Biquandle Bracket Quivers

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Abstract

Biquandle brackets define invariants of classical and virtual knots and links using skein invariants of
biquandle-colored knots and links. Biquandle coloring quivers categorify the biquandle counting invariant
in the sense of defining quiver-valued enhancements which decategorify to the counting invariant. In this
paper we unite the two ideas to define biquandle bracket quivers, providing new categorifications of
biquandle brackets. In particular, our construction provides an infinite family of categorifications of the
Jones polynomial and other classical skein invariants.

Keywords: Biquandles, Biquandle Brackets, Quiver Enhancements, Enhancements of Counting
Invariants

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1 Introduction

In [8], skein invariants of biquandle-colored knots and links known as biquandle brackets were introduced.
A biquandle bracket associates an element of a commutative ring with identity $R$ to a biquandle-colored
knot or link diagram via skein relations which vary depending on the biquandle colors at the crossing, in a
way which is unchanged by biquandle-colored Reidemeister moves. The multiset of such ring elements over
the complete set of biquandle colorings (i.e., over the biquandle homset $\text{Hom}(B(L), X)$ where $B(L)$ is the
fundamental biquandle of the link $L$ and $X$ is the coloring biquandle) forms an enhanced invariant of knots
and links that determines the biquandle counting invariant and includes the classical skein invariants and
the biquandle 2-cocycle invariant as special cases.

In [4], the quandle homset invariant was categorized (in the sense of defining a category-valued invariant
from which the original invariant can be recovered) to directed graphs we call quandle coloring quivers
associated to subsets of the endomorphism ring of the coloring quandle $X$. Convenient polynomial enhancements
of the quandle counting invariant were obtained as decategorifications of these quivers. In [3] these quiver-valued
invariants were extended to the cases of pseudoknots, singular knots and virtual knots via psyquandles and
biquandles; in particular, biquandle coloring quivers were established for classical and virtual knots and
links.

In this paper we categorify biquandle brackets via biquandle counting quivers to obtain biquandle bracket
quivers, quiver-valued invariants of classical and virtual knots and links weighted with weights associated to
a subset of the endomorphism ring of the coloring biquandle together with a biquandle bracket structure with
coefficients in a commutative ring with identity $R$. The original biquandle bracket polynomial invariants as
well as new enhanced invariants are recovered as decategorifications. We note that this construction yields
an infinite family of alternative categorifications of the classical skein invariants and the biquandle 2-cocycle
invariants, one for each element of the power set of each biquandle in the biquandle bracket construction.

The paper is organized as follows. In Section 2 we review the basics of biquandles, biquandle brackets and
biquandle counting quivers. In Section 3 we introduce the new invariants and illustrate their computation
via examples. In Section 4 we provide additional computational examples, and we conclude in Section 5
with some questions for future research.

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2 Biquandles, Brackets and Quivers

In this section we briefly review biquandles, biquandle brackets and quivers; see [4], [5] and [8] for more details.

Definition 1. A \textit{biquandle} is a set \( X \) with two binary operations \( \sqcup, \sqcap : X \to X \) satisfying \( \forall x, y, z \in X \):

(i) \( x \sqcup x = x \sqcap x \),

(ii) The maps \( \alpha_y, \beta_y : X \to X \) and \( S : X \times X \to X \times X \) defined by \( \alpha_y(x) = x \sqcup y \), \( \beta_y(x) = x \sqcap y \) and \( S(x, y) = (y \sqcap x, x \sqcup y) \) are invertible,

(iii) The \textit{exchange laws} are satisfied:

\[
\begin{align*}
(x \sqcup y) \sqcup (z \sqcup y) &= (x \sqcup z) \sqcup (y \sqcap z) \\
(x \sqcap y) \sqcap (z \sqcup y) &= (x \sqcap z) \sqcap (y \sqcap z) \\
(x \sqcup y) \sqcap (z \sqcap y) &= (x \sqcap z) \sqcap (y \sqcap z)
\end{align*}
\]

If \( x \sqcup y = x \) for all \( x, y \in X \), we say \( X \) is a \textit{quandle}.

