Closed String Thermal Torus From Thermo Field Dynamics

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Abstract

In this Letter a topological interpretation for the string thermal vacuum in the Thermo Field Dynamics (TFD) approach is given. As a consequence, the relationship between the Imaginary Time and TFD formalisms is achieved when both are used to study closed strings at finite temperature. The TFD approach starts by duplicating the system’s degrees of freedom, defining an auxiliary (tilde) string. In order to lead the system to finite temperature a Bogoliubov transformation is implemented. We show that the effect of this transformation is to glue together the string and the tilde string to obtain a torus. The thermal vacuum appears as the boundary state for this identification. Also, from the thermal state condition, a Kubo-Martin-Schwinger condition for the torus topology is derived.
I. INTRODUCTION

Over the years string theory has been considered the best candidate for a theory that quantizes gravity. The two most pressing areas are those of black holes physics and cosmology. However, in despite of the significant progress which has been made in the last years, there is a lot of indications that to understand the non-perturbative regime of the theory (necessary to study black holes and cosmology), a more robust theoretical framework is needed. The formulation of string theory at finite temperature has given good hints that the perturbative formulation of the theory does not work at high temperature. Owing to the exponential growth of states as function of energy, there is an upper bound temperature above which the statistical partition function diverges (the Hagedorn temperature). The very existence of the Hagedorn temperature shows that the fundamental degrees of freedom of the theory could not be the ones of the perturbative string. If the specific heat at the Hagedorn temperature is finite it denotes a phase transition, and just as quark and gluons emerge as the basic ingredients of QCD at high temperature, the true degrees of freedom of the string theory may also emerge at high temperature. So, beyond phenomenological applications, string at finite temperature may also provide important pieces of evidence of the true degrees of freedom of the theory at non-perturbative regime.

It has to be mentioned that in the last two decades there have been interesting works studying string theory at finite temperature. The standard way used to lead the string theory from zero to finite temperature is the Imaginary Time formalism [1, 2, 3].

In general, the statistical average of an operator \( Q \), \( \langle Q \rangle \), is defined by the functional

\[
\omega(1) = \text{Tr} (\rho Q) = \frac{\text{Tr}[Q e^{-\beta H}]}{\omega(1)},
\]

where \( \rho = e^{-\beta H} \) and \( \omega(1) \) is the partition function. In statistical quantum field theory the trace is taken over the Fock space formed by the field operators equipped with some algebraic properties. As a consequence of the cyclic property of the trace, the Kubo-Martin-Schwinger condition follows: \( \omega(A(t) B) = \omega(B A(t + i \beta)) \), which is the basis of the Imaginary Time formalism (Matsubara formalism).

Using the Matsubara formalism, a quantum field at finite temperature relates the functional generator in \( R^{1,d-1} \) to a trace when the theory is defined in \( R^{d-1} \times S^1 \) with the length of the compactified circle equals to \( \beta \). When the Matsubara formalism is applied to a free
closed string, the theory is defined on a torus, where $\beta$ is related to the imaginary part of the modulus parameter $\tau$. The real part of $\tau$ is related to a Lagrange multiplier ($\lambda$), that imposes the $S^1$ isometry of the closed string to the Fock space. The gauge fixing of the $S^1$ isometry improves the level matching condition of the physical spectrum of the closed string. So, for the closed string, $\omega(Q)$ must be redefined as:

$$\omega(Q) = \int_0^1 d\lambda \text{Tr} \left[ Q e^{-\beta H + 2\pi i \lambda P} \right],$$

(2)

where $P$ is the momentum operator that generates translations in the world-sheet $\sigma$ coordinate. The integral over the Lagrange multiplier guarantees that the trace is taken only over the physical states. The partition function is defined on a torus with moduli space parameters defined by: $\tau = \lambda + i \frac{\beta}{2\pi}$. A different perspective in which the torus topology can be observed is perceived by noting that the operator $e^{(-2\pi i \beta H)}$ propagates a closed string through imaginary time $i 2\pi \beta$ and the operator $e^{i 2\pi \lambda P}$ rotates the closed string at an $2\pi \lambda$ angle. Hence the trace corresponds to a torus constructed by gluing together the ends of an open cylinder with a relative twist [4].

