ENERGY IDENTITY FOR APPROXIMATE HARMONIC MAPS FROM SURFACE TO GENERAL TARGETS

WENDONG WANG, DONGYI WEI, AND ZHIFEI ZHANG

Abstract. Let $u_n$ be a sequence of mappings from a closed Riemannian surface $M$ to a general Riemannian manifold $N$. If $u_n$ satisfies

$$\sup_n (\|\nabla u_n\|_{L^2(M)} + \|\tau(u_n)\|_{L^p(M)}) \leq \Lambda$$

for some $p > 1$,

where $\tau(u_n)$ is the tension field of $u_n$, then there hold the so called energy identity and neckless property during blowing up. This result is sharp by Parker’s example, where the tension fields of the mappings from Riemannian surface are bounded in $L^1(M)$ but the energy identity fails.

1. Introduction

Let $(M,g)$ be a closed Riemannian manifold and $(N,h)$ be a Riemannian manifold without boundary. Let $u$ be a mapping from $M$ to $N$ in $W^{1,2}(M,N)$. We define the Dirichlet energy of $u$ as follows

$$E(u) = \int_M e(u) dV,$$

where $dV$ is the volume element of $(M,g)$, and $e(u)$ is the density of $u$

$$e(u) = \frac{1}{2} |du|^2 = \text{Trace}_g u^* h,$$

where $u^* h$ is the pull-back of the metric tensor $h$.

A map $u \in C^1(M,N)$ is called harmonic if it is a critical point of the energy $E$. By the Nash embedding theorem, $(N,h)$ can be isometrically embedded into a Euclidean space $\mathbb{R}^k$ for some positive integer $k$ with the metric induced from the Euclidean metric. Hence, a map $u \in C^1(M,N)$ can be viewed as a map of $C^1(M,\mathbb{R}^k)$ whose image lies in $N$. Then we can obtain the Euler-Lagrange equation

$$\triangle u - A(u)(du,du) = 0, \quad \text{or} \quad P(u) \triangle u = 0,$$

where $A(u)(du,du)$ is the second fundamental form of $N$ in $\mathbb{R}^k$. Let $P(y) : \mathbb{R}^k \to T_y N$ be the orthogonal projection map. The tension field $\tau(u)$ is defined by

$$\tau(u) \stackrel{\text{def}}{=} \triangle u - A(u)(du,du) = P(u) \triangle u.$$

Then $u$ is harmonic if and only if $\tau(u) = 0$. We refer to [7] for the systematic study on the harmonic maps.

The harmonic maps are of special interest when $M$ is a Riemannian surface, because the Dirichlet energy is conformally invariant in two dimensions. It is an important question to understand the limiting behavior of sequences of harmonic maps. Let $u_n$ be a sequence of mappings from Riemannian surface $M$ to $N$ with bounded energy. It is clear that $u_n$ converges weakly to $u$ in $W^{1,2}(M,N)$ for some $u \in W^{1,2}(M,N)$. In general, it may not
converge strongly in $W^{1,2}(M,N)$ due to the concentration of the energy at finitely many points \cite{13}. Thus, it is natural to ask (1) whether the lost energy is exactly the sum of energies of some harmonic spheres (bubbles), which are defined as harmonic maps from $S^2$ to $N$; (2) whether attaching all possible bubbles to the weak limit gives uniform convergence. The first one is so called the energy identity, and the second one is called the bubble tree convergence.

When $\tau(u_n) = 0$, Jost and Parker \cite{10} independently proved the energy identity and neckless property during blowing up. When $\tau(u_n)$ is bounded in $L^2(M)$, the energy identity was proved by Qing \cite{11} for the sphere, by Ding and Tian \cite{1} and Wang \cite{16} for general target manifold. Qing and Tian \cite{12} also proved neckless property during blowing up. One can refer to \cite{14,15,5} for the related results of the heat flow of harmonic maps. Notice that $L^2(M)$ space for the tension field is not conformally invariant. So, a natural substitution of $L^2(M)$ space seems \cite{12}.

Q: whether the energy identity or the neckless property holds for a general target manifold when the tension field is bounded in $L^p(M)$ for $p > 1$?

This question has been solved by Lin and Wang \cite{6} when the target manifold is the sphere. Li and Zhu \cite{3} also proved the energy identity for the tension fields bounded in $L \ln^+ L$, and constructed a sequence of mappings with tension fields bounded in $L \ln^+ L$ so that there is a positive neck during blowing up. For general target manifolds, partial important progress has been made: Li and Zhu \cite{3} proved the energy identity and the neckless property for $p \geq \frac{2}{5}$; Luo \cite{8} obtained the same result under the following condition

$$\left( \int_{D_r \setminus D_{r/2}} |\tau(u_n)|^2 \, dx \right)^{\frac{1}{2}} \leq Cr^{-a}$$

for some $a \in (0,1)$ and any $r \in (0,1)$. Here we denote by $D(x,r)$ the ball with the center $x$ and the radius $r$ and $D_r = D(0, r)$.

The goal of this paper is to give a positive answer to Q. When $\tau(u_n)$ is bounded in $L^p(M)$ for some $p > 1$, the small energy regularity (see Lemma \cite{21}) implies that $u_n$ converges strongly in $W^{1,2}(M, N)$ outside a finite set of points. For simplicity, we assume that $M$ is the unit disk $D_1 = D(0,1)$ and $0$ is the only one singular point.

Our main result is stated as follows.

**Theorem 1.1.** Let $\{u_n\}$ be a sequence of mappings from $D_1$ to $N$ in $W^{1,2}(D_1, N)$ with tension field $\tau(u_n)$ satisfying

$$\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$$

for some $p > 1$, and for $0 < \delta < 1$,

$$u_n \to u \text{ strongly in } W^{1,2}(D_1 \setminus D_\delta, \mathbb{R}^k) \text{ as } n \to \infty.$$ 

Then there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and some nonnegative integer $k_0$ such that for any $i = 1, \cdots, k_0$, there exist points $x_{i,n}^i$, positive numbers $r_{i,n}^i$ and a nonconstant harmonic sphere $w_i$ (a map from $\mathbb{R}^2 \cup \{\infty\} \to N$), which satisfy

1. for any $i = 1, \cdots, k_0$, $x_{i,n}^i \to 0$, $r_{i,n}^i \to 0$ as $n \to \infty$. 

2. \[
\lim_{n \to \infty} \left( \frac{r_n^i}{r_n^j} + \frac{r_n^j}{r_n^i} + \frac{|x_n^i - x_n^j|}{r_n^i + r_n^j} \right) = \infty \quad \text{for any} \quad i \neq j.
\]
3. \(w^i\) is the weak limit or strong limit of \(u_n(x_n^i + r_n^i x)\) in \(W_{loc}^{1,2}(\mathbb{R}^2, N)\).

4. Energy identity
\[
\lim_{n \to \infty} E(u_n, D_1) = E(u, D_1) + \sum_{i=1}^{k_0} E(w^i).
\]

5. Neckless property: The image \(u(D_1) \cup \bigcup_{i=1}^{k_0} w^i(\mathbb{R}^2)\) is a connected set.

2. Bubble tree structure of approximate harmonic maps

Let us first recall the following small energy regularity result [11, 14].

**Lemma 2.1.** Let \(u\) be a mapping from \(D_1\) to \(N\) in \(W^{1,2}(D_1, N)\) with tension field \(\tau(u) \in L^p(D_1)\) for \(p > 1\). Then there exists a positive constant \(\epsilon_N\) depending on the target manifold \(N\) such that if \(E(u, D_1) \leq \epsilon_N^2\), then
\[
\|u - \bar{u}\|_{W^{2,p}(D_1)} \leq C \left( \|\nabla u\|_{L^2(D_1)} + \|\tau(u)\|_{L^p(D_1)} \right),
\]
where \(\bar{u}\) is the mean value of \(u\) on the disk \(D_\tau\).

Let \(u_n\) be a sequence of mapping from \(D_1\) to \(N\) satisfying
\[
\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda
\]
for some \(p > 1\). A point \(x \in D_1\) is called an energy concentration point (blow-up point) of \(u_n\) if for any \(r\) so that \(D(x, r) \subset D_1\), we have
\[
\limsup_{n \to \infty} E(u_n, D(x, r)) > \epsilon_N^2.
\]

Based on Lemma 2.1, using standard blow-up argument as in [11, 14], it can be proved that for fixed sufficiently small \(\epsilon \in (0, \epsilon_N)\), there exists \(k_0 \geq 0\) so that for any \(i = 1, \ldots, k_0\), there exist a point \(x_n^i\), a positive number \(r_n^i\), and a nonconstant harmonic sphere \(w^i\) satisfying the conclusions 1–3 in Theorem 1.1. Moreover, it holds that

1. \(w^i\) is the strong limit of \(u_n(x_n^i + r_n^i x)\) in \(W_{loc}^{1,2}(\mathbb{R}^2 \setminus Z_i, N)\), where \(Z_i\) is the set of blow-up points of this scaling sequence, thus \(Z_i\) is finite and for \(x \in Z_i\),
\[
m_i(x) \equiv \lim_{\delta \to 0} \lim_{n \to \infty} E(u_n, D(x_n^i + r_n^i x, \delta r_n^i)) \geq \epsilon_N.
\]

For the sake of completeness, we denote \(x_n^0 = 0,\ r_n^0 = 1,\ Z_0 = \{0\},\ w^0 = u.

2. \(\exists f : \{1, \ldots, k_0\} \to \mathbb{Z}\) and \(\delta_0, R_0 > 0\) so that
\[
0 \leq f(j) < j, \quad \lim_{n \to \infty} \frac{r_n^{f(j)}}{r_n^j} = 0, \quad \lim_{n \to \infty} \frac{x_n^j - x_n^{f(j)}}{r_n^j} = y_j \in Z_f(j),
\]
and \(E(u_n, D(x_n^j, r_n^{f(j)} \delta_0) \setminus D(x_n^j, r_n^j R_0)) \leq \epsilon\).

