.2.1. Case (i). This case is characterized by a flat \( \Omega_{AB} \). Coordinates \( \{ \theta, \varphi \} \) can therefore be chosen such that

\[
\Omega_{AB} dx^A dx^B = d\theta^2 + d\varphi^2. \tag{39}
\]

By changing \( x \rightarrow -x \) if necessary, we can take \( B = 0 \) and \( A \neq 0 \) without loss of generality, so that \( W = A^{-2} e^{-2x} \). The line element and electromagnetic potential are now obtained by introducing this into (33) and using (25). By performing the change \( x = \frac{1}{2} \ln(b/(r + b)) \)
with \( b \equiv (4A^2\sqrt{c^2 - 1})^{-1} > 0 \), together with \( \tau = 2\sqrt{c^2 - 1} t \), which induces the rescaling \( \Psi \rightarrow \Psi/(2\sqrt{c^2 - 1}) \), the line element can be finally cast as
\[
\begin{align*}
\mathrm{d}s^2 &= -\frac{(r + b)b}{r^2} \, \mathrm{d}r^2 + \frac{r^2}{(r + b)b} \, \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \mathrm{d}\phi^2),
\end{align*}
\]
(35)
after a further convenient rescaling of \( \theta, \phi \). The electromagnetic potential, after a trivial shift, reads
\[
\Phi = e^{\phi} \frac{b}{r}.
\]
Note that the only restriction of the ranges of the coordinates is on \( r \). Since we have taken \( b > 0 \) we are left with two different ranges, \(-b < r < 0\) and \( r > 0\). This family of solutions belong to the static plane-symmetric Einstein–Maxwell fields for which the surface element of the surfaces \( S_2 \) with metric \( e^{-2\Phi} \Omega_{AB} \) has a non-vanishing gradient (see chapter 15.4 in [7]). It can also be regarded as the flat counterpart of the Reissner–Nordström metric. Although that family of spacetimes in [7] presents, in principle, two parameters \( m \) and \( e \), whenever \( m \neq 0 \) a convenient change in \( r \) can bring both \( e \) and \( m \) in [7] into a single parameter. If \( m = 0 \) that family falls into the \( R_{ab} = 0 \) case.

5.2.2. Case (ii). This case is characterized by an \( \Omega_{AB} \) with positive constant curvature \(-4AB = 4|AB|\). Coordinates \( \{ \theta, \varphi \} \) can therefore be chosen such that
\[
\begin{align*}
\Omega_{AB} \mathrm{d}x^A \mathrm{d}x^B &= \frac{1}{4|AB|}(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2),
\end{align*}
\]
where \( \theta \in (0, \pi) \) and \( \varphi \in [0, 2\pi) \). After the change \( e^{2\xi} = R^2 \) and renaming \( a = 2\sqrt{c^2 - 1} \), the direct substitutions lead to the line element
\[
\begin{align*}
\mathrm{d}s^2 &= -\frac{a^2R^2}{(R^2 - 1)^2} \, \mathrm{d}r^2 + \frac{(R^2 - 1)^2}{a^2(R^2 + B)^2} \left[ \frac{1}{4|AB|} \left( \frac{(AR^2 + B)^2}{(AR^2 + B)^2 - 4|AB|} \right) \right] \left( \frac{1}{4|AB|} \left( (AR^2 + B)^2 - 4|AB| \right) \right) \mathrm{d}R^2 + (\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2)
\end{align*}
\]
(36)
for an electromagnetic potential given by
\[
\Phi = e^{\phi} \frac{a R^2 + 1}{2 R^2 - 1}.
\]
Note that although three parameters appear in the metric, one of them can be absorbed by applying a convenient change of coordinates, and therefore only two are relevant. Now, this metric contains two very different subfamilies, depending on whether the gradient of the surface element of the \( \{ \theta, \varphi \} \) surfaces (see above) vanishes or not. Direct computation shows that the gradient vanishes if and only if \( A + B = 0 \).

When \( A + B \neq 0 \) one must obtain the Reissner–Nordström solution. Indeed, the change \( \{ t, x \} \rightarrow \{ \tau, r \} \) given by
\[
\begin{align*}
e^{2\xi} &= \frac{1 - 4\sqrt{|AB|}(c^2 - 1)Br}{1 + 4\sqrt{|AB|}(c^2 - 1)Ar}, \\
\tau &= 2\epsilon \frac{\sqrt{|AB|}(c^2 - 1)}{A + B} t,
\end{align*}
\]
(37)
where \( \epsilon^2 = 1 \), followed by the rearranging of the constants \( A, B \) into
\[
\begin{align*}
Q_c &= \frac{\epsilon}{4|AB|\sqrt{c^2 - 1}}, \\
M &= \frac{B - A}{8|AB|^{3/2}\sqrt{c^2 - 1}},
\end{align*}
\]
(38)
leads to the Reissner–Nordström metric in canonical coordinates
\[
\begin{align*}
\mathrm{d}s^2 &= -\left( 1 - \frac{2M}{r} + \frac{Q_c^2}{r^2} \right) \mathrm{d}r^2 + \left( 1 - \frac{2M}{r} + \frac{Q_c^2}{r^2} \right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2),
\end{align*}
\]
(39)
in the ranges \(0 < r < M - \sqrt{M^2 - Q^2_c}\) and \(M + \sqrt{M^2 - Q^2_c} < r\), and its corresponding electromagnetic potential
\[
\Phi = e^{\vartheta Q_c/r}
\]
after a trivial shift. Note that \(M^2 - Q^2_c > 0\) by construction (see below) and that the usual \(Q\) and \(P\) [8] obviously correspond to \(\cos \theta Q_c\) and \(\sin \theta Q_c\), respectively.

The line element of the special family for which \(A + B = 0\) can be conveniently written as
\[
ds^2 = -\sinh^2 \left(\frac{z}{b}\right) \, dr^2 + dz^2 + b^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2)
\]
for
\[
\Phi = e^{i\vartheta \cosh \left(\frac{z}{b}\right)}
\]
after the changes \(\tau = \sqrt{e^2 - 1}r\) and \(\sinh x = [\sinh(y/b)]^{-1}\), where \(1/b \equiv 4A^2\sqrt{e^2 - 1}\). This is the well-known Bertotti–Robinson solution, which is also characterized by being the only homogeneous Einstein–Maxwell field with a homogeneous non-null Maxwell field, and the only conformally flat solution with a non-null Maxwell field [7]. Furthermore, the Bertotti–Robinson solution is known to describe the near-horizon limit of an extreme Reissner–Nordström black hole [24].

It is worth noting here that the relationship \(4|AB|(M^2 - Q^2_c) = (A + B)^2 Q^2_c\) implies that in class (ii) of solutions we only find the \(M^2 - Q^2_c > 0\) part of the Reissner–Nordström solution. Indeed, the extreme case \(M^2 - Q^2_c = 0\) is excluded in class (ii) because \(A + B = 0\) in (36) leads to the Bertotti–Robinson solution instead. This is due to the fact that in case (ii) we consider solutions with \(\hat{R} > 0\) whereas the extreme Reissner–Nordström solution has \(\hat{R} = 0\), thus falling into the Majumdar–Papapetrou class. The \(M^2 - Q^2_c < 0\) case implies \(\hat{R} < 0\), and therefore will appear in the case \(e^2 < 1\) below. To sum up, the line element (36) corresponds to the (static and \(M^2 - Q^2_c > 0\)) Reissner–Nordström solution containing the near-horizon Bertotti–Robinson metric as a limit instead of the extreme case. Note, again, that only two parameters in (36) are relevant, but for the sake of shortness we do not pursue the rewriting of (36) any further.

