Approximation of continuous periodic functions of two variables via power series methods of summability

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\textbf{Abstract:} We prove a Korovkin type approximation theorem via power series methods of summability for continuous $2\pi$-periodic functions of two variables and verify the convergence of approximating double sequences of positive linear operators by using modulus of continuity. An example concerning double Fourier series is also constructed to illustrate the obtained results.

\section{Introduction}

The classical Korovkin second theorem is stated as follows.

\textbf{Theorem.} Let $(L_n)$ be a sequence of positive linear operators from the space $C_{2\pi} (\mathbb{R})$ into itself. Then $\lim_{n} \|L_n(f) - f\| = 0$ for all $f \in C_{2\pi} (\mathbb{R})$ if and only if $\lim_{n} \|L_n(f_i) - f_i\| = 0$ for $i = 0, 1, 2$ where $f_0(x) = 1$, $f_1(x) = \cos x$, $f_2(x) = \sin x$.

Classical versions of which are introduced by P. P. Korovkin [9, 10], Korovkin type theorems deal with the approximation of functions by positive linear operators by means of providing subsets of test functions which guarantee the approximation in whole space. Following its invention, Korovkin theory has developed in many ways and found applications in various branches of mathematics. Researchers have investigated the approximation of functions in different spaces through various subsets of test functions. Besides, in connection with sequence transformations, weighted mean methods and power series methods of summability have been applied to Korovkin type theorems to recover the convergence of operators for which classical Korovkin theorems fail to work [5, 8, 11, 12, 13, 14, 15]. Furthermore, there are studies dealing with Korovkin type theorems via weighted mean summability methods for continuous functions of two variables [1, 4, 6, 7]. In this paper, we prove a Korovkin type approximation theorem for $2\pi$-periodic and real valued continuous functions on $\mathbb{R}^2$ by using power series methods of summability and also give an approximation theorem by using modulus of continuity. Besides we construct an illustrative example concerning double Fourier series of functions of two variables such that our new result works but classical Korovkin theorem for periodic functions of two variables does not work.

Now we give some preliminaries concerning the space of $2\pi$-periodic and real valued continuous functions on $\mathbb{R}^2$ and concerning the concept of power series methods of summability for double sequences. A function $f$ on $\mathbb{R}^2$ is $2\pi$-periodic if for all $(x, y) \in \mathbb{R}^2$

\[ f(x, y) = f(x + 2k\pi, y) = f(x, y + 2k\pi) \]

holds for $k = 0, \pm 1, \pm 2, \ldots$. The space of $2\pi$-periodic and real valued continuous functions on $\mathbb{R}^2$ is denoted by $C^\ast (\mathbb{R}^2)$ and is equipped with the norm

\[ \|f\|_{C^\ast (\mathbb{R}^2)} := \sup_{(x,y) \in \mathbb{R}^2} |f(x,y)| \quad (f \in C^\ast (\mathbb{R}^2)). \]

Suppose that $p = (p_{mn})$ is a double sequence of nonnegative numbers with $p_{00} > 0$ such that $\sum_{k,l=0}^{m,n} p_{kl} \to \infty$ as $m, n \to \infty$ and associated power series $p(r,s) = \sum_{m,n=0}^{\infty} p_{mn} r^m s^n$ is convergent for $r, s \in (0, 1)$, where

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within the paper convergence of double sequences and of double series is meant in Pringsheim’s sense. A sequence \((a_{mn})\) is said to be summable to \(l\) by power series method determined by \(p\) if \(\sum_{m,n=0}^{\infty} a_{mn}p_{mn}r^ms^n\) converges for \(r, s \in (0, 1)\) and

\[
\frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} a_{mn}p_{mn}r^ms^n = l \quad \text{as} \quad r, s \to 1^-,
\]

where we write \(a_{mn} \to l (J_p)\). Power series method \(J_p\) is b-regular if for any fixed \(m_0, n_0\)

\[
\frac{1}{p(r,s)} \sum_{m=0}^{\infty} p_{mn}r^m \to 0, \quad \frac{1}{p(r,s)} \sum_{n=0}^{\infty} p_{mn}s^n \to 0 \quad \text{as} \quad r, s \to 1^{-}
\]

holds (see [2, p. 84], [3, 13]). We note that in special cases \(p_{mn} = 1\) and \(p_{mn} = \frac{1}{(m+1)(n+1)}\) corresponding power series methods are the Abel summability method and logarithmic summability method, respectively.

