$L^p$-BOUNDEDNESS OF THE WAVE OPERATOR FOR THE ONE DIMENSIONAL SCHRODINGER OPERATOR

PIERO D’ANCONA, LUCA FANELLI

Abstract. Given a one dimensional perturbed Schrödinger operator $H = -d^2/dx^2 + V(x)$, we consider the associated wave operators $W_\pm$, defined as the strong $L^2$ limits
$$
\lim_{s \to \pm \infty} e^{isH} e^{-isH_0}.
$$
We prove that $W_\pm$ are bounded operators on $L^p$ for all $1 < p < \infty$, provided $(1 + |x|)^2 V(x) \in L^1$, or else $(1 + |x|) V(x) \in L^1$ and 0 is not a resonance. For $p = \infty$ we obtain an estimate in terms of the Hilbert transform. Some applications to dispersive estimates for equations with variable rough coefficients are given.

1. Introduction

Let $H_0 = -d^2/dx^2$ be the one-dimensional Laplace operator on the line, and consider the perturbed operator $H = H_0 + V(x)$. For a potential $V(x) \in L^1(\mathbb{R})$, the operator $H$ can be realized uniquely as a selfadjoint operator on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$. The absolutely continuous spectrum of $H$ is $[0, +\infty[$, the singular spectrum is absent, and the possible eigenvalues are all strictly negative and finite in number. Moreover, the wave operators

$$
W_\pm f = L^2 - \lim_{s \to \pm \infty} e^{isH} e^{-isH_0} f
$$

exist and are unitary from $L^2(\mathbb{R})$ to the absolutely continuous space $L^2_{ac}(\mathbb{R})$ of $H$. A very useful feature of $W_\pm$ is the intertwining property. If we denote by $P_{ac}$ the projection of $L^2$ onto $L^2_{ac}(\mathbb{R})$, the property can be stated as follows: for any Borel function $f$,

$$
W_\pm f(H_0) W_\pm^* f(H) = f(H) P_{ac}
$$

(see e.g. [10], [7]).

Thanks to [10], one can reduce the study of an operator $f(H)$, or more generally $f(t, H)$, to the study of $f(t, H_0)$ which has a much simpler structure. When applied to the operators $e^{tH}$, $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$, $\frac{\sin(t\sqrt{H+1})}{\sqrt{H+1}}$, this method can be used to prove decay estimates for the Schrödinger, wave and Klein-Gordon equations

$$
iu_t - \Delta u + Vu = 0, \quad u_{tt} - \Delta u + Vu = 0, \quad u_{tt} - u_{xx} - \Delta u + u + Vu = 0,$$

provided one has some control on the $L^p$ behaviour of $W_\pm$, $W_\pm^*$. Indeed, if the wave operators are bounded on $L^p$, the $L^q - L^q$ estimates valid for the free operators extend

2000 Mathematics Subject Classification. 58J50.

Key words and phrases. scattering theory, wave operator, decay estimates, Schrödinger equation.

Piero D’Ancona, Dipartimento di Matematica, Università "La Sapienza" di Roma, Piazzale Aldo Moro 2, I-00185 Roma, Italy. E-mail: dancona@mat.uniroma1.it.

Luca Fanelli, Dipartimento di Matematica, Università "La Sapienza" di Roma, Piazzale Aldo Moro 2, I-00185 Roma, Italy. E-mail: fanelli@mat.uniroma1.it.
immediately to the perturbed ones via the elementary argument

\[ \|e^{itH}\partial_x f\|_{L^q} = \|W_+ e^{itH_0} W_+^* f\|_{L^q} \leq C \|e^{itH_0} W_+^* f\|_{L^q} \leq C t^{-\alpha} \|W_+^* f\|_{L^q} \leq C t^{-\alpha} \|f\|_{L^q} \]

Such a program was developed systematically by K. Yajima in a series of papers [20], [21], [22] where he obtained the \( L^p \) boundedness for all \( p \) of \( W_\pm \), under suitable assumptions on the potential \( V \), for space dimension \( n \geq 2 \). The analysis was completed in the one dimensional case in Arzbazar-Yajma [3] and Weder [18]. We remark that in high dimension \( n \geq 4 \) the decay estimates obtained by this method are the best available from the point of view of the assumptions on the potential; only in low dimension \( n \leq 3 \) more precise results have been proved (see [11], [12], [16], [23] and [9]). We also mention [13] for an interesting class of related counterexamples.

In order to explain the results in more detail we recall a few notions. The relevant potential classes are the spaces

(1.3) \( L^\gamma_\ast (\mathbb{R}) \equiv \{ f : (1 + |x|)^\gamma f \in L^1(\mathbb{R}) \} \).

Moreover, given a potential \( V(x) \), the Jost functions are the solutions \( f_\pm(\lambda, x) \) of the equation \(-f'' + V f = \lambda^2 f \) satisfying the asymptotic conditions \( |f_\pm(\lambda, x) - e^{\pm i\lambda x}| \to 0 \) as \( x \to \pm \infty \). When \( V(x) \in L^1_\ast \), the solutions \( f_\pm \) are uniquely defined ([10]). Now consider the Wronskian

\[ W(\lambda) = f_+(\lambda, 0)\partial_x f_-(-\lambda, 0) - \partial_x f_+(\lambda, 0) f_-(\lambda, 0). \]

The function \( W(\lambda) \) is always different from zero for \( \lambda \neq 0 \), and it can only vanish at \( \lambda = 0 \). Then we say that 0 is a resonance for \( H \) when \( W(0) = 0 \), and that it is not a resonance when \( W(0) \neq 0 \). The first one is also called the exceptional case.

In [18] Weder proved that the wave operators are bounded on \( L^p \) for all \( 1 < p < \infty \), provided \( V \in L^1_\gamma \) for \( \gamma > 5/2 \). The assumption can be relaxed to \( \gamma > 3/2 \) provided 0 is not a resonance. It is natural to conjecture that these conditions may be sharpened, also in view of the \( L^\infty - L^1 \) decay estimate for the perturbed Schrödinger equation proved by Goldberg and Schlag [11] under the milder assumption \( \gamma = 2 \) in the general and \( \gamma = 1 \) in the nonresonant case.

Indeed, the main result of the present paper is the following:

**Theorem 1.1.** Assume \( V \in L^1_\gamma \) and 0 is not a resonance, or \( V \in L^1_2 \) in the general case. Then the wave operators \( W_\pm, W_+^* \) can be extended to bounded operators on \( L^p \) for all \( 1 < p < \infty \). Moreover, in the endpoint \( L^\infty \) case we have the estimate

(1.4) \( \|W_\pm g\|_{L^\infty} \leq C \|g\|_{L^\infty} + C \|\mathcal{H} g\|_{L^\infty} \),

for all \( g \in L^\infty \cap L^p \) for some \( p < \infty \) such that \( \mathcal{H} g \in L^\infty \), where \( \mathcal{H} \) is the Hilbert transform on \( \mathbb{R} \); the conjugate operators \( W_\pm^* \) satisfy the same estimate.

**Remark 1.1.** The appearence of the Hilbert trasform (see the beginning of Section [4] for a quick reminder) at the endpoint \( p = \infty \) is not a surprise. Indeed, the very precise analysis of Weder showed that the wave operator can be decomposed as the sum of a multiple of
the Hilbert transform, plus a term bounded on $L^\infty$. Thus a weaker estimate like (1.4) is actually optimal.

At the opposite endpoint $p = 1$, we get an even weaker result by duality (see Remark 1.1). We conjecture that by a suitable modification of our methods a stronger bound

$$\|W_{\pm}g\|_{L^1} \leq C\|g\|_{L^1} + C\|Hg\|_{L^1}$$

(1.5)
can be proved. Notice that (1.5) is equivalent to

$$\|W_{\pm}g\|_{L^1} \leq C\|g\|_{H_1}$$

(1.6)
where $H_1$ is the Hardy space; by duality this would imply

$$\|W_{\pm}g\|_{BMO} \leq C\|g\|_{L^\infty}.$$ 

(1.7)
A further evidence in this direction is that the above estimates are a consequence of Weder’s decomposition, of course under stronger assumptions on the potential.

Remark 1.2. Our proof is based on the improvement of some results of Deift and Trubowitz [10], combined them with the stationary approach of Yajima [20], [3], and some precise Fourier analysis arguments. Quite inspirational has been the paper [11], both for showing there was room for improvement in the assumptions on the potential, and for the very effective harmonic analysis approach. We mention in particular the idea of using Wiener’s lemma to estimate the $L^1$ norm of the Fourier transform of a quotient, essential in Section 3 below.

Remark 1.3. In the proof of Theorem 1.1 we split as usual the wave operator into high and low energy parts; the high energy part is known to be easier to handle since the resolvent is only singular at frequency $\lambda = 0$. Here we can prove that the high energy part is bounded on $L^p$ for all $p$, including the cases $p = 1$ and $p = \infty$, under the weaker assumption $V \in L^1(\mathbb{R})$ (see Section 2 and Lemma 2.1).

Remark 1.4. An essential step in the low energy estimate is a study of the Fourier properties of the Jost functions; this kind of analysis is classical (see [1]) and the fundamental estimates were obtained by Deift and Trubowitz in [10]. In Section 3 we improve their results by showing that the $L^1$ norms of the Fourier transforms of the Jost functions satisfy a linear bound as $|x| \to +\infty$ instead of an exponential one as in [10]. In the resonant case we can prove a quadratic bound (see Lemmas 3.2, 3.3 and Corollary 3.1).

Remark 1.5. It is possible to continue the analysis and prove that the wave operators are bounded on Sobolev spaces $W^{k,p}$, under the additional assumption $V \in W^{k,1}$ (see [18]), but we prefer not to pursue this question here.

Theorem 1.1 has several applications; here we shall focus on the dispersive estimates for the one dimensional Schrödinger and Klein-Gordon equations with variable rough coefficients.

Consider first the initial value problem

$$iu_t - a(x)u_{xx} + b(x)u_x + V(x)u = 0, \quad u(0,x) = f(x).$$

(1.8)
Then we obtain the following decay result, where the notation $f \in L^2_1$ means $(1 + |x|) f \in L^2$. Notice that the case $a = 1$, $b = 0$ is already considered in [11], where actually the endpoint $L^\infty - L^1$ is reached.

**Proposition 1.1.** Assume $V \in L^1_2$, $a \in W^{2,1}(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R})$ with

$$ (1.9) \quad a(x) \geq c_0 > 0 \quad a', b \in L^2_1, \quad a'', b' \in L^1_2 $$

for some constant $c_0$. Then the solution of the initial value problem (1.8) satisfies

$$ (1.10) \quad \| P_{ac} u(t, \cdot) \|_{L^q} \leq C t^{\frac{1}{q} - \frac{1}{2}} \| f \|_{L^{q'}}, \quad 2 \leq q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1. $$

The same result holds if $a = 1$, $b = 0$ and $V \in L^1_1$, provided 0 is not a resonance for $H$.

**Remark 1.6.** As discussed above, the case $q = \infty$ escapes this method since the wave operator is not bounded on $L^\infty$. It is however possible to recover the estimate also in this case by the direct approach of [11].

