Elementary constructive approach to the higher-rank numerical ranges of unitary matrices.

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Abstract

Some problems of the quantum error-correcting codes theory can be reduced to the investigation of the higher-rank numerical ranges of the operators related to the error operators. We constructively verify a conjecture on the structure of higher-rank numerical range for unitary matrices.

1 Introduction

Quantum error correction is one of the main directions in the developing of the quantum information theory since middle of the 1990, see [1]-[4]. Recently in series of papers [5]-[7] it was introduced one approach to the realization of error-correcting codes for quantum channels. Results of [5] - [7] give the possibility to reduce the realization of the correctable codes to the matrix analysis problem, namely, to the study of the "higher-rank numerical range" of the operators related to the "error operator" of the quantum channel. The "higher-rank numerical range" is a generalization of the usual notion of the operator spectrum. Namely, let $H$ be a finite-dimensional Hilbert space, $B(H)$ be a set of operators acting on $H$, $\sigma \in B(H)$. For $k \geq 1$ the rank-$k$ numerical range of $\sigma$ is the subset of complex plane $\Lambda_k(\sigma) = \{ \lambda \in C : P\sigma P = \lambda P \}$ for some $k$-dimensional orthogonal projections $P$ on $H$. The key problem is a description of $\Lambda_k(\sigma)$ for a given operator $\sigma$ in explicit terms. The next statement was proved in [7].

**Proposition 1.** Let $H$ be an $N$-dimensional Hilbert space, $k \geq 1$ be a positive integer, $\sigma$ be a normal matrix, then

$$\Lambda_k(\sigma) \subseteq \Omega_k(\sigma),$$

(1)
where

\[ \Omega_k(\sigma) = \cap \text{conv}(\Gamma), \]

and \( \Gamma \) runs through all \((N - k + 1)\)-point subsets (counting multiplicities) of the set of eigenvalues \( \text{spec}(\sigma) \) for \( \sigma \), \( \text{conv}(\Gamma) \) means the convex hulls of the set \( \Gamma \).

It was conjectured in [5], that the conversion of this statement is valid.

**Conjecture.** For the normal matrix \( \sigma \)

\[ \Lambda_k(\sigma) = \Omega_k(\sigma). \]  \hspace{1cm} (2)

For the brevity we will denote Conjecture for given \( N, k \) as \((N, k)\). This statement is not proved in general case, some particular cases were discussed in [7]. In particular, the next propositions were checked in [7]

**Proposition 2.** Conjecture holds if and only if the corresponding statement holds for all unitary matrices.

**Proposition 3.** Conjecture (2) is valid for \( N \geq 3k, k \geq 2 \), \((5, 2)\), \((8, 3)\), and, generally, \((3k - 1, k), k \geq 2 \).

Conjecture \((N, k), N \geq 3k\), was verified in [7] explicitly. The corresponding construction was presented with help of simple and elementary terms, see discussion below. However, the verification of the Conjecture \((5, 2)\), \((8, 3)\), and \((3k - 1, k)\) was given [7] non-constructively. But at the realization of the quantum error-correcting codes it is necessary to get an explicit description of the corresponding objects, such as projector \( P \). Note, that full proof of the Conjecture was obtained in [8] with help of more advanced technique.

The aim of this note is to modify an elementary approach of the [7] and to suggest a constructive verification of the Conjecture \((3k - 1, k), k \geq 2 \) and \((3k - 2, k), k \geq 5 \). We will consider here the mathematical details only, initial motivation and discussion of possible applications in the theory of quantum error-correcting codes can be found in [5] - [7].

## 2 General considerations

As it follows from Proposition 2, we can discuss a unitary matrix \( \sigma \), so its spectrum belongs to the unit circle. Let eigenvalues of \( \sigma \) are \( \lambda_j = \exp(i\theta_j), j = 1, 2, ..., N \), such that \( 0 \leq \theta_1 \leq \theta_2 \leq ... \theta_N < 2\pi \). We extend the numbering of the \( \lambda_j \) and \( |\psi_j| \) cyclically if it is necessary.
For multiple eigenvalues the numbering is arbitrary, and we choose an orthonormal system of eigenvectors $|\psi_j> \in \mathbf{H}$,

$$\sigma |\psi_j> = \lambda_j |\psi_j>,$$ \(j = 1,2,\ldots,N\). \(3\)