3. If \(f(i) = f(j)\) and \(y_i = y_j\), then \(i = j,\ Z_i = \{y_j | f(j) = i\}\.\)
Let us just present a sketch for the construction of \((x_n^i, r_n^i)\) (see P.118-121 in [9] for similar construction). As \(m_0(0) = \lim_{\delta \to 0} \lim_{n \to \infty} E(u_n, D_\delta) \geq \epsilon N\), there exists \(\delta > 0\) so that
\[
|E(u_n, D_\delta) - m_0(0)| \leq \frac{\epsilon}{4}
\]
for \(n\) sufficiently large. Let \(Q_n(t) = \sup_{D(z,t) \subseteq D_\delta} E(u_n, D(z,t))\) for \(0 \leq t \leq \delta\). Then \(Q_n(t)\) is continuous and non-decreasing in \(t\), \(Q_n(0) = 0\) and \(Q_n(\delta) = E(u_n, D_\delta)\). Therefore, there exists \(0 < r_n^\ast < \delta\) such that \(Q_n(r_n^\ast) = \max(Q_n(\delta) - \epsilon, \frac{\epsilon}{2})\) and there exists \(D(x_n^i, r_n^i) \subseteq D_\delta\) such that \(E(u_n, D(x_n^i, r_n^i)) = Q_n(r_n^i)\). Thus, we have \(x_n^i \to 0, r_n^i \to 0\) and
\[
E(u_n, D(x_n^i, r_n^i)) \leq E(u_n, D_\delta) - E(u_n, D(x_n^i, r_n^i)) = Q_n(\delta) - Q_n(r_n^i) \leq \epsilon
\]
for \(n\) sufficiently large. Hence, we can take \(f(1) = 0\). Moreover, by Lemma 2.1, \(u_n(x_n^i + r_n^i x)\) has a subsequence, which strongly converges in \(W^{1,2}(\mathbb{R}^2 \setminus Z_1, N)\) to \(w^1\), and for \(x \in Z_1\),
\[
m_1(x) \leq \limsup_{n \to \infty} Q_n(r_n^i) \leq \max(m_0(0) - \frac{3}{4}\epsilon, \frac{\epsilon}{2}),
\]
which implies that this construction can only happen finite times.

**Remark 2.2.** In fact, Zhu [18] proved the bubble tree theorem under the weaker condition
\[
\|u_n\|_{W^{1,2}(D_\delta)} + \|\tau(u_n)\|_{L^{\ln^+ L}(D_\delta)} \leq \Lambda,
\]
where \(\|f\|_{L^{\ln^+ L}(D_\delta)} \overset{\text{def}}{=} \int_{D_\delta} |f(x)| \ln(2 + |f(x)|) dx\).

Now, by Property 1 and Lemma 2.1 for fixed \(0 < \delta < 1 < R\) and \(1 \leq i \leq k_0\), we have
\[
\begin{align*}
&u_n(x_n^i + r_n^i x) \to w^i \text{ strongly in } W^{1,2}(D_R \setminus \cup_{x \in Z_1} D(x, \delta)) \cap C^0(D_R \setminus \cup_{x \in Z_1} D(x, \delta)), \\
u_n \to u \text{ strongly in } W^{1,2}(D_1 \setminus D_\delta) \cap C^0(D_1 \setminus D_\delta),
\end{align*}
\]
as \(n \to \infty\). Therefore, as \(n \to \infty\),
\[
E(u_n, D(x_n^i, r_n^i) \setminus \cup_{x \in Z_1} D(x_n^i + r_n^i x, r_n^i \delta)) \to E(w^i, D_R \setminus \cup_{x \in Z_1} D(x, \delta)), \\
E(u_n, D_1 \setminus D_\delta) \to E(u, D_1 \setminus D_\delta).
\]
Moreover,
\[
\limsup_{n \to \infty} \text{osc } (u_n, D(x_n^i, r_n^i \delta) \setminus D(x_n^i, r_n^i R)) \geq \text{diam}(w^{f(i)}(\partial D_\delta) \cup w^i(\partial D_R)),
\]
and for \(n\) sufficiently large,
\[
\begin{align*}
&\left|E(u_n, D_1) - E(u_n, D_1 \setminus D_\delta) - \sum_{i=1}^{k_0} E(u_n, D(x_n^i, r_n^i) \setminus \cup_{x \in Z_1} D(x_n^i + r_n^i x, r_n^i \delta))\right| \\
&\leq \sum_{i=1}^{k_0} E(u_n, D(x_n^i, r_n^i \delta) \setminus D(x_n^i, r_n^i R)).
\end{align*}
\]
Thus, the energy identity is equivalent to show that there is no energy on the neck during blow-up process, i.e.,
\[
\lim_{\delta \to 0} \lim_{R \to \infty} \limsup_{n \to \infty} E(u_n, D(x_n^i, r_n^i \delta) \setminus D(x_n^i, r_n^i R)) = 0 \quad \text{for } i = 1, \cdots, k_0.
\]
While, the neckless property is equivalent to show that there is no oscillation on the neck, i.e.,
\[
(2.2) \lim_{\delta \to 0} \lim_{R \to \infty} \lim_{n \to \infty} \sup \text{osc}\{u_n, D(x_n^i, r_n^R(\delta)) \setminus D(x_n^i, r_n^R(\delta))\} = 0 \quad \text{for } i = 1, \ldots, k_0.
\]

In order to prove (2.1) and (2.2), our key idea is to show that the Hopf differential of the approximate harmonic map \(u\) can be approximated by a holomorphic function, where the error is quantized by the tension field of \(u\). The result is trivial in the case when \(\tau(u) = 0\), because the Hopf differential of harmonic map \(u\) is holomorphic.

3. The Coulomb gauge frame

Consider \(\Omega = D_1 \setminus D_r\) for \(0 < r \leq \frac{1}{2}\). Assume that
\[
E(u, \Omega) \leq \epsilon_0,
\]
where \(\epsilon_0 > 0\) will be determined later.

Recall that \((N, h)\) can be isometrically embedded into \(\mathbb{R}^k\). Let \(\overline{N}\) be a submanifold of \(\mathbb{R}^{2k}\) defined by
\[
\overline{N} \overset{\text{def}}{=} \{(y, y') \in \mathbb{R}^k \times \mathbb{R}^k : y \in N, \ y' \perp T_y N\}.
\]
Then \(N = N \times \{0\}\) is a totally geodesic submanifold of \(\overline{N}\). As in [2, 7], we may introduce the Coulomb gauge frame of \(u^* T \overline{N}\). Let us present the construction of Coulomb gauge.

The following lemma makes use of Hardy space (for example, see [2, 7]).

Lemma 3.1. If \(\triangle \Phi = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1}\), then we have
\[
\|d\Phi\|_{L^{2,1}(\mathbb{R}^2)} \leq C \|df\|_{L^2(\mathbb{R}^2)} \|dg\|_{L^2(\mathbb{R}^2)}.
\]
Here \(L^{p,q}(\mathbb{R}^2)\) is the Lorentz space.

We also need the following extension lemma.

Lemma 3.2. Let \(f \in W^{1,2}(\Omega)\). Then there exists an extension \(\tilde{f} \in W^{1,2}(\mathbb{R}^2)\) of \(f\) so that
\[
E(\tilde{f}, \mathbb{R}^2) \leq CE(f, \Omega).
\]
Here \(C\) is a constant independent of \(r\).

Proof. We only consider the case \(\Omega = D_1 \setminus D_r\). First of all, we can find \(f_j \in W^{1,2}(\mathbb{R}^2), j = 1, 2\) such that \(E(f_j, \mathbb{R}^2) \leq CE(f_j, D_1 \setminus D_{\frac{1}{2}})\), and \(f_1(z) = f(z), \ f_2(z) = f(2rz)\) in \(D_1 \setminus D_{\frac{1}{2}}\). Then we can extend \(f\) to \(W^{1,2}(\mathbb{R}^2)\) by taking \(\tilde{f}(z) = f_1(z)\) for \(|z| > 1\) and \(\tilde{f}(z) = f_2(z/2r)\) for \(|z| < r\). Then we find that
\[
E(\tilde{f}, \mathbb{R}^2) = E(\tilde{f}, D_{2r}) + E(\tilde{f}, D_1 \setminus D_{2r}) + E(\tilde{f}, \mathbb{R}^2 \setminus D_{\frac{1}{2}}) = E(f_2, D_1) + E(f_1, D_{\frac{1}{2}} \setminus D_{2r}) + E(f_1, \mathbb{R}^2 \setminus D_{\frac{1}{2}}) \leq CE(f_2, D_1 \setminus D_{\frac{1}{2}}) + CE(f_1, D_{\frac{1}{2}} \setminus D_{2r}) + CE(f, D_1 \setminus D_{\frac{1}{2}}) \leq CE(f, D_1 \setminus D_r).
\]
This completes the proof.
Let $\mathcal{A}(\Omega)$ be the set of $R = (e_1, \cdots, e_k) \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{2k \times k})$ such that
\[
R^T R = I_k, \quad RR^T = \begin{pmatrix} P(u) & I_k - P(u) \end{pmatrix} \quad \text{in } \Omega.
\]

First of all, we show that $\mathcal{A}(\Omega)$ is nonempty and $E(R) = \frac{1}{2} \int_{\mathbb{R}^2} |dR|^2$ attains a minimum in $\mathcal{A}(\Omega)$.

Indeed, let $R_0 = \begin{pmatrix} P(u) & I_k - P(u) \end{pmatrix}$ in $\Omega$ and extend $R_0$ to $\mathbb{R}^2$ so that $\int_{\mathbb{R}^2} |dR_0|^2 = C \int_{\Omega} |dR_0|^2$. Then $R_0 \in \mathcal{A}(\Omega)$ and $E(R_0) \leq CE(u, \Omega)$. Let $R_n \in \mathcal{A}(\Omega)$ so that
\[
\lim_{n \to \infty} E(R_n) = E_0 = \inf_{R \in \mathcal{A}(\Omega)} E(R).
\]

Then there exists a subsequence of $\{R_n\}$ (still denoted by $\{R_n\}$) and $R \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{2k \times k})$ such that $R_n \to R$ strongly in $L^2(\Omega)$ as $n \to \infty$, and $dR_n \to dR$ weakly in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Then $R_n^T R_n \to R^T R$ and $R_n R_n^T \to RR^T$ strongly in $L^1(\Omega)$ as $n \to \infty$, and $E(R) \leq E_0$. Therefore, $R \in \mathcal{A}(\Omega)$ and $E(R) = E_0 \leq E(R_0) \leq CE(u, \Omega)$.

