Gradient-Descent for Randomized Controllers under Partial Observability

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Abstract. Randomization is a powerful technique to create robust controllers, in particular in partially observable settings. The degrees of randomization have a significant impact on the system performance, yet they are intricate to get right. The use of synthesis algorithms for parametric Markov chains (pMCs) is a promising direction to support the design process of such controllers. This paper shows how to define and evaluate gradients of pMCs. Furthermore, it investigates varieties of gradient descent techniques from the machine learning community to synthesize the probabilities in a pMC. The resulting method scales to significantly larger pMCs than before and empirically outperforms the state-of-the-art, often by at least one order of magnitude.

1 Introduction

Markov chains (MCs) are the common operational model to describe closed-loop systems with probabilistic behavior, i.e., systems together with their controllers whose behavior is described by a stochastic process (Fig. 1(a)). Examples include self-stabilizing protocols for distributed systems \cite{29} and exponential back-off mechanisms in wireless networks. Randomization is also important for robustness in autonomous systems with noisy sensors \cite{59}, obfuscation and (fuzz) test-coverages \cite{19}. Such systems are typically subject to temporal specifications, e.g., with high probability an autonomous system should not crash, and a self-stabilizing protocol should reach a stable configuration in few expected steps. Checking system models against these specifications can be efficiently done using state-of-the-art probabilistic model checking \cite{27, 37}. We highlight that while controllers for these systems operate under partial information, the analysis of a system with controller does not need to take partial observability into account.

One step beyond verification is the correct-by-construction synthesis of controllers for such systems via Partially Observable Markov Decision Processes (POMDPs) (Fig. 1(b)). In general, the synthesis for partial-information controllers is undecidable \cite{4, 21, 41}. Syntax-guided synthesis \cite{2} takes a simpler perspective and synthesizes only particular system aspects starting from a user-provided

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template. In this paper, we focus on being provided with a template controller with a fixed memory structure (influencing the number of indistinguishable states) and a fixed set of potential actions that we want to randomize over. This setting is useful, as in many systems one randomizes on purpose, e.g., in distributed protocols to break symmetry or for robustness. In particular, the randomization is controllable, but selecting a (near-)optimal way to randomize is non-trivial.

The synthesis task reduces to randomize appropriately in a system with a fixed topology (Fig. 1(c)). In this context, a controller selects a fixed set of actions (of the POMDP) \( \alpha_1, \ldots, \alpha_n \) with probabilities \( p_1, \ldots, p_n \). The aim is to synthesize a realizable controller, that is, the result of the synthesis should not enforce to randomize differently in indistinguishable states — such a controller depends on information which is not available at runtime and therefore cannot be implemented. Consequently, for indistinguishable states, a realizable controller must take an action \( \alpha \) with the same probability \( p_i \). Synthesizing such controllers can be formally described \[33\] as feasibility synthesis in parametric Markov chains (pMCs), i.e., MCs with symbolic probabilities \( p_1, \ldots, p_n \) \[14,38\]. The feasibility synthesis task asks to find values \( u_1, \ldots, u_n \) for the parameters such that the MC satisfies a given property. This problem has been studied extensively in the literature, e.g. in \[11,12,20,22,49\], see also the related work section.

Example 1. Fig. 2(a) depicts a POMDP. The colors match the observations at a state. When observing a red state, \( s_1 \) or \( s_3 \), with probability \( q_1 \) action \( \alpha_1 \) is taken and with probability \( q_2 \) action \( \alpha_2 \). At state \( s_0 \) action \( \alpha_i \) is taken with probability \( p_i \). This directly results in the pMC of Fig. 2(b).

The challenge in applying parameter synthesis is twofold: whereas the problem is ETR-complete \[34\], the number of parameters grows linear in the number of different observations and the number of actions available to the controller. For many real-life applications we must thus deal with thousands of parameters. This scale is out of reach for exact or complete methods \[15\]. Heuristic methods have shown some promise. These methods either rely on efficient model checking but are heavily sample-inefficient \[11\], or rely on the efficiency of convex solvers to search the parameter space in a more principled way \[13\].

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4 ETR = Existential Theory of the Reals. ETR-complete decision problems are as hard as finding the roots of a multivariate polynomials.
This paper presents a novel method that advances the state-of-the-art in feasibility synthesis often by one or more orders of magnitude. The method is rooted in two key observations:

– gradient-based search methods, i.e., variants of gradient search, scale to high-dimensional search spaces, and
– in pMCs, the gradient at a parameter evaluation can be efficiently evaluated.

In this paper, we show a principled way to evaluate gradients in parametric MCs. We characterize gradients as solutions of a linear equation system over the field over rational functions and alternatively as expected rewards of an automaton that is easily derived from the pMC at hand. Using the efficient computation of gradients, we evaluate both classical (Plain GD, Momentum GD [51], and Nesterov accelerated GD [46, 58]) and adaptive (RMSProp [60], Adam [36], and RAdam [39]) gradient descent methods. We also consider the classical gradient descent methods where we only respect the sign of the gradient. Furthermore, we investigate various methods (projection, barrier function, logistic function) to deal with restrictions on the parameter space (e.g. parameters should represent probabilities). Using an empirical evaluation, we show that 1) projection outperforms the other restriction methods, 2) Momentum-Sign outperforms the other gradient descent methods, and 3) Momentum-Sign often outperforms state-of-the-art methods QCQP and PSO. Moreover, we discuss some domain-specific properties and the consequences for gradient descent.

We formalize our problem statement in Sec. 2.3, discuss the evaluation of gradient in Sec. 3, consider the use of gradient descent in Sec. 4, give an empirical evaluation in Sec. 5 and discuss related work in Sec. 6. Sec. 7 concludes and provides pointers for future work.
2 Preliminaries

2.1 Parametric Markov Chains

Let \( V \) be a set of \( n \) real-valued parameters (or variables) \( p_1, \ldots, p_n \). Let \( \mathbb{R}[V] \) denote the set of multivariate polynomials over \( V \).

A parameter instantiation is a function \( u: V \to \mathbb{R} \). We often denote \( u \) as a vector \( \vec{u} \in \mathbb{R}^n \) by ordering the set of variables \( V = \{ p_1, \ldots, p_n \} \) and setting \( u_i = u(p_i) \). We assume that all parameters are bounded, i.e., \( lb_i \leq u(p_i) \leq ub_i \) for each parameter \( p_i \). Let \( R_i = [lb_i, ub_i] \) denote the bounds for parameter \( p_i \) in region \( R \). The parameter space of \( V \), denoted \( \mathcal{U} \subseteq \mathbb{R}^V \), is the set of all possible parameter values, i.e. the hyper-rectangle spanned by the intervals \( [lb_i, ub_i] \). A set \( R \subseteq \mathcal{U} \) of instantiations is called a region.

