SOME UNIVERSAL NONLINEAR INEQUALITIES

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Abstract

In this paper, new versions of Chebyshev’s, Minkowski’s and Hölder’s type inequalities are studied by using a monotone measure-base universal integral on an arbitrary measurable space. This paper generalizes some previous results obtained by many researchers.

Keywords: Monotone measure; Universal integral; Chebyshev’s inequality; Minkowski’s inequality; Hölder’s inequality.

1 Introduction

Observe that in the last few years, there were introduced and discussed several inequalities for non-classical integrals, thus developing a theoretical background for further applications. Inequalities are at the heart of the mathematical analysis of various problems in machine learning and made it possible to derive new efficient algorithms.

In this paper, new versions of Chebyshev’s, Minkowski’s and Hölder’s type inequalities for universal integral on abstract spaces are studied in rather general form, thus generalizing the results of [1, 2, 8, 14, 16, 17]. Many nonlinear systems are built by non-classical techniques, and thus we believe that our results will prove their usefulness in flourishing areas, such as the economy and decision making, among others.

The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarization of some previous known results. In Section 3, we will focus on some interesting integral inequalities, including Chebyshev’s inequality, Hölder’s inequality and Minkowski’s inequality for universal integral. Section 4 includes reverse previous inequalities for semiconormed fuzzy integrals. Finally, a conclusion is given.

2 Universal integral

In this section, we are going to review some well-known known results from universal integral. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [11]).

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Definition 2.1 \[11\] A monotone measure \( m \) on a measurable space \((X, A)\) is a function \( m : A \rightarrow [0, \infty] \) satisfying

(i) \( m(\emptyset) = 0 \),
(ii) \( m(X) > 0 \),
(iii) \( m(A) \leq m(B) \) whenever \( A \subseteq B \).

Note that a monotone measure is not necessarily \(\sigma\)-additive. This concept goes back to M. Sugeno \[22\] (where also the continuity of the measures was required). To be precise, normed monotone measures on \((X, A)\), i.e., monotone measures satisfying \( m(X) = 1 \), are also called fuzzy measures \[9, 22, 24\], depending on the context.

For a fixed measurable space \((X, A)\), i.e., a non-empty set \(X\) equipped with a \(\sigma\)-algebra \(A\), recall that a function \( f : X \rightarrow [0, \infty] \) is called \(A\)-measurable if, for each \( B \in B([0, \infty]) \), the \(\sigma\)-algebra of Borel subsets of \([0, \infty]\), the preimage \( f^{-1}(B) \) is an element of \(A\). We shall use the following notions:

Definition 2.2 \[11\] Let \((X, A)\) be a measurable space.

(i) \( F(X, A) \) denotes the set of all \(A\)-measurable functions \( f : X \rightarrow [0, \infty] \);
(ii) For each number \( a \in (0, \infty] \), \( M_a(X, A) \) denotes the set of all monotone measures (in the sense of Definition 2.1) satisfying \( m(X) = a \); and we take

\[
M^{(X, A)} = \bigcup_{a \in (0, \infty]} M_a^{(X, A)}.
\]

Let \(S\) be the class of all measurable spaces, and take

\[
D_{[0, \infty]} = \bigcup_{(X, A) \in S} M^{(X, A)} \times F^{(X, A)}.
\]

The Choquet \[3\], Sugeno \[22\] and Shilkret \[20\] integrals (see also \[4, 18\]), respectively, are given, for any measurable space \((X, A)\), for any measurable function \( f \in F^{(X, A)} \) and for any monotone measure \( m \in M^{(X, A)} \), i.e., for any \((m, f) \in D_{[0, \infty]} \), by

\[
\text{Su}(m, f) = \sup \{ \min (t, m(\{ f \geq t \})) \mid t \in (0, \infty) \}, \quad (2.1)
\]
\[
\text{Sh}(m, f) = \sup \{ t \cdot m(\{ f \geq t \}) \mid t \in (0, \infty) \}, \quad (2.2)
\]

where the convention \(0 \cdot \infty = 0\) is used. All these integrals map \( M^{(X, A)} \times F^{(X, A)} \) into \([0, \infty]\) independently of \((X, A)\). We remark that fixing an arbitrary \( m \in M^{(X, A)} \), they are non-decreasing functions from \( F^{(X, A)} \) into \([0, \infty]\), and fixing an arbitrary \( f \in F^{(X, A)} \), they are non-decreasing functions from \( M^{(X, A)} \) into \([0, \infty]\).

We stress the following important common property for all three integrals from (2.1) and (2.2). Namely, these integrals does not make difference between the pairs \((m_1, f_1), (m_2, f_2) \in D_{[0, \infty]} \) which satisfy, for all for all \( t \in (0, \infty] \),

\[
m_1(\{ f_1 \geq t \}) = m_2(\{ f_2 \geq t \}).
\]

Therefore, such equivalence relation between pairs of measures and functions was introduced in \[11\].
Definition 2.3  Two pairs \((m_1, f_1) \in \mathcal{M}(X_1, A_1) \times \mathcal{F}(X_1, A_1)\) and \((m_2, f_2) \in \mathcal{M}(X_2, A_2) \times \mathcal{F}(X_2, A_2)\) satisfying
\[
m_1(\{f_1 \geq t\}) = m_2(\{f_2 \geq t\}) \text{ for all } t \in (0, \infty],
\]
will be called integral equivalent, in symbols
\[
(m_1, f_1) \sim (m_2, f_2).
\]

To introduce the notion of the universal integral we shall need instead of the usual plus and product more general real operations.

Definition 2.4 \([23]\) A function \(\otimes: [0, \infty]^2 \to [0, \infty]\) is called a pseudo-multiplication if it satisfies the following properties:
\begin{enumerate}[(i)]
  \item it is non-decreasing in each component, i.e., for all \(a_1, a_2, b_1, b_2 \in [0, \infty]\) with \(a_1 \leq a_2\) and \(b_1 \leq b_2\) we have \(a_1 \otimes b_1 \leq a_2 \otimes b_2\);
  \item 0 is an annihilator of \(e\), i.e., for all \(a \in [0, \infty]\) we have \(a \otimes 0 = 0 \otimes a = 0\);
  \item has a neutral element different from 0, i.e., there exists an \(e \in (0, \infty]\) such that, for all \(a \in [0, \infty]\), we have \(a \otimes e = e \otimes a = a\).
\end{enumerate}

There is neither a smallest nor a greatest pseudo-multiplication on \([0, \infty]\). But, if we fix the neutral element \(e \in (0, \infty]\), then the smallest pseudo-multiplication \(\otimes_e\) and the greatest pseudo-multiplication \(\otimes^e\) with neutral element \(e\) are given by
\[
a \otimes_e b = \begin{cases} 
0 & \text{if } (a, b) \in [0, e)^2, \\
\max(a, b) & \text{if } (a, b) \in [e, \infty)^2, \\
\min(a, b) & \text{otherwise},
\end{cases}
\]
and
\[
a \otimes^e b = \begin{cases} 
\min(a, b) & \text{if } \min(a, b) = 0 \text{ or } (a, b) \in (0, e]^2, \\
\infty & \text{if } (a, b) \in (e, \infty)^2, \\
\max(a, b) & \text{otherwise}.
\end{cases}
\]

Restricting to the interval \([0, 1]\) a pseudo-multiplication and a pseudo-addition with additional properties of associativity and commutativity can be considered as the \(t\)-norm \(T\) and the \(t\)-conorms \(S\) (see [10]), respectively.

For a given pseudo-multiplication on \([0, \infty]\), we suppose the existence of a pseudo-addition \(\oplus: [0, \infty]^2 \to [0, \infty]\) which is continuous, associative, non-decreasing and has 0 as neutral element (then the commutativity of follows, see [10]), and which is left-distributive with respect to \(\otimes\) i.e., for all \(a, b, c \in [0, \infty]\) we have \((a \oplus b) \otimes c = (a \otimes c) \otimes (b \oplus c)\). The pair \((\oplus, \otimes)\) is then called an integral operation pair, see [4, 11].

Each of the integrals mentioned in (2.1) and (2.2) maps \(\mathcal{D}_{[0, \infty]}\) into \([0, \infty]\) and their main properties can be covered by the following common integral given in [11].

Definition 2.5 A function \(I: \mathcal{D}_{[0, \infty]} \to [0, \infty]\) is called a universal integral if the following axioms hold:
\begin{enumerate}[(I1)]
  \item For any measurable space \((X, \mathcal{A})\), the restriction of the function \(I\) to \(\mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})\) is non-decreasing in each coordinate;
\end{enumerate}
(I2) there exists a pseudo-multiplication $\otimes: [0, \infty]^2 \to [0, \infty]$ such that for all pairs $(m, c.1_A) \in \mathcal{D}_{[0,\infty]}$,

$$I(m, c.1_A) = c \otimes m(A);$$

(I3) for all integral equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$ we have $I(m_1, f_1) = I(m_2, f_2)$.

By Proposition 3.1 from [1] we have the following important characterization.

**Theorem 2.6** Let $\otimes: [0, \infty]^2 \to [0, \infty]$ be a pseudo-multiplication on $[0, \infty]$. Then the smallest universal integral $I$ based on $\otimes$ is given by

$$I_\otimes(m, f) = \sup \{t \otimes m(\{f \geq t\}) \mid t \in (0, \infty)\}.$$

Specially, we have $Su = I_{Min}$ and $Sh = I_{Prod}$, where the pseudo-multiplications $Min$ and $Prod$ are given (as usual) by $Min(a, b) = \min(a, b)$ and $Prod(a, b) = a.b$. Note that the nonlinearity of the Sugeno integral $Su$ (see, e.g., [12, 13]) implies that universal integrals are also nonlinear, in general.

**Proposition 2.7** There exists the smallest universal integral $I_\otimes$ among all universal integrals satisfying the conditions

(i) for each $m \in \mathcal{M}^{(X,A)}$ and each $c \in [0, \infty]$ we have $I(m, c.1_X) = c$,

(ii) for each $m \in \mathcal{M}^{(X,A)}$ and each $A \in \mathcal{A}$ we have $I(m, e.1_X) = m(A)$, given by

$$I_\otimes(m, f) = \max \{m(\{f \geq e\}), essinf_m f\}$$

where $essinf_m f = \sup \{t \in [0, \infty] \mid m(\{f \geq t\}) = m(X)\}$.

