On the support of a body by a surface with random roughness

D. Treschev

May 20, 2013

Abstract

Suppose an interval is put on a horizontal line with random roughness. With probability one it is supported at two points, one from the left, and another from the right from its center. We compute probability distribution of support points provided the roughness is fine grained. We also solve an analogous problem where a circle is put on a rough plane. Some applications in static are given.

1 Introduction

1.1 Motivations

The Amonton-Coulomb law of friction (dry friction) says that if the motion of a body is a translation along a fixed plane, the friction force is up to a constant multiplier (the dry friction coefficient) equals total normal load. If the body slides along a plane with nonzero angular velocity, to obtain total friction force and total friction momentum, one has to integrate infinitesimal friction forces over the contact spot. This makes the problem of sliding of a body along a plane in the presence of dry friction non-trivial.

There is a series of publications where dynamical problems of this kind are studied: [10, 13, 5, 9, 11, 12]. A key role in these models is played by the hypothesis on the distribution of the normal load on the contact spot. All such hypotheses are essentially phenomenological although some quasistatic argument is usually attached. The uniform distribution [5, 11, 12] or rotational symmetric ones (for cylindrical bodies with rotational symmetric base) [6, 7] are compatible with dynamics only for bodies of infinitesimal height. Dynamically compatible deformations of the above distributions are considered in [4], see also [1, 2, 3], where qualitative analysis of the motion is presented.

Very careful experiments [12], where a plastic disk slides along nylon, stretched over the surface of a flat table, essentially confirm (even quantitatively) theoretical predictions. Other experiments, where a rigid disk slides along a rigid surface [8, 11] produce much more noisy data which correspond to the above theoretical works only qualitatively. We believe that the main reason for such noisy and unstable data is that when both the disk and the support surface are sufficiently rigid, it is hard to expect that their surfaces are perfectly flat: very small deviations from ideal flatness can change unpredictably the distribution of the normal load and break any deterministic hypothesis on the distribution of a load over the contact spot. In this
case one should use some probabilistic assumptions. For example, it is possible to consider a (perfectly) flat body on a rough surface with random roughness.

Instead of a disk on a plane in this paper we consider two simpler problems: an interval on a rough line and a circle on a rough plane. We also consider some static problems which appear in this context.

1.2 An interval on a line

Consider the points

\[ w_j = (x_j, 0), \quad x_j = -1 + 2/N, \quad j = 1, \ldots, N \]

on the horizontal interval

\[ I = \{(x, z) \in \mathbb{R}^2 : x \in [-1, 1], z = 0\}. \]

Each point \( w_j \) is supposed to be the lower end of a vertical interval whose length \( \xi_j \) is uniformly distributed on \([0, 1]\). We call any such vertical interval a tooth and the whole set of these intervals a random comb, see Fig. 1

An interval \( J \), lying on this random comb and projecting exactly on \( I \), with probability 1 is supported by two teeth with horizontal coordinates

\[ a_1 = x_{j_1} \in I_- = [-1, 0], \quad a_2 = x_{j_1} \in I_+ = [0, 1], \quad 1 \leq j_1, j_2 \leq N. \]

We say that in this case the event \( S^a \) takes place.

**Theorem 1.1.** In the limit \( N \to \infty \) the density \( p : I_- \times I_+ \to \mathbb{R}_+ \) of probability distribution of the random event \( S^a \) is

\[ p(a) = (a_2 - a_1) \left( \frac{4}{3(1 + a_2)^3} + \frac{4}{3(1 - a_1)^3} + \frac{1}{6} \right). \quad (1.1) \]

Graph of the function \( p \) is presented in Fig. 2 We see that \( p(0, 0) = 0 \) and \( p \) attains global maximum at the points \((-1, 0)\) and \((0, 1)\).

To get “mechanical” interpretations, suppose that the heavy interval \( J \) is drawn along a rough line. Where it will be scratched more: near ends or in the middle?

Density of probability distribution for the right support point \( a_2 \) is as follows:

\[ p_2 = \int_{-1}^{0} p(a) \, da_1 = \frac{4}{3(1 + a_2)^2} - \frac{2}{3(1 + a_2)^3} + \frac{2a_2}{3} + \frac{1}{4}. \]
We have: \( p_2(1)/p_2(0) = 14/11 \). Therefore endpoints of \( J \) are support points 14/11 times more frequently than points near the center of \( J \). However we should take into account that the rate of scratching depends also on the normal load. Hence we have to perform another calculation.

Suppose that the rate of scratching is proportional to the normal load. If \( J \) is supported at the points \( a_1 < 0 < a_2 \), the left and right tooth carries the weight

\[
l_1(a) = \frac{Pa_2}{a_2 - a_1} \quad \text{and} \quad l_2(a) = \frac{Pa_1}{a_1 - a_2}
\]

respectively, where \( P \) is the weight of \( J \). Therefore the rate of scratching at the left support point is proportional to

\[
\text{scr}_1(a_1) = \int_0^1 \frac{a_2}{a_2 - a_1} p(a) \, da_2 = \frac{2}{3(1 - a_1)^3} + \frac{1}{2}.
\]

Analogously \( \text{scr}_2(a_2) = \frac{2}{3(1 + a_2)^3} + \frac{1}{2} \). Since \( \frac{\text{sc}_2(0)}{\text{sc}_2(1)} = \frac{10}{7} \), we see that the middle point will be scratched stronger than the end.

Another application of (1.1) is as follows. Suppose that \( J \) is a heavy beam of mass \( M \) lying on an uneven surface. A man of mass \( m \) walks along the beam. At some moment it may happen that under the weight of the man the beam will leave its initial equilibrium, starting to rotate on one of the support points. We compute the probability

\[
p_* = p_*(\mu), \quad \mu = \frac{m}{m + M} \in [0, 1]
\]
of the random event that this does not happen. This event is equivalent to the following two inequalities:

\[-a_1 > \mu, \quad a_2 > \mu.\]

Therefore

\[
p_* = \int_{-1}^{1} \int_{\mu}^{1} (a_2 - a_1) \left( \frac{4}{3(1 + a_2)^3} + \frac{4}{3(1 - a_1)^3} + \frac{1}{6} \right) da_2 da_1 = \frac{(1 - \mu)^2}{6} \left( 6 + \mu - \left( \frac{2\mu}{1 + \mu} \right)^2 \right).
\]

In particular, if \( m = M \), we have \( p_* \approx 1/4 \). Graph of the function \( p_*(\mu) \) is presented in Fig. 3.

