Representations of CCR algebras in Krein spaces of entire functions

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Abstract
Representations of CCR algebras in spaces of entire functions are classified on the basis of isomorphisms between the Heisenberg CCR algebra $A_H$ and * algebras of holomorphic operators. To each representation of such algebras, satisfying a regularity and a reality condition, one can associate isomorphisms and inner products so that they become Krein * representations of $A_H$, with the gauge transformations implemented by a continuous $U(1)$ group of Krein space isometries. Conversely, any holomorphic Krein representation of $A_H$, having the gauge transformations implemented as before and no null subrepresentation, are shown to be contained in a direct sum of the above representations. The analysis is extended to CCR algebras with $[a_i, a^*_j] = \delta_{i,j} \eta_i$, $\eta_i = \pm 1$, $i = 1, \ldots, M$, the infinite dimensional case included, under a spectral condition for the implementers of the gauge transformations.

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0 **Introduction**

After the Von Neumann classification of the Hilbert space representations of the CCR algebras, a major step was the Fock-Bargmann-Segal (FBS) analysis of the representations in Hilbert spaces of holomorphic functions [1], which has played a crucial role for the solution of mathematical and physical problems, especially in connection with the classical limit and the coherent state representations [2]. As it is well known, covariant quantizations of the electromagnetic CCR algebra, of non-abelian gauge fields and of covariant strings require representations in indefinite (Krein) spaces [3, 4, 5]. E.g. the commutation relations $[A^\mu(x), A^\nu(y)] = -g^\mu\nu \delta(x-y)$ exclude a Hilbert space realization with a Lorentz invariant vacuum. Indefinite representations also arise in the Pauli-Villars renormalization scheme and in general in regularization procedures [6].

The analog of Von Neumann classification for the case of Krein space representations of the Heisenberg algebra $A_H$, has been discussed in [7, 8]. The aim of this note is to analyze the Krein representations of the Heisenberg algebra on spaces of entire functions, with the same motivations and applications of the FBS analysis. A complete classification of holomorphic Hilbert-Krein * representations will be obtained under a regularity condition which amounts to the implementability of the group of gauge transformations and generalizes the condition that the spectrum of $N = a^* a$ is discrete.

The regularity condition leads to a representation of an extended Heisenberg Lie algebra isomorphic to the Lie algebra $G(0, 1)$, with discrete spectrum of one of its generators. Representations of $G(0, 1)$ with this property have been extensively studied in the literature, on (Hilbert) spaces of holomorphic functions (of two variables), [9] but a classification of the * representations of the Heisenberg algebra in Krein spaces of entire functions (of one variable) seems to be lacking.

Our analysis is based on the following steps: $i)$ the characterization of the isomorphisms $\sigma$ between $A_H$ and the * algebras $A^K$ obtained from the algebra $A$, generated by the holomorphic operators $z, \partial_z$, through the introduction of a conjugation $K$, any two such isomorphisms being related by a transformation in a group $G \sim SL(2, \mathbb{C})$ of automorphisms of $A$; $ii)$ a classification of representations $\pi$ of $A$ under the regularity condition that a $U(1)$ subgroup of $G$ is implementable in $\pi$; the identification of the subgroup $S \subset G$, which is implementable in the space $F$ of entire functions leads to a reduction of the regularity subgroups to either the Bargmann case, namely
\[ z \to e^{i\overline{s}z}, \quad \partial \to e^{-i\overline{s}\partial}, \text{ or the Schroedinger case, } (z \pm \partial) \to e^{\mp i\overline{s}} (z \pm \partial); \]

*(iii)* to each irreducible representation of \( \mathcal{A} \), satisfying a reality condition, one can associate isomorphisms \( \sigma \) and inner products \( \langle , \rangle \) so that they become Krein * representations of \( \mathcal{A}_H \) with the gauge transformations implemented by a \( U(1) \) group of Krein space isometries \( U(s) \), continuous in a Hilbert-Krein topology; *(iv)* conversely, any holomorphic Krein representation of \( \mathcal{A}_H \), in which the gauge transformations are implemented by a group of operators \( U(s) \), continuous in \( s \) in a Hilbert-Krein topology, with no null subrepresentation, is contained in a direct sum of the representations obtained above.

As in the Hilbert space case, all the irreducible Krein regular representations of \( \mathcal{A}_H \), classified in [7, 8], are Krein equivalent to holomorphic Krein representations.

The above analysis, for simplicity done in the case of one degree of freedom, can be extended to the representations of CCR algebras \( \mathcal{A}_H(\eta) \) for \( M \) degrees of freedom, generated by elements \( a_i, a_i^*; i = 1, \ldots, M \), (including the case \( M = \infty \)) satisfying

\[ [a_i, a_j] = 0 = [a_i^*, a_j^*], \quad [a_i, a_j^*] = \delta_{i,j} \eta_i, \quad \eta_i = \pm 1.\]

In this way one covers the field theory cases mentioned above. The classification, discussed in Sect. 4, is obtained under a *spectral condition* for the implementers of the gauge transformations.

## 1 Holomorphic Heisenberg algebras

We consider the vector space \( \mathcal{F} \) of entire functions of one variable, endowed with the standard topology \( \tau \) of the sup over compact sets.

