Permutations of the Haar system

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Abstract

General permutations acting on the Haar system are investigated. We give a necessary and sufficient condition for permutations to induce an isomorphism on dyadic BMO. Extensions of this characterization to Lipschitz spaces $\Lambda(\frac{1}{p})$, $(0 < p \leq 1)$ are obtained. When specialized to permutations which act on one level of the Haar system only, our approach leads to a short straightforward proof of a result due to E.M. Semyonov and B. Stoeckert.

Let us briefly describe the setting in which we are working. $\mathcal{D}$ denotes the set of all dyadic intervals contained in the unit interval. $\pi : \mathcal{D} \to \mathcal{D}$ denotes a permutation of the dyadic intervals. The operator induced by $\pi$ is determined by the equation

$$T_\pi h_I = h_{\pi(I)}$$

where $h_I$ denotes the $L_\infty$-normalised Haar function supported on the dyadic interval $I$. The main result of this paper treats general permutations on BMO and on Lipschitz spaces. The condition on $\pi$ which controls the boundedness of $T_\pi$ is given in terms of the Carleson constant of collections of dyadic intervals. The proof of the general result given below is quite complicated. We start therefore by considering first a special class of permutation operators. To study these operators on $L_p$ E.M. Semyonov introduced the parameter,

$$K = \sup \left\{ \frac{1}{|B^*|} : B \subseteq \mathcal{D} \right\}$$

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where for example $B^*$ denotes the pointset covered by the collection $B$

E.M Semyonov and B.Stöckert proved the following result.

**Theorem 1** If for every $I \in \mathcal{D}$ we have $|\pi(I)| = |I|$ and $K < \infty$ then for $2 \leq p < \infty$ the operator $T_\pi$ is bounded on $L_p$.

We will obtain this result from

**Theorem 2** If for every $I \in \mathcal{D}$ we have $|\pi(I)| = |I|$ and $K < \infty$ then the operator $T_\pi$ is bounded on dyadic-BMO.

**Proof**: Recall that for formal series $f = \sum_{I \in \mathcal{D}} a_I h_I$ the dyadic - BMO norm is given by

$$
\left( \sup_B \frac{1}{|B^*|} \sum_{I \in B} a_I^2 |I| \right)^{\frac{1}{2}}
$$

where the supremum is extended over all collections of dyadic intervals $B$. We fix now $x = \sum x_I h_I$ and obtain $T_\pi x = \sum x_{\pi^{-1}(I)} h_I$. Then choose $B \subseteq \mathcal{D}$ such that

$$
\frac{1}{2} ||T_\pi x||_{BMO}^2 \leq \frac{1}{|B^*|} \sum_B x_{\pi^{-1}(I)}^2 |I|
$$

By hypothesis this expression equals with

$$
\frac{1}{|B^*|} \sum_B x_{\pi^{-1}(I)}^2 |\pi^{-1}(I)|
$$

Which equals trivially with

$$
\frac{|\pi^{-1}(B)^*|}{|B^*|} \frac{1}{|\pi^{-1}(B)^*|} \sum_B x_{\pi^{-1}(I)}^2 |\pi^{-1}(I)|
$$

The last expression is of course bounded by $K ||x||_{BMO}^2$. This finishes the proof of Theorem 2.

**Remark**: 1) For every permutation which satisfies $|\pi(I)| = |I|$ there exist $E \subseteq \mathcal{D}$ and $x \in BMO$ for which the above chain of inequalities can be reversed. Hence for such permutations the condition $K < \infty$ is implied by the boundedness of $T_\pi$.

2) As $T_\pi$ is bounded on $L_2$ we obtain from Corollary 2 p60 that $T_\pi$ is bounded on $L_p$ for $2 < p < \infty$. We thus obtained Theorem 1 from Theorem 2 by interpolation.

Up to this point we considered permutations which act on one level of the Haar system only. We now turn to arbitrary permutations.

