The "Hot Spots" Conjecture on Homogeneous Hierarchical Gaskets

Xiaofen Qiu*

Division of Fundamental Education, Shanghai Industry and Commerce Foreign Language College, Shanghai 201399, China and School of Mathematical Science, Zhejiang University, Hangzhou 310027, Zhejiang, China

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Abstract. In this paper, using spectral decimation, we prove that the "hot spots" conjecture holds on a class of homogeneous hierarchical gaskets introduced by Hambly, i.e., every eigenfunction of the second-smallest eigenvalue of the Neumann Laplacian (introduced by Kigami) attains its maximum and minimum on the boundary.

Key Words: Neumann Laplacian, "hot spots" conjecture, homogeneous hierarchical gasket, spectral decimation, analysis on fractals.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

The "hot spots" conjecture was posed by J. Rauch at a conference in 1974. Informally speaking, it was stated in [3] as follows: Suppose that \( D \) is an open connected bounded subset of \( \mathbb{R}^d \) and \( u(t,x) \) is the solution of the heat equation in \( D \) with the Neumann boundary condition. Then for "most" initial conditions, if \( z_t \) is a point at which the function \( x \to u(t,x) \) attains its maximum, then the distance from \( z_t \) to the boundary of \( D \) tends to zero as \( t \) tends to \( \infty \). In other words, the "hot spots" move towards the boundary. Formally, there are several versions of the hot spots conjecture. See [3] for details. In this paper, we will use the following version: every eigenfunction of the second-smallest eigenvalue of the Neumann Laplacian attains its maximum and minimum on the boundary.

The "hot spots" conjecture holds in many typical domains in Euclidean space, especially for certain convex planar domains and lip domains. For examples, please see [1, 3, 11]. On the other hand, Burdzy and Werner [5] and Burdzy [4] constructed interesting planar domains such that the "hot spots" conjecture fails.

*Corresponding author. Email address: 21106081@zju.edu.cn (X. F. Qiu)
The underlying spaces in above works are domains in Euclidean space. Since we can do analysis on fractals (see [12, 13, 21]), it is natural to ask whether the conjecture holds for p.c.f. fractals. Recently, there are some works on this topic. On the one hand, Ruan [17], Ruan and Zheng [18], Li and Ruan [15] proved that the conjecture hold on the Sierpinski gasket (SG₂ for short), the level-3 Sierpinski gasket (SG₃ for short) and higher dimensional Sierpinski gaskets. On the other hand, Lau, Li and Ruan [14] proved that the conjecture does not hold on the hexagasket. The basic tool used in these this paper is spectral decimation.

The above fractals studied are all p.c.f. self-similar. Thus it is interesting to ask whether the conjecture holds for non p.c.f. self-similar fractals. In this paper, we will consider homogeneous hierarchical gaskets, which were introduced by Hambly [8, 9]. These gaskets are non p.c.f. Fortunately, they admit spectral decimation so that we can use similar method to prove that the conjecture holds on these gaskets.

Roughly speaking, the subdivision scheme for homogeneous hierarchical gaskets is a variant of the one for the usual Sierpinski gasket and constructed level by level. Each cell of level \( m \) is contained in a triangle and that triangle is split into triangles of sides \( 1/b_{m+1} \) times the side of the original triangle, where \( b_{m+1} \in \{2,3,\cdots\} \). If \( b_{m+1} = 2 \), we will have the cell of level \( m+1 \) as the same construction of \( SG₂ \), if \( b_{m+1} = 3 \), we will have the cell of level \( m+1 \) as the same construction as \( SG₃ \). The resulting gasket is denoted by \( HH(b) \) for \( b = (b_1,b_2,\cdots) \). In this paper, we will restrict that \( b_m \) equals 2 or 3 for each \( m \). See Fig. 1 for an example.

Notice that \( SG₂ \) and \( SG₃ \) are typical p.c.f. self-similar sets, while generally \( HH(b) \) is not a self-similar set. Meanwhile, the Dirichlet Laplacian and the Neumann Laplacian of these gaskets have already been discussed by Drenning and Strichartz [6]. Thus, it is natural to ask whether the hot spots conjecture holds on certain homogeneous hierarchical gaskets.

The rest of the paper is organized as follows. Basic concepts are recalled in Section 2.
Spectral decimation on $HH(b)$ are described in Section 3. In Section 4, we prove that the “hot spots” conjecture holds on $HH(b)$.

2 Preliminaries

In this section, we recall some basic notations in [6, 13, 21].

Let $q_i, i=1, 2, 3$, be non-collinear points in $\mathbb{R}^2$. Define functions $S_i, i=1, \ldots, 3$, on $\mathbb{R}^2$ as follows:

$$S_i(x) = \frac{x + q_i}{2}, \quad i = 1, 2, 3. \quad (2.1)$$

The Sierpinski gasket is the attractor of the iterated function system $\{S_i\}_{i=1}^3$.

