On the Averaged Green’s Function of an Elliptic Equation with Random Coefficients

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Abstract

We consider a divergence-form elliptic difference operator on the lattice $\mathbb{Z}^d$, with a coefficient matrix that is an i.i.d. perturbation of the identity matrix. Recently, Bourgain introduced novel techniques from harmonic analysis to prove the convergence of the Feshbach-Schur perturbation series related to the averaged Green’s function of this model. Our main contribution is a refinement of Bourgain’s approach which improves the key decay rate from $-2d + \epsilon$ to $-3d + \epsilon$. (The optimal decay rate is conjectured to be $-3d$.) As an application, we derive estimates on higher derivatives of the averaged Green’s function which go beyond the second derivatives considered by Delmotte–Deuschel and related works.

1. Introduction

In the late 1950s, De Giorgi, Nash and Moser [16,29,31] completed the classical regularity theory for elliptic and parabolic equations with bounded and measurable coefficients. Their results include the Hölder regularity of weak solutions $u$ to the divergence-form elliptic equation $\nabla^* A(x) \nabla u = 0$ with rough coefficient matrix $A(x)$. Subsequently, it was also shown that the Green’s function $G_A(x, y)$ is controlled by the Green’s function of the ordinary Laplacian. Specifically, when $d \geq 3$, it holds that

$$|G_A(x, y)| \leq C|x - y|^{2-d}$$

for all $x, y \in \mathbb{R}^d$; see [3,4,26].

When the coefficient matrix $A(x)$ is generated by a stationary random process, one may consider regularity properties that hold on average or with high probability; see, e.g., [2,19,21,27]. Here we focus on the averaged (or “annealed”) Green’s function $\mathbb{E}[G_A(x, y)]$, which is translation-invariant in the sense that

$$\mathbb{E}[G_A(x, y)] = G(x - y)$$
for some function $G$. In this setting, Conlon–Naddaf [12] (see also [10]) observed that the averaged Green’s function $G(x)$ is continuously differentiable for $x \neq 0$ and its derivative satisfies the decay estimate

$$|\nabla G(x)| \leq C|x|^{1-d},$$

when working on either $\mathbb{R}^d$ or $\mathbb{Z}^d$ with $d \geq 3$. Note that the decay rate $1-d$ is optimal in view of the Green’s function of the ordinary Laplacian. In the discrete setting, [12] also proved that the second derivatives of $G$ are controlled by $C\delta(1 + |x|)^{-d+\delta}$ for arbitrarily small $\delta > 0$.

The result of Conlon-Naddaf was extended by Delmotte–Deuschel [15] who adapted the classical regularity theory to the random setting. They showed that the second derivatives of the averaged Green’s function can actually be controlled with the optimal decay rate:

$$|
abla^\alpha G(x)| \leq C|x|^{2-d-|\alpha|}, \quad \text{for any multi-index } |\alpha| \leq 2, \quad (1.2)$$

again on $\mathbb{R}^d$ or $\mathbb{Z}^d$ when $d \geq 3$. In fact, [15] establish a stronger version of (1.2) where one takes the absolute value before taking expectation. Moreover, they have a similar result for $d = 2$, i.e., (1.2) holds with $1 \leq |\alpha| \leq 2$, if the first and second derivatives of $G$ are properly interpreted. In the discrete case (i.e., on $\mathbb{Z}^d$), there is no singularity near the origin and so $|x|^{2-d-|\alpha|}$ can be replaced by $(1 + |x|)^{2-d-|\alpha|}$ in (1.2).

We mention that the elliptic results presented here have parabolic analogs; see, e.g., [9,12,15,31].

In the last few years, the derivative estimate (1.2) on the averaged Green’s function has been generalized to higher moments and to the non-scalar case [11,18,27,28]. One reason for the continued interest in these Green’s function estimates is that they have applications to the quantitative theory of stochastic homogenization. Consider for example a family of equations of the form

$$\nabla^* A \left( \frac{x}{\epsilon} \right) \nabla u_\epsilon = f, \quad (1.3)$$

indexed by $\epsilon > 0$, with a random coefficient matrix $A(x)$ and a deterministic function $f$. Under certain assumptions on $A(x)$, it is known that, as $\epsilon \to 0$, a solution $u_\epsilon$ to (1.3) can be approximated by a solution $u$ to a “homogenized” deterministic constant coefficient equation. This general phenomenon is called stochastic homogenization and has been extensively studied; see [24,25,30,32,36] and the more recent works [1,2,6,7,13,14,17,20–23]. While stochastic homogenization furnishes part of our general motivation, we will not directly discuss it anymore in what follows.

Despite the recent research activities on the averaged Green’s function, it has been unknown, to the best of our knowledge, whether the optimal decay rate in (1.2) holds true beyond the second derivatives. One consequence of our results is that the estimate (1.2) indeed extends to higher order derivatives for all $|\alpha| \leq d+1$, in the discrete setting when $A(x)$ is an i.i.d. perturbation of the identity matrix.

Our argument is different from those in [12,15] and is based on the line of research recently initiated by Sigal [33] and Bourgain [8]. Bourgain gave a rather
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precise description of an averaged operator $L$ defined in (1.7) (whose Green’s function is exactly the averaged Green’s function $G$ from above). His main result establishes the convergence of the Feshbach-Schur perturbation series and shows, in particular, in which way the deterministic operator $L$ retains some spatial dependence. Our main result improves a key decay estimate for the spatial dependence of $L$ obtained in [8]; see Theorem 1.1 below. The estimate on the higher derivatives of the averaged Green’s function is a corollary of this main result and is obtained by using standard tools from Fourier analysis.

We organize this paper as follows: in the remainder of this section, we give our setup, the precise statements of our main results, and an outline of the argument. In Sect. 2, we provide background: (a) We precise operator-theoretic aspects of the setup, and (b) we recall two key tools introduced in [8] and state abstract versions to be used later on. We prove our main results, Theorems 1.1 and 1.9, in Sects. 3 and 4, respectively. We prove the new derivative estimates on the averaged Green’s function, Corollary 1.5, in the Appendix. In addition, we give proofs of the statements in Sect. 2 in the Appendix for completeness.

Notations. Let $-\Delta = \nabla^*\nabla$ be the standard Laplacian on $\mathbb{Z}^d$, where $\nabla = (\nabla_1, \nabla_2, \ldots, \nabla_d)^T$ is the discrete derivative. For a function $u : \mathbb{Z}^d \to \mathbb{R}$ or $\mathbb{C}$, it is defined by

$$\nabla_j u(x) := u(x + e_j) - u(x)$$

for the $j$-th standard unit vector $e_j$. We denote by $\nabla^* = (\nabla_1^*, \ldots, \nabla_d^*)$ the adjoint of $\nabla$, where

$$\nabla_j^* u(x) := u(x - e_j) - u(x).$$

For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, $\alpha_j \geq 0$, we write $|\alpha| = \sum_{j=1}^d \alpha_j$ and $\nabla^\alpha = \nabla_1^{\alpha_1} \cdots \nabla_d^{\alpha_d}$.

1.1. Statement of Main Results

We continue with the precise setup of the model. Let $d \geq 2$. For each $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, an underlying probability space, let $A(x) = A(x, \omega)$ be an i.i.d. perturbation of the identity matrix $I_d$, i.e.,

$$A(x, \omega) := (1 + \delta \sigma(x, \omega))I_d,$$

where $0 < \delta < 1$ is a small parameter and $\{\sigma(x, \cdot)\}_{x \in \mathbb{Z}^d}$ is a bounded family of real-valued i.i.d. random variables which we normalize by the condition $||\sigma||_{L^\infty(\mathbb{Z}^d \times \Omega)} \leq 1$. (Our results extend to non-diagonal i.i.d. perturbations; see Remark 1.4.) The i.i.d. model (1.4) is perhaps the simplest possible choice for the random coefficient matrix; see [13, Theorem 1.2], where a corresponding homogenization problem is addressed.

Let $L$ be the corresponding i.i.d. perturbation of the Laplacian on $\mathbb{Z}^d$, i.e.,

$$L := \nabla^* A \nabla = -\Delta + \delta \nabla^* \sigma I_d \nabla.$$
We also write \( L_\omega := \nabla^* A(\cdot, \omega) \nabla \) to emphasize the dependence on \( \omega \in \Omega \). Assume that \( u_\omega \) is the solution to the equation
\[
L_\omega u_\omega(x) = f(x),
\]
(1.6)
where \( f : \mathbb{Z}^d \to \mathbb{R} \) is a deterministic function.

Sigal [33] considered the averaged operator \( L \) which describes the averaged solution \( E u_\omega \) via the equation
\[
L[E u_\omega](x) = f(x).
\]
(1.7)
We note that \( L \) can be written formally as \( (E L - 1/\Omega) - 1 \).

To motivate this operator further, consider the case when \( f \) is the discrete Dirac delta function \( \delta_y \) on \( \mathbb{Z}^d \) for some \( y \in \mathbb{Z}^d \) and assume that \( u_y \omega \) is the solution to (1.6) in an appropriate \( \ell^p(\mathbb{Z}^d) \) space. We recall that the function \( G_\omega(x, y) := u_y \omega(x) \) is the Green’s function for the operator \( L_\omega \). Therefore, in a sense, Sigal considers an operator \( L \) whose Green’s function is equal to the averaged Green’s function associated with the operator \( L \). We defer a further discussion to Sect. 1.2.

By introducing several novel techniques from harmonic analysis to the problem, Bourgain [8] recently established the remarkable result that the operator \( L \) can be expressed as a convergent perturbation series for sufficiently small \( \delta > 0 \), and it admits the representation
\[
L = (1 + \delta E \sigma)(-\Delta) + \nabla^* K^\delta \nabla.
\]
(1.8)
Here, \( E \sigma \in \mathbb{R} \) is the expectation of any copy of \( \sigma(x, \cdot) \) and \( K^\delta = (K^\delta_{i,j})_{1 \leq i, j \leq d} \) is an operator-valued matrix whose matrix elements \( K^\delta_{i,j} \) are convolution operators on \( \mathbb{Z}^d \). Bourgain also proved that the convolution kernel \( K^\delta_{i,j}(x) \) is controlled by the decaying function \( (1 + |x|)^{-2d+\epsilon} \), with \( \epsilon > 0 \) depending on \( \delta \). This implies that its Fourier transform \( \hat{K}^\delta_{i,j} \) belongs to the Hölder space \( C^{d-1,1-\epsilon}(\mathbb{T}^d) \).

The main result of this paper is the following improved decay estimate for the convolution kernel:

**Theorem 1.1.** (Main result) Let \( d \geq 3 \) and \( 0 < \epsilon < 1 \). There is a constant \( c_d > 0 \) such that the representation (1.8) is valid for any \( 0 < \delta < c_d \epsilon \). Moreover, the convolution kernel of \( K^\delta_{i,j} \) obeys the decay estimate
\[
|K^\delta_{i,j}(x - y)| \leq C_d \delta^2 (1 + |x - y|)^{-3d+\epsilon}
\]
(1.9)
for all \( x, y \in \mathbb{Z}^d \), with an additional factor of \( \delta^2 \) when \( x \neq y \).

Consequently, the Fourier transform \( \hat{K}^\delta_{i,j} \) is an element of the Hölder space \( C^{2d-1,1-\epsilon}(\mathbb{T}^d) \); in particular, it has \( 2d - 1 \) continuous derivatives.

**Remark 1.2.** (i) The exponent \(-3d + \epsilon\) in (1.9) improves the exponent \(-2d + \epsilon\) obtained in [8]. Theorem 1.1 also yields additional factors of \( \delta \) and quantifies the dependence on \( \epsilon \) for the allowed range of \( \delta \). However, it is an interesting open question whether this dependence on \( \epsilon \) can be completely removed.
(ii) Our work is motivated by a conjecture of Tom Spencer (private communication), which says that $-3d$ should be the optimal decay rate in (1.9). Note that our bound (1.9) establishes the conjecture up to an arbitrarily small $\epsilon > 0$. The conjecture is supported by an examination of the $n = 3$ term in the perturbation series (1.14), which is the leading contribution in $\delta$ when $x \neq y$.

(iii) The proof of Theorem 1.1 also shows that the function $\delta \mapsto K_\delta(0)$, defined on $\delta \in (0, c_d)$, is analytic and given by the convergent perturbation series (1.14) below. One consequence of this fact is that the diffusion matrix of $L$, more precisely the collection

$$\{(1 + \delta E_\sigma)I_d + K_\delta(0)\}_{0 < \delta < c_d},$$

uniquely determines the probability distribution of the random variable $\sigma$; see Proposition 1.8. This indicates that the averaged operator $L$ still contains a considerable amount of information on the probability distribution of $\sigma$.

Our proof yields a similar result for some regularized versions of $L_\omega$, with bounds that are uniform in the regularizing parameter and in this case one can include $d = 2$. For instance, consider the operator $L_\omega^\mu := L_\omega + \mu I$ for each $\mu > 0$. The operator $L_\omega^\mu$ is strictly positive and therefore invertible on $\ell^2(\mathbb{Z}^d)$. We refer the reader to [22, Lemma 4] for a pointwise decay estimate for the Green’s function of $L_\omega^\mu$. We state a version of Theorem 1.1 for the corresponding averaged operator $L^\mu$.

**Theorem 1.3.** Let $d \geq 2$, $\mu > 0$ and $0 < \epsilon < 1$. There is a constant $c_d > 0$ such that, any $0 < \delta < c_d\epsilon$, we may write

$$L^\mu = (1 + \delta E_\sigma)(-\Delta) + \mu I + \nabla^* K_\delta^\mu \nabla$$

for a convolution operator $K_\delta^\mu = (K_{i,j}^\delta,\mu)_{1 \leq i, j \leq d}$. The convolution kernel of $K_{i,j}^\delta,\mu$ obeys the decay estimate

$$|K_{i,j}^\delta,\mu(x - y)| \leq C_d \delta^2 (1 + |x - y|)^{-3d+\epsilon}$$

uniformly in $\mu > 0$ for all $x, y \in \mathbb{Z}^d$, with an additional factor of $\delta^2$ when $x \neq y$. Consequently, the Fourier transform $\hat{K}_{i,j}^{\delta,\mu}$ is an element of the Hölder space $C^{2d-1.1-\epsilon}(\mathbb{T}^d)$.

We omit the proof of Theorem 1.3 since it is identical to the proof of Theorem 1.1 modulo the replacement of the positive operator $-\Delta$ by the strictly positive operator $-\Delta^\mu := -\Delta + \mu I$ in each step of the proof.

**Remark 1.4.** It is straightforward to generalize Theorems 1.1 and 1.3 for the coefficient matrix of the form

$$A(x, \omega) := A_0 + \delta \Sigma(x, \omega).$$

Here, $A_0$ is a deterministic, positive definite matrix satisfying

$$\sum_{1 \leq i, j \leq d} a_i(A_0)_{i,j} \geq c \sum_{1 \leq i \leq d} |a_i|^2$$
for some constant \( c > 0 \) for any \( a_i \in \mathbb{C} \) and \( \Sigma(x, \omega) = (\sigma_{i,j}(x, \omega))_{i,j} \) is a symmetric \( d \times d \) matrix and
\[
\left\{ \sigma_{i,j}(x, \cdot) : 1 \leq i, j \leq d, \ x \in \mathbb{Z}^d \right\}
\]
is a family of identically distributed random variables satisfying the following independence condition: for any \( 1 \leq i, j, i', j' \leq d \), the random variables \( \sigma_{i,j}(x, \cdot) \) and \( \sigma_{i',j'}(y, \cdot) \) are independent if \( x \neq y \). (If complex-valued functions \( u : \mathbb{Z}^d \to \mathbb{C} \) are considered, then it is necessary to restrict to Hermitian matrices \( \Sigma \), in order to ensure ellipticity.) In this setting, one finds
\[
\mathcal{L} = \nabla^* \mathbb{E} A \nabla + \nabla^* K^\delta \nabla
\]
and the operator kernel of \( K^\delta \) satisfies the bound (1.9).

As a corollary of Theorems 1.1 and 1.3, we establish decay estimates for the discrete derivatives of the averaged Green’s functions associated with the operators \( L \) and \( L^\mu \), which coincide with the Green’s functions \( G(x - y) \) and \( G^\mu(x - y) \) for the translation invariant operators \( \mathcal{L} \) and \( \mathcal{L}^\mu \), respectively. These estimates extend the result (1.2) from [12,15] to higher order derivatives for our choice of random environment.

**Corollary 1.5.** (Bounds on the averaged Green’s function) There is a constant \( c_d > 0 \) such that the following holds for any \( 0 < \delta < c_d \):

(i) If \( d \geq 3 \), then
\[
|\nabla^\alpha G(x)| \leq C_d (1 + |x|)^{-(d-2 + |\alpha|)}
\]
holds for any multi-index \( 0 \leq |\alpha| \leq d + 1 \). Similar estimates hold for \( \nabla^\alpha G^\mu \) uniformly in \( \mu > 0 \).

(ii) If \( d = 2 \), then
\[
|\nabla^\alpha G^\mu(x)| \leq C_d (1 + |x|)^{-|\alpha|}
\]
unifomly in \( \mu > 0 \) for any multi-index \( 1 \leq |\alpha| \leq 3 \).

As a consequence of Theorem 1.1 and Corollary 1.5, we obtain an estimate on the derivatives of the averaged solution \( \mathbb{E}[u_\omega] \) to \( L_\omega u_\omega = f \).

**Corollary 1.6.** Let \( d \geq 3 \) and let \( p_d \) be the Hardy–Littlewood–Sobolev exponent, i.e., \( p_d^{-1} = 2^{-1} + d^{-1} \). Assume that \( f \in \ell^{p_d}(\mathbb{Z}^d) \). For each \( \omega \in \Omega \), there exists a unique solution \( u_\omega \in \ell^{q_d}(\mathbb{Z}^d) \), \( q_d^{-1} := 2^{-1} - d^{-1} = (p_d')^{-1} \), to
\[
L_\omega u_\omega = f,
\]
such that \( |\nabla u_\omega| \in \ell^2(\mathbb{Z}^d) \). The averaged solution can be represented by
\[
\mathbb{E}[u_\omega] = L^{-1} f = G \ast f.
\]
Moreover, there is a constant $c_d$ such that for any $0 < \delta < c_d$, the derivatives of the average can be estimated pointwise by

$$|\nabla^\alpha \mathbb{E}[u_\omega](x)| \leq C \sum_{y \in \mathbb{Z}^d} \frac{|f(y)|}{(1 + |x - y|)^{d-2+|\alpha|}}$$

for any multi-index $0 \leq |\alpha| \leq d + 1$.

We prove Corollaries 1.5 and 1.6 in the Appendix. For the former, we use that the Fourier space representations of $\nabla^\alpha G$ and $\nabla^\alpha G^\mu$ can be controlled via Theorems 1.1 and 1.3.

