Improved nearly minimax prediction for independent Poisson processes under Kullback–Leibler loss

Xiao Li¹,* and Fumiyasu Komaki¹,²

¹Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan
²RIKEN Center for Brain Science, 2-1 Hirosawa, Wako City, Saitama, 351-0198, Japan

*Corresponding Author: lixiaoms@163.com

Abstract

The problem of predicting independent Poisson random variables is commonly encountered in real-life practice. Simultaneous predictive distributions for independent Poisson observables are investigated, and the performance of predictive distributions is evaluated using the Kullback–Leibler (K-L) loss. This study introduces intuitive sufficient conditions, based on superharmonicity of priors, to improve the Bayesian predictive distribution based on the Jeffreys prior. The sufficient conditions exhibit a certain analogy with those known for the multivariate normal distribution. Additionally, this study examines the case where the observed data and target variables to be predicted are independent Poisson processes with different durations. Examples that satisfy the sufficient conditions are provided, including point and subspace shrinkage priors. The K-L risk of the improved predictions is demonstrated to be less than 1.04 times a minimax lower bound.

Keywords: Predictive distribution; Jeffreys prior; Superharmonic function; Shrinkage prior; Multivariate Poisson

1 Introduction

The problem of predicting independent Poisson random variables $y = (y_1, y_2, ..., y_d)$ using independent observations $x = (x_1, x_2, ..., x_d)$ is commonly encountered in real-life practice, such as sales forecasting. For example, if a store sells $d$ brands of products, the sales volume of the $i$th product within $r$ days is assumed to follow a Poisson distribution with parameter $r\lambda_i$. The store owner can use sales records from one month to predict the sales volume of each product for the next week by formulating the above problem with $r = 30$ and $s = 7$, where $r$ and $s$ are defined below. Similar applications are widespread in other fields, such as transportation volume forecasting and predicting the numbers of accidents and strikes. It is assumed that $x = (x_1, x_2, ..., x_d)$ and $y = (y_1, y_2, ..., y_d)$ are respectively distributed according to:

$$p(x \mid \lambda) = \prod_{i=1}^{d} p(x_i \mid \lambda_i) = \exp\{-r(\lambda_1 + \lambda_2 + \cdots + \lambda_d)\} \frac{(r\lambda_1)^{x_1}}{x_1!} \cdots \frac{(r\lambda_d)^{x_d}}{x_d!}$$

and

$$p(y \mid \lambda) = \prod_{i=1}^{d} p(y_i \mid \lambda_i) = \exp\{-s(\lambda_1 + \lambda_2 + \cdots + \lambda_d)\} \frac{(s\lambda_1)^{y_1}}{y_1!} \cdots \frac{(s\lambda_d)^{y_d}}{y_d!},$$

where $r$ and $s$ are known positive real numbers. Here, $\lambda = (\lambda_1, \ldots, \lambda_d)$ is an unknown parameter.

The Bayesian prediction distribution based on prior $\pi(\lambda)$ is expressed as:

$$p_{\pi}(y \mid x) = \frac{\int p(x, y \mid \lambda)\pi(\lambda)d\lambda}{\int p(x \mid \lambda)\pi(\lambda)d\lambda} = \frac{\int p(x \mid \lambda)p(y \mid \lambda)\pi(\lambda)d\lambda}{\int p(x \mid \lambda)\pi(\lambda)d\lambda}.$$
The Kullback–Leibler (K-L) loss of the predictive distribution $p_\pi(y \mid x)$ is used in this study, which is expressed as:

$$D(p(y \mid \lambda), p_\pi(y \mid x)) = \sum_y p(y \mid \lambda) \log \frac{p(y \mid \lambda)}{p_\pi(y \mid x)}.$$

A natural class of priors is expressed as:

$$\pi_\beta(\lambda) d\lambda_1 d\lambda_2 \cdots d\lambda_d \propto \lambda_1^{\beta_1-1} \lambda_2^{\beta_2-1} \cdots \lambda_d^{\beta_d-1} d\lambda_1 d\lambda_2 \cdots d\lambda_d,$$

where $\beta_i > 0$ and $i = 1, 2, \ldots, d$. $\pi_\beta(\lambda)$ is equal to the Jeffreys prior $\pi_\beta(\lambda)$ when $\beta_i = 1/2$ for any $i$, which is a frequently used noninformative prior. The prior $\pi_\beta(\lambda)$ has an advantage in that the K-L risk of the Bayesian predictive distribution $p_\beta(y \mid x)$ based on $\pi_\beta(\lambda)$ has an upper bound for any $\lambda$. In contrast, the K-L risk of the Bayesian predictive distribution has no upper bound if a gamma prior distribution is adopted. The corresponding proofs are provided in the Appendix. Therefore, a main focus of this study is the construction of the Bayesian predictive distribution, which has no upper bound if a gamma prior distribution is adopted. The estimation of a multivariate normal mean was introduced by Brown (1971). Recently, Hamura and Kubokawa (2020) studied the Bayesian predictive distribution for a Poisson model with parametric restriction under K-L loss. Yano et al. (2021) presented a class of Bayesian predictive distributions and estimators based on the priors were shown to dominate the Bayesian predictive distribution based on the Jeffreys prior (Komaki, 2006). There are several counterparts for distributions that attain asymptotic minimaxity in sparse Poisson sequence models.

Numerous studies have been performed on the simultaneous estimation of Poisson parameters. Clewonson and Zidek (1975) proposed generalized Bayes estimators dominating the maximum likelihood estimator when $d \geq 2$ under the standardized squared error loss $\sum \lambda_i^{-1} (\lambda_i - \lambda_i)^2$. Tsui and Press (1982) studied the estimation under the generalized loss function $\sum (\lambda_i - \lambda_i)^2 / \lambda_i^k$, where $k$ is a given positive integer. Ghosh and Yang (1988) characterized admissible linear estimators of multiple Poisson parameters under K-L loss. Estimation of parameters under K-L loss can be generalized to a predictive distribution problem, which is important for several statistical scenarios. The predictive method was shown to be preferable in Aitchison (1975). Noninformative prior or vague prior distributions are often used for constructing Bayesian predictive distributions. The Jeffreys prior has been widely used in various problems under K-L loss, such as in Akaake (1978) and Clarke and Barron (1994).

Compared with the large number of estimation studies, decision theory regarding predictive distributions on the Poisson model has been developed relatively recently. A class of prior distributions, $\pi_{\alpha, \beta}(\lambda) d\lambda_1 d\lambda_2 \cdots d\lambda_d \propto \lambda_1^{\beta_1-1} \lambda_2^{\beta_2-1} \cdots \lambda_d^{\beta_d-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_d)^{\alpha} d\lambda_1 d\lambda_2 \cdots d\lambda_d$ was proposed in Komaki (2004). Let $\theta_i := \sqrt{n_i}$, $i = 1, \ldots, d$. The Bayesian predictive distribution based on the prior $\|\theta\|^{2-d} d\theta$ was shown to dominate that based on the Jeffreys prior when $d \geq 3$ (Komaki, 2004). Komaki (2015) considered the problem of independent Poisson processes with different durations and introduced a class of improper prior densities that is a generalization of $\pi_{\alpha, \beta}(\lambda)$. The corresponding Bayesian predictive distribution was shown to dominate that based on the Jeffreys prior. A class of proper priors was proposed, and Bayesian predictive distributions and estimators based on the priors were shown to dominate the Bayesian predictive distribution and estimator based on the Jeffreys prior under K-L loss (Komaki, 2006). The proper priors with respect to $\theta$ coincides with the function of Strawderman’s prior in a normal model (Strawderman, 1971). Recently, Hamura and Kubokawa (2020) studied the Bayesian predictive distribution for a Poisson model with parametric restriction under K-L loss. Yano et al. (2021) presented a class of Bayesian predictive distributions that attain asymptotic minimaxity in sparse Poisson sequence models.

It is natural that similar results hold simultaneously for the multivariate normal and Poisson models from the viewpoint of a model manifold with the Fisher metric (Komaki, 2006). There are several counterparts for these two models. The Bayesian predictive distribution for a multivariate normal model $N_d(\mu, \sigma^2 I)$ based on Stein’s harmonic prior (Stein, 1974), i.e.,

$$\pi(\mu) = \|\mu\|^{2-d},$$

dominate that based on the Jeffreys prior (Komaki, 2001). This result is similar to that reported in Komaki (2004). Johnstone (1984) studied the admissibility and recurrence in estimating a Poisson mean under the standardized squared error loss, which is a counterpart to the diffusion characterization of admissibility in the estimation of a multivariate normal mean that was introduced by Brown (1971).
George et al. (2006) generalized the result presented in Komaki (2001) for the multivariate normal model and proved that Bayesian predictive distributions based on superharmonic priors dominate those based on the Jeffreys prior. Thus, it is natural to speculate that a similar result exists for the independent Poisson observable model. This speculation is confirmed in the present study, which demonstrates the relationship between the superharmonic function and improved Bayesian prediction in the Poisson model.

**Contribution and structure of the paper:** Theorem 1 provides sufficient conditions for prior \( \pi(\lambda) \) to make \( p_\pi(y \mid x) \) dominate the Bayesian predictive distribution \( p_\beta(y \mid x) \) based on \( \pi_\beta(\lambda) \). Sufficient conditions are also provided for prior \( \pi(\lambda) \) to make \( p_\pi(y \mid x) \) dominate \( p_\beta(y \mid x) \) for the simultaneous prediction of independent Poisson processes with different durations (Theorem 3). Let function \( f \) denote \( \pi(\lambda)/\pi_\beta(\lambda) \); it is shown that the key points for prior \( \pi(\lambda) \) are to study functions \( f(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d}) \) and \( f(\sqrt{\lambda_1}\gamma_1, \ldots, \sqrt{\lambda_d}/\gamma_d) \), where \( \gamma \) is obtained from the durations \( r \) and \( s \). The proof of the theorems may be applicable to the study of improved Bayesian predictive distribution for other distributions.

In Theorems 2 and 3, the Bayesian predictive distribution \( p_\beta(y \mid x) \) based on the Jeffreys prior. Thus, it is natural to speculate that a similar result exists for the independent Poisson observable model. This speculation is confirmed in the present study, which demonstrates the relationship between the superharmonic function and improved Bayesian prediction in the Poisson model. Therefore, it is shown in Theorem 4, the K-L risk of \( p_\beta(y \mid x) \) is shown to be less than 1.04 times a minimax lower bound. Therefore, these improved Bayesian predictive distributions are nearly minimax.

This paper is organized as follows. Theorem 1, i.e., the result of sufficient conditions for prior \( \pi(\lambda) \) to improve the Bayesian predictive distribution \( p_\beta(y \mid x) \) is presented in Section 2. In addition, Theorem 2 i.e., the main result on using superharmonic functions to improve the Bayesian predictive distribution \( p_\lambda(y \mid x) \) is provided. Similar results for independent Poisson processes with different durations are presented in Section 3. Examples are provided in Section 4, including point and subspace shrinkage priors. Here, the Bayesian predictive distributions based on the example priors are proven to dominate those based on the Jeffreys prior. Numerical experiments and a real data application are presented in Section 5. Finally, the results and example priors are discussed in Section 6. The proof of the theorems and propositions are provided in the Appendix.

## 2 Improved Bayesian prediction using superharmonic function

In this section, we define \( \theta_i := \sqrt{\lambda_i}, \ i = 1, \ldots, d \). The goal is to find function \( f(\theta_1, \ldots, \theta_d) \) such that for prior

\[
\pi_{f,\beta}(\lambda) \propto f(\theta_1, \ldots, \theta_d)\lambda_1^{\beta_1-1}\lambda_2^{\beta_2-1}\cdots\lambda_d^{\beta_d-1}d\lambda_1d\lambda_2\cdots d\lambda_d,
\]

the Bayesian predictive distribution \( p_{f,\beta}(y \mid x) \) based on \( \pi_{f,\beta}(\lambda) \) dominates the Bayesian predictive distribution \( p_\beta(y \mid x) \) based on \( \pi_\beta(\lambda) \) under K-L loss. \( \pi_{f,\beta} \) denotes the prior \( \pi_{f,\beta} = \pi_{(1/2,\ldots,1/2)} \), which is expressed as:

\[
\pi_{f,\beta}(\lambda) = f(\theta_1, \ldots, \theta_d)\lambda_1^{-1/2}\lambda_2^{-1/2}\cdots\lambda_d^{-1/2}d\lambda_1d\lambda_2\cdots d\lambda_d.
\]

Therefore, prior \( \pi_{f,\beta}(\lambda)d\lambda \) is equivalent to prior \( f(\theta)d\theta \).

In Theorem 1 we show that if \( f \) satisfies certain conditions, the Bayesian predictive distribution \( p_{f,\beta}(y \mid x) \) dominates the Bayesian predictive distribution \( p_\beta(y \mid x) \). These conditions include three regularity and two essential conditions. As a corollary, for any superharmonic function satisfying certain regularity conditions, we can construct \( f \) such that \( p_{f,\beta}(y \mid x) \) based on prior \( f(\theta)d\theta \) dominates \( p_\lambda(y \mid x) \) based on the Jeffreys prior. The specific description of the result is presented in Theorem 2.

In brief, \( p_{f,\beta}(y \mid x) \) dominates \( p_\beta(y \mid x) \) if \( f(\theta) = \sum_{a \in \{1, -1\}^d} h(a_1\theta_1, a_2\theta_2, \ldots, a_d\theta_d) \) and \( h(\theta) \) is a superharmonic function. Here,

\[
\sum_{a \in \{1, -1\}^d} h(a_1\theta_1, a_2\theta_2, \ldots, a_d\theta_d) := h(\theta_1, \theta_2, \ldots, \theta_d) + h(-\theta_1, \theta_2, \ldots, \theta_d) + \cdots + h(\theta_1, -\theta_2, \ldots, -\theta_d) + h(-\theta_1, -\theta_2, \ldots, -\theta_d).
\]

However, \( p_{f,\beta}(y \mid x) \) based on the superharmonic prior \( f(\theta) = h(\theta) \) does not necessarily dominate \( p_\lambda(y \mid x) \), which is different from the result in the normal model (George et al. 2006). For example, the K-L risk of
prior study. The Bayesian predictive distribution based on the Jeffreys prior could be improved using superharmonic
The similarity and difference between the Poisson and normal models are consistent with the results of this
µ model with parameter θ defined on z 56, which is larger than that of p1(y | x). The K-L risk of p_{f,1}(y | x) is 0.56 if
is 0.62, which is larger than that of p1(y | x). The K-L risk of p_{f,1}(y | x) is 0.56 if
f(θ) = \sum_{a \in \{1,-1\}^d} \left( (a_1θ_1 - 2)^2 + (a_2θ_2 - 2)^2 + (a_3θ_3 - 2)^2 \right)^{-1/2}.

Note that the domain of definition for the mean parameter µ of the normal model is \mathbb{R}^d, whereas the
domain of definition for parameter θ of the Poisson model is \mathbb{R}_+^d. In addition, the Fisher metric of the normal
model with parameter µ and that of the Poisson model with parameter θ are Euclidean metric (Figure 1).
The similarity and difference between the Poisson and normal models are consistent with the results of this
study. The Bayesian predictive distribution based on the Jeffreys prior could be improved using superharmonic
functions in both models because the Fisher metric is Euclidean. \( f(θ) = \sum_{a \in \{1,-1\}^d} h(a_1θ_1, a_2θ_2, \ldots, a_dθ_d) \) is
set to ensure that the prior \( f(θ)dθ \) is superior to the Jeffreys prior. This is consistent with the domain of
definition for the parameter θ, which is \mathbb{R}_+^d.

Figure 1: Poisson model with parameter θ and normal model with parameter µ exhibit
the same Fisher metric even though their domains of definition are different.

**Theorem 1.**

1) Let \( θ_i = \sqrt{\lambda_i}, i = 1, \ldots, d \). The Bayesian predictive distribution \( p_{f,β}(y | x) \) dominates the Bayesian
predictive distribution \( p_β(y | x) \) if for every \( r > 0 \), the function
\[
F(z, r) = F(z_1, z_2, \ldots, z_d, r) := \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^d r^{z_i + θ_i} \exp(-rλ_i) \text{d}λ
\]
defined on \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{N}^d \) is not a constant function, and for every \( z \in \mathbb{N}^d, r > 0 \),
\[
\sum_{i=1}^d z_i(F(z, r) - F(z - δ_i, r)) + \sum_{i=1}^d (z_i + θ_i)(F(z, r) - F(z + δ_i, r)) ≥ 0, \tag{2.1}
\]
where \( δ_i \) denotes the d-dimensional vector whose \( i^{th} \) element is 1, all other elements are 0, and \( F(z - δ_i, r) \) is
defined as 1 if \( z_i = 0 \).

2) Condition \( 2.1 \) is satisfied if \( f \in C^2([0, ∞)^d) \) satisfies the regularity conditions provided below and the following conditions:
\[
\sum_{i=1}^d \frac{∂}{∂θ_i} \left( \frac{∂f}{∂θ_i} (θ) \prod_{j=1}^d θ_j^{2β_j - 1} \right) ≤ 0; \tag{2.2a}
\]
and
\[ \lim_{\theta_i \to 0} \frac{\partial f}{\partial \theta_i} (\theta) \theta_i^{2\beta_i - 1} \leq 0, \] (2.2b)
for every \( i \).

The proof of Theorem \( \Pi \) is presented in the Appendix.

The regularity conditions are expressed as:
\[
\int f(\theta) \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.3a)
\]
\[
\int \left| \frac{\partial f}{\partial \theta_i} (\theta) \right| \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.3b)
\]
and
\[
\int \left| \frac{\partial^2 f}{\partial \theta_i^2} (\theta) \right| \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.3c)
\]
for every \( z \in \mathbb{N}^d \), \( r > 0 \), and \( i \). Conditions \( (2.3a) \), \( (2.3b) \), and \( (2.3c) \) are related to the integrability of the expressions involving \( f(\theta) \), \( \frac{\partial f}{\partial \theta_i} (\theta) \), and \( \frac{\partial^2 f}{\partial \theta_i^2} (\theta) \), respectively. These conditions are necessary to ensure that each part is integrable when integration by parts is used in the proof of Theorem \( \Pi \). These regularity conditions hold when \( f(\theta) \), \( \frac{\partial f}{\partial \theta_i} (\theta) \), and \( \frac{\partial^2 f}{\partial \theta_i^2} (\theta) \) are less than an exponential function \( \exp(o(\sum_{j=1}^{d} \theta_j^2)) \). This is because \( \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(o(\sum_{j=1}^{d} \theta_j^2) - r \sum_{j=1}^{d} \theta_j^2) \) is integrable on \([0, \infty)^d \). Here, \( o(\sum_{j=1}^{d} \theta_j^2) \) implies a function \( g(\theta) \) such that \( g(\theta) / (\sum_{j=1}^{d} \theta_j^2) \to 0 \) when \( \sum_{j=1}^{d} \theta_j^2 \to \infty \).

These conditions yield the following conditions when \( \beta_i = 1/2 \), \( \forall i \):
\[
\int f(\theta) \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.4a)
\]
\[
\int \left| \frac{\partial f}{\partial \theta_i} (\theta) \right| \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.4b)
\]
and
\[
\int \left| \frac{\partial^2 f}{\partial \theta_i^2} (\theta) \right| \prod_{j=1}^{d} \theta_j^{2\beta_j} \exp(-r\theta_j^2) \text{d}\theta < \infty, \quad (2.4c)
\]
for every \( z \in \mathbb{N}^d \), \( r > 0 \), and \( i \). Subsequently, these regularity conditions hold when \( f(\theta) \), \( \frac{\partial f}{\partial \theta_i} (\theta) \), and \( \frac{\partial^2 f}{\partial \theta_i^2} (\theta) \) are less than an exponential function \( \exp(o(\sum_{j=1}^{d} \theta_j^2)) \).

Conditions \( (2.2a) \) and \( (2.2b) \) are related to the signs of the expressions involving the derivative and second derivative of \( f \), respectively. The Jeffreys prior, i.e.,
\[
\pi_1(\lambda) \propto \frac{1}{(\lambda_1\lambda_2 \cdots \lambda_d)^{1/2}} \text{d}\lambda_1 \text{d}\lambda_2 \cdots \text{d}\lambda_d,
\]
corresponds to the case that \( \beta_i = 1/2 \), \( \forall i \). In this case, condition \( (2.2a) \) indicates that \( f \) is a superharmonic function.

Condition \( (2.2b) \) indicates that the derivative of \( f \) on the boundary is nonpositive. For any superharmonic function \( h \in C^2(\mathbb{R}^d) \), we note that
\[
f(\theta) = \sum_{a \in \{1,-1\}^d} h(a_1\theta_1, a_2\theta_2, \ldots, a_d\theta_d)
\]
satisfies the two conditions. Therefore, the relationship between the superharmonic function and improvement of the Bayesian predictive distribution based on the Jeffreys prior is obtained, as provided below.

