Multiplicity Free Spaces with a One-Dimensional Quotient

Hubert Rubenthaler

Communicated by J. Faraut

Abstract. The multiplicity free spaces with a one dimensional quotient were introduced by Thierry Levasseur in [12]. Recently, the author has shown that the algebra of differential operators on such spaces which are invariant under the semi-simple part of the group is a Smith algebra ([18]). We give here the classification of these spaces which are indecomposable, up to geometric equivalence. We also investigate whether or not these spaces are regular or of parabolic type as a prehomogeneous vector space.

Mathematics Subject Classification 2010: 14L30, 11S90, 22E46.

Key Words and Phrases: Multiplicity free spaces, one dimensional quotient, prehomogeneous vector spaces.

1. Introduction

A multiplicity free space is a representation of a connected reductive group $G$ on a finite dimensional vector space $V$ (everything is defined over $\mathbb{C}$) such that every irreducible representation of $G$ appears at most once in the associated representation of $G$ on the space $\mathbb{C}[V]$ of polynomials on $V$ (see Section 2 for details). For a survey on multiplicity free spaces we refer to [1]. Multiplicity free spaces, which were introduced by V. Kac in [7], play now an important role in invariant theory and harmonic analysis (see for example [5], [6], [10], the references in [1], see also [9] for a more general concept). Various characterizations of multiplicity free spaces, which are summarized in Theorem 2.6 below, were obtained by Vinberg-Kimelfeld ([24]), Howe-Umeda ([6]), Knop ([10]). A corollary of these characterizations is that a multiplicity free space is always a prehomogeneous space (in fact even under a Borel subgroup). This is the reason why prehomogeneous vector spaces occur so often in this paper. The classification of multiplicity free spaces was achieved independently by Benson-Ratcliff ([2]) and Leahy ([11]) after partial classifications by Brion ([4]) and Kac ([7]).

In this paper we are interested in a specific family of multiplicity free spaces, the so-called multiplicity free spaces with a one dimensional quotient which were introduced by T. Levasseur in [12]. This means roughly speaking that the categorical quotient $V//G'$ has dimension one, where $G'$ is the semi-simple part of
In his paper T. Levasseur proves that if \((G, V)\) is a multiplicity free space with a one dimensional quotient, then the radial component of the (non-commutative) algebra \(D(V)^{G'}\) of \(G'\)-invariant differential operators is a Smith algebra over \(\mathbb{C}\). In a recent paper ([18]) we showed that the full algebra \(D(V)^{G'}\) is a Smith algebra over its center, which is a polynomial algebra.

The purpose of this paper is to give the complete classification of all multiplicity free spaces with a one dimensional quotient, including irreducible and non irreducible representations. Our classification is obtained up to geometric equivalence, which is the natural equivalence relation among multiplicity free spaces. It is worthwhile noticing that the list of irreducibles already appears in [12]. Moreover our investigations lead in all cases (irreducibles and non irreducibles) to some extra informations like parabolicity, regularity, and explicit fundamental relative invariants of the underlying prehomogeneous vector spaces.

In section 2 we recall general facts about multiplicity free space. We give first a brief account of the theory of prehomogeneous vector spaces and also recall the definition of parabolic type prehomogeneous spaces (2.2), including their weighted Dynkin diagrams which encode many informations (see Definition 2.2 and Remark 2.3). General definitions and results about multiplicity free spaces can be found in 2.3. In 2.5, as an example, we describe an important family of irreducible multiplicity free space with a one dimensional quotient, namely the irreducible regular prehomogeneous vector spaces of commutative parabolic type.

Section 3 contains the main result, the classification theorem of the multiplicity free spaces with a one dimensional quotient (Theorem 3.3). The corresponding lists (Tables 2 and 3) take place at the end of the paper.

Section 4 is devoted to the proof of Theorem 3.3. The proof uses case by case examinations from the list by Benson and Ratcliff ([2]) and some tools from the theory of prehomogeneous vector spaces.

Notations: In this paper we will denote by \(GL(n)\), \(SL(n)\), \(SO(n)\) the general linear group, the special linear group, the special orthogonal group of complex matrices of size \(n\) respectively. As usual we will denote by \(Sp(n)\) the symplectic group of \(2n \times 2n\) complex matrices. We will also denote by \(gl(n)\), \(sl(n)\), \(o(n)\), \(sp(n)\) the corresponding Lie algebras. The vector space of \(m \times n\) complex matrices will be denoted by \(M_{m,n}\), and the space of square \(n \times n\) matrices will be denoted by \(M_n\). Finally \(Sym(n)\) will denote the space of \(n \times n\) symmetric matrices and \(AS(n)\) will denote the space of skew symmetric matrices. If \(n\) is even and if \(x \in AS(n)\), then \(Pf(x)\) stands for the pfaffian of the matrix \(x\).

2. Multiplicity free spaces. Basic definitions and properties

2.1. Prehomogeneous Vector Spaces.

Let \(G\) be a connected algebraic group over \(\mathbb{C}\), and let \((G, \rho, V)\) be a rational representation of \(G\) on the (finite dimensional) vector space \(V\). Then the triplet \((G, \rho, V)\) is called a prehomogeneous vector space (abbreviated to \(PV\)) if the action of \(G\) on \(V\) has a Zariski open orbit \(\Omega \subseteq V\). For the general theory of \(PV\)'s, we
refer the reader to the book of Kimura [8] or to [21]. The elements in \( \Omega \) are called \textit{generic}. The \( PV \) is said to be \textit{irreducible} if the corresponding representation is irreducible. The \textit{singular set} \( S \) of \( (G,\rho,V) \) is defined by \( S = V \setminus \Omega \). Elements in \( S \) are called \textit{singular}. If no confusion can arise we often simply denote the \( PV \) by \( (G,V) \). We will also write \( g.x \) instead of \( \rho(g)x \), for \( g \in G \) and \( x \in V \). It is easy to see that the condition for a rational representation \( (G,\rho,V) \) to be a \( PV \) is in fact an infinitesimal condition. More precisely let \( \mathfrak{g} \) be the Lie algebra of \( G \) and let \( d\rho \) be the derived representation of \( \rho \). Then \( (G,\rho,V) \) is a \( PV \) if and only if there exists \( v \in V \) such that the map:

\[
\begin{array}{ccc}
\mathfrak{g} & \longrightarrow & V \\
X & \mapsto & d\rho(X)v
\end{array}
\]

is surjective (we will often write \( X.v \) instead of \( d\rho(X)v \)). Therefore we will call \((\mathfrak{g},V)\) a \( PV \) if the preceding condition is satisfied.

Let \((G,V)\) be a \( PV \). A rational function \( f \) on \( V \) is called a \textit{relative invariant} of \((G,V)\) if there exists a rational character \( \chi \) of \( G \) such that \( f(g.x) = \chi(g)f(x) \) for \( g \in G \) and \( x \in V \). From the existence of an open orbit it is easy to see that a character \( \chi \) which is trivial on the isotropy subgroup of an element \( x \in \Omega \) determines a unique relative invariant \( P_\chi \). Let \( S_1, S_2, \ldots, S_k \) denote the irreducible components of codimension one of the singular set \( S \). Then there exist irreducible polynomials \( P_1, P_2, \ldots, P_k \) such that \( S_i = \{ x \in V | P_i(x) = 0 \} \). The \( P_i \)'s are unique up to nonzero constants. It can be proved that the \( P_i \)'s are relative invariants of \((G,V)\) and any nonzero relative invariant \( f \) can be written in a unique way \( f = cP_{i_1}^{n_1}P_{i_2}^{n_2}\ldots P_{i_k}^{n_k} \), where \( n_i \in \mathbb{Z} \) and \( c \in \mathbb{C}^* \). The polynomials \( P_1, P_2, \ldots, P_k \) are called the \textit{fundamental relative invariants} of \((G,V)\). Moreover if the representation \((G,V)\) is irreducible then there exists at most one irreducible polynomial (up to multiplication by a non zero constant) which is relatively invariant.

