Modulational Instability of Spin-Orbit Coupled Bose-Einstein Condensates in Discrete Media

S. Sabari1,* , R. Tamiltiruvalluvar2,† and R. Radha1,#

1Centre for Nonlinear Science, Department of Physics, Government College for Women(A), Kumbakonam 612001, India
2Department of Physics, Pondicherry University, Puducherry - 605014, India
*ssabar01@gmail.com, †tapthy1905@gmail.com, ‡vittal.cnls@gmail.com

Abstract

We address the impact of intra-site spin-orbit (SO) coupling and associated inter-component Rabi coupling on the modulational instability (MI) of plane-wave states in two-component discrete Bose-Einstein condensates (BECs). Conditions for the onset of the MI and the respective instability are found analytically. SO coupling allows us to produce the MI even for a small initial wavenumber \((q < \pi/2)\) for miscible states. In particular, SO coupling introduces MI even in the absence of hopping coefficient, a concept which may have wider ramifications in heavy atomic BECs. We have also shown how our results of the linear stability analysis can be corroborated numerically. The fact that we have brought out the stability criteria in different domains of system parameters means that our model is tailor made experiments.

Keywords: Bose-Einstein condensates, Spin-orbit coupling, Modulational instability, Discrete media, Linear stability analysis.

1. Introduction

Even though the ubiquitous nature of the spin-orbit (SO) coupling, which involves the interaction between quantum particle’s spin and its momentum is quite well known, it came to the fore only after the experimental realization of Bose-Einstein condensates (BECs), a new state of matter comprising of ultracold atoms getting piled up in the ground state like a giant matter wave. It must be emphasized that BECs provide an ideal platform (launching pad) to engineer SO coupling in a neutral atomic BEC by dressing two atomic spin states with parts of laser. In fact, in the presence of laser coupling [1], the interaction between the two dressed atomic spin states are modified driving a quantum phase transition from a spatially spin mixed state (laser off) to a phase separated state(stripped phase). The stripped phase which occurs above a critical laser density initiates a phase transition by engineering SO coupling thereby leading to the rich possibility of generating synthetic electric and magnetic fields. Recent investigations have explored the possibility of identifying tunable spin-orbit coupled (SOC) BECs with various trapping potentials and stable domains of condensates have been observed. The SOC-BECs involving the interplay between the nonlinear two-body interaction with the linear interaction of SOC have been investigated in the last few years. For instance, it has been reported that the new ground-state phase can be created in two-component nonlinear systems such as stripes, phase separation, etc [3], tricritical points [3], different type of solitons [4, 5, 6, 7], 2D solitons with embedded vorticity and vortex lattice [8, 12, 13].

The combinations of optical lattice (OL) potential with the interaction of SO coupling was shown to posses interesting phenomena like flattening of Bloch potential [12], atomic Zitterbewegung [13] and new topological phases [14]. Generally, BECs become fragmented on each lattice when it is trapped by deep OL potential. Such a system has been effectively described in the tight-binding approximation by discrete version of the corresponding Gross-Pitaevskii (GP) equation [15, 16, 17]. Recent investigations on composite solitons and localized modes are carried out on Rashba type SOC-BEC trapped in deep OL [11, 18].

The modulational instability (MI) is a generic phenomenon which occurs in a dynamical system like fluid dynamics, nonlinear optics, plasma physics due to the interplay between nonlinearity and dispersion (or diffraction in spatial domain) resulting in the fragmentation of carrier waves into a train of localized waves [19]. Recently, the onset of MI in scalar [20, 21, 22] and vector BECs [23, 24, 25] within the framework of continuous NLS type (GP) equations has been investigated. The above scenario could drastically change in discrete multicomponent settings [26, 27, 28, 29, 30].

