Proof diagrams for multiplicative linear logic

Matteo Acclavio

I2M
Marseille, France
Aix-Marseille Université
matteo.acclavio@univ-amu.fr

The original idea of proof nets can be formulated by means of interaction nets syntax. Additional machinery as switching, jumps and graph connectivity is needed in order to ensure correspondence between a proof structure and a correct proof in sequent calculus.

In this paper we give two models of proof structures in the syntax of string diagrams. Even though we lose standard proof equivalence, our construction allows to prove a uniform correction criterion which is internal to the syntax and of constant time complexity.

Introduction

Proof nets are a geometrical representation of linear logic proofs introduced by J-Y. Girard [5]. The building blocks of proof nets are called proof structures that have been generalized by Y. Lafont [11] in the so-called interaction nets. To recognize if a proof structure is a proof net one needs to verify its sequentializability property, that is, whether it corresponds to a linear logic proof derivation. Following Girard’s original correction criterion, others methods have been introduced, notably by Danos-Regnier [4], that ensures graph acyclicity by a notion of switchings on ⊗ cells, and by Guerrini [7], that reformulates correction by means of graph contractability.

Proof structures allow to recover the semantic equivalence of derivation under commutation and permutation of some inference rules. Unfortunately this property makes ineffective the aforementioned criteria in presence of the multiplicative constant ⊥. In order to recover a sequentialization condition for the multiplicative fragment with constants, Girard has introduced the notion of jumps [6]. These are untyped edges between two cells which express a dependency relation of the respective rules in sequentialization.

In this work we reformulate proof nets by replacing the underlying interaction nets syntax with that of string diagrams in order to achieve a new sequentialization criterion. String diagrams [2] are a syntax for 2-arrows (or 2-cells) of a 2-category with a rigid structure. Although the two syntaxes may graphically look similar, string diagrams’ strings do not just denote connections between cells but they represent morphisms. Since crossing strings is not allowed without the introduction of twisting operators, we introduce the notion of twisting relations in order to equate diagrams by permitting cells to cross certain strings.

We study several diagram rewriting systems given by twisting polygraphs, a particular class of polygraph [3] where string crossings are restrained by the introduction of some non-crossing control strings in the syntax.

As soon as one considers proof derivations as sequences of n-ary operators applications over lists of formulas, then control strings intuitively represent their correct parenthesization. In particular these strings disallow non-sequentializable diagram compositions, lastly resulting, thanks to negative constants’ fixed position, into a correction criterion based on diagram inputs and outputs pattern only. More-
over, the underlying categorical semantic drawing out from this model (a monoidal category) result different from the standard one [15].

1 String diagrams

1.1 Monochrome String Diagrams

We now recall some basic notions in string diagram rewriting by considering the monochrome string diagrams settings, where there are no labels on backgrounds or strings. For an introduction to string diagrams, see J. Baez’s notes [2].

Given \( p \) and \( q \) natural numbers, a diagram \( \phi : p \Rightarrow q \) with \( p \) inputs and \( q \) outputs is pictured as follows:

\[
\begin{array}{c}
p \\
\phi \\
q \\
\end{array}
\]

Diagrams may be composed in two different ways. If \( \phi : p \Rightarrow q \) and \( \phi' : p' \Rightarrow q' \) are diagrams, we define:

- **sequential composition:** if \( q = p' \), the diagram \( \phi' \circ \phi : p \Rightarrow q' \) corresponds to usual composition of maps:

\[
\begin{array}{c}
p \\
\phi \\
\phi' \\
q' \\
\end{array}
\]

This composition is associative with unit \( \text{id}_p : p \Rightarrow p \) for each \( p \in \mathbb{N} \). In other words, we have \( \phi \circ \text{id}_p = \phi = \text{id}_q \circ \phi \). The identity diagram \( \text{id}_p \) is pictured as follows:

\[
\begin{array}{c}
p \\
\end{array}
\]

- **parallel composition:** the diagram \( \phi \ast \phi' : p + p' \Rightarrow q + q' \) is pictured as follows:

\[
\begin{array}{c}
p + p' \\
\phi \ast \phi' \\
q + q' \\
\end{array}
\]

This composition is associative with unit \( \text{id}_0 : 0 \Rightarrow 0 \). In other words, we have \( \text{id}_0 \ast \phi = \phi = \phi \ast \text{id}_0 \). This \( \text{id}_0 \) is called the empty diagram.

Our two compositions satisfy the interchange rule: if \( \phi : p \Rightarrow q \) and \( \phi' : p' \Rightarrow q' \), so \((\text{id}_q \ast \phi') \circ (\phi \ast \text{id}_{p'}) = \phi \ast \phi' = (\phi \ast \text{id}_{q'}) \circ (\text{id}_p \ast \phi')\) that corresponds to the following picture:

\[
\begin{array}{c}
\ldots \\
\phi \\
\phi' \\
\ldots \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \\
\phi \\
\phi' \\
\ldots \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \\
\phi \\
\phi' \\
\ldots \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \\
\phi \\
\phi' \\
\ldots \\
\end{array}
\]
Monochrome string diagrams can be interpreted as morphisms in a PRO, that is a strict monoidal category whose objects are natural numbers and whose product on objects is addition. To be coherent with the cellular notation we use in next sections, diagrams represent 2-arrows in the $2-\text{PRO}$ obtained by suspension of a regular PRO (see [?]).

Definition 1 (Signature). A signature $\mathcal{S}$ is a finite set of atomic diagrams (or gates type). Given a signature, a diagram $\phi : p \Rightarrow q$ is a morphism in the PRO $\mathcal{S}^*$ freely generated by $\mathcal{S}$, i.e. by the two compositions and identities. A gate is an occurrence of an atomic diagram, we note $g : \alpha$ if $g$ is an occurrence of $\alpha \in \mathcal{S}$.

Definition 2. We say that $\phi$ is a subdiagram of $\phi'$ whenever there exist $\psi_u, \psi_d \in \mathcal{S}^*$ and $k,k' \in \mathbb{N}$ such that $\phi' = \psi_d \circ (\text{id}_{\Gamma} \ast \phi \ast \text{id}_{\Delta}) \circ \psi_u$.