On the other hand, the functional $\omega(Q)$ is called a state in algebraic statistical quantum field theory and the operators form a $C^*$ algebra [5, 6]. The functional $\omega$ resembles a vector space, so that the algebra equipped with a particular functional admits a reducible representation on a Hilbert space such as a Fock space. This is the basis of an alternative formalism to study quantum field at finite temperature developed by Takahashi and Umezawa, named Thermo Field Dynamics (TFD) [7, 8, 9, 10, 11, 12]. The TFD was developed in order to handle finite temperature with a real time operator formalism [13, 14]. The main idea is to interpret the statistical average as the expectation value of $Q$ in a thermal vacuum:

$$\frac{\omega(Q)}{\omega(1)} = \langle 0(\beta) | Q | 0(\beta) \rangle.$$

(3)

Concerning string theory, the idea of building a thermal Fock space is particularly tempting as new degrees of freedom could be identified in this Fock space at some temperature. Although the TFD approach was adopted in the past to study first quantized bosonic string [18, 19], heterotic string [20, 21] and string field theory [22], it was employed within a path integral formulation. The idea of using the Fock space formulation in string theory came up in Refs. [23, 24, 25, 26, 27, 28], where the thermal space was used to construct bosonic thermal boundary states interpreted as D-branes at finite temperature. To further explore
the algebraic characteristics of the TFD, and to upgrade it to a powerful tool to understand
string theory at finite temperature, it is first necessary to set up the connections between
the TFD and the Imaginary Time formalisms, when both are applied to strings at thermal
equilibrium. In [15], the TFD was used to derive thermodynamical quantities for type IIB
superstring in a pp-wave background. It was shown that the free energy, calculated from
the world-sheet torus partition function in the Imaginary Time formalism, can be derived
from the thermal expectation value in TFD. In subsequent works the $SU(1,1)$ and $SU(2)$
thermal groups were used to generalize the TFD for applications in closed string theories
[16, 17]. Such a generalization again reproduces the free energy obtained via Imaginary Time
formalism. A question arises: how the torus defined by taking the trace on the right-hand
side of equation (3) can be interpreted on the left-hand side? The aim of this work is to
answer this question. It will be shown how a torus can appear when the TFD is used to
treat the closed string at finite temperature at thermal equilibrium\(^1\). Also, it will be shown
that the thermal vacuum for the free closed string represents a kind of boundary state for
string theory. As a consequence, a Kubo-Martin-Schwinger (KMS) relation arises with a
topological interpretation.

II. TFD APPROACH FOR THE CLOSED BOSONIC STRING

The TFD is introduced by first duplicating the degrees of freedom of the system. To this
end a copy of the original Hilbert space, denoted by $\tilde{H}$, is constructed. The tilde Hilbert
space for the closed bosonic string in the light-cone gauge is built with a set of oscillators
that have the same commutation properties as the original ones. The total Hilbert space
is the tensor product of the two spaces $\mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}}$ with elements $|\Phi\rangle = |\phi, \tilde{\phi}\rangle$ and the
vacuum is defined by

$$
\begin{align*}
a_n^L|0\rangle &= \tilde{a}_n^L|0\rangle = 0, \\
\tilde{a}_n^L|0\rangle &= \tilde{\bar{a}}_n^L|0\rangle = 0
\end{align*}
$$

(4)

for $n > 0$ and $|0\rangle = |\tilde{0}\rangle = |0\rangle \otimes |\tilde{0}\rangle$ as usual. The bar and non-bar operators denote, here
and in the following, the left- right-moving modes, respectively. The map between the tilde

\(^1\) The closed bosonic string was chosen for the sake of simplicity, but the results presented here can easily
be extended for the light-cone Green-Schwarz superstring.
and non-tilde operators is defined by the following tilde (or dual) conjugation rules \[29\]:

\[
(A_i A_j) \tilde{=} \tilde{A}_i \tilde{A}_j,
\]

\[
(c A_i + A_j) \tilde{=} c^* \tilde{A}_i + \tilde{A}_j,
\]

\[
(A_i^\dagger) \tilde{=} (\tilde{A}_i)^\dagger,
\]

\[
(\tilde{A}_i) \tilde{=} A_i,
\]

\[
[\tilde{A}_i, A_j] = 0.
\] (5)