Let \((L_{mn})\) be a double sequence of positive linear operators from \(C^*(\mathbb{R}^2)\) into itself such that for every \(r, s \in (0, 1)\)

\[
\sum_{m,n=0}^{\infty} \|L_{mn}(f_0)\|_{C^*(\mathbb{R}^2)} p_{mn}r^ms^n < \infty
\]

(1.1)

where \(f_0(x, y) = 1\). Then for all \(f \in C^*(\mathbb{R}^2)\) double series \(\sum_{m,n=0}^{\infty} L_{mn}(f; x, y)p_{mn}r^ms^n\) is convergent for \(r, s \in (0, 1)\).

2 Main Results

**Theorem 2.1.** Let \((L_{mn})\) be a double sequence of positive linear operators from \(C^*(\mathbb{R}^2)\) into \(C^*(\mathbb{R}^2)\) such that (1.1) is satisfied. Then for all \(f \in C^*(\mathbb{R}^2)\),

\[
L_{mn}(f) \to f (J_p)
\]

(2.1)

if and only if

\[
L_{mn}(f_i) \to f_i (J_p) \quad (i = 0, 1, 2, 3, 4)
\]

(2.2)

with \(f_0(x, y) = 1, f_1(x, y) = \sin x, f_2(x, y) = \sin y, f_3(x, y) = \cos x, f_4(x, y) = \cos y\).

**Proof.** Let \((L_{mn})\) be a double sequence of positive linear operators from \(C^*(\mathbb{R}^2)\) into \(C^*(\mathbb{R}^2)\) satisfying (1.1). First suppose that (2.1) is satisfied for all functions in \(C^*(\mathbb{R}^2)\). Then in particular it is satisfied for \(f_0, f_1, f_2, f_3, f_4\) since \(f_i \in C^*(\mathbb{R}^2)\) and this completes the necessity part of the proof of theorem. Now we shall prove the sufficiency part. Let \(f \in C^*(\mathbb{R}^2)\) and (2.2) be satisfied. Let \(I\) and \(J\) be closed subintervals of length \(2\pi\) of \(\mathbb{R}\). Fix \((x, y) \in I \times J\). It follows from the continuity of \(f\) that for given \(\varepsilon > 0\) there is a number \(\delta > 0\) such that

\[
|f(u, v) - f(x, y)| < \varepsilon + \frac{2M_f}{\sin^2 \left(\frac{x}{2}\right)} \left\{\sin^2 \left(\frac{u - x}{2}\right) + \sin^2 \left(\frac{v - y}{2}\right)\right\}
\]

where \(M_f = \|f\|_{C^*(\mathbb{R}^2)}\) in view of the the proof of Theorem 2.1 in [7]. Hence we obtain

