On transforms of timelike isothermic surfaces in pseudo-Riemannian space forms

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Abstract

The basic theory on the conformal geometry of timelike surfaces in pseudo-Riemannian space forms is introduced, which is parallel to the classical framework of Burstall etc. for spacelike surfaces. Then we provide a discussion on the transforms of timelike (±)-isothermic surfaces (or real isothermic, complex isothermic surfaces), including c– polar transforms, Darboux transforms and spectral transforms. The first main result is that c–polar transforms preserve timelike (±)-isothermic surfaces, which are generalizations of the classical Christoffel transforms. The next main result is that a Darboux pair of timelike isothermic surfaces can also be characterized as a Lorentzian $O(n - r + 1, r + 1)/O(n - r, r) \times O(1, 1)$–type curved flat. Finally two permutability theorems of c–polar transforms are established.

Keywords: (±)-timelike isothermic surfaces, curved flats, Darboux transforms, polar transforms as generalized Christoffel transforms, spectral transforms

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1 Introduction

The study of isothermic surfaces in $\mathbb{R}^3$ can be traced back to the early development of the classical differential geometry. It was revealed only 30 years ago in [5] that, there is a structure of integrable system underlying the theory about isothermic surfaces, which has its root in the transforms of isothermic surfaces. This theory was explained later as a special kind of curved flat in [3] and as a $G^1_{m,1}$–system in [1] (see also [24]). For a complete theory concerning the geometry and integrable system of isothermic surfaces in $\mathbb{R}^n$ as well as the interpretation of what is an integrable system, we refer to [2].

In recent years, the integrable system theory of isothermic surfaces is used to the study transforms of timelike isothermic surfaces in pseudo-Riemannian space forms ([7, 8, 20, 28]), mainly using the methods developed by Burstall in [2] and Bruck et al in [1].

On the other hand, a geometric observation initiated in the works of [18, 26], where it was shown that there exists a new kind of natural transforms for spacelike surfaces in
n-dimensional Lorentzian space forms. The key observation is that any parallel normal section of the conformal flat normal bundle yields a conformal map. So we can define the polar transforms locally. The new isothermic surface produced in this way is neither the spectral transform nor the Darboux transform of the original isothermic surface. It is the generalization of the Christoffel transforms for isothermic surfaces.

We also note that in [22], the globally isothermic surfaces, named also as soliton surfaces, appeared in the mechanical equilibrium of closed membranes. Since timelike surfaces arise naturally in the study of surfaces in Lorentzian space forms, it does make sense to have a detailed study on the geometry of timelike isothermic surfaces in pseudo-Riemannian space forms. Moreover, some new examples of global isothermic surfaces are also presented in our paper, see Section 2.3.

In this paper, we want to provide a similar theory concerning the conformal geometry and integrable system of timelike isothermic surfaces in pseudo-Riemannian space forms, following the treatment in [4], [2] and [3]. For this purpose, we first derive the surfaces theory for timelike surfaces in pseudo-Riemannian space forms. Then we unify these into the conformal geometry of timelike surfaces in the projective light cone, following the framework in [4]. After these preparation, we define the $c$–polar transforms of timelike $(\pm)$–isothermic surfaces and prove that $c$–polar surfaces are also timelike $(\pm)$–isothermic surfaces when they are immersions. Moreover, we derive a geometric description of the Darboux transforms for timelike $(\pm)$–isothermic surfaces. As an application, we show that a Darboux pair of timelike $(\pm)$–isothermic surfaces is equivalent to a Lorentzian $O(n−r+1,r+1/O(n−r,r)\times O(1,1))$–curved flat. Since $c$–polar transforms can be viewed as generalized Christoffel transforms, it is natural to expect that $c$–polar transforms commute with the spectral transform and the Darboux transform. Similar to [18], we obtain two of such permutability theorems, see Section 5.

This paper is organized as follows. In Section 2 we derive the main theory of timelike surfaces both in the isometric case and in the conformal case, together with the relations between them. The definition and basic properties of isothermic surfaces are also discussed here. A Bonnet-type theorem is also provided. Then we introduce the $c$–polar transforms of timelike isothermic surfaces and the description of them in Section 3. In Section 4 we prove that a Darboux pair of timelike isothermic surfaces are equivalent to a $O(n−r+1,r+1/O(n−r,r)\times O(1,1))$–curved flat. Finally, we provide a brief proof of the permutability theorems between $c$–polar transforms and the spectral transforms, the Darboux transforms in Section 5.

2 Surface Theory for timelike isothermic surface

In this section we first review the timelike surface theory in space forms. Then we consider the conformal geometry of timelike surface theory in pseudo-Riemannian space forms. Furthermore, the relations between the isometric invariants and the conformal invariants of a timelike surface are established. Finally we end this section by providing some new examples of timelike isothermic surfaces.
2.1 Timelike isothermic surfaces in $N^n_r(c)$

Let $\mathbb{R}^n_s$ be the linear space $\mathbb{R}^m$ equipped with the quadric form

$$\langle x, x \rangle = \sum_{1}^{m-s} x_i^2 - \sum_{m-s+1}^{m} x_i^2.$$ 

Let $N^n_r(c)$ be the $n$-dimensional pseudo-Riemannian space form with constant curvature $c$, $c = 0, 1, -1$, with its pseudo-Riemannian metric of signature $(n - r, r)$, $n - r \geq 2, r \geq 1$. It is well-known that $N^n_r(c)$ can be described as

$$\begin{align*}
\{ x \in N^n_r(c) = \mathbb{R}^n_1, &\ c = 0, \\
\{ x \in N^n_r(c) = S^n_r \subset \mathbb{R}^{n+1}_r, &\ c = 1, \\
\{ x \in N^n_r(c) = H^n_r \subset \mathbb{R}^{n+1}_r, &\ c = -1,
\end{align*}$$

We recall the basic properties of timelike surfaces. A surface $x : M \to N^n_r(c)$ is called a timelike surface or Lorentzian surface if the induced metric $x^*\langle , \rangle$ on $M$ is a Lorentzian metric. For the two dimensional Lorentzian surface $x$, there are two (conformally) invariant lightlike directions and integration along them gives an (locally) asymptotic coordinate system on $M$. Now let $x : M \to N^n_r(c)$ be an oriented timelike surface with local asymptotic coordinate $(u, v)$ and metric $|dx|^2 = e^{2\omega}du^2dv$. Choose an orthonormal frame $\{n_\alpha\}$ of the normal bundle with $(n_\alpha, n_\alpha) = \varepsilon_\alpha = \pm 1$, and let $D_u$ and $D_v$ denote the normal connection.

The structure equations are as follows

$$\begin{align*}
x_{uu} &= 2\omega_u x_u + \Omega_1, \\
x_{vv} &= 2\omega_v x_v + \Omega_2, \\
x_{uv} &= \frac{1}{2}e^{2\omega}H - \frac{1}{2}e^{2\omega}x, \\
\langle n_\alpha, u \rangle &= D_u n_\alpha - \langle n_\alpha, H \rangle x_u - 2e^{-2\omega}\langle n_\alpha, \Omega_1 \rangle x_v, \\
\langle n_\alpha, v \rangle &= D_v n_\alpha - \langle n_\alpha, H \rangle x_v - 2e^{-2\omega}\langle n_\alpha, \Omega_2 \rangle x_u, \
\end{align*}$$

(1)

Here $H = \sum_\alpha \omega_\alpha n_\alpha$ is the mean curvature vector and the two $\Gamma(T^\perp M)$—valued 2-forms $\Omega_1 du^2 = \sum_{\alpha=3}^{n} \Omega_{1\alpha} n_{\alpha} dz^2$ and $\Omega_1 du^2 = \sum_{\alpha=3}^{n} \Omega_{1\alpha} n_{\alpha} dz^2$ are the vector-valued Hopf differentials. The Gauss equation, Codazzi equations and Ricci equations as integrable equations are as below

$$\begin{align*}
\langle H, H \rangle - K + c &= 4e^{-4\omega}\langle \Omega_1, \Omega_2 \rangle, \\
D_u H &= 2e^{-2\omega}D_u \Omega_1, \quad D_v H = 2e^{-2\omega}D_v \Omega_2, \\
R^{\perp_{n_\alpha}}_{uv} := D_v D_u n_\alpha - D_u D_v n_\alpha &= 2e^{-2\omega}(\langle n_\alpha, \Omega_1 \rangle \Omega_2 - \langle n_\alpha, \Omega_2 \rangle \Omega_1).
\end{align*}$$

(2)

There are several equivalent definition of timelike isothermic surfaces, concerning different properties of them. Here first we define the $(\pm)$—timelike isothermic surfaces following the terms of Fujioka and Inoguchi.

**Definition 2.1.** [Li] Let $x : M \to N^n_r(c)$ be a timelike surface. It is called $(\pm)$—isothermic if around each point of $M$ there exists an asymptotic coordinate $(u, v)$ such that the vector-valued Hopf differentials $\Omega_1 = \pm \Omega_2$. Such coordinate $(u, v)$ is called an adapted (asymptotic) coordinate.
Another equivalent definition is given by Dussan and Magid in \[7\], the so-called timelike real isothermic and complex isothermic surfaces.

