Improved resolvent bounds for radial potentials. II

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Abstract. We prove semiclassical resolvent estimates for the Schrödinger operator in \( \mathbb{R}^d \), \( d \geq 3 \), with real-valued radial potentials \( V \in L^\infty(\mathbb{R}^d) \). We show that if \( V(x) = O\left(\langle x \rangle^{-\delta} \right) \) with \( \delta > 4 \), then the resolvent bound is of the form \( \exp\left(C h^{-\frac{\delta}{\delta-1}} \left(\log(h^{-1})\right)^{\frac{1}{\delta-1}}\right) \) with some constant \( C > 0 \).

If \( V(x) = O\left(e^{-\tilde{C}\langle x \rangle^\alpha} \right) \) with \( \tilde{C}, \alpha > 0 \), we get better resolvent bounds of the form \( \exp\left(Ch^{-1} \left(\log(h^{-1})\right)^{\frac{\alpha}{2}} \right) \).

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1. Introduction and statement of results. Our goal in this note is to improve some of the resolvent bounds proved in [11] for radial real-valued potentials. We also give a different, shorter proof of the sharp resolvent bounds proved recently in [2] for radial compactly supported real-valued potentials. Consider the Schrödinger operator

\[
P(h) = -h^2 \Delta + V(x)
\]

where \( 0 < h \ll 1 \) is a semiclassical parameter, \( \Delta \) is the negative Laplacian in \( \mathbb{R}^d \), \( d \geq 3 \), and \( V \in L^\infty(\mathbb{R}^d) \) is a real-valued short-range potential satisfying the condition

\[
|V(x)| \leq C(|x| + 1)^{-\delta}
\]  

(1.1)

where \( C > 0 \) and \( \delta > 1 \) are some constants. We are interested in bounding the quantity

\[
g_s^\pm(h, \theta) := \log \|(|x| + 1)^{-s} (P(h) - E \pm i\theta)^{-1} (|x| + 1)^{-s} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}
\]

from above by an explicit function of \( h \), independent of \( \theta \). Here \( 0 < \theta < 1 \), \( s > 1/2 \) is independent of \( h \), and \( E > 0 \) is a fixed energy level independent of \( h \). When \( \delta > 2 \), it has been proved in [4] that
\[ g^\pm_s(h, \theta) \leq C h^{-4/3} \log(h^{-1}). \] (1.2)

The bound (1.2) was first proved in [6] and [7] for compactly supported potentials, and in [8] when \( \delta > 3 \). It was also shown in [9] that (1.2) still holds for more general asymptotically Euclidean manifolds. On the other hand, it is shown in [11] that the logarithmic term in the right-hand side of (1.2) can be removed for real-valued potentials \( V \) depending only on the radial variable \( r = |x| \), provided \( V \) satisfies (1.1) with \( \delta > 2 \). Furthermore, for compactly supported radial potentials, a much better bound has been recently proved in [2], namely the following one

\[ g^\pm_s(h, \theta) \leq C h^{-1}. \] (1.3)

The bound (1.3) was previously proved in [1], [5], [10] for slowly decaying Lipschitz potentials \( V \) with respect to the radial variable \( r \). Note also that when \( d = 1 \), the bound (1.3) is proved in [3] for \( V \in L^1(\mathbb{R}) \). We show in the present paper that better resolvent bounds than those obtained in [11] can be proved for non-compactly supported radial potentials, too. Our main result is the following

**Theorem 1.1.** Let \( d \geq 3 \) and suppose that the potential \( V \) depends only on the radial variable. If \( V \) satisfies (1.1) with \( \delta > 4 \), then there exist constants \( C > 0 \) and \( h_0 > 0 \) independent of \( h \) and \( \theta \) but depending on \( s \), \( E \), such that the bound

\[ g^\pm_s(h, \theta) \leq C h^{-\delta} \left( \log(h^{-1}) \right)^{\frac{1}{\delta-1}} \] (1.4)

holds for all \( 0 < h \leq h_0 \). If the potential satisfies the condition

\[ |V(x)| \leq C_1 e^{-C_2 |x|^{\alpha}} \] (1.5)

with some constants \( C_1, C_2, \alpha > 0 \), then we have the better bound

\[ g^\pm_s(h, \theta) \leq C h^{-1} \left( \log(h^{-1}) \right)^{\frac{1}{\alpha}}. \] (1.6)

