Roman Domination in Convex Bipartite Graphs

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Abstract

In the Roman domination problem, an undirected simple graph $G(V, E)$ is given. The objective of Roman domination problem is to find a function $f : V \rightarrow \{0, 1, 2\}$ such that for any vertex $v \in V$ with $f(v) = 0$ must be adjacent to at least one vertex $u \in V$ with $f(u) = 2$ and $\sum_{u \in V} f(u)$, called Roman domination number, is minimized. It is already proven that the Roman domination problem (RDP) is NP-complete for general graphs and it remains NP-complete for bipartite graphs. In this paper, we propose a dynamic programming based polynomial time algorithm for RDP in convex bipartite graph.

1 Introduction

The concept of domination has a significant role in graph theory. It has many practical importance in several areas of computer science such as networking, facility location problem, wireless sensor networking, social networking, etc. Let $G(V, E)$ be an undirected graph. A set $S \subseteq V$ is said to be a dominating set, if for each vertex $v \in V \setminus S$, there exist at least one vertex $u \in S$, such that the edge $uv \in E$. The dominating set with minimum cardinality is known to be the minimum dominating set, and the corresponding cardinality is known to be the domination number, denoted by $\gamma(G)$ of the graph $G(V, E)$. The domination problem studied intensively in the literature [3, 4, 7]. Recently, researchers started exploring variations on domination to meet the requirement and demand of domination with some additional constraints in several other fields. Some of the variations on domination are Roman domination, Italian domination, perfect Roman domination, perfect Italian domination [1, 8, 9] etc.

This paper mainly focuses on Roman domination. It was first introduced by Cockayne et al. [2] and was motivated from an article, which was based on legion deployment for
better security with limited resources \[15\]. A Roman dominating function (RDF) on graph \(G(V, E)\) is defined as a function \(f : V \rightarrow \{0, 1, 2\}\) satisfying the condition that every vertex \(v\) with \(f(v) = 0\) is adjacent to at least one vertex \(u\) with \(f(u) = 2\). The weight of a RDF is the value \(f(V) = \sum_{v \in V} f(v)\). The Roman domination number (RDN) of a graph \(G\), denoted by \(\gamma_R(G)\), is the minimum weight among all possible RDFs on \(G\). In the Roman domination problem, for a given undirected simple graph \(G(V, E)\), the objective is to find a Roman domination function \(f : V \rightarrow \{0, 1, 2\}\) such that \(f(V)\) is minimized. Roman domination problem (RDP) is NP-complete for general graphs \[4\]. It is also NP-complete when restricted to bipartite graphs, split graphs, and planar graphs \[3\]. Let \(G(X, Y, E)\) be a bipartite graph. The graph \(G\) is said to be a tree-convex bipartite graph if there exist a tree \(T = (X, E)\) such that for each \(v \in Y\), the neighborhood of \(v\) induces a subtree of \(T\). See \[14\] for linear time algorithm to recognize tree-convex bipartite graph and corresponding tree construction. A graph is a star (comb) convex-bipartite graph if it is a tree-convex bipartite graph and the corresponding tree is a star (comb). A graph is a line convex-bipartite graph if it is a tree-convex-bipartite graph and the corresponding tree is a line graph. In some papers, line convex-bipartite graph is known as convex-bipartite graph. The NP-completeness of star convex bipartite graphs and comb convex bipartite graphs, can be found in \[11\]. In \[11\], authors also gave linear time algorithms for bounded treewidth graphs, chain graphs, and threshold graphs. The RDP is linear-time solvable for interval graphs and co-graphs \[10\]. In \[10\], authors also gave polynomial-time algorithms for D-octopus graphs and AT-free graphs. The RDP is also studied on circulant graphs, generalized Peterson graphs, and Cartesian product graphs \[16\]. In the literature, we observed that some problems are NP-complete for bipartite graphs, but the same problems are solvable in polynomial time in some subclasses of bipartite graphs. So, it will be interesting to see the behavior of the Roman domination problem (RDP) in different subclasses of bipartite graphs as mentioned in \[12\].

The remaining part of this paper is organized as follows. In Section 2, we introduce some relevant preliminaries along with some of the important observations and lemmas related to line convex bipartite graph and Roman domination. In Section 3, we detail the approach for finding the Roman domination function of a line convex bipartite graph and the corresponding algorithm. Finally, we conclude the paper in section 4.

