THE TAP-PLEFKA VARIATIONAL PRINCIPLE FOR THE SPHERICAL SK MODEL

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Abstract. We reinterpret the Thouless-Anderson-Palmer approach to mean field spin glass models as a variational principle in the spirit of the Gibbs variational principle and the Bragg-Williams approximation. We prove this TAP-Plefka variational principle rigorously in the case of the spherical Sherrington-Kirkpatrick model.

1. Introduction

There are several approaches in theoretical physics and mathematics to study the Sherrington-Kirkpatrick (SK) mean field spin glass model [20] and its variants. The most successful in physics is the replica approach, which with Parisi’s replica symmetry breaking Ansatz led him to his celebrated formula for the free energy [15]. The mathematically rigorous proofs of the formula due to Guerra, Talagrand and Panchenko are based on a subtle combination of interpolation, recursion, the Ghirlanda-Guerra identities and an invariance property for the limiting Gibbs measure [13, 16, 17, 24]. A further approach in the physics literature is the one due to Thouless, Anderson and Palmer (TAP) and Plefka. It originates in [25] as a diagrammatic expansion of the partition function of the Ising SK model relating the free energy to the so called TAP free energy, which is a disorder-dependent function defined on the space of magnetizations of the spins. It claims that the free energy equals the TAP free energy at magnetizations that solve a set of mean field equations and satisfy certain convergence conditions, which have not been completely clarified. Plefka’s condition [18, 19] is believed to be necessary, but it is not clear if it is also sufficient. The high temperature analysis of [25] has been made rigorous in [1]. The physicist’s TAP approach has been adapted to spherical models in [10].

In this paper we reinterpret the TAP approach as a variational principle for the free energy, which states that the free energy equals the maximum of the TAP free energy taken over magnetizations satisfying appropriate conditions. We make this rigorous in the case of the spherical Sherrington-Kirkpatrick model, and show that for this model Plefka’s condition is the only condition needed to formulate the variational principle.
Let $H_N(\sigma), \sigma \in \mathbb{R}^N$, be the 2-spin spherical SK Hamiltonian which is a centered Gaussian process on $\mathbb{R}^N$ with covariance

$$\mathbb{E}[H_N(\sigma)H_N(\sigma')] = N(\sigma \cdot \sigma')^2,$$

which can be constructed by setting

$$H_N(\sigma) = \sqrt{N} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j$$

for iid standard Gaussian random variables $J_{ij}$ and $\sigma \in \mathbb{R}^N$. Let $E$ be the uniform measure on the unit ball in $\mathbb{R}^N$ and let $Z_N(\beta, h_N) = E[e^{\beta H_N + Nh_N \cdot \sigma}]$ and $F_N(\beta, h_N) = \frac{1}{N} \log Z_N(\beta, h_N)$ be the partition function and free energy in the presence of an external field $h_N \in \mathbb{R}^N$. The TAP free energy for this model is given by [10, 25]

$$H_{TAP}(m) = \beta H_N(m) + Nm \cdot h_N + \frac{N}{2} \log (1 - |m|^2) + N \frac{\beta^2}{2} (1 - |m|^2)^2,$$

and Plefka’s condition [18, 19] reads

$$\beta(m) \leq \frac{1}{\sqrt{2}},$$

where

$$\beta(m) = \beta(1 - |m|^2).$$

We refer to the approximation

$$F_N(\beta, h_N) \approx \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{\sqrt{2}}} H_{TAP}(m)$$

as the TAP-Plefka variational principle and prove it in the following form.

**Theorem 1.** For any $\beta > 0$, $h \geq 0$ and any sequence $h_1, h_2, \ldots$ with $|h_N| = h$ one has

$$F_N(\beta, h_N) - \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{\sqrt{2}}} H_{TAP}(m) \to 0 \text{ in probability.}$$

We also include a solution of the TAP-Plefka variational problem that reduces it from a random $N$-dimensional optimization problem to one which is deterministic and one dimensional.
Lemma 2. For any $\beta, h, h_1, h_2, \ldots$, as in Theorem 1 one has

$$\left| \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| \leq 1, \beta(m) \leq \frac{1}{\sqrt{2}}} H_{TAP}(m) - \sup_{q \in [0,1]: \beta(1-q) \leq \frac{1}{\sqrt{2}}} B(q) \right| \to 0 \text{ in probability},$$

where

$$B(q) = B(q; \beta, h) = \sqrt{h^2 q + \frac{2 \beta^2 q^2}{2} + \frac{1}{2} \log (1 - q) + \frac{\beta^2}{2} (1 - q)^2}.$$

Together, Theorem 1 and Lemma 2 show that

$$(1.3) \quad F_N(\beta, h_N) \to \sup_{q \in [0,1]: \beta(1-q) \leq \frac{1}{\sqrt{2}}} B(q).$$

For comparison, the Parisi formula in this context [11, 23] states that

$$(1.4) \quad F_N(\beta, h_N) \to \inf_{q \in [0,1]} P(q),$$

where

$$P(q) = \frac{1}{2} h^2 (1 - q) + \frac{1}{2} \frac{q}{1 - q} + \frac{1}{2} \log (1 - q) + \frac{1}{2} \beta^2 (1 - q^2).$$

1.1. Discussion.

1.1.1. The TAP-Plefka variational principle. The TAP-Plefka variational principle (1.1) should be compared to the classical Gibbs variational principle which states that

$$(1.5) \quad F_N(\beta, h_N) = \frac{1}{N} \sup_{\mathcal{G}} \left\{ \mathcal{G}(\beta H_N(\sigma) + N \sigma \cdot h_N) - H(\mathcal{G}||E) \right\},$$

where the supremum is over all probability measures which are absolutely continuous with respect to $E$, and $H(\mathcal{G}||E)$ is the relative entropy of $\mathcal{G}$ with respect to $E$. The first term is the internal energy and the second is the entropy.

In the classical Bragg-Williams approximation [7, 26, Section 4.1.2] in non-disordered statistical physics one restricts this sup to simple measures $\mathcal{G}$ that are parameterized by a mean magnetization $m \in \mathbb{R}^N$; in the case of $\pm 1$ spins one considers measures under which the spins $\sigma_i$ are independent with mean $m_i$. For any $m$ the corresponding measure gives a lower bound for the free energy, because of the Gibbs variational principle. If the Bragg-Williams approximation is successful, maximizing over $m$ yields the true free energy (at least to leading order). If applied to approximate the free energy of the Curie-Weiss Hamiltonian $\frac{\beta}{N} \sum_{i,j} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$ one obtains a variational problem over $m \in \mathbb{R}^N$ that is equivalent to

$$\frac{1}{N} \sup_{m \in [-1,1]} \left\{ \beta \bar{m}^2 + h \bar{m} - \frac{1 + \bar{m}}{2} \log \left( \frac{1 - \bar{m}}{2} \right) - \frac{1 - \bar{m}}{2} \log \left( \frac{1 + \bar{m}}{2} \right) \right\},$$
which also appears in the classical solution of the model via the large deviation rate function of the binomial distribution, and is thus indeed an accurate approximation.

In the spherical setting a product measure on the spins is not absolutely continuous with respect to $E$, but a natural family of measures is provided by exponential tilts of the uniform distribution given by $e^{\lambda \sigma \cdot m} dE$ appropriately normalized, for $\lambda = \lambda (m)$ chosen so that the mean magnetization is $m$. For such a measure the internal energy will be close to $\beta H_N (m) + N m \cdot h_N$ and the entropy will be close to $N/2 \log (1 - |m|^2)$. Thus the Bragg-Williams approximation of the free energy is

$$\frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1} \left\{ \beta H_N (m) + N m \cdot h_N + \frac{N}{2} \log (1 - |m|^2) \right\},$$

which is in fact inaccurate, in light of \((1.1)\). However, the TAP-Plefka variational principle can be seen as the appropriate modification of the Bragg-Williams approximation to obtain an accurate approximation for this disordered system, by adding the Onsager correction term $\frac{N}{2} \beta^2 (1 - |m|^2)^2$ and restricting the sup to $m$-s satisfying Plefka’s condition.

1.1.2. The 2-spin model. The 2-spin spherical spin glass is a much simpler model than the other Ising and spherical SK variants. It is always replica symmetric, and the Parisi formula can be written as a one parameter variational principle (see \((1.4)\)). For $h = 0$ an explicit closed form (non-variational) formula for $\lim_{N \to \infty} F_N$ exists, also in low temperature.

