Coherence simplices

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Abstract. Coherence simplices are generic topological correlation-function defects supported by a hierarchy of coherence functions. We classify coherence simplices based on their topology and discuss their structure and dynamics, together with their relevance to several physical systems.

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1. Introduction

The quantized vortex is an archetypal topological defect that arises in a variety of physical systems including superfluids, superconductors, optical speckle fields, coherent fields in the focal volumes of lenses, coherent optical fields scattered from sharp edges, eigenmodes of waveguides and angular momentum eigenstates of the hydrogenic atom [1–3]. The structure and dynamics of these conventional quantized vortices are similar to those of classical eddies, while enabling some unique material properties such as crystallization of magnetic flux in type-II superconductors and quantized circulation in rotating superfluids [4].

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Let us focus on this quantization of circulation, restricting the discussion to complex scalar fields such as the wavefunction of Schrödinger wave mechanics and the complex analytic signal associated with a scalar monochromatic electromagnetic wave. Quantized vortices in such fields are very generic, as was emphasized by Dirac in a visionary paper from the 1930s [5]. When studying vortices from a topological perspective one can ignore the details of the particular differential equation obeyed by a given complex-valued field $\psi$, including whether the said equation is linear or nonlinear, instead relying on the very general physical property that $\psi$ must be a continuous and single-valued function of position and time. Using this assumption alone, Dirac was able to show that quantized vortices will in general exist in complex fields, and that they are stable with respect to perturbation. The essential idea is that while the complex field is single valued, its phase need not necessarily be single valued; hence the integral of the phase around a closed loop need not vanish, and may in fact be an integer multiple $m$ of $2\pi$ radians. While we explore this point in further detail in section 2, the key point to note here is that nonzero $m$ heralds the presence of a quantized vortex.

As the previous list of examples shows, quantized vortices in complex fields have long been known in physics. Examples include Schrödinger's angular-momentum eigenstates of the hydrogenic atom, Wolther's 1950 treatment of vortices in the context of the Goos–Hänchen effect, and the 1952 study by Braunbek and Laukien investigating the exact solution due to Sommerfeld, for the diffraction of a plane electromagnetic wave from an infinite half plane [6]. Nye and Berry’s work [7] on dislocations in wave trains was pivotal in bringing the study of quantized vortices to the attention of the scalar optics community. The study of such optical vortices has now blossomed into the area of singular optics, with several key reviews including those of Berry [8], Nye [9], Soskin and Vasnetsov [10] and Dennis et al [6]. The nodal lines that thread vortex cores can form closed loops or extend to infinity [5]; they can terminate at points where the potential is discontinuous; and they can even form knots [11–15].

Now, partially coherent optical fields or mixed-state wavefunctions are often studied via correlation functions such as the mutual intensity or the cross-spectral density (for partially coherent optical fields) [16, 17] and the density matrix (for mixed-state wavefunctions) [18]. These correlation functions are very often continuous single-valued complex functions of pairs of spatial coordinates. If one recalls Dirac’s argument for the existence of vortices in complex functions, one has the logical possibility of vortex-type structures in field correlation functions. Such vortical structures, in two-point correlation functions of both classical and quantum-mechanical complex fields, are known as coherence vortices.

Accordingly, quantized vorticity has emerged as a topic of interest in the context of optical and matter field coherence [19–31]. For a recent review of coherence vortices, see Gbur and Visser [32]. Here the term ‘coherence vortex’ has been coined to describe phase singularities or nodal lines in the cross-spectral density matrix, and related coherence functions such as the spectral degree of coherence [20]. Coherence vortices may undergo topological reactions such as splitting, fusing and pairwise creation/annihilation [28]. They may be spontaneously formed in a Young-type three-pinhole interferometer [27], and via Mie scattering from one or several spheres [29, 30]. Coherence vortices have also been studied in the context of linear optical imaging systems and the focal volume of aberrated lenses [26]. Recently, it was shown that coherence vortices may emerge even in systems with only one spatial dimension where conventional vortices are manifestly absent [31]. Coherence vortices have been observed experimentally in optical fields [21, 23, 24].
All of the previously cited work refers to screw-type and/or edge-type topological defects in two-point coherence functions. The desire to investigate both higher-order correlation functions and more complex topological correlation defects has motivated the present investigation. Here we develop the notion of a ‘coherence simplex’ as a generalization of the concept of a coherence vortex, thereby categorizing topological defects emerging in generic \( p \)th order complex correlation functions.

