Analysis of one-dimensional Yut-Nori game: winning strategy and avalanche-size distribution

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In the Korean traditional board game Yut-Nori, teams compete by moving their pieces on the
two-dimensional game board, and the team whose all pieces complete a round trip on the board
wins. In every round, teams throw four wooden sticks of the shape of half-cut cylinder and the
number of sticks that show belly sides, i.e., the flat sides, determines the number of steps the team’s
piece can advance on the board. It is possible to pile up one team’s pieces if their sites are identical
so that pieces as a group can move together afterwards (piling). If a piece of the opponent team
is at the new site of one team’s piece, the piece is caught and removed from the board, and the
team is given one more chance to throw sticks and proceed (catching). For a simplicity, we simulate
this game on one-dimensional board with the same number of sites as the original game, and show
that catching is more advantageous strategy than piling to win. We also study the avalanche-size
distribution in thermodynamic limit to find that it follows an exponential form.

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I. INTRODUCTION

Yut-Nori with Nori meaning a game, is a Korean tradi-
tional board game. Each team is given four pieces, usu-
ally small stones, called Mal, and teams share four
sticks called Yut and the game board called Yut-Pan to
play. The set of four sticks acts like a dice (see Fig.1):
Each wooden stick has the shape of a cylinder half-cut in
a longitudinal direction and has the back side (the round
side) and the belly side (the flat side). The number of
belly sides of four thrown sticks determines the number
of steps that a piece on the board can move. If only one
stick shows the belly side and others back sides, we call
this Do. When one gets Do, the team can advance its
piece by one step on the game board. Likewise, Gae and
Geol are for two and three sticks of belly sides, and the
piece proceeds by two and three steps, respectively. If all
four sticks show belly sides, which is called Yut (the same
term Yut is used both for the outcome of stick throwing
and for the name of sticks), the team’s piece is advanced
by four steps, whereas for four back side sticks (no belly
sided stick) called Mo, the piece moves by five steps on
the game board. Yut sticks are cut in such a way that the
probability p for the belly side is larger than the proba-
bility 1 − p for the back side. For a given value of p,
it is straightforward to compute the probabilities for Do,
Gae, Geol, and Mo as \(P_{\text{Do}} = (\frac{4}{4})p(1-p)^3\), \(P_{\text{Gae}} = (\frac{4}{2})p^2(1-p)^2\), \(P_{\text{Geol}} = (\frac{4}{3})p^3(1-p)^1\), \(P_{\text{Yut}} = (\frac{4}{4})p^4(1-p)^0\), and \(P_{\text{Mo}} = (\frac{4}{4})p^0(1-p)^4\), respectively. In Fig.2 we dis-
play the above probabilities as functions of p. It is clear
that p must be larger than 0.5 to have \(P_{\text{Mo}} < P_{\text{Yut}}\) for
example. Also, if \(p > 1/(1+4^{-1/3})\), \(P_{\text{Yut}}\) becomes larger
than \(P_{\text{Do}}\), which needs to be avoided since Yut is much
better than Do and thus should occur less frequently than
Do. Consequently, we expect that the probability p for
the belly side must satisfy \(p \lesssim 0.6\).

There still remains a debate about the historical ori-
igin of Yut-Nori, but the argument by Chae-Ho Shin, who

![](https://example.com/fig1.png)

FIG. 1: (Color online) A Yut stick can show (a) the round
back side (marked by crosses) up or (b) the flat belly side up.
A Yut stick is shaped in such a way that the belly side has
higher chance to occur than the back side, i.e., \(P_\text{Gae} < P_\text{Do} \equiv p\).
(c) Four Yut sticks can have five different states, depending
on the number of belly sides: Do, Gae, Geol, and Yut for one
to four belly sides, respectively. Mo is when all four Yut-sticks
are of the back sides.
was a Korean historian and independence movement activist during Japan’s colonization period of Korea, has been widely accepted. According to him, Yut-Nori originated from an ancient Korean kingdom Gojoseon, which existed from 2333 BC according to the history book Samguk Yusa written by a Buddhist monk Ilyeon in the 13th century AD. In Gojoseon era, there were five influential tribes ruling five regions in the kingdom, and they joined forces during wartime. The game board Yut-Pan mimics how these five tribes spatially arranged their positions during battles. Except for the king’s tribe, all four tribes had their names from livestock; pigs, dogs, cows, and horses. These names are reflected as the terms in Yut-Nori as Do, Gae, Yut, and Mo, respectively. Chae-Ho Shin argues that the king’s tribe is represented as Geol in Yut-Nori. It is interesting to note that the distance a piece can move appears to correspond to the sizes and the step widths of animals. The speed of a pig is the smallest and a horse the fastest, giving the increasing step size in Yut-Nori. Even in modern Korean language, a dog is still called Gae, and a pig Doe-Ji, probably sharing the same origin as Do in Yut-Nori.