The **algebra of holomorphic operators** \( \mathcal{A} \), is the polynomial algebra generated by \( z \) and \( \partial/\partial z \equiv \partial_z \). We shall denote by \( \mathcal{A}^K \) the * algebra obtained by associating to \( \mathcal{A} \) an antilinear involution \( K \), which leaves stable the vector space \( \mathcal{A}_1 \) generated by \( z \) and \( \partial_z \). In the following, we shall use the notation (\( C_K \) a \( 2 \times 2 \) matrix)

\[ \mathcal{Z} = \begin{pmatrix} z \\ \partial_z \end{pmatrix}; \quad K(\mathcal{Z}) \equiv \mathcal{Z}^* = \begin{pmatrix} z^* \\ \partial_z^* \end{pmatrix} = C_K \mathcal{Z}. \tag{1} \]

The **Heisenberg algebra** \( \mathcal{A}_H \) is the polynomial *-algebra generated by an element \( a \), satisfying \( [a, a^*] = 1 \).
Special cases of isomorphisms between the Heisenberg algebra and the * algebras \( \mathcal{A}^K \) are i) the FBS realization given by \( a = \partial_z, \ a^* = z, \ C_B = \sigma_1, \) ii) the Schroedinger realization corresponding to \( a = (z + \partial_z)/\sqrt{2}, \ a^* = (z - \partial_z)/\sqrt{2}, \ C_S = \sigma_3. \) The isomorphisms between the Heisenberg algebra \( \mathcal{A}_H \) and the * algebras \( \mathcal{A}^K \) are characterized by

**Proposition 1** The group \( \mathcal{G} \) of automorphisms \( \beta \) of \( \mathcal{A} \), which leave invariant the subspace \( \mathcal{A}_1 \), is isomorphic to the group \( SL(2, \mathbb{C}) \):

\[
\beta(Z) = TZ, \ T \in SL(2, \mathbb{C}).
\]

All the * algebras \( \mathcal{A}^K \) are isomorphic to \( \mathcal{A}_H \) and all isomorphisms between \( \mathcal{A}_H \) and the * algebras \( \mathcal{A}^K \), mapping the linear span of \( z, \partial_z \) into \( \mathcal{A}_1 \), are given by

\[
\begin{pmatrix} a^* \\ a \end{pmatrix} = VZ, \ V \in SL(2, \mathbb{C}), \ C_K = V^{-1} \sigma_1 V,
\]

with \( \sigma_1 \) the first Pauli matrix (having one on the off diagonal).

The group of Bogoliubov transformations of \( \mathcal{A}_H \) corresponds, for each \( K \), to the group \( \mathcal{G}^K \), isomorphic to \( SL(2, \mathbb{R}) \sim Sp(2, \mathbb{C}) \), of *-automorphisms of \( \mathcal{A}^K \), given by the matrices \( T \) which satisfy \( det T = 1, \ T^T C = CT \).

The analysis of the representation of \( \mathcal{A} \) in \( \mathcal{F} \) crucially involves the implementation of the above automorphisms. An automorphism \( \beta \in \mathcal{G} \) is said **implementable** in \( \mathcal{F} \) if there exists an operator \( T_\beta \) on \( \mathcal{F} \) such that

\[
\beta(Z)f = T_\beta Z T_\beta^{-1} f, \ \forall f \in \mathcal{F}.
\]

A subgroup \( \mathcal{G} \subseteq \mathcal{G}^K \) is said to be **represented** in \( \mathcal{F} \) if eq.(4) holds for each \( \beta \in G \) and the operators \( T_\beta \) obey the group law of \( G \). The Lie algebra of \( \mathcal{G} \) is represented in \( \mathcal{F} \) by

\[
\pi(\sigma_3) = z \partial_z + \frac{1}{2}, \ \pi(\sigma_1) = \frac{i}{2}(\partial_z^2 - z^2), \ \pi(\sigma_2) = i \frac{1}{2}(\partial_z^2 + z^2).
\]

**Proposition 2** The subgroup \( S \) of \( SL(2, \mathbb{C}) \) defined by

\[
\sigma(Z) = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} z \\ \partial_z \end{pmatrix} \equiv S(\alpha, \beta)Z, \ \alpha, \beta \in \mathbb{C},
\]

is represented in \( \mathcal{F} \) in the following way

\[
f(z) \rightarrow \Gamma_S f(z) = f_S(z) = f(\alpha z)e^{-\alpha^{-1}\beta z^2/2}.
\]
The operators $\Gamma_S$ are $\tau$-continuous; the representation is $\tau$-differentiable in $S$ with generators in $A$ and therefore $\Gamma_S(a, b) f$ is $C^\infty$ in $a, b$ in the $\tau$-topology; the (Lie algebra) generators of the representation of $S$ are $z \partial_z = \pi(\sigma_3) - \frac{z^2}{2}$ and $-z^2/2 = \pi(\sigma_1 + i\sigma_2)/2 \equiv \pi(\sigma_+)$. Only the automorphisms belonging to $S$ are implementable in $F$.

**Proof.** In fact, $(\sigma(S) f)(z) = (\Gamma_S S \Gamma_S^{-1} f)(z)$. The identification of the (Lie algebra) generators of $S$ is easily obtained from the above equations, (on $S$ their action is given by the transposed matrices).

If an automorphism $\gamma \in G$, $\gamma \notin S$ is implementable, so is the subgroup generated by $\gamma$ and $S$, which always contains the subgroup $\beta_-(s)$, $s \in C$, generated by $\sigma_- \equiv (\sigma_1 - i\sigma_2)/2$; now $\beta_-(s)(z) = z + s \partial_z$ and, since, $\forall s \neq 0$, there is a $g \in F$ satisfying $(z + s \partial_z)g = 0$, eq.(15) cannot hold.

The (Lie algebra) generators of the one parameter subgroups of $G$ are of the form $\sigma \cdot n = \sigma_i n_i$, with $n$ a complex vector. $\sigma \cdot n$ generates a two (real) parameter subgroup $G_n$ of the form $\exp(\lambda \sigma \cdot n)$, $\lambda \in C$, and, if $n^2 = n'^2$, $n, n' \neq 0$, the two subgroups $G_n$, $G_n'$ are $SL(2, C)$ conjugate.

