Realizations of non-commutative rational functions around a matrix centre, I: synthesis, minimal realizations and evaluation on stably finite algebras

Motke Porat and Victor Vinnikov

Abstract

In this paper we generalize classical results regarding minimal realizations of non-commutative (nc) rational functions using nc Fornasini–Marchesini realizations which are centred at an arbitrary matrix point. We prove the existence and uniqueness of a minimal realization for every nc rational function, centred at an arbitrary matrix point in its domain of regularity. Moreover, we show that using this realization we can evaluate the function on all of its domain (of matrices of all sizes) and also with respect to any stably finite algebra. As a corollary we obtain a new proof of the theorem by Cohn and Amitsur, that equivalence of two rational expressions over matrices implies that the expressions are equivalent over all stably finite algebras. Applications to the matrix valued and the symmetric cases are presented as well.

Contents

Introduction ............... 1250
1. Preliminaries .............. 1253
2. Realizations of NC rational expressions ........ 1256
3. Realizations of NC rational functions ......... 1282
4. Realizations of matrix-valued NC rational functions ..... 1286
5. Realizations of hermitian NC rational functions ...... 1289
References ............... 1296

Introduction

Noncommutative (nc, for short) rational functions are a skew field of fractions — more precisely, the universal skew field of fractions — of the ring of nc polynomials, that is, polynomials in non-commuting indeterminates (the free associative algebra). Essentially, they are obtained by starting with nc polynomials and applying successive arithmetic operations; a considerable amount of technical details is necessary here since in contrast to the commutative case, there is no canonical coprime fraction representation for a nc rational function. NC rational functions originated from several sources: the general theory of free rings and of skew fields (see [20, 22, 23, 45, 52–54], [21, 24–26] for comprehensive expositions and [55, 65] for good surveys); the theory of rings with rational identities (see [6], also [17] and [66, Chapter 8]); and rational former power series in the theory of formal languages and finite automata (see [30–32, 51, 71, 72] and [18] for a good survey).
Much like in the case of rational functions of a single variable [13, 50] (and unlike the case of several commuting variables [35, 46]), nc rational functions that are regular at 0 admit a good state space realization theory, see in particular Theorem 1 below. This was first established in the context of finite automata and recognizable power series, and more recently reformulated, with additional details, in the context of transfer functions of multidimensional systems with evolution along the free monoid (see [4, 8–12]). State space realizations of nc rational functions have figured prominently in work on robust control of linear systems subjected to structured possibly time-varying uncertainty (see [14, 15, 56]). Another important application of nc rational functions appears in the area of linear matrix inequalities (LMIs, see, for example, [58, 59, 73]). Most optimization problems of system theory and control are dimensionless in the sense that the natural variables are matrices, and the problem involves nc rational expressions in these matrix variables which have therefore the same form independent of matrix sizes (see [19, 36, 37]).

Coming from a different direction, the method of state space realizations, also known as the linearization trick, found important recent applications in free probability, see [16, 41, 74, 75]. Here it is crucial to evaluate nc rational expressions on a general algebra — which is stably finite in many important cases — rather than on matrices of all sizes. Stably finite algebras appeared in this context in the work of Cohn [26] and they play an important and not surprising role in our analysis.

Here is a full characterization of nc rational functions which are regular at 0 and their (matrix) domains of regularity, in terms of their minimal realizations (for the proofs, see [8, 10, 30–32, 47, 49, 71, 72]).

**Theorem 1.** If \( \mathcal{R} \) is an nc rational function of \( x_1, \ldots, x_d \) and \( \mathcal{R} \) is regular at 0, then \( \mathcal{R} \) admits a unique (up to unique similarity) minimal nc Fornasini–Marchesini realization

\[
\mathcal{R}(x_1, \ldots, x_d) = D + C \left( I_L - \sum_{k=1}^{d} A_k x_k \right)^{-1} \sum_{k=1}^{d} B_k x_k,
\]

where \( A_1, \ldots, A_d \in \mathbb{K}^{L \times L}, B_1, \ldots, B_d \in \mathbb{K}^{L \times 1}, C \in \mathbb{K}^{1 \times L}, D = \mathcal{R}(0) \in \mathbb{K} \) and \( L \in \mathbb{N} \). Moreover, for all \( m \in \mathbb{N} : (X_1, \ldots, X_d) \in (\mathbb{K}^{m \times m})^d \) is in the domain of regularity of \( \mathcal{R} \) if and only if det\((I_{Lm} - X_1 \otimes A_1 - \ldots - X_d \otimes A_d) \neq 0\); in that case

\[
\mathcal{R}(X_1, \ldots, X_d) = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^{d} X_k \otimes A_k \right)^{-1} \sum_{k=1}^{d} X_k \otimes B_k.
\]

Here a realization is called minimal if the state space dimension \( L \) is as small as possible; equivalently, the realization is observable, that is,

\[
\bigcap_{0 \leq k} \bigcap_{1 \leq i_1, \ldots, i_k \leq d} \ker(CA_{i_1} \cdots A_{i_k}) = \{0\},
\]

and controllable, that is,

\[
\bigcup_{0 \leq k} \bigcup_{1 \leq i_1, \ldots, i_k, j \leq d} A_{i_1} \cdots A_{i_k} B_j = \mathbb{K}^L.
\]

Theorem 1 is strongly related to expansions of nc rational functions which are regular at 0 into formal nc power series around 0; that is why it is not applicable for all nc rational functions. For example, the nc rational expression \( R(x_1, x_2) = (x_1 x_2 - x_2 x_1)^{-1} \) is not defined at 0, nor at any pair \((y_1, y_2) \in \mathbb{K}^2\); therefore, one cannot consider realizations of \( R \) which are centred at 0 as in Theorem 1, nor at any scalar point (a tuple of scalars). A realization theory
for such expressions (and hence functions) is required in particular for all of the applications mentioned above. Such a theory is presented here, using the ideas of the general theory of nc functions.

The theory of nc functions has its roots in the works by Taylor \[76, 77\] on nc spectral theory. It was further developed by Voiculescu \[78–80\] and Kalyuzhnyi-Verbovetskyi–Vinnikov \[48\], including a detailed discussion on nc difference-differential calculus. See also the work of Helton–Klep–McCullough \[38, 39\], of Popescu \[61, 62\], of Muhly–Solel \[57\] and of Agler–McCarthy \[1–3\].

A crucial fact \[48\], Chapters 4–7\] is that nc functions admit power series expansions, called Taylor–Taylor series in honour of Brook Taylor and of Joseph L. Taylor, around an arbitrary matrix point in their domain. This motivates us to generalize realizations as in Theorem 1 to the case where the centre is a $d$-tuple of matrices rather than 0 or a $d$-tuple of scalars.

This is the first in a series of papers with the goal of generalizing the theory of (Fornasini–Marchesini) realizations centred at 0 (or at a scalar point), to the case of (Fornasini–Marchesini) realizations centred at an arbitrary matrix point in the domain of regularity of an nc rational function. In particular, we present a generalization of Theorem 1 (see Theorem 2 below), namely, the existence and uniqueness of a minimal realization, together with the inclusion of the domain of the nc rational function in the domain of any of its minimal realizations. (The other inclusion and hence the equality of the two domains is presented in a follow-up paper \[63\]).

Other types of realizations of nc rational functions that are not necessary regular at 0 have been considered in \[27, 28\] and in \[82\], see also the recent papers \[67–70\]. We will consider further the relation between our representations and those of \[27, 28\] in our follow-up paper \[63\].

Here is an outline of the paper: In Section 1 we give some preliminaries on nc rational functions and evaluations over general algebras.

In Section 2 we present the setting of nc Fornasini–Marchesini realizations centred at a matrix point $Y \in (K^{s \times s})^d$ and generalize classical results which are well known in the scalar case ($s = 1$) to the case where $s \geq 1$. We prove, using synthesis, the existence of such realizations for any nc rational expression (Theorem 2.4), and introduce the terms of observability and controllability (Subsection 2.2) analogously to the scalar case as in \[50\]. The uniqueness of minimal realizations, up to unique similarity, is then proved (Theorems 2.13 and 2.16), followed by a Kalman decomposition argument (Theorem 2.15). An example of an explicit construction of a minimal realization is presented in Subsection 2.5 for the nc rational expression $(x_1 x_2 - x_2 x_1)^{-1}$. During the whole section we carry on the results also in a more general setting of evaluations with respect to arbitrary unital stably finite $K$-algebra; as a corollary we obtain a new proof of a theorem of Cohn that equivalence of two rational expressions over matrices implies their equivalence over all stably finite algebras (Theorem 2.19). Finally, in Subsection 2.7, we define the McMillan degree of an nc rational expression using minimal Fornasini–Marchesini realizations and show that it does not depend on the centre of the realization.

Section 3 contains the main result of the paper, which is a partial generalization of Theorem 1 for nc rational functions not necessary regular at a scalar point:

**Theorem 2** (Theorem 3.3, Corollary 2.18). If $\mathfrak{R}$ is an nc rational function of $x_1, \ldots, x_d$ over $K$, then for every $Y = (Y_1, \ldots, Y_d) \in \text{dom}_s(\mathfrak{R})$ there exists a unique (up to unique similarity) minimal (observable and controllable) nc Fornasini–Marchesini realization

\[
\mathfrak{R}_{\mathcal{FM}}(X_1, \ldots, X_d) = D + C \left( I_L - \sum_{k=1}^{d} A_k (X_k - Y_k) \right)^{-1} \sum_{k=1}^{d} B_k (X_k - Y_k)
\]
centred at $\mathbf{Y}$, such that for every $m \in \mathbb{N}$ and $(X_1, \ldots, X_d) \in \text{dom}_{sm}(\mathcal{R})$:

$$\mathcal{R}(X_1, \ldots, X_d) = I_m \otimes D + (I_m \otimes C) \left( I_{lm} - \sum_{k=1}^d (X_k - I_m \otimes Y_k)A_k \right)^{-1} \sum_{k=1}^d (X_k - I_m \otimes Y_k)B_k.$$ 

Moreover, using the realization $\mathcal{R}_{FM}$ we can evaluate $\mathcal{R}$ on every matrix point in the domain of regularity of $\mathcal{R}$ as well as with respect to any unital stably finite $\mathbb{K}$-algebra.

The strength of Theorem 3.3 is that we can evaluate any nc rational function on all of its domain and also with respect to any unital stably finite $\mathbb{K}$-algebra, by using a minimal realization of any nc rational expression which represents the function, which is centred at any point from its domain. As a corollary (Corollary 3.4) we provide a proof of Theorem 1 which — unlike the original proof in [49] — does not make any use of the difference-differential calculus of nc functions, but only the results from Sections 2 and 3.

Generalizations of the main results from Sections 2 and 3 to matrix-valued nc rational functions are briefly summarized in Section 4.

Finally, in Section 5 we provide a full and precise parameterization (5.3) in Theorem 5.2) of hermitian nc rational functions in terms of their minimal nc Fornasini–Marchesini realizations centred at a matrix point. A short discussion and some parameterizations are given for descriptor realizations as well.

One of the difficulties which arises when moving from a scalar to a matrix centre is that a minimal nc Fornasini–Marchesini realization $\mathcal{R}_{FM}$ of an nc rational expression is no longer an nc rational expression by itself (cf. Remark 2.2). However, in the sequel paper [63], we show that under some constraints (called the linearized lost abbey conditions) on the coefficients of the realization — which follow immediately when $\mathcal{R}_{FM}$ is a minimal nc Fornasini–Marchesini realization of an nc rational expression — $\mathcal{R}_{FM}$ is actually the restriction of an nc rational function $\mathcal{R}$ with $\text{DOM}_s(\mathcal{R}_{FM}) = \text{dom}_s(\mathcal{R})$. This will imply the opposite inclusion of the domains in Theorem 2 and thereby complete the proof that the domain of an nc rational function coincides with the domain of any of its minimal realizations, centred at an arbitrary matrix point. As a corollary, also in [63], we will prove that the domain of an nc rational function is equal to its stable extended domain. In a slightly different direction, we will use the theory of realizations with a matrix centre developed in this paper, together with the results from [63], to present an explicit construction of the free skew field $\mathbb{K}\langle \mathcal{R} \rangle$, with a self-contained proof that it is the universal skew field of fractions of the ring of nc polynomials. Moreover, we will construct a functional model and use it to provide a different one step proof for the existence of a realization formula for nc rational functions, without using synthesis. Furthermore, we will establish a generalization of the Kronecker–Fliess theorem, which gives a full characterization of nc rational functions in terms of their formal nc-generalized power series expansions around a matrix point. These results will appear in [64].

Finally, we point out that instead of working with Fornasini–Marchesini realizations (for the original version in the commutative setting, see [33, 34]), one can consider structured realizations as in [8] and obtain similar results. This is also true for descriptor realizations; for more details, see Remark 5.5.

1. Preliminaries

Notations: $d$ will stand for the number of non-commuting variables, which will be usually denoted by $x_1, \ldots, x_d$, we often abbreviate non-commuting by nc. For a positive integer $d$, we denote by $G_d$ the free monoid generated by $d$ generators $g_1, \ldots, g_d$, we say that a word $\omega = g_{i_1} \ldots g_{i_\ell} \in G_d$ is of length $|\omega| = \ell$ if $\ell \geq 1$ and $\omega = \emptyset$ is of length 0. For a field $\mathbb{K}$ and
\textbf{Motke Porat and Victor Vinnikov}

$n \in \mathbb{N}$, let $\mathbb{K}^{n \times n}$ be the vector space of $n \times n$ matrices over $\mathbb{K}$, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{K}^n$ and let $E_n = \{E_{ij} = e_i e_j^T : 1 \leq i, j \leq n\}$ be the standard basis of $\mathbb{K}^{n \times n}$. The tensor (Kronecker) product of two matrices $P \in \mathbb{K}^{m \times n_2}$ and $Q \in \mathbb{K}^{n_3 \times n_4}$ is the $n_1 n_3 \times n_2 n_4$ block matrix $P \otimes Q = [p_{ij}]_{1 \leq i \leq n_1, 1 \leq j \leq n_2}$. The range of a matrix $P$, which is the span of all of its columns, is denoted by $\text{Im}(P)$.

We denote operators on matrices by bold letters such as $\mathbf{A}, \mathbf{B}$ and the action of $\mathbf{A}$ on $X$ by $\mathbf{A}(X)$. If $\mathbf{A}$ is defined on $s \times s$ matrices, we extend $\mathbf{A}$ to act on $sm \times sm$ matrices for any $m \in \mathbb{N}$, by viewing an $sm \times sm$ matrix $X$ as an $m \times m$ matrix with $s \times s$ blocks and by evaluating $\mathbf{A}$ on the $s \times s$ blocks. In that case we denote the evaluation by $(X)\mathbf{A}$. If $C$ is a constant matrix and $\mathbf{A}$ is an operator, then $C \cdot \mathbf{A}$ and $\mathbf{A} \cdot C$ are two operators, defined by $(C \cdot \mathbf{A})(X) := C \mathbf{A}(X)$ and $(\mathbf{A} \cdot C)(X) := \mathbf{A}(X)C$. For every $n_1, n_2 \in \mathbb{N}$, we define the permutation matrix

$$E(n_1, n_2) = [E_{ij}^T]_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \in \mathbb{K}^{n_1 n_2 \times n_1 n_2}$$

and use these matrices to change the order of factors in the Kronecker product of two matrices by the following rule:

$$P \otimes Q = E(n_1, n_3)(Q \otimes P)E(n_2, n_4)^T,$$

for all $n_1, n_2, n_3, n_4 \in \mathbb{N}, Q \in \mathbb{K}^{n_3 \times n_2}$ and $P \in \mathbb{K}^{n_1 \times n_4}$; for more details, see [44, pp. 259–261]. If $P = [P_{ij}]_{1 \leq i, j \leq m} \in (\mathbb{K}^{s \times s})^{m \times m}$ and $Q = [Q_{ij}]_{1 \leq i, j \leq m} \in (\mathbb{K}^{s \times s})^{m \times m}$, then we use the notation

$$P \odot_s Q := [\sum_{k=1}^m P_{ik} \otimes Q_{kj}]_{1 \leq i, j \leq m}$$

for the so-called faux product of $P$ and $Q$, viewed as $m \times m$ matrices over the tensor algebra of $\mathbb{K}^{s \times s}$, where $P_{ik} \otimes Q_{kj}$ denotes the element of $\mathbb{K}^{s \times s} \otimes \mathbb{K}^{s \times s}$, rather than the Kronecker product of the matrices; see [60, p. 241] for the exact definition and [29] for its origins in operator spaces. If $X = (X_1, \ldots, X_d) \in (\mathbb{K}^{sm \times sm})^d$ and $\omega = g_1 \cdots g_\ell$, then

$$X^{\odot_s \omega} := X_{i_1} \odot_s \cdots \odot_s X_{i_\ell}.$$  

We use $\mathfrak{R}, \mathfrak{R}, R$ and $r$ for nc rational function, nc Fornasini–Marchesini realization, nc rational expression and matrix valued nc rational function, respectively. Likewise, we use $\mathfrak{a}$ to denote elements in an algebra $\mathcal{A}$ and $\mathfrak{A}$ to denote matrices over $\mathcal{A}$. Throughout the paper, we use underline to denote vectors or $d$-tuples.

1.1. **NC rational functions**

If $V$ is a vector space over a field $\mathbb{K}$, then $V_{nc}$, the nc space over $V$, consists of all square matrices over $V$, that is,

$$V_{nc} = \bigoplus_{n=1}^{\infty} V^{n \times n}.$$  

For every $\Omega \subseteq V_{nc}$ and $n \in \mathbb{N}$, we use the notation $\Omega_n := \Omega \cap V^{n \times n}$. A subset $\Omega \subseteq V_{nc}$ is called an nc set if it is closed under direct sums, that is, if $X \in \Omega_n, Y \in \Omega_m$, then $X \oplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$, $\forall m, n \in \mathbb{N}$. In the special case where $V = \mathbb{K}^d$, we have the identification

$$(\mathbb{K}^d)^{nc} = \bigoplus_{n=1}^{\infty} (\mathbb{K}^d)^{n \times n} = \bigoplus_{n=1}^{\infty} (\mathbb{K}^{n \times n})^d,$$  

which is the nc space of all $d$-tuples of square matrices over $\mathbb{K}$. Let $V, W$ be vector spaces over a field $\mathbb{K}$ and $\Omega \subseteq V_{nc}$ be an nc set, then $f : \Omega \to W_{nc}$ is called an nc function if $f$ is graded, that is, if $n \in \mathbb{N}$ and $X \in \Omega_n$, then $f(X) \in W_{n \times n}$, and

1. $f$ respects direct sums, that is, if $X, Y \in \Omega$, then $f(X \oplus Y) = f(X) \oplus f(Y)$;
2. $f$ respects similarities, that is, if $n \in \mathbb{N}$, $X \in \Omega_n$ and $T \in \mathbb{K}^{n \times n}$ is invertible such that $T \cdot X \cdot T^{-1} \in \Omega$, then $f(T \cdot X \cdot T^{-1}) = T \cdot f(X) \cdot T^{-1}$.

Notice that if $X \in \Omega_n$ and $T \in \mathbb{K}^{n \times n}$, by the products $T \cdot X$ and $X \cdot T$ we mean the standard matrix multiplication and we use the action of $\mathbb{K}$ on $\mathcal{V}$. In particular, if $\mathcal{V} = \mathbb{K}^d$, $X = (X_1, \ldots, X_d) \in (\mathbb{K}^{n \times n})^d$ and $T \in \mathbb{K}^{n \times n}$, the products are given by

$$T \cdot X := (TX_1, \ldots, TX_d) \quad \text{and} \quad X \cdot T := (X_1T, \ldots, X_dT).$$

An important and central example of nc functions is nc rational expressions. We denote by $\mathbb{K}(x_1, \ldots, x_d)$ the $\mathbb{K}$-algebra of nc polynomials in the $d$ nc variables $x_1, \ldots, x_d$ over $\mathbb{K}$. We obtain nc rational expressions by applying successive arithmetic operations (addition, multiplication and taking inverse) on $\mathbb{K}(x_1, \ldots, x_d)$. For an nc rational expression $R$ and $n \in \mathbb{N}$, let $\text{dom}_n(R)$ be the set of all $d$-tuples of $n \times n$ matrices over $\mathbb{K}$ for which all the inverses in $R$ exist; the domain of regularity of $R$ is then defined by

$$\text{dom}(R) := \bigcap_{n=1}^{\infty} \text{dom}_n(R).$$

An nc rational expression $R$ is called non-degenerate if $\text{dom}(R) \neq \emptyset$. For example, $R(x) = (x_2 + (1 - x_1)^{-1}(x_3^{-1}x_1 - x_2))x_1$ is an nc rational expression in $x_1, x_2, x_3$, while its domain of regularity is given by

$$\text{dom}(R) = \bigcap_{n=1}^{\infty} \{(X_1, X_2, X_3) \in (\mathbb{K}^{n \times n})^3 : \det(I_n - X_1), \det(X_3) \neq 0\}.$$ 

Every nc rational expression $R$ is an nc function from $\text{dom}(R) \subseteq (\mathbb{K}^d)_{nc}$ to $\mathbb{K}_{nc}$. For a detailed discussion of nc rational expressions and their domains of regularity, see [48].

What comes now is the definition of an nc rational function. Let $R_1$ and $R_2$ be nc rational expressions in $x_1, \ldots, x_d$ over $\mathbb{K}$. We say that $R_1$ and $R_2$ are $(\mathbb{K}^d)_{nc}$-evaluation equivalent, if $R_1(X) = R_2(X)$ for every $X \in \text{dom}(R_1) \cap \text{dom}(R_2)$. An nc rational function is an equivalence class of non-degenerate nc rational expressions. For every nc rational function $\mathfrak{R}$, define its domain of regularity

$$\text{dom}(\mathfrak{R}) := \bigcup_{R \in \mathfrak{R}} \text{dom}(R). \quad (1.3)$$

The $\mathbb{K}$-algebra of all nc rational functions of $x_1, \ldots, x_d$ over $\mathbb{K}$ is denoted by $\mathbb{K}\langle x_1, \ldots, x_d \rangle$ and it is a skew field, called the free skew field. Moreover, $\mathbb{K}\langle x_1, \ldots, x_d \rangle$ is the universal skew field of fractions of $\mathbb{K}(x_1, \ldots, x_d)$. See [6, 17, 22, 23, 66] for the original proofs and [26] for a more modern reference, while a proof of the equivalence with the evaluations over matrices is presented in [47, 49].

1.2. Evaluations over algebras

Let $\mathcal{A}$ be a unital $\mathbb{K}$-algebra. If $\underline{a} = (a_1, \ldots, a_d) \in \mathcal{A}^d$ and $\omega = g_{i_1} \cdots g_{i_\ell} \in \mathcal{G}_d$, then we use the notations $\underline{a}^\omega := a_{i_1} \cdots a_{i_\ell}$ and $\underline{a}^0 = 1_{\mathcal{A}}$, where $1_{\mathcal{A}}$ is the unit element in $\mathcal{A}$. We recall the definitions of evaluation and domain of nc rational expressions over $\mathcal{A}$. For more details see [41].

**Definition 1.1 (\mathcal{A}-domains and evaluations).** For any nc rational expression $R$ in $x_1, \ldots, x_d$ over $\mathbb{K}$, its $\mathcal{A}$-domain $\text{dom}^\mathcal{A}(R) \subseteq \mathcal{A}^d$ and its evaluation $R^\mathcal{A}(\underline{a})$ at any $\underline{a} = (a_1, \ldots, a_d) \in \text{dom}^\mathcal{A}(R)$ are defined by the following.
1. If \( R = \sum_{\omega \in \mathcal{G}} r_{\omega} x^{\omega} \) is an nc polynomial (\( r_{\omega} \in \mathbb{K} \)), then
   \[
   \text{dom}^A(R) = \mathcal{A}^d \quad \text{and} \quad R^A(a) = \sum_{\omega \in \mathcal{G}} r_{\omega} a^{\omega}.
   \]

2. If \( R = R_1 R_2 \) where \( R_1 \) and \( R_2 \) are nc rational expressions, then
   \[
   \text{dom}^A(R) = \text{dom}^A(R_1) \cap \text{dom}^A(R_2) \quad \text{and} \quad R^A(a) = R_1^A(a) R_2^A(a).
   \]

3. If \( R = R_1 + R_2 \) where \( R_1 \) and \( R_2 \) are nc rational expressions, then
   \[
   \text{dom}^A(R) = \text{dom}^A(R_1) \cap \text{dom}^A(R_2) \quad \text{and} \quad R^A(a) = R_1^A(a) + R_2^A(a).
   \]

4. If \( R = R_1^{-1} \) where \( R_1 \) is an nc rational expression, then
   \[
   \text{dom}^A(R) = \{ a \in \text{dom}^A(R_1) : R_1^A(a) \text{ invertible in } \mathcal{A} \} \quad \text{and} \quad R^A(a) = (R_1^A(a))^{-1}.
   \]

**Remark 1.2.** Let \( n \in \mathbb{N} \) and consider the \( \mathbb{K} \)-algebra \( \mathcal{A}_n = \mathbb{K}^{n \times n} \). Then, it is easily seen that \( \text{dom}^A_n(R) = \text{dom}_n(R) \) and \( R(a) = R^A_n(a) \) for every \( a \in \text{dom}_n(R) \).

As it will be pointed out later (cf. Theorem 2.15), we are interested in a certain family of algebras, called stably finite algebras. A unital \( \mathbb{K} \)-algebra \( \mathcal{A} \) is called stably finite if for every \( m \in \mathbb{N} \) and \( A, B \in \mathcal{A}^{m \times m} \), we have
\[
\mathfrak{A} \mathfrak{B} = I_m \otimes 1_\mathcal{A} \iff \mathfrak{B} \mathfrak{A} = I_m \otimes 1_\mathcal{A}.
\]

If \( \mathcal{A} \) is a unital \( \mathbb{C}^* \)-algebra with a faithful trace, then \( \mathcal{A} \) is stably finite. The following is a characterization of stably finite algebras that we find useful at a later stage of the paper; see [41, Lemma 5.2] for its proof.

**Lemma 1.3.** Let \( \mathcal{A} \) be a unital \( \mathbb{K} \)-algebra. The following are equivalent.

1. \( \mathcal{A} \) is stably finite.
2. For every \( n \in \mathbb{N} \), \( m_1, \ldots, m_n \in \mathbb{N} \) and \( \mathfrak{A}_{i,j} \in \mathcal{A}^{m_i \times m_j} \), \( i, j = 1, \ldots, n \), if the upper (or lower) triangular block matrix
   \[
   \begin{bmatrix}
   \mathfrak{A}_{11} & * & * & * \\
   0 & \mathfrak{A}_{22} & * & * \\
   \vdots & \ddots & \ddots & * \\
   0 & \cdots & 0 & \mathfrak{A}_{nn}
   \end{bmatrix}
   \]
   is invertible, then the matrices \( \mathfrak{A}_{11}, \ldots, \mathfrak{A}_{nn} \) are invertible.

2. **Realizations of NC rational expressions**

NC Fornasini–Marchesini realizations, see [8, 49] and [33, 34] for the original commutative version, apply to nc rational expressions which are regular at 0. By translation, the point 0 can be replaced by any scalar point. In this section we develop analogous realization formulas for nc rational expressions, centred at an arbitrary matrix point in the domain of regularity of the expression.