Let $q_i, i=1, 2, 3$, be non-collinear points in $\mathbb{R}^2$. Define functions $F_i, i=1, \ldots, 6$, on $\mathbb{R}^2$ as follows:

$$\begin{align*}
F_1(x) &= \frac{x + 2q_i}{3}, \quad i = 1, 2, 3, \\
F_4(x) &= \frac{x + q_1 + q_2}{3}, \\
F_5(x) &= \frac{x + q_2 + q_3}{3}, \\
F_6(x) &= \frac{x + q_1 + q_3}{3}.
\end{align*} \quad (2.2)$$

The level-3 Sierpinski gasket is the attractor of the iterated function system $\{F_i\}_{i=1}^6$.

First we define a sequence of graphs $\Gamma_0, \Gamma_1, \ldots$ with vertices $V_0 \subseteq V_1 \subseteq \cdots$, and $V_* = \bigcup_{m=1}^\infty V_m$. The initial graph $\Gamma_0$ is just the complete graph on $V_0 = \{q_1, q_2, q_3\}$, the vertices of a triangle which is considered as the boundary of $HH(b)$. At stage $m$ of the construction of $HH(b)$, all the cells of level $m-1$ lie in triangles whose vertices make up $V_{m-1}$. If $b_m = 2$, then each cell of level $m-1$ splits into three cells of level $m$, adding three new vertices to $V_m$, connected exactly as in the $SG_2$ construction. If $b_m = 3$, then each cell splits into six cells of level $m$, adding seven vertices in $V_m$, connected exactly as in the $SG_3$ construction. See Fig. 2. For $x, y \in V_m$, we use $x \sim_m y$ to denote that $x$ and $y$ is connected in $\Gamma_m$.

Figure 2: Building block for $SG_2$ and $SG_3$. 
Definition 2.1. For any continuous function \( u \) on \( HH(b) \), we define the graph Laplacian \( \Delta_m \) for positive integers \( m \) by

\[
\Delta_m u(x) = \frac{4}{\deg x} \sum_{y \sim_m x} (u(y) - u(x)), \quad x \in V_m \setminus V_0,
\]

where \( \deg x \) is the cardinality of the set \( \{ y : y \sim_m x \} \). Let \( f \) be a continuous function on \( HH(b) \). We say that \( u \in \text{dom} \Delta \) with \( \Delta u = f \) if

\[
\frac{3}{2} 5^{m_2}(\frac{90}{7})^{m_3} \Delta_m u(x)
\]

converges uniformly to \( f \) on \( V_\ast \setminus V_0 \) as \( m \) goes to infinity, where \( m_2 = m \) and \( m_3 \) is the cardinality of the set \( \{ j \leq m : b_j = 2 \} \) and \( m_3 \) is the cardinality of the set \( \{ j \leq m : b_j = 3 \} \).

Definition 2.2. The normal derivative at \( p \in V_0 \) of a function \( u \) on \( HH(b) \) is defined to be

\[
\partial_n u(p) = \lim_{m \to \infty} \left( \frac{5}{3} \right)^{m_2} \left( \frac{15}{7} \right)^{m_3} \sum_{x \sim_m p} (u(p) - u(x))
\]

if the limit exists, where \( m_2 \) and \( m_3 \) are defined as in Definition 2.1.

Definition 2.3. A function \( u \in \text{dom} \Delta \) is called an eigenfunction of Neumann Laplacian with eigenvalue \( \lambda \) if

\[
-\Delta u = \lambda u \quad \text{on} \ V_s \setminus V_0, \quad \text{and} \quad \partial_n u = 0 \quad \text{on} \ V_0.
\]

For simplicity, we call \( \lambda \) an N-eigenvalue and \( u \) an N-eigenfunction if (2.4) holds.

### 3 Spectral decimation on \( HH(b) \)

The main tool to prove the hot spots conjecture on p.c.f. self-similar fractals is the spectral decimation, which was studied in [7, 16, 19, 20]. Drenning and Strichartz [6] pointed out that we can also use this method to analyze all Neumann (or Dirichlet) eigenvalues and eigenfunctions. Relative discussions on Laplacian and spectral decimation on \( SG_3 \) can also be found in [2] and [10].

