Non-linear stability of a brane wormhole

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Abstract

We analytically study the non-linear stability of a spherically symmetric wormhole supported by an infinitesimally thin brane of negative tension, which has been devised by Barcelo and Visser. We consider a situation in which a thin spherical shell composed of dust falls into an initially static wormhole; The dust shell plays a role of the non-linear disturbance. The self-gravity of the falling dust shell is completely taken into account through Israel’s formalism of the metric junction. When the dust shell goes through the wormhole, it necessarily collides with the brane supporting the wormhole. We assume the interaction between these shells is only gravity and show the condition under which the wormhole stably persists after the dust shell goes through it.

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I. INTRODUCTION

The wormhole is a fascinating spacetime structure by which shortcut trips or travels to disconnected world are possible. Active theoretical studies of this subject began by influential papers written by Morris, Thorne and Yurtsever[1] and Morris and Thorne[2]. The earlier works are shown in the book written by Visser[3] and review paper by Lobo[4].

We should note that it is not a trivial task to define a wormhole in mathematically rigorous and physically reasonable manner, although we may easily find a wormhole structure in each individual case. Hayward gave an elegant definition of the wormhole as an extension of the “black hole” defined by using trapping horizon[5, 6]. Recently, more sophisticated definition has been proposed by Tomikawa, Izumi and Shiromizu, and showed that the violation of the null energy condition is a necessary condition for the existence of the traversable stationary wormhole in the framework of general relativity, where the null energy condition means that $T_{\mu\nu}k^\mu k^\nu \geq 0$ holds for any null vector $k^\mu$.[7]

1 Several researchers have pointed out an intriguing fact that stationary wormhole solutions exist even without the violation of the null energy condition, if they have non-vanishing NUT charge which causes closed timelike curves[8, 9].

But where does the exotic matter violating the null energy condition appear? In Refs.[1] and [2], the possibilities of quantum effects were discussed. Alternatively, such an exotic matter is often discussed in the context of cosmology. The phantom energy, whose pressure $p$ is given through the equation of state $p = w\rho$ with $w < -1$ and positive energy density, $\rho > 0$, does not satisfy the null energy condition, and a few researchers showed the possibility of the wormhole supported phantom-like matter[10–12]. Recently, theoretical studies from observational point of view on a compact object made of the exotic matter, possibly wormholes, have also reported[13–17], whereas the observational constraint has been reported by Takahashi and Asada[18].

It is important to study the stability of wormhole model in order to know whether it is really traversable or not. The stability against linear perturbations is a necessary condition for the traversable wormhole, but it is insufficient. The investigation of non-linear dynamical situation is necessary, and there are a few studies in this direction[19–22]. In this paper, we also study the non-linear stability of a wormhole in the similar way as that in Ref.[22].

In Ref. [22], the wormhole is assumed to be spherically symmetric and be supported by
an infinitesimally thin spherical shell. The largest merit of a spherical thin shell wormhole is
the finite number of its dynamical degrees of freedom, and hence we can analyze this model
analytically even in highly dynamical cases. The thin shell wormhole was first devised by Visser
[23], and then its stability against linear perturbations was investigated by Poisson
and Visser [24]. Recently, the linear stability of the thin shell wormhole in more general
situation has been investigated by Garcia, Lobo and Visser [25].

We assume that the spherical shell supporting the wormhole is a brane whose equation
of state is \( P = -\sigma \), where \( P \) is the tangential pressure and \( \sigma \) is the energy per unit area.
Furthermore, we assume the existence of spherically symmetric electric field. This wormhole
model has been devised by Barcelo and Visser [26], and its higher dimensional extension has
been studied by Kokubu and Harada [27]. The brane wormhole has a positive gravitational
mass; This is an important difference between the present study and the previous one in
Ref. [22] in which the gravitational mass of the wormhole is negative. The sign of the
mass will be significant for the stability, since the positive mass may cause the gravitational
collapse to form a black hole. It is worthwhile to notice that the positivity of the mass
avoids the observational constraint given in Ref. [18]. Then as in Ref. [22], we consider a
situation in which a infinitesimally thin spherical dust shell concentric with the wormhole
falls into the wormhole, or in other words, plays a role of a non-linear disturbance in the
wormhole spacetime. These spherical shells are treated by Israel’s formulation of metric
junction [28]. When the dust shell goes through the wormhole, it necessarily collides with
the brane supporting the wormhole. The collision between thin shells has already studied
by several researchers [29–31], and we follow them. Then, we show the condition that the
wormhole persists after the passage of a spherical shell.

This paper is organized as follows. In Sec. II, we derive the equations of motion for the
brane supporting the wormhole and the spherical dust shell falling into the wormhole, in
accordance with Israel’s formalism of metric junction. In Sec. III, we derive a static solution
of the wormhole supported by the brane, which is the initial condition. In Sec. IV, we
investigate the condition that a dust shell freely falls from infinity and reaches the wormhole
throat. In Sec. V, we study the motion of the shells and the change in the gravitational mass
of the wormhole after collision. In Sec. VI, we show the condition that the wormhole persists
after the dust shell goes through it. Some complicated manipulations and discussions on this
subject are given separately in Appendix A. Sec. VII is devoted to summary and discussion.
In this paper, we adopt the geometrized unit in which the speed of light and Newton’s gravitational constant are one. However, if necessary, they will be recovered.

II. EQUATION OF MOTIONS FOR SPHERICAL SHELLS

We consider two concentric spherical shells which are infinitesimally thin. As mentioned in the previous section, one is the brane supporting the wormhole and the other is composed of the dust which will cause a non-linear perturbation for the wormhole.

The trajectories of these shells in the spacetime are timelike hypersurfaces: One formed by the brane is denoted by \( \Sigma_1 \), and the other formed by the dust shell is denoted by \( \Sigma_2 \). These hypersurfaces divide the spacetime into three domains denoted by \( D_1 \), \( D_2 \) and \( D_3 \), respectively; \( \Sigma_1 \) divides the spacetime into \( D_1 \) and \( D_2 \), whereas \( \Sigma_2 \) divides the spacetime into \( D_2 \) and \( D_3 \). We also call \( \Sigma_1 \) and \( \Sigma_2 \) the shell-1 and the shell-2, respectively. This configuration is depicted in Fig. I.

The geometry of the domain \( D_i \) \((i = 1, 2, 3)\) is assumed to be described by the Reissner-Nordström solution: the infinitesimal world interval is given by

\[
    ds^2 = -f_i(r)dt_i^2 + \frac{1}{f_i(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

with

\[
    f_i(r) = 1 - \frac{2M_i}{r} + \frac{Q_i^2}{r^2},
\]

where \( M_i \) and \( Q_i \) are the mass parameter and the charge parameter, respectively, whereas the gauge one-form is given by

\[
    A_\mu = \left( -\frac{Q_i}{r}, 0, 0, 0 \right).
\]

We should note that the coordinate \( t_i \) is not continuous at the shells, whereas \( r, \theta \) and \( \phi \) are everywhere continuous.

If \( M_i > |Q_i| \) holds, two horizons can exist, and their locations are given by real roots of the algebraic equation \( f_i(r) = 0 \):

\[
    r = r_{i\pm} := M_i \pm \sqrt{M_i^2 - Q_i^2}.
\]

If \( M_i = |Q_i| \), there can be one degenerate horizon at \( r = M_i \). If \( M_i < |Q_i| \) holds, the roots of \( f_i(r) = 0 \) are complex or real negative, and hence there is no horizon.
Since finite energy and finite momentum concentrate on the infinitesimally thin domains, the stress-energy tensor diverges on these shells. This means that these shells are categorized into the so-called curvature polynomial singularity through the Einstein equations. Even though $\Sigma_A (A = 1, 2)$ are spacetime singularities, we can derive the equation of motion for each spherical shell which is consistent with the Einstein equations by so-called Israel’s formalism, since each of these singularities is so weak that its intrinsic metric exists and the extrinsic curvature defined on each side of $\Sigma_A$ is finite.