Note that when \( V \) is compactly supported, the proof of Theorem 1.1 leads to the bound (1.3) already proved in [2] in a different way. Note also that our proof of (1.4) works for all \( \delta > 1 \), but the bound (1.4) is stronger than the bounds proved in [11] only when \( \delta > 4 \). That is why we state (1.4) only in this case.

The fact that the potential is radial plays an important role in the proof of the above theorem. It allows us to reduce the \( d \)-dimensional resolvent bound to one-dimensional ones. In other words, we have to bound the resolvent of an infinite family of one-dimensional Schrödinger operators depending on an additional parameter denoted by \( \nu \geq 0 \) below, which can be expressed in terms of the eigenvalues of the Laplace–Beltrami operator on the \((d-1)\)-dimensional unit sphere (see Section 2). To do so, we make use of some bounds already proved in [11] (see Proposition 2.2) and we show that [11, Proposition 3.2] can be improved significantly for potentials decaying at infinity sufficiently fast (see Proposition 2.3). It is not clear if the bounds in Theorem 1.1 still hold for non-radial \( L^\infty \) potentials since neither a proof nor counterexamples are
available. To the author’s best knowledge, the best resolvent bound for such potentials is (1.2), which seems hard to improve without extra conditions even if the potential is supposed to be compactly supported.

2. Preliminaries. We will use the fact that the potential is radial to reduce the resolvent bound to infinitely many one-dimensional resolvent bounds (see also [11, Section 2]). To this end we will write the operator $P(h)$ in polar coordinates $(r, w) \in \mathbb{R}^+ \times S^{d-1}$, $r = |x|$, $w = x/|x|$ and we will use that $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^+ \times S^{d-1}, r^{d-1} dr dw)$. We have the identity

$$r^{(d-1)/2} \Delta r^{-(d-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}$$

(2.1)

where $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(d-1)(d-3)$ and $\Delta_w$ denotes the negative Laplace–Beltrami operator on $S^{d-1}$. Using (2.1), we can write the operator $P^\pm(h)$ as follows

$$P^\pm(h) = r^{(d-1)/2}(P(h) - E \pm i\theta)r^{-(d-1)/2}$$

in the coordinates $(r, w)$ as follows

$$P^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} + V(r) - E \pm i\theta$$

where we have put $\mathcal{D}_r = -ih\partial_r$ and $\Lambda_w = -h^2\tilde{\Delta}_w$. Let $\lambda_j \geq 0$ be the eigenvalues of $-\Delta_w$ repeated with the multiplicities and let $e_j \in L^2(S^{d-1})$ be the corresponding orthonormal eigenfunctions. Set

$$\nu = h^{1/2}(\lambda_j + \frac{1}{4}(d-1)(d-3))$$

and

$$Q^\pm_\nu(h) = \mathcal{D}_r^2 + \frac{\nu^2}{r^2} + V(r) - E \pm i\theta.$$ 

Let $v \in L^2(\mathbb{R}^+ \times S^{d-1}, dr dw)$ and set

$$v_j(r) = \langle v(r, \cdot), e_j \rangle_{L^2(S^{d-1})}.$$ 

We can write

$$v = \sum_j v_j e_j,$$

$$P^\pm(h)v = \sum_j Q^\pm_\nu(h)v_j e_j,$$

so we have

$$\|v\|^2_{L^2(\mathbb{R}^+ \times S^{d-1})} = \sum_j \|v_j\|^2_{L^2(\mathbb{R}^+)};$$

$$\|(r+1)^{-s}v\|^2_{L^2(\mathbb{R}^+ \times S^{d-1})} = \sum_j \|(r+1)^{-s}v_j\|^2_{L^2(\mathbb{R}^+)};$$

$$\|(r+1)^s P^\pm(h)v\|^2_{L^2(\mathbb{R}^+ \times S^{d-1})} = \sum_j \|(r+1)^s Q^\pm_\nu(h)v_j\|^2_{L^2(\mathbb{R}^+)}.$$ 