Furthermore, the Hamiltonian can be written as $H_N (\sigma) = \sqrt{N} \sigma^T S_N \sigma$ for a random matrix $S_N$ from the Gaussian orthogonal ensemble, and by the rotational invariance of the sphere we can work in the diagonalizing basis of $S_N$, in which case $H_N (\sigma) = \sqrt{N} \sum_{i=1}^{N} \lambda_i \sigma_i^2$ where $\lambda_i$ are the eigenvalues of $S_N$. Because of this the free energy can be computed by a random matrix approach, without using the Parisi formula \([4, 12, 14]\). Part of our analysis also relies on random matrix considerations. The resulting formulas \((1.2)\) and \((1.3)\) are not related to previously obtained formulas for the free energy. Our proof is the first rigorous derivation of a TAP variational principle based on a microcanonical analysis that yields bounds valid for finite $N$, and where Plefka’s condition appears naturally.

1.1.3. Previous work in the mathematical literature. Recently in \([9]\) Chen and Panchenko used the Parisi formula to verify a TAP variational principle for mixed Ising SK models in the thermodynamic limit, that is an equality after taking the limit $N \to \infty$, with a different condition replacing Plefka’s condition. In \([21]\) Subag constructs for very low temperatures a decomposition of the Gibbs measure of pure $p$-spin spherical models into pure states in a microcanonical fashion, and notices that the log of the
weight of each pure state coincides with its TAP free energy. Further mathematical results concern the TAP equations, which are usually derived as self-consistency equations for the mean magnetization but can also be seen as the critical point equations of the TAP free energy. Bolthausen has developed an iterative scheme for solving the TAP equations for the Ising SK model \[6\] that converges in the whole conjectured high temperature regime. Talagrand \[22\] and Chatterjee \[8\] showed that in high enough temperature the mean magnetization of the Ising SK Gibbs measure satisfies the TAP equations. Auffinger and Jagganath have used the Parisi formula to prove that solutions of the TAP equations describe the magnetization inside appropriately defined pure states of generic mixed Ising models for all temperatures \[3\]. Auffinger, Ben Arous & Cerny have studied the (annealed) complexity of TAP solutions for pure \(p\)-spin spherical Hamiltonians \[2\].

1.2. A word on the proof. The proof of Theorem \[4\] splits into a proof of a lower bound and a proof of an upper bound for the partition function \(Z_N(\beta, h_N)\). Both are based on recentering the Hamiltonian around magnetizations \(m\) of potential pure states (a similar recentering has been used by TAP \[25\], Bolthausen \[3\] and Subag \[21\]). In general, recentering around a given \(m\) gives rise to an effective external field for the recentered Hamiltonian.

The lower bound is presented in Section \[3\] and is proved by considering a recentering around any magnetization \(m\) that satisfies Plefka’s condition. We then restrict the integral in \(Z_N(\beta, h_N)\) to a subset of the sphere centered on \(m\) that is perpendicular to both \(m\) and to the effective external field. The mean energy (value of Hamiltonian and external field) on this subset is \(\beta H_N(m) + N m \cdot h_N\), cf. the first two terms of \(H_{TAP}(m)\). The log of the measure of the subset is approximately \(\frac{N}{2} \log (1 - |m|^2)\), cf. the third term of \(H_{TAP}(m)\). Finally the recentered Hamiltonian on this subset turns out to be a \(2\)-spin Hamiltonian on a lower dimensional sphere without external field at inverse temperature \(\beta(m) = \beta(1 - |m|^2)\). If Plefka’s condition is satisfied this is less than the critical inverse temperature \(\beta_c = \frac{1}{\sqrt{2}}\) and it is therefore natural that if we use the uniform measure on the subset as a reference measure the free energy of the recentered Hamiltonian is \(\frac{1}{2} \beta (m)^2 = \frac{1}{2} \beta^2 (1 - |m|^2)^2\), cf. the last term of \(H_{TAP}(m)\) (the Onsager correction). In this way we show that the subset contributes approximately \(\exp (H_{TAP}(m))\) to \(Z_N(\beta, h)\). This shows that \(H_{TAP}(m)\) is a lower bound of the free energy for any \(m\) satisfying Plefka’s condition. Note that it also provides a natural interpretation of the terms in \(H_{TAP}(m)\), and of Plefka’s condition as the condition that a pure state should effectively be in high temperature.

The upper bound is significantly harder and is proved in Section \[4\]. It involves the construction of a low-dimensional subspace of magnetizations \(M_N\) with the property that after recentering around any \(m \in M_N\), the effective external field is again almost
completely contained in $\mathcal{M}_N$. We write the integral in $Z_N (\beta, h_N)$ as a double integral first over $\mathcal{M}_N$ and then over the perpendicular space $\mathcal{M}_N^\perp$. For a fixed $m \in \mathcal{M}_N$ the integral over the perpendicular space $\mathcal{M}_N^\perp$ is seen to be related to a partition function without external field at a higher effective temperature, and is shown to be close to the exponential of a modified TAP energy, with the Onsager correction $\frac{\beta^2}{2} (1 - |m|^2)^2$ replaced by a different, not entirely explicit, expression. The integral in $Z_N (\beta, h_N)$ thus reduces to an integral of the exponential of the modified TAP energy over the low-dimensional space $\mathcal{M}_N$, and by the Laplace method the log of the integral turns into the supremum over the modified TAP energy over all $m$. We then show that if the Hessian at a critical point of the modified TAP energy is negative semi-definite, as it must be at any local maximum, then $m$ satisfies Plefka’s condition and furthermore the modified TAP energy and the original TAP energy $H_{TAP} (m)$ are close. From this the upper bound on $Z_N (\beta, h_N)$ is seen to follow.

In Section 5 we prove Lemma 2. In the next section we fix notation and recall some basic facts.

Acknowledgments. The first author thanks Erwin Bolthausen and Giuseppe Genovese for valuable discussions on a draft of this article. The second author wishes to express his gratitude to Markus Petermann for a long-standing discussion on spin glasses, and to Anton Wakolbinger for encouragement.

2. Notation and basic facts

The letter $c$ denotes a constant that does not depend on $N$, possibly a different one each time it is used.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with random variables $J_{ij}, i, j \geq 1$ that are iid standard Gaussians. Define

$$H_N (\sigma) = \sqrt{N} \sum_{i,j=1}^{N} J_{i,j} \sigma_i \sigma_j \text{ for } \sigma \in \mathbb{R}^N.$$ 

For any $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}^N$ we have

$$H_N (\lambda \sigma) = \lambda^2 H_N (\sigma).$$

Let $S_N$ be the $N \times N$ matrix given by

$$(S_N)_{ij} = \frac{J_{ij} + J_{ji}}{2}.$$ 

Note that

$$H_N (\sigma) = \sqrt{N} \sigma^T S_N \sigma,$$

and

$$\nabla H_N (\sigma) = 2 \sqrt{N} S_N \sigma.$$
For this reason the 2-spin Hamiltonian gradient is linear, i.e.
\[(2.2) \quad \nabla H_N (\sigma_1 + \sigma_2) = \nabla H_N (\sigma_1) + \nabla H_N (\sigma_2) \text{ for all } \sigma_1, \sigma_2 \in \mathbb{R}^N.\]

We will use, especially in the upper bound, that the empirical spectral distribution of $S_N$ converges to the semi-circle law. Let $\sqrt{N} \theta_1^N < \ldots < \sqrt{N} \theta_N^N$ be the eigenvalues of the matrix $S_N$. We have that
\[
\frac{1}{N} \sum_{i=1}^{N} \theta_i^N \to \mu \text{ in distribution, } \mathbb{P} - a.s.,
\]
where
\[(2.3) \quad \mu (dx) = \frac{1}{\pi} \sqrt{2 - x^2} 1_{[-\sqrt{2}, \sqrt{2}]}.
\]
In addition if we let
\[(2.4) \quad \theta_u = \inf \left\{ \theta : \int_{-\theta}^{\theta} \mu (dx) = u \right\},
\]
then
\[(2.5) \quad \theta_i^N = \theta_i^\frac{1}{N} + o(1) \text{ for } i = 1, \ldots, N,
\]
where the $o(1)$ terms tend to zero $\mathbb{P}$-a.s. uniformly in $i$.

