SPACE-TIME TRANSITIONS IN STRING THEORY*

EDWARD WITTEN†
School of Natural Sciences,
Institute for Advanced Study,
Olden Lane,
Princeton, N.J. 08540

ABSTRACT

Simple mean field methods can be used to describe transitions between different space-time models in string theory. These include transitions between different Calabi-Yau manifolds, and more exotic things such as the Calabi-Yau/Landau-Ginzberg correspondence.

* Lecture at Strings, 1993 (Berkeley, May, 1993)
† Research supported in part by NSF Grant PHY-92-45317.
Today I will be talking about transitions between different space-time models in string theory. In string theory, space-time is represented by a two dimensional quantum field theory. We will therefore study transitions among such two dimensional theories that occur as the parameters are varied. I will describe very simple mean field methods that can be used [1] to relate Calabi-Yau models with different target spaces (as recently developed in a different way by P. Aspinwall, D. Morrison, and B. Greene [2]). The same methods also give a new explanation of the familiar relation between certain Calabi-Yau models and certain Landau-Ginzburg models.

In fact, I believe that in this way we get the generalization of the Calabi-Yau/Landau-Ginzburg correspondence to arbitrary Calabi-Yau sigma models. The general story involves an extension of the sigma model moduli space beyond the classical region. If \( X \) is a Calabi-Yau manifold, the Kahler class of a Ricci-flat Kahler metric defines a point in \( H^2(X, \mathbb{R}) \). This point lies in a conical region of \( H^2(X, \mathbb{R}) \) called the Kahler cone. In classical field theory, the region of \( H^2(X, \mathbb{R}) \) outside the Kahler cone does not have much use. In string theory, however, (or in a world-sheet quantum field theory), nothing is wasted. The quantum moduli space is all of \( H^2(X, \mathbb{R}) \) – and in fact, \( H^2(X, \mathbb{C}) \), to allow for theta angles. \( H^2(X, \mathbb{R}) \) is divided into cones, of which the classical Kahler cone of \( X \) is only one. In each cone, the theory has a different "geometrical" description. For instance, the traditional Calabi-Yau/Landau-Ginzburg correspondence arises when \( H^2(X, \mathbb{R}) \) is one dimensional. There are then only two cones: the positive half-line, which is the Kahler cone of \( X \); and the negative half-line, which is the Landau-Ginzburg region.

Let me illustrate these ideas in the Calabi-Yau/Landau-Ginzburg case. Let \( z_1, \ldots, z_5 \) be complex variables, and let \( F(z_1, \ldots, z_5) \) be a homogeneous quanta polynomial, with the transversality property that the equations

\[
0 = \frac{\partial F}{\partial z_1} = \ldots = \frac{\partial F}{\partial z_5}
\]

have a common solution only at the origin. A generic homogeneous \( F \) has that property.

Consider the hypersurface \( X \) of solutions of \( F = 0 \) in \( \mathbb{C}P^4 \). It is a smooth Calabi-Yau manifold by virtue of the transversality of \( F \). The sigma model with target space \( X \) is an \( N = 2 \) superconformal field theory.
On the other hand, we can introduce chiral superfields $\Phi_1, \ldots, \Phi_5$ in $N = 2$ superspace in two dimensions, with superpotential

$$W(\Phi_1, \ldots, \Phi_5) = F(\Phi_1, \ldots, \Phi_5).$$

With this superpotential, we can make an $N = 2$ model that is not conformally invariant

$$\mathcal{L} = \int d^2 x \, d^4 \theta \sum_i \Phi_i \Phi_i - \left( \int d^2 x \, d^2 \theta F(\Phi_i) + h.c. \right)$$

The ordinary potential is then

$$V = \sum_i \left| \frac{\partial F}{\partial \Phi_i} \right|^2.$$ (4)

This model has (by virtue of transversality of $F$) an isolated vacuum at $\Phi_i = 0$ with massless particles; it is called a Landau-Ginzburg model.

It is claimed (by Martinec and by Greene, Vafa and Warner) that a suitable orbifold of this Landau-Ginzburg model is “equivalent” in the infrared to the Calabi-Yau model determined by the same $F$.

To obtain a new insight about this relationship, we consider a $U(1)$ gauge theory in $N=2$ superspace, described by a vector superfield. The gauge invariant field strength is $\Sigma = \{ \overline{D}_+, D_- \}/2\sqrt{2}$ and the gauge kinetic energy is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4e^2} \int d^2 x \, d^4 \theta \Sigma \Sigma.$$ (5)

Such a $U(1)$ gauge theory in two dimensions has two additional interactions possible, the $\theta$ angle and the Fayet-Iliopoulos $D$ term. These can be written as

$$L_{D,\theta} = \frac{it}{2\sqrt{2}} \int d^2 x \, d\theta^+ d\theta^- \Sigma + h.c.$$ (6)

with

$$t = ir + \frac{\theta}{2\pi}.$$ (7)

$\theta$ is the usual $\theta$ angle, and $r$ is the coefficient of the $D$ term.
To this we add chiral superfields – the five superfields $\Phi_i$ of the earlier discussion, which we now take to have charge 1, and a new superfield $P$ of charge $-5$. Their kinetic energy is

$$L_{kin} = \int d^2x d^4\theta \left\{ \sum_i \bar{\Phi}_i \Phi_i + \bar{P} P \right\}.$$  

(8)

Finally, we introduce the superpotential, which we take to be the gauge invariant function $W = PF(\Phi_i)$. The corresponding piece of the Lagrangian is hence

$$L_W = -\int d^2x d^2\theta PF(\Phi_i) - \text{h.c.}$$  

(9)

After performing the $\theta$ integrals and eliminating the auxiliary fields, the ordinary potential of the model is

$$V = \frac{e^2}{2} \left( \sum \bar{\varphi}_i \varphi_i - 5\bar{p}p - r \right)^2 + |F(\varphi_i)|^2 + \sum_i \bar{p}p \left| \frac{\partial F}{\partial \varphi_i} \right|^2.$$  

(10)

Here $\varphi_i, p$ are the bosonic components of superfields $\Phi_i, P$. What remains is to study the vacuum structure as a function of $r$.