Since $\sigma \cdot n$ and $\sigma \cdot \lambda n$, $0 \neq \lambda \in C$, generate the same subgroup, the two (real) parameter subgroups $G_n$ fall in two equivalence classes with respect to $SL(2, C)$: those corresponding to $n^2 = 0$, called degenerate, and those with $n^2 \neq 0$. This implies that all $G_n$, with $n^2 \neq 0$, are isomorphic to $U(1) \times R$. The other $G_n$, with $n^2 = 0$, are isomorphic to $R^2$.

The equivalence classes of the subgroups $G_n$, with respect to $S$, are given by the orbits under $S$ in the adjoint representation of $SL(2, C)$. Since the generic matrix $S \in S$ can be written in the form $\exp(a \sigma_3) \exp(b \sigma_+)$, $a, b \in C$, the (adjoint) action of $S$ on $\sigma \cdot n = n_3 \sigma_3 + n_- \sigma_+ + n_+ \sigma_-$, $n_3, n_\pm \in C$, is of the form $(n_3, n_-, n_+) \rightarrow (n_3 + bn_+, e^{2a}(n_- - 2bn_3 - b^2 n_+), e^{-2a} n_+)$. Hence, we have

**Proposition 3** The orbits defined by $S$ in the adjoint representation of $SL(2, C)$ are the following:

1) the orbit $\{\sigma_+\}$; it corresponds to a subgroup $G_n$ with $n^2 = 0$;
2) the orbit $\{S \sigma_3\}$; it consists of the set $\{\sigma_3 + n_- \sigma_+, n_- \in C\}$
3) the orbit $\{S \sigma_1\}$; it consists of the set $\{n_3 \sigma_3 + n_- \sigma_+ + n_+ \sigma_-, n_+ \neq 0, n_3^2 + n_+ n_- = 1\}$, since for a generic $S$, $(0, 1, 1) \rightarrow (b, e^{2a}(1 - b^2), e^{-2a})$
4) the orbit $\{S \sigma_-\}$, which corresponds to a subgroup $G_n$ with $n^2 = 0$, and consists of the set $\{be^{2a} \sigma_3 - (be^{2a})^2 \sigma_+ + \sigma_-\}$.
2 Regular representations on entire functions

By Proposition 2 the group $S$ is implementable in $F$, eq.(7), and two representation are called $S$-equivalent if the corresponding representation spaces are related by a transformation of $S$. Any representation $\pi$ in $V_\pi \subseteq F$ is automatically infinite dimensional and faithful.

In analogy with Nelson’s strategy of analyzing Lie algebra representations in terms of exponentiability of quadratic operators [10] we introduce

**Definition 1** A $U(1)$ subgroup $\beta_s$, $s \in [0, 2\pi)$, of the automorphisms $G \simeq SL(2,\mathbb{C})$, is said to be **regularly represented** in a representation $\pi$ of $A$ contained in $F$ if it is implementable in $\pi$ by a $U(1)$ group $U(s)$, $s \in [0, 2\pi)$, $\tau$-continuous in $s$ and generated by an element $N \in A$, i.e. $\forall f \in V_\pi$

$$\frac{d}{ds} U(s) f = i N U(s) f, \ N \in A,$$

the derivative being taken in the $\tau$-topology.

A representation $\pi$ of $A$ is said to be **regular** if there are $U(1)$ subgroups of $G$ regularly represented in $\pi$, called **regularity subgroups** of $\pi$.

Eq.(8) implies that actually $U(s) f$ is $C^\infty$ in $s$ and therefore a regular representation is the close analog of Gårding domain for the generators of $U(1)$ groups, (however, in general, for fixed $s$, $U(s)$ is not a $\tau$-continuous operator). The above regularity condition generalizes the regularity condition on $N = a^* a$, which plays a crucial rôle in the analysis of Hilbert space representations of the Heisenberg algebra. As we shall see, such generalized purely algebraic regularity condition leads to a classification of the irreducible Hilbert-Krein $*$ representations of the Heisenberg algebra on entire functions in terms of those with discrete spectrum of the $N$ operator. The regularity condition implies that the representations of $A$ extend to representations of the Lie algebra of the group $G(0,1)$; by Proposition 4, their classification follows from the analysis of those with discrete spectrum of one of its generators, in standard (Bargmann or Schroedinger) form, covering in this way a much wider class of representations with respect to Ref. [9].

The one parameter $U(1)$ subgroups $U(s)$ of $SL(2,\mathbb{C})$ are generated by $\sigma \cdot n$, $n$ a unit real vector. By Proposition 3, $U(s)$ or $U(-s)$ is conjugated by transformations of $S$ either to the subgroup $\beta_B^s$ generated by $\sigma_3$, represented in $F$ by $z \partial_z + \frac{i}{2} \equiv N_B + \frac{i}{2}$, or to the subgroup $\beta_S^s$ generated by $\sigma_1$, represented by $\frac{i}{2}(\partial_z^2 - z^2) \equiv -N_S - \frac{i}{2}$. Accordingly, the regularity subgroups will be called of Bargmann or of Schroedinger type.
Proposition 4  Let $\pi$ be a regular representation of $A$, $V_\pi$ its representation space and $U(s)$ the representative of a regularity subgroup, then

\begin{equation}
(2\pi)^{-1}\int_0^{2\pi} ds U(s) e^{-iks} f \equiv f_k
\end{equation}

exists in the $\tau$ topology and belongs to $F$,

ii) $f(z) = \sum_k f_k(z)$, the series being convergent in the $\tau$ topology, so that at least one $f_k \neq 0$; the sequence $\{f_k\}$ is of fast decrease in $k$, in all the $\tau$-topology seminorms

iii) $f_k$ is an eigenvector of the generator of $U(s)$ and, modulo $S$ transformations, satisfy either one of the following equations

\begin{equation}
z\partial_z f_k = kf_k,
\end{equation}

\begin{equation}
\frac{i}{2}(-\partial_z^2 + z^2 - 1) f_k = (k + \theta) f_k, \quad \text{Re} \theta \in (-1, 0].
\end{equation}