In the next subsection, we sketch the proof of Theorem 1.1. This ultimately motivates an alternative approach towards Theorem 1.1. We raise a question regarding that approach and partially answer the question by our second main result, Theorem 1.9.

In the following, we commonly abuse notation and identify operators with their kernels, i.e., we do not distinguish notationally between a function $K : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$ and the operator $Kf(x) = \sum_{y \in \mathbb{Z}^d} K(x, y)f(y)$. From now on, $C$ denotes a positive constant that is uniform in all the parameters except dimension and whose numerical value may change from line to line.

### 1.2. Outline of the Argument

Our proof of Theorem 1.1 relies on the techniques introduced in [8,33]. Specifically, it starts with the Feshbach-Schur approach to the averaged Green’s function introduced in [33], while the analysis of the perturbation series is performed by refining the argument of Bourgain [8]. In this section, we review the Feshbach-Schur series and Bourgain’s approach, and in the end we briefly sketch how we can go beyond Bourgain’s result.

#### 1.2.1. Derivation of the Perturbation Series Via Feshbach-Schur

Let us sketch the derivation of (1.8); for further details see [8]. We may view \(L = \nabla^* A \nabla\) as a map acting on functions on the product space $\mathbb{Z}^d \times \Omega$ via $Lu(x, \omega) = \omega_\omega \mu(x)$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be appropriate function spaces so that $L : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and its inverse $L^{-1} : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ are bounded operators. (The precise definition of these function spaces is postponed to Sect. 2.1.)

Let $P = \mathbb{E}[-]$ be the projection operator acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, and $P^\perp := I - P$. The equation (1.6) may be written as $Lu = f = Pf$ and we have $Pu = PL^{-1}Pf$. Therefore, we may identify the operator $L$ with the inverse of $PL^{-1}P : P\mathcal{H}_2 \rightarrow P\mathcal{H}_1$.

To compute $(PL^{-1}P)^{-1}$, one may decompose the operator $L$ into blocks

$$L = \begin{pmatrix}
PPL & PL^\perp \\
P^\perp LP & P^\perp LP^\perp
\end{pmatrix}$$

under the identification $u = (Pu, P^\perp u)^T$. When $P^\perp LP^\perp : P^\perp \mathcal{H}_1 \rightarrow P^\perp \mathcal{H}_2$ is invertible, the inverse of $PL^{-1}P$ exists and is given by the Feshbach-Schur map
(also called Schur complement formula)

\[ \mathcal{L} = (PL^{-1}P)^{-1} = PLP - PLP^\perp (P^\perp LP^\perp)^{-1} P^\perp LP. \]  

(1.13)

This formula was used in the present context in [33], inspired by a previous application of the Feshbach-Schur map in quantum many-body theory [5]. See Sect. 2.1 for a discussion on the invertibility of \( P^\perp LP^\perp \).

The upshot of these considerations, which we do not repeat here, is the expression for the operator \( K^\delta \)

\[ K^\delta = \delta \sum_{n=1}^{\infty} (-\delta)^n P \sigma (KP^{-\sigma})^n, \]  

(1.14)

where we introduced the operator-valued matrix \( K := \nabla(\Delta)^{-1} \nabla^* \). We emphasize that by the operator \( P^\perp \sigma \) we mean the composition of \( P^\perp \) and the multiplication operator associated with \( \sigma I_d \).

1.2.2. Analysis of the Perturbation Series (1.14) Note that each entry of the matrix \( K \) is a singular integral operator of convolution type. However, the reader is invited to think of \( K \) as a usual scalar singular integral operator acting on \( \mathbb{Z}^d \). See the beginning of Sect. 2.2 for a discussion on such operators.

Our key result, Proposition 3.2, says that

\[ |P \sigma (KP^{-\sigma})^n(x, y)| \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n |x - y|^{-3d+\epsilon}, \]  

(1.15)

where we write \( \langle \cdot \rangle = 1 + |\cdot| \). This shows that the series in (1.14) is convergent for sufficiently small \( \delta > 0 \) and, therefore, implies Theorem 1.1.

To show (1.15), one writes the kernel of \( P \sigma (KP^{-\sigma})^n \) as

\[ P \sigma (KP^{-\sigma})^n(x_0, x_n) = P \sum_{x \in (\mathbb{Z}^d)^{n-1}} \sigma(x_0)K(x_0 - x_1)P^{-\sigma}(x_1) \ldots K(x_{n-1} - x_n)P^{-\sigma}(x_n). \]  

(1.16)

where we denote by \( x \) the vector \((x_1, x_2, \ldots, x_{n-1})\). We interpret this expression as a sum over paths \( x \) in \( \mathbb{Z}^d \) connecting \( x_0 \) to \( x_n \). Along these paths, one evaluates the random variables \( \sigma(x_k, \cdot) \). In between two such evaluations, one uses the (matrix-valued) “propagator” \( K(x_k - x_{k+1}) \) to travel from site to site.

The first idea is then to treat (1.16) as a composition of deterministic operators. For each \( 1 \leq j \leq n \), define \( K^j(x, y) := K_j(x - y)b_j(y) \) for a singular integral operator \( K_j \) and a bounded function \( b_j \) on \( \mathbb{Z}^d \). For a given subset \( S \subset (\mathbb{Z}^d)^{n-1} \), define the deterministic operator

\[ T^\sigma_S(x_0, x_n) = \sum_{x \in S} K^1(x_0, x_1)K^2(x_1, x_2) \ldots K^n(x_{n-1}, x_n). \]  

(1.17)
Note that $T^n_{(Z^d)^{n-1}}(x_0, x_n)$ is the kernel of the operator $K^1 K^2 \cdots K^n$. After replacing $P^\perp$ by $I - P$ in (1.16), we may control (1.16) by a bound for $T^n_{(Z^d)^{n-1}}(x_0, x_n)$. (In the application to (1.16), $K_j$ is a matrix element of $K = \nabla(\Delta)^{-1}\nabla^*$ and $b$ is a realization of the random variable $\sigma$.) It follows from [8, Lemma 1] (see also Lemma 2.2) that

$$\left| T^n_{(Z^d)^{n-1}}(x_0, x_n) \right| \leq \epsilon \left( \frac{C}{\epsilon} \right)^n \langle x_0 - x_n \rangle^{-d+\epsilon}, \quad (1.18)$$

with the decay rate of $-d + \epsilon$.

### 1.2.3. Reducible Paths

The above considerations show that the deterministic estimate (1.18) is not enough—the randomness must be utilized. In order to discuss the role of randomness and the projection operators $P^\perp$ in the sum (1.16), we state

**Definition 1.7.** (Reducible paths) Let $n \geq 2$ and fix $x_0, x_n \in Z^d$ with $x_0 \neq x_n$. We say that $\underline{x} = (x_1, x_2, \ldots, x_{n-1}) \in (Z^d)^{n-1}$ is a reducible path (from $x_0$ to $x_n$) if there exists $0 \leq j < n$ such that

$$\{x_0, \ldots, x_j\} \cap \{x_{j+1}, \ldots, x_n\} = \emptyset.$$

Otherwise we say that $\underline{x}$ is an irreducible path.

The importance of this notion stems from the fact that we may discard any portion of reducible paths $\underline{x}$ from the summation in (1.16). Indeed, if $\underline{x} = (x_1, \ldots, x_{n-1})$ is a reducible path (from $x_0$ to $x_n$), then

$$P \sigma(x_0, \cdot) P^\perp \sigma(x_1, \cdot) P^\perp \sigma(x_2, \cdot) \cdots P^\perp \sigma(x_n, \cdot) = 0. \quad (1.19)$$

This fundamental vanishing property follows from the assumption that the random variables are independent. In principle, this is a promising observation because it allows one to discard terms from the summation in (1.16). In effect, the sum over $(Z^d)^{n-1}$ in (1.16) can be replaced by one over appropriate subsets $S \subset (Z^d)^{n-1}$. The discarding of reducible paths is the only way in which the randomness is utilized. Afterwards, the remaining task is to bound the deterministic quantity $T^n_S(x_0, x_n)$ for the selected $S \subset (Z^d)^{n-1}$.

This touches upon a central, but subtle, issue: precisely which reducible paths should be discarded from the sum (1.16) (in other words, which $T^n_S(x_0, x_n)$ one should aim to bound) is not at all clear a priori. To avoid confusion, we emphasize that once one has reduced matters to the deterministic quantity $T^n_S(x_0, x_n)$, one may no longer drop reducible paths. Moreover, $T^n_S(x_0, x_n)$ does not depend on $S$ in a monotone way. In fact, the summation involves significant cancellations due to the presence of the singular integral operators and one should avoid taking absolute values inside the sum if possible. (From this perspective, Bourgain’s deterministic bound (1.18) is already non-trivial; see Remark 2.5.) To summarize, the main technical difficulty is the delicate matter of bounding the oscillatory object $T^n_S(x_0, x_n)$ for a subset $S$ obtained by discarding an appropriate subset of reducible paths.
Bourgain uses the vanishing property in the following way: by a dyadic decomposition, one may focus on the sum in (1.16) over paths $\mathbf{x}$ such that, for some fixed $0 \leq j_0 < n$, the length of their longest segment $\max_{0 \leq j < n} |x_j - x_{j+1}|$ is equal to $|x_{j_0} - x_{j_0+1}|$ and is comparable to $R$ for some large $R > 0$. Let $S_{j_0}$ be the collection of such paths. Next, using the identity (1.19), he discards from $S_{j_0}$ exactly those reducible paths where the sub-paths $(x_0, \ldots, x_{j_0})$ and $(x_{j_0+1}, \ldots, x_n)$ are not connected. In other words, Bourgain only keeps paths in the set

$$\tilde{S}_{j_0} := \bigcup_{j_1 \leq j_0 < j_2} S_{j_1, j_2}, \text{ where } S_{j_1, j_2} := \{x \in S_{j_0} : x_{j_1} = x_{j_2}\}. \quad (1.20)$$

Thanks to (1.18) and the structure of $S_{j_1, j_2}$, it is possible to control the sum (1.16) restricted to the subset $S_{j_1, j_2}$ by

$$\left|T^n_{S_{j_1, j_2}}(x_0, x_n)\right| \leq \epsilon^2 \left(\frac{C}{\epsilon}\right)^n R^{-d} |x_0 - x_n|^{-d+\epsilon}. \quad (1.21)$$

Since $R \geq C|x_0 - x_n|/n$, this already displays the decay rate of $-2d + \epsilon$ obtained in [8]. However, a key point is that the union in (1.20) is not disjoint and therefore a bound on the individual $T^n_{S_{j_1, j_2}}$ does not directly imply a bound on $T^n_{\tilde{S}_{j_0}}$. (We emphasize that this issue is a consequence of the oscillatory nature of the problem. If the definition (1.17) of $T^n_S$ would only involve positive terms, this step would follow by a simple union bound.) This a priori serious technical problem is solved in a highly original way in [8] by introducing Steinhaus systems and appealing to the Markov brothers’ inequality for polynomials. We call this “Bourgain’s disjointification trick” and abstract it to Lemma 2.7. Altogether, Bourgain’s argument gives the decay rate $-2d + \epsilon$.

1.2.4. A Second Long Segment Our improved decay rate starts with a simple observation: for each path $\mathbf{x} = (x_1, \ldots, x_{n-1}) \in S_{j_1, j_2}$, we have

$$|x_0 - x_n| \leq |x_0 - x_{j_1}| + |x_{j_2} - x_n|,$$

by the triangle inequality and $x_{j_1} = x_{j_2}$. This observation implies that there exists another “long” segment among the sub-paths $(x_0, \ldots, x_{j_1})$ or $(x_{j_2}, \ldots, x_n)$. Exploiting this additional information, we further decompose the set $S_{j_1, j_2}$ and discard certain reducible paths using the identity (1.19) once more. These steps amount to specifying even smaller subsets $S \subset (\mathbb{Z}^d)^{n-1}$ for which (1.17) is to be controlled. We show an improved bound for $T^n_S(x_0, x_n)$ using additional structures in $S$ and then obtain (1.9), i.e., the decay rate $-3d + \epsilon$.

1.3. The Diffusion Matrix of $\mathcal{L}$ Determines $\sigma$ Uniquely

Recall the definition of the operator $L$ in (1.4) and (1.5). It shows that the parameter space of our model is given by the probability distribution of the random variable $\sigma$, call it $\mathbb{P}_\sigma$. By Theorem 1.1, for any $0 < \delta < c_d$, we have

$$\mathcal{L} = \nabla^* (\mathbf{1} + \delta \mathbb{E}\sigma) \mathbf{1}_d + \mathbf{K}^\delta) \nabla.$$
Moreover, it is part of the proof of Theorem 1.1 (following Proposition 3.2) that the matrix-valued convolution kernel $K^\delta(x)$ is given by the convergent perturbation series (1.14).

We recall that the series for $K^\delta(x)$ depends on $\sigma$, but only on the probability distribution $\mathbb{P}_\sigma$ of $\sigma$. Thus, we may consider a map that takes the datum $\mathbb{P}_\sigma$ to the collection of diffusion matrices of $\mathcal{L}$ for $\delta \in I$ for an open interval $I \subset (0, c_d)$. We observe that this map is injective. More precisely, we have

**Proposition 1.8.** The map

$$\Phi : \mathbb{P}_\sigma \mapsto \{(1 + \delta \mathbb{E}\sigma)I_d + K^\delta(0)\}_{\delta \in I}$$

is injective for any open interval $I \subset (0, c_d)$.

This result shows that the formulae for $\mathcal{L}$, still contain a lot of non-trivial information, even though $\mathcal{L}$ is an averaged operator.

We may phrase Proposition 1.8 more concretely, as follows. If we take two i.i.d. families of random variables $\sigma, \tilde{\sigma}$ with different probability distributions $\mathbb{P}_\sigma \neq \mathbb{P}_{\tilde{\sigma}}$, then, either $\mathbb{E}\sigma \neq \mathbb{E}\tilde{\sigma}$, or, for any choice of interval $I \subset (0, c_d)$ there exists $\delta \in I$ such that $K^\delta$ is different for $\sigma$ and $\tilde{\sigma}$.

**Proof of Proposition 1.8.** The proof is constructive, i.e., we show how $\mathbb{P}_\sigma$ can be determined from $\Phi(\mathbb{P}_\sigma)$. Recall that we assume $\sigma \in [-1, 1]$. Since the Hausdorff moment problem has a unique solution (by Stone-Weierstrass), it suffices to show that $\Phi(\mathbb{P}_\sigma)$ determines all moments of $\sigma$, i.e. $\mu_n := \mathbb{E}[\sigma^n]$ for all $n \in \mathbb{N}$.

The convergence of the series (1.14) for all $\delta \in (0, c_d)$ implies that the function $\delta \mapsto (1 + \delta \mathbb{E}\sigma)I_d + K^\delta(0)$ is analytic on the interval $I$. Therefore, $\Phi(\mathbb{P}_\sigma)$ uniquely determines its (matrix-valued) power series coefficients, which we call $\{a_n\}_{n \geq 0}$. According to (1.14), the coefficients are given by

$$a_0 = I_d, \quad a_1 := \mu_1 I_d, \quad a_{n+1} := (-1)^n \mathbb{P}_\sigma(KP^\perp\sigma)^n(0, 0), \quad (n \geq 1).$$

We may write out $\mathbb{P}_\sigma(KP^\perp\sigma)^n(0, 0)$ as a sum as in (1.16). The key observation is that, by singling out the path $x = (0, \ldots, 0)$ from that sum, we obtain

$$a_{n+1} = (-1)^n \mu_{n+1} [K(0)]^n + p_n(\mu_1, \ldots, \mu_n)$$

for a suitable map $p_n : \mathbb{R}^n \to \mathbb{R}^{d \times d}$ which depends polynomially on $\mu_1, \ldots, \mu_n$. This equation determines $\mu_{n+1}$ uniquely, if all the lower moments and $a_{n+1}$ are known. Therefore, we can inductively obtain the values of all the moments $\{\mu_n\}_{n \geq 1}$.

This completes the proof. \(\Box\)

### 1.4. A Related Question and a Partial Result

As described above, any successful argument has to negotiate how many reducible paths to discard from the summation—because afterwards one needs to control $T^S_n(x_0, x_n)$ on the resulting set $S$ of paths. Bourgain implements the cancellation (1.19) once in his argument and we implement it twice to prove Theorem 1.1.
Now, what happens if we discard all the reducible paths from the outset? Our result in this direction, Theorem 1.9, succeeds almost in yielding another proof of Theorem 1.1 (upto logarithm).

Let \( n \geq 2 \) and fix \( x_0, x_n \in \mathbb{Z}^d \) with \( x_0 \neq x_n \). We denote by \( U = U_{x_0,x_n} \) the set of all irreducible paths from \( x_0 \) to \( x_n \), i.e.,

\[
U := \{ x \in (\mathbb{Z}^d)^{n-1} : \{ x_0, \ldots, x_j \} \cap \{ x_{j+1}, \ldots, x_n \} \neq \emptyset, \text{ for any } 0 \leq j < n \}.
\]

Note that \( U_{x_0,x_2} = \emptyset \) when \( n = 2 \), and \( U_{x_0,x_3} = \{ (x_3, x_0) \} \) when \( n = 3 \). The set \( U_{x_0,x_n} \) becomes more complicated when \( n \geq 4 \).

Note that by the vanishing property, (1.19), we have

\[
P \sigma(K \mathbb{P}^\perp \sigma)^n(x_0, x_n) = P \sum_{x \in U} \sigma(x) K(x_0 - x_1) \mathbb{P}^\perp \sigma(x_1) \ldots K(x_{n-1} - x_n) \mathbb{P}^\perp \sigma(x_n). \tag{1.22}
\]

Each matrix element of the right-hand side of (1.22) can be controlled by a sum of deterministic terms \( T_U^n(x_0, x_n) \) defined in (1.17); see Sect. 3.1.

Our second main result provides a non-trivial estimate for \( T_U^n(x_0, x_n) \).

**Theorem 1.9.** Let \( n \geq 3 \) and \( x_0, x_n \in \mathbb{Z}^d, x_0 \neq x_n \). Then there is an absolute constant \( C > 1 \) such that

\[
|T_U^n(x_0, x_n)| \leq C^n \log n \epsilon^{3-n} \langle x_0 - x_n \rangle^{-3d+\epsilon}
\]

for all sufficiently small \( \epsilon > 0 \).

One may compare Theorem 1.9 with the trivial estimate

\[
|T_U^n(x_0, x_n)| \leq C^n \log n \epsilon^{1-n} \langle x_0 - x_n \rangle^{-d+\epsilon},
\]

which holds for any \( U \subset (\mathbb{Z}^d)^{n-1} \), see (2.6).