Now, for $\psi \in C_0^\infty(\mathbb{R}^2, so(k))$, we have $R \exp t \psi \in \mathcal{A}(\Omega)$ for $t \in \mathbb{R}$. Therefore, $E(R \exp t \psi) \geq E_0 = E(R)$ and
\[
0 = \frac{d}{dt} \bigg|_{t=0} E(R \exp t \psi) = \int_{\mathbb{R}^2} \langle dR, \frac{d}{dt} \big|_{t=0} d(R \exp t \psi) \rangle = \int_{\mathbb{R}^2} \langle dR, d(R \psi) \rangle
\]
\[
= \int_{\mathbb{R}^2} \langle \langle dR, R(d\psi) \rangle + \langle dR, (dR)\psi \rangle \rangle = \int_{\mathbb{R}^2} \langle dR, R(d\psi) \rangle = \int_{\mathbb{R}^2} \langle R^T dR, d\psi \rangle,
\]
as $\langle dR, (dR)\psi \rangle = 0$. For $\psi \in C_0^\infty(\mathbb{R}^2, R^{k \times k})$, we have $\psi^T - \psi \in C_0^\infty(\mathbb{R}^2, so(k))$. So,
\[
0 = \int_{\mathbb{R}^2} \langle R^T dR, d(\psi^T - \psi) \rangle = \int_{\mathbb{R}^2} \langle (R^T dR)^T - R^T dR, d\psi \rangle
\]
\[
= \int_{\mathbb{R}^2} \langle (dR^T)^T R - R^T dR, d\psi \rangle.
\]

Therefore, $d^*((dR^T)^T R - R^T dR) = 0$ and there exists $\Phi \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^{k \times k})$ so that
\[
\frac{1}{2}((dR^T)^T R - R^T dR) = \langle \frac{\partial \Phi}{\partial x_2} dx_1 - \frac{\partial \Phi}{\partial x_1} dx_2, dx \rangle.
\]

Noticing that
\[
\Delta \Phi = \frac{\partial R^T}{\partial x_1} \frac{\partial R}{\partial x_2} - \frac{\partial R^T}{\partial x_2} \frac{\partial R}{\partial x_1},
\]
we infer from Lemma 3.1 that
\[
\|d\Phi\|_{L^2(\mathbb{R}^2)} \leq C \|dR^T\|_{L^2(\mathbb{R}^2)} \|dR\|_{L^2(\mathbb{R}^2)} = C \|dR\|^2_{L^2(\mathbb{R}^2)} \leq CE(u, \Omega).
\]

Thanks to $0 = dI_k = d(R^T R) = (dR^T)^T R + R^T dR$ in $\Omega$, we have
\[
(dR^T)^T R = \frac{\partial \Phi}{\partial x_2} dx_1 - \frac{\partial \Phi}{\partial x_1} dx_2, \quad d^*((dR^T)^T R) = 0 \text{ in } \Omega,
\]
\[
\| (dR^T)^T R \|_{L^2(\Omega)}^2 = \|d\Phi\|_{L^2(\Omega)}^2 \leq CE(u, \Omega).
\]

We introduce
\[
A = R^T \begin{pmatrix} \frac{\partial u}{\partial x} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau^i = \frac{1}{4} R^T \begin{pmatrix} \tau \\ 0 \end{pmatrix}, \quad w = \frac{\partial R^T}{\partial z} R \in so(k) \otimes \mathbb{C} \text{ for } z \in \Omega.
\]
Then the system of (1.1) is equivalent to

$$\frac{\partial A}{\partial \bar{z}} = wA + \tau^1,$$

where $w$ satisfies

$$\|w\|_{L^{2,1}(\mathbb{R}^2)} \leq C\epsilon_0.$$

Let us introduce a linear operator $T : L^\infty(\mathbb{C}, \mathbb{C}^{k\times k}) \to L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ defined by

$$T(B)(z) = \left( \frac{1}{\pi z} * (wB) \right)(z) = \int_{\mathbb{C}} \frac{w(\zeta)B(\zeta)}{\pi(z - \zeta)} d\zeta.$$

Thanks to $\frac{1}{\pi z} \in L^{2,\infty}(\mathbb{C})$ and $w \in L^{2,1}(\mathbb{C})$, we deduce that $T$ maps $L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ continuously to itself with the bound

$$\|T\| \leq C\|w\|_{L^{2,1}(\mathbb{R}^2)} \leq C\|\nabla u\|_{L^2(\Omega)}^2,$$

Moreover, it holds that

$$\frac{\partial}{\partial \bar{z}} T(B) = wB.$$

**Lemma 3.3.** If $\|T\| \leq \frac{1}{3}$, then there exists a nonsingular matrix $B \in L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ so that

$$B - T(B) = I_k, \quad B^T B = I_k.$$

Here $I_k$ is the $k \times k$ identity matrix.

*Proof.* Due to $\|T\| \leq \frac{1}{3}$, we get by the fixed point theorem that

$$B - T(B) = I_k$$

has a unique solution $B \in L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ with

$$\|B - I_k\|_{L^\infty(\mathbb{C})} \leq \frac{3}{2} \|T\| \leq \frac{1}{2}.$$

Recall that $w^T = -w$ and $\frac{\partial}{\partial \bar{z}} B = \frac{\partial}{\partial \bar{z}} T(B) = wB$. Thus, we get

$$\frac{\partial}{\partial \bar{z}} B^T = B^T w^T,$$

which implies that

$$\frac{\partial}{\partial \bar{z}} (B^T B) = B^T (w^T + w) B = 0.$$

That means that $B^T B$ is holomorphic. On the other hand, $\lim_{z \to \infty} B = I_k$. Then

$$B^T B = I_k.$$

The proof is finished. \[\square\]
4. Holomorphic Approximation of Hopf Differential

Throughout this section, let us assume that \( u \) is a mapping from \( D_1 \) to \( N \) in \( W^{1,2}(D_1, N) \) with the tension field \( \tau(u) \in L^p(D_1) \) for some \( p \in (1, 2) \). We denote by \( h(z) \) the Hopf differential of \( u \), i.e.,

\[
h(z) \overset{\text{def}}{=} \left< \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \right> = \sum_{j=1}^{k} \frac{\partial u^j}{\partial z} \frac{\partial u^j}{\partial \bar{z}}.
\]

It was well-known that if \( u \) is harmonic, then \( h(z) \) is holomorphic. In this section, we will show that in general case, \( h(z) \) can be approximated by a holomorphic function, where the error is quantized by \( \|\tau(u)\|_{L^p(D_1)} \). This result may be independent of interest.

**Proposition 4.1.** Assume that \( E(u, D_1) \leq m \) for some \( m \in \mathbb{N} \), where \( \epsilon \) is the minimum of \( \epsilon_0 \) given by Lemma 4.2 and Lemma 4.3. Then there exist \( C_m \) and a holomorphic function \( h_0 \) in \( D_{1/4} \) such that

\[
\|h - h_0\|_{L^1(D_{1/4})} \leq C_m \|\tau(u)\|_{L^p(D_1)}^{3 - m}.
\]

The proof of Proposition 4.1 is based on the following lemmas.

**Lemma 4.2.** There exists \( \epsilon_0 > 0 \) such that if \( E(u, \Omega) \leq \epsilon_0 \), \( \Omega = D_1 \) or \( D_1 \setminus D_r \) for some \( 0 < r \leq \frac{1}{4} \), then there exists a holomorphic function \( h_0 \) in \( \Omega \) so that

\[
\|h - h_0\|_{L^1(\Omega)} \leq C \|\tau(u)\|_{L^p(\Omega)}.
\]

**Proof.** Thanks to \( E(u, \Omega) \leq \epsilon_0 \), by (3.3) we can take \( \epsilon_0 \) small enough so that \( T \leq C \epsilon_0^2 \leq \frac{1}{3} \). Then by Lemma 3.3 there exists a nonsingular matrix \( B \in L^\infty(C, \mathbb{C}^{k \times k}) \) so that

\[
B - T(B) = I_k, \quad B^TB = I_k, \quad \|B - I_k\|_{L^\infty(C)} \leq \frac{3}{2}\|T\| \leq C \epsilon_0^2.
\]

Let \( A = BG \). Then \( \frac{\partial}{\partial z} G = B^{-1}T \). We write \( G = G_1 + G_2 \) with \( G_2 = \frac{1}{\pi z} \ast (B^{-1}T) \). Then it holds that

\[
\frac{\partial G_2}{\partial z} = B^{-1}T, \quad \frac{\partial G_1}{\partial z} = 0 \quad \text{in} \ \Omega.
\]

Moreover, it holds that

\[
\|G_2\|_{L^{\frac{2p}{p-2p}}(\mathbb{R}^2)} \leq C \|\tau\|_{L^p(\mathbb{R}^2)} \leq \|\tau(u)\|_{L^p(\Omega)}.
\]

Let \( h_1(z) = G_1(z)^T G_1(z) \), which is holomorphic in \( \Omega \). Notice that

\[
h(z) = A^T A = G^T B^TBG = G^T G.
\]

Hence, by \( G = B^{-1}A \) and \( \|A\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \), we deduce that

\[
\|h - h_1\|_{L^p(\Omega)} = \|G^T G - G_1^T G_1\|_{L^p(\Omega)} \\
\leq \|G_2\|_{L^{\frac{2p}{p-2p}}(\Omega)} \left( \|G\|_{L^2(\Omega)} + 2\|G_2\|_{L^2(\Omega)} \right) \\
\leq C \|\tau(u)\|_{L^p(\Omega)} \left( \|\nabla u\|_{L^2(\Omega)} + \|\tau(u)\|_{L^p(\Omega)} \right) \\
\leq C \|\tau(u)\|_{L^p(\Omega)} (1 + \|\tau(u)\|_{L^p(\Omega)}).
\]

