The hitting distribution of a line segment for two dimensional random walks

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Abstract  Asymptotic estimates of the hitting distribution of a long segment on the real axis for two dimensional random walks on \( \mathbb{Z}^2 \) of zero mean and finite variances are obtained: some are general and exhibit its apparent similarity to the corresponding Brownian density, while others are so detailed as to involve certain characteristics of the random walk.

1 Introduction and Results

Let \( S_n = z + \xi_1 + \cdots + \xi_n \) be a two dimensional random walk of i.i.d. increments \( \xi_1, \xi_2, \ldots \) and initial position \( S_0 = z \) moving on the square lattice \( \mathbb{Z}^2 \), which we suppose to be embedded in the complex plane \( \mathbb{C} \). Let \( n \) be a positive integer and denote by \( H_{z}^{I(n)}(s) \) the probability that the first visit (after time 0) to the interval \( \{-n+1, \ldots, n-1\} \) of the random walk \( S \) starting at \( z \) takes place at \( s \). For the later use it is convenient to define a positive number \( n^* \) and an interval \( I(n) \) by

\[
I(n) = (-n^*, n^*) = \{ u \in \mathbb{R} : |u| < n^* \}, \quad n^* = n - 1/2.
\]

Then \( H_{z}^{I(n)}(s), s \in I(n), \) is written as

\[
H_{z}^{I(n)}(s) = P_{z}[\exists j \geq 1, S_j = s \text{ and } S_k \notin I(n) \text{ for } 1 \leq k < j],
\]

where \( P_{z} \) stands for the probability of the walk starting at \( z \in \mathbb{Z} + i\mathbb{Z} \). An explicit expression of the corresponding distribution for Brownian motion is readily derived from the Poisson kernel for the unit disc in view of the conformal invariance of harmonic measures. Let \( h_{x}^{I(n)} \) denote the Brownian analogue of \( H_{z}^{I(n)}(s) \), namely the density of hitting distribution of the interval \( I(n) \) for the two dimensional standard Brownian motion starting at \( z \). Then, for \( x \in \mathbb{R} \setminus [-n^*, n^*] \)

\[
h_{x}^{I(n)}(s) = \frac{\sqrt{x^2 - n_{-}^2}}{\pi|x - s|} \cdot \frac{1}{\sqrt{n_{+}^2 - s^2}}, \quad (s \in I(n))
\]  \hspace{1cm} (1)

(see Appendix (A), in which we compute \( h_{z}^{I(n)}(s) \) for general \( z \)). From the Donsker’s invariance principle it is expected that \( H_{x}^{I(n)}(s) \) behaves similarly to \( h_{x}^{I(n)}(s) \) if the covariance matrix of \( \xi_1 \) is isotropic, but it is not clear at all in what sense they are similar. In the present paper we compute exact asymptotic forms of \( H_{x}^{I(n)}(s) \) for all \( x \in \mathbb{Z} \) as \( |x - s| \wedge n \to \infty \): the first of them exhibits its apparent similarity to \( h_{x}^{I(n)}(s) \) and the others give finer estimates that involve certain characteristics of the random walk. The case of non-real initial sites will be briefly discussed in Appendix (D).

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The problem with real initial sites reduces to a one-dimensional one. Indeed, if \( S \) starts at a point \( x \in \mathbb{Z} \subset \mathbb{C} \) and \( X = (X_n) \) is its trace on the real axis, namely \( X \) is a one-dimensional random walk on \( \mathbb{Z} \) imbedded in \( (S_n) \) with \( X_n \) being the position of the \( n \)-th return of \( S \) to the real axis, then \( H_x^{(n)}(\cdot) \) equals the hitting distribution of \( I(n) \) for \( X_n \). It is remarked that the increment distribution of \( X \) is almost Cauchy in the sense that its tails are asymptotically \( C/|x| \) in both directions [15].

For the symmetric simple random walk H. Kesten has obtained the upper bound

\[
H^{(n)}_\infty(s) := \lim_{|z| \to \infty} H^{(n)}_z(s) \leq C[n(n-s)]^{-1/2} \quad (0 \leq s < n)
\]  

(in [4] (the limit on the left-hand side exists [11]:Theorem 14.1, p.141)) and applied it to a study of the DLA model in [5] (cf. also [6]; a unified exposition is found in [8]). For a rectangle with a side on the real axis Lawler and Limic [9] give an explicit expression for the hitting distribution of its boundary for a simple random walk started inside it and, by taking limits, derive from it the corresponding ones for a half-infinite strip and a quadrant. For a quadrant of the plane, one half of it split along its diagonal line and the complements of these regions as well Fukai [2] obtains very detailed evaluations of the hitting distributions by exploiting the properties special to simple random walk.

Throughout this paper we suppose that the walk \( S_n \) is irreducible, \( E_0[S_1] = 0 \) and

\[
E_0[|S_1|^{2+\delta}] < \infty \quad \text{either for } \delta = 0 \text{ or for some } \delta > 1/2;
\]  

we make an explicit reference to \( \delta \) in the latter case, while no reference is understood to mean the case \( \delta = 0 \).

**Theorem 1** Let \( \delta > 1/2 \) in (3). Then uniformly for integers \( s \in I(n) \) and \( x, |x| \geq n \), as \( n \to \infty \)

\[
H^{(n)}_x(s) = h^{(n)}_x(s) \left[ 1 + O\left( \frac{1}{\sqrt{(|x| - n)^\wedge (n - |s|)}} \right) \right].
\]  

From Theorem [11] it follows that

\[
H^{(n)}_\infty(s) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{n^2 - s^2}} \left[ 1 + O\left( \frac{1}{\sqrt{n - |s|}} \right) \right] \quad (s \in I(n));
\]  

(2) is thus refined. Indeed, the probability that the walk starting at \( z \) hits the real line in the interval \( I(N) \) tends to zero for any \( N > n \) as \( |z| \to \infty \) so that \( H^{(n)}_\infty(s) \) is represented as the limit of a convex combination of \( H^{(n)}_x(s), |x| > N \), which with, e.g., \( N = 2n \) shows the relation above in view of (4).

When either \( |x| - n \) or \( n - |s| \) remains bounded, (4) does not determine the asymptotic form of \( H^{(n)}(\cdot) \). The next theorem improves the estimate in this respect in the case \( \delta = 0 \); in particular it determines a precise asymptotic form of \( H^{(n)}_\infty(s) \) valid uniformly for \( s \in I(n) \), which is not provided by (3). (See Section 4 (Theorems [2] [11]) for the case \( \delta > 1/2 \).) The result is expressed by means of a pair of renewal functions, \( \mu(y) \) and \( \nu(y), \ y \geq 0 \), associated with the imbedded random walk \((X_n)\) — the trace of \( S\) — on the real line mentioned above. They are characterized as positive solutions of the Wiener-Hopf equations

\[
\mu(y) = E_{-y}[\mu(-X_1); X_1 \leq 0] \quad \text{and} \quad \nu(y) = E_y[\nu(X_1); X_1 \geq 0]
\]
together with the pairing condition \( \mu(0)\nu(0) = \pi\sigma^{-2}e^{\sum_{k=1}^{\infty}k^{-1}p_k[X_k=0]} \), except for determination of \( \mu(0) > 0 \) (or \( \nu(0) \)) that is in our disposal: we are to single out \( \mu(0) \) appropriately for the present purpose. Here \( \sigma^2 \) is the square root of the determinant of the covariance matrix \( Q \) of the i.i.d. increments \( \xi_k: \sigma := (\det Q)^{1/2} \); the quadratic form of \( Q \) is given by \( E_0[(S_1 \cdot \theta)^2] \). The equations above plainly say that \( \mu \) and \( \nu \) are harmonic for, respectively, the walks \( -X \) and \( X \) killed on hitting the negative half-line. We extend \( \mu \) and \( \nu \) to \( y < 0 \) by these equations. It then follows that \( \mu(y) \) and \( \nu(y) \), \( y \in \mathbb{Z} \) are (strictly) increasing. We can and do choose \( \mu(0) \) so that

\[
\frac{\mu(y)}{\sqrt{y}} \longrightarrow \frac{2}{\sigma^2} \quad \text{and} \quad \mu(-y)\sqrt{y} \longrightarrow 1 \quad \text{as} \quad y \to \infty,
\]

which entails the same property for \( \nu \) in place of \( \mu \) (\[14\]: Theorem 1.1). (For more details see Appendix (C).)