5.2.3. Case (iii). This case is characterized by an \(\Omega_{AB}\) with negative constant curvature \(-4AB\). Coordinates \(\{\vartheta, \varphi\}\) can therefore be chosen such that
\[
\Omega_{AB} \, dx^A \, dx^B = \frac{1}{4AB} (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2),
\]
where \(\vartheta \in (0, \infty)\) and \(\varphi \in (-\infty, \infty)\). As in the previous case (ii), after the change \(e^{2\vartheta} = R^2\) and \(a \equiv 2\sqrt{e^2 - 1}\), the direct substitutions lead to the same line element (36) with \(\sin \vartheta\) changed by \(\sinh \vartheta\).

Since \(AB > 0\) in this case, \(A + B\) cannot vanish, and therefore the change (37) is always possible. After performing the same parameter redefinitions (38) one obtains the metric
\[
ds^2 = -\left(\frac{1 - 2M}{r} + \frac{Q^2_c}{r^2}\right) \, dr^2 + \left(\frac{1 - 2M}{r} + \frac{Q^2_c}{r^2}\right)^{-1} \, dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2)
\]
and its corresponding electromagnetic potential
\[
\Phi = e^{i\vartheta \cosh \left(\frac{z}{b}\right)}
\]
after a trivial shift. In this case the only constraint on the values of the parameters \(M\) and \(Q_c\) is \(M^2 + Q^2_c \neq 0\). The range for the coordinate \(r\) for which the metric is static is given
by \(-M - \sqrt{M^2 + Q^2} < r < -M + \sqrt{M^2 + Q^2}\). This is the hyperbolic counterpart of the Reissner–Nordström solution.

5.3. Case \(c^2 < 1\)

The equation that differs from the previous case \(c^2 > 1\) is

\[
\hat{R}_{ab} = -2V_aV_b. \tag{42}
\]

We proceed in an analogous way to solve the system (42), (31) and (32). The difference in sign in (42) compared to (30) only affects the equation for \(W\), whose solution is given now by

\[
W = (Ae^{ix} + A^*e^{-ix})^{-1}, \tag{43}
\]

where \(A\) is a complex number. The same previous procedure shows now that the surfaces \((S_2, \Omega_{AB})\) are of positive constant curvature \(4A\). Coordinates \(\{\vartheta, \varphi\}\) can therefore be chosen such that

\[
\Omega_{AB}dxdy = \frac{1}{4AA}(d\vartheta^2 + \sin^2\vartheta d\varphi^2),
\]

where \(\vartheta \in (0, \pi)\) and \(\varphi \in [0, 2\pi)\). The complete line element of the solution is found using (43) on (33) and taking into account (26) for \(V = \chi\). This case is analogous to case (ii) above.

When \(A + A^* \neq 0\), as expected, the change of coordinates

\[
e^{2ix} = \frac{i - 4\sqrt{AA}(1 - c^2)Ar}{i + 4\sqrt{AA}(1 - c^2)Ar}, \quad \tau = \frac{\sqrt{A\bar{A}(1 - c^2)}t}{A + \bar{A}},
\]

and the renaming

\[
Q = \frac{\epsilon}{4AA\sqrt{1 - c^2}}, \quad M = \frac{i(A - \bar{A})}{8(A\bar{A})^{1/2}\sqrt{1 - c^2}},
\]

is what takes us to the Reissner–Nordström metric (39), but for \(M^2 - Q^2 < 0\). Note that with the above definitions \(4AA(M^2 - Q^2) = -(A - \bar{A})^2Q^2\).

If \(A + \bar{A} = 0\) the change \(z = a\cot x\) with \(a^{-1} \equiv -(A - \bar{A})^2\sqrt{1 - c^2}\) and \(T = \sqrt{1 - c^2}t\), which induces the change \(\Psi \rightarrow \Psi / \sqrt{1 - c^2}\), leads to

\[
dz^2 = -\left(1 + \frac{z^2}{a^2}\right)dT^2 + \left(1 + \frac{z^2}{a^2}\right)^{-1}dz^2 + a^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{44}
\]

and the electromagnetic potential (after a trivial shift) \(\Phi = -e^{ix}z/a\). The metric corresponds again to the near-horizon Bertotti–Robinson spacetime (40), now in different coordinates \(\{T, z\}\).

Let us stress on the fact that the ‘intrinsic’ difference that has led to (40) and (44) in the present setting lies in the different sign of the scalar curvature \(\hat{R}\) of the scaled quotient space \(\hat{h}_{ab}\) with respect to the Killing vectors \(\partial_x\) and \(\partial_T\), respectively, but it is not an intrinsic property of the spacetime. In other words, the difference lies in the possibility of choosing timelike Killing vector fields in the Bertotti–Robinson spacetime with associated positive and negative curved scaled quotient spaces \(\hat{h}_{ab}\). Note, however, that in the Reissner–Nordström case the \(\partial_t\) Killing is intrinsically defined (unit at infinity) and that the sign of \(\hat{R}\) corresponds to the sign of \(M^2 - Q^2\), which leads to two globally different spacetimes.
### Table 1. Possible quotient spaces \( (\Sigma_1, \tilde{h}_{ab}) \) in conformastat electrovacuum solutions.

| \( \tilde{h}_{ab} \) Ricci scalar | \( \Omega_{AB} \)                           |
|-----------------------------------|------------------------------------------|
| \( \tilde{R} = 0 \)              | Majumdar–Papapetrou                       |
| \( \tilde{R} > 0 \)              | Plane-symmetric fields                    |
| Spherical                         | Bertotti–Robinson                         |
| Hyperbolic                        | hyperbolic Reissner–Nordström             |
| \( \tilde{R} < 0 \)              | Spherical                                 |
| \( \Rightarrow \)                | Bertotti–Robinson                         |
|                                   | Reissner–Nordström exterior               |
|                                   | \( M^2 - Q^2 > 0 \)                      |
|                                   | Reissner–Nordström exterior               |
|                                   | \( M^2 - Q^2 < 0 \)                      |

### 6. Results

The combination of the above theorems and the classification of the functionally dependent conformastat electrovacuum solutions in section 5 leads to the following final result.

**Theorem 3.** All conformastat electrovacuum spacetimes either belong to the Majumdar–Papapetrou class or correspond to either

- the Bertotti–Robinson conformally flat solutions (40),
- the non-extreme exterior Reissner–Nordström solution (39),
- or its flat (35) or hyperbolic (41) counterparts.

Let us stress that the five classes are exclusive, and that the extreme Reissner–Nordström case is included in the Majumdar–Papapetrou class. For completeness we include table 1 with a classification of the conformastat electrovacuum solutions in terms of the geometrical properties of the timelike static congruence defined by \( \partial_t \) in (1) with \( A_a = 0 \).

The first corollary of this theorem and the classification presented in table 1 constitutes an improved local characterization of Majumdar–Papapetrou. The original local characterization (see e.g. [7]) states that it is the class of static electrovacuum spacetimes with flat \( \tilde{h}_{ab} \). Here we have relaxed the requirement on \( \tilde{h}_{ab} \) by showing that

**Corollary 3.1.** The Majumdar–Papapetrou class of solutions are the static electrovacuum spacetimes with conformally flat \( \tilde{h}_{ab} \) and \( \tilde{R} = 0 \).