## 2 Preliminaries

Let \(G(X, Y, E)\) be a bipartite graph, where \(X = \{x_1, x_2, \ldots, x_m\}\) and \(Y = \{y_1, y_2, \ldots, y_n\}\) are ordered from top to bottom. \(G(X, Y, E)\) is said to be a line convex bipartite graph if it is a tree convex bipartite graph and the corresponding tree is a line graph. In other words,
a bipartite graph $G(X, Y, E)$ is said to be a line convex over the vertices of partite set $Y$ if there exist a linear ordering of the vertices of $X$ such that for each vertex $v \in Y$, neighbors of $v$ form an interval in $X$ i.e., $N(v) = \{x_{l(v)}, x_{l(v)+1}, \ldots, x_{h(v)}\} = I(v)$, where $l(v)$ is the index associated with the lowest indexed neighbor of $v$, $h(v)$ is the index associated with the highest indexed neighbor of $v$ and $I(v)$ is the interval associated with the vertex $v$. The bipartite graph $G$ in Fig. 1(a) is a line convex bipartite graph as the vertices of the partite set $X$ can be rearranged in such a way that the $N(v)$ is an interval, for each $v \in Y$ as shown in Fig. 1(b). In this paper, whenever, we refer to line convex bipartite graph that means the graph is convex with respect to $Y$. The line convex bipartite graph is interchangeably referred to as convex bipartite graph.

![Figure 1: Line convex bipartite graph](image)

**Lemma 2.1.** Every line convex bipartite graph (with respect to partite set $Y$) can be represented in such a way that for any two vertices $y_i$ and $y_j$ in the partite set $Y$, if $i < j$, then $h(y_i) \leq h(y_j)$, where $h(y_i)$, $h(y_j)$ are the indices of the highest indexed neighbors of $y_i$ and $y_j$, respectively (See Fig. 2).

![Figure 2: Line convex bipartite graph satisfying $h(y_i) \leq h(y_j)$](image)

**Observation 2.2.** Given a graph $G$, each isolated vertex of the graph $G$ carries Roman value 1 in any optimal solution of the Roman dominating function.

**Observation 2.3.** Let $G(X, Y, E)$ be a line convex bipartite graph. If $G'(X \cup \{x_0\}, Y \cup \{y_0\}, E)$ is a bipartite graph obtained from $G$ by adding two isolated vertices $x_0$ and $y_0$, then $G'$ is also a line convex bipartite graph and $\gamma_R(G) = \gamma_R(G') - 2$. 

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Lemma 2.4. Given a line convex bipartite graph $G(X,Y,E)$ satisfying Lemma 2.1 any induced graph $G[X_i,Y_j]$, where $X_i = \{x_0, x_1, \ldots, x_i\}$, $Y_j = \{y_0, y_1, \ldots, y_j\}$, $x_0$, and $y_0$ are two isolated vertices. The vertices that appear in between the neighbors of $x_i \in X_i$ but are not the neighbors of $x_i$ are isolated.

Proof. The induced graph $G[X_i,Y_j]$ is a line convex bipartite graph with respect to $Y_j$ not necessarily with $X_i$. So all the neighbors of $x_i \in X_i$ may not be consecutive. Suppose there exist a vertex, say $u$ that appears in between the neighbors of $x_i$ and is not the neighbor of $x_i$ but is not isolated. In that case, $u \in Y_j$ must be connected (through an edge) to a vertex $v \in X_i$, where $v \neq x_i$, i.e., $uv \in E_{ij}$ as shown in Fig. 3(a). That means there exist at least one neighbor (say, $w$) of $x_i$ which lies above $u$. As $w$ is connected to $x_i$ with an edge, so the interval $I(u)$ must contain $x_i$ (see Fig. 3(b)), i.e., $u$ is also connected to $x_i$ with an edge (Lemma 2.1). Therefore, $u$ becomes one of the neighbor of $x_i$, which leads to a contradiction.

![Figure 3: (a) uv \in E_{ij}, and (b) xi \in I(u)](image)

Lemma 2.5. Given a line convex bipartite graph $G(X,Y,E)$ satisfying Lemma 2.1, where $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_n\}$. If $x_m y_n \in E$, then in an optimal solution the Roman value associated with the pair $(x_m, y_n)$ will never be $(2,1)$ or $(1,2)$.

Proof. Suppose in an optimal solution, $x_m y_n \in E$ and $R(x_m) = 2$, $R(y_n) = 1$. Now, we can always get a better solution by reassigning 0 to $y_n$, which is a contradiction. The same can be claimed for $R(x_m) = 1$, $R(y_n) = 2$ also.

Lemma 2.6. Given a line convex bipartite graph $G(X,Y,E)$ satisfying Lemma 2.1, where $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_n\}$. If $x_m y_n \notin E$, then in an optimal solution the Roman value associated with the pair $(x_m, y_n)$ will never be $(0,0)$.

