HOMOGENIZATION FOR THE STOKES EQUATIONS IN RANDOMLY PERFORATED DOMAINS UNDER ALMOST MINIMAL ASSUMPTIONS ON THE SIZE OF THE HOLES

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ABSTRACT. We prove the homogenization to the Brinkman equations for the incompressible Stokes equations in a bounded domain which is perforated by a random collection of small spherical holes. The fluid satisfies a no-slip boundary condition at the holes. The balls generating the holes have centres distributed according to a Poisson point process and i.i.d. unbounded radii satisfying a suitable moment condition. We stress that our assumption on the distribution of the radii does not exclude that, with overwhelming probability, the holes contain clusters made by many overlapping balls. We show that the formation of these clusters has no effect on the limit Brinkman equations. In contrast with the case of the Poisson equation studied in [A. Giunti, R. Höfer, and J.J.L. Velázquez, Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes], the incompressibility condition requires a more detailed study of the geometry of the random holes generated by the class of probability measures considered.

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1. Introduction

In this paper we consider the steady incompressible Stokes equations

\[
\begin{cases}
-\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D_\varepsilon \\
\nabla \cdot u_\varepsilon = 0 & \text{in } D_\varepsilon \\
u_\varepsilon = 0 & \text{on } \partial D_\varepsilon
\end{cases}
\] (1.1)

in a domain $D_\varepsilon$, that is obtained by removing from a bounded set $D \subseteq \mathbb{R}^d$, $d > 2$, a random number of small balls having random centres and radii. More precisely, for $\varepsilon > 0$, we define

\[
D_\varepsilon = D \setminus H_\varepsilon, \quad H_\varepsilon := \bigcup_{z_i \in \Phi \cap D \varepsilon} B_{\varepsilon \frac{x-z_i}{\rho_i}}(\varepsilon z_i),
\] (1.2)
where \( \Phi \) is a Poisson point process on \( \mathbb{R}^d \) with homogeneous intensity rate \( \lambda > 0 \), and the radii \( \{ \rho_i \}_{i \in \Phi} \subseteq \mathbb{R}_+ \) are identically and independently distributed unbounded random variables. We comment on the exact assumptions on the distribution of each \( \rho_i \) later in this introduction. Our main result states that, for almost every realization of \( H^\varepsilon \) in (1.2), the solution \( u_\varepsilon \) to (1.1) weakly converges in \( H^1_0(D) \) to the solution \( u_h \) of the Brinkman equations

\[
\begin{align*}
-\Delta u_h + \mu u_h + \nabla p_h &= f \quad \text{in } D, \\
\nabla \cdot u_h &= 0 \quad \text{in } D, \\
u_h &= 0 \quad \text{on } \partial D. 
\end{align*}
\]

(1.3)

The constant matrix \( \mu \) appearing in the equations above satisfies

\[ \mu = \mu_0 I, \quad \mu_0 = C_d \lambda (\rho^{d-2}), \]

(1.4)

where \( \langle \cdot \rangle \) denotes the expectation under the probability measure on the radii \( \rho_i \), and the constant \( C_d > 0 \) depends only on the dimension \( d \). In the case \( d = 3 \), we have \( C_d = 6\pi \).

From a physical point of view, the equations in (1.1) represent the motion of an incompressible viscous fluid among many small obstacles; the additional term \( \mu u_h \) appearing in (1.3) corresponds to the effective friction force of the obstacles acting on the fluid. In the physical literature, the term \( \mu \) is usually referred to as the “Stokes resistance”; in this paper, we mostly adopt for \( \mu \) the term “Stokes capacity density” to emphasize the analogy with the harmonic capacity density which appears in the analogue homogenization problem for the Poisson equation [5, 10]. More precisely, for a smooth and bounded set \( E \subseteq \mathbb{R}^d \), let us define its Stokes capacity as the symmetric and positive-definite matrix given by

\[ \xi^t \cdot M \xi = \inf_{w \in E^t} \int_{\mathbb{R}^d \setminus E} |\nabla w|^2, \quad \text{for all } \xi \in \mathbb{R}^d. \]

(1.5)

Here,

\[ E^t = \{ w \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) : \nabla \cdot w = 0, \ w = \xi \text{ in } E, \ w \rightarrow 0 \text{ for } |x| \uparrow +\infty \}. \]

Then, in the case \( E = B_r \), we obtain \( M = C_d r^{d-2} I \) (see e.g. [1]). The definition (1.4) of \( \mu \) is thus an averaged version of the previous formula where we take into account the intensity rate of the Process \( \Phi \) according to which the balls of \( H^\varepsilon \) are generated.

This work is an adaptation to the Stokes equations of the homogenization result obtained in [10] for the Poisson equation. In particular, the class of random holes considered in the current paper is included in the class studied in [10]. In the latter, it is assumed that the identically distributed radii \( \rho_i \) in (1.2) satisfy

\[ \langle \rho^{d-2} \rangle < +\infty. \]

(1.6)

In the current paper, we require the slightly stronger condition

\[ \langle \rho^{d-2+\beta} \rangle < +\infty, \quad \text{for some } \beta > 0. \]

(1.7)

Before further commenting on (1.7) in the next paragraph, we recall that in the case of the Poisson problem, the analogue of the term \( \mu \) appearing in the homogenized equation (1.3) is the asymptotic harmonic capacity density generated by the holes \( H^\varepsilon \). Assumption (1.6) is minimal in order to have that this quantity is finite in average, but does not exclude that with overwhelming probability some balls generating \( H^\varepsilon \) overlap. For further comments on this, we refer to the introduction in [10].

The main challenge in proving the results of this paper is related to the regions of \( H^\varepsilon \) where there are clustering effects. More precisely, the main goal is to estimate their contribution to the Stokes capacity density, and thus to the limit term \( \mu \) appearing in (1.3). In the case of the Poisson equation in [10], the analogue is done by relying on the sub-additivity of the harmonic capacity, together with (1.6) and a Strong Law of Large Numbers. In the case of the Stokes capacity (1.4), though, sub-additivity fails due to the incompressibility of the fluid (i.e. the divergence-free condition). We thus need to cook up a different method to deal with the balls in \( H^\varepsilon \) which overlap or are too close. Heuristically speaking, the main challenge is that the incompressibility condition yields that big velocities are needed to squeeze a fixed volume of fluid through a possible narrow opening. The main reason for the strengthened assumption (1.7) is that it allows us to obtain a certain degree of information on the geometry of the clusters of \( H^\varepsilon \). In particular, (1.7) rules out the occurrence of clusters made of too many holes of
similar size. We emphasize, however, that it neither prevents the balls generating $H^\varepsilon$ from overlapping, nor it implies a uniform upper bound on the number of balls of very different size which combine into a cluster (see Section 6). The main technical effort of this paper goes into developing a strategy to deal with these geometric considerations and succeed in controlling the term in (1.3). We refer to Subsection 2.3 for a more detailed discussion on our strategy.

We also mention that, to avoid further technicalities, we only treat the case where the centres of the balls in (1.2) are distributed according to a homogeneous Poisson point process. It is easy to check that our result applies both to the case of periodic centres and to any (short-range) correlated point process for which the results contained in Appendix C hold.

After Brinkman proposed the equations (1.3) in [3] for the fluid flow in porous media, an extensive literature has been developed to obtain a rigorous derivation of (1.3) from (1.1) in the case of periodic configuration of holes [2, 15, 20, 16]. We take inspiration in particular from [1], where the method used in [5] for the Poisson equations is adapted to treat the case of the Stokes equations in domains with periodic holes of arbitrary and identical shape. In [1], by a compactness argument, the same techniques used for the Stokes equations also provide the analogous result in the case of the stationary Navier-Stokes equations. The same is true also in our setting (see Remark 2.2 in Section 2).

In [6], with methods similar to [1] and [5], the homogenization of stationary Stokes and Navier-Stokes equations has been extended also to the case of spherical holes where different and constant Dirichlet boundary conditions are prescribed at the boundary of each ball. This corresponds to the quasi-static regime of holes slowly moving in a fluid, and gives rise in (1.3) to an additional source term $\mu_j$, with $j$ being the limit flux of the holes. In [6], the holes have all the same radius, are not necessarily periodic, but satisfy a uniform minimal distance condition of the same order of $\varepsilon$ as in the periodic setting. In [11], this last condition has been weakened but not completely removed. In particular it is still assumed that, asymptotically for $\varepsilon \downarrow 0$, the radius of each hole is much smaller than its distance to any other hole.

In [12], the quasi-static Stokes equations are considered in perforated domains with holes of different shapes which are both translating and rotating. Due to the shapes of the holes, the problem becomes non-isotropic, i.e. the matrix $\mu$ in (1.3) is not a multiple of the identity. Moreover, since also the rotations of the holes are included into the model, a more complicated source term $F$ arises on the right hand side of the limit problem. The result in [12] is proved under the same uniform minimal distance assumption as in [6].

Finally, we also mention that the homogenization in the Brinkman regime for evolutionary Navier-Stokes in a bounded domain of $\mathbb{R}^3$ has been considered in [7]. In this paper, the holes are assumed to be disjoint, have arbitrary shape and uniformly bounded diameter. A condition on the minimal distance between the holes is substituted by a weaker assumption implying that, for $\varepsilon$ small enough, the diameter of the holes is much smaller than the distance between them.

There are fewer results in the literature concerning the case of randomly distributed holes: In [19], the case of $N$ randomly distributed spherical holes of size $N^{-1}$ in $\mathbb{R}^3$ is considered. Starting from the Brinkman equation (1.3) with the term $\mu$ sufficiently large, it is shown that in the limit $N \to \infty$ an additional zero-order term appears in the limit equation. This result has been recently generalized in [4] to the case of the Stokes equations in the quasi-static regime.

The derivation of the Brinkman equations can be viewed as a very first step in deriving the so-called Vlasov-(Navier-)Stokes equations, a model for the coupled dynamics of particles suspended in a fluid. A rigorous derivation of these equations for the full problem is completely open. Homogenization results for such dynamic problems have only been achieved in the case when the inertia of the particles is neglected. In that case, an external constant gravitation field is considered, and the friction caused by the particles is only related to gravity. For inertialless particles, [14] identified the regime that is so dilute that particles effectively do not interact. In [13], the homogenization result for the inertialless problem has been obtained under a uniform minimal distance assumption. A related result has been obtained in [18] where convergence to the same limit equation is proven also when rotations of the particles are taken into account. The assumptions on the initial particle distributions in [18] do not contain the uniform minimal distance assumption from [13], but they are similar to those in [11]. However, the convergence is only proved for small times and for initial particle distributions that are sufficiently dilute.
We emphasize that the main novelty of our paper is that we consider spherical holes whose radii are not uniformly bounded and only satisfy (1.7). As already mentioned above, for small $\beta$ in (1.7), with probability tending to one as $\varepsilon \to 0$, the perforated domain $D^\varepsilon$ in (1.2) contains many holes that overlap. In all the deterministic results listed above, overlapping balls are either excluded or asymptotically ruled out for $\varepsilon \not\to 0$. Similarly, in the random settings of [19] and [4], the overlapping are negligible in probability: Since the radii of the holes are chosen to be identically $N^{-1}$, it is shown that, with probability tending to one as $N \to \infty$, the minimal distance between them is bounded below by $N^{-\alpha}$ for $\alpha < 1$.

We finally mention that in this paper we also give a convergence result for the pressures $\{p_\varepsilon\}_{\varepsilon > 0}$. In all the papers mentioned above except for [1], the convergence of the pressure is not considered. In fact, the problem may be reformulated so that the pressure only plays the role of a Lagrange multiplier for the incompressibility of the fluid. As a physical quantity, though, the pressure is important in itself and obtaining bounds may turn out to be a challenging problem. In [1] it is shown that for a suitable extension $P_\varepsilon(p_\varepsilon)$ for $p_\varepsilon$ on the whole domain $D$, the functions $P_\varepsilon p_\varepsilon$ converge to $p_h$ weakly in $L^2(D)$. Since $u_\varepsilon$ converges weakly in $H^1$, this is the optimal result that one could expect. In our work, we prove a sub-optimal convergence result for a suitable modification $\tilde{p}_\varepsilon$ of the pressures $p_\varepsilon$. The main difficulty in our case is again given by the presence of the clusters of $H^\varepsilon$ that prevents us from finding suitable bounds for $p_\varepsilon$ close to those regions. Roughly speaking, the definition of $\tilde{p}_\varepsilon$ allows us to cut-off a small neighbourhood $E^\varepsilon$ of $H^\varepsilon$ and show that, away from it, the pressures convergence to $p_h$ in $L^q$, $q < \frac{d-1}{d-2}$. The neighbourhood $E^\varepsilon$ is small in the sense that the harmonic capacity of the difference $E^\varepsilon \setminus H^\varepsilon$ almost surely vanishes in the limit $\varepsilon \not\to 0^+$.

This paper is organized as follows: In Section 2 we state the two main theorems, namely the convergence of the fluid velocity $u_\varepsilon$ and a partial convergence result for the pressure $p_\varepsilon$. In Subsection 2.4 we formulate Lemma 2.4 which provides a rich class of test-functions for (1.1) and characterizes their behaviour in the limit $\varepsilon \to 0$. We then show how the convergence of $u_\varepsilon$ follows from this result. In Section 3, we give some geometric properties for the realization of the holes $H^\varepsilon$ that are needed in order to prove Lemma 2.4. These properties are split into two lemmas. The first one is analogous to the corresponding lemma in [10], the other one gives more detailed informations on the geometry of the clusters of $H^\varepsilon$ and is the result which requires the strengthened version (1.7) of (1.6). In subsection 3.2, we prove the results stated in Section 3. In Section 4, we prove Lemma 2.4. In Section 5, we prove the main result concerning the convergence of pressure. In Section 6, we prove some probabilistic result on the number of comparable balls which may combine into a cluster of $H^\varepsilon$. These are the key ingredients used in subsection 3.2 to show the geometric results of Section 3. Finally, the appendix is divided into three parts: In Appendix A, we show how to extend the convergence result from the Stokes equations to the Stationary Navier-Stokes equations. In Appendix B, we give some standard estimates for the solutions of the Stokes equations in annuli and exterior domains. In Appendix C, we recall some results concerning the Strong Law of Large Numbers, which have been proved in detail in [10] and which are used also throughout this paper.

2. SETTING AND MAIN RESULT

Let $D \subseteq \mathbb{R}^d$, $d > 2$, be an open and bounded set that is star-shaped with respect to the origin. For $\varepsilon > 0$, we denote by $D^\varepsilon \subseteq D$ the domain obtained as in (1.2), namely by setting $D^\varepsilon = D \setminus H^\varepsilon$ with

$$H^\varepsilon := \bigcup_{z_j \in \Phi \cap \frac{1}{\varepsilon}D} B_{\frac{\varepsilon}{\varepsilon+\varepsilon^q z_j}}(\varepsilon z_j).$$

(2.1)

Here, $\Phi \subseteq \mathbb{R}^d$ is a homogeneous Poisson point process having intensity $\lambda > 0$ and the radii $\mathcal{R} := \{\rho_\varepsilon\}_{\varepsilon \in \Phi}$ are i.i.d. random variables which satisfy condition (1.7) for a fixed $\beta > 0$. Since assumption (1.7) with $\beta_1 > 0$ implies (1.7) for every other $0 < \beta \leq \beta_1$, with no loss of generality we assume that $\beta \leq 1$.

Throughout the paper we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space associated to the marked point process $(\Phi, \mathcal{R})$, i.e. the joint process of the centres and radii distributed as above. We refer to [10] for a detailed introduction of marked point processes as the one introduced in this paper.
2.1. Notation. For a point process $\Phi$ on $\mathbb{R}^d$ and any bounded set $E \subseteq \mathbb{R}^d$, we define the random variables
\[
\Phi(E) := \Phi \cap E, \quad \Phi^c(E) := \Phi \cap \left( \mathbb{R}^d \setminus \frac{1}{\varepsilon} E \right),
\]
(2.2)
For $\eta > 0$, we denote by $\Phi_\eta$ a thinning for the process $\Phi$ obtained as
\[
\Phi_\eta(\omega) := \{ x \in \Phi(\omega) : \min_{y \in \Phi(\omega)} |x - y| \geq \eta \},
\]
(2.3)
i.e. the points of $\Phi(\omega)$ whose minimal distance from the other points is at least $\eta$. Given the process $\Phi_\eta$, we set $\Phi_\eta(E), \Phi^c_\eta(E), N_\eta(E)$ and $N^c_\eta(E)$ for the analogues for $\Phi_\eta$ of the random variables defined in (2.2).

For a bounded and measurable set $E \subseteq \mathbb{R}^d$ and any $1 \leq p < +\infty$, we denote
\[
L^p_0(E) := \{ f \in L^p(E) : \int_E f = 0 \}.
\]

As in [10], we identify any $v \in H^1_0(D^\varepsilon)$ with the function $\tilde{v} \in H^1_0(D)$ obtained by trivially extending $v$ in $H^\varepsilon$.

Throughout the proofs in this paper, we write $a \lesssim b$ whenever $a \leq C b$ for a constant $C = C(d, \beta)$ depending only on the dimension $d$ and $\beta$ from assumption (1.7). Moreover, when no ambiguity occurs, we use a scalar notation also for vector fields and vector-valued function spaces, i.e. we write for instance $C_0^\infty(D), H^1(\mathbb{R}^d), L^p(\mathbb{R}^d)$ instead of $C_0^\infty(D; \mathbb{R}^d), H^1(\mathbb{R}^d; \mathbb{R}^d), L^p(\mathbb{R}^d; \mathbb{R}^d)$.

2.2. Main results. Let $(\Phi, \mathcal{R})$ be a marked point process as above, and let $H^\varepsilon$ be defined as in (2.1). Then, we have:

**Theorem 2.1.** For $f \in H^{-1}(D; \mathbb{R}^d)$ and $\varepsilon > 0$, let $(u_\varepsilon, p_\varepsilon) = (u_\varepsilon(\omega, \cdot), p_\varepsilon(\omega, \cdot)) \in H^1_0(D^\varepsilon; \mathbb{R}^d) \times L^2_0(D^\varepsilon; \mathbb{R})$ be the solution of
\[
\begin{cases}
-\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D^\varepsilon \\
\nabla \cdot u_\varepsilon = 0 & \text{in } D^\varepsilon \\
u_\varepsilon = 0 & \text{on } \partial D^\varepsilon.
\end{cases}
\]
(2.4)

Then, for $\mathbb{P}$-almost every $\omega \in \Omega$
\[
u_\varepsilon(\omega, \cdot) \rightharpoonup u_h \quad \text{in } H^1_0(D; \mathbb{R}^d), \quad \text{for } \varepsilon \downarrow 0^+,
\]

where $(u_h, p_h) \in H^1_0(D; \mathbb{R}^d) \times L^2_0(D; \mathbb{R})$ is the solution of
\[
\begin{cases}
-\Delta u_h + \nabla p_h + C_d \lambda (\rho^{d-2}) u_h = f & \text{in } D \\
\nabla \cdot u_h = 0 & \text{in } D \\
u_h = 0 & \text{on } \partial D,
\end{cases}
\]
(2.5)
with $C_d$ as in (1.4).

**Remark 2.2** (Stationary Navier-Stokes equations). As in the case of periodic holes [1], we remark that the same result of Theorem 2.1 holds in dimension $d = 3$ for the solutions $u_\varepsilon$ to the stationary Navier-Stokes system
\[
\begin{cases}
u_\varepsilon \cdot \nabla u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } D^\varepsilon \\
\nabla \cdot u_\varepsilon = 0 & \text{in } D^\varepsilon, \\
u_\varepsilon = 0 & \text{on } \partial D^\varepsilon
\end{cases}
\]
(2.6)
with homogenized equations
\[
\begin{cases}
u_h \cdot \nabla u_h - \Delta u_h + C_d \lambda (\rho^{d-2}) u_h + \nabla p_h = f & \text{in } D \\
\nabla \cdot u_h = 0 & \text{in } D \\
u_h = 0 & \text{on } \partial D,
\end{cases}
\]
(2.7)
We argue in the appendix how the same argument that we give in the next section for Theorem 2.1 allows also to treat the non-linear term in (2.6).
The previous theorem shows that the holes of $H^ε$ which overlap do not destroy the homogenization process and that their effect on the value of the Brinkman term is negligible. On the other hand, the complicated geometries which may arise from the clustering effects in $H^ε$ prevent us from obtaining a suitable extension of the pressure terms $p_ε$ to the whole domain $D$ which converges to $p_b$. Nonetheless, in the next theorem we prove a convergence result for $p_ε$ to $p_b$, as long as we remove from $D$ an exceptional set $E^ε$ containing $H^ε$. This set mostly coincides with $H^ε$ in the sense that the difference $E^ε\setminus H^ε$ has vanishing harmonic capacity.

**Theorem 2.3.** For almost every $ω \in Ω$, there exists a set $E^ε \subseteq \mathbb{R}^d$ such that $E^ε \supset H^ε$ and for $ε \downarrow 0^+$

$$\text{Cap}(E^ε\setminus H^ε) \to 0,$$

(2.8)

where Cap denotes the harmonic capacity in $\mathbb{R}^d$. Moreover, for every compact set $K \subseteq D$, the modification of the pressure

$$\tilde{p}_ε = \begin{cases} p_ε - f_{K\setminus E^ε} p_ε & \text{in } K\setminus E^ε \\ 0 & \text{in } D\setminus K \cup E^ε \end{cases}$$

(2.9)

satisfies for all $q < \frac{d}{d-1}$

$$\tilde{p}_ε \rightharpoonup p_b \text{ in } L^q_0(K; \mathbb{R}).$$

Since this result relies on some of the tools which will be developed along the proof of Theorem 2.1, we give the argument for Theorem 2.3 in Section 5.

### 2.3. Main ideas in proving Theorem 2.1 and Theorem 2.3.

As already mentioned above, the structure and many arguments of this paper are an adaptation of [10] to the case of the Stokes equations. In this subsection, we point out the main differences and the challenges that we encountered along the process.

In contrast with [10], we prove the convergence of the fluid velocities $u_ε$ by using an implicit version of the method of oscillating test-functions, which is similar to the one of [6]: We construct an operator $R_ε$ which acts on divergence-free test-functions $v$ such that $R_ε v \in H^0_0(D^ε)$ is an admissible test function for (2.4), $R_ε v \to v$ in $H^0_0(D)$ and $\nabla \cdot R_ε v = 0$ in $D$. This last condition in particular implies that we may test the equation (2.4) with $R_ε v$ and do not need any bounds on the pressure $p_ε$. We emphasize that, as done in [1], a convergence result on the pressure terms $\{p_ε\}_{ε>0}$ is required if one constructs divergence-free oscillating functions $w_ε \in H^0_0(D^ε)$ and tests the equation (1.1) for $u_ε$ with the products $\phi w_ε$, for arbitrary $\phi \in C^∞_0(D)$. We remark that, in principle, the partial result that we obtain on the convergence of the pressure is strong enough to allow us to follow also this last approach. However, as we show in Section 7, obtaining bounds on the pressure in our setting strongly relies on the geometric properties of the clusters and requires a fairly (and further) technical argument. We thus find easier to first give a proof for the homogenization of $u_ε$, which does not rely on any bounds on the sequence $\{p_ε\}_{ε>0}$, and only afterwards show how to extract a convergence result also for $p_ε$.

As in [10] with the construction of the oscillating test-functions $w_ε$, the construction of the operator $R_ε$ relies on a lemma dealing with the geometric properties of the set of holes $H^ε$ which perforate $D$ in (1.2). This lemma allows us to split the set $H^ε$ into a “good” set $H^ε_g$, which contains holes which are small and well-separated, and a “bad” set $H^ε_b$, which contains big and overlapping holes. On the one hand, we construct $R_ε v$ such that it vanishes on $H^ε_g$ by closely following the ideas in [1] and [6]. On the other hand, to define $R_ε v$ in such a way that it vanishes also on $H^ε_b$, we need to improve the arguments used in [10]. In fact, as pointed out in the introduction, in contrast with [10], by the incompressibility condition it is not enough to prove that the harmonic capacity of $H^ε_b$ vanishes in the limit $ε \downarrow 0^+$.

In order to overcome this problem, we use the following strategy to construct $R_ε v$ such that, for any divergence-free $v \in C^∞_0(D, \mathbb{R}^d)$, the function $R_ε v$ vanishes on the “bad” set $H^ε_b$, remains divergence-free in $D$ and converges to $v$ in $H^0_0(D; \mathbb{R}^d)$. We recall that in the set $H^ε_b$ the balls may overlap; the challenge is therefore to find a suitable truncation for $v$ on this set, which preserves the divergence-free condition and which remains bounded in an $H^1$-sense. A first approach to construct $R_ε v$ would then
be to solve the Stokes problem in a large enough neighbourhood \(D_b^\varepsilon\) of \(H_b^\varepsilon\)

\[
\begin{cases}
-\Delta w + \nabla \pi = \Delta v & \text{in } D_b^\varepsilon \setminus \overline{H_b^\varepsilon} \\
\nabla \cdot w = 0 & \text{in } D^\varepsilon \setminus \overline{B_b^\varepsilon} \\
w = 0 & \text{on } \partial H_b^\varepsilon \\
w(x) = v & \text{on } \partial D_b^\varepsilon.
\end{cases}
\tag{2.10}
\]

The connection with the concept of "Stokes capacity" generated by the set \(H_b^\varepsilon\) thus becomes apparent; namely, at least in the case of sets \(E\) regular enough, the minimizer in (1.5) solves

\[
\begin{cases}
-\Delta w + \nabla \pi = 0 & \text{in } \mathbb{R}^d \setminus \overline{E} \\
\nabla \cdot w = 0 & \text{in } \mathbb{R}^d \setminus \overline{E} \\
w = \xi & \text{on } \partial E \\
w(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]

However, getting \(H^1\)-estimates on the solution \(v^\varepsilon\) of (2.10) which depend explicitly on \(\varepsilon\), requires more informations than we have on the geometry of the set \(H_b^\varepsilon\). In fact, condition (1.7) does not prevent the balls from overlapping nor provides an upper bound on the number of balls in each of the clusters (cf. Lemma 6.1). The approach that we adopt to construct \(R\varepsilon v\) is therefore different and is based on finding a suitable covering \(\bar{H}_b^\varepsilon\) of the set \(H_b^\varepsilon\). The set \(\bar{H}_b^\varepsilon\) is obtained by selecting some of the balls that constitute \(H_b^\varepsilon\) and dilating them by a uniformly bounded factor \(\lambda_\varepsilon \leq \Lambda\). The main, crucial, feature of this covering is that it allows us to construct \(\bar{R}\varepsilon v\) vanishing on \(H_b^\varepsilon \subseteq \bar{H}_b^\varepsilon\) by solving different Stokes problems in disjoint annuli of the form \(B_{\theta \lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i) \setminus B_{\lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i), \theta > 1\), and iterating this procedure a finite number of steps. The advantage in this is that we construct \(R\varepsilon v\) iteratively and obtain bounds by applying a finite number of times some standard and rescaled estimates for solutions to Stokes equations in the annulus \(B_{9b^\varepsilon} \setminus B_1\).

