The hyperbolic $BC_n$ Sutherland and the rational $BC_n$ Ruijsenaars–Schneider–van Diejen models: Lax matrices and duality

B.G. Pusztai
Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, H-6720 Szeged, Hungary
e-mail: gpusztai@math.u-szeged.hu

Abstract

In this paper, we construct canonical action-angle variables for both the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ Ruijsenaars–Schneider–van Diejen models with three independent coupling constants. As a byproduct of our symplectic reduction approach, we establish the action-angle duality between these many-particle systems. The presented dual reduction picture builds upon the construction of a Lax matrix for the $BC_n$-type rational Ruijsenaars–Schneider–van Diejen model.

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1 Introduction

The Calogero–Moser–Sutherland (CMS) and the Ruijsenaars–Schneider–van Diejen (RSvD) interacting many-particle models play a distinguished role among the integrable Hamiltonian systems, having numerous relationships to important fields of mathematics and physics. They have profound applications in the theory of solitons (see e.g. [1], [2], [3], [4]), and recently they appeared in the context of random matrix theory as well (see e.g. [5], [6]). Quite surprisingly, these intriguing relationships are well-understood only for the models associated with the $A_n$ root system. It appears that the main technical obstacle for developing analogous theories in association with the non-$A_n$-type root systems is the lack of knowledge of explicit action-angle variables for the non-$A_n$-type CMS and RSvD models. In this paper we wish to narrow this gap by constructing action-angle systems of canonical coordinates for both the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models.

In order to define the $BC_n$-type hyperbolic Sutherland and the rational RSvD many-particle systems, we first introduce the subset

$$\mathfrak{c} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ldots > x_n > 0 \} \subset \mathbb{R}^n,$$

which can be seen as an appropriate model for the open Weyl chamber of type $BC_n$. Let us recall that the phase space of the Sutherland model is the cotangent bundle of $\mathfrak{c}$, for which we have the natural identification

$$\mathcal{P}^S = \mathfrak{c} \times \mathbb{R}^n = \{ (q, p) \mid q \in \mathfrak{c}, p \in \mathbb{R}^n \}.$$  \hfill (1.2)

Recall also that the hyperbolic $BC_n$ Sutherland dynamics is generated by the interacting many-particle Hamiltonian

$$H^S = \frac{1}{2} \sum_{c=1}^n p^2_c + \sum_{1 \leq a < b \leq n} \left( \frac{g^2}{\sinh^2(q_a - q_b)} + \frac{g^2}{\sinh^2(q_a + q_b)} \right) + \sum_{c=1}^n \frac{g_1^2}{\sinh^2(q_c)} + \sum_{c=1}^n \frac{g_2^2}{\sinh^2(2q_c)},$$

where the so-called coupling parameters $g$, $g_1$, and $g_2$ are arbitrary real numbers satisfying the inequalities $g^2 > 0$ and $g_1^2 + g_2^2 > 0$. In other words, we are interested only in the hyperbolic $BC_n$ Sutherland model with purely repulsive interaction. Since the strength of the interaction is governed by the numbers $g^2$, $g_1^2$, and $g_2^2$, they are usually called the coupling constants.

Though the non-$A_n$-type CMS models have received a lot of attention in the last couple of decades (see e.g. the fundamental papers [7], [8] and the book [9]), the symplectic reduction understanding of the $BC_n$ Sutherland model with three independent coupling constants is a quite recent development [10]. Besides providing a Lax representation of the dynamics, the symplectic reduction approach has the advantage that it naturally leads to a fairly simple

1Notice that with the specialization $g_2 = 0$ we recover the hyperbolic $B_n$ Sutherland particle system, meanwhile with $g_1 = 0$ we obtain the $C_n$-type model.
solution algorithm of purely algebraic nature. By pushing forward the reduction picture, in this paper we are able to furnish action-angle variables for the standard $BC_n$ Sutherland model with repulsive interaction.

The non-$A_n$-type deformations of the classical Ruijsenaars–Schneider many-particle systems have been introduced by van Diejen [11]. Just as for the Sutherland model, the phase space of the rational $BC_n$ RSvD model is the cotangent bundle $T^*c$, which is naturally identified with the manifold

$$\mathcal{P}^R = c \times \mathbb{R}^n = \{ (\lambda, \theta) \mid \lambda \in c, \theta \in \mathbb{R}^n \}. \quad (1.4)$$

Recall that the rational RSvD dynamics is characterized by the Hamiltonian

$$H^R = \sum_{c=1}^{n} \cosh(2\theta_c) \left( 1 + \frac{\nu^2}{\lambda^2_c} \right) \left( 1 + \frac{\kappa^2}{\lambda^2_c} \right) \prod_{d=1, \ d \neq c}^{n} \left( 1 + \frac{4\mu^2}{(\lambda_c - \lambda_d)^2} \right) \left( 1 + \frac{4\mu^2}{(\lambda_c + \lambda_d)^2} \right)$$

$$+ \frac{\nu \kappa}{4\mu^2} \prod_{c=1}^{n} \left( 1 + \frac{4\mu^2}{\lambda^2_c} \right) - \frac{\nu \kappa}{4\mu^2}, \quad (1.5)$$

where $\mu$, $\nu$ and $\kappa$ are arbitrary real parameters satisfying $\mu \neq 0 \neq \nu$. Although the Liouville integrability of the non-$A_n$-type RSvD models has been verified [11], the Lax representation of their dynamics is still missing. However, by generalizing our results on the $C_n$-type RSvD model [12], in this paper we provide a Lax matrix and an elementary solution algorithm for the rational RSvD model (1.5) with $\nu \kappa \geq 0$. Moreover, the proposed reduction approach permits us to construct action-angle variables as well.

The organization of the paper can be outlined as follows. Section 2 is devoted to a brief account on the necessary group theoretic and symplectic geometric background underlying the derivation of the Sutherland and the RSvD models from a unified symplectic reduction framework. In Section 3 we review the symplectic reduction understanding of the $BC_n$ Sutherland model. Although this is a standard material (see [10]), our new contribution on the spectral properties of the Lax matrix of the Sutherland model, formulated in Lemma 1, seems to be crucial in advancing the reduction approach to cover the RSvD model, too. Starting with Section 4 we present our new results on the rational $BC_n$ RSvD model. By fitting the $BC_n$ RSvD model into a convenient symplectic reduction picture, we are able to provide a Lax matrix and an elementary solution algorithm as well. The main technical result is Theorem 5, in which we confirm that the parametrization of the Lax matrix of the RSvD model does provide a Darboux system on the reduced phase space. In Section 5 we elaborate on the consequences of the proposed reduction approach. In particular, a natural construction of canonical action-angle variables for both the Sutherland and the RSvD model comes for free. Furthermore, the action-angle duality between the repulsive $BC_n$ Sutherland model and the $BC_n$ RSvD system with $\nu \kappa \geq 0$ becomes also transparent.

This paper is a continuation of our recent work [12] on the hyperbolic Sutherland and the rational RSvD models associated with the $C_n$ root system. It is a very fortunate situation that many results for the models associated with the $BC_n$ root system can be derived almost effortlessly by generalizing the analogous results of the $C_n$-type particle systems. Since in [12] we have carried out a very detailed analysis on the particle systems of type $C_n$, in this paper
we can be brief on many aspects of the $BC_n$-type models. Though our presentation tries to be self-contained, in this paper we rather focus on differences between the $C_n$-type and the $BC_n$-type models, and we provide proofs only for those facts that have no natural analogs in the $C_n$ case. Therefore the reader may find it useful to have a copy of [12] on hand while reading the paper.

2 Preliminaries

In this section we gather the necessary group theoretic and symplectic geometric material underlying the unified symplectic reduction derivation of the hyperbolic $BC_n$ Sutherland and the rational $BC_n$ RSvD models. Throughout the paper our group theoretic conventions try to be consistent with the book [13], whereas the symplectic geometric conventions come mainly from [14]. To facilitate the comparison with our work on the $C_n$-type models, the majority of the notations are directly borrowed from paper [12].

Take an arbitrary positive integer $n \in \mathbb{N} = \{1, 2, \ldots\}$ and let $N = 2^n$. With the aid of the $N \times N$ unitary matrix

$$C = \begin{bmatrix} 0_n & 1_n \\ 1_n & 0_n \end{bmatrix} \in U(N)$$

(2.1)

we define the non-compact real reductive matrix Lie group

$$G = U(n, n) = \{y \in GL(N, \mathbb{C}) \mid y^*Cy = C\}.$$  

(2.2)

The corresponding real matrix Lie algebra has the form

$$\mathfrak{g} = u(n, n) = \{Y \in \mathfrak{gl}(N, \mathbb{C}) \mid Y^* + CY = 0\},$$

(2.3)

on which the map

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad (Y_1, Y_2) \mapsto \langle Y_1, Y_2 \rangle = \text{tr}(Y_1Y_2)$$

(2.4)

provides a symmetric Ad-invariant non-degenerate bilinear form.

Let us remember that the fixed-point set of the Cartan involution $\Theta(y) = (y^*)^{-1} (y \in G)$ naturally selects a maximal compact subgroup

$$K = \{y \in G \mid \Theta(y) = y\} = \{y \in G \mid y \text{ is unitary} \} \cong U(n) \times U(n)$$

(2.5)

of the Lie group $G$. Also, the Lie algebra involution $\theta(Y) = -Y^* (Y \in \mathfrak{g})$ corresponding to $\Theta$ naturally induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

(2.6)

with the Lie subalgebra and the complementary subspace

$$\mathfrak{k} = \ker(\theta - \text{Id}) = \{Y \in \mathfrak{g} \mid Y^* = -Y\} \quad \text{and} \quad \mathfrak{p} = \ker(\theta + \text{Id}) = \{Y \in \mathfrak{g} \mid Y^* = Y\},$$

(2.7)

respectively. Due to the Cartan decomposition (2.6), each $Y \in \mathfrak{g}$ can be decomposed uniquely as

$$Y = Y_+ + Y_- \quad (Y_+ \in \mathfrak{k}, Y_- \in \mathfrak{p}).$$

(2.8)
Next, notice that the set of diagonal matrices

\[ a = \{ Q = \text{diag}(q_1, \ldots, q_n, -q_1, \ldots, -q_n) \in p \mid (q_1, \ldots, q_n) \in \mathbb{R}^n \} \]  \hspace{1cm} (2.9)

forms a maximal Abelian subspace in \( p \). Let \( a^\perp \) denote the subspace of the off-diagonal elements of \( p \); then we have the orthogonal decomposition \( p = a \oplus a^\perp \). Let us also consider the centralizer of \( a \) inside \( K \), which is the Abelian subgroup

\[ M = \mathcal{Z}_K(a) = \{ \text{diag}(e^{i\chi_1}, \ldots, e^{i\chi_n}) \in K \mid (\chi_1, \ldots, \chi_n) \in \mathbb{R}^n \}. \]  \hspace{1cm} (2.10)

Obviously its Lie algebra has the form

\[ m = \{ \text{diag}(i\chi_1, \ldots, i\chi_n, i\chi_1, \ldots, i\chi_n) \in \mathfrak{k} \mid (\chi_1, \ldots, \chi_n) \in \mathbb{R}^n \}. \]  \hspace{1cm} (2.11)

If \( m^\perp \) denotes the subspace of the off-diagonal elements of \( \mathfrak{k} \), then we can write \( \mathfrak{k} = m \oplus m^\perp \).