Given $k \in \mathbb{Z}$, $\theta \in (-1, 0]$, eq. (11) has two non trivial solutions, given by the parabolic cylinder functions $D_{\theta+k}(\sqrt{2}z)$ and $D_{-\theta-k-1}(i\sqrt{2}z)$.

iv) the vector space $\bar{V}_\theta$, $\text{Re} \theta \in (-1, 0]$ consisting of the $\tau$-convergent series $\sum c_k D_{\theta+k}$ is the carrier of a regular representation of $A$ with a regularity group of Schroedinger type; conversely, every such representation is $S$ equivalent to a representation contained in $\bar{V}_\theta + S(i)\bar{V}_\theta$, $S(i) \equiv S(i, 0) \in S$.

Proof.  The existence of the integral (9) follows from the $\tau$-continuity of $U(s)$; this also implies that $f_k \in F$. Point ii) follows from the standard properties of the inversion of the Fourier series, the periodic function $U(s)e^{iks}f$ being $C^\infty$ in $s$. The generator of $U(s)$ can be taken of the form

\begin{equation}
N = \sum_i n_i \pi(\sigma_i) - \frac{i}{2} - \theta \equiv N(n) - \theta, \quad \text{Re} \theta \in (-1, 0].
\end{equation}

In fact, since $\pi$ is faithful, the center of $\pi(A)$ consists of the multiples of the identity and in $\pi$ the operator $N - N(n)$ belongs to the center of $\pi(A)$.

Eqs. (8),(9) and the $\tau$-continuity of $z$ and $\partial_z$ imply that $f_k$ satisfies $N f_k = k f_k$. In the Bargmann case, $N = z\partial_z - \theta$, such an equation has solutions in $F$ only for $\theta \in \mathbb{Z}$ and therefore $\theta = 0$. The solutions of eq.(11) are parabolic cylinder functions [11, 9].

By $\tau$-continuity the operators $z$, $\partial_z$ send $\tau$-convergent sequences into $\tau$-convergent ones and, as we have seen, $z \pm \partial_z$ act as raising/lowering operators with respect to $k$; therefore all the series $\sum c_k D_{\theta+k} k^n$, $n \in \mathbb{N}$ are
τ-convergent so that stability under $\mathcal{A}$ and regularity follows. Then, point iv) follows from ii)-iii).

Since $N_B = z \partial_z$, every Bargmann regular representation is $\mathcal{S}$ equivalent to a representation invariant under rotations. Modulo $\mathcal{S}$ transformations, $f_k = z^k$ and the set of eigenvalues of $N_B$ is $\mathbb{N}$.

In the case of eq.(11), the raising and lowering operators are $a_+ \equiv (z - \partial_z)/\sqrt{2}$, $a_- \equiv (z + \partial_z)/\sqrt{2}$, $[a_-, a_+] = 1$. Putting $F_\lambda(z) \equiv D_\lambda(\sqrt{2}z)$, with $D_\lambda$ the unique solution of eq.(11) with $\lambda = \theta + k$, vanishing at $+\infty$ on the real line, one has [9]

$$a_+ F_\lambda(z) = F_{\lambda+1}(z) \quad , \quad a_- F_\lambda(z) = \lambda F_{\lambda-1}(z) \ , \quad (13)$$

If $\lambda$ is not an integer, the right hand sides of eqs.(13) cannot vanish and $F_\lambda(z)$ and $F_\mu(z)$ define the same cyclic representation space $V_\lambda$ iff $\lambda - \mu$ is an integer. The representations of $\mathcal{A}$ in $V_\lambda$ are irreducible, since every vector is cyclic.

For integer $\lambda = n$, the vector spaces $V_n$ obtained from the cyclic vector $F_n(z)$, $n \in \mathbb{Z}$, $n < 0$ define the same representations of $\mathcal{A}$, which is reducible as a consequence of eqs.(13) with integer index. In fact, $V_n$ contains the invariant subspace $V_0$, generated by $F_0(z)$, actually the only non trivial one, since in eqs.(13) only one coefficient vanishes, and $V_0$ carries an irreducible representation of $\mathcal{A}$.

The other solutions of eq.(11) define $\mathcal{S}$ equivalent representation spaces $V_{i-\lambda-1}$ related to $V_\lambda$ by the transformation $S(i) : f(z) \rightarrow f(iz)$.

In conclusion, the representation spaces $V_S$ spanned by the solution of the eigenvalue equation (11) contain the following irreducible representation:

i) $V_0$, with cyclic vector $F_0(z)$, satisfying $a_- F_0(z) = 0$ ;

ii) $V_{-1} = S_i V_0$, with cyclic vector $F_0(iz)$, satisfying $a_+ F_0(iz) = 0$ ;

iii) $V_\theta$, $\Re \theta \in (-1, 0)$, $\theta \neq 0$, with cyclic vector $F_\theta(z)$;

iv) $V_{i-\theta-1} = S_i V_\theta$, $\Re \theta \in (-1, 0)$, $\theta \neq 0$, with cyclic vector $F_\theta(iz)$.