More precisely, \(\bar{H}_b^\varepsilon\) is chosen to satisfy the following properties:

\((a)\) \(H_b^\varepsilon\) is the union of \(M < +\infty\) families of balls such that, inside the same family, the balls \(B_{\lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i)\) are disjoint even if dilated by a further factor \(\theta^2 > 0\), i.e. by considering \(B_{\theta^2 \lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i)\);

By this property, if we want to construct \(R\varepsilon v\) vanishing only in the holes of the same family, it suffices to solve (2.10) in the disjoint annuli \(B_{\theta \lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i) \setminus B_{\lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i)\) and stitch the solutions together. This suffices to construct \(R\varepsilon v\) vanishing on the balls \(B_{\lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i)\) of the same family, and thus on the subset of \(H_b^\varepsilon\) covered by them. In order to obtain \(R\varepsilon v\) vanishing on the whole set \(H_b^\varepsilon\), one may try to iterate the previous procedure: Let the families of balls constituting \(\bar{H}_b^\varepsilon\) be ordered with an index \(k = 1, \ldots , M\). Then:

- We construct a first solution \(v_1^\varepsilon\) which solves (2.10) in all the (disjoint) annuli generated by the first family;
- We construct \(v_2^\varepsilon\) solving (2.10) with \(v\) substituted by \(v_1^\varepsilon\) in the (disjoint) annuli of the second family;
- We iterate the procedure up to the \(M\)-th family and set \(R\varepsilon v = v_M^\varepsilon\).

However, property \((a)\) alone does not ensure that the final solution constructed in this fashion vanishes on \(H_b^\varepsilon\). Since annuli generated by different families may still intersect, at each step the zero-boundary conditions of the previous steps may be destroyed (as an example, see Figure 1). This is the reason why we need that the covering \(\bar{H}_b^\varepsilon\) satisfies an additional property. This property should ensure that, if at step \(k\) the function \(v^k\) vanishes on a certain subset of \(H_b^\varepsilon\), then also \(v^{k+1}\) vanishes on that same subset. We thus construct \(\bar{H}_b^\varepsilon\) in such a way that

\((b)\) all the balls \(B_{\theta \lambda_\varepsilon \frac{\partial \varepsilon}{\rho_i}}(\varepsilon z_i)\) belonging to the \(k\)-th family do not intersect the balls of \(H_b^\varepsilon\) contained in the previous families (cf. property (3.8) of the Lemma 3.2).\(^1\)

The construction of \(\bar{H}_b^\varepsilon\) satisfying \((a)-(b)\) is given in Lemma 3.2 of Section 4 and constitutes the most technically challenging part of this paper.

---

\(^1\)Strictly speaking, this is a simplification of the statement of Lemma 3.2 (cf. Remark 3.3 in Section 3).
2.4. Lemma 2.4 and proof of Theorem 2.1. The proof of Theorem 2.1 relies on the following lemma:

**Lemma 2.4.** For almost every $\omega \in \Omega$ and for all $\varepsilon \leq \varepsilon_0(\omega)$ there exists a linear map

$$R_\varepsilon : \{ v \in C_0^\infty(D) : \nabla \cdot v = 0 \} \to H^1(D)$$

with the following properties:

(i) $R_\varepsilon v = 0$ in $H^\varepsilon$ and, for $\varepsilon$ small enough, also $R_\varepsilon v \in H^1_0(D)$;

(ii) $\nabla \cdot R_\varepsilon v = 0$ in $\mathbb{R}^d$;

(iii) $R_\varepsilon v \to v$ in $H^1_0(D)$;

(iv) $R_\varepsilon v \to v$ in $L^p(D)$ for all $1 \leq p < \infty$;

(v) For all $u_\varepsilon \in H^1_0(D)$ such that $\nabla \cdot u_\varepsilon = 0$ in $D$ and $u_\varepsilon \to u$ in $H^1_0(D)$, we have

$$\int \nabla R_\varepsilon v : \nabla u_\varepsilon \to \int \nabla v : \nabla u + C_d \lambda(\rho^{d-2}) \int v \cdot u,$$

with $C_d$ as in Theorem 2.1.

**Proof of Theorem 2.1.** Let us fix $\omega \in \Omega$ such that the operator $R_\varepsilon$ of Lemma 2.4 exists and satisfies all the properties (i) - (v). We trivially extend $u^\varepsilon$ to the whole set $D$. Since by the standard energy estimate we have $\|u_\varepsilon\|_{H^1_0(D)} \leq \|f\|_{H^{-1}(D)}$, then up to a subsequence $\varepsilon_j$, we have $u_\varepsilon \to u^*$ in $H^1_0(D)$. Note that also $\nabla \cdot u^* = 0$ in $D$. We show that $u^*$ solves (2.5) and, by uniqueness, that $u^* = u_h$ in $H^1_0(D)$. We thus may extend the convergences above to the whole limit $\varepsilon \downarrow 0^+$.

For any divergence-free $v \in C_0^\infty(D)$, we consider $\varepsilon$ small enough such that the divergence-free vector field $R_\varepsilon v$ obtained by means of Lemma 2.4 is in $H^1_0(D)$. By testing (2.4) with this vector field, we obtain

$$\int \nabla R_\varepsilon v : \nabla u_\varepsilon = \langle R_\varepsilon v, f \rangle_{H^1, H^{-1}}.$$

We now apply (iii) and (v) of Lemma 2.4 to the left- and right-hand side of the above identity, respectively, and conclude that $u^*$ satisfies

$$\int \nabla v : \nabla u^* + C_d \lambda(\rho^{d-2}) \int v \cdot u^* = \langle v, f \rangle_{H^1, H^{-1}}.$$
Since $v \in C_0^\infty(D)$ is an arbitrary divergence-free test function, we conclude that $u^*$ is the solution $u_h$ of (2.5).

## 3. Geometric properties of the holes

This section is the core of the argument of Theorem 2.1 and Theorem 2.3 and provides some almost sure geometrical properties on $H^\varepsilon$. These allow us to construct the operator of Lemma 2.4. The results contained in this section rely on assumption (1.7) and may be considered as an upgrade of Section 4 of [10]. Since (1.7) is stronger than the one assumed in [10] (see (1.6)), the marked point process $(\Phi, R)$ considered in this work is included in the class of processes studied in [10]. Therefore, all the results for $H^\varepsilon$ contained in Section 4 of [10] hold also in our case. Bearing this in mind, we introduce the first main result of this section: This is almost a rephrasing of Lemma 4.2 of [10], where, thanks to (1.7), we are allowed to choose the sequence $r_\varepsilon$ appearing in the statement of Lemma 4.2 in [10] as a power law $r_\varepsilon = \varepsilon^d$, for $\delta = \delta(d, \beta) > 0$.

**Lemma 3.1.** There exists a $\delta = \delta(d, \beta) > 0$ such that for almost every $\omega \in \Omega$ and all $\varepsilon \leq \varepsilon_0 = \varepsilon_0(\omega)$, there exists a partition $H^\varepsilon = H^\varepsilon_b \cup H^\varepsilon_g$ and a set $D^\varepsilon_b \subseteq \mathbb{R}^d$ such that $H^\varepsilon_b \subseteq D^\varepsilon_b$ and

$$\text{dist}(H^\varepsilon_b, D^\varepsilon_b) > \varepsilon^{1+\delta}, \quad |D^\varepsilon_b| \downarrow 0^+.$$  

(3.1)

Furthermore, $H^\varepsilon_g$ is a union of disjoint balls centred in $n^\varepsilon \subseteq \Phi^\varepsilon(D)$, namely

$$H^\varepsilon_g = \bigcup_{z_i \in n^\varepsilon} B_{\varepsilon^{1+\delta} \rho_i}(\varepsilon z_i), \quad \varepsilon^d \# n^\varepsilon \rightarrow |\lambda|,$$

(3.2)

$$\min_{z_i \neq z_j \in n^\varepsilon} \varepsilon |z_i - z_j| \geq 2\varepsilon^{1+\frac{\delta}{2}}, \quad \varepsilon^{\frac{d}{2}} \rho_i \leq \varepsilon^{1+\delta}.$$  

Finally, if for $\eta > 0$ the process $\Phi^\varepsilon(D)$ is defined as in (2.3), then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^d \#(\{z_i \in \Phi^\varepsilon(D) : \text{dist}(\varepsilon z_i, D^\varepsilon_b) \leq \eta \varepsilon\}) = 0.$$  

(3.3)

The next result upgrades the previous lemma and is the key result on which relies the construction of the operator $R^\varepsilon$ of Lemma 2.4. We introduce the following notation: We set $I^\varepsilon := \Phi^\varepsilon(D) \setminus n^\varepsilon$, so that, by the previous lemma, we may write

$$H^\varepsilon_b := \bigcup_{z_i \in I^\varepsilon} B_{\varepsilon^{1+\delta} \rho_i}(\varepsilon z_i).$$  

(3.4)

As already discussed in Subsection 2.1, the main aim of the next result is to show that there exists a suitable covering for $H^\varepsilon_b$, which is of the form

$$\tilde{H}^\varepsilon := \bigcup_{z_j \in J^\varepsilon} B_{\lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j}(\varepsilon z_j), \quad J^\varepsilon \subseteq I^\varepsilon, \quad \sup_{z_j \in J^\varepsilon} \lambda^\varepsilon_j \leq \Lambda,$$

and which satisfies (a) and (b) of Subsection 2.1. More precisely, we have:

**Lemma 3.2.** Let $\theta > 1$ be fixed. Then for almost every $\omega \in \Omega$ and $\varepsilon \leq \varepsilon_0(\omega, \beta, d, \theta)$ we may choose $H^\varepsilon_b, H^\varepsilon_g$ of Lemma 3.1 in such a way that the following:

- There exist $\Lambda(\beta, d) > 0$, a sub-collection $J^\varepsilon \subseteq I^\varepsilon$ and constants $\{\lambda_j\}_{z_j \in J^\varepsilon} \subseteq [1, \Lambda]$ such that

$$H^\varepsilon_b \subseteq \tilde{H}^\varepsilon := \bigcup_{z_j \in J^\varepsilon} B_{\lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j} (\varepsilon z_j), \quad \lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j \leq \Lambda \varepsilon^{d+\delta}.$$  

(3.5)

- There exists $k_{\max} = k_{\max}(\beta, d) > 0$ such that we may partition

$$I^\varepsilon = \bigcup_{k=-3}^{k_{\max}} I^\varepsilon_k, \quad J^\varepsilon = \bigcup_{i=-3}^{k_{\max}} J^\varepsilon_k,$$

with $I^\varepsilon_k \subseteq J^\varepsilon_k$ for all $k = 1, \ldots, k_{\max}$ and

$$\bigcup_{z_i \in I^\varepsilon_k} B_{\varepsilon^{\frac{d}{2}} \rho_i}(\varepsilon z_i) \subseteq \bigcup_{z_j \in J^\varepsilon_k} B_{\lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j}(\varepsilon z_j);$$  

(3.6)

- For all $k = -3, \ldots, k_{\max}$ and every $z_i, z_j \in J^\varepsilon_k, z_i \neq z_j$

$$B_{\varepsilon^{\frac{d}{2}} \lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j} (\varepsilon z_i) \cap B_{\varepsilon^{\frac{d}{2}} \lambda^\varepsilon_j \varepsilon^{\frac{d}{2}} \rho_j} (\varepsilon z_j) = \emptyset;$$  

(3.7)
For each $k = -3, \cdots, k_{\text{max}}$ and $z_i \in T_k^c$ and for all $z_j \in \bigcup_{l=-3}^{k-1} J_l^c$ we have
\[ B_{\frac{1}{\varepsilon^{d-2} \rho_i}} (\varepsilon z_i) \cap B_{\frac{1}{\theta \lambda^j \varepsilon^{d-2} \rho_j}} (\varepsilon z_j) = \emptyset. \] (3.8)

Finally, the set $D_k^c$ of Lemma 3.1 may be chosen as
\[ D_k^c = \bigcup_{z_i \in J_k^c} B_{\theta \lambda^j \varepsilon^{d-2} \rho_i} (\varepsilon z_i). \] (3.9)

Remark 3.3. As explained in Subsection 2.3, property (3.8) is crucial for the construction of the operator $R_\varepsilon$ of Lemma 2.4. However, it slightly differs from property (b) stated in that section. Namely, the balls $B_{\frac{1}{\varepsilon^{d-2} \theta \lambda^j \rho_i}} (z_j)$, $z_j \in J_k^c$ might intersect with some of the balls in $H_k^c$ that are contained in $B_{\frac{1}{\theta \lambda^j \varepsilon^{d-2} \rho_i}} (\varepsilon z_i)$ for $z_i \in J_k^c$, $k > l$. This is why the additional index sets $T_k^c$ are introduced.

In these index sets, the balls are not ordered by size, but in such a way that (3.8) holds. More precisely, if a ball in $H_k^c$ is contained in several of the dilated balls in $J_k^c$, we will put it into the index set $I_k^c$ with $k$ minimal such that it is contained in a dilated ball in $J_k^c$.

### 3.1. Structure and main ideas in the proof of Lemma 3.1 and Lemma 3.2.

Since the proof of Lemma 3.2 requires different steps and technical constructions, we give a sketch of the ideas behind it. It is clear that Lemma 3.1 follows immediately from Lemma 3.2; we thus only need to focus on the proof of this last result.

To this end we introduce the following notation, which we will also use throughout the rigorous proof of Lemma 3.2 in Section 5: Let
\[ \delta := \frac{\beta}{2(d-2)(d-2+\beta)} \wedge \frac{\beta}{2d} \] (3.10)
and
\[ I_k^c := \begin{cases} \{ z_i \in \Phi^c(D) : \frac{1}{\varepsilon^{d-2}} \rho_i < \frac{1}{\varepsilon^{d-2(k+1)}} \} & k \geq -2 \\ \{ z_i \in \Phi^c(D) : \frac{1}{\varepsilon^{d-2}} \rho_i < \frac{1}{\varepsilon^{d+2 \delta}} \} & k = -3 \end{cases} \] (3.11)

Note that $\Phi^c(D) = \bigcup_{k=-3}^{k_{\text{max}}} I_k^c$. We remark that the sets $I_k^c$ correspond to $I_{\delta,k}^c$ in (6.1) of Section 6 with $\delta$ as in (3.10). Since we chose $\delta$ above such that $\delta < \frac{\beta}{2d}$, we may apply Lemma 6.1 with this choice of $\delta$ and infer that there exists $k_{\text{max}} \in \mathbb{N}$ such that $I_k^c = \emptyset$ for all $k > k_{\text{max}}$. From now on, we assume that $k_{\text{max}}$ is chosen in this way and thus that
\[ \Phi^c(D) = \bigcup_{k=-3}^{k_{\text{max}}} I_k^c. \]

In addition, since we may bound
\[ \varepsilon^{\frac{d-2}{\Phi^c(D)}} \rho_i \leq \varepsilon^{\frac{d-2}{\Phi^c(D)}} \left( \varepsilon^d \sum_{z_i \in \Phi^c(D)} \rho_i^{d-2+\beta} \right)^{\frac{1}{d-2+\beta}}, \]
we use (1.7) and the Strong Law of Large Numbers, to infer that almost surely and for $\varepsilon$ small enough
\[ \varepsilon^{\frac{d-2}{\Phi^c(D)}} \rho_i \lesssim \varepsilon^{\frac{d-2}{d-2+\beta}} \rho_i^{d-2+\beta} \left( \frac{1}{d-2+\beta} \right)^{\frac{1}{d-2+\beta}}, \]
This implies by (3.10) that
\[ \max_{z_i \in \Phi^c(D)} \varepsilon^{\frac{d-2}{\Phi^c(D)}} \rho_i \lesssim \varepsilon^{2d \delta}. \] (3.12)

**Step 1: Combining clusters of holes of similar size:** We begin obtaining a first covering of $H^c$ made by a union of balls which, if of comparable size, are disjoint even if dilated by a constant factor $\alpha > 1$. Roughly speaking, we do this by merging the balls of $H^c$ generated each family $I_k^c \cup I_{k-1}^c$, in holes of similar size which are also disjoint. More precisely, we prove:
Claim: Let $\alpha > 1$. Then, there exists $\tilde{\Lambda} = \tilde{\Lambda}(d, \beta, \alpha) > 0$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$ and all $\varepsilon < \varepsilon_0(\omega)$ and all $-3 \leq k \leq k_{\text{max}}$ there are $\tilde{I}_k \subseteq I_k$ and $\{\tilde{\lambda}^j\}_{z_j \in \tilde{I}_k} \subseteq [1, \tilde{\Lambda}]$ with the following properties:

$$\forall z_i \in I_k \exists z_j \in \bigcup_{l \geq k} \tilde{I}_l : B_{\varepsilon \frac{d}{\alpha \tilde{\lambda}^j \rho_i}} (\varepsilon z_i) \subseteq B_{\varepsilon \frac{d}{\alpha \tilde{\lambda}^j \rho_j}} (\varepsilon z_j).$$

For each $-3 \leq k \leq k_{\text{max}}$ the balls

$$\left\{ B_{\varepsilon \frac{d}{\alpha \tilde{\lambda}^j \rho_i}} (\varepsilon z_i) \right\}_{z_i \in I_k \cup I_{k-1}}$$

are pairwise disjoint.

To construct the sets $\tilde{I}_k$ above we adopt the following strategy (see Figure 2 for a sketch):

- Let $\alpha > 1$ and $-2 \leq k \leq k_{\text{max}}$ be fixed. We multiply each one of the radii $\{\rho_i\}_{z_i \in I_k \cup I_{k-1}}$ by $\alpha$ and consider the set of balls

$$\left\{ B_{\varepsilon \alpha \frac{d}{\tilde{\lambda}^j \rho_i}} (\varepsilon z_i) \right\}_{z_i \in I_k \cup I_{k-1}}.$$  

For each point $z_i \in I_k \cup I_{k-1}$ we now define a new radius $R_i^k$ in the following way: For each disjoint ball in the previous collection we set $R_i^k := \rho_i$. We now consider the balls which are not disjoint: For each connected component $C_k^i$ of (3.15), we pick on of the largest balls belonging to $C_k^i$, say $B_{\varepsilon \frac{d}{\alpha \tilde{\lambda}^j \rho_i}} (\varepsilon z_i)$, and set $R_i^k$ as the minimal one such that $C_k^i \subseteq B_{\varepsilon \frac{d}{\tilde{\lambda}^j \rho_i}} (\varepsilon z_i)$. We set $R_i^k = 0$ for all the $z_i \neq z_i$ generating the balls contained in $C_k^i$. We thus have a new collection of radii $\{R_i^k\}_{z_i \in I_k \cup I_{k-1}}$.

- We multiply each $R_i^k$ above by the same factor $\alpha$ of the previous step and repeat the construction sketched above with $\rho_i$ substituted by $R_i^k$.

- We show that, almost surely, after a number $M = M(d, \beta) < +\infty$ of iterations of the previous two steps, all the radii $R_i^k$ obtained at the $M^{th}$-step do not change any further. This means
that the balls $B_{\varepsilon e^{-d\bar{\lambda}^*} R^*_{\rho i}}(\varepsilon z_j)$, for $R^*_{\rho i} \neq 0$, satisfy (3.13) and (3.14). Moreover, we may easily bound each ratio $\frac{w}{R^*_{\rho i}} =: \tilde{\lambda}_i^* \leq \bar{\lambda}$.

The key idea to prove the existence of the threshold $M$ is that the configurations $\omega \in \Omega$ for which the radii $R_i$'s obtained after $M$ iterations continue to change is related to events of the form

"There exist $M+1$ balls in $I^*_k \cup I^*_{k-1}$ which are connected when dilated by $C(\alpha, M)$".

By Lemma 6.1, this event has zero probability for $\varepsilon$ sufficiently small.

- The construction above can be expressed by a dynamical system (cf. (3.19)).
- We iterate this process for $I^*_k \cap I^*_{k-1}$, with $-2 \leq k \leq k_{\text{max}}$ starting from $k = -2$, each time working with the dilated radii that we got from the previous step.

**Step 2: Construction of the sets $I^\varepsilon$ and $J^\varepsilon$:** Let us set $\theta = \frac{\alpha}{1+\alpha} \geq 1$, with $\alpha \geq 1$ as in Step 1 (see (3.14)). In the previous step we extracted from each family $I^*_k$ generating the whole $\Phi^\varepsilon(D)$ a sub-collection $I^*_k$. These sub-collections provide a covering for the whole set $H^\varepsilon$ and satisfy (3.14).

The aim of this step is to use the previous result to find a way to extract from $\Phi^\varepsilon(D)$ the subset $I^\varepsilon$ generating the bad holes and to construct the covering $H^\varepsilon$.

We remark that, if we set $\lambda_i = \theta^2 \bar{\lambda}_i$, the covering

$$\bigcup_{k=-3}^{k_{\text{max}}} \bigcup_{z_j \in I^*_k} B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_i}(\varepsilon z_j) \supseteq H^\varepsilon$$

satisfies (3.7) thanks to (3.14).

The construction of this step is based on the following simple geometric fact: Let $z_1 \in I^*_k$ and $z_2 \in I^*_k$ with $k_1 < k_2 - 1$. Since by construction we had $I^*_k \subseteq I^*_k$, this means by definition (3.11) of the sets $I^*_k$ that $\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1 \leq \varepsilon e^{-d\bar{\lambda}^*_i} \rho_2$ and thus that the ball $B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_1)$ is much smaller than $B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_2}(\varepsilon z_2)$.

Therefore, for $\varepsilon \leq \varepsilon_0(d, \beta, \theta)$ we have that

$$B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_1) \cap B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_2}(\varepsilon z_2) \neq \emptyset \Rightarrow B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_1) \subseteq B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_2}(\varepsilon z_2).$$

Indeed, if the inequality on the left-hand side is true, for all $z \in B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_1)$ we have

$$\varepsilon |z - z_2| \leq \varepsilon |z - z_1| + \varepsilon |z_1 - z_2| \leq \varepsilon^{-d} \theta \bar{\lambda}^*_i \rho_1 + \varepsilon^{-d} \theta \bar{\lambda}^*_i \rho_1 + \varepsilon^{-d} \theta \bar{\lambda}^*_i \rho_2.$$

Since $\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1 \leq \varepsilon e^{-d\bar{\lambda}^*_i} \rho_2$ and all $1 \leq \bar{\lambda}_i^* \leq \bar{\lambda}$, we may choose $\varepsilon^d < \frac{\theta^2 e^{-1}}{\theta \bar{\lambda}^*(1+\theta^2)}$ and obtain that

$$\varepsilon |z - z_2| \leq \varepsilon^{-d} \theta^2 \bar{\lambda}^*_i \rho_2,$$

i.e. the right-hand side in (3.16).

By relying on (3.16), we construct the covering $J^\varepsilon$ in the following way:

- We start with $k_{\text{max}}$ and set $J^\varepsilon_{k_{\text{max}}} = I^*_k$ and $J^\varepsilon_{k_{\text{max}}-1} = I^*_k$. We know that all the balls of the form $B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_i)$ generated by $z_i \in I^*_k \cup I^*_k$ are disjoint in the sense of (3.14) (recall that $\theta^2 = \alpha$). The same holds for the balls $B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_2}(\varepsilon z_j)$ generated by the centres in $I^*_k \cup I^*_k$. We thus focus on the intersections between the balls generated by $I^*_k$ and $I^*_k$.

- We show how to obtain the set $J^\varepsilon_{k_{\text{max}}-2}$ from $I^*_k$ in such a way that (3.8) is satisfied by this family. We begin by dilating the balls generated by the centres in $J^\varepsilon_{k_{\text{max}}}$ of a factor $\theta^2$ and thus obtain the set

$$E^\varepsilon_{k_{\text{max}}} = \bigcup_{z_j \in I^*_k} B_{\varepsilon e^{-d\bar{\lambda}^*_i} \rho_1}(\varepsilon z_j)$$

(recall that $\lambda^*_i^* = \theta^2 \bar{\lambda}^*_i$). We define

$$J^\varepsilon_{k_{\text{max}}-2} := \{z_i \in I^*_k: B_{\varepsilon e^{-d\theta^2 \bar{\lambda}^*_i \rho_1}}(\varepsilon z_i) \notin E^\varepsilon_{k_{\text{max}}}\}.$$
We finally define and partition the set $E_{k_{\text{max}}-2}$.

The circles with the full line in the second picture represent the balls whose centres are in the set $J_{k_{\text{max}}-2}$. The third picture shows the set $E_{k_{\text{max}}-2}$.

