ON SEMISIMPLICITY OF JANTZEN MIDDLES FOR THE PERIPLECTIC LIE SUPeralgebra

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Abstract. We prove that an integral block of the category $\mathcal{O}$ of the periplectic Lie superalgebra contains a non-semisimple Jantzen middle if and only if it contains a simple module of atypical highest weight. As a consequence, every atypical integral block of $\mathcal{O}$ does not admit a Kazhdan-Lusztig theory in the sense of Cline, Parshall and Scott.

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1. Introduction

1.1. For a finite-dimensional complex semisimple Lie algebra, it is shown by Andersen and Stroppel [AS, Section 7] that the validity of Kazhdan-Lusztig conjecture [KL] is equivalent to the semisimplicity of Jantzen middles for the regular blocks of the category $\mathcal{O}$. Later on, Coulembier [Co1, Theorem 6.4] developed an analogous connection for basic classical Lie superalgebras. Therefore, the problem of the semisimplicity of Jantzen middles turns out to be interesting and important.

An earlier achievement is the semisimplicity of Jantzen middles for the general linear Lie superalgebras $\mathfrak{gl}(m|n)$ established in [Co1, Theorem 6.10], which is based on the Brundan-Kazhdan-Lusztig theory formulated in [Br] and proved in [BLW, CLW]. However, it is still an open question whether the Jantzen middle is always semisimple for Lie superalgebras arising from Kac’s classification [Ka1].

Recently, the representation theory for the periplectic Lie superalgebra $\mathfrak{pe}(n)$ has been studied extensively; see, e.g., [Se1], [Ch], [B+9], [Co3], [CMS], [EAS], [IRS], [IS], [KB] and references therein. In addition, basic aspects and partial solutions to the irreducible character problem of the category $\mathcal{O}$ were given in [CC] and [CP]. In order to have a complete picture, it is natural to ask whether there exists a Kazhdan-Lusztig pattern for $\mathfrak{pe}(n)$.

In [CPS2], Cline, Parshall and Scott introduced a formulation of abstract Kazhdan-Lusztig theory in order to provide an appropriate axiomatic framework encompassing numerous important examples in representation theory. In [CS, Corollary 3.3], it is shown that the Brundan-Kazhdan-Lusztig theory for the category $\mathcal{O}^Z$ of $\mathfrak{gl}(m|n)$-modules of integral weights is an abstract Kazhdan-Lusztig theory. The goal of this paper is to start the investigation into the semisimplicity of Jantzen middles and connection with Kazhdan-Lusztig theory in the sense of [CPS2, Definition 3.3] for the
periplectic Lie superalgebra $\mathfrak{pe}(n)$; see Section 2.1 and Section 3.2.3 for the definitions. This type of connection is generalized to the so-called Lie superalgebras of type I in the full generality.

1.2. To explain the results of the paper in more detail, we start by explaining our precise setup. Following [Se2], we consider $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a quasireductive Lie superalgebra (or classical in [Ma]) throughout the present paper, namely, $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_1$ is semisimple over $\mathfrak{g}_0$ under the adjoint action. In addition, we will make three assumptions in our setup. First, we assume that $\mathfrak{g}$ has a type-I gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is induced by a grading operator from a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$. Next, we choose a triangular decomposition in the sense of [Ma]

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \tag{1.1}$$

such that the odd parts of $\mathfrak{n}^\pm$ are $\mathfrak{g}_{\pm 1}$, respectively. We refer to these assumptions as (A1)–(A3) in the paper; see Section 2.1. Lie superalgebras $\mathfrak{g}$ satisfying these assumptions fit into the framework of [CC] under the name Lie superalgebras of type I-0. For such Lie superalgebras, there is a number of basic properties of the twisting functors developed in [CC] Section 4.3 that are to be used in the present paper. We will mainly focus on the case of the periplectic Lie superalgebra $\mathfrak{pe}(n)$, which is really the main topic of the paper.

Let $\mathcal{O}$ denote the BGG category associated to the triangular decomposition (1.1). Let $\mathcal{O}^Z$ be the full subcategory of $\mathcal{O}$ consisting of modules with integral weights. Our first main result is the following.

**Theorem A.** Consider $\mathfrak{g}$ a Lie superalgebra of type I-0. If a (indecomposable) block of $\mathcal{O}^Z$ contains a (non-zero) non-semisimple Jantzen middle then it does not admit a Kazhdan-Lusztig theory in the sense of [CPS2, Definition 3.3].

For $\mathfrak{g}$ a (not necessarily type I) basic classical Lie superalgebra from Kac’s list [Ka1], the Theorem A has been established in the earlier work of Coulembier [Co1].

The notion of typicality of weights for $\mathfrak{pe}(n)$ has been introduced in [Se1, Section 5]; see (4.1) for its definition. A block of $\mathcal{O}^Z$ is said to be atypical, in case it contains a simple module of atypical highest weight, and typical otherwise. The following is our second main result.

**Theorem B.** Consider $\mathfrak{g} = \mathfrak{pe}(n)$. Then a block of $\mathcal{O}^Z$ contains a (non-zero) non-semisimple Jantzen middle if and only if it is atypical. As a consequence, every atypical block of $\mathcal{O}^Z$ does not admit a Kazhdan-Lusztig theory in the sense of [CPS2, Definition 3.3].

1.3. The paper is organised as follows. In Section 2 we provide some background materials on quasireductive Lie superalgebras. We review the representation categories and introduce our assumptions in the Section 2.1. The standard matrix realization of $\mathfrak{pe}(n)$ is reviewed in Section 2.3.

In Section 3 we study the relevance between semisimplicity of Jantzen middles and abstract Kazhdan-Lusztig theory in the sense of [CPS1, Definition 3.3]. Section 3.2 is devoted to the proof of Theorem A. Then, we put these results together to obtain the proof of Theorem B in Section 4.
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2. Preliminaries

Throughout the paper the symbols $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq 0}$ stand for the sets of all complex numbers, real numbers, integers and non-negative integers. We always work over the ground field $\mathbb{C}$. Denote the abelian group of order two by $\mathbb{Z}_2 = \{0, 1\}$. For a homogeneous element $x$ of a vector superspace $V = V_0 \oplus V_1$, we denote its parity by $\overline{x} \in \mathbb{Z}_2$. In the paper, we let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional quasireductive Lie superalgebra, namely, $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_1$ is a semisimple $\mathfrak{g}_0$-module under the adjoint action. We denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$ and its center by $Z(\mathfrak{g})$.

In this section, we collect preliminaries and assumptions on quasireductive Lie superalgebras.

2.1. Assumptions and notations. Throughout the present paper, we assume that the Lie superalgebra $\mathfrak{g}$ is of type I-0 in the sense of [CC, Section 2.3.1], which we shall explain as follows.

2.1.1. Fix a triangular decomposition of $\mathfrak{g}_0$:

(2.1) $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+.$

In the present paper, we assume that $\mathfrak{g}$ is a quasireductive Lie superalgebra with a compatible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ induced by a grading element $H \in \mathfrak{h}$, that is,

(A1) $\mathfrak{g}_0 = \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ with $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$.

(A2) $[H, x] = kx,$ for $x \in \mathfrak{g}_k$ with $k = \pm 1$.

We refer to such a Lie superalgebra as a Lie superalgebra of type I-0. We will use notations $\mathfrak{g}_{\leq 0} := \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ and $\mathfrak{g}_{\geq 0} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

For an element $h \in \mathfrak{h}$, we define the following subalgebras:

(2.2) $\mathfrak{l} := \bigoplus_{\text{Re}(h) = 0} \mathfrak{g}^0,$ $\mathfrak{u}^+ := \bigoplus_{\text{Re}(h) > 0} \mathfrak{g}^0,$ $\mathfrak{u}^- := \bigoplus_{\text{Re}(h) < 0} \mathfrak{g}^0$

where $\mathfrak{g}^0 := \{X \in \mathfrak{g} | [h, X] = \alpha(h)X, \text{ for all } h \in \mathfrak{h}\}$. We claim that (A1) and (A2) imply the following assertion:

(A3) There exists an element $h' \in \mathfrak{h}$ giving rise to $\mathfrak{l} = \mathfrak{h}$, $\mathfrak{u}_0^+ = \mathfrak{n}_0^+$ and $\mathfrak{u}_1^+ = \mathfrak{g}_{\pm 1}$.

