Another generalization of the box–ball system with many kinds of balls

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A cellular automaton that is a generalization of the box–ball system with either many kinds of balls or finite carrier capacity is proposed and studied through two discrete integrable systems: nonautonomous discrete KP lattice and nonautonomous discrete two-dimensional Toda lattice. Applying reduction technique and ultradiscretization procedure to these discrete systems, we derive two types of time evolution equations of the proposed cellular automaton, and particular solutions to the ultradiscrete equations.

Keywords: box–ball systems; nonautonomous discrete two-dimensional Toda lattice; biorthogonal polynomials; spectral transformations.

1. Introduction

In this paper, we propose a novel soliton cellular automaton that is a generalization of the box–ball system (BBS) with either many kinds of balls or finite carrier capacity, but not same as the known BBS with both rules proposed by Hatayama et al. [1]. It is known that the time evolution equations of the BBSs are derived by applying a limiting procedure called ultradiscretization to discrete integrable systems [2]. There are two types of evolution equations for the BBS. One is derived from the discrete KdV lattice which corresponds to the Euler representation and another is from the discrete Toda lattice with finite boundary condition [3] which corresponds to the “difference form” of the Lagrange representation, we call the finite Toda representation in this paper. Note that these Euler–Lagrange notions for cellular automata come from hydrodynamics [4]. In the previous paper [5], it was clarified that the BBS with both many kinds of balls and finite carrier capacity, originally proposed and analyzed through the \((M+1)\)-reduced nonautonomous discrete KP lattice (nd-KP lattice) [6], which corresponds to the Euler representation, also has the finite Toda representation derived from the nonautonomous discrete hungry Toda lattice (ndh-Toda lattice). The proposed cellular automaton also has a simple evolution rule and comes from the nd-KP lattice and the ndh-Toda lattice via ultradiscretization. Furthermore, there is an interesting particular solution especially for the finite Toda representation.

The paper is organized as follows. In Section 2, we explain the time evolution rule of the cellular automaton studied in this paper, and write two types of time evolution equations on the min-plus algebra, i.e. the Euler representation and the finite Toda representation. In Section 3, we show that the time evolution equation of the Euler representation of the cellular automaton written in Section 2 is derived from the nd-KP lattice by applying reduction and ultradiscretization procedures. Imposing the reduction condition to an \(N\)-soliton solution of the nd-KP lattice, we also give an \(N\)-soliton solution to the Euler representation equation. In Section 4, we show that the time evolution equation of the finite Toda representation of the cellular automaton written in Section 2 is derive from the ndh-Toda lattice with finite lattice condition by applying ultradiscretization procedures. We also give a particular solution to the finite Toda representation. Section 5 is devoted to concluding remarks.
Fig. 1. Step-by-step illustration of the rewriting rule of the case in which $M = 3$ and $S^{(t)} = 2$. Underline indicates the position of the machine.

2. A cellular automaton

2.1. Definition

Let us consider a finite-state machine that moves in one direction on an infinite tape. The machine has a state $(V^{(0)}, V^{(1)}, \ldots, V^{(M)}) \in \mathbb{Z}^{M+1}$ and its initial state is $(0, S, S, \ldots, S)$, where $S$ is a positive integer or $+\infty$. We call the state of $V^{(k)}$ **rewritable times of** $k$. The tape is divided into infinitely many cells and an integer from 0 to $M$ is written on each cell. Here, we assume that there are a finite number of positive integers on the tape. The moving machine reads the integer at each cell, and if the integer is $k$ and $V^{(k)} \geq 1$, then the machine rewrites the integer $k$ on the cell to $k - 1$, subtracts 1 from $V^{(k)}$, and adds 1 to $V^{(k-1)}$, where $V^{(-1)} = V^{(M)}$ identically; if the integer is $k$ and $V^{(k)} = 0$, then do nothing.

For example, let us consider the case where $M = 3$, $S = 2$, and the given tape is

$$\ldots 000111230001130000\ldots$$

(See also Fig. 1.) Hereafter, we suppose that the machine on the tape moves from left to right. Since the initial state of the machine is $(0, 2, 2, 2)$, i.e. the rewritable times of 0 is zero, the machine do nothing until reaching the first ‘1’. The machine rewrites the first and second ‘1’s to ‘0’s, and transits its state as $(0, 2, 2, 2) \rightarrow (1, 1, 2, 2) \rightarrow (2, 0, 2, 2)$. Now the rewritable times of 1 becomes zero. Therefore, the third ‘1’ is not rewritten by the machine. Next, the machine rewrites ‘2’, ‘3’, ‘0’ to ‘1’, ‘2’, ‘3’, respectively, and transits its state as $(2, 0, 2, 2) \rightarrow (2, 1, 1, 2) \rightarrow (2, 1, 2, 1) \rightarrow (1, 1, 2, 2)$. Finally, the next integers ‘001130’ are rewritten to ‘300123’ and the state of the machine becomes $(0, 0, 3, 3)$. After that, the machine does not rewrite remaining infinite ‘0’s. We obtain the tape

$$\ldots 000001123300113000\ldots$$

We introduce a discrete time variable $t$ as the number of iterations of the process above. We can choose the parameter $S$ at each $t$, write $S^{(t)}$. Fig. 2 (a) shows an example of the time evolution in which the initial values are given by (1), where ‘.’ is used instead of ‘0’. A more complicated example is shown in Fig. 2 (b). In these examples, we can observe that the blocks of positive integers propagate and interact with each other like solitons. The observation may be reminiscent of the BBS with many kinds of balls and finite carrier capacity. In fact, the proposed cellular automaton is a generalization of the BBS.

2.2. Time evolution equation of the Euler representation

Let

$$U^{(k,t)}_n = \begin{cases} 1 & \text{if an integer } k \text{ is written on the } n\text{th cell at time } t, \\ 0 & \text{otherwise,} \end{cases}$$

and $V^{(k,t)}_n \in \{0, 1, \ldots, MS^{(t)}\}$ be the rewritable times of $k$ just before the machine reads the integer on the $n$th cell from time $t$ to $t + 1$. Then, we have

$$\min(U^{(k,t)}_n, V^{(k,t)}_n) = \begin{cases} 1 & \text{if } U^{(k,t)}_n = 1 \text{ and } V^{(k,t)}_n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$
where \( U \) and \( V \) are given respectively by

\[
U_n^{(k,t+1)} = U_n^{(k,t)} - X_n^{(k,t)} + X_n^{(k+1,t)},
\]

\[
V_n^{(k,t+1)} = V_n^{(k,t)} - X_n^{(k,t)} + X_n^{(k+1,t)},
\]

\[
X_n^{(k,t)} = \min(U_n^{(k,t)}, V_n^{(k,t)}).
\]

where \( U_n^{(k+M+1,t)} = U_n^{(k,t)} \) and \( V_n^{(k+M+1,t)} = V_n^{(k,t)} \) for all \( k, t \) and \( n \). The boundary condition is given by

\[
U_n^{(k,t)} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, \ldots, M. \end{cases}
\]

\[
V_n^{(k,t)} = \begin{cases} 0 & \text{if } k = 0, \\ S(t) & \text{if } k = 1, 2, \ldots, M. \end{cases}
\]

for \( n \ll -1 \). The relations

\[
\sum_{k=0}^{M} U_n^{(k,t)} = 1, \quad \sum_{k=0}^{M} V_n^{(k,t)} = M S(t)
\]

always hold obviously.

Remark 2.1. If \( M = 1 \), then \( U_n^{(0,t)} = 1 - U_n^{(1,t)} \) and \( V_n^{(1,t)} = S(t) - V_n^{(0,t)} \) hold by (3). Hence, the time evolution equation (2a) is rewritten as

\[
U_n^{(1,t+1)} = U_n^{(1,t)} - \min(U_n^{(1,t)}, S(t) - V_n^{(0,t)}) + \min(1 - U_n^{(1,t)}, V_n^{(0,t)})
\]

\[
= \min(1 - U_n^{(1,t)}, V_n^{(0,t)}) + \max(0, U_n^{(1,t)} + V_n^{(0,t)} - S(t)),
\]

where we used the following simple formulae

\[ A + \min(B, C) = \min(A + B, A + C), \]

\[ -\min(-A, -B) = \max(A, B). \]
Subtraction of (2a) from (2b) yields
\[ V_{n+1}^{(0,l)} = V_n^{(0,l)} + U_n^{(1,l)} - U_n^{(1,l+1)} \]
\[ = V_{n-1}^{(0,l)} + U_{n-1}^{(1,l)} - U_{n-1}^{(1,l+1)} + U_{n}^{(1,l)} - U_{n}^{(1,l+1)} \]
\[ = \ldots \]
\[ = \sum_{j=-\infty}^{n} \left( U_j^{(1,l)} - U_j^{(1,l+1)} \right). \] (5)

Substituting (5) into (4), we obtain
\[ U_n^{(1,l+1)} = \min\left( 1 - U_n^{(1,l)}, \sum_{j=-\infty}^{n-1} \left( U_j^{(1,l)} - U_j^{(1,l+1)} \right) \right) + \max\left( 0, \sum_{j=-\infty}^{n} U_j^{(1,l)} - \sum_{j=-\infty}^{n-1} U_j^{(1,l+1)} - S(t) \right). \]

This equation is well known as the time evolution equation of the BBS with carrier capacity \( S(t) \). In this case, the variable \( V_n^{(0,l)} \) denotes the number of balls in the carrier at the \( n \)th box from time \( t \) to \( t+1 \).

