A GEOMETRIC PROOF OF A FAITHFUL LINEAR-CATEGORICAL SURFACE MAPPING CLASS GROUP ACTION

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ABSTRACT. We give completely combinatorial proofs of the main results of [3] using polygons. Namely, we prove that the mapping class group of a surface with boundary acts faithfully on a finitely-generated linear category. Along the way we prove some foundational results regarding the relevant objects from bordered Heegaard Floer homology.

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1. Introduction

An important open question in topology is whether the mapping class group of a surface with boundary is linear. In [3], the authors use the tools of bordered Heegaard Floer homology to answer a categorified version of this question, showing that the mapping class group of a surface with boundary \( F \) acts faithfully on a finitely-generated linear category. Specifically, the category is the derived category of finitely-generated left \( B(F) \)-modules, where \( B(F) \) is an algebra associated to the surface \( F \). There is also a standard bimodule \( \mathcal{D} \left( \frac{1}{2} \right) \), and a version \( \otimes \) of the tensor product, which lets us define a bimodule \( \mathcal{D} \left( \frac{1}{2} \right) \otimes M(\phi) \). The action comes...
from assigning to $\phi$ the functor
\[(DD(\frac{1}{2}) \boxtimes M(\phi)) \otimes_{\mathcal{B}(F)} \cdot,\]
and then passing to the derived category. The bimodule $M(\phi)$ can be defined in terms of curves on $F$ and polygons formed between them, and this gives very concrete geometric interpretations to the results involved. Incidentally, $M(\phi)$ is not actually an ordinary bimodule but a more general $A_\infty$-bimodule, and checking $A_\infty$ identities tends to involve verifying infinitely many relations.

While [3] gives combinatorial definition of the bimodules $M(\phi)$, it refers proofs about their structure to [4,6], which rely on hard analysis. In this paper we give completely combinatorial proofs of the main results in [3] in terms of polygons and operations on them. In light of [2], it is interesting to compare our results with [1].

The paper is organized as follows. In Section 2 we define some of the basic objects that will be used throughout the paper, especially $A_\infty$-algebras and modules, and we define $\mathcal{B}(F)$ and $M(\phi)$. In Section 3 we prove some foundational results:

**Theorem 1.1.** $M(\phi_0)$ is a $\mathcal{B}(\mathcal{Z}')$-$\mathcal{B}(\mathcal{Z})$ $A_\infty$-bimodule.

**Theorem 1.2.** Let $\phi_b$ and $\phi_{nb}$ be diffeomorphisms representing the same element of $\text{MCG}_0(F)$. Then $M(\phi_b)$ and $M(\phi_{nb})$ are homotopy equivalent as $A_\infty$-bimodules.

In Section 4 we give the key ingredient for an action:

**Corollary 1.3.** The bimodules $M(\psi \circ \phi)$ and $M(\phi) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi))$ are $A_\infty$-homotopy equivalent.

We then complete the proof by showing that the identity mapping class gives the identity functor and that the action is faithful:

**Theorem 1.4.** $DD(\frac{1}{2}) \boxtimes M(\mathbb{I})$ is isomorphic as a type DA bimodule to $A^1\mathbb{I}_A$.

**Theorem 1.5.** If $DD(\frac{1}{2}) \boxtimes M(\phi_0)$ is quasi-isomorphic to $A^1\mathbb{I}_A$, then $\phi_0$ is isotopic to $\mathbb{I}$.

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### 2. SOME DEFINITIONS

In this section we give definitions for some of the objects that will play key roles in this paper.

**2.1. $A_\infty$-algebras and $A_\infty$-bimodules.** We begin by defining $A_\infty$-algebras, and $A_\infty$-bimodules over $A_\infty$-algebras. These are like associative algebras and ordinary differential bimodules except with associativity replaced by a weaker condition involving infinitely many terms. Actually all of our $A_\infty$-bimodules will be defined over ordinary differential algebras, but we define $A_\infty$-algebras here for completeness. We fix a ground ring $k$, which for our purposes will always be a direct sum of copies of $\mathbb{F}_2$. 
Definition 2.1. An $A_\infty$-algebra $A$ over $k$ is a $k$-module $A$, together with $k$-linear maps 
\[ \mu_i : A^\otimes i \to A \]
for each $i \geq 1$, satisfying the following structure equation for each $n \geq 1$ and $a_1, ..., a_n \in A$:
\[
\sum_{i+j=n+1} \sum_{l=1}^{n-j+1} \mu_i(a_1 \otimes ... \otimes a_{l-1} \otimes \mu_j(a_l \otimes ... \otimes a_{l+j-1}) \otimes a_{l+j} \otimes ... \otimes a_n) = 0.
\]

By setting $\mu_0 \equiv 0$ we can combine the $\mu_i$’s to define a single map 
\[ \mu := \sum_{i=0}^{\infty} \mu_i : T^*(A) \to A \]
on the tensor algebra
\[ T^*(A) := \bigoplus_{n=0}^{\infty} A^\otimes n. \]

Defining $\mathcal{D}^A : T^*(A) \to T^*(A)$ by
\[ \mathcal{D}^A(a_1 \otimes ... \otimes a_n) := \sum_{j=1}^{n} \sum_{l=1}^{n-j+1} a_1 \otimes ... \otimes a_{l-1} \otimes \mu_j(a_l \otimes ... \otimes a_{l+j-1}) \otimes a_{l+j} \otimes ... \otimes a_n, \]
the $A_\infty$-algebra relations (2.1) can be written more succinctly as $\mu \circ \mathcal{D}^A = 0$.

Define $\Delta_2 : T^*(A) \to T^*(A) \otimes T^*(A)$ by
\[ \Delta_2(a_1 \otimes ... \otimes a_n) := \sum_{i=0}^{n} (a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes ... \otimes a_n). \]
Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be $A_\infty$-algebras over $k$. Then an $A_\infty$-bimodule $\mathcal{A}M_B$ over $\mathcal{A}$ and $\mathcal{B}$ is a $k$-bimodule $M$ together with $k$-linear maps
\[ m_{i,1,j} : \mathcal{A}^\otimes i \otimes M \otimes B^\otimes j \to M \]
for each $i, j \geq 0$ satisfying the following:

\[ \Delta_2 \Delta_2 + \bar{D}^A \bar{D}^B = 0. \]

We will also refer to $A_\infty$-bimodules as type AA bimodules.

2.2. Arc diagrams and the algebra $\mathcal{B}(\mathcal{Z})$. Let $F$ be a connected, oriented surface of genus $g$ with $b > 0$ boundary components $Z_1, ..., Z_b$, with each $Z_i$ divided into two closed arcs $S_i^+$ and $S_i^-$ which overlap only at their endpoints. Consider a collection of pairwise-disjoint, embedded paths $\alpha_1, ..., \alpha_{2(g+b-1)}$ in $F$ with $\partial \alpha_i \subset \cup_i S_i^-$, such that $F \setminus (\cup_i \alpha_i)$ is a union of disks, and the boundary of each such disk contains exactly one $S_i^+$. Assume we also have a basepoint $z_i$ in each $S_i^+$.

Putting $\{a_i, a_i'\} = \partial \alpha_i$, we call the tuple
\[ \mathcal{Z} := ((Z_1, ..., Z_b), \{a_1, a_1'\}, ..., \{a_{2(g+b-1)}, a_{2(g+b-1)'}\}, (z_1, ..., z_b)) \]
an arc diagram for $F$.

Associated to $\mathcal{Z}$ we have an algebra $\mathcal{B}(\mathcal{Z})$ over $k = \bigoplus_{i=1}^{2(g+b-1)} \mathbb{F}_2$ with a basis over $\mathbb{F}_2$ consisting of one element $I_i$ for each pair $\{a_i, a_i'\}$ and one element $\rho$ for each chord, i.e. nontrivial interval in $Z_i \setminus \{z_i\}$ with endpoints in $\{a_1, a_1', ..., a_{2(g+b-1)}, a_{2(g+b-1)}'\}$. For each chord $\rho$ we denote the initial and terminal points (with respect to the orientation of $Z_i$) by $\rho^-$ and $\rho^+$ respectively.

The product on $\mathcal{B}(\mathcal{Z})$ is defined as follows:

- The $I_i$'s are orthogonal idempotents, i.e. $I_i I_j = \delta_{ij} I_i$.
- $I_i \rho = \rho$ if $a_i$ or $a_i'$ is $\rho^-$, and $I_i \rho = 0$ otherwise. Similarly, $\rho I_j = \rho$ if $a_j$ or $a_j'$ is $\rho^+$, and $\rho I_j = 0$ otherwise.
- For chords $\rho$ and $\rho'$, $\rho \rho'$ is the chord from $\rho^-$ to $(\rho')^+$ if $\rho^+ = (\rho')^-$ and $\rho \rho' = 0$ otherwise.

Note that the sum of the idempotents $1_{\mathcal{B}(\mathcal{Z})} := \sum_i I_i$ provides a multiplicative identity for $\mathcal{B}(\mathcal{Z})$.

From $\mathcal{Z}$ we can build a surface $F^\circ(\mathcal{Z})$ by thickening each boundary circle $Z_i$ of $\mathcal{Z}$ to an annulus $Z_i \times [0, 1]$ and then attaching 2-dimensional 1-handles to each pair $\{a_i, a_i'\}$ in the inner boundaries of the annuli. The surface $F(\mathcal{Z})$ obtained by
cutting along each \( z_i \times [0, 1] \subset Z_i \times [0, 1] \) is canonically identified with \( F \) (up to isotopy).

Associated to each \( Z \) is a dual arc diagram \( Z' \) coming from \( \{ \eta_i \cap \partial F(Z) \} \), where \( \{ \eta_i \} \) is the (unique up to isotopy) dual set of curves in \( F^\circ(Z) \) such that

- \( \eta_i \) is contained in the handle of \( F^\circ(Z) \) corresponding to \( \alpha_i \) and
- \( \eta_i \) intersects \( \alpha_i \) in a single point.

From \( Z' \) we get an algebra \( B(Z') \).

### 2.3. The bimodule \( M(\phi) \)

We now wish to define a \( B(Z') - B(Z) \cdot A_\infty \)-bimodule \( M(\phi) \) associated to each mapping class \( \phi \in \text{MCG}_0(F(Z)) \). Here \( \text{MCG}_0(F) \) is the group of isotopy classes of diffeomorphisms of \( F \) which fix the boundary of \( F \) pointwise. Let \( \phi_0 \) be a representative of \( \phi \), and let \( \phi_0 \) act on the \( \eta \)-curves \( \eta_1, ..., \eta_{2(g+b-1)} \) of \( F^\circ(Z) \), giving a new set of curves \( \beta_1, ..., \beta_{2(g+b-1)} \). We put \( \alpha = \alpha_1 \cup ... \cup \alpha_{2(g+b-1)} \) and \( \beta = \beta_1 \cup ... \cup \beta_{2(g+b-1)} \) and let

\[
D(\phi_0) := (F^\circ(Z), \alpha, \beta).
\]

We can always assume that \( \alpha \) and \( \beta \) intersect transversally.

In order to define \( M(\phi) \), we will first need the notion of polygons in \( D(\phi_0) \). Let

\[
\mathbb{D}^2 := \{ x + iy \in \mathbb{C} \mid x \geq 0, x^2 + y^2 \leq 1 \}
\]

\[
\gamma_L := \partial \mathbb{D}^2 \cap \{ x + iy \in \mathbb{C} \mid x = 0 \}
\]

\[
\gamma_R := \partial \mathbb{D}^2 \cap \{ x + iy \in \mathbb{C} \mid x \geq 0 \}.
\]

Note the non-standard definition of \( \mathbb{D}^2 \). We orient \( \gamma_L \) and \( \gamma_R \) from \(-i\) to \(i\) and refer to them as the left and right sides of \( \mathbb{D}^2 \) respectively.

**Definition 2.3.** Let \( \sigma_1, ..., \sigma_m \) and \( \rho_1, ..., \rho_n \) be chords in \( Z \) and \(-Z'\) (the orientation reversal of \( Z' \)) respectively, and let \( x, y \in \alpha \cap \beta \) be intersection points in \( D(\phi_0) \). A polygon in \( D(\phi_0) \) connecting \( x \) to \( y \) through \( (\sigma_1, ..., \sigma_m) \) and \( (\rho_1, ..., \rho_n) \) is a map \( P : \mathbb{D}^2 \rightarrow D(\phi_0) \) such that

- \( P(\gamma_L) \subset (\beta \cup \partial D(\phi_0)) \) and \( P(\gamma_R) \subset (\alpha \cup \partial D(\phi_0)) \).
We allow $\gamma$ be equivalent to a polygon and its image when no confusion will arise.

sides of $\gamma$ through $(x_0, y_0, x_1, y_1)$. By abuse of notation we will often not distinguish between a polygon and its image when no confusion will arise.

We are only interested in equivalence classes of polygons, where we define $P$ to be equivalent to $P'$ if and only if $P$ and $P'$ are topologically equivalent. We show that the definition given in the previous section does indeed make $M$ well-defined up to $\sim$-homotopy equivalence.

Remark 2.5. Because of the last paragraph above, we say that $M$ is strictly unital.

Since the $A_\infty$-homotopy type of $M(\phi_0)$ depends only on the isotopy class of $\phi_0$ (see Section 3.2 below), we will write $M(\phi) = M(\phi_0)$.

3. Some Preliminary Results

In this section we establish two fundamental facts about $M(\phi)$. In Section 3.1 we show that the definition given in the previous section does indeed make $M(\phi_0)$ into an $A_\infty$-bimodule, and in Section 3.2 we show that $M(\phi_0)$ is $A_\infty$-homotopy invariant under isotopies of $\phi_0$ (this is made precise below), and therefore $M(\phi)$ is well-defined up to $A_\infty$-homotopy equivalence.