**Definition 2.2.** [7] Let \( x : M \to N^m_r(c) \) be a timelike surface. Suppose that the normal bundle of \( x \) is flat and that \( n_\alpha \) is a parallel orthonormal frame of the normal bundle. Then \( x \) is called real isothermic surface, if there exist coordinates \((\tilde{u}, \tilde{v})\), such that its first and second fundamental forms is of the form

\[
I = e^{2\omega}(d\tilde{u}^2 - d\tilde{v}^2), \quad II = \sum_\alpha (b_{1\alpha}d\tilde{u}^2 - b_{2\alpha}d\tilde{v}^2)n_\alpha. \tag{3}
\]

And \( x \) is called complex isothermic surface, if there exist coordinates \((\tilde{u}, \tilde{v})\), such that the first and second fundamental forms of \( x \) is of the form

\[
I = e^{2\omega}(d\tilde{u}^2 - d\tilde{v}^2), \quad II = \sum_\alpha (b_{1\alpha}(d\tilde{u}^2 - d\tilde{v}^2) + b_{0\alpha}d\tilde{v}^2)n_\alpha. \tag{4}
\]

From the structure equations above, it is direct to see the equivalence between these two definitions.

**Lemma 2.3.** For a timelike surface, it is \((+)\)–isothermic if and only if it is real isothermic. It is \((−)\)–isothermic if and only if it is complex isothermic.

**Proof.** \( \Omega_1 = \pm \Omega_2 \) together with the conformal Ricci equations in \((2)\) shows that the normal bundle of \( x \) is flat. This is an important property of isothermic surfaces, which guarantees that all shape operators commute and the curvature lines could still be defined. Setting \( u = s + t, v = s - t \), the two fundamental forms of an isothermic surface, with respect to some parallel normal frame \( \{n_\alpha\} \), are of the form

\[
I = e^{2\omega}(ds^2 - dt^2), \quad II = \sum_\alpha (b_{1\alpha}ds^2 - b_{2\alpha}dt^2)n_\alpha \tag{5}
\]

if \( x \) is \((+)\)–isothermic and

\[
I = e^{2\omega}(ds^2 - dt^2), \quad II = \sum_\alpha (b_{1\alpha}(ds^2 - dt^2) - 2b_{2\alpha}dsdt)n_\alpha \tag{6}
\]

if \( x \) is \((−)\)–isothermic. \( \square \)

Since timelike isothermic surfaces are also conformally invariant as the spacelike cases, it will be better to deal with them by using a conformally invariant frame. This idea leads the next subsection.

### 2.2 Timelike isothermic surfaces in \( Q^n_r \)

Let \( C^{m-1}_s \) be the light cone of \( \mathbb{R}^m_s \). Here \( \mathbb{R}^m_s \) is the same space defined in Section 2.1. The metric of \( \mathbb{R}^{n+2}_r \) induces a \((n - r, r)\)–type pseudo-Riemannian metric on

\[
S^{n-r} \times S^r = \{ x \in \mathbb{R}^{n+2}_r \mid \sum_{i=1}^{n-r+1} x_i^2 = \sum_{i=n-r+2}^{n+2} x_i^2 = 1 \} \subset C^{n+1}_r \setminus \{0\},
\]
which can be written as \( g(S^{n-r}) \oplus (-g(S^r)) \) with \( g(S^{n-r}) \) and \( g(S^r) \) the standard Riemannian metrics on \( S^{n-r} \) and \( S^r \). Let
\[
Q^r_n = \{[x] \in \mathbb{R}^{P+1} | x \in C_r^{n+1} \setminus \{0\} \}
\]
be the projective light cone. The projection \( \pi : S^{n-r} \times S^r \rightarrow Q^r_n \) therefore induces a conformal structure \([h]\) on \( Q^r_n \), with \( h \) locally a \((n-r, r)\)-type pseudo-Riemannian metric. The conformal group of \((Q^r_n, [h])\) is the orthogonal group \( O(n-r+1, r+1) \) which keeps the inner product of \( \mathbb{R}_{r+1}^{n+1} \) invariant and acts on \( Q^r_n \) by
\[
T([x]) = [xT], \quad T \in O(n-r+1, r+1).
\]
The space forms \( N^n_r(c), c = 0, 1, -1 \) can be conformally embedded as a proper subset of \( Q^r_n \) via
\[
x \mapsto \left( \frac{\langle x, x \rangle - 1}{2}, x, \frac{\langle x, x \rangle + 1}{2} \right).
\]
Basic conformal surface theory shows that for a timelike surface \( y : M \rightarrow Q^r_n \) with local asymptotic coordinate \((u, v)\), there exists a local lift \( Y : U \rightarrow C_r^{n+1} \setminus \{0\} \) of \( y \) such that \( \langle dY, dY \rangle = \frac{1}{2}(du \otimes dv + dv \otimes du) \) in an open subset \( U \) of \( M \). We denote by
\[
V = \text{Span}\{Y, Y_u, Y_v, Y_{uv}\}
\]
the conformal tangent bundle, and by \( V^\perp \) its orthogonal complement or the conformal normal bundle. Set \( N = 2Y_{uv} + 2(\kappa_1, \kappa_2)Y \), we have
\[
\langle Y, N \rangle = -1, \quad \langle Y_u, N \rangle = \langle Y_v, N \rangle = \langle N, N \rangle = 0.
\]
Let \( D \) denote the normal connection and let \( \psi \in \Gamma(V^\perp) \) be an arbitrary section. Then the structure equation of \( y \) can be derived as follows [25]
\[
\begin{align*}
Y_{uu} &= -\frac{\kappa_1}{2}Y + \kappa_1, \\
Y_{uv} &= -\frac{\kappa_2}{2}Y + \kappa_2, \\
Y_{vv} &= -\langle \kappa_1, \kappa_2 \rangle Y + \frac{1}{2}N, \\
N_u &= -2\langle \kappa_1, \kappa_2 \rangle Y_u - s_1Y_v + 2D_s\kappa_1, \\
N_v &= -s_2Y_u - 2\langle \kappa_1, \kappa_2 \rangle Y_v + 2D_u\kappa_2, \\
\psi_u &= D_u\psi + 2\langle \psi, D_u\kappa_1 \rangle Y - 2\langle \psi, \kappa_1 \rangle Y_v, \\
\psi_v &= D_v\psi + 2\langle \psi, D_v\kappa_2 \rangle Y - 2\langle \psi, \kappa_2 \rangle Y_u.
\end{align*}
\]
The first two equations above define four basic invariants \( \kappa_1, \kappa_2 \) and \( s_1, s_2 \) dependent on \((u, v)\). Similar to the case in Möbius geometry, \( k_i \), and \( s_i \) are called the conformal Hopf differential and the Schwarzian derivative of \( y \), respectively (see [4], [17] and [25]).

The conformal Hopf differential defines a conformal invariant metric
\[
g := \langle \kappa_1, \kappa_2 \rangle du dv
\]
and the Willmore functional
\[
W(y) := \frac{1}{2} \int_M \langle \kappa_1, \kappa_2 \rangle du dv
\]
For more details on timelike Willmore surfaces we refer to [6]. The conformal Gauss Codazzi and Ricci equations as integrable conditions are:

\[
\begin{align*}
\{ (s_1)_u \} &= 3(D_u \kappa_2, \kappa_1) + (D_u \kappa_1, \kappa_2), \\
\{ (s_2)_u \} &= (\kappa_1, D_v \kappa_2) + 3(D_v \kappa_1, \kappa_2), \\
D_u D_v \kappa_1 + \frac{s_2 \kappa_1}{2} &= D_u D_u \kappa_2 + \frac{s_1 \kappa_2}{2}
\end{align*}
\]  

\begin{equation}
R^D_u \psi := D_v D_u \psi - D_u D_v \psi = 2(\psi, \kappa_1) \kappa_2 - 2(\psi, \kappa_2) \kappa_1.
\end{equation}

Let \((\tilde{u} = \tilde{u}(u), \tilde{v} = \tilde{v}(v))\) be another asymptotic coordinate with \(\langle Y_\tilde{u}, Y_\tilde{v} \rangle = \frac{1}{2} e^{2 \rho}, \) then \(\tilde{Y} = e^{-\rho} Y, \) and correspondingly,

\[
\tilde{\kappa}_1 \frac{d\tilde{u}^2}{\sqrt{dudv}} = \kappa_1 \frac{d\tilde{u}^2}{\sqrt{dudv}}, \quad \tilde{\kappa}_2 \frac{d\tilde{v}^2}{\sqrt{dudv}} = \kappa_2 \frac{d\tilde{v}^2}{\sqrt{dudv}}.
\]

And

\[
\tilde{s}_1 = s_1 \left( \frac{\partial u}{\partial \tilde{u}} \right)^2 + 2 \rho_\tilde{u} u - 2(\rho_\tilde{u})^2, \quad \tilde{s}_2 = s_2 \left( \frac{\partial v}{\partial \tilde{v}} \right)^2 + 2 \rho_\tilde{v} v - 2(\rho_\tilde{v})^2,
\]

With these at hand, we can state the conformal version of Bonnet theorem on the existence and uniqueness of conformally timelike surface as follows. (This is directly derived as the usual submanifold theory. Since we can not find a reference for Bonnet theorem of timelike surfaces, we refer to [6]. The conformal Gauss Codazzi and Ricci equations are given as integrable conditions.