It would be very interesting to know whether it is possible to improve the constant \( C^n \log n \) to \( C^n \) in Theorem 1.9, which would then imply Theorem 1.1 arguing as in Sect. 3.1. In fact, we show that we may write \( U = \bigcup_{\alpha \in A} U_{\alpha} \) for some index set \( A \) with \( \#A \leq 2^n \) such that

\[
|T_{U_{\alpha}}^n(x_0, x_n)| \leq \epsilon^n (C/\epsilon)^n \langle x_0 - x_n \rangle^{-3d+\epsilon}.
\]

Since the sets \( U_{\alpha} \) are not disjoint, this does not immediately yield a bound on \( T_U^n(x_0, x_n) \). Nonetheless, we can perform an appropriate “disjointification” to write \( U = \bigsqcup_{\alpha \in A} U'_{\alpha} \). (Here \( \sqcup \) denotes disjoint union.) Unfortunately, the most efficient way to implement this disjointification that we have found still produces the \( C^n \log n \) bound in Theorem 1.9.
2. Preliminaries

2.1. Invertibility of $L$ on Some Function Spaces

In this subsection, we discuss the invertibility of the operators $L_\omega$ and $L$ on appropriate domains of definition; see Proposition 2.1 below. To this end, we introduce function spaces which play the role of Sobolev spaces in the discrete setting. This section furnishes the formal operator-theoretic foundation for the study of various objects in this paper and can be skipped upon a first reading.

We start the discussion with the identity $-\hat{\Delta} f(\theta) = \sum_{j=1}^{d} 2(1 - \cos \theta_j) \hat{f}(\theta)$ for $\theta \in \mathbb{T}^d = [-\pi, \pi]^d$, where the symbol $\sum_{j=1}^{d} 2(1 - \cos \theta_j)$ of $-\Delta$ is comparable to $|\theta|^2$. For $f \in \ell^2(\mathbb{Z}^d)$ and $s > -d/2$, define the Riesz potential $\Lambda^s = (-\Delta)^{s/2}$ by

$$\Lambda^s f(\theta) = \left( \sum_{j=1}^{d} 2(1 - \cos \theta_j) \right)^{s/2} \hat{f}(\theta).$$

We shall work with $\Lambda^1$ and $\Lambda^{-1}$ for $d \geq 3$.

We note that $\Lambda^{-1}$ is a bounded map from $\ell^2(\mathbb{Z}^d)$ to $\ell^{q_d}(\mathbb{Z}^d)$ for $q_d^{-1} := 2^1 - d^{-1}$. This is a consequence of a discrete version of the Hardy–Littlewood–Sobolev inequality:

$$||\Lambda^{-1} f||_{\ell^{q}(\mathbb{Z}^d)} \leq C_{p,d} ||f||_{\ell^p(\mathbb{Z}^d)} \quad (2.1)$$

for $1 < p \leq 2$ and $q^{-1} = p^{-1} - d^{-1}$. The estimate (2.1) follows from an estimate for a discrete analogue of fractional integrals on $\mathbb{Z}^d$ (see, e.g., [35, Proposition (a)]) and the fact that $\Lambda^{-1} f = K * f$ for a convolution kernel $K \in \ell^2(\mathbb{Z}^d)$ satisfying the bound $K(x) = O((1 + |x|)^{-d+1})$. See the proof of Corollary 1.5 in Appendix for a related computation.

We specify the domain of the map $L_\omega$. Let $H^1(\mathbb{Z}^d)$ be the image of $\ell^2(\mathbb{Z}^d)$ by the injection $\Lambda^{-1} : \ell^2(\mathbb{Z}^d) \rightarrow \ell^{q_d}(\mathbb{Z}^d)$. Namely,

$$H^1(\mathbb{Z}^d) := \Lambda^{-1}[\ell^2(\mathbb{Z}^d)] = \{ \Lambda^{-1} f : f \in \ell^2(\mathbb{Z}^d) \}.$$

We equip $H^1(\mathbb{Z}^d)$ with the norm $||\Lambda^{-1} f||_{H^1(\mathbb{Z}^d)} := ||f||_{\ell^{q_d}(\mathbb{Z}^d)}$. In fact, $H^1(\mathbb{Z}^d)$ is a Hilbert space equipped with the inner product $\langle \Lambda^{-1} f_1, \Lambda^{-1} f_2 \rangle_{H^1(\mathbb{Z}^d)} := \langle f_1, f_2 \rangle_{\ell^{q_d}(\mathbb{Z}^d)}$. We define $\Lambda : H^1(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ to be the inverse of the map $\Lambda^{-1}$. By definition, the maps $\Lambda^{-1} : \ell^2(\mathbb{Z}^d) \rightarrow H^1(\mathbb{Z}^d)$ and $\Lambda : H^1(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ are isometries.

The range of $L_\omega$ can be identified with $H^{-1}(\mathbb{Z}^d)$ defined by

$$H^{-1}(\mathbb{Z}^d) := \{ f \in \ell^2(\mathbb{Z}^d) : \Lambda^{-1} f \in \ell^2(\mathbb{Z}^d) \}.$$

Note that $H^{-1}(\mathbb{Z}^d)$ is a Hilbert space equipped with the inner product $\langle f, g \rangle_{H^{-1}(\mathbb{Z}^d)} := \langle \Lambda^{-1} f, \Lambda^{-1} g \rangle_{\ell^2(\mathbb{Z}^d)}$. The map $\Lambda^{-1} : H^{-1}(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ and its inverse $\Lambda^1 : \ell^2(\mathbb{Z}^d) \rightarrow H^{-1}(\mathbb{Z}^d)$ are isometries. By (2.1), we have $\ell^{q_d}(\mathbb{Z}^d) \subset H^{-1}(\mathbb{Z}^d)$ for

$$p_d^{-1} := 2^{-1} + d^{-1}.$$
Proposition 2.1. Let $0 < \delta < 1$. The operator $L_\omega : H^1(\mathbb{Z}^d) \to H^{-1}(\mathbb{Z}^d)$ is a bounded operator with the bounded inverse $L_\omega^{-1}$. The operator norms for $L_\omega$ and $L_\omega^{-1}$ are bounded uniformly in $\omega$.

Proof. The proof is standard. First, observe that $\nabla_j : H^1(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ and $\nabla_j^* : \ell^2(\mathbb{Z}^d) \to H^{-1}(\mathbb{Z}^d)$ are bounded maps by Plancherel’s theorem. Thus, $L_\omega$ is a bounded map from $H^1(\mathbb{Z}^d)$ to $H^{-1}(\mathbb{Z}^d)$ uniformly in $\omega \in \Omega$.

Next, we observe that $-\Delta = \Lambda^{1} \Lambda$ on $H^1(\mathbb{Z}^d)$. This is because

$$-\Delta(\Lambda^{-1} f) = \Lambda^{1} f = \Lambda^{1} \Lambda(\Lambda^{-1} f)$$

for any $f \in \ell^2(\mathbb{Z}^d)$. This allows us to factorize $L_\omega$ as

$$L_\omega = \Lambda^{1}(I + \delta M_\omega) \Lambda,$$

where $M_\omega := \Lambda^{-1} \nabla^* \sigma(\cdot, \omega) \mathbf{I}_d \nabla \Lambda^{-1}$. The operator $M_\omega$ is bounded on $\ell^2(\mathbb{Z}^d)$ with the operator norm bounded by 1. Therefore, when $0 < \delta < 1$, the inverse of $I + \delta M_\omega$ exists and is bounded on $\ell^2(\mathbb{Z}^d)$. This implies that $L_\omega$ is invertible and $L_\omega^{-1}$ is bounded (uniformly in $\omega$). \(\Box\)

This prototypical result also applies in a slightly different context which we will occasionally consider and which is therefore made precise next.

Recall that we may view $L = \nabla^* \mathbf{A} \nabla$ as a map acting on functions on the product space $\mathbb{Z}^d \times \Omega$ via $Lu(x, \omega) = L_\omega u_\omega(x)$. One can show a completely analogous proposition, where the relevant function spaces are replaced by the following ones: we first let $L^2(\mathbb{Z}^d \times \Omega)$ be the Hilbert space equipped with the inner product induced from $\ell^2(\mathbb{Z}^d)$ and $L^2(\Omega)$. By letting $\Lambda^{-1}$ act on the lattice variable, we may regard it as a bounded injection from $L^2(\mathbb{Z}^d \times \Omega)$ to the mixed norm space $L^2(\Omega, \ell^q(\mathbb{Z}^d))$. Define the Hilbert spaces $H^1(\mathbb{Z}^d \times \Omega) = \Lambda^{-1}[L^2(\mathbb{Z}^d \times \Omega)]$ and

$$H^{-1}(\mathbb{Z}^d \times \Omega) = \{ f \in L^2(\mathbb{Z}^d \times \Omega) : \Lambda^{-1} f \in L^2(\mathbb{Z}^d \times \Omega) \}.$$

Arguing as in the proof of Proposition 2.1, one verifies that

$$L : H^1(\mathbb{Z}^d \times \Omega) \to H^{-1}(\mathbb{Z}^d \times \Omega)$$

is a bounded operator with the bounded inverse $L^{-1}$.

Next, we discuss the invertibility of $P^\perp L P^\perp$, which is used in (1.13). We write

$$P^\perp L P^\perp = \Lambda^{1}(I + \delta M^\perp) \Lambda P^\perp,$$

where $M^\perp := \Lambda^{-1} \nabla^* P^\perp \sigma(\cdot, \omega) \mathbf{I}_d \nabla \Lambda^{-1}$. Note that the $L^2(\mathbb{Z}^d \times \Omega)$ operator norm of $M^\perp$ is bounded by 1. Therefore $(I + \delta M^\perp)$ is invertible and the inverse of $P^\perp L P^\perp$ is given by

$$(P^\perp L P^\perp)^{-1} = \Lambda^{-1}(I + \delta M^\perp)^{-1} \Lambda^{-1} P^\perp.$$

Finally, we also have that $\mathcal{L} : H^1(\mathbb{Z}^d) \to H^{-1}(\mathbb{Z}^d)$ is a bounded operator with bounded inverse, whenever $0 < \delta < 1$. This follows from the expression (1.13) and the boundedness of the operators $L$ and $L^{-1}$.
2.2. Bourgain’s Lemmas

In this subsection, we state abstract versions of two main tools introduced in [8]: a deterministic bound on composition of singular integral operators and Bourgain’s disjointification trick.

Before we proceed, we briefly recall some well-known properties of singular integral operators to be used later.

By a singular integral operator (of convolution type) acting on \( \mathbb{Z}^d \), we mean, in this paper, a Fourier multiplier transformation \( K \) of the form \( \hat{K}u(\theta) = m(\theta)\hat{u}(\theta) \) associated with a multiplier \( m \) on the \( d \)-torus \( \mathbb{T}^d \) satisfying the bounds

\[
|\partial^\alpha m(\theta)| \leq C_\alpha |\theta|^{-|\alpha|}
\]

for all multi-indices \( |\alpha| \geq 0 \). Here \( \hat{u}(\theta) = \sum_{x \in \mathbb{Z}^d} u(x)e^{-i\theta \cdot x} \) denotes the Fourier transform of a function \( u \) on \( \mathbb{Z}^d \). The convolution kernel \( K(x) \) of such \( K \) satisfies the decay estimate \( |K(x)| \leq C (1 + |x|)^{-d} \) and the “gradient” estimate \( |\nabla K(x)| \leq C (1 + |x|)^{-(d+1)} \). The \( \ell^2(\mathbb{Z}^d) \) boundedness of the convolution operator \( K \) follows from the boundedness of \( m \). It is also well-known, by the Calderón–Zygmund theory, that \( K \) is of weak-type \( (1, 1) \), hence bounded on \( \ell^p(\mathbb{Z}^d) \) for all \( 1 < p < \infty \) by interpolation and duality with the operator norm \( O((p - 1)^{-1}) \) as \( p \to 1 \). See [34] for a treatment of singular integrals in the continuous setting.

2.2.1. Deterministic Bounds

We recall that we identify an operator \( K \) with its kernel. We write \( K^* \) for the adjoint of \( K \) and so \( K^*(x, y) = \hat{K}(y, x) \). For an interval \( I \subset [0, \infty) \), we set

\[
K_I(x, y) := K(x, y)\chi_I(|x - y|).
\]

**Lemma 2.2.** Let \( A > 0 \) and let \( \epsilon > 0 \) be sufficiently small. There exists a constant \( C = C_{d, A} \) such that the following holds. Let \( I_m = [0, 2^m) \) and let \( \{K_j\}_{1 \leq j \leq n} \) be a collection of operators acting on functions on \( \mathbb{Z}^d \) satisfying the assumptions

1. \( |K^j(x, y)| \leq A(x - y)^{-d}; \)
2. \( \sup_{m \geq 0} \left( ||K_I^j||_{\ell^p \to \ell^p} + ||(K_I^j)^\ast||_{\ell^p \to \ell^p} \right) \leq A/(p - 1) \) for all sufficiently small \( p > 1 \).

Then the operator \( T^n := K^n = K_1 \cdots K_n \) satisfies the pointwise bound

\[
|T^n(x_0, x_n)| \leq \epsilon \left( \frac{C}{\epsilon} \right)^n \langle x_0 - x_n \rangle^{-d + \epsilon}
\]

for all sufficiently small \( \epsilon > 0 \).

Lemma 2.2 is an abstract version of Lemma 1 in [8], where the operators \( K_j^j \) are of the form \( K^j(x, y) = K(x - y)b(y) \) for a singular integral operator \( K \) and a bounded function \( b \), see Example 2.3. The main motivation to state Lemma 2.2 in this generality is that it makes it easier to apply the result for variants of \( K^j \); see Corollary 2.4. While the bound in Lemma 1 of [8] features \( (C/\epsilon)^n \), we note here that its proof in fact yields an additional factor of \( \epsilon \). This gain allows for the improvements described in Remark 1.2 (i). We present the proof of Lemma 2.2 in the Appendix for completeness.

We give the main example for operators \( \{K_j^j\}_{1 \leq j \leq n} \) for Lemma 2.2.
Example 2.3. Consider the example from Lemma 1 in [8]:

\[ K^j(x, y) = K(x - y)b(y) \]  

(2.3)

for a singular integral operator \( K \) of convolution type acting on \( \mathbb{Z}^d \) (e.g., \( K = \nabla_j (-\Delta)^{-1} \nabla_k^* \)) and a function \( b \in L^\infty(\mathbb{Z}^d) \). To check Assumption (i) and (ii), it is enough to assume that \( K^j(x, y) = K(x - y) \) since \( b \in L^\infty(\mathbb{Z}^d) \). Assumption (i) is immediate from our assumption on \( K \). To verify Assumption (ii), we compare \( K_{I_m} \) with its variant \( \tilde{K}_{I_m} \), where the sharp cutoff \( \chi_{I_m}(|x - y|) \) is replaced by a smooth cutoff \( \psi(2^{-m}|x - y|) \). Here, \( \psi \) is a smooth even function supported on \([-2, 2]\) and equal to 1 on \([-1, 1]\). Note that \( ||K_{I_m} - \tilde{K}_{I_m}||_{L_1(\mathbb{Z}^d)} \leq C \) and thus the operator \( K_{I_m} - \tilde{K}_{I_m} \) is uniformly bounded on \( \ell^p(\mathbb{Z}^d) \) for \( 1 \leq p \leq \infty \). Therefore, it suffices to check Assumption (ii) for \( \tilde{K}_{I_m} \), which is well-known by the Calderón–Zygmund theory.

In fact, Lemma 2.2 has a slightly wider scope than its statement suggests. We state a specific version in the following corollary:

Corollary 2.4. Let \( A > 0 \) and let \( \epsilon > 0 \) be sufficiently small. There exists a constant \( C = C_{d, A} \) such that the following holds. Let \( \{K^j\}_{1 \leq j \leq n} \) be a collection of operators as in (2.3) or Lemma 2.2. Let \( I = \{I_j\}_{1 \leq j \leq n} \) be a collection of intervals \( I_j \subset [0, \infty) \). Then

\[ |K_1^j K_2^j \ldots K_n^j(x_0, x_n)| \leq C \left( \frac{1}{\epsilon} \right)^n (x_0 - x_n)^{-d+\epsilon} \]  

(2.4)

for all sufficiently small \( \epsilon > 0 \). Moreover, the bound (2.4) is invariant under the change

\[ K^j_j \rightarrow e_1^j(x) K^j_j(x, y) e_2^j(y) \]  

(2.5)

for any \( e_1^j, e_2^j \in \ell^\infty(\mathbb{Z}^d) \) such that \( ||e_1^j||_{\ell^\infty} \leq 1 \) and \( ||e_2^j||_{\ell^\infty} \leq 1 \).

Proof. We first note that \( K^j_j \) satisfies Assumptions (i) and (ii) with another constant \( A' > 0 \) independent of \( I_j \). This can be shown by choosing an interval of the forms \( I_m \) or \( I_m \setminus I_{m+1} \) for certain \( m \geq m_0 \geq 1 \) that best approximates \( I_j \) and arguing as in Example 2.3 using Assumption (ii) for \( K^j_{I_m} \). This verifies (2.4) by Lemma 2.2. Moreover, this bound is invariant under (2.5) since the bounds in Assumption (i) and (ii) are preserved under the replacement (2.5). \( \square \)

Remark 2.5. The constant \( (C/\epsilon)^n \) obtained in [8, Lemma 1] (and also stated in Corollary 2.4) is rather non-trivial. To compare, we note that there is a much weaker bound that only requires (a weaker version of) the size assumption \( |K^j(x, y)| \leq A (x - y)^{-d} : |T^n(x_0, x_n)| \) is bounded by

\[ \sum_{x \in \mathbb{Z}^d} |K^1(x_0, x_1) K^2(x_1, x_2) \ldots K^n(x_{n-1}, x_n)| \leq \epsilon \left( \frac{nC}{\epsilon} \right)^n (x_0 - x_n)^{-d+\epsilon} \]  

(2.6)
for all $0 < \epsilon < d/4$. Note that Lemma 2.2 saves a factor of $n^n$ compared to (2.6). (We still use (2.6) to justify the use of Fubini’s theorem on various occasions throughout this note.)

To see that (2.6) holds, one may use the bound $|K^j(x, y)| \leq A(x - y)^{-d + (\epsilon/4)}$ and then apply the following elementary inequality $(n - 1)$-times.

**Lemma 2.6.** Assume that $\alpha, \beta$ are positive numbers such that $3d/4 \leq \alpha, \beta \leq d - \epsilon$ for some $\epsilon > 0$. Then there exists a constant $C$ such that for any $a, b \in \mathbb{Z}^d$, we have

$$\sum_{x \in \mathbb{Z}^d} (a - x)^{-\alpha}(x - b)^{-\beta} \leq C \epsilon^{-(\alpha + \beta - d)}.$$ 

One way to verify Lemma 2.6 is to make a dyadic decomposition $\mathbb{Z}^d = \bigcup_{m \geq 0} \{x \in \mathbb{Z}^d : \max(|a - x|, |x - b|) \sim 2^m\}$ as in the proof of Lemma 2.2 (see the Appendix). We leave the details to the interested reader.