Let \( m \) be a nonnegative integer and \( u_m \) a function on \( V_m \) and \( \lambda_m \) a real number. We call \( u_m \) a discrete N-eigenfunction and \( \lambda_m \) a discrete N-eigenvalue on \( V_m \) if

\[
\begin{align*}
-\Delta_m u_m(x) &= \lambda_m u_m(x), & x &\in V_m \setminus V_0, \\
(4 - \lambda_m) u_m(q_i) &= 2 \sum_{y \sim_m q_i} u_m(y), & i &\in \{1, 2, 3\}.
\end{align*}
\]

We denote by \( \Lambda_m \) the set of all discrete N-eigenvalue of \( \Delta_m \).
Define
\[ R_2(x) = x(5-x), \quad R_3(x) = \frac{3x(x-5)(x-4)(x-3)}{3x-14}. \]

For \( \lambda \not\in \{2,5\} \), we define
\[ \zeta(\lambda) = \frac{4-\lambda}{(2-\lambda)(5-\lambda)}, \quad \eta(\lambda) = \frac{2}{(2-\lambda)(5-\lambda)}. \quad (3.2) \]

For \( \lambda \not\in \{3,5,3-\sqrt{5},3+\sqrt{5}\} \), we define
\[
\begin{align*}
\alpha(\lambda) &= \frac{1}{3-\lambda} + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}, \\
\beta(\lambda) &= \frac{16-3\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}, \\
\gamma(\lambda) &= \frac{4}{3(4-6\lambda+\lambda^2)}, \\
\delta(\lambda) &= \frac{4}{3(4-6\lambda+\lambda^2)}. \quad (3.3a)
\end{align*}
\]

\[
\begin{align*}
\alpha(\lambda) &= \frac{1}{3-\lambda} + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}, \\
\beta(\lambda) &= \frac{16-3\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}, \\
\gamma(\lambda) &= \frac{4}{3(4-6\lambda+\lambda^2)}, \\
\delta(\lambda) &= \frac{4}{3(4-6\lambda+\lambda^2)}. \quad (3.3b)
\end{align*}
\]

**Theorem 3.1** (Spectral decimation theorem I, see [6, 20]). Let \( m > 0 \), we assume that \( \lambda_{m-1} = R_{b_m}(\lambda_m) \), and \( \lambda_m \not\in \{2,5,6\} \) if \( b_m = 2 \), and \( \lambda_m \not\in \{3,5,3-\sqrt{5},3+\sqrt{5},6\} \) if \( b_m = 3 \).

(i). If \( u \) is a discrete \( N \)-eigenfunction of \( \Delta_{m-1} \) with eigenvalue \( \lambda_{m-1} \), then there exists a unique extension \( \tilde{u} \) on \( V_m \) such that \( \tilde{u} \) is a discrete \( N \)-eigenfunction of \( \Delta_m \) with eigenvalue \( \lambda_m \). Furthermore, \( \tilde{u} \) take values on \( V_m \) in one \( V_{m-1} \) cell shown in Fig. 3 with
\[ x = \zeta(\lambda_m)(a+b) + \eta(\lambda_m)c \quad (3.4) \]
and similarly for the other vertices if \( b_m = 2 \), and
\[ w = \delta(\lambda_m)(a+b+c), \quad x = \alpha(\lambda_m)a + \beta(\lambda_m)b + \gamma(\lambda_m)c, \quad (3.5) \]
and similarly for the other vertices if \( b_m = 3 \).

(ii). Conversely, if \( u \) is a discrete \( N \)-eigenfunction of \( \Delta_m \) with eigenvalue \( \lambda_m \), then \( u|_{V_{m-1}} \) is a discrete \( N \)-eigenfunction of \( \Delta_{m-1} \) with eigenvalue \( \lambda_{m-1} \).

(iii). If \( \lambda_m \in \Lambda_m \), then the multiplicity of \( \lambda_m \) on \( \Delta_m \) equals that of \( \lambda_{m-1} \) on \( \Delta_{m-1} \).

**Theorem 3.2** (Spectral decimation theorem II, [6, 20]). (i). Let \( m_0 \geq 0 \), let \( u \) be a discrete \( N \)-eigenfunction of \( \Delta_{m_0} \) with eigenvalue \( \lambda_{m_0} \). Assume that \( \{\lambda_m\}_{m \geq m_0} \) is an infinite sequence related by \( \lambda_{m-1} = R_{b_m}(\lambda_m) \), with all but a finite number of \( \lambda_m = \min R_{b_m}(\lambda_{m-1}) \). If we define
\[ \lambda = \frac{3}{2} \lim_{m \to \infty} 5^{m_2} \left( \frac{90}{7} \right)^{m_3} \lambda_{m_0} \quad (3.6) \]
and extend \( u \) to \( V_\ast \) by successively using (3.4) and (3.5), then \( u \) is an \( N \)-eigenfunction of \( \Delta \) with eigenvalue \( \lambda \).

(ii). Every \( N \)-eigenvalue and its corresponding \( N \)-eigenfunctions of \( \Delta \) can be obtained by the process described in (i).