\[
\left| \frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f; x, y)p_{mn}r^ms^n - f(x, y) \right| = \left| \frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f; x, y)p_{mn}r^ms^n - \frac{f(x, y)}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0; x, y)p_{mn}r^ms^n \right|
\]
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\[ \frac{f(x, y)}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0; x, y)p_{mn}r^m s^n - f(x, y) \]
\[ \leq \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f(u, v) - f(x, y); x, y)p_{mn}r^m s^n \]
\[ + M_f \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0; x, y)p_{mn}r^m s^n - f_0(x, y) \right| \]
\[ \leq \frac{\varepsilon}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0; x, y)p_{mn}r^m s^n \]
\[ + \frac{2M_f}{\sin^2 \left( \frac{\delta}{2} \right)} \sum_{m,n=0}^{\infty} L_{mn} \left( \sin^2 \left( \frac{u - x}{2} \right) + \sin^2 \left( \frac{v - y}{2} \right) ; x, y \right)p_{mn}r^m s^n \]
\[ + M_f \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0; x, y)p_{mn}r^m s^n - f_0(x, y) \right| \]
\[ \leq \varepsilon + (\varepsilon + M_f) \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_1; x, y)p_{mn}r^m s^n - f_1(x, y) \right| \]
\[ + |\sin x| \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_1; x, y)p_{mn}r^m s^n - f_1(x, y) \right| \]
\[ + |\sin y| \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_2; x, y)p_{mn}r^m s^n - f_2(x, y) \right| \]
\[ + |\cos x| \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_3; x, y)p_{mn}r^m s^n - f_3(x, y) \right| \]
\[ + |\cos y| \left| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_4; x, y)p_{mn}r^m s^n - f_4(x, y) \right| \]
\[ \leq \varepsilon + K \sum_{i=0}^{4} \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_i; x, y)p_{mn}r^m s^n - f_i(x, y) \]

where \( K = \varepsilon + M_f + \frac{2M_f}{\sin^2(\delta/2)} \) by using the fact that \( \sin^2 \left( \frac{\delta}{2} \right) = \frac{1}{2}(1 - \cos a \cos b - \sin a \sin b) \). Then taking supremum over \((x, y)\) we conclude
\[ \left\| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f)p_{mn}r^m s^n - f \right\|_{C^*(\mathbb{R}^2)} \leq \varepsilon + K \sum_{i=0}^{4} \left\| \frac{1}{p(r, s)} \sum_{m,n=0}^{\infty} L_{mn}(f_i)p_{mn}r^m s^n - f_i \right\|_{C^*(\mathbb{R}^2)} \]

and this completes the proof. \( \square \)

Now we give a theorem concerning the convergence of the sequence of positive linear operators acting on the space \( C^*(\mathbb{R}^2) \) with the help of modulus of continuity. For \( f \in C^*(\mathbb{R}^2) \) and for \((u, v), (x, y) \in \mathbb{R}^2\) the modulus of
continuity of \( f \) is defined by

\[
\omega(f; \delta) := \sup \left\{ |f(u, v) - f(x, y)| : \sqrt{(u - x)^2 + (v - y)^2} \leq \delta \right\}
\]

for any \( \delta > 0 \).

**Theorem 2.2.** Let \((L_{mn})\) be a double sequence of positive linear operators from \(C^*(\mathbb{R}^2)\) into \(C^*(\mathbb{R}^2)\) such that (1.1) is satisfied. If

(i) \( L_{mn}(f_0) \to f_0 (J_p) \)

(ii) \( \lim_{r, s \to 1^-} \omega(f; \gamma(r, s)) = 0 \) where \( \gamma(r, s) = \sqrt{\left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(\varphi) p_{mn} r^m s^n \right|_{C^*(\mathbb{R}^2)}^2} \)

with \( \varphi(u, v) = \sin^2 \left( \frac{u - x}{2} \right) + \sin^2 \left( \frac{v - y}{2} \right) \), then for all \( f \in C^*(\mathbb{R}^2) \)

\( L_{mn}(f) \to f (J_p) \).