Example 1. Let \( X = \mathbb{Z}_n \), \( x \sqcup y = x \) for all \( x, y \in X \) and \( x \sqcap y \equiv 2y - x \) (mod \( n \)). Then \( X \) is called a \textit{dihedral quandle}, also sometimes called a \textit{cyclic quandle} or a \textit{Takasaki kei}.

Example 2. Let \( G \) be a group, \( x \sqcup y = x \) for all \( x, y \in X \) and let \( \sqcap \) be \textit{n}-fold conjugation: \( x \sqcap y = y^n xy^{-n} \). Then \( G \) is a quandle.

Example 3. For a non-quandle family of examples, let \( R \) be a commutative ring with identity and let \( t, s \in R^\times \) be units in \( R \). Then \( R \) has a biquandle structure known as an \textit{Alexander biquandle} defined by \( x \sqcup y = tx + (s - t)y \) and \( x \sqcap y = sx \).

Example 4. Let \( L \) be an oriented knot or link. The fundamental biquandle of \( L \), denoted \( \mathcal{B}(L) \), is the set of equivalence classes of biquandle words in a set of generators corresponding to the semiarcs in a diagram of \( L \) under the equivalence relation generated by the crossing relations of \( L \) and the biquandle axioms. The Hopf link \( L2a1 \)

\[ x \quad v \quad w \quad y \]

has the fundamental biquandle presentation

\[ \mathcal{B}(L2a1) = \langle x, y, v, w \mid x \sqcap v = w, v \sqcup x = y, v \sqcap x = y, x \sqcup y = w \rangle \]

Remark 1. The fundamental biquandle of a classical knot or link interpreted as an Alexander biquandle determines the Alexander polynomials of the knot or link, for virtual knots and links determines the Sawollek polynomial, also called the generalized Alexander polynomial. See [7] for more details.

Definition 2. Let \( X, Y \) be biquandles with binary operations \( \sqcup_X, \sqcap_X, \sqcup_Y, \sqcap_Y \) respectively. A \textit{biquandle homomorphism} is a map \( f : X \to Y \) such that for any \( a, b \in X \) we have

\[
\begin{align*}
f(a \sqcup_X b) &= f(a) \sqcup_Y f(b) \\
f(a \sqcap_X b) &= f(a) \sqcap_Y f(b)
\end{align*}
\]
Definition 3. Let $X$ be a finite biquandle, called the *coloring biquandle* and $L$ be an oriented knot or link diagram. Then a *biquandle coloring* of $L$ is an assignment of elements of $X$ to the semiarcs in $L$ such that the following crossing relations are satisfied at every crossing:

\[
\begin{align*}
\xrightarrow{y \not \rightarrow x} & \quad \begin{array}{c}
\xrightarrow{y \not \rightarrow x} \\
\xleftarrow{y \not \rightarrow x}
\end{array} \\
\xleftarrow{x \not \rightarrow y} & \quad \begin{array}{c}
\xleftarrow{x \not \rightarrow y} \\
\xleftarrow{x \not \rightarrow y}
\end{array}
\end{align*}
\]

Remark 2. The set of biquandle colorings of $L$ can be identified with the set of biquandle homomorphisms from the fundamental biquandle of $L$ to $X$, denoted $\text{Hom}(\mathcal{B}(L), X)$. If $L$ is tame and $n$ denotes the number of semiarcs in $L$ then $\mathcal{B}$ is finitely generated with $2n$ generators; hence $|\text{Hom}(\mathcal{B}(L), X)| \leq |X|^{2n}$. The *biquandle counting invariant* is the cardinality of the coloring space, denoted $\Phi_X(L) = |\text{Hom}(\mathcal{B}(L), X)|$.

Definition 4. Let $X$ be a finite biquandle, $R$ a commutative ring with identity and let $R^X$ denote the units of $R$. A *biquandle bracket* over $R$ is a pair of maps $A, B : X \times X \to R^X$ assigning $A_{x,y}B_{x,y} \in R^X$ to the pairs of elements $(x, y) \in X \times X$ such that the following conditions are satisfied:

(i) $\forall x \in X$, the elements $-A_{x,x}^{-1}B_{x,x}$ are equal with their common value denoted as $w \in R^X$, i.e. $w$ is the value of a positive crossing.

