Relaxing safety for metric first-order temporal logic via dynamic free variables

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Abstract

We define a fragment of metric first-order temporal logic formulas that guarantees the finiteness of their table-representations. We extend our fragment’s definition to cover the temporal dual operators trigger and release and show that our fragment is strictly larger than those previously used in the literature. We integrate these additions into an existing runtime verification tool and formally verify in Isabelle/HOL that the tool correctly outputs the table of constants that satisfy the monitored formula. Finally, we provide some example specifications that are now monitorable thanks to our contributions.

1 Introduction

Runtime verification (RV) complements other techniques for system quality-assurance such as testing or model checking [16]. It allows monitoring properties during a system’s execution by indicating when they are violated. Metric first-order temporal logic (MFOTL) is among the most expressive temporal and declarative specification languages for RV [7]. It adds intervals to the logic’s temporal operators to model quantitative descriptions of time [5, 14]. For these reasons, it is often used in monitor implementations [4, 11, 19].

Besides being expressive, monitors should also be efficient and trustworthy. Efficiency allows them to be deployed more invasively by course-correcting the evolution of a system [12], while trustworthiness makes them reliable in safety-critical applications. Recently, Verimon, an MFOTL-based RV-algorithm, has achieved high expressivity [22] and efficiency [3] while staying trustworthy because of its formally verified implementation in the Isabelle/HOL proof assistant [19]. It uses finite relations to represent the set of valuations that make a specification true which contributes to its efficiency. Nevertheless, this feature also makes it inherit well-known issues from relational databases [8] forcing it to operate inside a fragment whose formula-evaluation guarantees finite outputs. This fragment is defined inductively on the structure of the formula via a predicate safe-formula which should not be confused with the notion of safety property from model checking [15]. The set of safe formulas that Verimon admits is rather restrictive. The fact that well-known temporal operators such as historically are immediately dubbed unsafe if they have free variables, is evidence of this. Here, we address this particular issue.

Our main contribution consists of the definition (§3) of a larger fragment of MFOTL-safe formulas, its formalisation in the Isabelle/HOL proof assistant, and its integration into Verimon’s first implementation. This generalisation of safety enables us to monitor a wider variety of future and past operators including globally and historically. To address safety of formulas involving these operators in conjunction with other connectives, we introduce the set of dynamic free variables (dfv, α) of a formula α at time-point i. It approximates the set of those free variables that contribute to the satisfiability of a formula at a specific point of a system’s execution. Safety is then decided by computing a set of allowed sets of dynamic free variables (ssfv) in such a way that whenever this set is nonempty, meaning that α is safe, dfv; α ∈ ssfv α.

Furthermore, with the view of integrating our generalisation into the latest optimised version of Verimon [22], we explicitly add conjunctions to the syntax of our implementation. This involves defining safety
for specific cases where one of the conjuncts is an equality, a constraint, or the negation of a safe formula. To go beyond the developments in the optimised version, we also add since and until’s dual operators, trigger and release respectively, to Verimon’s syntax. We therefore also extend Verimon’s monitoring algorithm with functions to evaluate these operators (§4) and prove them correct.

Our formalisation and proof of correctness (§6) of the extended monitoring algorithm is also a major contribution of this work. It involves redefining proof-invariants to accommodate dynamic free variables which largely reverberates in the proofs of correctness for each connective. The formalisation and proofs are available online, and corresponding definitions are linked throughout the paper and indicated with the (clickable) Isabelle-logo.\textsuperscript{1} We add to the relevance of our safety relaxation by providing examples in §5 that can now be monitored due to our contributions. We discuss future work and our conclusions in §7.

Related work. In terms of expressivity of other monitors, our work directly extends the oldest Verimon version [19] by the above-mentioned additions. However, our generalisation has not been applied to the latest Verimon+ [22] as we do not deal with aggregators, recursive rules, or regular expressions. We foresee no issues to adapt our approach to these other extensions. Verimon+ does not have an explicit version of dual operators and deems historically and globally unsafe. A recent extension [10], adds the dual operators and describes lengthy encodings to monitor historically and globally. Yet, our work is complementary because of our larger safety fragment. Another past-only, first-order monitor is DejaVu [11]. It uses binary decision diagrams which can model infinite relations and thus, does not require a notion of safety. In contrast, our work supports both past and bounded future operators and, in general, it is hard to compare performance between both RV-approaches [11, 22].

Safety has been well studied for relational databases [1, 6, 8, 13, 21]. Kifer [13] organises and relates various definitions of safety, and Avron and Hirshfeld [2] complement his work by answering some of Kifer’s conjectures at the end of his paper. More recently, evaluation of queries that are relatively safe has been explored [18]. Thereby, there is an approach to translate any relational calculus query into a pair of safe queries where, if the second holds, then the original is unsafe (produces an infinite output). Otherwise, the original query’s output is the same as the output of the first one. A recent extension to the temporal setting [17] remains to be formally verified and integrated into Verimon+. Arguably, using it as black boxes for the monitoring algorithm would be less efficient than the direct integration we provide.

2 Metric First-Order Temporal Logic

In this section, we briefly describe syntax and semantics of MFOTL for the Verimon implementation and introduce some concepts for later explanations. Following Isabelle/HOL notation and conventions, we use $x :: \alpha$ to state that variable $x$ is of type $\alpha$. The type of lists and the type of sets over $\alpha$ are $\langle \alpha \text{ list} \rangle$ and $\langle \alpha \text{ set} \rangle$ respectively. We implicitly use “$s$” to indicate a list of terms of some type, for instance, if $x :: \alpha$, then $xs :: \langle \alpha \text{ list} \rangle$. The standard operations on lists that apply a function $f$ to each element (map $f$ $xs$), get the $n$th element ($xs ! n$), output the list’s length (length $xs$) or add an element to the left ($x \# xs$) also appear throughout the paper. The expression $f ' X$ denotes the set-image of $X$ under $f$. We freely use binary operations inside parenthesis as functions, that is we can write $x + y$ or $(+) x y$. Finally, the natural numbers have type $\mathbb{N}$.

The type of terms $\langle \alpha \text{ trm} \rangle$ simply consists of variables $\mathbf{v} x$ and constants $\mathbf{c} a$ with $x :: \mathbb{N}$ and $a :: \alpha$. The syntax of MFOTL formulas $\langle \alpha \text{ frm} \rangle$ that we use is

\[
\alpha ::= p | t = t | \neg \alpha | \alpha \land \alpha | \alpha \lor \alpha | \exists I \alpha \mid Y I \alpha \mid X I \alpha \mid \alpha S I \alpha \mid \alpha U I \alpha \mid \alpha T I \alpha \mid \alpha R I \alpha ,
\]

where $p :: \text{string}$, $t :: \langle \alpha \text{ trm} \rangle$, $ts :: \langle \alpha \text{ trm list} \rangle$ and $I$ is a non-empty interval of natural numbers. We distinguish MFOTL connectives and quantifiers from meta-statements via the subscript $F$ and freely use well-known

\textsuperscript{1} Readers wishing to download and see the files in Isabelle, must ensure they download commit \texttt{b4663034ecadcc5783085dececd4d6e47f6/01/52} of branch \texttt{ssfvs}.
interval notation, e.g. \([a, b) = \{n \mid a \leq n < b\}\) or \([a, b] = \{n \mid a \leq n \leq b\}\). Furthermore, we also write \(m > I + n\) (and similar abbreviations) to state that \(m\) is greater than \(n\) plus any other element in \(I\). Finally, the above syntax implies usage of De Bruijn indices, that is, \(\exists_F p \triangleright [v, v']\) represents the formula \(\exists y. p \triangleright y\).

Verimon encodes valuations as lists of natural numbers, \(v :: \mathbb{N}\) list where the \(x\)th element of the list is the value of \(v_x\). That is, the evaluation of terms (\(\lambda_j\)) is given by \(v ! (\alpha a) = a\) and \(v ! (v x) = v ! x\). For the semantics, a trace \(\sigma\), meaning an infinite time-stamped sequence of sets, models the input from the monitored system. The function \(\tau \sigma i :: \mathbb{N}\) outputs the time-stamp at time-point \(i :: \mathbb{N}\), whereas \(\Gamma \sigma i\) outputs the corresponding set. When clear from context we use their abbreviated forms \(\tau_i\) and \(\Gamma_i\), respectively. The time-stamps are monotone \(\forall i \leq j. \tau_i \leq \tau_j\) and eventually increasing \(\forall n. \exists i. n \leq \tau_i\). The sets \(\Gamma_i\) contain pairs \((p, xs)\) where \(p\) is a name for a predicate \(p :: \text{string}\) and \(xs\) is a list of values “satisfying” \(p\). Formally, the semantics are

\[
\langle \sigma, v, i \rangle \models p \triangleright t \quad \Leftrightarrow \quad \langle p, \text{map}((\lambda_j) v) ts \rangle \in \Gamma_i, \quad \langle \sigma, v, i \rangle \models \exists_F \alpha \Leftrightarrow \exists a. \langle \sigma, a \# v, i \rangle \models \alpha
\]

\[
\langle \sigma, v, i \rangle \models \alpha \land_F \beta \Leftrightarrow \langle \sigma, v, i \rangle \models \alpha \land \langle \sigma, v, i \rangle \models \beta, \quad \langle \sigma, v, i \rangle \models t_1 \triangleright t_2 \Leftrightarrow \langle v \triangleright t_1 \rangle = \langle v ! t_2 \rangle
\]

\[
\langle \sigma, v, i \rangle \models \alpha \lor_F \beta \Leftrightarrow \langle \sigma, v, i \rangle \models \alpha \lor \langle \sigma, v, i \rangle \models \beta, \quad \langle \sigma, v, i \rangle \models \neg \alpha \Leftrightarrow \langle \sigma, v, i \rangle \not\models \alpha
\]

\[
\langle \sigma, v, i \rangle \models X_\alpha \Leftrightarrow \langle \sigma, v, i + 1 \rangle \models \alpha \land (\tau_{i+1} - \tau_i) \in I
\]

\[
\langle \sigma, v, i \rangle \models Y_\alpha \Leftrightarrow \text{if } i = 0 \text{ then false else } \langle \sigma, v, i - 1 \rangle \models \alpha \land (\tau_i - \tau_{i-1}) \in I
\]

\[
\langle \sigma, v, i \rangle \models \alpha \lor F_1 \beta \Leftrightarrow \exists j \leq i. (\tau_i - \tau_j) \in I \wedge (\forall (v, j) \models \beta \land (\forall k \in [j, i]). (\sigma, v, i) \models \alpha)
\]

\[
\langle \sigma, v, i \rangle \models \alpha \lor R_1 \beta \Leftrightarrow \forall j \leq i. (\tau_j - \tau_i) \in I \Rightarrow (\sigma, v, j) \models \beta \land (\exists k \in [j, i]). (\sigma, v, i) \models \alpha
\]

Other operators can be encoded, e.g. true \((\top \equiv \text{e } a \Rightarrow \text{e } a)\), eventually \((F_1 \alpha \equiv \top U_1 \alpha)\), or historically \((H_1 \alpha \equiv \neg F_1 \neg \alpha)\). Moreover, trigger and release satisfy their dualities with since and until, that is, \(\langle \sigma, v, i \rangle \models \alpha \lor F_1 \beta \Leftrightarrow \langle \sigma, v, i \rangle \models \neg F_1 ((\neg F_1 \alpha) \lor F_1 \beta)\) and \(\langle \sigma, v, i \rangle \models \alpha \lor R_1 \beta \Leftrightarrow \langle \sigma, v, i \rangle \models \neg R_1 ((\neg F_1 \alpha) \lor R_1 \beta)\).

