Regularity of $\infty$ for Elliptic Equations with Measurable Coefficients and Its Consequences

UGUR G. ABDULLA

DEPARTMENT OF MATHEMATICS, FLORIDA INSTITUTE OF TECHNOLOGY, MELBOURNE, FLORIDA 32901

Abstract. This paper introduces a notion of regularity (or irregularity) of the point at infinity ($\infty$) for the unbounded open set $\Omega \subset \mathbb{R}^N$ concerning second order uniformly elliptic equations with bounded and measurable coefficients, according as whether the $A$-harmonic measure of $\infty$ is zero (or positive). A necessary and sufficient condition for the existence of a unique bounded solution to the Dirichlet problem in an arbitrary open set of $\mathbb{R}^N$, $N \geq 3$ is established in terms of the Wiener test for the regularity of $\infty$. It coincides with the Wiener test for the regularity of $\infty$ in the case of Laplace equation. From the topological point of view, the Wiener test at $\infty$ presents thinness criteria of sets near $\infty$ in fine topology. Precisely, the open set is a deleted neighborhood of $\infty$ in fine topology if and only if $\infty$ is irregular.

Key words: uniformly elliptic equations, metric compactification of $\mathbb{R}^{N+1}$, measurable coefficients, regularity (or irregularity) of $\infty$, $A$-harmonic measure, Dirichlet problem, PWB solution, $A$-super- or subharmonicity, Newtonian capacity, Wiener test, fine topology, $A$-thinness, Brownian motion, characteristic Markov process, differential generator

AMS subject classifications: 35J25, 31C05, 31C15, 31C40, 60J45, 60J60
1 Description of Main Results

1.1 Introduction and Motivation

This paper introduces the notion of regularity of the point at infinity ($\infty$) and establishes a necessary and sufficient condition for the unique solvability of the Dirichlet problem (DP) in an arbitrary open set of $\mathbb{R}^N$ for the uniformly elliptic equations in divergence form

$$\mathcal{A}u = -(a_{ij}(x)u_{x_i})_{x_j} = 0$$  \hspace{1cm} (1.1)

when the coefficients and boundary values are only supposed to be bounded and measurable. The criterion is the same as Wiener test for the regularity of $\infty$ concerning the classical DP for harmonic functions in an arbitrary open set of $\mathbb{R}^N$ [1].

In order to formulate our result, we first introduce some terminology. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) denote any unbounded open subset, and $\partial \Omega$ its topological boundary. We consider the differential operator $\mathcal{A}u$, with $a_{ij} = a_{ji}$ being real bounded measurable functions defined in $\Omega$.

Throughout this paper, we use the summation convention and assume that $\mathcal{A}$ is uniformly elliptic. That is, there is a constant $\lambda \geq 1$, such that

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$  \hspace{1cm} (1.2)

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$. We will assume that the coefficients $a_{ij}$ are defined and satisfy (1.2) for all $x \in \mathbb{R}^N$. This can always be achieved by putting $a_{ij} = \delta_{ij}$ outside of $\Omega$.

Throughout, we use the standard notation of Sobolev spaces [4]. A function $u$ in $H^{1,2}_{loc}(\Omega)$ is a weak solution of the equation (1.1) in $\Omega$ if

$$\int_{\Omega} a_{ij}u_{x_i}\phi_{x_j}dx = 0$$  \hspace{1cm} (1.3)

whenever $\phi \in C_0^\infty(\Omega)$. A function $u$ in $H^{1,2}_{loc}(\Omega)$ is a supersolution of (1.1) in $\Omega$ if

$$\int_{\Omega} a_{ij}u_{x_i}\phi_{x_j}dx \geq 0$$  \hspace{1cm} (1.4)

whenever $\phi \in C_0^\infty(\Omega)$ is nonnegative. A function $u$ is a subsolution of (1.1) in $\Omega$ if $-u$ is a supersolution of (1.1).

The weak solution of (1.1) is locally Hölder continuous [6, 21, 20]. Continuous weak solution of (1.1) in $\Omega$ is called $\mathcal{A}$-harmonic in $\Omega$.

A function $u$ is called a $\mathcal{A}$-superharmonic in $\Omega$ if it satisfies the following conditions:

(b) $u$ is lower semicontinuous (l.s.c.);

(c) for each open $U \Subset \Omega$ and each $\mathcal{A}$-harmonic $h \in C(\overline{\Omega})$, the inequality $u \geq h$ on $\partial U$ implies $u \geq h$ in $U$. 


We use the notation $S(\Omega)$ for a class of all $A$-superharmonic functions in $\Omega$. Similarly, $u$ is $A$-subharmonic in $\Omega$ if $-u$ is $A$-superharmonic in $\Omega$; the class of all $A$-subharmonic functions in $\Omega$ is $-S(\Omega)$.

It is well known (\cite{17, 9}) that $u \in H^{1,2}_{loc}$ is $A$-superharmonic if and only if it is a supersolution with

$$u(x) = \text{ess} \liminf_{y \to x} u(y) \quad \text{for all } x \in \Omega. \quad (1.5)$$

Moreover, in the $L^{1}$-equivalence class of every supersolution, there is an $A$-superharmonic representative which satisfies (1.5).

Given boundary function $f$ on $\partial \Omega$, consider a Dirichlet problem (DP)

$$Au = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial \Omega \quad (1.6)$$

Consider a compactified space $\overline{\mathbb{R}^N} = \mathbb{R}^N \cup \{\infty\}$. Denote by $\Omega$ and $\partial \Omega$ respectively, the metric compactification of $\Omega$ and $\partial \Omega$. Throughout, we always assume that $\infty \in \partial \Omega$. For example, if $\Omega$ is an exterior of compact, then $\infty$ is an isolated boundary point of $\Omega$.

Assuming for a moment that $\partial \Omega$ is non-compact, $f \in C(\partial \Omega)$, and $f$ has a limit $f_\infty$ at infinity, prescribe $f(\infty) = f_\infty$. The generalized upper (or lower) solution (in the sense of Perron-Wiener-Brelot) of the DP is defined as

$$\Pi^\Omega_f \equiv \inf\{u \in S(\Omega) : \liminf_{x \to y, x \in \Omega} u \geq f(y) \quad \text{for all } y \in \partial \Omega\} \quad (1.7)$$

$$H^\Omega_f \equiv \sup\{u \in -S(\Omega) : \limsup_{x \to y, x \in \Omega} u \leq f(y) \quad \text{for all } y \in \partial \Omega\} \quad (1.8)$$

According to classical theory (\cite{7, 9}), $f$ is a resolutive boundary function, in the sense that

$$\overline{H}^\Omega_f \equiv \Pi^\Omega_f \equiv H^\Omega_f.$$

Being $A$-harmonic in $\Omega$, $H^\Omega_f$ is called a generalized solution of the DP (1.5).

The generalized solution is unique by construction, and it coincides with the classical, or Hilbert, or Sobolev space solution whenever the latter exists. Also, note that the construction of the generalized solution is accomplished by prescribing the behavior of the solution at $\infty$. The elegant theory, while identifying a class of unique solvability, leaves the following questions open:

- Would a unique solution still exist if its limit at infinity were not specified? That is, could it be that the solutions would pick up the “boundary value” $f_\infty$ without being required?

- What if the boundary datum $f$ on $\partial \Omega$, while being continuous, does not have a limit at infinity, for example, it exhibits bounded oscillations. Is the DP uniquely solvable?