The following lemma is proved in [11, Section 2] using the above identities.
Lemma 2.1. Let $s > 1/2$ and suppose that for all $\nu$ the estimates
\[
\|(r+1)^{-s}u\|^2_{L^2(\mathbb{R}^+)} \leq M_\nu \|(r+1)^s Q^\pm_\nu (h)u\|^2_{L^2(\mathbb{R}^+)}
+ M_\nu \theta \|u\|^2_{L^2(\mathbb{R}^+)} + M_\nu \theta \|\mathcal{D}_r u\|^2_{L^2(\mathbb{R}^+)}
\]
hold for every $u \in H^2(\mathbb{R}^+)$ such that $u(0) = 0$ and $(r+1)^s Q^\pm_\nu (h)u \in L^2(\mathbb{R}^+)$, with $M_\nu > 0$ independent of $\theta$ and $u$. Suppose also that
\[
M := (2 + E + \|V\|_{L^\infty}) \sup_{\nu^2 \in \text{spec } \Lambda_\nu} M_\nu < \infty.
\]
Then we have the bound
\[
g^\pm_s (h, \theta) \leq \log(M + 1).
\] (2.3)

Thus we reduce our problem to proving estimates like (2.2) with as good bounds $M_\nu$ as possible. We will make use of the following proposition proved in [11, Section 3] (see [11, Proposition 3.1]).

**Proposition 2.2.** The estimate (2.2) holds for all $\nu$ with $M_\nu = e^{C(\nu+1)/h}$, where $C > 0$ is a constant independent of $\nu$ and $h$.

Therefore, we only need to bound $M_\nu$ for large $\nu$. Set $\tau = 1$ if $V$ is compactly supported, $\tau = (\epsilon h)^{-\frac{1}{1-\alpha}}$ if $V$ satisfies (1.1), and $\tau = \epsilon^{-1/\alpha}$ if $V$ satisfies (1.5), where $\epsilon = (\log(h^{-1}))^{-1} \ll 1$. In what follows in this paper, we will prove the following

**Proposition 2.3.** The estimate (2.2) holds for all $\nu \geq c\tau$ with $M_\nu = C(\epsilon h)^{-2}$, where $C, c > 0$ are constants independent of $\nu$ and $h$.

Clearly, the bounds (1.4) and (1.6) follow from (2.3) and Propositions 2.2 and 2.3.

3. A priori estimates. Let $\phi_0 \in C^\infty(\mathbb{R})$ be a real-valued function such that $0 \leq \phi_0 \leq 1$, $\phi_0' \geq 0$, $\phi_0(\sigma) = 0$ for $\sigma \leq 1$, $\phi_0(\sigma) = 1$ for $\sigma \geq 2$, and set $\phi(r) = \phi_0(r/\lambda)$, where $\lambda \gg 1$. We also set
\[
Q^\pm_{\nu,0}(h) = \mathcal{D}_r^2 + \frac{\nu^2}{r^2} - E \pm i\theta.
\]
In this section, we will prove the following

**Proposition 3.1.** For every $u \in H^2(\mathbb{R}^+)$ such that $(r+1)^1+\epsilon Q^\pm_{\nu,0}(h)u \in L^2(\mathbb{R}^+)$, we have the estimate
\[
\int_0^\infty (r+1)^{-1-\epsilon} \left( |\phi u(r)|^2 + |\mathcal{D}_r (\phi u)(r)|^2 \right) dr 
\leq C\lambda^{-2} \epsilon^{-1} h \int_\lambda^{2\lambda} (|u(r)|^2 + |\mathcal{D}_r u(r)|^2) dr
\]
\[ +C(\epsilon h)^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}^{\pm}(h)u(r)|^2 dr \]

\[ +C(\epsilon h)^{-1} \theta \int_0^\infty (|u(r)|^2 + |D_r u(r)|^2) dr \]

with a constant \( C > 0 \) independent of \( \epsilon, \theta, \nu, \lambda, \) and \( h \).