On the other hand, we have

\[
\|h\|_{L^1(\Omega)} \leq \left\| \frac{\partial u}{\partial z} \right\|_{L^2(\Omega)}^2 \leq E(u, \Omega) \leq 1.
\]
This concludes that
\[
\min\{\|h - h_1\|_{L^1(\Omega)}, \|h\|_{L^1(\Omega)}\} \leq \min\{C\|\tau(u)\|_{L^p(\Omega)}(1 + \|\tau(u)\|_{L^p(\Omega)}), 1\} \\
\leq C\|\tau(u)\|_{L^p(\Omega)}.
\]
Thus, the lemma is true for either \(h_0 = h_1\) or \(h_0 = 0\). \(\square\)

**Lemma 4.3.** There exist \(\epsilon_0 > 0\) so that if
\[
E(u, D_1 \setminus D_r) \leq \epsilon_0 \quad \text{for some } 0 < r \leq \frac{1}{4}, \quad \|\tau(u)\|_{L^p(D_1)} \leq 1,
\]
and there exists a holomorphic function \(h_{0,2r}\) in \(D_{2r}\) satisfying
\[
\|h - h_{0,2r}\|_{L^1(D_{2r})} + r^{2p-2} \|\tau(u)\|_{L^p(D_1)} \leq A_0.
\]
Then there exists a holomorphic function \(h_0\) in \(D_1\) such that
\[
\|h - h_0\|_{L^1(D_1)} \leq C \left( A_0 \ln\frac{1}{r} + \min\left\{\frac{A_0}{r}, A_0^\frac{1}{p} + r^2 \right\} + \|\tau(u)\|_{L^p(D_1)} \right).
\]
Here \(C\) is a constant independent of \(A_0\) and \(r\).

**Proof.** Using the same notations as in Lemma 4.2 with \(\Omega = D_1 \setminus D_r\), we have
\[
\|h - h_1\|_{L^1(D_1 \setminus D_r)} \leq C \|h - h_1\|_{L^p(D_1 \setminus D_r)} \leq C \|\tau(u)\|_{L^p(D_1)}.
\]
We denote by \(\sum_{n \in \mathbb{Z}} a_n z^n (a_n \in \mathbb{C}^k)\) the Laurent expansion of \(G_1(z)\) in \(D_1 \setminus D_r\). Then we have
\[
h_1(z) = \sum_{n \in \mathbb{Z}} b_n z^n \quad \text{with} \quad b_n = \sum_{m \in \mathbb{Z}} (a_m, a_{n-m}),
\]
and we define
\[
h_0(z) = \sum_{n=0}^{\infty} b_n z^n.
\]
Hence, \(h_0(z)\) is holomorphic in \(D_1\). Since \(h_{0,2r}\) is holomorphic in \(D_{2r}\), we may write
\[
h_{0,2r}(z) = \sum_{n=0}^{\infty} b_n' z^n \quad \text{in } D_{2r}.
\]
Let \(r_1 = \frac{r}{4}, \ r_2 = \frac{3}{4} r, \ r_3 = \frac{7}{4} r\). Then we obtain
\[
\|h - h_0\|_{L^1(D_1)} \leq \|h - h_0\|_{L^1(D_1 \setminus D_{2r})} + \|h - h_0\|_{L^1(D_{2r})} \\
\leq \|h - h_1\|_{L^1(D_1 \setminus D_{2r})} + \|h - h_{0,2r}\|_{L^1(D_{2r})} \\
+ \|h_1 - h_0\|_{L^1(D_1 \setminus D_{2r})} + \|h_0 - h_{0,2r}\|_{L^1(D_{2r})} \\
\leq \|h - h_1\|_{L^1(D_1 \setminus D_{r})} + \|h - h_{0,2r}\|_{L^1(D_{2r})} \\
+ \sum_{n=1}^{\infty} \|b_{-n}\|_{L^1(D_1 \setminus D_{2r})} + \sum_{n=0}^{\infty} \|b_n - b_n'\|_{L^1(D_{2r})} \\
\leq \|h - h_1\|_{L^1(D_1 \setminus D_{r})} + \|h - h_{0,2r}\|_{L^1(D_{2r})} \\
+ C \left( |b_{-1}| + |b_{-2}| \ln\frac{1}{r_2} + \sum_{n=3}^{\infty} |b_{-n}| \frac{1}{r_2^{n-2}} + \sum_{n=0}^{\infty} |b_n - b_n'| r_2^{n+2} \right).
\]

**Estimate of \(b_{-n}\).**
Thanks to the assumption, we get
\[
\|h_{0,2r} - h_1\|_{L^1(D_{2r} \setminus D_r)} \leq \|h - h_{0,2r}\|_{L^1(D_{2r} \setminus D_r)} + C r^{2r-\rho} \|h - h_1\|_{L^p(D_{2r} \setminus D_r)} \leq C A_0,
\]
which implies that for \( j = 1, 2, 3 \),
\[
\frac{1}{|z|} \int_{|z| = r_j} |z||h_{0,2r} - h_1||dz| \leq C \|h_{0,2r} - h_1\|_{L^1(D_{2r} \setminus D_r)} \leq C A_0.
\]
Hence, we deduce that for \( n \geq 1 \),
\[
|b_{-n}| = \frac{1}{2\pi} \int_{|z| = r_1} z^{n-1} h_1 dz \leq r_1^{n-2} \int_{|z| = r_1} |z||h_{0,2r} - h_1||dz| \leq C r_1^{n-2} A_0,
\]
and for \( n \geq 0 \),
\[
|b_n - b'_n| = \frac{1}{2\pi} \int_{|z| = r_3} z^{n-1} h_1 dz \leq r_3^{n-2} \int_{|z| = r_3} |z||h_{0,2r} - h_1||dz| \leq C r_3^{n-2} A_0.
\]

**Refined estimate of \( b_{-1} \).**

Recalling \( b_n = \sum_{m \in \mathbb{Z}} \langle a_m, a_{n-m} \rangle \), we get
\[
|b_{-1}| \leq 2 \left( |a_{-1}| |a_0| + \sum_{n=1}^{\infty} |a_n| |a_{-n-1}| \right),
\]
\[
|\langle a_{-1}, a_{-1} \rangle| \leq |b_{-2}| + 2 \sum_{n=0}^{\infty} |a_n| |a_{-n-1}|.
\]

A direct calculation yields that for \( r \leq \rho_2 < \rho_1 \leq 1 \), we have
\[
\|z^n\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} = \frac{\rho_1^{2n+2} - \rho_2^{2n+2}}{n+1} \quad \text{for } n \neq -1,
\]
\[
\|z^{-1}\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} = 2\pi \ln \frac{\rho_1}{\rho_2},
\]
\[
\|z^n\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} \leq \rho_1^2 \|z^n\|^2_{L^2(D_{1} \setminus D_{r})} \quad \text{for } n \geq 0,
\]
\[
\|z^n\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} \leq (r/\rho_2)^2 \|z^n\|^2_{L^2(D_{1} \setminus D_{r})} \quad \text{for } n \leq -2.
\]

Then, using the fact that
\[
\|G_1\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} = \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})},
\]
we infer that
\[
\|G_1\|^2_{L^2(D_{\rho_1} \setminus D_{\rho_2})} \leq 2\pi |a_{-1}|^2 \ln \frac{\rho_1}{\rho_2} + \max \left( \rho_1, \frac{r}{\rho_2} \right)^2 \|G_1\|^2_{L^2(D_{1} \setminus D_{r})}.
\]

Using the fact that \( \ln \|z^n\|^2_{L^2(D_{1} \setminus D_{r})} \) is a convex function of \( n \), we obtain
\[
\|z^n\|^2_{L^2(D_{1} \setminus D_{r})} \|z^{-1-n}\|^2_{L^2(D_{1} \setminus D_{r})} \geq \|z^n\|^1_{L^2(D_{1} \setminus D_{r})} \|z^{-2}\|^2_{L^2(D_{1} \setminus D_{r})} \geq C (1 - r^2)^2 \frac{(1 - r^4)(1 - r^2)}{2r^2}
\]

for \( n \geq 1 \), and for \( n \geq 0 \),
\[
\|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-2-n}\|_{L^2(D_1 \setminus D_r)} \geq \|z^0\|_{L^2(D_1 \setminus D_r)} \|z^{-2}\|_{L^2(D_1 \setminus D_r)} = \frac{1 - r^2}{r}.
\]

Then we conclude that
\[
\|G_1\|_{L^2(D_1 \setminus D_r)}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2
\]
\[
\geq |a_0|^2 \|z^0\|_{L^2(D_1 \setminus D_r)}^2 + 2 \sum_{n=1}^{\infty} |a_n| |a_{-1-n}| \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-1-n}\|_{L^2(D_1 \setminus D_r)}
\]
\[
\geq |a_0|^2 \pi (1 - r^2) + 2 \sum_{n=1}^{\infty} |a_n| |a_{-1-n}| \pi \sqrt{\frac{(1 - r^4)(1 - r^2)}{2r^2}}
\]
\[
\geq |a_0|^2 + \frac{2}{r} \sum_{n=1}^{\infty} |a_n| |a_{-1-n}|,
\]
and
\[
\|G_1\|_{L^2(D_1 \setminus D_r)}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \|z^n\|_{L^2(D_1 \setminus D_r)}^2
\]
\[
\geq 2 \sum_{n=0}^{\infty} |a_n| |a_{-2-n}| \|z^n\|_{L^2(D_1 \setminus D_r)} \|z^{-2-n}\|_{L^2(D_1 \setminus D_r)}
\]
\[
\geq 2 \sum_{n=1}^{\infty} |a_n| |a_{-2-n}| \pi \frac{1 - r^2}{r}
\]
\[
\geq \frac{2}{r} \sum_{n=1}^{\infty} |a_n| |a_{-2-n}|,
\]
which along with \[(4.2)\] and \[(4.3)\] yield that
\[
|b_{-1}| \leq 2 |a_{-1}| \|G_1\|_{L^2(D_1 \setminus D_r)} + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2, \tag{4.5}
\]
\[
|(a_{-1}, a_{-1})| \leq |b_{-2}| + r \|G_1\|_{L^2(D_1 \setminus D_r)}^2. \tag{4.6}
\]

It remains to estimate \( a_{-1} \). For this, we denote
\[
q(z) = q(|z|) = \frac{1}{2\pi} \int_0^{2\pi} R(ze^{i\theta}) d\theta, \quad u = \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathbb{R}^{2k}.
\]