The prehomogeneous vector space \((G,V)\) is called \textit{regular} if there exists a relative invariant polynomial \( P \) whose Hessian \( H_P(x) \) is nonzero on \( \Omega \). If \( G \) is reductive, then \((G,V)\) is regular if and only if the singular set \( S \) is a hypersurface, or if and only if the isotropy subgroup of a generic point is reductive. If the \( PV \) \((G,V)\) is regular, then the contragredient representation \((G,V^*)\) is again a \( PV \). Regular \( PV \)'s are of particular interest, due to the zeta functions that one can associate to their real forms ([22]).

\textbf{Remark 2.1.} Let us mention a well known Lemma from the Theory of \( PV \)'s, which will be used in section 4. If \((G,V)\) is a \( PV \), and if \( X_0 \) is a generic point, then the characters arising as characters of relative invariants are the characters of the quotient group \( G/H \) where \( H \) is the normal subgroup of \( G \) generated by the derived group \([G,G]\) and the generic isotropy subgroup \( G_{X_0} \). This group does not depend on \( X_0 \). For details, see [8], Proposition 2.12. p.28.

\textbf{2.2. PV's of parabolic type.}

A \( PV \) \((G,V)\) is called reductive if the group \( G \) is reductive. Among
the reductive PV’s there is a family of particular interest, the so-called PV’s of parabolic type. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Sigma$ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We fix once and for all a system of simple roots $\Psi$ for $\Sigma$. We denote by $\Sigma^+$ (resp. $\Sigma^-$) the corresponding set of positive (resp. negative) roots in $\Sigma$. Let $\theta$ be a subset of $\Psi$ and let us make the standard construction of the parabolic subalgebra $p_\theta \subset \mathfrak{g}$ associated to $\theta$. As usual we denote by $\langle \theta \rangle$ the set of all roots which are linear combinations of elements in $\theta$, and put $\langle \theta \rangle^\pm = \langle \theta \rangle \cap \Sigma^\pm$. Set $h_\theta = \theta^\perp = \{ X \in \mathfrak{h} | \alpha(X) = 0 \ \forall \alpha \in \theta \}$, $l_\theta = \mathfrak{z}_\mathfrak{g}(h_\theta) = \mathfrak{h} \oplus \sum_{\alpha \in \langle \theta \rangle} \mathfrak{g}^\alpha$, $n_\theta^\pm = \sum_{\alpha \in \Sigma^\pm \setminus \langle \theta \rangle^\pm} \mathfrak{g}^\alpha$.

Then $p_\theta = l_\theta \oplus n_\theta^+$ is the standard parabolic subalgebra associated to $\theta$. There is also a standard $\mathbb{Z}$-grading of $\mathfrak{g}$ related to these data. Define $H_\theta$ to be the unique element of $h_\theta$ satisfying the linear equations

$$\alpha(H_\theta) = 0 \ \forall \alpha \in \theta \ \text{and} \ \alpha(H_\theta) = 2 \ \forall \alpha \in \Psi \setminus \theta.$$ 

The above mentioned grading is just the grading obtained from the eigenspace decomposition of $\text{ad} H_\theta$:

$$d_p(\theta) = \{ X \in \mathfrak{g} | [H_\theta, X] = 2pX \}.$$ 

Then we obtain easily:

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} d_p(\theta), \ \ l_\theta = d_0(\theta), \ \ n_\theta^+ = \sum_{p \geq 1} d_p(\theta), \ \ n_\theta^- = \sum_{p \leq -1} d_p(\theta).$$ 

It is known (using a result of Vinberg [23]) that $(l_\theta, d_1(\theta))$ is a prehomogeneous vector space. In fact all the spaces $(l_\theta, d_p(\theta))$ with $p \neq 0$ are prehomogeneous, but there is no loss of generality if we only consider $(l_\theta, d_1(\theta))$. These spaces have been called prehomogeneous vector spaces of parabolic type ([14]). They are in general neither irreducible nor regular. But they are of particular interest, because in the parabolic context, the group (or more precisely its Lie algebra $l_\theta$) and the space (here $d_1(\theta)$) of the PV are embedded into a rich structure, namely the simple Lie algebra $\mathfrak{g}$. For example the derived representation of the PV is just the adjoint representation of $l_\theta$ on $d_1(\theta)$. Moreover the Lie algebra $\mathfrak{g}$ also contains the dual PV, namely $(l_\theta, d_{-1}(\theta))$.

There is an easy criterion to decide whether or not an irreducible PV of parabolic type is regular and in fact most of the reduced irreducible reductive regular PV’s from Sato-Kimura list are of parabolic type (for details we refer to [15],[16] and [17]).

As these PV’s are in one to one correspondence with the subsets $\theta \subset \Psi$, we will describe them by means of the following weighted Dynkin diagram:
Definition 2.2. The diagram of the $PV$ $(\mathfrak{l}_\theta, d_1(\theta))$ is the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ (or $\Sigma$) where the vertices corresponding to the simple roots of $\Psi \setminus \theta$ are circled (see an example below).

This very simple classification by means of diagrams contains nevertheless some immediate and interesting information concerning the $PV$ $(\mathfrak{l}_\theta, d_1(\theta))$ (for all these facts, see [14], [15], [16] or [17]):

Remark 2.3. a) The Dynkin diagram of $\mathfrak{l}_\theta = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ (i.e. the semi-simple part of the Lie algebra of the group) is the Dynkin diagram of $\mathfrak{g}$ where we have removed the circled vertices and the edges connected to these vertices.

b) In fact as a Lie algebra $\mathfrak{l}_\theta = \mathfrak{l}_\theta' \oplus \mathfrak{h}_\theta$ and $\dim \mathfrak{h}_\theta = \text{number of circled vertices}.

c) The number of irreducible components of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ is also equal to the number of circled roots, and hence the parabolic $PV$ $(\mathfrak{l}_\theta, d_1(\theta))$ is irreducible if and only if $p_\theta$ is maximal. More precisely, if $\alpha$ is a (simple) circled root, then any nonzero root vector $X_\alpha \in \mathfrak{g}^\alpha$ generates an irreducible $\mathfrak{l}_\theta$-module $V_\alpha$, and $d_1(\theta) = \oplus_{\alpha \in \Psi \setminus \theta} V_\alpha$ is the decomposition of $d_1(\theta)$ into irreducibles.

The decomposition of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ into irreducibles can also be described by using the eigenspace decomposition with respect to $\text{ad}(\mathfrak{h}_\theta)$, as we will explain now. For each $\alpha \in \mathfrak{h}_\theta^*$, let $\overline{\alpha}$ be the restriction of $\alpha$ to $\mathfrak{h}_\theta$ and define

$$\mathfrak{g}^{\overline{\alpha}} = \{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}_\theta, [H, X] = \overline{\alpha}(H)X \}. $$

Then $\mathfrak{g}^{\overline{\alpha}} = \mathfrak{l}_\theta$ and for $\alpha \in \Psi \setminus \theta$, we have $V_\alpha = \mathfrak{g}^{\overline{\alpha}}$. Hence we can write $d_1(\theta) = \oplus_{\alpha \in \Psi \setminus \theta} \mathfrak{g}^{\overline{\alpha}}$.