**Proof.** Let \( f \in C^*(\mathbb{R}^2) \) and \((x, y) \in [-\pi, \pi] \times [-\pi, \pi] \). Then in view of the proof of Theorem 9 in [6] we have

\[
|f(u, v) - f(x, y)| \leq \left( 1 + \pi^2 \sin^2 \left( \frac{u - x}{2} \right) + \sin^2 \left( \frac{v - y}{2} \right) \right) \omega(f; \delta)
\]

and followingly we get

\[
\left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f; x, y) p_{mn} r^m s^n - f(x, y) \right|
\]

\[
\leq \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn} \left( |f(u, v) - f(x, y)|; x, y \right) p_{mn} r^m s^n
\]

\[
+ M_f \left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f_0; x, y) p_{mn} r^m s^n - f_0(x, y) \right|
\]

\[
\leq \frac{\omega(f; \delta)}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn} \left( 1 + (\pi/\delta)^2 \left\{ \sin^2 \left( \frac{u - x}{2} \right) + \sin^2 \left( \frac{v - y}{2} \right) \right\}; x, y \right) p_{mn} r^m s^n
\]

\[
+ M_f \left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f_0; x, y) p_{mn} r^m s^n - f_0(x, y) \right|
\]

\[
= \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f_0; x, y) p_{mn} r^m s^n - f_0(x, y) \right| \omega(f; \delta) + \omega(f; \delta)
\]

\[
+ M_f \left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(\varphi; x, y) p_{mn} r^m s^n
\]

\[
+ \frac{\pi^2 \omega(f; \delta)}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(\varphi; x, y) p_{mn} r^m s^n
\]

\[
+ M_f \left| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f_0; x, y) p_{mn} r^m s^n - f_0(x, y) \right|
\]

as in the proof of Theorem 2.1 Taking supremum over \((x, y)\) and putting \( \delta := \gamma(r, s) \) we conclude that

\[
\left\| \frac{1}{p(r, s)} \sum_{m, n=0}^{\infty} L_{mn}(f) p_{mn} r^m s^n - f \right\|_{C^*(\mathbb{R}^2)}
\]
by the help of Abel-Poisson kernel $P$.

Let $L_{mn}(f_0) = \left\| \frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0) p_{mn} r^m s^n - f_0 \right\|_{C^*(\mathbb{R}^2)}$,

$$+ M_f \left\| \frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0) p_{mn} r^m s^n - f_0 \right\|_{C^*(\mathbb{R}^2)}$$

$$\leq K \left\{ \left\| \frac{1}{p(r,s)} \sum_{m,n=0}^{\infty} L_{mn}(f_0) p_{mn} r^m s^n - f_0 \right\|_{C^*(\mathbb{R}^2)} \omega(f; \gamma(r,s)) + \omega(f; \gamma(r,s)) + \omega(f; \gamma(r,s)) \right\}$$

where $K = \max\{1 + \pi^2, M_f\}$. Finally by taking limit as $r, s \to 1^-$ the proof is completed by the assumptions (i) and (ii) of the theorem. 

3 Illustrative example

Let $f \in C^*(\mathbb{R}^2)$ and $(S_{mn}(f))$ be the sequence of $mn$—th partial sums of double Fourier series

$$\frac{a_{00}}{4} + \frac{1}{2} \sum_{j=1}^{\infty} (a_{j0} \cos jx + c_{j0} \sin jx) + \frac{1}{2} \sum_{k=1}^{\infty} (a_{0k} \cos ky + b_{0k} \sin ky)$$

$$+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ a_{jk} \cos jx \cos ky + b_{jk} \cos jx \sin ky \right\} + c_{jk} \sin jx \cos ky + d_{jk} \sin jx \sin ky \right\}$$

of $f$ where $a_{jk}, b_{jk}, c_{jk}, d_{jk}$’s are the real double Fourier coefficients. We know that sequence $(S_{mn}(f))$ does not have to converge to $f$ neither uniformly nor pointwise in general and many studies have been done concerning the conditions ensuring the convergence. Besides some averaging processes have been applied to recover the convergence of double Fourier series. Now we apply Theorem 2.1 with $p_{mn} = 1$ to the operator

$$L_{mn}(f; x, y) = (1 + (-1)^{m+n}) T_{mn}(f; x, y)$$

(3.1)

