Cohomological extension of $\text{Spin}(7)$–invariant super Yang–Mills theory in eight dimensions

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Abstract

It is shown that the $\text{Spin}(7)$–invariant super Yang–Mills theory in eight dimensions, which relies on the existence of the Cayley invariant, permits the construction of a cohomological extension, which relies on the existence of the eight–dimensional analogue of the Pontryagin invariant arising from a quartic chiral primary operator.

1 Introduction

Topological quantum field theory (TQFT) has attracted a lot of interest over the last years, both for its own sake and due to their connection with string theory (for a review, see, e.g., [1]). Particularly interesting are TQFT’s in $D = 2$ [2], because of their connection with $N = 2$ superconformal theories and with Calabi–Yau moduli spaces [3]. Considerable impact on physics and mathematics has had Witten’s construction of topological Yang–Mills theory in $D = 4$ and the discovery of its relation with the Donaldson map, which relates the de Rham cohomology groups on four–manifolds with those on the moduli space of quaternionic instantons, as well as its relation to the topologically twisted super Yang–Mills theory [4].

Recently, the construction of cohomological gauge theories on manifolds of special holonomy in $D > 4$ have received considerable attention, too [5, 6, 7]. These theories, which arise without the necessity for a topological twist, acquire much of the characteristics of a TQFT. However, such theories are not fully topological, since they are only invariant under such metric variations which do not change the reduced holonomy structure. For $D = 8$ examples of cohomological gauge theories have been constructed for the cases when the holonomy group in $SO(8)$ is either $\text{Spin}(7)$ [3, 5, 6] (Joyce manifolds) or $\text{Spin}(6) \sim SU(4)$ [5] (Calabi–Yau four–folds) or $\text{Spin}(5) \sim Sp(4)$ [8] (hyper–Kähler eight–folds).

At present, the physical content of these theories has not been entirely revealed, especially, since they are not renormalizable. But, recent developments of string theory have renewed the interest in super Yang–Mills theories (SYM) in $D > 4$, particularly because of their crucial role in the study of D–branes and in the matrix approach to M–theory. It is widely believed that the low–energy effective world volume theory of D–branes obtains through dimensional reduction of $N = 1$, $D = 10$ SYM [9]. Hence, the above mentioned cohomological theories in $D = 8$ provide effective field theories on the world volume of Euclidean Dirichlet 7–branes wrapping around manifolds of special holonomy. In order to improve their renormalizibility, counterterms are needed at high energies arising from string theory compactifications down to eight dimensions. Such terms were computed in [10]. In [11] it was argued that the complete string–corrected counterterms for the $\text{Spin}(7)$–invariant theory, whose construction relies on the existence of the Cayley invariant, should be still cohomological.

In this paper we show — without going beyond the scope of a cohomological theory — that the $\text{Spin}(7)$–invariant theory, in fact, has a cohomological extension. It relies on the existence
of the eight–dimensional analogue of the Pontryagin invariant which arises from the primary operator $W_0 = \frac{1}{4} \text{tr} \phi^4$. This extension, which in flat space is uniquely determined by the shift and vector supersymmetries, can be related to the one–loop string–corrected counterterms.

The paper is organized as follows: In Sect. 2, we briefly describe the formulation of the $\text{Spin}(7)$–invariant $N_T = 1$, $D = 8$ SYM. By analyzing the structure of the observables it is argued that this theory permits the construction of a cohomological extension which relies on the existence of the eight–dimensional analogue of the Pontryagin invariant. In Sect. 3, by performing a dimensional reduction to $D = 4$, it is shown that the matter–independent part of the resulting half–twisted theory [12] can be obtained from the operator $\hat{W}_0 = \frac{1}{2} \text{tr} \phi^2$ by means of a relationship associated with the vector supersymmetry. In Sect. 4, by generalizing this relationship to $D = 8$, we give the cohomological extension of the $\text{Spin}(7)$–invariant theory in the Landau type gauge. Appendix A contains the Euclidean spinor conventions which are used. Appendix B lists the off–shell transformation rules for the matter–independent part of the half–twisted theory. Appendix C gives, in some detail, the derivation of the cohomological extension in the Feynman type gauge.

2 $\text{Spin}(7)$–invariant, $D = 8$ Euclidean super Yang–Mills theory

An eight–dimensional analogue of Donaldson–Witten theory on $\text{Spin}(7)$ holonomy Joyce manifold, which localizes onto the moduli space of octonionic instantons, has been constructed in [5, 6]. This theory, just as Donaldson–Witten theory on $SU(2)$ holonomy Joyce–Kähler two–fold, is topological without twisting. In fact, it is invariant under metric variations preserving the $\text{Spin}(7)$ structure. In flat space the theory arises from the $N = 2$, $D = 8$ Euclidean SYM by reducing the $SO(8)$ rotation invariance to $\text{Spin}(7)$. This reduction has been explicitly carried out in [13] (preserving hermiticity and imposing some extra constraints) and in [14] (relaxing the reality conditions on fermions without introducing extra constraints). Thereby, extensive use is made of the octonionic algebra together with their compatibility with $\text{Spin}(7)$ invariance, generalized self–duality and chirality.

The action of the $\text{Spin}(7)$–invariant theory reads

\[ S = \int_E d^8 x \text{tr} \left\{ \frac{1}{8} \Theta^{abcd} F_{ab} F_{cd} - 2 D^a \bar{\phi} D_a \phi - 2 \chi^{ab} D_a \psi_b + 2 \eta D^a \psi_a ight\} + 2 \bar{\phi} \{ \psi^a, \psi_a \} + \frac{1}{4} \phi \{ \chi^{ab}, \chi_{ab} \} + 2 \phi \{ \eta, \eta \} - 2 \{ \phi, \bar{\phi} \}^2 \],

(1)

where $F_{ab} = \partial_a A_b + [A_a, A_b]$ and $D_a = \partial_a + [A_a, \cdot ]$. Here, the (unnormalized) projector $\Theta_{abcd}$ projects any antisymmetric second rank tensor onto its self–dual part $\eta$ according to the decomposition $28 = 7 \oplus 21$ of the adjoint representation of $SO(8) \sim SO(8)/\text{Spin}(7) \otimes \text{Spin}(7)$. It
satisfies the relations \[14\]
\[
\frac{1}{2} (\Theta_{abc} \Theta_{cfg}^g - \Theta_{abf} \Theta_{cdg}^g) = -\Theta_{efc} \delta_{bd} + \Theta_{efd} \delta_{bc} + \Theta_{efb} \delta_{cd} - \Theta_{efb} \delta_{ac} \\
+ \Theta_{abc} \delta_{df} - \Theta_{abf} \delta_{ce} - \Theta_{abf} \delta_{de} + \Theta_{abf} \delta_{ce} \\
- \Theta_{cdg} \delta_{bf} + \Theta_{cdg} \delta_{af} + \Theta_{cdg} \delta_{be} - \Theta_{cdg} \delta_{ae},
\]
enclosing all the properties of the structure constants entering the octonionic algebra \[15\].

On–shell, upon using the equation of motion for \( \chi^{ab} \), the action (1) can be recast into the \( \mathcal{Q} \)-exact form, \( S = Q \Psi \), where the gauge fermion
\[
\Psi = \int d^8 x \, \text{tr} \left\{ \chi^{ab} (F_{ab} - \frac{1}{16} \Theta_{abcd} F^{cd}) - 2 \psi^a D_a \phi - 2 [\eta, \tilde{\phi}] \phi \right\}
\]
is uniquely fixed by requiring its invariance under vector supersymmetry, \( Q_a \Psi = 0 \).

