The bound state wave functions for a wide class of exactly solvable potentials are found utilizing the quantum Hamilton-Jacobi formalism of Leacock and Padgett. It is shown that, exploiting the singularity structure of the quantum momentum function, until now used only for obtaining the bound state energies, one can straightforwardly find both the eigenvalues and the corresponding eigenfunctions. After demonstrating the working of this approach through a few solvable examples, we consider Hamiltonians, which exhibit broken and unbroken phases of supersymmetry. The natural emergence of the eigenspectra and the wave functions, in both unbroken and the algebraically non-trivial broken phase, demonstrates the utility of this formalism.

I Introduction

Amongst various approaches to quantum mechanical problems, the Hamilton-Jacobi formalism has been shown to be ideally suited for determining the bound state energies of exactly solvable problems. For a lucid review of various formulations of quantum mechanics and their relative merits and demerits, the reader is referred to Styer et. al. In classical mechanics, the action-angle variables are the most appropriate ones for the description of the dynamical systems exhibiting periodic motion. The corresponding demonstration, for the quantum mechanical bound state problems, is of recent origin. In 1984, Leacock and Padgett defined an exact quantum action variable through a quantum momentum function (QMF), whose quantization condition led to the bound state energies. The fact that only the residues of the QMF are required for finding the eigenvalues, and the same can be evaluated rather easily from the quantum Hamilton-Jacobi (QHJ) equation, explains the economy of the QHJ formalism in finding the bound state spectra. In the limit \( \hbar \to 0 \) the QMF goes over to the classical momentum, \( \sqrt{2m(E-V)} \) and one can derive the well-known WKB quantization rule. The demonstration of the connection between the supersymmetric WKB approximation scheme, known to be exact for the so called shape-invariant potentials and the exact Hamilton-Jacobi quantization condition further clarified the working of various semi-classical quantization methods.

QHJ formalism is formulated in the complex plane, making it an ideal tool for studying quasi-exactly solvable and recently discovered PT-symmetric potentials, since these are also naturally studied in the same domain. It is well suited for studying chaotic systems, where the periodic orbits are known to play a significant role. In this connection it is worth mentioning a recent demonstration of the close relationship between the Kolmogorov-Sinai entropy and the action, both in classical and quantum domains. Before proceeding to the analysis of the above mentioned non-trivial quantal problems, it is necessary to develop a precise understanding of the procedure for finding the wave functions from the quantum Hamilton-Jacobi equation. Furthermore, in light of the close similarity between the QHJ and the WKB type quantization conditions and the well-known difficulties one encounters in extracting the wave functions in the semi-classical approaches, it is of deep interest to find the same in the QHJ scheme.

The present work is devoted to the study of bound state wave functions through the quantum Hamilton-Jacobi formalism. It is shown that the singularity structure of the quantum momentum
function, which has been so far used only for finding the bound state energies, can straightforwardly yield both the eigenvalues and the corresponding eigenfunctions. In the earlier works of, two of the present authors and those of Leacock and Padgett, the singularity structure of the QMF and the boundary conditions it needs to satisfy have been carefully studied. Since the same potentials are being analyzed here, we use the singularity structure of the QMF found earlier to determine the eigenstates. We will also not deal with the details of the QHJ formalism and refer the interested reader to earlier papers.

We first demonstrate the working of this approach through a number of exactly solvable potentials, which have been studied extensively in the literature through a number of approaches, ranging from factorization method, path integral techniques, group theoretical method to SUSY operator method. We then analyze Hamiltonians which exhibit broken and unbroken phases of supersymmetry. The natural emergence of the eigenspectra and the wave functions, in both the phases of supersymmetry, demonstrates the utility of this formalism. It should be mentioned that Hamiltonians exhibiting broken phase of supersymmetry are not straightforwardly amenable to an algebraic treatment.

The paper is organized as follows. We first give a brief description of the QHJ formalism using the well-studied harmonic oscillator problem as a prototype. We explicate the procedure to obtain the bound state wave functions and its connection with the singularity structure of the QMF. This is an example of the confluent hypergeometric class. We then proceed to the study of Rosen-Morse and the Scarf-I potentials, belonging to the hypergeometric class. It is interesting to note that, Scarf-I potential, although similar to the Rosen-Morse, exhibits both broken and unbroken phases of supersymmetry, for different ranges of potential parameters. The role of the proper implementation of the boundary conditions, satisfied by the QMF in Hamilton-Jacobi formalism, is clearly brought out in this section. It is shown that, imposing the boundary conditions correctly leads automatically to the appropriate bound state wave functions, for the broken and the unbroken phases of supersymmetry (SUSY).

