Exact solution for a matrix dynamical system with usual and Hadamard inverses

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Abstract

Let \( A \) be an \( n \times n \) matrix with entries \( a_{ij} \) in the field \( \mathbb{C} \). Consider the following two involutive operations on such matrices: the matrix inversion \( I: A \mapsto A^{-1} \) and the element-by-element (or Hadamard) inversion \( J: a_{ij} \mapsto a_{ij}^{-1} \). We study the algebraic dynamical system generated by iterations of the product \( J \circ I \). In the case \( n = 3 \), we give the full explicit solution for this system in terms of the initial matrix \( A \). In the case \( n = 4 \), we provide an explicit ansatz in terms of theta-functions which is full in the sense that it works for a Zariski open set of initial matrices. This ansatz also generalizes for higher \( n \) where it gives partial solutions.

1 Introduction

A composition of two noncommuting involutions acting on square matrices generates sometimes an interesting dynamical system (more specifically: an algebraic dynamical system with discrete time). A good example can be found in paper [1] (see also references there), where block matrices \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) were considered, \( A, B, C \) and \( D \) being themselves matrices \( n \times n \). The first involution consisted in taking the usual matrix inverse (of the whole block matrix), and the second one was the following block transposing: \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & C \\ B & D \end{pmatrix} \). Such a system (at least when considered to within some natural gauge freedom) was shown in [1] to be a typical solitonic system solved by usual algebraic-geometrical methods.

There is a natural desire to find solutions to more general (than just solitonic) dynamical systems. Note that the dynamical system generated by the following transformation: \( z \mapsto z^2 \), where \( z \) belongs to the unit circle in the complex plane, is obviously solvable but exhibits (as much obviously) a chaotic, not solitonic, behavior.
The aim of this paper is to present solutions to the following algebraic dynamical system (see [2] about its origin and some results for its particular cases). Let $\mathcal{A}$ be a $n \times n$ matrix whose entries $a_{ij}$ belong to the complex field $\mathbb{C}$. We consider two involutive operations on such matrices: the matrix inversion $I: \mathcal{A} \mapsto \mathcal{A}^{-1}$ and the element-by-element (or Hadamard) inversion $J: a_{ij} \mapsto a_{ij}^{-1}$. Our dynamical system is generated by iterations of the product $J \circ I$.

The case $n = 2$ is trivial but serves us as a useful warm-up exercise. For the case $n = 3$, we give the full explicit solution for this system in terms of the initial matrix $\mathcal{A}$. For the case $n = 4$, we provide an explicit ansatz in terms of one-dimensional theta-functions. This ansatz provides a full solution in the sense that it encompasses a Zariski open set of matrices. For the case $n \geq 5$, the same ansatz also works but encompasses only a subvariety of matrices of nonzero codimension.

2 2 × 2 matrices

Despite the triviality of this system, its solution supplies us with some tool that will play a key role also in the case $n = 3$. This is the multiplicative basis of evolution.

We denote the matrix and its elements as $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the determinant of $\mathcal{A}$ as $\Delta = ad - bc$. The crucial point is that both transformations $I$ and $J$ of the following six values:

$$a, b, c, d, \Delta \text{ and } (-1)$$

are described in purely multiplicative terms. Namely, $I$ act like this:

$$a \mapsto d\Delta^{-1}, \ b \mapsto (-1)b\Delta^{-1}, \ c \mapsto (-1)c\Delta^{-1}, \ d \mapsto a\Delta^{-1}, \ \Delta \mapsto \Delta^{-1}, \ (-1) \mapsto (-1);$$

and $J$ like this:

$$a \mapsto a^{-1}, \ b \mapsto b^{-1}, \ c \mapsto c^{-1}, \ d \mapsto d^{-1}, \ \Delta \mapsto (-1)a^{-1}b^{-1}c^{-1}d^{-1}\Delta, \ (-1) \mapsto (-1).$$

One can say that there are two matrices $T_I, T_J \in \text{GL}(6, \mathbb{Z})$ which act multiplicatively (in the following sense: the number 2 acts on a variable $x$ multiplicatively by $x \mapsto x^2$) on columns of values (1), where

$$T_I = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_J = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Of course, the answer for values (1) after $N$ steps of evolution is given in the similar way by the matrix $(T_J T_I)^N$. This latter is given by two slightly different expressions
for $N$ odd and even. For example, if $N = 2k$, then

\[
(T_jT_l)^N = \begin{pmatrix}
-k + 1 & -k & -k & 2k & -k \\
-k & -k + 1 & -k & 2k & -k \\
-k & -k & -k + 1 & 2k & -k \\
-2k & -2k & -2k & 4k + 1 & -2k \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

