Signaling to Relativistic Observers: 
An Einstein–Shannon–Riemann Encounter

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A communication scenario is described involving a series of events triggered by a transmitter and observed by a receiver experiencing relativistic time dilation. The message selected by the transmitter is assumed to be encoded in the events' timings and is required to be perfectly recovered by the receiver, regardless of the difference in clock rates in the two frames of reference. It is shown that the largest proportion of the space of all \(k\)-event signals that can be selected as a code ensuring error-free information transfer in this setting equals \(\zeta(k)^{-1}\), where \(\zeta\) is the Riemann zeta function.

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INTRODUCTION

Information theory [1, 7] – one of the major scientific achievements of the second half of the 20th century – was developed by Shannon as a formal framework for the study of transmission and processing of information in the classical domain. In this paper, we introduce and analyze a problem which brings information theory in the relativistic context and is, in particular, meant to illustrate the effects of time dilation [3, 4] on the ultimate limits of information transfer.

Model description

Consider the following model of communication between two parties: Alice triggers \(k\) events at moments \(t_1, t_2, \ldots, t_k\), which are selected from the set of integers \(\{1, 2, \ldots, N\}\) according to a clock in her reference frame, that is \(t_i \in \{1, 2, \ldots, N\}, 1 \leq t_1 < t_2 < \cdots < t_k \leq N\), and they are detected by Bob at moments \(\alpha t_1, \alpha t_2, \ldots, \alpha t_k\) according to his own clock. The factor \(\alpha\), modeling the difference in clock rates in the two frames of reference, is not a priori known by either side. (The two parties are assumed to be synchronized in the frames of reference, is not a priori known by either side. For example, for \(k = 2\), if Bob were to detect the two events at the moments \((2, 4)\), it would be impossible for him to deduce with certainty whether \(t = (1, 2)\) or \(t = (2, 4)\). In this case, the two parties have to agree beforehand to restrict the set of allowed signals \(t\) to a proper subset of \(T_{N,k}\) (a code) in order to enable Bob to always infer the transmitted \(k\)-tuple correctly, regardless of \(\alpha\). In other words, in the above scenario time dilation represents signal distortion, and the following question then naturally arises: how many bits of information can be reliably conveyed to the receiver by using the simple form of communication just described?

We adopt an information-theoretic approach and model the signal distortion – the unknown factor \(\alpha\) – as an absolutely continuous random variable with probability density function \(p_\alpha(\cdot)\) whose support is either \([1, \overline{\alpha}]\), for some constant \(\overline{\alpha} \in (1, \infty)\), or \([1, \infty)\). The seemingly more general model where the support of \(p_\alpha(\cdot)\) is \([\alpha, \overline{\alpha}]\), is equivalent to the case \([1, \overline{\alpha}/\alpha]\) from the communication viewpoint, so the lower end of the interval may be taken to be 1 without loss of generality.

Remark 1. The difference in clock rates that is assumed in the model may be caused by various physical effects, such as relative motion between Alice and Bob, difference in the gravitational potential between them, clock imperfections, etc. We note that our results apply to all models with a linear change of time scale \(t \mapsto \alpha t\) and do not depend on the physical reasons causing this change.

Zero-error codes

We say that two input signals \(t', t'' \in T_{N,k}\) are confusable if the receiver can confuse one with the other with
positive probability, meaning that the corresponding sets of output signals \( \{ \alpha' t \} \) and \( \{ \alpha'' t' \} \) have infinite intersection (here \( \alpha' \) and \( \alpha'' \) vary through the support of \( p_0(\cdot) \)). A subset \( S \subseteq T_{N,k} \) is called a zero-error code [8] if any two distinct elements of \( S \) are non-confusable. Elements of a code are called codewords. Hence, a zero-error code is a set of permissible signals that can be unambiguously distinguished by the receiver. In other words, based on the observed signal, Bob will be able to identify the codeword which produced that signal, and thereby recover the transmitted information, with probability 1.