**Definition 2.1.** Let \( s, L \in \mathbb{N} \), \( \mathbf{Y} = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s \times s})^d \),
\[
\mathbf{A}_1, \ldots, \mathbf{A}_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L} \quad \text{and} \quad \mathbf{B}_1, \ldots, \mathbf{B}_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s}
\]
be linear mappings, \( C \in \mathbb{K}^{s \times L} \) and \( D \in \mathbb{K}^{s \times s} \). Then

\[
\mathcal{R}(X_1, \ldots, X_d) = D + C \left( I_L - \sum_{k=1}^{d} A_k(X_k - Y_k) \right)^{-1} \sum_{k=1}^{d} B_k(X_k - Y_k) \tag{2.1}
\]

is called an nc Fornasini–Marchesini realization centred at \( Y \) and it is defined for every \( X = (X_1, \ldots, X_d) \in DOM_s(\mathcal{R}) \), where

\[
DOM_s(\mathcal{R}) := \{ X \in (\mathbb{K}^{s \times s})^d : \det \left( I_L - \sum_{k=1}^{d} A_k(X_k - Y_k) \right) \neq 0 \}.
\]

In that case we say that the realization \( \mathcal{R} \) is described by the tuple \((L, D, C, A, B)\).

**Remark 2.2.** If \( s = 1 \), then \( \mathcal{R} \) is a \( 1 \times 1 \) matrix-valued nc rational expression (see Remark 4.4 for details) and \( DOM_s(\mathcal{R}) = dom_s(\mathcal{R}) \). However, this is not the case for \( s > 1 \) and that is why we use the notation \( DOM_s(\mathcal{R}) \) instead of \( dom_s(\mathcal{R}) \).

Let \( s_1, s_2, s_3, s_4 \in \mathbb{N} \). If \( T : \mathbb{K}^{s_1 \times s_2} \to \mathbb{K}^{s_3 \times s_4} \) is a linear mapping and \( m \in \mathbb{N} \), then \( T \) can be naturally extended to a linear mapping \( T : \mathbb{K}^{s_1m \times s_2m} \to \mathbb{K}^{s_3m \times s_4m} \), by the following rule:

\[
X = \left[ X_{ij} \right]_{1 \leq i, j \leq m} \in \mathbb{K}^{s_1m \times s_2m} \implies (X)T = \left[ T(X_{ij}) \right]_{1 \leq i, j \leq m},
\]

that is, \( (X)T \) is an \( m \times m \) block matrix with entries in \( \mathbb{K}^{s_3 \times s_4} \). Therefore, we can extend the realization (2.1) to act on \( d \)-tuples of \( sm \times sm \) matrices: for every \( X = (X_1, \ldots, X_d) \) in

\[
DOM_{sm}(\mathcal{R}) := \{ X \in (\mathbb{K}^{sm \times sm})^d : \det \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right) \neq 0 \},
\]

define

\[
\mathcal{R}(X) := I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k.
\]

In addition, if \( \mathcal{A} \) is a unital \( \mathbb{K} \)-algebra, a linear mapping \( T : \mathbb{K}^{s_1 \times s_2} \to \mathbb{A}^{s_3 \times s_4} \) can be also naturally extended to a linear mapping \( T^\mathcal{A} : \mathcal{A}^{s_1 \times s_2} \to \mathcal{A}^{s_3 \times s_4} \) by the following rule:

\[
\mathfrak{A} = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} E_{ij} \otimes a_{ij} \in \mathcal{A}^{s_1 \times s_2} \implies (\mathfrak{A})T^\mathcal{A} = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} T(E_{ij}) \otimes a_{ij} \in \mathcal{A}^{s_3 \times s_4},
\]

where \( E_{ij} = e_i e_j^T \in \mathbb{K}^{s_1 \times s_2} \) and \( a_{ij} \in \mathcal{A} \). If \( \mathcal{R} \) is an nc Fornasini–Marchesini realization centred at \( Y \), as in (2.1), define its \( \mathcal{A} \)-domain to be the subset of \((\mathcal{A}^{s \times s})^d \) given by

\[
DOM^\mathcal{A}(\mathcal{R}) := \bigg\{ \mathfrak{A} \in (\mathcal{A}^{s \times s})^d : \left( I_L \otimes 1_\mathcal{A} - \sum_{k=1}^{d} (\mathfrak{A}_k - Y_k \otimes 1_\mathcal{A})A^\mathcal{A}_k \right) \text{ is invertible in } \mathcal{A}^{L \times L} \bigg\}
\]

and for every \( \mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) \in DOM^\mathcal{A}(\mathcal{R}) \) define the evaluation of \( \mathcal{R} \) at \( \mathfrak{A} \) by

\[
\mathcal{R}^\mathcal{A}(\mathfrak{A}) := D \otimes 1_\mathcal{A} + (C \otimes 1_\mathcal{A}) \left( I_L \otimes 1_\mathcal{A} - \sum_{k=1}^{d} \left[ (\mathfrak{A}_k)A^\mathcal{A}_k - A_k(Y_k) \otimes 1_\mathcal{A} \right] \right)^{-1} \sum_{k=1}^{d} \left[ (\mathfrak{A}_k)B^\mathcal{A}_k - B_k(Y_k) \otimes 1_\mathcal{A} \right].
\]
2.1. Existence

The way we define what is a realization of an nc rational expression is different than the usual definition. In the usual case, the expression and the realization coincide whenever they are both defined, while in our definition, we include the fact that the domain of the expression is contained in the domain of the realization. We begin with the definition of an nc rational expression admitting a realization, both in the usual way (over matrices) and in the case of evaluations with respect to an algebra. We say that:

**Definition 2.3.** Let $R$ be an nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$, $\underline{Y} = (Y_1, \ldots, Y_d) \in \text{dom}_s(R)$, $\mathcal{R}$ be an nc Fornasini–Marchesini realization centred at $\underline{Y}$ and $\mathcal{A}$ be a unital $\mathbb{K}$-algebra. We say that:

1. $R$ admits the realization $\mathcal{R}$, or that $\mathcal{R}$ is a realization of $R$, if
   
   $$\text{dom}_{sm}(R) \subseteq \text{DOM}_{sm}(\mathcal{R}) \text{ and } R(\underline{X}) = \mathcal{R}(\underline{X}), \forall \underline{X} \in \text{dom}_{sm}(\mathcal{R})$$

   for every $m \in \mathbb{N}$.

2. $R$ admits the realization $\mathcal{R}$ with respect to $\mathcal{A}$, or that $\mathcal{R}$ is a realization of $R$ with respect to $\mathcal{A}$, if for every $\underline{a} = (a_1, \ldots, a_d) \in \text{dom}^A(R)$:

   $$I_s \otimes \underline{a} := (I_s \otimes a_1, \ldots, I_s \otimes a_d) \in \text{DOM}^A(\mathcal{R})$$

   and $I_s \otimes R^A(\underline{a}) = \mathcal{R}^A(I_s \otimes \underline{a})$.

We begin by showing the existence of an nc Fornasini–Marchesini realization for every nc rational expression $R$, centred at any $\underline{Y} \in \text{dom}_s(R)$, which is also a realization of $R$ with respect to any unital $\mathbb{K}$-algebra.

**Theorem 2.4.** Let $R$ be an nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$ and let $\underline{Y} = (Y_1, \ldots, Y_d) \in \text{dom}_s(R)$. There exists an nc Fornasini–Marchesini realization $\mathcal{R}$ of $R$ centred at $\underline{Y}$, such that $\mathcal{R}$ is a realization of $R$ with respect to any unital $\mathbb{K}$-algebra.

The proof is done by synthesis, which is going back to ideas from automata theory [18, 71, 72] and system theory [27, 28]. We also use the following technical fact: let $\underline{X} = (X_1, \ldots, X_d) \in (\mathbb{K}^{sm \times sm})^d$ and write

$$X_k = \sum_{i,j=1}^{m} E_{ij} \otimes X_{ij}^{(k)}, \text{ with } E_{ij} \in \mathbb{K}^{m \times m}, X_{ij}^{(k)} \in \mathbb{K}^{s \times s}, 1 \leq k \leq d, \quad (2.2)$$

then $(X_k - I_m \otimes Y_k)\mathcal{A}_k = \sum_{i,j=1}^{m} E_{ij} \otimes \mathcal{A}_k(X_{ij}^{(k)}) - I_m \otimes \mathcal{A}_k(Y_k)$ and hence

$$(X_k - I_m \otimes Y_k)\mathcal{A}_k = P_2 \left( \sum_{i,j=1}^{m} \mathcal{A}_k(X_{ij}^{(k)}) \otimes E_{ij} - \mathcal{A}_k(Y_k) \otimes I_m \right) P_2^{-1}$$

and similarly

$$(X_k - I_m \otimes Y_k)\mathcal{B}_k = P_2 \left( \sum_{i,j=1}^{m} \mathcal{B}_k(X_{ij}^{(k)}) \otimes E_{ij} - \mathcal{B}_k(Y_k) \otimes I_m \right) P_2^{-1},$$

where $P_1 = E(m, s)$ and $P_2 = E(m, L)$ are the shuffle matrices defined in (1.1). Therefore for every $\underline{X} \in \text{DOM}_{sm}(\mathcal{R})$ we have
\[ P_1^{-1} \mathcal{R}(X) P_1 \]
\[ = D \otimes I_m + P_1^{-1}(I_m \otimes C)P_2 \left( I_{Lm} - \sum_{k=1}^{d} \left[ \sum_{i,j=1}^{m} A_k(X_{ij}^{(k)}) \otimes E_{ij} - A_k(Y_k) \otimes I_m \right] \right)^{-1} \]
\[ \sum_{k=1}^{d} \left[ \sum_{i,j=1}^{m} B_k(X_{ij}^{(k)}) \otimes E_{ij} - B_k(Y_k) \otimes I_m \right] = D \otimes I_m + (C \otimes I_m) \left( I_{Lm} - \sum_{k=1}^{d} \left[ \sum_{i,j=1}^{m} B_k(X_{ij}^{(k)}) \otimes E_{ij} - B_k(Y_k) \otimes I_m \right] \right) \quad (2.3) \]

**Proof.** We first show that the theorem is true for all monomials \( x_1, \ldots, x_d \) and constants, then we show that if it is true for two rational expressions, so it is also true for their summation, multiplication and their inversion, when they exist.

1. **Constants:** Let \( R_0(\mathbb{K}) = K \subset \mathbb{K} \). If \( \mathfrak{g} \in \text{dom}^A(R_0) = A^d \), then \( I_s \otimes \mathfrak{g} \in \text{DOM}^A(R_0) = (A^{s \times s})^d \), where the realization \( \mathcal{R}_0 \) is centred at \( Y \in \text{dom}_s(R_0) = (\mathbb{K}^{s \times s})^d \) and described by

\[ L = 1, D = I_s \otimes K, C = 0, A_1 = \cdots = A_d = 0, B_1 = \cdots = B_d = 0 \quad (2.4) \]

and \( I_s \otimes R_0^\mathfrak{g}(\mathfrak{a}) = I_s \otimes (K \otimes 1_A) = D \otimes 1_A = \mathcal{R}_0^A(I_s \otimes \mathfrak{g}) \). Moreover,

\[ \text{dom}_{sm}(R_0) = (\mathbb{K}^{sm \times sm})^d = \text{DOM}_{sm}(R_0) \]

and for every \( X \in \text{dom}_{sm}(R_0) \) we have \( R_0(X) = I_{sm} \otimes K = I_m \otimes D = R_0(X) \).

2. **Monomials:** Let \( R_j(x_j) = x_j \) for \( 1 \leq j \leq d \). If \( \mathfrak{g} \in \text{dom}^A(R_j) = A^d \), then \( I_s \otimes \mathfrak{g} \in \text{DOM}^A(R_j) = (A^{s \times s})^d \), where the realization \( \mathcal{R}_j \) is centred at \( Y \in \text{dom}_s(R_j) = (\mathbb{K}^{s \times s})^d \) and described by

\[ L = s, D = Y_j, C = I_s, A_1 = \cdots = A_d = 0, B_j = Id, B_k = 0 (\forall k \neq j) \quad (2.5) \]

and

\[ I_s \otimes R_j^\mathfrak{g}(\mathfrak{a}) = I_s \otimes a_j = Y_j \otimes 1_A + (I_s \otimes a_j - Y_j \otimes 1_A)B_j^A = \mathcal{R}_j^A(I_s \otimes \mathfrak{g}). \]

Moreover, \( \text{dom}_{sm}(R_j) = (\mathbb{K}^{sm \times sm})^d = \text{DOM}_{sm}(\mathcal{R}_j) \) and for every \( X \in \text{dom}_{sm}(R_j) \) we have \( R(X) = X_j = I_m \otimes Y_j + (X_j - I_m \otimes Y_j)B_j = \mathcal{R}_j(X) \).

3. **Addition:** Suppose that \( R_1 \) and \( R_2 \) are two nc rational expressions admitting realizations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) both centred at \( Y \), described by the tuples \((L_1, D^1, C^1, A_1^1, B_1^1)\) and \((L_2, D^2, C^2, A_2^1, B_2^1)\), respectively, and also with respect to any unital \( \mathbb{K} \)-algebra \( A \).

Thus, \( \mathfrak{g} \in \text{dom}^A(R_1 + R_2) = \text{dom}^A(R_1) \cap \text{dom}^A(R_2) \) implies that \( I_s \otimes \mathfrak{g} \in \text{DOM}^A(R_1) \cap \text{DOM}^A(R_2) \),

\[ I_s \otimes (R_1 + R_2)^\mathfrak{g}(\mathfrak{a}) = \mathcal{R}_1^A(I_s \otimes \mathfrak{g}) + \mathcal{R}_2^A(I_s \otimes \mathfrak{g}) = (\mathcal{R}_1^A + \mathcal{R}_2^A)(I_s \otimes \mathfrak{g}) = D_{\text{par}} \otimes 1_A \]

\[ + (C_{\text{par}} \otimes 1_A) \left( I_L \otimes 1_A - \sum_{k=1}^{d} [A_k^\text{par}(I_s) \otimes a_k - A_k^\text{par}(Y_k) \otimes 1_A] \right)^{-1} \sum_{k=1}^{d} [B_k^\text{par}(I_s) \otimes a_k - B_k^\text{par}(Y_k) \otimes 1_A] := (\mathcal{R}_{\text{par}})^A(I_s \otimes \mathfrak{g}) \]
and \( I_s \otimes a \in DOM^A(\mathcal{R}_{\text{par}}) \), when \( \mathcal{R}_{\text{par}} \) is the nc Fornasini–Marchesini realization centred at \( \overline{\mathcal{Y}} \) described by

\[
L = L_1 + L_2, \quad D_{\text{par}} = D^1 + D^2, \quad C_{\text{par}} = \begin{bmatrix} C^1 & C^2 \end{bmatrix}, \quad A_k^{\text{par}} = \begin{bmatrix} A_k^1 & 0 \\ 0 & A_k^2 \end{bmatrix}
\]

and \( B_k^{\text{par}} = \begin{bmatrix} B_k^1 \\\nB_k^2 \end{bmatrix}, \quad k = 1, \ldots, d. \tag{2.6} \)

Also, for every \( m \in \mathbb{N}, \quad X \in dom_{sm}(R_1 + R_2) = dom_{sm}(R_1) \cap dom_{sm}(R_2) \) implies \( X \in DOM_{sm}(\mathcal{R}_i) \) and \( R_i(X) = \mathcal{R}_i(X) \) for \( i = 1, 2 \) and hence \( (R_1 + R_2)(\mathcal{X}) = R_1(\mathcal{X}) + R_2(\mathcal{X}) \).

Write \( \mathcal{X} = (X_1, \ldots, X_d) \in (\mathbb{A}^{sm \times sm})^d \) and use (2.3) to obtain that

\[
P_1^{-1}(\mathcal{R}_1(\mathcal{X}) + \mathcal{R}_2(\mathcal{X}))P_1 = D_{\text{par}} \otimes I_m + (C_{\text{par}} \otimes I_m)P_2^{-1}
\left(I_{L_m} - \sum_{k=1}^d \sum_{i,j=1}^m A_k^{\text{par}}(X_{ij}^{(k)}) \otimes E_{ij} - A_k^{\text{par}}(Y_k) \otimes I_m\right)
^{-1}
\times P_2P_2^{-1}
\left(\sum_{k=1}^d \sum_{i,j=1}^m E_{ij} \otimes B_k^{\text{par}}(X_{ij}^{(k)}) - I_m \otimes B_k^{\text{par}}(Y_k)\right)P_1 = P_1^{-1}\mathcal{R}_{\text{par}}(\mathcal{X})P_1,
\]

that is, \( \mathcal{R}_1(\mathcal{X}) + \mathcal{R}_2(\mathcal{X}) = \mathcal{R}_{\text{par}}(\mathcal{X}) \) while it is easily seen that \( \mathcal{X} \in DOM_{sm}(\mathcal{R}_{\text{par}}) \).

4. **Multiplication**: Suppose that \( R_1 \) and \( R_2 \) are two nc rational expressions admitting realizations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) both centred at \( \overline{\mathcal{Y}} \), described by the tuples \((L_1, D^1, C^1, A^1, B^1)\) and \((L_2, D^2, C^2, A^2, B^2)\), respectively, and also with respect to any unital K-algebra \( \mathcal{A} \).

Thus, \( a \in dom^A(R_1R_2) = dom^A(R_1) \cap dom^A(R_2) \) implies that \( I_s \otimes a \in DOM^A(\mathcal{R}_1) \cap DOM^A(\mathcal{R}_2) \),

\[
I_s \otimes (R_1R_2)^A(a) = \mathcal{R}_1^A(I_s \otimes a)\mathcal{R}_2^A(I_s \otimes a) = (\mathcal{R}_1\mathcal{R}_2)^A(I_s \otimes a) = D_{\text{ser}} \otimes 1_A + (C_{\text{ser}} \otimes 1_A)
\left(I_L \otimes 1_A - \sum_{k=1}^d \left[A_k^{\text{ser}}(I_s) \otimes a_k - A_k^{\text{ser}}(Y_k) \otimes 1_A\right]\right)^{-1}
\sum_{k=1}^d \left[B_k^{\text{ser}}(I_s) \otimes a_k - B_k^{\text{ser}}(Y_k) \otimes 1_A\right]
\]

\[
:= (\mathcal{R}_{\text{ser}})^A(I_s \otimes a),
\]

and \( I_s \otimes a \in DOM^A(\mathcal{R}_{\text{ser}}) \), when \( \mathcal{R}_{\text{ser}} \) is the nc Fornasini–Marchesini realization centred at \( \overline{\mathcal{Y}} \) described by

\[
L = L_1 + L_2, \quad D_{\text{ser}} = D^1D^2, \quad C_{\text{ser}} = \begin{bmatrix} C^1 & D^1C^2 \end{bmatrix}, \quad A_k^{\text{ser}} = \begin{bmatrix} A_k^1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} B_k^1 \\
B_k^2 \end{bmatrix}, \quad k = 1, \ldots, d. \tag{2.7} \]
Also, for every \( m \in \mathbb{N} \), \( X \in \text{dom}_{sm}(R_1R_2) = \text{dom}_{sm}(R_1) \cap \text{dom}_{sm}(R_2) \) implies that \( X \in \text{DOM}_{sm}(R) \) and \( R_i(X) = R_i(X) \). Now, let \( X = (X_1, \ldots, X_d) \in (\mathbb{K}^{s \times s})^d \) as in (2.2), so similar computation shows that \( X \in \text{DOM}_{sm}(R) \) and \( (R_1R_2)(X) = R_1(X)R_2(X) \).

5. Inverses: Suppose that \( R \) is an nc rational expression admitting a realization \( R \) centred at \( Y \), described by the tuple \( (L, D, C, A, B) \), also with respect to any unital \( \mathbb{K} \)-algebra \( A \) and \( R(Y) = D \) is invertible.

Thus, \( a \in \text{dom}^{A}(R^{-1}) \) implies that \( a \in \text{dom}^{A}(R) \), \( I_s \otimes a \in \text{DOM}^{A}(R) \), \( R^{A}(a) \) is invertible and \( \text{DOM}^{A}(I_s \otimes a) = I_s \otimes R^{A}(a) \), so
\[
I_s \otimes (R^{-1})^{A}(a) = (I_s \otimes R^{A}(a))^{-1} = (\text{DOM}^{A}(I_s \otimes a))^{-1} = (\text{DOM}^{A}(I_s \otimes a))^{-1} = (\text{DOM}^{A}(I_s \otimes a))^{-1}
\]
and \( I_s \otimes a \in \text{DOM}^{A}(R^{inv}) \), when \( R^{inv} \) is the nc Fornasini–Marchesini realization centred at \( Y \) described by
\[
D^{inv} = D^{-1}, \ C^{inv} = D^{-1}C, \ \ A_{k}^{inv} = A_{k} - B_{k} \cdot (D^{-1}C) \quad \text{and} \quad B_{k}^{inv} = -B_{k} \cdot D^{-1}, \ k = 1, \ldots, d.
\]

Moreover, if \( X \in \text{dom}_{sm}(R^{-1}) \), then \( X \in \text{dom}_{sm}(R) \) and \( R(X) \) is invertible, so \( X \in \text{DOM}_{sm}(R) \) and \( R'(X) = R(X) \) is invertible; therefore, the matrices \( M \) and \( I_m \otimes D + (I_m \otimes C)M^{-1}N \) are invertible, where
\[
M := I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \quad \text{and} \quad N := \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k.
\]

Consider the matrix
\[
E := \begin{bmatrix} -M & N \\ I_m \otimes C & I_m \otimes D \end{bmatrix} \in \mathbb{K}(L+s)m \times (L+s)m
\]

such that its two Schur complements decompositions
\[
E = \begin{bmatrix} I_{Lm} & \text{dom}_{sm} & 0 \\ -I_{m} \otimes C)M^{-1} & I_{sm} \end{bmatrix} \begin{bmatrix} -M & 0 \\ 0 & I_{m} \otimes D + (I_{m} \otimes C)M^{-1}N \end{bmatrix} \begin{bmatrix} I_{Lm} & -M^{-1}N \\ 0 & I_{sm} \end{bmatrix} = \begin{bmatrix} I_{Lm} & N(I_{m} \otimes D)^{-1} \\ 0 & I_{sm} \end{bmatrix} \begin{bmatrix} -M - N(I_{m} \otimes D^{-1}C) & 0 \\ 0 & I_{m} \otimes D \end{bmatrix} \begin{bmatrix} I_{Lm} & 0 \\ I_{m} \otimes (D^{-1}C) & I_{sm} \end{bmatrix}.
\]

As \( M \) and \( I_m \otimes D + (I_m \otimes C)M^{-1}N \) are invertible, it follows that \( E \) is invertible and hence
\[
M + N(I_{m} \otimes (D^{-1}C)) = I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k^{inv}
\]
is invertible, that is, \( X \in \text{DOM}_{sm}(R^{inv}) \). Thus, \( \text{dom}_{sm}(R^{-1}) \subseteq \text{DOM}_{sm}(R^{inv}) \) and for every \( X \in \text{dom}_{sm}(R^{-1}) \), we have \( R^{-1}(X) = R^{inv}(X) \). \( \square \)

We finish this subsection by comparing the two parts of Definition 2.3 for the \( \mathbb{K} \)-algebra \( A_n = \mathbb{K}^{n \times n} \) (cf. Remark 1.2). This will imply (see Corollary 2.6) that for every nc rational expression \( R \), the realization that we have constructed in Theorem 2.4 — centred at a \( d \)-tuple of \( s \times s \) matrices — allows us to evaluate \( R \) at every point in its domain of regularity and not only at the points whose dimension is a multiple of \( s \). An alternative way to evaluate an nc rational expression on all of its domain of regularity will be given later in Theorem 3.3. We define \( P_1 = E(n, s) \) and \( P_2 = E(n, L) \), correspondingly to (1.1).
Proposition 2.5. Let \( n \geq 1 \), \( A_n = \mathbb{K}^{n \times n} \), \( R \) be an nc Fornasini–Marchesini realization centred at \( Y \in (\mathbb{K}^{s \times s})^d \) and \( X \in (\mathbb{K}^{n \times n \times s})^d \). Then
\[
X \in DOM^{A_n}(R) \iff P_1 \cdot X \cdot P_1^{-1} \in DOM_{sn}(R) \tag{2.9}
\]
and \( R^{A_n}(X) = P_1^{-1}R(P_1 \cdot X \cdot P_1^{-1})P_1 \), whenever (2.9) holds.

Proof. Let \( X = (X_{ij}) \in (\mathbb{K}^{n \times n \times s})^d \) and consider the decomposition (2.2), where \( X_{ij}^{(k)} \in \mathbb{K}^{n \times n} = A_n \) for \( 1 \leq i, j \leq s \) and \( 1 \leq k \leq d \). As
\[
I_L \otimes 1_A - \sum_{k=1}^d (X_k - Y_k \otimes 1_A)A_k^{A_n}
\]
\[
= I_Ln - \sum_{k=1}^d \left[ \sum_{i,j=1}^s A_k(E_{ij}) \otimes X_{ij}^{(k)} - A_k(Y_k) \otimes I_n \right]
\]
\[
= P_2^{-1} \left( I_Ln - \sum_{k=1}^d \left[ \sum_{i,j=1}^s X_{ij}^{(k)} \otimes A_k(E_{ij}) - I_n \otimes A_k(Y_k) \right] \right) P_2
\]
\[
= P_2^{-1} \left( I_n \otimes I_L - \sum_{k=1}^d \left[ \sum_{i,j=1}^s (X_{ij}^{(k)} \otimes E_{ij})A_k - (I_n \otimes Y_k)A_k \right] \right) P_2
\]
\[
= P_2^{-1} \left( I_n \otimes I_L - \sum_{k=1}^d (P_1X_kP_1^{-1} - I_n \otimes Y_k)A_k \right) P_2,
\]
we have \( X \in DOM^{A_n}(R) \) if and only if \( P_1 \cdot X \cdot P_1^{-1} \in DOM_{sn}(R) \).

Similar computation shows that
\[
(X_k)B_k^{A_n} - B_k(Y_k) \otimes I_n = \sum_{i,j=1}^s B_k(E_{ij}) \otimes X_{ij}^{(k)} - B_k(Y_k) \otimes I_n
\]
\[
= P_2^{-1} \left( \sum_{i,j=1}^s X_{ij}^{(k)} \otimes B_k(E_{ij}) - I_n \otimes B_k(Y_k) \right) P_1
\]
\[
= P_2^{-1}(P_1X_kP_1^{-1} - I_n \otimes Y_k)B_k P_1,
\]
and hence \( X \in DOM^{A_n}(R) \) implies
\[
R^{A_n}(X) = D \otimes I_n + (C \otimes I_n) \left( I_Ln - \sum_{k=1}^d \left[ (X_k)A_k^{A_n} - A_k(Y_k) \otimes I_n \right] \right)^{-1}
\]
\[
\times \sum_{k=1}^d \left[ (X_k)B_k^{A_n} - B_k(Y_k) \otimes I_n \right]
\]
is defined by 

\[ X = A_n \otimes R \] for the later ones, see \[ \text{in the case of nc Fornasini–Marchesini realizations centred at a scalar point.} \] For a reference

\[ A \text{ and the multilinear mapping} \ X \otimes I_s. \]

Corollary 2.6. If \( n \in \mathbb{N} \) and \( \mathcal{R} \) is an nc Fornasini–Marchesini realization of an nc rational expression \( R \) with respect to \( \mathcal{A}_n = \mathbb{K}^{n \times n} \), then for every \( \mathcal{X} \in \text{dom}_n(R) \) we have

\[ \mathcal{X} \otimes I_s \in \text{DOM}_{sn}(\mathcal{R}) \text{ and } R(\mathcal{X}) \otimes I_s = \mathcal{R}(\mathcal{X} \otimes I_s). \]

Proof. Let \( \mathcal{X} \in \text{dom}_n(R) \), thus \( \mathcal{X} \in \text{dom}^{A_n}(R) \) and that implies by Definition 2.3 that \( I_s \otimes \mathcal{X} \in \text{DOM}^{A_n}(\mathcal{R}) \) and \( \mathcal{R}^{A_n}(I_s \otimes \mathcal{X}) = I_s \otimes \mathcal{R}^{A_n}(\mathcal{X}). \) Using Proposition 2.5, we get

\[ \mathcal{X} \otimes I_s = P_1 \cdot (I_s \otimes \mathcal{X}) \cdot P_1^{-1} \in \text{DOM}_{sn}(\mathcal{R}) \]

and

\[ P_1^{-1}\mathcal{R}(\mathcal{X} \otimes I_s)P_1 = \mathcal{R}^{A_n}(I_s \otimes \mathcal{X}) = I_s \otimes R(\mathcal{X}), \]

which implies that \( \mathcal{R}(\mathcal{X} \otimes I_s) = R(\mathcal{X}) \otimes I_s. \)

2.2. Controllability and observability

To consider the notion of minimal realizations as in the classical realization theory, we first introduce the definitions of controllability and observability in the case of nc Fornasini–Marchesini realizations centred at a matrix point, which are generalizations of the definitions in the case of nc Fornasini–Marchesini realizations centred at a scalar point. For a reference for the later ones, see [8].