Finally, recalling the Cartan decomposition \( (2.6) \), we end up with the refined decomposition

\[ g = m \oplus m^\perp \oplus a \oplus a^\perp. \]  \hspace{1cm} (2.12)

Having equipped with the above group theoretic objects, in the rest of the section we review some basic notions from symplectic geometry. Recall that the cotangent bundle \( T^*G \) of the Lie group \( G \) can be trivialized, say, by left translations. Upon identifying the dual space \( \mathfrak{g}^* \) with the Lie algebra \( \mathfrak{g} \) via the bilinear form \( (2.4) \), it is clear that the product manifold

\[ \mathcal{P} = G \times \mathfrak{g} = \{(y, Y) \mid y \in G, Y \in \mathfrak{g}\} \]  \hspace{1cm} (2.13)

provides a convenient model for \( T^*G \). Furthermore, the tangent spaces of \( \mathcal{P} \) can be naturally identified as

\[ T_{(y,Y)} \mathcal{P} = T_{(y,Y)}(G \times \mathfrak{g}) \cong T_yG \oplus T_Y \mathfrak{g} \cong T_yG \oplus \mathfrak{g} \]  \hspace{1cm} ((y, Y) \in \mathcal{P}). \hspace{1cm} (2.14)

Let us observe that on the model space \( \mathcal{P} \cong T^*G \) the canonical one-form \( \vartheta \in \Omega^1(\mathcal{P}) \) reads

\[ \vartheta_{(y,Y)}(\delta y \oplus \delta Y) = \langle y^{-1}\delta y, Y \rangle \hspace{1cm} ((y, Y) \in \mathcal{P}, \delta y \oplus \delta Y \in T_yG \oplus \mathfrak{g}) \]  \hspace{1cm} (2.15)

whereas for the canonical symplectic form we use the convention \( \omega = -d\vartheta \in \Omega^2(\mathcal{P}) \).

Now note that the smooth left action of the product Lie group \( K \times K \) on the group manifold \( G \) defined by the formula

\[ (k_L, k_R) \cdot y = k_L y k_R^{-1} \hspace{1cm} (y \in G, (k_L, k_R) \in K \times K) \]  \hspace{1cm} (2.16)

naturally lifts onto \( T^*G \). Working with the model space \( \mathcal{P} \) \( (2.13) \) of the cotangent bundle, the lift of the above \( K \times K \)-action \( (2.16) \) takes the form

\[ (k_L, k_R) \cdot (y, Y) = (k_L y k_R^{-1}, k_R Y k_R^{-1}) \hspace{1cm} ((y, Y) \in \mathcal{P}, (k_L, k_R) \in K \times K). \]  \hspace{1cm} (2.17)

This action is clearly symplectic, admitting the \( K \times K \)-equivariant momentum map

\[ J : \mathcal{P} \to (\mathfrak{k} \oplus \mathfrak{k})^* \cong \mathfrak{k} \oplus \mathfrak{k}, \hspace{0.5cm} (y, Y) \mapsto J(y, Y) = (yYy^{-1})_+ \oplus (-Y_+). \]  \hspace{1cm} (2.18)
Without any further notice, in the rest of the paper we shall frequently use the natural dual space identification \((\mathfrak{k} \oplus \mathfrak{k})^* \cong \mathfrak{k} \oplus \mathfrak{k}\) induced by the bilinear form (2.3).

To proceed further, with each column vector \(V \in \mathbb{C}^N\) subject to the conditions \(V^*V = N\) and \(CV + V = 0\) we associate the Lie algebra element

\[
\xi(V) = i\mu(VV^* - 1_N) + i(\mu - \nu)C \in \mathfrak{k},
\]

where \(\mu, \nu\) are arbitrary real parameters satisfying \(\mu \neq 0 \neq \nu\). Also, let \(E \in \mathbb{C}^N\) denote the distinguished column vector with components

\[
E_a = -E_{n+a} = 1 \quad (a \in \mathbb{N}_n = \{1, \ldots, n\}),
\]

and consider the Lie algebra element

\[
J_0 = (-\xi(E)) \oplus i\kappa C \in \mathfrak{k} \oplus \mathfrak{k},
\]

where \(\kappa\) is an arbitrary real parameter. In order to derive the hyperbolic \(BC_n\) Sutherland and the rational \(BC_n\) RSvD models from symplectic reduction, we wish to reduce the symplectic manifold \((\mathcal{P}, \omega)\) at the very special value \(J_0\) (2.21) of the momentum map \(J\) (2.18). We mention in passing that the parametrizations of the Lie algebra elements \(\xi(V)\) (2.19) and \(J_0\) (2.21) turn out to be very natural in the sense that, after performing the reduction, the parameter triple \((\mu, \nu, \kappa)\) can be identified with the coupling parameters of the rational \(BC_n\) RSvD model (1.5).

Our experience with the CMS and the RSvD models convinces us that, in general, the application of the shifting trick leads to a shorter and neater derivation of these particle systems from symplectic reduction. As the initial step of the shifting trick (see e.g. [15]), we have to identify the adjoint orbit passing through \(-J_0\) (2.21). Since \(C\) commutes with each element of \(K\), for the adjoint orbit in question we have the natural identification

\[
\mathcal{O} \oplus \{-i\kappa C\} \cong \mathcal{O},
\]

where

\[
\mathcal{O} = \mathcal{O}(\xi(E)) = \{\xi(V) \in \mathfrak{k} \mid V \in \mathbb{C}^N, V^*V = N, CV + V = 0\}.\]

Following the prescription of the shifting trick, we also introduce the extended phase space

\[
\mathcal{P}^\text{ext} = \mathcal{P} \times \mathcal{O} = \{(y, Y, \rho) \mid y \in G, Y \in \mathfrak{g}, \rho \in \mathcal{O}\},
\]

and endow it with the product symplectic structure

\[
\omega^\text{ext} = \omega + \omega^\mathcal{O},
\]

where, of course, \(\omega^\mathcal{O}\) is the standard Kirillov–Kostant–Souriau symplectic form carried by the orbit \(\mathcal{O}\) (2.22). The natural extension of the \(K \times K\)-action (2.17) onto \(\mathcal{P}^\text{ext}\) is given by the diagonal action

\[
(k_L, k_R) \cdot (y, Y, \rho) = (k_L y k_R^{-1}, k_R Y k_R^{-1}, k_L \rho k_L^{-1}),
\]

and the corresponding \(K \times K\)-equivariant momentum map takes the form

\[
J^\text{ext} : \mathcal{P}^\text{ext} \to \mathfrak{k} \oplus \mathfrak{k}, \quad (y, Y, \rho) \mapsto J^\text{ext}(y, Y, \rho) = ((yY y^{-1})_+ + \rho) \oplus (-Y_+ - i\kappa C).
\]
As a matter of fact, it is clear that $J^\text{ext}$ takes its values in the subalgebra
\[ s(\mathfrak{g} \oplus \mathfrak{g}) = \{ X_L \oplus X_R \in \mathfrak{g} \oplus \mathfrak{g} \mid \text{tr}(X_L) + \text{tr}(X_R) = 0 \} \leq \mathfrak{g} \oplus \mathfrak{g}. \] (2.27)

Now, the shifting trick guarantees that
\[ \mathcal{P} / \sim_j(K \times K) \cong \mathcal{P}^\text{ext} / \sim_j(K \times K), \] (2.28)
i.e. for our purposes it is an equally valid approach to perform the Marsden–Weinstein reduction of the symplectic manifold $(\mathcal{P}^\text{ext}, \omega^\text{ext})$ at the zero value of the momentum map $J^\text{ext}$ (2.26).

### 3 The hyperbolic $BC_n$ Sutherland model

In this section we review the Lax matrix and the symplectic reduction understanding of the hyperbolic $BC_n$ Sutherland model with three independent coupling constants. Our discussion on the reduction aspects of the model is mainly based on the ideas presented in [10], adapted to the conventions of [12].

#### 3.1 The Lax matrix of the Sutherland model

The main goal of this subsection is to introduce the Lax matrix of the Sutherland model, and to analyze some of its spectral properties that prove to be pertinent in the symplectic geometric understanding of the $BC_n$ RSvD model. As a preparatory step, with each point $(q, p) \in \mathcal{P}^S$ we associate the diagonal matrices
\[ Q = \text{diag}(q_1, \ldots, q_n, -q_1, \ldots, -q_n) \in \mathfrak{a} \quad \text{and} \quad P = \text{diag}(p_1, \ldots, p_n, -p_1, \ldots, -p_n) \in \mathfrak{a}. \] (3.1)

Let $\tilde{\text{ad}}_Q$ denote the restriction of the linear operator $\text{ad}_Q = [Q, \cdot] \in \mathfrak{gl}(\mathfrak{g})$ onto the off-diagonal part of the Lie algebra $\mathfrak{g}$ (2.18). Notice that the regularity condition $q \in c$ ensures the invertibility of the linear operator $\tilde{\text{ad}}_Q$. Therefore, making use of the standard functional calculus, the matrix
\[ L_p(q, p) = P - \sinh(\tilde{\text{ad}}_Q)^{-1}\xi(E) + \coth(\tilde{\text{ad}}_Q)(\iota \kappa C) \in \mathfrak{p} \] (3.2)
is well-defined. Let us note that the above introduced $N \times N$ matrix $L_p = L_p(q, p)$ is Hermitian with block-matrix structure
\[ L_p = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix}, \] (3.3)
where $A$ and $B$ are $n \times n$ matrices satisfying $A^* = A$ and $B^* = -B$. More concretely, for their matrix entries we have
\[ A_{a,b} = -\frac{i\mu}{\sinh(q_a - q_b)}, \quad A_{c,c} = p_c, \quad B_{a,b} = \frac{i\mu}{\sinh(q_a + q_b)}, \quad B_{c,c} = \frac{i\nu}{\sinh(2q_c)} + i\kappa \coth(2q_c), \] (3.4)
where $a, b, c \in \mathbb{N}_n$ and $a \neq b$. Now, with the parameter triple $(\mu, \nu, \kappa)$ we associate the map
\[ L: \mathcal{P}^S \to \mathfrak{g}, \quad (q, p) \mapsto L(q, p) = L_p(q, p) - i\kappa C. \] (3.5)
As one can see in [10], the above map $L$ provides a Lax matrix for the hyperbolic $BC_n$ Sutherland model. The exact relationship between the parameters $(\mu, \nu, \kappa)$ and the Sutherland coupling parameters $(g, g_1, g_2)$ appearing in (1.3) will be clarified later (see (3.65) and (3.66)).