An alternative statement of the above theorem is that the static charged black-hole-related geometries, that is, the Majumdar–Papapetrou, the Reissner–Nordström exterior and the near-horizon Bertotti–Robinson geometry, together with the trivial plane and hyperbolic generalizations of Reissner–Nordström, are locally characterized by being the only conformastat electrovacuum spacetimes. A global argument regarding asymptotic flatness can then be used to establish that

**Corollary 3.2.** The conformastat electrovacuum asymptotically flat spacetimes are either isometric

- to the asymptotically flat subset of the Majumdar–Papapetrou class
- or to the exterior Reissner–Nordström solution.

Further global considerations may be finally used to single out the black hole geometries within the Majumdar–Papapetrou class, the so-called standard Majumdar–Papapetrou, favoured by the uniqueness results in [1]. In order to do that one should ask for the global
requirements that single out the standard Majumdar–Papapetrou among the complete class
that appear as hypotheses in the results shown in [25], which basically consist of demanding
a non-empty black hole region and a non-singular domain of outer communications.

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Appendix A. Proof of $\beta_3 = 0$

In this appendix we present the proof of proposition 1, as indicated in section 4: the proof
that the integrability conditions of equations (21), (22) and (23) imply an additional isometry.
This follows, exactly up to a couple of points and modulo some typos and missing terms
in intermediate steps, sections 4 and 5 of [15]. Let us recall that the two differences of
our proof with that in [15] come simply from the two aspects in which the treatment of
the static electrovacuum case differs to that of the stationary vacuum case, as explained in
section 3.

The first is the fact that our functions $\alpha$ and $\gamma = \tilde{\alpha}$ are two composed functions, one
conjugate to the other, and not one complex function and its complex conjugate. The same
goes for the coordinates $x^1 = \sigma$ and $x^2 = \tilde{\sigma}$. Although the product $\alpha \gamma$ is positive (see
subsection 4.1), other factors such as $\alpha_2 \gamma_1 (= \alpha_2 \tilde{\alpha}_2)$ can be negative in general. The positiveness
of $\alpha_2 \gamma_1 (= \alpha_2 \tilde{\alpha}_2)$ in the complex case is used precisely in the final step of the proof in [15],
section 5. Therefore we will need some further steps to complete the proof in our case.

The second difference comes from the number $N$ (see (8)), which infers a different numeric
factor in one composed equation. This difference will only imply different combinations to
produce the equations needed in each step of the proof. We will indicate all the calculations
keeping an arbitrary $N$. The purpose is twofold. Apart from the usual completeness reason, we
also want to reproduce the proof in [15], and by doing so, indicate (and fix) some intermediate
errors (typos and some missing factors) that we have found in [15], section 5. Therefore we
will keep using the notation $\tilde{\sigma}$ for the conjugate operation that particularizes to the complex
conjugate in the complex case.

The starting point is the set of equations (21), (22) and the equations for the Ricci tensor
(23). Note that $N$ only enters one equation in (23), the (1, 2) component. The aim is to prove
that $\beta$ does not depend on $x^3$. To do so, we assume $\beta_3 \neq 0$ in order to find a contradiction.
Recall that neither $\alpha$ nor $\gamma$ depends on $x^3$. The first step is to strictly follow the arguments in
section 3, where the integrability conditions for the functions $\beta$ and $\rho$ in equations (21),
(22) are obtained. The integrability conditions eventually yield three differential equations,
namely (29b), (29c) and (29d) in [15], together with their conjugates, for the functions $\alpha$ and $\gamma$.

The second step follows section 4 in [15], in which the equations for the Ricci tensor
(23) are used. The generalization to include an arbitrary $N$ is straightforward and we simply
indicate the equation involved and the result. $N$ appears in the $\hat{R}_{12} = N / (x^1 + x^2)^2$ equation
component, and therefore contributes (only) to equation (31)—with (32)—in [15], which now reads
\[
- \left( \dddot{R}_{11} + \rho^{-2} \frac{\gamma}{D} \dddot{R}_{13} \right) D^{-1} + 4 \frac{\gamma}{D} \left( \frac{\alpha}{D} \dddot{R}_{11} + \frac{\beta}{D} \dddot{R}_{12} \right) = 4 \frac{\beta \gamma}{D^2} \frac{N}{(x^1 + x^2)^2}.
\]  
(A.1)

This equation (and its conjugate) is convenient because, after using the equations for the derivatives of \( \rho \) and \( \beta \) (equations (24) and (25) in [15]), it provides the only combination in which no \( \partial_{\chi^3} \) derivatives appear, leading to a polynomial of degree 9 in \( \beta \). The ten coefficients of the polynomial must thus vanish, providing, in principle, ten differential equations for \( \alpha \) and \( \gamma \). Nevertheless, those ten equations are proportional to two independent composed equations plus one imaginary equation. Indeed, a straightforward calculation shows that the equations corresponding to the odd powers of \( \beta \) are all multiples of the composed equation
\[
(\sigma + \tilde{\sigma})^2 \gamma_{12} - 4(\sigma + \tilde{\sigma}) \gamma_1 - 2(2N + 1)\gamma = 0.
\]  
(A.2)

\( N \) only affects the odd coefficients, and thus this is in fact the only equation where \( N \) appears. The equations for the even powers of \( \beta \) provide the composed equation
\[
(\sigma + \tilde{\sigma})^2(\alpha \gamma_1 + 5\gamma \gamma_2 - 3\alpha_1 \gamma_1 - 3\gamma_2^2) + 12\gamma^2 + (\sigma + \tilde{\sigma})(14\alpha \gamma_1 - 6\gamma_2) = 0
\]  
(A.3)

plus the imaginary equation
\[
(\sigma + \tilde{\sigma})^2(\gamma_22 - \alpha_11) - 6(\sigma + \tilde{\sigma})(\gamma_2 - \alpha_1) - 12(\alpha - \gamma) = 0.
\]  
(A.4)

Equation (A.2) particularizes to (33) in [15] for \( N = 1 \), and (A.3) and (A.4) correspond to (34) and (29b) in [15], respectively. As claimed in [15], the composed equation (A.3) implies (29d) in [15] and one can easily check that (A.2) implies (29c) in [15]. As stated in [15], there may appear another combination of the equations for the Ricci tensor, namely \( \alpha \dddot{R}_{11} - \gamma \dddot{R}_{11} (= \alpha \dddot{R}_{11} - \gamma \dddot{R}_{22}) = 0 \). However, this equation provides no new information. All in all we are finally left with equations (A.2), (A.3) and (A.4).