**Proof.** It is easy to see that the first derivative of the function

\[ F(r) = (E - \nu^2 r^{-2})|\phi u(r)|^2 + |D_r(\phi u)(r)|^2 \]

is given by

\[ F'(r) = 2\nu^2 r^{-3} |\phi u|^2 - \Phi(r) \]

where

\[ \Phi(r) = 2h^{-1} \text{Im} Q_{\nu,0}^{\pm}(h) \frac{\phi u D_r(\phi u)}{D_r(\phi u)^2} \]

\[ = 2h^{-1} \text{Im} \phi Q_{\nu,0}^{\pm}(h) u D_r(\phi u) \]

\[ \leq \gamma(r + 1)^{-1-\epsilon} |D_r(\phi u)|^2 + \gamma^{-1} h^{-2} (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}^{\pm}(h)u|^2 \]

\[ + \theta h^{-1} (|u|^2 + |D_r u|^2) + \Psi(r), \]

\( \gamma > 0 \) being arbitrary, where

\[ \Psi(r) = 2h^{-1} \text{Im} [D_r^2, \phi] u D_r(\phi u) \]

\[ = -2 \text{Im} (2i\phi' D_r u + h\phi'' u) (\phi D_r u + ih\phi'u) \]

\[ = -4 \phi |D_r u|^2 + 2h(2\phi'^2 + \phi''^2) \text{Im} \bar{u}D_r u - 2h^2 \phi' \phi'' |u|^2 \]

\[ \leq C h \lambda^{-2} (\phi_0'(r/\lambda) + |\phi_0''(r/\lambda)|) (|u(r)|^2 + |D_r u(r)|^2) \]

with some constant \( C > 0 \) independent of \( h \) and \( \lambda \). Hence, given any \( t > 0 \), we get

\[ F(t) = -\int_t^\infty F'(r) dr \leq \int_t^\infty \Phi(r) dr \]

\[ \leq \gamma \int_0^\infty (r + 1)^{-1-\epsilon} |D_r(\phi u)|^2 dr \]

\[ + \gamma^{-1} h^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}^{\pm}(h)u|^2 dr \]

\[ + h^{-1} \theta \int_0^\infty (|u|^2 + |D_r u|^2) dr \]

\[ + Ch \lambda^{-2} \int_{\lambda}^{2\lambda} (|u|^2 + |D_r u|^2) dr. \]
Multiplying this inequality by \((t + 1)^{-1-\epsilon}\) and integrating with respect to \(t\) lead to the estimate
\[
\int_0^\infty (t + 1)^{-1-\epsilon} F(t) dt \leq \gamma \int_0^\infty (r + 1)^{-1-\epsilon} |D_r(\phi u)|^2 dr
\]
\[
+ \gamma^{-1} (\epsilon h)^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}(h) u|^2 dr
\]
\[
+ (\epsilon h)^{-1} \theta \int_0^\infty (|u|^2 + |D_r u|^2) dr
\]
\[
+ Ch\lambda^{-2}\epsilon^{-1} \int_\lambda^{2\lambda} (|u|^2 + |D_r u|^2) dr
\]
for any \(\gamma > 0\), where we have used that
\[
\int_0^\infty (t + 1)^{-1-\epsilon} dt = \epsilon^{-1}.
\]