It has been found that the onset of MI not only depends on the sign of diffraction (or dispersion), but also on the sign of inter-species and intra-species interaction strengths. Recently, MI in SOC-BECs has been investigated in continuous media [31, 32]. However, the condition for the onset MI in discrete media (or settings) has not yet been completely brought out, particularly after the advent of SOC-BECs. In the light of the above developments, it would be interesting to study the onset of MI in SOC-BEC in discrete media. The freedom associated with the SOC-BECs by virtue of Rashba coupling, SO coupling, intra/inter species interaction and dispersion could lead to the identification of several interesting stable domains from an ex-
perimental perspective.

In view of the above, we have made an attempt to analyze the impact of SO coupling on MI in discrete BECs. This paper is structured as follows. In sec. 2, the SO coupling model is constructed in terms of coupled discrete GP equations and associate dispersion relations are found via linear stability analysis. The results of semi analytical analysis are discussed with different signs of intra- and inter component fragmentation with SO coupling interaction in sec. 3. Finally in sec. 4, we elaborate on the salient features of the present investigation.

2. Theoretical model and dispersion relation for MI

We consider a BEC comprising of atoms in different hyperfine ground states corresponding to \( |F = 1, m_F = 0 \) and \( |F = 1, m_F = -1 \). These settings induce the SO coupling resulting in the formation of two dressed states as a basis. The dynamics of such a system described by mean-field GP equation in the presence of deep OL in spinor form is governed by

$$\frac{\partial \Psi}{\partial t} = \left[ \frac{\hbar^2}{2m} V_{OL} + \hat{H}_{SOC} + \hat{H}_{mix} + \hat{H}_{int} \right] \Psi \tag{1}$$

where \( \Psi = (\Psi^+, \Psi^-)^T \) represents the spinor wave function in the dressed atomic state and the term \( \frac{\partial^2}{\partial x} \) represents the kinetic energy of the quasi-one-dimensional condensate along the \( x \)-direction. The Hamiltonian \( \hat{H} \) which describes the SOC interaction has the following form,

$$\hat{H}_{SOC} = -\frac{\hbar^2}{m} \gamma \sigma_z \frac{\partial}{\partial x} \tag{2}$$

where \( \gamma \) is the strength of SO coupling and \( \sigma_z \) is \( z \)-component Pauli’s spin matrix. The linear coupling term is represented by \( \hat{H}_{mix} = \hbar \delta \sigma_x \), which originates through magnetic field in intra-SOC system where \( \delta \) is strength of Rabi coupling. \( \hat{H}_{int} \) accounts for the intra- and inter-component interactions in the BEC for the mixture which are nonlinear in nature.

Such a system in deep OL leads to the replacement of the continuous GP equation by the discrete counterpart to single component BECs in earlier studies [15, 16, 17] and two-component BECs [27, 28, 29, 30]. Thus, we arrive at a system of coupled discrete GP equation with intra-SOC and Rabi coupling of the following form [11, 18],

$$\frac{\partial \Psi_n^\pm}{\partial t} = -\Gamma^\pm (\Psi_n^{n+1} + \Psi_{n-1}^\pm - 2\Psi_n^\pm) \mp i \gamma (\Psi_n^{n+1} - \Psi_n^\pm) \tag{3}$$

where, \( \Gamma \) represents hopping coefficient of adjacent lattice sites while \( g^\pm \) and \( G^\pm \) represent intra- and inter-component collision strengths respectively. This system conserves two quantities for infinite lattices namely

$$N = \sum_n |\Psi_n^+|^2 + |\Psi_n^-|^2$$

and the Hamiltonian which is

$$H = \sum_n \left( -\Gamma^+ (\Psi_n^{n+1} + \Psi_{n-1}^- - i\gamma (\Psi_n^{n+1} - \Psi_n^-) - i\gamma (\Psi_n^{n+1} + \Psi_{n-1}^- - \Psi_n^-) \right)$$

$$+ \frac{1}{4} (g^+ |\Psi_n^+|^4 + g^- |\Psi_n^-|^4) + \frac{1}{4} (G^+ |\Psi_n^+|^4 - G^- |\Psi_n^-|^4) \tag{4}$$