We cover the neighborhood of the singular hypersurface $\Sigma_A$ by a Gaussian normal coordinate $\lambda$, where $\partial/\partial \lambda$ is a unit vector normal to $\Sigma_A$ and directs from $D_A$ to $D_{A+1}$. Then, the sufficient condition to apply Israel’s formalism is that the stress-energy tensor is written in the form

$$T_{\mu\nu} = S_{\mu\nu} \delta(\lambda - \lambda_A), \quad (5)$$

where $\Sigma_A$ is located at $\lambda = \lambda_A$, $\delta(x)$ is Dirac’s delta function, and $S_{\mu\nu}$ is the surface stress-energy tensor on $\Sigma_A$.

The junction condition of the metric tensor is obtained as follows. We impose that the metric tensor $g_{\mu\nu}$ is continuous even at $\Sigma_A$. Hereafter, $n^\mu$ denotes the unit normal vector of $\Sigma_A$, instead of $\partial/\partial \lambda$. The intrinsic metric of $\Sigma_A$ is given by

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \quad (6)$$

and the extrinsic curvature is defined as

$$K^{(i)}_{\mu\nu} = -h^\alpha_\mu h^\beta_\nu \nabla^{(i)}_\alpha n_\beta, \quad (7)$$

FIG. 1: The initial configuration is depicted.
where $\nabla^{(i)}_\alpha$ is the covariant derivative with respect to the metric in the domain $D_i$. This extrinsic curvature describes how $\Sigma_A$ is embedded into the domain $D_i$. In accordance with Israel’s formalism, the Einstein equations lead to

$$K_{\mu\nu}^{(A+1)} - K_{\mu\nu}^{(A)} = 8\pi \left( S_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \text{tr} S \right),$$

Where $\text{tr} S$ is the trace of $S_{\mu\nu}$. Equation (8) gives us the condition of the metric junction.

By the spherical symmetry of the system, the surface stress-energy tensors of the shells should be the perfect fluid type;

$$S_{\mu\nu} = \sigma_A u_\mu u_\nu + P_A (h_{\mu\nu} + u_\mu u_\nu),$$

where $\sigma_A$ and $P_A$ are the energy per unit area and the pressure on $\Sigma_A$, respectively, and $u^\mu$ is the 4-velocity.

By the spherical symmetry, the motion of the shell-$A$ is described in the form of $t_i = T_{A,i}(\tau)$ and $r = R_A(\tau)$, where $i = A$ or $i = A + 1$, that is to say, $i$ represents one of two domains divided by the shell-$A$, and $\tau$ is the proper time of the shell. The 4-velocity is given by

$$u^\mu = \left( \dot{T}_{A,i}, \dot{R}_A, 0, 0 \right),$$

where a dot means the derivative with respect to $\tau$. Then, $n_\mu$ is given by

$$n_\mu = \left( -\dot{R}_A, \dot{T}_{A,i}, 0, 0 \right).$$

Together with $u^\mu$ and $n^\mu$, the following unit vectors form an orthonormal frame;

$$\hat{\theta}^\mu = \left( 0, 0, \frac{1}{r}, 0 \right),$$

$$\hat{\phi}^\mu = \left( 0, 0, 0, \frac{1}{r \sin \theta} \right).$$

The extrinsic curvature is obtained as

$$K^{(i)}_{\mu\nu} u^\mu u^\nu = \frac{1}{f_i T_{A,i}} \left( \ddot{R}_A + \frac{f_i'(R_A)}{2} (R_A) \right),$$

$$K^{(i)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = K^{(i)}_{\mu\nu} \hat{\phi}^\mu \hat{\phi}^\nu = -n^\mu \partial_\mu \ln r|_{D_i} = -\frac{f_i(R_A)}{R_A} \dot{T}_{A,i}$$

and the other components vanish, where a prime means a derivative with respect to its argument. By the normalization condition $u^\mu u_\mu = -1$, we have

$$\dot{T}_{A,i} = \pm \frac{1}{f_i(R_A)} \sqrt{\dot{R}_A^2 + f_i(R_A)}.$$
Substituting the above equation into Eq. (15), we have

\[ K^{(i)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = \mp \frac{1}{R_A} \sqrt{\dot{R}_A^2 + f_i(R_A)}. \] (17)

From the \( u-u \) component of Eq. (8), we obtain the following relations.

\[ \frac{d(\sigma_A R_A^2)}{d\tau} + P_A \frac{dR_A^2}{d\tau} = 0. \] (18)

In the case of the following equation of state

\[ P_A = w_A \sigma_A, \] (19)

where \( w_A \) is constant, by substituting Eq. (19) into Eq. (18), we obtain

\[ \sigma_A \propto R_A^{-2(w_A+1)}. \] (20)

A. The shell-1: The brane

As mentioned, we assume that the shell-1 is a brane, i.e.,

\[ w_1 = -1. \]

Without loss of generality, we assume \( Q_2 \geq 0 \). Furthermore, we focus on the case of

\[ Q_2 = |Q_1| = Q \geq 0. \]

Since the electric charge of the shell-1 is equal to \( Q_2 - Q_1 \), the electric charge of the shell-1 is zero in the case of \( Q_2 = Q_1 \), whereas the electric charge of the shell-1 may not vanish in the case of \( Q_2 = -Q_1 \). As will be shown later, the results in both cases are identical to each other.

By the assumption, the union of the domains \( D_1 \) and \( D_2 \) should have the wormhole structure by the shell-1. This means that \( n^a \partial_a \ln r|_{D_1} < 0 \) and \( n^a \partial_a \ln r|_{D_2} > 0 \) (see Fig. 2), and we have

\[ K^{(1)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = + \frac{1}{R_{1}} \sqrt{\dot{R}_{1}^2 + f_1} \quad \text{and} \quad K^{(2)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = - \frac{1}{R_{1}} \sqrt{\dot{R}_{1}^2 + f_2}. \] (21)

Here, note that Eq. (21) implies \( \dot{T}_{1,1} \) is negative, whereas \( \dot{T}_{1,2} \) is positive. Hence, the direction of the time coordinate basis vector in \( D_1 \) is opposite with that in \( D_2 \).
From θ-θ component of Eq. (8), we obtain the following relations.

$$\sqrt{\dot{R}_1^2 + f_2(R_1)} + \sqrt{\dot{R}_1^2 + f_1(R_1)} = -4\pi \sigma_1 R_1.$$  \hfill (22)

Equation (22) is satisfied only if $\sigma_1$ is negative, and hence we assume so. From Eq. (20), we have

$$\sigma_1 = -\frac{\mu}{4\pi},$$  \hfill (23)

where $\mu$ is a positive constant, and, hereafter, we call it the stress constant.

Let us rewrite Eq. (22) into the form of the energy equation for the shell-1. First, we write it in the form

$$\sqrt{\dot{R}_1^2 + f_2(R_1)} = -\sqrt{\dot{R}_1^2 + f_1(R_1)} + \mu R_1,$$  \hfill (24)

and then take a square of the both sides of the above equation to obtain

$$\sqrt{\dot{R}_1^2 + f_1(R_1)} = \frac{1}{2\mu R_1} \left[ f_1(R_1) - f_2(R_1) + (\mu R_1)^2 \right].$$  \hfill (25)

By taking a square of the both sides of the above equation, we have

$$\dot{R}_1^2 + V_1(R_1) = 0,$$  \hfill (26)

where

$$V_1(r) = 1 - \frac{1}{r^4} \left( \frac{M_2 - M_1}{\mu} \right)^2 - \frac{M_1 + M_2}{r} + \frac{Q^2}{r^2} - \left( \frac{\mu}{2} \right)^2 r^2.$$  \hfill (27)

Equation (26) is regarded as the energy equation for the shell-1. The function $V_1$ corresponds to the effective potential. In the allowed domain for the motion of the shell-1, an inequality $V_1 \leq 0$ should hold. But, this inequality is not a sufficient condition of the allowed region.
The left hand side of Eq. (24) is non-negative, and hence the right hand side of it should also be non-negative. Then, substituting Eq. (25) into the right hand side of Eq. (24), we have

$$0 \leq -\sqrt{\dot{R}_1^2 + f_1(R_1)} + \mu R_1 = \frac{\mu R_1}{2} - \frac{M_2 - M_1}{\mu R_1^2}. \quad (28)$$

Further manipulation leads to

$$R_1^3 \geq \frac{2}{\mu^2} (M_2 - M_1). \quad (29)$$

By the similar argument, we obtain

$$-\sqrt{\dot{R}_2^2 + f_2(R_1)} + \mu R_1 \geq 0. \quad (30)$$

Then, by the similar procedure, we have

$$R_1^3 \geq \frac{2}{\mu^2} (M_1 - M_2). \quad (31)$$

Hence, we have the following constraint;

$$R_1 \geq \left(\frac{2|M_1 - M_2|}{\mu^2}\right)^{\frac{1}{3}}. \quad (32)$$

In order to find the allowed domain for the motion of the shell-1, we need to take into account the constraint (32) in addition to the condition $V_1 \leq 0$.