3.1. Checking the $A_\infty$ relations for $M(\phi_0)$.

Theorem 3.1. $M(\phi_0)$ is a $B(Z')-B(Z)$ $A_\infty$-bimodule.
Proof. Since $\mathcal{B}(\mathcal{Z}')$ and $\mathcal{B}(\mathcal{Z})$ are in fact ordinary associative algebras (a much simplified case of $\mathcal{A}_\infty$-algebras), many of the terms in the structure equations drop out. In fact, the $\mathcal{A}_\infty$ relations (2.2) can be written as

$$0 = \sum_{k=0}^{i} \sum_{l=0}^{j} m(\sigma_i, \ldots, \sigma_{k+1}, m(\sigma_k, \ldots, \sigma_1, x, \rho_1, \ldots, \rho_l), \rho_{l+1}, \ldots, \rho_j)$$

$$+ \sum_{k=1}^{i-1} m(\sigma_i, \ldots, \sigma_{k+2}, \mu_2(\sigma_{k+1}, \sigma_k), \sigma_{k-1}, \ldots, \sigma_1, x, \rho_1, \ldots, \rho_j)$$

$$+ \sum_{l=1}^{j-1} m(\sigma_i, \ldots, \sigma_1, x, \rho_1, \ldots, \rho_{l-1}, \mu_2(\rho_l, \rho_{l+1}), \rho_{l+2}, \ldots, \rho_j)$$

(3.1)

for each $x \in M(\phi_0), \sigma_1, \ldots, \sigma_i \in \mathcal{B}(\mathcal{Z}'), \rho_1, \ldots, \rho_j \in \mathcal{B}(\mathcal{Z})$.

To establish (3.1), we will show that the nontrivial summands in the expansion of the right side cancel pairwise. We first introduce some notation. Let $S^L_1$ denote the set of polygons $\mathbb{D}^2 \to \mathcal{D}(\phi_0)$ starting at $x$ through $(\sigma_1, \ldots, \sigma_{k-1}, \mu_2(\sigma_{k+1}, \sigma_k), \sigma_{k+2}, \ldots, \sigma_1)$ and $(\rho_1, \ldots, \rho_i)$ for some $1 \leq k \leq i - 1$, and let $S^R_1$ denote the set of polygons $\mathbb{D}^2 \to \mathcal{D}(\phi_0)$ starting at $x$ through $(\sigma_1, \ldots, \sigma_i)$ and $(\rho_1, \ldots, \rho_{l+1}, \mu_2(\rho_l, \rho_{l+1}), \rho_{l+2}, \ldots, \rho_j)$ for some $1 \leq l \leq j - 1$. For a polygon $P$, let $\text{IP}(P)$ and $\text{TP}(P)$ denote its initial point and terminal point respectively, let $\text{Dom}(P)$ denote the domain\footnote{In this paper the word “domain” is always synonymous with “source”; this differs from the terminology elsewhere in the Heegaard Floer literature.} of $P$, and let $\gamma_L(\text{Dom}(P))$ and $\gamma_R(\text{Dom}(P))$ denote the left and right sides of $\text{Dom}(P)$ respectively. Let $S_2$ denote the set of pairs $(P_1, P_2)$ of polygons $\mathbb{D}^2 \to \mathcal{D}(\phi_0)$, where $P_1$ is a polygon starting at $x$ through $(\sigma_1, \ldots, \sigma_k)$ and $(\rho_1, \ldots, \rho_i)$ and $P_2$ is a polygon starting at $\text{TP}(P_1)$ through $(\sigma_{k+1}, \ldots, \sigma_1)$ and $(\rho_{l+1}, \ldots, \rho_j)$, for some $0 \leq k \leq i$, $0 \leq l \leq j$. Set

$$S := S^L_1 \cup S^R_1 \cup S_2.$$

It will suffice to construct a fixed point free involution $I: S \to S$.

To begin, suppose $(P_1, P_2) \in S_2$. Since $\text{TP}(P_1) = \text{IP}(P_2)$, there is an embedding $s_1: [0, 1] \to \partial \text{Dom}(P_1)$ with $s_1(0) = i$ and an embedding $s_2: [0, 1] \to \partial \text{Dom}(P_2)$ with $s_2(0) = -i$, such that $P_1 \circ s_1 = P_2 \circ s_2$. We choose $s_1$ and $s_2$ to be maximal in length. Let

$$P_G : \text{Dom}(P_1) \cup_{s_1 \sim s_2} \text{Dom}(P_2) \to \mathcal{D}(\phi_0)$$

denote the map obtained by gluing together $P_1$ and $P_2$ along $s_1$ and $s_2$. Let $p \in \text{Dom}(P_G)$ denote the point $s_1(1) = s_2(1)$. Then $P_G(p)$ is either an intersection point of an $\alpha$ arc and a $\beta$ arc, or else an endpoint of an $\alpha$ or $\beta$ arc. In the former case, $p$ is either $i \in \text{Dom}(P_2)$ or $-i \in \text{Dom}(P_1)$. We will define $I$ in steps as follows:

- **Step 1** defines $I((P_1, P_2))$ for $(P_1, P_2) \in S_2$ when $p = i \in \text{Dom}(P_2)$.
- **Step 2** defines $I((P_1, P_2))$ for $(P_1, P_2) \in S_2$ when $p = -i \in \text{Dom}(P_1)$.
- **Step 3** defines $I((P_1, P_2))$ for $(P_1, P_2) \in S_2$ when $P_G(p)$ is an intersection point of two adjacent short chords.
Step 1: Note that $p$ lies either on a) the left side of $\text{Dom}(P_1)$ or b) the right side of $\text{Dom}(P_1)$. In case a) (resp. b)) there is an embedding $s_3 : [0,1] \to \text{Dom}(P_G)$ with $s_3((0,1)) \subset \text{Int}(\text{Dom}(P_G))$, $s_3(0) = p$, and $s_3(1) \in \gamma_L(\text{Dom}(P_1)) \setminus \gamma_L(\text{Dom}(P_2))$ (resp. $s_3(1) \in \gamma_R(\text{Dom}(P_1)) \setminus \gamma_R(\text{Dom}(P_2))$), such that $P_G \circ s_3$ lies along an $\alpha$ arc (resp. $\beta$ arc). We observe that $s_3$ cuts $\text{Dom}(P_G)$ into two pieces, and the restrictions of $P_G$ to these pieces define polygons $P'_1$ from $\text{IP}(P_1)$ to $P_G(s_3(1))$ and $P'_2$ from $P_G(s_3(1))$ to $\text{TP}(P_2)$. One easily checks that $(P'_1, P'_2) \in S_2$ and $(P'_1, P'_2) \neq (P_1, P_2)$, and we define $I((P_1, P_2)) := (P'_1, P'_2)$. See Figure 2.

Step 2: Similar to Step 1, $p$ lies either a) on the left side of $\text{Dom}(P_2)$ or b) the right side of $\text{Dom}(P_2)$. In case a) (resp. b)) there is an embedding $s_3 : [0,1] \to \text{Dom}(P_G)$ with $s_3((0,1)) \subset \text{Int}(\text{Dom}(P_G))$, $s_3(0) = p$, and $s_3(1) \in \gamma_L(\text{Dom}(P_2)) \setminus \gamma_L(\text{Dom}(P_1))$, such that $P_G \circ s_3$ lies along an $\alpha$ arc (resp. $\beta$ arc). We observe that $s_3$ cuts $\text{Dom}(P_G)$ into two pieces, and the restrictions of $P_G$ to these pieces define polygons $P'_1$ from $\text{IP}(P_1)$ to $P_G(s_3(1))$ and $P'_2$ from $P_G(s_3(1))$ to $\text{TP}(P_2)$. Again, one easily checks that $(P'_1, P'_2) \in S_2$ and $(P'_1, P'_2) \neq (P_1, P_2)$, and we define $I((P_1, P_2)) := (P'_1, P'_2)$. See Figure 2.

Step 3: In this case $p$ lies on either a) the left side of $\text{Dom}(P_1)$ or b) the right side of $\text{Dom}(P_1)$. In case a) $P_G$ itself defines a polygon $I((P_1, P_2))$ in $S^L_1$, whereas in case b) $P_G$ defines a polygon $I((P_1, P_2))$ in $S^R_1$.

Step 4: Let $P \in S^L_1$ be a polygon through $(\sigma_1, ..., \sigma_{k-1}, \mu_2(\sigma_{k+1}, \sigma_k), \sigma_{k+2}, ..., \sigma_i)$ and $(\rho_1, ..., \rho_j)$. Let $p \in \gamma_L(P)$ be the point corresponding to $\sigma_k \cap \gamma_{k+1}$. There must be an embedding $s_3 : [0,1] \to \text{Dom}(P)$ with $s_3((0,1)) \subset \text{Int}(\text{Dom}(P))$, $s_3(0) = p$, and $s_3(1) \in \gamma_R(\text{Dom}(P))$. We can assume that $s_3(1)$ lies between $(\rho_1, ..., \rho_i)$ and $(\rho_{i+1}, ..., \rho_j)$. Then $s_3$ cuts $\text{Dom}(P)$ into two pieces, and the restrictions of $P$ to these pieces define polygons $P'_1$ from $\text{IP}(P)$ to $P(s_3(1))$ through $(\sigma_1, ..., \sigma_k)$ and $(\rho_1, ..., \rho_i)$ and $P'_2$ from $P(s_3(1))$ to $\text{TP}(P)$ through $(\sigma_{k+1}, ..., \sigma_i)$ and $(\rho_{i+1}, ..., \rho_j)$. We define $I(P) := (P'_1, P'_2) \in S_2$.

$I(P) \in S_2$ is defined similarly when $P \in S^R_1$.

We have now defined $I$ on all of $S$. It follows immediately that $I$ is fixed point free, and we leave it to the reader to verify that $I$ is an involution.

\[ \Box \]

3.2. Homotopy invariance of $M(\phi_0)$ under isotopies. Recall that for ordinary chain complexes $(M,d)$ and $(M',d')$, a chain map is a linear map $f : M \to M'$ such that $d' \circ f = f \circ d$, i.e. $f$ commutes with the differential. Two chain maps $f,f' : M \to M'$ are called homotopic if there exist a linear map $h : M \to M'$ such that $f - f' = d' \circ h + h \circ d$. Defining a differential $\partial$ on linear maps $h : M \to M'$ by $\partial(h) = d' \circ h + h \circ d$, we see that a linear map $f$ is a chain map if $\partial(f) = 0$, and linear maps $f$ and $f'$ are homotopic if $f - f' \in \text{im}(\partial)$. Observe that $(\text{Hom}(M,M'),\partial)$ is itself a chain complex, since we have

\[ (\partial \circ \partial)(h) = \partial(d' \circ h + h \circ d) = d' \circ d' \circ h + d' \circ h \circ d + d' \circ h \circ d + h \circ d \circ d = 0. \]
Recall that a chain map $f$ is called a homotopy equivalence if there exists a chain map $g : M' \rightarrow M$ such that $g \circ f$ is homotopic to $I_M$ and $f \circ g$ is homotopic to $I_{M'}$.

We now proceed to define analogous terms for the $A_\infty$ case, which will allow us to make precise the statement that $\mathcal{M}(\phi_0)$ is homotopy invariant under isotopies of $\phi_0$. Let $\mathcal{A}$ and $\mathcal{B}$ be $A_\infty$-algebras over $k$ and let $\mathcal{A}M_B$ and $\mathcal{A}M'_B$ be $A_\infty$-bimodules over $\mathcal{A}$ and $\mathcal{B}$. The analogues of linear maps are called morphisms.

**Definition 3.2.** A morphism $\mathcal{A}M_B \rightarrow \mathcal{A}M'_B$ is a collection of $k$-linear maps

$$f_{i,j} : A^{\otimes i} \otimes M \otimes B^{\otimes j} \rightarrow M'$$

for $i, j \geq 0$.

Let $f$ be the total map $f : T^*(A) \otimes M \otimes T^*(B) \rightarrow M'$. With the notation of Section 2.1, we define a differential on morphisms by

$$\partial(f) = \Delta_2 \downarrow f \downarrow \Delta_2 + \Delta_2 \downarrow m \downarrow \Delta_2 + \overline{D}^A \downarrow f \downarrow \overline{D}_B.$$

This makes $(\text{Mor}(\mathcal{A}M_B, \mathcal{A}M'_B), \partial)$ into a chain complex. The $A_\infty$ analogue of chain maps are morphisms $f$ such that $\partial(f) = 0$, and we call these $A_\infty$-homomorphisms. We say that morphisms $f$ and $f'$ are $A_\infty$-homotopic if $f - f' \in \text{im}(\partial)$. Given
another morphism \( g : \mathcal{A}M'_B \rightarrow \mathcal{A}M''_B \), define

\[
\begin{align*}
g \circ f &= \Delta_2 \quad (3.3)
\end{align*}
\]

We say that an \( \mathcal{A}_\infty \)-homomorphism \( f \) is an \( \mathcal{A}_\infty \)-homotopy equivalence if there exists some \( \mathcal{A}_\infty \)-homomorphism \( g \) such that \( g \circ f \) is \( \mathcal{A}_\infty \)-homotopic to \( I_M \) and \( f \circ g \) is \( \mathcal{A}_\infty \)-homotopic to \( I_M' \). Here \( I_M \) is the morphism \( \mathcal{A}M_B \rightarrow \mathcal{A}M_B \) with \( \mathbb{I}_{\mathcal{A}M_B}(a_1 \otimes \ldots \otimes a_i \otimes x \otimes b_1 \otimes \ldots \otimes b_j) \) defined to be \( x \) if \( i = j = 0 \) and zero otherwise (and \( \mathbb{I}_{\mathcal{A}M_B}' \) is defined similarly).

**Theorem 3.3.** Let \( \varphi_b \) and \( \varphi_{nb} \) be diffeomorphisms representing the same element of \( \text{MCG}_0(F) \). Then \( M(\varphi_b) \) and \( M(\varphi_{nb}) \) are homotopy equivalent as \( \mathcal{A}_\infty \)-bimodules.

The main ingredient of the proof will be Lemma 8.6 of [5], suitably modified for \( \mathcal{A}_\infty \)-bimodules:

**Lemma 3.4.** Let \( \mathcal{A}M_B \) be an \( \mathcal{A}_\infty \)-bimodule over \( \mathcal{A}_\infty \)-algebras \( \mathcal{A}, \mathcal{B} \) over \( k \), let \( M \) denote its underlying chain complex over \( k \), and let \( \tilde{f}_1 : N \rightarrow M \) be a homotopy equivalence of chain complexes. Then we can find

- an \( \mathcal{A}_\infty \)-bimodule structure \( \mathcal{A}N_B \) on \( N \) and
- an \( \mathcal{A}_\infty \)-homotopy equivalence \( F : \mathcal{A}N_B \rightarrow \mathcal{A}M_B \) with the property that \( F_1 = \tilde{f}_1 \).