II Harmonic Oscillator in the QHJ formalism

We begin this section with a brief summary of the QHJ formalism for one dimensional potentials. As pointed out by Leacock and Padgett, the QHJ equation,

$$p^2 - i\hbar \frac{dp}{dx} = 2m(E - V(x)),$$

is a special case having the well-known Riccati equation,

$$A(x)p^2 + B(x)p + C(x) + i\frac{dp}{dx} = 0.$$

can be connected to the Schrödinger equation for a given potential $V(x)$ as

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x),$$

provided the QMF $p$ is related to the wave function $\psi(x)$ by

$$p = -i\hbar \frac{d}{dx} \ln \psi(x).$$

The above relation can be inverted to yield $\psi(x) = \exp \left( \frac{i}{\hbar} \int p dx \right)$. We first analytically continue $p$ to the complex plane by taking $x$ as a complex variable. In what follows, the QMF plays a central role in finding the bound state wave functions. The knowledge of the singularities of the QMF is sufficient to determine its form, which consequently lead to the wave functions. The solutions of Riccati equation has two types of singularities, the fixed and the moving singularities. The fixed singular points of $p(x)$ reflect the singularities of the potential and are energy independent. Besides these, the QMF has other singular points called the moving singularities of the type,

$$p \approx \frac{\hbar}{i} \frac{1}{x - a}.$$
which cancel amongst the terms in the left hand side of (2). It can be seen clearly from (2), that the residue is given by $-i\hbar$. These poles, unlike the fixed singularities, are dependent on energy and the initial conditions. It is known that, for the solutions of the Riccati equation, a moving singularity can only be a pole. It is a well-known fact that, the wave function of the $n$th excited state will have $n$ zeros on the real line in the classical region, which implies that the QMF will have $n$ poles. In all the solvable models studied within the QHJ approach, no other moving singularity was found except the $n$ moving poles. This observation has been very useful in solving for the bound state energies. In all the potentials analyzed here, it has been assumed that, the QMF has finite number of singularities in the complex plane and that the point at infinity is an isolated singular point. The knowledge of the location of poles and the corresponding residues are sufficient to fix the form and to obtain the bound state wave functions.

The QHJ equation can be used to obtain the residues at different poles. Owing to the fact that the Riccati equation is quadratic, one gets two answers for the residues. Hence, one needs a boundary condition to select the correct value. The original boundary condition

$$\lim_{\hbar \to 0} p \to p_{cl}, \quad (7)$$

which was given by Leacock and Padgett can be cast in several equivalent forms, which will be given as and when they are used. We will use one of the convenient forms to identify the correct residue.

For the purpose of illustration, we start with the harmonic oscillator $V(x) = 1/2(kx^2)$. The Schrödinger equation, with $\xi = \alpha x$, $\lambda = 2E/\hbar\omega_c$, $\omega_c = \sqrt{k/m}$ and $mk = \hbar^2 \alpha^4$,

$$\frac{\partial^2 \tilde{\psi}(\xi)}{\partial \xi^2} + (\lambda - \xi^2) \tilde{\psi}(\xi) = 0, \quad (8)$$

has the corresponding QHJ equation in the form

$$p^2 - i\frac{dp}{d\xi} = \lambda - \xi^2. \quad (9)$$

From the above equation it appears reasonable to assume that for large $\xi$, $p$ goes as $\pm i\xi$. To fix the right behavior of $p$ at infinity one can use the boundary condition or instead use the relation between $p$ and $\psi$

$$\psi(x) = \exp(i \int p dx). \quad (10)$$

Demanding the square integrability of the wave function, the correct behavior for large $\xi$ is seen to be

$$\lim_{\xi \to \infty} p \approx i\xi. \quad (11)$$

The QMF $p$ has $n$ moving poles on the real line with residue $-i$. There are no fixed singularities in the finite domain in the complex plane, since the potential is not singular. It is possible to show that there are no other moving poles of the QMF in the finite complex plane, thus making $p$ analytic everywhere except at the $n$ moving poles. Therefore we can write $p$ as