**B. The shell-2: The dust shell**

As mentioned, we assume that the shell-2 is composed of non-exotic dust, i.e., $w_2 = 0$ and $\sigma_2 > 0$. The proper mass of the shell-2 is defined as

$$m_2 \equiv 4\pi \sigma_2 R_2^2. \quad (33)$$

We find that $m_2$ is constant by Eq. (20) and positive by $\sigma_2 > 0$. We also assume

$$Q_3 = Q_2 = Q.$$ 

This assumption means that the shell-2 is electrically neutral.

The wormhole structure does not exist around the shell-2 due to $\sigma_2 > 0$. Hence, the extrinsic curvature of the shell-2 is given by

$$K^{(2)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = -\frac{1}{R_2} \sqrt{\dot{R}_2^2 + f_2(R_2)} \quad \text{and} \quad K^{(3)}_{\mu\nu} \hat{\theta}^\mu \hat{\theta}^\nu = -\frac{1}{R_2} \sqrt{\dot{R}_2^2 + f_3(R_2)}. \quad (34)$$
By using the above result, the $\theta - \theta$ component of the junction condition leads to
\[
\sqrt{\dot{R}_2^2 + f_3(R_2)} - \sqrt{\dot{R}_2^2 + f_2(R_2)} = -\frac{m_2}{R_2}. \tag{35}
\]

Since $m_2$ is positive, we find from the above equation that $f_2(R_2) > f_3(R_2)$, or equivalently, $M_3 > M_2$. From Eq. (35), we have
\[
\sqrt{\dot{R}_2^2 + f_3(R_2)} = \sqrt{\dot{R}_2^2 + f_2(R_2)} - \frac{m_2}{R_2}. \tag{36}
\]

By taking the square of the both sides of Eq. (36), we have
\[
\dot{R}_2^2 + f_2(R_2) = \frac{M_3 - M_2}{m_2} + \frac{m_2}{2R_2}. \tag{37}
\]

By taking a square of both sides of Eq. (37), we obtain an energy equation for the shell-2,
\[
\dot{R}_2^2 + V_2(R_2) = 0, \tag{38}
\]
where
\[
V_2(r) = 1 - E^2 - \frac{2M_4}{r} + \frac{Q_2^2}{r^2} - \left(\frac{m_2}{2r}\right)^2, \tag{39}
\]
with
\[
E \equiv \frac{M_3 - M_2}{m_2} \quad \text{and} \quad M_4 \equiv \frac{1}{2}(M_2 + M_3). \tag{40}
\]
Note that $E$ is a constant which corresponds to the specific energy of the shell-2.

In the allowed domain for the motion of the shell-2, the effective potential $V_2$ should be non-positive. But, as in the case of the shell-1, it is not a sufficient condition for the allowed domain. Since the left hand side of Eq. (36) is non-negative, the following inequality should be satisfied.
\[
\sqrt{\dot{R}_2^2 + f_2(R_2)} - \frac{m_2}{R_2} \geq 0. \tag{41}
\]
Substituting Eq. (37) into the left hand side of Eq. (41), we have
\[
R_2 \geq R_b := \frac{m_2^2}{2(M_3 - M_2)}. \tag{42}
\]
The above inequality should also be taken into account as a condition for the allowed domain.

As mentioned, in the case of $M_3 \geq Q$, the horizon may appear in the domain $D_3$; When the radius $R_2$ of the shell-2 becomes smaller than or equal to
\[
R_H := r_{3+} = M_3 + \sqrt{M_3^2 - Q^2}, \tag{43}
\]
a black hole including both the wormhole and the shell-2 forms. Here it should be noted that Eq. (42) is derived by using $u_2^2$ is positive, but $u_2^2$ can change its sign within the black hole $R_2 < R_H$. Hence, if $R_b$ is smaller than $R_H$, Eq. (42) looses its validity, and thus the allowed domain for the motion of the shell-2 is determined by the only condition $V_2 \leq 0$. The allowed domain for the motion of the shell-2 satisfies $V_2 \leq 0$ and furthermore Eq. (42) only if $R_b \geq R_H$.

### III. STATIC WORMHOLE SOLUTION

We consider a situation in which the brane supporting the wormhole is initially static and located at $r = a$. Furthermore, we assume that the wormhole is initially mirror symmetric with respect to $r = a$, i.e., $f_1(r) = f_2(r) = f(r)$, or equivalently, $M_1 = M_2 = M_w$. In order that the shell-1 is in a static configuration, its areal radius $R_1 = a$ should satisfy $V_1(a) = 0 = V_1'(a)$. Furthermore, $V_1''(a) > 0$ should hold so that this structure is stable.

The condition $V_1(a) = 0$ leads to the following relation between the stress constant $\mu$ and the areal radius $a$:

$$\mu^2 = \frac{4}{a^2} f(a), \quad (44)$$

whereas, together with the above condition, the condition $V_1'(a) = 0$ leads to

$$a^2 - 3M_w a + 2Q^2 = 0. \quad (45)$$

The roots of the above equation are given by

$$a = a_\pm := \frac{1}{2} \left( 3M_w \pm \sqrt{9M_w^2 - 8Q^2} \right).$$

The following inequality should hold so that $a$ is real and positive;

$$M_w \geq \frac{2\sqrt{2}}{3} Q. \quad (46)$$

Equation (46) implies that $M_w$ is non-negative.

Together with Eqs. (44) and (45), the condition $V_1''(a) > 0$ leads to

$$a < \sqrt{2}Q. \quad (47)$$

The above condition implies that the charge parameter $Q$, cannot vanish so that the areal radius $a$ is positive. Since we have

$$a_\pm - \sqrt{2}Q = \frac{1}{2} \sqrt{3M_w - 2\sqrt{2}Q} \left( \sqrt{3M_w - 2\sqrt{2}Q} \pm \sqrt{3M_w + 2\sqrt{2}Q} \right),$$

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\( a = a_+ \) does not satisfy Eq. (47), but \( a = a_- \) does.

Since \( \mu^2 \) should be positive, Eq. (44) implies that \( f(a_-) > 0 \) should be satisfied. By using Eq. (45), the condition \( f(a_-) > 0 \) leads to

\[
3M_w^2 - 2Q^2 > M_w \sqrt{9M_w^2 - 8Q^2}.
\]

By taking the square of both sides of the above inequality, we obtain \( M_w < Q \).

To summarize this section, the areal radius \( a \) and the stress constant \( \mu \) of the static wormhole are given as a function of \( M_w \) and \( Q \);

\[
a = \frac{1}{2} \left( 3M_w - \sqrt{9M_w^2 - 8Q^2} \right), \tag{48}
\]

\[
\mu = \frac{a}{2} \sqrt{1 - \frac{2M_w}{a} + \frac{Q^2}{a^2}}, \tag{49}
\]

with a constraint

\[
M_w < Q < \frac{3}{2\sqrt{2}} M_w. \tag{50}
\]

Equations (48) and (50) lead to

\[
M_w < a < \frac{3}{2} M_w. \tag{51}
\]

IV. CAN THE SHELL-2 REACH THE WORMHOLE THROAT?

We consider the condition that the shell-2 enters the wormhole supported by the shell-1. The allowed domain for the motion of the shell-2 is determined by the conditions (42) and \( V_2 \leq 0 \). The shell-2 is assumed to come from the spatial infinity. By this assumption, \( E \geq 1 \) should be satisfied so that \( V_2(r) < 0 \) for sufficiently large \( r \).