Moreover, the structure map \( \overline{\mu} \) of \( \mathcal{A}N_B \) is given as follows. Let \( \tilde{g}_1 : M \rightarrow N \) be a homotopy inverse to \( \tilde{f}_1 \) and let \( T : M \rightarrow M \) be a homotopy between \( \tilde{f}_1 \circ \tilde{g}_1 \) and \( \mathbb{I}_M \). For \( a_1, \ldots, a_m \in A \), \( b_1, \ldots, b_n \in B \), and \( r \geq 1 \), let

\[
\begin{align*}
\mu_r^* ((a_1 \otimes \ldots \otimes a_m) \otimes (b_1 \otimes \ldots \otimes b_n)) := \\
\sum ((a_{i_0+1} \otimes \ldots \otimes a_{i_1}) \otimes (a_{i_1+1} \otimes \ldots \otimes a_{i_2}) \otimes \ldots \otimes (a_{i_{r-1}+1} \otimes \ldots \otimes a_{i_r})) \\
\otimes ((b_{j_0+1} \otimes \ldots \otimes b_{j_1}) \otimes (b_{j_1+1} \otimes \ldots \otimes b_{j_2}) \otimes \ldots \otimes (b_{j_{r-1}+1} \otimes \ldots \otimes b_{j_r})),
\end{align*}
\]

where the sum is over all \( 0 = i_0 \leq i_1 \leq \ldots \leq i_r = m \), \( 0 = j_0 \leq j_1 \leq \ldots \leq j_r = n \) such that \( (i_k, j_k) \neq (i_{k-1}, j_{k-1}) \) for \( 1 \leq k \leq r \). Let \( a = (a_1 \otimes \ldots \otimes a_m) \) and
\( b = (b_1 \otimes \ldots \otimes b_n) \). Then \( \overline{m} \) is given by

\[
\overline{m}(a \otimes y \otimes b) = a \rightarrow b + a \leftarrow b + a \rightarrow b + \ldots
\]

(3.4)

By a standard result of curves on surfaces (see for example [7]), \( D(\phi_b) \) and \( \mathcal{D}(\phi_{nb}) \) differ by a sequence of finger moves, i.e. isotopies adding or removing an innermost bigon. Then it suffices to consider the case that \( \mathcal{D}(\phi_{nb}) \) is obtained from \( D(\phi_b) \) by removing an innermost bigon \( B \). In particular we assume that an isotopy between \( D(\phi_b) \) and \( \mathcal{D}(\phi_{nb}) \) is the identity outside of a small neighborhood of \( B \), and that the local picture inside the neighborhood is as in Figure 3. It will also be convenient to further assume that the symmetric difference between the \( \beta \) curves of \( D(\phi_b) \) and those of \( \mathcal{D}(\phi_{nb}) \) has exactly two components, i.e. there are no superfluous bigons created between the \( \beta \) curves of the two diagrams.

Let \( \{x_1, \ldots, x_n\} \) denote the canonical basis for \( M(\phi_b) \), where the bigon \( B \) to be removed is from \( x_1 \) to \( x_2 \). We will write \( x_i \rightarrow x_j \) if \( x_j \) is a summand of \( m_1(x_i) \) (in the basis implied by the context). We begin by introducing a new basis to isolate \( B \). Specifically, let

\[
\begin{align*}
\tilde{x}_1 & := x_1 \\
\tilde{x}_2 & := m_1(x_1) \\
\tilde{x}_i & := x_i + x_1 \quad \text{whenever } x_i \rightarrow x_2 \quad \text{(for } 3 \leq i \leq n) \\
\tilde{x}_i & := x_i \quad \text{otherwise}
\end{align*}
\]

The reader can easily verify that \( \{\tilde{x}_1, \ldots, \tilde{x}_n\} \) again forms a basis, and that in this basis the only differential relation involving \( \tilde{x}_1 \) or \( \tilde{x}_2 \) is \( m_1(\tilde{x}_1) = \tilde{x}_2 \). That is, representing the chain complex of \( M(\phi_b) \) as a directed graph as in Figure 4, there are no arrows entering or leaving \( \tilde{x}_1 \) or \( \tilde{x}_2 \) except for a single arrow \( \tilde{x}_1 \rightarrow \tilde{x}_2 \).
Figure 3. A “finger move” showing how $\mathcal{D}(\phi_b)$ and $\mathcal{D}(\phi_{nb})$ are related.

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$x_1$};
\node (B) at (1,0) {$x_2$};
\node (C) at (2,0) {$x_3$};
\node (D) at (3,0) {$x_4$};
\node (E) at (4,0) {$x_5$};
\node (F) at (0,-1) {$x_1$};
\node (G) at (1,-1) {$x_2$};
\node (H) at (2,-1) {$x_3$};
\node (I) at (3,-1) {$x_4$};
\node (J) at (4,-1) {$x_5$};
\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (D) -- (E);
\draw[->] (F) -- (G);
\draw[->] (G) -- (H);
\draw[->] (H) -- (I);
\draw[->] (I) -- (J);
\node at (2,0) {$\triangle$};
\node at (2,-1) {$\triangle$};
\end{tikzpicture}
\end{center}

Figure 4. Changing basis in a chain complex to isolate a bigon. The arrows here represent the differential.

Now let $\{y_3, ..., y_n\}$ denote the canonical basis for $M(\phi_{nb})$, and observe that we have a natural correspondence $y_i \leftrightarrow x_i$ between the generators for $i \geq 3$. We define maps $f_1 : M(\phi_{nb}) \to M(\phi_b)$ and $g_1 : M(\phi_b) \to M(\phi_{nb})$ by

\[
f_1(y_i) := x_i \quad 3 \leq i \leq n \quad \quad g_1(x_i) := y_i \quad 3 \leq i \leq n \quad \quad g_1(x_i) := 0 \quad i = 1, 2.
\]

We also define maps $\tilde{f}_1 : M(\phi_{nb}) \to M(\phi_b)$ and $\tilde{g}_1 : M(\phi_b) \to M(\phi_{nb})$ by

\[
\tilde{f}_1(y_i) := \tilde{x}_i \quad 3 \leq i \leq n \quad \quad \tilde{g}_1(\tilde{x}_i) := y_i \quad 3 \leq i \leq n \quad \quad \tilde{g}_1(\tilde{x}_i) := 0 \quad i = 1, 2.
\]

Lemma 3.5. The maps $\tilde{f}_1$ and $\tilde{g}_1$ are inverse homotopy equivalences of the underlying chain complexes of $M(\phi_{nb})$ and $M(\phi_b)$.

Before proving this lemma, it will be useful to give a general result relating polygons in $\mathcal{D}(\phi_{nb})$ to polygons in $\mathcal{D}(\phi_b)$. Let $S_{i}^{(\sigma_1, ..., \sigma_m), (\rho_1, ..., \rho_n)}(x_i, x_j)$ denote the set of polygons in $\mathcal{D}(\phi_b)$ from $x_i$ to $x_j$ through $(\sigma_1, ..., \sigma_m)$ and $(\rho_1, ..., \rho_n)$. For $r \geq 2$, let $S_{i}^{(\sigma_1, ..., \sigma_m), (\rho_1, ..., \rho_n)}(x_i, x_j)$ denote the set of tuples $(P_1, ..., P_r)$ of polygons in $\mathcal{D}(\phi_b)$ such that

- $P_1$ is a polygon, distinct from $B$, from $x_i$ to $x_2$ through $(\sigma_{s_0+1}, ..., \sigma_{s_1})$ and $(\rho_{t_0+1}, ..., \rho_{t_1})$, 

Lemma 3.6. There is a natural bijective correspondence between the elements of $S(\sigma_1, \ldots, \sigma_m) \cdot (\rho_1, \ldots, \rho_n)(x_i, x_j)$ and polygons in $D(\phi_{nb})$ from $y_i$ to $y_j$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\rho_1, \ldots, \rho_n)$ (for $i, j \geq 3$).

Proof. For a polygon $P : \mathbb{D}^2 \to D(\phi_{nb})$ from $y_i$ to $y_j$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\rho_1, \ldots, \rho_n)$, let $P' : \mathbb{D}^2 \to D(\phi_b)$ be the corresponding map into $D(\phi_b)$. Observe that the components of $(P')^{-1}(\beta)$ intersecting the left side of $\text{Dom}(P')$ cut $\text{Dom}(P')$ into various connected domains (here we are implicitly relying on the symmetric difference assumption above to avoid unnecessary “ripples” between the $\beta$ curves). Let $D_1, \ldots, D_r$ denote those domains whose boundaries map entirely into $\alpha \cup \beta \cup \partial D(\phi_b)$. Then the restrictions $(P'|_{D_1}, \ldots, P'|_{D_r})$ define an element of $S(\sigma_1, \ldots, \sigma_m) \cdot (\rho_1, \ldots, \rho_n)(x_i, x_j)$.

Conversely, suppose $(P_1, \ldots, P_r) \in S(\sigma_1, \ldots, \sigma_m) \cdot (\rho_1, \ldots, \rho_n)(x_i, x_j)$. Superimposing $D(\phi_b)$ and $D(\phi_{nb})$, there is a rectangular region $R$ which has three $\beta$ sides and one $\alpha$ side, where the $\alpha$ side is also a side of $B$. There is a natural way to glue a copy of $R$ along one of its sides to a segment of the left side of $P_k$, and along another side to a segment of the left side of $P_{k+1}$, for $1 \leq k < r$, to obtain a polygon

$P : \text{Dom}(P_1) \cup R \cup \text{Dom}(P_2) \cup \ldots \cup R \cup \text{Dom}(P_r) : \to D(\phi_{nb})$

from $g_1(\text{IP}(P_1))$ to $g_1(\text{TP}(P_r))$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\rho_1, \ldots, \rho_n)$.

The reader can easily check that these operations are inverses. □

Suppose for each $i$ we have

$m_1(x_i) = a^i_1 x_1 + a^i_2 x_2 + \ldots + a^i_n x_n,$

where $a^i_k \in \mathbb{F}_2$ for $1 \leq k \leq n$.

Corollary 3.7. For $i \geq 3$, we have

$m_1(y_i) = g_1(m_1(x_i)) + a^i_2 g_1(m_1(x_1)).$

Proof. This follows from the fact that $S_r^{(i)}(x_i, x_j)$ is the empty set for $r \geq 3$ since $B$ is the unique bigon from $x_1$ to $x_2$. The first term comes from $S_1^{(i)}(x_i, x_j)$ and the second term comes from $S_2^{(i)}(x_i, x_j)$. □
since we have only added an even number of \(x_1\)'s. Then we have
\[
m_1(\tilde{x}_i) = m_1(x_i + a^i_1x_1) = a^i_2(x_2 + m_1(x_1)) + a^i_3\tilde{x}_3 + a^i_4\tilde{x}_4 + \ldots + a^i_n\tilde{x}_n
\]
and therefore
\[
\tilde{g}_1(m_1(\tilde{x}_i)) = a^i_2\tilde{g}_1(x_2 + m_1(x_1)) + a^i_3y_3 + a^i_4y_4 + \ldots + a^i_ny_n
= a^i_2\tilde{g}_1(a^i_1x_1 + a^i_3x_3 + a^i_4x_4 + \ldots + a^i_nx_n) + g_1(m_1(x_i))
= a^i_2\tilde{g}_1(a^i_1[x_1 + x_1] + a^i_3\tilde{x}_3 + a^i_4\tilde{x}_4 + \ldots + a^i_n\tilde{x}_n) + g_1(m_1(x_i))
= a^i_2(a^i_1y_3 + a^i_4y_4 + \ldots + a^i_ny_n) + g_1(m_1(x_i))
= a^i_2g_1(m_1(x_i)) + g_1(m_1(x_i))
= m_1(\tilde{g}_1(\tilde{x}_i)),
\]
where the third line follows from another evenness argument and the last line follows from Corollary 3.7. Then evidently \(\tilde{g}_1 \circ m_1 = m_1 \circ \tilde{g}_1\), and it easily follows from the definitions of \(\tilde{f}_1\) and \(\tilde{g}_1\) that \(\tilde{f}_1 \circ m_1 = m_1 \circ \tilde{f}_1\).

Since \(\tilde{g}_1 \circ \tilde{f}_1\) is the identity, to complete the proof it suffices to find a homotopy \(T : M(\phi_h) \to M(\phi_h)\) from \(\tilde{f}_1 \circ \tilde{g}_1 - 1\) to \(T \circ m_1 + m_1 \circ T\). Since \(\tilde{f}_1 \circ \tilde{g}_1 - 1\) fixes \(\tilde{x}_1\) and \(\tilde{x}_2\) and sends every other \(\tilde{x}_i\) to zero, one easily checks that \(T\) can be chosen to be the map sending \(\tilde{x}_2\) to \(\tilde{x}_1\) and sending \(\tilde{x}_1, \tilde{x}_3, \ldots, \tilde{x}_n\) to zero.

We are now finally in a position to prove Theorem 3.3. To aid the proof, let \(\#x_k : M(\phi_h) \to \mathbb{F}_2\) and \(\#y_k : M(\phi_{nb}) \to \mathbb{F}_2\) be functions counting the number of \(x_k\) and \(y_k\) summands respectively of the expansion of the input in the corresponding basis.