We now explore the stability of stationary plane-wave solutions of Eq. (3) with the following ansatz

$$\Psi_n^\pm = u^\pm \exp[\omega^\pm t + i m q^\pm] \tag{5}$$

Employing the ansatz given by Eq. (5), one can obtain the following dispersion relation

$$\omega = -2\Gamma^\pm \cos(q) + g^\pm u^2 + G^\pm u_c^2 - \frac{u^2}{u^2} \delta \pm 2 \gamma \sin(q) \tag{6}$$

Next, we slightly modify the plane wave ansatz given by Eq. (5) of the following form

$$\Psi_n^\pm = (u^+ + q^\mp) \exp[\omega^\pm t + i m q^\pm] \tag{7}$$

Subsequently, perturbed solution of the above ansatz was substituted in Eq. (3) to extract the discrete differential equation only in linear terms of \( \xi^\pm (\xi^\mp) \) is complex conjugate as

$$\frac{\partial \xi_n^\pm}{\partial t} = -\Gamma^\pm (\xi_n^{n+1} + \xi_{n-1}^\mp - 2\xi_n^\pm) \mp i \gamma (\xi_n^{n+1} - \xi_{n-1}^\mp - \xi_n^\pm$$

$$+ G^\mp (\xi_n^\pm + \xi_n^\mp) - i\gamma (\xi_n^{n+1} - \xi_{n-1}^\mp - \xi_n^\pm) \tag{7}$$

The perturbed amplitudes are taken as plane waves in discrete coordinates of the following form

$$\xi_n^+ = \psi^+ \exp[iQ \Omega + m Q^1] + \psi^- \exp[-iQ \Omega - m Q^1] \tag{8}$$

A straightforward substitution of equation (8) in (7) yields the following dispersion relation for eigenfrequency \( \Omega \) as

$$\Omega^4 + P_2 \Omega^3 + P_2 \Omega^2 + P_1 \Omega + P_0 = 0 \tag{9}$$

where

$$P_0 = [c + (c + 2G - \gamma) + A_\pm (\gamma - G) - \Lambda_\pm (G - G + \Lambda_\pm)]$$

$$[c + (c + 2G - \gamma) + A_\pm (\gamma - G) - \Lambda_\pm (\gamma - G + \Lambda_\pm)]$$

$$P_1 = 2\Lambda_\pm (c + 2G - \gamma) - \Lambda_\pm (2G - 2G) + \frac{\gamma}{2} (\gamma - 2G)]$$

$$P_2 = 2c^2 + 4cG - 2c^2 + A_\pm^2 + \Lambda_\pm^2 - 2\Lambda_\pm$$

$$P_3 = 2\Lambda_\pm (\gamma - G) \tag{10}$$

and the instability growth rate is defined as \( \xi = 2|\text{Im}(\Omega_\pm)| \).
Table 1: Review of the MI in discrete SOC-BECs at unstaggered mode

| Hopping Strength | SOC | Rabi coup. | Possible Cases | Conjecture |
|------------------|-----|------------|----------------|------------|
| $\gamma = 0$     | $\delta = 0$ | $g > 0, g_{12} > 0$ | Unstable | |
|                  |     | $g > 0, g_{12} < 0$ | Unstable | |
|                  |     | $g < 0, g_{12} > 0$ | Stable | |
|                  |     | $g < 0, g_{12} < 0$ | Stable | |
| $\Gamma = 1$     | $\delta = 1$ | $g > 0, g_{12} > 0$ | Unstable | |
|                  |     | $g > 0, g_{12} < 0$ | Unstable | |
|                  |     | $g < 0, g_{12} > 0$ | Stable | |
|                  |     | $g < 0, g_{12} < 0$ | Stable | |
| $\gamma = 1$     | $\delta = 0$ | $g > 0, g_{12} > 0$ | Unstable | |
|                  |     | $g > 0, g_{12} < 0$ | Unstable | |
|                  |     | $g < 0, g_{12} > 0$ | Stable | |
|                  |     | $g < 0, g_{12} < 0$ | Stable | |
| $\Gamma = 0$     | $\delta = 0$ | $g > 0, g_{12} > 0$ | Unstable | |
|                  |     | $g > 0, g_{12} < 0$ | Unstable | |
|                  |     | $g < 0, g_{12} > 0$ | Stable | |
|                  |     | $g < 0, g_{12} < 0$ | Stable | |