$$p(\xi) = \sum_{k=1}^{n} \frac{-i}{\xi - \xi_k} + i\xi + Q(\xi), \quad (12)$$

where the summation term describes the sum of all the principal parts of the individual Laurent expansion of $p$ around each moving pole. $Q(\xi)$ is an entire function bounded for large $\xi$, with $i\xi$ describing the behavior of $p$ at $\infty$. Thus from Liouville’s theorem $Q(\xi)$ is a constant $C$. Hence the expression can be cast in a useful form

$$p(\xi) = -i\frac{P'}{P} + i\xi + C \quad (13)$$
where $P$ is an $n^{th}$ degree polynomial $\prod_{k=1}^{n} (\xi - \xi_k)$ and is seen to satisfy
\begin{equation}
P''_n - 2\xi P'_n - 2CiP'_n - (2i\xi C + C^2 - \lambda + 1)P_n = 0. \tag{14}
\end{equation}
Equating the coefficients of $\xi^{n+1}$ and $\xi^n$ to zero gives
\begin{equation}
C = 0, \quad \lambda = 2n + 1 \tag{15}
\end{equation}
and hence $E_n = (n + \frac{1}{2})\hbar\omega_c$. With these values (14) becomes the Hermite equation and $P_n$ coincides with Hermite polynomial $H_n(\xi)$ apart from an overall constant and one obtains the known eigenfunctions

\begin{equation}
\psi(\xi) = \exp \left(i \int pd\xi \right) = \exp \left(-\frac{\xi^2}{2} + \ln(H_n(\xi)) \right) \tag{16}
\end{equation}

One can solve the radial part of hydrogen atom in the same way as the above problem except that one encounters a fixed pole at the origin along with the $n$ moving poles. Thus one needs to take this pole into account, when one writes the behaviour of the QMF of the hydrogen atom in the entire complex plane similar to (15). For more details one is referred to [31] which is an earlier version of this paper. The bound state wave functions for other solvable potentials, belonging to confluent hypergeometric case, e.g., Morse and the radial oscillator potentials can be obtained in the same way. In some cases like Morse potential, one needs a suitable change of variable, before solving the QHJ equation.

### III Rosen-Morse Potential

This section is devoted to a study of solvable potentials belonging to the hypergeometric class. Note that, in general one does not come across potentials which are meromorphic in nature. As mentioned earlier to make the QMF a rational function one needs to do a suitable change of variable $y = f(x)$. For a general potential the QHJ equation
\begin{equation}
p^2 - ip' - (E - V(x)) = 0 \tag{17}
\end{equation}
after a change of variable $y = f(x)$ and introduction of $q$ by $q(y) \equiv ip(x(y))$, becomes
\begin{equation}
q^2(y) + F(y)\frac{dq(y)}{dy} + E - \tilde{V}(y) = 0. \tag{18}
\end{equation}
Here $F(y)$ equals the derivative $\frac{d(f(x))}{dx}$ expressed as a function of $y$ and $\tilde{V}(y) = V(x(y))$. We see that [18] does not have the convenient form of [2]. To bring [18] to the QHJ form we introduce $\chi$ by means of the following equations.
\begin{equation}
q = F(y)\phi, \quad \chi = \phi + \frac{1}{2} \frac{d}{dy}(\ln F(y)), \tag{19}
\end{equation}
and obtain
\begin{equation}
\chi^2 + \frac{d\chi}{dy} + \frac{E - \tilde{V}(y)}{(F(y))^2} - \frac{1}{2} \left( \frac{F''(y)}{F(y)} \right) + \frac{1}{4} \left( \frac{F'(y)}{F(y)} \right)^2 = 0 \tag{20}
\end{equation}

One can observe from (20), that the residues at the $n$ moving poles of $\chi$ will be unity. Therefore, we can make use of (20), instead of (17), for any general potential and proceed in the same way as for the harmonic oscillator.