**Proof of Theorem 3.3.** By Lemma 3.4, \(M(\phi_h)\) is \(\mathcal{A}_\infty\)-homotopy equivalent to an induced \(\mathcal{A}_\infty\) structure \(\overline{m}\) on the underlying chain complex of \(M(\phi_{nb})\), and therefore it will suffice to show that the map \(\overline{m}\) is the same as the usual structure map \(m\) for \(M(\phi_{nb})\). In fact, since \(\overline{m}\) is given by (3.4), where \(T\) is the map defined at the end of the proof of Lemma 3.5 sending \(\tilde{x}_2\) to \(\tilde{x}_1\) and \(\tilde{x}_1, \tilde{x}_3, \ldots, \tilde{x}_n\) to zero, unwinding the diagram shows that we have
\[
\overline{m}(\sigma_1, \ldots, \sigma_m, y_i, \rho_1, \ldots, \rho_n) = \sum_{j=3}^n y_j \#y_j(\tilde{g}_1(m(\sigma_1, \ldots, \sigma_m, \tilde{x}_i, \rho_1, \ldots, \rho_n)))
\]
\[
+ \sum_{j=3}^n y_j \sum_{r=0}^n \#\tilde{x}_2^{m(\sigma_{s_0+1}, \ldots, \sigma_{s_1}, \tilde{x}_i, \rho_{t_0+1}, \ldots, \rho_{t_1}))} \#\tilde{x}_2^{m(\sigma_{s_1+1}, \ldots, \sigma_{s_2}, \tilde{x}_1, \rho_{t_1+1}, \ldots, \rho_{t_2}))} \cdots \#\tilde{x}_2^{m(\sigma_{s_{r-2}+1}, \ldots, \sigma_{s_{r-1}}, \tilde{x}_1, \rho_{t_{r-2}+1}, \ldots, \rho_{t_{r-1}}))} \#y_j(\tilde{g}_1(m(\sigma_{s_{r-1}+1}, \ldots, \sigma_{s_r}, \tilde{x}_1, \rho_{t_{r-1}+1}, \ldots, \rho_{t_r}))),
\]
where the last sum is over all \(r \geq 2\) and all \(0 = s_0 \leq s_1 \leq \ldots \leq s_r = m,\) \(0 = t_0 \leq t_1 \leq \ldots \leq t_r = n\) such that \((s_k, t_k) \neq (s_{k-1}, t_{k-1})\) for \(1 \leq k \leq r\).
Lemma 3.8. For any \( i, j \geq 3 \), \( \sigma = (\sigma_1, ..., \sigma_s) \), and \( \rho = (\rho_1, ..., \rho_t) \), we have

\[
\#_{y_j}(\hat{g}_1(m(\sigma, x_i, \rho))) = \\
\#_{x_j}(m(\sigma, x_i, \rho)) + \#_{x_j}(m(\sigma, x_1, \rho))\#_{x_i}(m(x_i)) + \\
\#_{x_j}(m(x_1))\#_{x_i}(m(\sigma, x_i, \rho)) + \#_{x_j}(m(\sigma, x_i, \rho))\#_{x_i}(m(x_i)).
\]

Proof. From the definitions of \( \hat{g}_1 \) and the basis \( \{\hat{x}_k\} \) (and in particular the fact that, for \( j \leq 3 \), \( \hat{x}_j \) appears only in the expansion of \( x_j \) and possibly in the expansion of \( x_2 \) if \( x_1 \to x_j \)), we have

\[
\#_{y_j}(\hat{g}_1(m(\sigma, x_i, \rho))) = \#_{\hat{x}_j}(m(\sigma, x_i, \rho)) \\
= \#_{x_j}(m(\sigma, x_i, \rho)) + \#_{x_j}(m(x_1))\#_{x_i}(m(\sigma, x_i, \rho)),
\]

as well as

\[
\hat{x}_i = x_i + \#_{x_2}(m(x_i))x_1,
\]

and these combine to give the desired result. \( \square \)

With a little work, we can now simplify (3.5) using the result from Lemma 3.8, as well as the facts \( \hat{x}_1 = x_1 \) and \( \#_{\hat{x}_2} = \#_{x_2} \), to obtain

\[
\overline{m}(\sigma_1, ..., \sigma_m, y_i, \rho_1, ..., \rho_n) = \sum_{j=3}^{n} y_j \#_{x_j}(m(\sigma_1, ..., \sigma_m, x_i, \rho_1, ..., \rho_n))
\]

\[
+ \sum_{j=3}^{n} y_j \sum \#_{x_2}(m(\sigma_{s_0+1}, ..., \sigma_{s_1}, x_i, \rho_{0+1}, ..., \rho_t))\#_{x_2}(m(\sigma_{s_1+1}, ..., \sigma_{s_2}, x_{i}, \rho_{1+1}, ..., \rho_{t_2})) ... \\
\#_{x_2}(m(\sigma_{s_{r-2}+1}, ..., \sigma_{s_{r-1}}, x_{1}, \rho_{t_{r-2}+1}, ..., \rho_{t_{r-1}}))\#_{x_j}(m(\sigma_{s_{r-1}+1}, ..., \sigma_{s_r}, x_1, \rho_{t_{r-1}+1}, ..., \rho_{t_r})),
\]

where the last sum is over all \( r \geq 2 \) and all \( 0 = s_0 \leq s_1 \leq ... \leq s_r = m \), \( 0 = t_0 \leq t_1 \leq ... \leq t_r = n \) such that \( (s_k, t_k) \neq (s_{k-1}, t_{k-1}) \) for \( 2 \leq k \leq r - 1 \) (note the difference in indexing from (3.5)). But Lemma 3.6 shows that this last sum also computes \( m \) (note that the condition \( (s_k, t_k) \neq (s_{k-1}, t_{k-1}) \) for \( 2 \leq k \leq r - 1 \) corresponds to the stipulation in the definition of \( S_{(\sigma_1, ..., \sigma_m), (\rho_1, ..., \rho_n)}(x_i, x_j) \) that the polygons be distinct from \( B \)), and this completes the proof. \( \square \)

4. The Main Result

In Section 4.1 we define type DD bimodules and DA bimodules, and in particular we define the type DD bimodule \( DD(\frac{1}{2}) \), which plays a key role. We also define the box product \( \boxtimes \), a kind of tensor product which can be performed on the various types of bimodules. In Section 4.2 we show how to obtain a representation of \( \mathcal{D}(\psi_0 \circ \phi_0) \) by gluing together \( \mathcal{D}(\phi_0), \mathcal{D}((I), \text{ and } \mathcal{D}(\psi_0) \). In Section 4.3 we prove the crucial ingredient for our mapping class group action, which is that \( M(\psi \circ \phi) \) and \( M(\phi) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi)) \) are \( \mathcal{A}_r \)-homotopy equivalent. Finally, in Section 4.4 we complete the proof that we have a faithful action.
4.1. Various types of bimodules. We have already defined type AA bimodules, which are just $A_\infty$-bimodules. We now define type DD and type DA bimodules.

Definition 4.1. Let $A = (A, \partial_A)$ and $B = (B, \partial_B)$ be differential algebras over $k$ (i.e. $\partial_A$ and $\partial_B$ are $k$-linear and satisfy the Leibniz rule) such that $\partial_A \circ \partial_A = 0$ and $\partial_B \circ \partial_B = 0$. A type DD bimodule $^A M^B$ over $A$ and $B$ is a $k$-bimodule $M$ and a map $\delta_{DD}^1 : M \to A \otimes M \otimes B$ such that the following compatibility condition holds:

\[
\delta_{DD}^1 + \partial_A \delta_{DD}^1 + \partial_B \delta_{DD}^1 = 0,
\]

where $\mu_2$ denotes algebra multiplication.

For later purposes we also define an auxiliary map $\delta_{DD} : M \to T^*(A) \otimes M \otimes T^*(B)$ by

\[
\delta_{DD} := \sum_{i \geq 0} \delta_{DD}^i
\]

Let $\delta_{DD}^0 = I$, let $\delta_{DD}^2 := (I \otimes \delta_{DD}^1 \otimes I) \circ \delta_{DD}^1$, and more generally define $\delta_{DD}^n$ inductively by $\delta_{DD}^n := (I \otimes \delta_{DD}^{n-1} \otimes I) \circ \delta_{DD}^1$. The above equation can then also be written as

\[
\delta_{DD} = \sum_{i \geq 0} \delta_{DD}^i
\]

Type DA bimodules, which we define presently, are a sort of hybrid of type DD and type AA bimodules.

Definition 4.2. Let $A$ and $B$ be $A_\infty$-algebras over $k$. A type DA bimodule $^A M^B$ over $A$ and $B$ consists of a $k$-bimodule $M$ and $k$-linear maps

\[
\delta_{DA}^{1,1+j} : M \otimes B^{\otimes j} \to A \otimes M
\]

satisfying a compatibility condition given as follows. Let $\delta_{DA}^1 : M \otimes T^*(B) \to A \otimes M$ be the total map given by $\delta_{DA}^1 = \sum_{i} \delta_{DA}^{1,i}$. Define a map $\delta_{DA} : M \otimes T^*(B) \to$
$T^*(A) \otimes M$ by

$$\delta_{DA} := \oplus \delta^1_{DA} \oplus \delta^1_{DA} \oplus \delta^1_{DA} \oplus \delta^1_{DA} \oplus \ldots$$

Here $\Delta_3 := (\Delta_2 \otimes \mathbb{I}) \circ \Delta_2$ (see Section 2.1), and in general $\Delta_n$ is defined inductively by $\Delta_n := (\Delta_{n-1} \otimes \mathbb{I}) \circ \Delta_2$.

The compatibility condition is then given by

$$\delta_{DA} \oplus \overline{D}^A = 0.$$

Remark 4.3. One can also define type AD bimodules, which are essentially the mirror images of type DA bimodules. Since we will not need these here, we omit them.

With these definitions at hand we now define the type DD bimodule $DD(\frac{1}{2})$. Call a chord in $B(\mathbb{Z})$ short if it connects adjacent points, and let $SC(\mathbb{Z})$ denote the set of short chords in $B(\mathbb{Z})$. For a short chord $\xi$ in $B(\mathbb{Z})$, there is a corresponding short chord $\xi'$ in $B(\mathbb{Z}')$ defined such that $\xi$ and $\xi'$ lie on the boundary of the same connected component of $D(\mathbb{I})$.

Definition 4.4. For each generator $x_i \in M(\mathbb{I})$, let $I(x_i)$ denote the corresponding idempotent in $B(\mathbb{Z})$ and let $J(x_i)$ denote the corresponding idempotent in $B(\mathbb{Z}')$. Let

$$DD\left(\frac{1}{2}\right) := \bigoplus_i B(\mathbb{Z})I(x_i) \otimes J(x_i)B(\mathbb{Z}').$$
We will abuse notation and let \(x_i\) denote a generator of the summand corresponding to \(x_i\). We define a differential by
\[
\partial(x_i) = \sum_j \sum_{\xi \in SC(Z)} I(x_i) \cdot \xi \cdot x_j \cdot \xi' \cdot J(x_i),
\]
and we extend this to all of \(DD(\frac{1}{2})\) by the Leibniz rule.

Next we define \(\boxtimes\) for bimodules. Specifically,
- \(DD \boxtimes AA\) is a type \(DA\) bimodule.
- \(AA \boxtimes DA\) is a type \(AA\) bimodule.

Remark 4.5. We can of course define \(\boxtimes\) for other combinations of bimodules as well, although we will not need them here.

Definition 4.6. Let \(^A\mathcal{M}^B\) be a \(DD\) bimodule and let \(^B\mathcal{N}^C\) be an \(AA\) bimodule. Then \((^A\mathcal{M}^B) \boxtimes (^B\mathcal{N}^C)\) as a \(k\)-bimodule is \(M \otimes_k N\), with type \(DA\) structure map \(\delta_{DA}\) given by

\[
\begin{aligned}
\delta_{DD} & \downarrow \quad \downarrow \\
\Pi & \downarrow \quad \downarrow \\
\delta_{DA} & \downarrow \quad \downarrow \quad \downarrow \\
\end{aligned}
\]

where \(\Pi\) denotes the algebra multiplication in \(A\).

Definition 4.7. Let \(^A\mathcal{M}^B\) be an \(AA\) bimodule and let \(^B\mathcal{N}^C\) be a \(DA\) bimodule. Then \((^A\mathcal{M}^B) \boxtimes (^B\mathcal{N}^C)\) as a \(k\)-bimodule is \(M \otimes_k N\), with type \(AA\) structure map given by

\[
\begin{aligned}
\delta_{DA} & \downarrow \quad \downarrow \\
\delta_{DD} & \downarrow \quad \downarrow \\
\Pi & \downarrow \quad \downarrow \\
\end{aligned}
\]

Remark 4.8. The product \(\boxtimes\) enjoys a number of desirable properties for tensor products. The reader is encouraged to consult Section 2.3 of [4] for a detailed treatment. We also mention here that \(\boxtimes\) is not strictly associative, which is why we must choose a parenthesization in Theorem 4.16.
4.2. The gluing construction for \( \mathcal{D}(\psi_0 \circ \phi_0) \). Since our goal is to relate \( M(\psi \circ \phi) \) to \( M(\phi) \) and \( M(\psi) \), we will need a way to relate polygons in \( \mathcal{D}(\psi_0 \circ \phi_0) \) to polygons in \( \mathcal{D}(\phi_0) \) and \( \mathcal{D}(\psi_0) \). We will think of \( \mathcal{D}(\phi_0) \) (or \( \mathcal{D}(\mathbb{I}) \) or \( \mathcal{D}(\psi_0) \)) as a space with distinguished \( \alpha \) and \( \beta \) curves on it, so that for example cutting or gluing \( \mathcal{D}(\phi_0) \) means cutting or gluing the \( \alpha \) and \( \beta \) arcs as well.

Recall that for a fixed arc diagram \( Z \) on a surface \( F \), \( F^c(\mathcal{Z}) \) is a surface with the same genus as \( F \) and twice as many boundary components. We can view \( \partial(F^c(\mathcal{Z})) \) as the union of \( \partial F(\mathcal{Z}) \) and \( \partial F(\mathcal{Z}') \), and we call these the \( \rho \) and \( \sigma \) boundaries respectively. Let \( \mathcal{D}(\phi_0) \cup_{\sigma} \mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \) be obtained from the disjoint union \( \mathcal{D}(\phi_0) \mathbb{I} - \mathcal{D}(\mathbb{I}) \mathbb{I} \mathcal{D}(\psi_0) \) by identifying the \( \rho \) boundary of \( \mathcal{D}(\phi_0) \) with the \( \mathcal{D}(\psi_0) \), \( \mu \) boundary of \( -\mathcal{D}(\mathbb{I}) \) and by identifying the \( \sigma \) boundary of \( -\mathcal{D}(\mathbb{I}) \) with the \( \mathcal{D}(\psi_0) \). For convenience, we will assume throughout that there are no bigons in \( \mathcal{D}(\phi_0) \) or \( \mathcal{D}(\psi_0) \) (by Theorem 3.3 there is no loss of generality).