The answer to these fundamental questions depends on whether $\Omega$ is sufficiently sparse, or equivalently $\Omega^c$ is sufficiently thin near $\infty$. In recent papers \cite{1, 2, 3} the answer is expressed in terms of the Wiener test for the regularity of $\infty$ for Laplace and heat equations. The principal purpose of this paper is
to prove that the Wiener test for the regularity of $\infty$, and accordingly for the
uniqueness of the bounded solution of the DP, is the same as that for Laplace
equation. This is the result in the spirit of [17] concerning the regularity of
finite boundary points.

1.2 Formulation of Problems and Main Result

Furthermore, we assume that $f : \partial \Omega \rightarrow \mathbb{R}$ is a bounded Borel measurable
function. Bounded Borel measurable functions are resolutive [7]. Concerning
$\Omega$, we don’t exclude the case when $\partial \Omega$ is compact. Without loss of generality,
we only assume that $\Omega$ has at least one connected component $\Omega_e$ such that
$\infty \in \partial \Omega_e$. Obviously if this assumption is not satisfied, then the unbounded
open set $\Omega$ is a union of (at most countable) connected bounded components.
In this case, within each bounded connected component, the solution is defined
uniquely from boundary values on the boundary of this component. This
makes the problem uniquely solvable in the whole region without prescribing
the boundary function at $\infty$.

By fixing an arbitrary finite real number $f_0$, extend a function $f$ as $f(\infty) = f_0$.
Obviously, the extended function is a bounded Borel measurable on $\partial \Omega$.
Since bounded Borel measurable functions are resolutive ([7]), there exists a
unique bounded generalized solution $H_f^\Omega$. It is natural to call it a generalized
solution of the DP (1.6). The major question now becomes:

Problem 1: How many bounded solutions do we actually have, or does the
constructed solution depend on $f$ ?

Note that if in particular, $\partial \Omega$ is non-compact and $f$ has a limit $f_0$ at $\infty$, the
above constructed generalized solution is included here by choosing $f = f_0$.

It is well possible that $H_f^\Omega$ does not take continuously on the boundary
values prescribed by $f$ at the finite boundary points. Therefore, the finite
boundary point $w \in \partial \Omega$ is called regular for $\Omega$ if

$$\lim_{z \rightarrow w, z \in \Omega} H_f^\Omega(z) = f(w) \quad \text{for all bounded } f \in C(\partial \Omega). \quad (1.9)$$

The regularity of a boundary point is a problem of local nature, and depends
on the measure-geometric properties of the boundary in a neighborhood of the
point, which is in turn dictated by the differential operator. In his pioneering
works [23, 24] Wiener discovered the necessary and sufficient condition for
the regularity of finite boundary points for harmonic functions. Remarkably,
the regularity criteria for finite boundary points with respect to the elliptic
equation (1.1) is the same [17].

For a given set $A$ from $\mathbb{R}^N$, denote as $1_A$ the indicator function of $A$. Since
the indicator function $1_\infty$ is a resolutive boundary function, the $A$-harmonic
measure of $\infty$ is well defined ([7, 9]):

$$\mu_\Omega(\cdot, \infty) = H_f^{\Omega}(\cdot).$$
It is said that \( \infty \) is an \( \mathcal{A} \)-harmonic measure null set if \( \mu_\Omega(z, \infty) \) vanishes identically in \( \Omega \). If this is not the case, \( \infty \) is a set of positive \( \mathcal{A} \)-harmonic measure. Since \( H^\Omega_\infty \) is \( \mathcal{A} \)-harmonic in \( \Omega \), from the strong maximum principle it follows that if \( \mu_\Omega(x, \infty) > 0 \), then it is positive in the whole connected component of \( \Omega \) which contains \( x \). We can now formulate the measure-theoretical counterpart of the Problem 1:

**Problem 2:** Given \( \Omega \), is the \( \mathcal{A} \)-harmonic measure of \( \infty \) null or positive?

Note that the assumption \( \infty \in \Omega \) doesn’t cause a loss of generality in this context, and \( \infty \) is an \( \mathcal{A} \)-harmonic measure null set otherwise.

In fact, both major problems are equivalent, and the next definition expresses the connection between them.

**Definition 1.1.** \( \infty \) is said to be **regular** for \( \Omega \) if it is an \( \mathcal{A} \)-harmonic measure null set. Conversely, \( \infty \) is **irregular** if it has a positive \( \mathcal{A} \)-harmonic measure.

The notion of the regularity of \( \infty \) is, in particular, related to the notion of continuity of the solution at \( \infty \).

**Problem 3:** Given \( \Omega \) with non-compact \( \partial \Omega \), whether or not

\[
\liminf f \leq \liminf(H^\Omega_f) \leq \limsup(H^\Omega_f) \leq \limsup f
\]

as \( z \to \infty \) for all bounded \( f \in C(\partial \Omega) \). (1.10)

Note that if \( f \) has a limit at \( \infty \), (1.10) simply means that the solution \( H^\Omega_f \) is continuous at \( \infty \).

The notion of the regularity of \( \infty \) introduced in Definition 1.1 and earlier in [1, 2, 3] fits naturally in the framework of \( \mathcal{A} \)-fine topology. \( \mathcal{A} \)-fine topology is the coarsest topology of \( \mathbb{R}^N \) which makes every \( \mathcal{A} \)-superharmonic function continuous [7, 9]. \( \mathcal{A} \)-fine topology is finer than the Euclidean topology. A major problem is to find the structure of the neighborhood base in \( \mathcal{A} \)-fine topology. It is well-known that there is an elegant connection between this problem and the problem on the regularity of finite boundary points. Namely, given open set \( \Omega \subset \mathbb{R}^N \), its finite boundary point \( x_0 \) is irregular if and only if \( \Omega \) is a deleted neighborhood of \( x_0 \) in fine topology [7, 15, 9]. Our definition of the regularity of \( \infty \) reveals a similar connection for the point at infinity.

Let \( u : \mathbb{R}^N \to (-\infty, +\infty] \) be an arbitrary \( \mathcal{A} \)-superharmonic function. Extend it to \( \mathbb{R}^N \) as follows:

\[
u(\infty) = \liminf_{x \to \infty} u(x).
\]

What is the coarsest topology in \( \mathbb{R}^N \) which makes every extended \( \mathcal{A} \)-superharmonic function continuous? How should \( \mathcal{A} \)-fine topology of \( \mathbb{R}^N \) be extended to \( \infty \) with “minimum increase” of the Euclidean neighborhood base of \( \infty \), to guarantee continuity at \( \infty \) of every extended \( \mathcal{A} \)-superharmonic function? The
following definition is helpful in understanding the structure of the neighborhood base of \( \infty \) in \( \mathcal{A} \)-fine topology.

**Definition 1.2.** A subset \( E \) of \( \mathbb{R}^N \) is called \( \mathcal{A} \)-thin at \( \infty \) in the following two cases:

(a) \( E \) is bounded

(b) \( E \) is unbounded and there exists an \( \mathcal{A} \)-superharmonic function \( u \) in \( \mathbb{R}^N \) such that

\[
 u(\infty) < \liminf_{x \to \infty, x \in E} u(x). \tag{1.11}
\]

A set \( E \) is \( \mathcal{A} \)-thin at \( \infty \) if and only if \( E^c \) is a deleted \( \mathcal{A} \)-fine neighborhood of \( \infty \) (see Lemma 2.4). Furthermore, the \( \mathcal{A} \)-fine derived set of \( E \), that is, the \( \mathcal{A} \)-fine closed set of \( \mathcal{A} \)-fine limit points of \( E \), will be denoted by \( E^f \). Hence, \( E \) is \( \mathcal{A} \)-thin at \( \infty \) if and only if \( \infty \not\in E^f \). As in the case of the finite points, \( \mathcal{A} \)-fine topology is strictly finer than Euclidean topology near \( \infty \). The exterior of any compact set is a deleted neighborhood of \( \infty \) both in \( \mathcal{A} \)-fine and Euclidean topology. However, in \( \mathcal{A} \)-fine topology, there are deleted neighborhoods of \( \infty \), which are unbounded open sets with non-compact boundaries.

