Rasmussen $s$-invariants of satellites do not detect slice knots

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Dedicated to the memory of Tim Cochran

Abstract. We present a large family of knots for which the Rasmussen $s$-invariants of arbitrary satellites do not detect sliceness. This answers a question of Hedden. The proof hinges on work of Kronheimer-Mrowka and Cochran-Harvey-Horn.

1. Introduction

In [Ras10], Rasmussen introduced a smooth knot concordance invariant $s(K)$ using a deformed version of Khovanov homology. In general, while invariants from Khovanov homology have common aspects with and are related to those from Heegaard Floer homology, it is expected and often confirmed that they behave very distinctively. For instance, the volume conjecture tells us that Jones polynomials of cables contain significantly more information than the Alexander polynomials of cables (which are completely determined by the Alexander polynomial of the initially given knot). In case of the $s$-invariant, it shares several properties with the $\tau$-invariant of Ozsváth-Szabó and Rasmussen, which may be viewed as its Heegaard Floer “analog”, but Hedden and Ording showed that $s$ is independent from $\tau$ [HO08]. Regarding the cabling, Hedden asked the following question:

**Question** ([Hed09, Question 1.4]). Does the Rasmussen $s$-invariant, applied to all iterated cables of $K$, determine whether $K$ is smoothly slice?

We remark that the behavior of $\tau$ under cabling is well understood by work of Hedden and Hom [Hed09] [Hom14a]. See also [VC10, Pet13]. In particular the $\tau$ version of the above question was answered in the negative, in a similar fashion to the Alexander polynomial case (but in a more sophisticated way) [Hed09] [Hom14a]. The $s$-invariant case was left open, mainly because of the difficulty of analyzing the Khovanov chain complex of cables.

The goal of this note is to answer Hedden’s question on the $s$-invariant by presenting a large family of counterexamples. To state it, we use the following condition for a knot $K$ in $S^3$, motivated by work of Kronheimer and Mrowka [KM13] and Cochran, Harvey, and Horn [CHH13]:

(KM) There exist pairs $(V_+, D_+)$, $(V_-, D_-)$ of a compact smooth 4-manifold $V_\pm$ and a smoothly embedded disk $D_\pm$ in $V_\pm$ such that $\partial(V_\pm, D_\pm) = (S^3, K)$, $b_1(V_\pm) = 0$, $V_\pm$ is $\pm$-definite, i.e., $b_2^+(V_\pm) = b_2(V_\pm)$, and $[D_\pm, \partial D_\pm] = 0$ in $\pi_2(V_\pm, S^3)$.

By [KM13 Corollary 1.1], $s(K) = 0$ if $K$ satisfies (KM).

For knots $K \subset S^3$ and $P \subset S^1 \times D^2$, denote by $P(K)$ the satellite knot with pattern $P$ and companion $K$. Denote the unknot by $U$.

**Theorem A.** If $K$ is a knot satisfying (KM), then $s(P(K)) = s(P(U))$ for any pattern $P \subset S^1 \times D^2$. 
Consequently, if $K$ satisfies (KM), any iterated cable of $K$ has the same $s$-invariant as the corresponding iterated cable of the unknot. This answers Hedden’s question in the negative.

The collection of knots satisfying (KM) is large. For instance, 0-bipolar knots in the sense of [CHH13] satisfy (KM). Especially if a knot has a diagram from which a slice knot is obtained by changing positive crossings and has a (possibly different) diagram from which a slice knot is obtained by changing negative crossings, then the knot satisfies (KM) [CL86, Lemma 3.4].

To describe explicit examples, let $K(a, -b)$ be the knot shown in Figure 1. Then $K(1, -n)$ and $K(n, -n)$ satisfy (KM) for any $n > 0$ by the above. It is known that $K(1, -n)$ for $2 \neq n > 0$ and $K(n, -n)$ for $n > 0$ are not slice, even topologically (e.g., see [CG86, Jia81, Kim05, Cha07]). In fact, they generate a subgroup isomorphic to $\mathbb{Z}_\infty \oplus \mathbb{Z}_2^\infty$ in the smooth and topological knot concordance group.