Having defined the Lax matrix of the $BC_n$ Sutherland model, we now turn our attention to its spectral properties. Remembering that the only difference between $L$ and $L_p$ is the anti-Hermitian constant term $i\kappa C$, it is clear that the spectral properties of the non-Hermitian Lax matrix $L$ (3.5) can be understood by analyzing the spectrum of the Hermitian matrix $L_p$ (3.2). Since $L_p$ belongs to the complementary subspace $p$ (2.7), we know from general principles that it can be conjugated into the maximal Abelian subspace $a$ (2.9) by some element of the maximal compact subgroup $K$ (2.5). However, this diagonalization procedure becomes much more explicit by exploiting the singular value decomposition of the sum of the matrices $A$ and $B$ introduced in the block-matrix decomposition (3.3). More precisely, we can write

$$A + B = usv^*, \quad (3.6)$$

where $u$ and $v$ are $n \times n$ unitary matrices, meanwhile $s = \text{diag}(s_1, \ldots, s_n)$ is a diagonal matrix filled in with the singular values $s_1 \geq \ldots \geq s_n \geq 0$ of the matrix $A + B$. Now, upon defining the $N \times N$ block-matrices

$$k = \frac{1}{2} \begin{bmatrix} v + u & v - u \\ v - u & v + u \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}, \quad (3.7)$$

one can easily verify that $k \in K$, $S \in a$ and $L_p = kSk^{-1}$. Having diagonalized the matrix $L_p$ (3.2), from the definition (3.5) we see at once that

$$L^2 = k(S^2 - \kappa^2 1_N)k^{-1}, \quad (3.8)$$

therefore the spectrum of the Hermitian matrix $L^2$ can be identified as

$$\sigma(L^2) = \sigma(S^2 - \kappa^2 1_N) = \sigma(s^2 - \kappa^2 1_n). \quad (3.9)$$

On the other hand, remembering the singular value decomposition (3.6), we can also write

$$(A + B)(A - B) = (A + B)(A + B)^* = usv^*vus^* = u\sigma(u)^{-1}, \quad (3.10)$$

from where we obtain the spectral identification

$$\sigma(s^2) = \sigma(A^2 - B^2 - [A, B]). \quad (3.11)$$

Now, the comparison of the equations (3.9) and (3.11) immediately leads to the formula

$$\sigma(L^2) = \sigma(A^2 - B^2 - [A, B] - \kappa^2 1_n). \quad (3.12)$$

Since $L^2$ is an Hermitian matrix, the spectral mapping theorem guarantees that each eigenvalue of $L$ is either a real number or a purely imaginary number. However, under certain technical assumptions, the relationship (3.12) permits us to provide a more accurate description for the spectrum of $L$. 

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Lemma 1. Suppose that $\nu \neq 2\mu$ and $\nu \kappa \geq 0$; then for each point $(q, p) \in P^S$ we have $L(q, p)^2 > 0$, i.e. the matrix $L(q, p)^2$ is positive definite. In particular, the eigenvalues of the Lax matrix $L(q, p)$ are non-zero real numbers.

Proof. Take an arbitrary point $(q, p) \in P^S$ and keep it fixed. First, notice that if $\kappa = 0$, then the Lax matrix $L = L(q, p)$ is of type $C_n$. However, we have a fairly complete knowledge on the spectrum of the Lax matrix of the $C_n$ Sutherland model. Namely, since $\nu \neq 2\mu$, from Lemma 1 in [16] we see that the Hermitian matrix $L$ is invertible, whence $L^2 > 0$ is immediate.

In the following we assume that $\kappa \neq 0$. As an important auxiliary object in our proof, let us consider the $BC_n$-type Lax matrix $\hat{L} = \hat{L}(q, p)$ associated with the parameters $(\mu, \nu - \kappa, 0)$. Recalling (3.5) and (3.3), we see that $\hat{L}$ is an Hermitian matrix with block-matrix decomposition

$$\hat{L} = \hat{L}_p = \begin{bmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{bmatrix} \in \mathfrak{p}. \quad (3.13)$$

Moreover, remembering (3.4), for the matrix entries of $\hat{A}$ and $\hat{B}$ we have

$$\hat{A}_{a,b} = -\frac{i\mu}{\sinh(q_a - q_b)}, \quad \hat{A}_{c,c} = \frac{1}{\sinh(q_c)}, \quad \hat{B}_{a,b} = -\frac{i\mu}{\sinh(q_a + q_b)}, \quad \hat{B}_{c,c} = \frac{i(\nu - \kappa)}{\sinh(2q_c)}, \quad (3.14)$$

where $a, b, c \in \mathbb{N}_n$ and $a \neq b$. Since $\hat{L}$ is Hermitian, it is clear that $\hat{L}^2 \geq 0$. Thus the direct application of (3.12) on the Lax matrix $\hat{L}$ yields immediately that

$$\hat{A}^2 - \hat{B}^2 - [\hat{A}, \hat{B}] \geq 0. \quad (3.15)$$

In order to find the connection between the Lax matrix $L$ (3.5) and the auxiliary Lax matrix $\hat{L}$ (3.13), we introduce the diagonal matrix $q = \text{diag}(q_1, \ldots, q_n)$. Comparing the equations (3.4) and (3.14), it is obvious that

$$A = \hat{A} \quad \text{and} \quad B = \hat{B} + i\kappa \coth(q). \quad (3.16)$$

Thus it is immediate that

$$A^2 - B^2 - [A, B] - \kappa^2 1_N = A^2 - \hat{A}^2 - \hat{B}^2 + \kappa[\coth(q), \hat{A}] - \kappa[\coth(q), \hat{B}] + \hat{B} \coth(q) + \kappa^2 \sinh(q)^{-2}. \quad (3.17)$$

Upon introducing the column vector $V \in \mathbb{C}^n$ with components $V_c = 1/\sinh(q_c)$ $(c \in \mathbb{N}_n)$, the right hand side of the above equation can be simplified considerably. Indeed, by applying the standard hyperbolic identity

$$\frac{\coth(x) + \coth(y)}{\sinh(x) + \sinh(y)} = \frac{1}{\sinh(x) \sinh(y)}, \quad (3.18)$$

one can easily verify the relations

$$i\kappa[\coth(q), \hat{A}] = -\mu \kappa V V^* + \mu \kappa \sinh(q)^{-2}, \quad (3.19)$$

$$i\kappa[\coth(q), \hat{B}] = -\mu \kappa V V^* + (\mu \kappa - \nu \kappa + \kappa^2) \sinh(q)^{-2}. \quad (3.20)$$
Plugging these formulae into (3.17), we end up with the concise expression

$$A^2 - B^2 - [A, B] - \kappa^2 1_N = \tilde{A}^2 - \tilde{B}^2 - [\tilde{A}, \tilde{B}] + \nu \kappa \sinh(q)^2.$$  \hfill (3.21)

However, due to (3.15) and the assumption $\nu \kappa \geq 0$, the matrix on the right hand side of the above equation is manifestly positive definite. Therefore, according to the spectral identification (3.12), we conclude that $L^2 > 0$. \hfill \Box

### 3.2 The phase space of the Sutherland model

In this subsection we perform the Marsden–Weinstein reduction of the extended symplectic manifold $(\mathcal{P}^{\text{ext}}, \omega^{\text{ext}})$ at the zero value of the momentum map $J^{\text{ext}}$ (2.26). As a first step of the reduction, we have to solve the constraint

$$J^{\text{ext}}(y, Y, \rho) = 0$$

for $(y, Y, \rho) \in \mathcal{P}^{\text{ext}}$. In other words, we have to understand the differential geometric properties of the closed level set

$$\mathcal{L}_0 = (J^{\text{ext}})^{-1}(\{0\}) = \{(y, Y, \rho) \in \mathcal{P}^{\text{ext}} \mid J^{\text{ext}}(y, Y, \rho) = 0\} \subset \mathcal{P}^{\text{ext}}.$$  \hfill (3.23)

In order to derive the phase space of the Sutherland model from the proposed reduction picture, we are looking for a special parametrization of $\mathcal{L}_0$ induced by the $KAK$ decomposition of the group elements $y \in G$. (For background information on the $KAK$ decomposition see e.g. the book [13].) As it can be seen from the lemma below, besides the $KAK$ decomposition, the most important ingredient of the parametrization is the Lax operator (3.5).

**Lemma 2.** Suppose that $\nu + \kappa \neq 0$; then for each point $(y, Y, \rho) \in \mathcal{L}_0$ of the level set there are some $q \in \mathfrak{c}$, $p \in \mathbb{R}^n$ and $\eta_L, \eta_R \in K$, such that

$$y = \eta_L e^Q \eta_R^{-1}, \quad Y = \eta_R L(q, p) \eta_R^{-1}, \quad \rho = \eta_L \xi(E) \eta_L^{-1}. \quad (3.24)$$

**Proof.** Take an arbitrary point $(y, Y, \rho) \in \mathcal{L}_0$. The $KAK$ decomposition tells us precisely that the Lie group element $y \in G$ can be decomposed as

$$y = k_L e^Q k_R^{-1}, \quad (3.25)$$

where $k_L, k_R \in K$ and

$$Q = \text{diag}(q_1, \ldots, q_n, -q_1, \ldots, -q_n) \in \mathfrak{a} \quad (3.26)$$

with some $q_1 \geq \ldots \geq q_n \geq 0$. Also, by (2.22), we can write $\rho = \xi(V)$ with some column vector $V \in \mathbb{C}^N$ satisfying $V^*V = N$ and $CV + V = 0$.

Plugging the above parametrizations into the constraint $J^{\text{ext}}(y, Y, \rho) = 0$, the explicit form of the momentum map $J^{\text{ext}}$ (2.26) immediately leads to the relationship $Y_+ = -i \kappa C$, together with

$$0 = (y Y y^{-1})_+ + \rho = k_L \left( \sinh(ad_Q)(k_R^{-1} Y_- k_R) - \cosh(ad_Q)(i \kappa C) + \xi(k_L^{-1} V) \right) k_L^{-1}. \quad (3.27)$$
Upon introducing the shorthand notations
\[
\hat{Y}_- = k_R^{-1} Y_-, k_R \in \mathfrak{p} \quad \text{and} \quad \hat{V} = k_L^{-1} V \in \mathbb{C}^N,
\] (3.28)
from the equation (3.27) it readily follows that
\[
\sinh(\text{ad}_Q) \hat{Y}_- = -\xi(\hat{V}) + \cosh(\text{ad}_Q)(i\kappa C) = -\xi(\hat{V}) + i\kappa \cosh(2Q)C.
\] (3.29)

Spelling out the components of the above matrix equation, for all \( k, l \in \mathbb{N}_N \) we have
\[
\sinh(q_k - q_l)(\hat{Y}_-)_{k,l} = -i\mu(\bar{V}_k \bar{V}_l - \delta_{k,l}) - i(\mu - \nu)C_{k,l} + i\kappa \cosh(2q_k)C_{k,l},
\] (3.30)
where it is understood that \( q_{n+c} = -q_c \) for all \( c \in \mathbb{N}_n \).

Now take an arbitrary \( c \in \mathbb{N}_n \). With the specialization \( k = l = c \) the above equation (3.30) takes the form
\[
0 = -i\mu|\bar{V}_c|^2 - 1),
\] (3.31)
whence it is obvious that \( \hat{V}_c = -\hat{V}_{n+c} = e^{i\chi_c} \) with some real parameter \( \chi_c \in \mathbb{R} \). Notice also that with \( k = c \) and \( l = n + c \) the equation (3.30) translates into
\[
\sinh(2q_c)(\hat{Y}_-)_{c,n+c} = i\nu + i\kappa \cosh(2q_c).
\] (3.32)

Therefore, under the assumption \( \nu + \kappa \neq 0 \), from the above relationship it is clear that \( q_c \neq 0 \).