Moreover one can notice (always for $\alpha \in \Psi \setminus \theta$) that $V_\alpha = \mathfrak{g}^{\overline{\alpha}} = \sum_{\beta \in \sigma_\alpha^1} \mathfrak{g}^{\beta}$, where $\sigma_\alpha^1$ is the set of roots which belong to $\alpha + \text{span}(\theta)$.

d) One can also directly read the highest weight of $V_\alpha$ from the diagram. The highest weight of $V_\alpha$ relatively to the negative Borel sub-algebra $\mathfrak{b}^{-}_\alpha = \mathfrak{h} \oplus \sum_{\alpha \in \theta} \mathfrak{g}^{\alpha}$ is $\overline{\alpha} = \alpha|_{\mathfrak{h}(\theta)}$. Let $\omega_\beta (\beta \in \theta)$ be the fundamental weights of $\mathfrak{l}_\theta'$ (i.e. the dual basis of $(H_\beta)_{\beta \in \theta}$). For each circled root $\alpha$ (i.e. for each $\alpha \in \Psi \setminus \theta$), let $J_\alpha = \{(\beta_i)\}$ be the set of roots in $\theta$ (= non-circled) which are connected to $\alpha$ in the diagram. From elementary diagram considerations we know that $J_\alpha$ may be empty and that there are always no more than 3 roots in $J_\alpha$.

If $J_\alpha = \emptyset$, then $V_\alpha$ is the trivial one dimensional representation of $\mathfrak{l}_\theta$.

If $J_\alpha \neq \emptyset$, then the highest weight $\overline{\alpha}$ of $V_\alpha$ is given by $\overline{\alpha} = \sum_{i \in J_\alpha} c_i \omega_{\beta_i}$, where $c_i = \alpha(H_{\beta_i})$ and where $\alpha(H_{\beta_i})$ can be computed as follows:

$$(R) \begin{cases} 
\text{if } ||\alpha|| \leq ||\beta_i||, \text{ then } \alpha(H_{\beta_i}) = -1; \\
\text{if } ||\alpha|| > ||\beta_i|| \text{ and if } \alpha \text{ and } \beta_i \text{ are connected by } j \text{ arrows } (1 \leq j \leq 3), \\
\text{ then } \alpha(H_{\beta_i}) = -j .
\end{cases}$$

Let us illustrate this with an example.

Example 2.4. Consider the following diagram:
This diagram is the diagram of a $PV$ of parabolic type inside $\mathfrak{g} \simeq \mathfrak{sp}(7) \simeq C_7$. The Lie algebra $\mathfrak{b}_o$ is isomorphic to $A_5 \oplus \mathfrak{h}_o \simeq \mathfrak{sl}(6) \oplus \mathfrak{h}_o$ where $\dim \mathfrak{h}_o =$ number of circled roots $= 2$. There are two irreducible components $V_{\alpha_1}$ and $V_{\alpha_2}$, and the highest weight of $(A_5, V_{\alpha_1})$ (resp. $(A_5, V_{\alpha_2})$) relatively to the Borel subalgebra $\mathfrak{b}_{-\theta}$ is $\omega_1$ (resp. $2\omega_5$), where $\omega_i$ ($i=1,\ldots,5$) are the fundamental weights of $A_5$ corresponding respectively to $\beta_1,\ldots,\beta_5$.

2.3. Multiplicity free spaces.

For the results concerning multiplicity free spaces we refer the reader to the survey by Benson and Ratcliff ([1]) or to [10]. Let $(G, V)$ be a finite dimensional rational representation of a connected reductive algebraic group $G$. Let $\mathbb{C}[V]$ be the algebra of polynomials on $V$. Then $G$ acts on $\mathbb{C}[V]$ by

$$g.\varphi(x) = \varphi(g^{-1}x) \quad (g \in G, \varphi \in \mathbb{C}[V]).$$

As the space $\mathbb{C}[V]^n$ of homogeneous polynomials of degree $n$ is stable under this action, the representation $(G, \mathbb{C}[V])$ is completely reducible. Let $D(V)$ be the algebra of differential operators with polynomial coefficients. The group $G$ acts also on $D(V)$ by

$$(g.D)(\varphi) = g.(D(g^{-1}.\varphi)) \quad (g \in G, D \in D(V), \varphi \in \mathbb{C}[V]).$$

**Definition 2.5.** Let $G$ be a connected reductive algebraic group, and let $V$ be the space of a finite dimensional (complex) rational representation of $G$. The representation $(G, V)$ is said to be multiplicity free if each irreducible representation of $G$ occurs at most once in the representation $(G, \mathbb{C}[V])$.

From now on ”multiplicity-free” will be abbreviated to ”MF”.

Let us give some results concerning MF spaces (see [1], [6], [10]):

**Theorem 2.6.**

1) A finite dimensional representation $(G, V)$ is MF if and only if $(B, V)$ is a prehomogeneous vector space for any Borel subgroup $B$ of $G$ (and hence each MF space $(G, V)$ is a PV).

2) A finite dimensional representation $(G, V)$ is MF if and only if the algebra $D(V)^G$ of invariant differential operators with polynomial coefficients is commutative.

3) If $(G, V)$ is a MF space, then the dual space $(G, V^*)$ is also MF. More generally if $(G, W \oplus V)$ is a representation of $G$ where $W$ and $V$ are $G$-stable, then $(G, W \oplus V)$ is MF if and only if $(G, W \oplus V^*)$ is MF.

**Proof.** (Indications) Part 1) is due to Vinberg and Kimelfeld ([24]), another proof can be found in [10]. Part 2) is due to Howe and Umeda ([6], Proposition 7.1). The first assertion of Part 3), also noted in [6], is a consequence of the
G-equivariant isomorphism \( \mathbb{C}[V^*] \simeq (\mathbb{C}[V])^* \). The second assertion of 3) is Corollary 3.3 in [11].

Note that the commutativity of \( D(V)^G \) for a MF space is just a consequence of the definition, since we have a simultaneous diagonalization of all the operators in \( D(V)^G \).

If \( (G,V) \) is a MF space, and if \( B \) is a Borel subgroup of \( G \), then, as we have seen \( (G,V) \) and \( (B,V) \) are prehomogeneous spaces. Let us denote by \( \Delta_0, \Delta_1, \ldots, \Delta_k, \ldots, \Delta_r \) the fundamental relative invariants of the \( PV \) \( (B,V) \), indexed in such a way that \( \Delta_0, \Delta_1, \ldots, \Delta_k \) are the relative invariants of the \( PV \) \( (G,V) \) and such that the other invariants are ordered by decreasing degree. It is worthwhile noticing that at least \( \Delta_r \) is of degree one as the highest weight vectors of the irreducible components of \( V^* \) must occur. The nonnegative integer \( r + 1 \) (= the number of fundamental relative invariants under \( B \)) is called the rank of the MF space \( (G,V) \).

2.4. Multiplicity free spaces with a one dimensional quotient.

Let us now define the main objects this paper deals with.

**Definition 2.7.** (T. Levasseur [12])

1) A prehomogeneous vector space \( (G,V) \) is said to be of rank one \(^1\) if there exists an homogeneous polynomial \( \Delta_0 \) on \( V \) such that \( \Delta_0 \notin \mathbb{C}[V]^G \) and such that \( \mathbb{C}[V]^G = \mathbb{C}[\Delta_0] \).

2) A multiplicity free space \( (G,V) \) is said to have a one-dimensional quotient if it is a \( PV \) of rank one. (This implies that the categorical quotient \( V//G' \) has dimension 1.)

