Vector-valued Jack polynomials and wavefunctions on the torus

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Abstract

The Hamiltonian of the quantum Calogero–Sutherland model of \( N \) identical particles on the circle with \( 1/r^2 \) interactions has eigenfunctions consisting of Jack polynomials times the base state. By use of the generalized Jack polynomials taking values in modules of the symmetric group and the matrix solution of a system of linear differential equations one constructs novel eigenfunctions of the Hamiltonian. Like the usual wavefunctions each eigenfunction determines a symmetric probability density on the \( N \)-torus. The construction applies to any irreducible representation of the symmetric group. The methods depend on the theory of generalized Jack polynomials due to Griffeth, and the Yang–Baxter graph approach of Luque and the author.

Keywords: Calogero–Sutherland model on torus, generalized Jack polynomials, representations of the symmetric group

1. Introduction

The quantum Calogero–Sutherland model for \( N \) identical particles with \( 1/r^2 \) interactions on the unit circle has the Hamiltonian

\[
\mathcal{H} = -\sum_{i=1}^{N} \left( \frac{\partial}{\partial \theta_i} \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq N} \frac{\kappa (\kappa - 1)}{\sin^2 \left( \frac{1}{2} (\theta_i - \theta_j) \right)}
\]

\[
= \sum_{i=1}^{N} \left( x_i \frac{\partial}{\partial x_i} \right)^2 - 2\kappa \sum_{1 \leq i < j \leq N} \frac{x_i x_j (\kappa - 1)}{(x_i - x_j)^2},
\]

where \( x_j = e^{i \theta_j} \) and \(-\pi < \theta_j \leq \pi\) for \( 1 \leq j \leq N \). The time-independent Schrödinger equation \( \mathcal{H} \psi = E \psi \) has solutions expressible as the product of the base-state.
\[ \psi_0(x) = \prod_{1 \leq i < j \leq N} \left( -\frac{(x_i - x_j)^2}{x_i x_j} \right)^{\kappa/2} \]

with a Jack polynomial, Lapointe and Vinet 1996, Awata 1997. The base-state is a solution of the first-order linear differential system

\[ \frac{\partial}{\partial x_i} \psi_0(x) = \kappa \psi_0(x) \left\{ \sum_{j \neq i} \frac{1}{x_i - x_j} - \frac{N - 1}{2x_i} \right\}, 1 \leq i \leq N; \]

and \( \mathcal{H} \psi_0 = \frac{1}{\kappa^2} \kappa^2 N (N^2 - 1) \psi_0 \). The theory of Jack polynomials has been generalized to polynomials taking values in modules of the symmetric group, Griffeth 2010. In this paper the Hamiltonian \( \mathcal{H} \) will be interpreted in that context. The base state \( \psi_0 \) is replaced by a matrix function satisfying an analogous differential system and the generalized wavefunctions are vector-valued. Nevertheless for an interval of parameter values depending on the module the wavefunctions do give rise to symmetric probability density functions on the torus. The interval is symmetric about \( \kappa = 0 \) hence this is qualitatively different from the usual scalar case where \( \kappa \) is unbounded above.

Section 2 is a brief overview of representation theory for the symmetric groups, and the commutative set of operators on polynomials of which the nonsymmetric Jack polynomials are simultaneous eigenfunctions. Section 3 concerns the first-order linear differential system defining the basic matrix function needed to map the polynomials to eigenfunctions of the Hamiltonian modified with twisted exchange operators. In section 4 there is a description of the Hermitian form related to integration of vector-valued polynomials on the torus, and the Yang–Baxter graph technique for constructing the nonsymmetric Jack polynomials. Section 5 presents the adaptation of the method of Baker and Forrester 1999 to form symmetric Jack polynomials; the analysis involves tableaux with certain properties. Also this section contains the formulae for the squared norms of the Jack polynomials. Then section 6 uses the vector-valued Jack polynomials and the matrix function from section 3 to construct vector-valued eigenfunctions of the Hamiltonian \( \mathcal{H} \) and the associated probability density. Also the Jack polynomial of minimal degree is described, and finally there is a brief description of the matrix function in the case of the two-dimensional representation of \( \mathcal{S}_4 \).

2. The generalized Jack polynomials and associated operators

The symmetric group \( \mathcal{S}_N \), the set of permutations of \( \{1, 2, \ldots, N\} \), acts on \( \mathbb{C}^N \) by permutation of coordinates. For \( \alpha \in \mathbb{Z}^N \) the norm is \( |\alpha| := \sum_{i=1}^N |\alpha_i| \) and the monomial is \( x^\alpha := \prod_{i=1}^N x_i^{\alpha_i} \). Denote \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \). The space of polynomials \( \mathcal{P} := \text{span}_{\mathbb{C}} \{ x^\alpha : \alpha \in \mathbb{N}_0^N \} \). Elements of \( \text{span}_{\mathbb{C}} \{ x^\alpha : \alpha \in \mathbb{Z}^N \} \) are called Laurent polynomials. The action of \( \mathcal{S}_N \) is extended to polynomials by \( w p(x) = p(xw) \) where \( (xw)_i = x_{w(i)} \) (consider \( x \) as a row vector and \( w \) as a permutation matrix, \( [w]_{ij} = \delta_{w(i),j} \), then \( xw = x[w] \)). This is a representation of \( \mathcal{S}_N \), that is, \( w_1 (w_2 p) (x) = (w_2 p) (xw_1) = p (xw_1 w_2) = (w_1 w_2) p(x) \) for all \( w_1, w_2 \in \mathcal{S}_N \).

Furthermore \( \mathcal{S}_N \) is generated by reflections in the mirrors \( \{x : x_i = x_j\} \) for \( 1 \leq i < j \leq N \). These are transpositions, denoted by \( (i,j) \), interchanging \( x_i \) and \( x_j \). Define the \( \mathcal{S}_N \)-action on \( \alpha \in \mathbb{Z}^N \) so that \( (xw)^\alpha = x^{\alpha \circ w} \)

\[
(xw)^\alpha = \prod_{i=1}^N x_{w(i)}^{\alpha_i} = \prod_{j=1}^N x_j^{\alpha_{w^{-1}(j)}}.
\]
that is \((\omega w) \alpha = c_{\omega w} \alpha\) (consider \(\alpha\) as a column vector, then \(w \alpha = [w] \alpha\)).

The simple reflections \(s_i := (i,i+1), 1 \leq i < N\), suffice to generate \(S_N\). They are the key devices for applying inductive methods, and satisfy the braid relations:

\[
   s_i s_j = s_j s_i \text{ if } |i-j| > 2,
   s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.
\]

We consider the situation where the group \(S_N\) acts on the range as well as on the domain of the polynomials. We use vector spaces, called \(S_N\)-modules, on which \(S_N\) has an irreducible unitary (orthogonal) representation: \(\tau : S_N \to O_m(\mathbb{R})\) \((\tau(w)^{-1} = \tau(w)^{t})\).

See James and Kerber 1981 for representation theory and a modern discussion of Young’s methods.

Denote the set of partitions \(N^+_0 = \{\lambda \in \mathbb{N}^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}\). We identify \(\tau\) with a partition of \(N\) given the same label, that is \(\tau \in N^+_0\) and \(|\tau| = N\). Thus \(\tau = (\tau_1, \tau_2, \ldots)\) and often the trailing zero entries are dropped when writing \(\tau\). The length of \(\tau\) is \(\ell(\tau) = \max \{i : \tau_i > 0\}\).

There is a Ferrers diagram of shape \(\tau\) (also given the same label), with boxes at points \((i,j)\) with \(1 \leq i \leq \ell(\tau)\) and \(1 \leq j \leq \tau_i\). A tableau of shape \(\tau\) is a filling of the boxes with numbers, and a reverse standard Young tableau (RSYT) is a filling with the numbers \(\{1, 2, \ldots, N\}\) so that the entries decrease in each row and each column. We exclude the one-dimensional representations corresponding to one-row \((N)\) or one-column \((1, 1, \ldots, 1)\) partitions (the trivial and determinant representations, respectively). The hook-length of the node \((i,j)\) is \(\tau\) is

\[
   h(i,j) := \tau_i - j + \# \{k : j \leq \tau_k, i < k \leq \ell(\tau)\} + 1,
   h_\tau := h(1,1) = \tau_1 + \ell(\tau) - 1
\]

and \(h_\tau\) is the maximum hook-length of \(\tau\). Denote the set of RSYT’s of shape \(\tau\) by \(\mathcal{Y}(\tau)\) and let \(V_\tau = \text{span}_\mathbb{C} \{T : T \in \mathcal{Y}(\tau)\}\) with orthogonal basis \(\mathcal{Y}(\tau)\). The hook-length formula is \(\# \mathcal{Y}(\tau) = N!/\prod_{(i,j) \in \tau} h(i,j)\). Set \(n_\tau := \dim V_\tau = \# \mathcal{Y}(\tau)\). For \(1 \leq i \leq N\) and \(T \in \mathcal{Y}(\tau)\) the entry \(i\) is at coordinates \((rw(i,T), cm(i,T))\) and the content of the entry is \(c(i,T) := cm(i,T) - rw(i,T)\). Each \(T \in \mathcal{Y}(\tau)\) is uniquely determined by its content vector \([c(i,T)]_{i=1}^N\). For example let \(\tau = (4,3)\) and \(T = \begin{bmatrix} 7 & 6 & 5 & 2 \\ 4 & 3 & 1 \end{bmatrix}\) then the content vector is \([1, 3, 0, -1, 2, 1, 0]\).

The representation is described by explicit formulae for \(\tau(s_i)\) stated in definition 1 below; the matrix for \(\tau(s_i)\) with respect to the basis \(\mathcal{Y}(\tau)\) has a block decomposition of \(1 \times 1\) and \(2 \times 2\) blocks. The \(2 \times 2\) blocks come from tableaux in which the entries \(i\) and \(i+1\) can be interchanged to produce another RSYT; this is possible whenever \(i \) and \(i+1\) are not in the same row or in the same column. This occurs when \(|c(i,T) - c(i+1,T)| \geq 2\). As example consider the 3-dimensional representation \((3,1)\) of \(S_4\); the elements of \(\mathcal{Y}(\tau)\) are

\[
   T_0 = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 1 \end{bmatrix}, T_1 = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 1 & 2 \\ 1 \end{bmatrix}, T_2 = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 2 \\ 1 \end{bmatrix},
\]

then \(\tau(s_3)T_0 = -T_0\), \(\tau(s_1)T_0 = T_0\), and \(\tau(s_2)T_0 = T_1 = \frac{1}{2}T_0 + \frac{1}{2}T_2\), \(\tau(s_2)T_2 = T_2\).