Intuitively, for the formula \(\alpha\), a monitor outputs the set \([\alpha]_i = \{v \mid \langle \sigma, v, i \rangle \models \alpha\}\) at time-point \(i :: \mathbb{N}\). However, these sets are redundant and infinite, e.g. \(v_1 = [4, 5]\), \(v_2 = [4, 5, 1]\) and \(v_3 = [4, 5, 2, 7]\) are all elements of \([p \triangleright [v, v] \triangleright 1]_i\). If \(\exists_F \alpha \equiv \top U_1 \alpha\), then \(x < n = \text{nfv } \alpha\). For example, \(v_4 = [4, 5, None]\) satisfies \(p \triangleright [v, v] \triangleright 1\) if it is a subformula of \(\alpha\). A formula \(\alpha\) is not generally considered safe by itself. Extending MFOTL’s syntax to include trigger and release, encoding \(H_1 \alpha \equiv (\neg F_1 \top) F_1 \alpha\), and defining safe-formula for these cases [10] is still unsatisfactory. Following the semantics above, formulas \(\alpha \lor F_1 \beta\) remain unsafe when \(0 \notin I\) because they could be vacuously true, i.e. if all \(j \leq i\) satisfy \(\tau_i - \tau_j \notin I\). Yet, crucially for our purposes, older Verimon versions deem some trivially true formulas like \(\text{e } a \Rightarrow \text{e } a\) safe. They evaluate to unit tables \(1_n\), where \(1_n = \{1\}_n\) and \(\{1\}_n :: \text{a option list}\).
only has None repeated n times. In fact, \((\alpha)_n\) is the only valuation \(v\) that satisfies \(w\)-tuple \(n\emptyset v\). In the next section, we take advantage of this and define a notion of safety that allows an encoding of \(\Pi \alpha \neq 0 \notin I\) to be safe.

The formalisation of syntax and semantics of the dual-operators is a technical contribution from our work. It involves routine extensions of definitions such as \(fv\) \(\alpha\) but it also requires adding properties about satisfiability of both operators. We add more than 400 lines of code to Verimon’s formalisation of syntax and semantics [19, 20].

3 Relaxation of Safety

If \(\alpha \top \beta\) is vacuously true at \(i\), then Verimon’s output is correct if it is equal to \([\alpha \top \beta]_i^n = 1_n\). However, in its current implementation, this would only be provable for formulas such that \(fv(\alpha \top \beta) = \emptyset\). Therefore, to define a larger fragment of evaluable formulas, it is convenient to choose the correct set of attributes at time-point \(i\). Here, we generalise Verimon’s fragment of safe formulas by introducing the dynamic free variables \(dfv\sigma i\alpha\) of \(\alpha\) at \(i\) and its set of safe sets of free variables \(ssfv\alpha\). The function \(dfv\sigma i\alpha\) approximates the set of free variables that influence the satisfiability of \(\alpha\) at \(i\) in the trace \(\sigma\). As with other trace-functions \(\Gamma\) and \(\tau\), we often use its abbreviated form \(dfv\alpha\). It is a semantic concept and we only need it to prove the algorithm’s correctness: outputs are exactly the sets \([\alpha]_i^{dfv\alpha}\) for the monitored formula \(\alpha\) at each \(i\). The function \(ssfv\alpha\) approximates all the possible combinations of attributes that tables at \(\alpha\) might have at different time-points. It is recursively defined so that we can decide \(\alpha\)’s safety by checking \(ssfv\alpha \neq \emptyset\), that is, we define \(is\)-safe \(\alpha \leftrightarrow ssvf\alpha \neq \emptyset\). We describe our reasoning behind the definition for each connective below and enforce various properties with our definitions: on one hand, as they are sets of free variables, (i) \(dfv\sigma i\alpha \subseteq fv\alpha\) and (ii) \(\bigcup\) \(ssfv\alpha \subseteq fv\alpha\). On the other hand, we define \(dfv\sigma i\alpha \in ssvf\alpha\). For the full formal definitions please see our Appendix A.

Atomic formulas. All atoms \(p \vdash ts\) are safe and their attributes do not change over time. Thus, we define \(ssvf(p \vdash ts) = \{fv(p \vdash ts)\}\) and \(dfv_i(p \vdash ts) = fv(p \vdash ts)\). Following Verimon, we do not make \(v x =_F v x\) safe as it is not practically relevant for us. Therefore, \(ssvf\) maps equalities \(\alpha \in \{v x =_F t, t =_F v x, t_1 =_F t_2\} \rightarrow \{fv\alpha\}\) whenever \(fv t = \emptyset\) (resp. \(fv t_1 = fv t_2 = \emptyset\)) and to \(\emptyset\) otherwise. Similarly, we define \(dfv_i(t_1 =_F t_2) = fv(t_1 =_F t_2)\) for all \(t_1, t_2 \vdash 'a trm\) and \(i \vdash N\).

Conjunctions. If safe, each conjunct may have many combinations of attributes. Moreover, a join \((\&\&\) outputs a table with all the attributes from its operands. Thus, if \(ssvf\alpha \neq \emptyset\) and \(ssvf\beta \neq \emptyset\), then \(ssvf(\alpha \& \beta) = ssvf\alpha \cup ssvf\beta\) where \((\emptyset)\) is the pairwise union \(A \cup B = \{a \cup b \mid a \in A \land b \in B\}\). We follow Verimon+ and define safety for cases when only the left conjunct \(\alpha\) is safe.\(^2\) If \(\beta\) is an equality \(t_1 =_F t_2\) with \(fv t_1 \subseteq X\) or \(fv t_2 \subseteq X\) for each \(X \in ssv\alpha\), then we can safely add any single variable on the other side of the equation, possibly not in \(fv\alpha\), to the elements of \(ssvf\alpha\). Therefore, in this case \(ssvf(\alpha \& \beta) = ((\cup) fv\beta)\) ’ (ssvf\(\alpha\)). Finally, if \(\beta\) is a negation \(~_F\beta\) of a safe formula \(\beta'\) and every \(Y \in ssv \beta'\) satisfies \(Y \subseteq X\) for each \(X \in ssv\alpha\) then we can compute antijoins, and thus \(ssvf(\alpha \& \beta) = ssvf\alpha\). If neither of these cases holds, then \(ssvf(\alpha \& \beta) = \emptyset\). Given the behaviour of join on columns, the dynamic free variables are simply \(dfv_i(\alpha \& \beta) = dfv_i \alpha \cup dfv_i \beta\).

Disjunctions. Due to union-compatibility, we can only take unions of tables with the same attributes. Yet, we can generalise for cases when formulas might be vacuously true at some time-points. To evaluate disjunctions we use the function \(eval-or\) \(n R_1 R_2\) that outputs \(1_n\) if either \(R_1 = 1_n\) or \(R_2 = 1_n\), and \(R_1 \cup R_2\) otherwise. Similarly, to ensure \(w\)-tuple \(n (dfv_i(\alpha \& \beta))\), we state that there are no “relevant” variables \((dfv_i(\alpha \& \beta) = \emptyset)\) for the satisfiability of \(\alpha \& \beta\) when either \(\alpha\) or \(\beta\) are logically valid at \(i\). If the variables of \(\alpha\) are “irrelevant” \((dfv_i \alpha = \emptyset)\) because \(\alpha\) is unsatisfiable at \(i\) \((\mathcal{A}_i = \emptyset)\), then we just need the variables of \(\beta\): \(dfv_i(\alpha \& \beta) = dfv_i \beta\). The symmetric case also holds. If both disjuncts are relevant \((dfv_i \alpha \neq \emptyset \neq dfv_i \beta)\), we need both sets of variables: \(dfv_i(\alpha \& \beta) = dfv_i \alpha \cup dfv_i \beta\).

The behaviour of \(dfv\) on disjunctions means that if \(\emptyset \in ssvf\alpha\) or \(\emptyset \in ssvf\beta\), then \(\emptyset\) should also be an element of \(ssvf(\alpha \& \beta)\). This may happen in various ways. First, if \(fv \alpha = \emptyset = fv \beta\), then by (ii) above, we know that \(ssvf \alpha = \{\emptyset\}\) or \(ssvf \beta = \{\emptyset\}\). In this case, we can define \(ssvf(\alpha \& \beta) = ssvf \alpha \cup ssvf \beta\) assuming both \(\alpha\) and \(\beta\) satisfy \(is\)-safe. Next, notice that if we allow attributes \(X_\alpha \in ssvf\alpha\) and \(X_\beta \in ssvf\beta\) such that \(\emptyset \neq X_\alpha \neq X_\beta \neq \emptyset\), the corresponding table for \(\alpha \& \beta\) would need to have infinite values. Therefore, at most we may allow \(ssvf \alpha \subseteq \{\emptyset, fv \alpha\}\),

\(^2\)Adding the symmetric case increases the number of proofs in the formalisation. It is easier to assume a formula rewrite can commute conjuncts if necessary.
ssfv $\beta \subseteq \{0, \text{fv } \beta\}$ and fv $\alpha = \text{fv } \beta$ with both $\alpha$ and $\beta$ having non-empty ssfv. In this case, if $\emptyset \in \text{ssfv } \alpha$ or $\emptyset \in \text{ssfv } \beta$ then ssfv$(\alpha \lor \beta) = \{0\} \cup (\text{ssfv } \alpha \cup \text{ssfv } \beta)$, otherwise ssfv$(\alpha \lor \beta) = \{\text{fv } \alpha\}$.

Negations. If $\alpha$ is safe and closed (ssfv$(\alpha) = \{0\}$ by (ii)), then we can safely evaluate its negation ssfv$(\neg \alpha) = \emptyset$. We also allow ssfv$(\neg t (t = t)) = \{\text{fv } (t = t)\}$ for arbitrary term $t$ to encode the constantly false formula. Otherwise negations are unsafe ssfv$(\neg \alpha) = \emptyset$. For dynamic free variables, we define dfv$_i(\neg \alpha) = \text{dfv } \alpha$.

Quantifiers. When interpreting De Bruijn indices, quantifiers remove 0 from fv $\alpha$ and subtract 1 to all its elements. Thus, we define dfv$_i(\exists \alpha \cdot \beta) = (\lambda x.x - 1)’(\text{dfv } (\alpha - \{0\}))$ and $(\lambda x.x - 1)’(X - \{0\}) \in \text{ssfv } (\exists \alpha \cdot \beta)$ for each $X \in \text{ssfv } \alpha$.

