Geodesics in the Generalized Schwarzschild Solution

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Since Schwarzschild discovered the point-mass solution to Einstein’s equations that bears his name, many equivalent forms of the metric have been catalogued. Using an elementary coordinate transformation, we derive the most general form for the stationary, spherically-symmetric vacuum metric, which contains one free function. Different choices for the function correspond to common expressions for the line element. From the general metric, we obtain particle and photon trajectories, and use them to specify several time coordinates adapted to physical situations. The most general form of the metric is only slightly more complicated than the Schwarzschild form, which argues effectively for teaching the general line element in place of the diagonal metric.

I. INTRODUCTION

For many decades, general relativity was considered a highly abstract subject, the purview of sophisticated mathematical physics far removed from the undergraduate curriculum. This situation has changed in recent years as general relativity has matured into a true experimental science: gravitational lensing now routinely probes dark matter in distant galaxies, gravitational wave detection thresholds are steadily marching into the range of realistic sources, and the Global Positioning System must compensate for gravitational time delays. In parallel to this evolving status of general relativity, a number of excellent textbooks at the undergraduate level have been published in recent years, demonstrating that teaching general relativity to physics majors is not only possible but exciting and relevant.

One of the elementary results of general relativity is the spacetime arising from a point mass $M$, which is spherically symmetric and stationary, or independent of the time coordinate. Close to the mass, the spacetime represents a black hole with an event horizon, while at a large distance it reproduces the results of Newtonian gravity. This spacetime is most commonly represented by the Schwarzschild metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \chi(r)dt^2 - \chi^{-1}(r)dr^2 - r^2d\Omega^2,$$  

where we have used units $c = G = 1$ and abbreviated $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Historically, this was the first widely-used form of the metric, and virtually all texts begin (and often end) the discussion of spherically symmetric metrics with Eq. (1). Formally, this form of the metric is in one sense the simplest possible, since it is the only diagonal form of the metric, and this property results in simplified calculations (e.g., the smallest possible number of non-zero connection coefficients).

However, metrics in general relativity are not physically measurable quantities, and numerous metrics employing different sets of coordinates can be used to describe the same physical spacetime. The usual form for the Schwarzschild metric possesses some peculiar properties which make it far from ideal pedagogically. Most notably, it contains an unphysical coordinate singularity at the event horizon $r = 2M$, the interpretation of which is confusing enough that its true nature was debated for many years. A second more subtle point is that the metric is the same under time-reversal: a particle traveling forward in time is equivalent to one traveling backward in time, but physically particles can cross the event horizon in only one direction.

Here we perform a general coordinate transformation of Eq. (1) to obtain the most general form of the stationary, spherically-symmetric metric:

$$ds^2 = \chi(r)du^2 + 2B(r)du dr - \chi^{-1}(1-B(r)^2)dr^2 - r^2d\Omega^2. \quad (2)$$

This form contains one free function of radius, $B(r)$, which explicitly displays the coordinate freedom in the metric. Through various choices for $B(r)$, all of the familiar (and not-so-familiar) forms of the metric can be obtained, including the Schwarzschild form and the Eddington-Finkelstein metric often used to investigate the physical nature of the event horizon. We advocate the use of Eq. (2) as a much clearer and pedagogically versatile form of the metric for a number of reasons:

1. Eq. (2) emphasizes the fact that the same physical spacetime is described by many different metrics;

2. It unifies the disparate expressions for the metric in the general relativity literature into a single simple form;

3. It naturally suggests variables describing motion of particles which are closely tied to physically measurable quantities, and investigating the reduction to different metric forms clarifies the spurious nature of the singularity in Eq. (1); and

4. Mathematical manipulations involving Eq. (2) are only slightly more complex than those using the Schwarzschild form.
While Eq. (1) may be the simplest form of the metric, it does not give the simplest or most physically transparent forms for equations of particle motions in the spacetime.