### 3 Holomorphic Krein representations

We can now obtain a classification of * representations of the Heisenberg algebra on inner product spaces of entire functions. An inner product $< , >$ on a vector space $V$ will be called a **Krein inner product** if it is non degenerate, it is continuous in a Hilbert space topology and the corresponding Hilbert completion of $V$ is a Krein space, i.e. $< f, g > = (f, \eta g)$, $\eta^2 = 1$. 

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A Krein representation of a \(*\) algebra \(\mathcal{B}\) is a representation of \(\mathcal{B}\) on a vector space \(V\) with Krein inner product such that the \(*\) conjugation is represented by the inner product adjoint. Two Krein representations \(\pi_1, \pi_2\) on vector spaces \(V_1, V_2\) are Krein equivalent if there is an invertible linear transformation \(U : V_1 \to V_2, U\pi_1 = \pi_2U\), which preserves the Krein inner product.

**Definition 2** A holomorphic Krein representation of \(\mathcal{A}_H\) is a Krein representation of \(\mathcal{A}_H\) on a space \(V\) of entire functions, defined by one of the \(*\) isomorphisms classified in Proposition 1 between \(\mathcal{A}_H\) and a \(*\) algebra \(\mathcal{A}^K\), the algebra of holomorphic operators \(z, \partial_z\), with conjugation \(K\).

A holomorphic (Krein) representation \(\pi\) of \(\mathcal{A}_H\) is said to be regular if

i) the \(U(1)\) group of gauge transformations \(\gamma_s, s \in [0, 2\pi)\)

\[\gamma_s(a) = e^{-is} a, \quad \gamma_s(a^*) = e^{is} a^*\]

is regularly represented (Definition 1); the implementers are generated by \(\pi(a^*a) + \mu, \mu \in \mathbb{C}\) and will be denoted by \(U(s)\);

ii) no null representation of \(\mathcal{A}_H\) is contained in the common domain of the closures of \(\pi(a), \pi(a^*)\).

The above notion of regularity is at the basis of the classification of compact Lie groups in Krein spaces [8]. The representation space of the above definitions is only required to be a dense domain in a Krein space (for Krein spaces see [12]), since the (unbounded) representation of the Heisenberg algebra can only be densely defined; all the operators \(\pi(A), A \in \mathcal{A}_H\) are closable since their adjoints are densely defined. It follows from Theorem 1 below that Krein completions of such domains can consist of holomorphic functions only for the Fock and anti-Fock representations of the Bargmann type.

From the discussion of Sect.2 one easily gets a relation between regular representations of \(\mathcal{A}\) and regular representations of \(\mathcal{A}_H\), as an algebra. In fact, each regular representation \(\pi\) of \(\mathcal{A}\) on a space \(V_\pi \subset \mathcal{F}\) is of the form \(\pi(A) = S \pi^0(A) S^{-1}\), for a suitable \(S \in \mathcal{S} \subset SL(2, \mathbb{C})\), with \(\pi^0\) regular with respect to a \(U(1)\) group with implementer generated by either \(N_B = z \partial_z\) or \(N(\theta) = \frac{i}{2}(-\partial_z^2 + z^2 - 1) - \theta\). The representation \(\pi\) gives rise to regular representations \(\pi_\sigma \equiv \pi \circ \sigma\) for \(\mathcal{A}_H\) as an algebra, for any isomorphism \(\sigma \in SL(2, \mathbb{C}), \sigma : \mathcal{A}_H \to \mathcal{A}\), with

\[S^{-1} \sigma(a^*a - \mu) S = \pm N, \quad N = N_B, \text{ or } N = N(\theta).\]  \hspace{1cm} (14)
We shall show that if $\pi$ is one of the irreducible representations of $\mathcal{A}$ classified in Section 2, with $\theta$ real, there are Krein inner products in $V_\pi$ such that the corresponding representations of $\mathcal{A}_H$ are Krein representations; conversely, regular holomorphic Krein representations of $\mathcal{A}_H$ can be classified in terms of such representations.

**Theorem 1** To each irreducible representation of $\mathcal{A}$, $\pi_B$ in $V_B$, $\pi_\theta$ in $V_\theta$, $\theta \in (-1,0]$, one can respectively associate isomorphisms between $\mathcal{A}$ and $\mathcal{A}_H$, $\sigma^\pm_B$, $\sigma^\pm_\theta$, $\gamma \in \mathbb{R}^+$, (independent of $\theta$), and (unique) Krein inner products $<$, $>$ such that $\pi \circ \sigma$ are holomorphic Krein regular representations of $\mathcal{A}_H$, with the regularity subgroup represented by a group $U(s)$ of Krein inner product isometries, weakly continuous in $s$ with respect to a suitable Hilbert-Krein topology.

Representations arising from different $\gamma$'s are Krein equivalent and $\pi_B \circ \sigma^\pm_B$ are Krein equivalent to $\pi_0 \circ \sigma^{\pm,1}$. For each of the above $\pi$ and $\sigma$, $\forall S \in \mathbb{S}$, $S \pi \circ \sigma \ S^{-1}$ defines a holomorphic Krein representation of $\mathcal{A}_H$ in $S V_{\pi \circ \sigma}$, with inner product $< S f, S g > \equiv < f, g >_{V_{\pi \circ \sigma}}$.

Conversely, for any holomorphic Krein regular representation $\pi$ of $\mathcal{A}_H$, with a regularity subgroup represented by $U(s)$ weakly continuous in $s$ with respect to a Hilbert-Krein topology $\kappa$, $V_\pi$ is contained in the $\kappa$ completion of one of the representation spaces $S V_{\pi_B \circ \sigma^+_B}$, $S (V_{\pi(\theta) \circ \sigma} + S(i) V_{\pi(\theta) \circ \sigma})$, $S \in \mathbb{S}$, with inner product modified in general by a $2 \times 2$ positive matrix in the commutant of the representation.

