Should Metric Signature Matter in Clifford Algebra Formulations of Physical Theories? *

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Abstract

Standard formulation is unable to distinguish between the (+ + + −) and (− − − +) spacetime metric signatures. However, the Clifford algebras associated with each are inequivalent, \( \mathbb{R}(4) \) in the first case (real 4 by 4 matrices), \( \mathbb{H}(2) \) in the latter (quaternionic 2 by 2). Multivector reformulations of Dirac theory by various authors look quite inequivalent pending the algebra assumed. It is not clear if this is mere artifact, or if there is a right/wrong choice as to which one describes reality. However, recently it has been shown that one can map from one signature to the other using a tilt transformation. The broader question is that if the universe is signature blind, then perhaps a complete theory should be manifestly tilt covariant. A generalized multivector wave equation is proposed which is fully signature invariant in form, because it includes all the components of the algebra in the wavefunction (instead of restricting it to half) as well as all the possibilities for interaction terms.

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I. Introduction

It is a well-known fact that physical predictions of relativity are independent of the absolute choice of (+ + + −) vs. (− − − +) metric signatures. If this is generally true for all physical laws, it would conversely follow that no physical measurement can determine the absolute metric signature. This is akin to the Einstein’s cosmological metaprinciple: that one cannot determine an absolute inertial reference frame of the universe. The truth of such broad statements cannot be deduced or derived from the accepted principles of physics, anymore than one can “derive” Einstein’s weak equivalence principle (the requirement that theories be locally Lorentz covariant). The laws which all the laws of physics must obey, called metalaws, must be imposed at the onset.

Many of the current treatments have been concerned with the possibility of the sign of the signature changing from one place to another[1] or even changing from Lorentzian to Euclidean[2]. At minimum this raises various topological issues at the boundaries between the regions, and perhaps some exotic physics such as pair production[3]. However, in our opinion it is premature to argue for physical interpretation of phenomena at the local discontinuities of metric change if the broader issues have not first been addressed. Specifically we should be asking if a new metalaw should be stated: Physical laws must be globally invariant under the point transformation $g_{\mu\nu} \rightarrow -g_{\mu\nu}$. For example, in section II we consider the behavior of non-relativistic classical electrodynamics under the change of 3D metric signature: (+ + +) $\rightarrow$ (− − −). The observable physics appears to be completely signature blind, however the Gibbs vector algebra (n.b. the curl in Maxwell’s equations) is not well suited to this type of abuse. The alternate mathematical language of Clifford Geometric Algebra[4] is better behaved, allowing for a formulation of Maxwell’s equations that is completely signature invariant in form. In section III however we find that Quantum Mechanics cannot be put into signature invariant form. Specifically the expectation values of the physical observables cannot be made invariant unless the wavefunction is allowed to change in form, as well as the wave equation and Dirac spin algebra.

In contrast to standard formulation, there has been a plethora of claims and results attributed to the absolute choice of metric signature in Clifford algebra “Multivector” formulations of physical theories. Hestenes[5] in particular champions the (− − − +) choice while others[6-7] use the complement (+ + + −). It is a fact that the algebraic structure of the Clifford algebra for the two different signatures is completely inequivalent. It follows that physical theories formulated with Clifford algebra are therefore potentially inequivalent pending the underlying choice of signature. In section IV we consider the dilemma: Are the multivectorial formulations of Dirac theory based upon the different metric signatures inequivalent in their physical description hence there is a testable absolute right or wrong one to match to reality? Or is there a metalaw of signature invariance, which restricts the form of both such that the differences are mere artifacts with no physically realizable results? If it is indeed the latter, what impact does this have on theories which hope to do ‘new physics’ based on using Clifford algebra?
II. Metric Signature in Classical Electrodynamics

Classical physics can be formulated within the language of vector (more generally tensor) algebra/calculus. These mathematical systems have an explicit (or at least implicit) dependence upon the “metric” associated with the geometry of the underlying physical space. In this section we ask: Can classical electrodynamics be formulated such that it is independent of the “signature” (absolute sign) of the metric? First we need to investigate which mathematical structures are preserved under the map which inverts the sign of the signature. We find in particular that the Gibbs Curl is not well behaved. Hence for example, Faraday’s law cannot be cast in metric signature invariant form using Gibbs algebra. However, the 3D Clifford algebra formulation allows for a signature invariant equation, suggesting that this language is better suited to express physical theories within which signature invariance is desired.