### A.1. The system of PDEs for \( \alpha \) and \( \gamma \)

Summing up, the complete system of equations for \( \alpha \) and \( \gamma \) which decouples from the rest of the field equations is given by (A.2), (A.3) and (A.4), which are conveniently rewritten as
\[
E_1 \equiv -\gamma_{12} + 2 \frac{\gamma}{r} \gamma_1 + M \gamma \frac{\gamma}{r^2} = 0
\]  
(A.5)

\[
E_2 \equiv -\gamma_{22} + \alpha_{11} + \frac{3}{r}(\gamma_2 - \alpha_1) + \frac{3}{r^2}(\alpha - \gamma) = 0
\]  
(A.6)

\[
E_3 \equiv -\gamma_{11} + 5\gamma \gamma_2 - 3(\alpha_1 \gamma_1 + \gamma_2^2) + \frac{1}{r} (\gamma_2\gamma_1 - 3\gamma_2^2) + \frac{3}{r^2} \gamma^2 = 0,
\]  
(A.7)

where \( r \equiv (\sigma + \tilde{\sigma})/2 \) and \( M \equiv 2N + 1 \). Since we will be only interested in \( M = 3 \) and \( M = 9 \) we will implicitly assume at some points that certain polynomials in \( M \) with other roots do not vanish and, in fact, that \( M > 0 \). The `accent is used here to keep an analogous notation to that in [15], and the only purpose is to denote differently certain equations. Note, however, that the `here corresponds to the tilde in [15]. Note that \( E_2 = -\tilde{E}_2 \), and therefore the above system of equations contains five real equations.

The procedure consists of generating new differential equations by computing the integrability conditions of the system \( (E_1, \tilde{E}_1, E_2, \tilde{E}_3, \tilde{E}_3) \). This procedure will be fixed by the use of very specific sets of rules, which must be applied in strict order. Before setting
the rules, let us produce two useful combinations after using $E_1$ and $\tilde{E}_1$ to eliminate $\gamma_{12}$ and $\alpha_{12}$, respectively:

$$E_3 \equiv 5\gamma E_2 + \tilde{E}_3 = \alpha \gamma_{11} + 5\gamma \alpha_{11} - 3(\alpha_1 \gamma_1 + \gamma_2^2)$$

$$+ \frac{1}{r}(7\alpha \gamma_1 + 12\gamma \gamma_2 - 15\gamma \alpha_1) + \frac{\gamma}{r^2}(15\alpha - 12\gamma) = 0,$$

(A.8)

$$E_4 \equiv \partial_2 E_3 + \left(\frac{2}{r} \gamma - 6\gamma_2\right) E_2 - \frac{2}{r} \tilde{E}_3 = \alpha_2 \gamma_{11} - \gamma_2 \alpha_{22} + \frac{1}{r}(2\gamma \alpha_{11} + 3\alpha_1 \gamma_2 + \alpha_2 \gamma_1)$$

$$+ \frac{1}{r^2} \left(\frac{2M + 3}{2} \gamma \alpha_1 - \frac{2M + 9}{2} \alpha \gamma_1 - 3\alpha \gamma_2\right) - (2M + 9) \frac{\alpha \gamma}{r^3} = 0.$$  

(A.9)

The first and main rule is as follows.

(i) Multiplication by unknown functions (or their derivatives) is allowed only when the resulting equation does not exceed the cubic degree in the unknown functions.

This rule only affects the choice of combinations to generate new equations. As we will indicate these combinations explicitly, this rule does not need to be implemented in the algorithm. It must also be stressed that in all the equations the factors that will be isolated (and thence 'eliminated') appear linearly and with a non-zero multiplicative factor. The first set of rules, \textit{Rules}$_1 = \{\text{(ii), (iii), (iv), (v), (vi)}\}$ reads

(ii) eliminate $\gamma_{12}$ and $\alpha_{12}$ using $E_1$ and $\tilde{E}_1$ respectively;

(iii) eliminate the product $\alpha_2 \gamma_{11}$ by using $E_4$;

(iv) eliminate the product $\gamma \alpha_{22}$ by using $\tilde{E}_3$;

(v) eliminate the product $\alpha \gamma_{11}$ by using $\tilde{E}_3$;

(vi) eliminate $\gamma_{22}$ by using $E_2$;

(vii) eliminate the product $\gamma \alpha_{22}$ by using rule (vi) applied to $\tilde{E}_4$.

In what follows we simply indicate the chain of equations used, and the explicit expressions will only be given when needed. For the sake of concreteness we prefer to specify whenever any set of rules is applied to any expression $f$ by \textit{Rules}(f).

The sequence of equations begins with

$$E_5 \equiv \text{Rules}_1[\gamma \partial_2 E_4],$$

and follows with

$$\tilde{E}_6 \equiv \text{Rules}_1[r^2 \partial_2 E_5 - 2r E_5 - 12\alpha_2 E_3],$$

$$E_6 \equiv \text{Rules}_1 \left[\frac{3}{51 - 2M} \tilde{E}_6 + \frac{3}{2} (\gamma \tilde{E}_4 - \alpha_1 \tilde{E}_3)\right],$$

$$E_7 \equiv r^2 \frac{2}{3(33 - M)} \text{Rules}_1 \left[2 \partial_2 \tilde{E}_6 - \frac{4}{r} \partial_2 E_6 + (M - 18) \partial_2 (\text{Rules}_1[\gamma_1 \tilde{E}_3]) - \frac{30}{r} \gamma_1 \tilde{E}_3^2\right].$$

Note that in the third factor, as indicated, one must apply some rules before differentiating.

The chain of equations follows with

$$E_8 \equiv \text{Rules}_1 \left[\frac{1}{2} \partial_2 E_7 - \frac{7M + 27}{6} \alpha_2 E_3 + \frac{(5M + 18)(2M - 51)}{9(33 - M)} \tilde{E}_6 - \frac{1}{r} E_7\right],$$

which results in a first order equation. From this point onwards it is convenient to define a new set of rules (keeping the first rule (i)): \textit{Rules}$_2 = \{\text{(ii), (vii), (iii), (iv), (vi)}\}$. The chain follows with

$$E_9 \equiv \text{Rules}_2[\partial_1 \tilde{E}_3],$$

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and a new sequence given by

\[ F_5 = Rules_2[\gamma_1 \partial_2 E_4], \]
\[ F_6 = Rules_2[\partial_2 F_5 - \frac{2}{r} F_5], \]
\[ F_9 = Rules_2[\partial_2 \tilde{E}_8], \]
\[ F_7 = Rules_2[\partial_2 E_6 - \frac{2}{r} E_6], \]

used to construct

\[ E_{10} = Rules_2 \left[ 4 \partial_2 E_9 + \partial_2 \left( r^2 F_6 \right) - 2 r F_0 - \frac{1}{8} E_9 - \frac{4}{r} F_9 \right], \]
\[ E_{11} = Rules_2 \left[ 2 \partial_2 \left( r^2 E_{10} \right) - 4 r E_{10} + 4 \left( 13 M + 56 \right) \left( -F_9 + \frac{2}{r} \tilde{E}_8 \right) - \frac{\left( 2 M^2 - M - 141 \right) \left( 2 M - 51 \right)}{6 \left( M - 33 \right)} \tilde{E}_7 - \frac{M + 2}{4} \left( M^2 + 10 M + 5 \right) \frac{1}{r^2} \alpha E_3 \\
+ \frac{\left( 9 M^3 + 106 M^2 + 645 M + 876 \right)}{4 \left( M + 5 \right)} \frac{1}{r^2} E_7 \right]. \]