**Proof.** Define the functions

$$ (1.11) \quad c(x) = \int_0^x a(s)^{-1/2} ds, \quad \sigma(x) = a(x)^{1/4} \exp \left( \int_0^x b(s) \frac{ds}{2a(s)} \right), $$

and apply the change of variables

$$ (1.12) \quad u(t, x) = \sigma(x) w(t, c(x)). $$

Then the problem is transformed to

$$ (1.13) \quad iw_t(t, y) - w_{yy} + \tilde{V}(y) w(t, y) = 0, \quad w(0, y) = \frac{f}{\sigma} \bigg|_{c^{-1}(y)}, $$

where the potential $\tilde{V}$ is defined by

$$ (1.14) \quad \tilde{V}(c(x)) = V(x) + \frac{1}{16a(x)} (2b(x) + a'(x))(2b(x) + 3a'(x)) - \frac{1}{4} (2b(x) + a''(x)). $$

It is elementary to check that $\tilde{V}$ satisfies the assumptions of Theorem 1.1. Hence by the intertwining property and the $L^p$ boundedness of the wave operator for $H_0 + \tilde{V}$ we obtain that $w(t, y)$ satisfies a dispersive estimate like (1.10). Coming back to the original variables we conclude the proof. \[\square\]

**Remark 1.7.** The range of indices allowed in (1.10) is sufficient to deduce the full set of Strichartz estimates, as it is well known. It is interesting to compare this with the result of Burq and Planchon [6] who proved the Strichartz estimates for the variable coefficient equation

$$ iu_t - \partial_x (a(x) \partial_x u) = 0 $$

assuming only that $a(x)$ is of BV class and bounded from below.

**Remark 1.8.** In view of the next application, we recall the definition of nonhomogeneous Besov spaces. Choose a Paley-Littlewood partition of unity, i.e., a sequence of smooth
cutoffs $\phi_j \in C^\infty_c(\mathbb{R})$ with $\sum_{j \geq 0} \phi_j(\lambda) = 1$ and $\text{supp} \phi_j = [2^{j-1}, 2^{j+1}]$, $\text{supp} \phi_0 = [-2, 2]$. Then the $B^s_{p,r}$ Besov norm is defined by

$$\|g\|_{B^s_{p,r}} \equiv \sum_{j \geq 0} 2^{jsr} \|\phi_j(\sqrt{H_0})g\|_{L^p}$$

with obvious modification for $r = \infty$. It is then natural to define the perturbed Besov norm corresponding to the selfadjoint operator $H = H_0 + V$ as

$$\|g\|_{B^s_{p,r}(V)} \equiv \sum_{j \geq 0} 2^{jsr} \|\phi_j(\sqrt{H})g\|_{L^p}.$$

Now, from the $L^p$ boundedness of the wave operators and the intertwining property in the form

$$\phi_j(\sqrt{H})W_\pm = W_\pm \phi_j(\sqrt{H_0})$$

we obtain immediately the Besov space bounds

$$\|W_\pm f\|_{B^s_{p,r}(V)} \leq C\|f\|_{B^s_{p,r}}, \quad \|W^\pm f\|_{B^s_{p,r}} \leq C\|f\|_{B^s_{p,r}(V)}.$$  \hspace{1cm} (1.15)

We now consider the initial value problem for the one dimensional Klein-Gordon equation

$$u_{tt} - a(x)u_{xx} + u + b(x)u_x + V(x)u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = g(x).$$  \hspace{1cm} (1.16)

Our second application is the following:

**Proposition 1.2.** Assume $V, a, b$ are as in Proposition 1.1. Then the solution of the initial value problem (1.16) satisfies

$$\|P_{ac}u(t, \cdot)\|_{L^q} \leq C t^{\frac{1}{2} - \frac{1}{q}} \|f\|_{B^\frac{1}{2} - \frac{1}{q'}(V)},$$  \hspace{1cm} (1.17)

$2 \leq q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$.

The same result holds if $a = 1$, $b = 0$ and $V \in L^1_1$, provided 0 is not a resonance for $H$.

**Proof.** For the free Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0, \quad u(0, x) = 0, \quad u_t(0, x) = g(x).$$

we know that the solution satisfies the energy estimate and the dispersive estimate, which we can write in the form

$$\|u(t, \cdot)\|_{L^2} \leq C\|g\|_{H^{-1}}, \quad \|u(t, \cdot)\|_{L^\infty} \leq C t^{-\frac{1}{2}} \|g\|_{B^{1/2}_{1,1}}.$$  \hspace{1cm} (1.18)

By real interpolation this implies

$$\|u(t, \cdot)\|_{L^q} \leq C t^{\frac{1}{2} - \frac{1}{q}} \|g\|_{B^\frac{1}{2} - \frac{1}{q'}_{1,q'}(V)},$$  \hspace{1cm} (1.19)

$2 \leq q \leq \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Consider now the variable coefficient problem (1.16): applying the same change of variables (1.11)–(1.12) as above, we transform the problem into

$$w_{tt}(t, y) - w_{yy} + w + \tilde{V}(y)w(t, y) = 0, \quad w(0, y) = 0, \quad w_t(0, y) = \frac{g}{\sigma_c^{-1}(y)}$$  \hspace{1cm} (1.20)

with $\tilde{V}$ as in (1.14). Applying as above the intertwining property, and using the bounds (1.15), we conclude the proof. \hfill \Box
The rest of the paper is devoted to the proof of Theorem 1.1. We first analyze the high energy part, in Section 2; Section 3 contains a detailed study of the Fourier properties of the Jost functions, necessary for the analysis of the low energy part which is the subject of Section 4.

Acknowledgments. We would like to thank Kenji Yajima for several very useful discussions concerning the subject of this paper.

2. The high energy analysis

In the estimate of the high frequency part of the wave operator we shall use the standard representation as a distorted Fourier transform; considering e.g. the operator $W_-$, we have

\begin{equation}
W_- g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \varphi(\lambda, x) e^{-i\lambda y} d\lambda \right) g(y) \, dy,
\end{equation}

where the generalized eigenfunction $\varphi(\lambda, x)$ is defined as the solution to the Lippman-Schwinger equation

\begin{equation}
\varphi(\lambda, x) = e^{i\lambda x} + \frac{1}{2i|\lambda|} \int e^{i|\lambda||x-y|} V(y) \varphi(\lambda, y) \, dy, \quad \lambda, x \in \mathbb{R}
\end{equation}

(see e.g. [3], [18]). An equivalent form of the equation (2.2) is the following:

\begin{equation}
\varphi(\lambda, x) = e^{i\lambda x} - R_0(\lambda^2 + i0) V \varphi(\lambda, x).
\end{equation}

We recall that the free resolvent $R_0(z) = (-\Delta - z)^{-1}$ admits the explicit representation

\begin{equation}
R_0(z^2) f(x) = \frac{1}{2i} \int \frac{e^{i|x-y|}}{z} f(y) \, dy,
\end{equation}

for $z \not\in [0, +\infty[$. Moreover, the limits

\begin{equation}
R_0(\lambda \pm i0) = \lim_{\epsilon \to 0} R_0(\lambda \pm i\epsilon)
\end{equation}

exist in the norm of bounded operators from the weighted $L^2_{1/2+\epsilon}$ to the weighted $L^2_{-1/2-\epsilon}$ spaces, for any $\lambda \in ]0, \infty[$ (see e.g. [2]). Thus we have the explicit formula

\begin{equation}
R_0(\lambda \pm i0) f(x) = \frac{1}{2i} \int \frac{e^{\pm i\lambda |x-y|}}{\lambda} f(y) \, dy,
\end{equation}

for any $\lambda > 0$ and $f$ at least in $L^1$. The strong singularity at $\lambda = 0$ is the main source of difficulties in the study of the wave operator.

The perturbed resolvent $R_V(z) = (-\Delta + V - z)^{-1}$ is related to $R_0$ by the identity

\begin{equation}
R_V = R_0(I + VR_0)^{-1}.
\end{equation}

We recall that under the assumption $V \in L^1_1$ the limiting absorption principle (2.4) holds also for $R_V$ (see [1], [3]).

By the representation (2.6) it is clear that for $\lambda \geq \lambda_0 = \|V\|_{L^1}$ the operator $R_0 V$ is bounded on $L^\infty$ with norm

\[\|R_0(\lambda^2 + i0) V\|_{L(L^\infty)} \leq \frac{1}{2}.\]
In particular, for \( \lambda \) large enough, \( I + R_0(\lambda^2 + i0)V \) can be inverted by a Neumann series, the solution \( \phi(\lambda, x) \) of (2.3) is well defined and it can be represented by a uniformly convergent series
\[
\varphi(\lambda, x) = \sum_{n \geq 0} (-1)^n \left( R_0(\lambda^2 + i0) \right)^n e^{ikx}, \quad |\lambda| \geq \lambda_0 := \|V\|_{L^1}, \quad x \in \mathbb{R}.
\]

Now take a smooth cutoff function \( \Phi \in C^\infty(\mathbb{R}^+) \) such that
\[
0 \leq \Phi \leq 1, \quad \Phi(\lambda^2) = 0 \text{ for } 0 \leq \lambda^2 \leq \lambda_0, \quad \Phi(\lambda^2) = 1 \text{ for } \lambda^2 \geq \lambda_0 + 1
\]
and consider the high energy part of the wave operator
\[
W_\Phi(H_0)g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(\lambda, x) e^{-i\lambda y} g(y) \Phi(\lambda^2) d\lambda dy.
\]
We split this operator into positive and negative frequencies, i.e., writing
\[
\chi(\lambda) = \begin{cases} 
\Phi(\lambda^2) & \text{for } \lambda > 0, \\
0 & \text{for } \lambda \leq 0
\end{cases}, \quad \psi(\lambda) = \begin{cases} 
\Phi(\lambda^2) & \text{for } \lambda < 0, \\
0 & \text{for } \lambda \geq 0
\end{cases}
\]
we define the operators
\[
Ag(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(\lambda, x) e^{-i\lambda y} g(y) \chi(\lambda) d\lambda dy
\]
and
\[
Bg(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(\lambda, x) e^{-i\lambda y} g(y) \psi(\lambda) d\lambda dy.
\]
In the following we shall study the positive part \( Ag \); clearly the estimate of the negative piece \( Bg \) is completely analogous. By (2.1) and (2.8), the integral kernel \( K(x, y) \) of \( A \)
\[
Ag(x) = \frac{1}{2\pi} \int K(x, y) g(y) dy,
\]
can be represented as
\[
K(x, y) = \sum_{n \geq 0} (-1)^n \int (R_0(\lambda^2 + i0)V)^n e^{i\lambda(x-y)} \chi(\lambda) d\lambda.
\]
We shall estimate the terms of the series (2.12) separately. Notice that for \( n \geq 2 \) we can write
\[
K_n(x, y) = \left( \frac{i}{2} \right)^n \int \frac{\chi(\lambda)}{\lambda^n} e^{i\lambda(x-y_1+y_1-y_2+\cdots+y_{n-1}-y_n+y_{n+1})} \prod_{j=1}^{n} V(y_j) dy_1 \ldots dy_n d\lambda.
\]
On the other hand, for \( n = 0, 1 \) we have the formal expressions
\[
K_0(x, y) = \int e^{i\lambda(x-y)} \chi(\lambda) d\lambda,
\]
\[
K_1(x, y) = \frac{i}{2} \int \frac{\chi(\lambda)}{\lambda} e^{i\lambda(x-y_1+y_1-y)V(y_1)} dy_1 d\lambda.
\]
which can defined precisely by adding a cutoff on \([0, L]\) and then sending \(L \to +\infty\) (see below). Denoting by \(A_n\) the operator with kernel \(K_n(x, y)\), we have

\[
Ag(x) = \frac{1}{2\pi} \left[ \int K_0(x, y)g(y)\,dy + \int K_1(x, y)g(y)\,dy - \sum_{n\geq 2} (-1)^n \int K_n(x, y)g(y)\,dy \right]
\]

(2.16) \[
= \frac{1}{2\pi} \left( A_0g(x) - A_1g(x) + \sum_{n\geq 2} (-1)^n A_n g(x) \right).
\]

Then we have:

**Lemma 2.1.** Assume \(V \in L^1(\mathbb{R})\) and let \(0 \leq \Phi \leq 1\) be a smooth function such that \(\Phi(\lambda^2) = 0\) for \(\lambda^2 < \|V\|_{L^1}\) and \(\Phi(\lambda^2) = 1\) for \(\lambda^2 > \|V\|_{L^1} + 1\). Then the high energy parts of the wave operators \(W_\pm\) are bounded on \(L^p\) for all \(1 \leq p \leq \infty\):

\[
\|W_\pm \Phi(H_0)g\|_{L^p} \leq C\|g\|_{L^p}.
\]

The same holds for the conjugate operators \(\Phi(H_0)W_\pm^*\).