A. Algebra and Signature

Given a set of basis vectors $e_i$, the metric is defined in terms of the inner “dot” product,

$$g_{ij} = e_i \cdot e_j = e_j \cdot e_i. \quad (1a)$$

Under a change of metric signature: $g_{ij} \to -g_{ij}$, it is presumed that coordinates, differentials $dx^j$ and physical observables (e.g. charge density $\rho$) will not change. The dot product of two physical vectors (e.g. electric and magnetic fields) will transform: $E \cdot B \to -E \cdot B$, while the gradient: $\nabla = e_j g^{jk} \partial_k \to -\nabla$ must acquire a sign change. Putting it together, Gauss’ law: $\nabla \cdot E = \rho$, is invariant under change of signature, hence we call it signature form invariant. In contrast, Faraday’s law expressed in Gibbs vectors: $\nabla \times E = -\partial_t B$ is not signature invariant, an anomalous change of sign will appear on the left side, but not the right. The potential equations are even more problematic when Gibbs vectors are used,

$$E = -\nabla V + \partial_t A, \quad (1b)$$

$$B = \nabla \times A. \quad (1c)$$

In order to keep the electric field unchanged, we must have $A$ invariant (and $V \to -V$), but then the magnetic field would not be invariant.

The definition of a Clifford algebra is,

$$g_{ij} = \frac{1}{2} \{e_i, e_j\} = e_i \cdot e_j. \quad (2a)$$

In this paper we will always restrict ourselves to $g_{ij} = \pm \delta_{ij}$, an orthonormal basis. Hence $e_i e_j = -e_j e_i = e_i \wedge e_j$ for $i \neq j$, so we can just leave out the $\wedge$ in products of basis elements and use the compact notation $e_{12} = e_1 e_2$. In 3D space, the products of the 3 basis vector generators create the full 8 element basis for the algebra: $\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\}$. Note the unit trivector $I = e_{123} = e_1 e_2 e_3$ commutes with all the elements. In usual Euclidean space
we have $g_{ij} = \delta_{ij}$, abbreviated as $(+++)\ signature$. The associated Clifford algebra is $\mathcal{C}(2) = \text{End} \mathbb{R}^{3,0}$, commonly known as the Pauli algebra of 2 by 2 complex matrices. It is completely inequivalent to the $\mathcal{H} = \text{End} \mathbb{R}^{0,3}$ (block diagonal quaternionic matrices) associated with the $(-+-)$ signature\[14]. In particular, in $(+++)\), the unit trivector $I^2 = -1$, hence behaves like the usual abstract $i$, while in the $(-+-)$ signature $I^2 = +1$ and no element plays the role of $i$. Classical physics can apparently be expressed in either metric, possibly because both have the same bivector subalgebra hence quadratic forms in each are invariant under the rotation symmetry group $O(3)$.

Although the algebras are inequivalent, it is possible for algebraic formulas to still be invariant in form. For example, eq. (2a) as well as the wedge product of two vectors: $a \wedge b = \frac{1}{2}[a, b]$ are valid in both $(+++)\) and $(-+-)$. To make general statements about signature form invariance we need the algebraic map associated with the replacement $g_{ij} \rightarrow -g_{ij}$. It is a common trick to let $e_j \rightarrow ie_j$, such that $e_j \cdot e_k \rightarrow -e_j \cdot e_k$, however this is “cheating” as there is no $i$ in the $(-+-)$ algebra. In mapping from $\mathcal{C}(2) \rightarrow \mathcal{H}$ we use the tilt transformation introduced by Lounesto\[8],

$$ab \rightarrow b_oe_ea_e + b_oa_e - b_oe_o,$$

(2b)

where the subscripts ‘$e$’ and ‘$o$’ refer to the even and odd parts respectively. Under eq. (2b) the Clifford definition of the dot product of two vectors: $a \cdot b = \frac{1}{2}[a, b]$ will transform: $a \cdot b \rightarrow -a \cdot b$ as desired. The norms of the odd elements will change sign: $I^2 \rightarrow -I^2, e_j^2 \rightarrow -e_j^2$, while the even elements (e.g. bivectors) will not. While the basis elements such as $I = e_{123}$ map unchanged in form, the form of the relation between a bivector and the dual of a vector (in 3D) changes: $Ia \rightarrow -Ia$. One can now construct a nonstandard definition of the Gibbs cross product which is metric signature form invariant:

$$C = A \times B = -I A \wedge B.$$

(2c)

The Clifford algebra analogies of Faraday’s law and eq. (1b),

$$\nabla \wedge E = -I \partial_t B,$$

(2d)

$$IB = \nabla \wedge A,$$

(2e)

are completely signature invariant whereas the Gibb’s vector forms were not! Apparently Clifford algebra is a better language in which to express principles of physics which should be metric signature invariant.