Previous and until. The definition of the dynamic free variables follows that of their semantics: dfv$_i(\chi \cdot \alpha) = \text{dfv } \alpha + 1$ while dfv$_i(\eta \cdot \chi) = \text{dfv } \chi - 1$ if $i > 0$ and dfv$_i(\eta \cdot \chi) = \text{dfv } \chi$ if $i = 0$. For safety, all combinations of attributes of $\alpha$ might be used in its one-step temporal versions ssfv$(\chi \cdot \alpha) = \text{ssfv } \alpha = \text{ssfv } (\chi \cdot \eta \cdot \alpha)$.

Since and until. Let us follow the semantics for since and until to define their dfvs at $i$. The definition for one operator emerges by dualising the time-point order and flipping subtractions of time-stamps in the other’s definition, hence we omit the description for until. We must collect all the dynamic free variables dfv$\beta$, $\alpha$ and dfv$\beta$ that influence the satisfiability of $\alpha \beta$. Start by defining $i = \{j \mid j < i \land (\tau_i - \tau_j) \in I\}$ to identify the indices $j$ for $\beta$, and the predicate saf$\beta$-sat $j = \exists \chi \cdot (\chi \cdot v, j) \mid \beta \land (\forall k \in (j, i), (\chi \cdot v, i) = \alpha)$ so that $\alpha \beta$ is unsatisfiable at $i$ if $\forall j < i \cdot \neg \text{saf$\beta$-sat } j$. When this happens we let dfv$_i(\alpha \beta) = \text{dfv } (\alpha \beta)$. Otherwise, the indices $j$ for $\beta$ are $\mathcal{J} = \{j \mid j < i \mid \text{saf$\beta$-sat } j\}$ while those $k$ for $\alpha$ are $\mathcal{K} = \bigcup_{j \in \mathcal{J}} (j, i)$. Having identified the indices, we obtain dfv$_i(\alpha \beta) = (\bigcup_{k \in \mathcal{K}} \text{dfv } k) \cup (\bigcup_{j \in \mathcal{J}} \text{dfv } \beta)$.

As we have seen for disjunctions, our definitions of saf$\beta$-sat depend on the operations in the formula-evaluation. In particular, to define ssfv$\beta$ for since, it is convenient to understand Verimon’s [19] implementation roughly represented with the equations:

\[
[\alpha \beta]_i = \bigcup_{j \in |j|} [\alpha S_{\tau_j \rightarrow j}]_i, \text{ and } [\alpha S_{\tau_j \rightarrow j}]_i = \bigcup_{k \in j \in |j|} \beta \cap \bigcap_{k \in \mathcal{K}} [\alpha]_k. 
\]

That is, the algorithm obtains the valuations in $[\alpha \beta]_i$, by iteratively updating those in $[\alpha S_{\tau_j \rightarrow j}]_i$ until it has visited all time-points $j \in |j|$, when it outputs their union. In the implementation, intersections are replaced with (anti)joins and sets of $|\alpha|$, with tables $[\beta]_{\mathcal{K}, \mathcal{J}}$ having attributes $\chi_i$ at index $l$ for safe $\alpha$. If $0 \in I$, one of the tables involved in the output union is $[\beta]_{\mathcal{K}, \mathcal{J}}$. Hence, by union-compatibility, all the other table-operators (represented by $[\alpha S_{\tau_j \rightarrow j}]_i$) must have the same attributes. To ensure this, we require ssfv$(\alpha \beta)$ to be non-empty only when ssfv$\beta = \{0\}$. Then, we define ssfv$(\alpha \beta) = \{\text{fv } \beta\}$ if $\text{fv } \alpha \subseteq \text{fv } \beta$ with ssfv$\alpha \neq \emptyset$ because the joins in the construction of the tables represented by $[\alpha S_{\tau_j \rightarrow j}]_i$ would always be guarded by the attributes $\text{fv } \beta$ of $[\beta]_{\mathcal{K}, \mathcal{J}}$. Similarly, $\text{ssfv } (\alpha \beta) = \{\text{fv } \beta\}$ for $\alpha = \neg \alpha’$ with $\text{ssfv } \alpha’ \neq \emptyset$ and $\text{fv } \alpha \subseteq \text{fv } \beta$ because of the corresponding antijoins.

The Verimon implementation for until is different and intuitively corresponds to

\[
[\alpha \ U]_i = \bigcup_{j \in |j|} [\beta]_j \cap \bigcap_{k \in \mathcal{K}} [\alpha]_k. 
\]

As before, one of the operators may be $[\beta]_{\mathcal{K}, \mathcal{J}}$, therefore we require $\text{ssfv } \beta = \{\text{fv } \beta\}$. If $\text{fv } \alpha \subseteq \text{fv } \beta$ with $\text{ssfv } \alpha \neq \emptyset$, then $\text{ssfv } (\alpha \ U) = \{\text{fv } \beta\}$. But for $\alpha = \neg \alpha’$, the tables for $\alpha’$ are united separately and antijoined to each table for $\beta$ at $j$. Thus, we need to take union-compatibility into account when $\alpha = \neg \alpha’$. Hence, in this case, we define $\text{ssfv } (\alpha \ U) = \{\text{fv } \beta\}$ if $\text{ssfv } \beta = \{\text{fv } \beta\}$ and $\text{ssfv } \alpha’ = \{\text{fv } \alpha’\}$.

Trigger and release. Our definition of dfvs for these dual operators is very similar to that of since and until. Assume $D \in \{T, R\}$ and let idz $= \lambda i \cdot \eta i$ and idet $= \lambda i \cdot \eta i$ if $D = T$; otherwise, if $D = R$, then idz $= \lambda i \cdot \eta i$ and idet $= \lambda [i, j]$. The key difference in the definition of dfvs for $D$ is that if idz $= \emptyset$, then $\alpha D \beta$ is vacuously true. Hence, we define dfv$_i(\alpha D \beta) = \emptyset$ because $\alpha D \beta$ will evaluate to the unit table. As before, we also define a predicate saf$\beta$-sat $v j$ that makes the formula unsatisfiable whenever $\forall v, \exists j \in \text{idz } \neg \text{ saf$\beta$-sat } v j$. In this case, dfv$_i(\alpha D \beta) = \text{dfv } (\alpha D \beta)$. The predicate saf$\beta$-sat also allows us to define sets of indices $\mathcal{K}$ and $\mathcal{J}$ to collect dfvs over time for the remaining cases: dfv$_i(\alpha D \beta) = (\bigcup_{k \in \mathcal{K}} \text{dfv } k) \cup (\bigcup_{j \in \mathcal{J}} \text{dfv } \beta)$.
Our definition of safety for dual operators is intuitively understood by observing
\[ [\alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta]_i = \bigcap_{j \in \text{iderv}_i} \beta_j \cup \left( \bigcup_{k \in \text{rel}_i} \text{rel}_k \cap [\alpha]_k \right), \]
where \( \text{rel}_k = [\beta]_k \) if \( 0 \in I \), and \( \text{rel}_k = [\alpha]_k \) otherwise. As before, whenever \( 0 \in I \), we define \( \text{ssfv}(\alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta) = \{ \beta \} \) assuming \( \text{ssfv} \beta = \{ \beta \} \) and \( \text{fv} \alpha \subseteq \text{fv} \beta \), both if \( \text{ssfv} \alpha \neq \emptyset \) or if \( \text{ssfv} \alpha = \emptyset \) but \( \alpha = \neg \alpha' \) with \( \text{ssfv} \alpha' \neq \emptyset \). When \( 0 \notin I \), \( \alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta \) might be vacuously true and union-compatibility is relevant. Therefore, we define \( \text{ssfv}(\alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta) = \{ \emptyset, \text{fv} \alpha \} \) whenever \( \text{ssfv} \alpha = \{ \text{fv} \alpha \} \), \( \text{ssfv} \beta = \{ \text{fv} \beta \} \) and \( \text{fv} \alpha = \text{fv} \beta \). Otherwise \( \text{ssfv}(\alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta) = \emptyset \).

We formalise all cases above together with the definition \( \text{is-safe} \alpha \Leftrightarrow \text{ssfv} \alpha \neq \emptyset \) in Isabelle/HOL. Next, by induction on the definition of ssfvs, we derive properties (i) \( \Box \), (ii) \( \Diamond \) and (iii) \( \Box \) above. Additionally, we also prove that for any formula \( \alpha \), \( \text{is-safe} \alpha \Leftrightarrow \text{fv} \alpha \in \text{ssfv} \alpha \).

In classical logic, an important property for syntactic substitutions of terms and formulas states that if two valuations \( v \) and \( v' \) coincide in \( \text{fv} \alpha \), then the value of \( \alpha \) is the same under both valuations. Similarly, we have that if \( v \vdash x = v' \vdash x \) for all \( x \in \text{dfv} \sigma \land \alpha \), then \( \langle \sigma, v, i \rangle \models \alpha \Leftrightarrow \langle \sigma, v', i \rangle \models \alpha \). This is useful for us because it ratifies that if \( \langle \sigma, (\cdot)(\cdot), i \rangle \models_M \alpha \) for \( \alpha \) with \( \text{dfv} \alpha = \emptyset \), then \( \alpha \) is logically valid at \( i \).

Let us compare our definition of safety with previous ones. To do this directly, we combine \( \text{safe-formula} \) predicates from Verimon [19] and Verimon+ [10, 22], restrict them to the syntax in §2 and add them to our Appendix A. Structurally, our definition of safety for conjunctions, negations, existential quantifiers, previous, next and until operators resembles that of Verimon+ [22] which is already more general than that of Verimon. However, in combination with other operators our definition deems more formulas safe (see §5). We discuss in §7 possible generalisations for these cases that involve dfvs and ssfvs. For equalities, \( \text{ssfv} \) generalises \( \text{safe-formula} \) even when incorporating Verimon+’s more expressive term language that includes some arithmetic operations. By using \( \text{eval-or} \), our definition of safety for disjunctions differs from \( \text{safe-formula} \) by admitting vacuously true formulas satisfying \( \text{is-safe} \) in either disjunct. Additionally, \( \text{is-safe} \) (\( \alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta \)) allows \( \alpha \) to have many ssfvs which generalises \( \text{safe-formula} \) (\( \alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta \)). Dual operators are only safe in an unintegrated extension [10] of Verimon+; but we also generalise this work by making them safe when \( 0 \notin I \), even when they are not part of a conjunction with a safe formula. In general, we show that \( \text{is-safe} \) generalises previous monitoring fragments: on one hand we prove that \( \text{is-safe} \alpha \) if \( \text{safe-formula} \alpha \) for any \( \alpha \). On the other hand, the formula \( \alpha \equiv \neg \exists v x. x = v \iff v x \) \( T_{[1,2]} (p \upharpoonright \lceil v x \rceil) \), which is equivalent to \( H_{[1,2]} p \upharpoonright \lceil v x \rceil \), satisfies \( \text{is-safe} \alpha \) but also \( \neg \text{safe-formula} \alpha \).