(iii). Let \( \{\lambda_m\}_{m \geq m_0} \) and \( \lambda \) be defined as in (i). Then the multiplicity of \( \lambda \) of \( \Delta \) equals that of \( \lambda_{m_0} \) of \( \Delta_{m_0} \).
Define
\[ \lambda_1 = 3 \quad \text{if} \quad b_1 = 2 \quad \text{and} \quad \lambda_1 = 3 - \sqrt{2} \quad \text{if} \quad b_1 = 3. \] (3.7)

For each \( m \geq 2 \), we inductively define
\[ \lambda_m = \min \{ R_{b_m}^{-1}(\lambda_{m-1}) \}. \] (3.8)

Using the similar method and results in [17, 18], it is easy to see that \( 0 < \lambda_m < 1 \) for all \( m \geq 2 \), and for all positive integer \( m \), we have
\[ \frac{1}{5} \lambda_{m-1} < \lambda_m < 5 \lambda_{m-1} \quad \text{if} \quad b_1 = 2 \quad \text{and} \quad \frac{7}{90} \lambda_{m-1} < \lambda_m < \frac{11}{72} \lambda_{m-1} \quad \text{if} \quad b_1 = 3. \] (3.9)

**Theorem 3.3.** Let \( \{ \lambda_m \}_{m \geq 1} \) be defined as in (3.7) and (3.8). Define \( \lambda = \frac{3}{2} \lim_{m \to \infty} 5^{m^2} (\frac{90}{72})^{m^3} \lambda_m \). Then \( \lambda \) is the second-smallest \( N \)-eigenvalue of \( \Delta \). Furthermore, the multiplicity of \( \lambda \) of \( \Delta \) equals 2.

**Proof.** It is clear that 0 is the smallest \( N \)-eigenvalues of \( \Delta \) with multiplicity 1. Thus, in order to prove the lemma, it suffices to prove that \( \lambda_m \) defined in the lemma is the smallest element in \( \Lambda_m \setminus \{ 0 \} \) for all \( m \geq 1 \). We will show this by induction.

In case that \( m = 1 \) and \( b_1 = 2 \), we can directly compute all \( N \)-eigenvalues of \( \Delta_1 \) from (3.1). We can obtain that \( \Lambda_1 = \{ 6, 3, 0, 6, 3, 6 \} \). Furthermore, the multiplicities for eigenvalues 0, 3, 6 are 1, 2, 3, respectively. Thus 3 is the second-smallest \( N \)-eigenvalues of \( \Delta_1 \), while the multiplicity of 3 is 2.

In case that \( m = 1 \) and \( b_1 = 3 \), we can directly compute all \( N \)-eigenvalues of \( \Delta_1 \) from (3.1). We can obtain that \( \Lambda_1 = \{ 0, 3 - \sqrt{2}, 4, 3 + \sqrt{2}, 6 \} \). Furthermore, the multiplicities for eigenvalues \( 0, 3 - \sqrt{2}, 4, 3 + \sqrt{2}, 6 \) are 1, 2, 1, 2, 4, respectively. Thus \( 3 - \sqrt{2} \) is the second-smallest \( N \)-eigenvalues of \( \Delta_1 \), while the multiplicity of \( 3 - \sqrt{2} \) is 2.

Assume that \( \lambda_k \) is the second-smallest \( N \)-eigenvalues of \( \Delta_k \) for some positive integer \( k \). Set \( m = k + 1 \). Let \( \tau \neq 0 \) is an \( N \)-eigenvalues of \( \Delta_{k+1} \). In case that \( \tau \in \{ 3, 5, 3 - \sqrt{5}, 3 + \sqrt{5}, 6 \} \), we have \( \lambda_{k+1} \leq \lambda_2 < 3 - \sqrt{5} \leq \tau \) from (3.9). In case that \( \tau \notin \{ 0, 3, 5, 3 - \sqrt{5}, 3 + \sqrt{5}, 6 \} \), from Spectral decimation theorem, there exists \( \tau' \in \Lambda_k \) and \( i \in \{ 2, 3 \} \) such that \( \tau = R_i(\tau') \). From the inductive assumption, we have \( \tau' \geq \lambda_k \). Since \( R_i \) (where \( i \in \{ 2, 3 \} \)) is strictly increasing in \((0, 1]\), we know that \( \tau \geq \lambda_{k+1} \). Thus \( \lambda_{k+1} \) is the second-smallest \( N \)-eigenvalue of \( \Delta_{k+1} \).
By induction, \( \lambda \) is the second-smallest \( N \)-eigenvalues of \( \Delta \). Since the multiplicity of \( \lambda_1 \) of \( \Delta_1 \) equals 2, we obtain from the Spectral decimation theorem that the multiplicity of \( \lambda \) of \( \Delta \) is also 2.

4 Proof of the main result

In this section, we always assume that \( \lambda \) and \( \{ \lambda_m \}_{m \geq 1} \) are defined as in (3.7), (3.8) and Theorem 3.3.