A. The case of \( Q \leq m_2/2 \)

In this case, \( V_2(r) \) is negative for \( r \geq a \). It should be noted that, in this case,

\[
M_3 = M_2 + Em_2 > M_w + 2EQ > Q
\]
is satisfied, and hence $R_H$ is real and positive. As explained in the paragraph including Eq. (43), since we have

$$R_H - R_b = M_3 + \sqrt{M_3^2 - Q^2} - \frac{m_2^2}{2(M_3 - M_2)}$$

$$= M_w + \frac{m_2(2E^2 - 1)}{2E} + \sqrt{M_3^2 - Q^2}$$

$$> 0,$$

the allowed domain for the motion of the shell-2 is determined by the only condition $V_2 < 0$, and hence the shell-2 can reach the wormhole throat $r = a$ in this case.

**B. The case of $Q > m_2/2$**

We consider the case of $E = 1$ and that of $E > 1$, separately.

1. The case of $E = 1$

In this case, the positive real root of $V_2(R_z) = 0$ is given by

$$R_z = \frac{4Q^2 - m_2^2}{4(2M_w + m_2)}.$$

The allowed domain for the motion of the shell-2 is $R_2 \geq R_z$. We have

$$a - R_z = \frac{1}{2} \left( 3M_w - \sqrt{9M_w^2 - 8Q^2} \right) - \frac{4Q^2 - m_2^2}{4(2M_w + m_2)}$$

$$= \frac{1}{2M_w + m_2} \left[ \left( \sqrt{9M_w^2 - 8Q^2} - \frac{2M_w + m_2}{4} \right)^2 - \frac{25}{4} M_w^2 + 7Q^2 + \frac{3}{16} m_2^2 + \frac{5}{4} M_w m_2 \right]$$

$$> \frac{1}{2M_w + m_2} \left[ \left( \sqrt{9M_w^2 - 8Q^2} - \frac{2M_w + m_2}{4} \right)^2 + \frac{3}{4} Q^2 + \frac{3}{16} m_2^2 + \frac{5}{4} M_w m_2 \right]$$

$$> 0,$$

where we have used $M_w < Q$ in Eq. (50). The above inequality implies that the shell-2 can reach the wormhole throat $r = a$. 


2. The case of $E > 1$

In this case, the positive real root of $V_2(R_z) = 0$ is given by

$$R_z = \frac{1}{E^2 - 1} \left[ -M_w - \frac{m_2 E}{2} + \sqrt{\left( M_w + \frac{m_2 E}{2} \right)^2 + (E^2 - 1) \left( Q^2 - \frac{m_2^2}{4} \right)} \right].$$

The allowed domain for the motion of the shell-2 is $R \geq R_z$. We can easily see that $R_z \to 0$ and so $a > R_z$, in the limit of $E \to \infty$. The derivative of $R_z$ with respect to $E$ with $M_w$, $m_2$ and $Q$ fixed is given by

$$\frac{\partial R_z}{\partial E} = \frac{X - Y}{(E^2 - 1)^2 \sqrt{\left( M_w + \frac{1}{2} m_2 E \right)^2 + (E^2 - 1) \left( Q^2 - \frac{m_2^2}{4} \right)}},$$

where

$$X = \left( \frac{1}{2} m_2 E^2 + \frac{1}{2} m_2 + 2 M_w E \right) \sqrt{\left( M_w + \frac{1}{2} m_2 E \right)^2 + (E^2 - 1) \left( Q^2 - \frac{m_2^2}{4} \right)},$$

$$Y = Q^2 E^3 + \frac{3}{2} M_w m_2 E^2 - \left( Q^2 - \frac{1}{4} m_2^2 - 2 M_w^2 \right) E + \frac{1}{2} M_w m_2.$$

It is not so difficult to see that $Y$ is positive for $E \geq 1$, whereas $X$ is trivially positive. Since we have

$$Y^2 - X^2 = \left( Q^2 - \frac{1}{4} m_2^2 \right) \left( Q^2 E^2 + M m_2 E + \frac{1}{4} m_2^2 \right) (E^2 - 1)^2 > 0,$$

we find

$$\frac{\partial R_z}{\partial E} < 0$$

for $E > 1$. As a result, since, as already shown, $a > R_z$ holds for both $E = 1$ and $E \to \infty$, we have $a > R_z$ even for $E > 1$.

In the case of $M_3 \geq Q$, as already shown in the case of $Q < m_2/2$, since $R_b < R_H$ holds, the allowed domain for the motion of the shell-2 is determined by the only condition $V_2 \leq 0$. Hence, the shell-2 can reach the wormhole throat $r = a$.

In the case of $M_3 < Q$, or equivalently, $M_w < Q - m_2 E$, no horizon forms in $D_3$, and hence we need to study whether $R_z$ is larger than $R_b$. In the case of $E = 1$, we have

$$R_z - R_b = \frac{4Q^2 - 4m_2 M_w - 3m_2^2}{4(2M_w + m_2)} > \frac{(2Q - m_2)^2}{4(2M_w + m_2)} > 0.$$
In the case of $E > 1$, we have

$$R_z - R_b = \frac{1}{2E(E^2 - 1)} \left[ -2ME + m_2 - 2m_2E^2 
+ 2E \sqrt{\left( M_w + \frac{m_2E}{2} \right)^2 + (E^2 - 1) \left( Q^2 - \frac{m_2^2}{4} \right)} \right]$$

$$> \frac{1}{2E(E^2 - 1)} \left[ -2ME + m_2 - 2m_2E^2 
+ 2E \sqrt{\left( M_w + \frac{m_2E}{2} \right)^2 + (E^2 - 1) \left( (M_w + m_2E)^2 - \frac{m_2^2}{4} \right)} \right]$$

$$= \frac{1}{2E(E^2 - 1)} \left[ 2ME(E - 1) + m_2(2E^2 - 1)(E - 1) \right]$$

$$> 0. \quad (56)$$

Since now we have $R_b < R_z$ for $E \geq 1$, Eq. (42) gives no additional constraint on the allowed domain for the motion of the shell-2. As a result, the shell-2 can reach the wormhole throat $r = a$ also in $M_3 < Q$.

To summarize this section, the shell-2 reaches the wormhole throat $r = a$ from infinity if it moves inward initially. This result is different from the case of the wormhole with the negative mass studied in Ref. [22]: In the negative mass case, $E$ should be larger than unity, or in other words, the larger initial ingoing velocity than the present positive mass case is necessary so that the shell-2 reaches the wormhole throat, since the gravity produced by the wormhole with the negative mass is repulsion.

V. COLLISION BETWEEN THE SHELLS

When the shell-2 goes through the wormhole, it necessarily collides with the shell-1 located at the wormhole throat $r = a$. The situation may be recognized by Fig. 3. Then, in this section, we show how the mass parameter in the domain between the shells changes by the collision.

We assume that the interaction between these shells is gravity only, or in other words, these shells merely go through each other: Both of the 4-velocity and the proper mass $4\pi \sigma_A R_A^2$ of each shell are continuous at the collision event.

