ON CYCLIC QUIVER PARABOLIC KOSTKA-SHOJI POLYNOMIALS

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ABSTRACT. We obtain an explicit combinatorial formula for certain parabolic Kostka-Shoji polynomials associated with the cyclic quiver, generalizing results of Shoji and of Liu and Shoji.

INTRODUCTION

In [OS] an analogue of parabolic Hall-Littlewood (HL) symmetric function was defined for general quivers. For the single loop quiver this recovers the parabolic Hall-Littlewood symmetric functions defined in [SW] [SZ] which in turn generalize the classical modified HL functions denoted $Q'_\mu(X;t)$ in [Mac]. For cyclic quivers this produces a modified (and parabolic) form of Shoji’s Hall-Littlewood functions for the complex reflection group $G(r,1,n)$ [Sho1, Sho2, Sho3] in the case of limit symbols. The parabolic Hall-Littlewood functions for general quivers encode the graded multiplicities in $GL$-equivariant Euler characteristics of vector bundles on Lusztig’s convolution diagrams [Lu]. Based on a higher vanishing conjecture of [OS], certain quiver Hall-Littlewood functions are expected—and in some cases known by [P]—to expand positively in the tensor Schur basis.

In this article we give a positive combinatorial formula for certain parabolic quiver Hall-Littlewood symmetric functions living on the cyclic quiver (Theorem 6); the Schur positivity of these functions was not previously known by other means such as [P]. Our formula expresses the Schur expansion coefficients of these functions, i.e., their quiver Kostka-Shoji polynomials, as a sum over certain multitableaux weighted by charge. This generalizes a result of [LiSho] in the non-parabolic case of the cyclic quiver with two nodes. The latter was derived from results of [AH] on the intersection cohomology of the enhance nilpotent cone. We give an independent, combinatorial proof of our formula. For a single node we recover the graded character of tensor products of Kirillov-Reshetikhin (KR) modules for affine $\mathfrak{sl}_n$ or equivalently the Euler characteristic of global sections of a line bundle twist of the cotangent bundle to a partial flag variety [Br] [Sh1] [Sh2].

Our formula implies that the cyclic quiver Hall-Littlewood functions to which it applies are the images of ordinary (single loop) parabolic Hall-Littlewood functions under a plethystic substitution (Corollary 8). We interpret this in the representation theory of the wreath product groups $\Gamma_n = \Gamma^n \rtimes S_n$, where $\Gamma$ is a cyclic group of order $r$, by showing that the plethystic substitution is realized by a graded form of induction from $S_n$ to $\Gamma_n$ (Proposition 9). We deduce that the graded induction of the (singly-graded) Garsia-Procesi module $R_\mu$ [GP] is a graded $\Gamma_n$-module whose Frobenius characteristic can be identified with a cyclic quiver parabolic Hall-Littlewood function (Corollary 10).
In a future work [OS2] we will study the relationship between the parabolic Hall-Littlewood functions for the cyclic quiver and Haiman’s wreath Hall-Littlewood functions (wreath Macdonald polynomials at \( q = 0 \)) [H].

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1. Statement of Main Results

1.1. Cyclic quiver symmetric functions. In this article we work with the cyclic quiver \( Q \) on \( n \) nodes. It has node set \( Q_0 = \mathbb{Z}/r\mathbb{Z} \) and arrow set \( Q_1 = \{(i, i + 1) \mid i \in Q_0\} \) where expressions involving elements of \( Q_0 \) are understood modulo \( r \). We set \( Q_1 = Q_1 \setminus \{(r - 1, 0)\} \).

Let \( T^{Q_i} \cong (\mathbb{C}^*)^r \) be an algebraic torus with a copy of \( \mathbb{C}^* \) for each arrow \( a \in Q_1 \). Write \( R(T^{Q_i}) = \mathbb{Z}[t_{a}^{-1}] \mid a \in Q_1 \) where \( t_a \) is the exponential weight of the \( a \)-th copy of \( \mathbb{C}^* \). Let \( \Lambda^Q = \bigotimes_{i \in Q_0} \Lambda^{(i)} \) be the tensor power of the algebra of symmetric functions \( \Lambda \) with a tensor factor \( \Lambda^{(i)} \cong \Lambda \) for each vertex \( i \in Q_0 \), with coefficients in \( R(T^{Q_i}) \). We use notation \( f[X^{(i)}] \) for \( f \in \Lambda \) and \( i \in Q_0 \), to denote the tensor in \( \Lambda^Q \) having \( f \) in the \( i \)-th tensor factor and 1’s elsewhere. Let \( Y \) be Young’s lattice of partitions. Then \( \Lambda^Q \) has a basis given by the tensor Schur functions \( s_{\lambda^\bullet} = \prod_{i \in Q_0} s_{\lambda^0}[X^{(i)}] \) where \( \lambda^\bullet = (\lambda^{(0)}, \ldots, \lambda^{(r - 1)}) \in Y^n \) is a \( Q_0 \)-multipartition.

1.2. Lusztig data. A Lusztig datum is a sequence of triples \( \{(i_k, a_k, \mu(k)) \mid 1 \leq k \leq m\} \) where \( i_k \in Q_0 \), \( a_k \in \mathbb{Z}_{> 0} \), and \( \mu(k) \in X_+(\text{GL}_{a_k}) \) is a dominant weight. Let \( i = (i_1, i_2, \ldots, i_m), a = (a_1, a_2, \ldots, a_m) \) and \( \mu(\bullet) = (\mu(1), \mu(2), \ldots, \mu(m)) \).

We say the Lusztig datum \( (i, a, \mu(\bullet)) \) is periodic if \( i_m = r - 1 \) and \( i_{k+1} = i_k + 1 \) for all \( 1 \leq k < m \), even if \( i_1 = 0 \), Borel if \( a_k = 1 \) for all \( k \), rectangular if the weight \( \mu(k) \) is a rectangle for all \( k \), that is, has the form \( \gamma^{a_k} \) for some \( \gamma_k \in \mathbb{Z}_{\geq 0}, \) concentrated at \( i \in Q_0 \) if \( \mu(k) = (0^{a_k}) \) for all \( k \) such that \( i_k \neq i \), and dominant if, for every \( i \in Q_0 \), the sequence of weights \( \mu(k) \) for \( i_k = i \), concatenates to a dominant weight. A periodic Lusztig datum will be called balanced if \( a_k = a_{k+1} \) whenever \( i_k \neq r - 1 \).

For even periodic Borel Lusztig data, the quiver Kostka-Shoji polynomial was defined by Finkelberg and Ionov [FI]. If all \( t_a \) are set to a single variable this recovers Shoji’s Green functions for the complex reflection group given by the wreath product of \( S_n \) with the cyclic group \( \mathbb{Z}/r\mathbb{Z} \), in the case of limit symbols [Sho1, Sho2].

We consider dominant periodic balanced rectangular Lusztig data that are concentrated at node \( r - 1 \). Such data can be specified by a triple \( (\mu, \eta, i_1) \) where \( \mu \in Y_s \) is a partition with \( s \) parts which are allowed to be zero, \( \eta = (\eta_1, \ldots, \eta_s) \in \mathbb{Z}_{\geq 0}^s \) with \( \sum_k \eta_k = n \) (which defines a standard parabolic subgroup of \( \text{GL}_n \)), and \( i_1 \in Q_0 \). Given \( (\mu, \eta, i_1) \) a Lusztig datum is constructed using \( s \) passes around the cyclic quiver \( Q \) with node set \( \mathbb{Z}/r\mathbb{Z} \). The first pass places \( \text{GL}_{\eta_1} \)-weights at nodes \( i_1, i_1 + 1 \), up to \( r - 1 \). For \( k \) going from 2 up to \( s \), the \( k \)-th pass places \( \text{GL}_{\eta_k} \)-weights at all the nodes in order from 0 to \( r - 1 \). For \( i \neq r - 1 \), all weights placed at node \( i \) are the zero weight. At node \( r - 1 \) the \( k \)-th pass places the rectangular weight \( (\mu_k^{\eta_k}) \) having \( \eta_k \) rows and \( \mu_k \) columns.

Example 1. Let \( r = 2 \), \( i_1 = 1, \eta = (\eta_1, \eta_2) = (2, 2), \) and \( \mu = (2, 1) \). We have Lusztig datum with first pass \( (1, 2, (2, 2)) \) and second pass \( (0, 2, (0, 0)), (1, 2, (1, 1)) \) so that \( i = (1, 0, 1), a = (2, 2, 2) \) and \( \mu(\bullet) = ((2, 2), (0, 0), (1, 1)) \). This Lusztig datum is
dominant: the weights at 0 ∈ Q_0 concatenate to (0, 0) ∈ X_+(GL_2) and at 1 ∈ Q_0 concatenate to (2, 2, 1, 1) ∈ X_+(GL_4).

1.3. Cyclic quiver HL functions and Kostka-Shoji polynomials. Let Ω[X] = \prod_{i∈X} (1-x)_{-1} be the Cauchy kernel. For a symmetric function \( f \) let \( f[X^{(i)}]^{-1} \) be the operator adjoint (with respect to the Hall inner product) to multiplication by \( K \). Define \( \Omega[Z] \) be the Cauchy kernel. For a symmetric function \( f \) let \( f[X^{(i)}]^{-1} \) be the operator adjoint (with respect to the Hall inner product) to multiplication by \( f \), all with respect to the tensor factor \( \Lambda^{(i)} \). For \( i ∈ Q_0 \) define the generating series of operators \( [OS] \):

\[
H^{(i)}(z) = \sum_{d∈\mathbb{Z}} z^d H^{(i)}_d = \Omega[zX^{(i)}]_1^{-1} H^{(i)}(X^{(i)} - t_{i,i+1} X^{(i+1)})^{-1}.
\]

For all \( d ∈ \mathbb{Z} \), \( H^{(i)}_d \) is a linear endomorphism of \( \Lambda^Q \) of degree \( d \). It is the cyclic quiver analogue of Garsia’s variant of Jing’s Hall-Littlewood creation operator \( \Omega[Z] \).

For \( i ∈ Q_0 \) and \( a ∈ \mathbb{Z}_{>0} \) let \( Z = (z_1, z_2, \ldots, z_a) \) and \( R(Z) = \prod_{1 ≤ k < t ≤ a} (1 - z_t / z_k) \). Define \( [OS] \):

\[
H^{(i,a)}(Z) = \sum_{β ∈ \mathbb{Z}^a} z^β H^{(i,a)}_{β} = R(Z) H^{(i)}(z_1) \cdots H^{(i)}(z_a).
\]

This is the cyclic quiver analogue of the parabolic Hall-Littlewood creation operator of \( [SZ] \).

Given a Lusztig datum \( Λ(i, a, µ(•)) \), the associated cyclic quiver parabolic Hall-Littlewood function \( \mu^{(i,a)}[X^{•}, t_{Q_1}] ∈ \Lambda^Q \) is defined by \( \mu^{(i,a)}[X^{•}, t_{Q_1}] = 1 \) and

\[
\mu^{(i,a)}[X^{•}, t_{Q_1}] = \mu^{(i,a)}[X^{•}, t_{Q_1}] \mu^{(i,a)}[X^{•}, t_{Q_1}]
\]

where \( i = (i_2, i_3, \ldots), a = (a_2, a_3, \ldots) \), and \( µ(•) = (µ(2), µ(3), \ldots) \).