**Theorem 2.**

1) Let \( \theta_i := \sqrt{\lambda_i} \), \( i = 1, \ldots, d \). The Bayesian predictive distribution \( p_{f,1}(y \mid x) \) dominates the Bayesian
predictive distribution \( p_3(y \mid x) \) based on the Jeffreys prior if for every \( r > 0 \), the function

\[
F(z, r) = F(z_1, z_2, \ldots, z_d, r) := \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{r^{z_i+1/2} \lambda_i^{z_i-1/2} \exp(-r \lambda_i)}{\Gamma(z_i + 1/2)} \, d\lambda
\]
defined on \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{N}^d \) is not a constant function, and for every \( z \in \mathbb{N}^d, r > 0, \)

\[
\sum_{i=1}^{d} z_i \{ F(z, r) - F(z - \delta_i, r) \} + \sum_{i=1}^{d} (z_i + 1/2) \{ F(z, r) - F(z + \delta_i, r) \} \geq 0. \tag{2.5}
\]

2) Condition \( (2.5) \) is satisfied if

\[
f(\theta) = \sum_{a \in \{1,\ldots, r\}^d} h(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d),
\]

where \( h \in C^2(\mathbb{R}^d) \) denotes a superharmonic function and \( f \) satisfies regularity conditions \( (2.4a), (2.4b) \), and \( (2.4c) \).

For the multivariate normal model, George et al. \( [2006] \) proved that Bayesian predictive distributions based on the superharmonic prior \( h(\mu_1, \ldots, \mu_d) \, d\mu \) dominate those based on the Jeffreys prior. Here, we proved that the Bayesian predictive distributions based on \( \sum_{a \in \{1,\ldots, r\}^d} h(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d) \, d\theta \) dominate those based on the Jeffreys prior for the multivariate Poisson model. We have shown that Bayesian predictive distributions based on \( h(\theta_1, \ldots, \theta_2) \, d\theta \) does not necessarily dominate those based on the Jeffreys prior using a numerical example provided at the beginning of this section. Hence, the results indicate the similarity and difference between the multivariate Poisson and normal models.

The theorem is divided into two parts to make it applicable to prior \( f(\theta) \) that is not differentiable. For example, we can construct a series of differentiable priors

\[
f_n(\theta) = \sum_{a \in \{1,\ldots, r\}^d} \left( (a_1 \theta_1 - 2)^2 + (a_2 \theta_2 - 2)^2 + (a_3 \theta_3 - 2)^2 + \frac{1}{n} \right)^{-1/2},
\]

for the prior

\[
f(\theta) = \sum_{a \in \{1,\ldots, r\}^d} \left( (a_1 \theta_1 - 2)^2 + (a_2 \theta_2 - 2)^2 + (a_3 \theta_3 - 2)^2 \right)^{-1/2}.
\]

\( f_n \) satisfies the conditions of the second part, thus \( f_n \) satisfies the conditions of the first part. Subsequently, it is proven that \( f \) also satisfies the conditions of the first part by using \( f_n \overset{a.s.}{\rightarrow} f \).

The K-L risk of estimator \( \hat{\lambda} \) is defined as the K-L divergence of \( p(x \mid \lambda) \) and plug-in density \( p(x \mid \hat{\lambda}) \), which is expressed as:

\[
\sum_x \left[ \sum_{i=1}^{d} \left( r \lambda_i \log \left( \frac{\lambda_i}{\hat{\lambda}_i} \right) - r \lambda_i + r \hat{\lambda}_i \right) \right] \prod_{j=1}^{d} \frac{(r \lambda_j)^{x_j} \exp(-r \lambda_j)}{x_j!}.
\]

The Bayesian estimators based on \( \pi_3(\lambda) \) and \( \pi_{f,\beta}(\lambda) \) are known to be \( \left( \frac{x_1 + \beta_1}{r}, \frac{x_2 + \beta_2}{r}, \ldots, \frac{x_d + \beta_d}{r} \right) \) and \( \left( \frac{x_1 + \beta_1}{r} F(x + \beta_1, r), \frac{x_2 + \beta_2}{r} F(x + \beta_2, r), \ldots, \frac{x_d + \beta_d}{r} F(x + \beta_d, r) \right) \), respectively. Therefore, the difference between the K-L risks of the Bayesian estimators based on \( \pi_3(\lambda) \) and \( \pi_{f,\beta}(\lambda) \) has the same sign as Eq. \( (A.10) \), which is provided in the Appendix. Using the proof of Theorem \( \Pi \) and considering the case that \( \beta_i = 1/2, \forall i \), the following can be obtained.

**Corollary 1.**

1) Let \( \theta_i := \sqrt{\lambda_i}, i = 1, \ldots, d \). The Bayesian estimator based on \( \pi_{f,3} \) dominates the Bayesian estimator based on \( \pi_3 \) if for every \( r > 0 \), the function

\[
F(z, r) = F(z_1, z_2, \ldots, z_d, r) := \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{r^{z_i+1/2} \lambda_i^{z_i-1/2} \exp(-r \lambda_i)}{\Gamma(z_i + 1/2)} \, d\lambda
\]
Therefore, prior durations which is discussed in the previous section.

Theorem 3.

2) Condition \(2.6\) is satisfied if

\[
 f(\theta) = \sum_{a \in \{1, \ldots, d\}} h(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d),
\]

where \(h \in C^2(\mathbb{R}^d)\) denotes a superharmonic function and \(f\) satisfies regularity conditions \(2.4a\), \(2.4b\), and \(2.4c\).

From the corollary, the Bayesian estimator based on the Jeffreys prior can also be improved using superharmonic functions.

3 Improved prediction for independent Poisson processes with different durations

In this section, we consider the case of independent Poisson processes with different durations. Suppose that \(x_i\) and \(y_i\) \((i = 1, \ldots, d)\) are independently distributed according to Poisson distributions

\[
p(x | \lambda) = \prod_{i=1}^{d} \frac{(r_i \lambda_i)^{x_i}}{x_i!} e^{-r_i \lambda_i}
\]

and

\[
p(y | \lambda) = \prod_{i=1}^{d} \frac{(s_i \lambda_i)^{y_i}}{y_i!} e^{-s_i \lambda_i}
\]

with mean \(r_i \lambda_i\) and \(s_i \lambda_i\), respectively. If \(r_1 = r_2 = \cdots = r_d\) and \(s_1 = s_2 = \cdots = s_d\), it is a case of same durations which is discussed in the previous section.

Define \(\gamma_i := \frac{1}{r_i} - \frac{1}{r_i + s_i}\) and \(\theta_i := \sqrt[\gamma_i]{\lambda_i}, i = 1, \ldots, d\). Prior \(\pi_{f, \beta}(\lambda)\) is still defined as

\[
\pi_{f, \beta}(\lambda) d\lambda_1 d\lambda_2 \cdots d\lambda_d \propto f(\theta_1, \ldots, \theta_d) \lambda_1^{\beta_1 - 1} \lambda_2^{\beta_2 - 1} \cdots \lambda_d^{\beta_d - 1} d\lambda_1 d\lambda_2 \cdots d\lambda_d.
\]

Therefore, prior \(\pi_{f, \beta}(\lambda) d\lambda = \pi_{f, \beta = (1/2, \ldots, 1/2)} d\lambda\) is equivalent to prior \(f(\theta)d\theta\). Note that \(\gamma_i\) is given and does not change in the following results.

We show that if \(f\) satisfies certain conditions that are similar to the conditions in Section 2, the Bayesian predictive distribution \(p_{f, \beta}(y | x)\) dominates the Bayesian predictive distribution \(p_\beta(y | x)\).

Theorem 3.

1) Let \(\theta_i := \sqrt[\gamma_i]{\frac{1}{\lambda_i}}, i = 1, \ldots, d\). The Bayesian predictive distribution \(p_{f, \beta}(y | x)\) dominates the Bayesian predictive distribution \(p_\beta(y | x)\) if for every \(r > 0\), the function

\[
F(z, r) = F(z_1, z_2, \ldots, z_d, r) := \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{r_i^{z_i} \lambda_i^{z_i + \beta_i - 1} \exp(-r_i \lambda_i)}{\Gamma(z_i + \beta_i)} d\lambda
\]

defined on \(z = (z_1, z_2, \ldots, z_d) \in \mathbb{N}^d\) is not a constant function, and for every \(z \in \mathbb{N}^d, r > 0\),

\[
\sum_{i=1}^{d} \gamma_i r_i z_i \left\{ F(z, r) - F(z - \delta_i, r) \right\} + \sum_{i=1}^{d} \gamma_i r_i (z_i + \beta_i) \left\{ F(z, r) - F(z + \delta_i, r) \right\} \geq 0,
\]

where \(F(z - \delta_i, r) := 1\) if \(z_i = 0\).

2) Condition \(3.1\) is satisfied if \(f \in C^2([0, \infty)^d)\) satisfies regularity conditions \(2.3a\), \(2.3b\), and \(2.3c\), and conditions \(2.2a\) and \(2.2b\).
The proof of Theorem 3 is a generalization of that of Theorem 1, which is presented in the Appendix. The relationship between the superharmonic function and improvement of the Bayesian predictive distribution based on the Jeffreys prior is obtained considering the case of \( \beta = 1/2 \), \( \forall i \), as provided below.

**Theorem 4.**

1) Let \( \lambda_i := \sqrt{X_i/\gamma_i} \), \( i = 1, \ldots, d \). The Bayesian predictive distribution \( p_{f,1}(y \mid x) \) dominates the Bayesian predictive distribution \( p_1(y \mid x) \) based on the Jeffreys prior if for every \( r > 0 \), the function

\[
F(z, r) = F(z_1, z_2, \ldots, z_d, r) := \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{r_i^{z_i+1/2} \lambda_i^{a_i-1/2} \exp(-r_i \lambda_i)}{\Gamma(z_i + 1/2)} d\lambda
\]

defined on \( z = (z_1, z_2, \ldots, z_d) \in \mathbb{N}^d \) is not a constant function, and for every \( z \in \mathbb{N}^d \), \( r > 0 \),

\[
\sum_{i=1}^{d} \gamma_i r_i z_i \left\{ F(z, r) - F(z - \delta_i, r) \right\} + \sum_{i=1}^{d} \gamma_i r_i (z_i + 1/2) \left\{ F(z, r) - F(z + \delta_i, r) \right\} \geq 0. \tag{3.2}
\]

2) Condition 3.2 is satisfied if

\[
f(\theta) = \sum_{a \in \{1, -1\}^d} h(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d),
\]

where \( h \in C^2(\mathbb{R}^d) \) denotes a superharmonic function and \( f \) satisfies regularity conditions (2.4a), (2.4b), and (2.4c).

Therefore, the Bayesian predictive distribution based on prior \( \sum_{a \in \{1, -1\}^d} h(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d) d\theta \) dominates that based on the Jeffreys prior. We focus on the construction of a prior that is superior to the Jeffreys prior because of the results provided below.

**Theorem 5.**

1) For any \( \lambda \), the K-L risk of \( p_1(y \mid x) \) is less than 0.52 \( \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right) \).

2) For any predictive distribution \( q(y \mid x) \) and positive number \( \epsilon \), there exists \( \lambda \) such that the K-L risk of \( q(y \mid x) \) is greater than 0.5 \( \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right) - \epsilon \).

The result is a generalization of Theorems 1 and 2 in Li (2023). The proof is presented in the Appendix. According to the theorem, the upper bound of the K-L risk of \( p_1(y \mid x) \) is less than 1.04 times the minimax lower bound. The definition of a nearly minimax predictive distribution is presented below. Hence, the Bayesian predictive distribution based on the Jeffreys prior is nearly minimax.

**Definition 1.** A predictive distribution \( q(y \mid x) \) is labeled as nearly minimax if for any \( \lambda \), the K-L risk of \( q(y \mid x) \) is less than 1.04 times the minimax lower bound.

## 4 Examples

We provide examples that satisfy the conditions presented in Sections 2 and 3, including point and subspace shrinkage priors. The proofs of the propositions are presented in the Appendix.

### 4.1 Shift point shrinkage prior

The class of priors in Komaki (2004) and Komaki (2015) are considered in this study. These are

\[
\pi_{\alpha, \beta}(\lambda)d\lambda_1 d\lambda_2 \cdots d\lambda_d \propto \frac{\lambda_1^{\alpha_1 - 1} \lambda_2^{\alpha_2 - 1} \cdots \lambda_d^{\alpha_d - 1}}{(\lambda_1 + \lambda_2 + \cdots + \lambda_d)^\alpha} d\lambda_1 d\lambda_2 \cdots d\lambda_d
\]

for the prediction of independent Poisson processes with the same duration, and

\[
\pi_{\alpha, \beta, \gamma}(\lambda)d\lambda_1 d\lambda_2 \cdots d\lambda_d \propto \frac{\lambda_1^{\alpha_1 - 1} \lambda_2^{\alpha_2 - 1} \cdots \lambda_d^{\alpha_d - 1}}{(\lambda_1/\gamma_1 + \lambda_2/\gamma_2 + \cdots + \lambda_d/\gamma_d)^\alpha} d\lambda_1 d\lambda_2 \cdots d\lambda_d
\]
for the prediction of independent Poisson processes with different durations, where \(0 < \alpha \leq \sum_{i=1}^{d} \beta_i - 1\). The two priors are the same as \(\pi_{f,\beta}(\lambda)\) in this study, where \(f(\theta) = (\sum_{i=1}^{d} \theta_i^2)^{-\alpha}\).

**Proposition 1.** The Bayesian predictive distribution \(p_{f,\beta}(y \mid x)\) dominates the Bayesian predictive distribution \(p_{\beta}(y \mid x)\) for the prediction of independent Poisson processes with the same or different durations when

\[
f(\theta) = \left(\sum_{i=1}^{d} \theta_i^2 + \eta\right)^{-\alpha},
\]

where \(\eta \geq 0\) and \(0 < \alpha \leq \sum_{i=1}^{d} \beta_i - 1\).

### 4.2 Point shrinkage prior

The Bayesian predictions based on prior \(\pi_{f,\beta}(\lambda)\) with \(f(\theta) = (\sum_{i=1}^{d} \theta_i^2 + \eta)^{-\alpha}\) shrink \(\theta\) toward the origin. Therefore, it is natural to investigate the priors that shrink \(\theta\) toward a general point \((\eta_1, \ldots, \eta_d)\). Note that function \((\sum_{i=1}^{d} (\theta_i - \eta_i)^2)^{-\alpha}\) is superharmonic when \(0 < \alpha \leq (d - 2)/2\). Proposition 2 is obtained using Theorem 4.

**Proposition 2.** The Bayesian predictive distribution based on \(\pi_{f,\beta}(\lambda)\) dominates that based on the Jeffreys prior and is thus nearly minimax for the prediction of independent Poisson processes with the same or different durations when

\[
f(\theta) = \sum_{a \in \{1, -1\}^d} \left(\sum_{i=1}^{d} (a_i \theta_i - \eta_i)^2\right)^{-\alpha},
\]

where \(0 < \alpha \leq (d - 2)/2\).

Prior \(\pi_{f,\beta}(\lambda)\) is equivalent to prior \(f(\theta)\). Thus, prior \(\pi_{f,\beta}(\lambda)\) shrinks \(\theta\) toward point \((\eta_1, \ldots, \eta_d)\) if \(\eta_i \geq 0, \, i = 1, \ldots, d\). From Corollary 1, it is known that the Bayesian estimators based on the point shrinkage prior dominates that based on the Jeffreys prior for a case with the same duration.

### 4.3 Subspace shrinkage prior

In the previous examples, the Bayesian predictions shrink \(\theta\) toward a point. Therefore, it is natural to investigate the subspace shrinkage prior that is constructed by the function \((sv(\theta))^{-\alpha}\), where \(sv(\theta)\) represents the squared distance from \(\theta\) to a linear subspace \(V \subset \mathbb{R}^d\).

The complementary space is assumed to be \(V^\perp = \text{span}(v_1, v_2, \ldots, v_{d-k})\), where \(\{v_1, v_2, \ldots, v_{d-k}\}\) denotes a standard orthonormal basis and \(k\) denotes the dimension of \(V\). We have

\[
s_v(\theta) = \sum_{i=1}^{d-k} (\theta_i v_i)^2.
\]

Note that function \((sv(\theta))^{-\alpha}\) is superharmonic when \(0 < \alpha \leq (d - k - 2)/2\). Proposition 3 is obtained using Theorem 4.

**Proposition 3.** The Bayesian predictive distribution based on \(\pi_{f,\beta}(\lambda)\) dominates that based on the Jeffreys prior and is thus nearly minimax for the prediction of independent Poisson processes with the same or different durations when

\[
f(\theta) = \sum_{a \in \{1, -1\}^d} (sv(a_1 \theta_1, a_2 \theta_2, \ldots, a_d \theta_d))^{-\alpha} = \sum_{a \in \{1, -1\}^d} (sv(a \theta))^{-\alpha},
\]

where \(0 < \alpha \leq (d - k - 2)/2\).

Prior \(\pi_{f,\beta}(\lambda)\) is equivalent to prior \(f(\theta)\). Thus, prior \(\pi_{f,\beta}(\lambda)\) shrinks \(\theta\) toward subspace \(V\) if \(V \cap \mathbb{R}^d_+ \neq \emptyset\).
4.4 Mix subspace shrinkage prior

The function $F$ in the first half of Theorem 4 has additivity, which implies that $F$ equals the sum of $F_1$ and $F_2$ if $f = f_1 + f_2$, which is expressed as:

$$F(z, r) = \int (f_1 + f_2) \prod_{i=1}^{d} r_i z_i^{1/2} \lambda_i^{z_i-1/2} \exp(-r_i \lambda_i) \frac{\exp(-r_i \lambda_i)}{\Gamma(z_i + 1/2)} d\lambda = F_1(z, r) + F_2(z, r).$$

Therefore, we can take the sum over different priors to obtain a new prior and ensure that the inequality condition (3.2) in the first half of Theorem 4 is still satisfied. For example, if there are $n$ pairs of subspace $V_i$ and $\alpha_i$ satisfying the conditions of Proposition 3, Proposition 4 is obtained using Theorem 4.

Proposition 4. The Bayesian predictive distribution based on $\pi_f, J(\lambda)$ dominates that based on the Jeffreys prior and is thus nearly minimax for the prediction of independent Poisson processes with the same duration or different durations when

$$f(\theta) = \sum_{a \in \{1, -1\}^d} (s_{V_1}(a\theta))^{-\alpha_1} + \sum_{a \in \{1, -1\}^d} (s_{V_2}(a\theta))^{-\alpha_2} + \cdots + \sum_{a \in \{1, -1\}^d} (s_{V_n}(a\theta))^{-\alpha_n},$$

where $0 < \alpha_i \leq (d - k_i - 2)/2$ and $k_i$ denotes the dimension of $V_i$, $i = 1, 2, \ldots, n$.

We call this type of prior a “mix subspace shrinkage prior.” A specific example is presented in the next section. A similar prior distribution has been studied for the normal model (George, 1986; George et al., 2012). From Corollary 4, it is known that the Bayesian estimators based on the subspace and mix subspace shrinkage priors dominate that based on the Jeffreys prior for a case with the same duration.

5 Numerical experiments and real data application

We perform numerical experiments to demonstrate the difference between the risk of the Bayesian predictive distribution based on the Jeffreys prior and those based on the shrinkage priors discussed in Section 4. We then use an application involving real data to compare the performance of different types of priors. Only cases with the same durations are considered in the experiments.

Experiment 1.

We set $r = s = 1$, $d = 3$, and $\theta_i = \sqrt{\lambda_i}$, $i = 1, \ldots, 3$. The first prior is a point shrinkage prior with

$$f(\theta) = \left( \frac{3}{\sqrt{\lambda_i}} \right)^{-(3-2)/2}.$$

The second prior is a “shift point shrinkage prior” with

$$f(\theta) = \left( \frac{3}{\sqrt{\lambda_i}} + 1 \right)^{-(3-2)/2}.$$

Figure 2 shows the differences between the risks of the Bayesian predictive distributions based on the two priors and the Jeffreys prior when $\lambda = \Lambda \times (1/3, 1/3, 1/3)$. The Bayesian predictive distribution based on the point shrinkage prior performs better when $\Lambda$ is small. The Bayesian predictive distribution based on the shift point shrinkage prior performs better when $\Lambda$ is large.
Figure 2: Log value of the difference between the K-L risks, i.e.,
\[
\log E \left[ D(p(y \mid \lambda), p_J(y \mid x)) - D(p(y \mid \lambda), p_{f,J}(y \mid x)) \right] \lambda.
\]

**Experiment 2.**

We set \( r = s = 1, d = 3, \) and \( \theta_i = \sqrt{\lambda_i}, \) \( i = 1, \ldots, 3. \) The first prior is “point shrinkage prior 1” with
\[
f(\theta) = \left( \sum_{i=1}^{3} \theta_i^2 \right)^{-(3-2)/2},
\]
which shrinks \( \theta \) toward the origin. The second prior is “point shrinkage prior 2” with
\[
f(\theta) = \sum_{a \in \{1, -1\}^d} \left( \sum_{i=1}^{3} (a_i \theta_i - 2)^2 \right)^{-(3-2)/2},
\]
which shrinks \( \theta \) toward point \((2, 2, 2)\). The third prior is a harmonic prior with
\[
f(\theta) = \left( \sum_{i=1}^{3} (\theta_i - 2)^2 \right)^{-(3-2)/2}.
\]

Figure 3 shows the differences between the risks of the Bayesian predictive distributions based on the three priors and the Jeffreys prior when \( \lambda = \Lambda \times (0.4, 0.4, 0.4) \). \( \theta \) is close to the origin and the Bayesian predictive distribution based on the point shrinkage prior 1 performs better when \( \Lambda \) is small. \( \theta \) is close to point \((2, 2, 2)\) and the Bayesian predictive distribution based on the point shrinkage prior 2 performs well when \( \Lambda \) is close to 10.