**Definition 1.** Suppose \(T \in \mathcal{Y}(\tau)\) and \(1 \leq i < N\). If \(c(i,T) = c(i+1,T) + 1\) then \(\tau(s_i)T = T\); if \(c(i,T) = c(i+1,T) - 1\) then \(\tau(s_i)T = -T\). If \(c(i,T) - c(i+1,T) \geq 2\) let \(T^{(i)}\) be \(T\) with \(i,i+1\) interchanged and set \(b = \frac{1}{(c(i,T) - c(i+1,T))}\) then \(\tau(s_i)T = T^{(i)} + bT\) if \(b > 0\) and \(\tau(s_i)T = (1-b^2)T^{(i)} + bT\) if \(b < 0\).
Let $S_1(\tau) := \sum_{i=1}^{N} c(i, T)$ (this sum depends only on $\tau$) and $\gamma := S_1(\tau) / N$. The $S_N$-invariant inner product on $V_\tau$ is defined by

$$
\langle T, T' \rangle_0 := \delta_{T,T'} \times \prod_{i \in \mathbb{N}, (i,j) \in (\tau)_{i-2}} \left( 1 - \frac{1}{(c(i,T) - c(j,T))^2} \right), \quad T, T' \in \mathcal{Y}(\tau).
$$

(2)

It is unique up to multiplication by a constant.

The Jucys–Murphy elements $\omega_i := \sum_{j \neq 1}^{N} \tau(i,j)$ satisfy $\sum_{j \neq 1}^{N} \tau(i,j) T = c(i,T) T$ and thus the central element $\sum_{i \leq j \leq N} \tau(i,j)$ (in the group algebra $\mathbb{R}[S_N]$) satisfies $\sum_{i \leq j \leq N} \tau(i,j) T = S_1(\tau) T$

for each $T \in \mathcal{Y}(\tau)$. We abbreviate $\tau((i,j))$ to $\tau(i,j)$.

The generalized Jack polynomials are elements of $\mathcal{P}_\tau = \mathcal{P} \otimes V_\tau$, the space of $V_\tau$-valued polynomials, which is equipped with the $S_N$ action:

$$
w(x^\alpha \otimes T) = (xw)^\alpha \otimes \tau(w) T, \quad \alpha \in \mathbb{N}_0^N, T \in \mathcal{Y}(\tau),
$$

$$wp(x) = \tau(w)p(xw), \quad p \in \mathcal{P}_\tau,$

extended by linearity. A symmetric polynomial $p$ satisfies $wp = p$, that is, $p(xw) = \tau(w)^{-1} p(x)$ for all $w \in S_N$. The following describes the transformation rules for $\tau(s_i)$ acting on symmetric polynomials.

**Proposition 2.** Suppose $p \in \mathcal{P}_\tau$ is symmetric, $1 \leq i < N$, and $T \in \mathcal{Y}(\tau)$. Express $p(x) = \sum_{T' \in \mathcal{Y}(\tau)} \langle T', T \rangle_0 p_{T'}(x) \otimes T'$. If $c(i,T) = c(i+1,T) + 1$ then $s_{p_{T'}} = p_{T'}$; if $c(i,T) = c(i+1,T) - 1$ then $s_{p_{T'}} = -p_{T'}$; if $c(i,T) - c(i+1,T) \geq 2$ and $T^{(i)}$ is $T$ with $i, i+1$ interchanged then $s_{p_{T'}} = p_{T^{(i)}} + b_{p_{T'}}$, where $b = \frac{1}{c(i,T) - c(i+1,T)}$.

**Proof.** The transformation properties of $p_{T}$ follow from $s_i (p_{T}(x) \otimes T) = p_{T^{(i)}} (xs_i) \otimes \tau(s_i) T$. The first case is when $r \tau(i,T) = r \tau(i+1,T)$ and $\tau(s_i) T = T$; the second case is when $\gamma \tau(i,T) = \gamma (i+1,T)$ and $\tau(s_i) T = T$. The in the case $c(i,T) - c(i+1,T) \geq 2$ the relations $\tau(s_i) T^{(i)} = (1 - b^2) T - b T^{(i)}$ and $\langle T^{(i)}, T^{(i)} \rangle_0 = (1 - b^2) \langle T, T \rangle_0$ hold. By hypothesis

$$
\frac{1}{\langle T, T \rangle_0} p_{T}(x) \otimes T + \frac{1}{\langle T^{(i)}, T^{(i)} \rangle_0} p_{T^{(i)}}(x) \otimes T^{(i)}
$$

$$
= \frac{1}{\langle T, T \rangle_0} p_{T^{(i)}} (xs_i) \otimes \tau(s_i) T + \frac{1}{\langle T^{(i)}, T^{(i)} \rangle_0} p_{T^{(i)}} (xs_i) \otimes \tau(s_i) T^{(i)}
$$

$$
= \frac{1}{\langle T, T \rangle_0} \left\{ p_{T^{(i)}} (xs_i) \otimes T^{(i)} + b p_{T^{(i)}} (xs_i) \otimes T^{(i)} + \left( (1 - b^2) T - b T^{(i)} \right) \right\}
$$

$$
= \frac{1}{\langle T, T \rangle_0} \left\{ (bp_{T^{(i)}} (xs_i) + p_{T^{(i)}} (xs_i)) \otimes T + \left( (1 - b^2) p_{T^{(i)}} (xs_i) - bp_{T^{(i)}} (xs_i) \right) \otimes T^{(i)} \right\}.
$$

Replace $x$ by $xs_i$ and conclude that $s_{p_{T}} = p_{T^{(i)}} + bp_{T^{(i)}}$ and $s_{p_{T^{(i)}}} = (1 - b^2) p_{T^{(i)}} - bp_{T^{(i)}}$. 

The linear space span $\{ p_{T} : T \in \mathcal{Y}(\tau) \}$ is invariant and irreducible under the action of $S_N$ and is called an $S_N$-module of isotype $\tau$. More information about such modules is found in section 5.1. The polynomials $p_{T}$ can be derived from $p_{T_0}$ where $T_0$ is the root RSYT (with
$N, N-1, \ldots$ entered column by column), but determining which polynomials can serve as $p_{\lambda}$ is nontrivial in general.

There is a parameter $\kappa \in \mathbb{R}$ (in general, $\kappa$ could be transcendental).

**Definition 3.** The Dunkl and Cherednik-Dunkl operators are $(1 \leq i \leq N, p \in \mathcal{P}_\tau)$$$
\mathcal{D}_{\lambda} p(x) := \frac{\partial}{\partial x_i} p(x) + \kappa \sum_{j \neq i} \tau(i,j) \frac{p(x) - p(x_{ij})}{x_i - x_j},$$$

\mathcal{U}_{\lambda} p(x) := \mathcal{D}_{\lambda} (s_i p(x)) - \kappa \sum_{j=1}^{i-1} \tau(i,j) p(x_{ij}).$$(4)

The commutation relations analogous to the scalar case hold:

$$\mathcal{D}_{\lambda} \mathcal{D}_{\lambda'} = \mathcal{D}_{\lambda'} \mathcal{D}_{\lambda}, \mathcal{U}_{\lambda} \mathcal{U}_{\lambda'} = \mathcal{U}_{\lambda'} \mathcal{U}_{\lambda}, \quad 1 \leq i, j \leq N,$$

$$w \mathcal{D}_{\lambda} = \mathcal{D}_{\omega(w) \lambda}, \forall w \in S_N; \quad s_i \mathcal{U}_{\lambda} = \mathcal{U}_{\lambda} s_i, \quad j \neq i-1, j;$$

$$s_i \mathcal{U}_{\lambda} = \mathcal{U}_{\lambda+1} + \kappa s_i, \quad \mathcal{U}_{\lambda} s_i = s_i \mathcal{U}_{\lambda+1} + \kappa, \quad \mathcal{U}_{\lambda+1} s_i = s_i \mathcal{U}_{\lambda} - \kappa.$$ (3)

The commutation properties for the $\mathcal{U}_i$ and $s_i$ are derived as follows:

1. if $j > i$ then $s_j$ commutes with each term in $\mathcal{U}_i$,
2. if $j < i - 1$ then $\mathcal{D}_{\lambda} x_j s_i = -\kappa \sum_{k < j} (k,i) s_j = s_i \mathcal{D}_{\lambda} x_j = -\kappa \sum_{k < j} (i,j) s_j$ because $(k,i) s_j = s_j (k,i)$ unless $k = j$ or $j + 1$ and then $\{i,j\} + (i,j+1) \} s_j = s_j \{(i,j+1) + (i,j)\}$, 
3. if $j = i$ then $s_i \mathcal{U}_{\lambda} = \mathcal{D}_{\lambda+1} + \kappa s_i, \mathcal{U}_{\lambda} s_i = s_i \mathcal{U}_{\lambda+1} + \kappa, \mathcal{U}_{\lambda+1} s_i = s_i \mathcal{U}_{\lambda} - \kappa$ follow from right and left multiplication by $s_i$.

From the commutation $\mathcal{D}_{\lambda} x_i - x_i \mathcal{D}_{\lambda} = 1 + \kappa \sum_{j \neq i} (i,j)$ we obtain

$$\mathcal{U}_i = \mathcal{D}_{\lambda} x_i - x_i \mathcal{D}_{\lambda} = 1 + \kappa \sum_{j \neq i} (i,j) = x_i \mathcal{D}_{\lambda} + 1 + \kappa \omega_i.$$ (4)

**Proposition 4.** If $q(x_1, x_2, \ldots, x_N)$ is a symmetric polynomial then $q(\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_N)$ commutes with each $w \in S_N$, as an operator on $\mathcal{P}_\tau$.

**Proof.** It suffices to prove the commutativity for each $s_i$ with $1 \leq i \leq N$ and each elementary symmetric polynomial in $\{\mathcal{U}_i\}$, that is, for $\prod_{j=1}^{N} (1 + \mathcal{U}_j)$. By the above formulae it suffices to show $s_i$ commutes with $(1 + \mathcal{U}_i)(1 + \mathcal{U}_{i+1}) = 1 + t (\mathcal{U}_i + \mathcal{U}_{i+1}) + t^2 \mathcal{U}_i \mathcal{U}_{i+1}$. Indeed

$$s_i (\mathcal{U}_i + \mathcal{U}_{i+1}) s_j = \mathcal{U}_{i+1} + \kappa s_i + \mathcal{U}_i - \kappa s_j = \mathcal{U}_{i+1} + \mathcal{U}_i,$$

$$s_i \mathcal{U}_i s_j = (\mathcal{U}_{i+1} s_j + \kappa) (\mathcal{U}_i - \kappa) = \mathcal{U}_{i+1} \mathcal{U}_i + \kappa s_j s_i - \kappa s_j = \mathcal{U}_{i+1} \mathcal{U}_i.$$

The nonsymmetric (vector-valued) Jack polynomials (NSJP) are defined to be simultaneous eigenfunctions of the commuting set $\{\mathcal{U}_i : 1 \leq i \leq N\}$. The symmetric vector-valued Jack polynomials are simultaneous eigenfunctions of the symmetric polynomials in $\{\mathcal{U}_i\}$. If $p$ is a NSJP then the sum $\sum_{w \in S_N} w p$ is either a scalar multiple of a symmetric Jack polynomial or zero.

