Noether-Lefschetz locus and a special case of the variational Hodge conjecture: Using elementary techniques

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Abstract

Fix integers $n \geq 1$ and $d$ such that $nd > 2n + 2$. The Noether-Lefschetz locus $\text{NL}_{d,n}$ parametrizes smooth projective hypersurfaces in $\mathbb{P}^{2n+1}$ such that $H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q}) \neq \mathbb{Q}$. An irreducible component of the Noether-Lefschetz locus is locally a Hodge locus. One question is to ask under what choice of a Hodge class $\gamma \in H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q})$ does the variational Hodge conjecture hold true? In this article we use methods coming from commutative algebra and Hodge theory to give an affirmative answer in the case $\gamma$ is the class of a complete intersection subscheme in $X$ of codimension $n$. Another problem studied in this article is: In the case $n = 1$ when is an irreducible component of the Noether-Lefschetz locus nonreduced? Using the theory of infinitesimal variation of Hodge structures of hypersurfaces in $\mathbb{P}^3$, we determine all non-reduced components with codimension less than or equal to $3d$ for $d \gg 0$. Here again our primary tool is commutative algebra.

Notation 0.1. Throughout this article, $X$ will denote a smooth hypersurface in $\mathbb{P}^{2n+1}$. Denote by $H^{n,n}(X, \mathbb{Q})$ the intersection $H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q})$ and $H_X$ the very ample line bundle on $X$.

1 Introduction

It was first stated by M. Noether and later proved by S. Lefschetz that for a general smooth surface $X$ in $\mathbb{P}^3$, the rank of the Néron-Severi group, denoted $\text{NS}(X)$ is of rank 1. We can then

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define the *Noether-Lefschetz locus*, denoted $\text{NL}_{d,1}$, to be the space of smooth degree $d$ surfaces in $\mathbb{P}^3$ with Picard rank greater than 1. Using Lefschetz $(1,1)$-theorem, one can see that $\text{NL}_{d,1}$ is the space of smooth degree $d$ surfaces $X$ such that $H^{1,1}(X, \mathbb{Q}) \neq \mathbb{Q}$. Similarly, we can define *higher Noether-Lefschetz locus* as follows: Let $n > 1$ and $d$ another integer such that $nd > 2n+2$. Denote by $\text{NL}_{d,n}$ the space of smooth degree $d$ hypersurfaces $X$ in $\mathbb{P}^{2n+1}$ such that $H^{n,n}(X, \mathbb{Q}) \neq \mathbb{Q}$. The orbit of the action of the monodromy group on a rational class is finite (see [CDK95]). Consequently, $\text{NL}_{d,n}$ is an uncountable union of algebraic varieties (see [Voi03, §3.3] for more details).

Let $L$ be an irreducible component of $\text{NL}_{d,n}$. Then $L$ can be locally studied as the Hodge locus corresponding to a Hodge class. In particular, take $X \in L$, general and consider the space of all smooth degree $d$ hypersurfaces in $\mathbb{P}^{2n+1}$, denoted $U_{d,n}$. For $X \in L$, general, there exists $\gamma \in H^{n,n}(X, \mathbb{Q})$ and an open (analytic) simply connected set $U$ in $U_{d,n}$ containing $X$ such that $L \cap U$ is the Hodge locus corresponding to $\gamma$, denoted $\text{NL}_{d,n}(\gamma)$ (see [Voi02, §5.3] for more details).