We will obtain a diagram \( \mathcal{D}(\psi_0 \circ \phi_0) \) by destabilizing \( \mathcal{D}(\phi_0) \cup_{\sigma} \mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \), a process which we now explain. Notice that in \( \mathcal{D}(\phi_0) \cup_{\sigma} \mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \), the \( \alpha \) arcs of \( \mathcal{D}(\phi_0) \) are glued to the \( \alpha \) arcs of \( -\mathcal{D}(\mathbb{I}) \) to form \( \alpha \) circles, and similarly the \( \beta \) arcs of \( -\mathcal{D}(\mathbb{I}) \) are glued to the \( \beta \) arcs of \( \mathcal{D}(\psi_0) \) to form \( \beta \) circles. Each generator \( w_j \in DD(\mathbb{I}) \) corresponds to a unique intersection point of an \( \alpha \) circle and a \( \beta \) circle in \( \mathcal{D}(\phi_0) \cup_{\sigma} -\mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \). Let \( T_{w_j} \) denote a tubular neighborhood of this pair of intersecting \( \alpha \) and \( \beta \) circles in \( \mathcal{D}(\phi_0) \cup_{\sigma} -\mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \), where \( T_{w_j} \) is sufficiently small that \( \partial T_{w_j} \cap \partial(\mathcal{D}(\phi_0) \cup_{\sigma} -\mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0)) = \emptyset \) and such that further shrinking \( T_{w_j} \) does not reduce the number of intersection points of \( \partial T_{w_j} \) with the \( \alpha \) and \( \beta \) arcs of \( \mathcal{D}(\phi_0) \) or \( \mathcal{D}(\psi_0) \), or \( \mathcal{D}(\mathbb{I}) \). Notice that \( T_{w_j} \) is a torus with one boundary component.

Let \( \{x_i\}, \{w_j\}, \) and \( \{y_k\} \) denote the canonical bases of \( M(\phi_0), DD(\mathbb{I}) \), and \( M(\psi_0) \) respectively. Let \( T := \cup_j T_{w_j} \), and let us further suppose that the \( T_{w_j} \)s are chosen to be pairwise disjoint.

**Construction 4.9.** The construction of \( \mathcal{D}(\psi_0 \circ \phi_0) \) involves removing each \( T_{w_j} \) from \( \mathcal{D}(\phi_0) \cup_{\sigma} -\mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0) \) and gluing in a corresponding disk \( D_{w_j} \) along \( \partial T_{w_j} \). That is, consider the space

\[
((\mathcal{D}(\phi_0) \cup_{\sigma} -\mathcal{D}(\mathbb{I}) \cup_{\rho} \mathcal{D}(\psi_0)) \setminus T) \cup_{\sim} (\mathbb{I} \setminus D_{w_j})
\]

where \( \sim \) identifies each \( \partial T_{w_j} \) diffeomorphically with \( \partial D_{w_j} \). Let \( D \) denote \( \cup_j D_{w_j} \).

Observe that each \( x_i \) lies on a \( \beta \) arc segment (i.e. connected subset) \( t_{x_i} \subset T_{w_j} \), for some \( T_{w_j} \), with \( \partial t_{x_i} \subset \partial T_{w_j} \). We additionally replace each \( t_{x_i} \) with a \( \beta \) arc segment \( d_{x_i} \) in \( D_{w_j} \), such that \( \partial d_{x_i} \) and \( \partial d_{x_i} \) are identified under \( \sim \). Similarly, each \( y_k \) lies on an \( \alpha \) arc segment \( t_{y_k} \subset T_{w_j} \), for some \( T_{w_j} \), with \( \partial t_{y_k} \subset \partial T_{w_j} \). We replace each \( t_{y_k} \) with an \( \alpha \) arc segment \( d_{y_k} \) in \( D_{w_j} \), such that \( \partial d_{y_k} \) and \( \partial d_{y_k} \) are identified under \( \sim \). The \( d_{x_i} \)s and \( d_{y_k} \)s are chosen such that they intersect each other minimally (we can picture them as chords in the \( D_{w_j} \)s). Let \( \mathcal{D}(\psi_0 \circ \phi_0) \) denote the resulting space with \( \alpha \) and \( \beta \) curves.

**Example 4.10.** See Figure 5 for the construction of \( \mathcal{D}(\psi_0 \circ \phi_0) \) when \( \phi_0 = \psi_0 \) is a Dehn twist on the once punctured torus.
Figure 5. The construction of $\tilde{D}(\psi_0 \circ \phi_0)$ when $\phi_0 = \psi_0$ is a Dehn twist on the once punctured torus. Note that here the orientations of $D(\phi_0)$ and $D(\psi_0)$ are the opposite of the standard orientation of the plane.

**Lemma 4.11.** The $\beta$ arc segments $\{d_{x_i}\}$ (resp. $\alpha$ arc segments $\{d_{y_k}\}$) in Construction 4.9 are pairwise disjoint. Two arcs in $\{d_{x_i}\}$ and $\{d_{y_k}\}$ respectively intersect if they lie in the same disk of $\{D_{w_j}\}$.

*Proof.* This follows more or less directly from the construction. □

**Lemma 4.12.** For some $\xi_0$ representing an element $\xi \in MCG_0(F(\mathcal{Z}))$, there is a diffeomorphism $\tilde{D}(\psi_0 \circ \phi_0) \cong D(\xi_0)$ sending $\alpha$ curves to $\alpha$ curves and $\beta$ curves to $\beta$ curves.

*Proof.* We first compute the genus of $\tilde{D}(\psi_0 \circ \phi_0)$ as follows. Let $\chi(\cdot)$ denote Euler characteristic and let $g'$ be the genus of $D(\phi_0) \cup_{\sigma} -D(\mathbb{I}) \cup_{\rho} D(\psi_0)$. Then we have

$$2 - 2g' - 2b = \chi(D(\phi_0) \cup_{\sigma} -D(\mathbb{I}) \cup_{\rho} D(\psi_0)) = 3\chi(D(\mathbb{I})) = 3(2 - 2g - 2b),$$

and therefore $g' = 3g + 2b - 2$. Since each destabilization lowers the genus by one, and we perform one for each of the $2(g + b - 1)$ $\alpha$ arcs of $D(\mathbb{I})$, we conclude that the genus of $\tilde{D}(\psi_0 \circ \phi_0)$ is $g$. 
Next, observe that $\mathcal{D}(\mathbb{I}) \setminus \alpha$ and $\tilde{\mathcal{D}}(\psi_0 \circ \phi_0) \setminus \alpha$ have the same Euler characteristic. Then since $\mathcal{D}(\mathbb{I}) \setminus \alpha$ is a collection of $b$ annuli, so is $\tilde{\mathcal{D}}(\psi_0 \circ \phi_0) \setminus \alpha$, and we can find a diffeomorphism $\mathcal{D}(\psi_0 \circ \phi_0) \setminus \alpha \cong \mathcal{D}(\mathbb{I}) \setminus \alpha$ respecting $\alpha$ arcs and boundary segments. This quotients to a diffeomorphism $\tilde{\mathcal{D}}(\psi_0 \circ \phi_0) \cong \mathcal{D}(\mathbb{I})$ respecting $\alpha$ arcs and boundaries, and the same reasoning shows that the resulting image of the $\beta$ arcs differ from those of $\mathcal{D}(\mathbb{I})$ by some $\xi_0$.

**Lemma 4.13.** The mapping class $\xi$ above is equal to $\psi \circ \phi$.

Let $M(\tilde{\mathcal{D}}(\psi_0 \circ \phi_0))$ denote the $\mathcal{B}(\mathcal{Z}) - \mathcal{B}(\mathcal{Z}) \mathcal{A}_\infty$-bimodule defined by $\tilde{\mathcal{D}}(\psi_0 \circ \phi_0)$.

**Corollary 4.14.** $M(\tilde{\mathcal{D}}(\psi_0 \circ \phi_0))$ is $\mathcal{A}_\infty$-homotopy equivalent to $M(\psi_0 \circ \phi_0)$.

**Sketch of proof of Lemma 4.13.** Lemma 4.13 can be understood in terms of $\alpha - \beta$-bordered Heegaard diagrams. $\mathcal{D}(\phi_0)$ and $\mathcal{D}(\psi_0)$ can be understood as representing mapping cylinders for $\phi$ and $\psi$ respectively, and gluing $\mathcal{D}(\phi_0)$ to $\mathcal{D}(\psi_0)$ gives a representation of the mapping cylinder for $\psi \circ \phi$. Moreover, it follows from the general theory of bordered Heegaard diagrams that performing Heegaard moves (isotopies, handleslides and (de)stabilizations) leave the represented 3-manifold invariant, which is why our construction of $\tilde{\mathcal{D}}(\psi \circ \phi)$ by gluing, destabilizing, and replacing arc segments is successful. The reader is encouraged to consult Section 3 of [6] for more details.

**4.3. Checking the composition behavior.** The following definition gives the natural notion of isomorphism in the category of $\mathcal{A}_\infty$-bimodules.

**Definition 4.15.** An $\mathcal{A}_\infty$-homomorphism $F = \{F_{i,j}\}$ from $\mathcal{A}M_B$ to $\mathcal{A}N_B$ is an $\mathcal{A}_\infty$-isomorphism if there exists an $\mathcal{A}_\infty$-homomorphism $G = \{G_{i,j}\}$ from $\mathcal{A}N_B$ to $\mathcal{A}M_B$ such that $G \circ F = \mathbb{I}_B$ and $F \circ G = \mathbb{I}_A$. Note that in particular $F$ is an $\mathcal{A}_\infty$-homotopy equivalence.

**Theorem 4.16.** There is an $\mathcal{A}_\infty$-isomorphism $F : M(\tilde{\mathcal{D}}(\psi_0 \circ \phi_0)) \to M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0))$ with $F_{i,j} = 0$ unless $i = j = 0$.

**Corollary 4.17.** The bimodules $M(\psi \circ \phi)$ and $M(\phi) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi))$ are $\mathcal{A}_\infty$-homotopy equivalent.

**Proof.** This is immediate from Theorem 3.3 and Corollary 4.14. □

**Proof of Theorem 4.16.** The morphism

$$F : M(\tilde{\mathcal{D}}(\psi_0 \circ \phi_0)) \to M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0))$$

has $F_{i,j} := 0$ unless $i = j = 0$, with $F_{0,0}$ defined in the following way. Recall that $M(\tilde{\mathcal{D}}(\psi_0 \circ \phi_0))$ is generated by the intersection points between $\alpha$ and $\beta$ arcs in $\tilde{\mathcal{D}}(\psi_0 \circ \phi_0)$. Let us denote this set by $\{z_l\}$. Each $z_l$ occurs in some $D_{w_j}$ as $d_{x_i} \cap d_{y_k}$ for some $d_{x_i}$ and $d_{y_k}$. In this case, following the definitions, we have $x_i \otimes (w_j \otimes y_k) \in \text{GEN}$, where GEN is the canonical $\mathbb{F}_2$-basis for $M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0))$. We define $F_{0,0}$ by

$$F_{0,0}(z_l) := x_i \otimes (w_j \otimes y_k).$$
We claim that, for each \( x_i \otimes (w_j \otimes y_k) \in \text{GEN} \), the arc segments \( d_{x_i} \) and \( d_{y_k} \) intersect at some point \( z_l \) in some \( D_{w_j} \). It then follows that \( F_{0,1,0} \) sets up a bijective correspondence between \( \{z_l\} \) and \( \text{GEN} \), and we extend it by linearity to a \( k \)-bimodule isomorphism.

We let

\[
G : M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0)) \to M(\hat{\mathcal{D}}(\psi_0 \circ \phi_0))
\]

be the morphism defined by \( G_{0,1,0} := F_{0,1,0}^{-1} \) and \( G_{i,1,j} := 0 \) otherwise. Our claim is that \( F \) and \( G \) are inverse \( \mathcal{A}_\infty \)-isomorphisms. Using the definition of morphism composition (see (3.3)), we see that \( F \) and \( G \) satisfy the conditions of Definition 4.15 if \( F \) and \( G \) are \( \mathcal{A}_\infty \)-homomorphisms. This is in turn equivalent to the following reduction of (3.2):

That is, we must show that \( F_{0,1,0} \) sets up bijective correspondence between the \( m \) relations of either bimodule. We verify the above equation by breaking it into two steps. **Direction 1** shows that for each nontrivial summand of \( m \) in \( M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0)) \) there is the corresponding summand of \( m \) in \( M(\hat{\mathcal{D}}(\psi_0 \circ \phi_0)) \), and **Direction 2** shows that for each nontrivial summand of \( m \) in \( M(\hat{\mathcal{D}}(\psi_0 \circ \phi_0)) \) there is the corresponding summand of \( m \) in \( M(\phi_0) \boxtimes (DD(\frac{1}{2}) \boxtimes M(\psi_0)) \).

**Direction 1:**

Let us consider \( m(\sigma_m, ..., \sigma_1, x_a \otimes (w_b \otimes y_c), \rho_1, ..., \rho_n) \) for \( x_a \otimes (w_b \otimes y_c) \in \text{GEN} \). To see what a summand means in terms of \( M(\phi_0), DD(\frac{1}{2}), \) and \( M(\psi_0) \), observe that the structure map \( \delta_{DA} \) on \( DD(\frac{1}{2}) \boxtimes M(\psi_0) \) is given by
Thus each summand of \( m(\sigma_1, ..., \sigma_m, x_a \otimes (w_b \otimes y_c), \rho_1, ..., \rho_n) \) comes from a diagram of the form

\[
\begin{align*}
\sigma_m & \rightarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_1 \downarrow m \\
& \downarrow \delta_{DD}^{s_1} \quad \Pi \\
& \downarrow \delta_{DD}^{s_2} \quad \Pi \\
& \downarrow m \quad m \quad m \quad \cdots \\
\end{align*}
\]

where \( 0 = t_0 \leq t_1 \leq \cdots \leq t_r = n, \ s_1, ..., s_r \geq 0, \) and \( r \geq 0. \)

In particular, the case \( r = 0 \) manifests itself as

\[
\begin{align*}
\sigma_m & \rightarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_1 \downarrow m \\
& \downarrow \delta_{DD}^{s_1} \quad \Pi \\
& \downarrow \delta_{DD}^{s_2} \quad \Pi \\
& \downarrow m \quad m \quad m \quad \cdots \\
\end{align*}
\]

(4.1)

On the other hand, if \( r > 0 \) and any of \( s_1, ..., s_r \) is zero, then we must have \( r = 1 \) and \( s_1 = 0, \) since otherwise the \( m \) in the left of (4.2) would be trivial. Then this case corresponds to a diagram of the form

\[
\begin{align*}
x_a & \rightarrow w_b \rightarrow y_c \rightarrow \cdots \rightarrow \rho_n \\
& \downarrow m \quad m \quad m \quad \cdots \\
\end{align*}
\]

(4.3)

Accordingly, we break our work into three cases:

- **Case 1**: \( r = 0 \)
- **Case 2**: \( r = 1 \) and \( s_1 = 0 \)
- **Case 3**: \( r \geq 1 \) and \( s_1, ..., s_r \geq 1 \)
For convenience let $\text{EQ}_{\partial T} : \partial D \to \partial T$ denote the diffeomorphism defining $\sim$, and let $\text{EQ}_{\partial D} : \partial T \to \partial D$ denote its inverse. In what follows $d(\cdot)$ and $t(\cdot)$ will always be corresponding segments as in Construction (4.9), and similarly for $D(\cdot)$ and $T(\cdot)$. We will use $\text{Dom}(\cdot)$ to denote the domain of a map.