For the Rosen-Morse potential, where the expression for the potential is
\begin{equation}
V(x) = A^2 - A(A + \alpha)\text{sech}^2\alpha x \tag{21}
\end{equation}
with $A > 0$, we first change the variable to $y = \tanh(\alpha x)$. Using (19), one obtains the equation for $\chi$ as follows:

$$
\chi^2 + \frac{d\chi}{dy} + \frac{E - A^2 + \alpha^2}{\alpha^2(1 - y^2)^2} + \frac{A(A + \alpha)}{\alpha^2(1 - y^2)} = 0. \quad (22)
$$

For the $n^{th}$ excited state, besides having $n$ moving poles with residue equal to one, QMF will have fixed poles at $y = \pm 1$. We assume that there are no other singular points in the finite complex-$y$ plane. Thus proceeding in the same way as in the harmonic oscillator, one can write $\chi(y)$ as the sum of the principal parts of the individual Laurent expansions around different singular points, plus an analytic part which will turn out to be a constant $C$. Thus

$$
\chi(y) = \frac{b_1}{y - 1} + \frac{b_1'}{y + 1} + \frac{P'}{P} + C; \quad (23)
$$

here $b_1$ and $b_1'$ are the residues at $y = \pm 1$ respectively. One obtains the residues at these poles by doing a Laurent expansion of $\chi(y)$ around each pole and substituting it in the QHJ equation for $\chi(y)$. This leads to the values

$$
b_1 = \frac{1}{2} \pm \frac{\sqrt{A^2 - E}}{2\alpha} \quad \text{and} \quad b_1' = \frac{1}{2} \pm \frac{\sqrt{A^2 - E}}{2\alpha}. \quad (24)
$$

The ambiguity in the signs of the square roots can be eliminated by applying the boundary condition (1). Since we are dealing with the SUSY potential, we can make use of the boundary condition,

$$
\lim_{E \to 0} q(y) = -W(y), \quad (25)
$$

instead of the boundary condition \([20]\) proposed by Leacock and Padgett, with $W = A\tanh(\alpha x)$, the superpotential\([9]\). Thus one obtains the correct residues as follows

$$
b_1 = \frac{1}{2} + \frac{\sqrt{A^2 - E}}{2\alpha}, \quad b_1' = \frac{1}{2} + \frac{\sqrt{A^2 - E}}{2\alpha}. \quad (26)
$$

Substituting $\chi(y)$ with the right values of $b_1$ and $b_1'$ in (22), and for large $y$ comparing the coefficients of $1/y$ and the coefficients of $1/y^2$, one gets

$$
C = 0, \quad E = A^2 - (A - n\alpha)^2 \quad (27)
$$

respectively and is left with the differential equation

$$
\frac{P''}{P} + \frac{2P'y}{P} \left(\frac{\alpha + \sqrt{A^2 - E}}{\alpha(y^2 - 1)}\right) + \frac{(\alpha + \sqrt{A^2 - E})^2}{2\alpha^2(y^2 - 1)} - \frac{E - A^2 + \alpha^2}{2\alpha^2(y^2 - 1)} - \frac{A(A + \alpha)}{\alpha^2(y^2 - 1)} = 0. \quad (28)
$$

Substitution of the energy eigenvalue in the above equation gives the Jacobi differential equation for $P$

$$
(1 - y^2)P'' - y(\mu + \nu + 2)P' + n(\mu + \nu + n + 1)P = 0. \quad (29)
$$

Thus $P$ is a Jacobi polynomial represented by $P^{\mu,\nu}_n(y)$, where $\mu = \nu = \frac{A}{\alpha} - n$. The bound state wave functions can be obtained by doing the change of variable and using (19), in (13) gives

$$
\psi(y) = \exp \left(\int \chi(y)dy + \frac{y}{1 - y^2}dy\right), \quad (30)
$$

where $\chi(y)$ is

$$
\chi(y) = \frac{\alpha + \sqrt{A^2 - E}}{2\alpha(y - 1)} + \frac{\alpha + \sqrt{A^2 - E}}{2\alpha(y + 1)} + \frac{P'}{P}. \quad (31)
$$
Thus the bound state wave function from (30) turns out to be

$$
\psi_n(y) = N(1 - y)^{\frac{\mu}{2}}(1 + y)^{\frac{\nu}{2}}P_n^{\mu,\nu}(y)
$$

(32)

which matches with the known results [9] for $A$, real and positive.