Before we state the first main result in this article, we fix some notations. Given a Hilbert polynomial $P$, of a subscheme $Z$, in $\mathbb{P}^{2n+1}$, denote by $H_P$ the corresponding Hilbert scheme. Denote by $Q_d$ the Hilbert polynomial of a degree $d$ hypersurface in $\mathbb{P}^{2n+1}$. The flag Hilbert scheme $H_{P,Q_d}$ parametrizes all pairs $(Z, X)$, where $Z \in H_P$, $X$ is a smooth degree $d$ hypersurface in $\mathbb{P}^{2n+1}$ containing $Z$. For any $n \geq 1$ we prove the following theorem which is a special case of the variational Hodge conjecture:

**Theorem 1.1.** Let $Z$ be a complete intersection subscheme in $\mathbb{P}^{2n+1}$ of codimension $n + 1$. Assume that there exists a smooth hypersurface in $\mathbb{P}^{2n+1}$, say $X$, containing $Z$, of degree $d > \deg(Z)$. For the cohomology class $\gamma = a[Z] \in H^{n,n}(X, \mathbb{Q})$, $a \in \mathbb{Q}$, $\gamma$ remains of type $(n, n)$ if and only if $\gamma$ remains an algebraic cycle. In particular, $\text{NL}_{d,n}(\gamma)$ (closure taken in $U_{d,n}$) is isomorphic to an irreducible component of $\text{pr}_2 H_{P,Q_d}$ which parametrizes all degree $d$ hypersurfaces in $\mathbb{P}^{2n+1}$ containing a complete intersection subcheme with Hilbert polynomial $P$, where $P$ (resp. $Q_d$) is the Hilbert polynomial of $Z$ (resp. $X$).

In [Otw03], Otwinowska proves this statement for $d \gg 0$. Furthermore, in the case $n = 1$, we prove:

**Theorem 1.2.** Let $d \geq 5$ and $\gamma$ is a divisor in a smooth degree $d$ surface of the form $\sum_{i=1}^r a_i[C_i]$
with $C_i$ distinct integral curves for all $i = 1, \ldots, r$ and $d > \sum_{i=1}^r a_i \deg(C_i)+4$. Then the following are true:

(i) If $r = 1$ and $\deg(C_1) < 4$ then $\text{NL}_{d,1}(\gamma)$ (closure taken under Zariski topology on $U_{d,1}$) is reduced. In particular, $\text{NL}_{d,1}(\gamma)$ is an irreducible component of $\text{pr}_2(H_{P,Q,d})$, the space parametrizing all degree $d$ surfaces containing a reduced curve with the same Hilbert polynomial as $C_1$, which we denote by $P$.

(ii) Suppose that $r > 1$. For $d \gg 0$, every irreducible component $L$ of $\text{NL}_{d,1}$ of codimension at most $3d$ is locally of the form $\text{NL}_{d,1}(\gamma)$ with $\gamma$ as above, $\deg(C_i) \leq 3$ and $\text{NL}_{d,1}(\gamma)_{\text{red}} = \bigcap_{i=1}^r \text{NL}_{d,1}([C_i])_{\text{red}}$. Moreover, $\text{NL}_{d,1}(\gamma)$ is non-reduced if and only if there exists a pair $(i,j)$, $i \neq j$ such that $C_i.C_j \neq 0$.

2 Proof of Theorem 1.1

Notation 2.1. Denote by $S^k_n$ the degree $k$-graded piece of $H^0(O_{\mathbb{P}^{2n+1}}(k))$. Define $S^n := \oplus_{k \geq 0} S^k_n$. Let $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^{2n+1}$, defined by an equation $F$. Denote by $J_F$, the Jacobian ideal of $F$ generated as an $S^n$-module by the partial derivatives of $F$ with respect to $\frac{\partial}{\partial X_i}$ for $i = 1, \ldots, 2n+1$, where $X_i$ are the coordinates of $\mathbb{P}^{2n+1}$. Define, $R_F := S^n/J_F$. For $k \geq 0$, let $J^k_F$ (resp. $R^k_F$) symbolize the degree $k$-graded piece of $J_F$ (resp. $R_F$).