For instance from the fact the eigenvalue of largest magnitude is of order $\sqrt{N}$ one can deduce that
\[(2.6) \quad \sup_{\sigma \in \mathbb{R}^N \colon |\sigma| \leq 1} |H_N (\sigma)| \leq cN \quad \text{and} \quad \sup_{\sigma \in \mathbb{R}^N \colon |\sigma| \leq 1} |\nabla H_N (\sigma)| \leq cN,
\]
for all $N$ large enough, almost surely. The latter implies that
\[(2.7) \quad |H_N (\sigma^1) - H_N (\sigma^2)| \leq cN |\sigma^1 - \sigma^2| \quad \text{for all } \sigma^i \in \mathbb{R}^N, |\sigma^i| \leq 1, i = 1, 2.
\]

We let $E^M$ denote the uniform measure on the unit sphere of $\mathbb{R}^M$. When $M = N$ we drop the superscript and write $E$. If $\mathcal{U}$ is a linear subspace of $\mathbb{R}^N$ we let $E^{\mathcal{U}}$ denote the uniform measure on the unit ball of $\mathbb{R}^N$ intersected with $\mathcal{U}$.

The surface area of the $N$-dimensional sphere of radius $r$ is $\frac{2\pi^\frac{N}{2}}{\Gamma(\frac{N}{2})} r^{N-1}$, and for any unit vector $v$ the inner product $\sigma \cdot v$ has a density under $E$ given by
\[(2.8) \quad E [\sigma \cdot v = dx] = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-1}{2} \right)} \left( 1 - x^2 \right)^{\frac{N-3}{2}}.
\]
More generally for any linear subspace $\mathcal{U} \subset \mathbb{R}^N$ of dimension $M$ the the projection $\tilde{\sigma}$ of $\sigma$ onto $\mathcal{U}$ has density
\[(2.9) \quad E [d\tilde{\sigma}] = \frac{1}{\pi^\frac{M}{2}} \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-M}{2} \right)} \left( 1 - |\tilde{\sigma}|^2 \right)^{\frac{N-M-2}{2}}.
\]
with respect to the standard Lebesgue measure on $\mathbb{R}^N$ restricted to $\mathcal{U}$.

3. LOWER BOUND

In this section we show the following lower bound for the free energy.

**Proposition 3.** For $\beta, h, h_1, h_2, \ldots$ as in Theorem 1 one has

\[
F_N (\beta, h_N) \geq \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{\sqrt{2}}} H_{\text{TAP}} (m) + o(1),
\]

where the $o(1)$ term tends to zero $\mathbb{P}$-a.s.

We prove this by noting that the partition function is certainly larger than the integral of $e^{\beta H_N (\sigma) + N \sigma \cdot h_N}$ over a slice $\{\sigma : |\sigma \cdot m - |m|^2| < \varepsilon\}$ for any $m$ inside the unit ball and $\varepsilon > 0$. On this slice we recenter the spins

\[
\hat{\sigma} = \sigma - m,
\]

and use the decomposition

\[
H_N (\sigma) = H_N (m) + \nabla H_N (m) \cdot \hat{\sigma} + H_N (\hat{\sigma}),
\]

which holds deterministically, to note that the integral over the slice is essentially the partition function of a 2-spin Hamiltonian on an $N - 1$-dimensional sphere of radius $1 - |m|^2$ with mean $\beta H_N (m)$ and external field $\beta \nabla H_N (m) + Nh_N$. By further restricting the integral to a subspace where the external field vanishes the Onsager correction term $\frac{1}{2} \beta^2 (1 - |m|^2)^2$ of the TAP free energy arises as the free energy of the partition function of this recentered Hamiltonian without external field. Plefka’s condition arises as the condition that the recentered Hamiltonian is in high temperature.

By the second moment method and concentration of measure one can show the following.

**Lemma 4.** It holds that

\[
\sup_{\beta \in [0, \frac{1}{\sqrt{2}}]} \left| \frac{1}{N} \log E \left[ \exp \left( \beta H_N (\sigma) \right) \right] - \frac{\beta^2}{2} \right| \to 0, \quad \mathbb{P} - a.s.
\]

It will be important to consider the partition function restricted to the intersection of the unit sphere with a hyperplane of dimension $N - 2$ (or $N - 1$). The next lemma shows that (3.3) remains true uniformly over all such restrictions. Recall that $E^{(u, v)}$ denotes the uniform measure on the unit sphere in the subspace $(u, v)^\perp$ perpendicular to $u$ and $v$. 
Lemma 5. We have

\begin{equation}
\sup_{\beta \in [0, \frac{1}{\sqrt{2}}], u, v \in \mathbb{R}^N} \left| \frac{1}{N} \log E^{(u,v)^\perp} [\exp (\beta H_N (\sigma))] - \frac{\beta^2}{2} \right| \rightarrow 0, \ \mathbb{P}\text{-a.s.}
\end{equation}

Proof. Recall that $H_N (\sigma) = \sqrt{N} \sigma^T S_N \sigma$ where $S_N$ is a real symmetric matrix. For any $u, v \in \mathbb{R}^N$ that are linearly independent, let $w_1, \ldots, w_N$ be an orthonormal basis such that $\langle u, v \rangle = \langle w_{N-1}, w_N \rangle$, and let $A$ be the top right $(N-2) \times (N-2)$ minor of $S_N$ when written in basis $w_1, \ldots, w_N$. For $\sigma \in \langle u, v \rangle^\perp$ we have $H_N (\sigma) = \sigma^T A \sigma$. Let $\sqrt{N} a_1, \ldots, \sqrt{N} a_{N-2}$ be the eigenvalues of $A$. Then

\begin{equation}
E^{(u,v)^\perp} [\exp (\beta H_N (\sigma))] = E^{N-2} \left[ \exp \left( N \beta \sum_{i=1}^{N-2} a_i \sigma_i^2 \right) \right].
\end{equation}

Let $B$ be the top right $(N-2) \times (N-2)$ minor of $S_N$ when written in the standard basis and let $\sqrt{N} b_1, \ldots, \sqrt{N} b_{N-2}$ be its eigenvalues. Note that $H_{N-2} (\sigma) = \sqrt{N-2} \sigma^T B \sigma$ for $\sigma \in \mathbb{R}^{N-2}$, and by (3.3) with $N-2$ in place of $N$ we have

\begin{equation}
E^{N-2} \left[ \exp \left( \sqrt{N} \beta \sigma^T B \sigma \right) \right] = e^{N \left( \frac{\beta^2}{2} + o(1) \right)},
\end{equation}

where the $o(1)$ term tends to zero almost surely. Also

\begin{equation}
E^{N-2} \left[ \exp \left( \sqrt{N} \beta \sigma^T B \sigma \right) \right] = E^{N-2} \left[ \exp \left( N \beta \sum_{i=1}^{N-2} b_i \sigma_i^2 \right) \right].
\end{equation}

Let $\theta_1^N, \ldots, \theta_{N-2}^N$ be the eigenvalues of $S_N$. By the eigenvalue interlacing inequality

$\theta_i^N \leq a_i, b_i \leq \theta_{i+2}^N$ for $i = 1, \ldots, N-2$,

so by (2.5) we have

$\sup_{i=1, \ldots, N} |a_i - b_i| = o(1)$.

Therefore

$\left| \beta \sum_{i=1}^{N-2} a_i \sigma_i^2 - \beta \sum_{i=1}^{N-2} b_i \sigma_i^2 \right| = o(1),$

so from (3.5), (3.6) and (3.7) it follows that

$E^{(u,v)^\perp} [\exp (\beta H_N (\sigma))] = e^{N \frac{\beta^2}{2} (1 + o(1))},$

uniformly over all linearly independent $u, v$, where the $o(1)$ terms tend to zero almost surely. The above argument but with $(N-1) \times (N-1)$ minors easily extends this to $u$ and $v$ that are linearly dependent. This proves (3.4). \quad \Box

We can now prove the lower bound Proposition 3.
Proof of Proposition 3. For any $m$ and $\sigma$, recenter the spins $\sigma$ around $m$ by letting $\hat{\sigma} = \sigma - m$. Recentering the Hamiltonian (see (3.2)) and the external field one obtains
\begin{equation}
\beta H_N(\sigma) + N \sigma \cdot h_N = \beta H_N(m) + N m \cdot h_N + N h^m \cdot \hat{\sigma} + \beta H_N(\hat{\sigma}),
\end{equation}
where
\begin{equation}
h^m = \frac{\beta}{N} \nabla H_N(m) + h_N,
\end{equation}
is the effective external field after recentering. Note that by our assumption $|h_N| = h$ and (2.6) we have that for $N$ large enough
\begin{equation}
|h^m| \leq c,
\end{equation}
for a constant $c$ depending only on $\beta$ and $h$. 