It can be observed that the Bayesian predictive distribution based on the harmonic prior does not dominate that based on the Jeffreys prior. Specifically, the harmonic prior performs worse than the Jeffreys prior when \( \lambda \) is close to the origin. However, Bayesian predictive distributions based on the harmonic prior dominates that based on the Jeffreys prior in the multivariate normal model (George et al., 2006). This example demonstrates the difference between the multivariate Poisson and normal models.
Figure 3: (a) The K-L risk difference, i.e., $E[D(p(y | \lambda), p_J(y | x)) - D(p(y | \lambda), p_J(y | x)) | \lambda]$. (b) The K-L risks of Bayesian predictive distributions based on different priors. The red horizontal line denotes the minimax lower bound $0.5d \log((r + s)/r)$.

Experiment 3.

We set $r = s = 1$, $d = 4$, and $\theta_i = \sqrt{\lambda}$, $i = 1, \ldots, 3$. The first prior is a point shrinkage prior with

$$f(\theta) = \left( \sum_{i=1}^{4} \theta_i^2 \right)^{-\frac{(4-2)}{2}},$$

which shrinks $\theta$ toward the origin. The second prior is a subspace shrinkage prior with

$$f(\theta) = \sum_{a \in \{1, -1\}^d} (s_V(a\theta))^{-\frac{(4-3)}{2}},$$

which shrinks $\theta$ toward subspace $V = \text{span}((1, 1, 1, 1))$.

Figure 4 shows the differences between the risks of the Bayesian predictive distributions based on the two priors and the Jeffreys prior when $\lambda = \Lambda \times (0.4, 0.4, 0.4, 0.4)$. $\theta$ is close to the origin and the Bayesian predictive distribution based on the point shrinkage prior performs better when $\Lambda$ is small. Note that $\theta$ is always in the subspace $V = \text{span}((1, 1, 1, 1))$. Therefore, the Bayesian predictive distribution based on the subspace shrinkage prior still performs well when $\Lambda$ is large.
Figure 4: Log value of the difference between the K-L risks, i.e.,
\[ \log \mathbb{E} \left[ D(p(y \mid \lambda), p_J(y \mid x)) - D(p(y \mid \lambda), p_{f,J}(y \mid x)) \right] \lambda. \]

**Experiment 4.**

We set \( r = s = 1 \), \( d = 4 \), and \( \theta_i = \sqrt{\lambda_i} \), \( i = 1, \ldots, 4 \). The first prior is a point shrinkage prior with
\[ f(\theta) = (\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2)^{-(4-2)/2}. \]

The second prior is “subspace shrinkage prior 1” with
\[ f(\theta) = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{-(4-3)/2}, \]
which shrinks \( \theta \) toward subspace \( V_1 = \text{span}((0, 0, 0, 1)) \). The third prior is “subspace shrinkage prior 2” with
\[ f(\theta) = (\theta_1^2 + \theta_2^2 + \theta_4^2)^{-(4-3)/2}, \]
which shrinks \( \theta \) toward subspace \( V_2 = \text{span}((0, 0, 1, 0)) \). The fourth prior is a “mix subspace shrinkage prior” with
\[ f(\theta) = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{-1/2} + (\theta_1^2 + \theta_2^2 + \theta_4^2)^{-1/2} + (\theta_1^2 + \theta_3^2 + \theta_4^2)^{-1/2} + (\theta_2^2 + \theta_3^2 + \theta_4^2)^{-1/2}. \]

Figure 5 shows the differences between the risks of the Bayesian predictive distributions based on the four priors and the Jeffreys prior when \( \lambda = \Lambda \times (1, 1, 1, 100)/20 \). \( \lambda \) is close to \( 0 \), and the Bayesian predictive distributions based on the point and subspace shrinkage priors perform well when \( \Lambda \) is small. \( \theta \) is close to \( V_1 = \text{span}((0, 0, 0, 1)) \) but not close to \( V_2 = \text{span}((0, 0, 1, 0)) \) because \( \lambda_4 \) is significantly larger than the others. Therefore, the Bayesian predictive distribution based on subspace shrinkage prior 1 performs better than the others, and the Bayesian predictive distribution based on the mix subspace shrinkage prior is second best.

If it is unknown which \( \lambda_i \) is large, the performance of the subspace shrinkage prior \( (\theta_1^2 + \theta_2^2 + \theta_3^2)^{-1/2} \) has \( 1/4 \) and \( 3/4 \) probabilities of being the performance of subspace shrinkage priors 1 and 2, respectively. In this case, the performance of the subspace shrinkage prior is inferior to that of the mix subspace shrinkage prior.

For example, when \( \Lambda = 5 \), the risk reduction \( \mathbb{E} \left[ D(p(y \mid \lambda), p_J(y \mid x)) - D(p(y \mid \lambda), p_{f,J}(y \mid x)) \right] \lambda \) of subspace shrinkage priors 1 and 2 and the mix subspace shrinkage prior are 0.15, 0.002, and 0.10, respectively. Note that 0.1 > 0.15 \times 1/4 + 0.002 \times 3/4. Therefore, using the mix subspace shrinkage prior is recommended for this type of data. A similar example is considered in the context of a real data application.
Figure 5: Log value of the difference between the K-L risks, i.e.,
\[
\log \mathbb{E} \left[ D(p(y | \lambda), p_j(y | x)) - D(p(y | \lambda), p_{f, j}(y | x)) \right].
\]

Real data application.

We utilized real data from an official database called the number of crimes in Tokyo by type and town (Tokyo-Metropolitan-Police-Department [2023]), which reports the annual number of crimes in Tokyo. More appropriate measures for preventing crime can be implemented if the number of future crimes can be accurately predicted using past crime data.

Shoplifting data for Shinjuku Ward was used from 2020–2022. After excluding towns with incomplete data, 131 towns were included. Figure 6 shows the number of shoplifting incidents in Shinjuku Ward for 2020–2021 and 2022. Notably, one town had a significantly larger number of incidents than the others.

Figure 6: Total number of shoplifting incidents in Shinjuku Ward for: (a) 2020 to 2021, and (b) 2022. Towns with incomplete data are shown in black.

The experimental settings were as follows. The shoplifting data for 131 towns from 2020 to 2021 were set as observed data \(x\) and the data from 2022 were set as future data \(y\). The goal was to use \(x\) to predict \(y\). Therefore, the parameters in the prediction problem were \(r = 2\), \(s = 1\), and \(d = 131\). The Bayesian predictive distribution \(p_\pi(y | x)\) based on prior \(\pi\) was evaluated in three ways.
Log-likelihood $-K$-L distance $107.8$ $104.7$ Max:$105.3$, Min:$101.1$, Mean:$104.7$ $101.1$

Table 1: Comparison of Bayesian predictive distributions based on different priors for shoplifting data

the point shrinkage prior with $f$ and normal models. with different durations. The results indicate the similarity and difference between the multivariate Poisson $p$ dominate $\pi$ A point shrinkage prior $\theta$ $P$ Experiment 1, the shift point shrinkage prior outperforms the point shrinkage prior when $P$ well only when $f$ $P$ by prior $f$ $(\theta)$ = $(d/2)\alpha$ $\theta$ $\theta_i$ $\eta_i$ $\alpha$ $\beta_i$ $\forall i$. However, the underlying nature of the result was unclear. For example, the condition $0 < \alpha \leq d/2 - 1$ corresponds exactly to $f(\theta)$ being a superharmonic function, but the superharmonic function was not mentioned in the results of these two papers.

Additionally, the Bayesian predictive distribution based on the Jeffreys prior was shown to be dominated by $\pi$ $\alpha$ $\beta$ $\theta$ $h$ is a superharmonic function. In addition, sufficient conditions for the general prior $\pi(\lambda)$ are provided to make the Bayesian predictive distribution $p_{\beta}(y \mid x)$ dominate $p_{\beta}(y \mid x)$ based on $\pi_{\beta}(\lambda)$. Sufficient conditions are also provided for independent Poisson processes with different durations. The results indicate the similarity and difference between the multivariate Poisson and normal models.

The results of this study help to discover different classes of priors that dominate the Jeffreys prior, such as the point shrinkage prior with $f(\theta) = (\sum_{i=1}^d \theta_i^2)^{-\alpha}$ and the subspace shrinkage prior with $f(\theta) = \sum_{a \in \mathbb{S}^d} (s_{\lambda}(a\theta))^\alpha$. The point shrinkage prior $f(\theta) = (\sum_{i=1}^d \theta_i^2)^{-\alpha}$ in the previous study performs well only when $\sum_{i=1}^d \lambda_i$ is small. In other cases, other priors discussed in Section 4 are more effective. In Experiment 1, the shift point shrinkage prior outperforms the point shrinkage prior when $\sum_{i=1}^d \lambda_i$ is large. In

131 subspace shrinkage priors which shrink $\theta$ toward the subspaces $\{\theta \mid \theta_i = 0, \forall i \neq j\}$, $j = 1, \ldots, 131$, i.e.,

\[
\left( \sum_{i \neq j} \theta_i^2 \right)^{-(d-3)/2} d\theta, \; j = 1, \ldots, 131;
\]

and the mix subspace shrinkage prior

\[
\sum_{j=1}^d \left( \sum_{i \neq j} \theta_i^2 \right)^{-(d-3)/2} d\theta.
\]

Table 1 presents a summary of the comparisons. As shown, the point and subspace shrinkage priors outperform the Jeffreys prior. For all the evaluation methods, the Bayesian predictive distribution based on the mix subspace shrinkage prior achieves the best scores.

Table 1: Comparison of Bayesian predictive distributions based on different priors for shoplifting data using the K-L distance, W-S distance, and predictive log-likelihood.

| Prior Type                        | Jeffreys | Point shrinkage | Subspace shrinkage | Mix subspace shrinkage |
|-----------------------------------|----------|-----------------|--------------------|------------------------|
| K-L distance                      | 107.8    | 104.7           | Max:105.3, Min:101.1, Mean:104.7 | 101.1 |
| W-S distance                      | 259.5    | 235.4           | Max:239.9, Min:218.7, Mean:235.5 | 218.7 |
| Log-likelihood                    | -169.7   | -167.6          | Max:-165.1, Min:-168.0, Mean:-167.6 | -165.1 |

6 Discussion

A point shrinkage prior $\pi_{f,\beta}(\lambda)$ with $f(\theta) = (\sum_{i=1}^d \theta_i^2)^{-\alpha}$ was proposed in Komaki (2004) and Komaki (2015). Additionally, the Bayesian predictive distribution based on the Jeffreys prior was shown to be dominated by that based on the point shrinkage prior if $0 < \alpha \leq d/2 - 1$, $\beta_i = 1/2$, $\forall i$. However, the underlying nature of the result was unclear. For example, the condition $0 < \alpha \leq d/2 - 1$ corresponds exactly to $f(\theta)$ being a superharmonic function, but the superharmonic function was not mentioned in the results of these two papers.

In this study, the relationship between the superharmonic functions and improved predictive distribution are demonstrated. We show that the Bayesian predictive distribution based on the Jeffreys prior could be improved by prior $f(\theta)d\theta$, if $f(\theta) = \sum_{a \in \mathbb{S}^d} h(a\theta_1, a\theta_2, \ldots, a\theta_d)$ and $h$ is a superharmonic function. In addition, sufficient conditions for the general prior $\pi(\lambda)$ are provided to make the Bayesian predictive distribution $p_{\beta}(y \mid x)$ dominate $p_{\beta}(y \mid x)$ based on $\pi_{\beta}(\lambda)$. Sufficient conditions are also provided for independent Poisson processes with different durations. The results indicate the similarity and difference between the multivariate Poisson and normal models.
Experiment 2, the point shrinkage prior performs well when \( \theta \) is close to a point (i.e., \( \lambda = \lambda_2 = \cdots = \lambda_d \)). In Experiment 3, the subspace shrinkage prior performs well when \( \theta \) is close to a subspace (e.g., \( \lambda_1 = \lambda_2 = \cdots = \lambda_d \)). Moreover, the mix subspace shrinkage prior does not require knowledge of the index \( i \) of the large \( \lambda_i \). These improved predictions based on different priors are nearly minimax, i.e., their K-L risk is less than 1.04 times the minimax lower bound.

On the basis of the results of this study, other types of prior distributions that dominate the Jeffreys prior will be constructed in future. Different types of prior distributions are expected to perform well under different values of \( \lambda \). A focus of further studies will be determining if there is a practical and relaxed condition for Theorems 1 and 3.

Acknowledgments

The first author is grateful for support from the China Scholarship Council. We thank Keisuke Yano for his helpful comments. This work was supported by JSPS KAKENHI Grant Number 22H00510, and AMED Grant Numbers JP23dm0207001 and JP23dm0307009.

Appendix A Proofs

Contents

A.1 Proof of the two statements in Introduction ............................................ 16
A.2 Proof of Theorem 1 .................................................................................. 17
A.3 Proof of Theorem 3 .................................................................................. 20
A.4 Lemmas used in the proofs of Theorem 1 and Theorem 3 ......................... 24
A.5 Proof of Proposition 1 ............................................................................ 32
A.6 Proof of Proposition 2 ............................................................................ 33
A.7 Proof of Proposition 3 ............................................................................ 35
A.8 Proof of Proposition 4 ............................................................................ 36
A.9 Proof of Theorem 5 ................................................................................ 36

A.1 Proof of the two statements in Introduction

Proof of the statement that the K-L risk of \( p_\beta(y \mid x) \) has an upper bound.

Using Theorem 1 in Komaki (2004), we have

\[
p_\beta(y \mid x) = \left( \frac{r}{r+s} \right)^{\sum_{i=1}^{d}(x_i+\beta_i)} \left( \frac{s}{r+s} \right)^{\sum_{i=1}^{d} y_i} \prod_{i=1}^{d} \Gamma(x_i + y_i + \beta_i) \Gamma(x_i + \beta_i) y_i!.
\] (A.1)

Thus, the K-L risk \( E(D(p(y \mid \lambda) , p_\beta(y \mid x)) \) is equal to

\[
\sum_{i=1}^{d} \left( -s\lambda_i + s\lambda_i \log(s\lambda_i) - (r\lambda_i + \beta_i) \log \left( \frac{r}{r+s} \right) - s\lambda_i \log \left( \frac{s}{r+s} \right) - E \left( \log \Gamma(x_i + y_i + \beta_i) - \log \Gamma(x_i + \beta_i) \right) \right)
\] (A.2)

Let Po(\( \lambda \)) denote the Poisson distribution with parameter \( \lambda \). If we consider the function

\[
f(t) := \sum_{i=1}^{d} \left( t\lambda_i \log \lambda_i + \beta_i \log t + \lambda_i(t \log t - t) - E \left( \log \Gamma(x + \beta_i) \mid x \sim \text{Po}(t\lambda_i) \right) \right),
\]
Therefore, (A.2) has an upper bound for any based on $\pi$.

**Proof of part 1.**

A.2 Proof of Theorem 1

We note that $\log \left( \frac{t\lambda_i}{x + \beta_i} \right) \leq -1 + \frac{t\lambda_i}{x + 1} \leq -1 + \frac{2t\lambda_i}{\beta_i(x + 1)(x + 2)}$. Thus, $f'(t) \leq \sum_{i=1}^{d} \left( \frac{\beta_i}{t} + \frac{2t}{\beta_i^2} \right)$. Therefore, (A.2) has an upper bound for any $\lambda$.

**Proof of the statement that the K-L risk of $p_\pi(y \mid x)$ based on a gamma prior has no upper bound.**

Considering a gamma prior $\pi(\lambda) \sim \prod_{i=1}^{d} \lambda_i^{\beta_i-1} \exp(-\alpha_i\lambda_i)$, we have

$$p_\pi(y \mid x) = \frac{\int p(x \mid \lambda)p(y \mid \lambda) \prod_{i=1}^{d} \lambda_i^{\beta_i-1} \exp(-\alpha_i\lambda_i) d\lambda}{\int p(x \mid \lambda) \prod_{i=1}^{d} \lambda_i^{\beta_i-1} \exp(-\alpha_i\lambda_i) d\lambda}$$

$$= \prod_{i=1}^{d} \frac{\int \exp(-(r+s+\alpha_i)\lambda_i)\lambda_i^{x_i+y_i+\beta_i-1} d\lambda_i}{\int \exp(-(r+\alpha_i)\lambda_i)\lambda_i^{x_i+\beta_i-1} d\lambda_i} \prod_{i=1}^{d} \frac{s^{y_i}}{y_i!}$$

$$= \left( \frac{r+\alpha_i}{r+s+\alpha_i} \right)^{\sum_{i=1}^{d} (x_i+\beta_i)} \left( \frac{s}{r+s+\alpha_i} \right)^{\sum_{i=1}^{d} y_i} \prod_{i=1}^{d} \frac{\Gamma(x_i+y_i+\beta_i)}{\Gamma(x_i+\beta_i)y_i}. \quad (A.3)$$

Thus, the K-L risk difference between $p_\pi(y \mid x)$ and $p_{\beta}(y \mid x)$ is equal to

$$\sum_{i=1}^{d} \left( (r\lambda_i + \beta_i) \left( \log \left( \frac{r}{r+s} \right) - \log \left( \frac{r+s+\alpha_i}{r+s+\alpha_i} \right) + s\lambda_i \log \left( \frac{r+s+\alpha_i}{r+s} \right) \right) \right). \quad (A.4)$$

The coefficient of $\lambda_i$ in (A.4) is $r \log(r/(r+s)) - r \log((r+\alpha_i)/(r+s+\alpha_i)) + s \log((r+s+\alpha_i)/(r+s)) > 0$. Thus, (A.4) has no upper bound. Because the K-L risk of $p_\beta(y \mid x)$ has an upper bound, the proof is complete.

A.2 Proof of Theorem 1

Proof of part 1. We show this in two steps. Let $\text{Po}(\lambda)$ denote the Poisson distribution with parameter $\lambda$. In Step 1, we show that the difference between the K-L risks of Bayesian predictive distributions based on $\pi_{f,\beta}$ and $\pi_\beta$ is

$$E \left( \log(\text{F}(z,r)) \mid z_i \sim \text{Po}(r\lambda_i), i = 1, \ldots, d \right) - E \left( \log(\text{F}(z,r+s)) \mid z_i \sim \text{Po}((r+s)\lambda_i), i = 1, \ldots, d \right).$$

In Step 2, we prove that the partial derivative of $E \left( \log(\text{F}(z,r)) \mid z_i \sim \text{Po}(r\lambda_i), i = 1, \ldots, d \right)$ with respect to $r$ is positive. The details of each step are presented below.

Step 1. Let $\theta_j := \sqrt{\lambda_j} (j = 1, \ldots, d)$. The difference between the K-L risks of Bayesian predictive distributions based on $\pi_{f,\beta}$ and $\pi_\beta$ is

$$E \left( \log \frac{p_{\beta}(y \mid x)}{p_{f,\beta}(y \mid x)} \right) = E \left( \log \frac{\int p(x \mid \lambda)\pi_{f,\beta}(\lambda) d\lambda}{\int p(x \mid \lambda)\pi_\beta(\lambda) d\lambda} \right) \lambda \right) = E \left( \log \frac{\int p(x \mid \lambda)\pi_{f,\beta}(\lambda) d\lambda}{\int p(x \mid \lambda)\pi_\beta(\lambda) d\lambda} \right)$$

$$= E \left( \log \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{\exp(-(r\lambda_i))}{\Gamma(x_i+\beta_i)} d\lambda \right) \lambda \right)$$

$$- E \left( \log \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{\exp((-r+s)\lambda_i)}{\Gamma(x_i+y_i+\beta_i)} d\lambda \right) \lambda \right)$$

$$= E \left( \log(\text{F}(z,r)) \mid z_i \sim \text{Po}(r\lambda_i), i = 1, \ldots, d \right) - E \left( \log(\text{F}(z,r+s)) \mid z_i \sim \text{Po}((r+s)\lambda_i), i = 1, \ldots, d \right). \quad (A.5)$$

From Lemma 1, we know that the risk difference (A.5) is finite.
Step 2. The risk difference (A.5) is negative if

$$
\mathbb{E}\left( \log F(z, r) \mid z_i \sim \text{Po}(r \lambda_i), i = 1, \ldots, d \right) = \sum_z \log(F(z, r)) \left\{ \prod_{i=1}^{d} \frac{(r \lambda_i)^{z_i} \exp(-r \lambda_i)}{z_i!} \right\} (A.6)
$$

is an increasing function of $r$.