Notation. Given $\phi \in \mathcal{S}^*$ and $\mathcal{S}' \subseteq \mathcal{S}$, we write $|\phi|_{\mathcal{S}'}$ the number of gates in $\phi$ with gate type $\alpha \in \mathcal{S}'$.

Definition 3. We call horizontal a diagram $\phi$ generated by parallel composition (and identities) only in $\mathcal{S}^*$. It is elementary if $|\phi|_{\mathcal{S}'} = 1$.

1.2 Diagram rewriting

Definition 4 (Diagram Rewriting System). A diagram rewriting system is a couple $(\mathcal{S}, \mathcal{R})$ given by a signature $\mathcal{S}$ and a set $\mathcal{R}$ of rewriting rules of the form

\[
\frac{p}{\phi}
\]

where $\phi, \phi' : p \Rightarrow q$ are diagrams in $\mathcal{S}^*$.

Definition 5. We allow each rewriting rules under any context, that is, if $\phi \equiv \phi'$ in $\mathcal{R}$ then, for every $\chi_u, \chi_d \in \mathcal{S}^*$,

\[
\frac{\chi_u \phi \chi_d}{\chi_u \phi' \chi_d}
\]

We say that $\psi$ reduces, or rewrites, to $\psi'$ (denoted $\psi \equiv \psi'$) if there is a rewriting sequence $P : \psi = \psi_0 \equiv \psi_1 \equiv \ldots \equiv \psi_n = \psi'$.

We here recall some classical notions in rewriting:

- A diagram $\phi$ is irreducible if there is no $\phi'$ such that $\phi \equiv \phi'$;
- A rewriting system terminates if there is no infinite rewriting sequence;
- A rewriting system is confluent if for all $\phi_1, \phi_2$ and $\phi$ such that $\phi \equiv \phi_1$ and $\phi \equiv \phi_2$, there exists $\phi'$ such that $\phi_1 \equiv \phi'$ and $\phi_2 \equiv \phi'$;
- A rewriting system is convergent if both properties hold.
2 Polygraphs

In this section we formulate some basic notion by using the language of polygraphs. Introduced by Street [16] as computads, later reformulated and extended by Burroni [3], polygraphs can be considered as the generalization, for higher dimensional categories, of the notion of monoid presentation.

Here we study some diagram rewriting systems with labels on strings in terms of 3-polygraphs, which are denoted $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$. In particular, we consider polygraphs with just one 0-cell in $\Sigma_0$ in order to avoid background labeling. The set of 1-cells $\Sigma_1$ represents string labels, the 2-cells in $\Sigma_2$ are the signature $S_{\Sigma}$ of our rewriting system with rules $R_{\Sigma} = \Sigma_3$, the set of 3-cells. We say that a polygraph $\Sigma$ exhibits some computational properties when the relative diagram rewriting system does.

**Notation.** We denote $\phi \in \Sigma$ whenever $\phi$ is a diagram generated by the associated signature $S_{\Sigma}$.

2.1 Twisting Polygraph

In this section we introduce a notion of polygraph which generalizes polygraphic presentations of symmetric monoidal categories.

**Definition 6 (Symmetric polygraph).** We call the polygraph of permutation the following monochrome 3-polygraph:

$$\mathcal{S} = \left( \Sigma_0 = \{\square\}, \Sigma_1 = \{\}\}, \Sigma_2 = \{\cdot\cdot\cdot\}, \Sigma_3 = \left\{ \begin{array}{l} \cdot\cdot\cdot \Rightarrow | \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \end{array} \right\} \right).$$

We call symmetric a 3-polygraph $\Sigma$ with one 0-cell, one 1-cell (i.e. $\Sigma_1 = \{\}$), containing one 2-cell $\cdot\cdot\cdot \in \Sigma_2$ and such that the following holds

$$\cdot\cdot\cdot = | \cdot\cdot\cdot \Rightarrow | \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot$$

for all $\alpha \in \Sigma_2$ in the 2-category $\Sigma^\ast$.

**Theorem 1 (Convergence of $\mathcal{S}$).** The polygraph $\mathcal{S}$ is convergent.

**Proof.** See [12] for a proof.

Each diagram in $\mathcal{S}$ can be interpreted as a permutation in the group of permutations over $n$ elements $S_n$ with product $\circ$ defined as their function composition. On the other hand, each $\sigma \in S_n$ corresponds to some diagrams in $\mathcal{S}$. In particular, we interpret the diagram $\text{id}_{k-1} \ast \cdot\cdot\cdot \ast \text{id}_{n-(k+1)} : n \to n$ as the transposition $(k,k+1) \in S_n$.

**Notation.** We call left ladder over $n$ elements a diagram of the form

$$\text{Lad}^l_n = | : 1 \Rightarrow 1 \text{ and } \text{Lad}^l_n = \begin{array}{c} \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \end{array} : n \Rightarrow n,$$

corresponding to the permutation $\text{Lad}^l_n = (1,n,n-1,\ldots,2) \in S_n$. In a similar way, a right ladder over $n$ elements

$$\text{Lad}^r_n = | : 1 \Rightarrow 1 \text{ and } \text{Lad}^r_n = \begin{array}{c} \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \cdot\cdot\cdot \Rightarrow \end{array} : n \Rightarrow n$$

corresponds to the permutation $\text{Lad}^r_n = (n,1,2,\ldots,n-1) \in S_n$.

**Proposition 1.** For any permutation $\sigma \in S_n$ there is a unique diagram in normal form $\hat{\phi}_\sigma : n \Rightarrow n \in \mathcal{S}$ corresponding to $\sigma$. We call it the canonical diagram of $\sigma$. 


Proof. We define $S_1 = \{\}$ and $S_{n+1}$ the set of diagrams in $S$ of the form:

$$
\begin{array}{c}
\begin{array}{c}
\vdots \\
\sigma \\
\vdots \\
\end{array}
\end{array}
$$

with $\sigma \in S_n$ and $\begin{array}{c}
\vdots \\
\sigma \\
\vdots \\
\end{array} = \begin{array}{c}
\vdots \\
\cdots \\
\sigma \\
\vdots \\
\end{array}$, $|S_n| = n!$ since $|S_1| = 1$ and $|S_{n+1}| = (n+1)|S_n|$ on account of $n + 1 = |\{\text{Lad}_k\}_{1 \leq k \leq n+1}| = |\{\text{Lad}_k \ast \text{id}_{(n+1-k)}\}_{1 \leq k \leq n+1}|$.