**Theorem 2.4.** For an oriented timelike surface \(M\) with local local asymptotic coordinate \((u, v), \) the differentials \(\frac{du^2}{\sqrt{dudv}}\) and \(\frac{dv^2}{\sqrt{dudv}}\) define two real line bundles \(L_1\) and \(L_2\) over \(M.\)

For a timelike surface \(M\) with local asymptotic coordinate \((u, v), \) let \(NM\) be an \(n - 2\) dimensional flat sub-vector bundle of the trivial bundle \(M \times \mathbb{R}^{n+2}\) over \(M,\) with an induced \((n, 1, r - 1)\) type flat pseudo Riemannian metric on each fiber of \(NM.\) Denote by \(D_u\) and \(D_v\) the connection over \(NM.\) Chose a section \(\kappa_1 \in \Gamma(NM \otimes L_2)\) and a section \(\kappa_2 \in \Gamma(NM \otimes L_2),\) and let \(s_1, s_2, D_u, D_v\) satisfy the integrable equations (13), (14), (15), then there exists a conformal immersion \(y : M \to Q^n\) with conformal data \(\{ \kappa_1, s_1, s_2, D_u, D_v, NM \}\) as given.

Moreover, let \(y, \hat{y} : M \to Q^n\) be two timelike surfaces with (same) local asymptotic coordinate \((u, v).\) Assume that for some canonical lift \(Y\) of \(y\) and \(\hat{Y}\) of \(\hat{y},\)

(i). \(Y\) and \(\hat{Y}\) have the same Schwarzians, that is, \(s_1 = \hat{s}_1, s_2 = \hat{s}_2,\)

(ii). There exist normal frames \(\{ \phi_\alpha \}\) of \(y\) and normal frames \(\{ \hat{\phi}_\alpha \}\) of \(\hat{y},\) such that for these frames,

\[
D_u \psi_\alpha = D_u \hat{\psi}_\alpha, \quad D_v \psi_\alpha = D_v \hat{\psi}_\alpha.
\]

(iii). Under the frames (ii), the Hopf differential are of the form

\[
\kappa_1 = \sum_\alpha k_{1\alpha} \psi_\alpha, \quad \kappa_2 = \sum_\alpha k_{2\alpha} \psi_\alpha, \quad \hat{\kappa}_1 = \sum_\alpha k_{1\alpha} \hat{\psi}_\alpha, \quad \hat{\kappa}_2 = \sum_\alpha k_{2\alpha} \hat{\psi}_\alpha.
\]

Then there is a Möbius transformation (See [7]) \(T \in O(n + 2, r + 1) : Q^n_r \to Q^n_r,\) such that \(Ty = \hat{y}.\)
Now let us turn to timelike isothermic surfaces. A timelike surface \( y : M \rightarrow Q^n_r \) is called \((\pm)-\)isothermic if its Hopf differentials satisfy that \( \kappa_1 = \pm \kappa_2 \) with respect to some asymptotic coordinate \((u, v)\).

To show the equivalence of this definition with the previous ones, we need to derive the relations between the isothermic invariants and conformal invariant of a timelike surface \( x : M \rightarrow N^n_r(c) \). We verify this for the case \( c = 0 \). The other cases are omitted. Let

\[
X = \left( \frac{(x,x) - 1}{2}, x, \frac{(x,x) + 1}{2} \right).
\]

(19)

Then \( y = [X] \) is the corresponding surface into \( Q^n_r \). It is direct to obtain a canonical lift \( Y = e^{-\omega}X \) with respect to \((u, v)\). So

\[
\begin{align*}
Y &= \frac{1}{2}e^{-\omega}(\langle x, x \rangle - 1, 2x, \langle x, x \rangle + 1), \\
Y_u &= -\omega_uY + e^{-\omega}(\langle x_u, x \rangle, x_u, \langle x_u, x \rangle), \\
Y_v &= -\omega_vY + e^{-\omega}(\langle x_v, x \rangle, x_v, \langle x_v, x \rangle) \quad \text{(20)} \\
N &= e^{\omega}(\langle H, x \rangle + 1, \langle H, x \rangle + 1) \\
& - 2\omega_uY_u - 2\omega_vY_v + 2(\kappa_1, \kappa_2) - \omega_{uu} - \omega_{uv}Y, \\
\psi_\alpha &= e^{-\omega}(\langle n_\alpha, x \rangle, n_\alpha, \langle n_\alpha, x \rangle) + \langle n_\alpha, H \rangle Y.
\end{align*}
\]

Then we see that the normal connection of \( \psi_\alpha \) is the same as the normal connection of \( \{n_\alpha\} \). We also calculate the Schwarzian derivatives and conformal differentials as

\[
\begin{align*}
s_1 &= 2\omega_{uu} - 2\omega_u^2 + 2\langle \Omega_1, H \rangle, \\
s_2 &= 2\omega_{vv} - 2\omega_v^2 + 2\langle \Omega_2, H \rangle, \\
\kappa_1 &= e^{-\omega}(\langle \Omega_1, x \rangle, \langle \Omega_1, x \rangle) + \langle \Omega_1, H \rangle Y, \\
\kappa_2 &= e^{-\omega}(\langle \Omega_2, x \rangle, \langle \Omega_2, x \rangle) + \langle \Omega_2, H \rangle Y.
\end{align*}
\]

(21)

Therefore \( \kappa_1 = \pm \kappa_2 \) if and only if \( \Omega_1 = \pm \Omega_2 \). So a timelike \((\pm)-\)isothermic surface in \( N^n_r(c) \) can be conformally embedded into \( Q^n_r \) and must be a conformal timelike \((\pm)-\)isothermic surface in \( Q^n_r \).

\[ \text{2.3 Examples} \]

**Example 2.5.** Let \( x \) be a timelike \( H = 0 \) surface or a CMC \( H \) timelike surface in a 3-dimensional Lorentzian space form. If \( \langle \Omega_1, \Omega_2 \rangle \neq 0 \), then \( x \) is isothermic (compare [17], [14], [12], [25]). To see this, from the codazzi equation of \( x \) in (2), we have

\[ D_v\Omega_1 = 0, \quad D_u\Omega_2 = 0. \]

So if we assume that \( \Omega_1 = a_1n, \Omega_2 = a_2n \). Then \( a_1a_2 \neq 0 \) and \( a_1 = a_1(u), a_2 = a_2(v) \). So by suitable coordinate changing \( u \mapsto \bar{u} = \bar{u}(u) \) and \( v \mapsto \bar{v} = \bar{v}(v) \), one will have the new Hopf differential

\[ \bar{\Omega}_1 = \pm \bar{\Omega}_2 = \pm n. \]

Note that if \( \langle \Omega_1, \Omega_2 \rangle = 0 \), \( x \) may not be isothermic. One can find counterexamples from the null scrolls in [10].
Example 2.6. Timelike surfaces by rotation on a timelike plane: Let
\[ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, 0) \]
be a timelike curve in \( \mathbb{R}^n_r \) with \( \gamma_1 \neq 0 \). Then a similar procedure as [18] verifies that
\[ x_\gamma = (\gamma_1 \cosh \theta, \gamma_2, \gamma_3, \ldots, \gamma_{n-1}, \gamma_1 \sinh \theta) \]
is a timelike (+)–isothermic surface.

Example 2.7. Timelike surfaces by rotation on a spacelike plane: Let \( \gamma = (0, \gamma_2, \gamma_3, \ldots, \gamma_n) \) be a timelike curve in \( \mathbb{R}^n_r \) with \( \gamma_2 \neq 0 \). Then a similar procedure as [18] verifies that
\[ x_\gamma = (\gamma_2 \cos \theta, \gamma_2 \sin \theta, \gamma_3, \ldots, \gamma_n) \]
is a timelike (+)–isothermic surface.

Example 2.8. [19] Set
\[ e_1 = (\cos \frac{t \theta}{\sqrt{1 - t^2}} \cos \phi, \cos \frac{t \theta}{\sqrt{1 - t^2}} \sin \phi, \sin \frac{t \theta}{\sqrt{1 - t^2}} \cos \phi, \sin \frac{t \theta}{\sqrt{1 - t^2}} \sin \phi), \]
\[ e_2 = \partial e_1 / \partial \phi = e_{1\phi}, \quad e_3 = \frac{\sqrt{1 - t^2}}{t} e_{1\theta}, \quad e_4 = \frac{\sqrt{1 - t^2}}{t} e_{2\theta}, \]
with \( t < 1 \). Let
\[ Y_t(\theta, \phi) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^6_2 \]
\[ Y_t(\theta, \phi) = (e_1, \cos \theta \sqrt{1 - t^2}, \sin \theta \sqrt{1 - t^2}). \]
For simplicity, we omit the subscript "\( t \)" of \( Y_t \). We have that \( y = [Y] : \mathbb{R} \times \mathbb{R} \rightarrow Q^4_1 \) is a timelike (−)–isothermic Willmore cylinder of \( Q^4_1 \) for any \( |t| < 1 \) and it is a torus when \( t \) is rational.