Restricting now to the unit interval $[0, 1]$ we shall consider functions $f \in \mathcal{F}^{(X,A)}$ satisfying $Ran(f) \subseteq [0, 1]$ (in which case we shall write shortly $f \in \mathcal{F}^{(X,A)}_{[0,1]}$). Observe that, in this case, we have the restriction of the pseudo-multiplication $\otimes$ to $[0, 1]^2$ (called a semicopula or a conjunctor, i.e., a binary operation $\otimes: [0, 1]^2 \to [0, 1]$ which is non-decreasing in both components, has $1$ as neutral element and satisfies $a \otimes b \leq \min(a, b)$ for all $(a, b) \in [0, 1]^2$, see [3, 7]), and universal integrals are restricted to the class $\mathcal{D}_{[0,1]} = \bigcup_{(X,A) \in \mathcal{S}} \mathcal{M}^{(X,A)} \times \mathcal{F}^{(X,A)}_{[0,1]}$. In a special case, for a fixed strict $t$-norm $T$, the corresponding universal integral $I_T$ is the so-called Sugeno-Weber integral [25]. The smallest universal integral $I_\otimes$ on the $[0, 1]$ scale related to the semicopula $\otimes$ is given by

$$I_\otimes(m, f) = \sup \{t \otimes m(\{f \geq t\}) \mid t \in [0, 1]\}.$$ 

This type of integral was called seminormed integral in [21].

Before starting our main results we need the following definitions:

**Definition 2.8** Functions $f, g: X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

and $f$ and $g$ are said to be countermonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0.$$
Then we say that $\varphi, \psi$ satisfies

$$A(B(a,b), B(c,d)) \geq B(A(a,c), A(b,d))$$

holds for any $a, b, c, d \in [0, \infty]$.

**Definition 2.10** Let $\ast : [0, \infty]^2 \to [0, \infty]$ be a binary operation and consider $\varphi : [0, \infty) \to [0, \infty]$. Then we say that $\varphi$ is subdistributive over $\ast$ if

$$\varphi(x \ast y) \leq \varphi(x) \ast \varphi(y)$$

for all $x, y \in [0, \infty]$. Analogously, we say that $\varphi$ is superdistributive over $\ast$ if

$$\varphi(x \ast y) \geq \varphi(x) \ast \varphi(y)$$

for all $x, y \in [0, \infty]$.

### 3 On some advanced type inequalities for universal integral

Now, we state the main result of this paper.

**Theorem 3.1** Let a non-decreasing $n$-place function $H : [0, \infty)^n \to [0, \infty)$ such that $H$ be continuous. If $\otimes : [0, \infty]^n \to [0, \infty]$ is the pseudo-multiplication with neutral element $e \in (0, \infty]$, satisfies

$$U_0^{-1}[U_0(H(\psi_1(a_1), \psi_2(a_2), ..., \psi_n(a_n))) \otimes c] \geq \begin{bmatrix}
H(\psi_1(U_1^{-1}[(U_1(a_1)) \otimes c]), \psi_2(a_2), ..., \psi_n(a_n)) \\
\vee H(\psi_1(a_1), \psi_2(U_2^{-1}[(U_2(a_2)) \otimes c]), \psi_3(a_3), ..., \psi_n(a_n)) \\
\vee ... \vee H(\psi_1(a_1), \psi_2(a_2), ..., \psi_{n-1}(a_{n-1}), \psi_n(U_n^{-1}[(U_n(a_n)) \otimes c]))
\end{bmatrix}$$

then for any system $U_0, U_1, ..., U_n : [0, \infty) \to [0, \infty)$ of continuous strictly increasing functions, and any system $\psi_1, \psi_2, ..., \psi_n : [0, \infty) \to [0, \infty)$ of continuous increasing functions and any comonotone system $f_1, f_2, ..., f_n \in \mathcal{F}(X,A)$ and a monotone measure $m \in \mathcal{M}(X,A)$ such that $b \otimes m(X) \leq b$ for all $b \in [0, \infty]$ and $I_\otimes(m, U_1(f_1)) < \infty$ for all $i = 1, 2, ..., n$, it holds

$$U_0^{-1}[I_\otimes(m, U_0[H(\psi_1(f_1), ..., \psi_n(f_n))] \geq H[\psi_1(U_1^{-1}(I_\otimes(m, U_1(f_1)))), ..., \psi_n(U_n^{-1}(I_\otimes(m, U_n(f_n)))).$$
Proof. Let $\varepsilon \in (0, \infty]$ be the neutral element of $\otimes$ and $I_{\otimes} (m, U_i (f_i)) = p_i < \infty$ for all $i = 1, 2, ..., n$. So, for any $\varepsilon > 0$, there exist $p_i(\varepsilon)$ such that

$$m \{ U_i (f_i) \geq p_i(\varepsilon) \} = m \{ f_i \geq U_i^{-1} (p_i(\varepsilon)) \} = M_i,$$

where $p_i(\varepsilon) \otimes M_i \geq p_i - \varepsilon$ for all $i = 1, 2, ..., n$. Then,

$$\psi_i (U_i^{-1} [p_i(\varepsilon) \otimes M_i]) \geq \psi_i (U_i^{-1} [p_i - \varepsilon]), \text{ for all } i = 1, 2, ..., n.$$

Then,

$$\psi_i (U_i^{-1} [p_i(\varepsilon)]) \geq \psi_i (U_i^{-1} [p_i(\varepsilon) \otimes m (X)]) \geq \psi_i (U_i^{-1} [p_i - \varepsilon]), \text{ for all } i = 1, 2, ..., n.$$

The comonotonicity of $f_1, f_2, ..., f_n$ and the monotonicity of $H$ imply that

$$m \{ \{ U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n)) \geq U_0 (H (\psi_1 (U_1^{-1} (p_1(\varepsilon))), ..., \psi_n (U_n^{-1} (p_n(\varepsilon)))))) \} \otimes (M_1 \wedge M_2 \wedge ... \wedge M_n) \}
\geq U_0^{-1} \left[ \left[ m \{ U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) \geq U_0 (H (\psi_1 (U_1^{-1} (p_1(\varepsilon))), ..., \psi_n (U_n^{-1} (p_n(\varepsilon)))))) \} \otimes (M_1 \wedge M_2 \wedge ... \wedge M_n) \right] \right]
= \left( \begin{array} {c}
\left( \begin{array} {c}
U_0^{-1} \left[ \left[ \left[ m \{ U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) \geq U_0 (H (\psi_1 (U_1^{-1} (p_1(\varepsilon))), ..., \psi_n (U_n^{-1} (p_n(\varepsilon)))))) \} \right] \otimes (M_1 \wedge M_2 \wedge ... \wedge M_n) \right] \right]
\end{array} \right)
\end{array} \right)
\geq \left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1(\varepsilon) \otimes M_1]), \psi_2 (U_2^{-1} [p_2(\varepsilon)]), ..., \psi_n (U_n^{-1} [p_n(\varepsilon)]))
\end{array} \right) \wedge 
\left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1(\varepsilon) \otimes M_1]), \psi_2 (U_2^{-1} [p_2(\varepsilon)]), ..., \psi_n (U_n^{-1} [p_n(\varepsilon)]))
\end{array} \right) \wedge 
\left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1(\varepsilon) \otimes M_1]), \psi_2 (U_2^{-1} [p_2(\varepsilon)]), ..., \psi_n (U_n^{-1} [p_n(\varepsilon)]))
\end{array} \right)
\geq \left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1 - \varepsilon]), \psi_2 (U_2^{-1} [p_2 - \varepsilon]), ..., \psi_n (U_n^{-1} [p_n - \varepsilon]))
\end{array} \right) \wedge 
\left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1 - \varepsilon]), \psi_2 (U_2^{-1} [p_2 - \varepsilon]), ..., \psi_n (U_n^{-1} [p_n - \varepsilon]))
\end{array} \right) \wedge 
\left( \begin{array} {c}
H (\psi_1 (U_1^{-1} [p_1 - \varepsilon]), \psi_2 (U_2^{-1} [p_2 - \varepsilon]), ..., \psi_n (U_n^{-1} [p_n - \varepsilon]))
\end{array} \right)
\geq H (\psi_1 (U_1^{-1} [p_1 - \varepsilon]), \psi_2 (U_2^{-1} [p_2 - \varepsilon]), ..., \psi_n (U_n^{-1} [p_n - \varepsilon]));
$$

whence $U_0^{-1} |I_{\otimes} (m, U_0[H (\psi_1 (f_1), ..., \psi_n (f_n))]|) \geq H (\psi_1 (U_1^{-1} [p_1]), \psi_2 (U_2^{-1} [p_2]), ..., \psi_n (U_n^{-1} [p_n]))$ follows from the continuity of $H, \psi_i, U_i$ for all $i$, and the arbitrariness of $\varepsilon$. And the theorem is proved. \(\square\)

Remark 3.2 (i) If $m(X) = \infty$, then the condition $b \otimes m (X) \leq b$ for all $b \in [0, \infty]$ holds readily. 
(ii) We can replace the condition “$b \otimes m (X) \leq b$ for all $b \in [0, \infty]$” with “$b \otimes a \leq b$ for all $a, b \in [0, \infty]$.”

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Corollary 3.3 Let \( f, g \in \mathcal{F}(X, A) \) be two comonotone measurable functions and \( \otimes : [0, \infty]^2 \rightarrow [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty) \) and \( m \in \mathcal{M}(X, A) \) be a monotone measure such that \( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \), \( \mathbf{I}_\otimes (m, U_1(f)) \) and \( \mathbf{I}_\otimes (m, U_2(g)) \) are finite and \( U_i : [0, \infty) \rightarrow [0, \infty) \), \( i = 0, 1, 2 \) be continuous strictly increasing functions. Let \( * : [0, \infty)^2 \rightarrow [0, \infty) \) be continuous and nondecreasing in both arguments and \( \psi : [0, \infty) \rightarrow [0, \infty) \) be continuous and strictly increasing function. If

\[
U_0^{-1} [U_0(\psi (a) \ast \psi (b)) \otimes c] \geq \left[ \psi \left( U_1^{-1} [U_1(a) \otimes c] \right) \ast \psi (b) \right] \vee \left[ \psi (a) \ast \psi \left( U_2^{-1} [U_2(b) \otimes c] \right) \right],
\]

then the inequality

\[
U_0^{-1} [\mathbf{I}_\otimes (m, U_0((\psi (f) \ast \psi (g))))] \geq \psi \left( U_1^{-1} (\mathbf{I}_\otimes (m, U_1(f)))) \right) \ast \psi \left( U_2^{-1} (\mathbf{I}_\otimes (m, U_2(g)))) \right)
\]

holds.