**Theorem 2**

(i) Uniformly for \( 0 \leq s < n \) and \( x \geq n \), as \( n \to \infty \) and \( x - s \to \infty \)

\[
H_x^{(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\nu(x-n)\mu(-n+s)}{x-s} \cdot \sqrt{\frac{x+n}{n+s}(1 + o(1))}.
\]

(ii) Uniformly for \( -n < s \leq 0 \) and \( x \geq n \), as \( n \to \infty \)

\[
H_x^{(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\nu(x-n)\nu(-n-s)}{x-s} \cdot \sqrt{\frac{x+n}{n-s}(1 + o(1))}.
\]

**Theorem 2’**

(i) Uniformly for \( -n < s \leq 0 \) and \( x \leq -n \), as \( n \to \infty \) and \( s - x \to \infty \)

\[
H_x^{(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\mu(-x-n)\nu(-n-s)}{s-x} \cdot \sqrt{\frac{-x+n}{n-s}(1 + o(1))}.
\]

(ii) Uniformly for \( 0 \leq s < n \) and \( x \leq -n \), as \( n \to \infty \)

\[
H_x^{(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\mu(-x-n)\mu(-n+s)}{s-x} \cdot \sqrt{\frac{-x+n}{n+s}(1 + o(1))}.
\]

By using the asymptotic form of the hitting distribution of the real line we can readily deduce asymptotic forms of \( H_x^{(n)}(s) \) for \( z \notin \mathbb{R} \) from Theorems 2 and 2’ (see Appendix (D)). Here we record only the case when \( z = \infty \). Since \( 1/\sqrt{n+s} \) may be replaced by \( \mu(-n+s) \) as \( n \to \infty \) we obtain the following

**Corollary 3** Uniformly for \( s \in I(n) \), as \( n \to \infty \)

\[
H_{\infty}^{(n)}(s) = \pi^{-1}\mu(-n+s)\nu(-n-s)(1 + o(1)).
\]

From Theorems 2 and 2’ we obtain the second corollary:
Corollary 4 \textit{Uniformly for integers }n \geq 1, \textit{and }x \in \mathbb{I}(n) \textit{and }x \in \mathbb{Z} \setminus \mathbb{I}(n), \textit{let }H_{x}(n)(s) \asymp h_{x}(n)(s), \textit{namely there exists a positive constant }C \textit{independent of }n, s \textit{and }x \textit{such that}

\[ C^{-1} h_{x}(n)(s) \leq H_{x}(n)(s) \leq C h_{x}(n)(s). \]

The next theorem provides the asymptotic form of \( H_{x}(n)(s) \) when \( x \in I(n) \). In view of the corresponding result for the first visit of the real axis, that may reads

\[ P_{x}[\exists j \geq 1, S_{j} = s \text{ and } S_{k} \notin \mathbb{R} \text{ for } 1 \leq k < j] \sim \sigma^{2} \lim_{y \to 0} \frac{1}{|y|} h_{x+iy}(s) \]

with \( h_{x}(s) = |y|/\pi(y^{2} + (x - s)^{2}) \) (see (32)), we extend \( h_{x}(n)(s) \) to the variables \( x \in I(n), x \neq s \) by

\[ h_{x}(n)(s) = \lim_{y \to 0} \frac{1}{|y|} h_{x+iy}(s). \]

In Appendix (B) we compute this limit and find that

\[ h_{x}(n)(s) = \frac{n_{x}^{2} - xs}{\pi(x - s)^{2} \sqrt{(n_{x}^{2} - x^{2})(n_{x}^{2} - s^{2})}} \quad x, s \in I(n), x \neq s. \]

(See (42); also the identities (31) and (30) for an underlying idea.) Let \( S_{1}^{(1)} \) and \( S_{1}^{(2)} \) be the real and imaginary parts of \( S_{1} \), respectively and let \( Y \) be the component of \( S^{(1)} \) that is perpendicular to \( S_{1}^{(2)} \) under \( P_{0} \), namely \( Y = S_{1}^{(1)} - \omega S_{1}^{(2)} \) where \( \omega = E_{0}[S_{1}^{(1)} S_{1}^{(2)}]/E_{0}[(S_{1}^{(2)})^{2}] \).

Theorem 5 \textit{Let }Y \textit{be as above and suppose the moment condition}

\[ E_{0}[|Y|^{2} \log |Y|] < \infty. \]  \hfill (7)

\textit{Let }x, s \in I(n). \textit{Then}

\begin{itemize}
  \item[(i)] as \((n - |s|) \land (n - |x|) \land |x - s| \to \infty\)
  \[ H_{x}(n)(s) = \sigma^{2} h_{x}(n)(s)(1 + o(1)); \]
  \item[(ii)] if \( s < x \), as \((n - x)/(n - s) \to 0\)
  \[ H_{x}(n)(s) = \frac{\sigma^{2}}{\pi} \cdot \frac{\nu(-n + x)\nu(-n - s)\sqrt{n}}{\sqrt{2} (x - s)^{3/2}} (1 + o(1)); \]
  \item[(ii')] if \( s > x \), as \((n - s)/(n - x) \to 0\)
  \[ H_{x}(n)(s) = \frac{\sigma^{2}}{\pi} \cdot \frac{\mu(-n + s)\mu(-n - x)\sqrt{n}}{\sqrt{2} (s - x)^{3/2}} (1 + o(1)). \]
\end{itemize}

Observe first that the condition \((n - x)/(n - s) \to 0\) in \( (ii) \) entails
\[ x - s \to \infty, \quad \frac{x}{n} \to 1, \quad \frac{x - s}{n - s} \to 1 \quad \text{and} \quad \frac{n^{2} - xs}{n(n - s)} \to 1, \]
and then that the formula of \( (ii) \) implies (and is actually finer than) the formula of \( (i) \). Under the condition \(|x - s| \to \infty\) the cases \( (ii) \) and \( (ii') \) together exhaust the case when \((x \lor s)/n \to 1\). We have an obvious analogue for \( (ii') \), which is a dual statement of \( (ii) \). Also observe that \( h_{x}(n)(s) \) as well as \( H_{x}(n)(s) \) is bounded away from zero and infinity whenever \(|x - s| \) is bounded above by any constant. These observations lead to the following corollary.
Corollary 6 Suppose $E_0[|Y|^2 \log |Y|] < \infty$. Then, for $x, s \in I(n)$, $C^{-1}h_x^{I(n)}(s) \leq H_x^{I(n)}(s) \leq Ch_x^{I(n)}(s)$. In particular if $x$ is kept within any bounded distance from $n$ and so is $s$ from $-n$, then $H_x^{I(n)}(s) \approx 1/n$.

Remark 1. Under the same supposition as in Theorem 5 the formulae obtained above can be extended to the general starting positions $x + iy$ as in [13] but with the resulting formula somewhat complicated (see (39)).

Denote by $H_x^+(s)$ the probability that the first visit (after time 0) to the positive real axis of the walk starting at $z \in \mathbb{C}$ takes place at $s \in \{1, 2, 3, \ldots \}$:

$$H_x^+(s) = P_z[ \exists j \geq 1, S_j = s \quad \text{and} \quad S_k \notin \{1, 2, 3, \ldots \} \text{ for } 1 \leq k < j].$$

Similarly let $H_x^-(s)$ denote the distribution of the first visiting sites (after time 0) of the set $\{-1, -2, -3, \ldots \}$. The proofs of Theorems 1 and 2 rest on the results on $H_x^\pm(s)$ obtained in [13] (Theorem 1.1; see also [14] for (9)) as given in the following theorem (and also in (23), (24) and (25) later).

Theorem (13, 14) Let $s < 0$. Then for $x \geq 0$, as $x \vee (-s) \to \infty$

$$H_x^-(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\nu(x)\mu(s)}{|x - s|} \left(1 + o(1)\right). \quad (8)$$

If $E_0[|Y|^2 \log |Y|] < \infty$ in addition, then as $|x - s| \to \infty$ under $x < 0, s < 0$,

$$H_x^-(s) = \frac{\sigma^2}{2\pi} \cdot \frac{|x + s|\nu(x)\mu(s)}{|x - s|^2} \left(1 + o(1)\right). \quad (9)$$

It is warned that in [13] the condition $E_0[|S_1^{(1)}|^2 \log |S_1^{(1)}|] < \infty$ is erroneously assumed where it should be (7) as in Theorem above.

Comparing the formulae given in Theorem with those in Theorems 2(i) and 5 we find the quite reasonable conclusion that $H_x^{I(n)}(s)/H_x^{I}(s - n) \to 1$ if and only if $x/n \to 1$ and $s/n \to 1$ (whether $x$ is larger than $n$ or not). For the symmetric simple random walk (i.e., $P_0[S_1 = x] = 1/4$ for $x \in \{\pm 1, \pm i\}$) we can improve the error estimate in [13] for $H_x^\pm(s)$ and accordingly that in Theorem 1 for $H_x^{I(n)}(s)$.