To see this, let $t \in \mathfrak{h}$ such that

$\mathfrak{h} = \bigoplus_{\text{Re}(t) = 0} \mathfrak{g}_0^0,$ $\mathfrak{n}_0^+ = \bigoplus_{\text{Re}(t) > 0} \mathfrak{g}_0^0,$ $\mathfrak{n}_0^- = \bigoplus_{\text{Re}(t) < 0} \mathfrak{g}_0^0,$

then there exists a positive real number $\epsilon$ such that $h' := H + \epsilon t$ gives the desired subalgebras in (2.2). Define $\mathfrak{n}^\pm := \mathfrak{u}^\pm$. We refer to the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ satisfying (A3) as the (distinguished) triangular decomposition of $\mathfrak{g}$. 
Also, we refer to the subalgebras $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+$ and $\mathfrak{b}^r := \mathfrak{b}_0 + \mathfrak{g}_-1$ as standard Borel subalgebra and reverse Borel subalgebra, respectively. There subalgebras are all Borel-Penkov-Serganova subalgebras in the sense of [PS1] and [Mu2, Section 3.2]; see also [CCC] Sections 1.3, 1.4 for more details.

We will make conventional definitions as follows. An element $\alpha \in \mathfrak{h}^* \setminus \{0\}$ is called a root if $g^\alpha \neq 0$. We denote the set of roots by $\Phi \subset \mathfrak{h}^*$. Let $\Phi^+$ be the set of roots in $\mathfrak{n}^+$. Let $\Phi^+_0$ be the positive system coming from the triangular decomposition \[2.1\] of $g_0$. We let $\Pi_0$ be the corresponding simple system for $\Phi^+_0$. We are mainly interested in the following Lie superalgebras:

Example. Each of the following quasireductive Lie superalgebra is of type I-0:

- Reductive Lie algebras $\mathfrak{g} = g_0$.
- The general linear Lie superalgebra $\mathfrak{gl}(m|n)$; see [CW] Section 1.1.2.
- The ortho-symplectic Lie superalgebras $\mathfrak{osp}(2|2n)$; see [CW] Section 1.1.3.
- The periplectic Lie superalgebra $\mathfrak{pe}(n)$; see Section 2.3.
- A semisimple extension $g := (\mathfrak{s} \otimes \Lambda(\xi)) \rtimes \mathfrak{d}$ of the Takiff superalgebra induced by a simple Lie algebra $\mathfrak{s}$ studied in [CCa] Section 2.1.

2.1.2. The Weyl group $W$ is defined as the Weyl group of $g_0$. We let $w_0 \in W$ denote the longest element in $W$. We fix a $W$-invariant bilinear form $\langle \cdot , \cdot \rangle$ on $\mathfrak{h}^*$. Let $\rho$ denote the half-sum of all roots in $\Phi^+_0$. Let $s_\alpha$ be the reflection associated with the root $\alpha \in \Phi^+_0$. The dot action of $W$ on $\mathfrak{h}^*$ is defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$, for any $\lambda \in \mathfrak{h}^*$.

For any $\alpha \in \Pi_0$, we set $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ to be the co-root to $\alpha$; see [Hu, Section 0.2]. A weight is called integral if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$, for any $\alpha \in \Phi^+_0$. We denote by $\mathcal{P} \subset \mathfrak{h}^*$ the set of integral weights. A weight $\lambda$ is said to be dominant (resp. anti-dominant) if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0}$ (resp. $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$), for any $\alpha \in \Phi^+_0$. For a given weight $\lambda \in \mathfrak{h}^*$, we let $W_\lambda$ be the stabilizer subgroup of $\lambda$ under the dot action of $W$.

2.2. BGG category $\mathcal{O}$.

2.2.1. The BGG category $\mathcal{O}$ associated to the triangular decomposition \[A3\] is defined as the category of finitely-generated $g$-modules on which $\mathfrak{h}$ acts semisimply and $\mathfrak{b}$ acts locally finitely. Therefore $\mathcal{O}$ is the category of $g$-modules restricted to $g_0$-modules by Res in the classical BGG category $\mathcal{O}_0$ of $g_0$-modules as defined in [BGG]. Also, we let $\mathcal{F}$ and $\mathcal{F}$ denote the category of finite dimensional $g$-modules and $g_0$-modules, respectively.

We have the exact Kac functor $K(-) : \mathcal{O}_0 \to \mathcal{O}$ defined as

$$K(N) := U(g) \otimes_{g_0 + g_1} N,$$

for any $N \in \mathcal{O}_0$ by letting $g_1$ acts on $N$ trivially.

For any $M \in \mathcal{O}$, we will freely use $[M : L]$ to denote the Jordan-Hölder decomposition multiplicities of a simple module $L$ in a composition series of $M$. In addition, we will use $\text{soc}(M), \text{rad}(M)$ to denote the socle and radical of $M$, respectively. The top $M$ is defined as $\text{top}(M) := M/\text{rad}M$. 
2.2.2. We recall that the category \( \mathcal{O} \) has a natural structure of highest weight category with respect to the triangular decomposition in \([A3]\). We define the partial order \( \leq \) on \( \mathfrak{h}^* \) as the transitive closure of the relations

\[
\lambda + \alpha \leq \lambda, \quad \text{for } \alpha \in \Phi(\mathfrak{n}^\pm),
\]

where \( \Phi(\mathfrak{n}^\pm) \) denotes the set of all roots in \( \mathfrak{n}^\pm \), respectively. For any \( \lambda \in \mathfrak{h}^* \), we define the Verma module over \( \mathfrak{g}_0 \) as follows

\[
M(\lambda) := U(\mathfrak{g}_0) \otimes_{\mathfrak{g}_0} \mathbb{C}_\lambda,
\]

by letting \( \mathfrak{n}^+ \) acts on \( \mathbb{C}_\lambda \) trivially. Also, the corresponding Verma (super)module over \( \mathfrak{g} \) is defined as

\[
\hat{M}(\lambda) := (U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda) \cong K(M(\lambda)).
\]

The (simple) tops of \( M(\lambda) \) and \( \hat{M}(\lambda) \) are denoted by \( L(\lambda) \) and \( \hat{L}(\lambda) \), respectively. Then \( \{L(\lambda) | \lambda \in \mathfrak{h}^* \} \) (resp. \( \{\hat{L}(\lambda) | \lambda \in \mathfrak{h}^* \} \) ) forms the complete list of simple modules in \( \mathcal{O}_0 \) (resp. \( \mathcal{O} \)). For any \( \lambda \in \mathfrak{h}^* \), the Kac induced module \( K(L(\lambda)) \) is an epimorphic image of \( \hat{M}(\lambda) \). By \([CCC]\) Theorem 3.1 \( (\mathcal{O}, \leq) \) is a highest weight category with standard objects \( \hat{M}(\lambda) \). Also, we denote by \( P(\lambda) \) and \( \hat{P}(\lambda) \) the projective covers of \( L(\lambda) \) and \( \hat{L}(\lambda) \) in \( \mathcal{O}_0 \) and \( \mathcal{O} \), respectively. Finally, for \( M \in \mathcal{O} \), we use \( \text{ch} M \) to denote the formal character of \( M \).