Proof. Suppose in an optimal solution $x_m y_n \notin E$ and $R(x_m) = 0$, $R(y_n) = 0$, then there must exist at least one vertex $p$ in $Y[1, \ldots, n-1]$ and another vertex $q$ in $X[1, \ldots, m-1]$ which will Roman dominate $x_m$ and $y_n$, respectively. If it is so, then there must exist two edges $x_m p$ and $y_n q$, and they must cross each other. Due to Lemma 2.1, $y_n$ must have edge with each vertex starting from $q$ to $x_m$, i.e., $x_m y_n \in E$, which is a contradiction.
Lemma 2.7. Given a line convex bipartite graph \( G(X, Y, E) \) satisfying Lemma 2.1, where \( X = \{x_0, x_1, \ldots, x_m\}, Y = \{y_0, y_1, \ldots, y_n\} \). If \( x_m y_n \notin E \), then in an optimal solution the Roman value associated with the pair \((x_m, y_n)\) will never be \((0, 2)\) or \((2, 0)\).

Proof. If \( x_m y_n \notin E \), \( R(x_m) = 0 \) and \( R(y_n) = 2 \), then in an optimal solution \( y_n \) will never be isolated, otherwise the solution is not optimal. Now, since \( y_n \) is not isolated, it is connected (with an edge) to at least one vertex (say, \( q \)) in \( X[1, \ldots, m - 1] \) and dominates one/more vertices from the partite set \( X[1, \ldots, m - 1] \) but not \( x_n \) (as \( x_m y_n \notin E \)). Unlike \( y_n \), \( x_m \) should be dominated by at least one vertex (say, \( p \)) in \( Y[1, \ldots, n - 1] \) otherwise the graph is not Roman dominated. That means the edges \( x_m p \) and \( q y_n \) are crossing each other and \( p \) lies above to \( y_n \) (index associated with vertex \( p \) is less than \( n \)). Due to Lemma 2.1, \( y_n \) forms an interval including \( x_m \), i.e., \( x_m y_n \in E \), which leads to contradiction. The same can be claimed for \((2, 0)\) also. \( \square \)

Lemma 2.8. Given a line convex bipartite graph \( G(X, Y, E) \) satisfying Lemma 2.1, where \( X = \{x_0, x_1, \ldots, x_m\}, Y = \{y_0, y_1, \ldots, y_n\} \). If \( x_m y_n \notin E \), then in an optimal solution the Roman value associated with the pair \((x_m, y_n)\) will never be \((2, 2)\).

Proof. If \( R(y_n) = 2 \) and \( R(x_m) = 2 \), then \( x_m \) and \( y_n \) will never be isolated. Otherwise the solution will not be optimal. That means \( y_n \) must be dominating at least one vertex (say, \( q \)) from the partite set \( X[1, \ldots, m - 1] \) but not \( x_m \) (as \( x_m y_n \notin E \)) and similarly, \( x_m \) must be dominating at least one (say, \( p \)) from \( Y[1, \ldots, n - 1] \) but not \( y_n \), i.e., \( x_mp, qy_n \in E \). Therefore, the edges \( x_mp \) and \( qy_n \) should be crossing each other. Due to Lemma 2.1, \( y_n \) must form an interval including \( x_m \), i.e., \( x_m y_n \in E \), which leads to a contradiction. \( \square \)

### 3 Algorithm for Roman domination in convex bipartite graphs

In this section, we present a polynomial time algorithm to find an optimal Roman domination function for a given convex bipartite graph \( G(X, Y, E) \).

#### 3.1 Notations

Let \( G(X, Y, E) \) be a convex bipartite graph. Let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) such that \( x_1, x_2, \ldots, x_m \) (respectively, \( y_1, y_2, \ldots, y_n \)) arranged from top to bottom. We add two vertices \( x_0 \) and \( y_0 \) into the set \( X \) and \( Y \), respectively such that \( x_0 \) (respectively, \( y_0 \)) is above \( x_1 \) (respectively, \( y_1 \)). In this paper, the given graph is assumed to be line convex with respect to \( Y \). For \( p \in \{0, 1, 2\} \), let \( R^p_{x_i} \) assigns Roman value \( p \) to
the vertex \( x_i \), \( R(x_i) = p \) means \( x_i \) has Roman value \( p \). In this paper, \( R() \) and \( f() \) are interchangeably used. The lowest index neighbor of \( y_j \) is denoted by \( l(y_j) \), and defined by \( l(y_j) = \min\{i : x_i \in N(y_j)\} \). The highest index neighbor of \( y_j \) is denoted by \( h(y_j) \), and defined by \( h(y_j) = \max\{i : x_i \in N(y_j)\} \), where \( N(y_j) \) is the open neighborhood of \( y_j \). Let \( G[X_i, Y_j] \) be the induced subgraph consisting of vertices \( x_0, x_1, \ldots, x_i \) in the partite set \( X_i \subseteq X \) and \( y_0, y_1, \ldots, y_j \) in the partite set \( Y_j \subseteq Y \). Let \( E_{ij} \subseteq E \) be the set of edges in the induced subgraph \( G[X_i, Y_j] \). Assume \( R(i, j) \) is the optimal Roman domination number (RDN) of \( G[X_i, Y_j] \). Some other important notations related to the induced graph \( G[X_i, Y_j] \) are as follows:

\[
\gamma_{eR}^{0,0}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the pair } (x_i, y_j) \text{ is } (0, 0) \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{012,1}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the vertex } y_j \text{ is } 1 \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{02}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the pair } (x_i, y_j) \text{ is } (0, 2) \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{0102}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the vertex } x_i \text{ is } 1 \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{2,0}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the pair } (x_i, y_j) \text{ is } (2, 0) \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{22}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the pair } (x_i, y_j) \text{ is } (2, 2) \text{ and } x_iy_j \in E_{ij}.
\]

\[
\gamma_{eR}^{1012}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the vertex } x_i \text{ is } 1 \text{ and } x_iy_j \notin E_{ij}.
\]

\[
\gamma_{eR}^{012,1}(i, j) = \text{Optimum Roman domination number of } G[X_i, Y_j] \text{ when the Roman value associated with the vertex } y_j \text{ is } 1 \text{ and } x_iy_j \notin E_{ij}.
\]

### 3.2 Overlapping subproblem and optimal substructure

Each induced graph \( G[X_i, Y_j] \) is a subgraph of the given line convex bipartite graph \( G(X, Y, E) \) and the Roman domination problem (RDP) on the induced subgraphs \( G[X_i, Y_j] \) can be viewed as a subproblem of RDP on \( G(X, Y, E) \) and their minimum RDF can be used to calculate the RDF of \( G(X, Y, E) \). Given a line convex bipartite graph, the RDF corresponding to each induced graph \( G[X_i, Y_j] \) (where \( i = 0, 1, \ldots, m \) and \( j = 0, 1, \ldots, n \)) can be calculated recursively and stored for further use. As a preliminary step, the vertices of the line convex bipartite graph are reordered based on Lemma 2.1. We added 2 isolated vertices intentionally to the graph \( G(X, Y, E) \); by doing so, the convexity property is not hampered (Observation 2.3). It helps to meet the base condition.
To begin with, the pair of isolated vertices, $x_0$ and $y_0$ are taken and $G[X_0, Y_0]$ (where $X_0 = \{x_0\}$, $Y_0 = \{y_0\}$) is the first subgraph under consideration. As $x_0$ and $y_0$ are isolated vertices, so $R(x_0) = 1$ and $R(y_0) = 1$, and hence RDN of $G[X_0, Y_0]$ is 2 (Observation 2.2). Next onwards, each time a new vertex (consecutive to the previously added vertex) is added and the corresponding induced graph $G[X_i, Y_j]$ is considered. While finding the RDF of the induced graph $G[X_i, Y_j]$, the pair of vertices with highest indices in each partite set, i.e., $(x_i, y_j)$ is considered. In $G[X_i, Y_j]$, either (A) $x_i y_j \in E_{ij}$ or (B) $x_i y_j \notin E_{ij}$. Now, we consider both cases separately.

**Case A** $(x_i y_j \in E_{ij})$: We consider all possible 9 Roman domination values of $x_i$ and $y_j$ and choose the best solution.

Case 1: $R(x_i) = 0, R(y_j) = 0$

Since Roman domination values of both $x_i$ and $y_j$ are 0, $x_i$ (respectively, $y_j$) is dominated by a vertex $y_p$ (respectively, $x_q$) in the partite set $Y_j$ (respectively, $X_i$), where $p < j$ (respectively, $q < i$) as shown in Fig. 4(a). If such $p$ or $q$ does not exist, then case 1 is invalid. Here, we consider Roman value of each vertex in the pair $(u, v)$, where $u \in N(x_i)$ and $v \in N(y_j)$ equal to 2 to solve the problem. In worst case there will be $mn$ such pairs, i.e., $mn$ number of subproblems. In each subproblem Roman value 2 to $u$ will dominate all the vertices in $N(u)$ and all the vertices in $N(u)$ are consecutive, whereas Roman value 2 to $v$ will dominate all the vertices in $N(v)$ but $N(v)$ may not be consecutive (see Fig. 4(a)). So those vertices (if any) that appear in between the neighbors of $v$ need to be handled separately (see Algorithm [1]) ensuring optimality (see Theorem 3.2). The sub-problems embedded within each pair of $(u, v)$ will be different from each other due to the variation in edges in their respective open neighborhood. For a specific pair of $(u, v)$, the number of sub-problems under consideration depends on their open neighborhood.