**B. Nonrelativistic Electrodynamics**

What conditions would insure a metric signature invariant formulation of non-relativistic electrodynamics? Let us review how the metaprinciple of isotropy (no preferred direction) is imposed. Equations of motion are derived from a Lagrangian (via the principle of least action). It is sufficient to require the
Lagrangian to be invariant under the rotation group $O(3)$. The equations of motion derived from such a Lagrangian will be rotationally form invariant.

As an example, let us consider the Lagrangian for a non-relativistic charged particle in a classical electromagnetic field, 

$$\mathcal{L} = \frac{1}{2} m v^k v_j g_{jk} + ev^j A^k g_{jk} - eV + \frac{1}{2} (E^j E^k - B^j B^k) g_{jk}, \quad (3a)$$

$$E^k = -g^{kj} \partial_j V + \partial_k A^j, \quad (3b)$$

$$B^k = g g_{jn} \epsilon^{mjk} \partial_m A^n, \quad (3c)$$

where the factor of $g = \det(g_{jn})$ in eq. (3c) is needed to overcome the problems with the Gibbs curl mentioned in the previous section in eq. (1b). Since the terms are all quadratic in the vectors, the Lagrangian is invariant under the rotation group $O(3)$.

Now lets consider if this Lagrangian is invariant under the signature transformation. Under the replacement of $g_{jk} \rightarrow -g_{jk}$ we see that we must also reflect the scalar potential $V \rightarrow -V$ in order for the electric field eq. (3b) to be invariant. The Lagrangian is not really invariant, but transforms $\mathcal{L} \rightarrow -\mathcal{L}$, acquiring an overall minus sign which does not change the equations of motion. Hence the Lagrangian eq. (3a) will generate a signature invariant form of electrodynamics as desired.

**C. Relativistic Electrodynamics**

In relativity, the metaprinciple of isotropy is generalized to include time as the fourth dimension. Either $(+++-)$ or $(-+-+)$ signatures can be used; apparently neither special nor general relativity can distinguish between the two signatures. Perhaps this is because relativity is only concerned with the group structure $SL(2,C) = SO(1,3) = SO(3,1)$, not from where the group derives\[12\].

In the $(+++-)$ signature, the proper time is defined,

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu, \quad (4)$$

which is a scalar, the same value in all reference frames connected by a Lorentz transformation. The four-velocity is defined in terms of this invariant,

$$u^\alpha = \frac{dx^\alpha}{d\tau}, \quad (5a)$$

$$u^\alpha u^\beta g_{\alpha\beta} = -1. \quad (5b)$$

Under the transformation $g_{\mu\nu} \rightarrow -g_{\mu\nu}$, eq. (4) is not invariant, unless we propose: $d\tau \rightarrow id\tau$. We are perhaps cheating abit here by introducing an abstract $i$, but it will completely vanish in the end. It follows that the four-velocity definition transforms: $u^\alpha \rightarrow -iu^\alpha$, such that eq. (5b) is "signature form invariant".
In the Lorentz gauge, Maxwell’s equations in the $(+++\ldots)$ signature become,

\[ g^{\alpha\beta} \partial_\alpha \partial_\beta A^\nu = J^\nu. \tag{6a} \]

If we wish the current to be invariant under the signature change \( g_{\mu\nu} \to -g_{\mu\nu} \), then the potential must transform: \( A^\mu \to -A^\mu \), which is different than we had for the non-relativistic case. Consider now the relativistic electrodynamic Lagrangian in the $(+++\ldots)$ signature,

\[ \mathcal{L} = m \sqrt{-g_{\alpha\beta} u^\alpha u^\beta} + eA^\alpha u^\beta g_{\alpha\beta}. \tag{6b} \]

Under the signature change, the Lagrangian will transform: \( \mathcal{L} \to -i\mathcal{L} \). The factor of \( i \) is a surprise, however since the proper time transforms \( d\tau \to id\tau \), the action \( A = \int \mathcal{L} d\tau \) will be invariant. Again, signature invariant equations of motion will be generated.