A zero-error code \( S \subseteq T_{N,k} \) is said to be optimal if it has the largest cardinality among all such codes in \( T_{N,k} \). This maximum cardinality is in general difficult to determine exactly for arbitrary parameters \( N, k \), and it is instructive to focus on its asymptotic behavior. To that end, we define the maximum asymptotic density of codes in the space of \( k \)-event signals:

\[
\delta_\alpha(k) = \lim_{N \to \infty} \max_{S \subseteq T_{N,k}} \frac{|S|}{N^k}, \tag{2}
\]

where the maximum is taken over all zero-error codes \( S \subseteq T_{N,k} \), and where \( \alpha \in (1, \infty) \) is the supremum of the support of \( p_0(\cdot) \). (It will be evident from the analysis that the cardinality of optimal codes depends on \( p_0(\cdot) \) only through \( \alpha \), justifying the notation in (2) ). Our aim is to characterize the quantity \( \delta_\alpha(k) \) for every \( k \) and \( \alpha \).

**OPTIMAL SIGNAL SETS AND THEIR DENSITY**

We first consider the case of unbounded indeterminacy of the time dilation factor, meaning that the support of the probability density function \( p_0(\cdot) \) is the entire half-line \([1, \infty)\). Time dilation distorts the signal represented by the point \( t \in T(N, k) \) by multiplying this point by a random factor \( \alpha \), or equivalently, by moving the point \( t \) by a random amount along its “line of sight” from the origin; see Fig. 1. Since any two points lying on the same line of sight from the origin are confusable (because \( \Pr\{ \alpha \geq \alpha_0 \} > 0 \) for any fixed \( \alpha_0 \), no two such points can belong to the same zero-error code. Therefore, an optimal code is obtained by selecting exactly one point on each line of sight, and the simplest choice is to select the first point encountered on each line. The points \( t \) in the grid \( \mathbb{N}^k \) that are encountered first when going along the straight lines from the origin are those that satisfy the condition \( \gcd(t') = \gcd(t_1, \ldots, t_k) = 1 \), where \( \gcd \) denotes the greatest common divisor. This shows that an optimal zero-error code for this model is:

\[
C_{N,k} = \bigg\{ t \in T_{N,k} : \gcd(t) = 1 \bigg\}. \tag{3}
\]

In the case when the support of \( p_0(\cdot) \) is a finite interval \([1, \overline{\alpha}]\), a code can be constructed by applying the following greedy procedure on every line of sight from the origin: select the first point \( t \) on the line (the one satisfying \( \gcd(t) = 1 \) ) as a codeword, and exclude the points in \( \{ \alpha t : \alpha \in [1, \overline{\alpha}] \} \) which are confusable with \( t \); then select the next point encountered on the line, \( [\overline{\alpha}] t \), as a codeword, and exclude the points in \( \{ \alpha[t] t : \alpha \in [1, \overline{\alpha}] \} \); and so on. This iterative procedure results in the following set:

\[
D_{N,k} = \left( \bigcup_{n=1}^{\infty} d_n C_{N,k} \right) \cap T_{N,k}, \tag{4}
\]

where \((d_n)_{n=1}^{\infty}\) is the sequence defined by the following recursion:

\[
d_n = [\overline{\alpha} d_{n-1}], \quad d_1 = 1, \tag{5}
\]

\(d_n C_{N,k}\) stands for \( \{ d_n t : t \in C_{N,k} \} \), and the notation \( \cup \) emphasizes that the sets in the union are pairwise disjoint. For reasons of simplicity, we do not make the dependence of \( d_n \) on \( \overline{\alpha} \) explicit in the notation. Note that, for \( \overline{\alpha} \in \mathbb{N}, \overline{\alpha} \geq 2 \), (5) simplifies to:

\[
d_n = \overline{\alpha}^{n-1}. \tag{6}
\]

