Mixing and Decay Constants of Pseudoscalar Mesons: The Sequel

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Abstract

We present further tests and applications of the new \( \eta - \eta' \) mixing scheme recently proposed by us. The particle states are decomposed into orthonormal basis vectors in a light-cone Fock representation. Because of flavor symmetry breaking the mixing of the decay constants can be identical to the mixing of particle states at most for a specific choice of this basis. Theoretical and phenomenological considerations show that the quark flavor basis has this property and allows, therefore, for a reduction of the number of mixing parameters. A detailed comparison with other mixing schemes is also presented.

In a recent reinvestigation [¹] of processes involving \( \eta \) and \( \eta' \) mesons we pointed out that a proper treatment of the \( \eta - \eta' \) system requires a sharp distinction between the mixing properties of the meson states from the mixing properties of the decay constants. While the particle state mixing involves the global wave functions, the decay constants probe the quark distributions at zero spatial separation. Conventionally, \( \eta \) and \( \eta' \) are expressed as superpositions of an \( SU(3) \) flavor octet and a flavor singlet state corresponding to an orthogonal transformation with mixing angle \( \theta \). The decay constants of \( \eta \) and \( \eta' \) defined by their matrix elements with singlet and octet axial vector currents will in general not show the same mixing since flavor symmetry breaking manifests itself differently at small and large distances. Because \( SU(3) \) breaking is solely caused by the current quark masses a simpler picture can be expected for properly defined decay constants in the quark-flavor basis. Indeed, a dramatic simplification is achieved by taking two orthogonal basis states which are assumed to have in a Fock state description the parton composition

\[
\begin{align*}
|\eta_q\rangle &= \Psi_q |u\bar{u} + d\bar{d}\rangle /\sqrt{2} + \ldots \\
|\eta_s\rangle &= \Psi_s |s\bar{s}\rangle + \ldots
\end{align*}
\]

(1)

Here \( \Psi_i \) denote (light-cone) wave functions of the corresponding parton states, and the dots stand for higher Fock states which also include \( |c\bar{c}\rangle \) components. These higher Fock states play no explicit role in the following discussions where we are mainly interested in mesonic states and decay constants. The physical meson states are related to the basis by an orthogonal transformation

\[
\begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \end{pmatrix} = U(\phi) \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \end{pmatrix}, \quad U(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},
\]

(2)

¹In principle, the two-state basis should be extended by states of higher energy, for instance by adding a \( c\bar{c} \) state. Energy considerations indicate, however, that the mixing with these states is small. The small charm components in the \( \eta \) and \( \eta' \) is discussed in [¹].
where $\phi$ is the mixing angle. Ideal mixing corresponds to the case $\phi = 0$. It is to be emphasized that our definition of meson states is given in terms of parton degrees of freedom without introducing model-dependent concepts like constituent quarks.

Our central ansatz \cite{1,2,3} has important consequences for the weak decay constants which probe the short-distance properties of the quark-antiquark Fock states. To see this in detail, let us define the decay constants \cite{4} by ($f_{\pi} = 131$ MeV)

$$
\langle 0| J^i_{\mu3} | P \rangle \equiv i f_{\mu} p_{\mu} ,
$$

where $P = \eta, \eta'$. Here $J^i_{\mu3}$ denotes the axial-vector currents with quark content $i = q, s$. The decay constants are related to the quark-antiquark wave functions at the origin of configuration space. Because of the fact that light-cone wave functions do not depend on the hadron momentum we can define two basic decay constants $f_q$ and $f_s$ arising from $\eta_q$ and $\eta_s$, respectively,

$$
f_i = 2\sqrt{6} \int \frac{dx \, d^2k_\perp}{16\pi^3} \Psi_i(x, k_\perp) .
$$

Here $x$ denotes the usual (light-cone +) momentum fraction of the quark and $k_\perp$ its transverse momentum with respect to its parent meson’s momentum. Eq. (4) is exact, only the quark-antiquark Fock state contributes to the decay constant, higher Fock states do not contribute. Using Eqs. (1–3), one immediately observes that our ansatz for the Fock decomposition naturally leads to decay constants in the quark-flavor basis which simply follow the pattern of state mixing:

$$
\begin{pmatrix}
  f_\eta \\
  f_{\eta'}
\end{pmatrix}
= U(\phi) \text{diag}[f_q, f_s] .
$$