To do this we need a scale invariant measure for the size of collections of dyadic Intervals $B$ : the so called Carleson condition. This notion was studied and carefully analyzed by P.W. Jones in his work on the uniform approximation property of BMO.
Definition 1  1. $B$ is said to satisfy the K-Carleson condition if

$$\sup_{J} \frac{1}{|T|} \sum_{J \subseteq I} |J| \leq K$$

The infimum over all such $K$ is called the Carleson constant of $B$ and will be denoted $\text{CC}(B)$.

2. $\text{max}B$ denotes those intervals in $B$ which are not contained in other intervals of $B$. As dyadic intervals are nested, the collection $\text{max}B$ contains only pairwise disjoint dyadic intervals.

Theorem 3 Let $\pi : \mathcal{D} \rightarrow \mathcal{D}$ be any permutation. The operator $T_\pi$ is an isomorphism on BMO if and only if there exists $M \in \mathbb{R}^+$ such that for all $B \subset \mathcal{D}$

$$\frac{1}{M} \text{CC}(B) \leq \text{CC}(\pi(B)) \leq \text{CC}(B) M$$

We will prove Theorem 3 by decomposing $\mathcal{D}$ so as to control the norm of $T_\pi$ on smaller parts. During the decomposition process we will collect additional information concerning the interaction of the small pieces. Each iteration step is based on

Lemma 1 Let $\pi : \mathcal{D} \rightarrow \mathcal{D}$ be a permutation which satisfies the condition of Theorem 3. Then for any $D(I) \subset \mathcal{D}$ with $D(I)^* \subset I$, $K \in \mathbb{R}^+$, and $x \in \text{BMO}$ we obtain a decomposition $D(I) = G(I) \cup S(I)$

1. where $G(I)$ satisfies

$$\sum_{J \in G(I)} x_J^2 \pi(J)^* \leq \|x\|_{\text{BMO}}^2 K$$

2. and $S(I)$ satisfies

$$\sum_{J \in \text{max}S(I)} |J| \leq \frac{M}{K}$$

Consequently for $\mathcal{N}(I) = \pi^{-1}(\text{max} \pi(D(I)))$ and $\mathcal{O}(I) = (\mathcal{N}(I) \cap S(I)) \cup \text{max}S(I)$ we obtain

$$\sum_{J \in \mathcal{O}(I)} \frac{|J|}{|T|} \leq \frac{M(M + 1)}{K}$$

Proof: We define

$$S(I) = \left\{ J \in D(I) : \frac{\pi(J)^*}{|\pi(D(I))^*|} \geq K \frac{|J|}{|T|} \right\}$$
and let \( G(I) = D(I) \setminus S(I) \). The defining inequality for \( J \in G(I) \) implies that

\[
\sum_{J \in G(I)} x_J^2 |\pi(J)| \leq \sum_{J \in G(I)} x_J^2 \frac{|J|}{|I|} K |\pi(D(I))^*|
\]

This expression has \( ||x||_{BMO}^2 K |\pi(D(I))^*| \) as upper bound. It remains to analyze \( S(I) \): by definition of \( S(I) \) we get

\[
\sum_{J \in \max S(I)} |J| \leq \frac{1}{K} \sum_{J \in \max S(I)} |\pi(D(I))^*|
\]

The sum on the right hand side is dominated by the Carleson constant of \( \pi(\max S(I)) \), which by assumption is bounded by \( M \) times the Carleson constant of \( \max S(I) \). However \( \max S(I) \) being a collection of pairwise satisfies the 1 Carleson condition. Summing up we obtain

\[
\sum_{J \in \max S(I)} \frac{|J|}{|I|} \leq \frac{M}{K}
\]

The collection \( \max \pi(D(I)) \) like any other collection of disjoint dyadic intervals has Carleson constant equal to one. Hence \( \pi^{-1}(\max \pi(D(I))) \) satisfies the M Carleson condition. In particular for \( L \in \max S(I) \)

\[
\sum_{\{J \in N(I) : J \subset L\}} |J| \leq M|L|
\]

Let us combine this estimate with the previous analysis of \( S(I) \) to describe the size of \( O(I) \).

\[
\sum_{J \in O(I)} |J| \leq \sum_{L \in \max S(I)} \sum_{\{J \in N(I) : J \subset L\}} |J| \leq \sum_{L \in \max S(I)} M|L| \leq \frac{M^2}{K}|I|
\]

The generations of the index set \( O(I) \) are used to form a stopping time decomposition of \( S(I) \).