To exhibit a one-to-one correspondence between $S_{n+1}$ and $S_{n+1}$, for any $\sigma \in S_{n+1}$ we define $Er(\sigma) \in S_n$ the permutation

$$
Er(\sigma) = \begin{cases} 
\sigma(i+1) & \text{if } \sigma(i+1) < \sigma(1) \\
\sigma(i) & \text{if } \sigma(1) < \sigma(i+1) 
\end{cases}
$$

and $\hat{\phi}_\sigma = (\text{Lad}_k \ast \text{id}_{(n+1-i\sigma(1))}) \circ (\text{id}_1 \ast \hat{\phi}_{Er(\sigma)})$.

Any element in $S_n$ contains no subdiagram of the form $\begin{array}{c}
\vdots \\
\sigma \\
\vdots \\
\end{array}$ nor $\begin{array}{c}
\vdots \\
\cdots \\
\sigma \\
\vdots \\
\end{array}$ meaning that it is irreducible and so, by the confluence of $S$, in normal form. □

Definition 7 (Twisting polygraph). A twisting polygraph is a 3-polygraph $\Sigma$ with one 0-cell equipped with a set $T_\Sigma \subseteq \Sigma$ called twisting family such that for each $A, B \in T_\Sigma$ there is a twisting operator $T_{A,B} : A \ast B \Rightarrow B \ast A \in \Sigma_2$ and $\Sigma_3$ includes the following families $T_{\Sigma}^R$ of twisting relations:

- For all $A, B, C \in T_\Sigma$

$$
\begin{array}{c}
\begin{array}{c}
A \\
\vdots \\
B \\
\vdots \\
C \\
\vdots \\
\end{array}
\end{array} \Rightarrow
\begin{array}{c}
\begin{array}{c}
A \\
\vdots \\
B \\
\vdots \\
C \\
\vdots \\
\end{array}
\end{array}
$$

and

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A \\
\vdots \\
B \\
\vdots \\
C \\
\vdots \\
\end{array}
\end{array}
\end{array} \Rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A \\
\vdots \\
B \\
\vdots \\
C \\
\vdots \\
\end{array}
\end{array}
\end{array}
$$

\hspace{5cm} ;

- For all $\alpha : \Gamma \rightarrow \Gamma' \in \Sigma_2$ with $\Gamma, \Gamma' \in T_\Sigma^R$, $A \in T_\Sigma$, at least one of the two possible orientation of the following rewriting rules is in $\Sigma_3$.

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\alpha \\
\vdots \\
\end{array}
\end{array}
\end{array} \Rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\alpha \\
\vdots \\
\end{array}
\end{array}
\end{array}
$$

and

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\alpha \\
\vdots \\
\end{array}
\end{array}
\end{array} \Rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\alpha \\
\vdots \\
\end{array}
\end{array}
\end{array}
$$

\hspace{5cm} .

Moreover, if $\phi, \psi$ are twisting diagrams (i.e. diagrams made only of twisting operators) $\phi \xRightarrow{\Sigma_2} \psi$ iff $\phi \xRightarrow{\Sigma_2} \psi$ where $\mathcal{R}_R$ is the set given by rewriting rules of (1). A total-twisting polygraph is a twisting polygraph with $T_\Sigma = \Sigma_1$.

The idea behind twisting polygraphs is to present diagram rewriting systems where, in equivalence classes modulo rewriting, the crossings of strings labeled by the twisting family are not taken into account. In fact, the family of relations (1) says that these crossings are involutive and satisfy Yang-Baxter equation (10) for braidings, while relations in (2) allow gates to “cross” a string in case of fitting labels.

We interpret a twisting diagram $\hat{\phi}_\sigma : \Gamma \Rightarrow \sigma(\Gamma)$ as the permutations in $S_{|\Gamma|}$ acting over the order of occurrence of 1-cells in the word $\Gamma \in T_\Sigma^R$. For this reason, as in $S$, we define left ladders, right ladders and the standard diagrams $\hat{\phi}_\sigma : \Gamma \Rightarrow \sigma(\Gamma)$ (or simply $\hat{\phi}_\sigma$) with source and target in $T_\Sigma^R$. In conformity with the twisting polygraph restrictions over $\Sigma_3$, we can prove the uniqueness of $\hat{\phi}_\sigma$ as in Proposition 1.
3 Multiplicative Linear Logic sequent calculus

In this paper we focus on the multiplicative fragment of linear logic sequent calculus with or without constants. We here we recall the usual inference rules:

| Structural  | Identity or Axiom | Cut |
|-------------|-------------------|-----|
|             | \(\vdash A, A \parallel\) | \(\vdash \Sigma, A \vdash A \parallel\) |

| Multiplicative | Tensor | Par |
|---------------|--------|-----|
|               | \(\vdash \Sigma, A \vdash B, \Gamma \parallel\) | \(\vdash \Sigma, A, B \vdash \Gamma \parallel\) |

| Constants | Bottom | 1 |
|-----------|--------|---|
|           | \(\vdash \Sigma, \bot\) | \(\vdash 1\) |

We also consider the usually omitted exchange rule:

\(\vdash A_1, \ldots, A_k \parallel\) \(\vdash A_{\sigma(1)}, \ldots, A_{\sigma(k)}\) \(\sigma \in S_k\)

We finally recall that the *multiplicative linear logic fragment with constants* (MLLc) is given by the aforementioned inference rules while the *multiplicative fragment* (MLL) is the one given by the inference rules \(\text{Ax}, \text{Cut}, \parallel, \otimes\) (and exchange) only.

**Remark 1 (On Negation).** We assume negation is involutive, i.e. \(A \parallel \parallel = A\) and the De-Morgan laws apply with respect to \(\otimes\) and \(\parallel\), i.e. \((A \otimes B) \parallel = B \parallel \otimes A\parallel\) for any formulas \(A, B\) where \(\otimes = \otimes\) and \(\parallel = \parallel\) or vice versa \(\parallel = \otimes\) and \(\parallel = \otimes\). Moreover \(1 \parallel = \bot\).

**Remark 2 (On Rules).** In this work we interpret inference rules as operations with specific arities over the set of sequents: \(\text{Ax}\) and \(1\) are 0-ary, \(\otimes\) and \(\parallel\) are unary and \(\otimes\) and \(\text{Cut}\) are binary.