The general explicit expression of \( E_{11} \) reads

\[-24 r^2 \left( M + 5 \right) E_{11} = \left( \alpha_1 - \frac{2}{r} \alpha \right)^2 \gamma \left( 322 M^3 + 3771 M^2 + 12018 M + 7713 \right) + 2 \gamma_1 \alpha \left( 319 M^3 + 4560 M^2 + 19473 M + 24488 \right) + \left( \gamma_2 - \frac{2}{r} \gamma \right) \alpha \left( 148 M^3 + 7797 M^2 + 64416 M + 147303 \right) + 2 \left( \gamma_2 - \frac{2}{r} \gamma \right) \alpha \left( 346 M^3 + 3831 M^2 + 14262 M + 20433 \right). \]

The procedure follows by taking the imaginary part

\[ \dot{E}_{11} = \frac{4}{3 \left( 33 - M \right) \left( M + 1 \right)} r^2 \left( E_{11} - \tilde{E}_{11} \right) \]
from where

\[ E_{12} = Rules_3[\partial_1 \dot{E}_{11}], \]
\[ F_{12} = Rules_3[\partial_2 \dot{E}_{11}], \]
\[ \dot{E}_{13} = Rules_2 \left[ 2 r^3 \left( \partial_2 E_{12} - \frac{2}{r} \left( E_{12} + F_{12} \right) \right) - \frac{23 M^2 - 236 M - 483}{9 \left( M + 1 \right)} \dot{E}_{11} \right]. \]

\( \dot{E}_{13} \) reads, explicitly,

\[ E_b = \frac{27 \left( M + 1 \right)^2}{8 \left( 55 M^3 + 1187 M^2 + 6087 M + 8091 \right)} \dot{E}_{13} \]
\[ = \left( \alpha_1 - \frac{2}{r} \alpha \right)^2 - \gamma_2 \left( \gamma_2 - \frac{2}{r} \gamma \right)^2 = 0. \]

The next equation is given by

\[ E_a = \frac{1}{3} \left( \dot{E}_{11} - \frac{29 M + 141}{3 \left( M + 1 \right)} E_b \right) = \left( \gamma_2 - \frac{2}{r} \gamma \right) \alpha_2 \gamma - \left( \alpha_1 - \frac{2}{r} \alpha \right) \gamma_1 \alpha = 0. \]
Let us recall here that the only positiveness property we can use in the general case is $\alpha \gamma > 0$.

Let us set up a new set of rules $\text{Rules}_{ab} = \{(a), (b)\}$, where

(a) eliminate the factor $\alpha \gamma^2$ using $E_b$,
(b) eliminate the factor $\alpha_2 \gamma_2 \gamma$ using $E_a$,

which applied to $E_{11}$ leads to

$$r^2(M + 5) \text{Rules}_{ab}[E_{11}] = \left( \alpha_1 - \frac{2}{r} \alpha \right) \left[ \alpha \gamma \left( \alpha_1 - \frac{2}{r} \alpha \right) + 2B \gamma_1 \alpha \right], \quad (A.10)$$

with

$$B = \frac{1}{28}(-665M^3 - 8391M^2 - 33735M - 45321),$$
$$A = \frac{1}{12}(-235M^3 - 5784M^2 - 38217M - 77508). \quad (A.11)$$

Note that $A < 0$ and $B < 0$ since $M > 0$. The factor $(\alpha_1 - \frac{2}{r} \alpha)$ cannot vanish, since otherwise $\tilde{E}_1$ would lead to $\alpha = 0$, and the same argument holds for the factor $(\gamma_2 - \frac{2}{r} \gamma)$ using $E_1$ and $\gamma \neq 0$. As a result, $(A.10)$ and its conjugate lead to the next pair of equations:

$$E_0 \equiv A \alpha \left( \gamma_2 - \frac{2}{r} \gamma \right) + 2B \alpha_2 \gamma = 0.$$ 

We now use this equation to set up the next set of rules $\text{Rules}_0 = \{(0i), (0ii)\}$ where

(0i) eliminate $\alpha \gamma$ using $E_0$,
(0ii) eliminate $\gamma \alpha_1$ using $\tilde{E}_0$.

The chain of equations follows with

$$E_{14} \equiv \text{Rules}_2 \left[ \partial_1 E_0 - \frac{2}{r} E_0 \right], \quad F_{14} \equiv \text{Rules}_2 \left[ \partial_2 E_0 - \frac{2}{r} E_0 \right],$$

from where we get

$$\dot{E}_{14} \equiv \frac{1}{\alpha \gamma} \text{Rules}_0 [\alpha \gamma E_{14}],$$

which explicitly reads

$$\dot{E}_{14} = 2\alpha_2 \gamma_1 B (2A^{-1} B + 1) + \alpha \gamma \frac{1}{r^2} \left( \frac{1}{2} A N + A + B N \right) = 0. \quad (A.12)$$

Note that $\dot{E}_{14} = \tilde{\dot{E}}_{14}$. Since $\alpha \gamma > 0$ this equation implies $\alpha_2 \gamma_1 < 0$. What we will really need later is simply $\alpha_2 \gamma_1 \neq 0$. From $(A.12)$ we set up the new rule $\text{Rules}_{14} = \{(14i)\}$, where

(14i) eliminate $\alpha_2 \gamma_1$ using $\dot{E}_{14}$.

The next equation reads

$$F_{15} \equiv \text{Rules}_0 [F_{14}],$$

which explicitly reads

$$F_{15} = \alpha_1 \alpha (A - 10B) + 6\alpha_1^2 B + \alpha_2 \gamma_2 (A + 8B) + \frac{1}{r} (3\alpha_1 \alpha (2B - A) - 2\alpha_2 \gamma_2 (A + 8B)) + 3 \frac{1}{r^2} \alpha^2 (A - 2B) = 0,$$

which, since $A - 10B > 0$, we use to set up the last rule $\text{Rules}_{15} = \{(15i)\}$, where

(15i) eliminate $\alpha_1 \alpha$ using $F_{15}$.
The final step consists of using the previous $E_{a}$, differentiating it, and using the sets of rules we have just defined in a very specific order. The precise algorithm begins with

$$E_{16} \equiv \gamma \text{Rules}_{15}[\alpha \text{Rules}_{1}[\partial_{2}E_{a}]].$$

Note that at this point we have ignored rule (i), but the outcome will precisely be the desired result, because

$$E_{\text{final}} \equiv \text{Rules}_{14}[\text{Rules}_{0}[\text{Rules}_{14}[\text{Rules}_{ab}[E_{16}]]]]$$

reads explicitly

$$E_{\text{final}} = \gamma \alpha^{3} \gamma \frac{1}{2r^{2}} \frac{4 ((M - 22) A - 2BM) B^{2} + (3AM + 6A + 10BM + 16B) A^{2}}{A (A + 2B) (10B - A)} = 0.$$

The last factor, after using (A.11) to introduce the values of $A$ and $B$ in terms of $M$, is a fraction containing polynomials in $M$ in which all the coefficients are positive numbers. Therefore, the only solution to $E_{\text{final}} = 0$ would be $\gamma \alpha^{3} \gamma = 0$, which is not allowed by virtue of (A.12) and $\alpha \gamma > 0$.

We have thus shown that $\alpha \gamma > 0 \Rightarrow \beta_{3} = 0$ for any positive $N$, and in particular, in the stationary vacuum case ($N = 1$), recovering the result in [15], and in the static electrovacuum case ($N = 4$).