Note that with this definition, for all $z_j \in J^c_{k_{\text{max}}-2}$ and every $z_i \in J^c_{k_{\text{max}}}$ we have that

$$
B_{\varepsilon_{d-2} \hat{\lambda}_j^c z_i} (\varepsilon z_j) \nsubseteq B_{\varepsilon_{d-2} \lambda_j^c z_i} (\varepsilon z_j)
$$

and thus by property (3.16) (with $z_i = z_1$ and $z_j = z_2$) that

$$
B_{\varepsilon_{d-2} \hat{\lambda}_j^c z_i} (\varepsilon z_i) \cap B_{\varepsilon_{d-2} \lambda_j^c z_i} (\varepsilon z_j) = \emptyset.
$$

Since $\hat{\lambda}_j^c \geq 1$, the previous equality implies that the collection $J^c_{k_{\text{max}}-2}$ satisfies condition (3.8).

- We now iterate the previous construction: We define

$$
E^c_{k_{\text{max}}-1} = E^c_{k_{\text{max}}} \cup \bigcup_{z_i \in J^c_{k_{\text{max}}-1}} B_{\varepsilon_{d-2} \lambda_j^c z_i}(\varepsilon z_i)
$$

and

$$
E^c_{k_{\text{max}}-2} = (E^c_{k_{\text{max}}-1}) \setminus \bigcup_{z_i \in J^c_{k_{\text{max}}-2}} B_{\varepsilon_{d-2} \lambda_j^c z_i}(\varepsilon z_i) \cup \bigcup_{z_i \in J^c_{k_{\text{max}}-2}} B_{\varepsilon_{d-2} \lambda_j^c z_i}(\varepsilon z_i).
$$

Note that in the definition of this last set we need to remove the annuli

$$
B_{\varepsilon_{d-2} \lambda_j^c z_i}(\varepsilon z_i) \setminus B_{\varepsilon_{d-2} \hat{\lambda}_j^c z_i}(\varepsilon z_i)
$$

in order to be able to iterate the argument of the previous step (see Figure 3 for an illustration of the construction of the set $E_{k_{\text{max}}-2}$).

- We iterate the previous procedure and construct the sets $J^c_k$, up to $-2 \leq k \leq k_{\text{max}}$. In the last step $k = -3$, we define $J^c_{-3}$ as the set of those elements which either intersect $E^c_{-2}$ or that are too close to each other. Thanks to this construction, some elements of $I^c_{-3}$, i.e. the holes which are small and well-separated from the clusters and from each others, do not belong to any of the sets $J^c_k$ nor are covered by any of the dilated balls generated by these centres. We then show that the remaining elements in $I^c_{-3}$ constitute the set $n^c$ generating the holes $H^c_k$.

- We finally define and partition the set $I^c$ generating the holes of $H^c_k$ by using the sets $\{J^c_k\}_{-3 \leq k \leq k_{\text{max}}}$: We insert in each $I^c_k$ the centres of the balls of $H^c$ such that $k$ is the smallest integer for which $J^c_k$ provides a covering.

**Step 3. Conclusion.** We show that with these definitions of $J^c, I^c_k$, and $\lambda^c_k$, the covering obtained in the previous step satisfies all the properties of Lemma 3.1 and Lemma 3.2.
3.2. Proof of Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.2. In the sake of a leaner notation, when no ambiguity occurs we drop the index $\varepsilon$ in the sets of points (e.g. $I_k^\varepsilon, J_k^\varepsilon, \cdots$) and holes which are generated by them.

Proof of Step 1. We start by fixing a (total) ordering $\leq$ of the points in $\Phi^\varepsilon(D)$ such that
\[
z_i \leq z_j \Rightarrow \rho_i \leq \rho_j,
\]
with $\rho_i$ and $\rho_j$ the radii of the balls in $H^\varepsilon(D)$ centred in $z_i$ and $z_j$, respectively. We fix $\alpha > 1$ and set $C_0(\alpha, M) = (2\alpha M)^M(\delta_{\text{max}} + 3) < +\infty$, where $M = M(\beta, d) \in \mathbb{N}$ is as in Lemma 6.1. We only consider $\omega \in \Omega$ belonging to the full-probability subset of $\Omega$ satisfying Lemma 6.1 with $\alpha = C_0$ and $\delta$ as in (3.10).

We introduce some more notation which is needed to implement the construction sketched in Step 1: Let $\Psi \subseteq \Phi^\varepsilon(D)$ be any sub-collection of centres and let $\mathcal{R}^\varepsilon = \{R_i\}_{z_i \in \Psi} \subseteq \mathbb{R}^{\#\Psi}_+$ be their associated radii. Throughout this proof, unless there is danger of ambiguity, we forget about the dependence of both $\Psi$ and $\mathcal{R}$ on $\varepsilon$. For any two centres $z_i, z_j \in \Psi$ with radii $R_i$ and $R_j$, respectively, we write
\[
z_i - \alpha z_j \Leftrightarrow B_{\alpha z_i - \alpha z_j}(\varepsilon z_j) \cap B_{\alpha z_i - \alpha z_j}(\varepsilon z_i) \neq \emptyset.
\]
We define a notion of connection between points and associated radii in the following way: We say that $(z_i, R_i)$ and $(z_j, R_j)$ are connected, and we write that $z_i \sim_{(\Psi, \mathcal{R}), \alpha} z_j$ whenever
\[
\exists z_1, \ldots, z_m \in \Psi \text{ s.t. } z_i - \alpha z_1 \cdots - \alpha z_m = z_j.
\]
This equivalence relation depends on $\varepsilon$, but we forget about it in the notation. We use the notation $[z_i]_{\Psi, \mathcal{R}, \alpha}$ for each equivalence class with respect to the previous equivalence relation $\sim_{(\Psi, \mathcal{R}), \alpha}$. Each equivalence class constitutes a cluster of balls in the sense of (3.17).

By using this notation we may reformulate the result of Lemma 3.2. In the sake of a leaner notation, when no ambiguity occurs we drop the index $\varepsilon$ in the sets of points (e.g. $I, J \subseteq \mathbb{R}^{\#\Psi}_+$), respectively. We fix $\omega \in \Omega$ and $\varepsilon \leq \varepsilon_0(\omega, d, \beta)$ and any $k \geq -2$, if we choose $\Psi = I_k \cup I_{k-1}$, and $\mathcal{R} = \{\rho_i\}_{z_i \in \Psi}$, we have
\[
\sup_{z_i \in \Psi}(\#[z_i]_{\Psi, \mathcal{R}, C_0}) \leq M,
\]
i.e. every equivalence class contains at most $M$ elements of $\Psi$. From now on, we thus fix $\omega \in \Omega$ and $\varepsilon \leq \varepsilon_0(\omega, d, \beta)$ satisfying this bound.

Given $\Psi \subseteq \Phi^\varepsilon(D)$, we introduce the map $T^\varepsilon : \mathbb{R}^{\#\Psi}_+ \to \mathbb{R}^{\#\Psi}_+$ which acts on $\mathcal{R} = \{R_i\}_{z_i \in \Psi}$ as
\[
(T^\varepsilon(R))_j := \begin{cases} 0 & \text{if } \max\{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}\} \neq z_j \\
\max_{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}}(e^1 - \frac{1}{\alpha z_j - z_i} + R_i) & \text{if } \max\{z_i \in [z_j]_{\Psi, \mathcal{R}, \alpha}\} = z_j
\end{cases}
\]
We recall that the maximum above is taken with respect to the ordering $\leq$ between centres of $\Phi^\varepsilon(D)$. We observe that (3.19) implies that, if $[z_j]_{\Psi, \mathcal{R}, \alpha} = \{z_j\}$, then
\[
T^\varepsilon(R)_j = R_j.
\]
By relying on (3.18), we use an iteration of the previous map to implement the construction sketched at Step 1. We begin by considering $k = -2$ and setting $\Psi = I_{-2} \cup I_{-3}$ and $\mathcal{R} = \{\rho_i\}_{z_i \in \Psi}$. We define the dynamical system
\[
\begin{cases} 
\mathcal{R}(n) = T^\varepsilon(R(n-1)) & n \in \mathbb{N} \\
\mathcal{R}(0) = \mathcal{R}
\end{cases}
\]
and claim that
\[
\begin{cases} 
\mathcal{R}(n) = \mathcal{R}(M) & \forall n \geq M \\
(\mathcal{R}(n))_j \leq (2\alpha M)^n \rho_j & \forall z_j \in \Psi, \forall n \leq M.
\end{cases}
\]
We start with (3.22) and prove it by induction over $n \leq M$. By definition (cf. (3.20)), the inequality trivially holds for $n = 0$. Let us now assume that (3.22) holds for some $0 \leq n < M$. We claim that at step $n + 1$, each equivalence class $[z_j]_{\Psi, \mathcal{R}(n), \alpha}$ contains at most $M$ elements: If otherwise, by the inductive hypothesis (3.22) for $n$ and the choice of the constant $C_0(M, \alpha)$, also the equivalence class $[z_i]_{\Psi, \mathcal{R}(0), C_0}$ contains more than $M$ elements. Since we chose $\mathcal{R}(0) = \{\rho_i\}_{z_i \in \Psi}$ by our choice of
\( \omega \in \Omega \) and \( \varepsilon \leq \varepsilon(\omega, C_0) \), property (3.18) is contradicted. Thus, each equivalence class \([z_i](\Psi, R(n), \alpha)\) contains at most \( M \) elements. This allows us to bound

\[
(\mathcal{R}(n+1))_j \overset{(3.20)}{\leq} 2\alpha \sum_{z_i \in [z_j](\Psi, R(n), \alpha)} R(n)_i, \quad (3.22) \quad \sum_{z_i \in [z_j](\Psi, R(n), \alpha)} \rho_i
\]

We now observe that by construction (3.20) and definition (3.19), either \( \mathcal{R}(n+1)_j = 0 \), and thus the bound (3.22) holds trivially, or \( \rho_j \geq \rho_i \) for all \( z_i \in [z_j](\Psi, R(n), \alpha) \). Thus, the previous inequality implies that

\[
(\mathcal{R}(n+1))_j \leq (2\alpha M)^{n+1} \rho_j, \quad (3.23)
\]

i.e. inequality (3.22) for \( n + 1 \). The induction proof for (3.22) is complete.

We now show (3.21): We begin by remarking that, by construction, if we have \( \mathcal{R}(M) \neq \mathcal{R}(M+1) \), then there exist \( z_1, \ldots, z_{M+1} \) such that

\[
\bigcup_{k=1}^{M+1} B_{\varepsilon \rho_k}(\varepsilon z_k) \subseteq B_{\varepsilon \rho_1}(\varepsilon z_1).
\]

This, together with estimate (3.22) for \( n = M \), implies that the equivalence class \([z_j](\Psi, R(0), C_0)\) contains more than \( M \) elements. As above, this contradicts our choice of the realization \( \omega \in \Omega \) and \( \varepsilon \).

We established (3.21).

Equipped with properties (3.22) and (3.21) we may set for every \( z_i \in \Phi^i(D) \)

\[
\mathcal{R}_j^{(-2)} := \begin{cases} \mathcal{R}(M) & \text{if } z_i \in I_{-2} \cup I_{-3} \\ \rho_i & \text{otherwise} \end{cases}
\]

and define

\[
I_{-3} := \{ z_i \in I_{-3} : \mathcal{R}_j^{(-2)} > 0 \}.
\]

Note that this definition of \( \mathcal{R}^{(-2)} \) implies that the balls

\[
\{ B_{\varepsilon \rho_i}(\varepsilon z_i) \}_{z_i \in I_{-2} \cup I_{-3}}
\]

are pairwise disjoint.

We now iterate the previous step up to \( k = k_{max} \): For each \(-1 \leq k \leq k_{max} \) we define recursively

\[
\mathcal{R}_j^{(k)} := \begin{cases} \mathcal{R}(M) & \text{if } z_i \in I_k \cup I_{k-1} \\ \mathcal{R}^{(k-1)} & \text{otherwise}, \end{cases}
\]

where \( \mathcal{R}(M) \) is obtained by solving (3.19) with \( \Psi = I_k \cup I_{k-1} \) and \( \mathcal{R}(0) = \mathcal{R}^{(-1)} \). We note that for a general \(-1 \leq k \leq k_{max} \), (3.22) turns into

\[
(\mathcal{R}^{(k)}(n))_j \leq (2\alpha M)^{(k+2)M+n} \rho_j \quad \forall z_j \in \Psi, \ \forall n \leq M.
\]

In fact, since for \( n \leq M \) we have \((2\alpha M)^{(k+2)M+n} \leq C_0\), property (3.21) follows by this inequality exactly as in the case \( k = -2 \) shown above. We emphasize that, by definition (3.24), at each step \( k \) we have that the balls

\[
\{ B_{\varepsilon \rho_i}(\varepsilon z_i) \}_{z_i \in I_k \cup I_{k-1}, \mathcal{R}_i^{(k)} > 0}
\]

are pairwise disjoint.

From the previous construction we construct the sets \( \tilde{I}_k \) and the parameters \( \{ \tilde{\lambda}_i \}_{z_i \in \bigcup_{k=-3}^{k_{max}} \tilde{I}_k} \) of Step 1: For every \(-3 \leq k \leq k_{max} \), let

\[
\tilde{I}_k := \{ z_i \in I_k : (\mathcal{R}^{(k+1)}(M))_i > 0 \},
\]

\[
\tilde{\lambda}_i = \frac{(\mathcal{R}^{(k+1)}(M))_i}{\rho_i} \quad \text{for } z_i \in \tilde{I}_k.
\]

By (3.25) and the definition of the sets \( \tilde{I}_k \), we immediately have that each \( \tilde{\lambda}_i \geq 1 \) and is bounded by \( \tilde{\Lambda} := (2\alpha M)^{(k_{max}+3)M} \). It remains to argue that \( \tilde{I}^k \) satisfy (3.13) and (3.14): Property (3.13) follows
immediately from the construction and the definition of the operator $T^{q,\alpha}$. To prove (3.14), we claim that is enough to show that for every $k = -2, \ldots, k_{\text{max}}$ and $z_i \in \tilde{I}_k$,
\[
\lambda_i = \frac{R_i^{(k)}}{\rho_i}.
\] (3.28)

Indeed, if this is true, then (3.14) follows immediately from (3.26).

Let $-2 \leq k \leq k_{\text{max}}$ be fixed. By (3.24), to show (3.28) it enough to prove that
\[
R_i^{(k)} = R_i^{(k+1)}, \quad \text{for all } z_i \in \tilde{I}_k.
\]

Since by (3.24) we have for all $z_i \in \tilde{I}_k$ that $R_i^{(k+1)} = \mathcal{R}(M)_i$, with $\mathcal{R}(M)$ solving
\[
\begin{cases}
\mathcal{R}(n) = T^{q,\alpha}((\mathcal{R}(n-1)) & n \in \mathbb{N} \\
\mathcal{R}(0) = \mathcal{R}(k),
\end{cases}
\]
we need to make sure that $\mathcal{R}(n)_i = R_i^{(k)}$ for each $1 \leq n \leq M$. By induction we show that for $z_i \in I_k$ we have
\[
\mathcal{R}(n)_i \neq R_i^{(k)} \Rightarrow \mathcal{R}(n+1)_i = R_i^{(k+1)} = 0
\] (3.29)

This implies (3.28) by definition (3.27).

For $n = 1$, property (3.29) is an easy consequence of (3.26) for the balls generated by points $z_i \in I_k$.

Let us assume that (3.27) holds at step $n$. Then, again by (3.27), we have that for $z_i \in I_k$ either $\mathcal{R}(n)_i = 0$, or $\mathcal{R}(n)_i = R_i^{(k)}$. Thus, if $\mathcal{R}(n+1)_i \neq \mathcal{R}(n)_i$, we necessarily have again by (3.26) that there exists $z_j \in I_{k+1}$ such that
\[
B_{\alpha \varepsilon^{\frac{q}{\alpha}} R_j^{(n)}(\varepsilon z_j)} \cap B_{\alpha \varepsilon^{\frac{q}{\alpha}} R_j^{(n-1)}(\varepsilon z_i)} \neq \emptyset.
\]

This implies that $\rho_j \geq \rho_i$ and in turn that $z_j \geq z_i$. By definition of the map $T^{q,\alpha}$, this yields necessarily that $\mathcal{R}(n+1)_i = 0$. The proof of (3.29) is complete. This establishes (3.28) and concludes the proof of (3.14).

We conclude this step with the following remark: Let $\Phi^\varepsilon_{23\varepsilon/2}(D)$ be the thinned process (see (2.3)) with $\delta$ fixed as in (3.10). Moreover, let $S^\varepsilon := \Phi^\varepsilon(D) \setminus \Phi^\varepsilon_{23\varepsilon/2}(D)$ and
\[
I^\varepsilon_{3} = I_{3} \cap \Phi^\varepsilon_{23\varepsilon/2}(D), \quad I^\varepsilon_{3} = I_{3} \setminus I^\varepsilon_{3} = I_{3} \cap S^\varepsilon.
\] (3.30)

We claim that, up to taking $\varepsilon_0 = \varepsilon_0(d, \beta)$ smaller than above, we have
\[
I^\varepsilon_{3} \subseteq \tilde{I}_{3}, \quad \lambda_i = 1 \quad \text{for all } z_i \in I^\varepsilon_{3}.
\] (3.31)

As will be shown in the next step, the set $I^\varepsilon_{3}$ contains the set $n^\varepsilon$ generating $H^\varepsilon_{y}$.

To show (3.31), we observe that whenever $z_i$, $z_j \in I^\varepsilon_{3} \cup I_{-2}$ with $z_i \neq z_j$, then we may choose $\varepsilon$ small enough to infer that
\[
B_{\alpha \varepsilon^{\frac{q}{\alpha}} \rho_j} (\varepsilon z_i) \cap B_{\alpha \varepsilon^{\frac{q}{\alpha}} \rho_j} (\varepsilon z_j) = \emptyset.
\]

Indeed, for $\varepsilon^{\frac{q}{\alpha}} \leq (\alpha \tilde{\Lambda})^{-1}$, we bound
\[
\varepsilon |z_i - z_j| \geq 2\varepsilon^{1+\frac{q}{\alpha}} \geq 2\alpha \tilde{\Lambda} \varepsilon^{1+\delta} \geq \varepsilon^{\frac{q}{\alpha}} (\alpha \rho_i + \tilde{\Lambda} \rho_j).
\]

This implies that after $M$ iterations of the dynamical system (3.23), we have $\mathcal{R}(M) = \rho_i$ for all $z_i \in I^\varepsilon_{-3}$. Thanks to (3.27) we obtain (3.31).

**Proof of Step 2.** In this step we rigorously implement the method sketched in Step 2 and construct the sets $J_{k}$ as subsets of $\tilde{I}_k$, $-3 \leq k \leq k_{\text{max}}$. We define $\lambda_j = \theta^2 \lambda_j$, with $\lambda_j \in [1, \tilde{\Lambda}]$ constructed in Claim 1 of Step 1, and $\theta^4 = \alpha$. Clearly, we may choose the upper bound $\Lambda$ in the statement of Lemma 3.2 as $\Lambda := \theta \Lambda$. We start by setting
\[
J_{k_{\text{max}}} := \tilde{I}_{k_{\text{max}}},
\]
\[
E_{k_{\text{max}}} := \bigcup_{z_j \in J_{k_{\text{max}}}} B_{\lambda_j \varepsilon^{\frac{q}{\alpha}} \rho_j} (\varepsilon z_j),
\]
and inductively define for \(-1 \leq l \leq k_{max}\)

\[
J_{l-1} := \left\{ z_j \in \hat{I}_{l-1} : B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \not\subseteq E_l \right\},
\]

(3.32)

\[
E_{l-1} := \left( E_l \setminus \bigcup_{z_j \in I_{l-1}} B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \right) \cup \bigcup_{z_j \in I_{l-1}} B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j).
\]

(3.33)

To construct the remaining sets \(J_{-3}\) and \(E_{-3}\), we need an additional step: We recall the definition of \(S^e\) and \(I^e_{-3}\) from (2.3) and (3.30), respectively. We first set

\[
\hat{J}_{-3} := \left\{ z_j \in I_{-3} \cap S^e : B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \not\subseteq E_{-2} \right\},
\]

(3.34)

\[
\hat{E}_{-3} := \left( E_{-2} \setminus \bigcup_{z_j \in J_{-3}} B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \right) \cup \bigcup_{z_j \in J_{-3}} B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j).
\]

Finally, for \(z_i \in \Phi^e(D)\) we define the set

\[
K^e := \left\{ z_j \in I^e_{-3} : B_{2^{g+1} \varepsilon} (\varepsilon z_j) \cap \bigcup_{z_i \in \bigcup_{k=3}^{k_{max}} J_k \cup \hat{J}_{-3}} B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_i) = \emptyset \right\},
\]

(3.35)

and finally consider

\[
J_{-3} := \hat{J}_{-3} \cup \left\{ z_j \in K^e : B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \not\subseteq \hat{E}_{-3} \right\},
\]

(3.36)

\[
\hat{E}_{-3} := \left( E_{-2} \setminus \bigcup_{z_j \in J_{-3}} B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \right) \cup \bigcup_{z_j \in J_{-3}} B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j).
\]

We remark that in the definitions of \(E_l\), the annuli \(B_{\theta \lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \setminus B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j)\) are cut out in order to satisfy (3.8). Moreover, we observe that each connected component of the set \(E_k\) is a subset of \(B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j)\) for some \(z_j \in J_l\), for \(k \geq l\). This follows from the the definition of \(E_k\) and (3.14).

We finally denote

\[
J := \bigcup_{k=-3}^{k_{max}} J_k.
\]

(3.37)

and define the set \(\mathcal{I}\) of the centres generating \(H^e_{\tilde{\phi}}\) as

\[
\mathcal{I} := \left\{ z_i \in \Phi^e(D) : B_{\varepsilon} (\varepsilon z_i) \subseteq B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \text{ for some } z_j \in J \right\},
\]

(3.38)

\[
\mathcal{I}_k := \left\{ z_i \in \mathcal{I} : k \text{ is minimal such that } B_{\varepsilon} (\varepsilon z_i) \subseteq B_{\lambda_j \varepsilon \frac{d_j}{\rho_j}} (\varepsilon z_j) \text{ for } z_j \in J_k \right\}.
\]

(3.39)

Equipped with the previous definition, we construct \(H^\phi_{\tilde{\rho}}, H^\phi_{\hat{\phi}}\) and \(D^\phi_{\hat{\phi}}\) as shown in (3.4), (3.5), and (3.9).

**Proof of Step 3.** We first argue that the sets \(H^\phi_{\tilde{\rho}}, H^\phi_{\hat{\phi}}\) and \(D^\phi_{\hat{\phi}}\) constructed in the previous step satisfy the conditions of Lemma 3.1.

We begin by claiming that

\[
n_{c} = I^g_{-3} \setminus K^e,
\]

(3.40)

with \(K^e\) defined in (3.35). Since, by construction we set \(H^g_{\tilde{\phi}} = H^e \setminus H^\phi_{\tilde{\rho}}\), by (3.4) this also reads as

\[
\Phi^e(D) \setminus \mathcal{I} = I^g_{-3} \setminus K^e.
\]

(3.41)

The \(\supseteq\)-inclusion is a consequence of the fact that by (3.31) we have by construction \(I^g_{-3} \cap \hat{J}_{-3} = \emptyset\) (see (3.34), (2.3)). This yields that in the definition (3.36) of \(J_{-3}\) the only elements of \(I^g_{-3}\) in \(J_{-3}\) are the ones contained in \(K^e\). By (3.32) and (3.37), this yields that \((I^g_{-3} \setminus K) \cap J = \emptyset\). We now use (3.39) to infer that also \((I^g_{-3} \setminus K) \cap \mathcal{I} = \emptyset\), i.e. the \(\supseteq\)-inclusion in (3.41).
For the $\subseteq$ inclusion we argue the complementary statement which, by (3.30), also reads as
\[
K^\varepsilon \cup \bigcup_{k\geq -2} I_k^\varepsilon \subseteq I.
\] (3.42)
We show how to argue that $I_k \subseteq I$, for some $k \geq -2$. The argument for the other sets is analogous.

Let $z_i \in I_k$. Then, by (3.13), there exists $l \geq k$, $z_i \in \tilde{I}_l$ such that
\[
B_{\frac{\varepsilon}{\delta_{\lambda_j}}} (\varepsilon z_i) \subseteq B_{\frac{\varepsilon}{\delta_{\lambda_j}}} (\varepsilon z_j).
\]
By definition (3.32), this yields that either $z_i \subseteq l_1$ or
\[
B_{\frac{\varepsilon}{\delta_{\lambda_j}}} (\varepsilon z_i) \subseteq E_{l+1}.
\]
In the first case, it is immediate that $z_i \in I$ (see (3.38)); in the second case, since each connected component of the set $E_{l+1}$ is a subset of a ball $B_{\lambda_{z_2}} (\varepsilon z_2)$ for some $z_2 \in J_l$ with $l_2 > l_1$, it follows that
\[
B_{\frac{\varepsilon}{\delta_{\lambda_j}}} (\varepsilon z_i) \subseteq B_{\lambda_{z_2}} (\varepsilon z_2).
\]
Hence, also in this case $z_i \in I$. We established $I_k \subseteq I$. This concludes the proof of (3.42) and thus also of (3.41) and (3.40).

From identity (3.40), the second line of (3.2) immediately follows by (3.30) and definition (3.11) for the set $I_{-3}$. In addition, since $K^\varepsilon$ is not contained in $n^\varepsilon$, also the first inequality in (3.1) holds. The remaining claims in (3.1), (3.2), and (3.3) may be obtained from (3.42) similarly to [10][Lemma 4.2], thanks to the very explicit definition of the sets $H_k^\varepsilon$ and $D_k^\varepsilon$.