Let \( \Omega = D_1 \setminus D_r \). We have
\[
\left\| \frac{R - q}{z} \right\|_{L^2(\Omega)}^2 = \int_r^1 \int_0^{2\pi} \frac{|R(te^{i\theta}) - q(t)|^2}{t^2} t dt d\theta
\]
\[
\leq \int_r^1 \int_0^{2\pi} \frac{|\partial_\theta R(te^{i\theta})|^2}{t^2} t dt d\theta
\]
\[
\leq \|\nabla R\|_{L^2(\Omega)}^2 \leq C \varepsilon_0.
\]

Moreover, we also have
\[
A = R^T \frac{\partial u}{\partial z}, \quad G = B^T A = B^T R^T \frac{\partial u}{\partial z},
\]
which gives

\[ G_1 = G - G_2 = B^T R^T \frac{\partial u}{\partial z} - G_2 \]

\[ = q^T \frac{\partial u}{\partial z} + (R - q)^T \frac{\partial u}{\partial z} + (B - I_k)^T R^T \frac{\partial u}{\partial z} - G_2. \]

Therefore for \( \rho = \sqrt{r/2} \), we have

\[(2\pi \ln 2)a_{-1} = \int_{D_2} G_1(z) \frac{dz}{z} = I_1 + I_2,\]

where

\[ I_1 = \int_{D_2} \frac{q^T \frac{\partial u}{\partial z} dz}{z}, \]

\[ I_2 = \int_{D_2} \left( \frac{(R - q)^T \frac{\partial u}{\partial z} + (B - I_k)^T R^T \frac{\partial u}{\partial z}}{z} - G_2 \right) dz. \]

Notice that

\[ I_1 = \int_0^{2\rho} \int_0^{2\pi} \frac{q(t)^T e^{-i\theta}}{2} (\partial_t - i \frac{1}{t} \partial_\theta) u(te^{i\theta}) t dt d\theta \]

\[ = \int_0^{2\rho} \int_0^{2\pi} \frac{q(t)^T}{2} \partial_t u(te^{i\theta}) dt d\theta \in \mathbb{R}^k. \]

Thus, we obtain

\[(2\pi \ln 2) |\text{Im} a_{-1}| \leq |I_2| \leq \left\| \frac{R - q}{z} \right\|_{L^2(\Omega)} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_2 \setminus D_\rho)} + \left\| \frac{1}{z} \right\|_{L^2(D_2 \setminus D_\rho)} \| G_2 \|_{L^2(D_2 \setminus D_\rho)} \]

\[ + \| B - I_k \|_{L^\infty(\Omega)} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_2 \setminus D_\rho)} \]

\[ \leq C\epsilon_0^\frac{1}{2} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_2 \setminus D_\rho)} + C\epsilon_0 \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_2 \setminus D_\rho)} + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)} \]

\[ \leq C\epsilon_0^\frac{1}{2} \| G \|_{L^2(D_2 \setminus D_\rho)} + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)} \]

\[ \leq C\epsilon_0^\frac{1}{2} \| G_1 \|_{L^2(D_2 \setminus D_\rho)} + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)} \]

\[ \leq C\epsilon_0^\frac{1}{2} \left( 2\pi \ln 2 |a_{-1}|^2 + (2\rho)^2 \| G_1 \|_{L^2(D_1 \setminus D_\rho)}^2 \right)^\frac{1}{2} + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)}, \]

where we used (4.4) in the last inequality. This along with (4.6) gives

\[ |a_{-1}|^2 = \text{Re} \langle a_{-1}, a_{-1} \rangle + 2 |\text{Im} a_{-1}|^2 \]

\[ \leq |b_{-2}| + r \| G_1 \|_{L^2(D_1 \setminus D_\rho)}^2 + C\epsilon_0 \left( |a_{-1}|^2 + r \| G_1 \|_{L^2(D_1 \setminus D_\rho)}^2 \right) + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)}^2 \]

\[ \leq |b_{-2}| + C r \| G_1 \|_{L^2(D_1 \setminus D_\rho)}^2 + C \| G_2 \|_{L^2(D_2 \setminus D_\rho)}^2 + C\epsilon_0 |a_{-1}|^2. \]

Taking \( \epsilon_0 \) small such that \( C\epsilon_0 \leq \frac{1}{2} \), we obtain

\[ |a_{-1}|^2 \leq C \left( |b_{-2}| + r \| G_1 \|_{L^2(D_1 \setminus D_\rho)}^2 + \| G_2 \|_{L^2(D_2 \setminus D_\rho)}^2 \right). \]
Using the facts that
\[ \|G_1\|_{L^2(D_1 \setminus D_r)} + \|G_2\|_{L^2(D_1 \setminus D_r)} \leq C \|\nabla u\|_{L^2(D_1 \setminus D_r)} + C\|\tau\|_{L^p(D_1 \setminus D_r)} \leq C, \]
and \( |b_-| \leq CA_0 \) and
\[ \|G_2\|^2_{L^2(D_2 \setminus D_r)} \leq C\rho \sqrt{p} \|G_2\|^2_{L^{2p}(D_1 \setminus D_r)} \leq C\rho \sqrt{p} \|\tau\|^2_{L^p(D_1 \setminus D_r)}, \]
we deduce that
\[ |a_-|^2 \leq C(A_0 + r), \]
\[ |b_-| \leq C(|a_-| + r) \leq C(A_0^{\frac{1}{2}} + r^{\frac{1}{2}}). \]
On the other hand, we also have \( |b_-| \leq C\frac{A_0}{r} \), hence,
\[ |b_-| \leq C \min \left( \frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}} \right). \]
Collecting the estimates of \( b_n \), we finally conclude that
\[ \|h - h_0\|_{L^1(D_1)} \leq C \left( \|\tau(u)\|_{L^p(D_1)} + A_0 \ln \frac{1}{r} + |b_-| \right) \]
\[ \leq C \left( \|\tau(u)\|_{L^p(D_1)} + A_0 \ln \frac{1}{r} + \min \left( \frac{A_0}{r}, A_0^{\frac{1}{2}} + r^{\frac{1}{2}} \right) \right). \]
This completes the proof of Lemma 4.3. \( \Box \)

**Lemma 4.4.** Let \( 0 < r < \rho \). If \( h \in L^1(D(z_0, 2r + \rho)) \) and for any \( z \in D(z_0, r + \rho) \), there exists a holomorphic function \( h_{z,r} \) in \( D(z, r) \) so that
\[ \|h - h_{z,r}\|_{L^1(D(z,r))} \leq A_0, \]
then there exists a holomorphic function \( h_0 \) in \( D(z_0, \rho) \) so that
\[ \|h - h_0\|_{L^1(D(z_0, \rho))} \leq C\frac{A_0^3}{r^3}, \]
Here \( C \) is a constant independent of \( r, \rho, \) and \( A_0 \).

**Proof.** Let \( \phi \) be a radial cut-off function satisfying
\[ \text{supp } \phi \subseteq D_{\frac{1}{2}}, \quad \int_{\mathbb{R}^2} \phi \, dx = 1. \]
Let \( \phi_{(r)}(x) = r^{-2} \phi \left( \frac{x}{r} \right) \) and
\[ h_{(r)}(z) = \phi_{(r)} * h(z) = \int_{D(z_0, 2r + \rho)} \phi_{(r)}(z - y) h(y) \, dy \]
for \( z \in D(z_0, r + \rho) \). We have \( \phi_{(r)} * h_{z,r} = h_{z,r} \) in \( D(z, \frac{r}{2}) \) due to the mean value equality of holomorphic function. Therefore,
\[ \|h - h_{(r)}\|_{L^1(D(z, \frac{r}{2}))} \leq \|h - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} + \|h_{(r)} - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} \]
\[ = \|h - h_{z,r}\|_{L^1(D(z, \frac{r}{2}))} + \|\phi_{(r)} * (h - h_{z,r})\|_{L^1(D(z, \frac{r}{2}))} \]
\[ \leq \|h - h_{z,r}\|_{L^1(D(z,r))} + \|\phi_{(r)}\|_{L^1} \|h - h_{z,r}\|_{L^1(D(z,r))} \]
\[ = 2A_0. \]
Using Fubini theorem, we get
\[
\int_{D(z_0, \frac{r}{2} + \rho)} \|h - h(r)\|_{L^1(D(z, \frac{r}{2}))} dz = \int_{D(z_0, \frac{r}{2})} \|h - h(r)\|_{L^1(D(z, \frac{r}{2} + \rho))} dz \\
\geq \int_{D(z_0, \frac{r}{2})} \|h - h(r)\|_{L^1(D(z, \frac{r}{2} + \rho))} dz \\
= \pi \left( \frac{r}{2} \right)^2 \|h - h(r)\|_{L^1(D(z_0, \rho))},
\]
which implies
\[
\|h - h(r)\|_{L^1(D(z_0, \rho))} \leq \left( 1 + \frac{2\rho}{r} \right)^2 \sup_{z \in D(z_0, \frac{r}{2} + \rho)} \|h - h(r)\|_{L^1(D(z, \frac{r}{2} + \rho))} \\
\leq \left( 1 + \frac{2\rho}{r} \right)^2 (2A_0).
\]
(4.9)

Notice that in \(D(z, r/2)\), we have
\[
\frac{\partial}{\partial z} h(r) = \frac{\partial}{\partial z} (h(r) - h_{z,r}) = \frac{\partial}{\partial z} (\phi(r) \ast (h - h_{z,r})) \\
= \left( \frac{\partial}{\partial z} \phi(r) \right) \ast (h - h_{z,r}).
\]
Therefore,
\[
\left\| \frac{\partial}{\partial z} h(r) \right\|_{L^1(D(z, \frac{r}{2}))} \leq \left\| \frac{\partial}{\partial z} \phi(r) \right\|_{L^1} \left\| h - h_{z,r} \right\|_{L^1(D(z,r))} \leq \frac{C}{r} A_0,
\]
which implies (similar to (4.9))
\[
\left\| \frac{\partial}{\partial z} h(r) \right\|_{L^1(D(z_0, \rho))} \leq \left( 1 + \frac{2\rho}{r} \right)^2 \sup_{z \in D(z_0, \frac{r}{2} + \rho)} \left\| \frac{\partial}{\partial z} h(r) \right\|_{L^1(D(z, \frac{r}{2} + \rho))} \\
\leq \left( 1 + \frac{2\rho}{r} \right)^2 \frac{C}{r} A_0.
\]