2.2.3. Define an involution \( \hat{\phi} := h^* \to \mathfrak{h}^* \) by letting \( \hat{\lambda} = -w_0 \lambda \), for \( \lambda \in \mathfrak{h}^* \). As observed in \([CCC]\) Section 1.3], there is an anti-involution \( \sigma \) on \( \mathfrak{g} \) satisfying that

\[
\sigma(\mathfrak{h}) = \mathfrak{h}, \quad \sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\mp,
\]

\[
(\hat{\lambda})(h) = \lambda(\sigma(h)), \quad \text{for } h \in \mathfrak{h}.
\]

This involution \( \sigma \) leads to a natural duality functor \( D \) on the category \( \mathcal{O} \) as follows. For any \( M \in \mathcal{O} \), let \( M^\circ \) be the restricted dual space of \( M \). Then \( M^\circ \) is a \( \mathfrak{g} \)-submodule of \( \text{Hom}_\mathbb{C}(M, \mathbb{C}) \). We now give a new \( \mathfrak{g} \)-module structure of \( M^\circ \) by letting

\[
xf(v) = (-1)^{\varphi(h)} f(\sigma(x)v),
\]

for any homogeneous elements \( x \in \mathfrak{g} \), \( f \in M^\circ \) and any \( v \in M \). Then denote this resulting module by \( DM \). This gives the endofunctor \( D \) on \( \mathcal{O} \); see \([CC]\) Section 2.2.4 for more details.

By \([CCC]\) Section 3], the functor \( D \) intertwines the standard and costandard objects of \( \mathcal{O} \) with respect to the two highest weight category structures via \( \mathfrak{b} \) and \( \mathfrak{b}^* \). We briefly recall this effect below.

For any \( \mu \in \mathfrak{h}^* \), let \( \hat{M}^\circ(\mu) \) be the maximal submodule of the \( \text{Coind}_{\mathfrak{n}^-_{\mu} + \mathfrak{g}^+_{\mu}}^\circ(C_\mu) \) on which \( \mathfrak{h} \) acts semisimply and locally finitely; see \([CCC]\) Definition 3.2, Theorem 3.1]. Next, we put \( \hat{M}^\circ_{\mathfrak{b}^*}(\mu) := U(\mathfrak{g}) \otimes_{\mathfrak{b}^*} \mathbb{C}_\mu \), and define \( \hat{M}^\circ_{\mathfrak{b}}(\mu) \) as the maximal submodule of the coinduced module \( \text{Coind}_{\mathfrak{n}^-_{\mu} + \mathfrak{g}^+_{\mu}}^\circ(C_\mu) \) on which \( \mathfrak{h} \) acts semisimply and locally finitely. Then by \([CCC]\) Proposition 3.4], we have the following isomorphisms

\[
D\hat{M}(\mu) \cong \hat{M}^\circ_{\mathfrak{b}^*}(\mu), \quad D\hat{M}^\circ(\mu) \cong \hat{M}^\circ_{\mathfrak{b}^*}(\mu), \quad \text{and } D\hat{L}(\mu) \cong \hat{L}_{\mathfrak{b}^*}(\mu),
\]
where $\bar{L}_\alpha(\hat{\mu})$ denotes the simple highest weight module of highest weight $\hat{\mu}$ with respect to $b'$. For any $\alpha \in \Pi_0$, we consider $f_\alpha \in g_0^a$ to be a non-zero root vector of root $\alpha$. A $g$-module $M$ is said to be $\alpha$-finite (resp. $\alpha$-free) if the action of $f_\alpha$ on $M$ is locally finite (resp. injective). The following useful lemma is a consequence of [CoM1, Lemma 2.1]

**Lemma 1.** Let $\lambda \in b^*$ and $\alpha \in \Pi_0$. Then we have

$$L(\lambda) \text{ is } \alpha \text{-finite } \iff \bar{D}L(\lambda) \text{ is } \hat{\alpha} \text{-finite } \iff \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}.$$  

2.3. **The periplectic Lie superalgebra** $\mathfrak{pe}(n)$. In this subsection, we introduce the periplectic Lie superalgebras $\mathfrak{pe}(n)$; see also [CW, Section 1.1] for more details.

2.3.1. **Matrix realization.** For any positive integer $n$, the standard matrix realization of the periplectic Lie superalgebra $\mathfrak{pe}(n)$ inside the general linear Lie superalgebra $\mathfrak{gl}(n|n)$ is given by

$$\mathfrak{pe}(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathbb{C}^{n \times n}, \ B \text{ symmetric and } C \text{ skew-symmetric} \right\}.$$

The type-I gradation of $\mathfrak{pe}(n)$ inherits that of $\mathfrak{gl}(n|n)$, namely,

$$\mathfrak{pe}(n)_1 := \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B^t = B \right\} \quad \text{and} \quad \mathfrak{pe}(n)_{-1} := \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C^t = -C \right\}.$$

The standard Cartan subalgebra $h \subset \mathfrak{pe}(n)$ consists of diagonal matrices. Let $E_{ab}$ denote the elementary matrix in $\mathfrak{gl}(n|n)$, for $1 \leq a, b \leq 2n$. We denote by $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ the dual basis of $h^*$ with respect to the following standard basis of $h$

$$\{H_i := E_{i,i} - E_{n+i,n+i} \mid 1 \leq i \leq n\} \subset \mathfrak{pe}(n).$$

In particular, we have

$$\Phi = \{\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j) \mid 1 \leq i \neq j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\},$$

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\},$$

$$\Pi_0 = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n - 1\}.$$  

The Weyl group $W$ is isomorphic to the symmetric group on $n$ letters. We fix a non-degenerate $W$-invariant bilinear form $\langle \cdot, \cdot \rangle : h^* \times h^* \to \mathbb{C}$ by letting $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$, for all $1 \leq i, j \leq n$. Fix the Borel subalgebra $b_0$ of $g_0 \cong \mathfrak{gl}(n)$ consisting of matrices in (2.10) with $B = C = 0$ and $A$ upper triangular. Without loss of generality, we shift the Weyl vector $\rho$ of $g_0$ by letting

$$\rho := (n - 1)\epsilon_1 + (n - 2)\epsilon_2 + \cdots + \epsilon_{n-1}.$$  

Also, we define $\omega_n := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$. For any $k \in \mathbb{C}$, we denote by $C_{k\omega_n}$ the one-dimension $g$-module of weight $k\omega_n$. Note that $C_{k\omega_n} \otimes -$ leads to an auto-equivalence of $\mathcal{O}$; see [CC, Section 5.10].

2.3.2. **Odd reflections.** In this subsection, we recall the notion of odd reflections for $\mathfrak{pe}(n)$ from [PS2, Lemma 1 and Section 2.2]. For a given Borel subalgebra $b'$, we denote the set of roots of $b'$ (i.e. non-zero weights of $b'$) by $\Phi(b')$. Consider the following
Let \( \lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \mathfrak{h}^* \). Suppose that
\[
\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^k \in \mathfrak{h}^*
\]
are \( \mathfrak{b}^0, \mathfrak{b}^1, \mathfrak{b}^2, \ldots, \mathfrak{b}^k \)-highest weights of \( \bar{L}(\lambda) \). For each \( 0 \leq \ell \leq k \), we set \( \lambda^\ell = \sum_{i=1}^{n} \lambda_i^\ell \epsilon_i \). Then we have the following rules:

(A) For each \( \alpha_\ell = \epsilon_p + \epsilon_q \) with \( p \neq q \), the \( \mathfrak{b}^{\ell+1} \)-highest weight of \( \bar{L}(\lambda) \) is given by
\[
(2.17) \quad \lambda^{\ell+1} = \begin{cases} 
\lambda^\ell + \alpha_\ell & \text{if } \lambda^p_\ell \neq \lambda^q_\ell, \\
\lambda^\ell & \text{otherwise}.
\end{cases}
\]

(B) If \( \alpha_\ell = 2\epsilon_p \), then the \( \mathfrak{b}^{\ell+1} \)-highest weight of \( \bar{L}(\lambda) \) is given by \( \lambda^{\ell+1} = \lambda^\ell \).

For a given \( \lambda \in \mathfrak{h}^* \) and \( 0 \leq \ell \leq k \), we denote by \( \bar{L}_{\mathfrak{b}^\ell}(\lambda) \) the simple module of \( \mathfrak{b}^\ell \)-highest weight \( \lambda \).