For any $m$ for any $m \in \mathbb{R}^N$ with $|m| < 1$ and $\varepsilon > 0$ let
\[
A = \{\sigma : \hat{\sigma} \cdot m, \hat{\sigma} \cdot h^m \in (-\varepsilon, \varepsilon)\}.
\]
Certainly we have
\[
Z_N(\beta, h_N) \geq E[1_A \exp (\beta H_N(\sigma) + N \sigma \cdot h_N)].
\]
Rewriting in terms of $\hat{\sigma}$ and using (3.8) the right hand-side can be bounded below by
\[
\exp (\beta H_N(m) + N m \cdot h_N - \varepsilon N) E[1_A \exp (\beta H_N(\hat{\sigma}))].
\]
Let $\gamma$ be the magnitude of the the projection of $\hat{\sigma}$ onto the hyperplane $\langle m, h^m \rangle^\perp$ perpendicular to $m$ and $h^m$, and let $\sigma^\perp$ be the normalized projection onto this hyperplane. If $\hat{\sigma} \cdot m \in (-\varepsilon, \varepsilon)$ then $\gamma = 1 - |m|^2 + O(\varepsilon)$. If also $\hat{\sigma} \cdot h^m \in (-\varepsilon, \varepsilon)$ then $|\hat{\sigma} - \gamma \sigma^\perp| \leq c \varepsilon$. Using this, (2.1) and (2.6)-(2.7), the above is at least
\[
\exp (\beta H_N(m) + N m \cdot h_N - c \varepsilon N) E[1_A \exp (\beta (1 - |m|^2) H_N(\sigma^\perp))],
\]
Since $\sigma^\perp$ is independent of $\sigma \cdot m, \sigma \cdot h^m$ under $E$, and is uniform on the unit sphere intersected with $\langle m, h^m \rangle^\perp$ this in fact equals
\[
\exp (\beta H_N(m) + N m \cdot h_N - c \varepsilon N) E[A] E^{(m,h^m)^\perp} \left[ \exp \left( \beta (1 - |m|^2) H_N(\sigma) \right) \right].
\]
If $m$ and $h^m$ are linearly independent then using (2.9) and (3.10) and it holds that
\[
E[A] \geq \frac{c \varepsilon^2}{N} \left( 1 - |m|^2 - c \varepsilon^2 \right) \frac{1}{N^{1/4}},
\]
and setting e.g. $\varepsilon = \frac{1}{\sqrt{N}}$ this equals
\[
\exp \left( \frac{N}{2} \log \left( 1 - |m|^2 \right) + o(N) \right).
\]
If \( m \) and \( h^m \) are linearly dependent the same lower bound follows from (2.8). Thus \( Z_N \) is at least

\[
\exp \left( \beta H_N (m) + N m \cdot h_N + \frac{N}{2} \log (1 - |m|^2) + o(N) \right) 
\times E^{(m,h^m)^\perp} \left[ \exp \left( \beta \left( 1 - |m|^2 \right) H_N (\sigma) \right) \right],
\]

for any \( m \) with \( |m| < 1 \), where the error term is \( o(N) \) uniformly in \( m \), almost surely.

By Lemma 5 this is in turn at least

\[
\exp \left( \beta H_N (m) + N m \cdot h_N + \frac{N}{2} \log (1 - |m|^2) + N \frac{\beta^2}{2} (1 - |m|^2)^2 + o(N) \right),
\]

provided

\[
(3.11) \quad \beta \left( 1 - |m|^2 \right) \leq \frac{1}{\sqrt{2}}, \quad \text{i.e. } \beta (m) \leq \frac{1}{\sqrt{2}},
\]

where the error term is \( o(N) \) almost surely, uniformly in \( m \) that satisfy (3.11). The claim (3.1) follows. \( \square \)

4. Upper bound

In this section we prove the following upper bound on the free energy.

**Proposition 6.** For \( \beta, h, h_1, h_2, \ldots \) as in Theorem 7 one has

\[
F_N (\beta, h_N) \leq \frac{1}{N} \sup_{m \in \mathbb{R}^N: |m| < 1, \beta (m) \leq \frac{1}{\sqrt{2}}} \ T_{\text{AP}} (m) + o(1),
\]

where the \( o(1) \) term tends to zero \( \mathbb{P} \)-a.s.

As for the lower bound, our proof is based on considering the Hamiltonian recentered around certain \( m \)-s inside the unit ball. However, for an upper bound we are not free to simply restrict the integral in the partition function to slices around an \( m \) and ignore the complement. Neither can we further restrict the integral inside the slice to a space where the effective external field vanishes. Lasty we can not ignore slices for which Plefka’s condition is not satisfied.

We get around these issues by constructing a low-dimensional subspace \( \mathcal{M}_N \) of \( m \)-s, such that the recentered Hamiltonian restricted to the space of configurations perpendicular to \( \mathcal{M}_N \) has almost vanishing external field for any \( m \in \mathcal{M}_N \), without further restriction. Using the Laplace method we are able to upper bound the free energy by a sup of the free energy contribution of each of these restricted Hamiltonians, since the dimension of \( \mathcal{M}_N \) is \( o(N) \). Lastly a coarse-graining of the recentered Hamiltonian gives a sequence of approximations to the free energy of the restricted Hamiltonians in a form that allows to show that the supremum must be attained at an \( m \) that satisfies Plefka’s condition.
4.1. **Diagonalization.** To prove the upper bound Proposition 6 we are obliged to make stronger use the diagonalized Hamiltonian

\[
N \sum_{i=1}^{N} \theta_i^N \sigma_i^2,
\]

and the semi-circle law. Let \( \tilde{h}_N \) be the vector \( h_N \) written in the diagonalizing basis of the matrix \( S_N \). By rotational symmetry we have

\[
F_N (\beta, h_N) = \frac{1}{N} \log E \left[ \exp \left( N \beta \sum_{i=1}^{N} \theta_i^N \sigma_i^2 + N \tilde{h}_N \cdot \sigma \right) \right].
\]

For convenience we also replace the diagonalized Hamiltonian (4.2) by its deterministic counterpart

\[
\tilde{H}_N (\sigma) = N \sum_{i=1}^{N} \theta_{i/N} \sigma_i^2,
\]

where each eigenvalue \( \theta_i^N \) is replaced by its typical position \( \theta_{i/N} \) (recall (2.4)-(2.5)). Let

\[
\tilde{F}_N (\beta, h_N) = \frac{1}{N} \log E \left[ \exp \left( N \beta \sum_{i=1}^{N} \theta_{i/N} \sigma_i^2 + N \tilde{h}_N \cdot \sigma \right) \right].
\]

and let

\[
\tilde{H}_{TAP} (\sigma) = \beta \tilde{H}_N (\sigma) + N m \cdot \tilde{h}_N + \frac{N}{2} \log (1 - |m|^2) + N \beta^2 \left( 1 - |m|^2 \right)^2.
\]

By (2.5) we have that

\[
\lim_{N \to \infty} \frac{1}{N} \sup_{\sigma \in [\sigma]} \left| \tilde{H}_N (\sigma) - N \sum_{i=1}^{N} \theta_i^N \sigma_i^2 \right| = 0, \quad \mathbb{P} - a.s.,
\]

and therefore the upper bound Proposition 6 follows from the following deterministic bound.

**Proposition 7.** For \( \beta, h, h_1, h_2, \ldots \) as in Theorem 7 one has

\[
\tilde{F}_N (\beta, h_N) \leq \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{N}} \tilde{H}_{TAP} (m) + o(1).
\]

The rest of this section is devoted to the proof of Proposition 7.
4.2. Free energy of coarse-grained Hamiltonian without external field. We will approximate \( \tilde{H}_N (\sigma) \) by a coarse-grained Hamiltonian where the \( \theta_{i/N} \) are replaced by a bounded number of distinct coefficients. For such a Hamiltonian it will be straight-forward to bound the free energy using the Laplace method. To this end consider for each \( K \geq 2 \) equally spaced numbers \( x_1, \ldots, x_K \) in \([-\sqrt{2}, \sqrt{2}]\), so that,

\[
-\sqrt{2} = x_1 < x_2 < \ldots < x_K = \sqrt{2} - \frac{2\sqrt{2}}{K} \text{ and } x_{k+1} - x_k = \frac{1}{K},
\]

and a partition \( I_1, \ldots, I_K \) of \( \{1, \ldots, N\} \) given by

\[
I_k = \{ i : x_k \leq \theta_{i/N} < x_{k+1} \}, \quad k = 1, \ldots, K - 1 \quad \text{and} \quad I_K = \{ i : x_K \leq \theta_{i/N} \}.
\]

Let

\[
\sigma^2_{[k]} = \sum_{i \in I_k} \sigma^2_i \quad \text{and} \quad \mu_k = \frac{|I_k|}{N}.
\]

We first show the following variational principle for the free energy of the coarse-grained Hamiltonians in the absence of an external field.