It is evident from the construction that \( D_{N,k} \) is indeed a zero-error code: the only way two different codewords \( d_n t \) and \( d_{n+1} t \) can produce the same signal at the receiver’s end is if \( \alpha \) takes the value \( \overline{\alpha} \) (and if \( \overline{\alpha} \in \mathbb{N} \), which is a
zero-probability event. We demonstrate in the following theorem that the code $D_{N,k}$ is in fact optimal and, based on this observation, we obtain a characterization of the maximum asymptotic density $\delta(k)$. In order for the statement to be valid for the trivial case $\alpha = 1$ as well, we define $d_n = n$ for $\alpha = 1$, which is justified by a continuity argument (taking the limit $\alpha \to 1$ in (5)).

Recall the definition of the Riemann zeta function [6]:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}. \quad (7)$$

**Theorem 2.** Fix $k \in \mathbb{N}$, $k \geq 2$, and $\alpha \in [1, \infty)$, and let $(d_n)_{n=1}^{\infty}$ be the sequence defined in (5). We then have:

$$\delta(\alpha, k) = \zeta(k)^{-1} \sum_{n=1}^{\infty} d_n^{-k}. \quad (8)$$

For $\alpha = \infty$, we have:

$$\delta(\infty, k) = \zeta(k)^{-1}. \quad (9)$$

**Proof.** Optimality of the code $D_{N,k}$ follows directly from the result of Shannon [8, Thm 3], which states that a zero-error code $S \subseteq T$ is optimal if there exists a mapping $f : T \to S$ with the property that $f(t') \neq f(t'')$ for any two non-confusable $t', t'' \in T$. The required function $f : T_{N,k} \to D_{N,k}$ in our case is defined by mapping all the points in the set $\{ad_n t : \alpha \in [1, \alpha]\}$ to $d_n \mathbf{t}$, for every $t \in T_{N,k}$ with $\gcd(t) = 1$, and every $n \geq 1$. Since any two non-confusable points in $T_{N,k}$ belong to different sets of the form $\{ad_n t : \alpha \in [1, \alpha]\}$ (i.e., either $n$ is different, or $t$ is, or both), they have different images under $f$. Therefore, it follows from the quoted result of Shannon that $D_{N,k}$ is optimal, implying that:

$$\delta(\alpha, k) = \lim_{N \to \infty} \frac{|D_{N,k}|}{(N!)} \quad (10)$$

Now consider the relation (4) and note that the subcodes $(d_n C_{N,k}) \cap T_{N,k}$ can be written as:

$$(d_n C_{N,k}) \cap T_{N,k} = \left\{ t \in T_{N,k} : \gcd(t) = d_n \right\} = d_n C_{[N/d_n],k}. \quad (11)$$

Their asymptotic density therefore equals:

$$\lim_{N \to \infty} \frac{|(d_n C_{N,k}) \cap T_{N,k}|}{|T_{N,k}|} = \lim_{N \to \infty} \frac{|C_{[N/d_n],k}|}{(N!)} = \lim_{N \to \infty} \frac{|C_{N,k}|}{(d_n N)!} = \lim_{N \to \infty} \frac{|C_{N,k}|}{N!} \frac{d_n N!}{d_n^k N!} = d_n^{-k} \delta(\infty, k), \quad (12)$$

where we have used the fact that the code $C_{N,k}$ from (3) is optimal for the case $\alpha = \infty$ and hence:

$$\delta(\infty, k) = \lim_{N \to \infty} \frac{|C_{N,k}|}{(N!)} \quad (13)$$

Now (4), (10) and (12) imply:

$$\delta(\alpha, k) = \delta(\infty, k) \sum_{n=1}^{\infty} d_n^{-k}. \quad (14)$$