**Proof.** The construction of $*$ representations of $\mathcal{A}_H$ starting from $V_B$ and $V(\theta)$, with $\theta$ real, begins with classifying the isomorphisms $\sigma$ such that

$$\sigma(a^* a) = N, \quad N = N_B, \text{ or } N = N_S. \quad (15)$$

A possible additive constant in eq.(15) is excluded at the algebraic level, so that the classification of the $\sigma$'s is independent of $\theta$; a possible minus sign in eq. (15), see eq.(14), only leads to a redefinition of the inner product, leading to a new class of representations. A "scaling" factor $\sigma(a) \rightarrow \gamma \sigma(a)$, $\sigma(a^*) \rightarrow \gamma^{-1} \sigma(a^*)$ is reduced to the case $\gamma > 0$ by the implementation of the gauge transformations and in the Bargmann case disappears by $S$-covariance. We are therefore reduced to the following cases.

The isomorphisms $\sigma^+_B : \sigma^+_B(a) = \partial_z$, $\sigma^+_B(a^*) = z$, and $\sigma^{1,+} : \sigma^{1,+}(a) = (z + \partial_z)/\sqrt{2}$, $\sigma^{1,+}(a^*) = (z - \partial_z)/\sqrt{2}$, give rise to the standard FBS and Schroedinger representations in spaces of entire functions, with the standard
positive inner products $< f, g >^+_B$, $< f, g >^+_0$, respectively. For any $\gamma > 0$, one has the isomorphisms

$$\sigma^{\gamma,+} = \sigma^{1,+} \circ \rho_\gamma, \quad \rho_\gamma(a) = \gamma a, \quad \rho_\gamma(a^*) = \gamma^{-1} a^*,$$

and the corresponding inner product in $V_0$ is given by

$$< f, g >^{\gamma,+} = < f, \gamma^{2N_S} g >^+.$$  

so that $< f, \sigma^{\gamma,+}(a) g >^{\gamma,+} = < \sigma^{\gamma,+}(a^*) f, g >^{\gamma,+}$. All the resulting * representations of $\mathcal{A}_H$ are unitarily equivalent.

In the representation space $V(\theta)$ the inner products

$$< F_{\theta+n}, F_{\theta+m} >^{\gamma,+} = \delta_{n,m} \gamma^{2(n+m)} \Gamma(\theta + n + m + 1),$$

with Hilbert majorants $(F_{\theta+n}, F_{\theta+m})^{\gamma,+} = |< F_{\theta+n}, F_{\theta+m} >^{\gamma,+}|$ ($\Gamma$ the gamma function), satisfy

$$< F_{\theta+n}, \sigma^{\gamma,+}(a) F_{\theta+m} >^{\gamma,+} = \delta_{n,m-1} \gamma^{2(n+m+1)} (\theta + n) \Gamma(\theta + n + m) =$$

$$= < \sigma^{\gamma,+}(a^*) F_{\theta+n}, F_{\theta+m} >^{\gamma,+},$$

so that one has Krein * representations.

In all cases, the $U(1)$ group of gauge transformations is regularly represented, by a strongly continuous unitary group, because $\sigma_B^+(a^* a) = N_B$, $\sigma^{\gamma,+}(a^* a) = N_S$. For fixed $\theta$, all above representations are Krein equivalent.

Other representations are obtained by the isomorphisms

$$\sigma^- \equiv \sigma \circ \rho^-, \quad \rho^-(a) = a^*, \quad \rho^-(a^*) = -a,$$

with $\sigma$ any of the isomorphisms introduced before. In fact, in the above spaces the inner products

$$< f, g >^- \equiv < f, (-1)^N g >^+, \quad N = N_B, N_S - \theta$$

give rise to Krein representations. As a result, $\sigma_B^-$ in $V_B$ and $\sigma^{1,-}$ in $V_0$ give rise to anti-Fock representations of the Heisenberg algebra [7, 8] on entire functions, of the Bargmann and of the Schroedinger type respectively; $\sigma^{1,-}$ in $V(\theta)$ gives rise to a representation which is Krein equivalent to that given by $\sigma^{1,+}$ in $V(-\theta - 1)$.

Conversely, let $\pi$ be a holomorphic Krein regular representation of $\mathcal{A}_H$, in $V_{\pi} \subset \mathcal{F}$ with inner product $< , >_{\pi}$, defined by the isomorphism $\sigma : \mathcal{A}_H \to \mathcal{A}$.  

11
By regularity, for some \( \mu \in \mathbb{C} \), \( \sigma(a^*a) + \mu \) generates, see eq.(8), the implementers \( U(s) \) of a \( U(1) \) subgroup of \( SL(2, \mathbb{C}) \). Therefore, by Proposition 4, there exists \( S \in \mathcal{S} \) such that

\[
S^{-1} \sigma(a^*a + \mu) S \equiv \rho_S^{-1} \circ \sigma(a^*a + \mu) = \pm N
\]  

(21)

(the generator of a one parameter subgroup being determined up to a sign), with \( N = N_B \) or \( N = N(\theta) \). The space \( V_{\pi^0} \equiv S^{-1}V_{\pi} \), with the inner product \( < f, g >_{\pi^0} \equiv < Sf, Sg >_{\pi} \), carries a holomorphic Krein regular representation \( \pi^0(A)f \equiv S^{-1}\pi(A)Sf, f \in V_{\pi^0} \), defined by the isomorphism \( \sigma^0 = \rho_S^{-1} \circ \sigma \). Eq.(14) reads

\[
\sigma^0(a^*a + \mu) = \pm z \partial_z \quad \text{or} \quad \sigma^0(a^*a + \mu) = \frac{1}{2}((\partial_z^2 - z^2 - 1) + \theta).
\]  

(22)

\( \sigma^0 \) is defined, as in Proposition 1, by a matrix \( V^{-1} \in SL(2, \mathbb{C}) \) and this immediately implies that eq.(22) only has the solutions \( \sigma^0 = \sigma^\pm_B \circ \rho_\gamma \), \( \sigma^0 = \sigma^\pm_S \circ \rho_\gamma \), respectively in the Bargmann and Schroedinger case, with \( \mu = 0 \) in the Bargmann case, \( \mu = \theta \) in the Schroedinger case.