1.3 Circle on a plane

Consider the points

\[ w_j = w(\alpha_j) = (x_j, y_j, z_j) = (\cos \alpha_j, \sin \alpha_j, 0), \quad \alpha_j = \frac{2\pi j}{N}, \quad j = 1, \ldots, N \]

on the horizontal circle

\[ c = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \quad z = 0 \}. \]

Each point \( w_j \) is supposed to be the lower end of a vertical interval whose length \( \xi(w_j) \) is uniformly distributed on \( c \). We call any such vertical interval a tooth and the whole set of these intervals a random circular comb, see Fig. 4.

A thin hoop \( J \), lying on this random comb, with probability 1 is supported by three teeth

\[ w_{n_i} = (\cos \alpha_{n_i}, \sin \alpha_{n_i}, 0), \quad 1 \leq n_i \leq N, \quad i = 1, 2, 3. \]

We say that in this case the event \( S^\varphi \) takes place \( S^\varphi \) takes place, where

\[ \varphi = (\varphi_1, \varphi_2, \varphi_3), \quad \varphi_i = \frac{2\pi n_i}{N} \mod 2\pi, \quad i = 1, 2, 3. \]

We are interested in probability distribution of the random event \( S^\varphi \).
We assume that orientation of the triangle $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\varphi_i = \alpha n_i$ is positive i.e.,

$$\varphi_{i+1} - \varphi_i = \theta_{i-1} \text{ mod } 2\pi,$$

for some real $\vartheta_1, \vartheta_2, \vartheta_3 > 0$, $\vartheta_1 + \vartheta_2 + \vartheta_3 = 2\pi$,

where it is convenient to assume the subscript $i$ to lie in the cyclic group $Z_3$.

The mass center of $J$ should lie inside the triangle with vertices $w_{n_1}, w_{n_2}, w_{n_3}$ (otherwise $J$ can not be in equilibrium on the teeth $n_1, n_2, n_3$). This condition is equivalent to the inequalities $0 < \vartheta_i < \pi$. Moreover, the events $S^{(\varphi_1, \varphi_2, \varphi_3)}, S^{(\varphi_2, \varphi_3, \varphi_1)}, S^{(\varphi_3, \varphi_1, \varphi_2)}$ are the same. Therefore

$$\varphi \in \hat{S} = S/Z_3,$$ $S = \{ \varphi \in T^3 : 0 < \vartheta_i(\varphi) < \pi, i = 1, 2, 3 \},$

where $Z_3$ acts on $T^3$ by cyclic permutations:

$$(\varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_2, \varphi_3, \varphi_1) \mapsto (\varphi_3, \varphi_1, \varphi_2).$$

In the limit $N \to \infty$ distribution of the random event $S^\varphi$ has density $p_S: \hat{S} \to \mathbb{R}_+$. This density is invariant with respect to the action $R_\alpha$ of the circle $T$:

$$\hat{S} \ni \hat{\varphi} \mapsto R_\alpha(\hat{\varphi}) = \varphi + \alpha \mathbf{1}, \quad \mathbf{1} = (1, 1, 1)^T \in \mathbb{R}^3, \quad \alpha \in T. \quad (1.3)$$

Therefore it is natural to consider this distribution on the quotient

$$\hat{T} = \hat{S}/T = T/Z_3,$$ $T = \{ \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) : 0 < \vartheta_i < \pi, \vartheta_1 + \vartheta_2 + \vartheta_3 = 2\pi \}. \quad (1.4)$

More precisely, let $pr : \hat{S} \to \hat{T}$ be the natural projection. Then there exists a function $p_T: \hat{T} \to \mathbb{R}_+$ such that $p_T \circ pr = p_S$. The space $\hat{T}$ should be considered with the measure $\mu_T$:

$$d\mu_T = \frac{1}{3} |d\vartheta_3 \wedge d\vartheta_2 + d\vartheta_1 \wedge d\vartheta_3 + d\vartheta_2 \wedge d\vartheta_1|. \quad (1.5)$$

Then $d\hat{\varphi} = d\hat{\varphi}_1 d\hat{\varphi}_2 d\hat{\varphi}_3$ is the pull-back of $d\mu_T$: $pr_*(d\mu_T) = d\hat{\varphi}$.

**Theorem 1.2.** The density $p_T$ satisfies the equation

$$p_T(\hat{\vartheta}) = 2\pi \sin \frac{\vartheta_1}{2} \sin \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2} \left( \frac{1}{\pi^2} + \sum_{i=1}^3 f(\hat{\vartheta}_i) \right), \quad f(\xi) = \int_0^{\xi/2} \frac{(\xi - 2\varphi) \sin \varphi}{((\pi - \varphi) \cos \varphi + \sin \varphi)^3} d\varphi.$$
Graph of the function $p_T$ is presented in Fig. 5. Here we take $\vartheta_1, \vartheta_2$ as coordinates on $\hat{T}$. Hence $\hat{T}$ can be regarded as the triangle

$$\{(\vartheta_1, \vartheta_2) : 0 < \vartheta_1 < \pi, 0 < \vartheta_2 < \pi, \vartheta_1 + \vartheta_2 < \pi\}$$

with identification

$$(\vartheta_1, \vartheta_2) \sim (\vartheta_2, 2\pi - \vartheta_1 - \vartheta_2) \sim (2\pi - \vartheta_1 - \vartheta_2, \vartheta_1).$$

We see that $p = 0$ if one of the angles $\vartheta_i$ vanishes. Maximal value of $p$ is attained at points $\vartheta$ such that for some $i \in \mathbb{Z}_3$ $\vartheta_i = \pi$ and $\vartheta_{i+1} = \pi/2$.