### 2.2.2. Bourgain’s Disjointification Trick

One of the main technical challenges that [8] overcomes is bounding $T^n_S$ for rather small $S \subset (\mathbb{Z}^d)^{n-1}$. In the proof, after this is achieved for certain sets $S$, it remains to add appropriate disjointness conditions, resulting in even smaller sets $S' \subset S$. Bourgain’s trick then gives a way to bound $|T^n_{S'}(x_0, x_n)|$ in terms of a bound for $|T^n_S(x_0, x_n)|$, up to a factor of $C^n$.

We slightly generalize Bourgain’s trick in the following lemma:

**Lemma 2.7.** Let $\{K^j\}_{1 \leq j \leq n}$ be a collection of operators as in Lemma 2.2 and define $T^n_S$ as in (1.17) for $S \subset (\mathbb{Z}^d)^{n-1}$. For given subsets $\{E_l, F_l\}_{1 \leq l \leq m}$ of $\{0, 1, 2, \ldots, n\}$, define

$$S' := S \cap \bigcap_{1 \leq l \leq m} \{x \in (\mathbb{Z}^d)^{n-1} : \{x_u : u \in E_l\} \cap \{x_v : v \in F_l\} = \emptyset\}.$$ 

Assume that $S$ is a finite set and we have $|T^n_S(x_0, x_n)| \leq M(x_0, x_n)$ for some function $M$ and that the estimate remains invariant under the change

$$K^j(x, y) \to e_j^1(x)K^j(x, y)e_j^2(y). \quad (2.7)$$

Then we have

$$|T^n_{S'}(x_0, x_n)| \leq 2\sum_{1 \leq l \leq m} |E_l| + |F_l| M(x_0, x_n). \quad (2.8)$$

When $S$ is not finite, consider the truncation $S \cap X_k$, where

$$X_k := \left\{ x \in (\mathbb{Z}^d)^{n-1} : \max_{0 \leq j < n} |x_j - x_{j+1}| < 2^k \right\}.$$ 

Then (2.8) holds if

$$|T^n_{S \cap X_k}(x_0, x_n)| \leq M(x_0, x_n)$$

holds for all large $k \geq 1$ and the estimate remains invariant under the change (2.7).

The proof, which we relegate to the Appendix, follows [8] and uses Steinhaus systems and the Markov brothers’ inequality for polynomials.
3. Proof of the Main Result

3.1. The Key Estimate

We express the right-hand side of (1.14) in terms of paths in $\mathbb{Z}^d$. We let $K_1, \ldots, K_n$ be singular integral operators of the form $\nabla_i (-\Delta)^{-1} \nabla_i^*$, for some $1 \leq i, i' \leq d$. This specific choice, however, is not important for the argument. For every $n \geq 1$ and every subset $X \subset (\mathbb{Z}^d)^{n-1}$, define a function $f_X : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ by

$$f_X(x_0, x_n) := P\sigma(x_0) \sum_{\underline{x} \in X} K_1(x_0 - x_1) P^\perp \sigma(x_1) \cdots K_n(x_{n-1} - x_n) P^\perp \sigma(x_n). \quad (3.1)$$

Here and in the following, we denote $x_0 = (x_1, \ldots, x_{n-1})$ and we suppress the randomness from the notation.

It should be noted that $T^n_X$, studied in Sect. 2, is a deterministic version of $f_X$. In the following, we indicate how to obtain a bound for $f_X(x_0, x_n)$ from a bound for $T^n_X(x_0, x_n)$.

**Lemma 3.1.** Let $X \subset (\mathbb{Z}^d)^{n-1}$ and $b_1, \ldots, b_n : \mathbb{Z}^d \to \mathbb{C}$ be functions such that $||b_j||_\infty \leq 1$. Define operators $K^j$ via their kernels $K^j(x, y) := K_j(x - y)b_j(y)$. Assume that

$$T^n_X(x_0, x_n) = \sum_{\underline{x} \in X} K^1(x_0, x_1) K^2(x_1, x_2) \cdots K^n(x_{n-1}, x_n)$$

satisfies the bound

$$|T^n_X(x_0, x_n)| \leq M(x_0, x_n)$$

for some function $M$ independent of the choice of $\{b_j\}_{1 \leq j \leq n}$. Then

$$|f_X(x_0, x_n)| \leq 2^n M(x_0, x_n).$$

**Proof.** We replace each $P^\perp$ with $I - P = I - \mathbb{E}$ in (3.1). This allows us to write $f_X(x_0, x_n)$ as a sum of $2^n$ terms of the form (3.1), where each $P^\perp$ is replaced by either $I$ or $-\mathbb{E}$. For each of these terms, we use Fubini’s theorem to move all the integrations corresponding to $\mathbb{E}$ outside of the sum $\sum_{\underline{x} \in X}$. The proof is completed by bounding the sum over $\underline{x} \in X$ using the assumption on $T^n_X$ with $b_j(x) = \sigma(x, \omega_j)$ for some $\omega_j \in \Omega$. (The assumption $||b_j||_\infty \leq 1$ is guaranteed because $|\sigma(x, \omega)| \leq 1$ holds for almost every $\omega$ and all the $\omega_j$ appear under an integral.) □

Lemma 2.2, Example 2.3, and Lemma 3.1 imply that

$$|f_{(\mathbb{Z}^d)^{n-1}}(x_0, x_n)| \leq \epsilon \left(\frac{C}{\epsilon}\right)^n \langle x_0 - x_n \rangle^{-d + \epsilon}. \quad (3.2)$$

Our main result is a consequence of the following improved estimate:
Proposition 3.2. (Key estimate) There exists a constant $C > 0$ such that the following holds. For every $1 \leq j \leq n$, let $K_j = \nabla i_j (-\Delta)^{-1} \nabla^*_i j$ for some $1 \leq i_j, i'_j \leq d$. Then, for every $\epsilon > 0$, $n \geq 3$ and distinct $x_0, x_n \in \mathbb{Z}^d$, we have

$$|f_{(\mathbb{Z}^d)^{n-1}}(x_0, x_n)| \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n \langle x_0 - x_n \rangle^{-3d+\epsilon}.$$  

(3.3)

Proof of Theorem 1.1 assuming Proposition 3.2. We recall what (1.14) says element-wise, i.e.,

$$K_{i,i'}^\delta = \delta \sum_{n=1}^\infty (-\delta)^n P \sigma (K P \sigma)^n_{i,i'}$$

for every $1 \leq i, i' \leq d$. When we write out the matrix product $(K P \sigma)^n_{i,i'}$, we obtain a sum of terms defined as in (3.1), with $K_j = \nabla i_j (-\Delta)^{-1} \nabla^*_i j$ for some $1 \leq i_j, i'_j \leq d$. For each choice of the “outside indices” $i, i'$, there are $d^{n-1}$ choices of $K_j$ (for every $n$), and so

$$|K_{i,i'}^\delta(x, y)| = \delta \sum_{n=1}^\infty (-\delta)^n P \sigma (K P \sigma)^n_{i,i'}(x, y)$$

$$\leq \delta \sum_{n=1}^\infty \delta^n d^{n-1} \max_{K_1, \ldots, K_n} |f_{(\mathbb{Z}^d)^{n-1}}(x, y)|.$$  

(3.4)

The maximum is taken over operators $K_1, \ldots, K_n$ of the form $\nabla i_j (-\Delta)^{-1} \nabla^*_i j$ for some $1 \leq i_j, i'_j \leq d$. Using (3.2), we see that

$$|K_{i,i'}^\delta(x, y)| \leq \delta \epsilon \sum_{n=1}^\infty \left( \frac{C d}{\epsilon} \right)^n \langle x - y \rangle^{-d+\epsilon} = C \delta^2 \langle x - y \rangle^{-d+\epsilon},$$

whenever $0 < \delta < c \epsilon$ with $c = (Cd)^{-1}$. This estimate, in particular, verifies the case $x = y$ in Theorem 1.1.

When $x \neq y$, we use Proposition 3.2 (instead of (3.2) as above) together with the following observation: $f_{(\mathbb{Z}^d)^{n-1}}(x, y) = 0$ for $n = 1, 2$ because

$$P \sigma (x, \cdot) P^\perp \sigma(y, \cdot) = 0, \quad P \sigma (x, \cdot) P^\perp \sigma(x_1, \cdot) P^\perp \sigma(y, \cdot) = 0,$$

where the second equality holds for all $x_1 \in \mathbb{Z}^d$. (Equivalently, when $n = 2$, all paths connecting $x \neq y$ are reducible in the sense of Definition 1.7.)

The fact that $K^\delta$ is a convolution operator, i.e., that $K_{i,i'}^\delta(x_0, x_n) = K_{i,i'}^\delta(x_0 - x_n, 0)$ follows from (3.1): We change the summation variables $x_k \rightarrow x_k - x_n$ and recall that the random variables $\{\sigma(x, \omega)\}_{x \in (\mathbb{Z}^d)^{n-1}}$ are identically distributed.

Finally, one can derive the regularity properties of the Fourier transform $\hat{K}_{i,j}^\delta$ from the decay estimate (1.9) by standard arguments (mainly integration by parts). This finishes the proof of Theorem 1.1. □
In the remainder of this section, we prove Proposition 3.2 by successively reducing it to simpler statements.

As we mentioned before, the basic observation behind our proof is that a second “long” segment exists in every path analyzed in [8] by the triangle inequality. Our contribution starts at the conclusion of Bourgain’s argument. Therefore we repeat Bourgain’s argument here, and we include some additional details, before we show how to go a step further.

**Preparations.** From now on, we fix $\epsilon > 0$, $n \geq 3$, and $x_0, x_n \in \mathbb{Z}^d$ with $x_0 \neq x_n$. For every $1 \leq j \leq n$, we let $K_j = \nabla_{i_j} (\Delta)^{-1} \nabla_{i_j}^*$ for some $1 \leq i_j, i_j' \leq d$ whose values do not matter in what follows.

### 3.2. Dyadic Decomposition by Longest Segment

We begin by making precise the dyadic decomposition used by Bourgain to prove Lemma 1. It decomposes the paths $\chi = (x_1, \ldots, x_{n-1})$ according to the dyadic scale of their “longest” segment $|x_j - x_{j+1}|$.

We recall Definition (3.1):

$$f_{(\mathbb{Z}^d)^{n-1}}(x_0, x_n) = P \sigma (x_0) \sum_{\chi \in (\mathbb{Z}^d)^{n-1}} K_1(x_0 - x_1)P_\perp \sigma (x_1)K_2(x_1 - x_2) \ldots \sigma (x_n),$$

where we write $\chi = (x_1, \ldots, x_{n-1})$. Using a dyadic decomposition according to the size of $\max_{0 \leq j < n} |x_j - x_{j+1}|$, we may decompose the sum over paths as follows:

$$f_{(\mathbb{Z}^d)^{n-1}}(x_0, x_n) = \sum_{m=0}^{\infty} \sum_{j_0=0}^{n-1} f_{S_{j_0}^m}(x_0, x_n). \quad (3.5)$$

Here we introduced the family of disjoint sets

$$S_{j_0}^m := \left\{ \chi \in (\mathbb{Z}^d)^{n-1} : \max_{0 \leq j < n} |x_j - x_{j+1}| < 2^{m+1}, \quad |x_{j_0} - x_{j_0+1}| \geq 2^m \right\}.$$  

The last condition says that $j_0$ is minimal: it is the first time that the path achieves the (dyadic scale of) the longest segment. The main objective is to estimate the sum over $S_{j_0}^m$ in (3.5). In other words, we fix $m \geq 0$, $0 \leq j_0 < n$ and focus on paths $\chi$ such that

$$\max_{0 \leq j < n} |x_j - x_{j+1}| \leq 2^{m+1}, \quad |x_{j_0} - x_{j_0+1}| \geq 2^m \quad \text{and} \quad j_0 \text{ is minimal}. \quad (3.7)$$

(Compare eq. (4.2) in [8].)
3.3. Discarding Reducible Paths

As we mentioned in the introduction, the main use of the probabilistic structure of the problem is that the contribution to (3.1) of every “reducible” path (i.e., a path that can be split into disjoint pieces) vanishes, by Fubini’s theorem.

We define the family of disjoint sets

$$\tilde{S}_m^{j_0} := \left\{ x \in S_m^{j_0} : \{x_0, \ldots, x_{j_0}\} \cap \{x_{j_0+1}, \ldots, x_n\} \neq \emptyset \right\}. \quad (3.8)$$

**Lemma 3.3.** For all $m \geq 0$ and $0 \leq j_0 < n$ we have

$$f_{\tilde{S}_m^{j_0}}(x_0, x_n) = f_{S_m^{j_0}}(x_0, x_n).$$

**Proof.** This holds because paths in $S_m^{j_0} \setminus \tilde{S}_m^{j_0}$ are reducible and thus do not contribute to the sum in (3.1). \qed

Hence it suffices to prove

**Proposition 3.4.** (Reduction 1) Let $x_0 \neq x_n$. For all $m \geq 0$ and $0 \leq j_0 < n$, we have

$$| f_{\tilde{S}_m^{j_0}}(x_0, x_n) | \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n 2^{-m(d-\epsilon)} (x_0 - x_n)^{-2d+\epsilon}. \quad (3.9)$$

Before we go on, we show that Proposition 3.4 implies the key estimate.

**Proof of Proposition 3.2 assuming Proposition 3.4.** By (3.5), Lemma 3.3 and Proposition 3.4, we have

$$| f_{(Z^d)^{n-1}}(x_0, x_n) | \leq \sum_{m=0}^{\infty} \sum_{j_0=0}^{n-1} | f_{\tilde{S}_m^{j_0}}(x_0, x_n) |$$

$$\leq n \epsilon^3 \left( \frac{C}{\epsilon} \right)^n \langle x_0 - x_n \rangle^{-2d+\epsilon} \sum_{m=0}^{\infty} \sum_{\bigcup_{j_0=0}^{n-1} \tilde{S}_m^{j_0} \neq \emptyset} 2^{-m(d-\epsilon)}. \quad (3.10)$$

From the Definition (3.6), we see that $\bigcup_{j_0=0}^{n-1} \tilde{S}_m^{j_0} \neq \emptyset$ implies, by the triangle inequality, $|x_0 - x_n| \leq n 2^{m+1}$, or equivalently,

$$2^m \geq \frac{|x_0 - x_n|}{2n}.$$

We distinguish cases. If $|x_0 - x_n| \leq 2n$, then we have $1 \leq (1 + 2n) \langle x_0 - x_n \rangle^{-1}$ and the claim follows easily from (3.10). If $|x_0 - x_n| > 2n$, then, letting $q_n := \lceil \log_2(|x_0 - x_n|/(2n)) \rceil$,

$$\sum_{m=0}^{\infty} 2^{-m(d-\epsilon)} \leq \sum_{m=q_n}^{\infty} 2^{-m(d-\epsilon)} \leq Cn^{d-\epsilon} \langle x_0 - x_n \rangle^{-d+\epsilon}.$$

Combining this with (3.10) yields the bound in Proposition 3.2. \qed
We are left with the task of proving Proposition 3.4. In the following, we always fix \( m \geq 0 \) and \( 0 \leq j_0 < n \) and therefore we suppress them from the notation

\[ S = \tilde{S}^m_{j_0}. \]

### 3.4. Decomposing the Set \( S \)

Observe that any path \( \bar{x} \in S (= \tilde{S}^m_{j_0} \text{ from } (3.8)) \) contains at least one coincidence point \( x_{j_1} = x_{j_2} \), with \( 0 \leq j_1 \leq j_0 \) and \( j_0 < j_2 \leq n \). Following [8], we decompose the set \( S \) according to where this coincidence occurs; see (3.12) below. (Afterwards, we show how an application of the triangle inequality implies that there exists a second “long” segment and so this procedure can be basically repeated; see (3.14) below.)

We define the sets

\[
S_{j_1, j_2} := \{ \bar{x} \in S : x_{j_1} = x_{j_2} \}, \\
S'_{j_0, j_1, j_2} := S_{j_1, j_2} \setminus \left( \bigcup_{j < j_1 \atop j_0 < j' \leq n} S_{j, j'} \cup \bigcup_{j_0 < j' < j_2} S_{j_1, j'} \right). 
\]

The set \( S_{j_1, j_2} \) implicitly depends on \( j_0 \) as well, due to (3.8). Recall also that \( x_0 \neq x_n \), and so \( S_{0, n} = \emptyset \).

The second family of sets is a “disjointification” of the first one. We split the path between \( x_{j_0} \) and \( x_{j_0+1} \), obtaining a “left piece” and a “right piece”. The disjointness is achieved by taking \( j_1 \) and \( j_2 \) to be extremal: a path \( \bar{x} \in S_{j_1, j_2} \) lies in \( S'_{j_0, j_1, j_2} \) iff \( j_1 \) is the first coincidence with the second piece and \( j_2 \) is the first coincidence with \( j_1 \).

We have the preliminary decomposition

\[ S = \bigcup_{0 \leq j_1 \leq j_0 \atop j_0 < j_2 \leq n} S_{j_1, j_2} = \bigcup_{0 \leq j_1 \leq j_0 \atop j_0 < j_2 \leq n} S'_{j_0, j_1, j_2}, \]

(3.12)

where \( \sqcup \) denotes a disjoint union.

Now we depart from the line of argument in [8] and decompose each set \( S'_{j_0, j_1, j_2} \) further. We denote

\[ r := |x_0 - x_n|. \]

Recall that \( x_0 \neq x_n \) and so \( r > 0 \). The central observation is

**Lemma 3.5.** Let \( \bar{x} \in S_{j_1, j_2} \). Then there exists

\[ k \in \{0, \ldots, j_1 - 1\} \cup \{j_2, \ldots, n - 1\} \]

such that

\[ |x_k - x_{k+1}| \geq \frac{r}{n}. \]
Proof. From $x_{j_1} = x_{j_2}$ and the triangle inequality, we have

$$r = |x_0 - x_n| \leq \sum_{k=0}^{j_1-1} |x_k - x_{k+1}| + \sum_{k=j_2}^{n-1} |x_k - x_{k+1}|.$$ 

Hence, at least one of the (at most $n$) terms on the right-hand side must exceed $r/n$.

Thanks to Lemma 3.5, we can decompose the set $S'_{j_0,j_1,j_2}$ further, according to the minimal $k$ satisfying $|x_k - x_{k+1}| \geq \frac{r}{n}$. Namely, we define

$$S_{k_0} := \left\{ x \in (\mathbb{Z}^d)^n : |x_{k_0} - x_{k_0+1}| \geq \frac{r}{n} \text{ and } (*) \right\}. \tag{3.13}$$

Here (*) encodes the minimality of $k_0$, i.e.,

$$(*) := \begin{cases} \max_{0 \leq j < k_0} |x_j - x_{j+1}| < \frac{r}{n}, & \text{if } 0 \leq k_0 < j_1, \\ \max \left\{ \max_{0 \leq j < j_1} |x_j - x_{j+1}|, \max_{j_2 \leq j < k_0} |x_j - x_{j+1}| \right\} < \frac{r}{n}, & \text{if } j_2 \leq k_0 < n. \end{cases}$$

Using this, we may refine the preliminary decomposition (3.12) as follows:

$$S = \bigsqcup_{0 \leq j_1 \leq j_0 \in ([0, j_1-1] \cup [j_2, n-1]) \cap \mathbb{Z}} S'_{j_0,j_1,j_2} \cap S_{k_0}, \tag{3.14}$$

Note that the union over $k_0$ is indeed disjoint because $k_0$ is chosen minimally.

