Quasi-exactly solvable quartic Bose Hamiltonians

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We consider Hamiltonians, which are even polynomials of the forth order with the respect to Bose operators. We find subspaces, preserved by the action of Hamiltonian. These subspaces, being finite-dimensional, include, nonetheless, states with an infinite number of quasi-particles, corresponding to the original Bose operators. The basis functions look rather simple in the coherent state representation and are expressed in terms of the degenerate hypergeometric function with respect to the complex variable labeling the representation. In some particular degenerate cases they turn (up to the power factor) into the trigonometric or hyperbolic functions, Bessel functions or combinations of the exponent and Hermite polynomials. We find explicitly the relationship between coefficients at different powers of Bose operators that ensure quasi-exact solvability of Hamiltonian.

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The conception of quasi-exactly solvable (QES) systems, discovered in 1980s \[1\] - \[12\], received in recent years much attention both from viewpoint of physical applications and their inner mathematical beauty. It turned out that in quantum mechanics there exists a peculiar class of objects that occupy an intermediate place between exactly solvable and non-solvable models in the sense that in an infinite Hilbert space a finite part of a spectrum is singled out within which eigenvectors and eigenvalues can be found from an algebraic equation of a finite degree - in other words, a partial algebraization of the spectrum occurs. For one-dimensional QES models corresponding QES Hamiltonians possess hidden group structure based on \( sl(2, R) \) algebra. Thus, they have direct physical meaning, being related to quantum spin systems \[13\].

Meanwhile, the notion of QES systems is not constrained by potential models and can have nontrivial meaning for any kind of infinite-dimensional systems. In the first place, it concerns Bose Hamiltonians whose physical importance is beyond doubt. Here one should distinguish two cases. First, it turns out that some systems of two interacting particles or quasi-particles with Bose operators of creation and annihilation \( a, a^+ \) and \( b, b^+ \) can be mapped on the problem for a particle moving in a certain type of one-dimensional potentials and, remarkably, these potentials belong just to the QES type \[14\] - \[16\]. In particular, such a type of Hamiltonians is widely spread in quantum optics and physics of magnetism \[13\]. The aforementioned mapping works only for a special class of Bose Hamiltonians which possess an integral of motion. Then the procedure is performed in three steps: (i) all Hilbert space splits in a natural way to different pieces with respect to the values of an integral of motion, (ii) in each piece the Schrödinger equation takes a finite-difference form, (iii) it is transformed into the differential equation by means of introducing a generating function. In so doing, the integral of motion under discussion represents a linear combination of numbers of particles \( a^+a \) and \( b^+b \).

The second kind of Bose systems looks much more usual - it is simply some polynomial
with respect to Bose operators of creation and annihilation of one particle. The fact that
only one pair $a, a^+$ enters Hamiltonian, deprives us, by contrast with the first case, of the
possibility to construct a simple integral of motion - in this sense the eigenvalue problem
becomes more complicated. In general, the solutions of the Shrödinger equation contain in-
finite numbers of quasi-particles and only approximate or numerical methods can be applied
to such systems. However, as was shown recently [17], if the coefficients at different powers
of $a, a^+$ are selected in a proper way, in some cases a finite-dimensional closed subspace is
singled out and algebraization of the spectrum occurs similar to what happens in ”usual”
QES potential models or differential equations. In so doing, the eigenvectors belonging to
the subspace under discussion, can be expressed as a finite linear combination of eigenvectors
of an harmonic oscillator and, thus, contain a finite number of quasi-particles [17].

In the present article we extend the approach of [17] and consider much more general
classes of Hamiltonians. Their distinctive feature consists in that the relevant basis functions
that compose a finite-dimensional subspace, look very much unlike the wave functions of a
harmonic oscillator. As a result, we obtain QES models with an infinite numbers of quasi-
particles in this finite-dimensional subspace. Bearing in mind physical application, we make
emphasis on Hermitian Hamiltonians, although our approach is applicable to more general
QES Bose operators without demand of Hermiticity.

II. BOSE HAMILTONIANS AS DIFFERENTIAL OPERATORS AND
STRUCTURE OF INVARIANT SUBSPACES

Consider the operator which is the even polynomial of the forth degree with respect to
Bose operators of creation $a^+$ and annihilation $a$. It can be written in the form

$$H = a_{++}K_+^2 + a_{--}K_-^2 + a_{00}K_0^2 + a_{0}K_0K_- + a_{+0}K_+K_0 + a_0K_0 + a_-K_- + a_+K_+,$$  \hspace{1cm} (1)

where

$$K_0 = \frac{1}{2} \left( a^+a + \frac{1}{2} \right), \quad K_- = \frac{a^2}{2}, \quad K_+ = \frac{a^{+2}}{2},$$  \hspace{1cm} (2)
\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \]  

(3)

The Casimir operator \( C = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) \equiv -\frac{3}{16}. \)

We will use the coherent state representation in which

\[ a \rightarrow \frac{\partial}{\partial z}, \quad a^+ \rightarrow z \]  

(4)

After substitution into (1) Hamiltonian \( H(a^+, a) \) becomes a differential operator \( H(z, \frac{\partial}{\partial z}). \)

In the previous article [17] we discussed Bose systems that possess the invariant subspace of the form \( F = \text{span}\{z^n\} \) or \( \text{span}\{z^{2n}\} \). The first natural step towards generalization consists in considering subspaces (with \( N \) fixed)

\[ F = \text{span}\{u_n\}, \quad u_n = z^{2n}u, \quad n = 0, 1, 2...N, \]  

(5)

for which the following procedure should be realized. (i) The action of operators of \( K_i \) on the functions \( u_n \) should lead to the linear combinations of functions from the same set \( \{u_n\} \), (ii) by the selection of appropriate coefficients in (1), we achieve the subspace \( F \) to be closed under the action of Hamiltonian \( H \). We would like to stress that the condition (i) does not forbid \( u_n \) with \( n > N \) to appear in terms like \( K_iu_n \) but the condition (ii) rules out such functions from \( Hu_n \) (recall that we consider Hamiltonians which are quadratic-linear combinations of \( K_i \)).