2.2. We now recall some standard facts about Hodge locus. Let $X$ be a smooth projective hypersurface in $\mathbb{P}^{2n+1}$ of degree $d$. Recall, there is a natural morphism from $H^{n,n}(X)$ to $H^{n,n}(X)_{\text{prim}}$, where $H^{n,n}(X)_{\text{prim}}$ denotes the primitive cohomology on $H^{n,n}(X)$ (see [Vo02, §6.2, 6.3] for more on this topic). Denote by $\gamma_{\text{prim}}$ the image of $\gamma$ under this morphism. Using the Lefschetz decomposition theorem, one can see that $\text{NL}_{d,n}(\gamma)$ coincides with $\text{NL}_{d,n}(\gamma_{\text{prim}})$ i.e., $\gamma$ remians of type $(n,n)$ if and only if so does $\gamma_{\text{prim}}$.

2.3. Now, $K_{\mathbb{P}^{2n+1}} = O_{\mathbb{P}^{2n+1}}(-2n-2)$, $H^0(K_{\mathbb{P}^{2n+1}}(2n+2)) = H^0(O_{\mathbb{P}^{2n+1}}) \cong \mathbb{C}$ generated by

$$\Omega := X_0 \cdots X_{2n+1} \sum_i (-1)^i \frac{dX_0}{X_0} \wedge \cdots \wedge \frac{dX_i}{X_i} \wedge \cdots \wedge \frac{dX_{2n+1}}{X_{2n+1}},$$

where the $X_i$ are homogeneous coordinates on $\mathbb{P}^{2n+1}$. Recall, for the closed immersion $j :$
\( X \to \mathbb{P}^{2n+1} \), denote by \( H^{2n}(X, \mathbb{Q})_{\text{van}} \), the kernel of the Gysin morphism \( j_* \) from \( H^{2n}(X, \mathbb{Q}) \) to \( H^{2n}(\mathbb{P}^{2n+1}, \mathbb{Q}) \). Now, [Voi03] Theorem 6.5 tells us that there is a surjective map,

\[
\alpha_{n+1} : H^0(\mathbb{P}^{2n+1}, O_{\mathbb{P}^{2n+1}}((n+1)d - 2n - 2)) \to F^{n+1}H^{2n+1}(\mathbb{P}^{2n+1} \setminus X, \mathbb{C}) \cong F^nH^{2n}(X, \mathbb{C})_{\text{van}}
\]

which sends a polynomial \( P \) to the residue of the meromorphism form \( P\Omega/F^{n+1} \), where \( F \) is the defining equation of \( X \) (see [Voi03] §6.1 for more). Finally, [Voi03] Theorem 6.10] implies that \( \alpha_{n+1} \) induces an isomorphism between \( H^{(n+1)d-(2n+2)}_F \) and \( H^{n,n}(X)_{\text{prim}} \).

2.4. We now recall a theorem due to Macaulay which will be used throughout this article. A sequence of homogeneous polynomials \( G_i \in S^d_n, i = 0, \ldots, 2n+1 \) with \( d_i > 0 \) is said to be \textit{regular} if the \( G_i \) have no common zero. Denote by \( I_G \) the ideal in \( S_n \) generated by the polynomials \( P_i \) for \( i = 0, \ldots, 2n+1 \). Denote by \( H_G \) the quotient \( S_n/I_G \) and by \( H^i_G \) the degree \( i \) graded piece in \( H_G \).

**Theorem 2.5** (Macaulay). Let \( N := \sum_{i=0}^{2n+1} d_i - 2n - 2 \). Then, the rank of \( H^N_G = 1 \) and for every integer \( k \), the pairing, \( H^k_G \times H^{N-k}_G \to H^N_G \) is perfect.

See [Voi03] Theorem 6.19] for the proof of the statement.

2.6. Denote by \( P \in S^{(n+1)d-(2n+2)}_n \) such that \( \alpha_{n+1}(P) = \gamma \). Using [Voi03] Theorem 6.17], we observe that \( T_X \NL_{d,n}(\gamma) \) is isomorphic to the preimage of \( \ker(\bar{P} : R^d_F \to R^{(n+1)d-(2n+2)}_F) \) under the natural quotient morphism from \( S^d_n \to S^d_n/J^d_F \).