Figure 1: (color online) Upper row: Two-dimensional (2D) plot showing the MI gain as a function of $Q$ and $\gamma$. Left and right columns correspond to $\delta = 1$ and $\delta = 0$, respectively. Remaining parameters are $g = 1$, $G = -1$ and $\Gamma = 1$.

3. Results And Discussion

3.1. MI in unstaggered mode

In this section, we study the onset of MI in an unstaggered mode. It has been found that the MI gives rise to unstaggered mode at $g^+ > 0$ ($q = 0$), and staggered mode for $g^+ < 0$ ($q = \pi$). We have brought out the onset of MI domains in the absence/presence of all system parameters in Table 1.

In Fig 1, upper two panels show the MI gain as a function of $Q$ for five different values of SO coupling strengths, $\gamma = 0, 0.25, 0.5, 0.75, 1.0$. Left and right columns correspond to $\delta = 1$ and $\delta = 0$. As the strength of the SO coupling increases from 0 to 0.5, the MI gain decreases for both $\delta = 1$ and $\delta = 0$. As we increase the strength of SO coupling further, one observes no variation in the gain for $\delta = 1$. But, for $\delta = 0$, the MI gain increases for $\gamma > 0.5$. The three-dimensional (3D) surface plot clearly illustrates this behaviour in the lower panels.

3.1.1. Miscible / Immiscible condensates

Next, we study the onset of MI in Miscible / Immiscible condensates.

3D surface plot shows the existence of MI domain as a function of $Q$ and $\gamma$ for upper row: partially miscible, $(g = 1, G = -1)$, middle row: immiscible $(g = 1, G = -0.5)$ and lower row: miscible $(g = 1, G = -1.5)$ condensates.

3.2. MI in Staggered Mode

In this section, we study the onset of MI in a staggered mode. It is worth mentioning at this juncture that $q = 0$, our results completely agree with the Ref. [27], and could reproduce the results of one of the cases (in the absence of three-body interaction) discussed in Ref. [28]. These results are reproduced by our investigation.

Next, to study the impact of SO coupling on MI i.e $\gamma \neq 0$, we first discuss the MI gain for $g > 0$ and $G > 0$. To facilitate our understanding and to make the investigation straightforward, we have reviewed the possible MI domains in table

Figure 2: (color online) 3D surface plot showing the MI region as a function of $Q$ and $\gamma$ for upper row: partially miscible, $(g = 1, G = -1)$, middle row: immiscible $(g = 1, G = -0.5)$ and lower row: miscible $(g = 1, G = -1.5)$ condensates.
2. We have thoroughly brought out the MI domains in the absence/presence/ of hopping strength, SO coupling and Rabi coupling for different possible combinations of signs of intra- and inter-component interactions strength.