We close this introduction with an overview of the remainder of the paper. Section 2 gives a brief background to conventional quantized vortices as screw-type phase singularities in continuous single-valued complex functions. In section 3 we define a coherence simplex as a generalized form of coherence vortex, and in section 4 we categorize a hierarchy of coherence simplices. Finally, in section 5 we discuss the structure and dynamics of coherence simplices, followed by a discussion and concluding remarks in section 6.

2. Conventional quantized vortex

To be self-contained and to facilitate further discussion, we briefly review some mathematical properties of conventional quantized vortices. A generic continuous single-valued complex scalar function \( f \) of two real variables \((x, y)\) can, without loss of generality, be cast in the Madelung form

\[
 f(x, y) = |f(x, y)| e^{i \theta(x, y)},
\]

where \( \theta(x, y) \) is a real valued phase function \([33]\). If in the vicinity of some point \((x_0, y_0)\) the phase function has the form

\[
 \theta(x - x_0, y - y_0) = m \arctan \left( \frac{y - y_0}{x - x_0} \right) + B(x, y),
\]

where \( m \) is an integer and \( B(x, y) \) is any real function that is analytic in the vicinity of \((x_0, y_0)\), then the phase singularity at \((x = x_0, y = y_0)\) is called a point vortex. Physically, we can view \( B(x, y) \) as a continuous background deformation, upon which sits the screw-type phase dislocation given by the term proportional to \( m \). The amplitude \(|f(x_0, y_0)|\) must vanish at the location of the phase singularity to ensure single valuedness of the complex function. In three-dimensional (3D) systems such zero-amplitude phase singularities are nodal lines or line vortices that may close on themselves, forming loops or vortex rings \([1, 5, 9, 34–36]\) or even vortex knots \([11–15]\).

In superfluids, the gradient of the phase function \( \nabla \arg[\psi(r)] \) of the complex order parameter field \( \psi(r) \) describing the system can be associated with the velocity field \( v_\psi(r) \) of the superfluid. Since the curl of the gradient \( \nabla \times \nabla A(r) \) of any vector field \( A(r) \) is identically zero unless the field contains singularities, applying Stokes’s law directly leads to the quantization of circulation of the superfluid

\[
 \oint_{\Omega} \nabla v_\psi(r) \cdot dl = mk,
\]

where \( m \) is an integer and \( k \) is the quantum of circulation. In helium II and atomic Bose–Einstein condensates it is the circulation of atoms that is quantized \([1, 35, 37]\). In superconductors magnetic flux is quantized and vortices form due to the motion of Cooper pairs \([2, 38, 39]\),
and in coherent optical fields the photons can form optical vortices [9, 36]. It is also possible to convert optical to matter-wave vortices and vice versa [40, 41].

From a mathematical perspective, any single-valued continuous complex function of at least two real variables may possess screw-type phase singularities of this kind [5] and hence vortices should be expected whenever a model of a physical system involves continuous complex functions. It is particularly interesting to investigate the vortex properties of complex $p$-point correlation functions of quantum fields, to which topic we now turn.

3. Coherence simplices

The previous section only considered vortices in field functions, considered as a function of position. Other control parameters such as time may be present, but this does not change the fact that the vortices of section 2 ‘swirl’ in physical space; in order to swirl in physical space, either two $(x, y)$ or three dimensions $(x, y, z)$ are required. However, as emphasized in the first two sections of this paper, the existence of vortices follows directly from one having a continuous single-valued complex function. This raises the logical possibility that complex two-point correlation functions, which are continuous functions of two spatial coordinates, $(x_1, x_2)$, $(x_1, y_1, x_2, y_2)$ or $(x_1, y_1, z_1, x_2, y_2, z_2)$, together with any other relevant control parameters such as time coordinates, may also possess vortical structures in the phase of the said correlation function. However, these vortical correlation-function phases—termed ‘coherence vortices’ [19–28, 32]—will ‘swirl’ in spaces of high-dimensionality, rather than in physical space. Moreover, these topological correlation-function phase maps are not restricted to two-point correlation functions, but may evidently be generalized to $p$-point correlation functions. Such structures, termed coherence simplices, are the topic of this section.