Let \( U_i (x) = \varphi_i (x) \) for all \( i = 0, 1, 2 \) and \( \psi (x) = x \) in Corollary 3.3. Then we have the following result.

Corollary 3.4 Let \( f, g \in \mathcal{F}(X, A) \) be two comonotone measurable functions and \( \otimes : [0, \infty]^2 \rightarrow [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty) \) and \( m \in \mathcal{M}(X, A) \) be a monotone measure such that \( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \), \( \mathbf{I}_\otimes (m, \varphi_1(f)) \) and \( \mathbf{I}_\otimes (m, \varphi_2(g)) \) are finite. Let \( * : [0, \infty)^2 \rightarrow [0, \infty) \) be continuous and nondecreasing in both arguments and \( \varphi_i : [0, \infty) \rightarrow [0, \infty) \) \( i = 0, 1, 2 \) be continuous strictly increasing functions. If

\[
\varphi_0^{-1} [\varphi_0 (p_1 \ast p_2) \otimes c] \geq \left[ \varphi_1^{-1} ((\varphi_1(p_1)) \otimes c) \ast p_2 \right] \vee \left[ p_1 \ast \varphi_2^{-1} (\varphi_2(p_2) \otimes c) \right],
\]

then the inequality

\[
\varphi_0^{-1} [\mathbf{I}_\otimes (m, \varphi_0 (f \ast g))] \geq \varphi_1^{-1} (\mathbf{I}_\otimes (m, \varphi_1 (f))) \ast \varphi_2^{-1} (\mathbf{I}_\otimes (m, \varphi_2(g)))
\]

holds.

In an analogous way as in the proof of Theorem 3.1 we have the following result.

Theorem 3.5 Let \( H : [0, \infty)^n \rightarrow [0, \infty) \) be a continuous and nondecreasing \( n \)-place function. If \( \otimes : [0, \infty]^n \rightarrow [0, \infty] \) is the pseudo-multiplication on \( [0, \infty] \) with neutral element \( e \in (0, \infty) \) such that \( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \), satisfies

\[
\left[ (H(p_1, p_2, \ldots, p_n))^{\xi_0} \otimes c \right]^{\omega_0} \geq H\left( \left( p_1^{\xi_1} \otimes c \right)^{\omega_1}, p_2, \ldots, p_n \right) \vee \ldots \vee H\left( p_1, \left( p_2^{\xi_2} \otimes c \right)^{\omega_2}, p_3, \ldots, p_n \right) \vee \ldots \vee H\left( p_1, p_2, \ldots, p_{n-1}, \left( p_n^{\xi_n} \otimes c \right)^{\omega_n} \right),
\]

then for any comonotone system \( f_1, f_2, \ldots, f_n \in \mathcal{F}(X, A) \) and a monotone measure \( m \in \mathcal{M}(X, A) \) such that \( \mathbf{I}_\otimes \left( m, f_i^{\xi_i} \right) < \infty \) and \( x^{\xi_i \omega_i / \omega_0} \geq x \) for all \( x \in [0, \infty) \) and \( i = 1, 2, \ldots, n \), it holds

\[
\left[ \mathbf{I}_\otimes \left( m, (H(f_1, \ldots, f_n))^{\xi_0} \right) \right]^{\omega_0} \geq H\left[ \left( \mathbf{I}_\otimes \left( m, f_1^{\xi_1} \right) \right)^{\omega_1}, \left( \mathbf{I}_\otimes \left( m, f_2^{\xi_2} \right) \right)^{\omega_2}, \ldots, \left( \mathbf{I}_\otimes \left( m, f_n^{\xi_n} \right) \right)^{\omega_n} \right]
\]

for all \( \omega_j, \xi_j \in (0, \infty), j = 0, 1, 2, \ldots, n \).
Proof. Let $e \in (0, \infty)$ be the neutral element of $\otimes$ and $I_\otimes \left( m, f^i_\xi \right) = p^i_\xi < \infty$ for all $i = 1, 2, \ldots, n$. So, for any $\varepsilon > 0$, there exist $p^i_\varepsilon$ such that $m \left( \left\{ f^i_\xi \geq p^i_\varepsilon \right\} \right) = m \left( \left\{ f^i_\xi \geq p^i_\varepsilon \right\} \right) = M_i$, where $p^i_\varepsilon \otimes M_i \geq (p_i - \varepsilon) M_i$ for all $i = 1, 2, \ldots, n$. The comonotonicity of $f_1, f_2, \ldots, f_n$ and the monotonicity of $H$ imply that

$$m \left( \left\{ H (f_1, f_2, \ldots, f_n) \geq H(p^1_\varepsilon, p^2_\varepsilon, \ldots, p^n_\varepsilon) \right\} \right) \geq m \left( \left\{ f_1 (x) \geq p^1_\varepsilon \right\} \right) \cap m \left( \left\{ f_2 (x) \geq p^2_\varepsilon \right\} \right) \cap \cdots \cap m \left( \left\{ f_n (x) \geq p^n_\varepsilon \right\} \right) = M_1 \cap M_2 \cap \cdots \cap M_n.$$

Since $p^i_\varepsilon \geq p_i$, then we have

$$\left[ \sup \left( t \otimes m \left( \left\{ H (f_1, f_2, \ldots, f_n) \right\} \right) \geq t \right) \mid t \in (0, \infty) \right] \right)^\omega \geq \left[ \left( H(p^1_\varepsilon, p^2_\varepsilon, \ldots, p^n_\varepsilon) \right) \otimes (M_1 \cap M_2 \cap \cdots \cap M_n) \right] \right)^\omega$$

$$= \left( \left[ \left( H(p^1_\varepsilon, p^2_\varepsilon, \ldots, p^n_\varepsilon) \right) \otimes M_1 \right] \right)^\omega \cap \left( \left[ \left( H(p^1_\varepsilon, p^2_\varepsilon, \ldots, p^n_\varepsilon) \right) \otimes M_2 \right] \right)^\omega \cap \cdots \cap \left( \left[ \left( H(p^1_\varepsilon, p^2_\varepsilon, \ldots, p^n_\varepsilon) \right) \otimes M_n \right] \right)^\omega$$

$$\geq \left( H \left( \left( p^1_\varepsilon \otimes M_1 \right), \ldots, p^n_\varepsilon \right) \right)^\omega$$

$$\geq \left( H \left( (p_1 - \varepsilon), \ldots, p^n_\varepsilon \right) \right)^\omega.$$
Remark 3.6 (i) If \( m \in \mathcal{M}^{(X,A)}_e \), then the condition \( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \) holds readily.
(ii) We can replace the condition “\( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \)” with “\( a \otimes b \leq a \) for all \( a, b \in [0, \infty] \)”.

Corollary 3.7 Let \( f, g \in \mathcal{F}^{(X,A)} \) be two comonotone measurable functions and \( \otimes: [0, \infty]^2 \to [0, \infty] \) be a smallest pseudo-multiplication on \([0, \infty] \) with neutral element \( e \in (0, \infty] \) and \( m \in \mathcal{M}^{(X,A)} \) be a monotone measure such that \( a \otimes m(X) \leq a \) for all \( a \in [0, \infty) \), \( I_\otimes (m, g^{\xi_1}) < \infty \) and \( I_\otimes (m, f^{\xi_1}) < \infty \). Let \( \star : [0, \infty)^2 \to [0, \infty) \) be continuous and nondecreasing in both arguments. If

\[
[I_\otimes (m, (f \star g)^{\xi_0})]^{\omega_0} \geq [I_\otimes (m, f^{\xi_1})]^{\omega_1} \star [I_\otimes (m, g^{\xi_2})]^{\omega_2}
\]

then the inequality

\[
[I_\otimes (m, (f \star g)^{\xi_0})]^{\omega_0} \geq [I_\otimes (m, f^{\xi_1})]^{\omega_1} \star [I_\otimes (m, g^{\xi_2})]^{\omega_2}
\]

holds, where \( x^\frac{1}{\xi_0} \geq x \) for all \( x \in [0, \infty), i = 1, 2 \) and \( \omega_j, \xi_j \in (0, \infty), j = 0, 1, 2 \).

The following example shows that the condition of \( x^\frac{1}{\xi_0} \geq x \) for all \( x \in [0, \infty) \) and \( i = 1, 2 \) in Corollary 3.7 (and thus the condition \( x^\frac{1}{\xi_0} \geq x \) for all \( x \in [0, \infty) \) and \( i = 1, 2, \ldots n \) in Theorem 3.5) is inevitable.

Example 3.8 Let \( X = [0, 1], \star = \wedge, \xi_0 = \omega_0 = 1, \xi_i = \frac{1}{2}, \omega_i = 1 \) for \( i = 1, 2 \). Let \( f(x) = x, g(x) = 1 \) for all \( x \in [0, 1] \) and the monotone measure \( m \) be the Lebesgue measure. If \( \otimes: [0, 1]^2 \to [0, 1] \) is minimum (i.e., for Sugeno integral), then (3.4) holds readily for all \( t_1, t_2, c \in [0, 1] \) and a straightforward calculus shows that

\( i \) \( I_{\text{Min}} (m, f^\frac{1}{2}) = \text{Su} (m, f^\frac{1}{2}) = \bigvee_{\alpha \in [0, 1]} [\alpha \wedge m (\{ \sqrt{x} \geq \alpha \})] = \frac{1}{2} (\sqrt{5} - 1) \),

\( ii \) \( I_{\text{Min}} (m, g^\frac{1}{2}) = \text{Su} (m, g^\frac{1}{2}) = 1 \),

\( iii \) \( I_{\text{Min}} (m, (f \wedge g)) = \text{Su} (m, f) = \bigvee_{\alpha \in [0, 1]} [\alpha \wedge m (\{ x \geq \alpha \})] = \frac{1}{2} \).

