Holography and the Geometry of Certain Convex Cocompact Hyperbolic 3-Manifolds

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1 Introduction

Applying the idea of AdS/CFT correspondence, Krasnov [Kra00] studied a class of convex cocompact hyperbolic 3-manifolds. In physics literature they are known as Euclidean BTZ black holes. Mathematically they can be described as $\mathbb{H}^3/\Gamma$, where $\Gamma \subset PSL(2, \mathbb{C})$ is a Schottky group. His main result, roughly speaking, identifies the renormalized volume of such a manifold with the action for the Liouville theory on the conformal infinity. See Takhtajan and Teo [TT] for a rigorous proof and related topics.

This is a nice result establishing another holography correspondence. But the Liouville theory is not yet fully established and the action which was proposed by Takhtajan and Zograf [ZT87] is quite complicated, so it is desirable to clarify the meaning of the renormalized volume in a more geometric and transparent way. This question was first raised by Manin and Marcolli [MM01] and they speculated that the renormalized volume could be calculated through the volume of the convex core of the bulk space based on an explicit example and a recent result by Brock [Bro] in a different but related situation.

In this paper we try to compute the renormalized volume in terms of geometric data. As the first step, we compute the renormalized volume using a different normalization which geometrically is very natural as it uses the distance function to the convex core. The result is very simple and geometric. We first describe the result in the Fuchsian case. Let $\Gamma \subset PSL(2, \mathbb{R})$ be a Fuchsian Schottky group with $2g$
generators. Let $\Omega(\Gamma) \subset S^2$ be its ordinary set. In physics, $X = \mathbb{H}^3/\Gamma$ is known as the Euclidean version of a non-rotating BTZ black hole. Mathematically $X$ is a convex cocompact hyperbolic 3-manifold with the conformal infinity $\Sigma = \Omega(\Gamma)/\Gamma$ which is a compact Riemann surface of genus $g$. $X$ has a totally geodesic surface $M = \mathbb{H}^2/\Gamma$. In analogy with general relativity, we can view $M$ as the $t = 0$ slice and $X$ as obtained by evolving $M$. The noncompact hyperbolic surface $M$ has a number of ends. For each end $E_i$ there is an “event horizon” $C_i$, which is a closed geodesic. Outside $C_i$ the geometry is totally understood as we know the end is $\mathbb{R}^+ \times S^1$ with the metric $dt^2 + \cosh^2(t) d\theta^2$, where $\theta$ is periodic with period $L_i =$ length$(C_i)$. The region inside all the event horizons is precisely the convex core $C$ and may have some wormholes. This is a compact hyperbolic surface with a totally geodesic boundary consisting of the closed geodesics $C_i$. By Gauss-Bonnet theorem and some topological consideration the area of $C$ is a topological invariant ($= 2\pi(g - 1)$) and does not capture the geometric information. The result turns out to be given in terms of $L_i$.

**Theorem 1.1** The renormalized volume of $X$ by the distance function to the convex core is given by

$$V = -\frac{\pi}{2} \sum_i L_i. \quad (1)$$

Unfortunately the normalization we use does not give rise to the hyperbolic metric on the boundary, so what we compute is not the canonical renormalized volume $V_c$, which according to Krasnov [Kra00] is the Liouville action on the boundary. We will comment on the difference.

In the general case when $\Gamma$ is non-Fuchsian, the picture is much more complicated. The convex core $C$ is then a compact domain whose boundary is a “pleated” hyperbolic surface, i.e. a hyperbolic surface with a measured geodesic lamination. It has only finitely many closed leaves $C_i$ with non-zero bending angle $\theta_i$. Let $L_i$ be the length of $C_i$. The result is

**Theorem 1.2** The renormalized volume of $X$ by the distance function to the convex core on $\Sigma$ is given by

$$V = \text{Vol}(C) - \frac{1}{2} \sum_i (\pi - \theta_i)L_i. \quad (2)$$

Again, we emphasize this is not the canonical renormalized volume. We hope these results are helpful in understanding the geometry and the canonical renormalized
volume. Our results reduce it to a problem on the convex core. Understanding the difference between the two normalizations raises a lot of interesting questions.

The paper is organized as follows. In Section 2, we summarize briefly some background knowledge. In Section 3, we discuss the geometry of Schottky group. The computation for the Fuchsian case is then carried out in Section 4. In the last section we discuss the non-Fuchsian case.