Let \( \text{EF}_2 \) be the set of all \( N \)-eigenfunctions on \( HH(b) \) corresponding to the eigenvalue \( \lambda \) of \( \Delta \). In case that \( b_1 = 2 \) (or \( b_1 = 3 \)), we define \( u_1 \) and \( u_2 \) to be functions in \( \text{EF}_2 \) such that \( u_1|_{V_1} \) and \( u_2|_{V_1} \) are functions as in Fig. 4 (or Fig. 5). It is easy to check that \( u_1|_{V_1} \) and \( u_2|_{V_1} \) are \( N \)-eigenfunctions corresponding to the eigenvalue \( \lambda_1 \) of \( \Delta_1 \). Thus \( u_1 \) and \( u_2 \) are well-defined by Spectral decimation theorem. Furthermore, it is easy to see that \( u_1 \) and \( u_2 \) are linearly independent, and so \( \{ u_1, u_2 \} \) is a base of \( \text{EF}_2 \). In the sequel of the paper, we will always use \( u_1 \) and \( u_2 \) to refer to these functions.

In the sequel, we define

\[
\zeta_m = \zeta(\lambda_m), \quad \eta_m = \eta(\lambda_m), \quad \alpha_m = \alpha(\lambda_m), \quad \beta_m = \beta(\lambda_m), \quad \gamma_m = \gamma(\lambda_m),
\]

for all \( m \geq 1 \), where functions \( \zeta, \eta, \alpha, \beta, \gamma \) are defined by (3.3a) and (3.3b).

Recall that \( 0 < \lambda_m < 1 \) for all \( m \geq 2 \). From above equalities, \( \zeta_m > \eta_m \) for all \( m \geq 2 \).

![Figure 4: The functions u1 and u2 on V1 in case that b1 = 2.](image)

**Lemma 4.1.** Define \( \{ z_m \}_{m \geq 1} \) inductively as follows: \( z_1 = \frac{1}{6} \) if \( b_1 = 2 \), \( z_1 = \frac{1 + \sqrt{7}}{6} \) if \( b_1 = 3 \), and for \( m \geq 1 \),

\[
z_{m+1} = \begin{cases} 
\frac{2}{3} \zeta_{m+1} + (\zeta_{m+1} + \eta_{m+1}) z_m, & \text{if } b_m = 2, \\
\frac{2}{3} \alpha_{m+1} + (\beta_{m+1} + \gamma_{m+1}) z_m, & \text{if } b_m = 3.
\end{cases}
\]

Then

\[
z_m = \frac{2}{3} \left( 1 - \frac{\lambda_m}{4} \right), \quad m \geq 1.
\]
In case that $b = m$, by inductive assumption, we have

$$z_{k+1} = \frac{2}{3} \zeta_{k+1} + \frac{2}{3} (\bar{\xi}_{k+1} + \eta_{k+1}) \left( 1 - \frac{\lambda_k}{4} \right) = \frac{2}{3} \left[ (2\bar{\xi}_{k+1} + \eta_{k+1}) - (\bar{\xi}_{k+1} + \eta_{k+1}) \frac{\lambda_k}{4} \right]
$$

$$= \frac{2}{3} \left[ \frac{10 - 2\lambda_{k+1}}{(2 - \lambda_{k+1})(5 - \lambda_{k+1})} - \frac{6 - \lambda_{k+1}}{(2 - \lambda_{k+1})(5 - \lambda_{k+1})} \cdot \frac{\lambda_{k+1}(5 - \lambda_{k+1})}{4} \right]
$$

$$= \frac{2}{3} \left( 1 - \frac{\lambda_{k+1}}{4} \right).$$

In case that $b_{k+1} = 3$, we have

$$z_{k+1} = \alpha_{k+1} + (\beta_{k+1} + \gamma_{k+1})z_k = \frac{2}{3} \left( \alpha_{k+1} + \beta_{k+1} + \gamma_{k+1} \right) - \frac{1}{4} \lambda_k (\beta_{k+1} + \gamma_{k+1})
$$

$$= \frac{2}{3} \left( \frac{4 - \lambda_{k+1}}{\lambda_{k+1}^2 - 6\lambda_{k+1} + 4} - \frac{\lambda_{k+1}(\lambda_{k+1} - 4)(\lambda_{k+1} + 6)}{4(\lambda_{k+1}^2 - 6\lambda_{k+1} + 4)} \right) = \frac{2}{3} \left( 1 - \frac{\lambda_{k+1}}{4} \right).$$