Therefore:

\[
[I_\otimes (m, (f \star g)^{\xi_0})]^{\omega_0} = I_{\text{Min}} (m, (f \wedge g)) = \frac{1}{2} < [I_\otimes (m, f^{\xi_1})]^{\omega_1} \star [I_\otimes (m, g^{\xi_2})]^{\omega_2}
\]

\[
= I_{\text{Min}} (m, f^\frac{1}{2}) \wedge I_{\text{Min}} (m, g^\frac{1}{2}) = \frac{1}{2} (\sqrt{5} - 1),
\]

which violates Corollary 3.7.
Remark 3.9 If \((x \ast e) \lor (e \ast x) \leq x \text{ and } x_0 \leq x \leq x_i\), \(x \geq x_0\) for any \(x \in [0, \infty)\) and \(\omega_i, \xi_i \in (0, \infty), i = 1, 2\) and \((\cdot)^{\omega_i}\) is superdistributive over \(\otimes\) and \((\cdot)^{\omega_i}, i = 1, 2\) are subdistributive over \(\otimes\) and \(\otimes\) dominates \(\ast\), then (3.4) holds readily. Indeed,

\[
(t_1 \ast t_2)_{\omega_0} \ast c \geq [(t_1 \ast t_2)_{\omega_0} \otimes c]_{\omega_1} \geq [(t_1 \ast t_2) \otimes c]_{\omega_1}
\]

and \([t_1 \ast (t_2 \otimes c)]_{\omega_2}\)

follows similarly, i.e.,

\[
([t_1 \ast t_2]_{\omega_0} \otimes c)_{\omega_2} \geq [(t_1 \ast t_2)_{\omega_0} \otimes c]_{\omega_2} \geq [(t_1 \ast t_2) \otimes c]_{\omega_2}
\]

We get an inequality related to the Hölder type inequality whenever \(\xi_0 = \omega_0 = 1, \xi_1 = p, \omega_1 = \frac{1}{p}, \xi_2 = q\) and \(\omega_2 = \frac{1}{q}\) for all \(p, q \in (0, \infty)\).

Corollary 3.10 Let \(f, g \in \mathcal{F}(X, A)\) be two comonotone measurable functions and \(\otimes: [0, \infty]^2 \to [0, \infty]\) be a smallest pseudo-multiplication on \([0, \infty]\) with neutral element \(e \in (0, \infty)\) and \(m \in \mathcal{M}(X, A)\) be a monotone measure such that \(a \otimes m(X) \leq a\) for all \(a \in [0, \infty]\), \(I_\otimes (m, g^q) < \infty\) and \(I_\otimes (m, f^p) < \infty\). Let \(\ast: [0, \infty]^2 \to [0, \infty]\) be continuous and nondecreasing in both arguments. If

\[
[a \ast b] \geq \left[a^p \otimes c\right]^{\frac{1}{p}} \ast b \lor \left[a \ast (b^q \otimes c)\right]^{\frac{1}{q}}
\]

then the inequality

\[
[I_\otimes (m, (f \ast g))]^{\frac{1}{s}} \geq [I_\otimes (m, (f^p))]^{\frac{1}{s}} \ast (I_\otimes (m, (f^s))^\frac{1}{s})
\]

holds for all \(p, q \in (0, \infty)\).

Again, we get an inequality related to the Minkowski type whenever \(\xi_0 = \xi_1 = \xi_2 = s\) and \(\omega_0 = \omega_1 = \omega_2 = \frac{1}{s}\) for all \(s \in (0, \infty)\).

Corollary 3.11 Let \(f, g \in \mathcal{F}(X, A)\) be two comonotone measurable functions and \(\otimes: [0, \infty]^2 \to [0, \infty]\) be a smallest pseudo-multiplication on \([0, \infty]\) with neutral element \(e \in (0, \infty)\) and \(m \in \mathcal{M}(X, A)\) be a monotone measure such that \(a \otimes m(X) \leq a\) for all \(a \in [0, \infty]\), \(I_\otimes (m, f^s) < \infty\) and \(I_\otimes (m, g^s) < \infty\). Let \(\ast: [0, \infty]^2 \to [0, \infty]\) be continuous and nondecreasing in both arguments. If

\[
[(a \ast b)^s \otimes c]^{\frac{1}{s}} \geq \left[(a^s \otimes c)^{\frac{1}{s}} \ast b\right] \lor \left[a \ast (b^s \otimes c)^{\frac{1}{s}}\right]
\]

then the inequality

\[
(I_\otimes (m, (f \ast g)^s))^{\frac{1}{s}} \geq (I_\otimes (m, f^s))^{\frac{1}{s}} \ast (I_\otimes (m, g^s))^{\frac{1}{s}}
\]

holds for all \(s > 0\).
Specially, when $s = 1$ we have the Chebyshev inequality.

**Corollary 3.12** Let $f, g \in F^{(X,A)}$ be two comonotone measurable functions and $\otimes: [0, \infty]^2 \to [0, \infty]$ be a smallest pseudo-multiplication on $[0, \infty]$ with neutral element $e \in (0, \infty]$ and $m \in \mathcal{M}^{(X,A)}$ be a monotone measure such that $a \otimes m(X) \leq a$ for all $a \in [0, \infty]$, $I_\otimes(m,f) < \infty$ and $I_\otimes(m,g) < \infty$. Let $*: [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments. If

$$(a \ast b) \otimes c \geq [(a \otimes c) \ast b] \lor [a \ast (b \otimes c)],$$

then the inequality

$$I_\otimes(m, (f \ast g)) \geq I_\otimes(m, f) \ast I_\otimes(m, g)$$

holds.

**Remark 3.13** If $\otimes$ is minimum (i.e., for Sugeno integral) and $n$-place function $H: [0, \infty)^n \to [0, \infty)$ is continuous and nondecreasing and bounded from above by minimum, then (3.6) holds readily whenever $x \otimes \omega_0 \leq x \leq x \otimes \omega_1$ and $x \geq x \otimes \omega_0$ for all $x \in [0, \infty)$ and $\omega_i, \xi_i \in (0, \infty), i = 1, 2, ..., n$. Indeed,

$$\left[H^{\omega_0} (p_1, p_2, ..., p_n) \land c \right]^{\omega_0} \geq \left[H^{\omega_0} (p_1, p_2, ..., p_n) \land c^{\omega_0} \right] \geq \left[H^{\omega_0} (p_1, p_2, ..., p_n) \land c^{\omega_0} \right] = H \left(\left(p_1^{\omega_1}, p_2, ..., p_n\right) \land \left(p_1^{\omega_1}, p_2, ..., p_n\right) \right).$$

and the others follow similarly. Thus the following results hold.

**Corollary 3.14** Let $n$-place function $H: [0, \infty)^n \to [0, \infty)$ be continuous and nondecreasing and bounded from above by minimum. Then for any comonotone system $f_1, f_2, ..., f_n \in F^{(X,A)}$ and a monotone measure $m \in \mathcal{M}^{(X,A)}$ such that $Su(m, f_i) < \infty, x \otimes \omega_0 \leq x \leq x \otimes \omega_1$ and $x \geq x \otimes \omega_0$ for all $x \in [0, \infty)$ and $\omega_i, \xi_i \in (0, \infty), i = 1, 2, ..., n$, it holds

$$\left[Su(m, (H(f_1, ..., f_n))^{\xi_0})\right]^{\omega_0} \geq H \left[Su(m, f_1), \left(Su(m, f_2)\right)^{\omega_1}, ..., (Su(m, f_n))^{\omega_n}\right]$$

for all $\omega_j, \xi_j \in (0, \infty), j = 0, 1, 2, ..., n$.

**Corollary 3.15** Let $f_1, f_2 \in F^{(X,A)}$ be two comonotone measurable functions. Let $*: [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $m \in \mathcal{M}^{(X,A)}$ be a monotone measure such that $Su(m, f_i) < \infty, x \otimes \omega_0 \leq x \leq x \otimes \omega_1$ and $x \geq x \otimes \omega_0$ for all $x \in [0, \infty)$ and $\omega_i, \xi_i \in (0, \infty), i = 1, 2$, it holds

$$\left[Su(m, (f_1 \ast f_2)^{\xi_0})\right]^{\omega_0} \geq \left[Su(m, f_1)\right]^{\omega_1} \ast \left[Su(m, f_2)\right]^{\omega_2}$$

for all $\omega_j, \xi_j \in (0, \infty), j = 0, 1, 2$. 

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Corollary 3.16 (\textsuperscript{10}) Let $f, g \in \mathcal{F}(X, A)$ be two comonotone measurable functions. Let $\ast : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $m \in \mathcal{M}(X, A)$ be a monotone measure such that $\text{Su}(m, f^*) < \infty$, $\text{Su}(m, g^*) < \infty$. Then the inequality

$$[\text{Su}(m, (f \ast g)^*)]^\frac{1}{s} \geq [\text{Su}(m, f^*)]^\frac{1}{s} \ast [\text{Su}(m, g^*)]^\frac{1}{s}$$

holds for all $0 < s < \infty$.

Corollary 3.17 Let $f, g \in \mathcal{F}(X, A)$ be two comonotone measurable functions. Let $\ast : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $m \in \mathcal{M}(X, A)$ be a monotone measure such that $\text{Su}(m, f^p) < \infty$, $\text{Su}(m, g^q) < \infty$. Then the inequality

$$\text{Su}(m, (f \ast g)) \geq [\text{Su}(m, f^p)]^\frac{1}{p} \ast [\text{Su}(m, g^q)]^\frac{1}{q}$$

holds, where $x \geq x^\frac{1}{r}$, $x \leq x^\frac{1}{r}$ for all $x \in [0, \infty)$ and $p, q \in (0, \infty)$.

Corollary 3.18 (\textsuperscript{13}) Let $f, g \in \mathcal{F}(X, A)$ be two comonotone measurable functions. Let $\ast : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $m \in \mathcal{M}(X, A)$ be a monotone measure such that $\text{Su}(m, f) < \infty$, $\text{Su}(m, g) < \infty$. Then the inequality

$$\text{Su}(m, f \ast g) \geq \text{Su}(m, f) \ast \text{Su}(m, g)$$

holds.

Notice that when working on $[0, 1]$ in Theorem 3.7 we mostly deal with $e = 1$, then $\otimes = \oplus$ is semicopula (t-seminorm) and the following results hold.