In the sake of completeness we give these arguments explicitly: We claim
\[
\lim_{\varepsilon \downarrow 0} \varepsilon^d \#(I) = 0.
\] (3.43)
By taking the complement with respect to $\Phi^\varepsilon(D)$ in (3.41), we have
\[
I = \bigcup_{k=-2}^{k_{max}} I_k \subseteq I_{-3} \cup K^\varepsilon.
\]
We estimate the limit for $\varepsilon \downarrow 0^+$ for the first sets on the right-hand side by appealing to Lemma C.1 and (3.10) (we recall that we assumed $\beta \leq 1$): Indeed, we have
\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^d \#(\bigcup_{k=-2}^{k_{max}} I_k) = \limsup_{\varepsilon \downarrow 0} \varepsilon^d \# \{ z_i \in \Phi^\varepsilon(D) : \varepsilon^{\frac{d}{\delta_{\lambda}}} \rho_i \geq \varepsilon^{1+2\delta} \}
\leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{d-(d-2)(1+2\delta)} \sum_{z_i \in \Phi^\varepsilon(D)} \rho_i^{d-2} \to 0
\lesssim \limsup_{\varepsilon \downarrow 0} \varepsilon^{2(1-(d-2)\delta)} = 0.
\]
We now turn to $I_{-3}^b$: Let $\{\delta_k\}_{k \in \mathbb{N}}$ be any sequence such that $\delta_k \downarrow 0^+$. Since $2\varepsilon^{\delta/2} \to 0$, we estimate for any $\delta_k > 0$
\[
\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) \leq \limsup_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - N_{\delta_k}^\varepsilon(D)) \leq \lim_{\varepsilon \downarrow 0^+} \varepsilon^d (N^\varepsilon(D) - \langle N_{\delta_k}^\varepsilon(D) \rangle).
\]
We now apply Lemma C.1 to $\Phi$ and each $\Phi_{\delta_k}$, $k \in \mathbb{N}$, to deduce that almost surely and for every $\delta_k > 0$
\[
\limsup_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) \leq \lambda |D| - \langle N_{\delta_k}^\varepsilon(D) \rangle.
\]
By sending $\delta_k \downarrow 0^+$, we use once more Lemma C.1 on the last term on the right-hand side above and obtain
\[
\lim_{\varepsilon \downarrow 0^+} \varepsilon^d \#(I_{-3}^b) = 0.
\]
To conclude the proof of (3.43), it thus remains to show that almost surely also
\[ \varepsilon^d \#(K^\varepsilon) \to 0 \quad \varepsilon \downarrow 0^+. \]  
(3.44)

We have for all \( z_i \in K^\varepsilon \subseteq I_{-3}' \)
\[ \min_{z_j \in \Phi^\varepsilon(D) \setminus \{z_i\}} |z_j - z_i| \geq 2\varepsilon^{1+\delta/2}, \quad \varepsilon^{d-\delta} \rho_i < \varepsilon^{1+2\delta}. \]  
(3.45)

In particular, by the first inequality above, the balls \( \{B_{z_i+2\delta}(\varepsilon z_i)\}_{z_i \in K^\varepsilon} \) are all disjoint, and therefore
\[ \varepsilon^d \#(K^\varepsilon) \lesssim \varepsilon^d \sum_{z_i \in K^\varepsilon} \varepsilon^{-(d+1+2\delta)} |B_{z_i+2\delta}(\varepsilon z_i)| = \varepsilon^{-2\delta} \sum_{z_i \in I_{-3}^\varepsilon} |B_{z_i+2\delta}(\varepsilon z_i)|. \]  
(3.46)

In addition, we observe that by definition of \( K^\varepsilon \), for any \( z_i \in K^\varepsilon \) there exists \( z_j \in \bigcup_{k=-2}^{k_{\max}} J_k \) such that
\[ B_{z_j+\delta}(\varepsilon z_j) \cap B_{\theta \lambda_j \varepsilon^{d-\delta} \rho_j}(\varepsilon z_j) \neq \emptyset. \]  
(3.47)

Here we used \( K_{\varepsilon} \subseteq \tilde{I}_{-3} \) and (3.14) to rule out that \( z_j \in J_{-3} \subseteq \tilde{I}_{-3} \). In particular, (3.45) and (3.47) imply
\[ 2\varepsilon^{1+\delta/2} \leq |z_i - z_j| \leq 2\varepsilon^{1+\delta} + \theta \lambda_j \varepsilon^{d-\delta} \rho_j, \]
we obtain that \( \theta \lambda_j \varepsilon^{d-\delta} \rho_j \geq 2\varepsilon^{1+\delta} \). We combine this inequality with condition (3.47) to infer that
\[ B_{z_j+\delta}(\varepsilon z_j) \subseteq B_{2\theta \lambda_j \varepsilon^{d-\delta} \rho_j}(\varepsilon z_j) \]
and, by (3.46), to estimate
\[ \varepsilon^d \#(K^\varepsilon) \lesssim \varepsilon^{-2\delta} \sum_{z_j \in \bigcup_{k=-2}^{k_{\max}} J_k} |B_{2\theta \lambda_j \varepsilon^{d-\delta} \rho_j}(\varepsilon z_j)| \]
\[ \lesssim \varepsilon^{-2\delta} \left( \frac{\varepsilon^{d}}{\varepsilon^{d-\delta}} \max_{z_j \in \Phi^\varepsilon(D)} \frac{\varepsilon^{d}}{\varepsilon^{d-\delta} \rho_j} \right)^2 \sum_{z_j \in \bigcup_{k=-2}^{k_{\max}} J_k} (\varepsilon^{d-2} \rho_j)^{d-2} \]
\[ \lesssim \varepsilon^{2\delta} \sum_{z_j \in \Phi^\varepsilon(D)} (\varepsilon^{d-2} \rho_j)^{d-2}. \]

Thanks to Lemma C.1, the right-hand side vanishes almost surely in the limit \( \varepsilon \downarrow 0^+ \). This concludes the proof of (3.43).

The limit in the first line of (3.2) is a direct consequence of (3.43). Moreover, the second inequality in (3.1) follows from (3.43) and Lemma C.2.

To show (3.3), we resort to the definition of \( D^\varepsilon_k \) to estimate
\[ \{ z_i \in \Phi^\varepsilon(D)(\omega) : \text{dist}(z_i, D^\varepsilon_k) \leq \eta \varepsilon \} \]
\[ \subseteq \mathcal{I} \cup \left\{ z_i \in n^\varepsilon(\omega) : \text{dist} \left( z_i, \bigcup_{z_j \in \bigcup_{k=-2}^{k_{\max}} J_k} B_{\Lambda \varepsilon^{d-\delta} \rho_j}(\varepsilon z_j) \right) \leq \eta \varepsilon \right\} \]
\[ \cup \left\{ z_i \in n^\varepsilon(\omega) \cap \Phi^\varepsilon_{2\eta}(D)(\omega) : \text{dist} \left( z_i, \bigcup_{z_j \in J_{-3}} B_{\Lambda \varepsilon^{d-\delta} \rho_j}(\varepsilon z_j) \right) \leq \eta \varepsilon \right\} \]
\[ := I_{k}^\varepsilon \cup F^\varepsilon \cup C^\varepsilon. \]

We already know \( \varepsilon^d \#(I_{k}^\varepsilon) \to 0 \). Next, we argue that
\[ \varepsilon^d \#(F^\varepsilon) \to 0. \]

This follows by an argument similar to the one for (3.44): We may choose \( \varepsilon_0 = \varepsilon_0(d) \) such that for all \( \varepsilon \leq \varepsilon_0, \varepsilon^{d/2} \leq \eta \cdot \). By definition of \( J_k \) and of \( F^\varepsilon \) above, we infer that for such \( \varepsilon \leq \varepsilon_0 \), for all \( z_j \in F^\varepsilon \) there exists \(-2 \leq k \leq k_{\max} \) and \( z_i \in J_k \) such that
\[ B_{z_i+\delta}(\varepsilon z_j) \subseteq B_{2\eta \varepsilon^{d-\delta} \Lambda \rho_j}(\varepsilon z_i) \subseteq B_{2\eta \varepsilon^{d-2} \Lambda \rho_j}(\varepsilon z_i), \]  
(3.48)
where in the second inequality we use that $\varepsilon^{-2\delta} \eta \geq 1$ and $\varepsilon^{\frac{d}{2\rho_i}} \rho_i \geq \varepsilon^{1+2\delta}$. We note that by (3.45) the balls $B_{\varepsilon^{1+\delta/2}}(\varepsilon z_j)$ with $z_j \in n^c$ are all disjoint. Hence,

$$
\varepsilon^d \#(E^c) \lesssim \varepsilon^{-d\delta} \left| \bigcup_{z_i \in \bigcup_{k=2}^{k_{\max}} L_k} B_{2\Lambda \varepsilon^{-2\delta} \varepsilon^{\frac{d}{2\rho_i}}}(\varepsilon z_i) \right|
$$

$$
\lesssim \eta^d \varepsilon^{-d(\delta+2\delta)} \max_{z_j \in \Phi^c(D)} (\varepsilon^{\frac{d}{2\rho_j}} \rho_j)^2 \sum_{z_j \in \Phi^c(D)} (\varepsilon^{\frac{d}{2\rho_j}} \rho_j)^{-d-2}
$$

(3.12)

The right-hand side vanishes almost surely in the limit $\varepsilon \downarrow 0^+$ thanks to (1.7) and Lemma C.1.

We conclude the argument for (3.3) by showing that the set $C^c$ is empty when $\varepsilon$ is small: In fact, by construction, if $z_i \in n^c$ satisfies

$$
dist(\varepsilon z_i, \bigcup_{z_j \in J_{-3}} B_{\frac{\varepsilon}{\Lambda \varepsilon^{-2\delta} \rho_j}}(\varepsilon z_j)) \leq \eta \varepsilon,
$$

then there exists a $z_j \in J_{-3} \subseteq I_{-3}$ such that for $\varepsilon \leq \varepsilon_0$ with $\varepsilon^{2\delta} \leq \eta$

$$
\varepsilon |z_i - z_j| \leq \text{dist}(\varepsilon z_i, B_{\frac{\varepsilon}{\Lambda \varepsilon^{-2\delta} \rho_j}}(\varepsilon z_j)) + \Lambda \varepsilon^{1+2\delta} \leq 2\eta \varepsilon.
$$

This yields $C^c \subseteq \Phi^c(D) \setminus \Phi^c_{2\eta}(D)$ and thus that it is empty since by definition we also have $C^c \subseteq \Phi^c_{2\eta}(D)$. This finishes the proof of (3.3).

We hence have shown that $H^c_\delta, H^c_\delta$ and $D^c_\delta$ in Lemma 3.1 may be chosen as in Step 2 (see (3.4), (3.5), and (3.9)). We also remark that it immediately follows by (3.12) and the bounds on $\lambda^i \leq \Lambda$ obtained at the beginning of Step 2, that the radii $\lambda^i \varepsilon^{\frac{d}{2\rho_j}} \rho_j$ generating the balls of $H^c_\delta$ satisfy the second inequality in (3.5).

It remains to argue (3.7) and (3.8). The first property follows directly from (3.14) for $J_k \subseteq \tilde{I}_k$ and the choice of the parameters $\lambda = \theta \lambda^i$, and $\theta^4 = \alpha$.

We now turn to (3.8) and begin by showing that it suffices to prove the following:

**Claim:** For all $-3 \leq k < l \leq k_{\max}$ and every $z_k \in J_k, z_l \in \tilde{I}_l$ we have

$$
B_{\frac{\varepsilon}{\lambda^i \varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_l) \cap B_{\frac{\varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_k) = \emptyset.
$$

(3.49)

We first prove (3.54) provided this claim holds. To do so, for any $k < l$ and $z_l \in \tilde{I}_l$ we begin by denoting by $E^{z_l}_k$ the set

$$
E^{z_l}_k := B_{\frac{\varepsilon}{\lambda^i \varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_l) \setminus \bigcup_{m=k}^{l-1} \bigcup_{z_m \in J_m} B_{\frac{\varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_m).
$$

(3.50)

and arguing that

$$
B_{\frac{\varepsilon}{\lambda^i \varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_l) \subseteq E^{z_l}_k \subseteq E_k,
$$

(3.51)

$$
E_k = \bigcup_{l \geq k} \bigcup_{z_j \in J_l} E^{z_j}_k,
$$

(3.52)

where each union above is between disjoint sets.

By (3.33) for $E_{l-1}$ and (3.32) for $J_l$, we clearly have that

$$
B_{\frac{\varepsilon}{\lambda^i \varepsilon^{\frac{d}{2\rho_j}} \rho_j}}(\varepsilon z_l) \subseteq E_{l-1}.
$$

Note that, by construction, this ball is a connected component of the set $E_{l-1}$. From the previous inclusion, the second inclusion in (3.51) is an easy application of the recursive definition (3.33) of $E_k$. Similarly, (3.52) is an easy consequence of the definition (3.33) of the sets $E_k$. Furthermore, since each $J_m \subseteq \tilde{I}_m$, we apply claim (3.49) to $z_j$ and all $z_k \in J_m$ with $m \leq l - 1$, and conclude also the first
We are now ready to argue (3.51). We conclude that definition (3.50) immediately yields the monotonicity property $E_{k-1}^{z_1} \subseteq E^{z_1}_k$ for all $z_j \in J_l$ and $-3 \leq k \leq l$.

Equipped with (3.51)-(3.52), we now turn to (3.8): Let $z_0 \in I_{k_0}$ for some $-2 \leq k_0 \leq k_{\text{max}}$. By definition (3.39), there exists $z_1 \in J_{k_0}$ such that

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \subseteq B_{\lambda_1 \epsilon \frac{d}{d-z_1}} (z_1).$$  \hspace{1cm} (3.53)

By this, property (3.8) follows immediately if we prove that for any $l < k_0$ and all $z_3 \in J_l$ we have

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \cap B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) = \emptyset.$$  \hspace{1cm} (3.54)

Let $-3 \leq k_2 \leq k_{\text{max}}$ be minimal such that there exists $z_2 \in I_{k_2}$ with the property that

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \subseteq B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2).$$  \hspace{1cm} (3.55)

Note that, by (3.13), we may always find such $k_2$. If $k_0 \leq k_2$, we use the above claim (3.49) on $z_2 \in I_{k_2}$ and $z_3 \in J_l$ with $l < k_2$ and conclude (3.54). Let us now assume that $k_0 > k_2$: Since $z_0 \in I_{k_0}$, by definition (3.39) we have that $z_2 \notin I_{k_2}$. This implies by (3.32) that

$$B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) \subseteq E_{k_2+1}.$$  \hspace{1cm} (3.56)

In particular, by (3.55) and (3.50) there exists a $k_0 > k_2$ and $z_1 \in J_{k_0}$ such that

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \subseteq B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) \subseteq E_{k_2+1}^{z_1}.$$  \hspace{1cm} (3.57)

Moreover, by (3.50) and the assumption $k_2 < k_0$, we also have

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \subseteq E_{k_2+1}^{z_1} \subseteq E_{k_0}^{z_1}.$$  \hspace{1cm} (3.58)

On the other hand, by (3.53) also

$$B_{\epsilon \frac{d}{d-z_0}} (z_0) \subseteq B_{\lambda_1 \epsilon \frac{d}{d-z_1}} (z_1) = E_{k_0}^{z_1}.$$  \hspace{1cm} (3.59)

By combining the previous two inequalities and using that the sets $E_{k_0}^{z_1}, E_k^{z_1}$ are whenever $z_i \neq z_j \in J_l$, we conclude that $z_1 = z_3$. Thus, since $z_1 \in J_{k_0}$, definition (3.50) applied to $E_{k_2+1}^{z_1}$ yields that for all $k_2 < l < k_0$ we have for all $z_i \in J_l$

$$E_{k_2+1}^{z_1} \cap B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) = \emptyset.$$  \hspace{1cm} (3.60)

By using (3.56), the above inequality implies (3.54) with $z_1 = z_3$ and for all $k_2 < l < k_0$. To extend (3.54) also to the indices $l \leq k_2$ it suffices to observe that for $l < k_2$ we may argue as above in the case $k_0 \leq k_2$. Finally, if $l = k_2$, we obtain (3.54) by applying (3.55) and (3.14) to $z_2 \in I_{k_2}$ and $z_3 \in J_{k_2} \subseteq I_{k_2}$.

It remains to prove claim (3.49). Let $z_1 \in I_f^l$, $-2 \leq l \leq k_{\text{max}}$. We begin by arguing that

$$B_{\theta \lambda_1 \epsilon \frac{d}{d-z_1}} (z_1) \subseteq E_l.$$  \hspace{1cm} (3.57)

Indeed, if $z_1 \in J_l$, this follows immediately from the definition of $E_l$. If $z_1 \notin J_l$, then by (3.32) we have $B_{\lambda_1 \epsilon \frac{d}{d-z_1}} (z_1) \subseteq E_{l+1}$. We now use (3.14) on the family $J_l$ and definition (3.33) of $E_l$ to conclude (3.57). From (3.57) we may use again (3.14) to the families $J_l, J_{l-1}$ and also obtain that

$$B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) \subseteq E_{l-1}.$$  \hspace{1cm} (3.58)

We are now ready to argue (3.49) by contradiction: Let us assume that there exists a $k < l$ and $z_k \in J_k$ such that (3.49) fails, i.e.

$$B_{\lambda_1 \epsilon \frac{d}{d-z_1}} (z_1) \cap B_{\theta \lambda_2 \epsilon \frac{d}{d-z_2}} (z_2) \neq \emptyset.$$  \hspace{1cm} (3.59)

Then, again by (3.14) applied to $J_l$ and $J_{l-1}$, we necessarily have $k \leq l - 2$. Let us now assume that $z_k \in J_{l-2}$: Then by (3.32) we have

$$B_{\epsilon \frac{d}{d-z_k}} (z_k) \subseteq E_{l-1}.$$  \hspace{1cm} (3.60)
This, together with (3.58) for $z_l$ and (3.59) yields
\[
B_{\lambda_k \varepsilon \rho_k} (\varepsilon z_l) \cap \partial B_{\lambda_k \varepsilon \rho_k} (\varepsilon z_l) \neq \emptyset. \tag{3.61}
\]
For a general $k < l - 2$, we claim that we may iterate the previous argument and obtain that (3.59) implies the existence of an integer $m \leq 1 + \lceil \log_2 l \rceil$ and a collection $k_0, \ldots, k_m \leq l - 2$, such that $k = k_0$ and for all $0 \leq n \leq m - 1$ we have $k_n \leq k_{n+1} - 2$ and there exist $z_{k_n} \in J_{k_n}$ and $z_m \in J_{k_m}$ satisfying (see Figure 4)
\[
B_{\lambda_k \varepsilon \rho_k} (\varepsilon z_{k_n}) \cap \partial B_{\lambda_k \varepsilon \rho_k} (\varepsilon z_l) \neq \emptyset, \tag{3.62}
\]
Indeed, for $z_k \in J_k$ with $k < l - 2$, we know that by (3.32)
\[
B_{\varepsilon \lambda_k \varepsilon \rho_k} (\varepsilon z_k) \subseteq E_{k+1}. \tag{3.63}
\]
If also (3.61) is true, then we obtain (3.62) with $k_0 = k_m = k$. Let us assume, instead, that (3.61) does not hold and thus, by (3.59) that
\[
B_{\varepsilon \lambda_k \varepsilon \rho_k} (\varepsilon z_k) \subseteq B_{\varepsilon \lambda_k \varepsilon \rho_k} (\varepsilon z_l) \subseteq E_{l-1}. \tag{3.64}
\]
Then, by (3.63) and (3.33) there exists an index $k_1 \leq l - 2$ and $z_{k_1} \in J_{k_1}$ such that
\[
B_{\varepsilon \lambda_{k_1} \varepsilon \rho_{k_1}} (\varepsilon z_{k_1}) \cap \partial B_{\varepsilon \lambda_{k_1} \varepsilon \rho_{k_1}} (\varepsilon z_{k_1}) \neq \emptyset. \tag{3.65}
\]
Moreover, by (3.14), we necessarily have $k_1 \geq k + 2$. We thus recovered the second line in (3.62). Since $z_{k_1} \in J_{k_1}$, we use again (3.32) to infer that
\[
B_{\varepsilon \lambda_{k_1} \varepsilon \rho_{k_1}} (\varepsilon z_{k_1}) \subseteq E_{k_1+1}. \tag{3.66}
\]
Therefore, if $k_1 = l - 2$, we argue as in (3.60) and conclude that (3.61) is satisfied with $z_k$ substituted by $z_{k_1}$. This and (3.65) yield (3.62) with $m = 1$. Clearly, the same holds if $k_1 < l - 2$ but (3.61) nonetheless satisfied by $z_{k_1}$. Let us now assume, instead, that $z_{k_1}$ does not satisfy the first line in (3.62): By (3.65) and (3.64) this implies that
\[
B_{\varepsilon \lambda_{k_1} \varepsilon \rho_{k_1}} (\varepsilon z_{k_1}) \subseteq E_{k_1+1}. \tag{3.67}
\]
We may now argue as for (3.63) above and obtain the existence of a new index $k_2 \geq k_1 + 2$ satisfying (3.65) with $k$ and $k_1$ substituted by $k_1$ and $k_2$ respectively. By repeating the same argument above we iterate and conclude (3.62) for a general $m$. We remark that, since at each step $n$ the index $k_n$
increases of at least 2 this procedure necessarily stops whenever $k_n = l - 2$. In other words, we obtain (3.62) after at most $1 + \lceil \frac{k_{\text{max}}}{2} \rceil$ iterations. We thus established (3.62).

Equipped with (3.62) we finally argue (3.49): Since for all $0 \leq n \leq m \leq 1 + \lceil \frac{k_{\text{max}}}{2} \rceil$, $1 \leq \lambda_n \leq \Lambda$ and $k_0 \leq \cdots \leq k_m \leq l - 2$, we estimate

$$\varepsilon |z_l - z_k| \geq \varepsilon |z_l - z_{k_m}| - \sum_{n=1}^{m} \varepsilon |z_{k_n} - z_{k_{n-1}}| \geq (3.62) \theta \lambda_l \frac{d}{\varepsilon^2} \rho_l - (1 + 2m) \Lambda \varepsilon \frac{d}{\varepsilon^2} \rho_{k_m}$$

$$\theta \geq 1 \geq \lambda_l \frac{d}{\varepsilon^2} \rho_l + (\theta - 1) \varepsilon \frac{d}{\varepsilon^2} \rho_l - (k_{\text{max}} + 4) \Lambda \varepsilon \frac{d}{\varepsilon^2} \rho_{k_m}.$$ We now use the fact that since $z_l \in \tilde{I}_l$ and $z_{k_m} \in J_{k_n} \subseteq \tilde{I}_{k_m}$, we have by (3.11) and the assumptions on the indices $k_n$ that $\frac{\rho_{k_n}}{\rho_{k_m}} \geq \varepsilon^{-\delta}$. From this inequality it follows that

$$\varepsilon |z_l - z_k| \geq \lambda_k \frac{d}{\varepsilon^2} \rho_l + (\theta - 1) \varepsilon \frac{d}{\varepsilon^2} \rho_l - (k_{\text{max}} + 4) \Lambda \varepsilon \frac{d}{\varepsilon^2} \rho_{k_m}$$

and for $\varepsilon$ small enough we bound

$$\varepsilon |z_l - z_k| \geq \lambda_k \frac{d}{\varepsilon^2} \rho_l + 2 \lambda \frac{d}{\varepsilon^2} \rho_{k_m},$$

where $\lambda$ is the factor associated to $z_k$. We now observe that if $k_m = k_0 = k$, then the above inequality contradicts (3.59). If, otherwise $k = k_0 \neq k_m$, then by construction we have $k_0 \leq k_m - 2$ and thus by (3.11) that $\rho_k \leq \rho_{k_m}$. This and the above inequality contradict (3.59) also in this case. This proves claim (3.49) and establishes (3.8). The proof of Lemma 3.2 and Lemma 3.1 are complete.

4. PROOF OF LEMMA 2.4

**Proof of Lemma 2.4.** For a $\theta > 1$ fixed, let $H^\varepsilon = H^\varepsilon_{\tilde{b}} \cup H^\varepsilon_{\tilde{g}}$ and the sets $H^\varepsilon_{\tilde{b}}$, $D^\varepsilon_{\tilde{b}}$ be as introduced in Lemma 3.1 and Lemma 3.2. Throughout this proof, we use the notation $\lesssim$ for $\leq C$ with the constant depending on $d$, $\beta$, $\theta$.

