The tree of good semigroups in \( \mathbb{N}^2 \) and a generalization of the Wilf conjecture

N. Maugeri\(^*\), G. Zito\(^†\)

Abstract

In this work, we study good semigroups of \( \mathbb{N}^n \) introducing the definition of length and genus for these objects. We show how to count the local good semigroup with a fixed genus. Furthermore we study the relationships of these concepts with other ones previously defined in the case of good semigroups with two branches.

Keywords: good semigroups, genus of a good semigroup, type of a good semigroup, Wilf conjecture.

Mathematics Subject Classification 2010: 13A18, 14H99, 13H99, 20M25.

Introduction

The study of good semigroups was formerly motivated by the fact that they are the value semigroups of one-dimensional analytically unramified rings (such as the local rings of an algebraic curve). The definition appeared the first time in [1] and these objects were widely studied in several works [2, 5, 10, 11, 18]. In [1], the authors proved that the class of good semigroups is actually larger than the one of value semigroups. Thus, such semigroups can be seen as a natural generalization of numerical semigroups and studied using a more combinatorial approach without necessarily referring to the ring theory context. In recent works [9], [20], [8], some of notable elements and properties of numerical semigroups has been generalized to the case of good semigroups. The main purpose of this work is to generalize the definitions of length and genus of an ideal of a numerical semigroup to the case of good ideals of a good semigroup by identifying the relationships between them and those already defined in the previous works in

\(^*\)e-mail: nicola.maugeri.1992@gmail.com

Part of this work was done while the first author visited the Universities of Almeria and Granada supported by the project MTM2014-55367-P, which is funded by Ministerio de Economía y Competitividad and Fondo Europeo de Desarrollo Regional FEDER, and by the Junta de Andalucía Grant Number FQM-343

\(^†\)e-mail: giuseppezito@hotmail.it

Both the authors was funded by the project "Proprietà algebriche locali e globali di anelli associati a curve e ipersuperfici" PTR 2016-18 - Dipartimento di Matematica e Informatica - Università di Catania". 
the case of subsemigroups of \( \mathbb{N}^2 \).

If \( R \) is an analytically unramified ring, the value semigroup \( v(R) = \{ v(r) \mid r \text{ is not a zero-divisor of } R \} \) is a good semigroup [7]. If \( I \) is a relative good ideal of \( R \), the extension \( I \subseteq \bar{I} \) is of finite type and the conductor ideal is \( C(I) = I : \bar{I} \), where both the closure and the colon operation are considered in the ring of total fractions. Fixed \( \alpha \in \mathbb{Z}^d \), we denote by \( I(\alpha) = \{ r \in R \mid v(r) \geq \alpha \} \). If \( c(v(I)) \) is the conductor of the ideal \( v(I) \) of \( v(R) \), we have \( v(I : \bar{I}) = v(I(c(I))) \). In the one-branch case, given a relative ideal \( I \) of \( R \), we have that the length of the \( R \)-module \( l_R(I/C(I)) = n(v(I)) \), where \( n(v(I)) \) is the cardinality of the set of small elements of the numerical semigroup \( v(I) \). For this reason, given a relative ideal \( E \) of a numerical semigroup \( S \), it is natural to call length of \( E \) the number \( n(E) \). On the other hand, the genus of \( E \) is defined as the number of gaps in \( E \) and it is denoted by \( g(E) \). It is straightforward that \( g(E) + n(E) = c(E) \).

In Section 1 we recall the definition of good semigroup and we fix the basic notations. In [7] it is defined a function of distance \( d \) between relative good ideals in a good semigroup \( S \) and it is proved that if \( S \) is a semigroup of values of a ring \( R \), given a good relative ideal \( I \) of \( R \), we have \( l_R(I/C(I)) = d(v(I) \setminus v(C(I))) \). For this reason, taking in account the additivity of function \( d \), we can generalize in a natural way the definition of length and genus to the case of good ideals of \( \mathbb{N}^d \) (not necessarily in case of semigroup of values of a ring) as it was done in [26] for Arf semigroups. Given a relative good ideal \( E \subseteq S \), we define respectively length of \( E \) and genus of \( E \) as \( l(E) = d(E \setminus E(c(E))) \) and \( g(E) = d(N^d \setminus E(c(E))) \). We conclude the section giving a slightly different version of an explicit method to compute length and genus introduced in [11]. In [26], it was computed the number of good Arf semigroups with \( n \) branches having a fixed genus using the untwisted multiplicity trees defined in [25]. Our aim is to obtain a similar result for a general good semigroup. In [3], it is presented a method to compute all numerical semigroups up to a fixed genus building a tree where each new level is obtained removing minimal generators larger than the Frobenius number from the semigroups of the previous level. In Section 2, we repeat the same idea for good semigroup \( S \subseteq \mathbb{N}^2 \); in this case, the tracks of the good semigroup, defined in [20], will have the role that minimal generators played in case of numerical one. In order to do this, we prove that every good semigroup of genus \( g \) can be obtained removing a track from one of genus \( g - 1 \) (Theorem 2.4) and that by removing a track from a good semigroup of genus \( g \) we obtain a good semigroup of genus \( g + 1 \) (Theorem 2.5). Then, we explain how to build the tree of good semigroups, underlining the differences with the numerical case. We report the results regarding the computation of the number of local good semigroup with a fixed genus until genus 27, produced with an algorithm written in [17] using the package [12]. In Section 3 we study the relationship of the genus and the length with others notable elements of a good semigroup. In [15], it was proved the inequality \( c(S) \leq (t(S) + 1)l(S) \) for numerical semigroups. The type of a good semigroup was originally defined in [1] for the good semigroups such that \( S - M \) is a good relative ideal of \( S \) and it was recently generalized in [8]. We conclude the paper asking if this inequality holds for good semigroup with respect the generalized version of definition of type. We definitely prove that length, genus and type satisfy the relationships \( t(S) + l(S) - 1 \leq g(S) \leq t(S)l(S) \) also in
the case of good semigroups (Proposition 3.6 and Corollary 3.7). To conclude the paper we observe that the definitions and the algorithm given in the previous section give us the possibility of introducing an analogous of the Wilf conjecture for good semigroups. In this case, we found counterexamples for the conjecture (Example 3.8) but the problem seems to stay open for good semigroups which are value semigroups of a ring.

1 Length and Genus of a Good Ideal

We begin recalling the definition of good semigroup introduced in [1].

**Definition 1.1.** A submonoid $S$ of $(\mathbb{N}^d, +)$ is a good semigroup if it satisfies the following properties:

(G1) If $\alpha, \beta \in S$, then $\min(\alpha; \beta) = (\min\{\alpha_1, \beta_1\}, \ldots, \min\{\alpha_d, \beta_d\}) \in S$;

(G2) There exists $\delta \in \mathbb{N}^d$ such that $\delta + \mathbb{N}^d \subseteq S$;

(G3) If $(\alpha, \beta) \in S; \alpha \neq \beta$ and $\alpha_i = \beta_i$ for some $i \in \{1, \ldots, d\}$; then there exists $\epsilon \in S$ such that $\epsilon_i > \alpha_i = \beta_i$ and $\epsilon_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$ (and if $\alpha_j \neq \beta_j$, the equality holds).

Furthermore, we always suppose to work with a local good semigroup $S$, that is, if $\alpha = (\alpha_1, \ldots, \alpha_d) \in S$ and $\alpha_i = 0$ for some $i \in \{1, \ldots, d\}$, then $\alpha = 0$. As a consequence of property (G2), the element $c(S) = \min\{\delta | S \supseteq \delta + \mathbb{N}^d\}$ is well defined and it is called conductor of the good semigroup. The element $f(S) = c(S) - 1$, where $1 = (1, \ldots, 1)$, is said Frobenius vector of the good semigroup. Furthermore, we denote by $\text{Small}(S) = \{(\alpha, \beta) \in S | (\alpha, \beta) \leq c(S)\}$.

Notice that the good semigroup with small elements $\{(0, 0), (1, 1)\}$ is a local good semigroup containing all the other ones. We denote it by $\mathbb{N}^2(1, 1)$.

Following the notations reported in [7], we recall some definitions.

Let $S$ be a good semigroup. If $E \subseteq \mathbb{Z}^d$ is such that $E + S \subseteq E$ and $\alpha + E \subseteq S$ for some $\alpha \in S$, then $E$ is called a relative ideal of $S$. A relative ideal of $S$ need not satisfy the properties (G1) and (G3) of good semigroups. A relative ideal $E$ that does satisfy properties (G1) and (G2) will be called a good relative ideal. Given a good semigroup $S$ and an its relative good ideal $E$, two elements $\alpha, \beta$ in $E$ are said consecutive if there are no elements $\gamma \in E$ such that $\alpha < \gamma < \beta$.

An ordered sequence of $n + 1$ elements in $E$: 

$\alpha = \alpha^{(0)} < \alpha^{(1)} < \ldots < \alpha^{(n)}$

is said a chain of length $n$ in $E$; furthermore it is called saturated in $E$ if all its elements are consecutive in $E$.

In [7] it is proved that, if $\alpha, \beta \in E$ with $\alpha < \beta$, then all the saturated chains between $\alpha$ and $\beta$ have the same length. This common length is denoted by $d_E(\alpha, \beta)$. 

3
Given two relative good ideals $E \supseteq F$, denoting by $e_E, e_F$ the minimal elements of $E$ and $F$ respectively, we can define:

$$d(E \setminus F) = d_E(e_E, \alpha) - d_F(e_F, \alpha);$$

in [7] it is proved that, providing to take $\alpha$ sufficiently large, this definition is independent of the choice of $\alpha$.

Given $\alpha \in \mathbb{N}^d$, we denote by $E(\alpha) = \{ \beta \in E : \beta \geq \alpha \}$. The function $d(- \setminus -)$ satisfies the following properties:

**Proposition 1.2.** [7]

1) If $E \supseteq F \supseteq G$ are good relative ideals of $S$, then we have $d(E \setminus G) = d(E \setminus F) + d(F \setminus G)$.