Let $\lambda \in \Lambda_k(\sigma)$, it means, that $k$-dimensional orthogonal projection $P$ exist, for which

$$P\sigma P = \lambda P.$$ \(4\)

Let

$$P = \sum_{s=1}^{k} |\varphi_s><\varphi_s|$$ \(5\)

for some set of orthonormal vectors $\{|\varphi_1>, |\varphi_2>, \ldots, |\varphi_k>\}$ and

$$|\varphi_s> = \sum_m z_{sm} |\psi_m>,$$ \(6\)

then normalization means, that

$$\sum_m |z_{sm}|^2 = 1,$$ \(7\)

and orthogonality means, that

$$\sum_m z_{sm} \overline{z_{pm}} = 0, s \neq p.$$ \(8\)

Relation (4) reads in our notations:

$$\sum_m \lambda_m |z_{sm}|^2 = \lambda, s = 1,2,\ldots,k.$$ \(9\)

For the convenience of following discussions we formulate the inversion of these considerations as a proposition.

**Proposition 4.** If for given $\lambda$ we can find a set of vectors $\{|\varphi_1>, |\varphi_2>, \ldots, |\varphi_k>\}$ which satisfy relations (6)-(9), then $\lambda \in \Lambda_k(\sigma)$ and corresponding projector $P$ is described by relation (5).

First of all we cite here one more result of [7] which is useful in what follows.

**Proposition 5.** Given integers $i, j$ with $i < j < i + N$, let $D(i,j)$ denote the convex subset of $\mathbf{C}$ bounded by the line segment from $\lambda_i$ to $\lambda_j$ and the counterclockwise circular arc from $\lambda_j$ to $\lambda_i$. Then

$$\Omega_k(\sigma) = \cap_{i=1}^{N} D(i, i + k).$$ \(10\)

The next simple result which will be exploited below follows from Proposition 5.
Corollary. Let $T(\lambda_i, \lambda_j, \lambda_m), i < j < m$, is a triangle with vertexes $\{(\lambda_i)\}, \{(\lambda_j)\}, \{(\lambda_m)\}$. If $| j - i | < k, | m - j | < k, | N + i - m | < k$, then $\Omega_k(\sigma) \subset T(\lambda_i, \lambda_j, \lambda_m)$, so for any $\lambda \in \Omega_k(\sigma)$ exist nonnegative numbers $p_i, p_j, p_m$, for which the following relations are valid:

$$p_i + p_j + p_m = 1,$$

$$\lambda_i p_i + \lambda_j p_j + \lambda_m p_m = \lambda.$$ 

Note, that these relations are a special case of relations (7) and (9). We will associate with such triangle a normalized vector

$$| \varphi_s > = \sqrt{p_i} | \psi_i > + \sqrt{p_j} | \psi_j > + \sqrt{p_m} | \psi_m > .$$ (11)

As was mentioned above, Conjecture $(N, k), N \geq 3k$, was proved in [7]. Namely, corresponding procedure includes a construction of $k$ triangles satisfying conditions of Corollary. These triangles have not common vertexes, so corresponding vectors (11) are orthogonal each other. Full set of these vectors satisfy Proposition 4.

Here we present some modification of an elementary approach [7]. Namely, we use a set of $k$ triangles too, but we permit existence of common vertex either for one pair of triangles (Conjecture $((3k - 1), k)$) or for two pairs of triangles (Conjecture $((3k - 2), k)$). The key result in our considerations is the following statement.

Proposition 6. Let we have two set of numbers $p_t, q_r \geq 0, t \in T \subset \{1, 2, ..., N\}, r \in R \subset \{1, 2, ..., N\}, T \cap R = \emptyset$, which satisfy the next conditions:

$$p_1 + \sum_{t \in T} p_t = 1$$ (12)

$$q_1 + \sum_{r \in R} q_r = 1$$ (13)

$$\lambda_1 p_1 + \sum_{t \in T} \lambda_t p_t = \lambda,$$ (14)

$$\lambda_1 q_1 + \sum_{r \in R} \lambda_r q_r = \lambda.$$ (15)

If either $p_1 \leq 1/2$ or $q_1 \leq 1/2$, then there are two orthonormal vectors

$$| \varphi_1 > = z_{11} | \psi_1 > + \sum_{s \in T \cup R} z_{1s} | \psi_s >,$$ (16)
\[ |\varphi_2| = z_{21} |\psi_1| + \sum_{s \in T \cup R} z_{2s} |\psi_s|, \]  

(17)
satisfying relations (7)-(9).