We can now formulate the topological counterpart of problems 1, 2 and 3:

**Problem 4:** Is given open set \( \Omega \) a deleted neighborhood of \( \infty \) in \( \mathcal{A} \)-fine topology? Conversely, is \( \Omega^c \) \( \mathcal{A} \)-thin at \( \infty \)?

The principal result of this paper expresses the solutions to equivalent Problems 1-4 in terms of the Wiener test for the regularity of \( \infty \). Recall that if \( K \subset \mathbb{R}^N \) is compact, the Newtonian capacity of \( K \) is

\[
 \text{cap}(K) \equiv \sup\{\mu(\mathbb{R}^N) : \mu \in \mathcal{M}(K), V_\mu \leq 1 \text{ in } \mathbb{R}^N\},
\]

where \( \mathcal{M}(K) \) denotes the collection of all nonnegative Radon measures on \( \mathbb{R}^N \) with support in \( K \), and

\[
 V_\mu(x) \equiv \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x - y|^{N-2}}, \quad x \in \mathbb{R}^N
\]

is the Newtonian potential of \( \mu \). Let

\[
 \Gamma_n \equiv \text{cap}(E_n), \quad E_n = \Omega^c \cap \{x : 2^{n-1} \leq |x| \leq 2^n\}
\]

Our main theorem reads:

**Theorem 1.1** The following conditions are equivalent:

1. \( \infty \) is regular (or irregular)
2. DP has a unique (or infinitely many) bounded solution(s)
3. If \( \partial \Omega \) is non-compact, \( (1.10) \) is satisfied (respectively isn’t satisfied).
Wiener series

\[ \sum_{n} 2^{-n(N-2)} \Gamma_n \]  

(1.12)

diverges or converges.

\[ \infty \in (\Omega^c)^{\mathcal{F}} \text{ (or } \Omega^c \text{ is } \mathcal{A}-\text{thin at } \infty) \]

An equivalent characterization is valid if one replaces (1.12) with the series

\[ \sum_{n} \lambda^{-n} \gamma_n \]

where \( \lambda > 1, \gamma_n = \text{cap}(e_n), e_n = \Omega^c \cap \{x : \lambda^{-n} \leq \Phi(x) \leq \lambda^{-n+1}\} \) and

\[ \Phi(x) = \frac{\Gamma(N/2)}{(N-2)(4\pi)^{n/2}} |x|^{2-N} \]

is the fundamental solution of the Laplace equation. Another equivalent characterization is valid if one replaces the series (1.12) with the Wiener integral

\[ \int_{1}^{+\infty} \frac{c(\rho)}{\rho^{N-1}} d\rho \]

where

\[ c(\rho) \equiv \text{cap}(\Omega^c \cap \{x : |x| \leq \rho\}). \]

**Counterpart at the finite boundary point:** In a paper [22] a generalization of the well-known Kelvin transformation for Laplace equation was introduced for the equation (1.1). This transformation allows to transform the uniformly elliptic equation (1.1) near \( \infty \) to another uniformly elliptic equation near finite boundary point \( x_0 \) which is the inversion of \( \infty \). Moreover, \( \mathcal{A} \)-harmonic functions which are bounded near \( \infty \) will be transformed to \( \overline{\mathcal{A}} \)-harmonic functions of class

\[ u(x) = O(|x - x_0|^{2-N}), \quad \text{as } x \to x_0 \]  

(1.13)

where \( \overline{\mathcal{A}} \) is an elliptic operator of type (1.1), (1.2). Counterpart of the **Problem 1** is whether or not singular Dirichlet problem in a bounded open set \( \Omega \), with \( x_0 \in \partial \Omega \), is uniquely solvable in a class (1.13). Otherwise speaking, whether or not fundamental solution kind singularity is removable in a non-isolated boundary point \( x_0 \). Since divergence or convergence of the Wiener series is preserved under the inversion transformation, Theorem 1.1 implies that the classical Wiener test at \( x_0 \) is a necessary and sufficient condition for uniqueness in a singular Dirichlet problem, as well as for the removability of the singularity according to (1.13).

**Probabilistic Counterpart:** From the probabilistic standpoint, Wiener test of Theorem 1.1 presents an asymptotic probability law for Markov processes in \( \mathbb{R}^N \). Assume that \( \{X_t\} \) is a continuous time, time-homogeneous Markov process with infinitesimal Dynkin generator being a differential operator \( -\mathcal{A}[S] \).
Conversely, $X_t$ is the characteristic process for the differential operator $-\mathcal{A}$. The characteristic process of the operator $-\mathcal{A} \equiv \frac{1}{2} \Delta$ is the Wiener process or Brownian motion. Assume that, $B$ is a closed set in $\mathbb{R}^N$ clustering to $\infty$, and $B_n$ is the intersection of $B$ with the spherical shell $2^{n-1} \leq |x| < 2^n$. Let $\mathbf{B}$ be an event that \{$t : X_t \in B$\} clusters to $+\infty$. Then

$$P_\bullet(\mathbf{B}) = 0 \quad \text{or} \quad 1 \quad \text{according as} \quad \sum_n 2^{-n(N-2)} \text{cap}(B_n) < \quad \text{or} \quad = +\infty.$$ 

The same result for an $N$-dimensional standard random walk on the $N$-dimensional lattice of points with integer coordinates was proved in [12]. In [13] (p.257) the law is mentioned for standard $N$-dimensional Brownian motion.

### 1.3 Historical Comments

It will be convenient to make some remarks concerning the Dirichlet problem for uniformly elliptic equations. The solvability, in some generalized sense, of the classical DP in bounded open set $E \subset \mathbb{R}^N$, with prescribed data on $\partial E$, is realized within the class of resolutive boundary functions, identified by Perron’s method and its Wiener [23, 24] and Brelot [5] refinements. Such a method is referred to as the PWB method, and the corresponding solutions are PWB solutions. The main tool of the PWB method consists of Harnack estimates for the Laplace equation.

Wiener, in his pioneering works [23, 24], proved a necessary and sufficient condition for the finite boundary point $x_\circ \in \partial E$ to be regular in terms of the “thinness” of the complementary set in the neighborhood of $x_\circ$. Cartan pointed out that the thinness could be characterized by means of fine topology – the coarsest topology of $\mathbb{R}^N$ which makes every superharmonic function continuous. In fact, a finite boundary point is irregular if and only if $E$ is a deleted neighborhood of $x_\circ$ in fine topology [7].

De Giorgi [6] and Nash [21] almost simultaneously proved that any local weak solution of (1.1) is locally Hölder continuous. Moser [20] gave a simpler proof of this fact as well as the Harnack inequality. In [17] it is proved that the Wiener test for the regularity of finite boundary points with respect to elliptic operator (1.1) coincides with the classical Wiener test for the boundary regularity of harmonic functions. Hence, the fine-topological neighborhood base of the finite boundary point is independent of elliptic operator (1.1). The Wiener test for the regularity of finite boundary points for linear degenerate elliptic equations is proved in [10]. Wiener test for the regularity of finite boundary points for quasilinear elliptic equations was settled due to [19] [11]. [10]. Nonlinear potential theory was developed along the same lines as classical potential theory for the Laplace operator and we refer to monographs [17] [18].