Next, let \( a, b \in \mathbb{N}_n \) be arbitrary numbers satisfying \( a \neq b \). With the specialization \( k = a \) and \( l = b \) the equation (3.30) has the form
\[
\sinh(q_a - q_b)(\hat{Y}_-)_{a,b} = -i\mu \bar{V}_a \bar{V}_b \neq 0,
\] (3.33)
therefore \( q_a \neq q_b \). Putting the above considerations together, we see that \( q_1 > \ldots > q_n > 0 \), whence the regularity property \( q \in \mathfrak{c} \) is immediate.

Now let us introduce the diagonal matrix
\[
m = \text{diag}(e^{i\chi_1}, \ldots, e^{i\chi_n}, e^{i\chi_1}, \ldots, e^{i\chi_n}) \in M.
\] (3.34)
Due to the construction of \( m \), it is clear that \( m^{-1} \hat{V} = E \) (2.20). Therefore, by applying the linear operator \( \text{Ad}_{m^{-1}} \) on the equation (3.29), we obtain
\[
\sinh(\text{ad}_Q)\text{Ad}_{m^{-1}}(\hat{Y}_-) = -\xi(E) + \cosh(\text{ad}_Q)(i\kappa C).
\] (3.35)
By inspecting the diagonal and the off-diagonal parts of \( \text{Ad}_{m^{-1}}(\hat{Y}_-) \) separately, and utilizing the regularity of \( q \), we can write
\[
\text{Ad}_{m^{-1}}(\hat{Y}_-) = P - \sinh(\text{ad}_Q)^{-1}\xi(E) + \coth(\text{ad}_Q)(i\kappa C)
\] (3.36)
with some diagonal matrix \( P = \text{diag}(p_1, \ldots, p_n, -p_1, \ldots, -p_n) \in \mathfrak{a} \). Thus, remembering the definition (3.2), we can simply write \( \hat{Y}_- = mL_p(q, p)m^{-1} \); therefore the relationship
\[
Y_- = k_R m L_p(q, p)m^{-1}k_R^{-1}
\] (3.37)
is immediate. Since \( V = k_L \hat{V} = k_L m E \), we also have
\[
\rho = \xi(V) = k_L m \xi(E)m^{-1}k_L^{-1}.
\] (3.38)

Therefore, with the group elements \( \eta_L = k_L m \in K \) and \( \eta_R = k_R m \in K \), the lemma follows. \( \square \)
To proceed further, let us notice that the Abelian group
\[ U(1)_* = \{(e^{i\chi}1_N, e^{i\chi}1_N) \in K \times K \mid \chi \in \mathbb{R}\} \cong U(1) \] (3.39)
is a closed normal subgroup of the product Lie group \( K \times K \), whence the coset space \((K \times K)/U(1)_*\) inherits a natural (real) Lie group structure from \( K \times K \). Let us also consider the smooth product manifold
\[ \mathcal{M}^S = \mathcal{P}^S \times (K \times K)/U(1)_*. \] (3.40)
Having equipped with the above objects, now we can introduce a natural parametrization of the level set \( \mathcal{L}_0 \) motivated by Lemma 2. Indeed, by imitating the proof of Lemma 2 in [12], one can easily verify that the map
\[ \Upsilon^S : \mathcal{M}^S \to \mathcal{P}^\text{ext}, \quad (q, p, (\eta_L, \eta_R)U(1)_*) \mapsto (\eta_L e^{Q\eta_R^{-1}}, \eta_R L(q, p)\eta_R^{-1}, \eta_L \xi(E)\eta_L^{-1}) \] (3.41)
is a well-defined injective immersion with image \( \Upsilon^S(\mathcal{M}^S) = \mathcal{L}_0 \). Also, just as in the proof of Lemma 3 in [12], the parametrization \( \Upsilon^S \) makes it easy to verify directly that the zero element of the Lie algebra \( \mathfrak{s}(k \oplus k) \) (2.27) is a regular value of the momentum map \( J^\text{ext} \) (2.26). Therefore the level set \( \mathcal{L}_0 \) is an embedded submanifold of \( \mathcal{P}^\text{ext} \) in a natural manner. More precisely, there is a unique smooth manifold structure on \( \mathcal{L}_0 \) such that the pair \((\mathcal{L}_0, \iota_0)\) with the tautological injection
\[ \iota_0 : \mathcal{L}_0 \hookrightarrow \mathcal{P}^\text{ext} \] (3.42)
is an embedded submanifold of \( \mathcal{P}^\text{ext} \). At this point let us notice that, due to the relationship \( \Upsilon^S(\mathcal{M}^S) = \iota_0(\mathcal{L}_0) \), the map \( \Upsilon^S \) factors through \((\mathcal{L}_0, \iota_0)\); therefore there is a well-defined map
\[ \Upsilon^S_0 : \mathcal{M}^S \to \mathcal{L}_0, \] (3.43)
such that \( \Upsilon^S = \iota_0 \circ \Upsilon^S_0 \). Let us observe that, since \( \iota_0 \) is an embedding, the map \( \Upsilon^S_0 \) is automatically smooth (see e.g. Theorem 1.32 in [17]). Now, since \( \Upsilon^S_0 \) is a smooth bijective immersion from \( \mathcal{M}^S \) onto \( \mathcal{L}_0 \), and since it acts between manifolds of the same dimension, it is immediate that \( \Upsilon^S_0 \) is a diffeomorphism. The above ideas can be summarized by saying that the diagram
\[ \begin{array}{ccc}
\mathcal{M}^S & \xrightarrow{\cong} & \mathcal{L}_0 \\
\Upsilon^S_0 \downarrow & & \downarrow \iota_0 \\
\mathcal{P}^\text{ext} & \xrightarrow{\Upsilon^S} & \mathcal{L}_0
\end{array} \] (3.44)
is commutative. In other words, the pair \((\mathcal{M}^S, \Upsilon^S)\) provides an equivalent model for the smooth embedded submanifold \((\mathcal{L}_0, \iota_0)\).

Utilizing the model \((\mathcal{M}^S, \Upsilon^S)\) of the level set \( \mathcal{L}_0 \), in the following we complete the symplectic reduction of \((\mathcal{P}^\text{ext}, \omega^\text{ext})\) at the zero value of the momentum map \( J^\text{ext} \). For this purpose let us note that on the model space \( \mathcal{M}^S \) (3.40) the residual \( K \times K \)-action takes the form
\[ (k_L, k_R) \cdot (q, p, (\eta_L, \eta_R)U(1)_* ) = (q, p, (k_L \eta_L, k_R \eta_R)U(1)_*). \] (3.45)
Therefore it is obvious that the orbit space $\mathcal{M}^S/(K \times K)$ can be naturally identified with the base manifold of the trivial principal $(K \times K)/U(1)_\ast$-bundle

$$\pi^S: \mathcal{M}^S \to \mathcal{P}^S, \quad (q, p, (\eta_L, \eta_R)U(1)_\ast) \mapsto (q, p). \quad (3.46)$$

An immediate consequence of the above observation is that the reduced symplectic manifold can be identified as

$$\mathcal{P}^{\text{ext}}/_{0}(K \times K) \cong \mathcal{M}^S/(K \times K) \cong \mathcal{P}^S. \quad (3.47)$$

As is known from the theory of symplectic reductions, the reduced symplectic form $\omega^S \in \Omega^2(\mathcal{P}^S)$ is uniquely determined by the condition

$$(\pi^S)^*\omega^S = (\Upsilon^S)^*\omega^{\text{ext}}. \quad (3.48)$$

However, since the derivative of $\Upsilon^S$ can be worked out explicitly, the computation of the pull-backs in (3.48) is almost trivial. Either doing the calculations by hand, or remembering the results presented in [10], the following theorem is immediate.

**Theorem 3.** The reduced symplectic form can be written as $\omega^S = 2 \sum_{c=1}^{n} dq_c \wedge dp_c$. That is to say, up to some trivial rescaling, the globally defined coordinate functions $q_c, p_c (c \in \mathbb{N}_n)$ form a Darboux system on the reduced manifold $\mathcal{P}^S$.  

### 3.3 Solution algorithm for the Sutherland model

The goal of this subsection is to present a solution algorithm for a class of Hamiltonian systems in association with the family of the Ad-invariant smooth functions defined on the Lie algebra $\mathfrak{g}$. For, take an arbitrary Ad-invariant smooth function $F: \mathfrak{g} \to \mathbb{R}$, i.e. we require

$$F(yY y^{-1}) = F(Y) \quad (\forall Y \in \mathfrak{g}, \forall y \in G). \quad (3.49)$$

Now, let

$$\text{pr}_\mathfrak{g}: \mathcal{P}^{\text{ext}} = G \times \mathfrak{g} \times \mathcal{O} \to \mathfrak{g} \quad (3.50)$$

denote the canonical projection onto $\mathfrak{g}$; then it is obvious that the composite function $\text{pr}_\mathfrak{g}^*F = F \circ \text{pr}_\mathfrak{g}$ is a smooth $K \times K$-invariant function on $\mathcal{P}^{\text{ext}}$; therefore it survives the reduction. More precisely, the corresponding reduced Hamiltonian $(\text{pr}_\mathfrak{g}^*F)^S \in C^\infty(\mathcal{P}^S)$ has the form

$$(\text{pr}_\mathfrak{g}^*F)^S = F \circ L, \quad (3.51)$$

as can be readily seen from the defining relationship

$$(\pi^S)^*(\text{pr}_\mathfrak{g}^*F)^S = (\Upsilon^S)^*\text{pr}_\mathfrak{g}^*F. \quad (3.52)$$

It is a standard fact in reduction theory that the Hamiltonian flows of the ‘unreduced’ Hamiltonian system $(\mathcal{P}^{\text{ext}}, \omega^{\text{ext}}, \text{pr}_\mathfrak{g}^*F)$ staying on the level space $\mathfrak{L}_0$ project onto the flows of the reduced system $(\mathcal{P}^S, \omega^S, F \circ L)$. However, finding the ‘unreduced’ flows is a relatively simple exercise;
therefore this projection method gives rise to a natural and efficient solution algorithm for the reduced Hamiltonian system.

To make the above observation precise, we need the integral curves of the Hamiltonian vector field $X_{pr^s_gF} \in \mathfrak{X}(\mathcal{P}^\text{ext})$ generated by $pr^s_gF$. Recalling $\omega^\text{ext}$ (2.24), from the defining relationship

$$X_{pr^s_gF} \cdot \omega^\text{ext} = d(pr^s_gF) \quad (3.53)$$

we find easily that at each point $(y, Y, \rho) \in \mathcal{P}^\text{ext}$ the Hamiltonian vector field has the form

$$(X_{pr^s_gF})_{(y,Y,\rho)} = (y\nabla F(Y))_y \oplus 0_Y \oplus 0_\rho \in T_{(y,Y,\rho)}\mathcal{P}^\text{ext}, \quad (3.54)$$

where the gradient $\nabla F(Y) \in \mathfrak{g}$ is defined by the condition

$$\langle \nabla F(Y), \delta Y \rangle = (dF)_Y(\delta Y) \quad (\forall \delta Y \in T_Y\mathfrak{g} \cong \mathfrak{g}). \quad (3.55)$$

From (3.54) it is clear that the induced Hamiltonian flows are complete, having the form

$$\mathbb{R} \ni t \mapsto (y_0 e^{t\nabla F(Y_0)}, Y_0, \rho_0) \in \mathcal{P}^\text{ext} \quad (3.56)$$

with some $(y_0, Y_0, \rho_0) \in \mathcal{P}^\text{ext}$. Therefore the reduced Hamiltonian flows are also complete.