The definition and formalisation of safety for MFOTL-formulas is a major contribution of this work. It involves defining ssfvs, dfvs, pairwise unions and proving their corresponding properties. The developments on this section add more than 1600 lines of code to the Verimon formalisation [19, 20].

4 Implementation of Dual Operators

Intuitively, the monitoring algorithm takes as inputs a safe MFOTL-formula \( \alpha \) and an event trace \( \sigma \) and outputs the table of satisfactions \( [\alpha]_{i,n}^{\text{dfv},\alpha} \) at each time-point \( i \). Concretely, Verimon provides two functions \( \text{minit} \) and \( \text{mstep} \) to initialise and update the monitor’s state respectively. The model for this monitor’s state at \( i \) is a three-part record \( \langle \rho_i^\sigma, \alpha_M^\sigma, n \rangle \). Here, \( n = \text{nfv} \alpha \) and \( \rho_i^\sigma :: \text{nat} \) is the progress or the earliest time-point for which satisfactions of \( \alpha \) cannot yet be evaluated for lack or information. Finally, \( \alpha_M^\sigma :: \alpha \) a mformula is a recursively defined structure associated with \( \alpha \) to store all the information needed to compute \( [\alpha]_{i,n}^{\text{dfv},\alpha} \). We describe our extensions to all these functions and structures in order to monitor dual operators trigger and release. In the sequel, we use \( A_i = [\alpha]_{i,n}^{\text{dfv},\alpha}, B_i = [\beta]_{i,n}^{\text{dfv},\beta} \) and \( C_i = [\gamma]_{i,n}^{\text{dfv},\gamma} \) to simplify notation.

The function \( \text{minit} \) is just a wrapper calling \( \text{minit0} \) to set the initial monitor’s state to \( 0, \alpha_M^\sigma, \text{nfv} \alpha \). Accordingly, \( \text{minit0} \) takes a safe MFOTL-formula \( \alpha :: \alpha \) and transforms it into \( \alpha_M:: \alpha \). The datatype \( \text{mformula} \) describes the information needed at each \( i \) to compute \( A_i \). For instance, if \( \alpha \)’s main connective is a binary operator with direct subformulas \( \beta \) and \( \gamma \), the set of satisfactions for either of them are only available up to \( \rho_i^\beta \) and \( \rho_i^\gamma \) respectively at \( i \). This is why \( \alpha_M \) includes a buffer \( \text{buf} : \alpha \) mbuf2 to store yet, unused tables \( B_j \) (resp. \( C_j \)) such that \( j \in [\rho_i^\beta, \rho_i^\gamma] \) (resp. \( j \in [\rho_i^\gamma, \rho_i^\beta] \)). Whenever both \( B_j \) and \( C_j \) are in the buffer, the algorithm operates them and either outputs the result or stores it in an auxiliary state for future processing. To help the algorithm know whether to do joins or antijoins, \( \alpha_M \) may also include a boolean indicating whether one of its direct sub-formulas is not negated. More specifically, the state-representations of an \( \alpha \circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \beta \) with \( \beta D \in \{ T, R \} \) include their sub-mformulas \( \alpha_M \) and \( \beta_M \), a boolean indicating if \( \alpha \) is not negated, the interval \( I \vdash I \), the buffer \( \text{buf} \), a corresponding list of unused time-stamps \( \tau_s \), and its auxiliary state. Formally, we add the last two lines below to the definition of the \( \text{mformula} \) datatype:
where the second boolean indicates whether $0 \in I$, and $'a mtaux$ and $'a mraux$ are the type-abbreviations we use for trigger and release’s auxiliary states respectively. Let us describe these in detail next.

Our implementation of the auxiliary state for trigger $Ts :: 'a mtaux$ resembles Verimon’s [19] auxiliary state for since. That is, it combines the intuitions in eqs. (1) and (2): at time-point $i$, it is a list of time-stamped tables $(\tau_j, T^i_{\tau_j})$ that the monitor joins after it has passed all $j \notin \downarrow_i i$. Abbreviating $\iota = (\min \rho^i_{\alpha} \rho^i_{\beta}) - 1$ and using $\infty$ to denote a join with non-negated subformula $\alpha$ and an antijoin with $\alpha$ for the direct subformula $\neg_F \alpha$, we intend

\[
T^i_{\tau_j} = \bigotimes_{k \in (\tau_i - \tau_j), i} B_k \cup \bigcup_{i \in [k,i]} B_i \infty^\iota A_i,
\]

if $0 \notin I$ and

\[
T^i_{\tau_j} = [\alpha T_{(\tau_i - \tau_j)} \beta]_{\iota}^{|\iota, n|},
\]

if $0 \notin I$.

The tables described by eq. (3) would coincide with the tables for $\alpha T_{(\tau_i - \tau_j)} \beta$ as in eq. (4) if we remove the $B_i$s inside the union, but we use them to simplify our definition of safety as discussed after eq. (2).

Likewise, following the intuition provided by eq. (2), the auxiliary state for release $Rs :: 'a mraux$ at $i$ is a list of ternary tuples $(\tau_j, R^i_L, R^i_R)$ such that

\[
R^i_L = \begin{cases}
if 0 \in I \text{ then } \bigcup_{k \in [j,i]} B_k \infty^\alpha A_k & \text{else } \bigcup_{k \in [j,i]} A_k
\end{cases}
\]

and

\[
R^i_R = \begin{cases}
if 0 \in I \text{ then } B_i \infty^\alpha 1 & \text{else } 1
\end{cases}
\]

The function that transforms $\alpha :: 'a frm \text{ to } \alpha_M :: 'a mformula$ is $\text{minit0}$. Our additions for this function on trigger and release follow our definition of safety. That is, $\text{minit0}$ maps $\alpha D_I \beta$ with $D \in \{T, R\}$ to

\[MDual (\alpha \neq \neg_F \alpha') (\text{minit0} \ n \ \alpha) (0 \in I) I (\text{minit0} \ n \ \beta) ([], [], [], []),\]

for some $\alpha'$ and $MDual \in \{MTrigger, MRelease\}$ accordingly.

When the monitor processes the $i$th time-point, the function $\text{mstep}$ outputs the new state $(\rho^i, \alpha_M, n)$. However, it also is a wrapper for the function $\text{meval}$ in charge of updating $\alpha_M :: 'a mformula$. Intuitively, $\text{meval}$ takes $\alpha_M$ and the trace information $\Gamma \sigma i$ and $\tau \sigma i$ and outputs the updated mformula $\alpha_M$ and the evaluated tables $A_j$ from $j = \rho^i - 1$ to $j = \rho^i - 1$. We describe its behaviour on auxiliary states below.

Assuming $\text{meval}$ has produced tables $A$, and $B_i$, it uses the function $\text{update-trigger}$ to update the auxiliary state $Ts :: 'a mtaux$ and to possibly output $[\alpha T_I \beta]_{\iota}^{|\iota, n|}$. This function first filters $Ts$ by removing all elements whose time-stamps $\tau_i$ are not relevant for future time-points, i.e. $\tau_i < \tau_i - I$. Then, following eqs. (3) and (4), it takes the union of the latest $B_i \infty A_i$ or $A_i$, with all elements in $Ts$, depending on whether $0 \in I$ or not respectively. Next, it adds $B_i$ either as a new element $(\tau_i, B_i) \# Ts$ if this is the first time $\tau_i$ is seen, or by joining it with the table in $Ts$ time-stamped with $\tau_i - 1 = \tau_i$, to obtain $T^i_{\tau_i}$. Finally, it joins all the tables $T^i_{\tau_j}$ in $Ts$ such that $j \in \downarrow_i i$.

The function $\text{update-release}$ takes the union of the latest $B_i \infty A_i$ or $A_i$, with each $R^i_{L,F}$ in $Rs :: 'a mraux$, depending on whether $0 \in I$ (see eqs. (5) and (6)). On the third entries $R^i_{L,F}$, it joins $B_i \cup R^i_{L,F} \# I$ if $(\tau_i - \tau_j) < I$, otherwise, it leaves $R^i_{L,F}$ as it is. Finally, it adds at the end of $Rs$ the tuple $(\tau_i, B_i \infty A_i, B_i)$ if $0 \in I$, or $(\tau_i, A_i, 1)$ if $0 \notin I$. To output the final results $[\alpha R_I \beta]_{\iota}^{d \infty A_i \# R_I \beta}$, $\text{meval}$ calls the function $\text{eval-future}$ which traverses $Rs$ and outputs all $R^i_{L,F}$ with $\tau_j + I < \tau_i$ and removes them from $Rs$.

The only other modification we do on $\text{meval}$ for the remaining logical connectives is in the disjunction case, where we have replaced every instance of the traditional union ($\cup$) with our function $\text{eval-or}$. This is the only place where we perform this substitution.

For our purposes, this concludes the description of the monitoring algorithm. It mainly consists of the initialisation function $\text{minit}$ and the single-step function $\text{mstep}$. Together, they form Verimon’s online interface with the monitored system. Our additions in this regard are mostly conceptual since we only implement the evaluation of trigger and release as functions $\text{update-trigger}$ and $\text{update-release}$.
5 Monitoring Examples

We describe various formulas that are safe according to our definition of ssfv but that are not safe in previous Verimon implementations. We present them through case-studies. Our examples occur not only in monitoring but also in relational databases.