**Definition 2**

1. We first recall how generations are formed: let \( G_0(O(I)) = \max O(I) \). Having defined \( G_0(O(I)), \ldots, G_l(O(I)) \) we put

\[
G_{l+1}(O(I)) = \max \left( O(I) \setminus \bigcup_{k \leq l} G_k(O(I)) \right)
\]

2. We now form the crucial decomposition of \( S(I) \):

For \( k \in \{0, 1, 2, \ldots \} \) and \( L \in G_k(O(I)) \) we define \( D(L) = \{ J \in S(I) : J \subset L \} \setminus \{ J \subset P : P \in G_{k+1}(O(I)) \} \)
Comment 1) Consider the following identity
\[ \sum_{J \in D(I)} x_J^2 |\pi(J)| = \sum_{J \in G(I)} x_J^2 |\pi(J)| + \sum_{L \in O(I)} \sum_{J \in D(L)} x_J^2 |\pi(J)| \]

In view of Lemma 1, the sum indexed by G(I) admits a good upper bound. We shifted the bad behavior of the permutation into the sum indexed by O(I). However, we used the hypothesis to show that this index set is geometrically small compared to I. This remark indicates that Lemma 1 permits us to show that a repeated application of the identity defines a converging algorithm.

2) One’s first idea might be to choose \( \max_{S(I)} \) as index set O(I). However, when one tries to prove convergence for the associated decomposition procedure, one meets serious technical difficulties. The way to get around these complications is to choose an index set which contains information about \( \max_{S(I)} \) and \( \pi^{-1}(\max_{\pi}(D(I))) \).

Proof of Theorem 3 The necessity of our condition is implied by the following relation between the Carleson condition and BMO: \( CC(B) = \| \sum_{I \in B} h_I \|_{BMO}^2 \).

The rest of the paper is used to show that the condition of Theorem 3 is also sufficient. We first choose \( J_0 \in D \) such that \( \frac{1}{2} \| T_{\pi} x \|_2 \) is bounded by
\[ \frac{1}{|\pi(J_0)|} \sum_{\{J : \pi(J) \subset \pi(J_0)\}} x_J^2 |\pi(J)| \]

We now let \( B = \pi^{-1}((J : \pi(J) \subset \pi(J_0))) \) and \( O_0 = \max B \). For \( I \in O_0 \) we put \( D(I) = \{ J \in B : J \subset I \} \) having produced \( O_1 \ldots O_l, D_1 \ldots D_l \) and \( N_1 \ldots N_l \). We choose \( I \in O_l \) and \( D(I) \in D_l \). Lemma 1 is now applied to \( D(I) \) and we obtain \( G(I), S(I), N(I) \), and \( O(I) \). Finally, \( S(I) \) is decomposed according to the Definition 2. This gives us \( \{ D(L) \subset S(I) : L \in O(I) \} \). Doing this for each \( I \in O_l \) allows us to define \( O_{l+1} = \bigcup_{I \in O_l} O(I), D_{l+1} = \bigcup_{I \in O_{l+1}} D(I), N_{l+1} = \bigcup_{I \in O_l} N(I) \).

We thereby completed the induction step. The next two claims describe the behaviour of our construction. From now on, \( K \) denotes any number bigger than \( 4M^2 \).

Claim 1 \( O = \bigcup_{l \in \mathbb{N}} O_l \) satisfies the 2 Carleson condition.