A polynomial \( f \) can be interpreted as a function \( f: \mathbb{R}^n \to \mathbb{R} \) where \( f(u) \) is obtained by substitution, i.e. in \( f(u) \) each occurrence of \( p_i \) in \( f \) is replaced by \( u(p_i) \). To make clear where substitution occurs, we write \( f[u] \) instead of \( f(u) \) from now on. We let \( \partial_p f \) denote the partial derivative of \( f \) with respect to \( p \).

Let \( X \) be any set and let \( \text{pFun}(X) = \{ f \mid f: X \to \mathbb{R}[V] \} \) denote the set of generalized functions. Now, let \( \text{pDistr}(X) \subseteq \text{pFun}(X) \) denote the set of parametric probability distributions over \( X \), i.e., the set of functions \( \mu: X \to \mathbb{R}[V] \) such that \( 0 \leq \mu(x)[u] \leq 1 \) and \( \sum_{x \in X} \mu(x)[u] = 1 \) for all \( u \) in the parameter space \( \mathcal{U} \).

Definition 1. A parametric Markov chain (pMC) is a tuple \( \mathcal{M} = (S, s_I, T, V, \mathcal{P}) \) with a finite set \( S \) of states, an initial state \( s_I \in S \), a finite set \( T \subseteq S \) of target states, a finite set \( V \) of real-valued variables (parameters) and a transition function \( \mathcal{P}: S \to \text{pDistr}(S) \).

The parametric probability of going from state \( s \) to \( t \), denoted \( \mathcal{P}(s, t) \), is given by \( \mathcal{P}(s)(t) \). A pMC with \( V = \emptyset \) is a Markov chain (MC). We will use \( \mathcal{M} \) to range over pMCs and \( \mathcal{D} \) to range over MCs. Applying an instantiation \( u \) to a pMC \( \mathcal{M} \) yields MC \( \mathcal{M}[u] \) by replacing each transition \( f \in \mathbb{R}[V] \) in \( \mathcal{M} \) by \( f[u] \). An instantiation \( u \) is graph-preserving (for \( \mathcal{M} \)) if the topology of \( \mathcal{M} \) is preserved, i.e., \( \mathcal{P}(s, s') \neq 0 \) implies \( \mathcal{P}(s, s')[u] \neq 0 \) for all states \( s, s' \). A region \( R \) is graph-preserving if all \( u \in R \) are graph-preserving.

Example 2. Fig. 3(a) depicts pMC \( \mathcal{M} \) with a single parameter \( p \). Region \( R = [0.1, 0.9] \) is graph-preserving, while \( R = [0, 0.9] \) is not graph-preserving.

We fix an MC \( \mathcal{D} \). Let \( \text{Paths}(s) \) denote the set of all infinite paths in \( \mathcal{D} \) starting from \( s \), i.e., infinite sequences of the form \( s_0s_1s_2 \ldots \) with \( s_0 = s \) and \( \mathcal{P}(s_i, s_{i+1}) > 0 \). A probability measure \( \text{Pr}_D \) is defined on measurable sets of infinite paths using a standard cylinder construction; for details, we refer to, e.g., [6] Ch. 10. For \( T \subseteq S \) and \( s \in S \), let

\[
\text{Pr}_D(s \models \diamond T) = \text{Pr}_D\{ s_0s_1s_2 \ldots \in \text{Paths}(s) \mid \exists i, s_i \in T \}
\]  

(1)

denote the probability to eventually reach some state in \( T \) from \( s \). For a pMC \( \mathcal{M} \), the reachability probability depends on the parameters and so we define it
as a function $\Pr_{\mathcal{M}}^{s\rightarrow T}: \mathcal{U} \rightarrow [0,1]$ given by $\Pr_{\mathcal{M}}^{s\rightarrow T}[u] = \Pr_{\mathcal{M}[u]}(s = \triangledown T)$ \cite{14}. For conciseness we typically omit the subscript $\mathcal{M}$ and write $\Pr^{s\rightarrow T}$. Zero and one reachability probabilities are preserved for graph-preserving instantiations, i.e., for all graph-preserving $u, u' \in \mathcal{U}$, we have $\Pr^{s\rightarrow T}[u] = 0$ implies $\Pr^{s\rightarrow T}[u'] = 0$ and analogously for $= 1$. In these cases, we just write $\Pr^{s\rightarrow T} = 0$ or $= 1$. Let $\mathcal{S}$ denote all states $s \in S$ with $\Pr^{s\rightarrow T} = 0$. W.l.o.g., we assume that there is at most one $\triangledown$ state (this is standard preprocessing \cite[Ch. 10]{6}). Furthermore, we merge all states $s \in T$ into a single $\triangledown$ state.

Example 3. For all states $s \in S$ in pMC $\mathcal{M}$ from Fig. 3(a), we have $\Pr^{s\rightarrow \triangledown} = 1$. Therefore, the pMC $\mathcal{M}$ has no $\triangledown$ state.

2.2 Expected Rewards

We are not only concerned with reachability probabilities but also with expected rewards. Let state reward function $\text{rew}: S \rightarrow \mathbb{R}$ associate a reward to each state. The cumulative reward for a finite path $\hat{\pi} = s_0s_1\ldots s_n$ is defined by:

$$\text{rew}(\hat{\pi}) = \text{rew}(s_0) + \text{rew}(s_1) + \ldots + \text{rew}(s_{n-1}).$$

For infinite paths $\pi = s_0s_1s_2\cdots$ the reward to eventually reach $\triangledown$ in $\mathcal{M}$ is:

$$\text{rew}(\pi, \triangledown) = \begin{cases} \text{rew}(s_0s_1\ldots s_n) & \text{if } s_i \neq \triangledown \text{ for } 0 \leq i < n \text{ and } s_n = \triangledown \\ \infty & \text{if } \pi \not\equiv \triangledown. \end{cases}$$

Remark 1. For the sake of simplicity, we restrict ourselves to constant rewards. However, all notions and concepts considered in the remainder of this paper can be generalized to parametric reward functions in a straightforward manner.

Remark 2. From now on, we only consider graph-preserving regions and we restrict ourselves to pMCs where every state $s$ eventually reaches $\triangledown$ almost surely, i.e., $\Pr^{s\rightarrow \triangledown} = 1$. 

Fig. 3. A (left) sample parametric MC and (right) its derived weighted automaton.
Definition 2 (Expected reward). The expected reward until reaching ♦ from $s \in S$ for an MC $D$ is defined as follows:

$$\text{ER}_D(s \models ♦) = \int_{\pi \models ♦} \text{rew}(\pi, ♦) \cdot \text{Pr}(\pi).$$

The expected reward for a pMC $M$ is defined analogously, but as a function $\text{ER}_{s_0\rightarrow ♦} : U \rightarrow \mathbb{R}$, given by $\text{ER}_{s_0\rightarrow ♦}(u) = \text{ER}_{M[u]}(s \models ♦)$. Again, for conciseness we typically omit the subscript $M$.