### III. Spin Space and Spin Metrics

Quantum physics requires the introduction of a new space associated with the spin degrees of freedom of a wavefunction. In order for physical observables to be invariant under the change of signature of real space, it is found that spin space must transform non-trivially. Further it is found that Dirac theory cannot be written in signature invariant form within standard formalism.

#### A. Dirac Algebra

Historically, the Dirac equation was derived by factoring the Klein-Gordon operator. In the $(+++-)$ signature,

\[ (\Box^2 - m^2) = (\Box - m)(\Box + m), \tag{7a} \]

where \( \Box = \gamma^\mu \partial_\mu \) and \( \Box^2 = g^{\mu\nu} \partial_\mu \partial_\nu \) is the D'Alembertian. The Dirac matrices \( \gamma^\mu \) are related to the metric,

\[ 2g^{\mu\nu} = \{ \gamma^\mu, \gamma^\nu \} = \gamma^{\mu A} B_A \gamma^B, \tag{7b} \]

where the indices “A” and “B” of \( \gamma^{\mu A}_B \) refer to the row and column of the matrix \( \gamma^\mu \).

Standard Dirac algebra is \( \mathbb{C}(4) \), meaning \( 4 \times 4 \) complex matrices. This corresponds to a 5D Clifford algebra where \( i = \gamma^{12345} = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 \) is the unit 5-volume. This algebra admits three possible 5D metric signatures: \((-+--), (++-+)\) and \((-+-++)\). The latter two of these show that both 4D signatures of \((+++\ldots)\) and \((-+-++)\) are contained as subalgebras of \( \mathbb{C}(4) \). Choosing one 4D metric signature over the other is simply taking a different “slice” of global 5D space.

The 5D signature change is really perhaps only a reshuffling of \((+++\ldots) \to (-+-++)\) which is all still in the same \( \mathbb{C}(4) \) algebra. According to eq. (7b),
under a signature change \( g_{\mu\nu} \rightarrow -g_{\mu\nu} \) we must have something like \( \gamma^\mu \rightarrow i \gamma^\mu \) (except \( \gamma^5 \) is invariant). This is actually a 5D duality transformation, trading vectors for their dual quadvectors. Perhaps the metaprinciple at work here is something like: The laws of physics are invariant under a global 5D duality transformation.

B. Wavefunctions

It is easy to show that wavefunctions cannot be invariant under the signature change. In order for the momentum density,

\[
p^\mu = -\frac{i\hbar}{2}(\Psi^\dagger(\partial_\nu \Psi) - (\partial_\nu \Psi^\dagger)\Psi)g^{\mu\nu},
\]

(8a)
to be invariant under \( g^{\mu\nu} \rightarrow -g^{\mu\nu} \), we must have something like \( \Psi \rightarrow \Psi^* \). The situation is more complicated when spin degrees of freedom are included. The solution \( \psi^A \) to the Dirac equation is a 4 complex component column bispinor. In the \((+++)\) signature the observable Dirac current is bar invariant,

\[
j^\mu = i\overline{\Psi} \gamma^\mu \Psi = \Psi^\dagger \gamma^4 \gamma^\mu \Psi,
\]

(8b)
where \( \overline{\gamma} = -\gamma \) and \( \overline{\Psi} = \Psi^\dagger i\gamma^4 \) in the standard matrix representation. Under the signature change \( g_{\mu\nu} \rightarrow -g_{\mu\nu} \), the current \( j^\mu \) should be invariant, and remain bar invariant. In the \((-+++\) signature the current should have the form: \( j^\mu = \overline{\Psi} \gamma^\mu \Psi \) where, contrary to the sensibilities of algebraists, the definition of the bar is different: \( \overline{\gamma} = +\gamma \) and \( \overline{\Psi} = \Psi^\dagger \gamma^4 \). The problem is to choose the map for \( \Psi \) such that eq. (8a) will be signature invariant, as well as the Dirac current eq. (8b). One possibility is,

\[
\Psi \rightarrow i\gamma^2 \Psi^*,
\]

(9)
which corresponds to the charge conjugation operator. Note that under eq. (9) the norm transforms,

\[
\overline{\Psi} \Psi = |\psi^1|^2 + |\psi^2|^2 - |\psi^3|^2 - |\psi^4|^2 \rightarrow -\overline{\Psi} \Psi.
\]

(10)
A possible physical interpretation would be that there is a connection between signature change and charge conjugation symmetries.