Corollary 3.19 Let a non-decreasing $n$-place function $H : [0, \infty)^n \to [0, \infty)$ such that $H$ is continuous. If semicopula $\otimes$ satisfies

$$[(H(p_1, p_2, \ldots, p_n))^{\xi_0} \otimes c]^{\omega_0} \geq H\left((p_1^{\xi_1} \otimes c)^{\omega_1}, p_2, \ldots, p_n\right) \lor 
H\left(p_1, (p_2^{\xi_2} \otimes c)^{\omega_2}, p_3, \ldots, p_n\right) \lor \ldots \lor H\left(p_1, p_2, \ldots, p_{n-1}, (p_n^{\xi_n} \otimes c)^{\omega_n}\right),$$

then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X, A)}_1$ and a monotone measure $m \in \mathcal{M}_1^{(X, A)}$, it holds

$$\left[I_\oplus\left(m, (H(f_1, \ldots, f_n))^{\xi_0}\right)\right]^{\omega_0} \geq H\left[I_\oplus\left(m, f_1^{\xi_1}\right)^{\omega_1}, I_\oplus\left(m, f_2^{\xi_2}\right)^{\omega_2}, \ldots, I_\oplus\left(m, f_n^{\xi_n}\right)^{\omega_n}\right],$$

where $\omega_i \geq 1$ for all $\omega_j, \xi_j \in (0, \infty), i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n$.

Corollary 3.20 Let $f, g \in \mathcal{F}^{(X, A)}_1$ be two comonotone measurable functions. Let $\ast : [0, 1]^2 \to [0, 1]$ be continuous and nondecreasing in both arguments. If semicopula $\otimes$ satisfies

$$[(a \ast b)^\alpha \otimes c]^{\lambda} \geq [(a^\beta \otimes c)^\nu \ast b] \lor [a \ast (b^\gamma \otimes c)^\tau], \quad (3.7)$$

then the inequality

$$[I_\oplus(m, (f \ast g)^\alpha)]^{\lambda} \geq [I_\oplus(m, f^\beta)]^{\nu} \ast [I_\oplus(m, g^\gamma)]^{\tau}$$

holds for all $\alpha, \beta, \gamma, \lambda, \nu, \tau \in (0, \infty), \gamma \tau \geq 1$, $\beta \nu \geq 1$ and for any $m \in \mathcal{M}_1^{(X, A)}$. 

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Let $\alpha = \beta = \gamma = s$ and $\lambda = \nu = \tau = \frac{1}{s}$ for all $s \in (0, \infty)$, then we get the reverse Minkowski type inequality for seminormed fuzzy integrals.

**Corollary 3.21** Let $f, g \in \mathcal{F}^{(X,A)}_{[0,1]}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If semicopula $\odot$ satisfies
\[
[(a \star b) \odot c]^{\frac{1}{s}} \geq [(a^s \odot c)^{\frac{1}{s}} \star b] \lor [a \star (b^s \odot c)]^{\frac{1}{s}},
\]
then the inequality
\[
(I_\odot (m, (f \star g)^s))^{\frac{1}{s}} \geq (I_\odot (m, f^s))^{\frac{1}{s}} \star (I_\odot (m, g^s))^{\frac{1}{s}}
\]
holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $0 < s < \infty$.

Again, we get the Chebyshev type inequality for seminormed fuzzy integrals whenever $s = 1$ [17].

**Corollary 3.22** Let $f, g \in \mathcal{F}^{(X,A)}_{[0,1]}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If semicopula $\odot$ satisfies
\[
[(a \star b) \odot c] \geq [(a \odot c) \star b] \lor [a \star (b \odot c)],
\]
then the inequality
\[
I_\odot (m, (f \star g)) \geq I_\odot (m, f) \star I_\odot (m, g)
\]
holds for any $m \in \mathcal{M}_1^{(X,A)}$.

**Remark 3.23** We can use an example in [17] to show that the condition of $[(a \star b) \odot c] \geq [(a \odot c) \star b] \lor [a \star (b \odot c)]$ in Corollary 3.22 (and thus in Theorem 3.5) cannot be abandoned, and so we omit it here.

Suppose the semicopula $\odot$ further satisfies monotonicity and associativity (i.e., it is a $t$-norm). Then, we have the following result:

**Corollary 3.24** Let $f, g \in \mathcal{F}^{(X,A)}_{[0,1]}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If semicopula $\odot$ be a continuous $t$-norm, then
\[
[I_\odot (m, (f \odot g)^\alpha)]^{\lambda} \geq [(I_\odot (m, f^\beta)]^{\nu} \odot [I_\odot (m, g^\gamma)]^{\tau}
\]
holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $\alpha, \beta, \gamma, \lambda, \nu, \tau \in (0, \infty), 0 < \alpha \lambda \leq 1, 1 \leq \beta \nu < \infty, 1 \leq \gamma \tau < \infty, \lambda \leq \tau, \nu \leq \sigma$ and $\alpha \leq \beta, \gamma$, where $(\cdot)^\alpha$ is superdistributive over $\odot$, $\odot^{\lambda}$ dominates $\odot$ and $(f \odot g)(x) = f(x) \odot g(x)$ for any $x \in X$.

Let $\alpha = \beta = \gamma = \lambda = \nu = \tau = 1$, then $\odot$ is obviously dominated by itself and we have the following result:
Corollary 3.25 Let \( f, g \in F^{(X,A)} \) be two comonotone measurable functions. Let \( \star : [0,1]^2 \to [0,1] \) be continuous and nondecreasing in both arguments. If semicopula \( \otimes \) be a continuous \( t \)-norm, then
\[
I_\otimes (m, (f \otimes g)) \geq (I_\otimes (m, f) \otimes I_\otimes (m, g))
\]
holds for any \( m \in M^{(X,A)}_1 \) and \((f \otimes g)(x) = f(x) \otimes g(x)\) for any \( x \in X \).

Notice that if the semicopula \( (t\)-seminorm) \( \otimes \) is minimum (i.e., for Sugeno integral) and \( \star \) is bounded from above by minimum, then \( \star \) is dominated by minimum. Thus the following result holds.

Corollary 3.26 Let \( f, g \in F^{(X,A)} \) be two comonotone measurable functions. Let \( \star : [0,1]^2 \to [0,1] \) be continuous and nondecreasing in both arguments and bounded from above by minimum. Then the inequality
\[
[Su (m, (f \star g)\alpha)]^\beta \geq [Su (m, f\beta)]^\alpha \star [Su (m, g\gamma)]^\tau
\]
holds for any \( m \in M^{(X,A)}_1 \) and for all \( \alpha, \beta, \gamma, \lambda, \nu, \epsilon, \tau \in (0, \infty), 0 < \alpha \lambda \leq 1, \beta \nu \geq 1, \gamma \tau \geq 1, \lambda \leq \tau, \nu \).

Theorem 3.27 Let \( f \in F^{(X,A)} \) be a measurable function and \( \otimes : [0, \infty]^2 \to [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty] \) and \( m \in M^{(X,A)} \) be a monotone measure such that \( I_\otimes (m, \varphi_2 (f)) \) is finite. Let \( \varphi_i : [0, \infty) \to [0, \infty), i = 1, 2 \) be continuous strictly increasing functions. If
\[
\varphi_i^{-1} (\varphi_1 (a) \otimes c) \geq \varphi_2^{-1} (\varphi_2 (a) \otimes c),
\]
then the inequality
\[
\varphi_i^{-1} (I_\otimes (m, \varphi_1 (f))) \geq \varphi_2^{-1} (I_\otimes (m, \varphi_2 (f)))
\]
holds.

Proof. Let \( e \in (0, \infty] \) be the neutral element of \( \otimes \) and \( I_\otimes (m, \varphi_2 (f)) = p < \infty \). So, for any \( \epsilon > 0 \), there exists \( p_\epsilon \) such that \( m (\{ \varphi_2 (f) \geq p_\epsilon \}) = M, \) where \( p_\epsilon \otimes M \geq p - \epsilon \). Hence,
\[
\varphi_i^{-1} (I_\otimes (m, \varphi_1 (f))) \geq \varphi_1^{-1} (\{ \varphi_1 (\varphi_2^{-1} (p_\epsilon)) \otimes m (\{ \varphi_1 (f) \geq \varphi_1 (\varphi_2^{-1} (p_\epsilon)) \}) \})
\]
\[
= \varphi_1^{-1} (\{ \varphi_1 (\varphi_2^{-1} (p_\epsilon)) \otimes m (\{ \varphi_2 (f) \geq \varphi_2^{-1} (p_\epsilon) \}) \})
\]
\[
\geq \varphi_2^{-1} (\{ \varphi_2 (\varphi_2^{-1} (p_\epsilon)) \otimes m (\{ \varphi_2 (f) \geq \varphi_2^{-1} (p_\epsilon) \}) \})
\]
\[
= \varphi_2^{-1} ([p_\epsilon \otimes M]) \geq \varphi_2^{-1} (p - \epsilon)
\]
whence \( \varphi_i^{-1} (I_\otimes (m, \varphi_1 (f))) \geq \varphi_2^{-1} (p) \) follows from the continuity of \( \varphi_2 \) and the arbitrariness of \( \epsilon \). And the theorem is proved. □

If we take \( \varphi_2 (x) = x \) in Theorem 3.14, then the the following Jensen inequality for universal integral is recaptured.

Corollary 3.28 Let \( f \in F^{(X,A)} \) be a measurable function and \( \otimes : [0, \infty]^2 \to [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty] \) and \( m \in M^{(X,A)} \) be a monotone measure such that \( I_\otimes (m, f) \) is finite. Let \( \varphi : [0, \infty) \to [0, \infty) \) be continuous strictly increasing function. If
\[
\varphi (a) \otimes c \geq \varphi (a \otimes c),
\]
(3.8)
then the inequality
\[ I_\otimes (m, \varphi(f)) \geq \varphi (I_\otimes (m, f)) \]
holds.

Again, if we take \( \varphi_1 (x) = x \) in Theorem 4.14 then we have the reverse Jensen inequality for universal integral.

**Corollary 3.29** Let \( f \in \mathcal{F}(X, A) \) be a measurable function and \( \otimes : [0, \infty]^2 \to [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty) \) and \( m \in \mathcal{M}(X, A) \) be a monotone measure such that \( I_\otimes (m, \varphi(f)) \) is finite. Let \( \varphi : [0, \infty) \to [0, \infty) \) be continuous strictly increasing function. If
\[ \varphi(a \otimes c) \geq (\varphi(a) \otimes c), \quad (3.9) \]
then the inequality
\[ \varphi (I_\otimes (m, f)) \geq I_\otimes (m, \varphi (f)) \]
holds.