Hence, the Roman value associated with $\gamma_{eR}^{0,0}(i, j)$ is as follows:

$$
\gamma_{eR}^{0,0}(i, j) = \min \{R_u^2 + R_v^2 + RDS_1(G[X_i, Y_j], v) + \min \{R(s, t) - |N(u) \cap \Psi_1| - |N(v) \cap \Psi_1| : l(u) - 1 \leq s < i, l(v) - 1 \leq t < j : u \in N(x_i) \setminus \{y_j\}, v \in N(y_j) \setminus \{x_i\} \}
$$

where $\Psi_1 = \{p : p \in V(G[s, t]), R(p) = 1\}$, $l(u) = \min \{a : x_a \in N(u)\}$, $l(v) = \min \{b : y_b \in N(v)\}$. $RDS_1(G[X_i, Y_j], v)$ (refer Algorithm [1]) function finds the optimal Roman cover of the vertices that are not the neighbors of $v$ but physically appear in between the neighbors of $v$ as the vertices in each partite set in $G(X, Y, E)$ follows a particular order i.e. $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$.

Let the set of vertices that appears in between the neighbors of $v$ but are not the neighbors of $v$ be $S_1$, $S_2$ be the neighbors of $S_1$ except $x_i$ (as $R(x_i) = 0$), i.e., $S_2 = N(S_1) \setminus \{x_i\}$ and $G[S]$ be the corresponding induced graph, where $S = S_1 \cup S_2$. See
Algorithm 1, which finds the optimal Roman domination of $S_1$ with respect to the induced graph $G[S]$.

**Algorithm 1** $RDS_1(G[X_i, Y_j], v)$  
**Input:** Graph $G[X_i, Y_j]$ and a vertex $v$  
**Output:** Minimum Roman domination for $S_1$ with respect to $G[S_1 \cup S_2]$.

1: $U = \{y_j : j = l(v), l(v) + 1, \ldots, h(v)\}$
2: $S_1 = U \setminus N(v)$
3: $S_2 = \{N(y_j) : y_j \in S_1\} \setminus \{x_i\}$
4: for each isolated vertex, $t \in S_1$ do
5: \hspace{1em} $R(t) = 1$; \hspace{1em} \(\triangleright\) Assigns Roman value 1 to each isolated vertex $t \in S_1$
6: \hspace{1em} $S_1 = S_1 \setminus \{t\}$
7: end for
8: $Q = \emptyset$, $I = \{I_z = N(z) : z \in S_1\}$
9: while $I \neq \emptyset$ do
10: \hspace{1em} pick an interval (say, $I_k$) with lowest finish index (say, $fIndex$)
11: \hspace{1em} remove all the intervals containing vertex $x_{fIndex}$
12: \hspace{1em} $Q = Q \cup x_{fIndex}$
13: end while
14: for each $q \in Q$ do
15: \hspace{1em} if $d(q) \geq 2$ then
16: \hspace{2em} $R(q) = 2$; \hspace{1em} \(\triangleright\) Assigns Roman value 2 to $q \in S_2$
17: \hspace{1em} else
18: \hspace{2em} $R(t) = 1$, where $tq \in G[S_1 \cup S_2]$; \hspace{1em} \(\triangleright\) Assigns Roman value 1 to $t \in S_1$
19: \hspace{1em} end if
20: end for

For each vertex $u \in S_1$ (except the isolated vertices, if any), $N(v)$ is an interval in $X$ and the intervals are already sorted in non-decreasing order with respect to their last index (Lemma 2.1). Let $I$ be the set of all such intervals, i.e., $I = \{I_z = N(z) : z \in S_1\}$ and let $fIndex$ be the index associated with the vertex where an interval ends. Lines 8–13 in Algorithm 1 find the minimum number of vertices in $S_2$ which stabs all intervals (see Theorem 3.1) and lines 14–20 assign the optimal Roman value (see Theorem 3.1).

**Theorem 3.1.** The set $Q$ in Algorithm 1 finds the minimum number of vertices in $S_2$, which stabs all intervals.

**Proof.** On contrary, suppose the vertex $x_{fIndex}$ is not in the optimum solution. Then the first interval might be stabbed by a vertex that is present either above or below $x_{fIndex}$ in the optimal solution. It can not lie below to $x_{fIndex}$ because if it is so, then the first interval is still not stabbed as $x_{fIndex}$ is the last vertex of the first interval. If it is present above to it, then it can always be replaced by $x_{fIndex}$ and is an optimal solution. Hence, the vertex $x_{fIndex}$ must be in at least one optimal solution. \(\square\)
Theorem 3.2. Algorithm 1 gives optimal Roman coverage of $S_1$ with respect to $S_1 \cup S_2$.