**Proof.** By standard duality arguments, it will be sufficient to prove the estimates for \(p = \infty\); since the proof is completely analogous for any of the four operators \(W_\pm, W_\pm^*\), we shall consider only \(W_-\). By the discussion above, we see that it is sufficient to estimate the operator \(A\) defined in (2.11) - (2.16).

We shall estimate each term \(A_n\) in the series (2.16) separately. For the term \(A_0\), we can write by (2.14)

\[
A_0g(x) = \int \left( \int e^{i\lambda(x-y)} \chi(\lambda)g(y)\,d\lambda \right) dy
\]

(2.18) \[
\leq \left( 1 + \|(1 - \chi)\|_{L^1} \right) \|g\|_{L^\infty} \leq C_0\|g\|_{L^\infty}.
\]

Consider now the term \(A_1\), which by (2.15) can be written formally

\[
A_1g(x) = \frac{i}{2} \int \left( \int \frac{\chi(\lambda)}{\lambda} e^{i\lambda|x-z+y|} V(z)g(y)\,dz \right) d\lambda dy.
\]

(2.19) \[
A_{1,L}g = \frac{i}{2} \int \left( \int \gamma_L(\lambda)e^{i\lambda|x-z+y|} V(z)g(y)\,dz \right) d\lambda dy,
\]

where \(\gamma_L(\lambda) = \frac{1}{\lambda} \chi(\lambda)\psi_L(\lambda),\quad \psi_L(\lambda) \equiv \psi(\lambda/L)\).
We claim that the operators $A_{1,L}$ are uniformly bounded on $L^\infty$, and that for each $g \in L^\infty$ the limit
\begin{equation}
A_1g = \lim_{L \to +\infty} A_{1,L}
\end{equation}
eexist in the norm of $L^\infty$. To prove this, we notice that by Fubini’s theorem \ref{eq:2.20} can be rewritten as
\begin{equation}
A_{1,L}g = \int \hat{\gamma}_L(|x-z| + z-y)V(z)g(y) \, dz \, dy.
\end{equation}
It is clear that the claim follows as soon as we can prove that $\hat{\gamma}_L$ converges in $L^1(\mathbb{R})$ when $L \to +\infty$: indeed, we have
\[ \|A_{1,L}g - A_{1,M}g\|_{L^\infty} \leq \|V\|_{L^1} \|\hat{\gamma}_L - \hat{\gamma}_M\|_{L^1} \|g\|_{L^\infty}. \]
To prove the claim, decompose $\gamma_L$ as follows:
\begin{equation}
\gamma_L(\lambda) = \psi_L(\lambda) \cdot \left( \frac{\lambda}{1 + \lambda^2} + (\chi - 1) \frac{\lambda}{1 + \lambda^2} + \frac{\chi(\lambda)}{\lambda(1 + \lambda^2)} \right).
\end{equation}
The function
\begin{equation}
\eta(\lambda) = \frac{\lambda}{1 + \lambda^2} + (\chi - 1) \frac{\lambda}{1 + \lambda^2} + \frac{\chi(\lambda)}{\lambda(1 + \lambda^2)}
\end{equation}
has a Fourier transform in $L^1$: this follows immediately from the standard formula
\begin{equation}
\hat{\lambda} e^{-|\xi|} d\xi = C \frac{\xi}{|\xi|^2} e^{-|\xi|}
\end{equation}
and the fact that the last term in \ref{eq:2.24} is smooth and decays faster than $|\lambda|^{-3}$. Since $\hat{\psi}_L$ is a $\delta$-sequence, we conclude that $\hat{\gamma}_L = \hat{\psi}_L * \hat{\eta}$ converges to $\hat{\eta}$ in $L^1(\mathbb{R})$. As a consequence, $A_{1,L}g$ converge uniformly to
\begin{equation}
A_1g = \int \int \hat{\eta}(|x-z| + z-y)V(z)g(y) \, dz \, dy.
\end{equation}
which is then a bounded operator on $L^\infty$:
\begin{equation}
\|A_1g\|_{L^\infty} \leq \|V\|_{L^1} \|\hat{\eta}\|_{L^1} \|g\|_{L^\infty}.
\end{equation}
To conclude the proof, it remains to estimate the operators $A_n$ for $n \geq 2$. By the explicit formula \ref{eq:2.13} we obtain
\[ A_n g(x) = \left( \frac{i}{2} \right)^n \int \int \cdots \int \psi_n(\lambda) e^{i\lambda(|x-y_1| + |y_1-y_2| + \cdots + |y_{n-1}-y_n| + y_n-y)} \times \prod_{j=1}^n V(y_j)g(y) \, dy_1 \ldots dy_n \, d\lambda \, dy, \]
where $\psi_n(\lambda) := \chi(\lambda)/\lambda^n$. By Fubini’s Theorem this can be written
\[ A_n g(x) = \left( \frac{i}{2} \right)^n \int \int \cdots \int \hat{\psi}_n(|x-y_1| + |y_1-y_2| + \cdots + |y_{n-1}-y_n| + y_n-y) \times \prod_{j=1}^n V(y_j)g(y) \, dy_1 \ldots dy_n, \]
and then we immediately get the inequality
\begin{equation}
(2.27) \quad \|A_ng(x)\|_{L^\infty} \leq \frac{1}{2^n}\|V\|_{L^1}^n\|\hat{\psi}_n\|_{L^1}\|g\|_{L^\infty}.
\end{equation}

To compute the norm of $\hat{\psi}_n$, introduce the scaling operators $S_h$ defined as $S_hg(x) = g(hx)$; then writing
\[
\chi_0(\lambda) := \chi(\lambda \cdot \lambda_0), \quad \lambda_0 = \|V\|_{L^1},
\]
we have
\[
\psi_n(\lambda) = \lambda_0^{-n} \cdot S_{1/\lambda_0} \left( \frac{\chi_0(\lambda)}{\lambda^n} \right)
\]
and hence
\[
\|\hat{\psi}_n\|_{L^1} = \lambda_0^{-n} \|\mathcal{F}S_{1/\lambda_0} (\chi_0/\lambda^n)\|_{L^1} = \lambda_0^{-n} \|\lambda_0 \mathcal{F} (\chi_0/\lambda^n)\|_{L^1}
\]
\[
= \lambda_0^{1-n} \|\mathcal{F} (\chi_0/\lambda^n)\|_{L^1} \leq C\lambda_0^{1-n} \|\xi^2 \mathcal{F} (\chi_0/\lambda^n)\|_{L^\infty}
\]
\[
\leq C\lambda_0^{1-n} \|(1 - \Delta) (\chi_0/\lambda^n)\|_{L^1} \leq C_0 \lambda_0^{1-n} = C_0 \lambda_0^{1-n} \|V\|_{L^1}^{1/n}
\]
for some constant $C_0$ independent of $n$ and $\lambda_0$. This inequality together with (2.27) gives
\begin{equation}
(2.28) \quad \|A_ng(x)\|_{L^\infty} \leq C_0 \frac{n^2}{2^n} \|V\|_{L^1} \|g\|_{L^\infty}.
\end{equation}

By the estimates (2.18), (2.26), (2.28) and by formula (2.16) we conclude the proof of the Lemma. □

3. Fourier properties of the Jost Functions

Throughout this section we shall assume that $V \in L^1_1(\mathbb{R})$ (at least).

The \textit{Jost functions} $f_\pm(z, x)$ are defined as the solutions of
\begin{equation}
(3.1) \quad -f''_\pm(z, x) + V(x)f_\pm(z, x) = z^2 f_\pm(z, x)
\end{equation}
satisfying the asymptotic conditions
\begin{equation}
(3.2) \quad |f_\pm(z, x) - e^{\pm i z x}| \to 0,
\end{equation}
for $x \to \pm \infty$. It is well known (see [10]) that $f_\pm(\lambda, x)$ are well defined for all $\lambda, x \in \mathbb{R}$.

Using the Jost functions it is possible to represent the kernel of the perturbed resolvent $R_V(\lambda^2 \pm i0)$; indeed, writing
\[
R_V(\lambda^2 \pm i0)g = \int K_\pm(x, y)g(y)dy,
\]
one has
\begin{equation}
(3.3) \quad K_\pm(x, y) = \frac{1}{2\pi i} \begin{cases} 
\frac{f_\pm(\pm \lambda, y)f_\pm(\pm \lambda, x)}{W(\pm \lambda)} & \text{for } x < y, \\
\frac{f_\pm(\pm \lambda, x)f_\pm(\pm \lambda, y)}{W(\pm \lambda)} & \text{for } x > y;
\end{cases}
\end{equation}
here
\[
W(\lambda) = f_+(\lambda, 0) \cdot \partial_x f_-(\lambda, 0) - \partial_x f_-(\lambda, 0) \cdot f_+(\lambda, 0)
\]
denotes the Wronskian of $f_+$ and $f_-$. It is always true (see [10]) that $W(\lambda) \neq 0$ for any $\lambda \neq 0$; thus the only possible zero of the Wronskian is at $\lambda = 0$, and when $W(0) = 0$ we say that 0 is a resonance for $-\Delta + V$.