The rest of this paper derives the general form of the metric, demonstrates its reduction to various specific forms, and discusses the corresponding expressions for the geodesic paths followed by light and particles moving in the radial direction. In the next Section, Eq. (2) is derived from the familiar Eq. (1) via a straightforward coordinate transformation. Section III establishes expressions for radial geodesics in the general case, and Sec. IV examines how the geodesic expressions are simplified for specific choices of the arbitrary function B(r), including all of the most commonly used forms of the metric.

Three brief appendices cover relevant but somewhat more technical material: Appendix A gives the connection and Ricci curvature components associated with Eq. (2) for convenience in doing calculations; Appendix B discusses why a change in time coordinate does not affect the gravitational redshift; and Appendix C expresses the general line element in vierbein form. The derivations and arguments in the body of the paper require no mathematics beyond elementary calculus, and as such are suitable for inclusion in any undergraduate general relativity class.

II. THE GENERAL LINE ELEMENT

In the Schwarzschild form of the metric, Eq. (1), the time direction is orthogonal to all other directions (that is, \( g_{0k} = 0 \) when \( k \neq 0 \)) so the metric is called static; Eq. (1) is the unique static spherically-symmetric vacuum line-element. The radial coordinate \( r \) in the Schwarzschild geometry has a clear physical interpretation: it is the proper distance measured by a massive particle falling into a black hole\(^7\). The time coordinate \( t \) is more problematic. The proper time for a particle (either massive or massless) falling through the event horizon is finite, but the coordinate time \( t \) diverges—it takes an infinite amount of time in the Schwarzschild system of coordinates to fall into a black hole\(^2\). Thus the singularity of Eq. (1) at the point \( r = 2M \) is not physical, and can be transformed away with a different choice of coordinates. We are interested in forms of the metric that are stationary but are not necessarily diagonal.

A transformation of the time coordinate of the type

\[
u = \alpha t + \beta(r),
\]

for some function \( \beta \) and constant \( \alpha \), ensures that the metric remains stationary, while keeping the spherical symmetry manifest. In other words, if \( \alpha \) and \( \beta \) remain unspecified, we obtain the most general stationary vacuum line element in spherical coordinates. Without loss of generality we take \( \alpha = 1 \); a simple rescaling of the time variable is equivalent to choosing a different time unit. Since \( u \) is a function of \( t \) and \( r \), its differential is

\[
du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial r} dr = dt + \frac{\partial \beta}{\partial r} dr.
\]

\[
du^2 = dt^2 + 2\beta(r)dt dr + \beta^2(r) dr^2.
\]

Rather than going through the complicated algebra of solving for \( dt \) and substituting into Eq. (1), note that a general stationary line element must have the form

\[
ds^2 = A(r)du^2 + 2B(r)du dr - C(r)dr^2 - r^2d\Omega^2;
\]

setting this equal to Eq. (1) and using the relations in Eq. (4),

\[
A(r) = \chi(r),
\]

\[
B(r) = -\chi(r)\beta'(r),
\]

\[
C(r) = \chi(r)\beta'(r)^2 + 2B(r)\beta'(r) + \chi(r)^{-1}.
\]

We can use Eq. (5b) to eliminate \( \beta' = -B(r)/\chi(r) \), which leads to a condition on \( B \) and \( C \) via Eq. (5c):

\[
C(r)\chi(r) + B(r)^2 = 1,
\]

and Eq. (2) follows. As Appendix A shows explicitly, the Einstein field equations do not specify \( B(r) \), so we will call Eq. (2) the general line element for the spherically-symmetric vacuum equations. We are free to choose \( B(r) \): for example, the Schwarzschild form of the metric is obtained from the general line element by letting \( B(r) = 0 \). We will discuss other coordinate choices below. Appendix B examines how a change in time coordinate, specified by \( B(r) \) and \( \chi(r) \), leaves some of the standard tests of general relativity unaffected.