Pieces move along the sites marked on the board counterclockwise, starting from the bottom right corner. Allowed sites are drawn as circles, and there are three forks, two upper corners and center [see Fig. 3(a)]. There are four possible ways to return to the starting point to make a round trip; the shortest path, two middle distance paths, and the longest path as shown in Fig. 3(a). Rules of the game are almost common across the whole country with a small variation for different regions. Each team, one by one in turn, rolls the sticks and moves pieces on the board. The team, whose all pieces make round trips around the board returning to the starting point, is the winner. If other team’s pieces are at the site of the stick-rolling team’s piece, other team’s pieces are caught and the team is asked to roll sticks once more time. We call this situation as catching in the present paper. Yut and Mo have less chance to occur in comparison with Do, Gae, and Geol (see Fig. 2) and when the team has either Yut or Mo, the team is given another chance to throw sticks. Consequently, one team often can throw sticks several times in one turn. Suppose that a team gets Yut, and thus the team throws sticks once more to get Mo, which gives the team Geol in next rolling. If the last Geol catches the opponent piece(s), the team can throw once more. Consequently, there can occur a variety of different situations even in a single turn, and it is possible in principle that one team can win the game in its first turn although present authors have not seen this in their entire lives. If the same team’s pieces meet at the same site on the board, they can pile up (we call this piling). Piled up pieces can move together afterwards so that piling reduces the number of turns that is needed to win. On the other hand, piling can be risky, since if piled pieces are caught by an opponent team, all pieces should start the game again from the starting position. Accordingly, catching is a more aggressive strategy in Yut-Nori, but piling is also a good strategy in some situation in which the chance of being caught is slim.

The original two-dimensional game board [see Fig. 3(a)] is too complicated to analyze. Only for the sake of simplicity, we consider only the longest path in the game board, which allows us to analyze the one-dimensional board displayed in Fig. 3(b). The advantage of using the one-dimensional board lies on the fact that we do not need to choose which path to take among the four possibilities in Fig. 3(a). Within the limitation, we believe that our game theoretic approach described below can give us some clue to answer the question which strategy between piling and catching yields bigger win-

\[
p \approx 0.6135\]
The remaining part of the present paper is organized as follows: In Sec. II we first consider the case in which one player with one piece throws Yut sticks. We perform a simple statistical analysis of our simplified one-dimensional Yut-Nori to compute the average step size and the distribution of number of turns to finish the game. Our main result for the winning strategy is reported in Sec. III, and the avalanche-size distribution in the infinitely long one-dimensional board is discussed in Sec. IV which is followed by summary in Sec. V.

II. MOVEMENT OF A PIECE IN ONE-DIMENSIONAL YUT-NORI

We first consider the simplest case that one player plays the game alone with only one piece in one-dimensional Yut-Pan [see Fig. 2(b)]. We can ask how long distance the piece advances in one turn on average. Let us denote this expectation value of the distance as $\Phi$. Since the player can throw Yut sticks once more if she has Yut or Mo, $\Phi$ satisfies the following self-consistent equation

$$\Phi = P(1) + 2P(2) + 3P(3) + (4 + \Phi)P(4) + (5 + \Phi)P(5),$$

where $P(m)$ is the probability for the step size $m$, e.g., $m = 3$ for Geol and $m = 5$ for Mo, respectively, and we get

$$\Phi = \frac{4p + 5(1 - p)^4}{1 - p^4 - (1 - p)^4}.$$  \hspace{1cm} (2)

When $p = 0.6$ (see Fig. 2), Eq. (2) yields $\Phi = 2.992424$, which we confirm through numerical simulation: From $10^6$ runs, the average value is found to be $\Phi = 2.992(2)$.