Proof. Let

\[ |\varphi_1| = [\sqrt{p_1} \cos \theta + i \sqrt{q_1} \sin \theta] |\psi_1| + \exp(i\alpha) \cos \theta \sum_t \sqrt{p_t} |\psi_t| + \exp(i\beta) \sin \theta \sum_r \sqrt{q_r} |\psi_r|, \]  

(18)
\[ |\varphi_2| = [\sqrt{p_1} \cos \tau + i \sqrt{q_1} \sin \tau] |\psi_1| + \cos \tau \sum_t \sqrt{p_t} |\psi_t| + \sin \tau \sum_r \sqrt{q_r} |\psi_r|. \]  

(19)
Simple calculations with help of (12)-(15) confirm, that relations (7) and (9) are valid for these vectors. We have to find values \(\theta, \tau, \alpha, \beta\) in order to get orthogonality of vectors \(|\varphi_1|, |\varphi_2|\).

We obtain the following condition:

\[ <\varphi_1, \varphi_2> = p_1 \cos \theta \cos \tau + q_1 \sin \theta \sin \tau + \exp(i\alpha) \cos \theta \cos \tau (1 - p_1) + \exp(i\beta) \sin \theta \sin \tau (1 - q_1) + \]
\[ i\sqrt{p_1 q_1} (\sin \theta \cos \tau - \cos \theta \sin \tau) = 0, \]

here we take into account relations (12) and (13). Let \(x = \tan \theta, y = \tan \tau\). Separating real and imaginary parts of the last expression, one can get the following pair of equations:

\[ p_1 + q_1 xy + \cos \alpha (1 - p_1) + \cos \beta (1 - q_1) xy = 0, \]  

(20)
\[ \sqrt{p_1 q_1} (x - y) + \sin \alpha (1 - p_1) + \sin \beta (1 - q_1) xy = 0. \]  

(21)
Excluding \(y\), we obtain quadratic equation for \(x\):

\[ x^2 + Ax + B = 0, \]  

(22)
where

\[ A = \frac{(p_1 - 1) \sin \alpha [q_1 + (1 - q_1) \cos \beta] + (q_1 - 1) \sin \beta [(p_1 - 1) \cos \alpha - p_1]}{\sqrt{p_1 q_1} (q_1 + (1 - q_1) \cos \beta)}, \]  

(23)
\[ B = \frac{p_1 + (1 - p_1) \cos \alpha}{q_1 + (1 - q_1) \cos \beta}. \]  

(24)
Equation (22) has real root, if the next condition holds:

\[ A^2 \geq 4B, \]  

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or, in more details,

\[
\{(p_1 - 1) \sin \alpha [q_1 + (1 - q_1) \cos \beta] + (q_1 - 1) \sin \beta [(p_1 - 1) \cos \alpha - p_1]\}^2 \geq \]

\[4p_1 q_1 [p_1 + (1 - p_1) \cos \alpha] [q_1 + (1 - q_1) \cos \beta].\]

Note, that if either \(p_1 \leq 1/2\) or \(q_1 \leq 1/2\) we can get non-positive right-hand side of the last expression by the corresponding choice of the parameters \(\alpha, \beta\). Then condition (25) holds, we can calculate corresponding (real) values \(x, y\) or, in other words, \(\theta, \tau\) and construct the pair of orthonormal vectors \(|\varphi_1>, |\varphi_2>\) in explicit terms. The result follows.

**Definition.** Let \(p_t \geq 0, t \in T \subset \{1, 2, ..., N\}\), and for given \(\lambda\)

\[p_{t_1} + \sum_{t \neq t_1} p_t = 1\]

\[\lambda_{t_1} p_{t_1} + \sum_{t \neq t_1} \lambda_t p_t = \lambda,\]

and \(p_{t_1} \leq 1/2\). We call the point \(\{t_1\}\) weak vertex of the polygon generated by \(T\).

In what follows we will construct \(k\) triangles \(T \{\lambda_i, \lambda_j, \lambda_m\}\), each of them will satisfy condition of the Corollary. Note, that each such triangle contains two weak vertexes. For the Conjecture \((3k - 1, k)\) only one pair of triangles (only two pairs for the Conjecture \((3k - 2, k)\)) will have one common vertex, weak for one of triangles. So, we can apply Proposition 6 and get a pair of necessary vectors \(|\varphi_1>, |\varphi_2>\) (two pairs for Conjecture \((3k - 2, k)\), respectively). For the remaining \((k - 2)\) \((k - 4)\) for Conjecture \((3k - 2, k)\) triangles corresponding vectors will be defined by relation (11). These vectors will be normalized and orthogonal each other, and this set of vectors will satisfy conditions (7)-(9).