Of course, it is also possible to change the spatial coordinates \( (r, \theta, \phi) \), but it is nice to leave them as written for two reasons: spherical symmetry is explicit, and as mentioned in the Introduction, the radial coordinate \( r \) is a measure of the proper distance measured by a massive particle falling radially into the black hole.

III. RADIAL GEODESICS

Since free particles (including photons) follow geodesics in general relativity, the Lagrangian for free motion can be written simply as the squared norm of the velocity \( \dot{x} = dx/d\lambda \) with respect to some parameter \( \lambda \):

\[
2L(x, \dot{x}, \lambda) = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu = \dot{u}^2 + 2B\dot{u}r - \chi^{-1}(1 - B^2)\dot{\theta}^2 - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2).
\]

The tests of general relativity that depend on the angular variables (light deflection, for example) do not explicitly involve the time coordinate, so we will look strictly at radial geodesics by letting \( \theta = \pi/2 \) and \( \phi = 0 \).

The magnitude of a geodesic vector (and hence the Lagrangian itself) is a constant of motion, so we have

\[
g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \dot{u}^2 + 2B\dot{u}r - \chi^{-1}(1 - B^2)r^2 \equiv \kappa\]

(8)
where $\kappa = 1$ for massive particles and $\kappa = 0$ for photons. The time coordinate $u$ is cyclic, so we know that

$$\frac{\partial \mathcal{L}}{\partial \dot{u}} = \chi \ddot{u} + B \dot{r} \equiv \epsilon$$ (9)

is constant, with $\epsilon$ related to the energy of the particle (with respect to an observer at infinity) as follows:

$$\epsilon = \begin{cases} \frac{E}{m} & \text{massive particles} \\ \frac{E}{\sqrt{\hbar}} & \text{photons} \end{cases}$$

Solving Eq. (9) for $\dot{u}$ and substituting it into Eq. (8) allows us to solve for $\dot{r}$:

$$\dot{r}^2 = \epsilon^2 - \kappa \chi.$$ (9)

Since this is independent of the choice of $B$, the radial geodesic equation is as well:

$$\dot{r}^2 + \kappa M/r^2 = 0,$$ (10)

which is the Newtonian expression (upon restoration of $G$)—although $r$ is not the flat spacetime radial coordinate, and when $\kappa = 1$, $\lambda$ is not the coordinate time but the proper time shown by clocks moving with the massive particle along the geodesics.

Using our expression for $\dot{r}$ in Eq. (9) allows us to write the velocities with respect to the proper time:

$$\dot{r} = w \sqrt{\epsilon^2 - \kappa \chi}$$ (11a)

$$\dot{u} = \chi^{-1} \left\{ \epsilon - wB \sqrt{\epsilon^2 - \kappa \chi} \right\}$$ (11b)

where $w = +1$ for outgoing geodesics, while $w = -1$ for incoming. From these, we can find the coordinate time as a function of radius,

$$u(r) = \int \frac{\dot{u}}{r} dr = \int \frac{dr}{\chi} \left\{ \frac{w}{\sqrt{1 - \kappa \chi / \epsilon^2}} - B \right\},$$ (12)

which tells us whether a particle crosses the event horizon in finite coordinate time, for a particular choice of $B(r)$. In the Schwarzschild case $B = 0$, for example, the time required diverges logarithmically for both values of $u$. Once $B$ is chosen, two parameters dictate the type of geodesic: $\kappa = 0$ or 1, which specifies lightlike (photons) versus timelike (massive particles) geodesics, and $\epsilon$, which determines the energy of the particle. In the case of null geodesics ($\kappa = 0$), the photon energy is irrelevant:

$$u(r) = \int \frac{dr}{\chi} (w - B).$$ (13)

### IV. COORDINATE SYSTEMS FROM GEODESICS

In the following subsections, we will examine choices of $B(r)$ that simplify the geodesic expressions. Some of these examples are familiar, others less so. Our central conceptual point is that the general form of the metric, Eq. (4), allows us to choose a specific system of coordinates which leads to simple and physically intuitive forms for the geodesics. Since the geodesics, unlike the metric, represent physically observable quantities, picking a coordinate system based on the properties of the geodesics instead of the metric makes sense, as long as the coordinate system does not lead to unduly complex computations.