By induction, the lemma holds for $m \geq 1$. \qed

**Lemma 4.2.** Define $\{x_m\}_{m \geq 1}$ and $\{y_m\}_{m \geq 1}$ inductively as follows: $x_1 = \frac{1}{6}$ and $y_1 = -\frac{1}{3}$ if $b_1 = 2$, $x_1 = \frac{\sqrt{5} - 2}{6}$ and $y_1 = \frac{1 - 2\sqrt{5}}{6}$ if $b_1 = 3$, and for all $m \geq 1$,

$$x_{m+1} = \begin{cases} 
-\frac{1}{3} \zeta_{m+1} + \bar{\xi}_{m+1} + \eta_{m+1} y_m, & \text{if } b_m = 2, \\
-\frac{1}{3} \xi_{m+1} \beta_{m+1} + \eta_{m+1} y_m, & \text{if } b_m = 3,
\end{cases} \tag{4.4a}$$

$$y_{m+1} = \begin{cases} 
-\frac{1}{3} \zeta_{m+1} + \bar{\xi}_{m+1} y_m + \eta_{m+1} x_m, & \text{if } b_m = 2, \\
\frac{1}{3} \xi_{m+1} \beta_{m+1} y_m + \eta_{m+1} x_m, & \text{if } b_m = 3.
\end{cases} \tag{4.4b}$$
Let \(\{z_m\}_{m \geq 1}\) be the sequence defined as in Lemma 4.1. Then for all \(m \geq 1\), we have

\[
2x_m + z_m = -(2y_m + z_m) \quad \text{and} \quad 0 < 2x_m + z_m \leq \frac{\lambda_m}{6}.
\]  

(4.5)

Proof. We will prove the lemma by induction. In case that \(b_1 = 2\), we have \(2x_1 + z_1 = -(2y_1 + z_1) = \frac{3}{5} = \lambda_1\). In case that \(b_1 = 3\), we have \(2x_1 + z_1 = -(2y_1 + z_1) = \frac{3 - 3\sqrt{2}}{5} < \frac{3}{5} = \lambda_1\).

Thus the lemma holds for \(m = 1\).

Assume that the lemma holds for \(m \leq k\), where \(k\) is a positive integer. Let \(m = k + 1\). In case that \(b_{k+1} = 2\), from (4.2), (4.4a), (4.4b) and using inductive assumption, we have

\[
\begin{align*}
2x_{k+1} + z_{k+1} &= \zeta_{k+1}(2x_k + z_k) + \eta_{k+1}(2y_k + z_k) = (\zeta_{k+1} - \eta_{k+1})(2x_k + z_k), \\
2y_{k+1} + z_{k+1} &= \zeta_{k+1}(2y_k + z_k) + \eta_{k+1}(2x_k + z_k) = (\eta_{k+1} - \zeta_{k+1})(2y_k + z_k).
\end{align*}
\]

Thus \(2x_{k+1} + z_{k+1} = -(2y_{k+1} + z_{k+1})\). From (4.1) and using inductive assumption, we have

\[
0 < 2x_{k+1} + z_{k+1} = \frac{2 - \lambda_{k+1}}{(2 - \lambda_{k+1})(5 - \lambda_{k+1})}(2x_k + z_k) \leq \frac{1}{5 - \lambda_{k+1}} \cdot \frac{\lambda_k}{6} = \frac{\lambda_{k+1}}{6}.
\]

In case that \(b_{k+1} = 3\), we have

\[
\begin{align*}
2x_{k+1} + z_{k+1} &= \beta_{k+1}(2x_k + z_k) + \gamma_{k+1}(2y_k + z_k) = (\beta_{k+1} - \gamma_{k+1})(2x_k + z_k), \\
2y_{k+1} + z_{k+1} &= \beta_{k+1}(2y_k + z_k) + \gamma_{k+1}(2x_k + z_k) = -(\beta_{k+1} - \gamma_{k+1})(2y_k + z_k).
\end{align*}
\]

Thus \(2x_{k+1} + z_{k+1} = -(2y_{k+1} + z_{k+1})\). From (4.1) and using inductive assumption, we have

\[
0 < 2x_{k+1} + z_{k+1} = \frac{1}{(\lambda_{k+1} - 3)(\lambda_{k+1} - 5)}(2x_k + z_k) \leq \frac{\lambda_k}{6(\lambda_{k+1} - 3)(\lambda_{k+1} - 5)} \cdot \frac{\lambda_{k+1}}{6} = \frac{1}{6} < \frac{\lambda_{k+1}}{6}.
\]

By induction, the lemma holds for \(m \geq 1\). 