2 Renormalized volume and conformal anomaly

We first give a very brief introduction to conformally cocompact Einstein manifolds, the mathematical framework for AdS/CFT correspondence. Let $\mathcal{X}$ be a compact manifold of $n + 1$ dimensions with boundary $\Sigma$. If $r$ is a smooth function on $\mathcal{X}$ with a first order zero on the boundary of $\mathcal{X}$, positive on $X$, then $r$ is called a defining function. A Riemannian metric $g$ on $X = \text{Int}\mathcal{X}$ is called conformally compact if for any defining function $r$, $\bar{g} = r^2g$ extends as a smooth metric on $\mathcal{X}$. The restriction of $\bar{g}$ to $\Sigma$ gives a metric on $\Sigma$. This metric changes by a conformal factor if the defining function is changed, so $\Sigma$ has a well-defined conformal structure $c$. We call $(\Sigma, c)$ the conformal infinity of $(X, g)$. If $g$ satisfies the Einstein equation $\text{Ric}(g) + ng = 0$ we say $(X, g)$ is a conformally compact Einstein manifold.

A metric $h \in c$ on $\Sigma$ determines a unique “good” defining function $r$ in a collar neighborhood of $\Sigma$ such that

$$g = r^{-2}(dr^2 + h_r),$$

where $h_r$ is an $r-$dependent family of metrics on $\Sigma$ with $h_r|_{r=0} = h$. By the Einstein equation the expansion of $h_r$ is of the following form (see e.g. Graham [Gra00]). For $n$ odd,

$$h_r = h_{(0)} + h_{(2)}r^2 + (\text{even powers}) + h_{(n-1)}r^{n-1} + h_{(n)}r^n + \ldots, \quad (3)$$

where the $h_{(j)}$ are tensors on $\Sigma$, and $h_{(n)}$ is trace-free with respect to $h$. The tensors $h_{(j)}$ for $j \leq n - 1$ are locally formally determined by the metric $h$, but $h_{(n)}$ is formally undetermined.

For $n$ even the analogous expansion is

$$h_r = h_{(0)} + h_{(2)}r^2 + (\text{even powers}) + kr^n \log r + h_{(n)}r^n + \ldots, \quad (4)$$

where the $h_{(j)}$ are locally determined for $j \leq n - 2$, $k$ is locally determined and trace-free, but $h_{(n)}$ is formally undetermined.
Consider now the asymptotics of Vol \( \{ r > \epsilon \} \) as \( \epsilon \to 0 \). By the above expansions for \( h \), we obtain for \( n \) odd

\[
\text{Vol} \left( \{ r > \epsilon \} \right) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \text{odd powers} + c_{n-1} \sigma^{-1} + V + o(1)
\]  

(5)

and for \( n \) even

\[
\text{Vol} \left( \{ r > \epsilon \} \right) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \text{even powers} + c_{n-2} \sigma^{-2} - L \log \epsilon + V + o(1).
\]  

(6)

The constant term \( V \) is called the renormalized volume, which a-priori depends on the choice \( h \) in the conformal class \( c \) on \( M \).

Actually for \( n \) odd it is not difficult to show that \( V \) is independent of the choice of \( h \) and thus defines an absolute invariant of the conformal compact Einstein manifold \((X, g)\). But this is not so if \( n \) is even.

In dimension \( 2 + 1 \), conformally compact hyperbolic Einstein manifolds are precisely the so called convex cocompact hyperbolic 3-manifolds, objects which have been much studied by geometers since Poincare.

Let \( \Gamma \subset PSL(2, \mathbb{C}) \) be a torsion-free discrete subgroup such that \( X = \mathbb{H}^3/\Gamma \) is noncompact. Let \( \Lambda(\Gamma) \subset S^2 \) be the limit set and \( \Omega(\Gamma) \) its complement. The convex core \( \mathcal{C} \) of \( M \) is the closed set \( CH(\Gamma)/\Gamma \), where \( CH(\Gamma) \) is the convex hull of \( \Lambda(\Gamma) \) in \( \mathbb{H}^3 \). It is easy to see that \( \mathcal{C} \) is a deformation retract of \( X \). \( X \) is convex cocompact if \( \mathcal{C} \) is compact. In this case \( X \) is conformally compact and the conformal infinity is the compact Riemannian surface \( \Sigma = \Omega(\Gamma)/\Gamma \).