**Notation.** We indicate with \(\mathcal{F}_{\text{MLL}}\) and \(\mathcal{F}_{\text{MLLc}}\) the set of formulas respectively in \(\text{MLL}\) and \(\text{MLLc}\).

4 String diagram syntax for proof net

In this section we give two particular 3-polygraphs for \(\text{MLL}\) and \(\text{MLLc}\) respectively, i.e. string diagrams representing linear logic derivations that we call *proof diagrams*. To these latter, we then add two non-twisting colors and we replace certain 2-cells in order to define what we call *control polygraphs*. In these polygraphs we are able to characterize diagrams corresponding to correct proof structures by just checking their inputs and outputs patterns.

**Notation.** In order to unify sequent and 1-cell composition notations, we replace the * symbol of parallel composition with a comma.

4.1 Proof diagrams for \(\text{MLL}\)

**Definition 8.** The 3-polygraph \(\Sigma_{\text{MLL}}\) is the polygraph of multiplicative linear logic with cut-elimination. It is given by the following sets of cells:
\[ \Sigma_0^M = \{ \Box \}; \]

\[ \Sigma_1^M = \tilde{\Sigma}_{\text{M}1}; \]

\[ \Sigma_2^M = \begin{cases} \otimes_{A,B}: A, B \Rightarrow A \otimes B = \begin{array}{c} A \\
B \end{array} & \text{if } A \not\perp B \\
A \otimes B & \text{if } A \perp B \end{cases}, \]

\[ \varnothing_{A,B}: A, B \Rightarrow A \varnothing B = \begin{array}{c} A \\
\varnothing \end{array} & \text{if } A \not\perp B \\
A \varnothing B & \text{if } A \perp B \end{cases}, \]

\[ A_{x_A}: \Box \Rightarrow A, A_\perp = \begin{array}{c} A \\
A_\perp \end{array}, \]

\[ \text{Cut}_{A}: A, A_\perp \Rightarrow \Box = \begin{array}{c} A \\
A_\perp \end{array}, \]

\[ T_{A,B}: A, B \Rightarrow B, A = \begin{array}{c} B \\
A \end{array}. \]

If there is no ambiguity we note \( \equiv \) and \( \equiv \equiv \) instead of \( \Box \) and \( \Box \).

\[ \Sigma_3^M = \Sigma_{\text{Twist}}^M \cup \Sigma_{\text{Cut}}^M \] where:

- \( \Sigma_{\text{Twist}}^M \) is given by the following twisting relations:

\[ A \] \hspace{1cm} \( A \) \hspace{1cm} \( A \)

\[ B \] \hspace{1cm} \( B \) \hspace{1cm} \( B \)

\[ C \] \hspace{1cm} \( C \) \hspace{1cm} \( C \)

\[ D \] \hspace{1cm} \( D \) \hspace{1cm} \( D \)

\[ E \] \hspace{1cm} \( E \) \hspace{1cm} \( E \)

\[ F \] \hspace{1cm} \( F \) \hspace{1cm} \( F \)

\[ G \] \hspace{1cm} \( G \) \hspace{1cm} \( G \)

\[ H \] \hspace{1cm} \( H \) \hspace{1cm} \( H \)

\[ I \] \hspace{1cm} \( I \) \hspace{1cm} \( I \)

\[ J \] \hspace{1cm} \( J \) \hspace{1cm} \( J \)

\[ K \] \hspace{1cm} \( K \) \hspace{1cm} \( K \)

\[ L \] \hspace{1cm} \( L \) \hspace{1cm} \( L \)

\[ M \] \hspace{1cm} \( M \) \hspace{1cm} \( M \)

\[ N \] \hspace{1cm} \( N \) \hspace{1cm} \( N \)

\[ O \] \hspace{1cm} \( O \) \hspace{1cm} \( O \)

\[ P \] \hspace{1cm} \( P \) \hspace{1cm} \( P \)

\[ Q \] \hspace{1cm} \( Q \) \hspace{1cm} \( Q \)

\[ R \] \hspace{1cm} \( R \) \hspace{1cm} \( R \)

\[ S \] \hspace{1cm} \( S \) \hspace{1cm} \( S \)

\[ T \] \hspace{1cm} \( T \) \hspace{1cm} \( T \)

\[ U \] \hspace{1cm} \( U \) \hspace{1cm} \( U \)

\[ V \] \hspace{1cm} \( V \) \hspace{1cm} \( V \)

\[ W \] \hspace{1cm} \( W \) \hspace{1cm} \( W \)

\[ X \] \hspace{1cm} \( X \) \hspace{1cm} \( X \)

\[ Y \] \hspace{1cm} \( Y \) \hspace{1cm} \( Y \)

\[ Z \] \hspace{1cm} \( Z \) \hspace{1cm} \( Z \)
together with two rules representing the involution $A^\perp = A$:

\[
\text{Ax} : \quad A \quad \quad \quad \quad \quad \quad A^\perp \quad , \quad \quad \quad \quad \quad \quad A \quad \quad \quad \quad \quad \quad A\text{;} \\
\]

- $\Sigma_{MLL}^M$ is the set of rules for the cut elimination:

\[
\text{Cut} \text{ is the set of rules for the cut elimination:} \\
\]

\[
\text{Ax and Cut rules are not wirings but cells;} \\
\text{Wiring may cross only by means of twisting operators, that is, only if both wires are labeled by colors of the twisting family;} \\
\text{Diagrams have a top-to-bottom orientation, that is, there are not diagrams like } \quad \not\otimes \otimes A \quad \text{nor upside-down tensor like } \quad \not\otimes \otimes \\
\text{Twisting relations and rewriting rules concerning cut elimination have the same semantical status.} \\
\]

**Theorem 2** (Interpretation of proofs in $\Sigma_{MLL}$). For any derivation $d(\Gamma)$ of $\Gamma$ in MLL there is a proof diagram $\phi_d(\Gamma) : \square \Rightarrow \Gamma \in \Sigma_{MLL}$.