Note that \( y_t \) is also a conformally homogeneous surface in \( Q^4_1 \). We refer to [19] for details.

Example 2.9. [19] Set
\[ e_1 = (\cosh \frac{t \theta}{\sqrt{1 - t^2}} \cosh \phi, \sinh \frac{t \theta}{\sqrt{1 - t^2}} \sinh \phi, 0, \cosh \frac{t \theta}{\sqrt{1 - t^2}} \sinh \phi, \sinh \frac{t \theta}{\sqrt{1 - t^2}} \cosh \phi, 0), \]
\[ e_2 = \partial e_1 / \partial \phi = e_{1\phi}, \quad e_3 = \frac{\sqrt{1 - t^2}}{t} e_{1\theta}, \quad e_4 = \frac{\sqrt{1 - t^2}}{t} e_{2\theta}, \]
\[ f_1 = (0, 0, \sinh \frac{\theta}{\sqrt{1 - t^2}}, 0, 0, \cosh \frac{\theta}{\sqrt{1 - t^2}}), f_2 = \sqrt{1 - t^2} f_{1\theta}. \]
with \( 0 < t < 1 \). Let
\[ Y_t(\theta, \phi) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^6_3 \]
\[ Y_t(\theta, \phi) := e_1 + f_1. \]
For simplicity, we omit the subscript "\( t \)" of \( Y_t \). We have that \( y = [Y] : \mathbb{R} \times \mathbb{R} \rightarrow Q^4_1 \) is a timelike (−)–isothermic Willmore plane of \( Q^4_1 \) for any \( 0 < t < 1 \). Note that \( y_t \) is also a conformally homogeneous surface in \( Q^4_2 \). We refer to [19] for details.
3 Generalized c-polar transforms as Christoffel transforms

Definition 3.1. Let \( y : M \rightarrow Q^n_r \) be a timelike isothermic surface. Suppose that \( \psi \) is a parallel section of the normal bundle of \( y \) with length \( c \), i.e. \( \psi \in \Gamma(V^\perp) \), \( D_u \psi = 0 \), \( D_v \psi = 0 \), and \( \langle \psi, \psi \rangle = c \). Then we call \( \psi : M \rightarrow N^{n+1}_r(\frac{1}{c}) \) a c-polar transform of \( y \) when \( c \neq 0 \), and \( [\psi] : M \rightarrow Q^N_r \) a 0-polar transform of \( y \) when \( c = 0 \).

Polar transforms can also be described as below:

Proposition 3.2. \( \psi : M \rightarrow \mathbb{R}^{n+2}_r \) derives a c-polar transform of \( y \) if it satisfies the following conditions:

(i). \( \langle \psi, Y \rangle = \langle \psi, Y_u \rangle = \langle \psi, Y_v \rangle = 0 \);

(ii). \( \psi_u \in \text{Span}\{Y,Y_v\} \) and \( \psi_v \in \text{Span}\{Y,Y_u\} \).

Proof. From (ii) we see that \( \langle \psi_u, Y_v \rangle = 0 \), together with (i), yields \( \langle \psi, Y_{uv} \rangle = 0 \). So \( \psi \) is a parallel section of the normal bundle by (ii). Also from (ii), \( \langle \psi_u, \psi \rangle = \langle \psi_v, \psi \rangle = 0 \), yielding \( \langle \psi, \psi \rangle = c \) for some constant \( c \). \( \square \)

Recall the definition of the classical Christoffel transforms. The Christoffel transform of a timelike \((\pm)-\)isothermic surface \( x : M \rightarrow \mathbb{R}^n_\rho \) is defined as a map \( x^C : M \rightarrow \mathbb{R}^n_\rho \) such that \( x^C_u \parallel x_v \) and \( x^C_v \parallel x_u \) (see [2], [25]). So \( c-\)polar transforms are generalizations of Christoffel transforms.

In [2], Burstall introduced the notion of generalized \( H \)-surface to discuss properties of isothermic surfaces and to produce new examples. For timelike surfaces one can take verbatim from [2].

Definition 3.3. Let \( x : M \rightarrow N^n_1(c) \) be a timelike surface with mean curvature vector \( H \). \( x \) is called a generalized \( H \)-surface if there exists a parallel isoperimetric section of \( x \). Here an isoperimetric section denotes a unit normal vector field \( N \) of \( x \) with \( \langle N, H \rangle \equiv \text{constant} \). \( N \) is called a minimal section when \( \langle N, H \rangle \equiv 0 \).

Theorem 3.4. Let \( y : M \rightarrow Q^n_r \) be a timelike \((\pm)-\)isothermic surface, i.e. \( \kappa_1 = \pm \kappa_2 \). We conclude that:

(i). Any c-polar transform \( \psi : M_0 \rightarrow N^{n+1}_r(\frac{1}{c}) \) is also a timelike \((\pm)-\)isothermic surface if \( \langle \psi, \kappa_1 \rangle \neq 0 \) on an open dense subset \( M_0 \) of \( M \). And \( \psi \) shares the same adapted coordinate with \( y \). Moreover, \( \psi \) is a generalized \( H \)-surface admitting a parallel minimal section. To be concrete, there exists a local lift \( Y^\psi \) of \( y \) (i.e., \( [Y^\psi] = y \)) with \( Y^\psi_u \parallel \psi_u \), \( Y^\psi_v \parallel \psi_v \), that is, \( Y^\psi \) is dual to \( \psi \).

(ii). Let \( \psi \) be a c-polar transform of \( y \). The conformal invariant metric of \( \psi \) is of the form

\[
g^\psi = \left( \langle \kappa_1, \kappa_2 \rangle + \left( \frac{\langle \psi, D_u \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} \right) \right) du dv \tag{22} \]

Furthermore, suppose that \( M \) is a closed surface. If \( \psi \) is globally immersed, then \( W(\psi) = W(y) \), i.e., in this case, the Willmore functional is polar transform invariant.
Proof. (i) Here we retain the notions in Section 2. Let \( \psi \in \Gamma(V^\perp) \) be a parallel section with \( \langle \psi, \psi \rangle = c \). So we obtain that

\[
\psi_u = 2\langle \psi, D_u \kappa_1 \rangle Y - 2\langle \psi, \kappa_1 \rangle Y_u,
\]

\[
\psi_v = 2\langle \psi, D_u \kappa_2 \rangle Y - 2\langle \psi, \kappa_2 \rangle Y_u.
\]

Since \( \kappa_2 = \varepsilon \kappa_1 \), \( \varepsilon = \pm 1 \), we derive that

\[
\langle \psi_u, \psi_v \rangle = 2\varepsilon \langle \psi, \kappa_1 \rangle^2.
\]

If \( \langle \psi, \kappa_1 \rangle \neq 0 \) on an open dense subset \( M_0 \) of \( M \), then \( \psi \) is an immersion. We calculate

\[
\psi_{uu} = 2\langle \psi, D_u D_u \kappa_1 \rangle Y - 2\langle \psi, D_u \kappa_1 \rangle Y_u + 2\langle \psi, D_v \kappa_1 \rangle Y_u - 2\langle \psi, \kappa_1 \rangle Y_{uu}
\]

\[
= \frac{2\langle D_u \kappa_1, \psi \rangle}{\langle \psi, \kappa_1 \rangle} \psi_u + \Omega^\psi_1
\]

with

\[
\Omega^\psi_1 = -\langle \psi, \kappa_1 \rangle N + 2\langle \psi, D_u \kappa_1 \rangle Y_u + 2\langle \psi, D_u \kappa_1 \rangle Y_v
\]

\[
+ 2\langle \psi, D_u D_v \kappa_1 \rangle + \langle \psi, \kappa_1 \rangle \langle \kappa_1, \kappa_2 \rangle - \frac{2\langle \psi, D_u \kappa_1 \rangle \langle \psi, D_v \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} Y.
\]

Similarly we get

\[
\psi_{uu} = 2\langle \psi, D_v D_v \kappa_2 \rangle Y - 2\langle \psi, D_v \kappa_2 \rangle Y_u + 2\langle \psi, D_u \kappa_2 \rangle Y_u - 2\langle \psi, \kappa_2 \rangle Y_{uv}
\]

\[
= \frac{2\langle D_v \kappa_2, \psi \rangle}{\langle \psi, \kappa_2 \rangle} \psi_v + \Omega^\psi_2
\]

with

\[
\Omega^\psi_2 = -\langle \psi, \kappa_2 \rangle N + 2\langle \psi, D_u \kappa_2 \rangle Y_v + 2\langle \psi, D_v \kappa_2 \rangle Y_u
\]

\[
+ 2\langle \psi, D_v D_u \kappa_2 \rangle + \langle \psi, \kappa_2 \rangle \langle \kappa_1, \kappa_2 \rangle - \frac{2\langle \psi, D_u \kappa_2 \rangle \langle \psi, D_v \kappa_2 \rangle}{\langle \psi, \kappa_2 \rangle} Y.
\]

It follows from \( \kappa_1 = \varepsilon \kappa_2 \) that

\[
\Omega^\psi_1 = \varepsilon \Omega^\psi_2, \quad \varepsilon = \pm 1.
\]

Thus \( \psi : M_0 \to N^{n+1}_{r}(\tfrac{1}{c}) \) is also a timelike (\( \pm \))--isothermic surface sharing the same adapted coordinate with \( y \).