3.5. Discarding More Reducible Paths

We employ the decomposition (3.14) and discard more reducible paths to make a further reduction from Proposition 3.4.

Proposition 3.6. (Reduction 2) Let $m \geq 0$, $0 \leq j_0 < n$. Let $j_1$, $j_2$, $k_0$, $k_1$, $k_2$ be integers such that $0 \leq j_1 \leq j_0 < j_2 \leq n$, $k_0 \in ([0, j_1-1] \cup [j_2, n-1]) \cap \mathbb{Z}$ and $0 \leq k_1 \leq k_0 < k_2 \leq n$. Then

$$\left| f_{S'_{j_0,j_1,j_2} \cap S_{k_0} \setminus S'_{k_0,k_1,k_2}} (x_0, x_n) \right| \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n 2^{-m(d-\epsilon)} (x_0 - x_n)^{-2d+\epsilon}. \tag{3.15}$$

We show that Reduction 2 implies Reduction 1 (and hence the main claim).

Proof of Proposition 3.4 assuming Proposition 3.6. We define the set

$$\tilde{S} := \bigsqcup_{0 \leq j_1 \leq j_0 \in ([0, j_1-1] \cup [j_2, n-1]) \cap \mathbb{Z}} \left( S'_{j_0,j_1,j_2} \cap S_{k_0} \right) \cap \left\{ x : \{x_0, \ldots, x_{k_0}\} \cap \{x_{k_0+1}, \ldots, x_n\} \neq \emptyset \right\}. \tag{3.16}$$

We have the following analog of Lemma 3.3:
Lemma 3.7. For all $m \geq 0$ and $0 \leq j_0 < n$, we have

$$f_S(x_0, x_n) = f_{\tilde{S}}(x_0, x_n).$$

This lemma holds because $S \setminus \tilde{S}$ consists of reducible paths and therefore does not contribute to the sum in (3.1).

Now we recall Definition (3.11). We have the finer decomposition

$$\tilde{S} = \bigcup_{0 \leq j_1 \leq j_0} \bigcup_{k_0 \in ([0, j_1 - 1] \cup [j_2, n - 1]) \cap \mathbb{Z}} \bigcup_{0 \leq k_1 \leq k_0, k_2 \leq n} \left( S_{j_0, j_1, j_2} \cap S_{k_0} \cap S_{k_1, k_2} \right)$$

Combining Lemma 3.7 with this gives

$$f_S(x_0, x_n) = f_{\tilde{S}}(x_0, x_n) = \sum_{0 \leq j_1 \leq j_0} \sum_{k_0 \in ([0, j_1 - 1] \cup [j_2, n - 1]) \cap \mathbb{Z}} \sum_{0 \leq k_1 \leq k_0, k_2 \leq n} f_{S_{j_0, j_1, j_2}}(x_0, x_n).$$

Since the total number of summands is bounded by $C^n$, (3.15) implies (3.9) and hence Proposition 3.4. \qed

In the following section, we give the proof of Proposition 3.6, and this will imply Theorem 1.1.

3.6. Proof of Proposition 3.6

In this section, we finally see the computation where the gain of $-d$ in the decay exponent comes from. At this point, we have used the randomness sufficiently to our advantage and it is enough to prove a deterministic statement. Indeed, Lemma 3.1 reduces Proposition 3.6 to the estimate

$$|T_{X'}^n(x_0, x_n)| \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n 2^{-m(d-\epsilon)}(x_0 - x_n)^{-2d+\epsilon},$$

where $X' := S'_{j_0, j_1, j_2} \cap S_{k_0} \cap S'_{k_0, k_1, k_2}$. For this, we use a two-step strategy as in [8]. First, we prove the claimed bound for $T_{X'}^n$ using Lemma 2.7. Next, we lift this to the bound for $T_{X'}^n$ using Lemma 2.7.

Lemma 3.8. Let $T_{X'}^n$ be as in Lemma 3.1 and $m, j_0, j_1, j_2, k_0, k_1, k_2$ be as in Proposition 3.6. Then

$$|T^n_{S_{j_1, j_2} \cap S_{k_0} \cap S_{k_1, k_2}}(x_0, x_n)| \leq \epsilon^3 \left( \frac{C}{\epsilon} \right)^n 2^{-m(d-\epsilon)}(x_0 - x_n)^{-2d+\epsilon}.$$

Moreover, this bound is stable under the choice of functions $\{b_j\}_{1 \leq j \leq n}$ with $\|b_j\|_{\infty} \leq 1$ and under the replacement

$$K_j(x, y) \rightarrow e_1^j(x)K_j(x, y)e_2^j(y)$$

for any $e_1^j, e_2^j$ satisfying $\|e_1^j\|_\infty \leq 1$ and $\|e_2^j\|_\infty \leq 1$. 
We first show that (3.17) follows easily from Lemma 3.8 via Lemma 2.7.

**Proof of (3.17).** We use Lemma 2.7. We define the sets

\[
E_1 := \{0, 1, \ldots, j_1 - 1\} \quad \text{and} \quad F_1 := \{j_0 + 1, \ldots, n\},
\]
\[
E_2 := \{j_1\} \quad \text{and} \quad F_2 := \{j_0 + 1, \ldots, j_2 - 1\},
\]
\[
E_3 := \{0, 1, \ldots, k_1 - 1\} \quad \text{and} \quad F_3 := \{k_0 + 1, \ldots, n\},
\]
\[
E_4 := \{k_1\} \quad \text{and} \quad F_4 := \{k_0 + 1, \ldots, k_2 - 1\}.
\]

These are chosen such that

\[
X' = X \cap \bigcap_{l=1}^{4} \{ x \in (\mathbb{Z}^d)^{n-1} : \{ x_u : u \in E_l \} \cap \{ x_v : v \in F_l \} = \emptyset \},
\]

where \( X := S_{j_1,j_2} \cap S_{k_0} \cap S_{k_1,k_2} \). Note that \( X \) is a finite set and \( \sum_{l=1}^{4} |E_l| + |F_l| \leq 4n \). Therefore, (3.17) follows from Lemmas 3.8 and 2.7. \( \Box \)

Finally, we prove Lemma 3.8, completing the proof of Theorem 1.1.

**Proof of Lemma 3.8.** To bound \( Q := T_n^{m} \cap S_{j_1,j_2} \cap S_{k_0} \cap S_{k_1,k_2} \), we have to implement the constraints arising from \( x \in S_{j_1,j_2} \cap S_{k_0} \cap S_{k_1,k_2} \). Two of them are trivial: writing \( \mathbb{1}_X \) for the indicator function of a set \( X \), we have

\[
\mathbb{1}_{S_{j_1,j_2} \cap S_{k_0} \cap S_{k_1,k_2}} = \mathbb{1}_{\{ x : x_{j_1} = x_{j_2} \}} \mathbb{1}_{\{ x : x_{k_1} = x_{k_2} \}} \mathbb{1}_{S_{j_0} \cap S_{k_0}}.
\]

The remaining constraint is that \( x \in S_{j_0}^{m} \cap S_{k_0} \); these sets were defined in (3.6) and (3.13). We write these constraints as intersections of “local” constraints, i.e., ones that only depend on a single segment \( |x_j - x_{j+1}| \). Namely, we have

\[
S_{j_0}^{m} = \bigcap_{0 \leq j < j_0} \{ x : |x_j - x_{j+1}| < 2^m \} \cap \bigcap_{j_0 < j < n} \{ x : |x_j - x_{j+1}| < 2^{m+1} \}
\]
\[
\cap \{ x : 2^m \leq |x_{j_0} - x_{j_0+1}| < 2^{m+1} \}.
\]

and

\[
S_{k_0} = \left\{ x : |x_{k_0} - x_{k_0+1}| \geq \frac{r}{n} \right\}
\]
\[
\cap \left( \bigcap_{0 \leq j < k_0} \{ x : |x_j - x_{j+1}| < \frac{r}{n} \}, \quad \text{if } 0 \leq k_0 < j_1, \right.
\]
\[
\cap \left( \bigcap_{0 \leq j < j_1, j_2 \leq j < k_0} \{ x : |x_j - x_{j+1}| < \frac{r}{n} \}, \quad \text{if } j_2 \leq k_0 < n. \right)
\]

These expressions imply that

\[
\mathbb{1}_{S_{j_0}^{m} \cap S_{k_0}}(x) = \prod_{j=0}^{n-1} \mathbb{1}_{I_j}(|x_j - x_{j+1}|)
\]

for appropriate intervals \( I_j \) (which also depend on \( m, n, r, j_0, k_0 \)).
We have
\[ Q = \sum_{\xi \in (\mathbb{Z}^d)^{n-1}} L^1(x_0, x_1) L^2(x_2, x_3) \ldots L^n(x_{n-1}, x_n), \quad (3.18) \]
where we introduced the operators \( L_j \) with kernels \( L_j(x, y) = \mathbb{1}_{I_j}(|x-y|) K^j(x, y) \).

Recall that any path \( \gamma \) under consideration contains the two “long” segments
\[ |x_{j_0} - x_{j_0+1}| \geq R, \quad |x_{k_0} - x_{k_0+1}| \geq \frac{r}{n}, \quad (3.19) \]
and \( \max_j |x_j - x_{j+1}| \leq 2R \), where we set \( R := 2^m \). From here on, the only data that matters is the collection of relevant times \( \{0, n, j_0, j_1, j_2, k_0, k_1, k_2\} \) and their ordering (subject to the usual constraints). By symmetry (we may invert the path), we can assume that \( k_0 \in [j_2, n-1] \cap \mathbb{Z} \).

Case 1: Assume that \( j_0 + 1 \leq k_1 \leq j_2 \), so the “relevant times” are ordered as follows
\[ 0 \leq j_1 \leq j_0 < j_0 + 1 \leq k_1 \leq j_2 \leq k_0 < k_0 + 1 \leq k_2 \leq n. \quad (3.20) \]
We first consider the case where all the relevant times are different, i.e.,
\[ 0 < j_1 < j_0 < j_0 + 1 < k_1 < j_2 < k_0 < k_0 + 1 < k_2 < n \quad (3.21) \]
and then later indicate necessary modifications for the general case (3.20). Recall that \( x_{j_1} = x_{j_2} \) and \( x_{k_1} = x_{k_2} \). We denote by \( v \) the vector
\[ v = (x_{j_0}, x_{j_0+1}, x_{j_1}, x_{k_0}, x_{k_0+1}, x_{k_1}) \in (\mathbb{Z}^d)^6 \]
and then group the propagators \( L_j \) together so that each group corresponds to a time interval in (3.21), i.e.,
\[ Q = \sum_{v \in (\mathbb{Z}^d)^6} \left( \prod_{j=1}^{j_1} L^j \right) (x_0, x_{j_1}) \left( \prod_{j=j_1+1}^{j_0} L^j \right) (x_{j_1}, x_{j_0}) L^{j_0+1} (x_{j_0}, x_{j_0+1}) \]
\[ \left( \prod_{j=j_0+2}^{k_1} L^j \right) (x_{j_0+1}, x_{k_1}) \left( \prod_{j=k_1+1}^{j_2} L^j \right) (x_{k_1}, x_{j_1}) \left( \prod_{j=j_2+1}^{k_0} L^j \right) (x_{j_1}, x_{k_0}) \]
\[ L^{k_0+1} (x_{k_0}, x_{k_0+1}) \left( \prod_{j=k_0+2}^{k_2} L^j \right) (x_{k_0+1}, x_{k_1}) \left( \prod_{j=k_2+1}^{n} L^j \right) (x_{k_1}, x_n). \quad (3.22) \]

It is important that we retain some of the information contained within the constraints that \( |x_j - x_{j+1}| \in I_j \) for all \( j \). Namely, we need the fact that all the action takes place within some large ball in \( \mathbb{Z}^d \). Let \( B_\rho(x_0) \subset \mathbb{Z}^d \) be the ball of radius \( \rho > 0 \) around \( x_0 \). By \( \max_j |x_j - x_{j+1}| \leq 2R \) and the triangle inequality, we have
\[ x_j \in B_{2nR} (x_0) \quad (3.23) \]
for all $1 \leq j \leq n$. Therefore, we may replace the sum $\sum_{v \in (\mathbb{Z}^d)^6}$ by $\sum_{v \in \mathcal{B}_n}$ in (3.22), where $\mathcal{B}_n := (B_{2nR}(x_0))^6$.

We are now in a position to apply Corollary 2.4. Note that the operators $L_j(x,y) = 1_{I_j}(|x - y|)K_j(x,y)$ are equal to $K_j^I$ from (2.2). From Corollary 2.4 and (3.19), we get

$$|\mathcal{Q}| \leq \left( \frac{C}{\epsilon} \right)^n \epsilon^5 R^{-d+4\epsilon} r^{-d} \times \sum_{v \in \mathcal{B}_n} \langle x_0 - x_{j_1} \rangle^{-d+\epsilon} \langle x_{j_1} - x_{j_0} \rangle^{-d+\epsilon} \langle x_{j_0} - x_{j_1} \rangle^{-d+\epsilon} \langle x_{j_1} - x_{j_1+1} \rangle^{-d+\epsilon} \langle x_{j_1+1} - x_{j_2} \rangle^{-d+\epsilon} \langle x_{j_2} - x_n \rangle^{-d+\epsilon}. \quad (3.24)$$

We can bound the sums over $x_{j_0}, x_{j_0+1}, x_{k_0}, x_{k_0+1}$ all in the same way. E.g., using that $|x_{j_1} - x_{j_0}| \leq |x_{j_1} - x_0| + |x_0 - x_{j_0}| \leq 4nR$, we have

$$\sum_{x_{j_0} \in B_{2nR}(x_0)} \langle x_{j_1} - x_{j_0} \rangle^{-d+\epsilon} \leq \sum_{y \in B_{4nR}(0)} \langle y \rangle^{-d+\epsilon} \leq \frac{Cn}{\epsilon} R^{\epsilon}. \quad (3.25)$$

From the bound (3.25) and its analogs for $x_{j_0+1}, x_{k_0}, x_{k_0+1}$, we get

$$|\mathcal{Q}| \leq \left( \frac{C}{\epsilon} \right)^n \epsilon^5 R^{-d+4\epsilon} r^{-d} \times \sum_{x_{j_1}, x_{k_1} \in B_{2nR}(x_n)} \langle x_0 - x_{j_1} \rangle^{-d+\epsilon} \langle x_{k_1} - x_{j_1} \rangle^{-d+\epsilon} \langle x_{k_1} - x_n \rangle^{-d+\epsilon}. \quad (3.26)$$

Using Lemma 2.6 twice, we get

$$|\mathcal{Q}| \leq \left( \frac{C}{\epsilon} \right)^n \epsilon^3 R^{-d+4\epsilon} r^{-2d+3\epsilon}. \quad (3.27)$$

(We mention that it is possible to replace Lemma 2.6 by an elementary observation: Since $|x_0 - x_{j_1}| + |x_{k_1} - x_{j_1}| + |x_{k_1} - x_n| \geq r$, at least one of these three distances is $\geq r/3$. Implementing this and summing over $x_{j_1}, x_{k_1} \in B_{2nR}(x_n)$ gives (3.27) with an additional, and irrelevant, $R^{2\epsilon}$ factor on the right-hand side.)

Next, we turn to the general case (3.20), where some of the relevant times may coincide. We note that (3.22) is still valid in the general case under the convention that

$$\left( \prod_{j=a+1}^{a} L_j \right) (x_a, x_a) \equiv 1. \quad (3.28)$$

Now we argue why the occurrence of any such coincidences does not change the final bound, (3.27).

Let $A_1, A_2, A_3, A_4$ denote the cases $j_0 = j_1, j_0+1 = k_1, k_0 = j_2, k_0+1 = k_2$, respectively. In addition, let $B_1, B_2, B_3$ denote the cases $j_1 = 0, k_2 = n, k_1 = j_2$, respectively. Then each possible combination of coincidences of the relevant
times in (3.20) corresponds to a subset of \( \{A_1, A_2, A_3, A_4, B_1, B_2, B_3\} \). So far, we considered the case of no coincidences, (3.21).

For each occurrence of \( A_1, A_2, A_3, A_4, B_1, B_2, B_3 \), we have the trivial identity (3.28) instead of having to apply Corollary 2.4. Effectively, this amounts to multiplying each summand in (3.24) by \( \varepsilon^{-1} \delta_{x_a}(x_b) \) for appropriate \( a, b \). (Here we denoted by \( \delta_x(y) \) the delta function: \( \delta_x(y) = 1 \) if \( x = y \) and 0 otherwise.) For instance, \( A_1 \) and \( B_1 \) produce the factors \( \varepsilon^{-1} \delta_{x_{j,0}}(x_{j,1}) \) and \( \varepsilon^{-1} \delta_{x_{j,1}}(x_0) \), respectively.

Thus we need to show that the factor \( \varepsilon^{-1} \delta_{x_a}(x_b) \) leads to the same bound as before, (3.27).

Consider the case \( A_1 \), which gives \( \varepsilon^{-1} \delta_{x_{j,1}}(x_{j,1}) \). This is to be compared with how we treated the original expression in (3.25), where the disappearance of the sum over \( x_{j,0} \) may alternatively expressed as a bound in terms of \( C^n R^\varepsilon \varepsilon^{-1} \delta_{x_{j,0}}(x_0) \). Since \( 1 \leq C^n R^\varepsilon \), we get the same bound, no matter whether \( A_1 \) occurs or not. The same argument works for \( A_2, A_3, A_4 \).

To summarize this part, we always get (3.26) (modified by the appropriate delta functions coming from the cases \( B_1, B_2, B_3 \)), no matter which subset of cases the \( A_1, A_2, A_3, A_4 \) occurs.

Finally, we come to the cases \( B_1, B_2, B_3 \). Notice that at most two of them may occur simultaneously because \( x_0 \neq x_n \). Consider the case where just \( B_1 \) occurs, i.e., (3.26) comes with an additional factor \( \varepsilon^{-1} \delta_{x_{j_1}}(x_0) \):

\[
|Q| \leq \left( \frac{C}{\varepsilon} \right)^n \varepsilon^4 R^{-d+4\varepsilon} R^{-d} \times \sum_{x_{j_1}, x_{k_1} \in B_{2nR}(x_n)} \delta_{x_{j_1}}(x_0) (x_0 - x_{j_1})^{-d+\varepsilon} (x_{k_1} - x_{j_1})^{-d+\varepsilon} (x_{k_1} - x_n)^{-d+\varepsilon}.
\]

Lemma 2.6 then yields (3.27). Similar considerations imply (3.27) for all the other cases as well.