**Step 1.** We recall that the set $D^\varepsilon_{\tilde{b}}$ is related to the partitioning of $H^\varepsilon = H^\varepsilon_{\tilde{b}} \cup H^\varepsilon_{\tilde{g}}$ and is such that $H^\varepsilon_{\tilde{b}} \subseteq H^\varepsilon_{\tilde{b}} \subseteq D^\varepsilon_{\tilde{b}}$. We construct $R_\varepsilon v$ by distinguishing between the parts of domain $D$ containing "small" holes (i.e. $H^\varepsilon_{\tilde{g}}$) and the ones containing the clusters of holes (i.e. $H^\varepsilon_{\tilde{b}}$). We set

$$R_\varepsilon v := \begin{cases} v^\varepsilon_{\tilde{b}} & \text{in } D^\varepsilon_{\tilde{b}} \\ v^\varepsilon_{\tilde{g}} & \text{in } D\setminus D^\varepsilon_{\tilde{b}}, \end{cases}$$

where the functions $v^\varepsilon_{\tilde{b}}$ and $v^\varepsilon_{\tilde{g}}$ satisfy

$$\begin{cases} v^\varepsilon_{\tilde{b}} = 0 & \text{in } H^\varepsilon_{\tilde{b}}, \\ \nabla \cdot v^\varepsilon_{\tilde{b}} = 0 & \text{in } D \setminus D^\varepsilon_{\tilde{b}}, \\ v^\varepsilon_{\tilde{g}} \in H^1_0(D) & \text{for } \varepsilon \text{ small enough and } v^\varepsilon_{\tilde{g}} \to v \text{ in } H^1_0(D), \\ \|v^\varepsilon_{\tilde{g}}\|_{C^0} \lesssim \|v\|_{C^0(D)}, \end{cases} \quad (4.2)$$

and

$$\begin{cases} v^\varepsilon_{\tilde{g}} = v & \text{in } D^\varepsilon_{\tilde{b}}, \\ v^\varepsilon_{\tilde{g}} = 0 & \text{in } H^\varepsilon_{\tilde{b}}, \\ v^\varepsilon_{\tilde{g}} \text{ satisfies properties (i) - (v) with } H^\varepsilon \text{ substituted by } H^\varepsilon_{\tilde{g}}. \end{cases} \quad (4.3)$$

In particular, this means

$$R_\varepsilon v = v^\varepsilon_{\tilde{b}} + v^\varepsilon_{\tilde{g}} - v. \quad (4.4)$$

Before constructing the functions $v^\varepsilon_{\tilde{g}}$ and $v^\varepsilon_{\tilde{b}}$, we argue that $R_\varepsilon v$ defined in (4.1) satisfies all the properties (i) - (v) enumerated in the lemma. Properties (i) and (ii) are immediately satisfied. We turn to properties (iii) and (iv). By (4.4), we rewrite

$$\|R_\varepsilon v - v\|_{L^p(\mathbb{R}^d)} = \|v^\varepsilon_{\tilde{g}} - v\|_{L^p(\mathbb{R}^d)} + \|v^\varepsilon_{\tilde{b}} - v\|_{L^p(D^\varepsilon_{\tilde{b}})}.$$
The first term on the right-hand side vanishes almost surely in the limit thanks to the second line of (4.3) (property (iv) for \(v^\varepsilon_g\)). We bound the second term by using Hölder’s inequality and the last estimate in (4.2):

\[
\|v^\varepsilon_b - v\|_{L^p(D^\varepsilon)}^p \leq \|v - v^\varepsilon_b\|_{C^0(D)} D^\varepsilon_b^p \lesssim \|v\|_{C^0(D)} D^\varepsilon_b,
\]

Thanks to (3.9), also this last line almost surely vanishes in the limit \(\varepsilon \downarrow 0^+\). Thus, almost surely the whole norm \(\|R^\varepsilon v - v\|_{L^p(\mathbb{R}^d)} \to 0\) when \(\varepsilon \downarrow 0^+\). This yields property (iv) for \(R^\varepsilon v\). To establish Property (iii) we use a similar argument to bound the \(L^2\)-norm of \(\nabla (R^\varepsilon v - v)\), this time using that by (4.2) the gradient \(\nabla (v^\varepsilon_b - v)\) converges strongly to zero in \(L^2(\mathbb{R}^d)\). Properties (i) - (iv) for \(R^\varepsilon v\) are hence established.

It remains to argue property (v): Let \(u_\varepsilon \in H^1_0(D_\varepsilon)\) be such that \(u_\varepsilon \rightharpoonup u\) in \(H^1(D)\) and \(\nabla \cdot u_\varepsilon = 0\) in \(D\). By (4.4), we have

\[
\int \nabla R^\varepsilon v \cdot \nabla u_\varepsilon = \int \nabla v^\varepsilon_b \cdot \nabla u_\varepsilon + \int \nabla (v^\varepsilon_b - v) \cdot \nabla u_\varepsilon.
\]

By (4.2) and the assumptions on \(u_\varepsilon\), the second integral on the right-hand side almost surely converges to zero in the limit \(\varepsilon \downarrow 0^+\). We treat the first integral term by observing that \(H^1_0(D^\varepsilon) \subseteq H^1_0(D \setminus \bar{H}^\varepsilon_g)\) and applying (4.3) (i.e. property (v) for \(v^\varepsilon_g\)). This implies property (v) for \(R^\varepsilon v\) and concludes the proof of the lemma provided we construct \(v^\varepsilon_g\) and \(v^\varepsilon_b\) as above.

**Step 2. Construction of \(v^\varepsilon_b\) satisfying (4.2).**

To construct \(v^\varepsilon_b\) on \(D^\varepsilon_b\), we exploit the construction of the covering \(\bar{H}^\varepsilon\) of Lemma 3.2, as sketched in Section 2.3. The main advantage in working with \(\bar{H}^\varepsilon\) instead of \(H^\varepsilon\) is related to the geometric properties satisfied by \(\bar{H}^\varepsilon\) which allow to define \(v^\varepsilon_b\) via a finite number of iterated Stokes problems on rescaled annuli.

Throughout this step, we skip the upper index \(\varepsilon\) and write \(v_b\) instead of \(v^\varepsilon_b\). Let \(J = \bigcup_{i=3}^{k_{max}} J_i\) be the sub-collection of the centres of the balls generating \(\bar{H}^\varepsilon\) in the proof of Lemma 3.2. For each \(z_j \in J\), we write

\[
\begin{align*}
R^\varepsilon_j &:= \lambda^\varepsilon_j \rho_j, \quad B_j := B_{\varepsilon^\frac{d-2}{2} R_j} (\varepsilon z_j), \\
B_{\theta,j} &:= B_{\varepsilon^\frac{d-2}{2} \theta R_j} (\varepsilon z_j), \quad A_j := B_{\theta,j} \setminus B_j,
\end{align*}
\]

with \(\lambda^\varepsilon_j \in [1, \Lambda]\) the factors defined in Lemma 3.2.

As a first step, we consider the set \(J_{k_{max}}\) and define the function \(v_0^\varepsilon\) on \(D\) as

\[
\begin{align*}
v_0^\varepsilon &= v \quad \text{in} \ D \setminus \bigcup_{z_j \in J_{k_{max}}} B_{\theta,j} \\
v_0^\varepsilon &= 0 \quad \text{in} \ B_j \quad \text{for all} \ z_j \in J_{k_{max}} \\
v_0^\varepsilon &= v^\varepsilon_j \quad \text{in} \ A_j \quad \text{for all} \ z_j \in J_{k_{max}},
\end{align*}
\]

where each \(v_0^\varepsilon\) solves

\[
\begin{cases}
-\Delta v_0^\varepsilon + \nabla p_0^\varepsilon = -\Delta v & \text{in} \ A_j, \\
\nabla \cdot v_0^\varepsilon = 0 & \text{in} \ A_j, \\
v_0^\varepsilon = 0 & \text{on} \ \partial B_j, \\
v_0^\varepsilon = v & \text{on} \ \partial B_{\theta,j}.
\end{cases}
\]

This is well-defined since \(\text{div} \ v = 0\). In particular, each function \(v_0^\varepsilon - v\) solves the first problem in (B.1) in \(A_j\), and we apply to it the estimates (B.2) with the choice \(R = \theta\) and after a rescaling by \(\varepsilon^\frac{d-2}{2} R_j\) and a translation of \(\varepsilon z_j\). This yields

\[
\begin{align*}
\|\nabla v_0^\varepsilon\|_{L^2(A_j)}^2 &\lesssim \left( \|\nabla v\|_{L^2(B_{\theta,j})}^2 + \frac{1}{(\varepsilon^\frac{d-2}{2} R_j)^2} \|v\|_{L^2(B_{\theta,j})}^2 \right) , \\
\|v_0^\varepsilon\|_{C^0(B_{\theta,j})} &\lesssim \|v\|_{C^0(B_{\theta,j})}.
\end{align*}
\]
We now use the definition (4.5) of $R_j$ to obtain
\[ \|\nabla v^0\|^2_{L^2(A_j)} \lesssim (\|\nabla v\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \lambda_j \rho_j^{d-2} \|v\|_{L^\infty}), \]
\[ \|v^0\|_{C^0(B_{\theta,j})} \lesssim \|v\|_{C^0(B_{\theta,j})}. \]
Note that thanks to (3.7) of Lemma 3.2, we have that $B_{\theta,i} \cap B_{\theta,j} = \emptyset$ for all $z_i \neq z_j \in J_{k_{\text{max}}}$ and $\lambda_i \leq \Lambda$ for all $z_i \in J$. Thus, this also implies by (4.6) that
\[ \|\nabla v^0\|^2_{L^2(D)} \lesssim \|\nabla v\|^2_{L^2(D)} + \varepsilon^d \sum_{z_j \in J_{k_{\text{max}}}} \rho_j^{d-2} \|v\|_{L^\infty(D)}, \]
\[ \|v^0\|_{C^0(D)} \lesssim \|v\|_{C^0(D)}. \]
Furthermore, since $v^0 - v$ is supported only in the balls $B_{\theta,j}$, the triangle inequality and (4.8) imply also that
\[ \|\nabla (v^0 - v)\|^2_{L^2(D)} \lesssim \|\nabla v\|^2_{L^2(D)} + \varepsilon^d \sum_{z_j \in J_{k_{\text{max}}}} \rho_j^{d-2} \|v\|_{L^\infty(D)}. \]
We observe also that, by using again the fact that by Lemma 3.2 all the balls $B_j$ are disjoint, the function $v^0$ vanishes on
\[ \bigcup_{z_j \in J_{k_{\text{max}}}} B_j \supseteq \bigcup_{z_j \in I_{k_{\text{max}}}} B_{\varepsilon \delta z_j} (\varepsilon z_j). \]
We now proceed iteratively and for $1 \leq i \leq k_{\text{max}} + 3$ we consider the subsets $J_{k_{\text{max}} - i} \subseteq J$. For each $i$ in the range above, let $v^i$ be defined as in (4.6) and (4.7), with $v^{i-1}$ instead of $v$ and the domains $B_j$ and $A_j$ generated by the elements $z_j \in J_{k_{\text{max}} - i}$. We now argue that at each step $i$ we have
\[ \|\nabla v^i\|^2_{L^2(D)} \lesssim \|\nabla v\|^2_{L^2(D)} + \varepsilon^d \sum_{z_j \in \bigcup_{k=0}^{i} J_{k_{\text{max}} - k}} \rho_j^{d-2} \|v\|_{L^\infty(D)}, \]
\[ \|v^i\|_{C^0(D)} \lesssim \|v\|_{C^0(D)}, \]
and
\[ v^i = 0 \quad \text{in} \quad \bigcup_{z_j \in \bigcup_{k=0}^{i} I_{k_{\text{max}} - k}} B_{\varepsilon \delta z_j} (\varepsilon z_j). \]
Moreover,
\[ \|\nabla (v^i - v)\|^2_{L^2(D)} \lesssim \sum_{z_j \in \bigcup_{k=0}^{i} J_{k_{\text{max}} - k}} \left( \|\nabla v\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{d-2} \|v\|_{L^\infty(D)} \right). \]
We prove the previous estimates by induction over $0 \leq i \leq k_{\text{max}} + 3$. It is easy to prove the estimates in (4.12) by induction: For $i = 0$, (4.9) is exactly (4.12). We now observe that at each step $i$ we may argue as for $v^0$ and obtain (4.9) with $v^0$, $v$ and $J_{k_{\text{max}}}$ substituted by $v^i$, $v^{i-1}$ and $J_{k_{\text{max}} - i}$, respectively. Therefore, if we now assume (4.12) holds at step $i - 1$, we only need to combine the analogue of (4.9) for $v^i$ with (4.12) for $v^{i-1}$. We now consider (4.13): For $i = 0$, this is implied immediately by (4.11). Let us now assume that (4.13) holds for $i - 1$. By definition of $v^i$ (cf. (4.7)), the function vanishes on
\[ \bigcup_{z_j \in J_{k_{\text{max}} - i}} B_j \supseteq \bigcup_{z_j \in I_{k_{\text{max}} - i}} B_{\varepsilon \delta z_j} (\varepsilon z_j) \]
and equals $v^{i-1}$ on $D \setminus \bigcup_{z_j \in J_{k_{\text{max}} - i}} B_{\theta,j}$. By the induction hypothesis (4.13) for $i - 1$, (4.13) for $i$ follows provided
\[ \left( \bigcup_{z_j \in J_{k_{\text{max}} - i}} B_{\theta,j} \right) \cap \left( \bigcup_{z_j \in I_{k_{\text{max}} - i}} B_{\varepsilon \delta z_j} (\varepsilon z_j) \right) = \emptyset. \]
By recalling the definitions of the balls $B_{\theta,j}$, this identity is a consequence of property (3.8) of Lemma 3.2. We established (4.13) and (4.12) for each $0 \leq i \leq k_{max} + 3$.

Finally, we turn to the claims in (4.14): For $i = 0$, both lines of (4.14) hold by construction and (4.10), respectively. If we now assume that (4.14) is true for $i - 1$, then $v^i$ is by construction equal to $v^{i-1}$ outside the set

$$\bigcup_{z_j \in J_{k_{max} - i}} B_{\theta,j}.$$ 

It now suffices to apply the induction hypothesis on $v^{i-1}$ to conclude the first statement in (4.14). In addition, by the triangle inequality we estimate

$$\|\nabla (v^i - v)\|^2_{L^2(D)} \leq \|\nabla (v^i - v^{i-1})\|^2_{L^2(D)} + \|\nabla (v^{i-1} - v)\|^2_{L^2(D)}.$$ 

We apply the induction hypothesis to the second term on the right-hand side above and get

$$\|\nabla (v^i - v)\|^2_{L^2(D)} \leq \|\nabla (v^i - v^{i-1})\|^2_{L^2(D)} + \sum_{z_j \in J_{k_{max} - i}} \left(\|\nabla v\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{-d-2} \|v\|^2_{L^\infty(D)}\right). \quad (4.15)$$

We now use the analogue of (4.8) with $v^0$ and $v$ substituted by $v^{i-1}$ and $v^i$ to infer that

$$\|\nabla (v^i - v^{i-1})\|^2_{L^2(D)} \leq \sum_{z_j \in J_{k_{max} - i}} \left(\|\nabla v^{i-1}\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{-d-2} \|v^{i-1}\|^2_{L^\infty(D)}\right),$$

and, by (4.12) for $v^{i-1}$, that

$$\|\nabla (v^i - v^{i-1})\|^2_{L^2(D)} \lesssim \sum_{z_j \in J_{k_{max} - i}} \left(\|\nabla v^{i-1}\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{-d-2} \|v^{i-1}\|^2_{L^\infty(D)}\right) \lesssim \sum_{z_j \in J_{k_{max} - i}} \|\nabla (v^{i-1} - v)\|^2_{L^2(B_{\theta,j})} + \sum_{z_j \in J_{k_{max} - i}} \left(\|\nabla v\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{-d-2} \|v\|^2_{L^\infty(D)}\right).$$

Since all $B_{\theta,j}$, $z_j \in J_{k_{max} - i}$, are disjoint, this implies that

$$\|\nabla (v^i - v^{i-1})\|^2_{L^2(D)} \lesssim \|\nabla (v^{i-1} - v)\|^2_{L^2(D)} + \sum_{z_j \in J_{k_{max} - i}} \left(\|\nabla v\|^2_{L^2(B_{\theta,j})} + \varepsilon^d \rho_j^{-d-2} \|v\|^2_{L^\infty(D)}\right).$$

We may apply the induction hypothesis on $v^{i-1}$ again and combine the above estimate with (4.15) to conclude (4.14) for $v^i$. The proof of (4.14) is complete.

Equipped with (4.12), (4.13) and (4.14), we finally set $v_0^i := v^{k_{max} + 3}$ and show that this choice fulfils all the conditions in (4.2): The first and the second line in (4.2) follow immediately by construction and the definition (3.9) of $D_0^\varepsilon$. The second estimate in (4.12) with $i = k_{max} + 3$ yields also the last inequality in (4.2). It thus only remain to show that, almost surely, $v_0^i \in H_0^1(D)$ for $\varepsilon$ small enough and $v_0^i \to v$ in $H_0^1(D)$.

To do this, we begin by showing that $\nabla (v_0^i - v) \to 0$ in $L^2(D)$. By (4.14) with $i = k_{max} + 3$ and the fact that $v \in C_0^\infty(D)$, we indeed obtain

$$\|\nabla (v_0^i - v)\|_{L^2(D)} \lesssim \|v\|_{C_0^1(D)} \sum_{z_j \in J} (\varepsilon_{\rho_j} + 1) \varepsilon^d \rho_j^{-d-2}.$$ 

We recall that the set $J$ depends on $\varepsilon$, i.e. $J = J^\varepsilon$. In addition, since $J \subseteq \mathcal{I}$ (cf. Lemma 3.2) and $n^\varepsilon = \Phi^\varepsilon(D) \setminus J^\varepsilon$, the limit in (3.2) of Lemma 3.1 yields that almost surely $\varepsilon^d \# J^\varepsilon \to 0$ when $\varepsilon \downarrow 0^+$. This, together with (3.5), (1.7) and the Strong Law of Large numbers (cf. Lemma (C.2) in the Appendix) implies that the right-hand side above almost surely vanishes in the limit $\varepsilon \downarrow 0^+$. Hence, we showed that $\nabla (v_0^i - v) \to 0$ in $L^2(\mathbb{R}^d)$. By Poincaré’s inequality, it now suffices to argue that almost surely and for $\varepsilon$ small enough $v_0^i \in H_0^1(D)$ to infer that $v_0^i \to v$ in $H_0^1(D)$ and thus conclude the proof of (4.2).

Let $K \in D$ be a compact set containing the support of $v$, and set $r = \text{dist}(K, D) > 0$. We show that, almost surely, $v_0^i \in H_0^1(D)$ for all $\varepsilon \leq \tilde{\varepsilon}$, with $\tilde{\varepsilon} = \varepsilon(r, \omega) > 0$. To do so, we fix any realization $\omega \in \Omega$ (which is independent from $v$) for which we have (3.12), and resort to the construction of $v_0^i$ via the
solutions $v^0, v^1, \ldots, v^{k_{\text{max}} + 3}$ obtained by iterating (4.7). We claim that for all $i = 0, \ldots, k_{\text{max}} + 3$ we have
\[
supp(v^i) =: K_i^\varepsilon \subseteq D, \quad \text{dist}(K_i^\varepsilon, D) \geq r - 2(i + 1)\theta \Lambda \varepsilon^{2d}, \tag{4.16}
\]
for all $\varepsilon$ such that the right-hand side in the last inequality is positive. Since $v_b^\varepsilon := v^{k_{\text{max}} + 3}$, we may choose $\varepsilon(r, \omega)$ such that $\varepsilon^{2d} \leq \frac{r}{4(k_{\text{max}} + 4)\theta \Lambda}$ and use the above estimate to infer that $v_b^\varepsilon$ is compactly supported in $D$ for all $\varepsilon \leq \varepsilon(r, \omega)$.

We prove (4.16) iteratively and begin with $i = 0$: By (4.7) and the assumption on the support of $v$, it follows that, if for $z_i \in J_{k_{\text{max}}}$ the ball $B_{\theta, i}$ does not intersect the support $K$ of $v$, then $v^0 = v \equiv 0$ on $B_{\theta, i}$. This, together with property (3.7) of Lemma 3.2, implies that
\[
supp(v^0) \subseteq K \bigcup_{\bar{z}_i \in J_{k_{\text{max}}}, B_{\theta, j} \cap K \neq \emptyset} B_{\theta, j}.
\]
By recalling that thanks to Lemma 3.2 each ball $B_{\theta, j}$ has radius
\[
\theta \varepsilon^{\frac{d}{d - 2}} \rho_j \leq \theta \varepsilon^{\frac{d}{d - 2}} \rho_i \leq \theta \varepsilon^{2d}, \tag{3.12}
\]
we observe that (4.17) yields estimate (4.16) for $v^0$. Let us now assume (4.16) for $v^i$. Then, since $v^{i+1}$ solves (4.7) with boundary datum $v_i$, we may argue as above to infer that
\[
K_{i+1}^\varepsilon \subseteq K_i^\varepsilon \bigcup_{\bar{z}_i \in J_{k_{\text{max}}}, B_{\theta, j} \cap K \neq \emptyset} B_{\theta, j}
\]
and thus that
\[
\text{dist}(K_{i+1}^\varepsilon, D) \geq \text{dist}(K_i^\varepsilon, D) - 2\theta \varepsilon^{2d}(4.16) \geq r - 2(i + 1)\theta \varepsilon^{2d}.
\]
This concludes the iterated estimate (4.16), which completes the proof of this step.

**Step 3. Construction of $v_g^\varepsilon$ satisfying (4.3).** We now turn to the remaining set $D \setminus D_b^\varepsilon$ and construct the vector field $v_g^\varepsilon$ in a way similar to [1][Subsection 2.3.2] and [6].

For every $z_i \in n^\varepsilon$, we write
\[
a_{\varepsilon, i} := \varepsilon^{\frac{d}{d - 2}} \rho_i, \quad d_i := \min \left\{ \text{dist}(\varepsilon z_i, D_b^\varepsilon), \frac{1}{2} \min_{z_j \in n^\varepsilon, \bar{z}_j \neq z_i} (\varepsilon |z_i - z_j|), \varepsilon \right\} \tag{4.18}
\]
and
\[
T_i = B_{a_{\varepsilon, i}}(\varepsilon z_i), \quad B_i := B_{d_i}(\varepsilon z_i), \quad B_{2,i} := B_{d_i}(\varepsilon z_i), \quad C_i := B_i \setminus T_i, \quad D_i := B_{2,i} \setminus B_i.
\]
We remark that, since $z_i \in n^\varepsilon$, Lemma 3.1 implies that for $\delta > 0$
\[
a_{\varepsilon, i} \leq \varepsilon^{1 + 2\delta}, \quad d_i \geq \varepsilon^{1 + \delta}, \tag{4.19}
\]
and that all the balls $B_{2,i}$ are pairwise disjoint.

For each $z_i \in n^\varepsilon$, we define the function $v_g^\varepsilon$ in $B_{2,i}$ in the following way:
\[
\begin{cases}
v_g^\varepsilon = 0 & \text{in } T_i \\
v_g^\varepsilon = v - \tilde{v}_i^\varepsilon & \text{in } C_i,
\end{cases}
\]
where $\tilde{v}_i^\varepsilon$ solves
\[
\begin{cases}
-\Delta \tilde{v}_i^\varepsilon + \nabla \pi_i^\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus T_i \\
\nabla \cdot \tilde{v}_i^\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus B_1 \\
\tilde{v}_i^\varepsilon = v & \text{on } \partial T_i \\
\tilde{v}_i^\varepsilon \to 0 & \text{for } |x| \to +\infty.
\end{cases} \tag{4.20}
\]
Finally, we require that on $D_i$, $v^\varepsilon_g$ solves
\[
\begin{cases}
-\Delta v^\varepsilon_g + \nabla q^\varepsilon_g = \Delta v & \text{in } D_i \\
\nabla \cdot v^\varepsilon_g = 0 & \text{in } D_i \\
v^\varepsilon_g = v & \text{on } \partial B_{2,i} \\
v^\varepsilon_g = v - \bar{v}_i^\varepsilon & \text{on } \partial B_i,
\end{cases}
\] (4.21)
and we then extend $v^\varepsilon_g$ by $v$ on $\mathbb{R}^d \setminus \bigcup_{z_i \in \mathbb{N}^d} B_{2,i}$. By Lemma 3.1 and the definition (4.18) of $d_i$, we have that $D^\varepsilon_i \subseteq \mathbb{R}^d \setminus \bigcup_{z_i \in \mathbb{N}^d} B_{2,i}$. Therefore, this definition of $v^\varepsilon_g$ satisfies the first line of (4.3) and property (i) with $H^\varepsilon$ substituted by $H^\varepsilon_g$. It is immediate that by construction $\nabla \cdot v^\varepsilon_g = 0$ in $D$, i.e. $v^\varepsilon_g$ satisfies also property (ii).

We observe that by uniqueness of the solution to (4.20), we may rescale the domains $C_i$ and rewrite
\[
v^\varepsilon_g = v - \phi^\varepsilon_g,i \left( \frac{-\varepsilon z_i}{a_{i,e}} \right) \text{ in } C_i,
\]
(4.22)
with $\phi^\varepsilon_g,i$ solving the second system in (B.1) in the annulus $\mathbb{R}^d \setminus B_1$ and with boundary datum $\psi(x) = v(a_{i,e} x - \varepsilon z_i)$. Similarly, by uniqueness of the solutions to (4.21) we may rescale the domains $D_i$ and write
\[
v^\varepsilon_g = v - \phi^\varepsilon_i \left( \frac{-\varepsilon z_i}{d_i} \right) \text{ in } D_i,
\]
(4.23)
with $\phi^\varepsilon_i$ solving the first system in (B.1) in the annulus $B_2 \setminus B_1$ and with boundary datum $\psi(x) = \phi^\varepsilon(\frac{d_i(a_{i,e} x - \varepsilon z_i)}{a_{i,e}})$.