**Case 1:** For a summand $x_a \otimes (w_b \otimes y_c)$ of $m(\sigma_m, \ldots, \sigma_1, x_a \otimes (w_b \otimes y_c), \rho_1, \ldots, \rho_n)$ coming from (4.2), it is clear that $w_b = w_b$, $y_c = y_c$, and $n = 0$. The $m$ in (4.2) implies the existence of a polygon $P_L : \mathbb{D}^2 \to \mathcal{D}(\phi_0)$ from $x_a$ to $x_a'$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\cdot)$. A little thought shows that the connected components of $P_L^{-1}(T)$ are neighborhoods of $\{A_1\}$ of $\alpha$ segments in $\text{Int}(\mathbb{D}^2)$ with both endpoints on the left side of $\mathbb{D}^2$, as well as a neighborhood $A_{\text{init}}$ of the right side of $\mathbb{D}^2$. Let $\mathcal{P}_L$ denote the restriction of $P_L$ to $P_L^{-1}(\mathcal{D}(\phi_0) \setminus T)$. Our plan is to upgrade $\mathcal{P}_L$ to a polygon $P_{\text{tot}} : \mathbb{D}^2 \to \mathcal{D}(\psi_0 \circ \phi_0)$ from $F_{0,1,0}^{-1}(x_a \otimes (w_b \otimes y_c))$ to $F_{0,1,0}^{-1}(x_a' \otimes (w_b \otimes y_c))$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\cdot)$, and we do this by gluing regions of disks to $\mathcal{P}_L$.

Consider the $\beta$ segments $d_{x_a}, d_{x_{a'}}, d_{y_c}$ in $D_{w_b}$ and the $\alpha$ segment $d_{y_c} \in D_{w_b}$. By Lemma 4.11 we know that $d_{x_a}$ and $d_{x_{a'}}$ are disjoint and both intersect $d_{y_c}$. Observe that $\mathcal{P}_L$ maps $\partial \mathcal{P}_L \cap \partial A_{\text{init}}$ diffeomorphically to a segment $s$ of $\partial \mathcal{D}_{w_b}$ connecting an endpoint of $t_{x_a}$ to an endpoint of $t_{x_{a'}}$, and it corresponds under $\sim$ to a segment $\text{EQ}_{\partial D}(s)$ of $\partial D_{w_b}$ connecting an endpoint of $d_{x_a}$ to an endpoint of $d_{x_{a'}}$. Together $d_{x_a}, d_{x_{a'}}, d_{y_c}$, and $\text{EQ}_{\partial D}(s)$ define a rectangular region $R_{\text{init}}$ of $D_{w_b}$. We proceed by gluing $R_{\text{init}}$ to $\mathcal{P}_L$ along $R_{\text{init}} \cap \partial D_{w_b}$.

Similarly, the intersection of each $A_i$ with the left side of $P_L$ is two nonintersecting $\beta$ segments $t^-(A_i), t^+(A_i) \in \{t_{x_i}\}$. The corresponding $\beta$ segments $d^-(A_i), d^+(A_i) \in \{d_{x_i}\}$ define a rectangular region $R_i$ of some disk $D(R_i) \in \{D_{w_j}\}$. We proceed by gluing $R_i$ to $\mathcal{P}_L$ along $R_i \cap \partial D(R_i)$.

The result after these gluings is the desired polygon $P_{\text{tot}}$.

**Case 3:** This case follows similarly to Case 1, except with the roles of the left and right sides switched.

**Case 4:** Finally, suppose we have a summand $x_{a'} \otimes (w_b \otimes y_c)$ coming from (4.1) with $r \geq 1$ and $s_1, \ldots, s_r \geq 1$. Assume that for each $i$ the contributing output of $\xi^{i,1}_{DD}$ is $\xi^{i,1} \otimes \cdots \otimes \xi^{i,s_i}$ on the left and $\xi^{i,1'} \otimes \cdots \otimes \xi^{i,s_i'}$ on the right, where $\xi^{i,j}$ and $\xi^{i,j'}$ are corresponding short chords in the sense of Section 4.1. Then we have

- a polygon $P_L : \mathbb{D}^2 \to \mathcal{D}(\phi_0)$ from $x_a$ to $x_{a'}$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\Pi(\xi^{1,1}, \ldots, \xi^{1,s_1}), \ldots, \Pi(\xi^{r,1}, \ldots, \xi^{r,s_r}))$,
- a compact connected component $P^i_M$ of $\mathcal{D}(\ell) \setminus (\alpha \cup \beta)$ containing $\xi^{i,j}$ and $\xi^{i,j'}$ for each $1 \leq i \leq r$ and $1 \leq j \leq s_i$, and
- a polygon $P^i_R : \mathbb{D}^2 \to \mathcal{D}(\psi_0)$ through $(\xi^{i,1'}, \ldots, \xi^{i,s_i'})$ and $(\rho_{t_{i-1}+1}, \ldots, \rho_{t_i})$ for each $1 \leq i \leq r$.

We begin by analyzing the preimage of $T$ in each $P^i_R$ and $P_L$. Since the left side of $P^i_R$ only includes short chords, the components of $(P^i_R)^{-1}(T)$ are neighborhoods of $\{A^1_i\}$ of $\beta$ segments in $\text{Int}(P^i_R)$ with both endpoints on the right side of $P^i_R$ and

\[^2\text{Here } P^i_M \text{ is really the closure of a component. We will often think of } P^i_M \text{ as the identity map on itself.}\]
neighborhoods of $\beta$ segments of the left side of $P^i_R$. On the other hand, the components of $P'^{i-1}_L(T)$ are

- neighborhoods $\{\bar{A}_i\}$ of $\alpha$ segments in $\text{Int}(P_L)$ with both endpoints on the left side of $P_L$,
- neighborhoods $\{A_i\}$ of $\alpha$ segments in $\text{Int}(P_L)$ with one endpoint on the left side of $P_L$ and one endpoint on the right side of $P_L$, and
- neighborhoods of $\alpha$ segments of the right side of $P_L$.

Remark 4.18. Notice that we have excluded neighborhoods of $\alpha$ segments in $\text{Int}(P_L)$ with endpoints on each side of $P_L$, since this would imply a component of $F \setminus \alpha$ whose boundary lies entirely in $\cup_i S_i^-$, contradicting the conditions given in Section 2.2.

Let $\mathcal{P}_L, \mathcal{P}_M, \mathcal{P}_R$ denote the restrictions of $P_L, P_M, P_R$ to $P'^{i-1}_L(D(\phi_0) \setminus T), (P'_M)^{i-1}(D(\phi) \setminus T)$, and $(P^i_R)^{-1}(D(\psi_0) \setminus T)$ respectively. We proceed by gluing each $\mathcal{P}^i_M$ along its $\epsilon^i_M$ side to $\mathcal{P}_L$ and along its $\epsilon^i_R$ side to $\mathcal{P}_R$ in the obvious way implied by (4.1), the gluing taking place along (slight reductions of) short chords. The result is a map $\mathcal{P}_{\text{tot}}$ into $(D(\phi_0) \cup \phi_0 = D(\phi) \cup \phi_0 = D(\psi_0) \setminus T)$. Our plan is to upgrade $\mathcal{P}_{\text{tot}}$ to a polygon $P_{\text{tot}} : \mathbb{D}^2 \to D(\psi_0, \psi_0)$ from $F_{0,1,0}(x_0 \otimes (w_b \otimes y_c))$ to $F_{0,1,0}(x_a \otimes (w_r \otimes y_c))$ through $(\sigma_1, \ldots, \sigma_m)$ and $(\rho_1, \ldots, \rho_n)$ by gluing in various regions of the disks $\{D_w\}$.

Firstly, we have already seen in cases 1 and 2 how to glue in regions $R^i_l$ and $\bar{R}_l$ corresponding to each $A_l^i$ and $\bar{A}_l$ respectively.

Denote the resulting map after this gluing by $\mathcal{P}'_{\text{tot}}$. Observe that, for each $l$, $\partial \text{Dom}(\mathcal{P}'_{\text{tot}}) \setminus \partial A$ has two components, one of which intersects the left side of $P_L$ and the right side of each $P^i_R$. Let us denote the union of the other component and $\partial \text{Dom}(\mathcal{P}'_{\text{tot}}) \cap \partial A_l$ by $B(\bar{A}_l)$. For coherence we break our remaining gluing into four steps.

1. The intersection of each $\partial A_l$ with the left side of $P_L$ is a $\beta$ segment $t(\bar{A}_l) \in \{t_{x_k}\}$, and $\mathcal{P}'_{\text{tot}}$ maps $\text{BS}(\bar{A}_l)$ diffeomorphically onto a segment $\mathcal{P}'_{\text{tot}}(\text{BS}(\bar{A}_l))$ of some $\partial T(\bar{A}_l) \in \{\partial T_{w_j}\}$ with endpoints $\partial t(\bar{A}_l)$. Together $d(\bar{A}_l)$ and $\text{EQ}_{\partial D}(\mathcal{P}'_{\text{tot}}(\text{BS}(\bar{A}_l)))$ define a bigonal region $R_l$ of $D(\bar{A}_l)$, which we glue along $\partial R_l \cap \partial D(\bar{A}_l)$ to $\text{BS}(\bar{A}_l)$.

2. Now let $\mathcal{P}''_{\text{tot}}$ denote the resulting map, and observe that $\text{Dom}(\mathcal{P}''_{\text{tot}})$ is simply-connected. Let $\gamma_L(\mathcal{P}_L)$ and $\gamma_R(\mathcal{P}_R)$ denote the left and right sides of $P_L$ and $P_R$ respectively. Let $C_{\text{init}}, C_1, \ldots, C_{r-1}, C_{\text{term}}$ denote the components of $\text{Dom}(\mathcal{P}''_{\text{tot}}) \setminus (\gamma_L(\mathcal{P}_L) \cup_i \gamma_R(\mathcal{P}^i_R) \cup \partial R_l \cup \partial \bar{R}_l \cup \partial R_l \cup \partial \bar{R}_l)$, ordered such that $C_{\text{init}}$ intersects $\mathcal{P}^1_R$, $C_i$ intersects $\mathcal{P}^i_R$ and $\mathcal{P}^{i+1}_R$ for $1 \leq i \leq r - 1$, and $C_{\text{term}}$ intersects $\mathcal{P}'_R$. Then for $1 \leq i \leq r - 1$, $\mathcal{P}''_{\text{tot}}$ maps $C_i$ diffeomorphically onto a segment $\mathcal{P}''_{\text{tot}}(C_i)$ of some $\partial T(C_i) \in \{\partial T_{w_j}\}$ with endpoints $\partial t(C_i) \in \{t_{y_k}\}$. Together $d(C_i)$ and $\text{EQ}_{\partial D}(\mathcal{P}''_{\text{tot}}(C_i))$ define a bigonal region $R_l$ of $D(C_i)$, which we glue along $\partial R_l \cap \partial D(C_i)$ to $C_l$.

3. Finally, we consider $C_{\text{init}}$ and $C_{\text{term}}$. $\mathcal{P}''_{\text{tot}}$ maps $C_{\text{init}}$ diffeomorphically onto a segment $\mathcal{P}''_{\text{tot}}(C_{\text{init}})$ of $\partial T_{w_b}$ with endpoints in $\partial t_{x_a}$ and $\partial t_{y_c}$ respectively.
Then $\text{Eq}_{\alpha D}(P''_{\text{tot}}(C_{\text{init}}))$ together with $d_{x_i}$ and $d_{y_k}$ define a triangular region $R_{\text{init}}$ of $D_{\text{tot}}$, which we glue along $\partial R_{\text{init}} \cap \partial D_{\text{tot}}$ to $C_{\text{init}}$. The vertex $d_{x_i} \cap d_{y_k}$ is of course none other than $F_{0,1,0}^{-1}(x_a \otimes (w_b \otimes y_c))$. We proceed similarly for $C_{\text{term}}$, gluing a triangular region $R_{\text{term}}$ of $D_{\text{tot}}'$ along $\partial R_{\text{term}} \cap \partial D_{\text{tot}}'$ to $C_{\text{term}}$.

The result after all of this gluing is the promised polygon $P_{\text{tot}}$.

**Direction 2:**

Now let $P_{\text{tot}} : \mathbb{D}^2 \to \mathcal{D}(\psi_0 \circ \phi_0)$ be a polygon from $F_{0,1,0}^{-1}(x_a \otimes (w_b \otimes y_c))$ to $F_{0,1,0}^{-1}(x_b' \otimes (w_b' \otimes y_{c'}))$ through $(\sigma_1, ..., \sigma_m)$ and $(p_1, ..., p_n)$. Our task is to reverse the process of **Direction 1** by breaking up $P_{\text{tot}}$ into smaller components which fit together as in (4.1). We first examine the components of $P_{\text{tot}}^{-1}(D)$.