Proof. From Observation 2.2, each isolated vertex (if any) in $S_1$ has Roman value 1 (lines 4–7). The remaining non-isolated vertices in $S_1$ can be covered by choosing vertices (i.e., the set $Q$) from $S_2$ optimally (see Theorem 3.1). Lines 8–13 in Algorithm 1 finds the minimum number of vertices i.e., $Q$ in $S_2$ that covers all vertices in $S_1$ except the isolated vertices if any. So the optimal Roman value associated with $S_1$ can be calculated by assigning $R(q) = 2$ and $R(t) = 1$, where $d(q) \geq 2$, $q \in Q$, and $d(t) = 1$, $tq \in G[S_1 \cup S_2]$.

Lemma 3.3. The set of vertices (that are neighbors of $u$ or $v$) with Roman value 1 in subproblem $R(s, t)$ will have Roman value 0 in $\gamma_{eR}^{0,0}(i, j)$ subject to $R(u) = R(v) = 2$.

Proof. WLOG, let $p$ be a vertex with Roman value 1 in $R(s, t)$ and also a neighbor of $u$. Since $R(u) = 2$, the Roman value of $p$ is 0 (Lemma 2.5) in $\gamma_{eR}^{0,0}(i, j)$.

The terms $|N(u) \cap \Psi_1|$ and $|N(v) \cap \Psi_1|$ ensure the optimality (see Lemma 3.3) of $\gamma_{eR}^{0,0}(i, j)$ by resetting the Roman value of all such vertices that are neighbors of $u$ or $v$ with Roman value 1 in $R(s, t)$ to 0.

case 2: $R(x_i) = k$, $R(y_j) = 1$ for $k = 0, 1, 2$

In this case, $y_j$ dominates itself only as $R(y_j) = 1$, so the optimal solution of $R(i, j)$ can be directly calculated from $R(i, j - 1)$. The sub-problem $R(i, j - 1)$ decides whether the Roman value of $x_i$ is 0, 1 or 2 in the optimal solution if $R(y_j) = 1$. However, due to Lemma 2.5, $k = 2$ will never arise. Hence, the optimal Roman value for $\gamma_{eR}^{012,1}(i, j)$ can be expressed in terms of $R(i, j - 1)$ as follows:

$$\gamma_{eR}^{012,1}(i, j) = R_{y_j}^1 + R(i, j - 1)$$

case 3: $R(x_i) = 0$, $R(y_j) = 2$

Since $y_j$ has Roman value 2, it dominates all the vertices in $N(y_j) \subseteq \{x_1, x_2, \ldots, x_i\}$

Figure 4: (a) $R(x_i) = 0, R(y_j) = 0$, and (b) $R(x_i) = 0, R(y_j) = 2$
as shown in Fig. 3(b) and all vertices in \( N(y_j) \) will be consecutive due to convexity of \( G(X_i, Y_j, E_{ij}) \). The neighbors of \( y_j \) may have Roman values 0 or 2 but not 1 (see Lemma 2.5). We set Roman value 0 instead of 1 to those vertices in the subproblem \( R(r, j - 1) \) with Roman value 1, where \( l(y_j) \leq r < i - 1 \) and are the neighbors of \( N(y_j) \). Hence, the term \( |N(y_j) \cap \Psi_2| \) is subtracted from \( R(r, j - 1) \), where \( \Psi_2 = \{ u : u \in V(G[r, j - 1]) , R(u) = 1 \} \). Thus, in this case \( \gamma_{eR}^{0.2}(i, j) \) can be expressed as follows:

\[
\gamma_{eR}^{0.2}(i, j) = \min \{ R_{y_j}^2 + R(r, j - 1) - |N(y_j) \cap \Psi_2| : l(y_j) - 1 \leq r < i - 1 \}
\]

where \( l(y_j) = \min \{ s : x_a \in N(y_j) \} \)

Case 4: \( R(x_i) = 1, R(y_j) = k \) for \( k = 0, 1, 2 \):

In this case, \( x_i \) dominates itself only as \( R(x_i) = 1 \). The optimal solution of \( R(i, j) \) can be calculated directly from \( R(i - 1, j) \). The solution of the sub-problem \( R(i - 1, j) \) sets the Roman value of \( y_j \) to either 0, 1 or 2 in the optimal solution. However, due to Lemma 2.5, \( k = 2 \) will never arise. Hence, \( \gamma_{eR}^{1.012}(i, j) \) can be expressed as:

\[
\gamma_{eR}^{1.012}(i, j) = R_{x_i}^1 + R(i - 1, j)
\]

Figure 5: (a) \( R(x_i) = 2, R(y_j) = 0 \), and (b) \( R(x_i) = 2, R(y_j) = 2 \)