As an illustration consider a man of mass $m$ going around the hoop of mass $M$. Let

$$p_* = p_*(\mu), \quad \cos \alpha = \mu = \frac{m}{m + M}, \quad 0 \leq \alpha \leq \pi/2$$

be the probability of the random event that the hoop stands motionless during all the walk.

This event is equivalent to the 3 inequalities

$$0 < \vartheta_i < 2\alpha, \quad i = 1, 2, 3.$$ 

Hence

$$p_* = \int_{D(\alpha)} p_T(\vartheta) \, d\vartheta_1 d\vartheta_2, \quad D(\alpha) = \{(\vartheta_1, \vartheta_2) : \vartheta_1 < 2\alpha, \vartheta_1 < 2\alpha, 2\pi - 2\alpha < \vartheta_1 + \vartheta_2\}/\sim,$$

where $\sim$ is the equivalence relation (1.6).
Figure 6: Graph of the function \( \mu \mapsto p_\ast(\mu) \)

Since \( D(\alpha) \) is empty for \( \alpha < \pi/3 \), we only have to consider the case \( \pi/3 < \alpha < \pi/2 \). Graph of the function \( \mu \mapsto p_\ast(\mu) \),

\[
p_\ast = \frac{2\pi}{3} \int_{2\pi-4\alpha}^{2\alpha} d\vartheta_1 \int_{2\pi-2\alpha-\vartheta_1}^{2\alpha} \sin \frac{\vartheta_1}{2} \sin \frac{\vartheta_2}{2} \sin \frac{\vartheta_1 + \vartheta_2}{2} \left( \frac{1}{\pi^2} + f(\vartheta_1) + f(\vartheta_2) + f(2\pi - \vartheta_1 - \vartheta_2) \right) d\vartheta_2
\]

is presented in Fig. 6. In particular, \( p_\ast = 1/2 \) for \( \mu \approx 1/6 \).

## 2 Proof of Theorem 1.1

Let \( \Omega \) be the configuration space of the random comb: \( \Omega = [0, 1]^N \). We consider large integer \( L \), and put

\[
n = (n_1, n_2), \quad 1 \leq n_1 \leq N/2 < n_2 \leq N.
\]

Then we define two random events \( \nu \) and \( Q_n \), where by definition

- \( \nu = n \) iff \( J \) is supported by the teeth \( n_1 \) and \( n_2 \),
- \( Q_n = (K_1, K_2) \) iff length of the tooth with number \( n_i \) equals

\[
\xi_{n_i} \in \left( 1 - (K_i - 1)/(NL), 1 - K_i/(NL) \right), \quad i = 1, 2.
\]

For any \( K \in \{1, \ldots, NL\}^2 \) we have:

\[
P\{Q_n = K\} = (NL)^{-2}.
\]

Therefore by the formula of total probability

\[
P\{\nu = n\} = \sum_K P\{\nu = n | Q_n = K\} P\{Q_n = K\} = \sum_K \frac{P\{\nu = n | Q_n = K\}}{(NL)^2}.
\]

In the limit \( L \to \infty \) we obtain:

\[
P\{\nu = n\} = \frac{1}{N^2} \int_0^N \int_0^N P\{\nu = n | \xi_n = 1 - A/N\} dA, \quad \xi_n = (\xi_{n_1}, \xi_{n_2}), \quad 1 = (1, 1)^T. \quad (2.1)
\]
In the limit $N \to \infty$ we obtain densities of probability distributions
\[ p : I_- \times I_+ \to \mathbb{R}_+, \quad p_{\nu|Q} : I_- \times I_+ \times [0,N]^2 \to \mathbb{R}_+, \]
\[ p(a) = \lim_{N \to \infty} \frac{N^2}{4} P\{\nu = n\}, \quad p_{\nu|Q}(a, A) = \lim_{N \to \infty} P\{\nu = n, \xi_n = 1 - A/N\}, \]
\[ a = (a_1, a_2) = (1 + 2n_1/N, -1 + 2n_2/N). \]

Equation (2.1) implies
\[ p(a) = \frac{1}{4} \int_0^N \int_0^N p_{\nu|Q}(a, A) dA. \] (2.2)

Now we turn to computation of $p_{\nu|Q}$. The interval $J = J(a, A)$ is determined by the equation
\[ z = 1 - \frac{A_1 a_2 - A_2 a_1}{N(a_2 - a_1)} - \frac{A_2 - A_1}{N(a_2 - a_1)} x, \quad x \in I. \]

We have to consider two cases.

(1) The interval $J$ does not intersect the line segment $I_+$ joining the points $(-1, 1)$ and $(1, 1)$. This happens provided
\[ |A_1 - A_2| < A_1 a_2 - A_2 a_1. \]

(2) $J \cap I_+ = (x_*, 1)$. In this case $|A_1 - A_2| \geq A_1 a_2 - A_2 a_1$ and
\[ x_* = \frac{A_1 a_2 - A_2 a_1}{A_1 - A_2}. \]

In case (1) probability for the point $w_j = (x_j, 0)$ to have the tooth (entirely) under $J$ is
\[ z_j = 1 - \frac{A_1 a_2 - A_2 a_1}{N(a_2 - a_1)} - \frac{A_2 - A_1}{N(a_2 - a_1)} x_j < 1. \] (2.3)

Therefore probability for the whole comb to be under $J$ is
\[ p_{\nu|Q}^{(1)} = \prod_{j=1}^N \left( 1 - \frac{A_1 a_2 - A_2 a_1}{N(a_2 - a_1)} - \frac{A_2 - A_1}{N(a_2 - a_1)} \left( 1 + \frac{2j}{N} \right) \right) = e^{F_1}, \]
\[ F_1 = \sum_{j=1}^N \log \left( 1 - \frac{A_1 a_2 - A_2 a_1}{N(a_2 - a_1)} - \frac{A_2 - A_1}{N(a_2 - a_1)} \left( 1 + \frac{2j}{N} \right) \right). \]

In the limit $N \to \infty$ we have: $F_1 = -\frac{A_1 a_2 - A_2 a_1}{a_2 - a_1}$. Hence
\[ p_{\nu|Q}^{(1)} = e^{-\frac{A_1 a_2 - A_2 a_1}{a_2 - a_1}}. \]

Consider case (2). For definiteness we assume that $A_1 > A_2$ i.e., $x_* > 0$. Then probability for any point $w_j$ to have the tooth under $J$ is determined by (2.3) if $x_j \in [-1, x_*]$ and equals 1 if $x_j \in [x_*, 1]$.