**Remark 3.30** If \( \varphi : [0, \infty) \to [0, \infty) \) is continuous strictly increasing function such that \( \varphi(x) \leq x \) for all \( x \in [0, \infty) \) and \( \varphi \) is subdistributive over \( \otimes \), then (3.8) holds readily. Indeed,
\[ \varphi(a \otimes c) \leq \varphi(a) \otimes \varphi(c) \leq \varphi(a) \otimes c. \]
Also, if \( \varphi(x) \geq x \) for all \( x \in [0, \infty) \) and \( \varphi \) is superdistributive over \( \otimes \), then (3.9) holds similarly, i.e.,
\[ \varphi(a \otimes c) \geq \varphi(a) \otimes \varphi(c) \geq \varphi(a) \otimes c. \]

**Corollary 3.31** Let \( f \in \mathcal{F}(X, A) \) be a measurable function and \( \otimes : [0, \infty]^2 \to [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty) \) and \( m \in \mathcal{M}(X, A) \) be a monotone measure such that \( I_\otimes (m, f) \) is finite. Let \( \varphi : [0, \infty) \to [0, \infty) \) be continuous strictly increasing function such that \( \varphi(x) \leq x \) for all \( x \in [0, \infty) \). Then the inequality
\[ I_\otimes (m, \varphi(f)) \geq \varphi (I_\otimes (m, f)) \]
holds, where \( \varphi \) is subdistributive over \( \otimes \).

**Corollary 3.32** Let \( f \in \mathcal{F}(X, A) \) be a measurable function and \( \otimes : [0, \infty]^2 \to [0, \infty] \) be the pseudo-multiplication with neutral element \( e \in (0, \infty) \) and \( m \in \mathcal{M}(X, A) \) be a monotone measure such that \( I_\otimes (m, \varphi(f)) \) is finite. Let \( \varphi : [0, \infty) \to [0, \infty) \) be continuous strictly increasing function such that \( \varphi(x) \geq x \) for all \( x \in [0, \infty) \). Then the inequality
\[ \varphi (I_\otimes (m, f)) \geq I_\otimes (m, \varphi (f)) \]
holds, where \( \varphi \) is superdistributive over \( \otimes \).

Notice that if the pseudo-multiplication \( \otimes \) is minimum (i.e., for Sugeno integral), then the following results hold (see [19] for a similar result).
Corollary 3.33 Let $f \in \mathcal{F}(X,\mathcal{A})$ be a measurable function and $m \in \mathcal{M}(X,\mathcal{A})$ be a monotone measure such that $\text{Su}(m,f)$ is finite. Let $\varphi : [0,\infty) \to [0,\infty)$ be continuous strictly increasing function such that $\varphi(x) \leq x$ for all $x \in [0,\infty)$. Then the inequality
\[
\text{Su}(m,\varphi(f)) \geq \varphi(\text{Su}(m,f))
\]
holds.

Corollary 3.34 Let $f \in \mathcal{F}(X,\mathcal{A})$ be a measurable function and $m \in \mathcal{M}(X,\mathcal{A})$ be a monotone measure such that $\text{Su}(m,\varphi(f))$ is finite. Let $\varphi : [0,\infty) \to [0,\infty)$ be continuous strictly increasing function such that $\varphi(x) \geq x$ for all $x \in [0,\infty)$. Then the inequality
\[
\varphi(\text{Su}(m,f)) \geq \text{Su}(m,\varphi(f))
\]
holds.

When $\varphi_1(x) = x^s$ and $\varphi_2(x) = x^r$ for all $r, s \in (0, \infty)$ in Theorem 4.14, then we have the following Lyapunov inequality for universal integral.

Corollary 3.35 Let $f \in \mathcal{F}(X,\mathcal{A})$ be a measurable function and $m \in \mathcal{M}(X,\mathcal{A})$ be a monotone measure such that $\text{I}_{\otimes}(m,f^r)$ is finite. If
\[
(a^s \otimes c)^\frac{1}{s} \geq (a^r \otimes c)^\frac{1}{r},
\]
then the inequality
\[
(\text{I}_{\otimes}(m,f^s))^\frac{1}{s} \geq (\text{I}_{\otimes}(m,f^r))^\frac{1}{r}
\]
holds for all $r, s \in (0, \infty)$.

Notice that when working on $[0, 1]$ in Theorem 4.14 we mostly deal with $e = 1$, then $\otimes = \odot$ is semicopula ($t$-seminorm) and the following results hold.

Corollary 3.36 Let $f \in \mathcal{F}_{[0,1]}(X,\mathcal{A})$ be a measurable function and $m \in \mathcal{M}_1(X,\mathcal{A})$ be a monotone measure. Let $\varphi_i : [0,\infty) \to [0,\infty), i = 1, 2$ be continuous strictly increasing functions. If semicopula $\odot$ satisfies
\[
\varphi_1^{-1}(\varphi_1(a) \odot c) \geq \varphi_2^{-1}(\varphi_2(a) \odot c),
\]
then the inequality
\[
\varphi_1^{-1}(\text{I}_{\otimes}(m,\varphi_1(f))) \geq \varphi_2^{-1}(\text{I}_{\otimes}(m,\varphi_2(f)))
\]
holds.

Corollary 3.37 Let $f \in \mathcal{F}_{[0,1]}(X,\mathcal{A})$ be a measurable function and $m \in \mathcal{M}_1(X,\mathcal{A})$ be a monotone measure. Let $\varphi : [0,1] \to [0,1]$ be continuous strictly increasing function such that $\varphi(x) \leq x$ for all $x \in [0,1]$. Then the inequality
\[
\text{I}_{\otimes}(m,\varphi(f)) \geq \varphi(\text{I}_{\otimes}(m,f))
\]
holds, where $\varphi$ is subdistributive over semicopula $\odot$. 

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Corollary 3.38  Let \( f \in \mathcal{F}_{[0,1]}^{(X,A)} \) be a measurable function and \( m \in \mathcal{M}_1^{(X,A)} \) be a monotone measure. Let \( \varphi : [0,1] \rightarrow [0,1] \) be continuous strictly increasing function such that \( \varphi (x) \geq x \) for all \( x \in [0,1] \). Then the inequality
\[
\varphi \left( I_\otimes (m, f) \right) \geq I_\otimes (m, \varphi (f))
\]
holds, where \( \varphi \) is superdistributive over semicopula \( \otimes \).

Corollary 3.39  Let \( f \in \mathcal{F}_{[0,1]}^{(X,A)} \) be a measurable function and \( m \in \mathcal{M}_1^{(X,A)} \) be a monotone measure. If semicopula \( \otimes \) satisfies
\[
(a^s \otimes c)^\frac{1}{s} \geq (a^r \otimes c)^\frac{1}{r}, \tag{3.10}
\]
then the inequality
\[
(I_\otimes (m, f^s))^\frac{1}{s} \geq (I_\otimes (m, f^r))^\frac{1}{r}
\]
holds for all \( r, s \in (0, \infty) \).

Corollary 3.40  Let \( f \in \mathcal{F}_{[0,1]}^{(X,A)} \) be a measurable function and \( m \in \mathcal{M}_1^{(X,A)} \) be a monotone measure. then the inequality
\[
(Su (m, f^s))^\frac{1}{s} \geq (Su (m, f^r))^\frac{1}{r}
\]
holds for all \( 0 < r \leq s < \infty \).

4  On reverse inequalities

By using the concepts of t-seminorm and t-semonicorm, Suárez and Gil proposed the a family of semiconormed integrals [21]. Define
\[
I_\oplus (m, f) = \inf \left\{ t \oplus m \left( \{ f > t \} \right) \mid t \in (0, \infty) \right\}.
\]
Hence, we get the following theorems.

Theorem 4.1  Let a non-decreasing \( n \)-place function \( H : [0,\infty)^n \rightarrow [0,\infty] \) such that \( H \) be continuous. If \( \oplus : [0,\infty]^n \rightarrow [0,\infty] \) is the pseudo-addition with neutral element \( 0 \), satisfies
\[
U_0^{-1} \left[ U_0 \left( H \left( \psi_1 (p_1), \psi_2 (p_2), ..., \psi_n (p_n) \right) \right) \oplus c \right] \leq H \left( \psi_1 \left( U_1^{-1} \left[ \left( U_1 (p_1) \right) \oplus c \right] \right), \psi_2 (p_2), ..., \psi_n (p_n) \right) \\
\land H \left( \psi_1 (p_1), \psi_2 \left( U_2^{-1} \left[ \left( U_2 (p_2) \right) \oplus c \right] \right), \psi_3 (p_3), ..., \psi_n (p_n) \right) \\
\land ... \land H \left( \psi_1 (p_1), \psi_2 (p_2), ..., \psi_{n-1} (p_{n-1}), \psi \left( U_n^{-1} \left[ \left( U_n (p_n) \right) \oplus c \right] \right) \right),
\]
then for any system \( U_0, U_1, ..., U_n : [0,\infty) \rightarrow [0,\infty] \) of continuous strictly increasing functions, and any system \( \psi_1, \psi_2, ..., \psi_n : [0,\infty) \rightarrow [0,\infty] \) of continuous increasing functions and any comonotone system \( f_1, f_2, ..., f_n \in \mathcal{F}^{(X,A)} \) and a monotone measure \( m \in \mathcal{M}^{(X,A)} \), \( I_\oplus (m, U_i (f_i)) < \infty \) for all \( i = 1, 2, ... n \), it holds
\[
U_0^{-1} \left[ I_\oplus (m, U_0 [H (\psi_1 (f_1), ..., \psi_n (f_n))]) \right] \leq H \left[ \psi_1 \left( U_1^{-1} \left( I_\oplus (m, U_1 (f_1)) \right) \right), ..., \psi_n \left( U_n^{-1} \left( I_\oplus (m, U_n (f_n)) \right) \right) \right].
\]
Proof. Let $I_{\otimes} (m, U_i (f_i)) = p_i < \infty$ for all $i = 1, 2, ..., n$. So, for any $\varepsilon > 0$, there exist $p_{i(\varepsilon)}$ such that

$$m(\{U_i (f_i) > p_{i(\varepsilon)}\}) = M_i,$$

where $p_{i(\varepsilon)} \oplus M_i \leq p_i + \varepsilon$ for all $i = 1, 2, ..., n$. Then,

$$\psi_i (U_i^{-1} [p_{i(\varepsilon)} \oplus M_i]) \leq \psi_i (U_i^{-1} [p_i + \varepsilon]) , \text{ for all } i = 1, 2, ..., n.$$