Appendix B. Proof of proposition 2

The starting point is equations (21), (22) and (23) when all functions depend only on $\sigma$ and $\tilde{\sigma}$. Equation (22) is used to isolate the two derivatives of $\rho$:

$$2D \frac{\partial_{1}}{\rho} = \beta \beta_{1} - \gamma \beta_{2} + \frac{1}{r} (\gamma - \beta) \beta,$$

$$2D \frac{\partial_{2}}{\rho} = \beta \beta_{2} - \alpha \beta_{1} + \frac{1}{r} (\alpha - \beta) \beta,$$  \hspace{1cm} (B.1)

where $r \equiv (\sigma + \tilde{\sigma})/2$. These two equations, which are of course related by conjugation, will be used to eliminate $\rho$ in what follows. The integrability condition will be dealt with later.

From equation (21) and taking into account that $\alpha \gamma > 0$, we can now isolate $\alpha_{2}$, and its conjugate $\gamma_{1}$, to obtain

$$\alpha_{2} = \gamma^{-1} \left[ -2\beta \alpha_{1} + \alpha \gamma \alpha_{2} + \alpha \beta_{1} + \beta \beta_{2} + \frac{1}{r} (3\alpha \beta - 2\alpha \gamma - \beta^{2}) \right],$$

$$\gamma_{1} = \alpha^{-1} \left[ -2\beta \gamma_{2} + \gamma \alpha \gamma_{1} + \gamma \beta_{1} + \beta \beta_{2} + \frac{1}{r} (3\gamma \beta - 2\alpha \gamma - \beta^{2}) \right].$$  \hspace{1cm} (B.2)

We use these two expressions to compute the second derivatives $\alpha_{12}, \alpha_{22}, \gamma_{11}, \gamma_{22}$ in terms of $\alpha_{1}, \gamma_{2}, \gamma_{12}, \alpha_{11}$, and the first and second derivatives of $\beta$. We use their substitutions in what follows.

We concentrate now on the Ricci equations (23). From the equation $\hat{R}_{33} = 0$ we isolate $\beta_{12}$, which reads

$$2D \beta_{12} = \gamma (\alpha_{1} + \beta_{1}) \beta_{2} + \alpha (\gamma_{2} + \beta_{1}) \beta_{1} - \beta (\alpha \beta_{1} + \gamma \beta_{2} + 2 \beta_{1} \beta_{2})$$

$$+ \frac{\beta}{r} \left[ (\beta - \gamma) \alpha \gamma_{1} + (\beta - \alpha) \gamma \alpha_{2} \right] + \frac{\beta}{r^{2}} (\alpha \gamma + \beta^{2} - \beta \alpha - \beta \gamma).$$  \hspace{1cm} (B.3)

Now, from the real combination $a \hat{R}_{11} + \gamma \hat{R}_{22} = 0$ we isolate $\beta_{11}$, which yields
\[ \alpha D\beta_{11} = \gamma(\alpha\gamma' - \beta\alpha_1)\beta_2 - \beta(\alpha(\beta_1 + \gamma_2)\gamma_2 + (\gamma\alpha_1 - 2\beta\gamma_2)\alpha_1) + (2\alpha\gamma - \beta^2)\alpha_1\beta_1 \]
\[ + \frac{1}{r}(\gamma - \beta)\alpha(3\beta\gamma_2 - 2\gamma\beta_2) + \alpha(2\alpha\beta_1 - 3\beta\alpha_1) \]
\[ + \frac{\alpha}{r} \left[ (5\beta^2 - \alpha\gamma - 2\beta\gamma') - \frac{1}{2} \beta(\alpha\gamma + 3\beta^2)\right]. \] (B.4)

The complex conjugate equation provides \( \beta_{22} \), which solves in turn the imaginary equation \( \alpha \hat{R}_{11} - \gamma \hat{R}_{22} = 0 \). This equation is in fact equivalent to the compatibility condition of the above system (B.1) for \( \rho \). It is straightforward to check that the compatibility condition \( \beta_{112} = \beta_{121} \) is automatically satisfied. The first important consequence of (B.4) is that if \( \beta = 0 \) the equation reduces to \( \gamma = 0 \), which contradicts \( \alpha\gamma > 0 \). We must therefore take \( \beta \neq 0 \) in what follows.

The equation \( \hat{R}_{13} = 0 \) is identically satisfied, so it only remains to consider the equation \( \hat{R}_{12} = N/(4r^2) \). It is convenient first to substitute \( \alpha_1 \) and \( \gamma_2 \) by two new functions \( Z \) and \( W \equiv \tilde{Z} \) defined by the relations
\[ \alpha_1 = \frac{1}{4\beta} \left[ 2\beta\beta_2 + 3\alpha\beta_1 + \frac{5}{r} - \frac{2}{r}\beta^2 + W \right], \]
\[ \gamma_2 = \frac{1}{4\beta} \left[ 2\beta\beta_1 + 3\gamma\beta_2 + \frac{5}{r} \gamma\beta - \frac{2}{r}\beta^2 + Z \right], \] (B.5)
from where we will also obtain \( \alpha_{11} \) and \( \gamma_{12} \) in terms of \( Z_1 \) and \( W_1 \). Let us also use the substitutions
\[ b_1 = \beta_1 - \frac{\beta}{r}, \quad b_2 = \beta_2 - \frac{\beta}{r}. \]

The equation \( \hat{R}_{12} = N/(4r^2) \) thus leads to the real relation
\[ F \equiv 9\alpha\gamma \left( \alpha\alpha_1^2 + \gamma\gamma_2^2 - 2\beta\beta_1\beta_2 - \frac{8\beta}{9\gamma^2}(N + 2)D \right) - \alpha Z^2 - \gamma W^2 + 2\beta WZ = 0. \] (B.6)

### B.1. Case A

Let us assume \( Z \neq 0 \), so that \( i^2(\alpha Z^2 + \gamma W^2 - 2\beta WZ) > 0 \) (this is a positive definite product due to \( i^2D = i^2(\alpha\gamma - \beta^2) > 0 \)). The procedure now consists of finding new equations on \( Z, W, Z_1, W_1, b_1, b_2 \) and \( \alpha, \beta, \gamma \) by first differentiating \( F \) and then using the substitutions above.

In the following expressions we also use (B.6) by isolating \( \beta_2^2 \). From the first two derivatives \( \partial_1 F \) and \( \partial_2 F \) one can isolate \( Z_1 \) and \( W_1 \) and find the explicit expressions
\[ 24\alpha\gamma\beta D \left( W_1 - \frac{2}{r} W \right) = 3\alpha\gamma \left[ (9\alpha\gamma - 12\beta^2)W - \alpha\beta Zb_1 + (5\alpha\gamma - 4\beta^2)Z + 3\gamma\beta W \right] \]
\[ + (9\alpha\gamma - 6\beta^2)\gamma W^2 - (14\alpha\gamma - 12\beta^2)\beta WZ + (5\alpha\gamma - 6\beta^2)\alpha Z^2 \]
\[ + 4(16 - N)D \frac{\alpha^2\gamma^2\beta 2(\alpha Z - \beta W)Z + 3\alpha\gamma(Zb_2 - Wb_1)}{\alpha Z^2 + \gamma W^2 - 2\beta WZ}. \] (B.7)
\[ 24\alpha\beta D \left( Z_1 - \frac{1}{r} Z \right) = (4\alpha\gamma + 6\beta^2)WZ - 3\gamma\beta W^2 - 7\alpha\beta Z^2 \]
\[ + 3\alpha[3\gamma\beta W + (9\alpha\gamma - 16\beta^2)Z]b_1 + 3[(11\alpha\gamma - 4\beta^2)\beta Z - (5\alpha\gamma - 2\beta^2)\gamma W]b_2 \]
\[ + 4(16 - N)\alpha\gamma D \frac{\beta 2\alpha\beta Z^2 - 2\alpha\gamma WZ - 3\alpha^2\gamma Zb_1 + 3\alpha\gamma(2\beta Z - \gamma W)b_2}{\alpha Z^2 + \gamma W^2 - 2\beta ZW}. \] (B.8)
We continue by taking the $\partial_2$ derivatives of (B.5) and use the above expressions to obtain $Z_2$ and $W_2$ in terms of $Z, W, b_1, b_2$ and $\alpha, \beta, \gamma$. One can therefore investigate the compatibility condition $Z_{12} = Z_{21}$ (equivalent to $W_{12} = W_{21}$), which provides one real equation