**Example 3.** Consider a weight \( \lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \bigoplus_{i=1}^{n} \mathbb{Z} \epsilon_i \) such that \( \lambda_1 = \lambda_2 \) and \( \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_n \). By a direct computation, it follows that
\[
\bar{L}_{\mathfrak{b}^\ell}(\lambda) = \bar{L}(\lambda + (n-1)\omega_n - \epsilon_1 - \epsilon_2),
\]
namely, \( \lambda^{\ell+1} = \lambda^\ell + \alpha_\ell \) at each step given in Part (A) of Lemma 2 for Borel subalgebras \( \mathfrak{b}^\ell \) and \( \mathfrak{b}^{\ell+1} \) that are connected by an odd reflection \( \alpha_\ell \neq \epsilon_1 + \epsilon_2 \).

3. The Jantzen middles

We continue to assume that \( \mathfrak{g} \) is a quasireductive Lie superalgebra of type I-0 (i.e., \( \mathfrak{g} \) satisfies assumptions (A1)–(A3)). Recall that \( O^Z \) denotes the full subcategory of \( O \) consisting of modules of integral weights. Similarly, we define \( O^Z_0 \subset O^Z_0 \). We will follow [CMW, Section 6] and define the Jantzen middles as the modules of twisted simple modules. Before giving the precise definitions, we recall the twisting functors as follows.

3.1. Twisting functors. Let \( \alpha \in \Pi_0 \) and \( s := s_\alpha \), we recall the corresponding Arkhipov’s twisting functor \( T_s \) (resp. \( T^0_s \)) on \( O \) (resp. \( O_0 \)) introduced in [CMW].
Section 3.6] and [CoM1 Section 5]. This functor was originally defined by Arkhipov in [Ar] and further studied in [AS, AL, CMW, KM, MS].

Let $G_s$ be the right adjoint to $T_s$. Let us recall some basic properties of $T_s$; see also [AS, CMW], [CoM1 Section 5] and [CC, Theorem 4.5].

1. The functor $T_s$ is right exact.
2. The functor $G_s$ is left exact and isomorphic to the Joseph’s version of Enright completion functor as introduced in [CC Section 4.2].
3. Let $G^0_s$ denote the Joseph’s Enright completion functor on $O_0$ introduced in [Jo2 Section 2]. Then we have
   \[
   \text{Ind} \circ T^0_s = T_s \circ \text{Ind} \quad \text{and} \quad \text{Res} \circ T_s = T^0_s \circ \text{Res}.
   \]
   \[
   \text{Ind} \circ G^0_s = G_s \circ \text{Ind} \quad \text{and} \quad \text{Res} \circ G_s = G^0_s \circ \text{Res}.
   \]
4. We have $D \circ G_{s_{a_i}} \circ D \cong T_{s_{a_i}}$ on $O^\mathbb{Z}$.
5. Denote the left derived functor of $T_s$ by $LT_s$. Then $L_i T_s = 0$, for $i > 1$. For any $M \in O$, the $L_1 T_s(M)$ is the maximal $\alpha$-finite submodule of $M$. If $\tilde{L}(\lambda)$ is $\alpha$-finite, then we have $T_1 \tilde{L}(\lambda) = 0$ and $L_1 T_1 \tilde{L}(\lambda) = \tilde{L}(\lambda)$.
6. Denote the right derived functor of $G_s$ by $RG_s$. Then $LT_s$ is an auto-equivalence of the bounded derived category $\mathcal{D}^b(O)$ with $RG_s$ as its inverse.

Since twisting functors satisfy the braid relations, see, for example [KM, CoM1], it follows that for any $\lambda \in \mathcal{P}$ with a reduced expression $\lambda = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ ($\alpha_1, \ldots, \alpha_k \in \Pi_0$) the associated twisting functor $T_w := T_{s_{\alpha_1}} \circ T_{s_{\alpha_2}} \circ \cdots \circ T_{s_{\alpha_k}}$ is well-defined. We use $T^w_0$ to denote the corresponding twisting functor on $O_0$. Then we have $\text{Res} \circ T_w = T^w_0 \circ \text{Res}$. The completion functors $G_w$ and $G^0_w$ are defined in similar fashion.

3.2. The Jantzen middles for type-I Lie superalgebras. Let $\alpha \in \Pi_0$. Following [CoM1 Section 6], we define the Jantzen middle $U_\alpha(\lambda)$ for $\tilde{L}(\lambda)$ associated with $\alpha$ as the radical of $T_{s_{\alpha}} \tilde{L}(\lambda)$, for any $\lambda \in \mathfrak{h}^*$. The following realization of Jantzen middles is an analogue of [CoM1 Proposition 6.2], where the case of basic classical Lie superalgebras were considered.

**Proposition 4.** Consider $\mathfrak{g}$ a quasireductive Lie superalgebra of type I-0. Suppose that $\lambda \in \mathcal{P}$ such that $\tilde{L}(\lambda)$ is $\alpha$-free. Then $T_{s_{\alpha}} \tilde{L}(\lambda)$ has a simple top isomorphic $\tilde{L}(\lambda)$. Furthermore, the Jantzen middle $U_\alpha(\lambda)$ is isomorphic to the largest $\alpha$-finite quotient of $\text{rad}\tilde{P}(\lambda)$.

3.2.1. Semisimplicity of Jantzen middles. For $\mathfrak{g}$ a Lie superalgebras with the category $O$ that admits a simple-preserving duality, the structure of $T_{s_{\alpha}} \tilde{L}(\lambda)$ has been studied in [CoM1 Theorem 5.12, Corollary 5.14]. It is shown that $\text{soc}(U_\alpha(\lambda)) \cong \text{top}(U_\alpha(\lambda))$ in the case when $\mathfrak{g}$ is either reductive or basic classical; see [AS Theorem 6.3] and [CoM1 Corollary 5.14]. For $\mathfrak{g} = \mathfrak{pe}(2)$, we give an example below showing that the socle and radical of a Jantzen middle are not necessarily isomorphic. Instead, we have the following proposition, which is an analogue of [AS Part (3) of Thoerem 6.3] and [CoM2 Part (ii) of Theorem 5.12] for any Lie superalgebra $\mathfrak{g}$ of type I-0. In particular, this applies to the case of $\mathfrak{pe}(n)$. 
Proposition 5. Suppose that $\alpha \in \Pi_0$ and $\lambda \in \mathcal{P}$ such that $\widetilde{L}(\lambda)$ is $\alpha$-free. Then we have

\begin{align}
\text{(3.1) } \text{soc}(U_\alpha(\lambda)) &\cong \bigoplus_{\widetilde{L}(\nu) \text{: } \alpha \text{-finite}} \widetilde{L}(\nu)^{\oplus \dim \text{Ext}^1_{\mathcal{O}}(\widetilde{L}(\nu), \widetilde{L}(\lambda))}, \\
\text{(3.2) } \text{top}(U_\alpha(\lambda)) &\cong \bigoplus_{\widetilde{L}(\nu) \text{: } \alpha \text{-finite}} \widetilde{L}(\nu)^{\oplus \dim \text{Ext}^1_{\mathcal{O}}(\widetilde{L}(\lambda), \widetilde{L}(\nu))}.
\end{align}

Proof. Mutatis mutandis the proof of [AS, Theorem 6.3]. \hfill \Box

Example 6. Consider $\mathfrak{g} = \mathfrak{pe}(2)$ and $\lambda = 2\epsilon_2 \in \mathfrak{h}^*$. Set $\alpha := \epsilon_1 - \epsilon_2$. We are going to show that the socle and radical of $U_\alpha(\lambda)$ are not isomorphic. Recall that $\omega_2 := \epsilon_1 + \epsilon_2$. By [CC, Lemma 5.11] it follows that $\widetilde{M}(\lambda) = K(L(\lambda)) = \widetilde{L}(\lambda)$. We provide two methods to prove the conclusion.