Let us adopt the language of second quantization [42] and introduce Heisenberg field operators $\hat{\psi}(\mathbf{r}, t)$, $\hat{\psi}^\dagger(\mathbf{r}, t)$, which respectively create and annihilate an excitation of the field, such as a particle, at space–time point $(\mathbf{r}, t)$. Furthermore, a bosonic field $\hat{\psi}(\mathbf{r}, t)$ obeys the canonical commutation relations

\[
[\hat{\psi}(\mathbf{r}, t), \hat{\psi}^\dagger(\mathbf{r}', t')] = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'),
\]

\[
[\hat{\psi}(\mathbf{r}, t), \hat{\psi}(\mathbf{r}', t')] = [\hat{\psi}^\dagger(\mathbf{r}, t), \hat{\psi}^\dagger(\mathbf{r}', t')] = 0,
\]

with similar relations holding for fermionic fields with the commutators replaced by anticommutators [43].

Consider the $p$th order field correlation [18, 37, 42, 43] (cf the many-body density matrix, Glauber quantum coherence functions, Green functions or Feynman propagators)

\[
g^{(p)}(x_1, x_2, \ldots, x_{2p}) = \langle \hat{\psi}^\dagger(\mathbf{r}_1, t_1)\hat{\psi}^\dagger(\mathbf{r}_2, t_2)\cdots \hat{\psi}^\dagger(\mathbf{r}_{2p}, t_{2p})\rangle
\]

\[
\times \hat{\psi}(\mathbf{r}_p, t_p)\hat{\psi}(\mathbf{r}_{p+1}, t_{p+1})\hat{\psi}(\mathbf{r}_{p+2}, t_{p+2})\cdots \hat{\psi}(\mathbf{r}_{2p}, t_{2p}),
\]

where $x_i$ refers to both space and time coordinates and the angular brackets denote quantum statistical average. We augment this by further defining

\[
\langle \hat{\psi}(\mathbf{r}, t) \rangle = g^{(0)}(x_0),
\]

which may acquire finite values through the spontaneous symmetry breaking mechanism [39].
With these prerequisites, we may define a \( p \)th order vortical coherence simplex as a generalized vortex in the complex valued correlation function \( g^{(p)}(x_1, x_2, \ldots, x_{2p}) \). For such vortical coherence simplices, coherence circulation is quantized via

\[
\oint_O \nabla \arg[g^{(p)}(x_1, x_2, \ldots, x_{2p})] \cdot d\mathbf{l}_{2pD} = m2\pi = \kappa_p,
\]

where \( d\mathbf{l}_{2pD} \) is a line element along an oriented single smoothed closed curve \( O \) in a \( 2pD \)-dimensional space, where \( D \) is the physical dimension of the space, and the constant \( \kappa_p \) is dimensionless quantized coherence circulation.

As in the case of conventional vortices [35, 44, 45], coherence simplices with winding number \(|m| > 1\) are likely to be topologically unstable with respect to perturbations (cf [28]), and split into a number of lower winding-number excitations. By the term ‘coherence simplex’ we refer generically to any kind of topological defect in any correlation function and thus include e.g. soliton-like structures.

In particular, the \( p = 0 \) vortical coherence simplex (VCS) reduces to a conventional quantized vortex in 2D or 3D space. It is immediately clear that if the field \( \hat{\psi}(r) \) contains a conventional vortex, it may also show up as a coherence simplex in the associated correlation functions (see [20, 21, 23, 24]). However, coherence simplices and VCSs in particular may also appear when the underlying field itself possesses no conventional vortices. In fact, VCSs emerge even when \( \hat{\psi}(r) \) has only one spatial dimension and therefore could not exhibit conventional vorticity even in principle [31].

In the generalized core, or hypercore, of the VCS \( g^{(p)} \) vanishes and the phase winds an integer \( m \) times \( 2\pi \) around it. While a conventional material vortex is a zero probability of finding a particle in some point which defines the location of the vortex core, the hypercore of a VCS is a zero probability of finding coherence between two (or more) space–time points. The coherence simplex can also be described as a topologically unavoidable loss of coherence between sets of space–time coordinates due to quantization of circulation of the flow of coherence. In particular, consider a line integral, (7), which yields a nonzero coherence circulation winding number \( m \). Even if the quantum field maintains full coherence at every point along and outside the closed curve \( O \), the topology of the problem dictates that there be at least one generalized point inside \( O \) at which the system is fully incoherent, as is further clarified in section 5.