If we exchange the integration and differentiation in $\frac{\partial F}{\partial r}(z, r)$, we have

$$
\frac{\partial F}{\partial r}(z, r) = \int f(\theta_1, \ldots, \theta_d) d\theta_1 \ldots d\theta_d \prod_{i=1}^{d} \frac{r^{z_i + \beta_i \lambda_i^{z_i + \beta_i - 1}} \exp(-r \lambda_i)}{\Gamma(z_i + \beta_i)} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{r} - \sum_{i=1}^{d} \lambda_i \right) d\lambda_i.
$$

From Lemma 2, we can exchange the integration and differentiation in $\frac{\partial F}{\partial r}(z, r)$.

If we differentiate (A.6) item-by-item, the partial differential function of (A.6) with respect to $\partial F$ is

$$
\sum_{z} \log(F(z, r)) \left\{ \prod_{i=1}^{d} \frac{(r \lambda_i)^{z_i} \exp(-r \lambda_i)}{z_i!} \right\} \left( \sum_{i=1}^{d} \frac{z_i}{r} - \sum_{i=1}^{d} \lambda_i \right) + \sum_{z} \left\{ \int f(\theta_1, \ldots, \theta_d) d\theta_1 \ldots d\theta_d \prod_{i=1}^{d} \frac{r^{z_i + \beta_i \lambda_i^{z_i + \beta_i - 1}} \exp(-r \lambda_i)}{\Gamma(z_i + \beta_i)} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{r} - \sum_{i=1}^{d} \lambda_i \right) d\lambda_i \right\} \prod_{i=1}^{d} \frac{(r \lambda_i)^{z_i} \exp(-r \lambda_i)}{z_i!}.
$$

(A.7)

From Lemma 3, we can differentiate (A.6) by terms under the condition (2.1).

We note that

$$
\sum_{z} \left\{ \log F(z_1, z_2, \ldots, z_d, r) \right\} \left\{ \prod_{i=1}^{d} \frac{(r \lambda_i)^{z_i} \exp(-r \lambda_i)}{z_i!} \right\} \sum_{i=1}^{d} \lambda_i = \sum_{z} \sum_{i} \left[ \frac{z_i}{r} \left\{ \log F(z_1, z_2, \ldots, z_d, r) \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!} \right].
$$

(A.8)

where $\delta_{ij}$ is defined as 1 if $i = j$ and 0 if $i \neq j$, $\delta_i$ is defined as the $d$-dimensional vector whose $i^{th}$ element is 1 and all other elements are 0, and $F(z - \delta_i, r)$ is defined as 1 if $z_i = 0$. Furthermore, we note that

$$
\int f(\theta_1, \ldots, \theta_d) \prod_{j=1}^{d} \frac{r^{z_j + \beta_j \lambda_j^{z_j + \beta_j - 1}} \exp(-r \lambda_j)}{\Gamma(z_j + \beta_j)} \lambda_i d\lambda_i = F(z + \delta_i, r) \frac{z_i + \beta_i}{r}.
$$

(A.9)

Thus, from (A.8) and (A.9), the partial differential function (A.7) of (A.6) with respect to $r$ is

$$
\sum_{z} \sum_{i=1}^{d} \left[ \frac{z_i}{r} \left\{ \log F(z, r) \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!} \right] - \sum_{z} \sum_{i=1}^{d} \left[ \frac{z_i}{r} \left\{ \log F(z - \delta_i, r) \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!} \right]
$$

$$+ \sum_{z} \left\{ \sum_{i=1}^{d} \frac{F(z, r)}{F(z - \delta_i, r)} \frac{z_i + \beta_i}{r} - \sum_{i=1}^{d} \frac{F(z + \delta_i, r)}{F(z, r)} \frac{z_i + \beta_i}{r} \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!}
$$

$$= \sum_{z} \sum_{i=1}^{d} \left[ \frac{z_i}{r} \left\{ \log F(z, r) \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!} \right] + \sum_{z} \sum_{i=1}^{d} \left[ \frac{z_i + \beta_i}{r} \left\{ 1 - \frac{F(z + \delta_i, r)}{F(z, r)} \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!} \right].
$$

(A.10)

By assumption, $F(z, r)$ is not a constant function of $z$; hence, $F(z - \delta_i, r) \equiv F(z, r)$ does not hold. Therefore, from the inequality $\log \epsilon > 1 - \frac{1}{\epsilon}, \epsilon \neq 1$, (A.10) is strictly larger than

$$
\sum_{z} \sum_{i=1}^{d} \left[ \frac{z_i}{r} \left\{ 1 - \frac{F(z - \delta_i, r)}{F(z, r)} \right\} \right] + \sum_{i=1}^{d} \frac{z_i + \beta_i}{r} \left\{ 1 - \frac{F(z + \delta_i, r)}{F(z, r)} \right\} \prod_{j=1}^{d} \frac{(r \lambda_j)^{z_j} \exp(-r \lambda_j)}{z_j!}.
$$

(A.11)
From (2.1), we know that \(\text{(A.11)}\) is nonnegative. Thus, \(\text{(A.10)}\) is positive. Therefore, \(\text{(A.6)}\) is an increasing function, and the first half of Theorem 1 is proved.

**Proof of part 2.** We prove that \(\text{(2.1)}\) is satisfied if \(f\) satisfies the conditions of the second half of Theorem 1. Let \(\theta_j := \sqrt{\lambda_j} (j = 1, \ldots, d)\). We show this in three steps. \(\text{(2.1)}\) is obtained by combining Steps 2 and 3. In Step 1, through integration by parts on \(\theta_i\), we prove that

\[
F(z + \delta_i, r) - F(z, r) = 2^{d-1} \int \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_i. \tag{A.12}
\]

In Step 2, by performing integration by parts on \(\theta_i\) again and using the condition \(\text{(2.2a)}\) of the second derivative, we prove that

\[
\sum_{i=1}^{d} z_i \{F(z, r) - F(z - \delta_i, r)\} + \sum_{i=1}^{d} (z_i + \beta_i) \{F(z, r) - F(z + \delta_i, r)\}
\geq \sum_{i=1}^{d} 2^{d-2} \lim_{r \to \infty} \lim_{\theta_i \to \infty} \left[ \int_{(a,b) \setminus \{\theta_i\}} \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j 
eq i} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_j \right]^v \tag{A.13}
\]

In Step 3, using the condition \(\text{(2.2b)}\) of the derivative on the boundary, we show that \(\text{(A.13)}\) \(\geq 0\). The details of each step are presented below.

**Step 1.** Using the substitution \(\theta = \sqrt{\lambda}\) and the definition of the function \(F\), we obtain

\[
F(z + \delta_i, r) - F(z, r) = -2^{d-1} \left\{ \int f(\theta) \prod_{j \neq i} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_j \right\} \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_i.
\]

By performing integration by parts on \(\theta_i\), we obtain

\[
F(z + \delta_i, r) - F(z, r) = \lim_{r \to \infty} \lim_{\theta_i \to \infty} \left[ 2^{d-1} \int_{(a,b) \times \{\theta_i\}} f(\theta) \prod_{j=1}^{d} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta \right] - \left\{ \int f(\theta) \prod_{j \neq i} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_j \right\} \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_i. \tag{A.14}
\]

Here, we use auxiliary variables \(a, b, u, v\) and Lemma 4 to ensure that the above equation \(\text{(A.14)}\) holds. Because of Lemma 5

\[
\lim_{r \to \infty} \lim_{\theta_i \to \infty} \left[ 2^{d-1} \int_{(a,b) \times \{\theta_i\}} f(\theta) \prod_{j=1}^{d} \frac{\theta_j^{z_j + \beta_j} \theta_i^{z_i + \beta_i - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta \right] = 0. \tag{A.15}
\]

Thus, using \(\text{(A.14)}\), Lemma 4 and \(\text{(A.15)}\), we obtain \(\text{(A.12)}\).

**Step 2.** Using \(\text{(A.12)}\) and auxiliary variables \(a, b, u, v\), we obtain

\[
\sum_{i=1}^{d} z_i \{F(z, r) - F(z - \delta_i, r)\} + \sum_{i=1}^{d} (z_i + \beta_i) \{F(z, r) - F(z + \delta_i, r)\}
= \sum_{i=1}^{d} 2^{d-1} \int \frac{\partial f}{\partial \theta_i}(\theta) z_i \prod_{j=1}^{d} \frac{\theta_j^{z_j - \delta_j} \theta_i^{z_i - \delta_i} \theta_j^{2(z_j - \delta_j) + \delta_j} \theta_i^{2(z_i - \delta_i) + \delta_i} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j) \Gamma(z_i + \beta_i)} d\theta_i. \tag{A.12}
\]
\[- \frac{\partial f}{\partial \theta_i}(\theta)(z_i + \beta_i) \prod_{j=1}^{n} \frac{\theta_j^{2z_j + \delta_j + 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \delta_j + \beta_j)} d\theta \]

\begin{align*}
&= \sum_{i=1}^{d} \frac{2^{d-2}}{r} \lim_{\theta_i \to \infty} \lim_{\theta_i \to \infty} \left\{ \int_u^v \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \frac{\theta_i^{2z_i - 1} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \right. \\
&\quad + \left. \int_u^v \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \frac{\theta_i^{2z_i - 1} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \right\}. \quad (A.16)
\end{align*}

By performing integration by parts on each parameter again, we find that (A.16) equals

\begin{align*}
&= \sum_{i=1}^{d} \frac{2^{d-2}}{r} \lim_{\theta_i \to \infty} \lim_{\theta_i \to \infty} \left\{ - \int_u^v \int_{[a,b]^{d-1}} \frac{\partial^2 f}{\partial \theta_i^2}(\theta) \frac{\theta_i^{2z_i - 1} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \\
&\quad + \left[ \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \frac{\theta_i^{2z_i - 1} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \right] \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} \right\}. \quad (A.17)
\end{align*}

From (2.3c), we have

\begin{align*}
\int \frac{\partial f}{\partial \theta_i}(\theta) \frac{d \theta_i^{2\beta_i - 1}}{\Gamma(z_i + \beta_i)} d\theta < \infty, \forall i. \quad (A.18)
\end{align*}

From (A.18) and condition (2.2a), we have

\begin{align*}
&\sum_{i=1}^{d} \lim_{\theta_i \to \infty} \lim_{\theta_i \to \infty} \left\{ - \int_u^v \int_{[a,b]^{d-1}} \frac{\partial^2 f}{\partial \theta_i^2}(\theta) \frac{\theta_i^{2z_i - 1} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \\
&\quad + \left[ \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \frac{d \theta_i^{2\beta_i - 1}}{\Gamma(z_i + \beta_i)} d\theta_i \right] \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} \right\} \\
&= - \sum_{i=1}^{d} \int_u^v \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \frac{d \theta_i^{2\beta_i - 1}}{\Gamma(z_i + \beta_i)} d\theta_i \frac{r^{z_i + \beta_i} \frac{\partial^2 \theta_i^{2z_i}}{\partial \theta_i^2} \exp(-r \theta_i^2)}{\Gamma(z_i + \beta_i)} d\theta_i \geq 0. \quad (A.19)
\end{align*}

Using (A.16), (A.17), and (A.19), we obtain the inequality (A.13).

**Step 3.** From Lemma 6, we know that (A.13) ≥ 0, which completes the proof. \(\square\)

### A.3 Proof of Theorem 3

**Proof of part 1.** For every \(i\) and \(\tau \in [0, 1]\), let \(\frac{1}{t_i(\tau)} := \frac{1}{t_i}(1 - \tau) + \frac{1}{t_i} + s_i \tau\). Then, \(t_i(\tau)\) is a smooth monotonically increasing function of \(\tau \in [0, 1]\) satisfying \(t_i(0) = r_i, t_i(1) = r_i + s_i, \) and \(\dot{t}_i/t_i = \gamma_i t_i\). Let \(\text{Po}(\lambda)\) denote the Poisson distribution with parameter \(\lambda\).

We prove the first half of Theorem 3 in two steps. In Step 1, we show that the difference between the K-L risks of Bayesian predictive distributions based on \(\pi_f,\beta\) and \(\pi_\beta\) is

\[E \left( \log F(z, t(0)) \mid z_i \sim \text{Po}(t_i(0)\lambda_i), i = 1, \ldots, d \right) - E \left( \log F(z, t(1)) \mid z_i \sim \text{Po}(t_i(1)\lambda_i), i = 1, \ldots, d \right).\]

In Step 2, we prove that the partial derivative of \(E \left( \log F(z, t(\tau)) \mid z_i \sim \text{Po}(t_i(\tau)\lambda_i), i = 1, \ldots, d \right)\) with respect to \(\tau\) is positive. The details of each step are presented below.

**Step 1.** Let \(\hat{\theta}_j := \sqrt{\frac{\lambda_j}{\gamma_j}}, j = 1, \ldots, d\). The difference between the K-L risks of Bayesian predictive distributions based on \(\pi_f,\beta\) and \(\pi_\beta\) is

\[E \left( \log \frac{p(y \mid x)}{p_f(y \mid x)} \right) \mid \lambda \]

\[= E \left( \log \frac{\int p(x \mid x, \lambda)\pi_f,\beta(\lambda) d\lambda}{\int p(x \mid x, \lambda)\pi_\beta(\lambda) d\lambda} \right) \mid \lambda \]

\[= E \left( \log \frac{\int p(x \mid x, \lambda)\pi_f,\beta(\lambda) d\lambda}{\int p(x \mid x, \lambda)\pi_\beta(\lambda) d\lambda} \right) \mid \lambda \]
Using \( \dot{\tau} \) is an increasing function of \( \tau \).

From Lemma 9, we can differentiate (A.21) by terms under condition (3.1).

From Lemma 8, we can exchange the integration and differentiation in (A.21).

From Lemma 7, we know that the risk difference (A.20) is finite. The risk difference (A.20) is negative if

\[
E \left( \log F(z, t(\tau)) \right| z_i \sim Po(t_i(\tau)\lambda_i), i = 1, \ldots, d \right) - E \left( \log F(z, t(1)) \right| z_i \sim Po(t_i(1)\lambda_i), i = 1, \ldots, d \right).
\]

From Lemma 7, we can exchange the integration and differentiation in \( \frac{\partial F}{\partial \tau} (z, t(\tau)) \).

From \( \dot{t}_i/t_i = \gamma_it_i \) and (A.22), the partial differential function of (A.22) with respect to \( \tau \) is

\[
\frac{\partial F}{\partial \tau} (z, t(\tau)) = \int f(\tilde{\theta}_1, \ldots, \tilde{\theta}_d) \prod_{i=1}^{d} t_i(\tau) z_{i}^{\gamma_i t_i(\tau)} \exp(-t_i(\tau)\lambda_i) \left\{ \sum_{j=1}^{d} (z_j + \beta_j) \gamma_j t_j(\tau) - \sum_{k=1}^{d} \lambda_k \gamma_k t_k^2(\tau) \right\} d\lambda.
\]

From Lemma 8, we can differentiate (A.21) by terms under condition (3.1).

We note that

\[
\sum_{z} \left\{ \log F(z, t(\tau)) \right\} \left\{ \prod_{i=1}^{d} t_i(\tau) \lambda_i \right\} z_{i}^{\gamma_i t_i(\tau)} \exp(-t_i(\tau)\lambda_i) \left\{ \sum_{j=1}^{d} (z_j + \beta_j) \gamma_j t_j(\tau) - \sum_{k=1}^{d} \lambda_k \gamma_k t_k^2(\tau) \right\} d\lambda.
\]

From Lemma 9, we can differentiate (A.21) by terms under condition (3.1).

We note that

\[
\sum_{z} \left\{ \log F(z, t(\tau)) \right\} \left\{ \prod_{i=1}^{d} t_i(\tau) \lambda_i \right\} z_{i}^{\gamma_i t_i(\tau)} \exp(-t_i(\tau)\lambda_i) \left\{ \sum_{j=1}^{d} (z_j + \beta_j) \gamma_j t_j(\tau) - \sum_{k=1}^{d} \lambda_k \gamma_k t_k^2(\tau) \right\} d\lambda.
\]

where \( \delta_{ij} \) is defined as 1 if \( i = j \) and 0 if \( i \neq j \), \( \delta_i \) is defined as the \( d \)-dimensional vector whose \( i \)-th element is 1 and all other elements are 0, and \( F(z - \delta_i, t(\tau)) \) is defined as 1 if \( z_i = 0 \). Furthermore, we note that

\[
\int f(\tilde{\theta}_1, \ldots, \tilde{\theta}_d) \prod_{j=1}^{d} t_j(\tau) z_{i}^{\gamma_j t_j(\tau)} \exp(-t_j(\tau)\lambda_j) \lambda_j t_j^2(\tau) d\lambda = F(z + \delta_i, t(\tau))(z_i + \beta_i) \gamma_i t_i(\tau).
\]

Thus, from (A.24) and (A.25), the partial differential function (A.23) of (A.21) with respect to \( \tau \) is

\[
\sum_{z} \sum_{i=1}^{d} \left\{ \log F(z, t(\tau)) \right\} \frac{d}{d} \prod_{j=1}^{d} t_j(\tau) \lambda_j z_{j}^{\gamma_j t_j(\tau)} \exp(-t_j(\tau)\lambda_j) z_j!
\]

- \[
\sum_{z} \sum_{i=1}^{d} \left\{ \log F(z - \delta_i, t(\tau)) \right\} \frac{d}{d} \prod_{j=1}^{d} t_j(\tau) \lambda_j z_{j}^{\gamma_j t_j(\tau)} \exp(-t_j(\tau)\lambda_j) z_j!
\]

21
\[ + \sum_{z} \left\{ F(z, t(\tau)) \sum_{i=1}^{d} (\gamma_i t_i(\tau) - \gamma_i t_i(\tau)) \right\} \frac{\prod_{j=1}^{d} (t_j(\tau)\lambda_j)^{z_i} \exp(-t_j(\tau)\lambda_j)}{z_j!} \]

\[ = \sum_{z} \left[ \sum_{i=1}^{d} z_i \gamma_i t_i(\tau) \right] \log \frac{F(z, t(\tau))}{F(z - \delta_i, t(\tau))} \frac{\prod_{j=1}^{d} (t_j(\tau)\lambda_j)^{z_i} \exp(-t_j(\tau)\lambda_j)}{z_j!} \]

\[ + \sum_{z} \left[ \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau) \right] \left\{ 1 - \frac{F(z + \delta_i, t(\tau))}{F(z, t(\tau))} \right\} \frac{\prod_{j=1}^{d} (t_j(\tau)\lambda_j)^{z_i} \exp(-t_j(\tau)\lambda_j)}{z_j!}, \quad (A.26) \]

By assumption, \(F(z, t(\tau))\) is not a constant function of \(z\); hence, \(F(z - \delta_i, t(\tau)) \equiv F(z, t(\tau))\) does not hold. Therefore, from the inequality \(\log \epsilon > 1 - \frac{1}{\epsilon}, \epsilon \neq 1\), \(A.26\) is strictly larger than

\[ \sum_{z} \left[ \sum_{i=1}^{d} z_i \gamma_i t_i(\tau) \right] \left\{ 1 - \frac{F(z - \delta_i, t(\tau))}{F(z, t(\tau))} \right\} + \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau) \left\{ 1 - \frac{F(z + \delta_i, t(\tau))}{F(z, t(\tau))} \right\} \frac{\prod_{j=1}^{d} (t_j(\tau)\lambda_j)^{z_i} \exp(-t_j(\tau)\lambda_j)}{z_j!}. \quad (A.27) \]

From (3.1), we have

\[ \sum_{i=1}^{d} \gamma_i t_i(\tau) z_i \{ F(z, t(\tau)) - F(z - \delta_i, t(\tau)) \} + \sum_{i=1}^{d} \gamma_i t_i(\tau) (z_i + \beta_i) \{ F(z, t(\tau)) - F(z + \delta_i, t(\tau)) \} \geq 0. \]

Thus, \((A.27)\) is nonnegative, and \((A.26)\) is positive. Therefore, \((A.21)\) is an increasing function, and the first half of Theorem 3 is proved.