The modified Jost functions $m_{\pm}$ are defined via

$$f_{\pm}(\lambda, x) = e^{\pm i \lambda x} m_{\pm}(\lambda, x);$$

equivalently, the functions $m_{\pm}$ can be characterized as the unique solutions of the equations

$$m''_{\pm}(\lambda, x) \pm 2i \lambda m'_{\pm}(\lambda, x) = V(x) m_{\pm}(\lambda, x)$$

satisfying the asymptotic conditions

$$m_{\pm}(\lambda, x) \to 1 \quad \text{for} \quad x \to \pm \infty.$$ 

Moreover, we can also obtain $m_{\pm}(\lambda, x)$ as the unique solutions of the Volterra integral equations

$$m_{\pm}(\lambda, x) = 1 \pm \int_{x}^{+\infty} D_\lambda(\pm(t - x)) V(t) m_{\pm}(\lambda, t) dt,$$

where

$$D_\lambda(x) := \int_{0}^{x} e^{2i \lambda t} dt = \frac{e^{2i \lambda x} - 1}{2i \lambda}.$$

The functions $m_{\pm}(\lambda, x)$ have a rich set of properties, studied in detail in [10]. Here we shall only need the following basic facts:

- if $V \in L^1_1$, then $m_{\pm}(\lambda, x) \in C(\mathbb{R}^2)$;
- if $V \in L^2_2$, then $m_{\pm}(\lambda, x) \in C^1(\mathbb{R}^2)$ and $\frac{\lambda}{W(\lambda)} \in C(\mathbb{R})$.

In scattering theory an essential role is played by the Fourier transform w.r. to $\lambda$ of the functions $m_{\pm} - 1$, which are usually written in the form

$$B_{\pm}(\xi, x) = \int_{\mathbb{R}} e^{-2i \lambda \xi} (m_{\pm}(\lambda, x) - 1) d\lambda.$$

(notice the factor 2 in the exponential). For each $x \in \mathbb{R}$ the function $B_+(\xi, x)$ is well defined, real valued, belongs to $L^2(\mathbb{R})$ and actually vanishes for $\xi < 0$; this means that $m_+(-, x) - 1$ belongs to the Hardy space $H^{2+}$ (see [10] for details). Analogously, $B_-(-, x)$ belongs to $L^2(\mathbb{R})$ and vanishes for $\xi > 0$, i.e., $m_-(-, x) - 1 \in H^{2-}$.

If we take the Fourier transform of equation (3.7), we obtain that $B_+(\xi, x)$ satisfies the Marchenko equation

$$B_+(\xi, x) = \int_{x+\xi}^{\infty} V(t) dt + \int_{0}^{\xi} dz \int_{x+\xi-z}^{\infty} V(t) B_+(z, t) dt$$

while $B_-(\xi, x)$ satisfies the symmetric equation

$$B_-(\xi, x) = \int_{-\infty}^{x+\xi} V(t) dt + \int_{\xi}^{0} dz \int_{x+\xi-z}^{-\infty} V(t) B_-(z, t) dt.$$
The functions $B_\pm(\xi, x)$ have many additional properties of boundedness and regularity; however we shall only be concerned here with the properties of the $L^1$ norms $\|B_\pm(\cdot, x)\|_{L^1}$. Writing for $x \in \mathbb{R}$

\[
\eta(x) = \int_x^\infty |V(t)| dt, \quad \gamma(x) = \int_x^\infty (t - x)|V(t)| dt = \int_x^\infty \int_y^\infty |V(t)| dtdy,
\]

the well-known estimate of Deift and Trubowitz is the following:

**Lemma 3.1.** Assume $V \in L^1_1$. Then, for all $\xi, x \in \mathbb{R}$, the solution $B_+(\xi, x)$ to (3.10) is well defined and satisfies the estimates

\[
|B_+(\xi, x)| \leq e^{\gamma(x)} \eta(\xi + x), \quad |\partial_x B_+(\xi, x) + V(x + \xi)| \leq e^{\gamma(x)} \eta(x + \xi).
\]

In particular, $B(\cdot, x)$ is in $L^1 \cap L^\infty$ for any $x$ and

\[
\|B_+(\cdot, x)\|_{L^1} \leq e^{\gamma(x)} \gamma(x), \quad \|\partial_x B_+(\cdot, x)\|_{L^1} \leq \eta(x) + e^{\gamma(x)} \gamma(x).
\]

The function $B_-$ has similar properties, with the behaviours at $\pm \infty$ reversed. Notice that $\gamma(x) \leq \|V\|_{L^1}$ for $x \geq 0$, while for negative $x$ the behaviour of $\gamma(x)$ is

\[
\gamma(x) \sim |x| \cdot \|V\|_{L^1} + \|V\|_{L^1}, \quad x \to -\infty.
\]

In other words, the estimate shows that $\|B_+(\cdot, x)\|_{L^1}$ is bounded by a constant depending on $\|V\|_{L^1}$ for $x > 0$, but it gives only an exponential bound for negative $x$. A similar estimate holds for the function $B_-$, exchanging the behaviours as $x \to +\infty$ and $x \to -\infty$.

A crucial tool in the study of the low energy case will be an essential improvement of the (3.13): indeed, we can prove that the norm of $B_+$ (resp. $B_-$) has at most a linear growth as $x \to -\infty$ (resp. $x \to +\infty$).

**Lemma 3.2.** Assume $V \in L^1_1$; then the functions $B_\pm(\xi, x)$ satisfy the estimates

\[
\|B_\pm(\cdot, x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \quad \|B_\pm(\cdot, x)\|_{L^1} \leq C(x) \quad \text{for } \pm x \leq 0
\]

for some constant $C$ depending on $\|V\|_{L^1}$.

**Proof.** We prove the result for $B_+$, the proof for $B_-$ is identical. The behaviour for positive $x$ is already contained in the Deift-Trubowitz estimate. Now, starting from the Marchenko equation (3.10), we integrate with respect to $\xi$ from 0 to $\infty$ (recall that $B_\pm$ vanish for $\xi < 0$) and we have

\[
\|B_+(\cdot, x)\|_{L^1} \leq \|V\|_{L^1} + |x| \cdot \|V\|_{L^1} + \int_0^\infty d\xi \int_0^\xi dz \int_x^{\infty} |V(t)| \cdot |B_+(z, t)| dt \\
\quad \leq \langle x \rangle \cdot \|V\|_{L^1} + \int_0^\infty d\xi \int_{-\infty}^\xi dz \int_x^{\infty} |V(t)| \cdot |B_+(z, t)| dt.
\]

We set $z' := \xi - z$ and exchange the order of integration: in this way we obtain

\[
\|B_+(\cdot, x)\|_{L^1} \leq \langle x \rangle \cdot \|V\|_{L^1} + \int_x^\infty dt \int_0^{t-x} \|V(t)\| \cdot \|B_+(\cdot, t)\|_{L^1} dz' \\
= \langle x \rangle \cdot \|V\|_{L^1} + \int_x^\infty |V(t)| \cdot (t-x) \cdot \|B_+(\cdot, t)\|_{L^1} dt.
\]

(3.15)
Now we remark that
\[
\int_{x}^{\infty} t|V(t)| \cdot \|B_+(-t^*, t)\|_{L^1} dt \leq \int_{0}^{\infty} t|V(t)| \cdot \|B_+(-t^*, t)\|_{L^1} dt
\]
which is obvious when \(x > 0\) and is also evident for \(x < 0\) since the integral from \(x\) to 0 is negative. Using the Deift-Trubowitz estimate \(3.13\) we see that\[
\|B_+(-t^*, t)\| \leq C_0(\|V\|_{L^1}) \quad \text{for } t > 0,
\]
and hence in conclusion
\[
\int_{x}^{\infty} t|V(t)| \cdot \|B_+(-t^*, t)\|_{L^1} dt \leq C_1 \equiv C_1(\|V\|_{L^1}) \quad \text{for all } x \in \mathbb{R}.
\]
Thus inequality \(3.15\) gives
\[
\|B_+(-t^*, t)\|_{L^1} \leq \langle x \rangle \cdot \|V\|_{L^1} + C_1(\|V\|_{L^1}) + |x| \int_{x}^{\infty} |V(t)| \cdot \|B_+(-t^*, t)\|_{L^1} dt
\]
which implies
\[
(3.16) \quad \frac{1}{\langle x \rangle}\|B_+(-t^*, t)\|_{L^1} \leq C_2(\|V\|_{L^1}) + \int_{x}^{\infty} \langle t \rangle|V(t)| \frac{\|B_+(-t^*, t)\|_{L^1}}{\langle t \rangle} dt.
\]
Applying Gronwall’s lemma for \(x < 0\) we finally obtain the required estimate
\[
\|B_+(-t^*, t)\|_{L^1} \leq C_3(\|V\|_{L^1}) \cdot \langle x \rangle.
\]

In the resonant case \(W(0) = 0\) it will be necessary to make the stronger assumption \(V \in L^2\). In this case, we know that the Jost functions are \(C^1\) in both variables and we shall study the behaviour of the functions
\[
(3.17) \quad C_\pm(\xi, x) = \int_{\mathbb{R}} e^{-2i\lambda \xi} \partial_\lambda m_\pm(\lambda, x) d\lambda \equiv 2i\xi B_\pm(\xi, x).
\]
As above, a direct application of the Deift-Trubowitz estimate gives an optimal bound only on a half line. Indeed, if we multiply \(3.12\) by \(2\xi\) and integrate in \(\xi\) we obtain
\[
(3.18) \quad \|C_+(\cdot, t)\|_{L^1} \leq 2e^{\gamma(x)} \int_{0}^{\infty} \xi \int_{x+\xi}^{\infty} |V(t)| dt d\xi
\]
and after exchanging the order of integration we get
\[
(3.19) \quad \|C_+(\cdot, t)\|_{L^1} \leq 2e^{\gamma(x)} \int_{x}^{\infty} (t-x)^2 |V(t)| dt.
\]
Notice that
\[
(3.20) \quad \gamma(x) \leq \|V\|_{L^1} \quad \text{for } x \geq 0, \quad \gamma(x) \leq \|V\|_{L^1} + |x| \cdot \|V\|_{L^1} \quad \text{for } x \leq 0;
\]
thus we obtain
\[
(3.21) \quad \|C_+(\cdot, t)\|_{L^1} \leq C(\|V\|_{L^2}) \quad \text{for } x \geq 0
\]
but we can only get an exponential growth for negative \(x\) (similar computations for \(C_-\)).

We can improve this estimate by a different argument:
Lemma 3.3. Assume \( V \in L^1_2 \); then the functions \( C_\pm(\xi, x) = 2i\xi B_\pm(\xi, x) \) satisfy

\[
\|C_\pm(\cdot, x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \quad \|C_\pm(\cdot, x)\|_{L^1} \leq C(x)^2 \quad \text{for } \pm x \leq 0
\]

for some constant \( C \) depending on \( \|V\|_{L^1_2} \).