This means that the new value of $a$ after $2k$ steps is

\[
a(2k) = a^{-k+1}b^{-k}c^{-k}d^{-k}\Delta^{2k}(-1)^{-k},
\]

and similarly for $b$, $c$ and $d$. In the r.h.s. of (4), of course, the initial values of the variables are taken.

## 3 $3 \times 3$ matrices: one special formula

This section provides a formula necessary for building the multiplicative basis for the evolution of $3 \times 3$ matrices, in analogy with section 2.

For a matrix

\[
A = \begin{pmatrix}
a & b & c \\
f & g & h \\
r & s & t
\end{pmatrix}
\]

we denote $\mathrm{dh} \ A$ the determinant of its Hadamard inverse multiplied, for convenience, by the product of all elements of $A$:

\[
\mathrm{dh} \ A = agtbhr + fscagt + bhrfs = rcfbt - fbtsha - sharc.
\]

It turns out that $\mathrm{dh} \ A$ behaves very nicely under the (usual) inversion of $A$:

\[
\mathrm{dh} \ A^{-1} = - \mathrm{dh} \ A \ (\det \ A)^{-4}.
\]

The direct proof of formula (7) consists simply in applying computer algebra. This, however, does not explain how one can arrive at such formula. In the rest of this section we present heuristic argument which clearly shows that a formula of such kind must exist. Our argument was suggested by paper [3].

Let $\mathrm{dh} \ A = 0$. This means that the matrix

\[
B = J(A) = \begin{pmatrix}
a^{-1} & b^{-1} & c^{-1} \\
f^{-1} & g^{-1} & h^{-1} \\
r^{-1} & s^{-1} & t^{-1}
\end{pmatrix}
\]

is degenerate (remember that we do not care about the rigor!). This means that

\[
B \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = 0
\]

for some (nonzero) column $\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}$. In terms of initial matrix $A$, this
yields:
\[
\mathcal{A} \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & 0 & \gamma' \end{pmatrix} \mathcal{A}^T = \begin{pmatrix} \alpha'' & 0 & 0 \\ 0 & \beta'' & 0 \\ 0 & 0 & \gamma'' \end{pmatrix}.
\] (8)

Here the superscript $T$ means matrix transposing; the values $\alpha', \ldots, \gamma''$ are given by
\[
\alpha' = \frac{\alpha}{af}, \quad \beta' = \frac{\beta}{bgs}, \quad \gamma' = \frac{\gamma}{cht},
\]
\[
\alpha'' = \alpha'a^2 + \beta'b^2 + \gamma'c^2, \quad \beta'' = \alpha'f^2 + \beta'g^2 + \gamma'h^2, \quad \gamma'' = \alpha'r^2 + \beta's^2 + \gamma't^2.
\]

It is clear that a relation similar to (8) holds for $I(\mathcal{A}) = \mathcal{A}^{-1}$ as well. This means that $(J \circ I)(\mathcal{A}) = J(\mathcal{A}^{-1})$ is degenerate. In other words,
\[
\text{dh} \mathcal{A} = 0 \Rightarrow \text{dh}(\mathcal{A})^{-1} = 0.
\]

There is no need to make this argument rigorous because, as has been said, formula (7) admits a direct verification.

4 3 × 3 matrices: the solution

We proceed along the same lines as in section 2, which is possible due to formula (7) from section 3. We use notation (5) for the entries of matrix $\mathcal{A}$.

The multiplicative basis of evolution comprises now 21 values: the matrix elements $a, b, c, f, g, h, r, s, t$; the determinant $\Delta = \det \mathcal{A}$; nine cofactors of matrix elements denoted by corresponding capital letters: $A = \begin{vmatrix} g & h \\ s & t \end{vmatrix}$, $B = \begin{vmatrix} f & h \\ r & t \end{vmatrix}$ and so on; the value $\Xi = \text{dh} \mathcal{A}$ defined by (6) and, finally, the value $(-1)$. Here is how our two involutions act on these values.