C. Wave Equation

It is immediately clear that one cannot write a signature invariant Klein-Gordon equation. Consider that eq. (7a) in the opposite signature of \((-+++\) has the form,

\[
(\Box^2 + m^2) = (\Box + i m)(\Box - i m),
\]

(11)
which differs from eq. (7a) by the factors of \( i \). Since the Dirac equation is based on this factorization, it follows that one cannot write a signature invariant form of the Dirac equation (nor a Lagrangian) in standard notation.
IV. Multivector Quantum Mechanics and Signature

Multivector Quantum Mechanics formulates wave equations entirely within the real Clifford algebra of 4D spacetime. There is no spin space separate from real space; a spinor wavefunction is now represented by a multivector (an aggregate of scalar, vector, bivector, etc.). Under a signature change, the wave function must hence transform by the same rule eq. (2b) as the underlying geometry of spacetime; it cannot have its own rule like eq. (9). Unlike standard theory, one can now formulate a signature invariant Dirac-like equation under certain restrictions. Further, under the tilt transformation the generalized multivector equation suggests a new interchange symmetry between spin and isospin.

A. The Algebraic Dirac Equation

The Clifford algebra associated with (+ + + −) signature is: \( \mathbf{R}(4) \), isomorphic to real 4 \( \times \) 4 matrices\(^{14} \), otherwise known as the Majorana algebra. It is inequivalent to the \( \mathbf{H}(2) \) algebra (2 \( \times \) 2 matrices with quaternionic entries) associated with metric (− − − +). While both algebras have 16 basis elements, they have very different properties and substructures. In the \( \mathbf{R}(4) \) algebra there are ten elements: \( \mathbf{R}_+ = \{1, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_{41}, \gamma_{42}, \gamma_{43}, \gamma_{412}, \gamma_{423}, \gamma_{431}\} \) for which if \( \Gamma \in \mathbf{R}_+ \) then \( \Gamma^2 = +1 \), while for \( \Gamma \in \mathbf{R}_- = \{\gamma_4, \gamma_{123}, \gamma_{1234}, \gamma_{12}, \gamma_{23}, \gamma_{31}\} \) we have \( \Gamma^2 = -1 \). In the \( \mathbf{H}(2) \) algebra however we have a set of 6 elements with positive squares: \( \mathbf{H}_+ = \{\gamma_1, \gamma_4, \gamma_{14}, \gamma_{24}, \gamma_{34}, \gamma_{123}\} \) and negative squares for the remaining 10 elements: \( \mathbf{H}_- = \{\gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23}, \gamma_{31}, \gamma_{412}, \gamma_{423}, \gamma_{431}, \gamma_{1234}\} \). Nevertheless, the even subalgebra of both of these algebras contain the same (special) Lorentz group \( \mathrm{SL}(2, \mathbb{C}) \).

Clifford algebras associated with even dimensional spaces (such as 4D) always lack a global commuting \( i \), regardless of metric signature. Hence one cannot do the factorization of eq. (11) within the Clifford algebra associated with (− − − +) space. This has led many authors to argue that the (+ + + −) signature is the only “correct” model of spacetime because the factorization of eq. (7a) is possible. In this viewpoint, there is an absolute signature of the universe. However, one of the new freedoms we have in these theories is the ability (indeed at times necessity) of dextrad multiplication\(^{9} \), i.e. operation on the right side of the wavefunction. Regardless of the metric signature, consider the generalized form,

\[
\Box \Psi = m \Psi \Gamma, \tag{12a}
\]

\[
\Psi = \Box \phi + m \phi \Gamma, \tag{12b}
\]

where \( \Gamma \) is some basis element of the Clifford algebra. Substituting eq. (12b) into eq. (12a), we recover the Klein-Gordon equation,

\[
\Box^2 \phi = m^2 \phi \Gamma^2. \tag{12c}
\]
For the $(++-)$ signature choose $\Gamma \in \mathbb{R}_+$. In the case $(--+)$ choose $\Gamma \in \mathbb{H}_-$. In both situations there are 10 choices. One can only construct a signature invariant formulation by choosing one of the 6 elements which is the intersection of these two dissimilar sets: $\Gamma \in \mathbb{R}_+ \cap \mathbb{H}_- = \{\gamma_1, \gamma_2, \gamma_3, \gamma_{412}, \gamma_{423}, \gamma_{431}\}$, all which are odd geometry.