Proof. We will consider only \( C_+ \); the proof for \( C_- \) is identical. We have already estimated \( C_+ \) on the positive half-line in (3.21). To prove the estimate for \( x < 0 \) we start again from Marchenko’s equation (3.10); if we multiply both sides by \( 2\xi \) and integrate in \( \xi \) we obtain

\[
\|C_+(\cdot, x)\|_{L^1} \leq 2\int_0^\infty \int_{x+\xi}^\infty |V(t)| \cdot \xi \, dt \, d\xi + 2\int_0^\infty d\xi \int_0^\infty d\sigma \int_{x+\xi-\sigma}^\infty |V(t)| \cdot |B_+(\sigma, t)| \cdot \xi \, dt.
\]

The first term gives, after exchanging the order of integration,

\[
2\int_0^\infty \int_{x+\xi}^\infty |V(t)| \cdot \xi \, dt \, d\xi = 2\int_x^\infty (t-x)^2 |V(t)| \, dt.
\]

Moreover, if we remark that

\[
\int_0^\infty d\xi \int_0^\xi d\sigma \int_{x+\xi-\sigma}^\infty F \, dt = \int_0^\infty d\sigma \int_x^\infty \int_0^{\sigma+t-x} F \, dt \, d\xi
\]

we see that the last integral in (3.23) is equal to

\[
\int_0^\infty d\sigma \int_x^\infty |V(t)| \cdot |B_+(\sigma, t)| \cdot [(t-x)^2 + 2\sigma(t-x)] \, dt
\]

and hence by (3.23), (3.24) we get

\[
\|C_+(\cdot, x)\|_{L^1} \leq \Phi(x)
\]

where

\[
\Phi(x) \equiv 2\int_x^\infty (t-x)^2 |V(t)| \, dt + \int_x^\infty |V(t)| \cdot [(t-x)^2 \|B_+(\cdot, t)\|_{L^1} + (t-x)\|C_+(\cdot, t)\|_{L^1}] \, dt.
\]

In particular we see that for all \( x \)

\[
\Phi(x) \geq 0, \quad \Phi'(x) \leq 0, \quad \Phi''(x) \geq 0
\]

i.e., \( \Phi \) is a nonnegative, decreasing and convex function on \( \mathbb{R} \). By differentiating twice \( \Phi \) we obtain

\[
\Phi''(x) = \int_x^\infty |V(t)| \left(4 + 2\|B_+(\cdot, t)\|_{L^1}\right) \, dt + |V(x)| \cdot \|C_+(\cdot, x)\|_{L^1}
\]

\[
\leq C_0(\|V\|_{L^1_2}) + |V(x)| \cdot \|C_+(\cdot, x)\|_{L^1}
\]

where we have used the estimate (3.14) already proved for \( B_+ \). By (3.26) we obtain that \( \Phi(x) \) satisfies the differential inequality

\[
\Phi''(x) \leq C_0(\|V\|_{L^1_2}) + |V(x)| \cdot \Phi(x).
\]

To make the following argument more clear we apply the simple change of variables

\[
\Psi(x) \equiv \Phi(-x)
\]
so that the function $\Psi$ is nonnegative, convex and increasing:

$$\Psi(x) \geq 0, \quad \Psi'(x) \geq 0, \quad \Psi''(x) \geq 0$$

and satisfies the inequality

$$(3.29) \quad \Psi''(x) \leq C_0(\|V\|_{L^1}) + |V(-x)| \cdot \Psi(x).$$

Integrating (3.29) from 0 to $x > 0$ we obtain

$$\Psi'(x) \leq \Psi'(0) + C_0 x + \int_0^x |V(-s)| \cdot \Psi(s) ds;$$

writing

$$\int_0^x |V(-s)| \cdot \Psi(s) ds = \int_0^x |V(-s)| \cdot \int_0^s \Psi'(t) dt ds + \int_0^x |V(-s)| \cdot \Psi(0) ds$$

and noticing that $\Psi'(t)$ is increasing and positive, we obtain

$$\int_0^x |V(-s)| \cdot \Psi(s) ds \leq \int_0^x |V(-s)| \cdot s \cdot \Psi'(s) ds + \Psi(0) \int_0^x |V(-s)| ds$$

In conclusion we have proved that

$$(3.30) \quad \Psi'(x) \leq C_0 x + C_1 \Psi(0) + \int_0^x |V(-s)| \cdot s \cdot \Psi'(s) ds$$

with $C_0, C_1$ depending on $\|V\|_{L^1}$.

We need now the following version of Gronwall's Lemma: if

$$\phi(x) \leq a + bx + \int_0^x a(s) \phi(s) ds, \quad x \geq 0$$

for some constants $a, b$ and some $L^1$ function $a(s) \geq 0$, then

$$(3.31) \quad \phi(x) \leq C(\|a\|_{L^1}) \cdot (a + bx), \quad x \geq 0.$$

To check (3.31), just consider the auxiliary function

$$\psi(x) = \left( a + bx + \int_0^x a(s) \phi(s) ds \right) e^{-\int_0^x a(s) ds}$$

which has the property

$$\psi'(x) \equiv \left( b + a(x) \left( \phi(x) - a - bx - \int_0^x a \phi ds \right) \right) e^{-\int_0^x a(s) ds} \leq be^{-\int_0^x a(s) ds} \leq b;$$

then it is clear that

$$\psi(x) \leq \psi(0) + bx \equiv a + bx$$

and recalling that $\phi(x) \leq \psi(x) e^{\int_0^x a(s) ds}$ we obtain (3.31).

Thus if we apply (3.31) to (3.30) with the choice $\phi \equiv \Psi'$, we obtain

$$(3.32) \quad \Psi'(x) \leq C_2(\|V\|_{L^1}) [\|\cdot\|_{L^1} + \Psi(0) + \Psi'(0)]$$

and integrating one last time from 0 to $x > 0$ we arrive at

$$(3.33) \quad \Psi(x) \leq \left[ C_3(\|V\|_{L^1}) + \Psi(0) + \Psi'(0) \right] \cdot \langle x \rangle^2, \quad x \geq 0.$$
To conclude the proof it is sufficient to estimate \( \Psi(0) \) and \( \Psi'(0) \). Indeed, using the preceding estimates for \( B_+ \) and \( C_+ \) on the positive half line we have immediately
\[
\Psi(0) \equiv \Phi(0) \equiv \int_0^\infty |V(t)| \cdot [4t^2 + 2t^2 \|B_+(\cdot, t)\|_{L^1} + t\|C_+(\cdot, t)\|_{L^1}] \, dt \leq C(\|V\|_{L^2})
\]
and
\[
\Psi'(0) \equiv -\Phi'(0) \equiv \int_0^\infty |V(t)| \cdot [4t + 2t\|B_+(\cdot, t)\|_{L^1} + \|C_+(\cdot, t)\|_{L^1}] \, dt \leq C(\|V\|_{L^1}).
\]
Recalling (3.26), we conclude that
\[
\text{the integral Minkowski inequality now gives}
\]
which are clearly related to the derivatives \( \partial_\lambda m_\pm \); the usefulness of these quantities in the resonant case had already been remarked in [3].

**Corollary 3.1.** Assume \( V \in L^1_2 \); then the functions \( \tilde{C}_\pm(\xi, x) = \int_\mathbb{R} e^{-2i\lambda \xi} n_\pm(\lambda, x) d\lambda \) satisfy
\[
\|\tilde{C}_\pm(\cdot, x)\|_{L^1} \leq C \quad \text{for } \pm x \geq 0, \quad \|\tilde{C}_\pm(\cdot, x)\|_{L^1} \leq C\langle x \rangle^2 \quad \text{for } \pm x \leq 0
\]
for some constant \( C \) depending on \( \|V\|_{L^1_2} \).

**Proof.** We can write
\[
n_\pm(\lambda, x) = \frac{m_\pm(\lambda, x) - m_\pm(0, x)}{\lambda} = \int_0^1 \partial_\lambda m_\pm(\lambda s, x) ds
\]
and this implies, by Fubini’s theorem and the rescaling properties of the one dimensional Fourier transform,
\[
\tilde{C}_\pm(\xi, x) = \int_0^1 F_{\lambda \to \xi} (\partial_\lambda m_\pm(\lambda s, x)) \, ds = \int_0^1 s^{-1} C_\pm(\xi/s, x) \, ds.
\]
The integral Minkowski inequality now gives
\[
\|\tilde{C}_\pm(\cdot, x)\|_{L^1} \leq \int_0^1 s^{-1} \|C_\pm(\cdot, x)\|_{L^1} ds \equiv \int_0^1 \|C_\pm(\cdot, x)\|_{L^1} ds \equiv \|C_\pm(\cdot, x)\|_{L^1}
\]
and by (3.22) the proof is concluded. \( \square \)

We conclude this section by studying the Fourier properties of the Wronskian \( W(\lambda) \)
\[
W(\lambda) = f_+(\lambda, 0)\partial_x f_-(\lambda, 0) - \partial_x f_+(\lambda, 0) f_-(\lambda, 0)
\]
which can be equivalently written
\[
W(\lambda) = m_+(\lambda, 0)\partial_x m_-(\lambda, 0) - \partial_x m_+(\lambda, 0) m_-(\lambda, 0) - 2i\lambda m_+(\lambda, 0) m_-(\lambda, 0).
\]
Notice that the following result is also proved in [11] by partly different arguments.
Lemma 3.4. Let $\chi(\lambda) \in C^\infty_0(\mathbb{R})$ be a smooth cutoff. If $V \in L^1_1(\mathbb{R})$ and $W(0) \neq 0$ then
\[(3.36)\quad \mathcal{F}\left(\frac{\chi(\lambda)}{W(\lambda)}\right) \in L^1(\mathbb{R}).\]

On the other hand, if $V \in L^1_2(\mathbb{R})$ and $W(0) = 0$ then
\[(3.37)\quad \mathcal{F}\left(\frac{\chi(\lambda)\lambda}{W(\lambda)}\right) \in L^1(\mathbb{R}).\]

Proof. Let $\chi_1 \in C^\infty_0(\mathbb{R})$ be a second cutoff such that $\chi_1 \equiv 1$ on the support of $\chi$. By the Deift-Trubowitz estimates (see Lemma 3.1) we know that both $m_{\pm}(\lambda,0) - 1$ and $\partial_x m_{\pm}(\lambda,0)$ have Fourier transform in $L^1$; then writing
\[
\chi_1 W(\lambda) \equiv \chi_1(\lambda)m_+(\lambda,0)\partial_x m_-(\lambda,0) - \chi_1(\lambda)\partial_x m_+(\lambda,0)m_-(\lambda,0) - 2i\lambda\chi_1(\lambda)m_+(\lambda,0)m_-(\lambda,0)
\]
we see that $\chi_1 W$ can be written as a sum of products in which each factor has a Fourier transform in $L^1$, and we conclude that $\chi_1 W$ has Fourier transform in $L^1$.