Quality assessment (operations on globally). A company passes its products through sequential processes \( p_1, p_2 \) and \( p_3 \). Every item passes through all processes. Every minute, the company logs the time \( \tau \) and the identification number ID of each item in process \( p_i \). It uses an online monitor to classify its products according to their quality. The best ones are those that pass through process \( p_i \) for exactly \( n_i \) minutes and that move to \( p_{i+1} \) immediately afterwards. The second-best ones are those that pass through at least one process \( p_i \) in at least \( n_i \) minutes. The remaining items need to be corrected after production. To identify the best ones, the company uses the following specification

\[
\text{best} \equiv \left( G_{[0,n_1]}(p_1 \uparrow [v \ x]) \land F \left( G_{[n_1,n_2]}(p_2 \uparrow [v \ x]) \land F \left( G_{[n_2,n_3]}(p_3 \uparrow [v \ x]) \right) \right) \right),
\]

where \( n_2^* = n_1 + n_2 \) and \( n_3^* = n_1 + n_2 + n_3 \). Similarly, to identify those products with just good quality, they use the specification

\[
\text{good} \equiv \left( G_{[0,n_1]}(p_1 \uparrow [v \ x]) \lor F \left( G_{[n_1,n_2]}(p_2 \uparrow [v \ x]) \lor F \left( G_{[n_2,n_3]}(p_3 \uparrow [v \ x]) \right) \right) \right).
\]

Our relaxation of safety, makes both formulas monitorable by encoding \( G_{[a,b]}(p \uparrow [v \ x]) \) as \( \neg \neg F(\neg x = F \ x \lor x)G_{[a,b]}(p \uparrow [v \ x]) \). Specifically, the safe sets of free variables are \( \text{ssfv}(\text{best}) = \{ \{x\} \} \neq \emptyset \) and \( \text{ssfv}(\text{good}) = \{\emptyset, \{x\}\} \neq \emptyset \). We use the functions \( \text{minit} \) and \( \text{mstep} \) to monitor best through a manually made trace. We assume four products with IDs 0, 1, 2 and 3 and fix \( n_1 = n_2 = n_3 = 2 \). The table below represents said trace and shows the monitor’s output at each time-point.

| time | product_id 0 | product_id 1 | product_id 2 | product_id 3 | output for best |
|------|--------------|--------------|--------------|--------------|----------------|
| 0    | \( p_1 \)    | \( p_1 \)    | \( p_1 \)    | \( p_1 \)    | \( \emptyset \) |
| 1    | \( p_1 \)    | \( p_1 \)    | \( p_1 \)    | \( p_1 \)    | \( \emptyset \) |
| 2    | \( p_2 \)    | \( p_2 \)    | \( p_1 \)    | \( p_1 \)    | \( \emptyset \) |
| 3    | \( p_2 \)    | \( p_2 \)    | \( p_2 \)    | \( p_2 \)    | \( \emptyset \) |
| 4    | \( p_3 \)    | \( p_2 \)    | \( p_2 \)    | \( p_3 \)    | \( \emptyset \) |
| 5    | \( p_3 \)    | \( p_3 \)    | \( p_2 \)    | \( p_3 \)    | \( \emptyset \) |
| 6    | \( - \)      | \( p_3 \)    | \( p_2 \)    | \( - \)      | \( \{0\}, \{3\} \) \( \odot \tau = 0 \) |

The monitor correctly classifies the products with IDs 0 and 3 as the best ones after the first 6 minutes have passed. Before that, it outputs the empty set indicating that no product was one of the best yet. Our Appendix \( \Lambda \) includes the formalisation of this trace. These encodings for \textit{best} and \textit{good} are not safe in any previous implementation. Assuming \( 0 < a < b \), the equivalence [10]

\[
G_{[a,b]}(\alpha) \equiv (F_{[a,b]}(\alpha) \land F \neg F (F_{[a,b]}(\alpha) \lor F (G_{[a,b]}(\alpha)))) \land F \neg F \alpha
\]

also produces encodings of \textit{best} and \textit{good} that satisfy \textit{safe-formula} but these are clearly longer than simply using release \( R \) as above.

Vaccine refrigeration times (conjunction of negated historically). A company that manufactures dry-ice thermal shipping containers has just reported a loss of their refrigeration effectiveness after \( m \) hours. This means that some vaccines transported in those containers are not effective because the vaccines are very sensitive to thermal conditions. Vaccination centres need to know which vaccines they can apply. They ask for help from the shipping company in charge of transporting the vaccines. This company needs to take into account its unpacking time that consistently requires \( n > 0 \) minutes. Fortunately, the shipping company has a log, that among other things, registers \( "\odot \tau (\text{travelling, id})" \) if package with ID-number id is travelling at time \( \tau \), and annotates \( "\odot \tau (\text{arrived, id})" \) when the package identified with id arrives at a centre at time \( \tau \). The company can deploy a monitor of the following specification over their log to know which packages contain vaccines that are safe to use:

\[
(\text{arrived} \uparrow [v \ x]) \land F \neg F H_{[n,m]}(\text{travelling} \uparrow [v \ x]),
\]

8
where \( m_* \) is \( m \) in minutes plus some margin of error \( \epsilon \), and \( \Pi_I (\text{travelling} \uparrow [v.x]) \equiv \neg_F (v.x = F v.x) \Pi_I (\text{travelling} \uparrow [v.x]) \).

Previous work on Verimon+ could tackle an equivalent specification, but it would require the less straightforward encoding [10]:

\[
H_{[a,b)} \alpha \equiv (P_{[a,b)} \alpha) \land_F \neg_F (P_{[a,b)} ((P_{[0,b)} \alpha) \lor_F (F_{[0,b)} \alpha))) \land_F \neg_F \alpha, \quad \text{with } 0 < a < b.
\]

Financial crime investigation (historically with many variables). Data scientists suspect a vulnerability in the security system of their employer, a new online bank. They notice the following pattern: various failed payment attempts of the same amount from one account to another for 5 consecutive minutes. Then, 30 minutes later, a successful payment of the same amount between the same accounts. One of the queries to the database that the scientists can issue to confirm their suspicions is

\[
(\text{approved.trans_from_paid_to} \uparrow [v.0, v.1, v.2, v.3]) \land_F H_{[30,34]} (\exists_F (\text{failed.trans_from_paid_to} \uparrow [v.0, v.1, v.2, v.3])).
\]

In both predicates, \( v.0 \) represents the transaction ID, \( v.1 \) is the account that pays, \( v.2 \) corresponds to the amount of money transferred, and \( v.3 \) denotes the receiving account. The query itself finds all transactions that were successful between two accounts \( v.1 \) and \( v.3 \) at a given time-point but that were attempted for 5 consecutive minutes, 30 minutes earlier. This not only provides all suspicious receivers, but also all possible victims and the amount of money they lost per transaction.

We can codify \( H_I \alpha \equiv (\bot_F \alpha) \Pi_I \alpha \), where the expression \( \bot_F \alpha \) denotes the formula \( \neg_F (v.0 = F v.0) \land_F \cdots \land_F \neg_F (v.n = F v.n) \) and \( n = \text{nfv} \alpha - 1 \). Monitoring such a simple encoding of historically (with \( 0 \notin I \)) is possible due to the integration of our safety relaxation into the algorithm. Simplifying this further to \( \bot_F \alpha \equiv (\neg_F (c.0 = F c.0)) \) produces an unsafe formula because the free variables on both sides of \( T_I \) do not coincide. We discuss generalisations involving ssfs and dfvs to achieve this simplification in §7.

Monitoring piracy (release operator). To deal with a recent increase in piracy, a shipping company integrates a monitor into their tanker tracking system. By standard, their vessels constantly broadcast a signal with their location through their automatic identification system, which in good conditions arrives every minute, but in adverse ones, can take more than 14 hours to update. This signalling system is one of the first turned off by pirates because it allows them to sell the tanker’s contents in nearby unofficial ports. Thus, the company regularly registers the ID of its planned route. This lets the company collaborate with local authorities and try to locate their vessels through alternative means. Due to our explicit implementation of the bounded release operator, the specification above is now straightforwardly monitorable. Indeed, running the monitor through a manual trace illustrated in the table below and assuming for simplicity \( n = 2 \) correctly identifies ships with IDs 1 and 2 as those possibly pirated. The formalisation of the trace is available in our Appendix A.

| time | ship_id 1 | ship_id 2 | ship_id 3 | output for best |
|------|-----------|-----------|-----------|-----------------|
| 0    | no_sign   | no_sign   | sign      | \( \emptyset \) |
| 1    | no_sign   | no_sign   | sign      | \( \emptyset \) |
| 2    | no_sign   | no_sign   | sign      | \( \emptyset \) |
| 3    | off_route | no_sign   | sign      | \{ [1], [2] \} @\tau = 0 |
| 4    | off_route | no_sign   | sign      | \{ [2] \} @\tau = 1 |

6 Correctness

In this section, we show that the integration of our relaxation of safety into Verimon’s monitoring algorithm is correct. We focus specifically on describing the proof of correctness for our implementation of dual operators. That is, we show that the algorithm described in previous sections outputs exactly the tables \( \alpha^n_{\text{nfv} \alpha} \) at time-point \( i \) for safe
α. We also comment on the overall additions and adaptations required in the proof of Verimon’s correctness to accommodate our definition of safety.

First, an Isabelle predicate to describe that a given set of valuations $R$ is a proper table with $n$ attributes in $X$ is $table\ n\ X\ R \Leftarrow (\forall v \in R. \ \text{wf-tuple} \ n \ X \ R)$. The formalisation of Verimon [20], uses the predicate $qtable\ n\ X\ P\ Q\ R$ to state correctness of outputs, where $n :: \mathbb{N}$, $X :: \mathbb{N}$ set, $P$ and $Q$ are predicates on valuations, and $R$ is a table. It is characterised by

\[
qtable\ n\ X\ P\ Q\ R \Leftarrow (table\ n\ X\ R) \land (\forall v. \ P\ v \Rightarrow (v \in R \Rightarrow Q\ v \land \text{wf-tuple} \ n \ X\ v)).
\]

In our case, it is typically evaluated to $qtable\ n\ (dfv,\ \alpha)\ P(\lambda v. (\sigma, v, i) \models_M \alpha)\ R$ with $n \geq \text{nf}v\ \alpha$. If $R$ is Verimon’s output for $\alpha$ at $i$, it roughly states that $R = [\alpha]_{n,\text{nf}v\ \alpha}^\alpha$ modulo $P$ and that $table(\text{nf}v\ \alpha) (dfv,\ \alpha)\ R$. In Verimon’s and our proof of correctness, $P$ is instantiated to a trivially true statement. We do not omit $P$ here because our general results require assumptions about it.

Given our use of empty dfvs for vacuously true formulas, the behaviour of $qtable$ on empty sets of variables is relevant for us. The only tables that satisfy $qtable\ n\ \emptyset\ P\ Q\ R$ are $R = 1_n$ and $\emptyset$ so that, assuming $P(\emptyset)_n$, if $Q(\emptyset)_n$ then $R = 1_n$; otherwise, if $(Q(\emptyset)_n)$ then $R = \emptyset$. In fact, the only way that $table\ n\ X\ 1_n$ holds is if $X \subseteq \{x | x \geq n\}$.

Given that we only use sets $X$ such that $X \subseteq \{x | x < n\}$, only $X = \emptyset$ fits $table\ n\ X\ 1_n$.

Since we must join all the tables in the auxiliary states for trigger and release, we also relate $qtable$ and joins ($\bowtie$).

Provided $dfv\ \sigma\ i\ \alpha \subseteq X$ and $dfv\ \sigma\ i\ \beta \subseteq Y$:

\[
[[\alpha \land F\ \beta]_{1_n}^{X,Y} = [[\alpha]_{1_n}^{X} \bowtie [[\beta]_{1_n}^{Y}], \text{ and}
\]

\[
[[\alpha \land F\ \neg\ \beta]_{1_n}^{X,Y} = [[\alpha]_{1_n}^{X} \bowtie [[\beta]_{1_n}^{Y}], \text{ assuming } Y \subseteq X.
\]

More general results in terms of $qtable$ are also available. For instance, if $\pi_X$ is the projection $\pi_X\ v = \text{map}(\lambda i. \text{if } i \in X \text{ then } v(i) \text{ else None})[0, \ldots, \text{length } v-1]$, then it holds that $qtable\ n\ Z\ P\ Q\ (R_1 \bowtie R_2)$ if $qtable\ n\ X\ P\ Q\ R_1$, $qtable\ n\ Y\ P\ Q\ R_2$, $Z = X \cup Y$ and $\forall v. \ \text{wf-tuple} \ n\ Z\ v\ \bowtie \ P\ v \Rightarrow (Q\ v \Rightarrow Q_1(\pi_X\ v) \land Q_2(\pi_Y\ v))$. A similar statement is true for joins.