**Proof of part 2.** Next, we prove that if \(s\) satisfies the conditions of the second half of Theorem 3, \((3.1)\) is satisfied. Let \(\theta_j := \frac{\lambda_j}{\gamma_j}, \ j = 1, \ldots, d\). We show this in three steps. \((3.1)\) is obtained by combining Steps 2 and 3. In Step 1, by performing integration by parts on \(\theta_i\), we prove that

\[ F(z + \delta_i, r) - F(z, r) = 2^{d-1} \int \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + \delta_j, i + 2\beta_j - 1} \exp(-\gamma_j r_j^{\theta_j}^2) \Gamma(z_j + \delta_j + \beta_j) d\theta. \quad (A.28) \]

In Step 2, by performing integration by parts on each \(\theta_i\) again and using the condition \((2.2a)\) of the second derivative, we prove that

\[ \sum_{i=1}^{d} \gamma_i z_i \Gamma(z_j + \delta_j + \beta_j) \Gamma(z_j + \delta_j + \beta_j) - \sum_{i=1}^{d} \gamma_i z_i \Gamma(z_j + \delta_j + \beta_j) \Gamma(z_j + \delta_j + \beta_j) \]

\[ \geq \sum_{i=1}^{d} 2^{d-2} \lim_{\theta_i \to \infty} \lim_{\theta_i \to -\infty} \left\{ \int_{[a, b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_i^{2z_j + \delta_j, i - 1} \exp(-\gamma_j r_j^{\theta_j}^2) \right\} \frac{1}{\Gamma(z_j + \delta_j + \beta_j)} d\theta_j \right\} \quad (A.29) \]

In Step 3, using the condition \((2.2b)\) of the derivative on the boundary, we show that \((A.29)\) \(\geq 0\). The details of each step are presented below.

**Step 1.** Using the substitution \(\lambda_j = \gamma_j \theta_j^2\), \(d\lambda_j = 2\theta_j \gamma_j d\theta_j\) and the definition of the function \(F\), we obtain

\[ F(z + \delta_i, r) - F(z, r) \]

\[ = -2^{d-1} \int f(\theta) \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + \delta_j, i - 1} \exp(-\gamma_j r_j^{\theta_j}^2) \frac{1}{\Gamma(z_j + \delta_j + \beta_j)} d\theta_j \]

\[ - 2^{d-1} \int f(\theta) \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + \delta_j, i - 1} \exp(-\gamma_j r_j^{\theta_j}^2) \frac{1}{\Gamma(z_j + \delta_j + \beta_j)} d\theta_j. \]
Here, we use auxiliary variables $a, b, u, v$ and Lemma 10 to ensure that the above equation holds.

Because of Lemma 11

$$\lim_{v \to \infty} \left[ \int_{[a,b]} f(\theta) \prod_{j=1}^{d-1} \left( \frac{\partial f}{\partial \theta_i} (\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_i)^{z_i + \beta_i} \theta_j^{2z_j + \delta_j + 2\beta_j - 1} \exp(-\gamma_j r_i \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i + \delta_i - 1} \exp(-\gamma_i r_i \theta_i^2)}{\Gamma(z_i + \beta_i)} \right] = 0.$$  

Thus, using (A.30), Lemma 10 and (A.31), we obtain (A.28).

Step 2. Using (A.28) and auxiliary variables $a, b, u, v$, we obtain

$$\sum_{i=1}^{d} \gamma_i r_i \{ F(z, r) - F(z - \delta_i, r) \} + \sum_{i=1}^{d} \gamma_i r_i (z_i + \beta_i) \{ F(z, r) - F(z + \delta_i, r) \}$$

$$= \sum_{i=1}^{d} \left[ \int_{\gamma_i r_i}^{\infty} \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_i)^{z_i + \beta_i} \theta_j^{2z_j + \delta_j + 2\beta_j - 1} \exp(-\gamma_j r_i \theta_j^2)}{\Gamma(z_j + \beta_j)} \right] \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i + \delta_i - 1} \exp(-\gamma_i r_i \theta_i^2)}{\Gamma(z_i + \beta_i)} \right] = 0.$$  

By performing integration by parts on $\theta_i$, we obtain

$$\int \frac{\partial}{\partial \theta_i} \left( \prod_{j=1}^{d} \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + \delta_j + 2\beta_j - 1} \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) d\theta < \infty, \forall i.$$  

(A.34)
From (A.34) and condition (2.2a), we have
\[
\sum_{i=1}^{d} \lim_{u \to 0} \lim_{v \to 0} \left[ -\int_{a}^{b} \int_{a}^{b} \frac{\partial f}{\partial \theta_i} \left( \frac{\partial f}{\partial \theta_i} \right) \prod_{j \neq i}^{d} \frac{(\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2} \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} d\theta_j \right] = -\int_{a}^{b} \frac{\partial f}{\partial \theta_i} \left( \prod_{j=1}^{d} \frac{(\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2} \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) \geq 0. \tag{A.35}
\]

Using (A.32), (A.33), and (A.35), we obtain the inequality (A.29).

**Step 3.** From Lemma 1, we know that (A.29) \( \geq 0 \), which completes the proof. \( \square \)

### A.4 Lemmas used in the proofs of Theorem 1 and Theorem 3

**Lemma 1.** Under condition (2.1), for a given \( r \), there exists \( \epsilon > 0 \) such that
\[
\sum_{i} \max_{r \in [r-\epsilon, r+\epsilon]} \left\{ \left| \log F(z, t) \right| \left( \sum_{i=1}^{d} z_i + 1 \right) \prod_{i=1}^{d} \frac{(\lambda_i)^{z_i} \exp(-t\lambda_i)}{z_i!} \right\} < \infty. \tag{A.36}
\]
In particular, \( E \left( \left| \log F(z, t) \right| \right) \) \( \sim P(\lambda_i), i = 1, \ldots, d \) < \infty.

**Proof**

From (2.1), we know that
\[
\sum_{i=1}^{d} z_i (F(z, t) - F(z - \delta_i, t)) + \sum_{i=1}^{d} (z_i + \beta_i) (F(z, t) - F(z + \delta_i, t)) \geq 0, \forall z. \tag{A.37}
\]

Let \( \beta_0 := \min\{\beta_1, \ldots, \beta_d\} \). Using (A.37), we have \( F(z + \delta_i, t) < \frac{2}{\beta_0} \sum_{i=1}^{d} (z_i + \beta_i) F(z, t), \forall z \). Hence,
\[
F(z, t) \leq \left( \frac{2}{\beta_0} \sum_{i=1}^{d} (z_i + \beta_i) \right)^{\sum_{i=1}^{d} z_i} F(\bar{t}, t). \tag{A.38}
\]

From (A.37), we have \( F(z - \delta_i, t) < 2\sum_{i=1}^{d} (z_i + \beta_i) F(z, t) \) if \( z_i > 0 \). Hence,
\[
F(z, t) \geq \left( 2 \sum_{i=1}^{d} (z_i + \beta_i) \right)^{-\sum_{i=1}^{d} z_i} F(\bar{t}, t). \tag{A.39}
\]

We choose \( \epsilon = r/2 \). Then, we have
\[
\max_{t \in [r/2, 3r/2]} F(\bar{0}, t) \leq \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{(3r/2)^{\beta_i} \lambda_i^{\beta_i-1} \exp(-3r/2 \lambda_i)}{\Gamma(\beta_i)} d\lambda = 3^{\sum_{i=1}^{d} \beta_i} F(\bar{0}, r/2). \tag{A.40}
\]

We also have
\[
\min_{t \in [r/2, 3r/2]} F(\bar{0}, t) \geq \int f(\theta_1, \ldots, \theta_d) \prod_{i=1}^{d} \frac{(r/2)^{\beta_i} \lambda_i^{\beta_i-1} \exp(-3r/2 \lambda_i)}{\Gamma(\beta_i)} d\lambda = (1/3)^{\sum_{i=1}^{d} \beta_i} F(\bar{0}, 3r/2). \tag{A.41}
\]

From (A.38), (A.39), (A.40), and (A.41), there exists a constant \( C \) such that
\[
\max_{t \in [r/2, 3r/2]} \left| \log F(z, t) \right| < C \left( \sum_{i=1}^{d} z_i + C \right)^2, \forall z.
\]

24
Thus, the left-hand side of (A.36) is less than
\[
\sum_{z} \left[ C(\sum_{i=1}^{d} z_i + C)^2 (\sum_{i=1}^{d} z_i + 1) \prod_{i=1}^{\frac{3r}{2\lambda_i}} \frac{z_i!}{z_i!} \right] \\
= \exp(r \sum_{i=1}^{d} \lambda_i) \mathbb{E}\left( C(\sum_{i=1}^{d} z_i + C)^2 (\sum_{i=1}^{d} z_i + 1) \big| z_i \sim \text{Po}(3r/2\lambda_i) \right) < \infty.
\]

\[\square\]

Lemma 2. If the function \( F(z, r) \) defined in Theorem 1 is finite, then for any given \( r \) and \( z \), there exists \( \epsilon > 0 \) such that
\[
\int \max_{t \in [r-\epsilon, r+\epsilon]} \left\{ f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{z_i + \beta_i}{t} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) d\lambda < \infty. \tag{A.42}
\]

Proof
We choose \( \epsilon = r/2 \). Then, the left-hand side of (A.42) is not greater than
\[
\int \max_{t \in [r/2, r+2r]} \left\{ f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{z_i + \beta_i}{t} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) d\lambda \\
\leq \int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{(2r) z_i + \beta_i}{t} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) d\lambda \\
= \frac{1}{r/\epsilon} \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} + \frac{1}{r} \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} \sum_{j=1}^{d} F(z, r/2) (z_j + \beta_j) \\
= \frac{1}{r/\epsilon} \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} + \frac{1}{r} \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} \sum_{j=1}^{d} F(z, r/2) (z_j + \beta_j).
\]

Because \( F(z, r/2) \) and \( F(z, r/2) (z_j + \beta_j) \) are finite, the proof is complete.

\[\square\]

Lemma 3. Under condition 2.1, for a given \( r \), there exists \( \epsilon > 0 \) such that
\[
\sum_{z} \max_{t \in [r-\epsilon, r+\epsilon]} \left[ \log(F(z, t)) \left\{ \prod_{i=1}^{d} \frac{(t\lambda_i)^{z_i} \exp(-t\lambda_i)}{z_i!} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) \right] \\
+ \sum_{z} \max_{t \in [r-\epsilon, r+\epsilon]} \left[ \int F(z, t) \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) d\lambda \right] \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) < \infty. \tag{A.43}
\]

Proof
We know that
\[
\int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{j=1}^{d} \frac{z_j + \beta_j}{t} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) d\lambda = F(z, \delta_i, t) \frac{z_i + \beta_i}{t}.
\]

Therefore, the lemma is equivalent to
\[
\sum_{z} \max_{t \in [r-\epsilon, r+\epsilon]} \left[ \log F(z, t) \left\{ \prod_{i=1}^{d} \frac{(t\lambda_i)^{z_i} \exp(-t\lambda_i)}{z_i!} \right\} \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) \right] \\
+ \sum_{z} \max_{t \in [r-\epsilon, r+\epsilon]} \left[ \frac{F(z, t)}{t} \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} F(z, \delta_i, t) \frac{z_i + \beta_i}{t} \right] \frac{d}{d\lambda_i} \left( \sum_{i=1}^{d} \frac{z_i + \beta_i}{t} - \sum_{i=1}^{d} \lambda_i \right) < \infty. \tag{A.43}
\]

From Lemma 2 we have \( \epsilon \in (0, r) \) such that the first term on the left-hand side of (A.43) is not greater than
\[
\sum_{z} \max_{t \in [r-\epsilon, r+\epsilon]} \left[ \log F(z, t) \left\{ \prod_{i=1}^{d} \frac{(t\lambda_i)^{z_i} \exp(-t\lambda_i)}{z_i!} \right\} \frac{\sum_{i=1}^{d} z_i + 1}{\min\{r - \epsilon, 1/\sum_{i=1}^{d} \lambda_i\}} \right] < \infty.
\]
Therefore, we need only prove that the second term on the left-hand side of (A.43) is finite. Using (2.1), we have \( \sum_{i=1}^{d} (z_i + \beta_i) F(z + \delta_i, t) < 2 \sum_{i=1}^{d} (z_i + \beta_i) F(z, t) \). Therefore, the second term on the left-hand side of (A.43) is less than

\[
\sum_z \max_{t \in [r - \epsilon, r + \epsilon]} \left[ \frac{F(z, t) \sum_{i=1}^{d} z_i + \beta_i}{F(z, t) \sum_{i=1}^{d} (t \lambda_i) z_i!} \prod_{i=1}^{d} (t \lambda_i) z_i! \right] 
\leq \exp(2r \sum_{i=1}^{d} \lambda_i) \sum_z \left\{ \sum_{i=1}^{d} (z_i + \beta_i) \prod_{i=1}^{d} ((r + \epsilon) \lambda_i) z_i! \right\} < \infty.
\]

\[\square\]

**Lemma 4.** Under conditions (2.3a) and (2.3b), we have

\[
F(z, r) < \infty, \quad F(z + \delta_i, r) < \infty,
\]

\[
\int \left\{ \int f(\theta) \prod_{j \neq i} \frac{r \theta_j^{2z_j + 2\beta_j - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j)} \right\} \frac{r \theta_i^{2z_i + 2\beta_i} \exp(-r \theta_i^2)}{\Gamma(z_i + 1 + \beta_i)} d\theta_i < \infty, \quad \text{(A.44)}
\]

and

\[
\int \left| \frac{\partial}{\partial \theta_i} f(\theta) \right| \prod_{j=1}^{d} \frac{r \theta_j^{2z_j + 2\beta_j + 2\delta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \delta_j + \beta_j)} d\theta_i < \infty, \quad \text{(A.45)}
\]

**Proof**

From (2.3a) and the substitution \( \lambda = \theta^2 \), we have \( F(z, r) < \infty \) and

\[
\int f(\theta) \prod_{j=1}^{d} \frac{r \theta_j^{2z_j + 2\beta_j - 1} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j + \delta_{ij})} d\theta < \infty. \quad \text{(A.46)}
\]

Substituting \( z + \delta_i \) for \( z \) yields \( F(z + \delta_i, r) < \infty \) and

\[
\int f(\theta) \prod_{j=1}^{d} \frac{r \theta_j^{2z_j + 2\beta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j + \delta_{ij})} d\theta < \infty. \quad \text{(A.47)}
\]

Using (A.46), (A.47), and \( \theta_j^{2z_j + 2\beta_j - 1} + \theta_j^{2z_j + 2\beta_j + 1} \geq \theta_j^{2z_j + 2\beta_j} \), we obtain (A.44). From (2.3b), we have

\[
\int \left| \frac{\partial}{\partial \theta_i} f(\theta) \right| \prod_{j=1}^{d} \frac{r \theta_j^{2z_j + 2\beta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j + \delta_{ij})} d\theta_i < \infty. \quad \text{(A.48)}
\]

From (2.3b), we have

\[
\int \left| \frac{\partial}{\partial \theta_i} f(\theta) \right| \prod_{j=1}^{d} \frac{r \theta_j^{2z_j + 2\beta_j + 2\delta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j + \delta_{ij})} d\theta_i < \infty. \quad \text{(A.49)}
\]

Using (A.48), (A.49), and \( \theta_j^{2z_j + 2\beta_j - 1} + \theta_j^{2z_j + 2\beta_j + 1} \geq \theta_j^{2z_j + 2\beta_j} \), we obtain (A.45).

\[\square\]

**Lemma 5.** If \( f \in C^2([0, \infty)^d) \) and conditions (2.3a) and (2.3b) are satisfied, then

\[
\left\{ \int_{[a, b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r \theta_j^{2z_j + 2\beta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j)} d\theta_j \right\} \frac{r \theta_i^{2z_i + 2\beta_i} \exp(-r \theta_i^2)}{\Gamma(z_i + 1 + \beta_i)} \quad \text{(A.50)}
\]

converges to 0 as \( \theta_i \to 0 \) or \( \theta_i \to \infty \).

**Proof**

By performing integration by parts on \( \theta_i \), we obtain

\[
\int_{u}^{v} \left\{ \int_{[a, b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r \theta_j^{2z_j + 2\beta_j} \exp(-r \theta_j^2)}{\Gamma(z_j + \beta_j)} d\theta_j \right\} \frac{r \theta_i^{2z_i + 2\beta_i} \frac{\partial}{\partial \theta_i} \exp(-r \theta_i^2)}{\Gamma(z_i + 1 + \beta_i)} d\theta_i
\]

\[26\]
is bounded for Lemma 6.

Proof

From (2.3b), we have

\[ + \int_u^v \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \right\} \frac{r^3 + \beta \theta^2_{i} + 2 \beta_i - 1}{\Gamma(z_i + 1 + \beta_i)} \frac{d\theta_i}{\theta_i} \]

\[ + \int_{[a,b]}^{[a,b]^{d-1}} \left\{ \frac{\partial}{\partial \theta_i} f(\theta) \right\} \prod_{j=1}^{d} \frac{r^3 + \beta \theta^2_{j} + 2 \delta_j + 2 \beta_j - 1}{\Gamma(z_j + \delta_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta}{\theta} \]

\[ = \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \right\} \frac{r^3 + \beta \theta^2_{i} + 2 \beta_i - 1}{\Gamma(z_i + 1 + \beta_i)} \frac{d\theta_i}{\theta_i} \right\}_{\theta_i = u} \cdot (A.51) \]

From Lemma 4 we know that all three terms on the left-hand side of (A.51) converge as \( v \to \infty \). Hence, the right-hand side of (A.51) converges as \( v \to \infty \). From Lemma 3 we have

\[ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} < \infty. \]

Hence, (A.50) converges to 0 as \( \theta_i \to \infty \).

Because \( f \in C^2([0, \infty)^d) \), \( f \) is bounded on \([0,1] \times [a,b]^{d-1}\). Thus,

\[ \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \right\} \frac{r^3 + \beta \theta^2_{i} + 2 \beta_i - 1}{\Gamma(z_i + 1 + \beta_i)} \frac{d\theta_i}{\theta_i} \]

is bounded for \( \theta_i \leq 1 \). Thus, (A.50) converges to 0 as \( \theta_i \to 0 \).

\[ \Box \]

Lemma 6. If \( f \in C^2([0, \infty)^d) \) and conditions 2.3a, 2.3b, 2.3c, and 2.3b are satisfied, then

\[ \lim_{v \to \infty} \int_{[a,b]^{d-1}} \frac{\partial^d f}{\partial \theta_i^d} (\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} < \infty. \]

Proof

From (2.3b), we have

\[ \int \left| \frac{\partial^d f}{\partial \theta_i^d} (\theta) \right| \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} < \infty. \]

Hence,

\[ \int \int_{[a,b]^{d-1}} \frac{\partial^d f}{\partial \theta_i^d} (\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} \]

\[ = \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \right\} \frac{r^3 + \beta \theta^2_{i} + 2 \beta_i - 1}{\Gamma(z_i + 1 + \beta_i)} \frac{d\theta_i}{\theta_i} \right\}_{\theta_i = u} \cdot (A.52) \]

By performing integration by parts on \( \theta_i \), we obtain

\[ \int_u^v \int_{[a,b]^{d-1}} \frac{\partial^d f}{\partial \theta_i^d} (\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} \]

\[ + \int_u^v \int_{[a,b]^{d-1}} \frac{\partial^d f}{\partial \theta_i^d} (\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \frac{d\theta_j}{\theta_j} \]

\[ = \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{r^3 + \beta \theta^2_{j} + 2 \beta_j - 1}{\Gamma(z_j + \beta_j)} \exp(-r \theta_j^2) \right\} \frac{r^3 + \beta \theta^2_{i} + 2 \beta_i - 1}{\Gamma(z_i + 1 + \beta_i)} \frac{d\theta_i}{\theta_i} \right\}_{\theta_i = u} \cdot (A.53) \]

From (A.12) and \( z_i \{ F(z, r) - F(z - \delta_i, r) \} < \infty \), we know that the first term in (A.53) converges as \( u \to 0 \) or \( v \to \infty \). From (A.12) and \( (z_i + \beta_i) \{ F(z, r) - F(z + \delta_i, r) \} < \infty \), we know that the second term in (A.53)
converges as \( u \to 0 \) or \( v \to \infty \). From (A.18), we know that the third term in (A.53) converges as \( u \to 0 \) or \( v \to \infty \). Thus, all three terms in (A.53) converge as \( u \to 0 \) or \( v \to \infty \). Hence,

\[
\left[ \int_{[a,b]^d} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{z_i-1} \prod_{j \neq i} \left( \frac{r_j + \beta_j \theta_j^{2z_j+2\beta_j-1} \exp(-r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) \right]_{\theta_j = \bar{\theta}}
\]

converges as \( u \to 0 \) or \( v \to \infty \). Accordingly, using (A.52), we obtain

\[
\lim_{\theta_i \to \infty} \int_{[a,b]^d} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{z_i-1} \prod_{j \neq i} \left( \frac{r_j + \beta_j \theta_j^{2z_j+2\beta_j-1} \exp(-r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) = 0.
\]

From (2.2b), we have

\[
\lim_{\theta_i \to 0} \int_{[a,b]^d} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{z_i-1} \prod_{j \neq i} \left( \frac{r_j + \beta_j \theta_j^{2z_j+2\beta_j-1} \exp(-r_j \theta_j^2)}{\Gamma(z_j + \beta_j)} \right) \leq 0.
\]

Using (A.54) and (A.55), we complete the proof.