During numerical experiment repeated $10^6$ times, we also measure how many turns needs to be spent to finish the game and call it $t$. The distribution function $H(t)$ is very well approximated by the Gaussian form as shown in Fig. 4. Among the $10^6$ runs, we find $t = 1$ occurs only about 100 times. Likewise, in playing the original Yut-Nori with the game board in Fig. 3(a), it is almost impossible to see the starting team win the game without ever giving its turn to other teams.

III. WINNING STRATEGY

Although more than two teams can play Yut-Nori, two-team play appears to be most popular. In every round, each team throws sticks in turn, and the player who actually throws sticks is chosen sequentially: Suppose that two teams $A$ and $B$ have players $A_1, A_2, A_3$ and $B_1, B_2, B_3$, respectively. In the first round $A_1$ and $B_1$ throw sticks, and in the second round $A_2$ and $B_2$ play, and so on. In our setting, the number of players in each team is not important if all players in the team use the same strategy. To further simplify the game, we assume that two teams use only two pieces each (instead of four as in the original Yut-Nori) and play the game in the one-dimensional board in Fig. 3(b). Even in this simple setting a variety of different situations can arise. Team’s strategy is crucial to win the game since there are different ways to move pieces on the board even in the same situation. A team can be aggressive and tries to catch the opponent’s pieces whenever possible. Otherwise, it can pile up their pieces to speed up the movement of pieces afterwards.

Our simplified game proceeds as follows: A team throws Yut sticks. Depending on the outcome (see Fig. 1), the number $m$ of steps the team’s piece can advance is decided. For given $m$, the team can decide what to do for the current configuration of the game board. If the piece of the opponent can be caught for given $m$ we denote this situation as ’C’, while we denote ’P’ if the team can pile up their two pieces together. If neither catching nor piling is possible, we denote this as ’N’. Since each team has two pieces, the team’s strategy must decide what to do for all possible cases $(X_1, X_2)$ with $X_1, X_2 = C, P, N$, where $X_1(2)$ is the situation for the first (second) piece. For example, $(X_1, X_2) = (C, P)$ means that the team’s first piece can catch the opponent’s piece and the team’s second piece can pile up with its first piece. We denote the decision made by a given strategy as $S(X_1, X_2)$. If the strategy $S$ decides to move the 1st piece, $S(X_1, X_2) = (X_1, 0)$, while $S(X_1, X_2) = (0, X_2)$ if the 2nd piece is decided to be moved. For example $S(C, P) = (C, 0)$ means that the strategy $S$ chooses to catch by using the first piece, while $S(C, P) = (0, P)$ means that the 2nd piece will be moved to pile up with the 1st piece. Among all $9(= 3 \times 3)$ possible combinations for $(X_1, X_2)$, it is clear...
that \((P, P)\) is impossible since only one piece that is behind the other piece can pile up. Among remaining 8 cases, \((P, N), (C, N), (N, P),\) and \((N, C)\) are easy to handle: We naturally assume that teams always favor \(P\) and \(C\) over \(N\), giving us \(S(P, N) = (P, 0), S(C, N) = (C, 0), S(N, P) = (0, P), S(N, C) = (0, C)\), irrespective of given strategy. For \((C, C)\), it is always better to catch the opponent’s piece that is located closer to the destination and thus \(S(C, C) = (C, 0)\) or \((0, C)\) depending on the locations of opponent’s pieces. From the above consideration, the only cases to be taken carefully are \((P, C), (C, P),\) and \((N, N)\). Among these three, \((N, N)\) is not related with the choice between catching and piling, and thus we fix the strategy outcome for this: We move the piece which has bigger chance to be caught by the opponent team. Suppose that the distance to the opponent team’s piece behind is \(d\). We put the highest priority for the piece if \(d = 2\) or 3, and the second highest priority for \(d = 1\), and then for \(d = 4\) (see Fig. 2. If \(d \geq 5\), we move the piece that is behind of the other piece.