The conventional octet-singlet basis states are obtained from the quark-flavor basis states by performing a rotation with the ideal mixing angle. The physical states are then related to the octet-singlet basis states by

$$
\begin{pmatrix}
  |\eta\rangle \\
  |\eta'\rangle
\end{pmatrix}
= U(\theta) \begin{pmatrix}
  |\eta_q\rangle \\
  |\eta_s\rangle
\end{pmatrix} ,
$$

with $\theta = \phi - \arctan \sqrt{2}$. The corresponding Fock decompositions of these octet-singlet basis states, following from Eqs. (1–3), read

$$
\begin{align*}
|\eta_q\rangle &= \frac{\Psi_q + 2 \Psi_s}{3} \frac{|u\bar{u} + d\bar{d} - 2s\bar{s}|}{\sqrt{6}} + \sqrt{2} \frac{|u\bar{u} + d\bar{d} + s\bar{s}|}{3} + \ldots \\
|\eta_s\rangle &= \frac{\sqrt{2} (\Psi_q - \Psi_s)}{3} \frac{|u\bar{u} + d\bar{d} - 2s\bar{s}|}{\sqrt{6}} + \frac{2 \Psi_q + \Psi_s}{3} \frac{|u\bar{u} + d\bar{d} + s\bar{s}|}{\sqrt{3}} + \ldots
\end{align*}
$$

Obviously, it is unavoidable that the so-defined octet (singlet) meson state contains an $SU(3)$ singlet (octet) admixture, except for identical wave functions $\Psi_q = \Psi_s$, an equality which holds in the flavor symmetry limit only. Only then one would find pure octet and singlet states in Eq. (6). Certainly, these results are based on our central ansatz, namely that $\eta_q$ and $\eta_s$ can be decomposed into two orthogonal states where one state has no $s\bar{s}$ and the other no $qq$ component. One may alternatively start from the assumption that $\eta_q$ and $\eta_s$ defined in Eq. (1) have Fock decompositions with only parton octet or singlet combinations in the quark-antiquark sector, respectively. Rotating these states back by the ideal mixing angle, the resulting $\eta_q$ ($\eta_s$) state has an $s\bar{s}$ ($qq$) component, unless the octet and singlet wave functions are equal. However, from both, theoretical and phenomenological considerations performed in \cite{1,2,3}, the quark-flavor basis is to be favored.

\footnote{We stress that occasionally used decay constants “$f_{\eta,\eta'}$” are ill-defined quantities.}
One may also define decay constants through matrix elements of octet and singlet axial-vector currents, analogously to Eq. (6). Using Eqs. (1) and (2), one easily sees that these decay constants cannot be expressed as $U(\theta)$ diag$[f_8, f_1]$. Rather one has

$$
\begin{pmatrix}
  f_8^\phi & f_1^\phi \\
  f_8^q & f_1^q
\end{pmatrix} = \begin{pmatrix}
  f_8 \cos \theta_8 & f_1 \sin \theta_1 \\
  f_8 \sin \theta_8 & f_1 \cos \theta_1
\end{pmatrix},
$$

(8)

where we use the new and general parametrization introduced in Ref. [3]. The parameters appearing in Eq. (6) are related to the basic parameters $\phi$, $f_q$ and $f_s$, characterizing the quark flavor mixing scheme as follows [1].

$$
\theta_8 = \phi - \arctan \frac{\sqrt{2} f_8}{f_q}, \quad f_8 = \frac{f_8^2 + 2 f_s^2}{3},
$$

$$
\theta_1 = \phi - \arctan \frac{\sqrt{2} f_q}{f_s}, \quad f_1 = \frac{2 f_q^2 + f_s^2}{3}.
$$

(9)

The decay constants $f_{i'}$ do not follow the pattern of state mixing in the octet-singlet basis; only in the $SU(3)_F$ symmetry limit one would have $\theta_8 = \theta = \theta_1$. This is a consequence of the non-trivial Fock decomposition in Eq. (7). The difference between $\theta_8$ and $\theta_1$ following from Eq. (8) leads to the same formula as derived within chiral perturbation theory [3]. In our approach the quantities $\theta_8$ and $\theta_1$ are parameters determined by the fundamental quantities $\phi$, $f_q$ and $f_s$. They are not to be used as $|\eta\rangle = \cos \theta_q |\eta_q\rangle - \sin \theta_1 |\eta_1\rangle$ etc.