Proving (8) is thus reduced to proving (9). The density $\delta(\infty, k)$ can be determined from (3), (13), and the fact that the probability of $k$ random positive integers being relatively prime equals $\zeta(k)^{-1}$ [5], but we give here a direct derivation as well. To this end, note that $\delta(1, k) = 1$ for every $k$. This is because the condition $\alpha = 1$ means that the time dilation factor is known exactly at the receiver, and hence the optimal code for this case is trivially the set of all possible input signals, $T_{N,k}$, which has density 1. We then get from (14) and the fact that $d_n = n$ when $\alpha = 1$:

$$1 = \delta(\infty, k) \sum_{n=1}^{\infty} n^{-k} = \delta(\infty, k) \zeta(k), \quad (15)$$

which completes the proof of the theorem. $lacksquare$

For integral values of $\alpha$, the resulting density can be expressed explicitly due to (6). Namely, for $\alpha \in \mathbb{N}, \alpha \geq 2$:

$$\delta(\alpha, k) = \frac{\alpha^k}{\zeta(k)(\alpha^k - 1)}. \quad (16)$$

As we have seen, when nothing is known about the factor $\alpha$, the relation (9) is obtained as an important special case of (8) (or, rather, the limiting case as $\alpha \to \infty$). In particular, the largest asymptotic density of a set of 2-event signals that are distinguishable by a receiver experiencing time dilation is:

$$\delta(\infty, 2) = \frac{6}{\pi^2}. \quad (17)$$

More generally, for any even $k = 2m$:

$$\delta(\infty, 2m) = \frac{(-1)^{m+1}2(2m)!}{(2m)^{2m}B_{2m}}, \quad (18)$$

where $B_{2m}$ are the Bernoulli numbers [2, Sec. 1.5].

**Remark 3.** A model closely related to the one presented in this paper, where, instead of relativistic time dilation, signals are distorted by a synchronization error known as clock drift, was analyzed in [9, 10]. Optimal zero-error codes for this model were determined in [9], though no estimate of their cardinality was given. It should be mentioned that the notion of “confusability” was defined in [9] by the requirement that the signals in
question cannot produce the same signal at the receiver's end. This definition is slightly different from ours which requires that the two input signals causing confusion at the receiver should be a zero-probability event. The consequent characterization of optimal codes in [9] is similar to (4), but with the sequence \((b_n)_{n=1}^\infty\) in place of \((d_n)_{n=1}^\infty\), where:

\[
b_n = \lfloor \alpha b_{n-1} + 1 \rfloor, \quad b_1 = 1. \tag{19}\]

In particular, for \(\alpha \in \mathbb{N}\):

\[
b_n = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{\alpha^n - 1}{\alpha - 1}. \tag{20}\]

It can be shown, in the same way as in the proof of Theorem 2, that the largest asymptotic density of codes defined as in [9] equals:

\[
\zeta(k)^{-1} \sum_{n=1}^\infty b_n^{-k}. \tag{21}\]

Notice that \(b_n \geq d_n\) for every \(n \geq 1\). In particular, for \(\alpha \in \mathbb{N}\), \(\alpha \geq 2\), the inequality is strict for every \(n \geq 2\), so the density in (21) is strictly smaller than the density \(\delta_{\alpha}(k)\) from (8). However, this is not always the case; for irrational \(\alpha\), we have \(b_n = d_n\) for every \(n \geq 1\), and so the two densities are equal.

**CLOSING REMARKS**

Quantifying information content and determining the fundamental limits of information transmission are two of the main directions of study in information theory. The lines of research that explore such questions in various physical systems have a long history in science, most notably in quantum information theory. In this paper, we have described a scenario where information transfer is considered in the relativistic context, and we have presented a result that quantifies the limits of communication in this model. The obtained solution, though simple, is interesting in that it provides a link between Shannon’s information theory, special relativity, and number theory. As such, it calls for further investigation of this and related problems, in particular of the much more difficult case of non-linear change of time scale, which may be caused by, e.g., accelerated observers.

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