The cyclic quiver parabolic Kostka-Shoji polynomials \( K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) ∈ R(T^{Q_1}) \) are defined by

\[
\mu^{(i,a)}[X^{•}, t_{Q_1}] = \sum_{λ^{•} ∈ \mathbb{V}^{Q_0}} K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) s_{λ^{•}}[X^{•}],
\]

Since the quiver \( Q \) has one cycle, the polynomials \( K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) \) essentially have only one variable. To make this precise let \( α^{(i)} = α^{(i)} - α^{(i+1)} \) for \( 0 ≤ i ≤ r - 2 \) be the simple roots of \( GL(\mathbb{C}^Q) \) where \( α^{(i)} \) is the standard basis of \( \mathbb{Z}^Q \). By the proof of \( [OS] \) Lemma 2.20], \( K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) = 0 \) unless \( \sum_{i ∈ Q_0} |λ^{(i)}|_ε^{(i)} - \sum_{k} |µ(k)|_ε^{(i_k)} \) is root lattice of \( GL(\mathbb{C}^Q) \), which means it has the form \( \sum_{i=0}^{r-2} a_i α^{(i)} \) for some integers \( a_i \). In that case we define the Laurent monomial

\[
t^{λ^{•} - µ(•)}_{Q_1} = \prod_{i=0}^{r-2} t^{a_i}_{i,i+1}.
\]

By \( [OS] \) Lemma 2.20] there is a polynomial \( K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) ∈ \mathbb{Z}[t] \) called the reduced Kostka-Shoji polynomial, such that

\[
K^{λ^{•}}_{i,a,µ(•)}(t_{Q_1}) = t^{λ^{•} - µ(•)}_{Q_1} \mu^{λ^{•}}_{i,a,µ(•)}(t_{Q_1} t_{12} \cdots t_{r-1,0}).
\]

We see that the single essential variable is the product of the arrow variables going around the cycle.
When \((i, a, \mu(\bullet))\) is the rectangular Lusztig datum associated to the triple \((\mu, \eta, i_1)\) as in [1.2] we use the notation

\[
\mathcal{H}_{\mu, \eta; i_1}[X^\bullet; t_{Q_1}] = \mathcal{H}_{i, \bullet}^{(i)}[X^\bullet; t_{Q_1}]
\]

(1.7)

\[
K_{\mu, \eta; i_1}^{\lambda_1^\bullet}(t_{Q_1}) = K_{i, \eta; (\bullet)}^{\lambda_1^\bullet}(t_{Q_1})
\]

(1.8)

\[
\mathcal{X}_{\mu, \eta; i_1}(t) = \mathcal{X}_{i, \eta; (\bullet)}(t).
\]

(1.9)

1.4. LR multitableaux. The definitions in the next several subsections follow [Sh1]. We refer the reader to Appendix A for conventions and further details on our tableau constructions. Let \(A_1, \ldots, A_s\) be the consecutive subintervals of \([n] = \{1, 2, \ldots, n\}\) where \(|A_k| = \eta_k\) and let \(Y_k\) be the rectangular tableau of width \(\mu_k\) and height \(\eta_k\) whose rows are constant and contain the values of \(A_k\).

Example 2. Continuing the previous example we have \(A_1 = \{1, 2\}\), \(A_2 = \{3, 4\}\) and

\[
Y_1 = \begin{array}{c}
1 & 1 \\
2 & 2
\end{array}, \quad Y_2 = \begin{array}{c}
3 \\
4
\end{array}
\]

Say that a word \(u\) in the alphabet \([n]\) is \((\mu, \eta)\)-LR (Littlewood-Richardson) if for all \(1 \leq k \leq s\), we have the Knuth equivalence \(u|_{A_k} \equiv Y_k\) (see \([A.1]\)) where \(u|_{A_k}\) is the subword obtained by erasing letters of \(u\) not in \(A_k\).

We identify (semistandard) tableaux with their reading words (see \([A.1]\)). By a multitableau we mean a tuple \(T^\bullet = (T^{(r)})_{r \in \Omega_0}\) of tableau. The word of a multitableau is defined as the concatenation word\(T^\bullet = T^{(0)}T^{(1)}\cdots T^{(r-1)}\) of reading words of each tableau in the multitableau. Say that a tableau or multitableau is \((\mu, \eta)\)-LR if its word is.

Let \(LR_{\mu, \eta}\) denote the set of \((\mu, \eta)\)-LR words. Let \(LR_{\mu, \eta}^\lambda\) be the set of \((\mu, \eta)\)-LR tableaux of shape \(\lambda \in \mathcal{Y}\).

Remark 1. The name LR tableau comes from the fact (see Cor. 24) that the LR coefficient of \(s_h\) in the product of the Schur functions \(s_{(\mu^\bullet)}\), is equal to \(|LR_{\mu, \eta}^\lambda|\).

1.5. Rotation of LR words. Let \(w_0^n\) be the long element of the subgroup \(S_n = S_{\eta_1} \times S_{\eta_2} \times \cdots \times S_{\eta_s}\) of \(S_n\). The following operation allows the “rotation” of LR words. This must be used every time letters pass between the 0-th and \((r - 1)\)-th tableaux in a multitableau.

Proposition 3. For words \(u\) and \(v\), \(uv \in LR_{\mu, \eta}\) if and only if \((w_0^n v)(w_0^n u) \in LR_{\mu, \eta}\), where \(w_0^n\) acts via the composition of \(GL_n\) crystal reflection operators (see \([A.3]\)). This induces an action of the cyclic group \(Z/NZ\) on \(LR_{\mu, \eta}\) where \(N = \sum \mu_k \eta_k\).

Remark 2. (1) If \(\eta_k = 1\) for all \(k\) then \(S_\eta\) is the identity subgroup and the action is usual rotation of a word by positions.

(2) Since the crystal reflection operators \(s_i\) are well-defined on Knuth classes (that is, if \(u \equiv v\) then \(s_i u \equiv s_i v\)), \(Z/NZ\) acts on the set \(\bigcup \lambda \in \mathcal{Y} LR_{\mu, \eta}\). This leads to the cyclage poset structure on LR tableaux [Sh1] which generalizes the cyclage poset on tableaux [LS].

Example 4. Let \(u = 2\) and \(v = 42131\). Then \(uv \in LR_{\mu, \eta}\) because \(uv|_{A_1} = 2211 = Y_1\) and \(uv|_{A_2} = 43 = Y_2\). We have \(w_0^n(u) = 1\), \(w_0^n v = 42132\). We have \((w_0^n v)(w_0^n u) = 421321 \in Y_{\mu, \eta}\) because \(421321|_{A_1} = 2121 = Y_1\) and \(421321|_{A_2} = 43 = Y_2\).
1.6. A charge statistic for LR words. Let \(|u|\) denote the length of the word \(u\).

**Proposition 5.** \([Sh1]\) There is a unique function \(\text{charge}_{\mu,\eta} : \text{LR}_{\mu,\eta} \to \mathbb{Z}_{\geq 0}\) such that

1. If \(\mu\) and \(\eta\) are empty then \(\text{charge}_{\mu,\eta}(\emptyset) = 0\) where \(\emptyset\) is the empty word.
2. \(\text{charge}_{\mu,\eta}\) is constant on Knuth classes.
3. If \(uY_1v \in \text{LR}_{\mu,\eta}\) then writing \(\mu = (\mu_1, \mu_2)\) and \(\eta = (\eta_1, \eta_2)\), we have
   \[
   \text{charge}_{\mu,\eta}(uY_1v) = |v| + \text{charge}_{\mu,\eta}(w_0^3v)(w_0^3u).
   \]

**Remark 3.** If \(\eta_k = 1\) for all \(k\) (the Lusztig datum is Borel) then \(\text{charge}_{\mu,\eta}\) is the charge statistic of Lascoux and Schützenberger \([LS]\).

1.7. Tableau formula. For \(\lambda^* \in \mathbb{Z}_0^n\) let \(\text{LR}^{\lambda^*}_{\mu,\eta,i_1}\) be the set of multitableaux \(T^*\) of shape \(\lambda^*\) (i.e., each \(T^*(i)\) has shape \(\lambda^{(i)}\)) such that \(\text{word}(T^*) \in \text{LR}_{\mu,\eta}\) and \(T^*(i)|_{A_i} = \emptyset\) for \(0 \leq i < i_1\). Note that for \(i_1 = 0\), \(\text{LR}^{\lambda^*}_{\mu,\eta,0}\) is just the set of multitableaux of shape \(\lambda^*\) whose word is \((\mu,\eta)\)-LR.

For \(T^* \in \text{LR}^{\lambda^*}_{\mu,\eta,i_1}\) we set \(\text{charge}_{\mu,\eta}(T^*) = \text{charge}_{\mu,\eta}(\text{word}(T^*))\).

**Theorem 6.** We have

\[
K^{\lambda^*}_{\mu,\eta,i_1}(t) = \sum_{T^* \in \text{LR}^{\lambda^*}_{\mu,\eta,i_1}} t^{\text{charge}_{\mu,\eta}(T^*)}.
\]

**Remark 4.** In the special case of \(r = 2\) nodes and \(\eta_k = 1\) for all \(k\) (Borel case), an equivalent formula was obtained in \([LiSho]\) using results of \([AH]\). We give a combinatorial proof of Theorem 6 in Section 3 which is independent of these results.

**Remark 5.** The higher vanishing criterion of \([P]\) is sufficient to establish the positivity of \(K^{\lambda^*}_{\mu,\eta,i_1}(t)\) in the Borel case for any \(r\), but not in general.

For \(T^* \in \text{LR}^{\lambda^*}_{\mu,\eta,i_1}\), using notation similar to \([1.5]\) let

\[
\text{wt}_{\mu,\eta}(T^*) = t^{\lambda^* - \sum_{i_0} (t_{1,2} \cdots t_{r-1,0})^\eta} \text{charge}_{\mu,\eta}(T^*),
\]

so that by \([1.9]\), Theorem 6 is equivalently expressed as

\[
K^{\lambda^*}_{\mu,\eta,i_1}(t_{Q_1}) = \sum_{T^* \in \text{LR}^{\lambda^*}_{\mu,\eta,i_1}} \text{wt}_{\mu,\eta}(T^*).
\]

For the single node version, let \(\mathcal{H}_{\mu,\eta}[X;t]\) be the parabolic Hall-Littlewood symmetric function associated with the sequence of rectangles \((\mu_k^{(n)})\) for \(1 \leq k \leq s\) \([SZ]\). Equivalently it is the graded character of a Kirillov-Reshetikhin module \([Sh2]\) and the cyclic quiver Hall-Littlewood symmetric function of \([OS]\) for the single loop quiver with Lusztig data \((0,\eta_k,(\mu_k^{(n)}))\) for \(1 \leq k \leq s\) and \(t = t_{00}\).