#### A. Two Simple Systems

Equation (12) gives the coordinate time as a function of radius for a null geodesic. Future-pointing paths of light in special relativity are straight lines defined by $u(r) = -r + \text{const.}$; to replicate this, we let $B = \chi - 1 = -2M/r$, which means that

$$u_{\text{EF}}(r) = \begin{cases} -r + u_0 & \text{incoming} \\ r + 4M \ln |r - 2M| + u_0 & \text{outgoing} \end{cases}$$

where $u_0$ is a constant of integration. This is one version of the familiar Eddington-Finkelstein coordinates; ingoing geodesics are straight lines, while outgoing must “climb out” of the gravity well formed by the black hole. Nothing starting within the event horizon can get out, since the “outward” geodesics starting from $r < 2M$ actually fall inward as well.

The line element for this choice of coordinates is

$$ds^2_{\text{EF}} = du^2 - dv^2 - r^2 d\Omega^2 - \frac{2M}{r} (du - dr)^2 = (\eta_{\mu\nu} - 2V_{\mu}V_{\nu}) dx^\mu dx^\nu,$$

where in the second line we have written the metric as the sum of the flat metric (in spherical coordinates) and a contribution from a null vector

$$[V^\mu] = (1, -1, 0, 0).$$

Metrics of this type are often called Kerr-Schild metrics, and in their more general form are used to find the rotating black hole solution (known as the Kerr solution; see Ref. [10] for more details).

A related class of coordinates can be obtained by setting the proper time (for a massive particle) or affine parameter (for a photon) equal to the coordinate time for an inbound geodesic. This means we must specify in advance the energy $\epsilon'$ of the particle, and whether it is massless or massive by setting the parameter $\kappa'$:

$$\dot{u} = 1 = \frac{\epsilon'}{\chi} \left\{ 1 + B \sqrt{1 - \kappa' \chi / \epsilon'^2} \right\}.$$ (12)

Selecting the retarded solution, we find

$$B = \frac{1}{\sqrt{1 - p\chi}} \left\{ \chi \sqrt{\frac{p}{\kappa'} - 1} \right\}$$ (14)
where
\[ p \equiv \kappa' / \epsilon^2 \]
is the metric parameter that determines the time coordinate associated with a particular energy. When \( \kappa' = p = 0, B = \chi / \epsilon' - 1 \); if \( \epsilon' \to 1 \) (so that \( |\epsilon| = 1 \)), we get back the Eddington-Finkelstein time coordinate. Thus, despite the appearance of both \( p \) and \( \kappa' \) in Eq. (13), they are not independent parameters. Since we have already discussed the Eddington-Finkelstein case, we will consider only \( \kappa' = 1 \) and \( p \neq 0 \) for the remainder of the section.

The line element for this geometry with \( \kappa' = 1 \) is
\[
ds^2_{GT} = \chi du^2 + \frac{2\sqrt{\rho - 1}}{\sqrt{1 - p\chi}} dr d\Omega^2
- \frac{2\sqrt{\rho - p(1 + \chi)}}{1 - p\chi} dr^2 - r^2 d\Omega^2
\tag{15}
\]
where the subscript “GT” indicates geodesic time. For an arbitrary geodesic, the trajectory is described by
\[
u_p(r) = \int \frac{dr}{\chi} \left\{ \frac{w}{\sqrt{1 - \kappa\chi / \epsilon^2}} \frac{\sqrt{\rho - 1}}{\sqrt{1 - p\chi}} \right\}
\]
Plots for different values of \( p \) and \( \epsilon \) are shown in Fig. 1. When \( \epsilon = \epsilon' \) (and \( \kappa = 1 \)), the time for an infalling geodesic takes the simple form
\[
u_p(r) = \int dr \sqrt{\frac{p}{1 - p\chi}}
\]
which reduces to the Eddington-Finkelstein case in the limit \( p \to 0 \).