Let \( h \) be a metric on \( \Sigma \) compatible with the conformal structure and \( V_h \) the corresponding renormalized volume. For another metric \( \hat{h} = e^{2u} h \), it can be shown

\[
V_{\hat{h}} = V_h - \frac{1}{4} \int_S \left( |\nabla u|^2 + Ru \right) d\mu_h.
\]  

(7)

If we know the renormalized volume for one metric \( h \) then the above can be used to calculate the renormalized volume with respect any other metric \( \hat{h} \) in the same conformal class.

If \( \Sigma \) has genus \( g > 1 \), then there is a canonical choice of \( h \), namely the hyperbolic metric.

**Proposition 2.1** Let \( X \) be a convex compact hyperbolic 3-manifold with the conformal infinity a compact Riemann surface \( \Sigma \) of genus \( g > 1 \). Let \( h \) be the hyperbolic metric on \( \Sigma \). Then for any metric \( \hat{h} = e^{2u} h \) with \( \text{Area}(\Sigma, \hat{h}) = \text{Area}(\Sigma, h) = 4\pi(g-1) \), we have

\[
V_{\hat{h}} \leq V_h
\]  

(8)

and the identity holds iff \( u \equiv 1 \).
Proof. We want to show that on the hyperbolic surface \((\Sigma, h)\)

\[
E(u) = \int_S (|\nabla u|^2 - 2u) \, dv_h \geq 0
\]

for any function \(u\) with \(\int_{\Sigma} e^{2u} \, dv_h = 1\). By the convexity of the exponential function we have

\[
\int_{\Sigma} u \leq \log \int_{\Sigma} e^{2u} \, dv_h = 0
\]

and hence \(E(u) \geq 0\). It is obvious that \(E(u) = 0\) iff \(u \equiv 1\). \(\square\)

From the above discussion it is natural to expect that the renormalized volume calculated by taking the hyperbolic metric on the conformal infinity can be expressed in terms of geometric invariants.

3 The geometry of Schottky 3-manifolds

We use the ball model \(B^3\) for the hyperbolic 3-space and denote its isometry group by \(M(B^3)\), the Möbius transformations preserving \(B^3\). A Schottky polyhedron in \(B^3\) is a convex polyhedron \(P\) with an even number of sides such that no two sides of \(P\) meet at infinity. Let \(\Phi\) be a \(M(B^3)\)-side-pairing for a Schottky polyhedron \(P\), with \(2g\) sides, such that no side of \(P\) is paired to itself. The group \(\Gamma\) generated by \(\Phi\) is called a classical Schottky group of genus \(g\). It is a torsion free discrete subgroup of \(M(B^3)\) and has \(P\) as a fundamental domain. Let \(\Omega(\Gamma) \subset S^2\) be the ordinary set. It is easy to see that \(\Sigma = \Omega(\Gamma)/\Gamma\) is a compact Riemann surface of genus \(g\). \(X = B^3/\Gamma\) is a convex cocompact hyperbolic 3-manifold with \(\Sigma\) as its conformal infinity. Topologically \(X\) is handle body of genus \(g\). For details we refer to the book [Rat94].

Now we focus on the special case when the Schottky group \(\Gamma \subset PSL(2, \mathbb{R})\). Then \(X = \mathbb{H}^3/\Gamma\) contains a totally geodesic surface \(M = \mathbb{H}^2/\Gamma\). By considering the exponential map on the normal bundle of \(M \subset X\), we can write \(X = \mathbb{R} \times M\) such that the metric on \(X\) takes the form

\[
g = dr^2 + \cosh^2(r)h, \tag{9}
\]

where \(h\) is the hyperbolic metric on \(M\). This explicit description will make the computation very transparent.

The surface \(M\) is noncompact with a finite number of ends. The genus of the surface and the number of ends depend on the Schottky group \(\Gamma\). The following
figures, which appear in both [Kra00] and [MM01], show two such surfaces. They have the same polyhedron but the side-pairings are different, and consequently the resulting surfaces are topologically different.