Proposition 7 Let $S_n$ be the symmetric simple random walk. Then

$$H_x^-(s) = \frac{1}{\pi} \cdot \frac{1}{x - s} \sqrt{\frac{x \vee 1}{-s}} \times \left[1 + O\left(\frac{1}{x \vee 1}\right) + O\left(\frac{1}{x \vee 1}\right)\right] \quad (x \geq 0, s < 0) \quad (10)$$

and as $n \to \infty$

$$H_x^{I(n)}(s) = h_x^{I(n)}(s) \left[1 + O\left(\frac{1}{(|x| - n_\ast) \wedge (n - |s|)}\right)\right] \quad (|x| \geq n, s \in I(n)). \quad (11)$$

Here the error estimates are uniform for integers $x, s$ subject to the respective constraints indicated in parentheses.

Theorem 1 is proved in Section 2 by taking for granted certain several results whose proofs are given in Section 3. Proof of Proposition 7 is given at the end of Section 3. In Section 4 we make detailed estimation of $H_x^{I(n)}(s)$ when $x$ or $s$ are near the edges of $I(n)$ under the moment condition (3); in particular Theorem 2 is proved. Theorem 5 is proved in Section 5.
2 Proof of Theorem 1

The proof of Theorem 1 primarily rests on the asymptotic estimates obtained in [13] of hitting distributions for half-real-lines. The first visit of \(I(n)\) occurs after the consecutive overshoots in the first \(k\) alternating entrances of the walk \(X\) into the half-lines \((-\infty, -n)\) and \((n, \infty)\) for some \(k = 0, 1, 2, \ldots\) and we simply sums up the relevant probabilities over \(k\). Motivated by this we bring in the sequence of probability kernels in below.

Throughout this section we pick and fix a (large) positive integer \(n\), which we shall not designate in the notation introduced in this section even though it depends on \(n\). Let \(1(S)\) stand for the indicator of a statement \(S\): \(1(S) = 1\) or \(0\) according as \(S\) is true or not. Define for integers \(x \geq n\) and \(y > -n\),

\[
Q(x, y) = \sum_{s=-\infty}^{-n} H_{x-n}^-(s-n)H_{s+n}^+(y+n),
\]

\[
Q_I(x, y) = Q(x, y)1(-n < y < n),
\]

\[
K_I(x, y) = H_{x-n}^-(y-n)1(-n < y < n),
\]

and \(Q^0 = 1\) (the identity matrix), \(Q^1 = Q\) and inductively

\[
Q^k(x, y) = \sum_{u=n}^\infty Q^{k-1}(x, u)Q(u, y) \quad (k = 1, 2, \ldots),
\]

and finally

\[
\Lambda(x, y) = \sum_{k=1}^\infty Q^k(x, y)1(y \geq n).
\]

Then for \(x \geq n\), \(-n < s < n\),

\[
H_I^n(x)(s) = (1 + \Lambda)(Q_I + K_I)(x, s)
\]

\[
= \sum_{y=n}^\infty [1(y = x) + \Lambda(x, y)][Q_I(y, s) + K_I(y, s)].
\]

The kernels \(Q, K_I, Q_I\) and \(\Lambda\) are probabilities with self-evident meaning. We are to compare them with the corresponding ones, denoted by \(q, k_I, q_I\) and \(\lambda\), for the standard two dimensional Brownian motion \(B(t)\). In doing this it is recalled that the interval \(I(n)\) is defined to be \((-n + 1/2, n - 1/2)\) instead of \([-n + 1, n - 1]\), which makes difference in the associated probabilities of the Brownian motion. Put \(L_\pm = \{t \in \mathbb{R} : \pm t > 0\}\) and \(\tau_{L_\pm} = \inf\{t > 0 : B(t) \in L_\pm\}\), and define

\[
h_x^\pm(s) = P_{BM}^z[B(\tau_{L_\pm}) \in ds]/ds \quad (x \in L_\pm, \pm s > 0),
\]

where \(P_{BM}^z\) denotes the law of \(B(t)\) starting at \(z\). Then for real \(x > n_*, y > -n_*\),

\[
q(x, y) = \int_{-\infty}^{-n_*} h_{x-n_*}^-(s-n_*)h_{s+n_*}^+(y+n_*)ds,
\]

\[
q_I(x, y) = q(x, y)1(-n_* < y < n_*),
\]

\[
k_I(x, y) = h_{x-n_*}^-(y-n_*)1(-n_* < y < n_*)
\]
and \(q^k\) and \(\lambda\) are given in analogous ways; in particular \(q^1 = q\) and

\[
\lambda(x, y) = \sum_{k=1}^{\infty} q^k(x, y) 1(y > n_*).
\]

We know that

\[
\begin{align*}
    h^{-}_x(s) &= \frac{\sqrt{x}}{\pi(x-s)} \cdot \frac{1}{\sqrt{-s}} \quad (x > 0, s < 0), \\
    h^{I(n)}_x(s) &= \frac{\sqrt{x^2 - n_*^2}}{\pi(x-s)} \cdot \frac{1}{\sqrt{n_*^2 - s^2}} \quad (x > n_*, -n_* < s < n_*).
\end{align*}
\]

The function \(Q\) is extended to that of reals by

\[
Q(u, v) = Q(x, y) \quad \text{for} \quad (u, v) \in (x - \frac{1}{2}, x + \frac{1}{2}] \times (y - \frac{1}{2}, y + \frac{1}{2}],
\]

and similarly for \(\Lambda\) and \(K_I\). (The summation in (12) can be then replaced by the integration over \(y > n_*\).) With \(Q\) thus extended put

\[
\eta = Q - q.
\]

We shall prove the relations (I) through (VII) given below. The symbol \(f \asymp g\) means that \(C^{-1}g \leq f \leq Cg\). Here and in what follows \(C\) denotes a positive constant which may depend on the law \(P[\mathcal{S}_1 = \cdot]\) but is independent of any variables \(x, n, y, s\) contained therein explicitly or inexplicitly and may change from line to line. The products of two functions (of two variables) are understood to be that of integral operators in an analogous way to (12): e.g.,

\[
\eta q(x, y) = \int_{n_*}^{\infty} \eta(x, u)q(u, y)du
\]

(the range of the integration is always the half-line \(u > n_*\) where \(q(u, \cdot)\) is defined).

Let \(x > n_*, y > -n_*\) and \(-n_* < s < n_*\); \(x, y, s\) are real numbers in (I) through (III).

(I) \(q(x, y) \asymp \frac{\sqrt{x - n_*}}{\sqrt{n_* + y}} \cdot \frac{1}{x} \log \frac{x + 2n}{y + 2n}\).

The function \(t^{-1} \log(1 + t)\) is understood to be continuously extended to \(t = 0\). On using the inequalities \(1/b < (b-a)^{-1} \log(b/a) < 1/a (0 < a < b)\) we infer that \((x - y)^{-1} \log[(x + 2n)/(y + 2n)] \asymp x^{-1}\) if \(|x - y| \leq 3n\), which combined with (I) yields the bound of \(q_I\) given in the next item where we also display the explicit form of \(k_I\) for convenience.

(I') \(q_I(x, s) \asymp \frac{\sqrt{x - n_*}}{\sqrt{n_* + s}} \cdot \frac{1}{x} \left(1 + \log \frac{x}{n_*}\right)\);

\[
k_I(x, s) = \frac{1}{\pi(x-s)\sqrt{n_* - s}}.
\]

(II) \[
\frac{\vert \eta(x, y) \vert}{q(x, y)} = o(1) \quad \text{as} \quad (x - n) \wedge (n + y) \to \infty \quad \text{if} \quad \delta = 0,
\]

\[
\leq \frac{C}{\sqrt{(x - n_*) \wedge (n + y)}} \quad \text{if} \quad \delta > \frac{1}{2}.
\]
(III) \[ |\eta|(1 + \lambda)(q_I + k_I)(x, s) \]
\[
= \left[ \frac{\lambda}{\sqrt{n_s^2 - s^2}} \wedge h^{(n)}_x(s) \right] \times o(1) \quad \text{as} \quad n \to \infty \quad \text{if} \quad \delta = 0,
\]
\[
\leq C \frac{1 + (\log x/n_s)^2}{\sqrt{x}\sqrt{n_s^2 - s^2}} \quad \text{if} \quad \delta > \frac{1}{2}.
\]

Here \(|\eta|\) in (III) stands for the integral operator of kernel \(|\eta(x, y)|\) as in (15), \(1\), also in (III), for the identity operator, and \(\delta\) in (II) and (III) for the constant in (3).