(ii) $\forall x, y \in X$, the elements $-A_{x,y}^{-1}B_{x,y} - A_{x,y}B_{x,y}$ are equal with their common value denoted as $\delta \in R$, i.e. $\delta$ is the value of a simple closed curve.

(iii) $\forall x, y, z \in X$ we have

\[
\begin{align*}
A_{x,y}A_{y,z}A_{x \not \rightarrow z, y \not \rightarrow z} & = A_{x,z}A_{y,x \not \rightarrow z, y \not \rightarrow z}A_{x \not \rightarrow z, y \not \rightarrow z} \\
A_{x,y}B_{y,z}B_{x \not \rightarrow z, y \not \rightarrow z} & = B_{x,z}B_{y,x \not \rightarrow z, y \not \rightarrow z}A_{x \not \rightarrow z, y \not \rightarrow z} \\
B_{x,y}A_{y,z}B_{x \not \rightarrow z, y \not \rightarrow z} & = B_{x,z}A_{y,x \not \rightarrow z, y \not \rightarrow z}B_{x \not \rightarrow z, y \not \rightarrow z} \\
A_{x,y}A_{y,z}B_{x \not \rightarrow z, y \not \rightarrow z} & = A_{x,z}B_{x,y \not \rightarrow z, x \not \rightarrow z} + A_{x,z}A_{y,x \not \rightarrow z, y \not \rightarrow z}A_{x \not \rightarrow z, y \not \rightarrow z} + \delta A_{x,z}B_{x,y \not \rightarrow z, x \not \rightarrow z} + B_{x,z}B_{y,x \not \rightarrow z, y \not \rightarrow z} + B_{x,z}B_{y,x \not \rightarrow z, y \not \rightarrow z}
\end{align*}
\]

These biquandle bracket axioms are chosen in such a way that the state-sum expansion of a $X$-colored oriented knot or link diagram $L$ using the skein relations

\[
\begin{align*}
\xrightarrow{x} & = A_{x,y} \\
\xleftarrow{y} & + B_{x,y}
\end{align*}
\]
is invariant under $X$-colored Reidemeister moves with $w$ as the value of a positive crossing and $\delta$ as the value of a simple closed curve.

We specify biquandle brackets over $X = \{x_1, \ldots, x_n\}$ with a block matrix $[A|B]$ with blocks $A$ and $B$ whose $(j,k)$ entries are the skein coefficients $A_{j,k}$ and $B_{j,k}$.

Examples of biquandle brackets include:

- Classical skein invariants including the Conway, Jones, HOMFLYPT and Kauffman polynomials with trivial biquandle,

- Biquandle 2-cocycle invariants (see [2,5]) are biquandle brackets such that the matrices of coefficients $A$ and $B$ are equal, and

- Other potentially new invariants.

**Remark 3.** Biquandle brackets representing biquandle 2-coboundaries can be used to modify other biquandle brackets over the same biquandle and ring by entrywise multiplication to obtain new biquandle brackets, with the resulting invariant equivalent to the original. Curiously, this operation extends the notion of biquandle brackets being cohomologous to other brackets beyond the set of biquandle cocycles. In particular, in [6] it is shown that many examples in other works are cohomologous to brackets representing the Jones polynomial.

**Definition 5.** Let $X$ be a finite biquandle and $L$ an oriented knot or link. For any set of biquandle endomorphisms $S \subset \text{Hom}(X, X)$ the associated *biquandle coloring quiver*, denoted $Q^X_S(L)$, is the directed graph whose vertices are all elements $f \in \text{Hom}(B(L), X)$. An edge is directed from $f$ to $g$ if $g = \phi f$ for an element $\phi \in S$.