The details are presented in section 5.
3. The matrix analogue of the base state

This is a summary of the pertinent results from Dunkl 2017. Vectors and matrices throughout are of size \( n \times n \) and \( n \times n \times n \) and are expressed with respect to the orthonormal basis \( \{ T, T \}_{0}^{-1/2} T : T \in \mathcal{Y}(\tau) \}. \) With \( I \) denoting the identity matrix and \( \partial_{i} := \frac{\partial}{\partial x_{i}} \) for \( 1 \leq i \leq N \) the differential system for the matrix function \( L \) is

\[
\partial_{i} L(x) = \kappa L(x) \left\{ \sum_{j \neq i} \frac{1}{x_{i} - x_{j}} \tau(i,j) - \frac{\gamma}{x_{i}} \right\}, \quad 1 \leq i \leq N,
\]

\[
\gamma := \frac{S_{1}(\tau)}{N} = \frac{1}{2N} \sum_{i=1}^{\ell(\tau)} (\tau_{i} - 2i + 1).
\]

The effect of the term \( \frac{\gamma}{x_{i}} \) is to make \( L(x) \) homogeneous of degree zero, that is, \( \sum_{i=1}^{N} x_{i} \partial_{i} L(x) = 0 \). The differential system is defined on \( C_{\text{reg}}^{N} := C_{\times}^{N} \setminus \bigcup_{1 \leq i < j \leq N} \{ x : x_{i} = x_{j} \} \) (where \( C_{\times} := C \setminus \{ 0 \} \)), and it is Frobenius integrable and analytic, thus any local solution can be continued analytically to any point in \( C_{\text{reg}}^{N} \). The equation is a modified version of the Knizhnik-Zamolodchikov equation. For the trivial representation \( \tau = (N) \) the solution \( L(x) = \psi_{0}(x) \) (up to scalar multiplication) because \( \tau(i,j) = I \) and \( \gamma = \frac{N-1}{2} \). The notations for the torus and its surface measure in terms of polar coordinates are

\[
\mathbb{T}^{\mathbb{N}} := \{ x \in C^{N} : |x_{j}| = 1, 1 \leq j \leq N \},
\]

\[
dm(x) = (2\pi)^{-N} d\theta_{1} \cdots d\theta_{N}, \quad x_{j} = \exp(i\theta_{j}), -\pi < \theta_{j} \leq \pi, 1 \leq j \leq N.
\]

Let \( \mathbb{T}_{\text{reg}}^{\mathbb{N}} := \mathbb{T}^{\mathbb{N}} \cap C_{\text{reg}}^{N} \), then \( \mathbb{T}_{\text{reg}}^{\mathbb{N}} \) has \( (N-1)! \) connected components and each component is homotopic to a circle; if \( x \) is in some component then so is \( ux = (ux_{1}, \ldots, ux_{N}) \) for each \( u \in \mathbb{T} \).

**Definition 5.** Let \( x_{0} := (1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}) \) and denote the connected component of \( \mathbb{T}_{\text{reg}}^{\mathbb{N}} \) containing \( x_{0} \) by \( C_{0} \), called the fundamental chamber.

Thus \( C_{0} \) is the set consisting of \( \{ e^{\theta_{1}}, \ldots, e^{\theta_{N}} \} \) with \( \theta_{1} < \theta_{2} < \cdots < \theta_{N} < \theta_{1} + 2\pi \). The homogeneity \( L(ux) = L(x) \) for \( |u| = 1 \) shows that \( L(x) \) has a well-defined analytic continuation to all of \( C_{0} \) starting from \( x_{0} \). Let \( w_{0} := (1, 2, 3, \ldots, N) = (12)(34) \cdots (N-1, N) \), an \( N \)-cycle, and let \( \{ w_{0} \} \) denote the cyclic group generated by \( w_{0} \). There are two components of \( \mathbb{T}_{\text{reg}}^{\mathbb{N}} \) which are set-wise invariant under \( \{ w_{0} \} \) namely \( C_{0} \) and the reverse \( \{ \theta_{N} < \theta_{N-1} < \cdots < \theta_{1} < \theta_{N} + 2\pi \} \). Indeed \( \{ w_{0} \} \) is the stabilizer of \( C_{0} \) as a subgroup of \( \mathcal{S}_{N} \). A list of properties of \( L(x) \) (from Dunkl 2017):

1. If \( L(x) \) is a solution of (5) in some connected open subset \( U \) of \( C_{\text{reg}}^{N} \), then \( L(xw)\tau(w)^{-1} \) is a solution in \( Uw^{-1} \).
2. If \( L(x_{0}) \) is nonsingular then \( L \) is nonsingular on all of \( C_{0} \); this follows from

\[
\det L(x) = c \prod_{1 \leq i < j \leq N} \left( \frac{(x_{i} - x_{j})^{2}}{x_{i}x_{j}} \right)^{\lambda/2}, \quad \lambda := \frac{\gamma n \tau}{2(N-1)} = \tr(\tau(1,2)),
\]

3. Suppose \( L(x) \) is normalized by \( L(x_{0}) = I \) then \( L(xw_{0}^{m}) = \tau(w_{0})^{-m} L(x) \tau(w_{0})^{m} \) for all \( x \in C_{0} \) and \( m \in \mathbb{Z} \).
For \( w \in S_N \) and for \( x \in T_N^{\text{reg}} \) define \( w_i \in S_N \) such that \( xw_i^{-1} \in C_0 \) and \( w_i(1) = 1 \); then \( w_i \) is uniquely defined and is constant on connected components. Then define \( L(x) \) on the other connected components of \( T_N^{\text{reg}} \) by

\[
L(x) := L(xw_i^{-1}) \tau(w_i).
\]

In order to derive a formula for the relation of \( L(xw)\tau(w)^{-1} \) to \( L(x) \) we need a twist: for \( w \in S_N \) and for \( x \in T_N^{\text{reg}} \) define

\[
M(w, x) := \tau(w_0)^{1-w(w(1))}.
\]

Henceforth the assumption \( L(x_0) = I \) is relaxed to \( L(x_0) \) commuting with \( \tau(w_0) \) and being nonsingular (so \( L(xw_0^m) = \tau(w_0)^{-m} L(x) \tau(w_0)^m \) still holds for \( x \in C_0 \)). Then \( M(w, x) \) and \( L(x) \) have the following properties \( (x \in T_N^{\text{reg}}) \):

\[
M(I, x) = I;  \\
M(w_1w_2, x) = M(w_2, xw_1) M(w_1, x), \quad w_1, w_2 \in S_N,  \\
L(xw) = M(w, x)L(x)\tau(w), \quad w \in S_N
\]

With the goal of analyzing vector functions of the form \( f(x) = L(x)p(x) \) where \( p \in P_{\tau} \) consider

\[
L(x)wp(x) = L(x)\tau(w)p(xw) = L(x)\tau(w)L(xw)^{-1}f(xw)  \\
= L(x)\tau(w)\{M(w, x)L(x)\tau(w)\}^{-1}f(xw)  \\
= M(w, x)^{-1}f(xw),
\]

in other words \( L(x)wL(x)^{-1}f(x) = M(w, x)^{-1}f(xw) \). Accordingly define a twisted action of \( S_N \) on vector-valued functions \( f(x) \) (defined on \( T_N^{\text{reg}} \)) by

\[
\sigma^M(w)f(x) = M(w, x)^{-1}f(xw).
\]

**Proposition 6.** Suppose \( w \in S_N \) then \( L(x)wL(x)^{-1} = \sigma^M(w) \) and \( \sigma^M \) is a representation of \( S_N \).

**Proof.** Using formula (7) let \( g(x) = \sigma^M(w_2)f(x) = M(w_2, x)^{-1}f(xw_2) \), then

\[
\sigma^M(w_1)\sigma^M(w_2)f(x) = \sigma^M(w_1)g(x) = M(w_1, x)^{-1}g(xw_1)  \\
= M(w_1, x)^{-1}M(w_2, xw_1)^{-1}f(xw_1w_2)  \\
= M(w_1w_2, x)^{-1}f(xw_1w_2) = \sigma^M(w_1w_2)f(x).
\]

If \( p(x) \) is symmetric \( (\tau(w)p(xw) = p(x)) \) then by formula (7)

\[
\sigma^M(w)L(x)p(x) = M(w, x)^{-1}L(xw)p(xw)  \\
= M(w, x)^{-1}\{M(w, x)L(x)\tau(w)\}p(xw)  \\
= L(x)\tau(w)p(xw) = L(x)p(x).
\]

This is crucial in the sequel where the operator \( L(x) \sum_{i=1}^N (\mathcal{H}_i - 1 - \kappa \gamma)^2 L(x)^{-1} \) is related to the Hamiltonian \( \mathcal{H} \).
**Proposition 7.** For $1 \leq i \leq N$

\[
L(x) (U_i - 1 - \kappa \gamma) L(x)^{-1} = x_i \partial_i - \kappa \sum_{j < i} \frac{x_i}{x_i - x_j} \sigma^M (i, j) - \kappa \sum_{j > i} \frac{x_j}{x_i - x_j} \sigma^M (i, j).
\] (9)

**Proof.** Write the differential system as $\partial_i L(x) = \kappa L(x) A_i (x)$ then

\[
0 = \partial_i (L^{-1} L) = (\partial_i L^{-1}) L + L^{-1} \partial_i L = (\partial_i L^{-1}) L + \kappa L^{-1} L A_i,
\]

thus $\partial_i L(x)^{-1} = -\kappa A_i (x) L(x)^{-1}$. Next by formula (8) $L(x) (i, j) L(x)^{-1} = \sigma^M (i, j)$. For the other term in $U_i = x_i D_i + 1 + \kappa \omega_i$ (formula (4)) we obtain $L(x) \omega_i L(x)^{-1} = \sum_{j>i} \sigma^M (i, j)$.

Consider

\[
D_i L(x)^{-1} f(x) = \left( \partial_i L(x)^{-1} f(x) + L(x)^{-1} \partial_i f(x) \right)
\]

\[
\begin{aligned}
+ \kappa \sum_{j \neq i} \frac{\tau (i, j)}{x_i - x_j} \left\{ L(x)^{-1} f(x) - L(x (i, j))^{-1} f (x (i, j)) \right\} \\
= -\kappa \left\{ \sum_{j \neq i} \frac{\tau (i, j)}{x_i - x_j} \frac{\gamma_f}{x_i} \right\} L(x)^{-1} f(x) + L(x)^{-1} \partial_i f(x) \\
+ \kappa \sum_{j \neq i} \frac{\tau (i, j)}{x_i - x_j} \left\{ L(x)^{-1} f(x) - \tau (i, j)^{-1} L(x)^{-1} M ((i, j), x)^{-1} f (x (i, j)) \right\}
\end{aligned}
\]

\[
= L(x)^{-1} \left\{ \frac{\kappa \gamma_f}{x_i} f(x) + \partial_i f(x) - \kappa \sum_{j \neq i} \frac{1}{x_i - x_j} \sigma^M (i, j) f(x) \right\}.
\]

Thus

\[
L(x) \left\{ x_i D_i + 1 + \kappa \omega_i \right\} L(x)^{-1} - 1 - \kappa \gamma
\]

\[
= x_i \partial_i - \kappa \sum_{j \neq i} \frac{x_i}{x_i - x_j} \sigma^M (i, j) + \kappa \sum_{j > i} \sigma^M (i, j)
\]

\[
= x_i \partial_i - \kappa \sum_{j < i} \frac{x_i}{x_i - x_j} \sigma^M (i, j) - \kappa \sum_{j > i} \frac{x_j}{x_i - x_j} \sigma^M (i, j).
\]

We will use an elementary double sum formula: suppose $g (i, j)$ is a function defined on all pairs $(i, j)$ with $1 \leq i, j \leq N$ then

\[
\sum_{i=1, j=1}^N g (i, j) = \sum_{1 \leq i < j \leq N} \{ g (i, j) + g (j, i) \}.
\] (10)

**Corollary 8.** $\sum_{i=1}^N U_i = \sum_{i=1}^N x_i \partial_i + N + \kappa S_1$.

**Proof.** From formula (9)

\[
L(x) \sum_{i=1}^N (U_i - 1 - \kappa \gamma) L(x)^{-1} = \sum_{i=1}^N x_i \partial_i - \kappa \sum_{i=1}^N \sum_{j \neq i} \frac{x_i^{\max (i, j)}}{x_i - x_j} \sigma^M (i, j).
\]
The double sum is of the form (10) and for $i \neq j$ one obtains $g(i,j) + g(j,i) = \frac{\kappa - \gamma}{(x_i - x_j)^2} \sigma^M(i,j) + \frac{\gamma}{(x_i - x_j)^2} \sigma^M(j,i) = 0$. From $\sum_{i=1}^N x_i \partial_i L(x) = 0$ it follows that $\sum_{i=1}^N x_i \partial_i$ commutes with $L(x)$ and together with $\gamma = S_1(\tau) / N$ completes the proof to get the stated formula.