The matrix inverse $I$:
\[
a \mapsto A\Delta^{-1}, \quad b \mapsto F\Delta^{-1}, \ldots, \quad h \mapsto H\Delta^{-1};
\]
\[
A \mapsto a\Delta^{-1}, \quad B \mapsto f\Delta^{-1}, \ldots, \quad H \mapsto h\Delta^{-1};
\]
\[
\Delta \mapsto \Delta^{-1}, \quad \Xi \mapsto (-1)\Xi\Delta^{-4}, \quad (-1) \mapsto (-1).
\]

The element-by-element inverse $J$:
\[
a \mapsto a^{-1}, \ldots, \quad h \mapsto h^{-1};
\]
\[
A \mapsto (-1)Ag^{-1}h^{-1}s^{-1}t^{-1}, \ldots, \quad H \mapsto (-1)Ha^{-1}b^{-1}f^{-1}g^{-1};
\]
\[
\Delta \mapsto \Xi^{-1}a^{-1}c^{-1}f^{-1}g^{-1}h^{-1}r^{-1}s^{-1}t^{-1},
\]
\[
\Xi \mapsto \Delta a^{-1}b^{-1}c^{-1}f^{-1}g^{-1}h^{-1}r^{-1}s^{-1}t^{-1},
\]
\[
(-1) \mapsto (-1).
\]

A matrix $T_jT_i$ corresponding to a step of evolution can now be calculated in analogy with section 2, but now it has sizes $21 \times 21$, and we do not write it out here. Still,
it makes no difficulty for a computer to handle such matrices. The remarkable fact is that all the eigenvalues of \( T^J T^I \) are sixth roots of unity, as the following table shows:

| eigenvalues | \( \frac{1+\sqrt{-3}}{2} \) | \( \frac{1-\sqrt{-3}}{2} \) | \( \frac{-1+\sqrt{-3}}{2} \) | \( \frac{-1-\sqrt{-3}}{2} \) | 1 | -1 |
|-------------|----------------|----------------|----------------|----------------|---|---|
| multiplicity | 1              | 1              | 4              | 4              | 7  | 4  |

Of course, \( T^J T^I \) does have nontrivial Jordan boxes. An interesting thing with them is that they all correspond only to eigenvalues \( \pm 1 \).

Here is the explicit answer for the variable \( a \) after \( 3k \) steps:

\[
a(3k) = a^{-2k+1}b^{-k}c^{-k}f^{-k}r^{-k}\Delta^{2k}A^{-2k}B^{-k}C^{-k}F^{-k}R^{-k}E^{2k}(-1)^{-3k(2k-1)}. \tag{9}
\]

Again, in the r.h.s. of formula (9) the initial values of all variables are taken.

5 4 \times 4 matrices: the conservation laws and computer algebra results

In this section we describe the way that has actually led to the ansatz presented in subsequent sections. Formally, however, the construction of our ansatz does not rely on the material of this section.

First, the evolution of a \( 4 \times 4 \) matrix \( \mathcal{A} \) can be considered to within the following gauge freedom: we can consider not \( \mathcal{A} \) itself but its equivalence class with respect to its multiplication both from the left and from the right by some diagonal matrices \( \mathcal{B} \) and \( \mathcal{C} \). Clearly, if

\[
\mathcal{A}' = \mathcal{B} \mathcal{A} \mathcal{C}
\]

then

\[
(J \circ I)(\mathcal{A}') = \mathcal{C}(J \circ I)(\mathcal{A})B. \tag{11}
\]

Using a transformation (10), we can reduce (almost any) \( \mathcal{A} \) to the following form:

\[
\mathcal{A} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & a & b & c \\
1 & f & g & h \\
1 & r & s & t
\end{pmatrix}. \tag{12}
\]

So, in a sense, there remains in \( \mathcal{A} \) nine parameters. If we find eight conservation laws this will be a strong argument suggesting that the evolution goes just along elliptic curves (because only elliptic and rational curves have infinite number of automorphisms).