For \( \psi \) being a generalized \( H \)-surface, we first get

\[
\psi_{uv} = -2\langle \psi, \kappa_1 \rangle \kappa_2 + \langle \psi, 2D_u D_v \kappa_1 + s_2 \kappa_1 \rangle Y.
\]

Let

\[
Y^\psi = \frac{1}{\langle \psi, \kappa_1 \rangle} Y.
\]

We compute that

\[
Y^\psi_u = \frac{1}{2\varepsilon \langle \psi, \kappa_1 \rangle^2} \psi_v, \quad Y^\psi_v = -\frac{1}{2\varepsilon \langle \psi, \kappa_1 \rangle^2} \psi_u.
\]
and
\[
\langle Y^\psi, Y^\psi \rangle = \langle Y^\psi, \psi \rangle = \langle Y^\psi, \psi_u \rangle = \langle Y^\psi, \psi_v \rangle = \langle Y^\psi, \psi_{uv} \rangle = 0 \Rightarrow \langle Y^\psi, H^\psi \rangle = 0.
\]
Here \(H^\psi\) is the mean curvature vector of \(\psi\). This finishes the proof of (i).

(ii). To compute \(g^\psi\), one derives directly by (15), (18) and (20) that
\[
\langle \kappa_1^\psi, \kappa_2^\psi \rangle = \frac{1}{4\varepsilon}\langle \Omega_1^\psi, \Omega_2^\psi \rangle
= \langle \kappa_1, \kappa_2 \rangle + \frac{\langle \psi, D_u D_u \kappa_1 \rangle \langle \psi, \kappa_1 \rangle - \langle \psi, D_u \kappa_1 \rangle \langle \psi, D_u \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle^2}
= \langle \kappa_1, \kappa_2 \rangle + \left( \frac{\langle \psi, D_u \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} \right)_v
\]

For the Willmore functional, it is just a consequence of
\[
\left( \frac{\langle \psi, D_u \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} \right)_v du \wedge dv = -d \left( \frac{\langle \psi, D_u \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} du \right)
\]
by using the Stokes formula.

\[\square\]

4 Timelike isothermic surfaces as an integrable system

There are several equivalent methods to treat isothermic surfaces as integrable systems, see [1], [2], [3], [5]. In each method, Darboux transform plays an essential role. So in this section, we first give a geometric definition of Darboux transforms for timelike isothermic surfaces.

For a detailed discussion of Darboux transforms, we refer to [11]. Then we introduce curved flat for timelike isothermic surfaces. In the end of this section we discuss briefly the relation between our treatment and the \(O(n - r + 1, r + 1)/O(n - r, r) \times O(1, 1)\)—system II (in the sense of Terng, etc. [1]) methods which has been discussed in details by Dussan and Magid in [7].

4.1 Darboux transforms of timelike isothermic surfaces

We define the Darboux transforms and give the basic properties of Darboux transforms as below (compare [7], [18], [28]).

**Definition 4.1.** Let \(y : M \to Q^n_r\) denote a timelike \((\pm)\)-isothermic surface with canonical lift \(\hat{Y}\) with respect to the adapted coordinate \((u,v)\). A timelike immersion \(\hat{y} : M \to Q^n_r\) is called a Darboux transform of \(y\) if its local lift \(\hat{\hat{Y}}\) satisfies
\[
\langle \hat{\hat{Y}}, \hat{\hat{Y}} \rangle \neq 0, \ \hat{\hat{Y}}_u \in \text{Span}\{\hat{\hat{Y}}, \hat{Y}, \hat{Y}_u\} \text{ and } \hat{\hat{Y}}_v \in \text{Span}\{\hat{\hat{Y}}, \hat{Y}, \hat{Y}_u\}
\]
where \(\hat{\hat{Y}}\) is not necessarily the canonical lift, and this definition is independent of the choice of local lift.
Remark 4.2. Geometrically, two isothermic surfaces $y$ and $\hat{y}$ in $Q^n_r$ form a Darboux pair if they envelope the same sphere congruence at the corresponding point and this transform preserves their conformal curvature lines (see for example [11], [3], [1], [16], [18], [7], [28], [25]).

Proposition 4.3. Let $y : M \to Q^n_r$ be a timelike $(\pm)$--isothermic surface. Assume that a timelike immersion $\hat{y} : M \to Q^n_r$ is the Darboux transformation of $y$ with a lift $\hat{Y}$. We have the following conclusions:

(i) $y$ and $\hat{y}$ share the same adapted coordinate $(u, v)$ and they envelope the same 4-dimensional $(2, 2)$--type subspace given by $\text{Span}\{Y, \hat{Y}, dY\} = \text{Span}\{Y, \hat{Y}, d\hat{Y}\}$ (also called “Lorentzian 2-sphere” congruence).

(ii) Set $\langle Y, \hat{Y} \rangle = -1$. We have

\[
\begin{align*}
\hat{Y}_u &= b\hat{Y} + \theta_1(Y_u + aY), \\
\hat{Y}_v &= a\hat{Y} + \theta_2(Y_u + bY)
\end{align*}
\]  

(29)

where $\theta_1 = \pm \theta_2$ is a non-zero constant.

(iii) $\hat{Y}$ is also a $(\pm)$--isothermic immersion sharing the same adapted coordinate $(u, v)$. As a consequence, the curvature lines of $y$ and $\hat{y}$ correspond to each other.

Proof. The conclusion (i) is obvious from the assumption (28). (Recall that a round 2-sphere in $R^n_r$ is identified to a 4-dimensional Lorentzian subspace in $R^{n+2}_r$. ) The normalization $\langle Y, \hat{Y} \rangle = -1$ ensures $\langle Y_u, \hat{Y} \rangle = -\langle Y, \hat{Y}_u \rangle = b$ and $\langle Y_v, \hat{Y} \rangle = -\langle Y, \hat{Y}_v \rangle = a$. Then $\hat{Y}_u \in \text{Span}\{\hat{Y}, Y, Y_u\}$ and $\hat{Y}_v \in \text{Span}\{\hat{Y}, Y, Y_v\}$ is explicitly expressed by (29) Respectively differentiate these two formulas in (29), we obtain

\[
\hat{Y}_{uv} = (b_v + ab)\hat{Y} + (b\theta_2)Y_u + (\theta_1 v + a\theta_1)Y_v + \theta_1 \kappa_2 + (\cdots)Y
\]  

(30)

and

\[
\hat{Y}_{vu} = (a_u + ab)\hat{Y} + (a\theta_1)Y_v + (\theta_2 u + b\theta_2)Y_u + \theta_2 \kappa_1 + (\cdots). \tag{31}
\]

Since $y$ is a timelike $(\pm)$--isothermic surface, $\kappa_1 = \pm \kappa_2$. Comparing (30) and (31), we have the following

\[
\begin{align*}
\theta_1 &= \pm \theta_2, \\
\theta_1 v + a\theta_1 &= 0, \\
a_u &= b_v.
\end{align*}
\]  

(32)

Hence $\theta_1$ and $\theta_2$ must be constant. They are both non-zero since $[\hat{Y}]$ is an immersion. This verifies (ii).

To show conclusion (iii) we need only to show that $\hat{Y}^{\perp}_{uu} = \pm \hat{Y}^{\perp}_{vv}$. Differentiate (29), we obtain

\[
\hat{Y}_{uu} = \theta_1(Y_{uv} + a_u Y + aY_u) + b_u \hat{Y} + b\hat{Y}
\]

\[
\begin{align*}
\hat{Y}_{vu} &= \theta_1(Y_{vu} + (\theta_1 a_u)Y - (\theta_1 ab)Y) \mod \{\hat{Y}, \hat{Y}_u, \hat{Y}_v\}
\end{align*}
\]
4.2 Maurer-Cartan forms of a Darboux pair of timelike isothermic surfaces.