Given linear mappings \( A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L}, B_k : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s} \) and a word \( \omega = g_{i_1} \cdots g_{i_\ell} \in \mathcal{G}_d \) of length \( |\omega| = \ell \), define the multilinear mapping \( \mathbf{A}^\omega : (\mathbb{K}^{s \times s})^\ell \to \mathbb{K}^{L \times L} \) by

\[ \mathbf{A}^\omega(X_1, \ldots, X_\ell) := A_{i_1}(X_1) \cdots A_{i_\ell}(X_\ell) \]

and the multilinear mapping \( \mathbf{A}^\omega \cdot B_k : (\mathbb{K}^{s \times s})^{\ell+1} \to \mathbb{K}^{L \times s} \) by

\[ (\mathbf{A}^\omega \cdot B_k)(X_1, \ldots, X_{\ell+1}) := \mathbf{A}^\omega(X_1, \ldots, X_\ell)B_k(X_{\ell+1}). \]

Definition 2.7. Let \( A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L} \) and \( B_1, \ldots, B_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s} \) be linear mappings, and \( C \in \mathbb{K}^{s \times L} \).

1. The controllable subspace \( \mathcal{C}_{\mathbf{A}, \mathbf{B}} \) is defined by

\[ \bigvee_{\omega \in \mathcal{G}_d, X_1, \ldots, X_{|\omega|+1} \in \mathbb{K}^{s \times s}, 1 \leq k \leq d} \text{Im} (\mathbf{A}^\omega(X_1, \ldots, X_{|\omega|})B_k(X_{|\omega|+1})). \]

If \( \mathcal{C}_{\mathbf{A}, \mathbf{B}} = \mathbb{K}^{L} \), the tuple \( (\mathbf{A}, \mathbf{B}) \) is called controllable.

2. The un-observable subspace \( \mathcal{N}\mathcal{O}_{C, \mathbf{A}} \) is defined by

\[ \bigcap_{\omega \in \mathcal{G}_d, X_1, \ldots, X_{|\omega|} \in \mathbb{K}^{s \times s}} \ker (C\mathbf{A}^\omega(X_1, \ldots, X_{|\omega|})). \]

If \( \mathcal{N}\mathcal{O}_{C, \mathbf{A}} = \{0\} \), the tuple \( (C, \mathbf{A}) \) is called observable.

The multilinear mapping \( \mathbf{A}^\omega \) can be viewed as a linear mapping from \( (\mathbb{K}^{s \times s})^\ell \) to \( \mathbb{K}^{L \times L} \). Then one can use the faux product, as introduced in (1.2), to define controllability and observability
not only on the level of $s \times s$ matrices, but also on the levels of $sm \times sm$ matrices, for every $m \in \mathbb{N}$, using the subspaces
\[
C^{(m)}_{\Delta \mathcal{B}} = \bigvee_{\omega \in \mathcal{G}_d, X_1, \ldots, X_{|\omega|+1} \in \mathbb{K}^{sm \times sm}, 1 \leq k \leq d} \operatorname{Im}((X_1 \circ_s \cdots \circ_s X_{|\omega|}A^\omega(X_{|\omega|+1}B_k))
\]
and
\[
NC^{(m)}_{\Delta \mathcal{A}} = \bigcap_{\omega \in \mathcal{G}_d, X_1, \ldots, X_{|\omega|} \in \mathbb{K}^{sm \times sm}} \ker((I_m \otimes C)(X_1 \circ_s \cdots \circ_s X_{|\omega|}A^\omega)).
\]

**Proposition 2.8.** If $m \in \mathbb{N}$, then $C^{(m)}_{\Delta \mathcal{B}} = C^{(1)}_{\Delta \mathcal{B}} \otimes \mathbb{K}^m$ and $NC^{(m)}_{\Delta \mathcal{A}} = NC^{(1)}_{\Delta \mathcal{A}} \otimes \mathbb{K}^m$.

**Proof.** Let $m \in \mathbb{N}$.

- If $u \in C^{(m)}_{\Delta \mathcal{B}}$, then $u$ is a linear combination of vectors of the form
  \[(X_1)A_{i_1} \cdots (X_k)A_{i_k} (X_{k+1})B_{i_{k+1}, u_{i_{k+1}}},\]
  where $1 \leq i_1, \ldots, i_{k+1} \leq d$, $X_1, \ldots, X_{k+1} \in \mathbb{K}^{sm \times sm}$ and $u \in \mathbb{K}^{sm}$. As the mappings $A_{i_1}B_{i_1} \mathbbm{1} \leq i \leq d$ act on $sm \times sm$ matrices by acting on their $s \times s$ blocks, we get that $(X_1)A_{i_1} \cdots (X_k)A_{i_k} (X_{k+1})B_{i_{k+1}, u_{i_{k+1}}} \in C^{(1)}_{\Delta \mathcal{B}} \otimes \mathbb{K}^m$ and as a linear combination of such vectors, we get that $u \in C^{(1)}_{\Delta \mathcal{B}} \otimes \mathbb{K}^m$.

- On the other hand, let $u \in C^{(1)}_{\Delta \mathcal{B}} \otimes \mathbb{K}^m$ and write $u = [u^T_1 \ldots u^T_m]^T$ where $u_1, \ldots, u_m \in C^{(1)}_{\Delta \mathcal{B}}$. Thus,
  \[
u = \sum_{i=1}^{m} u_i \otimes e_i = \sum_{i=1}^{m} \left( \sum_{j=1}^{k_i} \mathbb{A}^{\omega_j,i} \left( X_{j,1}^{(i)}, \ldots, X_{j,|\omega_j,i|}^{(i)} \right) B_{\ell,i,j} \left( \tilde{X}_j^{(i)} \right) w_{j,i} \right) \otimes e_i = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \left[ (I_m \otimes \mathbb{A}^{\omega_j,i}) \left( X_{j,1}^{(i)}, \ldots, X_{j,|\omega_j,i|}^{(i)} \right) B_{\ell,i,j} \left( \tilde{X}_j^{(i)} \right) w_{j,i} \right] e_i = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \left[ (I_m \otimes X_{j,1}^{(i)}) \otimes_s \cdots \otimes_s (I_m \otimes X_{j,|\omega_j,i|}^{(i)}) \mathbb{A}^{\omega_j,i} \left( I_m \otimes \tilde{X}_j^{(i)} \right) B_{\ell,i,j} \left( I_m \otimes w_{j,i} \right) e_i \right] \in C^{(m)}_{\Delta \mathcal{B}},\]
  where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{K}^m$, $k_i \in \mathbb{N}$, $\omega_j,i \in \mathcal{G}_d$, $1 \leq \ell,i,j \leq d$ and $X_{j,1}^{(i)}, \ldots, X_{j,|\omega_j,i|}^{(i)}, \tilde{X}_j^{(i)} \in \mathbb{K}^{s \times s}$ for $1 \leq j \leq k_i$ and $1 \leq i \leq m$.

- Next, let
  \[
u = [\nu^T_1 \ldots \nu^T_m]^T \in NC^{(m)}_{\Delta \mathcal{A}},\]
  then for all $\omega \in \mathcal{G}_d$, $1 \leq j \leq m$ and $X_1, \ldots, X_{|\omega|} \in \mathbb{K}^{s \times s}$, we have
  \[
u = (I_m \otimes C) \left( (E_{jj} \otimes X_1) \circ_s \cdots \circ_s (E_{jj} \otimes X_{|\omega|})A^\omega u \right) = (I_m \otimes C) \left( E_{jj} \otimes A^\omega (X_1, \ldots, X_{|\omega|}) \right) u = CA^\omega (X_1, \ldots, X_{|\omega|})u_j,\]
  that is, $u_j \in NC^{(1)}_{\Delta \mathcal{A}}$ and hence $u \in NC^{(1)}_{\Delta \mathcal{A}} \otimes \mathbb{K}^m$.

- On the other hand, let
  \[
u = [\nu^T_1 \ldots \nu^T_m]^T \in NC^{(1)}_{\Delta \mathcal{A}} \otimes \mathbb{K}^m,\]
  then for every $\omega = g_1 \cdots g_k \in \mathcal{G}_d$ and $Z_1, \ldots, Z_k \in \mathbb{K}^{sm \times sm}$, we have
\((I_m \otimes C)(Z_1 \otimes_s \cdots \otimes_s Z_{|\omega|})A_\omega u = (I_m \otimes C)(Z_1)A_{i_1} \cdots (Z_k)A_{i_k}u\)

\[
= (I_m \otimes C) \left[ A_{i_1} \left( Z_{(1)} \right) \right]_{1 \leq p, q \leq m} \cdots \left[ A_{i_k} \left( Z_{(k)} \right) \right]_{1 \leq p, q \leq m} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = 0
\]

as \(u_1, \ldots, u_m \in N(C.A)\) and each of the entries in the product is a linear combination of vectors of the form \(CA_{i_1}(Z_{(1)}) \cdots A_{i_k}(Z_{(k)})u\), where \(1 \leq i \leq m\), which are all 0. \(\square\)

Two immediate consequences of Proposition 2.8 are the following. If \((A, B)\) is controllable, then \(C^{(m)}_{A,B} = K^{Lm}\) for all \(m \in \mathbb{N}\); whereas if \(C^{(m)}_{A,B} = K^{Lm}\) for some \(m \in \mathbb{N}\), then \((A, B)\) is controllable. Similarly, if \((C, A)\) is observable, then \(N^{(m)}_{C,A} = \{0\}\) for all \(m \in \mathbb{N}\); whereas if \(N^{(m)}_{C,A} = \{0\}\) for some \(m \in \mathbb{N}\), then \((C, A)\) is observable.

Next, we show how the original definitions of controllability and observability may be reformulated using the standard basis \(E_s\) of \(K^{s \times s}\).

**Proposition 2.9.** Let \(A_1, \ldots, A_d : K^{s \times s} \to K^{L \times L}\) and \(B_1, \ldots, B_d : K^{s \times s} \to K^{L \times s}\) be linear mappings, and \(C \in K^{s \times L}\). Then

1. \((A, B)\) is controllable if and only if

\[
\bigvee_{\omega \in \mathcal{G}_d, x_1, \ldots, x_{|\omega|+1} \in \mathcal{E}_s, 1 \leq k \leq d} \text{Im} \left( A^{(\omega)}(X_1, \ldots, X_{|\omega|})B_k(X_{|\omega|+1}) \right) = K^L.
\]

2. \((C, A)\) is observable if and only if

\[
\bigcap_{\omega \in \mathcal{G}_d, x_1, \ldots, x_{|\omega|} \in \mathcal{E}_s} \ker \left( C A^{(\omega)}(X_1, \ldots, X_{|\omega|}) \right) = \{0\}.
\]

**Proof.** Since \(E_s \subseteq K^{s \times s}\), the direction \(\Leftarrow\) of part 1 is trivial. To prove the other direction, suppose that \((A, B)\) is controllable, let \(X_1, \ldots, X_{\ell+1} \in K^{s \times s}\) and \(\omega = g_1, \ldots, g_{\ell} \in \mathcal{G}_d\). Thus, one can write \(X_t = \sum_{p,q=1}^s E_{pq} \otimes x_{pq}^{(t)}\) for \(1 \leq t \leq \ell + 1\) and by linearity of \(A_k, B_k\) we get

\[
A^{(\omega)}(X_1, X_t)B_k(X_{t+1}) = \left( \prod_{t=1}^{\ell} \sum_{p,q=1}^s x_{pq}^{(t)} A_{j_t}(E_{pq}) \right) \sum_{p,q=1}^s x_{pq}^{(t+1)} B_k(E_{pq})
\]

\[
\in \bigvee_{\nu \in \mathcal{G}_d, z_1, \ldots, z_{\ell+1} \in \mathcal{E}_s, 1 \leq k \leq d} \text{Im} \left( A^{(\nu)}(Z_1, \ldots, Z_{\ell+1})B_k(Z_{\ell+1}) \right).
\]

Therefore \(C_{A,B} \subseteq \bigvee_{\nu \in \mathcal{G}_d, z_1, \ldots, z_{\ell+1} \in \mathcal{E}_s, 1 \leq k \leq d} \text{Im} \left( A^{(\nu)}(Z_1, \ldots, Z_{\ell+1})B_k(Z_{\ell+1}) \right)\), whereas \(C_{A,B} = K^L\) implies the wanted equality. Similar proof holds for part 2. \(\square\)

The last part of this subsection discusses observability and controllability matrices. The infinite block matrix

\[
C_{A,B} := \text{row} \left[ C^{(\omega,k)}_{A,B} \right]_{(\omega,k) \in \mathcal{G}_d \times \{1, \ldots, d\}}
\]

is called the controllability matrix associated with the tuple \((A, B)\), where

\[
C^{(\omega,k)}_{A,B} \in K^{L \times s^3(|\omega|+1)}\]

is given by \(C^{(\omega,k)}_{A,B} := \text{row} \left[ (A^{(\omega,k)} \cdot B_k)(Z) \right]_{Z \in \mathcal{E}_s^{|\omega|+1}}\).

\[
(1_m \otimes C)(Z_1 \otimes_s \cdots \otimes_s Z_{|\omega|})A_\omega u = (1_m \otimes C)(Z_1)A_{i_1} \cdots (Z_k)A_{i_k}u
\]
for each \((\omega, k) \in \mathcal{G}_d \times \{1, \ldots, d\}\) and the infinite block matrix

\[
\mathcal{O}_{C, A} := \text{col} \left[ \mathcal{O}^{(\omega)}_{C, A} \right]_{\omega \in \mathcal{G}_d}
\]

is called the observability matrix associated to the tuple \((C, A)\), where

\[
\mathcal{O}^{(\omega)}_{C, A} \in \mathbb{K}^{s|\omega| \times L} \text{ is given by } \mathcal{O}^{(\omega)}_{C, A} := \text{col} \left[ C \cdot A^{\omega}(Z) \right]_{Z \in \mathcal{E}^{(\omega)}}
\]

for each \(\omega \in \mathcal{G}_d\). The following is a characterization of controllability and observability using the controllability and observability matrices. Most of the arguments in the proof are taken from linear algebra.

**Proposition 2.10.** Let \(A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L}\) and \(B_1, \ldots, B_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s}\) be linear mappings, and \(C \in \mathbb{K}^{s \times s}\). The following are equivalent:

1. \((A, B)\) is controllable [respectively, \((C, A)\) is observable].
2. The matrix \(C_{A, B}\) [respectively, \(C_{C, A}\)] is right [respectively, left] invertible.
3. The finite block matrix row \[C^{(\omega, k)}_{A, B} \mid \omega \leq \ell, 1 \leq k \leq d\] [respectively, col \(C^{(\omega)}_{C, A} \mid |\omega| \leq \ell\)] is right [respectively, left] invertible for some \(\ell \in \mathbb{N}\).

In that case, we can choose \(\ell \leq L - 1\).

**Proof.** \(1 \implies 2\): If \((A, B)\) is controllable, then \(e_j \in \mathbb{K}^L\) can be written as

\[
e_j = \sum_{i=1}^{k_j} A^{e_j,i} \left( X^{(j)}_{i,1}, \ldots, X^{(j)}_{i,|\omega,j,i|} \right) B^{e_j,i} \left( X^{(j)}_{i,|\omega,j,i|+1} \right) u^{e_j,i},
\]

where \(k_j \in \mathbb{N}\), \(\omega,j,i \in \mathcal{G}_d\), \(X^{(j)}_{i,1}, \ldots, X^{(j)}_{i,|\omega,j,i|+1} \in \mathbb{K}^{s \times s}\) and \(1 \leq \ell, j \leq d\), for every \(1 \leq i \leq k_j\) and \(1 \leq j \leq L\). Thus \(e_1, \ldots, e_L\) belong to the column span of the matrix \(C_{A, B}\) and hence \(C_{A, B}\) is right invertible.

\(2 \implies 3\): If the infinite matrix \(C_{A, B}\) is right invertible, it means that its column span is equal to \(\mathbb{K}^L\); however, this span of infinitely many vectors of length \(L\) must coincide with a span of finitely many of the columns, which easily implies part 3.

\(3 \implies 1\): If 3 holds, then the column span of \(C_{A, B}\) contains \(e_1, \ldots, e_L\) and thus is equal to \(\mathbb{K}^L\), that is, \(C_{A, B} = \mathbb{K}^L\) and the tuple \((A, B)\) is controllable.

• Suppose next that \((A, B)\) is controllable, and define

\[
C_\ell := \bigvee_{|\omega| \leq \ell, 1 \leq k \leq d} \text{Im} \left( C^{(\omega, k)}_{A, B} \right) = \bigvee_{|\omega| \leq \ell, 1 \leq k \leq d} \text{Im}(A^\omega \cdot B_k)
\]

for \(\ell \geq 0\). Then \(C_0 \subset C_1 \subset \ldots \subset C_\ell \subset C_{\ell+1} \subset \ldots\) are all subspaces of \(\mathbb{K}^L\), whereas the controllability of \((A, B)\) implies that \(\bigcup_{\ell=0}^\infty C_\ell = \mathbb{K}^L\) and \(C_0 \neq \{0\}\). Moreover, it is easily seen that if \(C_{\ell_0} = C_{\ell_0+1}\) for some \(\ell_0 \geq 0\), then \(C_{\ell_0+k} = C_{\ell_0}\) for all \(k \geq 0\) and hence

\[
\mathbb{K}^L = \bigcup_{\ell=0}^\infty C_\ell = C_0 \cup \ldots \cup C_{\ell_0} = C_{\ell_0}.
\]

Therefore, the sequence \(1 \leq \dim(C_0) \leq \dim(C_1) \leq \ldots \leq \dim(C_L) \leq \ldots \leq L\) must stabilize after at most \(L - 1\) inequalities, that is, \(\dim(C_{L-1}) = L\) which means that \(\mathbb{K}^L = C_{A, B} = C_{L-1}\).
• As for observability, to prove that $1 \implies 2 \implies 3 \implies 1$ we use the same arguments as above; so we only show how to get the bound on the size of the matrix. Suppose that $(C, A)$ is observable and define

$$\mathcal{NO}_\ell := \bigcap_{|\omega| \leq \ell} \ker (C A^\omega(X))$$

for $\ell \geq 0$. Then $\mathcal{NO}_0 \supseteq \mathcal{NO}_1 \supseteq \ldots \supseteq \mathcal{NO}_\ell \supseteq \mathcal{NO}_{\ell+1} \supseteq \ldots$ are all subspaces of $\mathbb{K}^L$, whereas the observability of $(C, A)$ implies that $\bigcap_{\ell \geq 0} \mathcal{NO}_\ell = \{0\}$ and $\mathcal{NO}_0 \neq \mathbb{K}^L$. It is easily seen that if $\mathcal{NO}_{\ell_0} = \mathcal{NO}_{\ell_0+1}$ for some $\ell_0 \geq 0$, then $\mathcal{NO}_{\ell_0+k} = \mathcal{NO}_{\ell_0}$ for all $k \geq 0$ and hence

$$\{0\} = \bigcap_{\ell \geq 0} \mathcal{NO}_\ell = \mathcal{NO}_{\ell_0}.$$  

Therefore, the sequence $L \geq \dim(\mathcal{NO}_0) \geq \dim(\mathcal{NO}_1) \geq \ldots \geq \dim(\mathcal{NO}_L) \geq \ldots$ must stabilize after at most $L - 1$ inequalities, that is, $\mathcal{NO}_{L-1} = \{0\}$.  

2.3. Minimal realizations

An nc Fornasini–Marchesini realization of the form (2.1) is said to be

- **controllable** if the tuple $(A, B)$ is controllable;
- **observable** if the tuple $(C, A)$ is observable.

If $\mathcal{R}$ is an nc Fornasini–Marchesini realization of an nc rational expression $R$ centred at $Y$, then it is said to be

- **minimal** if the dimension $L$ is the smallest integer for which $R$ admits such a realization, that is, if $\mathcal{R}'$ is an nc Fornasini–Marchesini realization of $R$ centred at $Y$ of dimension $L'$, then $L \leq L'$.

**Remark 2.11.** In fact, the minimality of a realization $\mathcal{R}$ is with respect to rational functions, meaning that if $\mathcal{R}$ is a minimal nc Fornasini–Marchesini realization of $R$ centred at $Y$, then it is also a minimal nc Fornasini–Marchesini realization of any nc rational expression $\tilde{R}$ which is $(\mathbb{K}^d)_{nc}$-evaluation equivalent to $R$ (cf. Lemma 3.1).

We proceed by showing that every two controllable and observable nc Fornasini–Marchesini realizations centred at $Y$ of $(\mathbb{K}^d)_{nc}$-evaluation equivalent nc rational expressions must be similar, where most of the ideas of the proof are taken from [7, 13]. We will also use the following facts, see [49, Theorem 4.8] and [48]:

- If $\tilde{R}$ is an nc rational expression and $Y \in \text{dom}_{sm}(\tilde{R})$, then

  $$\text{Nilp}(Y; sm) := \{X \in (\mathbb{K}^{sm \times sm})^d : X - I_m \otimes Y \text{ is jointly nilpotent}\} \subseteq \text{dom}_{sm}(\tilde{R})$$

  for every $m \in \mathbb{N}$, where a tuple $Z = (Z_1, \ldots, Z_d) \in (\mathbb{K}^{sm \times sm})^d$ is called jointly nilpotent if there exists $\kappa \in \mathbb{N}$ such that $Z^{|\omega|} = 0$ for all $\omega \in G_d$ satisfying $|\omega| \geq \kappa$.

- $\tilde{R} |_{\text{Nilp}(Y)}$ is an nc function on the nilpotent ball around $Y$, that is,

  $$\text{Nilp}(Y) := \prod_{m=1}^{\infty} \text{Nilp}(Y; sm).$$

- Every nc function on $\text{Nilp}(Y)$ has a power series expansion around $Y$ of the form

  $$\sum_{\omega \in G_d} (X - I_m \otimes Y)^{\omega} \pi_{\omega} \mathcal{R}_\omega, \ X \in (\mathbb{K}^{sm \times sm})^d,$$
where $R_\omega$ are $|\omega|$-linear mappings from $(\mathbb{K}^{s\times s})^{|\omega|}$ to $\mathbb{K}^{s\times s}$, called the Taylor–Taylor coefficients and are uniquely determined, see [48, Theorem 5.9]; notice that the sum is actually finite.

**Lemma 2.12.** If $R$ is an nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$ and $R$ is an nc Fornasini–Marchesini realization of $R$ centred at $Y \in \text{dom}_s(R)$, of the form (2.1), then the Taylor–Taylor coefficients of $R$ are given by $R_\omega := D$ and the multilinear mappings $R_{\omega g_k} := C \cdot A^{\omega} \cdot B_k : (\mathbb{K}^{s\times s})^{l+1} \to (\mathbb{K}^{s\times s})$ which act as

$$R_{\omega g_k}(Z_1, \ldots, Z_{l+1}) = C A_{i_1}(Z_1) \cdots A_{i_k}(Z_l) B_k(Z_{l+1})$$

for $Z_1, \ldots, Z_{l+1} \in \mathbb{K}^{s\times s}, \omega = g_{i_1} \cdots g_{i_k} \in \mathcal{G}_d$ and $1 \leq k \leq d$. Moreover, if $Z_1, \ldots, Z_{l+1} \in \mathbb{K}^{s\times sm}$, then

$$(Z_1 \circ \cdots \circ Z_{l+1}) R_{\omega g_k} = (I_m \otimes C)(Z_1) A_{i_1} \cdots (Z_l) A_{i_k}(Z_{l+1}) B_k.$$

**Proof.** Let $X \in \text{Nilp}(Y; sm)$, then we use the Neumann series with respect to nilpotent elements in $(T(\mathbb{K}^{s\times s}))^{m \times m}$ — where $T(\mathbb{K}^{s\times s})$ is the tensor algebra of $\mathbb{K}^{s\times s}$ — to obtain that

$$R(X) = R(Y) = (I_m \otimes D) + (I_m \otimes C) \left( \sum_{\omega \in \mathcal{G}_d} (X - I_m \otimes Y)^{\circ \omega} A^{\omega} \right) \sum_{k=1}^d (X_k - I_m \otimes Y) B_k$$

$$= I_m \otimes D + \sum_{\omega \in \mathcal{G}_d, 1 \leq k \leq d} (X - I_m \otimes Y)^{\circ \omega} R_{\omega g_k} = \sum_{\nu \in \mathcal{G}_d} (X - I_m \otimes Y)^{\circ \nu} R_{\nu},$$

where the multilinear mappings $(R_{\omega g_k})$ are given by (2.12). However, $R \big|_{\text{Nilp}(Y)}$ is an nc function on $\text{Nilp}(Y)$ and so it has a unique Taylor–Taylor expansion, given by the coefficients $(R_{\nu})_{\nu \in \mathcal{G}_d}$. \hfill \Box

**Theorem 2.13** (Similarity of minimal realizations). Let $R_1$ and $R_2$ be two nc rational expressions in $x_1, \ldots, x_d$ over $\mathbb{K}$, which admit nc Fornasini–Marchesini realizations

$$R_1(X) = D_1^1 + C_1^1 \left( I_{L_1} - \sum_{k=1}^d A_k^{1}(X_k - Y_k) \right)^{-1} \sum_{k=1}^d B_k^{1}(X_k - Y_k)$$

and

$$R_2(X) = D_2^2 + C_2^2 \left( I_{L_2} - \sum_{k=1}^d A_k^{2}(X_k - Y_k) \right)^{-1} \sum_{k=1}^d B_k^{2}(X_k - Y_k),$$

respectively, both centred at $Y \in (\mathbb{K}^{s\times s})^d$. Assume that both $R_1$ and $R_2$ are controllable and observable.