The full set of supersymmetry transformations, generated by the supercharges \( Q, Q_a \) and \( Q_{ab} = \frac{1}{6} \Phi_{abcd} Q^{cd} \), which leave the action (1) invariant, are given by \[14\],
\[
\begin{align*}
Q A_a & = \psi_a, & Q \psi_a & = D_a \phi, \\
Q \phi & = 0, & Q \bar{\phi} & = \eta, \\
Q \eta & = [\bar{\phi}, \phi], & Q \chi_{ab} & = \frac{1}{4} \Theta_{abcd} F^{cd}, \\
Q_a A_b & = \delta_{ab} \eta + \chi_{ab}, & Q_a \psi_b & = F_{ab} - \frac{1}{4} \Theta_{abcd} F^{cd} + \delta_{ab} [\phi, \tilde{\phi}], \\
Q_a \phi & = \psi_a, & Q_a \bar{\phi} & = 0, \\
Q_a \bar{\eta} & = D_a \bar{\phi}, & Q_a \chi_{cd} & = \Theta_{abcd} D^b \bar{\phi}
\end{align*}
\]
and
\[
\begin{align*}
Q_{ab} A_c & = -\Theta_{abcd} \psi^d, & Q_{ab} \psi_c & = \Theta_{abcd} D^d \phi, \\
Q_{ab} \phi & = 0, & Q_{ab} \bar{\phi} & = \chi_{ab}, \\
Q_{ab} \bar{\eta} & = -\frac{1}{4} \Theta_{abcd} F^{cd}, & Q_{ab} \chi_{cd} & = \frac{1}{4} \Theta_{abcg} \Theta_{cdg}^g F^{ef} + \Theta_{abcd} [\tilde{\phi}, \phi].
\end{align*}
\]

The \textit{on–shell} algebra of the supercharges \( Q, Q_a \) and \( Q_{ab} \) reads
\[
\begin{align*}
\{Q, Q\} & \equiv -2 \delta_G (\phi), & \{Q, Q_a\} & \equiv \partial_a + \delta_G (A_a), & \{Q_a, Q_b\} & \equiv -2 \delta_{ab} \delta_G (\bar{\phi}), \\
\{Q_a, Q_b\} & \equiv 0, & \{Q_{ab}, Q_c\} & \equiv \Theta_{abcd} (\partial^d + \delta_G (A^d)), & \{Q_{ab}, Q_{cd}\} & \equiv -2 \Theta_{abcd} \delta_G (\phi),
\end{align*}
\]
where \( \delta_G (\varphi) \) denotes a gauge transformation with field–dependent parameter \( \varphi = (A_a, \phi, \bar{\phi}) \), being defined by \( \delta_G (\varphi) A_a = -D_a \varphi \) and \( \delta_G (\varphi) X = [\varphi, X] \) for all the other fields. (The symbol \( \equiv \) means that the corresponding relation is fulfilled only \textit{on–shell}.)

The crucial hint, supporting our claim that (1) allows for a cohomological extension, comes from the structure of the topological observables. Namely, in terms of differential forms, the method of descent equation implies, starting from the primary operator \( \tilde{W}_0 = \frac{1}{2} \text{tr} \phi^2 \), the following ladder of \( k \)-forms \( \tilde{W}_k \) \( (4 \leq k \leq 8) \) with ghost number \( 8 - k \) \[5\] \[6\],
\[
\begin{align*}
\tilde{W}_4 & = \Phi \wedge \text{tr} \left( \frac{1}{2} \phi^2 \right), \\
\tilde{W}_5 & = \Phi \wedge \text{tr} \left( \psi \phi \right), \\
\tilde{W}_6 & = \Phi \wedge \text{tr} \left( F \phi + \frac{1}{2} \psi \wedge \psi \right), \\
\tilde{W}_7 & = \Phi \wedge \text{tr} \left( F \wedge \psi + \psi \wedge \psi \right), \\
\tilde{W}_8 & = \Phi \wedge \text{tr} \left( \frac{1}{2} F \wedge F \right),
\end{align*}
\]
with $\psi = \psi_a e^a$, $F = \frac{1}{2} F_{ab} e^a \wedge e^b$ and $\Phi = \frac{1}{24} \Phi_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$ ($e^a = e_\mu^a dx^\mu$), where $e_\mu^a$ is the 8–bein on the $\text{Spin}(7)$–holonomy Joyce manifold $\mathcal{M}$ endowed with metric $g_{\mu\nu}$. The trace is taken in $\text{Lie}(G)$. These $k$–forms obey the following descent equations,

$$\begin{align*}
0 &= Q\tilde{W}_4, \\
 d\tilde{W}_k &= Q\tilde{W}_{k+1}, \quad 4 \leq k \leq 7, \\
 d\tilde{W}_8 &= 0,
\end{align*}$$

which are typical for any cohomological gauge theory.

Hence, if $\gamma$ is a $k$–dimensional homology cycle on $\mathcal{M}$ then the integrated descendants $\tilde{I}_k(\gamma) = \int_\gamma \tilde{W}_k$ ($4 \leq k \leq 7$) and $\tilde{I}_8 = \int_\mathcal{M} \tilde{W}_8$ are $Q$–invariant, $Q\tilde{I}_k(\gamma) = \int_\gamma Q\tilde{W}_k = \int_\gamma d\tilde{W}_{k-1} = 0$. They depend, up to a $Q$–exact term, only upon the homology class of $\gamma$, i.e., when adding to $\gamma$ a boundary term $\partial\alpha$, then $\tilde{I}_k(\gamma)$ remains unaltered, modulo a $Q$–exact term, $\tilde{I}_k(\gamma + \partial\alpha) = \int_{\gamma + \partial\alpha} \tilde{W}_k = \tilde{I}_k(\gamma) + \int_\alpha d\tilde{W}_k = \tilde{I}_k(\gamma) + \int_\alpha Q\tilde{W}_{k+1} = \tilde{I}_k(\gamma)$. Here, the integrated descendant $\tilde{I}_8$ is the Cayley invariant, being invariant under a certain class of metric variations which do not change the reduced $\text{Spin}(7)$ structure.

By means of the same method, starting from the primary operator $W_0 = \frac{1}{4} \text{tr} \phi^4$, one can derive the following ladder of $k$–forms $W_k$ ($0 \leq k \leq 8$) with ghost number $8 - k$, \footnote{In order to condense the notation, we introduced a symmetrized trace, $\text{tr}_{(s)}$, which is defined in Appendix C.}

$$\begin{align*}
W_0 &= \text{tr}_{(s)} \left( \frac{1}{4} \phi^4 \right), \\
W_1 &= \text{tr}_{(s)} \left( \psi \phi^3 \right), \\
W_2 &= \text{tr}_{(s)} \left( F \phi^3 + \frac{1}{2} \psi \wedge \psi \phi^2 \right), \\
W_3 &= \text{tr}_{(s)} \left( F \wedge \psi \phi^2 + \psi \wedge \psi \wedge \psi \right), \\
W_4 &= \text{tr}_{(s)} \left( \frac{1}{2} F \wedge F \phi^2 + F \wedge \psi \wedge \psi \phi + \frac{1}{4} \psi \wedge \psi \wedge \psi \wedge \psi \right), \\
W_5 &= \text{tr}_{(s)} \left( \frac{1}{2} F \wedge F \wedge \psi \phi + F \wedge \psi \wedge \psi \wedge \psi \right), \\
W_6 &= \text{tr}_{(s)} \left( F \wedge F \wedge F \phi + \frac{1}{2} F \wedge F \wedge \psi \wedge \psi \right), \\
W_7 &= \text{tr}_{(s)} \left( F \wedge F \wedge F \wedge \psi \right), \\
W_8 &= \text{tr}_{(s)} \left( \frac{1}{4} F \wedge F \wedge F \wedge F \right),
\end{align*}$$

(9)

which obey similar descendant equations as before. Again, we observe that the integrated descendant $\tilde{I}_8 = \int_\mathcal{M} \tilde{W}_8$ yields a topological invariant which is now unchanged under arbitrary metric variations and which may be regarded as the eight–dimensional analogue of the Pontryagin invariant. This suggests that the $\text{Spin}(7)$–invariant theory permits, in fact, a cohomological extension, $S_{\text{ext}} = Q\Psi_{\text{ext}}$, which possibly should be constructed by the help of the primary operator $W_0 = \frac{1}{4} \text{tr} \phi^4$ and with $\Psi_{\text{ext}}$ being uniquely fixed by the requirement under vector supersymmetry, $Q_a \Psi_{\text{ext}} = 0$.