$$(16 - N)(\gamma W - \beta Z)D^2 K F_{11} = 0,$$

where

$$K \equiv \alpha Z^2 - 2\beta Z W + \gamma W^2$$

and

$$F_{11} \equiv 6\alpha\gamma \left[ K + \frac{16 - N}{r^2} - \alpha\gamma \beta D \right] \Im[b_1\alpha(\beta Z - \gamma W)]$$

$$+ K \left[ (2\beta^2 - \alpha\gamma) - \alpha\beta\gamma \frac{D}{r^2}(3(N + 8)\alpha\gamma + 2(N + 2)\beta^2) \right]$$

$$- 4 ((14 - N)N + 32) \frac{D^2}{r^2} \alpha^3 \gamma^3 \beta^2.$$

For the cases we are interested in we can assume $N \neq 16$, and since $\gamma W - \beta Z \neq 0$ as otherwise $K = 0$, we necessarily have $F_{11} = 0$. In the complex case (stationary vacuum) studied in [16, 17], one resorts to the fact that $\beta \equiv (\sigma + \bar{\sigma})^2(\gamma^2 + \bar{\gamma}^2) > 0$ and $N = 1$ to establish that $K + \frac{1}{r^2}(16 - N)\alpha\beta\gamma D > 0$. In the general case, however, one must still consider two subcases.

**Subcase A1:** $K + \frac{1}{r^2}(16 - N)\alpha\beta\gamma D \neq 0$. We use (B.9) to isolate $b_1$ and consider the imaginary combination $\Im[(\beta Z - \gamma W)\alpha\partial_1]F_{11} = 0$, which leads to

$$\alpha\beta\gamma(\gamma W - \beta Z)D^2 G_I = 0,$$

where the factor $G_I$ satisfies

$$\frac{1}{r^4} G_I - W F_{11} = -\alpha(\gamma W - \beta Z) \left( K + \frac{16 - N}{r^2} - \alpha\beta\gamma D \right) 2\Im(3\alpha\gamma b_2 Z - \alpha Z^2).$$

(Note that for this last step one must not use $F_{11}$ explicitly and leave $b_1$ unsubstituted.) As a result, the equation

$$F_{111} \equiv \Im(3\alpha\gamma b_2 Z - \alpha Z^2) = 0$$

follows. We proceed with a twin combination to the previous, $\Im[(\alpha Z - \beta W)\alpha\partial_1]F_{11} = 0$, to complete recovering the two derivatives of $F_{11}$. This combination, after neglecting non-vanishing terms, leads to $G_{111} = 0$, where $G_{111}$ satisfies

$$\frac{1}{r^4} G_{111} - Z F_{11} = 6\alpha\gamma(\gamma W - \beta Z) \left( K + \frac{16 - N}{r^2} - \alpha\beta\gamma D \right) \Im(\alpha Z b_1),$$

and therefore yields to

$$\Im(\alpha Z b_1) = 0.$$  

(B.11)

If $\gamma W^2 - \alpha Z^2 \neq 0$, equations (B.10) and (B.11) lead to $b_1 = W/(3\alpha)$ (and $b_2 = Z/(3\gamma)$), which substituted on $F = 0$ (B.6) implies $D\alpha\beta\gamma(N + 2) = 0$, which is impossible in the present case. Since we are interested in the cases $N = 1$ and $N = 4$ we will also assume in the following that $N + 2 \neq 0$. Therefore we need

$$\gamma W^2 - \alpha Z^2 = 0$$

(B.12)

to make (B.10) and (B.11) linearly dependent.
We continue by taking the imaginary combination \((Z\partial_2 - W\partial_1)\) (B.12) and substituting \(b_2\) from (B.11). Using (B.12), and after neglecting non-vanishing terms, that combination is shown to lead to the following equation:

\[
\begin{align*}
\left[(15\alpha\gamma - 3\beta^2)\alpha Z^2 + (9\beta^2 - 21\alpha\gamma)\beta WZ + 6(16 - N)\alpha^2\gamma^2\beta \frac{D}{r^2}\right]ab_1 \\
+ \left[(5\alpha\gamma - 9\beta^2)(\alpha Z^2 - \beta WZ) + 4(16 - N)\alpha^2\gamma^2\beta \frac{D}{r^2}\right]W = 0. \quad (B.13)
\end{align*}
\]

The real combination \((Z\partial_2 + W\partial_1)\) (B.12) is proportional to \(\gamma W^2 - \alpha Z^2\) and therefore bears no information.

On the other hand, let us take \(\delta_1\) (B.10) and apply the following chain of substitutions: first \(Z_1\) and \(W_1\) from (B.8) and (B.7), followed by \(\alpha z\) and \(\gamma_1\) from (B.2), then use (B.5) and follow by first substituting the first derivatives of \(\beta\) by \(b_1\) and \(b_2\) and then the first derivatives of \(b_1\) and \(b_2\) by the corresponding expressions in terms of \(Z\), \(W\), \(b_1\), \(b_2\), \(\alpha\), \(\gamma\), \(\beta\) that come from the above relations for \(b_{12}\), \(\beta_{11}\) (and \(\beta_{22}\)). Next substitute \(b_2\) from (B.10) and use the combination of (B.6) with (B.11) so that \(b_2^2\) can be isolated in terms of \(Z\), \(W\), \(\alpha\), \(\gamma\), \(\beta\) only. Finally, use (B.12) to first eliminate the factor \(\gamma W^2\) and then to express the equation in the form \(f b_1 + g W = 0\) for some factors \(f\) and \(g\) not depending on \(b_1\), just like equation (B.13). At this point it is convenient to introduce the definition

\[K = \alpha Z^2 + \gamma W^2 - 2\beta ZW,\]

so that \(K = 2Z(\alpha Z - \beta W)\) because of (B.12) and use it within the factors \(f\) and \(g\) to express first \(Z^2 W^2\) in terms of \(K^2\) and \(ZW K\) and then \(Z^2\) and \(W^2\) (separately) in terms of \(K\) and \(ZW\). Using this procedure the expression for \(\delta_1\) (B.10) can be cast as

\[\begin{align*}
-3[(2\alpha\gamma + \beta^2)K^2 - 2\beta DZW K]b_1 \alpha + \left[(5\beta^2 - 2\alpha\gamma)K^2 - 2\beta DZW K\right]b_1 \beta \\
+ 4\alpha\gamma\beta \frac{D}{r^2}[ -K((10 - 4N)\alpha\gamma - (2 + N)\beta^2) - 4(N + 2)\beta DZW] \\
+ \frac{8(16 - N)\alpha^2\gamma^2\beta^2 D^2}{r^4}\right) W = 0. \quad (B.14)
\end{align*}\]

We already have the equations needed to end the proof: (B.9), (B.13) and (B.14). On top of the above-defined \(K\), we will now make use of the following extra useful definitions:

\[n \equiv \frac{16 - N}{2 + N} \neq 0 \quad \text{(and} \neq -1), \quad \delta \equiv (2 + N)\alpha\gamma\beta \frac{D}{r^2},\]

so that in the stationary vacuum case \(n = 5\) and in the static electrovacuum case \(n = 2\). Let us stress that in the general case \(\beta\) does not have a fixed sign, and thence neither \(\delta\) has, even for \(N > 2\). It is only the complex case that ensures us that \(\beta > 0\) and therefore \(\delta < 0\) for \(N > 2\) (recall that \(\alpha\gamma > 0\), \(\nu^2 D > 0\)).