**Example 5.** The Hopf link $L2\alpha1$ has four colorings by the biquandle $X$ given by the operation tables

| $\triangleright$ | $1$ | $2$ | $\triangleright$ | $1$ | $2$ |
|-----------------|-----|-----|-----------------|-----|-----|
| $1$             | 1   | 1   | $1$             | 1   | 1   |
| $2$             | 2   | 2   | $2$             | 2   | 2   |

An important special case is $S = \text{Hom}(X, X)$, which is called the *full biquandle coloring quiver* of $L$ with respect to $X$, denoted $Q^X_X(L)$.

**Example 5.** The Hopf link $L2\alpha1$ has four colorings by the biquandle $X$ given by the operation tables
Choosing $S = \{\phi_1, \phi_2\}$ where $\phi_1(1) = \phi_1(2) = 1$ and $\phi_2(1) = \phi_2(2) = 2$ yields the following biquandle coloring quiver:

Biquandle coloring quivers provide categorifications of the biquandle counting invariant in the sense that they are quivers and hence small categories, with vertices as objects and directed paths as morphisms (with empty paths based at vertices as identity elements). Any further invariants of quivers then automatically provide invariants of knots and links via this construction.

### 3 Biquandle Bracket Quivers

In this section we introduce our main definition, the biquandle bracket quiver, and illustrate its computation with an example.

**Definition 6.** Let $X$ be a finite biquandle, $R$ a commutative ring with identity, $\beta = \{A, B : X \times X \rightarrow R^\times\}$ a biquandle bracket and $S \subset \text{Hom}(X, X)$ a set of biquandle endomorphisms. For any oriented link $L$, we define the biquandle bracket quiver of $L$ with respect to $X, R, \beta$ and $\phi$, denoted $\text{BBQ}^S_{X, \beta}(L)$ to be the biquandle coloring quiver of $L$ with vertices $v$ weighted with the biquandle bracket values $\beta(v)$ of the $X$-colorings of $L$ they represent.

Since both the biquandle coloring quiver and the biquandle bracket multiset are unchanged by Reidmeister moves, we have our main result:

**Theorem 1.** The isomorphism class of $\text{BBQ}^S_{X, \beta}(L)$ as a weighted graph is an invariant of oriented knots and links.

**Example 6.** Let $X$ be the biquandle given by the operation tables

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 1 \\
3 & 3 & 3 & 4 \\
4 & 4 & 4 & 3 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 4 & 4 & 4 \\
4 & 3 & 3 & 3 \\
\end{array}
\]

let $\beta$ be the biquandle bracket over $X$ with coefficients in $\mathbb{Z}_3$ given by

\[
\begin{bmatrix}
1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 \\
2 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
2 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 \\
\end{bmatrix}
\]
which has \( \delta = -(1)(2) - 2(1) = -4 = -1 = 2 \) and \( w = -1^2(2) = -2 = 1 \). Let \( \phi : X \to X \) be the biquandle endomorphism defined by \( \phi(1) = 2, \phi(2) = 1, \phi(3) = 3 \) and \( \phi(4) = 4 \) and consider the Hopf link \( L = L_{2a1} \). \( L \) has eight \( X \)-colorings determining a biquandle coloring quiver as shown.

Taking one of these colorings and expanding to the set of Kauffman states, we find the biquandle bracket value for this coloring

\[
1 \, \begin{array}{c} 1 \end{array} = A_{1,1}A_{1,1} \begin{array}{c} \circ \end{array} + A_{1,1}B_{1,1} \begin{array}{c} \bigcirc \end{array} \\
+ B_{1,1}A_{1,1} \begin{array}{c} \bigcirc \end{array} + B_{1,1}B_{1,1} \begin{array}{c} \bigcirc \end{array} \\
= (1)(1)2^2 + (1)(2)2 + (2)(1)2 + (2)(2)2^2 \\
= 1 + 1 + 1 + 1 \\
= 1
\]

Repeating for each coloring, we obtain the biquandle bracket quiver \( \mathcal{BBQ}^S_{X,\beta}(L) \).