### 3 Dimensional reduction to four dimensions

In order to get an idea how such an extension $S_{\text{ext}}$ could be constructed with the help of the primary operator $W_0 = \frac{1}{4} \text{tr} \phi^4$, we look at the matter–independent part of the topologically half–twisted theory which, similarly to Ref. \cite{12}, is obtained by reducing to four dimensions the $\text{Spin}(7)$–invariant theory. To this end, for $1 \leq a, b \leq 4$, we group the components of $A_a$, $\psi_a$ and $\chi_{ab}$ into a vector isosinglet $A_A$, a vector isosinglet $\psi_A$, and a self–dual tensor isosinglet $\chi_{AB} = \frac{1}{2} \epsilon_{ABCD} \chi^{CD}$ ($A, B = 1, 2, 3, 4$), respectively, and for $5 \leq a, b \leq 8$, into a left–handed two–spinor isodoublent $G_{a i}^i$, a left–handed isodoublent $\lambda^i_a$ and a right–handed isodoublent $\zeta^{i a}_{\dot{a}}$ ($i = 1, 2$), respectively (for the two–spinor conventions, see, Appendix A). The index $i$ is raised and lowered as follows: $\epsilon^i \varphi^j = \varphi^i$ and $\varphi^i \epsilon_{ij} = \varphi_j$, where $\epsilon_{ij}$ is the invariant tensor of the group $\text{SU}(2)$, $\epsilon^{12} = \epsilon_{12} = 1$.
Then, for that action of the half-twisted theory one gets, according to the prediction \[9\],

\[
S_{\text{red}} = S_0 + \int_E d^4x \left\{ \frac{1}{2} D^A G^i_A D_A G^i_A + \frac{1}{2}[G^i_A, G^j_B] [G^i_A, G^j_B] - 2 [G^i_A, \bar{\phi}] [G^i_A, \phi] \\
+ 2 \phi \{\lambda^\alpha_i, \lambda^\alpha_i\} + 2 \phi \{\zeta^{\alpha \dot{\alpha}}, \zeta^{\alpha \dot{\alpha}}\} - \frac{1}{2} \chi^{AB} (\sigma_{AB})^{\alpha \beta} [G^i_A, \lambda^{\beta i}] \\
- 2 i \zeta^{\alpha \dot{\alpha}} (\sigma^A)_{\alpha \beta} D_A \lambda^{\beta i} + 2 i \zeta^{\alpha \dot{\alpha}} (\sigma^A)_{\alpha \beta} [G^i_A, \psi_A] + 2 \eta [G^i_A, \lambda^{\beta i}] \right\},
\]

(10)

where the matter-independent part \(S_0\) is just the Donaldson–Witten action \[6\],

\[
S_0 = \int_E d^4x \left\{ \frac{1}{8} \Theta^{ABCD} F_{AB} F_{CD} - 2 D^A \bar{\phi} D_A \phi - 2 \chi^{AB} D_A \psi_B + 2 \eta D^A \psi_A \\
+ 2 \bar{\phi} \{\psi^A, \psi_A\} + \frac{1}{2} \phi \{\chi^{AB}, \chi_{AB}\} + 2 \phi \{\eta, \eta\} - 2 [\phi, \bar{\phi}]^2 \right\}.
\]

(11)

Hence, the result of compactifying the \(Spin(7)\)-invariant theory \[11\] to four dimensions gives the Donaldson–Witten theory with matter in the adjoint representation. On the other hand, Donaldson–Witten theory with matter, after some rearrangements of the spinor fields, so that \(S_{\text{red}}\) becomes real, gets unified in eight dimensions (the resulting theory is very similar to the non–Abelian version of the Seiberg–Witten monopole theory).

The (unnormalized) projector

\[
\Theta^{ABCD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + \epsilon_{ABCD}, \quad \frac{1}{4} \Theta^{ABEF} \Theta_{CD}^{\ EF} = \Theta^{ABCD},
\]

(12)

projects any antisymmetric second rank tensor onto its self-dual part \(3\) according to the decomposition \(6 = 3 \oplus 3\) of the adjoint representation of \(SO(4) \sim Spin(3) \otimes Spin(3)\). It obeys the relations

\[
\frac{1}{2} (\Theta^{ABEG} \Theta_{CDF}^{\ G} - \Theta^{ABFG} \Theta_{CDE}^{\ G}) = \Theta_{EFAC} \delta_{BD} - \Theta_{EFAD} \delta_{BC} \\
- \Theta_{EFBC} \delta_{AD} + \Theta_{EFBD} \delta_{AC} \\
= \Theta_{ABCE} \delta_{DF} - \Theta_{ABDF} \delta_{CE} \\
- \Theta_{ABCF} \delta_{DE} + \Theta_{ABDE} \delta_{CF},
\]

(13)

with \(\epsilon_{ABCD}\) being the Levi–Civita tensor in four dimensions.

Furthermore, for the dimensionally reduced transformation rules, generated by \(Q, Q_A\) and \(Q_{\dot{A}} = \frac{1}{2} \epsilon_{ABCD} Q^{CD}\), from \[11\, - \, 11\) one gets

\[
Q A_A = \psi_A, \\
Q \bar{\phi} = D_A \phi, \\
Q \phi = 0, \\
Q \bar{\phi} = \eta, \\
Q \eta = [\bar{\phi}, \phi], \\
Q \chi_{AB} = \frac{1}{4} \Theta_{ABCD} F^{CD} + \frac{1}{4} (\sigma_{AB})^{\alpha \beta} [G^i_A, G^i_B], \\
Q G^i_A = \lambda^i_A, \\
Q \lambda^i_A = [G^i_A, \phi], \\
Q \zeta^{\alpha \dot{\alpha}} = \frac{1}{2} i (\sigma^A)_{\alpha \beta} D_A G^{\dot{\beta}i},
\]

(14)
\[ Q_{AB} = \delta_{AB} \eta + \chi_{AB}, \]
\[ Q_A \psi_B = F_{AB} - \frac{1}{4} \Theta_{ABCD} F^{CD} + \delta_{AB} [\phi, \bar{\phi}] - \frac{1}{4} (\sigma_{AB})^{\alpha\beta} [G_{\alpha}^i, G_{\beta i}], \]
\[ Q_A \phi = \psi_A, \]
\[ Q_A \bar{\phi} = 0, \]
\[ Q_A \eta = D_A \bar{\phi}, \]
\[ Q_A \chi_{CD} = \Theta_{ABCD} D^B \bar{\phi}, \]
\[ Q_A G_{\alpha}^i = i (\sigma_A)_{\alpha\beta} \bar{C}_{\beta i}, \]
\[ Q_A \lambda_{\alpha} = \frac{1}{2} D_A G_{\alpha}^i - \frac{1}{2} (\sigma_{AB})_{\alpha\beta} D^B G_{\beta i}, \]
\[ Q_A \zeta_{\alpha} = i (\sigma_A) \bar{\alpha}_{\alpha} [G_{\alpha}^i, \bar{\phi}], \]
\[ \text{(15)} \]
\[ Q_{AB} A_C = -\Theta_{ABCD} \psi_D, \]
\[ Q_{AB} \psi_C = \Theta_{ABCD} D^D \phi, \]
\[ Q_{AB} \phi = 0, \]
\[ Q_{AB} \bar{\phi} = \chi_{AB}, \]
\[ Q_{AB} \eta = -\frac{1}{4} \Theta_{ABCD} F^{CD} - \frac{1}{4} (\sigma_{AB})^{\alpha\beta} [G_{\alpha}^i, G_{\beta i}], \]
\[ Q_{AB} \chi_{CD} = \frac{1}{4} \Theta_{ABCD} \Theta_{DFG} F^{EF} + \Theta_{ABCD} [\bar{\phi}, \phi] + \frac{1}{4} (\sigma_{AB})^{\alpha\gamma} (\sigma_{CD})^{\beta\gamma} [G_{\alpha}^i, G_{\beta i}], \]
\[ Q_{AB} G_{\alpha}^i = (\sigma_{AB})_{\alpha\beta} \lambda_{\beta i}, \]
\[ Q_{AB} \lambda_{\alpha} = -(\sigma_{AB})_{\alpha\beta} [G_{\beta i}, \phi], \]
\[ Q_{AB} \zeta_{\alpha} = \frac{1}{2} i (\sigma_A) \bar{\alpha}_{\alpha} \Theta_{ABCD} D^D G_{\beta i}. \]
\[ \text{(16)} \]
Notice that by introducing an auxiliary field \( B_{AB} = \frac{1}{2} \epsilon_{ABCD} B^{CD} \) the matter–independent part of (14)–(16) can be closed off–shell (see, Appendix B).