**Lemma 4.19.** The components of $P_{\text{tot}}^{-1}(D)$ consist of:

- a domain $R_{\text{init}} \subset D_{\text{tot}}$ containing $-i$, and possibly a distinct domain $R_{\text{term}} \subset D_{\text{tot}}'$ containing $i$
- domains $R_i \cong D(R_i) \in \{D_{\text{tot}}\}$ in the interior of Dom($P_{\text{tot}}$)
- domains $\overline{R}_i \subset D(\overline{R}_i) \in \{D_{\text{tot}}\}$ such that $\overline{R}_i \cap \partial(\text{Dom}(P_{\text{tot}}))$ is a single segment of the left side of Dom($P_{\text{tot}}$)
- domains $\overline{R}_i \subset D(\overline{R}_i) \in \{D_{\text{tot}}\}$ such that $\overline{R}_i \cap \partial(\text{Dom}(P_{\text{tot}}))$ is a single segment of the right side of Dom($P_{\text{tot}}$)
- domains $\overline{R}_i \subset D(\overline{R}_i) \in \{D_{\text{tot}}\}$ such that $\overline{R}_i \cap \partial(\text{Dom}(P_{\text{tot}}))$ is two disjoint segments of the left side of Dom($P_{\text{tot}}$)
- domains $\overline{R}_i \subset D(\overline{R}_i) \in \{D_{\text{tot}}\}$ such that $\overline{R}_i \cap \partial(\text{Dom}(P_{\text{tot}}))$ is two disjoint segments of the right side of Dom($P_{\text{tot}}$)

where (by abuse of notation) we have used $\subset$ to denote an embedding under $P_{\text{tot}}$.

Moreover, we have one of the following:

- **Case 1:** $R_{\text{init}} = R_{\text{term}}$ is Dom($P_{\text{tot}}$) minus a half disk with equator along the left side of Dom($P_{\text{tot}}$),
- **Case 2:** $R_{\text{init}} = R_{\text{term}}$ is Dom($P_{\text{tot}}$) minus a half disk with equator along the right side of Dom($P_{\text{tot}}$), or
- **Case 3:** $R_{\text{init}}$ and $R_{\text{term}}$ are distinct, and $R_{\text{init}} \cap \partial(\text{Dom}(P_{\text{tot}}))$ and $R_{\text{term}} \cap \partial(\text{Dom}(P_{\text{tot}}))$ are both connected.

**Proof.** The first part of the lemma follows essentially from Lemma 4.11. In particular, a component of $P_{\text{tot}}^{-1}(D)$ whose intersection with $\partial(\text{Dom}(P_{\text{tot}}))$ has components on both sides of Dom($P_{\text{tot}}$) would imply a pair of nonintersecting arcs $d_{x_i}$ and $d_{y_k}$ in the same disk $D_{\text{tot}}$. On the other hand, a component of $P_{\text{tot}}^{-1}(D)$ whose intersection with $\partial(\text{Dom}(P_{\text{tot}}))$ has more than two components on the same side of Dom($P_{\text{tot}}$) would imply three segments of $\{d_{x_i}\}$ (resp. $\{d_{y_k}\}$) in a disk $D_{\text{tot}}$ with the wrong combinatorics (i.e. disagreeing with Lemma 4.11). Namely, no element of $\{d_{y_k}\}$ (resp. $\{d_{x_i}\}$) could intersect all three in minimal position.

The second part of the lemma follows similarly.

**Example 4.20.** See Figure 8 for an example of Case 3.
We consider Cases 1 - 3 from Lemma 4.19 separately. Let \( \alpha(x_i) \) denote the \( \alpha \) arc of \( \mathcal{D}(\phi_0) \) containing \( x_i \), and let \( \beta(y_k) \) denote the \( \beta \) arc of \( \mathcal{D}(\psi_0) \) containing \( y_k \).

**Case 1:** In this case the entire right side of \( P_{\text{tot}} \) is a segment of some \( \alpha \) segment from \( \{d_{y_i}\} \), and therefore \( u_{wb} = w_b, y_c = y_c, \) and \( n = 0 \). Let \( \mathcal{P}_L \) denote the restriction of \( P_{\text{tot}} \) to \( \text{Dom}(P_{\text{tot}}) \setminus P_{\text{tot}}^{-1}(D) \). Then \( \mathcal{P}_L \) maps \( \partial \mathcal{P}_L \cap \partial \mathcal{P}_{\text{init}} \) diffeomorphically to a segment \( \mathcal{P}_L(\partial \mathcal{P}_L \cap \partial \mathcal{P}_{\text{init}}) \) of \( \partial D_{wb} \) with endpoints in \( d_{x_a} \) and \( d_{x_{a'}} \), respectively. Together \( \text{EQ}_{\partial T}(\mathcal{P}_L(\partial \mathcal{P}_L \cap \partial \mathcal{P}_{\text{init}})), t_{xa}, t_{xa'}, \) and \( \alpha(x_a) = \alpha(x_{a'}) \) define a rectangular region \( A_{\text{init}} \) of \( T_{wb} \). We proceed by gluing as follows:

- Glue \( A_{\text{init}} \) to \( \mathcal{P}_L \) along \( \partial A_{\text{init}} \cap \partial T_{wb} \).
- For each \( \tilde{R}_i \), the intersection \( \tilde{R}_i \cap \partial(\text{Dom}(P_{\text{tot}})) \) is two \( \beta \) segments \( d^-(\tilde{R}_i), d^+(\tilde{R}_i) \in \{d_{x_i}\} \). The corresponding \( \beta \) segments \( t^-\tilde{R}_i, t^+\tilde{R}_i \in \{t_{x_i}\} \), together with the two components of \( \text{EQ}_{\partial T}(\mathcal{P}_L(\tilde{R}_i \cap \mathcal{P}_L)) \), define a rectangular region \( \tilde{A}_i \) of \( T(\tilde{R}_i) \), which we glue to \( \mathcal{P}_L \) along \( \partial \tilde{R}_i \).

Let \( P_{\text{L}} \) denote the result after the above gluing. The following lemma shows that \( P_L \) is a polygon in \( \mathcal{D}(\phi_0) \) from \( x_a \) to \( x_{a'} \) through \( (\sigma_1, ..., \sigma_m) \) and (\), as in (4.2).

**Lemma 4.21.** \( \{R_i\} = \{\tilde{R}_i\} = \{R_i\} = \{\tilde{R}_i\} = \emptyset \).

**Proof.** The fact that \( \{R_i\} \) and \( \{R_i\} \) are empty is immediate, since the entire right side of \( P_{\text{tot}} \) is contained in \( R_{\text{init}} \). We note that \( \mathcal{P}_L \) has image in \( \mathcal{D}(\phi_0) \), and therefore \( \{R_i\} \) and \( \{\tilde{R}_i\} \) must be empty as well. \( \square \)

**Case 2:** This case follows similarly to Case 1, except with the roles of the left and right sides switched.

**Case 3:** In this case, let \( \bar{\rho} \) and \( \bar{\sigma} \) denote the images of the \( \rho \) and \( \sigma \) boundaries of \( \mathcal{D}(\phi_0) \setminus T \) in \( \mathcal{D}(\psi_0 \circ \phi_0) \). Observe that each component of \( P_{\text{tot}}^{-1}(\mathcal{D}(\phi_0)) \) is a rectangular region \( P_{\mathcal{M}} \) (with the indices \( i, j \) to be defined shortly) of \( \text{Dom}(P_{\text{tot}}) \). Let \( \xi^{i,j} = P_{\mathcal{M}} \cap \bar{\rho} \) and \( \xi^{i,j'} = P_{\mathcal{M}} \cap \bar{\sigma} \). Let \( \mathcal{P}_{\text{tot}} \) denote the restriction of \( P_{\text{tot}} \) to \( \text{Dom}(P_{\text{tot}}) \setminus P_{\text{tot}}^{-1}(D) \).

**Lemma 4.22.** \( \{R_i\} = \emptyset \).

**Proof.** Let \( r \) and \( s \) denote the number of components of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \setminus \bigcup_{i,j} P_{\mathcal{M}}^{i,j} \) and \( P_{\text{tot}}^{-1}(\bar{\rho} \cup \bar{\sigma}) \) respectively. We first claim that each component of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \setminus \bigcup_{i,j} P_{\mathcal{M}}^{i,j} \) intersects \( \partial \text{Dom}(P_{\text{tot}}) \). This is because the image of such a component would violate the conditions of Section 2.2, as in Remark 4.18. Then a little thought shows that we must have

\[
\begin{align*}
\frac{r}{s} &\leq |\{R_i\}| + |\{\tilde{R}_i\}| + 2 \\
\frac{r}{s} &\leq 2 - |\{R_i\}| + 1 \\
\frac{r}{s} &\leq 2 + 2|\{\tilde{R}_i\}| + 2|\{R_i\}| + 4|\{R_i\}|.
\end{align*}
\]

These combine to yield

\[
1 + |\{\tilde{R}_i\}| + |\{R_i\}| + 2|\{R_i\}| - |\{R_i\}| + 1 \leq |\{\tilde{R}_i\}| + |\{R_i\}| + 2.
\]
i.e. \[|\{R_i\}| \leq 0.\]

Now let \( \mathcal{P}_L \) denote the restriction of \( \mathcal{P}_{\text{tot}} \) to the union of the components of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \setminus \bigcup_{i,j} \mathcal{P}_{M}^{i,j} \) which intersect the left side of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \). Similarly, let \( \mathcal{P}_R^1, \ldots, \mathcal{P}_R^r \) denote the restrictions of \( \mathcal{P}_{\text{tot}} \) to the components of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \setminus \bigcup_{i,j} \mathcal{P}_{M}^{i,j} \) which intersect the right side of \( \text{Dom}(\mathcal{P}_{\text{tot}}) \) (ordered from \(-i\) to \(i\)). Then \( \mathcal{P}_{M}^{i,j} \) is the \( j \)th element of \( \{\mathcal{P}_{M}^{i,j}\} \) intersecting \( \mathcal{P}_R^i \).

By gluing regions of the tori \( \{T_{w_j}\} \), we will upgrade

- \( \mathcal{P}_L \) to a polygon \( \mathcal{P}_L : \mathbb{D}^2 \to \mathcal{D}(\psi_0) \) from \( x_a \) to \( x_{a'} \) through \( (\sigma_1, \ldots, \sigma_m) \) and \( (\Pi(\xi_1^{r_1}, \ldots, \xi_{r_1}^{s_r}), \ldots, \Pi(\xi_1^{r_1}, \ldots, \xi_{r_1}^{s_r})) \)
- each \( \mathcal{P}_{M}^{i,j} \) to a connected component \( \mathcal{P}_{M}^{i,j} \) of \( \mathcal{D}(\mathbb{I}) \setminus (\alpha \cup \beta) \) defined by \( \xi_{i,j}^{r,s} \) and \( \xi_{i,j}^{t,s} \), and
- each \( \mathcal{P}_R^i \) to a polygon \( \mathcal{P}_R^i : \mathbb{D}^2 \to \mathcal{D}(\psi_0) \) through \( (\xi_1^{r_1}, \ldots, \xi_{i,j}^{s_i}) \) and \( (\rho_{i_{i+1}}, \ldots, \rho_{r_i}) \) for each \( 1 \leq i \leq r \),

where \( 0 = t_0 \leq t_1 \leq \ldots \leq t_r = n \).

The gluing to \( \mathcal{P}_L \) is as follows:

- For each \( \bar{R}_i \), the intersection \( \bar{R}_i \cap \partial(\text{Dom}(\mathcal{P}_{\text{tot}})) \) maps to \( \beta \) segment \( d(\bar{R}_i) \in \{d_x\} \). Its corresponding \( \beta \) segment \( t(\bar{R}_i) \in \{t_x\} \), together with the two components of \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \cap \mathcal{D}(\phi_0) \), define a rectangular region \( \bar{A}_i \) of \( T(\bar{R}_i) \cap \mathcal{D}(\phi_0) \), which we glue to \( \mathcal{P}_L \) along \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \cap \mathcal{D}(\phi_0) \).
- For each \( \bar{R}_i \), the intersection \( T(\bar{R}_i) \cap \mathcal{D}(\phi_0) \) is split in half by \( \alpha \). We glue the half containing \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \cap \mathcal{D}(\phi_0) \) to \( \mathcal{P}_L \) along \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \cap \mathcal{D}(\phi_0) \).
- For each \( \bar{R}_i \), the intersection \( \bar{R}_i \cap \partial \text{Dom}(\mathcal{P}_{\text{tot}}) \) maps to two \( \beta \) segments \( d^-(\bar{R}_i), d^+(\bar{R}_i) \in \{d_x\} \). The corresponding \( \beta \) segments \( t^-(\bar{R}_i), t^+(\bar{R}_i) \in \{t_x\} \), together with the two components of \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \), define a rectangular region \( \bar{A}_i \) of \( T(\bar{R}_i) \). We glue \( \bar{A}_i \) to \( \mathcal{P}_L \) along \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_L))) \).

Similarly, the gluing to the \( \mathcal{P}_R^i \)’s is as follows:

- For each \( \bar{R}_i \) intersecting \( \mathcal{P}_{M}^{i,j} \), the intersection \( T(\bar{R}_i) \cap \mathcal{D}(\psi_0) \) is split in half by \( \beta \). We glue the half containing \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_R^i))) \cap \mathcal{D}(\psi_0) \) to \( \mathcal{P}_R^i \) along \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_R^i))) \cap \mathcal{D}(\psi_0) \).
- For each \( \bar{R}_i \) intersecting \( \mathcal{P}_R^i \) and \( \mathcal{P}_R^{i+1} \), the intersection \( T(\bar{R}_i) \cap \mathcal{D}(\psi_0) \) is split in half by \( \beta \). We glue these two halves to \( \mathcal{P}_R^i \) and \( \mathcal{P}_R^{i+1} \) along \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_R^i))) \) and \( \mathcal{Q}_{\partial T}(\mathcal{P}_{\text{tot}}(\bar{R}_i \cap \text{Dom}(\mathcal{P}_R^{i+1}))) \) respectively.
For each $R_i$ intersecting $P^i_R$, the intersection $R_i \cap \partial \text{Dom}(P^i_R)$ maps to two $\alpha$ segments $d^-(R_i), d^+(R_i) \in \{d_{y_k}\}$. The corresponding $\alpha$ segments $t^-(R_i), t^+(R_i) \in \{t_{y_k}\}$, together with the two components of $\text{EQ}_{\partial T}(P_{\text{init}}(R_i \cap \partial \text{Dom}(P^i_R)))$, define a rectangular region $A_i$ of $T(R_i)$. We glue $A_i$ to $P^i_R$ along $\text{EQ}_{\partial T}(P_{\text{init}}(R_i \cap \partial \text{Dom}(P^i_R)))$.