Taking \(\gamma\) small enough, independent of \(\epsilon, h, \text{ and } \lambda\), we can absorb the first term in the right-hand side of the above inequality. Thus we get
\[
\int_0^\infty (r + 1)^{-1-\epsilon} \left( |\phi u(r)|^2 + |D_r(\phi u)(r)|^2 \right) dr
\]
\[
\leq \int_0^\infty \nu^2 r^{-3} |\phi u|^2 dr
\]
\[
+ (\epsilon h)^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}(h) u|^2 dr
\]
\[
+ (\epsilon h)^{-1} \theta \int_0^\infty (|u|^2 + |D_r u|^2) dr
\]
\[
+ h\lambda^{-2} \epsilon^{-1} \int_\lambda^{2\lambda} (|u|^2 + |D_r u|^2) dr.
\]

On the other hand, we have
\[
2 \int_0^\infty \nu^2 r^{-3} |\phi u|^2 dr = \int_0^\infty F'(r) dr + \int_0^\infty \Phi(r) dr = \int_0^\infty \Phi(r) dr
\]
\[
\leq \gamma \int_0^\infty (r + 1)^{-1-\epsilon} |D_r(\phi u)|^2 dr
\]
\[ + \gamma^{-1} h^{-2} \int_{0}^{\infty} (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}^{\pm}(h)u|^2 \, dr \]
\[ + h^{-1} \theta \int_{0}^{\infty} (|u|^2 + |D_r u|^2) \, dr \]
\[ + C h \lambda^{-2} \int_{\lambda}^{2\lambda} (|u|^2 + |D_r u|^2) \, dr \]

for any \( \gamma > 0 \). Combining the above inequalities and taking \( \gamma \) small enough, independent of \( \epsilon, h, \) and \( \lambda \), in order to absorb the corresponding term, we get (3.1).

\[ \square \]

4. Proof of Proposition 2.3. We will first derive from Proposition 3.1 the following

**Proposition 4.1.** Let \( u \in H^2(\mathbb{R}^+) \) be such that \( (r + 1)^{1+\epsilon} Q_{\nu}^{\pm}(h)u \in L^2(\mathbb{R}^+) \). There exists a constant \( \lambda_0 > 0 \) such that if \( \lambda \geq \lambda_0 \tau \), then we have the estimate

\[ \int_{2\lambda}^{\infty} (r + 1)^{1-\epsilon} \left( |u(r)|^2 + |D_r u(r)|^2 \right) \, dr \leq C \lambda^{-2} \epsilon h \int_{\lambda}^{2\lambda} (|u(r)|^2 + |D_r u(r)|^2) \, dr \]
\[ + C(\epsilon h)^{-2} \int_{0}^{\infty} (r + 1)^{1+\epsilon} |Q_{\nu}^{\pm}(h)u(r)|^2 \, dr \]
\[ + C(\epsilon h)^{-1} \theta \int_{0}^{\infty} (|u(r)|^2 + |D_r u(r)|^2) \, dr \]  
(4.1)

with a constant \( C > 0 \) independent of \( \epsilon, \theta, \nu, \lambda, \) and \( h \).

**Proof.** We apply the estimate (3.1) and observe that

\[ \frac{1}{2} \int_{0}^{\infty} (r + 1)^{1+\epsilon} |\phi Q_{\nu,0}^{\pm}(h)u(r)|^2 \, dr \]
\[ \leq \int_{0}^{\infty} (r + 1)^{1+\epsilon} |\phi Q_{\nu}^{\pm}(h)u(r)|^2 \, dr \]
\[ + \int_{0}^{\infty} (r + 1)^{1+\epsilon} |V(r)(\phi u)(r)|^2 \, dr \]
\begin{align}
&\leq \int_0^\infty (r + 1)^{1+\epsilon} |Q_\nu^\pm (h) u(r)|^2 dr \\
&\quad + \rho(\lambda) \int_0^\infty (r + 1)^{-1-\epsilon} |(\phi u)(r)|^2 dr \tag{4.2}
\end{align}

where

\[ \rho(\lambda) = \sup_{r \geq \lambda} (r + 1)^{2+2\epsilon} |V(r)|^2. \]