The crucial point is that, on–shell, the matter–independent part \( S_0 \) can be obtained from the operator \( \hat{W}_0 = \frac{1}{2} \text{tr} \phi^2 \) as follows,
\[ S_0 \doteq S_{\text{top}}^{4D} - \frac{1}{4!} \epsilon_{ABCD} Q_A Q_B Q_C Q_D \int_E d^4 x \frac{1}{2} \text{tr} \phi^2 \doteq Q \Psi_0, \]
\[ \text{(17)} \]
where
\[ S_{\text{top}}^{4D} = \int_E d^4 x \text{tr} \left\{ \frac{1}{2} \epsilon_{ABCD} F_{AB} F_{CD} \right\} \]
\[ \text{(18)} \]
is the four–dimensional Pontryagin invariant. Moreover, on–shell, the gauge fermion \( \Psi_0 \) can be obtained from the prepotential \( V_0 = \frac{1}{2} \text{tr} \phi^2 \) according to
\[ \Psi_0 \doteq \frac{1}{4!} Q_A B^C Q^C B^A \int_E d^4 x \frac{1}{2} \text{tr} \bar{\phi}^2. \]
\[ \text{(19)} \]
On the other hand, the topological observables of the Donaldson–Witten theory can also be obtained from \( \hat{W}_0 \) by means of the following ladder of \( k \)-forms \( \hat{W}_k \) (\( 0 \leq k \leq 4 \)) with ghost number \( 4 - k \) \[ 4 \],
\[ \hat{W}_0 = \text{tr} \left( \frac{1}{2} \phi^2 \right), \]
\[ \hat{W}_1 = \text{tr} (\phi \phi), \]
\[ \hat{W}_2 = \text{tr} (F \phi + \frac{1}{2} \psi \wedge \psi), \]
\[ \hat{W}_3 = \text{tr} (F \wedge \psi + \psi \wedge \psi), \]
\[ \hat{W}_4 = \text{tr} \left( \frac{1}{2} F \wedge F \right). \]
\[ \text{(20)} \]
Hence, because \([3]\) is just the eight–dimensional analogue of \([20]\), the cohomological extension \(S_{\text{ext}}\) of the action \([1]\) should be precisely the eight–dimensional extension of \([17]–[19]\).

## 4 Cohomological extension \(S_{\text{ext}}\) in the Landau type gauge

As anticipated, for the cohomological extension \(S_{\text{ext}}\) we are looking for we make the following ansatz

\[
S_{\text{ext}} = S_{\text{top}}^{8D} - \frac{1}{8!} \epsilon^{abcd} \epsilon^{gh} Q_a Q_b Q_c Q_d Q_e Q_f Q_g Q_h \int_E d^8 x \frac{1}{4} \text{tr} \phi^4 = Q \Psi_{\text{ext}},
\]

with

\[
\Psi_{\text{ext}} = \frac{1}{8!} Q_a Q_b Q_c Q_d Q_e Q_f Q_g Q_h \int d^8 x \frac{1}{4} \text{tr} \phi^4,
\]

where

\[
S_{\text{top}}^{8D} = \int_E d^8 x \text{tr} \left\{ \frac{1}{64} \epsilon^{abcd} \epsilon^{gh} F_{ab} F_{cd} F_{ef} F_{gh} \right\}
\]

is the eight–dimensional analogue of the Pontryagin invariant.

Let us briefly comment on the possibilities to determine \(\Psi_{\text{ext}}\) either from \([21]\) or from \([22]\). The most favorable way seems to be to construct \(S_{\text{ext}}\) by means of \(Q\) and \(Q_{ab}\) (and not by means of \(Q_a\)) because then \(\Psi_{\text{ext}}\) can be obtained directly from the prepotential \(V_0 = \frac{1}{4} \text{tr} \phi^4\). However, despite the fact that the transformation rules \([4]–[6]\) look very similar to those of the matter–independent part of \([14]–[16]\), it is impossible to close the latter off–shell with a finite number of auxiliary fields. Apart from the general arguments given in \([16]\), this may be traced back to the fact that in the former case there are fewer algebraic identities among the \(S\)–exact form. However, proceeding in that less ideal way one is confronted with the tricky problem to verify the \(Q\)–exactness of \([21]\) and, therefore, the exposition of \(\Psi_{\text{ext}}\) becomes very complex (see, Appendix C).

Obviously, when evaluating \([21]\) in the Feynman type gauge one gets a huge number of terms belonging to the non–minimal sector which, however, are of no particular interest. Therefore, focusing only on terms belonging to minimal sector, we shall restrict ourselves to the Landau type gauge. In that gauge the action \([1]\) simplifies as follows:

\[
S' = Q \Psi', \quad \Psi' = \int_E d^8 x \text{tr} \left\{ \chi^{ab} F_{ab} - 2 \psi^a D_a \phi \right\},
\]

where the first term of \(\Psi'\) enforces the localization into the moduli space whereas the second term ensures that pure gauge degrees of freedom are projected out. One easily verifies that \(\Psi'\) is invariant under the vector supersymmetry \(Q_a\) (in the Landau type gauge)

\[
\begin{align*}
Q_a A_b &= 0, & Q_a \bar{\phi} &= 0, \\
Q_a \psi_b &= \bar{\psi}_a, & Q_a \eta &= D_a \bar{\phi}, \\
Q_a \bar{\psi}_b &= F_{ab}, & Q_a \chi_{cd} &= \Theta_{abcd} D^b \bar{\phi}.
\end{align*}
\]

For the cohomological extension \(S'_{\text{ext}}\) of \([21]\), whose computation is postponed to Appendix C, one obtains

\[
S'_{\text{ext}} = Q \Psi'_{\text{ext}}, \quad \Psi'_{\text{ext}} = \alpha \int_E d^8 x \text{tr} \left\{ \Phi^{[abcd] \chi_e f} F_{ab} F_{cd} F_{ef} - 12 \psi^a \Phi^{[abcd] D^b \bar{\phi} F_{ab} F_{cd}} \right\},
\]

\(\alpha\) being a constant.

\[\text{7}\]
α being an arbitrary constant. \( \Psi'_{\text{ext}} \) is invariant under the vector supersymmetry \([25]\) as well.  

