A Model for Optimal Reserve Inventory Between two Machines with Reference to Truncation Point of the Repair Time

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Abstract

In Inventory control theory, many suitable models for real life systems are constructed with the objective of determining the optimal inventory level. In a system where the machines are in series for producing the finished products, the reserve of semi-finished products between two machines becomes unavoidable to minimize the idle time of machines in series. In this model the repair time of machines is assumed to be a random variable and it follows exponential distribution which satisfies the so called Setting the Clock Back to Zero (SCBZ) property. Also, the truncation point of the repair time distribution is itself a random variable and it follows mixed exponential distribution. Under this assumption an optimal reserve inventory is obtained.

Key words: Reserve Inventory, SCBZ property, Repair time, Truncation point and Optimal reserve.

1. Introduction

Describing the behaviour of the inventory system in terms of the mathematical model carries out a determination of optimal inventory under different real life circumstances. When working systems are considered, avoiding the breakdown of the system if quiet possible by keeping reserve inventory between the machines are in series. A system in which there are two machines M₁ and M₂ are in series. The output of the machine M₁ is the input of the machine M₂. The breakdown of M₁ causes the idle time of M₂ since there is no input to M₂ from M₁. The idle time of M₂ is very costly and hence to avoid it, a reserve inventory is maintained in between M₁ and M₂. The repair time of M₁ is considered as a random variable and after the repair time of M₁ is exceeded, it supplies to the reserve inventory. If the reserve inventory is surplus in quantity, there is an inventory holding cost. If the reserve inventory slacks in its quantity, then it assumes the high idle time cost. Hence, in order to balance these costs, an optimal reserve inventory must be maintained between these two machines. Therefore, the problem is to determine the optimal reserve between M₁ and M₂. The very basic model has been discussed by Hanssmann. F. The extension over this model is discussed by Ramachandran and Sathyamoorthi2.

This type of models has been discussed by many authors under the assumptions that the repair time is a random variable. Sachithanantham. S et al.3 discussed this...
model with the assumption that the probability function of the repair time undergoes a parametric change after the truncation point. The very basic concept of parametric change known as SCBZ property was discussed by Raja Rao, B, and Talwalker, S.\textsuperscript{4}. The model for optimal reserve inventory between two machines under the assumption that the repair time of machine M\textsubscript{1} satisfies the SCBZ property with the truncation point being a random variable is derived by Sachithanantham, S. et al.\textsuperscript{5}. And the same model with a modification of the probability function of truncation point is discussed by Ramathilagam, S. et al.\textsuperscript{6}. The improvement over this model is being discussed in this paper, in which the truncation point is a random variable, which follows the mixed exponential distribution with parameters $\theta_1$ and $\theta_2$.

2. Notations

$$h$$: Inventory holding cost/ unit/ unit time.

$$d$$: Idle time cost due to M\textsubscript{2} / unit time.

$$\mu$$: Mean time interval between successive breakdown of machine M\textsubscript{1}, assuming exponential distribution of inter-arrival times.

$$t$$: Continuous random variable denoting the repair time of M\textsubscript{1} with pdf $g(.)$ and c.d.f $G(.)$.

$$r$$: Constant consumption rate of M\textsubscript{2} per unit of time.

$$S$$: Reserve inventory between M\textsubscript{1} and M\textsubscript{2}.

$$T$$: Random variable denoting the idle time of M\textsubscript{2}.

$$\theta$$: Parameter of exponential distribution before the truncation point.

$$\theta^*$$: Parameter of exponential distribution after the truncation point.

$$\beta$$: Probability value involved in the mixed exponential distribution.

$$\theta_1$$ & $$\theta_2$$: Parameters of mixed exponential distribution.

3. Model I:

If $T$ is a random variable denoting idle time of M\textsubscript{2}, it can be seen that

$$T = \begin{cases} 0 & \text{if } \frac{s}{r} > t \\ t - \frac{s}{r} & \text{if } \frac{s}{r} \leq t \end{cases}$$

(1)

The expected total cost of inventory holding and idle time of M\textsubscript{2} per unit of time is given by

$$E(c) = hs + \frac{d}{\mu} \int_{\frac{s}{r}}^{\infty} (t - \frac{s}{r}) g(t, \theta) dt$$

(2)

The optimal reserve $\bar{S}$ can be obtained by solving the equation $\frac{dE(c)}{ds} = 0$.