In the present work, we aim to find which strategy between catching and piling gives bigger chance to win, and thus consider two different strategies denoted by \(S_C\) and \(S_P\). The former prefers catching and the latter piling. Accordingly, \(S_C(P, C) = (C, 0)\) and \(S_P(P, C) = (0, C)\), whereas \(S_P(C, P) = (0, P)\) and \(S_P(C, C) = (P, 0)\). We also need to consider the effect of who starts first, which makes us consider all four different combinations of competition: \([S_1, S_2] = [S_C, S_C], [S_C, S_P], [S_P, S_C], [S_P, S_P]\), where \([S_1, S_2]\) is the strategy chosen by the first (the second) team. We calculate the winning rate of the first team from \(10^9\) independent runs, and report the result in Table I.

We find that if the 2nd team’s strategy is \(S_C\), the 1st team is better off if \(S_C\) is used (the top left corner of Table I). If the 2nd team is using \(S_P\) instead, the 1st team again has a higher winning rate for \(S_C\). We thus conclude that \(S_C\) always gives better chance to win regardless of the opponent’s strategy. Other seemingly interesting finding one can draw from Table I is that always the first team has bigger chance to win. In other words, being the first team to throw Yut sticks is more important than the strategy choice, within the limitation of simplification made in the present work.

For more detailed understanding of the dominance of \(S_C\), let us compare advantages of two strategies, \(S_C\) and \(S_P\), in terms of dominance. We define dominance of one team, say \(A\), against the other team, say \(B\), as

\[
D = x_1^A + x_2^A - x_1^B - x_2^B, \tag{3}
\]

due to the fact that the team \(A\) is given one more chance to throw sticks. If the team \(A\) uses the strategy \(S_P\), then the piece at \(x_1^A\) will be piled on the top of the piece at \(x_2^A\). Piling itself does not contribute to the increment of \(D\) at this turn. Instead, the team \(A\) will benefit from the piling at later turns unless they are caught by the piece of team \(B\) behind them. Thus, we get \(\Delta D_P = m + \Phi\), where \(\epsilon\) is the expected number of free movement of the piled piece. The condition for the dominance of \(S_C\), i.e. \(\Delta D_C > \Delta D_P\), reads \(x_2^A > (c-1)\Phi\). This inequality holds only for sufficiently large \(x_2^A\) because larger \(x_2^A\) leads to both larger \(x_2^B\), which is \(x_2^A + m\), and smaller \(\epsilon\). Precisely, the expectation of \(m\) is \(\Phi\), and one gets \(c\Phi = 20 - x_2^A\). If the team \(A\) is not interrupted by the team \(B\) after piling, resulting in the condition as \(x_2^A > 10 - \Phi \approx 7\) for \(p \approx 0.6\). Despite of the oversimplification, our argument gives a hint for understanding the results.

To conclude the present section, in the present numerical experiment of strategy competition, an aggressive team who starts the game first has a higher chance to win.

### IV. AVALANCHE-SIZE DISTRIBUTION

Suppose that there are many pieces on the board. If the density of opponent team’s pieces is very high, your piece can easily catch those pieces, giving you one more chance to throw Yut sticks and thus to catch more. Accordingly, one can expect that if the density of opponent team’s pieces is high, the density will soon be reduced. On the other hand, if the density is low, it will gradually increase because most of pieces will survive. The above reasoning suggests a possibility that the board configuration can be self-organized to some state, which might be critical (or not).

We construct a simple model of the one-dimensional game with infinite number of teams and infinite number of sites. We restrict each team to have only one piece, and the piece can move only one (at probability \(p\)) or two (at probability \(1-p\)) steps. If the piece is decided to move two steps at probability \(1-p\), or if it catches other piece, the team has one more chance to move. This simplified
model can be considered as Yut-Nori with one Yut stick, with only Do and Mo. Of course, differently from the original Yut-Nori, Mo with no belly sided stick gives a team two steps, instead of five steps, in this simplified version of the game.

The number of pieces on the board varies in time. It first increases with fluctuation [see Fig. 5(a)], and it sometimes decreases drastically [see Fig. 5(b)]. Avalanche size $s$ is defined as the number of caught pieces removed from the board in one turn, and we display the avalanche-size distribution (see Ref. [2] for a general discussion) in Fig. 6. Irrespective of $p$, the avalanche-size distribution is shown to follow an exponential form $f(s; p) \sim e^{-s/\xi_p}$ with $\xi_p$ decreasing with $p$.