Although the following result is implicit in [12] we provide a proof here.

**Proposition 2.8.** If \( (G,V) \) is a \( PV \) of rank one, then the polynomial \( \Delta_0 \) is the unique fundamental relative invariant of \( (G,V) \). More precisely a \( PV \) \( (G,V) \) is of rank one if and only if it has a unique fundamental relative invariant.

**Proof.** We can write \( G = G'C \) where \( G' = [G,G] \) is the derived group of \( G \) and where \( C = Z(G)^0 \simeq (\mathbb{C}^*)^p \) is the connected component of the center of \( G \). Let \( g \in C \). Then \( \Delta_0(g^{-1}x) \) is again \( G' \)-invariant. As \( \mathbb{C}[V]^G = \mathbb{C}[\Delta_0] \) and as \( \Delta_0(g^{-1}x) \) has the same degree as \( \Delta_0 \) we obtain that \( \Delta_0(g^{-1}x) = \chi(g)\Delta_0(x) \) with \( \chi(g) \in \mathbb{C}^* \). Therefore \( \Delta_0 \) is a relative invariant. Suppose that \( \Delta_0 \) is not irreducible. Then \( \Delta_0 = P_1 \ldots P_m \), where the polynomials \( P_i \) are irreducible relative invariants and \( \partial^\circ(P_i) < \partial^\circ(\Delta_0) \) (where \( \partial^\circ(P) \) denotes the degree of the polynomial \( P \)). We should have \( P_i \in \mathbb{C}[\Delta_0] \), which is impossible. Hence \( \Delta_0 \) is irreducible. If \( f \) is another fundamental relative invariant then we would have \( f \in \mathbb{C}[\Delta_0] \) which is impossible.

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\(^1\)It is worth noticing that if \( (G,V) \) is multiplicity free, then its rank as a \( PV \) is not at all the same as its rank as a \( MF \) space.
It remains to prove that if a $PV$ $(G, V)$ has a unique fundamental relative invariant $\Delta_0$ then it is of rank one. As $\Delta_0$ is non constant we have of course that $\Delta_0 \notin \mathbb{C}[V]^G$. Let $P \in \mathbb{C}[V]^G$. If $P = P_0 + P_1 + \cdots + P_m$ with $\partial^i(P_i) = i$, then each $P_i$ is $G'$-invariant. Therefore we can suppose that $P$ has fixed degree $n$ (i.e. $P \in \mathbb{C}[V]^{G'} \cap \mathbb{C}[V]^n$).

Let 
\[
\mathbb{C}[V]^n = \bigoplus_{i=0}^{\infty} M_i
\]
be the decomposition of $\mathbb{C}[V]^n$ into $G'$-isotypic components. We suppose that $M_0$ is the isotypic component of the trivial $G'$-module $(M_0 = \mathbb{C}[V]^{G'} \cap \mathbb{C}[V]^n)$. Hence $P \in M_0$. As $G = CG'$, the group $G$ stabilizes each $M_i$. Therefore we can write
\[
M_0 = \bigoplus M_\chi
\]
where the $G$-isotypic components $M_\chi$ of $M_0$ are indexed by characters $\chi$ of $G$ and given by
\[
M_\chi = \{ \varphi \in M_0 | \varphi(z^{-1}x) = \chi(z) \varphi(x), \forall z \in C, x \in V \}.
\]
Hence $P = \sum P_\chi$. $P_\chi \in M_\chi$, and for $x \in C, g' \in G'$ and $x \in V$ we have $P_\chi(g'x) = \chi^{-1}(z)P_\chi(g'x) = \chi^{-1}(z)P_\chi(x)$. Therefore each $P_\chi$ is a relative invariant. But $(G, V)$ has a unique fundamental relative invariant namely $\Delta_0$. Hence $P_\chi = c_\chi \Delta_0^j$ ($c_\chi \in \mathbb{C}$). The exponent $j$ does not depend on $\chi$, since all the $P_\chi$’s have the same degree. Therefore all the characters $\chi$ are the same, namely $\chi = \lambda_0$ where $\lambda_0$ is the character of $\Delta_0$. This implies that $M_0 = M_{\lambda_0}$, and that $P = c \Delta_0^j$. Hence $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$.

The following result gives a criterion to decide whether or not a $PV$ has rank one. It will be useful in section 4 for the classification of the $MF$ spaces with a one dimensional quotient.

**Proposition 2.9.** Let $G$ be a connected algebraic group and let $(G, V)$ be a $PV$. We suppose that $V = V_1 \oplus V_2$ where $V_1$ and $V_2$ are $G$-stable subspaces, and that $(G, V_1)$ is a rank one $PV$. Let $(x_0, y_0)$ be a generic element in $V$, with $x_0 \in V_1$ and $y_0 \in V_2$. Let $G_{x_0}$ be the isotropy subgroup of $x_0$. We suppose also that the prehomogeneous vector space $(G_{x_0}, V_2)$ has no nontrivial relative invariant (this property does not depend on the choice of $(x_0, y_0)$). Then $(G, V)$ is a rank one $PV$.

**Proof.** For the fact that $(G_{x_0}, V_2)$ is a $PV$ and that $y_0$ is generic for this $PV$ we refer to [19]. As $(G, V_1)$ is a rank one $PV$ it has a unique fundamental relative invariant $f(x)$ by Proposition 2.8. Define $\Delta_0(x, y) = f(x) (x \in V_1, y \in V_2)$. Then, as it is irreducible, $\Delta_0(x, y)$ is a fundamental relative invariant of $(G, V)$. Let $\varphi(x, y)$ be a fundamental relative invariant of $(G, V)$ and consider the function $y \mapsto \varphi(x_0, y)$. For $g \in G_{x_0}$, we have $\varphi(gx_0, y) = \varphi(x_0, gy) = \chi_s(g) \varphi(x_0, y)$. Hence $y \mapsto \varphi(x_0, y)$ is a relative invariant of $(G_{x_0}, V_2)$. But by hypothesis $(G_{x_0}, V_2)$ has no nontrivial relative invariant, hence for all $y \in V_2$, we have
\( \varphi(x_0, y) = \psi(x_0) \) (constant with respect to \( y \)). But as this is true for any generic \( x_0 \in V_1 \), we obtain that \( \varphi(x, y) = \psi(x) \), for all \( x \in V_1 \) and all \( y \in V_2 \). In other words \( \varphi \) does only depend on the \( x \) variable. As \( \varphi \) is irreducible, so is also \( \psi \). And \( \psi \) is then a relative invariant of \( (G, V_1) \), hence \( \psi = cf \) \( (c \in \mathbb{C}) \), or equivalently \( \varphi(x, y) = c\Delta_0(x, y) \). Then Proposition 2.8 implies that \( (G, V) \) is a rank one \( PV \). 

**Notation:** If \( (G, V) \) is a \( MF \) space with a one dimensional quotient, we will sometimes say that \( (G, V) \) is \( QD1 \).