IV Scarf - I (trigonometric)

There exist potentials [26, 28] for which SUSY can be exact or broken, depending on the range of potential parameters. As mentioned earlier in the introduction, Bhalla et al. [29] have studied the eigenspectra of Scarf - I potential. In this section, we study this potential and obtain the expressions for the wave functions. The expression for the potential is

$$
V_\alpha(x) = -A^2 + (A^2 + B^2 - A\alpha\hbar)\sec^2 \alpha x - B(2A - \alpha\hbar)\tan \alpha x \sec \alpha x.
$$

(33)

For this potential, SUSY is known to be exact in the parameter range

$$
(A - B) > 0, \ (A + B) > 0,
$$

(34)

and broken in the range

$$
(A - B) > 0, \ (A + B) < 0.
$$

(35)

Since broken SUSY is one of the possibilities, unlike the previous cases, we cannot assume SUSY to be exact and hence cannot use the superpotential to find the correct residue at the fixed poles. We therefore, fall back upon the boundary condition originally proposed by Leacock and Padgett [11, 12]

$$
\lim_{\hbar \to 0^+} p \to p_{cl}.
$$

(36)

It turns out that imposing the boundary condition consistently for different ranges of the parameters $A$ and $B$, automatically gives rise to the correct wave functions and energy eigenvalues for both broken and unbroken phases of SUSY. Retaining $\hbar$ with $(2m = 1)$, replacing $p$ with $-iq$ in the QHJ equation, one gets,

$$
q^2 + \hbar \frac{dq}{dx} + E + A^2 - (A^2 + B^2 - A\alpha\hbar)\sec^2 \alpha x + B(2A - \alpha\hbar)\tan \alpha x \sec \alpha x = 0.
$$

(37)

Making a change of variable

$$
y = \sin \alpha x,
$$

(38)

and switching to $\chi$ in place of $q$ as explained previously, we get

$$
\chi^2 + \hbar \frac{d\chi}{dy} + \frac{y^2\hbar^2}{4(1 - y^2)^2} + \frac{E + A^2}{\alpha^2(1 - y^2)} + \frac{\alpha^2\hbar^2 - 2(A^2 + B^2 - A\alpha\hbar)}{2\alpha^2(1 - y^2)^2} + \frac{B(2A - \alpha\hbar)y}{\alpha^2(1 - y^2)^2} = 0.
$$

(39)

Proceeding in the same way as before, we assume that there are no singular points in the finite complex plane, other than the fixed poles at $y = \pm 1$ and the $n$ moving poles on the real line. Therefore, we write $\chi(y)$ as

$$
\chi(y) = \frac{b_1}{y - 1} + \frac{b'_1}{y + 1} + \hbar P' + C,
$$

(40)

where $b_1$ and $b'_1$ are the residues at $y = \pm 1$ respectively. The values of the residues $b_1$ and $b'_1$ are obtained as in the previous case. The values of $b_1$ at $y = 1$ are

$$
b_1 = \frac{(A - B)}{2\alpha} + \frac{\hbar}{4} \quad \text{and} \quad -\frac{(A - B)}{2\alpha} + \frac{3\hbar}{4};
$$

(41)

The two values of the residue at $y = -1$ are

$$
b'_1 = \frac{(A + B)}{2\alpha} + \frac{\hbar}{4} \quad \text{and} \quad -\frac{(A + B)}{2\alpha} + \frac{3\hbar}{4}.
$$

(42)
Different pairs of $b_1$ and $b'_1$ give physically acceptable solutions in the two phases of SUSY. We proceed to obtain these solutions by taking the correct choice of the residues using (36), the details of which are similar to those in Bhalla et.al. The boundary condition for this range of values of $A$ and $B$ implies, $b_1 > 0$ and $b'_1 > 0$ in the limit $\hbar \to 0$.

Case: 1 $A + B > 0, A - B > 0$

For this particular range of parameters the right choice of the residues is

$$b_1 = \frac{(A - B)}{2\alpha} + \frac{\hbar}{4}, \quad b'_1 = \frac{(A + B)}{2\alpha} + \frac{\hbar}{4}. \tag{43}$$

Thus $\chi$ in (40) becomes

$$\chi(y) = \left( \frac{A - B}{2\alpha} + \frac{\hbar}{4} \right) \frac{1}{y - 1} + \left( \frac{A + B}{2\alpha} + \frac{\hbar}{4} \right) \frac{1}{y + 1} + \frac{P'}{P}. \tag{44}$$

Substitution of which in (39) and putting $\hbar = 1$ gives the energy eigenvalue

$$E_n = (A + n\alpha)^2 - A^2. \tag{45}$$

The equation for $P$ assumes the form

$$(1 - y^2)P'' + (\mu - \nu - (\mu + \nu + 1)y)P' + n(n + \mu + \nu)P = 0. \tag{46}$$

where $\mu = \frac{A - B}{\alpha}$ and $\nu = \frac{A + B}{\alpha}$. The above differential equation is of the form of the Jacobi differential equation and hence $P$ coincides with $P_n^{\mu-1/2, \nu-1/2}$. The bound state wave function when SUSY is exact, is

$$\psi_n(y) = N(1 - y)^{\frac{\mu}{2}}(1 + y)^{\frac{\nu}{2}}P_n^{\mu-1/2, \nu-1/2}(y). \tag{51}$$

These wave functions correspond to the solutions of the unbroken phase of SUSY.