It is seen from (2), (4) that the operators \( K_i \) contain \( z \) and \( \frac{\partial}{\partial z} \). Therefore, it is convenient to assume that differentiation of \( u(z) \) gives rise to \( u \) up to the factor that contains powers of \( z \). The corresponding choice is not unique. In the present article we restrict ourselves to one of the simplest possibilities that leads to nontrivial solutions. To this end, we choose \( u \) that obeys the differential equation

\[ u' = A(z)u, \quad A(z) = (\frac{\beta}{z} + 2\rho z). \]  

(6)

We will show below that the choice (6) relates \( K_iu_n \) to \( u_n, \quad u_{n\pm1} \) that, in turn, allows us to formulate the conditions of cut off for Hamiltonian in the form of algebraic equations which its coefficients obey. It follows from (6) that \( u = z^\beta \exp(\rho z^2) \). To ensure asymptotic analytic
behavior near $z = 0$, we demand that $\beta = 0, 1, 2...$ Now let us take into account some basic properties of coherent states (see, e.g. Ch. 7 of Ref. [18]). Our functions $u_n(z)$ must belong to the Bargmann-Fock space. It means that they should obey the conditions of integrability and analyticity. The condition of integrability for any two functions $f, g$ from our space

$$\int dzdz^* f^* g e^{-zz^*} < \infty,$$

entails, for our choice of $u$, $|\rho| < 1/2$.

Taking into account eq. (3), it is straightforward to show that

$$K_+ u_n = C_+ u_{n+1},$$

$$K_- u_n = A_-(n) u_n + B_-(n) u_{n-1} + C_- u_{n+1},$$

$$K_0 u_n = A_0(n) u_n + C_0 u_{n+1},$$

where $C_+ = \frac{1}{2}$, $C_- = 2\rho^2$, $C_0 = \rho$, $A_-(n) = (2\beta + 4n + 1)\rho$, $A_0(n) = \frac{2\beta + 4n + 1}{4}$, $B_-(n) = \frac{(\beta + 2n)(\beta + 2n - 1)}{2}$.

Using eqs. (8) - (10), one can present the action of the operator (11) in the form

$$H u_n = D_2 u_{n+2} + D_1(n) u_{n+1} + \tilde{D}_0(n) u_n + \tilde{D}_1(n) u_{n-1} + \tilde{D}_2(n) u_{n-2},$$

where

$$D_2 = \frac{a_{+0}}{2} C_0 + a_{0-} C_0 + a_{00} C_0^2 + a_{-+} C_- + \frac{a_{++}}{4}$$

$$D_1(n) = a_{-+} [A_-(n) C_- + C_- A_-(n + 1)] + a_{00} [A_0(n) C_0 + C_0 A_0(n + 1)] + \frac{a_{+0}}{2} A_0(n) + a_{0-} [A_0(n) C_0 + C_0 A_0(n + 1)] + a_0 C_0 + a_{-+} + \frac{a_+}{2},$$

$$\tilde{D}_0 = a_{00} A_0^2(n) + a_{0-} [A_-(n) A_0(n) + B_-(n) C_0] + a_0 A_0(n) + a_{-+} A_-(n),$$

$$\tilde{D}_1(n) = a_{-+} [A_-(n) B_-(n + 1) + B_-(n) A_-(n - 1)] + a_{0-} B_-(n) A_0(n - 1) + a_{-+} B_-(n),$$

$$\tilde{D}_2 = a_{-+} B_-(n) B_-(n - 1).$$

For the operator (11) to be quasi-exactly solvable with the invariant subspace (5), it is necessary that the following conditions of cut off be satisfied:
\[ D_2 = 0, \] 
\[ D_1(N) = 0, \] 
\[ \tilde{D}_1(0) = 0, \] 
\[ \tilde{D}_2(0) = 0, \] 
\[ \tilde{D}_2(1) = 0. \] 

In general, this system is rather cumbersome. However, it is simplified greatly if we consider Hermitian Hamiltonians with \( a_{--} = 0 = a_{++} \). Then \( \tilde{D}_2 \equiv 0 \) and we get three equations

\[ \rho^2 a_{00} + 2\rho^3 a_{0-} + \frac{1}{2}\rho a_{0-} = 0 \] 
\[ \frac{a_0}{4}(2\beta - 3) + a_{-} \beta(\beta - 1) = 0 \] 
\[ \frac{1 + 4\rho^2}{2} a_{--} + \rho a_{00} + \left[ \frac{2\beta + 4N + 1}{8} + \rho^2 \frac{7 + 6\beta + 12N}{2} \right] a_{0-} + \frac{a_0}{2} \rho(2\beta + 4N + 3) = 0. \]

It is assumed that \( a_+ = a_- \), \( a_{+0} = a_{0-} \), all coefficients are real. The analysis leads to the following table of possible solutions:

| \( a_{++} = 0 \) | \( \rho \) | \( \beta \) | \( a_- \) | \( a_{00} \) | \( a_0 \) |
|-----------------|-----|-----|-----|-----|-----|
| 1 \( 0 \) \( 0 \) | \(-\frac{4N+1}{4}a_{0-}\) | a.v. | a.v. |
| 2 \( 0 \) \( 1 \) | \(-\frac{4N+3}{4}a_{0-}\) | a.v. | a.v. |
| 3 a.v. a.v. \( \frac{3-2\beta}{4}a_{0-}\) | \( f(\rho)a_{0-}\) | \( \frac{2\beta+2N+1}{4\rho} - 2(N+1)\rho \) \( a_{0-} \) |

| \( a_{++} \neq 0 \) | \( 4 \) a.v. | \( 0 \) a.v. | \( f(\rho)a_{0-}\) | \( f(\rho)a_{0-} + f_1(\rho, N)a_{0-} + f_2(\rho, N)a_{++} \) |
|-----------------|-----|-----|-----|-----|
| 5 a.v. \( 1 \) a.v. | \( f(\rho)a_{0-}\) | \( f(\rho)a_{0-} + f_1(\rho, N + \frac{1}{2})a_{0-} + f_2(\rho, N + \frac{1}{2})a_{++} \) |

Here "a.v." denotes "arbitrary value" with the reservation that \( \beta \) is a positive integer or zero and \(|\rho| < 1/2\), as is explained above. By definition,

\[ f(\rho) = -\frac{1 + 4\rho^2}{2\rho}, \quad f_1(\rho, N) = \frac{4N + 5}{8\rho} - \frac{4N + 1}{2} \rho, \quad f_2(\rho, N) = -\frac{4N + 3}{8\rho^2}(16\rho^4 - 1). \] 

One can check that the solution 1, when \( u = 1 \), corresponds to even states for the example considered in eq. (7) of [17] provided in that equation the coefficient \( A_2 = 0 \). In a
similar way, the case 2 \((u = z)\) corresponds to odd states from the same example. However, the cases (3)-(5) represent new solutions that were not contained in [17].