**Proof.** Let $d(\Gamma)$ be a derivation in MLL of $\Gamma$. First we observe that, if there is a diagram $\phi : \Delta \Rightarrow \Gamma$ so there is a diagram $\phi^\sigma = \phi_\sigma \circ \phi : \Delta \Rightarrow \sigma(\Gamma)$ for all permutation $\sigma \in S_{|\Gamma|}$. By this fact we can proceed by induction on the number of inference rules appearing in $d(\Gamma)$:

- If just one inference rule occurs in $d(\Gamma)$, it must be an Ax rule, $\Gamma = A, A^\perp$ and $\phi_d(\Gamma) = \text{Ax}_A : \square \Rightarrow A, A^\perp$. 

**Remark 3.** The 2-cells in $\Sigma_{MLL}$ are similar to MLL proof structures. We remark the following important differences:

- Ax and Cut rules are not wirings but cells;
- Wiring may cross only by means of twisting operators, that is, only if both wires are labeled by colors of the twisting family;
- Diagrams have a top-to-bottom orientation, that is, there are not diagrams like $\not\otimes \otimes A$ nor upside-down tensor like $\not\otimes \otimes$;
- Twisting relations and rewriting rules concerning cut elimination have the same semantical status.
In order to have a correctness criterion for \( d(\Gamma) \), then we consider the last one and we distinguish two cases in base of its arity (see Rem. 2):

- If it is unary and \( \Gamma = \Gamma', A \otimes B \), then, by inductive hypothesis, there is a diagram \( \phi_{d(\Gamma',A,B)} : \square \rightarrow \Gamma',A,B \) of the derivation \( d(\Gamma',A,B) \) with \( n \) inference rules. Therefore

\[
\phi_{d(\Gamma)} = (\mathbf{id}_{\Gamma'},\otimes_{A,B}) \circ \phi_{d(\Gamma',A,B)} : \square \Rightarrow \Gamma;
\]

- If it is binary and \( \Gamma = \Delta,A \otimes B,\Delta' \), then, by inductive hypothesis, there are two diagrams

\[
\phi_{d(\Delta,A)} : \square \Rightarrow \Delta,A \quad \text{and} \quad \phi_{d(B,\Delta')} : \square \Rightarrow B,\Delta' \quad \text{relative to the two derivations} \quad d(\Delta,A) \quad \text{and} \quad d(B,\Delta') \quad \text{with at most} \quad n \quad \text{inference rules. Therefore}
\]

\[
\phi_{d(\Gamma)} = (\mathbf{id}_{\Delta},\otimes_{A,B},\mathbf{id}_{\Delta'}) \circ (\phi_{d(\Delta,A)},\phi_{d(B,\Delta')}) : \square \Rightarrow \Gamma
\]

Similarly, if it is binary and \( \Gamma = \Delta,\text{Cut}(A,A^\perp),\Delta' \), then

\[
\phi_{d(\Gamma)} = (\mathbf{id}_{\Delta},\text{cut}_A,\mathbf{id}_{\Delta'}) \circ (\phi_{d(\Delta,A)},\phi_{d(A^\perp,\Delta')}) : \square \Rightarrow \Gamma.
\]

\[\square\]

### 4.2 Proof diagram with control for MLL

In order to have a correctness criterion for MLL proof diagrams, we enrich the set of string labels with two new non-twisting colors \( L \) (left) and \( R \) (right) and re-define some 2-cells.

The idea is to use these latter to introduce a notion of well-paranthesization in a setting where a proof derivation can be seen as a sequence of operations over lists of sequents: unary derivation rules act on single sequents (as in the case of \( \otimes \)), binary ones act on two sequent (as in the case of \( \otimes \) and \( \text{Cut} \)) and the 0-ary one, that is \( Ax \), generates a new sequent.

**Definition 9.** The control polygraph of multiplicative linear logic \( \mathcal{M}^2 \) is given by the following sets of cells:

- \( \mathcal{M}_0 = \{ \square \} \);
- \( \mathcal{M}_1 = \mathfrak{F}_{\mathbb{M}L} \cup \{ L = \downarrow, R = \uparrow \} \);
- \( \mathcal{M}_2 = \mathfrak{R}_\mathbb{M}L \cup \{ \begin{array}{c}
\otimes_{A,B} : A,R,L,B \Rightarrow A \otimes B = A \downarrow B \\
\otimes_{A,B} : A,B \Rightarrow A \otimes B = A \downarrow B \\
Ax_A : \square \Rightarrow L,A,A^\perp,R = A \downarrow A^\perp \\
\text{Cut}_A : A,R,L,A^\perp \Rightarrow \square = A \downarrow A^\perp \\
T_{A,B} : A,B \Rightarrow B,A = B \downarrow A \\
x, y, z, u, v, w, A, B \in \mathfrak{R}_\mathbb{M}L \end{array} \} \)
• $\mathcal{M}_3 = \mathcal{M}_{Twist}$ is given by the following twisting relations:

\[
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\quad
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\]

\[
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\quad
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\]

\[
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\quad
\begin{array}{c}
\vdash A B C \\
\downarrow \quad \downarrow \\
\vdash A B C
\end{array}
\]

together with one rule representing the involution $A^\perp \perp = A$:

\[
\begin{array}{c}
\vdash A \\
\downarrow \\
\vdash A^\perp A
\end{array}
\quad
\begin{array}{c}
\vdash A \\
\downarrow \\
\vdash A^\perp A
\end{array}
\]

**Remark 4.** The polygraph $\mathcal{M}$ is twisting with twisting family $\mathfrak{F}_{MLL}$.

**Theorem 3** (Proof diagrams correspondence in $\mathcal{M}$).

\[\vdash_{MLL} \Gamma \iff \exists \phi \in \mathcal{M} \text{ such that } \phi : \Box \Rightarrow L, \Gamma, R.\]

**Proof.** To prove the left-to-right implication $\Rightarrow$, as in Teor. 2, we remark that, if there is a diagram $\phi : \Box \Rightarrow L, \Gamma, R$ with $\Gamma$ sequent in $MLL$, so there is a diagram

\[
\phi^\sigma = (\text{id}_L, \hat{\phi}_\sigma, \text{id}_R) \circ \phi : \Box \Rightarrow L, \sigma(\Gamma), R
\]

for any permutation $\sigma \in S_{|\Gamma|}$. Then we proceed by induction on the number of inference rules in a derivation $d(\Gamma)$ in $MLL$:

- If just one inference rule occurs $d(\Gamma)$, then it is an $Ax$ and $\Gamma = A, A^\perp$ and $\phi_{d(\Gamma)} = Ax_A : \Box \Rightarrow L, A, A^\perp, R$;

- If $n+1$ inference rules appear, then we consider the last one and we distinguish two cases in base of its arity:
  - If it is an unary $\exists$ and $\Gamma = \Gamma', A \exists B$, then, by inductive hypothesis, there is a diagram $\phi_{d(\Gamma', A, B)} : \Box \Rightarrow L, \Gamma', A, B, R$ of the derivation $d(\Gamma', A, B)$ and
    \[
    \phi_{d(\Gamma)} = (\text{id}_L, \exists_{A, B}, \hat{\phi}_R) \circ \phi_{d(\Gamma, A, B)} : \Box \Rightarrow L, \Gamma, R;
    \]
  - If it is a binary $\otimes$ and $\Gamma = \Delta, A \otimes B, \Delta'$, then, by inductive hypothesis, there are two diagrams $\phi_{d(\Delta, A)} : \Box \Rightarrow L, \Delta, A, R$ and $\phi_{d(\Delta, A') : \Box \Rightarrow L, B, \Delta', R}$ relative to the two derivations $d(\Delta, A)$ and $d(\Delta, B')$ with at most $n$ inference rules. Therefore
    \[
    \phi_{d(\Gamma)} = (\text{id}_L, \Delta, \otimes_{A, B}, \exists_{\Delta', R}) \circ (\phi_{d(\Delta, A)} \otimes \phi_{d(\Delta, A')}) : \Box \Rightarrow L, \Gamma, R
    \]
    Similarly, if it is a binary $\text{Cut}$ and $\Gamma = \Delta, \text{Cut}(A, A^\perp), \Delta'$, then
    \[
    \phi_{d(\Gamma)} = (\text{id}_L, \Delta, \text{Cut}_{A^\perp}, \exists_{\Delta', R}) \circ (\phi_{d(\Delta, A)} \otimes \phi_{d(\Delta, A')}) : \Box \Rightarrow L, \Gamma, R
    \]
In order to prove sequentialization, i.e. the right-to-left implication \( \leftrightharpoons \), we proceed by induction on the number \( |\phi|_n \) of gates in \( \phi \):

- If \( |\phi|_n = 0 \) so \( \phi : \text{id}_\Gamma : \Gamma \Rightarrow \Gamma \). By hypothesis \( \phi \) has no input (i.e. \( s_2(\phi) = \Box \)) so it is the identity diagram over the empty string, this is the empty diagram \( \text{id}_0 : \Box \Rightarrow \Box \) which it is not sequentializable since \( t_2(\phi) = \Box \neq L,R \);
- If \( |\phi|_n = 1 \) then \( \phi \) is an elementary diagram. The elementary diagrams with source \( \Box \) and target \( L,\Gamma,R \) with \( \Gamma \in \mathcal{S}_{MLL}^c \) are atomic made of a unique 2-cell of type \( A \chi_A : \Box \Rightarrow L,A,\perp,R \) for some \( A \in \mathcal{S}_{MLL} \). The associated sequent \( \vdash A,\perp \) is derivable in \( MLL \);
- Otherwise there is 2-cell of type \( \alpha : \Gamma \Rightarrow \alpha(\Gamma') \in \mathcal{S}_{MLL}^c \) and \( \Gamma = \Delta,\alpha(\Gamma'),\Delta' \). In this case \( \phi = (\text{id}_{L,\Delta},\alpha,\text{id}_{A,R}) \circ \phi' \) where \( \phi' : \Box \Rightarrow L,\Delta,\Gamma',\Delta',R \). We have the following cases:
  - If \( \alpha = T_{A,B}, \Gamma' = A,B \) and \( \alpha(\Gamma') = B,A \). The diagram \( \phi' \) is sequentializable by inductive hypothesis since \( |\phi'|_{|\phi|_n} = |\phi'|_{|\phi|_n} + 1 \);
  - Similarly if \( \alpha = \eta_{A,B}, \Gamma' = A,B \) and \( \alpha(\Gamma') = A \eta B \);
  - If \( \alpha = \otimes_{A,B} \) so \( \Gamma' = A,R,L,B \). \( \alpha(\Gamma') = A \otimes B \) and
    \[
    \phi' : \Box \Rightarrow L,\Delta,A,R,L,B,\Delta',R.
    \]
    This diagram is a parallel composition \( \phi = \phi', \phi' \) with
    \[
    \phi'_L : \Box \Rightarrow L,\Delta,A,R \quad \text{and} \quad \phi'_R : \Box \Rightarrow L,B,\Delta',R
    \]
    of two diagrams which satisfy inductive hypothesis since \( |\phi|_{|\phi|_n} = |\phi'|_{|\phi|_n} + |\phi'|_{|\phi|_n} + 1 \);
  - Similarly if \( \alpha = \text{Cut}_A \) with \( B = A' \) we have \( \Gamma' = A,R,L,\perp,A \) and \( \alpha(\Gamma') = \emptyset \).

\[\Box\]

### 4.3 Proof diagrams for \( MLLc \)

In this section we extend the signatures of the two previous polygraphs in order to accommodate multiplicative constants in our syntax of proof diagrams and we enunciate some relation between this syntax and the multiplicative proof structure’s one.

**Definition 10.** The polygraph of multiplicative linear logic with constants and cut-elimination \( \Sigma_{MLLc} \) is given by the following sets of cells:

- \( \Sigma_0^c = \{ \Box \} \);
- \( \Sigma_1^c = \mathcal{S}_{MLLc} \);
- \( \Sigma_2^c = \left\{ 1 : \Box \Rightarrow 1 = \begin{array}{cc}
\top & 1 \\
1 & \top 
\end{array} \right\} \cup \Sigma_2^M \)
- \( \Sigma_3^c = \Sigma_{Twist}^c \cup \Sigma_{Cut}^c \) where:
  - \( \Sigma_{Twist}^c \) is \( \Sigma_{Twist}^M \) along with the following twisting relations:
Remark 5. The polygraph $\Sigma_{MLL}$ is total-twisting.

Theorem 4 (Interpretation of proofs in $\Sigma_{MLLc}$). For any derivation $d(\Gamma)$ of $\Gamma$ in $MLLc$ there is a proof diagram $\phi_{d(\Gamma)}: \square \Rightarrow \Gamma \in \Sigma_{MLLc}$.