Case: 2 $A - B > 0, A + B < 0$

For this range of the parameters, the right choice of the residues is

$$b_1 = \frac{(A - B)}{2\alpha} + \frac{\hbar}{4}, \quad b'_1 = -\frac{(A + B)}{2\alpha} + \frac{3\hbar}{4}. \tag{48}$$

Substitution of $\chi$ with these values of $b_1$ and $b'_1$ in (39) and putting $\hbar = 1$ gives the energy eigenvalue as

$$E = \left( B - \left( n + \frac{1}{2}\right)\alpha \right)^2 - A^2. \tag{49}$$

and the differential equation for $P$ turns out to be

$$(1 - y^2)P'' + (\nu - (\mu + 1) - y(\nu + \mu + 2))P' + n(n + 1 + \mu + \nu)P = 0 \tag{50}$$

and the bound state wave function is found to be

$$\psi_n(y) = N(1 - y)^{\frac{\mu}{2}}(1 + y)^{\frac{\nu}{2}}P_n^{\mu+1, \nu-1}(y). \tag{51}$$

where $\mu = \frac{A - B}{\alpha}, \nu = \frac{A + B}{\alpha}$. These bound state wave functions correspond to the broken phase of SUSY of the Scarf-I potential. The Scarf-I potential can be related to the Pöschl-Teller-I potential by redefinition of the potential parameters as

$$\alpha x \to \alpha x + \frac{\pi}{2}, \quad A = \delta + \beta \text{ and } B = \delta - \beta. \tag{52}$$
With this redefinition of the parameters and then putting $\alpha = 1$, we see that the bound state wave functions in (51) matches with the bound state wave functions of the broken SUSY phase of the Pöschl-Teller-I potential given in (28). Thus we see that the QHJ formalism in one dimension gives the correct expressions of the bound state wave functions, when there are different phases of SUSY. It may be remarked here that, in the range $A - B < 0, A + B < 0$, SUSY is exact but the roles of $H_-$ and $H_+$ are interchanged. In the range $A - B < 0, A + B > 0$ SUSY is again broken. The methods outlined above, has been applied to many other potentials like the Morse, Eckart, Pöschl-Teller(hyperbolic) and Scarf-II (hyperbolic) and the results agree with those given in (2).

V Conclusions

In conclusion, the singularity structure of the quantum momentum function, in the complex domain, can be effectively utilized for obtaining both eigenfunctions and eigenvalues in an efficient manner. Besides illustrating the procedure to find the wave functions, through the analysis of the exactly solvable problems, we have clarified the role of an appropriate implementation of the boundary condition, necessary to solve the quantum Hamilton-Jacobi equation. The same played an important role in yielding the eigenvalues and eigenfunctions of potentials exhibiting broken and unbroken phases of SUSY. As has been pointed out earlier, a close connection between Kolmogorov-Sinai entropy and the action has been obtained recently, making use of a Hamilton-Jacobi approach through Bohm’s formulation. The QHJ formalism of Leacock and Padgett and the above are not identical. Hence, a deeper exploration of the above connection, as well as the extension of the Hamilton-Jacobi formalism to intrinsically non-separable systems, like Hamiltonians whose classical dynamics reveal chaotic behavior, need thorough investigation. Although, we have dealt with solvable systems here, one needs to extend this approach, to non-solvable cases, like various anharmonic oscillators. We would like to mention that, during the course of our study, we have analyzed scattering states of the Rosen-Morse potential. It was found, by assuming that QMF has finite number of moving singularities, one can obtain the scattering state wave functions, provided the potential parameter $A$ is equal to an integer $n$, a special case when the potential is reflectionless. The case when $A \neq n$ needs further study, as also a more careful analysis of the general scattering problem. The continuous energy functions, of interest for scattering, are currently under study.

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[1] Due to typographical errors in the table 3 of [29], the results do not match the results obtained in this study.