Let \( y \) and \( \hat{y} = [\hat{y}] \) be a Darboux pair of isothermic surfaces. For the notion of curved flats, we need to deform the structure equations by using frames related with the Darboux pairs. To begin with, first we denote by \( O(n - r + 1, r + 1) \) the group defined as below (compare \([3]\))

\[
\begin{align*}
O(n - r + 1, r + 1) &:= \{ A \in Mat(n+2, \mathbb{R}) | A^t I A = I \}, \\
\mathfrak{o}(n - r + 1, r + 1) &:= \{ A \in Mat(n+2, \mathbb{R}) | A^t \bar{I} + \bar{I} A = 0 \},
\end{align*}
\]

with

\[
\bar{I} = \begin{pmatrix} J_2 & J_2 \\ I_{n-r-1,r-1} & \end{pmatrix}.
\]

Here

\[
J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad I_{n-r-1,r-1} = \text{diag}\{1, \ldots, 1, -1, \ldots, -1\}.
\]

Now let us turn to the Darboux pairs. We can write out \( \hat{Y} \) explicitly by using the frame of \( y \):

\[
\hat{Y} = N + 2aY_u + 2bY_v + (2ab + \frac{1}{2} \langle \xi, \xi \rangle)Y + \xi. \tag{34}
\]

Set

\[
P_1 = Y_v + aY \quad \text{and} \quad P_2 = Y_u + bY. \tag{35}
\]

The structure equations \([10]\) of \( Y \) can be rewritten with respect to the frame \( \{Y, \hat{Y}, P_1, P_2, \psi\} \) as below

\[
\begin{align*}
\hat{Y}_u &= b\hat{Y} + \theta_1 P_1, \\
\hat{Y}_v &= a\hat{Y} + \theta_2 P_2, \\
P_{1u} &= -bP_1 + \frac{1}{2}\hat{Y} - \frac{1}{2} \langle \xi, \xi \rangle Y, \\
P_{1v} &= aP_1 + \theta_1 Y + (\kappa_2 + \langle \kappa_2, \xi \rangle)Y, \\
P_{2u} &= bP_2 + \frac{1}{2}P_1 + (\kappa_1 + \langle \kappa_1, \xi \rangle)Y, \\
P_{2v} &= -aP_2 + \frac{1}{2}\hat{Y} - \frac{1}{2} \langle \xi, \xi \rangle Y - \frac{1}{2} \xi, \\
\psi_u &= -2 \langle \psi, \kappa_1 \rangle P_1 - \langle \psi, D_u \xi - b\xi \rangle Y, \\
\psi_v &= -2 \langle \psi, \kappa_2 \rangle P_2 - \langle \psi, D_v \xi - a\xi \rangle Y,
\end{align*}
\]
Let $\tilde{\psi}_j = \psi_j + \langle \psi_j, \xi \rangle Y$, we have
\[
\begin{align*}
\begin{cases}
\tilde{\psi}_{ju} &= -2\langle \tilde{\psi}_j, \kappa_1 \rangle P_1 + \langle \tilde{\psi}_j, \xi \rangle P_2, \\
\tilde{\psi}_{jv} &= -2\langle \tilde{\psi}_j, \kappa_2 \rangle P_2 + \langle \tilde{\psi}_j, \xi \rangle P_1.
\end{cases}
\end{align*}
\]
Set $F = \left( Y, -\tilde{Y}, P_1, 2P_2, \tilde{\psi}_3, \ldots, \tilde{\psi}_n \right)$, and assume that
\[
k_{1j} = \langle \kappa_1, \psi_j \rangle, \ k_{2j} = \langle \kappa_2, \psi_j \rangle, \ \epsilon_j = \langle \psi_j, \psi_j \rangle, \ \epsilon_j = \langle \xi, \psi_j \rangle, \ j = 3, \ldots, n.
\]
Note that $\epsilon_j = 1$, $3 \leq j \leq n - r + 1$, $\epsilon_j = -1$, $n - r + 2 \leq j \leq n$. Then we obtain that
\[
\alpha = F^{-1} F_u = \begin{pmatrix}
-b & 0 & 0 & \theta_1 & 0 & \cdots & 0 \\
0 & b & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & -\theta_1 & -b & 0 & -k_{13} \epsilon_3 & \cdots & -k_{1n} \epsilon_n \\
\frac{1}{2} & 0 & 0 & b & \frac{1}{2} \epsilon_3 \epsilon_3 & \cdots & \frac{1}{2} \epsilon_n \epsilon_n \\
0 & 0 & -\frac{1}{2} \epsilon_3 & 2k_{13} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -\frac{1}{2} \epsilon_n & 2k_{1n} & 0 & \cdots & 0 \\
\end{pmatrix}, \quad (37)
\]
and
\[
\beta = F^{-1} F_v = \begin{pmatrix}
-a & 0 & \frac{1}{2} \theta_2 & 0 & 0 & \cdots & 0 \\
0 & a & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & a & 0 & \epsilon_3 \epsilon_3 & \cdots & \epsilon_n \epsilon_n \\
0 & -\frac{1}{2} \theta_2 & 0 & -a & -k_{23} \epsilon_3 & \cdots & -k_{2n} \epsilon_n \\
0 & 0 & k_{23} & -c_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & k_{2n} & -c_n & 0 & \cdots & 0 \\
\end{pmatrix}, \quad (38)
\]
Note that now $F$ provides a frame in $O(n - r + 1, r + 1)$.
Assume moreover that
\[
F^{-1} dF = \alpha du + \beta dv = (\alpha_0 + \alpha_1) du + (\beta_0 + \beta_1) dv.
\]
with
\[
\alpha_0 = \begin{pmatrix}
-b & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & b & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & b & 0 & -k_{13} \epsilon_3 & \cdots & -k_{1n} \epsilon_n \\
0 & 0 & 0 & b & \frac{1}{2} \epsilon_3 \epsilon_3 & \cdots & \frac{1}{2} \epsilon_n \epsilon_n \\
0 & 0 & -\frac{1}{2} \epsilon_3 & 2k_{13} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & -\frac{1}{2} \epsilon_n & 2k_{1n} & 0 & \cdots & 0 \\
\end{pmatrix}, \quad (40)
\]
and
\[
\alpha_1 = \begin{pmatrix}
0 & 0 & 0 & \theta_1 & 0 & \cdots & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & -\theta_1 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad (41)
\]
and

\[
\beta_0 = \begin{pmatrix}
-a & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & a & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & a & 0 & c_3 \epsilon_3 & \ldots & c_n \epsilon_n \\
0 & 0 & 0 & -a & -k_{23} \epsilon_3 & \ldots & -k_{2n} \epsilon_n \\
0 & 0 & k_{23} & -c_3 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & k_{2n} & -c_n & 0 & \ldots & 0
\end{pmatrix}
\] (42)

and

\[
\beta_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\] (43)

It is straightforward to verify that

\[\alpha_1 \beta_1 - \beta_1 \alpha_1 = 0.\]

On the other hand, the integrability of \(F\) yields

\[\alpha_v - \beta_u - \alpha \beta + \beta \alpha = 0.\]

Altogether we derive that

\[
\begin{cases}
\alpha_{0v} - \beta_{0u} - \alpha_0 \beta_0 + \beta_0 \alpha_0 = 0, \\
\alpha_{1v} - \beta_{1u} - \alpha_1 \beta_0 + \beta_0 \alpha_1 - \alpha_0 \beta_1 + \beta_1 \alpha_0 = 0, \\
\alpha_1 \beta_1 - \beta_1 \alpha_1 = 0.
\end{cases}
\] (44)

4.3 Timelike isothermic surfaces as Curved flats

One of the basic relations of isothermic surfaces with integrable system involves the so-called curved flats discovered by Ferus-Pedit [9]. In [3], they showed that an isothermic surface in \(\mathbb{R}^3\) together with its Darboux transform defines a curved flat in \(SO(1, 4)/SO(3) \times SO(1, 1)\). For the higher co-dimensional case, see Burstall’s summary paper on isothermic surfaces [2]. Here we will show that timelike isothermic surface together with its Darboux transform can also be related with a curved flat.

First we recall the definition of curved flats. Let \(G/K\) be a symmetric space with an involution \(\sigma : G \to G\) of (semi-simple) Lie group \(G\) and \(K\) as the fixed subgroup of \(\sigma\). Then we have the decomposition

\[g = \mathfrak{k} \oplus \mathfrak{p}\]

with \(\mathfrak{k}\) the \(+1\)-eigenspace of \(\sigma\), and \(\mathfrak{p}\) the \(-1\)-eigenspace of \(\sigma\). Note the famous conditions

\[\mathfrak{k} \mathfrak{k} \subset \mathfrak{k}, \, \mathfrak{p} \mathfrak{p} \subset \mathfrak{k}, \, \mathfrak{k} \mathfrak{p} \subset \mathfrak{p}.
\]

Then we have
Definition 4.4. \cite{3}, \cite{9} Let $\phi : M \to G/K$ be an immersion with a frame $F : M \to G$. Suppose the Maurer-Cartan form of $F$ is decomposed according $\sigma$ as
\[ \Phi = F^{-1} dF = \alpha_t + \alpha_p. \]
Then $\phi$ is called a curved flat, if each $\alpha_p$ is contained in an Abelian subalgebra of $\mathfrak{g}$.

