The Cardinality of the Second Uniform Indiscernible

Greg Hjorth
Group in Logic, University of California, Berkeley, CA94720

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Abstract

When the second uniform indiscernible is $\aleph_2$, the Martin-Solovay tree only constructs countably many reals; this resolves a number of open questions in descriptive set theory.

1. Introduction and Definitions.

From now on we work in the theory ZFC + $\forall x \in \omega \omega (x^\# \exists)$.

Assuming $\forall x \in \omega \omega (x^\# \exists)$, Martin and Solovay showed that every $\Sigma^1_3$ set is the projection of a simply definable tree, $T_2$. This extended several earlier results, but in some instances only with the further assumption that $T_2$ has size less than $\aleph_2$.

The present paper smooths the way for a further analysis by showing that if $u_2 = \aleph_2$ (a necessary and sufficient condition for $T_2$ to have size bigger than $\aleph_1$) then the smallest inner model of set theory containing $T_2$ and all the ordinals, $\mathbf{L}[T_2]$, has only countably many reals.

1.1. Definition: $\gamma \in \text{Ord}$ is a uniform indiscernible if $\forall x \in \omega \omega (\exists \gamma \in \text{L}[x]$ indiscernible).

1.2. Notation: For $\alpha \in \text{Ord}$, let $u_\alpha$ be the $\alpha^{th}$ uniform indiscernible, beginning with $u_1 = \aleph_1$. For $x \in \omega^\omega$, $\tau \in \text{L}([x])$ indicates that $\tau$ is a skolem function over $\text{L}[x]$, definable from no parameters.

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1.3. Representation Lemma (Solovay): For all $\gamma \in \text{Ord}$, $\beta < u_\gamma$, there exists $x \in \omega$, $\tau \in L(L[x])$, and $\gamma_1, \ldots, \gamma_n < \gamma$ such that $\tau(u_{\gamma_1}, \ldots, u_{\gamma_n}) = \beta$. $\blacksquare$

A similar result holds for finite strings of ordinals; any finite string of ordinals less than $u_\omega$ can be coded from a single real and finitely many of the uniform indiscernibles less than $u_\omega$. These lemmas are significant because they enable us to phrase questions about what occurs inside $L[x]$ at $\alpha < u_\omega$ in a $\Delta^3_3(x,y)$ manner, for any $y \in \omega$ from which $\alpha$ can be defined using the uniform indiscernibles.

For our purposes it will be unimportant how the Martin-Solovay tree is defined. It can, for example, be extracted from the scale discussed in Moschovakis [3]. A closely related tree arises from the “shift maps”, as implicit in the construction in 2.1. below. Both derive from [2]. The only properties of the tree required are that:

(i) it is no worse than $\Sigma^1_3$ in the codes for the ordinals less than $u_\omega$;
(ii) it projects to the complete $\Sigma^1_3$ set;
(iii) it has the same cardinality as $u_2$.

Let us fix such a tree and call it $T_2$.

1.4. Notation: For $\alpha \in \text{Ord}$, $x \in \omega$, set $\text{Next}(\alpha, x)$ to be the least $L[x]$ indiscernible above $\alpha$.

2. $u_2 = N_2$.

2.1. Theorem (ZFC+$\forall x \in \omega (\exists^x \exists))$: If $u_2 = N_2$, then there are only countably many reals in $L[T_2]$.

Proof:

Suppose otherwise, and now we will derive a contradiction. Let $\theta$ be a big, regular cardinal, so that $V_\theta \models \text{"$T_2$ constructs } \geq \omega_1 \text{ many reals"}$. Now choose

(i) $N_0 < N_1 < \ldots < N_i \ldots < V_\theta$, such that $\forall i \in \omega$, $N_1 \subset N_i$, $|| N_i ||=N_1$;
(ii) $(y_i)_{i \in \omega} \subset \omega^\omega$, $y_i \in N_{i+1}$, $y_i$ recursive in $y_{i+1}$, such that $\text{Next}(N_1, y_i) > N_2 \cap N_i$.