In order to check this crucial property one needs the following identities:

\[
\begin{align*}
\frac{1}{2} e_{abcd} fgh \Phi_{m ngh} &= \Phi_{abcd} \delta^e_{[m} \delta^f_{n]} - \Phi_{bod[e} \delta^f_{[m} \delta^a_{n]} + \frac{1}{2} \Phi_{cde[f} \delta^a_{[m} \delta^b_{n]}, \\
- \frac{1}{6} \Phi_{d[ef} \delta^b_{[m} \delta^c_{n]} + \frac{1}{4} \Phi_{e[ab} \delta^c_{[m} \delta^d_{n]} &= \frac{1}{18} \Phi_{abcd} \delta^e_{[m} \delta^f_{n]}, \\
\frac{1}{24} e_{abcd} fgh \Phi_{m fgh} &= \Phi_{abcd}, \\
\frac{1}{24} e_{abcd} fgh \Phi_{efgh} &= \Phi_{abcd},
\end{align*}
\]

where the last one expresses the self–duality of \( \Phi_{abcd} \).

It is amusing to see that, in the Landau type gauge, the extension of \( \Psi' \) into \( \Psi'_{\text{ext}} \) is rather simple. Namely, it obtains either by performing in \([24]\) the replacements

\[
\Phi_{abcd} F_{ab} \rightarrow \Phi_{abcd} F_{ab} + \alpha \Phi_{abcd} F_{[ab} \Phi_{cd]ef}], \\
D^e \psi_e \rightarrow D^e \psi_e + 6 \alpha \Phi_{abcd} D^e (\psi_{ab} \Phi_{cd} + F_{ab} \psi_e F_{cd} + F_{ab} \Phi_{cd} \psi_e),
\]

or, equivalently, by changing (formally) in the same manner the self–duality gauge condition \( \Phi_{abcd} F^{ab} = 0 \) and the ghost gauge condition \( D^e \psi_e = 0 \).

Summarizing, we have shown that the Spin(7)–invariant super Yang–Mills theory, which relies on the existence of the Cayley invariant, permits the construction of a cohomological extension by the help of the operator \( W_0 = \frac{1}{4} \text{tr} \phi^4 \), which relies on the existence of the eight–dimensional analogue of the Pontryagin invariant.

With regard to this, a couple of interesting questions is left still open deserving a further study. So far, the Spin(7)–invariant theory was considered in flat space only. But, according to Berger’s classification \([17]\) a metric with Spin(7) holonomy on the simply connected eight–dimensional Riemannian manifold \( M \) with Euclidean signature admits a covariantly–constant spinor \( \zeta \). If such \( \zeta \) exists, the metric is automatically Ricci–flat. In addition, such metric has the Spin(7)–invariant closed Cayley four–form \( \Phi \) (for a given choice of the orientation of \( M \)). Conversely, if \( \Phi \) with respect to a metric on \( M \) is closed, then the metric has Spin(7) holonomy. Hence, the action \([24]\) and its extension \([20]\) can be considered on a curved manifold \( M \) with Spin(7) holonomy.

As is widely believed, super Yang–Mills theory in eight dimensions may arise as low–energy effective world volume theory on a Euclidean 7–brane in Type IIB string theory \([9]\). Thus, as it was pointed out in \([9]\), the Spin(7)–invariant theory in curved space can be considered as a theory being obtained by wrapping an Euclidean 7–brane of Type IIB string theory around a manifold with Spin(7) holonomy. However, such a theory is not renormalizable and, therefore, extra degrees of freedom are needed at high energies. It seems to be natural to assume that such extra degrees of freedom are given by the counterterms arising from string theory after compactification to eight dimensions. In another context, in \([11]\) some arguments were given that all the string–corrected eight–dimensional counterterms are still \( Q \)-exact. So, one may ask whether in the Landau type gauge the cohomological extension \([24]\) in curved space differs from the string–corrected one–loop counterterm (identifying \( \alpha \) with the string tension) only by replacing the Levi–Civita tensor \( \epsilon_{abcdefgh} \) through a certain \( SO(8) \)-invariant tensor \( t_{abcdefgh}(\phi^2) \) involving the operator \( W_0 = \frac{1}{2} \text{tr} \phi^2 \) (see, e.g., \([13]\)). Another question is whether, in a similar way, a cohomological extension can be constructed from the operator \( W_0 = \frac{1}{6} \text{tr} \phi^6 \), too, which should be related to the string–corrected two–loop counterterms, and so on.

\footnote{The square bracket antisymmetrization is iteratively defined as

\[
[ab] = ab - ba, \quad [abc] = a[bc] + b[ca] + c[ab], \quad [abcd] = a[bc]d - b[cd]a + c[dab] - d[abc],
\]

etc.}
Appendix A

The Euclidean two–spinor conventions adopted in this paper are similar to those of Ref. [19], Appendix E. The numerically invariant tensors $(\sigma_A)^{\alpha\beta}$ and $(\sigma_A)_{\dot{\alpha}\dot{\beta}}$ are the Clebsch–Gordon coefficients relating the $(\frac{1}{2}, \frac{1}{2})$ representation of $SL(2, \mathbb{C})$ to the vector representation of $SO(4)$,

$$(\sigma_A)^{\dot{\alpha}\dot{\beta}} = (-i\sigma_1, -i\sigma_2, -i\sigma_3, I_2), \quad (\sigma_A)^{\dot{\alpha}\dot{\beta}} := (\sigma_A)^{\dot{\gamma}\dot{\delta}} \epsilon_{\dot{\gamma}\dot{\delta}} = (\sigma_A)^{\dot{\alpha}\dot{\beta}},$$

$$(\sigma_A)_{\alpha\beta} = (i\sigma_1, i\sigma_2, i\sigma_3, I_2), \quad (\sigma_A)^{\alpha\beta} := \epsilon^{\alpha\gamma\delta\beta}(\sigma_A)^{\dot{\gamma}\dot{\delta}} = (\sigma_A)_{\alpha\beta},$$

$(\sigma_A)^{\dot{\alpha}\dot{\beta}}$ and $(\sigma_A)^{\alpha\beta}$ being the corresponding complex conjugate coefficients. Thereby, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, which satisfy the Clifford algebra

$$(\sigma_A)^{\alpha\gamma}(\sigma_B)^{\beta\dot{\gamma}} + (\sigma_B)^{\alpha\gamma}(\sigma_A)^{\beta\dot{\gamma}} = 2\delta_{AB}^\alpha \delta^\alpha \beta,$$

$$(\sigma_A)^{\dot{\alpha}\gamma}(\sigma_B)^{\dot{\beta}\dot{\gamma}} + (\sigma_B)^{\dot{\alpha}\gamma}(\sigma_A)^{\dot{\beta}\dot{\gamma}} = 2\delta_{AB}^\dot{\alpha} \delta^{\dot{\alpha}} \dot{\beta},$$

and, in addition, the completeness relations,

$$(\sigma_A)^{\dot{\alpha}\dot{\beta}}(\sigma_B)^{\dot{\beta}\dot{\gamma}} = 2\delta_{A}^B, \quad (\sigma_A)^{\dot{\alpha}\dot{\beta}}(\sigma_A)^{\dot{\gamma}\dot{\delta}} = 2\delta_{A}^\dot{\alpha} \delta^{\dot{\alpha}} \dot{\delta},$$

$$(\sigma_A)^{\alpha\beta}(\sigma_A)^{\gamma\dot{\gamma}} = 2\epsilon_{\alpha\gamma\delta\beta}, \quad (\sigma_A)^{\alpha\beta}(\sigma_A)^{\gamma\dot{\delta}} = 2\epsilon^{\alpha\gamma\delta\beta}.$$  

The spinor index $\alpha$ (analogously $\dot{\alpha}$) is raised and lowered by $\epsilon^{\alpha\gamma\varphi \dot{\gamma}} \varphi = \varphi^{\alpha\beta}$ and $\varphi_{\alpha\gamma} \epsilon_{\gamma\delta} = \varphi_{\alpha\beta}$, where $\epsilon_{\alpha\beta}$ (analogously $\epsilon_{\dot{\alpha}\dot{\beta}}$) is the invariant tensor of the group $SU(2), \epsilon_{12} = \epsilon_{12} = \epsilon_{i2} = 1$.