Probability for the whole comb to be under $J$ is $p_{\nu|Q}^{(2)} = e^{F_2}$,
\[ F_2 = \sum_{j \geq 1, -1 + 2j/N \leq x_*} \log \left( 1 - \frac{A_1 a_2 - A_2 a_1}{N(a_2 - a_1)} - \frac{A_2 - A_1}{N(a_2 - a_1)} \left( 1 + \frac{2j}{N} \right) \right) \]
\[ = -\frac{1}{2} \int_{-1}^{x_*} \left( \frac{A_1 a_2 - A_2 a_1}{a_2 - a_1} + \frac{A_2 - A_1}{a_2 - a_1} x \right) dx + O(1/N). \]
For \( N \to \infty \) we obtain:

\[
P^{(2)}_{\nu|Q} = e^{-\frac{(A_1(a_2+1)-A_2(a_1+1))^2}{4(a_2-a_1)(A_1-A_2)}} \quad \text{if } A_1 > A_2.
\]

The case \( A_1 < A_2 \) can be obtained from this one by the exchange \( A_1 \leftrightarrow A_2, a_1 \leftrightarrow -a_2 \). Therefore

\[
P^{(2)}_{\nu|Q} = e^{-\frac{(A_1(-a_1+1)-A_2(-a_2+1))^2}{4(a_2-a_1)(A_2-A_1)}} \quad \text{if } A_2 > A_1.
\]

Considering in (2.2) the limit \( N \to \infty \) we see that

\[
p(a)|_{N \to \infty} = Q_1 + Q_2^+ + Q_2^-,
\]

\[
Q_1 = \frac{1}{4} \int_{D_1} p^{(1)}_{\nu|Q} dA, \quad Q_2^+ = \frac{1}{4} \int_{D_2^+} p^{(2)}_{\nu|Q} dA, \quad Q_2^- = \frac{1}{4} \int_{D_2^-} p^{(2)}_{\nu|Q} dA,
\]

where

\[
D_1 = \{ A \in \mathbb{R}^2 : |A_1 - A_2| < A_1a_2 - A_2a_1 \},
\]

\[
D_2^\pm = \{ A \in \mathbb{R}^2 : |A_1 - A_2| > A_1a_2 - A_2a_1, \pm(A_1 - A_2) > 0 \}.
\]

Change of the variables

\[a_2A_1 - a_1A_2 = r, \quad -A_1 + A_2 = q\]

transforms the integrals as follows:

\[
Q_1 = \frac{1}{4} \int_{|q| < r} \frac{1}{a_2 - a_1} e^{\frac{-r}{a_2 - a_1}} drdq = \frac{1}{2} (a_2 - a_1),
\]

\[
Q_2^+ = \frac{1}{4} \int_{D^+} \frac{1}{a_2 - a_1} e^{\frac{-r}{a_2 - a_1}} drdq, \quad D^+ = \{ -(1 + a_2)q < r < -2q \}.
\]

The quantity \( Q_2^- \) is obtained from \( Q_2^+ \) by the exchange \( a_1 \leftrightarrow -a_2 \).

It is convenient to compute \( Q_2^+ \) in the variables \( u = -r^2/q, v = -r/q \). Direct computation gives:

\[
Q_2^+ = \frac{4(a_2 - a_1)}{3} \left( \frac{1}{(1 + a_2)^3} - \frac{1}{8} \right), \quad Q_2^- = \frac{4(a_2 - a_1)}{3} \left( \frac{1}{(1 - a_1)^3} - \frac{1}{8} \right).
\]

Now equation (1.1) follows from (2.4), (2.5), and (2.6).

\[\square\]

### 3 Proof of Theorem 1.2

Let \( \Omega \) be the configuration space of the circular random comb: \( \Omega = [0,1]^N \). The teeth that support \( J \) are determined by equation (1.2).

We consider large integer \( L \), and define two random events \( \nu \) and \( Q_n \), where by definition

- \( \nu = n \) iff \( J \) is supported by the teeth \( n = (n_1, n_2, n_3) \),

- \( Q_n = K = (K_1, K_2, K_3) \) iff length of the tooth with number \( n_i \) equals

\[
\xi_{n_i} \in (1 - (K_i - 1)/(NL), 1 - K_i/(NL)), \quad i = 1, 2, 3.
\]

For any \( K \in \{1, \ldots, NL\}^3 \) we have: \( P\{Q_n = K\} = (NL)^{-3} \). Therefore by the formula of total probability

\[
P\{\nu = n\} = \sum_K P\{\nu = n | Q_n = K\} P\{Q_n = K\} = \sum_K \frac{P\{\nu = n | Q_n = K\}}{(NL)^3}.
\]
Putting $\mathbf{1} = (1,1,1)^T \in \mathbb{R}^3$, in the limit $L \to \infty$ we obtain:

$$P\{\nu = n\} = \frac{1}{N^3} \int_{0}^{N} \int_{0}^{N} \int_{0}^{N} P\{\nu = n|\xi_n = \mathbf{1} - A/N\} \, dA, \quad \xi_n = (\xi_{n_1}, \xi_{n_2}, \xi_{n_3}). \quad (3.1)$$

In the limit $N \to \infty$ we obtain densities of probability distributions

$$p_S: \mathcal{S} \to \mathbb{R}_+, \quad \tilde{p}_{\nu|Q}: \mathcal{S} \times [0, N]^3 \to \mathbb{R}_+,$$

$$p_S(\hat{\varphi}) = \lim_{N \to \infty} \left(\frac{N}{2\pi}\right)^3 P\{\nu = n\}, \quad \tilde{p}_{\nu|Q}(\varphi|A) = \lim_{N \to \infty} P\{\nu = n|\xi_n = 1 - A/N\}, \quad (3.2)$$

where $\hat{\varphi} \in \hat{\mathcal{S}}$ and $\varphi \in \mathcal{S}$.