As with the biquandle coloring quiver, \( \mathcal{BBQ}^S_{X,\beta}(L) \) is a small category with vertices as objects and directed paths as morphisms (with identity morphisms given by “empty” paths which stay at a vertex); hence, it is a categorification of the biquandle bracket polynomial invariant. Invariants of these graphs then give us new invariants of knots as decategorifications.
Definition 7. Let $BBQ_{X,\beta}(L)$ be the biquandle bracket quiver of an oriented link $L$ with vertex set $V$ and edge set $E$. We define the in-degree biquandle bracket polynomial to be

$$\Phi^{S,\text{deg}^+}_{X,\beta}(L) = \sum_{v \in V} u^{\beta(v)} v^{\deg^+(v)}$$

and we define the biquandle bracket 2-variable polynomial to be

$$\Phi^{S,2}_{X,\beta}(L) = \sum_{e \in E} s^{\beta(s(e))} t^{\beta(t(e))}$$

where $s(e)$ and $t(e)$ are the source and target vertices of each edge $e \in E$.

Corollary 2. The polynomials $\Phi^{S,\text{deg}^+}_{X,\beta}(L)$ and $\Phi^{S,\text{deg}^+}_{X,\beta}(L)$ are invariants of oriented classical knots and links.

Remark 4. If $S = \{\phi\}$ is a singleton, we will often write $\phi$ instead of $\{\phi\}$ in the superscript for simplicity.

Example 7. The Hopf link $L2a1$ has in-degree biquandle bracket and 2-variable biquandle bracket polynomials

$$\Phi^{\phi,\text{deg}^+}_{X,\beta}(L2a1) = 4u^2v + 1uv$$

and

$$\Phi^{\phi,2}_{X,\beta}(L2a1) = 4s^2t^2 + 4st$$

with respect to the biquandle $X$, biquandle bracket $\beta$ and biquandle endomorphism $\phi$ in Example 6.

Everything in our construction applies without modification to virtual knots and links by the usual convention of ignoring the virtual crossings when determining biquandle colorings; hence we have the following:

Corollary 3. Biquandle bracket quivers $BBQ^S_{X,\beta}(L)$ and the polynomials $\Phi^{\phi,2}_{X,\beta}(L)$ and $\Phi^{\phi,\text{deg}^+}_{X,\beta}(L)$ are invariants of oriented virtual knots and links.

4 Examples

In this section we collect a few examples of the new invariants.

Example 8. Let $X$ and $\beta$ be the biquandle and biquandle bracket on $R = \mathbb{Z}_6$ defined by the operation tables and matrix

$$\begin{array}{c|cccc}
\triangleright & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 2 & 3 & 4 \\
2 & 1 & 1 & 1 & 1 \\
3 & 3 & 4 & 4 & 4 \\
4 & 3 & 3 & 3 & 4 \\
\end{array}$$

and let $\phi_1, \phi_2 : X \to X$ be the endomorphism defined by $\phi_1(1) = 2, \phi_2(2) = 1, \phi_3(3) = 4$ and $\phi(4) = 3$. We computed via python the $\Phi^{\phi_i,2}_{X,\beta}$-values defined in Section 3 for a choice of orientation for each of the prime links with up to seven crossings in the table at $\text{H}$; the results are in the table.

| $\Phi^{\phi_1,2}_{X,\beta}(L)$ | $L$ |
|-----------------------------|----------|
| $6s^4t^4 + 2s^2t^4$       | $L2a1, L6a2, L6a3, L7a5, L7a6,$ |
| $8s^4t^4 + 8s^2t^2$       | $L4a1$ |
| $8s^4t^4 + 2s^2t^2 + 2s^2t^4 + 4s^2t^2$ | $L5a1, L7a1, L7a3, L7a4$ |
| $12s^4t^4 + 2s^2t^4 + 2s^2t^4$ | $L7a1,$ |
| $12s^4t^4 + 4s^2t^2$       | $L6a1, L7a2, L7a2$ |
| $12s^4t^2 + 12s^2t^4 + 40s^2t^2$ | $L6a2$ |
| $16s^2t^2$                | $L6a5, L6n1, L7a7$ |
In particular we note that most of the links in the table have 16 $X$-colorings but are distinguished by the enhancement information.