As the operators of the two systems commute among themselves, the doubled system is described by two independent strings defining two world-sheets. Using the Euclidian time \((\tau = -it)\), the mode expansions for the two strings in the light-cone gauge are:

\[
X^I = x_0^I - i\alpha^I p_0^I t + \sqrt{\alpha'} \sum_{n>0} \frac{1}{\sqrt{n}} \left[ (a_n^I e^{-n(t-i\sigma)} + a_n^I e^{n(t-i\sigma)})
\right.
\]

\[
+ (\tilde{a}_n^I e^{-n(t+i\sigma)} + \tilde{a}_n^I e^{n(t+i\sigma)}) \bigg],
\] (6)

and

\[
\tilde{X}^I = \tilde{x}_0^I + i\alpha^I p_0^I t + \sqrt{\alpha'} \sum_{n>0} \frac{1}{\sqrt{n}} \left[ (\tilde{a}_n^I e^{-n(i-\sigma)} + \tilde{a}_n^I e^{n(i-\sigma)})
\right.
\]

\[
+ (\tilde{\tilde{a}}_n^I e^{-n(i+\sigma)} + \tilde{\tilde{a}}_n^I e^{n(i+\sigma)}) \bigg],
\] (7)

where the expansion for \(\tilde{X}(t, \sigma)\) was obtained from the tilde conjugation rules. It can be seen that the tilde string can be interpreted as a string propagating backwards in Euclidian time.

The oscillators satisfy the extended algebra:

\[
\left[ a_n^I, a_{m}^{I,J} \right] = \left[ \tilde{a}_n^I, \tilde{a}_m^{I,J} \right] = \delta_{n,m} \delta^{I,J},
\]

\[
\left[ \tilde{a}_n^I, a_{m}^{I,J} \right] = \left[ \tilde{a}_n^I, \tilde{a}_m^{I,J} \right] = \delta_{n,m} \delta^{I,J},
\]

\[
\left[ a_{n}^{I,J}, \tilde{a}_m^I \right] = \left[ a_{n}^{I,J}, a_{m}^{I,J} \right] = \left[ a_{n}^{I,J}, \tilde{a}_m^I \right] = \left[ a_{n}^{I,J}, a_{m}^{I,J} \right] = 0.
\] (8)

Now, the thermal vacuum and the thermal Fock space for the closed string can be constructed. This is achieved by implementing a Bogoliubov transformation in the total Hilbert space. At this point it is necessary to emphasize that the physical variables are described by the non-tilde operators. The tilde operators are auxiliary degrees of freedom necessary to
provide enough room to accommodate the thermal properties of the system. All the thermodynamical quantities are derived from expectation values of $T = 0$ non-tilde operators in the thermal vacuum to be defined.

The generator for the Bogoliubov transformation of the closed string is given by

\[
G = G + \bar{G},
\]

where

\[
G = -i \sum_n \theta_n \left( a_n \cdot \tilde{a}_n - \tilde{a}_n^\dagger \cdot a_n^\dagger \right),
\]

\[
G = -i \sum_n \bar{\theta}_n \left( a_n \cdot \tilde{a}_n - \tilde{a}_n^\dagger \cdot a_n^\dagger \right).
\]

Here the dots represent the inner products and $\theta, \bar{\theta}$ are the transformation’s parameters. As we shall see, at thermal equilibrium they are related to the Bose-Einstein distribution of the oscillator $n$.

The thermal vacuum is given by the following relation

\[
|0 (\theta)\rangle = e^{-iG} |0\rangle = \prod_{n=1}^{\infty} \left[ \left( \frac{1}{\cosh(\theta_n)} \right)^{D-2} \left( \frac{1}{\cosh(\bar{\theta}_n)} \right)^{D-2} e^{\tanh(\theta_n) (a_n^\dagger \tilde{a}_n) + \tanh(\bar{\theta}_n) (\tilde{a}_n^\dagger a_n^\dagger)} \right] |0\rangle. \tag{11}
\]

The creation and annihilation operators at $T \neq 0$ are given by the Bogoliubov transformations as follows

\[
a_n^I(\theta_n) = e^{-iG} a_n^I e^{iG} = \cosh (\theta_n) a_n^I - \sinh (\theta_n) \tilde{a}_n^\dagger I,
\]

\[
\bar{a}_n^I(\bar{\theta}_n) = e^{-iG} \bar{a}_n^I e^{iG} = \cosh (\bar{\theta}_n) \bar{a}_n^I - \sinh (\bar{\theta}_n) a_n^\dagger I. \tag{13}
\]