Lemma 4.3. Let \(\{x_m, y_m, z_m\}_{m \geq 1}\) be defined as in Lemmas 4.1, 4.2, and \(\{S_i\}_{i=1}^3\), \(\{F_i\}_{i=1}^6\) be defined as (2.1), (2.2). Define \(\{G^m_i\}_{i=1}^3 = \{S_i\}_{i=1}^3\) if \(b_m = 2\), \(\{G^m_i\}_{i=1}^6 = \{F_i\}_{i=1}^6\) if \(b_m = 3\), Then for all \(m \geq 1\),

\[
\begin{align*}
&u_1(G^m_1(q_2)) = u_1(G^m_3(q_3)) = z_m, \\
&u_1(G^m_2(q_1)) = x_m, \quad u_1(G^m_3(q_3)) = y_m, \\
&u_1(q_2) = \frac{-1}{3} < y_m < x_m < u_1(q_1) = \frac{2}{3}.
\end{align*}
\]  

(4.6a) (4.6b) (4.6c)

Proof. We will prove the lemma by induction. From Figs. 4 and 5, we know that the lemma holds for \(m = 1\).
Assume that the lemma holds for \( m \leq k \), where \( k \) is a positive integer. Let \( m = k + 1 \) and \( b_{k+1} = 2 \). From Spectral decimation theorem and inductive assumption, we have
\[
u_1(s^{k+1}_1(q_3)) = \tilde{\xi}_{k+1} u_1(s^1_1(q_1)) + \xi_{k+1} u_1(s^1_1(q_2)) + \eta_{k+1} u_1(s^1_1(q_3)) = \frac{2}{3} \xi_{k+1} + (\tilde{\xi}_{k+1} + \eta_{k+1}) z_k = z_{k+1}.
\]

In the case that \( b_{k+1} = 3 \), we have
\[
u_1(s^{k+1}_1(q_2)) = \alpha_{k+1} u_1(s^1_1(q_1)) + \beta_{k+1} u_1(s^1_1(q_2)) + \gamma_{k+1} u_1(s^1_1(q_3)) = \frac{2}{3} \alpha_{k+1} + (\beta_{k+1} + \gamma_{k+1}) z_k = z_{k+1}.
\]

Similarly, we can prove that other equalities in (4.6a) and (4.6b) also hold for \( m = k + 1 \).

By definition of \( x_1 \) and \( y_1 \), we know that (4.6c) holds for \( m = 1 \). Thus it suffices to show that (4.6c) holds for \( m \geq 2 \).

Let \( \{z_m\}_{m \geq 1} \) be defined as in Lemma 4.1. From Lemma 4.2, we have \(|2x_m + z_m| \leq \frac{\lambda_m}{6}\).

Substituting \( z_m \) by \( \frac{2}{3} (1 - \frac{2}{3} \lambda_m) \) and noticing that \( 0 < \lambda_m < 1 \) for \( m \geq 2 \), we obtain that
\[-\frac{1}{3} \leq x_m \leq -\frac{1}{3} + \frac{\lambda_m}{6} < 0.
\]
Similarly, we have \(-\frac{1}{3} \leq y_m < 0\). From (4.5), we have \(2x_m + z_m > 0 > 2y_m + z_m\) so that \( x_m > y_m \). Hence (4.6c) holds for \( m \geq 2 \).

The following lemma plays an essential role in our proof.

**Lemma 4.4.** Let \( \{x_m, y_m, z_m\}_{m \geq 1} \) be defined as in Lemmas 4.1 and 4.2. Let \( p_1, p_2, p_3 \) be three distinct vertices of one cell of \( V_m \) where \( m \geq 1 \). Assume that \( u_1(p_1) \leq u_1(p_2) \leq u_1(p_3) \). Then
\[
\nu_1(p_1) \leq u_1(p_2) \leq z_m, \quad u_1(p_3) \leq \frac{2}{3} \quad \text{and} \quad u_1(p_1) \geq -\frac{1}{3}, \quad u_1(p_2) \geq y_m, \quad u_1(p_3) \geq x_m.
\]

**Proof.** We will prove the lemma by induction. From Figs. 4 and 5, we know that the lemma holds for \( m = 1 \).

Assume that the lemma holds for \( m \leq k \), where \( k \) is a positive integer. Let \( m = k + 1 \) and \( p_1, p_2, p_3 \) be three distinct vertices of one cell \( C \) of \( V_{k+1} \). Then there exists a unique cell \( C' \) of \( V_k \) which contains \( C \). Let \( p'_1, p'_2, p'_3 \) be three distinct vertices of \( C' \).

In case that \( b_{k+1} = 2 \), from Fig. 3, we know that there exists a permutation \((i_1, i_2, i_3)\) of \((1, 2, 3)\) such that \( p_{i_1} \in \{p'_1, p'_2, p'_3\} \), while \( p_{i_2}, p_{i_3} \in V_{k+1} \setminus V_k \).