Now take an arbitrary flow

$$\mathbb{R} \ni t \mapsto (q(t), p(t)) \in \mathcal{P}^S \quad (3.57)$$

induced by the reduced Hamiltonian $F \circ L$, and let $L_0 = L(q(0), p(0))$. Since the unreduced flow passing through the point $(e^{Q(0)}, L_0, \xi(E)) \in \Sigma_0$ at $t = 0$ projects onto the reduced flow (3.57), from the definition of the parametrization $Y^S$ (3.41) it is clear that for each $t \in \mathbb{R}$ there is some $(\eta_L(t), \eta_R(t)) \in \tilde{K} \times \tilde{K}$ such that

$$(e^{Q(0)} e^{t\nabla F(L_0)}, L_0, \xi(E)) = (\eta_L(t) e^{Q(t)} \eta_R(t)^{-1}, \eta_R(t) L(q(t), p(t)) \eta_R(t)^{-1}, \eta_L(t) \xi(E) \eta_L(t)^{-1}). \quad (3.58)$$

By comparing the $\mathfrak{g}$-components of the above equation we see that

$$L_0 = \eta_R(t) L(q(t), p(t)) \eta_R(t)^{-1}, \quad (3.59)$$

from where we conclude that during the time evolution of the reduced dynamics the Lax matrix $L$ (3.55) undergoes an isospectral deformation. What is even more important, the $G$-component of equation (3.58) leads to the relationship

$$e^{Q(0)} e^{t\nabla F(L_0)} e^{t\nabla F(L_0)^*} e^{Q(0)} = \eta_L(t) e^{2Q(t)} \eta_L(t)^{-1}, \quad (3.60)$$

which entails the spectral identification

$$\sigma(e^{2Q(t)}) = \sigma(e^{Q(0)} e^{t\nabla F(L_0)} e^{t\nabla F(L_0)^*} e^{Q(0)}) = \sigma(e^{2Q(0)} e^{t\nabla F(L_0)} e^{t\nabla F(L_0)^*}). \quad (3.61)$$

Somewhat more informally one can say that the matrix flow $Q(t)$, and so the trajectory $q(t)$, can be recovered simply by diagonalizing the matrix flow

$$t \mapsto e^{2Q(0)} e^{t\nabla F(L_0)} e^{t\nabla F(L_0)^*}. \quad (3.62)$$
Besides providing a nice solution algorithm, the above observation can also be seen as the starting point of the scattering theoretic analysis of the reduced Hamiltonian system $\psi^S, \omega^S, F \circ L$. Indeed, by analyzing the temporal asymptotics of the matrix flow (3.62), one can understand the temporal asymptotics of the trajectory $q(t)$ as well.

In order to establish the connection with the Sutherland many-particle systems, notice that the Hamiltonian of the $BC_n$ Sutherland model (1.3) with three independent coupling constants can be realized as the reduced Hamiltonian induced by the Ad-invariant quadratic function

$$F_2(Y) = \frac{1}{4} \langle Y, Y \rangle = \frac{1}{4} \text{tr}(Y^2) \quad (Y \in g).$$

Indeed, a simple computation reveals that

$$(pr^s_g F_2)^S = H^S,$$

with coupling constants

$$g^2 = \mu^2, \quad g_1^2 = \frac{1}{2} \nu \kappa, \quad g_2^2 = \frac{1}{2} (\nu - \kappa)^2.$$ 

Note that if $\nu \kappa \geq 0$, then the interaction is purely repulsive. Conversely, if one starts with an arbitrary triple of the non-negative Sutherland coupling constants $(g^2, g_1^2, g_2^2)$ satisfying the inequalities $g^2 > 0$ and $g_1^2 + g_2^2 > 0$, then the corresponding repulsive Sutherland model can be recovered e.g. by choosing the parameter triple $(\mu, \nu, \kappa)$ with components

$$\mu = -|g|, \quad \nu = \frac{|g_2| + \sqrt{g_2^2 + 4g_1^2}}{2}, \quad \kappa = \frac{2\sqrt{2}g_2^2}{|g_2| + \sqrt{g_2^2 + 4g_1^2}}.$$ 

Notice that the above defined parameters satisfy the inequalities $\mu < 0$, $\nu > 0$ and $\kappa \geq 0$, therefore the assumptions made in Lemmas [1] and [2] are automatically met. As for the solution algorithm of the Sutherland model, note that $\nabla F_2(L_0) = L_0/2$, whence from equation (3.62) we see at once that the trajectories can be determined by diagonalizing the matrix flow

$$t \mapsto e^{2Q(0)} e^{\frac{1}{2} L_0 t} e^{\frac{1}{2} L_0^\ast t}.$$ 

We close this section with some remarks on the range of the attainable Sutherland coupling constants coming from the proposed symplectic reduction picture. From equation (3.66) it is clear that any non-negative triple of the coupling constants $(g^2, g_1^2, g_2^2)$ with $g^2 > 0$ and $g_1^2 + g_2^2 > 0$ can be realized by an appropriate choice of the parameters $(\mu, \nu, \kappa)$ satisfying $\nu \kappa \geq 0$. Due to the repulsive nature of the interaction, in these cases the Sutherland model has only scattering states. It conforms with Lemma [1] according which the eigenvalues of the Lax operator $L$ are real. Thus the scattering properties of the model can be understood explicitly by analyzing the temporal asymptotics of the truly exponential-type matrix flow (3.67).

However, looking at (3.65), it is obvious that from the proposed reduction picture we can derive the Sutherland model even with $g_1^2 < 0$, provided that $\nu \kappa < 0$. Since Lemma [1] does not apply in these cases, in certain non-empty region of the phase space $\psi^S$ some of the eigenvalues
of the Lax operator may be purely imaginary. Therefore, as can be conjectured from (3.67), some of the Sutherland particles may exhibit oscillatory behavior with bounded trajectories. To fully understand the details of this phenomenon, one should sharpen Lemma 1 to characterize the spectral properties of the Lax matrix for $\nu \kappa < 0$. Nevertheless, since our primary interest is to study the duality properties of the standard hyperbolic Sutherland $BC_n$ model with purely repulsive interaction, we leave this interesting exercise for a future study.

4 The rational $BC_n$ RSvD model

In this section we work out the rational $BC_n$ RSvD model with three independent coupling constants from the proposed symplectic reduction framework. Let us keep in mind that Section 3 and the present section provide the necessary technical background to establish the action-angle duality between the standard repulsive Sutherland model (1.3) and the rational RSvD (1.5) model with $\nu \kappa \geq 0$. Therefore, according to our discussion in the concluding remarks of Section 3, we may assume at the outset that $\mu < 0$, $\nu > 0$ and $\kappa \geq 0$.

4.1 The phase space of the RSvD model

First of all, we need some objects introduced in the study [12] of the $C_n$-type rational RSvD model associated with the pair of non-zero parameters $(\mu, \nu)$. For each $a \in \mathbb{N}_n$ let us consider the complex-valued rational function

$$c \ni \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto z_a(\lambda) = - \left( 1 + \frac{i\nu}{\lambda_a} \right) \prod_{d=1}^{n} \left( 1 + \frac{2i\mu}{\lambda_a - \lambda_d} \right) \left( 1 + \frac{2i\mu}{\lambda_a + \lambda_d} \right) \in \mathbb{C}. \quad (4.1)$$

Recall also that the Lax matrix of the rational $C_n$ RSvD model is defined by the matrix-valued function

$$\mathcal{A}: \mathbb{P}^R \to \exp(p), \quad (\lambda, \theta) \mapsto \mathcal{A}(\lambda, \theta), \quad (4.2)$$

where the matrix entries lying in the diagonal $n \times n$ blocks are given by the formulae

$$\mathcal{A}_{a,b}(\lambda, \theta) = e^{\theta_a + \theta_b} |z_a(\lambda)z_b(\lambda)|^{\frac{1}{2}} \frac{2i\mu}{2i\mu + \lambda_a - \lambda_b}, \quad (4.3)$$

$$\mathcal{A}_{n+a,n+b}(\lambda, \theta) = e^{-\theta_a - \theta_b} \frac{z_a(\lambda)z_b(\lambda)}{|z_a(\lambda)z_b(\lambda)|} \frac{2i\mu}{2i\mu - \lambda_a + \lambda_b}, \quad (4.4)$$

meanwhile the matrix entries belonging to the off-diagonal $n \times n$ blocks have the form

$$\mathcal{A}_{a,n+b}(\lambda, \theta) = \overline{\mathcal{A}_{n+b,a}(\lambda, \theta)} = e^{\theta_a - \theta_b} |z_a(\lambda)z_b(\lambda)|^{-\frac{1}{2}} \frac{2i\mu}{2i\mu + \lambda_a + \lambda_b} + \frac{i(\mu - \nu)}{i\mu + \lambda_a} \delta_{a,b}, \quad (4.5)$$

for any $a, b \in \mathbb{N}_n$. Besides the Lax matrix $\mathcal{A}$, from the theory of the $C_n$ RSvD model we need the column vector $\mathcal{F}(\lambda, \theta) \in \mathbb{C}^N$ with components

$$\mathcal{F}_a(\lambda, \theta) = e^{\theta_a} |z_a(\lambda)|^{\frac{1}{2}} \quad \text{and} \quad \mathcal{F}_{n+a}(\lambda, \theta) = e^{-\theta_a} \overline{z_a(\lambda)} |z_a(\lambda)|^{-\frac{1}{2}}, \quad (4.6)$$
where \( a \in \mathbb{N}_n \). Also, in the forthcoming computations we shall frequently encounter the column vector
\[
V(\lambda, \theta) = A(\lambda, \theta)^{-\frac{1}{2}} F(\lambda, \theta) \in \mathbb{C}^N.
\]
(4.7)
Notice that \( V(\lambda, \theta) \) is well-defined, since the positive definite Lax matrix \( A(\lambda, \theta) \in \exp(p) \) has a unique square root belonging to \( \exp(p) \). Furthermore, as it can be seen from Proposition 8 in [12], we have \( V^*V = N \) and \( CV + V = 0 \).