**Remark 2.2.** If \( S(t) = +\infty \) for all \( t \), then \( V_n^{(k,l)} = +\infty \) and \( X_n^{(k,l)} = U_n^{(k,l)} \) hold for \( k = 1, 2, \ldots, M \). Then, (2a) is rewritten as
\[ U_n^{(k,l+1)} = U_n^{(k+1,l)}, \quad k = 1, 2, \ldots, M - 1, \]
\[ U_n^{(M,l+1)} = \min(U_n^{(0,l)}, V_n^{(0,l)}). \] (6)

From (2b), we have
\[ V_n^{(0,l)} = V_n^{(0,l)} - \min(U_n^{(0,l)}, V_n^{(0,l)}) + U_n^{(1,l)} \]
\[ = V_n^{(0,l)} - U_n^{(M,l+1)} + U_n^{(1,l)} \]
\[ = V_{n-1}^{(0,l)} - U_{n-1}^{(M,l+1)} + U_{n-1}^{(1,l)} - U_n^{(M,l+1)} + U_n^{(1,l)} \]
\[ = \ldots \]
\[ = \sum_{j=-\infty}^{n} \left( U_j^{(1,l)} - U_j^{(M,l+1)} \right). \]

Hence, we obtain
\[ U_n^{(M,l+1)} = \min\left( 1 - \sum_{k=1}^{M} U_n^{(k,l)}, \sum_{j=-\infty}^{n-1} \left( U_j^{(1,l)} - U_j^{(M,l+1)} \right) \right). \]

Further, from (6),
\[ U_n^{(j,l+M)} = U_n^{(k+1,l+M-1)} \]
\[ = \ldots \]
\[ = U_n^{(M,l+k)} \]
\[ = \min\left( 1 - \sum_{j=1}^{M} U_n^{(j,l+k-1)}, \sum_{j=-\infty}^{n-1} \left( U_j^{(1,l+k-1)} - U_j^{(M,l+k)} \right) \right) \]
\[ = \min\left( 1 - \sum_{j=1}^{k-1} U_n^{(j,l+M)} - \sum_{j=k}^{M} U_n^{(j,l)}, \sum_{j=-\infty}^{n-1} \left( U_j^{(k,l)} - U_j^{(k,l+M)} \right) \right). \]

This equation is known as the time evolution equation of the BBS with \( M \) kinds of balls [6].
2.3. Time evolution equation of the finite Toda representation

It is known that there is another type of time evolution equation for the BBSs. Let

- \( Q_n^{(k,t)} \): the number of cells written \( k \) in the \( n \)th block of positive integers at time \( t \), \( k = 1, 2, \ldots, M \);
- \( E_n^{(1,t)} \): the number of cells (written 0) between the \( n \)th and \((n + 1)\)st blocks of positive integers at time \( t \);
- \( A_n^{(k,t)} \): the re writable times of \( k \) just before the machine arrives at the \( n \)th block of positive integers from time \( t \) to \( t + 1 \), \( k = 1, 2, \ldots, M \);
- \( B_n^{(1,t)} \): the re writable times of 0 just before the machine arrives at the \( n \)th block of 0, i.e. the block between the \( n \)th and \((n + 1)\)st blocks of positive integers, from time \( t \) to \( t + 1 \).

Then, from time \( t \) to \( t + 1 \), \( \min(Q_n^{(k,t)}, A_n^{(k,t)}) \) cells written \( k \) in the \( n \)th block of positive integers are re written to \( k - 1 \) by the machine, where \( k = 1, 2, \ldots, M \), and \( \min(B_n^{(1,t)}, E_n^{(1,t)}) \) cells in the \( n \)th block of 0 are re written to \( M \) by the machine. Note that the integers in each block of positive integers must be arranged in ascending order from left to right. For example, ‘12233112’ is composed of two blocks ‘122333’ and ‘112’. Hence, we can write the time evolution equation of the proposed cellular automaton as

\[
Q_n^{(k,t+1)} = Q_n^{(k,t)} - Q_n^{(k,t)} + O_n^{(k,t+1)}, \quad A_n^{(k,t)} = A_n^{(k,t)} - O_n^{(k,t)} + O_n^{(k+1,t)},
\]

\[
E_n^{(1,t+1)} = E_n^{(1,t)} - O_n^{(M+1,t)} + O_n^{(1,t+1)}, \quad B_n^{(1,t+1)} = B_n^{(1,t)} - O_n^{(M+1,t)} + O_n^{(1,t+1)},
\]

\[
\bar{Q}_n^{(k,t)} = \min(Q_n^{(k,t)}, A_n^{(k,t)}), \quad \bar{Q}_n^{(M+1,t)} = \min(B_n^{(1,t)}, E_n^{(1,t)}),
\]

for \( k = 1, 2, \ldots, M \) and \( n = 0, 1, 2, \ldots, N - 1 \) with the boundary condition

\[
A_0^{(k,t)} = S(t), \quad k = 1, 2, \ldots, M, \quad (7d)
\]

\[
B_0^{(1,t)} = \min(Q_0^{(1,t)}, S(t)), \quad E_n^{(1,t)} = +\infty, \quad (7e)
\]

for all \( t \in \mathbb{Z} \).
Remark 2.3. If \( M = 1 \), then we have
\[
A^{(1,1)}_n = A^{(1,1)}_{n-1} - Q^{(1,1)}_{n-1} + Q^{(2,1)}_{n-1} \\
= A^{(1,1)}_{n-1} - Q^{(1,1)}_{n-1} + Q^{(1,2)}_{n-1} \\
= A^{(1,1)}_{n-2} - Q^{(1,1)}_{n-2} + Q^{(1,2)}_{n-1} - Q^{(1,1)}_{n-1} + Q^{(1,2)}_{n-1} \\
= \ldots \\
= A^{(1,1)}_0 - \sum_{j=0}^{n-1} (Q^{(1,1)}_j - Q^{(1,1+1)}_j) \\
= S^{(t)} - \sum_{j=0}^{n-1} (Q^{(1,1)}_j - Q^{(1,1+1)}_j).
\]

Hence, we obtain
\[
\tilde{Q}^{(1,1)}_n = \min \left( Q^{(1,1)}_n, S^{(t)} - \sum_{j=0}^{n-1} (Q^{(1,1)}_j - Q^{(1,1+1)}_j) \right) \\
= \min \left( \sum_{j=0}^{n} Q^{(1,1)}_j - \sum_{j=0}^{n-1} Q^{(1,1+1)}_j, S^{(t)} \right) - \sum_{j=0}^{n-1} (Q^{(1,1)}_j - Q^{(1,1+1)}_j). \tag{8}
\]

Similar recursive calculation yields
\[
B^{(1,1)}_n = \sum_{j=0}^{n} \tilde{Q}^{(1,1)}_j - \sum_{j=0}^{n-1} \tilde{Q}^{(2,1)}_j, \\
A^{(1,1)}_n = S^{(t)} - \sum_{j=0}^{n-1} (\tilde{Q}^{(1,1)}_j - \tilde{Q}^{(2,1)}_j) = S^{(t)} - B^{(1,1)}_n + \tilde{Q}^{(1,1)}_n = S^{(t)} - B^{(1,1)}_{n-1} + \tilde{Q}^{(2,1)}_{n-1}.
\]

By using these relations, we obtain
\[
\sum_{j=0}^{n-1} (Q^{(1,1)}_j - Q^{(1,1+1)}_j) = B^{(1,1)}_n - \tilde{Q}^{(1,1)}_n = B^{(1,1)}_{n-1} - \tilde{Q}^{(2,1)}_{n-1}.
\]

Substitution of this relation into (8) yields
\[
Q^{(1,1)}_n - \tilde{Q}^{(1,1)}_n = \max(0, B^{(1,1)}_{n-1} - \tilde{Q}^{(2,1)}_{n-1} + Q^{(1,1)}_n - S^{(t)}), \\
B^{(1,1)}_n = \min(B^{(1,1)}_{n-1} - \tilde{Q}^{(2,1)}_{n-1} + Q^{(1,1)}_n, S^{(t)}).
\]

Let \( \tilde{Q}^{(1,1)}_n := Q^{(1,1)}_n - \tilde{Q}^{(1,1)}_n \). Then, the system (7) of the case \( M = 1 \) is rewritten as
\[
\begin{align*}
\tilde{Q}^{(2,1)}_n &= \min(B^{(1,1)}_n, E^{(1,1)}_n), \\
B^{(1,1)}_{n+1} &= \min(B^{(1,1)}_n - \tilde{Q}^{(2,1)}_n + Q^{(1,1)}_{n+1}, S^{(t)}), \\
\tilde{B}^{(1,1)}_{n+1} &= \max(0, B^{(1,1)}_n - \tilde{Q}^{(2,1)}_n + Q^{(1,1)}_{n+1} - S^{(t)}), \\
Q^{(1,1+1)}_n &= \tilde{Q}^{(2,1)}_n + \tilde{B}^{(1,1)}_n, \\
E^{(1,1+1)}_n &= E^{(1,1)}_n - \tilde{Q}^{(2,1)}_n + Q^{(1,1)}_{n+1} - \tilde{B}^{(1,1)}_n.
\end{align*}
\]

This system is known as the nonautonomous ultradiscrete Toda lattice, which gives the time evolution equation of the BBS with finite carrier capacity [7].
Remark 2.4. If $S(t) = +\infty$, then $A_n^{(k,j)} = +\infty$ and $\bar{Q}_n^{(k,j)} = Q_n^{(k,j)}$ hold for all $k = 1, 2, \ldots, M$, $n = 0, 1, \ldots, N - 1$ and $t \in \mathbb{Z}$. Hence, the system (7) becomes

$$Q_n^{(k,j+1)} = Q_n^{(k+1,j)}, \quad k = 1, 2, \ldots, M - 1,$$

$$Q_n^{(M,j+1)} = \min(B_n^{(1,j)}, E_n^{(1,j)}),$$

$$E_n^{(1,j+1)} = E_n^{(1,j)} - Q_n^{(M+1,j)} + Q_n^{(1,j)} + Q_{n+1}^{(1,j)}$$

with the boundary condition

$$B_0^{(1,j)} = Q_0^{(1,j)}, \quad E_{N-1}^{(1,j)} = +\infty.$$

Using the recurrence relations recursively, we can rewrite $Q_n^{(M,j+1)}$ and $B_n^{(1,j)}$ as

$$Q_n^{(M,j+1)} = Q_n^{(M-1,j+2)} = Q_n^{(M-2,j+3)} = \cdots = Q_n^{(1,j+M)},$$

$$B_n^{(1,j)} = \sum_{j=0}^{n} Q_j^{(1,j)} - \sum_{j=0}^{n-1} Q_j^{(1,j+M)},$$

respectively. Therefore, we obtain

$$Q_n^{(1,j+M)} = \min\left(\sum_{j=0}^{n} Q_j^{(1,j)} - \sum_{j=0}^{n-1} Q_j^{(1,j+M)}, E_n^{(1,j)}\right),$$

$$E_n^{(1,j+1)} = E_n^{(1,j)} - Q_n^{(1,j+M)} + Q_{n+1}^{(1,j)}.$$

This system is known as the ultradiscrete hungry Toda lattice, which gives the time evolution equation of the BBS with $M$ kinds of balls [8].

Remark 2.5. Remarks 2.1, 2.2, 2.3 and 2.4 say that the proposed cellular automaton is a generalization of the BBS with either finite carrier capacity or many kinds of balls. However, the proposed cellular automaton is not the known BBS with both finite carrier capacity and many kinds of balls. Figs. 4 and 5 give a comparison between the proposed cellular automaton and the BBS with both finite carrier capacity and many kinds of balls. In the two examples, the same initial state is given, the parameters in Fig. 4 are chosen as $M = 3$ and $S(t) = 2$ for all $t$, and carrier capacity in Fig. 5 is chosen as $MS(t) = 6$ for all $t$. We can observe that the states in Fig. 4 at time $t = 3, 6, 9, 12, 15, \ldots$ are very similar, but slightly different, to the states in Fig. 5 at time $t = 1, 2, 3, 4, 5, \ldots$, respectively. On the other hand, we can find that there are initial states and parameter choices for which the states after evolution coincide with each other. These observations may suggest that there exists a connection between the proposed cellular automaton and the BBS with both finite carrier capacity and many kinds of balls.