The figure eight knot is the simplest case ($n = 1$). We remark that $K(n, -n)$ with $n > 0$ has order 2 in the knot concordance group, and is smoothly rationally slice, that is, bounds a smoothly embedded disk in a rational homology ball whose boundary is $S^3$. See [Cha07] Theorem 4.16 and Figure 6] for a proof. Consequently $P(K(n, -n))$ is rationally concordant to $P(U)$ for any $P$. This relates the case of $K(n, -n)$ to an intriguing open question (e.g., see [CPT4] Question 2.1): if $K$ is rationally slice, does $s(K)$ vanish? If so, then one would conclude immediately that $s(P(K(n, -n))) = s(P(U))$.

Our proof of Theorem A which is given in Section 2 shows the conclusion without the invariance of $s$ under rational concordance. Our method is largely influenced by work of Cochran, Harvey, and Horn [CHH13].

We remark that our argument for the $s$-invariant applies to the case of the $\tau$-invariant, the $\epsilon$-invariant and the knot Floer chain complex invariant $[CFK^\infty(K)]$ of Hom [Hom14a, Hom14b], and the $\delta_p$-invariant of Manolescu-Owens [MO07] and Jabuka [Jab12] as well. Using this, we observe that these invariants of arbitrary satellites, even when considered all together, do not detect slice knots:

**Theorem B.** There are knots $K$ which generate a subgroup isomorphic to $\mathbb{Z}_\infty \oplus \mathbb{Z}_2^\infty$ in the smooth and topological knot concordance groups and satisfy $\bullet(P(K)) = \bullet(P(U))$ for any pattern $P$ and $\bullet = s, \tau, \epsilon, \delta_p, [CFK^\infty(-)]$.

The proof of Theorem B is given in Section 3.

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2. Proof of Theorem A: $s$-invariant of satellites

Henceforth we assume that everything is smooth. We say that a pattern $Q \subset S^1 \times D^2$ is slice if $Q$ viewed as a knot in $S^1 \times D^2 \cup D^2 \times S^1 = \partial(D^2 \times D^2) = S^3$ is slice, that is, $Q(U)$ is slice. Let $C$ be the knot concordance group. The following innocent observation reduces the investigation of the effect of satellite operations on a homomorphism of $C$ to the case of slice patterns. We state it as a lemma for later use as well:

**Lemma 2.1.** Suppose $f: C \to A$ is an abelian group homomorphism and $K$ is a knot in $S^3$ satisfying $f(Q(K)) = 0$ for any slice pattern $Q$. Then $f(P(K)) = f(P(U))$ for any pattern $P$.

*Proof.* For a given pattern $P$, let $Q_P$ be the pattern $P\# - P(U)$, that is, $(S^1 \times D^2, Q_P)$ is the connected sum of two pairs $(S^1 \times D^2, P)$ and $(S^3, -P(U))$. Then $Q_P(J) = P(J)\# - P(U)$ for any knot $J$. So $Q_P(U) = P(U)\# - P(U)$ is a ribbon knot in $S^3$ and thus $Q_P$ is a slice pattern. For the given $K$, by the hypotheses, we have

$$0 = f(Q_P(K)) = f(P(K)\# - P(U)) = f(P(K)) - f(P(U)).$$

The following is a variation of arguments appeared in [CHHI13 Proposition 3.3], [CPI14 Theorem 2.6 (6)].

**Lemma 2.2.** Suppose that a knot $K$ in $S^3$ satisfies (KM) and $Q \subset S^1 \times D^2$ is a slice pattern. Then the satellite knot $Q(K)$ satisfies (KM).

*Proof.* Suppose $K$ satisfies (KM) via $(V_{\pm}, D_{\pm})$. Choose a slice disk $\Delta \subset D^2 \times D^2$ bounded by $Q \subset S^1 \times D^2 \subset \partial(D^2 \times D^2)$. Choose a diffeomorphism $D^2 \times D^2 \to \nu(D_{\pm})$ onto the normal bundle $\nu(D_{\pm})$ which sends $D^2 \times 0$ to $D_{\pm}$ and $p \times D^2$ to a normal disk fiber for each $p \in D^2$. Let $\Delta_{\pm}$ be the image of the disk $\Delta$ under the diffeomorphism. Then $\partial\Delta_{\pm} = Q(K)$, that is, $\Delta_{\pm}$ is a slice disk for $Q(K)$ in $V_{\pm}$.