Case 2: Assume that either \( 0 \leq k_1 \leq j_1 \) or \( j_1 < k_1 \leq k_0 \). We may follow exactly the same steps as in Case 1, unless \( 0 \leq k_1 < j_1 \), so we assume this in the following. We start by discussing the case, where all the relevant times are different, i.e.,

\[
0 < k_1 < j_1 < j_0 < j_0 + 1 < j_2 < k_0 < k_0 + 1 < k_2 < n.
\]

Arguing as in Case 1 and after summing over \( x_{j_0}, x_{j_0+1}, x_{k_0}, x_{k_0+1} \), we may bound \( |Q| \) by a slightly different expression (compared to what we got in (3.26)):

\[
|Q| \leq \left( \frac{C}{\varepsilon} \right)^n \varepsilon^5 R^{-d+4\varepsilon} R^{-d} \times \sum_{x_{j_1}, x_{k_1} \in B_{2nR}(x_n)} (x_0 - x_{k_1})^{-d+\varepsilon} (x_{k_1} - x_{j_1})^{-d+\varepsilon} (x_{k_1} - x_n)^{-d+\varepsilon}.
\]
\[
\leq \left( \frac{C}{\epsilon} \right)^n \epsilon^4 R^{-d+5\epsilon} r^{-d} \sum_{x_{k_1} \in B_{2nR}(x_n)} \langle x_0 - x_{k_1} \rangle^{-d+\epsilon} (x_{k_1} - x_n)^{-d+\epsilon}
\]
\[
\leq \left( \frac{C}{\epsilon} \right)^n \epsilon^3 R^{-d+5\epsilon} r^{-2d+2\epsilon}.
\]

We may also treat the case where some of the relevant times may coincide as in Case 1. Finally, the stability of the bound is a consequence of Corollary 2.4. This finishes the proof of Lemma 3.8. \(\square\)

4. Proof of Theorem 1.9—Partitioning the Set of Irreducible Paths

In this section, we prove Theorem 1.9, which is an immediate consequence of the following decomposition result for \(U\).

**Proposition 4.1.** Let \(n \geq 3\) and \(T_S^n\) be as in Definition (1.17). For each \(x_0, x_n \in \mathbb{Z}^d\), \(x_0 \neq x_n\), there is a partition of the set of irreducible paths \(U\) into \(O(2^n)\) many disjoint subsets \(\{U'_\alpha\}_{\alpha \in A}\) such that

\[
\max_{\alpha \in A} |T_{U'_\alpha}(x_0, x_n)| \leq C n \log n \epsilon^{-n} \langle x_0 - x_n \rangle^{-3d+\epsilon} \quad (4.1)
\]

for all sufficiently small \(\epsilon > 0\).

We prove Proposition 4.1 in the following subsections by explicitly constructing the sets \(U'_\alpha\).

4.1. First Decomposition

In the following, we write “\(\{\underline{x} : \ldots\}\)” for \(\{x = (x_1, \ldots, x_{n-1}) \in (\mathbb{Z}^d)^{n-1} : \ldots\}\). Define the sets

\[
V_{i,j} := \{\underline{x} : x_i = x_n, x_j = x_0\},
\]
\[
V'_{i,j} := V_{i,j} \cap \{\underline{x} : x_0 \notin \{x_{j+1}, x_{j+2}, \ldots, x_{n-1}\}\}
\]
\[
\cap \{\underline{x} : x_n \notin \{x_1, x_2, \ldots, x_{i-1}\}\}.
\]

In other words, if \(\underline{x} \in V'_{i,j}\), then

\[i = \min \{l : x_l = x_n\}\quad \text{and}\quad j = \max \{l : x_l = x_0\},.\]

The sets \(\{V'_{i,j}\}_{0<i,j<n}\) are disjoint, and we have

\[
\bigcup_{0<i<j<n} V'_{i,j} \subset U \subset \bigcup_{0<i,j<n} V_{i,j} = \bigcup_{0<i,j<n} V'_{i,j}.
\]

Therefore, we can decompose

\[
U = \left( \bigcup_{0<i,j<n} V'_{i,j} \right) \sqcup \left( \bigcup_{1<j<i<n-1} U \cap V'_{i,j} \right) \quad (4.2)
\]

with the observation that \(U \cap V'_{i,1} = U \cap V'_{n-1,j} = \emptyset\).
4.2. A Further Decomposition

In this subsection, we further decompose the set

\[ U' := \bigcup_{1 < j < i < n-1} U \cap V'_{i,j}. \]

**Procedure.** Fix \( \bar{x} \in U \cap V'_{i,j} \) for some \( j < i \). Since \( \bar{x} \) is irreducible, there should exist \( i_1 < j \) and \( j_1 > j \) such that \( x_{i_1} = x_{j_1} \). We define

\[ j_1 = \max\{l : l > j \text{ and } x_l = x_{i_1} \text{ for some } 0 < i_1 < j\} \]

and then

\[ i_1 = \min\{l : 0 < l < j \text{ and } x_l = x_{j_1}\}. \]

Note that, by definition,

\[ \{x_1, x_2, \ldots, x_{j-1}\} \cap \{x_{j_1+1}, x_{j_1+2}, \ldots, x_{n-1}\} = \emptyset. \tag{4.3} \]

We have the following two alternatives (\( j_1 \neq i \) due to the condition imposed on \( V'_{i,j} \)):

1. \( j_1 > i \): we stop with a single pair \((i_1, j_1)\).
2. \( j_1 < i \): we continue to choose \((i_2, j_2)\) as follows. Since \( \bar{x} \) is irreducible, there should be some \( i_2 < j_1 \) and \( j_2 > j_1 \) such that \( x_{i_2} = x_{j_2} \). We choose \( j_2 \) as the maximum of all such \( j_2 \) and then choose \( i_2 \) as the minimum of all \( l \) such that \( x_l = x_{j_2} \). From (4.3), \( j < i_2 < j_1 \).

Having chosen \((i_2, j_2)\), we again have the alternatives

(a) \( j_2 > i \): we stop with \((i_1, j_1), (i_2, j_2)\);
(b) \( j_2 < i \): we continue to choose the next pair \((i_3, j_3)\) for some \( j_1 < i_3 < j_2 \) and \( j_3 > j_2 \), following the same procedure. We repeat this procedure until we obtain \((i_1, i_1), \ldots, (i_m, j_m)\) for some \( m \) satisfying \( j_{m-2} < i_m < j_{m-1} \) and \( j_m > i \). We write \( m(\bar{x}) \) for this \( m \). By a simple counting argument, we see that \( m(\bar{x}) \leq (n - 3)/2 \) for any \( \bar{x} \in U' \).

From the Procedure, we may write \( U \cap V'_{i,j} \) as a disjoint union. We first define some basic building blocks. For \( 0 < i < j < n \), define

\[ S_{i,j} := \{x : x_i = x_j\}. \]

We note that the definition is different from the definition in (3.11) and [8]—it does not require a further restriction regarding the dyadic decomposition. It is convenient to set \( j_0 = j \) and \( j_{-1} = 0 \). For \( m \geq 1 \), define, for a given \((i_m, j_m)\) and fixed \( j_{m-2}, j_{m-1}, \)

\[ S'_{i_m,j_m} := S_{i_m,j_m} \setminus \left( \bigcup_{j_{m-2} < u < j_{m-1}} S_{u,v} \right) \cup \left( \bigcup_{j_{m-2} < u < i_m} S_{u,j_m} \right). \]
We shall write \( S'_{im,jm} \) for \( S'_{im,jm} \) \((jm-2, jm-1)\) for the sake of simplicity. The set \( S'_{im,jm} \) is chosen so that if \((im, jm)\) is selected in the Procedure for \( x \in U \cap V'_{i,j} \), then \( x \in S'_{im,jm} \).

Note that the first step of the Procedure gives

\[
U \cap V'_{i,j} = \bigcup_{0 < i_1 < j \atop j < j_1 < n} U \cap V'_{i_1,j_1} \cap S'_{i_1,j_1}.
\]

When \( j_1 < i \), the Procedure decomposes \( U \cap V'_{i,j} \cap S'_{i_1,j_1} \) further; this corresponds to the case 2. We have

\[
U \cap V'_{i,j} \cap S'_{i_1,j_1} = \bigcup_{j < j_2 < n} U \cap V'_{i,j} \cap S'_{i_1,j_1} \cap S'_{i_2,j_2}.
\]

Each \( U \cap V'_{i,j} \cap S'_{i_1,j_1} \cap S'_{i_2,j_2} \) is decomposed further when \( j_2 < i \); this corresponds to the case 2.(b) in the Procedure.

Repeating this yields the desired decomposition of the set \( U' \). To describe this decomposition in a compact way, we set

\[
I_m(j, jm-2, jm-1, i) := \{(im, jm) : jm-2 \leq im < jm-1, i < jm < n\}
\]

\[
I'_m(j, jm-2, jm-1, i) := \{(im, jm) : jm-2 < im < jm-1, jm-1 < jm < i\}.
\]

We shall simply write \( I_m \) and \( I'_m \) assuming that we work with fixed indices \( j, jm-2, jm-1, i \). Note that if \((i_1, j_1), \ldots, (im(\bar{x}), jm(\bar{x}))\) are obtained from the Procedure for some \( \bar{x} \in U \cap V'_{i,j} \), then

\[
(i_m, jm) \in I'_m \text{ for } 1 \leq m < m(\bar{x})
\]

\[
(i_m(\bar{x}), jm(\bar{x})) \in I_m(\bar{x}),
\]

and

\[
\bar{x} \in S'_{i_1,j_1} \cap S'_{i_2,j_2} \cap \ldots \cap S'_{im(\bar{x}),jm(\bar{x})}.
\]

In conclusion, combined with (4.2), we can write

\[
U = \bigcup_{i < j < n} V'_{i,j}
\]

\[
\bigcup_{m} \bigcup_{1 \leq i < j < n - 1} (i_j, j_1) \in I'_m (im, jm) \in I_m \quad 1 \leq l \leq m V'_{i,j} \cap S'_{i_1,j_1} \cap S'_{i_2,j_2} \cap \ldots \cap S'_{im,jm},
\]

(4.4)

where the union in \( m \) is taken over \( 1 \leq m \leq (n - 3)/2 \).

We write \( U = \bigcup_{\alpha \in \mathcal{A}} U'_{\alpha} \) after renaming all disjoint sets involved in (4.4). We claim that \#\( \mathcal{A} \leq 2^{n-1} \). First note that there are \( \binom{n-1}{2} \) sets \( V'_{i,j} \), \( 0 < i < j < n \).
Moreover, for each $1 \leq m \leq (n - 3)/2$, the disjoint union $\bigcup_{1 < j < i < n - 1} \bigcup_{(i_j, j) \in I'_l} \bigcup_{1 \leq l < m} (i_m, j_m) \in I_m$, involves $\left(\frac{n - 1}{2m + 2}\right)$ sets. Therefore,

$$\#A = \sum_{0 \leq m \leq (n - 3)/2} \left(\frac{n - 1}{2m + 2}\right) \leq 2^{n - 1}.$$  

For the proof of Proposition 4.1, it only remains to prove (4.1).

**4.3. Proof of (4.1)**

In this subsection, we prove estimates for each set appearing in the partition (4.4) using Lemmas 2.2 and 2.7. Recall that the set $X_k$ is defined by

$$X_k = \left\{ x \in (\mathbb{Z}^d)^{n - 1} : \max_{0 \leq j < n} |x_j - x_{j+1}| < 2^k \right\}.$$  

Note that the truncation $S \rightarrow S \cap X_k$ amounts to the replacement $K^j \rightarrow K^j_k$, where $I_k = [0, 2^k)$.

**Lemma 4.2.** Assume that $0 < i < j < n$. Then

$$|T^n_{V_{i,j}}(x_0, x_n)| \leq \epsilon^3 \left(\frac{C}{\epsilon}\right)^n \langle x_0 - x_n \rangle^{-3d + \epsilon}.$$  

**Proof.** First, note that $V_{i,j}' = V_{i,j} \cap A_V$, where

$$A_V = \{ x : \{ x_0 \} \cap \{ x_{j+1}, x_{j+2}, \ldots, x_{n-1} \} = \emptyset \}$$

and

$$\cap \{ x : \{ x_n \} \cap \{ x_1, x_2, \ldots, x_{i-1} \} = \emptyset \} \quad (4.5)$$

Therefore, by Lemma 2.7, it is enough to show that

$$|T^n_{V_{i,j}}(x_0, x_n)| \leq \epsilon^3 \left(\frac{C}{\epsilon}\right)^n \langle x_0 - x_n \rangle^{-3d + \epsilon}$$

for all large $k \geq 1$. This is a consequence of the factorization

$$T^n_{V_{i,j} \cap X_k}(x_0, x_n) = K^1_{k} K^2_{k} \ldots K^i_{k} (x_0, x_n) K^{i+1}_{k} \ldots K^j_{k} (x_n, x_0)$$

and Corollary 2.4.  \(\square\)

**Lemma 4.3.** Let $1 \leq m \leq (n - 3)/2$ and

$$S = V_{i,j} \cap S_{i_1,j_1} \cap S_{i_2,j_2} \cap \ldots \cap S_{i_m,j_m}$$

for some $(i_l, j_l) \in I'_l$ for $1 \leq l < m$ and $(i_m, j_m) \in I_m$. Then

$$|T^n_S(x_0, x_n)| \leq \epsilon^{3 + 2m} \left(\frac{C}{\epsilon}\right)^n \langle x_0 - x_n \rangle^{-3d + \epsilon}.$$
Moreover, we have the bound

\[ |T^n_{S \cap X_k}(x_0, x_n)| \leq \varepsilon^{3+2m} \left( \frac{C}{\varepsilon} \right)^n (x_0 - x_n)^{-3d+\varepsilon} \]  

(4.6)

uniformly in \( k \geq 1 \).

**Proof.** We only prove the bound for \( T^n_S(x_0, x_n) \). The argument for the truncated version (4.6) is the same.

The proof uses an induction on \( m \). We start with the base case \( m = 1 \). Note that \( 0 < i_1 < j < i_1 < j_1 < n \). We may factor \( T(S) \) as

\[ T^n_S(x_0, x_n) = \sum_{x_{i_1}} T_1(x_0, x_{i_1}) T_2(x_{i_1}, x_0) T_3(x_0, x_n) T_4(x_n, x_{i_1}) T_5(x_{i_1}, x_n), \]

where \( T_1 = K^{i_1} K^{i_1+1} \ldots K^{i_1+j} \), \( T_2 = K^{i_1+1} \ldots K^{j_1} \), \( T_3 = K^j \ldots K^{i_1} \), \( T_4 = K^{j_1} \ldots K^{j} \), and \( T_5 = K^{i} \ldots K^{i_2} \).

From Lemma 2.2, we have

\[ |T^n_S(x_0, x_n)| \leq \varepsilon^5 \left( \frac{C}{\varepsilon} \right)^n (x_0 - x_n)^{-d+\varepsilon} \sum_{x_{i_1}} (x_0 - x_{i_1})^{-2(d-\varepsilon)} (x_{i_1} - x_n)^{-2(d-\varepsilon)}. \]

(4.7)

Here and in what follows, all sums are over \( \mathbb{Z}^d \). The claimed estimate then follows from

\[ |x_0 - x_n| \leq |x_0 - x_{i_1}| + |x_{i_1} - x_n|, \]

which allows us to decompose the summation into two parts:

\[ \mathbb{Z}^d = \{ x_{i_1} : |x_0 - x_{i_1}| \geq |x_0 - x_n|/2 \} \cup \{ x_{i_1} : |x_{i_1} - x_n| \geq |x_0 - x_n|/2 \}. \]

Next, we shall derive the claimed estimate for \( m = 2 \) from the estimate for \( m = 1 \). Here, \( 0 < i_1 < j < i_2 < j_1 < i < j_2 < n \). Following the above argument, we have

\[ |T^n_S(x_0, x_n)| \leq \varepsilon^7 \left( \frac{C}{\varepsilon} \right)^n \sum_{x_{i_1}, x_{i_2}} (x_0 - x_{i_1})^{-2(d-\varepsilon)} (x_0 - x_{i_2})^{-d+\varepsilon} (x_{i_2} - x_{i_1})^{-d+\varepsilon} \times (x_{i_1} - x_n)^{-d+\varepsilon} (x_{i_2} - x_n)^{-2(d-\varepsilon)}. \]

We first take the sum over \( x_{i_2} \) using the Cauchy-Schwarz inequality to get:

\[ \sum_{x_{i_2}} (x_0 - x_{i_2})^{-d+\varepsilon} (x_{i_2} - x_n)^{-d+\varepsilon} (x_{i_1} - x_{i_2})^{-d+\varepsilon} (x_{i_2} - x_n)^{-d+\varepsilon} \leq C (x_0 - x_n)^{-d+\varepsilon} (x_{i_1} - x_n)^{-d+\varepsilon}. \]

Then we get the expression (4.7) up to a multiplicative factor \( C \varepsilon^2 \).

Passing from \( m - 1 \) to \( m \) is similar. We omit the details. \( \square \)
Finally, we pass from (4.6) to an estimate for the “primed” sets. This is the part where we lose a constant factor bounded by $Cn \log n$.

**Lemma 4.4.** Let $S' = V_{i, j} \cap S'_{i_1, j_1} \cap S'_{i_2, j_2} \cap \ldots \cap S'_{i_m, j_m}$ for some $(i, j) \in I'$ for $1 \leq l < m$ and $(i_m, j_m) \in I_m$. Then

$$|T_n S'(x_0, x_n)| \leq C n \log n \epsilon^{3 + 2m - n} |x_0 - x_n|^{-3d + \epsilon}$$

for some constant $C > 0$.