Despite the importance of the Wiener criterion in Analysis, its meaning for the point at infinity was not correctly understood. In the framework of Brelot’s theory and its generalizations, the regularity of $\infty$ was associated with the existence of the solution to the Dirichlet problem with the same limit at
∞ as the boundary function. This approach implied that ∞ is always regular if \( N \geq 3 \), and accordingly, the geometric nature of the Wiener criterion was ignored. “Labeling” of ∞ as “always regular” became a standard result in classical and modern potential theory. The deficiency of this approach is clear in the context of fine topology. The connection mentioned above between the irregularity of the boundary point and fine topological thinness falls apart for the point at infinity, since otherwise fine topology would be trivial at ∞. Another deficiency comes out in the context of the asymptotic properties of Markov processes. A well-known elegant connection between the irregularity of boundary points for the Laplacian and non-escaping property of Wiener processes starting at boundary point is ignored for the point ∞.

In [1, 2, 3] we introduced a correct notion of regularity of the point at infinity on \( \partial E \) for the Laplace and heat equations. Basically, the Dirichlet problem for \( E \) with continuous data \( \phi \), has either one and only one bounded solution, or infinitely many. If the DP has a unique solution, the point at infinity on \( \partial E \) is regular, otherwise it is irregular. In [1] we characterize the regularity of ∞ through the Wiener test at ∞. The principal result of this paper proves that the Wiener test for the regularity of ∞ is independent of elliptic operators (1.1) with bounded measurable coefficients. The Wiener test at ∞ characterizes fine topological thinness at ∞. It also characterizes the asymptotic properties of the characteristic Markov processes with differential generator \(-A\).

2 Preliminary Results on Potential Theory

This section formulates basic known facts about \( A \)-capacity, \( A \) potentials and Green functions for differential operator (1.1) which we need to prove the main Theorem 1.1. Lemmas 2.1–2.3 are due to [17]. In Lemma 2.4 we formulate \( A \)-thinness criteria at ∞ in \( A \)-fine topology.

Let \( \Sigma \) be a fixed open sphere and \( E \) be a compact subset of \( \Sigma \). Then the \( A \)-capacity of \( E \) with respect to operator \( A \) and sphere \( \Sigma \) is defined as

\[
\text{cap}_A(E) = \inf \left\{ D_A : \phi \in H_0^{1,2}(\Sigma), \phi \geq 1 \text{ on } E \text{ in the sense of } H_0^{1,2}(\Sigma) \right\}
\]

where

\[
D_A = \int_{\Sigma} a_{ij} \phi_x^i \phi_x^j dx
\]

The function \( u \) giving the infimum to \( D_A(\phi) \) is called \( A \)-capacitary potential (with respect to \( A \) and \( \Sigma \)).

**Lemma 2.1**

1. There exists one and only one \( A \)-capacitary potential \( u \in H_0^{1,2}(\Sigma) \) such that \( u = 1 \text{ on } E \text{ in the sense of } H_0^{1,2}(\Sigma) \) and \( \text{cap}_A(E) = D_A(u) \).

2. \( A \)-capacitary potential is \( A \)-supersolution in \( \Sigma \), and \( A \)-harmonic in \( \Sigma - E \).

For any Radon measure \( \mu \) with compact support in \( \Sigma \), consider a problem

\[ A u = \mu \text{ in } \Sigma, \quad u = 0 \text{ on } \partial \Sigma. \tag{2.1} \]
$u \in L^1(\Sigma)$ is a weak solution of (2.1) if
\[ \int_\Sigma u \mathcal{A} \Phi dx = \int_\Sigma \Phi d\mu \]
for every $\Phi \in H^{1,2}_0(\Sigma) \cap C(\Sigma)$ such that $\mathcal{A} \Phi \in C(\Sigma)$.

**Lemma 2.2**
1. There exists a unique weak solution $u$ of (2.1) such that $u \in H^{1,p}_0(\Sigma)$ for every $p < n/(n-1)$.
2. Weak solution $u \in H^{1,2}_0(\Sigma)$, if and only if $\mu \in H^{-1,2}(\Sigma)$.

The Green’s function $g(x,y)$ of the operator $\mathcal{A}$ on $\Sigma$ is defined as the weak solution of the problem (2.1) with $\mu = \delta_y$, where $\delta_y$ is the Dirac measure of $y$.

**Lemma 2.3**
1. For every Radon measure with compact support in $\Sigma$, the integral
\[ u(x) = \int_\Sigma g(x,y) d\mu(y). \] (2.2)
exists and finite a.e., and is a weak solution of (2.1).
2. $\mathcal{A}$-capacitary potential is a weak solution of (2.1) with the nonnegative ($\mathcal{A}$-capacitary) measure supported on the exterior boundary of compact $E$, and accordingly the representation (2.3) is valid. Moreover, $\mu(E) = \text{cap}_\mathcal{A}(E)$.
3. Let $g$ and $\overline{g}$ be the Green functions for any uniformly elliptic operators $\mathcal{A}$ and $\overline{\mathcal{A}}$ with the ellipticity constant $\lambda$ on a sphere $\Sigma$. Then, for any compact subset $E$ of $\Sigma$, there exists a constant $K$ depending on $E, \Sigma$ and $\lambda$ such that
\[ K^{-1} \overline{g}(x,y) \leq g(x,y) \leq K \overline{g}(x,y), \text{ for all } x, y \in \Sigma. \] (2.3)
4. If $\mu$ is a nonnegative Radon measure with compact support $E$ in $\Sigma$, and $u$ and $\overline{u}$ are the weak solutions of (2.1) for differential operators $\mathcal{A}$ and $\overline{\mathcal{A}}$ respectively, then
\[ \lambda^{-2} \text{cap}_{\mathcal{A}}(E) \leq \text{cap}_{\mathcal{A}}(E) \leq \lambda^2 \text{cap}_{\mathcal{A}}(E) \] (2.4)
\[ K^{-1} \pi(x) \leq u(x) \leq K \pi(x), \text{ a.e. on } E \] (2.5)
If $\mu \in H^{-1,2}(\Sigma)$, then (2.5) is valid in the sense of $H^{1,2}_0(\Sigma)$.
5. $\text{cap}_{\mathcal{A}}(\cdot)$ is a monotone, subadditive set function; it is a Choquet capacity ([7, 2]) and for any differential operators $\mathcal{A}$ and $\overline{\mathcal{A}}$, $\text{cap}_{\mathcal{A}}(\cdot)$ and $\text{cap}_{\overline{\mathcal{A}}}(\cdot)$ are mutually absolutely continuous (see (2.4)).
6. In the $L^1$-equivalence class of every capacitary potential, there is an $\mathcal{A}$-superharmonic representative which satisfies (1.3).
7. $g(\cdot, y) \geq 0$ is $\mathcal{A}$-harmonic and Hölder continuous in $\Sigma - y$, $\lim_{x \to y} g(x, y) = +\infty$. 
If $\mathcal{A}$ is taken to be the Laplace operator, then from (2.5) it follows that for any compact subset $E$ of the open sphere $\Sigma$,

$$K^{-1}|x - y|^{2-N} \leq g(x, y) \leq K|x - y|^{2-N} \quad \text{for all } x, y \in E \quad (2.6)$$

where $K$ depends on $E, \Sigma, \lambda$ and $R$ is a radius of $\Sigma$.