Moreover, the join of two tables with the same attributes is simply their intersection. By our definition of ssfvs, our $n$-ary join on the auxiliary states is made on tables with the same attributes, thus we use the following fact: for a finite non-empty set of indices $I$, $qtable\ n\ X\ P\ Q\ (\bigcap_{i \in I} R_i)$ holds if $qtable\ n\ X\ P\ Q\ R_i$ and $\forall v. \ \text{wf-tuple} \ n\ X\ v\ \bowtie \ P\ v \Rightarrow (Q\ v \Rightarrow (\forall i \in I. Q_i(v)))$.

The relationship between eval-or and $qtable$ is also relevant for our correctness proof due to our modifications to the algorithm. We state here the specific statement: if $qtable\ n\ (dfv,\ \alpha)\ P(\lambda v. (\sigma, v, i) \models_M \alpha)\ R_1$ and $qtable\ n\ (dfv,\ \beta)\ P(\lambda v. (\sigma, v, i) \models_M \beta)\ R_2$ then $qtable\ n\ (dfv(\alpha \land F\ \beta))\ P(\lambda v. (\sigma, v, i) \models_M \alpha \lor F\ \beta)\ R_1 \bowtie R_2$, provided $P(\emptyset)_n$ and $dfv,\ \alpha = \emptyset, \ dfv,\ \beta = \emptyset$ or $dfv,\ \alpha = dfv,\ \beta$. See more results in the Isabelle code.

To state the correctness of the algorithm, Verimon defines an inductive predicate $wf-mformula\ \sigma\ i\ n\ U\ \alpha_{M}\ \beta\ \alpha$ where $i, n :: \mathbb{N}$, $\alpha_{M} :: \alpha$ is a mformula, $\alpha :: \alpha$ a frm and $U$ is a set of valuations. The set $U$ is always instantiated to the universal set $UNIV$, or the set of all terms of a given type. In fact, $P$ in $qtable$ just checks for membership in $U = UNIV$. The predicate $wf-mformula$ is an invariant that holds after initialisation with $\text{minit}$ and that remains true after each application of $\text{mstop}$. It carries all the information to prove correctness of outputs $R$ at $i$ via the predicate $qtable\ n\ (dfv,\ \alpha)\ P(\lambda v. (\sigma, v, i) \models_M \alpha)\ R$. We describe here only our additions for dual operators.

inductive $wf-mformula$ where . . .

| Trigger: \text{wf-mformula} \ \sigma\ i\ n\ U\ \alpha_{M}\ \alpha \Rightarrow \text{wf-mformula} \ \sigma\ i\ n\ U\ \beta_{M}\ \beta |
| \Rightarrow \alpha' = (pos_{\downarrow} A, \alpha) \land \text{mem}_{\downarrow} \alpha \Rightarrow \text{mem}, \ \emptyset \in I |
| is-safe (\alpha'\ T_1\ \beta) \Rightarrow \text{wf-mbuf2}^{\sigma} \ i\ n\ U\ \alpha\ \beta\ \text{buf} \Rightarrow \text{wf-ts} \ \sigma\ i\ \alpha\ \beta\ \text{nts} |
| \Rightarrow \text{wf-trigger-aux} \ \sigma\ i\ n\ U\ \alpha\ \text{mem}_0 I\ \beta\ \text{aux} (\text{progress} \ \sigma (\alpha'\ T_1\ \beta)) |
| \Rightarrow \text{wf-mformula} \ \sigma\ i\ n\ U (MTrigger\ pos_{\downarrow} \alpha_{M}\ \text{mem}_0 I\ \beta_{M}\ \text{buf}\ \text{aux}) (\alpha'\ T_1\ \beta) |
| Release: \text{wf-mformula} \ \sigma\ i\ n\ U\ \alpha_{M}\ \alpha \Rightarrow \text{wf-mformula} \ \sigma\ i\ n\ U\ \beta_{M}\ \beta |
| \Rightarrow \alpha' = (pos_{\downarrow} A, \alpha) \land \text{mem}_{\downarrow} \alpha \Rightarrow \text{mem}, \ \emptyset \in I |
| is-safe (\alpha'\ R_1\ \beta) \Rightarrow \text{wf-mbuf2}^{\sigma} \ i\ n\ U\ \alpha\ \beta\ \text{buf} \Rightarrow \text{wf-ts} \ \sigma\ i\ \alpha\ \beta\ \text{nts} |
| \Rightarrow \text{wf-release-aux} \ \sigma\ i\ n\ U\ \alpha\ \text{mem}_0 I\ \beta\ \text{aux} (\text{progress} \ \sigma (\alpha'\ R_1\ \beta)) |
| \Rightarrow \text{progress} \ \sigma (\alpha'\ R_1\ \beta) \ i + \text{length} \ \text{aux} = \text{min} (\text{progress} \ \sigma (\alpha\ i)) (\text{progress} \ \sigma (\beta\ i)) |
| \Rightarrow \text{wf-mformula} \ \sigma\ i\ n\ U (MRelease\ pos_{\downarrow} \alpha_{M}\ \text{mem}_0 I\ \beta_{M}\ \text{buf}\ \text{aux}) (\alpha'\ R_1\ \beta) |

The code above states that if all of the conditions before the last arrow ($\Rightarrow$) are satisfied, then we can assert $\text{wf-mformula}$ for trigger or release respectively. The function $\text{progress} \ \sigma\ \alpha\ i$ is $\rho_{\alpha}^i$, while $pos_{\downarrow} A$ is our Isabelle
abbreviation to state that \( \alpha \) is not-negated according to the boolean pos. Similarly, \( \alpha \land \) test \( \lor \beta \) is just \( \alpha \) if test is true, otherwise it is \( \beta \). The predicates \( \psi-
abla mbuf2' \) and \( \psi-tl \) check that the buffer and the corresponding list of time-stamps are well-formed in the sense that the buffer has every visited but yet unused table for \( \alpha \) and \( \beta \) while \( nts \) has all the corresponding time-stamps \( \tau_1 \) with \((\min \rho_{\alpha}^i \rho_{\beta}^j) \leq j \leq (\max \rho_{\alpha}^i \rho_{\beta}^j)\). Additionally, we describe below the corresponding invariants \( \psi-
abla trigger-aux \) and \( \psi-
abla release-aux \) for the auxiliary states.

Recall from eqs. (3) and (4) that trigger’s auxiliary state \( Ts \) at time-point \( i \) is a list of pairs \((\tau_j, T^*_j)\). In the formalisation (see also Appendix A), we split our definition of its invariant \( \psi-
abla trigger-aux \) into two parts. First, we state the properties of the time-stamps \( \tau_j \); they are strictly ordered, less than the latest \( \tau_i \) and satisfy that \( \tau_j \in \tau_i-I \) for \( i = (\min \rho_{\alpha}^i \rho_{\beta}^j)-1 \). Conversely, it also affirms that every time-stamp satisfying these properties appears in \( Ts \). The second part asserts correctness. That is, \( qtable n(\psi-
abla \beta \) P \( Q_i^T \) \( T^*_j \) where \( Q_i^T \) \( v \) \( \Leftrightarrow (\langle \sigma, v, \iota \rangle = M \alpha T_{(\tau_i - \tau_j)} \beta \) if \( 0 \notin I \), and \( Q_i^T \) \( v \) \( \Leftrightarrow (\forall k \leq i. \tau_k = \tau_j \Rightarrow (\sigma, v, k) \models M \beta \lor (\exists l \in (k, i). (\sigma, v, l) \models M) \) if \( 0 \in I \).

The invariant \( \psi-
abla release-aux \) for release’s auxiliary state \( Rs \) is more verbose. Assuming \( 0 \in I \), it asserts \( qtable n(\psi-
abla \beta \) P \( Q_{\L \circ \L}^{ij} R_{L}^{ij} \) and \( qtable n(\psi-
abla \beta \) P \( Q_{\L \circ \L}^{ij} R_{K}^{ij} \) where \( Q_{\L \circ \L}^{ij} \) and \( Q_{\L \circ \L}^{ij} \) describe the first parts of eqs. (5) and (6). That is,

\[
Q_{\L \circ \L}^{ij} v \Leftrightarrow (\exists k \in [j, i). (\sigma, v, k) \models M \beta \land (\sigma, v, k) \models M \alpha)
\]

\[
Q_{\L \circ \L}^{ij} v \Leftrightarrow (\forall k \in [j, i). (\tau_k - \tau_j) \in I \Rightarrow (\sigma, v, k) \models M \beta \lor Q_{\L \circ \L}^{ij} v).
\]

However, when \( 0 \notin I \), the invariant asserts \( qtable n(\psi-
abla \beta \) P \( Q_{\L \circ \L}^{ij} R_{L}^{ij} \) where

\[
Q_{\L \circ \L}^{ij} v \Leftrightarrow (\exists k \in [j, i). (\sigma, v, k) \models M \alpha).
\]

For the right table, if \( \tau_i < \tau_j + I \), it simply asserts \( R_{K}^{ij} = 1_a \). However, if \( (\tau_i - \tau_j) \in I \), the invariant states that

\[
qtable n(\psi-
abla \beta \) P \( Q_{\L \circ \L}^{ij} R_{K}^{ij} \) where
\]

\[
Q_{\L \circ \L}^{ij} v \Leftrightarrow (\forall k \in [j, i). (\tau_k - \tau_j) \in I \Rightarrow (\sigma, v, k) \models M \beta \lor Q_{\L \circ \L}^{ij} v).
\]

Finally, the case when \( \tau_i + I < \tau_j \) asserts \( qtable n(\psi-
abla \beta \) P \( Q_{\L \circ \L}^{ij} R_{K}^{ij} \).

We then adapt Verimon’s proof of correctness [19] for the monitored formula \( \alpha \). It consists of two facts: (a) after initialisation, \( \alpha_0^M \) satisfies \( \psi-
abla mformula \), and (b) whenever \( \alpha_{\Lambda-1}^M \) satisfies \( \psi-
abla mformula \), then after an execution of \( meval \), the new \( \alpha_M^M \) also satisfies \( \psi-
abla mformula \) and all the outputs of \( meval \) are correct.