Proof: Fix \( I \in O_{k_0} \). If \( J \in O_k \), and \( J \subset I \) then by construction we obtain \( k \geq k_0 \). Applying the estimates of Lemma 1, we obtain for \( k \geq k_0 \)
\[ \sum_{\{J \in O_k : J \subset I\}} |J| \leq \left( \frac{M(M + 1)}{K} \right)^{k-k_0} |I| \]
For \( I \in \mathcal{O}_{k_0} \)

\[
\sum_{\{J \in \mathcal{N} : J \not\subset I\}} |J| = \sum_{k \geq k_0} \sum_{\{J \in \mathcal{O}_k : J \subset I\}} |J|
\]

Invoking the observation above we obtain for this sum the following majorization:

\[
|I| \sum_{k \geq k_0} \left( \frac{M(M+1)}{K} \right)^{k-k_0} \leq |I|^2
\]

**Claim 2** \( \mathcal{N} = \bigcup_{l \in \mathbb{N}} \mathcal{N}_l \) satisfies the \((3M)\) Carleson condition.

**Proof:** When we look back at the construction we see that each dyadic interval lies in at most one of the collections \( \mathcal{N}(J) \). Fix now \( I \in \mathcal{N} \). Hence \( I \in \mathcal{N}_{k_0} \) for some \( k_0 \), and there exists exactly one dyadic interval \( P \) such that \( I \in \mathcal{N}(P) \).

This remark gives the representation

\[
\{J \in \mathcal{N} : J \subset I\} = \{J \in \mathcal{N}(P) : J \subset P\} \cup \{\mathcal{N}(L) : L \in \mathcal{O}, L \subset P\}
\]

We thus obtain the following identity:

\[
\frac{1}{|I|} \sum_{\{J \in \mathcal{N} : J \not\subset I\}} |J| = \frac{1}{|I|} \sum_{J \in \mathcal{N}(P)} |J| + \frac{1}{|I|} \sum_{\{L \in \mathcal{O} : L \subset P\}} \sum_{\{J \in \mathcal{N}(L)\}} |J|
\]

The first summand is simply estimated by \( \text{CC}(\mathcal{N}(P)) \) which in turn is less than \( M \). The second term is majorized by

\[
\frac{1}{|I|} \sum_{\{L \in \mathcal{O} : L \subset P\}} M|L| \leq \text{MCC}(\mathcal{O})
\]

Having proved claim 2, we resume the proof of Theorem 3.

We know now that \( \mathcal{N} \) satisfies the \((3M)\) Carleson condition. Observe that \( \pi(\mathcal{N}) = \bigcup_{I \in \mathcal{O}} \text{max}\pi(D(I)) \). Therefore

\[
\frac{1}{|\pi(J_0)|} \sum_{I \in \mathcal{O}} |\text{max}\pi(D(I))| \leq \text{CC}(\pi(\mathcal{N}))
\]

By hypothesis on \( \pi \) we get \( \text{CC}(\pi(\mathcal{N})) \leq \text{MCC}(\mathcal{N}) \). Recall next that \( \{G(I) : I \in \mathcal{O}\} \) is a decomposition of \( B \). We thus obtain the following final estimates.

\[
\frac{1}{2} \|T_{\mu x}\|_{BMO}^2 \leq \frac{1}{|\pi(J_0)|} \sum_{\{\pi(J) : \pi(J) \in B\}} x^3_3 |\pi(J)| = \frac{1}{|\pi(J_0)|} \sum_{I \in \mathcal{O}} \sum_{\{\pi(J) : \pi(J) \in G(I)\}} x^3_3 |\pi(J)|
\]

\[
\leq \frac{1}{|\pi(J_0)|} \sum_{I \in \mathcal{O}} |\text{max}\pi(D(I))| \|x\|^2
\]
As we observed above this sum is bounded from above by $3M^2||x||^2_{BMO}$
This proves that $T_\pi$ is a bounded operator on $BMO$. As the hypothesis of the
theorem is symmetric in $\pi$ and $\pi^{-1}$ we conclude that $T_{\pi^{-1}} = T_{\pi}^{-1}$ is bounded
as well.