2.7. It is easy to see that for any \( \gamma' \in H^{n,n}(X, \mathbb{Q}), \NL_{d,n}(\gamma') = \NL_{d,n}(a'\gamma') \) for any \( a' \in \mathbb{Q} \), non-zero. For the rest of this section, we assume \( \gamma = [Z] \), where \( Z \) is as in the statement of the theorem.

**Notation 2.8.** Denote by \( N := (n+1)d - (2n+2) \). Since \( X \) is smooth, the corresponding Jacobian ideal \( J_F \) can be generated by a regular sequence of \( 2n+2 \) polynomials \( G_i \) of degree \( d - 1 \). Using Theorem 2.5 we see that there exists a perfect pairing \( R^k_F \times R^{2N-k}_F \to R^{2N}_F \) for all \( k \leq 2N \) and \( R^{2N}_F \) is one dimensional complex vector space. Denote by \( T'_0 \), the subspace of \( R^{2N}_F \) which is the kernel under the multiplication map, \( .P : R^{2N}_F \to R^{2N}_F \). Denote by \( T_0 \) the preimage of \( T'_0 \) in \( S^n_n \) under the natural projection map from \( S^n_n \to R^{2N}_F \). Define \( T_1 \) the subspace
of $S_n$, a graded $S_n$-module such that for all $t \geq 0$, the $t$-graded piece of $T_1$, denoted $T_{1,t}$ is the largest subvector space of $S_n^t$ such that $T_{1,t} \otimes S_n^{N-t}$ is contained in $T_0$ for $t < N$, $T_{1,N} = T_0$ and $T_{1,N+t} = T_0 \otimes S_n^t$ for $t > 0$.

2.9. It follows from the perfect pairing above that $\dim S_n^N / T_{1,N} = 1$. Using the definition of $T_1$, it follows,

$$S_n^k / T_{1,k} \times S_n^{N-k} / T_{1,N-k} \rightarrow S_n^N / T_{1,N}$$

is a perfect pairing. Hence, $\dim S_n^d / T_{1,d} = \dim S_n^{N-d} / T_{1,N-d}$.

**Lemma 2.10.** The tangent space $T_X(NL_{d,n}(\gamma))$ coincides with $T_{1,d}$.

**Proof.** Note that $H \in T_{1,d}$ if and only if $\bar{H} \otimes R_n^{N-d}$ is contained in $T'_0$ which by definition is equivalent to $\bar{H} \otimes R_n^{N-d}$ is contained in $T'_0$. Using the perfect pairing 2.8 we can conclude that $\bar{H} = 0$ in $R_n^{N+d}$. This is equivalent to $H \in T_X(NL_{d,n}(\gamma))$.

2.11. Suppose that $Z$ is defined by $n+1$ polynomials $P_0, ..., P_n$. Since $Z \subset X$, we can assume that there exist polynomials $Q_0, ..., Q_n$ of degree $d - \deg P_i$, respectively such that $X$ is defined by a polynomial of the form $P_0Q_0 + ... + P_nQ_n$. Let $I$ be the ideal in $S_n$ generated by $P_0, ..., P_n$ and $Q_0, ..., Q_n$.

**Proposition 2.12.** The $k$-graded pieces, $T_{1,k} = I_k$ for all $k \leq N$.

**Proof.** Denote by $Z_1$ the subschemes in $\mathbb{P}^{2n+1}$, defined by $Q_0 = P_1 = ... = P_n = 0$. Since $Z \cup Z_1$ is the intersection of $X$ and $\{P_1 = ... = P_n = 0\}$, then $[Z] = -[Z_1]$ mod $QH^n_X$ in the cohomology group $H^{n,n}(X, \mathbb{Q})$. So, $[Z]_{\text{prim}} = -[Z_1]_{\text{prim}}$. Denote by $Z_2$ the subvariety defined by $Q_0 = ... = Q_n = 0$. Proceeding similarly, we get $[Z]_{\text{prim}} = a[Z_2]_{\text{prim}}$ for some integer $a$. Using [GH83 4.a.4], we have $(P_0, ..., P_n, Q_0, ..., Q_n) \subset T_1$. Since $X$ is smooth the sequence $\{P_0, ..., P_n, Q_0, ..., Q_n\}$ is a regular sequence. Using Theorem 2.5 we can conclude that $\dim S_n^N / I_N = 1$, where $I_N$ denotes the degree $N$ graded piece of $I$ and