If $N \geq 3$ and the radius of $\Sigma$ goes to infinity, the Green function increases to a Green function $G(x, y)$ of $\mathbb{R}^N$, which is locally Hölder continuous and $\mathcal{A}$-harmonic in $\mathbb{R}^N - y$. In fact, $G(\cdot, y) \in H^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap H^{1,2}_{\text{loc}}(\mathbb{R}^N - y)$, $p < n/(n-1)$. Moreover, we have

$$K^{-1}|x - y|^{2-N} \leq G(x, y) \leq K|x - y|^{2-N}, \quad (2.7)$$

uniformly in $\mathbb{R}^N$. Accordingly, $\mathcal{A}$-capacity and related $\mathcal{A}$-capacitary potential are well defined if $\Sigma$ and $g(x, t)$ are replaced with $\mathbb{R}^N$ and $G(x, t)$ respectively.

In particular, the $\mathcal{A}$-capacitary potential $\zeta_E$ of the compact subset $E \subset \mathbb{R}^N$ is a weak solution of the problem (2.1) with $\Sigma$ replaced by $\mathbb{R}^N$. Accordingly, we have

$$\zeta_E(x) = \int_{\mathbb{R}^N} G(x, y) d\mu(y). \quad (2.8)$$

where $\mu$ is a $\mathcal{A}$-capacitary measure with support on the exterior boundary of $E$ with $\mu(\mathbb{R}^N) = \text{cap}_\mathcal{A}(E)$. As before, in the $L_1$-equivalence class, there is a lower semicontinuous representative which satisfies (1.5), and it is $\mathcal{A}$-superharmonic in $\mathbb{R}^N$. Using different terminology, this representative is a smoothed $\mathcal{A}$-reduction of 1 on $E$.

Furthermore, we will assume that the $\mathcal{A}$-capacity and $\mathcal{A}$-potential of the compact subsets of $\mathbb{R}^N$ are defined with respect to $\mathbb{R}^N$. We will drop subscript $\mathcal{A}$ when differential operator $\mathcal{A}$ is Laplacian. In this case $\mathcal{A}$-capacity and $\mathcal{A}$-potential coincide with Newtonian capacity and Newtonian potential. Hence, in view of (2.7) and (2.8), (2.5) becomes

$$K^{-1}V_\nu(x) \leq \zeta_E(x) \leq KV_\nu(x), \quad (2.9)$$

a.e. on $E$ and everywhere on $E^c$ with $\nu$ being the Newtonian capacitary measure of $E$.

Throughout, we will say that a property holds quasieverywhere, if it holds except on a set of $\mathcal{A}$-capacity (or Newtonian capacity) zero. Sets of capacity zero are called $\mathcal{A}$-polar sets (or simply polar sets) in potential theory. Polar sets are essential for the description of the singularities of $\mathcal{A}$-superharmonic functions. In fact, for any polar set $E \in \mathbb{R}^N$, there is an $\mathcal{A}$-superharmonic function $u$ in $\mathbb{R}^N$ such that $u = +\infty$ on $E$. We refer to [7, 9] for the essential properties of polar sets.

Let $D_e$ be the connected component of $E^c$ which contains $\infty$ and $E_e = \partial D_e$. The restriction of $\zeta_E$ to $D_e$ is $\mathcal{A}$-harmonic and solves the exterior Dirichlet problem with boundary value 1 on $E_e$ and 0 at infinity. More precisely, $\zeta_E \equiv H^E_{1,e}$. From (2.7), (2.8) it follows that it vanishes at infinity with the same rate as a fundamental solution of the Laplacian. For example, if $E = \Sigma \equiv \{x : |x| < R\}$ then

$$K^{-1}R^{N-2}|x|^{N-2} \leq \zeta_\Sigma(x) \leq KR^{N-2}|x|^{2-N} \quad \text{for } |x| > R \quad (2.10)$$

Remembering that the Newtonian capacity of $\Sigma$ is $R^{N-2}$, we also have
\[ \lambda^{-2}R^{N-2} \leq \text{cap}_A(\Sigma) \leq \lambda^2 R^{N-2} \] (2.11)

In the next lemma, we formulate the $A$-thinness criteria of sets at $\infty$.

**Lemma 2.4** (1) If the sets $E_1, E_2, \ldots, E_n$ are $A$-thin at $\infty$, then $E = \cup_{i=1}^n E_i$ is also $A$-thin at $\infty$.

(2) A set $E$ is $A$-thin at $\infty$ if and only if $E^c$ is a deleted neighborhood of $\infty$ in $A$-fine topology.

The proof of these facts is standard ([7, 15, 9]). For (1), it is easy to observe that if $v_i(x)$ is the $A$-superharmonic function related to $E_i$ by the relation (1.11), then
\[ \sum_{i=1}^n v_i(x) \]
will, by the lower semicontinuity of $v_i$, satisfy (1.11).

To prove (2), first note that $A$-fine topology is the coarsest topology where the sets of the form $\{u \geq \beta\}, \{u < \beta\}$ are open, for all $A$-superharmonic functions $u$ and for all real numbers $\beta$. The family formed by the finite intersections of these sets forms a neighborhood base of $A$-fine topology. Since $u$ is lower semicontinuous, the sets of the form $\{u \geq \beta\}$ are open in Euclidean topology. Accordingly, we can consider a neighborhood base of $\infty$ consisting of the sets
\[ \cap_{i=1}^m \{x \in \Sigma^c : u_i \leq \beta\} \] (2.12)
where $m$ is an integer, $\beta > 0$, $\Sigma$ is an open sphere in $\mathbb{R}^N$, $u_i$ is a locally bounded $A$-superharmonic function in $\mathbb{R}^N$ such that $u_i(\infty) = 0$.

Assume that $E^c$ is a deleted $A$-fine neighborhood of $\infty$. Then it contains some element of the neighborhood base:
\[ \cap_{i=1}^m \{x \in \Sigma^c : u_i \leq \beta\} \subset E^c \]
which means that
\[ E \subset \cup_{i=1}^m E_i \cup \Sigma \]
where $E_i = \{u_i > \beta\}$. By Definition 2.1, both $\Sigma$ and each of the sets $E_i$ are $A$-thin at $\infty$. By assertion (1) of Lemma 2.4, the union, and accordingly its subset $E$, is $A$-thin at $\infty$.

Assume that $E$ is $A$-thin at $\infty$. If $E$ is bounded in $\mathbb{R}^N$, then the assertion of the lemma is trivial. Assume $E$ is unbounded and choose an $A$-superharmonic function in $\mathbb{R}^N$ according to Definition 2.1 and a real number $\beta$ such that
\[ u(\infty) < \beta < \liminf_{x \to \infty, x \in E} u(x) \]
Accordingly, there is an open sphere $\Sigma$ such that
\[ u(x) \geq \beta \quad \text{for all } x \in E \cap \Sigma^c \]
This implies that $A$-fine neighborhood $\{x \in \Sigma^c : u(x) < \beta\}$ of $\infty$ is contained in $E^c$. Hence, $E^c$ is a deleted $A$-fine neighborhood of $\infty$. 

12
3 Proof of Theorem 1.1

(1) \iff (2): Assume that \( \infty \) is regular and let \( u_1 \) and \( u_2 \) be two bounded solutions of DP. Then \( v = u_1 - u_2 \) is a bounded solution of DP with zero boundary data on finite boundary points. Since a generalized solution is order preserving (1.9), we have

\[ |v| \leq H_{M,1,\infty}^\Omega \equiv MH_{1,\infty}^\Omega \equiv 0, \quad \text{with} \quad M = \sup |v|. \]

On the other hand, if \( \infty \) is irregular, then for an arbitrary real number \( r, rH_{1,\infty}^\Omega \) is a generalized solution of the DP with zero boundary data on \( \partial \Omega \).