The expression for optimal reserve inventory is given by

$$G\left(\frac{\bar{S}}{r}\right) = 1 - \frac{r\mu h}{d}$$

This result is discussed in Hanssmann, F.\textsuperscript{1}. It may be observed that the above expression for optimal value of $S$ has a constraint that $\frac{hr}{d} < 1$, otherwise the solution is not a feasible one. Hence, a slight modification in the expression for the expected total cost can be incorporated as below.

1. Model II:

In this model it is assumed that the repair time distribution satisfies the SCBZ property is basically discussed by Raja Rao and Talwalker(1991). Under this assumption

$$g(t) = \begin{cases} g(t, \theta), & t \leq X_0 \\ g(t, \theta^*), & t > X_0 \end{cases}$$

where $X_0$ is a random variable denoting the truncation point of repair time and it is distributed as exponential with parameter $\lambda$.

$$g(t, \theta) = \begin{cases} \theta e^{-\theta t}, & \text{if } t \leq X_0 \\ \theta^* e^{-\theta^* t} e^{\lambda e^{(\theta^* - \theta)}}, & \text{if } t > X_0 \end{cases}$$

(3)

Thus

$$E(c) = hs + \frac{d}{\mu} \left[ \int_{\frac{s}{r}}^{\infty} (t - \frac{s}{r}) g(t, \theta) dt \right. + \left. \int_{0}^{\infty} (t - \frac{s}{r}) g(t, \theta^*) dt \right] f(x_0) dx_0$$

$$+ \frac{d}{\mu} \left[ \int_{\frac{s}{r}}^{\infty} g(t, \theta^*) dt \right] f(x_0) dx_0$$

Under this model the optimal reserve inventory obtained by solving the equation $\frac{dE(c)}{ds} = 0$

$$(\theta - \theta^*)e^{-\lambda e^{(\theta^*)}} e^{-\theta^*} + \lambda e^{-\theta^*} \frac{hr}{d} = \frac{hr}{d}$$

This model discussed by Sachithananandam et. al.\textsuperscript{5}

1. Results

In this model, it is assumed that the repair time of machine M\textsubscript{1} is a random variable and undergoes a parametric change. That is the pdf of the repair time is exponential and it undergoes a parametric change.
g(t, \theta) = \begin{cases} 
\theta e^{-\theta t}, & \text{if } t \leq X_0 \\
\theta^* e^{-\theta t} e^{x_0(\theta^* - \theta)}, & \text{if } t > X_0 
\end{cases} 
(4)

where X_0 is a random variable denoting that truncation point and it is distributed as mixed exponential with parameter \( \theta_1 \) and \( \theta_2 \). Hence the pdf of repair time can be rewritten as

\[ f(t) = g(t, \theta) P(t \leq X_0) + g(t, \theta^*) P(t > X_0) \]

Here \( P(t \leq X_0) = P(X_0 \geq t) = \int_t^\infty f(x_0) \, dx_0 \]

\[ = \int_t^\infty \theta_1 e^{-\theta_1 y} + (1 - \beta) \theta_2 e^{-\theta_2 y} \, dy \]

It may be observed that the random variable ‘T’ defined in equation (1) also undergoes a parametric change and the average idle time of \( M_2 \) is

\[ E(T) = \int_{S/\tau}^{\infty} (t - S/\tau) f(t) \, dt \]

\[ = \int_{S/\tau}^{\infty} (t - S/\tau) \left\{ g(t, \theta) \left[ (\beta e^{-\theta_1 t}) + (1 - \beta) (e^{-\theta_2 t}) \right] + \int_0^t g(t, \theta^*) \left[ (\beta \theta_1 e^{-\theta_1 x_0} + (1 - \beta) \theta_2 e^{-\theta_2 x_0} \right] \right\} dt \]