Case 5: \( R(x_i) = 2, R(y_j) = 0 \)

Since \( x_i \) has Roman value 2, it covers all the vertices \( N(x_i) \subseteq \{ y_1, y_2, \ldots, y_j \} \) including \( y_j \) as shown in Fig. 5(a). Since, \( x_i \) may not follow convexity, so the vertices (if any) that appear in between the neighbors but are not the neighbors of \( x_i \) are isolated (Lemma 2.4) in the induced graph \( G[X_i, Y_j] \). The explanation for the term \( |N(x_i) \cap \Psi_3| \) is similar to case 3, where \( \Psi_3 = \{ u : u \in V(G[r - 1, w]) , R(u) = 1 \} \). Hence, \( \gamma_{eR}^{2.0}(i, j) \) can be expressed in terms of \( R(i - 1, w) \) as:

\[
\gamma_{eR}^{2.0}(i, j) = \min \{ R_{x_i}^2 + AssignR1(x_i) + \min \{ R(i - 1, w) - |N(x_i) \cap \Psi_3| : l(x_i) - 1 \leq w < j \} \}
\]

where \( l(x_i) = \min \{ s : x_b \in N(x_i) \} \), \( AssignR1(x_i) \) assigns Roman value 1 to the isolated
vertices that fall in between the neighbors of \( x_i \) but are not the neighbors of \( x_i \).

**Case 6:** \( R(X_i) = 2, R(Y_j) = 2 \)

In this case, \( x_i \) and \( y_j \) will cover all their respective neighbors as shown in Fig. 5(b). Now, there may exist some isolated vertices in between the neighbors of \( x_i \) (Lemma 2.4). Assign \( R_1(x_i) \) assigns Roman value 1 to those vertices. The explanation for the subtracted terms \( |N(x_i) \cap \Psi_4| \) and \( |N(y_j) \cap \Psi_4| \) is similar to Case 1, where \( \Psi_4 = \{p : p \in V(G[J, g]), R(p) = 1\} \). In this case, \( \gamma_R^{2.2}(i, j) \) can be expressed as follows:

\[
\gamma_R^{2.2}(i, j) = R_{x_i}^2 + R_{y_j}^2 + \text{Assign}R_1(x_i) + \min\{R(f, g) - |N(x_i) \cap \Psi_4| - |N(y_j) \cap \Psi_4| : l(y_j) - 1 \leq f < i, l(x_i) - 1 \leq g < j\}
\]

where Assign \( R_1(x_i) \) assigns Roman value 1 to the isolated vertices that fall in between the neighbors of \( x_i \) but are not the neighbors of \( x_i \).

**Case B \((x_i, y_j \notin E)\):** The possible Roman function assignments from the set \( \{0, 1, 2\} \) to \( x_i \) and \( y_j \) may be one of the followings:

- **Case 1:** \( R(x_i) = 0, R(y_j) = k \) for \( k = 0, 2 \):
  - This case will never arise due to Lemma 2.6 when \( k = 0 \) and Lemma 2.7 when \( k = 2 \).
- **Case 2:** \( R(x_i) = k, R(y_j) = 1 \) for \( k = 0, 1, 2 \):
  - This case can be handled similar to Case A2: \( \gamma_R^{012.1}(i, j) = R_{y_j}^1 + R(i, j - 1) \).
- **Case 3:** \( R(x_i) = 1, R(y_j) = k \) for \( k = 0, 1, 2 \):
  - This case can be handled similar to Case A4: \( \gamma_R^{1012}(i, j) = R_{x_i}^1 + R(i - 1, j) \).
- **Case 4:** \( R(x_i) = 2, R(y_j) = k \) for \( k = 0, 2 \):
  - This case will never arise due to Lemma 2.7 when \( k = 0 \) and Lemma 2.8 when \( k = 2 \).

Hence, by observing all the cases, we have the following recursive equation:

\[
R(i, j) = \begin{cases} 
2, & \text{if } i, j = 0 \\
\min\{\gamma_R^{0.0}(i, j), \gamma_R^{012.1}(i, j), \gamma_R^{0.2}(i, j), \gamma_R^{1012}(i, j)\}, & \text{else if } x_i, y_j \in E \\
\gamma_R^{2.0}(i, j), \gamma_R^{2.2}(i, j)\}, & \text{otherwise}
\end{cases}
\]

### 3.3 Algorithm

This algorithm \( MRDN-ConBipGraph \) (see Algorithm 2) finds the minimum Roman domination number of a line convex bipartite graph \( G(X, Y, E) \), where \( X = \{x_1, x_2, \ldots, x_m\} \), \( Y = \{y_1, y_2, \ldots, y_n\} \), \( |X| = m \) and \( |Y| = n \), with the assumption that the graph \( G(X, Y, E) \) satisfies Lemma 2.1.
Algorithm 2 \textit{MRDN-ConBipGraph}