III. DOUBLED INVARIANT SUBSPACES

Much more rich family of new classes of Bose QES quartic Hamiltonians can be obtained if we generalize the structure of the invariant subspace introducing, in addition to \(u_n\), a second subset of independent functions. Consider the set of functions

\[ u_n = z^{2n}u, \quad v_n = z^{2n+1}v, \quad n = 0, 1, 2... \]  

(26)

We are interested in such function which form a set, defined for a fixed \(N = 0, 1, 2...\),

\[ F = \text{span} \{u_n, v_n\} = \text{span} \{z^{2n} \cdot u(z), z^{2n+1} \cdot v(z)\}, \quad n = 0, 1, 2..., N; \]  

(27)

invariant with respect to the action of the operator \(H\). The dimension of \(F\) is equal to \(2(N + 1)\). It may happen that, for some values of parameters, the functions \(v_n\) may be proportional or even exactly equal to \(u_n\). Then our subspace reduces to the \(N + 1\) one (3) considered in a previous section. To gain qualitatively new QES models, in what follows we will consider the functions \(u_n\) and \(v_n\) as, generally speaking, independent.

Let \(u\) and \(v\) obey the system of differential equations that generalizes the relation (3):

\[ u' = Au + Bv, \]  

(28)

\[ v' = Cu + Dv. \]  

(29)

Here prime denotes differentiation with respect to \(z\), \(A, B, C, D\) are functions of \(z\).

It follows from (28), (29) that

\[ u'' - u' \left( S + \frac{B'}{B} \right) + u[\Delta + \frac{W(B, A)}{B}] = 0, \]  

(30)

\[ v'' - v' \left( S + \frac{C'}{C} \right) + v[\Delta + \frac{W(C, D)}{C}] = 0. \]  

(31)
Here \( S = A + D = SpL \), where \( L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), \( W(f_1, f_2) \equiv f'_1f_2 - f_1f'_2 \) is a Wronskian, \( \Delta = AD - BC \) is a determinant of \( L \). In what follows we assume for simplicity that quantities in denominators in (30), (31) \( B(z) \equiv \alpha = const \) and \( C(z) = \gamma = const \). Then we have

\[
\begin{align*}
    u'' - u'S + u(\Delta - A') &= 0, \\ 
v'' - v'S + v(\Delta - D') &= 0.
\end{align*}
\]

We assume also, by analogy with (8), that \( A \) and \( D \) contain only terms of the order \( z \) and \( z^{-1} \) in the Loran series: \( A = 2\rho z + \beta z^{-1} \), \( D = 2\tau z + \delta z^{-1} \). Then we have

\[
\begin{align*}
    \frac{d}{dz} u(z) &= \alpha v(z) + \frac{\beta}{z} u(z) + 2\rho u(z), \\ 
    \frac{d}{dz} v(z) &= \gamma u(z) + \frac{\delta}{z} v(z) + 2\tau v(z).
\end{align*}
\]

One obtains from (32), (33)

\[
\begin{align*}
    u'' - u' \left[ 2(\rho + \tau)z + \beta \left( \frac{1 + \delta}{z^2} \right) + 4\rho\tau z^2 + 2(\rho + \delta) - 2\rho - \alpha\gamma \right] &= 0, \\ 
v'' - v' \left[ 2(\rho + \tau)z + \delta \left( \frac{1 + \beta}{z^2} \right) + 4\rho\tau z^2 + 2(\rho + \beta) - 2\rho - \alpha\gamma \right] &= 0.
\end{align*}
\]

Our functions \( u_n(z) \) and \( v_n(z) \) must belong to the Bargmann-Fock space that entails, similarly to what is obtained in the previous section, the conditions \( |\rho| < 1/2, |\tau| < 1/2 \).

To elucidate what constraints are imposed by the demand of analyticity, consider separately several different cases. If \( \gamma = \alpha = 0 \), eqs. (34) can be integrated and one easily finds that \( u = z^\beta \exp(\rho z^2), \ v = z^\delta \exp(\tau z^2), \) whence it is obvious that \( \beta = 0,1,2... \) and \( \delta = -1,0,1,2... \) If \( \alpha = 0 \) but \( \gamma \neq 0 \), one can make the substitution \( v = z^\delta \exp(\tau z^2) w \). Then

\[
w' = \gamma z^{\beta - \delta} \exp[(\rho - \tau)z^2]
\]

It is clear that \( \beta = 0,1,2... \), whereas \( \delta \) is arbitrary except \( \delta = \beta + 1 \) since the latter would have led to the logarithmic terms in \( v(z) \). The similar situation occurs when \( \gamma = 0 \) but \( \alpha \neq 0 \). Then \( \delta = -1,0,1,2... \) and forbidden values of \( \beta \) are \( \beta = \delta + 1 \).
Let now $\alpha \gamma \neq 0$. First, consider the case $\rho \neq \tau$. Then by substitutions

\begin{align*}
    u(z) &= y \left( z^2 (\tau - \rho) \right) \cdot \exp \left( \rho z^2 \right) \cdot z^\beta \\
    v(z) &= \tilde{y} \left( z^2 (\rho - \tau) \right) \cdot \exp \left( \tau z^2 \right) \cdot z^\delta
\end{align*}

eqs. (35), (36) are reduced to the form, typical of a degenerate hypergeometric function

\begin{equation}
    x \frac{d^2}{dx^2} y(x) + (\eta - x) \frac{d}{dx} y(x) - \xi y(x) = 0,
\end{equation}

where $\eta = \frac{1}{2} (\beta - \delta + 1)$, $\xi = \frac{\alpha \gamma}{4(\tau - \rho)}$. The function $\tilde{y}$ satisfies the equation of the same form (40) but with parameters $\tilde{\eta} = 1 - \eta$, $\tilde{\xi} = -\xi$.