Now we let \( h_1 = \left( \frac{1}{\pi z} \right) \ast \left( \frac{\partial}{\partial z} h(r) \chi_{D(z_0, \rho)} \right) \) and \( h_0 = h(r) - h_1 \). Then \( h_0 \) is holomorphic in \( D(z_0, \rho) \) and by (4.9),
\[
\|h - h_0\|_{L^1(D(z_0, \rho))} = \|h - h(r) + h_1\|_{L^1(D(z_0, \rho))} \\
\leq \|h - h(r)\|_{L^1(D(z_0, \rho))} + \|h_1\|_{L^1(D(z_0, \rho))} \\
\leq \left( 1 + \frac{2\rho}{r} \right)^2 \left( 2A_0 + \frac{C\rho}{r} A_0 \right) \\
\leq \frac{C\rho^2}{r^3} A_0,
\]
here we used
\[
\|h_1\|_{L^1(D(z_0, \rho))} \leq \left\| \frac{1}{\pi z} \right\|_{L^1(D(2\rho))} \left\| \frac{\partial}{\partial z} h(r) \right\|_{L^1(D(z, \frac{r}{2} + \rho))} \leq \left( 1 + \frac{2\rho}{r} \right)^2 \frac{C\rho}{r} A_0.
\]
This completes the proof of Lemma 4.4. □
Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. We use the induction argument. The case of $m = 1$ follows from Lemma 4.2. Let us assume that the case of $m - 1$ is true. The proof of the assertion for $m$ is split into many cases.

Case 1. $\|\tau(u)\|_{L^p(D_1)} \geq 1$.

In this case, we may take $h_0 = 0$, since

$$\|h\|_{L^1(D_1)} \leq \left\| \frac{\partial u}{\partial z} \right\|_{L^2(D_1)} \leq E(u, D_1) \leq m\epsilon_0 \leq m\epsilon_0 \|\tau(u)\|_{L^p(D_1)}^{31-m}.$$  

Case 2. $\|\tau(u)\|_{L^p(D_1)} \leq 1$ and $E(u, D_\frac{1}{4}) \leq \epsilon_0$.

We first consider the function $v(z) = u(z/t^2)$, which satisfies

$$E(v, D_1) = E(u, D_\frac{1}{4}) \leq \epsilon_0, \quad \tau(v)(z) = \frac{1}{16} \tau(u)(\frac{z}{4}),$$

hence,

$$\|\tau(v)\|_{L^p(D_1)} = \left(\frac{1}{4}\right)^{\frac{p-2}{p}} \|\tau(u)\|_{L^p(D_\frac{1}{4})} \leq \|\tau(u)\|_{L^p(D_1)} \leq 1.$$  

Then Lemma 4.2 ensures that there exists a holomorphic function $\tilde{h}_0$ in $D_1$ so that

$$\|\tilde{h} - \tilde{h}_0\|_{L^1(D_1)} \leq C\|\tau(v)\|_{L^p(D_1)},$$

where $\tilde{h}(z) = (\frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}}) = \frac{1}{16} h(z/t^2)$. Therefore, $h_0(z) = 16\tilde{h}_0(4z)$ is holomorphic in $D_{\frac{1}{4}}$ and satisfies

$$\|h - h_0\|_{L^1(D_{\frac{1}{4}})} = \|\tilde{h} - \tilde{h}_0\|_{L^1(D_1)} \leq C\|\tau(v)\|_{L^p(D_1)} \leq C\|\tau(u)\|_{L^p(D_1)} \leq C\|\tau(u)\|_{L^p(D_1)}^{31-m}.$$  

Case 3. $\|\tau(u)\|_{L^p(D_1)} \leq 1$ and $Q(\frac{1}{8}) \leq (m-1)\epsilon_0$, where

$$Q(t) \overset{\text{def}}{=} \sup_{D(z,t) \subseteq D_1} E(u, D(z,t)) \text{ for } t \in [0,1].$$

Obvious, $Q(t)$ is continuous and non-decreasing in $t$ and $Q(0) = 0$. In this case, we let $v(z) = u(z' + \frac{z}{8})$ for $z' \in D_{1/2}$. Then $v$ satisfies

$$E(v, D_1) = E(u, D(z', \frac{1}{8})) \leq Q(\frac{1}{8}) \leq (m-1)\epsilon_0,$$

and $\tau(v)(z) = \frac{1}{64} \tau(u)(z' + \frac{z}{8})$, hence,

$$\|\tau(v)\|_{L^p(D_1)} = \left(\frac{1}{8}\right)^{\frac{p-2}{p}} \|\tau(u)\|_{L^p(D(z', \frac{1}{8}))} \leq \|\tau(u)\|_{L^p(D_1)}.$$  

Then the induction assumption ensures that there exists a holomorphic function $\tilde{h}_0$ in $D_{\frac{1}{4}}$ so that

$$\|\tilde{h} - \tilde{h}_0\|_{L^1(D_{\frac{1}{4}})} \leq C_m \|\tau(v)\|_{L^p(D_1)}^{32-m},$$

where $\tilde{h}(z) = (\frac{\partial v}{\partial z}, \frac{\partial v}{\partial \bar{z}}) = \frac{1}{64} h(z' + \frac{z}{8})$. Set $r = \frac{1}{32}$. Then $h_{z',r}(z) = 64\tilde{h}_0(8(z-z'))$ is holomorphic in $D(z', \frac{1}{8})$ and satisfies

$$\|h - h_{z',r}\|_{L^1(D(z', \frac{1}{32}))} = \|\tilde{h} - \tilde{h}_0\|_{L^1(D_{\frac{1}{4}})} \leq C_m \|\tau(v)\|_{L^p(D_1)}^{32-m} \leq C_m \|\tau(u)\|_{L^p(D_1)}^{32-m}.$$
Therefore, $h$ satisfies the conditions in Lemma 4.3 with $z_0 = 0, r = \frac{1}{4}, \rho = \frac{1}{4}$ and $A_0 = C_{m-1}\|\tau(u)\|_{L^p(D_1)}^{3^2-m}$. Thus, Lemma 4.4 ensures the existence of a holomorphic function $h_0$ in $D_{1/4}$ so that

$$\|h - h_0\|_{L^1(D_{1/4})} \leq C_1\rho_3 A_0 = 8^3 CC_{m-1}\|\tau(u)\|_{L^p(D_1)}^{3^2-m} \leq CC_{m-1}\|\tau(u)\|_{L^p(D_1)}^{3^2-m}.$$

**Case 4.** $\|\tau(u)\|_{L^p(D_1)} \leq 1$, $Q(\frac{1}{8}) > (m-1)\epsilon_0$ and $E(u, D_{1/4}) > \epsilon_0$.

In this case, there exists $0 < r_0 < \frac{1}{8}$ such that $Q(r_0) = (m-1)\epsilon_0$. Thus, there exists $D(z_0, r_0) \subseteq D_1$ so that $E(u, D(z_0, r_0)) = (m-1)\epsilon_0$. Hence,

$$E(u, D(z_0, r_0)) + E(u, D_{1/4}) > E(u, D_1),$$

which implies that $D(z_0, r_0) \cap D_{1/4} \neq \emptyset$, thus $|z_0| \leq \frac{1}{4} + r_0$ and then $D(z_0, 4r_0) \subseteq D_1$. For $z' \in D(z_0, 3r_0)$, the function $v(z) = u(z' + r_0z)$ satisfies

$$E(v, D_1) = E(u, D(z', r_0)) \leq Q(r_0) = (m-1)\epsilon_0,$$

and $\tau(v)(z) = r_0^2 \tau(u)(z' + r_0z)$, hence,

$$\|\tau(v)\|_{L^p(D_1)}^2 = \frac{2^{2p-2}}{r_0^{2p}} \|\tau(u)\|_{L^p(D(z', r_0))} \leq \frac{2^{2p-2}}{r_0^{2p}} \|\tau(u)\|_{L^p(D_1)}.$$

Then following argument of Case 3(using Lemma 4.4 with $A_0 = C_{m-1}\left(\frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}\right)^3 2^2-m$, $r = \frac{1}{4}, \rho = 2r_0$ ), we can conclude the existence of a holomorphic function $\tilde{h}_0$ in $D(z_0, 2r_0)$ such that

$$\|h - \tilde{h}_0\|_{L^1(D(z_0, 2r_0))} \leq C_2\rho_3 A_0 = 8^3 CC_{m-1}\left(\frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}\right)^3 2^2-m.$$

Now set $\rho_0 = \frac{1}{2} + r_0$, $r = r_0/\rho_0$ and consider the function $v(z) = u(\rho_0z + z_0)$. Then we have

$$E(v, D_1 \setminus D_r) = E(u, D(z_0, \rho_0) \setminus D(z_0, r_0)) \leq E(u, D_1) - E(u, D(z_0, r_0)) \leq \epsilon_0,$$

$$\|\tau(v)\|_{L^p(D_1)} = \frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D(z', r_0))} \leq \|\tau(u)\|_{L^p(D_1)} \leq 1.$$

For $0 < r_0 < r < \frac{1}{4}$, the function $h_{0, 2r}(z) = \rho_0^2 \tilde{h}_0(\rho_0z + z_0)$ is holomorphic in $D_{2r}$ and satisfies

$$\|\tilde{h} - h_{0, 2r}\|_{L^1(D_{2r})} + r^{2p-2} \frac{2p-2}{r_0^p} \|\tau(v)\|_{L^p(D_1)} = \|h - \tilde{h}_0\|_{L^1(D(z_0, 2r_0))} + r^2 \frac{2p-2}{r_0^p} \|\tau(v)\|_{L^p(D_1)}$$

$$\leq 8^3 CC_{m-1}\left(\frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}\right)^3 2^2-m + \frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}$$

$$\leq CC_{m-1}\left(\frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}\right)^3 2^2-m,$$

here $\tilde{h} = \langle \frac{\partial u}{\partial z} \cdot \frac{\partial u}{\partial z} \rangle = \rho_0^2 h(\rho_0z + z_0)$. Therefore, $v$ and $\tilde{h}$ satisfies the conditions of Lemma 4.3 with

$$(4.10) \quad A_0 = CC_{m-1}\left(\frac{2p-2}{r_0^p} \|\tau(u)\|_{L^p(D_1)}\right)^3 2^2-m.$$