Method 1. We have $\text{ch} T_{s_\alpha} \widetilde{L}(\lambda) = \text{ch} \widetilde{M}(s_\alpha \cdot \lambda) = \text{ch} \widetilde{L}(\lambda) + \text{ch} \widetilde{L}(s_\alpha \cdot \lambda - \Delta)$ by [CC, Lemma 6.5]. It follows that

\begin{align}
\text{(3.3) } \text{ch} U_\alpha(\lambda) = \text{ch} \widetilde{L}(s_\alpha \cdot \lambda) + \text{ch} \widetilde{L}(s_\alpha \cdot \lambda - \omega_2) = \text{ch} \widetilde{L}(\omega_2) + \text{ch} \widetilde{L}(0).
\end{align}

We claim that the socle of $U_\alpha(\lambda)$ is $\widetilde{L}(0)$. To see this, we consider the following short exact sequence

\begin{align}
0 \to \widetilde{L}(\lambda) \to \widetilde{M}(\omega_2) \to K(L(\omega_2)) \to 0,
\end{align}

obtained by applying the Kac functor $K(-)$ to the short exact sequence $0 \to L(\lambda) \to M(\omega_2) \to L(\omega_2) \to 0$. Since the socle of $K(L(\omega_2))$ is isomorphic to $\widetilde{L}(0)$ and $\widetilde{M}(\omega_2)$ has simple socle (see, e.g., [CCM, Theorem 51]) isomorphic to $\widetilde{L}(\lambda)$, we may conclude that $\text{Ext}^1_{\mathcal{O}}(\widetilde{L}(0), \widetilde{L}(\lambda)) \neq 0$. By Proposition 5 it follows that $\widetilde{L}(0)$ is isomorphic to a submodule of $U_\alpha(\lambda)$.

Now, let

\begin{align}
0 \to \widetilde{L}(\lambda) \to E \xrightarrow{f} \widetilde{L}(\omega_2) \to 0,
\end{align}

be a short exact sequence. Since $\lambda$ can not be written as a sum of $\omega_2$ and positive roots, we may conclude that the preimage of the highest weight vector of $\widetilde{L}(\omega_2)$ under $f$ is again a highest weight vector of $E$, namely, $E$ is a quotient of $\widetilde{M}(\omega_2)$. By [CC, Lemma 6.1] and [CC, Lemma 5.11], the socle of $\widetilde{M}(\omega_2)$ is $\widetilde{L}(\lambda)$, and we have a non-split short exact sequence of the radical of $\widetilde{M}(\omega_2)$:

\begin{align}
0 \to \widetilde{L}(\lambda) \to \text{rad} \widetilde{M}(\omega_2) \to \widetilde{L}(0) \to 0,
\end{align}

which implies that (3.3) is split. Namely, we have $\text{Ext}^1_{\mathcal{O}}(\widetilde{L}(\omega_2), \widetilde{L}(\lambda)) = 0$. Consequently, the socle and top of $U_\alpha(\lambda)$ are isomorphic to $\widetilde{L}(0)$ and $\widetilde{L}(\omega_2)$, respectively.

Method 2. By [AS, Theorem 2.3] we have $T^0_{s_\alpha}(L(\lambda)) = M(s_\alpha \cdot \lambda)^\vee$, where $M(s_\alpha \cdot \lambda)^\vee$ denotes the dual Verma module in the sense of [HM, Section 3.3]. We claim that

\begin{align}
T_{s_\alpha} \widetilde{L}(\lambda) = T_{s_\alpha} K(L(\lambda)) \cong K(M(s_\alpha \cdot \lambda)^\vee).
\end{align}
To see this, we note that $T^0 g \cdot \mu L(\mu) = T^0 g \cdot \mu M(\mu) \cong M(\mu)$, which has a unique $g \geq 0$-module structure, and so $T^0 g M(\mu)$ contains $T_s^0 g M(\mu) \cong Ind_{g \geq 0} g T_s^0 g M(\mu)$, as desired. This implies that there is a short exact sequence

$$0 \to L(\mu) \to E \to M(\mu) \to 0$$

Consequently, $L(\mu)$ is isomorphic to $M(\mu)$. We will generalize this result and show the existence of non-semisimple Jantzen middles for arbitrary $\mu$. The main goal of this section is to prove an analogue for Lie superalgebras in our setting, including $\mathfrak{pe}(n)$. We then use this result to complete the proof of Theorem B introduced in Section II.

The following lemma is an analogue of [Co1, Corollary 5.7] for any Lie superalgebra of type I-0.

**Lemma 7.** Let $\alpha \in \Pi_0$. Then for any $\alpha$-free modules $M, V \in \mathcal{O}$ and $\tilde{\alpha}$-free module $N \in \mathcal{O}$, we have

$$\text{Hom}_\mathcal{O}(T_{\tilde{s}_\alpha} M, DN) \cong \text{Hom}_\mathcal{O}(T_{\tilde{s}_\alpha} N, DM).$$

$$\text{Ext}_\mathcal{O}^k(T_{\tilde{s}_\alpha} M, T_{\tilde{s}_\alpha} V) \cong \text{Ext}_\mathcal{O}^k(M, V), \text{ for any } k \geq 0.$$

**Proof.** We shall adapt the proof of [Co1, Corollary 5.7] to establish (3.8). We compute

$$\text{Hom}_\mathcal{O}(T_{\tilde{s}_\alpha} M, DN) \cong \text{Hom}_\mathcal{O}(\mathcal{T} T_{\tilde{s}_\alpha} M, DN)$$

$$\cong \text{Hom}_\mathcal{D}^{\mathcal{O}}(M, \mathcal{R} G_{s_a} DN) \cong \text{Hom}_\mathcal{D}^{\mathcal{O}}(M, D \mathcal{T} T_{\tilde{s}_\alpha} N)$$

$$\cong \text{Hom}_\mathcal{O}(M, DT_{\tilde{s}_\alpha} N) \cong \text{Hom}_\mathcal{O}(T_{\tilde{s}_\alpha} N, DM).$$

For any $k \in \mathbb{Z}$, let $(-)[k]$ be the corresponding shift functor on $\mathcal{D}^{\mathcal{O}}$. To obtain (3.9), we use the argument as in proof of [Co1, Corollary 5.7 (2)] and compute

$$\text{Ext}_\mathcal{O}^k(T_{\tilde{s}_\alpha} M, T_{\tilde{s}_\alpha} V) \cong \text{Hom}_\mathcal{D}^{\mathcal{O}}(\mathcal{T} T_{\tilde{s}_\alpha} M, \mathcal{T} T_{\tilde{s}_\alpha} V[k])$$

$$\cong \text{Hom}_\mathcal{D}^{\mathcal{O}}(M, V[k]) \cong \text{Ext}_\mathcal{O}^k(M, V).$$

The conclusion follows. \qed

The following non-vanishing property of Ext-group will be helpful.

**Lemma 8.** Suppose that $\text{Ext}_\mathcal{O}^1(M(\lambda), L(\mu)) \neq 0$, for some $\lambda, \mu \in \mathfrak{h}^*$. Then we have $\text{Ext}_\mathcal{O}^1(M(\lambda), \tilde{L}(\mu)) \neq 0$.

**Proof.** Let

$$0 \to L(\mu) \to E \to M(\mu) \to 0$$

be a non-split short exact sequence in $\mathcal{O}_0$. We note that every maximal submodule of $E$ contains $L(\mu)$ (otherwise (3.10) is split). Since $M(\lambda)$ has a unique maximal submodule,
we may conclude $E$ has a simple top. Applying the Kac functor $K(-)$, we obtain a short exact sequence in $O$

\[(3.11) \quad 0 \rightarrow K(L(\mu)) \rightarrow K(E) \rightarrow \tilde{M}(\lambda) \rightarrow 0.\]

By [CCM, Theorem 51], the module $K(E)$ has a simple top. Now, consider the short exact sequence

\[(3.12) \quad 0 \rightarrow \tilde{L}(\mu) \rightarrow K(E)/\text{rad}K(L(\mu)) \rightarrow \tilde{M}(\lambda) \rightarrow 0.\]

Since $K(E)$ has a simple top, we may conclude that (3.12) is non-split. The conclusion follows.

\[\square\]

3.2.3. Abstract Kazhdan-Lusztig theory. We recall the definition of an abstract Kazhdan-Lusztig theory formulated by Cline, Parshall and Scott in [CPS2, Definition 3.3]. Let $\mathcal{C}$ be a highest weight category with weight poset $\Lambda$, simple objects $S(\lambda)$, induced objects $A(\lambda)$, Weyl objects $V(\lambda)$ and a length function $\ell: \Lambda \rightarrow \mathbb{Z}$ in the sense of [CPS2, Section 1] (see also [CPS3, Definition 2.1]). Then $\mathcal{C}$ is said to have an abstract Kazhdan-Lusztig theory relative to $\ell$ provided that

\[(3.13) \quad \text{Ext}^1_{\mathcal{C}}(S(\lambda), A(\mu)) \neq 0 \Rightarrow \ell(\lambda) - \ell(\mu) \equiv n \pmod{2},\]

\[(3.14) \quad \text{Ext}^1_{\mathcal{C}}(V(\lambda), S(\mu)) \neq 0 \Rightarrow \ell(\lambda) - \ell(\mu) \equiv n \pmod{2},\]

for any $n$, $\lambda, \mu \in \Lambda$; see [CPS2, CPS3, Sc, Pa] for the background, examples and discussions.