Example 4. Reconsider the pMC $M$ from Fig. 3(a) with a state reward function $\text{rew}(s_i) = i$ for $s_i \in S \setminus \{♦\}$. The expected reward function $\text{ER}^{s_0\rightarrow ♦}$ is given by:

$$3 \cdot p^2 + 4 \cdot p \cdot (1-p) + 2 \cdot (1-p) = -p^2 + 2 \cdot p + 2.$$

On a graph-preserving region, the function $\text{ER}^{s_0\rightarrow ♦}$ is always continuously differentiable and admits a closed-form as a rational function over $V$.

Remark 3. Reachability probabilities are obtained by using expected rewards by letting $\text{rew}(s) = 0$ for $s \in S \setminus \{♦\}$ and $\text{rew}(♦) = 1$. We add one sink state $s'$ s.t. $\mathcal{P}(s, s') = 0$ if $s \in S \setminus \{♦, ♦\}$ and $\mathcal{P}(s, s') = 1$ otherwise. The quantity $\text{ER}^{s_0\rightarrow s'}$ now equals the reachability probability of eventually reaching ♦.

2.3 Problem Statement

This paper is concerned with the question of synthesising a randomized controller under partial observability. Synthesizing these controllers can formally be described as feasibility synthesis in pMCs. Therefore, we consider the following question on the expected reward of eventually reaching a target state ♦ in a given pMC $M$ and a graph-preserving region $R$:

Given $\lambda \geq 0$, and comparison operator $\sim$, find an instantiation $u \in R$ with:

$$\text{ER}_{M[u]}(s \models ♦) \sim \lambda.$$

To solve this problem, we first show how to compute the derivative of $\text{ER}^{s_0\rightarrow ♦}$ and introduce a derived weighted automaton. Then, we exploit this derivative by considering several gradient descent methods and applying them to solve our problem. Finally, we show how our approach experimentally compares to existing methods from.

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5 Technically, we use graph-preserving to ensure continuously differentiability of $\text{ER}^{s_0\rightarrow ♦}$.

For acyclic pMCs, these functions are continuously differentiable without assuming graph-preservation.
3 Computing Gradients for Expected Rewards

In this section, we show that we can efficiently evaluate the gradient of the function \( ER^{s \rightarrow \omega} \) with respect to a parameter \( p \) at an instantiation \( u \). We note that first computing \( ER^{s \rightarrow \omega} \) and deriving this function symbolically is intractable: the function can be exponentially large in the number of parameters [5]. A tractable construction follows from taking the derivative of the equation system that characterizes the expected reward [6, Ch. 10]. Alternatively, it can be obtained as an equation system for the expected rewards of a “derived” pMC. Let \( M = (S, s_I, \{ \omega \}, V, \mathcal{P}) \) with reward function \( rew \) and parameter \( p \in V \).

3.1 Equation-System Based Characterisation

Definition 3. The system of equations for the partial derivative of \( ER^{s \rightarrow \omega}_M \) w.r.t. \( p \) \( \in V \) is given by:

\[
x_s = 0, \quad \partial_p x_s = 0 \quad \text{if } s = \omega
\]

\[
x_s = rew(s) + \sum_{s' \in S} \mathcal{P}(s, s') \cdot x_{s'} \quad \text{for } s \in S \setminus \{ \omega \}
\]

\[
\partial_p x_s = \sum_{s' \in S} \left( \partial_s \mathcal{P}(s, s') \cdot x_{s'} + \mathcal{P}(s, s') \cdot \partial_p x_{s'} \right) \quad \text{for } s \in S \setminus \{ \omega \}.
\]

where \( \partial_s \mathcal{P}(s, s') \) is the derivative of the probability function \( \mathcal{P}(s, s') \) w.r.t. \( p \).

Note that we obtain the derivative for \( x_s \), i.e. \( \partial_p x_s \), by applying the sum rule and the product rule to \( x_s \). This equation system is equivalent to an equation system for POMDPs in [1] p.47-48. We remark that the equation system is linear with coefficients in a polynomial ring. However, if the parameters are considered to be variables, then the system of equations is nonlinear (and nonconvex) [12]. Observe that the equations for \( x_s \) do not depend on the equations for \( \partial_p x_s \) and thus can be solved independently first. The equations for \( x_s \) have a unique solution which coincides with \( ER^{s \rightarrow \omega}_M \). This is a known result for MCs [6, Ch. 10] and carries over to pMCs [31]. We show below that the equation system for \( \partial_p x_s \) has a unique solution as well and yields the derivative \( \partial_p ER^{s \rightarrow \omega}_M \).

Example 5. For our running example we obtain the following equation system:

\[
\begin{align*}
x_0 &= 0 + p \cdot x_1 + (1-p) \cdot x_2 & \partial_p x_0 &= 1 \cdot x_1 + p \cdot \partial_p x_1 + \partial_p x_2 + (1-p) \cdot \partial_p x_2 \\
x_1 &= 1 + p \cdot x_2 + (1-p) \cdot x_3 & \partial_p x_1 &= 1 \cdot x_2 + p \cdot \partial_p x_2 + \partial_p x_3 + (1-p) \cdot \partial_p x_3 \\
x_2 &= 2 + 1 \cdot x_\omega & \partial_p x_2 &= 1 \cdot \partial_p x_\omega \\
x_3 &= 3 + 1 \cdot x_\omega & \partial_p x_3 &= 1 \cdot \partial_p x_\omega \\
x_\omega &= 0 & \partial_p x_\omega &= 0.
\end{align*}
\]

Solving these equations yields \( x_0 = -p^2 + 2 \cdot p + 2 \), the expected reward function \( ER^{\omega \rightarrow \omega}_M \), see Example 4 and \( \partial_p x_0 = -2 \cdot p + 2 \), i.e., \( \partial_p ER^{\omega \rightarrow \omega}_M \).
Theorem 1. The equation system of Def. 3 has exactly one solution: $x_s$ equals $\text{ER}^{s\rightarrow u}$ and $\partial_p x_s$ equals $\partial_p \text{ER}^{s\rightarrow u}$ for each $s \in S$.

The proof is given in Appendix A.

From a computational point, we notice that computing $\partial_p \text{ER}^{s\rightarrow u}$ by solving the equation system (over the field of rational functions $\mathbb{R}(V)$) is intractable, as this function may be exponential in the number of parameters. Matters appear worse as we aim to compute the derivative w.r.t. to a subset of the parameters $V' \subseteq V$, rather than with respect to a single parameter. However, we observe that, for a gradient descent, we are only interested in computing $\left(\partial_p \text{ER}^{s\rightarrow u}\right)[u]$, and the equation system can be solved efficiently when we substitute all $P(s,s')$ by $P(s,s')[u]$ and solve for $\left(\partial_p \text{ER}^{s\rightarrow u}\right)[u]$ using constant coefficients from the rationals or reals. Furthermore, as the $x_s$ variables can be solved independently of the $\partial_p x_s$ variables, we first solve the $x_s$-equation system with $|S|$ variables and equations. In a second step, we construct for every $p \in V'$ an equation system (with $|S|$ variables and equations) by directly substituting the $x_s$ variables with the expected reward $\text{ER}^{s\rightarrow u}[u]$. In total, this means that we evaluate $(|V'|+1)$ equation systems with $|S|$ equations and variables each.