Define their coefficients by

\[
\mathcal{H}_{\mu,\eta}[X;t] = \sum_{\lambda \in \mathcal{Y}} K^{\lambda}_{\mu,\eta}(t) s_{\lambda}[X].
\]

**Theorem 7.** \([Sh1]\)

\[
K^{\lambda}_{\mu,\eta}(t) = \sum_{T \in \text{LR}^{\lambda}_{\mu,\eta}} t^{\text{charge}_{\mu,\eta}(T)}
\]
Remark 6. In [Sh1] the above sum over $T$ was shown to give a graded isotypic component of a line bundle twist of the coordinate ring of a nilpotent adjoint orbit closure [SW]. Subsequently the parabolic Jing operators were defined [SZ] and the above isotypic component was shown to agree with the coefficient of a Schur function in the parabolic Hall-Littlewood function, which by definition is created by the parabolic Jing operators. The connection of these notions to the Kirillov-Reshetikhin character was proved in [Sh2]; independently similar but dual combinatorics were developed in [ScWa] for KR characters.

Corollary 8. With

\[ Y = \sum_{i=0}^{r-1} t_{i,i+1} t_{i+1,i+2} \cdots t_{r-2,r-1} X^{(i)} \]

we have

\[ \mathcal{H}_{\mu,\nu,i_1=0}[X^*; tQ_1] = \mathcal{H}_{\mu,\nu}[Y; t_0 t_1 \cdots t_{r-1,0}] . \]

Proof. Iterating the coproduct formula \[ \Delta(s_\lambda) = \sum_{\mu, \nu \in Y} (s_\mu, s_\nu s_\mu) s_\nu \otimes s_\mu \] and appropriately scaling alphabets we have

\[ s_\lambda[Y] = \sum_{\lambda^* \in YQ_0} t^{\lambda^* - N_\lambda^{\nu(r-1)}} c^{\lambda^*}_{\lambda} s_\lambda[X^*] \]

where \[ c^{\lambda^*}_{\lambda} = (s^{\lambda^*}, \prod_{i \in Q_0} s^{\lambda(i)}) \]. Applying this to \( \mathcal{H}_{\mu,\nu}[X; t] \) we have

\[ \mathcal{H}_{\mu,\nu}[Y; t_0 \cdots t_{r-1,0}] = \sum_{\lambda^* \in YQ_0} s_\lambda[X^*] t^{\lambda^* - N_\lambda^{\nu(r-1)}} \sum_{\lambda \in Y} c^{\lambda^*}_{\lambda} \mathcal{H}_{\mu,\nu}[Y_1 \cdots t_{r-1,0}] . \]

Consider the map \( T^* \mapsto \eta \mapsto (P(\eta), Q(\eta)) \) where \( \eta \) is the sequence of row words obtained from the rows of \( T^{r-1} \), then the rows of \( T^{r-2} \), up to the rows of \( T^{(0)} \) and \( (P(\eta), Q(\eta)) \) is the column insertion Robinson-Schensted-Knuth tableau pair (see [A2]). By Theorem [25] and Cor. [27] and their notation, the image consists of all tableau pairs \( (P, Q) \) such that \( Q \) is \( D \)-compatible where \( D \) is the skew shape \( \nu(r-1) \ast \cdots \ast \nu^{(0)} \). Restricting this bijection to \( (\mu, \eta) \)-LR words, we obtain a bijection

\[ LR_{\mu,\eta,i_1=0} \mapsto \bigcup_{\lambda} LR_{\mu,\eta}^\lambda \times \{ D \text{-compatible tableaux of shape } \lambda \} . \]

Since Knuth equivalence preserves \( \mu, \eta \), the result follows from Theorems [8] and [4].

2. Graded representations of wreath products

In this section we discuss the meaning of our results in the representation theory of the wreath product group \( \Gamma_n = \Gamma^r \rtimes S_n \) where \( \Gamma \) is a cyclic group of order \( r \). We show in particular that the plethystic substitution of Corollary [8] is, on the level of Frobenius characterisitics, a type of graded induction from \( S_n \) to \( \Gamma_n \). It follows that the induction of the Garsia-Procesi module \( R_{\mu} \) results in a graded \( \Gamma_n \)-module whose Frobenius characteristic is equal, up to the involution \( \omega \) on symmetric functions, to a cyclic quiver parabolic Hall-Littlewood function.

We assume in this section that \( t_{i,i+1} = t \) for all \( i \in Q_0 \), with \( t \) serving as the grading parameter. (Note the the difference between this \( t \) and \( t \) above which played the role of the product of all arrow variables \( t = t_0, t_{1,2} \cdots t_{r-1,0} \).)
2.1. Frobenius characteristics of graded $S_n$-modules. Let $R(S_n)$ denote the Grothendieck ring of the category of graded $S_n$-modules $M = \oplus_{d \geq 0} M_d$ with $M$ finite-dimensional over $\mathbb{C}$. The graded Frobenius characteristic map

$$M \mapsto \text{Frob}(M; t) = \frac{1}{n!} \sum_{w \in S_n} \text{Tr}_M(w; t) p_{r(w)}$$

gives a linear isomorphism of $R(S_n)$ with degree $n$ symmetric functions over $\mathbb{Z}[t]$. Here we set $\text{Tr}_M(x; t) = \sum_d t^d \text{Tr}_M(x)$ for any graded endomorphism $x$ of a finite-dimensional graded vector space $M = \oplus_{d \geq 0} M_d$. For a permutation $w \in S_n$, we let $\tau(w)$ denote its cycle type, i.e., the partition of $n$ with parts given by the lengths of cycles in $w$. And for $\lambda \in \mathbb{Y}$, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ is the corresponding power sum basis element in $\Lambda$.

2.2. The ring $\Lambda^\Gamma$. We realize $\Gamma$ as the group of $r$-th roots of unity and fix a generator $\zeta \in \mathbb{C}^\times$. Accordingly, we let $\Lambda^\Gamma$ denote the tensor power $\Lambda \otimes^{\mathbb{Q}^0}$ of symmetric functions with coefficients in $\mathbb{C}[t]$. Let $\Lambda^\Gamma_\mathbb{Y} \subset \Lambda^\Gamma$ denote the $\mathbb{C}[t]$-submodule of elements of total degree $n$.

There are two natural sets of power sum generators in $\Lambda^\Gamma$, one indexed by conjugacy classes in $\Gamma$ and the other by irreducible representations of $\Gamma$. In our present situation, we abuse notation and identify both index sets with $\mathbb{Y}$. For a cycle $\tau$ in $\Gamma$, we write $\text{Tr}_M(\tau)$. Then, for $\lambda \in \mathbb{Y}$, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. The ring $\Lambda^\Gamma = \mathbb{C}[t][p_\lambda(i) : \lambda \in \mathbb{Y}, i \geq 0] = \mathbb{C}[t][\hat{p}_\lambda(i) : i \in \mathbb{Q}_0, s \geq 0]$ with $\deg p_\lambda(i) = \deg \hat{p}_\lambda(i) = s$. The two sets of generators are related by the Fourier transform on $\Gamma$ as follows:

$$p_\lambda^{(j)} = r^{-1} \sum_{i=0}^{r-1} \zeta^{ij} \hat{p}_\lambda^{(i)}.$$  

For $\lambda^\bullet \in \mathbb{Y}^{\mathbb{Q}_0}$ we set

$$p_{\lambda^\bullet} = \prod_{i \in \mathbb{Q}_0} p_{\lambda(i)}^{(i)}; \quad \hat{p}_{\lambda^\bullet} = \prod_{i \in \mathbb{Q}_0} \hat{p}_{\lambda(i)}^{(i)}$$

where $p_{\lambda(i)}^{(i)} = p_{\lambda_1(i)}^{(i)} p_{\lambda_2(i)}^{(i)}$ as usual, and similarly for $\hat{p}_{\lambda(i)}^{(i)}$.

2.3. Conjugacy in $\Gamma_n$. Suppose $g = (g_1, \ldots, g_n, w) \in \Gamma_n$. For a cycle $C = (ab \cdots c)$ of $w \in S_n$, we write $g_C = g_{a} g_{b} \cdots g_{c}$, and say that $C$ has color $i$ (with respect to $g$) if $g_C = \zeta^i$ where $0 \leq i < r$. We define the cycle type $\tau^\bullet(g)$ to be the $\mathbb{Q}_0$-tuple of partitions of total size $n$ given as follows: $\tau^{(i)}(g)$ lists the sizes of cycles of $w$ having color $i$. This gives a bijection between conjugacy classes in $\Gamma_n$ and $\mathbb{Q}_0$-tuples of partitions of total size $n$.

2.4. Frobenius characteristics of graded $\Gamma_n$-modules. Let $R(\Gamma_n)$ be the complexified Grothendieck ring of the category of graded $\Gamma_n$-modules $M = \oplus_{d \geq 0} M_d$ with $M$ finite-dimensional over $\mathbb{C}$. In this setting, the Frobenius characteristic map

$$M \mapsto \text{Frob}^\Gamma(M; t) = |\Gamma_n|^{-1} \sum_{g \in \Gamma_n} \text{Tr}_M(g; t) \hat{p}_{\tau^\bullet(g)}$$
gives an isomorphism $R(\Gamma_n) \cong \Lambda^n$ sending irreducible $\Gamma_n$-modules to the tensor Schur functions (relative to the $p_s^{(j)}$ of degree $n$ [Mac, I, Appendix B].

2.5. Graded induction. Let $S = \text{Sym}(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n]$, with $\Gamma_n$ acting by the natural algebra automorphisms given by $g \cdot x_k = g(w(k))x(w(k))$ for $g$ as above. We take the standard grading on $S$ with $\deg x_k = 1$. Consider the $\Gamma_n$-submodule

$$S_{<r} = \bigoplus_{0 \leq m_1, \ldots, m_n < r} \mathbb{C} x_1^{m_1} \cdots x_n^{m_n}.$$  

Note that $S_{<r} \cong \mathbb{C}[\Gamma^n]$ as ungraded $\Gamma_n$-modules, with an isomorphism given by

$$x_1^{m_1} \cdots x_n^{m_n} \mapsto \sum_{g = (g_1, \ldots, g_n) \in \Gamma^n} g_{m_1}^{-1} \cdots g_n^{-m_n} \cdot g.$$

For any finite-dimensional graded $S_n$-module $M$ we define a graded $\Gamma_n$-module $\text{Ind}(M)$ as the following tensor product of $\Gamma_n$-modules:

$$(2.4) \quad \text{Ind}(M) = S_{<r} \otimes_{\mathbb{C}} \tilde{M}.$$  

Here $\tilde{M} = M$ as ungraded vector spaces, with the $\Gamma_n$-action on $\tilde{M}$ trivially extended from that of $S_n$ on $M$; we choose the grading on $\tilde{M}$ which is dilated from that of $M$ by a factor of $r$, i.e., $\tilde{M} = \oplus_{d \geq 0} \tilde{M}_d$ where $\tilde{M}_r = M$. As ungraded $\Gamma_n$-modules we clearly have $\text{Ind}(M) \cong \text{Ind}_{S_n}(M)$.

Remark 7. In the case when $M = S/I$ is a graded $S_n$-module determined by an $S_n$-equivariant homogeneous ideal $I \subset S$, we have the following equivalent description of $\text{Ind}(M)$. Let $J$ be the image of $I$ under the $r$-th power ring homomorphism given by $x_k \mapsto x_k^r$ on the generators of $S$. Then $SJ \subset S$ is a $\Gamma_n$-equivariant homogeneous ideal and $\text{Ind}(M) \cong S/SJ$ as graded $\Gamma_n$-modules.