### 4. Hierarchies of coherence simplices

Figure 1 illustrates the simplest topologies in the infinite hierarchy of possible coherence simplices. The dimensionality of the physical space is depicted on the vertical axis and the horizontal axis labels the order \( p \) of the correlation function \( g^{(p)}(x_1, x_2, \ldots, x_{2p}) \) in which the coherence simplex is embedded (see e.g. (6)). The dimension \( M \) of the embedding space of a VCS is \( M = D \) for \( p = 0 \) and \( M = 2p \times D \) otherwise, where the dimension of the physical space is denoted by \( D \). A generic coherence simplex has a codimension

\[
C = M - N,
\]

where \( N \) is the number of degrees of freedom consumed by the topological defect type. Vortical coherence simplices categorized in figure 1 therefore have a codimension \( C_v = M - 2 \), which is denoted in the lower left corner of each pane. The diagram in each pane illustrates the simplex
Figure 1. Hierarchies of vortical coherence simplices in the space spanned by spatial dimension versus the order \( p \) of the coherence function. The lower left corner of each matrix element shows the codimension \( C_v \) of the vortical coherence simplex and the diagrams depict the simplex structure of the underlying coherence function. The numbers inside the bullets denote the physical dimension of the space.

structure of the correlation function underlying the topological defect. Solitonic coherence simplices have a codimension \( C_s = M - 1 \) and, unlike coherence vortices [19, 20], also exist in the \( D = 1, p = 0 \) case. However, solitons are not in general topologically protected in the way vortices are and therefore their stability with respect to perturbations [9] is determined by energetic considerations rather than their topology. Higher order \( N > 2 \) coherence simplices can be categorized using similar diagrams to that shown in figure 1.

The first (shaded) column in figure 1 corresponds to the conventional vortices, which are manifestly absent in systems with one spatial dimension. In 2D systems these conventional vortices are point charges, whereas in the 3D case they correspond to line-like vortex filaments. The second column in figure 1 contains all possible VCSs supported by the two-point correlation functions. In particular, in 1D these correspond to coherence vortices [31] and in 2D they are sheet-like structures.

Coherence simplices can be constrained to lower dimensional spaces, revealing a correspondingly simpler topological structure. For example, slicing through a conventional 3D vortex line results in a 2D point vortex. Similarly, fixing three of the six coordinates \((x, y, z, x', y', z')\) of a first order \( p = 1 \) VCS in a physical space with three spatial dimensions \( D = 3 \) reveals the VCSs as a 1D nodal line object, whereas if all of the coordinates are left free the VCS is a 4D object. Most of the coherence simplices discussed in the literature so far [19, 20, 24, 25, 29, 30] correspond to such reduced-dimensional or constrained VCSs. Curiously, as is evident in figure 1, a line-like \( C_v = 1 \) VCS only exists for \( p > 0 \) in the presence of constraints fixing one or more degrees of freedom, but never as a free object. On the other hand, under suitable constraints any VCS with finite codimension \( C_v > 0 \) can be constrained to appear as a line-like coherence vortex.
5. Structure and dynamics of coherence simplices

The zeroth-order simple vortical coherence simplices are simply conventional vortices with vanishing field amplitude in the vicinity of a screw-type phase singularity. In other words, they correspond to a hole around which the entities described by the field, such as atoms or photons, circulate. In contrast, the higher order \((p > 0)\) VCSs typically have a finite probability of finding corpuscles inside their hypercores, despite the fact that the corresponding coherence function vanishes there. As in the case of conventional vortices, due to energetic considerations multi-quantum \(|m| > 1\) VCSs are likely to split into multiple lower order VCSs and multiple VCSs in equilibrium may arrange in a VCS lattice whose structure may be that of an Abrikosov-type triangle shape, square etc depending on the details of the defect topology and interactions.

If the complex field supporting the VCSs has multiple internal degrees of freedom, the field operator becomes a multicomponent spinor \(\hat{\psi}_\sigma(r, t) = (\hat{\psi}_1(r, t), \hat{\psi}_2(r, t), \ldots \hat{\psi}_\sigma(r, t))^T\), where \(T\) denotes transpose and \(\sigma\) is a spin index, and hence the coherence functions have a tensor structure \(g^{(p)}(x_1, x_2, \ldots, x_{2p})\). Through this construction such coherence functions will in general admit coherence singularities analogous to monopoles, skyrmions, fractional vortices, hedgehogs, sheets, textures, knots etc [46–49]. In such spinor fields with vector order parameter, continuous coherence simplices for which even the coherence density does not have to vanish in the spinor VCS hypercores are possible.