The inclusion of electromagnetic interactions requires some new element $\Lambda^2 = -1$ to play the role of the $U(1)$ gauge generator. Since no element in 4D will commute with all four $\gamma_\mu$, this element must be applied “dextrally” (on the right side). The generalization of eq. (12a) is,

$$\Box \Psi - m \Psi \Gamma - e A^\mu \gamma_\mu \Psi \Lambda = 0,$$

(eq. 13a)

$$[\Lambda, \Gamma] = 0.$$  \hspace{1cm} (eq. 13b)

Although constrained by eq. (13b), the choice of $\Lambda$ is not unique. It is up to the author to show that given his preselection of metric signature, his choice of $\Gamma$ and $\Lambda$ will allow for a wavefunction solution with 8 degrees of freedom which is isomorphic to standard Dirac theory (e.g. has proper parity, charge conjugation and Lorentz transformation properties). In the $(--+)$ signature, Hestenes favors $\Lambda = \gamma_{12}$ with $\Gamma = \gamma_{412}$. Lounesto points out that the same choices will work in the opposite tilted $(+++)$ signature. However, as we shall see in the next section, it does not necessarily follow that the formulation is signature form covariant under the tilt transformation eq. (2b).

**B. Weak Tilt Covariance and the Dirac Equation**

Unfortunately, there is no general agreement on the proper generalization of a Lagrangian for multivector quantum mechanics. We choose a form which reduces to the standard in the $(++-)$ signature for $\Gamma = 1$ and is also generally Dirac-bar invariant,

$$L = \overline{\Psi} (\Box \Psi) - (\overline{\nabla} \Box \Psi - m \overline{\Psi} \Gamma, \Gamma) + \{\overline{\nabla} \gamma^\mu \Psi, \Lambda\} A_\mu.$$  \hspace{1cm} (14)

The constraint $\overline{L} = L$ insures invariance under the Lorentz group, at the cost of requiring $\overline{\Gamma} = +\Gamma$ and $\overline{\Lambda} = -\Lambda$. In either metric signature this restricts $\Gamma$ to 4 choices; if one further insists on signature invariance it limits one to $\Gamma \in \{\gamma_{412}, \gamma_{423}, \gamma_{431}\}$. Having made a selection for $\Gamma$, the choice of $\Lambda$ is fixed, e.g. if $\Gamma = \gamma_{412}$ one must have $\Lambda = \gamma_{12}$ in order to satisfy eq. (13b). Different from standard theory, in real Clifford algebra we have, $\overline{\gamma}_\mu = -\gamma_\mu$ in either metric.

Regardless of these restrictions, we find that eq. (13a) can have the same form in both metric signatures. However it does not necessarily follow that eq. (13a) will be form invariant under the tilt transformation. In fact, the application of eq. (2b) to (the bar of) eq. (13a) returns the same eq. (13a), except that the wavefunction transforms: $\Psi \rightarrow \tilde{\Psi} = -\gamma_{1234} \overline{\Psi} \gamma_{1234}$. Full signature invariance can be obtained only if the wavefunction is restricted such that $\Psi = \pm \tilde{\Psi}$. The choice of $\Psi = +\tilde{\Psi}$ limits the solution to only 6 components, is too restrictive for
Dirac theory (which requires 8 degrees of freedom). The other choice \( \Psi = -\tilde{\Psi} \) has 10 components, but the 6 odd geometric parts are completely decoupled from the 4 even parts if \( \Gamma \) is odd geometry as previously argued. Dirac theory necessarily has all 8 components coupled. **Therefore we cannot have a fully signature invariant Dirac equation under the tilt transformation of eq. (2b).**

This is consistent with the arguments presented in section III.B, showing that wavefunctions (in standard theory) could not be invariant under a signature change. A weaker condition would be to see if the physical observables are invariant under the tilt transformation, allowing \( \Psi \to \tilde{\Psi} \). We shall call this **Weak Tilt Covariance.** Let us consider a Greider-Ross[6] multivector current which for eq. (13a) obeys a generalized conservation equation,

\[
\partial_\mu j^\mu = m \left[ \overrightarrow{\nabla} \Phi, \Gamma \right],
\tag{15a}
\]

\[
j^\mu = \overrightarrow{\nabla} \gamma^\mu \Psi = -\gamma_4 \Psi \gamma_1 \gamma_3 \gamma^\mu \Psi.
\tag{15b}
\]