To prepare for the proof of Lemma 4.4, we first prove the following weaker estimate:

$$|T_n S'(x_0, x_n)| \leq C n^2 \epsilon^{3 + 2m - n} |x_0 - x_n|^{-3d + \epsilon}. \tag{4.8}$$

**Proof of (4.8).** We first write $S'_{i_m, j_m}$ as

$$S'_{i_m, j_m} = S_{i_m, j_m} \cap \bigcap_{1 \leq l \leq m} \{x : \{x_u : j_{l-2} < u < i_l \} \cap \{x_{j_l} = \emptyset\} \cap \{x_u : j_{l-2} < u < j_{l-1}\} \cap \{x_v : j_l < v < n\} = \emptyset\}. \tag{4.9}$$

Let $S = V_{i_1, j_1} \cap S_{i_1, j_1} \cap S_{i_2, j_2} \ldots \cap S_{i_m, j_m}$. Then we may write

$$S' = S \cap A_V \cap A^1_S \cap A^2_S,$$

where $A_V$ is as in (4.5) and

$$A^1_S = \bigcap_{1 \leq l \leq m} \{x : \{x_u : j_{l-2} < u < i_l \} \cap \{x_{j_l} = \emptyset\} \cap \{x_u : j_{l-2} < u < j_{l-1}\} \cap \{x_v : j_l < v < n\} = \emptyset\},$$

$$A^2_S = \bigcap_{1 \leq l \leq m} \{x : \{x_u : j_{l-2} < u < j_{l-1} \} \cap \{x_v : j_l < v < n\} = \emptyset\}.$$

We apply Lemmas 4.3 and 2.7. We count the number of terms $x_j$ needed to define intersections in $A_V$, $A^1_S$, and $A^2_S$. First, $A_V$ involves at most $n - j + i \leq 2n$ terms. In addition, $A^1_S$ involves at most $\sum_{l=1}^m (i_l - j_{l-2}) \leq \sum_{l=1}^m (j_l - j_{l-2}) \leq 2n$ terms. Finally, $A^2_S$ involves at most $\sum_{l=1}^m (n - j_l + j_{l-1} - j_{l-2}) \leq nm \leq n^2 / 2$. In total, we lose a factor bounded by $2^{m^2 + n^2 / 2} \leq 2^{5n^2}$ in the application of Lemma 2.7. This finishes the proof. \qed

Next, we indicate how to modify the proof of (4.8) to obtain Lemma 4.4. First, recall that the intersection with $A^2_S$ is the only part that we lose a factor larger than $C^n$. We lost a factor of $C^{n^2}$ from the bound

$$\sum_{l=1}^m |E_l| + |F_l| \leq n^2 / 2,$$

where

$$E_l := (j_{l-2}, j_{l-1}) \cap \mathbb{Z} \quad \text{and} \quad F_l := (j_l, n) \cap \mathbb{Z}. \tag{4.9}$$

We show that we can rewrite $A^2_S$ in a more efficient way, which implies Lemma 4.4.
Lemma 4.5. Let $0 = j_{-1} < j_0 < j_1 < \ldots < j_m < n$ be an increasing sequence of integers such that the sets $E_l, F_l$ defined in (4.9) are non-empty. Then there exist subsets $\{E'_l, F'_l\}_{1 \leq l \leq m}$ of $(0, n) \cap \mathbb{Z}$ such that

$$A := \bigcap_{l=1}^{m}\{x : \{x_u : u \in E_l\} \cap \{x_v : v \in F_l\} = \emptyset\}$$

$$= \bigcap_{l=1}^{m}\{x : \{x_u : u \in E'_l\} \cap \{x_v : v \in F'_l\} = \emptyset\}, \quad (4.10)$$

and

$$\sum_{l=1}^{m} |E'_l| + |F'_l| = O(n \log n).$$

Proof. We first give an informal discussion. The idea is to choose $l_0 := \max\{l : j_l \leq \frac{n}{2} - 1\}$ (4.11)

and then write $A$ as an intersection of three parts

$$A = (A'_{l_0} \cap A_{l_0+1} \cap A_{l_0+2}) \cap \bigcap_{l=1}^{l_0-1} A'_l \cap \bigcap_{l=l_0+3}^{m} A_l, \quad (4.12)$$

where

$$A_l := \{x : \{x_u : u \in E_l\} \cap \{x_v : v \in F_l\} = \emptyset\},$$

$$A'_l := \{x : \{x_u : u \in E_l\} \cap \{x_v : v \in F_l \setminus F_{l_0}\} = \emptyset\}, \quad \text{for } l < l_0$$

$$A'_{l_0} := \{x : \{x_u : u \in \bigcup_{l=1}^{l_0} E_l\} \cap \{x_v : v \in F_{l_0}\} = \emptyset\}.$$

The saving comes from that now $A'_l$ involves $x_v$ for $v \in F_l \setminus F_{l_0}$ when $l < l_0$. We iterate this manipulation to $\cap_{l=1}^{l_0-1} A'_l$ and $\cap_{l=l_0+3}^{m} A_l$.

We turn to a rigorous argument. For a given $0 < j_0 < j_1 < \ldots < j_m < n$, define

$$\tilde{C}(n; j_0, j_1, \ldots, j_m) := \min_{l=1}^{m}(|E'_l| + |F'_l|),$$

where the minimum is taken over all collection of subsets $\{E'_l, F'_l\}_{1 \leq l \leq m}$ of $(0, n) \cap \mathbb{Z}$, satisfying (4.10). Here the parameter $n$ is associated with the largest element $n-1$ in $F_l$. In addition, define

$$C(n) := \max \tilde{C}(n; j_0, j_1, \ldots, j_m),$$

where the maximum is taken over all $(j_0, j_1, \ldots, j_m)$ satisfying the assumption of Lemma 4.5. Certainly, $C(n)$ is non-decreasing and $C(n) = O(1)$ when $n = O(1)$.
We claim that \( C(n) = O(n \log n) \). Without loss of generality, we may assume that \( n \) is a power of 2. We will show that
\[
C(n) \leq 3n + 2C(n/2). \tag{4.13}
\]
Iterating (4.13) \( k \) times, we get \( C(n) \leq 3kn + 2^k C(n/2^k) \), from which we obtain the claim by choosing \( k \sim \log n \).

Let \( 0 < j_0 < j_1 < \ldots < j_m < n \) be given. We need to show that
\[
\tilde{C}(n; j_0, j_1, \ldots, j_m) \leq 3n + 2C(n/2). \tag{4.14}
\]
Let \( l_0 \) be as in (4.11) and write \( A \) as in (4.12). Observe that the sets \( E_l \) and \( F_l \) for \( l < l_0 \) are contained in the set \([1, j_{l_0}] \cap \mathbb{Z} \). Therefore, there are subsets \( \{E'_l, F'_l\}_{1 \leq l \leq l_0-1} \) of \([1, j_{l_0}] \cap \mathbb{Z} \) such that
\[
\bigcap_{l=1}^{l_0-1} A'_l = \bigcap_{l=1}^{l_0-1} \{x : \{x_u : u \in E'_l\} \cap \{x_v : v \in F'_l\} = \emptyset\}
\]
and
\[
\sum_{l=1}^{l_0-1} |E'_l| + |F'_l| = \tilde{C}(j_{l_0}, j_0, j_1, \ldots, j_{l_0-1}) \leq C(j_{l_0} + 1) \leq C(n/2),
\]
since \( j_{l_0} + 1 \leq n/2 \) by the choice of \( l_0 \).

The situation for \( l \geq l_0 + 3 \) is essentially the same as the case for \( l < l_0 \) since the sets \( E_l \) and \( F_l \), for \( l \geq l_0 + 3 \), are contained in the interval \((j_{l_0+1}, n) \cap \mathbb{Z} \) of length less than or equal to \( n/2 \). Notice that we have translation invariance, i.e., we may work with translated sets of \( E_l - j_{l_0+1} \) and \( F_l - j_{l_0+1} \) for the purpose of choosing the sets \( E'_l \) and \( F'_l \). Thanks to this, we may find sets \( \{E'_l, F'_l\}_{l_0+3 \leq l \leq m} \) such that
\[
\bigcap_{l=l_0+3}^m A_l = \bigcap_{l=l_0+3}^m \{x : \{x_u : u \in E'_l\} \cap \{x_v : v \in F'_l\} = \emptyset\}
\]
and
\[
\sum_{l=l_0+3}^m |E'_l| + |F'_l| \leq C(n - j_{l_0+1}) \leq C(n/2).
\]

For the remaining part, \( A'_{l_0} \cap A_{l_0+1} \cap A_{l_0+2} \), we just set \( E'_l = E_l \) and \( F'_l = F_l \) for \( l_0 \leq l \leq l_0 + 2 \) except that \( E'_{l_0} := \bigcup_{l=1}^{l_0} E_l \).

So far, we have found \( \{E'_l, F'_l\}_{1 \leq l \leq m} \) satisfying (4.10) such that
\[
\sum_{1 \leq l \leq m} |E'_l| + |F'_l| \leq 3n + 2C(n/2),
\]
which verifies (4.14). This completes the proof. \( \square \)
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A Proof of Corollary 1.5 on Derivatives of the Averaged Green’s Function

The proof is based on the standard fact that existence of derivatives in Fourier space (which we get from Theorem 1.1) can be translated to decay in physical space via integration by parts. For the endpoint case $|\alpha| = d + 1$, we use a variant of the argument which only requires the Fourier transform to be Hölder continuous.

Let $d \geq 2$ and assume that $\alpha$ is a multi-index such that $|\alpha| > 2 - d$. This condition ensures that the symbol of $\nabla^\alpha G$ (and $\nabla^\alpha G^\mu$) is integrable on $\mathbb{T}^d$. We shall prove the first statement for $d \geq 3$ as the proof of the second statement is identical.

Fix $0 < \varepsilon < 1$ and let $0 < \delta < c\varepsilon$, where $c$ is the constant $\tilde{c}_d$ from Theorem 1.1. Note that the operator $\mathcal{L}$ is a convolution operator whose symbol is given by

$$m(\theta) = (1 + \delta E^\sigma) \sum_{j=1}^d 2(1 - \cos \theta_j) + \sum_{1 \leq j,k \leq d} (e^{-i\theta_j} - 1) \hat{K}_{j,k}(\theta)(e^{i\theta_k} - 1)$$

for $\theta \in \mathbb{T}^d$. By Theorem 1.1, we have

$$||\hat{K}_{j,k}||_{C^{2d-1,1-\varepsilon}(\mathbb{T}^d)} \leq C\delta^2.$$ 

In particular, we may find $0 < c_d \leq c\varepsilon$ such that for any $0 < \delta < c_d$, we have the lower bound

$$|m(\theta)| \geq C|\theta|^2$$  \hspace{1cm} (A.1)

for some constant $C > 0$ for any $\theta$ in $\mathbb{T}^d$ which we identify with $[-\pi, \pi]^d$.

Next, let $m^\sigma$ be the symbol of $\nabla^\sigma$, i.e. $m^\sigma(\theta) = \prod_{j=1}^d (e^{i\theta_j} - 1)^{\sigma_j}$. Since $|m^\sigma(\theta)| \leq \prod_{j=1}^d |\theta_j|^{\sigma_j} \leq |\theta|^{|\sigma|}$, we see that

$$\left| \frac{m^\sigma(\theta)}{m(\theta)} \right| \leq C|\theta|^{|\sigma|-2},$$  \hspace{1cm} (A.2)

which is integrable on $\mathbb{T}^d$ provided that $|\sigma| > 2 - d$.

The kernel $\nabla^\sigma G(x)$ is the Fourier inverse of $m^\sigma(\theta)[m(\theta)]^{-1}$. We estimate $\nabla^\sigma G(x)$ using a dyadic decomposition of $\mathbb{T}^d$ as follows. Let $\phi$ be a smooth even function compactly supported on $[-2, 2]$ and $\phi(r) = 1$ for $r \in [-1, 1]$. Let $\psi(r) := \phi(r) - \phi(2r)$ and $\psi_l(r) := \psi(2^l r)$ for $l \geq 1$ and $\psi_0(r) := 1 - \phi(2r)$. Note that $\sum_{l \geq 0} \psi_l(r) = 1$ for any $r \neq 0$. We write $\nabla^\sigma G(x) = \sum_{l \geq 0} \nabla^\sigma G_l(x)$, where we denote by $\nabla^\sigma G_l(x)$ the Fourier inverse of $\psi_l(|\theta|)m^\sigma(\theta)[m(\theta)]^{-1}$. 

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Define
\[ g^\alpha_l(\theta) := \frac{\phi(\theta)m^\alpha(2^{-l}\theta)}{m(2^{-l}\theta)}, \]
where \( \phi(\theta) := \psi(|\theta|) \). Then for \( l \geq 1 \), we may write by a change of variable
\[ \nabla^\alpha G_l(x) = 2^{-ld} \int_{\mathbb{R}^d} g^\alpha_l(\theta) e^{i2^{-l}x \cdot \theta} d\theta \frac{d\theta}{(2\pi)^d}. \]

First note that, by (A.2), \( |\nabla^\alpha G_l(x)| \leq C 2^{-l(d-2+|\alpha|)} \) for any \( x \in \mathbb{Z}^d \). This bound may be improved when \( 2^{-l} |x| \geq 1 \). We claim that when \( |x| \geq 2^l \) and \( l \geq 1 \), we have
\[ |\nabla^\alpha G_l(x)| \leq \frac{C 2^{-l(d-2+|\alpha|)}}{(2^{-l}|x|)^{2d-\epsilon}}. \] (A.3)

Given the estimate (A.3), Corollary 1.5 follows quickly. First of all, one can check, using integration by parts, that
\[ |\nabla^\alpha G_0(x)| \leq C (1 + |x|)^{-2d-1}. \]

Using this bound and (A.3), we get
\[ |\nabla^\alpha G(x)| \leq C \sum_{l \geq 0} 2^{-l(d-2+|\alpha|)} \leq C \] (A.4)

for any \( x \in \mathbb{Z}^d \), since we assume \( |\alpha| > 2 - d \). Next we assume that \( |x| \geq 100 \) and study the sum over \( 2^l > |x| \) and \( 2^l \leq |x| \) separately. We have
\[ \sum_{l \geq 0: 2^l > |x|} |\nabla^\alpha G_l(x)| \leq C \sum_{l \geq 0: 2^l > |x|} 2^{-l(d-2+|\alpha|)} \leq C|x|^{-(d-2+|\alpha|)}. \] (A.5)

On the other hand, if \( |\alpha| \leq d + 1 \), we have
\[ \sum_{l \geq 0: 2^l \leq |x|} |\nabla^\alpha G_l(x)| \leq C|x|^{-(2d-1)} + C \sum_{l \geq 1: 2^l \leq |x|} 2^{l(d+2-|\alpha|\epsilon)} |x|^{-2d+\epsilon} \]
\[ \leq C|x|^{-(d-2+|\alpha|)}. \] (A.6)

Observe that (A.4), (A.5) and (A.6) implies Corollary 1.5.

It remains to verify (A.3). We need the following lemma:

**Lemma A.1.** For \( 0 < \delta < c_d \) and \( l \geq 1 \), we have
\[ \||g^\alpha_l||_{C^{2d-1.1-\epsilon}(\mathbb{R}^d)} \leq C 2^{-l(|\alpha|-2)}. \]
**Proof.** When \( \theta \in \text{supp} \phi, |m(2^{-l}\theta)| \) is comparable to \( 2^{-2l} \) as \( |\theta| \sim 1 \). In addition, we have the estimates

\[
\|(2^{-l}\partial)\beta m(2^{-l}\theta)| \leq C_\beta 2^{-2l} \\
\|(2^{-l}\partial)\beta m^\alpha(2^{-l}\theta)| \leq C_{\beta, \alpha} 2^{-l|\alpha|}
\]

for all multi-index \( \beta \) with \( |\beta| \leq 2d - 1 \). From these estimates, if follows that

\[
g^\alpha_l \in C^{2d-1}(\mathbb{R}^d) \leq C 2^{-l|\alpha| - 2}.
\]

For the H"older estimate, we note that when \( |\beta| = 2d - 1 \), we may write

\[
\partial_\beta g^\alpha_l (\theta) = \frac{\chi(\theta)2^{-l(2d-1)}\partial_\beta m(2^{-l}\theta)m^\alpha(2^{-l}\theta)}{m(2^{-l}\theta)^2} + R(\theta),
\]

where \( ||R||_{C^1} \leq C 2^{-l|\alpha| - 2} \). This implies that the \( C^{0,1-\epsilon} \) norm of the first term is \( O(2^{-l|\alpha| - 2}) \). This again reduces to quantify the \( C^{0,1-\epsilon} \) norm of the functions resulting from replacing \( \partial_\beta m(2^{-l}\theta) \) in the first term by

\[
(e^{-i2^{-l}\theta_j} - 1)(e^{i2^{-l}\theta_k} - 1)\partial_\beta K^\beta_{j,k}(2^{-l}\theta)
\]

for each \( 1 \leq j, k \leq d \). One can verify that \( C^{0,1-\epsilon} \) norm of the resulting functions are \( O(2^{-l|\alpha| - 2}) \). \( \square \)

Finally, we may deduce (A.3) from Lemma A.1 by a standard argument. Let \( |x| \geq 2^l \). Without loss of generality, we may assume that \( |x_1| = \max_j |x_j| \), hence \( |x_1| \sim |x| \). Using integration by parts, we see that

\[
\nabla^\alpha G_l(x) = \frac{C2^{-ld}}{(2^{-l}x_1)^{2d-1}} \int (\partial_1)^{2d-1} g^\alpha_l (\theta) e^{i2^{-l}x \cdot \theta} d\theta.
\]

After the change of variable \( \theta_1 \to \theta_1 + \frac{\pi}{2^{-l}x_1} \) in the integral, we also see that

\[
\nabla^\alpha G_l(x) = -\frac{C2^{-ld}}{(2^{-l}x_1)^{2d-1}} \int (\partial_1)^{2d-1} g^\alpha_l \left( \theta_1 + \frac{\pi}{2^{-l}x_1}, \theta' \right) e^{i2^{-l}x \cdot \theta} d\theta,
\]

where we write \( \theta' = (\theta_2, \ldots, \theta_d) \). Estimating the average of these expressions for \( \nabla^\alpha G_l(x) \) using Lemma A.1, we obtain (A.3) which finishes the proof.

### B Proof of Corollary 1.6 on Averaged Solutions

We have \( f \in H^{-1}(\mathbb{Z}^d) \) by (2.1) and \( u_\omega = L^{-1}_\omega f \) is the unique solution in \( H^1(\mathbb{Z}^d) \) to the equation \( L_\omega u_\omega = f \).

By the definition of \( L \), we have \( \mathbb{E}[u_\omega] = L^{-1} f \). In addition, we have

\[
L(G \ast f) = (LG) \ast f = \delta_0 \ast f = f \quad (B.1)
\]
which yields \( L^{-1} f = G \ast f \). We need to verify the first equality of (B.1), which is trivial when \( f \) is compactly supported. For general \( f \in \ell^p d (\mathbb{Z}^d) \), it suffices to show that
\[
\nabla_i^* K_{i,j}^\delta \nabla_j (G \ast f) = (\nabla_i^* K_{i,j}^\delta \nabla_j G) \ast f.
\]

(B.2)

To see this, first note that the sum defining the convolution \( G \ast f \) converges absolutely since \( G \in \ell^q d (\mathbb{Z}^d) \) and \( f \in \ell^p d (\mathbb{Z}^d) \) and \( 1/p + 1/q = 1 \). This shows that \( \nabla_j (G \ast f) = (\nabla_j G) \ast f \) with \( \nabla_j G \in \ell^q d (\mathbb{Z}^d) \). In fact, \( \nabla_j G \in L^2 (\mathbb{Z}^d) \) since \( G \in H^1 (\mathbb{Z}^d) \), but we do not use this fact here. Moreover, the kernel of \( K_{i,j} \) belongs to \( \ell^1 (\mathbb{Z}^d) \) and we have \( K_{i,j} (\nabla_j G) \ast f \) by Fubini’s theorem and \( K_{i,j} \nabla_j G \in \ell^q d (\mathbb{Z}^d) \). The argument for \( \nabla_i^* \) is the same and this establishes (B.2).