To determine the admissible range of parameters $\beta$, $\delta$ one can appeal directly to the well-known properties of this function and take into account that the general solution of eq. (40) has the form $y = A y_1 + B y_2$, where $y_1 = \Phi(\xi, \eta; x)$ and $y_2 = x^{1-\eta} \Phi(\xi - \eta + 1, 2 - \eta; x)$ and the standard notation for the degenerate hypergeometric function is used (see Ch. 6 of Ref. [19]). First consider the case when $\eta$ is non-integer. Then $\Phi \to 1$ when $x \to 0$ and from (38) we obtain the function $u$ can have two possible asymptotic forms: $u_1 \sim z^\beta$ and $u_2 \sim z^{\delta+1}$. The function $v(z)$ behaves, correspondingly, like $v_1 \sim z^{\beta+1}$ and $v_2 \sim z^\delta$. Therefore, it turns out that there are two cases:

\begin{equation}
    \beta = 0, 1, 2..., \delta \text{ is arbitrary}
\end{equation}

or

\begin{equation}
    \delta = -1, 0, 2..., \beta \text{ is arbitrary.}
\end{equation}

If $\eta$ is integer, there exists only one independent solution of eq. (40), regular at $x \to 0$. The corresponding solution is known to be $\Phi^*(\xi, \eta; x) = \frac{\Phi(\xi, \eta; x)}{\Gamma(\eta)}$. In the limit $x \to 0$ $\Phi^* \sim x^{1-\eta}$ that does not affect the conclusion about admissible range of $\beta$ and $\delta$.

Let now $\rho = \tau$, $\alpha \gamma \neq 0$. By substitution

\begin{equation}
    u(z) = y \left( 2\sqrt{\alpha \gamma} z \right) \cdot \exp \left( z (\rho z - \sqrt{\alpha \gamma}) \right) \cdot z^\beta
\end{equation}
we obtain that the function \( y(x) \) obeys the equation
\[
 x \frac{d^2}{dx^2} y(x) + (\beta - \delta - x) \frac{d}{dx} y(x) - \frac{(\beta - \delta)}{2} y(x) = 0
\]
that has the same form as (40) and admissible \( \beta \) and \( \delta \) satisfy one of criteria (41), (42).

Differential equations for our functions can be also written in the symmetric form. Let us make the substitution
\[
u = \Psi z^{\beta + \delta} \exp \left[ \frac{(\rho + \tau)}{2} z^2 \right]. \tag{43}
\]
Then
\[
\Psi'' + (\varepsilon - V_{\text{eff}}) \Psi = 0, \tag{44}
\]
where
\[
V_{\text{eff}} = \frac{k}{z^2} + (\rho - \tau)^2 z^2, \tag{45}
\]
\[
k = \frac{(\delta - \beta)(\delta - \beta + 2)}{4}. \tag{46}
\]
\[
\varepsilon = (\rho - \tau)(\delta - \beta - 1) - \alpha \gamma. \tag{47}
\]
In a similar way,
\[
v = \tilde{\Psi} z^{\beta + \delta} \exp \left[ \frac{(\rho + \tau)}{2} z^2 \right], \tag{48}
\]
where \( \tilde{\Psi} \) obeys the equation (44) with the same structure of \( V_{\text{eff}} \) but with another \( \tilde{k} = \frac{(\beta - \delta)(\beta - \delta + 2)}{4} \). \( \tilde{\varepsilon} = (\tau - \rho)(\beta - \delta - 1) - \alpha \gamma. \) It is seen that \( \tilde{\varepsilon} \) can be obtained from \( \varepsilon \) by interchange between \( \rho \) and \( \tau, \beta \) and \( \delta \). It is also seen that \( \tilde{\varepsilon} - \varepsilon = 2(\rho - \tau) \).

Thus, formally, we obtain the harmonic oscillator with a barrier \( z^{-2} \) (Kratzer Hamiltonian). Let us remind, however, that the variable \( z \) in our context is complex.

It follows from (44) and (43), (48) that the functions \( \Psi, \tilde{\Psi} \) obey the system of equations
\[
\left[ \frac{d}{dz} + (\tau - \rho)z + \frac{\delta - \beta}{2z} \right] \Psi = \alpha \tilde{\Psi}, \tag{49}
\]
\[
\left[ \frac{d}{dz} + (\rho - \tau)z + \frac{\beta - \delta}{2z} \right] \tilde{\Psi} = \gamma \Psi. \tag{50}
\]
IV. CONDITIONS OF QUASI-EXACT SOLVABILITY

The action of operators $K_i$ in the subspace has the same structure but now the corresponding quantities $A_i, B_i, C_i$ becomes $2 \times 2$ matrices:

\[ K_+ \vec{f}_n = C_+ \vec{f}_{n+1}, \quad (51) \]
\[ K_- \vec{f}_n = A_-(n) \vec{f}_n + B_-(n) \vec{f}_{n-1} + C_- \vec{f}_{n+1}, \quad (52) \]
\[ K_0 \vec{f}_n = A_0(n) \vec{f}_n + C_0 \vec{f}_{n+1}, \quad (53) \]

where \( \vec{f}_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \),

\[ C_+ = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_- = \begin{pmatrix} 2\rho^2 & 0 \\ \gamma(\tau + \rho) & 2\tau^2 \end{pmatrix}, \quad C_0 = \frac{1}{2} \begin{pmatrix} 2\rho & 0 \\ \gamma & 2\tau \end{pmatrix}, \quad (54) \]

\[ A_-(n) = \frac{1}{2} \begin{pmatrix} 4\beta \rho + 8n \rho + 2\rho + \alpha \gamma & 2\alpha(\tau + \rho) \\ \gamma(\beta + \delta + 2 + 4n) & 4\delta \tau + 8n \tau + 6 \tau + \alpha \gamma \end{pmatrix}, \quad (55) \]

\[ A_0(n) = \frac{1}{4} \begin{pmatrix} 2\beta + 4n + 1 & 2\alpha \\ 0 & 2\delta + 4n + 3 \end{pmatrix}, \quad (56) \]

\[ B_-(n) = \frac{1}{2} \begin{pmatrix} (\beta + 2n)(\beta + 2n - 1) & \alpha(\beta + \delta + 4n) \\ 0 & (\delta + 2n + 1)(\delta + 2n) \end{pmatrix}. \quad (57) \]