At initialisation (a), our relaxation of safety allows us to replace \( safe-formula \) with \( is-safe \). Also, due to the condition \( P(\gamma) \), in our results about \( qtable \) and \( eval-or \), we need to assume \( \gamma \in U \) where \( U \) is the set referred in \( \psi-
abla mformula \). Formally, our correctness of initialisation states that if \( is-safe \alpha, \gamma \) then \( \alpha \) is an element of the set \( U \), and the free variables of \( \alpha \) are all less than \( n \), then \( \psi-
abla mformula \) at \( 0 \in U \) \( (minit) n \) \( \gamma \) \( \alpha \). The proof is a typical application of inductive reasoning but not fully automatic since we need case distinctions for negations, conjunctions and dual operators.

The addition of \( dfv \)s and \( sfv \)s produces more changes in Verimon’s invariant preservation proof (b) than in the initialisation proof (a). In many preliminary definitions and lemmas, including that of \( \psi-
abla mbuf2' \), we replace the argument \( fv\alpha \) with \( dfv\alpha \). In others, a straightforward substitution is necessary. For instance, in the auxiliary state for until, we do not simply use \( dfv\alpha \) but the union of various \( dfv \)s. This reverberates in the proof of correctness of the auxiliary state which quintuples its size from 28 to 140 lines of code due to the various cases generated by both \( dfv \)s and \( sfv \)s.

Our proof of correctness for trigger’s auxiliary state consists of a step-wise decomposition of \( update-trigger \) and stating, at each step, what the tables in the auxiliary state satisfy in terms of \( qtable \). It is 440 lines of code long, double the size of the proof for since due to the case distinctions \( 0 \in I \) and \( 0 \notin I \). These also appear in the corresponding proof of correctness for release and dictate its main structure. On one hand, the case \( 0 \in I \) for release is further split into whether the auxiliary state was previously an empty list or not. On the other hand, the assumption \( 0 \notin I \) requires analysing the different cases \( \tau_i < \tau_j + I, (\tau_i - \tau_j) \in I \) and \( \tau_j + I < \tau_i \) as above. Each of these also considers the emptiness of the auxiliary state at the previous time-point.

Finally, the theorem that uses all of these correctness results and modifications is the invariant preservation proof (b) above. In more detail, it states that if we start with \( \alpha_M \) such that \( \psi-
abla mformula \sigma i n U \alpha_M \alpha \) and \( meval n \tau_i \Gamma \alpha_M = (outputs, \alpha_M) \), then \( \psi-
abla mformula \sigma i + 1 n U \alpha_M \alpha \) and \( qtable n(\psi-
abla \alpha) P (\lambda v. (\sigma, v, j) \models M \alpha) R_j \) for each \( R_j \in outputs \) with \( j \in [\rho_{\alpha}^i, \rho_{\beta}^j+1] \). We do its proof over the structure of \( \alpha_M : \psi-
abla mformula \). The base step for equalities and inequalities requires some case distinctions and our results about \( qtable \) and \( 1_a \). The inductive
steps require mostly the same argument: from wf-mformula we know most of the information to prove qtable, we supply it to our preliminary lemmas like the correctness of auxiliary states and buffers, finally we use these results and the inductive definition of wf-mformula to obtain our desired conclusion. Our separation of preliminary lemmas from the main body of the proof of (b) highly increases readability of this long argument.

This concludes our description of the correctness argument from definitions to explanations on the proof structure. Their formalisation is one of our major contributions. Our additions on properties about qtable and dfvs consists of approximately 350 lines of code, while those to the proof of correctness are more than 1500. This still does not take into account the additions on other already existing results, like the modifications to the proof of correctness of until’s auxiliary state. In total, the correctness argument changed from approximately 1000 lines of code to more than 3000.

7 Conclusion

We defined a fragment of MFOTL-formulas guaranteeing their relational-algebra representations to be computed through well-known table operations. For this, we introduced the set of safe sets of free variables (ssfvs) of a formula which collects all possible allowed attributes of the formula’s table-representations over time. The fragment required this set to be non-empty. We argued that this safe fragment is larger than others from previous work on temporal properties and pointed to our Isabelle/HOL proof of this fact. We integrated our relaxation of safety into a monitoring algorithm. The formal verification of this integration was possible due to our newly introduced concept of dynamic free variables (dfvs) of the monitored specification. We also extended the algorithm with concrete syntax and functions to monitor MFOTL dual operators trigger and release. The combination of ssfvs, dfvs and dual operators enabled the algorithm to monitor more specifications, some of which we illustrated via examples.

Future work. Our relaxation of safety can be generalised in various ways. The simplest of these add cases to our definition of ssfvs. For instance, asserting ssfv(t = F t) = {0} is possible since we can map it to 1v. However, doing this has unintended consequences that forces us to rethink other cases, e.g., the conjunction v x = F v x ∧ F v y = F v y would become safe under the current definition. Furthermore, we lose some “nice” properties like is-safe α ⇒ fv α ∈ ssfv α. It is also unsatisfactory that safety for (¬p0 α)Sjβ only requires ssfv β = {β} and fv α ⊆ fv β while that for (¬p0 α)Ujβ needs the stronger condition ssfv β = {β} and ssfv α = {fv α}. A reimplementation of the monitoring functions for until would alleviate this situation.

An orthogonal development replaces every instance of union, (∪) or (), in the implementation with our generalised eval-or. This would allow us to change our definition of safety so that more attributes are available for the right-hand-side formula in temporal operators (i.e., ssfv β ⊆ {0, fv β}). Consequently, this would allow us to write combinations of them, e.g., Pj α (p q ∧ Hq q ⊆ Hq q), where Pj α ⊆ Pj α.

A different avenue of research follows the standard approach in logic and the database community and defines a series of transformations that determine if a formula is equivalent to a safe one [1, 8, 17, 18]. If such a transformation is obtained, formally verified and implemented, its integration into Verimon would mean that many more future-bounded formulas would be monitorable.

With the long-term view of developing a more trustworthy, expressive and efficient monitor than other non-verified tools, we intend to integrate our relaxation of safety into Verimon+ [22]. This requires adding more complex terms inside equalities and inequalities that contain additions, multiplications, divisions and type castings. Additionally, safety would need to be defined for aggregations like sum or average, dynamic operators from metric first-order dynamic logic, and recursive let operations. A first attempt and its not-yet complete integration into an old Verimon+ version [3] are available online.

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A Appendix: Formal Definitions

We supply our Isabelle/HOL definitions of dynamic free variables, safe sets of free variables, a combination of Verimon’s `safe-formula` predicates [10, 19, 22], and trigger and release’s invariants for auxiliary states. We also provide the formalisation of the traces in §5 to showcase the working monitoring algorithm.

**Dynamic free variables.** In the code below, the notation `mem I 0` represents `0 ∈ I`, for interval `I`. Also, `Suc` is the successor function on natural numbers. The first two definitions correspond to the sets `↓_i` and `↑_i` of §3 respectively.

```plaintext
definition down-cl-ivl α i ≡ {j | j ≤ i ∧ mem I ((τ σ i − τ σ j))}
definition up-cl-ivl α i ≡ {j | i ≤ j ∧ mem I ((τ σ j − τ σ i))}
```

```plaintext
definition ssvf :: ('a list × 'a list) set trace ⇒ nat ⇒ 'a MFOTL-Formula.formula ⇒ nat set
where ssvf σ i (p ↑ ts) = FV (p ↑ ts)
| ssvf σ i (if t1 = P t2) = FV (if t1 = P t2)
| ssvf σ i (~P α) = ssvf σ i α
| ssvf σ i (α ∧ P β) = ssvf σ i α ∪ ssvf σ i β
| ssvf σ i (α ∨ P β) = (if ssvf σ i α = {})
  then if ⟨v. (σ, v, i) = β⟩ = {} then ssvf σ i β else {}
  else if ssvf σ i β = {} then if ⟨v. (σ, v, i) = β⟩ = {} then ssvf σ i α else {}
  else ssvf σ i α ∪ ssvf σ i β
| ssvf σ i (∃ P α) = (∃τ. nat. x = i) (∪(ssvf σ i α ∩ {}))
| ssvf σ i (Y I α) = (if i=0 then FV α else ssvf σ (i-1) α)
| ssvf σ i (X I α) = ssvf σ (Suc i) α
| ssvf σ i (α S I β) =
  (let satisf-at = λi. ∃ v. (σ, v, j) = β ∧ (∃ k ∈ (j..i)). (σ, v, k) = α) in
  (if (∀ j e down-cl-ivl α I i. satisf-at j) then FV (α S I β)
  else let J = {(j∈down-cl-ivl α I i. satisf-at j); K = (Un {v. j ∈ J})
   in (UUn {v. j ∈ J})});
| ssvf σ i (α U I β) =
  (let satisf-at = λi. ∃ v. (σ, v, j) = β ∧ (∃ k ∈ (i..<j)). (σ, v, k) = α) in
  (if (∀ j e up-cl-ivl α I i. satisf-at j) then FV (α U I β)
  else let J = {j∈up-cl-ivl α I i. satisf-at j}; K = (Un {v. j ∈ J})
   in (UUn {v. j ∈ J})});
| ssvf σ i (α T I β) =
  (let satisf-at = λi. ∃ j v. (σ, v, j) = β ∧ (∃ k ∈ (i..<j)). (σ, v, k) = α) in
  (if (∀ j e down-cl-ivl α I i. satisf-at v j) then FV (α T I β)
  else let J = { j∈down-cl-ivl α I i. satisf-at v j}; K = (k. ∃ v. j e down-cl-ivl α I i. j < k ∧ k ≤ i ∧ (σ, v, k) = α)
   in (UUn {v. j ∈ J})});
| ssvf σ i (α R I β) =
  (let satisf-at = λi. ∃ j v. (σ, v, j) = β ∧ (∃ k ∈ (i..<j)). (σ, v, k) = α) in
  (if (∀ j e up-cl-ivl α I i. satisf-at v j) then FV (α R I β)
  else let J = { j∈up-cl-ivl α I i. satisf-at v j}; K = (k. ∃ v. j e up-cl-ivl α I i. j ≤ k ∧ k < j ∧ (σ, v, k) = α)
   in (UUn {v. j ∈ J})});

Safe sets of free variables.