Second, there are many conserved quantities which are, moreover, invariant under (10). Consider a decomposition of a \( 4 \times 4 \) matrix in four \( 2 \times 2 \) submatrices, for instance, one of the following:

\[
\begin{pmatrix}
\diamond & \diamond & \ast & \ast \\
\diamond & \diamond & \ast & \ast \\
\| & \| & \$ & \$ \\
\| & \| & \$ & \$
\end{pmatrix} \quad \text{or} \quad 
\begin{pmatrix}
\diamond & \$ & \diamond & \$ \\
\| & \ast & \ast & \| \\
\diamond & \$ & \diamond & \$ \\
\| & \ast & \ast & \|
\end{pmatrix},
\]
where the entries denoted by the same symbol belong to the same submatrix.

For any such decomposition $p$, we construct the value $\Pi_p$ — the product of four corresponding minors. One can verify that under one step of evolution, the ratio of any two such values $\Pi_p/\Pi_q$, where $\Pi_p'$ is the product of the cofactors of the minors of decomposition $p$ (in the new matrix $A$). So, $\Pi_p/\Pi_q$ is invariant under two steps of evolution.

Computer experiments show that there are eight algebraically independent invariants of such kind. Moreover, given fixed values of these invariants and using the form (12) for the matrix, one can exclude all except two parameters in (12) from the equations and get the curve as given by just one equation in two variables. Its genus turns out to be one, as expected. Besides, one can see from that equation that, for example, there are four points in the curve where the function $a(z)$ ($z$ being a parameter for the elliptic curve) takes value 1, and the function $b(z)$ takes value 1 in exactly the same points, as well as some information about the coincidence of some poles and zeros of those functions.

This was exactly what led us to the ansatz presented in the following sections.

6 A determinant of theta-function ratios

The key formula for our ansatz is the formula for the determinant of the $n \times n$ matrix

$$K = (k_{ij}), \quad \text{where} \quad k_{ij} = \frac{\vartheta(y - \lambda_i - \mu_j)}{\vartheta(x + \lambda_i + \mu_j)}. \tag{13}$$

Here $i$ stays of course for the number of a row and $j$ for the number of a column. So, there are complex variables $x$ and $y$ and, moreover, two arrays, $(\lambda_i)$ and $(\mu_j)$, each of $n$ complex variables. By $\vartheta$ we denote here the odd Jacobian theta-function:

$$\vartheta(u) = 2q^{1/4} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos \frac{\pi u}{K} + q^{4n})(1 - q^{2n}),$$

where $q = \exp(-\pi K'/K)$; $K$ and $K'$ are called half-periods. Our formula states that

$$\det K = -\left(\vartheta(x + y)\right)^{n-1} \vartheta\left((n - 1)x - y + \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{n} \mu_j\right)$$

$$\prod_{1 \leq i_1 < i_2 \leq n} \vartheta(\lambda_{i_2} - \lambda_{i_1}) \cdot \prod_{1 \leq j_1 < j_2 \leq n} \vartheta(\mu_{j_2} - \mu_{j_1}) \cdot \prod_{i=1}^{n} \prod_{j=1}^{n} \vartheta(x + \lambda_i + \mu_j). \tag{14}$$

Proof of formula (14) goes by induction on $n$. Consider $\det K$ as a function of $\lambda_n$. It is obliged to have the following form:

$$\det K = F \frac{\vartheta(\lambda_n - G) \prod_{i=1}^{n-1} \vartheta(\lambda_n - \lambda_i)}{\prod_{j=1}^{n} \vartheta(x + \lambda_n + \mu_j)}. \tag{15}$$
where $F$ and $G$ are some quantities that do not depend on $\lambda_n$ but can depend on other variables. Here are the reasons why exactly such factors appear in formula (15), typical for proofs of the like formulas for theta-functions.

Each factor $\vartheta(\lambda_n - \lambda_i)$ in the numerator is responsible for the fact that $\det \mathcal{K}$ obviously vanishes when $\lambda_n$ coincides with any other $\lambda_i$ (because of two identical rows). The denominator of (15) is simply the common denominator of all elements of $\mathcal{K}$. Finally, the factor $\vartheta(\lambda_n - G)$ is necessary to ensure that the whole expression has the same number of zeros and poles as a function of $\lambda_n$.