We now turn to properties (iii) and (iv) for $v^\varepsilon_g$: We write
\[
\|v^\varepsilon_g - v\|_{L^p(\mathbb{R}^d)} = \sum_{z_i \in \mathbb{N}^d} \|v^\varepsilon_g - v\|_{L^p(B_{2,i})},
\]
(4.24)
\[
\|
abla (v^\varepsilon_g - v)\|_{L^2(\mathbb{R}^d)} = \sum_{z_i \in \mathbb{N}^d} \|\nabla (v^\varepsilon_g - v)\|_{L^2(B_{2,i})},
\]
and, since $B_{2,i} = D_i \cup C_i \cup T_i$, we may further split each norm on the right hand side into the contributions on each set $D_i$, $C_i$ and $T_i$. We begin by focussing on the domains $D_i$: By (4.23), we apply (B.2) to $\phi^\varepsilon_i$ and infer that
\[
\|
abla (v^\varepsilon_g - v)\|_{L^2(D_i)} \lesssim \|\nabla \phi^\varepsilon_i\|_{L^2(D_i)}^2 + a_{i,e}^{-2} \|v^\varepsilon_i\|_{L^2(D_i)}^2,
\]
(4.25)
\[
\|v^\varepsilon_g - v\|_{C^0(D_i)} \lesssim \|v^\varepsilon_i\|_{C^0(\partial B_{d_i a_{i,e}^{-1}})}.
\]
By using (4.22) and changing variables, we rewrite the second line above as
\[
\|v^\varepsilon_g - v\|_{C^0(B_{2,i})} \lesssim \|\phi^\varepsilon(\frac{d_i(a_{i,e} x - \varepsilon z_i)}{a_{i,e}})\|_{C^0(\partial B_{d_i a_{i,e}^{-1}})}.
\]
and use (B.4) on $\phi^\varepsilon_i$ to infer that
\[
\|v^\varepsilon_g - v\|_{C^0(B_1)} \lesssim \|v\|_{C^0} \left( \frac{a_{i,e}}{d_i} \right)^{d-2} \lesssim \|v\|_{C^0} \varepsilon^{d(d-2)}.
\]
In particular,
\[
\|v^\varepsilon_g - v\|_{L^p(D_i)} \lesssim a_{i,e} \|v\|_{C^0} \varepsilon^{d(d-2)} \lesssim \|v\|_{C^0} \varepsilon^{d+\delta(d-2)}. \tag{4.26}
\]
We now turn to the first inequality in (4.25), use (4.22) on the right-hand-side, and change variables to estimate
\[
\|
abla (v^\varepsilon_g - v)\|_{L^2(D_i)} \lesssim a_{i,e}^{d-2} \|\nabla \phi^\varepsilon_i\|_{L^2(B_{d_i a_{i,e}^{-1}} \setminus B_\frac{1}{2} d_i a_{i,e}^{-1})}^2 + a_{i,e} d_i^{-2} \|\phi^\varepsilon_i\|_{L^2(B_{d_i a_{i,e}^{-1}} \setminus B_\frac{1}{2} d_i a_{i,e}^{-1})}^2 \tag{B.5}
\]
\[
\lesssim \|v\|_{C^0} \varepsilon^{d+\delta(d-2)} \lesssim \|v\|_{C^0} \varepsilon^{d+\delta(d-2)} \rho_i^{d-2}.
\]
(4.27)
We consider the sets $C_i$: We use the definition (4.22) for $v^\varepsilon_g$ on $C_i$ and a change of variables to rewrite
\[
\|
abla (v^\varepsilon_g - v)\|_{L^2(C_i)} = a_{i,e}^{d-2} \|\nabla \phi^\varepsilon_i\|_{L^2(B_\frac{1}{2} d_i a_{i,e}^{-1} \setminus B_1)}^2,
\]
\[
\lesssim \|v\|_{C^0} \varepsilon^{d+\delta(d-2)} \rho_i^{d-2}.
\]
Hence, using (B.3) for $\phi_{\varepsilon,i}^{\varepsilon}$, we obtain
\[
\|\nabla(v_g^\varepsilon - v)\|_{L^2(C_1)}^2 \lesssim \|\nabla v\|_{L^2(B_{2a\rho_i}(\varepsilon z_i))}^2 + a_{\varepsilon,i}^2 \|v\|_{L^2(B_{2a\rho_i}(\varepsilon z_i))}^2 \lesssim a_{\varepsilon,i}^d \|v\|_{L^2(C_1)}^2 = \varepsilon^d \rho_i^{d-2} \|v\|_{L^2(C_1)}^2.
\]
(4.28)
Similarly, by (4.22) and a change of variables, for each $2 \leq p < +\infty$ we have
\[
\|v_g^{\varepsilon,i} - v\|_{L^p(C_1)}^p = a_{\varepsilon,i}^d \|\phi_{\varepsilon,i}^{\varepsilon} \|_{L^p(B_{2a\rho_i}(\varepsilon z_i) \setminus B_1)}^p,
\]
and, thanks to the pointwise estimate (B.4) for $\phi_{\varepsilon,i}^{\varepsilon}$, we have that for all $p > \frac{d}{d-2}$
\[
\|v_g^{\varepsilon} - v\|_{L^p(C_1)}^p \lesssim \|v\|_{C^{(\varepsilon,2+4\delta) \rho_i^{d-2}}}
\]
(4.19)
(4.29)
We finally turn to $T_i$, on which we easily bound
\[
\|\nabla(v_g^{\varepsilon})\|_{L^2(T_i)}^2 = \|\nabla v\|_{L^2(T_i)}^2 \lesssim \|v\|_{C^1}^2 \rho_i^{d-2},
\]
(4.30)
and for all $p > \frac{d}{d-2}$
\[
\|v_g^{\varepsilon} - v\|_{L^p(T_i)}^p \lesssim \|v\|_{C^{(\varepsilon,2+4\delta) \rho_i^{d-2} + \delta(p(d-2))}}
\]
(4.31)
We insert these estimates in (4.24) and apply (1.7) and the Strong Law of Large Numbers on the right-hand sides to conclude that almost surely
\[
\|\nabla v_g^{\varepsilon}\|_{L^2(D)} \lesssim 1
\]
and that $v_g^{\varepsilon} \rightarrow v$ in $L^p(D)$ for $p > \frac{d}{d-2}$. Since $v, v_g^{\varepsilon}$ are supported in the bounded domain $D$ for $\varepsilon$ small enough, we conclude properties (iii) and (iv) for $v_g^{\varepsilon}$.

We finally turn to property (v). We use an argument very similar to the one for Lemma 3.1 of [10]. For any $N \in \mathbb{N}$ fixed and all $z_i \in n^\varepsilon$, let us define
\[
n_N^\varepsilon := \left\{ z_i \in n^\varepsilon : \text{dist}(z_i, N) \geq \frac{\varepsilon}{N} \right\},
\]
where $Q \subseteq \mathbb{R}^d$ is a unit cube. Moreover, let $\mathcal{R}^N := \{z_i \in n^\varepsilon\}$ be the truncated environment given by $\rho_i^N := \rho_i \wedge N$ and let $H^{\varepsilon,N}_g$ be the set of holes generated by $n_N^\varepsilon$ with $\mathcal{R}^N$. Let $v_g^{\varepsilon,N}$ be the analogues of $v_g^{\varepsilon}$ for $H^{\varepsilon,N}_g$. We begin by showing that $v_g^{\varepsilon,N}$ satisfy property (v) on $H^{\varepsilon,N}_g$ with
\[
\mu^N = C_d(\rho^N)^{d-2}(\#(N_2^\varepsilon(Q))),
\]
where $Q$ is a unit ball and $N_2^\varepsilon$ is defined in Subsection (2.1).

Before showing this, we argue how to conclude also property (v) for $v_g^{\varepsilon}$: Let $u_e \in H^1_0(D_\varepsilon)$ such that $u_e \rightarrow u$ in $H^1(D)$. For each $N \in \mathbb{N}$ fixed we bound
\[
\limsup_{\varepsilon \downarrow 0^+} \left| \nabla v_g^{\varepsilon} \cdot \nabla u_e - \left( \int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \leq \limsup_{\varepsilon \downarrow 0^+} \left| \nabla v_g^{\varepsilon,N} \cdot \nabla u_e - \left( \int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| + \limsup_{\varepsilon \downarrow 0^+} \left| \nabla (v_g^{\varepsilon} - v_g^{\varepsilon,N}) \cdot \nabla u_e \right|.
\]
Since $H^{\varepsilon,N}_g \subseteq H^{\varepsilon}_g$, property (v) for $v_g^{\varepsilon,N}$ yields
\[
\limsup_{\varepsilon \downarrow 0^+} \left| \nabla v_g^{\varepsilon} \cdot \nabla u_e - \left( \int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \leq \int v \cdot (\mu - \mu^N) u + \limsup_{\varepsilon \downarrow 0^+} \left| \nabla (v_g^{\varepsilon} - v_g^{\varepsilon,N}) \cdot \nabla u_e \right|.
\]
(4.32)
We now appeal to the explicit construction of the functions \( v_g^\varepsilon, v_g^\varepsilon,N \) to observe that
\[
\text{supp}(v_g^\varepsilon - v_g^\varepsilon,N) \subseteq \bigcup_{z_i \in n_N^\varepsilon} B_{2,i} \cup \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{2,i},
\]
\[
v_g^\varepsilon - v_g^\varepsilon,N = v_g^\varepsilon \quad \text{in} \quad \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{2,i}.
\]
Therefore,
\[
\|\nabla(v_g^\varepsilon - v_g^\varepsilon,N)\|_{L^2(D)}^2 \leq \sum_{z_i \in n_N^\varepsilon} \|\nabla(v_g^\varepsilon - v_g^\varepsilon,N)\|_{L^2(B_{2,i})}^2 + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \|\nabla v_g^\varepsilon\|_{L^2(B_{2,i})}^2.
\]
We smuggle in the norms on the right-hand side the function \( v \) and appeal to (4.31) for \( v_g^\varepsilon \) (and the analogue for \( v_g^\varepsilon,N \)) to get that
\[
\|\nabla(v_g^\varepsilon - v_g^\varepsilon,N)\|_{L^2(D)}^2 \lesssim \|v\|_{C^1(D)}^d \left( \sum_{z_i \in n^\varepsilon} \rho_i^{d-2} 1_{\rho_i \geq N} + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} (1 + \rho_i^{d-2}) \right).
\]
Assumption (1.7) and the Strong Law of the Large Numbers yield that almost surely
\[
\sum_{z_i \in n^\varepsilon} \rho_i^{d-2} 1_{\rho_i \geq N} \to \langle \rho 1_{\rho \geq N} \rangle.
\]
Moreover, by (3.2) and (3.3) of Lemma 3.1, and (C.3) of Lemma C.1, we have that almost surely
\[
\lim_{N \uparrow + \infty, \varepsilon \downarrow 0^+} \lim_{\varepsilon^d \#(n^\varepsilon \setminus n_N^\varepsilon) = 0.}
\]
This yields by Lemma C.2 that
\[
\lim_{N \uparrow + \infty, \varepsilon \downarrow 0^+} \lim \|\nabla(v_g^\varepsilon - v_g^\varepsilon,N)\|_{L^2(D)} = 0.
\]
Since \( \nabla u_\varepsilon \) is uniformly bounded in \( L^2(D) \), we can insert this in (4.32) to conclude
\[
\limsup_{\varepsilon \downarrow 0^+} \left| \int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon - \left( \int \nabla v \cdot \nabla u + \int v \cdot \mu u \right) \right| \lesssim \limsup_{N \uparrow + \infty} \left| \int v \cdot (\mu - \mu^N)u \right|.
\]
By using again assumption (1.7) and (4.33) we infer that the right-hand side above vanishes almost surely and conclude property (v) for \( v_g^\varepsilon \) with \( \mu \) as in Theorem 2.1.

We now turn to property (v) for \( v_g^\varepsilon,N \). When no ambiguity occurs, we drop the upper index \( N \). For every \( u_\varepsilon \) as above, we split the integral
\[
\int \nabla v_g^\varepsilon \cdot \nabla u_\varepsilon = \int \nabla v \cdot \nabla u_\varepsilon - \int \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon.
\]
The first term converges to \( \int \nabla v \cdot \nabla u \) by the assumption on the sequence \( u_\varepsilon \). To conclude property (v) it thus remains to argue that
\[
\int \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon \to \int v \cdot \mu^N u.
\]
To prove this, we recall the construction of \( v_g^\varepsilon \), and we split the integral into
\[
\int \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{T_i} \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \int_{D_i} \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon.
\]
Note that the integral on each \( T_i \) vanishes by the assumption \( u_\varepsilon \in H_0^1(D^\varepsilon) \). We first focus on the second sum on the right-hand side above and use Cauchy-Schwarz and (4.27) to bound
\[
\sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \int_{D_i} \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon \lesssim \|\nabla u_\varepsilon\|_{L^2(D)} \left( \varepsilon^{d-\delta(d-2)} \sum_{z_i \in n^\varepsilon} \rho_i^{d-2} \right)^{\frac{1}{2}} \|v\|_{C^\infty}.
\]
By the assumption on the weak convergence for the sequence \( \nabla u_\varepsilon \) and the Strong Law of Large Numbers, the right-hand side almost surely vanishes in the limit \( \varepsilon \downarrow 0^+ \). Thus,
\[
\int \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^\varepsilon} \int_{C_i} \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon + o(1).
\]
We turn to the remaining term above: For each $z_i \in n^e$, let $(\bar{\phi}_{\infty}^{z_i}, \bar{\pi}_{\infty}^{z_i})$ solve the Stokes problem (B.1) in the exterior domain $\mathbb{R}^d \setminus B_1$ and with constant boundary datum $\nu(\varepsilon z_i)$. We define

$$\bar{\phi}_{\infty} = \bar{\phi}_{\infty}(\varepsilon z_i)_{a_{z_i,i}}, \quad \bar{\pi}_{\infty} = a_{z_i,i} \pi_{\infty}(\varepsilon z_i)_{a_{z_i,i}},$$

and smuggle these functions in each one of the integrals over $C_i$. This yields

$$\sum_{z_i \in n^e} \int_{C_i} \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^e} \int_{C_i} \nabla (v_g^\varepsilon - \bar{\phi}_{\infty}^{z_i}) \cdot \nabla u_\varepsilon + \sum_{z_i \in n^e} \int_{C_i} \nabla (\bar{\phi}_{\infty}^{z_i}) \cdot \nabla u_\varepsilon. \quad (4.37)$$

We claim that the first integral on the right-hand side vanishes in the limit $\varepsilon \downarrow 0^+$: By (4.22) and (4.36), each difference $v_g^\varepsilon - \bar{\phi}_{\infty}^{z_i}$ solves the second system in (B.1) in $\mathbb{R}^d \setminus T_i$ with boundary datum $\psi = v - v(\varepsilon z_i)$. Therefore, by the first inequality in (B.3),

$$\|\nabla (v_g^\varepsilon - \bar{\phi}_{\infty}^{z_i})\|_2^2(C_i) \lesssim \|\nabla v\|_2^2(B_{2a_{z_i,i}}(\varepsilon z_i) \cap T_i) + a_{z_i,i}^2 \|v - v(\varepsilon z_i)\|_2^2(B_{2a_{z_i,i}}(\varepsilon z_i) \setminus T_i).$$

As the vector field $v$ is smooth, we use a Lipschitz estimate on the last term, and conclude that

$$\|\nabla (v_g^\varepsilon - \bar{\phi}_{\infty}^{z_i})\|_2^2 \leq \|v\|_C^2 a_{z_i,i}^4 \|\nabla u_\varepsilon\|_2^2 \lesssim \|v\|_C^2 \varepsilon^{2+4\delta} \varepsilon^{d-2} \rho_i^2.$$ \hspace{1cm} (4.19)

By Cauchy-Schwarz inequality and this last estimate we find

$$\sum_{z_i \in n^e} \int_{C_i} \nabla (v_g^\varepsilon - \bar{\phi}_{\infty}^{z_i}) \cdot \nabla u_\varepsilon \leq \|\nabla u_\varepsilon\|_2^2 \left( \varepsilon^{2+4\delta} \sum_{z_i \in n^e} \rho_i^{d-2} \right),$$

and use the the Strong Law of Large Numbers to conclude that almost surely the above right-hand side vanishes. This, together with (4.37) and (4.35), yields

$$\int \nabla (v_g^\varepsilon - v) \cdot \nabla u_\varepsilon = \sum_{z_i \in n^e} \int_{C_i} \nabla \bar{\phi}_{\infty}^{z_i} \cdot \nabla u_\varepsilon + o(1). \quad (4.38)$$

We now integrate the first integral on the right-hand side above by parts and, since $u_\varepsilon$ vanishes in $T_i$, we obtain

$$\int_{C_i} \nabla \bar{\phi}_{\infty}^{z_i} \cdot \nabla u_\varepsilon = - \sum_{z_i \in n^e} \int_{C_i} \Delta \bar{\phi}_{\infty}^{z_i} u_\varepsilon + \int_{\partial B_i} \partial_n \bar{\phi}_{\infty}^{z_i} u_\varepsilon,$$

where $\nu$ denotes the outer unit normal. By using (4.36), the equation (B.1) for $(\bar{\phi}_{\infty}^{z_i}, \bar{\pi}_{\infty}^{z_i})$ and the fact that $\nabla \cdot u_\varepsilon = 0$ in $D$, we obtain

$$\int_{C_i} \nabla \bar{\phi}_{\infty}^{z_i} \cdot \nabla u_\varepsilon = \sum_{z_i \in n^e} \int_{\partial B_i} (\partial_n \bar{\phi}_{\infty}^{z_i} - \bar{\pi}_{\infty}^{z_i} \nu) \cdot u_\varepsilon.$$ \hspace{1cm} (4.39)

By wrapping this up with (4.38), we conclude that to show (4.34) it suffices to prove that

$$\sum_{z_i \in n^e} \int_{\partial B_i} (\partial_n \bar{\phi}_{\infty}^{z_i} - \bar{\pi}_{\infty}^{z_i} \nu) \cdot u_\varepsilon \to \int v \cdot \mu^N u.$$

We establish (4.39) as in [1]: We remark, indeed, that by the uniqueness of the solutions in (B.1), for each $z_i \in n^e$, we have

$$\bar{\phi}_{\infty}^{z_i} = \frac{d}{k=1} v_k(\varepsilon z_i) w_k^\varepsilon, \quad \bar{\pi}_{\infty}^{z_i} = \frac{d}{k=1} v_k(\varepsilon z_i) q_k^\varepsilon,$$

with $(w_k^\varepsilon, q_k^\varepsilon)$ the analogues of the oscillating test functions constructed in [1][Proposition 2.1.4]. We remark that the only difference is that in this setting, the scales $a_{z,i}$ (i.e. the size of the holes $T_i$) depend on the index $z_i$ and are not constant but bounded by $N$ (we recall that we are considering the truncated environment $\mathcal{R}^N$). Therefore, by arguing as in Lemma 2.3.7 of [1] we use Lemma 2.3.5 of [1] and linearity to rewrite

$$\sum_{z_i \in n^e} \int_{\partial B_i} (\partial_n \bar{\phi}_{\infty}^{z_i} - \bar{\pi}_{\infty}^{z_i} \nu) u_\varepsilon = \left( \mu^N \varepsilon, u_\varepsilon \right)_{H^{-1}, H^1} + r_\varepsilon,$$

with

$$\mu^N = \frac{C_d}{|B_1|} \sum_{z_i \in n^e} v(\varepsilon z_i) (\rho_i^{d-2} (2\varepsilon)^d d_i^4 1_{B_i}, \quad r_\varepsilon \to 0 \text{ in } H^{-1}(D).$$
Since \( v \in C_0^\infty(D) \) and the radii \( \rho_i^N \) are uniformly bounded, we can also replace \( \mu_i^N \) by
\[
\tilde{\mu}_i^N = \frac{C_d}{|B_1|} \sum_{z_i \in u^\varepsilon} (\rho_i^N)^{d-2} (2\varepsilon)^d \mathbf{1}_{B_i,v}.
\]
To establish (4.39), it remains to argue as in [10][Lemma 3.1, case (b)] (see from formula (4.75) on) and appeal to Lemma C.3 in [10]. This yields property (v) for \( \tilde{v}_\varepsilon \) and thus completes the proof of this step and of the whole lemma. \( \square \)

5. Estimates for the pressure (Proof of Theorem 2.3)

We begin this section by defining the set \( E^\varepsilon \) appearing in the statement of Theorem 2.3. In order to do so, we recall and introduce some notation. In order to keep the notation simpler we again often omit the index \( \varepsilon \) when no ambiguity occurs. From Lemma 3.1 and Lemma 3.2, we recall the definition of the index sets \( n^\varepsilon \) and \( J \) and the factors \( \lambda_j, j \in J \). We use the notation
\[
B_j = B_{\varepsilon\rho_j}(\varepsilon z_j), \quad B_{j,\theta} = B_{\theta\varepsilon\rho_j}(\varepsilon z_j) \quad \text{for } j \in n^\varepsilon
\]
\[
B_j = B_{\lambda_j\varepsilon\rho_j}(\varepsilon z_j), \quad B_{j,\theta} = B_{\theta\lambda_j\varepsilon\rho_j}(\varepsilon z_j) \quad \text{for } j \in J,
\]
and we denote \( A_j = B_{j,\theta} \setminus B_j \).

Moreover, we recall the definition of the set \( E_l \) for \(-3 \leq l \leq k_{\text{max}} + 1\) from the proof of Lemma 3.2:
\[
E_{k_{\text{max}} + 1} = \emptyset,
\]
\[
E_{l-1} := \left( E_l \setminus \bigcup_{z_j \in J_{l-1}} A_j \right) \cup \bigcup_{z_j \in J_{l-1}} B_j.
\]

We now define
\[
E^\varepsilon := E_{-3} \cup H^\varepsilon,
\]
where \( H^\varepsilon \) denotes the set of “good” holes as in Lemma 3.1. We remark that \( E^\varepsilon \) is precisely the set where the operator \( R_\varepsilon \) from Lemma 5.2 truncates to zero, i.e. \( R_\varepsilon v = 0 \) in \( E^\varepsilon \) for all \( v \in C_0^\infty(D) \), and \( E^\varepsilon \) is the largest set with this property.

For the proof of Theorem 2.3, we will rely on some properties of the set \( E_{-3} \) that follow from the explicit construction in the proof of Lemma 3.2. We collect them in the following Lemma.

**Lemma 5.1.** For \( j \in J \), let \( E^\varepsilon_j \) be the connected component of \( E_{-3} \) which contains \( \varepsilon z_j \). Then,
\[
E_{-3} = \bigcup_{j \in J} E^\varepsilon_j.
\]

Moreover, for \( j \in J_k \), let \( \tilde{E}^\varepsilon_j = E^\varepsilon_j \setminus E_{k+1} \). Then, \( E^\varepsilon_j \supset B_{\varepsilon\rho_j}(\varepsilon z_j) \) and
\[
|\tilde{E}^\varepsilon_j| \geq |B_{\varepsilon\rho_j}(\varepsilon z_j)|.
\]

Furthermore, there exists \( N_1 \in N_0 \) and \( z_{in} \in \cup_{l=-3}^{k-2} J_l, 1 \leq n \leq N_1 \) such that
\[
E^\varepsilon_j = B_j \setminus \left( \bigcup_{n=1}^{N_1} E^{\varepsilon n} \right),
\]
and there exists \( N_2 \in N_0 \) and \( z_{jn} \in \cup_{l=-3}^{k-2} J_l, 1 \leq n \leq N_2 \) such that
\[
A_j \cap E_{k+1} \cap E = \bigcup_{n=1}^{N_2} E^{\varepsilon n} \cap A_j \cap E_{k+1}.
\]

**Proof.** As mentioned above, the proof of this lemma follows from the construction in the proof of Lemma 3.2. First of all, the sets \( E^\varepsilon_j \) have been defined in that proof after (3.49). Moreover, (5.3) is a direct consequence of (3.52), and (5.5) follows from (3.50).

We turn to the proof of (5.6): Since by construction of \( E_k \) and \( D^\varepsilon_b, E_{k+1} \subseteq D^\varepsilon_b \), (3.1) implies \( E_{k+1} \cap E = E_{k+1} \cap E_{-3} \). Moreover, by (5.1), \( E_k \cap A_j = \emptyset \). Hence,
\[
A_j \cap E_{-3} \subseteq A_j \cap \left( E_k \cup \bigcup_{l<k, z_j \in J_l} E^\varepsilon_j \right) = A_j \cap \bigcup_{l<k, z_j \in J_l} E^\varepsilon_j
\]
This implies (5.6).

It remains to prove (5.4). To this end, we note that if \( z_i \in J_k \) and \( B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \cap E_{k+1} \neq \emptyset \), then there are unique \( l > k \) and \( z_i \in J_l \) such that
\[
B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \cap E_{k+1} = B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \cap E_{k+1}.
\]

(5.7)

Indeed, let \( l_1 > k \) be minimal such that there is \( z_i \in J_{l_1} \) with
\[
B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \cap B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_1) \neq \emptyset.
\]

Then, since by (3.14) \( l_1 \geq k + 2 \) we have \( \rho_1 \ll \rho_i \),
\[
B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \subseteq B_{\theta \lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_1).
\]

Now assume there is \( l_2 \geq l_1 \) and \( z_2 \in J_{l_2} \) such that
\[
B_{\lambda \varepsilon \frac{d}{d} \rho_i} (\varepsilon z_i) \cap E_{k+1} \neq \emptyset
\]

(5.8)

Then, applying (3.14), \( l_1 \leq l_2 - 2 \). In particular
\[
E_{k+1} \subseteq B_{\lambda \varepsilon \frac{d}{d} \rho_2} (\varepsilon z_2) \setminus B_{\theta \lambda \varepsilon \frac{d}{d} \rho_2} (\varepsilon z_1)
\]

which contradicts (5.8) and thus proves (5.7). We remark, that this gives the set \( J \) the structure of a forest.

Furthermore, going through the proof of the claim (3.49) we see that actually for any \( \gamma < \theta^2 \) there exists \( \varepsilon \) sufficiently small such that for all \( z_j \in J^c \)
\[
B_{\varepsilon \frac{d}{d} \gamma \lambda \rho_j} (\varepsilon z_j) \subseteq E^{z_j}.
\]

Therefore, choosing \( \theta < \gamma < \theta^2 \), for \( z_j \in J_k \),
\[
|E^{z_j} \setminus E_{k+1}| \geq |B_{\varepsilon \frac{d}{d} \gamma \lambda \rho_j} (\varepsilon z_j) \setminus E_{k+1}| \geq |B_{\varepsilon \frac{d}{d} \gamma \lambda \rho_j} (\varepsilon z_j)|
\]

where the last inequality follows from (5.7) and the fact that \( z_j \notin J_k \) if \( B_{\varepsilon \frac{d}{d} \gamma \lambda \rho_j} (\varepsilon z_j) \subseteq E_{k+1} \).
\[ \square \]

The proof of Theorem 2.3 relies on the following two results. The first lemma below is an adaptation of Lemma 2.4 of Section 2.4 to the case of the reduction operator \( R_v \), applied to the function \( v = e_k \), where \( e_k, k = 1, \cdots, d \) are the canonical vectors of \( \mathbb{R}^d \). The second lemma below is a variant of the standard Bogovski lemma to the set \( D \setminus E^c \) which allows to obtain estimates for the pressure in the Stokes equations (1.1). The non-trivial aspect of that Lemma is that the estimate is uniform in \( \varepsilon \) for small \( \varepsilon \). A priori, any such estimate highly depends on the exact geometry of the set of holes. To prove this result, we therefore again use an iteration scheme similar to the one in the construction of the operator \( R_v \).