For integers \(x \geq n, -n < s < n\),

(IV) \[ \sum_{y=n}^{\infty} \Lambda(x, y) \leq C \sqrt{\frac{x - n_s}{x}}, \]

(V) \[ \sum_{y=n}^{\infty} \Lambda(x, y) \frac{1}{y - s} \leq C \frac{1}{n} \sqrt{\frac{x - n_s}{x}} \cdot \log \frac{3n}{n - s}, \]

(VI) \[ \sum_{y=n}^{n+N} \Lambda(x, y) \leq C \sqrt{\frac{x - n_s}{x}} \cdot \frac{N}{n} \quad (N = 1, 2, \ldots). \]

There exist functions \(\varepsilon_j(t), j = 1, 2\), of (a single variable) such that as \(t \to \infty\), \(\varepsilon_j(t) = o(1)\) or \(O(1/\sqrt{t})\) according as \(\delta = 0\) or \(\delta > 1/2\) in (3) and that

(VII) \[ |k_I - K_I|(x, s) \leq k_I(x, s)[\varepsilon_1(x - n_s) + \varepsilon_2(n - s)]. \]

The proofs of these results are postponed to the next section. In the rest of this section we prove Theorem 1 taking them for granted.

By symmetry we may suppose \(x \geq n\). From the identity

\[ H^{(n)}_x = (1 + \Lambda)(K_I + Q_I)(x, \cdot) \]

and a similar one for \(h^{(n)}_x\) it follows that

\[ H^{(n)}_x - h^{(n)}_x = (1 + \Lambda)(K_I - k_I + Q_I - q_I) + (\Lambda - \lambda)(k_I + q_I)(x, \cdot). \]

Writing \(Q = \Lambda - \Lambda Q\) and \(q = \lambda - q\lambda\) (valid on \([n_s, \infty)^2\)) one finds the identity \(\Lambda q - Q\lambda = \Lambda \eta \lambda\), which yields

\[ \Lambda - \lambda = \eta + \Lambda \eta + \eta \lambda + \Lambda \eta \lambda = (1 + \Lambda)\eta(1 + \lambda). \quad (16) \]

Let \(q_I + k_I\) act on the both sides from the right. Let \(x\) and \(s\) be integers such that \(x \geq n\) and \(-n < s < n\). Using (III), (IV) and the simple inequality

\[ x^{-\alpha} \log x/n \leq (\epsilon \alpha)^{-1} n^{-\alpha} \quad (\alpha > 0), \]

first observe that

\[ \lambda|\eta|(1 + \lambda)(q_I + k_I)(x, s) \leq \sqrt{\frac{x - n_s}{x}} \cdot \frac{\varepsilon(n)}{\sqrt{n^2 - s^2}}. \]
where \( \varepsilon(t) \) is a function of the same meaning as \( \varepsilon_j(t) \) in (VII), and then, further using (16) and (III), that

\[
|\Lambda - \lambda|(q_I + k_I)(x, s) \leq \left[ \varepsilon(n) \sqrt{x - n_\ast} + C \left( 1 + \log \frac{x}{n} \right)^2 \right] \frac{1}{\sqrt{x(n^2 - s^2)}} \\
\leq C' \left( \varepsilon(n) + \frac{1}{\sqrt{x - n_\ast}} \right) h_x^{I(n)}(s).
\]

(17)

The last inequality in particular implies

\[
\Lambda(k_I + q_I)(x, s) \leq C h_x^{I(n)}(s).
\]

(18)

Let \( \delta > 1/2 \) in (3). Combined with (V) and (18) the bound (VII) shows that

\[
\Lambda|k_I - K_I|(x, s) \leq C \left( \frac{1}{\sqrt{n}} \log \frac{3n}{n - s} + h_x^{I(n)}(s) \right) \frac{1}{\sqrt{n - s}}.
\]

(19)

On the other hand by (II) and (I') we have

\[
|q_I - Q_I|(y, s) \leq C \left( \frac{1}{\sqrt{y - n}} + \frac{1}{\sqrt{n + s}} \right) \cdot q_I(y, s) \leq C \frac{1 + \log y/n}{\sqrt{y}} \cdot \frac{1}{n + s}
\]

(20)

and \( \Lambda|q_I - Q_I| \leq C \sqrt{x - n_\ast}/\sqrt{x n}(n + s) \leq C' h_x^{I(n)}(s)/\sqrt{n + s} \), which in conjunction with (19) gives

\[
(1 + \Lambda) \left( |q_I - Q_I| + |k_I - K_I| \right)(x, s) \leq C h_x^{I(n)}(s) \left( \frac{1}{\sqrt{n - s}} + \frac{1}{\sqrt{n + s}} \right).
\]

(21)

The bounds (VII), (17), (20) and (21) together yield the formula of Theorem I in the case \( \delta > 1/2 \).

3 Proofs of (I) through (VII)

Proof of (I). Let \( x \geq n_\ast \) and \( y > -n_\ast \). It follows from (14) that

\[
q(x, y) = \frac{1}{\pi^2} \int_{-\infty}^{-n_\ast} \frac{\sqrt{x - n_\ast}}{(x - u)\sqrt{-u - n_\ast}} \frac{\sqrt{-u - n_\ast}}{(y - u)\sqrt{n_\ast + y}} \, du \\
= \frac{1}{\pi^2} \sqrt{\frac{x - n_\ast}{y + n_\ast}} J_n(x + n_\ast, y + n_\ast),
\]

(22)

where

\[
J_n(a, b) = \int_0^{\infty} \frac{\sqrt{t} \, dt}{\sqrt{t + 2n_\ast(t + a)(t + b)}} \quad (a = y + n_\ast > 0, \ b = x + n_\ast > 2n_\ast).
\]
If $n_* \leq a < b$, then

$$J_n(a, b) = \frac{1}{a} \int_0^\infty \sqrt{t} dt \frac{\sqrt{t + 2n_*/a}}{t + 2n_*/a(t + b/a)}$$

$$\leq \frac{1}{b} + \frac{1}{a} \int_1^\infty \frac{dt}{(t + 1)(t + b/a)}$$

$$= \frac{1}{b} + \frac{1}{b - a} \log \frac{a + b}{2a}$$

$$\leq \frac{1}{b - a} \log \frac{b}{a}.$$

This shows (I) in the case $y \geq n_*$. In the case $-n_* < y < n_*$, a similar computation gives

$$J_n(a, b) = \frac{1}{2n_*} \int_0^\infty \sqrt{t} dt \frac{\sqrt{t + 2n_*/a}}{t + 2n_*/a(t + b/2n_*)}$$

$$\leq \frac{1}{b} + \frac{1}{b - a} \log \frac{2n_* + b}{2n_* + a}.$$

Thus (I) has been proved.

**Proof of (II).** First suppose that $\delta > 1/2$ in (3). Then it is shown in [13] (Theorem 1.3) that there exists a constant $C$ such that for $x \geq n$ and $s < n$,

$$\left| H_{x-n}^-(s - n) - \frac{1}{\pi} \cdot \frac{1}{x - s} \sqrt{\frac{x - n_*}{n - s}} \right| \leq C h_{x-n_*}^-(s - n_*) \left( \frac{1}{\sqrt{n_* - s}} + \frac{1}{\sqrt{x - n_*}} \right). \quad (23)$$

In making application of this and its obvious analogue for $H^+$ there arise four terms to be estimated for computation of the difference $Q - q = H^- H^+ - h^- h^+ = (H^- - h^-)H^+ + h^- (H^+ - h^+)$ (the right side of (23) is counted two terms), which are equal to those obtained by inserting the factors

$$\frac{1}{\sqrt{x - n_*}} + \frac{1}{\sqrt{-u + n_*}} \quad \text{and} \quad \frac{1}{\sqrt{n_* + y}} + \frac{1}{\sqrt{-u - n_*}}$$

under the integral symbol of the integral of (22). Among them only two terms require computation, which we are to show to be not larger than the sum of the other two. To this end, we make the same change of variables that led to the second equality of (22) and find that it suffices in view of (I) to verify the following inequalities

$$\int_0^\infty \sqrt{t} dt \frac{\sqrt{t + 2n_*/(t + a)(t + b)}}{(t + 2n_*/(t + a)(t + b))} \leq \int_0^\infty \frac{dt}{\sqrt{t + 2n_*/(t + a)(t + b)}} \leq \pi \frac{1}{(a \lor b) \sqrt{a \land b}}$$

($a = y + n_* > 0, b = x + n_*$ as before). The first one is trivial. The second one is verified by letting $(a \lor b)^{-1} \int_0^\infty \frac{dt}{\sqrt{t + a \land b}}$ dominate the integral in the middle. This completes the proof in the case $\delta > 1/2$.