Lemma 7. Under condition (3.1), for a given \( r \), there exists \( \epsilon > 0 \) such that

\[
\sum_{z} \left\{ \max_{\theta \in B(r, \epsilon)} \left( \frac{\log F(z, t) \left( \sum_{i=1}^d z_i + 1 \prod_{i=1}^d (t_i \lambda_i)^{z_i} \exp(-t_i \lambda_i)}{z_i!} \right) \right) \right\} < \infty,
\]

where \( B(r, \epsilon) \) is a closed ball with center \( r = (r_1, \ldots, r_d) \) and radius \( \epsilon \). In particular, \( E \left( \frac{\log F(z, r)}{z_i} \right) \sim P \theta(r, \lambda_i), i = 1, \ldots, d < \infty \).

Proof

From (3.1), we know that

\[
\sum_{i=1}^d \gamma_i t_i \left\{ F(z, t) - F(z - \delta_i, t) \right\} + \sum_{i=1}^d \gamma_i t_i (z_i + \beta_i) \left\{ F(z, t) - F(z + \delta_i, t) \right\} \geq 0, \forall z.
\]

Let \( \beta_0 := \min\{\beta_1, \ldots, \beta_d\} \) and \( \gamma_0 := \min\{\gamma_1, \ldots, \gamma_d\} \).

Using (A.57), we obtain \( F(z + \delta, t) < \frac{2}{\beta_0 \gamma_0} \max\{t_1, \ldots, t_d\} \sum_{i=1}^d (z_i + \beta_i) F(z, t), \forall z \). Hence,

\[
F(z, t) \leq \left( \frac{2}{\beta_0 \gamma_0} \max\{t_1, \ldots, t_d\} \sum_{i=1}^d (z_i + \beta_i) \right)^{\sum_{i=1}^d z_i} F(\bar{\theta}, t).
\]

Using (A.57), we obtain \( F(z - \delta, t) < \frac{2}{\gamma_0} \max\{t_1, \ldots, t_d\} \sum_{i=1}^d (z_i + \beta_i) F(z, t) \) if \( z_i > 0 \). Hence,

\[
F(z, t) \geq \left( \frac{2}{\gamma_0} \max\{t_1, \ldots, t_d\} \sum_{i=1}^d (z_i + \beta_i) \right)^{-\sum_{i=1}^d z_i} F(\bar{\theta}, t).
\]

We choose \( \epsilon = \min\{r_1, \ldots, r_d\}/2 \). Then, we have

\[
\max_{t \in B(r, \epsilon)} F(\bar{\theta}, t) \leq \int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^d (3r_i/2)^{\beta_i} \lambda_i^{\beta_i-1} \exp(-r_i \lambda_i) d\lambda = 3^{\sum_{i=1}^d \beta_i} F(\bar{\theta}, r/2).
\]
We also have

\[
\min_{t \in B(r, s)} F(\bar{0}, t) \geq \int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{(r_i/2)^{\beta_i} \lambda_i^{\beta_i-1} \exp(-3r_i/2\lambda_i)}{\Gamma(\beta_i)} d\lambda = (1/3)\sum_{i=1}^{d} \beta_i F(\bar{0}, 3r/2). \tag{A.61}
\]

From (A.58), (A.59), (A.60), and (A.61), there exists a constant $C$ such that

\[
\max_{t \in B(r, s)} \left| \log F(z, t) \right| < C\left( \sum_{i=1}^{d} z_i + C \right)^2, \forall z.
\]

Thus, the left-hand side of (A.56) is less than

\[
\sum_{z} \left[ C\left( \sum_{i=1}^{d} z_i + C \right)^2 \prod_{i=1}^{d} \frac{(3r_i/2\lambda_i)^{z_i} \exp(-r_i/2\lambda_i)}{z_i!} \right] = \exp\left( \sum_{i=1}^{d} r_i \lambda_i \right) \mathbb{E}\left[ C\left( \sum_{i=1}^{d} z_i + C \right)^2 \left( \sum_{i=1}^{d} z_i + 1 \right) \right| z_i \sim \text{Po}(3r_i/2\lambda_i) \right] < \infty.
\]

\[\square\]

**Lemma 8.** If the function $F(z, r)$ defined in Theorem 3 is finite, then for any given $r = (r_1, \ldots, r_d)$, $s = (s_1, \ldots, s_d)$, and $z = (z_1, \ldots, z_d)$, we have

\[
\int \max_{\tau \in [0, 1]} \left\{ f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{t_i(\tau)^{z_i + \beta_i} \lambda_i^{z_i + \beta_i - 1} \exp(-t_i(\tau) \lambda_i)}{\Gamma(z_i + \beta_i)} \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau) - \sum_{i=1}^{d} \lambda_i \gamma_i^2 t_i^2(\tau) \right\} d\lambda < \infty. \tag{A.62}
\]

**Proof**

We define $r_0 = \min\{r_1, \ldots, r_d\}$ and $s_0 = \max\{r_1 + s_1, \ldots, r_d + s_d\}$. Because $t_i(\tau) \in [r_i, r_i + s_i]$, we have $t_i(\tau) \in [r_0, s_0]$. Then, the left-hand side of (A.62) is not greater than

\[
\int \left[ f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{\lambda_i^{z_i + \beta_i} \exp(-r_0 \lambda_i)}{\Gamma(z_i + \beta_i)} \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i s_0 + \sum_{i=1}^{d} \lambda_i s_0^2 \right] d\lambda = \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i s_0 \left( \frac{s_0}{r_0} \right)^{z_i + \beta_i} F(z, (r_0, \ldots, r_0)) + \sum_{j=1}^{d} \gamma_j s_0 \left( \frac{s_0}{r_0} \right)^{z_j + \beta_j + 1} F(z + \delta_j, (r_0, \ldots, r_0))(z_j + \beta_j).
\]

Because $F(z, (r_0, \ldots, r_0))$ and $F(z + \delta_j, (r_0, \ldots, r_0))$ are finite, the proof is complete. \[\square\]

**Lemma 9.** Under condition (3.1), for a given $\tau_0$, there exists $\epsilon > 0$ such that

\[
\sum_{z} \max_{|\tau - \tau_0| \leq \epsilon} \left[ \left\{ \log F(z, t(\tau)) \right\} \left\{ \prod_{i=1}^{d} \frac{(t_i(\tau) \lambda_i)^{z_i} \exp(-t_i(\tau) \lambda_i)}{z_i!} \right\} \left( \sum_{i=1}^{d} z_i \gamma_i t_i(\tau) - \sum_{i=1}^{d} \lambda_i \gamma_i t_i^2(\tau) \right) \right] + \sum_{z} \max_{|\tau - \tau_0| \leq \epsilon} \left[ \int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{i=1}^{d} \frac{t_i(\tau)^{z_i + \beta_i} \lambda_i^{z_i + \beta_i - 1} \exp(-t_i(\tau) \lambda_i)}{\Gamma(z_i + \beta_i)} \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau) - \sum_{i=1}^{d} \lambda_i \gamma_i^2 t_i^2(\tau) d\lambda \right] F(z, t(\tau)) d\lambda \left\{ \prod_{i=1}^{d} \frac{(t_i(\tau) \lambda_i)^{z_i} \exp(-t_i(\tau) \lambda_i)}{z_i!} \right\} \right] < \infty.
\]

**Proof**

We know that

\[
\int f(\bar{\theta}_1, \ldots, \bar{\theta}_d) \prod_{j=1}^{d} \frac{t_j(\tau)^{z_j + \beta_j} \lambda_j^{z_j + \beta_j - 1} \exp(-t_j(\tau) \lambda_j)}{\Gamma(z_j + \beta_j)} \lambda_j \gamma_j^2 t_j^2(\tau) d\lambda = F(z + \delta_j, t(\tau))(z_j + \beta_j) \gamma_j t_j(\tau).
\]
Therefore, the lemma is equivalent to

\[
\sum_z \max_{|\tau - \tau_0| \leq \epsilon} \left[ \log F(z, t(\tau)) \left\{ \prod_{i=1}^{d} \frac{(t_i(\tau) \lambda_i)^{z_i} \exp(-t_i(\tau) \lambda_i)}{z_i!} \right\} \sum_{i=1}^{d} z_i \gamma_i t_i(\tau) - \sum_{i=1}^{d} \lambda_i \gamma_i t_i^2(\tau) \right]
\]

\[
+ \sum_z \max_{|\tau - \tau_0| \leq \epsilon} \left\{ \frac{F(z, t(\tau)) \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau) - \sum_{i=1}^{d} F(z + \delta_i, t(\tau))(z_i + \beta_i) \gamma_i t_i(\tau)}{F(z, t(\tau))} \prod_{i=1}^{d} \frac{(t_i(\tau) \lambda_i)^{z_i} \exp(-t_i(\tau) \lambda_i)}{z_i!} \right\} \leq \infty. \tag{A.63}
\]

From Lemma 7 when we set \( r = t(\tau_0) \), we know that there exists \( \delta > 0 \) such that

\[
\sum_z \left[ \max_{t \in B(t(\tau_0), \delta)} \left\{ \log F(z, t) \left\{ \prod_{i=1}^{d} z_i \right\} + 1 \right\} \prod_{i=1}^{d} \frac{(t_i(\lambda))^z \exp(-t_i(\lambda))}{z_i!} \right] < \infty.
\]

Because \( t(\tau) \) is continuous, there exists \( \epsilon > 0 \) such that for any \( \tau \in [\tau_0 - \epsilon, \tau_0 + \epsilon] \), \( t(\tau) \in B(t(\tau_0), \delta) \). Thus, the first term on the left-hand side of (A.63) is not greater than

\[
\sum_z \max_{t \in B(t(\tau_0), \delta)} \left[ \log F(z, t) \left\{ \prod_{i=1}^{d} \frac{(t_i(\lambda))^z \exp(-t_i(\lambda))}{z_i!} \right\} \sum_{i=1}^{d} z_i \gamma_i t_i - \sum_{i=1}^{d} \lambda_i \gamma_i t_i^2 \right] \leq \sum_z \max_{t \in B(t(\tau_0), \delta)} \left[ \log F(z, t) \left\{ \prod_{i=1}^{d} \frac{(t_i(\lambda))^z \exp(-t_i(\lambda))}{z_i!} \right\} \left( \sum_{i=1}^{d} z_i + 1 \right) \prod_{i=1}^{d} \gamma_i \left( ||t(\tau_0)|| + \delta \right) + \sum_{i=1}^{d} \lambda_i \gamma_i \left( ||t(\tau_0)|| + \delta \right)^2 \right] < \infty. \tag{A.64}
\]

Therefore, we need only prove that the second term on the left-hand side of (A.63) is finite. Using (3.1), when we set \( r = t(\tau) \), we obtain \( \sum_{i=1}^{d} \gamma_i t_i(\tau)(z_i + \beta_i)F(z + \delta_i, t(\tau)) < 2 \sum_{i=1}^{d} \gamma_i t_i(\tau)(z_i + \beta_i)F(z, t(\tau)). \) We define \( r_0 = \min\{r_1, \ldots, r_d\} \) and \( s_0 = \max\{s_1 + s_1, \ldots, s_d + s_d\}. \) Because \( t_i(\tau) \in [r_i, r_i + s_i] \), we have \( t_i(\tau) \in [r_0, s_0]. \) Therefore, the second term on the left-hand side of (A.63) is less than

\[
\sum_z \max_{|\tau - \tau_0| \leq \epsilon} \left\{ \frac{F(z, t(\tau)) \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i t_i(\tau)}{F(z, t(\tau))} \prod_{i=1}^{d} \frac{(t_i(\lambda))^z \exp(-t_i(\lambda))}{z_i!} \right\}
\]

\[
\leq \exp((s_0 - r_0) \sum_{i=1}^{d} \lambda_i) \sum_z \left( \sum_{i=1}^{d} (z_i + \beta_i) \gamma_i s_0 \right) \prod_{i=1}^{d} \frac{(s_0 \lambda_i)^z \exp(-s_0 \lambda_i)}{z_i!} < \infty.
\]

Lemma 10. Under conditions (2.3a) and (2.3b), we have

\[
F(z, r) < \infty, \quad F(z + \delta_i, r) < \infty,
\]

\[
\int \left\{ \int f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^z \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \delta_j + \beta_j)} \right\} \frac{(\gamma_i r_i)^z \exp(-\gamma_i r_i \theta_i^2)}{\Gamma(z_i + 1 + \beta_i)} d\theta_i < \infty, \quad \tag{A.65}
\]

and

\[
\int \frac{\partial}{\partial \theta_i} f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^z \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \delta_j + \beta_j)} d\theta_j < \infty. \quad \tag{A.66}
\]

Proof

From (2.3a) and the substitution \( \lambda = \gamma \theta^2 \), we have \( F(z, r) < \infty \) and

\[
\int f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^z \exp(-\gamma_j r_j \theta_j^2)}{\Gamma(z_j + \delta_j + \beta_j)} d\theta < \infty. \quad \tag{A.67}
\]
Substituting \( z + \delta_i \) for \( z \) yields \( F(z + \delta_i, r) < \infty \) and
\[
\int f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta < \infty. \tag{A.68}
\]
Using (A.67), (A.68), and \( \theta_i^{2+\beta_i - 1} + \theta_i^{2+\beta_i + 1} \geq \theta_i^{2+\beta_i} \), we obtain (A.65). From (2.3b), we have
\[
\int \frac{\partial}{\partial \theta_i} f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta < \infty. \tag{A.69}
\]
From (2.3b), we have
\[
\int \frac{\partial}{\partial \theta_i} f(\theta) \prod_{j=1}^{d} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta < \infty. \tag{A.70}
\]
Using (A.69), (A.70), and \( \theta_i^{2+\beta_i - 1} + \theta_i^{2+\beta_i + 1} \geq \theta_i^{2+\beta_i} \), we obtain (A.66).

Lemma 11. If \( f \in \mathbb{C}^2([0, \infty)^d) \) and conditions (2.3a), and (2.3b) are satisfied, then
\[
\left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \right\} \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \tag{A.71}
\]
converges to 0 as \( \theta_i \to 0 \) or \( \theta_i \to \infty \).

Proof By performing integration by parts on \( \theta_i \), we obtain
\[
\int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \, d\theta_i
\]
\[
+ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \, d\theta_i
\]
\[
+ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \, d\theta_i
\]
\[
= \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j + \delta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \right\} \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \tag{A.72}
\]
From Lemma 10, we know that all three terms on the left-hand side of (A.72) converge as \( v \to \infty \). Hence, the right-hand side of (A.72) converges as \( v \to \infty \). Using Lemma 10, we obtain
\[
\int \left\{ \int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \right\} \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \, d\theta_i < \infty.
\]
Hence, (A.71) converges to 0 as \( \theta_i \to \infty \). Because \( f \in \mathbb{C}^2([0, \infty)^d) \), \( f \) is bounded on \([0, 1] \times [a, b]^{d-1}\). Thus,
\[
\int_{[a,b]^{d-1}} f(\theta) \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j
\]
is bounded for \( \theta_i \leq 1 \). Thus, (A.71) converges to 0 as \( \theta_i \to 0 \).

Lemma 12. If \( f \in \mathbb{C}^2([0, \infty)^d) \) and conditions (2.3a), (2.3b), (2.3c), and (2.2b) are satisfied, then
\[
\lim_{v \to \infty} \left[ \int_{[a,b]^{d-1}} \frac{\partial f(\theta)}{\partial \theta_i} \theta_i^{2\gamma_i - 1} \prod_{j \neq i} \frac{(\gamma_j r_j)^{z_j+\beta_j} \theta_j^{2\gamma_j + 2\beta_j - 1}}{\Gamma(z_j + \beta_j)} \exp(-\gamma_j r_j \theta_j^2) \, d\theta_j \right] \frac{(\gamma_i r_i)^{z_i+\beta_i} \theta_i^{2\gamma_i + 2\beta_i}}{\Gamma(z_i + 1 + \beta_i)} \exp(-\gamma_i r_i \theta_i^2) \geq 0.
\]

31
Proof

From (2.31), we have

$$\int |\frac{\partial f}{\partial \theta_i}(\theta)| \prod_{j=1}^{d} \theta_j^{2z_j - 2 + \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \, d\theta < \infty.$$ 

Thus,

$$\int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i < \infty. \tag{A.73}$$

By performing integration by parts on $\theta_i$, we obtain

$$\int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i + \int \int_{[a,b]^{d-1}} \frac{\partial^2 f}{\partial \theta_i^2}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i + \int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i \tag{A.74}$$

From (A.25) and $\gamma_i r_i (F(z,r) - F(z - \delta_i, r)) < \infty$, we know that the first term in (A.74) converges as $u \to 0$ or $v \to \infty$. From (A.25) and $\gamma_i r_i(z_i + \beta_i)(F(z,r) - F(z + \delta_i, r)) < \infty$, we know that the second term in (A.74) converges as $u \to 0$ or $v \to \infty$. Hence, all three terms in (A.74) converge as $u \to 0$ or $v \to \infty$. Therefore,

$$\int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i \Rightarrow \theta_i \to \infty$$

converges as $u \to 0$ or $v \to \infty$. Thus, using (A.73), we obtain

$$\lim_{\theta_i \to \infty} \int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i \tag{A.75}$$

From (2.21b), we have

$$\lim_{\theta_i \to 0} \int \int_{[a,b]^{d-1}} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2z_i - 2 + \beta_i - 1} \left\{ \prod_{j \neq i} (\gamma_j r_j)^{z_j + \beta_j} \theta_j^{2z_j + 2 \beta_j - 1} \exp(-\gamma_j r_j \theta_j^2) \right\} \frac{d\gamma_j r_j}{\Gamma(z_j + \beta_j)} \frac{d\theta_j}{\Gamma(z_i + \beta_i)} \, d\theta_i \tag{A.76}$$

Using (A.75) and (A.76), we complete the proof. \qed

A.5 Proof of Proposition A.1

Independent Poisson processes with the same duration correspond to $r_1 = r_2 = \cdots = r_d$ and $s_1 = s_2 = \cdots = s_d$. Therefore, we need only prove that $f(\theta) = (\sum_{i=1}^{d} \theta_i^2 + \eta)^{-\alpha}$ satisfies the conditions in the first half of Theorem 3. We show this in two steps. In Step 1, we prove that the condition (3.1) is satisfied. In Step 2, we prove that $F(z,r)$ is not a constant function of $z$. The details of each step are presented below.