**Extensions to Lipschitz functions:** The dyadic BMO condition appears
as a natural limit of some Lipschitz condition for martingales. We describe
subsequently an extension of our main result to Lipschitz spaces. For
$\varphi \in L^2$ say and $0 < p \leq 1$ the Lipschitz condition assumes the form

$$||\varphi||_{\Lambda(\frac{1}{p} - 1)} = \sup_{I \in D} \left( \int_I |\varphi - \varphi_I|^2 \frac{dt}{|I|} \right)^{\frac{1}{2}} |I|^{1 - \frac{1}{p}}$$

where $\varphi_I$ denotes the meanvalue of $\varphi$ over $I$. The particular interest in these
class of functions stems from a duality relation due to C. Herz \[2\], which general-
izes C.Fefferman’s duality theorem. Herz’s theorem identifies $\Lambda(\frac{1}{p} - 1)$ with the
dual space of dyadic $H^p(0 < p \leq 1)$.

We shall now discuss permutations of Haar functions which are norm-
alised in $\Lambda(\frac{1}{p} - 1)$. The norm of the $L^\infty$ normalised functions $h_I$ in $\Lambda(\frac{1}{p} - 1)$ equals $|I|^{1 - \frac{1}{p}}$. Hence the operator induced by the permutation $\pi : D \to D$ is given by the relation

$$T_{p,\pi} \left( \frac{h_I}{|I|^{1 - \frac{1}{p}}} \right) = \frac{h_{\pi(I)}}{|\pi(I)|^{1 - \frac{1}{p}}}$$

It is useful to observe that the $\Lambda(\frac{1}{p} - 1)$ norm of $f = \sum a_I h_I$ can be expressed
in terms of the coefficients $a_I$. In fact we obtain

$$||f||^2_{\Lambda(\frac{1}{p} - 1)} = \sup_{B} \left( \frac{1}{|I|^{2(\frac{1}{p} - \frac{1}{2})}} \sum_{J \subset I} a_J^2 |J|^{2(\frac{1}{p} - \frac{1}{2})} \right)$$

This formula suggests how to extend properly the notion of Carleson - constant.

**Definition 3** For a collection $\mathcal{B}$ of dyadic intervals, the Carleson p-constant is
given by

$$\sup_{I \in \mathcal{B}} \left( \frac{1}{|I|^{2(\frac{1}{p} - \frac{1}{2})}} \sum_{J \subset I} |J|^{2(\frac{1}{p} - \frac{1}{2})} \right)$$

It will be denoted by $CC_p(\mathcal{B})$

Our extension of Theorem 3 reads now as follows

**Theorem 4** Let $\pi : D \to D$ be a permutation. Let $0 < p \leq 1$. The operator
$T_{p,\pi}$ is an isomorphism on $\Lambda(\frac{1}{p} - 1)$ if and only if there exists $M \in \mathbb{R}^+$ such that
for every $\mathcal{B} \subseteq D$

$$\frac{1}{M} CC_p(\mathcal{B}) \leq CC_p(\pi(\mathcal{B})) \leq CC_p(\mathcal{B}) M$$
The proof of theorem 4 follows the pattern explained above. The first step is again given by the following

**Lemma 2** Let $\pi : D \rightarrow D$ be a permutation which satisfies the condition of Theorem 4. Then for any $D(I) \subset D$ with $D(I)^* \subset I$, $K \in \mathbb{R}^+$, and $x \in \Lambda_{\frac{k}{p} - 1}$ we obtain a decomposition $D(I) = G(I) \cup S(I)$

1. where $G(I)$ satisfies
   \[
   \sum_{\{\pi(J) : J \in G(I)\}} x_J^2 \pi(J)^2 \left(\frac{1}{2} - \frac{1}{k} \right) \leq \sum_{L \in \max \pi(D(I))} |L|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} |x|^{2} \Lambda_{\frac{k}{p} - 1} K
   \]