The case $\delta = 0$ is similarly dealt with based on the corresponding result for $H_x^-(s)$ (Theorem 1 of [13]).

**Proof of (III).** Consider the case $\delta > 1/2$. Set

$$A(y) = \frac{1}{\sqrt{x - n_*}} q(x, y) h_{y}^{(n)}(s) \sqrt{n_*^2 - s^2},$$

10
\[ B(y) = q(x,y) \frac{1}{\sqrt{y + n_2}} \frac{h^{I(n)}_y(s)}{\sqrt{n_2^2 - s^2}} \]

and
\[ I_A = \int_{(x-2n)\lor n_2}^{\infty} A(y)dy \quad \text{and} \quad I_B = \int_{n_2}^{(x-2n)\lor n_2} B(y)dy. \]

Notice that \(1/\sqrt{x-n_2} \leq 1/\sqrt{y+n_2}\) if and only if \(y \leq x-2n_2\) and that according to (II)
\[ |\eta| h^{I(n)}(x, s) \leq C(I_A + I_B)/\sqrt{n_2^2 - s^2}. \]

By (I)
\[ A(y) \asymp \frac{\sqrt{y-n_2}}{y-s} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n}. \]

A simple computation shows that
\[ \int_{(x-2n)\lor n_2}^{(x-2n)\lor (2n)} A(y)dy = O(1/\sqrt{x}) \]

and
\[ \int_{(x-2n)\lor n_2}^{\infty} A(y)dy \leq C \int_{(x-2n)\lor (2n)}^{\infty} \frac{1}{y^{3/2}} \left(1 + \log \frac{y}{x}\right) dy = O(1/\sqrt{x}). \]

Thus \(I_A = O(1/\sqrt{x})\) (uniformly in \(s < n_2\)).

As for \(I_B\) first we observe
\[ B(y) \asymp \frac{\sqrt{x-n_2}}{\sqrt{y+n_2}} \frac{\sqrt{y-n_2}}{y-s} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n}. \]

Let \(x \geq 3n\). It is easy to see that \(\int_{n_2}^{2n} B(y)dy = \text{const} \ (1/\sqrt{x}) \log(x/n_2),\) while
\[ \int_{2n}^{x} B(y)dy \leq C\sqrt{x} \int_{2n}^{x} \frac{1}{y} \log \frac{x+2n}{y+2n} dy \]
\[ \leq C \frac{2\sqrt{x}}{x+2n} \int_{4n/(x+2n)}^{1} \frac{1}{u(1-u)} \log \frac{1}{u} du \]

and the last member is dominated by a constant multiple of \((1/\sqrt{x})(\log x/n_2)^2\) owing to the equality \(\int_{a}^{1} u^{-1} \log(1/u)du = \frac{1}{2} \left[ \log a \right]^2\) \(0 < a < 1\). This verifies (III) when \(\delta > 1/2\). For the case \(\delta = 0\) the same argument as above leads to the upper bound \(o(1)/\sqrt{n_2^2 - s^2}\); the identity \(q^{I} = h^{I} - (k_I + q_I) (\leq h^{I})\) gives the other bound \(h^{I} \times o(1)\). The proof of (III) is complete.

**Proof of (IV).** Put \(p_n = \sup_{x,y \geq n} Q(x,y)\). Then \(p_n = O(1/\sqrt{n})\) and
\[ \sum_{y \geq n} Q^k(x,y) \leq p_n^{k-1} \sum_{y \geq n} Q(x,y) \leq p_n^{k-1} \sum_{y \geq n} H_{x-n}(y) \leq p_n^{k-1} C \sqrt{\frac{x-n_2}{x}}, \]

hence \(\sum_{y=n}^{\infty} \Lambda(x,y) \leq \sum_k \sum_{y \geq n} Q^k(x,y) \leq C' \sqrt{(x-n_2)/x}.\)
Lemma 8 \textit{Uniformly for integers }x \geq n, -n < s < n,\textit{ }

\[
\sum_{y=n}^{\infty} Q(x,y) \frac{1}{y-s} \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{1}{n}\right) \log \frac{3n}{n-s}.
\]

\textbf{Proof.} By employing (I) and the bound \((x-y)^{-1} \log \frac{x+2n}{y+2n} \asymp x^{-1} [1 + \log(x/y)]\) valid for \(n \leq y \leq x + n\) one sees that

\[
\sum_{y=n}^{2n} Q(x,y) \frac{1}{y-s} \asymp \int_n^{2n} \frac{\sqrt{x-n_*}}{(y-s)\sqrt{n+y}} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n} dy
\]

\[
\asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right) \log \frac{2n-s}{n-s}
\]

as well as

\[
\sum_{y=2n+1}^{\infty} Q(x,y) \frac{1}{y-s} \leq C \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right)
\]

(break the summation according as \(y\) is larger than \(x \lor (2n)\) or not and consider the cases \(x \leq 2n\) and \(x > 2n\) separately). These together yield the estimate of the lemma. 

\textbf{Proof of (V).} Letting \(2\sqrt{x/n}\) dominate \(1 + \log(x/n)\) in the right-hand side of the asymptotic formula of Lemma 8 and employing (IV), we have

\[
\sum_{y=n}^{\infty} \Lambda(x,y) \frac{1}{y-s} = \sum_{y=n}^{\infty} (Q + \Lambda Q)(x,y) \frac{1}{y-s} \leq C \int \frac{\sqrt{x-n_*}}{x\sqrt{n}} \log[3n/(n-s)].
\]

Thus (V) is proved.

\textbf{Proof of (VI).} As in the proof of Lemma 8 we have

\[
\sum_{y=n}^{n+N} Q(x,y) \leq C \sqrt{(x-n_*)/x} N/n;
\]

the estimate of (VI) is then follows from (IV) as in the preceding proof.

\textbf{Proof of (VII):} This follows immediately from (23) if \(\delta > 1/2\). In the case \(\delta = 0\) use (8).

\textbf{Proof of Proposition 7.} Both formulae (10) and (11) of Proposition 7 are proved in a similar way as Theorem 1 and in their proofs given below we omit details. We first show the deduction of (11) from (10). For the symmetric simple random walk the right-hand side of (23) can be replaced by \(C h^-_x(s)(-s)^{-1} + (x \lor 1)^{-1}\) (cf. [13]) and accordingly we deduce that

\[
(|k_I - K_I| + |q_I - Q_I|)(y,s) \leq \frac{C k_I(y,s)}{[(y-n_*) \land (n-s) \land (n+s)]},
\]

\[
|\eta|(x,y) \leq \frac{C q(x,y)}{(x-n_*) \land (n_* + y)},
\]

\[
\sum_{y=n}^{\infty} \frac{Q(x,y)}{(y-s)\sqrt{y-n_*}} \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right) \frac{1}{\sqrt{n-s}},
\]

\[
\sum_{y=n}^{\infty} \frac{\Lambda(x,y)}{(y-s)\sqrt{y-n_*}} \leq \frac{C \sqrt{x-n_*}}{n\sqrt{x}\sqrt{n-s}}
\]
and
\[ \sum_{y=n}^{\infty} \frac{\Lambda(x, y)k_I(y, s)}{y - n} \leq \frac{C\sqrt{x - n_*}}{\sqrt{x n(n - s)}}, \]
and with these bounds we can proceed as above to obtain (11).

Proof of (10): The proof is based on an asymptotic expansion of the potential function of the random walk (cf. [3], [12], [7]) from which an application of the reflection principle immediately yields
\[ H_{im}(s) = \frac{|m|}{\pi(|s|^2 + m^2)} + O\left(\frac{1}{|s|^3 + |m|^3}\right) \quad (m \neq 0), \]
where \( H_{im}(s) \) stands for the probability that the first visit to the real axis of the simple random walk starting at \( im \in i\mathbb{Z} \) takes place at \( s \in \mathbb{Z} \). We proceed as in Section 2. Bearing symmetry of the walk in mind, this time we define for \( y \in \mathbb{Z} \) and \( x \geq 0 \),
\[ Q(x, y) = \sum_{m=-\infty}^{\infty} H_{ix}(m)H_{im}(y) \quad \text{if} \quad x > 0 \quad \text{and} \quad = H_0(y) \quad \text{if} \quad x = 0 \]
and inductively \( Q^k(x, y) = \sum_{u=0}^{\infty} Q^{k-1}(x, u)Q(u, y) \quad (k = 1, 2, \ldots) \). We have the corresponding quantities \( h, h^-, q \) and \( q^k \) for the standard Brownian motion. Then for \( s < 0, x \geq 0 \),
\[ H_x^-(s) = \Lambda(x, s) := \sum_{k=1}^{\infty} Q^k(x, s), \quad h_x^-(s) = \lambda(x, s) := \sum_{k=1}^{\infty} q^k(x, s). \]
We know that \( C^{-1} \lambda \leq \Lambda \leq C\lambda \) for some constant \( C > 0 \) (cf. [13]). We suitably extending \( Q \) to the real variables and put \( \eta = Q - q \) as before. An elementary computation then gives in turn
\[ \eta(x, y) = O(|xy^2|^{-1} \wedge |x^2y|^1), \quad |\eta|\lambda(x, y) = O(|xy|^{3/2} \wedge |x^2y|^{-1}), \quad \Lambda|\eta|(x, y) = O(|x^{1/2}y^2|^1) \]
and
\[ \Lambda|\eta|(x, s) \leq \frac{C}{|x^{1/2}s^{3/2}|^+_1} \asymp \frac{1}{x - s} \sqrt{\frac{x \vee 1}{-s}} \times \left[ \frac{1}{-s} + \frac{1}{x \vee 1} \right] \quad (s \leq -1, x \geq 0), \]
where \( |a|_+ = |a| \vee 1 \). Thus (10) follows because of the identity
\[ \Lambda - \lambda = (1 + \Lambda)\eta(1 + \lambda). \]
The proof of Proposition 7 is complete.