3. Euler representation

3.1. From the nd-KP lattice

First, we give a brief exposition on the nd-KP lattice [9]. Let us consider the Lax pair

$$\alpha_n^{(k)} \varphi_n^{(k+1,j)} = \beta_n \varphi_{n+1}^{(k,j)} + (\alpha_n^{(k)} - \beta_n) \bar{u}_n^{(k,j)} \varphi_n^{(k,j)}, \quad (9a)$$

$$\gamma_n(\varphi_n^{(k,j+1)}) = \beta_n \varphi_n^{(k,j)} + (\gamma_n - \beta_n) \bar{x}_n^{(k,j)} \varphi_n^{(k,j)}, \quad (9b)$$
where \( k, n, t \in \mathbb{Z} \) are independent variables, \( \alpha^{(k)}, \beta_n \) and \( \gamma^{(t)} \) are functions of \( k, n \) and \( t \), respectively, and \( u_n^{(k,t)}, x_n^{(k,t)} \), and \( \varphi_n^{(k,t)} \) are some nonzero functions. The nd-KP lattice is derived as the compatibility condition for the Lax pair (9a) and (9b). Subtraction of (9a) from (9b) yields

\[
\gamma^{(t)} \varphi_n^{(k,t+1)} = \alpha^{(k)} \varphi_n^{(k+1,t)} + (\gamma^{(t)} - \alpha^{(k)}) \varphi_n^{(k,t+1)},
\]

where \( \varphi_n^{(k,t+1)} \) is a function satisfying

\[
(\gamma^{(t)} - \alpha^{(k)}) \varphi_n^{(k,t+1)} = (\gamma^{(t)} - \beta_n) \bar{\varphi}_n^{(k,t)} - (\alpha^{(k)} - \beta_n) \bar{\varphi}_n^{(k+1,t)}.
\]

In addition, the following relations are derived as the compatibility conditions for the linear equations (9):

\[
\bar{u}_n^{(k,t+1)} \bar{x}_n^{(k,t)} = \bar{u}_n^{(k,t)} \bar{x}_n^{(k+1,t)}, \quad \bar{g}_n^{(k,t)} \bar{x}_n^{(k,t)} = \bar{g}_n^{(k,t)} \bar{x}_n^{(k+1,t)},
\]

Consider the transformation of dependent variables

\[
\bar{u}_n^{(k,t)} = \frac{f_{n+1}^{(k,t+1)} f_{n+1}^{(k+1,t)}}{f_{n}^{(k,t+1)} f_{n}^{(k+1,t)}}, \quad \bar{g}_n^{(k,t)} = \frac{f_{n+1}^{(k,t+1)} f_{n}^{(k+1,t)}}{f_{n}^{(k,t+1)} f_{n}^{(k+1,t)}}, \quad \bar{x}_n^{(k,t)} = \frac{f_{n+1}^{(k,t+1)} f_{n}^{(k+1,t)}}{f_{n}^{(k,t+1)} f_{n}^{(k+1,t)}}.
\]

Then, these dependent variables indeed satisfy the relations (10b) and (10c), and equation (10a) is transformed into the bilinear equation of the nd-KP lattice

\[
(\alpha^{(k)} - \beta_n) f_{n+1}^{(k+1,t)} f_{n}^{(k,t+1)} + (\beta_n - \gamma^{(t)}) f_{n+1}^{(k,t+1)} f_{n}^{(k+1,t)} + (\gamma^{(t)} - \alpha^{(k)}) f_{n+1}^{(k,t)} f_{n}^{(k+1,t+1)} = 0.
\]
It is known that an $N$-soliton solution of the nd-KP lattice is given by

$$f_n^{(k,t)} = 1 + \sum_{J \subseteq \{0,1, \ldots, N-1\}} \left( \prod_{r_0 < r_1} \alpha_{r_0} \prod_{r \in J} \phi_{r,n}^{(k,t)} \right),$$

where

$$\alpha_{r_0} = \frac{(p_{r_0} - p_{r_1})(\tilde{p}_{r_1} - \tilde{p}_{r_0})}{(p_{r_0} - \tilde{p}_{r_1})(p_{r_1} - \tilde{p}_{r_0})}, \quad \phi_{r,n}^{(k,t)} = \theta_r \prod_{k'=0}^{k-1} \frac{\alpha^{(k')}}{\alpha^{(k')}} - \tilde{p}_{r} \prod_{n' = 0}^{n-1} \beta_{n'} - \tilde{p}_{r} \prod_{l'=0}^{l-1} \gamma^{(l')} - \tilde{p}_{r},$$

and

$$p_r, \tilde{p}_r$$

are some constants satisfying $p_{r_0} \neq \tilde{p}_{r_1}$ for all $r_0, r_1 \in \{0, 1, \ldots, N-1\}$. Next, we impose the $(M + 1)$-reduction condition

$$f_n^{(k+M+1,t)} = f_n^{(k,t)}, \quad \alpha^{(k+M+1)} = \alpha^{(k)}$$

for all $k, n, t$, which is the same condition as the one for deriving the time evolution equation of the BBS with $M$ kinds of balls. We define new functions $\delta_n = -1 + \tilde{p}_n$ and $s^{(i)} = -\gamma^{(i)}$, and set

$$\alpha^{(k)} = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, M - 1, \\ 1 & \text{if } k = M. \end{cases}$$

Further, let us define the new dependent variables

$$u_n^{(k,t)} = \begin{cases} \delta_n u_n^{(M+1,t)} & \text{if } k = 0, \\ (1 + \delta_n) u_n^{(k-1,t)} & \text{if } k = 1, 2, \ldots, M, \end{cases} \quad v_n^{(k,t)} = \begin{cases} (1 + s^{(i)}) v_n^{(M,t)} & \text{if } k = 0, \\ s^{(i)} v_n^{(k-1,t)} & \text{if } k = 1, 2, \ldots, M, \end{cases}$$

Then, we obtain the equations

$$u_n^{(k+1,t)} = u_n^{(k,t)} \frac{x_n^{(k+1,t)}}{x_n^{(k,t)}}, \quad v_n^{(k+1,t)} = v_n^{(k,t)} \frac{x_n^{(k+1,t)}}{x_n^{(k,t)}}, \quad x_n^{(k,t)} = u_n^{(k,t)} + v_n^{(k,t)}$$

with the boundary conditions

$$u_n^{(k,t)} = \begin{cases} \delta_n & \text{if } k = 0, \\ 1 + \delta_n & \text{if } k = 1, 2, \ldots, M, \end{cases} \quad v_n^{(k,t)} = \begin{cases} 1 + s^{(i)} & \text{if } k = 0, \\ s^{(i)} & \text{if } k = 1, 2, \ldots, M, \end{cases}$$

for $n \ll -1$. We also have

$$\prod_{k=0}^{M} u_n^{(k,t)} = \delta_n (1 + \delta_n)^M, \quad \prod_{k=0}^{M} v_n^{(k,t)} = (s^{(i)})^M (1 + s^{(i)})$$

for all $n$ and $t$.

### 3.2. Particular solutions

According to the previous studies [1, 6], we construct an $N$-soliton solution to (2). The construction is complicated, but similar to the previous studies; the only difference is the choice of the parameter $\gamma^{(i)}$. Therefore, we give only the result here. Set $u_n^{(k,t)} = e^{-u_n^{(k,t)}/\epsilon}, v_n^{(k,t)} = e^{-v_n^{(k,t)}/\epsilon}, x_n^{(k,t)} = e^{-x_n^{(k,t)}/\epsilon}$.
\[ \delta_n = e^{-\Delta_n/\epsilon}, \ s^{(t)} = e^{-S^{(t)}/\epsilon}, \] and suppose that \( \Delta_n \geq 0 \) and \( S^{(t)} \geq 0 \). Taking the limit \( \epsilon \to +0 \), then the system (14) goes to the time evolution equation of Euler representation (2) and the relations (15) also go to (3) through the ultradiscretization formula

\[ \lim_{\epsilon \to +0} -\epsilon \log(e^{-A/\epsilon} + e^{-B/\epsilon}) = \min(A, B). \]

**Remark 3.1.** To obtain the time evolution equation (2), we should set \( \Delta_n = 1 \) identically. In general, the parameter \( \Delta_n \) gives the capacity of \( n \)th box (cell). This generalization is easily obtained by changing the initial and boundary conditions to

\[ \sum_{k=0}^{M} U^{(k,0)}_n = \Delta_n \]

for all \( n \in \mathbb{Z} \) and

\[ U^{(k,t)}_n = \begin{cases} \Delta_n & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, \ldots, M, \end{cases} \]

for \( n \ll -1 \), respectively.

Soliton solutions are given as follows. From (11) and (13), \( f^{(k,t)}_n = e^{-\epsilon f^{(k,t)}/\epsilon} \) and \( \epsilon \to +0 \) yield

\[ U^{(k,t)}_n = \begin{cases} F^{(M,t)}_n - F^{(M,t+1)}_{n+1} + F^{(0,t)}_{n+1} - F^{(0,t)}_n + \Delta_n & \text{if } k = 0, \\ F^{(k-1,t)}_n - F^{(k-1,t+1)}_{n+1} + F^{(k,t)}_{n+1} - F^{(k,t)}_n & \text{if } k = 1, 2, \ldots, M, \end{cases} \]

\[ V^{(k,t)}_n = \begin{cases} F^{(M,t)}_n - F^{(M,t+1)}_{n+1} + F^{(0,t+1)}_{n+1} - F^{(0,t)}_n & \text{if } k = 0, \\ F^{(k-1,t)}_n - F^{(k-1,t+1)}_{n+1} + F^{(k,t+1)}_{n+1} - F^{(k,t)}_n + S^{(t)} & \text{if } k = 1, 2, \ldots, M, \end{cases} \]

\[ X^{(k,t)}_n = F^{(k-1,t)}_n - F^{(k-1,t+1)}_{n+1} + F^{(k-1,t+1)}_{n+1} - F^{(k-1,t+1)}_n. \]