Thus the expected total cost is

\[ E(c) = hs + \frac{d}{\mu} \int_{S/\tau}^{\infty} \left\{ \int_{t_0}^{x_0} (t - S/\tau) g(t, \theta) \, dt + \int_{x_0}^{\infty} (t - S/\tau) g(t, \theta^*) \, dt \right\} f(x_0) \, dx_0 \]

\[ + \frac{d}{\mu} \int_0^{S/\tau} \left\{ \int_{t_0}^{\infty} (t - S/\tau) g(t, \theta^*) \, dt \right\} f(x_0) \, dx_0 \]

\[ \frac{dE(c)}{ds} = 0 \]

\[ h + \frac{d}{\mu} \int_{S/\tau}^{\infty} \left\{ \int_{t_0}^{x_0} (t - S/\tau) g(t, \theta) \, dt + \int_{x_0}^{\infty} (t - S/\tau) g(t, \theta^*) \, dt \right\} \left[ (\beta \theta_1 e^{-\theta_1 x_0} + (1 - \beta) \theta_2 e^{-\theta_2 x_0} \right] \, dx_0 + \frac{d}{\mu} \int_0^{S/\tau} \left( \int_{t_0}^{x_0} (t - S/\tau) g(t, \theta^*) \, dt \right) \left( (\beta \theta_1 e^{-\theta_1 x_0} + (1 - \beta) \theta_2 e^{-\theta_2 x_0} \right) \, dx_0 = 0 \]

\[ h + \frac{d}{\mu} \int_{S/\tau}^{\infty} \frac{d}{ds} (l_1 + l_2) (\beta \theta_1 e^{-\theta_1 x_0} + (1 - \beta) \theta_2 e^{-\theta_2 x_0}) \, dx_0 + \frac{d}{\mu} \int_0^{S/\tau} \frac{d}{ds} l_3 (\beta \theta_1 e^{-\theta_1 x_0} + (1 - \beta) \theta_2 e^{-\theta_2 x_0}) \, dx_0 = 0 \]

Consider

\[ \frac{d}{ds} (l_1 + l_2) = \frac{d}{ds} \int_{S/\tau}^{x_0} (t - S/\tau) \theta e^{-\theta t} \, dt + \frac{d}{ds} \int_{x_0}^{\infty} (t - S/\tau) \theta^* e^{-\theta^* t} e^{x_0(\theta^* - \theta)} \, dt \]

\[ = \int_{S/\tau}^{x_0} \left( -\frac{1}{\tau} \right) \theta e^{-\theta t} \, dt + \int_{x_0}^{\infty} \left( -\frac{1}{\tau} \right) \theta^* e^{-\theta^* t} e^{x_0(\theta^* - \theta)} \, dt \]
\[
\begin{align*}
\frac{d}{ds}(I_1 + I_2) &= \frac{1}{r} e^{-\theta \tau(r)} \\
\text{And } \frac{dI_3}{ds} &= \frac{d}{ds} \int_{s_f/r}^{s_f/r} \left(1 - s_f/r\right) e^{-\theta t} e^{x_0(\theta - \theta')} dt \\
&= \frac{\theta e^{x_0(\theta')}}{r} \int_{s_f/r}^{s_f/r} \left(1 - \frac{1}{r}\right) e^{-\theta t} dt \\
&= \frac{\theta e^{x_0(\theta')}}{r} \left[\frac{e^{-\theta t}}{-\theta}\right]_{s_f/r}^{s_f/r} \\
&= \frac{\theta e^{x_0(\theta')}}{r} \left(0 - e^{-\theta s_f/r}\right) \\
&= \frac{dI_3}{ds} = \frac{-e^{-\theta s_f/r}}{r} e^{x_0(\theta - \theta')}
\end{align*}
\]
Using the appropriate values of $h$, $d$, $r$, the optimal value of $S$ can be obtained.

6. Special Case:

Let $\beta = 1$, then

$$\frac{b\mu r}{d} \left[ e^{-\theta^*/r} + \frac{\theta_1}{(\theta_1 + \theta - \theta^*)} \right]$$

$$\frac{b\mu r}{d} (\theta_1 + \theta - \theta^*) = e^{\theta^*/r} \left[ e^{-\theta^*/r} + \frac{\theta_1}{(\theta_1 + \theta - \theta^*)} \right]$$

This equation obtained by Sachithanandham et al., in which it is assumed that the truncation point follows exponential and it could be seen that the model (11) is reduced to the above equation when $\beta = 1$.