By inductive assumption, we have
\[
u_1(p_{i_1}) \leq \max\{u_1(p'_1), u_1(p'_2), u_1(p'_3)\} \leq \frac{2}{3}, \quad \text{(4.8a)}
\]
and
\[
u_1(p_{i_1}) \geq \min\{u_1(p'_1), u_1(p'_2), u_1(p'_3)\} \geq -\frac{1}{3}. \quad \text{(4.8b)}
\]
Without loss of generality, we assume that \( u_1(p_{i_2}) \leq u_1(p_{i_3}) \). By Spectral decimation theorem,
\[
\begin{align*}
u_1(p_{i_2}) &= \zeta_{k+1}(u_1(p_{i_2}^b) + u_1(p_{i_2}^g)) + \eta_{k+1}u_1(p_{i_2}^h), \\
u_1(p_{i_3}) &= \zeta_{k+1}(u_1(p_{i_3}^b) + u_1(p_{i_3}^g)) + \eta_{k+1}u_1(p_{i_3}^h),
\end{align*}
\]
where \((j_1,j_2,j_3)\) and \((\ell_1,\ell_2,\ell_3)\) are two distinct permutations of \((1,2,3)\). Notice that \( \zeta_m > \eta_m \) for all \( m \geq 2 \) and \( z_m \leq \frac{2}{3} \) for all \( m \geq 1 \). By inductive assumption,
\[
u_1(p_{i_3}) \leq \frac{2}{3}\zeta_{k+1} + \zeta_{k+1} + \eta_{k+1} = z_{k+1}.
\]
Combining this with (4.8a), we know that (4.7a) holds for \( m = k+1 \). \(\square\)

By Lemma 4.3, we know that \(-\frac{1}{3} \leq y_m < x_m\) for all \( m \geq 1 \). By inductive assumption,
\[
u_1(p_{i_2}) \geq -\frac{1}{3}\zeta_{k+1} + x_k\zeta_{k+1} + x_k\eta_{k+1} = y_{k+1}. \tag{4.9}
\]
Since \((\ell_1,\ell_2,\ell_3) \neq (j_1,j_2,j_3)\) is another permutation of \((1,2,3)\) and \( u_1(p_{i_2}) \leq u_2(p_{i_3}) \),
\[
u_1(p_{i_3}) \geq -\frac{1}{3}\zeta_{k+1} + x_k\zeta_{k+1} + y_k\eta_{k+1} = x_{k+1}.
\]
Combining this with (4.8b) and (4.9), we know that (4.7b) holds for \( m = k+1 \).

In case that \( b_{k+1} = 3 \), we know from the proof of Lemma 4.4 in [18] that (4.7a) and (4.7b) also holds for \( m = k+1 \).

It directly follows from the above lemma that we have:

**Theorem 4.1.** \( u_1 \) attains its maximum and minimum on \( V_0 \).

Define
\[
f = u_1 + \frac{1}{3}, \quad g = u_2 + \frac{1}{3}, \quad h = -u_1 - u_2 + \frac{1}{3}. \tag{4.10}
\]
Noticing that \( u_2 \) is a rotation of \( u_1 \), and \(-(u_1 + u_2)\) is a symmetry of \( u_1 \), we know that \( u_2 \) and \(-(u_1 + u_2)\) also attains its maximum and minimum on \( V_0 \). Clearly, \( 0 \leq f, g, h \leq 1 \).

Now we can show that the “hot spots” conjecture holds on \( HH(b) \).

**Theorem 4.2.** Every eigenfunction of the second-smallest eigenvalue of Neumann Laplacian on \( HH(b) \) attains its maximum and minimum on the boundary \( V_0 \).

**Proof.** Let \( u \) be an \( N \)-eigenfunctions with respect to the second-smallest \( N \)-eigenvalue of \( \Delta \). Since \( \{u_1,u_2\} \) is a base of \( EF_2 \), there exist constants \( c_1, c_2 \) such that \( u = c_1u_1 + c_2u_2 \). By (4.10), we have \( f + g + h = 1 \) so that \( u_1 = f - \frac{1}{3}(f + g + h) \) and \( u_2 = g - \frac{1}{3}(f + g + h) \). It follows that
\[
u = c_1u_1 + c_2u_2 = \left( \frac{2}{3}c_1 - \frac{1}{3}c_2 \right)f + \left( -\frac{1}{3}c_1 + \frac{2}{3}c_2 \right)g + \left( -\frac{1}{3}c_1 - \frac{1}{3}c_2 \right)h.
\]
Notice that \(0 \leq f, g, h \leq 1\), \(f + g + h = 1\) and \(\{ u(q_i) | i = 1, 2, 3 \} = \{ \frac{2}{3} c_1 - \frac{1}{3} c_2, -\frac{1}{3} c_1 + \frac{2}{3} c_2, -\frac{1}{3} c_1 - \frac{1}{3} c_2 \} \). Hence,

\[
\min \{ u(q_1), u(q_2), u(q_3) \} \leq u(x) \leq \max \{ u(q_1), u(q_2), u(q_3) \}
\]

for all \(x \in HH(b)\).

\[ \square \]

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