**Lemma 5.2.** Let \( k = 1, \cdots, d \) be fixed. Then, for almost every \( \omega \in \Omega \) and any \( \varepsilon \leq \varepsilon_0 (\omega) \) and all \( k = 1, \cdots, d \), there exist \( w_k^\varepsilon \in H^1(D; \mathbb{R}^d) \cap L^\infty (D; \mathbb{R}^d) \), \( k = 1, \cdots, d \), such that

(\text{H1}) \( w_k^\varepsilon \) is 0 on \( E^c \) and \( \nabla \cdot w^\varepsilon = 0 \) in \( D \);

(\text{H2}) \( w_k^\varepsilon \rightarrow e_k \) in \( H^1(D) \) and \( w_k^\varepsilon \rightarrow e_k \) in \( L^p(D) \) for any \( 1 \leq p < +\infty \);

(\text{H3}) For any \( \phi \in C_0^\infty (D) \) and sequence \( v_\varepsilon \rightarrow v \) in \( H^1_0(D; \mathbb{R}^d) \) with \( \nabla \cdot v_\varepsilon = 0 \) on \( D \) we have
\[
\lim_{\varepsilon \downarrow 0^+} \int_0^\mu \phi \nabla w_k^\varepsilon \cdot \nabla v_\varepsilon = \int_0^\mu \phi e_k \cdot \mu v,
\]

with \( \mu \) defined in Theorem 2.1.

**Lemma 5.3.** Let \( q > d \) and let \( K \subset D \). Then, almost surely, there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon < \varepsilon_0 \) and all \( g \in L^d_0(K \setminus E^c) \) there exists \( v \in H^1_0(D \setminus E^c) \) such that
\[
\text{div} v = g,
\]
\[
\|v\|_{H^1} \leq C \|g\|_{L^q},
\]

(5.9)

where \( C = C(d, \beta, q) \).
Proof of Theorem 2.3. We first observe that (2.8) holds with the choice of $E^\varepsilon$ as in (5.2). Indeed, $E^\varepsilon \setminus H_\varepsilon \subset D^\varepsilon_0$ and by (3.9), sub-additivity of the harmonic capacity, and Lemma C.2

$$\text{Cap}(E^\varepsilon \setminus H_\varepsilon) \leq \sum_{z_j \in J^\varepsilon} \text{Cap}(B_{\Lambda \varepsilon^d \rho_j}^j(\varepsilon z_j)) \leq \varepsilon^d \sum_{z_j \in J^\varepsilon} \rho_j^{d-2} \to 0$$

almost surely as $\varepsilon \to 0$.

Let $K \Subset D$ and let $\varepsilon_0 > 0$ be as in Lemma 5.3. Let $g \in L^q_0(K \setminus E^\varepsilon)$ and let $v \in H^1_0(D \setminus E^\varepsilon)$ satisfy (5.9). Then, testing (2.4) with $v$ yields

$$\int_{K \setminus E^\varepsilon} p_\varepsilon g = \int_{D \setminus E^\varepsilon} p_\varepsilon \text{div} v = (\nabla u_\varepsilon, \nabla v)_{L^2(D^\varepsilon)} + \langle f, v \rangle_{H^{-1}, H^1} \leq 2\|v\|_{H^1} \|f\|_{H^{-1}} \lesssim \|g\|_{L^{q'}} \|f\|_{H^{-1}}.$$

Since $g \in L^q(K \setminus E^\varepsilon)$ was arbitrary, this implies that, up to a subsequence, $\hat{p}_\varepsilon$ defined in (2.9) converges to $p^*$ weakly in $L^{q'}(D)$, where $q'$ is the Hölder conjugate of $q$. It remains to identify the limit $p^*$ and extend the above convergence to the whole family $\varepsilon \downarrow 0^+$. To do so, it suffices to fix any smooth vector field $\phi \in C_0^\infty(\mathbb{R}^d)$ and test the equation (2.4) for $u_\varepsilon$ with the admissible test function $\sum_{k=1}^d w_k^\varepsilon \phi_k$. The integral terms containing $\nabla u_\varepsilon$ and $f$ may be treated as in the proof of Theorem 2.1 by relying on Lemma 5.2 instead of Lemma 2.4. It thus remains to show that also

$$\sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi)p_\varepsilon \to \int \nabla \phi \cdot p^*.$$

This indeed yields that $(u_\varepsilon, p^*)$ solve (2.5) and, by uniqueness, that $p^* = p_h$ in $L^q_0(D)$.

Let $K \Subset D$ be the support of $\phi$. Then, by (H1) of Lemma 5.2 each product $w_k^\varepsilon \phi_k$ is supported in $K \setminus E^\varepsilon$ and therefore

$$\sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi)p_\varepsilon = \sum_{k=1}^d \int \nabla \cdot (w_k^\varepsilon \phi)\hat{p}_\varepsilon = \sum_{k=1}^d \int w_k^\varepsilon \nabla \phi \varepsilon,$$

where in the last identity we used Leibniz rule and the divergence-free condition for $w_k^\varepsilon$ in (H1) of Lemma 5.2. It now remains to combine the convergence of $\hat{p}_\varepsilon$ with (H2) of Lemma 5.2 and send $\varepsilon \downarrow 0^+$ in the right-hand side above. This establishes (5.10) and concludes the proof of Theorem 2.3. $\square$

Proof of Lemma 5.2. We construct $w_k^\varepsilon$ as $R^\varepsilon e_k$ by mimicking the proof of Step 1 and Step 2 of Lemma 2.4, with the smooth vector field $v \in C_0^\infty(D, \mathbb{R}^d)$ substituted by $e_k$. We remark that the construction does not require that $v$ is compactly supported in $D$. This yields from property (ii) of Lemma 2.4 that $\nabla \cdot w_k^\varepsilon = 0$ in $D$. Moreover, a careful look to the construction of Step 2 on the set $D^\varepsilon_0$ shows that $R^\varepsilon e_k$ vanishes in the set $E^\varepsilon \cap D^\varepsilon_0 \subset H_\varepsilon^1$ and, since $E^\varepsilon = H_\varepsilon^0$ on $D \setminus D^\varepsilon_0$, we may upgrade property (i) of Lemma 2.4 to obtain (H1) of Lemma 5.2. Property (H2) follows from (iii) and (iii) of Lemma 2.4. Similarly, we argue that (H3) for $w_k^\varepsilon$ may be proven as (v) of Lemma 2.4, since the term on the left-hand side of (H3) may be rewritten as

$$\int \phi \nabla w_k^\varepsilon \cdot \nabla v = - \int \nabla \phi \cdot \nabla w_k^\varepsilon v - (\Delta w_k^\varepsilon, \phi v)_{H^{-1}, H^1}.$$

Thanks to (H2) of Lemma 5.2 and the assumption on $v$, the first term on the right-hand side vanishes almost surely in the limit $\varepsilon \downarrow 0^+$. The remaining term may be treated analogously to (4.39) in the proof of Lemma 2.4 (see also [1][Subsection 2.3.2]). $\square$

Proof of Lemma 5.3. Step 1: Strategy: Let $g_0 \in L^q_0(K \setminus E^\varepsilon)$ and extend it by zero to a function $g_0 \in L^q_0(D \setminus E^\varepsilon)$. The idea is to first solve the problem to find $v_0 \in H^1_0(K)$ such that

$$\text{div} v_0 = g_0, \quad \|v_0\|_{H^1} \lesssim \|g_0\|_{L^q}.$$

(5.11)

Clearly, since $K$ does not depend on $\varepsilon$, this just follows from the classical estimates for the Bogovski operator (see e.g. [8]). Then, we want to do corrections in order to have $v = 0$ in $E$. For $j \in \mathbb{N}$ the
correction is straightforward by taking \( v = v_0 + v_j \) in \( B_{\theta,j} \), where \( v_j \) solves the problem
\[
\begin{cases}
-\Delta v_j + \nabla p_j = 0 & \text{in } A_j \\
\operatorname{div} v_j = 0 & \text{in } A_j \\
v_j = 0 & \text{on } \partial B_{j,\theta} \\
v_j = -v_0 & \text{in } B_j.
\end{cases}
\] (5.12)

By (B.2), we have
\[
\|v_j\|_{H^1(B_{\theta,j})} \lesssim \left(\|v_0\|_{H^1(B_{\theta,j})} + R_j^{d/2} \|v_0\|_{L^\infty}\right),
\]
\[
\|v_j\|_{C^0} \lesssim \|v_0\|_{C^0},
\] (5.13)

where \( R_j = \varepsilon^{d/2} \rho_j \).

We would like to do this also for \( z_j \in J \). We should start with \( z_j \in J_{\text{max}} \). However, recall the complementary condition for existence of a solution to equation (5.12)
\[
\int_{\partial B_j} v_0 \cdot \nu = 0.
\]

This is in general not satisfied for those \( z_j \) since we have
\[
\int_{\partial B_j} v_0 \cdot \nu = \int_{B_j} g_0,
\]
and the latter integral might be nonzero if \( B_j \not\subseteq E \) and we simply extended \( g_0 \) by zero inside \( E \). (Clearly, \( B_j \subseteq E \) holds for \( z_j \in n^\varepsilon \).) Moreover, note that for \( z_j \in J_k \), \(-3 \leq k \leq k_{\text{max}}\), instead of the problem (5.12), we need to find a corrector \( v_j \) that solves
\[
\begin{cases}
\operatorname{div} v_j = g_0 & \text{in } A_j \cap E_{k+1} \\
\operatorname{div} v_j = 0 & \text{in } A_j \setminus E_{k+1} \\
v_j = 0 & \text{on } \partial B_{j,\theta} \\
v_j = -v^{(k+1)} & \text{in } B_j,
\end{cases}
\] (5.14)

where \( v^{(k)} \) is inductively defined by
\[
v^{(k_{\text{max}}+1)} := v_0, \\
v^{(k)} := v^{(k+1)} + \sum_{z_j \in J_k} v_j.
\]

By Lemma B.2, we can find a solution \( v_j \) to (5.14) with
\[
\|v_j\|_{H^1} \lesssim \|v^{(k+1)}\|_{H^1(B_{\theta,j})} + \|g\|_{L^2(B_{\theta,j})} + R_j^{d/2} \left(\|v^{(k+1)}\|_{C^0} + \|\operatorname{div} v^{(k+1)}\|_{L^\infty(B_j)} + \|g\|_{L^\infty}\right),
\]
\[
\|v_j\|_{C^0} \lesssim \|v^{(k+1)}\|_{C^0} + \|\operatorname{div} v^{(k+1)}\|_{L^\infty(B_j)} + \|g\|_{L^\infty},
\] (5.15)

with \( R_j = \varepsilon^{d/2} \rho_j \), provided the complementary condition holds, namely
\[
\int_{A_j \cap E_{k+1}} g_0 - \int_{\partial B_j} v^{(k+1)} \cdot \nu = 0.
\] (5.16)

Again, this is not satisfied in general, since
\[
\int_{A_j \cap E_{k+1}} g_0 - \int_{\partial B_j} v^{(k+1)} \cdot \nu = \int_{A_j \cap E_{k+1}} g_0 - \int_{B_j \setminus E_{k+1}} g_0.
\]

For this reason, instead of simply extending \( g_0 \) by zero, we need to extend it in a nontrivial way to a function \( g \in L^0_0(D) \).

**Step 2: Extension of the function \( g_0 \):** First, we extend \( g_0 \) by \( g = 0 \) to \( \mathbb{R}^d \setminus E \). As seen above, for \( z_j \in n^\varepsilon \), we can also simply choose \( g = 0 \) in \( B_j \). For \( z_j \in J \) let \( N_1 \in \mathbb{N}_0 \) and \( z_{ln} \in \cup_{l=3}^{k-2} J_l \), \( 1 \leq n \leq N_1 \) such (5.5) holds, and let \( N_2 \in \mathbb{N}_0 \) and \( z_{ln} \in \cup_{l=3}^{k-2} J_l \), \( 1 \leq n \leq N_2 \) such that (5.6) holds. We now
choose \( g = g_j = \text{const} \) in \( \tilde{E}^{z_j} \) and \( g = 0 \) in \( E^{z_j} \setminus \tilde{E}^{z_j} \), where the constants \( g_j \) are uniquely determined by satisfying

\[
0 = \int_{A_i \cap E_{k+1}^i} g - \int_{B_j \setminus E_{k+1}^i} g
\]

\[
= \int_{A_i \cap E_{k+1}^i} g_0 + \sum_{n=1}^{N_2} |\tilde{E}^{z_n} \cap A_j \cap E_{k+1}^i| g_{j_n}
\]

\[
- \int_{B_j \setminus (E_{k+1}^i \cup E)} g_0 - |\tilde{E}^{z_j} | g_j - \sum_{n=1}^{N_1} |\tilde{E}^{z_n} \cap B_j \setminus E_{k+1}^i| g_{j_n}.
\]

(5.17)

Indeed, since \( z_{i_n}, z_{j_n} \in \cup_{l=3}^{k-2} J_l \), this formula yields \( g_j \) for all \( z_j \in J_k \), provided we already know \( g_i \) for \( z_i \in \cup_{l=3}^{k-2} J_l \). Therefore, all \( z_j, j \in J \) are inducibly defined by (5.17).

We observe that by this procedure we might extend the function \( g_0 \) non-trivially also in holes that are not contained in \( K \), namely if they are within a cluster that intersects with \( K \). Therefore, we fix some \( K \in K' \subset D \) and argue that for \( \varepsilon \) sufficiently small, \( g = 0 \) in \( D \setminus K' \). Indeed, this follows by induction very similarly to the argument at the end of Step 2 in the proof of Lemma 2.4, only that here we start from the small holes towards the big holes. Indeed, \( g_j = 0 \) for all \( j \in J_{-3} \) with \( B_{\theta,j} \subset D \setminus K \), and \( g_j = 0 \) for \( j \in J_k \) if \( B_{\theta,j} \subset D \setminus K \) and \( B_{\theta,j} \cap B_{\theta,j} = \emptyset \) for all \( i \in \cup_{l=3}^{k-1} \) with \( g_i \neq 0 \).

Hence, instead of (5.11), we can find \( v_0 \in H_0^1(K') \) with

\[
\text{div} v_0 = g,
\]

\[
\|v_0\|_{H^1} \lesssim \|g\|_{L^1}.
\]

(5.18)

and extend \( v_0 \) by zero to a function in \( D \). In order to find such a \( v_0 \), we need to check the complementary condition \( \int g = 0 \). By (5.17)

\[
\int_{K'} g = \int_{K} g_0 + \sum_{E_{-3}} g = \int_{E_{-3}} g
\]

\[
= \int_{E_{-2}} g + \sum_{j \in J_{-3}} \int_{B_j \setminus E_{-3}} g - \int_{A_j \cap E_{-3}} g = \int_{E_{-2}} g.
\]

By induction, this indeed yields \( g = 0 \) since \( E_{k_{\max} + 1} = \emptyset \).

**Step 3: Solving \( \text{div} v = g \) and obtaining the desired estimates:** We need to show that by the extension of \( g_0 \) to \( g \), we do not increase its norm too much, i.e.,

\[
\|g\|_{L^q(K')}^q \lesssim \|g_0\|_{L^q(K)}^q.
\]

(5.19)

We claim that with the above definition of \( g_j \), we have for all \( z_j \in J_k \)

\[
|\tilde{E}^{z_j}| \|g_j\| \leq (2k_{\max} + 3)^{k+2} |g_0|_{L^q(B_{\theta, z_j} \setminus E)}
\]

(5.20)

where \( B_{\theta, z_j} := B_{\theta, \lambda_j z_j, \frac{d}{2}} (\varepsilon z_j) \). We prove (5.20) by induction over \( k \). For \( z_j \in J_{-3} \), we have

\[
|\tilde{E}^{z_j}| \|g_j\| = \int_{A_j \cap E_{k+1}^i \setminus E} g_j
\]

so (5.20) holds for \( k = -3 \). Assume that (5.20) holds for all \( 1 \leq l \leq k-1 \) and consider \( z_j \in J_k \). Let \( N_1, N_2 \in \mathbb{N} \) and \( z_{i_n}, z_{j_n} \in \cup_{l=3}^{k-2} J_l \) such that (5.5) and (5.6) hold. Then,

\[
|\tilde{E}^{z_j}| \|g_j\| \leq \int_{B_{\theta,j} \setminus E} |g_0| + \sum_{n=1}^{N_1} |B_{\theta,j} \cap \tilde{E}^{z_{i_n}}| |g_{i_n}| + \sum_{n=1}^{N_2} |B_{\theta,j} \cap \tilde{E}^{z_{j_n}}| |g_{j_n}|
\]

\[
\leq \|g_0\|_{L^q(B_{\theta,j} \setminus E)} + \sum_{n=1}^{N_1} (2k_{\max} + 3)^{k+1} \|g_0\|_{L^q(B_{\theta, z_j, i_n} \setminus E)}
\]

\[
+ \sum_{n=1}^{N_2} (2k_{\max} + 3)^{k+1} \|g_0\|_{L^q(B_{\theta, z_j, j_n} \setminus E)}.
\]

(5.21)
We observe that $B_{g^2,i_n} \subseteq B_{g^2,j}$ since $B_{i_n} \cap B_{\theta,j} \neq \emptyset$ and the radius of the ball $B_{i_n}$ is much smaller than the one of $B_j$ since $z_{i_n} \in J_l$ with $l \leq k - 2$. Moreover, for every $x \in B_{g^2,j}$,

$$\# \{ z_{i_n} : 1 \leq n \leq N : x \in B_{g^2,i_n} \} \leq k + 1,$$

since, by (3.7), $B_{g^2,i_n} \cap B_{g^2,i_m} = \emptyset$ whenever $z_{i_n} \neq z_{i_m} \in J_l$ for some $1 \leq m, n, \leq N$, $-3 \leq l \leq k - 2$

Using this in (5.21) yields (5.20).

By definition of $g$, we have

$$\| g \|_{L^q(K')} = \| g_0 \|_{L^q(K')} + \sum_{z_j \in J_K} \| \tilde{E}^{z_j} \|_{L^q(K')}^q g_j.$$  

We estimate for $z_j \in J$, using (5.20) and (5.4),

$$\| \tilde{E}^{z_j} \|_{L^q(K')}^q g_j \lesssim \frac{1}{|E^{z_j}|} \| g_0 \|_{L^q(B_{g^2,j}\setminus E')}^q \lesssim \| g_0 \|_{L^q(B_{g^2,j}\setminus E')}^q.$$  

Using similar as above that for all $x \in K'$

$$\# \{ z_j \in J : x \in B_{\theta,j} \} \leq k_{\text{max}} + 1,$$

this yields (5.19).

Hence, the function $v_0$ solving (5.18) satisfies

$$\text{div} v_0 = g,$$

$$\| v_0 \|_{H^\frac{d}{2}} \lesssim \| g \|_{L^\frac{d}{2}} \lesssim \| g_0 \|_{L^\frac{d}{2}}.$$  

Now we just proceed by adding correctors as sketched in Step 1: First, let $v_j$ be the solutions to (5.12) for $z_j \in n^\varepsilon$ and define

$$v^{(k_{\text{max}} + 1)} := v_0 + \sum_{z_j \in n^\varepsilon} v_j.$$  

Then, $v^{(k_{\text{max}} + 1)} \in H^1_0(D)$,

$$\text{div} v^{(k_{\text{max}} + 1)} = g,$$

$$v^{(k_{\text{max}} + 1)} = 0 \quad \text{in } H^\frac{d}{2},$$  

and, since $v_j$ have disjoint support, using (5.13)

$$\| v^{(k_{\text{max}} + 1)} \|_{C^0} \lesssim \| g_0 \|_{L^\frac{d}{2}}$$  

and

$$\| v^{(k_{\text{max}} + 1)} \|^2_{H^1} = \sum_{z_j \in n^\varepsilon} \| v_j \|_{H^1}^2 \lesssim \sum_{z_j \in n^\varepsilon} \| v_0 \|_{H^1(B_{\theta,j})}^2 + \varepsilon \frac{d}{d - 2} \rho_j \| v_0 \|_{L^\infty} \lesssim \| g_0 \|_{L^\frac{d}{2}},$$  

almost surely, for $\varepsilon$ small enough.

Then, inductively for $k = k_{\text{max}}, \ldots, -3$, for all $z_j \in J_k$, we claim that we find solutions to $v_j$ (5.14) that satisfy (5.15), and defining

$$v^{(k)} := v^{(k + 1)} + \sum_{z_j \in J_k} v_j,$$

we have $v^{(k)} \in H^1_0(D)$ with

$$\text{div} v^{(k)} = g \quad \text{in } D \setminus E_k,$$

$$v^{(k)} = 0 \quad \text{in } H^\frac{d}{2} \cup E_k,$$

$$\| v^{(k)} \|_{H^1} + \| v^{(k)} \|_{C^0} \lesssim \| g_0 \|_{L^\frac{d}{2}}.$$  

It remains to prove this claim. Indeed, if (5.23) holds, then setting $v = v^{(-3)}$ yields the assertion.

The proof proceeds by induction in $k$. Indeed, for $k = k_{\text{max}} + 1$, (5.22) yields (5.23). Assume (5.23) holds for some $k + 1$. Then, we recall that the complementary condition for solving (5.14) is (5.16), which is equivalent to (5.17) since $\text{div} v^{(k + 1)} = g$ in $D \setminus E_{k + 1}$. However, (5.17) holds, because this is exactly how we chose the values of $g_i$, $i \in J$. Therefore, $v_j$ is well defined, and satisfies (5.15). In particular $v^{(k)}$ is well defined, and, using that $|\text{div} v^{(k + 1)}| \leq |g|$ pointwise together with
the estimates for $v^{(k+1)}$, we get the estimate in (5.23) analogously as we obtained the estimates for $v^{(k_{\max}+1)}$. Moreover, by construction, $\text{div} \, v^{(k)} = g$ in $D \setminus E_k$.

Furthermore, $v^{(k)} \in H^1_0(D)$, since we only changed $v^{(k+1)}$ in $B_{\theta,j}$ for holes that are in certain cluster that overlaps with $K'$. These balls are contained in $D$ by an argument analogous to the one at the end of Step 2 in the proof of Lemma 2.4. It remains remark that by construction $v^{(k)} = 0$ in $H^*_g \cup E_k$, since $v^{(k)} = 0$ in $E_{k+1} \setminus \bigcup_{z_j \in I_k} B_{\theta,j}$ and in $\bigcup_{z_j \in I_k} B_{\theta,j}$.

\[ \square \]

6. Probabilistic results

The aim of this section is to give some probabilistic results on the random set $H^*$, in terms of the size of the clusters generated by overlapping balls of comparable size; these results are used in Section 3 to obtain a good covering for $H^*$ and to estimate its size.

We introduce the following notation: For $\alpha \geq 1$, let

$$H^*_\alpha = \bigcup_{z_i \in \Phi^\alpha(D)} B_{\varepsilon \frac{d}{d-\beta}}(\varepsilon z_i).$$

For a step-size $\delta > 0$, we partition the (random) collection of points $\Phi^\delta(D)$ in terms of the order of magnitude of the associated radii: We define

$$I^\varepsilon_{k,\delta} := \{z_i \in \Phi^\delta(D) : \varepsilon^{1-\delta k} \leq \varepsilon^{d-\beta} \rho_i \leq \varepsilon^{1-\delta} \} \quad \text{for } k \geq -2,$$

and for every $k \geq -2$ also

$$\Psi^{k,\varepsilon}_{\delta} = I^\varepsilon_{k} \cup I_{k-1}^{\varepsilon} \subseteq \Phi^\varepsilon(D).$$

Each collection $\Psi^{k,\varepsilon}_{\delta}$ thus generates the set

$$H^*_{k,\alpha} := \bigcup_{z_i \in \Psi^{k,\varepsilon}_{\delta}} B_{\varepsilon \frac{d}{d-\beta}}(\varepsilon z_i) \subseteq H^*_\alpha$$

which is made of balls having radii which differ by at most two orders $\delta$ of magnitude.

\textbf{Lemma 6.1.} Let $\alpha \geq 1$ and $0 < \delta < \frac{\beta}{d-\beta}$ be fixed. Then, there exists $M(d, \beta), k_{\max}(\beta, d) \in \mathbb{N}$ such that for almost every $\omega \in \Omega$ and every $\varepsilon \leq \varepsilon_0(\omega)$

(I) For every $k > k_{\max}$ we have

$$I^\varepsilon_{k,\delta} = \emptyset;$$

(II) For every $-2 \leq k \leq k_{\max}$, each connected component of $H^*_{k,\alpha}$ defined in (6.2) is made of at most $M$ balls.