Figure 2 is an illustration of vector coherence defects produced by the mixture of \((a, b)\) two and \((c, d)\) three traveling two-component Gaussian wave packets of the form 
\[
\psi_n = A_n e^{-((x - x_n)/\sqrt{2} \mu_n)^2} (e^{ik_n x}, e^{ik_n y})^T / \mu_n,
\]
where \(A_n\) is a normalization constant. To produce figures 2(a) and (b) we used dimensionless values \(x_1 = -1, k_1^1 = k_1^2 = 0.5, \mu_1 = 1\) and \(x_2 = 1, k_2^1 = k_2^2 = -0.4, \mu_2 = 1\) and for \((c, d)\) \(x_1 = 3.0, k_1^1 = -2.0, k_1^2 = 2.0, \mu_1 = 3.0, x_2 = 0, k_2^1 = -1.0, \mu_2 = 2.0\) and \(x_3 = -3.0, k_3^1 = 1.5, k_3^2 = -1.5, \mu_2 = 1.5\). Frames (a) and (c) show the resulting coherence densities \(g_{\sigma\sigma}^{(1)}(x, x')\) and (b) and (d) are the corresponding phase functions. The circles in (b) and (c), respectively mark the location of winding number \(w = \{1, 1, 1, 1\}\) and \(w = \{0, -1, 1, 0\}\) defects where the generalized winding number 
\[
w = \{m_{11}, m_{12}, \ldots, m_{\sigma\sigma}\}
\]
is expressed as a vector of winding numbers corresponding to the different spin–spin correlation functions.

Coherence simplices are generically dynamical objects flowing with the coherence current [24]. When perturbed, equilibrium coherence simplices are expected to reveal elementary excitation modes such as helical Kelvin waves [37, 50, 51] and Tkachenko shear waves [52–55] and their VCS displacement-wave generalizations. In this context, figure 3 of Marasinghe et al [30] displays a clear signature of oscillatory dynamics, similar to the Crow instability of antiparallel conventional vortices [56, 57], between coherence vortex–antivortex pairs. Non-equilibrium coherence simplex dynamics may reveal VCS reactions such as VCS intercommutation or reconnection events and coherence-simplex turbulence (cf figure 4).

In contrast to the topological arguments given earlier, a study of the dynamics of coherence simplices presupposes knowledge of the equations of motion underlying a given field \(\hat{\psi}(r, t)\). Suppose that this equation, for the field supporting a given coherence simplex or simplices, is given by the Heisenberg equation of motion [42]
\[
\hat{H} = i\hbar \frac{\partial}{\partial t} \hat{\psi}(r, t) = [\hat{\psi}(r, t), \hat{H}],
\]

\[ \text{(9)} \]
Figure 2. Vector coherence simplices in two-member (a,b) and three-member (c,d) ensembles of two-state wavefunctions defined in the text. The four different spin-spin correlations are denoted in the upper corners of each frame in each subfigure. Panels (a) and (c) show the coherence density and (b) and (d) show the phase of the coherence function. Two different kinds of vector coherence defects are circled in frames (b) and (c). The constant $a$ is an arbitrary spatial length scale.

where $\hat{H}$ is the Hamiltonian operator. From here on we focus on a free-field case (interactions alter the dynamics of coherence simplices but do not invalidate the topological considerations) $\hat{H} = -\hbar^2 \nabla^2 / 2m_0$, where $m_0$ is a mass of a particle, and define the linear differential operator

$$\mathcal{L} = i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m_0},$$

in terms of which (9) may be expressed as

$$\mathcal{L} \hat{\psi}(\mathbf{r}, t) = 0.$$

Acting with the components $k$ of $\mathcal{L}$ on the coherence function $g^{(p)}(x_1, x_2, \ldots, x_{2p})$, we obtain a set of Wolf-like equations [16, 17, 58]

$$\mathcal{L}_k g^{(p)}(x_1, x_2, \ldots, x_{2p}) = 0.$$
These equations can be cast in a ‘hydrodynamic’ form [33] by separating the real and imaginary parts, which yields two sets of coupled equations
\[
\text{Re}[g^{(p)} L_k g^{(p)}] = 0, \tag{13}
\]
\[
\text{Im}[g^{(p)} L_k g^{(p)}] = 0.
\]
These correspond to an Euler-like equation and a continuity equation for coherence flux, which govern the evolution of coherence in the system and hence the dynamics of the coherence simplices that move with the coherence flow [25]. The spectrum of coherence excitations or coherence quasi-particles can be found by linearizing these equations [35, 37].