**Lemma 9.** For $1 \leq i \leq N$

$$L(x) \left(U_\gamma - 1 - \kappa \gamma\right)^2 L(x)^{-1} = \left(x_i \partial_i\right)^2 - \kappa \sum_{j \neq i} \frac{x_i x_j}{(x_i - x_j)^2} \left(\kappa - \sigma^M(i,j)\right)$$

$$- \kappa \sum_{j > i} \left\{ \frac{x_j^2}{x_i - x_j} \sigma^M(i,j) \partial f + \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j) \partial f \right\}$$

$$- \kappa \sum_{j < i} \left\{ \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j) \partial f + \frac{x_i^2}{x_i - x_j} \sigma^M(i,j) \partial f \right\}$$

$$+ \kappa^2 \sum_{\# \{i,j,k\} = 3} \frac{x_{\max(i,j)}}{x_i - x_j} \sigma^M(i,j) \frac{x_{\max(j,k)}}{x_j - x_k} \sigma^M(i,k).$$

**Proof.** In the square of formula (9) group the terms as

$$- \kappa x_i \partial_i \sum_{j \neq i} \frac{x_{\max(i,j)}}{x_i - x_j} \sigma^M(i,j) - \kappa \sum_{j \neq i} \frac{x_{\max(i,j)}}{x_i - x_j} \sigma^M(i,j) x_i \partial_i$$

$$= \kappa \sum_{j \neq i} \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j) - \kappa \sum_{j \neq i} \frac{x_{\max(i,j)}}{x_i - x_j} \left\{ x_j \sigma^M(i,j) \partial_j + x_j \sigma^M(i,j) \partial_i \right\},$$

because

$$\partial_i \left(\sigma^M(i,j) f(x) = M((i,j), x)^{-1} \partial_j f(x(i,j)) = M((i,j), x)^{-1} (\partial_j f)(x(i,j)) \right)$$

$$= \sigma^M(i,j) (\partial_j f)(x),$$

and $M(w,x)$ is locally constant in $x$. Next consider

$$\kappa^2 \sum_{j \neq i} \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j) - \kappa \sum_{j \neq i} \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j) + \kappa^2 \sum_{j > i} \frac{x_j}{x_i - x_j} \sigma^M(i,j) \frac{x_j}{x_i - x_j} \sigma^M(i,j)$$

$$= - \kappa^2 \sum_{j < i} \frac{x_j x_i}{x_i - x_j} \sigma^M(i,j)^2 - \kappa^2 \sum_{j < i} \frac{x_j x_i}{(x_i - x_j)^2} \sigma^M(i,j)^2 = - \kappa^2 \sum_{j < i} \frac{x_j x_i}{(x_i - x_j)^2},$$

because $\sigma^M(i,j) \frac{x_j}{x_i - x_j} = \frac{x_j}{x_i - x_j} \sigma^M(i,j)$ and $\sigma^M(i,j)^2 = I$ (by proposition 6).

More detailed analysis of the terms in line (14) shows that there are four different coefficients of $\frac{\kappa^2}{(x_i - x_j)^2} \sigma^M((i,j),(i,k))$ depending on the numerical order of $i,j,k$:

1. $x_j x_i$ if $j < k < i$ or $k < j < i$,
2. $x_j x_k$ if $j < i < k$,
3. $x_i x_k$ if $i < j < k$ or $i < k < j$,
4. $x_j^2$ if $k < i < j$.

The next step is to sum over $1 \leq i \leq N$. Lines (12) and (13) sum to zero by using formula (10). We show that all the terms in line (14) sum to zero. This is a sum over all cycles of order 3.
Any 3-cycle is of the form \((a, b, c)\) with \(1 \leq a < b < c \leq N\) or \(1 \leq a < c < b \leq N\). Each 3-cycle appears three times in the sum since
\[
(a, c) (a, b) = (b, a) (b, c) = (c, b) (c, a) = (a, b, c).
\]
If \(a < b < c\) then by the above formulæ the coefficient of \(\kappa^2 \sigma^M ((a, b, c))\) is
\[
\frac{x_{b,c}}{(x_a - x_c) (x_c - x_b)} + \frac{x_{c,a}}{(x_b - x_a) (x_a - x_b)} + \frac{x_{a,b}}{(x_c - x_b) (x_b - x_c)} = 0.
\]
If \(a < c < b\) then the coefficient of \(\kappa^2 \sigma^M ((a, b, c))\) is
\[
\frac{x_{a,c}}{(x_b - x_c) (x_c - x_a)} + \frac{x_{c,b}}{(x_a - x_b) (x_b - x_a)} + \frac{x_{b,a}}{(x_c - x_a) (x_a - x_c)} = 0.
\]
Each pair \(\{i,j\}\) appears twice in the sum of the terms in (11). We have proven the following:

**Theorem 10.** The \(L(x)\) conjugate of \(\sum_{i=1}^N (U_i - 1 - \kappa \gamma)^2\) is
\[
\mathcal{H}_M := L(x) \sum_{i=1}^N (U_i - 1 - \kappa \gamma)^2 L(x)^{-1}
\]
\[
= \sum_{i=1}^N (x_i \partial_i)^2 - 2\kappa \sum_{1 \leq i < j \leq N} \frac{x_i x_j}{(x_i - x_j)^2} (\kappa - \sigma^M (i, j)).
\]
Thus \(\mathcal{H}_M\) agrees with \(\mathcal{H}\) when applied to \(L(x)p(x)\) where \(p\) is a symmetric \((\tau(w)p(xw) = p(x))\) polynomial. By proposition 4 the operators \(L(x) \sum_{i=1}^N U_i L(x)^{-1}\) commute with \(\mathcal{H}_M\) and with \(\sigma^M(w)\) for \(w \in S_N\), for \(m = 1, 2, 3, \ldots\) and \(L(x)p(x)\) is an eigenfunction of \(\mathcal{H}_M\) for any NSJP \(p(x)\).

**4. Hermitian forms and nonsymmetric Jack polynomials**

The results in this section come from Dunkl 2017, Dunkl and Luque 2011, Griffeth 2010. These papers rely on induction arguments based on the adjacent transpositions and a degree raising operator. The Yang–Baxter graph technique is used: this refers to a directed graph whose nodes correspond to the nonsymmetric Jack polynomials. One aspect of the development is to demonstrate that different directed paths in the graph with the same end-points produce the same result. To obtain square-integrable and mutually orthogonal wavefunctions we start with a Hermitian form \(\langle \cdot, \cdot \rangle_{\mathcal{P}_\tau}\) for \(\mathcal{P}_\tau\) with the properties \((f, g) \in \mathcal{P}_\tau; 1 \leq i \leq N; c \in \mathbb{C}; T, T' \in \mathcal{Y}(\tau)\)
\[
\langle 1 \otimes T, 1 \otimes T' \rangle_{\mathcal{P}_\tau} = \langle T, T' \rangle_{\mathcal{P}_\tau},
\]
\[
\langle f, g \rangle_{\mathcal{P}_\tau} = \overline{\langle g, f \rangle}_{\mathcal{P}_\tau},
\]
\[
\langle wf, wg \rangle_{\mathcal{P}_\tau} = \langle f, g \rangle_{\mathcal{P}_\tau}, w \in S_N,
\]
\[
\langle x_i D f, g \rangle_{\mathcal{P}_\tau} = \langle f, x_i D g \rangle_{\mathcal{P}_\tau},
\]
\[
\langle x f, x g \rangle_{\mathcal{P}_\tau} = \langle f, g \rangle_{\mathcal{P}_\tau}.
\]
(15)
The properties define the form uniquely and imply \(\langle U_i f, g \rangle_{\mathcal{P}_\tau} = \langle f, U_i g \rangle_{\mathcal{P}_\tau}\) and thus the orthogonality of the NSJP’s (theorem 14 below). The form is not defined for some \(\kappa\) and need not be positive-definite. The key results from Dunkl 2017 (recall the maximum hook-length \(h_\tau\) from (1)) are:

**Theorem 11.** Suppose \(-1/h_\tau < \kappa < 1/h_\tau\) and \(L_\kappa(x)\) is the solution of (5) satisfying \(L_\kappa(x_0) = I\) and extended to \(\mathbb{T}_N^{\text{reg}}\) by (6) then there exists a unique positive-definite matrix \(B\)...
such that \( B \tau (w_0) = \tau (w_0) B \) and

\[
\langle f, g \rangle_\tau = \int_{\mathbb{T}^N} f(x)^* L_\alpha(x)^* B L_\alpha(x) g(x) dm(x).
\]

Each \( f \in \mathcal{P}_\tau \) has the expansion \( \sum_{T \in \mathcal{Y}(\tau)} \langle T, T \rangle^{-1/2} f_T(x) \otimes T \) with \( f_T \in \mathcal{P} \) and \( f(x) \) is considered as a column vector \([f_T]_{T \in \mathcal{Y}(\tau)} \) in the integral formula. It is implicit in the theorem that \( L_\alpha(x)^* B L_\alpha(x) \) is integrable, and \( B \) depends on \( \kappa \). Henceforth we use \( \| p \|^2 := \langle p, p \rangle_\tau \) (which need not be positive for \( \kappa \) outside the above interval).

There is a unique positive-definite matrix \( C \) such that \( C^2 = B \); as a consequence \( C \) commutes with \( \tau (w_0) \) (because there is real polynomial \( r(t) \) such that \( r(B) = C \)). We apply the results of the previous section to

\[
L(x) := CL_\alpha(x),
\]

and the integral formula becomes

\[
\int_{\mathbb{T}^N} \{ L(x) f(x) \}^* L(x) g(x) dm(x) = \langle f, g \rangle_\tau.
\]

Here is an outline of the structure and properties of NSJP’s: The operators \( \mathcal{U}_\alpha \) have a triangularity property with respect to a partial order on \( \mathbb{N}_0^N \). For \( \alpha \in \mathbb{N}_0^N \) let \( \alpha^+ \) denote the nonincreasing rearrangement of \( \alpha \) so that \( \alpha^+ \) is a partition.