In the Bargmann case, the automorphisms of \( A \), \( \rho_B^\gamma \equiv \sigma^\pm_B \circ \rho_\gamma \circ (\sigma^\pm_B)^{-1} \), are implemented by transformations in \( S \) and therefore

\[
\sigma^\pm_B \circ \rho_\gamma(A) = \rho^B_\gamma \circ \sigma^\pm_B = S^{-1}_\gamma \sigma^\pm_B(A)S_\gamma,
\]  

(23)

so that one is reduced, by a transformations in \( S \), to \( \sigma^\pm_B \). In the Schroedinger case, for \( \gamma = \exp -is \), the corresponding automorphisms \( \rho_S^\gamma \) are implemented by \( U_0(s) \equiv S^{-1}U(s)S \), and therefore one has to consider only the case \( \gamma > 0 \).

Representations of \( A_H \) defined by \( \sigma^\pm_B, \sigma^\pm_S \circ \rho_\gamma, \gamma > 0 \) will now be reduced to the representations introduced above on spaces spanned by eigenvectors of \( N_B \) and \( N_S \). Weak continuity of \( U(s) \) in a Hilbert Krein topology \( \kappa \) is equivalent to weak continuity of \( U_0(s) \) in \( V_{\pi^0} \), in a Hilbert-Krein topology \( \kappa^0 \), defined from \( \kappa \) by the isometry \( S \). This implies that the integrals defining eigenvectors, eq.(9), exist also as weak limit in the corresponding Krein space, and that the eigenvector expansion converges weakly in the same space. By \( \tau \) continuity of \( z \) and \( \partial_z \) and their closability in the weak Hilbert-Krein topology, one obtains a Krein representation of \( A_H \) on spaces spanned by eigenvalues of \( N_B \), or of \( N_S \).

In the Bargmann case, eigenvalues are not degenerate; therefore, given the isomorphisms \( \sigma^\pm \), there is only one (up to a constant) inner product, in the space spanned by the eigenvectors, which gives a Krein representation. By
weak convergence of the eigenfunction expansion, the space $V_\pi$ is a subspace of a Krein completion of the space spanned by the eigenvectors, and the result follows since $S$ extends to a Krein space isometry.

In the Schrödinger case, the eigenvalues of $N_S$ are real, since

$$(\lambda + n) \langle f_\lambda, g_{\lambda+n} \rangle = \langle \sigma(a^*a)f_\lambda, g_{\lambda+n} \rangle = \overline{\lambda} \langle f_\lambda, g_{\lambda+n} \rangle$$

for $\text{Im} \lambda \neq 0$ implies the vanishing of the inner product. The parameter $\theta$, eq. (12), is therefore real. For $\theta \neq 0$, one has a Krein representation on the space spanned by the eigenvalues, which is reducible into two irreducible equivalent representations, so that the Krein inner product is determined, and coincides with the inner product introduced above, apart from an hermitean $2 \times 2$ matrix $M$ in the commutant of the representation; $M$ must be positive (or negative) definite, since otherwise null subrepresentations would appear. For $\theta = 0$, the eigenvectors are $F_n(z), F_n(iz), n \in \mathbb{Z}$, and only $n \geq 0$ is admitted; otherwise

$$\langle F_0(z), F_0(z) \rangle = \langle a_+ F_{-1}, F_0 \rangle = \langle F_{-1}, a_- F_0 \rangle = 0$$

which implies $\langle F_n(z), F_m(z) \rangle = 0 \ \forall n, m \geq 0$, i.e. a null subrepresentation in the closure of $\pi$. The same applies to $F_n(iz)$, so that all eigenvectors belong to $V_0 + S(i)V_0$; therefore, the representation is contained in a Krein completion of the unique (up to two irrelevant constants in the inner products) representation in $V_0 + S(i)V_0$. For all $\theta \in (-1, 0]$, the representation is reconstructed by weak convergence of the eigenfunction expansion and by the fact that $S$ extends to a Krein space isometry.

4 Krein representations of infinite dimensional CCR algebras

In this Section we extend our analysis to representations of CCR algebras for $M$ degrees of freedom including the infinite dimensional case.