### 2.5. An example: the regular commutative \( PV \)'s of parabolic type.

Among the \( PV \)'s of parabolic type there is a family, the so-called regular commutative \( PV \)'s of parabolic type, which are \( MF \) spaces with a one dimensional quotient. We will give here a brief description of these objects. Notations and conventions are the same as in section 2.2. The \( PV \)'s of parabolic type we are going to describe are irreducible. Therefore there is only one circled root which we denote by \( \alpha_0 \) (and then \( \theta = \Psi \setminus \{ \alpha_0 \} \)). In this section we will impose the extra condition that the coefficient of \( \alpha_0 \) in the highest root is 1. This implies that \( d_p(\theta) = \{ 0 \} \) for \( p \neq 0, 1, -1 \). Hence \( p_\theta = l_\theta \oplus d_1(\theta) \), and the nilradical \( d_1(\theta) \) of \( p_\theta \) is a commutative subalgebra. Therefore the spaces \( (l_\theta, d_1(\theta)) \) are called commutative \( PV \)'s of parabolic type. It is known that these \( PV \)'s are all \( MF \) spaces ([13]). By Proposition 2.8 those which have a one dimensional quotient are exactly those which have a non trivial relative invariant. From [13] these are also exactly those which are regular, and the list is given in Table 1 below. As they are irreducible, and as the center of \( l_\theta \) is one-dimensional, these spaces are automatically indecomposable and saturated (see Definition 3.1 below).

**Table 1**

| \( PV \) | \( g \) | \( l_\theta \) | \( d_1(\theta) \) |
| --- | --- | --- | --- |
| \( A_{2n+1} \) | \( \alpha_1 \ldots \alpha_{n+1} \alpha_{2n+1} \) | \( sl(n+1) \times gl(n+1) \) | \( M_{n+1} \) |
| \( B_n \) | \( \alpha \) | \( \text{so}(2n-1) \times \mathbb{C} \) | \( \mathbb{C}^{2n-1} \) |
| \( C_n \) | \( \alpha \) | \( gl(n) \) | \( \text{Sym}(n) \) |
| \( D^1_n \) | \( \alpha \) | \( \text{so}(2n-2) \times \mathbb{C} \) | \( \mathbb{C}^{2n-2} \) |
| \( D^2_{2n} \) | \( \alpha \) | \( gl(2n) \) | \( \text{AS}(2n) \) |
| \( E_7 \) | \( \alpha \) | \( E_6 \times \mathbb{C} \) | \( \mathbb{C}^{27} \) |
3. The classification

Let us now explain the classification of $MF$ spaces with a one dimensional quotient. We begin to describe briefly the classification of all $MF$ spaces. Kac ([7]) determined all the cases where the representation $(G, V)$ is irreducible. Brion ([4]) did the case where $G' = [G, G]$ is (almost) simple. Finally Benson-Ratcliff and Leahy did the rest, independently ([1], [2], [11], [10]).

Definition 3.1. (see [10])

1) Two representations $(G_1, \rho_1, V_1)$ and $(G_2, \rho_2, V_2)$ are called geometrically equivalent if there is an isomorphism $\Phi : V_1 \rightarrow V_2$ such that $\Phi(\rho_1(G_1))\Phi^{-1} = \rho_2(G_2)$.

2) A representation $(G, V)$ is called decomposable if it is geometrically equivalent to a representation of the form $(G_1 \times G_2, V_1 \oplus V_2)$, where $V_1$ and $V_2$ are non-zero. It is called indecomposable if it is not decomposable.

3) A representation $(G, V)$ is called saturated if the dimension of the center of $\rho(G)$ is equal to the number of irreducible summands of $V$.

Remark 3.2.

The notion of geometric equivalence is quite natural, once one has remarked that the notion of $MF$ space depends only on $\rho(G)$. Is is worthwhile pointing out that any representation is geometrically equivalent to its dual representation (see Theorem 2.6). Finally note that any representation can be made saturated by adding a torus.

Theorem 3.3. The complete list, up to geometric equivalence, of indecomposable saturated $MF$ spaces with a one dimensional quotient is given by Table 2 (irreducibles) and Table 3 (non irreducibles) at the end of the paper.

Remark 3.4.

A $MF$ space $(G, V)$ can fail to be $QD1$ in one of three ways:

(a) $(G', V)$ is $MF$ (and hence $C[V]^{G'} = C$) or

(b) $(G', V)$ is not $MF$ but $C[V]^{G'} = C$ or

(c) $C[V]^{G'}$ has two or more generators (and hence also $(G', V)$ is not $MF$).

Our paper, together with the lists in [2], [4], [7], [10], [11], shows the following.

- The irreducible $MF$ spaces (Section 4.1) which are not $QD1$ each satisfy (a) except for example 4.1.8, which satisfies (b).

- The indecomposable non-irreducible $MF$ spaces (Section 4.2) which are not $QD1$ each satisfy (b) except for example 4.2.8 with $n, m \geq 3$ which satisfies (a) and the following four examples which satisfy (c): example 4.2.8 with $n = m = 2$, example 4.2.9 with $n = 2$, example 4.2.10 and example 4.2.12. These last four have two dimensional quotients.
4. Proof

This section is devoted to the proof of Theorem 3.3. The classification tables show that indecomposable saturated MF spaces are either irreducible (see for example Table I p. 153 in [2]) or they have two irreducible summands (see Table II p. 154 in [2]). Using Proposition 2.8, we have only to decide whether or not a given MF-space has a unique fundamental invariant.

For the irreducible MF spaces we have checked this by a case by case computation. For the non-irreducible MF spaces, we also check which one have a one dimensional quotient by a case by case examination, using sometimes the criterion given by Proposition 2.9.

We also indicate whether or not these spaces are regular or of parabolic type as a prehomogeneous vector spaces.

4.1. Irreducible MF spaces.

As said above we examine case by case the MF spaces occurring in Table 1 of the paper by Benson-Ratcliff ([2]). Of course the similar tables of Leahy ([11]) or of Knop ([10]) could have been used instead. Except that we explicitly mention the center $\mathbb{C}^*$, we adopt the notations of Benson and Ratcliff. The notations $SL(n), SO(n), Sp(n)$ will not only stand for the groups but also for the natural representation of the corresponding group on $\mathbb{C}^n, \mathbb{C}^n, \mathbb{C}^{2n}$ respectively. The notation $S^2(SL(n))$ denotes the ”natural” representation of $SL(n)$ on the space $Sym(n)$ of symmetric matrices of size $n$, whereas $\Lambda^2(SL(n))$ stands for the ”natural” representation of $SL(n)$ on the space $AS(n)$ of skew-symmetric matrices of size $n$. Also $G_2$ stands for the 7-dimensional representation of $G_2$, and $E_6$ denotes the 27-dimensional representation of $E_6$. The notation $H^*$ denotes the contragredient representation of the representation $H$ and the notation $H_1 \otimes H_2$ denotes the tensor product of the corresponding representations of $H_1$ and $H_2$.

Recall also that we say that a MF space is QD1 if it has a one dimensional quotient.

4.1.1. $SL(n) \times \mathbb{C}^*(n \geq 1)$

It is well known that, for $n > 1$, this representation has no non-trivial relative invariant. For $n = 1$, the corresponding representation is $\mathbb{C}^*$ acting on $\mathbb{C}$, which has obviously one fundamental relative invariant, and hence is QD1. It is of commutative parabolic type, corresponding to the diagram $\circ$. This is the case $m = n = 1$ of 4.1.6. below. It is case $A_1$ in Table 1, and a particular case of (4) in Table 2.

4.1.2. $SO(n) \times \mathbb{C}^*(n \geq 3)$.

It is well known that the natural representation of $SO(n) \times \mathbb{C}^*$ on $\mathbb{C}^n$ has a unique fundamental relative invariant, namely the nondegenerate quadratic form $Q$. Therefore it is QD1. It is also well known that this space is a PV of commutative parabolic type, corresponding to the diagrams $\circ$ if $n = 2p + 1$ and $\circ$ if $n = 2p$.