Let us now briefly review the determination of the mixing parameters performed in Ref. [4]. For this purpose we considered the divergences of axial-vector currents which incorporate the $U(1)_A$ anomaly $(q_t = u, d, s)$

$$
\partial^\mu q_i \gamma_\mu \gamma_5 q_i = 2 m_i \bar{q}_i i \gamma_5 q_i + \frac{\alpha_s}{4\pi} G \tilde{G},
$$

(10)

Sandwiching Eq. (10) between the vacuum and the meson states and using the definition of the decay constants (3) together with (3) one obtains the $\eta-\eta'$ mass matrix in the quark flavor basis. Its elements are composed of the gluonic matrix elements $\langle 0 | \frac{\alpha_s}{4\pi} G \tilde{G} | \eta\rangle$ and matrix elements of the quark mass terms contained in (10), which can be expressed by the pion and the $K$ meson masses using flavor symmetry and its breaking to first order. The known eigenvalues of this mass matrix (the masses of $\eta$ and $\eta'$) give then the value of the gluonic matrix elements and, in particular, the value $42.4^\circ$ for the mixing angle $\phi$.

Alternatively, the mixing parameters can be determined from phenomenology without using $SU(3)_F$ relations. The mixing angle $\phi$ can be determined by considering appropriate ratios of decay widths or cross sections, in which either the $\eta_q$ or the $\eta_s$ components are probed. The decay constants $f_8$ and $f_s$ can be evaluated from the $\eta$, $\eta' \rightarrow \gamma\gamma$ decay widths, relying on the chiral anomaly prediction. The analysis of a number of decay and scattering processes leads to the phenomenological set of parameters $f_q/f_\pi = 1.07$, $f_s/f_\pi = 1.34$, $\phi = 39.3^\circ$ which we will use in the following. We like to point out that the phenomenological values for the mixing angle $\phi$ from different experiments are all consistent with each other within a rather small uncertainty. The resulting differences between $\theta_8$, $\theta$ and $\theta_1$ (although only caused by $SU(3)_F$ breaking effects) are enormous, see Table [4]. We also list in this table (in anti-chronological order) the parameter values obtained in previous approaches.

It is possible to take decompositions of $\eta$ and $\eta'$ where either the octet or the singlet state is pure — in the sense that admixtures of the orthogonal quark-antiquark combination are absent — but not both. Note that one may, for instance, write the weak decay constants in Eq. (8) in the form

$$
\begin{pmatrix}
  f_8^\phi & f_1^\phi \\
  f_8^q & f_1^q
\end{pmatrix} = U(\theta_8) \begin{pmatrix}
  f_8 & f_1 \sin(\theta_8 - \theta_1) \\
  0 & f_1 \cos(\theta_8 - \theta_1)
\end{pmatrix}.
$$

(11)

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We remark at this point, that the flavor singlet axial-vector current is not conserved in QCD. Consequently, the singlet decay constant $f_1$, and, hence, $f_q$ and $f_s$ too, are renormalization scale dependent, although only mildly since the corresponding anomalous dimension is of order $\alpha_s^2$ [3]. Varying the scale $\mu$ between $M_\pi$ and $M_{t_\mu}$, the value of $f_1(\mu)$ changes by $5\%$ only, an effect which we discard.
Then it is tempting to introduce a new basis by
\[
\left( \begin{array}{c}
|\eta\rangle \\
|\eta'\rangle
\end{array} \right) = U(\theta_8) \left( \begin{array}{c}
|\tilde{\eta}_8\rangle \\
|\bar{\eta}\rangle
\end{array} \right).
\] (12)

The elements of the second matrix on the r.h.s. of Eq. (11) can now be viewed as the decay constants of \(\tilde{\eta}_8\), \(\bar{\eta}\) through octet or singlet axial vector currents, respectively. This matrix is still non-diagonal but triangular. The new basis has the special feature that the anomaly contributes to the singlet (\(\bar{\eta}\)) mass alone, i.e. one has
\[
\langle 0 | \frac{\alpha_s}{4\pi} \tilde{G} \tilde{G} | \tilde{\eta}_8 \rangle = 0.
\] (13)