\textbf{Proof of Lemma 6.1.} We begin with (I) and observe that assumption (1.7) and Chebyshev’s inequality imply that for a constant $C < +\infty$

$$\langle \rho^{d-2+\beta} \rangle \leq C, \quad \mathbb{P}(\rho \geq r) \leq C r^{-(d-2+\beta)}.$$  \hspace{1cm} (6.3)

In addition, as already argued in Section 4 (see (3.12)),(1.7) and the Strong Law of Large Numbers (see Lemma C.1) imply that for almost every $\omega \in \Omega$ and all $\varepsilon$ sufficiently small

$$\max_{z_i \in \Phi^\varepsilon(D)} \varepsilon^{d-\beta} \rho_i \leq 2 \varepsilon^{d-\beta} + \varepsilon^{d-2+\beta} (\rho^{d-2+\beta}) x^{d-2+\beta}.$$  \

Hence, for the same choice of $\omega$ and $\varepsilon$ we have $I^k = \emptyset$ whenever $k > k_{\max}$ with

$$\varepsilon^{1-\delta(k_{\max}+1)} < \varepsilon^{d-\beta} x^{d-2+\beta},$$  \

namely if

$$1 - \delta(k_{\max} + 1) < \frac{d}{d-2} - \frac{d}{d-2+\beta}.  \hspace{1cm} (6.4)$$

We may thus choose the minimal $k_{\max}$ satisfying the inequality above and conclude the proof for (II).
We now turn to (II) and fix $-2 \leq k \leq k_{\max}$: For any $m \in \mathbb{N}$ we consider the event
\[ A_{\varepsilon,\delta,k}^{\alpha,m} := \{ \omega : \text{There exist } m \text{ intersecting balls in } H_{\varepsilon,\alpha}^{\delta,k} \}. \]
Then, (II) is equivalent to show that there exists an integer $M = M(\beta, d) \geq 2$ such that
\[ \mathbb{P}\left( \bigcap_{\varepsilon > 0} \bigcup_{0 \leq \varepsilon < \varepsilon_0} \bigcup_{k \geq -2} A_{\varepsilon,\delta,k}^{\alpha,M} \right) = 0. \] (6.5)
Furthermore, we begin by arguing that it suffices to prove that
\[ \mathbb{P}\left( \bigcap_{l \in \mathbb{N}} \bigcup_{-1 \leq l \leq k \geq -2} A_{\varepsilon,\delta,k}^{\alpha,M} \right) = 0, \] (6.6)
i.e. statement (6.5) for the sequence $\varepsilon_l = 2^{-l}$ and $\alpha, \delta$ substituted by $\tilde{\alpha} = 2^{\frac{\varepsilon_l}{2}} \alpha$ and $3\delta$.
Suppose, indeed, that (6.6) holds: For any $\varepsilon > 0$, let $l \in \mathbb{N}$ be such that $\varepsilon_{l+1} \leq \varepsilon \leq \varepsilon_l$. Then for every two $z_i, z_j \in \Psi^{\delta,\varepsilon}$ with $\rho_i \geq \rho_j$, definition (6.1) yields that
\[ \rho_i - \rho_j \leq \rho_j \frac{(\rho_i - 1)}{(\rho_j - 1)} \leq \rho_j (\varepsilon_{l+1} - 1) \leq \rho_j \varepsilon_{l+1}. \]
This implies that if $\rho_j \in I_{k-1}^{\varepsilon_{l+1},3\delta}$ for some $\tilde{k} \in \mathbb{Z}$, then $\rho_i \in I_{k}^{\varepsilon_{l+1},3\delta}$. This is equivalent to
\[ \Psi_{\tilde{k}}^{\delta,\varepsilon} \subseteq \Psi_{k}^{3\delta,\varepsilon_{l+1}}. \] (6.7)
Equipped with this inclusion, we now show that
\[ A_{\varepsilon,\delta,k}^{\alpha,m} \subseteq A_{\varepsilon_{l+1},3\delta,\tilde{k}}^{\delta,m}. \] (6.8)
To do so, let us assume that $z_i, z_j \in \Psi_{k}^{\delta,\varepsilon}$ satisfy
\[ B_{\alpha \varepsilon^{\frac{2d}{d-2}} \rho_j}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{\frac{2d}{d-2}} \rho_i}(\varepsilon z_i) \neq \emptyset. \]
Then,
\[ \varepsilon |z_i - z_j| \leq \alpha \varepsilon^{\frac{2d}{d-2}} (\rho_i + \rho_j) \]
which yields
\[ |z_i - z_j| \leq \alpha \varepsilon^{\frac{2d}{d-2}} (\rho_i + \rho_j) \leq \alpha \varepsilon^{\frac{2d}{d-2}} (\rho_i + \rho_j) = 2^{\frac{\varepsilon_l}{2}} \alpha \varepsilon_{l+1} (\rho_i + \rho_j). \]
This is equivalent to
\[ B_{\tilde{\alpha} \varepsilon_{l+1} z_j}(\varepsilon_{l+1} z_i) \cap B_{\tilde{\alpha} \varepsilon_{l+1} z_i}(\varepsilon_{l+1} z_i) \neq \emptyset. \]
Since the previous argument holds for any choice of two elements in $\Psi^{\delta,\varepsilon}$, this and (6.7) imply (6.8).
This last statement also allows to conclude that for every $m \in \mathbb{Z}$
\[ \bigcup_{k \geq -2} A_{\varepsilon,\delta,k}^{\alpha,m} \subseteq \bigcup_{k \geq -2} A_{\varepsilon_{l+1},3\delta,\tilde{k}}^{\delta,m}. \]
This establishes that (6.6) implies (6.5).
To conclude the proof of (II), it only remains to show (6.6): We begin by deriving a basic estimate for the probability of having a certain number of close points in a Poisson point process. We recall indeed that the centres $\Psi^{\varepsilon}(D)$ are distributed according to a Poisson point process in $1\varepsilon D$ with intensity $\lambda$. We also recall that, for a general set $A \subseteq \mathbb{R}^d$ we denote by $N(A)$ the random variable providing the number of points of the process which are in $A$.
For $0 < \eta < 1$, let
\[ Q_{\eta} := \{ [-\eta 2, \eta 2]^d + y \mid y \in (\eta \mathbb{Z})^d \}, \]
i.e. the set of cubes of length $\eta$ centered at the points of the lattice $(\eta \mathbb{Z})^d$. Let $S_{\eta}$ be the set containing the edges of the cube $[0, \eta 2]^d$, i.e.
\[ S_{\eta} := \{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d : z_k \in \{0, \frac{\eta}{2}\} \text{ for all } k = 1, \ldots, d \}. \]
Then, for any \( x \in \mathbb{R}^d \) there always exists \( z \in S_\eta \) and \( B_2^\eta(x) \subseteq Q \) for some \( Q \in \mathcal{Q}_\eta + z \). Thus, if \( \eta \) is chosen such that \( \lambda \eta^d \leq 1 \), we use this geometric consideration to estimate
\[
\mathbb{P}(\exists x \in \frac{1}{\varepsilon}D : N(B_2^\eta(x)) \geq m) \lesssim \mathbb{P}(\exists Q \in \mathcal{Q}_\eta, z \in S_\eta : (Q + z) \cap \frac{1}{\varepsilon}D \neq \emptyset, N(Q + z) \geq m),
\]
and the distribution for \( N(A) \) to conclude that
\[
\mathbb{P}(\exists x \in \frac{1}{\varepsilon}D : N(B_2^\eta(x)) \geq m) \lesssim \varepsilon^{-d} \lambda e^{-\lambda \eta^d} \sum_{k=m}^{\infty} \frac{(\lambda \eta^d)^k}{k!} \lesssim (\varepsilon)^{-d} (\lambda \eta^d)^m.
\] (6.9)

Equipped with (6.9), we estimate each \( P(A_{\varepsilon,k}^{\alpha,m}) \): Let us assume that \( z_i, z_j \in \Psi^{\delta,\varepsilon} \) are such that
\[
B_{\alpha \varepsilon^{-1/\rho_j}}(\varepsilon z_j) \cap B_{\alpha \varepsilon^{-1/\rho_i}}(\varepsilon z_i) \neq \emptyset.
\]
Then,
\[
|z_i - z_j| \leq 2 \alpha \varepsilon^{\delta,\varepsilon}, \quad A_{\varepsilon,k}^{\alpha,m} \subseteq \{ \exists x \in \frac{1}{\varepsilon}D : \#(\Psi^{\delta,\varepsilon} \cap B_{\max \kappa}(x)) \geq m \}.
\] (6.11)

We now want to estimate the event in the right-hand side above by appealing to (6.9) for each \( \varepsilon \) and \( k \) fixed and with \( \eta = \eta_k \) given by
\[
\eta_k := m \alpha \varepsilon^{\delta,\varepsilon}.
\] (6.12)

We observe indeed that by definition (6.1), for every \( \varepsilon \) the processes \( \Psi^{\delta,\varepsilon} \) are Poisson processes on \( \frac{1}{\varepsilon}D \) with intensity given by
\[
\lambda_k^\varepsilon = \lambda \mathbb{P}(\varepsilon^{-\frac{d}{2}} - \delta(k-1) \leq \rho \leq \varepsilon^{-\frac{d}{2}} - \delta(k+1))) \lesssim \varepsilon^{-(d+2+\beta)}(\frac{\lambda \eta^d}{\varepsilon^{d/2} + \delta(k-1)}
\] (6.13)
for any \( k \geq -1 \), and
\[
\lambda_{-2}^\varepsilon = \lambda \mathbb{P}(\rho \leq \varepsilon^{-\frac{d}{2}} - \delta(-1)) \lesssim \lambda
\] (6.14)
for \( k = 2 \).

We first argue that, provided that for every \( k \) and \( \varepsilon \) small enough, there exists \( \mu_k > 0 \) such that
\[
\lambda_k^\varepsilon(\eta_k)^d \leq \varepsilon^{\mu_k},
\] (6.15)
then we conclude the proof of (6.6). Indeed, by the previous inequality we may apply (6.9) to the right-hand side of (6.11) and bound by (6.12) and (6.15)
\[
\mathbb{P}(A_{\varepsilon,k}^{\alpha,m}) \lesssim \varepsilon^{m \mu_k - d(1 + \kappa_k)}.
\]

By choosing \( m = M \), \( M \) sufficiently large, we thus get
\[
\mathbb{P}(A_{\varepsilon,k}^{\alpha,m}) \lesssim \varepsilon^{\mu_k}.
\]
Since by (I) we only have to consider finitely many values of \( k = -3, \ldots, k_{\max} \), \( M \) can be chosen independently of \( k \). Therefore, recalling that \( \varepsilon_l = 2^{-l} \) in (6.6), we use the previous estimate and assumption (6.15) to infer
\[
\sum_{l \in \mathbb{N}} \mathbb{P} \left( \bigcup_{k \geq -2} A_{\varepsilon_l,\delta,k}^{\alpha,M} \right) < \infty.
\]

I thus remains to apply Borel-Cantelli’s lemma to obtain (6.6) and thus (6.5) as well as (II).

To conclude the proof of the lemma, it thus remains to show (6.15). To do so, we recall the definitions (6.12) and (6.10) of \( \eta_k \) and \( \kappa_k \) and we also set for every \( -1 \leq k \leq k_{\max} \)
\[
\gamma_k := (d - 2 + \beta) \left( \frac{2}{d - 2} + \delta(k - 1) \right).
\] (6.16)
By (6.13), this definitions allows us to bound for each $\varepsilon$
\[\lambda_k^\varepsilon \leq \varepsilon^\gamma_k.\]  
(6.17)

We first show (6.15) for $k = -2$: In this case, by (6.12), (6.10) and (6.14), we have
\[\lambda_{-2}^\varepsilon \lesssim \varepsilon^d\]
and we may thus simply choose $\mu_{-2} = d\delta > 0$. We now turn to the case $k > -2$: Again by (6.12) and, this time, by (6.17) we have
\[\lambda_k^\varepsilon \lesssim \varepsilon^{\gamma_k + d\mu_k}.\]
Therefore we need
\[\mu_k = \gamma_k + d\kappa_k \overset{(6.16), (6.10)}{=} \frac{2(d - 2 + \beta)}{d - 2} - (2 - \beta)\delta(k - 1) - 2d\delta > 0.\]
Since we assumed that $\beta \leq 1$, we may use (6.4) on the second term in the right-hand side above and, after a short calculation, obtain that
\[\mu_k > 2 - (2 - \beta) - 2d\delta \geq \beta - 2d\delta.\]
Thanks to our assumption $\delta < \frac{\beta}{2d}$, we thus conclude that $\mu_k > 0$. This establishes (6.15) and completes the proof of the lemma.

\[\square\]

**Appendix A. Proof of Remark 2.2**

The proof of the homogenization result in this case is analogous to the case of the Stokes equations, provided we prove the convergence of the non-linear term $u_\varepsilon \nabla \cdot u_\varepsilon$. We recall the weak formulation of (2.6). We define the space $V_\varepsilon := \{w \in H^1_0(D_\varepsilon) : \text{div } w = 0\}$ equipped with the norm $\|\nabla \cdot \|_{L^2}$. Then, we call $u_\varepsilon \in V$ a weak solution to (2.6) if
\[\mu \int \nabla u_\varepsilon \cdot \nabla \phi + \int u_\varepsilon \cdot \nabla u_\varepsilon \cdot \phi = \langle f, \phi \rangle \quad \forall \phi \in \hat{V}_\varepsilon := \{w \in H^1_0(D_\varepsilon) \cap L^d : \text{div } w = 0\},\]
where the space $\hat{V}_\varepsilon$ is chosen such that the nonlinear term makes sense. Furthermore, by Sobolev embedding we observe $\hat{V}_\varepsilon = V_\varepsilon$ for $d \leq 4$. The weak formulation of (2.7) is analogous. Existence of solutions to (2.7) is well-known. However, the solution is only known to be unique if $d \leq 4$ and
\[\|f\|_{V'} < C(d, D).\]  
(A.1)

If $d \leq 4$ testing with the solution $u$ yields the energy estimate
\[\|\nabla u_\varepsilon\|_{L^2} \leq \|f\|_{V'}.\]  
(A.2)

For more details on the stationary Navier-Stokes equations see for example [21] and [9].

The proof of the convergence $u_\varepsilon \to u_k$ in $H^1(D)$ in the case $d = 3$ is now straightforward provided (A.1) holds. Indeed, thanks to (A.2), the sequence $u_\varepsilon$ is bounded in $H^1$, and by the uniqueness of the solutions to (2.7), it therefore suffices to prove that the weak limit $u^*$ of any subsequence of $u_\varepsilon$ satisfies (2.7). To this end, let $v \in C_0^\infty(D)$ with div $= 0$. Then, applying Lemma 2.4, we know
\[\int \nabla u_\varepsilon \cdot \nabla (R_\varepsilon v) \to \int \nabla u^* \cdot \nabla v + \mu u^* \cdot v,\]
\[\langle f, R_\varepsilon v \rangle \to \langle f, v \rangle.\]

Therefore, it remains to show
\[\int u_\varepsilon \cdot \nabla u_\varepsilon \cdot (w_\varepsilon^* \phi) \to \int u^* \cdot \nabla u_\varepsilon^* \phi.\]

However, since $2^* = 6 > 4$ both $u_\varepsilon$ and $R_\varepsilon v$ converge strongly in $L^4$ and $\nabla u_\varepsilon$ converges weakly in $L^2$. Thus, the convergence above follows immediately.

In the case $d = 4$ this argument just fails, since the embedding from $H^1$ to $L^4$ is not compact. However, since by Lemma 2.4 also $R_\varepsilon v \to v$ strongly in $L^q$, for any $4 < q < \infty$, the argument works again.
Appendix B. Estimates for the Stokes equations in annuli and in the exterior of balls

In this section we summarize some standard results for the solutions to the Stokes equation in annular and exterior domains (see, e.g., [8, 1]).

Lemma B.1. Let \( R > 1 \), denote \( A_R := B_R \setminus B_1 \), and let \( \psi \in H^1(B_\theta) \cap C^0(\bar{B}_\theta) \) satisfy \( \int_{\partial B_1} \psi \cdot \nu = 0 \). Let \((\phi_R, \pi_R)\) and \((\phi_\infty, \pi_\infty)\) be the (weak) solutions of

\[
\begin{align*}
\Delta \phi_R - \nabla \pi_R &= 0 \quad \text{in } A_R \\
\nabla \cdot \phi_R &= 0 \quad \text{in } A_R \\
\phi_R &= \psi \quad \text{on } \partial B_1 \\
\phi_R &= 0 \quad \text{on } \partial B_R,
\end{align*}
\] (B.1)

Then,

\[
\|\phi_R\|_{L^2(A_R)} + \|\nabla \phi_R\|_{L^2(A_R)} \leq C_1 (\|\nabla \psi\|_{L^2(A_R)} + \|\psi\|_{L^2(A_R)}),
\]

\[
\|\phi_R\|_{C^0(\partial B_1)} \leq C_1 \|\psi\|_{C^0(\partial B_1)},
\] (B.2)

with \( C_1 = C_1(d, R) \). Moreover,

\[
\|\phi_\infty\|_{L^2(\mathbb{R}^d \setminus B_1)} + \|\nabla \phi_\infty\|_{L^2(\mathbb{R}^d \setminus B_1)} \leq C_2 (\|\nabla \psi\|_{L^2(A_2)} + \|\psi\|_{L^2(A_2)}),
\]

\[
\|\phi_\infty\|_{C^0} \leq C_2 \|\psi\|_{C^0(\partial B_1)},
\] (B.3)

with \( C_2 = C_2(d) \). Furthermore,

\[
\|\phi_\infty(x)\|^2 \leq C_3 \|\psi\|_{C^0(\partial B_1)} |x|^{2-d},
\] (B.4)

and, if \( \nabla \cdot \psi = 0 \) in \( B_1 \),\(^2\)

\[
|\nabla \phi_\infty(x)| \leq C_2 \|\psi\|_{H^1(B_1)} |x|^{1-d} \quad \text{for all } |x| \geq 3.
\] (B.5)

Proof. The existence and uniqueness of solutions to both problems in (B.1) together with the first estimate in both (B.2) and (B.3) is a standard result [8][Section IV and V]. The second estimate in both (B.2) and (B.3) can be found in [17][Theorem 5.1 and Theorem 6.1]. Estimate (B.4) can be found in [17][Theorem 6.1], too.

To prove (B.5), we extend \( \phi_\infty \) by \( \psi \) inside \( B_1 \) and \( \pi_\infty \) by 0 inside \( B_1 \). Then, by (B.3)

\[
\begin{align*}
-\Delta \phi_\infty + \nabla \pi_\infty &= f \quad \text{in } \mathbb{R}^d \\
\nabla \cdot \phi_\infty &= 0 \quad \text{in } \mathbb{R}^d
\end{align*}
\]

for some \( f \in \dot{H}^{-1}(\mathbb{R}^d) \), with

\[
\text{supp } f \subseteq \overline{B_1},
\]

\[
\|f\|_{\dot{H}^{-1}(\mathbb{R}^d)} \lesssim \|\psi\|_{H^1(B_2)}.
\]

Here, \( \dot{H}^{-1}(\mathbb{R}^d) \) is the dual of the homogeneous Sobolev space

\[
\dot{H}^1(\mathbb{R}^d) := \left\{ v \in L^2(\mathbb{R}^d)^d : \nabla v \in L^2(\mathbb{R}^d) \right\}, \quad \| \cdot \|_{\dot{H}^1(\mathbb{R}^d)} := \| \nabla \cdot \|_{L^2(\mathbb{R}^d)}.
\]

Hence, with \( U \) being the fundamental solution of the Stokes equations we have

\[
\phi_\infty(x) = (U * f)(x).
\]

The fundamental solution satisfies

\[
|D^\alpha U(x)| \lesssim C(d, |\alpha|) |x|^{2-d-|\alpha|}.
\]

Using the compact support of \( f \), and letting \( \eta \in C_c^\infty(B_2) \) be a cut-off function with \( \eta = 1 \) in \( B_1 \), we deduce for all \( |x| > 3 \)

\[
|\nabla \phi_\infty(x)| = |\langle \eta \nabla U(x - \cdot), f \rangle_{\dot{H}^1, \dot{H}^{-1}}| \leq \| \eta \nabla U(x - \cdot) \|_{\dot{H}^1(\mathbb{R}^d)} \| f \|_{\dot{H}^{-1}(\mathbb{R}^d)} \lesssim C_5 \| \psi \|_{H^1(B_2)} |x|^{1-d}.
\]

This proves (B.5). \( \square \)

\(^2\)This assumption is not needed, but makes the proof slightly simpler.
Lemma B.2. Let $q > d$ and let $0 < r < 1$, $\theta > 1$, $B_r := B_r(0)$, $B_{r\theta} := B_{r\theta}(0)$, $A_{r\theta} := B_{r\theta} \setminus B_r$. Assume $g \in L^q(B_{r\theta})$ and $v \in H^1(B_{r\theta}) \cap C^0(B_{r\theta})$ with div $v \in L^q(B_r)$ satisfy

$$
\int_{A_{r\theta}} g + \int_{\partial B_r} v \cdot \nu = 0.
$$

Then, there exists $u \in H^1_0(B_{r\theta}) \cap C^0(B_{r\theta})$ solving

$$
\begin{aligned}
    \text{div } u &= g & \text{ in } A_{r\theta} \\
    u &= 0 & \text{ on } \partial B_{r\theta} \\
    u &= v & \text{ in } B_r,
\end{aligned}
$$

with

$$
\|u\|_{H^1} \leq C \|v\|_{H^1} + \|g\|_{L^2} + r^{\frac{d-2}{2}} (\|v\|_{C^0} + \|\text{div } v\|_{L^q(B_r)} + \|g\|_{L^q}),
$$

$$
\|u\|_{C^0} \leq C \|v\|_{C^0} + \|\text{div } v\|_{L^q(B_r)} + \|g\|_{L^q}.
$$

with $C = C(\theta, d, q)$.

Proof. We will define $u = u_1 + u_2$, where $u_1$ solves

$$
\begin{aligned}
    \text{div } u_1 &= g & \text{ in } A_{r\theta} \\
    \text{div } u_1 &= \text{div } v & \text{ in } B_r \\
    u_1 &= 0 & \text{ on } \partial B_{r\theta},
\end{aligned}
$$

and $u_2$ is the solution to

$$
\begin{aligned}
   -\Delta u_2 + \nabla p &= 0 & \text{ in } A_{r\theta} \\
   \text{div } u_2 &= 0 & \text{ in } A_{r\theta} \\
   u &= 0 & \text{ on } \partial B_{r\theta} \\
   u &= v - u_1 & \text{ in } B_r,
\end{aligned}
$$

(B.6)

As it is well known (see e.g. [8][Theorem 3.1]), the first problem has a solution with

$$
\|u_1\|_{H^1} \lesssim \|\text{div } v\|_{L^2(B_r)} + \|g\|_{L^2},
$$

$$
\|u_1\|_{W^{1,q}} \lesssim \|\text{div } v\|_{L^q(B_r)} + \|g\|_{L^q}.
$$

By Sobolev inequality,

$$
\|u_1\|_{C^0} \lesssim \|\text{div } v\|_{L^q(B_1)} + \|g\|_{L^q}.
$$

Using estimate (B.2) rescaled with $r$ for the solution to (B.6), we find

$$
\|\nabla u_2\|_{L^2} \lesssim \|\nabla (v - u_1)\|_{L^2} + \frac{1}{r} \|v - u_1\|_{L^2} \lesssim \|v\|_{L^2} + \|u_1\|_{L^2} + r^{\frac{d-2}{2}} \|v - u_1\|_{C^0}
$$

$$
\lesssim \|\nabla v\|_{L^2} + \|g\|_{L^2} + r^{\frac{d-2}{2}} (\|v\|_{C^0} + \|\text{div } v\|_{L^q(B_1)} + \|g\|_{L^q}),
$$

and

$$
\|u_2\|_{C^0} \lesssim \|v - u_1\|_{C^0} \lesssim \|v\|_{C^0} + \|\text{div } v\|_{L^q(B_1)} + \|g\|_{L^q}.
$$

Combining these inequalities for $u_1$ and $u_2$ (and the Poincare inequality) yields the desired estimate for $u$. \qed

Appendix C. Some results on Strong Law of Large Numbers

For the reader’s convenience, we list below some of the results proven in [10][Section 5] on Strong Law of Large Numbers for a general marked point process and which we use throughout this paper. We adapt these statements to our special case of $\Phi$ being a Poisson process with intensity $\lambda > 0$ (see also Section 2).

Lemma C.1. Let $(\Phi, \mathcal{R})$ be as in Section 2. Then, for every bounded set $B \subseteq \mathbb{R}^d$ which is star-shaped with respect to the origin, we have

$$
\lim_{\varepsilon \downarrow 0^+} \varepsilon^d N^\varepsilon(B) = \lambda |B| \quad \text{almost surely},
$$

(C.1)
Then, almost surely, we have
\[ \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in \Phi^\varepsilon(B)} \rho_i^{d-2} = \lambda \langle \rho^{d-2} \rangle |B| \quad \text{almost surely.} \quad (C.2) \]

Furthermore, for every $\delta < 0$ the process $\Phi_\delta$ obtained from $\Phi$ as in (2.3) satisfies the analogues of (C.2), (C.1) and
\[ \lim_{\delta \downarrow 0^+} \langle N_\delta(A) \rangle = \lambda |A| \quad (C.3) \]

for every bounded set $A \subseteq \mathbb{R}^d$.

**Lemma C.2.** In the same setting of Lemma C.1, let $\{I_\varepsilon\}_{\varepsilon > 0}$ be a family of collections of points such that $I_\varepsilon \subseteq \Phi^\varepsilon(B)$ and
\[ \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \# I_\varepsilon = 0 \quad \text{almost surely.} \]

Then,
\[ \lim_{\varepsilon \downarrow 0^+} \varepsilon^d \sum_{z_i \in I_\varepsilon} \rho_i^{d-2} \rightarrow 0 \quad \text{almost surely.} \]

**Lemma C.3.** In the same setting of Lemma C.1, let us assume that in addition the marks satisfy $\langle \rho^2(d - 2) \rangle < +\infty$. For $z_i \in \Phi$ and $\varepsilon > 0$, let $r_{i,\varepsilon} > 0$, and assume there exists a constant $C > 0$ such that for all $z_i \in \Phi$ and $\varepsilon > 0$
\[ r_{i,\varepsilon} \leq C \varepsilon. \]

Then, almost surely, we have
\[ \lim_{\varepsilon \downarrow 0^+} \sum_{z_i \in \Phi^\varepsilon(B)} \rho_i^{d-2} \frac{\varepsilon^d}{r_{i,\varepsilon}} \int_{B_{r_{i,\varepsilon}(z_i)}} \zeta(x) \, dx = |B_1| \langle \rho^{d-2} \rangle \int_B \zeta(x) \, dx, \]

for every $\zeta \in C^1_0(B)$, where $B_1 \subseteq \mathbb{R}^d$ denotes the unit ball.

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