(2) \implies (3): Assume that (3) is not satisfied. That is to say, there is a bounded function \( f \) such that for some generalized solution \( H_f^\Omega \), one of the inequalities in (1.10) is violated. In this case, by choosing a number \( \overline{f} \) satisfying

\[ f_* = \liminf_{z \to \infty} f(z) \leq \overline{f} \leq \limsup_{z \to \infty} f(z) \equiv f^*, \]

and by extending \( f(\infty) = \overline{f} \), we can always construct a generalized solution \( H_f^\Omega \) which satisfies (1.10). Indeed, since a generalized solution is order preserving, we clearly have

\[ |H_f^\Omega| \leq M \equiv \sup |f|. \]  \hspace{1cm} (3.1)

Then for an arbitrary \( \epsilon > 0 \) we choose \( R > 0 \) such that

\[ f_* - \epsilon \leq f \leq f^* + \epsilon, \quad \text{on} \quad \partial \Omega \cap \Sigma^c, \]  \hspace{1cm} (3.2)

where \( \Sigma \equiv \{ z \mid |z| < R \} \). Since capacitary potential \( \zeta_{\Sigma} \equiv H_{1,\infty}^{\Sigma \cap \Omega} \) we have

\[ f_* - \epsilon - 2M\zeta_{\Sigma} \leq H_f^\Omega \leq f^* + \epsilon + 2M\zeta_{\Sigma} \quad \text{on} \quad \Omega \cap \Sigma^c. \]  \hspace{1cm} (3.3)

This follows from the fact that (3.3) is satisfied on \( \partial(\Omega \cap \Sigma^c) \) and a generalized solution is order preserving. Passing to limit, first as \( x \to \infty \), and then as \( \epsilon \downarrow 0 \), from (3.3) and (2.10) it follows that the constructed generalized solution satisfies (1.10). Contradiction with uniqueness.

(3) \implies (2): Let \( u_1 \) and \( u_2 \) be two bounded solutions of the DP. Their difference is a generalized solution with a zero boundary function, and accordingly, it vanishes identically in view of (1.10). On the other hand, if at least for one \( f \) (1.10) is violated, then from the relation (2) \implies (3) for direct assertions, it follows that there are at least two solutions of the DP. This implies that the DP with zero boundary data on \( \partial \Omega \) has a non-trivial solution. That means it has infinitely many solutions.

(1) \implies (4): Assume that the series (1.12) converges, and prove that \( H_{1,\infty}^\Omega \neq 0 \). In fact, we are going to prove that

\[ \limsup_{z \to \infty} H_{1,\infty}^\Omega = 1. \]  \hspace{1cm} (3.4)
Let $0 < \epsilon < 1$ be an arbitrary small number. Choose a positive integer $m$ so large that
\[
\sum_{n=m+1}^{\infty} 2^{-n(N-2)} \Gamma_n < \frac{2\epsilon}{K 4^{N-1}}. \tag{3.5}
\]
Since $\Omega$ has at least one connected component $\Omega_\epsilon$ such that $\infty \in \partial \Omega_\epsilon$, we can choose $m$ so large that there is a point $x^*$ with
\[
x^* \in \Omega_\epsilon \cap \Sigma_{2m-1}
\]
Consider a sequence of increasing $A$-harmonic functions
\[
\vartheta_M(x) = \sum_{n=m+1}^{M} \zeta_{E_n}(x), \quad M = m + 1, m + 2, \ldots \quad x \in \Omega^m \equiv \left( \bigcup_{n=m+1}^{+\infty} E_n \right)^c,
\]
where $\zeta_{E_n}$ is a capacitary potential of $E_n$. Obviously we have
\[
\vartheta(x) \equiv \sum_{n=m+1}^{+\infty} \zeta_{E_n}(x) \equiv \sup_M \vartheta_M(x), \quad x \in \Omega_m
\]
Since $\vartheta$ is a limit of the increasing sequence of $A$-harmonic functions in each component of $\Omega^m$ which includes $\Omega$, either $\vartheta \equiv +\infty$, or $\vartheta$ is $A$-harmonic. Let us now estimate $\vartheta(x^*)$. From (2.7)-(2.9) it follows that
\[
\zeta_{E_n}(x^*) \leq K \int_{E_n} |y - x^*|^{2-N} d\mu_n(y) \leq K (2^n - |x^*|)^{2-N} \Gamma_n
\]
\[
\leq K 4^{N-2} 2^{-n(N-2)} \Gamma_n, \quad \text{for} \quad n \geq m + 1,
\]
\[
\vartheta(x^*) \leq K 4^{N-2} \sum_{n=m+1}^{+\infty} 2^{-n(N-2)} \Gamma_n
\]
and in view of (3.5) we have
\[
\vartheta(x^*) < \frac{\epsilon}{2}. \tag{3.6}
\]
Hence, $\vartheta$ is $A$-harmonic in $\Omega_m$ and $A$-superharmonic in $\mathbb{R}^N$. In fact, $\vartheta$ is a generalized solution of the DP in $\Omega^m$ under the boundary function
\[
\sum_{n=m+1}^{+\infty} \zeta_{E_n}(x) \geq 1 \quad \text{quasieverywhere on} \quad \partial \Omega^m. \tag{3.7}
\]
Since the generalized solution is order preserving, we have
\[
0 \leq H^{1}_{1-1} \leq \vartheta + \zeta_{\Sigma_2m} \quad \text{on} \quad \Omega \cap \Sigma_{2m}. \tag{3.8}
\]
Let $R$ be an arbitrary number satisfying
\[
R > 2^m K^{\frac{1}{N-2}} \left( \frac{\epsilon}{2} \right)^{\frac{1}{2-N}} \tag{3.9}
\]
14
Since $\vartheta$ is $\mathcal{A}$-harmonic in $\Omega^m$, from (3.6) and (3.7) we conclude that there must be a point $x_R \in \Omega \cap \{x : |x| = R\}$ such that
\[ \vartheta(x_R) < \frac{\epsilon}{2} \]

From (3.8), (3.9) and (2.10) it follows that
\[ 0 \leq H_{1-1,\infty}^{\Omega}(x_R) \leq \epsilon. \] (3.10)

Passing to the limit, first as $R \to +\infty$, and then as $\epsilon \to 0$ from (3.10) it follows that
\[ \liminf_{x \to \infty} H_{1-1,\infty}^{\Omega} = 0. \] (3.11)

Since
\[ H_{1-1,\infty}^{\Omega} \equiv 1 - H_{1,\infty}^{\Omega}, \]
we arrive at (3.4).

(4) $\Rightarrow$ (1): Assume that the series (1.12) diverges and prove that
\[ H_{1,\infty}^{\Omega} \equiv 0 \] (3.12)

Proof is based on the construction of the family $\{G_p\}$ of nonnegative $\mathcal{A}$-harmonic functions in $\Omega$ with the following properties:
\[ \lim_{x \to \infty} G_p(x) = 1 \quad \text{for any fixed } p \] (3.13)
\[ \lim_{p \to +\infty} G_p(x) = 0 \quad \text{for any fixed } x \in \Omega \] (3.14)

Indeed, in this case family $\{G_p\}$ is in the upper class $S(\Omega)$ with respect to the Perron solution $H_{1,\infty}^{\Omega}$. Accordingly, we have
\[ 0 \leq H_{1,\infty}^{\Omega}(x) \leq G_p(x), \quad x \in \Omega \]
and passing to the limit as $p \to +\infty$, (3.12) follows.