Recall now that by Wiener’s Lemma, if a function $a(\lambda)$ does not vanish on the support of $b(\lambda)$ and both $\hat{a}, \hat{b} \in L^1$, we have also $\mathcal{F}(b/a) \in L^1$. This implies that
\[
\left\| \mathcal{F}_{\lambda \to \xi} \left( \frac{\chi(\lambda)}{W(\lambda)} \right) \right\|_{L^1_\xi} \equiv \left\| \mathcal{F}_{\lambda \to \xi} \left( \frac{\chi(\lambda)}{\chi_1(\lambda)W(\lambda)} \right) \right\|_{L^1_\xi} < \infty.
\]

Consider now the resonant case with $V \in L^1_2$. Using the functions $n_{\pm}$ defined in (3.34) we can rewrite $W$ as follows:
\[
W(\lambda) = \lambda n_+(\lambda,0)\partial_x m_-(\lambda,0) + \lambda m_+(0,0)\partial_x n_-(\lambda,0) + m_+(0,0)\partial_x m_-(0,0)
+ 2i\lambda m_+(\lambda,0)m_-(\lambda,0);
\]
from this formula and the assumption $W(0) = 0$ we see that the term $m_+(0,0)\partial_x m_-(0,0)$ must vanish, hence we obtain
\[(3.38)\quad \frac{W(\lambda)}{\lambda} = n_+(\lambda,0)\partial_x m_-(\lambda,0) + m_+(0,0)\partial_x n_-(\lambda,0) + 2im_+(\lambda,0)m_-(\lambda,0).
\]
We know already that the functions $m_{\pm}(\lambda,0) - 1$, $\partial_x m_{\pm}(\lambda,0)$ and $n_{\pm}(\lambda,0)$ have Fourier transform in $L^1$; this follows as above from the Deift-Trubowitz estimate and from our Corollary 3.1 (see (3.35)). We can show that also $\partial_x n_{\pm}(\lambda,0)$ have the same property; indeed,
\[
n_{\pm}(\lambda,x) = \int_0^1 \partial_\lambda m_{\pm}(\lambda s,x)ds \quad \Rightarrow \quad \partial_x n_{\pm}(\lambda,x) = \int_0^1 \partial_\lambda \partial_x m_{\pm}(\lambda s,x)ds;
\]
then by Fubini’s theorem and the rescaling properties of the Fourier transform we have
\[
\mathcal{F}_{\lambda \to \xi} (\partial_x n_{\pm}(\lambda,0)) = \int_0^1 2i\xi s^{-1}\partial_x B_{\pm}(\xi/s,0)ds
\]
and, by the integral Minkowski inequality,
\[
\|\mathcal{F}_{\lambda \to \xi} (\partial_x n_{\pm}(\lambda,x))\|_{L^1_\xi} \leq 2 \int_0^1 \|\xi s^{-1}\partial_x B_{\pm}(\xi/s,0)\|_{L^1_\xi}s\,ds = 2 \int_0^1 \|\xi\partial_x B_{\pm}(\xi,0)\|_{L^1_\xi}s\,ds
\]
and we arrive at
\begin{equation}
\|F_{\lambda \to \xi} (\partial_x n_{\pm}(\lambda, x)) \|_{L^1_{\xi}} \leq \|\xi \partial_x B_{\pm}(\xi, 0)\|_{L^1_{\xi}}.
\end{equation}

Recalling now the Deift-Trubowitz estimate (3.12), we have immediately
\[ |\xi \partial_x B_{\pm}(\xi, 0)| \leq C|\xi| \cdot |\eta(\xi) + |V(\xi)|| \implies \|\xi \partial_x B_{\pm}(\xi, 0)\|_{L^1_{\xi}} \leq C\|V\|_{L^1_{\xi}}^2, \]
and this proves that the Fourier transform of $\partial_x n_{\pm}(\lambda, 0)$ belongs to $L^1(\mathbb{R})$.

Now, coming back to (3.38), and choosing a cutoff $\chi_1$ as above, we see that $\chi_1(\lambda)W(\lambda)/\lambda$ can be written as a sum of products of functions with Fourier transform in $L^1$ and hence it also has Fourier transform in $L^1$; applying Wiener’s Lemma exactly as before we conclude the proof. \(\square\)

4. The low energy analysis

In this section we shall study the low energy part of the wave operator $W_+$; the estimate for $W_-$ is completely analogous. By the stationary representation formula (see e.g. [20]), given a cutoff $\Phi(\lambda^2)$ supported near zero, we can represent the low energy part of $W_+$ as follows:
\begin{equation}
W_+ \Phi(H_0)g = \Phi(H_0)g - \frac{1}{\pi i} \int_0^{+\infty} R_\nu(\lambda^2 - i0) V \Im R_0(\lambda^2 + i0) \lambda \Phi(\lambda^2) g d\lambda.
\end{equation}
Thus it is sufficient to study the boundedness in $L^p$ of the operator
\begin{equation}
Ag := \int_0^{+\infty} R_\nu(\lambda^2 - i0) V \Im R_0(\lambda^2 + i0) \lambda \chi(\lambda) g d\lambda
\end{equation}
for an even cutoff function $\chi(\lambda) = \Phi_0(\lambda^2) \in C_0^\infty(\mathbb{R})$.

As remarked in the Introduction, an $L^\infty - L^\infty$ estimate will be impossible in general, owing to the presence of a Hilbert transform term in the wave operator. We recall that the Hilbert transform on $\mathbb{R}$ is the operator
\[ \mathcal{H}g(y) = \frac{1}{\pi} V.P. \int_{\mathbb{R}} \frac{g(s)}{y - s} ds, \]
which can be equivalently defined as a Fourier multiplier with symbol $(2\pi i)^{-1} \text{sgn} \xi$:
\[ \mathcal{H}g = \frac{1}{2\pi i} \int e^{iy\lambda} \frac{\lambda}{|\lambda|} \tilde{g}(\lambda) d\lambda. \]
We also recall that $\mathcal{H}^2 = -1$, and that $\mathcal{H}$ is a bounded operator on $L^p$ for all $1 < p < \infty$, but not on $L^1$ and on $L^\infty$.

In order to state a simple but useful interpolation lemma we introduce the Banach spaces
\begin{equation}
L^\infty_0 = \{ g \in L^\infty : g \to 0 \text{ as } |x| \to \infty \}, \quad \|g\|_{L^\infty_0} = \|g\|_{L^\infty}.
\end{equation}
and
\begin{equation}
L^p_\mathcal{H} = \{ g \in L^p : \mathcal{H}g \in L^p \}, \quad \|g\|_{L^p_\mathcal{H}} = \|g\|_{L^p} + \|\mathcal{H}g\|_{L^p}.
\end{equation}
Notice that the last definition is relevant only when \( p = 1 \) or \( p = \infty \), since we have otherwise

\[
(4.5) \quad L^p_H \simeq L^p \quad \text{for} \quad 1 < p < \infty.
\]

Our interpolation lemma is then the following:

**Lemma 4.1.** Let \( T \) be a bounded operator on \( L^2 \), and assume that

\[
(4.6) \quad \|Tg\|_{L^\infty} + \|T^*g\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|Hg\|_{L^\infty}, \quad \forall g \in L^\infty \cap L^\infty_H \cap L^1.
\]

Then \( T \) and \( T^* \) can be extended to bounded operators on \( L^p \) for all \( 1 < p < \infty \).

**Proof.** From the standard theory of complex interpolation it is known that

\[
(4.7) \quad [L^{p_0}, L^{p_1}]_\theta = L^{p_\theta}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1, \quad 1 \leq p_0 < p_1 \leq \infty
\]

and it is also well known that \( L^\infty \) can be replaced by \( L_0^\infty \) in the interpolation scale:

\[
(4.8) \quad [L^p, L_0^\infty]_\theta = L^{p_\theta}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{p}, \quad 0 < \theta < 1, \quad 1 \leq p < \infty
\]

(see [5]). By an elementary modification in the construction, keeping into account \( (4.5) \), it is easy to see that we have also

\[
(4.9) \quad [L^p, L_0^\infty \cap L^\infty_H]_\theta = L^{p_\theta}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{p}, \quad 0 < \theta < 1, \quad 1 < p < \infty
\]

where \( L_0^\infty \cap L^\infty_H \) is endowed with the norm \( \|g\|_{L^\infty \cap L^\infty_H} = \|g\|_{L^\infty} + \|g\|_{L^\infty_H} \). Now, by a density argument we see that \( (4.6) \) implies that \( T, T^* \) can be extended to bounded operators from \( L_0^\infty \cap L^\infty_H \) to \( L^\infty \), and on the other hand they are bounded on \( L^2 \) by assumption. Using \( (4.9) \), by interpolation we obtain that \( T, T^* \) are bounded on all \( L^p \) for \( 2 \leq p < \infty \), and by duality we conclude the proof. \( \square \)

**Remark 4.1.** In the endpoint case \( p = \infty \) we can modestly improve \( (4.6) \) to

\[
(4.10) \quad \|Tg\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|Hg\|_{L^\infty}, \quad \forall g \in L^\infty \cap L_0^\infty \cap L^1
\]

for some \( p < \infty \); this follows immediately by a density argument. Moreover, in the opposite endpoint \( p = 1 \), by duality, we obtain that

\[
(4.11) \quad \|Tg\|_{L^1 + L^1_H} \leq C\|g\|_{L^1}
\]

where \( L^1 + L^1_H \) is the Banach space with norm

\[
\|g\|_{L^1 + L^1_H} \equiv \inf \{ \|g_1\|_{L^1} + \|g_2\|_{L^1_H} : g = g_1 + g_2, \ g_1 \in L^1, g_2 \in L^1_H \}.
\]

We are now ready to prove our estimate of the low frequency part of the wave operator:

**Lemma 4.2.** Assume \( V \in L^1 \) and the nonresonant condition \( W(0) \neq 0 \) is satisfied. Let \( \Phi(\lambda^2) \) be a smooth compactly supported cutoff function. Then the low energy parts of the wave operators \( W_\pm \) satisfy the estimates

\[
(4.12) \quad \|W_\pm \Phi(H_0)g\|_{L^\infty} \leq C(\|g\|_{L^\infty} + \|Hg\|_{L^\infty}) \quad \forall g \in L^1 \cap L^\infty \cap L^\infty_H
\]

and hence can be extended to bounded operators on \( L^p \), for all \( 1 < p < \infty \). The same properties hold for the conjugate operators \( \Phi(H_0)W_\pm^* \).
**Proof.** The proof for the operators $W_\pm$ and $W_\pm^*$ is completely analogous, hence we shall focus on the estimate for $W_+$. By Lemma 4.11 it is sufficient to prove that $W_+\Phi(H_0)$ satisfies (4.12); moreover, using the stationary representation formula (4.1), we are reduced to estimate the operator $A$ defined by (4.2).

By the explicit expression of the kernel of $R_V$ in terms of the Jost functions (3.3), we can split $A$ as $A = A_1 + A_2$ where (forgetting constants)

\begin{equation}
A_1 g(x) = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{f_+(-\lambda, y)f_-(-\lambda, x)}{W(-\lambda)} V(y)\lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y)
\end{equation}

and

\begin{equation}
A_2 g(x) = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{f_+(-\lambda, x)f_-(-\lambda, y)}{W(-\lambda)} V(y)\lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y).
\end{equation}

Using the relations (3.4) and (4.13) as follows:

\begin{equation}
m_\pm(-\lambda, x) = m_\pm(\lambda, x), \quad W(-\lambda) = \overline{W(\lambda)}
\end{equation}

(see. e.g. [10]), we have

\begin{equation}
A_1 g(x) = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{m_+(\lambda, y)m_-(\lambda, x)}{W(\lambda)} V(y)e^{i\lambda(x-y)} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y)
\end{equation}

\begin{equation}
A_2 g(x) = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{m_+(\lambda, x)m_-(\lambda, y)}{W(\lambda)} V(y)e^{i\lambda(y-x)} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y).
\end{equation}

In the following we shall estimate the operator $A_1$; the proof for $A_2$ is completely analogous. By Fubini’s Theorem we can exchange the order of integration and rewrite (4.15) as follows:

\begin{equation}
A_1 g(x) = \int_{x<y} d\lambda \int d\lambda \chi(\lambda) 1_{(0, +\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y)
\end{equation}

where $\mathcal{F}$ denotes the standard Fourier transform from the $\lambda$ to the $\xi$ variable and $1_{(0, +\infty)}$ is the characteristic function of the half line $(0, +\infty)$.