Now, to show that $Q(K)$ satisfies (KM), it suffices to prove that $[\Delta_{\pm}, \partial\Delta_{\pm}] = 0$ in $\pi_2(V_{\pm}, S^3)$. Consider the following commutative diagram of inclusions, where $\nu(\partial D_{\pm}) = \nu(D_{\pm}) \cap S^3$.

$$\begin{array}{ccc}
(\Delta_{\pm}, \partial\Delta_{\pm}) & \xrightarrow{\nu(D_{\pm}), \nu(\partial D_{\pm})} & (V_{\pm}, S^3) \\
\downarrow i & & \downarrow j \\
(D_{\pm}, \partial D_{\pm})
\end{array}$$

The induced map $i_*$ on $\pi_2$ is zero by the condition (KM). Since $i$ is a homotopy equivalence, $k_* = 0$ on $\pi_2$ too. It follows that $[\Delta_{\pm}, \partial\Delta_{\pm}] = 0$ in $\pi_2(V_{\pm}, S^3)$.

Now we are ready to prove Theorem A.

*Proof of Theorem A.* Suppose that $K$ satisfies (KM). For any slice pattern $Q \subset S^1 \times D^2$, $Q(K)$ satisfies (KM) by Lemma 2.2 and consequently $s(Q(K)) = 0$ by [KMI13 Corollary 1.1]. Since $s: C \to Z$ is a homomorphism, $s(P(K)) = s(P(U))$ for all pattern $P \subset S^1 \times D^2$ by Lemma 2.1.

3. Proof of Theorem B: invariants from Floer homology

Our argument used in Section 2 can be applied to $\tau$, $\epsilon$, $[CFK^\infty(-)]$, and $\delta_{\rho^*}$ in a similar way. We begin with an observation based on Hom’s work, which will be used to reduce the cases of $\epsilon$ and $[CFK^\infty(-)]$ to the case of $\tau$. 


Theorem 3.1 (Hom). For two knots $K$ and $K'$, the following are equivalent:

1. $\epsilon(K\#-K')=0$.
2. $[CFK^\infty (K)]= [CFK^\infty (K')]$.
3. $\tau(P(K))=\tau(P(K'))$ for any pattern $P$.
4. $\epsilon(P(K))=\epsilon(P(K'))$ for any pattern $P$.
5. $\epsilon(P(K)#-P(K'))=0$ for any pattern $P$.
6. $[CFK^\infty (P(K))]= [CFK^\infty (P(K'))]$ for any pattern $P$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (5) $\Leftrightarrow$ (6) are definitions in \cite[Corollary 5]{Hom14} and (2) $\Leftrightarrow$ (3) is due to \cite[Corollary 5]{Hom14}.

For (3) $\Rightarrow$ (4), let $P_{\pm}$ be the pattern obtained by taking $(2, \pm 1)$ cable of $P \subset S^1 \times D^2$. Denote the $(p,q)$-cable of a knot $J$ by $J_{p,q}$. Then $P(K)_{2,\pm}=P_{\pm}(K)$. Hom showed that the value of $\epsilon(J)$ is determined by the pair of integers $(\tau(J_{2,1}), \tau(J_{2,-1}))$ \cite[Theorem 5.2]{Hom14}. Since $\tau(P_{\pm}(K))=\tau(P_{\pm}(K'))$ by (3), it follows that $\epsilon(P(K))=\epsilon(P(K'))$.

For (4) $\Rightarrow$ (3), define $Q_P:=P#-P(K')$ similarly to the proof of Theorem A. Since $Q_P(K')=P(K')#-P(K')$ is slice, $\epsilon(Q_P(K'))=0$. By (4), $\epsilon(Q_P(K))=\epsilon(Q_P(K'))=0$. Hom showed that $\tau(J)=0$ whenever $\epsilon(J)=0$ \cite[Theorem 5.2]{Hom14}. Applying this, it follows that $\tau(Q_P(K))=0$. Now $\tau(P(K))=\tau(P(K'))=\tau(P(K)#-P(K'))=\tau(Q_P(K))=0$.