These operators annihilate the state written in (11) defining it as the vacuum. By using the Bogoliubov transformation, the relations

\[
a_n^I(\theta_n) |0 (\theta)\rangle = \bar{a}_n^I(\theta_n) |0 (\theta)\rangle = 0,
\]

\[
\langle 0 (\theta) | a_n^{+I} (\theta_n) = \langle 0 (\theta) | \bar{a}_n^{+I} (\theta_n) = 0, \tag{14}
\]

give rise the so called thermal state conditions:

\[
[a_n^I - \tanh (\theta_n) \tilde{a}_n^\dagger I] |0 (\theta)\rangle = 0, \tag{15}
\]

\[
[\tilde{a}_n^I - \tanh (\theta_n) a_n^{+I}] |0 (\theta)\rangle = 0, \tag{16}
\]
\[
\begin{align*}
[\tilde{a}_n^I - \tanh (\bar{\theta}_n) \tilde{a}_n^{\dagger I}] |0 (\theta)\rangle &= 0, \\
[\bar{a}_n^I - \tanh (\bar{\theta}_n) \bar{a}_n^{\dagger I}] |0 (\theta)\rangle &= 0, \\
\langle 0 (\theta)| [\bar{a}_n^{\dagger I} - \tanh (\bar{\theta}_n) \bar{a}_n^I] &= 0, \\
\langle 0 (\theta)| [\tilde{a}_n^{\dagger I} - \tanh (\bar{\theta}_n) \tilde{a}_n^I] &= 0.
\end{align*}
\]
\[\tag{17} \]
\[\tag{18} \]
\[\tag{19} \]
\[\tag{20} \]
\[\tag{21} \]
\[\tag{22} \]

The thermal Fock space is constructed by applying the thermal creation operators to the vacuum \( |0 (\theta)\rangle \). As the Bogoliubov transformation is canonical, the thermal operators obey the same commutation relations as the operators at \( T = 0 \). It is easy to see that thermal states are not eigenstates of the original Hamiltonian but they are eigenstates of the combination:

\[
\hat{H} = H - \tilde{H},
\]
\[\tag{23} \]

in such a way that \( \hat{H} \) plays the rôle of the Hamiltonian, generating temporal translation in the thermal Fock space. Using the commutation relations we can prove that the Heisenberg equations are satisfied replacing \( H \) and \( \tilde{H} \) by \( \hat{H} \). Also we have

\[
\hat{P} = P - \tilde{P},
\]
\[\tag{24} \]

where \( P, \tilde{P} \) and \( \hat{P} \) are the world-sheet translation generators of the original, auxiliary and transformed systems, respectively.

The effect of the Bogoliubov transformation is to entangle the elements of the two Hilbert spaces. After the transformation, the image of the two independent strings defining two different cylinders is lost. In quantum field theory the thermal vacuum can be interpreted as a condensed state and the Bogoliubov transformation confines the fields in a restricted region of the time axis. In [30] the analytical continuation for \( \sinh^2 (\theta_n) \) is explored to demonstrate that the Bogoliubov operator can also produce confinement in spatial directions. In the next section, it will be demonstrated that the time confinement produced by the Bogoliubov operator, in a very special way, transforms the two cylinders in a torus and the state \(|0 (\theta)\rangle \) becomes a string boundary state.
III. THERMAL TORUS AND THERMAL STATE

Suppose that one wants to build a torus with the two initial cylinders defined by two independent strings. From the tilde conjugation rules one sees that the tilde string propagates backwards in the Euclidian time. In this way, considering the original string propagating by an amount of the value $\beta$, i.e., $X(\frac{\beta}{2})$, then the tilde string propagates by the same amount in the opposite direction, $\tilde{X}(\frac{-\beta}{2})$. One can construct a torus by gluing together the end of the original cylinder with the origin of the tilde cylinder, and vice-versa. This procedure will confine the fields to a restrict region $\beta$ in the Euclidian time. Also, before gluing, the identification $\tilde{\sigma} = \sigma - \pi \lambda$, must be done in order to take into account the Dehn twist in one cycle. The two parameters of the resulting torus moduli space will be related to $\beta$ and $\lambda$.