This corresponds to case (1) in Table 2.
4.1.3. $\text{Sp}(n) \times \mathbb{C}^*(n \geq 2)$.

This $PV$ is of parabolic type (corresponding to the diagram

According to the table in ([15] or [17]) it has no non-trivial relative invariant, hence it is not $QD1$.

4.1.4. $S^2(\text{SL}(n)) \times \mathbb{C}^*(n \geq 2)$.

Up to geometric equivalence this is the representation of $\text{GL}(n)$ on $\text{Sym}(n)$ given by $g.X = gX^t g$ $(g \in \text{GL}(n), X \in \text{Sym}(n))$. This $PV$ is of commutative parabolic type corresponding to the diagram

The unique fundamental relative invariant is the determinant, hence it is $QD1$. It corresponds to case (2) in Table 2.

4.1.5. $\Lambda^2(\text{SL}(n)) \times \mathbb{C}^*(n \geq 4)$.

Up to geometric equivalence this is the representation of $\text{GL}(n)$ on $\text{AS}(n)$ given by $g.X = gX^t g$ $(g \in \text{GL}(n), X \in \text{AS}(n))$. This $PV$ is of commutative parabolic type corresponding to the diagram

It is well known that there is no relative invariant if $n$ is odd, and that for $n$ even the unique fundamental relative invariant is the pfaffian. Therefore it is $QD1$ if and only if $n = 2p$. It corresponds to case (3) in Table 2.

4.1.6. $\text{SL}(n) \otimes \text{SL}(m)^* \times \mathbb{C}^*(n, m \geq 2)$.

By Remark 3.2, this representation is geometrically equivalent to case $\text{SL}(n) \otimes \text{SL}(m) \times \mathbb{C}^*(n, m \geq 2)$ which is considered in Table 1 of Benson-Ratcliff ([2]). This is the representation of $\text{SL}(n) \times \text{SL}(m)$ on the space $M_{n,m}$ given by $(g_1, g_2).X = g_1 X g_2^{-1}$, $g_1 \in \text{SL}(n), g_2 \in \text{SL}(m), X \in M_{n,m}$. This is a commutative parabolic $PV$ corresponding to the diagram

If $n \neq m$, then there is no fundamental relative invariant. If $n = m$, then the unique fundamental relative invariant is the determinant. Hence it is $QD1$ if and only if $n = m$. It corresponds to case (4) in Table 2.

4.1.7. $\text{SL}(2) \otimes \text{Sp}(n) \times \mathbb{C}^*(n \geq 2)$.

Up to geometric equivalence we can consider that this is the representation of $G = \text{GL}(2) \times \text{Sp}(n)$ acting on $V = M_{2n,2}$ by

This is a regular irreducible $PV$ of parabolic type (not commutative), corresponding to the diagram

(see [21], [15], [17]). Hence it is $QD1$. According to the computations in [21] (Proposition 17 p.100-101), the fundamental relative invariant is $f(X) = Pf((X^t J X))$, where $X \in M_{2n,2}(\mathbb{C})$, where $J = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}$, and where $Pf(.)$ is the pfaffian of a $2 \times 2$ skew symmetric matrix. It is case (6) in Table 2.
4.1.8. $SL(3) \otimes Sp(n) \times \mathbb{C}^*(n \geq 2)$.

This is a non regular irreducible PV of parabolic type corresponding to the diagram

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

of type $C_{n+3}$, it is known ([15] or [17]) that it has no non-trivial relative invariant, hence it is not QD1.

4.1.9. $SL(n) \otimes Sp(2) \times \mathbb{C}^*(n \geq 4)$.

This again is an irreducible PV of parabolic type corresponding to the diagram

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

of type $C_{n+2}$. It is known ([15] or [17]) that this PV has a non-trivial relative invariant if and only if $n = 4$. Hence this space is QD1 if and only if $n = 4$, and then it is regular. In this case the group $SL(4) \times Sp(2)$ acts on $M_4$ by $(g_1, g_2).X = g_1Xg_2^{-1}$, and the fundamental relative invariant is the determinant. It is case (7) in Table 2.

4.1.10. $Spin(7) \times \mathbb{C}^*$.

This space is known ([15] or [17]) to be an irreducible regular PV of parabolic type inside $F_4$ corresponding to the diagram $\bullet \quad \bullet \quad \bullet \quad \bullet$. Here the space has dimension 8 and the action is obtained by embedding $Spin(7)$ into $SO(8)$. The fundamental relative invariant is the nondegenerate quadratic form which defines $SO(8)$. It is case (8) in Table 2.

4.1.11. $Spin(9) \times \mathbb{C}^*$.

According to [21], p. 146, number (19) of Table, this is an irreducible regular PV whose fundamental relative invariant is a quadratic form, hence it is QD1. According to the diagrammatical rules in Remark 2.3 d) it is not of parabolic type. But it has nevertheless an interesting connection with PV’s of parabolic type, see [20], Theorem 5.1. p. 377. It is case (9) in Table 2.

4.1.12. $Spin(10) \times \mathbb{C}^*$.

This is a PV of parabolic type inside $E_6$, corresponding to the diagram $\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$.

According to the table in ([15] or [17]) it has no non-trivial relative invariant, hence it is not QD1.

4.1.13. $G_2 \times \mathbb{C}^*$.

According to [21], p. 146, number (25) of Table, this is an irreducible regular PV whose fundamental relative invariant is a quadratic form, hence it is QD1. According to the diagrammatical rules in Remark 2.3 d) it is not of parabolic type. But it has nevertheless an interesting connection with PV’s of parabolic type, see [20], Theorem 6.1. p. 381. It is case (10) in Table 2.

4.1.14. $E_6 \times \mathbb{C}^*$.

This space is known ([15] or [17]) to be an irreducible regular PV of parabolic type inside $E_7$ corresponding to the diagram $\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$. 


It is the 27-dimensional representation of $E_6$. Its fundamental relative invariant has degree 3, it is known as the Freudenthal cubic. This is the case (5) in Table 2.

4.2. Non-irreducible MF spaces.

Here we examine the cases arising in Table II of [2]. We keep the same notations for the representations as before. In addition we adopt also the following notation from [2]. If $(G_1, V_1)$ and $(G_2, V_2)$ are representations of two semi-simple groups $G_1$ and $G_2$ which share a common simple factor $H$, then the notation $G_1 \oplus_H G_2$ denotes the image of the representation on $V_1 \oplus V_2$ where the common factor $H$ acts diagonally. For example $SL(n) \oplus_{SL(n)} SL(n)$ denotes the direct sum representation $(SL(n), \mathbb{C}^n \oplus \mathbb{C}^n)$, and $Spin(8) \oplus_{Spin(8)} SO(8)$ denotes the action of $Spin(8)$ on $\mathbb{C}^8 \oplus \mathbb{C}^8$ via the Spin representation on the first $\mathbb{C}^8$ factor and via the natural representation of $SO(8)$ on the second factor.

4.2.1. $(SL(n) \oplus_{SL(n)} SL(n)) \times \mathbb{C}^2(n \geq 2)$.

This space is a parabolic $PV$ corresponding to the diagram

\[ \bullet \cdots \bullet \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
type. We refer the reader interested into details to Lemma 4.8 in [19]. It is case (1) in Table 3.

4.2.3. \((SL(n) \oplus_{SL(n)} \Lambda^2(SL(n)) \times \mathbb{C}^*; n \geq 4)\).