2) If $E \supseteq F$ are good relative ideals of $S$, then $d(E \setminus F) = 0$ if and only if $E = F$.

3) Let us consider $E$, a good relative ideal of $S$ and $\alpha \in \mathbb{Z}^d$. If $\alpha^i = \alpha + e^i$, where $e^i_j = 0$ if $j \neq i$ and $e^i_i = 1$, then $d(E(\alpha) \setminus E(\alpha^i)) \leq 1$, where the equality holds if and only if \( \{ \beta \in E | \beta_i = \alpha_i \text{ and } \beta_j \geq \alpha_j, \text{ if } j \neq i \} \neq \emptyset \).

Given a good ideal $E$, there always exists a minimal element $c(E)$ such that if $\alpha \geq c(E)$, then $\alpha \in E$. The element $c(E)$ is called conductor of the ideal $E$ and the ideal $C(E) := E(c(E))$ is called conductor ideal of $E$. If $c(E) = (c_1, \ldots, c_d)$, we denote by $c_E := c_1 + \ldots + c_d$.

**Definition 1.3.** Given a good ideal $E$ of a good semigroup $S \subseteq \mathbb{N}^d$, we define the genus of $E$ as the number $g(E) = d(\mathbb{N}^d \setminus E)$ and the length of $E$ as the number $l(E) = d(E \setminus C(E))$. In particular $g(S) = d(\mathbb{N}^d \setminus S)$ and $l(S) = d(S \setminus C(S))$.

**Remark 1.4.** By Proposition 1.2.1, $d(\mathbb{N}^d \setminus C(E)) = d(\mathbb{N}^d \setminus E) + d(E \setminus C(E))$. Thus, we can write: $g(E) = c_E - l(E)$.

Now we want to introduce an useful formula for the computation of the genus. As in [6] and in [20], we will work on the equivalent structure of semiring $\Gamma_S$ in order to simplify the notation. We set $\overline{\mathbb{N}} = \mathbb{N} \cup \{ \infty \}$ and extend the natural order and the sum over $\mathbb{N}$ to $\overline{\mathbb{N}}$, setting respectively, $a < \infty$ for all $a \in \mathbb{N}$ and $x + \infty = \infty + x = \infty$.

If $S \subseteq \mathbb{N}^d$ is a good semigroup with $c(S) = (c_1, \ldots, c_d)$, taking $k \in \{1, \ldots, d\}$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$, we introduce the sets $S^{\infty(i_1, \ldots, i_k)} \subseteq \overline{\mathbb{N}}^d$. We say that the vector $(\alpha_1, \ldots, \alpha_d) \in S^{\infty(i_1, \ldots, i_k)}$ if and only if $\alpha_j = \infty$ for all $j \notin \{i_1, \ldots, i_k\}$ and there exists $\beta \in S$ such that $\beta_j = \alpha_j$ if $j \in \{i_1, \ldots, i_k\}$ and $\beta_j = c_j$ if $j \notin \{i_1, \ldots, i_k\}$.

We set

$$S^{\infty} := \bigcup_{k=1}^d \bigcup_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}} S^{\infty(i_1, \ldots, i_k)},$$
and we notice that $\infty = (\infty, \ldots, \infty) \in S_{\infty}^{\infty(0)} \subseteq S_{\infty}$.

We define $\Gamma_S := S \cup S_{\infty}$: given $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \Gamma_S$, we introduce the operations

$$\alpha \oplus \beta := \min\{\alpha, \beta\} = (\min\{\alpha_1, \beta_1\}, \ldots, \min\{\alpha_d, \beta_d\}),$$
$$\alpha \odot \beta := \alpha + \beta.$$

It is easy to prove that $(\Gamma_S, \oplus, \odot)$ is a semiring.

Given a good relative ideal $E$ of a good semigroup $S \subseteq \mathbb{N}^d$ we denote by

$$E_j = \{ e_j \mid (e_1, \ldots, e_d) \in E \},$$

the $j$th projection of $E$. We introduce the following numbers:

$$l_S(E_j) = |\{ \alpha \in E_j \mid \alpha_j < c_j \}|,$$
$$g_S(E_j) = |\{ \alpha \in \mathbb{N} \setminus E_j \mid \alpha_j < c_j \}|.$$

**Remark 1.5.** It is easy to notice that, given $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$, the subset of $S$ defined as

$$E^{(i_1, \ldots, i_k)} = \{ \alpha = (\alpha_1, \ldots, \alpha_d) \in S \mid \alpha_j \geq c_j \forall j \neq i_1, \ldots, i_k \},$$

is actually a good relative ideal of $S$ (it is not a good semigroup because the zero vector is missing due to the locality of $S$). Furthermore, considered $j \in \{i_1, \ldots, i_k\}$, we have that $j$th projection $S_{\infty(j)}^{(i_1, \ldots, i_k)}$ of $S_{\infty(i_1, \ldots, i_k)}$ is equal to $E_j^{(i_1, \ldots, i_k)}$. Hence, it makes sense to compute $l_S(S_{\infty(j)}^{(i_1, \ldots, i_k)})$ and $g_S(S_{\infty(j)}^{(i_1, \ldots, i_k)})$.

In pursuit of an idea reported in [11], with the following recursive formulas we can reduce to compute the length and the genus of a good semigroup by only considering small elements of numerical ones.

**Proposition 1.6.** Given a good semigroup $S \subseteq \mathbb{N}^d$, we have:

$$l(S) = \sum_{i=1}^d l_S(S_{i}^{\infty(i_1, \ldots, i_d)}),$$
$$g(S) = \sum_{i=1}^d g_S(S_{i}^{\infty(i_1, \ldots, i_d)}).$$

**Proof.** Let us consider the chain from 0 to $c(S) = (c_1, \ldots, c_d)$, where the elements are $(i_1, \ldots, i_j, \ldots, i_d)$ with $0 \leq i_j \leq c_j$, ordered with respect the lexicographical order on $\mathbb{N}^d$.

We consider the elements of the chain having the form $(i_1, 0, \ldots, 0)$ with $i_1 \in \{0, \ldots, c_1 - 1\}$, as a consequence of Proposition 1.2.3):

$$d(S((i_1, 0, \ldots, 0)) \setminus S((i_1 + 1, 0, \ldots, 0))) = 1 \iff \{ \beta \in S \mid \beta_1 = i_1, \beta_k \geq 0, k \neq 1 \} \neq \emptyset \iff$$

there exists an element of the form $(i_1, \alpha_2, \ldots, \alpha_d) \in S$. 5
Thus, by Proposition 1.2.1), we have:

\[ d(S((0, 0, \ldots, 0)) \setminus S((1, 0, \ldots, 0))) = l(S_1) = l_S(S_1^{\infty(1,\ldots,d)}) \]

Now, let us consider the elements of the chain of the form \((c_1, \ldots, c_j, i_j, \ldots, 0)\) with \(i_j \in \{1, \ldots, c_j\}\). We observe that \((c_1, \ldots, c_j, i_j, \ldots, 0) \in S\) if and only if \((\infty, \ldots, \infty, i_j, \ldots, 0) \in \Gamma_S\); so we have:

\[ d(S((c_1, \ldots, c_j, i_j, \ldots, 0)) \setminus S((c_1, \ldots, c_j-1, i_j + 1, \ldots, 0))) = 1 \iff \{\beta \in \Gamma_S | \beta_j = i_j, \beta_k \geq 0, k > j \text{ and } \beta_k = \infty, k < j \} \neq \emptyset \iff \text{there exists an element of the form } (\infty, \ldots, i_j, \alpha_{j+1}, \ldots, \alpha_d) \in \Gamma_S. \]

We observe \(d(S((c_1, \ldots, c_j-1, 0, \ldots, 0)) \setminus S((c_1, \ldots, c_j-1, 1, \ldots, 0))) = 0\), otherwise there would be an element of the form \((c_1, \ldots, c_j-1, 0, \alpha_{j+1}, \ldots, \alpha_d) \in S\), but this contradicts the locality of \(S\). Hence, by Proposition 1.2.1), we have:

\[ d(S((c_1, \ldots, c_{j-1}, 1, \ldots, 0)) \setminus S((c_1, \ldots, c_{j-1}, c_j, \ldots, 0))) = l_S(S_j^{\infty(j,\ldots,d)}), \]

Using again Proposition 1.2.1), by the definition of length we obtain:

\[ l(S) = \sum_{i=1}^{d} l_S(S_i^{\infty(i,\ldots,d)}). \]

The proof of the second formula is analogous to the first one. \(\square\)

**Example 1.7.** Let us consider the good semigroup \(S \subseteq \mathbb{N}^4\) having small elements:

\[
\text{Small}(S) = \{(2, 3, 2, 9), (2, 4, 4, 9), (2, 4, 6, 9), (2, 4, 8, 9), (2, 4, 9, 9), (2, 4, 10, 9),
(4, 3, 2, 10), (4, 6, 4, 18), (4, 7, 6, 18), (4, 7, 8, 18), (4, 7, 9, 18), (4, 7, 10, 18),
(4, 8, 6, 18), (4, 8, 8, 18), (4, 8, 9, 18), (4, 8, 10, 18), (6, 3, 2, 10), (6, 6, 4, 18),
(6, 7, 6, 18), (6, 7, 8, 18), (6, 7, 9, 18), (6, 7, 10, 18), (6, 9, 6, 18), (6, 10, 8, 18),
(6, 10, 9, 18), (6, 10, 10, 18), (7, 3, 2, 10), (7, 6, 4, 18), (7, 7, 6, 18), (7, 7, 8, 18),
(7, 7, 9, 18), (7, 9, 6, 18), (7, 10, 8, 18), (7, 10, 9, 18), (8, 3, 2, 10), (8, 6, 4, 18),
(8, 7, 6, 18), (8, 7, 8, 18), (8, 7, 10, 18), (8, 9, 6, 18), (8, 10, 8, 18), (8, 10, 10, 18)\}.
\]