Let $y \in \omega^\omega$ uniformly code the sequence $(y_i)_{i \in \omega}$, let $N_\omega = \cup (N_i)_{i \in \omega}$, let $N$ be the transitive collapse of $N_\omega$, and let $T^*_2$ be $T_2$ as calculated in $N$. $N$ must be correct in calculating that $T^*_2$ constructs at least $N_1$ many reals. So it will suffice to show that $T^*_2 \in L[y]$. Let $(w_\alpha)_{\alpha < \omega_1}$ be a generic sequence of reals for $\text{Coll}(\omega, < \omega_1)$. In virtue of general facts about forcing, it will suffice to show that $T^*_2$ in $L[y, (w_\alpha)_{\alpha < \omega_1}]$. Since $(T_2)^N[(w_\alpha)_{\alpha < \omega_1}] = (T_2)^N \models T^*_2$ and since every ordinal less than $(u_\omega)^N$ is coded by some $(u_1)^N, \ldots, (u_n)^N$
and \((w_{\alpha_1}, \ldots, w_{\alpha_n}, y_0, \ldots, y_n)\), it suffices to show that \(\text{Th}_{\Sigma_3}^{N[(w_\alpha)_{\alpha<\omega_1}]}((w_\alpha)_{\alpha<\omega_1} \cup (y_\ell)_{\ell<\omega})\) can be calculated by \(L[y,(w_\alpha)_{\alpha<\omega_1}]\).

Let \(S\) be the set of \(\Sigma_3\) sentences such that \(\exists z \psi(w_{\alpha_1}, \ldots, w_{\alpha_n}, y_0, \ldots, y_n, z)\) ∈ \(S\) provided:

(i) \(\psi\) is \(\Pi_2^1\);

(ii) there exists \(x\) recursively above \((w_{\alpha_1}, \ldots, w_{\alpha_n}, y_0, \ldots, y_n)\), and a well-founded model \(m(x) \models \exists z \psi(w_{\alpha_1}, \ldots) \land \forall x \L[\L]\), along with indiscernibles \((e_i)_{i<\omega, i>0}\) which generate \(m(x)\);

(iii) and there exists \(f : \text{Ord}^{m(x)} \to (u_\omega)^N\), such that \(m(x) \models \gamma = \tau(c_1, \ldots, c_m, c_{m+1}) \iff f(\gamma) \in ([u_m]^N, (u_{m+1})^N)\), for all \(\tau \in \text{L}(M(x))\), and such that \(f(\tau(c_1, \ldots, c_n)) = \sigma([u_1]^N, \ldots, [u_m]^N)\) implies that \(f(\tau(c_1, \ldots, c_n) = \sigma([u_1]^N, \ldots, [u_m]^N)\), for all \(\tau \in \text{L}(m(x))\), \(\sigma \in \text{L}(L[r])\), \(r \in (\omega^\omega)^N[(w_\alpha)_{\alpha<\omega_1}]\), \(m < n\), \(0 < l_1 < l_2, \ldots, < l_n < \omega\).

Observe that membership in \(S\) can be phrased in terms of a tree construction, with a given \(f\) and \(m(x)\) corresponding to a branch. We can view the nodes of the tree as constructing larger and larger finite initial segments of the theory of \(m(x)\) of \((c_i)_{i<\omega}\), as in a consistency property, along with larger and larger finite initial segments of \(f\), that witnesses well-foundedness in a particularly strong form. Observe that the tree can in fact be calculated in \(L[y,(w_\alpha)_{\alpha<\omega_1}]\), since it can locate a set of reals \(X \subset (\omega^\omega)^N\) which is closed under the pairing and sharp operations and provides codes for the ordinals below \((u_\omega)^N\). In essence, we may take \(X\) to be the set of reals generated by \((w_\alpha)_{\alpha<\omega_1} \cup (y_\ell)_{\ell<\omega}\) by the operations of pairing and taking sharps, and observe that the proof of Solovay’s representation lemma goes through for reals resticted to this set and for ordinals.

Claim: if \(\exists z \psi(w_{\alpha_1}, \ldots)\) ∈ \(S\), then \(N[(w_\alpha)_{\alpha<\omega_1}] \models \exists z \psi(w_{\alpha_1}, \ldots)\).

Given a branch \((m(x), f)\), we can expand \(m(x)\) out along \(\omega_1\) many indiscernibles, inducing a model \(m(x)(c_\alpha)_{0<\alpha<\omega_1}\). By Soenfield absoluteness, it suffices to show that this model is well founded. Now we witness well-foundedness by canonically extending \(f\) to \(f_{\omega_1} : \text{Ord}^{m(x)(c_\alpha)_{0<\alpha<\omega_1}} \to (u_\omega)^N\).