Now choose a $C_0^\infty$ cutoff function $\psi(\lambda)$ such that $\psi \equiv 1$ on $\text{supp} \chi$; then the function

\begin{equation}
G(\lambda, x, y) = \frac{m_+(\lambda, y)m_-(\lambda, x)}{W(\lambda)} \chi(\lambda) 1_{(0, +\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y)
\end{equation}

can be written as a product

\begin{equation}
G(\lambda, x, y) = F_1(\lambda, y) F_2(\lambda, x) F_3(\lambda) F_4(\lambda, y)
\end{equation}

where

\begin{equation}
F_1(\lambda, y) = m_+(\lambda, y) \psi(\lambda), \quad F_2(\lambda, x) = m_-(\lambda, y) \psi(\lambda), \quad F_3(\lambda) = \frac{\psi(\lambda)}{(\lambda W(\lambda))}
\end{equation}

and

\begin{equation}
F_4(\lambda, y) = \chi(\lambda) 1_{(0, +\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y).
\end{equation}

We are interested in the Fourier transform of $G$ with respect to $\lambda$; this can be written as the convolution of the transforms $\mathcal{F}_j$, $j = 1, 2, 3, 4$. 
By Lemma 3.4 (see (3.36)) we already know that
\[ \| \hat{F}_3(\xi) \|_{L^1} = C_0 < \infty. \]  

Consider now \( \hat{F}_1(\xi, y) \), which can be written
\[ \hat{F}_1(\xi, y) = \mathcal{F}((m_+(\lambda, y) - 1)\psi_1 + \psi_1) = B_+(\xi/2, y) \ast \hat{\psi}_1 + \hat{\psi}_1 \]  
(the inessential factor \( 1/2 \) comes from the nonstandard Fourier transform used in Definition 3.9, and the minus sign from the conjugation). Recalling Lemma 3.2, we get
\[ \| \hat{F}_1(\cdot, y) \|_{L^1} \leq \begin{cases} C & \text{for } y > 0 \\ C\langle y \rangle & \text{for } y \leq 0 \end{cases} \]
for some \( C \) depending on \( \| V \|_{L^1} \). The same argument gives
\[ \| \hat{F}_2(\cdot, x) \|_{L^1} \leq \begin{cases} C & \text{for } x \leq 0 \\ C\langle x \rangle & \text{for } x > 0 \end{cases} \]
with the same \( C \). Recalling that in (4.17) we have \( x < y \), we can write
\[ \| \mathcal{F}(F_1 F_2) \|_{L^1} \leq \| \hat{F}_1(\cdot, x) \|_{L^1} \| \hat{F}_2(\cdot, y) \|_{L^1} \leq \begin{cases} C\langle y \rangle & \text{for } x < y < 0 \\ C & \text{for } x < 0 < y \\ C\langle x \rangle & \text{for } 0 < x < y \end{cases} \]
and in conclusion
\[ \| \mathcal{F}(F_1 F_2) \|_{L^1} \leq C(\| V \|_{L^1}) \cdot \langle y \rangle. \]

Coming back to \( G(\lambda, x, y) \), if we put together (4.19) and (4.20) and we use Young’s inequality, we have proved that, for \( x < y \),
\[ \| \hat{G}(\cdot, x, y) \|_{L^\infty} \leq \mathcal{C}(\| V \|_{L^1}) \cdot \langle y \rangle \cdot \| \hat{F}_4(\cdot, y) \|_{L^\infty}. \]

It remains to estimate
\[ \| \hat{F}_4(\cdot, y) \|_{L^\infty} \equiv \sup_\xi \left| \mathcal{F}_{\lambda \to \xi} (\chi(\lambda)1_{(0, +\infty)}(\lambda)\lambda \Im R_0(\lambda^2 + i0)g(y)) \right|. \]

We have
\[
\mathcal{F}(\chi(\lambda)1_{(0, +\infty)}(\lambda)\lambda \Im R_0(\lambda^2 + i0)g(y)) = \int_0^\infty e^{i\lambda \xi} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0)g(y) d\lambda 
\equiv C e^{i\xi \sqrt{H_0}} \chi(\sqrt{H_0})g
\]
by the spectral theorem. Now we remark that the function
\[ U(\xi, y) = e^{i\xi \sqrt{H_0}} \chi(\sqrt{H_0})g \]
is a solution of the one dimensional wave equation
\[ U_{\xi \xi} + H_0 U \equiv U_{\xi \xi} - U_{yy} = 0, \]
with initial data
\[ U(0, y) = U_0(y) = \chi(\sqrt{H_0})g, \quad U_\xi(0, y) = U_1(y) = i\sqrt{H_0} \chi(\sqrt{H_0})g. \]
By the explicit representation formula of the solution to the wave equation we have then
\[ U(\xi, y) = \frac{U_0(\xi + y) + U_1(\xi - y)}{2} + \frac{1}{2} \int_{\xi - y}^{\xi + y} U_1(\sigma) d\sigma. \]

The first term is easy to bound:
\[ \left| \frac{U_0(\xi + y) + U_1(\xi - y)}{2} \right| \leq \|U_0\|_{L^\infty} = \|\chi(\sqrt{H_0})g\|_{L^\infty} \leq C\|g\|_{L^\infty} \]
since \( \chi(\sqrt{H_0}) \) is bounded on \( L^\infty \) as it is well known. On the other hand, we can write
\[ U_1(y) = i\sqrt{H_0}\chi(\sqrt{H_0})g(y) = \int e^{i\lambda y} |\lambda| \chi(\lambda) \tilde{g}(\lambda) d\lambda \equiv \int e^{i\lambda y} (-i\lambda) \chi(\lambda) \frac{i\lambda}{|\lambda|} \tilde{g}(\lambda) d\lambda \]
and this implies, apart from a constant,
\[ U_1(y) = \frac{d}{dy} \chi(\sqrt{H_0})\mathcal{H}g \quad \Rightarrow \quad \int_{\xi - y}^{\xi + y} U_1(\sigma) d\sigma = \chi(\sqrt{H_0})\mathcal{H}g(\xi + y) - \chi(\sqrt{H_0})\mathcal{H}g(\xi - y). \]
In conclusion
\[ \left| \frac{1}{2} \int_{\xi - y}^{\xi + y} U_1(\sigma) d\sigma \right| \leq \|\chi(\sqrt{H_0})\mathcal{H}g\|_{L^\infty} \leq C\|\mathcal{H}g\|_{L^\infty} \]
and summing up we have proved that
\[ \|\tilde{F}_1\|_{L^\infty_{\xi,y}} \leq C\|U\|_{L^\infty_{\xi,y}} \leq C \left( \|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty} \right). \]
By (4.17), (4.21) and (4.23) we finally obtain
\[ \|A_1g\|_{L^\infty} \leq C(\|V\|_{L^1_1}) \cdot (\|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty}). \]
The operator \( A_2 \) can be estimated in a similar way, and this concludes the proof of the Lemma.

We pass to the analysis of the resonant case \( W(0) = 0 \).

**Lemma 4.3.** Assume \( V \in L^2_1 \) and we are in the resonant case \( W(0) = 0 \). Let \( \Phi(\lambda^2) \) be a smooth compactly supported cutoff function. Then the following estimate holds:
\[ \|W_+ \Phi(H_0)g\|_{L^\infty} \leq C \left( \|g\|_{L^\infty} + \|\mathcal{H}g\|_{L^\infty} \right) \forall g \in L^1 \cap L^\infty \cap L^\infty_{\mathcal{H}} \]
where \( \mathcal{H} \) is the Hilbert transform on \( \mathbb{R} \), and hence can be extended to bounded operators on \( L^p \), for all \( 1 < p < \infty \). The same estimate holds for the conjugate operators \( \Phi(H_0)W^\pm_1 \).

**Proof.** As in the proof of Lemma 4.2, we are reduced to estimate the \( L^\infty \) norm of \( Ag = A_1g + A_2g \) where \( A_jg \) are given by expressions (4.15) and (4.16). The new difficulty now is of course the denominator \( W(\lambda) \) which vanishes at \( \lambda = 0 \). Thus we decompose \( Ag \) into several terms:
\[ Ag = I_1 + I_2 + II_1 + II_2 + III_1 + III_2 \]
where, recalling the notation (3.34).
\[ I_1 = \int_{0}^{+\infty} d\lambda \int_{x>y} dy \frac{-\lambda}{W(-\lambda)} m_+(-\lambda,y)m_-(-\lambda,x)e^{i\lambda(y-x)}V(y)\lambda \chi(\lambda) \Im R_0(\lambda^2 + i0)g(y), \]
\[ I_2 = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{-\lambda}{W(-\lambda)} m_+(-\lambda, x)m_-(-\lambda, y)e^{i\lambda(x-y)}V(y)\chi(\lambda)\Re R_0(\lambda^2 + i0)g(y), \]

\[ II_1 = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{-\lambda}{W(-\lambda)} m_+(0, y)m_-(-\lambda, x)e^{i\lambda(y-x)}V(y)\chi(\lambda)\Re R_0(\lambda^2 + i0)g(y), \]

\[ II_2 = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{-\lambda}{W(-\lambda)} m_+(0, y)m_-(-\lambda, x)e^{i\lambda(x-y)}V(y)\chi(\lambda)\Re R_0(\lambda^2 + i0)g(y), \]

\[ III_1 = \int_0^{+\infty} d\lambda \int_{x>y} dy \frac{m_+(0, y)m_-(-\lambda, y)}{W(-\lambda)}V(y)e^{i\lambda(y-x)}\chi(\lambda)\Re R_0(\lambda^2 + i0)g(y), \]

\[ III_2 = \int_0^{+\infty} d\lambda \int_{x<y} dy \frac{m_+(0, y)m_-(-\lambda, y)}{W(-\lambda)}V(y)e^{i\lambda(y-x)}\chi(\lambda)\Re R_0(\lambda^2 + i0)g(y). \]

An essential remark is the following: since \( W(0) = 0 \), we know that for \( \lambda = 0 \) the Jost functions \( f_+(0, x) \equiv m_+(0, x) \) and \( f_-(0, x) \equiv m_-(0, x) \) are linearly dependent, i.e.,

\[ m_-(0, x) = c_0 \cdot m_+(0, x) \]

for some constant \( c_0 \neq 0 \). Moreover, by definition \( m_\pm(0, x) \to 1 \) as \( \pm x \to \infty \), and together with (4.31) this implies that \( m_\pm(0, x) \) are bounded on \( \mathbb{R} \):

\[ |m_\pm(0, x)| \leq c_1, \quad x \in \mathbb{R}. \]

Finally, when \( W(0) = 0 \) we have

\[ \int_{-\infty}^{+\infty} V(y)m_\pm(0, y)dy = 0 \]

(see e.g. [3]).