The relative complexity of the line element in Eq. 15 may be why it is not more widely known—except in the specific case \( p = 1 \), which is the Painlevé-Gullstrand metric (see Ref. 11), independently discovered by Lasenby, Doran, and Gull12.

\[
ds^2_{PG} = \chi du^2 + 2\sqrt{\frac{2M}{r}} dudr - dr^2 - r^2 d\Omega^2.
\tag{16}
\]
We will see this metric again as a special case in the following section.

### B. Relationship Between \( u \) and Proper Time

In the previous section, we have chosen coordinate times \( u \) which lead to physically intuitive forms for radial geodesics, specified by \( u(r) \). In one case we explicitly equate the coordinate time with the proper time for a massive particle moving along a radial geodesic, Eq. (14). More generally, we can derive the proper time \( \tau \) for any choice of coordinate time \( u \) (or equivalently, the function \( B(r) \)). Starting from Eq. (13), we multiply both sides by \( d\lambda / dr \) and rearrange somewhat:
\[
\kappa' \frac{d\lambda}{dr} = \left( \chi \frac{du}{dr} + 2B \right) \frac{du}{d\lambda} - \frac{1}{\chi} (1 - B^2) \frac{dr}{d\lambda},
\tag{17}
\]
Applying Eq. (11b) to the first term of the right-hand side eliminates \( du/d\lambda \)
\[
\left( \chi \frac{du}{dr} + 2B \right) \frac{\dot{u}}{\epsilon'} = \frac{du}{dr} w B \sqrt{1 - p\chi}
+ \frac{2B}{\chi} (1 - w B \sqrt{1 - p\chi})
= \frac{du}{dr} + B (2\chi - 1) \left( 1 - w B \sqrt{1 - p\chi} \right),
\]
which is a coordinate transform as in Eq. 8. For massive particles with \( p \neq 0 \), Eq. (18) gives the proper time \( \lambda \) in terms of the coordinate time \( u \); when \( p = 0 \), no proper time can be defined but the integral is the same as the one in Eq. (13).

For special choices of \( B \), the integral in Eq. (18) takes a simpler form. The obvious example is
\[
B = -\sqrt{1 - p\chi},
\tag{19}
\]
which gives rise to the Lake-Martel-Poisson11,13 metric
\[
ds^2 = \chi du^2 - 2\sqrt{1 - p\chi} dudr - dr^2 - r^2 d\Omega^2.
\]
Different values of \( p \) yield several well-studied solutions: when \( p = 0 \), we obtain the more familiar form of the retarded Eddington-Finkelstein metric (in which infalling null geodesics are lines of constant \( u \)), while \( p = 1 \) yields the Painlevé-Gullstrand metric of Eq. (16). We may also treat \( p \) in general as a positive parameter (which can be associated with the energy of a particle at infinity) in which case \( r \) can range from zero to infinity for \( 0 \leq p \leq 1 \), and
\[
r \leq 2Mp/(p - 1) \quad \text{for} \quad p > 1.
\]

The latter case is proportional to the Gautreau-Hoffmann coordinate system9, with the initial radial value defined by \( R_i = 2Mp/(p - 1) \) in our notation. Note that Eq. (19) is equivalent to Eq. (13) when \( p = \epsilon' = 1 \); in this limit, by construction, the proper time is equal to the coordinate time. This case is equivalent to setting \( \dot{u} = \dot{\epsilon'} p \) and repeating the analysis of Section 14.A.

### V. DISCUSSION

The general line element provides a clear and straightforward way to pick specific time coordinates for the spherically symmetric, stationary spacetime. Eq. (2) is a simpler starting point than the diagonal Schwarzschild
metric for determining the two forms of the Eddington-Finkelstein coordinate system, and provides both the geodesic time parameter and the Lake-Martel-Poisson coordinates directly, as opposed to performing a rather complicated coordinate transformation. In fact, the geodesic time line element is new to us, and obtaining it directly from the Schwarzschild solution involves a difficult integral.