The formal solution to (12) is
\[
g^{(p)}(x_1, x_2, \ldots, x_{2p}) = \int \cdots \int dx'_1 dx'_2 \cdots dx'_{2p} \times G(x_1, x_2, \ldots, x_{2p}, x'_1, x'_2, \ldots, x'_{2p}) g^{(p)}(x'_1, x'_2, \ldots, x'_{2p}), \tag{14}
\]
which evolves the coherence function into the future using the forward-time many-body propagator
\[
G(x_1, x_2, \ldots, x_{2p}, x'_1, x'_2, \ldots, x'_{2p}) = -i[I_1(x_1) \cdots I_2(x_{2p}) \hat{\psi}^\dagger(x'_1) \cdots \hat{\psi}^\dagger(x'_{2p})]. \tag{15}
\]
Here we assume a time-ordering in which all primed time coordinates lie in the past of their corresponding unprimed coordinates. Equation (15) may be viewed as a generalized Huygens-type formula for the propagation of coherence [16], the propagation occurring from one set of equal-time space–time points \((x'_1, x'_2, \ldots, x'_{2p})\), to a different set of equal-time future space–time points \((x_1, x_2, \ldots, x_{2p})\). For the special case of two-point correlation functions in one spatial dimension, this may be visualized as shown in figure 3(a). This depicts the propagation of the two-point equal-time one-spatial-dimensional correlation function \(g^{(1)}\), from its boundary value over the hyperplane \(t_1 = t_2 = t\), to its boundary value over the hyperplane \(t_1 = t_2 = t + \tau\), \(\tau \geq 0\). Spatial coordinates are denoted with uppercase \(X\), to distinguish from space–time coordinates denoted by a lowercase \(x\). In this low-dimensional example, the coherence function over an infinitesimal patch at \(A = (X'_1, t_1 = t, X'_2, t_2 = t) \equiv (x'_1, x'_2)\) with infinitesimal area \(d\tilde{X}_1 d\tilde{X}_2 = dx'_1 dx'_2\) in the \(X'_1 - X'_2\) plane (denoted \(\Pi_1\)), propagates to give the contribution \(dx'_1 dx'_2 G(x_1, x_2, x'_1, x'_2) g^{(1)}(x_1, x_2)\) to the coherence over the entire \(X_1 - X_2\) plane (denoted \(\Pi_2\)). This contribution is the coherence-theory analogue of the concept of a Huygens elementary wavelet, as embodied in the Huygens–Fresnel principle [16]. Every such contribution (from the points \(A, B, C\) etc in figure 3(a)) is then summed, via equation (14), to give the propagated coherence function \(g^{(1)}(x_1, x_2)\) over the plane \(\Pi_2\).

This superposition of elementary Huygens-type wavelets may lead to points \(D\) at which \(g^{(1)}\) vanishes, due to what may be termed ‘complete destructive interference of the coherence-function wavelets’ at such a point. Coherence vortices (\(N_1\) and \(N_2\) in figure 3(b), which appear as points \(F, G, L\) and \(M\) in the planes \(\Pi_1\) and \(\Pi_2\)), for which a zero of \(g^{(1)}\) is accompanied with a nonzero value of \(\kappa_1\), are a special case of such complete destructive interference of coherence. More ‘regular’ points such as \(E\) in figure 3(a) (and \(H, I, J\) and \(K\) in figure 3(b)) correspond to nonzero values for \(g^{(1)}\).