In order to handle the \( BC_n \) RSvD model associated with the parameter triple \((\mu, \nu, \kappa)\), we need some new objects, too. In particular, let us introduce the smooth functions \( \alpha \) and \( \beta \) defined on the positive half-line by the formulae
\[
\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 + \kappa^2}}}{\sqrt{2x}} \quad \text{and} \quad \beta(x) = i\kappa \frac{1}{\sqrt{2x} \sqrt{x + \sqrt{x^2 + \kappa^2}}}
\]
(4.8)
where \( x \in (0, \infty) \). Now, with each positive \( n \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (0, \infty)^n \) we associate the \( n \times n \) diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and consider the Hermitian \( N \times N \) matrix
\[
h(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{bmatrix}.
\]
(4.9)
Making use of the functional equation \( \alpha(x)^2 + \beta(x)^2 = 1 \), one can show that \( h(\lambda)Ch(\lambda) = C \), i.e. the matrix \( h(\lambda) \) is an Hermitian element of the Lie group \( G \) (2.2). Finally, let us introduce the shorthand notation
\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n) \in a.
\]
(4.10)
Having equipped with the above objects, we are now in a position to provide an appropriate parametrization of the level set \( \mathfrak{S}_0 \) (3.23) based on the diagonalization of the Lie algebra part of \( \mathfrak{S}_0 \).

**Lemma 4.** Suppose that \( \nu \neq 2\mu, \nu + \kappa \neq 0 \) and \( \nu \kappa \geq 0 \); then for each point \((y, Y, \rho) \in \mathfrak{S}_0 \) there are some \( \lambda \in c, \theta \in \mathbb{R}^n \) and \( \eta_L, \eta_R \in K \), such that
\[
y = \eta_L A(\lambda, \theta) h(\lambda)^{-1} \eta_R^{-1}, \quad Y = \eta_R h(\lambda) \Lambda h(\lambda)^{-1} \eta_R^{-1}, \quad \rho = \eta_L \xi(V(\lambda, \theta)) \eta_L^{-1}.
\]
(4.11)
**Proof.** Take an arbitrary point \((y, Y, \rho) \in \mathfrak{S}_0 \). Remembering the momentum map \( J^\text{ext} \) (2.26), it is clear that \( Y_+ = -i\kappa C \). On the other hand, since any element of the subspace \( p \) (2.7) can be conjugated into \( a \) (2.9) by some element of \( K \) (2.5), we can write
\[
Y_- = k_R D k_R^{-1}
\]
(4.12)
with some \( k_R \in K \) and
\[
D = \text{diag}(d_1, \ldots, d_n, -d_1, \ldots, -d_n) \in a
\]
(4.13)
satisfying \( d_1 \geq \ldots \geq d_n \geq 0 \). Therefore
\[
Y = k_R(D - i\kappa C)k_R^{-1},
\]
(4.14)
from where we obtain
\[ Y^2 = k_R(D^2 - \kappa^21_N)k_R^{-1}. \] (4.15)

However, by combining Lemmas 1 and 2, it is obvious that \( Y^2 \) is positive definite, i.e. \( Y^2 > 0 \). Giving a glance at the above equation (4.15), it is thus clear that \( \lambda_a = \sqrt{d_a^2 - \kappa^2} \) is a well-defined positive real number for each \( a \in N_n \), satisfying the inequalities \( \lambda_1 \geq \ldots \geq \lambda_n > 0 \). Now, recalling \( h(\lambda) \) (4.9) and utilizing the functional equations
\[ \alpha(x)^2 - \beta(x)^2 = \sqrt{1 + \frac{\kappa^2}{x^2}} \quad \text{and} \quad 2\alpha(x)\beta(x) = \frac{i\kappa}{x}, \] (4.16)
it is not hard to see that
\[ h(\lambda)\Lambda h(\lambda)^{-1} = D - i\kappa C. \] (4.17)

Plugging this observation into (4.14), we get
\[ Y = k_Rh(\lambda)\Lambda h(\lambda)^{-1}k_R^{-1}. \] (4.18)

To proceed further, notice that the momentum map constraint yields the relationship
\[ 0 = (y_Y y^{-1})_+ + \rho = \frac{y_Y y^{-1} - (y_Y y^{-1})^*}{2} + \rho \] (4.19)
as well. This is clearly equivalent to the equation
\[ y^*yY - Y^*y^*y + 2y^*\rho y = 0. \] (4.20)

However, due to (2.22) we can write \( \rho = \xi(V) \), where \( V \in \mathbb{C}^N \) is an appropriate column vector satisfying \( V^*V = N \) and \( CV + V = 0 \). Thus, recalling (2.19), the above constraint (4.20) can be cast into the form
\[ y^*yY - Y^*y^*y + 2i\mu y^*VV^*y - 2i\mu y^*y + 2i(\mu - \nu)C = 0. \] (4.21)

Plugging the parametrization (4.18) into the above equation, we get
\[ 2i\mu hk_Ry^*yk_Rh + \Lambda hk_Ry^*yk_Rh - hk_Ry^*yk_Rh\Lambda = 2i\mu(hk_Ry^*V)(hk_Ry^*V)^* + 2i(\mu - \nu)C. \] (4.22)

At this point let us note that the last equation can be naturally identified with equation (31) in [16]. Since this equation was the starting point of our scattering theoretic analysis of the hyperbolic \( C_n \) Sutherland model, we have full control over the structure of its ingredients. In particular, due to Lemma 1 in [16], we know that \( \Lambda \) must be a regular element of \( \mathfrak{a} \), whence \( \lambda_1 > \ldots > \lambda_n > 0 \), i.e. \( \lambda \in \mathfrak{c} \). Furthermore, by Lemma 2 in [16], we can write
\[ h(\lambda)k_Ry^*yk_Rh(\lambda) = mA(\lambda, \theta)m^* \] (4.23)

with some \( \theta \in \mathbb{R}^n \) and \( m \in M \). Noticing that the group element \( h(\lambda) \in G \) commutes with each element of \( M \) (2.10), we obtain
\[ (yk_Rmh)^*yk_Rmh = m^*hk_Ry^*yk_Rhm = A. \] (4.24)
Third, our analysis yields the relationship

$$h(\lambda)k^*_R y^* V = m\mathcal{F}(\lambda, \theta) \quad (4.25)$$

as well. Now let us notice that equation (4.24) allows us to give a characterization of the global Cartan decomposition (polar decomposition) of $yk_R m h(\lambda)$. Indeed, we have

$$yk_R m h(\lambda) = \eta_L A(\lambda, \theta)^{\frac{1}{2}} \quad (4.26)$$

with some $\eta_L \in K$; thus the parametrization

$$y = \eta_L A(\lambda, \theta)^{\frac{1}{2}} h(\lambda)^{-1}(k_R m)^{-1} \quad (4.27)$$

is immediate. Plugging this into (4.25) and remembering the definition (4.7), we obtain at once that $V = \eta_L V(\lambda, \theta)$; therefore the relationship

$$\rho = \xi(V) = \eta_L \xi(V(\lambda, \theta)) \eta_L^{-1} \quad (4.28)$$

also follows. Finally, upon setting $\eta_R = k_R m \in K$, from the equations (4.27), (4.18) and (4.28) we see that the proof is complete.

Motivated by the above lemma, we can introduce a natural parametrization of the level set $\mathfrak{L}_0 (3.23)$. To make this idea precise, first let us define the smooth product manifold

$$\mathcal{M}^R = \mathcal{P}^R \times (K \times K)/U(1)^* \quad (4.29)$$

Now, a trivial generalization of the proof of Lemma 10 in [12] immediately convinces us that

$$\Upsilon^R: \mathcal{M}^R \rightarrow \mathcal{P}^{\text{ext}} \quad (4.30)$$

defined by the assignment

$$(\lambda, \theta, (\eta_L, \eta_R) U(1)^*) \mapsto (\eta_L A(\lambda, \theta)^{\frac{1}{2}} h(\lambda)^{-1} \eta_R h(\lambda) A h(\lambda)^{-1} \eta_L^{-1}, \eta_L \xi(V(\lambda, \theta)) \eta_L^{-1}) \quad (4.31)$$

is a well-defined injective immersion with image $\Upsilon^R(\mathcal{M}^R) = \mathfrak{L}_0$. Repeating the same arguments that we applied in the Sutherland picture, it is clear that $\Upsilon^R$ factors through the embedded submanifold $(\mathfrak{L}_0, \iota_0)$, and the resulting smooth map $\Upsilon^R_0: \mathcal{M}^R \rightarrow \mathfrak{L}_0$ is a diffeomorphism. That is to say, the pair $(\mathcal{M}^R, \Upsilon^R)$ provides an equivalent model for the smooth embedded submanifold $(\mathfrak{L}_0, \iota_0)$.

In order to complete the reduction of $\mathcal{P}^{\text{ext}}$ at the zero value of the momentum map $J^{\text{ext}}$, let us observe that the residual $K \times K$-action on the model space $\mathcal{M}^R$ (4.29) takes the form

$$(k_L, k_R) \cdot (\lambda, \theta, (\eta_L, \eta_R) U(1)^*) = (\lambda, \theta, k_L \eta_L, k_R \eta_R U(1)^*). \quad (4.32)$$

Therefore the orbit space $\mathcal{M}^R/(K \times K)$ gets naturally identified with the base manifold of the trivial principal $(K \times K)/U(1)^*$-bundle

$$\pi^R: \mathcal{M}^R \rightarrow \mathcal{P}^R, \quad (\lambda, \theta, (\eta_L, \eta_R) U(1)^*) \mapsto (\lambda, \theta). \quad (4.33)$$
It is thus evident that for the reduced symplectic manifold we have the alternative identification
\[
\mathcal{P}^\text{ext} / \mathcal{O}(K \times K) \cong \mathcal{M}^R / (K \times K) \cong \mathcal{P}^R.
\] (4.34)

Our remaining task is to make explicit the reduced symplectic form \(\omega^R \in \Omega^2(\mathcal{P}^R)\) naturally induced on the reduced phase space \(\mathcal{P}^R\). Just as in the Sutherland case, it is a tempting idea to compute the reduced symplectic structure via a formula analogous to (3.48). However, due to the presence of the square root of \(A\) in \(\Upsilon^R\) (4.31), this approach seems to be hopeless. Instead, it is more expedient to invoke the alternative machinery presented in Subsection 4.3 of paper [12]. Namely, by analyzing the Poisson brackets of the auxiliary \(K \times K\)-invariant functions defined in equations (4.65) and (4.66) of [12], an almost verbatim computation as in the \(C_n\) case convinces us that the standard coordinates of \(\mathcal{P}^R\) are canonical.

**Theorem 5.** Utilizing the global coordinate functions \(\lambda_c\) and \(\theta_c\) \((c \in \mathbb{N})\) defined on the reduced manifold \(\mathcal{P}^R\), the reduced symplectic structure takes the form \(\omega^R = 2 \sum_{c=1}^n d\theta_c \wedge d\lambda_c\).