Step 1. We first show that $f(\theta) = (\sum_{i=1}^{d} \theta_i^2 + \eta)^{-\alpha}$ satisfies the conditions in the second half of Theorem 3 when $\eta > 0$, $0 < \alpha \leq \sum_{i=1}^{d} \beta_i - 1$. Here, $\theta_i = \frac{\lambda_i}{\gamma_i}$, $i = 1, \ldots, d$. 32
is satisfied because \( f(\theta) \leq \eta^{-\alpha} = \exp(\alpha(\sum_{j=1}^{d} \theta_j^{2})) \).

\[ \frac{\partial f}{\partial \theta_i}(\theta) \leq 2\alpha \eta^{-\alpha-1} \theta_i = \exp(\alpha(\sum_{j=1}^{d} \theta_j^{2})). \]

\[ \frac{\partial^2 f}{\partial \theta_i^2}(\theta) + (2\beta_i - 1) \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{-1} \leq 4\alpha(\alpha + 1) \eta^{-\alpha-2} \theta_i^{2} + 4\alpha \beta_i \eta^{-\alpha-1} = \exp(\alpha(\sum_{j=1}^{d} \theta_j^{2})). \]

\[ \sum_{i=1}^{d} \frac{\partial}{\partial \theta_i} \left( \frac{\partial f}{\partial \theta_i}(\theta) \prod_{j=1}^{d} \theta_j^{2\beta_j-1} \right) = -4\alpha \sum_{j=1}^{d} \theta_j^{2} + \eta^{-2} \prod_{j=1}^{d} \theta_j^{2\beta_j-1} \left\{ \left( \sum_{i=1}^{d} \beta_i - \alpha - 1 \right) \left( \sum_{j=1}^{d} \theta_j^{2} \right) + \sum_{i=1}^{d} \beta_i \eta \right\} \leq 0. \]

\[ \lim_{\theta_i \to 0} \frac{\partial f}{\partial \theta_i}(\theta) \theta_i^{2\beta_i-1} = \lim_{\theta_i \to 0} -2\alpha \theta_i^{2\beta_i} \left( \sum_{j=1}^{d} \theta_j^{2} + \eta \right)^{-\alpha-1} \leq 0. \]

From the statements above, we know that \( f(\theta) = (\sum_{i=1}^{d} \theta_i^{2} + \eta)^{-\alpha} \) satisfies the conditions of the second half of Theorem 3. Therefore, Proposition 2 holds when \( f(\theta) = (\sum_{i=1}^{d} \theta_i^{2} + \eta)^{-\alpha}, \eta > 0, 0 < \alpha \leq \sum_{i=1}^{d} \beta_i - 1. \)

Next, we prove that \( (3.1) \) still holds when \( f(\theta) = (\sum_{i=1}^{d} \theta_i^{2})^{-\alpha}. \) The key is to consider \( \eta \to 0. \) Let \( \zeta = \min\{\gamma_1 r_1, \ldots, \gamma_d r_d\} \) and \( \lambda := \theta^2. \) Because \( -\alpha + \sum_{j=1}^{d} (\beta_j + z_j) \geq 1, \) we have

\[ \int \left( \sum_{j=1}^{d} \theta_j^{2} \right)^{-\alpha} \prod_{j=1}^{d} \frac{\lambda^{\beta_j + \gamma_j} / (\gamma_j + \beta_j)}{\Gamma(z_j + \beta_j)} \exp(-r_j \lambda^{\gamma_j}) \, d\lambda = \int \left( \sum_{j=1}^{d} \lambda^{\beta_j + \gamma_j} / (\gamma_j + \beta_j) \right) \Gamma(z_j + \beta_j) \exp(-r_j \lambda^{\gamma_j}) \, d\lambda \]

\[ \leq \int \prod_{j=1}^{d} \frac{\lambda^{\beta_j + \gamma_j} / (\gamma_j + \beta_j)}{\Gamma(z_j + \beta_j)} \exp(-r_j \lambda^{\gamma_j}) \, d\lambda = \int \prod_{j=1}^{d} \frac{\lambda^{\beta_j + \gamma_j} / (\gamma_j + \beta_j)}{\Gamma(z_j + \beta_j)} \exp(-r_j \lambda^{\gamma_j}) \, d\lambda \]

Therefore, using the dominated convergence theorem, we have

\[ \int \left( \sum_{i=1}^{d} \theta_i^{2} + \eta \right)^{-\alpha} \prod_{i=1}^{d} \frac{\lambda^{\beta_i + \gamma_i} / (\gamma_i + \beta_i) \exp(-r_i \lambda^{\gamma_i}) \, d\lambda \to \int \left( \sum_{i=1}^{d} \theta_i^{2} \right)^{-\alpha} \prod_{i=1}^{d} \frac{\lambda^{\beta_i + \gamma_i} / (\gamma_i + \beta_i) \exp(-r_i \lambda^{\gamma_i}) \, d\lambda \]

when \( \eta \to 0. \) From the definition of \( F \) in \( (3.1) \), we know that \( (3.1) \) still holds when \( f(\theta) = (\sum_{i=1}^{d} \theta_i^{2})^{-\alpha}. \)

**Step 2.** \( F(z, r) \) is not a constant function of \( z \) because when \( z_1 \to \infty, \)

\[ F(z, r) = \int (\lambda_1 / \gamma_1 + \lambda_2 / \gamma_2 + \cdots + \lambda_d / \gamma_d + \eta)^{-\alpha} \prod_{i=1}^{d} \frac{\lambda_i^{\beta_i + \gamma_i} / (\gamma_i + \beta_i) \exp(-r_i \lambda_i) \, d\lambda}{\Gamma(z_i + \beta_i)} \]

\[ < \int (\lambda_1 / \gamma_1)^{-\alpha} \prod_{i=1}^{d} \frac{\lambda_i^{\beta_i + \gamma_i} / (\gamma_i + \beta_i) \exp(-r_i \lambda_i) \, d\lambda}{\Gamma(z_i + \beta_i)} \to 0. \]

Therefore, \( f(\theta) = (\sum_{i=1}^{d} \theta_i^{2} + \eta)^{-\alpha}, \eta \geq 0, 0 < \alpha \leq \sum_{i=1}^{d} \beta_i - 1 \) satisfies the conditions in the first half of Theorem 3. Thus, the Bayesian predictive distribution \( p_{f, \beta}(y \mid x) \) dominates the Bayesian predictive distribution \( p_{\beta}(y \mid x) \) for the prediction of independent Poisson processes with the same or different durations. \( \square \)

**A.6 Proof of Proposition 2**

Independent Poisson processes with the same duration correspond to \( r_1 = r_2 = \cdots = r_d \) and \( s_1 = s_2 = \cdots = s_d. \) Therefore, we need only prove that \( f(\theta) = \sum_{\alpha \in \{1, \ldots, 1\}^d} (\sum_{i=1}^{d} \alpha_i \theta_i - \eta_i)^{-\alpha} \) satisfies the conditions in the first
We show this in two steps. In Step 1, we prove that the condition (3.2) is satisfied. In Step 2, we prove that $F(z, r)$ is not a constant function of $z$. The details of each step are presented below.

**Step 1.** We first show that $f_r(\theta) = \sum_{a \in \{1, \ldots, 1\}} \left( \sum_{i=1}^d (a_i \theta_i - \eta_i)^2 + \epsilon \right)^{-\alpha}$ satisfies the conditions in the second half of Theorem 4 when $\epsilon > 0$, $0 < \alpha \leq d/2 - 1$. Here, $\theta_i = \sqrt{\frac{\lambda_i}{\gamma_i}}$, $i = 1, \ldots, d$.

(3.2) is satisfied because $f_r(\theta) \leq 2^d \epsilon^{-\alpha} = \exp(o(\sum_{j=1}^d \theta_j^2))$.

(2.4a) is satisfied because

$$\frac{\partial f_r}{\partial \theta_i}(\theta) \leq 2^{d+1} \alpha \epsilon^{-\alpha-1} (\theta_i + |\eta_i|) = \exp(o(\sum_{j=1}^d \theta_j^2)).$$

(2.4b) is satisfied because

$$\frac{\partial^2 f_r}{\partial \theta_i^2}(\theta) \leq 2^{d+2} \alpha (\alpha + 1) \epsilon^{-\alpha-2} (\theta_i + |\eta_i|)^2 + 2^{d+1} \alpha \epsilon^{-\alpha-1} = \exp(o(\sum_{j=1}^d \theta_j^2)).$$

$h(\theta) = (\sum_{i=1}^d (\theta_i - \eta_i)^2 + \epsilon)^{-\alpha}$ is superharmonic because

$$\sum_{i=1}^d \frac{\partial^2 h}{\partial \theta_i^2}(\theta) = 2\alpha \left( \sum_{i=1}^d (\theta_i - \eta_i)^2 + \epsilon \right)^{-\alpha-2} \left\{ (2\alpha + 1) - d \right\} \sum_{i=1}^d (\theta_i - \eta_i)^2 - de \leq 0.$$

From the statements above, we know that $f_r(\theta)$ satisfies the conditions of the second half of Theorem 4. Therefore, (3.2) holds when $f(\theta) = \sum_{a \in \{1, \ldots, d\}} \left( \sum_{i=1}^d (a_i \theta_i - \eta_i)^2 + \epsilon \right)^{-\alpha}$, $\epsilon > 0$, $0 < \alpha \leq d/2 - 1$.

Next, we prove that (3.2) still holds when $f(\theta) = \sum_{a \in \{1, \ldots, d\}} \left( \sum_{i=1}^d (a_i \theta_i - \eta_i)^2 \right)^{-\alpha}$. The key is to consider $\epsilon \to 0$. Let $\zeta := \min{\{\gamma_1 r_1, \ldots, \gamma_d r_d\}}$. By using inequality $u^{\alpha \epsilon} \exp(-2\sqrt{\pi} u) \leq v^{\alpha} \exp(-2v)$, we know that there exists constant $c$ such that

$$u^{\alpha \epsilon} \leq c \exp(2\sqrt{\pi} u - v), \forall u, v \geq 0.$$

Therefore, there exists constant $C$ such that for every $\theta$ and $z$,

$$\prod_{j=1}^d r_j^{z_j+1/2} \lambda_j^{z_j-1/2} \exp(-r_j \gamma_j) \frac{\Gamma(z_j + 1/2)}{\Gamma(z_j + 1/2)} \leq C \prod_{j=1}^d \exp(-\sqrt{2\pi \zeta} \zeta^{-1} z_j^2) \exp(-r_j \gamma_j \theta^2) \frac{\Gamma(z_j + 1/2)}{\Gamma(z_j + 1/2)} \frac{d\theta}{\sqrt{2\pi \zeta} \zeta^{-1} z_j^2}.$$

(A.77)

Therefore, we have

$$\int \left( \sum_{j=1}^d (a_j \theta_j - \eta_j)^2 \right)^{-\alpha} \prod_{j=1}^d r_j^{z_j+1/2} \lambda_j^{z_j-1/2} \exp(-r_j \gamma_j) \frac{\Gamma(z_j + 1/2)}{\Gamma(z_j + 1/2)} \frac{d\lambda}{\sqrt{2\pi \zeta} \zeta^{-1} z_j^2} \leq C \prod_{j=1}^d \exp(-\zeta \theta_j - \sqrt{\frac{2}{r_j \gamma_j} \zeta^{-1} z_j^2}) \frac{d\theta}{\sqrt{2\pi \zeta} \zeta^{-1} z_j^2}.$$

(A.78)

where $\sqrt{\frac{2}{r_j \gamma_j} \zeta^{-1} z_j^2}$ denotes the vector whose $i$-th element is $\sqrt{\frac{z_i}{r_i \gamma_i}}$ and $a \eta$ denotes the vector whose $i$-th element is $a_i \eta_i$. Therefore, using the dominated convergence theorem, for every $a$, we have

$$\int \left( \sum_{j=1}^d (a_j \theta_j - \eta_j)^2 + \epsilon \right)^{-\alpha} \prod_{j=1}^d r_j^{z_j+1/2} \lambda_j^{z_j-1/2} \exp(-r_j \gamma_j) \frac{d\lambda}{\Gamma(z_j + 1/2)} \to \int \left( \sum_{j=1}^d (a_j \theta_j - \eta_j)^2 \right)^{-\alpha} \prod_{j=1}^d r_j^{z_j+1/2} \lambda_j^{z_j-1/2} \exp(-r_j \gamma_j) \frac{d\lambda}{\Gamma(z_j + 1/2)}$$

when $\epsilon \to 0$. From the definition of $F$ in (3.2), we know that (3.2) still holds.
Step 2. \( F(z,r) \) is not a constant function of \( z \) because when \( z_1 \to \infty \), from the inequality \( \text{(A.78)} \),

\[
\int \left(\sum_{j=1}^{d} (a_j \theta_j - \eta_j)^2\right)^{-\alpha} \prod_{j=1}^{d} \frac{r_j^{z_j+1/2} \lambda_j^{z_j-1/2} \exp(-r_j \lambda_j)}{\Gamma(z_j + 1/2)} \, d\lambda \leq C E[||\theta - a\eta||^{-2\alpha} | \theta \sim N_d(\sqrt{\frac{z}{r \gamma}}, \zeta^{-1} I_d)] \to 0.
\]

Therefore, \( f(\theta) = \sum_{a \in \{1, \ldots, r\}} (\sum_{i=1}^{d} (a_i \theta_i - \eta_i)^2)^{-\alpha} \) satisfies the conditions in the first half of Theorem 4. Thus, the Bayesian predictive distribution \( p_{|z}(y | x) \) dominates the Bayesian predictive distribution \( p_{|z}(y | x) \) for the prediction of independent Poisson processes with the same or different durations.

A.7 Proof of Proposition 3

We need only prove that \( f(\theta) = \sum_{a \in \{1, \ldots, r\}} (s_{\mathcal{V}}(a\theta))^\alpha \) satisfies the conditions in the first half of Theorem 4. We show this in two steps. In Step 1, we prove that the condition \( \text{(3.2)} \) is satisfied. In Step 2, we prove that \( F(z,r) \) is not a constant function of \( z \).

Step 1. We construct a \( C^2 \) prior

\[
f_\epsilon(\theta) = \sum_{a \in \{1, \ldots, r\}} (s_{\mathcal{V}}(a\theta) + \epsilon)^{-\alpha}, 0 < \epsilon < 1.
\]

We first show that \( f_\epsilon(\theta) \) satisfies the conditions of the second half of Theorem 4.

First, we have \( f_\epsilon(\theta) \leq 2d^2 \epsilon^{-\alpha} = \exp(o(\sum_{j=1}^{d} \theta_j^2)) \),

\[
\left| \frac{\partial f_\epsilon}{\partial \theta_i}(\theta) \right| \leq \epsilon^{-\alpha-1} 2d^2 + (d - k)^2 \sum_{j=1}^{d} \theta_j^2 + \epsilon^{-\alpha-1} 2d^2 + (d - k) \sum_{j=1}^{d} \theta_j^2.
\]

Second, we know that \( h(\theta) = (s_{\mathcal{V}}(\theta) + \epsilon)^{-\alpha} \) is a superharmonic function because

\[
\frac{\partial h}{\partial \theta_i}(\theta) = \alpha \sum_{j=1}^{d-k} (\theta, v_j)^2 + \epsilon^{-\alpha-2} \left\{ 4(\alpha + 1) - 2(d - k) \sum_{j=1}^{d-k} (\theta, v_j)^2 - 2(d - k) \epsilon \right\} < 0.
\]

Therefore, from the second half of Theorem 4, we know that \( \text{(3.2)} \) holds when \( f(\theta) = \sum_{a \in \{1, \ldots, r\}} (s_{\mathcal{V}}(a\theta) + \epsilon)^{-\alpha} \), \( 0 < \alpha \leq (d - k - 2)/2 \). Next, we prove that \( \text{(3.2)} \) still holds when \( f(\theta) = \sum_{a \in \{1, \ldots, r\}} (s_{\mathcal{V}}(a\theta))^\alpha \). The key is to consider \( \epsilon \to 0 \).

From \( \text{(A.77)} \), we have

\[
\int (s_{\mathcal{V}}(a\theta))^{-\alpha} \prod_{i=1}^{d} \frac{r_{z_i+1/2} \lambda_i^{z_i-1/2} \exp(-r_i \lambda_i)}{\Gamma(z_i + 1/2)} \, d\lambda \leq \int \left(\sum_{i=1}^{d-k} (\theta, av_i)^2\right)^{-\alpha} C \left(\sum_{j=1}^{d-k} \theta_j^2\right)^{-\alpha} \frac{\exp(-\epsilon \theta_j)}{\sqrt{2\pi \gamma} \zeta^{-1}} \, d\theta
\]

\[
= CE \left[ \left(\sum_{i=1}^{d-k} (\theta, av_i)^2\right)^{-\alpha} | \theta \sim N_d(\sqrt{\frac{z}{r \gamma}}, \zeta^{-1} I_d) \right]
\]

\[
= CE[||\mu||^{-2\alpha} | \mu \sim N_{d-k}((av_1, \ldots, av_{d-k})^\top \sqrt{\frac{z}{r \gamma}}, \zeta^{-1} I_{d-k})] < \infty.
\]

Therefore, using the dominated convergence theorem, for every \( a \),

\[
\int (s_{\mathcal{V}}(a\theta) + \epsilon)^{-\alpha} \prod_{i=1}^{d} \frac{r_{z_i+1/2} \lambda_i^{z_i-1/2} \exp(-r_i \lambda_i)}{\Gamma(z_i + 1/2)} \, d\lambda \to \int (s_{\mathcal{V}}(a\theta))^{-\alpha} \prod_{i=1}^{d} \frac{r_{z_i+1/2} \lambda_i^{z_i-1/2} \exp(-r_i \lambda_i)}{\Gamma(z_i + 1/2)} \, d\lambda
\]

when \( \epsilon \to 0 \). From the definition of \( F \) in \( \text{(3.2)} \), \( \text{(3.2)} \) still holds when \( f(\theta) = \sum_{a \in \{1, \ldots, r\}} (s_{\mathcal{V}}(a\theta))^\alpha \).
Lemma 13. where \( \lambda > 0 \) and \( v \in \mathbb{R}^d \) such that \( \min_{a \in \{1, \ldots, d\}} |\langle \frac{z}{r_{\gamma}}, a v_1 \rangle| > c \) for any constant \( c \).

From the first half of Theorem 4, we know that the Bayesian predictive distribution based on \( \pi_{f,3}(\lambda) \) dominates that based on the Jeffreys prior.

A.8 Proof of Proposition 4

From Step 1 of the proof of Proposition 3 and additivity of \( F \), we know that the combination

\[
\begin{align*}
f(\theta) &= \sum_{a \in \{1, \ldots, d\}} (sv_1(a\theta))^{-\alpha_1} + \sum_{a \in \{1, \ldots, d\}} (sv_2(a\theta))^{-\alpha_2} + \cdots + \sum_{a \in \{1, \ldots, d\}} (sv_n(a\theta))^{-\alpha_n} \\
&= \sum_{a \in \{1, \ldots, d\}} (sv_1(a\theta))^{-\alpha_1} + \sum_{a \in \{1, \ldots, d\}} (sv_2(a\theta))^{-\alpha_2} + \cdots + \sum_{a \in \{1, \ldots, d\}} (sv_n(a\theta))^{-\alpha_n}
\end{align*}
\]

satisfies the inequality condition (3.2). From Step 2 of the proof of Proposition 3, \( F(z, r) \rightarrow 0 \) when

\[
\begin{align*}
\min_{a \in \{1, \ldots, d\}} |\langle \frac{z}{r_{\gamma}}, a v_1 \rangle| \rightarrow \infty,
\end{align*}
\]

where \( v_1 \) is the first vector of a standard orthonormal basis of the complementary space of \( V_i \). Thus, \( F(z, r) \) is not a constant function of \( z \). In summary, \( f(\theta) \) satisfies the conditions in the first half of Theorem 4. Therefore, the Bayesian predictive distribution \( p_{f,3}(y \mid x) \) dominates that based on the Jeffreys prior.

A.9 Proof of Theorem 5

We use the following lemma for the proof of the theorem. The proof of the lemma is presented at the end of the proof. Let \( \text{Po}(\lambda) \) denote the Poisson distribution with parameter \( \lambda \).

**Lemma 13.** For any \( \lambda > 0 \), \( f(\lambda) = \lambda E\left( \log \left( \frac{x + 0.5}{\lambda} \right) \right) \mid x \sim \text{Po}(\lambda) \right) > -0.02 \). Furthermore, \( \lim_{\lambda \to \infty} f(\lambda) = 0 \).

**Proof of part 1.** Similar to (A.1) in the case of same durations, we have

\[
\begin{align*}
p_j(y \mid x) &= \prod_{i=1}^{d} \left( \frac{r_i}{r_i + s_i} \right)^{x_i + 1/2} \left( \frac{s_i}{r_i + s_i} \right)^{y_i} \frac{\Gamma(x_i + y_i + 1/2)}{\Gamma(x_i + 1/2) y_i!}.
\end{align*}
\]

Therefore, the K-L risk \( E(D(p(y \mid \lambda), p_j(y \mid x))) \) is given by

\[
\begin{align*}
&= \sum_{i=1}^{d} \left( -s_i \lambda_i + s_i \lambda_i \log(s_i \lambda_i) - (r_i \lambda_i + \frac{1}{2}) \log \left( \frac{r_i}{r_i + s_i} \right) - s_i \lambda_i \log \left( \frac{s_i}{r_i + s_i} \right) \\
&\quad - E\left( \log \Gamma(x_i + y_i + \frac{1}{2}) - \log \Gamma(x_i + \frac{1}{2}) \mid \lambda_i \right) \right)
\end{align*}
\]

Considering the functions

\[
F_i(t) := t \lambda_i \log \lambda_i + \frac{1}{2} \log t + \lambda_i (t \log t - t) - E\left( \log \Gamma(x_i + \frac{1}{2}) \mid x \sim \text{Po}(\lambda_i) \right),
\]

the K-L risk \( A.80 \) is equal to

\[
\sum_{i=1}^{d} \left( F_i(r_i + s_i) - F_i(r_i) \right) = \sum_{i=1}^{d} \int_{r_i}^{r_i + s_i} F_i'(t)dt,
\]

where

\[
F_i'(t) = \frac{1}{2t} - E\left( \lambda_i \log \left( \frac{x + 0.5}{\lambda_i t} \right) \mid x \sim \text{Po}(\lambda_i) \right).
\]

From Lemma 13, \( F_i'(t) < 0.52/t \). Thus, \( A.81 \) is equal to \( \sum_{i=1}^{d} \left( \int_{r_i}^{r_i + s_i} F_i'(t)dt \right) < 0.52 \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right) \).
Proof of part 1. We only need to show that \(0.5 \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right)\) is the Bayes risk limit of a sequence of Bayes rules \(p_{\pi_n}\) with

\[
\pi_n(\lambda) = \prod_{i=1}^{d} \lambda_i^{-1/2} \exp \left( -\frac{\lambda_i}{n} \right) \frac{1}{n^{1/2} \Gamma(1/2)}.
\]

Similar to (A.3) in the case of same durations, we have

\[
p_{\pi_n}(y \mid x) = \prod_{i=1}^{d} \left( \frac{r_i + 1/n}{r_i + s_i + 1/n} \right)^{x_i + 1/2} \left( \frac{s_i}{r_i + s_i + 1/n} \right)^{y_i} \frac{\Gamma(x_i + y_i + 1/2)}{\Gamma(x_i + 1/2)y_i!}
\]

The Bayes risk of \(p_{\pi_n}\) is

\[
E\left( E \left( \sum_{y} p(y \mid \lambda) \log \frac{p(y \mid \lambda)}{p_{\pi_n}(y \mid x)} \right) \mid \lambda_i \sim \Gamma \left( \frac{1}{2}, \frac{1}{n} \right) \right)
\]

We use (A.81) in the proof of part 1 to show that the left term in (A.82) is equal to

\[
E \left( \sum_{i=1}^{d} \int_{r_i}^{r_i+s_i} \left( \frac{1}{2t} - E(\lambda_i \log \frac{x_i + 0.5}{t\lambda_i}) \right) dt \mid \lambda_i \sim \Gamma \left( \frac{1}{2}, \frac{1}{n} \right) \right) = 0.5 \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right) = 0.5 \sum_{i=1}^{d} \int_{r_i}^{r_i+s_i} \frac{1}{2t} E(t \lambda_i \log \frac{x_i + 0.5}{t \lambda_i}) dt \mid \lambda_i \sim \Gamma \left( \frac{1}{2}, \frac{1}{n} \right) \right) dt.
\]

According to \(\lim_{\lambda \to \infty} f(\lambda) = 0\) from Lemma 13, (A.83) converges to \(0.5 \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right)\) when \(n \to \infty\).