Suppose $M$ has genus $k$ and $e$ ends. It is easy to get $b_1(M) = k - e + 1$. On the other hand we know $X$ is a handlebody of genus $g$ and hence $b_1(X) = g$. Since $M$ is a deformation retract of $X$, their Betti numbers are equal, i. e.

$$g = k - e + 1. \quad (10)$$

For each end there is an outermost closed geodesic $C_i$ and we denote $E_i$ the part outside of $C_i$. Topologically $E_i$ is a cylinder. We introduce coordinates $(t, \theta)$ on $E_i$ where $t > 0$ is the distance to the boundary $C_i$ and $\theta$ is an arc-length parameter on $C_i$ and is periodic with period $L_i=$length of $C_i$. Then $E_i = \mathbb{R}^+ \times S^1$ with the metric

$$h = dt^2 + \cosh^2(t)d\theta^2. \quad (11)$$

Cutting off all the ends $E_1, \ldots, E_e$ along these closed geodesics $C_1, \ldots, C_e$, we get a compact hyperbolic surface $C$ with totally geodesic boundary. This is precisely the convex core of both $M$ and $X$. Therefore we obtain the following decomposition

$$M = C \cup \sqcup_i E_i. \quad (12)$$

4 Computation in the Fuchsian case

By (12) we also obtain a decomposition for $X$

$$X = (\mathbb{R} \times C) \cup \sqcup_i (\mathbb{R} \times E_i). \quad (13)$$

We define a function $f : X = \mathbb{R} \times M \to \mathbb{R}^+$ as follows. For $x \in C$ $f(r, x) = |r|$. For $x \in E_i$ we use the coordinates $(t, \theta)$ on $E_i$ described in last section and define $f(r, t, \theta) > 0$ such that $\cosh f = \cosh r \cosh t$. It is easy to see that $f$ is $C^{1,1}$ and piecewise smooth on $X - C$ and $|\nabla f| \equiv 1$ by (9) and (11). Geometrically $f$ is just the distance function to the convex core $C$.  

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Outside the convex core $C$, $X$ is foliated by the level sets $\Sigma_\lambda = \{ f = \lambda \}$ for $\lambda \in (0, \infty)$. We have the following decomposition

$$\Sigma_\lambda = \mathcal{C}^+(\lambda) \cup \mathcal{C}^-(\lambda) \cup \bigcup_i T_i(\lambda),$$

where

$$\mathcal{C}^\pm(\lambda) = \{ \pm \lambda \} \times C \subset \mathbb{R} \times M$$

and

$$T_i(\lambda) = \{ (r, x) \in \mathbb{R} \times E_i \mid \cosh r \cosh t = \cosh \lambda \}.$$ 

With the induced metric both $\mathcal{C}^+(\lambda)$ and $\mathcal{C}^-(\lambda)$ are isometric to $(C, \cosh^2 \lambda h)$. We compute the induced metric on $T_i(\lambda)$

$$dr^2 + \cosh^2 r (dt^2 + \cosh^2 td\theta^2) = dr^2 + \frac{\cosh^2 \lambda \sinh^2 r}{\cosh^2 \lambda - \cosh^2 r} dr^2 + \cosh^2 \lambda d\theta^2$$

$$= \frac{\cosh^2 \lambda \sinh^2 r}{\cosh^2 \lambda - \cosh^2 r} dr^2 + \cosh^2 \lambda d\theta^2$$

$$= \cosh^2 \lambda \left( \frac{\sinh^2 \lambda}{\cosh^2 \lambda} d\phi^2 + d\theta^2 \right),$$

where $\phi \in [-\pi/2, \pi/2]$ is the new variable such that $\sin \phi = \frac{\sinh r}{\sinh \lambda}$. Therefore in the new coordinates $(\phi, \theta)$

$$T_i(\lambda) = [-\pi/2, \pi/2] \times S^1$$

with the induced metric taking the following form

$$h_\lambda = \cosh^2 \lambda \left( \frac{\sinh^2 \lambda}{\cosh^2 \lambda} d\phi^2 + d\theta^2 \right).$$

This is a standard (flat) cylinder.