Proposition 9. For any finite-dimensional graded $S_n$-module $M$ we have

$$(2.5) \quad \text{Frob}^G(\text{Ind}(M); t) = \text{Frob}(M; t^r)[X^{(0)} + tX^{(1)} + \cdots + t^{r-1}X^{(r-1)}].$$

Proof. We compute $\text{Frob}^G(\text{Ind}(M); t)$ directly. By (2.4) we have

$$\text{Tr}_{\text{Ind}(M)}(g; t) = \text{Tr}_{S_{<r}}(g; t)\text{Tr}_{\tilde{M}}(w; t) = \text{Tr}_{S_{<r}}(g; t)\text{Tr}_{\tilde{M}}(w; t^r).$$

for any $g = (g_1, \ldots, g_n, w) \in \Gamma_n$.

Now fix such a $g$ and let $C_1, \ldots, C_\ell$ be the cycles of $w$ (in any fixed order) and $i_1, \ldots, i_\ell$ their colors (with respect to $g$). By considering the matrix of $g$ with respect to the basis of $S_{<r}$, given by the monomials in (2.2), one finds

$$\text{Tr}_{S_{<r}}(g; t) = \sum_{0 \leq j_1, \ldots, j_\ell < r} \zeta^{i_1j_1 + \cdots + i_\ell j_\ell} t^{j_1|C_{i_1}| + \cdots + j_\ell|C_{i_\ell}|}$$

where $|C|$ denote the length of a cycle $C$. Each summand above is a diagonal entry corresponding to a basis element of the form $x_{C_1}^{j_1} \cdots x_{C_\ell}^{j_\ell}$, where $x_C = x_{a_1}x_{b_1} \cdots x_{c_1}$.

---

1 One may be tempted to compute $\text{Tr}_{S_{<r}}(g; t)$ via the standard basis of $\mathbb{C}[\Gamma^n] \cong S_{<r}$. However, this basis is not homogeneous with respect to (2.3). The ungraded computation immediately yields $\text{Tr}_{\mathbb{C}[\Gamma^n]}(g; 1) = p_{r(w)}^{(0)}$, which is seen to equal $p_{r(w)}(X^{(0)} + X^{(1)} + \cdots + X^{(r-1)})$ by (2.1).
for a cycle $C = (ab\cdots c)$; the other diagonal entries vanish. Hence

$$r^n \sum_{g_1,\ldots,g_n \in \Gamma} \text{Tr}_{S_{\langle r \rangle}}(g; \mathfrak{t}) \overline{p}(g) = r^{-n} \prod_{s=1}^{\ell} \sum_{0 \leq j_s < r} \sum_{0 \leq \ell_s < r} r^{-1} \xi_{\ell_s} \tau_{\ell_s} \overline{p}(\ell_s) = p_{\tau(w)}[X^{(0)} + tX^{(1)} + \cdots + t^{r-1}X^{(r-1)}].$$

From this we immediately deduce (2.5). □

2.6. Induction of $R_\mu$ as a quiver Hall-Littlewood function. For a partition $\mu$ of size $n$, consider the ordinary parabolic Hall-Littlewood function $H_{(1,\ldots,1),\mu}$ indexed by the sequence of rectangles of sizes $\mu_k \times 1$ (i.e., columns of height $\mu_k$). By [SW] Example 2.3(2), we have $H_{(1,\ldots,1),\mu} = \omega \text{Frob}(R_\mu; \mathfrak{t})$ where $R_\mu = S/I_\mu$ is the graded $S_\mu$-module of $\left[S,\mu \right]$ and $\omega$ is the involution on $\Lambda$ such that $\omega(s_{\lambda}) = s_{\lambda'}$ for the transpose \( \lambda' \) of $\lambda$.

By abuse of notation, we use $\omega$ also to denote the involution on $\Lambda^\Gamma$ given by the tensor power $\Lambda^\Gamma \otimes \mathfrak{t}_0$. Combining Corollary 8 and Proposition 9 we see that $H_{(1,\ldots,1),\mu}$ is given by a cyclic quiver parabolic Hall-Littlewood function as follows:

**Corollary 10.** For any partition $\mu$ of size $n$ we have

$$H_{(1,\ldots,1),\mu} = H_{(1,\ldots,1),\mu}.$$

**Remark 8.** Generators for the defining ideal of $\text{Ind}(R_\mu)$ are obtained simply by replacing $x_1,\ldots,x_n$ in the the Tanisaki generators of $I_\mu$ [GP] 1.5 by their $r$-th powers, thanks to Remark 4. In the case of $\mu = (1^n)$, $R_n = R_{(1^n)}$ and $\text{Ind}(R_n)$ are the coinvariant algebras of $S_n$ and $\Gamma_n$, respectively. Steinbrügge [S] Theorem 6.6] gives the graded $\Gamma_n$-module structure of $\text{Ind}(R_n)$. Chan and Rhoades [CR] extend this to a generalized coinvariant algebra for $\Gamma_n$ depending on an additional parameter $k$ such that $0 \leq k \leq n$.

3. Proof of Theorem 6

3.1. Recurrence. We recall a recurrence for the parabolic Kostka-Shoji polynomials [OS] 4.

Let $(\mathfrak{i}, \mathfrak{a}, \mathfrak{b}(\bullet))$ denote the Lusztig data for $\mu$, $\eta$, $i_1$. So $\mathfrak{i} = (i_1, i_1 + 1, i_1 + 2, \ldots)$, $\mathfrak{a}$ consists of $\eta_1$ repeated $r - i_1$ times followed by $\eta_k$ repeated $r$ times for $2 \leq k \leq s$, and $\mathfrak{b}(\bullet)$ is the sequence starting with $r - i_1 - 1$ copies of $(0^{|\eta_1|})$ followed by $(\mu_1^n)$ and then for $2 \leq k \leq s$, $r - 1$ copies of $(0^{|\eta_k|})$ followed by $(\mu_k^n)$.

Let $(i_1, a_1, \mu(1)) = (i, a, \nu)$. We have $\nu = (\mu_1^{i})$ if $i = r - 1$ and $\nu = (0^{|\eta_1|})$ otherwise. For $w \in S_n$ define $\alpha(w) \in X(GL_n) = \mathbb{Z}^a$ and $\beta(w) \in X(GL_{n-a}) = \mathbb{Z}^{n-a}$ by

$$w^{-1}(\lambda^{(i)}) + \rho = (\alpha(w), \beta(w))$$
Let $S_n^a$ be the set of minimum length coset representatives for $S_n/(S_a \times S_{n-a})$. By [OS] we have

\begin{equation}
K_{\mu,\eta,i_1}^\ast(tQ_1) = \sum_{w \in S_n^a} (-1)^w t_i^{\alpha(w)} \sum_{\gamma^{(i+1)}} C_{\lambda^{(i+1)},\alpha(w)}^{\gamma^{(i+1)}} K_{\mu,\eta,i_1+1}^\ast(tQ_1),
\end{equation}

where $\gamma^{(j)} = \lambda^{(j)}$ for $0 \leq j < i$, $\gamma^{(i)} = \beta(w)$, and $\gamma^{(i+1)} \in \mathbb{Y}_n$.

Let $K_{\sigma}^{\lambda/\mu}$ be the Kostka number, the number of semistandard tableaux of shape $\gamma/\lambda$ and weight $\sigma$. Using the Jacobi-Trudi formula for the skew Schur function we have

\begin{equation}
K_{\mu,\eta,1}^\ast(tQ_1) = \sum_{\gamma^{(i)}} (-1)^w t_i^{\alpha(w)} K_{\alpha(w)}^{\gamma^{(i+1)}/\lambda^{(i+1)}} K_{\beta(w)}^{\gamma^{(i)+1}} K_{\mu,\eta,i_1+1}^\ast(tQ_1),
\end{equation}

where

\begin{equation}
\gamma^{(j)} = \lambda^{(j)} \quad \text{for } j \in Q_0 \setminus \{i, i+1\}.
\end{equation}

If $i_1 = r - 1$, the formula and proof is very similar. We have $\nu = (\mu_1^{q_1})$ and obtain

\begin{equation}
K_{\mu,\eta,r-1}^\ast(tQ_1) = \sum_{\gamma^{(i)}} (-1)^w t_i^{\alpha(w)/\nu} K_{\alpha(w)/\nu}^{\gamma^{(i+1)}/\lambda^{(i+1)}} K_{\beta(w)}^{\gamma^{(i)+1}} K_{\mu,\eta,0}^\ast(tQ_1),
\end{equation}

such that (3.4) holds.

Remark 9. The recurrence (3.2) is more efficient than (3.5) but the cancelling bijection is much simpler to define for the latter.

3.2. Morris data. Let $T_{\nu}^{\lambda/\mu}$ be the set of semistandard tableaux of shape $\gamma/\lambda$ and weight $\nu$.

A ($\lambda^\ast; \mu, \eta, i_1$)-Morris datum is a tuple $\Xi = (w, S^\ast, U^{(i)}, U^{(i+1)})$ where $w \in S_n$ is such that $\alpha(w) \geq \nu$ componentwise, $U^{(i)} \in T_{\lambda^{(i)}}^{(r)}, U^{(i+1)} \in T_{\lambda^{(i)+1},\nu}$, and $S^\ast \in LR_{\mu,\eta,1}^{\gamma^{(i)},\lambda^{(i)+1}}$ if $0 \leq i_1 < r - 1$ and $S^\ast \in LR_{\mu,\eta,1}^{\gamma^{(i)+1},\lambda^{(i)+1}}$ if $i_1 = r - 1$ with $\gamma^\ast$ as in (3.4).

Define the sign and weight of $\Xi = (w, S^\ast, U^{(i)}, U^{(i+1)})$ by

\begin{equation}
\sgn(\Xi) = (-1)^{t(w)}
\end{equation}

\begin{equation}
\wt Q_1(\Xi) = t_i^{\alpha(w)/\nu} \times \begin{cases} \wt_{\mu,\eta}(S^\ast) & \text{if } 0 \leq i_1 < r - 1 \\ \wt_{\mu,\eta}(S^\ast) & \text{if } i_1 = r - 1. \end{cases}
\end{equation}

Write $M_{\mu,\eta,i_1}^\ast$ for the set of ($\lambda^\ast; \mu, \eta, i_1$)-Morris data. It suffices to find a sign-reversing weight-preserving involution $\Phi$ on $M_{\mu,\eta,i_1}^\ast$ whose fixed point set has a weight-preserving bijection with $LR_{\mu,\eta,1}^{\gamma^{(i)},\lambda^{(i)+1}}$.

3.3. Embedding LR multitables into Morris data. We define a map

\begin{equation}
LR_{\mu,\eta,i_1}^{\ast} \longrightarrow M_{\mu,\eta,i_1}^\ast
\end{equation}

\begin{equation}
\iota(T^\ast) = (\text{id}, S^\ast, U^{(i)}, U^{(i+1)})
\end{equation}

as follows.