This property is related to the fact that the ratio \(\langle 0 | \frac{\alpha_s}{4\pi} \tilde{G} \tilde{G} | \eta' \rangle / \langle 0 | \frac{\alpha_s}{4\pi} \tilde{G} \tilde{G} | \eta \rangle\), is given by \(-\cot\theta_8\). It allows to determine a Gell-Mann–Okubo formula for the mass of the \(\tilde{\eta}_8\) basis state. Transforming the mass matrix found in the quark flavor basis (see above) to the new basis \(\tilde{\eta}_8\) one finds
\[
f_8^2 m_{\tilde{\eta}_8}^2 = \frac{f_s^2 M_{\tilde{\eta}_8}^2 + 2 f_s^2 (2 M_R^2 - M_\pi^2)}{3}.
\] (14)

This formula reminds of the suggestion put forward in Ref. [4] (see also [13]), namely to use the product \(f^2 M^2\) rather than \(M^2\) to determine the \(SU(3)_F\) breaking effects in the Gell-Mann–Okubo formula. Insertion of the relations in Eq. (13) into Eq. (14) yields
\[
m_{\tilde{\eta}_8}^2 \simeq \frac{4 M_R^2 - M_\pi^2}{3} - \Delta_{\text{GMO}} \frac{M_R^2 - M_\pi^2}{3}.
\] (15)

with \(\Delta_{\text{GMO}} = 4 (f_s^2 - f_\pi^2)/(3 f_\pi^2)\). The deviation from the standard Gell-Mann–Okubo relation \(\Delta_{\text{GMO}}\) can also be derived in chiral perturbation theory [4, 8].

The above discussion clearly shows: An analysis which implicitly uses equation (13) provides for an estimate of the parameter \(\theta_8\) rather than the angle \(\theta\). Indeed, previous treatments along these lines obtained mixing angles close to \(-20^\circ\), which is consistent with our value of \(\theta_8\) (see Table 1).

As a first test of our mixing approach we analyzed the \(\eta\eta\) and \(\eta\gamma\) transition form factors in Ref. [4]. A good description of the experimental data has been found from the phenomenological set of parameters. For details we refer to [4, 3]. Let us now turn to further tests and applications of our results which have not been discussed in Ref. [4].

Table 1: Comparison of different determinations of mixing parameters. The values given in parentheses are not quoted in the original literature but have been evaluated by us from information given therein. Crosses indicate approaches where the difference between \(\theta\), \(\theta_1\) and \(\theta_8\) has been ignored.

| \(\theta\) | \(\theta_8\) | \(\theta_1\) | \(f_s/f_\pi\) | \(f_1/f_\pi\) | method |
|----------------|----------------|----------------|----------------|----------------|----------------|
| \(-12.3^\circ\) | \(-21.0^\circ\) | \(-2.7^\circ\) | 1.28 | 1.15 | \(qs\)-scheme (theo.) [4] |
| \(-15.4^\circ\) | \(-21.2^\circ\) | \(-9.2^\circ\) | 1.26 | 1.17 | \(qs\)-scheme (phen.) [4] |
| \(-21.4^\circ\) | \(-20.5^\circ\) | \(-4^\circ\) | 1.28 | 1.25 | ChPT [4] |
| \(-15.5^\circ\) | \(-12.2^\circ\) | \(-30.7^\circ\) | 0.71 | 0.94 | GMO formula [4] |
| \(-19.7^\circ\) | \(-19.5^\circ\) | \(-5.5^\circ\) | 1.27 | 1.17 | model [4] |
| \(-12.6^\circ\) | \(-23^\circ - 17^\circ\) | \(-5^\circ\) | 1.2 \(-1.3\) | 1.0 \(-1.2\) | phenomenology [4] | 
| \(-9^\circ\) | \(-20^\circ\) | \(-5^\circ\) | 1.2 | 1.1 | \(U(1)_A\) anomaly [12] |
Radiative Decays of S-wave quarkonia:

We define the ratio of decay widths $R(3S_n) = \Gamma[3S_n \rightarrow \eta'\gamma]/\Gamma[3S_n \rightarrow \eta\gamma]$ where $3S_n$ represents one of the quarkonia $J/\psi,\psi',\Upsilon,\ldots$. According to [13] the photon is emitted by the $c$ quarks which then annihilate into lighter quark pairs through the effect of the anomaly. Thus, the creation of the corresponding light mesons is controlled by the matrix element $\langle 0 | \bar{c}Gc | P \rangle$, leading to

$$R(3S_n) = \cot^2 \theta_8 \left( \frac{k_{12}}{k_{\eta\gamma}} \right)^3$$  \hspace{1cm} (16)

where $k_{12} = \sqrt{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2/(2M)}$ denotes the final state’s three-momentum in the rest frame of the decaying particle. The experimental value of $R$ in the $J/\psi$ case [17] has already been used in the phenomenological determination of the basic mixing parameters in [1]. Using the phenomenological value of $\theta_8$ quoted in Table I, we predict $R(\psi') = 5.8$ and $R(\Upsilon) = 6.5$. The prediction for $R(\psi')$ agrees with the experimental value $2.9^{+5.8}_{-1.6}$, which still has uncomfortably large errors however [17]. For the radiative $\Upsilon$ decays only upper bounds exist at present.

$\chi_{cJ}$ decays into two pseudoscalars:

Because these are energetic decays the current quarks produced will be in an almost pure SU(3) singlet state. However, flavor symmetry violation can occur in the hadronisation process. The ratio of $\chi_{cJ}$ decay widths into different pairs of pseudoscalar mesons can be written ($J = 0, 2$)

$$\frac{\Gamma[\chi_{cJ} \rightarrow P_1P_2]}{\Gamma[\chi_{cJ} \rightarrow P_2P_1]} = \left( \frac{C_{12}}{C_{34}} \right)^2 \left( \frac{k_{12}}{k_{34}} \right)^{2J+1}.$$  \hspace{1cm} (17)

For the coefficients $C_{ij}$ two limiting cases can be considered. If the mesons are formed at hadronic distances the influence of the different decay constants will be a minor one and one expects to a good accuracy $C_{\eta\eta} = C_{\eta'\eta'} = C_{\pi^0\pi^0}$, $C_{\eta'\pi} = 0$ and, from isospin symmetry, $C_{\pi^+\pi^-} = \sqrt{2}C_{\pi^0\pi^0}$ (where we included the statistical factor $\sqrt{2}$). If, however, the meson generation starts already at a time at which the inter-quark distances are very small, the decay amplitudes are obtained from the convolution of a hard scattering process with the corresponding wave functions [18]. Assuming equal shapes of the wave functions, an assumption which is not in conflict with present experimental information, differences in the decay amplitudes are then solely due to the different decay constants and the mixing angle. One finds: $C_{\eta\eta} = f_\eta^2 \cos^2 \phi + f_\eta^2 \sin^2 \phi = 1.41 C_{\pi^0\pi^0}$, $C_{\eta'\eta'} = f_{\eta'}^2 \sin^2 \phi + f_{\eta'}^2 \cos^2 \phi = 1.53 C_{\pi^0\pi^0}$ and (with the statistical factor $\sqrt{2}$ included) $C_{\eta'\pi} = \sqrt{2}(f_\eta^2 - f_{\eta'}^2) \sin \phi \cos \phi = -0.45 C_{\pi^0\pi^0}$ and $C_{\pi^+\pi^-} = \sqrt{2}C_{\pi^0\pi^0}$. Numerically we obtain $\Gamma[\chi_{c0}(2) \rightarrow \eta\eta]/\Gamma[\chi_{c0}(2) \rightarrow \pi^0\pi^0] = 1.9 (1.7)$, $\Gamma[\chi_{c0}(2) \rightarrow \eta'\eta']/\Gamma[\chi_{c0}(2) \rightarrow \pi^0\pi^0] = 1.9 (1.3)$, $\Gamma[\chi_{c0}(2) \rightarrow \eta'\pi]/\Gamma[\chi_{c0}(2) \rightarrow \pi^0\pi^0] = 0.2 (0.1)$ and $\Gamma[\chi_{c0}(2) \rightarrow \pi^+\pi^-]/\Gamma[\chi_{c0}(2) \rightarrow \pi^0\pi^0] = 2$. From the differences between the two limiting cases it appears that $\chi_{cJ}$ decays are less suited for testing $\eta-\eta'$ mixing parameters, but, taking the mixing parameters from other processes, they will provide interesting information on meson formation in these reactions. Experimentally, only the ratio $\Gamma[\chi_{c0}(2) \rightarrow \eta\eta]/\Gamma[\chi_{c0}(2) \rightarrow \pi^0\pi^0]$ is known [16, 17]: $0.76^{+1.1}_{-0.5}$ ($0.76^{+0.9}_{-0.6}$) (for the $\pi^0\pi^0$ branching ratio we combined the data with the one for the $\pi^+\pi^-$ channel). At present, the large experimental errors prevent any definite conclusion.