### 3 Constructive verification of the Conjecture \((3k - 1, k)\).

So, our aim is to find a necessary system of triangles. In order to clarify details, we firstly consider \(N = 5, k = 2\). The spectrum of the unitary operator \(\sigma\) is depicted on figure 1. Note, that some eigenvalues can coincide, but we represent them as different points for more clearness. Let \(\lambda \in \Omega_2(\sigma)\). In accordance with Proposition 2,

\[\lambda \in T \{1, 3, 5\} \cap T \{1, 2, 4\} \cap T \{2, 4, 5\}.\]
As a first triangle, appearing in Proposition 6, we take \( T\{1,3,5\} \). Note, that either vertex \( \{1\} \) or vertex \( \{5\} \) is weak vertex of this triangle. In the first case we take \( T\{1,2,4\} \) as a second triangle appearing in Proposition 6. In this case \( T\{1,3,5\} \cap T\{1,2,4\} = \{1\} \). If the vertex \( \{5\} \) is weak vertex of triangle \( T\{1,3,5\} \), we take \( T\{2,4,5\} \) as a second triangle, \( T\{1,3,5\} \cap T\{2,4,5\} = \{5\} \). In both situations intersection of chosen triangles contains only one vertex, which is weak for triangle \( T\{1,3,5\} \) and we can apply Proposition 6. Then we obtain the pair of vectors satisfying relations (7)-(9).

As the second example we consider \( N = 3k - 1 \), \( k \geq 3 \). The spectrum in this situation is depicted on figure 2. If \( \lambda \in \Omega_k(\sigma) \), then, in accordance with Corollary, \( \lambda \) belongs to the triangle \( T\{1,k+1,2k+1\} \). In that triangle either vertex \( \{1\} \) or \( \{2k+1\} \) is weak one. Let, for distinctness, it is \( \{1\} \) (there is a symmetry of the picture). Then we take as the second triangle \( T\{1,k,2k\} \). The remaining \((k-2)\) triangles are \( T\{k-1,2k-1,3k-2\}, T\{k-2,2k-2,3k-3\} \) etc. As follows from the Corollary, \( \Omega_k(\sigma) \) belongs to intersection of all triangles. Note, that only triangles \( T\{1,k+1,2k+1\}, T\{1,k,2k\} \) have one common vertex (which is weak for the first triangle). Applying Proposition 6 to the triangles \( T\{1,k+1,2k+1\}, T\{1,k,2k\} \), we can construct a pair of orthogonal vectors \( |\varphi_k\rangle, |\varphi_{k-1}\rangle \), which satisfy conditions (7), (9). For each triangle \( T\{k-m,2k-m,3k-m-1\}, m = 1,2,...,k-2 \), we take associated by (11) vectors,

\[
|\varphi_m\rangle = \sqrt{p_{k-m}} |\psi_{k-m}\rangle + \sqrt{p_{2k-m}} |\psi_{2k-m}\rangle + \sqrt{p_{3k-m-1}} |\psi_{3k-m-1}\rangle,
\]

where positive numbers \( p_{k-m}, p_{2k-m}, p_{3k-m-1} \) are determined by relation

\[
\lambda = \lambda_{k-m} p_{k-m} + \lambda_{2k-m} p_{2k-m} + \lambda_{3k-m-1} p_{3k-m-1}.
\]

As was mentioned above, vectors \( \{|\varphi_m\rangle, m = 1,2,...,k\} \) are orthogonal each other. So, we have constructed the necessary set of vectors and the corresponding orthogonal projector is given by relation (5).

Note, that correctness of Conjecture (\((3k-1)m,km\), \( m \geq 2 \)) follows immediately from our results.