Having described all the pieces in the decomposition, we obtain the entire surface $\Sigma_\lambda$ by gluing them together as illustrated by the following picture.
If we scale the metric by dividing the constant factor \( \cosh^2 \lambda \), the surface \( \Sigma_\lambda \) consists of two copies of the compact hyperbolic surface \((C, h)\) and a number of cylinders \( T_i \) with the base a circle of length \( L_i \) and height \( \pi \sinh \lambda \cosh \lambda \). As \( \lambda \to \infty \) these height of these cylindrical pieces converges to \( \pi \) and we get a closed surface which consists of two copies of \((C, h)\) connected by these flat cylinders of height \( \pi \). It is a Riemann surface with a \( C^1 \) metric, denoted by \( h \). This must be the conformal infinity for \( X \).

Let \( \hat{\rho} = e^{-f} \). Then we can write \( X - C = (0, \infty) \times \Sigma \) with the metric \( g = \hat{\rho}^{-2}(d\hat{\rho}^2 + \hat{h}_{\hat{\rho}}/4) \). We have shown that \( \hat{h}_{\hat{\rho}}|_{\hat{\rho}=0} \) is the hyperbolic metric on the conformal infinity.

We now compute the renormalized volume. For \( \epsilon \in (0, 1) \) let \( X_\epsilon = \{(r, x) \in \mathbb{R} \times M | \hat{\rho}(r, x) > e\} \). We have

\[
\text{Vol}(X_\epsilon) = \text{Vol}(\{(r, x) \in X_\epsilon | x \in C\}) + \text{Vol}(\{(r, x) \in X_\epsilon | x \in \sqcup_i E_i\}). \quad (17)
\]

Denote the two summands by \( V_1 \) and \( V_2 \). We compute

\[
V_1 = \text{Area}(C) \int_0^{-\log \epsilon} \cosh^2 r dr = \frac{\pi(g-1)}{4}(e^{-2} + \log \frac{\epsilon}{2} - \epsilon^2)
\]
and

\[ V_2 = \sum_i \text{Vol}(\{(r, x) \in \mathbb{R} \times E_i \cap X_\epsilon\}) \]

\[ = \sum_i L_i \int_{\cosh r \cosh t \leq (\epsilon + \epsilon^{-1})/2} \cosh^2 \cosh t \, dr \, dt \]

\[ = \frac{\pi}{4} (\epsilon^{-2} - 2 + \epsilon^2) \sum_i L_i. \]

Therefore we obtain

\[ \text{Vol}(X_\epsilon) = \frac{\pi}{4} \left( \sum_i L_i + g - 1 \right) \epsilon^{-2} + \frac{\pi}{8} (g - 1) \log \epsilon - \frac{\pi}{2} \sum_i L_i + \frac{\pi}{4} (\sum_i L_i - g + 1) \epsilon^2. \] (18)

The constant term in the above expansion

\[ V = -\frac{\pi}{2} \sum_i L_i. \] (19)

is then the renormalized volume with respect to \((\Sigma, h)\).

To compute the renormalized volume \(V_c\) with respect to the hyperbolic metric \(h_0\) on \(\Sigma\), we can use formula (7). We write \(h_0 = e^{2\phi}h\). Then

\[ \Delta \phi + 1 - e^{2\phi} = 0, \text{ on } \mathcal{C}^+ \cup \mathcal{C}^-, \] (20)

\[ \Delta \phi - e^{2\phi} = 0, \text{ on the flat cylindrical pieces.} \] (21)

Then by (7) we have

\[ V_c = V - \frac{1}{4} \int_\Sigma (|\nabla \phi|^2 + R \phi) \, d\mu_h \] (22)

Note both terms on the right hand side are given on the convex core. But the second term is very inexplicit as we do not know much about the \(\phi\) which solves (20) and (21). It seems difficult to express it in terms of geometric quantities. It raises the following general question: Let \(S\) be a hyperbolic surface. We get a new Riemann surface by cutting it along a closed geodesic and then attaching a cylinder of height \(t\). How to describe the hyperbolic metric on the new surface?
5 The non-Fuchsian case

We now turn our attention to the general case where the geometry is much more complicated. The same method works, but the result is less explicit than the Fuchsian case. A good reference for the following discussion is [Eps87]. The original source is [Thu].

For a non-Fuchsian Schottky group $\Gamma \subset PSL(2, \mathbb{C})$, the limit set $\Lambda(\Gamma) \subset S^2$ is not contained in any great circle. The convex core $C$ is a compact domain in $X$. Its boundary $S = \partial C$ is a “pleated” surface according to Thurston. With the intrinsic distance $S$ is actually a hyperbolic surface and how it sits in $X$ is described by a measured geodesic lamination $K \subset S$. All the geometric information is encoded in $S$ with this measured geodesic lamination. An important fact is that $K$ is of measure zero and has only finitely many closed leaves $C_i$ with nonzero bending angles $\theta_i$.