In a similar way, the action of Hamiltonian in the invariant subspace can be represented in the form

\[ H \vec{f}_n = D_2 \vec{f}_{n+2} + D_1(n) \vec{f}_{n+1} + D_0(n) \vec{f}_n + D_1(n) \vec{f}_{n-1} + D_2(n) \vec{f}_{n-2}, \quad (58) \]
where matrices $D_i$ and $\tilde{D}_i$ have the form (12) - (16) with $A_i, B_i, C_i$ taken from eqs. (54) - (57) (the order of operators is taken into account properly in this form of writing).

For the operator (11) to be quasi-exactly solvable with the invariant subspace (27), it is necessary that the matrix version of the conditions of cut off (17) - (21) be satisfied. The corresponding system of equations is too cumbersome to be listed here. It can be simplified greatly if we assume the condition $a_{-\cdot} = 0$, in which case after simple calculations we get $\tilde{D}_2 = 0$,

$$D_2 = \begin{pmatrix}
\rho^2 a_{00} + 2 \rho^3 a_{0-} + \frac{1}{2} \rho a_{+0} + \frac{a_{++}}{4} & 0 \\
\gamma Y & \tau^2 a_{00} + 2 \tau^3 a_{0-} + \frac{1}{2} \tau a_{+0} + \frac{a_{++}}{4}
\end{pmatrix}$$

(59)

where $Y \equiv \left[ \frac{1}{4} (\rho + \tau) a_{00} + (\rho^2 + \tau^2 + \rho \tau) a_{0-} + \frac{1}{4} a_{0+} \right]$,

$$2\tilde{D}_1(0) = \begin{pmatrix}
\frac{\alpha}{4} (2\beta - 3) + a_{-} & \alpha \left[ \frac{\alpha}{4} (2\beta^2 + 2\delta^2 + 2\beta \delta - 3\beta - \delta) + \alpha (\beta + \delta) \right] \\
0 & \frac{\delta}{(4\delta + 4 \alpha)} \left[ (2\delta - 1) \frac{\alpha}{4} a_{-} + a_{-} \right]
\end{pmatrix}$$

(60)

$$D_1(N) = \begin{pmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4
\end{pmatrix}$$

(61)

$$d_1 = \frac{a_{+}}{2} + 2 \rho^2 a_{-} + \rho a_{0} + \frac{a_{+0}}{8} (2\beta + 4N + 1) + \frac{a_{0-}}{2} [\alpha \gamma (\tau + 2 \rho) + \rho^2 (7 + 6\beta + 12N)] + \frac{a_{00}}{4} [\rho (4\beta + 8N + 6) + \alpha \gamma],$$

(62)

$$d_2 = \alpha Y,$$

(63)

$$d_3 = \frac{\gamma}{2} Z, \quad Z = [2(\tau + \rho) a_{-} + a_{0} + \frac{a_{-0}}{2} \xi + \frac{a_{00}}{2} (\beta + \delta + 4N + 4)],$$

(64)

$$\xi = 12N (\rho + \tau) + \alpha \gamma + \tau (4\delta + 2\beta + 11) + \rho (4\beta + 2\delta + 9),$$

(65)

$$d_4 = \frac{a_{+}}{2} + 2 \tau^2 a_{-} + \tau a_{0} + \frac{a_{+0}}{8} (2\delta + 4N + 3) + \frac{a_{0-}}{2} [\alpha \gamma (\rho + 2 \tau) + \tau^2 (12N + 13 + 6\delta)] + \frac{a_{00}}{4} [\tau (4\delta + 8N + 10) + \alpha \gamma]$$

(66)
One can observe that

$$d_1 - d_4 = (\rho - \tau)Z + (\beta - \delta - 1)Y$$  \hspace{1cm} (69)$$

The system of equations (17) - (21) with (59) - (67) taken into account looks rather cumbersome but the relation (69) simplifies analysis significantly.

However, for a generic case \(a_{--} \neq 0\) algebraic calculation are so bulky that we had to resort to using a computer.

In what follows we restrict ourselves by the Hermitian case only which is the most interesting for physical applications. This implies that \(a_{++} = a_{--}, a_+ = a_-, a_{+0} = a_{0-}\), where all coefficients are real.

The full set of non-trivial Hermitian solutions of the system (17) - (21) and their classification are given in the next Section.

V. HERMITIAN SOLUTIONS OF ALGEBRAIC EQUATIONS

For the type of invariant subspaces (27) under consideration, we suggest below the classification of all QES Hermitian Hamiltonians, quadratic-linear with respect to operators \(K_i\) (i.e., those which represent even polynomial of the fourth order in terms of \(a, a^+\)).

In the tables below we list only qualitatively different cases in the following sense. If some solutions can be obtained by the limiting transition (\(a_{++} \to 0, \gamma \to 0\), etc.) from a more general case, we do not repeat them. As before, we use abbreviation ”a.v.” for ”arbitrary value”.