Here, a one-soliton solution to the \((M + 1)\)-reduced nd-KP lattice is given by

\[ f^{(k,t)}_n = 1 + \sum_{j=0}^{M-1} \psi_{0,j,n}^{(k,t)}, \quad (16) \]

\[ \psi_{r,j,n}^{(k,t)} = \theta_{r,j} \left( \frac{\tilde{z}_{r,j}}{1 - z_r} \right)^{k'} \prod_{n'=0}^{n-1} \prod_{l'=0}^{l-1} \frac{1 - \tilde{z}_{r,j} + \delta_{n'} l'+1}{1 - z_r + s^{(l')}} \]

\[ = \theta_{r,j} \sum_{l=0}^{n-1} \left( \frac{\tilde{z}_{r,j}}{1 - z_r} \right)^{k' + l} \frac{1 - z_r s^{(l')}}{z_r + \delta_{n'} l'+1} \prod_{n'=0}^{n-1} \prod_{l'=0}^{l-1} \frac{1 - z_r + s^{(l')}}{1 - z_r + s^{(l')}} \]

\[ = \theta_{r,j} \sum_{l=0}^{n-1} \left( \frac{\tilde{z}_{r,j}}{1 - z_r} \right)^{k' + l} \text{ mod } \frac{z_r}{1 - z_r} \left( \frac{z_r}{1 - z_r} \right)^{\lfloor (k' + l)/M \rfloor} \prod_{n'=0}^{n-1} \prod_{l'=0}^{l-1} \frac{1 - z_r + \delta_{n'} l'+1}{1 - z_r + s^{(l')}} \]

where \( 0 < z_r < 1, k' = k \mod (M + 1), \lfloor x \rfloor := \max\{ m \in \mathbb{Z} : m \leq x \} \) is the floor function, \( \tilde{z}_{r,0}, \tilde{z}_{r,1}, \ldots, \tilde{z}_{r,M-1} \in \mathbb{C} \) are the roots of the following algebraic equation except \( 1 - z_r \):

\[ \left( \frac{z}{1 - z_r} \right)^M = \frac{z_r}{1 - z_r}, \]
Applying the ultradiscretization procedure to this solution, we obtain an \(N\)-soliton solution to the ultradiscrete system (2):

\[
\hat{S}_i^{(t)} := \begin{cases} 
1 & \text{if } i = 0, \\
\sum_{0 \leq j_0 < j_1 < \cdots < j_{i-1} \leq M-1} \prod_{j=0}^{i-1} \hat{s}^{(j)} & \text{if } i = 1, 2, \ldots, t, \\
0 & \text{otherwise.}
\end{cases}
\]

Applying the method of the previous studies to (16), we obtain a one-soliton solution to the ultradiscrete system (2):

\[
F_n^{(k,l)} = \min_{0 \leq r_0 < r_1 < \cdots < r_n \leq M-1} \left( \sum_{j=0}^{M-1} \prod_{r=0}^{j-1} \hat{\omega}_{r_{0:j}} \right) \prod_{r=0}^{n} \hat{\Psi}_{r_{0:n}}^{(k,l)}.
\]

In general, an \(N\)-soliton solution to the \((M+1)\)-reduced nd-KP lattice is given by

\[
f_n^{(k,l)} = 1 + \sum_{0 \leq r_0 < r_1 < \cdots < r_n \leq M-1} \left( \prod_{r=0}^{n} \hat{\omega}_{r_{0:n}} \right) \prod_{r=0}^{n} \hat{\Psi}_{r_{0:n}}^{(k,l)}.
\]

Applying the ultradiscretization procedure to this solution, we obtain an \(N\)-soliton solution to the ultradiscrete system (2):

\[
\hat{F}_n^{(k,l)} = \min_{0 \leq r_0 < r_1 < \cdots < r_n \leq M-1} \left( \prod_{r=0}^{n} \hat{\Psi}_{r_{0:n}}^{(k,l)} \right).
\]

Note that the parameters must satisfy \(Z_0 \geq Z_1 \geq \cdots \geq Z_{M-1} \geq 0\) and \(\zeta_{0,j} \geq \zeta_{1,j} \geq \cdots \geq \zeta_{N-1,j} \geq 0\) for \(j = 1, 2, \ldots, M\). For example, the solution corresponding to Fig. 3 is given by setting \(\Theta_0 = 1, \Theta_1 = 7, \Theta_2 = 13, Z_0 = 8, Z_1 = 4, Z_2 = 2, \zeta_{0,1} = 2, \zeta_{0,2} = 3, \zeta_{0,3} = 3, \zeta_{1,1} = 1, \zeta_{1,2} = 2, \zeta_{1,3} = 1, \zeta_{2,1} = 1, \zeta_{2,2} = 0, \zeta_{2,3} = 1, \Delta_n = 1\) for all \(n\), and \(S^{(t)} = 2\) for all \(t\).

4. Finite Toda representation

4.1. \((M, 1)\)-biorthogonal polynomials and the ndh-Toda lattice

First, we prepare the notion of \((M, 1)\)-biorthogonal polynomials [5]. We introduce spectral transformations of the \((M, 1)\)-biorthogonal polynomials, and derive the ndh-Toda lattice as their compatibility condition.
Let $\mathcal{L}^{(k,t)} : \mathbb{C}[z] \to \mathbb{C}$ be a linear functional, where $k, t \in \mathbb{Z}$ are discrete time variables. Let us consider polynomial sequences $\{\phi_n^{(k,t)}(z)\}_{n=0}^\infty$ and $\{\psi_n^{(k,t)}(z)\}_{n=0}^\infty$ satisfying the following properties:

(i) $\deg \phi_n^{(k,t)}(z) = \deg \psi_n^{(k,t)}(z) = n$;

(ii) The polynomials $\phi_n^{(k,t)}(z)$ and $\psi_n^{(k,t)}(z)$ are monic;

(iii) There exists a positive integer $M$ such that the $(M, 1)$-biorthogonal relation with respect to $\mathcal{L}^{(k,t)}$

\[ \mathcal{L}^{(k,t)}[\phi_m^{(k,t)}(z)\psi_n^{(k,t)}(z^M)] = h_n^{(k,t)}\delta_{m,n}, \quad h_n^{(k,t)} \neq 0, \quad m, n = 0, 1, 2, \ldots, \quad (17) \]

holds, where $\delta_{m,n}$ is the Kronecker delta.

In this paper, we call the polynomial sequences $\{\phi_n^{(k,t)}(z)\}_{n=0}^\infty$ and $\{\psi_n^{(k,t)}(z)\}_{n=0}^\infty$ the pair of monic $(M, 1)$-biorthogonal polynomial sequences with respect to $\mathcal{L}^{(k,t)}$. Note that the $(M, 1)$-biorthogonal relation (17) is equivalent to

\[ \mathcal{L}^{(k,t)}[z^M\phi_n^{(k,t)}(z)] = \mathcal{L}^{(k,t)}[z^n\psi_n^{(k,t)}(z^M)] = h_n^{(k,t)}\delta_{n,M}, \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n. \quad (18) \]

Let us introduce time evolution into the linear functional by

\[ \mathcal{L}^{(k,t+1)}[z^m] := \mathcal{L}^{(k,t)}[z^{m+1}], \quad \mathcal{L}^{(k,t+1)}[z^m] := \mathcal{L}^{(k,t)}[z^m(z + s^{(t)})], \quad m = 0, 1, 2, \ldots, \quad (19) \]

where $s^{(t)}$ is a parameter depending on $t$. Define the moment of $\mathcal{L}^{(0,t)}$ by

\[ \mu_m^{(t)} := \mathcal{L}^{(0,t)}[z^m], \quad m = 0, 1, 2, \ldots. \]

Then we have

\[ \mathcal{L}^{(k,t)}[z^m] = \mu_k^{(t)} + \mu_m^{(t+1)} + s^{(t)}\mu_m^{(t)}. \]

By using the $(M, 1)$-biorthogonal relation (18) and the moments, one can easily show that the $(M, 1)$-biorthogonal polynomials have the following determinant representation:

\[
\phi_0^{(k,t)}(z) = 1, \quad \phi_n^{(k,t)}(z) = \frac{1}{\tau_n^{(k,t)}} \left| \begin{array}{cccc}
\mu_k^{(t)} & \mu_k^{(t+1)} & \cdots & \mu_k^{(t+n-1)} \\
\mu_k^{(t+1)} & \mu_k^{(t+1)+M} & \cdots & \mu_k^{(t+n-1)+M(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_k^{(t+n-1)} & \mu_k^{(t+n-1)+M} & \cdots & \mu_k^{(t+n-1)+M(n-1)} \\
\end{array} \right|, \quad n = 1, 2, 3, \ldots, 
\]

\[
\psi_0^{(k,t)}(z) = 1, \quad \psi_n^{(k,t)}(z) = \frac{1}{\tau_n^{(k,t)}} \left| \begin{array}{cccc}
\mu_k^{(t)} & \mu_k^{(t+1)} & \cdots & \mu_k^{(t+n-1)} \\
\mu_k^{(t+1)} & \mu_k^{(t+1)+M} & \cdots & \mu_k^{(t+n-1)+M(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_k^{(t+n-1)} & \mu_k^{(t+n-1)+M} & \cdots & \mu_k^{(t+n-1)+M(n-1)} \\
\end{array} \right|, \quad n = 1, 2, 3, \ldots, 
\]

where

\[ \tau_0^{(k,t)} := 1, \quad \tau_n^{(k,t)} := \left[ \mu_k^{(t+1)+M} \right]_{i,j=0}^{n-1}, \quad n = 1, 2, 3, \ldots. \quad (20) \]
From the theory of biorthogonal polynomials, it is shown that one of the pair of the \((M, 1)\)-biorthogonal polynomials \(\{\phi_n^{(k,t)}(z)\}_n\) satisfies the following relations:

\[
\begin{align*}
z \phi_n^{(k+1,t)}(z) &= \phi_{n+1}^{(k,t)}(z) + q_n^{(k,t)} \phi_n^{(k,t)}(z), \\
\phi_{n+1}^{(k-M,t)}(z) &= \phi_n^{(k,t)}(z) + e_n^{(k-M,t)} \phi_n^{(k,t)}(z), \\
(z + s^{(t)}) \phi_n^{(k,t+1)}(z) &= \phi_{n+1}^{(k,t)}(z) + q_n^{(k,t)} \phi_n^{(k,t)}(z), \\
(z + s^{(t)}) \phi_n^{(k+1,t)}(z) &= z \phi_n^{(k,t+1)}(z) + a_n^{(k,t)} \phi_n^{(k,t)}(z), \\
(z + s^{(t)}) \phi_n^{(k+1,t+1)}(z) &= \phi_{n+1}^{(k-M,t)}(z) + b_n^{(k-M,t)} \phi_n^{(k,t)}(z),
\end{align*}
\]

where

\[
\begin{align*}
q_n^{(k,t)} &= \frac{\tau_n^{(k,t)} \tau_{n+1}^{(k+1,t)}}{\tau_{n+1}^{(k+1,t)} \tau_n^{(k,t)}}, \\
e_n^{(k,t)} &= \frac{\tau_n^{(k,t)} \tau_{n+2}^{(k+1,t)}}{\tau_{n+1}^{(k,t)} \tau_{n+1}^{(k+1,t)}}, \\
q_n^{(k,t+1)} &= \frac{\tau_n^{(k+1,t)} \tau_{n+1}^{(k,t+1)}}{\tau_{n+1}^{(k,t+1)} \tau_n^{(k+1,t)}}, \\
b_n^{(k,t)} &= \frac{\tau_n^{(k+1,t)} \tau_{n+1}^{(k,t+1)}}{\tau_{n+1}^{(k,t+1)} \tau_n^{(k+1,t)}}.
\end{align*}
\]