### 4.2 Solution algorithm for the RSvD model

In this subsection we work out an efficient solution algorithm for the rational \(BC_n\) RSvD model built on the projection method naturally offered by the symplectic reduction framework. For, take an arbitrary \(K \times K\)-invariant real-valued smooth function \(f: G \to \mathbb{R}\) defined on the Lie group \(G\), i.e. we assume that
\[
f((k_L, k_R) \cdot y) = f(k_L y k_R^{-1}) = f(y) \quad (\forall y \in G, \forall (k_L, k_R) \in K \times K).
\] (4.35)

Let
\[
pr_G: \mathcal{P}^\text{ext} = G \times \mathfrak{g} \times \mathcal{O} \to G
\] (4.36)
denote the canonical projection onto \(G\); then the composite function \(pr_G^* f = f \circ pr_G\) is clearly \(K \times K\)-invariant on the unreduced phase space \(\mathcal{P}^\text{ext}\). Thus, based on the defining formula
\[
(\pi^R)^*(pr_G^* f)^R = (\Upsilon^R)^* pr_G^* f,
\] (4.37)
it is easy to see that the corresponding reduced Hamiltonian \((pr_G^* f)^R \in C^\infty(\mathcal{P}^R)\) has the form
\[
(pr_G^* f)^R(\lambda, \theta) = f(A(\lambda, \theta)^{1/2} h(\lambda)^{-1}) \quad ((\lambda, \theta) \in \mathcal{P}^R).
\] (4.38)

As we have discussed it in the Sutherland picture, the essence of the projection method is that the flows of the unreduced Hamiltonian system staying on the level space \(\Sigma_0\) project onto the flows of the reduced Hamiltonian system. However, from the defining formula
\[
X_{pr_G^* f} \cdot \omega^\text{ext} = d(pr_G^* f),
\] (4.39)
we see immediately that at each \((y, Y, \rho) \in \mathcal{P}^\text{ext}\) the Hamiltonian vector field \(X_{pr_G^* f}\) generated by \(pr_G^* f\) has the form
\[
(X_{pr_G^* f})_{(y, Y, \rho)} = 0_y \oplus (-\nabla f(y))_Y \oplus 0_\rho = 0_{y, Y, \rho} \in T_{(y, Y, \rho)}\mathcal{P}^\text{ext},
\] (4.40)
where the gradient $\nabla f(y) \in \mathfrak{g}$ is defined by the requirement

$$\langle \nabla f(y), y^{-1}\delta y \rangle = (df)_y(\delta y) \quad (\forall \delta y \in T_y G). \quad (4.41)$$

Therefore the Hamiltonian flows of pr$_G^* f$ are complete, having the very simple form

$$\mathbb{R} \ni t \mapsto (y_0, Y_0 - t\nabla f(y_0), \rho_0) \in \mathcal{P}^{ext}, \quad (4.42)$$

where $(y_0, Y_0, \rho_0) \in \mathcal{P}^{ext}$ is an arbitrary point. Hence the reduced flows are complete as well.

Now take an arbitrary flow

$$\mathbb{R} \ni t \mapsto (\lambda(t), \theta(t)) \in \mathcal{P}^R \quad (4.43)$$

of the reduced Hamiltonian system $(\mathcal{P}^R, \omega^R, (pr^*_G f)^R)$. For simplicity let us now introduce the shorthand notations

$$\mathcal{A}_0 = \mathcal{A}(\lambda(0), \theta(0)), \quad h_0 = h(\lambda(0)), \quad \Lambda_0 = \Lambda(0), \quad \mathcal{V}_0 = \mathcal{V}(\lambda(0), \theta(0)). \quad (4.44)$$

It is obvious that the unreduced flow

$$\mathbb{R} \ni t \mapsto (\mathcal{A}_0^{\frac{1}{2}}h_0^{-1}, h_0\Lambda_0h_0^{-1} - t\nabla f(\mathcal{A}_0^{\frac{1}{2}}h_0^{-1}), \xi(\mathcal{V}_0)) \in \mathcal{L}_0 \quad (4.45)$$

projects onto [4.43]. Recalling $\Upsilon^R$ (4.31), we see that for each $t \in \mathbb{R}$ we can find some pair of group elements $(\eta_L(t), \eta_R(t)) \in K \times K$ such that

$$\mathcal{A}_0^{\frac{1}{2}}h_0^{-1} = \eta_L(t)\mathcal{A}(\lambda(t), \theta(t))^{\frac{1}{2}}h(\lambda(t))^{-1}\eta_R(t)^{-1}, \quad (4.46)$$

$$h_0\Lambda_0h_0^{-1} - t\nabla f(\mathcal{A}_0^{\frac{1}{2}}h_0^{-1}) = \eta_R(t)h(\lambda(t))\Lambda(t)h(\lambda(t))^{-1}\eta_R(t)^{-1}, \quad (4.47)$$

$$\xi(\mathcal{V}_0) = \eta_L(t)\xi(\mathcal{V}(\lambda(t), \theta(t)))\eta_L(t)^{-1}. \quad (4.48)$$

From (4.46) we conclude that

$$h_0^{-1}\mathcal{A}_0h_0^{-1} = \eta_R(t)h(\lambda(t))^{-1}\mathcal{A}(\lambda(t), \theta(t))h(\lambda(t))^{-1}\eta_R(t)^{-1}, \quad (4.49)$$

which entails the spectral identification

$$\sigma(h_0^{-1}\mathcal{A}_0h_0^{-1}) = \sigma(h(\lambda(t))^{-1}\mathcal{A}(\lambda(t), \theta(t))h(\lambda(t))^{-1}). \quad (4.50)$$

Thus, during the time evolution of the reduced Hamiltonian system, the positive definite Hermitian matrix

$$\mathcal{A}^{BC}(\lambda, \theta) = h(\lambda)^{-1}\mathcal{A}(\lambda, \theta)h(\lambda)^{-1} \in G \quad (4.51)$$

undergoes an isospectral deformation. Meanwhile, from (4.47) it follows that

$$\sigma(h_0\Lambda_0h_0^{-1} - t\nabla f(\mathcal{A}_0^{\frac{1}{2}}h_0^{-1})) = \sigma(\Lambda(t)) = \{\lambda_1(t), \ldots, \lambda_n(t), -\lambda_1(t), \ldots, -\lambda_n(t)\}, \quad (4.52)$$

whence the trajectory $t \mapsto \lambda(t)$ can be recovered simply by diagonalizing the linear matrix flow

$$t \mapsto h_0\Lambda_0h_0^{-1} - t\nabla f(\mathcal{A}_0^{\frac{1}{2}}h_0^{-1}). \quad (4.53)$$
It is worth mentioning that, due to its linearity in \( t \), the temporal asymptotics of the above matrix flow (4.53) can be analyzed by elementary perturbation theoretic techniques. Note that this observation could serve as the starting point of the scattering theoretic analysis of the reduced Hamiltonian system \((P^R, \omega^R, (pr^*_G f)^R)\).

Our considerations so far apply to any reduced system associated with a \( K \times K \)-invariant smooth function \( f \). Note, however, that under the assumption \( \nu \kappa \geq 0 \) the rational \( BC_n \) RSvD model with three independent coupling constants can be nicely fitted into this picture. Indeed, upon introducing the \( K \times K \)-invariant function
\[
f_1(y) = \frac{1}{2} \text{tr}(yy^*) \quad (y \in G),
\]
(4.54)
one can verify that the corresponding reduced Hamiltonian coincides with the RSvD Hamiltonian (1.5) associated with the coupling parameters \((\mu, \nu, \kappa)\), i.e.
\[
(pr^*_G f_1)^R = H^R.
\]
(4.55)

Let us observe that the assumption \( \nu \kappa \geq 0 \) automatically guarantees the lower bound \( H^R > n \) on the RSvD Hamiltonian. We mention in passing that the verification of (4.55) is a quite tedious, but elementary calculation. Nevertheless, it can be done easily by utilizing the functional identities collected in the appendix of [12].

Turning to the solution algorithm based on (4.53), notice that for the gradient (4.41) of the function \( f_1 \) we have
\[
\nabla f_1(y) = \frac{1}{2} \left( y^* y - (y^* y)^{-1} \right) \in \mathfrak{p} \quad (y \in G).
\]
(4.56)
Therefore, from (4.51) and (4.53) we see at once that the trajectories of the rational \( BC_n \) RSvD model can be determined by diagonalizing the matrix flow
\[
t \mapsto h_0 A_0 h_0^{-1} - \frac{1}{2} t \left( A^{BC}_0 - (A^{BC}_0)^{-1} \right),
\]
(4.57)
where \( A^{BC}_0 = A^{BC}(\lambda(0), \theta(0)) \). We find it remarkable that the properties of dynamics generated by the highly non-trivial Hamiltonian \( H^R \) (1.5) can be captured by analyzing the linear matrix flow (4.57).

To sum up, we see that under the assumption \( \nu \kappa \geq 0 \) the rational \( BC_n \) RSvD model (1.5) can be derived from an appropriate symplectic reduction framework. Parallel to our discussion in Section 3 on the Sutherland model (1.3) with \( g_1^2 < 0 \), we expect that the RSvD model with \( \nu \kappa < 0 \) can also be understood from symplectic reduction by generalizing Lemma 4. We wish to come back to this issue in a later publication.

5 Discussion

In the previous two sections we derived both the standard hyperbolic \( BC_n \) Sutherland and the rational \( BC_n \) RSvD models from a unified symplectic reduction framework. The derivation of the Sutherland model relies on the \( KAK \) decomposition of the Lie group part of the level set \( \Sigma_0 \).
(5.2), meanwhile the symplectic geometric understanding of the RSvD model builds upon the diagonalization of the Lie algebra part of $L_0$. Thereby, by performing the Marsden–Weinstein reduction of the symplectic manifold $(\mathcal{P}^{\text{ext}}, \omega^{\text{ext}})$ at the zero value of the momentum map $J^{\text{ext}}(2.26)$, we end up with two equivalent realizations, $\mathcal{P}^S$ and $\mathcal{P}^R$, of the same symplectic quotient $\mathcal{P}^{\text{ext}}/\Gamma_0(K \times K)$. Thus it is obvious that there is a natural symplectomorphism $S: \mathcal{P}^S \to \mathcal{P}^R$ making the diagram

$$
\begin{array}{ccc}
\mathcal{M}^{\text{ext}} & \xrightarrow{\pi^{\text{ext}}} & \mathcal{P}^{\text{ext}} \\
\uparrow & & \uparrow \gamma_0 \\
\mathcal{M}^S & \xrightarrow{\pi^S} & \mathcal{P}^S \\
\downarrow & & \downarrow S \\
\mathcal{P}^S & \xrightarrow{\pi^R} & \mathcal{P}^R
\end{array}
$$

(5.1)

commutative. In the rest of this section we examine some of the immediate consequences of the dual reduction picture (5.1). First, let us define the functions

$$
\hat{\lambda}_c = S^* \lambda_c, \quad \hat{\theta}_c = S^* \theta_c, \quad \hat{q}_c = (S^{-1})^* q_c, \quad \hat{p}_c = (S^{-1})^* p_c,
$$

(5.2)

where $c \in \mathbb{N}_n$. Since $S$ is a symplectomorphism, from Theorems 3 and 5 it follows that

$$
\omega^S = S^* \omega^R = 2 \sum_{c=1}^n d\hat{\theta}_c \wedge d\hat{\lambda}_c \quad \text{and} \quad \omega^R = (S^{-1})^* \omega^S = 2 \sum_{c=1}^n d\hat{q}_c \wedge d\hat{p}_c.
$$

(5.3)

In other words, the globally defined functions $\hat{\lambda}_c$ and $\hat{\theta}_c$ provide a new Darboux system on the Sutherland phase space $\mathcal{P}^S$, meanwhile the global coordinates $\hat{q}_c$ and $\hat{p}_c$ are canonical on the RSvD phase space $\mathcal{P}^R$. Now we show that these new families of canonical coordinates give rise to natural action-angle variables for the Sutherland and the RSvD models, respectively.