A. \(a_{++} = 0\)

\(\gamma \neq 0\)
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\delta & \beta & \alpha & \rho & \tau & a_{-} & a_{00} \\
\hline
1 & \text{a.v.} & 0 & \text{a.v.} & 0 & \frac{\alpha \gamma}{2(2N+1-\delta)} & \frac{1-2\delta}{4}a_{0-} \\
\hline
2 & \text{a.v.} & 0 & \text{a.v.} & \frac{\alpha \gamma}{4(N+1)} & 0 & \frac{1-2\delta}{4}a_{0-} \\
\hline
3 & \text{a.v.} & 1 & \text{a.v.} & 0 & \frac{\alpha \gamma}{2(2N+2-\delta)} & \frac{1-2\delta}{4}a_{0-} \\
\hline
4 & \text{a.v.} & 1 & \text{a.v.} & \frac{\alpha \gamma}{4(N+1)} & 0 & \frac{1-2\delta}{4}a_{0-} \\
\hline
5 & 0 & \text{a.v.} & 0 & \frac{\alpha \gamma}{4N+1} & 0 & \frac{3-2\beta}{4}a_{0-} \\
\hline
6 & 0 & \text{a.v.} & 0 & \frac{\alpha \gamma}{2(2N+3-\beta)} & 0 & \frac{3-2\beta}{4}a_{0-} \\
\hline
7 & 0 & 0 & \text{a.v.} & 0 & \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{\alpha \gamma - \tau(4N+1)}{4\tau}a_{0-} \\
\hline
8 & 0 & 0 & \text{a.v.} & 0 & \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{3-2\beta}{4}a_{0-} \\
\hline
9^* & 0 & 0 & \text{a.v.} & 0 & 0 & - \frac{\alpha \gamma}{2}a_{00} \\
\hline
10 & 0 & 1 & 0 & 0 & \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{\alpha \gamma - \tau(4N+1)}{4\tau}a_{0-} \\
\hline
11 & 0 & 0 & \text{a.v.} & 0 & - \frac{4N+3}{4\rho}a_{0-} & \frac{3-2\beta}{4}a_{0-} \\
\hline
12 & -1 & 1 & \text{a.v.} & 0 & \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{\alpha \gamma - \tau(4N+1)}{4\tau}a_{0-} \\
\hline
13 & -1 & 1 & \text{a.v.} & - \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{3-2\beta}{4}a_{0-} & \frac{\alpha \gamma - \tau(4N+1)}{4\tau}a_{0-} \\
\hline
14 & -1 & 1 & \text{a.v.} & 0 & 0 & - \frac{3-2\beta}{4}a_{0-} \\
\hline
15 & -1 & 0 & 0 & 0 & \frac{\alpha \gamma - \rho(4N+3)}{4\rho} & \frac{3-2\beta}{4}a_{0-} \\
\hline
16 & -1 & 0 & 0 & \text{a.v.} & 0 & \frac{3-2\beta}{4}a_{0-} \\
\hline
17 & -1 & \text{a.v.} & 0 & \frac{\alpha \gamma}{4N+1} & 0 & \frac{3-2\beta}{4}a_{0-} \\
\hline
18 & -1 & \text{a.v.} & 0 & \frac{\alpha \gamma}{2(2N+2-\beta)} & 0 & \frac{3-2\beta}{4}a_{0-} \\
\hline
\end{array}
\]

where

\[ g(x) \doteq x - 3 - 4N \quad (70) \]

and we used definition of \( f \) according to (23).

*Note: in the case 9 the coefficients \( a_{+0} = a_{0-} = 0 \).

\[ \gamma = 0 \]
\[ \delta \quad \beta \quad \alpha \quad \rho \quad \tau \quad a_- \quad a_{00} \quad a_0 \]

| \delta | \beta | \alpha | \rho | \tau | a_- | a_{00} | a_0 |
|-------|------|-------|-----|-----|-----|-------|-----|
| 19    | a.v.| \delta + 1 | 0   | a.v. | \rho \frac{1 - 2\delta}{4} a_{0-} | f(\rho) a_{0-} | a_0 - \frac{[2\delta + 2N + 3]}{4\rho} - 2\rho(N + 1) |
| 20    | 0   | 1     | 0   | 0   | \frac{(4N + 3)}{4} a_{0-} | a.v. | a.v. |
| 21    | -1  | 0     | 0   | 0   | \frac{(1 + 4N)}{4} a_{0-} | 0     | 0 |

B. \( a_{++} \neq 0 \)

\[ \gamma \neq 0 \]

In all admissible cases 22 - 25 \( \rho \) and \( \tau \) take arbitrary values,

\[ a_{0-} = -\frac{1}{2\rho \tau} (\tau + \rho)(4\tau + 1) a_{++}, a_{00} = \frac{1}{4\rho \tau} ((4\tau + 1)^2 + 4(\rho^2 + \tau^2)) a_{++}. \]

The rest of relevant quantities is

| \delta | \beta | \alpha | \rho | \tau | a_- | a_{00} | a_0 |
|-------|------|-------|-----|-----|-----|-------|-----|
| 22    | 0    | 0     | a.v. | \rho | f_0(\rho, \tau) a_{++} | f_-(\rho, \tau) a_{++} |
| 23    | 0    | 1     | 0   | \rho | g_0(\rho, \tau, N) a_{++} | g_-(\rho, \tau, N) a_{++} |
| 24    | -1   | 1     | a.v. | \rho | f_0(\rho, \tau) a_{++} | f_-(\tau, \rho) a_{++} |
| 25    | -1   | 0     | 0   | \rho | g_0(\rho, \tau, N - \frac{1}{2}) a_{++} | g_-(\rho, \tau, N - \frac{1}{2}) a_{++} |

Here \( f_0(\rho, \tau) \equiv \frac{1}{2\rho \tau}[(16\tau^2 \rho^2 - 1)(N + 1) - \rho^2 + \tau^2 + \alpha \gamma (\tau + \rho)], \)

\[ g_0(\rho, \tau, N) = \frac{(5 + 4N)}{8\rho \tau}(16\tau^2 \rho^2 - 1), \]

\[ g_-(\rho, \tau, N) = -\frac{(\tau + \rho)}{8\rho \tau}[\tau\rho(16N + 28) - 4N - 3] \]

\[ f_-(\rho, \tau) \equiv -\frac{1}{8\rho \tau}[\tau^2 \rho(16N + 28) + \tau \rho^2(20 + 16N) + \alpha \gamma(4\rho \tau + 1) - \rho(4N + 3) - \tau(1 + 4N)] \]

\[ \gamma = 0 \]