The relations (21) are called spectral transformations. The compatibility conditions for (21) give the recurrence relations

\[
\begin{align*}
\tilde{q}_n^{(k,t)} &= q_n^{(k,t)} + a_n^{(k,t)}, \\
\tilde{q}_n^{(k+M,t)} &= b_n^{(k,t)} + \tilde{e}_n^{(k,t)}, \\
q_n^{(k,t+1)} &= \frac{q_n^{(k,t+1)} \tilde{q}_n^{(k,t)}}{q_n^{(k,t)}}, \\
b_n^{(k,t+1)} &= \frac{b_n^{(k,t)} + \tilde{q}_n^{(k,t+1)}}{\tilde{q}_n^{(k,t+1)}}, \\
a_{n+1}^{(k,t)} &= a_n^{(k,t)} \frac{q_n^{(k+1,t)}}{q_n^{(k,t)}}, \\
e_{n+1}^{(k,t+1)} &= e_n^{(k,t+1)} \frac{q_n^{(k+1,t+1)}}{\tilde{q}_n^{(k,t+1)}}.
\end{align*}
\]

for \(n = 0, 1, 2, \ldots\), with the boundary condition

\[
a_0^{(k,t)} = s^{(t)}, \quad b_0^{(k,t)} = q_0^{(k,t)} + s^{(t)}
\]

for all \(k, t \in \mathbb{Z}\). In this paper, we call the system (23) the ndh-Toda lattice, which is a reduced system of the nonautonomous discrete two-dimensional Toda lattice.

### 4.2. Finite lattice case

Hereafter, we consider the case in which the finite lattice boundary condition

\[
e_n^{(k,t)} = 0 \quad \text{if } n > N
\]

is imposed, where the lattice size \(N\) is a positive integer. This condition implies

\[
e_{N-1}^{(k,t)} = 0
\]

by (22). Hence, from (21b), we obtain the relation

\[
\phi_{N}^{(k+M,t)}(z) = \phi_{N}^{(k,t)}(z).
\]
Note that, although we omitted the relations for \( \{ \psi^{(k,f)}_n(z) \}_{n=0}^N \), there is also the “dual” relation

\[
\psi^{(k+1,j)}_N(z) = \psi^{(k,f+1)}_N(z) = \psi^{(k,j)}_N(z).
\]

By using the \( N \times N \) bidirectional matrices

\[
R^{(k,f)} := \begin{pmatrix}
q_0^{(k,f)} & 1 & & \\
q_1^{(k,f)} & q_0^{(k,f)} & 1 & \\
& \ddots & \ddots & \ddots \\
& & 1 & q_{N-1}^{(k,f)} \\
\end{pmatrix}, \quad L^{(k,f)} := \begin{pmatrix}
1 & e_0^{(k,f)} & & \\
e_1^{(k,f)} & 1 & \ddots & \\
& \ddots & \ddots & \ddots \\
& & \ddots & e_{N-2}^{(k,f)} & 1 \\
\end{pmatrix},
\]

and the \( N \)-dimensional vectors

\[
\phi^{(k,j)}(z) := \begin{pmatrix}
\phi_0^{(k,j)}(z) \\
\phi_1^{(k,j)}(z) \\
\vdots \\
\phi_{N-1}^{(k,j)}(z) \\
\end{pmatrix}, \quad \phi^{(k,j)}_N(z) := \begin{pmatrix}
0 \\
\vdots \\
0 \\
\phi_0^{(k,j)}(z) \\
\end{pmatrix},
\]

the spectral transformations (21a)–(21c) are written as

\[
(z + s^{(f)}) \phi^{(k+1,j)}(z) = R^{(k,f)} \phi^{(k,j)}(z) + \phi^{(k,j)}_N(z), \quad (27a)
\]

\[
\phi^{(k-M,j)}(z) = L^{(k-M,f)} \phi^{(k,j)}(z), \quad (27b)
\]

\[
(z + s^{(f)}) \phi^{(k,j+1)}(z) = \tilde{R}^{(k,f)} \phi^{(k,j)}(z) + \phi^{(k,j)}_N(z), \quad (27c)
\]

respectively. We thus have

\[
z(z + s^{(f)}) \phi^{(k+1,j+1)}(z) = R^{(k,j+1)} \tilde{R}^{(k,f)} \phi^{(k,j)}(z) + R^{(k,j+1)} \phi^{(k,j)}_N(z) + (z + s^{(f)}) \phi^{(k,j+1)}_N(z)
\]

\[
= R^{(k+1,j)} \tilde{R}^{(k,f)} \phi^{(k,j)}(z) + R^{(k+1,j)} \phi^{(k,j)}_N(z) + z \phi^{(k+1,j)}_N(z), \quad (28a)
\]

\[
(z + s^{(f)}) \phi^{(k,j+1)}(z) = L^{(k,j+1)} \tilde{R}^{(k+1,f)} \phi^{(k+1,j)}(z) + L^{(k,j)} \phi^{(k+1,j)}_N(z)
\]

\[
= \tilde{R}^{(k,f)} L^{(k,j)} \phi^{(k,j+1)}(z) + \phi^{(k,j)}_N(z), \quad (28b)
\]

We should remark that the relation

\[
a^{(k,j)}_n = a^{(k,j)}_{n-1} \frac{q^{(k+1,j)}_n}{q^{(k,j)}_{n-1}} = (q^{(k,j)}_{n-1} - q^{(k,j)}_n) \frac{q^{(k+1,j)}_n}{q^{(k,j)}_{n-1}} = q^{(k+1,j)}_n - q^{(k,j+1)}_{n-1}
\]

holds by the recurrence relations (23). Thus the relation (21d) for \( n = N \) is rewritten as

\[
q^{(k,j+1)}_{N-1} \phi^{(k,j)}_N(z) + (z + s^{(f)}) \phi^{(k+1,j+1)}_N(z) = q^{(k+1,j)}_{N-1} \phi^{(k,j)}_N(z) + z \phi^{(k+1,j)}_N(z).
\]
This relation is equivalent to
\[ R^{(k,t+1)}(z) + (z + s^{(t)}) \Phi_N^{(k,t+1)}(z) = R^{(k,t)}(z) + z \Phi_N^{(k+1,t)}(z). \]

Further, \( L^{(k,t+1)} \Phi_N^{(k+M,t)}(z) = \Phi_N^{(k,t)}(z) \) holds by (26). Hence, the compatibility conditions for (28) are simply written as
\[ R^{(k,t+1)} \tilde{R}^{(k,t)} = \tilde{R}^{(k+1,t)} R^{(k,t)}, \quad L^{(k,t+1)} \tilde{R}^{(k+M,t)} = \tilde{R}^{(k,t)} L^{(k,t)}. \]  (29)

Now consider the upper Hessenberg matrix
\[ H^{(k,t)} = L^{(k,t)} R^{(k+M-1,t)} R^{(k+M-2,t)} \ldots R^{(k,t)}. \]

By using (29), we find
\[ H^{(k,t+1)} \tilde{R}^{(k,t)} = L^{(k,t+1)} R^{(k+M-1,t+1)} \ldots R^{(k+2,t+1)} R^{(k+1,t+1)} \tilde{R}^{(k,t)} \]
\[ = L^{(k,t+1)} R^{(k+M-1,t+1)} \ldots R^{(k+2,t+1)} \tilde{R}^{(k+1,t+1)} R^{(k,t)} \]
\[ = L^{(k,t+1)} R^{(k+M-1,t+1)} \ldots R^{(k+2,t+1)} \tilde{R}^{(k+1,t+1)} R^{(k,t)} \]
\[ = \ldots \]
\[ = L^{(k,t+1)} \tilde{R}^{(k+M,t)} R^{(k+M-1,t)} \ldots R^{(k+2,t)} R^{(k+1,t)} R^{(k,t)} \]
\[ = R^{(k,t)} L^{(k,t)} R^{(k+M-1,t)} \ldots R^{(k+2,t)} R^{(k+1,t)} R^{(k,t)} \]
\[ = R^{(k,t)} H^{(k,t)}. \]

Hence, the ndh-Toda lattice (23) with the finite lattice boundary condition (24) also can compute iterations of similarity transformations of the upper Hessenberg matrices as with another nonautonomous version of the discrete hungry Toda lattice studied in the previous paper [5].

Next, we construct solutions to the finite ndh-Toda lattice (23) with (24). Suppose that all the eigenvalues \( z_0, z_1, \ldots, z_{N-1} \) of \( H^{(k,t)} \) are simple. Note that these eigenvalues are directly related to a polynomial constructed by \( \Phi_N^{(k,t)}(z), \Phi_N^{(k+1,t)}(z), \ldots, \Phi_N^{(k+M-1,t)}(z) \); see also Appendix A. By the result of the previous paper [5], there exist some complex-valued functions \( u_0^{(m)}, u_1^{(m)}, \ldots, u_{N-1}^{(m)} \) satisfying \( u_r^{(m)} = u_r^{(m \mod M)} \) for \( m \in \mathbb{Z}, r = 0, 1, \ldots, N - 1 \), such that the moments of \( L^{(0,0)} \) are written as
\[ \mu^{(0)}_m = L^{(0,0)}[z^m] = \sum_{r=0}^{N-1} u_r^{(m)} z_r^m. \]

We assume that \( z_r \neq 0 \) for \( r = 0, 1, \ldots, N - 1 \). The relation (19) yields
\[ \mu^{(t)}_m = L^{(0,0)}[z^m] \]
\[ = L^{(0,0)} \left[ z^m \prod_{j=0}^{t-1} (z + s^{(j)}) \right] \]
\[ = L^{(0,0)} \left[ z^{m+t} + \sum_{l=1}^{t} \left( \sum_{0 \leq j_0 < j_1 < \cdots < j_{l-1} \leq t-1} \prod_{j=l}^{t-1} s^{(j)} \right) z^{m+t-l} \right] \]
\[ = \sum_{r=0}^{N-1} u_r^{(m,t)} z_r^m. \]  (30)
where

\[ \tilde{u}_r^{(m,i)} := \sum_{i=0}^{l} u_r^{(m+i)} z_r^{i/M} s_r^{(i)}, \]

\[ \tilde{s}_i^{(l)} := \begin{cases} 1 & \text{if } i = 0, \\ \sum_{0 \leq j_0 < j_1 < \cdots < j_{l-1} \leq l-1} \prod_{j=0}^{l-1} \tilde{s}^{(j_i)} & \text{if } i = 1, 2, \ldots, l, \\ 0 & \text{otherwise.} \end{cases} \]

By definition, \( \tilde{u}_r^{(m,0)} = u_r^{(m)} \) holds. It is clear that \( \tilde{u}_r^{(m,i)} = u_r^{(m \mod M, i)} \) also holds.