Proof. The proof is much like the one we provided for Theorem 2. In order to accommodate units, we just need to slightly revisit our inductive reasoning by considering the following two additional cases (i.e. the remaining cases stay the same):

- If just one inference rule occurs in $d(\Gamma)$, then it may be a $1$ rule (in addition to $Ax$). It follows that $\Gamma = 1$ and $\phi_1 = 1 : \square \Rightarrow 1$;

- If the last of the $n + 1$ inference rules appearing in $d(\Gamma)$ is an unary $\perp$ and $\Gamma = \Gamma', \perp$, then, by inductive hypothesis, there is a diagram $\phi_{\Gamma'}: \square \Rightarrow \Gamma'$ and $\phi_{\Gamma} = \phi_{\Gamma'}, \perp$;

In order to state some relation between 2-cells of $\Sigma_{MLLc}$ are multiplicative proof structures with units, in this paper we do not recall formal definitions of proof structures [6] neither we give the definition of paths in a 2-cell of a twisting polygraph with the relative notion of adjacency. These latter result intuitively derivable form the following example: we say that in the 2-cell

![2-cell diagram]

there is a path connecting the $\perp$-gate with the first input of the $Cut$-gate. Similarly there is a path from unique input of $\phi$ to the second input of the $\otimes$-gate which, for these reasons, are considered adjacent. If follows that the $\otimes$-gate is adjacent with the $\gamma$-gates and the leftmost $Ax$-gate, the $\gamma$-gate with the $Cut$-gate and so on.

Proposition 2. It is possible to associate to any 2-cell $\phi \in \Sigma_{MLLc}$ a proof structure $P(\phi)$.

Proof. Given a $\phi \in \Sigma_{MLLc}$, the proof structure $P(\phi)$ is the one which exhibits a one-to-one correspondence between its cells and the gates in $\phi$ such that both type and adjacencies of cells in $P(\phi)$ correspond to type and adjacencies of gates in $\phi$.

In particular, we notice that this correspondence $P(-)$ is invariant under the equivalence relation $\simeq_{Twist}$ over 2-cells in $\Sigma_{MLLc}$ generated by the set of twisting relations $\Sigma^c_{Twist}$, i.e. given $\phi, \phi' \in \Sigma_{MLLc}$ such that $\phi \simeq_{Twist} \phi'$ then $P(\phi) = P(\phi')$.

Corollary 5. There is a one-to-one correspondence between the set of proof diagrams in $\Sigma_{MLLc}$ modulo the equivalence relation $\simeq_{Twist}$ and the set of multiplicative proof structures with constants.

Proof. It suffice to verify that for any rule $\phi \Rightarrow \phi'$ in $\Sigma^c_{Twist}$, $P(\phi) = P(\phi')$. 

If we note $\{P\}$ the class of $\sim_{Tw}$-equivalent 2-cells in $\Sigma_{MLLc}$ associate to the same the proof structure $P$, we formulate the following

**Proposition 3** (Cut elimination). If $P, P'$ are two proof nets $P \leadsto \text{cut-elim} P' \iff \exists \phi, \phi' \in \Sigma_{MLLc}, \phi \in [P], \phi' \in [P']$ such that $\phi \equiv_{\Sigma_{Cut}} \phi'$.

*Sketch of proof.* The left-to-right implication results trivial. In order to prove the converse we need to choose one $\phi \in \{P\}$ such that $\phi = \psi$ where $\psi$ is a diagram containing the Cut-gate corresponding to the Cut cell (or link depending on the chosen definition of proof net) we want to eliminate and $\psi$ is one of the premise of a rewriting rule in $\Sigma_{Cut}$. The prove follows the correspondence $P(-)$.

**Corollary 6** (Cut elimination). If $\phi : \Box \Rightarrow \Gamma \in \Sigma_{MLLc}$ is a 2-cell representing a derivation in MLLc, if $\hat{\phi} \in \Sigma_{MLLc}$ is irreducible and $\phi \equiv \hat{\phi}$, so $\hat{\phi}$ have no Cut-gate.

### 4.4 Proof diagrams with control for MLLc

We finally extend proof diagrams with control to the general case of MLLc.

**Definition 11.** The control polygraph of multiplicative linear logic with constants $\hat{\mathcal{M}}^c$ is given by

- $\hat{\mathcal{M}}^c_0 = \{ \Box \}$;
- $\hat{\mathcal{M}}^c_1 = \hat{\mathcal{M}}_{MLLc} \cup \{ L = \{ R = \parallel \} \}$;
- $\hat{\mathcal{M}}^c_2 = \left\{ 1 : \Box \Rightarrow L, 1, R = \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\} \cup \Sigma^M$
- $\hat{\mathcal{M}}^c_3 = \hat{\mathcal{M}}^{Twist}_{MLLc}$ is made of rules in $\hat{\mathcal{M}}^{Twist}_{MLLc}$ plus the following twisting relations:

\[
\begin{array}{c}
A \\
A \\
\end{array} \equiv 
\begin{array}{c}
A \\
\cdot \\
A \\
A \\
\end{array},
\begin{array}{c}
A \\
\cdot \\
A \\
A \\
\end{array} \equiv 
\begin{array}{c}
A \\
\cdot \\
A \\
A \\
\end{array} ;
\end{array}
\]

**Theorem 7** (Controlled proof diagram correspondence in $\hat{\mathcal{M}}^c$).

$\vdash_{MLLc} \Gamma \iff \exists \phi \in \hat{\mathcal{M}}^c$ such that $\phi : \Box \Rightarrow L, \Gamma, R$.