### 3.2 Derived Automaton

We now show that an alternative way to obtain $\partial_p \text{ER}^{s\rightarrow u}$ is by the standard equation system for $\text{ER}^{s\rightarrow u}$ on the “derivative” of pMC $M$. To that end, we mildly generalize pMCs to (parametric) weighted automata [16] and show that we can describe “taking the derivative” as an operation on these weighted automata. We do so by relaxing our parametric probability distributions by dropping the requirement that $0 \leq \mu(x)[u] \leq 1$; in particular, negative real values are allowed. These functions are called quasi-distributions as $\sum_{x \in X} \mu(x)[u] = 1$ still holds. Let $pQDist(X) \subset pFun(X)$ denote the set of quasi-distributions.

Definition 4. A weighted finite automaton (WFA) is a tuple $A = (S, s_I, T, V, E)$ where $S$, $s_I$, $T$, $V$ are as in Def. 1 and $E : S \rightarrow pQDist(S)$.

Example 6. Fig. 3(b) depicts WFA $A$ with single parameter $p$. Note that some of the transitions are labelled with $p$ and $1-p$ (as in Fig. 3(a)). We will later explain the relation of this WFA to the pMC in Fig. 3(a).

Instead of creating a system of equations to compute the derivative, we can alternatively construct an automaton which has the derivative as its semantics. This is called the derived weighted automaton. Intuitively, the automaton $\partial_p M$ of a pMC $M$ is constructed by applying product and sum rules directly to $M$.

Definition 5. Let $M = (S, s_I, T, V, \mathcal{P})$ be a pMC with reward function $\text{rew}$ and let $p \in V$ a parameter. The derived weighted automaton of $M$ w.r.t. $p$ is the WFA $\partial_p M = (S', \partial_p s_I, T, V, E)$ with the reward function $\text{rew}'$ where

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6 In our implementation, we support exact rationals or floating point arithmetic.
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– \( S' = S \cup \partial_p S \) with \( \partial_p S = \{ \partial_p \ s \mid s \in S \} \).

– the transition function \( E \) is given by:

\[
E(s,t) = \begin{cases} 
\mathcal{P}(s,t) & \text{if } s, t \in S, \\
\mathcal{P}(s',t') & \text{if } s, t \in \partial_p S \text{ and } s = \partial_p s' \text{ and } t = \partial_p t', \\
\partial_p \mathcal{P}(s',t) & \text{if } s \in \partial_p S \text{ and } s = \partial_p s' \text{ and } t \in S, \\
0 & \text{otherwise},
\end{cases}
\]

– the reward function \( \text{rew}' \) is given by \( \text{rew}'(s) = \text{rew}(s) \) for \( s \in S \) and \( \text{rew}'(s) = 0 \) for \( s \in \partial_p S \).

The intuition behind this derived automaton is as follows. “Deriving” the state \( s \in S \) with respect to \( p \in V \) yields the new state \( \partial_p s \). For every transition \( \mathcal{P}(s, s') \neq 0 \) for \( s, s' \in S \), we “use the product rule” and add the transitions \( \mathcal{P}(\partial_p s, \partial_p s') = \mathcal{P}(s, s') \) and \( \mathcal{P}(\partial_p s, s') = \partial_p \mathcal{P}(s, s') \) to \( \partial_p \mathcal{M} \).

Example 7. Applying Def. 5 to the pMC \( \mathcal{M} \) from Fig. 3(a) results in the derived weighted automaton \( \partial_p \mathcal{M} \) in Fig. 3(b).

Note that although \( \partial_p \mathcal{M} \) is not a pMC as some transitions have negative weights, the parametric expected reward \( \text{ER}^{\partial_p s_1 \to \mathcal{M}}_{\partial_p \mathcal{M}} \) can be computed as in Def. 3 as we restrict ourselves to graph-preserving regions, ensuring continuously differentiability of \( \text{ER}^{s \to \mathcal{M}}_u \). The derivative of the expected reward in \( \mathcal{M} \) can now be obtained as the parametric expected reward \( \text{ER}^{\partial_p s_1 \to \mathcal{M}}_{\partial_p \mathcal{M}} \) in \( \partial_p \mathcal{M} \).

Proposition 1. For each pMC \( \mathcal{M} \) we have: \( \text{ER}^{\partial_p s_1 \to \mathcal{M}}_{\partial_p \mathcal{M}} = \partial_p \text{ER}^{s \to \mathcal{M}}_u \).

Stated in words, the expected reward of the derived automaton \( \partial_p \mathcal{M} \) equals the partial derivative of the expected reward of the pMC \( \mathcal{M} \).

4 Gradient Descent

Gradient descent (GD) is a first-order optimisation technique to maximize an objective function \( f(u) \). It updates the GD parameters in the direction of its gradient \( \partial_p f(u) \). We want to use GD to solve the problem introduced in Sec. 2.3, i.e., given \( \lambda \geq 0 \) and comparison operator \( \sim \), find an instantiation \( u \in R \) with: \( \text{ER}^{\sim \to u}_{\mathcal{M}}[u] \sim \lambda \).

We consider several GD update methods (Plain GD, Momentum GD [51], and Nesterov accelerated GD [46,58], RMSProp [60], Adam [36], and RAdam [39]). Three variants of GD are common in the literature. Batch GD computes the gradient of \( f \) w.r.t. all parameters. In contrast, stochastic GD performs updates for each parameter separately. Mini-batch GD sits in between and performs an update for a subset of parameters. We describe the GD update methods w.r.t. 7 It is only based on the first derivative and not on higher ones.
Algorithm 1 GD

1: while \( f[u] \leq \lambda \) do
2:   if \( u \) is a local optimum then
3:      pick new \( u \)
4:   update \( u \) with GD-method
5: return \( u \)

4.1 Plain GD

Plain GD is the simplest type of GD. A fixed learning rate \( \eta \) is used to determine the step size taken to reach a (local) maximum. The parameter \( p_i \) gets updated in \( u \) based on \( \partial_{p_i} f[u] \) as follows:

\[
u_{t+1}^{i} = u_{t}^{i} + \eta \cdot \partial_{p_i} f[u_{t}^{i}],\]

where \( u_{t}^{i} = u^t(p_i), \) i.e., the value of \( p_i \) with instantiation \( u^t \).