If $R_1$ and $R_2$ are $(\mathbb{K}^d)_{nc}$-evaluation equivalent, then $R_1$ and $R_2$ are uniquely similar, that is, $L_1 = L_2, D_1 = D_2$ and there exists a unique invertible matrix $T \in \mathbb{K}^{L_1 \times L_2}$ such that

$$C_2^2 = C_1^1 T^{-1}, B_k^{2} = T \cdot B_k^{1} \quad \text{and} \quad A_k^{2} = T \cdot A_k^{1} \cdot T^{-1}, \quad 1 \leq k \leq d. \quad (2.13)$$

Moreover,

$$\text{DOM}_{sm}(R_1) = \text{DOM}_{sm}(R_2) \quad \text{and} \quad R_1(X) = R_2(X), \quad \forall X \in \text{DOM}_{sm}(R_1).$$
for every \( m \in \mathbb{N} \), and for any unital \( \mathbb{K} \)-algebra \( A \):

\[
\text{DOM}^A(R_1) = \text{DOM}^A(R_2) \quad \text{and} \quad \text{DOM}^A(R_1) = \text{DOM}^A(R_2), \quad \forall \mathcal{A} \in \text{DOM}^A(R_1).
\]

**Proof.** From Lemma 2.12, the Taylor–Taylor coefficients of the nc rational expressions \( R_1 \) and \( R_2 \) (with respect to the centre \( \mathcal{Y} \)) are

\[
R_{\omega g_k}^{(1)} = C^1 \cdot (A^1)_{\omega} \cdot B^1_k \quad \text{and} \quad R_{\omega g_k}^{(2)} = C^2 \cdot (A^2)_{\omega} \cdot B^2_k,
\]

respectively. Since \( R_1 \) and \( R_2 \) are \((\mathbb{K}^d)_{\omega}\)-evaluation equivalent, their restrictions to \( \text{Nilp}(\mathcal{Y}) \) produce the same nc function, and therefore, by the uniqueness of the Taylor–Taylor coefficients, \( R_{\omega}^{(1)} = R_{\omega}^{(2)} \) and \( R_{\omega g_k}^{(1)} = R_{\omega g_k}^{(2)} \) as multilinear mappings for every \( \omega \in \mathcal{G}_d \) and \( 1 \leq k \leq \ell \), that is, \( D^1 = D^2 \) and

\[
C^1 \cdot (A^1)_{\omega} \cdot B^1_k = C^2 \cdot (A^2)_{\omega} \cdot B^2_k. \tag{2.14}
\]

Define a mapping \( T \) in the following way: for every \( \omega \in \mathcal{G}_d \), \( 1 \leq k \leq \ell \), \( X = (X_1, \ldots, X_{|\omega|}) \in (\mathbb{K}^{s \times s})^d \), \( X_{|\omega|+1} \in \mathbb{K}^{s \times s} \) and \( u \in \mathbb{K}^s \), let

\[
T((A^1)_{\omega}(X))^1_k(X_{|\omega|+1})u := (A^2)_{\omega}(X)^2_k(X_{|\omega|+1})u \tag{2.15}
\]

and extend it by linearity. We proceed by showing some properties of \( T \).

- **The domain of \( T \) is \( \mathbb{K}^{L_1} \):** The domain of \( T \) consists of all the vectors in \( \mathbb{K}^{L_1} \) which are in

\[
\bigwedge_{\omega \in \mathcal{G}_d, X \in (\mathbb{K}^{s \times s})^{\omega}, X_{|\omega|+1} \in \mathbb{K}^{s \times s}, 1 \leq k \leq \ell, \ u \in \mathbb{K}^s}
\]

and that is exactly \( C_{A^1 B^1} \mathbb{K}^{L_1} \), by the controllability of \( R_1 \) and hence of \( (A^1, B^1) \).

- **\( T \) is well defined:** Let \( w_1, w_2 \in \mathbb{K}^{L_1} \), then they can be written as

\[
w_1 = \sum_{j=1}^{p_1} (A^1)_{\omega^1_{1,j}} (X_{j,1}^{(1)}, \ldots, X_{j,|\omega|_{1,j}}^{(1)}) B^1_{k_{1,j}} (X_{j,|\omega|_{1,j}+1}^{(1)}) u_{1,j}
\]

and

\[
w_2 = \sum_{i=1}^{p_2} (A^1)_{\omega^2_{i}} (X_{i,1}^{(2)}, \ldots, X_{i,|\omega|_{2,i}}^{(2)}) B^1_{k_{2,i}} (X_{i,|\omega|_{2,i}+1}^{(2)}) u_{2,i},
\]

where \( p_1, p_2 \in \mathbb{N}, \omega_{1,j}, \omega_{2,i} \in \mathcal{G}_d, 1 \leq k_{1,j}, k_{2,i} \leq \ell, u_{1,j}, u_{2,i} \in \mathbb{K}^s \) and \( X_{j,\alpha}^{(1)} \), \( X_{i,\beta}^{(2)} \in \mathbb{K}^{s \times s} \), for every \( 1 \leq \alpha \leq |\omega_{1,j}|, 1 \leq \beta \leq |\omega_{2,i}|, 1 \leq j \leq p_1 \) and \( 1 \leq i \leq p_2 \). Apply (2.14), so for every \( \omega \in \mathcal{G}_d, \)

\[
C^1 \cdot (A^1)_{\omega} \cdot (A^1)_{\omega^1_{1,j}} \cdot B^1_{k_{1,j}} = C^2 \cdot (A^2)_{\omega} \cdot (A^2)_{\omega^1_{1,j}} \cdot B^2_{k_{1,j}}
\]

and

\[
C^1 \cdot (A^1)_{\omega} \cdot (A^1)_{\omega^2_{i}} \cdot B^1_{k_{2,i}} = C^2 \cdot (A^2)_{\omega} \cdot (A^2)_{\omega^2_{i}} \cdot B^2_{k_{2,i}},
\]

which imply that for every \( X \in (\mathbb{K}^{s \times s})^{\omega}, \)

\[
C^2(A^2)_{\omega}(X) \left[ \sum_{j=1}^{p_1} (A^2)_{\omega^1_{1,j}} (X_{j,1}^{(1)}, \ldots, X_{j,|\omega|_{1,j}}^{(1)}) B^2_{k_{1,j}} (X_{j,|\omega|_{1,j}+1}^{(1)}) u_{1,j} \right] - C^2(A^2)_{\omega}(X) \left[ \sum_{i=1}^{p_2} (A^2)_{\omega^2_{i}} (X_{i,1}^{(2)}, \ldots, X_{i,|\omega|_{2,i}}^{(2)}) B^2_{k_{2,i}} (X_{i,|\omega|_{2,i}+1}^{(2)}) u_{2,i} \right]
\]
for all $= \in \\mathbb{R}$.

Finally, if $w_1 = w_2$, then $C^2(\mathbb{A}^2)^\omega(X)(T(w_1) - T(w_2)) = 0$ for all $\in \mathbb{G}_d$ and $X \in (\mathbb{K}^{s \times s})^{\omega}$, whereas the observability of $\mathcal{R}_2$ and hence of $(C^2, \mathbb{A}^2)$ guarantees that $T(w_1) = T(w_2)$.

- $T$ is 1-1: If $w_1, w_2 \in \mathbb{K}^{L_1}$ such that $T(w_1) = T(w_2)$, it follows from (2.16) that $C^1(\mathbb{A}^1)^\omega(X)(w_1 - w_2) = 0$ for every $\in \mathbb{G}_d$ and $X \in (\mathbb{K}^{s \times s})^{\omega}$, whereas the observability of $\mathcal{R}_1$ and hence of $(C^1, \mathbb{A}^1)$ guarantees that $w_1 = w_2$.

- $T$ is onto $\mathbb{K}^{L_2}$: The tuple $(\mathbb{A}^2, \mathbb{B}^2)$ is controllable and hence every $\in \mathbb{K}^{L_2}$ can be written as

$$\epsilon = \sum_{j=1}^{p} (\mathbb{A}^2)^\omega_j (X((j))) \mathbb{B}^2_{kj}(X(j+1)u_j) = T\left(\sum_{j=1}^{p} (\mathbb{A}^1)^\omega_j (X((j))) \mathbb{B}^1_{kj}(X(j+1)u_j)\right),$$

where $p \in \mathbb{N}$, $u_j \in \mathbb{G}_d$, $X((j)) \in (\mathbb{K}^{s \times s})^{\omega}$, $X(j+1) \in \mathbb{K}^{s \times s}$, $1 \leq j \leq d$ and $u_j \in \mathbb{K}^s$ for every $1 \leq j \leq p$, that is, $\mathbb{K}^{L_2} \subseteq \text{Im}(T)$ and hence $T$ is onto $\mathbb{K}^{L_2}$.

Therefore, $T : \mathbb{K}^{L_1} \rightarrow \mathbb{K}^{L_2}$ is an isomorphism, $L_1 = L_2 := L$, the representative matrix $T := [T]_{\epsilon_1 \epsilon_2} \in \mathbb{K}^{L \times L}$ is invertible and $T(w) = Tw$ for all $w \in \mathbb{K}^{L}$.

- The realizations $\mathcal{R}_1$ and $\mathcal{R}_2$ are similar: If $w_1 = 0$ and $\omega = 0$, then (2.16) implies

$$C^2Tw_2 = C^1w_1, \forall w_2 \in \mathbb{K}^L \implies C^1 = C^2T,$$

while applying (2.16) again with $w_1 = 0$ and using the observability of $(C^2, \mathbb{A}^2)$, lead to

$$A^2_k(X) = T^{-1}A^2_k(X)T, \forall X \in \mathbb{K}^{s \times s}, 1 \leq k \leq d.$$
• Finally, the relations in (2.13) also imply that \( C^2 \otimes 1_A = (C^1 \otimes 1_A)(T^{-1} \otimes 1_A) \) and that for every \( \mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) \in (\mathcal{A}^{s \times s})^d \), we have

\[
(\mathfrak{A}_k)(B_k^2)^A = (T \otimes 1_A)(\mathfrak{A}_k)(B_k^1)^A \quad \text{and} \quad (\mathfrak{A}_k)(A_k^2)^A = (T \otimes 1_A)(\mathfrak{A}_k)(A_k^1)^A(T^{-1} \otimes 1_A)
\]

for every \( 1 \leq k \leq d \). Therefore,

\[
I_L \otimes 1_A - \sum_{k=1}^{d} (\mathfrak{A}_k - Y_k \otimes 1_A)(A_k^2)^A
\]

\[
= I_L \otimes 1_A - \sum_{k=1}^{d} \left[ (T \otimes 1_A)(\mathfrak{A}_k - Y_k \otimes 1_A)(A_k^1)^A(T^{-1} \otimes 1_A) \right]
\]

\[
= (T \otimes 1_A) \left( I_L \otimes 1_A - \sum_{k=1}^{d} (\mathfrak{A}_k - Y_k \otimes 1_A)(A_k^1)^A \right)(T \otimes 1_A)^{-1},
\]

which implies that \( DOM^A(\mathcal{R}_1) = DOM^A(\mathcal{R}_2) \) and that \( \mathcal{R}^A_1(\mathfrak{A}) = \mathcal{R}^A_2(\mathfrak{A}) \) for every \( \mathfrak{A} \in DOM^A(\mathcal{R}_1) \). □

**Remark 2.14.** One can obtain the equality in (2.14) and hence prove Theorem 2.13 without using Lemma 2.12, by evaluating nc rational expressions on generic matrices and then considering power series in commuting variables (the entries of the generic matrices).

### 2.4. Kalman decomposition

We proceed next to obtain a Kalman decomposition for nc Fornasini–Marchesini realizations centred at a matrix point \( Y \in (\mathbb{K}^{s \times s})^d \), which generalizes the Kalman decomposition for nc Fornasini–Marchesini realizations centred at a scalar point (as in [8]), where the later decomposition is a generalization of the classical Kalman decomposition (see [13, 50]).

This is the first place in our analysis where \( \mathcal{A} \) is no longer an arbitrary unital \( \mathbb{K} \)-algebra, but has to be stably finite. The stably finiteness is used to deduce the invertibility of one of the blocks in a block upper triangular matrix that is invertible (cf. Lemma 1.3).

**Theorem 2.15** (Kalman decomposition). Let

\[
\mathcal{R}(\mathfrak{X}) = D + C \left( I_L - \sum_{k=1}^{d} A_k(X_k - Y_k) \right)^{-1} \sum_{k=1}^{d} B_k(X_k - Y_k)
\]

be an nc Fornasini–Marchesini realization centred at \( Y \in (\mathbb{K}^{s \times s})^d \). There exists an nc Fornasini–Marchesini realization \( \tilde{\mathcal{R}} \) centred at \( \tilde{Y} \), that is controllable and observable, of dimension \( \tilde{L} = \dim(\mathcal{C}_{\mathcal{A}, \mathcal{B}}) - \dim(\mathcal{C}_{\mathcal{A}, \mathcal{B}} \cap \mathcal{NOC}_{\mathcal{A}}) \), such that

\[
DOM_{sm}(\mathcal{R}) \subseteq DOM_{sm}(\tilde{\mathcal{R}}) \quad \text{and} \quad \mathcal{R}(\mathfrak{X}) = \tilde{\mathcal{R}}(\mathfrak{X}), \forall \mathfrak{X} \in DOM_{sm}(\mathcal{R})
\]

for every \( m \in \mathbb{N} \), and for any unital stably finite \( \mathbb{K} \)-algebra \( \mathcal{A} \),

\[
DOM^A(\mathcal{R}) \subseteq DOM^A(\tilde{\mathcal{R}}) \quad \text{and} \quad \mathcal{R}^A(\mathfrak{A}) = \tilde{\mathcal{R}}^A(\mathfrak{A}), \forall \mathfrak{A} \in DOM^A(\mathcal{R}).
\]

As the proof shows (see (2.19) below), \( \tilde{\mathcal{R}} \) is obtained from \( \mathcal{R} \) analogously to the classical case, by restricting to a joint invariant subspace of the operators \( \mathcal{A}_1, \ldots, \mathcal{A}_d \) and then compressing to a co-invariant subspace.
Proof. Using the controllability and un-observability subspaces of $\mathbb{K}^L$, which correspond to $(A, B)$ and $(C, A)$, define $C := C_{A,B}, N/O := N/O_{C,A}$,
\[ H_1 := N/O \cap C, \tag{2.17} \]
$H_1$ is a complementary subspace of $H_1$ in $C$ and $H_3$ is a complementary subspace of $H_1$ in $N/O$; thus,
\[ C = H_2 + H_1 \text{ and } N/O = H_1 + H_3. \tag{2.18} \]
If $h_1 + h_2 + h_3 = 0$ where $h_1 \in H_1, h_2 \in H_2$ and $h_3 \in H_3$, then $h_1 + h_2 = -h_3 \in C \cap N/O \cap H_3 = H_1 \cap H_3 = \{ 0 \}$, which implies that $h_1 + h_2 = -h_3 = 0$ and hence $h_1 = h_2 = h_3 = 0$. Therefore, the sum $H_1 + H_2 + H_3$ is a direct sum; let $H_4$ be a complementary subspace of $H_1 + H_2 + H_3$ in $\mathbb{K}^L$, thus
\[ \mathbb{K}^L = H_2 + H_1 + H_4 + H_3. \tag{2.19} \]
Notice that we are interested in $H_4$ as it is a subspace of a controllable (invariant) subspace and a complementary subspace of an un-observable subspace. Define the dimensions $L_j := \dim(H_j)$ for $1 \leq j \leq 4$, so $L = L_1 + \cdots + L_4$. From Proposition 2.8 we have $N/O^{(m)} = N/O \otimes \mathbb{K}^m$ and $C^{(m)} = C \otimes \mathbb{K}^m$ for every $m \in \mathbb{N}$, hence
\[ H_1^{(m)} := N/O^{(m)} \cap C^{(m)} = (N/O \otimes \mathbb{K}^m) \cap (C \otimes \mathbb{K}^m) = H_1 \otimes \mathbb{K}^m. \]
Define $H_2^{(m)} := H_2 \otimes \mathbb{K}^m, H_3^{(m)} := H_3 \otimes \mathbb{K}^m$ and $H_4^{(m)} := H_4 \otimes \mathbb{K}^m$, thus
\[ H_2^{(m)} + H_1^{(m)} = C^{(m)}, H_1^{(m)} + H_3^{(m)} = N/O^{(m)} \]
and
\[ \mathbb{K}^{Lm} = H_2^{(m)} + H_1^{(m)} + H_4^{(m)} + H_3^{(m)}. \tag{2.20} \]

- With respect to the decomposition (2.19) of $\mathbb{K}^L$, there exists $P \in \mathbb{K}^{L \times L}$ invertible such that for every $X \in \mathbb{K}^{sm \times sm}$ and $1 \leq k \leq d$, the matrices $(X)A_k \in \mathbb{K}^{Lm \times Lm}$, $(X)B_k \in \mathbb{K}^{Lm \times sm}$ and $I_m \otimes C \in \mathbb{K}^{sm \times Lm}$ can be decomposed, with respect to (2.20), as
\[
\begin{pmatrix} P^{(m)} \end{pmatrix}^{-1} (X)A_k P^{(m)} = (X) \begin{bmatrix} A_{1,k}^{1,1} & A_{1,k}^{1,2} & A_{1,k}^{1,3} & A_{1,k}^{1,4} \\ A_{2,k}^{1,1} & A_{2,k}^{1,2} & A_{2,k}^{1,3} & A_{2,k}^{1,4} \\ A_{3,k}^{1,1} & A_{3,k}^{1,2} & A_{3,k}^{1,3} & A_{3,k}^{1,4} \\ A_{4,k}^{1,1} & A_{4,k}^{1,2} & A_{4,k}^{1,3} & A_{4,k}^{1,4} \end{bmatrix},
\]
and $(I_m \otimes C)P^{(m)} = \begin{bmatrix} C^1 & C^2 & C^3 & C^4 \end{bmatrix}$,
where $P^{(m)} := I_m \otimes P$. If $u_1 \in H_1^{(m)}, u_2 \in H_2^{(m)}, u_3 \in H_3^{(m)}$ and $u \in \mathbb{K}^{sm}$, then
\[
(X)A_k u_1 \in N/O^{(m)} \cap C^{(m)} = H_1^{(m)}, (X)A_k u_2 \in C^{(m)}, (X)A_k u_3 \in N/O^{(m)},
\]
\[
(X)B_k u \in C^{(m)} \text{ and } (I_m \otimes C)u_1 = \emptyset,
\]
which imply that
\[
(X)A_{1,k}^{1,2}, (X)A_{1,k}^{2,2}, (X)A_{1,k}^{3,2}, (X)A_{1,k}^{4,2}, (X)A_{1,k}^{3,1}, (X)A_{1,k}^{4,1}, (X)A_{1,k}^{1,4}, (X)A_{1,k}^{3,4}, (X)B_k^1, (X)B_k^4, C^2, C^4
\]
all vanish. Therefore, we get

\[
(P^{(m)})^{-1}(X)A_k P^{(m)} = (X) \begin{bmatrix}
A_k^{1,1} & 0 & A_k^{1,3} & 0 \\
A_k^{2,1} & A_k^{2,2} & A_k^{2,3} & A_k^{2,4} \\
0 & 0 & A_k^{3,3} & 0 \\
0 & 0 & A_k^{4,3} & A_k^{4,4}
\end{bmatrix}, \quad (P^{(m)})^{-1}(X)B_k = (X) \begin{bmatrix}
B_k^1 \\
B_k^2 \\
0 \\
0
\end{bmatrix}
\]

and \((I_m \otimes C)P^{(m)} = \begin{bmatrix} C^1 & 0 & C^3 & 0 \end{bmatrix}^T\), hence for every \(X \in DOM_{sm}(R)\),

\[
\mathcal{R}(X) = I_m \otimes D + (I_m \otimes C)P^{(m)} \left( I_{L,m} - \sum_{k=1}^d \left( (P^{(m)})^{-1}(X_k - I_m \otimes Y_k)A_k P^{(m)} \right) \right)^{-1} \times \sum_{k=1}^d \left( (P^{(m)})^{-1}(X_k - I_m \otimes Y_k)B_k \right)
\]

\[
= I_m \otimes D + \begin{bmatrix} C^1 & 0 & C^3 & 0 \end{bmatrix} \begin{bmatrix}
\lambda_{1,1} & 0 & \lambda_{1,3} & 0 \\
\lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} \\
0 & 0 & \lambda_{3,3} & 0 \\
0 & 0 & \lambda_{4,3} & \lambda_{4,4}
\end{bmatrix}^{-1} \sum_{k=1}^d (X_k - I_m \otimes Y_k) \begin{bmatrix}
B_k^1 \\
B_k^2 \\
0 \\
0
\end{bmatrix},
\]

where

\[
\Lambda_X := \begin{bmatrix}
\lambda_{11} & 0 & \lambda_{13} & 0 \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\
0 & 0 & \lambda_{33} & 0 \\
0 & 0 & \lambda_{43} & \lambda_{44}
\end{bmatrix} = I_{L,m} - \sum_{k=1}^d (X_k - I_m \otimes Y_k) \begin{bmatrix}
A_k^{1,1} & 0 & A_k^{1,3} & 0 \\
A_k^{2,1} & A_k^{2,2} & A_k^{2,3} & A_k^{2,4} \\
0 & 0 & A_k^{3,3} & 0 \\
0 & 0 & A_k^{4,3} & A_k^{4,4}
\end{bmatrix}
\]

is invertible. Thus

\[
\det(\lambda_{11}) = \det \left( I_{L,m} - \sum_{k=1}^d (X_k - I_m \otimes Y_k)A_k^{1,1} \right) \neq 0, \quad (2.21)
\]

where \(\bar{L} = L_2 = \dim(C) = \dim(N \mathcal{O} \cap C)\) and the inverse of \(\Lambda_X\) is given by

\[
\Lambda_X^{-1} = \begin{bmatrix}
\lambda_{11}^{-1} & 0 & 0 & 0 \\
-\lambda_{22}^{-1}\lambda_{21}\lambda_{11}^{-1} & \lambda_{22}^{-1} & 0 & 0 \\
0 & 0 & -\lambda_{33}^{-1}\lambda_{23} + \lambda_{21}\lambda_{11}^{-1}\lambda_{33}^{-1} & 0 \\
0 & 0 & -\lambda_{43}^{-1}\lambda_{33}^{-1} & \lambda_{44}^{-1}
\end{bmatrix}.
\]

Therefore, for every \(X \in DOM_{sm}(R)\), we have

\[
\mathcal{R}(X) = \mathcal{R}(\bar{X}) := I_m \otimes D + (I_m \otimes C^1) \left( I_{\bar{L},m} - \sum_{k=1}^d (X_k - I_m \otimes Y_k)A_k^{1,1} \right)^{-1} \times \sum_{k=1}^d (X_k - I_m \otimes Y_k)B_k^1,
\]

where \(\mathcal{R}\) is the nc Fornasini–Marchesini realization described by \((\bar{L}, D, C^1, A_k^{1,1}, B_k^1)\) and centred at \(\bar{X}_k\), and (2.21) implies that \(X \in DOM_{sm}(\mathcal{R})\).
Next, let \( \mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) \in (A^{s \times s})^d \) and write \( \mathfrak{A}_k = \sum_{i,j=1}^s E_{ij} \otimes a_{ij}^{(k)} \), where \( E_{ij} \in \mathcal{E}_s \) and \( a_{ij}^{(k)} \in A \), then

\[
I_L \otimes 1_A - \sum_{k=1}^d (\mathfrak{A}_k - Y_k \otimes 1_A) A_k^A
\]

\[
= I_L \otimes 1_A - \sum_{k=1}^d \left[ \sum_{i,j=1}^s A_k(E_{ij}) \otimes a_{ij}^{(k)} - A_k(Y_k) \otimes 1_A \right]
\]

\[
= I_L \otimes 1_A - \sum_{k=1}^d \sum_{i,j=1}^s \left[ P \begin{bmatrix} A_{k,1}^{1,1} & 0 & A_{k,2}^{1,3} & 0 \\ A_{k,2,1}^{2,1} & A_{k,2,2}^{2,2} & A_{k,2,3}^{2,3} & A_{k,2,4}^{2,4} \\ 0 & 0 & A_{k,3,3}^{3,3} & 0 \\ 0 & 0 & 0 & A_{k,4,4}^{4,4} \end{bmatrix} (E_{ij})P^{-1} \right] \otimes a_{ij}^{(k)}
\]

\[
- \sum_{k=1}^d \left[ P \begin{bmatrix} A_{k,1}^{1,1} & 0 & A_{k,2}^{1,3} & 0 \\ A_{k,2,1}^{2,1} & A_{k,2,2}^{2,2} & A_{k,2,3}^{2,3} & A_{k,2,4}^{2,4} \\ 0 & 0 & A_{k,3,3}^{3,3} & 0 \\ 0 & 0 & 0 & A_{k,4,4}^{4,4} \end{bmatrix} (Y_k)P^{-1} \right] \otimes 1_A
\]

\[
= (P \otimes 1_A) - \sum_{k=1}^d \left[ \lambda_{1,A} \begin{bmatrix} 0 & * & 0 \\ * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix} \right] (P \otimes 1_A)^{-1},
\]

where

\[
\lambda_{1,A} := I_L \otimes 1_A - \sum_{k=1}^d \left[ \sum_{i,j=1}^s A_k^{1,1}(E_{ij}) \otimes a_{ij}^{(k)} - A_k^{1,1}(Y_k) \otimes 1_A \right].
\]

Therefore, if \( \mathfrak{A} \in DOM^A(\mathcal{R}) \), then

\[
(P \otimes 1_A)^{-1} \left( I_L \otimes 1_A - \sum_{k=1}^d (\mathfrak{A}_k - Y_k \otimes 1_A) A_k^A \right) (P \otimes 1_A) = \begin{bmatrix} \lambda_{1,A} & 0 & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}
\]

is invertible in \( A^{\tilde{L} \times \tilde{L}} \), while applying Lemma 1.3 to obtain that

\[
\lambda_{1,A} = I_L \otimes 1_A - \sum_{k=1}^d (\mathfrak{A}_k - Y_k \otimes 1_A) \left( A_k^{1,1} \right)^A
\]

is invertible in \( A^{\tilde{L} \times \tilde{L}} \), that is, that \( \mathfrak{A} \in DOM^A(\tilde{\mathcal{R}}) \). Moreover, a similar computation shows that if \( \mathfrak{A} \in DOM^A(\mathcal{R}) \), then \( \mathcal{R}^A(\mathfrak{A}) = \tilde{\mathcal{R}}^A(\mathfrak{A}) \).

• For every \( u \in \mathbb{K}^s \), \( 1 \leq k \leq d \), a word \( \omega = g_i_1 \ldots g_i_\ell \in G_d \) of length \( \ell \in \mathbb{N} \) and \( X_1, \ldots, X_{\ell+1} \in \mathbb{K}^{s \times s} \), we have

\[
A_\omega(X_1, \ldots, X_\ell)B_k(X_{\ell+1})u = P(P^{-1}A_{i_1}(X)P) \cdots (P^{-1}A_{i_\ell}(X)P)P^{-1}B_k(X_{\ell+1})u
\]

\[
= P \begin{bmatrix} (A_k)^{1,1} \omega(X_1, \ldots, X_\ell)B_k^*(X_{\ell+1})u \\ 0 \\ 0 \end{bmatrix}.
\]
Thus, for every \( \nu \in \mathcal{C} \), we have \( P^{-1}\nu \in \tilde{\mathcal{C}} \oplus K^{L_1} \oplus \{0\} \oplus \{0\} \subseteq K^L \) and hence \( \dim(\mathcal{C}) \leq \dim(\tilde{\mathcal{C}}) + L_1 \). As \( \dim(\mathcal{C}) = L_1 + L_2 \), we get that \( L_2 \leq \dim(\tilde{\mathcal{C}}) \), while \( \tilde{\mathcal{C}} \subseteq K^{L_2} \) yields that \( \tilde{\mathcal{C}} = K^{L_2} = K^L \), that is, the realization \( \tilde{R} \) is controllable.