The above considerations can be written as follows:

\[ X(t, \sigma) - \tilde{X}\left(-t - \frac{\beta}{2}, \sigma - \lambda \pi\right) = 0, \]
\[ X\left(-\tilde{t} - \frac{\beta}{2}, \tilde{\sigma} + \lambda \pi\right) - \tilde{X}\left(\tilde{t}, \tilde{\sigma}\right) = 0. \]  

(25)

Expanding $X(t, \sigma)$ and $\tilde{X}(\tilde{t}, \tilde{\sigma})$ in modes, the above identification turns out to be the following set of operatorial equations for a boundary state $|\Phi\rangle = |\phi, \tilde{\phi}\rangle$:

\[ \left[ a_n^I - e^{-n(\frac{\beta}{2} + i \lambda \pi)} \tilde{a}_{\tilde{n}}^I \right] |\Phi\rangle = 0, \]  
\[ \left[ \tilde{a}_n^I - e^{-n(\frac{\beta}{2} + i \lambda \pi)} a_{\tilde{n}}^I \right] |\Phi\rangle = 0, \]  
\[ \left[ \bar{a}_n^I - e^{-n(\frac{\beta}{2} - i \lambda \pi)} \bar{a}_{\tilde{n}}^I \right] |\Phi\rangle = 0, \]  
\[ \left[ \tilde{\bar{a}}_n^I - e^{-n(\frac{\beta}{2} - i \lambda \pi)} \bar{a}_{\tilde{n}}^I \right] |\Phi\rangle = 0. \]  

(26-29)

It is assumed that there is no center-of-mass momentum dependence on $|\Phi\rangle$, which means that the theory is being described in a particular frame. As the state $|\Phi\rangle$ will be related to the thermal state, there is no problem with that assumption since the temperature breaks the Lorentz invariance. Defining $\tau = \lambda + i \frac{\beta}{2\pi}$ and $q = e^{2\pi i \tau}$, the normalized solution of the equations (26)-(29) is

\[ |\Phi\rangle = \left[ (q \bar{q})^{\frac{D-2}{2 \pi}} |\eta(\tau)|^{-2(D-2)} \right]^{-\frac{1}{2}} \sum_{n > 0} (a_n \tilde{a}_{\tilde{n}}) q^{\frac{n}{2}} \times e^{\sum_{n > 0} \bar{a}_{\tilde{n}}^I \tilde{a}_n^I \bar{q}^{\tilde{n}} q^n} |0\rangle, \]  

(30)

where $\eta(\tau)$ is the Dedekind $\eta$ function

\[ \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \]  

(31)
and $\bar{q}$ is the complex conjugate of $q$.

Next it will be shown that the above state is precisely the thermal state \cite{11}. To this end it is necessary to find the explicit dependence of the $\theta$ parameters with respect to $\beta$ and $\lambda$. If the following identifications

$$\tanh (\theta_n) = q_n^{\frac{1}{2}}, \quad \tanh (\bar{\theta}_n) = \bar{q}_n^{\frac{1}{2}},$$

hold, one can see that equations \cite{26}-\cite{29} are in fact the thermal state conditions \cite{15}-\cite{18} that define the thermal vacuum. Equation (32) can be derived by minimizing the following potential $F$ \cite{7, 15, 16}:

$$F = E - \frac{1}{\beta} S, \quad (33)$$

with respect to the transformation’s parameters. Here $E$ is related with the thermal energy and $S$ with the entropy of the string. So, $F$ is a free energy like potential. In TFD, the thermal energy is given by computing the matrix elements of the $T = 0$ Hamiltonian in the thermal vacuum. In the same way, $S$ is computed by using the entropy operator $K$ defined for the closed string by:

$$K = -\sum_{n=1}^{\infty} \left\{ a_n^\dagger \cdot a_n \ln (\sinh^2 (\theta_n)) - a_n^\dagger \cdot a_n^\dagger \ln (\cosh^2 (\theta_n)) \right\}$$

$$-\sum_{n=1}^{\infty} \left\{ a_n^\dagger \cdot \bar{a}_n \ln (\sinh^2 (\bar{\theta}_n)) - \bar{a}_n \cdot a_n^\dagger \ln (\cosh^2 (\bar{\theta}_n)) \right\}. \quad (34)$$