It is well-known that one can introduce a loop of Maurer-Cartan forms as follows (Lemma 3 in \cite{9}, Proposition 3.1 \cite{2}):

**Proposition 4.5.** Let $F : M \to G$ with $F^{-1} dF = \alpha_t + \alpha_p$. Then $F$ frames a curved flat if and only if
\[ \alpha_\lambda = \alpha_t + \lambda \alpha_p \]
satisfies
\[ d\alpha_\lambda + \frac{1}{2} [\alpha_\lambda \wedge \alpha_\lambda] = 0, \ \forall \lambda \in \mathbb{R}. \]

For the existence of $F_\lambda$ framing a curved flat $f$, we refer to Theorem 3.2 \cite{2}.

For a Darboux pair of timelike isothermic surfaces, it is obvious that the related $G/K$ is described as
\[ G = O(n - r + 1, r + 1), \]
and
\[ K = O(n - r, r) \times O(1, 1) = \left\{ A \in G | \left( \begin{array}{cc} -I_2 & 0 \\ 0 & I_n \end{array} \right) \cdot A \cdot \left( \begin{array}{cc} -I_2 & 0 \\ 0 & I_n \end{array} \right) = A \right\}. \]

So
\[ t = \text{Lie}K = \left\{ \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) | \ A_1^tJ_2 + J_2A_1 = 0, \ A_2^t\tilde{I} + \tilde{I}A_2 = 0 \right\} \subset \mathfrak{o}(n - r + 1, r + 1), \tag{48} \]
and
\[ p = \left\{ \left( \begin{array}{cc} 0 & -b_1^t\tilde{I} \\ b_1 & 0 \end{array} \right) \right\} \subset \mathfrak{o}(n - r + 1, r + 1). \tag{49} \]

Here
\[ \tilde{I} = \left( \begin{array}{cc} J_2 & 0 \\ 0 & I_{n-r-1, r-1} \end{array} \right). \]

Looking $O(n - r + 1, r + 1)/O(n - r, r) \times O(1, 1)$ as a Grassmannian of two-dimensional Lorentzian plane in $\mathbb{R}^{n+2}_{r+1}$, the map $f = y \wedge \hat{y} : M \to O(n - r + 1, r + 1)/O(n - r, r) \times O(1, 1)$ defines a pair of surfaces $y, \hat{y} : M \to Q^n_\mathbb{R}$. $f$ is called Lorentzian non-degenerate if both $y$ and $\hat{y}$ are Lorentzian immersions. We have a timelike version of the main theorem in \cite{3}, and Theorem 3.3 in \cite{2}:

**Theorem 4.6.** The Lorentzian non-degenerate map $f = y \wedge \hat{y} : M \to O(n - r + 1, r + 1)/O(n - r, r) \times O(1, 1)$ is a curved flat if and only if $y$ and $\hat{y}$ is a Darboux pair of timelike $(\pm)$-isothermic surfaces.
Proof. For a Darboux pair \((y, \hat{y})\) as described in above subsection, we have a frame \(F\) such that
\[
F^{-1}dF = \alpha du + \beta dv = (\alpha_0 + \alpha_1)du + (\beta_0 + \beta_1)dv,
\]
with \(\alpha_0, \alpha_1, \beta_0\) and \(\beta_1\) of the form in \([40], [41], [42]\) and \([43]\). Set
\[
\alpha_\lambda = \alpha_0 + \lambda \alpha_1 \quad \text{and} \quad \beta_\lambda = \beta_0 + \lambda \beta_1
\]
with \(\lambda \in \mathbb{R}\). One verifies directly that
\[
\alpha_{\lambda u} - \beta_{\lambda u} - \alpha_\lambda \beta_\lambda + \beta_\lambda \alpha_\lambda = 0
\]
is equivalent to the equations in \([44]\). As a consequence, \(F\) induces a curved flat in \(G/K\).

Conversely, let \(f = y \wedge \hat{y}\) be a Lorentzian non-degenerate curved flat. Since \(y\) inherits a Lorentzian metric, there exists some local asymptotic coordinate \((u, v)\) and some lift \(Y\) of \(y\) such that \(\langle Y_u, Y_u \rangle = \langle Y_v, Y_v \rangle = 0\), \(\langle Y_u, Y_v \rangle = \frac{1}{2}\). Let \(\hat{Y}\) be a lift of \(\hat{y}\) such that \(\langle \hat{Y}, Y \rangle = -1\). Set
\[
P_1 = Y_v + aY \quad \text{and} \quad P_2 = Y_u + bY
\]
as above such that \(P_1 \perp \hat{Y}\) and \(P_2 \perp \hat{Y}\). Choose \(\tilde{\psi}_j\) such that
\[
F = (Y, -\hat{Y}, P_1, 2P_2, \tilde{\psi}_3, \ldots, \tilde{\psi}_n)
\]
represents a map into \(O(n - r + 1, r + 1)\) (Hence \(\epsilon_j = \langle \psi_j, \tilde{\psi}_j \rangle\) with \(\epsilon_j = 1, 3 \leq j \leq n - r + 1, \epsilon_j = -1, n - r + 2 \leq j \leq n\)). Then we may assume that
\[
\hat{Y}_u = a \hat{Y} + p_0 P_1 + \tilde{p}_0 P_2 = \sum_{j=3}^{n} p_j \tilde{\psi}_j, \quad \hat{Y}_v = b \hat{Y} + q_0 P_1 + \tilde{q}_0 P_2 = \sum_{j=3}^{n} q_j \tilde{\psi}_j.
\]
Now set
\[
F^{-1}dF = \alpha du + \beta dv = (\alpha_0 + \alpha_1)du + (\beta_0 + \beta_1)dv
\]
with \(\alpha_0, \beta_0 \in \mathfrak{f}\) and \(\alpha_1, \beta_1 \in \mathfrak{p}\). Then we obtain that
\[
\alpha_1 = \begin{pmatrix}
0 & 0 & \frac{1}{2} \tilde{p}_0 & p_0 & p_3 \epsilon_3 & \cdots & p_n \epsilon_n \\
0 & 0 & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & -p_0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & -\frac{1}{2} \tilde{p}_0 & 0 & 0 & 0 & \cdots & 0 \\
0 & -p_3 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -p_n & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
and
\[
\beta_1 = \begin{pmatrix}
0 & 0 & \frac{1}{2} \tilde{q}_0 & q_0 & q_3 \epsilon_3 & \cdots & q_n \epsilon_n \\
0 & 0 & 0 & -1 & 0 & \cdots & 0 \\
0 & -q_0 & 0 & 0 & 0 & \cdots & 0 \\
1 & -\frac{1}{2} \tilde{q}_0 & 0 & 0 & 0 & \cdots & 0 \\
0 & -\frac{1}{2} \tilde{q}_0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -q_n & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]
The conditions of curved flats are equivalent to (44). Now the last matrix equation of (44) yields
\[ p_3 = \cdots = p_n = q_3 = \cdots = q_n, \quad \tilde{p}_0 + q_0 = 0, \quad \tilde{p}_0 - q_0 = 0. \]
Therefore we obtain
\[ \hat{Y}_a \in \text{span}\{Y, \hat{Y}, Y_o\}, \quad \hat{Y}_v \in \text{span}\{Y, \hat{Y}, Y_u\} \quad \text{and} \quad \langle \hat{Y}_a, \hat{Y}_v \rangle = 0. \]
Since \( \hat{Y} \) is non-degenerate, \( \langle \hat{Y}_a, \hat{Y}_v \rangle \neq 0 \), by Theorem A of [25] (compare Proposition 4.3), \([Y]\) and \([\hat{Y}]\) are a Darboux pair of timelike \((\pm)\)–isothermic surfaces.

**Remark 4.7.** Note that when \( \langle \hat{Y}_a, \hat{Y}_v \rangle > 0 \), one obtains a pair of timelike \((+)\)–isothermic surfaces, when \( \langle \hat{Y}_a, \hat{Y}_v \rangle < 0 \), one obtains a pair of timelike \((-)\)–isothermic surfaces.

**Remark 4.8.** It was explained in [3] that Christoffel transforms can be obtained as a limit of Darboux transforms for isothermic surfaces in \( \mathbb{R}^3 \). While for \( c–\)polar transforms, usually it can not be derived as a limit of Darboux transforms. Therefore the classical Christoffel transforms can be looked as a very special kind of transforms of \( c–\)polar transforms which have closed relation with Darboux transforms.

**Remark 4.9.** \( O(n-j+1, j+1/O(n-j, j)) \times O(1,1) \)–**system and Timelike isothermic surfaces:**
Along an independent line, almost in the same time, Terng introduced the notion of \( U/K \)–system [24], which also has its roots on the study of transforms of submanifolds [23]. In [24], Terng showed that the \( U/K \)–system, as an integrable system, inherits Lax pair and one also can introduce loop parameter \( \lambda \), etc. Moreover, such systems are naturally related with several kinds of submanifolds together with their transforms. Such relationships are discussed in details in the later note [1]. As a special case, in [1], they found that \( G^1_{m,1} \)–system \( (O(m+1,1)/O(m) \times O(1,1)) \) is exactly related with a pair of dual isothermic surfaces in \( \mathbb{R}^m \). See Section 8 [1] for details.