The conductor is \(c(S) = (8, 10, 10, 18)\). We compute the sets \(S_i^{\infty(i,\ldots,d)}\).

\[
S_1^{\infty(1,2,3,4)} = S
\]

\[
S_1^{\infty(1,2,3,4)} = \{0, 2, 4, 6, 7, 8 = c_1, \rightarrow\} = S_1, l_S(S_1^{\infty(1,2,3,4)}) = 5, g_S(S_1^{\infty(1,2,3,4)}) = 3.
\]

\[
S_1^{\infty(2,3,4)} = \{(\infty, 3, 2, 10), (\infty, 6, 4, 18), (\infty, 7, 6, 18), (\infty, 7, 8, 18),
(\infty, 7, 10, 18), (\infty, 9, 6, 18), (\infty, 10, 8, 18), (\infty, 10, 10, 18)\}.
\]

\[\]
In case of good semigroups with two branches, we can write in a different way the formula for the computation of the genus. Now we introduce some notations regarding the good semigroups of \( \mathbb{N}^2 \) that will be useful also in the next section. Given a good semi group \( S \subseteq \mathbb{N}^d \) and an element \( \alpha \in \mathbb{N}^d \), following the notation in [1], we set:

\[
\Delta_i(\alpha) := \{ \beta \in \mathbb{Z}^d | \alpha_i = \beta_i \text{ and } \alpha_j < \beta_j \text{ for } j \neq i \}
\]

\[
\Delta(\alpha) := \bigcup_{i=1}^{d} \Delta_i(\alpha)
\]

\[
\Delta^S_i(\alpha) := S \cap \Delta_i(\alpha)
\]

\[
\Delta^S(\alpha) := S \cap \Delta(\alpha).
\]

Furthermore we define:

\[
i\Delta(\alpha) := \{ \beta \in \mathbb{Z}^d | \alpha_i = \beta_i \text{ and } \beta_j < \alpha_j \text{ for } j \neq i \}
\]

\[
i\Delta^S(\alpha) := S \cap_i \Delta(\alpha).
\]

Extending some of the previous definitions to infinite elements of \( \mathbb{N}^2 \), we set

\[
_1\Delta((\alpha_1, \infty)) := \{ \beta \in \mathbb{Z}^2 | \beta_1 = \alpha_1 \}
\]

\[
_2\Delta((\alpha_1, \infty)) := \emptyset
\]

\[
_1\Delta((\infty, \alpha_2)) := \emptyset
\]

\[
_2\Delta((\infty, \alpha_2)) := \{ \beta \in \mathbb{Z}^2 | \beta_2 = \alpha_2 \}
\]

\[
i\Delta^S(\alpha) := S \cap_i \Delta(\alpha).
\]

From this point onwards, with a little abuse of notation, we denote again by \( S \) the semiring associated to the good semigroup \( S \).
We will say that an element $\alpha \in S \setminus 0$ is irreducible if, from $\alpha = \beta + \gamma$, it follows $\alpha = \beta$ or $\alpha = \gamma$. An element that is not irreducible will be said reducible. We denote by $I(S)$ the set of irreducible elements of $S$. An element $\alpha \in S$ will be said absolute in $S$ if $\alpha \in S \setminus S^\infty$ and $\Delta_S(\alpha) = \emptyset$ (finite absolute), or if $\alpha \in S^\infty$ (infinite absolute).

We denote by $A_f(S)$ the set of finite absolutes in $S$ (these are also called maximal elements for example in [1],[11]). Furthermore, we denote by $A_\infty(S)$ the set of infinite absolutes in $S$ and with $A(S)$ the set of all irreducible absolutes in $S$. It is easy to see that the set of irreducible absolutes is finite and in [20] it is proved that $S$ is generated by these elements as a semiring.

Corollary 1.8. If $S \subseteq N^2$ is a good semigroup, then $g(S) = g(S_1) + g(S_2) + |A_f(S)|$.

Proof. In this case, by the last proposition, follows $g(S) = g(S_1) + g(S_2) + |A_f(S_2^{\infty}(2))|$. We notice that:

$$\mathbb{N} \setminus S^{\infty}(2) = (\mathbb{N} \setminus S^2) \cup \{ y \mid \max\{x \mid \exists (x, y) \in S\} < c_1 \}$$

and observing that the second set has the same cardinality of $A_f(S)$, we obtain the thesis.

\[ \square \]

2 The tree of good semigroups of $\mathbb{N}^2$ by genus

In [3], it is presented a method to compute all numerical semigroups until a fixed genus building a tree where each new level is obtained removing minimal generators larger than the Frobenius number from the semigroups of the previous level.

In this section we want to repeat the same process with the good semigroups, building a tree of local good semigroups of $\mathbb{N}^2$, where the $g$th level of the tree collects all local good semigroups with genus $g$. In order to follow this idea, we will show that, in this case, the analogous of minimal generators are the tracks of the good semigroup originally defined in [20].

Here we will recall the definition:

Definition 2.1. Given $\alpha, \beta \in I_A(S)$ we say that $\alpha$ and $\beta$ are connected by a piece of track if they are not comparable, i.e. $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, and denoted by $\gamma = \alpha \oplus \beta$, we have $\Delta^S(\gamma) \cap (S \setminus I(S)) = \emptyset$.

Definition 2.2. Given $\alpha_1, \ldots, \alpha_n \in I_A(S)$, with $\alpha_{11} < \ldots < \alpha_{n1}$ we say that $\alpha_1, \ldots, \alpha_n$ are connected by a track if we have:

- $2\Delta^S(\alpha_1) \cap (S \setminus I(S)) = \emptyset$;
- $1\Delta^S(\alpha_n) \cap (S \setminus I(S)) = \emptyset$;
- $\alpha_i$ and $\alpha_{i+1}$ are connected by a piece of track for all $i \in \{1, \ldots, n-1\}$. 

8
In this case, denoted by $\gamma_i = \alpha_i \oplus \alpha_{i+1}$ for $i \in \{1, \ldots, n-1\}$, we set:

$$T((\alpha_1, \ldots, \alpha_n)) = \{\alpha_1\} \cup 2^S(\alpha_1) \cup (\bigcup_{i=1}^{n-1} \Delta^S(\gamma_i)) \cup \Delta^S(\alpha_n) \cup \{\alpha_n\},$$

and we call this set the track connecting $\alpha_1, \ldots, \alpha_n$.

We will simply say that $T \subseteq S$ is a track in $S$ if there exist $\alpha_1, \ldots, \alpha_n \in I_A(S)$ such that $T$ is the track connecting $\alpha_1, \ldots, \alpha_n$. Notice that the previous definition implies that a track $T$ of $S$ never contains elements $\alpha$ such that $\alpha \geq c(S) + e(S)$.

The following statement holds:

**Lemma 2.3.** Given a good semigroup $S$, and a track $T = T((\alpha_1, \ldots, \alpha_n))$ in $S$, then, $S' = S \setminus T$ is a good semigroup strictly contained in $S$.

**Proof.** See [20].

Given $A, B \subseteq \mathbb{N}^d$ we set:

$$\min(A, B) = \{\min(\alpha, \beta) \mid \alpha \in A, \beta \in B\}$$

In the next two theorems we will establish a relationship between the tracks and the genus of a good semigroup.

**Theorem 2.4.** Each local good semigroup $S \neq \mathbb{N}^2(1, 1)$ with genus $g(S)$ can be obtained removing a track from a good semigroup with genus $g(S) - 1$.

**Proof.** Let us consider a semigroup $S$ with genus $g(S)$; we have $c(S) = (c_1, c_2)$ and $f(S) = (c_1 - 1, c_2 - 1) = (f_1, f_2)$. We will distinguish two cases.

**Case 1:** $f(S) \notin S$. In this case we introduce the following sets:

- $X = \{x \mid (x, f_2) \in S\}$ if $f_2 \neq 0$,
- $Y = \{y \mid (f_1, y) \in S\}$ if $f_1 \neq 0$.

Notice that $S \neq \mathbb{N}^2(1, 1)$ implies that at least one between $X$ and $Y$ can be always considered.

If $X$ and $Y$ are defined and not empty we consider respectively: $\tilde{x} = \max\{x \mid (x, f_2) \in S\}$ and $\tilde{y} = \max\{y \mid (f_1, y) \in S\}$. If $X = \emptyset$, we denote by

$$T_X = \Delta_2(f(S)) \cup [(\mathbb{N}^2 \setminus S) \cap \min (S, \Delta_2(f(S)))]$$

otherwise we denote by

$$T_X = \Delta_2(f(S)) \cup \Delta_1((\tilde{x}, f_2)) \cup [(\mathbb{N}^2 \setminus S) \cap \min (S, \Delta_2(f(S)))]$$

If $Y = \emptyset$, we denote by

$$T_Y = \Delta_1(f(S)) \cup [(\mathbb{N}^2 \setminus S) \cap \min (S, \Delta_1(f(S)))]$$

and if $X = \emptyset$, we denote by

$$T_X = \Delta_1(f(S)) \cup [(\mathbb{N}^2 \setminus S) \cap \min (S, \Delta_1(f(S)))]$$

where $\Delta_1(S)$ and $\Delta_2(S)$ are the sets of the first and the second coordinate of $S$, respectively.
otherwise we denote by
\[ T_Y = \Delta_1(f(S)) \cup \Delta_2(f_1, \tilde{y}) \cup [(\mathbb{N}^2 \setminus S) \cap (S, \Delta_1(f(S))]]. \]

We consider the following sets:
\[ S^1 = S \cup T_Y, \]
\[ S^2 = S \cup T_X. \]

In order to prove the thesis we can reduce to consider only the good semigroup \( S^2 \).

According to definition of tracks, it is easy to observe that \( T_X \) is a track of \( S^2 \) having the form \( T_X = T((\tilde{x}, \infty), (\infty, f_2)) \) if \( X \neq \emptyset \) and \( T_X = T((\infty, f_2)) \) otherwise.