We have emphasized stationary forms of the metric; however, it is possible to “maximally continue” any of these coordinate systems to make them diagonal. This is done by using

\[ u \equiv t + \beta(r) \quad \text{and} \quad v \equiv t - \beta(r) \]

in place of \( t \) and \( r \), and rewriting the line element in the new coordinates. The familiar example of this is the Kruskal-Szekeres extension, which uses the Eddington-Finkelstein coordinates as its starting point. However, all maximally-extended solutions include a non-physical region of spacetime (the “white hole” problem), and this poses as serious an interpretational difficulty as the coordinate singularity problem at the event horizon.

From a pedagogical point of view, we emphasize that Eq. (2) contains virtually all other discussed coordinate systems as special cases. The general metric is only slightly more complicated to use in calculations than the diagonal Schwarzschild case; for convenience we list the connection and Ricci tensor components explicitly in Appendix A. For reasons of pedagogical and conceptual clarity, We advocate presenting Eq. (2) as the line element describing the spherically symmetric and stationary vacuum spacetime when teaching general relativity, especially at the undergraduate level; the familiar Schwarzschild form of the metric can be easily obtained as a special case. Appendix C shows how to derive Eq. (2) in a straightforward way which is no more complicated than usual derivations of the Schwarzschild metric.

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APPENDIX A: SOLUTION OF EINSTEIN’S EQUATIONS

The Christoffel connection coefficients are found using the standard formula

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left( \partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right). \]
If we take Eq. (2) to be an ansatz for solving Einstein’s equations, with unknown functions $\chi(r)$ and $B(r)$, the nonzero connection coefficients are

\[
\begin{align*}
\Gamma^0_{00} &= -\Gamma^1_{01} = B\chi'/2 \\
\Gamma^0_{01} &= -\Gamma^1_{11} = C\chi'/2 \\
\Gamma^0_{11} &= CB^2 - BC'\chi'/2 \\
\Gamma^0_{22} &= \Gamma^0_{33} \csc^2 \theta = Br \\
\Gamma^1_{22} &= \Gamma^1_{33} \csc^2 \theta = -\chi r \\
\Gamma^2_{12} &= \Gamma^3_{13} = r^{-1} \\
\Gamma^3_{33} &= -\sin \theta \cos \theta \\
\Gamma^3_{23} &= \cot \theta \\
\end{align*}
\]

where we have used the shorthand (see Eq. (3))

\[C = \chi^{-1}(1 - B^2)\]

and $\chi' = d\chi/dr$. These expressions are not complicated, but there are more of them than in the diagonal case.

The Ricci tensor components,

\[R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\rho},\]

turn out to be quite simple:

\[
\begin{align*}
R_{00} &= \chi\Phi/2r \\
R_{01} &= B\Phi/2r \\
R_{11} &= -C\Phi/2r \\
R_{22} &= R_{33} \csc^2 \theta = 1 - \chi - \chi' r \\
\end{align*}
\]

where

\[\Phi \equiv \chi'' r + 2\chi'.\]

Thus, Einstein’s vacuum equations $R_{\mu\nu} = 0$ do not involve $B(r)$, and are easily solved to give Eq. (11), with $M$ as an integration constant which is identified with the total mass by ensuring the Newtonian limit.

**APPENDIX B: NEW TIME VARIABLES AND THE “STANDARD TESTS”**

Most derivations of the measureable effects due to a planet, star, or black hole start from Eq. (11); nevertheless, changing time variables does not change the predictions of general relativity. We will not bother to rederive all the familiar expressions, but instead will emphasize their independence of particular coordinate assumptions. In fact, most tests of GR (bending of light, perihelion advance of an object in orbit, etc.) depend on the radial velocity with respect to proper time, which is a generalization of Eq. (11). Since this equation does not involve $u$ in any way, we don’t need to discuss these tests here. The exceptions are time delay and gravitational redshift; demonstrating that the former is independent of time coordinate choice requires coverage of non-radial motion, so we will not discuss it here.