Equations (3.1)–(3.2) imply

$$p_S(\hat{\varphi}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3_+} \tilde{p}_{\nu|Q}(\varphi|A) \, dA, \quad \mathbb{R}^3_+ = \{A = (A_1, A_2, A_3) \in \mathbb{R}^3 : A_i > 0, \ i = 1, 2, 3\}, \quad (3.3)$$

where $\hat{\varphi}$ is the image of $\varphi$ under the natural map $\mathcal{S} \to \hat{\mathcal{S}}$.

Both densities $p_S$ and $\tilde{p}_{\nu|Q}$ are invariant with respect to the action $R_\alpha$ of the group $\mathbb{T}$, see (1.3). Hence we obtain the densities $p_T, p_{\nu|Q}$ on $\hat{T} = \hat{\mathcal{S}}/\mathbb{T}$ and $\mathcal{T} \times [0, N]^3$ respectively: $\mathcal{T} = \mathcal{S}/\mathbb{T}$,

$$p_T(\hat{\vartheta}) = 2\pi p_S(\hat{\varphi}), \quad p_{\nu|Q}(\vartheta|A) = \tilde{p}_{\nu|Q}(\varphi|A),$$

where measures on $\hat{T}$ and $\mathcal{T}$ are determined by (1.5). Then (3.3) implies

$$p_T(\hat{\vartheta}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3_+} p_{\nu|Q}(\vartheta|A) \, dA. \quad (3.4)$$

Now we turn to computation of $\tilde{p}_{\nu|Q}$.

The plane passing through $J = J(a(\varphi), A)$ is determined by the equation

$$z = 1 - \frac{\sigma_0}{N} - \frac{\sigma_x}{N} x - \frac{\sigma_y}{N} y,$$

$$\sigma_0 = \frac{1}{\Delta} \begin{vmatrix} \cos \varphi_1 & \sin \varphi_1 & A_1 \\ \cos \varphi_2 & \sin \varphi_2 & A_2 \\ \cos \varphi_3 & \sin \varphi_3 & A_3 \end{vmatrix} = \frac{A_1 \sin \vartheta_1 + A_2 \sin \vartheta_2 + A_3 \sin \vartheta_3}{\Delta} > 0, \quad (3.5)$$

$$\Delta = \begin{vmatrix} \cos \varphi_1 & \sin \varphi_1 & 1 \\ \cos \varphi_2 & \sin \varphi_2 & 1 \\ \cos \varphi_3 & \sin \varphi_3 & 1 \end{vmatrix} = \sin \vartheta_1 + \sin \vartheta_2 + \sin \vartheta_3, \quad (3.6)$$

$$\sigma_x = \frac{1}{\Delta} \begin{vmatrix} A_1 & \sin \varphi_1 & 1 \\ A_2 & \sin \varphi_2 & 1 \\ A_3 & \sin \varphi_3 & 1 \end{vmatrix}, \quad \sigma_y = \frac{1}{\Delta} \begin{vmatrix} \cos \varphi_1 & A_1 & 1 \\ \cos \varphi_2 & A_2 & 1 \\ \cos \varphi_3 & A_3 & 1 \end{vmatrix}. \quad (3.7)$$

We consider two cases.

(1) The disk $J$ does not intersect the disk $I_+$, obtained as a shift of the disk $I$ by the vector $(0,0,1)$. This happens provided $\sigma_x \cos \varphi + \sigma_y \sin \varphi + \sigma_0 > 0$ for all real $\varphi$ i.e.,

$$\sigma_x^2 + \sigma_y^2 < \sigma_0^2.$$
(2) $J \cap I_+ \neq \emptyset$. In this case $J$ is below $I_+$ over the domain

$$D_\sigma = \{x^2 + y^2 \leq 1, \quad \sigma_x x + \sigma_y y + \sigma_0 \geq 0\}.$$ 

In case (1) probability for the point $w_j = w(\alpha_j)$ to have a tooth (entirely) under $J$ is

$$1 - \frac{\sigma_0}{N} - \frac{\sigma_x}{N} \cos \alpha_j - \frac{\sigma_y}{N} \sin \alpha_j \leq 1.$$ 

Therefore probability for the whole comb to be under $J$ equals

$$p^{(1)}_A = \prod_{j \neq n_1, n_2, n_3} \left(1 - \frac{\sigma_0}{N} - \frac{\sigma_x}{N} \cos \alpha_j - \frac{\sigma_y}{N} \sin \alpha_j\right) = e^{F_1}, \quad \alpha_j = \frac{2\pi j}{N}, \quad j = 1, \ldots, N,$

$$F_1 = \sum_{j \neq n_1, n_2, n_3} \log \left(1 - \frac{\sigma_0}{N} - \frac{\sigma_x}{N} \cos \alpha_j - \frac{\sigma_y}{N} \sin \alpha_j\right) = -\sigma_0 + O(1/N).$$

For $N \to \infty$ we obtain:

$$p^{(1)}_A = e^{-\sigma_0}.$$ 

In case (2) the tooth is under $J$ with probability

$$1 - \frac{\sigma_0 + \sigma_x \cos \alpha_j + \sigma_y \sin \alpha_j}{N} \quad \text{if} \quad \alpha_j \in B^+, \quad \text{and} \quad 1 \quad \text{if} \quad \alpha_j \in B^-,$

$$B^\pm = \{\alpha \in \mathbb{T} : \pm (\sigma_x \cos \alpha + \sigma_y \sin \alpha + \sigma_0) \geq 0\}.$$ 

Therefore probability for the whole comb to be under $J$ equals

$$p^{(2)}_A = \prod_{\alpha_j \in B^+, j \neq n_1, n_2, n_3} \left(1 - \frac{\sigma_0 + \sigma_x \cos \alpha_j + \sigma_y \sin \alpha_j}{N}\right) = e^{F_2}, \quad \alpha_j = \frac{2\pi j}{N}, \quad j = 1, \ldots, N,$

$$F_2 = \sum_{\alpha_j \in B^+, j \neq n_1, n_2, n_3} \log \left(1 - \frac{\sigma_0 + \sigma_x \cos \alpha_j + \sigma_y \sin \alpha_j}{N}\right) = \sigma_0 A + O(1/N),$$

$$A = \frac{1}{2\pi \sigma_0} \int_{B^+} (\sigma_x \cos \varphi + \sigma_y \sin \varphi + \sigma_0) d\varphi.$$ 

**Proposition 3.1.** $A = \frac{1}{\pi} (\varphi_\sigma - \tan \varphi_\sigma)$, where $\varphi_\sigma = \arccos \left( -\sigma_0 / \sqrt{\sigma_x^2 + \sigma_y^2} \right)$. 