If \( X = \emptyset \), removing the track \( T_X \), we have that \( A_f(S^2) = A_f(S) \), \( g(S^2_2) = g(S_2) \) and \( g(S^2_1) - 1 = g(S_1) \). As a consequence of Corollary 1.8 we have that \( S \) is obtained removing a track by the good semigroup \( S_2 \) having genus \( g(S) - 1 \).

If \( X \neq \emptyset \), removing the track \( T_X \), we have that \( |A_f(S^2)| = |A_f(S)| + 1 \), \( g(S^2_2) = g(S_2) \) and \( g(S^2_1) = g(S_1) \). Hence, using again Corollary 1.8 we have that \( S \) is obtained removing a track by the good semigroup \( S^2 \) having genus \( g(S) - 1 \).

Figure 1: Representation of \( T_X, X = \emptyset \)  
Figure 2: Representation of \( T_X, X \neq \emptyset \)

**Case 2: \( f(S) \in S \).** In this case we define \( T = \Delta(f(S)) \). If we consider
\[ S' = S \cup T; \]
also in this case it easy to observe that \( S' \) is a good semigroup and \( T \) is a track of \( S' \).

Removing the track \( T_X \) from \( S' \), we have that \( |A_f(S')| = |A_f(S)| + 1 \), \( g(S'_2) = g(S_2) \) and \( g(S'_1) = g(S_1) \). Hence, using again Corollary 1.8 we have that \( S \) is obtained removing a track
by the good semigroup $S'$ having genus $g(S) - 1$.

\[ \text{Figure 3: Representation of } T \]

**Theorem 2.5.** Given a good semigroup $S$ with genus $g(S)$, removing a track of $S$ we obtain a semigroup with genus $g(S) + 1$.

**Proof.** Let be $S$ a good semigroup, we suppose to consider a track $T := T((\alpha_1, \ldots, \alpha_n))$. With the same notation used in Lemma 2.3, we consider $S' = S \setminus T$. We want to prove that $g(S') = g(S) + 1$. We observe that by the Definition 2.1, $\alpha_i \in A_f(S)$, for any $i \in \{2, \ldots, n-1\}$. In order to fix the notation, we will suppose $\alpha_1 = (x, \infty)$ and we distinguish two cases.

**Case 1:** $\alpha_n \in A_f(S)$. In this situation, removing the track $T$, we remove the elements $\alpha_i$ with $i = 1, \ldots, n$, hence we lose $n - 1$ finite absolutes, in $S'$. At the same time the elements $\gamma_i = \min \{\alpha_i, \alpha_{i+1}\} \in S'$ with $i = 1, \ldots, n - 1$ become finite absolutes of $S'$; therefore $|A_f(S)| = |A_f(S')|$. Since $\alpha_{i,2} = \gamma_{i-2}$, for any $i \in \{2, \ldots, n\}$, we have also $g(S'_2) = g(S_2)$. Now we observe that $\Delta_i^{S'}(\alpha_n) = \emptyset$, since $S' \subset S$ and $\alpha_n$ was an absolute of $S$, furthermore $\Delta_i^{S'}(\alpha_n) = \emptyset$, from the definition of $S'$. Hence in $S'$ we lose an element in the first projection; we have $g(S'_1) = g(S_1) + 1$. From Corollary 1.8:

\[ g(S') = g(S'_1) + g(S'_2) + |A_f(S')| = g(S_1) + 1 + g(S_2) + |A_f(S)| = g(S) + 1. \]

**Case 2:** $\alpha_n = (\infty, y)$. In this situation, removing the track $T$, we remove the elements $\alpha_i$ with $i = 1, \ldots, n$, so in $S'$ we lose $n - 2$ finite absolutes in $S'$. At the same time, the elements $\gamma_i = \min \{\alpha_i, \alpha_{i+1}\}$ with $i = 1, \ldots, n - 1$, become finite absolutes in $S'$; hence $|A_f(S)| = |A_f(S')| - 1$. Since $\alpha_{i,2} = \gamma_{i-2}$, for any $i \in \{2, \ldots, n\}$ and $\alpha_{i,1} = \gamma_{i,1}$, for any $i \in \{1, \ldots, n-1\}$; we have respectively $g(S'_2) = g(S_2)$ and $g(S'_1) = g(S_1)$. Therefore, again from Corollary 1.8:

\[ g(S') = g(S'_1) + g(S'_2) + |A_f(S')| = g(S_1) + g(S_2) + |A_f(S)| + 1 = g(S) + 1. \]
Now we suppose \( \alpha_1 \in A_f(S) \); in this case, if \( \alpha_n = (\infty, y) \) the proof of the Theorem is analogous at the Case 2 that we have seen above, so we can suppose \( \alpha_n \in A_f(S) \). In this case removing the track \( T \), we remove the elements \( \alpha_i, i = 1, \ldots, n \), so we lose \( n \) finite absolutes in \( S' \). At the same time the elements of \( \gamma_i, i = 1, \ldots, n-1 \) become finite absolutes in \( S' \); hence \( |A_f(S)| = |A_f(S')| + 1 \). With the same argument that we used in Case 1, it is easy to observe that \( g(S'_1) = g(S_1) + 1 \) and \( g(S'_2) = g(S_2) + 1 \). So we have:

\[
g(S') = g(S'_1) + g(S'_2) + |A_f(S')| = g(S_1) + 1 + g(S_2) + 1 + |A_f(S)| - 1 = g(S) + 1.
\]

We have observed that tracks play the same role of minimal generators in the case of numerical semigroups. Now we want to show that, as in the numerical case, it is not necessary to consider all the minimal generators in order to build the tree. In fact, as we are going to show, it is sufficient to consider some special tracks.

**Definition 2.6.** Given a local good semigroup \( S \subseteq \mathbb{N}^2 \), we say that a track \( T((\alpha_1, \ldots, \alpha_n)) \) is a beyond track, if \( \{\alpha \in S, \alpha \geq c(S)\} \cap T((\alpha_1, \ldots, \alpha_n)) \neq \emptyset \).

Furthermore, we denote by

\[
BT(S) = \{T = T((\alpha_1, \ldots, \alpha_n)) | T \text{ is a beyond track of } S\}.
\]

If \( S \) is a good semigroup of \( \mathbb{N}^2 \) obtained from a good semigroup \( S' \) removing a track, we say that \( S' \) is a parent of \( S \) (or equivalently \( S \) is a son of \( S' \)). We say that \( S' \) is a special parent of \( S \) (or equivalently \( S \) is a special son of \( S' \)), if \( S \) is obtained from \( S' \) removing a beyond track.

**Lemma 2.7.** If \( S' \) is a special parent of \( S \), then \( c(S') < c(S) \).

**Proof.** Since \( S' \) is a special parent of \( S \), we have \( S = S' \setminus T \), where \( T \) is a beyond track of \( S' \). In particular we have \( S \subseteq S' \), implying that \( c(S') \leq c(S) \). Thus, we need to prove that \( c(S') \neq c(S) \). Let us assume by contradiction that \( c(S') = c(S) \). Since \( T \) is a beyond track of \( S' \), there exists an element \( \beta \in T \) with \( \beta \geq c(S') \). We recall that for a good semigroup \( S \) we have that \( \alpha \in S \iff \min(\alpha, c(S)) \in S \). Thus, since \( \beta \notin S \), we have

\[
c(S) = c(S') = \min(\beta, c(S')) = \min(\beta, c(S)) \notin S,
\]

that is a contradiction. \( \square \)

From Theorem 2.4 we can deduce the following corollary.

**Corollary 2.8.** Let \( S \) be a good semigroup \( S \neq \mathbb{N}^2(1,1) \), with \( f(S) = (f_1, f_2) \). Denote by \( p = |\{i \in \{1, 2\} | f_i \neq 0\}| \).

1) If \( f(S) \in S \), then \( S \) has exactly one special parent.
2) If $f(S) \notin S$, then $S$ has exactly $p$ special parents.

Proof. 1) If we define the set $T$ as in the proof of Theorem 2.4, the good semigroup $S'$ is trivially a special parent of $S$. Now we want to prove that, if there exists a good semigroup $S'$ and a beyond track $T'$ of $S'$, such that $S = S' \setminus T'$, then $T' = T$.

If $f(S) \in S$ then, by the previous lemma, there exists $\beta \in \Delta(f(S)) \cap T'$. By property (G3) of good semigroups, it follows $\Delta(f(S)) \subseteq T'$. In this case, since $(f_1, \infty)$ and $(\infty, f_2)$, are respectively a point of start and a point of end, we have $T' = \Delta(f(S))$.

2) If we define, when it is possible, the sets $T_X$ and $T_Y$ as in the proof of Theorem 2.4, the good semigroups $S^1$ and $S^2$ are trivially special parents of $S$. Now we want to prove that, if $S'$ is a good semigroup and there exists a beyond track $T'$ of $S'$ such that $S = S' \setminus T'$, then $T' = T_X$ or $T' = T_Y$.

By the previous lemma, there exists $\beta \in \Delta(f(S)) \cap T'$. Thus, by property (G3) of good semigroups, it means either $\Delta_1(f(S)) \subseteq T'$ or $\Delta_2(f(S)) \subseteq T'$. We suppose $\Delta_1(f(S)) \subseteq T'$, and we prove that $T' = T_Y$. We start observing that $(f_1, \infty)$ is always a point of start. If the set $Y$ is empty, then it is also a point of end, hence $T' = \Delta_1(f(S)) = T_Y$.

If $Y \neq \emptyset$, we observe that since $\Delta_1(f(S)) \subseteq T'$, then $T' = T((f_1, \infty), \infty, y))$ of $S'$, with $(\infty, y) \in I_A(S')$. Therefore by the maximality of $y$, it follows $y = \tilde{y}$; so we have $T' = T((f_1, \infty), \infty, \tilde{y}) = T_Y$. Assuming $\Delta_2^p(f(S)) \subseteq T'$, repeating the same proof, it is easy to observe that $T' = T_X$. □

Now, we denote by $\mathcal{G}_g$ the set of good semigroups with genus $g$. We build the following family of sets of good semigroup:

- $A_1 = \mathbb{N}^2(1, 1)$;
- $A_{i+1} = \{S \mid S, \text{ is a special son of } S' \in A_i\}$.