Then,

$$\psi_i (U_i^{-1} [p_{i(\varepsilon)}]) = \psi_i (U_i^{-1} [p_{i(\varepsilon)} \oplus 0]) \leq \psi_i (U_i^{-1} [p_i + \varepsilon]) , \text{ for all } i = 1, 2, ..., n.$$ 

The comonotonicity of $f_1, f_2, ..., f_n$ and the monotonicity of $H$ imply that

$$m \left( \{U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) > U_0 (H (\psi_1 (U_1^{-1} (p_{1(\varepsilon)})), ..., \psi_n (U_n^{-1} (p_{n(\varepsilon)})))) \} \right) \leq m \left( \{U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) > U_0 (H (\psi_1 (U_1^{-1} (p_{1(\varepsilon)})), ..., \psi_n (U_n^{-1} (p_{n(\varepsilon)})))) \} \right) \leq M_1 \lor M_2 \lor \ldots \lor M_n.$$ 

Hence

$$U_0^{-1} \left[ \inf \{ (t \oplus m(\{U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) > t) \mid t \in (0, \infty)] \} \right) \leq U_0^{-1} \left( \left[ m(\{U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) > U_0 (H (\psi_1 (U_1^{-1} (p_{1(\varepsilon)})), ..., \psi_n (U_n^{-1} (p_{n(\varepsilon)})))) \}) \oplus (M_1 \lor M_2 \lor \ldots \lor M_n) \right] \right) \leq U_0^{-1} \left( \left[ m(\{U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))) > U_0 (H (\psi_1 (U_1^{-1} (p_{1(\varepsilon)})), ..., \psi_n (U_n^{-1} (p_{n(\varepsilon)})))) \}) \oplus (M_1 \lor M_2 \lor \ldots \lor M_n) \right] \right) \leq \left( \left[ H (\psi_1 (U_1^{-1} [p_{1(\varepsilon)} \oplus M_1]), ..., \psi_n (U_n^{-1} [p_{n(\varepsilon)} \oplus M_n])) \right] \right) \leq H (\psi_1 (U_1^{-1} [p_{1(\varepsilon)} \oplus M_1]), ..., \psi_n (U_n^{-1} [p_{n(\varepsilon)} \oplus M_n])) \leq H (\psi_1 (U_1^{-1} [p_{1(\varepsilon)}], ..., \psi_n (U_n^{-1} [p_{n(\varepsilon)}])) \leq \left( \left[ H (\psi_1 (U_1^{-1} [p_{1(\varepsilon)}]), ..., \psi_n (U_n^{-1} [p_{n(\varepsilon)}])) \right] \right) \leq H (\psi_1 (U_1^{-1} [p_{1(\varepsilon)}]), ..., \psi_n (U_n^{-1} [p_{n(\varepsilon)}])).$$

whence $U_0^{-1} [I_{\otimes} (m, U_0 (H (\psi_1 (f_1), ..., \psi_n (f_n))))] \leq H (\psi_1 (U_1^{-1} [p_1]), ..., \psi_n (U_n^{-1} [p_n]))$ follows from the continuity of $H, \psi_i, U_i$ for all $i$, and the arbitrariness of $\varepsilon$. And the theorem is proved. $\Box$

**Corollary 4.2** Let a non-decreasing $n$-place function $H : [0, 1]^n \to [0, 1]$ such that $H$ be continuous and a continuous non-decreasing $\psi : [0, 1] \to [0, 1]$ be given. If the $t$-seminorm $S$ satisfies

$$U_0^{-1} [S (U_0 (H (\psi (p_1), \psi (p_2), ..., \psi (p_n))), c)] \leq H (\psi (U_1^{-1} [S (U_1 (p_1), c)]), \psi (p_2), ..., \psi (p_n)) \land H (\psi (p_1), \psi (U_2^{-1} [S (U_2 (p_2), c)]), \psi (p_3), ..., \psi (p_n)) \land \ldots \land H (\psi (p_1), \psi (p_2), ..., \psi (p_{n-1}), \psi (U_n^{-1} [S (U_n (p_n), c)])),$$
then for any system \(U_0, U_1, \ldots, U_n : [0, 1] \rightarrow [0, 1]\) of continuous strictly increasing functions and any comonotone system \(f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X, \omega)}_{[0,1]}\) and a monotone measure \(m \in \mathcal{M}^{(X, \omega)}_1\), it holds
\[
U_0^{-1}\left[ I_S \left(m, U_0 \left( H \left( \psi \left( f_1 \right), \ldots, \psi \left( f_n \right) \right) \right) \right) \right] \leq H \left[ \psi \left( U_1^{-1} \left( I_S \left(m, U_1 \left( f_1 \right) \right) \right) \right), \ldots, \psi \left( U_n^{-1} \left( I_S \left(m, U_n \left( f_n \right) \right) \right) \right) \right].
\]

In an analogous way as in the proof of Theorem 4.1 we have the following results.

**Theorem 4.3** Let a non-decreasing \(n\)-place function \(H : [0, \infty)^n \rightarrow [0, \infty)\) such that \(H\) be continuous. If \(\oplus : [0, \infty)^n \rightarrow [0, \infty]\) is the pseudo-addition with neutral element 0, satisfies
\[
\left( (H \left( p_1, p_2, \ldots, p_n \right))^\xi_0 \oplus c \right)^\omega_0 \leq H \left( \left( p_1^{\xi_1} \oplus c \right)^\omega_1, p_2, \ldots, p_n \right) \land \nabla \right.
\]
then for any comonotone system \(f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X, \omega)}_{[0,1]}\) and a monotone measure \(m \in \mathcal{M}^{(X, \omega)}_1\), it holds
\[
\left[ I_\oplus \left(m, \left( H \left( f_1, \ldots, f_n \right) \right)^\xi_0 \right) \right]^{\omega_0} \leq H \left[ \left( I_\oplus \left(m, f_1^{\xi_1} \right) \right)^{\omega_1}, \left( I_\oplus \left(m, f_2^{\xi_2} \right) \right)^{\omega_2}, \ldots, \left( I_\oplus \left(m, f_n^{\xi_n} \right) \right)^{\omega_n} \right]
\]
for all \(\omega_j, \xi_j \in (0, \infty)\), \(\omega_i \xi_i \leq 1\), where \(i = 1, 2, \ldots, n\) and \(j = 0, 1, 2, \ldots, n\).

**Corollary 4.4** Let a non-decreasing \(n\)-place function \(H : [0, \infty)^n \rightarrow [0, \infty)\) such that \(H\) be continuous. If the \(t\)-seiconorm \(S\) satisfies
\[
S^{\omega_0} \left( (H \left( p_1, p_2, \ldots, p_n \right))^\xi_0, c \right) \leq H \left( S^{\omega_1} \left( p_1^{\xi_1}, c \right), p_2, \ldots, p_n \right) \land \nabla \right.
\]
then for any comonotone system \(f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X, \omega)}_{[0,1]}\) and a monotone measure \(m \in \mathcal{M}^{(X, \omega)}_1\), it holds
\[
\left[ I_S \left(m, \left( H \left( f_1, \ldots, f_n \right) \right)^\xi_0 \right) \right]^{\omega_0} \leq H \left[ \left( I_S \left(m, f_1^{\xi_1} \right) \right)^{\omega_1}, \left( I_S \left(m, f_2^{\xi_2} \right) \right)^{\omega_2}, \ldots, \left( I_S \left(m, f_n^{\xi_n} \right) \right)^{\omega_n} \right]
\]
for all \(\omega_j, \xi_j \in (0, \infty)\), \(\omega_i \xi_i \leq 1\), where \(i = 1, 2, \ldots, n\) and \(j = 0, 1, 2, \ldots, n\).

**Corollary 4.5** Let \(f, g \in \mathcal{F}^{(X, \omega)}_{[0,1]}\) be two comonotone measurable functions. Let \(* : [0, 1]^2 \rightarrow [0, 1]\) be continuous and nondecreasing in both arguments. If the seiconorm \(S\) satisfies
\[
S^\lambda \left( \left( a \ast b \right)^\alpha, c \right) \leq S^\nu \left( a^\beta, c \ast b \right) \land \left[ a \ast S^\tau \left( b^\gamma, c \right) \right],
\]
then the inequality
\[
\left[ I_S \left(m, \left(f \ast g\right)^\alpha \right) \right]^\lambda \leq \left[ I_S \left(m, f^\beta \right) \right]^\nu \ast \left[ I_S \left(m, g^\gamma \right) \right]^\tau
\]
holds for all \(\alpha, \beta, \gamma, \lambda, \nu, \tau \in (0, \infty)\), \(\gamma \tau \leq 1\), \(\beta \nu \leq 1\) and for any \(m \in \mathcal{M}^{(X, \omega)}_1\).
Let $\alpha = \beta = \gamma = k$ and $\lambda = \nu = \tau = \frac{1}{k}$ for all $k \in (0, \infty)$, then we get the Minkowski inequality for semiconormed fuzzy integrals (if $k = 1$, then we have the reverse Chebyshev inequality for semiconormed fuzzy integrals [17]).

**Corollary 4.6** Let $f, g \in \mathcal{F}_{[0,1]}^{(X,A)}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm $S$ satisfies

$$ \left[ S((a \star b)^k, c) \right]^\frac{1}{k} \leq \left[ (S(a^k, c))^\frac{1}{k} \star b \right] \wedge \left[ a \star (S(b^k, c))^\frac{1}{k} \right], $$

then the inequality

$$ \left( \text{I}_S \left( m, (f \star g)^k \right) \right)^\frac{1}{k} \leq \left( \text{I}_S \left( m, f^k \right) \right)^\frac{1}{k} \star \left( \text{I}_S \left( m, g^k \right) \right)^\frac{1}{k} $$

holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $0 < k < \infty$.

Notice that if the seminorm $S$ is maximum (i.e., for Sugeno integral) and $\star$ is bounded from below by maximum, then $S$ is dominated by $\star$. Thus the following results hold.

**Corollary 4.7** Let $f, g \in \mathcal{F}_{[0,1]}^{(X,A)}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality

$$ \left[ \text{Su} \left( m, (f \star g)^\alpha \right) \right]^\lambda \leq \left[ \text{Su} \left( m, f^\beta \right) \right]^\nu \star \left[ \text{Su} \left( m, g^{\gamma} \right) \right]^\tau $$

holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $\alpha, \beta, \gamma, \lambda, \nu, \tau \in (0, \infty)$, $1 \leq \alpha \lambda < \infty, 0 < \beta \nu \leq 1, 0 < \gamma \tau \leq 1, \lambda \geq \tau, \nu$.