**Lemma 8.** For all \( C > 0 \) we have uniformly in \( 0 < \beta \leq C \) and large enough \( K \) that

\[
\frac{1}{N} \log E \left[ \exp \left( N \beta \sum_{k=1}^{K} x_k \sigma^2_{[k]} \right) \right] = \sup_{0 \leq f_1, f_2 + \ldots + f_{K-1}} \left\{ \beta \sum_{k=1}^{K} x_k f_k + \frac{1}{2} \sum_{k=1}^{K} \mu_k \log \frac{f_k}{\mu_k} \right\} + O \left( e^{K^2 \log N} \right).
\]

**Proof.** For instance by applying (2.9) one sees that the \( E \)-distribution of the vector \( \left( \sigma^2_{[1]}, \ldots, \sigma^2_{[K-1]} \right) \) has a density on \( \mathbb{R}^{K-1} \) with respect to Lebesgue measure given by

\[
\Gamma \left( \frac{N}{2} \right) \prod_{k=1}^{K} \frac{\rho_k^{\frac{|I_k|-2}}}{\Gamma \left( \frac{|I_k|}{2} \right)} 1_A d\rho_1 \ldots d\rho_{K-1},
\]

where we write \( \rho_K = 1 - \rho_1 - \ldots - \rho_{K-1} \) and \( A = \{ \rho_1, \ldots, \rho_{K-1} \geq 0, \rho_1 + \ldots + \rho_{K-1} \leq 1 \} \).

Thus \( E \left[ \exp \left( N \beta \sum_{k=1}^{K} x_k \sigma^2_{[k]} \right) \right] \) equals

\[
\frac{\Gamma \left( \frac{N}{2} \right)}{\prod_{k=1}^{K} \Gamma \left( \frac{|I_k|}{2} \right)} \int_{[0,1]^{K-1}} 1_A \exp \left( N \left\{ \beta \sum_{k=1}^{K} x_k \rho_k + \sum_{k=1}^{K} \frac{1}{2} \left( \mu_k - \frac{2}{N} \right) \log \rho_k \right\} \right) d\rho_1 \ldots d\rho_{K-1}.
\]

By the Laplace method the integral equals

\[
\exp \left( N \left\{ \sup_{0 \leq f_1, f_2 + \ldots + f_{K-1}} \left\{ \beta \sum_{k=1}^{K} x_k f_k + \frac{1}{2} \sum_{k=1}^{K} \mu_k \log f_k \right\} + O \left( e^{K^2 \log N} \right) \right\} \right).
\]
where one uses that $\mu_k \geq \frac{c}{K^{3/2}}$, and that this implies that the maximizer in the sup
must satisfy $f_k \geq e^{-K^2}$.

Using the bounds $\Gamma(x) \simeq \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ for $x \geq \frac{1}{2}$ and $1 \leq \prod_{k=1}^K |I_k| \leq N^K$, one sees that
$$
\frac{1}{N} \log \frac{\Gamma \left( \frac{N}{2} \right)}{\prod_{k=1}^K \Gamma \left( \frac{|I_k|}{2} \right)} = -\frac{1}{2} \sum \mu_k \log \mu_k + O \left( \frac{K}{N} \right).
$$
This completes the proof. \qed

The variational problem on the bottom line of (4.4) can be solved. To state the result let

$$
g_K(\lambda) = \sum_{k=1}^K \frac{\mu_k}{\lambda - x_k}.
$$

For all $\beta > 0$ there is a unique $\lambda_K(\beta) > x_K$ such that

$$
g_K(\lambda_K(\beta)) = 2\beta.
$$

Let

$$
h_K(\lambda) = \sum_{k=1}^K \mu_k \log (\lambda - x_k),
$$

and

$$(4.5) \quad F_K(\beta) = \beta \lambda_K(\beta) - \frac{1}{2} - \frac{1}{2} \log (2\beta) - \frac{1}{2} h_K(\lambda_K(\beta)).$$

The next lemma shows that $F_K(\beta)$ is the supremum in the variational problem from (4.4).

**Lemma 9.** For each $K$ and $\beta > 0$ we have

$$(4.6) \quad \sup_{0 < f_1, f_2, \ldots, f_K} \left\{ \beta \sum_{k=1}^K x_k f_k + \frac{1}{2} \sum \mu_k \log \frac{f_k}{\mu_k} \right\} = F_K(\beta).$$

**Proof.** Using Lagrange multipliers and the KKT condition it follows that if for some $\lambda$ we have

$$(4.7) \quad \sum_{k=1}^K f_k = 1, \quad \text{where} \quad f_k = \frac{1}{2\beta} \frac{\mu_k}{\lambda - x_k},$$

then the supremum in (4.6) is attained at this $(f_1, \ldots, f_K)$. If $f_k$ takes this form

$$(4.8) \quad \sum_{k=1}^K x_k f_k = \sum_{k=1}^K x_k \frac{1}{2\beta} \frac{\mu_k}{\lambda - x_k} = \frac{1}{2\beta} \left( \lambda \sum_{k=1}^K \frac{\mu_k}{\lambda - x_k} - 1 \right) = \lambda - \frac{1}{2\beta},$$
and
\[
\frac{1}{2} \sum \mu_k \log \frac{f_k}{\mu_k} = \frac{1}{2} \sum \mu_k \log \frac{1}{2\beta} \frac{1}{\lambda - x_k} = -\frac{1}{2} \log (2\beta) - \frac{1}{2} h_K (\lambda).
\]
Since (4.7) holds with \( \lambda = \lambda_K (\beta) \) the claim (4.6) follows. \( \square \)

Note that Lemmas 8 and 9 show that the free energy of the coarse-grained Hamiltonians has no phase transition for any finite \( K \). Also those lemmas and the bound (4.9)
\[
\sum_{i=1}^{N} \theta_{i/N} \sigma_i^2 = \sum_{k=1}^{K} x_k \sigma_k^2 + O (K^{-1}),
\]
imply the following.

**Lemma 10.** For all \( C > 0 \) and \( K \geq 2 \) we have
\[
\limsup_{N \to \infty} \sup_{\beta \in [0, C]} \left| \frac{1}{N} \log E^N \left[ \exp \left( \beta N \sum_{i=1}^{N} \theta_{i/N} \sigma_i^2 \right) \right] - F_K (\beta) \right| \leq \frac{c}{K}.
\]

We now investigate the behavior of \( F_K (\beta) \) as \( K \to \infty \). Let
\[
g (\lambda) = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\mu (x)}{\lambda - x} \, dx \quad \text{for} \quad \lambda \geq \sqrt{2}.
\]
By standard estimates for Riemann sums
\[
\lim_{K \to \infty} g_K (\lambda) = g (\lambda) \quad \text{for} \quad \lambda > \sqrt{2}.
\]
The integral can be computed explicitly, and in fact
\[
g (\lambda) = \lambda - \sqrt{\lambda^2 - 2}.
\]
Note that \( g (\sqrt{2}) = \sqrt{2} \). If \( \beta \leq \frac{1}{\sqrt{2}} \) there is a unique \( \lambda (\beta) \geq \sqrt{2} \) such that \( g (\lambda (\beta)) = 2\beta \). In fact
\[
\lambda (\beta) = \frac{1}{\sqrt{2}} \left( \sqrt{2\beta} + \frac{1}{\sqrt{2\beta}} \right) \quad \text{for} \quad \beta \leq \frac{1}{\sqrt{2}}.
\]
The convergence (4.11) implies that
\[
\lim_{K \to \infty} \lambda_K (\beta) = \lambda (\beta) \quad \text{for} \quad \beta < \frac{1}{\sqrt{2}}.
\]
Also define
\[
h (\lambda) = \int_{-\sqrt{2}}^{\sqrt{2}} \mu (x) \log (\lambda - x) \, dx \quad \text{for} \quad \lambda \geq \sqrt{2},
\]
which can be computed explicitly as

\[
(4.14) 
\begin{align*}
    h(\lambda) &= \frac{\lambda^2}{2} - \frac{1}{2} - \frac{\lambda \sqrt{\lambda^2 - 2}}{2} + \log \left( \frac{\lambda + \sqrt{\lambda^2 - 2}}{2} \right).
\end{align*}
\]

By the convergence of the Riemann sum

\[
(4.15) 
\lim_{K \to \infty} h_K(\lambda) = h(\lambda) \quad \text{for} \quad \lambda > \sqrt{2}.
\]

Define

\[
(4.16) 
\mathcal{F}(\beta) = \beta \lambda(\beta) - \frac{1}{2} - \frac{1}{2} \log(2\beta) - \frac{1}{2} h(\lambda(\beta)), \beta \in \left[0, \frac{1}{\sqrt{2}}\right].
\]

Using the identities (4.12) and (4.14), this expression simplifies to

\[
(4.17) 
\mathcal{F}(\beta) = \frac{\beta^2}{2} \quad \text{for} \quad \beta \in \left[0, \frac{1}{\sqrt{2}}\right].
\]

Also it follows from (4.13) and (4.15) and the monotonicity of \( h_K(\lambda) \) that

\[
(4.18) 
\lim_{K \to \infty} \mathcal{F}_K(\beta) = \mathcal{F}(\beta) \quad \text{if} \quad \beta < \frac{1}{\sqrt{2}}.
\]

A posteriori it is clear that for \( \beta > \frac{1}{\sqrt{2}} \) the function \( \mathcal{F}_K(\beta) \) converges to the low-temperature free energy of the Hamiltonian \( H_N(\sigma) \) without external field, but this is not a step in the proof of our main results, but rather a consequence.