For gravitational redshift, let $[\xi^\mu] \equiv (1, 0, 0, 0)$ be a timelike (Killing) vector in whatever coordinate system we desire. The “direction of time” at a given value of $r$ is then indicated by

\[\hat{\xi}^\mu(r) = \xi^\mu / \sqrt{\epsilon^{\mu\nu} \xi_\nu} = \xi^\mu / \sqrt{\chi(r)}.\] (B1)

If we locate two stationary observers $O_1$ and $O_2$ at $R_1$ and $R_2$, respectively, and send a pulse of light between them, we obtain a shift in frequency due to the presence of $\chi$. The path of the light is indicated by the null momentum vector $k$, whose time component gives the frequency:

\[\omega_1 = k_\mu \hat{\xi}^\mu \bigg|_{r=R_1}, \quad \omega_2 = k_\mu \hat{\xi}^\mu \bigg|_{r=R_2}.\]

The projection of the timelike vector $\xi$ along the geodesic is constant (although proof of this conjecture requires a discussion of Killing vectors, which is beyond the scope of this paper). Using Eq. (B1), we obtain

\[\frac{\omega_1}{\omega_2} = \sqrt{\frac{\chi(R_2)}{\chi(R_1)}}.\] (B2)

The redshift formula is directly obtained from this ratio; since it doesn’t depend on the choice of time coordinate, it holds for all variations on the general line element. Thus, the redshift is infinite if the emitter is located at $r_1 = 2M$, even if the coordinate system allows for a crossing of the event horizon in finite time.

**APPENDIX C: VIERBEIN FORM OF THE GENERAL LINE ELEMENT**

The general line element expresses all the metric degrees of freedom that are possible by changing the time coordinate. However, some degrees of freedom lie “beneath” the metric; these are often expressed as objects called vierbeins $h^a_b$, which are like a “square-root” of the metric. The equivalence principle allows us to assign an orthonormal frame of vectors (whose metric is just the flat spacetime metric $\eta_{ab}$) to each point of spacetime, and the vierbeins take us between the orthonormal frame and the full metric of GR:

\[g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab} \rightarrow \sqrt{-\det g} = \det h.\]

Writing things out in terms of the vierbeins gives us not only the general line element, but also lets us express the isometries—coordinate transformations that do not change the form of the metric.

We write in matrix form the vierbeins associated with the most general spherically symmetric and stationary
spacetime as

$$\left[ h^\mu_\nu \right] = \begin{pmatrix}
g_1(r) & f_2(r) & 0 & 0 
g_2(r) & f_1(r) & 0 & 0 
0 & 0 & r & 0 
0 & 0 & 0 & r \sin \theta
\end{pmatrix}$$

so that

$$ds^2 = \left( g_1^2 - g_2^2 \right) du^2 + 2\left( f_1 g_2 - f_2 g_1 \right) du dr - \left( f_1^2 - f_2^2 \right) dr^2 - r^2 d\Omega^2.$$  \hfill (C1)

(This is the way we first obtained the general line element.\textsuperscript{15}) Eq. (2) follows from the identifications

$$\chi = g_1^2 - g_2^2, \quad B = f_1 g_2 - f_2 g_1, \quad C = f_1^2 - f_2^2,$$

and

$$f_1 g_1 - f_2 g_2 = 1.$$  \hfill (C2)

Einstein’s equations then give Eq. (11), and Eq. (C2) leads directly to Eq. (3), so at first glance it appears that these two conditions on the four free functions leave two free functions to be specified. Nevertheless, the condition $C \chi + B^2 = 1$ arises from $f_1 g_1 - f_2 g_2 = 1$, so that the metric still only contains one free function. The two free vierbein functions are both required to determine the metric, so individually they represent isometric degrees of freedom—a given expression for $B$ corresponds to a great many choices of vierbein fields.