In order to apply TFD in closed strings, it is necessary to redefine the thermal energy operator in such a way that the level matching conditions are improved. The procedure is to consider as the thermal energy operator, the shifted Hamiltonian in the sense of Refs. \cite{15, 16, 31},

$$H_s = \frac{P_s \cdot P_s}{2} + \sum_{n=1}^{\infty} n (N_n + \bar{\bar{N}}_n) + \frac{1}{\beta} i 2\pi \lambda \sum_{n=1}^{\infty} n (N_n - \bar{\bar{N}}_n) - 2, \quad (35)$$

where

$$N_n = a_n^\dagger \cdot a_n, \quad \bar{\bar{N}}_n = \bar{a}_n^\dagger \cdot \bar{a}_n, \quad (36)$$

and $\sum_{n=1}^{\infty} n (N_n - \bar{\bar{N}}_n)$ is the momentum operator of the world-sheet. By considering the expectation value of that shifted Hamiltonian as the thermal energy, the $S^1$ isometry is fixed and the expectation value \cite{3} reproduces the trace \cite{2, 15, 16}.

Now, using (36) and (34) $F$ is minimized and the following relations are found:

$$\sinh^2 (\theta_n) = \frac{1}{e^{n(\beta + i 2\pi \lambda)} - 1}, \quad \sinh^2 (\bar{\theta}_n) = \frac{1}{e^{n(\beta - i 2\pi \lambda)} - 1}, \quad (37)$$
These expressions fix the thermal vacuum (11), now reproducing the trace over the transverse sector. All the thermodynamical quantities at equilibrium can be derived from this state. However, a time dependence could be allowed on θ and out of equilibrium physics could be described. In this sense, this formalism is more general than the Imaginary Time formalism. Also, we can easily see that

$$\prod_{n=1}^{D-2} \left( \frac{1}{\cosh(\theta_n)} \right)^{D-2} \left( \frac{1}{\cosh(\bar{\theta}_n)} \right)^{D-2} = \left[ (q\bar{q})^{D-2} |\eta(\tau)|^{-2(D-2)} \right]^{-\frac{1}{2}},$$

(38)

so, equations (15)-(22) hold and the thermal vacuum is exactly the boundary state given in (30).

IV. KMS CONDITION FOR THE CLOSED BOSONIC STRING

In this section the thermal state (boundary) equations will be used in order to derive a KMS condition for closed bosonic string.

The operators $\hat{H}$ and $\hat{P}$ defined in (23) and (24) commute with the Bogoliubov generator (10). Therefore, it follows that

$$\hat{H} |0(\theta)\rangle = \hat{P} |0(\theta)\rangle = 0.$$  

(39)

Now, using the following commutation relations

$$e^{\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} a_n e^{-\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} = e^{-n\left(\frac{\beta}{2} + i\lambda\pi\right)} a_n, \quad (40)$$

and (32), the equations (15), (16), (19) and (20) for the right-moving modes become

$$\left[ a_n - e^{\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} \bar{a}_n \right] |0(\theta)\rangle = 0, \quad (41)$$

$$\left[ \bar{a}_n - e^{-\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} a_n \right] |0(\theta)\rangle = 0, \quad (42)$$

$$\langle 0(\theta) | \left[ a_n - e^{\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} a_n \right] = 0, \quad (43)$$

$$\langle 0(\theta) | \left[ \bar{a}_n - e^{-\left(\frac{\beta}{2} \hat{H} + i\pi \lambda \hat{P}\right)} \bar{a}_n \right] = 0. \quad (44)$$
The above expressions (41) and (44), for example, can be extended to

\[
\begin{align*}
A - e^{(\frac{2i}{\beta} \hat{H} + i\pi \lambda \hat{P})} \tilde{A}^\dagger |0(\theta)\rangle &= 0, \\
\langle 0(\theta) | [\tilde{A}^\dagger - Ae^{-\left(\frac{2i}{\beta} \hat{H} + i\pi \lambda \hat{P}\right)}] &= 0,
\end{align*}
\]

(45)

where \( A \) stands for any sum of normal ordered products of \( a_n \) and \( a_n^\dagger \).