The selfdual and anti-selfdual $SO(4)$ generators, $(\sigma_{AB})_{\alpha\beta}$ and $(\sigma_{AB})_{\dot{\alpha}\dot{\beta}}$, and the various Clebsch–Gordon coefficients are related by the properties

$$(\sigma_A)^{\alpha\gamma}(\sigma_B)^{\beta\dot{\gamma}} = (\sigma_{AB})^{\alpha\beta} - \delta_{AB} \epsilon^{\alpha\beta},$$

$$(\sigma_C)^{\alpha\gamma}(\sigma_A)^{\beta\dot{\gamma}} = (\delta_{AC} \delta_{BD} - \delta_{BC} \delta_{AD} - \epsilon_{ABCD})(\sigma_D)^{\alpha\beta},$$

$$(\sigma_{CD})^{\dot{\alpha}\dot{\gamma}}(\sigma_{AB})^{\dot{\beta}\dot{\gamma}} = (\delta_{AC} \delta_{BD} - \delta_{BC} \delta_{AD} - \epsilon_{ABCD}) \delta_{[\dot{C}\dot{A}]}(\sigma_{BD})^{\dot{\alpha}\dot{\beta}},$$

$$(\sigma_A)^{\dot{\alpha}\gamma}(\sigma_B)^{\dot{\beta}\dot{\gamma}} = (\sigma_{AB})^{\dot{\alpha}\dot{\beta}} + \delta_{AB} \epsilon_{\dot{\alpha}\dot{\beta}},$$

$$(\sigma_C)^{\dot{\alpha}\gamma}(\sigma_{AB})^{\dot{\beta}\dot{\gamma}} = (\delta_{AC} \delta_{BD} - \delta_{BC} \delta_{AD} + \epsilon_{ABCD})(\sigma_D)^{\dot{\alpha}\dot{\beta}},$$

Finally, some often used identities are

$$(\sigma_{AB})_{\dot{\alpha}\dot{\beta}}(\sigma_B)^{\dot{\beta}\dot{\gamma}} = -2(\sigma_B)^{\dot{\gamma}\dot{\delta}} - \epsilon_{\dot{\alpha}\dot{\beta}}(\sigma_A)^{\dot{\gamma}\dot{\delta}}, \quad (\sigma_{AB})^{\dot{\alpha}\dot{\beta}}(\sigma_B)^{\dot{\gamma}\dot{\delta}} = 8\epsilon_{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{\beta}\dot{\delta}} - 4\epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\gamma}\dot{\delta}},$$

$$(\sigma_{AB})_{\alpha\beta}(\sigma_B)^{\gamma\dot{\gamma}} = 2(\sigma_A)^{\alpha\delta} \epsilon_{\gamma\delta} + \epsilon_{\alpha\beta}(\sigma_A)^{\gamma\delta}, \quad (\sigma_{AB})^{\alpha\beta}(\sigma_B)^{\gamma\dot{\delta}} = 8\epsilon_{\alpha\gamma} \epsilon_{\beta\delta} - 4\epsilon_{\alpha\beta} \epsilon_{\gamma\delta}.$$

Appendix B

In this Appendix we give the off–shell transformation rules of the matter–independent part of [14–16].

$$\begin{align*}
QA_A &= \psi_A, \\
Q\psi_A &= DA\phi, \\
Q\phi &= 0, \\
Q\dot{\phi} &= \eta, \\
Q\eta &= [\phi, \phi], \\
Q\chi_{AB} &= B_{AB}, \\
QB_{AB} &= [\chi_{AB}, \phi],
\end{align*}$$

(B.1)
\[ Q_{AB}A_C = -\Theta_{ABCD}^E[F_{DE} - \frac{1}{2}B_{DE}] + \frac{1}{2}[\chi_{DE}, \phi], \]
\[ Q_{AB}B_{CD} = \Theta_{ABCD}^E[D_{[E}^C\psi_{DE]} - \frac{1}{2}[\chi_{CE}, \phi]] - \Theta_{ABCD}^E[\eta, \phi], \]

where the operator \( \Theta_{ABCD} \) was introduced in Eq. (12).

### Appendix C

In this Appendix we prove the off–shell \( Q \)-exactness of the cohomological extension (21) and we show that, by choosing the Landau type gauge, one exactly reproduces the expression (26) for the gauge fermion \( \Psi'_{\text{ext}} \).

As already emphasized in the Sect. 4, we are forced to start with the complicated expression

\[ S^\text{ext} = S^\text{top} - \frac{1}{12} \epsilon_{abcdefgh} Q_a Q_b Q_c Q_d Q_e Q_f Q_g Q_h \int_E d^8x \frac{1}{4} \text{tr} \phi^4, \]

where \( S^\text{top} \) is the eight–dimensional topological invariant (23). Namely, we found only the off–shell extension of the scalar and vector supersymmetry transformations (3) and (4),

\[ Q_A = \phi_a, \quad Q_A = \eta, \]
\[ Q_A = \phi_a, \quad Q_A = \eta, \]
\[ Q_A = \phi_a, \quad Q_A = \eta, \]

and

\[ Q_a = \delta_{ab} \eta + \chi_{ab}, \]
\[ Q_a = \psi_a, \quad Q_a = \eta, \]
\[ Q_a = \psi_a, \quad Q_a = \eta, \]
\[ Q_a = \psi_a, \quad Q_a = \eta, \]

...
Here \( B_{ab} = \frac{1}{6} \Phi_{abcd} B^{cd} \) is the anti-field of \( \chi_{ab} \). One simply verifies that \( Q \) and \( Q_a \) satisfy the following superalgebra off-shell,

\[
\{ Q, Q \} = -2\delta_G(\phi), \quad \{ Q, Q_a \} = \partial_a + \delta_G(A_a), \quad \{ Q_a, Q_b \} = -2\delta_{ab}\delta_G(\phi). \tag{C.4}
\]

In order to recast (C.1) into the \( Q \)-exact form \( S_{\text{ext}} = Q\Psi_{\text{ext}} \) it is convenient to exploit only the algebraic relations (C.4) among \( Q \) and \( Q_a \), but not their explicit realizations (C.2) and (C.3). Otherwise, owing to the increasing number of terms which arise by evaluating (C.1), it is nearly impossible to determine \( \Psi_{\text{ext}} \) explicitly.

To begin with, by repeated application of \( Q_a \) on the primary operator \( W_0 = \frac{1}{4} \text{tr} \phi^4 \) we decompose (C.1) into a form where the different operators \( Q_a \) act only on the scalar field \( \phi \). After straightforward calculations one obtains

\[
S_{\text{ext}} = S_{\text{top}}^{8D} - \frac{1}{8!} \varepsilon_{abcdefg} \int d^8 x \text{tr}_{(s)} \left\{ \psi_{abcdefg} \phi^3 + 8\psi_{abcdefg} \psi_h \phi^2 + 28\psi_{abcdefg} \psi_{gh} \phi^2 \right. \tag{C.5}
\]

\[
+ 56\psi_{abcdefg} \psi_f \psi_h \phi + 56\psi_{abcdefg} \psi_{fg} \phi^2 + 168\psi_{abcdefg} \psi_{fg} \psi_h \phi + 336\psi_{abcdefg} \psi_f \psi_g \psi_h
\]

\[
+ 35\psi_{abcdefg} \psi_{fg} \phi^2 + 280\psi_{abcdefg} \psi_{fgh} \phi + 420\psi_{abcdefg} \psi_{fgh} \phi + 840\psi_{abcdefg} \psi_{fgh} \phi
\]

\[
+ 280\psi_{abcdefg} \psi_{fgh} \phi + 560\psi_{abcdefg} \psi_{fgh} \phi + 1680\psi_{abcdefg} \psi_{fgh} \phi + 630\psi_{abcdefg} \psi_{fgh} \phi \right\},
\]

where

\[
\psi_a = Q_a \phi, \quad \psi_{ab} = Q_a \psi_b, \quad \psi_{abc} = Q_a \psi_{bc}, \quad \text{etc.,}
\tag{C.6}
\]

and where, for the sake simplicity, we have introduced a symmetrized trace, \( \text{tr}_{(s)} \). It is defined as follows: First, for every monomial \( X_1 X_2 X_3 X_4 \) in the integrand of (C.5) which, due to its origin from \( \phi^4 \), always consists of exactly 4 factors, we consider all those graded permutations (with repetitions) of the various factors \( X_i = (\phi, \psi_a, \psi_{ab}, \ldots) \) which do not correspond to an antisymmetrization of their space indices; owing to the presence of the Levi–Civita tensor in (C.5) we can ignore such permutations. After that, we take the trace of the sum of all these permuted polynomials and drop their largest common factor.