We want to show that all semigroups of genus $g + 1$ are special sons of semigroups of genus $g$, in other words:

**Proposition 2.9.** $A_g = \mathcal{G}_g$, for any $g \in \mathbb{N} \setminus \{0\}$.

Proof. We work by induction on $g$. We suppose $A_{g-1} = \mathcal{G}_{g-1}$ and trivially $A_g \subseteq \mathcal{G}_g$. If $S \in \mathcal{G}_g$, in Proposition 2.8 we have proved that there exists a special parent $S'$ that, by Theorem 2.5, belongs to $\mathcal{G}_{g-1} = A_{g-1}$; so $S \in A_g$. □

So we can build all the semigroups of genus $g$ by removing special tracks from semigroups of genus $g - 1$. 
Remark 2.10. We denote by \( n_g \) the number of local good semigroups of genus \( g \). The fact that a good semigroup of \( \mathcal{N}_g \) can have two distinct special parents in \( \mathcal{N}_{g-1} \) implies that the formula

\[
n_g = \sum_{S \in \mathcal{N}_{g-1}} |\{S' \mid S' \text{ is a special son of } S\}| = \sum_{S \in \mathcal{N}_{g-1}} |BT(S)|,
\]
does not hold in general.
From the computational point of view, it would be convenient to determine, for each good semigroup \( S \), a subset \( T(S) \subseteq BT(S) \) such that

\[
n_g = \sum_{S \in \mathcal{N}_{g-1}} |T(S)|,
\]
for each \( g \in \mathbb{N} \).
In order to do that we define, given a good semigroup \( S \) with conductor \( c(S) = (c_1, c_2) \),

\[
T(S) = \begin{cases} 
BT(S), & \text{if } c_2 = 1 \\
\{T(\langle \alpha_1, \ldots, \alpha_n \rangle) \in BT(S) \mid \alpha_n \geq (\infty, c_2)\}, & \text{if } c_2 \neq 1,
\end{cases}
\]
and we claim that \( T(S) \) satisfies the required property.
In order to do that it suffices to show that each good semigroup \( S \) has one and only one special parent \( S' \), such that \( S = S' \setminus T \) with \( T \in T(S') \). Thus, let us consider an arbitrary good semigroup \( S \), with conductor \( c(S) = (c_1, c_2) \).
Case 1: \( c_2 = 1 \). Corollary 2.8 tells us that \( S \) has only one special parent \( S' \), such that \( S = S' \setminus T \) with \( T \in BT(S') \). Furthermore, denoting by \( c(S') = (c_1', c_2') \), we still have \( c_2' = 1 \). Hence, by definition of \( T(S') \), we have \( BT(S') = T(S') \) and \( T \in T(S') \) as required.
Case 2: \( c_2 \neq 1 \) and \( f(S) = (f_1, f_2) \in S \). Corollary 2.8 tells us that \( S \) has only one special parent \( S' \), namely \( S' = S \cup \Delta(f(S)) \). Notice that \( \Delta(f(S)) \) is the track \( T = T((f_1, \infty), (\infty, f_2)) \in BT(S') \). Since we have \( c(S') \leq f(S) \), it follows that \( (\infty, f_2) \geq (\infty, c_2') \), implying that \( T \in T(S') \).
Case 3: \( c_2 \neq 1 \) and \( f(S) = (f_1, f_2) \notin S \). In this case we have exactly two special parents

\[
S^1 = S \cup T_Y, \\
S^2 = S \cup T_X,
\]
where

\[
T_Y = \begin{cases} 
T((f_1, \infty)) \in BT(S^1) & \text{if } Y = \emptyset \\
T((f_1, \infty), (\infty, \tilde{y})) \in BT(S^1) & \text{if } Y \neq \emptyset,
\end{cases}
\]
\[
T_X = \begin{cases} 
T((\infty, f_2)) \in BT(S^2) & \text{if } X = \emptyset \\
T((\tilde{x}, \infty), (\infty, f_2)) \in BT(S^2) & \text{if } X \neq \emptyset
\end{cases}
\]
Notice that \( T_Y \) never belongs to \( T(S^1) \), since \( c(S^1) = (c_1 - m, c_2) \) and \((\infty, \tilde{y}) < (\infty, c_2)\). On the other hand, \( T_X \) always belongs to \( T(S^2) \) because, in both the possible definitions, the end point is \((\infty, f_2) \geq (\infty, c(S^2)_2)\) (we recall that \( c(S^2) \) has the form \((c_1, c_2 - n)\)). Thus \( S^2 \) is the required unique special parent of \( S \).

Now we show with an example the construction of the tree up to genus \( g = 4 \).

**Example 2.11.** The starting point is the set

\[ \mathfrak{N}_1 = \{ \mathbb{N}^2(1,1) \} \]

The beyond tracks of \( \mathbb{N}^2(1,1) \) are: \( T_1 = T((\infty,1)), T_2 = T((1,\infty)), T_3 = T((1,\infty), (\infty, 1)) \).

If we denote by \( S_i \) the special son associated to \( T_i \), we have: \( S_1 = \{(0,0),(1,2)\}, S_2 = \{(0,0),(2,1)\}, S_3 = \{(0,0),(1,1),(2,2)\} \).

\[ \mathfrak{N}_2 = \{ S_1, S_2, S_3 \} \]

Let us consider \( S_1 = \{(0,0),(1,2)\}, \) we have the beyond tracks: \( T_{1,1} = T((1,\infty),(\infty,2)), T_{1,2} = T((1,\infty),(\infty,3)), T_{1,3} = T((1,\infty)), T_{1,4} = T((\infty,2)), T_{1,5} = T((\infty,3)) \). Notice that Remark 2.10 tells us that we can avoid to compute the son of \( S_1 \) with respect to the track \( T_{1,3} \notin T(S_1) \). We obtain:

\[ S_{1,1} = \{(0,0),(1,2),(2,3)\}, \quad S_{1,2} = \{(0,0),(1,2),(1,3),(2,2),(2,4)\}, \]

\[ S_{1,3} = \{(0,0),(2,2)\}, \quad S_{1,4} = \{(0,0),(1,3)\}, \quad S_{1,5} = \{(0,0),(1,2),(1,4)\}. \]

If we consider \( S_2 = \{(0,0),(2,1)\} \), we have the beyond tracks: \( T_{2,1} = T((2,\infty),(\infty,1)), T_{2,2} = T((3,\infty),(\infty,1)), T_{2,3} = T((\infty,1)), T_{2,4} = T((2,\infty)), T_{2,5} = T((3,\infty)) \). We obtain:

\[ S_{2,1} = \{(0,0),(2,1),(3,2)\}, \quad S_{2,2} = \{(0,0),(2,1),(2,2),(3,1),(4,2)\}, \]

\[ S_{2,3} = \{(0,0),(2,2)\} \quad S_{2,4} = \{(0,0),(3,1)\}, \quad S_{2,5} = \{(0,0),(2,1),(4,1)\}. \]

We notice that \( S_{1,3} = S_{2,3} \), but in this case \( T_{2,3} \notin T(S_2) \).

For what concerns the good semigroup \( S_3 = \{(0,0),(1,1),(2,2)\} \), we have only a beyond track: \( T_{3,1} = T((2,\infty),(\infty,2)) \). We obtain:

\[ S_{3,1} = \{(0,0),(1,1),(2,2),(3,3)\}. \]

\[ \mathfrak{N}_3 = \{ S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}, S_{1,5}, S_{2,1}, S_{2,2}, S_{2,3}, S_{2,4}, S_{2,5}, S_{3,1} \} \]

Now, we have:

- \( S_{3,1} \) has got only one beyond track, so only a special son

\[ S_{3,1,1} = \{(0,0),(1,1),(2,2),(3,3),(4,4)\}. \]
• $S_{2,1}$ has got two beyond tracks, thus two special sons

\[
S_{2,1,1} = \{(0, 0), (2, 1), (4, 2)\}, \quad S_{2,1,2} = \{(0, 0), (2, 1), (3, 2), (3, 3), (4, 2), (5, 3)\}.
\]

• $S_{2,2}$ has no beyond tracks.

• $S_{2,3}$ has got eight tracks. So it has got eight special sons:

\[
S_{2,3,1} = \{(0, 0), (2, 2), (3, 3)\}, \quad S_{2,3,2} = \{(0, 0), (2, 2), (2, 3), (3, 2), (3, 4)\},
\]

\[
S_{2,3,3} = \{(0, 0), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2), (4, 4)\},
\]

\[
S_{2,3,4} = \{(0, 0), (2, 2), (2, 3), (3, 2), (4, 3)\}, \quad S_{2,3,5} = \{(0, 0), (3, 2)\},
\]

\[
S_{2,3,6} = \{(0, 0), (2, 2), (4, 2)\}, \quad S_{2,3,7} = \{(0, 0), (2, 3)\}, \quad S_{2,3,8} = \{(0, 0), (2, 2), (2, 4)\}.
\]

• $S_{2,4}$ has got seven tracks. So it has got seven special sons:

\[
S_{2,4,1} = \{(0, 0), (3, 1), (4, 2)\}, \quad S_{2,4,2} = \{(0, 0), (4, 1)\},
\]

\[
S_{2,4,3} = \{(0, 0), (3, 1), (3, 2), (4, 1), (5, 2)\}, \quad S_{2,4,4} = \{(0, 0), (3, 1), (5, 1)\},
\]

\[
S_{2,4,5} = \{(0, 0), (3, 1), (3, 2), (4, 1), (5, 2)\}, \quad S_{2,4,6} = \{(0, 0), (3, 1), (4, 1), (6, 1)\},
\]

\[
S_{2,4,7} = \{(0, 0), (3, 2)\}.
\]

• $S_{2,5}$ has got four tracks. So it has got four special sons:

\[
S_{2,5,1} = \{(0, 0), (2, 1), (4, 2)\}, \quad S_{2,5,2} = \{(0, 0), (2, 1), (2, 2), (4, 1), (4, 2), (5, 1), (6, 2)\},
\]

\[
S_{2,5,3} = \{(0, 0), (2, 1), (4, 1), (6, 1)\}, \quad S_{2,5,4} = \{(0, 0), (2, 2), (4, 2)\}.
\]

• $S_{1,1}$ has got two tracks. So it has got two special sons:

\[
S_{1,1,1} = \{(0, 0), (1, 2), (2, 3), (2, 4), (3, 3), (3, 5)\}, \quad S_{1,1,2} = \{(0, 0), (1, 2), (2, 4)\}
\]

• $S_{1,2}$ has no tracks.