Substituting the moment representation (30) into the determinant (20), we find

\[ r_n^{(k,i)} = \det(\hat{V}_n D^{(k)} V_n^{(k,i)}), \]

where

\[ \hat{V}_n := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_0^{n-1} & z_1^{n-1} & \cdots & z_{N-1}^{n-1} \end{pmatrix}, \]

\[ D^{(k)} = \text{diag}\left( z_0^{k/M}, z_1^{k/M}, \ldots, z_{N-1}^{k/M} \right), \]

\[ V_n^{(k,i)} := \begin{pmatrix} u_r^{(k,i)} & u_r^{(k+1,i)} z_r^{1/M} & \cdots & u_r^{(k+n-1,i)} z_r^{(n-1)/M} \\ u_r^{(k,i)} & u_r^{(k+1,i)} z_r^{1/M} & \cdots & u_r^{(k+n-1,i)} z_r^{(n-1)/M} \\ \vdots & \vdots & \ddots & \vdots \\ u_r^{(k,i)} & u_r^{(k+1,i)} z_r^{1/M} & \cdots & u_r^{(k+n-1,i)} z_r^{(n-1)/M} \end{pmatrix}. \]

Let us introduce the minors of \( V_n^{(k,i)} \):

\[ V_n^{(k,i)} \left( r_0, r_1, \ldots, r_{n-1} \mid c_0, c_1, \ldots, c_{n-1} \right) := \begin{vmatrix} u_r^{(k+c_0,i)} z_r^{c_0/M} & u_r^{(k+c_1,i)} z_r^{c_1/M} & \cdots & u_r^{(k+c_{n-1},i)} z_r^{c_{n-1}/M} \\ u_r^{(k+c_0,i)} z_r^{c_0/M} & u_r^{(k+c_1,i)} z_r^{c_1/M} & \cdots & u_r^{(k+c_{n-1},i)} z_r^{c_{n-1}/M} \\ \vdots & \vdots & \ddots & \vdots \\ u_r^{(k+c_0,i)} z_r^{c_0/M} & u_r^{(k+c_1,i)} z_r^{c_1/M} & \cdots & u_r^{(k+c_{n-1},i)} z_r^{c_{n-1}/M} \end{vmatrix}. \]

We allow \( c_0, c_1, \ldots, c_{n-1} \) to take values larger than \( N - 1 \). By definition, we have

\[ V_n^{(k,i)} \left( r_0 \mid c_0 \right) = u_r^{(k+c_0,i)} z_r^{c_0/M} s_r^{(i)} = \sum_{i=0}^{l} u_r^{(k+c_0,i)} z_r^{c_0+i/M} s_r^{(i)} = \sum_{i=0}^{l} V_n^{(k,i)} \left( r_0 \mid c_0 + i \right) s_r^{(i)} = \sum_{i=0}^{l} V_n^{(k,i)} \left( r_0 \mid c_0 + i \right) s_r^{(i)} \]

and

\[ V_n^{(k,i)} \left( r_0, r_1, \ldots, r_{n-1} \mid c_0, c_1, \ldots, c_{n-1} \right) = \begin{vmatrix} V_n^{(k,i)} \left( r_0 \mid c_0 \right) & V_n^{(k,i)} \left( r_0 \mid c_1 \right) & \cdots & V_n^{(k,i)} \left( r_0 \mid c_{n-1} \right) \\ V_n^{(k,i)} \left( r_1 \mid c_0 \right) & V_n^{(k,i)} \left( r_1 \mid c_1 \right) & \cdots & V_n^{(k,i)} \left( r_1 \mid c_{n-1} \right) \\ \vdots & \vdots & \ddots & \vdots \\ V_n^{(k,i)} \left( r_{n-1} \mid c_0 \right) & V_n^{(k,i)} \left( r_{n-1} \mid c_1 \right) & \cdots & V_n^{(k,i)} \left( r_{n-1} \mid c_{n-1} \right) \end{vmatrix}. \]
Applying the Binet–Cauchy formula and the expansion formula for the Vandermonde determinant, we obtain

\[
\tau_n^{(k,t)} = \sum_{0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N-1} \mathcal{V}(k,t) \left( \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} z_{r_j}^{k/M} \prod_{0 \leq i < j \leq n-1} (z_{r_j} - z_{r_i}) \right). \tag{33}
\]

Here, from (31) and (32), we have

\[
V(k,t) \left( \left( \begin{array}{c} r_0, r_1, \ldots, r_{n-1} \\ 0, 1, \ldots, n-1 \end{array} \right) \right) = \sum_{i=0}^{t} \sum_{j=0}^{t} \cdots \sum_{c_{n-1}=n-1}^{t} \left( \begin{array}{c} r_0 \\ c_0 \\ \vdots \\ c_{n-1} \end{array} \right) \left( \begin{array}{c} r_1 \\ c_1 \\ \vdots \\ c_{n-1} \end{array} \right) \cdots \left( \begin{array}{c} r_{n-1} \\ c_{n-1} \end{array} \right) z_{c_0}^{t+1-c_1} \cdots z_{c_{n-1}}^{t+n-1-c_{n-1}},
\]

where

\[
z_{(k_0, k_1, \ldots, k_{n-1})}^{(l)} := \sum_{\sigma \in S_n} \text{sgn} \sigma \prod_{j=0}^{n-1} z_{\lambda^{(k,j)+j-\sigma(j)}}^{(l)} = \begin{vmatrix} z_{\lambda_0}^{(l)} & \cdots & z_{\lambda_{n-1}}^{(l)} \\ \vdots & \ddots & \vdots \\ z_{\lambda_{n-1}}^{(l)} & \cdots & z_{\lambda_0}^{(l)} \end{vmatrix}.
\]

and \(S_n\) is the symmetric group on \(\{0, 1, \ldots, n-1\}\). Since \(z_{\lambda_0}^{(l)}, z_{\lambda_1}^{(l)}, \ldots, z_{\lambda_{n-1}}^{(l)}\) are the elementary symmetric polynomials of \(s^{(0)}, s^{(1)}, \ldots, s^{(l-1)}\), by the Jacobi–Trudi formula, this is the Schur polynomial for the partition \((\lambda_0, \lambda_1, \ldots, \lambda_{n-1})'\), which is the conjugate of the partition \((\lambda_0, \lambda_1, \ldots, \lambda_{n-1})\):

\[
z_{(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})}^{(l)} = \sum_{Y} \prod_{j=0}^{l-1} s_{\gamma_j}^{y_j},
\]

where the summation is over all semistandard Young tableaux \(Y\) of the partition \((\lambda_0, \lambda_1, \ldots, \lambda_{n-1})'\), and \(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\) are the weights of \(Y\).

To derive a sufficient condition for the positivity of \(\tau_n^{(k,t)}\), we discuss recurrence relations among the
minors $V^{(k,d)}(r_0, r_1, \ldots, r_{n-1})$. The followings are readily derived by definition:

\begin{align}
V^{(k,0)}(r_0, c_0) &= u^{(k+\delta_0)}_{r_0} z_0^{c_0/M}, \\
V^{(k,0)}(r_0, r_1, c_0, c_1) &= \begin{vmatrix}
V^{(k,0)}(r_0, c_0) & V^{(k,0)}(r_0, c_1) \\
V^{(k,0)}(r_1, c_0) & V^{(k,0)}(r_1, c_1)
\end{vmatrix}.
\end{align}

(34a) (34b)

For $n \geq 3$, the Jacobi identity yields

\begin{align}
V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}, r_{n-2}, r_{n-1}, c_0, c_1, \ldots, c_{n-3}, c_{n-2}, c_{n-1}) &= \begin{vmatrix}
V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}, r_{n-2}) & V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}, r_{n-2}) \\
V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}, r_{n-2}) & V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}, r_{n-2}) \\
V^{(k,0)}(r_0, r_1, \ldots, r_{n-3}) & V^{(k,0)}(r_0, r_1, \ldots, r_{n-3})
\end{vmatrix}.
\end{align}

(35)

if $V^{(k+\delta_0,0)}(r_0, r_1, \ldots, r_{n-3}) \neq 0$. Further, by definition,

\begin{align}
V^{(k,0)}(r_0, r_1, \ldots, r_{n-1}, c_0, c_1, \ldots, c_{n-1}) &= V^{(k+\delta_0,0)}(r_0, r_1, \ldots, r_{n-1}, 0, c_1 - c_0, \ldots, c_{n-1} - c_0) \prod_{j=0}^{n-1} z_j^{c_0/M},
\end{align}

holds. Hence, the followings give a sufficient condition for $r^{(k,f)}_n > 0$ for all $k = 0, 1, \ldots, M - 1$, $t \in \mathbb{Z}$ and $n = 1, 2, \ldots, N - 1$:

(i) All the parameters $z_r$, $s^{(i)}$, $u^{(k)}_r$ are real and the real $M$th root $z_r^{1/M}$ are chosen;

(ii) $0 < z_0^{1/M} < z_1^{1/M} < \cdots < z_{N-1}^{1/M}$;

(iii) $s^{(i)} \geq 0$ for all $t \in \mathbb{Z}$;

(iv) $u^{(k)}_r > 0$ for all $k = 0, 1, \ldots, M - 1$ and $r = 0, 1, \ldots, N - 1$;

(v) The inequality

\begin{align*}
V^{(k,0)}(r_0, 0) V^{(k,0)}(r_1, c_1) &> V^{(k,0)}(r_1, 0) V^{(k,0)}(r_0, c_1),
\end{align*}

i.e.

\begin{align*}
u^{(k)}_{r_0} \nu^{(k+\epsilon_1)}_{r_1} z_{r_0}^{c_1/M} &> u^{(k+\epsilon_1)}_{r_0} \nu^{(k)}_{r_1} z_{r_1}^{c_1/M}
\end{align*}

holds for all $k = 0, 1, \ldots, M - 1$, $c_1 = 1, 2, \ldots, M - 1$ and pairs of indices $(r_0, r_1)$ satisfying $0 \leq r_0 < r_1 < N - 1$;
the inequality

\[ V^{(k,0)}(r_0, r_1, \ldots, r_{n-2}) < V^{(k,0)}(r_0, r_1, \ldots, r_{n-2}, r_{n-1}) \]

holds for all \( k = 0, 1, \ldots, M - 1, n = 3, 4, \ldots, N - 1, \) \( n \)-tuples \((r_0, r_1, \ldots, r_{n-1})\) satisfying \( 0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N - 1 \) and \((n - 1)\)-tuples \((c_1, c_2, \ldots, c_{n-1})\) satisfying \( 0 < c_1 < c_2 < \cdots < c_{n-1} \), \( c_1 \leq M \) and \( c_j - c_{j-1} \leq M, i = 2, 3, \ldots, n - 1 \).