Let $\pi : X - C \to S$ be the nearest point projection. Denote $X_\epsilon = \{ x \in X | d(x, C) \leq -\log \epsilon \}$. Then we have the following decomposition

$$X_\epsilon = C \cup F_\epsilon \cup T_i,$$

(23)

where,

$$F_\epsilon = \{ x \in X_\epsilon - C | \pi(x) \in S - K \},$$

and

$$T_i = \{ x \in X_\epsilon - C | \pi(x) \in C_i \}.$$

Let $\hat{\rho}(x) = \exp(-d(x, C))$ which is $C^{1,1}$ by [Eps87]. We claim that this is the defining function on $X$ that induces $h/4$ on the conformal infinity. To see this we look at the level set $\Sigma_\lambda = \{ x \in X | d(x, C) = \lambda \}$. This is a $C^{1,1}$ manifold and we give it the induced metric divided by $\cosh^2(\lambda)$. Since $S - K$ is smooth and totally geodesic in $X$, the piece $\{ x \in \Sigma_\lambda | \pi(x) \in S - K \}$ is smooth and hyperbolic. On the other hand $\{ x \in \Sigma_\lambda | \pi(x) \in \cup C_i \}$ consists of flat pieces which shrink to disappearance as $\lambda \to \infty$. Therefore $\Sigma_\lambda$ converges to a hyperbolic surface as $\lambda \to \infty$. The detail is parallel to the Fuchsian case.

Having shown that $\hat{\rho}(x)$ is the right defining function, the renormalized volume is just the constant term in the asymptotic expansion as $\epsilon \to 0$ of

$$\text{Vol}(X_\epsilon) = \text{Vol}(C) + \text{Vol}(F_\epsilon) + \sum_i \text{Vol}(T_i).$$

(24)

Since $S - K$ is totally geodesic in $X$, the second piece is very simple. The volume of $F_\epsilon$ is given by

$$V_2 = \text{Area}(S) \int_0^{-\log \epsilon} \cosh^2 r dr = \frac{\pi(g - 1)}{4} \left( e^{-2} + \frac{\log \epsilon}{2} - \epsilon^2 \right),$$

(25)
and there is no contribution to the renormalized volume. To visualize the pieces \( T_i \), we work on the universal covering \( \mathbb{H}^3 \). Assume the geodesic \( C \) is the \( z \)-axis in the upper half-space model and the two bending planes are \( y = 0 \) and \( y = \tan(\pi - \theta)x \). Then we consider the following region which is the set of points within distance \( r \) to the wedge and whose nearest point projection to the wedge lies on the \( z \) axis

\[
\{(x, y, z) \in \mathbb{H}^3| x \geq 0, y \leq \tan(\theta - \pi/2)x, \sqrt{x^2 + y^2 + z^2}/z \leq (\epsilon + \epsilon^{-1})/2\}. \quad (26)
\]

It is easy to compute the volume

\[
V = (\pi - \theta)(\epsilon - \epsilon^{-1})^2/4 \int_1^{e^L} \frac{dz}{z^2} = \frac{(\pi - \theta)L}{4}(\epsilon + \epsilon^{-2}) - \frac{(\pi - \theta)L}{2}. \quad (27)
\]

Therefore

\[
\text{Vol}(X_\epsilon) = \text{Vol}(\mathcal{C}) - \frac{1}{2} \sum_i (\pi - \theta_i)L_i + \frac{1}{4} \sum_i (\pi - \theta_i)L_i(\epsilon + \epsilon^{-2}) + \frac{\pi(g - 1)}{4} \left( \epsilon^{-2} + \frac{\log \epsilon}{2} - \epsilon^2 \right),
\]

and this gives the renormalized volume as

\[
\text{Vol}(\mathcal{C}) - \frac{1}{2} \sum_i (\pi - \theta_i)L_i.
\]

It will be intriguing to see what geometric information can be captured by the simple procedure of renormalization for other convex cocompact hyperbolic manifolds. In a sequel to this paper, we will study quasi-Fuchsian deformations.

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