**Example 9.** Let $X$ be the biquandle with operation tables

\[
\begin{array}{cccc|cccc}
\varphi & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 \\
3 & 4 & 4 & 4 & 4 & 3 & 4 & 4 & 4 \\
4 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 3 \\
\end{array}
\]

let $\beta$ be the $X$-bracket with values in $R = \mathbb{Z}[q^{\pm 1}]$ given by the matrix

\[
\begin{bmatrix}
1 & 1 & q & q \\
1 & 1 & q & q \\
1 & 1 & q & q \\
1 & 1 & q & q \\
\end{bmatrix}
\]

The endomorphism ring of $X$ has eight elements; let $S = \text{Hom}(X, X)$. Then the trefoil knot $3_1$ has

\[
\Phi_{X,\beta}^{S, \text{deg}^+}(3_1) = 16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4
\]

and

\[
\Phi_{X,\beta}^{S, 2}(3_1) = 16s^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4
\]

while the figure eight knot $4_1$ has

\[
\Phi_{X,\beta}^{S, \text{deg}^+}(4_1) = 16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4
\]

and

\[
\Phi_{X,\beta}^{S, 2}(4_1) = 32s^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4
\]

These two knots have the same biquandle bracket invariant value but are distinguished by the quiver enhancement information.

**Example 10.** The table contains the values of $\Phi_{X,\beta}^{S, \text{deg}^+}(L)$ for all prime knots with up to eight crossings with respect to the biquandle and biquandle bracket in Example 9 and $S = \text{Hom}(X, X)$.

| $K$ | $\Phi_{X,\beta}^{S, \text{deg}^+}(L)$ |
|-----|---------------------------------|
| $3_1$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $4_1$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $5_1$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $5_2$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $1_2$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $6_2$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $6_3$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_1$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_2$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_3$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_4$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_5$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_6$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
| $7_7$ | $16u^{-q^3-q^5}v^{12} + 16u^9q^{-q^3-q}v^4$ |
Example 11. Let $X$ be the biquandle with operation tables

\[
K
\begin{array}{|c|c|c|c|c|}
\hline
\Phi_{X,\beta}^s(L) & 1 & 2 & 3 & 4 \\
\hline
81 & 1 & 2 & 2 & 2 \\
82 & 1 & 1 & 1 & 1 \\
83 & 3 & 4 & 4 & 4 \\
84 & 4 & 3 & 3 & 3 \\
\hline
\end{array}
\]

let $\beta$ be the $X$-bracket with values in $R = \mathbb{Z}[q^{\pm 1}]$ given by the matrix

\[
\begin{bmatrix}
1 & 1 & q & q & q & q & 1 & 1 \\
1 & 1 & q & q & q & q & 1 & 1 \\
1 & 1 & q & q & q & q & 1 & 1 \\
1 & 1 & q & q & q & q & 1 & 1 \\
\end{bmatrix}
\]

and let $S = \text{Hom}(X, X)$. The virtual knots 3.1 and 3.7
both have trivial Jones polynomial but are distinguished by the biquandle bracket quiver invariants with
\[
\Phi_{S, \deg^+}(3.1) = 16u^{-q-q^{-1}}v^4.
\]
and
\[
\Phi_{X, \beta}(3.7) = 8u^{-q-q^{-1}}v^4 + 8u^{-q^2-q^{-2}}v^4.
\]
In particular, this example shows that biquandle bracket quivers and their decategorification invariants are not determined by the Jones polynomial.

5 Questions
We conclude with some questions for future research.

- One of the more exciting aspects of this project was the realization that while many biquandle brackets are cohomologous to classical skein invariants such as the Jones or HOMFLYPT polynomial, the biquandle bracket quivers for these can be different. In particular, this means our construction gives new and different categorifications of these invariants, allowing for further new invariants via decategorification.
- What other new invariants can be obtained from these quivers?
- How can these quivers be further enhanced?
- As always, faster method of computing biquandle brackets, particularly for large biquandles and large finite or infinite rings, are of great interest.

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