Starting with the Sutherland side of the dual reduction picture, take an arbitrary point $(q, p) \in \mathcal{P}^S$ and let $(\lambda, \theta) = S(q, p) \in \mathcal{P}^R$. Now, recalling the parametrizations $\Upsilon^S(3.41)$ and $\Upsilon^R(4.31)$, from the commutativity of the diagram (5.1) it is clear that

$$
(e^Q, L(q, p), \xi(E)) = (\eta_L A(\lambda, \theta) \frac{1}{\hbar} h(\lambda)^{-1} \eta_R^{-1}, \eta_R h(\lambda) A h(\lambda)^{-1} \eta_R^{-1}, \eta_L \xi(V(\lambda, \theta)) \eta_L^{-1})
$$

(5.4)

with some group elements $\eta_L, \eta_R \in K$. By inspecting the $g$-component of the above equation we obtain the spectral identification

$$
\sigma(L(q, p)) = \sigma(\Lambda) = \{\pm \hat{\lambda}_c(q, p) \mid c \in \mathbb{N}_n\}.
$$

(5.5)

Next, take an arbitrary real-valued Ad-invariant smooth function $F: g \to \mathbb{R}$ (3.49) defined on $g$, and consider the naturally associated reduced Hamiltonian system $(\mathcal{P}^S, \omega^S, (\text{pr}_g^* F)^S)$. Due to (3.51) and (3.4) it is clear that

$$
(\text{pr}_g^* F)^S = F \circ L = F \circ \Lambda = F(\text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_n, -\hat{\lambda}_1, \ldots, -\hat{\lambda}_n)),
$$

(5.6)

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the relationship between the action-angle variables for the RSvD model as well. Notice also that the positive definite matrix $L$ provides $n$ functionally independent first integrals in involution. Furthermore, the matrix $L$ naturally enters the solution algorithm of the Sutherland model (see equation (3.67)), whence the non-Hermitian matrix $L$ is indeed a Lax matrix for the Sutherland many-particle system.

To proceed further, take an arbitrary point $(\lambda, \theta) \in \mathcal{P}^R$ and let $(q, p) = \mathcal{S}^{-1}(\lambda, \theta) \in \mathcal{P}^S$. Recalling the mappings $\Upsilon^R$ (4.31) and $\Upsilon^S$ (3.41), it is clear that
\begin{equation}
(\mathcal{A}(\lambda, \theta) \frac{1}{2} h(\lambda)^{-1}, h(\lambda) \mathcal{A}(\lambda, \theta) h(\lambda)^{-1}, \xi(\mathcal{A}(\lambda, \theta))) = (\eta_L e^{Q} \eta_R^{-1}, \eta_R L(q, p) \eta_R^{-1}, \eta_L \xi(E) \eta_L^{-1})
\end{equation}
with some group elements $\eta_L, \eta_R \in K$. Now, remembering the definition of the positive definite matrix $\mathcal{A}^{BC}$ (4.51), notice that the $G$-component of the above equation immediately leads to the relationship
\begin{equation}
\mathcal{A}^{BC}(\lambda, \theta) = h(\lambda)^{-1} \mathcal{A}(\lambda, \theta) h(\lambda)^{-1} = \eta_R e^{Q} \eta_R^{-1},
\end{equation}
from where we get the spectral identification
\begin{equation}
\sigma(\mathcal{A}^{BC}(\lambda, \theta)) = \{ e^{\pm 2\lambda_c} | c \in \mathbb{N}_n \}.
\end{equation}
We see that $\mathcal{A}^{BC}(\lambda, \theta)$ has a simple spectrum, and the positive eigenvalues of the Hermitian matrix $\ln(\mathcal{A}^{BC})/2$ are exactly the coordinate functions $\lambda_c$ $c \in \mathbb{N}_n$).

To conclude the study of the Sutherland side of the dual reduction picture (5.1), let us recall that the hyperbolic $BC_n$ Sutherland model can be realized as the reduced Hamiltonian system generated by the quadratic Ad-invariant function $F_2$ (3.63). Therefore the above construction of action-angle coordinates applies to the repulsive Sutherland model equally well. Also, from (5.5) we see that the positive eigenvalues of $L$ (3.5) provide $n$ functionally independent first integrals in involution. Furthermore, the matrix $L$ naturally enters the solution algorithm of the Sutherland model (see equation (3.67)), whence the non-Hermitian matrix $L$ is indeed a Lax matrix for the Sutherland many-particle system.

In the following we turn our attention to the Ruijsenaars side of the dual reduction picture (5.1). For, take an arbitrary point $(\lambda, \theta) \in \mathcal{P}^R$ and let $(q, p) = \mathcal{S}^{-1}(\lambda, \theta) \in \mathcal{P}^S$. Recalling the mappings $\Upsilon^R$ (4.31) and $\Upsilon^S$ (3.41), it is clear that
\begin{equation}
(\mathcal{A}(\lambda, \theta) \frac{1}{2} h(\lambda)^{-1}, h(\lambda) \mathcal{A}(\lambda, \theta) h(\lambda)^{-1}, \xi(\mathcal{A}(\lambda, \theta))) = (\eta_L e^{Q} \eta_R^{-1}, \eta_R L(q, p) \eta_R^{-1}, \eta_L \xi(E) \eta_L^{-1})
\end{equation}
with some group elements $\eta_L, \eta_R \in K$. Now, remembering the definition of the positive definite matrix $\mathcal{A}^{BC}$ (4.51), notice that the $G$-component of the above equation immediately leads to the relationship
\begin{equation}
\mathcal{A}^{BC}(\lambda, \theta) = h(\lambda)^{-1} \mathcal{A}(\lambda, \theta) h(\lambda)^{-1} = \eta_R e^{Q} \eta_R^{-1},
\end{equation}
from where we get the spectral identification
\begin{equation}
\sigma(\mathcal{A}^{BC}(\lambda, \theta)) = \{ e^{\pm 2\lambda_c} | c \in \mathbb{N}_n \}.
\end{equation}
We see that $\mathcal{A}^{BC}(\lambda, \theta)$ has a simple spectrum, and the positive eigenvalues of the Hermitian matrix $\ln(\mathcal{A}^{BC})/2$ are exactly the coordinate functions $\lambda_c$ $c \in \mathbb{N}_n$).

To proceed further, take an arbitrary $K \times K$-invariant smooth function $f: G \to \mathbb{R}$ (4.38), and consider the naturally generated reduced Hamiltonian system $(\mathcal{P}^R, \omega^R, (\mathcal{P}^*_R f)_R)$. Recalling (4.38) and (5.7), for the reduced Hamiltonian we have
\begin{equation}
(\mathcal{P}^*_R f)_R(\lambda, \theta) = f(\mathcal{A}(\lambda, \theta) \frac{1}{2} h(\lambda)^{-1}) = f(\text{diag}(e^{-\lambda(\lambda, \theta)}, \ldots, e^{-\lambda(\lambda, \theta)}, e^{-\lambda(\lambda, \theta)}, \ldots, e^{-\lambda(\lambda, \theta)})),
\end{equation}
i.e. the reduced Hamiltonian depends only on the coordinates $\lambda_c$ $c \in \mathbb{N}_n$. Thus, it is immediate that the global canonical coordinates $\lambda_c$ and $\bar{\lambda}_c$ $c \in \mathbb{N}_n$ form an action-angle system for the mechanical system $(\mathcal{P}^R, \omega^R, (\mathcal{P}^*_R f)_R)$. Let us observe that the action and the angle coordinates of the Ruijsenaars picture are coming from the pull-backs of the canonical positions and the canonical momenta of the Sutherland picture.

Now remember that the reduced Hamiltonian system $(\mathcal{P}^R, \omega^R, (\mathcal{P}^*_R f)_R)$ generated by the $K \times K$-invariant function $f_1$ (4.51) coincides with the rational $BC_n$ RSvD model (1.5) with three independent coupling constants. Therefore the canonical coordinates $\lambda_c$ and $\bar{\lambda}_c$ $c \in \mathbb{N}_n$) provide action-angle variables for the RSvD model as well. Notice also that the positive definite matrix

i.e. the reduced Hamiltonian depends only on the coordinates $\lambda_c$ $c \in \mathbb{N}_n$. It follows that the global coordinates $\lambda_c$ and $\bar{\lambda}_c$ $c \in \mathbb{N}_n$ provide canonical action-angle variables for the reduced system $(\mathcal{P}^S, \omega^S, (\mathcal{P}^*_S f)_S)$. Note that the action and the angle coordinates of the Sutherland picture are exactly the pull-backs of the canonical positions and the canonical momenta of the Ruijsenaars picture.
plays a distinguished role in the theory of the rational RSvD model. Indeed, the positive eigenvalues of the Hermitian matrix \( \ln(\mathcal{A}^{BC})/2 \) give rise to \( n \) functionally independent first integrals in involution. In particular, the matrix \( \mathcal{A}^{BC} \) undergoes an isospectral deformation during the time evolution of the RSvD dynamics. Also, remember that \( \mathcal{A}^{BC} \) naturally appears in the solution algorithm of the model, as can be seen in equation (4.57). Therefore \( \mathcal{A}^{BC} \) meets all the criteria to call it the Lax matrix of the rational \( BC_n \) RSvD model.

To sum up, we constructed action-angle systems of canonical coordinates for both the repulsive hyperbolic \( BC_n \) Sutherland and the rational \( BC_n \) RSvD models with three independent coupling constants. The relationships between the coupling parameters of the corresponding particle systems are displayed in equations (3.65) and (3.66). As we have seen, the action and the angle coordinates of the Sutherland model can be naturally identified with the canonical positions and the canonical momenta of the RSvD model, and vice versa. That is to say, making use of the dual reduction picture (5.1), we established the action-angle duality between the standard \( BC_n \)-type Sutherland and RSvD models. This interesting phenomenon was originally discovered by Ruijsenaars in the context of the \( A_n \)-type particle systems [18]. Using advanced techniques from symplectic geometry, in the last years the \( A_n \)-type dualities have been reinterpreted in the reduction framework, too (see the papers [19], [20]). It appears to be an attractive research problem for the future to generalize these techniques to the non-\( A_n \)-type setup.

We conclude the paper with some remarks on the possible applications of our results. Besides the natural appearance of the Sutherland and the RSvD many-particle systems in the soliton scattering description of certain integrable field theories (see e.g. [1], [2], [3], [4]), we expect that our results find applications in the theory of random matrices as well. Indeed, by exploiting the existing dualities between the \( A_n \)-type particle systems, the authors of the recent papers [5] and [6] have introduced new classes of random matrix ensembles with novel spectral statistical properties. Built on the Lax matrices \( L \) (3.5) and \( \mathcal{A}^{BC} \) (4.51), the proposed dual reduction picture (5.1) seems to be indispensable in initiating the study of the integrable random matrix ensembles associated with non-\( A_n \)-type root systems.

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