The proof of Proposition 7 is complete.

\[ \square \]

4 Estimation of \( H_x^{I(n)}(s) \) near the edges

We continue the arguments of the preceding section to estimate \( H_x^{I(n)}(s) \) mainly in the case when \( \delta > 1/2 \) and either \( n - s \) or \( n + s \) is small in comparison with \( x - n \). The case when \( \delta = 0 \) or \( x - n \) is not large can be similarly dealt with and is only briefly discussed at the end of this section.
Theorems 9 and 10 given below are based on the following result from [13] ((10) of Theorem 1.3 and its dual): if $\delta > 1/2$ in (3), then for $x \geq s > 0$,

$$H_{-x}^-(s) = \frac{\sqrt{x}}{\pi(x+s)} \mu^-(s) + O\left(\frac{1}{x}\right),$$  \hspace{1cm} (24)

$$H_{-x}^+(s) = \frac{\sqrt{x}}{\pi(x+s)} \nu^-(s) + O\left(\frac{1}{x}\right),$$  \hspace{1cm} (25)

where $\mu^-(s) = \mu(-s)$ and $\nu^-(s) = \nu(-s)$. The following theorem concerns particularly to the case when $(x-n)/(n-s) \to \infty$ so that $h_{I(n)}^I > (x-s)^{-1}$.

**Theorem 9** If $\delta > 1/2$ in (3), then uniformly for integers $n > 1, 0 \leq s < n$ and $x \geq n$,

$$H_{I(n)}^I(x) = \sqrt{n_s - s} \mu^-(n-s)h_{I(n)}^I(s) + O\left(\log n\right),$$

**Proof.** Make decomposition $H_{I(n)}^I = K_I + Q_I + \Lambda(K_I + Q_I)$ and infer from (24) that

$$K_I = \sqrt{n_s - s} \mu^-(n-s)k_I + O(1/(x-s)),$$  \hspace{1cm} (26)

$$\Lambda K_I(x,s) = \sum_{y=n}^{\infty} \Lambda(x,y) \left[ \sqrt{n_s - s} \mu^-(n-s)k_I(y,s) + O\left(\frac{1}{y-s}\right) \right].$$

In view of (I) we have $\sup_{s \geq 0, x > n_s} q_I(x,s) \leq C/n$, which in particular shows that

$$\lambda q_I(x,s) = O(1/n) \quad \text{uniformly for} \quad 0 \leq s < n_s, x > n_s.$$

Thus, on employing (I'), for $s > 0$,

$$q_I + \lambda q_I = O(1/n) \quad \text{and} \quad Q_I + \Lambda Q_I \leq C(q_I + \lambda q_I) = O(1/n).$$  \hspace{1cm} (27)

By (V) of the preceding section we have

$$\sum_{y=n}^{\infty} \Lambda(x,y) \frac{1}{y-s} \leq C \sqrt{\frac{x-n_s}{x} \frac{\log n}{n}},$$  \hspace{1cm} (28)

so that

$$\Lambda K_I(x,s) = \sqrt{n_s - s} \mu^-(n-s)k_I(x,s) + O(n^{-1} \log n).$$

Here the factor $\sqrt{(x-n_s)/x}$ on the right side of (28) is replaced by 1: the loss of accuracy to the estimate of $H_{I(n)}^I$ caused by this replacement is small in comparison with the error term $O(1/(x-s))$ in (26). By (17) $\Lambda k_I = \lambda k_I + (\Lambda - \lambda) k_I = \lambda k_I + O(1/n)$; hence

$$\Lambda K_I = \sqrt{n_s - s} \mu^-(n-s)\lambda k_I + O(1/n^-1 \log n),$$

which together with (26), (27) yields the assertion of the theorem. \hfill \Box
Theorem 10 If $\delta > 1/2$ in (3), then uniformly for integers $n > 1, -n < s < 0$ and $x \geq n$,
\[
H_x^{I(n)}(s) = \sqrt{n_x + s}\nu(-n - s)h_x^{I(n)}(s)
\]
\[
\times \left[ 1 + O\left( \frac{s + n_x}{n} \cdot \log n \right) + O\left( \frac{x}{n(x - n_x)} \right) \right].
\]

Proof. We make decomposition
\[
H_x^{I(n)}(s) = K_1(x, s) + \sum_{y = -\infty}^{-n} H_x^{I(n)}(y - n)H_y^{I(n)}(s)
\]
\[
= K_1(x, s) + \sum_{y = -\infty}^{-n} (H_x^{I(n)} - h_x^{I(n)})(y - n)H_y^{I(n)}(s)
\]
\[
+ \sum_{y = -\infty}^{-n} h_x^{I(n)}(y - n)H_y^{I(n)}(s). \quad (29)
\]

For evaluation of the second sum of the last line we substitute the estimate of Theorem 9 for $H_x^{I(n)}(s)$ (with $S_n$ replaced by $-S_n$; hence $\sqrt{n_x - s} \mu^{-}(n - s)$ by $\sqrt{n_x + s} \nu^{-}(n + s)$), use the expression of $h_x^{I(n)}(s)$ analogous to the first expression in (29) of $H_x^{I(n)}(s)$ and observe that
\[
\sup_{-n < s \leq 0} k_1(x, s) \geq \frac{\sqrt{x - n_x}}{x \sqrt{n}}; \quad \int_{-\infty}^{-n} h_x^{-}(y - n)dy \leq \frac{\sqrt{x - n_x}}{\sqrt{x}};
\]
and 
\[
\int_{-\infty}^{-n} h_x^{-}(y - n)(s - y)^{-1}dy \leq \frac{\sqrt{x - n_x}}{x \sqrt{n}} \cdot \log \frac{n}{s + n}
\]
to obtain
\[
\sum_{y = -\infty}^{-n} h_x^{-}(y - n)H_y^{I(n)}(s) = \sqrt{n_x + s} \nu^{-}(n + s)h_x^{I(n)}(s) + O\left( \frac{\sqrt{x - n_x}}{\sqrt{x} n} \cdot \log n \right). \quad (30)
\]

For evaluation of the first sum apply (23) to have
\[
|H_x^{-}(y') - h_x^{-}(y')| \leq C((x - n_x)^{-1/2} + n_x^{-1/2})h_x^{-}(y')
\]
for $y' \leq -2n$; also use the bound $H_x^{I(n)}(s) \leq C h_x^{I(n)}(s)$ that follows from Theorem 11. These bounds as well as \[
\int_{-\infty}^{-n} h_x^{-}(y - n)h_y^{I(n)}(s)dy \leq h_x^{I(n)}(s)
\]
yield
\[
\sum_{y = -\infty}^{-n} \left| (H_x^{-} - h_x^{-})(y - n) \right| H_y^{I(n)}(s) \leq C \left( \frac{1}{\sqrt{x - n_x}} + \frac{1}{\sqrt{n}} \right) h_x^{I(n)}(s)
\]
\[
\approx \frac{\sqrt{x}}{(x - n_x)n} h_x^{I(n)}(s).
\]

Combined with (29) and (30) this completes the proof of the theorem. \hfill \Box

As being mentioned at the beginning of this section our estimation of $H^{I(n)}$ made above is appropriate if $x - n$ is large in comparison with $n \pm s$. When $x - n$ is not large, it is better to replace $h_x^{-}(y - n)$ by
\[
\frac{\sigma^2 \nu(x - n)}{2\sqrt{x - n_x}} h_x^{-}(y - n)
\]
in (29); also make use of the corresponding estimate of $H_x^\pm$ in [13].