For $p = 1$, the process becomes trivially deterministic, which allows us to get $f(s; p = 1)$ analytically. The first player starts and put her piece at the first site. We represent the configuration of the board as $\cdots 000000001$ (the first site is the rightmost one), which we denote as binary digit 1. Likewise, after the second turn we get the board $\cdots 00000010 = 10$, and 11 for the third turn, 100 for the fourth turn, 101 for the fifth turn, all in the binary format. In this binary notation, the board configuration after the $m$th turn is written as a binary number which is simply $m$ in decimal format. Consequently, in order to occupy all sites up to $n$th site, one needs $2^n - 1$ turns since the next player at the $2^n$-th turn will have her piece at the $(n + 1)$-th site removing all $n$ pieces on its way. In other words, the avalanche of the size $s = n$ occurs at time $t = 2^n$. Avalanches that occur in the time interval $t \in [2^{n-1} + 1, 2^n - 1]$ are exactly identical to the avalanches in $t \in [0, 2^{n-1} - 1]$. It is interesting to see that the above process is similar to the game of Hanoi towers, in which one is asked to move all disks to other pole within the constraint that a smaller disk must be put on a larger disk. For $t < 2^n$, the largest avalanche was of the size $n - 1$ and it occurred once. The second largest avalanche size was $n - 2$ and it occurred twice, and the avalanche of the size $n - 3$ occurred four times, and so on. Consequently, the distribution of the avalanche sizes $s$ is written as

$$f(s, p = 1) \sim 2^{-s}. \quad (4)$$

In Fig. 6, it is seen that for $p \geq 0.8$, the avalanche-size distribution almost overlaps with $2^{-s}$ in Eq. (4). We also present the avalanche-size distribution obtained for the simplified Yut-Nori (like the original Yut-Nori on infinitely long one-dimensional board). We assume that there is only one Yut stick, and if the stick shows belly side up (at probability $p$) the piece moves one step. Otherwise, for the back side, the piece is moved two steps and the team is given one more chance to play, like Mo in the original game. We also assume that each team is given one piece and the number of teams is not limited, which means that every piece on the board is put by a different team. For any value of the probability $p$ for the belly side, the avalanche-size distribution clearly follows an exponential function:

$$f(s; p) \sim e^{-s/\xi_p}$$

with $\xi_p$ decreasing with $p$. At $p = 1$, $f(s; p = 1) \sim 2^{-s}$, as explained in the text. We also include the result (triangles) for the four stick game (like the original Yut-Nori), which lies almost on top of curves for $p = 1$ and $p = 0.8$.
occur at probability about 0.15 in total, which presumably explains the agreement with the one stick game at \( p \approx 0.85 \), as shown in Fig. 6, since Yut and Mo give players a chance to rethrow sticks.

It has been well known that one-dimension Abelian sandpile models do not show self-organized criticality \[3\] whereas the avalanche-size distribution of the non-Abelian model follows a power law form \[4\]. Our Yut-Nori model is close to the one-dimensional non-Abelian model \[4\]. However, the avalanche-size distribution of our model does not follow the power law because dissipation of pieces can occur at any place in the board. In the presence of severe dissipation of pieces as in our model, the average avalanche size was shown not to diverge and the distribution has a cutoff \[2\], which explains the absence of the self-organized criticality in our model.

V. SUMMARY

One-dimensional version of the Korean game of Yut-Nori has been investigated. It has been shown that the probability \( p \) for one Yut stick to show the flat belly side up must be close to 0.6. When Yut-Nori is played, there can be two competing ways to decide how to move pieces on the game board Yut-Pan: A team can either prefer piling, or catching. By performing numerical simulations of huge number of simplified virtual Yut-Nori games, we have shown that the more aggressive strategy that favors catching yields a higher winning rate. It has been also found that the team who starts first has a higher chance to win. The avalanche-size distribution has been investigated and has been shown not to exhibit self-organized criticality. The exponential form of the distribution has been understood from the consideration of the limiting case that Yut stick always shows belly side up.

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