- For every \( \nu_1 \in K^{L_1}, \nu_2 \in K^{L_2}, \nu_3 \in K^{L_3} \), a word \( \omega = g_1 \ldots g_{i_\ell} \in \mathcal{G}_d \) of length \( \ell \in \mathbb{N} \) and \( X_1, \ldots, X_\ell \in \mathbb{K}^{\times \times s} \), we have

\[
C A^\omega(X_1, \ldots, X_\ell)P \begin{bmatrix} w_2^T & w_1^T & 0 & w_3^T \end{bmatrix}^T = CP \begin{bmatrix} A_{i_1}^{(1)}(X_1) & \cdots & A_{i_\ell}^{(1)}(X_\ell) \end{bmatrix} \begin{bmatrix} w_2 & w_1 & 0 \end{bmatrix}^T
\]

where \( \omega \in \widetilde{\mathcal{O}} := \overline{\mathcal{O}}_{C,1} A^{1:1} \) implies that \( P \begin{bmatrix} w_2^T & w_1^T & 0 & w_3^T \end{bmatrix}^T \in \mathcal{O} \). As \( P \) is invertible, \( \dim(\widetilde{\mathcal{O}}) \geq \dim(\widetilde{\mathcal{O}}) + L_1 + L_3 = \dim(\widetilde{\mathcal{O}}) + \dim(\mathcal{O}) \) which guarantees that \( \widetilde{\mathcal{O}} = \{0\} \), that is, that \( \tilde{R} \) is observable.

As a corollary we get that an nc Fornasini–Marchesini realization (of an nc rational expression) is minimal if and only if it is both controllable and observable:

**Theorem 2.16.** Let \( R \) be an nc rational expression in \( x_1, \ldots, x_d \) over \( \mathbb{K} \) and \( \mathcal{R} \) be an nc Fornasini–Marchesini realization of \( R \) centred at \( \mathcal{Y} \in (\mathbb{K}^{\times g})^d \). Then \( \mathcal{R} \) is minimal if and only if \( \mathcal{R} \) is controllable and observable.

**Proof.** If \( \mathcal{R} \) is minimal, then using Theorem 2.15 we must have

\[
L \leq \tilde{L} = \dim(C^{(1)}) - \dim(C^{(1)}) \cap \mathcal{O}^{(1)} \leq L,
\]

which implies that \( \tilde{L} = L, \dim(C^{(1)}) = L \) and \( \dim(N\mathcal{O}^{(1)}) = 0 \), that is, that \( \mathcal{R} \) is controllable and observable.

On the other hand, let \( \mathcal{R} \) be both controllable and observable, and suppose that \( \mathcal{R}' \) is a minimal nc Fornasini–Marchesini realization of \( R \) centred at \( \mathcal{Y} \) of dimension \( L' \). Thus, by the first part of the theorem, \( \mathcal{R}' \) is controllable and observable, as well as \( \mathcal{R} \), so Theorem 2.13 implies that \( L = L' \) and hence \( \mathcal{R} \) is minimal.

Minimal nc Fornasini–Marchesini realizations are playing a central role in the analysis of the domains; one of the reasons is that they admit the maximal domain among all nc Fornasini–Marchesini realizations — of a given nc rational expression — which are centred at the same point:

**Lemma 2.17.** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be two nc Fornasini–Marchesini realizations of an nc rational expression \( R \), both centred at \( \mathcal{Y} \in (\mathbb{K}^{\times s})^d \). If \( \mathcal{R}_2 \) is minimal, then

\[
\text{DOM}_{\text{sm}}(\mathcal{R}_1) \subseteq \text{DOM}_{\text{sm}}(\mathcal{R}_2) \quad \text{and} \quad \mathcal{R}_1(\underline{X}) = \mathcal{R}_2(\underline{X}), \forall \underline{X} \in \text{DOM}_{\text{sm}}(\mathcal{R}_1)
\]

for every \( m \in \mathbb{N} \), and for any unital stably finite \( \mathbb{K} \)-algebra \( \mathcal{A} \):

\[
\text{DOM}^A(\mathcal{R}_1) \subseteq \text{DOM}^A(\mathcal{R}_2) \quad \text{and} \quad \mathcal{R}_1^A(\underline{\mathcal{A}}) = \mathcal{R}_2^A(\underline{\mathcal{A}}), \forall \underline{\mathcal{A}} \in \text{DOM}^A(\mathcal{R}_1).
\]

**Proof.** Applying Theorem 2.15 for the nc Fornasini–Marchesini realization \( \mathcal{R}_1 \), there exists a minimal nc Fornasini–Marchesini realization \( \tilde{\mathcal{R}}_1 \) centred at \( \mathcal{Y} \) for which

\[
\text{DOM}_{\text{sm}}(\mathcal{R}_1) \subseteq \text{DOM}_{\text{sm}}(\tilde{\mathcal{R}}_1) \quad \text{and} \quad \mathcal{R}_1(\underline{X}) = \tilde{\mathcal{R}}_1(\underline{X}), \forall \underline{X} \in \text{DOM}_{\text{sm}}(\mathcal{R}_1)
\]
for every \( m \in \mathbb{N} \), and for any unital stably finite \( \mathbb{K} \)-algebra \( 
abla \):
\[
\text{DOM}^\mathbb{A}(\mathcal{R}_1) \subseteq \text{DOM}^\mathbb{A}(\tilde{\mathcal{R}}_1) \quad \text{and} \quad \mathcal{R}_1^\mathbb{A}(\mathfrak{A}) = \tilde{\mathcal{R}}_1^\mathbb{A}(\mathfrak{A}), \ \forall \mathfrak{A} \in \text{DOM}^\mathbb{A}(\mathcal{R}_1).
\]
In particular \( \tilde{\mathcal{R}}_1 \) is an nc Fornasini–Marchesini realization of \( R \). As both \( \mathcal{R}_2 \) and \( \tilde{\mathcal{R}}_1 \) are minimal nc Fornasini–Marchesini realizations of \( R \), both centred at \( \mathcal{Y} \), Theorem 2.13 implies that
\[
\text{DOM}_{sm}(\tilde{\mathcal{R}}_1) = \text{DOM}_{sm}(\mathcal{R}_2) \quad \text{and} \quad \tilde{\mathcal{R}}_1(\mathfrak{X}) = \mathcal{R}_2(\mathfrak{X}), \ \forall \mathfrak{X} \in \text{DOM}_{sm}(\tilde{\mathcal{R}}_1)
\]
and
\[
\text{DOM}^A(\tilde{\mathcal{R}}_1) = \text{DOM}^A(\mathcal{R}_2) \quad \text{and} \quad \tilde{\mathcal{R}}_1^A(\mathfrak{A}) = \mathcal{R}_2^A(\mathfrak{A}), \ \forall \mathfrak{A} \in \text{DOM}^A(\tilde{\mathcal{R}}_1).
\]
Therefore, \( \text{DOM}_{sm}(\mathcal{R}_1) \subseteq \text{DOM}_{sm}(\tilde{\mathcal{R}}_1) = \text{DOM}_{sm}(\mathcal{R}_2) \) and \( \mathcal{R}_1(\mathfrak{X}) = \tilde{\mathcal{R}}_1(\mathfrak{X}) = \mathcal{R}_2(\mathfrak{X}) \) for every \( \mathfrak{X} \in \text{DOM}_{sm}(\mathcal{R}_1) \). Moreover, \( \text{DOM}^A(\mathcal{R}_1) \subseteq \text{DOM}^A(\tilde{\mathcal{R}}_1) = \text{DOM}^A(\mathcal{R}_2) \) and \( \mathcal{R}_1^A(\mathfrak{A}) = \tilde{\mathcal{R}}_1^A(\mathfrak{A}) = \mathcal{R}_2^A(\mathfrak{A}) \) for every \( \mathfrak{A} \in \text{DOM}^A(\mathcal{R}_1) \).

The following is a summary of all the main results in this subsection.

**Corollary 2.18.** If \( R \) is an nc rational expression in \( x_1, \ldots, x_d \) over \( \mathbb{K} \) and \( \mathcal{Y} \in \text{dom}_s(R) \subseteq (\mathbb{K}^{s \times s})^d \), then \( R \) admits a unique (up to unique similarity) minimal nc Fornasini–Marchesini realization centred at \( \mathcal{Y} \) that is also a realization of \( R \) with respect to any unital stably finite \( \mathbb{K} \)-algebra.

Moreover, any minimal nc Fornasini–Marchesini realization of \( R \) centred at \( \mathcal{Y} \) is a realization of \( R \) with respect to any unital stably finite \( \mathbb{K} \)-algebra.

**Proof.** From Theorem 2.4, \( R \) admits an nc Fornasini–Marchesini realization \( \mathcal{R} \) centred at \( \mathcal{Y} \) that is also an nc Fornasini–Marchesini realization of \( R \) with respect to any unital stably finite \( \mathbb{K} \)-algebra \( \nabla \), while Theorem 2.15 guarantees the existence of a minimal nc Fornasini–Marchesini realization \( \tilde{\mathcal{R}} \) centred at \( \mathcal{Y} \), for which
\[
\text{DOM}_{sm}(\mathcal{R}) \subseteq \text{DOM}_{sm}(\tilde{\mathcal{R}}) \quad \text{and} \quad \mathcal{R}(\mathfrak{X}) = \tilde{\mathcal{R}}(\mathfrak{X}), \ \forall \mathfrak{X} \in \text{DOM}_{sm}(\mathcal{R})
\]
for every \( m \in \mathbb{N} \), and
\[
\text{DOM}^A(\mathcal{R}) \subseteq \text{DOM}^A(\tilde{\mathcal{R}}) \quad \text{and} \quad \mathcal{R}^A(\mathfrak{A}) = \tilde{\mathcal{R}}^A(\mathfrak{A}), \ \forall \mathfrak{A} \in \text{DOM}^A(\mathcal{R}).
\]
Therefore, for every \( m \in \mathbb{N} \),
\[
\text{dom}_{sm}(\mathcal{R}) \subseteq \text{DOM}_{sm}(\tilde{\mathcal{R}}) \quad \text{and} \quad R(\mathfrak{X}) = \tilde{R}(\mathfrak{X}), \ \forall \mathfrak{X} \in \text{dom}_{sm}(\mathcal{R}),
\]
that is, \( \tilde{\mathcal{R}} \) is a realization of \( R \), while the uniqueness of \( \tilde{\mathcal{R}} \) follows from Theorem 2.13.

- Moreover, if \( \mathfrak{a} \in \text{dom}^A(\mathcal{R}) \), then \( I_s \otimes \mathfrak{a} \in \text{DOM}^A(\tilde{\mathcal{R}}) \) and \( I_s \otimes R^A(\mathfrak{a}) = R^A(I_s \otimes \mathfrak{a}) = \tilde{R}^A(I_s \otimes \mathfrak{a}) \), that is, \( \tilde{\mathcal{R}} \) is a realization of \( R \) with respect to \( \nabla \).

- Furthermore, if \( \tilde{\mathcal{R}} \) is a minimal nc Fornasini–Marchesini realization of \( R \) centred at \( \mathcal{Y} \), then \( \tilde{\mathcal{R}} \) and \( \mathcal{R} \) are both minimal nc Fornasini–Marchesini realizations of \( R \) centred at \( \mathcal{Y} \), hence by Lemma 2.13, \( \text{DOM}^\mathbb{A}(\mathcal{R}) = \text{DOM}^\mathbb{A}(\tilde{\mathcal{R}}) \) and \( \mathcal{R}^\mathbb{A}(\mathfrak{A}) = \tilde{\mathcal{R}}^\mathbb{A}(\mathfrak{A}), \ \forall \mathfrak{A} \in \text{DOM}^\mathbb{A}(\tilde{\mathcal{R}}). \)

2.5. **Example**

Consider the nc rational expression \( R(x_1, x_2) = (x_1 x_2 - x_2 x_1)^{-1} \), with \( \mathbb{K} = \mathbb{C} \), \( s = 2 \) and \( \mathcal{Y} = (Y_1, Y_2) = ((0, 1), (0, 0)) \in \text{dom}_s(R) \). We use synthesis and follow the proof of Theorem 2.4, to find an nc Fornasini–Marchesini realization of \( R \) centred at \( \mathcal{Y} \):
\( R_1(x) = x_1 \) admits an nc Fornasini–Marchesini realization centred at \( Y \) (see (2.5)), described by
\[
L_1 = 2, \quad D_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = I_2, \quad A_1^1 = A_2^1 = 0_2, \quad B_1^1 = Id_2, \quad B_2^1 = 0_2.
\]

\( R_2(x) = x_2 \) admits an nc Fornasini–Marchesini realization centred at \( Y \) (see (2.5)), described by
\[
L_2 = 2, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = I_2, \quad A_1^2 = A_2^2 = 0_2, \quad B_1^2 = 0_2, \quad B_2^2 = Id_2.
\]

\( R_3(x) = R_1(x)R_2(x) = x_1x_2 \) admits an nc Fornasini–Marchesini realization centred at \( Y \) (see (2.7)), described by
\[
L_3 = 4, \quad D_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3^3(X) = (I_2 \otimes X) \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix}, \quad A_2^3(X) = 0_4, \quad B_3^1(X) = (I_2 \otimes X) \begin{bmatrix} 1 & 0 \\ 0_2 & 0_2 \end{bmatrix}, \quad B_2^1(X) = (I_2 \otimes X) \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}.
\]

\( R_4(x) = -R_2(x)R_1(x) = -x_2x_1 \) admits an nc Fornasini–Marchesini realization centred at \( Y \) (see (2.7)), described by
\[
L_4 = 4, \quad D_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_4^4(X) = 0_4, \quad A_2^4(X) = (I_2 \otimes X) \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix}, \quad B_4^1(X) = (I_2 \otimes X) \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}.
\]

\( R_5(x) = R_3(x) + R_4(x) = x_1x_2 - x_2x_1 \) admits an nc Fornasini–Marchesini realization centred at \( Y \) (see (2.6)), described by
\[
L_5 = 8, \quad D_5 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1^5(X) = (I_4 \otimes X) \begin{bmatrix} 0_2 & I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}, \quad A_2^5(X) = (I_4 \otimes X) \begin{bmatrix} 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}, \quad B_1(X) = (I_4 \otimes X) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_2(X) = (I_4 \otimes X) \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}.
\]

\( R(x) = R_5(x)^{-1} = (x_1x_2 - x_2x_1)^{-1} \) admits an nc Fornasini–Marchesini realization \( R \) centred at \( Y \) (see (2.8)), described by
\[
L = 8, \quad D = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.
\]
\[ A_1(X) = \frac{1}{2}(I_4 \otimes X) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} I_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \]
\[ A_2(X) = \frac{1}{2}(I_4 \otimes X) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]
\[ B_1(X) = \frac{1}{2}(I_4 \otimes X) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} 2, \quad B_2(X) = \frac{1}{2}(I_4 \otimes X) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

The nc Fornasini–Marchesini realization $R$ is controllable; however, $R$ is not observable, as
\[
\mathcal{NO}_{C,A} = \bigcap_{\omega \in \mathcal{G}_2, Z_1, \ldots, Z_t \in \mathbb{C}^{2 \times 2}} \ker (C \mathcal{A}^\omega(Z_1, \ldots, Z_t)) = \text{span}\{e_1 + e_5, e_2 + e_6\},
\]
hence $R$ is not minimal. By the Kalman decomposition (see Theorem 2.15) argument, we obtain a minimal nc Fornasini–Marchesini realization of $R$ centred at $Y$ of dimension $\tilde{L} = \dim(CA) - \dim(\mathcal{NO}_{C,A} \cap \mathcal{C}_{A,B}) = 6$, that is,
\[
\tilde{R}(X_1, X_2) = \tilde{D} + \tilde{C} \left( I_6 - \tilde{A}_1(X_1 - Y_1) - \tilde{A}_2(X_2 - Y_2) \right)^{-1} \left( \tilde{B}_1(X_1 - Y_1) + \tilde{B}_2(X_2 - Y_2) \right),
\]
with
\[
\tilde{D} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \tilde{C} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & -2 & 0 \end{bmatrix},
\]
\[
\tilde{A}_1(X) = \frac{1}{2}(I_3 \otimes X) \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
\[
\tilde{A}_2(X) = \frac{1}{2}(I_3 \otimes X) \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \]
\[
\begin{align*}
\tilde{B}_1(X) &= \frac{1}{2}(I_3 \otimes X) \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad \tilde{B}_2(X) = \frac{1}{2}(I_3 \otimes X) \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}.
\end{align*}
\]

In this example, it is easy to check directly that
\[
\det \left( I_{6m} - \sum_{k=1}^{2} (X_k - I_m \otimes Y_k)\tilde{A}_k \right) \neq 0 \iff \det(X_1X_2 - X_2X_1) \neq 0
\]
for \( X = (X_1, X_2) \in (\mathbb{C}^{2m \times 2m})^2 \), that is, that \( \text{dom}_{2m}(R) = \text{DOM}_{2m}(\tilde{R}) \) and also \( R(X) = \tilde{R}(X) \). Furthermore, for any unital stably finite \( \mathbb{C} \)-algebra \( A \) and \( a = (a_1, a_2) \in A^2 \), the element \( a_1a_2 - a_2a_1 \) is invertible in \( A \) if and only if the matrix
\[
\begin{bmatrix}
I_6 \otimes 1_A - \sum_{k=1}^{2} \tilde{A}_k(I_2) \otimes a_k - \tilde{A}_k(Y_k) \otimes 1_A
\end{bmatrix}
\]
is invertible in \( A^{6 \times 6} \), that is, \( \text{dom}^A(R) = \text{DOM}^A(\tilde{R}) \) and also \( I_2 \otimes R^A(a) = R^A(I_2 \otimes a) \).

### 2.6. Cohn’s theorem

As a corollary of the results in Subsection 2.4, we get a new proof of a theorem of Cohn, stating that if two nc rational expressions represents the same nc rational function, then they are \( A \)-evaluation equivalent for any unital stably finite \( \mathbb{K} \)-algebra \( A \). Cohn’s theorem was proved originally in [24] and afterwards in his book [25, Theorem 7.3.2]. In [41], the authors proved a weaker version of the theorem, applicable only for nc rational expressions which are regular at the origin, using their theory of realizations for nc rational functions regular at the origin. We omit their assumption on regularity at the origin and prove the theorem in its full version.

**Theorem 2.19 (Cohn’s theorem).** Let \( R \) and \( \tilde{R} \) be \((\mathbb{K}^d)_{nc}\)-evaluation equivalent nc rational expressions in \( x_1, \ldots, x_d \) over \( \mathbb{K} \), that is, \( R(X) = \tilde{R}(X) \) for all \( X \in \text{dom}(R) \cap \text{dom}(\tilde{R}) \). Then \( R \) and \( \tilde{R} \) are \( A \)-evaluation equivalent for any unital stably finite \( \mathbb{K} \)-algebra \( A \), that is,
\[
R^A(a) = \tilde{R}^A(a), \quad \forall a \in \text{dom}^A(R) \cap \text{dom}^A(\tilde{R}).
\]

**Proof.** Assume that \( R \) and \( \tilde{R} \) are non-degenerate nc rational expressions, so there exists \( s \in \mathbb{N} \) such that \( \text{dom}_s(R), \text{dom}_s(\tilde{R}) \neq \emptyset \). As \( \text{dom}_s(R), \text{dom}_s(\tilde{R}) \) are Zariski open sets in \( \mathbb{K}^{s \times s} \), there exists \( \ell \in \mathbb{N} \) such that \( \text{dom}_s(R) \cap \text{dom}_s(\tilde{R}) \neq \emptyset \). The reasoning for that is clear when \( \mathbb{K} \) is infinite (with \( \ell = 1 \)), while if \( \mathbb{K} \) is finite, we use a similar argument as in [47, Remark 2.6]. Let
\[
Y = (Y_1, \ldots, Y_d) \in \text{dom}_s(R) \cap \text{dom}_s(\tilde{R}).
\]
From Corollary 2.18, \( R \) and \( \tilde{R} \) admit minimal nc Fornasini–Marchesini realizations \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \), respectively, both centred at \( Y \) with the special properties:
\[
\begin{align*}
\forall a \in \text{dom}^A(R) & \implies I_{s \ell} \otimes a \in \text{DOM}^A(\mathcal{R}) \quad \text{and} \quad I_{s \ell} \otimes R^A(a) = \mathcal{R}^A(I_{s \ell} \otimes a), \\
\forall a \in \text{dom}^A(\tilde{R}) & \implies I_{s \ell} \otimes a \in \text{DOM}^A(\tilde{\mathcal{R}}) \quad \text{and} \quad I_{s \ell} \otimes \tilde{R}^A(a) = \tilde{\mathcal{R}}^A(I_{s \ell} \otimes a).
\end{align*}
\]
Moreover, Theorem 2.13 implies that
\[
\text{DOM}^A(\mathcal{R}) = \text{DOM}^A(\tilde{\mathcal{R}}) \quad \text{and} \quad \mathcal{R}^A(a) = \tilde{\mathcal{R}}^A(a), \quad \forall a \in \text{DOM}^A(\mathcal{R}).
\]
Finally, if
\[ a \in \text{dom}^A(R) \cap \text{dom}^A(\tilde{R}), \]
then \( I \omega \times a \in \text{DOM}^A(\mathcal{R}) = \text{DOM}^A(\tilde{R}) \) and \( I \omega \times R^A(a) = R^A(I \omega \times a) = \tilde{R}^A(I \omega \times a) \), therefore \( R^A(a) = \tilde{R}^A(a) \).

**Remark 2.20.** Theorem 2.19 implies that one can evaluate any nc rational function by evaluating a minimal realization of the function. This proves that \( \mathbb{K}\langle x \rangle \) is the universal skew field of fractions of \( \mathbb{K}\langle x \rangle \), see [22, 66] for the original proofs and [47] for a modern proof. We postpone a detailed discussion and an application to an explicit construction of \( \mathbb{K}\langle x \rangle \) to [64].

### 2.7. The McMillan degree

For an nc rational expression \( R \) in \( x_1, \ldots, x_d \) over \( \mathbb{K} \) and \( Y \in \text{dom}(R) \), we define by \( L_R(Y) \) the dimension of a minimal nc Fornasini–Marchesini realization of \( R \) centred at \( Y \). The first part of the next theorem is an analogue of [82, Theorem 5.10].

**Theorem 2.21.** Let \( R \) be a non-degenerate nc rational expression in \( x_1, \ldots, x_d \) over \( \mathbb{K} \) and let \( Y \in \text{dom}_s(R) \).

1. If \( \tilde{Y} \in \text{dom}_s(R) \), then \( L_R(Y) = L_R(\tilde{Y}) \).
2. If \( n \in \mathbb{N} \), then \( I_n \times Y \in \text{dom}_s(R) \) and \( L_R(I_n \times Y) = n L_R(Y) \).
3. If \( s' \in \mathbb{N} \) and \( Y' \in \text{dom}_{s'}(R) \), then \( s' L_R(Y) = s L_R(Y') \).

**Proof.** Applying Corollary 2.18, the expression \( R \) admits a minimal nc Fornasini–Marchesini realization \( \mathcal{R} \) centred at \( Y \), described by a tuple \((L, D, C, A, B)\). Let
\[
T_1 := I_L - \sum_{k=1}^{d} A_k (\tilde{Y}_k - Y_k) \quad \text{and} \quad T_2 := \sum_{k=1}^{d} B_k (Y_k - \tilde{Y}_k),
\]
as \( \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_d) \in \text{dom}_s(R) \subseteq \text{DOM}_s(\mathcal{R}) \), the matrix \( T_1 \) is invertible. Therefore, for every \( X \in \text{DOM}_{sn}(\mathcal{R}) \) and \( m \in \mathbb{N} \):
\[
\mathcal{R}(X) = I_m \times D + (I_m \times C) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \times \tilde{Y}_k) A_k - \sum_{k=1}^{d} (I_m \times (\tilde{Y}_k - Y_k)) A_k \right)^{-1}
\]
\[
\times \left( \sum_{k=1}^{d} (X_k - I_m \times \tilde{Y}_k) B_k - \sum_{k=1}^{d} (I_m \times (Y_k - \tilde{Y}_k)) B_k \right)
\]
\[
= I_m \times D + (I_m \times CT_1^{-1}) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \times \tilde{Y}_k) \tilde{A}_k \right)^{-1}
\]
\[
\times \sum_{k=1}^{d} (X_k - I_m \times \tilde{Y}_k) B_k - (I_m \times CT_1^{-1}) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \times \tilde{Y}_k) \tilde{A}_k \right) \left( I_m \times T_2 \right),
\]
where $\tilde{A}_k = A_k \cdot T_1^{-1}$, and since
\[
\left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{A}_k \right)^{-1} (I_m \otimes T_2) \nabla I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{A}_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{B}_k,
\]
where $\tilde{B}_k = \tilde{A}_k \cdot T_2$, we conclude that
\[
\mathcal{R}(X) = I_m \otimes D - I_m \otimes (CT_1^{-1} T_2) + (I_m \otimes CT_1^{-1}) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{A}_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{A}_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes \tilde{Y}_k)\tilde{B}_k,
\]
that is, that $\mathcal{R}(X) = \tilde{R}(X)$ where $\tilde{R}$ is an nc Fornasini–Marchesini realization of $R$, centred at $\tilde{Y}$, described by\n\[
\tilde{D} = D - CT_1^{-1} T_2, \ \tilde{C} = CT_1^{-1}, \ \tilde{A}_k = A_k \cdot T_1^{-1}, \ \tilde{B}_k = B_k - A_k \cdot (T_1^{-1} T_2).
\] (2.23)
Thus, $L_{\tilde{R}}(\tilde{Y}) \subseteq L = L_{\tilde{R}}(\tilde{Y})$ and by symmetry we get that $L_{\tilde{R}}(\tilde{Y}) = L_{\tilde{R}}(\tilde{Y})$, hence $\tilde{R}$ is a minimal nc Fornasini–Marchesini realization of $R$ centred at $\tilde{Y}$.

- Suppose next that $n \in \mathbb{N}$. As $\text{dom}(R)$ is closed under direct sums, it follows that $I_n \otimes \tilde{Y} \in \text{dom}_{\text{sn}}(R)$, while for every $p \in \mathbb{N}$ letting $m = np$ yields for every $X \in \text{dom}_{\text{sn}}(R)$:
\[
\mathcal{R}(X) = I_{np} \otimes D + (I_{np} \otimes C) \left( I_{Lnp} - \sum_{k=1}^{d} (X_k - I_{np} \otimes Y_k)A_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_{np} \otimes Y_k)B_k
\]
\[
= I_p \otimes D^{(n)} + (I_p \otimes C^{(n)}) \left( I_{L^{(n)}p} - \sum_{k=1}^{d} (X_k - I_p \otimes Y_k^{(n)})A_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_p \otimes Y_k^{(n)})B_k,
\]
where
\[
L^{(n)} = L, \ D^{(n)} = I_n \otimes D, \ C^{(n)} = I_n \otimes C \text{ and } Y_k^{(n)} = I_n \otimes Y_k.
\] (2.24)
We obtained an nc Fornasini–Marchesini realization $\mathcal{R}^{(n)}$ — described by the tuple $(L^{(n)}, D^{(n)}, C^{(n)}, A, B)$ — of $R$ that is centred at $Y^{(n)} := I_n \otimes \tilde{Y}$ and it is easily seen that controllability and observability of $\mathcal{R}$ imply the controllability and observability of $\mathcal{R}^{(n)}$ as well, thus $\mathcal{R}^{(n)}$ is minimal and hence $L_{\mathcal{R}^{(n)}}(I_n \otimes \tilde{Y}) = nL_{\mathcal{R}}(\tilde{Y})$.