\textbf{Input:} \( G(X, Y, E) : B[1, 2, \ldots, m][1, 2, \ldots, n] \)

\textbf{Output:} Roman domination number, \( \gamma_R(G) \)

1. \( R[0, \ldots, m][0, \ldots, n] \) is a matrix of size \((m + 1) \times (n + 1)\)
2. \( R[0, 0] = 2 \) \quad \( \triangleright R(x_0) = 1, \ R(y_0) = 1 \)
3. \textbf{for} \( i = 1 \) to \( m \) \textbf{do}
   4. \( R[i, 0] = 2 + i \) \quad \( \triangleright \) Initializes \( 0^{th} \) row by assigning Roman value 1 to each vertex, since each induced graph is a empty graph
5. \textbf{end for}
6. \textbf{for} \( j = 1 \) to \( n \) \textbf{do}
7. \( R[0, j] = 2 + j \) \quad \( \triangleright \) Initializes \( 0^{th} \) column by assigning Roman value 1 to each vertex, since each induced graph is a empty graph
8. \textbf{end for}
9. \textbf{for} \( i = 1 \) to \( m \) \textbf{do}
10. \textbf{for} \( j = 1 \) to \( n \) \textbf{do}
11. \quad \textbf{if} \( B[i, j] = 1 \) \textbf{then}
12. \quad \quad \( R[i, j] = \min \{ \gamma_{eR}^{0,0}(i, j), \gamma_{eR}^{0,12}(i, j), \gamma_{eR}^{0,2}(i, j), \gamma_{eR}^{1,012}(i, j), \gamma_{eR}^{1,2}(i, j), \gamma_{eR}^{2,0}(i, j), \gamma_{eR}^{2,2}(i, j) \} \)
13. \quad \textbf{else}
14. \quad \quad \( R[i, j] = \min \{ \gamma_{eR}^{1,012}(i, j), \gamma_{eR}^{012,1}(i, j) \} \)
15. \quad \textbf{end if}
16. \textbf{end for}
17. \textbf{end for}
18. \textbf{return} \( \gamma_R(G) = R[m, n] - 2 \);
Lemma 3.4. The time complexity of Algorithm 2 is polynomial.

Proof. The complexity of MRDN-ConBipGraph (Algorithm 2) is primarily dominated by the nested for loop (lines 9 – 17). Each term used in line 14 and 15 can be calculated in polynomial time. Hence, Algorithm 2 runs in polynomial time.

Theorem 3.5. The Roman domination number obtained from MRDN-ConBipGraph for the graph $G(X,Y,E)$ is an optimal solution.

Proof. We prove the theorem using induction on number of vertices. Given a convex bipartite graph, $G(X,Y,E)$ satisfying Lemma 2.1, where $V = X \cup Y$ is the set of vertices with $|X| = m$, $|Y| = n$ and $E$ as the set of edges. The graph obtained by adding two isolated vertices $x_0$ and $y_0$ to each partite set of $G(X,Y,E)$ is also a convex bipartite graph (Observation 2.3), say $G'(X', Y', E')$ with $|X'| = m + 1$, $|Y'| = n + 1$ and $E' = E$. While proving the theorem, we have used the lexicographic ordering\(^1\). Base case: $R(0,0)$: $x_0$ and $y_0$ are two isolated vertices, hence their optimal Roman values are 1 (Observation 2.2), i.e., $R(x_0) = 1$ and $R(y_0) = 1$. Hence, $R(0,0) = 2$ and is optimal. Inductive step: Let $R(p,q) = OPT_{p,q}$ be the optimal RDN of $G[X_p, Y_q]$, then we can always find the optimal solution for $R(p',q')$, where $(p,q) < (p',q')$ in lexicographic ordering and $(p',q') \leq (m,n)$. Here, we are considering all possible Roman values of $x'_p$ and $y'_q$, where $p' \leq m$ and $q' \leq n$ and find optimal solution in each of the cases (see Section 3.2). Finally, we are choosing best solution. Hence, the theorem follows.

4 Conclusion

Here, we have considered Roman domination problem in line convex bipartite graph and proposed a polynomial time algorithm for it. There exist some other subclasses of bipartite graph such as circular convex bipartite graphs, chordal bipartite graphs, triad convex bipartite graphs etc. for which the status of Roman domination is still unknown, and it will be interesting to see whether poly-time algorithms exist for these graphs or not.

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\(^1\)Lexicographic ordering: The ordered pair $(p,q)$ is less than or equal to $(r,s)$ if either $p < r$, or $p = r$ and $q < s$. 
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