Suppose first that $0 \leq i_1 < r - 1$. Let $u_k$ be the $k$-th row of $T^{(i)}$, denoted $\text{row}_k(T^{(i)})$, for $1 \leq k \leq a$ and let $S^{(i)}$ be the last $n - a$ rows of $T^{(i)}$ so that
Let $\Psi(u_a \cdot \cdot \cdot u_1, T(i+1)) = (S^{(i+1)}, U^{(i+1)})$ where $\Psi$ is defined in \(A.2\) via Schensted column insertion. Let $S^{(j)} = T^{(j)}$ for $j \in Q_0 \setminus \{i, i+1\}$. Then

$$T^{(0)} \cdot \cdot \cdot T^{(r-1)} \equiv S^{(0)} \cdot \cdot \cdot S^{(i)}u_a \cdot \cdot \cdot u_1 T^{(i+1)} S^{(i+1)} \cdot \cdot \cdot S^{(r-1)}$$

so that $S^\bullet$ is $(\mu, \eta)$-LR. For $0 \leq j < i$ we have $S^{(j)}|_{A_1} = T^{(j)}|_{A_1} = \emptyset$ since $T^\bullet \in LR_{\mu, \eta, i_1}^\lambda$. $S^{(i)}|_{A_1} = \emptyset$ because $A_1$ consists of the smallest $a$ values and such values must occur in the first $a$ rows of a tableau of partition shape such as $T^{(i)}$. These values are removed from the $i$-th tableau and put into the $(i+1)$-th tableau.

It follows that $S^\bullet \in LR_{\mu, \eta, i_1+1}$ where $\gamma^\bullet = \text{shape}(S^\bullet)$.

If $i_1 = r - 1$ there are two additional issues: there is a copy of $Y_1$ that must be removed, and the action of the cyclic group on $(\mu, \eta)$-words must be employed to preserve the LR property. Since $T^\bullet$ is $(\mu, \eta)$-LR and has no letters of $A_1$ except in $T^{(r-1)}$ it follows that $T^{(r-1)} \supset Y_1$. Let $u_a \cdot \cdot \cdot u_{2y_1}$ be the first $a$ rows and $S^{(r-1)}$ be the last $n - a$ rows of $T^{(r-1)}$. Then for $1 \leq k \leq a$, $u_k = k^{\mu_1} v_k$ for a row word (weakly increasing word) $v_k$. We have

$$T^{(0)} T^{(1)} \cdot \cdot \cdot T^{(r-1)} \equiv T^{(0)} T^{(1)} \cdot \cdot \cdot T^{(r-2)} S^{(r-1)} Y_1 v_a \cdot \cdot \cdot v_1.$$

This word is $(\mu, \eta)$-LR. Let us “rotate” this word by $|v_a| + \cdot \cdot \cdot + |v_1|$ positions to the right. By Lemma 29 there are tableaux $T^{(0)}, S^{(1)}, \ldots, S^{(r-1)}$ of the same shapes as $T^{(0)}, T^{(r-2)}, S^{(r-1)}$ and row words $v'_k$ with $|v'_k| = |v_k|$ for $1 \leq k \leq a$, such that

$$w_0^{\eta}(T^{(0)} T^{(1)} \cdot \cdot \cdot T^{(r-2)} S^{(r-1)} Y_1) = T^{(0)} S^{(1)} \cdot \cdot \cdot S^{(r-2)} S^{(r-1)} Y_1$$

$$w_0^{\eta}(v_a \cdot \cdot \cdot v_1) = v'_a \cdot \cdot \cdot v'_1.$$

Define $\Psi(v'_a \cdot \cdot \cdot v'_1, T^{(0)}) = (S^{(0)}, U^{(0)})$. Then

$$v'_a \cdot \cdot \cdot v'_1 T^{(0)} S^{(1)} \cdot \cdot \cdot S^{(r-1)} Y_1 \equiv S^{(0)} S^{(1)} \cdot \cdot \cdot S^{(r-1)} Y_1$$

is $(\mu, \eta)$-LR, and therefore $S^\bullet \in LR_{\mu, \eta, 0}^\lambda$ for $\gamma^\bullet = \text{shape}(S^\bullet)$.

In either case define $U^{(i)} = Y^\text{shape}(S^{(i)})$ to be the tableau of shape $\text{shape}(S^{(i)})$ filled with letter $k$ in each row $k$. This completes the definition of $\iota$.

Lemma 11. For $T^\bullet \in LR_{\mu, \eta, i_1}^\lambda$ we have

$$\text{wt}_{\mu, \eta}(T^\bullet) = \text{wt}_{Q_1}(\iota(T^\bullet)).$$

Proof. Suppose $0 \leq i_1 < r - 1$. Here $\nu = \emptyset$. For $u = u_a \cdot \cdot \cdot u_1$ as in the definition of $\iota$ we have $|u| = |\alpha(w)/\nu|$. Also word($T^\bullet$) $\equiv$ word($S^\bullet$) so that charge$^{\eta, \iota} (T^\bullet) = \text{charge}_{\mu, \eta} (S^\bullet)$. Letting $\gamma^\bullet = \text{shape}(S^\bullet)$ and $N = \sum_{k=1}^s \mu_k\eta_k$ we have

$$\sum_{j \in Q_0} (|\lambda(j) - |\gamma(j)|) e(j) = |\alpha(w)/\nu| (\epsilon(j) - \epsilon^{(i+1)})$$

from which we deduce that $t_{Q_1}^{\lambda^\bullet - N e^{(r-1)}} = t_{Q_1}^{\lambda - N e^{(r-1)}}$ as required.

Suppose $i_1 = r - 1$. Let $\hat{N} = \sum_{k=2}^s \mu_k\eta_k$. We have $|\nu| = \mu_1\eta_1$, $|u| = |\alpha(w)/\nu|$, $|\lambda^{(r-1)} - |\gamma^{(r-1)}| = |u| + \mu_1\eta_1$, $|\lambda^{(0)} - |\gamma^{(0)}| = -|u|$ so that

$$-N e^{(r-1)} + \sum_{j \in Q_0} |\lambda(j)| e(j) = -\hat{N} e^{(r-1)} + \sum_{j \in Q_0} |\gamma(j)| e(j) + |\alpha(w)/\nu| (\epsilon^{(r-1)} - \epsilon^{(0)}).$$
By Proposition \[3\] we have charge\(_\mu\,\eta\)(\(T^*\)) = charge\(_\mu\,\eta\)(\(S^*\)) + |\(u|\). Therefore

\[
t^\bigstar_{\mu^*\eta^*}(\alpha(w)/\nu)\bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} \equiv \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} = \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} = \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} = \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} = \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|} = \bigstar_{\mu^*\eta^*}(t_0, 1 \cdots t_{r-2}, r-1)^{-|\alpha(w)/\nu|}
\]

as required. \(\square\)

3.4. Cancellation. Let \(\Xi = (w, S^*, U^{(i)}, U^{(i+1)}) \in M^\bigstar_{\mu^*\eta^*}\). We define a sign-reversing weight-preserving involution \(\Phi\) on \(M^\bigstar_{\mu^*\eta^*}\) with fixed point set the image of \(\iota\). Let \(\Phi(\Xi) = \Xi = (\tilde{w}, \tilde{S}^*, \tilde{U}^{(i)}, \tilde{U}^{(i+1)})\) with \(\Xi\) to be specified.

The cancellation begins by trying to find a \(\iota\)-preimage for \(\Xi\).

Suppose \(0 \leq i < r - 1\). Let

\[
\Psi^{-1}(S^{(i)}, U^{(i)}) = (u_n \cdots u_{a+1}, \emptyset) \quad (3.11)
\]

\[
\Psi^{-1}(S^{(i+1)}, U^{(i+1)}) = (u_n \cdots u_1, T^{(i+1)}) \quad (3.12)
\]

Say that \(\Xi\) has a violation if \(u_n \cdots u_1\) is not the row word factorization of a tableau of partition shape. By Proposition \[31\] this is equivalent to saying that the tableau \(Q(u)\) of \(\lambda_{\mu^*\eta^*}\) is not Yamanouchi. If \(w \neq \text{id}\) then \(Q(u)\) cannot be Yamanouchi as its weight \((\alpha(w), \beta(w))\) is not dominant.

Suppose \(\Xi\) does not have a violation. Then \(w = \text{id}\), \(u_n \cdots u_1\) is the word of a tableau, say, \(T^{(i)}\), and \(u_n \cdots u_1\) and \(u_n \cdots u_{a+1}\) are also tableau words. This implies that \(U^{(i)}\) and \(U^{(i+1)}\) are Yamanouchi (but \(U^{(i+1)}\) has skew shape).

Let \(T^{(j)} = S^{(j)}\) for \(j \in Q_0 \setminus \{i, i + 1\}\).

Then \(\iota(T^*) = (w, S^*, U^{(i)}, U^{(i+1)})\). Define \(\Phi(\Xi) = \Xi\).

Otherwise suppose \(\Xi\) has a violation. Take the rightmost letter \(p+1\) in \(\text{word}(Q(u))\) that is \(p\)-unpaired. We define \(\tilde{w} = ws_p\) and

\[
\Psi(\tilde{w}, \emptyset) = (P(u), s_p e_p(Q(u))) \quad (3.13)
\]

\[
\Psi(\tilde{u}_n \cdots \tilde{u}_{a+1}, \emptyset) = (\tilde{S}^{(i)}, \tilde{U}^{(i)}) \quad (3.14)
\]

\[
\Psi(\tilde{u}_n \cdots \tilde{u}_1, T^{(i+1)}) = (\tilde{S}^{(i+1)}, \tilde{U}^{(i+1)}) \quad (3.15)
\]

and \(\tilde{S}^{(j)} = S^{(j)}\) for \(j \in Q_0 \setminus \{i, i + 1\}\). Since \(\text{word}(\tilde{S}^*) \equiv P(u) \equiv \text{word}(S^*)\) and \(s_p e_p(Q(u))\) has weight \(\tilde{w}(\lambda^{(i)} + \rho) - \rho\), it follows readily that \(\Xi \in M^\bigstar_{\mu^*\eta^*}\) and \(w_{Q_1}(\Xi) = w_{Q_1}(\Xi)\). Note that \(\Xi\) has a violation due to Lemma \[39\] with the rightmost \(p\)-unpaired letter \(p+1\) playing the same role in \(\text{word}(Q(u))\) as it did in \(\text{word}(Q(u))\). The latter ensures that the map \(\Xi \rightarrow \Xi\) is an involution.

This defines \(\Xi\) and establishes its properties for the case \(0 \leq i_1 < r - 1\).