$$S / I_k \times S / I_{N-k} \rightarrow S / I_N$$
is perfect pairing. So, \( I \) is Gorenstein of socle degree \( N \) contained in \( T_1 \) which is Gorenstein of the same socle degree. So, \( T_{1,k} = I_k \) for all \( k \leq N \). \( \square \)

2.13. The parameter space, say \( H \) of complete intersection subschemes in \( \mathbb{P}^{2n+1} \) of codimension \( n+1 \), defined by \( n+1 \) polynomials of degree \( \deg(P_i) \), respectively is irreducible. In particular, it is an open subscheme of

\[
\mathbb{P}(S_n^{\deg P_0}) \times \cdots \times \mathbb{P}(S_n^{\deg P_n})
\]

which is irreducible. Denote by \( R_0 \) the Hilbert polynomial of \( Z \) as a subscheme in \( \mathbb{P}^{2n+1} \). Consider the flag Hilbert scheme \( H_{R_0,Q_d} \) and the projection map \( \text{pr}_1 \) which is the projection onto the first component. Since the generic fiber of \( \text{pr}_1 \) is isomorphic to \( \mathbb{P}(I_d(Z)) \) for the generic subscheme \( Z \) on \( \text{pr}_1 H_{R_0,Q_d} \), it is irreducible, where \( I_d(Z) \) is the degree \( d \) graded piece of the ideal, \( I(Z) \), of \( Z \). So, there exists an unique irreducible component in \( H_{R_0,Q_d} \) such that the image under \( \text{pr}_1 \) of this component coincides with \( H \). For simplicity of notation, we denote by \( H_{R_0,Q_d} \) this irreducible component, since we are interested only in this scheme.

2.14 (Proof of Theorem 1.1). Using basic deformation theory and Hodge theory, we can conclude that \( \text{pr}_2(H_{R_0,Q_d}) \) is contained in \( \overline{\text{NL}_{d,n}(\gamma)} \). So,

\[
\text{codim } \text{pr}_2(H_{R_0,Q_d}) \geq \text{codim } \overline{\text{NL}_{d,n}(\gamma)} \geq \text{codim } T_X \overline{\text{NL}_{d,n}(\gamma)}.
\]

Now, there is a natural morphism, denoted \( p \) from \( T_{1,d} \) to \( H_{Q_d} \) which maps \( F_1 \) to the zero locus of \( F_1 \). Since every element of \( T_{1,d} \) defines a hypersurface containing a subscheme with Hilbert polynomial \( R_0 \), \( \text{pr}_2(H_{R_0,Q_d}) \) contains \( \overline{\text{Im } p} \). Since the zero locus of a polynomial is invariant under multiplication by a scalar,

\[
\dim T_{1,d} = \dim \overline{\text{Im } p} + 1.
\]

Finally,

\[
\text{codim } \text{pr}_2(H_{R_0,Q_d}) = \dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d))) - \dim \text{pr}_2(H_{R_0,Q_d}) \leq
\]

\[
\leq (h^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d)) - 1) - \dim \overline{\text{Im } p} \leq h^0(\mathcal{O}_{\mathbb{P}^{2n+1}}(d)) - \dim T_{1,d} = \text{codim } T_X \overline{\text{NL}_{d,n}(\gamma)}
\]
where the last equality follows from Lemma 2.10. This proves Theorem 1.1.

2.15. Furthermore, note that $NL_{d,1}(\gamma)$ is reduced and parametrizes all degree $d$ surfaces in $\mathbb{P}^3$ containing a complete intersection curve with the Hilbert polynomial $P$. This is a part of Theorem 1.2(ii).