Example 8. Consider \( f(p) = \frac{1}{2}p^4 - 4p^3 + 9p^2 - 4p + 2 \) on a region \( R = [0, 3] \). Assume that our initial instantiation is \( u^0(p) = 1 \) and that we take \( \eta = 0.1 \) and \( \lambda = 5.9 \). The red halfdots in Fig. 4(a) illustrate how the value of \( p \) changes over time when using Plain GD. The blue dot indicates the optimum. At \( t = 0 \), the gradient is 4 and so \( p \) is updated to 1.4. For \( t = 1 \), the gradient is 3.17, increasing \( p \) again. This is repeated until at \( t = 3 \), we have \( f[u^t] = 5.96 \). As this value exceeds \( \lambda \), a feasible instantiation \( (p = 2.08) \) is found.
4.2 GD Update Methods

Intuitively, all GD methods attempt to “guess” how the gradient will change by guiding the search for maxima based upon the past behaviour of the gradient. Many GD optimization methods exist and a recent overview is given by Ruder [50]. We consider the following methods: Momentum, Nesterov accelerated GD (NAG), RMSProp, Adam, and RAdam. Momentum and NAG are classical and very similar to Plain GD. The latter three are adaptive algorithms, i.e., their learning rate is changing over time and each parameter has its own learning rate. Parameters with larger gradients have smaller learning rates than the ones with smaller gradients. The latter three have been developed for machine learning purposes [39]. We will elaborate on the Momentum and NAG method and briefly sketch the other methods.

**Momentum** [51]. Instead of only considering the current derivative, the Momentum method also takes into consideration previous derivatives. They are weighted by the average decay factor $\gamma \in [0, 1]$ (typically at least 0.9). This method uses an additional update vector $v$. Momentum GD adjusts the parameter value according to the following equation. (Note that, if $\gamma = 0$, Momentum GD is equal to Plain GD.)

$$v_{i+1}^t = \gamma \cdot v_i^t + \eta \cdot \partial p_i[f[u_i^t]] \tag{2}$$
$$u_{i+1}^t = u_i^t + v_{i+1}^t \tag{3}$$

**Nesterov accelerated GD (NAG)** [46, 58]. As for Momentum GD, NAG weighs the past steps by $\gamma$. Additionally, it attempts to predict the future by guessing the next instantiation of $u$, denoted $u'$ (Eq. (4)). This should prevent us from moving to the other side of the optimum (Example 9). As for Momentum, the instantiation is updated according to Eq. (3), whereas the update vector is obtained as in Eq. (5):

$$u_j' = \begin{cases} u_j^t - \gamma \cdot v_j^t & \text{if } j = i \\ u_j^t & \text{otherwise} \end{cases} \tag{4}$$
$$v_{i+1}^t = \gamma \cdot v_i^t + \eta \cdot \partial p_i[f[u']]. \tag{5}$$

**Example 9.** Reconsider our running example. Figs. 4(b) and 4(c) show how the value of $p$ changes over time using Momentum GD and NAG respectively. Note that for both methods we need one step less compared to Plain GD, i.e., a feasible instantiation is found at $t = 2$. This is due to taking results of previous steps into account. Furthermore, observe that for Momentum GD at $t = 2$ the instantiation of $p$ actually passed the optimum, whereas for NAG this does not occur.

**Adaptive methods.** RMSProp (Root Mean Square Propagation) [60] is akin to Momentum and NAG, but its learning rate is adapted based on the previous squared gradient (Eq. (7)). This squared gradient is recursively defined as the
sum of $\beta \in [0, 1)$ times the past squared gradient, and $1 - \beta$ times the current squared gradient (Eq. (6)). $\beta$ is called the squared average decay. In Eq. (7) a small amount $\epsilon > 0$ is added to the update vector at $p_i$ to avoid division by zero.

\[
v_{i+1}^t = \beta \cdot v_i^t + (1 - \beta) \cdot (\partial_{p_i} f[u])^2
\]

\[
u_{i+1} = u_i^t + \frac{\mu}{\sqrt{v_i^t + \epsilon}} \cdot \partial_{p_i} f[u]. \tag{7}
\]

In addition to the mean, Adam (Adaptive Moment Estimation) [36] takes the second moment (the uncentered variance) of the gradients into account. RAdam (Rectified Adam) [39] solves an issue with Adam in which the variance of learning rate is too large in the initial steps of the algorithm.

Sign methods [43]. For the non-adaptive methods, we additionally implemented variants that only respect the signs of the gradients and not their magnitudes. That is, we update the parameter as

\[
u_{i+1}^t = u_i^t + \eta \cdot \text{sgn}(\partial_{p_i} f[u]).
\]

Note that this implies we don’t need to calculate the full gradient.

4.3 Dealing with parameter regions

So far we dealt with unconstrained GD. However, as a graph-preserving region $R$ is given, we need to deal with parameter values getting out of $R$. To do so, we discuss the following methods: Projection, Penalty Function, Barrier Function, and logistic Function. Recall that, $R_i = [lb_i, ub_i]$ denotes the bound for parameter $p_i$ in region $R$.

Projection. The projection method acts as a hard wall around the region. As soon as $u_i \notin R_i$, $u_i$ gets set to the bound of the region, i.e., $u_i^{t+1} = \min(\max(u_i^t, lb_i), ub_i)$. Furthermore, if the parameter $p_i$ got out of the given region, we set its past gradients to 0, i.e. $v_i^{t+1} = 0$.

Example 10. Reconsider our running example. However, now consider region $R' = [0.5, 1.5]$. For $t = 0$, the gradient is 4, and $p$ is updated to 1.4. For $t = 1$, the gradient is 3.17, yielding $p$ to be updated to 1.72. As this is out of the region $R'$, $p$ is projected to 1.5.

Penalty function. The penalty method [53] transforms the constrained problem into an unconstrained one, by adding a penalty function to $f[u_i]$. This penalty depends on how bad the violation is, e.g. what the difference is between $u_i$ and the bounds of $R_i$. It can be interpreted as a red warning zone outside of the region. As this might yield non-graph-preserving instantiations, we do not further look into this.
Barrier function. The barrier function [61] (also called indicator function) works as a soft wall inside of the region, discouraging one to get close to the wall. It is independent of how bad the violation is. We consider the log-barrier function for maximizing \( f \) (see Eqs. (10)-(11c)) as this yields a differentiable function. The barrier function is weighted by \( \mu \in [0, 1] \). The equations are:

\[
\begin{align*}
    f[u^t] &= f[u^t] + \mu \cdot \text{bar}[u^t] \\
    \partial_p f'[u^t] &= \partial_p f[u^t] + \mu \cdot \partial_p \text{bar}[u^t] \\
    \text{bar}[u^t] &= \sum_i \text{bar}_i[u^t]
\end{align*}
\]

\[
\text{bar}_i[u] = \begin{cases} 
    \log(u_i - lb_i) & \text{if } lb_i + \frac{ub_i - lb_i}{2} < u_i \text{ and } u_i \in R_i \\
    \log(ub_i - u_i) & \text{if } lb_i + \frac{ub_i - lb_i}{2} \geq u_i \text{ and } u_i \in R_i \\
    -\infty & \text{otherwise.}
\end{cases}
\]

\[
\partial_p \text{bar}_i[u] = \begin{cases} 
    1 & \text{if } lb_i + \frac{ub_i - lb_i}{2} < u_i \text{ and } u_i \in R_i \\
    \frac{1}{ub_i - u_i} & \text{if } lb_i + \frac{ub_i - lb_i}{2} \geq u_i \text{ and } u_i \in R_i \\
    \infty & \text{otherwise.}
\end{cases}
\]

Note that for higher learning rates, the barrier function might not be strong enough to prevent \( u_i \not\in R \), see also the upcoming example.