For the category $O_\mathbb{F}$ of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, it is proved in [CS, Corollary 3.3] that the Brundan-Kazhdan-Lusztig theory formulated in [Br] and established in [CLW, BLW] is an abstract Kazhdan-Lusztig theory. For a weight $\eta \in \mathfrak{h}^*$, we denote by $O_\eta$ the indecomposable block of $O$ that contains $\tilde{L}(\eta)$. The following theorem is a restatement of Theorem A, which is an analogue of [Co1, Theorem 6.4] for Lie superalgebras of type I-0.

**Theorem 9.** Let $\eta$ be integral. Suppose that $O_\eta$ contains a non-semisimple Jantzen middle. Then $O_\eta$ does not admit a Kazhdan-Lusztig theory in the sense of [CPS2, Definition 3.3].

**Proof.** We shall adapt the proof of [Co1, Theorem 6.4] to obtain the conclusion for Lie superalgebra $\mathfrak{g}$ of type I-0 by establishing all essential ingredients. To see this, let $\alpha \in \Pi_0$ and $\lambda, \mu, \gamma \in \mathcal{P}$ with $\tilde{\lambda} \neq s_\alpha \cdot \tilde{\mu}$. We claim that if $\tilde{L}(\lambda)$, $\tilde{L}(\gamma)$ are $\alpha$-free and $\tilde{L}(\mu)$ is $\alpha$-finite then we have

\[(3.15) \quad \text{Hom}_O(U_\alpha(\lambda), \tilde{M}(\gamma)) = \text{Hom}_O(\tilde{M}(\gamma), U_\alpha(\lambda)) = 0,\]

and inclusions

\[(3.16) \quad \text{Hom}_O(U_\alpha(\lambda), \tilde{M}(\mu)) \hookrightarrow \text{Ext}^1_{O}(\tilde{L}(\lambda), \tilde{M}(\mu)),\]

\[(3.17) \quad \text{Hom}_O(\tilde{M}(\mu), U_\alpha(\lambda)) \hookrightarrow \text{Ext}^1_{O}(\tilde{M}(\mu), \tilde{L}(\lambda)).\]

The equality (3.15) is an immediate consequence of Proposition 4. Now we are going to show (3.16). To see this, we apply the functor $\text{Hom}_O(-, \tilde{M}(\mu))$ to the short exact
sequence

\[(3.18) \quad 0 \to U_\alpha(\lambda) \to T_{s_\alpha} \bar{L}(\lambda) \to \bar{L}(\lambda) \to 0,\]

and obtain a long exact sequence which contains

\[(3.19) \quad \text{Hom}_O(T_{s_\alpha} \bar{L}(\lambda), \bar{M}^\vee(\mu)) \to \text{Hom}_O(U_\alpha(\lambda), \bar{M}^\vee(\mu)) \to \text{Ext}^1_O(\bar{L}(\lambda), \bar{M}^\vee(\mu)).\]

We recall the following isomorphisms

\[\bar{M}^\vee(\mu) \cong D\bar{M}_{\bar{v}}(\bar{\mu}), \quad \bar{L}_{\bar{v}}(\bar{\lambda}) \cong D\bar{L}(\lambda)\]

from \((2.8)\). Also, we note that \(\bar{\mu}\) is \(\hat{\alpha}\)-finite and so \(T^0_{s_\alpha} M(\bar{\mu}) \cong M(s_{\hat{\alpha}} \cdot \bar{\mu})\), which implies that \(T_{s_\alpha} \bar{M}_{\bar{v}}(\bar{\mu}) \cong \bar{M}_{\bar{v}}(s_{\hat{\alpha}} \cdot \bar{\mu})\) for the same reason as that given for \((3.7)\). Therefore by Lemma \(7\) we have

\[(3.20) \quad \text{Hom}_O(T_{s_\alpha} \bar{L}(\lambda), \bar{M}^\vee(\mu)) \cong \text{Hom}_O(\bar{M}_{\bar{v}}(s_{\hat{\alpha}} \cdot \bar{\mu}), \bar{L}_{\bar{v}}(\bar{\lambda})) = 0.\]

This proves \((3.16)\).

Next, we proceed with the proof of \((3.17)\). By \([AL\) Lemma 6.2] and \([CoM1\) Lemma 5.7], we have a four term exact sequence

\[(3.21) \quad 0 \to M(s_{\alpha} \cdot \mu) \to M(\mu) \to T^0_{s_\alpha} M(s_{\alpha} \cdot \mu) \to M(s_{\alpha} \cdot \mu) \to 0.\]

Applying the Kac functor \(K(-)\), we obtain a four term exact sequence of \(\mathfrak{g}\)-modules

\[(3.22) \quad 0 \to \bar{M}(s_{\alpha} \cdot \mu) \to \bar{M}(\mu) \to K(T^0_{s_\alpha} M(s_{\alpha} \cdot \mu)) \to \bar{M}(s_{\alpha} \cdot \mu) \to 0.\]

We claim that \(K(T^0_{s_\alpha} M(s_{\alpha} \cdot \mu)) \cong T_{s_\alpha} K(M(s_{\alpha} \cdot \mu)).\) To see this, let \(M\) denote the \(\mathfrak{g}_{\geq 0}\)-module which is \(M(s_{\alpha} \cdot \mu)\) with trivial \(\mathfrak{g}_1\)-action. With slightly abusing notations, we again denote by \(T_{s_\alpha}\) the twisting functor for \(\mathfrak{g}_{\geq 0}\). Since

\[\text{Res}^\mathfrak{g}_{\geq 0} T_{s_\alpha} M \cong T^0_{s_\alpha} M(s_{\alpha} \cdot \mu)\]

is a quotient of \(P(s_{\alpha} \cdot \mu)\), it follows that \(\text{Res}^\mathfrak{g}_{\geq 0} T_{s_\alpha} M\) is indecomposable. Therefore, we have either \((T_{s_\alpha} M)_{\mathfrak{g}_0} = 0\) or \((T_{s_\alpha} M)_{\mathfrak{g}_1} = 0\), and so the \(\mathfrak{g}_1\)-action on \(T_{s_\alpha} M\) is trivial. Consequently, we have \(K(T^0_{s_\alpha} M(s_{\alpha} \cdot \mu)) \cong \text{Ind}^\mathfrak{g}_{\geq 0} T_{s_\alpha} M \cong T_{s_\alpha} K(M(s_{\alpha} \cdot \mu)).\) Using \((3.22)\), the equality \((3.17)\) follows from an argument identical to the proof of \([Co1\) Corollary 6.5 (2)].