2. and $S(I)$ satisfies
   \[
   \sum_{J \in \max S(I)} \left(\frac{|J|}{|I|}\right)^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} \leq \frac{M}{K}
   \]

Consequently for $N(I) = \pi^{-1}(\max \pi(D(I)))$ and $O(I) = (N(I) \cap S(I)) \cup \max S(I)$ we obtain
   \[
   \sum_{J \in O(I)} \left(\frac{|J|}{|I|}\right)^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} \leq \frac{M(M + 1)}{K}
   \]

**Proof:** We define
   \[S(I) = \left\{ J \in D(I) : \frac{|\pi(J)|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}}{\sum_{L \in \max \pi(D(I))} |L|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}} \geq K \left(\frac{|J|}{|I|}\right)^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} \right\}\]

and let $G(I) = D(I) \setminus S(I)$. The defining inequality for $J \in G(I)$ implies that
\[
\sum_{J \in G(I)} x_J^2 \pi(J)^2 \left(\frac{1}{2} - \frac{1}{k} \right) \leq \sum_{J \in G(I)} x_J^2 \left(\frac{|J|}{|I|}\right)^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} \sum_{L \in \max \pi(D(I))} |L|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}
\]

This expression has
\[
||x||^{\frac{2}{\left(\frac{k}{p} - 1 \right)}} K \sum_{L \in \max \pi(D(I))} |L|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}
\]
as upper bound. It remains to analyze $S(I)$: by definition of $S(I)$ we get
\[
\sum_{J \in \max S(I)} \left(\frac{|J|}{|I|}\right)^{2 \left(\frac{1}{2} - \frac{1}{k} \right)} \leq \frac{1}{K} \sum_{J \in \max S(I)} \frac{|\pi(J)|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}}{\sum_{L \in \max \pi(D(I))} |L|^{2 \left(\frac{1}{2} - \frac{1}{k} \right)}}
\]
The sum on the right hand side is dominated by the Carleson p constant of \( \pi(\max S(I)) \), which by assumption is bounded by \( M \) times the Carleson p constant of \( \max S(I) \). This in turn is bounded by one. We obtained:

\[
\sum_{J \in \max S(I)} \left( \frac{|J|}{|I|} \right)^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq \frac{M}{K}
\]

The collection \( \max \pi(D(I)) \) has Carleson p constant equal to one. Hence \( \pi^{-1}(\max \pi(D(I))) \) satisfies the \( M \) Carleson p condition. In particular for \( L \in \max S(I) \)

\[
\sum_{\{J \in \mathcal{N}(I) : J \subset L\}} |J|^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq M|L|^{2\left(\frac{1}{p} - \frac{1}{2}\right)}
\]

This estimate and the previous analysis of \( S(I) \) gives an upper bound for the size of \( O(I) \).

\[
\sum_{J \in \mathcal{O}(I)} |J|^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq \sum_{L \in \max S(I)} \sum_{\{J \in \mathcal{N}(I) : J \subset L\}} |J|^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq \sum_{L \in \max S(I)} M|L|^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq \frac{M^2}{K}|I|
\]

**Proof of Theorem 4**

The necessity of our condition is implied by the following fact

\[
\text{CC}_p(B) = \left\| \sum_{I \in \mathcal{B}} \frac{h_I}{|I|^{1 - \frac{1}{p}}} \right\|_{A_p(\mathcal{B})}^2.
\]

We show now that the condition of Theorem 4 is also sufficient. We first choose \( J_0 \in \mathcal{D} \) such that \( \frac{1}{2}||T_p, \pi x||_{A_p(\mathcal{B})}^2 \) is bounded by

\[
\frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p} - \frac{1}{2}\right)}} \sum_{\{J : \pi(J) \subset \pi(J_0)\}} x_J^2|\pi(J)|^{2\left(\frac{1}{p} - \frac{1}{2}\right)}
\]