The $\alpha$ segment $t_{y_k}$, together with $\beta(y_c)$ and $\text{EQ}_{\partial T}(P_{\text{term}}(R_{\text{term} } \cap \partial \text{Dom}(P^i_R))) \cap \mathcal{D}(\psi_0)$, define a rectangular region of $T_{w_k} \cap \mathcal{D}(\psi_0)$, which we glue to $P^i_R$ along $\text{EQ}_{\partial T}(P_{\text{term}}(R_{\text{term} } \cap \partial \text{Dom}(P^i_R))) \cap \mathcal{D}(\psi_0)$.

Similarly, the $\alpha$ segment $t_{y_{l'}}$, together with $\beta(y_c')$ and $\text{EQ}_{\partial T}(P_{\text{init}}(R_{\text{init} } \cap \partial \text{Dom}(P^i_R))) \cap \mathcal{D}(\psi_0)$, define a rectangular region of $T_{w_{l'}} \cap \mathcal{D}(\psi_0)$, which we glue to $P^i_R$ along $\text{EQ}_{\partial T}(P_{\text{init}}(R_{\text{init} } \cap \partial \text{Dom}(P^i_R))) \cap \mathcal{D}(\psi_0)$.

Finally, we let $P_{\alpha M}^{i,j}$ be the component of $\mathcal{D}(\mathbb{I}) \setminus (\alpha \cup \beta)$ containing $P^i_R$, and this completes the proof. \hfill $\square$

**Example 4.23.** Continuing Example 4.10, there is a relation in $M(\phi) \otimes (DD\left(\frac{1}{2}\right) \otimes M(\psi))$ (see Figure 5) coming from a diagram as in Figure 6. Figure 7 illustrates how the corresponding polygons in $\mathcal{D}(\phi_0), \mathcal{D}(\mathbb{I}), \text{and } \mathcal{D}(\psi_0)$, together with regions of $D_{w_1}$ and $D_{w_2}$, are glued together as in **Direction 1** to form the corresponding polygon in $\mathcal{D}(\psi_0 \circ \phi_0)$. Figure 8 illustrates how **Direction 2** recovers the polygons of Figure 6 from a polygon in $\mathcal{D}(\psi_0 \circ \phi_0)$.

**Remark 4.24.** Note that our choice of the parenthesization $M(\phi) \otimes (DD\left(\frac{1}{2}\right) \otimes M(\psi))$ (rather than $(M(\phi) \otimes DD\left(\frac{1}{2}\right)) \otimes M(\psi)$) manifests itself in **Direction 1** and **Direction 2**. Namely, in **Direction 1** we have a single polygon $P_L$ on the left and multiple polygons $P^i_R$ on the right, instead of vice versa, and this results in our gluing the left ends of $P^{i,j}_M$ and $P^{i,j+1}_M$ directly adjacent to each other on the right side of $P_L$, while the right sides of $P^{i,j}_M$ and $P^{i,j+1}_M$ are separated by a $\beta$ arc on the left side of $P^i_R$. In **Direction 2**, we replace each left domain $R_i$ with a
Figure 7. Direction 1 removes the preimage of $T_{w_1}$ and $T_{w_2}$ from $P_L, P_{M}^{1,1}, P_{M}^{1,2}, P_{M}^{2,2}, P_{R}^{1}$, and $P_{R}^{2}$ and glues in the pieces $\overline{R}_1, R_1, R_{1R}, R_{init}$, and $R_{term}$ to form a polygon in $\mathcal{D}(\psi_0 \circ \phi_0)$.

single domain $\overline{A}_l$ in order to construct a single polygon $P_L$ on the left, whereas we replace each right domain $R_l$ with two domains in order to split the corresponding region on the right side into multiple polygons.

4.4. Completing the proof. In this section we prove two theorems. The first shows that the identity mapping class gives the identity module, and the second shows that no other mapping class gives the identity module.

Definition 4.25. For an $\mathcal{A}_\infty$-algebra $\mathcal{A}$ over $k$, the identity bimodule $\mathcal{A}_\mathcal{I}_{\mathcal{A}}$ is given as follows. As a $k$-module $\mathcal{A}_\mathcal{I}_{\mathcal{A}}$ is isomorphic to $k$. For $k \neq 2$, $\delta_{DA}^{1,k} = 0$, while $$\delta_{DA}^{1,2}(\iota, a) = a \otimes \iota,$$

where $\iota$ is the generator of $\mathcal{A}_\mathcal{I}_{\mathcal{A}}$.

Lemma 4.26. For a chord $\rho \in \mathcal{B}(\mathcal{Z})$, let $|\rho|$ denote the number of short chords composing $\rho$, and similarly for chords in $\mathcal{B}(\mathcal{Z}')$. Let $P$ be a polygon in $\mathcal{D}(\mathcal{I})$ through $(\sigma_1, ..., \sigma_m)$ and $(\rho_1, ..., \rho_n)$. We have

$$\left( \sum_{i=1}^{m} |\sigma_i| \right) - m = n - 1 \quad (4.4)$$
Figure 8. Direction 2 performs the reverse process of Direction 1 to recover \( P_L, P_M^{1,1}, P_M^{1,2}, P_M^{2,2}, P_R^1, \) and \( P_R^2, \) and hence the relation of Figure 61, from a polygon in \( D(\psi_0 \circ \phi_0). \)

and

\[
\left( \sum_{i=1}^{n} |\rho_i| \right) - n = m - 1. \tag{4.5}
\]

Proof. Recall that in \( D(\mathbb{I}), \) each \( \alpha \) arc intersects its dual \( \beta \) arc exactly once and is disjoint from every other \( \beta \) arc. It follows that each component of \( C \) of \( P^{-1}(\beta) \) must have endpoints on both sides of \( \text{Dom}(P). \) To see this, note that if both endpoints of \( C \) were on the right side of \( \text{Dom}(P), \) we could cut \( P \) along \( C \) to obtain a polygon in \( D(\mathbb{I}) \) passing through no chords on the left side, which violates the first sentence of this proof. On the other hand, \( C \) cannot have both endpoints on the left side of \( \text{Dom}(P) \) as this would violate the conditions in the first paragraph of Section 2.2.

Note also that the left endpoint of \( C \) lies at an intersection point of two adjacent \( B(\mathcal{Z}') \) short chord preimages, whereas the right side is the unique intersection point of the \( \alpha \) arc connecting \( \rho_i \) and \( \rho_{i+1} \) for some \( 1 \leq i < n. \) Then (4.4) follows by noting that the number of left endpoints of components of \( P^{-1}(\beta) \) is \( (\sum_{i=1}^m |\sigma_i|) - m, \) whereas the number of right endpoints is \( n - 1. \)

(4.5) is proved similarly. \( \square \)

Theorem 4.27. \( DD(\frac{1}{2}) \boxtimes M(\mathbb{I}) \) is isomorphic as a type \( DA \) bimodule to \( A_{\mathbb{I},A}. \)

Proof. First observe that \( DD(\frac{1}{2}) \boxtimes M(\mathbb{I}) \) has one generator \( w_i \otimes x_i \) for each idempotent \( I_i \) of \( B(\mathcal{Z}), \) and hence its generators are in a natural bijective correspondence
with the generators of $A \mathbb{I}_A$. The structure map $\delta^1_{BD}$ of $DD\left(\frac{1}{2}\right) \boxtimes M(I)$ is given by

$$\delta^1_{BD} = \frac{1}{2} \sum_{i} \alpha_i \beta_i$$

for $\rho_{i_1,i_2}, ..., \rho_{i_{k-1},i_k} \in {\mathcal B}(Z)$. Since the left inputs of the $m$ in the diagram are short chords, we must have $k = 2$ by (4.4) of Lemma 4.26. Thus it suffices to show that there is a unique polygon $P : D^2 \to D\mathbb{I}$ with initial point $x_i$ passing through entirely short chords on the left and $\rho_{i_1,i_2}$ on the right, and that $P$ fits into a diagram of the form (4.6).

Let $\rho_{i_1,i_1+1}, \rho_{i_1+1,i_1+2}, ..., \rho_{i_2-1,i_2} \in {\mathcal B}(Z)$ be short chords such that

$$\rho_{i_1,i_2} = \rho_{i_1,i_1+1}\rho_{i_1+1,i_1+2} ... \rho_{i_2-1,i_2}.$$  

For $0 \leq j < i_2 - i_1$, let $P_{i_1+j,i_1+j+1}$ denote the component of $D\mathbb{I} \setminus (\alpha \cup \beta)$ defined by $\rho_{i_1+j,i_1+j+1}$ and the corresponding short chord $\sigma_{i_1+j,i_1+j+1}$. Observe that $P_{i_1+j,i_1+j+1}$ is a hexagon with one side corresponding to $\rho_{i_1+j,i_1+j+1}$, one side corresponding to $\sigma_{i_1+j,i_1+j+1}$, two sides $\alpha^{\text{init}}_{i_1+j,i_1+j+1}$ and $\alpha^{\text{term}}_{i_1+j,i_1+j+1}$ intersecting the initial and terminal points of $\rho_{i_1+j,i_1+j+1}$ respectively, and two sides $\beta^{\text{init}}_{i_1+j,i_1+j+1}$ and $\beta^{\text{term}}_{i_1+j,i_1+j+1}$ intersecting the initial and terminal points of $\sigma_{i_1+j,i_1+j+1}$ respectively. We proceed by setting

$$P := P_{i_1,i_1+1} \cup P_{i_1+1,i_1+2} \cup ... \cup P_{i_2-1,i_2},$$

where $P_{i_1+j,i_1+j+1}$ is glued to $P_{i_1+j+1,i_1+j+2}$ along $\alpha^{\text{term}}_{i_1+j,i_1+j+1}$ and $\alpha^{\text{init}}_{i_1+j+1,i_1+j+2}$. The reader can easily check that $P$ is a polygon of the desired form.

As for uniqueness, suppose $P' : D^2 \to D\mathbb{I}$ is a polygon with initial point $x_i$ passing through entirely short chords on the left and $\rho_{i_1,i_2}$ on the right. As in the proof of Lemma 4.26, there must be pairwise disjoint embedded paths $\alpha_{i_1+j,i_1+j+1} : [0,1] \to \text{Dom}(P')$ for $0 \leq j < i_2 - i_1 - 1$ with $P'([\alpha_{i_1+j,i_1+j+1}]) \subset \alpha$. $P'([\alpha_{i_1+j,i_1+j+1}(0)]) = \rho^+_{i_1+j,i_1+j+1}$ and $\alpha_{i_1+j,i_1+j+1}(1) \in \text{Dom}(P')$. These $\{\alpha_{i_1+j,i_1+j+1}\}$ divide $\text{Dom}(P')$ into regions, the restriction of $P'$ to which are diffeomorphisms $P'_{i_1,i_1+1}, ..., P'_{i_2-1,i_2}$ onto $P_{i_1,i_1+1}, ..., P_{i_2-1,i_2}$ respectively. It then follows that $P'$ differs from $P$ by some diffeomorphism $D^2 \to D^2$, i.e. $P'$ and $P$ are equivalent.

Theorem 4.28. If $DD\left(\frac{1}{2}\right) \boxtimes M(\phi_0)$ is quasi-isomorphic to $A \mathbb{I}_A$, then $\phi_0$ is isotopic to $I$.  

\[
\text{Diagram (4.6)}
\]
Proof. Suppose $DD(\frac{1}{2}) \boxtimes M(\phi_0)$ is quasi-isomorphic to $A^\bullet \mathbb{A}$. Then by Corollary 4.17 and Theorem 4.27 we have $M(\phi_0) \simeq M(\mathbb{I})$. Let $I_i \in \mathcal{B}(Z)$ and $J_i \in \mathcal{B}(Z')$ be idempotents corresponding to $\alpha_i$ and $\beta_i$ respectively. Observe that $J_i H_*(M(\phi_0))I_i$ is the Floer homology $HF(\alpha_i, \beta_j)$ of $\alpha_i$ and $\beta_j$, i.e. the homology of the chain complex generated by intersection points of $\alpha_i$ and $\beta_j$ whose differential counts (equivalence classes of) immersed bigons between $\alpha_i$ and $\beta_j$. Since $HF(\alpha_i, \beta_i)$ is an isotopy invariant of $\alpha_i$ and $\beta_j$, we can assume there are no bigons between $\alpha$ and $\beta$, and we have

$$J_i H_*(M(\phi_0))I_i = i(\alpha_i, \beta_j),$$

where $i(\alpha_i, \beta_j)$ is the geometric intersection number of $\alpha_i$ and $\beta_j$, i.e. the minimal number of intersection points over all isotopic representatives of $\alpha_i$ and $\beta_j$.

But then since $J_i H_*(M(\phi_0))I_i$ is a quasi-isomorphism invariant, we must have $i(\alpha_i, \beta_j) = \delta_{ij}$. It follows that, up to isotopy, $\phi_0$ fixes the dual curves $\{\eta_i\}$. Since $F \setminus \cup_i \eta_i$ is a collection of disks, $\phi_0$ must be isotopic to the identity. □

References

[1] Mohammed Abouzaid, On the fukaya categories of higher genus surfaces, 2006, arXiv:0606598v2.
[2] Denis Auroux, Fukaya categories of symmetric products and bordered Heegaard-Floer homology, 2010, arXiv:1001.4323.
[3] Robert Lipshitz, Peter S. Ozváth, and Dylan P. Thurston, A Faithful linear-categorical action of the mapping class group of a surface with boundary, 2010, arXiv:1012.1032.
[4] Robert Lipshitz, Peter S. Ozváth, and Dylan P. Thurston, Bimodules in bordered Heegaard Floer homology, 2010, arXiv: 1003.0598.
[5] Robert Lipshitz, Peter S. Ozváth, and Dylan P. Thurston, Computing $\hat{HF}$ by factoring mapping classes, 2010, arXiv: 1010.2550.
[6] Robert Lipshitz, Peter S. Ozváth, and Dylan P. Thurston, Heegaard Floer homology as morphism spaces, 2010, arXiv: 1005.1248.
[7] Benson Farb and Dan Margalit, A primer on mapping class groups, 2010 book draft, available at http://www.math.utah.edu/~margalit/primer/.

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