The representation is given by \((g, \lambda, \mu). (u, x) = (\lambda gu, \mu gx^t g)\) where \(\lambda, \mu \in \mathbb{C}^*,\ g \in SL(n),\ u \in \mathbb{C}^n,\ x \in AS(n)\). This \(PV\) is not of parabolic type except for the cases where \(n = 5, 6, 7\) which correspond respectively to the following diagrams:

- Suppose first that \(n = 2p\) is even. It is well known that the restriction of the representation to \(AS(n)\) is a regular \(PV\) of parabolic type, and that its unique fundamental relative invariant is the pfaffian. Moreover the generic isotropy subgroup of this component at the point \(J = \begin{bmatrix} 0 & Id_p \\ -Id_p & 0 \end{bmatrix}\) is the subgroup \(Sp(p) \times \mathbb{C}^*\). The restriction of the ”natural” representation of \(SL(2p) \times \mathbb{C}^*\) on \(\mathbb{C}^{2p}\) to \(Sp(p) \times \mathbb{C}^*\) is well known to have no non-trivial relative invariant. Hence, by Proposition 2.9 this space is \(QD1\). As the fundamental relative invariant does not depend on all variables, it is not regular. It is case \((2)(a)\) in Table 3.

- Suppose now that \(n = 2p + 1\) is odd. Rather than the group \(SL(n) \times \mathbb{C}^2\), we will here consider the Lie algebra \(g = gl(n) \times \mathbb{C}\) acting on \(V = \mathbb{C}^n \oplus AS(n)\) by \((U, \lambda)(v, x) = (\lambda v + Uv, Ux + x^t U)\) where \(\lambda \in \mathbb{C}, U \in gl(n), v \in \mathbb{C}^n, x \in AS(n)\). Once we identify \(gl(n)\) with \(sl(n) \times \mathbb{C}\) this is essentially the derived representation of \((SL(n) \oplus_{SL(n)} \Lambda^2(SL(n)) \times \mathbb{C}^*; n \geq 4)\). Consider the point \(X_0 = (v_0, x_0) \in \mathbb{C}^n \oplus AS(n)\) where \(v_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\) and where \(x_0 = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}\). An easy computation shows that the isotropy subalgebra \(g_{X_0}\) is given by

\[
g_{X_0} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -\lambda \end{bmatrix}, \lambda \right\}, \quad \lambda \in \mathbb{C}, A \in sp(p)\right\}.
\]

As \(\dim g - \dim g_{X_0} = \dim V\), the point \(X_0\) is generic. The Lie algebra \(g_{X_0}\) is the Lie algebra of a reductive subgroup. Hence this space is regular. As \([g, g] = gl(n) \times \{0\}\), the Lie algebra generated by \(g_{X_0}\) and \([g, g]\) is equal to \(sl(n) \times \mathbb{C}\), and hence \(g/(sl(n) \times \mathbb{C}) \simeq \mathbb{C}\). According to Remark 2.1, there exists exactly one (up to constants) fundamental relative invariant, and hence this space is \(QD1\). Keeping the same notation as above it is easy to see that the polynomial

\[
f(v, x) = Pf \left( \begin{bmatrix} x \\ -v \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} \right), \quad v \in \mathbb{C}^n, x \in AS(2p + 1)
\]

is a fundamental relative invariant. These spaces are \(Q\)-irreducible in the sense of [19] (see Remark 4.15 in [19]). It is case \((2)(b)\) in Table 3.

4.2.4. \((SL(n)^* \oplus_{SL(n)} \Lambda^2(SL(n)) \times \mathbb{C}^*; n \geq 4)\).
This $PV$ is always of parabolic type. The corresponding diagram is the following:

$$D_{n+1}$$

Up to geometric equivalence we can take here $G = GL(n) \times \mathbb{C}^*$ acting on $V = M_{1,n} \oplus AS(n)$ by $(g, \lambda)(v, x) = (\lambda vU^{-1}, Ux^tU)$.

- Suppose first that $n = 2p$ is even. The restriction of the representation to $AS(n)$ is a regular $PV$ whose fundamental relative invariant is the pfaffian. The partial isotropy of $J = \begin{bmatrix} 0 & Id_p \\ -Id_p & 0 \end{bmatrix} \in AS(n)$ is equal to $Sp(p) \times \mathbb{C}^*$, and it is well known that the action of $Sp(p) \times \mathbb{C}^*$ on $M_{1,n}$ has no non-trivial relative invariant. Therefore, from Proposition 2.9, we obtain that this $MF$ space is $QD1$. As the fundamental relative invariant does not depend on all variables, it is not regular. It is case (3) in Table 3.

- Suppose now that $n = 2p + 1$ is odd. Rather than the group action, we will consider here the infinitesimal action. In other words we consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) \times \mathbb{C}$ acting on $V = M_{1,2p+1} \oplus AS(2p + 1)$ by $(U, \lambda)(v, x) = (\lambda vU - UvU^t + x^tU)$ where $\lambda \in \mathbb{C}, U \in \mathfrak{gl}(n), v \in \mathbb{C}^n, x \in AS(n)$. Consider the element $X_0 = (v_0, x_0) \in V$ which is defined by $v_0 = (1, 0, \ldots, 0)$ and $x_0 = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \in AS(2p + 1)$ with $J = \begin{bmatrix} 0 & Id_p \\ -Id_p & 0 \end{bmatrix}$. A computation shows that the isotropy subalgebra $\mathfrak{g}_{X_0}$ is the set of couples of the form $(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \lambda)$,

- where $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -t^t \alpha \end{bmatrix}$ with $\alpha = \begin{bmatrix} \lambda & 0 & \ldots & 0 \\ 0 & A_1 \end{bmatrix}$, $A_1 \in M_{p-1,p};$ $\beta = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, b \in Sym(p-1); \gamma \in Sym(p)$.

- where $B = \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix}, \tilde{B} \in \mathbb{C}^{2p-1}$

- and where $D, \lambda \in \mathbb{C}$.

Then one verifies that $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0} = \dim V$, and hence $X_0$ is generic. Moreover the Lie subalgebra generated by $\mathfrak{g}_{X_0}$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(n) \times \{0\}$ is equal to $\mathfrak{g}$. According again to Remark 2.1, this shows that there is no non-trivial relative invariant, and therefore this space is not $QD1$.

4.2.5. $(SL(n) \oplus SL(n)) (SL(n) \otimes SL(m)) \times \mathbb{C}^{r2}(n, m \geq 2)$.

It is convenient here to replace this representation by the representation $(G, V)$ where $G = GL(n) \times GL(m)$ acts on $V = M_{n,1} \oplus M_{n,m}$ by

$$(g_1, g_2)(v, x) = (g_1v, g_1xg_2^{-1}), g_1 \in GL(n), g_2 \in GL(m), v \in M_{n,1}, x \in M_{n,m}.$$  

Due to Remark 3.2, this representation is geometrically equivalent to the first one. It is not of parabolic type except for the following cases:

$$n = 3, m \in \mathbb{N} : D_{m+3}$$
• Suppose $n = m$. Then the component $M_{n,n} = M_n$ has a unique fundamental relative invariant, namely the determinant. The point $X_0 = (e_1, Id_n)$, where $e_1$ is the first vector of the canonical basis of $M_{n,1} \simeq \mathbb{C}^n$, is generic. And the partial isotropy subgroup $G_{(0,Id_n)}$ is the diagonal subgroup $\{(g,g) \in GL(n) \times GL(n)\}$. Therefore the action of $G_{(0,Id_n)}$ on $M_{n,1}$ has no relative invariant. According to Proposition 2.9 this space is $QD1$ in the case $m = n$. As the fundamental relative invariant does not depend on all variables, it is not regular. It is case (4)(a) in Table 3.