We should remark on the condition (36) that, since \( w_r^{(k+M)} = w_r^{(k)} \) and \( z_{r_1} > z_{r_0} \), this condition implies

\[ w_r^{(k)} w_{r_1}^{(k+c_1)M} z_{r_1} > w_r^{(k+c_1)} w_{r_1}^{(k+c_1)M} z_{r_1} \]

for all positive integers \( c_1 \).

4.3. Ultradiscretization

Let us consider the transformations of variables \( q_n^{(k,j)} = e^{-Q_n^{(k,j)}/\epsilon}, q_n^{(k,j)} = e^{-Q_n^{(k,j)}/\epsilon}, c_n^{(k,j)} = e^{A_n^{(k,j)}/\epsilon}, a_n^{(k,j)} = e^{-A_n^{(k,j)}/\epsilon}, b_n^{(k,j)} = e^{-B_n^{(k,j)}/\epsilon} \) and \( s^{(i)} = e^{-S^{(i)}/\epsilon} \), where \( \epsilon \) is a positive parameter. Since there is an ultradiscretization formula

\[ \lim_{\epsilon \to +0} -\epsilon \log(p_1 e^{-C_1/\epsilon} + p_2 e^{-C_2/\epsilon}) = \min(C_1, C_2), \]

where \( p_1 \) and \( p_2 \) are positive numbers, applying these transformations and taking a limit \( \epsilon \to +0 \) yields

\[ \tilde{Q}_n^{(k,j)} = \min(Q_n^{(k,j)}, A_n^{(k,j)}), \quad \tilde{Q}_n^{(k+M,j)} = \min(B_n^{(k,j)}, E_n^{(k,j)}), \]

\[ Q_n^{(k+1,j)} = Q_n^{(k,j)} - \tilde{Q}_n^{(k+1,j)} + \tilde{Q}_n^{(k,j)} + \tilde{Q}_n^{(k+1,j)}, \quad A_n^{(k+1)} = A_n^{(k,j)} - \tilde{Q}_n^{(k,j)} + \tilde{Q}_n^{(k+1,j)}, \]

\[ E_n^{(k+1,j)} = E_n^{(k,j)} - \tilde{Q}_n^{(k+1,j)} + \tilde{Q}_n^{(k+1,j)} - \tilde{Q}_n^{(k+1,j)} + \tilde{Q}_n^{(k+1,j)}, \]

for \( n = 0, 1, 2, \ldots \) with the boundary condition

\[ A_0^{(k,j)} = S^{(j)}, \quad B_0^{(k,j)} = \min(Q_0^{(k,j)}, S^{(j)}) \]

for all \( k, j \in \mathbb{Z} \). In addition, we also impose the finite lattice condition corresponding to (25):

\[ E_n^{(k,j)} = +\infty. \]

The derived ultradiscrete system (38) coincides with the time evolution equation of finite Toda representation (7).

A solution to the ultradiscrete system (38) with the finite lattice condition (39) is constructed from the solution (22) and (33) to the ndh-Toda lattice (23). Consider the transformations of variables \( z_r = p_r e^{-Z_r/\epsilon}, w^{(m)} = e^{-W^{(m)}/\epsilon} \) and \( V^{(k,j)}(r_0, r_1, \ldots, r_{n-1}) = \exp(-\nabla^{(k,j)}(r_0, r_1, \ldots, r_{n-1})/\epsilon) \) and the limit \( \epsilon \to +0 \), where \( p_r \) is a positive constant satisfying \( p_r < p_{r+1} \) if \( Z_r = Z_{r+1} \). Since we assumed that the inequality \( 0 < z_0 < z_1 < \cdots < z_{N-1} \) holds, the new variables \( Z_r \) must satisfy \( Z_0 \geq Z_1 \geq \cdots \geq Z_{N-1} \). To apply the transformations, \( Z_r \) must be positive; i.e., by (36) and (37), the followings must be satisfied:
The inequality
\[
W_{r_0}^{(k)} + W_{r_1}^{(k+c_1)} + \frac{c_1 Z_{r_1}}{M} \leq \frac{W_{r_0}^{(k+c_1)}}{M} + \frac{c_1 Z_{r_1}}{M}
\] (40)
holds for all \( k = 0, 1, \ldots, M - 1, c_1 = 1, 2, \ldots, M - 1 \) and pairs of indices \((r_0, r_1)\) satisfying \( 0 \leq r_0 < r_1 \leq N - 1 \);

The inequality
\[
\gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-2}}{0, c_1, \ldots, c_{n-3}, c_{n-2}}\right) + \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-1}}{0, c_1, \ldots, c_{n-3}, c_{n-1}}\right)
\leq \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-2}}{0, c_1, \ldots, c_{n-3}, c_{n-2}}\right) + \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-2}}{0, c_1, \ldots, c_{n-3}, c_{n-1}}\right)
\] (41)
holds for all \( k = 0, 1, \ldots, M - 1, n = 2, 3, \ldots, N - 1, n\)-tuples \((r_0, r_1, \ldots, r_{n-1})\) satisfying \( 0 \leq r_0 < r_1 < \cdots < r_{n-1} \leq N - 1 \) and \((n - 1)\)-tuples \((c_1, c_2, \ldots, c_{n-1})\) satisfying \( 0 < c_1 < c_2 < \cdots < c_{n-1} \), \( c_1 \leq M \) and \( c_i - c_{i-1} \leq M, i = 2, 3, \ldots, n - 1 \).

We should remark that the following formula holds:
\[
\lim_{\epsilon \to +0} - \epsilon \log(p_1 e^{-C_i/\epsilon} - p_2 e^{-C_i/\epsilon}) = C_i \quad \text{if } A < B \text{ or } A = B \text{ and } p_1 > p_2 > 0.
\]

Hence, under the conditions above, by (34), we obtain
\[
\gamma^{(k,0)}\left(\frac{r_0}{c_0}\right) = W_{r_0}^{(k+c_0)} + \frac{c_0 Z_{r_0}}{M},
\]
\[
\gamma^{(k,0)}\left(\frac{r_0, r_1}{c_0, c_1}\right) = W_{r_0}^{(k+c_0)} + W_{r_1}^{(k+c_1)} + \frac{c_0 Z_{r_0} + c_1 Z_{r_1}}{M}.
\]

Further, (35) yields the relation
\[
\gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-2}}{c_0, c_1, \ldots, c_{n-3}, c_{n-2}, c_{n-1}}\right) = \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-2}}{c_0, c_1, \ldots, c_{n-3}, c_{n-2}}\right) + \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}, r_{n-1}}{c_0, c_1, \ldots, c_{n-3}, c_{n-1}}\right) - \gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-3}}{c_0, c_1, \ldots, c_{n-3}}\right)
\]
for \( n \geq 3 \). Hence, we obtain
\[
\gamma^{(k,0)}\left(\frac{r_0, r_1, r_2}{c_0, c_1, c_2}\right) = \gamma^{(k,0)}\left(\frac{r_0, r_1}{c_0, c_1}\right) + \gamma^{(k,0)}\left(\frac{r_0, r_2}{c_0, c_2}\right) - \gamma^{(k,0)}\left(\frac{r_0}{c_0}\right)
= \sum_{j=0}^{2} \left( W_{r_j}^{(k+c_j)} + \frac{c_j Z_{r_j}}{M} \right).
\]

In general, the following equation is proved by induction on \( n \):
\[
\gamma^{(k,0)}\left(\frac{r_0, r_1, \ldots, r_{n-1}}{c_0, c_1, \ldots, c_{n-1}}\right) = \sum_{j=0}^{n-1} \left( W_{r_j}^{(k+c_j)} + \frac{c_j Z_{r_j}}{M} \right).
\]
Substituting this result into the inequality (41), we obtain the simpler condition

\[
W_{r_0}^{(k+c_0)} + W_{r_1}^{(k+c_1)} + \frac{c_0 Z_{r_0} + c_1 Z_{r_1}}{M} \leq W_{r_0}^{(k+c_1)} + W_{r_1}^{(k+c_0)} + \frac{c_1 Z_{r_0} + c_0 Z_{r_1}}{M}
\]

for all pairs \((r_0, r_1)\) satisfying \(0 \leq r_0 < r_1 \leq N - 1\), and all pairs \((c_0, c_1)\) satisfying \(0 \leq c_0 < c_1\) and \(c_1 - c_0 \leq M\). This result means that the conditions (40) ensure other conditions (41) hold.

Suppose that all the conditions above are satisfied. Then, \(\tau_n^{(k,t)}\) (33) is ultradiscretized as

\[
T_n^{(k,t)} = \min_{0 \leq r_0 < r_1 < \ldots < r_{n-1} \leq N-1} \left( \sum_{j=0}^{n-1} \left( W_{r_j}^{(k+c_j)} + \frac{k + M j + c_j}{M} Z_{r_j} + S^{(t)}(r_{t+c_0}, r_{t+c_1}, \ldots, r_{t+c_{n-1}}) \right) \right),
\]

where \(S^{(t)}(y_0, y_1, \ldots, y_{n-1})\) indicates the minimum value over all semistandard Young tableaux \(Y\) of the partition \((\lambda_0, \lambda_1, \ldots, \lambda_{n-1})\), and \(y_0, y_1, \ldots, y_{n-1}\) are the weights of \(Y\). Finally, ultradiscretization of (22) yields

\[
\begin{align*}
Q_n^{(k,t)} &= T_n^{(k,t)} - T_n^{(k+1,t)} - T_n^{(k+1,t)}, \\
E_n^{(k,t)} &= T_n^{(k,t)} - T_n^{(k+1,t)} - T_n^{(k+1,t)}, \\
\hat{Q}_n^{(k,t)} &= T_n^{(k,t)} - T_n^{(k+1,t)} - T_n^{(k+1,t)} + T_n^{(k+1,t+1)}, \\
\hat{A}_n^{(k,t)} &= T_n^{(k,t)} - T_n^{(k+1,t)} + T_n^{(k+1,t+1)} - T_n^{(k+1,t+1)}, \\
B_n^{(k,t)} &= T_n^{(k+1,t)} - T_n^{(k+1,t+1)} + T_n^{(k+1,t+1)} - T_n^{(k+1,t+1)}.
\end{align*}
\]

For example, the solution corresponding to Fig. 3 is given by setting \(Z_0 = 8, Z_1 = 4, Z_2 = 2, W_0^{(0)} = 1, W_0^{(1)} = 4/3, W_0^{(2)} = 2/3, W_1^{(0)} = 9, W_1^{(1)} = 26/3, W_1^{(2)} = 25/3, W_2^{(0)} = 15, W_2^{(1)} = 46/3, W_2^{(2)} = 47/3,\) and \(S^{(t)} = 2\) for all \(t\).