Example 11. Reconsider our running example with \( \mu = 0.1 \). We observe that at all \( t \) where \( u_i \in R_i \) case Eq. (11a) applies, so the barrier function is given by \( \text{bar}^t = \log(1.5 - p) \). For learning rate 0.1, at \( t = 0 \), the gradient is \( 4 - \mu \cdot \frac{1}{1.5 - p} \), so \( p \) is updated to 1.38. For \( t = 1 \), the gradient is 0.24. So \( p \) is updated to 1.62, which is outside region \( R' \). When considering a smaller learning rate, e.g. 0.01, at \( t = 0 \) \( p \) is updated to 1.038. This converges around \( t = 30 \) with \( p \approx 1.46 \in R' \).

Logistic function. For the logistic function, we map each restricted parameter \( p_i \) to unrestricted parameter \( q_i \) by using a sigmoid function [24] (see Eq. (13)) tailored to \( R_i \). We denote instantiations of \( q \) with \( u' \). \( u',_0 \) is the value of the sigmoid’s midpoint. \( u' \) gets updated according to the GD method. The gradient \( (v'_i) \) at \( u' \) is computed according to Eq. (14).

\[
\begin{align*}
    u',_0 &= \frac{ub_i - lb_i}{2} \\
    u_i &= \frac{ub_i - lb_i}{1 + e^{-(u',_0 - u',)}} + lb_i \\
    v'_i[u'] &= \frac{e^{u'_i} \cdot v_i[u]}{(1 + e^{u'_i})^2}.
\end{align*}
\]

\[^8\text{When considering a minimization problem, \( \text{bar} \) is subtracted from } f.\]
Example 12. Reconsider our running example. Let the learning rate be 0.1, and \( u^0(q) = 0.5 \). The sigmoids midpoint is \( u'_{0,i} = 0.5 \). For \( t = 0 \), we have \( u^0_i = 1 \). The gradient at this point \( v^0_i[u^0] = 0.94 \), so \( q \) is updated to 0.59. Therefore, \( p \) is set to 1.02. At each iteration \( p \) and \( q \) get updated. E.g. at \( t = 100 \), \( q = 3.63 \) and \( p = 1.45 \).

5 Empirical Evaluation

We implemented all gradient descent methods from Sec. 4 in the probabilistic model checker Storm [27]. All parameters, i.e. batch-size, learning rate, average decay and squared average decay, are configurable via Storm’s command line interface. We evaluate the different gradient descent methods and compare them to two baselines: One approach based on Quadratically-Constrained Quadratic Programming (QCQP) [13], a convex optimization-based method, and the sampling-based approach Particle Swarm Optimization (PSO) [11]. These baselines are implemented in the tool PROPhESY [15]. All methods use the same version of Storm for model building, simplification, model checking, and solving of linear equation systems. We specifically answer the following questions experimentally:

Q1 Which region restriction method works best?
Q2 Which GD methods works best?
Q3 How does GD compare to previous techniques (QCQP and PSO)?

5.1 Set-up

We took the approach as described in Sec. 3.1, i.e., one sparse matrix is created per parameter and instantiated at the current position. Our implementation works with Mini-Batch GD as described above. This means that we compute the derivative w.r.t. \( k \) parameters and then perform one step. We allow for stochastic GD and batch GD by setting \( k \) to 1 or \( |V| \), respectively.

For the experiments, we solve equation systems with GMRES from the \texttt{gmm++} linear equation solver library included in Storm, which uses floating-point arithmetic. All experiments run on a single thread and perform some preprocessing (e.g. bisimulation minimization). The times reported are the runtimes for GD, PSO and QCQP and do not include preprocessing. We set a time-out of two hours. We have used machines with an Intel Xeon Platinum 8160 CPU and 32GB of RAM. In the comparisons with QCQP and PSO, we report the average runtime over five runs.

Settings. For all constants except the learning rate, we chose the default from the literature (e.g., [36, 39, 50, 60]), i.e. we set the batch size \( k \) to 32, average decay \( \gamma \) to 0.9 and squared average decay \( \beta \) to 0.999. Whereas in the literature the learning rate is often set between 0.001 and 0.1, we stick to 0.1. As we are interested in finding a feasible instantiation, we can take the risk of jumping over
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a local optimum due to a too high learning rate. Also, our experiments show that lower learning rates slow down the search process (see Fig. 5). Furthermore, we start at \( u_i = 0.5 + \varepsilon \) for all parameter \( p_i \) with \( \varepsilon = 10^{-6} \), to overcome possible saddle points at \( p_i = 0.5 \). After every parameter has performed a step of less than \( 10^{-6} \) in sequence, we conclude a local optimum has been found (we are aware this is an impatient criterion, tweaking this is a matter for further research). When an infeasible local optimum is found, a new starting point is selected randomly (see Algorithm 1, line 3). Consequently, the GD methods may yield different runtimes on different invocations on the same benchmark, though in practice we observe only a small deviation in the runtimes. For the barrier region restriction method, we initially set \( \mu \) to 0.1. If no feasible solution is found, we divided \( \mu \) by 10. We continue this procedure until a feasible solution is found, or \( \mu < 10^{-6} \).

Benchmarks. We consider pMCs obtained from POMDPs (cf. [33]) and Bayesian networks (cf. [52]) with a large number of parameters. We took at least one variant of all POMDPs with reachability or expected reward properties from [8, 47], except for the dining cryptographer’s protocol which has a constant reachability probability. Furthermore, we took a medium and large Bayesian network from [53]. We excluded the typical pMC examples [25] with only two or four parameters. We observed that for some benchmarks (e.g., drone and refuel) the optimum for some parameters is often at its bound. We refer to these parameters as “easy-parameters”.

Table 1 shows the benchmarks. The first seven benchmarks consider reachability properties, whereas the latter four consider expected rewards. The table includes the required property (Bound) and the instance of the benchmark. For network2-prios, “ps” refers to successfully delivered packets and “dp” refers to dropped packets. For each benchmark we denote the number of states, transitions and parameters after minimization, as well as the number of “easy-parameters”. The entry N/A for “easy-parameters” means that all runs for GD timed out, therefore, no feasible instantiation was found and the number of “easy-parameters” could not be determined.

To obtain bounds for the feasible instantiations, we considered values close to known optima from the literature. For those benchmarks where the optimal was not available, we approximated it by applying GD several times and picking the optimum solution found. We checked feasibility against the optimum-bounds, and the relaxed bounds, where we relaxed all bounds by 10% and 20%, respectively. The plots for 10% are similar to those for 20% and therefore omitted.

5.2 Results

Our experiments show that GD can be used to find feasible parameter instantiations. In the following, we provide the numerical results and then answer the questions Q1–Q3 in the next paragraphs.