On the other hand, in the neighborhood of the value $\lambda_n = -x - \mu_n$, where $k_{nn}$ has a pole, our determinant behaves as

$$\det \mathcal{K} \approx \det \mathcal{K}_{\text{smaller}} \frac{\vartheta(x + y)}{\vartheta(x + \lambda_n + \mu_n)},$$

where $\mathcal{K}_{\text{smaller}}$ is the same matrix $\mathcal{K}$ but without its $n$th row and $n$th column. Comparing (16) with (15) (where we at this moment also substitute $\lambda_n = -x - \mu_n$), we can get the quantity $F$ in the following form:

$$F = \det \mathcal{K}_{\text{smaller}} \frac{\vartheta(x + y) \prod_{j=1}^{n-1} \vartheta(\mu_n + \mu_j)}{\vartheta(-x - \mu_n - G) \prod_{i=1}^{n-1} \vartheta(-x - \mu_n - \lambda_i)} \cdot \frac{\vartheta(\lambda_n - G) \prod_{i=1}^{n-1} \vartheta(\lambda_n - \lambda_i)}{\prod_{j=1}^{n} \vartheta(x + \lambda_n + \mu_j)}.$$  \quad (17)

Now, substituting (17) in (15), we get the following, almost final, formula expressing $\det \mathcal{K}$ through $\det \mathcal{K}_{\text{smaller}}$:

$$\det \mathcal{K} = \det \mathcal{K}_{\text{smaller}} \frac{\vartheta(x + y) \prod_{j=1}^{n-1} \vartheta(-\mu_n + \mu_j)}{\vartheta(-x - \mu_n - G) \prod_{i=1}^{n-1} \vartheta(-x - \mu_n - \lambda_i)} \cdot \frac{\vartheta(\lambda_n - G) \prod_{i=1}^{n-1} \vartheta(\lambda_n - \lambda_i)}{\prod_{j=1}^{n} \vartheta(x + \lambda_n + \mu_j)} \cdot \frac{\vartheta(\lambda_n - \lambda_i)}{\prod_{i=1}^{n} \vartheta(x + \lambda_n + \mu_j)}. \quad (18)$$

It remains to calculate the value $G$. It can be deduced from the following reasoning. According to the inductive hypothesis, $\det \mathcal{K}_{\text{smaller}}$ contains the multiplier $\vartheta \left( (n-2)x - y + \sum_{i=1}^{n-1} \lambda_i + \sum_{j=1}^{n-1} \mu_j \right)$, so it must have a zero at such $\lambda_n$ when the argument of that theta-function equals zero. On the other hand, $\det \mathcal{K}$, generally, does not have a zero at such $\lambda_n$. So, the mentioned theta-function must cancel with the same factor in the denominator of (18), which role only $\vartheta(-x - \mu_n - G)$ can assume. This implies

$$-x - \mu_n - G = \pm \left( (n-2)x - y + \sum_{i=1}^{n-1} \lambda_i + \sum_{j=1}^{n-1} \mu_j \right),$$

and the sign here, namely plus, can be fixed for example by considering the expression (18) as a function of $x$ (which must have the right difference, namely $n$, between the number of poles and zeros in a parallelogram of periods).

Once we have got the proper formula for transition from $\det \mathcal{K}$ to $\det \mathcal{K}_{\text{smaller}}$, the inductive step is over. As for the induction basis, the formula (14) does obviously hold for $n = 1$. 

7
7 Expression for a matrix element of \((J \circ I)A\)

Let us apply the transformation \(J \circ I\), i.e. one step of our evolution, to the \(n \times n\) matrix \(A = K\) given by formula (13). A matrix element \(a_{ji}^{\text{new}}\) of the obtained matrix \(A^{\text{new}}\) is given by formula

\[
a_{ji}^{\text{new}} = \frac{\det A}{A_{ij}} ,
\]

where \(A_{ij}\) is the cofactor for the element \(a_{ij}\) of \(A\). Both the numerator and denominator in (19) are determinants of the form (14). The calculation yields:

\[
a_{ji}^{\text{new}} = a^\text{global} a^\text{row}_j a^\text{column}_i a^\text{element}_{ji},
\]

where

\[
a^\text{global} = \vartheta(x + y)\vartheta\left((n - 1)x - y + \sum_{k=1}^{n} \lambda_k + \sum_{l=1}^{n} \mu_l\right)
\]

is a factor which depends neither on \(j\) nor on \(i\);

\[
a^\text{row}_j = \frac{\prod_{l=1, l \neq j}^{n} \vartheta(\mu_j - \mu_l)}{\prod_{l=1}^{n} \vartheta(x + \lambda_k + \mu_j)}
\]