After using (B.11) to get rid of \(b_2\) and (B.12) together with the above procedure for expressions of the form \(f b_1 + g W = 0\) so that \(f\) and \(g\) depend on \(W\) and \(Z\) only through the factors \(K\) and \(ZW\), equation (B.9) reads

\[3\alpha^2\gamma K(K + n\delta)b_1 + [\alpha\gamma(K + 4\delta)(K + n\delta) - 2\beta^2 K(K - \delta)]W = 0. \quad (B.15)\]

Analogously, equations (B.13) and (B.14) read, respectively,

\[3[(5\alpha\gamma - \beta^2) K - 4DZW K + 4n\delta\alpha\gamma]ab_1 + [(5\alpha\gamma - 9\beta^2) K + 8n\delta\alpha\gamma]W = 0, \quad (B.16)\]
Let us now rewrite (B.6) conveniently as
\[ 9αb_1^2 K = W^2(K + 8δ). \]  
(B.18)
Since \( i^2 K > 0 \) this equation implies that \( i^2(K + 8δ) \geq 0 \). For \( K + 8δ = 0 \) it is necessary and sufficient that \( b_1 = 0 (= b_2) \). In that case, though, (B.15) together with (B.16) lead to \( n = 4 \) (\( N = 8/5 \)). Since we are not interested in that case we can assume \( n \neq 4 \) in the following, so that \( b_1 \neq 0 \) and thus \( i^2(K + 8δ) > 0 \).

The combination of (B.15) and (B.16) that cancels the terms \( δb_1 \) reads
\[ -3αb_1 K(DK - 4βDW) + [-(K^2D + 4[2β^2 - (n - 4)αγ])δK + 16nδ^2αγ]W = 0. \]  
(B.19)
The combination (B.19) – (B.17) leads to
\[ 9K^2αb_1(αγ + β^2) + [3(αγ - 3β^2)K^2 + 4βDW(K + 8δ) + 4αγKδ(n - 2)]W = 0. \]  
(B.20)
On the other hand, let us isolate \( WZ \) from (B.19), use that on (B.20), multiply the result by \( 3Kαb_1 \) and then use (B.18) to get rid of \( b_1^2 \). Again, multiply the result by \( b_1/(2W) \) and use (B.18) to get rid of \( b_1^2 \). The resulting equation, after neglecting the multiplying factors \( W \) and \( K + 8δ \), reads
\[ -3αb_1[(K - 4δ)(2αγ + β^2)K + 2nδαγ] - K[2(nαγ - 2β^2)δ + (2αγ - 5β^2)K]W = 0. \]  
(B.21)
Another useful combination consists of taking (B.15), multiplying it by \( ab_1/W \) and using (B.18) to get rid of \( b_1^2 \) to get
\[ 3αb_1[(αγ - 2β^2)K^2 + (αγ(n + 4) + 2β^2)Kδ + 4nαγδ^2] + αγ(nδ + K)(K + 8δ)W = 0. \]  
(B.22)
Now, the combination \(-4\times (B.15)-(B.21)+2\times (B.22)\) multiplied by 1/(3Kβ^2) leads to
\[ 3αb_1K - (K - 4δ)W = 0. \]  
(B.23)
Proceeding once more by multiplying this equation by \( b_1 \) and using (B.18) to eliminate \( b_1^2 \), we obtain a different relation between \( b_1 \) and \( W \):
\[ 3αb_1(K - 4δ) - (K + 8δ)W = 0. \]  
(B.24)
Finally, isolating \( b_1 \) from the latter and substituting in (B.23) we finally obtain
\[ δW^2(δ - K) = 0, \]  
(B.25)
which now implies \( δ = K \) because we are assuming \( Z \neq 0 \) and \( β \neq 0 \). Now we only need to isolate \( b_1 \) from (B.23) and substitute that onto (B.15) also using \( δ = K \) to obtain
\[ KWαγ(n + 1) = 0, \]  
which contradicts our assumption \( Z \neq 0 \) in the present case. This completes subcase A1.

**Subcase A2:** \( K + \frac{1}{2}(16 - N)αβγD = 0 \). With the above definitions this is \( K + nδ = 0 \). We only have to go back to equation (B.9) and express it in terms of \( K \) to obtain
\[ Kβ(n + 1) = 0, \]  
which contradicts our assumption \( Z \neq 0 \) in the present case. This completes subcase A2 and therefore case A completely.
B.2. Case B

We deal now with $Z = 0$. Let us take $\alpha_1$ from (B.5) and use it on the first equation in (B.2) to get

$$\gamma \alpha_2 - \alpha \gamma_2 + \frac{1}{2} \frac{\alpha \beta_1}{r} + \frac{\alpha}{r} \left(2 \gamma - \frac{1}{2} \beta\right) = 0,$$  \hspace{1cm} (B.26)

followed by $\gamma_2$ from (B.5) to obtain

$$4 \alpha_2 \beta r - 3 \beta_2 \alpha r + 3 \alpha \beta = 0,$$ \hspace{1cm} (B.27)

or equivalently $(\alpha^4 \beta^{-3} \beta')_2 = 0$. The solution is thus of the form $\alpha^4 \beta^{-3} \beta' = \zeta^4(x^1)$ for some analytic (or hyperbolic analytic [23]) function $\zeta(x^1)$. From this equation we have

$$\alpha = \zeta \beta^{3/4} \gamma^{-3/2}, \hspace{1cm} \gamma = \bar{\zeta} \beta^{3/4} r^{-3/2},$$

which used back into (B.26) leads first to

$$\beta^{3/4} r^{9/2} \left(\frac{\beta}{r^{1/2}}\right)_1 = -4 \bar{\zeta}.$$  

The first thing this equation implies is that $\bar{\zeta}$ is real, and thence, by the (generalized) Cauchy–Riemann equations $\partial_1 \Re(\zeta) = \partial_2 \Im(\zeta), \partial_2 \Re(\zeta) = i^2 \partial_2 \Im(\zeta)$, $\zeta$ must be constant. The result

$$\beta = r^2 (ar^{-2} + b)^{3/4}$$

for real constants $a$ and $b$ thus follows. Introducing this solution together with the above expression for $\alpha$ (and $\gamma$) into (B.3) leads to $ab(a + br^2) = 0$, which contradicts $\beta \neq 0$. This completes case $B$ and therefore the proof.

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