**Proof of Theorem 2 (the case $\delta = 0$).** The proof is based on an estimate of $H_x^-(s)$ verified in [13] (Theorem 1.1). The case when $(x - n) \wedge (n - s) \wedge (n + s) \to \infty$ the assertion is included in Theorem 1. The other case is dealt with as in the proofs of Theorems 9 and 10 by employing (VI). The computations to be carried out this time are much simpler in either case.

### 5 Proof of Theorem 5

We always have the relation

$$H_x^{I(n)}(s) = H_x(s) + \sum_{|x_1| \geq n} H_x(x_1)H_x^{I(n)}(s) \quad s \in I(n),$$

where $H_x(s), x, s \in \mathbb{Z}$, denote the probability that $s$ is the site where the real axis is hit for the first time after time 0 by the walk $S_n$ started at $x$. Suppose $E_0[|Y|^2 \log |Y|] < \infty$. Then

$$H_x(s) = H_0(s - x) = \frac{\sigma^2}{\pi(s - x)^2}(1 + o(1)) \quad \text{as } |s - x| \to \infty$$

(cf. [15]). The proof of Theorem 5 is relatively simple owing to (32). Before proceeding with the proof we mention a few points to be recognized. For the estimate of $H_x^{I(n)}(s), x, s \in I(n)$ we may suppose that $x \geq 0$ for obvious reason and then, $s < x$, by considering the time-reversed walk. Also (ii') is a dual statement of (ii) as already noted. There is some possibility of improving the estimates in certain cases that we do not take up in this paper, and for that purpose some details given below would be helpful.

From what is noticed above we may suppose

$$x \geq 0 \quad \text{and} \quad x - s \to \infty.$$ (33)

Then, according to Theorem 2, for $x_1 \geq n$,

$$H_x^{I(n)}(s) = \left[ \frac{\nu(-n - s)}{\sqrt{n - s}} \right] \frac{\sigma^2 \nu(x_1 - n) \sqrt{n + x_1}}{2\pi|x_1 - s|} (1 + o(1)) \quad (34)$$

$$= \left[ \sqrt{n + s \nu(-n - s)} \right] h_x^{I(n)}(s) \frac{\sigma^2 \nu(x_1 - n)}{2\sqrt{x_1 - n}} (1 + o(1)).$$ (35)

(This is obtained directly from Theorem 2(ii), which also covers the case $s > 0$ under (33).)

**Proof of (ii).** The Wiener-Hopf equation for $\nu$ may be written as

$$\sum_{x_1=n}^{\infty} H_x(x_1) \nu(x_1 - n) = \nu(-n + x)$$

(see (44)). We claim that as $(n - x)/(n - s) \to 0$,

$$J_{n,s,x} := \sum_{x_1 \geq n} H_x(x_1)H_x^{I(n)}(s) = \left[ \frac{\nu(-n - s)}{\sqrt{n - s}} \right] \frac{\sigma^2 \nu(n + x) \sqrt{2n}}{2\pi(n - s)} (1 + o(1)).$$ (37)
Put $\xi = n - x$ and observe that owing to (36) the summation in (36) may be restricted to $x_1 \leq n + K\xi$ by choosing $K$ large enough. Then, on looking at (34), the claim (37) follows if we show that for each $\varepsilon > 0$ we can find $K > 1$ such that

$$\sum_{x_1 \geq n + K\xi} H_x(x_1)H_{x_1}^{I(n)}(s) \leq \varepsilon J_{n,s,x}.$$ 

However, by simple consideration this reduces to

$$\int_{K\xi}^{\infty} \frac{\sqrt{2n + u}}{u^n \sqrt{u}} \, du \leq \frac{\varepsilon \sqrt{n}}{(n-s)\sqrt{\xi}},$$

which is certainly true if $K$ is large enough. Thus the claim is verified. One can easily check that

$$\sum_{x_1 \leq -n} H_x(x_1)H_{x_1}^{I(n)}(s) = O\left(\frac{1}{(n+s) \wedge \sqrt{n}^2}\right) = o(J_{n,s,x}).$$

Finally notice that $(n-s)/(x-s) \to 1$ and hence the right side of (37) may be identical to that of the required formula. The proof of (ii) is complete. \qed

**Proof of (i).** We may suppose that $(n-x)/(n-s)$ is bounded away from zero, the case $(n-x)/(n-s) \to 0$ being included in (ii) that is proved above. Under this condition we see

$$\sum_{n \leq |x_1| \leq n + K} H_x(x_1)H_{x_1}^{I(n)}(s) = o\left(\frac{1}{(s-x)^2}\right)$$

(indeed this is valid if $(n-x)^4/(n-s) \to \infty$; the contribution of $-n-K \leq x_1 \leq -n$ is easy to estimate), namely the sum above is negligible if compared with $H_x(s)$. Hence one can replace the ratio appearing last in (35) by 1. Also $\nu(-n-s)$ may be replaced by $1/\sqrt{n+s}$ and, substituting the resulting expression into (31) and applying Lemma 11 of Appendix (B), we conclude the formula of (i). \qed

**Remark 2.** We could have employed the identity

$$H_x^{I(n)}(s) = H_{x-n}(s-n) + \sum_{x_1 \geq -n} H_{x-n}(x_1 - n)H_{x_1}^{I(n)}(s) \quad s \in I(n), \quad (38)$$

instead of (31). This way is simpler owing to (9) except for a tedious computation for exact evaluation of the definite integral of a certain rational function.

### 6 Appendices

**A** Let $D$ be the complement of the line segment with edges at ±1:

$$D = C \setminus \{s : -1 \leq s \leq 1\}$$

and denote the Poisson kernel (density of harmonic measure) for $D$ by $h_D(z, s \pm i0)$. Putting $h_z^{[-1,1]}(s) = h_D(z, s + i0) + h_D(z, s - i0)$, we have

$$h_z^{I(n)}(s) = h_z^{[-1,1]}(s/n_*)/n_*.$$
We compute \( h_D(z, s \pm i0) \) by using the conformal invariance of harmonic measures. The function \( z = \frac{1}{2}(w + w^{-1}) \) univalently maps the exterior of the unit circle onto \( D \). Denote by \( f(z) \) its inverse map, which may be represented by

\[
f(z) = z + \sqrt{z^2 - 1}, \quad z \in D
\]

with the standard choice of a branch of the square root (so that \( f(\pm s) = \pm s \pm \sqrt{s^2 - 1} \) for \( s > 1 \) and \( f(s \pm i0) = s \pm i\sqrt{1 - s^2} \) for \( -1 < s < 1 \)). As \( w = f(z) \) moves on a circle centered at the origin counter-clockwise starting at a point \( R > 1 \), \( z \) describes the ellipse \([2x/(R + R^{-1})]^2 + [2y/(R - R^{-1})]^2 = 1 \) (which surrounds the segment \(-1 \leq s \leq 1\) and shrinks to it as \( R \downarrow 1 \)) rotating also counter-clockwise and starting at the point \( f(R) = \frac{1}{2}(R + R^{-1}) \in (1, R) \). (Cf. [1]:p.94 or [10]:p.269). The Poisson kernel for the exterior of the unit circle is given by

\[
K(Re^{i\theta}, e^{i\theta'}) = \frac{R^2 - 1}{2\pi(R^2 - 2R\cos(\theta - \theta') + 1)}, \quad R > 1.
\]

Put

\[
\theta(z) = \arg f(z).
\]

Then, for \(-1 \leq s \leq 1\), \( \theta(s \pm i0) = \pm \arccos s \in (-\pi, \pi) \), so that \(|d\theta(s \pm i0)| = ds/\sqrt{1 - s^2}\); thus the conformal invariance shows that

\[
h_D(z, s \pm i0) = \frac{1}{2\pi} \cdot \frac{|f(z)|^2 - 1}{|f(z)|^2 - 2|f(z)|\cos[\theta(z) - \theta(s \pm i0)] + 1} \cdot \frac{1}{\sqrt{1 - s^2}}
\]

(39)

which, for \( z = x \in \mathbb{R} \setminus [-1, 1] \), reduces to

\[
h_D^{-1,1}(s) = 2h_D(x, s + i0) = \frac{\sqrt{x^2 - 1}}{\pi |x - s|} \cdot \frac{1}{\sqrt{1 - s^2}}.
\]

Let \( Q = (\sigma_{ij}) \) be a \( 2 \times 2 \) matrix that is symmetric and positive definite and \( \tilde{h}_D(z, s \pm i0) \) the corresponding hitting density for the process \( Q^{1/2}B_t \), a two-dimensional Brownian motion of mean zero and the covariance matrix \( tQ \). Then for \( z \in D \),

\[
\tilde{h}_D(z, s \pm i0) = h_D(\tilde{z}, s \pm i0), \quad \tilde{z} = (x - \omega y) + i\lambda y,
\]

(40)

where \( \omega = \sigma_{12}/\sigma_{22} \) and \( \lambda = \sigma^2/\sigma_{22} = \sqrt{\sigma_{11}/\sigma_{22} - \omega^2} \). If \( z \) is real, the identity above follows immediately from the rotation invariance of the standard Brownian motion. In view of the strong Markov property of \( S \) its full validity is deduced from the identity thus restricted in conjunction of the corresponding identity for the Poisson kernel of the upper half plane (see (E) below).