7. Numerical Illustrations:

The variations in the values of optimal reserve inventory $S$, consequent on the changes in the parameters $0$, $\theta$, $r$, $d$, and $\mu$, have been studied by taking numerical illustrations. The tables and the corresponding graphs are given.

Case (i)

For $h=5$, $\mu=2$, $r=30$, $d=5000$, $\beta=0.5$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $\theta$.

| $\theta$ | 2 | 3 | 4 | 5 | 6 |
|----------|---|---|---|---|---|
| $\hat{S}$ | 25.859 | 20.009 | 17.270 | 15.686 |

Case (ii)

For $h=5$, $\mu=2$, $r=30$, $d=5000$, $\beta=0.5$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $\beta^*$.

| $\beta^*$ | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|
| $\hat{S}$ | 19.940 | 17.270 | 15.223 | 13.583 |
Case (iii)
For $h=5$, $\mu=2$, $d=3000$, $\beta=0.5$, $\theta=4$, $\theta^*=4$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $r$.

| $r$ | 30   | 35   | 40   | 45   |
|-----|------|------|------|------|
| $\hat{S}$ | 17.270 | 18.799 | 20.149 | 21.343 |

Case (iv)
For $h=5$, $\mu=2$, $r=30$, $\beta=0.5$, $\theta=4$, $\theta^*=4$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $d$.

| $d$   | 2500 | 3000 | 3500 | 4000 |
|-------|------|------|------|------|
| $\hat{S}$ | 15.902 | 17.270 | 18.426 | 19.427 |

Case (v)
For $\mu=2$, $r=30$, $d=3000$, $\beta=0.5$, $\theta=4$, $\theta^*=4$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $h$.

| $h$ | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|
| $\hat{S}$ | 24.142 | 21.101 | 18.943 | 17.270 | 15.902 |

Case (vi)
For $h=5$, $r=30$, $d=3000$, $\beta=0.5$, $\theta=4$, $\theta^*=4$, $\theta_1=2$, $\theta_2=3$, the optimal value of $S$ is obtained and the variations in $\hat{S}$ for the changes in the value of $\mu$.

| $\mu$ | 2   | 2.5 | 3   | 3.5 |
|-------|-----|-----|-----|-----|
| $\hat{S}$ | 17.270 | 15.596 | 14.229 | 13.073 |
From the tables and graphs it is observed that,
As $\theta$, the parameter of the repair time increases, the optimal reserve inventory $\hat{S}$ decreases.
As $\theta^*$, the repair time increases, the optimal reserve inventory decreases.
As $r$, the consumption rate of $M_2$ increases, the optimal reserve inventory $\hat{S}$ increases.
As $h$, the parameter of the holding cost increases, the optimal reserve inventory $\hat{S}$ decreases. It is understood that, the model suggests small size inventory, when the inventory holding cost increases.

8. Reference

1. Hanssmann F., Operations Research in Production and Inventory Control. John Wiley and sons, inc. Newy York (1962).
2. Ramachandran,V and Sathiyamoorthi. R., Optimal Reserves for Two Machines. IEEE Trans. On Reliability. Vol. R. 30. No.4. p. 397.
3. Sachithanantham. S., Ganesan. V . and Sathiyamoorthi. R., Optimal Reserves for Two Machines with Repair time having SCBZ Property. Bulletin of Pure and Applied Sciences. Vol. 25E(No.2) PP.487-297 (2006).
4. Raja Rao, B., Life Expectancy for a class of life distributions having the setting the Clock back to Zero property. Mathematical Bio-Sciences, 98, pp. 251-271 (1990).
5. Sachithanantham. S., Ganesan. V . and Sathiyamoorthi. R., A model for Optimal Reserve inventory Between Two Machines in Series. Journal of Indian Acad. Math. Vol. 29. No. 1 pp 59-70 (2007).
6. Ramathilagam.S, Henry.L and Sachithanantham. S., A model for optimal Reserve inventory between two machines in series with repair time undergoes a parametric change. Ultra scientist vol.26(3) P: 227-237 (2014).