Given \(\tau(c_{\beta_1}, \ldots, c_{\beta_n})\), where \(\tau \in \text{L}(m(x))\), we find \(r \in (\omega^\omega)^N, \sigma \in \text{L}(L[r])\), \(m \leq n\) such that \(f(\tau(c_1, \ldots, c_n)) = \sigma([u_1]^N, \ldots, (u_m)^N) \in ([u_m]^N, (u_{m+1})^N)\), and set \(f_{\omega_1}(\tau(c_{\beta_1}, \ldots, c_{\beta_n})) = \sigma([u_{\beta_1}]^N, \ldots, (u_{\beta_n})^N)\).

Subclaim: \(f_{\omega_1}\) is well defined.

Suppose \(\tau_1(c_{\beta_1}, \ldots, c_{\beta_n}) = \tau_2(c_{\beta_1}, \ldots, c_{\beta_n})\), \(f(\tau_1(c_1, \ldots, c_n)) = \sigma_1([u_1]^N, \ldots, (u_n)^N)\) and \(f(\tau_2(c_1 \ldots, c_{n+1-k})) = \sigma_2([u_1]^N, \ldots, (u_{n+1-k})^N)\) where \(\sigma_1\) and \(\sigma_2\) are skolem
functions definable from reals in \( N \). Then since \( f \) respected the shift maps and the \((u_i)_{0<i<\omega}^N\) are joint indiscernibles for the reals used in \( \sigma_1 \) and \( \sigma_2 \), we have that
\[
 f(\tau_2(c_k \ldots c_n)) = f(\tau_1(c_1 \ldots c_n)) = \sigma_2((u_k)^N \ldots (u_n)^N) = \sigma_1((u_1)^N \ldots (u_n)^N),
\]
and, consequently, \( \sigma_2((u_\beta_k)^N \ldots (u_\beta_n)^N) = \sigma_1((u_\beta_1)^N \ldots (u_\beta_n)^N) \), as required.

The other cases follow in an exactly similar fashion.

Subclaim: \( f_{\omega_1} \) is order preserving.

This is immediate given the previous claim and the requirement that \( f \) respect the order on \( \text{Ord}^m(x) \).

Claim: if \( N[(w_\alpha)_{\alpha<\omega}] \models \exists z \psi(w_{\alpha_1}, \ldots) \) then \( \exists z \psi(w_{\alpha_1}, \ldots) \in S \).

This follows by considering the appropriate sharp, and the natural map into \((u_\omega)^N\).

But since \( S \) can be calculated in \( L[y, (w_\alpha)_{\alpha<\omega}] \), these last two claims suffice. \( \square \)

2.2. Corollaries (ZFC+\( \forall x \in \omega(x^3 \exists) \)):

(i) every \( \Sigma^1_3 \) set of size greater than \( \aleph_1 \) contains a perfect set;
(ii) Martin’s Axiom implies \( \Sigma^1_3 \) Lebesgue measurability;
(iii) if there is a \( \Sigma^1_3 \) well ordering of the reals, then the continuum hypothesis holds;
(iv) if \( K_0 \subset V \) is an inner model with a good \( \Sigma^1_3 \) well ordering of the reals, and if \( u_2 = \aleph_2 \), then \( K_0 \prec \Sigma^1_3 \), or the goodness of the well ordering is in some sense sufficiently robust, then \( K_0 \) does not calculate \( \omega_1 \) correctly.

Proof:

These results are all proved using standard techniques along with the fact that \( T_2 \) never constructs more than \( \aleph_1 \) many reals: if \( u_2 = \aleph_2 \), then this follows by the theorem; if \( u_2 < \aleph_2 \) then, after coding \( T_2 \) as subset of the ordinals, this follows by the entirely general observation that a subset of \( \omega_1 \) never constructs more than \( \aleph_1 \) many reals. For instance, (iii) follows as in Kechris’s proof of Mansfield theorem that if there is a \( \Sigma^1_2 \) well ordering of the reals then every real is in \( L \). (ii) follows by considering that “almost every” real must be random over \( L[T_2] \). This answers a question of Judah’s. \( \square \)

2.3. Remark: 2.2.(iii) is rather strange sounding, but the point is that the standard canonical inner models of large cardinals below \( \Pi^1_2 \) determinacy have such “robust” well orders. This indicates difficulties in forcing models of \( u_2 = \aleph_2 \).
References

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