For $r >> 0$, vanishing of $V_1 = \frac{e^2}{2} (\sum \bar{\varphi}_i \varphi_i - 5\bar{p}p - r)^2$ requires $\varphi_i \neq 0$. Vanishing of $V_2 = \bar{p}p \sum_i \left| \frac{\partial F}{\partial \varphi_i} \right|^2$ then implies (given transversality of $F$) that $p = 0$. Vanishing of $V_1$ then requires

$$\sum \bar{\varphi}_i \varphi_i = r$$  

(11)

and after dividing by the gauge group $U(1)$, this gives a copy of $\mathbb{C}\mathbb{P}^4$ with Kahler class proportional to $r$. Finally, vanishing of $V_3 = |F(\varphi_i)|^2$ shows that the space of classical ground states is the Calabi-Yau hypersurface $F = 0$ in $\mathbb{C}\mathbb{P}^4$.

Now for $r << 0$, vanishing of $V_1$ gives $p \neq 0$. Vanishing of $V_2$ gives then $\varphi_i = 0$, so there is up to gauge transformation a unique classical ground state with $p = \sqrt{-r/5}$. In expanding around this ground state, the $\varphi_i$ are massless and governed by an effective superpotential $\bar{W} = \langle p \rangle F(\varphi_i)$. The expectation value of $p$ breaks the gauge group to $\mathbb{Z}_5$, and the low energy theory is a $\mathbb{Z}_5$ orbifold of a Landau-Ginzburg theory with superpotential $W$. 

4
So Calabi-Yau and Landau-Ginzburg arise as two different "phases" of one system. To probe more deeply, one must understand what happens near $r = 0$. For this I refer to my paper [1]. Suffice it to say that as long as the $\theta$ angle is non-zero and at least for quantities such as Yukawa couplings, there is a smooth continuation from $r > 0$ to $r < 0$. The transition involves passing through a situation where stringy effects are big and field theory is not a good approximation.

So at least to this extent, Calabi-Yau and Landau-Ginzburg are two different limits of the same system, rather than two different phases in the strict sense. They are related like water to steam. This is the sense in which Calabi-Yau and Landau-Ginzburg are "equivalent."

By changing the gauge group and the quantum numbers of the chiral superfields, one can work out many generalizations of this. Among other things one gets transitions among space-times of different topology. For more detail, I refer to [1, 2].

I find these results fascinating because along with phenomena such as the $R \leftrightarrow 1/R$ duality, they are among relatively few examples of really "stringy" phenomena that we know. I am sure there is much more lurking under the surface, waiting to be unearthed once we have understood the geometrical tools and language appropriate to string theory. For the time being, not very much of this is accessible, though we can hopefully do more even with the methods we already have.

The phenomenon of topology change makes me wonder about the relation of the diffeomorphism group $\text{diff}(X)$ of a classical space-time to the corresponding symmetry group – call it $\text{str}(X)$ – of string theory with target space $X$. (It could be that $\text{str}(X)$ is an equivalence relation not generated by a group action, and conceivably well-defined only on shell. I will ignore such questions here.) Can $\text{diff}(X)$ be embedded in $\text{str}(X)$? One might suppose so, but there is no evident way to make this embedding in any formalism I know of. I would tend to believe that $\text{diff}(X)$ cannot be embedded in $\text{str}(X)$; such an embedding is certainly possible in the long distance limit, but there may be deviations of order $\alpha'$. The topology-changing phenomena show that under certain conditions $\text{str}(X) = \text{str}(X')$ for distinct space-times $X$ and $X'$. So if $\text{diff}(X)$ can be embedded in $\text{str}(X)$, so can be $\text{diff}(X')$, presumably, as $X$ and $X'$ seem to be on an entirely equal footing.

One can ask the opposite question: is there a homomorphism from $\text{str}(X)$ onto $\text{diff}(X)$? I would presume the answer is no; the existence of such a homomorphism would more or less give
a way to recover field theoretic physics from string theoretic physics, and any such attempt ought to be thwarted by terms of order $\alpha'$. 

At any rate, the topology-changing phenomena ought to mean that if there is a homomorphism from $\text{str}(X)$ onto $\text{diff}(X)$, there should also be one onto $\text{diff}(X')$. This seems even less plausible.

The topology-changing phenomena really ought to mean that the relation between $\text{str}(X)$ and $\text{diff}(X)$ is purely a long-distance approximation that is relevant in a particular limit of the string theory moduli space. It is tantalizing to think that the deviation of the symmetry group of the world from $\text{diff}(X)$ might have some observable effect in the low energy world; but it is hard to see how that would be.

REFERENCES

1. E. Witten, “Phases of $N = 2$ Models in Two Dimensions,” (IASSNS-HEP-93/3), to appear in Nucl. Phys. B.

2. P. Aspinwall, D. Morrison, and B. Greene, “Multiple Mirror Manifolds and Topology Change in String Theory” (IASSNS-HEP-93/4), Phys. Lett. 303B, 249 (1993.)