**Definition 12.** The dominance order \( \prec \) and the derived order \( \prec \) on \( \mathbb{N}_0^N \) are given by

(i) \( \alpha \prec \beta \) if and only if \( \sum_{j=1}^N \alpha_j \leq \sum_{j=1}^N \beta_j, \) for \( 1 \leq i \leq N \) and \( \alpha \neq \beta \); (ii) \( \alpha \prec \beta \) if and only if \( |\alpha| = |\beta|, \alpha^+ \prec \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \prec \beta \).

For example: \((3, 2, 1) \prec (0, 2, 4) \prec (4, 0, 2) \); while \((4, 1, 1), (3, 3, 0) \) are not \( \prec \)-comparable.

The NSJP’s are labeled by pairs \( (\alpha, T) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) but the leading term involves a twist.

**Definition 13.** For \( \alpha \in \mathbb{N}_0^N \) the rank function on \( \{1, \ldots, N\} \) is given by

\[
r_\alpha(i) = \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \},
\]

then \( r_\alpha \in \mathcal{S}_N \) and \( r_\alpha(\alpha) = \alpha^+ \) the nonincreasing rearrangement of \( \alpha \).

For example if \( \alpha = (1, 2, 1, 4) \) then \( r_\alpha = [3, 2, 4, 1] \) and \( r_\alpha(\alpha) = \alpha^+ = (4, 2, 1, 1) \) (recall \( \omega_\alpha = \omega_{\alpha^{-1}(i)} \).

**Theorem 14.** For \( (\alpha, T) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) and for all \( \kappa \) except for a discrete subset of \( \mathbb{Q} \) there is a unique simultaneous eigenfunction \( \zeta_{\alpha, T} \in \mathcal{P}_\tau \) of \( \{\mathcal{U}_\alpha\} \), homogeneous of degree \( |\alpha| \), such that

\[
\mathcal{U}_\alpha \zeta_{\alpha, T} = (\alpha_1 + 1 + \kappa r_\alpha(i, T)) \zeta_{\alpha, T}, 1 \leq i \leq N.
\]

The \( \zeta_{\alpha, T} \) are called nonsymmetric Jack polynomials. The condition on \( \kappa \) for existence is satisfied if each pair \( (\alpha, T) \) is determined by its spectral vector \( \xi_{\alpha, T} := [\alpha_1 + 1 + \kappa r_\alpha(i, T)]_{i=1}^N \); this includes the interval \( -1/\hbar_\tau \leq \kappa \leq 1/\hbar_\tau \). There is an algorithmic approach to the construction based on the Yang–Baxter (directed) graph. The edges involve the adjacent transpositions \( s_i \), which act by transposition on the spectral vector, and a degree-raising operation which shifts and increments the spectral vector. The nodes of the graph are of the form.
(α, T, ξ, α, r_α, ζ, α, T),

(abbreviated to (α, T)) the root is (0^N, T_0, [1 + κc (i, T_0)]^N, I, 1 ⊗ T_0) where T_0 is formed by entering N, N − 1, . . . , 1 column-by-column in the Ferrers diagram. The degree-raising edge uses the map Φ: (c_1, c_2, . . . , c_N) → (c_2, c_3, . . . , c_N, c_1 + 1) on N-tuples. It is called an affine step and is defined by

(α, T, ξ, α, r_α, ζ, α, T) ↦ Φ(α, T, Φ(α, T, r_α w_0, ζ, α, T)),

(ζ, α, T) = x_0 w_0^{-1} ζ, α, T:

(recall w_0 = (1, 2, . . . , N), an N-cycle) the leading term is x^φ_α ⊗ τ (w_0^{-1} r_α^{-1}) T and w_0^{-1} r_α^{-1} = r_0^{-1} because r_Φ = r_α w_0 for any α: r_α w_0(i) = r_α (w_0(i)) = r_α (i + 1) for 1 ≤ i ≤ N, r_α w_0(N) = r_α (1). For example: α = (0, 3, 5, 0), r_α = [3, 2, 1, 4]; Φ = (3, 5, 0, 1), r_Φ = [2, 1, 4, 3].

The other edges are called steps or jumps, both labeled by s_i: the formulae for both rely on the commutation relations (3) and the coefficient b is determined by the condition that s_iζ, α, T = bζ, α, T is an eigenfunction of U_i.

If α_i < α_{i+1}, then the step s_i is

(α, T, ξ, α, r_α, ζ, α, T) ↦ (s_iα, T, s_iξ, α, r_α s_iζ, α, T)

ζ, α, T = s_iζ, α, T − κ ζ, α, T (i) − ζ, α, T (i + 1) ζ, α, T.

If α_i = α_{i+1}, set j = r_α (i), so that j + 1 = r_α (i + 1) and s_j^{-1} r_α^{-1} = r_α^{-1} s_j. Thus ζ, α, T (i) = α_i + 1 + κc (j, T) and ζ, α, T (i + 1) = α_i + 1 + κc (j + 1, T). Set

b' = \frac{1}{c (j, T) − c (j + 1, T)};

If b' = 1 (case: rw (j, T) = rw (j + 1, T)) or −1 (case: cm (j, T) = cm (j + 1, T)) then s_iζ, α, T or −ζ, α, T respectively. Otherwise let T^{(j)} denote the result of interchanging j and j + 1 in T. If 0 < b' ≤ \frac{1}{2}, that is, rw (j, T) < rw (j + 1, T) (and cm (j, T) > cm (j + 1, T)), then the jump s_i is (‘jump’ suggests jumping from one tableau to another)

(α, T, ξ, α, r_α, ζ, α, T) ↦ (α, T^{(j)}, s_iξ, α, r_α, ζ, α, T^{(j)}),

ζ, α, T^{(j)} = s_iζ, α, T − b'ζ, α, T.

The leading term is transformed (x^α ⊗ τ (r_α^{-1}) T) = (x^α) T + τ (r_α^{-1}) T = x^α ⊗ τ (r_α^{-1}) T and τ (s_j) T = T^{(j)} + b'T. The jump applies to the situation α = 0^V and the transformation formulae for s_i acting on T ∈ \mathcal{Y} (τ) stated in definition 1 are recovered (that is, on I ⊗ T and r_α (i) = i).

Example 15. Let N = 3, τ = (2, 1) and T_0 = \frac{3}{2}, T_1 = \frac{3}{2}, T_2 = \frac{3}{2}, and consider (α, T) = ((0, 1, 1), T_0). Then r_α = [3, 1, 2] and ξ, α, T_0 = [1, 2 + κ, 2 − κ]. The step s_1 is ζ,(1,0,1),T_0 = s_1 ξ, α, T_0 + \frac{1}{1+τ} ζ, α, T_0 for the jump s_2 one finds j = 1, b' = \frac{1}{2} and ζ,(0,1,1),T_1 = s_2 ξ, α, T_0 − \frac{1}{2} ζ, α, T_0. Note ζ,(0,1,1),T_1 = [1, 2 − κ, 2 + κ].
The hypotheses on the Hermitian form (15) imply 
\[ \langle \zeta_{\alpha,T}(i), \zeta_{\beta,T} \rangle_T = \langle \xi_{\alpha,T}, U \xi_{\beta,T} \rangle_T = \xi_{\beta,T}(i) \langle \zeta_{\alpha,T}, \zeta_{\beta,T} \rangle_T \]
and thus \( \langle \zeta_{\alpha,T}, \zeta_{\beta,T} \rangle_T = 0 \), (for permitted values of \( \kappa \)). The orthogonality provides an inductive process for computing \( \langle \zeta_{\alpha,T}, \zeta_{\alpha,T} \rangle_T \) for the step \( s \) with \( \alpha_i < \alpha_{i+1} \) we have \( s \zeta_{\alpha,T} = \zeta_{\alpha,T} + b \zeta_{\alpha,T} \) (where \( b = \frac{\xi_{\alpha,T}(i)}{\xi_{\alpha,T}(i+1)} \)) and

\[
\|\zeta_{\alpha,T}\|^2 = \|s \zeta_{\alpha,T}\|^2 = \|\zeta_{\alpha,T}\|^2 + b^2 \|\zeta_{\alpha,T}\|^2,
\]

\[ \|\zeta_{\alpha,T}\|^2 = (1 - b^2) \|\zeta_{\alpha,T}\|^2. \]

A similar formula holds for the jump (\( \alpha_i = \alpha_{i+1} \)). For the affine step, the hypotheses (15) imply \( \| \zeta_{\alpha,T} \|^2 = \| \zeta_{\alpha,T} \|^2 \). Together with \( (1 \otimes T, 1 \otimes T') = \langle T, T' \rangle_T \) this procedure leads to formulae for all \( \| \zeta_{\alpha,T} \|^2 \).

**Theorem 16.** For \( \lambda \in \mathbb{N}_0^N \) (\( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_N \)) and \( T \in \mathcal{Y}(\tau) \)

\[
\| \zeta_{\alpha,T} \|^2 = \langle T, T \rangle_T \prod_{1 \leq i < \ell \leq N} \left( 1 - \left( \frac{\kappa}{\varepsilon + \kappa (c \langle \alpha, T \rangle - c \langle \alpha, i, T \rangle)} \right)^2 \right).
\]

There is an additional factor for nonpartition indices.

**Definition 17.** For \( \alpha \in \mathbb{N}_0^N, T \in \mathcal{Y}(\tau) \) and \( \varepsilon = \pm 1 \) set

\[ \mathcal{E}_\varepsilon \langle \alpha, T \rangle := \prod_{1 \leq i < \ell \leq N} \left( 1 - \frac{\varepsilon \kappa}{\alpha_j - \alpha_i + \kappa (c \langle r_\alpha (j), T \rangle - c \langle r_\alpha (i), T \rangle)} \right). \]

**Theorem 18.** Suppose \( \alpha \in \mathbb{N}_0^N, T \in \mathcal{Y}(\tau) \) then \( \| \zeta_{\alpha,T} \|^2 = \langle \mathcal{E}_1 \langle \alpha, T \rangle \mathcal{E}_{-1} \langle \alpha, T \rangle \rangle^{-1} \| \zeta_{\alpha,T} \|^2. \)

It is important that \( -1/\kappa < \kappa < 1/\kappa \) implies \( \| \zeta_{\alpha,T} \|^2 > 0 \) for all \( \alpha, T \) and thus \( \langle \zeta, \cdot \rangle_T \) is positive-definite. Observe that the value of \( \| \zeta_{\alpha,T} \|^2 \) depends only on the differences \( \alpha_i - \alpha_j \).