Table 2: Review of the MI in discrete SOC-BECs at Staggered mode

| Hopping Strength | SOC | Rabi coup. | Possible Cases | Conjecture |
|------------------|-----|------------|----------------|------------|
| $\Gamma = 1$     | $\gamma = 0$ | $\delta = 0$ | $g > 0, g_{12} > 0$ | Stable |
|                  |      |            | $g > 0, g_{12} < 0$ | Stable |
|                  |      |            | $g < 0, g_{12} > 0$ | Unstable |
|                  |      |            | $g < 0, g_{12} < 0$ | Unstable |
|                  | $\gamma = 1$ | $\delta = 1$ | $g > 0, g_{12} > 0$ | Stable |
|                  |      |            | $g > 0, g_{12} < 0$ | Unstable |
|                  |      |            | $g < 0, g_{12} > 0$ | Stable |
|                  |      |            | $g < 0, g_{12} < 0$ | Unstable |
|                  | $\Gamma = 0$ | $\gamma = 1$ | $g > 0, g_{12} > 0$ | Unstable |
|                  |      |            | $g > 0, g_{12} < 0$ | Unstable |
|                  |      |            | $g < 0, g_{12} > 0$ | Stable |
|                  |      |            | $g < 0, g_{12} < 0$ | Stable |

Figure 3: (color online) Left and right columns correspond to $\delta = 1$ and $\delta = 0$, respectively. The remaining parameters are $g = 1$, $G = 1$ and $\Gamma = 1$.

In Fig.3 upper two panels show the MI gain as a function of $Q$ for five different values of SO coupling strengths, $\gamma = 0.0, 0.25, 0.50, 0.75$, and $1.0$. Left and right columns correspond to $\delta = 1$ and $\delta = 0$. As the strength of the SO coupling increases from 0 to 0.5, the MI gain decreases for both $\delta = 1$ and $\delta = 0$. As we increase the strength of SO coupling further, one observes no variation in the gain for $\delta = 1$. But, for $\delta = 0$, the MI gain increases for $\gamma > 0.5$. The three-dimensional (3D) surface plot clearly illustrates this behaviour in the lower panels.

Figure 4: (color online) MI gain as a function of $Q$ for (top left to bottom right) (i) $g = 1$, $G = 1$, (ii) $g = 1$, $G = -1$, (iii) $g = -1$, $G = 1$ and (iv) $g = -1$, $G = -1$. Remaining parameters are $\delta = 1$ and $\Gamma = 1$.

Next, we discuss the MI domains for other three different possible combinations of intra- and inter-species fragmentation, i.e $g$ and $G$. When the inter-component interaction is repulsive
respectively. Remaining parameters are the other parameters picked from Fig. 4 (ii). Figure 5 shows the modulationally stable (MS) and modulationally unstable (MU) behaviour for 2\textsuperscript{nd} and 3\textsuperscript{rd} cases corresponding to $g > 0$, $g_{12} < 0$ and $g < 0$, $g_{12} > 0$, in Table 1. Upper and middle panels show the 3D space-time evolution of the density for both condensates (left and right) and for MS and MU cases, respectively. In the upper two panels, during the time evolution of the wave in condensates, the wave amplitudes (periodicity) exhibit small oscillations and remain close to their initial values demonstrating the stability of the system. On the contrary, in the lower two panels, the wave amplitude in the condensates grows exponentially due to their instability. The comparative evolution of the wave amplitudes in both cases is plotted in lower panel. The maximum density shows mild oscillations nearly around its initial value 1 for the MS case in both condensates (dashed blue and dash-dotted red lines). Blue and red solid lines show the exponential growth of the maximum densities for the MU mode in both the condensates.

Figure 6: The space-time evolution of the density in both the condensates (left and right). Panels from upper to lower show the density with increasing strengths of the SOC, $\gamma = 0.1$, $\gamma = 0.2$, $\gamma = 0.3$, $\gamma = 0.4$ and $\gamma = 0.5$, respectively. Remaining parameters are $g = 1.0$, $g_{12} = −1.0$ and $\delta = 1.0$ with the other parameters picked from Fig. 4(ii).

Figure 6 shows the 3D space-time evolution of the density in both the condensates for the unstable modes. The instability increases on increasing the strengths of the SOC. The 3D space-time evolution of the density remains stable up to time interval $t = 60$ with small disturbances in both the condensates (left and right) for $\gamma = 0.1$. However, when we increase the strength of the SOC to $\gamma = 0.3$, the 3D space-time evolution of the densities starts to collapse around time $t = 20$. Also, if we increase the strength of the SOC to $\gamma = 0.5$ further, the maximum densities increase exponentially with large oscillations.