(B) In Section 5 (at the end of it) we have used the following lemma.

\textbf{Lemma 11} For \( x, s \in (-n_*, n_*) \) with \( s \neq x \),

\[
\frac{1}{(s - x)^2} + \int_{|\xi| \geq n_*} \frac{1}{(\xi - x)^2} h_{\xi}^{(n)}(s)d\xi = \frac{n_*^2 - xs}{(s - x)^2 \sqrt{(n_*^2 - x^2)(n_*^2 - s^2)}}.
\]

(41)
Proof. By the scaling property we may suppose the interval \( I(n) \) to be \([-1, 1] \). Let \( h_D \) be as in (A) and \( h_L(s) \) be the Poisson kernel on the upper half plane: \( h_L(s) = y/\pi(y^2 + (s - x)^2) \). Then

\[
h^{[-1,1]}(s) = h_D(z, s + i0) + h_D(z, s - i0) = h_L(s) + \int_{|\xi|>1} h_L(\xi)h^{[-1,1]}(s) d\xi,
\]

which shows \( \lim_{y \downarrow 0} \pi y^{-1} h^{[-1,1]}(s) \) equals L.H.S. of (41). The lemma therefore follows if we verify that if \( |x| < 1 \) and \( |s| < 1 \),

\[
\lim_{y \downarrow 0} \frac{\pi [h_D(x + iy, s + i0) + \pi h_D(x + iy, s - i0)]}{y} = \text{R.H.S. of (41).}
\]

If \( w = -(1 - x^2 + y^2) + i2xy \) and \( \phi = \pi - \arg w \in (-\pi/2, \pi/2) \), then

\[
|f(z)|^2 = (x + |w|^{1/2} \sin \phi)^2 + (y + |w|^{1/2} \cos \phi)^2
\]

and we see that \( y^{-1}|f(z)|^2 - 1) \to 2/\sqrt{1 - x^2} \). In view of (39) this shows that

\[
\lim_{y \downarrow 0} \frac{\pi h_D(x + iy, s + i0)}{y} = \frac{1}{2(1 - \cos(\theta_x \mp \theta_s))\sqrt{(1 - x^2)(1 - s^2)}},
\]

where \( \cos \theta_t = t \) with \( \theta_t \in (0, \pi) \) for \(-1 < t < 1 \). Now (42) follows from the identity

\[
\frac{1}{1 - \cos(\theta_x - \theta_s)} + \frac{1}{1 - \cos(\theta_x + \theta_s)} = \frac{2 - 2 \cos \theta_x \cos \theta_s}{(\cos \theta_x - \cos \theta_s)^2} = \frac{2(1 - x s)}{(x - s)^2}.
\]

(C) Let \( (X_n) \) be the imbedded walk on the real axis mentioned in Section 1. In other words \( (X_n) \) is the one-dimensional random walk with the transition probability \( p^X(x, y) = H_0(y - x) \), where \( H_0(x) \) be the hitting distribution of the real line for our random walk starting at the origin as being introduced in (31). Let \( \mu(x), x \geq 0 \) be a renewal function for the ascending ladder-height variables of the walk \( (X_n) \) and \( \nu \) its dual; they are normalized so as to satisfy \( \lim_{x \to \infty} \mu(x)/\nu(x) \equiv 1 \) and given by

\[
\mu(x) = \frac{\sqrt{\pi} e^{\theta_+}}{\sigma} (v_0 + \cdots + v_x), \quad \nu(x) = \frac{\sqrt{\pi} e^{-\theta_+}}{\sigma} (u_0 + \cdots + u_x)
\]

for \( x = 0, 1, 2, \ldots \) ([11]:p. 212), where if \( c = \exp \left(-\sum_{k=1}^{\infty} \frac{1}{k} P_0[X_k = 0] \right) \),

\[
\theta_+ = \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} - P_0[X_k > 0] \right) + \frac{1}{2} \log c = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left( P_0[X_k < 0] - P_0[X_k > 0] \right),
\]

\( v_0 = u_0 = 1/\sqrt{c} \) and \( \sqrt{c} v_k \) (resp. \( \sqrt{c} u_k \)) equals the probability that the ascending (resp. descending) ladder-height process visits \( k \) (resp. \(-k\) ([11]:p.202, 203). Then \( \mu \) and \( \nu \) are positive solutions of the Wiener-Hopf integral equations associated with the kernels \( p^X(x, y) \) and \( p^X(y, x) \)(\( x, y \geq 0 \)), respectively to the effect that for \( x = 0, 1, 2, \ldots \),

\[
\mu(x) = \sum_{k=0}^{\infty} \mu(k) H_0(x - k) \quad \text{and} \quad \nu(x) = \sum_{k=0}^{\infty} \nu(k) H_0(-x + k)
\]
find, with the notation analogous to \( U \), algebraic manipulation leads to \( h \) shows that
\[
\text{let } z = x + iy, \text{ so that the matrix } H_z(s) = P_z[\exists j, S_n = s \text{ and } S_k \notin \mathbb{R} \text{ for } 1 \leq k < j] \quad (z = x + iy)
\]
it is shown in [15] (Theorem 1.2) that as \(|x - s| + |y| \to \infty
\[
H_z(s) = \frac{1}{\pi} \frac{\sigma_2^2 a(y)}{\|s - z\|^2} (1 + o(1)).
\]
(45)
(\text{Note that } \sigma_2^2 a(y) \geq |y| \text{ [11]:p332, for uniqueness of positive solutions}).

(D) Suppose that \( E_0[|Y|^2 \log |Y|] < \infty \) (as in Theorem 5). Put \( \sigma_j^2 = E[(S_i^{(j)})^2] \) \((j = 1, 2)\) and \( \sigma_{12} = E[S_1^{(1)} S_1^{(2)}] \). Then putting for \( s, x, y \in \mathbb{Z} \)
\[
H_z(s) = P_z[\exists j, S_n = s \text{ and } S_k \notin \mathbb{R} \text{ for } 1 \leq k < j]
\]
it is shown in [15] (Theorem 1.2) that as \(|x - s| + |y| \to \infty
\[
H_z(s) = \frac{1}{\pi} \frac{\sigma_2^2 a(y)}{|y|} h_{x-\omega y+i\lambda y}(s)(1 + o(1)).
\]
(46)
Using this we can readily deduce from Theorems 2 and 2' that as \(|z| \wedge |z - s| \to \infty
\[
H_z^{(n)}(s) = \frac{\sigma_2^2 a(y)}{|y|} \mu(-n + s) \nu(-n + s) \sqrt{\frac{n^2}{2} - s} h_{x-\omega y+i\lambda y}^{(n)}(s)(1 + o(1))
\]
(47)
for \( y \neq 0 \).

(E) In view of Donsker’s invariance principle the relation (46) (resp. (47)) incidentally shows that \( h_{x-\omega y+i\lambda y}(s) \) (resp. \( h_{x-\omega y+i\lambda y}^{[-1,1]}(s) \)) is the density of the hitting distribution of the real line (resp. the interval \( I(n) \)) for the process \( Q^{1/2} B_t \).

We give a direct algebraic verification. We may replace \( B_t \) by \( U B_t \) with any orthogonal matrix \( U \) and choose \( U \) so that the matrix \( Q^{1/2} U \) sends the real line to itself. A simple algebraic manipulation leads to
\[
(Q^{1/2}U)^{-1} = \frac{\sigma_2}{\sigma^2} \begin{pmatrix} 1 & -\omega \\ 0 & \lambda \end{pmatrix}.
\]
Let \( z = x + iy, \bar{z} = x - \omega y + i\lambda y \) and \( c = \sigma_2/\sigma^2 \). Noting that \( c\bar{z} \) corresponds to \( z \), we then find, with the notation analogous to \( \tilde{h}_D(z, s) \) in (40), that
\[
\tilde{h}_z(s) = c h_{c\bar{z}}(cs),
\]
of which the right-hand side equals \( h_{\bar{z}}(s) \), yielding the analogue of (40) as desired.

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