Weak Field Phase Diagram for an Integer Quantum Hall Liquid

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We study the localization properties in the transition from a two-dimensional electron gas at zero magnetic field into an integer quantum Hall (QH) liquid. By carrying out a direct calculation of the localization length for a finite size sample using a transfer matrix technique, we systematically investigate the field and disorder dependences of the metal-insulator transition in the weak field QH regime. We obtain a different phase diagram from the one conjectured in previous theoretical studies. In particular, we find that: (1) the extended state energy $E_c$ for each Landau level (LL) is always linear in magnetic field; (2) for a given Landau level and disorder configuration there exists a critical magnetic field $B_c$ below which the extended state disappears; (3) the lower LLs are more robust to the metal-insulator transition with smaller $B_c$. We attribute the above results to strong LL coupling effect. Experimental implications of our work are discussed.

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It is very important to understand the localization properties in the transition from two-dimensional electron gas (at zero magnetic field) into an integer quantum Hall liquid\textsuperscript{[1]}. According to the scaling theory of localization\textsuperscript{[2]} all electrons in a two-dimensional system are localized in the absence of magnetic field. When the two-dimensional electron system is subject to a strong perpendicular magnetic field, the energy spectrum becomes a series of impurity broadened Landau levels. Extended state appears in the center of each Landau band, while states at other energies are localized. This gives rise to the integer quantum Hall effect. The interesting issue is to understand the evolution of the extended states in the weak field regime as the magnetic field goes to zero where all extended states disappear.

There could be two scenarios for the fate of the extended states as $B \to 0$. The first one was proposed by Kivelson, Lee, and Zhang\textsuperscript{[3]} in their global phase diagram of the quantum Hall effect. According to this phase diagram, in a strongly disordered quantum Hall system, the extended states stay with the center of the Landau bands at strong magnetic field, but float up in energy at small magnetic field and go to infinity as $B \to 0$. This phase diagram is consistent with the semiclassical argument put forth by Khmelnitskii\textsuperscript{[4]} and Laughlin\textsuperscript{[5]}. In this letter, we are proposing an alternative scenario for the behavior of the extended states at weak magnetic field limit. In our picture, each extended state is simply destroyed by strong disorder at a critical magnetic field instead of floating up in energy. By carrying out a direct calculation of the localization length for a finite size sample using a transfer matrix technique, we systematically investigate the field and disorder dependence of the metal-insulator transition in the weak field quantum Hall regime. We find that: (1) the extended state energy $E_c$ for each Landau level (LL) is always linear in magnetic field; (2) for a given Landau level and disorder configuration there exists a critical magnetic field $B_c$ below which the extended state disappears; (3) the lower LLs are more robust to the metal-insulator transition with smaller $B_c$. We attribute the above results to strong LL coupling effect.

FIG. 1. Sketch of our proposed weak field phase diagram for an integer quantum Hall liquid, where $E$ is the Fermi energy, and $B$ the magnetic field. Shaded area represents the metallic regime.

Our results can be summarized as the phase diagram presented in Fig. 1. At strong magnetic field, the extended states appear in the center of Landau bands ($E_n = (n + 1/2)\hbar \omega_c$, where $\omega_c = eB/mc$). As the magnetic field decreases, the energies for the extended states also decrease. In the weak magnetic field regime, the extended states disappear when the magnetic field is less than a critical magnetic field which depend on the parameters of the quantum Hall system (such as sample size and disorder strength). The higher the Landau level
is, the larger the critical magnetic field for destroying the extended states. The extended state associated with the lowest Landau band is the most robust and survives at very low magnetic field and is finally destroyed before the magnetic field reaches zero. In other words, in our scenario, the energies for the extended states never “float” up and there is a critical magnetic field to create any extended states for a given disorder configuration.

There have been a number of experimental attempts \cite{6–10} to address the transition of delocalized states at the weak magnetic field limit. Earlier experiments \cite{11} on strongly disordered two-dimensional electron gas (2DEG) have demonstrated a transition from an Anderson insulator at $B = 0$ (with all states localized) to quantum Hall conductor at strong magnetic field. These experiments serve as direct evidence for magnetic field induced delocalization in 2D systems. They are certainly consistent with our weak field phase diagram.