In the above paragraph we have shown that (4) implies $\epsilon(Q_P(K))=0$. Since $Q_P(K)=P(K)#-P(K')$, this shows (4) $\Rightarrow$ (5). The converse is a straightforward consequence of Hom’s result that $\epsilon(J#J')=\epsilon(J')$ whenever $\epsilon(J)=0$ \cite[Proposition 3.6 (6)]{Hom14}: indeed, from this and the concordance invariance of $\epsilon$, it follows that $\epsilon(J')=\epsilon(J#-J'#J')=\epsilon(J)$ if $\epsilon(J#-J')=0$. The implication (5) $\Rightarrow$ (4) is a special case of this. \hfill $\square$

For the cases of $\tau$, $\epsilon$, $[CFK^\infty]$, and $\delta_{\rho}$, it turns out to be natural to consider the class of $R$-homology $n$-bipolar knots defined in \cite{CP14}. Here $R$ is a subring of $\mathbb{Q}$. This is a homology version of the notion of $n$-bipolar knots introduced in \cite{CHH13}. We do not spell out the definition since we do not use it directly; the readers are referred to \cite[Definition 2.3]{CP14} for details. We will use the following facts only. For a prime $p$, denote by $\mathbb{Z}_p=(a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, p \nmid b)$, the localization of $\mathbb{Z}$ at $p$.

1. An $n$-bipolar knot is $R$-homology $n$-bipolar for any $R$ \cite[p. 1544]{CP14}.
2. If $K$ is a $R$-homology $n$-bipolar knot, then $Q(K)$ is $R$-homology $n$-bipolar for any slice pattern $Q$ \cite[Theorem 2.6 (6)]{CP14}.
3. $\tau(K)=0$ if $K$ is $\mathbb{Z}_p$-homology 0-bipolar for some prime $p$, or equivalently $Q$-homology bipolar \cite[Theorem 2.7]{CP14}.
4. $\delta_{\rho}(K)=0$ if $K$ is $\mathbb{Z}_p$-homology 1-bipolar \cite[Theorem 2.8]{CP14}.

We remark that (B2), (B3) and (B4) above are mild generalizations of \cite[Propositions 3.3, 1.2, and 2.8]{CHH13}.

Theorem 3.2. Suppose $K$ is a knot in $S^3$ and $p$ is a prime.

1. If $K$ is $\mathbb{Z}_p$-homology 0-bipolar, $\bullet(P(K))=\bullet(P(U))$ for any pattern $P$ and $\bullet=\tau$, $\epsilon$, $[CFK^\infty(-)]$.
2. If $K$ is $\mathbb{Z}_p$-homology 1-bipolar, $\delta_{\rho}(P(K))=\delta_{\rho}(P(U))$ for any pattern $P$.

Consequently, if $K$ is 1-bipolar, $\bullet(P(K))=\bullet(P(U))$ for any pattern $P$ and $\bullet=\tau$, $\epsilon$, $[CFK^\infty(-)]$.

Proof. This is shown by a variation of the proof of Theorem A using (B2) in place of Lemma 2.1. Also, we need to use (B3) and (B4) instead of \cite[Corollary 1.1]{KM13}. Details are as follows.
(1) By Theorem 3.1, it suffices to prove the $\tau$ case. Suppose $K$ is $\mathbb{Z}(p)$-homology 0-bipolar. For an arbitrary slice pattern $Q$, $Q(K)$ is $\mathbb{Z}(p)$-homology 0-bipolar by (12). It follows that $\tau(Q(K)) = 0$ by (13). By Lemma 2.1, $\tau(P(K)) = \tau(P(U))$ for any pattern $P$.

(2) Suppose that $K$ is $\mathbb{Z}(p)$-homology 1-bipolar. For any slice pattern $Q$, $Q(K)$ is $\mathbb{Z}(p)$-homology 1-bipolar by (12). It follows that $\delta_{p,k}(Q(K)) = 0$ by (14). By Lemma 2.1, $\delta_{p,k}(P(K)) = \delta_{p,k}(P(U))$ for any pattern $P$.

□

Proof of Theorem B. There exists a family of 1-bipolar knots, say $\{K_i\}$, which generates a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}^\infty_2$ in the smooth and topological knot concordance groups [CHH13, Theorem 7.1]. Since a 1-bipolar knot satisfies (KM) and is $\mathbb{Z}(p)$-homology 1-bipolar, by Theorem 3.2 we have $\bullet(P(K)) = \bullet(P(U))$ for any pattern $P$ and $\bullet = \tau$, $\epsilon$, $[CFK^\infty(-)]$, $\delta_{p,k}$ when $K$ is 1-bipolar, especially when $K = K_i$.

□

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