We now let \( B = \pi^{-1}(\{J : \pi(J) \subset \pi(J_0)\}) \). Using Lemma 2 we form as in the proof of Theorem 3 the collections \( \mathcal{O}, \mathcal{N}, \{G(I) : I \in \mathcal{O}\} \) and \( \{D(I) : I \in \mathcal{O}\} \) such that:

1. \( \text{CC}_p(\mathcal{N}) \leq 3M \)
2. \( \{G(I) : I \in \mathcal{O}\} \) is a decomposition of \( \mathcal{B} \)
3. \( \sum_{J \in G(I)} x_J^2|\pi(J)|^{2\left(\frac{1}{p} - \frac{1}{2}\right)} \leq \sum_{L \in \max \pi(D(I))} |L|^{2\left(\frac{1}{p} - \frac{1}{2}\right)}|x|_{A_p(\mathcal{B})}^2 \leq \frac{M}{K} \)
As \( \pi(\mathcal{N}) = \bigcup_{I \in \mathcal{O}} \max_\pi(D(I)) \) we obtain

\[
\frac{1}{|\pi(J_0)|^2(\frac{1}{p} - \frac{1}{2})} \sum_{I \in \mathcal{O}} \sum_{L \in \max_\pi(D(I))} |L|^{2(\frac{1}{p} - \frac{1}{2})} \leq \text{CC}_p(\pi(\mathcal{N}))
\]

By hypothesis on \( \pi \) we get \( \text{CC}_p(\pi(\mathcal{N})) \leq M \text{CC}_p(\mathcal{N}) \). Moreover \( \{G(I) : I \in \mathcal{O}\} \) is a decomposition of \( \mathcal{B} \). We thus obtain

\[
\frac{1}{2} \left| ||T_{p,\pi}x||_{A(\frac{1}{p} - 1)}^2 \right| \leq \frac{1}{|\pi(J_0)|^2(\frac{1}{p} - \frac{1}{2})} \sum_{(\pi(J)) : J \in \mathcal{B}} x_J^2 |\pi(J)|^{2(\frac{1}{p} - \frac{1}{2})} = \frac{1}{|\pi(J_0)|^2(\frac{1}{p} - \frac{1}{2})} \sum_{I \in \mathcal{O}} \sum_{L \in \max_\pi(D(I))} |L|^{2(\frac{1}{p} - \frac{1}{2})} ||x||_{(\frac{1}{p} - 1)}^2
\]

We observed already that this sum is bounded by \( 3M^2 ||x||_{(\frac{1}{p} - 1)}^2 \). This proves that \( T_{p,\pi} \) is a bounded operator on \( A(\frac{1}{p} - 1) \). Repeating this procedure with \( \pi \) replaced by \( \pi^{-1} \) we are able to conclude that \( T_{p,\pi^{-1}} = T_{p,\pi} \) is bounded as well.

**Remark**

In view of the above mentioned duality results due to C. Herz and C. Fefferman the transposed operator of \( T_{p,\pi} \), \( (0 < p \leq 1) \) is given by the operator \( S_{p,\pi} \) defined by \( \pi^{-1} \) acting on \( H^p \) normalised Haar functions. More precisely \( S_{p,\pi} : H^p \to H^p \) is given by the relation.

\[
S_{p,\pi} \left( \frac{h_I}{|I|^{\frac{1}{p}}} \right) = \frac{h_{\pi^{-1}(I)}}{|\pi^{-1}(I)|^{\frac{1}{p}}}
\]

An equivalent formulation of Theorem 4 is thus given by

**Theorem 5** Let \( \pi : \mathcal{D} \to \mathcal{D} \) be a permutation. Let \( 0 < p \leq 1 \). The operator \( S_{p,\pi} \) is an isomorphism on dyadic \( H^p \) if and only if there exists \( M \in \mathbb{R}^+ \) such that for every \( \mathcal{B} \subseteq \mathcal{D} \)

\[
\frac{1}{M} \text{CC}_p(\mathcal{B}) \leq \text{CC}_p(\pi(\mathcal{B})) \leq \text{CC}_p(\mathcal{B}) M
\]

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