Suppose \(i_1 = r - 1\). Let

\[
\Psi^{-1}(S^{(0)}, U^{(0)}) = (v_n \cdots v_1, W^{(0)})
\]

We have the \((\tilde{\mu}, \tilde{\eta})\)-LR word

\[
S^{(0)} S^{(1)} \cdots S^{(r-1)} = v_n \cdots v_1 W^{(0)} S^{(1)} \cdots S^{(r-1)}.
\]

Let \(v'_1 \cdots v'_k\) be the sequence of row words with \(|v_k| = |v'_k|\) for \(1 \leq k \leq a\), and \(W^{(0)}, S^{(1)}, \ldots, S^{(r-2)}, S^{(r-1)}\) the tableaux of the same shapes as \(W^{(0)}, S^{(1)}, \ldots, S^{(r-1)}\),
\[ \ldots, S^{(r-1)} \text{ such that} \]
\[ w_0^\eta(v_a \cdots v_1) = u'_a \cdots v'_1 \]
\[ w_0^\eta(W(0) S^{(1)} \ldots S^{(r-2)} S^{(r-1)}) = W'(0) S'(1) \ldots S'(r-2) S'(r-1). \]

Then the word
\[ W'(0) S'(1) \ldots S'(r-1) u'_a \cdots v'_1 \]
is \((\hat{\mu}, \eta)\)-LR. Let \( u'_k = k^{\mu_1} v'_k \) for \( 1 \leq k \leq a \). Let
\[ \Psi^{-1}(S'(r-1), U^{(r-1)}) = (u'_a \cdots u'_{a+1}, \emptyset). \]

We arrive at \( u' = (u'_n, \ldots, u'_1) \).

Say that \( \Xi \) has a violation if \( u'_n \cdots u'_1 \) is not the row word factorization of a tableau of partition shape.

Suppose \( \Xi \) does not have a violation. As before, \( w = \text{id} \), \( u'_n \cdots u'_1 \) is the reading word of a tableau \( T^{(r-1)} \). We have \( T^{(r-1)} = P(u'_n \cdots u'_1) = P(S^{(r-1)} u'_a \cdots u'_1) \).

Letting \( T(0) = W'(0) \), \( T(j) = S'(j) \) for \( 1 \leq j \leq r-2 \) we have
\[ T(0) T(1) \ldots T^{(r-2)} T^{(r-1)} u'_n \cdots u'_1 \equiv W'(0) S'(1) \ldots S'(r-2) S'(r-1) Y_1 v'_a \cdots v'_1 \]
which is \((\mu, \eta)\)-LR. As before we verify that \( \iota(T^{\ast}) = \Xi \) and declare that \( \Phi(\Xi) = \Xi \).

Suppose \( \Xi \) has a violation. The index \( p \) and \( \bar{w} \) are defined as before. Define
\[
\Psi(\bar{u}', \emptyset) = (P(\bar{u}'), s_p e_p(Q(\bar{u}'))) \\
\Psi(\bar{u}'_n \cdots \bar{u}'_{a+1}, \emptyset) = (S'(r-1), U^{(r-1)}).
\]

**Lemma 12.** For \( 1 \leq k \leq a \), we have \( \bar{u}'_k = k^{\mu_1} \bar{v}'_k \) for row words \( \bar{v}'_k \).

**Proof.** Rather than acting on the \( Q \) tableaux by \( s_p e_p \) we pass between the sequences of row words \( u'_n \cdots u'_1 \) and \( \bar{u}'_n \cdots \bar{u}'_1 \) by Proposition 32 acting directly on the sequences of row words using the dual crystal operator \( s_p^* e_p^* \). See also Example 33.

We have \( \bar{u}'_{p+1} \bar{u}'_j = u'_{p+1}u'_j \) and \( \bar{u}'_j = u'_j \) for \( j \notin \{p, p+1\} \). There is nothing to prove if \( p > a \). Suppose that \( 1 \leq p < a \). We have \( u'_{p+1}u'_p = (p+1)^{\mu_1} v'_p \) where \( v'_p \) and \( v'_{p+1} \) contain only letters greater than \( a \). Consider the two row skew tableau with lower row \( u'_{p+1} \) and upper row \( u'_p \), having maximum overlap (that is, with the rows partially slide over each other so as to make a skew tableau using the minimum total number of columns).

Suppose a letter \( p \) moves from row \( p \) to row \( p+1 \) during the computation of \( s_p^* e_p^* \). This can only happen if the two rows start in the same column. This would be a two row tableau of partition shape, which we need never encounter when computing \( s_p^* e_p^* \), yielding a contradiction. A letter \( p+1 \) can never move from row \( p+1 \) to row \( p \) to do so, right before the \( p+1 \) moves up the tableau must look like
\[
\begin{array}{c}
\vdots \\
p \bullet \\
\vdots \\
q \\
\vdots \\
\end{array}
\]
where we write \( q \) for \( p+1 \) and \( \bullet \) for the moving “hole” in the jeu-de-taquin. This is a contradiction to semistandardness since there are exactly \( \mu_1 \) \( p \)'s (all in the upper row to the left of the hole) and \( \mu_1 \) \( (p+1) \)'s (all in the lower row with one of them located just below the hole) and all other letters are greater than \( p+1 \).

The remaining case is \( p = a \). One must show that in passing from \( u'_{p+1}u'_p \) to \( \bar{u}'_{p+1} \bar{u}'_p \) no letter \( p \) goes from the \( p \)-th row to the \((p+1)\)-th. The proof is as in the previous case. \( \square \)
Let
\[ W'(0) = W'(0) \]
\[ \tilde{S}(j) = S(j) \quad \text{for} \quad 1 \leq j \leq r - 2. \]

For \( 1 \leq k \leq a \) let \( \tilde{v}_k \) be the row word such that \( |\tilde{v}_k| = |\hat{v}_k'| \) and let \( \hat{W}(0), \tilde{S}(1), \ldots, \tilde{S}(r-1) \) be the tableaux of the same shapes as \( \hat{W}'(0), \tilde{S}'(1), \ldots, \tilde{S}'(r-1) \) such that
\[ w_0^\eta(\tilde{v}_a' \cdots \tilde{v}_1') = \tilde{v}_a \cdots \tilde{v}_1 \]
\[ w_0^\eta(\hat{W}'(0) \tilde{S}'(1) \cdots \tilde{S}'(r-2) \tilde{S}'(r-1)) = \hat{W}'(0) \hat{S}'(1) \cdots \hat{S}'(r-1). \]

Let
\[ \Psi(\tilde{v}_a \cdots \tilde{v}_1, \hat{W}'(0)) = (\hat{S}'(0), \hat{U}'(0)) \]
It is straightforward to check that \( \hat{\Xi} = (\hat{\psi}', \hat{S}'^{(r-1)}, \hat{U}'(0)) \in M_{\mu, \eta, r-1}. \) Again, \( \hat{\Xi} \) has a violation by Lemma 30 and by our choice of \( p \) the map \( \Xi \to \hat{\Xi} \) is an involution.

We now verify that \( wt_{Q_1} \) is preserved. Let \( v = v_a \cdots v_1 \) and \( \tilde{v} = v_a' \cdots v_1' \). We have
\[
\text{charge}_{\mu, \eta}(S^*) + |v| = \text{charge}_{\mu, \eta}(W'(0) S'(1) \cdots S'(r-1) v') = \text{charge}_{\mu, \eta}((W'(0) S'(1) \cdots S'(r-1) v')) = \text{charge}_{\mu, \eta}(S^*) + |v'|
\]
\[
|\gamma^{(r-1)}| = |\text{shape}(S^{(r-1)})| = |\text{shape}(S'(r-1))| = \sum_{k=a+1}^n |v'_k| = |Q(u')| - \sum_{k=1}^a |v'_k|.
\]

Similarly \( |\tilde{\gamma}^{(r-1)}| = |\tilde{Q}(\tilde{u}')| - \sum_{k=1}^a |\tilde{v}'_k| \). But \( |\tilde{Q}(\tilde{u}')| = |Q(\tilde{u}')| \) so \( |\gamma^{(r-1)}| - |\tilde{\gamma}^{(r-1)}| = |v| - |\tilde{v}|. \) Similarly \( |\gamma^{(0)}| - |\tilde{\gamma}^{(0)}| = |v| - |\tilde{v}|. \) One now readily verifies \( wt_{\mu, \eta}(S^*) = wt_{\mu, \eta}(S^*). \)

4. ON THE CATABOLIZABLE TABLEAU CONJECTURE OF [OS]

In [OS] a tableau conjecture was given for the Kostka-Shoji polynomial for any quiver such that every vertex has in-degree at most one and out-degree at most one. This applies to the cyclic quiver. We explain how this conjecture holds when the above conditions intersect with the conditions of Theorem 9. In this section we assume the Lusztig data is even, periodic, Borel, but not necessarily concentrated at node \( r - 1 \). In particular let \( \eta_k = 1 \) for all \( i \). Such Lusztig data is parametrized by \( \mu^* \in \mathcal{W}_n \): \( i \) is \( 0, 1, 2, \ldots, r - 1 \) repeated \( n \) times, \( a \) is \( 1 \) repeated \( rn \) times, and \( \mu(*) \) is the sequence of single row partitions of the following sizes:
\[
\mu^{(0)}_1, \mu^{(1)}_1, \ldots, \mu^{(r-1)}_1, \mu^{(0)}_2, \mu^{(1)}_2, \ldots, \mu^{(r-1)}_2, \ldots, \mu^{(0)}_n, \mu^{(1)}_n, \ldots, \mu^{(r-1)}_n.
\]
Denote by \( H_{\mu^*}[X^*, t_{Q_1}] \) the corresponding cyclic quiver Hall-Littlewood function and by \( \mathcal{K}_{\mu^*}(t_{Q_1}) \) its coefficient at \( s_\lambda \), with \( \mathcal{K}_{\mu^*}(t) \) defined analogously.
4.1. **Single row catabolism.** For a word (or tableau or multitableau) $u$ let $|u|_k$ be the number of times the letter $k$ appears in $u$. For a tableau $T$ let $\hat{T}$ denote the tableau with its first row (denoted row$_1(T)$) removed. Similarly for $\lambda \in \mathcal{Y}$ let $\hat{\lambda} = (\lambda_2, \lambda_3, \ldots)$ be the partition $\lambda$ with its first part removed.

Given a multitableau $T^\bullet$, a node $i \in Q_0$ and nonnegative integer $p$, we say that $T^\bullet$ admits $\text{cat}^{(i)}_p$ if $|\text{row}_1(T^{(i)})| \geq p$. Suppose this is so: let $\text{row}_1(T^{(i)}) = 1^p v$ for a row word $v$. For $r > 1$ define $\text{cat}^{(i)}_p(T^\bullet)$ to be the multitableau $S^\bullet$ with $S^{(j)} = T^{(j)}$ for $j \in Q_0 \setminus \{i, i + 1\}$, $S^{(i)} = \hat{T}^{(i)}$, and $T^{(i+1)} = P(vT^{(i+1)})$ where $P$ is the Schensted $P$-tableau (see (A.1)). For $r = 1$ we have a single tableau, $i = 0$, and $\text{cat}^{(0)}_p(T^{(0)}) = P(v\hat{T}^{(0)})$.