In the limit $N \to \infty$ we obtain the probability

$$p^{(2)}_A = e^{-\sigma_0 A}.$$ 

By (3.4) we have the equation

$$p = p_1 + p_2, \quad p_1 = \frac{1}{4\pi^2} \int_{B_1} p^{(1)}_A dA, \quad p_2 = \frac{1}{4\pi^2} \int_{B_2} p^{(2)}_A dA,$$

$$B_1 = \{A \in \mathbb{R}_+^3 : \sigma_x^2 + \sigma_y^2 < \sigma_0^2\}, \quad B_2 = \{A \in \mathbb{R}_+^3 : \sigma_x^2 + \sigma_y^2 \geq \sigma_0^2\}.$$ 

Computation of the integrals $p_1, p_2$ requires some preliminary work. First, we introduce new coordinates

$$\tau_i = (1 - \cos \theta_i) A_i / \Delta, \quad i = 1, 2, 3$$

and put

$$\mathbf{1} = (1, 1, 1)^T \in \mathbb{R}^3, \quad c_i = \cot \frac{\theta_i}{2}, \quad i = 1, 2, 3.$$
Proposition 3.2. For any \( \vartheta \in \mathcal{T} \)

\[
c_1 c_2 + c_2 c_3 + c_3 c_1 = 1, \quad \frac{\sum \sin \vartheta_i}{\prod \sin(\vartheta_i/2)} = 4, \quad \frac{1}{\prod \sin(\vartheta_i/2)} = \langle 1, c \rangle - c_1 c_2 c_3. \tag{3.9}
\]

Combining (3.5)–(3.7) and (3.9), we have:

\[
\Delta = \frac{4}{\langle c, 1 \rangle - c_1 c_2 c_3}, \quad \sigma_0 = \langle c, \tau \rangle, \quad \sigma_x^2 + \sigma_y^2 - \sigma_0^2 = -\langle \tau, J \tau \rangle, \quad J = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = 1 \otimes 1 - 2. \tag{3.10}
\]

Integrals (3.8) in the new coordinates take the form

\[
p_i = \frac{\Delta^3 \hat{p}_i}{32\pi^2 \prod \sin^2 \vartheta_i} = \frac{2 \hat{p}_i}{\pi^2 (\langle c, 1 \rangle - c_1 c_2 c_3)} = \frac{2 \vartheta_i}{\pi^2} \sin \frac{\vartheta_1}{2} \sin \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2},
\]

\[
\hat{p}_1 = \int_{C_1} e^{-\langle c, \tau \rangle} d\tau, \quad \hat{p}_2 = \int_{C_2} e^{-\langle c, \tau \rangle} d\tau,
\]

\[
C_1 = \{ \tau \in \mathbb{R}^3 : \langle c, \tau \rangle > 0, \langle \tau, J \tau \rangle > 0 \}, \quad C_2 = \{ \tau \in \mathbb{R}^3 : \langle \tau, J \tau \rangle < 0 \}. \tag{3.11}
\]

3.1 Convenient variables

To compute integrals (3.12), it is convenient to introduce new variables. We put

\[
w = 1 - \frac{\langle \tau, J \tau \rangle}{\langle c, \tau \rangle^2}, \quad \lambda = \frac{|c \times 1|}{\sqrt{2}} = \sqrt{\langle c, 1 \rangle^2 - 3}.
\]

Equation (3.11) implies

\[
(\sigma_x^2 + \sigma_y^2)/\sigma_z^2 = w^2. \tag{3.14}
\]

In the domain \( C_2 \) (see (3.13)) we have: \( w > 1 \). The identity

\[
\lambda^2 w = \left( \frac{\langle 1, \tau \rangle}{\langle c, \tau \rangle} - \langle c, 1 \rangle \right)^2 + \frac{\langle c \times 1, \tau \rangle^2}{\langle c, \tau \rangle^2}
\]

suggests the following change of variables: \( (\tau_1, \tau_2, \tau_3) \mapsto (u, w, \psi) \),

\[
\langle c, \tau \rangle = u, \quad \frac{1}{\lambda} \left( \frac{\langle 1, \tau \rangle}{\langle c, \tau \rangle} - \langle c, 1 \rangle \right) = \sqrt{w} \cos \psi, \quad \frac{\langle c \times 1, \tau \rangle}{\lambda \langle c, \tau \rangle} = \sqrt{w} \sin \psi.
\]

The Jacobian \( \det \frac{\partial (u, w, \psi)}{\partial (\tau_1, \tau_2, \tau_3)} \) equals the product

\[
\det \frac{\partial (u, w, \psi)}{\partial (u, v_1, v_2)} \det \frac{\partial (u, v_1, v_2)}{\partial (\tau_1, \tau_2, \tau_3)}, \quad v_1 = \sqrt{w} \cos \psi, \quad v_2 = \sqrt{w} \sin \psi.
\]

These determinants equal 2 and \( 2u^{-2} \) respectively. Therefore

\[
\det \frac{\partial (u, w, \psi)}{\partial (\tau_1, \tau_2, \tau_3)} = \frac{4}{u^2}.
\]
Assuming \( i \) to be an element of the cyclic group \( \mathbb{Z}_3 \), we put
\[
a_i = \frac{c_i \langle c, 1 \rangle - c^2}{(c_{i-1} + c_{i+1})}, \quad b_i = \frac{c_{i-1} - c_{i+1}}{(c_{i-1} + c_{i+1})}.
\]