• $S_{1,3} = S_{2,3}$ was studied before.

• $S_{1,4}$ has got seven tracks. So it has got seven special sons:

\[
S_{1,4,1} = \{(0, 0), (1, 3), (2, 4)\}, \quad S_{1,4,2} = \{(0, 0), (1, 3), (1, 4), (2, 3), (2, 5)\},
\]

\[
S_{1,4,3} = \{(0, 0), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 6)\}, \quad S_{1,4,4} = \{(0, 0), (2, 3)\},
\]

\[
S_{1,4,5} = \{(0, 0), (1, 4)\}, \quad S_{1,4,6} = \{(0, 0), (1, 3), (1, 5)\}, \quad S_{1,4,7} = \{(0, 0), (1, 3), (1, 4), (1, 6)\}.
\]
• $S_{1,5}$ has got four tracks. So it has got four special sons:

$$S_{1,5,1} = \{(0, 0), (1, 2), (2, 4)\}, \quad S_{1,5,2} = \{(0, 0), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (2, 6)\}$$

$$S_{1,5,3} = \{(0, 0), (2, 2), (2, 4)\}, \quad S_{1,5,4} = \{(0, 0), (1, 2), (1, 4), (1, 6)\}.$$

Notice that

• $S_{2,1,1} = S_{2,5,1}$, but $T_{2,1,1} \notin T(S_{2,1})$ while $T_{2,5,1} \in T(S_{2,5})$;

• $S_{2,3,5} = S_{2,4,7}$, but $T_{2,3,5} \notin T(S_{2,3})$ while $T_{2,4,7} \in T(S_{2,4})$;

• $S_{2,3,6} = S_{2,5,4}$, but $T_{2,3,6} \notin T(S_{2,3})$ while $T_{2,5,4} \in T(S_{2,5})$;

• $S_{1,4,4} = S_{2,3,7}$, but $T_{1,4,4} \notin T(S_{1,4})$ while $T_{2,3,7} \in T(S_{2,3})$;

• $S_{1,5,1} = S_{1,1,2}$, but $T_{1,5,1} \notin T(S_{1,5})$ while $T_{1,1,2} \in T(S_{1,1})$;

• $S_{1,5,3} = S_{2,3,8}$, but $T_{1,5,3} \notin T(S_{1,5})$ while $T_{2,3,8} \in T(S_{2,3})$.

thus, the repeated semigroups can be computed only one time by taking in account the consequences of Remark 2.10.
Figure 4: Tree of local good semigroups until genus 4.
We implemented a function in "GAP" [17] using the package "NumericalSgps" [12] to find the special tracks and an algorithm to build the tree of good semigroups by genus (where each level is built starting from the previous one). The following table contains the obtained results regarding the value of \( n_g \) up to genus 27:

| \( g \) | \( n_g \) | \( \frac{n_g}{n_{g-1}} \) | \( \frac{n_g}{n_{g-1}} - \frac{n_g}{n_{g-1}} \) |
|--------|--------|----------------|----------------|
| 1      | 1      |                |                |
| 2      | 3      | 3              |                |
| 3      | 10     | 3.333333       | +0.333333      |
| 4      | 29     | 2.9            | -0.433333      |
| 5      | 78     | 2.689655       | -0.210345      |
| 6      | 211    | 2.705128       | +0.015473      |
| 7      | 555    | 2.630332       | -0.074796      |
| 8      | 1419   | 2.556757       | -0.073575      |
| 9      | 3658   | 2.577872       | +0.021115      |
| 10     | 9291   | 2.539913       | -0.037959      |
| 11     | 23559  | 2.53568        | -0.004233      |
| 12     | 59750  | 2.536186       | +0.000506      |
| 13     | 151489 | 2.535381       | -0.000805      |
| 14     | 384962 | 2.541188       | +0.005807      |
| 15     | 981175 | 2.548758       | +0.007570      |
| 16     | 2509148| 2.557289       | +0.008531      |
| 17     | 6446022| 2.569008       | +0.011719      |
| 18     | 16643410 | 2.581966 | +0.012958      |
| 19     | 43206759 | 2.596028 | +0.014062      |
| 20     | 112813434 | 2.611014 | +0.014986      |
| 21     | 296385223 | 2.627216 | +0.016202      |
| 22     | 783663199 | 2.644069 | +0.016854      |
| 23     | 2085649918 | 2.661411 | +0.017341      |
| 24     | 5588023752 | 2.679272 | +0.017861      |
| 25     | 15074196720 | 2.697759 | +0.018318      |
| 26     | 40945190707 | 2.71624 | +0.018654      |
| 27     | 111988822296 | 2.735091 | +0.018847      |

Table 1: Table reporting the number of local good semigroups by genus

In 2008 Bras Amorós observed that, in case of numerical semigroup the numerical sequence \( \{n_g\} \) seemed to have the same behaviour of Fibonacci sequence and conjectured that the ratio \( \frac{n_g}{n_{g-1}} \) converges to the golden ratio [3]. In 2013 this was actually proved by Zhai [24]. Looking to the previous table, also if we don’t have a great quantity of data, the tendency for local good
semigroups of $\mathbb{N}^2$ seems to be different. In fact the difference $\frac{n_g}{n_g-1} - \frac{n_{g-1}}{n_{g-2}}$ seems to be an increasing function, so the ratio seems to diverge.

### 3 Relationship between genus and other notable elements

#### 3.1 On the type of a good semigroup

In this subsection we want to relate the genus and the type of a good semigroup $S \subseteq \mathbb{N}^2$ by generalizing a well known inequality that holds in the case of numerical semigroups.

First of all, we recall the concept of type of a good semigroup by following the definition introduced in [8] that extends the one initially given in [1].

We write that $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$ if and only if $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$ or $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$ and in the same way we write $(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$ if and only if $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$.

Given a good semigroup $S \subseteq \mathbb{N}^2$, let us consider a set $A \subseteq S$ such that there exists $c \in \mathbb{N}^2$ with $c + \mathbb{N}^2 \subseteq S \setminus A$. As described in [8], it is possible to build up a partition of such set $A$, in the following way.

Let us define,

- $D^{(0)} = \emptyset$:
- $B^{(i)} = \{ \alpha \in A \setminus (\cup_{j<i} D^{(j)}) : \alpha$ is maximal with respect to $\leq \}$
- $C^{(i)} = \{ \alpha \in B^{(i)} : \alpha = \beta_1 \oplus \beta_2$ for some $\beta_1, \beta_2 \in B^{(i)} \setminus \{ \alpha \} \}$
- $D^{(i)} = B^{(i)} \setminus C^{(i)}$.

For a certain $N \in \mathbb{N}$, we have $A = \bigcup_{i=1}^{N} D^{(i)}$ and $D^{(i)} \cap D^{(j)} = \emptyset$. In according to notation of [8], we rename these sets in an increasing order setting $A_i = D^{(N+1-i)}$. Thus we have

$$A = \bigcup_{i=1}^{N} A_i,$$

and the $A_i$ are called *levels* of $A$.

It was proved [Thm. 2.7 [8]] that, if $E = S \setminus A$ is a proper good ideal of $S$, then $N = d(S \setminus E)$.

**Definition 3.1.** Let us consider a set $A \subseteq \mathbb{N}^2$ such that there exists $c \in \mathbb{N}^2$ with $c + \mathbb{N}^2 \subseteq \mathbb{N}^2 \setminus A$. We denote by $NL(A)$ the integer such that

$$A = \bigcup_{i=1}^{NL(A)} A_i,$$

is the partition in levels of $A$ described above.

Now we want to generalize to good ideals a result proved for good semigroups in [8]
Proposition 3.2. Let $I$ be a good relative ideal of a good semigroup $S \subseteq \mathbb{N}^2$. Consider $A \subseteq I$ satisfying the conditions of Definition 3.1, such that $E = I \setminus A$ is a good relative ideal of $S$. Then

$$
NL(A) = d(I \setminus E)
$$

Proof. If $I$ is a good relative ideal of $S$ there exists $\alpha \in S$ such that $\alpha + I \subseteq S$. We notice that $(\alpha + I) \cup \{0\}$ is a good semigroup. In fact, if $\alpha + i_1, \alpha + i_2 \in \alpha + I$,

$$(\alpha + i_1) + (\alpha + i_2) = \alpha + (\alpha + i_1 + i_2) \in \alpha + I$$

since $\alpha + (i_1 + i_2) \in S + I \subseteq I$. Furthermore, it is easy to check that $\alpha + E$ is a proper good ideal of $(\alpha + I) \cup \{0\}$. Setting $S' = (\alpha + I) \cup \{0\}$, $E' = \alpha + E$ and $A' = ((\alpha + I) \cup \{0\}) \setminus (\alpha + E)$, by [Thm. 2.7 [8]], we have

$$d(S' \setminus E') = NL(A')$$

so we can write:

$$1 + d(\alpha + I \setminus \alpha + E) = d(S' \setminus E') = NL(A') = 1 + NL(\alpha + I \setminus \alpha + E).$$

We can conclude that:

$$d(I \setminus E) = d(\alpha + I \setminus \alpha + E) = NL(\alpha + I \setminus \alpha + E) = NL(I \setminus E) = NL(A).$$

Corollary 3.3. Given a good semigroup $T$, $g(T) = NL(\mathbb{N}^2 \setminus T)$.