**Example 13.** Let $r = 2$, $\lambda^* = ((6, 4, 2, 0), (3, 2, 1, 0))$, $\mu = (5, 5, 4, 4)$, and $T^\bullet \in T(\lambda^*, \mu)$ given by

$$T^\bullet = \begin{array}{cccc|cc} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 4 & 4 & & \\ 4 & 4 & & & & \\ \end{array} \text{\rotatebox[origin=c]{90}{$\otimes$}} \begin{array}{cc} 1 & 1 \\ 2 & 3 \\ 3 & \end{array}$$

Any multitableau admits $\text{cat}^{(0)}_0$. Applying this to $T^\bullet$, the first row is removed from the 0-th tableau and is column inserted into the 1-th tableau.

$$\begin{array}{cccc|cc} 2 & 2 & 4 & 4 & \otimes & 1 & 1 \\ 4 & 4 & & & \end{array} \equiv \begin{array}{cccc} 2 & 2 & 4 & 4 \\ 4 & 4 & \end{array} \otimes \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 3 \\ 2 & 2 & \end{array}$$

The resulting multitableau admits $\text{cat}^{(1)}_5$. The first row is removed from the 1-th tableau, 5 of its ones are removed, and the rest of the word is column inserted into the 0-th tableau:

$$\begin{array}{cccc|cc} 3 & 3 & 3 & \mathbf{1} & 2 & 2 & 4 \end{array} \otimes \begin{array}{ccc} 2 & 2 & 2 \\ 4 & 4 & \end{array} \equiv \begin{array}{cccc} 3 & 3 & 3 & \mathbf{1} \\ 2 & 2 & 4 & 4 \\ 4 & 4 & \end{array} \otimes \begin{array}{ccc} 2 & 2 & 2 \\ 3 \end{array} = \text{cat}^{(1)}_5 \text{cat}^{(0)}_0(T^\bullet).$$

Let $T^\bullet$ be a multitableau of shape $\lambda^*$ and $d^* \in \mathbb{Z}_{\geq 0}^{Q_0}$ an effective dimension vector. The $d^*$-**cascading catabolism** of $T^\bullet$ is defined by

$$\text{ccat}_{d^*}(T^\bullet) = \text{cat}^{(r-1)}_{d^*_{r-1}}, \text{cat}^{(r-2)}_{d^*_{r-2}}, \ldots, \text{cat}^{(1)}_{d^*_{1}}, \text{cat}^{(0)}_{d^*_{0}}(T^\bullet).$$

Of course this need not be defined as the requisite ones may not be present.

The multipartition $\mu^* \in \mathcal{Y}^{Q_0}$ can also be regarded as a sequence of $n$ effective dimension vectors $\mu^*_1, \mu^*_2, \ldots, \mu^*_n$. Say that the multitableau $T^\bullet$ is $\mu^*$-cascading catabolizable if $|T^\bullet|_k = |\mu^*_k|$ for all $1 \leq k \leq n$ and $T^\bullet$ admits the composition of operators $\text{ccat}_{\mu^*_1} \circ \ldots \circ \text{ccat}_{\mu^*_{n-1}} \circ \text{ccat}_{\mu^*_n}$ (which will necessarily produce the empty multitableau when applied to $T^\bullet$). Here $\text{ccat}_{\mu^*_k}$ is acting on a multitableau containing letters $k$ through $n$ and is removing letters $k$.

Let $\text{CT}(\lambda^*, \mu^*)$ be the set of $\mu^*$-cascading catabolizable multitableaux of shape $\lambda^*$. 


Lemma 17. Let

\[ T^{\bullet} = \bigoplus_{\lambda \vdash t} \lambda^{\bullet} \]

\( T \in CT(\lambda^{\bullet}, \mu^{\bullet}) \)

Remark 11. In the special case that \( \lambda^{(i)} \) is also empty for \( i \in Q_0 \setminus \{r - 1\} \) so that \( \lambda^{\bullet} = (\emptyset^{r-1}, \lambda) \) for some \( \lambda \in \mathcal{Y}_n \), we have \( t_{Q_1}^{\lambda^{\bullet} - \mu^{\bullet}} = 1 \) and the reduced Kostka-Shoji polynomial is the usual Kostka-Foulkes polynomial \( K_{\lambda, \mu}(t) \). This is a theorem of Shoji [Sho3].

4.2. Proof of Theorem 15 For a multitableau \( T^{\bullet} \) and positive integer \( k \) define the dimension vector

\[ m_k(T^{\bullet}) = \sum_{i \in Q_0} |T^{(i)}|_k \epsilon^{(i)}. \]

It remembers how many letters \( k \) there are at the various vertices of a multitableau.

For dimension vectors \( d^{\bullet}, f^{\bullet} \in \mathbb{Z}^{Q_0} \) define \( d^{\bullet} \supseteq f^{\bullet} \) if \( d^{\bullet} - f^{\bullet} \in \bigoplus_{i=0}^{r-2} \mathbb{Z}_{\geq 0} \epsilon^{(i)}. \)

Lemma 17. Let \( |T^{\bullet}|_1 = |d^{\bullet}|. \) Then \( T^{\bullet} \) admits \( \text{ccat}_{d^{\bullet}} \) if and only if \( m_k^{\bullet}(T^{\bullet}) \supseteq_{Q_0} d^{\bullet}. \)

Proof. To check this it is enough to assume that \( T^{\bullet} \) consists of only ones. Let \( T^{(i)} \) consist of \( \epsilon^{(i)} \) ones so that \( m_k^{\bullet}(T^{\bullet}) = \epsilon^{\bullet}. \) Define \( f \in \mathbb{Z}^{Q_0} \) by \( f(i) = \sum_{j=0}^{i} (\epsilon^{(j)} - d^{(j)}). \)

By induction on \( i \), one proves that cat\((d^{(r-1)} \cdots d^{(0)})(T^{\bullet})\) is defined if and only if \( f(j) \geq 0 \) for \( 0 \leq j \leq i - 1 \) and in that case, the resulting multitableau consists of empty tableaux at nodes 0 through \( i - 1 \), \( f(i-1) + \epsilon^{(i)} \) ones at node \( i \), and \( \epsilon^{(j)} \) ones at node \( j \) for \( i < j < r - 1 \). By induction the statement holds at \( i = r \), in which case it says that \( T^{\bullet} \) admits \( \text{ccat}_{d^{\bullet}} \) if and only if \( \epsilon^{\bullet} \supseteq_{Q_0} d^{\bullet} \), in which case the result of \( \text{ccat}_{d^{\bullet}}(T^{\bullet}) \) is the empty multitableau.

Lemma 18. Suppose \( |T^{\bullet}|_1 = x \).

(a) \( T^{\bullet} \) admits \( \text{ccat}_{(0, \ldots, 0, x)}. \)

(b) Given \( d^{\bullet} \in \mathbb{Z}^{Q_0} \) let \( x = \sum_{i \in Q_0} d^{(i)}. \) Suppose \( T^{\bullet} \) admits \( \text{ccat}_{d^{\bullet}} \). Then \( T^{\bullet} \) admits \( \text{ccat}_{(0, \ldots, 0, x)} \) and

\[ \text{ccat}_{(0,0,\ldots,x)}(T^{\bullet}) = \text{ccat}_{d^{\bullet}}(T^{\bullet}). \]
Proof. For (a) define \( d(i) = |T(i)|_1 \) for \( i \in Q_0 \). Then \( x = |T^•|_1 = |d^•| \) and \( d^• \succeq Q_0 \).

Now let \( T^• \) and \( d^• \) be as in (b). By (a) \( T^• \) admits \( \text{ccat}_0(0,...,0,x) \).

Remark 12. If \( n = 1 \) then \( \text{CT}(\lambda^•, \mu^•) \neq \emptyset \) if and only if \( \lambda^1 \succeq Q_0 \mu^1 \). In that case \( \text{CT}(\lambda^•, \mu^•) \) is a singleton, the unique multitableau of shape \( \lambda^• \) containing only ones.

Remark 13. Lemmas 17 and 18 can be combined to give an alternative condition to \( \mu^•\)-cascade catabolizability. \( T^• \) is \( \mu^•\)-cascade-catabolizable if and only if \( m^1(\mu^•) \succeq Q_0 \mu^1 ; \) \( m^2(\text{ccat}_{0,...,0,\mu^•}(\mu^•)) \succeq Q_0 \mu^2 \), etc.

Let \( T(\lambda^•, \mu) \) be the set of multitableaux of shape \( \lambda^• \) and weight \( \mu \).

Lemma 19. For \( \mu^• = (0^{r-1}, \mu) \) we have \( \text{CT}(\lambda^•, \mu^•) = T(\lambda^•, \mu) \).

Proof. This follows from Lemma 18.

Since \( \text{LR}^•_{\mu,n,0} \) clearly also equals \( T(\lambda^•, \mu) \) in the Borel case, we see that Theorem 15 is a consequence of Theorem 6 and Lemma 19.

APPENDIX A. TABLEAU CONSTRUCTIONS

A.1. Knuth equivalence. Knuth equivalence \( \equiv \) on words is the transitive closure of the following relations, where \( u \) and \( v \) are words and \( x, y, z \) are values satisfying

\[
begin{align*}
uxzyv & \equiv uzyxv & \text{for } x \leq y < z \\
uyxzv & \equiv uyxzw & \text{for } x < y \leq z.
end{align*}
\]

Lemma 20. Let \( u \) and \( v \) be words. If \( u \equiv v \) then for all intervals \( I \) \( u|_I \equiv v|_I \).

The Ferrers diagram \( D(\lambda) \) of a partition \( \lambda \in \mathbb{Y} \) is the set of matrix-style pairs \((i, j) \in \mathbb{Z}^2 \) with \( j \leq \lambda_i \). By a tableau we mean a semistandard tableau of some partition shape \( \lambda \), a function \( T : D(\lambda) \to \mathbb{Z}_{>0} \) which weakly increases along rows \( T(i,j) \leq T(i,j+1) \) for \((i,j),(i,j+1) \in D(\lambda) \) and strictly increases down columns \( T(i,j) < T(i+1,j) \) for \((i,j),(i+1,j) \in D(\lambda) \).

The reading word (denoted \( \text{word}(T) \)) of a tableau \( T \) is the word obtained by reading the rows of \( T \) from left to right, starting with the bottom row and proceeding to earlier rows. We regard a tableau \( T \) as a word in this manner.
Example 21. 

\[
T = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & \\
\end{array}
\]

word(T) = 23112.

Theorem 22. For every word \( u \) (in symbols \( 1, 2, \ldots \)) there is a unique tableau denoted \( P(u) \), such that \( u \equiv \text{word}(P(u)) \).

\( P(u) \) can be computed by Schensted’s row or column insertion algorithms [Sch].

### A.2. Column insertion and RSK

For \( \lambda, \nu \in \mathbb{Y} \) we say that \( \nu/\lambda \) is a horizontal strip if \( D(\nu) \supset D(\lambda) \) and the set difference \( D(\nu) - D(\lambda) \) (which we denote by \( \nu/\lambda \)) has at most one box in each column.

**Proposition 23.** [Sch] Given \( p \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{Y} \), there is a bijection \( \psi_{\lambda} \) sending \( (u, T) \) to \( P \) where \( u \) is a row word with \( |u| = p \), \( T \) is a tableau of shape \( \lambda \), and \( P \) is a tableau of some shape \( \nu \) such that \( \nu/\lambda \) is a horizontal strip of size \( p \). It is uniquely specified by the condition \( uT \equiv P \).

The forward map is \( (u, T) \mapsto P(uT) \). We denote the inverse map by \( \psi_{\lambda}^{-1}(P) = (u, T) \). These may be computed directly using Schensted column insertion and its reverse [Sch].

Given a tableau \( T \) let \( T|_{\leq k} \) be the subtableau of \( T \) consisting of the entries of value at most \( k \). Iterating the above Lemma yields the following bijection.