- Finally, let $\tilde{Y}' \in \text{dom}_{\text{sn}}(R)$, then part 2 implies that $I_{s' \otimes \tilde{Y}'}, I_s \otimes \tilde{Y}' \in \text{dom}_{\text{sn}}(R)$, $L_{\mathcal{R}}(I_{s' \otimes \tilde{Y}'}) = s'L_{\mathcal{R}}(\tilde{Y}')$ and $L_{\mathcal{R}}(I_s \otimes \tilde{Y}') = sL_{\mathcal{R}}(\tilde{Y}')$, while from part 1, $L_{\mathcal{R}}(I_s \otimes \tilde{Y}') = L_{\mathcal{R}}(I_{s' \otimes \tilde{Y}'})$, therefore $s'L_{\mathcal{R}}(\tilde{Y}') = sL_{\mathcal{R}}(\tilde{Y}')$. \hfill \□

**Remark 2.22.** In the proof of Theorem 2.21, we built explicit minimal nc Fornasini–Marchesini realizations $\tilde{R}$ and $\mathcal{R}^{(n)}$ of $R$, centred at $\tilde{Y}$ and $I_n \otimes \tilde{Y}$, respectively, using a minimal realization $\mathcal{R}$ of $R$ centred at $\tilde{Y}$. From Corollary 2.18 it follows right away that $\tilde{R}$ and $\mathcal{R}^{(n)}$ are also nc Fornasini–Marchesini realizations of $R$ with respect to any unital stably finite $K$-algebra...
Moreover, direct computations — which are omitted — easily show that
\[ \text{DOM}^A(R) = \text{DOM}^A(\overline{R}) \text{ and } R^A(\mathfrak{A}) = \overline{R}^A(\mathfrak{A}), \quad \forall \mathfrak{A} \in \text{DOM}^A(R) \]
and
\[ \mathfrak{A} \in \text{DOM}^A(R) \iff I_n \otimes \mathfrak{A} \in \text{DOM}^A(\overline{R}^{(n)}) \text{ and } I_n \otimes R^A(\mathfrak{A}) = \left( \overline{R}^{(n)} \right)^A(I_n \otimes \mathfrak{A}). \]

The first part of Theorem 2.21 guarantees that the value \( L_R(Y) \) does not depend on \( Y \) but only on \( s \), so it will be denoted as \( L_R(s) := L_R(Y), \) while from the third part of the theorem, it follows that there exists \( m(R) > 0 \) such that
\[ L_R(s) = m(R)s, \quad \forall s \geq 1, \quad (2.25) \]
where \( m(R) \) depends only on \( R \); we define \( m(R) \) as the McMillan degree of \( R \).

In the next lemma we actually show that \( m(R) \in \mathbb{N} \). This is a direct corollary and yet separated from the arguments of Theorem 2.21, as it requires a non-trivial tool from PI-ring theory, that is, if \( R \) is a non-degenerate nc rational expression, then there exists \( n \in \mathbb{N} \) such that \( \text{dom}^{k_n}(R) \neq \emptyset \) for every \( k \geq n \); see [66, Chapter 8] and [49, Remarks 2.15 and 2.16] for a more detailed discussion.

**Lemma 2.23.** If \( R \) is a non-degenerate nc rational expression in \( x_1, \ldots, x_d \) over \( \mathbb{K} \) and \( \text{dom}_s(R) \neq \emptyset \), then \( s \mid L_R(s) \) and hence \( m(R) \in \mathbb{N} \).

**Proof.** As \( \text{dom}_s(R) \neq \emptyset \), let \( Y \in \text{dom}_s(R) \) and according to Corollary 2.18, let \( \mathcal{R} \) be a minimal nc Fornasini–Marchesini realization of \( R \), centred at \( Y \). Since \( R \) is non-degenerate, there exists \( n \in \mathbb{N} \) such that \( \text{dom}^{k_n}(R) \neq \emptyset \) for all \( k \geq n \). Consider the sequence \( (k_j)_{j \geq 1} \) given by \( k_j = s_j + 1 \), clearly for \( j \) large enough we get \( k_j \geq n \) and hence \( \text{dom}^{k_n}(R) \neq \emptyset \). Let \( W \in \text{dom}^{k_n}(R) \) and apply Theorem 2.21 for \( W \) and \( Y \); we obtain that \( k_j L_R(Y) = s L_R(W) \), but it is easily seen that \( s \) and \( k_j \) are co-prime integers, hence \( s \mid L_R(Y) \) and hence \( m(R) \in \mathbb{N} \). \( \square \)

**Remark 2.24.** If \( R \) is an nc rational expression in \( x_1, \ldots, x_d \) over \( \mathbb{K} \), \( Y_1 \in \text{dom}_{s_1}(R) \) and \( Y_2 \in \text{dom}_{s_2}(R) \), then \( Y_1 \oplus Y_2 \in \text{dom}_{s_1 + s_2}(R) \) and \( (2.25) \) implies that
\[ L_R(s_1 + s_2) = (s_1 + s_2)m(R) = s_1m(R) + s_2m(R) = L_R(s_1) + L_R(s_2). \]

If \( R \) admits two minimal nc Fornasini–Marchesini realizations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), centred at \( Y_1 \) and \( Y_2 \), respectively, which are described by the tuples \( (L_1, D^1, C^1, A^1, B^1) \) and \( (L_2, D^2, C^2, A^2, B^2) \), it is very tempting to consider \( \mathcal{R} \) to be an nc Fornasini–Marchesini realization of \( R \) centred at \( Y_1 \oplus Y_2 \), where \( \mathcal{R} \) is described by
\[ D = D^1 \oplus D^2, \quad C = C^1 \oplus C^2, \quad A_k = \begin{bmatrix} A^1_k & 0 \\ 0 & A^2_k \end{bmatrix} \text{ and } B_k = \begin{bmatrix} B^1_k \\ B^2_k \end{bmatrix}, \quad 1 \leq k \leq d, \]
but we only know that \( R(X) = \mathcal{R}(X) \) whenever \( X \in \text{dom}_{s_1 + s_2}(R) \) is of the form \( X = X^{(1)} \oplus X^{(2)} \), with \( X^{(1)} \in \text{dom}_{s_1}(R_1) \) and \( X^{(2)} \in \text{dom}_{s_2}(R_2) \).

3. Realizations of NC rational functions

From the previous section (cf. Theorem 2.13) we know that — given an nc rational function — all of its minimal nc Fornasini–Marchesini realizations which are centred at the same point, must have the same domain (and \( \mathcal{A} \)-domain) and the same evaluation (with respect to \( \mathcal{A} \) as well; here \( \mathcal{A} \) is a unital stably finite \( \mathbb{K} \)-algebra).

In this section we continue to establish connections between all minimal nc Fornasini–Marchesini realizations (with centres of all possible sizes) of a given nc rational function.
Using Theorem 2.21 and Remark 2.22, the general case — where the two centres of minimal realizations of a rational function are different — is considered and solved, which then will lead us to the main conclusion, that is, Theorem 3.3.

**Lemma 3.1.** Let $R_1$ and $R_2$ be nc rational expressions in $x_1, \ldots, x_d$ over $\mathbb{K}$, with $Y_1 \in \text{dom}_{s_1}(R_1)$ and $Y_2 \in \text{dom}_{s_2}(R_2)$, where $s_1, s_2 \in \mathbb{N}$. Suppose that $R_1$ and $R_2$ are minimal nc Fornasini–Marchesini realizations of $R_1$ and $R_2$, centred at $Y_1$ and $Y_2$, respectively. If $R_1$ and $R_2$ are $\mathbb{K}^d$-evaluation equivalent, then

$$\text{DOM}_{pm}(R_1) = \text{DOM}_{pm}(R_2) \text{ and } \text{R}_1(Y) = \text{R}_2(Y)$$

(3.1)

for every $m \in \mathbb{N}$ and $Y \in \text{DOM}_{pm}(R_1)$, where $p = l.c.m(s_1, s_2)$. Moreover, for any unital stably finite $\mathbb{K}$-algebra $A$ and $a \in A^d$,

$$I_{s_1} \otimes a \in \text{DOM}^A(R_1) \iff I_{s_2} \otimes a \in \text{DOM}^A(R_2)$$

and for every such $a$, we have

$$I_{s_2} \otimes \mathcal{R}^A(I_{s_1} \otimes a) = I_{s_1} \otimes \mathcal{R}^A(I_{s_2} \otimes a).$$

**Proof.** We know that $\text{dom}_{s_1}(R_1), \text{dom}_{s_2}(R_2) \neq \emptyset$, hence $\text{dom}_{lp}(R_1), \text{dom}_{lp}(R_2) \neq \emptyset$ and as they are both open Zariski sets in $(\mathbb{K}^p \otimes \mathbb{K}^q)^d$, there exists $\ell \in \mathbb{N}$ such that $\text{dom}_{lp}(R_1) \cap \text{dom}_{lp}(R_2) \neq \emptyset$, so let us fix

$$\mathcal{Y} \in \text{dom}_{lp}(R_1) \cap \text{dom}_{lp}(R_2) \subseteq \text{DOM}_{lp}(R_1) \cap \text{DOM}_{lp}(R_2).$$

Once again (as in the proof of Theorem 2.19), the reasoning for that is clear when $\mathbb{K}$ is infinite (with $\ell = 1$), while if $\mathbb{K}$ is finite, we use a similar argument as in [47, Remark 2.6].

• Let $n_1$ and $n_2$ be the integers for which $p\ell = s_1n_1 = s_2n_2$. From Remark 2.22, there exist minimal nc Fornasini–Marchesini realizations $\mathcal{R}^{(n_k)}$ of $R_k$, centred at $I_{n_k} \otimes \mathcal{Y}$, such that

$$\text{DOM}_{pm}(R_k) = \text{DOM}_{pm}\left(\mathcal{R}^{(n_k)}\right) \text{ and } \mathcal{R}^{(n_k)}(Y) = \mathcal{R}_k(Y), \forall Y \in \text{DOM}_{pm}(R_k)$$

for every $m \in \mathbb{N}, A \in \text{DOM}^A(R_k) \iff I_{n_k} \otimes A \in \text{DOM}^A(\mathcal{R}^{(n_k)})$ and

$$\left(\mathcal{R}^{(n_k)}\right)^A(I_{n_k} \otimes A) = I_{n_k} \otimes \mathcal{R}^A_k(A), \forall A \in \text{DOM}^A(R_k),$$

for $k = 1, 2$. In addition, there exist minimal nc Fornasini–Marchesini realizations $\mathcal{R}_k$ of $R_k$, centred at $\mathcal{Y}$, such that

$$\text{DOM}_{pm}(\mathcal{R}_k) = \text{DOM}_{pm}(\mathcal{R}_k) \text{ and } \mathcal{R}^{(n_k)}(Y) = \mathcal{R}_k(Y), \forall Y \in \text{DOM}_{pm}(\mathcal{R}_k)$$

for every $m \in \mathbb{N}, \text{DOM}^A(\mathcal{R}^{(n_k)}) = \text{DOM}^A(\mathcal{R}_k)$ and

$$\left(\mathcal{R}^{(n_k)}\right)^A(A) = \mathcal{R}_k^A(A), \forall A \in \text{DOM}^A(\mathcal{R}^{(n_k)}),$$

for $k = 1, 2$. Therefore $\mathcal{R}_1$ and $\mathcal{R}_2$ are minimal nc Fornasini–Marchesini realizations, both centred at $\mathcal{Y}$, of $R_1$ and $R_2$ — which are $(\mathbb{K}^d)_{nc}$-evaluation equivalent — hence Theorem 2.13 implies

$$\text{DOM}_{pm}(\mathcal{R}_1) = \text{DOM}_{pm}(\mathcal{R}_2) \text{ and } \mathcal{R}_1(Y) = \mathcal{R}_2(Y), \forall Y \in \text{DOM}_{pm}(\mathcal{R}_1)$$

for every $m \in \mathbb{N}$ and

$$\text{DOM}^A(\mathcal{R}_1) = \text{DOM}^A(\mathcal{R}_2) \text{ and } \mathcal{R}_1^A(A) = \mathcal{R}_2^A(A), \forall A \in \text{DOM}^A(\mathcal{R}_1),$$

for every $m \in \mathbb{N}$ and
which yield that

$$\text{DOM}_{p_{\ell m}}(R_1) = \text{DOM}_{p_{\ell m}}(\tilde{R}_1) = \text{DOM}_{p_{\ell m}}(\tilde{R}_2) = \text{DOM}_{p_{\ell m}}(R_2) \quad (3.2)$$

for every $m \in \mathbb{N}$ and

$$\tilde{R}_1(X) = \tilde{R}_1(X) = \tilde{R}_2(X) = R_2(X), \forall X \in \text{DOM}_{p_{\ell m}}(R_1). \quad (3.3)$$

It is easily seen that $X \in \text{DOM}_{p_{\ell m}}(R_k) \iff I_{\ell} \otimes X \in \text{DOM}_{p_{\ell m}}(R_k)$ and in that case $\tilde{R}_k(I_{\ell} \otimes X) = I_{\ell} \otimes \tilde{R}_k(X)$, where $k = 1$ or $k = 2$, and thus, from (3.2) and (3.3) one can get (3.1).

Moreover,

$$\text{DOM}^A\left(\tilde{R}^{(n_1)}\right) = \text{DOM}^A\left(\tilde{R}_1\right) = \text{DOM}^A\left(\tilde{R}_2\right) = \text{DOM}^A\left(\tilde{R}^{(n_2)}\right)$$

implies that for every $a \in A^d$,

$$I_{s_1} \otimes a \in \text{DOM}^A(R_1) \iff I_{s_1} \otimes (I_{s_1} \otimes a) \in \text{DOM}^A\left(\tilde{R}^{(n_1)}\right) \iff$$

$$I_{s_2} \otimes (I_{s_2} \otimes a) \in \text{DOM}^A\left(\tilde{R}^{(n_2)}\right) \iff I_{s_2} \otimes a \in \text{DOM}^A(R_2)$$

and for every such $a$:

$$I_{s_1} \otimes \tilde{R}^A_1(I_{s_1} \otimes a) = \left(\tilde{R}^{(n_1)}\right)^A(I_{p_{\ell}} \otimes a) = \tilde{R}_1^A(I_{p_{\ell}} \otimes a) =$$

$$= \tilde{R}^A_2(I_{p_{\ell}} \otimes a) = \left(\tilde{R}^{(n_2)}\right)^A(I_{p_{\ell}} \otimes a) = I_{s_2} \otimes \tilde{R}^A_2(I_{s_2} \otimes a),$$

which then, as $s_1n_1 = s_2n_2$, implies $I_{s_1} \otimes \tilde{R}^A_1(I_{s_1} \otimes a) = I_{s_1} \otimes \tilde{R}^A_2(I_{s_2} \otimes a)$. \hfill \Box

**Remark 3.2.** If $R_1$ and $R_2$ are $(\mathbb{K}_d)_{nc}$-evaluation equivalent nc (non-degenerate) rational expressions in $x_1, \ldots, x_d$ over $\mathbb{K}$, as explained in the beginning of the proof, there exists $\tilde{Y} \in \text{dom}_{\ell p}(R_1) \cap \text{dom}_{\ell p}(R_2)$ for some $\ell \in \mathbb{N}$; thus, Theorem 2.13 implies that $L_{R_1}(\ell p) = L_{R_2}(\ell p)$ and hence

$$m(R_1) = \frac{L_{R_1}(\ell p)}{\ell p} = \frac{L_{R_2}(\ell p)}{\ell p} = m(R_2).$$

Therefore, we define the McMillan degree of an nc rational function $\mathcal{R}$ to be $m(\mathcal{R}) := m(R)$ for every $R \in \mathcal{R}$.

Recall that an nc rational function $\mathcal{R}$ is an equivalence class of the form

$$\mathcal{R} = \{R : R \text{ is a non-degenerate representative of } \mathcal{R}\},$$

whose elements are $(\mathbb{K}_d)_{nc}$-evaluation equivalent nc rational expressions in $x_1, \ldots, x_d$ over $\mathbb{K}$, whereas the domain and $A$-domain of regularity of $\mathcal{R}$ are given by

$$\text{dom}(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} \text{dom}(R) \text{ and } \text{dom}^A(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} \text{dom}^A(R),$$

respectively. We now use Corollary 2.18 and Lemma 3.1, to show that the domain of regularity of an nc rational function $\mathcal{R}$ at the level of $n \times n$ matrices, that is,

$$\text{dom}_n(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} \text{dom}_n(R),$$

lives inside the domain of any minimal nc Fornasini–Marchesini realization of a representative in $\mathcal{R}$, up to a tensor product with the identity matrix.
THEOREM 3.3. Let $\mathcal{R} \in \mathbb{K} \langle x_1, \ldots, x_d \rangle$ be an nc rational function. For every nc rational expression $R \in \mathcal{R}$, integer $s \in \mathbb{N}$, point $Y \in \text{dom}_s(\mathcal{R})$, minimal nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $Y$ of $R$ and unital stably finite $\mathbb{K}$-algebra $\mathcal{A}$, we have the following properties.

1. If $Z \in \text{dom}_s(\mathcal{R})$, then $I_s \otimes Z \in \text{DOM}_{sn}(\mathcal{R})$ and $I_s \otimes \mathcal{R}(Z) = \mathcal{R}(I_s \otimes Z)$.
2. If $s \mid n$, then $\text{dom}_s(\mathcal{R}) \subseteq \text{DOM}_n(\mathcal{R})$ and $\mathcal{R}(Z) = \mathcal{R}(Z)$ for every $Z \in \text{dom}_n(\mathcal{R})$.
3. If $\mathfrak{a} \in \text{dom}^A(\mathcal{R})$, then $I_s \otimes \mathfrak{a} \in \text{DOM}^A(\mathcal{R})$ and $I_s \otimes \mathcal{R}^A(\mathfrak{a}) = \mathcal{R}^A(I_s \otimes \mathfrak{a})$.

Proof. Let $Z \in \text{dom}_n(\mathcal{R})$, so there exists an nc rational expression $\hat{R} \in \mathcal{R}$ such that $Z \in \text{dom}_n(\hat{R})$, while Corollary 2.18 implies the existence of a minimal nc Fornasini–Marchesini realization $\hat{\mathcal{R}}$ of $\hat{R}$, centred at $Z$. Then $\mathcal{R}$ and $\hat{\mathcal{R}}$ are minimal nc Fornasini–Marchesini realizations of $R$ and $\hat{R}$, with centres in $(\mathbb{K}^{s \times s})^d$ and $(\mathbb{K}^{n \times n})^d$, respectively. Since $R, \hat{R} \in \mathcal{R}$, it follows that $R$ and $\hat{R}$ are $(\mathbb{K}^d)_{nc}$-evaluation equivalent; therefore, Lemma 3.1 guarantees that

$$\text{DOM}_{pm}(\hat{\mathcal{R}}) = \text{DOM}_{pm}(\mathcal{R}) \text{ and } \hat{\mathcal{R}}(X) = \mathcal{R}(X), \forall X \in \text{DOM}_{pm}(\mathcal{R})$$

(3.4)

for every $m \in \mathbb{N}$, where $p = \text{l.c.m}(s, n)$. Thus, $p \mid sn$ implies that

$$Z \in \text{dom}_n(\hat{R}) \implies I_s \otimes Z \in \text{dom}_{sn}(\hat{R}) \subseteq \text{DOM}_{sn}(\hat{R}) = \text{DOM}_{sn}(\mathcal{R}),$$

and hence

$$I_s \otimes \mathcal{R}(Z) = I_s \otimes \hat{R}(Z) = \hat{R}(I_s \otimes Z) = \mathcal{R}(I_s \otimes Z),$$

which ends the proof of part 1.

- Suppose next that $s \mid n$; in that case we have $p = \text{l.c.m}(s, n) = n$. Thus, in view of (3.4) with $m = 1$, if $Z \in \text{dom}_n(\hat{R})$, then

$$Z \in \text{DOM}_n(\hat{R}) = \text{DOM}_n(\mathcal{R}) \text{ and } \mathcal{R}(Z) = \hat{R}(Z) = \mathcal{R}(Z),$$

which ends the proof of part 2.

- Finally, let $\mathfrak{a} \in \text{dom}^A(\mathcal{R})$, then there exists a non-degenerate nc rational expression $\hat{R} \in \mathcal{R}$ such that $\mathfrak{a} \in \text{dom}^A(\hat{R})$. As $\hat{R}$ is non-degenerate, there exists $t \in \mathbb{N}$ such that $\text{dom}_t(\hat{R}) \neq \emptyset$. Let $\hat{Y} \in \text{dom}_t(\hat{R})$ and apply Corollary 2.18: so there exists a minimal realization $\hat{\mathcal{R}}$ of $\hat{R}$, centred at $\hat{Y}$, such that

$$\mathfrak{a} \in \text{dom}^A(\hat{R}) \implies I_t \otimes \mathfrak{a} \in \text{DOM}^A(\hat{R}) \text{ and } \hat{\mathcal{R}}^A(I_t \otimes \mathfrak{a}) = I_t \otimes \hat{\mathcal{R}}^A(\mathfrak{a}).$$

As $\mathcal{R}$ and $\hat{\mathcal{R}}$ are minimal nc Fornasini–Marchesini realizations of $R \in \mathcal{R}$ and $\hat{R} \in \mathcal{R}$, respectively, Lemma 3.1 guarantees that

$$I_t \otimes \mathfrak{a} \in \text{DOM}^A(\hat{R}) \implies I_s \otimes \mathfrak{a} \in \text{DOM}^A(\mathcal{R})$$

and also that

$$I_t \otimes \mathcal{R}^A(I_s \otimes \mathfrak{a}) = I_s \otimes \hat{\mathcal{R}}^A(I_t \otimes \mathfrak{a}) = I_s \otimes \left( I_t \otimes \hat{\mathcal{R}}^A(\mathfrak{a}) \right),$$

thus $\mathcal{R}^A(I_s \otimes \mathfrak{a}) = I_s \otimes \hat{\mathcal{R}}^A(\mathfrak{a}) = I_s \otimes \mathcal{R}^A(\mathfrak{a})$. \hfill \Box

What we proved is that

$$\text{dom}_{sn}(\mathcal{R}) \subseteq \text{DOM}_{sn}(\mathcal{R}), \forall m \in \mathbb{N},$$

however in the case where $s = 1$ and $Y = (0, \ldots, 0)$, the nc Fornasini-Marchesini realization $\mathcal{R}$ is actually a $1 \times 1$ matrix-valued nc rational expression (this is not a priori the case if $s > 1$):
by viewing $\mathcal{R}$ as a $1 \times 1$ matrix-valued nc rational function (cf. Remark 4.1), it follows that the nc Fornasini–Marchesini realization $\mathcal{R}$ is a representative of $\mathcal{R}$, and therefore

$$\text{dom}_m(\mathcal{R}) \supseteq \text{DOM}_m(\mathcal{R}), \forall m \in \mathbb{N}.$$ 

In other words, by applying Theorem 3.3 we actually obtain a proof for Theorem 1 from the introduction which — unlike the original proof in [49] — does not make any use of the difference-differential calculus of nc functions.

**Corollary 3.4.** If $\mathcal{R}$ is a nc rational function of $x_1, \ldots, x_d$ over $\mathbb{K}$ and $\mathcal{R}$ is regular at $0$, then $\mathcal{R}$ admits a unique (up to unique similarity) minimal (observable and controllable) nc Fornasini–Marchesini realization

$$\mathcal{R}(x_1, \ldots, x_d) = D + C \left( I_L - \sum_{k=1}^d A_k x_k \right)^{-1} \sum_{k=1}^d B_k x_k,$$

where $A_1, \ldots, A_d \in \mathbb{K}^{L \times L}$, $B_1, \ldots, B_d \in \mathbb{K}^{L \times 1}$, $C \in \mathbb{K}^{1 \times L}$, $D = \mathcal{R}(0)$ and $L \in \mathbb{N}$,

$$\text{dom}_m(\mathcal{R}) = \{(X_1, \ldots, X_d) \in (\mathbb{K}^{m \times m})^d : \det (I_{Lm} - X_1 \otimes A_1 - \ldots - X_d \otimes A_d) \neq 0\}$$

for every $m \in \mathbb{N}$ and

$$\mathcal{R}(X_1, \ldots, X_d) = I_m \otimes D + (I_m \otimes C) \left( I_{mL} - \sum_{k=1}^d X_k \otimes A_k \right)^{-1} \sum_{k=1}^d X_k \otimes B_k$$

for every $(X_1, \ldots, X_d) \in \text{dom}_m(\mathcal{R})$.

4. **Realizations of matrix-valued NC rational functions**

All of the analysis and results up to now can be generalized to the settings of matrix-valued nc rational functions. In this section we describe the relevant definitions and main results in the matrix-valued case.

If $\alpha, \beta \in \mathbb{N}$, we say that $\mathfrak{r}$ is an $\alpha \times \beta$ matrix-valued nc rational function if $\mathfrak{r}$ is an $\alpha \times \beta$ matrix of nc rational functions, that is, if

$$\mathfrak{r} = [\mathcal{R}_{ij}]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta},$$

where $\mathcal{R}_{ij}$ are nc rational functions. The domain of regularity of $\mathfrak{r}$ is then defined by

$$\text{dom}(\mathfrak{r}) := \bigcap_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} \text{dom}(\mathcal{R}_{ij})$$

(4.1)

and for every $\mathbf{X} \in \text{dom}_n(\mathfrak{r})$ the evaluation $\mathfrak{r}(\mathbf{X})$ is given by

$$\mathfrak{r}(\mathbf{X}) := E(n, \alpha) [\mathcal{R}_{ij}(\mathbf{X})]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} E(n, \beta)^T,$$

where $E(\ell_1, \ell_2) \in \mathbb{K}^{\ell_1 \times \ell_2 \times \ell_1 \ell_2}$ defined in (1.1). The need for the correction terms, which are shuffle matrices, is coming simply because otherwise evaluating $\mathfrak{r}$ term by term does not yield an nc function (it does not preserve direct sums), see, for example, [47, pp. 17–18]. If $\mathcal{A}$ is a unital $\mathbb{K}$-algebra, then the $\mathcal{A}$-domain of $\mathfrak{r}$ is defined by

$$\text{dom}^{\mathcal{A}}(\mathfrak{r}) := \bigcap_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} \text{dom}^{\mathcal{A}}(\mathcal{R}_{ij})$$

and for every $\mathfrak{g} \in \text{dom}^{\mathcal{A}}(\mathfrak{r})$ the evaluation $\mathfrak{r}^{\mathcal{A}}(\mathfrak{g})$ is given by

$$\mathfrak{r}^{\mathcal{A}}(\mathfrak{g}) := [\mathcal{R}_{ij}^{\mathcal{A}}(\mathfrak{g})]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta}.$$
Remark 4.1. In [49] the authors define matrix-valued nc rational functions and their domains of regularity using equivalence classes of matrix-valued nc rational expressions. However, it follows from [81, Lemma 3.9] that one can also define matrix-valued nc rational functions as a matrix of nc rational functions, and the domains of regularity in both cases are equal.