The CCR algebras which arise in physically interesting models, in particular in the Gupta-Bleuler quantization of the electromagnetic field [3], are generated by elements $a_i, a_i^*, i = 1, \ldots, M$, satisfying in general commutation relations of the form

$$[a_i, a_j] = 0 = [a_i^*, a_j^*], \quad [a_i, a_j^*] = \eta_{ij},$$

(24)
with $\eta_{ij}$ a non-degenerate complex matrix. Thus, by a linear transformation one can reduce to the case

$$[a_i, a_j] = 0 = [a_i^*, a_j^*], \quad [a_i, a_j^*] = \delta_{i,j} \eta_i, \quad \eta_i = \pm 1. \quad (25)$$

The algebra $A_H(\eta)$ generated by $a_i, a_i^*, i = 1, \ldots, M$, (possibly $M = \infty$), satisfying eqs.(25), is isomorphic to the Heisenberg algebra $A_{H,M}$, for $M$ degrees of freedom (corresponding to $\eta_i = 1, i = 1, \ldots, M$), with isomorphism given by $\rho(a_i) = \frac{1}{2}(1 + \eta_i) a_i + \frac{i}{2}(1 - \eta_i) a_i^*$. However, the isomorphism does not commute with the gauge transformations

$$\gamma^s(a_i) = e^{-is} a_i, \quad s \in [0, 2\pi), \quad i = 1, \ldots, M \quad (26)$$

and therefore a classification of representations in terms of conditions on the gauge generators, $N = \sum_{i=1}^M \eta_i a_i^* a_i$, depends on $\eta_i$. In general the analysis depends on the specification of $M$ regularity $U(1)$ groups, leading to products of representations classified in Theorem 1.

A substantial simplification occurs if one requires positivity of the generator of gauge transformations, a property that in the physical applications is closely related to the energy spectral condition, that is stability. In fact, this condition leads to a unique Krein representation of the algebra defined by eqs.(25), up to multiplicities, which is a Hilbert space representation iff the spectrum of $\eta_{ij}$ is positive. This result covers the infinite dimensional case, for algebras defined by eqs.(25).

The regularity property of Definition 2 is easily adapted to the $M$-dimensional case. The implementers $U(s)$ are said to satisfy the spectral condition if, $\forall f, g \in V$, $< f, U(s) g >$ extends to a bounded analytic function in the upper half (complex) plane.

**Theorem 2** A regular Krein representation of $A_H(\eta)$, admitting implementers $U(s)$ satisfying the spectral condition is contained in a direct sum of representations of the form $\pi = \pi_H \circ \rho$, where $\pi_H$ is the cyclic subrepresentation of $A_{H,M}$ in the tensor product of Fock and antiFock representations of $A_H^{(i)}$, defined by a cyclic vector $\Psi_{\theta_i}$, which, for each $i$, defines a Fock/antiFock representation of $A_H^{(i)}$, for $\eta_i = \pm 1$.

The representation is Krein equivalent to a holomorphic representation of the Bargmann form

$$\pi(a_i) = \partial_{z_i}, \quad \pi(a_i^*) = z_i \quad (27)$$
on the space of polynomials in $M$ (possibly $M = \infty$) complex variables, with inner product given for $n_i, m_j \geq 0$ by

$$< z_{i_1}^{n_1} \ldots z_{i_k}^{n_k}, z_{i_1}^{m_1} \ldots z_{i_k}^{m_k} > = \prod_{i=1}^{k} n_i! (-1)^{n_i(1-\eta_i)/2} \delta_{n_i, m_i},$$

(28)

or of Schrödinger form

$$\pi(a_i) = \sqrt{\frac{1}{2}}(z_i + \partial_z_i), \quad \pi(a_i^*) = \sqrt{\frac{1}{2}}(z_i - \partial_z_i)$$

(29)

on the space of polynomials in $M$ (possibly $M = \infty$) variables, multiplied by $\Psi_0(z_1, \ldots z_M) = \prod_{i=1}^{M} e^{-z_i^2/2}$, with scalar product given similarly to eq.(28), in terms of Hermite polynomials.

**Proof.** Given $f \in V_\pi$, by the regularity property, eq.(8) defines, as in Theorem 1, at least one non zero element $f_k$ of the Krein closure of $V_\pi$, which is in the domain of the closures of all the elements $\pi(A), A \in \mathcal{A}_H(\eta)$. On the other hand, the spectral condition implies that $\forall g \in V_\pi$

$$< g, U(t) f > = \sum_k < g, f_k > e^{ikt}$$

is the Fourier-Laplace transform of a distribution $F(\omega)$ with $\text{supp } F(\omega) \subseteq \mathbb{R}^+$, so that $< g, f_k > \neq 0$ only for $k \geq 0$ and therefore $f_k = 0$ for $k < 0$. Hence, by closability of $\pi(A), A \in \mathcal{A}_H(\eta), \forall n > k$,

$$(2\pi) \pi(a_{i_1} \ldots a_{i_n}) f_k = \int_{0}^{2\pi} ds e^{-iks} \pi(a_{i_1} \ldots a_{i_n}) U(s) f =$$

$$\int_{0}^{2\pi} ds e^{-i(k-n)s} U(s) \pi(a_{i_1} \ldots a_{i_n}) f_k = (2\pi) (\pi(a_{i_1} \ldots a_{i_n}) f)_{k-n} = 0.$$

If $\pi(a_i) f_k = 0, \forall i$, we consider the cyclic representation of $f_k$; otherwise for some $i_1$, $\pi(a_{i_1}) f_k = (\pi(a_{i_1}) f)_{k-1} \neq 0$. Within $n$ steps, we obtain a non zero vector $\Psi_0$ such that $\pi(a_i) \Psi_0 = 0, \forall i$. Therefore $\pi_H \equiv (\pi \circ \rho^{-1})$ is a representation of $\mathcal{A}_{H,M}$ with cyclic vector $\Psi_0$ satisfying the Fock/antiFock condition

$$\rho(a_i) \Psi_0 = 0, \text{ if } \eta_i = 1, \quad \rho(a_i^*) \Psi_0 = 0, \text{ if } \eta_i = -1.$$

By the same argument as in the proof of Theorem 1, the inner product in the cyclic space is then unique up to a factor, which cannot vanish, since otherwise one would get a null subrepresentation. The isomorphism with the holomorphic representations characterized in Theorem 1 is straightforward.
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