depends only on \(j\) (the row number for \(a_{ji}^{\text{new}}\));

\[
a^\text{column}_i = \frac{\prod_{k=1, k \neq i}^{n-1} \vartheta(\lambda_i - \lambda_k)}{\prod_{l=1}^{n-1} \vartheta(x + \lambda_i + \mu_l)}
\]

depends only on \(i\) (the column number); and the last factor

\[
a^\text{element}_{ji} = \frac{\vartheta(x + \lambda_i + \mu_j)}{\vartheta\left((n - 2)x - y + \sum_{k=1}^{n} \lambda_k + \sum_{i=1}^{n} \mu_l - \lambda_i - \mu_j\right)}
\]

has much the same form as the initial \(a_{ij}\).

To be exact, the difference between the matrix \(a^\text{element}_{ji}\) made of matrix elements (24) and the initial matrix \(A = K\) can be described as follows: change

\[
x \mapsto x^{\text{new}} = y - (n - 2)x - \sum_{i=1}^{n} \lambda_i - \sum_{j=1}^{n} \mu_j , \quad y \mapsto y^{\text{new}} = -x,
\]

and then perform the matrix transposing. As for the factors (22) and (23), their effect consists in multiplying the matrix \(a^\text{element}_{ji}\) from two sides by diagonal matrices, i.e. doing a gauge transformation (10). The main point is that if we have done not one but \(N\) steps of evolution, the effect of all arising factors (22) and (23), as well as (21), consists just in the appearing of some products of theta-functions with their arguments changing according to a simple law. We do not write out here the corresponding
obvious but bulky formulas. What we see already is that the evolution of a matrix of the form (13) can be described by an explicit formula. The same applies, obviously, to a matrix that was obtained from such one by a transformation (10).

Remark. Here we do not mean just the evolution of gauge equivalence classes as in section 5. Formula (11) shows that if we know what happens with $A$, we also know what happens with $BAC$ after any number of evolution steps.

8 Comparing different values of $n$

8.1 $n = 3$

The ansatz (13) (taken together with the possibility of multiplying a matrix by two diagonal matrices as in formula (10)) is definitely superfluous for the case $n = 3$ where we have presented, in sections 3 and 4, a more direct approach which, by the way, gives the exhaustive information about the cases where the evolution cannot go ahead because of a division by zero.

8.2 $n = 4$

Here the ansatz (13) together with multiplication by two diagonal matrices contains exactly 16 independent parameters, i.e. gives a Zariski open set of matrices. To see this, note first that the 10 values $x, y, \lambda_i$ and $\mu_j$ produce really only 8 parameters, because nothing in (13) changes if we do one of the following translations ($\alpha$ being an arbitrary complex number):

$$\lambda_i \mapsto \lambda_i + \alpha \text{ for all } i, \quad \mu_j \mapsto \mu_j - \alpha \text{ for all } j$$

or

$$x \mapsto x + \alpha, \quad y \mapsto y - \alpha, \quad \lambda_i \mapsto \lambda_i - \alpha \text{ for all } i.$$

Second, the modulus of the elliptic curve is the 9th parameter. And finally, the two diagonal matrices produce the 7 remaining parameters.

8.3 $n \geq 5$

Our ansatz works for any $n$, but when $n > 4$ it corresponds only to a subvariety of a nonzero codimension in the space of all $n \times n$ matrices. For instance, for $n = 5$, the calculation of parameters similar to that done in subsection 8.2 shows that we can describe in the same way the evolution of a 20-parameter family of matrices.

9 A special case with period 4

Although the parameterisation (13) together with the possibility of multiplying the matrices by two diagonal ones as in (10) encompasses in the case $n = 4$ a Zariski open
subset of all $4 \times 4$ matrices, it does not include some interesting special cases (and it does not seem very easy to obtain them as any limiting cases). One specific feature of parameterisation (13) is that if some $2 \times 2$ minor of the matrix equals zero, then all other minors also do so. We present here a matrix

$$
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & xy & x \\
1 & zt & 1 & z \\
1 & t & y & 1 \\
\end{pmatrix}
$$

that does not obey such a requirement. Its interesting property is that if we consider its evolution up to gauge transformations (10), then after four steps we get back at the initial matrix (25). This statement is proved by a direct computer calculation.

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