Direct computations show that
\[
a_i^2 + b_i^2 = 1, \quad a_{i+1} b_{i-1} - a_{i-1} b_{i+1} = \frac{2c_i}{c_i^2 + 1} = \sin \vartheta_i, \quad b_{i+1} b_{i-1} + a_{i+1} a_{i-1} = \frac{c_i^2 - 1}{c_i^2 + 1} = \cos \vartheta_i.
\]
Therefore for some \( \psi_1, \psi_2, \psi_3 \in \mathbb{T} \)
\[
a_i = \sin \psi_i, \quad b_i = \cos \psi_i, \quad \psi_{i+1} - \psi_{i-1} = \vartheta_i. \tag{3.15}
\]

### 3.2 The integrals \( \hat{p}_1 \) and \( \hat{p}_2 \)

By using the variables \((u, v, \psi)\) in (3.12), we obtain:
\[
\hat{p}_1 = \int_0^\infty du \int_{G_1} \frac{u^2}{4} e^{-u} dw d\psi, \quad \hat{p}_2 = \int_0^\infty du \int_{G_2} \frac{u^2}{4} e^{-uA} dw d\psi, \tag{3.16}
\]
where the domains \(G_1, G_2\) are as follows:
\[
G_1 = \{(w, \psi) : 0 < w < 1\}, \quad G_2 = \{(w, \psi) : w > 1, \frac{\tau_i}{\langle c, \tau \rangle} > 0, \ i = 1, 2, 3\}.
\]

**Proposition 3.3.**
\[
\hat{p}_1 = \pi, \quad \hat{p}_2 = \pi^3 (f(\vartheta_1) + f(\vartheta_2) + f(\vartheta_3)). \tag{3.17}
\]

**Proof of Proposition 3.3** The first equation (3.17) is obvious. To prove the second one, we note that
\[
\hat{p}_2 = \frac{1}{2} \int_{w>1} dw \int_{\psi \in G(w)} A^{-3} d\psi, \tag{3.18}
\]
\[
G(w) = \{\psi \in \mathbb{T} : \text{sin}(\psi + \psi_i) < 1/\sqrt{w}, \ i \in \mathbb{Z}_3\}. \tag{3.19}
\]

Indeed, by Proposition 3.1 and equation (3.14) we have: \( A = A(w) \). Therefore we can perform integration in (3.10) in the variable \( u \) which implies (3.18).

To check that the domain \( G \) is determined by (3.19), we define
\[
\nu = \langle 1, \tau \rangle = u(\lambda \sqrt{w} \cos \psi + \langle c, 1 \rangle),
\]
\[
\beta = \langle c \times 1, \tau \rangle = u \lambda \sqrt{w} \sin \psi.
\]
Then
\[
\begin{pmatrix}
  u \\
  \nu \\
  \beta
\end{pmatrix} =
\begin{pmatrix}
  c_1 & c_2 & c_3 \\
  1 & 1 & 1 \\
  c_2 - c_3 & c_3 - c_1 & c_1 - c_2
\end{pmatrix}
\begin{pmatrix}
  \tau_1 \\
  \tau_2 \\
  \tau_3
\end{pmatrix},
\]
\[
\begin{pmatrix}
  \tau_1 \\
  \tau_2 \\
  \tau_3
\end{pmatrix} = \frac{1}{\lambda^2}
\begin{pmatrix}
  3c_1 - \langle c, 1 \rangle & c^2 - c_1 \langle c, 1 \rangle & c_2 - c_3 \\
  3c_2 - \langle c, 1 \rangle & c^2 - c_2 \langle c, 1 \rangle & c_3 - c_1 \\
  3c_3 - \langle c, 1 \rangle & c^2 - c_3 \langle c, 1 \rangle & c_1 - c_2
\end{pmatrix}
\begin{pmatrix}
  u \\
  \nu \\
  \beta
\end{pmatrix}.
\]
Hence the inequalities \(\tau_i > 0\) take the form
\[
3c_i - \langle c, 1 \rangle + (c^2 - c\langle c, 1 \rangle)(\lambda\sqrt{w}\cos \psi + \langle c, 1 \rangle) + (c_{i+1} - c_{i-1})\lambda\sqrt{w}\sin \psi > 0.
\]
After simple transformations we get: \(a_i \cos \psi + b_i \sin \psi < 1/\sqrt{w}\) which implies (3.19).

Equations (3.18)–(3.19) imply
\[
\hat{p}_2 = \frac{1}{2} \int_{w>1} \frac{|G(w)|}{A^3(w)} dw,
\]
where \(|G(w)|\) is the measure of the set \(G(w)\).
The set \(\mathbb{T} \setminus \{\pi/2 - \psi_1, \pi/2 - \psi_2, \pi/2 - \psi_3\}\) has 3 connected components: \(U_1, U_2,\) and \(U_3\), where the interval \(U_i\) has endpoints \(\pi/2 - \psi_{i-1}\) and \(\pi/2 - \psi_{i+1}\). Hence
\[
G(w) = G_1(w) + G_2(w) + G_3(w),\quad G_i(w) = G \cap U_i,
\]
\[
\hat{p}_2 = \hat{p}_2^{(1)} + \hat{p}_2^{(2)} + \hat{p}_2^{(3)},\quad \hat{p}_2^{(i)} = \int_{w>1} \frac{|G_i(w)|}{A^3(w)} dw.
\]
By using (3.15) and (3.19), we get:
\[
|G_i(w)| = \vartheta_i - 2(\pi - \varphi_\sigma(w))
\]
provided the right-hand side is non-negative.