In the recent work by Dussan and Magid [7], they generalized the methods in [1] to derive a detailed description on timelike isothermic surfaces in \( \mathbb{R}^n_j \). Comparing with the formulas in Theorem 3.1 and Theorem 3.2 in [7], one can see that these are exactly the same as the integrability conditions in (13), (14), (15) for the case \((+)\)–isothermic surfaces. And the formulas in Theorem 3.1 and Theorem 3.2 in [7], one can see that these are exactly the same as the integrability conditions in (13), (14), (15) for the case \((-)\)–isothermic surfaces. To get Darboux transforms, they need the methods of dressing actions, see [1], [8].

### 5 Two permutability theorems

#### 5.1 Spectral transforms of timelike isothermic surfaces

Let \( y: M \to Q^m_n \) be an immersed timelike \((+)\)–isothermic surface with adapt coordinate \((u, v)\) and invariants in Section 2. If one only change the Schwarzians from \( s_i \) to \( s_i^\hat{c} = s_i + \hat{c} \), with all the other coefficients invariant, then the conformal Gauss, Codazzi, and Ricci equations are remain satisfied, where \( \hat{c} \in \mathbb{R} \) is a parameter.

To be concrete, if we consider a new data as below
\[ s_i^\hat{c} = s_i + \hat{c}, \quad \langle k_i^\hat{c}, \psi^\hat{c} \rangle = \langle k_i, \psi \rangle, \quad \langle \psi^\hat{c}, \psi^\hat{c} \rangle = \langle \psi, \psi \rangle, \quad D^\hat{c}_z = D_z. \]
where \( \psi \) is arbitrary section of normal bundle with deforming normal section \( \psi^\tilde{c} \). The integrable equations are satisfied and then by Theorem 2.4 there will be an associate family of non-congruent timelike \((+)\)-isothermic surfaces \( [Y^\tilde{c}] \) with corresponding invariants.

Note that for a timelike \((-)\)-isothermic surface, the deformation should be \( s_1 \) to \( s_1^\tilde{c} = s_1 + \tilde{c} \) and \( s_2 \) to \( s_2^\tilde{c} = s_2 - \tilde{c} \).

**Definition 5.1.** The timelike isothermic surface \( [Y^\tilde{c}] \) is called a \((\tilde{c}-)\)-parameter spectral transform of the timelike isothermic surface \( y : M \to Q^n_r \).

**Theorem 5.2.** Let \( y^\tilde{c} \) be a \((\tilde{c}-)\)-parameter spectral transform of a timelike \((\pm)\)-isothermic surface \( y : M \to Q^n_r \). Denote their canonical lift as \( Y, Y^\tilde{c} \) for the same adapted coordinate \( (u,v) \). Let \( \psi \) be a non-degenerate \( c \)-polar surface of \( y \). Then there exists a \( c \)-polar surface \( \psi^\tilde{c} \) of \( y^\tilde{c} \) satisfying that \( \psi^\tilde{c} \) is a \((\tilde{c}-)\)-parameter spectral transform of \( \psi \). In other words, one has the commuting diagram:

\[
\begin{array}{ccc}
[\hat{Y}] & \to & [Y^\tilde{c}] \\
\downarrow & & \downarrow \\
\hat{\psi} & \to & \psi^\tilde{c}
\end{array}
\]

**Proof.** Note that the spectral transforms preserve normal connection and the corresponding \( k_i \). Let \( \psi^\tilde{c} \) denote the corresponding parallel normal section of \( \psi \), by Theorem 3.4, \( \psi^\tilde{c} \) is also a \( c \)-polar transform of \( y^\tilde{c} \). By using (15), (17)-(21), all the other conditions except Schwarzian are satisfied. For \( s_i \), we have that

\[
\begin{align*}
    s_1^\psi &= 2\omega_{uu}^\psi - 2(\omega_u^\psi)^2 + \frac{\langle \psi, D_u D_u \kappa_2 \rangle}{\langle \psi, \kappa_2 \rangle} + s_1, \\
    s_2^\psi &= 2\omega_{vv}^\psi - 2(\omega_v^\psi)^2 + \frac{\langle \psi, D_v D_v \kappa_1 \rangle}{\langle \psi, \kappa_1 \rangle} + s_2.
\end{align*}
\]

So \( s_i^\tilde{c} = s_i + \tilde{c} \) when \( y \) is \((+)\)-isothermic and \( \psi^\tilde{c} \) is a \((\tilde{c}-)\)-parameter spectral transform of \( \psi \). When \( y \) is \((-)\)-isothermic, we have \( s_1^\tilde{c} = s_1 + \tilde{c} \), \( s_2^\tilde{c} = s_2 - \tilde{c} \) and \( \psi^\tilde{c} \) is also a \((\tilde{c}-)\)-parameter spectral transform of \( \psi \). \( \square \)

### 5.2 Permutability with Darboux transforms

**Theorem 5.3.** Let \( y : M \to Q^n_r \) be a timelike \((\pm)\)-isothermic surface and \( [\hat{Y}] \) be a Darboux transform of \( y \). If \( \psi \) is a non-degenerate \( c \)-polar surface of \( y \), then there exits a \( c \)-polar surface \( \hat{\psi} \) of \( \hat{Y} \) such that \( \hat{\psi} \) is also a Darboux transform of \( \psi \).

**Proof.** Write out \( \hat{Y} \) explicitly:

\[
\hat{Y} = N + 2aY_u + 2bY_v + (2ab + \frac{1}{2}\langle \xi, \xi \rangle)Y + \xi.
\]  

(50)

Set

\[
P_1 = Y_v + aY, \quad P_2 = Y_u + bY
\]

(51)

The structure equations (14) of \( Y \) can be rewritten with respect to the frame \( \{Y, \hat{Y}, P_1, P_2, \psi\} \).
as below

\[
\begin{align*}
\dot{Y}_u &= b\dot{Y} + \theta_1 P_1, \\
\dot{Y}_v &= a\dot{Y} + \theta_2 P_2, \\
P_{1u} &= -bP_1 + \frac{1}{2}\dot{Y} - \frac{1}{2}\langle \xi, \xi \rangle Y - \frac{1}{2}\xi, \\
P_{1v} &= aP_1 + \left(\frac{\theta_1}{\theta_2} + \langle \kappa_2, \xi \rangle\right)Y + \kappa_2, \\
P_{2u} &= bP_2 + \left(\frac{\theta_1}{\theta_2} + \langle \kappa_1, \xi \rangle\right)Y + \kappa_1, \\
P_{2v} &= -aP_2 + \frac{1}{2}\dot{Y} - \frac{1}{2}\langle \xi, \xi \rangle Y - \frac{1}{2}\xi, \\
\psi_u &= -2\langle \psi, \kappa_1 \rangle P_1 - \langle \psi, D_u \xi - b\xi \rangle Y, \\
\psi_v &= -2\langle \psi, \kappa_2 \rangle P_2 - \langle \psi, D_v \xi - a\xi \rangle Y,
\end{align*}
\]

(52)

Now let us find out the \(c\)-polar transform of \(\dot{Y}\) corresponding to \(\psi\). From (29) we have

\[
\dot{Y}_{uv} = -\frac{1}{4}\langle \xi, \xi \rangle \dot{Y} + \frac{\theta_1 \theta_2}{2} N
\]

where

\[
N = \left(1 + \frac{2\langle \kappa_2, \xi \rangle}{\theta_2}\right)Y + \frac{2a}{\theta_2} P_1 + \frac{2b}{\theta_1} P_2 + 2\left(\frac{ab}{\theta_1 \theta_2} + \frac{2\langle \kappa_1, \kappa_2 \rangle}{\theta_1 \theta_2}\right)\dot{Y} + \frac{2}{\theta_2} \kappa_2.
\]

Denote

\[
\hat{\psi} = \psi + \langle \psi, \xi \rangle Y + \frac{2\langle \psi, \kappa_2 \rangle}{\theta_2} \dot{Y}.
\]

(53)

It is straightforward to verify that

\[
\hat{\psi} \perp \{\dot{Y}, \dot{Y}_u, \dot{Y}_v, \dot{Y}_{uv}\},
\]

and \(\hat{\psi}\) is a parallel section of \(\dot{Y}\) with length \(c\). Computing shows

\[
\dot{\hat{\psi}}_u = -\frac{\langle \psi, \xi \rangle}{2\langle \psi, \kappa_2 \rangle} \psi_u + \left(\frac{\langle \psi, D_u \kappa_2 \rangle}{\langle \psi, \kappa_2 \rangle} + b\right) (\hat{\psi} - \psi),
\]

(54)

and

\[
\dot{\hat{\psi}}_v = -\frac{\langle \psi, \xi \rangle}{2\langle \psi, \kappa_1 \rangle} \psi_u + \left(\frac{\langle \psi, D_v \kappa_2 \rangle}{\langle \psi, \kappa_2 \rangle} + a\right) (\hat{\psi} - \psi).
\]

(55)

This shows that \(\hat{\psi}\) is a Darboux transform of \(\psi\) by Definition 4.1.

\[\square\]

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