To construct the required family of $\mathcal{A}$-harmonic functions, we need to enter the positive integer parameter $p$ into the Wiener series and make some rearrangements. Let
\[ \Gamma_n^p \equiv \text{cap}(E_n^p), \quad E_n^p \equiv \Omega^c \cap \{2^{\frac{n-1}{p}} \leq |x| \leq 2^n\} \]

Assuming that $n \geq p$, from the subadditivity of the capacitary measure it follows that
\[ \Gamma_n \leq \Gamma_{np}^p + \Gamma_{np-1}^p + \cdots + \Gamma_{np-n+1}^p \]

Accordingly, one of the following series must be divergent:
\[ \sum_n 2^{-n(N-2)} \Gamma_{np}^p \sum_n 2^{-n(N-2)} \Gamma_{np-1}^p \cdots \sum_n 2^{-n(N-2)} \Gamma_{np-n+1}^p \]

It is easy to see that the divergence of any of these series implies that
\[ \sum_n 2^{-\frac{n(N-2)}{p}} \Gamma_n^p = +\infty. \]
By choosing \( p^2 \) successive values of \( n \) as

\[ kp^2, kp^2 + 1, \ldots, kp^2 + p^2 - 1, \]

it follows that one of the following series must also be divergent:

\[
\sum_{n} 2^{-np(N-2)} \frac{\Gamma_p}{np^2} \cdot \sum_{n} 2^{-(np+\frac{1}{p})(N-2)} \frac{\Gamma_p}{np^2+1} \cdots \sum_{n} 2^{-(n+1)p-\frac{1}{p})(N-2)} \frac{\Gamma_p}{(n+1)p^2-1}
\]

We may therefore assume without loss of generality, that

\[
\sum_{n} 2^{-np(N-2)} \frac{\Gamma_p}{np^2} = +\infty.
\]  \( (3.15) \)

The proof given for the contrary case is similar to the one presented. In view of \( (3.15) \), for an arbitrary large integer \( p \) we can find an integer \( N_p \) such that

\[
\sum_{n=1}^{N_p} 2^{-np(N-2)} \frac{\Gamma_p}{np^2} > p.
\]  \( (3.16) \)

If \( p > 2 \), the distance between \( E_{np^2}^p \) and its closest neighbors may be estimated as follows

\[
dist(E_{np^2}^p; E_{(n-1)p^2}^p) \geq 2^{np}(2-\frac{1}{p} - 2^p),
\]

\[
dist(E_{np^2}^p; E_{(n+1)p^2}^p) \geq 2^{np}(1 - 2^{-\frac{p}{2}}).
\]

Accordingly, we have

\[
dist(E_{np^2}^p; E_{kp^2}^p) \geq 2^{np} \alpha(p), \quad \text{for } \forall k \neq n,
\]

where

\[
\alpha(p) = \min(2^{-\frac{1}{p}} - 2^p; 1 - 2^{-\frac{p}{2}}).
\]

By using \( (2.9) \) we have

\[
\zeta_{E_{np^2}^p}(x) \leq K 2^{-np(N-2)} \frac{\Gamma_p}{np^2} \alpha^{2-N}(p), \quad x \in E_{kp^2}^p, k \neq n.
\]  \( (3.17) \)

Since the function \( \sum_{n=1}^{N_p} \zeta_{E_{np^2}^p} \) is \( \mathcal{A} \)-harmonic in \( \bigcup_{n=1}^{N_p} E_{np^2}^p \) including in \( \Omega \), and vanishes at \( \infty \), from \( (3.17) \) it follows that

\[
\sum_{n=1}^{N_p} \zeta_{E_{np^2}^p}(x) \leq K + K \alpha^{2-N}(p) \sum_{n=1}^{N_p} 2^{-np(N-2)} \frac{\Gamma_p}{np^2}, \quad x \in \left( \bigcup_{n=1}^{N_p} E_{np^2}^p \right)^c.
\]  \( (3.18) \)

Now, we choose the required family of \( \mathcal{A} \)-harmonic functions as follows:

\[
G_p(x) = - \frac{\sum_{n=1}^{N_p} \zeta_{E_{np^2}^p}(x)}{\gamma_p \sum_{n=1}^{N_p} 2^{-np(N-2)} \frac{\Gamma_p}{np^2}} + 1, \quad \gamma_p = K(p^{-1} + \alpha^{2-N}(p)).
\]

Nonnegativity of \( G_p \) follows from \( (3.16) \) and \( (3.18) \):

\[
G_p(x) \geq \frac{K \left( 1 - \alpha^{2-N}(p) \sum_{n=1}^{N_p} 2^{-np(N-2)} \frac{\Gamma_p}{np^2} \right)}{\gamma_p \sum_{n=1}^{N_p} 2^{-np(N-2)} \frac{\Gamma_p}{np^2}} + 1
\]

16
\[ \geq -\frac{K(p^{-1} + \alpha 2^{-N(p)})}{\gamma_p} + 1 = 0, x \in \Omega. \]

Since
\[ \lim_{x \to \infty} \zeta_{E_{n}^{p}}(x) = 0, \ n = 1, 2, ..., N_p, \]
(3.13) easily follows.

Let \( x \in \Omega \) is fixed. From (2.9) it follows that
\[ \zeta_{E_{n}^{p}}(x) \geq \frac{K^{-1} \Gamma_{n}^{p}}{2^{n}p + |x|^{N-2}} \geq \frac{K^{-1}2^{-np(N-2)} \Gamma_{n}^{p}}{1 + 2^{-p} |x|^{N-2}}. \]
Hence we have
\[ G_{p}(x) \leq -\frac{K^{-1}}{\gamma_p |1 + 2^{-p} |x|^{N-2} + 1}, \]
and (3.14) immediately follows. Regularity of \( \infty \) is proved.

(1) \( \Rightarrow \) (5): Assume that \( \Omega^c \) is \( \mathcal{A} \)-thin at \( \infty \). Prove that \( \infty \) is irregular for \( \Omega \). We will only consider the non-trivial case when \( \Omega^c \) is unbounded. It is sufficient to show that \( \infty \) is irregular for \( \Omega \cup \Sigma \), where \( \Sigma \) is some open sphere of large radius. This immediately follows from the equivalence of the irregularity of \( \infty \) to convergence of the Wiener series (1.12). Obviously the latter is not affected by adding a finite number of terms.

By Definition 1.2 and (1.11), it is easy to construct an \( \mathcal{A} \)-superharmonic function \( u \) such that
\[ 0 \equiv u(\infty) < 2 \equiv \lim_{x \to \infty, x \in \Omega^c} u(x) \]
Choose an open sphere \( \Sigma \) of large enough radius such that
\[ u(x) \geq 1 \quad \text{for} \quad x \in \Omega^c \cap \Sigma^c \]
It is easy to see that \( \mathcal{A} \)-subharmonic function \( v = 1 - u \) satisfies
\[ \lim_{x \to \infty} \sup v = 1 \]
and belongs to lower class \( -\mathcal{S}(\Omega \cup \Sigma) \) with respect to the Perron solution \( H_{1_{\infty}}^{\Omega \cup \Sigma} \). Therefore we have
\[ v(x) \leq H_{1_{\infty}}^{\Omega \cup \Sigma}(x) \quad \text{for} \quad x \in \Omega \cup \Sigma \]
which implies that
\[ \lim_{x \to \infty} \sup H_{1_{\infty}}^{\Omega \cup \Sigma}(x) = 1. \]
Hence \( \infty \) is irregular for \( \Omega \cup \Sigma \), and for \( \Omega \) as well.