Now \( \rho = \tau. \)

| \delta | \beta | \alpha | \rho | \tau | a_- | a_{00} | a_0 |
|-------|------|-------|-----|-----|-----|-------|-----|
| 26    | 0    | 0     | a.v. | \rho | f_0(\tau) a_{++} | 2f(\tau) a_{++} | -f^2(\tau) a_{++} + f(\tau) a_- |
| 27    | 0    | 1     | \tau | \rho | f(\tau) a_{0-} + f(2\tau^2) a_{++} | a.v. | f(\tau) a_- - g_0(\tau, \tau, N) a_{++} + f_1(\tau, N + \frac{1}{2}) a_{0-} |
| 28    | -1   | 1     | a.v. | \rho | f_0(\tau) a_{++} | 2f(\tau) a_{++} | -f^2(\tau) a_{++} + f(\tau) a_- |
| 29    | -1   | 0     | \tau | \rho | f(\tau) a_{0-} + f(2\tau^2) a_{++} | a.v. | f(\tau) a_- - g_0(\tau, \tau, N - \frac{1}{2}) a_{++} + f_1(\tau, N) a_{0-} |

\[ f_0(\tau) = \frac{1}{4\tau^2} \left(1 + 16\tau^4 + 16\tau^2\right), \]

the function \( f_1 \) is defined according to (25).
VI. EXPLICIT EXAMPLES OF INVARIANT SUBSPACES

In this section we list shortly the explicit form of solutions for $u(z)$ and $v(z)$.

1) The case 9: $\beta = \delta = \rho = \tau = 0$,

a) $\gamma = -\alpha = -\omega \neq 0$

Then it follows directly from (34) that the functions $u, v$ can be chosen as $u = \cos \omega z$, $v = \sin \omega z$. After simple calculations one finds that, apart from the Hermitian QES Hamiltonian, there exists also the non-Hermitian QES operator:

$$H = a_{00}K_0^2 + a_{0-}K_0K_- + \left[\frac{a_{0-}\omega^2}{2} - 2a_{00}(N + 1)\right]K_0 + a_-K_- + \frac{a_{00}}{2}\omega^2K_+$$  \hspace{1cm} (71)

b) In a similar way, we obtain for $\gamma = \alpha = \omega$ that $u = \cosh (\omega z)$, $v = \sinh \omega z$, and the operator $H$ is obtained by replacement $\omega^2 \rightarrow -\omega^2$ in the expression (71).

2) $\rho = \tau = 0$, $\delta = -1 - n$, $\beta = n$ ($n = 0,1...$), $\alpha = -1$, $\gamma = 1$

Then we have the following solutions of (34): $u = J_n(z)$, $v(z) = J_{n+1}(z)$ (Bessel functions).

$$a_- = \frac{(3 + 2n)}{4}a_{0-}, \quad a_{+0} = 0 = a_{++}, \quad a_+ = \frac{a_{00}}{2}, \quad a_0 = \frac{a_{0-}}{2} - \frac{a_{00}}{2}(4N + 3).$$ \hspace{1cm} (72)

The effective Hamiltonian is non-Hermitian.

3) Consider the case 8 of Hermitian Hamiltonians (the case 7 can be considered in a similar manner): $\beta = \delta = 0 = \tau$, then $k = 0 = \tilde{k}$ and formally we have in the $z$-representation the wave function that looks like that of a pure harmonic oscillator would look in the coordinate representation. Eqs. (53), (54) take the form

$$b\Psi = \frac{\alpha}{\sqrt{2\rho}}\tilde{\Psi},$$ \hspace{1cm} (73)

$$b^+\tilde{\Psi} = \frac{\gamma}{\sqrt{2\rho}}\Psi,$$ \hspace{1cm} (74)

where $b = \frac{1}{\sqrt{2\rho}}[\frac{d}{dz} - \rho z]$, $b^+ = \frac{1}{\sqrt{2\rho}}[\frac{d}{dz} + \rho z]$. It is obvious that $[b, b^+] = 1$.

The frequency is equal to $\omega = 2\rho$, $\varepsilon = -\frac{\omega}{2} - \alpha\gamma$, $\tilde{\varepsilon} - \varepsilon = 2\rho$. Let also $\varepsilon = \varepsilon_n \equiv \omega(n+1/2)$, $\alpha\gamma = -\omega(n + 1)$. Then we have the eigenvalue and $\tilde{\varepsilon} = \varepsilon_{n+1}$. Thus, our subspace is
\[ \text{span}\{\Psi_n z^{2n} \exp\left(\frac{\rho}{2} z^2\right), \Psi_{n+1} z^{2n+1} \exp\left(\frac{\rho}{2} z^2\right)\}, \]

where \(\Psi_n\) is the wave function of the n-th level of the harmonic oscillator, \(\Psi_n = \exp\left(-\frac{1}{2}\rho z^2\right) H_n(z\sqrt{\rho})\), \(H_n\) is the Hermite polynomial. Here \(\beta = 0, 1, \ldots\)

We would like to stress that our system represents an anharmonic (not harmonic!) Bose oscillator. The functions, which have the same form as those of an harmonic oscillator, appear in this context in the coherent state representation (not in the coordinate one, as would be the case for the usual harmonic oscillator) and represent auxiliary quantities.

**VII. GENERALIZATIONS**

In these section we describe shortly, on the basis of the suggested approach, some possible ways of generation of new invariant subspaces, suitable for constructing QES Bose Hamiltonians. As the method of constructing is the same as was used above, we only dwell upon the structure of subspaces.