Proof. It is sufficient to apply Proposition 3.2, considering $S = T$, $I = \mathbb{N}^2$, $A = \mathbb{N}^2 \setminus T$. □

In [8], the set of pseudo-frobenius elements of a good semigroup $S$ is defined as

$$PF(S) = \{ \alpha \in \mathbb{N}^2 \setminus S \mid \alpha + M \subseteq S \},$$

where $M = S \setminus \{0\}$ is the maximal ideal of $S$.

The set $PF(S)$ satisfies the condition of the set $A$ in Definition 3.1. The type of the good semigroup $S$ is defined as $t(S) = NL(PF(S))$, that is the number of levels of the pseudo-frobenius elements.

Remark 3.4. We recall that, given two ideals $E$ and $F$ of $S$, it is possible to consider the set

$$E - F = \{ \alpha \in \mathbb{Z}^2 \mid \alpha + F \subseteq E \}.$$ 

This set is not in general a good ideal. We have that $PF(S) = (S - M) \setminus S$, thus $t(S) = NL((S - M) \setminus S)$. In [8, Proposition 3.5], it is proved that if $S - M$ is a good ideal, then $t(S) = d(S - M \setminus S)$ as it was initially defined in [1].
Given a good semigroup $S$, we want to prove the inequality
\[ g(S) \leq t(S)l(S), \]
that generalizes the analogous one proved in [15] for numerical semigroups.
In order to do that we need some lemmas.

**Lemma 3.5.** Let us consider a subset $A \subseteq \mathbb{N}^2$ such that there exists $c \in \mathbb{N}^2$ with $c+\mathbb{N}^2 \subseteq \mathbb{N}^2\setminus A$. Suppose that $A = \bigcup_{j=1}^{h} B_j$ with $B_l \cap B_m = \emptyset$ if $l \neq m$. Then
\[
NL(A) \leq \sum_{j=1}^{h} NL(B_j).
\]

**Proof.** Denote by $n = NL(A)$ and by $m_j = NL(B_j)$. Furthermore we write
\[ A = \bigcup_{i=1}^{n} A_i, \quad B_j = \bigcup_{l=1}^{m_j} B_{j,l} \text{ for } j = 1, \ldots, h. \]

We want to find a chain
\[ \alpha_1 \leq \ldots \leq \alpha_n, \text{ where } \alpha_i \in A_i \text{ for all } i = 1, \ldots, n, \]
and such that each $B_{j,l}$ contains at most one of the $\alpha_i$’s. In order to do that we consider $\alpha_1$ as an arbitrary element of $A_1$, then we choose $\alpha_i$, with $i \geq 2$, by taking in account the following rule:

**Case 1:** Denote by $D = A_{i+1} \cap \{ \beta \in A | \alpha_i \ll \beta \}$. If $D$ is not empty, then we choose as $\alpha_{i+1}$ an arbitrary element of $D$.

**Case 2:** If $D = \emptyset$, then [8, Lemma 2.4 (1)] ensures that $\Delta_1(\alpha_i) \cap A_{i+1}$ and $\Delta_2(\alpha_i) \cap A_{i+1}$ are both non-empty. Furthermore, if we suppose that $\alpha_i \in B_{j,l}$, then there must exist a $k \in \{1, 2\}$ such that $\Delta_k(\alpha_i) \cap B_{j,l} = \emptyset$. In fact, otherwise, we would have $\alpha_i = \beta_1 \oplus \beta_2$, with $\alpha_i, \beta_1, \beta_2 \in B_{j,l}$ that it is a contradiction since $B_{j,l}$ is a level of $B_j$ and cannot contain such a configuration. Thus, in this case we choose an element of $\Delta_k(\alpha_i) \cap A_{i+1}$ as $\alpha_{i+1}$.

By construction and by the properties of the levels, it is clear that it is not possible to find $l \in \{1, \ldots, m_j\}$ and $j \in \{1, \ldots, h\}$ such that $|B_{j,l} \cap \{ \alpha_1, \ldots, \alpha_n \}| \geq 2$.

Thus $NL(A) \leq \sum_{j=1}^{h} NL(B_j)$ as required. \hfill \Box

Now we are ready to prove the main result of this subsection.

**Proposition 3.6.** Let $S$ be a good semigroup. Then
\[ g(S) \leq t(S)l(S). \]
Proof. Denote by \( n = l(S) \). We choose
\[
0 = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = c(S),
\]
an arbitrary saturated chain in \( S \) between \( 0 \) and \( c(S) \). We consider the following chain of ideals of \( S \):
\[
S \subseteq S - S(\alpha_1) = S - M \subseteq S - S(\alpha_2) \subseteq \ldots \subseteq S - S(\alpha_n) = S - C(S) = \mathbb{N}^2.
\]
We have
\[
\mathbb{N}^2 \setminus S = \bigcup_{i=1}^{n} (S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1})),
\]
and by Lemma 3.5 and Corollary 3.3 we can deduce:
\[
g(S) = \text{NL}(\mathbb{N}^2 \setminus S) \leq \sum_{i=1}^{n} \text{NL}((S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1}))). \tag{1}
\]
Now we claim that
\[
\text{NL}((S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1}))) \leq \text{NL}((S - S(\alpha_1)) \setminus (S - S(\alpha_0))) = \text{NL}((S - M) \setminus S) = t(S),
\]
for all \( i = 2, \ldots, n \). For each \( i \in \{2, \ldots, n\} \), we denote by
\[
\tilde{\alpha}_{i-1} = \begin{cases} 
\alpha_{i-1}, & \text{if } \alpha_i \notin \Delta^S(\alpha_{i-1}) \\
\max\{\Delta^S_{3-k}(\alpha_{i-1})\} \oplus c(S), & \text{if } \alpha_i \in \Delta^S_{3-k}(\alpha_{i-1}) \end{cases},
\]
and we consider the following function
\[
f : (S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1})) \leftrightarrow (S - M) \setminus S
\]
\[
\gamma \mapsto \gamma + \tilde{\alpha}_{i-1}.
\]
The function \( f \) is clearly injective, thus in order to prove our claim we need only to show that it is well defined.
Thus we fix an arbitrary \( \beta \in (S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1})) \) and we prove that \( \beta + \tilde{\alpha}_{i-1} \in (S - M) \setminus S \).

1. We prove that \( \beta + \tilde{\alpha}_{i-1} \in S - M \). Let us consider an element \( \gamma \in M = S(\alpha_1) \). We need to show that \( \beta + \tilde{\alpha}_{i-1} + \gamma \in S \). Since \( \beta \in S - S(\alpha_i) \), it suffices notice that \( \tilde{\alpha}_{i-1} + \gamma \geq \alpha_i \). In fact, otherwise \( (\tilde{\alpha}_{i-1} + \gamma) \oplus \alpha_i = \delta \in S \) would be an element such that \( \alpha_i - 1 < \delta < \alpha_i \), since \( \alpha_{i-1} \leq \tilde{\alpha}_{i-1} \leq \tilde{\alpha}_{i-1} + \gamma \) and \( \alpha_{i-1} \leq \alpha_i \). But this contradicts the fact that we considered a saturated chain in \( S \).
2. We prove that \( \beta + \tilde{\alpha}_{t-1} \not\in S \). Let us assume, by contradiction, that \( \beta + \tilde{\alpha}_{t-1} \in S \). We consider \( D := S(\alpha_{i-1}) \setminus S(\alpha_i) \). We have two cases:

**Case 1:** \( \alpha_i \not\in \Delta_S(\alpha_{i-1}) \). In this case we have \( D = \{ \alpha_{i-1} = \tilde{\alpha}_{t-1} \} \). In fact, if \( \beta \in D \) and \( \beta \neq \alpha_{i-1} \), it would follow \( \alpha_{i-1} < \beta \oplus \alpha_i < \alpha_i \), against the fact that \( \alpha_{i-1} \) and \( \alpha_i \) are consecutive in \( S \). Thus \( \beta + \tilde{\alpha}_{t-1} \in S \) implies \( \beta \in (S - S(\alpha_{i-1})) \) that is a contradiction.

**Case 2:** There exists \( k \in \{1, 2\} \) such that \( \alpha_i \in \Delta^S_S(\alpha_{i-1}) \). It is easy to notice that in this case \( D := S(\alpha_{i-1}) \setminus S(\alpha_i) = \Delta^S_{S-k}(\alpha_{i-1}) \cup \{ \alpha_{i-1} \} \). Let us consider an arbitrary \( \gamma \in D \) and let us show that \( \beta + \gamma \in S \). If \( \gamma \in \Delta^S_{S-k}(\alpha_{i-1}) \), then \( \beta + \gamma \in \Delta^S_{S-k}(\beta + \alpha_{i-1}) \). We notice that, by definition of \( \alpha_{i-1} \), in this case \( c(S) \in \Delta^S_k(\alpha_{i-1}) \). Since \( \beta + \alpha_{i-1} \in S \), it easily follows \( \beta + \gamma \in \Delta^S_{S-k}(\beta + \alpha_{i-1}) \subseteq S \). If \( \gamma < \alpha_{i-1} \), then property (G3) implies that there must exist a \( \delta \in \Delta^S_3(\gamma) \). Furthermore \( \delta \in S(\alpha_i) \), (otherwise \( \alpha_{i-1} < \delta \oplus \alpha_i < \alpha_i \)). Since \( \beta \in S - S(\alpha_i) \), we have \( \beta + \delta \in S \).

Finally we have:

\[
\gamma + \beta = (\beta + \delta) \oplus (\beta + \alpha_{i-1}) \in S.
\]

Thus, also in this case, we deduce \( \beta \in (S - S(\alpha_{i-1})) \) that is a contradiction.