In the proof of Proposition 7 at the end of the next section we will use the two lemmas that now follow to rule out \( m \) that do not satisfy Plefka’s condition. First note that \( g_K(\lambda), \lambda_K(\beta), h_K(\lambda) \) and thus \( \mathcal{F}_K(\beta) \) are all continuous and differentiable. We have the following identity.

**Lemma 11.** For all \( \beta > 0 \)

\[
(4.19) 
\mathcal{F}_K'(\beta) = \lambda_K(\beta) - \frac{1}{2\beta}.
\]

**Proof.** This follows from the definition (4.16) and the equalities that \( h_K' = g_K \) and \( g_K(\lambda(\beta)) = 2\beta \). \( \square \)

**Lemma 12.** For all \( K \geq 2 \) there is a \( \varepsilon \in \left(0, \frac{2\sqrt{2}}{K}\right) \) such that

\[
\lambda_K(\beta) \geq \sqrt{2} - \varepsilon \implies \beta \leq \frac{1}{\sqrt{2}}.
\]

**Proof.** We may set \( \sqrt{2} - \varepsilon = \sqrt{2} - \lambda_K\left(\frac{1}{\sqrt{2}}\right) \) since

\[
\lambda_K(\beta) \geq \lambda_K\left(\frac{1}{\sqrt{2}}\right) \implies \beta \leq \frac{1}{\sqrt{2}},
\]
and
\[ x_K < \lambda_K \left( \frac{1}{\sqrt{2}} \right) < \lambda \left( \frac{1}{\sqrt{2}} \right) = \sqrt{2}. \]

Lemma 11 also allows us to strengthen the pointwise convergence (4.18) to uniform convergence.

**Lemma 13.** We have
\[ \lim_{K \to \infty} \sup_{\beta \in [0, \frac{1}{\sqrt{2}}]} |F_K(\beta) - F(\beta)| = 0. \]  

**Proof.** The \( F_K(\beta) \) are increasing \( \beta \), and \( F(\beta) \) is increasing in \( \beta \) and uniformly continuous (recall (4.17)), and therefore (4.18) implies that for any \( \delta \),
\[ \lim_{K \to \infty} \sup_{\beta \in [0, \frac{1}{\sqrt{2}} - \delta]} |F_K(\beta) - F(\beta)| = 0. \]

For any \( \delta > 0 \) we have
\[ \sup_{\beta \in \left[ \frac{1}{\sqrt{2}} - \delta, \frac{1}{\sqrt{2}} \right]} |F(\beta) - F(\beta - \delta)| \leq c\delta, \]
and by (4.19) we have uniformly in \( K \) that
\[ \sup_{\beta \in \left[ \frac{1}{\sqrt{2}} - \delta, \frac{1}{\sqrt{2}} \right]} |F_K(\beta) - F_K(\beta - \delta)| \leq c\delta, \]
so the claim (4.20) follows.

**4.3. Proof of upper bound.** As for the lower bound (recall (3.8)-(3.9)), an important step in the proof of the upper bound is to recenter the Hamiltonian around an \( m \in \mathbb{R}^N \) which yields an effective external field \( \beta \frac{1}{N} \nabla \tilde{H}_N(m) + \tilde{h}_N \) (see (4.29)). The next lemma constructs a low-dimensional subspace \( \mathcal{M}_N \subset \mathbb{R}^N \), such that if we recenter around any \( m \in \mathcal{M}_N \), and restrict to the space perpendicular to \( \mathcal{M}_N \), the effective external field after recentering is small.

**Lemma 14.** Let \( \beta, h, h_1, h_2, \ldots \) be as in Theorem\( \square \). There exists a sequence of linear spaces \( \mathcal{M}_1, \mathcal{M}_2, \ldots \) such that \( \mathcal{M}_N \subset \mathbb{R}^N \) and
\[ \dim(\mathcal{M}_N) = \lceil N^{3/4} \rceil, \]
and
\[ \lim_{N \to \infty} \sup_{m \in \mathcal{M}_N, |m| \leq 1, \hat{\sigma} \in \mathcal{M}_N, |\hat{\sigma}| \leq 1} \left| \left( \beta \frac{1}{N} \nabla \tilde{H}_N(m) + \tilde{h}_N \right) \cdot \hat{\sigma} \right| = 0, \]
Proof. We remove small coordinates from the external field by setting
\[ (\bar{h}_N)_i = \left( h_N \right)_i 1 \{ |(\bar{h}_N)_i| \geq \frac{1}{N} \}, \ i = 1, \ldots, N - 1, \]
and ensure that the external field in the direction of the eigenvector of the largest eigenvalue is not too small by setting
\[ (4.21) \quad (\bar{h}_N)^N = \left( h_N \right)^N 1 \{ |(\bar{h}_N)^N| \geq \frac{1}{N} \} + \frac{1}{N} 1 \{ |(\bar{h}_N)^N| < \frac{1}{N} \}. \]
Since for $|\sigma| \leq 1$ we have
\[ \sum_{i: |(\bar{h}_N)_i| \leq \frac{1}{N} } (\bar{h}_N)_i \sigma_i \leq \sqrt{ \sum_{i: |(\bar{h}_N)_i| \leq \frac{1}{N} } (\bar{h}_N)_i^2 } |\sigma| \leq N^{-1/2}, \]
and
\[ (h_{N,N}\sigma_N) = (h_{N,N} + O (N^{-1})), \]
it suffices to construct the spaces $\mathcal{M}_N$ such that
\[ (4.22) \quad \lim_{N \to \infty} \sup_{m \in \mathcal{M}_N, \sigma \in \mathcal{M}_N^\perp, |\sigma| \leq 1} \left| \left( \beta \frac{1}{N} \nabla \bar{H}_N (m) + \bar{h}_N \right) \cdot \sigma \right| = 0. \]
Consider the sequence given by
\[ \bar{h}_N^1 = \bar{h}_N, \]
and
\[ (4.23) \quad \bar{h}_N^{k+1} = \nabla \bar{H}_N (\bar{h}_N^k). \]
Note that
\[ \nabla_i \bar{H}_N (m) = 2 \theta_{i/N} m_i, \]
so
\[ (4.24) \quad \frac{1}{2^{k-1}} (\bar{h}_N^k)_i = (\theta_{i/N})^{k-1} (\bar{h}_N^1)_i. \]
Normalize $\bar{h}_N^k$ by setting
\[ \bar{h}_N^k = \frac{\bar{h}_N^k}{|\bar{h}_N^k|}. \]
Let
\[ V = \sqrt{N} (\log N)^2. \]
We build $\mathcal{M}_N$ starting with the space $\langle \tilde{h}_N^1, \ldots, \tilde{h}_N^{(k-1)} \rangle$. Let
\[ (4.25) \quad J_N = \left\{ i : |\theta_{i/N}| \geq \sqrt{2} - \frac{1}{\sqrt{N}} \right\} \cup A, \]
where \( A \) is an arbitrary subset of \( \{1, \ldots, N\} \) such that \( V - 1 + A = \lfloor N^{3/4} \rfloor \), which can be done since from (2.4) it follows that
\[
\left| \left\{ i : |\theta_{i/N}| \geq \sqrt{2 - \frac{1}{\sqrt{N}}} \right\} \right| \leq c\sqrt{N}.
\]

Let
\[
\mathcal{M}_N = \left\langle \hat{h}_N^1, \ldots, \hat{h}_N^{V-1}, e_{i_1}, \ldots, e_{i_{|J_N|}} \right\rangle,
\]
where \( e_i \) are the standard basis vectors of \( \mathbb{R}^N \) and \( i_1, \ldots, i_{|J_N|} \) is an enumeration of \( J_N \).