Theorem 4.2. For every \( \alpha \times \beta \) matrix-valued nc rational function
\[
\tau = [\mathcal{R}_{ij}]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta}
\]
and let \( Y = (Y_1, \ldots, Y_d) \in \text{dom}_s(\tau) \), there exist unique (up to unique similarity) \( L \in \mathbb{K}^{\alpha \times \beta} \), \( D \in \mathbb{K}^{a \times s} \), \( C \in \mathbb{K}^{a \times L} \), linear mappings \( A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L} \) and \( B_1, \ldots, B_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times \beta} \) such that \( (A, B) \) is controllable and \( (C, A) \) is observable, for which
\[
\text{dom}_{sm}(\tau) \subseteq \left\{ X \in (\mathbb{K}^{sm \times sm})^d : \det \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right) \neq 0 \right\}
\]
for every \( m \in \mathbb{N} \) and \( \tau(X) = \mathcal{R}(X) \) for every \( X \in \text{dom}_{sm}(\tau) \), where
\[
\mathcal{R}(X) = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k.
\]
Moreover,
\[
\text{dom}_{n}(\tau) \subseteq \left\{ X \in (\mathbb{K}^{n \times n})^d : \det \left( I_{nL} - \sum_{k=1}^{d} (I_k \otimes X_k - I_n \otimes Y_k)A_k \right) \neq 0 \right\}
\]
for every \( n \in \mathbb{N} \) and \( I_n \otimes \tau(X) = \mathcal{R}(I_n \otimes X) \) for every \( X \in \text{dom}_{n}(\tau) \). Furthermore, for any unital stably finite \( \mathbb{K} \)-algebra \( \mathcal{A} \):
\[
\text{dom}^A(\tau) \subseteq \left\{ a \in \mathcal{A}^d : \left( I_{L} \otimes 1_{\mathcal{A}} - \sum_{k=1}^{d} (I_k \otimes a_k - Y_k \otimes 1_{\mathcal{A}})A_k^A \right) \text{ is invertible in } \mathcal{A}^{L \times L} \right\}
\]
and \( I_s \otimes \tau^A(a) = \mathcal{R}^A(I_s \otimes a) \) for every \( a \in \text{dom}^A(\tau) \), that is,
\[
I_s \otimes \tau^A(a) = D \otimes 1_{\mathcal{A}} + (C \otimes 1_{\mathcal{A}}) \left( I_{L} \otimes 1_{\mathcal{A}} - \sum_{k=1}^{d} (I_k \otimes a_k - Y_k \otimes 1_{\mathcal{A}})A_k^A \right)^{-1}
\]
\[
\times \sum_{k=1}^{d} (I_k \otimes a_k - Y_k \otimes 1_{\mathcal{A}})B_k^A.
\]

Similarity of nc Fornasini–Marchesini realizations of matrix-valued nc rational functions is defined analogously to the case of (scalar) nc rational functions (cf. Theorem 2.13), as well as controllability and observability, which are defined via the controllable and un-observable subspaces of \( \mathbb{K}^L \) (cf. Definition 2.7 and notice that the only difference is that the operators \( \mathcal{A}^\omega \cdot B_k \) and \( C \cdot \mathcal{A}^\omega \) return matrices in \( \mathbb{K}^{L \times \beta_s} \) and \( \mathbb{K}^{s \times L} \), respectively).

Proof. Suppose that \( \tau = [\mathcal{R}_{ij}]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} \) is an \( \alpha \times \beta \) matrix-valued nc rational function and let \( Y \in \text{dom}_s(\tau) \). For every \( 1 \leq i \leq \alpha \) and \( 1 \leq j \leq \beta \) we have \( Y \in \text{dom}_s(\mathcal{R}_{ij}) \), while applying Theorem 3.3, the nc rational function \( \mathcal{R}_{ij} \) admits a minimal nc Fornasini–Marchesini realization \( \mathcal{R}_{ij} \) centred at \( Y \), described by a tuple \( (L_{ij}, D^{ij}, C^{ij}, A^{ij}, B^{ij}) \), which is also a realization of \( \mathcal{R}_{ij} \) with respect to \( \mathcal{A} \).
• Let $X \in \text{dom}_{sm}(r)$, then $X \in \text{dom}_{sm}(\mathcal{R}_{ij})$ and hence $X \in \text{DOM}_{sm}(\mathcal{R}_{ij})$ and $\mathcal{R}_{ij}(X) = \mathcal{R}(X)$ for any $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. Therefore,

$$\mathbf{v}(X) = E(sm, \alpha) \begin{bmatrix} \mathcal{R}_{11} & \cdots & \mathcal{R}_{1\beta} \\ \vdots & \vdots & \vdots \\ \mathcal{R}_{\alpha 1} & \cdots & \mathcal{R}_{\alpha \beta} \end{bmatrix} (X) E(sm, \beta)^T = E(sm, \alpha) \begin{bmatrix} \mathcal{R}_{11} & \cdots & \mathcal{R}_{1\beta} \\ \vdots & \vdots & \vdots \\ \mathcal{R}_{\alpha 1} & \cdots & \mathcal{R}_{\alpha \beta} \end{bmatrix} (X)$$

$$\times E(sm, \beta)^T = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^d (X_k - I_m \otimes Y_k) A_k \right)^{-1}$$

$$\times \sum_{k=1}^d (X_k - I_m \otimes Y_k) B_k = \mathcal{R}(X),$$

where the nc Fornasini–Marchesini realization $\mathcal{R}$ is described by

$$L = \sum_{i=1}^\alpha \sum_{j=1}^\beta I_{ij}, \quad D = E(s, \alpha) \begin{bmatrix} D^{11} & \cdots & D^{1\beta} \\ \vdots & \vdots & \vdots \\ D^{\alpha 1} & \cdots & D^{\alpha \beta} \end{bmatrix} E(s, \beta)^T \in \mathbb{K}^{s \times \beta s},$$

$$C = E(s, \alpha) \begin{bmatrix} C^{11} & \cdots & C^{1\beta} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & C^{21} & \cdots & C^{2\beta} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \hdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & C^{\alpha 1} & \cdots & C^{\alpha \beta} \end{bmatrix} \in \mathbb{K}^{s \times L},$$

with the linear mappings

$$A_k = \begin{bmatrix} A^{11}_k & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & A^{1\beta}_k & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & A^{\alpha 1}_k & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & A^{\alpha \beta}_k \end{bmatrix}, \quad B_k = \begin{bmatrix} B^{11}_k & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & B^{1\beta}_k \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & B^{2\beta}_k \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ B^{\alpha 1}_k & \cdots & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & B^{\alpha \beta}_k \end{bmatrix} \in \mathbb{K}^{s \times \beta s},$$

It is easily seen, by the diagonal structure of $A_k$, that

$$\text{dom}_{sm}(t) \subseteq \bigcap_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} \text{DOM}_{sm}(\mathcal{R}_{ij}) = \text{DOM}_{sm}(\mathcal{R}).$$

• Moreover, if $X \in \text{dom}_n(t)$, then $I_s \otimes X \in \text{dom}_{sm}(t)$ and $\mathbf{v}(I_s \otimes X) = I_s \otimes \mathbf{v}(X)$, while by the first part of the theorem, we know that $I_s \otimes X \in \text{dom}_{sm}(\mathcal{R})$ and

$$I_s \otimes \mathbf{v}(X) = \mathbf{v}(I_s \otimes X) = \mathcal{R}(I_s \otimes X).$$

• Furthermore, let $\mathbf{a} \in \text{dom}^A(t)$, thus $\mathbf{a} \in \text{dom}^A(\mathcal{R}_{ij})$ and therefore $I_s \otimes \mathbf{a} \in \text{DOM}^A(\mathcal{R}_{ij})$ and $\mathcal{R}^A_{ij}(I_s \otimes \mathbf{a}) = I_s \otimes \mathcal{R}^A_{ij}(\mathbf{a})$, for every $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. A direct and careful computation — which is omitted — shows that
\( R^A(I_s \otimes \mathfrak{a}) = (E(s, \alpha) \otimes 1_{A}) \left[ R^A_{ij}(I_s \otimes \mathfrak{a}) \right]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} (E(s, \beta) \otimes 1_{A}) \)
\( = (E(s, \alpha) \otimes 1_{A}) \left[ I_s \otimes R^A_{ij}(\mathfrak{a}) \right]_{1 \leq i \leq \alpha, 1 \leq j \leq \beta} (E(s, \beta) \otimes 1_{A}) = I_s \otimes r^A(\mathfrak{a}). \)

- This proves the existence of an nc Fornasini–Marchesini realization for \( r \), centred at \( Y \), which is also a realization of \( r \) with respect to \( A \). To obtain a minimal nc Fornasini–Marchesini realization, we use the Kalman decomposition as in Theorem 2.13, corresponding to the controllable and un-observable subspaces of \( K^L \), whereas the uniqueness (up to unique similarity) of such a minimal realization is proved with the same ingredients as in the proof of Theorem 2.13. The details of the proofs of the Kalman decomposition and of the uniqueness are omitted.

\[ \square \]

**Remark 4.3.** It is not hard to see that the nc Fornasini–Marchesini realization built in the proof is not necessarily minimal, even if \( R_{ij} \) are all minimal nc Fornasini–Marchesini realizations. However, the opposite is true, that is, if \( R \) is a minimal nc Fornasini–Marchesini realization, then all of the nc Fornasini–Marchesini realizations \( R_{ij} \) must be minimal as well.

**Remark 4.4.** (cf. Remark 4.1). The proof of the existence part in Theorem 4.2 can be done using matrix-valued nc rational expressions and the usual process of synthesis, yielding (4.2) for the a priori bigger domain of \( r \), which uses matrix-valued nc rational expressions and once again by [81, Lemma 3.9] is equal to the domain in the sense of (4.1).

**Remark 4.5** (McMillan degree of a matrix-valued nc rational function). One can prove analogous versions of Theorem 2.21 and Remark 3.2, for matrix-valued nc rational expressions and functions, thereby there exists an integer \( m(r) \) such that for any \( Y \in \text{dom}_s(r) \), we have
\[ L_r(Y) = s \cdot m(r); \]
here \( L_r(Y) \) is the dimension of a minimal nc Fornasini–Marchesini realization of \( r \), centred at \( Y \). We call \( m(r) \) the McMillan degree of \( r \). It follows from Theorem 4.2 that
\[ m(r) = \frac{L_r(Y)}{s} \leq \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} L_{ij} = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m(R_{ij}). \]

5. Realizations of hermitian NC rational functions

In the case where \( K = \mathbb{R} \) or \( K = \mathbb{C} \), one often considers symmetric or hermitian nc rational expressions, specially with applications to free probability [16, 41, 74] and in optimization theory [5, 40, 42, 43]. Unlike the case of descriptor realizations (see [41, 48]), the expression for \( R_{FM}^\ast(X) \) does not have the form of an nc Fornasini–Marchesini realization, for an nc Fornasini–Marchesini realization \( R_{FM} \). Nevertheless, we can use our methods to obtain an analogue of Corollary 2.18 in the case where the function \( R \) is hermitian, that is, when
\[ R^\ast(X) := R(X^\ast) = R(X) \]
for all \( X \in \text{dom}(R) \), with the matrix pencil to be inverted having hermitian coefficients. We also get explicit (necessary and sufficient) conditions on the coefficients of the realization for the nc rational function to be hermitian.

**Remark 5.1.** One can define hermitian nc rational functions more precisely. First, one needs to define — using synthesis — an nc rational expression \( R^\ast \), for any nc rational expression \( R \).
Then, one can show that $R_1^* \sim R_2^*$, whenever $R_1 \sim R_2$ (that is, whenever $R_1$ and $R_2$ are $(\mathbb{K}^d)_{nc}$-evaluation equivalent). Finally, for every nc rational function $\mathcal{R}$, let $\mathcal{R}^* = \{ R^* : R \in \mathcal{R} \}$ and define $\mathcal{R}$ to be hermitian if $\mathcal{R}^* = \mathcal{R}$, as equivalence classes.

We use the following notions: if $T$ is a linear mapping on matrices, then $T^*$ is the linear mapping given by $T^*(X) := T(X^*)$ and $T$ is called hermitian if $T^* = T$. If $J$ is a square matrix of the form
\[
J = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ with } p, q, t \geq 0,
\]
then we say that $J$ is a semi-signature matrix; notice that if $t = 0$, then $J$ is a signature matrix.

**Theorem 5.2.** Let $\mathcal{R}$ be an hermitian nc rational function of $x_1, \ldots, x_d$ over $\mathbb{K}$, $Y^* = Y \in \text{dom}_s(\mathcal{R})$ and
\[
\mathcal{R}_{\mathcal{F},\mathcal{M}}(X) = D + C \left( I_L - \sum_{k=1}^{d} A_k (X_k - Y_k) \right)^{-1} \sum_{k=1}^{d} B_k (X_k - Y_k)
\]
be a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$, centred at $Y$.

1. There exists a unique $S = S^* \in \mathbb{K}^{L \times L}$ such that
\[
D^* = D, \ A_k^* \cdot S = S \cdot A_k, \ B_k^* \cdot S = C \cdot A_k \text{ and } C \cdot B_k = (C \cdot B_k)^*, \ 1 \leq k \leq d.
\]

2. Once the relations in (5.1) hold, we have
\[
\ker(S) = \bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s}} \ker(A_k(X)), \ \text{Im}(S) = \bigvee_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s}} \text{Im}(A_k^*(X))
\]
and
\[
\ker(C^*) = \bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s}} \ker(B_k(X)).
\]

3. Symmetry of the minimal nc Fornasini–Marchesini realization: There exist $\hat{C} \in \mathbb{K}^{s \times L}$, a semi-signature matrix $J \in \mathbb{K}^{L \times L}$ and hermitian linear mappings $\hat{A}_1, \ldots, \hat{A}_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L}$, such that
\[
\mathcal{R}_{\mathcal{F},\mathcal{M}}(X) = D + \hat{C} \left( I_L - \sum_{k=1}^{d} \hat{A}_k (X_k - Y_k) J \right)^{-1} \sum_{k=1}^{d} \hat{A}_k (X_k - Y_k) \hat{C}^*,
\]
with $(\hat{A} \cdot J, \hat{A} \cdot \hat{C})$ controllable and $(\hat{C}, \hat{A} \cdot J)$ observable. Conversely, if $\mathcal{R}$ admits a realization of the form (5.3) with the controllability and observability conditions, then $\mathcal{R}$ is an hermitian nc rational function.

4. Hermitian nc descriptor realization: There exist $D_D = D_D^* \in \mathbb{K}^{s \times s}$, $C_D \in \mathbb{K}^{s \times (L+s)}$, a signature matrix $J_D \in \mathbb{K}^{(L+s) \times (L+s)}$ and hermitian linear mappings $\hat{A}_{1,D}, \ldots, \hat{A}_{d,D} : \mathbb{K}^{s \times s} \to \mathbb{K}^{(L+s) \times (L+s)}$, such that
\[
\mathcal{R}_{\mathcal{F},\mathcal{M}}(X) = \mathcal{R}_D(X) := D_D + C_D \left( J_D - \sum_{k=1}^{d} \hat{A}_{k,D} (X_k - Y_k) \right)^{-1} C_D^*
\]

and
\[ \text{DOM}_s(\mathcal{R}_{FM}) = \text{DOM}_s(\mathcal{R}_D) := \left\{ X \in (\mathbb{K}^{s \times s})^d : \det \left( J_D - \sum_{k=1}^d A_{k,D}(X_k - Y_k) \right) \neq 0 \right\}. \]

Moreover, similarly to Theorem 3.3, this also applies — after a suitable tensoring — to domains and evaluations on \( n \times n \) matrices for all \( n \in \mathbb{N} \) and with respect to any unital stably finite \( \mathbb{K} \)-algebra \( A \).

5. The matrix \( S \) is invertible \( \iff \) \( \bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s} } \ker(A_k(X)) = \{0\} \iff \) there exists \( Q \in \mathbb{K}^{L \times s} \) such that
\[ B_k = A_k \cdot Q, \forall 1 \leq k \leq d. \] (5.5)

In that case, \( J \in \mathbb{K}^{L \times L} \) is invertible,
\[ \mathcal{R}_{FM}(X) = D + \tilde{C} J \left( J - \sum_{k=1}^d \hat{A}_k(X_k - Y_k) \right)^{-1} \sum_{k=1}^d \hat{A}_k(X_k - Y_k) \hat{C}^*, \] and
\[ \mathcal{R}_{FM}(X) = \tilde{D} + \hat{C} J \left( J - \sum_{k=1}^d \tilde{A}_k(X_k - Y_k) \right)^{-1} \hat{C}^*, \] (5.7)

where \( \tilde{A}_k = J \hat{A}_k J \) and \( \tilde{D} = D - \hat{C} J \hat{C}^* \).

**Proof.** 1. The proof of the first part of the theorem follows the same ideas as the proof of Theorem 2.13. As \( \mathcal{R}^* = \mathcal{R} \), we obtain that \( \mathcal{R}^*_{FM} = \mathcal{R}_{FM} \) and then compare the coefficients in the Taylor–Taylor power series expansions
\[ \mathcal{R}_{FM}(X) = \sum_{\nu \in G_d} (X - I_m \otimes Y)^{\otimes \nu} \mathcal{R}_\nu \]
and
\[ \mathcal{R}^*_{FM}(X) = \sum_{\nu \in G_d} (X - I_m \otimes Y)^{\otimes \nu} \mathcal{R}_{\nu, s}, \]
we obtain that \( \mathcal{R}_\nu(Z_1, \ldots, Z_\ell) = \mathcal{R}_{\nu, s}(Z_1, \ldots, Z_\ell) \) for every \( \nu = g_{i_1} \ldots g_{i_\ell} \in G_d \) and \( Z_1, \ldots, Z_\ell \in \mathbb{K}^{s \times s} \), where
\[ \mathcal{R}_\nu(Z_1, \ldots, Z_\ell) = \begin{cases} D & : \text{if } \ell = 0 \\ C B_{i_1}(Z_1) & : \text{if } \ell = 1 \\ C A_{i_1}(Z_1) \cdots A_{i_{\ell - 1}}(Z_{\ell - 1}) B_{i_\ell}(Z_\ell) & : \text{if } \ell > 1 \end{cases} \]
and
\[ \mathcal{R}_{\nu, s}(Z_1, \ldots, Z_\ell) = \begin{cases} D^* & : \text{if } \ell = 0 \\ B^*_{i_1}(Z_1) C^* & : \text{if } \ell = 1 \\ B^*_{i_1}(Z_1) A^*_{i_2}(Z_2) \cdots A^*_{i_\ell}(Z_\ell) C^* & : \text{if } \ell > 1. \end{cases} \]

For \( \nu = \emptyset \) we get that \( D = D^* \). Next, define a linear mapping
\[ S(A^{g_1}(W_1, \ldots, W_k) B_j(Z)u) = (A^{g_1})^{g_2}(W_1, \ldots, W_k) A^*_{j}(Z) C^* u \] (5.8)
for every \( \omega \in G_d \) of length \( k \geq 0 \), \( 1 \leq j \leq d \), \( W_1, \ldots, W_k, Z_j \in \mathbb{K}^{s \times s} \) and \( u \in \mathbb{K}^s \), then we extend \( S \) by linearity.
\[ w_1 = \sum_{l \in I_L} A^{\omega_l}(W)B_{j_l}(Z_l)u_l = \sum_{t \in I_T} A^{\eta_t}(Q)B_{i_t}(P_t)v_t = w_2, \]

for \( \omega_l, \eta_t \in G_d, W = (W_1^{(t)}, \ldots, W_k^{(t)}) \in (K^{s \times s})^k, Q = (Q_1^{(t)}, \ldots, Q_m^{(t)}) \in (K^{s \times s})^m, Z_l, P_t \in K^{s \times s}, 1 \leq j_l, i_t \leq d \) and \( u_l, v_t \in K^s \) for every \( l \in I_L \) and \( t \in I_T \). Thus for every \( 1 \leq n \leq d, \alpha \in G_d, \tilde{X} \in K^{s \times s} \) and \( X = (X_1, \ldots, X_{|\alpha|}) \in (K^{s \times s})^{|\alpha|} \):

\[
\sum_{l \in I_L} R_{g_n, \omega_l, g_{j_l}} (\tilde{X}, X, W, Z_l)u_l = \sum_{t \in I_T} R_{g_n, \eta_t, g_{i_t}} (\tilde{X}, X, Q, P_t)v_t = \sum_{t \in I_T} R_{g_n, \eta_t, g_{i_t}} (\tilde{X}, X, Q, P_t)v_t,
\]

which implies that

\[
B_n^*(\tilde{X}) (A^*)^\alpha(X) = \sum_{l \in I_L} (A^*)^{\omega_l}(W)A_{j_l}^{*}(Z_l)C^* u_l = B_n^*(\tilde{X}) (A^*)^\alpha(X) \sum_{t \in I_T} (A^*)^{\eta_t}(Q)A_{i_t}^{*}(P_t)C^* v_t,
\]

whereas the controllability of \((A, B)\) implies that \( S(w_1) = S(w_2) \).

- Let \( S \in K^{L \times L} \) be the matrix such that \( S(u) = Su \) for every \( u \in K^L \). We show that \( S \) is self-adjoint: for every \( u, v \in K^L \) we use the controllability of \((A, B)\) to write them as

\[
u = \sum_{l \in I_L} A^{\omega_l}(W)B_{j_l}(Z_l)u_l \quad \text{and} \quad v = \sum_{t \in I_T} A^{\eta_t}(Q)B_{i_t}(P_t)v_t,
\]

thus, using the notation \( Q^\# := (Q_{m_t}, \ldots, Q_1) \),

\[
\langle Su, v \rangle = \sum_{l \in I_L} \sum_{t \in I_T} \langle SA^{\omega_l}(W)B_{j_l}(Z_l)u_l, A^{\eta_t}(Q)B_{i_t}(P_t)v_t \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle (A^*)^{\omega_l}(W)A_{j_l}^{*}(Z_l)C^* u_l, A^{\eta_t}(Q)B_{i_t}(P_t)v_t \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle (A^*)^{\eta_t}(Q)A_{i_t}^{*}(P_t)C^* v_t, (A^*)^{\omega_l}(W)A_{j_l}^{*}(Z_l)C^* u_l \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle v_t^*R_{g_n, \eta_t, g_{i_t}}(P_t, Q^#, W, Z_l)u_l \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle v_t^*CA_{i_t}(P_t)A_{j_l}^{*}(Q^#)A^{\omega_l}(W)B_{j_l}(Z_l)u_l \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle (A^{\omega_l}(W)B_{j_l}(Z_l)u_l, (A^*)^{\eta_t}(Q)A_{i_t}(P_t)C^* v_t \rangle
\]

\[
= \sum_{l \in I_L} \sum_{t \in I_T} \langle (A^{\omega_l}(W)B_{j_l}(Z_l)u_l, SA^{\eta_t}(Q)B_{i_t}(P_t)v_t \rangle = \langle u, Sv \rangle,
\]

that is, \( S = S^* \).
• From (5.8) it follows that

\[ SB_k(Z) = A_k^*(Z)C^*, \forall 1 \leq k \leq d, Z \in \mathbb{K}^{s \times s} \]  

(5.9)

and for every \( w \),

\[ SA_k(Z)w = \sum_{l \in L} S A_k(Z)A_k^*(W)B_{j_l}(Z)u_l = \sum_{l \in L} A_k^*(Z)(A^*)^j (W)A_k^*(Z)C^* u_l \]

= \( A_k^*(Z) \sum_{l \in L} A_k^*(W)B_{j_l}(Z)u_l = A_k^*(Z)Sw \).

Once again, from the controllability of \((A, B)\) we have

\[ SA_k(Z) = A_k^*(Z)S, \forall 1 \leq k \leq d, Z \in \mathbb{K}^{s \times s}. \]  

(5.10)

• It is easily seen that every self-adjoint matrix \( S \) which satisfies the relations in (5.1) must satisfy (5.8) too. This implies the uniqueness of the matrix \( S \).

2. Suppose that all the relations in (5.1) hold.

• Let \( u \in \ker(S) \), thus (5.10) implies that \( u := A_k(Z)u \in \ker(S) \), while (5.9) implies that \( u \in \ker(C) \). So \( u \in \ker(S) \) implies that \( A^j(X_1, \ldots, X_{\omega})u \in \ker(C) \), but \((C, A)\) is observable, hence \( u = 0 \) and \( v \in \ker(A_k(Z) \forall 1 \leq k \leq d \text{ and } Z \in \mathbb{K}^{s \times s} \).

• On the other hand, if \( A_k(Z)u = 1 \) for all \( 1 \leq k \leq d \text{ and } Z \in \mathbb{K}^{s \times s} \), then \( A_k^*(Z)S = 0 \) and also \( B_k(Z)C^* = 0 \), whereas the controllability of \((A, B)\) implies that \( Sv = 0 \).

• If \( B_k(Z)u = 1 \) for all \( 1 \leq k \leq d \text{ and } Z \in \mathbb{K}^{s \times s} \), then \( A_k^*(Z)C^* = 0 \) and also \( B_k^*(Z)C^* = 0 \), which implies that \( C^*u = 0 \).

• On the other hand, if \( u \in \ker(C^*) \), then \( B_k(Z)u \in \ker(C) \) and \( SB_k(Z)u = 0 \), that is, \( B_k(Z)u \in \ker(S) \). Therefore \( B_k(Z)u \in \ker(A_k(Z)) \) for every \( 1 \leq j \leq d \) and \( Z \in \mathbb{K}^{s \times s} \), thus \( B_k(Z)u = 0 \).

3. The matrix \([S^T_C^*]\) is left invertible, as if \( u \in \ker([S^T_C^*]) \), then \( u \in \ker(C) \) and

\[ u \in \ker(S) = \bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s}} \ker(A_k(X)), \]

which implies \( u = 0 \), since \((C, A)\) is observable.

• Let \( K \in \mathbb{K}^{L \times (L+s)} \) be a left inverse of \([S^T_C^*]\) and \( \tilde{A}_k := K \cdot [A_k^T] \), thus

\[ A_k = K \cdot \left[ \begin{array}{cc} S & C \\ C & C^* \end{array} \right] \cdot \tilde{A}_k = \tilde{A}_k \cdot K = \left[ \begin{array}{cc} S & C \\ C & C^* \end{array} \right] \cdot B_k = \tilde{A}_k \cdot C^* \]  

(5.11)

and

\[
\begin{align*}
[S^T_C^*] \cdot \tilde{A}_k \cdot [S^T_C^*] &= \left[ \begin{array}{cc} S \cdot \tilde{A}_k \cdot S & S \cdot \tilde{A}_k \cdot C^* \\ C \cdot \tilde{A}_k \cdot S & C \cdot \tilde{A}_k \cdot C^* \end{array} \right] = \left[ \begin{array}{cc} S \cdot A_k \cdot S & S \cdot B_k \cdot C^* \\ C \cdot A_k \cdot S & C \cdot B_k \cdot C^* \end{array} \right] \\
&= \left[ \begin{array}{cc} (S \cdot A_k)^* & (C \cdot A_k)^* \\ (S \cdot B_k)^* & (C \cdot B_k)^* \end{array} \right] = \left[ \begin{array}{cc} S \cdot \tilde{A}_k^* \cdot S & S \cdot \tilde{A}_k^* \cdot C^* \\ C \cdot \tilde{A}_k^* \cdot S & C \cdot \tilde{A}_k^* \cdot C^* \end{array} \right] \\
&= \left[ \begin{array}{cc} S \cdot \tilde{A}_k^* & S \cdot \tilde{A}_k^* \cdot C^* \\ C \cdot \tilde{A}_k^* \cdot S & C \cdot \tilde{A}_k^* \cdot C^* \end{array} \right]
\end{align*}
\]

which implies that \( \tilde{A}_k = \tilde{A}_k^* \). Moreover, \( \tilde{A}_k \) is independent of the choice of the left inverse \( K \): if \( K' \) is another left inverse of \([S^T_C^*]\), then

\[
(K - K') \cdot \left[ \begin{array}{cc} A_k & B_k \\ B_k & C \end{array} \right] = \left[ \begin{array}{cc} A_k - A_k & B_k - B_k \end{array} \right] = \left[ \begin{array}{cc} 0 & 0 \end{array} \right]
\]
and the right invertibility of \([ S \ C^*]\) implies that \(K \cdot [A_k^\dagger B_k^\dagger] = K' \cdot [A_k^\dagger B_k^\dagger].\) Notice that from the controllability of \((\mathbf{A}, \mathbf{B})\) we get

\[
\bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{r \times s}} \ker \left( \mathbf{A}_k(X) \right) = \{0\}.
\]

- Next, as \(S = S^*,\) one can write \(S = TJJ^*,\) where \(T \in \mathbb{K}^{L \times L}\) is invertible and \(J \in \mathbb{K}^{L \times L}\) is a semi-signature matrix. Thus, using the relations in (5.11), we obtain

\[
R_{F,M}(X) = D + C \left( I_L - \sum_{k=1}^{d} \mathbf{A}_k(X_k - Y_k)S \right)^{-1} \sum_{k=1}^{d} \mathbf{A}_k(X_k - Y_k)C^*,
\]

\[
= D + \hat{C} \left( I_L - \sum_{k=1}^{d} \mathbf{A}_k(X_k - Y_k)J \right)^{-1} \sum_{k=1}^{d} \mathbf{A}_k(X_k - Y_k)C^*,
\]

where \(\hat{C} = C(T^*)^{-1} + \mathbf{A}_k = T^* \cdot \mathbf{A}_k \cdot T\) are hermitian, for \(1 \leq k \leq d.\) Moreover, it is easily seen that the controllability of \((\mathbf{A}, \mathbf{B})\) and the observability of \((C, \mathbf{A})\) imply the controllability of \((\mathbf{A} \cdot J, \mathbf{A} \cdot \hat{C}^*)\) and the observability of \((\hat{C}, \mathbf{A} \cdot J),\) respectively.