Suppose on the contrary that \(\mathcal{O}_\eta\) admits a Kazhdan-Lusztig theory relative to a length function \(\ell\) from the set of highest weights of simple modules of \(\mathcal{O}_\eta\) to \(\mathbb{Z}\). We claim that, for given \(\bar{L}(\mu_1), \bar{L}(\mu_2), U_\alpha(\lambda) \in \mathcal{O}_\eta,\) if

\[(3.23) \quad \text{Hom}_\mathcal{O}(U_\alpha(\lambda), \bar{M}^\vee(\mu_1)) \neq 0, \quad \text{Hom}_\mathcal{O}(U_\alpha(\lambda), \bar{M}^\vee(\mu_2)) \neq 0,\]

then we have

\[(3.24) \quad \text{Ext}^1_\mathcal{O}(\bar{M}(\mu_1), \bar{L}(\mu_2)) = 0.\]

To see this, we first assume that \(\bar{\lambda} \neq s_{\hat{\alpha}} \cdot \bar{\mu}_1, s_{\hat{\alpha}} \cdot \bar{\mu}_2\). Then \(\text{Ext}^1_\mathcal{O}(\bar{L}(\lambda), \bar{M}^\vee(\mu_1)) \neq 0, \quad \text{Ext}^1_\mathcal{O}(\bar{L}(\lambda), \bar{M}^\vee(\mu_2)) \neq 0\) by \((3.16)\), which implies that \(\ell(\mu_1) \equiv \ell(\mu_2) \pmod{2}\). Hence \((3.24)\) follows.

We now show that \((3.23)\) implies \((3.24)\) in the case when \(\bar{\lambda} \in \{s_{\hat{\alpha}} \cdot \bar{\mu}_1, s_{\hat{\alpha}} \cdot \bar{\mu}_2\}\). To see this, we only need to consider the case that \(\bar{\lambda} = s_{\hat{\alpha}} \cdot \bar{\mu}_1\) and \(\bar{\lambda} \neq s_{\hat{\alpha}} \cdot \bar{\mu}_2\). By
(3.16) again, we have $\text{Ext}_1^O(\tilde{L}(\lambda), \tilde{M}'(\mu_2)) \neq 0$, and so $\ell(\lambda) \equiv \ell(\mu_2) + 1 \pmod 2$. Also, we note that $\lambda = s_{\alpha} \cdot \mu_1$ is equivalent to $\lambda = s_{\alpha} \cdot \mu_1$. It then follows from Lemma 5 and [CoM2, Proposition 6.17] that $\text{Ext}_1^O(M(\lambda), \tilde{L}(\mu_1)) > 0$, which implies that $\ell(\lambda) \equiv \ell(\mu_1) + 1 \pmod 2$. Consequently, since $\ell(\mu_1) \equiv \ell(\mu_2) \pmod 2$ we have $\text{Ext}_1^O(M(\mu_1), \tilde{L}(\mu_2)) = 0$, as desired.

Similarly, if

$$
\ell(\mu_1) \equiv \ell(\mu_2) \pmod 2 \ \text{by} \ (3.17) \ \text{for the same reason. We may conclude that}
$$

$$
\text{Ext}_1^O(L(\mu_1), \tilde{M}'(\mu_2)) = 0. \ \text{Consequently, the Jantzen middle } U_\alpha(\lambda) \ \text{is always semisimple by [CPS2, Theorem 4.1], which contradicts to our assumption. This completes the proof.} \quad \Box
$$

4. The Jantzen Middles for $\mathfrak{pe}(n)$

In this section, we consider the periplectic Lie superalgebra $\mathfrak{g} := \mathfrak{pe}(n)$. Recall that $\omega_n := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$.

4.1. Jantzen Middles in Typical Blocks. We define the equivalence relation $\sim$ on $\mathcal{P}$ transitivity generated by $\lambda \sim w \cdot \lambda$ and $\lambda \sim \lambda \pm 2\epsilon_k$, for $w \in W$ and $1 \leq k \leq n$; see [CC, Section 5.2]. In particular, we have the following decomposition of $O^Z$ into indecomposable blocks.

**Lemma 10. ([CC Theorem 5.4])** For any $\lambda, \mu \in \mathcal{P}$, we have

$$
\tilde{L}(\mu) \in O_\lambda \iff \lambda \sim \mu.
$$

In particular, we have

$$
O^Z = \bigoplus_{i=0}^{n} O_{\partial^i + k\omega_n},
$$

with $\partial^i := i \epsilon_1 + (i - 1) \epsilon_2 + \cdots + \epsilon_i$.

For any $k \in \mathbb{C}$, we have an equivalence

$$
C_{k\omega_n} \otimes - : O_{\partial^i} \cong O_{\partial^i + k\omega_n},
$$

see [CC, Lemma 5.10]. Therefore, we shall focus on the blocks $O_{\rho^i}, \ldots, O_{\rho^n}$. We recall the notion of typicality of weights for $\mathfrak{pe}(n)$ from [Sel] Section 5 below. Let $T, T_+, T_-$ be the polynomials on $\mathfrak{h}^*$ given by

$$
T_\pm(\lambda) := \prod_{\alpha \in \Phi_0^+} (\langle \lambda + \rho, \alpha \rangle \pm 1) = \prod_{i<j} (\lambda_i - \lambda_j + j - i \pm 1), \ T(\lambda) := T_+(\lambda)T_-(\lambda).
$$

Then $\lambda$ is called *typical* if $T(\lambda) \neq 0$, and *atypical* otherwise. A block $O_\lambda$ is said to be *atypical* if $\tilde{L}(\mu) \in O_\lambda$, for some atypical $\mu \in \mathfrak{h}^*$, and $O_\lambda$ is called *typical* otherwise. By Lemma 10 these blocks

$$
O_{\rho_i}, \ \text{for } i = 0, \ldots, n - 2
$$

are atypical. Conversely, if $\tilde{L}(\lambda)$ lies in one of

$$
O_{\rho_{n-1}}, O_{\rho_n},
$$

then $\lambda$ is typical.}
then $\lambda + \rho = \sum_{i=1}^{n}(\lambda + \rho)_i \epsilon_i$ satisfying that $(\lambda + \rho)_1, (\lambda + \rho)_2, \ldots, (\lambda + \rho)_n$ are either all even or all odd. Therefore, $\lambda$ is typical, and so both $O_{\rho_n-1}$, $O_{\rho_n}$ are typical.

It is shown in [CP] Theorem C, Theorem 4.6] that the characters of tilting modules in $\mathcal{O}_{\rho_n-1}$ and $\mathcal{O}_{\rho_n}$ are completely controlled by the Kazhdan-Lusztig polynomials of type A Lie algebras. Therefore it is natural to ask whether $\mathcal{O}_{\rho_n-1}$ and $\mathcal{O}_{\rho_n}$ admit abstract Kazhdan-Lusztig theories. The following proposition provides another evidence by showing the semisimplicity of Jantzen middle of $\mathcal{O}_{\rho_n-1}$ and $\mathcal{O}_{\rho_n}$.

**Theorem 11.** Suppose that $\lambda \in \mathfrak{h}^*$ is typical. Then $U_\alpha(\lambda)$ is either zero or semisimple, for any $\alpha \in \Pi_0$. In particular, all (non-zero) Jantzen middle in $\mathcal{O}_{\rho_n-1}$ and $\mathcal{O}_{\rho_n}$ are semisimple.