3 Proof of Theorem 1.2

3.1. If $C_1$ is a complete intersection curve then reducedness of $NL_{d,1}([C_1])$ follows from 2.15.

If $C_1$ is an integral curve, $\deg(C_1) < 4$ and $C_1$ not complete intersection then $C_1$ is a twisted cubic. Recall, the twisted cubic $C_1$ is generated by 3 polynomials of degree 2 each. Suppose $P$ is a polynomial in $T_1$ such that $P$ is not contained in $I(C_1)$. Since the zero locus of the ideal, say $I'$ generated by $I(C_1)$ and $P$ is non-empty, $J_F$ (which is base point free) is not contained in $I'$. Since the Jacobian ideal, $J_F \subset T_1$, we can show that there is a regular sequence in $T_1$ consisting of 4 elements, two of which are the generators of $I(C_1)$, the third one is $P$ and the forth is an element in $J_F^{d-1}$. Since, codim $T_{1,d-4} = 1$, Theorem 2.5 implies $2+2+\deg(P)+d-1-4 \geq 2d-4$. So, $\deg(P) \geq d - 3$. By Proposition 2.12

$$\text{codim } T_X(NL_{d,1}(\gamma)) = \text{codim } T_{1,d} = \text{codim } T_{1,d-4} = \text{codim } I_{d-4}(C_1) = 3(d - 4) + 1 = 3d - 11.$$ 

3.2. Using basic deformation theory and Hodge theory we can conclude that there exists an unique irreducible component $H$ of $H_{P,Q_d}$ whose generic element is $(C,X)$, where $C$ is a twisted cubic contained in $X$ such that $\text{pr}_2(H)$ is contained in $NL_{d,1}(\gamma)$. It is easy to compute that codim$(\text{pr}_2(H)) = 3d - 11$. So,

$$3d - 11 \geq \text{codim } NL_{d,1}(\gamma) \geq \text{codim } T_X(NL_{d,1}(\gamma)) = 3d - 11.$$ 

So, $NL_{d,1}(\gamma)$ is reduced and parametrizes smooth degree $d$ surfaces containing a twisted cubic. This finishes the proof of (i).

3.3. We now recall a result due to Otwinowska that will help us make the characterization of
the irreducible components of $NL_{d,1}$ as in Theorem [12](ii).

**Theorem 3.4 ([OrfwH Theorem 1]).** Let $\gamma$ be a Hodge class on a smooth degree $d$ surface. There exists $C \in \mathbb{R}^*_+$ depending only on $r$ such that for $d \geq C(r-1)^8$ if $\text{codim } NL_{d,1}(\gamma) \leq (r-1)d$ then $\gamma_{\text{prim}} = \sum_{i=1}^{t} a_i[C_i]_{\text{prim}}$ where $a_i \in \mathbb{Q}^*$, $C_i$ are integral curves and $\text{deg}(C_i) \leq (r-1)$ for $i = 1, \ldots, t$ for some positive integer $t$.

This implies the following:

**Proposition 3.5.** Let $d \gg 0$, $\gamma$ be a Hodge class in a smooth degree $d$ surface in $\mathbb{P}^3$ such that $\text{codim } NL_{d,1}(\gamma) \leq 3d$. Then there exists integral curves $C_1, \ldots, C_t$ of degree at most 3 such that $\gamma = \sum_{i=1}^{t} a_i[C_i] + bH_X$ for some integers $a_i$, $b$ and $\text{NL}(\gamma)_{\text{red}}$ is the same as $\bigcap_{i=1}^{t} \text{NL}([C_i])_{\text{red}}$.