(5) \( \Rightarrow \) (1): Assume that \( \infty \) is irregular for \( \Omega \). Prove that \( \Omega^c \) is \( \mathcal{A} \)-thin at \( \infty \). As before, we will only consider the non-trivial case when \( \Omega^c \) is unbounded. It
is sufficient to demonstrate that \((\Omega \cup \Sigma)^c\) is \(\mathcal{A}\)-thin at \(\infty\), where \(\Sigma\) is a sphere of large radius.

While proving the relation \((1) \Rightarrow (4)\) above, we constructed function \(\vartheta\) which is \(\mathcal{A}\)-superharmonic in \(\mathbb{R}^N\) and satisfies \((3.6)\), and

\[
\vartheta \geq 1 \quad \text{quasieverywhere on } (\Omega^m)^c \equiv \left( \bigcup_{n=m+1}^{+\infty} E_n \right)
\]

We want to have this property everywhere on \((\Omega^m)^c\). Let

\[e_n = \{ x \in E_n \cap \Sigma_{2^n} : \vartheta(x) < 1 \}, \quad n = m + 1, m = 2, \ldots\]

Since \(\text{cap}_A(e_n) = 0\), for arbitrary positive \(\delta_n\), we can choose open cover \(c_n\) of \(e_n\) such that \(\text{cap}_A(d_n) < \delta_n\), where \(d_n = \overline{e_n}\). We can obviously choose open sets \(c_n\) in such a way that

\[c_n \subset \{ x : |x| \geq 2^{n-\frac{1}{2}} \}.
\]

Clearly, \(x^*\) lies outside \(\left( \bigcup_{n=m+1}^{+\infty} E_n \cup c_n \right)\). Consider \(\mathcal{A}\)-superharmonic function

\[
\zeta(x) \equiv \sum_{n=m+1}^{+\infty} \zeta_{d_n}(x)
\]

We estimate \(\zeta(x^*)\) similar to the estimation of \(\vartheta(x^*)\). From \((2.7)-(2.9)\) it follows that

\[
\zeta_{d_n}(x^*) \leq K \int_{d_n} |y - x^*|^{2-N} d\lambda_n(y)
\leq K \left( \frac{4}{\sqrt{2} - 1} \right)^{N-2} 2^{-n(N-2)} \text{cap}_A(d_n), \quad \text{for } n \geq m + 1,
\]

\[
\zeta(x^*) \leq K \left( \frac{4}{\sqrt{2} - 1} \right)^{N-2} \sum_{n=m+1}^{+\infty} 2^{-n(N-2)} \delta_n
\]

where \(\lambda_n\) is an \(\mathcal{A}\)-capacitary measure of \(d_n\). By choosing numbers \(\delta_n\) sufficiently small, we have

\[
\zeta(x^*) < \frac{\epsilon}{2}
\]

Consider a function \(\varpi = \zeta + \vartheta\). It is \(\mathcal{A}\)-superharmonic in \(\mathbb{R}^N\),

\[
\varpi(x) \geq 1 \quad \text{everywhere on } (\Omega^m)^c.
\]

and

\[
\varpi(x^*) < \epsilon
\]

Since \(\varpi\) is \(\mathcal{A}\)-superharmonic in \(\Omega^m\), by applying the \(\mathcal{A}\)-superharmonic minimum principle in \(\Omega^m \cap \{ x : |x| < R \}\) we conclude that for all sufficiently large \(R\), there must be a point \(x_R \in \Omega_e \cap \{ x : |x| = R \}\) such that

\[
\varpi(x_R) < \epsilon
\]
Hence, we deduce that
\[
\liminf_{x \to \infty} \varpi(x) \leq \epsilon.
\]
Therefore, \(\varpi\) is an \(\mathcal{A}\)-superharmonic function which guarantees the \(\mathcal{A}\)-thinness of \(\Omega^c\) according to the Definition 1.2 and (1.11).

References

[1] U.G. Abdulla, Wiener’s Criterion for the Unique Solvability of the Dirichlet Problem in Arbitrary Open Sets with Non-Compact Boundaries, Nonlinear Analysis, 67, 2 (2007), 563-578.

[2] U.G. Abdulla, Wiener’s Criterion at \(\infty\) for the Heat Equation, Advances in Differential Equations, 13, 5-6(2008), 457-488.

[3] U.G. Abdulla, Wiener’s Criterion at \(\infty\) for the Heat Equation and its Measure-Theoretical Counterpart, Electronic Research Announcements in Mathematical Sciences, 15, (2008), 44-51.

[4] R.A. Adams, Sobolev Spaces, Academic Press, 1975

[5] M. Brelot, On Topologies and Boundaries in Potential Theory, Lecture Notes in Mathematics, 175, Springer-Verlag, 1971.

[6] E. De Giorgi, Sulla differentiabilità e l’analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino, S. III, Parte I (1957), 25-43.

[7] J.L. Doob, Classical Potential Theory and its Probabilistic Counterpart, Springer, 1984.

[8] E.B. Dynkin, Markov Processes, Springer-Verlag, 1965.

[9] J Heinonen, T Kilpeläinen, O Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Clarendon Press, 1993.

[10] E.Fabes, D.Jerison and C.Kenig, The Wiener test for degenerate elliptic equations, Annales de l’Institut Fourier, 32, 3(1982), 151-182.

[11] R.Gariepy and W.P.Ziemer, A regularity condition at the boundary for solutions of quasilinear elliptic equations, Archive for Rational Mechanics and Analysis, 67 (1977), 25-39.

[12] K. Ito and H.P. McKean,Jr., Potential and random walk, Indiana Univ. Math. J., 4(1960), 119-132.

[13] K. Ito and H.P. McKean,Jr., Diffusion Processes and Their Sample Paths. Springer, 1996.

[14] T.Kilpeläinen and J. Maly, The Wiener test and potential estimates for quasilinear elliptic equations, Acta Mathematica, 172 (1994), 137-161.

[15] N.S. Landkof, Foundations of Modern Potential Theory, Springer, 1972.
[16] P. Lindqvist, O. Martio, Two theorems of N. Wiener for solutions of quasi-linear elliptic equations, *Acta Mathematica*, 155 (1985), 153-171.

[17] W. Littman, G. Stampacchia and H.F. Weinberger, Regular Points for Elliptic Equations with Discontinuous Coefficients, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 17 (3), (1963), 43–77.

[18] J. Malý and W.P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, American Mathematical Society, 1997.

[19] V.G. Maz’ya, On the continuity at a boundary point of solutions of quasi-linear elliptic equations, *Vestnik Leningrad University: Mathematics*, 3 (1976), 225-242.

[20] J. Moser, On Harnack’s theorem for elliptic differential equations, *Comm. Pure Appl. Math.*, XIV (1961), 577-591.

[21] J. Nash, Continuity of the solutions of parabolic and elliptic equations, *Amer. J. Math.*, 80 (1958), 931-954.

[22] J. Serrin and H.F. Weinberger, Isolated singularities of solutions of linear elliptic equations, *Amer. J. Math.*, 72 (1966), 258-272.

[23] N. Wiener, Certain Notions in Potential Theory, *J. Math. Phys.*, 3, (1924), 24–51.

[24] N. Wiener, The Dirichlet Problem, *J. Math. Phys.*, 3, (1924), 127–146.