Similar relations as given here (modified according to the correct charge factors) should hold for two-photon annihilations into pairs of pseudoscalar mesons.

$g^*g^* \rightarrow \eta, \eta'$ transition form factors:

These form factors offer, in principle, a way to measure the angle $\theta_1$. Allowing both the gluons to be virtual where at least one of the virtualities $q_1^2$ and $q_2^2$ is supposed to be very large, one may easily work out the leading-twist result for these form factors [20]. In this approximation one has ($i = q, s$)

$$F_{\eta, g^*}(q_1^2, q_2^2) = -C_i \alpha_s f_i \int dx \frac{\phi_i(x)}{x q_1^2 + (1 - x) q_2^2}$$  \hspace{1cm} (18)
where \( \phi_i \) is the \( \eta_i \) distribution amplitude, and \( C_i \) a numerical factor \((C_\eta = \sqrt{2}, C_s = 1)\). Combining both the form factors into those for the physical mesons and assuming the equality of the two distribution amplitudes (which at least holds in the formal limit \( q^2 \to \infty \) since both the distribution amplitudes evolve into the asymptotic one), one arrives at

\[
\frac{F_{\eta' g^*}(q_1^2, q_2^2)}{F_{\eta g^*}(q_1^2, q_2^2)} = \frac{\sqrt{2} f_\eta \cos \phi - f_s \sin \phi}{\sqrt{2} f_\eta \sin \phi + f_s \cos \phi} = -\tan \theta_1 .
\] (19)

Of course, at finite values of momentum transfer one may expect corrections from differences between the two distribution amplitudes and from transverse momentum. Such corrections can be worked out following, for instance, Ref. [3].

For a measurement of these form factors one may consider the process \( pp \to \text{jet + jet + } \eta(\eta') \) where the mesons are supposed to be produced in the central rapidity region [21]. According to Close [22] these form factors may also be of relevance in \( pp \to pp\eta(\eta') \), provided the Pomeron couples to quarks \( \propto \gamma^6 \) and/or \( gg \to \eta(\eta') \) is the elementary process in the Pomeron-Pomeron interaction. The explicit extraction of the form factors from such measurements may, however, be very difficult. Kilian and Nachtmann [23] discuss \( \gamma\text{-Odderon-} \eta(\eta') \) form factors which appear in diffractive \( e-p \) scattering processes. We find that the ratio of these form factors is given by \( \cot \theta_8 \) at large momentum transfer.

\( Z \to \eta(\eta')\gamma \) decay:

The treatment of these processes is rather academic since the expected branching ratios are far below the present experimental bounds (The \( Z \to \pi\gamma \) decay has e.g. been discussed in Ref. [24]). Nevertheless, they provide an additional example of reactions which are sensitive to the angle \( \theta_1 \). The ratio of the decay widths are given by

\[
R(Z) = \frac{\Gamma[Z \to \eta'\gamma]}{\Gamma[Z \to \eta\gamma]} = \left| \frac{F_{\eta'\gamma Z}}{F_{\eta\gamma Z}} \right|^2 \left( \frac{k_{\eta'\gamma}}{k_{\eta\gamma}} \right)^3
\] (20)

where \( F_{P\gamma Z} \) is the time-like form factor for \( P \gamma \) transitions mediated by the \( Z \) boson. That form factor can be calculated along the same lines as the \( P\gamma\gamma^* \) transition form factor [3]. Since the value of \( M_Z^2 \) is very large it suffices to consider the asymptotic limit of the form factor only. In terms of the octet and singlet decay constants, defined in Eq. [3], the result reads:

\[
F_{P\gamma Z}(M_Z^2) = \frac{6 C_{8\gamma Z} f_\mu^8 + 6 C_{1\gamma Z} f_\mu^1}{M_Z^2}
\] (21)

where \( C_{8\gamma Z} = (1-4 \sin^2 \theta_W)/(6\sqrt{6}) \) and \( C_{1\gamma Z} = (2-4 \sin^2 \theta_W)/(3\sqrt{3}) \). The weak coupling of the flavor octet current is strongly suppressed by \((1-4 \sin^2 \theta_W)\). Hence, \( R(Z) \simeq \cot^2 \theta_1 \).

The same transition form factors \( F_{P\gamma Z} \) appear in \( \eta(\eta') \to \gamma\mu^+\mu^- \) decays, but the momentum transfer is very small. In analogy to the \( P \to \gamma\gamma \) decays the amplitudes in this case involve the inverse decay constants, and the \( \eta \) to \( \eta' \) ratio is sensitive to the angle \( \theta_8 \). To measure these form factors in \( \eta(\eta') \to \gamma\mu^+\mu^- \) decays one has to extract the \( \gamma-Z \) interference term from suitably chosen asymmetries as discussed in detail in [25].

Radiative transitions between light vector and pseudoscalar mesons:

The relevant coupling constants are defined by [3]

\[
\langle P(p_P) | J_{\mu}^{EM} | V(p_V, \lambda) \rangle |_{q^2=0} = -g_{VPP} \epsilon_{\mu\nu\rho\sigma} p_P^\nu p_V^\rho \epsilon^{\sigma}(\lambda) .
\] (22)

In Ref. [1] these coupling constants are expressed in terms of meson masses and decay constants by exploiting the chiral anomaly prediction (at \( q^2 = 0 \)) and vector meson dominance. However, the difference between \( \theta_8 \) and \( \theta_1 \) has not been considered. Translating the expressions for \( g_{PV\gamma} \) correctly to the quark-flavor scheme, following otherwise Ref. [1], we arrive at the formulas and values listed in Table [2].
The numerical result for the coupling constants depend on the actual values of the vector mixing angle $\phi_V$, which is expected to amount to only a few degrees (see e.g. \cite{28}). The values in Table 2 are calculated for $\phi_V = 0$. For the vector meson decay constants we take \cite{27} $f_\rho = 210$ MeV, $f_\omega = 195$ MeV, $f_\phi = 237$ MeV. The predictions agree rather well with experiment. Indeed the relations

$$\frac{g_{\rho'\gamma}}{g_{\rho\gamma}} = \frac{g_{\omega'\gamma}}{g_{\omega\gamma}} = \frac{g_{\phi'\gamma}}{g_{\phi\gamma}} = \tan \phi = 0.82$$

are well confirmed by experiment. In Ref. \cite{11} results of similar quality could only be achieved by using a value of $\theta = \theta_8 - \theta_1 = -17^\circ$ which deviates from the values obtained from other applications substantially.

| $P$ | $V$ | $g_{V\rho'\gamma}$ (in units $m_{\rho'^2}$) | $|g_{V\rho'\gamma}|$(theo.) | $|g_{V\rho'\gamma}|$(exp.) |
|-----|-----|-----------------|-----------------|-----------------|
| $\eta$ | $\rho$ | $\frac{\cos \phi}{4f_\rho}$ | 1.52 | 1.85 $\pm$ 0.34 |
| $\eta'$ | $\rho$ | $\frac{\sin \phi}{4f_\rho}$ | 1.24 | 1.31 $\pm$ 0.12 |
| $\eta$ | $\omega$ | $\frac{\cos \phi \cos \phi_V}{4f_\rho} - \frac{2 \sin \phi \sin \phi_V}{4f_\rho}$ | 0.56 | 0.60 $\pm$ 0.15 |
| $\eta'$ | $\omega$ | $\frac{\cos \phi \cos \phi_V}{4f_\rho} + \frac{2 \sin \phi \sin \phi_V}{4f_\rho}$ | 0.46 | 0.45 $\pm$ 0.06 |
| $\eta$ | $\phi$ | $\frac{\sin \phi \cos \phi_V}{4f_\rho} + \frac{2 \sin \phi \sin \phi_V}{4f_\rho}$ | 0.78 | 0.70 $\pm$ 0.03 |
| $\eta'$ | $\phi$ | $\frac{\sin \phi \sin \phi_V}{4f_\rho} - \frac{2 \cos \phi \cos \phi_V}{4f_\rho}$ | 0.95 | 1.01 $\pm$ 0.25 |

Table 2: Various coupling constants $g_{V\rho'\gamma}$ from theory and experiment \cite{16}. The numerical values are quoted in units of GeV$^{-1}$.