In contrast to eq. (10), the even and odd multivector parts of quadratic forms unfortunately do not transform the same under the tilt transformation eq. (2b),

\[
(\overline{\Psi} \Psi)_e \to (\overline{\Psi} \tilde{\Psi})_e.
\tag{16a}
\]

\[
(\overline{\Psi} \Psi)_o \to (\overline{\tilde{\Psi}} \tilde{\Psi})_e \neq (\overline{\Psi} \Psi)_e.
\tag{16b}
\]

The conservation law eq. (15a) can be made invariant under the tilt only if the wavefunction is restricted to be either pure even or pure odd geometry and transforms \( \Psi \to \pm \tilde{\Psi} \). A pure even (or odd) wavefunction will have 8 degrees of freedom, therefore is sufficient to describe Dirac theory. Under these restrictions, noting that \( \overrightarrow{\nabla}_e = +\tilde{\Psi}_e \) and \( \overrightarrow{\nabla}_o = -\tilde{\Psi}_o \), the Lagrangian eq. (14) is found to be invariant under the tilt transformation, insuring that the “physics” will be independent of signature choice.

### C. Generalized Tilt Covariant Gauge Interactions

The lack of unique choice for \( \Gamma, \Lambda \) in eq. (13a) has two sources, generally unrecognized by authors. First, the full multivector solution \( \Psi \) to eq. (13a) has 16 degrees of freedom, while only 8 are needed to describe standard Dirac theory (i.e. isomorphic to a 4 complex component bispinor). Some author’s choices for \( \Gamma, \Lambda \) will only represent Dirac theory if the wavefunction is restricted by some criteria to a specific 8 geometric components (n.b. Hestenes[3] and Lounesto[8] require it to be pure odd or pure even). In contrast, it has been argued[7] that the full multivector solution of eq. (13a) represents an isospin doublet of bispinors, for which both of the isospin components have the correct electromagnetic interaction only for \( \Gamma = 1, \Lambda = \gamma_4 \) in (+ + + −). In the previous section, the choice of \( \Gamma = 1 \) was excluded because it would work only in the (+ + + −) signature. However, there is a counterpart in the (− − + +) signature which can be found by applying eq. (2b) to eq. (13a),

\[
\Box \Phi - m \tilde{\Phi} - A^\mu \gamma_\mu \Phi \gamma_4 = 0,
\tag{17a}
\]
\[ \hat{\Phi} = -\gamma_{1234}\Phi\gamma_{1234}, \quad (17b) \]

where \( \Phi = \tilde{\Psi} \). The mass term now involves the grade involution of eq. (17b), a form not considered in eq. (12a). Indeed, Lounesto\(^8\) obtained a similar result and was concerned about the lack of physical interpretation for the appearance of a grade involution.

This leads us to our second main point about the multitude of choices for \( \Gamma \) and \( \Lambda \) in eq. (13a). Consider that the electromagnetic interaction term \( (\gamma_\mu \Psi \Lambda) \) involved both left and right sided multiplication. Multivector theory allows for a variety of new gauge interactions based on this bilateral (two-sided) form\(^[6,7]\).

We have proposed the generalized equation\(^{11}\), which includes all possible couplings,

\[ \square \Psi = -E^{(i)}_\mu \Psi F^{(j)}_\mu \Omega^{(ij)}. \quad (18) \]

The factor \( E^{(i)}_\mu \) (or \( F^{(j)}_\mu \)) is one of the 16 basis elements of the Clifford algebra. For example let: \( E^{(0)}_\mu = 1 \), \( E^{(i)}_\mu = \gamma_\mu \) (for \( \mu = 1, 2, 3, 4 \)), and \( E^{(15)}_\mu = \gamma_{1234} \).