Finally, let us consider asymptotics for the autonomous case. If \(S^{(t)} = S\) for all \(t\), then

\[
\hat{S}^{(t)}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \left( \sum_{j=0}^{n-1} \lambda_j \right) S.
\]

Hence, in this autonomous case, a solution is given by

\[
T_n^{(k,t)} = \min_{0 \leq r_0 < r_1 < \ldots < r_{n-1} \leq N-1} \left( \sum_{j=0}^{n-1} \left( W_{r_j}^{(k+c_j)} + \frac{k + M j + c_j}{M} Z_{r_j} + (t + j - c_j) S \right) \right)
\]

\[
= \min_{0 \leq r_0 < r_1 < \ldots < r_{n-1} \leq N-1} \left( \sum_{j=0}^{n-1} \left( W_{r_j}^{(k+c_j)} + \frac{k + M j}{M} Z_{r_j} + (t + j) S + \frac{c_j(Z_j - M S)}{M} \right) \right).
\]

Suppose that \(Z_0 \geq \cdots \geq Z_{m-1} > MS \geq Z_m \geq \cdots \geq Z_{N-1}\) holds. Then, since \(W_{r_j}^{(k+M)} = W_{r_j}^{(k)}\),

\[
T_n^{(k+M,t)} - T_n^{(k,t)} = \begin{cases} 
\sum_{j=0}^{n-1} Z_{N-n+j} & \text{if } n \leq N - m, \\
(n-m)MS + \sum_{j=0}^{N-m-1} Z_{m+j} & \text{if } n > N - m.
\end{cases}
\]
hold for $t \gg 1$. Hence, we obtain

$$Q_n^{(k,t+M)} - Q_n^{(k,t)} = 0 \quad \text{for } t \gg 1,$$

and

$$\sum_{k=1}^{M} Q_n^{(k,t+M)} - \sum_{k=1}^{M} Q_n^{(k,t)} = 0 \quad \text{for } t \gg 1.$$

Therefore, in the autonomous case, the size of each soliton converges to some constant, and the arrangement of positive integers of each soliton changes in period $M$. In addition, we have

$$T_n^{(k+M,j)} - T_n^{(k,j)} = \sum_{j=0}^{n-1} Z_{N-n+j}$$

for $n \leq N - m$ and $t \gg 1$. Hence,

$$\sum_{k=1}^{M} Q_n^{(k,t)} = T_n^{(1,t)} - T_{n+1}^{(1,t)} + T_{n+1}^{(M+1,t)} - T_n^{(M+1,t)} = Z_{N-n-1}$$

holds for $n < N - m$ and $t \gg 1$. Further, we also have

$$\sum_{j=0}^{M-1} Q_n^{(k,t+j)} = T_n^{(k,t)} - T_{n+1}^{(k,t)} + T_{n+1}^{(k,t+M)} - T_n^{(k,t+M)} = \begin{cases} Z_{N-n-1} & \text{if } n \leq N - m, \\ MS & \text{if } n > N - m, \end{cases}$$

for $t \gg 1$. This means that the speed, which is the moving distance from time $t$ to $t + M$, of the $n$th soliton at sufficiently large time $t$ is given by $\min(Z_{N-n-1}, MS)$.

5. Concluding remarks

In this paper, we have proposed a novel soliton cellular automaton, derived two types of time evolution equations to it, and given particular solutions to the evolution equations. We have focused on only the time evolution equations and their solutions. Several important properties of the proposed cellular automaton, e.g. conserved quantities, linearization [10,11], and relations to the solvable lattice models or Yang–Baxter relation [12] should be investigated in detail. Applications of the ndh-Toda lattice studied in this paper to numerical algorithms like the one proposed by Fukuda et al. [13] are also important future works.

In the discussion for the derivation of particular solutions to both the Euler representation and the finite Toda representation, variables taking negative or complex values cause a difficulty for ultradiscretization. The methods used in the previous studies and this paper are based on analysis to show that dominant terms are positive under some restricted conditions for parameters. However, this analysis is complicated and a little hard to perform in general. For this problem, several methods, i.e. ultradiscretization for variables taking negative or complex values, have been proposed [14–16]. Another method is to use permanent solutions instead of determinant solutions [17,18]. It may be interesting to consider applying these methods for ultradiscretization of solutions to the systems appearing in this paper.

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A. Characteristic polynomial of $H^{(k,t)}$

In this appendix, we investigate a relationship between $\Phi_N^{(k,t)}(z)$ and $H^{(k,t)}$. By using (27a) and (27b) repeatedly, we obtain

$$z^M \Phi^{(k,t)}(z) = H^{(k,t)} \Phi^{(k,t)}(z) + \sum_{j=0}^{M-1} L^{(k,t)} R^{(k+j+1,t)} R^{(k+j+1,t)} \cdots R^{(k+M+1,t)} z^j \Phi^{(k+j,t)}(z).$$  \hspace{1cm} (42)

Substituting $z = ze^{-2\pi i/v}$, $\nu = 0, 1, \ldots, M - 1$, into (42), we obtain

$$z^M \Phi^{(k,t)}(ze^{-2\pi i/v}) = H^{(k,t)} \Phi^{(k,t)}(ze^{-2\pi i/v}) + \sum_{j=0}^{M-1} L^{(k,t)} R^{(k+j+1,t)} R^{(k+j+1,t)} \cdots R^{(k+M+1,t)} z^j e^{-2\pi i j/v} M \Phi^{(k+j,t)}(ze^{-2\pi i/v}).$$  \hspace{1cm} (43)

Consider the linear combination of (43):

$$z^M \sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} \Phi^{(k,t)}(ze^{-2\pi i/v}) = H^{(k,t)} \sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} \Phi^{(k,t)}(ze^{-2\pi i/v}) + \sum_{j=0}^{M-1} L^{(k,t)} R^{(k+j+1,t)} R^{(k+j+1,t)} \cdots R^{(k+M+1,t)} z^j \sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} e^{-2\pi i j/v} M \Phi^{(k+j,t)}(ze^{-2\pi i/v}).$$  \hspace{1cm} (44)

where $\hat{u}_{0}^{(k,t)}, \hat{u}_{1}^{(k,t)}, \ldots, \hat{u}_{M-1}^{(k,t)} \in \mathbb{C}$ are constants. If there exist a value $z_r \in \mathbb{C}$ and a nonzero vector $(\hat{u}_{0}^{(k,t)}, \hat{u}_{1}^{(k,t)}, \ldots, \hat{u}_{M-1}^{(k,t)})$ satisfying

$$\sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} e^{-2\pi i j/v} M \Phi^{(k+j,t)}(z_r e^{-2\pi i/v}) = 0, \quad j = 0, 1, \ldots, M - 1,$$  \hspace{1cm} (45)

then equation (44) gives

$$z_r^M \sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} \Phi^{(k,t)}(z_r e^{-2\pi i/v}) = H^{(k,t)} \sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} \Phi^{(k,t)}(z_r e^{-2\pi i/v}).$$

This equation means that $z_r^M$ and $\sum_{\nu=0}^{M-1} \hat{u}_{\nu}^{(k,t)} \Phi^{(k,t)}(z_r e^{-2\pi i/v})$ are an eigenvalue and a corresponding eigenvector of the matrix $H^{(k,t)}$, respectively. To exist nonzero solutions for the linear system (45), the value $z_r$ must be a zero of the following polynomial of degree $MN$:

$$\Phi_N^{(k,t)}(z) := \begin{vmatrix}
\phi_N^{(k,t)}(z) & \phi_N^{(k,t)}(ze^{-2\pi i/v}) & \ldots & \phi_N^{(k,t)}(ze^{-2\pi i(M-1)/v}) \\
\phi_N^{(k+1,t)}(z) & e^{-2\pi i/v} M \phi_N^{(k+1,t)}(ze^{-2\pi i/v}) & \ldots & e^{-2\pi i(M-1)/v} M \phi_N^{(k+1,t)}(ze^{-2\pi i(M-1)/v}) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_N^{(k+M-1,t)}(z) & e^{-2\pi i(M-1)/v} M \phi_N^{(k+M-1,t)}(ze^{-2\pi i/v}) & \ldots & e^{-2\pi i(M-1)/v} M \phi_N^{(k+M-1,t)}(ze^{-2\pi i(M-1)/v})
\end{vmatrix}.$$
Note that, since the determinant is a multilinear alternating map, it is readily shown that
\[
\Phi_N^{(k,j)}(ze^{2\pi i/M}) = (-1)^{M-1}e^{2\pi i/M}e^{2\pi i:2/M}\ldots e^{2\pi i(M-1)/M}\Phi_N^{(k,j)}(z) = \Phi_N^{(k,j)}(z).
\]
This implies that the relation
\[
\Phi_N^{(k,j)}(ze^{-2\pi iv/M}) = \Phi_N^{(k,j)}(z), \quad v = 0, 1, \ldots, M - 1,
\]
holds. Hence, \(\Phi_N^{(k,j)}(z^{1/M})\) is a polynomial of degree \(N\) and its zeros are the eigenvalues of \(H^{(k,j)}\).

In addition to the \((M, 1)\)-orthogonality relation (18), let us consider the discrete \((M, 1)\)-orthogonality relation
\[
\mathcal{L}^{(k,j)}[z^m\psi_N^{(k,j)}(z^M)] = 0, \quad m = 0, 1, 2, \ldots \tag{46}
\]
Suppose that all the zeros of \(\Phi_N^{(k,j)}(z)\) and \(\psi_N^{(k,j)}(z)\) are simple. Then, as discussed in the previous paper [5], \(\psi_N^{(k,j)}(z)\) is the characteristic polynomial of \(H^{(k,j)}\). Therefore, it is shown that the relation
\[
C_{M,n}^{-1}\Phi_N^{(k,j)}(z) = \psi_N^{(k,j)}(z^M)
\]
holds, where the constant \(C_{M,n}\) is the leading coefficient of \(\Phi_N^{(k,j)}(z)\), which is calculated by using the result on the eigenvalues of the discrete Fourier transform matrix [19]:
\[
C_{M,n} = (-1)^{(M-1)n+[(M+2)/4]i[(M-1)/4]}(-1)^{(M+1)/4}(\sqrt{M})^M.
\]
The discrete \((M, 1)\)-orthogonality relation (46) is now equivalent to
\[
\mathcal{L}^{(k,j)}[z^m\Phi_N^{(k,j)}(z)] = 0, \quad m = 0, 1, 2, \ldots \tag{46}
\]

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