$\eta$ and $\eta'$ admixtures to the pion:

As is well-known (see e.g. \cite{28}) an accurate prescription of the decays of $\eta(\eta')$ to three pions can only be achieved by taking isospin violation into account. This effect is usually parametrized in terms of $\eta$ and $\eta'$ admixtures to the pion,

$$\pi^0 = \phi_3 + \epsilon \eta + \epsilon' \eta'$$

where $\phi_3$ denotes the pure isospin-1 state. A straightforward generalization of our mixing scheme yields for the strength of $\eta$ and $\eta'$ admixtures in the pion

$$\epsilon = \cos \phi \frac{m_{\eta dd}^2 - m_{uu}^2}{2(M_\eta^2 - M_\rho^2)}, \quad \epsilon' = \sin \phi \frac{m_{\eta dd}^2 - m_{uu}^2}{2(M_{\eta'}^2 - M_\rho^2)},$$

where the difference $m_{\eta dd}^2 - m_{uu}^2$ can be estimated from $2(M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2)$ to amount to $0.0104$ GeV$^2$. A possible difference in $u-$ and $d-$ quark decay constants is ignored in the derivation of \cite{28}. The expressions for $\epsilon$ and $\epsilon'$ look rather simple in the quark flavor scheme and are intimately connected to physical quantities. Inserting our phenomenological number for the mixing angle $\phi$ we obtain $\epsilon = 0.014$ and $\epsilon' = 0.0037$. By exploiting the properties of the mass matrix \cite{1} the ratio $\epsilon/\epsilon'$ following from \cite{28} can also be expressed in terms of $\theta_8$ and $\theta$

$$\frac{\epsilon'}{\epsilon} = -\tan \theta_8 \left( \frac{\cos \theta + \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta - \sin \theta} \right)^2. $$

The numerical value, following from our phenomenological set of parameters, is 0.26. In contrast, the conventional approach (see e.g. \cite{28}), using $\theta = \theta_8 \simeq -20^\circ$ gives the much smaller value 0.17.

The values of the parameters $\epsilon$ and $\epsilon'$ have recently been shown to be of importance for the investigation of $CP$-violation in $B \to \pi\pi$ decays since it breaks the isospin triangle relation for the amplitudes of the three processes $B^+ \to \pi^+\pi^0$, $B^0 \to \pi^0\pi^0$ and $B^0 \to \pi^+\pi^- $ \cite{29}. The value of $\epsilon'$ used in Ref. \cite{29} ($\epsilon' = 0.0077$) is substantially larger than our value.
Summary
We discussed the mixing properties of the $\eta$ and $\eta'$ meson state vectors and of their decay constants and showed that there is, at most, only one basis where the mixing of the decay constants can follow the pattern of state mixing. Chiral perturbation theory as well as phenomenological analyses favor this proposition for the quark-flavor basis. In general, e.g. in the familiar octet-singlet basis, one needs two angles in order to parametrize the decay constants. However, when using our quark-flavor mixing scheme, these new angles are fixed by the basic parameters $\phi, f_q, f_s$, leading to a number of important consequences for many reactions. The results are quite different from conventional mixing schemes in which the subtleties discussed here are not considered and where, as a consequence of that, the mixing parameters often show a strong process dependence.

The improved knowledge of the mixing parameters is also of importance for the analysis of $B$ decays, like $B \rightarrow K\eta'$ (see e.g. [30]) or $B \rightarrow \pi\pi$ [29]. Further interesting applications of our approach refer to $\eta$ and $\eta'$ production processes in high energy hadron collisions where exotic form factors such as the $g^*g^*\eta(\eta')$ form factors play an important role.

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