This is a consequence of the torus property \( \langle x f, x g \rangle_T = \langle f, g \rangle_T \) and the commutation (where \( e_N := 1 \times 1 \times \cdots \times 1 \))

\[ U_i (e_N^\alpha) = me_N^\alpha f + e_N^\alpha U_i f, \quad 1 \leq i \leq N, m \in \mathbb{N}. \]

Thus \( U_i (e_N^\alpha \zeta_{\alpha,T}) = (m + \alpha_i + \kappa (r_\alpha(i), T)) e_N^\alpha \zeta_{\alpha,T} \) and \( e_N^\alpha \zeta_{\alpha,T} \) is a simultaneous eigenfunction of \( \{ U_i \} \) with the same eigenvalues and the same leading term as \( \zeta_{\alpha+1, T} \) for \( m \geq 0 \) (with \( 1 := (1, 1, \ldots, 1) \in \mathbb{N}_0^N \)). Hence \( \zeta_{\alpha+m, T} = e_N^m \zeta_{\alpha,T} \). There are Laurent polynomial eigenfunctions of \( \{ U_i \} \). The structure of NSJP’s is extended to \( V_\tau \)-valued Laurent polynomials, thereby producing a basis:

**Definition 19.** Suppose \( \alpha \in \mathbb{Z}^N \) then set \( \zeta_{\alpha,T} = e_N^{-m} \zeta_{\alpha+m,T} \) where \( m \in \mathbb{N}_0 \) and satisfies \( m \geq -\min \alpha \). This is well-defined since \( \alpha + m \mathbf{1} \in \mathbb{N}_0^N \) and \( \zeta_{\alpha+1,T} = e_N \zeta_{\alpha,T} \) for \( k \in \mathbb{N}_0 \).

5. Symmetric vector-valued polynomials

For an arbitrary \( \langle \alpha, T \rangle \in \mathbb{N}_0^N \times \mathcal{Y}(\tau) \) we can define a symmetric polynomial simply by averaging: \( p = \frac{1}{\lambda_1^\mathbf{1}} \sum_{w \in S_{\lambda_1^\mathbf{1}}} w \zeta_{\alpha,T} \). From proposition 4 it follows that \( p \) is an eigenfunction of \( \sum_{i=1}^N U_i^m \) for each \( m = 1, 2, 3, \ldots \). The idea of using this method to construct Jack polynomials from the scalar nonsymmetric Jack polynomials is due to Baker and Forrester 1999; the usual Jack parameter is \( \alpha = 1/\kappa \). It is possible that for some \( \langle \alpha, T \rangle \) the sum \( p = 0 \), and for...
some pairs \((\alpha, T)\) and \((\beta, T')\) that the sums agree up to multiplication by a constant. In this section we present the structure of Jack polynomials, the assignment of unique labels, and orthogonality properties (henceforth, unmodified ‘Jack’ implies symmetry). Multiplication by \(L(x)\) will yield symmetric eigenfunctions of \(\mathcal{H}\), the vector-valued wavefunctions. The results are mostly from [Dunkl and Luque 2011, section 5.2]. The Jack polynomials correspond to certain connected components of the Yang–Baxter graph after the affine jumps are removed. The proofs rely on requiring invariance under each \(s_i\) and the fact that the action of \(s_i\) on the nonsymmetric Jack polynomials decomposes nicely into \(1 \times 1\) and \(2 \times 2\) blocks just as for RSYT’s.

**Definition 20.** For \(\alpha \in \mathbb{N}_0^N, T \in \mathcal{Y}(\tau)\) define \([\alpha, T]\) to be the filling of the Ferrers diagram of \(\tau\) obtained by replacing \(i\) by \(\alpha_i^+\) in \(T\), for all \(i\).

Obviously \([\alpha, T] = [\alpha^+, T]\).

**Example 21.** Let \(\tau = (3, 2), \alpha = (1, 4, 2, 0, 3)\) and

\[
T = \begin{pmatrix} 5 & 4 & 1 \\ 3 & 2 \end{pmatrix}, [\alpha, T] = \begin{pmatrix} 0 & 1 & 4 \\ 2 & 3 \end{pmatrix}.
\]

**Proposition 22.** ([Dunkl and Luque 2011, proposition 5.2]) \((\alpha, T)\) and \((\beta, T')\) are connected by steps and jumps (without regard to the orientation) if and only if \([\alpha, T] = [\beta, T']\).

The properties of steps and jumps it follows that the spectral vectors of \((\alpha, T)\) and \((\beta, T')\) are permutations of each other. Set \(\mathcal{T}(\alpha, T) = \{ (\beta, T') : [\beta, T'] = [\alpha, T] \}\), the set of nodes in the connected component.

A tableau is column-strict if the entries are increasing in each column, and nondecreasing in each row.

**Theorem 23.** For \((\alpha, T) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau)\) the span \(\{ \zeta_{\beta, T'} : (\beta, T') \in \mathcal{T}(\alpha, T) \}\) contains a unique nonzero symmetric polynomial if and only if \([\alpha, T]\) is column-strict.

As usual in this context, unique means up to multiplication by a scalar. Suppose \(\lambda \in \mathbb{N}_0^{N,+}\) and consider the sum

\[
p = \sum_{(\beta, T') \in \mathcal{T}(\lambda, T)} a(\beta, T') \zeta_{\beta, T'},
\]

subject to the conditions \(s p = p\) for \(1 \leq i < N\) (sufficing for symmetry).

Suppose there is a step or jump \(s_i\) from \((\beta, T')\) to \((\gamma, T'')\) then \(\zeta_{(\gamma, T'')} = s_i \zeta_{(\beta, T')} = b \zeta_{(\beta, T')}\) for some \(b\); this implies \(s_i \zeta_{(\gamma, T'')} = -b \zeta_{(\beta, T')} + (1 - b^2) \zeta_{(\gamma, T'')}\). The condition \(s_i [a(\beta, T') \zeta_{\beta, T'} + a(\gamma, T'') \zeta_{\gamma, T''}] = a(\beta, T') \zeta_{\beta, T'} + a(\gamma, T'') \zeta_{\gamma, T''}\) implies \(a(\beta, T') = (1 + b) a(\gamma, T'')\). The column strictness hypothesis implies that \(b = -1\) can not occur. The relation is used in an inductive evaluation of \(a(\beta, T')\) once the beginning and end have been identified.

**Definition 24.** For \(\alpha \in \mathbb{N}_0^N\) and \(T \in \mathcal{Y}(\tau)\) let

\[
\text{inv}(\alpha) := \# \{(i, j) : 1 \leq i < j \leq N, \alpha_i < \alpha_j\},
\]

\[
\text{inv}(T) := \# \{(i, j) : 1 \leq i < j \leq N, c(i, T) \geq c(j, T) + 2\}.
\]

Thus a step reduces \(\text{inv}(\alpha)\) by 1 (that is, \(\text{inv}(s_i \alpha) = \text{inv}(\alpha) - 1\)) and a jump reduces \(\text{inv}(T)\) by 1 (in example 15 \(\text{inv}(T_0) = 1\) and \(\text{inv}(T_1) = 0\)). Hence the root \((\alpha, T)\) of \(\in \mathcal{T}(\alpha, T)\) has maximum \(\text{inv}(\alpha) + \text{inv}(T)\) and the sink has minimum \(\text{inv}(\alpha) + \text{inv}(T)\). Clearly \(\text{inv}(\alpha)\) is minimized at \(\alpha^+\) and maximized at \(\alpha^-\), the nondecreasing rearrangement of \(\alpha\). In [Dunkl and Luque 2011, definition 5.6] it is shown that there are unique tableaux \(T_R, T_S\) in \(\mathcal{T}(\alpha, T)\) such that \((\alpha^-, T_R)\) is the root and \((\alpha^+, T_S)\) is the sink (maximizes, respectively minimizes \(\text{inv}(\beta) + \text{inv}(T')\) for \((\beta, T') \in \mathcal{T}(\alpha, T))\). The formulae for \(T_R\) and \(T_S\) are with \(\mathcal{T} = [\alpha, T]\).
\[
T_R(i, j) = \# \{(k, l) : T(k, l) > T(i, j)\}
+ \# \{(k, l) : T(k, l) = T(i, j), (l > j) \lor (l = j \land k \geq i)\}; \quad (18)
\]
\[
T_S(i, j) = \# \{(k, l) : T(k, l) > T(i, j)\}
+ \# \{(k, l) : T(k, l) = T(i, j), (k > i) \lor (k = i \land l \geq j)\}. \quad (19)
\]

**Example 25.** \(\alpha = (3^3, 2^3, 1, 0)\),

\[
[\alpha, T] = \begin{pmatrix}
0 & 2 & 2 & 3 & 8 & 5 & 4 & 1 & 8 & 6 & 5 & 3 \\
2 & 6 & & & & & & & & & \\
\end{pmatrix}, \quad T_R = 7, 3, 2, \quad T_S = 7, 2, 1.
\]

As motivation for the formulae for \(a(\beta, T')\) in the sum suppose \(\beta_i \prec \beta_{i+1}\) (and \(\varepsilon = \pm 1\)) then

\[
E_\varepsilon(\beta, T) = 1 + \varepsilon \frac{\xi_{i+1}(\beta, T) - \xi(\beta, T)}{\xi_{i+1}(s_i \beta, T) - \xi(s_i \beta, T)}.
\]

where \(C_{i, \beta, T} = s_i \zeta_{i, \beta, T} - b \zeta_{i, \beta, T}\). We introduce two functions on \(\mathcal{Y}(\tau)\) to deal analogously with jumps:

**Definition 26.** For \(T \in \mathcal{Y}(\tau)\) and \(\varepsilon = \pm 1\) set

\[
C_\varepsilon(T) = \prod \left\{ 1 + \varepsilon \frac{c(i, T) - c(j, T)}{c(i, T) - c(j, T - 2)} : 1 \leq i < j \leq N, c(i, T) \leq c(j, T) - 2 \right\}.
\]

From (2) \(\langle T, T \rangle_0 = C_1(T)C_{-1}(T)\). In the jump with \(\beta_i = \beta_{i+1}, \tau_\beta(i) = j\) and \(c(j, T) - c(j + 1, T) \geq 2\) (as in 17) \(T^{(0)}\) has \(j\) and \(j + 1\) interchanged so that \(c(j, T^{(0)}) - c(j + 1, T^{(0)}) = c(j + 1, T) - c(j, T) \leq -2\). Then

\[
C_\varepsilon(T^{(0)}) = C_\varepsilon(T) \left(1 + \frac{\varepsilon}{c(j, T^{(0)}) - c(j + 1, T^{(0)})}\right)
\]

so that \(C_{-1}(T^{(0)}) = C_{-1}(1 + b)\) where \(b = (c(j, T) - c(j + 1, T))^{-1}\). From these relations it can be shown:

**Proposition 27.** Suppose \((\alpha, T) \in \mathbb{N}_0^N \times \mathcal{Y}(\tau)\) and \([\alpha, T]\) is column-strict then

\[
p = \sum_{(\beta, T') \in T(\alpha, T)} \frac{C_{-1}(\beta, T')}{C_{-1}(T')} \zeta(\beta, T')
\]

is symmetric and nonzero.