As an illustration let us give some examples:

\[
\text{tr}_{(s)} \left\{ \psi_{abcdefg} \psi_h \phi^2 \right\} = \text{tr} \left\{ \psi_{abcdefg} (\psi_h \phi^2 + \phi \psi_h \phi + \phi^2 \psi_h) \right\},
\]

\[
\text{tr}_{(s)} \left\{ \psi_{abcd} \psi_{efgh} \phi^2 \right\} = \text{tr} \left\{ \psi_{abcd} (\psi_{efgh} \phi^2 + \phi \psi_{efgh} \phi + \phi^2 \psi_{efgh}) \right\},
\]

\[
\text{tr}_{(s)} \left\{ \psi_{abcd} \psi_f \psi_g \psi_h \right\} = \text{tr} \left\{ \psi_{abcd} \psi_f \psi_g \psi_h \right\}.
\]

In the first example, the 24 possible graded permutations of the monomial \( \psi_{abcdefg} \psi_h \phi^2 \), after taking the trace over their sum lead to only 3 different monomials with the common factor 8 which has to be dropped. In the same way one performs the symmetrized trace of the other examples. Notice, that in the second example one has to ignore the permutation of \( \psi_{abcd} \) and \( \psi_{efgh} \phi^2 \) because it can be reversed through an antisymmetrization of their space indices. In the last example one even has to ignore all possible graded permutations.

With that definition of the symmetrized trace the various factors in front of the monomials in (C.5) agree precisely with the number of permutations which are necessary in order to recast the symmetrized trace of these polynomials into a fully antisymmetrized form. For example, in order to recast \( \psi_{abcdefg} \psi_{efgh} \phi^2 \) into a totally antisymmetrized form one has to perform \( 8! / 4! 4! = 70 \) permutations, thereby taking into account that, owing to the presence of the Levi–Civita tensor, only the completely antisymmetrized part of \( \psi_{abcd} \) and \( \psi_{efgh} \phi^2 \) appears in (C.5). By taking the symmetrized trace of that polynomial this number is further reduced to 35 since, under that trace, one can permute \( \psi_{abcd} \) and \( \psi_{efgh} \phi^2 \) thereby dividing the total number of permutations by
two. Hence, one has to perform only 35 permutations in accordance with the prefactor of that polynomial in (C.5).

As a next step, owing to the $Q$–exactness of $\psi_a$ we can split off from each of the higher rank objects $\psi_{ab}$, $\psi_{abc}$, ... in (C.6) an $Q$–exact term by making use of the second relation (C.4), as a result of which one gets the following decompositions,

$$\psi_{ab} = -Q\lambda_{ab} + F_{ab}, \quad (C.7)$$
$$\psi_{abc} = Q\lambda_{abc} - 3D_a\lambda_{bc},$$
$$\psi_{abcd} = -Q\lambda_{abcd} + 4D_a\lambda_{bcd} - 3\{\lambda_{ab}, \lambda_{cd}\},$$
$$\psi_{abcde} = Q\lambda_{abcde} - 5D_a\lambda_{bcde} + 10[\lambda_{ab}, \lambda_{cde}],$$
$$\psi_{abcdef} = -Q\lambda_{abcdef} + 6D_a\lambda_{bcdef} + 10[\lambda_{abc}, \lambda_{def}] - 15\{\lambda_{ab}, \lambda_{cdef}\},$$
$$\psi_{abcdefg} = Q\lambda_{abcdefg} - 7D_a\lambda_{bcdefg} - 35[\lambda_{abc}, \lambda_{defg}] + 21[\lambda_{ab}, \lambda_{cdefg}],$$
$$\psi_{abcdefgh} = -Q\lambda_{abcdefgh} + 8D_a\lambda_{bcdefgh} - 28\{\lambda_{ab}, \lambda_{cdegh}\} + 56[\lambda_{abc}, \lambda_{degh}] - 35\{\lambda_{abcd}, \lambda_{efgh}\},$$

where

$$\lambda_{ab} = Q_aA_b, \quad \lambda_{abc} = Q_a\lambda_{bc}, \quad \lambda_{abcd} = Q_a\lambda_{bcd}, \quad \text{etc.} \quad (C.8)$$

Thereby, for the sake simplicity, we still have performed some replacements on the right–hand side of (C.7). For example, let us derive the decomposition of $\psi_{abc}$,

$$\psi_a = Q_a\phi = QA_a,$$
$$\psi_{ab} = Q_a\psi_b = Q_a(QA_b) = -Q(Q_aA_b) + F_{ab} = -Q\lambda_{ab} + F_{ab},$$
$$\psi_{abc} = Q_a\psi_{bc} = -Q_a(Q\lambda_{bc} - F_{bc}) = Q(Q_a\lambda_{bc}) - D_a\lambda_{bc} - D_{[c}(Q_aA_{b]} \Rightarrow Q\lambda_{abc} - 3D_a\lambda_{bc},$$

where in the last relation we have replaced $D_{[c}(Q_aA_{b]} = D_c\lambda_{ab} - D_b\lambda_{ac}$ through $2D_a\lambda_{bc}$ since only the completely antisymmetric part of $\psi_{abc}$ enters into (C.5). In exactly the same way one can derive iteratively all the other decompositions in (C.7). Notice, that the various factors in front of the different terms in (C.7) agree precisely with the number of permutations which must be performed in order to fully antisymmetrize these terms, thereby taking into account that after inserting the decompositions (C.7) into (C.5) only the fully antisymmetric part of $\lambda_{ab}$, $\lambda_{abc}$, ... contribute.

We are now faced with the difficult problem to rewrite (C.5) in a $Q$–exact form. This is owing to the fact that after inserting the decompositions (C.7) into (C.5) one gets a huge number of symmetrized terms and, therefore, the computational effort in order to expose the gauge fermion is considerably. By making use of the first relation (C.4), after a straightforward
but lengthy algebraic computation for $\Psi_{ext}$ one obtains

$$\Psi_{ext} = \frac{1}{8!} \epsilon^{abcdfhg} \int_E \omega^s x \, \text{tr}(s) \left\{ \lambda_{abcdfhg} \phi^3 - 8 \lambda_{abcdfh} \psi_h \phi^2 - 28 \lambda_{abcdf} (Q \lambda_{gh} - F_{gh}) \phi^2 \right\} \quad (C.9)$$

$$+ 56 \lambda_{abcdfh} \psi_h \phi - 56 \lambda_{abcd} (Q \lambda_{fhg} - 3D_f \lambda_{gh}) \phi^2 - 336 \lambda_{abcdefh} \psi_h \phi$$

$$+ 168 \lambda_{abcdef} (Q \lambda_{gfh} - F_{gh}) \phi - 35 \lambda_{abcd} (Q \lambda_{efg} - 8D_e \lambda_{fg} + 6 \{\lambda_{ef}, \lambda_{gh}\}) \phi^2$$