**Proof.** By [Sel] Lemma 3.2] (see, also [CC] Lemma 5.11], we have $\tilde{L}(\lambda) = K(L(\lambda))$. Set $s := s_\alpha$. We first claim that $T_s K(L(\lambda)) \cong K(T^0_s L(\lambda))$. To see this, consider the $\mathfrak{g}_{0+}$-module $L_\lambda$, which is the $\mathfrak{g}_0$-module $L(\lambda)$ with trivial $\mathfrak{g}_1$-action. With slightly abusing notations, we denote by $T_s$ the twisting functor for $\mathfrak{g}_{0+}$-modules again. Note that $\text{Res}_{\mathfrak{g}_0} T_s L_\lambda \cong T^0_s L(\lambda)$ is either zero or an indecomposable $\mathfrak{g}_0$-module by Proposition 4. Since we have either $(T_s L_\lambda)_0 = 0$ or $(T_s L_\lambda)_1 = 0$, it follows that $\mathfrak{g}_1$ acts on the $\mathfrak{g}_{0+}$-module $T_s L_\lambda$ trivially. We compute

$$T_s K(L(\lambda)) \cong \text{Ind}_{\mathfrak{g}_{0+}}^{\mathfrak{g}_0} T_s L_\lambda \cong K(T^0_s L(\lambda)).$$

By [CCM] Theorem 51] the Kac functor $K(-)$ preserves the length of top. Consequently, we have

$$U_\alpha(\lambda) = \text{rad} T_s \tilde{L}(\lambda) \cong \text{rad} K(T^0_s L(\lambda)) \cong K(\text{rad} T^0_s L(\lambda)).$$

Since $T^0_s (-)$ is right exact and $\text{ch} T^0_s M(\lambda) = \text{ch} M(s \cdot \lambda)$, we may conclude that every composition factor of $\text{rad} T^0_s L(\lambda)$ is of the form $L(\mu)$ with $\mu \in W \cdot \lambda$. Therefore, if $m_{\lambda,\mu} := [\text{rad} T^0_s L(\lambda) : L(\mu)] > 0$, then $\mu$ is typical. This implies that $U_\alpha(\lambda) \cong \bigoplus_{\mu \in \mathfrak{h}^*} \tilde{L}(\mu)_{\mu + m_{\lambda,\mu}}$ is either zero or semisimple. The conclusion follows. 

**Example 12.** Consider $\mathfrak{g} = \mathfrak{pe}(2)$ with the unique positive even root $\alpha := \epsilon_1 - \epsilon_2$. We have computed Jantzen middles $U_\alpha(\lambda)$ for atypical weights $\lambda$ in Example 6. Now, we consider Jantzen middles $U_\alpha(\lambda)$ in the case when $\lambda$ are typical. Set $T := T_{\lambda_0}$.

Suppose that $\lambda = a \epsilon_1 + b \epsilon_2 \in \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2$ is typical, namely, $b \neq a, a + 2$. If $b > a + 2$ then $T^0_s M(\lambda)$ is the dual Verma module over $\mathfrak{g}_0$ with top and radical isomorphic to $L(\lambda)$ and $L(s \cdot \lambda)$, respectively. It then follows that

$$U_\alpha(\lambda) = \text{rad} T_s L(\lambda) \cong K(\text{rad} T^0_s M(\lambda)) \cong \tilde{L}(s \cdot \lambda).$$

If $b = a + 1$ then $U_\alpha(\lambda) = 0$ by the character formulas of twisted simple modules computed in [CC] Lemma 6.7]. We now collect our results in the following complete classification of Jantzen middles of $\mathfrak{pe}(2)$ (see also [CC] Lemma 6.7]):

$$U_\alpha(\lambda) = \begin{cases} 0 & \text{if } b \leq a + 1, \\ K(L(\lambda + \epsilon_1 - \epsilon_2)) & \text{if } b = a + 2, \\ \tilde{L}(s \cdot \lambda) & \text{if } b > a + 2. \end{cases}$$

(4.4)
4.2. The absence of Kazhdan-Lusztig theory for \( \mathfrak{p}(n) \). In this section, we turn to the semisimplicity of Jantzen middles for atypical blocks of \( \mathfrak{p}(n) \).

Suppose that \( \mathcal{O}_\eta \) is an atypical block of \( \mathcal{O}^\mathbb{Z} \). Applying the equivalence \((4.2)\) if necessary, we may assume that \( \eta \in \bigoplus_{i=1}^n \mathbb{Z} \varepsilon_i \). For any weight \( \gamma \in \mathcal{P} \), we let \( \gamma_i \) be determined by \( \gamma = \sum_{i=1}^n \gamma_i \varepsilon_i \). Since \( \mathcal{O}_\eta \) is atypical, there are \( 1 \leq i, j \leq n \) such that \((\eta + \rho)_i \not\equiv (\eta + \rho)_j \text{ (mod 2)}\) by \((4.1)\). In addition, there is an anti-dominant weight \( \lambda \in \mathcal{P} \) such that \( \bar{L}(\lambda) \in \mathcal{O}_\eta \) satisfying

\[
(\lambda + \rho)_1 = 0, \quad (\lambda + \rho)_2 = 1, \quad \text{and} \quad \lambda_{k+1} - \lambda_k > 1,
\]

for \( k = 2, 3, \ldots, n - 1 \). The following proposition shows that the corresponding Jantzen middle \( U_\alpha(\lambda) \) in \( \mathcal{O}_\eta \) is non-semisimple.

**Proposition 13.** Retain the notations above. Then \( U_\alpha(\lambda) \) is not semisimple.

**Proof.** Let \( \alpha := \varepsilon_1 - \varepsilon_2 \) and \( s := s_\alpha \). First, we note that the socle of \( K(L(s \cdot \lambda)) \) is \( \bar{L}_{bo}(s \cdot \lambda - (n - 1)\omega_n) \). Put \( \mu := s \cdot \lambda - \varepsilon_1 - \varepsilon_2 = \lambda - 2\varepsilon_2 \). It follows from Lemma \([2]\) (see also Example \([3]\)) that

\[
\text{soc} K(L(s \cdot \lambda)) = \bar{L}_{bo}(s \cdot \lambda - (n - 1)\omega_n) = \bar{L}(\mu).
\]

We firstly claim that \( \text{Ext}^1_{bo}(\bar{L}(\lambda), \bar{L}(\mu)) = 0 \), which will imply that \( \text{top}(U_\alpha(\lambda)) : \bar{L}(\mu) = 0 \) by Proposition \([5]\). To see this, we let

\[
0 \to \bar{L}(\mu) \to E \xrightarrow{\pi} \bar{L}(\lambda) \to 0,
\]

be a short exact sequence. Let \( v_\lambda \in E \) be the preimage of the highest weight vector of \( \bar{L}(\lambda) \) under \( \pi \). By weight consideration we have \( n^+ v_\lambda = 0 \). This means that \( E \) is an image of the Verma module \( \tilde{M}(\lambda) \). Since \( \lambda \) is anti-dominant, it follows that \( \tilde{M}(\lambda) = \bar{L}(\lambda) \) by \([CC\text{ Lemma 5.11}]\); see also \([Se1\text{ Lemma 3.2}]\). Consequently, the short exact sequence \((4.5)\) is split.

Next, we show that \( \text{Ext}^1_{bo}(\bar{L}(\mu), \bar{L}(\lambda)) \neq 0 \), which will imply that \( \text{soc} U_\alpha(\lambda) : \bar{L}(\mu) \neq 0 \) by Proposition \([5]\). We recall that every Verma module has a simple socle by \([CCM\text{ Theorem 51}]\). Applying the Kac functor \( K(-) \) to the short exact sequence \( 0 \to L(\lambda) \to M(s \cdot \lambda) \to L(s \cdot \lambda) \to 0 \), we then obtain a non-split short exact sequence

\[
0 \to \bar{L}(\lambda) \to \tilde{M}(s \cdot \lambda) \to K(L(s \cdot \lambda)) \to 0,
\]

and so \( \bar{L}(\lambda) = \text{soc} \tilde{M}(s \cdot \lambda) \).

Since \( \text{ch} T_s \bar{L}(\lambda) = \text{ch} \tilde{M}(s \cdot \lambda) \), it follows that \( \text{top}(U_\alpha(\lambda)) \neq U_\alpha(\lambda) \). Hence, we have already proved that \( U_\alpha(\lambda) \) is not semisimple.

We continue to show the stronger claim that \( \text{Ext}^1_{bo}(\bar{L}(\mu), \bar{L}(\lambda)) \neq 0 \). Since \( \tilde{M}(s \cdot \lambda) \) has simple socle, there is a submodule \( E \subseteq \tilde{M}(s \cdot \lambda) \) and a non-split short exact sequence

\[
0 \to \bar{L}(\lambda) \to E \to \bar{L}(\mu) \to 0.
\]

This completes the proof. \(\square\)

The conclusion of Theorem B is a direct consequence of Theorem \([9]\) Theorem \([11]\) and Proposition \([13]\).
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