To proceed with the analysis we impose a normalization and then find a closed formula for the squared-norm \(\| \cdot \|^2\). Replace \(\alpha\) by \(\lambda = \alpha^+\) and use the sink \((\lambda, T_\lambda)\) as normalization by requiring that the coefficient of \(x^\lambda \otimes T_\lambda\) is 1. From the sink property and the \(\triangleright\)-triangularity of the NSJP’s it follows that \(x^\lambda \otimes T_\lambda\) appears only in \(\zeta(\lambda, T)\) in the sum \(p\), with coefficient 1. Thus define (for \(\lambda \in \mathbb{N}_0^{\lambda^+}\))

\[
J_{\lambda, T_\lambda} := \sum_{(\beta, T') \in T(\lambda, T_\lambda)} \frac{C_{-1}(T_\lambda)}{C_{-1}(T')} \zeta(\beta, T') \zeta(\beta, T'). \quad (20)
\]
By orthogonality \( \|J_{\lambda,T}\|^2 = \sum_{(\beta,T') \in T(\lambda,T)} \left( \frac{c_1(T_j)}{c_1(T')} \right)^2 \|\xi_{\beta,T'}\|^2 \); fortunately there is a formula without summation. Suppose \((\beta,T') \in T(\lambda,T)\) and \(J_{\lambda,T} = \sum_{w \in S_\lambda} w \xi_{\beta,T'}\) then

\[
\|J_{\lambda,T}\|^2 = c \sum_{w \in S_\lambda} \langle J_{\lambda,T}, w \xi_{\beta,T'} \rangle_T = N!c \langle J_{\lambda,T}, \xi_{\beta,T'} \rangle_T ;
\]

if we write \(J_{\lambda,T} = \sum_{(\beta,T') \in T(\lambda,T)} a(\beta,T') \xi_{\beta,T'}\) then \(\|J_{\lambda,T}\|^2 = (N!c a(\beta,T') \|\xi_{\beta,T'}\|^2\)

and \(c\) can be determined by careful choice of \((\beta,T')\). Consider the stabilizer group \(G_{\lambda,T}\) of \(\xi_{\lambda,T}\) (\(w \in G_{\lambda,T}\) implies \(w \xi_{\lambda,T} = \xi_{\lambda,T}\)). The group is generated by \(\{s_i : \lambda_i = \lambda_{i+1}, rw(i,T_s) = rw(i + 1,T_s)\}\). It was shown ([Dunkl and Luque 2011, proposition 5.11]) that the coefficient of \(\xi_{\lambda,T}\) in \(\sum_{w \in S_\lambda} w \xi_{\lambda,T} - \xi_{\lambda,T}\) is \(#G_{\lambda,T}\), hence that \(\sum_{w \in S_\lambda} w \xi_{\lambda,T} - \xi_{\lambda,T} = \#G_{\lambda,T}J_{\lambda,T}\). We deduce

\[
\|J_{\lambda,T}\|^2 = \left( \frac{N!}{\#G_{\lambda,T}} \right) c_1(T_s) \sum_{v=0}^{C_1-1} \frac{c_1(T_s)^{v}}{c_1(T)} \|\xi_{\lambda,-T_s}\|^2 .
\]

Also

\[
\frac{\|\xi_{\lambda,-T_s}\|^2}{c_1(T)} = \frac{\|\xi_{\lambda} - T_s\|^2}{c_1(T)} = \left( E_{-1}(\lambda,-T_s) E_1(\lambda,-T_s) \right) \|\xi_{\lambda,-T_s}\|^2
\]

and \(\|\xi_{\lambda,-T_s}\|^2 = (\#S_{\lambda,T}) \frac{\|\xi_{\lambda} - T_s\|^2}{c_1(T)}\) (from theorem 16). From (2) it follows that \(\frac{(T_s)^{r_s}}{(T_s)^{r_s}} = c_1(T_s)^{c_1-1}(T_s)\).

\[
\|J_{\lambda,T}\|^2 = \left( \#G_{\lambda,T} \right) \frac{c_1(T_s)}{c_1(T)} E_1(\lambda,-T_s) \|\xi_{\lambda,-T_s}\|^2 .
\]

**Theorem 28.** Suppose \((\lambda,T) \in \mathbb{N}_0^N \times Y(\tau)\) and \([\lambda,T]\) is column-strict. Define \(T_r\) and \(T_s\) by formulae (18) and (19) then

\[
\|J_{\lambda,T_s}\|^2 = \left( \#G_{\lambda,T_s} \right) \frac{c_1(T_s)}{c_1(T)} E_1(\lambda,-T_s) \|\xi_{\lambda,-T_s}\|^2 .
\]

Suppose \([\lambda,T_s],[\lambda',T'_s]\) are unequal column-strict tableaux. By definition \(T(\lambda,T_s) \cap T(\lambda',T'_s) = 0\) and from the mutual orthogonality of the terms in the sums (20) it follows that \(\langle J_{\lambda,T_s}, J_{\lambda',T_s} \rangle_T = 0\).

Multiplication of \(J_{\lambda,T_s}\) by \(e^\mu_n\) produces the Jack polynomial \(J_{\lambda,T_s}\) (see definition (19) and recall \(e^\mu_n = \prod_{i=1}^{N} x_i\); here \(m = -1, -2, \ldots\) is valid and defines Jack Laurent polynomials. The expression \(e^\mu_nJ_{\lambda,T_s}\) is made unique by the requirement \(\lambda_N = 0\).

We see that there is a unique symmetric polynomial \(p_{\lambda,T}\) of minimum degree: the tableau \([\lambda,T]\) has the entry \(i - 1\) in each box in row \(#i\). The degree is \(n(\tau) = \sum_{j=1}^{N} (i - 1) \tau_i\).

If \((\beta,T') \in T(\lambda,T_s)\) then the spectral vector \(\xi_{\beta,T'}\) is a permutation of \(\xi_{\lambda,T_s}\) thus

\[
\sum_{s=1}^{N} \xi_{\lambda,T_s} = \sum_{s=1}^{N} \xi_{(s,1)}(\lambda,T_s) \xi_{\lambda,T_s} = \sum_{s=1}^{N} (\lambda_i + \gamma (\xi_{\lambda,T_s})^2) J_{\lambda,T_s}.
\]
Set $S_2 := \sum_{i=0}^{\ell(\tau)} c(i, T_\lambda) z^i = \frac{1}{3} \sum_{i=1}^{\ell(\tau)} \tau_i \{(\tau_i - 1) (\tau_i - 2) - 6 (\tau_i - i) (i - 1)\}$. The eigenvalue can be written as $\sum_{i=1}^{N} \lambda_i^2 + 2 \kappa \lambda_i (c(i, T_\lambda) - \gamma) + \kappa^2 \left( S_2 - N \gamma^2 \right)$. In the trivial case $\tau = (N)$ the last term becomes $\frac{1}{12} \kappa^2 N (N^2 - 1)$. The effect of multiplying by $e^\gamma_N$ on the eigenvalue is

$$
\sum_{i=1}^{N} (U_i - 1 - \kappa \gamma)^2 e^{\gamma}_N J_{\lambda, T_\gamma} = \sum_{i=1}^{N} \left( \lambda_i + m + \kappa (c(i, T_\lambda) - \gamma) \right)^2 e^{\gamma}_N J_{\lambda, T_\gamma}
$$

$$= \left\{ \sum_{i=1}^{N} (\lambda_i + \kappa (c(i, T_\lambda) - \gamma))^2 + 2m \sum_{i=1}^{N} \lambda_i + Nm^2 \right\} e^{\gamma}_N J_{\lambda, T_\gamma}.
$$

This is minimized over $m$ when $m$ is the nearest integer to $-\sum_{i=1}^{N} \lambda_i / N$.

5.1. Algebraic and combinatorial properties

The Hilbert (or Hilbert–Poincaré) series of a graded vector space $M$ is $\sum_{i=0}^{\infty} z^i \dim M_i$ where $M_i$ is the component of degree $n$. The Hilbert series for (scalar) polynomials in $(x_1, \ldots, x_N)$ is $(1 - z)^{-N}$ and for scalar symmetric polynomials the series is $\prod_{i=1}^{N} (1 - z)^{-1}$. The Hilbert series for the symmetric polynomials in $P_\tau$ can be found from a combinatorial result for the number of column-strict tableaux of shape $\tau$. A weak reverse tableau of shape $\tau$ and weight $n$ has its entries nondecreasing in each row and in each column and the sum of the entries is $n$. Such a tableau can be transformed to a column-strict tableau by adding $i - 1$ to each entry in row $i$ for $1 \leq i \leq \ell(\tau)$. The number of the weak reverse tableaux is the coefficient of $z^n$ in $H_\tau(z) := \prod_{(i,j) \in \tau} \left( 1 - z^{h(i,j)} \right)^{-1}$ (see [Stanley 1999, p 379, h(i,j) from Formula (1)]). Thus the desired Hilbert series is $z^{\ell(\tau)} H_\tau(z)$ and the number of Jack polynomials of degree $n$ is the coefficient of $z^n$ in the series.

There is another approach based on the coinvariant algebra, the linear space of (scalar) polynomials modulo the ideal generated by the ring of nonconstant symmetric polynomials. The coinvariant algebra is finite-dimensional with Hilbert series $\prod_{i=1}^{N} \left( \frac{1 - z^i}{1 - z} \right)$ and it is isomorphic to the regular representation of $S_N$ (for details see [Humphreys 1990, section 3.6]). Thus the number of linearly independent submodules of isotype $\tau$ equals $n_\tau = \dim V_\tau$. The Hilbert series for the polynomials of isotype $\tau$ in the coinvariant algebra is $n_\tau z^{\ell(\tau)} H_\tau(z) \prod_{i=1}^{N} (1 - z^i)$. Any module of isotype $\tau$ is a product of a symmetric polynomial with an isotype-$\tau$ submodule of the coinvariant algebra. The coefficient of $z^n$ in $z^{\ell(\tau)} H_\tau(z) \prod_{i=1}^{N} (1 - z^i)$ is the number of linearly independent isotype-$\tau$ modules of degree $n$ in the coinvariant algebra, hence these comprise a generating set for all the isotype-$\tau$ modules under multiplication by symmetric polynomials (in fact a minimal generating set, as can be seen by multiplying the respective Hilbert series).

As shown in proposition 2 there is a one-to-one correspondence between isotype-$\tau$ polynomial modules and symmetric polynomials in $P_\tau$. Hence the minimal generating set for these polynomials is described by the Hilbert series $z^{\ell(\tau)} H_\tau(z) \prod_{i=1}^{N} (1 - z^i)$. The cardinality of the set equals $\dim V_\tau$.

For example consider $S_4$ and the representation $2, 2$. The Hilbert series of the coinvariant algebra is $1 + 3z + 5z^2 + 6z^3 + 5z^4 + 3z^5 + z^6$ and for its submodules of isotype $2, 2$ the series is $z^2 + z^4$. Thus the symmetric polynomials in $P_\tau$ are generated by scalar symmetric
polynomials multiplying the polynomials labeled by \([\lambda, T] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1' & 2 \end{bmatrix}\); the second one can be replaced by \(\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\).