The feasibility problem remains a combinatorially hard problem, but the presence of easy parameters typically (but not always) indicates that the gradient remains (positive/negative) over the complete space.
**Table 1. Model characteristics**

| Model    | Bound          | Instance        | States | Trans. | $|V|$ | $|V_{easy}|$ |
|----------|----------------|-----------------|--------|--------|------|---------|
| hailfinder | $\geq 0.145$ (2000) | 1540 324982     | 1249   | 0      |
| nrp      | $\leq 0.001$ (16,2) | 787 16092       | 95     | 32     |
| drone    | $\geq 0.85$ (5,1) | 3678 27376      | 756    | 667    |
| 4x4grid  | $\geq 0.9$ (5)   | 1216 2495       | 99     | 42     |
| newgrid  | $\geq 0.99$ (8,10) | 30191 66410     | 399    | 244    |
| child    | $\leq 0.43$ (210) | 243 3277        | 223    | 170    |
| refuel   | $\geq 0.35$ (5,3) | 1564 4206       | 452    | 317    |
| network2 | $\leq 0.1$ (8,5, ps) | 397 2837       | 140    | 128    |
| samplerocks | $\leq 3.5$ (8,5, dp) | 2822 69688   | 888    | 537    |
| 4x4grid  | $\leq 4.2$ (5)   | 1410 2879       | 99     | 38     |
| maze2    | $\leq 6$ (15)    | 5340 10799      | 2624   | 1257   |

**Numerical results.** The scatter plots in Fig. 5 show how the different region restriction methods compare for Momentum-Sign and Adam. Point $(x, y)$ denotes that the restriction method projection took $x$ seconds and the alternative took $y$ seconds to find a feasible instantiation for the given GD method. The scatter plots in Figs. 6 and 7 show how the different GD methods and the baseline methods QCQP and PSO (y-axis) compare to Momentum-Sign (x-axis), respectively. Note that all scatter plots are log-log scale plots. Point $(x, y)$ denotes that Momentum-Sign took $x$ seconds and the alternative took $y$ seconds to find a feasible instantiation. All implicit vertical lines denote the same benchmark. Points on the TO/MO line denote that the method has timed out or used too much memory and the ERR line denotes that the method has encountered some internal error. The dashed lines denote differences of a factor 10 and 100.

**Comparison of region restriction methods.** Fig. 5(a) (Fig. 5(b)) displays how projection with learning rate 0.1 (x-axis) compares to all other restriction methods for the optimum-bounds of all benchmarks on Momentum-Sign (Adam). The ERR line indicates that we found an infeasible parameter instantiation. This occurs when the learning rate is too high, and thus the barrier function not strong enough (see also Example 11). Imagine a vertical line through $x = 0.1$. This line represents the benchmark for which momentum-sign needed $\approx 0.1$ seconds. We now obtain that the barrier function timed-out or threw an error for all learning rates.

First of all, we observe that for Momentum-Sign the logistic-function is slightly outperformed by projection. Secondly, we observe that for Adam the logistic-
function is outperformed by projection often up to orders of magnitude. Finally, we observe that for learning rate 0.1, the barrier function method is outperformed by projection. As many “easy-parameters” occur, the optima often lie at the edges of the region. Therefore, we choose a relatively large learning rate. The barrier function method tends to push us away from the edges, as the steps taken are too large, we cannot get close enough to the edge.

Comparison of GD methods. When comparing the different GD Methods, we fix the region restriction method to projection. Fig. 6(b) displays how Momentum-Sign (x-axis) compares to all other methods for the optimum-bounds of all benchmarks. First of all, we observe that Momentum-Sign typically obtains better runtimes compared to the adaptive methods (RMSProp, Adam, RAdam). As our parameters occur with almost the same frequency, the adaptive methods are less suited for our benchmarks. Secondly, we observe that for the non-adaptive
methods, the methods where only the sign of the gradient is respected (and not the value gradient itself) often outperform their alternative. This is caused by 1) the occurrence of the “easy-parameters” and 2) the influence a single parameter may have on the reachability probability/expected reward. If a more influential parameter gets changed at the first parameter batch, this might yield a feasible solution before we have even updated all parameters. Monotonicity could be a cause, and the ordering of parameters on influentiallity needs further investigation (see Sec. 7).

Comparison to state-of-the-art feasibility methods. Fig. 7 shows Momentum-Sign with projection versus QCQP and PSO respectively, on both the optimum-bounds (upper) and 20% relaxed-bounds (lower). First of all, our experiments reveal that Momentum-Sign always outperforms PSO, on both the optimum-bounds and the relaxed-bounds. Secondly, note that PSO throws an error during preprocessing of the MC on some benchmarks as they violate an implicit assumption by the PSO implementation. Thirdly, Momentum-Sign outperforms QCQP often by at least one order of magnitude. Finally, we observe that QCQP outperforms Momentum-Sign for the samplerocks benchmarks. Based on the structure of the

**Fig. 7.** Comparison of GD with QCQP and PSO against optimum-bounds (upper) and 20% relaxed-bounds (lower).
samplerocks benchmark, preprocessing with e.g. monotonicity checking might improve Momentum-Sign (see Sec. 7).

6 Related Work

Finding satisfying instantiations of parametric MCs. Parametric MCs \cite{14,38} have received quite some attention. The classical focus has been on computing closed forms for solution functions that map parameter values to expected rewards \cite{5,14,17,18,23,30,32}. Feasibility as considered in this paper — finding a satisfying instantiation — and its extension to model repair \cite{7} has been formulated as a search problem before: Chen et al. \cite{11} considered three different search methods: PSO, Markov Chain Monte Carlo and Cross-Entropy. In this context, PSO was most successful. Model repair and feasibility have also been studied as optimization problems: \cite{7} considered a one-shot encoding, whereas \cite{12,13} took iterative approaches in which the encoding was simplified around a point to guide the search. Spel et al. \cite{57} present a graph-based heuristic to determine whether a pMC is monotonic, i.e., whether the gradient w.r.t. some parameter is positive on the complete parameter space. Chen et al. \cite{10} analyze (non-controllable) perturbations in MCs from a robustness perspective. Fast sampling of the parameter space and evaluating the corresponding pMCs is also a preprocessing step to other methods \cite{22,32}. Storm offers optimized routines, and for large numbers of samples, just-in-time compilation is a feasible alternative \cite{20}.

Controller synthesis under partial observability. The standard model for controller synthesis under partial observability are partially observable MDPs (POMDPs) \cite{35}. Controller synthesis in finite POMDPs can equivalently be reformulated as controller synthesis for infinitely large belief-MDPs. Due to the curse of history, finding a feasible controller for a quantitative objective — the setting discussed in this paper — is undecidable \cite{41}. At the beginning of this millennium, this lead to trying to search for memoryless or small-memory controllers in POMDPs \cite{42}. Among others, the use of gradient descent methods to learn finite-state controllers for partially observable environments was explored by Meuleau et al. \cite{43}. This approach has further developed into deep learning for POMDPs, as e.g. used to learn Atari-games \cite{44}. Some methods allow explicit extraction of the finite-state controllers \cite{9}. Those approaches are generally model-free — they learn policies from sets of demonstrations or traces. Closest to our approach is the work by Aberdeen \cite{1} in using a model-based approach to find memoryless strategies in POMDPs via gradient descent. The major differences are in computing the gradients by using value-iteration and a softmax operation, and the use of stochastic gradient descent. The approach back then could and did not compare to the current state-of-the-art methods.