By using the change \(w = 1/\cos^2 \varphi_\sigma, \varphi_\sigma \in (\pi - \vartheta_i/2, \pi)\) in the integral
\[
\hat{p}_2^{(i)} = \frac{\pi^3}{2} \int_{w>1} \frac{\vartheta_i - 2(\pi - \varphi_\sigma(w))}{(\varphi_\sigma(w) - \tan \varphi_\sigma(w))^3} dw,
\]
we obtain the equation
\[
\hat{p}_2^{(i)} = \int_{\pi - \vartheta_i/2}^{\varphi_\sigma} \frac{\pi^3(\vartheta_i - 2(\pi - \varphi_\sigma)) \sin \varphi_\sigma}{(-\varphi_\sigma \cos \varphi_\sigma + \sin \varphi_\sigma)^3} d\varphi_\sigma = 2\pi^3 f(\vartheta_i).
\]

4 Several proofs

Proof of Proposition 3.1 We define \(\sigma_*\) and \(\varphi_*\) by the equations
\[
\sqrt{\sigma_x^2 + \sigma_y^2} = \sigma_*,\quad \cos \varphi_* = \sigma_x/\sigma_*,\quad \sin \varphi_* = \sigma_y/\sigma_*
\]
Then \(\mathcal{A}\) takes the form
\[
\mathcal{A} = \frac{1}{2\pi} \int_{\mathcal{B}_+} \left(1 + \frac{\sigma_*}{\sigma_0} \cos(\varphi - \varphi_*)\right) d\varphi,
\]
\[
\mathcal{B}_+ = \{\varphi \in \mathbb{T} : \sigma_0 + \sigma_* \cos(\varphi - \varphi_*) \geq 0\} = \{\varphi \in \mathbb{T} : -\varphi_\sigma \leq \varphi - \varphi_* \leq \varphi_\sigma\}.
\]
This implies the required assertion. □

Proof of Proposition 3.2 The first identity (3.9) follows from the equation
\[
c_3 = -\cot(\vartheta_1/2 + \vartheta_2/2) = \frac{1 - c_1c_2}{c_1 + c_2}.
\]
To prove the second one we note that

\[ \sin \vartheta_1 = -2 \sin(\vartheta_1/2) \cos(\vartheta_2/2 + \vartheta_3/2) = 2 \prod \sin(\vartheta_i/2) - 2 \sin(\vartheta_1/2) \cos(\vartheta_2/2) \cos(\vartheta_3/2). \]

Adding to this equation two analogous ones and dividing by \( \prod \sin(\vartheta_i/2) \), we get:

\[ \sum \frac{\sin \vartheta_i}{\prod \sin(\vartheta_i/2)} = 6 - 2(c_2c_3 + c_3c_1 + c_1c_2) = 4. \]

Finally, adding up the equations

\[
\begin{align*}
\sin^2 \frac{\vartheta_1}{2} &= \sin \frac{\vartheta_1}{2} \sin \left( \frac{\vartheta_2}{2} + \frac{\vartheta_3}{2} \right) = \sin \frac{\vartheta_1}{2} \left( \sin \frac{\vartheta_2}{2} \cos \frac{\vartheta_3}{2} + \cos \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2} \right), \\
\cos^2 \frac{\vartheta_1}{2} &= -\cos \frac{\vartheta_1}{2} \cos \left( \frac{\vartheta_2}{2} + \frac{\vartheta_3}{2} \right) = \cos \frac{\vartheta_1}{2} \left( \sin \frac{\vartheta_2}{2} \sin \frac{\vartheta_3}{2} - \cos \frac{\vartheta_2}{2} \cos \frac{\vartheta_3}{2} \right).
\end{align*}
\]

dividing by \( \prod \sin(\vartheta_i/2) \), we obtain the third identity \((3.9)\). \(\square\)

5 Discussion

Our computation of probability distributions in Theorems 1.1 and 1.2 are based on the assumptions that length of a tooth is uniformly distributed on \([0, 1]\) and the teeth are situated on the base of the comb with a constant step. However we believe that the answers (i.e., formulas for densities of these distributions) are not sensitive to these details. For example, the answers should be the same if the teeth are randomly uniformly distributed on the base and/or lengths of the teeth are identical independently distributed random values with a continuous distribution density on \([0, b]\), \(0 < b < \infty\). It would be interesting to obtain a proof of this conjecture.

We have already mentioned that it is interesting to consider analogous problems where base of a random comb is two-dimensional, for example, a disk. Also we would be happy to see dynamical applications of these problems.

References

[1] T.Salnikova, D.Treschev, S.Galliamov. Motion of a free puck on a rough horizontal surface. Nonlinear Dynamics, 2012, V.8, N.1, P.83-101 (in Russian)

[2] Burlakov D., Seslavina A.

[3] Erdakova N., Ivanova T., Treschev D.

[4] Ivanov A.P. A dynamically consistent model of the contact stresses in the plane motion of a rigid body. J. Appl. Math. Mech. 2009 N.2, P. 134-144

[5] Ishlinsky A.Yu., Sokolov B.N., Chernousko F.L. On motion of flat bodies with friction // Izvestiya of Russian Acad. Sci.: Rigid Body Mechanics. 1981. No. 4., P. 17-28.

[6] Kireenkov A.A. On the motion of a homogeneous rotating disk along a plane in the case of combined friction. J. Mechanics of Solids 2002, no.1, P. 47-53.
[7] Kireenkov A.A. A method for the calculation of the force and torque of friction in a combined model of dry friction for circular contact areas. J. Mechanics of Solids, 2003, no. 3 P. 39-43.

[8] Kireenkov A.A., Semendyaev S.V. and Filatov V.F. Experimental Study of Coupled Two-Dimensional Models of Sliding and Spinning Friction. J. Mechanics of Solids, 2010, no. 6, P. 921-930.

[9] Macmillan W.D., Dynamics of Rigid Bodies. McGraw-Hill, New York (1936).

[10] Contensou P. Couplage entre frottement de glissement et frottement de pivotement dans la thorie de la toupie. Kreiselsprobleme. Gyrodynamics. Symp. Celerina, 1962. Derlin etc.: Springer, 1963.

[11] Farkas Z., Bartels G., Unger T., Wolf D. Frictional coupling between sliding and spinning motion. Phys.Rev.Letters 2003, V.90, no. 24. 248302.

[12] Weidman P.D. Malhotra Ch.P. On the terminal motion of sliding spinning disks with uniform Coulomb friction // Physica D: Nonlinear Phenomena, 2007, vol. 233, pp. 1–13. 2007 Elsevier.

[13] Zhuravlev, V.F. On a model of dry friction in the problem of the rolling of rigid bodies. (Russian) Prikl. Mat. Mekh. 62 (1998), no. 5, 762–767; translation in J. Appl. Math. Mech. 62 (1998), no. 5, 705710 (1999) 70F40 (70E18)