```plaintext
definition safe-assignment X α = (case α of
  v z = P v y ⇒ (x ∈ X ↔ y ∈ X)
  | x = P t ⇒ (x ∈ X ∧ fv-trm t ⊆ X)
  | t = P v x ⇒ (x ∈ X ∧ fv-trm t ⊆ X)
  | _ ⇒ False)
```

```plaintext
definition ssvf :: 'a MFOTL-Formula.formula ⇒ nat set
where ssvf (p ↑ trms) = {FV (p ↑ trms)}
| ssvf (v x = P v y ⇒ (x ∈ X ↔ y ∈ X)
  | x = P t ⇒ (x ∈ X ∧ fv-trm t ⊆ X)
  | t = P v x ⇒ (x ∈ X ∧ fv-trm t ⊆ X)
  | _ ⇒ False)
```

```plaintext
fun ssfv :: 'a MFOTL-Formula.formula ⇒ nat set
where ssfv (p ↑ trns) = {FV (p ↑ trns)}
| ssfv (v x = P t) = (if FV x = t then {v} else {})
| ssfv (t = P v x = (if FV x = t then {v} else {})
| ssfv (t1 = P t2) = (if FV x = t1 ∪ FV x = t2 then {{}} else {})
| ssfv (~P t1 = P t2) = (let X = FV (t1 = P t2) in if t1 = t2 ∨ X = {} then X else {})
| ssfv (α ∧ P β) = (let A = ssfv α; B = ssfv β in
  if A ≠ {} then
    if B ≠ {} then A ⊔ B
```
else if \( \forall X \in A.\) safe-assignment \( X, \beta \) then ((\( \cup \)) (FV \( \beta \)) \( \beta \) \( \alpha \))
else if is-constraint \( \beta \) \& (\( \forall X \in A.\) FV \( \beta \) \( \subseteq \) \( X \)) then \( \alpha \)
else (case \( \beta \) of \( \neg \beta \) \( \Rightarrow \) (let \( \beta' = \text{ssfv} \) \( \beta \)) in
(if (\( \beta' \neq \{ \} \)) \& (\( \forall Y \in B'.\) \( \forall X \in A.\) Y \( \subseteq \) \( X \)) then \( \alpha \) else (\( \{ \} \)) | \( \ldotp \))
else (\( \{ \} \))

\( \text{ssfv} \) (\( \alpha \cup \beta \)) = (let \( A = \text{ssfv} \) \( \alpha \); B = \( \text{ssfv} \) \( \beta \); X = FV \( \alpha \); Y = FV \( \beta \) in
if (\( A \neq \{ \} \) \& B \( \neq \{ \} \)) then
\( X = Y \) \& A \( \subseteq \) \{\} \& B \( \subseteq \) \{\} \& \{\} \) then
(if (\( \{ \} \) \( \in \) \( \text{A} \) \& \( \{ \} \) \( \in \) \( \text{B} \) then (\( \{ \} \) \& (\( \text{A} \cup \text{B} \)) else \( \text{A} \cup \text{B} \))
else
(if \( X = \{ \} \) \& Y = \{ \} then \( \text{A} \cup \text{B} \) else (\( \{ \} \))
else (\( \{ \} \))

\( \text{ssfv} \) (\( \exists \beta \) \( \alpha \)) = ((\( (\ldotp) \) (\( \lambda \ldotp X \). \( x - 1 \)) \circ (\( \lambda X \). \( X - \{ \} \))) \& \( \text{ssfv} \) \( \alpha \)
\( \text{ssfv} \) (\( \text{Y I} \) \( \alpha \)) = \( \text{ssfv} \) \( \alpha \)
\( \text{ssfv} \) (\( \text{X I} \) \( \alpha \)) = \( \text{ssfv} \) \( \alpha \)
\( \text{ssfv} \) (\( \alpha \) \( \mathbf{I} \) \( \beta \)) = (let \( A = \text{ssfv} \) \( \alpha \); B = \( \text{ssfv} \) \( \beta \); X = FV \( \alpha \); Y = FV \( \beta \) in
if (\( B = \{ \} \)) then
if \( A \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)
else (case \( \alpha \) of
\( \neg \beta \) \( \alpha' \) \( \Rightarrow \) (let \( \alpha' = \text{ssfv} \) \( \alpha' \) in if \( \alpha' \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)) else (\( \{ \} \))
else (\( \{ \} \))

\( \text{ssfv} \) (\( \alpha \) \( \mathbf{U} \) \( \beta \)) = (let \( A = \text{ssfv} \) \( \alpha \); B = \( \text{ssfv} \) \( \beta \); X = FV \( \alpha \); Y = FV \( \beta \) in
if \( \text{mem I} \) \( 0 \) then
if (\( B = \{ \} \)) then
if \( A \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)
else (case \( \alpha \) of
\( \neg \beta \) \( \alpha' \) \( \Rightarrow \) (let \( \alpha' = \text{ssfv} \) \( \alpha' \) in if \( \alpha' \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)) else (\( \{ \} \))
else (\( \{ \} \))
else
if \( X = Y \) \& A = \{\} \& B = \{\} then (\( \{\} \), \( X \) else (\( \{\} \))

\( \text{ssfv} \) (\( \alpha \) \( \mathbf{R} \) \( \beta \)) = (let \( A = \text{ssfv} \) \( \alpha \); B = \( \text{ssfv} \) \( \beta \); X = FV \( \alpha \); Y = FV \( \beta \) in
if \( \text{mem I} \) \( 0 \) then
if (\( B = \{ \} \)) then
if \( A \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)
else (case \( \alpha \) of
\( \neg \beta \) \( \alpha' \) \( \Rightarrow \) (let \( \alpha' = \text{ssfv} \) \( \alpha' \) in if \( \alpha' \neq \{ \} \) \& \( X \subseteq \) Y then \( \{ \} \)) else (\( \{ \} \))
else (\( \{ \} \))
else
if \( X = Y \) \& A = \{\} \& B = \{\} then (\( \{\} \), \( X \) else (\( \{\} \))
\( \text{ssfv} \) (\( \neg \beta \) \( \alpha \)) = (if \( \text{ssfv} \) \( \alpha = \{\} \) then (\( \{\} \)) else (\( \{\} \))

Verimon’s safe-formula predicate. Below we provide the definition of safe-formula used in our proof that is-safe defines a larger fragment. This is also the predicate that does not hold for our examples in §5.

definition safe-dual where safe-dual conjoined safe-formula \( \alpha \) \( I \) \( \beta \) =
if (\( \text{mem I} \) \( 0 \)) then
(safe-formula \( \beta \) \& \( FV \) \( \alpha \) \( \subseteq \) \( FV \) \( \beta \)
\& (safe-formula \( \alpha \)
\& \( \text{conj} \)
\& (safe-formula \( \alpha \) \& safe-formula \( \beta \) \& \( FV \) \( \alpha \) \& \( FV \) \( \beta \)))
else

function safe-formula :: \( ^{\prime} \) a MFOOTL-Formula formula \( \Rightarrow \) bool
where safe-formula (\( t1 \) \( \equiv \) \( t2 \)) = (\( \text{trm.is-Const} \) \( t1 \) \& (\( \text{trm.is-Const} \) \( t2 \) \& trm.is-Var \( t2 \)))
\& (\( \text{trm.is-Var} \) \( t1 \) \& trm.is-Var \( t2 \))
\& (\( \text{safe-formula} \) (\( \neg \beta \) \( \alpha \)) = (\( \{ \} \) \& \( \text{safe-formula} \) \( \alpha \))
\& (\( \alpha \) \& \( \beta \)) = (\( \{ \} \) \& \( \text{safe-formula} \) \( \alpha \) \& \( \text{safe-formula} \) \( \beta \))
\& (\( \neg \beta \) \( \alpha \)) = (\( \{ \} \) \& \( \text{safe-formula} \) \( \alpha \) \& \( \text{safe-formula} \) \( \beta \))
\& (\( \text{safe-assignment} \) (\( FV \) \( \alpha \) \( \beta \)
\& safe-formula \( \beta \)
\& (\( FV \) \( \beta \) \( \subseteq \) \( FV \) \( \alpha \) \& \( \text{is-constraint} \) \( \beta \))
\& (\( \text{case} \) \( \beta \) \( \Rightarrow \) \( \text{safe-formula} \) \( \beta \) \) (safe-assignment \( \{ \} \) \( \beta \)
\& safe-formula \( \beta \)
\& (\( FV \) \( \beta \) \( \subseteq \) \( FV \) \( \alpha \) \& (\( \text{is-constraint} \) \( \beta \))
\& (\( \text{case} \) \( \beta \) \( \Rightarrow \) \( \text{safe-formula} \) \( \beta \) \)
Finally, we show the formalisation of the traces displayed as tables in wf-trigger-aux.

Auxiliary states for trigger and release. The predicate wf-past-aux below is the first part described in §6 of the invariant for wf-trigger-aux. We also abuse notation here and use the predicate $Q_t$ from §6 instead of the names mem0-taux-sat and nmem0-taux-sat in the formalisation.

The notation list-all2 below indicates universal pairwise quantification over its two list-arguments aux and $[n..<n+\text{length aux}]$. The first argument is the auxiliary state, while the notation $[a..<b]$ represents the list of all natural numbers greater or equal than $a$ and less than $b$. As before, we use notation $Q^\text{L,0\in I}_t$ and $Q^\text{R,0\in I}_t$ from §6 instead of that in the formalisation.

Example traces. Finally, we show the formalisation of the traces displayed as tables in §5. We also provide an abbreviated version of the monitor’s output via Isabelle/HOL’s command value that call’s its code generator [9], executes the generated code and displays the final result. The trace for the quality assessment example is the next one.
Below, we do not show the full output for the second argument in the monitor’s state because it is long and
difficult to parse. For a shorter version, see the next example. The monitor correctly identifies the best quality
products to have IDs 0 and 3.

\[
\text{value } \text{mbest6} = \left\{ \left( 0, \text{[Some 0]} \right), \left( 0, \text{[Some 3]} \right) \right\},
\]
\[(\text{mstate-i} = 1, \text{mstate-m} = \text{best}_6, \text{mstate-n} = 1)\]

The piracy trace is formalised with functions \text{minit} and \text{mstep} as shown below.

\[
\text{definition } \text{mpira} \equiv \text{minit pirated}
\]
\[
\text{definition } \text{mpira0} \equiv \text{mstep }\{\{\text{no-sign, [1]}\}, \{\text{no-sign, [2]}\}, \{\text{sign, [3]}\}\}, 0 \text{ mpira}
\]
\[
\text{definition } \text{mpira1} \equiv \text{mstep }\{\{\text{no-sign, [1]}\}, \{\text{no-sign, [2]}\}, \{\text{sign, [3]}\}\}, 1 \text{ (snd mpira0)}
\]
\[
\text{definition } \text{mpira2} \equiv \text{mstep }\{\{\text{no-sign, [1]}\}, \{\text{no-sign, [2]}\}, \{\text{sign, [3]}\}\}, 2 \text{ (snd mpira1)}
\]
\[
\text{definition } \text{mpira3} \equiv \text{mstep }\{\{\text{off-route, [1]}\}, \{\text{no-sign, [2]}\}, \{\text{sign, [3]}\}\}, 3 \text{ (snd mpira2)}
\]

We provide the monitor’s output at time-point 3 and show its full state.

\[
\text{value } \text{mpiracy3} = \left\{ \left( 0, \text{[Some 1]} \right), \left( 0, \text{[Some 2]} \right) \right\},
\]
\[(\text{mstate-i} = 1, \text{mstate-m} = \text{MRelease True } \text{MPred }'\text{off-route'} [\text{v 0}] \text{ True } (\text{Abs-I } (-, -, \text{True}) ) (\text{MPred }'\text{no-signat'} [\text{v 0}] ) [\text{}], [\text{}], [\text{}], [\text{}], [\text{}], \text{mstate-n} = 1)\]