When \( V \) is compactly supported we have \( \rho(\lambda) = 0 \), provided \( \lambda \geq \lambda_0 \), \( \lambda_0 \) being big enough, independent of \( h \). When \( V \) satisfies (1.1) with \( \delta > 4 \), we have

\[ \rho(\lambda) \lesssim \lambda^{-2\delta+2+2\epsilon} \lesssim \lambda_0^{-2\delta+2}(\epsilon h)^2 \]

where we have used that \( h^{-\epsilon} = e \) together with the inequality \( \epsilon^{-\epsilon} < e \) to bound \( \lambda^{2\epsilon} \) by a constant. When \( V \) satisfies (1.5), we have

\[ \rho(\lambda) \lesssim \lambda^{2+2\epsilon} e^{-C_2 \lambda^\alpha} \lesssim \lambda_0^2 \epsilon^{-2/\alpha} e^{C_2 \lambda_0^\alpha \log h} \lesssim \epsilon^{-2/\alpha} h^{C_2 \lambda_0^\alpha} \lesssim h(\epsilon h)^2 \]

provided \( \lambda_0 \) is big enough, independent of \( h \) and \( \epsilon \). Here we have used the inequality \( \epsilon^{-\epsilon} < e \) to bound \( \lambda^{2\epsilon} \) by a constant.

Thus, taking \( h \) small enough or \( \lambda_0 \) big enough, we can absorb the last term in the right-hand side of (4.2) and obtain (4.1). \( \square \)

To prove Proposition 2.3, we will combine Proposition 4.1 with the following

**Proposition 4.2.** Let \( u \in H^2(\mathbb{R}^+) \) be such that \( u(0) = 0 \). Then there exists a constant \( \kappa > 0 \) such that we have the estimate

\[
\int_0^{3\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \
\leq C h^2 \nu^{-2} \int_0^{4\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr + 4 \int_0^\infty |Q_\nu^\pm (h) u(r)|^2 dr \tag{4.3}
\]

with a constant \( C > 0 \) independent of \( \theta, \nu, \) and \( h \).

**Proof.** Let \( \psi_0 \in C^\infty(\mathbb{R}) \) be a real-valued function such that \( 0 \leq \psi_0 \leq 1 \), \( \psi_0(\sigma) = 1 \) for \( \sigma \leq 3 \), \( \psi_0(\sigma) = 0 \) for \( \sigma \geq 4 \), and set \( \psi(r) = \psi_0(r/\kappa \nu) \), where \( \kappa^{-1} = 4\sqrt{1 + E + \|V\|_{L^\infty}} \). The estimate (4.3) is a consequence of the following

**Lemma 4.3.** We have the estimate

\[
\int_0^\infty \left( |\psi u(r)|^2 + |D_r (\psi u)(r)|^2 \right) dr \leq 4 \int_0^\infty |Q_\nu^\pm (h)(\psi u)(r)|^2 dr. \tag{4.4}
\]

**Proof.** The choice of \( \kappa \) guarantees the inequality

\[
(v^2 r^{-2} + V(r) - E)|\psi u(r)|^2 \geq |\psi u(r)|^2
\]
for all $r$. Therefore, integrating by parts, we obtain

\[
\text{Re} \int_0^\infty Q_{\nu}^\pm(h)\psi u(r)\bar{\psi u(r)} dr
\]

\[
= \int_0^\infty |D_r(\psi u)(r)|^2 + \int_0^\infty (\nu^2 r^{-2} + V(r) - E)|\psi u(r)|^2 dr
\]

\[
\geq \int_0^\infty |D_r(\psi u)(r)|^2 dr + \int_0^\infty |\psi u(r)|^2 dr.
\]