Thus, there exists a holomorphic function $\tilde{h}_0$ in $D_1$ such that

$$\|\tilde{h} - \tilde{h}_0\|_{L^1(D_1)} \leq C\left(A_0 \ln \frac{1}{r} + \min \left\{ \frac{A_0}{r}, A_0 + r^2 \right\} + \|\tau(u)\|_{L^p(D_1)} \right).$$
For \( A_0 \) given by (4.10), we have
\[
A_0 \ln \frac{1}{r} \leq \frac{p}{2p-2} 3^{m-2} C_{m-1} \| \tau(u) \|_{L^p(D_1)}^{2-m} \leq C 3^m C_{m-1} \| \tau(u) \|_{L^p(D_1)}^{2-m} \leq A_0 \ln \frac{1}{r} + \min \left( \frac{A_0}{r}, A_0 \right) \leq A_0 \ln \frac{1}{r} + \min \left( \frac{A_0}{r}, A_0 \right) \leq A_0 \ln \frac{1}{r} + \min \left( \frac{A_0}{r}, A_0 \right) \leq C C_{m-1} \| \tau(u) \|_{L^p(D_1)}^{2-m}.
\]
This gives
\[
\| \tilde{h} - \tilde{h}_0 \|_{L^1(D_1)} \leq C 3^m C_{m-1} \| \tau(u) \|_{L^p(D_1)}^{2-m}.
\]
Let \( h_0(z) = \rho_0^{-2} \tilde{h}_0 \left( \frac{z - z_0}{\rho_0} \right) \). Then \( h_0(z) \) is holomorphic in \( D(z_0, \rho_0) \) and satisfies
\[
\| h - h_0 \|_{L^1(D_1 / \delta)} \leq \| h - h_0 \|_{L^1(D(z_0, \rho_0))} = \| \tilde{h} - \tilde{h}_0 \|_{L^1(D_1)} \leq C 3^m C_{m-1} \| \tau(u) \|_{L^p(D_1)}^{2-m}.
\]
Therefore, the assertion holds for \( m \) with \( C_m = C 3^m C_{m-1} \). The proof of Proposition 4.1 is completed.

5. **Proof of Theorem 1.1**

Let us first prove the following energy decay estimates for the map satisfying the assumptions of Lemma 4.3.

**Lemma 5.1.** Let \( u \) be as in Lemma 4.3. Then it holds that
\[
E(u, D_\rho \setminus D_{\ell/\rho}) \leq C \left( (A_0 + r) \ln \frac{\rho}{r} + \rho^{\frac{4p-4}{p}} \right) \quad \text{for} \quad \sqrt{r} \leq \rho \leq 1,
\]
\[
\text{osc}(u, D_\rho \setminus D_{\ell/\rho}) \leq C \left( (A_0 + r)^{\frac{1}{2}} \ln \frac{1}{r} + \rho^{\frac{2p-4}{p}} \right) \quad \text{for} \quad 2\sqrt{r} \leq \rho \leq \frac{1}{2}.
\]

**Proof.** Here we will use the notations in the proof of Lemma 4.3. By (4.11) and (4.12), we get
\[
E(u, D_\rho \setminus D_{\ell/\rho}) \leq C \| G \|_{L^2(D_\rho \setminus D_{\ell/\rho})} \leq C \left( \| G_1 \|_{L^2(D_\rho \setminus D_{\ell/\rho})}^2 + \| G_2 \|_{L^2(D_\rho \setminus D_{\ell/\rho})}^2 \right)
\leq C \left( (A_0 + r) \ln \frac{\rho}{r} + \rho^2 \| G_1 \|_{L^2(D_{\ell/\rho})}^2 + \rho^{\frac{4p-4}{p}} \| G_2 \|_{L^2(D_{\ell/\rho})}^2 \right)
\leq C \left( (A_0 + r) \ln \frac{\rho}{r} + \rho^2 + \rho^{\frac{4p-4}{p}} \| \tau \|_{L^p(D_{\ell/\rho})} \right)
\leq C \left( (A_0 + r) \ln \frac{\rho}{r} + \rho^{\frac{4p-4}{p}} \right).
\]
If \( 2\sqrt{r} \leq \rho \leq \frac{1}{2} \), there exists a positive integer \( \ell > 1 \) such that \( e^{-\ell} \rho < r/\rho \leq e^{-\ell+1} \rho \) and \( \ell \leq \ln \frac{\rho^2}{r} + 1 \). Let \( \rho_j = e^{-j} \rho \) for \( 0 \leq j < \ell \) and \( \rho_\ell = r/\rho \). Then by (4.14) and Sobolev...
embedding and the fact that $G_1$ is holomorphic, we deduce that for $0 < j \leq \ell$,

$$\text{osc}(u, D_{\rho_j} \setminus D_{\rho_j}) \leq C \rho_j^{\frac{2p-2}{p}} \| \nabla u \|_{L^{\frac{2p}{2p-2}}(D_{\rho_j} \setminus D_{\rho_j})} \leq C \rho_j^{\frac{2p-2}{p}} \| G \|_{L^{\frac{2p}{2p-2}}(D_{\rho_j} \setminus D_{\rho_j})}$$

$$\leq C \rho_j^{\frac{2p-2}{p}} \left( \| G_1 \|_{L^{\frac{2p}{2p-2}}(D_{\rho_j} \setminus D_{\rho_j})} + \| G_2 \|_{L^{\frac{2p}{2p-2}}(D_{\rho_j} \setminus D_{\rho_j})} \right)$$

$$\leq C \left( \| G_1 \|_{L^2(D_{2\rho_j} \setminus D_{\rho_j})} + \rho_j^{\frac{2p-2}{p}} \| G_2 \|_{L^{\frac{2p}{2p-2}}(D_{\rho_j} \setminus D_{\rho_j})} \right)$$

$$\leq C \left( |a_{-1}| + \max \{2e\rho_j, 2r/\rho_j\} \| G_1 \|_{L^2(D_{\rho_j} \setminus D_{\rho_j})} + \rho_j^{\frac{2p-2}{p}} \| \tau \|_{L^p(D_{\rho_j} \setminus D_{\rho_j})} \right)$$

$$\leq C \left( |a_{-1}| + \max \{2e\rho_j, 2r/\rho_j\} + \rho_j^{\frac{2p-2}{p}} \right),$$

which gives

$$\text{osc}(u, D_{\rho} \setminus D_{r/\rho}) \leq \sum_{j=1}^{\ell} \text{osc}(u, D_{\rho_{j-1}} \setminus D_{\rho_j})$$

$$\leq C \sum_{j=1}^{\ell} \left( |a_{-1}| + \max \{2e\rho_j, 2r/\rho_j\} + \rho_j^{\frac{2p-2}{p}} \right)$$

$$\leq C \left( |a_{-1}| \ell + \rho + \rho^{\frac{2p-2}{p}} \right)$$

$$\leq C \left( (A_0 + r)^{\frac{1}{2}} \ln \frac{1}{r} + \rho + \rho^{\frac{2p-2}{p}} \right).$$

The proof is finished. \qed

Now let us complete the proof of Theorem 1.1. By the construction of bubble tree, it is sufficient to prove (2.1) and (2.2) under the assumption of $E(u_n, D(x_n, r_n f_i^{(j)}(\delta_0)) \setminus D(x_n, r_n r_0 R_0)) \leq \epsilon$.

We can also assume $\delta_0^{1/p} \Lambda \leq 1$, $r_n \leq 1$. Let $m$ be a positive integer so that $m > \Lambda^2/\epsilon$. Let $r = R_0 r_n^{1/2}$ and consider the function $w_n(z) = u_n(x_n + \delta_0 r_n^{(i)} z)$. It holds that

$$E(w_n^i, D_1 \setminus D_r) = E(u_n, D(x_n, r_n f_i^{(j)}(\delta_0)) \setminus D(x_n, r_n r_0 R_0)) \leq \epsilon,$$

$$\| \tau(w_n^i) \|_{L^p(D_1)} = (\delta_0 r_n^{f_i^{(j)}})^{\frac{2p-2}{p}} \| \tau(u_n) \|_{L^p(D_0)} \leq \delta_0^{\frac{2p-2}{p}} \Lambda \leq 1.$$

For $n$ sufficiently large, $0 < r_n < r < \frac{1}{8}$. We consider the function $v_n(z) = w_n^i(8rz) = u_n(x_n + 8R_0 r_n^i z)$, which satisfies

$$E(v_n^i, D_1) = E \left( u_n, D(x_n, 8r_n R_0) \right) \leq \Lambda^2 \leq m\epsilon,$$

$$\| \tau(v_n^i) \|_{L^p(D_1)} = (8r)^{\frac{2p-2}{p}} \| \tau(w_n^i) \|_{L^p(D_{8r})} \leq (8r)^{\frac{2p-2}{p}} \leq 1.$$

Thus, Proposition 4.1 and scaling argument ensure that there exists a holomorphic function $h_{0,2r}^{n,i}$ in $D_{2r}$ such that

$$\| h_{0,2r}^{n,i} - h_{0,2r}^{n,i} \|_{L^1(D_{2r})} \leq C_m \| \tau(v_n^i) \|_{L^p(D_1)}^{3^{1-m}} \leq C_m (8r)^{\frac{2p-2}{p}3^{1-m}}.$$
Hence, we get
\[
\|h^{n,i} - h_{0,2r}^{n,i}\|_{L^1(D_{2r})} + r^{2p-2} p \|\tau(u_n^i)\|_{L^p(D_1)} = C_n(8r^{2p-2} + r^{2p-2} p)
\leq CC_m r^{2p-2} 3^{1-m},
\]
here \(h^{n,i} = \langle \frac{\partial w_{n,i}^i}{\partial z}, \frac{\partial w_{n,i}^i}{\partial z} \rangle = \langle \delta_{n,i} r_{n,i}^i \rangle \). Thus, the conditions in Lemma 4.3 are satisfied for \(w_n^i\) with \(A_0 = CC_m r^{2p-2} 3^{1-m}\). For this \(A_0\), it holds that
\[
\lim_{n \to \infty} (A_0 + r) \ln \frac{\rho^2}{r} = \lim_{n \to \infty} (CC_m r^{2p-2} 3^{1-m} + r) \ln \frac{\rho^2}{r} = 0,
\]
as \(r_n^i \to 0\) (recall \(r = R_0 r_n^i / (\delta_0 r_{n,i}^i) \to 0\)). Then we apply Lemma 5.1 to conclude that
\[
\limsup_{n \to \infty} E(u_n, D(x_n^i, \rho_{n,i} f(i) \delta_0) \setminus D(x_n^i, r_n^i R_0 / \rho)) \leq C \rho^{4p-4},
\]
\[
\limsup_{n \to \infty} \text{osc}(u_n, D(x_n^i, \rho_{n,i} f(i) \delta_0) \setminus D(x_n^i, r_n^i R_0 / \rho)) \leq C \rho^{2p-2},
\]
which yield (2.1) and (2.2) by taking \(\rho \to 0\). The proof of Theorem 1.1 is completed. \(\square\)

6. A NECESSARY AND SUFFICIENT CONDITION OF ENERGY IDENTITY

Let us first recall the following result [11 4].