4. Define \(E := F + \hat{C}J\hat{C}^*.\) The matrix

\[
\tilde{S} = \begin{bmatrix} J & \hat{C}^* \\ \hat{C} & E \end{bmatrix} \in \mathbb{K}^{(L+s) \times (L+s)}
\]

is hermitian and invertible, whenever \(F \succ 0\) or \(F \prec 0: it is easily seen that \(\tilde{S}^* = \tilde{S};\) let \([v, u] \in \ker(\tilde{S}),\) thus \(Ju + \hat{C}^*u = 0\) and \(\hat{C}u + Ev = 0,\) therefore \(\hat{C}^*u = -Ju\) and when we plug it in to the other equation we get \(Fv = \hat{C}(J^2 - I_L)u.\) Thus, \(-Ju = \hat{C}F^{-1}\hat{C}(J^2 - I_L)u\) and multiplying both sides by \(u^*(J^2 - I_L)\) on the left, to get

\[
u^*(J^2 - I_L)\hat{C}F^{-1}\hat{C}(J^2 - I_L)u = -u^*(J^2 - I_L)Ju = -u^*(J^3 - J)u = 0.
\]

Then \(v = \hat{C}(J^2 - I_L)u = 0,\) which implies that \(Ju = 0\) and \(\hat{C}u = 0,\) that is, that \([\hat{C}]u = 0.\) Recall that \([\hat{C}^*] = [\hat{C}T^*] = [T^0 I_1] = \hat{C}^*T^*\) is left invertible and hence \([\hat{C}]\) is left invertible and hence \(u = 0.\)

- As \(\tilde{S}\) is hermitian and invertible, there exist an invertible matrix \(\tilde{T} \in \mathbb{K}^{(L+s) \times (L+s)}\) and a signature matrix \(J_D \in \mathbb{K}^{(L+s) \times (L+s)}\) such that \(S^{-1} = TJ_D\tilde{T}^*,\) therefore

\[
R_{F,M}(X) = D - E + [\hat{C} \ E] \left( I_{L+s} - \sum_{k=1}^{d} \begin{bmatrix} \mathbf{A}_k^* & 0 \\ 0 & \mathbf{A}_k \end{bmatrix} \right) \begin{bmatrix} X_k - Y_k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_s \end{bmatrix}
\]

\[
= D - E + [0 \ I_s] \tilde{T} \begin{bmatrix} I_{L+s} - \sum_{k=1}^{d} \begin{bmatrix} \mathbf{A}_k & 0 \\ 0 & \mathbf{A}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} X_k - Y_k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_s \end{bmatrix}
\]

\[
= D - E + [0 \ I_s] \begin{bmatrix} \tilde{S}^{-1} - \sum_{k=1}^{d} \begin{bmatrix} \mathbf{A}_k & 0 \\ 0 & \mathbf{A}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} X_k - Y_k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_s \end{bmatrix}
\]

\[
= D_D + C_D \left( J_D - \sum_{k=1}^{d} \mathbf{A}_{k,D}(X_k - Y_k) \right)^{-1} C_D = R_D(X),
\]

for every \(X \in DOM_s(R_{F,M}),\) where

\[
D_D := D - F - \hat{C}J\hat{C}^*, \quad C_D := [0 \ I_s] \tilde{T}^{-*}\) and \(\mathbf{A}_{k,D} := \tilde{T}^{-1} \begin{bmatrix} \mathbf{A}_k & 0 \\ 0 & 0 \end{bmatrix} \tilde{T}^{-*},
\]
for every $1 \leq k \leq d$. We showed that $\mathcal{R}_{\mathcal{F}, \mathcal{M}}(X) = \mathcal{R}_D(X)$ and it is easily seen from the last computation that

$$X \in \text{DOM}_s(\mathcal{R}_{\mathcal{F}, \mathcal{M}}) \iff \det \left( J_D - \sum_{k=1}^{d} A_{k,D}(X_k - Y_k) \right) \neq 0,$$

that is, that $\text{DOM}_s(\mathcal{R}_D) = \text{DOM}_s(\mathcal{R}_{\mathcal{F}, \mathcal{M}})$. 

- Furthermore, straightforward computations show that for every $m \geq 1$:

$$X \in \text{DOM}_{sm}(\mathcal{R}_{\mathcal{F}, \mathcal{M}}) \iff \det \left( I_m \otimes J_D - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_{k,D} \right) \neq 0,$$

that is, that $\text{DOM}_{sm}(\mathcal{R}_{\mathcal{F}, \mathcal{M}}) = \text{DOM}_{sm}(\mathcal{R}_D)$ and also that $\mathcal{R}_{\mathcal{F}, \mathcal{M}}(X) = \mathcal{R}_D(X)$, as well as $\mathcal{R}_{\mathcal{F}, \mathcal{M}}^A(\mathcal{R}_{\mathcal{F}, \mathcal{M}}) = \text{DOM}^A(\mathcal{R}_D)$ for any unital stably finite $\mathbb{K}$-algebra $A$, that is, for every $\mathfrak{A} \in (A^{\times s})^d$:

$$\mathfrak{A} \in \text{DOM}^A(\mathcal{R}_{\mathcal{F}, \mathcal{M}}) \iff \left( J_D \otimes 1_A - \sum_{k=1}^{d} (\mathfrak{A}_k - Y_k \otimes 1_A)A^A_{k,D} \right) \text{ is invertible in } A^{(L+s) \times (L+s)}$$

and for such $\mathfrak{A}$ we have $\mathcal{R}_{\mathcal{F}, \mathcal{M}}^A(\mathfrak{A}) = \mathcal{R}_D^A(\mathfrak{A})$.

5. If $S$ is invertible, then $B_k^* \cdot S = C^* \cdot A_k$ implies that $B_k = S^{-1} \cdot A^*_k \cdot C^* = A_k \cdot S^{-1}C^*$ so we can choose $Q = S^{-1}C^*$ to get (5.5).

- On the other hand, suppose that there exists $Q \in \mathbb{K}^{L \times s}$ such that $B_k = A_k \cdot Q$ for all $1 \leq k \leq d$. Thus, $C \cdot A_k = B_k^* \cdot S = Q^* \cdot A_k^* \cdot S = Q^*S \cdot A_k$ and also $C \cdot B_k = C \cdot A_k \cdot Q = Q^*S \cdot A_k \cdot Q = Q^*S \cdot B_k$, but the controllability of $(A, B)$ implies that $C = Q^*S$. Therefore, $\text{ker}(S) \subseteq \text{ker}(C)$ but as

$$\text{ker}(S) = \bigcap_{1 \leq k \leq d, X \in \mathbb{K}^{s \times s}} \ker (A_k(X)),$$

it follows from the observability of $(C, A)$ that $\text{ker}(S) = \{0\}$, that is, that $S$ is invertible.

- If $S$ is invertible, then $J$ is invertible and the realization (5.6) is obtained from the realization (5.3) immediately.

- Finally, from (5.6) we get

$$\mathcal{R}_{\mathcal{F}, \mathcal{M}}(X) = D + \hat{C} J \left( J - \sum_{k=1}^{d} \tilde{A}_k(X_k - Y_k) \right)^{-1} \left( \sum_{k=1}^{d} \tilde{A}_k(X_k - Y_k) - J + J \right) \hat{C}^*$$

$$= D - \hat{C} J \hat{C}^* + \hat{C} \left( J - \sum_{k=1}^{d} \tilde{A}_k(X_k - Y_k) \right)^{-1} \hat{C}^*,$$

that is the realization in (5.7)

\[ \square \]

Remark 5.3. If $\mathcal{R}$ is an hermitian nc rational function, there exists $Y \in \text{dom}(\mathcal{R})$ such that $Y^* = \sum Y$; for a proof see [82, pp. 28–29].

Remark 5.4. We leave it to a future work, to describe connections between properties of the (semi-)signature matrices, $J$ and $J_D$, that appear in Theorem 5.2, and properties of the function $\mathcal{R}$, such as perhaps (matrix) convexity (cf. [43]).
Remark 5.5. We say that an nc rational function $\mathcal{R}$ admits a descriptor realization

$$\mathcal{R}_D(X) = C_D \left( I_{L_D} - \sum_{k=1}^{d} A_{k,D}(X_k - Y_k) \right)^{-1} B_D$$

centred at $Y \in (\mathbb{K}^{s \times s})^d$, if $\text{dom}_{sm}(\mathcal{R}) \subseteq \text{DOM}_{sm}(\mathcal{R}_D)$ and $\mathcal{R}(X) = \mathcal{R}_D(X)$ for every $X \in \text{dom}_{sm}(\mathcal{R})$, cf. Definition 2.3. We present some relations between Fornasini–Marchesini realizations and descriptor realizations (not necessarily in the symmetric case), without precise definitions of controllability and observability, as well as the McMillan degree (denoted by $\text{Deg}_{s,D}(\mathcal{R})$), of descriptor realization:

- If $\mathcal{R}$ admits a descriptor realization centred at $Y$, described by $(L_D, C_D, A_{k,D}, B_D)$, then it admits an nc Fornasini–Marchesini realization described by

$$L_{F,M} = L_D, D_{F,M} = C_D B_D, C_{F,M} = C_D, A_{k,F,M} = A_{k,D} \text{ and } B_{k,F,M} = A_{k,D} B_D,$$

for $1 \leq k \leq d$, $\mathcal{R}_D$ is observable if and only if $\mathcal{R}_{F,M}$ is observable, and if $\mathcal{R}_{F,M}$ is controllable, then $\mathcal{R}_D$ is controllable. Therefore we have the relation

$$m(\mathcal{R}) s \leq \text{Deg}_{s,D}(\mathcal{R}). \quad (5.12)$$

- If $\mathcal{R}$ admits an nc Fornasini–Marchesini realization centred at $Y \in (\mathbb{K}^{s \times s})^d$, described by $(L_{F,M}, D_{F,M}, C_{F,M}, A_{F,M}, B_{F,M})$, then $\mathcal{R}$ admits a descriptor realization described by

$$L_D = L_{F,M} + s, C_D = \begin{bmatrix} C_{F,M} & D_{F,M} \end{bmatrix}, A_{k,D} = \begin{bmatrix} A_{k,F,M} & B_{k,F,M} \end{bmatrix} \text{ and } B_D = \begin{bmatrix} 0_{L_{F,M} \times s} \\ I_s \end{bmatrix},$$

for $1 \leq k \leq d$, $\mathcal{R}_{F,M}$ is controllable if and only if $\mathcal{R}_D$ is controllable, and if $\mathcal{R}_D$ is observable, then $\mathcal{R}_{F,M}$ is observable. Therefore we have the relation

$$\text{Deg}_{s,D}(\mathcal{R}) \leq (m(\mathcal{R}) + 1)s. \quad (5.13)$$

- The inequalities (5.12) and (5.13) imply that

$$m(\mathcal{R}) s \leq \text{Deg}_{s,D}(\mathcal{R}) \leq (m(\mathcal{R}) + 1)s,$$

whereas an analogue of Lemma 2.23 for descriptor realizations guarantees that $s \mid \text{Deg}_{s,D}(\mathcal{R})$, which then imply that

$$\text{Deg}_{s,D}(\mathcal{R}) = m(\mathcal{R}) s \text{ or } \text{Deg}_{s,D}(\mathcal{R}) = (m(\mathcal{R}) + 1)s.$$

Acknowledgements. The authors would like to thank Joseph Ball, Bill Helton, Dmitry Kaluzhnyi-Verbovetskyi, Roland Speicher and Jurij Volčič for their helpful comments and discussions. The authors would also like to thank the referees for their valuable comments which helped to improve the paper.

References
1. J. Agler and J. E. McCarthy, ‘Global holomorphic functions in several non-commuting variables’, Canad. J. Math. 67 (2015) 241–285.
2. J. Agler and J. E. McCarthy, ‘Pick interpolation for free holomorphic functions’, Amer. J. Math. 137 (2015) 1685–1701.
3. J. Agler and J. E. McCarthy, ‘Aspects of non-commutative function theory’, Concr. Oper. 3 (2016) 15–24.
4. D. Alpay and D. S. Kaluzhnyi-Verbovetskyi, ‘On the intersection of null spaces for matrix substitutions in a non-commutative rational formal power series’, C. R. Math. 339 (2004) 533–538.
5. D. Alpay and D. S. Kaluzhnyi-Verbovetskyi, ‘Matrix $J$-unitary noncommutative rational formal power series’, The state space method generalizations and applications, Operator Theory: Advances and Applications 161 (eds D. Alpay and I. Gohberg; Birkhäuser-Verlag, Basel, 2005) 49–113.
6. S. A. AMITSUR, ‘Rational identities and applications to algebra and geometry’, J. Algebra 3 (1966) 304–359.
7. J. A. BALL and N. COHEN, ‘De Branges–Rovnyak operator models and systems theory: a survey’, Topics in Matrix and Operator Theory, Operator Theory: Advances and Applications 50 (eds H. Bart, Y. Z. Golberg and M. A. Kaashoek; Birkhäuser-Verlag, Boston, 1991) 93–136.
8. J. A. BALL, G. GROENEWALD and T. MALAKORN, ‘Structured noncommutative multidimensional linear systems’, SIAM J. Control Optim. 44 (2005) 1474–1528.
9. J. A. BALL, G. GROENEWALD and T. MALAKORN, ‘Bounded real lemma for structured noncommutative multidimensional linear systems and robust control’, Multidimens. Syst. Signal Process. 17 (2006) 119–150.
10. J. A. BALL, G. GROENEWALD and T. MALAKORN, ‘Conservative structured noncommutative multidimensional linear systems’, The state space method generalizations and applications, Operator Theory: Advances and Applications 161 (eds D. Alpay and I. Gohberg; Birkhäuser, Basel, 2006) 179–223.
11. J. A. BALL and D. S. KALUZHENYI-VERBOVETS’KIY, ‘Conservative dilations of dissipative multidimensional systems: the commutative and non-commutative settings’, Multidimens. Syst. Signal Process. 19 (2008) 79–122.
12. J. A. BALL and V. VYNNIKOV, ‘Lax–Phillips scattering and conservative linear systems: a Cuntz-algebra multidimensional setting’, Mem. Amer. Math. Soc. 178 (2005).
13. H. BART, I. GOHBERG and M. A. KAASHOEK, ‘Minimal factorization of matrix and operator functions’, Operator theory: advances and applications (Birkhäuser-Verlag, Basel, 1979).
14. C. BECK, ‘On formal power series representations for uncertain systems’, IEEE Trans. Automat. Control 46 (2001) 314–319.
15. C. L. BECK, J. DOYLE and K. GLOVER, ‘Model reduction of multidimensional and uncertain systems’, IEEE Trans. Automat. Control 41 (1996) 1466–1477.
16. S. T. BELINSCHI, T. MAI and R. SPEICHER, ‘Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem’, J. Reine Angew. Math. 2017 (2017) 21–53.
17. G. M. BERGMAN, ‘Skew fields of noncommutative rational functions, after Amitsur’, Séminaire Schützenberger–Lentin–Nivat, Année 16 (Paris, 1969/70).
18. J. BERSTEL and C. REUTENAUER, Rational series and their languages, EATCS Monographs on Theoretical Computer Science 12 (Springer, Berlin, 1988).
19. J. F. CAMINO, J. W. HELTON, R. E. SKELTON and J. YE, ‘Matrix inequalities: a symbolic procedure to determine convexity automatically’, Integral Equations Operator Theory 46 (2003) 399–454.
20. P. M. COHN, ‘On the embedding of rings in skew fields’, Proc. Lond. Math. Soc. 3 (1961) 511–530.
21. P. M. COHN, Free rings and their relations, London Mathematical Society Monographs 2 (Academic Press, London, 1971).
22. P. M. COHN, ‘The embedding of firs in skew fields’, Proc. Lond. Math. Soc. 3 (1971) 193–213.
23. P. M. COHN, Universal skew fields of fractions’, Symposia Mathematica 8 (1972) 135–148.
24. P. M. COHN, ‘The universal field of fractions of a semifir I. Numerators and denominators’, Proc. Lond. Math. Soc. 3 (1982) 1–32.
25. P. M. COHN, ‘Skew fields’, Theory of general division rings, Encyclopedia of Mathematics and its Applications 57 (Cambridge University Press, Cambridge, 1995).
26. P. M. COHN, Free ideal rings and localization in general rings, New Mathematical Monographs 3 (Cambridge University Press, Cambridge, 2006).
27. P. M. COHN and C. REUTENAUER, ‘A normal form in free fields’, Canad. J. Math. 46 (1994) 517–531.
28. P. M. COHN and C. REUTENAUER, ‘On the construction of the free field’, Internat. J. Algebra Comput. 9 (1999) 307–323.
29. E. G. EFFROS, ‘Advances in quantized functional analysis’, Proceedings of the International Congress of Mathematicians (Berkeley, 1986) 906–916.
30. M. FLIESS, ‘Sur le plongement de l’algèbre des séries rationnelles non commutatives dans un corps gauche’, C. R. Acad. Sci. Ser. A 271 (1970) 926–927.
31. M. FLIESS, ‘Matrices de Hankel’, J. Math. Pures Appl. 53 (1974) 197–222.
32. M. FLIESS, ‘Sur divers produits de series formelles’, Bull. Soc. Math. France 102 (1974) 181–191.
33. E. FORNASISI and G. MARCHESINI, ‘Doubly-indexed dynamical systems: state-space models and structural properties’, Math. Syst. Theory 12 (1978–1979) 59–72.
34. E. FORNASISI and G. MARCHESINI, ‘A critical review of recent results on 2-D system theory’, Control Science and Technology for the Progress of Society 1 (Kyoto, 1981), IFAC, Laxenburg, (1982) 255–261.
35. K. GALKOWSKI, ‘Minimal state-space realization for a class of linear, discrete, nD, SISO systems’, Internat. J. Control 74 (2001) 1279–1294.
36. J. W. HELTON, ‘“Positive” noncommutative polynomials are sums of squares’, Ann. of Math. (2) 156 (2002) 655–694.
37. J. W. HELTON, Manipulating matrix inequalities automatically’, Mathematical systems theory in biology, communications, computation, and finance (eds J. Rosenthal and D. S. Gilliam; Springer, New York, 2003) 237–256.
38. J. W. HELTON, I. KLEP and S. MCCULLOUGH, ‘Analytic mappings between noncommutative pencil balls’, J. Math. Anal. Appl. 376 (2011) 407–428.
39. J. W. HELTON, I. KLEP and S. MCCULLOUGH, ‘Proper Analytic Free Maps’, J. Funct. Anal. 260 (2011) 1476–1490.
40. J. W. Helton, I. Klep, S. McCullough and J. Volčič, ‘Noncommutative polynomials describing convex sets’, Found. Comput. Math. 21 (2021) 575–611.
41. J. W. Helton, T. Mai and R. Speicher, ‘Applications of realizations (aka linearizations) to free probability’, J. Funct. Anal. 274 (2018) 1–79.
42. J. W. Helton and S. McCullough, ‘Every convex free basic semi-algebraic set has an LMI representation’, Ann. of Math. (2) 176 (2012) 979–1013.
43. J. W. Helton, S. McCullough and V. Vinnikov, ‘Noncommutative convexity arises from linear matrix inequalities’, J. Funct. Anal. 240 (2006) 105–191.
44. R. A. Horn and C. R. Johnson, Topics in matrix analysis, Corrected reprint of the 1991 original (Cambridge University Press, Cambridge, MA, 1994).
45. I. Hughes, ‘Division rings of fractions for group rings’, Comm. Pure Appl. Math. 23 (1970) 181–188.
46. F. K. Jørgensen, Two dimensional linear systems, Lecture Notes in Control and Information Sciences 68 (Springer, Berlin, 1985).
47. D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, ‘Noncommutative rational functions, their difference-differential calculus and realizations’, Multidimens. Syst. Signal Process. 23 (2012) 49–77.
48. D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Foundations of free noncommutative function theory, Mathematical Surveys and Monographs 199 (American Mathematical Society, Providence, RI, 2014).
49. D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, ‘Singularities of rational functions and minimal factorizations: the noncommutative and the commutative settings’, Linear Algebra Appl. 430 (2009) 869–889.
50. R. E. Kalman, M. A. Arbib and P. L. Falb, Topics in mathematical systems theory (McGraw Hill, New York, NY, 1969).
51. S. C. Kleene, ‘Representation of events in nerve nets and finite automata’, Automata Studies, Annals of Mathematics Studies 34 (Princeton University Press, Princeton, NJ, 1956), 3–41.
52. J. Lewin, ‘Fields of fractions for group algebras of free groups’, Trans. Amer. Math. Soc. 192 (1974) 339–346.
53. A. I. Lichtman, ‘On universal fields of fractions for free algebras’, J. Algebra 231 (2000) 652–676.
54. P. A. Linnell, ‘Division rings and group von Neumann algebras’, Forum Math. 5 (1993) 561–576.
55. P. A. Linnell, ‘Noncommutative localization in group rings’, Non-commutative localization in algebra and topology, London Mathematical Society Lecture Note Series 330 (ed. Andrew Ranicki; Cambridge University Press, Cambridge, 2006) 40–59.
56. W. M. Lu, K. Zhou and J. C. Doyle, ‘Stabilization of uncertain linear systems: an LFT approach’, IEEE Trans. Automat. Control 41 (1996) 58–65.
57. F. M. Muhly and B. Solel, ‘Progress in noncommutative function theory’, Sci. China Math. 54 (2011) 2275–2294.
58. A. Nemirovskii, ‘Advances in convex optimization: conic programming’, Plenary Lecture, International Congress of Mathematicians 1, (Madrid, Spain, 2006) 413–444.
59. Y. Nesterov and A. Nemirovskii, Interior-point polynomial algorithms in convex programming, Studies in Applied Mathematics 13 (SIAM, Philadelphia, PA, 1994).
60. V. Paulsen, Completely bounded maps and operator algebras, Cambridge Studies in Advanced Mathematics 78 (Cambridge University Press, Cambridge, MA, 2002).
61. G. Popescu, ‘Free holomorphic functions on the unit ball of $B(H)^{n^\nu}$’, J. Funct. Anal. 241 (2006) 268–333.
62. G. Popescu, ‘Free holomorphic automorphisms of the unit ball of $B(H)^{n^\nu}$’, J. Reine Angew. Math. 638 (2010) 119–168.
63. M. Porat and V. Vinnikov, ‘Realizations of non-commutative rational functions around a matrix centre, II: the lost-abbey conditions’, to appear, https://urldefense.com/v3/__N11eVZ2wtsf59JXbrjft659pQxQeU985k7o0HBZRD7kCI9A3T9pYd6sidxggWwAf_R08kikhTg%8__https://arxiv.org/abs/2009.08527.
64. M. Porat and V. Vinnikov, ‘Realizations of non-commutative rational functions around a matrix centre, III: functional models, Kronecker–Fliess theorem and the free skew field’, to appear.
65. C. Reutenauer, ‘Malcev–Neumann series and the free field’, Expo. Math. 17 (1999) 469–478.
66. L. H. Rowen, Polynomial identities in ring theory, Pure and Applied Mathematics 84 (Academic Press, New York–London, 1980).
67. K. Schrempf, ‘A standard form in (some) free fields: How to construct minimal linear representations’, Open Math. 18 (2020) 1365–1386.
68. K. Schrempf, Free fractions: An invitation to (applied) free fields. Preprint, 2018, ArXiv preprint. https://arxiv.org/abs/1809.05425.
69. K. Schrempf, ‘Linearizing the word problem in (some) free fields’, Internat. J. Algebra Comput. 28 (2018) 1209–1230.
70. K. Schrempf, ‘On the factorization of non-commutative polynomials (in free associative algebras)’, J. Symbolic Comput. 94 (2019) 126–148.
71. M. P. Schützenberger, ‘On the definition of a family of automata’, J. Symbolic Comput. 4 (1981) 245–270.
72. M. P. Schützenberger, ‘Certain elementary families of automata’, Proceedings of the Symposium on Mathematical Theory of Automata (New York, 1962) (Polytechnic Press of Polytechnic Institute of Brooklyn, Brooklyn, New York, 1963) 139–153.
73. R. E. Skelton, T. Iwasaki and K. M. Grigoriadis, A unified algebraic approach to linear control design (Taylor & Francis, London, 1997).

74. R. Speicher, ‘Free probability theory’, The Oxford handbook of random matrix theory (eds G. Akemann, J. Baik and P. Di Francesco; Oxford University Press, Oxford, 2011) 452–470.

75. R. Speicher, ‘Polynomials in asymptotically free random matrices’, Acta Phys. Polon. B 46 (2015) 1611–1624.

76. J. L. Taylor, ‘A general framework for a multi-operator functional calculus’, Adv. Math. 9 (1972) 183–252.

77. J. L. Taylor, ‘Functions of several noncommuting variables’, Bull. Amer. Math. Soc. (N.S.) 79 (1973) 1–34.

78. D. V. Voiculescu, ‘Free analysis questions I: duality transform for the coalgebra of ∂X,B’, Int. Math. Res. Not. 16 (2004) 793–822.

79. D. V. Voiculescu, ‘Free analysis questions II: the Grassmannian completion and the series expansion at the origin’, J. Reine Angew. Math. 645 (2010) 155–236.

80. D. V. Voiculescu, K. J. Dykema and A. Nica, ‘Free random variables’, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups, CRM Monograph Series 1 (American Mathematical Society, Providence, RI, 1992).

81. J. Volčič, ‘On domains of noncommutative rational functions’, Linear Algebra Appl. 516 (2017) 69–81.

82. J. Volčič, ‘Matrix coefficient realization theory of noncommutative rational functions’, J. Algebra 499 (2018) 397–437.

Motke Porat and Victor Vinnikov
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
Beer-Sheva 84105
Israel
motpor@gmail.com
vinnikov@math.bgu.ac.il