### 3.2.1. Miscible / Immiscible condensates

Next, we study the MI gain in Miscible / Immiscible condensates.

Figure 7 displays the MI region in 3D surface plot as a function of $Q$ and $\gamma$ for partially miscible ($g = 1$, $G = 1$) condensates in upper row, miscible ($g = 1$, $G = 0.5$) case in middle row and immiscible ($g = 1$, $G = 1.5$) case in lower row.

Figure 7: (color online) 3D surface plot showing the MI region as a function of $Q$ and $\gamma$ for upper panel: partially miscible, ($g = 1$, $G = 1$) and lower row (left): immiscible ($g = 1$, $G = 0.5$) and (right): miscible ($g = 1$, $G = 1.5$) conditions.

Figure 8 shows the comparative evolution of the wave amplitudes in both cases discussed in Fig. 7 for miscible, immiscible and partially miscible cases. Mild and large oscillations of the maximum density illustrate MS and MU cases.

In the absence of both Rabi and SOC, the MI region exists only in the repulsive two-body inter-atomic interaction zone, as shown in Fig. 8 (a). When we add Rabi coupling alone, the MI region expands from repulsive to attractive two-body inter-atomic as well as intra-atomic interaction zone as shown in Fig. 8 (b). We also observe that in the presence of SOC and absence of Rabi coupling, the MI region exists only in repulsive two-body inter-atomic and both repulsive and attractive intra-atomic interactions zone, as displayed in Fig. 8 (c). Now, when we reinforce SOC along with Rabi coupling, the MI region exists mostly in attractive two-body inter-atomic and both...
repulsive and attractive intra-atomic interaction zone as shown in Fig.8 (d).

Further, from panels (a)-(c), where the strength of the Rabi coupling is absent, the stability states stay in the same region with small variations. However, from panels (a)-(b), where the strength of the SOC is absent, we observe that the repulsive inter-atomic region which was stable earlier becomes unstable while repulsive/attractive inter-atomic and intra-atomic regions which were unstable now shift to stable domain. All the above occurs at the behest of Rabi coupling. Hence, from Fig.8 it is pretty obvious that the impact of Rabi coupling is more pronounced compared to the other system parameters and this result underscores the dominance of Rabi coupling.

In Fig.10 panels (a) and (b) show the effect of SOC in $\langle |\Psi(x,t)|^2 \rangle$ plane. When there is no SOC, the MI regions exist only for $Q \geq 0.8$. But, in the presence of SOC, the onset of MI appears at $Q \geq 0.1$. In panel (c), we show the effect of the SOC when the hopping strength is absent ($\Gamma = 0$). In this case, there is no unstable island inside the stable region as shown in panel (b). In addition, from Fig.10 we observe MI region when $g \geq 1.7$.

4. Conclusion

We have investigated the impact of intra-site SO coupling and associated inter-component Rabi coupling on the MI of plane-wave states in two-component discrete BECs. We have brought out the existence of MI domains under the interplay between SOC, Rabi coupling, intra and inter species interactions. We have also found that the SO coupling initiates MI even for a small initial wavenumber in miscible states and also introduces MI even in the absence of hopping coefficient, a concept which may have wider ramifications in heavy atomic BECs. We have also shown how the results of linear stability analysis can be corroborated numerically.

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Figure 9: (color online) Regions of modulational stability / instability in the \((g,G)\) plane for all possible combinations of Rabi and SOCs.

Figure 10: (color online) \(Q\) Vs \(q\) for (a) \(\Gamma = 1\) and \(\gamma = 0\), (b) \(\Gamma = 1\) and \(\gamma = 1\), (c) \(\Gamma = 0\) and \(\gamma = 1\). Remaining parameters are \(g = 1\), \(G = 1\) and \(\delta = 1\).
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