Two very recent experiments on low mobility gated GaAs/AlGaAs heterostructure \cite{9} and high mobility Si samples \cite{10} have reported floating up of the delocalized states in the carrier-density – magnetic-field plane. The authors in Ref. \cite{9} claim that their experimental result “unambiguously” demonstrated that the “energy” of the delocalized state (extended states) floats up as $B \to 0$. However, as we will show in the latter part of this paper, floating up of the carrier density does not necessarily imply the same behavior of the energy. Strong disorder scattering and Landau level mixing effects (which are important at the weak field limit) could contribute to anomalous behavior of the carrier density while keeping the extended states at the center of lowest Landau band. In fact, we will demonstrate that these experimental results \cite{6–10} are all consistent with our weak field phase diagram (Fig. 1) for the integer quantum Hall liquid.

In the following, we briefly outline our model and technique to calculate the localization length. We model our two-dimensional system in a very long strip geometry with a finite width ($M$) square lattice with nearest neighbor hopping. Periodic boundary condition in the width direction is used to get rid of the edge extended states. The disorder potential is modeled by the on-site white-noise potential $V_{im}$ ($i$ denotes the column index, $m$ denotes the chain index) ranging from $-W/2$ to $W/2$. The effect magnetic field appears in the complex phase of the hopping term. The strength of the magnetic field is characterized by the flux per plaquette ($\phi$) in unit of magnetic flux quanta ($\phi_i = \hbar c/e$). The Hamiltonian of this system can be written as:

$$H = \sum_{i} \sum_{m=1}^{M} V_{im} |im><im|$$

$$+ \sum_{<im;jn>} \left[ t_{im;jn} |im><jn| + t_{im;jn}^\dagger |jn><im| \right],$$

where $<im;jn>$ indicates nearest neighbors on the lattice. The amplitude of the hopping term is chosen as the unit of the energy. For a specific energy $E$, a transfer matrix $T_i$ can be easily set up mapping the wavefunction amplitudes at column $i - 1$ and $i$ to those at column $i + 1$, i.e.,

$$\begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = T_i \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} = \begin{pmatrix} H_i - E & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix},$$

where $H_i$ is the Hamiltonian for the $i$th column, $I$ is a $M \times M$ unit matrix. Using a standard iteration algorithm \cite{11}, we can calculate the Lyapunov exponents for the transfer matrix $T_i$. The localization length $\lambda_{M}(E)$ for energy $E$ at finite width $M$ is then given by the inverse of the smallest Lyapunov exponent. In our numerical calculation, we choose the sample length to be over $10^4$ so that the self-averaging effect automatically takes care of the ensemble statistical fluctuations.

**FIG. 2.** Finite-size localization length ($\lambda_{M}$) at different weak magnetic field (from the lowest curve to the highest one. (a) $\phi = 1/13, 1/9, 1/7, 1/5$; (b) $\phi = 1/33, 1/31, 1/29, 1/27, 1/25$) for fixed disorder strength $W = 1$. The localization length for higher magnetic field is scaled by a factor proportional to the magnetic field from the one for next smaller field to show the linear dependence in magnetic field.

In Fig. 2 we present the localization length at various weak magnetic fields for a finite size sample ($M = 32$). Due to the symmetry of the lattice model, only the lower
energy branch results are shown here. In the absence of disorder, the energy band for a tight-binding lattice should break up into $q$ subbands in a magnetic field with $\phi = 1/q$ (where $q$ is an integer). In a continuous model, the subbands in the lower energy branch correspond to Landau levels. In the presence of disorder, for a continuous model, each Landau level evolves into impurity broadened Landau band with extended state in the center of each Landau band. We have similar effects in the lattice model as presented in Fig. 2. The maximas in the finite-size localization length are the locations of the subbands. The extended states appear at the centers of these subbands and their energies (count from the band edge $E = -4.0$) are linear in $B \propto \phi$, resembling the behavior of a continuous model.

FIG. 3. Finite-size localization length ($\lambda_M$) for different disorder strength at magnetic field $\phi = 1/11$ and $W = 1, 2, 3, 4, 5$. (from the highest curve to the lowest one). The localization length for weaker disorder strength is scaled by a factor of 500 from the one for next stronger disorder strength.

We now address the effect of disorder strength on the extended states at the weak magnetic field limit. As presented in Fig. 3, the localization length decreases as strength of disorder increases. In a fixed magnetic field, the extended states in higher energy subbands (Landau bands) are destroyed in the presence of strong enough disorder, while at the same time the width of the extended-state bands in the lower energy subbands (Landau bands) become narrower. The extended states in the lowest energy subband (Landau band) are the most robust and nevertheless destroyed at very strong disorder strength. In the entire range of the disorder strength shown in Fig. 3, even though the localization lengths are changed by several order of magnitude, the energies of the extended states still stay with corresponding energy band without floating up at strong disorder limit. We can conclude that in a continuous model with fixed disorder strength, the extended states always stay in their corresponding Landau bands, and are finally destroyed below certain critical magnetic field (the higher the Landau level is, the larger the critical field). This is the basis for our proposed weak field phase diagram (Fig. 1). We believe that the localization transition here is caused by Landau level coupling effect which is more severe at weak field. For our model in Eq.(1), the zero-field level broadening is