In the domain $D_2$, we may introduce two kinds of the orthonormal frame $(u^\alpha_A, n^\alpha_A, \hat{\theta}^\alpha, \hat{\phi}^\alpha)$
FIG. 3: The shell-1 supporting the wormhole is initially static. The shell-2 falls into the wormhole and collides with the shell-1. The interaction between these shells is assumed to be gravity only: The shells merely go through each other.

at the collision event, where $A = 1, 2$. We can express the 4-velocity $u_1^\alpha$ of the shell-1 by using the orthonormal frame $(u_2^\alpha, n_2^\alpha, \theta^\alpha, \phi^\alpha)$, and converse is also possible;

$$u_1^\alpha = \left(-u_1^\alpha u_{2\beta} + n_1^\alpha n_{2\beta} + \hat{\theta}^\alpha \hat{\theta}_\beta + \hat{\phi}^\alpha \hat{\phi}_\beta \right) u_2^\beta = -(u_1^\beta u_{2\alpha}) u_2^\alpha + (u_1^\beta n_{2\alpha}) n_2^\alpha, \quad (57)$$

$$u_2^\alpha = \left(-u_1^\alpha u_{1\beta} + n_1^\alpha n_{1\beta} + \hat{\theta}^\alpha \hat{\theta}_\beta + \hat{\phi}^\alpha \hat{\phi}_\beta \right) u_1^\beta = -(u_2^\beta u_{1\alpha}) u_1^\alpha + (u_2^\beta n_{1\alpha}) n_1^\alpha. \quad (58)$$

The components of $u_1^\alpha$ and $n_A^\alpha$ with respect to the coordinate basis in $D_2$ are given by

$$u_1^\alpha = \left(\frac{1}{\sqrt{f}}, 0, 0, 0 \right), \quad (59)$$

$$n_1^\alpha = \left(0, \sqrt{f}, 0, 0 \right), \quad (60)$$

$$u_2^\alpha = \left(\frac{1}{f} \sqrt{\dot{R}_2^2 + f}, \dot{R}_2, 0, 0 \right), \quad (61)$$

$$n_2^\alpha = \left(\frac{\dot{R}_2}{f}, \sqrt{\dot{R}_2^2 + f}, 0, 0 \right), \quad (62)$$
where \( f = f(a) \). Hence, we have

\[
{u_1}^\beta u_2^\beta = {u_2}^\beta u_1^\beta = - \sqrt{\frac{\dot{R}_2^2}{f}} + 1,
\]

(63)

\[
{u_1}^\beta n_2^\beta = - \frac{\dot{R}_2}{\sqrt{f}},
\]

(64)

\[
{u_2}^\beta n_1^\beta = \frac{\dot{R}_2}{\sqrt{f}}.
\]

(65)

**A. Shell-1 after the collision**

The orthonormal frame \((u_2^\alpha, n_2^\alpha, \hat{\theta}_\alpha, \hat{\phi}_\alpha)\) at the collision event is available also in the domain \(D_3\). The components of \(u_2^\alpha\) and \(n_2^\alpha\) with respect to the coordinate basis in \(D_3\) are given by

\[
u_2^\alpha = \left( \frac{1}{f_3} \sqrt{\dot{R}_2^2 + f_3}, \dot{R}_2, 0, 0 \right),
\]

(66)

\[
n_2^\alpha = \left( \frac{\dot{R}_2}{f_3}, \sqrt{\dot{R}_2^2 + f_3}, 0, 0 \right),
\]

(67)

where \( f_3 = f_3(a) \). By using the above equations, we obtain the components of \(u_1^\alpha\) with respect to the coordinate basis in \(D_3\) as

\[
u_1^t = -(u_1^\beta u_2^\beta)u_2^t + (u_1^\beta n_2^\beta)n_2^t = -(u_1^\beta u_2^\beta)\frac{1}{f_3} \sqrt{\dot{R}_2^2 + f_3} + (u_1^\beta n_2^\beta)\frac{\dot{R}_2}{f_3}
\]

= \[
\frac{1}{f_3 \sqrt{f}} \left[ \sqrt{(\dot{R}_2^2 + f)(\dot{R}_2^2 + f_3)} - \dot{R}_2^2 \right],
\]

(68)

\[
u_1^r = -(u_1^\beta u_2^\beta)u_2^r + (u_1^\beta n_2^\beta)n_2^r = -(u_1^\beta u_2^\beta)\dot{R}_2 + (u_1^\beta n_2^\beta)\sqrt{\dot{R}_2^2 + f_3}
\]

= \[
\frac{\dot{R}_2}{\sqrt{f}} \left( \sqrt{\dot{R}_2^2 + f} - \sqrt{\dot{R}_2^2 + f_3} \right),
\]

(69)

\[
u_1^\theta = u_1^\phi = 0.
\]

(70)

The above components are regarded as those of the 4-velocity of the shell-1 just after the collision event. By using Eqs. (35) and (69), we have

\[
u_1^r = \frac{m_2 \dot{R}_2}{a \sqrt{f}}.
\]

(71)

By taking the square of Eq. (35) and using Eq. (38), we have

\[
\sqrt{(\dot{R}_2^2 + f)(\dot{R}_2^2 + f_3)} = \dot{R}_2^2 + \frac{f + f_3}{2} - \frac{1}{2} \left( \frac{m_2}{a} \right)^2 = E^2 - \left( \frac{m_2}{2a} \right)^2.
\]

(72)
The above equation implies
\[ E^2 > \left( \frac{m_2}{2a} \right)^2. \] (73)

Then, we have
\[ u_1^{\alpha} = \frac{1}{f_3 \sqrt{f}} \left[ 1 - \frac{2M_4}{a} + \frac{Q^2}{a^2} - \frac{1}{2} \left( \frac{m_2}{a} \right)^2 \right]. \] (74)

We can check that the normalization condition \(-f_3(u_1^{\alpha})^2 + f^{-1}(u_1^{\alpha})^2 = -1\) is satisfied.

The above result implies that just after the collision, the derivative of the areal radius of the shell-1 with respect to its proper time becomes
\[ \hat{R}_1|_{\text{after}} = \frac{m_2 \hat{R}_2}{a \sqrt{f}}. \] (75)

Since the shell-2 falls into the wormhole just before the collision, \(\hat{R}_2\) is negative. This fact implies that the shell-1 or equivalently the radius of the wormhole throat begins shrinking just after the collision since \(m_2\) is assumed to be positive.

The domain between the shell-1 and the shell-2 after the collision is called \(D_4\). From the junction condition between \(D_3\) and \(D_4\), the shell-2 obeys the following equation just after the collision;
\[ \hat{R}_2^2|_{\text{after}} = -1 + \left( \frac{M_3 - M_4}{\mu R_1^2} \right)^2 + \frac{M_3 + M_4}{R_1} - \frac{Q^2}{R_1^2} + \left( \frac{\mu R_1}{2} \right)^2. \] (76)

From the above equation and Eq. (75), we obtain
\[ \hat{R}_2^2 = -f \left( \frac{a}{m_2} \right)^2 \left[ 1 - \left( \frac{M_3 - M_4}{\mu a^2} \right)^2 - \frac{M_3 + M_4}{a} + \frac{Q^2}{a^2} - \left( \frac{\mu a}{2} \right)^2 \right]. \] (77)

Here note that \(\hat{R}_2\) is the value of the shell-2 just before the collision.

B. Shell-2 after the collision

Since the orthonormal frame \((u_1^\alpha, n_1^\alpha, \hat{\theta}^\alpha, \hat{\phi}^\alpha)\) is available also in the domain \(D_1\). By using Eqs. (15), (16) and (21), the components of \(u_1^\alpha\) and \(n_1^\alpha\) with respect to the coordinate basis in \(D_1\) are given by
\[ u_1^\alpha = \left( -\frac{1}{\sqrt{f}}, 0, 0, 0 \right), \] (78)
\[ n_1^\alpha = \left( 0, -\sqrt{f}, 0, 0 \right). \] (79)
As already noted just below Eq. (21), the time component of \( u^t_\alpha \) with respect to the coordinate basis in \( D_1 \) is negative.

By using the above equations, we obtain the components of \( u^\alpha_2 \) with respect to the coordinate basis in \( D_1 \) as

\[
\begin{align*}
  u^t_1 &= -(u^\beta_2 u^t_1) + (u^\beta_2 n^t_1) n^t_1 = (u^\beta_2 u^t_1) \frac{1}{\sqrt{f}} = -\frac{1}{f} \sqrt{\hat{R}_2^2 + f}, \\
  u^r_2 &= -(u^\beta_2 u^r_1) + (u^\beta_2 n^r_1) n^r_1 = -(u^\beta_2 n^r_1) \sqrt{f} = -\hat{R}_2, \\
  u^\theta_2 &= u^\phi_2 = 0.
\end{align*}
\]

Since \( \dot{R}_2 \) is negative, the shell-2 begins expanding after the collision. This is a reasonable result because of the wormhole structure.