This means that \( \beta + \tilde{\alpha}_{t-1} \not\in S \) and the claim is proved. Finally from expression (1) and the claim it follows:

\[
g(S) \leq \sum_{i=1}^{n} \text{NL}((S - S(\alpha_i)) \setminus (S - S(\alpha_{i-1}))) \leq \sum_{i=1}^{l(S)} \text{NL}((S - M \setminus S) = t(S)l(S),
\]

and the proof is complete. \( \square \)

Given a good semigroup \( S \), its canonical ideal is the set \( K = \{ \alpha \in \mathbb{N}^2 | \Delta^S(\gamma - \alpha) = \emptyset \} \). In [Prop 2.17. [1]] it is proved that \( g(S) \geq d(K \setminus C(S)) \).

We observe that, if \( S - M \) is a good ideal, \( t(S) = d((S - M) \setminus S) \). Since \( S - M \subseteq K \cup \Delta(\gamma) \), we have:

\[
t(S) \leq d(K \cup \Delta(\gamma) \setminus S) \leq d(K \cup \Delta(\gamma) \setminus K) + d(K \setminus K) = 1 + d(K \setminus S) = d(K \setminus C(S)) - d(S \setminus C(S)) + 1 \leq g(S) - l(S) + 1
\]

Using Proposition 3.2 it is possible prove this inequality for any good semigroup, also if \( S - M \) is not a good semigroup:

**Corollary 3.7.** Given a good semigroup \( S \), \( g(S) \geq t(S) + l(S) - 1 \).

**Proof.** Since \( PF(S) \subseteq (K \setminus S) \cup \Delta(\gamma) \), by [8, Lemma 3.6] and Lemma 3.5, we deduce \( t(S) = \text{NL}(PF(S)) \leq \text{NL}((K \setminus S) \cup \Delta(\gamma)) \leq \text{NL}((K \setminus S)) + \text{NL}(\Delta(\gamma)) \leq \text{NL}(K \setminus S) + 1 \)

Applying Proposition 3.2 considering \( I = K, A = K \setminus S, E = S \), we have:

\[
\text{NL}(K \setminus S) + 1 = d(K \setminus S) + 1 = d(K \setminus C(S)) - d(S \setminus C(S)) + 1 \leq g(S) - l(S) + 1.
\]

\( \square \)

24
3.2 On the Wilf Conjecture

In [23], it was firstly introduced the well known Wilf conjecture regarding the numerical semigroups. In [13] it was rephrased in the context of numerical semigroups. It states that the number of minimal generators of a numerical semigroup $S$, i.e. the embedding dimension of the semigroup, always satisfies the inequality

$$\text{edim}(S) \geq \frac{c(S)}{c(S) - g(S)},$$

where $c(S)$ and $g(S)$ are respectively the conductor and the genus of the semigroup. The Wilf conjecture represents an important open problem in the context of the numerical semigroup theory, and it has been proved for many special cases [15], [19], [22], [21], [14], [4] and checked for numerical semigroups up to genus 50 in ([3]) and up genus 67 in [16].

In [20] the concept of embedding dimension of a good semigroup of $\mathbb{N}^2$ is introduced, therefore it makes sense to extend in a natural way the conjecture to these more general objects. Specifically we want to check if, for a good semigroup $S \subseteq \mathbb{N}^d$ of genus $g$, the inequality

$$\text{edim}(S) \geq \frac{c_S}{c_S - g(S)},$$

always holds, where $c_S$ is defined as in Section 1.

By exploring the tree of good semigroups of $\mathbb{N}^2$, introduced in the previous section, it is possible to check that the conjecture is satisfied for semigroups up to genus 22. However, starting from genus 23, examples where the conjecture is not verified begin to show up, as is shown in the following example.

**Example 3.8.** Consider the good semigroup $S$, represented by the following set of small elements:

$$\text{Small}(S) = \{(0, 0), (4, 3), (8, 6), (8, 9), (8, 11), (8, 12), (8, 13), (8, 14), (9, 6), (12, 9), (12, 11), (12, 12), (12, 13), (12, 14), (13, 9), (13, 11), (15, 9), (16, 12), (16, 13), (16, 14), (17, 12), (17, 13), (17, 14), (18, 12), (18, 13), (19, 12), (20, 14)\}.$$  

We have $c_S = 34$ and $g(S) = 23$. Using the algorithms presented in [20], it is possible to check that $\text{edim}(S) = 3$ (according to the terminology introduced in that paper, the set $\{(4, 3), (8, \infty), (13, 11)\} \subseteq I_A(S)$ constitutes a minimal set of representatives for $S$).

Finally we have:

$$3 = \text{edim}(S) < \frac{c_S}{c_S - g(S)} = \frac{34}{11},$$

disproving the Wilf conjecture for good semigroups of $\mathbb{N}^2$. 

25
Remark 3.9. It still makes sense to ask whether the Wilf conjecture is true for good semigroups that are value semigroups. In fact, at the moment, there are no known examples of value semigroups disproving the conjecture, since for all the known counterexamples it seems impossible to find suitable rings having them as value semigroups. This fact may suggest that the Wilf conjecture is more related to the structure inherited from the rings than on the combinatorical properties of these objects.

Acknowledgements

The authors would like to thank Marco D’Anna, Felix Delgado, Manuel Delgado and Pedro A. García Sánchez, for their helpful comments and suggestions during the development of this paper. The first author is supported by the projects MTM2014-55367-P, which is funded by Ministerio de Economía y Competitividad and Fondo Europeo de Desarrollo Regional FEDER, and by the Junta de Andalucía Grant Number FQM-343. He also thanks the "University of Granada" for host him and providing the machines necessary to complete some of the computations reported on the paper. Both the authors gratefully acknowledge support by the project "Proprietà algebriche locali e globali di anelli associati a curve e ipersuperfici" PTR 2016-18 - Dipartimento di Matematica e Informatica - Università di Catania".
References

[1] V. Barucci, M. D’Anna, and R. Fröberg. Analytically unramified one-dimensional semilocal rings and their value semigroups. *Journal of Pure and Applied Algebra*, 147(3):215–254, 2000.

[2] V. Barucci, M. D’Anna, and R. Fröberg. The semigroup of values of a one-dimensional local ring with two minimal primes. *Comm. Algebra*, 28(8):3607–3633, 2000.

[3] M. Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, 76(2):379–384, March 2008.

[4] W. Bruns, P. Garcia-Sanchez, C. O’Neill, and D. Wilburne. Wilf’s conjecture in fixed multiplicity. *arXiv preprint arXiv:1903.04342*, 2019.

[5] A. Campillo, F. Delgado, and K. Kiyek. Gorenstein properties and symmetry for one-dimensional local cohen-macaulay rings. *Manuscripta Math.*, 83:405–423, 1994.

[6] E. Carvalho and M. Escudeiro Hernandes. The Semiring of Values of an Algebroid Curve. *ArXiv e-prints*, April 2017.

[7] M. D’Anna. Canonical module of a one-dimensional reduced local ring. *Comm. Algebra*, 25(09):2939–2965, 1997.

[8] M. D’Anna, L. Guerrieri, and V. Micale. The type of a good semigroup and the almost symmetric condition. *(To Appear) Mediterranean Journal of Mathematics*.

[9] M. D’Anna, L. Guerrieri, and V. Micale. The Apéry Set of a Good Semigroup. *ArXiv e-prints*, December 2018.

[10] F. Delgado. The semigroup of values of a curve singularity with several branches. *Manuscripta Math.*, 59:347–374, 1987.

[11] F. Delgado. Gorenstein curves and symmetry of the semigroup of value. *Manuscripta Math.*, 61:285–296, 1988.

[12] M. Delgado, P.A. García-Sánchez, and J. Morais. *NumericalSgps, A GAP package for numerical semigroups, Version 1.1.10*, 2018.

[13] D.E. Dobbs and G.L. Matthews. On a question of wilf concerning numerical semigroups. *International Journal of Commutative Rings*, 3(2), 2003.

[14] S. Eliahou. Wilf’s conjecture and macaulay’s theorem. *arXiv preprint arXiv:1703.01761*, 2017.
[15] R. Fröberg, C. Gottlieb, and R. Häggkvist. On numerical semigroups. In *Semigroup forum*, volume 35, pages 63–83. Springer, 1986.

[16] J. Fromentin and F. Hivert. Exploring the tree of numerical semigroups. *Mathematics of Computation*, 85(301):2553–2568, 2016.

[17] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.0*, 2018.

[18] A. García. Gorenstein curves and symmetry of the semigroup of value. *J. Reine Angew. Math.*, 336:165–184, 1982.

[19] N. Kaplan. Counting numerical semigroups by genus and some cases of a question of wilf. *Journal of Pure and Applied Algebra*, 216(5):1016–1032, 2012.

[20] N. Maugeri and G. Zito. Embedding dimension of a good semigroup. In (To Appear) *Proceedings of the INDAM meeting: International meeting on numerical semigroups*, Springer INdAM Series. Springer.

[21] A. Moscariello and A. Sammartano. On a conjecture by wilf about the frobenius number. *Mathematische Zeitschrift*, 280(1-2):47–53, 2015.

[22] A. Sammartano. Numerical semigroups with large embedding dimension satisfy wilf’s conjecture. In *Semigroup Forum*, volume 85, pages 439–447. Springer, 2012.

[23] H.S. Wilf. A circle-of-lights algorithm for the “money-changing problem”. *The American Mathematical Monthly*, 85(7):562–565, 1978.

[24] A. Zhai. Fibonacci-like growth of numerical semigroups of a given genus. In *Semigroup Forum*, volume 86, pages 634–662. Springer, 2013.

[25] G. Zito. Arf good semigroups. *Journal of Algebra and Its Applications*, 17(10), 2018.

[26] G. Zito. Arf good semigroups with fixed genus. *arXiv preprint arXiv:1802.02897*, 2018.