Hence

\[
\int_0^\infty |D_r(\psi u)(r)|^2 dr + \int_0^\infty |\psi u(r)|^2 dr
\]

\[
\leq 2 \int_0^\infty |Q_{\nu}^\pm(h)\psi u(r)|^2 dr + \frac{1}{2} \int_0^\infty |\psi u(r)|^2 dr,
\]

which clearly implies (4.4). \hfill \Box

Since

\[
\int_0^\infty ||Q_{\nu}^\pm(h), \psi|| u(r)|^2 dr = \int_0^\infty ||D_r, \psi|| u(r)|^2 dr
\]

\[
\leq C h^2 \nu^{-2} \int_{3\kappa \nu}^{4\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr,
\]

the estimate (4.3) follows from (4.4). \hfill \Box

Let $u \in H^2(\mathbb{R}^+) \text{ be such that } u(0) = 0 \text{ and } (r + 1)^{1+\epsilon} Q_{\nu}^\pm(h)u \in L^2(\mathbb{R}^+)$. We apply Proposition 4.1 with $\lambda = \kappa \nu$ and suppose that $\nu \geq \lambda_0 \tau / \kappa$. By (4.1) and (4.3), we have

\[
(4\kappa \nu + 1)^{-1-\epsilon} \int_{3\kappa \nu}^{4\kappa \nu} (|u(r)|^2 + |D_r u(r)|^2) dr
\]

\[
\leq \int^{\infty}_{2\kappa \nu} (r + 1)^{-1-\epsilon} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr
\]

\[
\leq C \nu^{-2} \epsilon^{-1} h \int^{2\kappa \nu}_{\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr
\]

\[
+ C(\epsilon h)^{-2} \int_{0}^{\infty} (r + 1)^{1+\epsilon} |Q_{\nu}^\pm(h)u(r)|^2 dr.
\]
\begin{align*}
+C(\epsilon h)^{-1} \theta \int_0^\infty \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \\
\leq C \nu^{-4} \epsilon^{-1} h^3 \int_0^{4\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \\
+C(\epsilon h)^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |Q_\nu^\pm (h) u(r)|^2 dr \\
+C(\epsilon h)^{-1} \theta \int_0^\infty \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr.
\end{align*}

It is clear that, taking $h$ small enough, we can absorb the first term in the right-hand side of the above inequality. Thus we obtain

\begin{align*}
(4\kappa \nu + 1)^{-1-\epsilon} \int_0^{4\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \\
\leq C(\epsilon h)^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |Q_\nu^\pm (h) u(r)|^2 dr \\
+C(\epsilon h)^{-1} \theta \int_0^\infty \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr,
\end{align*}

which together with (4.3) yields

\begin{align*}
\int_0^{3\kappa \nu} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \\
\leq C \epsilon^{-2} \int_0^\infty (r + 1)^{1+\epsilon} |Q_\nu^\pm (h) u(r)|^2 dr \\
+C \theta \int_0^\infty \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr 
\tag{4.5}
\end{align*}

with a new constant $C > 0$. Combining (4.1) with (4.5) leads to the estimate
\[
\int_{2\kappa \nu}^{\infty} (r + 1)^{-1 - \epsilon} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr \leq C(\epsilon h)^{-2} \int_0^{\infty} (r + 1)^{1 + \epsilon} |Q_{\nu}^\pm(h) u(r)|^2 dr
\]
\[+ C(\epsilon h)^{-1} \theta \int_0^{\infty} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr. \tag{4.6}\]

By (4.5) and (4.6), we conclude
\[
\int_0^{\infty} (r + 1)^{-1 - \epsilon} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr
\]
\[\leq C(\epsilon h)^{-2} \int_0^{\infty} (r + 1)^{1 + \epsilon} |Q_{\nu}^\pm(h) u(r)|^2 dr
\]
\[+ C(\epsilon h)^{-1} \theta \int_0^{\infty} \left( |u(r)|^2 + |D_r u(r)|^2 \right) dr. \tag{4.7}\]

Taking \( h \) small enough, we can arrange that \( \epsilon < 2s - 1 \). Therefore, Proposition 2.3 follows from (4.7). \qed

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