The terms of type I and II are handled in a way very similar to the proof of Lemma 4.2. In order to estimate the term \( I_1 \), we write it as

\[ I_1 = \int_{x<y} F_{\lambda \to \xi}(G(\lambda, x, y))|_{\xi = y-x} V(y)dy \]

where

\[ G(\lambda, x, y) = F_1(\lambda, y)F_2(\lambda, x)F_3(\lambda)F_4(\lambda, y), \]

\( \psi \) is a \( C_c^\infty(\mathbb{R}) \) cutoff function such that \( \psi \equiv 1 \) on \( \text{supp} \chi \), and

\[ F_1(\lambda, y) = n_+(-\lambda, y), \quad F_2(\lambda, x) = m_-(-\lambda, x)\psi(\lambda), \quad F_3(\lambda) = \frac{\psi(\lambda)\lambda}{W(-\lambda)} \]

and

\[ F_4(\lambda, y) = \chi(\lambda)1_{(0, +\infty)}(\lambda)\Re R_0(\lambda^2 + i0)g(y). \]
Then we have
\[ \sup_{x<y} \| \hat{G}(\cdot, x, y) \|_{L^\infty} \leq \sup_{x<y} \left( \| \hat{F}_1(\cdot, y) \|_{L^1} \cdot \| \hat{F}_2(\cdot, x) \|_{L^1} \cdot \| \hat{F}_3 \|_{L^1} \cdot \| \hat{F}_4(\cdot, y) \|_{L^\infty} \right) \]
Using Lemma 3.3 we see that \( \| \hat{F}_3 \|_{L^1} = C_0 < \infty \), and by Lemma 3.1 and Corollary 3.1 we obtain as before (by considering the three cases \( x < y < 0 \), \( x < 0 < y \) and \( 0 < x < y \))
\[ \| \hat{F}_1(\cdot, y) \|_{L^1} \cdot \| \hat{F}_2(\cdot, x) \|_{L^1} \leq C(\| V \|_{L^2}^2) \cdot \langle y \rangle^2 \quad \text{for } x < y. \]
Thus we arrive at
\[ \| \hat{G}(\cdot, x, y) \|_{L^\infty} \leq C(\| V \|_{L^2}^2) \cdot \langle y \rangle^2 \cdot \| \hat{F}_4(\cdot, y) \|_{L^\infty} \]
and the remaining term \( \| \hat{F}_4(\cdot, y) \|_{L^\infty} \) has already been estimated in \((4.33)\). Summing up we have proved that
\[ |I_1| \leq C(\| V \|_{L^2}^2) \cdot (\| g \|_{L^\infty} + \| Hg \|_{L^\infty}). \]
The estimate of \( I_2 \) is completely analogous; the estimate of the terms \( II_1 \) and \( II_2 \) is even easier, keeping into account that the functions \( m_{\pm}(0, x) \) are bounded on \( \mathbb{R} \) (see \((4.32))\).

Consider now the more delicate terms \( III_1, III_2 \). Since \( m_-(0, x) = c_0 \cdot m_+(0, x) \) we can put the two integrals back together as follows:
\[ III_1 + III_2 = c_0 \int_0^{+\infty} d\lambda \int dy \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y) e^{i\lambda |y-x|} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y). \]
We decompose this integral in a different way:
\[ III_1 + III_2 = c_0 IV_1 + c_0 IV_2 \]
where
\[(4.34) \quad IV_1 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y) [e^{i\lambda |y-x|} - e^{i\lambda |x|}] \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y) \]
and
\[(4.35) \quad IV_2 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y) e^{i\lambda |x|} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(y). \]
Using the identity
\[e^{i\lambda |y-x|} - e^{i\lambda |x|} = \int_0^1 e^{i\lambda (s|x-y|+(1-s)|x|)} ds \cdot i\lambda \cdot (|x-y|-|x|)\]
and Fubini’s theorem we can rewrite \( IV_1 \) as follows:
\[ IV_1 = \int_0^1 ds \int F_{\lambda \to \xi} \left( \frac{\lambda}{W(-\lambda)} \lambda \chi(\lambda) 1_{(0, +\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y) \right)_{\xi = s|x-y|+(1-s)|x|} K dy \]
where
\[ K = K(x, y) = \Im m_+(0, y)m_+(0, x)V(y) (|x-y|-|x|); \]
notice that
\[(4.36) \quad |K(x, y)| \leq C \| y \| \cdot |V(y)|, \]
by \((4.32)\). At this point, we can proceed as above using Lemma 3.4 and \((4.23)\) to obtain
\[ \left\| F_{\lambda \to \xi} \left( \frac{\lambda}{W(-\lambda)} \lambda \chi(\lambda) 1_{(0, +\infty)}(\lambda) \Im R_0(\lambda^2 + i0) g(y) \right) \right\|_{L^\infty} \leq C(\| V \|_{L^2}^2) \cdot (\| g \|_{L^\infty} + \| Hg \|_{L^\infty}). \]
whence the estimate of $IV_1$ follows immediately.

To conclude the proof, it remains to estimate the term $IV_2$. By property (4.33) we have trivially

$$
(4.37) \quad \int_0^{+\infty} d\lambda \int dy \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y)e^{i\lambda |x|} \lambda \chi(\lambda) \Im R_0(\lambda^2 + i0) g(0) = 0
$$

(indeed, in the inner integral only $V(y)$ and $m_+(0, y)$ depend on $y$). Thus we can subtract (4.37) from $IV_2$ and rewrite it in the form

$$
IV_2 = \int_0^{+\infty} d\lambda \int dy \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y)e^{i\lambda |x|} \lambda \chi(\lambda) \left[ \Im R_0(\lambda^2 + i0) g(y) - \Im R_0(\lambda^2 + i0) g(0) \right] .
$$

We now use the elementary identity

$$
\Im R_0(\lambda^2 + i0) g(y) - \Im R_0(\lambda^2 + i0) g(0) = \int_0^y \partial_s \left( \Im R_0(\lambda^2 + i0) g(s) \right) ds
$$

and we obtain, after applying Fubini’s theorem,

$$
(4.38) \quad IV_2 = \int dy \int_0^y ds \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y) \int_0^{+\infty} e^{i\lambda |x|} \frac{\lambda \chi(\lambda)}{W(-\lambda)} \partial_s \left( \Im R_0(\lambda^2 + i0) g(s) \right) d\lambda
$$

Now we notice the following property: by the spectral theorem

$$
\int_0^{+\infty} \phi(\lambda) \lambda \partial_s \left( \Im R_0(\lambda^2 + i0) g(s) \right) d\lambda = \partial_s \phi(\sqrt{H_0}) g(s),
$$

but we have also, using the representation of $\phi(\sqrt{H_0})$ as a Fourier multiplier (forgetting constants)

$$
\partial_s \phi(\sqrt{H_0}) g(s) = \partial_s \int \mathbb{R} \mathbb{R} e^{i\xi s} \phi(|\xi|) \hat{g}(\xi) d\xi = \int \mathbb{R} e^{i\xi s} \phi(|\xi|) \frac{i\xi}{|\xi|} \hat{g}(\xi) d\xi
$$

and hence

$$
\partial_s \phi(\sqrt{H_0}) g(s) = C \cdot \sqrt{H_0} \phi(\sqrt{H_0}) H g(s).
$$

or, equivalently,

$$
(4.39) \quad \int_0^{+\infty} \phi(\lambda) \lambda \partial_s \left( \Im R_0(\lambda^2 + i0) g(s) \right) d\lambda = C \int_0^{+\infty} \phi(\lambda) \lambda^2 \Im R_0(\lambda^2 + i0) H g(s) d\lambda.
$$

Thus, by (4.38) and (4.39), we obtain (apart from a constant)

$$
IV_2 = \int dy \int_0^y ds \frac{m_+(0, y)m_+(0, x)}{W(-\lambda)} V(y) \int_0^{+\infty} e^{i\lambda |x|} \frac{\lambda \chi(\lambda)}{W(-\lambda)} \Im R_0(\lambda^2 + i0) H g(s) d\lambda
$$

and this can be estimated exactly as the other terms considered above.

The proof is concluded. □
References

[1] Z.S. Agranovich, V.A. Marchenko, The inverse scattering theory. Gordon and Breach, New York, 1963.
[2] S. Agmon, Spectral properties of Schrödinger operators and Scattering Theory, Ann. Sc. Norm. Sup. Pisa Cl. Sci. 2 (1975) no. 2, 151–218.
[3] G. Artbazar, K. Yajima, The $L^p$-continuity of wave operators for one dimensional Schrödinger operators, J. Math. Sci. Univ. Tokyo 7 (2000), 221–240.
[4] J. A. Barcelo, A. Ruiz, L. Vega, Weighted Estimates for the Helmholtz Equation and Some Applications, J. Funct. Anal. 150 (1997) no.2, 356–382.
[5] J. Bergh, J. Löfström, Interpolation spaces. Springer Verlag, Berlin, 1976.
[6] N. Burq, F. Planchon, Smoothing and dispersive estimates for 1d Schrödinger equations with BV coefficients and applications, Preprint 2004.
[7] M. Christ, A. Kiselev, Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials, GAFA 12 (2002), 1174–1234.
[8] P. D’Ancona, L. Fanelli, Decay estimates for the wave and Dirac equations with a magnetic potential, to appear on J. Funct. Anal.
[9] P. D’Ancona, V. Pierfelice, On the wave equation with a large rough potential, to appear on J. Funct. Anal.
[10] P. Deift, E. Trubowitz, Inverse scattering on the line, Comm. Pure and Appl. Math. 33 (1979), 121–251.
[11] M. Goldberg, W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, Comm. Math. Phys. 251 (2004) no. 1, 157–178.
[12] M. Goldberg, Dispersive estimates for the three-dimensional Schrödinger equation with rough potentials, Preprint 2004.
[13] M. Goldberg, M. Visan, A counterexample to dispersive estimates for Schrödinger operators in higher dimensions, Preprint 2005.
[14] A. Jensen, S. Nakamura, Mapping properties of functions of Schrödinger operators between $L^p$-spaces and Besov spaces, Adv. Stud. in Pure Math. 23 (1994), 187–209.
[15] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998) no. 5, 955–980.
[16] I. Rodnianski, W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, Invent. Math. 155 (2004) no. 3, 451–513.
[17] W. Schlag, Dispersive estimates for Schrödinger operators: A survey, preprint 2005.
[18] R. Weder, The $W^{k,p}$-continuity of the Schrödinger Wave Operators on the line, Comm. Math. Phys. 208 (1999), 507–520.
[19] R. Weder, $L^p – L^{p'}$ estimates for the Schrödinger equations on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, J. Funct. Anal. 170 (2000), 37–68.
[20] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators, J. Math. Soc. Japan 47 (1995), 551–581.
[21] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators III, even dimensional cases $m \geq 4$, J. Math. Sci. Univ. Tokyo 2 (1995), 311–346.
[22] K. Yajima, $L^p$-boundedness of wave operators for two-dimensional Schrödinger operators, Comm. Math. Phys. 208 (1999), 125–152.
[23] K. Yajima, Dispersive estimate for Schrödinger equations with threshold resonance and eigenvalue, Comm. Math. Phys. 259 (2005), 475–509.