### 4 Constructive verification of the Conjecture \((3k-2,k)\).

Now we consider Conjecture \((3k-2,k), k \geq 5 \). In order to verify this situation we have twice apply Proposition 6 .
For the convenience we begin from Conjecture (13, 5) (see figure 3). First of all we depict the triangle \( T\{1, 4, 9\} \). In this triangle either \( \{1\} \) or \( \{4\} \) is a weak vertex. There is an evident symmetry of our figure on this stage, and we choose vertex \( \{1\} \). Then next triangle will be \( T\{1, 6, 11\} \), which has one common vertex with triangle \( T\{1, 4, 9\} \). With help of Proposition 6 we can construct two orthonormal vectors \( \varphi_1 >, \varphi_2 > \), satisfying relations (7)-(9). As the next triangle we choose \( T\{3, 8, 12\} \), where either \( \{8\} \) or \( \{12\} \) is a weak vertex.

1) Let \( \{8\} \) is the weak vertex of the triangle \( T\{3, 8, 12\} \). Then we take triangle \( T\{5, 8, 13\} \) and for pair of triangles \( T\{3, 8, 12\}, T\{5, 8, 13\} \) we construct with help of Proposition 6 the pair of orthogonal vectors \( \varphi_3 >, \varphi_4 > \), satisfying relations (7)-(9). Residuary vertexes gives us the last triangle \( T\{2, 7, 10\} \), which generates the fifth necessary vector in accordance with (11).

2) Let \( \{12\} \) is the weak vertex of the triangle \( T\{3, 8, 12\} \). Then we choose triangle \( T\{2, 7, 12\} \) and for pair of triangles \( T\{3, 8, 12\}, T\{2, 7, 12\} \) we construct with help of Proposition 6 the pair of orthogonal vectors \( \varphi_3 >, \varphi_4 > \), satisfying relations (7)-(9). Residuary vertexes gives us the last triangle \( T\{5, 10, 13\} \) and we obtain the fifth vector, associated with this triangle by (11).

Let now consider Conjecture \((3k - 2, k)\) for \( k > 5 \), see figure 4. First triangle is triangle \( T\{1, k - 1, 2k - 1\} \), and either \( \{1\} \) or \( \{k - 1\} \) is a weak vertex. Due to symmetry we can choose any of them, and we choose \( \{1\} \). The next triangle is \( T\{1, k + 1, 2k + 1\} \), which has one common vertex with \( T\{1, k - 1, 2k - 1\} \). So, proposition 6 gives the possibility to construct a pair vectors \( \varphi_1 >, \varphi_2 > \) with necessary properties. Then we choose triangle \( T\{k - 2, 2k - 2, 3k - 3\} \). Here either \( \{2k - 2\} \) or \( \{3k - 3\} \) is a weak vertex.

1) Let \( \{2k - 2\} \) is a weak vertex. Then we choose as a next triangle \( T\{k, 2k - 2, 3k - 2\} \). With help of Proposition 6 we construct one more pair of vectors \( \varphi_3 >, \varphi_4 > \) with necessary properties. The triangle \( T\{2, k + 2, 2k\} \) gives the fifth vector \( \varphi_5 > \). Additional \((k - 5)\) vectors can be constructed with help of \( k - 5 \) triangles \( T\{m, k + m, 2k + m - 1\}, m = 3, 4, ..., k - 3 \). Each such triangle satisfies condition of Corollary and relation (11) gives us corresponding vector.

2) Let now \( \{3k - 3\} \) is a weak vertex. As a next triangle we choose \( T\{k - 3, 2k - 3, 3k - 3\} \). Applying Proposition 6 we construct one more pair of vectors \( \varphi_3 >, \varphi_4 > \) with necessary properties. The triangle \( T\{k, 2k, 3k - 2\} \) gives us the fifth vector \( \varphi_5 > \). Additional \((k -
5) vectors can be constructed with help of \((k - 5)\) triangles \(T\{m, k + m, 2k + m - 1\}, m = 2, 3, \ldots, k - 4\) and relation (11).

Evidently, that Conjecture \(((3k - 2)s, ks), k \geq 5, s = 2, 3, \ldots\), follows from this result.

5 Conclusion

We have discussed the "higher-rank numerical ranges" method of constructing error-correcting codes for quantum channels. The realization of the correctable codes is reduced in this framework to the matrix analysis problem, which was thoroughly considered in papers [5] - [7] and solved in [8]. Realization of the error-correcting codes is based on explicit description of higher-rank numerical ranges of operators related to the error operators. Corresponding constructive description was obtained in [7] for \(N \geq 3k\) in elementary terms. Here we have presented some modification of this construction, which can be applied for \(N = 3k - 1, k \geq 2\) and \(N = 3k - 2, k \geq 5\). These results can be useful at constructing error-correcting codes for a special classes of quantum channels CHKZ.

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