By encircling the coherence vortex \(G\) in figure 3(b), the associated simple smooth closed path sampled at points \(H, I, J\) and \(K\) picks up a phase factor which identifies a vortical coherence field. If such an integral yields nonzero \(\kappa_1\), even if the field is fully or partially coherent at points \(H, I, J\) and \(K\), there must be at least one point such as \(G\) in the space.
Figure 3. Coherence simplices illustrated. As discussed in the main text, panel (a) shows a Huygens-type construction for the propagation of two-point correlation functions from hyperplane $\Pi_1$ to hyperplane $\Pi_2$. Three Huygens wavelets are shown, emanating from points $A$, $B$ and $C$; $\tau$ is a control parameter, which can be pictured as a generalized propagation distance, whose increasing value continuously evolves $\Pi_1$ into $\Pi_2$. Panel (b) shows an open nodal line $N_1$, which threads coherence vortices located at points $F$ and $G$, with $N_2$ denoting a closed coherence-vortex loop. A third possibility, that of a nodal-line knot $N_3$ or a coherence Hopfion (cf [11–15]), is not shown.

enclosed by the path, where coherence vanishes. A low-dimensional example, of this, is studied in [31]. Generically, the hypercore of a coherence simplex corresponds to a zero in the function $g^{(p)}(x_1, x_2, \ldots, x_{2p})$ with an associated nonzero coherence circulation $\kappa_p$. Note that the propagator, (15), being a correlation function itself, may also be vortical.

Figure 3(b) also illustrates the conservation of the coherence circulation. Consider a coherence vortex $N_1$ which at time $t$ pierces the $X'_1$–$X'_2$ plane and is the only coherence vortex present at that time. As for conventional vortical systems [1, 39], the total circulation is a topological invariant and therefore must be preserved during the propagation of the coherence over any time interval $\tau$. Note, however, that coherence vortex–antivortex pairs $(L, M)$ may nucleate and/or annihilate in the course of propagation without affecting the value of the total coherence circulation. Such coherence vortex pairs appear as a manifestation of coherence vortex space–time loops, such as $N_2$, along which the coherence vanishes. Coherence-vortex nodal-line knots (not shown) are also possible; such knotted vortices have been studied both theoretically and experimentally, for the case of coherent fields [11–15].

Finally, we demonstrate the evolution of vector coherence simplices, shown in figure 4 and confined in a toroid of length $12\alpha$, by propagating the coherence function forward in time as described by (14). The nodal lines of coherence with $m = \pm 1$ are denoted by green and black lines. Coherence vortex waves and loops of the kind $N_2$ (the fundamental elements of quantum...
Figure 4. Vector coherence dynamics. The nodal lines corresponding to the propagation of the coherence field shown in figure 2(a). Due to the symmetry of the problem only the (↑, ↑) spin component of the coherence function is shown. Green and black lines correspond to $m = 1$ and $m = -1$ coherence circulations, respectively. Coherence vortex loops and coherence vortex displacement waves are clearly visible in the figure. Coherence phase function at time $t = 5$ is also plotted (the color map corresponds to that in figures 2(b) and (d)).

coherence turbulence) are clearly visible in the figure. Space–time points where green and black lines join correspond to coherence vortex pair creation and annihilation events.

6. Discussion

We have introduced a hierarchy of coherence simplices. A zeroth-order coherence simplex is nothing but a conventional quantized vortex in complex wavefunctions describing e.g. coherent optical fields or superfluid matter. Higher-order vortical coherence simplices emerge as quantized vortices in generic multi-dimensional correlation functions. We have discussed the topological structure and dynamics of such coherence simplices. A hypercore of a coherence simplex corresponds to a topologically inevitable total destructive interference of the coherence-function wavelets. The existence of first-order vortical coherence simplices have already been verified experimentally using interferometric measurements of light [24]. To our
knowledge, they are yet to be discovered in the quasi-particle and matter-wave counterparts of coherent light.

Furthermore, the Berezinskii–Kosterlitz–Thouless mechanism, which enables quasi-long-range order and superfluidity in 2D systems, relies upon the coherence of vortex–antivortex pairs [59, 60]. However, direct observation of such spontaneously forming vortex dipoles has remained elusive [61–65]. It would be interesting to apply the theory of coherence simplices to such systems and to investigate the role of VCSs in the superfluid–normal phase transition in low-dimensional systems. The recently created photon Bose–Einstein condensates [66] provide a natural platform from which to study coherence simplices and their relation to conventional quantized vortices, which are a hallmark of superfluidity and are responsible for many of the special properties of superfluids. It is therefore fascinating to contemplate the analogous idea that higher-order coherence simplices might underlie and signify some fundamental yet so far undiscovered physical properties of matter. Finally, it was pointed out in [31] that coherence simplices may be associated with the process of quantum mechanical decoherence—a suggestion that warrants further research.

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