Quickly afterwards, breakthroughs in point-based solvers \cite{48,56} and Monte-Carlo methods for finding solutions \cite{54} shifted attention back to the belief-MDP \cite{28,62} (although some of those ideas also influenced the deep-RL community). Likewise, most recent support in the probabilistic model checkers
PRISM \cite{47} and Storm \cite{8} is based on an abstraction of the belief-MDP \cite{40} and abstraction refinement. The use of game-based abstraction leads to non-randomized controllers. Winterer et al. \cite{64} support a finite set of uniform randomizations. In contrast, we consider an infinite combination of possibilities. Likewise, Andriushenko et al. \cite{3} recently consider syntax-guided synthesis for partial information controllers with a finite set of options.

7 Conclusion and Future Work

This paper has shown that gradient descent often outperforms state-of-the-art methods for tackling the feasibility problem: find an instance of a parametric Markov chain that satisfies a reachability objective. As synthesizing a realizable controller with a fixed memory structure and a fixed set of potential actions can formally be described as feasibility synthesis in pMCs \cite{?}. Our approach supports the correct-by-construction synthesis of controllers for systems whose behavior is described by a stochastic process. Experiments showed that 1) projection outperforms other region restriction methods, 2) basic gradient descent methods perform better on our problem than more sophisticated ones, and 3) Momentum-Sign often outperforms QCQP and PSO.

Outlook. As observed in the evaluation of the results, future work consists of extending the preprocessing of the parametric Markov chains with monotonicity checking and investigating a possible ordering of parameters based on the influence on the property. Also, models with a large state space could be handled by e.g. using value iteration to solve the system of equations. Furthermore, questions regarding the derived weighted automaton can be asked, e.g. regarding the applicability of bisimulation minimisation or parameter lifting \cite{49}.

Data availability. The tools used and data generated in our experimental evaluation are archived at DOI 10.5281/5568910 \cite{26}.
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A Proofs

Definition 3. The system of equations for the partial derivative of $ER_{M}^{s ightarrow \omega}$ w.r.t. $p \in V$ is given by:

\[
\begin{align*}
x_s &= 0, \quad \partial_p x_s = 0 \quad \text{if } s = \omega \\
x_s &= \text{rew}(s) + \sum_{s' \in S} P(s, s') \cdot x_s' \quad \text{for } s \in S \setminus \{\omega\} \\
\partial_p x_s &= \sum_{s' \in S} (\partial_p P(s, s') \cdot x_s' + P(s, s') \cdot \partial_p x_s') \quad \text{for } s \in S \setminus \{\omega\}
\end{align*}
\]

where $\partial_p P(s, s')$ is the derivative of the probability function $P(s, s')$ w.r.t. $p$.

First of all, we observe that an alternative way to write the equations in Def. 3 for $S \setminus \{\omega\} = \{s_0, \ldots, s_{n-1}\}$ is:

\[
\begin{pmatrix}
  x_0 \\
  \vdots \\
  x_{n-1}
\end{pmatrix}
= (A - \partial_p A A^{-1})
\begin{pmatrix}
  \text{rew}(s_0) \\
  \vdots \\
  \text{rew}(s_{n-1})
\end{pmatrix}
\]

where $A$ equals the transition probability function $P$ restricted to $S \setminus \{\omega\}$.

Theorem 1. The equation system of Def. 3 has exactly one solution: $x_s$ equals $ER_{M}^{s \rightarrow \omega}$ and $\partial_p x_s$ equals $\partial_p ER_{M}^{s \rightarrow \omega}$ for each $s \in S$.

Proof (of Theorem 1). Clearly $x_s$ equals $ER_{M}^{s \rightarrow \omega}$. Furthermore, observe that in order to obtain the derivative for $x_s$, i.e. $\partial_p x_s$, we apply the sum rule and the product rule to $x_s$.

For the variables $x_0, \ldots, x_{n-1}$ we have:

\[
(1 - A)
\begin{pmatrix}
  x_0 \\
  \vdots \\
  x_{n-1}
\end{pmatrix}
= \begin{pmatrix}
  \text{rew}(s_0) \\
  \vdots \\
  \text{rew}(s_{n-1})
\end{pmatrix}
\] \[\Leftrightarrow \begin{pmatrix}
  x_0 \\
  \vdots \\
  x_{n-1}
\end{pmatrix}
= (1 - A)^{-1}
\begin{pmatrix}
  \text{rew}(s_0) \\
  \vdots \\
  \text{rew}(s_{n-1})
\end{pmatrix}
= \begin{pmatrix}
  ER_{M}^{s_0 \rightarrow \omega} \\
  \vdots \\
  ER_{M}^{s_{n-1} \rightarrow \omega}
\end{pmatrix}
\]

\]
\((1 - A)^{-1}\) exists because \(1 - A\) is invertible (proof in \([6, \text{p. 821}]\)). For the variables \(\partial_p x_0, \ldots, \partial_p x_{n-1}\) we have:

\[
\begin{pmatrix}
\partial_p x_0 \\
\vdots \\
\partial_p x_{n-1}
\end{pmatrix}
= (\partial_p A) 
\begin{pmatrix}
x_0 \\
\vdots \\
x_{n-1}
\end{pmatrix} + A 
\begin{pmatrix}
\partial_p x_0 \\
\vdots \\
\partial_p x_{n-1}
\end{pmatrix}
\]

\[\iff\]

\[
(1 - A) \begin{pmatrix}
\partial_p x_0 \\
\vdots \\
\partial_p x_{n-1}
\end{pmatrix} = (\partial_p A) \begin{pmatrix}
\text{ER}^{s_0 \rightarrow \omega} \\
\vdots \\
\text{ER}^{s_{n-1-1} \rightarrow \omega}
\end{pmatrix}
\]

\[\iff\]

\[
(1 - A) \begin{pmatrix}
\partial_p x_0 \\
\vdots \\
\partial_p x_{n-1}
\end{pmatrix} = (\partial_p A) \begin{pmatrix}
\text{ER}^{s_0 \rightarrow \omega} \\
\vdots \\
\text{ER}^{s_{n-1-1} \rightarrow \omega}
\end{pmatrix}
\]

\[\iff\]

\[
\begin{pmatrix}
\partial_p x_0 \\
\vdots \\
\partial_p x_{n-1}
\end{pmatrix} = (1 - A)^{-1} (\partial_p A) \begin{pmatrix}
\text{ER}^{s_0 \rightarrow \omega} \\
\vdots \\
\text{ER}^{s_{n-1-1} \rightarrow \omega}
\end{pmatrix}
\]

Thus there exists a unique solution of the system of equations.