The pointwise estimate is a direct consequence of Corollary 1.5. □

C Proof of the Deterministic Bound in Lemma 2.2

We closely follow [8] and provide some details. Recall that
\[
T^n (x_0, x_n) = \sum_{\underline{x}} K^1 (x_0, x_1) K^2 (x_1, x_2) \ldots K^n (x_{n-1}, x_n),
\]
where \( \underline{x} = (x_1, x_2, \ldots, x_{n-1}) \in (\mathbb{Z}^d)^{n-1} \). When \( x_0 \neq x_n \), we may write
\[
\sum_{\underline{x}} = \sum_{m \geq 0} \sum_{\underline{x} \in S_m} \sum_{j_0 \in (0, 2^m)} = \sum_{j_0=0}^{n-1} \sum_{m \geq 0} \sum_{\underline{x} \in S_m}^{j_0},
\]
where \( S_m \) is defined in (3.6). When \( x_0 = x_n \), this yields a decomposition for the sum \( \sum_{\underline{x}} \), except for \( \underline{x} = (x_0, \ldots, x_0) \) for which we may invoke the bound
\[
|K^1 (x_0, x_0) \cdots K^n (x_0, x_0)| \leq A^n.
\]
Let \( I_m = [0, 2^m) \). Observe that
\[
\sum_{\underline{x} \in S_m}^{j_0} K^1 (x_0, x_1) \ldots K^n (x_{n-1}, x_n)
\]
\[
\begin{align*}
&= \sum_{\underline{x} \in (\mathbb{Z}^d)^{n-1}} \prod_{j=1}^{j_0} K^j_m (x_{j-1}, x_j) K_{m+1}^{j_0+1} (x_{j_0}, x_{j_0+1}) \prod_{j=j_0+2}^{n} K^j_{m+1} (x_{j-1}, x_j) \\
&= \sum_{x_{j_0+1}} T_m^{j_0} (x_0, x_{j_0}) K_{m+1}^{j_0+1} (x_{j_0}, x_{j_0+1}) \tilde{T}_{m+1}^{j_0} (x_{j_0+1}, x_n),
\end{align*}
\]
(C.1)

where \( T_m^{j_0} := \prod_{j=1}^{j_0} K^j_m \) and \( \tilde{T}_{m+1}^{j_0} := \prod_{j=j_0+2}^{n} K^j_{m+1} \). Here, the sum over \( x_{j_0} \) is in fact a finite sum; \(|x_0 - x_{j_0}| \leq |x_0 - x_1| + \ldots + |x_{j_0-1} - x_{j_0}| \leq n 2^{m+1} \). Similarly,
the sum over \( x_{j_0+1} \) is a finite sum over \( |x_{j_0+1} - x_n| \leq n2^m+1 \). From this, Hölder’s inequality, and Assumption (i),

\[
\begin{align*}
|\text{C.1}| & \leq A2^{-md} \sum_{x_{j_0}} |T_{j_0}^m(x_0, x_{j_0})| \sum_{x_{j_0+1}} |\tilde{T}_{j_0}^m(x_{j_0+1}, x_n)| \\
& \leq C A2^{-md} (n2^m)^{2d(p-1)/p} \left( \sum_{x_{j_0}} |T_{j_0}^m(x_0, x_{j_0})|^p \right)^{1/p} \left( \sum_{x_{j_0+1}} |\tilde{T}_{j_0}^m(x_{j_0+1}, x_n)|^p \right)^{1/p}
\end{align*}
\]

for \( p > 1 \) selected by \( 2d(p-1) = \epsilon \).

Let \( \delta_y \) be the delta function on \( \mathbb{Z}^d \); \( \delta_y(x) \) is equal to 1 if \( x = y \) and 0 otherwise. Note that the product of two \( \ell^p \) sums in the last inequality is bounded by

\[
||| (T_{j_0}^m)^* \delta_{x_0} |||_{\ell^p(\mathbb{Z}^d)} ||| \tilde{T}_{j_0}^m \delta_{x_n} |||_{\ell^p(\mathbb{Z}^d)},
\]

which is bounded by \( [A/(p-1)]^j_0 [A/(p-1)]^{n-j_0-1} = \left( \frac{2dA}{\epsilon} \right)^{n-1} \) by Assumption (2). Therefore, we get

\[
|\text{C.1}| \leq C n^\epsilon \left( \frac{2dA}{\epsilon} \right)^n \epsilon 2^{-m(d-\epsilon)}.
\] (C.2)

It only remains to sum (C.2) over \( m \geq 0 \) and \( j_0 \). For this, we distinguish the cases \( |x_0 - x_n| \geq 2n \) and \( |x_0 - x_n| < 2n \). For the first case, the sum over \( m \) is restricted to

\[
2^m \geq \frac{|x_0 - x_n|}{2n},
\]

which follows from (given that \( \max_j |x_j - x_{j+1}| < 2^{m+1} \))

\[
|x_0 - x_n| \leq \sum_{j=0}^{n-1} |x_j - x_{j+1}| \leq n2^{m+1}.
\]

In the first case, therefore, summing (C.2) over \( m \) and \( j_0 \) yields

\[
|T^n(x_0, x_n)| \leq C n^{1+\epsilon} \left( \frac{2dA}{\epsilon} \right)^n \epsilon \sum_{m: 2^m \geq |x_0 - x_n|/(2n)} 2^{-m(d-\epsilon)}
\]

\[
\leq (C_d A/\epsilon)^n \epsilon (x_0 - x_n)^-(d-\epsilon).
\]

When \( |x_0 - x_n| < 2n \), we sum (C.2) over \( m \geq 0 \) and \( j_0 \). Then we get

\[
|T^n(x_0, x_n)| \leq C n^{1+\epsilon} \left( \frac{2dA}{\epsilon} \right)^n \epsilon,
\]

which completes the proof since \( 1 \leq C n (x_0 - x_n)^{-1} \).
D Proof of Lemma 2.7 on Disjointification

Since $S' = \emptyset$ when $E_l \cap F_l \neq \emptyset$ for some $l$, we may assume that $E_l \cap F_l = \emptyset$ for all $1 \leq l \leq m$.

We closely follow the argument given in Section 4 of [8]. We introduce an additional set of variables (“Steinhaus system”) on the torus $T = \mathbb{R}/2\pi\mathbb{Z}$

$$\overline{\theta} := (\overline{\theta}^1, \overline{\theta}^2, \ldots, \overline{\theta}^m), \text{ where } \overline{\theta}^l := \{\theta^l_x \in T : x \in \mathbb{Z}^d\}.$$

We use these variables to define the complex-valued functions $e_j : \mathbb{Z}^d \to \mathbb{C}$,

$$e_j(x, \overline{\theta}) := \prod_{l=1}^m \exp\left(i v_j^l \theta^l_x\right),$$

where $v_j^l := \begin{cases} 1, & \text{if } j \in E_l, \\ -1, & \text{if } j \in F_l, \\ 0, & \text{otherwise.} \end{cases}$

Note that $\|e_j(\cdot, \overline{\theta})\|_{\infty} \leq 1$ for all $\overline{\theta}$.

Assume first that the set $S$ is finite. Define

$$\tilde{T}^n_S(x_0, x_n, \overline{\theta}) := \sum_{x \in S} \tilde{K}^1(x_0, x_1)\tilde{K}^2(x_1, x_2) \cdots \tilde{K}^n(x_{n-1}, x_n)$$

$$= \sum_{x \in S} K^1(x_0, x_1) \cdots K^n(x_{n-1}, x_n) \prod_{i=1}^m \exp\left(i \left(\sum_{j \in E_l} \theta^l_{x_j} - \sum_{k \in F_l} \theta^l_{x_k}\right)\right),$$

where we introduced the operators $\tilde{K}^j(x, y) := K^j(x, y)e_j(y, \overline{\theta})$ for $2 \leq j \leq n$ and $\tilde{K}^1(x, y) := e_0(x, \overline{\theta})K^1(x, y)e_1(y, \overline{\theta})$. It is important to observe that, by the assumption, we have

$$|\tilde{T}^n_S(x_0, x_n, \overline{\theta})| \leq M(x_0, x_n) \quad (D.1)$$

for all $\overline{\theta}$.

The next step is to average the bound (D.1) over the variables $\overline{\theta}$ with respect to specific probability measures to be chosen. Define the set

$$\mathbb{Z}^d_S := \{x_0, x_n\} \cup \{x \in \mathbb{Z}^d : x = x_j \text{ for some } (x_1, \ldots, x_{n-1}) \in S \text{ and } 1 \leq j \leq n - 1\},$$

which is finite since $S$ is finite by assumption. For each $-1 < t < 1$, let $P_t(\overline{\theta})$ be the Poisson kernel of the unit disk

$$P_t(\overline{\theta}) = \sum_{n=-\infty}^{\infty} t^n e^{in\theta}. $$
Note that $P_t(\theta) \frac{d\theta}{2\pi}$ is a probability measure on $\mathbb{T}$. For each $1 \leq l \leq m$ and $|t| < 1$, consider the product measure $d\mu^l_t$ on $\mathbb{T}^d \times \mathbb{T}$ given by

$$d\mu^l_t(\theta) := \prod_{x \in \mathbb{Z}^d} P_t(\theta^l_x) \frac{d\theta^l_x}{2\pi}.$$ 

We first average (D.2) over the probability space $\mathbb{T}^d \times \mathbb{T}$ equipped with the measure $d\mu^l_t$. On the one hand, since (D.1) holds pointwise in $\theta$, we have

$$\left| \int_{\mathbb{T}^d} \tilde{T}_S^n(x_0, x_n, \tilde{\theta}) d\mu^l_t(\tilde{\theta}^1) \right| \leq M(x_0, x_n) \quad (D.2)$$

for any $(\tilde{\theta}^2, \ldots, \tilde{\theta}^m)$ and $|t| < 1$. On the other hand, we may write the integral above as

$$\sum_{\tilde{x} \in S} K^1(x_0, x_1) K^2(x_1, x_2) \ldots K^n(x_{n-1}, x_n) \prod_{l=2}^m \exp \left( i \left( \sum_{j \in E_l} \theta^l_j - \sum_{k \in F_l} \theta^l_k \right) \right) \times \int_{\mathbb{T}^d} \exp \left( i \left( \sum_{j \in E_1} \theta^1_j - \sum_{k \in F_1} \theta^1_k \right) \right) \prod_{x \in \mathbb{Z}^d} P_t(\theta^1_x) \frac{d\theta^1_x}{2\pi}.$$ 

As was observed in [8], the integral in the above line is equal to $i^{w(x_0, x_1, \ldots, x_n)}$, where

$$w(x_0, \ldots, x_n) = \sum_{\tilde{x} \in \mathbb{Z}^d} \left| \left\{ j \in E_1 : x_j = x \right\} \right| - \left| \left\{ k \in F_1 : x_k = x \right\} \right| \leq |E_1| + |F_1|.$$ 

Moreover, $w(x_0, \ldots, x_n) = |E_1| + |F_1|$ if and only if $\tilde{x} \in S_1$, where

$$S_1 := S \cap \left\{ x : \{ j \in E_1 : x_j = x \} \cap \{ k \in F_1 : x_k = x \} = \emptyset \right\}.$$ 

Therefore, we may write $\int_{\mathbb{T}^d} \tilde{T}_S^n(x_0, x_n, \tilde{\theta}) d\mu^l_t(\tilde{\theta}^1)$ as a polynomial

$$f(t) = a_D t^D + a_{D-1} t^{D-1} + \ldots + a_0, \quad (D.3)$$

where $D = |E_1| + |F_1|$ and

$$a_D = \sum_{\tilde{x} \in S_1} K^1(x_0, x_1) K^2(x_1, x_2) \ldots K^n(x_{n-1}, x_n) \prod_{l=2}^m \exp \left( i \left( \sum_{j \in E_l} \theta^l_j - \sum_{k \in F_l} \theta^l_k \right) \right).$$

At this point, we recall a special case of the Markov brothers’ inequality.

**Lemma D.1.** Let $f(t)$ be a polynomial as in (D.3). Then we have

$$|a_D| \leq 2^{D-1} \max_{-1 \leq t \leq 1} |f(t)|.$$
Combined with (D.2), we get, with \( D = |E_1| + |F_1| \),
\[
|a_D| \leq 2^{D-1} M(x_0, x_n)
\]
for any \((\bar{\theta}^2, \ldots, \bar{\theta}^m)\). What comes next is a similar averaging argument for the top coefficient \(a_D\) over the measure \(d\mu^2\), which yields
\[
\left| \sum_{x \in S_2} K^1(x_0, x_1) K^2(x_1, x_2) \ldots K^n(x_{n-1}, x_n) \prod_{l=3}^m \exp \left( i \left( \sum_{j \in E_l} \theta_{x,j}^l - \sum_{k \in F_l} \theta_{x,k}^l \right) \right) \right| \leq \left( \prod_{1 \leq l \leq 2} 2^{|E_l|+|F_l|-1} \right) M(x_0, x_n)
\]
for any \((\bar{\theta}^3, \ldots, \bar{\theta}^m)\), where
\[
S_2 := S_1 \cap \{ x : \{ x_j : j \in E_2 \} \cap \{ x_k : k \in F_2 \} = \emptyset \}.
\]
A successive averaging over \(d\mu^3, \ldots, d\mu^m\) finishes the proof.

Next, assume that \(S\) is not a finite set. By dominated convergence and the a priori bound (2.6), we have \(T^n_S(x_0, x_n) = \lim_{k \to \infty} T^n_{S \cap X_k}(x_0, x_n)\). Therefore, it is sufficient to show that
\[
|T^n_{S \cap X_k}(x_0, x_n)| \leq 2^{\sum_{1 \leq l \leq m} |E_l|+|F_l|} M(x_0, x_n)
\]
for all large \(k \geq 1\), which follows from applying the result for the finite set to \(S \cap X_k\). 

\[\square\]

References

1. Armstrong, S., Kuusi, T., Mourrat, J.-C.: The additive structure of elliptic homogenization. Invent. Math. 208, 999–1154, 2017
2. Armstrong, S.N., Mourrat, J.-C.: Lipschitz regularity for elliptic equations with random coefficients. Arch. Ration. Mech. Anal. 219, 255–348, 2016
3. Aronson, D.G.: Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73, 890–896, 1967
4. Aronson, D.G.: Non-negative solutions of linear parabolic equations. Ann. Scuola Norm. Sup. Pisa 22(3), 607–694, 1968
5. Bach, V., Fröhlich, J., Sigal, I.M.: Renormalization group analysis of spectral problems in quantum field theory. Adv. Math. 137(2), 205–298, 1998
6. Bella, P., Giunti, A., Otto, F.: Effective Multipoles in Random media, arXiv:1708.07672
7. Bella, P., Giunti, A., Otto, F.: Quantitative stochastic homogenization: local control of homogenization error through corrector, Mathematics and materials, 301–327, IAS/Park City Math. Ser., 23, Amer. Math. Soc., Providence, RI, 2017
8. Bourgain, J.: On a homogenization problem. J. Stat. Phys. 172, 314–320, 2018
9. Carlen, E.A., Kusuoka, S., Stroock, D.W.: Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré Probab. Statist. 23(2), 245–287, 1987
10. Conlon, J.G.: Green’s functions for elliptic and parabolic equations with random coefficients. II. Trans. Amer. Math. Soc. 356(10), 4085–4142, 2004
11. Conlon, J.G., Giunti, A., Otto, F.: Green’s function for elliptic systems: existence and Delmotte–Deuschel bounds. *Calc. Var. Partial Differ. Equ.* 56(6), Art. 163, 51, 2017
12. Conlon, J.G., Naddaf, A.: Greens functions for elliptic and parabolic equations with random coefficients. *New York J. Math.* 6, 153–225, 2000
13. Conlon, J.G., Naddaf, A.: On homogenization of elliptic equations with random coefficients. *Electron. J. Probab.* 5(9), 58, 2000
14. Conlon, J.G., Naddaf, A.: Greens functions for elliptic and parabolic equations with random coefficients. *New York J. Math.* 6, 153–225, 2000
15. Conlon, J.G., Spencer, T.: Strong convergence to the homogenized limit of elliptic equations with random coefficients. *Probab. Theory Relat. Fields* 193(3), 358–390, 2005
16. De Giorgi, E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* 3, 25–43, 1957
17. Duerinckx, M., Gloria, A., Otto, F.: The structure of fluctuations in stochastic homogenization, arXiv:1602.01717
18. Gloria, A., Marahrens, D.: Annealed estimates on the Green functions and uncertainty quantification. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33(5), 1153–1197, 2016
19. Gloria, A., Neukamm, S., Otto, F.: A regularity theory for random elliptic operators, arXiv:1409.2678
20. Gloria, A., Neukamm, S., Otto, F.: Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.* 199(2), 455–515, 2015
21. Gloria, A., Otto, F.: An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* 39(3), 779–856, 2011
22. Gloria, A., Otto, F.: An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.* 22(1), 1–28, 2012
23. Gloria, A., Otto, F.: Quantitative results on the corrector equation in stochastic homogenization. *J. Eur. Math. Soc.* 19(11), 3489–3548, 2017
24. Jikov, V.V., Kozlov, S.M., Oleinik, O.A.: *Homogenization of Differential Operators and Integral Functionals*. Springer, New York 1994
25. Kozlov, S.M.: The averaging of random operators. *Mat. Sb.* 109(151)(2), 188–202, 1979
26. Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa* 17, 43–77, 1963
27. Marahrens, D., Otto, F.: Annealed estimates on the Green function. *Probab. Theory Related Fields* 163(3–4), 527–573, 2015
28. Marahrens, D., Otto, F.: On annealed elliptic Green’s function estimates. *Math. Bohemica* 140(4), 489–506, 2015
29. Moser, J.: A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations. *Commun. Pure Appl. Math.* 13, 457–468, 1960
30. Naddaf, A., Spencer, T.: On homogenization and scaling limit of some gradient perturbations of a massless free field. *Commun. Math. Phys.* 183(1), 55–84, 1997
31. Nash, J.: Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* 80, 931–954, 1958
32. Papanicolaou, G., Varadhan, S.: Boundary value problems with rapidly oscillating random coefficients. *Colloq. Math. Soc. János Bolyai* 27, 835–873, 1982
33. Sigal, I.M.: Homogenization problem, unpublished preprint
34. Stein, E.M.: *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press 1970
35. Stein, E.M., Wainger, S.: Discrete analogues in harmonic analysis. II. Fractional integration. *J. Anal. Math.* 80, 335–355, 2000
36. Yurinski, V.V.: Averaging of symmetric diffusion in a random medium. (Russian) *Sibirsk. Mat. Zh.* 27(4), 167–180, 1986