1) Let us introduce quantities

\[
\begin{align*}
&f^1_n = z^{2n} u^2, \quad f^2_n = z^{2n+1} u v, \quad f^3_n = z^{2n} v^2
\end{align*}
\]

and consider the subspace \(F_N = \text{span}\{f^1_n, f^2_n, f^3_n\}, \ n = 0, 1, \ldots, N\); \(N = 1, 2, \ldots\). Let \(\tilde{f}_n \equiv \begin{pmatrix} f^1_n \\ f^2_n \\ f^3_n \end{pmatrix}\). Then (51)-(53) take place, where, however, now the corresponding matrices have dimension \(3 \times 3\):

\[
A_0(n) = \begin{pmatrix}
\xi + \frac{1}{4} & \alpha & 0 \\
0 & \frac{2\xi + 2\eta + 3}{4} & 0 \\
0 & \gamma & \eta + \frac{1}{4}
\end{pmatrix}, \quad C_0 = \begin{pmatrix}
\gamma & \omega & \frac{\alpha}{2} \\
0 & 0 & 2\tau
\end{pmatrix}, \quad C_+ = \frac{1}{2}I,
\]

\(I\) is a unit matrix,

\[
A_-(n) = \begin{pmatrix}
2\rho (4\xi + 1) + \alpha \gamma & 2\alpha (\omega + 2\rho) & \alpha^2 \\
\gamma (2 + 3\xi + \eta) & \omega (2\xi + 2\eta + 3) + 2\alpha \gamma & \frac{\alpha}{2} (2 + \xi + 3\eta) \\
\gamma^2 & 2\gamma (\omega + 2\tau) & 2\tau (1 + 4\eta + \alpha \gamma)
\end{pmatrix},
\]

17
\[
B_-(n) = \begin{pmatrix}
\xi (2\xi - 1) & \alpha (3\xi + \eta) & 0 \\
0 & \frac{1}{2} (\eta + \xi + 1) (\eta + \xi) & 0 \\
0 & \gamma (3\eta + \xi) & \eta (2\eta - 1)
\end{pmatrix}
\] (78)

\[
C_-(n) = \begin{pmatrix}
8\rho^2 & 0 & 0 \\
\gamma (\omega + 2\rho) & 2\omega^2 & \alpha (2\tau + \omega) \\
0 & 0 & 8\tau^2
\end{pmatrix}
\] (79)

\[\xi = \beta + n, \ \eta = n + \delta, \ \omega = \rho + \tau.\]

2) Further generalization of consists in considering

\[
f_1^1 = z^{2n} u\tilde{u}, \ f_2^2 = z^{2n+1} u\tilde{u}v, \ f_3^3 = z^{2n+1} u\tilde{v}, \ z^{2n+1} u\tilde{v}, \ f_4^4 = z^{2n} v\tilde{v}
\] (80)

Here \(\tilde{u}, \tilde{v}\) refer to the functions obeying eqs. (34) with parameters \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\rho}, \tilde{\tau}\). Then we can construct \(\text{span} \{f_1^1, f_2^2, f_3^3, f_4^4\}\).

3) Consider \(\text{span}\{z^{2n+\xi(k)} u^{M-k} v^k\}\}, \xi(k) = 1, \text{if } k \text{ is odd and } \xi = 0, \text{if } k \text{ is even. Here } k = 0, 1, 2...M, n = 0, 1...N.

The functions \(\{u^i\}\) that appear in the subspaces 1)-3) obey the system of equations of the type \(\frac{du^i}{dz} = C^i_j(z) u^j\) that contains, as a particular case, eqs. (34).

**VIII. SUMMARY**

Let us summarize the basic features of our approach to constructing invariant subspaces for Bose Hamiltonians. We consider systems whose Hamiltonian can be expressed in terms of generators of \(K_i\) (2). Further, we (1) use the coherent state representation in which all Bose operators become differential ones, (2) split the space of states of an harmonic oscillator to even and odd states, (3) deform each of two pieces by introducing, as a factor, additional unknown function which is different for each piece, (4) demand that both these functions obey a coupled system of differential equations that the action of \(K_i\) convert each of basis vectors into a linear combination of vectors of the same type, (5) select coefficients, with
which \( K_i \) enter Hamiltonian \( H \), to ensure the cut off in the space of basis functions. In a sense, we introduce a kind of an additional degree of freedom - effective ”spin" \( s \). Then kinds of subspaces considered in our paper can be assigned the values \( s = 0 \) (Sec.II), \( 1/2 \) (Sec. III), \( 1 \) (Sec. VII), etc. However, we want to stress that this ”spin”, in contrast to matrix QES models \([20] \) - \([23] \), does not appear in Hamiltonian and serves to describe the structure of solutions only.

It is worth stressing that it is just \( H \) itself is quasi-exactly solvable, whereas generators \( K_i \) themselves, with the help of which \( H \) is built up, are, in general, not. This is in a sharp contrast with ”usual” QES Hamiltonians in quantum mechanics. Let us remind ourselves that in the latter case \( H \) can be expressed in terms of operators of an effective spin \( S_i \) that realize \( sl(2, R) \) algebra, each of them possessing finite-dimensional subspace.

The essential ingredient of our approach is using the coherent state representation. Formally, formulas \([1]\) look like those in the coordinate-momentum representation. However, for our Hamiltonian one should check carefully the normalizability of solutions that implies integration over whole complex plane in a scalar product and impose constraints on admissible values of parameters. Another constraints stem from the demand of analyticity.

We want to point out that non-Hermitian QES operators can also be of interest. They may be used, as auxiliary quantities, in physical applications for finding spectra of Hermite Hamiltonians. For instance, they may appear in mappings like \( HL = LH' \), where \( H \) is Hermitian. Knowing the spectrum of \( H' \) or its part due to its quasi-exact solvability, one can restore a part of spectrum of a physical Hamiltonian \( H \). Apart from this, the approach considered in the present article opens a way to the search and classification of linear differential operators with different invariant subspaces. This would enable one to generalize or extend the results obtained for such subspaces with a basis of monomials \([24], [25] \). In particular, for the case \([72] \) we obtained solutions in the form of combinations of Bessel functions and monomials.

In this paper we restricted ourselves to one-particle systems but the suggested approach is obviously extendable to many-particle Hamiltonians. It also enables one to generate new
QES Bose Hamiltonians by choosing another kinds of functions $A$, $B$, $C$, $D$ in eqs. (28), (29).

The suggested approach can be useful in the problems of solid state physics when interaction between phonons is essential, quantum optics, theory of molecules, etc. In this context, especially important is the fact that our approach is extendable to many-particle systems. Concrete elaboration and applications of the obtained results deserve special treatment.

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