**Corollary 4.8** ([2]) Let $f, g \in \mathcal{F}_{[0,1]}^{(X,A)}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality

$$ \left( \text{Su} \left( m, (f \star g)^k \right) \right)^\frac{1}{k} \leq \left( \text{Su} \left( m, f^k \right) \right)^\frac{1}{k} \star \left( \text{Su} \left( m, g^k \right) \right)^\frac{1}{k} $$

holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $0 < k < \infty$.

**Corollary 4.9** Let $f, g \in \mathcal{F}_{[0,1]}^{(X,A)}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality

$$ \text{Su} \left( m, (f \star g)^k \right) \leq \text{Su} \left( m, f^p \right)^\frac{1}{p} \star \text{Su} \left( m, g^q \right)^\frac{1}{q} $$

holds for any $m \in \mathcal{M}_1^{(X,A)}$ and for all $p, q \in [1, \infty)$.

**Corollary 4.10** Let $f, g \in \mathcal{F}_{[0,1]}^{(X,A)}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality

$$ \text{Su} \left( m, (f \star g) \right) \leq \text{Su} \left( m, f \right) \star \text{Su} \left( m, g \right) $$

holds for any $m \in \mathcal{M}_1^{(X,A)}$.  

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Remark 4.11 If \((x \star 0) \lor (0 \star x) \geq x\) for any \(x \in [0, 1]\) and if \(\Phi (x) = (.)^\alpha\) is subdistributive over \(\star\) and \(S^\lambda\) dominates \(\star\), then \([4.7]\) holds readily for all \(\alpha, \beta, \gamma, \lambda, v, \tau \in (0, \infty), 1 \leq \alpha \lambda < \infty, 0 < \beta v \leq 1, 0 < \gamma \tau \leq 1, \alpha \geq \beta, \gamma\) and \(\lambda \geq \tau, v\).

Suppose the seminorm \(S\) further satisfies monotonicity and associativity (i.e., it is a \(t\)-conorm). Then, we have the following result:

Corollary 4.12 Let \((X, \mathcal{F}, \mu)\) be a fuzzy measure space and \(f, g: X \to [0, 1]\) two comonotone measurable functions. If \(S\) be a continuous \(t\)-conorm, then

\[
[I_S(m, S^\alpha (f, g))]^\lambda \leq S ([I_S(m, f^\beta)]^\upsilon, [I_S(m, g^\gamma)]^\tau)
\]

holds for any \(m \in \mathcal{M}_1^{(X, \mathcal{A})}\) and for all \(\alpha, \beta, \gamma, \lambda, v, \tau \in (0, \infty), 1 \leq \alpha \lambda < \infty, 0 < \beta v \leq 1, 0 < \gamma \tau \leq 1, \alpha \geq \beta, \gamma\) and \(\lambda \geq \tau, v\), where \((.)^\alpha\) is subdistributive over \(S\), \(S^\lambda\) dominates \(S\) and \(S(f, g)(x) = S(f(x), g(x))\) for any \(x \in X\).

Let \(\alpha = \beta = \gamma = \lambda = v = \tau = 1\), then we have the following result:

Corollary 4.13 Let \((X, \mathcal{F}, \mu)\) be a fuzzy measure space and \(f, g: X \to [0, 1]\) two comonotone measurable functions. If \(S\) be a continuous \(t\)-conorm, then

\[
I_S(m, S(f, g)) \leq S (I_S(m, f), I_S(m, g))
\]

holds for any \(m \in \mathcal{M}_1^{(X, \mathcal{A})}\), where \(S(f, g)(x) = S(f(x), g(x))\) for any \(x \in X\).

Theorem 4.14 Let \(f \in \mathcal{F}^{(X, \mathcal{A})}\) be a measurable function and \(\oplus: [0, \infty]^n \to [0, \infty]\) be the pseudo-addition with neutral element \(0\), satisfies and \(m \in \mathcal{M}^{(X, \mathcal{A})}\) be a monotone measure such that \(I_\oplus(m, \varphi_1 (f))\) is finite. Let \(\varphi_i : [0, \infty) \to [0, \infty), i = 1, 2\) be continuous strictly increasing functions. If

\[
\varphi_1^{-1} (\varphi_1 (a) \oplus c) \leq \varphi_2^{-1} (\varphi_2 (a) \oplus c),
\]

then the inequality

\[
\varphi_1^{-1} (I_\oplus (m, \varphi_1 (f))) \leq \varphi_2^{-1} (I_\oplus (m, \varphi_2 (f)))
\]

holds.

Proof. Let \(I_\oplus (m, \varphi_2 (f)) = p < \infty\). So, for any \(\varepsilon > 0\), there exists \(p_\varepsilon\) such that \(m(\{\varphi_2 (f) \geq p_\varepsilon\}) = M\), where \(p_\varepsilon + M \leq p + \varepsilon\). Hence,

\[
\varphi_1^{-1} (I_\oplus (m, \varphi_1 (f))) \leq \varphi_1^{-1} (\{\varphi_1 (\varphi_2^{-1} (p_\varepsilon)) \oplus m(\{\varphi_2 (f) \geq \varphi_1 (\varphi_2^{-1} (p_\varepsilon))\})\})
= \varphi_1^{-1} (\{\varphi_2^{-1} (p_\varepsilon) \oplus m(\{\varphi_2 (f) \geq p_\varepsilon\})\})
\leq \varphi_2^{-1} (\{p_\varepsilon + M\}) \leq \varphi_2^{-1} (p + \varepsilon)
\]

whence \(\varphi_1^{-1} (I_\oplus (m, \varphi_1 (f))) \geq \varphi_2^{-1} (p)\) follows from the continuity of \(\varphi_2\) and the arbitrariness of \(\varepsilon\). And the theorem is proved. \framebox[0.5in]{ }
Corollary 4.15 Let $f \in \mathcal{F}_{[0,1]}^{(X,A)}$ be a measurable function and $m \in \mathcal{M}_{1}^{(X,A)}$ be a monotone measure. Let $\varphi_{i} : [0,1] \to [0,1], i = 1,2$ be continuous strictly increasing functions. If the semiconorm $S$ satisfies
\[
\varphi_{1}^{-1}(S(\varphi_{1}(a), c)) \leq \varphi_{2}^{-1}(S(\varphi_{2}(a), c)),
\]
then the inequality
\[
\varphi_{1}^{-1}(I_{S}(m, \varphi_{1}(f))) \leq \varphi_{2}^{-1}(I_{S}(m, \varphi_{2}(f)))
\]
holds.

5 Conclusion

We have introduced some interesting inequalities, including Chebyshev’s inequality, Hölder’s inequality and Minkowski’s inequality for universal integral on abstract spaces. Furthermore, the reverse previous inequalities for semiconormed fuzzy integrals are presented. For further investigation, it would be a challenging problem to determine the conditions under which (3.5) becomes an equality.

References

[1] H. Agahi, A. Mohammadpour, S. M. Vaezpour, A generalization of the Chebyshev type inequalities for Sugeno integrals, Soft Computing 16 (2012) 659-666.

[2] H. Agahi, R. Mesiar, Y. Ouyang, General Minkowski type inequalities for Sugeno integrals, Fuzzy Sets and Systems 161 (2010) 708-715.

[3] B. Bassan and F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, J. Multivariate Anal. 93 (2005) 313-339.

[4] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals In: E. Pap, editor, Handbook of Measure Theory, Vol II, Elsevier, (2002) 1329-1379.

[5] G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953-1954) 131-292.

[6] C. Dellacherie, Quelques commentaires sur les prolongements de capacités, in: Seminaire de Probabilites (1969/70), Strasbourg, Lecture Notes in Mathematics, Vol. 191 (Springer, Berlin, 1970) 77-81.

[7] F. Durante, C. Sempi, Semicopulae, Kybernetika 41 (2005) 315-328.

[8] A. Flores-Franulič, H. Román-Flores, A Chebyshev type inequality for fuzzy integrals, Applied Mathematics and Computation 190 (2007) 1178-1184.

[9] M. Grabisch, T. Murofushi, and M. Sugeno (eds.), Fuzzy measures and integrals. Theory and applications, Physica-Verlag, Heidelberg, 2000.
[10] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Trends in Logic. Studia Logica Library, Vol. 8, Kluwer Academic Publishers, Dordrecht, 2000.

[11] E.P. Klement, R. Mesiar, E. Pap, A universal integral as Common Frame for Choquet and Sugeno Integral, IEEE Transactions on Fuzzy Systems 18,1 (2010) 178 - 187.

[12] E. P. Klement and D. A. Ralescu, Nonlinearity of the fuzzy integral, Fuzzy Sets and Systems 11 (1983) 309-315.

[13] R. Mesiar and A. Mesiarová, Fuzzy integrals and linearity, International Journal of Approximate Reasoning 47 (2008) 352-358.

[14] R. Mesiar, Y. Ouyang, General Chebyshev type inequalities for Sugeno integrals, Fuzzy Sets and Systems 160 (2009) 58-64.

[15] Y. Ouyang, J. Fang, L. Wang, Fuzzy Chebyshev type inequality, International Journal of Approximate Reasoning 48 (2008) 829-835.

[16] Y. Ouyang, R. Mesiar, H. Agahi, An inequality related to Minkowski type for Sugeno integrals, Information Sciences 180 (2010) 2793-2801.

[17] Y. Ouyang, R. Mesiar, On the Chebyshev type inequality for seminormed fuzzy integral, Applied Mathematics Letters 22 (2009) 1810-1815.

[18] E. Pap (ed.), Handbook of Measure Theory, Elsevier Science, Amsterdam, 2002.

[19] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, A Jensen type inequality for fuzzy integrals, Information Sciences 177 (2007) 3192-3201.

[20] N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971), 109-116.

[21] F. Suárez García, P. Gil Álvarez, Two families of fuzzy integrals, Fuzzy Sets and Systems 18 (1986) 67-81.

[22] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.

[23] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, Journal of Mathematical Analysis and Applications 122 (1987) 197-222.

[24] Z. Wang and G. J. Klir, Fuzzy measure theory, Plenum Press, New York, 1992.

[25] S. Weber, Two integrals and some modified versions: critical remarks, Fuzzy Sets and Systems 20 (1986), 97-105.