By the spherical symmetry, \( D_4 \) is also described by the Reissner-Nordström geometry with the mass parameter \( M_4 \) and the unchanged charge parameter \( Q \). From the junction condition between \( D_1 \) and \( D_4 \), we have

\[
\dot{R}_2^2|_{\text{after}} = -1 + \left( \frac{M_1 - M_4}{m_2} \right)^2 + \frac{M_1 + M_4}{R_2} - \frac{Q^2}{R_2} + \left( \frac{m_2}{2R_2} \right)^2.
\]

From Eq. (81), since \( \dot{R}_2^2 \) is unchanged by the collision, we have

\[
\dot{R}_2^2 = -1 + \left( \frac{M_1 - M_4}{m_2} \right)^2 + \frac{M_1 + M_4}{a} - \frac{Q^2}{a^2} + \left( \frac{m_2}{2a} \right)^2.
\]

Here again note that \( \dot{R}_2 \) is the value of the shell-2 just before the collision.

### C. The mass parameter \( M_4 \) in \( D_4 \)

From Eqs. (38) and (39), we can write \( \dot{R}_2^2 \) just before the collision in the form

\[
\dot{R}_2^2 = -1 + \left( \frac{M_3 - M_2}{m_2} \right)^2 + \frac{M_2 + M_3}{a} - \frac{Q^2}{a^2} + \left( \frac{m_2}{2a} \right)^2.
\]

Then, Eqs. (77), (84) and (85) determine the unknown parameter \( M_4 \).

Since \( M_1 = M_2 = M_w \), Eqs. (84) and (85) lead to

\[
\left( \frac{M_w - M_3}{m_2} \right)^2 + \frac{M_w + M_3}{a} = \left( \frac{M_w - M_4}{m_2} \right)^2 + \frac{M_w + M_4}{a}.
\]

By solving the above equation with respect to \( M_4 \), we obtain two roots, \( M_4 = M_3 \) and \( M_4 = 2M_w - M_3 - \frac{m_2^2}{a} \). By using Eqs. (44) and (77), we find that the latter one, i.e.,

\[
M_4 = 2M_w - M_3 - \frac{m_2^2}{a}
\]
is a solution we need, where we have used $M_3 = M_w + m_2E$. Hence, after the collision, the wormhole does not have the mirror symmetry with respect to $r = a$.

VI. THE CONDITION THAT THE WORMHOLE PERSISTS

In this section, we consider the condition that the wormhole stably exists after the passage of the shell-2. From Eq. (76), the effective potential of the shell-1 after the collision is given by

$$V_{1|\text{after}}(r) = \frac{1}{r^4} \left[ -\frac{\mu^2}{4} r^6 + r^4 - (M_3 + M_4) r^3 + Q^2 r^2 - \left( \frac{M_4 - M_3}{\mu} \right)^2 \right]$$

(88)

By using Eqs. (44), (45) and (87), we have

$$\mu^2 = \frac{2(a - M_w)}{a^3},$$

(89)

$$Q^2 = \frac{a}{2} \left( 3M_w - a \right),$$

(90)

$$M_3 + M_4 = 2M_w - \frac{m_2^2}{a},$$

(91)

$$M_3 - M_4 = 2m_2E + \frac{m_2^2}{a}.$$  

(92)

Equations (89)–(92) imply that the effective potential $V_{1|\text{after}}$ is characterized by four parameters, $M_w$, $a$, $m_2$ and $E$. By regarding $M_w$ as a parameter to determine the unit of length, the motion of the wormhole after the passage of the shell-2 is characterized by three parameters $a$, $m_2$ and $E$.

A. No black-hole formation

First of all, $a > R_H$ must be satisfied in the case of $M_3 \geq Q$. If not, the wormhole is enclosed by an event horizon after the shell-2 enters the domain $r \leq R_H$, and hence the wormhole cannot stably persist.

The inequality $M_3 \geq Q$ leads to

$$m_2 \geq \frac{1}{E} \left( \sqrt{\frac{a(3M_w - a)}{2}} - M_w \right),$$

(93)

whereas the inequality $a > R_H$ leads to

$$m_2 < \frac{a - M_w}{4E}.$$  

(94)
If Eq. (93) holds, Eq. (94) should be satisfied. It is not so difficult to see that
\[
\frac{a - M_w}{4} > \sqrt{\frac{a(3M_w - a)}{2} - M_w},
\]
and hence both of Eqs. (93) and (94) can hold simultaneously. In the case of
\[
m_2 < \frac{1}{E}\left(\sqrt{\frac{a(3M_w - a)}{2} - M_w}\right),
\]
and hence no horizon appears in \(D_3\) even if the shell-2 enters the wormhole. As a result, the event horizon does not form by the passage of the shell-2 only if the inequality (94) holds.

Hereafter, we focus on the following bounded domain in the parameter space \((a, m_2)\);
\[
\mathcal{D} = \{(a, m_2) \mid M_w < a < \frac{3}{2}M_w \text{ and } 0 < m_2 < \frac{a - M_w}{4E}\}. \quad (95)
\]

B. **Allowed domain for the motion of the shell-1**

The allowed domain for the motion of the shell-1 after the collision should be restricted in \(r > 0\) and bounded so that the wormhole stably persists. We introduce a function defined as
\[
W(r) := r^4V_{1|\text{after}}(r) = -\frac{\mu^2}{4}r^6 + r^4 - (M_3 + M_4)r^3 + Q^2r^2 - \left(\frac{M_4 - M_3}{\mu}\right)^2. \quad (96)
\]
It is easy to see that the function \(W(r)\) has a negative minimum at \(r = 0\). Since \(W(r)\) has at most five extrema, \(W(r)\) should have two non-negative maxima and one negative minimum in \(r > 0\) and one maximum in \(r < 0\) so that there is a bounded domain of \(V_{1|\text{after}} < 0\) in \(r > 0\).

We introduce a function \(w(r)\) defined as
\[
\frac{dW(r)}{dr} = -\frac{3\mu^2}{2}rw(r) := -\frac{3\mu^2}{2}r \left[r^4 - \frac{8}{3\mu^2}r^2 + \frac{2(M_3 + M_4)}{\mu^2}r - \frac{4Q^2}{3\mu^2}\right]. \quad (97)
\]
The quartic equation \(w(r) = 0\) should have three positive real roots and one negative real root so that there is a bounded domain of \(V_{1|\text{after}} < 0\) in \(r > 0\). In Appendix A, we see that this is the case as long as the parameters \(a\) and \(m_2\) are restricted to the domain \(\mathcal{D}\). Thus, \(W(r)\) has two maxima and one minimum in \(r > 0\) and one maximum in \(r < 0\). The radial coordinates of the extrema of \(W(r)\) other than \(r = 0\), i.e., the roots of \(w(r) = 0\) are denoted
FIG. 4: Adopting the unit $M_w = 1$, the function $W(r)$ with $(a, m_2, E) = (1.3, 0.05, 1)$ is depicted. There is a maximum of $W(r)$ at $r = r_A < 0$. However, it is very large compared with extrema in $r \geq 0$, and hence we do not show it in this figure.

by $r_A$, $r_B$, $r_C$ and $r_D$, all of which are the functions of not $E$ but $a$ and $m_2$; the explicit forms of $r_A$, $r_B$, $r_C$ and $r_D$ are given through Ferrari’s formula for the roots of a quartic equation, but we will not show them here since the expressions of the roots are too complicated to get any information from them. We assume $r_A < 0 < r_B < r_C < r_D$, and hence $W(r)$ takes maxima at $r = r_A$, $r = r_B$ and $r = r_D$ whereas it takes minima at $r = 0$ and $r = r_C$. (See Fig. 4).

In the case of $m_2 = 0$, since $V_{1|\text{after}}(r)$ is equal to $V_1(r)$, we have $r_C = a$ and $W(r_C) = 0$, and both $W(r_B)$ and $W(r_D)$ are positive (see Fig. 5). By contrast, in the case of non-vanishing $m_2$, we have

$$W(a) = -\frac{m_2^2 a^2}{2(a - M_w)} \left[ 2(2E^2 - 1)a + 2M_w + 4Em_2 + \frac{m_2^2}{a} \right] < 0,$$

and hence $W(r_C)$ must be negative by the continuous dependence of $W(r)$ on the parameter $m_2$.