**Lemma 6.1.** If \(\tau(u_n)\) is bounded in \(L^p\) for some \(p > 1\), then the tangential energy on the neck domain is vanishing, i.e.,
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \int_{D(0, r_n^i \delta) \setminus D(0, r_n^i R)} |x|^{-2} |\partial_{\theta} u_n (x_n^i + x)|^2 dx = 0.
\]

Notice that
\[
h_n(x_n^i + x) = h_n(x_n^i + r e^{i\theta})
\]
\[
= \frac{1}{4} e^{-2i\theta} (|\partial_{r} u_n|^2 - r^{-2} |\partial_{\theta} u_n|^2 - 2i \frac{r}{r} (\partial_{r} u_n, \partial_{\theta} u_n)) (x_n^i + r e^{i\theta})
\]
which motivates the following equivalent statement of the energy identity.

**Proposition 6.2.** Let \(h_n = \langle \frac{\partial u_n}{\partial z}, \frac{\partial u_n}{\partial z} \rangle\). Then the energy identity holds if and only if
\[
(6.1) \quad \lim_{\delta \to 0} \limsup_{n \to \infty} \|h_n\|_{L^1(D_\delta)} = 0.
\]

**Proof.** On the one hand, if (6.1) holds, then we have by Lemma 6.1 that
\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \int_{D(0, x_n^i \delta) \setminus D(0, r_n^i R)} (|\partial_{r} u_n|^2 - |x|^{-2} |\partial_{\theta} u_n|^2) (x_n^i + x) dx
\]
\[
\leq \lim_{\delta \to 0} \limsup_{n \to \infty} 2 \int_{D(0, x_n^i \delta) \setminus D(0, r_n^i R)} |h_n(x_n^i + x)| dx
\]
\[
= \lim_{\delta \to 0} \limsup_{n \to \infty} 2 \|h_n\|_{L^1(D(0, x_n^i \delta) \setminus D(0, r_n^i R))}
\]
\[
\leq \lim_{\delta \to 0} \limsup_{n \to \infty} 2 \|h_n\|_{L^1(D_\delta)} = 0,
\]

which gives (2.1), thus the energy identity.

On the other hand, if the energy identity holds, we denote

\[ w^i_n(z) = u_n(x^i_n + r^i_n z), \quad h^i_n = \left( \frac{\partial w^i_n}{\partial z}, \frac{\partial w^i_n}{\partial \bar{z}} \right), \quad h^i = \left( \frac{\partial w^i}{\partial z}, \frac{\partial w^i}{\partial \bar{z}} \right) \]

for \( i = 1, \ldots, k_0 \). Then \( h^i_n(z) = (r^i_n)^2 h_n(x^i_n + r^i_n z) \), and \( h^i \) is a \( L^1 \) holomorphic function in \( \mathbb{C} \), thus \( h^i = 0 \). Thanks to

\[ u^i_n \to u^i \text{ strongly in } W^{1,2}(D_0, \cup_{z \in \Omega} D(x, \delta)) \quad \text{as } n \to \infty, \]

we infer that as \( n \to \infty \),

\[ h^i_n \to h^i = 0 \quad \text{strongly in } L^1(D_0, \cup_{z \in \Omega} D(x, \delta)), \]

\[ \|h_n\|_{L^1(D(x, \delta))} = \|h^i_n\|_{L^1(D(x, \delta))} \to 0. \]

Notice that

\[ \|h_n\|_{L^1(D_0 \setminus D_\delta)} \leq \sum_{i=1}^{k_0} \|h^i_n\|_{L^1(D(x^i_n, r^i_n \delta))} \cup_{z \in \Omega} D(x^i_n + r^i_n z, \delta)} + \sum_{i=1}^{k_0} \|h^i_n\|_{L^1(D(x^i_n, r^i_n \delta))} \cup_{z \in \Omega} D(x^i_n + r^i_n z, \delta)} \]

\[ \leq \sum_{i=1}^{k_0} \left| E(u_n, D(x^i_n, r^i_n \delta)) \cup D(x^i_n, r^i_n \delta)) \right| + \sum_{i=1}^{k_0} E(u_n, D(x^i_n, r^i_n \delta)) \cup D(x^i_n, r^i_n \delta)). \]

Thus, (6.1) follows easily from (2.1). \( \square \)

Using Lemma 6.2 and Proposition 4.1, let us present another proof of the energy identity.

Let \( m \) be a positive integer so that \( m > \Lambda^2/\epsilon \). Then for fixed \( 0 < \delta_1 < 1 \), we consider the function \( w_n(z) = u_n(\delta_1 z) \), which satisfies

\[ E(w_n, D_1) = E(u_n, D_{\delta_1}) \leq \Lambda^2 \leq m \epsilon, \]

\[ \|\tau(w_n)\|_{L^p(D_1)} = \delta_1^{2p-2} \|\tau(u_n)\|_{L^p(D_{\delta_1})} \leq \delta_1^{2p-2} \Lambda. \]

Thus, Proposition 4.1 and scaling argument ensure that there exists a holomorphic function \( h_{0,n} \) in \( D_{\delta_1/4} \) such that

\[ \|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} \leq C_m \|\tau(w_n)\|_{L^p(D_1)}^{3^1-m} \leq CC_m(\delta_1)^{\frac{2p-2}{p} 3^{1-m}}. \]

Therefore, for \( 0 < \delta < \delta_1/4 \),

\[ \|h_n\|_{L^1(D_\delta)} \leq \|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} + \|h_{0,n}\|_{L^1(D_{\delta_1/4})} \]

\[ \leq \|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} + (4\delta/\delta_1)^2 \|h_{0,n}\|_{L^1(D_{\delta_1/4})} \]

\[ \leq 2 \|h_n - h_{0,n}\|_{L^1(D_{\delta_1/4})} + (4\delta/\delta_1)^2 \|h_n\|_{L^1(D_{\delta_1/4})} \]

\[ \leq CC_m(\delta_1)^{\frac{2p-2}{p} 3^{1-m}} + (4\delta/\delta_1)^2 \Lambda^2. \]

Then (6.1) follows by first letting \( \delta \to 0 \), then letting \( \delta_1 \to 0 \).
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References

[1] W. Ding and G. Tian, Energy identity for a class of approximate harmonic maps from surfaces, Comm. Anal. Geom., 3(1995), 543-554.
[2] F. Hélein, Harmonic Maps, Conservation Laws and Moving Frames, Diderot, Paris, 1997.
[3] J. Li and X. Zhu, Small energy compactness for approximate harmonic mappings, Comm. Contemp. Math., 13(2011), 741-763.
[4] J. Li and X. Zhu, Energy identity for the maps from a surface with tension field bounded in $L^p$, Pacific J. Math., 260 (2012), 181-195.
[5] F.-H. Lin and C. Wang, Energy identity of harmonic map flows from surfaces at finite singular time, Calc. Var. Partial Differential Equations, 6(1998), 369-380.
[6] F.-H. Lin and C. Wang, Harmonic and quasi-harmonic spheres, II, Comm. Anal. Geom., 10 (2002), 341-375.
[7] F.-H. Lin and C. Wang, The analysis of harmonic maps and their heat flows, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
[8] Y. Luo, Energy identity and removable singularities of maps from a riemann surface with tension field unbounded in $L^2$, Pacific J. Math., 256 (2012), 365-380.
[9] D. McDuff and D. Salamon, J-holomorphic curves and symplectic topology, American Mathematical Society Colloquium Publications, 52, American Mathematical Society, Providence, RI, 2012.
[10] T. H. Parker, Bubble tree convergence for harmonic maps, J. Differential Geom., 44 (1996), 595-633.
[11] J. Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres, Comm. Anal. Geom., 3(1995), 297-315.
[12] J. Qing and G. Tian, Bubbling of the heat flows for harmonic maps from surfaces, Comm. Pure Appl. Math., 50(1997), 295-310.
[13] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, Ann. of Math., 113(1981), 1-24.
[14] P. Topping, Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow, Ann. of Math., 159(2004), 465-534.
[15] P. Topping, Winding behaviour of finite-time singularities of the harmonic map heat flow, Math. Z., 247 (2004), 279-302.
[16] C. Wang, Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets, Houston J. Math., 22(1996), 559-590.
[17] X. Zhu, No neck for approximate harmonic maps to the sphere, Nonlinear Anal., 75(2012), 4339-4345.
[18] X. Zhu, Bubble tree for approximate harmonic maps, Proc. Amer. Math. Soc., 142 (2014), 2849-2857.

School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China
E-mail address: wendong@dlut.edu.cn

School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: jnwdyi@163.com

School of Mathematical Sciences and LMAM, Peking University, Beijing 100871, China
E-mail address: zfzhang@math.pku.edu.cn