where $(T_{mn})$ is the $(\frac{m}{m+1}, \frac{n}{n+1})$—th Abel-Poisson mean of double Fourier series of $f$. Then, it follows that

$$L_{mn}(f; x, y) = (1 + (-1)^{m+n}) \left\{ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x - u, y - v) P\left( \frac{m}{m+1}, u \right) P\left( \frac{n}{n+1}, v \right) du dv \right\}$$

by the help of Abel-Poisson kernel $P(r, t) = \frac{1}{1 - 2r \cos t + r^2}$. Condition (1.1) is satisfied since we have $L_{mn}(f_0; x, y) = 1 + (-1)^{m+n}$. Besides, since

$$L_{mn}(f_1; x, y) = (1 + (-1)^{m+n}) \left( \frac{m}{m+1} \right) \sin x, \quad L_{mn}(f_2; x, y) = (1 + (-1)^{m+n}) \left( \frac{n}{n+1} \right) \sin y,$$

$$L_{mn}(f_3; x, y) = (1 + (-1)^{m+n}) \left( \frac{m}{m+1} \right) \cos x, \quad L_{mn}(f_4; x, y) = (1 + (-1)^{m+n}) \left( \frac{n}{n+1} \right) \cos y,$$

we get

$$\left\| \frac{1}{(1-r)(1-s)} \sum_{m,n=0}^{\infty} L_{mn}(f_i) r^m s^n - f_i \right\|_{C^*(\mathbb{R}^2)} = \begin{cases} \frac{(1-r)(1-s)}{(1+r)(1+s)^2}, & i = 0 \\ E(r, s), & i = 1, 3 \\ E(s, r), & i = 2, 4 \end{cases}$$
where \( E(x, t) = \left\{ \frac{(1-x)(1-t)}{(1+x)(1+t)} + \frac{(1-x)\ln(1-x)}{x} - \frac{(1-t)(1-x)\ln(1+x)}{(1+t)x} \right\} \). Then we obtain

\[
\left\| (1-r)(1-s) \sum_{m,n=0}^{\infty} L_{mn}(f_i)r^m s^n - f_i \right\|_{C^*([0,1] \times [0,1])} \to 0 \quad \text{as} \quad r, s \to 1^{-}
\]

for \( i = 0, 1, 2, 3, 4 \) and by Theorem 2.1 we conclude that \( L_{mn}(f) \to f \ (J_1) \). However, classical Korovkin theorem for periodic functions of two variables fails to work for \( L_{mn} \) since even \( L_{mn}(f_0) = (1 + (-1)^{m+n}) \to f_0 \) as \( m, n \to \infty \).

Now we verify the approximation of sequence of linear operators \( (L_{mn}) \) in (3.1) by the help of Theorem 2.2 \( L_{mn}(f_0) \to f_0 \ (J_1) \) is satisfied since

\[
\left\| (1-r)(1-s) \sum_{m,n=0}^{\infty} L_{mn}(f_0)r^m s^n - f_0 \right\|_{C^*([0,1] \times [0,1])} = \frac{(1-r)(1-s)}{(1+r)(1+s)}
\]

and so (i) of Theorem 2.2 holds. Now consider (ii) of Theorem 2.2. Since we have

\[
\gamma(r, s) = \sqrt{\left\| (1-r)(1-s) \sum_{m,n=0}^{\infty} L_{mn} \left( \sin^2 \left( \frac{u-x}{2} \right) + \sin^2 \left( \frac{v-y}{2} \right) \right) r^m s^n \right\|_{C^*([0,1] \times [0,1])}}
\]

and since the power series method \( J_p \) is regular we get that \( \lim_{r,s \to 1^{-}} \gamma(r, s) = 0 \). Then from uniform continuity of \( f \) we conclude that \( \lim_{r,s \to 1^{-}} \omega(f; \gamma(r, s)) = 0 \) and (ii) of Theorem 2.2 is satisfied. So \( L_{mn}(f) \to f \ (J_1) \).

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