Since the shell-1 shrinks just after the passage of the shell-2 [see Eq. (75)], if $W(r_B)$ is negative, the shell-1, or equivalently, wormhole collapses to form a black hole. If $W(r_B)$ vanishes, the shell-1 asymptotically approaches $r = r_B$ and thus the size of the wormhole
remains finite. If \( W(r_B) \) is positive, the shell-1 bounces off the potential varier, and then \( R_1 \) increases. In this case, \( W(r_D) \) should be equal to or larger than zero so that the wormhole persists with its size finite. The domain in \((a, m_2)\)-space with \( E \) fixed in which the wormhole persists after the passage of the shell-2 is a curve \( W(r_B) = 0 \) and a domain restricted by \( W(r_B) > 0 \) and \( W(r_D) \geq 0 \). Hence the critical curves in \((a, m_2)\)-space with \( E \) fixed are given by the condition

\[
W(r_B) = 0 \quad \text{and} \quad W(r_D) = 0.
\]

In Fig. 6, we depict the domain in \((a, m_2)\)-space with \( E = 1 \), in which the wormhole persists after the passage of the shell-2, as an unshaded region. Figure 7 is Fig. 6 in close-up of the neighborhood of the intersections of the curves \( W(r_B) = 0, W(r_D) = 0 \) and \( a - M = 4m_2 \), i.e., upper bound of the domain \( \mathcal{D} \). The mass of the shell-2, \( m_2 \), is bounded from above by 0.0785026\( M_w \) at which the initial radius of the wormhole throat, \( a \), equals 1.31581\( M_w \). This result shows another physically significant difference from the case of the wormhole with the negative mass investigated in Ref. [22]: The upper bound on \( m_2 \) is of the order \( |M_w| \) in the negative mass case, since the gravitational collapse to form a black hole is prevented by the negative mass of the wormhole.

Here it should be noted that \( E \) appears only at the last term in the right hand side of Eq. (96) [see Eq. (22)], and the inclination of \( W(r) \) does not depend on \( E \). The area of the
FIG. 6: The \((a, m_2)\)-space with \(E = 1\) is depicted. The domain in which the wormhole stably persists is specified as an unshaded region.

domain in \((a, m_2)\)-space in which the wormhole persists decreases as \(E\) increases. However, there is a domain, in which the wormhole persists, for any \(E\) larger than unity. We depict the same as Fig. 6 but \(E = 2\) in Fig. 7.

VII. SUMMARY AND DISCUSSION

We analytically studied the non-linear stability of a wormhole supported by an infinitesimally thin spherical brane, i.e., a thin spherical shell whose tangential pressure is equal to its energy per unit area with an opposite sign; We consider a situation in which a thin spherical shell composed of dust concentric with the brane goes through the initially static wormhole in order to play a role of the non-linear disturbance. We took into account the self-gravities of both the brane and the dust shell completely through Israel’s formalism of metric junction. The wormhole was assumed to have a mirror symmetry with respect to the brane supporting it. As Barcelo and Visser has shown, in such a situation, the gravi-
Black hole formation by the shrinkage of the shell-2

Wormhole stably persists even after the passage of the shell-2.

FIG. 7: The close-up of the neighborhood of the intersections of the curves $W(r_B) = 0$, $W(r_D) = 0$ and $4m_2 = a - M_w$.

The gravitational mass of the static wormhole should be positive, and the static electric field should exist in order that the wormhole is stable against linear perturbations. Then we studied the condition that the wormhole persists after the dust shell goes through it. We assumed that the interaction between the brane and the dust shell is only gravity, or in other words, the 4-velocities of these shells are assumed to be continuous at the collision event. In this model, there are three free parameters; The initial areal radius, $a$, of the wormhole, the conserved specific energy $E$ and the proper mass $m_2$, of the dust shell, by regarding the initial gravitational mass, $M_w$, of the wormhole as a unit of length. Then, we showed that there is a domain of the non-zero measure in $(a, m_2)$-space for $E \geq 1$, in which the wormhole persists after the dust shell goes through it. In the case of $E = 1$, the maximum mass of the dust shell $m_2$ is almost equal to $0.08M_w$.

Assuming $a \simeq Q \simeq M_w$, through the geodesic deviation equations, the tidal acceleration
Black hole formation by the shrinkage of the shell-2

\[ W(\tau) = 0 \]

\[ B_W(r) = 0 \]

\[ D \]

Black hole formation by the collapse of the wormhole

Wormhole stably persists even after the passage of the shell-2.

FIG. 8: The same as Fig. 6 but \( E = 2 \).

\[
A_{\text{tidal}} = \frac{2M_w \ell}{a^3} \left( \frac{3Q^2}{2M_w a} - 1 \right) \approx \frac{\ell^6 \ell}{G^2 M_w^2} = 10 \left( \frac{M_w}{4 \times 10^5 M_\odot} \right)^{-2} \left( \frac{\ell}{40 \text{m}} \right) \text{m/s}^2,
\]

where \( \ell \) is the length of the spacecraft and \( M_\odot \) is the solar mass \( 2 \times 10^{30} \text{kg} \). The area of the wormhole throat with \( M_w = 4 \times 10^5 M_\odot \) is about \( 4\pi M_w^2 \approx 4.5 \times 10^{12} \text{km}^2 \). Here, let us imagine \( 10^{12} \) spacecrafts placed with almost equal spacing on a sphere concentric with a spherical brane wormhole with \( M_w = 4 \times 10^5 M_\odot \). The lump of them can be regarded as a dust shell if they are almost freely falling into the wormhole along the radial direction. If the size of the spacecraft is about 40m, the tidal acceleration on each spacecraft is the order of \( 10\text{m/s}^2 \) even at the throat of the wormhole. Then, since the average separation between the nearest spacecrafts is the order of 1km, they can safely go through the wormhole. Since the mass of each spacecraft will be about \( 2 \times 10^6 \text{kg} \), the total mass of the shell composed of these spacecrafts is \( 2 \times 10^{18} \text{kg} \approx 10^{-12} M_\odot \). The present result suggests that the wormhole supported by the negative tension brane stably persists even after the passage of these.
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Appendix A: On the roots of the quartic equation \( w(r) = 0 \)

In this appendix, we show that if the parameters \( a \) and \( m_2 \) are in the domain \( D \) of \( E \geq 1 \), the quartic equation \( w(r) = 0 \) has three positive real roots and one negative real root.

In accordance with Eqs. (89)–(92), we regard \( \mu \) and \( Q \) as functions of \( a \), \( M_3 + M_4 \) as a function of \( a \) and \( m_2 \) and \( M_3 - M_4 \) as a function of \( a \), \( m_2 \) and \( E \).

First, we show that \( M_3 + M_4 \) is bounded below by a positive value. Because of Eq. (94), by using Eq. (91), we have

\[
M_3 + M_4 = 2M_w - \frac{m_2^2}{a} > N(a, M_w, E),
\]

where

\[
N(a, M_w, E) := \frac{(16E^2 + 1)M_w}{8E^2} - \frac{1}{16E^2} \left( a + \frac{M_w^2}{a} \right).
\]

Because of Eq. (51),

\[
\frac{\partial N}{\partial a} = \frac{M_w^2 - a^2}{16E^2a^2} < 0
\]

is satisfied, and hence we have

\[
N(a, M_w, E) > N \left( \frac{3}{2}M_w, M_w, E \right) = \frac{1}{96} \left( 192 - \frac{1}{E^2} \right) M_w \geq \frac{191}{96} M_w,
\]

where we have used \( E \geq 1 \). As a result, we obtain

\[
M_3 + M_4 > \frac{191}{96} M_w. \tag{A1}
\]

The derivative of \( w(r) \) is given by

\[
\frac{dw(r)}{dr} = 4r^3 - \frac{16}{3\mu^2} r + \frac{2(M_3 + M_4)}{\mu^2}. \tag{A2}
\]
If the inequality
\[(M_3 + M_4)\mu < \frac{32}{27}\]  \hspace{1cm} (A3)
holds, the cubic equation \(dw/dr = 0\) has three real roots. We show that Eq. \(\text{(A3)}\) necessarily holds in the domain \(D\) of \(E \geq 1\). Because of Eq. \(\text{(51)}\), we have
\[\frac{d\mu}{da} = \frac{3M_w - 2a}{a^3} \sqrt{\frac{a}{2(a - M_w)}} > 0,\]  \hspace{1cm} (A4)
and hence
\[\mu < \mu_{a = \frac{2}{3}M_w} = \frac{1}{M_w} \sqrt{\frac{8}{27}}\]  \hspace{1cm} (A5)
holds. Equation \(\text{(A5)}\) leads to Eq. \(\text{(A3)}\) as follows;
\[(M_3 + M_4)\mu = \left(2M_w - \frac{m_2^2}{a}\right)\mu < 2M_w\mu < \sqrt{\frac{32}{27}} < \frac{32}{27}.\]

By virtue of Eqs. \(\text{(A1)}\) and \(\text{(A3)}\), we find that \(w(r)\) has one minimum in \(r < 0\) and one maximum and one minimum in \(r > 0\).