**Proof.** Let $X \in NL_{d,1}(\gamma)$. There exists a maximal $\mathbb{Q}$-vector space $\Lambda \subset H^2(X, \mathbb{Q})$ such that $\Lambda$ remains of type $(1,1)$ in $NL_{d,1}(\gamma)$ i.e., $\text{NL}_{d,1}(\gamma)_{\text{red}} = \bigcap_{\gamma_i \in \Lambda} \text{NL}_{d,1}(\gamma_i)_{\text{red}}$. There exists a surface $X' \in NL_{d,1}(\gamma)$ such that the Néron-Severi group $\text{NS}(X')$ is the translate (under deformation from $X$ to $X'$) of $\Lambda$ in $H^2(X', \mathbb{Z})$ which we again denote by $\Lambda$ for convenience. Then, Theorem [3.4] implies that any $\gamma \in \Lambda$ is of the form $\sum_{i} a_i[C_i] + bH_X$ with $\text{deg}(C_i) \leq 3$. So, $\Lambda$ is generated by classes of curves of degree at most 3 and $H_X$. Note that the classes of these curves are also contained in $\Lambda$ since $\Lambda$ is the complete Néron-Severi group of $X'$. This proves the proposition, which is also the first part of Theorem [12](ii).

3.6. We now come to the proof of the final part of the theorem. Suppose now that $\gamma$ is as in the above proposition i.e., of the form $\sum_{i=1}^{t} a_i[C_i] + bH_X$ such that $\text{NL}(\gamma)_{\text{red}} = \bigcap_{i=1}^{t} \text{NL}([C_i])_{\text{red}}$. Denote by $\tilde{P}_i$ the element in $R_F^{2d-4}$ such that $\alpha_2(\tilde{P}_i) = [C_i]_{\text{prim}}$ for $i = 1, \ldots, t$. Since $\alpha_2$ is a linear map, $\alpha_2(\sum_{i=1}^{t} a_i\tilde{P}_i) = \sum_{i} a_i[C_i]_{\text{prim}}$. Denote by $\tilde{P} := \sum_{i=1}^{t} a_i\tilde{P}_i$. So, $\alpha_2(\tilde{P}) = \gamma$.

Denote by $T_{1,d-4}^{[C_i]}$ the corresponding $T_{1,d-4}$ in [23] obtained by replacing $P$ by $\tilde{P}_i$ for $i = 1, \ldots, r$. Note that $\text{codim } T_X \text{NL}_{d,1}(\gamma) = \text{codim } T_{1,d} = \text{codim } T_{1,d-4}$, where the last equality is due to perfect pairing. Note that, $\bigcap_{i=1}^{t} T_{1,d-4}^{[C_i]} \subset T_{1,d-4}$ because $\tilde{P} = \sum_{i} a_i\tilde{P}_i$, so $\bigcap_{i=1}^{r} \ker \tilde{P}_i \subset \ker \tilde{P}$. Therefore, $\text{codim } T_X \text{NL}_{d,1}(\gamma) \leq \text{codim } I_{d-4}(\bigcup_{i=1}^{r} C_i)$.

Before we go to the last step of the proof we need the following computation:
Lemma 3.7. Let $d \geq 5$ and $C$ be an effective divisor on a smooth degree $d$ surface $X$ of the form $\sum_i a_i C_i$, where $C_i$ are integral curves with $\deg(C) + 4 \leq d$. Then, $\dim |C| = 0$, where $|C|$ is the linear system associated to $C$.

Proof. Let $C = \sum_i a_i C_i$ with $C_i$ integral. Then,

$$\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = a_i C_i^2 + \sum_{j \neq i} a_j C_i C_j.$$ 

Denote by $e_i := \deg(C_i)$. Using the adjunction formula and the fact that $K_X \cong \mathcal{O}_X(d - 4)$, we have that

$$\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) \leq a_i (e_i^2 - (d - 1)e_i) + \sum_{j \neq i} a_j C_i C_j$$