For another example consider the 6-dimensional representation \((3, 1, 1)\) of \(S_5\); the Hilbert series for the isotype \((3, 1, 1)\) submodules of the coinvariant algebra is

\[
\prod_{i=1}^5 \frac{(1 - z^i)}{(1 - z^i)^2 (1 - z^i)^2 (1 - z^i)} = z^3 + z^4 + 2z^5 + z^6 + z^7.
\]

6. Symmetric wavefunctions

In the notation of section 5 there is a set of mutually orthogonal wavefunctions \(L(x)J_{\lambda, T_\delta}(x)\) such that

\[
\mathcal{H}L(x)J_{\lambda, T_\delta}(x) = \sum_{i=1}^N (\lambda_i + \kappa (c (i, T_\delta) - \gamma))^2 L(x)J_{\lambda, T_\delta}(x)
\]

by theorem 10. Thus

\[
\frac{1}{||J_{\lambda, T_\delta}||} (L(x)J_{\lambda, T_\delta}(x))^* L(x)J_{\lambda, T_\delta}(x)
\]

is a probability density function on \(\mathbb{R}^N\). Since multiplication of \(L(x)J_{\lambda, T_\delta}(x)\) by powers of \(e_N\) does not change the density function, we can assume \(\lambda_N = 0\). In contrast to the scalar case the matrix \(L(x)\) has singularities of order \(|x_i - x_j|^{\pm \kappa}\) in neighborhoods of points \(x\) with \(x_i = x_j\) and all other \(x_k\) being pairwise distinct. Nevertheless we can show that the symmetric wavefunctions are bounded in such sets when \(0 < \kappa < \frac{1}{2}\). By the invariance it suffices to prove this near \(\{x : x_{N-1} = x_N\}\) in the fundamental chamber. More precisely let \(\delta > 0\) and define

\[
\Omega_\delta := \{x \in \mathbb{R}^N : 1 \leq i \leq N - 2 \quad \text{and} \quad |x_i - x_j| \geq 2 \delta, |x_{N-1} - x_1| \geq \delta\}.
\]

Theorem 29. Suppose \(0 < \kappa < \frac{1}{2}\) and \((\lambda, T_\delta)\) is as in theorem 28 then \(L(x)J_{\lambda, T_\delta}(x)\) is uniformly bounded in \(\Omega_\delta\).

The proof depends on a power series formulation for \(L\) proven in [Dunkl 2017, section 5]. Arrange \(\mathcal{Y}(\tau)\) linearly listing the tableaux \(T\) with \(c(N-1, T) = -1\) first (that is, \(cm\) \((N-1, T) = 1, \text{rw} \ (N-1, T) = 2\)). This results in the matrix representation of \(\tau (N-1, N)\) being

\[
\begin{bmatrix}
-I_{m_{\tau}} & O \\
O & I_{n_{\tau} - m_{\tau}}
\end{bmatrix}
\]

where \(n_{\tau} := \dim V_{\tau} = \#\mathcal{Y}(\tau)\) and \(m_{\tau}\) is given by \(\text{tr} (\tau (N-1, N)) = n_{\tau} - 2m_{\tau}\). We use \(O\) to denote a zero matrix of dimensions determined by the context. From the sum \(\sum_{i < j} \tau(i,j) = S_1(\tau)I\) it follows that \(\binom{N}{2} \text{tr} (\tau (N-1, N)) = S_1(\tau) n_{\tau}\) and \(m_{\tau} = n_{\tau} \left(\frac{1}{2} - \frac{S(\tau)}{N(N+1)}\right)\). (The traces of the transpositions are equal because they are conjugate to each other.) The property of a matrix commuting or anti-commuting with \(\sigma := \tau (N-1, N)\)
is used in the argument; to express this neatly we introduce the \( \sigma \)-block decomposition
\[
(\alpha + (n_x - m_x)) \times (\alpha + (n_x - m_x))
\]
of a matrix
\[
\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.
\]
Then \( \sigma \alpha \sigma = \alpha \) if and only if \( \alpha_{12} = O = \alpha_{21} \) and \( \sigma \alpha \sigma = -\alpha \) if and only if \( \alpha_{11} = O = \alpha_{22} \).

For \( z_1, z_2 \in \mathbb{C} \) let
\[
\rho(z_1, z_2) := \begin{bmatrix} z_1 I_{n_x} & O \\ O & z_2 I_{n_x - m_x} \end{bmatrix}.
\]

We showed in [Dunkl 2017, section 5] that there exist matrix coefficients \( \alpha_n (x') \) with
\[
x' := (x_1, \ldots, x_{N-2}, \frac{x_N - x_N - 1}{2})
\]
analytic on the closure of \( \Omega \cup \Omega_5 \) \((N - 1, N)\), such that (with
\[
z := \frac{x_N - x_N - 1}{2}
\]
for \( \kappa \geq 0 \) and the second part is of order \( \kappa \)
relation
\[
\kappa > 0
\]
and \( \sigma \alpha_n (x') \) is \((-1)^{n_x} \alpha_n (x') \). In particular the \( \sigma \)-block decomposition of \( \alpha_0 (x') \)
is
\[
\begin{pmatrix} \alpha_0 (x') \\ O \\ \alpha_{n,22} (x') \end{pmatrix}.
\]
The series converges absolutely for \( \frac{1}{2} |x_N - x_N - 1| < \min_{1 \leq i \leq N-2} \left| x_N - x_{N-1} \right|
\]
if and only if
\[
|\gamma| \leq 2\kappa
\]
for \( \gamma \geq 0 \) and the omitted terms are of order \( \kappa \)
while the behaviour of \( \Omega \) near a point with multiple repeated entries (say \( x_{N-2} = x_{N-1} = x_N \)) is complicated and the power series method used here does not apply.

\subsection{Minimal degree symmetric polynomials}

To produce the minimal degree \( J_{\lambda,T_5} \), or equivalently, the minimal degree column-strict tableau of shape \( \tau \), the entries in row \( \neq i \) all equal \( i - 1 \). The corresponding \( T_5 \) has the numbers
\[
N, N - 1, \ldots, 1
\]
entered row-by-row, and \( T_R = T_5 \). With \( l = \ell (\tau) \) and using superscripts to denote multiplicity
\[
\lambda = ((l - 1)^{\gamma_1}, (l - 2)^{\gamma_2}, \ldots, 1^\gamma, 0^\eta)
\]
there is an explicit formula for \( J_{\lambda,T_5} \). It is derived from proposition 2 and involves the Specht polynomials. For
\[ 1 \leq n_1 \leq n_2 \leq N \] define the alternating polynomial \( a(x; n_1, n_2) = \prod_{\ell=n_1+1}^{n_2} (x_\ell - x) \) (the empty product \( a(x; n_1, n_1) = 1 \)). Denote the transpose of the partition \( \tau \) by \( \tau^t \) then \( \tau_j^t = \ell(\tau) \). Form \( p_{\tau_0} \) as the product of the alternating polynomials for each column of \( T_0 \); that is, set \( k_j = N - \sum_{i=1}^{\tau_j^t} 0 \leq j \leq \tau_1 \) and

\[
p_{\tau_0}(x) := \prod_{j=1}^{\tau_1} a(x; k_j + 1, k_j-1).
\]

Then the other polynomials \( p_{\tau} \) are produced by the formulae in the proposition and the minimal \( J_{\lambda, T} = \langle T_S, T_S \rangle \sum_{T'} \frac{1}{\nu_{\tau}(S, T)_{\nu_{\tau}(T)}} p_{\nu_{\tau}(T)}(x) \otimes T' \) (see Dunkl and Luque 2011, section 5.4).

For example let \( \tau = (2, 2) \) then

\[
T_0 = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}, \quad |\lambda, T_1 | = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad (22)
\]

\[
p_{\tau_0} = (x_3 - x_4) (x_4 - x_2) p_{\tau_1} = s_2 p_{\tau_1} = \frac{1}{2} p_{\tau_0} = x_1 x_2 + x_3 x_4 - \frac{1}{2} (x_1 + x_2) (x_3 + x_4),
\]

because \( b = (c (2, T_0) - c (3, T_0))^{-1} = \frac{1}{2} \). Observe that \( s_{i+1} p_{\tau_1} = p_{\tau_1} = s_1 p_{\tau_1} \). Also \( \langle T_0, T_0 \rangle_0 = 1 \) and \( \langle T_1, T_1 \rangle_0 = \frac{3}{4} \) (see (2), \( \lambda = (1, 1, 0, 0) \) and \( J_{\lambda, T_1} = \frac{3}{4} p_{\tau_0} \otimes T_0 + p_{\tau_1} \otimes T_1 \).

There is a formula for the minimal \((\lambda, T_3)\) proven in [Dunkl 2010, theorem 8]:

\[
\|J_{\lambda, T_0}\|^2 = N! \left( \prod_{i=1}^{\ell(\tau)} \tau_i^! \right)^{-1} \langle T_S, T_S \rangle_0 \prod_{(ij) \in \tau} \frac{(1 - \kappa h(i, j)) \log(i, j)}{(1 + \kappa (j - i))},
\]

where \( \log(i, j) := \# \{l : l > i, \tau_l \geq j \} \) (equivalently \( \tau_j^t - i \)), and \( (m)_n \) is the Pochhammer symbol \( \prod_{i=1}^{n} (m + j - 1) \).

The energy eigenvalue \( \sum_{i=1}^{N} \lambda_i^2 + 2\kappa \sum_{i=1}^{N} \lambda_i (c(i, T_3) - 1) + \kappa^2 (S_2 - N_3^2) \) from (21) specializes to

\[
\sum_{i=1}^{\ell(\tau)} (i - 1) \tau_i + \kappa \sum_{i=1}^{\ell(\tau)} \tau_i (i - 1) (\tau_i + 1 - 2i - 2\gamma) + \kappa^2 (S_2 - N_3^2),
\]

for the minimal degree \( J_{\lambda, T_0} \).

6.2. Example

Let \( N = 4, \tau = (2, 2), V(\tau) = \{ T_0, T_1 \} \) see (22), \( h_\tau = 3 \). The system (5) can be reduced to a hypergeometric equation in one variable by the substitution \( \zeta(x) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)} \). The bounds \( 0 < \zeta(x) < 1 \) hold in the fundamental domain \( C_0 \). A fundamental solution \( L_\zeta \) is given in terms of the functions

\[
g_1(\kappa; \zeta) := 2F_1 \left( \begin{array}{cc} -\kappa, \kappa \\ 2\kappa \end{array}; \zeta \right),
\]

\[
g_2(\kappa; \zeta) := \frac{\kappa \zeta}{1 + 2\kappa} 2F_1 \left( \begin{array}{cc} 1 + \kappa, 1 - \kappa \\ 2 + 2\kappa \end{array}; \zeta \right); \]

\[
20
\]
Then the solution $L(x)$ which satisfies (16) is up to a positive multiplicative constant

$$L(x) = \begin{bmatrix} \gamma(\kappa)^{1/2} & 0 \\ 0 & \gamma(-\kappa)^{1/2} \end{bmatrix} L_F(\zeta(x)),$$

where $\gamma(\kappa) := \frac{\Gamma(1+2\kappa)}{\Gamma(1+\kappa)^2 \Gamma(1+3\kappa)}$; observe that $\gamma(\kappa) > 0$ for $\kappa > -\frac{1}{2}$. As yet the problem of determining the normalizing constant is still open. The purpose of the example is to demonstrate the qualitative difference of $L(x)$ from the scalar case, so the underlying computations are not presented here. The derivation uses the known transformation properties of the hypergeometric series for the change-of-variable $\zeta \to 1 - \zeta$, since $\zeta(xw_0) = 1 - \zeta(x)$ where $w_0 = (1, 2, 3, 4)$, a 4-cycle.

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