*Proof.* The proof can be given extending the one of Theorem[3] To prove the left-to-right implication $\Rightarrow$ we should to consider the following two additional cases:

- If just one inference rule occurs $d(\Gamma)$, then it could also be a $1$, $\Gamma = 1$ and $\phi_{d(\Gamma)} = 1 : \Box \Rightarrow L, 1, R$;
- If the last of the $n + 1$ inference rules appearing in $d(\Gamma)$ is a $\bot$ (unary), $\Gamma = \Gamma', \bot$, then, by inductive hypothesis, there is a diagram $\phi_{\Gamma'} : \Box \Rightarrow L, \Gamma', R$ and $\phi_T = (L, \bot, \id_{\Gamma'}, R) \circ \phi_{\Gamma'}$.
In order to prove sequentialization, i.e. the right-to-left implication $\iff$, we have to consider the following two additional cases:

- If $|\phi|_{\mathcal{D}} = 1$ than $\phi$ is an elementary diagram. The elementary diagrams with source $\Box$ and target $L, \Gamma, R$ with $\Gamma \in \mathcal{F}_{MLL_c}^*$ are atomic made of a unique 2-cell of type $Ax_A : \Box \rightarrow L, A, A^\perp, R$ for some $A \in \mathcal{F}_{MLL_c}$ but also $1 : 0 \rightarrow L, 1, R$. The associated sequent $\vdash 1$ is derivable in $MLL_c$;

- Otherwise we should consider the case if there is 2-cell of type $\perp$. Then $\phi = (\text{id}_{L,A}, \perp, \text{id}_{\Delta, R}) \circ \phi'$ with the diagram $\phi'$ sequentializable by hypothesis since $|\phi|_{\mathcal{D}} = |\phi'|_{\mathcal{D}} + 1$.

## 5 Conclusion and future work

We have presented proof diagrams, a particular class of string diagrams suitable for interpreting linear logic proof derivations. In particular, such settings exhibits an internal correction criterion as we have shown a one-to-one correspondence between $MLL$, with or without constants, (one-sided) sequent calculus proof derivations and proof diagrams. Moreover, the correctness of a proof diagram, i.e. whether it corresponds to a proof in sequent calculus, can be verified in constant time.

Our results raise an important question about the quotient set over proofs introduced by proof diagrams, and how it relates to that performed by proof nets.

For this, let $\sim$ be the equivalence relation over proof derivations induced by proof diagrams equivalence $\simeq$ in $(\mathcal{M}_{\mathcal{D}})^*$. Then, one the one hand, $\sim$ captures all commutations of reversible inference rules $\leftarrow$ and $\perp$ by the interchange rule and twisting relations. On the other hand, this is not the case for $\otimes$ and $\text{Cut}$: let $\alpha, \beta \in \{\otimes, \text{Cut}\}$, then $\sim$ equates only permutations of the kind that follows

$$
\frac{
\vdash \Sigma, A \\
\vdash B, \Gamma, C \\
\vdash D, \Delta
}{\vdash \Sigma, \alpha(A, B), \Gamma, \beta(C, D), \Delta}
\sim
\frac{
\vdash \Sigma, A \\
\vdash B, \Gamma, C \\
\vdash D, \Delta
}{\vdash \Sigma, \alpha(A, B), \Gamma, \beta(C, D), \Delta}
$$

that is, $\otimes$ or $\text{Cut}$ permutations that do not change the order of the leafs in a derivation tree.

It follows that proof nets equivalence is coarser than proof diagrams one, proof nets equate more. For an actual example, consider the linear logic sequent $B \otimes C, A \otimes D$: this latter exhibits two different derivations that correspond to the following two non-equivalent proof diagrams

On the other hand, the two proof derivations have the same proof net.

We conjecture that, in order to recover the whole proof equivalence induced by proof nets, we should extend control polygraph rewriting with the possibility to permute $Ax$ and $1$ gates’ position in a diagram. Anyway, this is not related to our complexity result for sequentialization. Indeed, proof diagrams exhibit a local sequentialization criterion which is ruled out in proof nets by complexity arguments (P. Lincoln
and T. Winkler [13], W. Heijltjes [9]), due to the number of jumps to check. Crucial in our settings is the fact that ⊥ gates have a specific position in diagrams, that one can interpret as a jump assignment: for example, given a ⊥ gate, we can point its jump to the unique gate of type Ax or 1 connected to the left-nearest L string. In particular, this means that equivalent proof diagrams in (M*)∗ may correspond to different jump assignments on the same proof net.

We believe this work suggests several future research directions. In particular, in the near future, we will focus on extending the present results to the multiplicative-exponential linear logic fragment.

References

[1] Matteo Acclavio. A complete proof of coherence for symmetric monoidal categories using rewriting.
[2] John C. Baez and Aaron Laud. A prehistory of n-categorical Physics.
[3] Albert Burroni. Higher dimensional word problems with application to equational logic. Theoretical computer Science 115 (1993), pp 43-62.
[4] Vincent Danos and Laurent Regnier. The structure of multiplicatives. Archive for Mathematical Logic 28 (1989), pp 181-203.
[5] Jean-Yves Girard. Linear Logic. In Theoretical Computer Science 50, pp 1-102, (1987).
[6] Jean-Yves Girard. Proof-nets : the parallel syntax for proof-theory. Logic and Algebra, eds. Ursini and Agliano (1996).
[7] Stefano Guerrini and Andrea Masini. Parsing MELL Proof Nets. Theoretical Computer Science 254, Issues 12 (2001), pp 317-335.
[8] Yves Guiraud and Philippe Malbos. Higher-dimensional categories with finite derivation type. Theory and Applications of Categories 22, n18 (2009), pp 420-478.
[9] Willem Heijltjes and Robin Houston. No Proof Nets for MLL with Units: Proof Equivalence in MLL is PSPACE-complete. In Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS 14, pages 50:1-50:10, New York, NY, USA, 2014. ACM.
[10] Michio Jimbo. Introduction to the Yang-Baxter equation. International Journal of Modern Physics A, Vol. 4, No 15 (1989), pp 3759- 3777.
[11] Yves Lafont. From proof nets to interaction nets. Advances in Linear Logic (1994).
[12] Yves Lafont. Towards an Algebraic Theory of boolean circuits. Journal of Pure and Applied Algebra 184 (2003).
[13] Patrick Lincoln and Timothy Winkler. Constant-only multiplicative linear logic is NP-complete. Theoretical Computer Science, 135 (1994), pp 155-169.
[14] Sauder Mac Lane. Categories for the working mathematicians (1978).
[15] Paul-Andé Mellies. Categorical semantics of linear logic. Interactive Models of Computation and Program Behaviour, Panoramas et Syntheses 27, Socit Mathmatique de France, pp 1-196 (2009).
[16] Ross Street. Limits indexed by category-valued 2-functors. Journal of Pure and Applied Algebra 8 (1976), pp149-181.