$$\leq a_i (e_i^2 - 3e_i - e_i \sum_j a_j e_j) + \sum_{j \neq i} a_j e_i e_j.$$ 

The first inequality follows from the bound on the genus of a curve in $\mathbb{P}^3$ in terms of its degree (see [Har77, Example 6.4.2]). The second inequality follows from the facts that $d \geq \deg(C) + 2$ and $C_i, C_j \leq e_i e_j$. It then follows directly that $\deg((\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) < 0$. This implies that $h^0(C_i, (\mathcal{O}_X(C)|_C \otimes \mathcal{O}_C)|_{C_i}) = 0$ for all $i$. This implies that $h^0(C, \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C) = 0$. Since $h^1(\mathcal{O}_X) = 0$ (by Lefschetz hyperplane section Theorem) and $h^0(\mathcal{O}_X) = 1$, using the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \otimes \mathcal{O}_C \to 0$$

we get that $h^0(\mathcal{O}_X(C)) = 1$. Since $|C| = \mathbb{P}(H^0(\mathcal{O}_X(C)))$, the lemma follows. \qed

3.8 (Proof of Theorem 1.2). Using Proposition 2.12 and 3.1, $\bigcap_{i=1}^t T_{1,d-4}^{[C_i]} = I_{d-4}(\bigcup_{i=1}^t C_i)$ is contained in $T_X \text{NL}_{d,1}(\gamma)$. Denote by $P_i$ the Hilbert polynomial of $C_i$ for $i = 1, \ldots, t$. By Theorem 1.2(i), there exists an irreducible component of $H_{P_1, Q_d}$ such that its image under the natural projection morphism $pr_2$ (onto the second component) is isomorphic to $\text{NL}_{d,1}(\{C_i\})_{\text{red}}$. So, there exists an irreducible component, say $H_\gamma$ of $H_{P_1, Q_d} \times H_{Q_d} \ldots \times H_{Q_d} H_{P_1, Q_d}$ such that
\[ \text{pr}_2(H_\gamma)_{\text{red}} = \overline{\text{NL}(\gamma)_{\text{red}}}, \] where \( \text{pr}_2 \) is the natural morphism from \( H_\gamma \) to \( H_{Q_d} \). Denote by \( L_\gamma := \text{pr}(H_\gamma) \), where \( \text{pr} \) is the natural projection morphism to \( H_{P_1} \times \ldots \times H_{P_t} \). A generic \( t \)-tuple of curves \( (C_1, \ldots, C_t) \in H_{P_1} \times \ldots \times H_{P_t} \) does not intersect each other. Since there exists \( i, j, i \neq j \) such that \( C_i \cap C_j \neq \emptyset \), we have \( \dim L_\gamma < \sum_{i=1}^t \dim H_{P_i} \). Lemma 3.7 implies that \( \dim |C_i| = 0 \) for \( i = 1, \ldots, t \). It is then easy to see that \( \text{codim NL}_{d,1}(\gamma) = \text{codim} I_d(\bigcup_{i=1}^t C_i) - \dim L_\gamma \). If \( \text{codim} I_{d-4}(\bigcup_{i=1}^t C_i) \leq \text{codim} I_d(\bigcup_{i=1}^t C_i) - \sum_{i=1}^t \dim H_{P_i} \) then
\[
\text{codim} T_X \text{NL}_{d,1}(\gamma) \leq \text{codim} I_{d-4}(\bigcup_{i=1}^t C_i) \leq \text{codim} I_d(\bigcup_{i=1}^t C_i) - \sum_{i=1}^t \dim H_{P_i} < \text{codim} I_d(\bigcup_{i=1}^t C_i) - \dim L_\gamma = \text{codim NL}_{d}(\gamma),
\]
where the first inequality follows from 3.6. Since \( d \gg 0 \), using the Hilbert polynomial of \( \bigcup C_i \), the inequality \( \dagger \) is equivalent to \( \sum_{i=1}^t \dim H_{P_i} = 4 \sum_{i=1}^t \deg(C_i) \). Since \( \deg(C_i) < 4 \) and \( C_i \) is integral, it is easy to compute that \( \dim H_{P_i} \) is in fact equal to \( 4 \deg(C_i) \). This proves (ii). Hence, completes the proof of the theorem.

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