Now, we are able to show that the thermal state conditions (15)-(22), that are in fact related with the identifications (25), give us a generalized KMS condition for closed string theory. In usual quantum field theory, defining two operators \( A(t) \) and \( B(t') \), the KMS condition reads

\[
\langle 0(\beta) | A(t) B(t') | 0(\beta) \rangle = \langle 0(\beta) | B(t') A(t + i\beta) | 0(\beta) \rangle.
\]

(47)

Let us derive this condition when \( A \) and \( B \) stand for closed string right-moving world-sheet fields. Since \( \hat{H} \) and \( \hat{P} \) generate time and \( \sigma \) translation, equations (45) and (46) can be written as

\[
\begin{align*}
\left[ A(\tau + i\beta, \sigma - 2\pi \lambda) - \tilde{A}^\dagger \left( \tilde{\tau} - i\frac{\beta}{2}, \tilde{\sigma} + \pi \lambda \right) \right] |0(\beta)\rangle &= 0, \\
\langle 0(\beta) | \tilde{A}^\dagger \left( \tilde{\tau} - i\frac{\beta}{2}, \tilde{\sigma} + \pi \lambda \right) - A(\tau, \sigma) &= 0.
\end{align*}
\]

(48)

By using the above expressions, one can derive the following result

\[
\langle 0(\beta) | A(\tau, \sigma) B(\tau', \sigma') | 0(\beta) \rangle = \langle 0(\beta) | \tilde{A}^\dagger \left( \tilde{\tau} - i\frac{\beta}{2}, \tilde{\sigma} + \lambda \pi \right) B(\tau', \sigma') | 0(\beta) \rangle \\
= \langle 0(\beta) | B(\tau', \sigma') \tilde{A}^\dagger \left( \tilde{\tau} - i\frac{\beta}{2}, \tilde{\sigma} + \lambda \pi \right) | 0(\beta) \rangle \\
= \langle 0(\beta) | B(\tau', \sigma') A(\tau + i\beta, \sigma - 2\pi \lambda) | 0(\beta) \rangle.
\]

(49)

The same procedure applied to the bar operators gives

\[
\langle 0(\beta) | \bar{A}(\tau, \sigma) \bar{B}(\tau', \sigma') | 0(\beta) \rangle = \langle 0(\beta) | \bar{B}(\tau', \sigma') \bar{A}(\tau + i\beta, \sigma + 2\pi \lambda) | 0(\beta) \rangle.
\]

(50)

The above results, (49) and (50), are a generalization of the KMS condition, when the torus topology is taken into account. It is a direct consequence of the thermal states conditions.

V. CONCLUSIONS

In this work the Thermo Field Dynamics (TFD) formalism was used to treat closed strings at thermal equilibrium. The main characteristic of TFD is the construction of a
thermal Fock space and thermal operators. We give a topological interpretation for the thermal vacuum, relating this approach with the Imaginary Time formalism.

The TFD algorithm starts by defining an auxiliary Hilbert space and a thermal vacuum is constructed by means of a Bogoliubov transformation, that entangles the elements of the two Hilbert spaces. We show that the auxiliary Hilbert space is related with a string that propagates backwards in Euclidean time. The effect of the thermal Bogoliubov transformation, in the equilibrium situation, is to identify the two strings in order to make a torus with moduli space parameters defined by \( \tau = \lambda + i \frac{\beta}{2\pi} \), where \( \lambda \) is a Lagrange multiplier fixing the closed string’s \( S^1 \) isometry. We show that these boundary equations are in fact the thermal state conditions and the thermal vacuum is the boundary state solution that comes from the identification of the two strings. With this we clarify why the TFD computations carried out in our previous works [15, 16, 17] reproduce the results coming from the Imaginary Time formalism for free closed strings, where the theory is defined on a torus. The interpretation for the torus in TFD presented here agrees with that given by Laflamme [32], where the original and tilde fields live in different surfaces. Also, as a consequence of the thermal state conditions, we provided a generalization of the KMS boundary condition that holds when the torus topology is taken into account.

There are many possible extensions of this work. We can try to use the techniques developed in [33] to further use TFD to study string theory at finite temperature at higher genus. The way how TFD works under these situation consists by itself in an interesting point for investigation. Also, as this is a real time formalism, it can be used in a more involved situation, taking into account for example, time dependent geometries. In fact, due to the evolution of the particle distribution, a real time formalism seems the appropriated one to be used in theories containing gravity [34]. Furthermore the possibility to go out of thermal equilibrium can be very useful for applications in string cosmology.

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