$$\Gamma = \frac{W^2}{6\pi E K(4t/E)},$$

where $t$ is hopping amplitude which is set to unity and $K(x)$ is the complete elliptical integral of first kind. The peaks in Fig. 3 start to disappear when $\Gamma \simeq \omega_c$, i.e. when the Landau levels start to couple together. We should mention that the global phase diagram [3] is based on the level floating up theory. However, that theory has a crucial assumption: the extended states are concentrated at discrete levels. This assumption is certainly true for strong field and smooth random potentials. To our knowledge, its validity for short range potentials has not been firmly established. The situation for weak field and strong disorder is totally unclear. There exists no theory showing that the extended states still have to remain at discrete levels when the Landau levels are strongly mixing.

In the experiments observing transition from Anderson localization to quantum Hall conductor [5, 6], only the lowest Landau level plateau was observed ($\nu = 2$ for spin unresolved 2DEG) which is consistent with our argument that the delocalization states in the lowest Landau band are the most robust. We propose similar experiment in strong magnetic field to test our scenario that higher Landau level QH plateau should emerge in sequence.

To explain the anomalous floating up of the carrier density in the recent experiment by Glazman et al [7], we should include the Landau level mixing effect at small magnetic field in the presence of strong disorder scattering. In the weak disorder limit, the Landau level broadening is much smaller comparing with the inter-Landau level energy difference as shown by Ando et al [11] in a simple Born-approximation calculation without any inter-Landau level coupling. However in a realistic situation, the disorder potential is Coulomb long range type and Landau levels are much more broadened than in the short-range impurity case, and therefore, the effect of Landau level mixing is much more important as demonstrated by Xie et al [12]. In another word, the localized tail of higher Landau bands could well extended into the lower Landau bands. So the electrons have to fill up the tail of higher Landau bands before reaching the extended states in the center of the lowest Landau band, and therefore, the real filling factor could be greater than 2 (for spin unresolved 2DEG) when the QH plateau for the lowest Landau level is observed. The critical point where the carrier density starts to float up is the moment
where inter-Landau level mixing shows up. The experimental fact that no floating was observed in higher mobility samples \[9\] strongly demonstrate the essential role of the Landau level mixing in the floating up of the carrier density. In the following we will demonstrate the above argument at a more quantitative level. Let us first consider the high field limit such that Landau levels are well separated. In this limit, the electron density $\rho_c$ below $E_c$ (energy of the extended state for the lowest Landau level) is proportional to the Landau level degeneracy. Thus, $\rho_c$ decreases linearly with decreasing $B$. At the weak field limit with strong Landau level coupling, the experimental density of states in GaAs samples can be well described by the Lorentzian form \[15\]

$$g(E) = \frac{1}{2\pi l^2} \sum \frac{\Gamma_n}{(E - E_n)^2 + \Gamma_n^2}$$

and

$$\rho_c = \int_{-\infty}^{E_0} g(E)dE \propto \omega_c \sum \frac{1}{n} \frac{2}{\pi} \arctan \frac{n\omega_c}{\Gamma_n}$$

where $l$ is the magnetic length, $\omega_c$ is the cyclotron energy and $E_n = (n + 1/2)\omega_c$ is the energy for the $n$th Landau level. $\Gamma_n$ is the level broadening for $n$th Landau level which is found experimentally \[15\] to be independent of magnetic field. As $B$ (or $\omega_c$) decreases with fixed $\Gamma_n$, more terms will be contained in the summation which makes $\rho_c$ to increase. In the limit of infinite summation over Landau levels, $\rho_c$ diverges. This simple model calculation shows that in the high field limit $\rho_c$ goes down linear with decreasing $B$, and in the zero field limit it approaches infinity. Therefore, $\rho_c$ has to float up at a certain magnetic field. We should mention that more realistic calculations \[14\] are needed in order to make quantitative comparisons with the experimental results.

In conclusion, we have proposed an alternative phase diagram for the integer quantum Hall system in the weak field limit. We have demonstrated that there exists a critical magnetic field to delocalize any two-dimensional disordered system. The extended states always stays with their corresponding Landau bands with those in the lowest Landau bands the most robust. In the strong disorder limit, Landau level mixing effect could contribute to the floating up of the carrier density even though the energies of the extended states never float up.

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[1] For a review, see The Quantum Hall Effect, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York, 1990).