Next we prove (4.22). From (4.21), (4.24) and (4.25) we have for \( i \notin J_N \)
\[
\left| \left( \hat{h}_N^V \right)_i \right| = \frac{|\theta_{i/N} (\hat{h}_N^i)_i|}{\sqrt{\sum_{i=1}^N \theta_{i/N}^2 (\hat{h}_N^i)_i}} \leq \frac{|\theta_{i/N}|^{V-1} h}{\sqrt{2 - \frac{1}{\sqrt{N}}}} \leq N\theta \left( 1 - \frac{1}{c\sqrt{N}} \right)^V \leq \frac{c}{N^2},
\]
so
\[
(4.26) \quad \sum_{i \notin J_N} \left( \hat{h}_N^V \right)_i = o(1).
\]

For any \( m \in \mathcal{M}_N \) with \( |m| \leq 1 \) we may decompose \( m \) as
\[
(4.27) \quad m = \sum_{k=1}^{V-1} \alpha_k \hat{h}_N^k + \sum_{j=1}^{|J_N|} \gamma_j e_j,
\]
for some \( \alpha_1, \ldots, \alpha_{V-1}, \gamma_1, \ldots, \gamma_{|J_N|} \in \mathbb{R} \) with \( |\alpha_{V-1}| \leq 1 \). Thus using the linearity of the gradient (cf. (2.2))
\[
\nabla \tilde{H}_N (m) = \sum_{k=1}^{V-1} \alpha_k \nabla \tilde{H}_N (\hat{h}_N^k) + \sum_{j=1}^{|J_N|} \gamma_j \nabla H_N (e_j)
\]
\[
= \sum_{k=1}^{K-1} \alpha_k \frac{\hat{h}_N^{k+1}}{|\hat{h}_N^k|} \hat{h}_N^{k+1} + \sum_{j=1}^{|J_N|} \gamma_j 2\theta_{ij}^N e_{i_j}.
\]

Therefore for \( \hat{\sigma} \in \mathcal{M}^\perp_N \) we have
\[
\left( \beta \frac{1}{N} \nabla \tilde{H}_N (m) + \tilde{h}_N \right) \cdot \hat{\sigma} = \alpha_{V-1} \frac{|\hat{h}_N^V|}{|\hat{h}_N^{V-1}|} \hat{h}_N^V \cdot \hat{\sigma}.
\]

Now by (4.26) and the fact that \( \hat{\sigma}_i = 0 \) for \( i \in J_N \) we have that
\[
\hat{h}_N^V \cdot \hat{\sigma} = o(1).
\]
By (4.23) we have
\[ \frac{\tilde{h}\nu}{\tilde{h}\nu^{-1}} \leq 2\sqrt{2}, \]
so the claim (4.22) follows.

We will need a version of Lemma 10 where we integrate over the subspace perpendicular to \( M_N \).

Lemma 15. For any \( C > 0 \) and \( K > 0 \)
\[ \limsup_{N \to \infty} \sup_{\beta \in [0,C]} \left| \frac{1}{N} \log E_{U_N} \left[ \exp \left( \beta N \sum_{i=1}^{N} \theta_i/N \sigma_i^2 \right) \right] - F_K(\beta) \right| \leq \frac{c}{K}, \]
where \( U_N = M_N^\perp \) and \( M_N \) is the sequence from Lemma 14.

Proof. This can be shown to follow from Lemma 10 similarly to how Lemma 5 follows from Lemma 4. Let
\[ M = \lfloor N^{3/4} \rfloor. \]
One considers an orthonormal basis of \( \mathbb{R}^{N-M} \) such that the space \( U_N \) is spanned by the first \( N-M \) basis vectors, and notes that the \((N-M) \times (N-M)\) minor of the matrix \( D \) which in the standard basis is diagonal with \( D_{ii} = \theta_i/N \) has eigenvalues \( a_1, \ldots, a_{N-M} \) that by the interlacing inequality satisfy
\[ a_i = \theta_i/N + o(1) = \theta_i/(N-M) + o(1), \]
so that an estimate for \( E_{U_N} [\cdot] \) follows from Lemma 10 with \( N-M \) in place of \( N \). \( \square \)

Define a modified TAP free energy by replacing the Onsager correction \( \frac{1}{2} \beta^2 (1 - |m|^2) \) by \( F_K(\beta (1 - |m|^2)) \) to obtain
\[ \tilde{H}^K_{TAP}(m) = \beta \tilde{H}_N(m) + N m \cdot \tilde{h}_N + \frac{N}{2} \log \left( 1 - |m|^2 \right) + NF_K \left( \beta \left( 1 - |m|^2 \right) \right). \]
We have the following version of the upper bound Proposition 7 with \( \tilde{H}^K_{TAP}(m) \) in place of \( \tilde{H}_{TAP}(m) \) and without a Plefka condition.

Proposition 16. For all \( K \geq 2 \) and \( \beta, h, h_1, h_2, \ldots \) as in Theorem 1 we have
\[ (4.28) \quad \tilde{F}_N(\beta, h_N) \leq \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1} \tilde{H}^K_{TAP}(m) + \frac{c}{K}, \]
for large enough \( N \).

Proof. Let \( M_N \) be the space from Lemma 14 and let \( U_N = M_N^\perp \).
Let $M = \lfloor N^{3/4} \rfloor$. For any $\sigma \in \mathbb{R}^N$ let $m$ be the projection of $\sigma$ onto $\mathcal{M}_N$ and $\hat{\sigma} = \sigma - m \in \mathcal{U}_N$. Recentering the Hamiltonian around $m$ (cf. (3.8)-(3.9)) we have that

\begin{equation}
E \left[ \exp \left( \beta \tilde{H}_N (\sigma) + N \tilde{h}_N \cdot \sigma \right) \right] = E \left[ \exp \left( N \beta H_N (m) + N \tilde{h}_N \cdot m + N \left( \frac{1}{N} \nabla \tilde{H}_N (m) + \tilde{h}_N \right) \cdot \hat{\sigma} + \beta \tilde{H}_N (\hat{\sigma}) \right) \right].
\end{equation}

By Lemma 14 this is at most

\begin{equation}
e^{o(N)} E \left[ \exp \left( N \beta H_N (m) + N \tilde{h}_N \cdot m + \beta \tilde{H}_N (\hat{\sigma}) \right) \right].
\end{equation}

Note that the the $E \left[ \cdot |m| \right]$-law of $\hat{\sigma}$ is the uniform distribution on sphere in the subspace $\mathcal{U}_N$ of radius $\sqrt{1 - |m|^2}$. Thus using also (2.1) this equals

\begin{equation}
E \left[ \exp \left( N \beta H_N (m) + N \tilde{h}_N \cdot m \right) E_{\mathcal{U}_N} \left[ \beta \left( 1 - |m|^2 \right) \tilde{H}_N (\sigma) \right] \right].
\end{equation}

By Lemma 15 this is at most

\begin{equation}
E \left[ \exp \left( N \beta H_N (m) + N \tilde{h}_N \cdot m + \mathcal{F}_K \left( \beta \left( 1 - |m|^2 \right) \right) \right) \right] e^{o(N)+\frac{\pi}{2}}.
\end{equation}

Using (2.9) the $E$-integral equals

\begin{equation}
a_N \int_{m:|m|<1} \left( 1 - |m|^2 \right)^{\frac{N-M-2}{2}} \exp \left( N \beta H_N (m) + N \tilde{h}_N \cdot m + N \mathcal{F}_K \left( \beta \left( 1 - |m|^2 \right) \right) \right) dm,
\end{equation}

where $a_N = \frac{1}{\sqrt{\frac{N-M-2}{2}} \Gamma \left( \frac{N-M-2}{2} \right)}$ and the integral is $M$-dimensional against Lebesgue measure on $\mathcal{M}_N$. Since the volume of the unit ball in dimension $M$ is bounded independently of $M$ the integral is most

\begin{equation}
c \exp \left( \sup_{m:|m|<1} \left\{ \tilde{H}^K_{TAP} (m) + \left( M + 2 \right) \log \left( 1 - |m|^2 \right) \right\} \right).
\end{equation}

There is a $\delta$ depending only on $\beta$ and $h$ such that the supremum is always achieved for $|m| < 1 - \delta$, since all terms in the supremum not involving log are bounded by $cN$. Thus (4.34) is at most

\begin{equation}
c \exp \left( \sup_{m:|m|<1} \tilde{H}^K_{TAP} (m) + cM \right).
\end{equation}

Combining (4.30)-(4.35) and noting that $\log a_N = o(N)$ the claim follows. $\square$
We can now prove the upper bound Proposition for free energy of the diagonal and deterministic Hamiltonian $\tilde{H}_N(\sigma)$, by showing that the sup in (4.28) is bounded above by that in (4.3).