The grade involution of eq. (17b) is simply a special case where the mass is identified with \( \Omega^{(15,15)} \). The bilateral connection \( \Omega^{(ij)} \) is a generalized set of gauge currents, subject to a Lagrangian constraint that non-zero components must have corresponding factors of \( E^{(i)}_\mu \), \( F^{(j)}_\mu \) both either bar negative or bar positive. Explicitly we rewrite eq. (18),

\[
\begin{align*}
\square \Psi &= \Psi (m_1 + \epsilon \gamma_\mu a_\mu + \epsilon \lambda) + \epsilon \Psi (\eta + \epsilon \gamma_\mu \pi^{\mu} + \epsilon m_2) \\
&+ \gamma^{\mu} \Psi (\gamma_\nu b_\nu + \epsilon \gamma_\mu \gamma_5 a_\mu + \epsilon \gamma_4 \gamma_\nu \rho_\nu A_\mu) + \epsilon \gamma^{\mu} \Psi (f_{1_{\mu}} + \epsilon \gamma_\nu a_{1_{\nu}} + \epsilon \phi_\mu) \\
&+ \gamma^{\alpha\beta} \Psi (\gamma_\mu S^{\mu}_{\alpha\beta} + \gamma_\mu R^{\mu}_{\alpha\beta} + \gamma_\mu R^{\mu}_{\alpha\beta}), \quad (19)
\end{align*}
\]

where \( A = 1, 2, 3 \) and \( \epsilon = \gamma_{1234} \). The symbology of the gauge fields in eq. (19) is deliberate, reflecting that nearly all the light unflavored mesonic interactions can be accomodated in the above scheme\(^{13}\). For full \( SU(2) \times U(1) \) electroweak theory in (+ + +), the correct correspondence has been demonstrated as\(^{7}\): mass \( m = m_1 \), electromagnetism \( A_\mu = b_\mu^4 \) (generator \( \gamma_4 \)), and vector bosons \( W^A_\mu = \rho_\mu^A \); in particular \( Z = W^0 \) has generator \( \gamma_{12} \) which commutes with \( \gamma_4 \).

On the other hand, the tilted form of the Hestenes-Dirac equation presented by Lounesto\(^8\) for (+ + +) has: \( m = a_0^3 \), and \( A_\mu = \rho_\mu^3 \). Comparing with above, it looks like they are using the Z boson for electromagnetism. Indeed as long as you restrict the wavefunction to a single (8 degree of freedom) Dirac bispinor, which is an isospin eigenstate, you cannot distinguish between the two. However this choice of \( \gamma_{12} \) for the generator of electromagnetism cannot accomodate a full electroweak theory. There is no set of elements which will commute with this and \( \Gamma = \gamma_{12} \) that have the necessary \( SU(2) \) group structure.

Our generalized eq. (18) is manifestly weak tilt covariant. Application of eq. (2b) to eq. (18) is reduced to a transformation among the various components \( \Omega^{(i,j)} \). Explicitly, in terms of eq. (19) we find under change of signature with \( \Psi \rightarrow \tilde{\Psi} \) the gauge fields transform,

\[ m_1 \leftrightarrow -m_2, \quad (20a) \]
$$b_\mu^\nu \leftrightarrow -a_1 \mu^\nu, \quad (20b)$$

$$R^{\alpha\beta\kappa\lambda} \leftrightarrow -\epsilon^{\alpha\beta\rho\sigma}\epsilon^{\kappa\lambda\theta}R_{\rho\sigma\theta}, \quad (20c)$$

while the rest are invariant. It is thought that some of these symmetries can be interpreted in terms of why we only see an electric monopole OR a magnetic monopole both, and a new symmetry between isospin and regular spin. A more complete treatment will show how the conserved currents map from one system to the other (hence the form of the observables different).

V. Summary and Conclusions

Classical mechanics (electrodynamics and geometrodynamics) does indeed appear to be unable to determine absolute metric signature. The formulations can be easily cast in signature invariant form. The situation is quantum mechanics is less clear. While the wave equation may not be signature form invariant, the observables can be put into such a form. The spin metric’s signature does not necessarily change with the space metric signature, nor would any signature associated with isospin space.

However, in multivector wavemecanics, spin space, indeed even isospin space is described in one single 4D spacetime algebra. Therefore the “spin basis” structure cannot be isolated from changes in the spacetime signature because of the inequivalence of the (+ + + −) and (− − − +) Clifford algebras. While at first it appears that the multivector formulations are not at all similar in the different signatures, this is because most authors only consider a subset of the full problem. Hence each is slicing a different way through a much richer reality, only seeing some features, e.g. only the electromagnetic interactions but not the weak interactions. We have proposed the most general form of the multivector wave equation, which includes all other author’s works as a subset. This equation does show full signature form invariance, when one considers all possible gauge interactions and all 16 components of the multivector wavefunction.

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