**Proposition 24.** [Sch] There is a unique bijection (the column insertion Robinson-Schensted-Knuth correspondence)

\[
(u_n, u_{n-1}, \ldots, u_1) \overset{\text{RSK}}{\mapsto} (P, Q)
\]

from sequences of \( n \) row words to pairs of (semistandard) tableaux (with \( Q \) on the alphabet \( \{1, 2, \ldots, n\} \)) such that for all \( 0 \leq k \leq n \), \( \text{shape}(Q|_{\leq k}) = \text{shape}(P(u_k \cdots u_2 u_1)) \). In particular \( |Q|_k = |u_k| \) for all \( 1 \leq k \leq n \).

We write \( \underline{u} \mapsto (P(\underline{u}), Q(\underline{u})) \) for this bijection.

More generally suppose \( T \) is a semistandard tableau of partition shape and \( \underline{u} = (u_n, \ldots, u_1) \) is a sequence of row words. Define \( \Psi(\underline{u}, T) = (P, U) \) where \( P = P(u_n \cdots u_1 T) \) and \( U \) is the semistandard skew tableau defined by the sequence of partitions \( \text{shape}(P(u_k \cdots u_2 u_1 T)) \) for \( 0 \leq k \leq n \). \( \Psi \) defines a bijection between pairs \( (\underline{u}, T) \) and \( (P, Q) \) where \( P \) is a semistandard tableau of partition shape and \( Q \) is a semistandard skew tableau with \( \text{shape}(Q) = \text{shape}(P)/\text{shape}(T) \) such that \( \text{wt}(Q) = (|u_1|, |u_2|, \ldots, |u_n|) \).

The following is a reformulation of a theorem of D. White [Wh].

Let \( y_\lambda \) be the Yamanouchi tableau of shape \( \lambda \), the unique tableau of shape and weight \( \lambda \). For partitions \( \mu \subset \lambda \in \mathbb{Y}_n \) say that the tableau \( Q \) is \( \lambda/\mu \)-compatible if \( y_\lambda \equiv Qy_\mu \).

**Theorem 25.** Let \( \mu, \lambda \in \mathbb{Y}_n \) with \( \mu \subset \lambda \) and let \( \underline{u} = u_n \cdots u_1 \) be a sequence of row words with \( |u_i| = \lambda_i - \mu_i \). Then \( \underline{u} \) is the sequence of rows of a semistandard tableau of shape \( \lambda/\mu \) if and only if, for any tableau \( T \) of partition shape, if \( \Psi(\underline{u}, T) = (P, Q) \) then \( Q \) is \( \lambda/\mu \)-compatible.

**Corollary 26.** For any \( \nu \in \mathbb{Y} \), \( c_{\mu, \nu}^\lambda = \langle s_\nu, s_{\lambda/\mu} \rangle \) is the number of \( \lambda/\mu \)-compatible tableaux of shape \( \nu \).
Corollary 27. [RW] For a sequence of partitions $\lambda(0), \lambda(1), \ldots, \lambda(r-1)$, the product of Schur functions $s_{\lambda(0)} \cdots s_{\lambda(r-1)}$ is a skew Schur function, associated with the skew shape $D = \lambda^{(r-1)} \cdots \lambda^{(0)}$ obtained by placing the partitions $\lambda^{(0)}$ up to $\lambda^{(r-1)}$ from northeast to southwest. Thus for $\lambda \in \mathcal{Y}$, $c_{\lambda}^{\ast} = \langle s_{\lambda}, s_{\lambda(0)} \cdots s_{\lambda(r-1)} \rangle$ is equal to the number of $D$-compatible tableaux of shape $\lambda$.

Example 28. The skew shape $(2,1) \ast (2,2) \ast (3)$ is pictured below.

A.3. $A_{n-1}$ crystal graphs. Words in $\{1,2,\ldots,n\}$ of a fixed length, form a type $A_{n-1}$ crystal graph. For such a word $u$, view each $i$ (resp. $i+1$) in $u$ as a right (resp. left) parenthesis. After matching parentheses, the unpaired parentheses form a subword $i^\phi(i+1)^\epsilon$. Define $\varphi_i(u) = \phi$ and $\varepsilon_i(u) = \epsilon$. If $\varepsilon_i(u) > 0$ then $\varepsilon_i(u)$ is defined by replacing the unpaired subword by $i^\phi(i+1)^{\epsilon-1}$. If $\varepsilon_i(u) = 0$ then $\varepsilon_i(u)$ is undefined. If $\varphi_i(u) > 0$ then $f_i(u)$ is defined by replacing the unpaired subword by $i^{\phi-1}(i+1)^{\epsilon+1}$. If $\varphi_i(u) = 0$ then $f_i(u)$ is undefined. The crystal reflection operator $s_i$ acts on a word $u$ by replacing the above unpaired subword by $i^\phi(i+1)^\epsilon$. The crystal graph is the directed graph (with edges colored by $1 \leq i \leq n-1$) having a directed arrow colored $i$ from each $u$ to $f_i(u)$. For a fixed $i$ the components of the graph are directed paths called $i$-strings. Along an $i$-string all the letters other than $i$ and $i+1$ and also all the $i$-paired letters, remain the same: only the substring of $i$-unpaired letters changes. $\varepsilon_i(u)$ (resp. $\varphi_i(u)$) is the distance from $u$ to the beginning (resp. end) of its $i$-string. If reference to $i$ is needed we will say that letters are $i$-paired or $i$-unpaired and so on.

Remark 14. The set of (semistandard) tableaux of a fixed (partition) shape $\lambda \in \mathcal{Y}$ with entries in $1,2,\ldots,n$ have a type $A_{n-1}$-crystal graph structure induced by inclusion into the crystal graph of words by sending a tableau to its row-reading word: the crystal operators $\varepsilon_i$ and $f_i$ stabilize the set of tableau words, those which are the reading words of tableaux. More generally the crystal operators preserve the set of tableaux of a fixed skew shape. The reason for these is that the crystal operators preserve the recording tableau $Q(u)$ of a word $u$ and the condition that a word $u$ be the reading word of a tableau of a fixed skew shape, is equivalent to saying that $Q(u)$ belongs to a given set of tableaux depending only on the skew shape $W_{\lambda}$.\[\]

Lemma 29. Let $D_1,\ldots,D_s$ be skew shapes and $T_1,\ldots,T_s$ be semistandard tableaux with $T_j$ of shape $D_j$. Let $s_i$ be a crystal reflection operator. Let $s_i(T_1 \cdots T_s) = u_1 \cdots u_s$ where $|u_j| = |T_j|$ for all $j$. Then $u_j$ is the reading word of a semistandard tableau of shape $D_j$ for all $j$.

Proof. This follows from Remark 14 since the word $T_1 \cdots T_s$ is the reading word of a tableau of a skew shape $D_s \ast \cdots D_2 \ast D_1$ obtained by placing $D_1, D_2,\ldots,D_s$ going from southwest to northeast on disjoint sets of northwest-southeast diagonals; see Example 28.\[\]

Remark 15. $\varepsilon_i(u) > 0$ if $|u|_{i+1} > |u|_i$ since some $i+1$ must be $i$-unpaired.
Lemma 30. $s_i = s_is_i$ is an involution on the set of words $u$ such that $\varepsilon_i(u) > 0$. It restricts to an involution on the set of tableaux $T$ of a given shape with $\varepsilon_i(T) > 0$.

A.4. Dual crystal graph structure on $n$-tuples of row words. Consider the set of $n$-tuples of row words $\bar{u} = (u_n, \ldots, u_2, u_1)$. It has a type $A_{n-1}$ crystal graph structure defined by acting on the column insertion RSK $Q$ tableau. That is, we define
\[
\varepsilon_i^*(\bar{u}) = \varepsilon_i(Q(\bar{u}))
\]
\[
\varphi_i^*(\bar{u}) = \varphi_i(Q(\bar{u})).
\]
For any $\bar{u}$, $s_i^*(\bar{u})$ is defined by
\begin{align*}
(A.1) & \quad P(s_i^*(\bar{u})) = P(\bar{u}) \\
(A.2) & \quad Q(s_i^*(\bar{u})) = s_i(Q(\bar{u})).
\end{align*}
If $\varepsilon_i^*(\bar{u}) > 0$ then $e_i^*(\bar{u})$ is defined by
\begin{align*}
(A.3) & \quad P(e_i^*(\bar{u})) = P(\bar{u}) \\
(A.4) & \quad Q(e_i^*(\bar{u})) = e_i(Q(\bar{u})).
\end{align*}
Given row words $v$ and $u$ define $ov(v, u)$ to be the maximum number of columns $c$ such that $v'u'$ is the word of a tableau of shape $(c, c)$ where $v'$ (resp. $u'$) is the subword of the last $c$ letters of $v$ (resp. first $c$ letters of $u$).

Say that a word $u$ in the alphabet $\{1, 2, \ldots, n\}$ is Yamanouchi (resp. almost Yamanouchi) if $\varepsilon_i(u) = 0$ for all $1 \leq i \leq n - 1$ (resp. $2 \leq i \leq n - 1$.)

Proposition 31. Let $\bar{u} = u_n \ldots u_1$ be a sequence of row words. Then the number of $i$-pairs in $Q(\bar{u})$ is equal to $ov(u_{i+1}, u_i)$. In particular, $\bar{u}$ is a tableau word if and only if $Q(\bar{u})$ is Yamanouchi, and $u_n \ldots u_2$ is a tableau word if and only if $Q(\bar{u})$ is almost Yamanouchi.

Proof. Follows from Theorem 25.

Proposition 32. Let $\bar{u} = u_n \ldots u_1$ be an $n$-tuple of row words and let $\bar{u}' = u'_n \ldots u'_2 u'_1 = st_i e_i^*(\bar{u})$.

1. We have $u'_j = u_j$ for $j \notin \{i, i+1\}$ and $u'_{i+1} u'_i = u_{i+1} u_i$.
2. Say $u_{i+1}$ and $u_i$ have lengths $b$ and $a$ respectively. Then $(u'_{i+1}, u'_i)$ is the unique pair of row words such that $u'_{i+1} u'_i = u_{i+1} u_i$ such that $u'_{i+1}$ has length $a + 1$ and $u'_i$ has length $b - 1$.

Example 33. Let $\bar{u} = (u_2, u_1)$ with $u_1 = 11334$ and $u_2 = 2234$. The rows have sizes $(5, 4)$. We compute $s_2 e_1^*(\bar{u}) = \bar{u} = (\bar{u}_2, \bar{u}_1)$. The new rows should have sizes $(3, 6)$. This is computed by the two-row skew tableau jeu-de-taquin, which is known to preserve Knuth equivalence $[LS]$. To move one number from the top row to the bottom row, we put a hole after the end of the bottom row and swap it left and up while preserving semistandardness. The exchange path of the hole is highlighted.

\[
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 2 & 3 & 1
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 2 & 3 & 1 \bullet
\end{array}
\]

We do it again:

\[
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 2 & 3 & 1 \bullet
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 1 & 3 \\
2 & 2 & 3 & 1 \\
\bullet
\end{array}
\]

The result is $s_1^* e_1^* \bar{u}$ with $\bar{u}_2 = 223444$ and $\bar{u}_1 = 113$. 

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