**Proof of Proposition**

Fix $K \geq 2$. For any $N \geq 1$, consider the variational problem

$$\sup_{m \in \mathbb{R}^N : |m| < 1} \tilde{H}_{TAP}^K(m).$$

Any local maximum $m$ of $\tilde{H}_{TAP}^K(m)$ must satisfy

$$\nabla \tilde{H}_{TAP}^K(m) = 0,$$

and

(4.36) $\nabla^2 \tilde{H}_{TAP}^K(m)$ is negative semi-definite.

The gradient of $\tilde{H}_{TAP}^K$ is

$$\nabla H_{TAP}^K(m) = \beta \nabla \tilde{H}_N(m) + \tilde{h}_N - Nm \left( \frac{1}{1 - |m|^2} + 2\beta F'_K(\beta (1 - |m|^2)) \right).$$

By Lemma we have for all $m$ that

$$\nabla \tilde{H}_{TAP}^K(m) = \beta \nabla \tilde{H}_N(m) - N2\beta m \lambda_K(\beta (1 - |m|^2)).$$

Thus the Hessian $\nabla^2 \tilde{H}_{TAP}^K(m)$ equals

$$\beta \nabla^2 \tilde{H}_N(m) - N2\beta I \lambda_K(\beta (1 - |m|^2)) + 4\beta^2 Nmm^T \lambda_K'(\beta (1 - |m|^2)).$$

For any local maximum $m$ let

$$A = \frac{1}{2N} \nabla^2 \tilde{H}_N(m) - I \lambda_K(\beta (1 - |m|^2)),$$

and

$$B = 2m(m)^T \lambda_K'(\beta (1 - |m|^2)).$$

Since $B$ is of rank one, the second largest eigenvalue $a_{N-1}$ of $A$ is bounded above by the largest eigenvalue of $A + B$. The latter matrix is the Hessian at $m$ multiplied by a positive scalar, so all its eigenvalues are non-positive. Thus $a_{N-1} \leq 0$. Furthermore $\frac{1}{2N} \nabla^2 \tilde{H}_N(m) = \frac{1}{N} D$ where $D$ is the diagonal matrix with $D_{ii} = \theta_{i/N}$, so the eigenvalues of $A$ are $\theta_{i/N} - \lambda_K(\beta (1 - |m|^2))$. This shows that

$$\lambda_K(\beta (1 - |m|^2)) \geq \theta_{1-\frac{1}{N}},$$

at $m$ which are local maxima. Since

$$\theta_{1-1/N} = \sqrt{2} + o(1),$$
it follows from Lemma 12 that we must have for such \( m \)
\[
\beta \left( 1 - |m|^2 \right) \leq \frac{1}{\sqrt{2}}, \text{ that is } \beta (m) \leq \frac{1}{\sqrt{2}},
\]
(provided \( N \) large enough depending on \( K \)), and by Lemma 13
\[
\mathcal{F}_K (\beta (1 - |m|^2)) \leq \frac{1}{2} \beta^2 (1 - |m|^2)^2 + \varepsilon_K,
\]
where \( \lim_{K \to \infty} \varepsilon_K = 0 \). Thus from (4.28) it holds for such \( N \) that
\[
\tilde{F}_N (\beta, h_N) \leq \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{\sqrt{2}}} \tilde{H}_{\text{TAP}} (m) + \varepsilon_K + cK.
\]
We have shown that
\[
\limsup_{N \to \infty} \left\{ \frac{1}{N} \sup_{m \in \mathbb{R}^N : |m| < 1, \beta(m) \leq \frac{1}{\sqrt{2}}} \tilde{H}_{\text{TAP}} (m) - \tilde{F}_N (\beta, h_N) \right\} \leq \varepsilon_K + \frac{c}{K},
\]
for all \( K \geq 2 \). Since the left-hand side is independent of \( K \), it is in fact at most 0. This implies (4.3).

This also completes the proof of the main upper bound Proposition 6. Together with the lower bound Proposition 3 this proves our main result Theorem 1.

5. Solution of the TAP-Plefka variational problem

In this section we prove Lemma 2. By (2.1) it follows from a result for the maximum of the Hamiltonian with external field on the unit sphere which we now state.

Lemma 17. For \( h, h_1, h_2, \ldots \) as in Theorem 2 we have
\[
(5.1) \quad \sup_{\sigma : |\sigma| = 1} \left\{ \beta \frac{1}{N} H_N (\sigma) + h_N \cdot \sigma \right\} \to \sqrt{h^2 + 2\beta^2},
\]
in probability.

Proof. We work in the diagonalizing basis of \( S_N \) and note that the left-hand side of (5.1) equals
\[
(5.2) \quad \sup_{\sigma : |\sigma| = 1} \left\{ \beta \sum_{i=1}^{N} \theta_i/N \sigma_i^2 + \tilde{h}_N \cdot \sigma \right\} + o(1),
\]
where, as in Section 4.1, \( \tilde{h}_N \) is the vector \( h_N \) written in the diagonalizing basis and we have used (2.5). The case \( h = 0 \) then follows trivially since \( \theta_1 = \sqrt{2} \), so we assume
in the sequel that $h > 0$. For any $\lambda > \sqrt{2}$ let

$$\sigma_i(\lambda) = \frac{1}{2\beta} \left( \frac{\overline{h}_N}{\lambda - \theta_i/N} \right).$$

Using Lagrange multipliers the maximizer of (5.2) can be shown to be $\sigma_i = \sigma_i(\lambda_N)$ where $\lambda_N > \sqrt{2}$ is the number such that $\sum_{i=1}^N \sigma_i^2(\lambda_N) = 1$. By rotational symmetry the $\mathbb{P}$-law of $(\overline{h}_N)_i$ is that of a uniform random vector on $\{x \in \mathbb{R}^N : |x| = h\}$. Using this one can show that for any $\lambda > \sqrt{2}$

$$\left| \sum_{i=1}^N \sigma_i(\lambda)^2 - \frac{1}{2\beta} \int \frac{1}{\overline{h}_N} \frac{\mu(dx)}{(\lambda - \theta_i/N)^2} \right| \to 0, \text{ in probability.}$$

Also for $\lambda > \sqrt{2}$

$$\sum_{i=1}^N \frac{1}{(\lambda - \theta_i/N)^2} \to \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\mu(dx)}{(\lambda - x)^2} = \frac{\lambda}{\sqrt{\lambda^2 - 2}} - 1,$$

and since for

$$\tilde{\lambda} = \sqrt{\frac{2}{1 - \left(1 + \frac{4\beta^2}{h^2}\right)}}^{-2},$$

and $\lambda = \tilde{\lambda}$ we have $\lambda/\sqrt{\lambda^2 - 2} - 1 = 2\beta/h^2$, it follows that

$$\lambda_N \to \tilde{\lambda}, \text{ in probability.}$$

Similarly for any $\lambda > \sqrt{2}$ we have that

$$\sum_{i=1}^N \theta_i/N \sigma_i(\lambda)^2 \to \frac{h^2}{2\beta} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x}{(\lambda - x)^2} \mu(dx) = \frac{h^2}{2\beta} \left( \frac{\lambda^2 - 1}{\sqrt{\lambda^2 - 2}} - \lambda \right),$$

and

$$\sum_{i=1}^N \left( \overline{h}_N \right)_i \sigma_i(\lambda) \to \frac{h^2}{2\beta} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\mu(dx)}{\lambda - x} = \frac{h^2}{2\beta} \left( \lambda - \sqrt{\lambda^2 - 2} \right),$$

both in probability. This shows that

$$\beta \sum_{i=1}^N \theta_i/N \sigma_i(\lambda_N)^2 + \overline{h}_N \cdot \sigma(\lambda_N) \to \frac{h^2}{2\beta} \left( \frac{\tilde{\lambda}^2 - 1}{\sqrt{\tilde{\lambda}^2 - 2}} - \tilde{\lambda} \right) + \frac{h^2}{2\beta} \left( \tilde{\lambda} - \sqrt{\tilde{\lambda}^2 - 2} \right),$$

in probability, and the right-hand side simplifies to $\sqrt{h^2 + 2\beta^2}$. \qed
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