Hereafter, the three real roots of \(dw/dr = 0\) are denoted by \(r = r_i\ (i = 1, 2, 3)\):
\[r_1 = \frac{4}{3\mu} \cos \left(\frac{\theta}{3}\right), \quad r_2 = \frac{4}{3\mu} \cos \left(\frac{\theta + 2\pi}{3}\right) \quad \text{and} \quad r_3 = \frac{4}{3\mu} \cos \left(\frac{\theta + 4\pi}{3}\right),\]  \hspace{1cm} (A6)
where
\[\theta = \arccos \left(-\frac{27}{32} (M_3 + M_4) \mu\right).\]  \hspace{1cm} (A7)
Since
\[-1 < -\frac{27}{32} (M_3 + M_4) \mu < 0\]
is satisfied by virtue of Eq. \(\text{(A3)}\),
\[\frac{\pi}{2} < \theta < \pi\]  \hspace{1cm} (A8)
holds. Equation \(\text{(A8)}\) leads to \(r_1 > r_3 > 0 > r_2\).

We introduce a function defined as
\[U(\rho) = -\frac{4}{3\mu^2} \left[\rho^2 - \frac{9(M_3 + M_4)}{8} \rho + Q^2\right].\]
Then, since \(r_i\) satisfies
\[r_i^4 = \frac{4}{3\mu^2} r_i^2 - \frac{M_3 + M_4}{2\mu^2} r_i,\]
we have
\[w(r_i) = U(r_i).\]
FIG. 9: We depict \((r_1 - \rho_+)\mu\) with \(m_2 = 0\) as a function of \(a\) in the unit of \(M_w = 1\).

Because of Eqs. (50) and (A1), the quadratic equation \(U(\rho) = 0\) has two real roots;

\[
\rho = \rho_{\pm} := \frac{9}{16} \left[ M_3 + M_4 \pm \sqrt{(M_3 + M_4)^2 - \left(\frac{16Q}{9}\right)^2} \right].
\]

If the inequalities,

\[
U(r_1) < 0, \quad U(r_2) < 0 \quad \text{and} \quad U(r_3) > 0,
\]

or equivalently,

\[
r_1 > \rho_+, \quad r_2 < \rho_- \quad \text{and} \quad \rho_- < r_3 < \rho_+ \quad (A9)
\]

are simultaneously satisfied, the quartic equation \(w(r) = 0\) has four real roots. We will see below that Eq. (A9) holds.

Since both \(\rho_{\pm}\) are positive, \(r_2 < \rho_-\) is trivially satisfied because of \(r_2 < 0\).

From Eq. (A7), we can see

\[
\frac{\partial \theta}{\partial m_2} = -\frac{27\mu m_2}{16a\sin \theta} < 0,
\]

where we have used Eq. (A8) in the inequality. Thus, we see

\[
\frac{\partial r_1}{\partial m_2} = -\frac{4}{9\mu} \sin \left(\frac{\theta}{3}\right) \frac{\partial \theta}{\partial m_2} > 0,
\]

and

\[
r_1 > r_1 \big|_{m_2=0} = \frac{4}{3\mu} \cos \left[ \frac{1}{3} \arccos \left( -\frac{27}{16} M_w \mu \right) \right] \quad (A10)
\]
It is not difficult to see that
\[ \frac{\partial \rho_+}{\partial m_2} < 0 \]
holds, and hence we have
\[ \rho_+ < \rho_+|_{m_2=0}. \]  (A11)

We depict \((r_1 - \rho_+)\mu\) for \(m_2 = 0\) in Fig. 3. Since, as shown in Fig. 3, \(r_1 > \rho_+\) holds for \(m_2 = 0\), we have from Eqs. (A10) and (A11)
\[ r_1 > \rho_+ \quad \text{for} \quad m_2 > 0. \]

We can easily see that \(\rho_+\) is an increasing function of \(M_3 + M_4\), whereas \(\rho_-\) is a decreasing function of \(M_3 + M_4\), in the domain \(D\) of \(E \geq 1\). Then, Eq. (A1) implies
\[ \rho_+ > \rho_+|_{M_3+M_4=\frac{M_w}{90}M_w} = \frac{9}{16} \left[ \frac{191}{96} + \sqrt{\left(\frac{191}{96}\right)^2 - \left(\frac{16Q}{9M_w}\right)^2} \right] M_w \]
\[ > \frac{9}{16} \left[ \frac{191}{96} + \sqrt{\left(\frac{191}{96}\right)^2 - \frac{32}{9}} \right] M_w > M_w. \]  (A12)

We have
\[ \frac{1}{4} \frac{dw}{dr} \bigg|_{r=M_w} = \frac{1}{a^3 \mu^2} \left( \frac{M_w a^3 \mu^2}{2} - \frac{1}{3} M_w a^3 - \frac{m_2 a^2}{2} \right) < -\frac{M_w}{3a^3 \mu^2} f(a), \]  (A13)
where
\[ f(a) = a^3 - 6M_w^2 a + 6M_w^3. \]
It is easy to see that \(f(a) > 0\) holds for \(a > 0\). This result implies that \(dw/dr|_{r=M_w} < 0\) holds. As a result, we have
\[ r_3 < M_w < \rho_+, \]
since \(r = r_3\) is the lower bound of the domain of \(dw/dr < 0\) in \(r > 0\).

It is easy to see that the following inequality holds in the domain \(D\) of \(E \geq 1\);
\[ \frac{\partial \rho_-}{\partial m_2} > 0, \]
and hence we have
\[ \rho_- < \rho_{ub}(a) := \rho_-|_{m_2=\frac{a-M_w}{4}} = \frac{9}{8} \left[ 1 - \frac{(a - M_w)^2}{32M_w a} - \sqrt{\left(1 - \frac{(a - M_w)^2}{32M_w a}\right)^2 - \left(\frac{8Q}{9M_w}\right)^2} \right] M_w \]  (A14)
FIG. 10: We depict $w'_{lb}$ as a function of $a$ in the unit of $M_w = 1$. It is positive in the domain of our interest.

where we have used Eq. (94) and $E \geq 1$ in the inequality. We can see

$$\frac{\mu^2}{4} \frac{dw}{dr} \bigg|_{r=\rho_{ub}} = \mu^2 \rho_{ub}^3 - \frac{4}{3} \rho_{ub} + M_w - \frac{m_2^2}{2a} > w'_{lb}(a) := \mu^2 \rho_{ub}^3 - \frac{4}{3} \rho_{ub} + M_w - \frac{(a - M_w)^2}{32a}, \quad (A15)$$

where we have used Eq. (94) and $E \geq 1$ in the inequality. In Fig. 10 we depict $w'_{lb}$ as a function of $a$. From this figure, we find that $dw/dr > 0$ at $r = \rho_{ub}$, and hence $r_3 > \rho_{ub} > \rho_-$ holds by the same reason as that leading to $r_3 < M_w < \rho_+$. As a result, we have $\rho_- < r_3 < \rho_+$.

The result obtained above implies that the quartic equation $w(r) = 0$ has four real roots. Here recall that the function $w(r)$ has one minimum in $r < 0$, whereas one maximum and one minimum exist in $r > 0$. Then, since $w < 0$ and $dw/dr > 0$ at $r = 0$, we find that one root of $w(r) = 0$ is negative and the other three are positive.

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