$$+ 280 \lambda_{abcde} (Q \lambda_{fgh} - 3D_f \lambda_{gh}) \phi + 420 \lambda_{abc} (Q \lambda_{ef} - F_{ef}) (Q \lambda_{g} - F_{gh}) \phi$$

$$- 840 \lambda_{ab} (Q \lambda_{d} - 6D_d \lambda_{ef}) (Q \lambda_{gh} - F_{gh}) \phi$$

$$- 560 \lambda_{a} (Q \lambda_{fg} - 6D_f \lambda_{gh}) - 1680 \lambda_{a} \psi_d (Q \lambda_{ef} - F_{ef}) (Q \lambda_{gh} - F_{gh})$$

$$+ 280 \lambda_{abc} \psi_d \{\lambda_{def}, \lambda_{gh}\} \phi^2 + 840 \lambda_{abc} \psi_d \{\lambda_{ef}, \lambda_{gh}\} \phi - 420 \lambda_{ab} \{\lambda_{cd}, \lambda_{ef}\} F_{gh} \phi$$

$$+ 315 \lambda_{ab} \{\lambda_{cd}, \lambda_{ef}\} Q \lambda_{gh} \phi - 630 \lambda_{ab} Q \lambda_{cd} Q \lambda_{ef} \lambda_{gh} - 560 \lambda_{abc} \psi_d D_e \lambda_{fgh} \phi$$

$$+ 1680 \lambda_{ab} \psi_d D_e \lambda_{fgh} \phi - 840 \lambda_{abc} \psi_d \{\lambda_{ef}, \lambda_{gh}\} + 840 \lambda_{abc} Q \lambda_{cd} \lambda_{ef} F_{gh}$$

$$+ 1680 \lambda_{ab} D_g \lambda_{cd} D_h \lambda_{ef} \phi - 1260 \lambda_{ab} Q \lambda_{cd} \psi_d F_{gh} - 2520 \lambda_{abc} D_e D_{ef} F_{gh}$$

$$+ 2520 \lambda_{abc} F_{ef} F_{gh},$$

which is the eight–dimensional extension of the gauge fermion $\Psi_0$ of the Donaldson–Witten theory.

Here, a remark is in order. In (C.5) the symmetrized trace was introduced for monomials consisting of exactly 4 factors $X_i$. After substituting for $\psi_{ab}, \psi_{abc}, \ldots$ the decompositions (C.7) we introduce, besides the field strength $F_{ab}$, also the covariant derivative $D_a$ and some graded commutators of the higher rank objects $\lambda_{ab}, \lambda_{abc},$ \ldots as well as their $Q$–transforms. Thus, in order not to spoil the definition of the symmetrized trace, one has to view these new objects, in particular all the graded commutators of $\lambda_{ab}, \lambda_{abc},$, as single factors, i.e., new objects like $X_i$! Let us now give some example of the correct use of the symmetrized trace in (C.9),

$$\text{tr}(s) \left\{ \lambda_{ab}, \lambda_{cde} | \lambda_{fgh} \phi^2 \right\} = \text{tr} \left\{ \lambda_{ab}, \lambda_{cde} | (\lambda_{fgh} \phi^2 + \phi \lambda_{fgh} \phi + \phi^2 \lambda_{fgh}) \right\},$$

$$\text{tr}(s) \left\{ \lambda_{abcd} \{\lambda_{ef}, \lambda_{gh}\} \phi^2 \right\} = \text{tr} \left\{ \lambda_{abcd} \{\lambda_{ef}, \lambda_{gh}\} \phi^2 + \phi \lambda_{ef} \phi + \phi^2 \lambda_{ef} \right\},$$

$$\text{tr}(s) \left\{ \lambda_{ab} \psi_c (D_d \lambda_{ef}) Q \lambda_{gh} \right\} = \text{tr} \left\{ \lambda_{ab} \psi_c (D_d \lambda_{ef}) Q \lambda_{gh} + (Q \lambda_{gh}) \psi_c D_d \lambda_{ef} - (D_d \lambda_{ef}) (Q \lambda_{gh}) \psi_c \right. \right.$$  

$$- (D_d \lambda_{ef}) \psi_c Q \lambda_{gh} + \psi_c (Q \lambda_{gh}) D_d \lambda_{ef} - (Q \lambda_{gh}) (D_d \lambda_{ef}) \psi_c \right\}.$$  

In order to pick out from (C.9) all the terms belonging to the minimal sector, we have to evaluate the higher rank objects $\lambda_{ab}, \lambda_{abc},$ \ldots in (C.8) explicitly. By making use of (C.3) after a simple calculation one obtains

$$\lambda_{ab} = \lambda_{ab},$$

$$\lambda_{abc} = \Phi_{abc} \, m D_m \phi,$$

$$\lambda_{abcd} = - \Phi_{abcd} \{\eta, \phi\} - \Phi_{abc} \, m [\lambda_{cdm}, \phi],$$

$$\lambda_{abcdc} = - 5 \Phi_{abcd} [D_c \phi, \phi],$$

$$\lambda_{abcdef} = - 5 \Phi_{abcd} [\lambda_{ef}, \phi],$$

$$\lambda_{abcde} = - 5 \Phi_{abc} \Phi_{efg} \, m \left[[D_m \phi, \phi], \phi\right],$$

$$\lambda_{ab} \psi_c (D_d \lambda_{ef}) Q \lambda_{gh} = 5 \Phi_{abcd} \Phi_{efgh} \left[[\eta, \phi], \phi\right] + 5 \Phi_{abcd} \Phi_{efg} \, m \left[[\chi_{hm}, \phi], \phi\right], \phi\right],$$  

(C.10)
with
\[ Q_{\lambda} = B_{ab}, \]
\[ Q_{\lambda_{abc}} = \Phi_{m}^{abc}([\psi_{m}, \phi], \{\eta, \eta\}) - \Phi_{ab}^{m}([B_{dm}, \phi] - \{\chi_{dm}, \eta\}), \]
\[ Q_{\lambda_{abcd}} = -\Phi_{abcd}([\phi, \phi], \{\phi\}) - \Phi_{m}^{abcd}([\psi_{dm}, \phi] - \{\chi_{dm}, \eta\}), \]
where we have carried out similar replacements as in (C.7) and omitted all the terms which do not contribute to (C.9) (remind that only the fully antisymmetric part of \( \lambda_{ab}, \lambda_{abc}, \ldots \) enters).

Then, by inserting into (C.9) for \( \lambda_{ab}, \lambda_{abc}, \ldots \) the expressions (C.10) and (C.11) and taking into account (27), together with the following basic identity [15],
\[ \Phi_{ab}^{cm} \Phi_{de}^{fm} = \delta_{d}^{[a} \delta_{f]}^{b} + \frac{1}{4} \Phi^{[ab}_{[de} \delta^{c]}_{f]}, \]
after a tedious calculation we could express \( \Psi_{ext} \) in terms of \( A_{a}, \psi_{a}, \chi_{ab}, \eta, \phi, \bar{\phi} \) and \( B_{ab} \). In order to select from \( \Psi_{ext} \) the terms belonging to the minimal sector, we rescale \( \chi_{ab}, \eta, \phi, \bar{\phi} \) and \( B_{ab} \) as well as \( \Psi_{ext} \) with the gauge parameter \( \xi \) and \( 1/\xi \), respectively. Then, by putting \( \xi \) equal to zero, i.e., by choosing the Landau type gauge, \( \Psi_{ext} \) considerably simplifies into
\[ \Psi_{ext}' = \frac{1}{8!} \epsilon_{abcdefg} \int_{E} d^{8}x \text{tr}_{(s)} \left\{ 2520 \lambda_{ab} F_{cd} F_{ef} F_{gh} - 1680 \lambda_{abc} \psi_{d} F_{ef} F_{gh} \right\}. \]
Moreover, it is easily seen that the further evaluation of the right–hand side of (C.12) reveals precisely the expression [14] we are looking for.

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