Based on

\[
p_{\pi_n}(y \mid x) = \prod_{i=1}^{d} \left( \frac{r_i}{r_i + 1/n} \right)^{x_i + 1/2} \left( \frac{r_i + s_i + 1/n}{r_i + s_i} \right)^{x_i + y_i + 1/2},
\]

when \(n \to \infty\), the right term in (A.82) is equal to

\[
\sum_{i=1}^{d} E \left( E \left( (x_i + 1/2) \log \left( \frac{r_i}{r_i + 1/n} \right) + (x_i + y_i + 1/2) \log \left( \frac{r_i + s_i + 1/n}{r_i + s_i} \right) \right) \mid \lambda_i \sim \Gamma \left( \frac{1}{2}, \frac{1}{n} \right) \right) = \sum_{i=1}^{d} E \left( r_i \lambda_i + 1/2 \log \left( \frac{r_i}{r_i + 1/n} \right) + (r_i + s_i) \lambda_i + 1/2 \log \left( \frac{r_i + s_i + 1/n}{r_i + s_i} \right) \right) \mid \lambda_i \sim \Gamma \left( \frac{1}{2}, \frac{1}{n} \right) \right) \to 0.
\]

Therefore, (A.82) converges to \(0.5 \sum_{i=1}^{d} \log \left( \frac{r_i + s_i}{r_i} \right)\) when \(n \to \infty\), which completes the proof.

Proof of Lemma 13

Proof of part 1. First, we prove that \(f(\lambda) = \lambda E(\log((x + 0.5)/\lambda) \mid x \sim \text{Po}(\lambda)) > -0.02, \forall \lambda > 0\) in two cases: \(\lambda \leq 1\) and \(\lambda > 1\). We present the outline of the proof’s flow as follows:

When \(\lambda \leq 1\), we prove \(f(\lambda) > 0\) using \(E(\log((x + 0.5)/\lambda) \mid x \sim \text{Po}(\lambda)) > \log(0.5/\lambda)P(x = 0) + \log(1.5/\lambda)P(x \geq 1)\).

When \(\lambda > 1\), we define the derivative of \(f(\lambda)/\lambda\) as \(g(\lambda)\). We derive a lower bound (A.87) and an upper bound (A.89) for \(g(\lambda)\). We used a computer to verify that \(f(3) > 0, f(4) > -0.0082, \text{ and } f(5) > -0.011\). Using these values and upper and lower bounds for \(g(\lambda)\), we can obtain \(f(\lambda) > -0.02\).

The details of each case are presented below.
Case 1: $\lambda \leq 1$.
When $\lambda \leq 1/2$, $(x + 0.5)/\lambda \geq 1$. Thus, $f(\lambda) \geq 0$.
When $1/2 < \lambda < 1$, $E(\log((x + 0.5)/\lambda) \mid x \sim \text{Po}(\lambda)) > \log(0.5/\lambda)P(x = 0) + \log(1.5/\lambda)P(x = 1) = \log(0.5/\lambda)e^{-\lambda} + \log(1.5/\lambda)(1-e^{-\lambda}) = \log(1.5/\lambda) - (\log 3)e^{-\lambda}$, which is positive because $(\log 1.5) - (\log 3)e^{-\lambda} = -1/\lambda + (\log 3)e^{-\lambda} < -1 + (\log 3)e^{-1/2} < 0$ and $(\log 1.5) - (\log 3)e^{-1} > 0$.

Case 2: $\lambda > 1$.
Let $g(\lambda)$ denote the derivative of $f(\lambda)/\lambda$.

$$g(\lambda) = \frac{\sum_{x=0}^{\infty} \log \frac{x + 0.5}{\lambda} e^{-\lambda \frac{x}{\lambda}}}{\lambda} = \sum_{x=0}^{\infty} \left( \log \frac{x + 0.5}{\lambda} e^{-\lambda \frac{x}{\lambda}} \right) = \sum_{x=0}^{\infty} \left( \log \frac{x + 1.5}{x + 0.5} e^{-\lambda \frac{x}{\lambda}} \right) - 1/\lambda = E \left( \log \frac{x + 1.5}{x + 0.5} \right) - 1/\lambda. \quad (A.84)$$

For any $t \geq 0$, note that the Taylor’s formula

$$\log \left( \frac{t + 1.5}{t + 0.5} \right) = \log \left( 1 + \frac{1}{2(t + 1)} \right) - \log \left( 1 - \frac{1}{2(t + 1)} \right) = \sum_{k=1}^{\infty} \frac{2}{2k - 1} \left( \frac{1}{2(t + 1)} \right)^{2k-1}. \quad (A.85)$$

Thus, $\log(t + 1.5) - \log(t + 0.5) > 1/(t + 1)$. Using (A.84), we obtain

$$g(\lambda) = E \left( \log \frac{x + 1.5}{x + 0.5} \right) - 1/\lambda = (\log 3)P(x = 0) + E \left( \log \frac{x + 1.5}{x + 0.5} \right)1(x \geq 1) - 1/\lambda > 1.09P(x = 0) + E \left( \frac{1}{x + 1} \right)1(x \geq 1) - 1/\lambda = 0.09P(x = 0) + E \left( \frac{1}{x + 1} \right) - 1/\lambda. \quad (A.86)$$

Because $E(\lambda/(x + 1)) = 1 - e^{-\lambda}$, from (A.86), we obtain a lower bound of $g(\lambda)$:

$$g(\lambda) > 0.09e^{-\lambda} - e^{-\lambda} - 1. \quad (A.87)$$

Using the Taylor’s formula (A.85), for any $t \geq 2$,

$$\log \left( \frac{t + 1.5}{t + 0.5} \right) < \frac{1}{t + 1} + \sum_{k=2}^{\infty} \left( \frac{2^{2k}}{2k - 1} \right)(t + 1)^{-3} = \frac{1}{t + 1} + \log 3 - \frac{1}{(t + 1)^{3}} \quad (A.88)$$

Because $\log(2.5/1.5) < 0.5 + 0.26/24$, combining (A.84) and (A.88), we obtain

$$g(\lambda) = E \left( \log \frac{x + 1.5}{x + 0.5} \right) - 1/\lambda = (\log 3)P(x = 0) + E \left( \log \frac{x + 1.5}{x + 0.5} \right)1(x \geq 1) - 1/\lambda < (\log 3 - 0.26/24)P(x = 0) + E \left( \frac{1}{x + 1} + \frac{0.26}{(x + 1)(x + 2)(x + 3)} \right) - 1/\lambda. \quad (A.89)$$

Because $E(\lambda/(x + 1)) = 1 - e^{-\lambda}$ and $E((x + 1)^{-1}(x + 2)^{-1}(x + 3)^{-1}) < \lambda^{-3}$, we get an upper bound of $g(\lambda)$:

$$g(\lambda) < 0.06e^{-\lambda} - e^{-\lambda} - 1 + 0.26\lambda^{-3}. \quad (A.89)$$

Using a computer, we can calculate the value of function

$$L(\lambda) = \sum_{x=0}^{20} \log (x + 0.5) \frac{\lambda^{x}}{x!} \exp(-\lambda) \lambda - \lambda \log \lambda$$

for $\lambda = 3, 4, 5$. We only calculate $x \leq 20$ to calculate only a finite number of terms. The code for the calculation and the analysis of potential numerical errors are available at [https://github.com/lxiaomn/EB-Poisson](https://github.com/lxiaomn/EB-Poisson). We obtained $f(3) > L(3) > 0$, $f(4) > L(4) > -0.0082$, and $f(5) > L(5) > -0.011$. Next, we use these inequalities and the upper and lower bounds of $g(\lambda)$ to prove that $f(\lambda) > -0.02$. We prove it in five cases as follows. The selection of 3, 4, 5, and 7 as the boundaries for different cases is because the inequality discussed in each case
holds in the corresponding interval, and the lower bounds of \( f(3), f(4), \) and \( f(5) \) are used.

1) Case of \( \lambda \geq 7 \). From (A.89), \( g(t) < 0.06e^{-t} + 0.26t^{-3} \). Because \( g(\lambda) = (f(\lambda)/\lambda)' \) and \( \lim_{\lambda \to \infty} f(\lambda) = 0 \) (the proof is presented in the second part of Appendix A), we have

\[
f(\lambda)/\lambda = -\int_0^\infty g(t)dt > -\int_0^\infty (0.06e^{-t} + 0.26t^{-3})dt = -0.06e^{-\lambda} - 0.13\lambda^{-2}.
\]

Thus, \( f(\lambda) > -0.06e^{-\lambda} - 0.13/\lambda \geq -0.06 \times e^{-7} \times 7 - 0.13/7 > -0.02 \).

2) Case of \( \lambda \in [5, 7] \). From (A.87), when \( t > 5 \), \( g(t) > 0.09e^{-t} - e^{-t-1} > -0.11e^{-t} \). Thus,

\[
f(\lambda)/\lambda = f(5)/5 + \int_5^\lambda g(t)dt > -0.011/5 - \int_5^\lambda 0.11e^{-t}dt > -0.00295 + 0.11e^{-\lambda}.
\]

Thus, \( f(\lambda) > -0.00295\lambda + 0.11e^{-\lambda} \geq -0.00295 \times 7 + 0.11e^{-7} \times 7 > -0.02 \).

3) Case of \( \lambda \in [4, 5] \). From (A.87), when \( t > 4 \), \( g(t) > 0.09e^{-t} - e^{-t-1} > -0.16e^{-t} \). Thus,

\[
f(\lambda)/\lambda = f(4)/4 + \int_4^\lambda g(t)dt > -0.0082/4 - \int_4^\lambda 0.16e^{-t}dt > -0.005 + 0.16e^{-\lambda}.
\]

Thus, \( f(\lambda) > -0.005\lambda + 0.16e^{-\lambda} \geq -0.005 \times 5 + 0.16e^{-5} \times 5 > -0.02 \).

4) Case of \( \lambda \in [3, 4] \). From (A.89), when \( t < 4 \), \( g(t) < 0.06e^{-t} - e^{-t-1} + 0.26t^{-3} < -0.19e^{-t} + 0.26t^{-3} \). When \( t \in (3, 4) \), using \( e^{-t^3} > e^{-4t^3} > 1 \), we have \( g(t) < -0.19t^{-3} + 0.26t^{-3} = 0.07t^{-3} \). Thus,

\[
f(\lambda)/\lambda = f(4)/4 - \int_4^\lambda g(t)dt > -0.0082/4 - \int_4^\lambda 0.07t^{-3}dt > -0.035\lambda^{-2}.
\]

Thus, \( f(\lambda) > -0.035\lambda^{-1} > -0.02 \).

5) Case of \( \lambda \in [1, 3] \). From (A.89), when \( t \in (1, 3) \), \( g(t) < 0.06e^{-t} - e^{-t-1} + 0.26t^{-3} < 0.2e^{-t-1} - e^{-t}t^{-1} + 0.26t^{-3} = -0.8e^{-t-1} + 0.26t^{-3} \). Because \( e^{-t^2} > \max(e^{-3} \times 3^2, e^{-1}) > 0.36 \) when \( t \in (1, 3) \), we obtain \( g(t) < -0.8 \times 0.36t^{-3} + 0.26t^{-3} < 0 \). Therefore, \( f(\lambda)/\lambda \) is decreasing in \([1, 3] \). Using \( f(3) > 0 \), we obtain \( f(\lambda) > 0 \) for any \( \lambda \in [1, 3] \).

**Proof of part 2.** Subsequently, we prove that \( \lim_{\lambda \to \infty} f(\lambda) = 0 \).

First, we prove \( \liminf_{\lambda \to \infty} f(\lambda) \geq 0 \). For any given \( \epsilon > 0 \), there exists \( \delta \in (0, 0.1) \) such that \( \log(1 + t) \geq t - (0.5 + \epsilon)t^2 \), \( \forall t \geq -2\delta \). Without loss of generality, we assume \( \lambda > 1/\delta \). Therefore, by setting \( t = (x+0.5-\lambda)/\lambda \), we obtain

\[
f(\lambda) = \lambda E\left( \log\left(\frac{x + 0.5}{\lambda}\right)1(x < (1 - \delta)\lambda)\right) + \lambda E\left( \log\left(\frac{x + 0.5}{\lambda}\right)1(x \geq (1 - \delta)\lambda)\right) \\
geq \lambda E\left( \log\left(\frac{x + 0.5}{\lambda}\right)1(x < (1 - \delta)\lambda)\right) + \lambda E\left( \left(\frac{x + 0.5 - \lambda}{\lambda} - (0.5 + \epsilon)\right)^2\left(\frac{x + 0.5 - \lambda}{\lambda}\right)^2\right)1(x \geq (1 - \delta)\lambda) \\
geq \lambda E\left( \log\left(\frac{x + 0.5}{\lambda}\right)1(x < (1 - \delta)\lambda)\right) + \lambda E\left( (x + 0.5 - \lambda)/\lambda - (0.5 + \epsilon)(x + 0.5 - \lambda)^2/\lambda^2 \right) \quad (\text{A.90})
\]

Using Chernoff bound for Poisson distribution, we obtain

\[
P(x < (1 - \delta)\lambda) \leq \frac{(e \lambda)\lambda^{1-\delta}e^{-\lambda}}{(1 - \delta)\lambda^{1-\delta}}. 
\]

When \( x < (1 - \delta)\lambda, |\log((x + 0.5)/\lambda)| \leq \log(2\lambda) \). Thus, the logarithm of the absolute value of the first term in the (A.90) is not greater than

\[
\log\left(\lambda \log(2\lambda)P(x < (1 - \delta)\lambda)\right) \leq \log\left(\lambda \log(2\lambda)\frac{(e \lambda)\lambda^{1-\delta}e^{-\lambda}}{(1 - \delta)\lambda^{1-\delta}}\right) \\
= \log(\lambda \log(2\lambda)) + (1 - \delta)\log \lambda + (1 - \delta)\lambda - (1 - \delta)\log((1 - \delta)\lambda) \\
= -(1 - \delta) \log((1 - \delta) - \delta)\lambda + o(\lambda) \to -\infty 
\]

when \( \lambda \to \infty \). Thus, the first term of (A.90) converges to 0 when \( \lambda \to \infty \). Because \( E((x + 0.5 - \lambda)^2) = \lambda + 0.25 \), the second term of (A.90) is \(-\epsilon - (0.5 + \epsilon)0.25/\lambda \). Thus, (A.90) \( \to -\epsilon \) when \( \lambda \to \infty \). Thus, \( \liminf_{\lambda \to \infty} f(\lambda) \geq -\epsilon \).
Because $\epsilon$ is an arbitrary positive value, $\liminf_{\lambda \to \infty} f(\lambda) \geq 0$.

Next, we prove $\limsup_{\lambda \to \infty} f(\lambda) \leq 0$. Note that $\log(1+t) \leq t - t^2/2 + t^3/3$, $\forall t$. Thus, using $\mathbb{E}((x-\lambda)^3) = \lambda$,

$$f(\lambda) = \lambda \mathbb{E}\left( \log \left( \frac{x + 0.5}{\lambda} \right) \mid x \sim \text{Po}(\lambda) \right) \leq \lambda \mathbb{E}\left( \frac{x + 0.5 - \lambda}{\lambda} - \frac{(x + 0.5 - \lambda)^2}{2\lambda^2} + \frac{(x + 0.5 - \lambda)^3}{3\lambda^3} \mid x \sim \text{Po}(\lambda) \right) = 0.5 - \frac{(\lambda + 0.5^2)/(2\lambda) + (\lambda + 1.5\lambda + 0.5^3)/(3\lambda^2)}{\lambda}.$$  \hspace{4cm} (A.91)

When $\lambda \to \infty$, \(\text{(A.91)}\) $\to 0$. Thus, $\limsup_{\lambda \to \infty} f(\lambda) \leq 0$.

\section*{References}

Aitchison, J. 1975. Goodness of prediction fit. \textit{Biometrika} 62, 3, 547–554.

Akaike, H. 1978. A new look at the bayes procedure. \textit{Biometrika} 65, 1, 53–59.

Brown, L. D. 1971. Admissible estimators, recurrent diffusions, and insoluble boundary value problems. \textit{The Annals of Mathematical Statistics} 42, 3, 855–903.

Clarke, B. S. and Barron, A. R. 1994. Jeffreys’ prior is asymptotically least favorable under entropy risk. \textit{Journal of Statistical planning and Inference} 41, 1, 37–60.

Clevenson, M. L. and Zidek, J. V. 1975. Simultaneous estimation of the means of independent poisson laws. \textit{Journal of the American Statistical Association} 70, 351a, 698–705.

George, E. I. 1986. Minimax multiple shrinkage estimation. \textit{The Annals of Statistics} 14, 1, 188–205.

George, E. I., Liang, F., and Xu, X. 2006. Improved minimax predictive densities under kullback-leibler loss. \textit{The Annals of Statistics}, 78–91.

George, E. I., Liang, F., and Xu, X. 2012. From Minimax Shrinkage Estimation to Minimax Shrinkage Prediction. \textit{Statistical Science} 27, 1, 82 – 94.

Ghosh, M. and Yang, M.-C. 1988. Simultaneous estimation of poisson means under entropy loss. \textit{The Annals of Statistics} 16, 1, 278–291.

Hamura, Y. and Kubokawa, T. 2020. Bayesian predictive distribution for a poisson model with a parametric restriction. \textit{Communications in Statistics-Theory and Methods} 49, 13, 3257–3266.

Johnstone, I. 1984. Admissibility, difference equations and recurrence in estimating a poisson mean. \textit{The annals of Statistics}, 1173–1198.

Komaki, F. 2001. A shrinkage predictive distribution for multivariate normal observables. \textit{Biometrika} 88, 3, 859–864.

Komaki, F. 2004. Simultaneous prediction of independent poisson observables. \textit{the Annals of Statistics} 32, 4, 1744–1769.

Komaki, F. 2006. A class of proper priors for bayesian simultaneous prediction of independent poisson observables. \textit{Journal of multivariate analysis} 97, 8, 1815–1828.

Komaki, F. 2015. Simultaneous prediction for independent poisson processes with different durations. \textit{Journal of Multivariate Analysis} 141, 35–48.

Li, X. 2023. Nearly minimax empirical bayesian prediction of independent poisson observables. \textit{arXiv preprint arXiv:2310.02004}.

Stein, C. M. 1974. Estimation of the mean of a multivariate normal distribution. In \textit{Proc. Prague Symposium on Asymptotic Statistics (J. Hájek, ed.)}. Vol. 2. Univ. Karlova, Prague, 345–381.
Strawderman, W. E. 1971. Proper bayes minimax estimators of the multivariate normal mean. *The Annals of Mathematical Statistics* 42, 1, 385–388.

Tokyo-Metropolitan-Police-Department. 2023. The number of crimes in tokyo prefecture by town and type. [https://www.keishicho.metro.tokyo.lg.jp/about_mpd/jokyo_tokei/jokyo/ninchikensu.html](https://www.keishicho.metro.tokyo.lg.jp/about_mpd/jokyo_tokei/jokyo/ninchikensu.html).

Tsui, K.-W. and Press, S. J. 1982. Simultaneous estimation of several poisson parameters under k-normalized squared error loss. *The Annals of Statistics*, 93–100.

Yano, K., Kaneko, R., and Komaki, F. 2021. Minimax predictive density for sparse count data. *Bernoulli* 27, 2, 1212–1238.