• Suppose that $n < m$. A simple calculation shows that the point $X_0 = (e_1, x_0)$ where $x_0 = \left[ \begin{array}{c} Id_n \\ 0 \end{array} \right]$ is generic and that its isotropy subgroup $G_{X_0}$ is the set of pairs of matrices of the form $(A, \left[ \begin{array}{l} A \\ B \\ C \end{array} \right])$, where $B \in M_{m,n,m}$ and $C \in GL(m - n)$, and where $A = \left[ \begin{array}{c} 1 \\ 0 \\ A_2 \end{array} \right]$, with $A_1 \in M_{1,n-1}$ and $A_2 \in GL(n - 1)$. This implies that the subgroup of $G = GL(n) \times GL(m)$ generated by $G_{X_0}$ and the derived group $SL(n) \times SL(m)$ is $G$ itself. Hence from Remark 2.1, we know that there is no non-trivial relative invariant, and therefore it is not $QD1$.

• Suppose that $n > m + 1$. Then the element $X_0 = (e_n, x_0)$ where $x_0 = \left[ \begin{array}{c} Id_m \\ 0 \end{array} \right]$ is generic and the isotropy subgroup $G_{X_0}$ is the set of pairs of matrices of the form $(A, \left[ \begin{array}{l} A \\ B \\ C \end{array} \right])$, where $A \in GL(m)$, where $B \in M_{m,n,m}$ is of the form $B = \left[ \begin{array}{c} B' \\ 0 \end{array} \right]$ with $B' \in M_{m,n,m-1}$, and where $C \in GL(n - m)$ is of the form $C = \left[ \begin{array}{c} C_1 \\ 0 \\ C_2 \end{array} \right]$ with $C_1 \in GL(n - m - 1)$ and $C_2 \in M_{1,n-m-1}$. Again this implies that the subgroup generated by $G_{X_0}$ and $[G,G]$ is equal to $G$. Hence by Remark 2.1, we obtain that this space has no non-trivial relative invariant, and hence it is not $QD1$.

• Suppose finally that $n = m + 1$. Then the same calculation as before holds. But now as $n - m = 1$, the isotropy subgroup $G_{X_0}$ is the set of pairs of matrices of the form $(A, \left[ \begin{array}{c} A \\ 0 \\ 1 \end{array} \right])$, where $A \in GL(m)$. The subgroup $\tilde{G}$
generated by $G_{X_0}$ and $[G,G]$ is equal to $\{(g_1, g_2) \in G \mid \det(g_1) = \det(g_2)\}$. This implies that dim $G/\overline{G} = 1$ and hence by Remark 2.1, we obtain that this space has one fundamental relative invariant, and therefore it is $QD1$. As the generic isotropy subgroup is reductive, it is regular. It is easy to see that $f(v, x) = \det(v; x)$, where $(v; x)$ is the $n \times n$ matrix obtained by putting the column vector $v$ left to the $m \times n$ matrix $x$, is the fundamental relative invariant. It is case (4)(b) in Table 3.

4.2.6. $(SL(n)^{*} \oplus SL(n)) (SL(n) \otimes SL(m)) \times \mathbb{C}^{*2}(n \geq 3, m \geq 2)$.

It is convenient here to consider the representation $(G, V)$ where $G = GL(n) \times GL(m)$ acts on $V = M_{1,n} \oplus M_{n,m}$ by

$$(g_1, g_2)(v, x) = (v g_{1}^{-1}, g_1 x g_2^{-1}), g_1 \in GL(n), g_2 \in GL(m), v \in M_{1,n}, x \in M_{n,m}.$$ This representation is geometrically equivalent to the original one. This $PV$ is of parabolic type and corresponds to the diagram:

$$\begin{array}{ccc}
\alpha_1 & \cdots & \alpha_{n+1} & \cdots & \alpha_{n+m} \\
& & A_{n+m}
\end{array}$$

The element $(e_1, x_0)$ where $e_1 = (1, 0, \ldots, 0)$ and where $x_0 = \text{Id}_n$ if $n = m$, $x_0 = \begin{bmatrix} \text{Id}_n & 0 \end{bmatrix}$ if $n < m$, and $x_0 = \begin{bmatrix} \text{Id}_m & 0 \end{bmatrix}$ if $n > m$, is generic and almost the same calculations as in 4.2.5 show that this MF space is $QD1$ if and only if $n = m$. Moreover by the same argument it is not a regular $PV$. It is case (5) in Table 3.

4.2.7. $(SL(2) \oplus SL(2)) (SL(2) \otimes Sp(n)) \times \mathbb{C}^{*2}(n \geq 2)$.

This $PV$ is of parabolic type and corresponds to the diagram:

$$\begin{array}{ccc}
& & C_{n+3}
\end{array}$$

It is convenient to consider here the group $G = \mathbb{C}^{*} \times GL(2) \times Sp(n)$ which acts on $V = M_{1,2} \oplus M_{2,n}$ by $(\lambda, g_1, g_2), (v, x) = (\lambda v g_1, g_2 x g_1)$, where $\lambda \in \mathbb{C}^{*}, g_1 \in GL(2), g_2 \in Sp(n), v \in M_{1,2}, x \in M_{2,n}$. This space is geometrically equivalent to the original one. The action of $G$ on $V_2$ is a regular parabolic $PV$ corresponding to the subdiagram

$$\begin{array}{ccc}
& & C_{n+2}
\end{array}$$

(see [21], [15], [17]). As we have already seen in section 4.1.7. its fundamental relative invariant is the function $x \mapsto Pf(t^t J x)$ where $J = \begin{bmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{bmatrix}$.

We know from [21] (p. 100-101) that the partial isotropy subalgebra of $(\mathfrak{g}, V_2)$ corresponding to a certain generic element $x_0$ in $V_2$ is given by

$$\mathfrak{g}_{x_0} = (\lambda, - A_1 \quad C_1
\begin{bmatrix}
A_1 & 0 \\
B_1 & -A_1
\end{bmatrix}
\begin{bmatrix}
A_4 & 0 \\
0 & A_4
\end{bmatrix}
\begin{bmatrix}
B_1 & 0 \\
0 & B_1
\end{bmatrix},
\begin{bmatrix}
C_1 & 0 \\
0 & C_1
\end{bmatrix}
\begin{bmatrix}
-\text{Id} & 0 \\
0 & -\text{Id}
\end{bmatrix}$$
where \( \lambda, A_1, B_1, C_1 \in \mathbb{C}, A_4 \in \mathfrak{gl}(n-1), B_4, C_4 \in \text{Sym}(n-1) \). This shows that \( g_{x_0} \simeq \mathbb{C} \times \mathfrak{sl}(2) \times \mathfrak{sp}(n-1) \). The action of \( g_{x_0} \) on \( M_{2,1} \) is then essentially the natural action of \( \mathfrak{gl}(2) \) on \( \mathbb{C}^2 \), which is known to have no non-trivial relative invariant. Therefore, using again Proposition 2.9, we obtain that this space is \( QD1 \). Its fundamental relative invariant is given by \( f(v, x) = Pf'(xJx) \). As this function depends only on \( x \), the corresponding \( PV \) is not regular. It is case (6) in Table 3.

4.2.8. \( (SL(n) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes SL(m)) \times \mathbb{C}^{*2}, (n, m \geq 2) \).

This space again is a \( PV \) of parabolic type corresponding to the diagram

\[
\begin{array}{cccccc}
\alpha_1 & \ldots & \alpha_n & \alpha_{n+2} & \ldots & \alpha_{n+m+1} \\
\end{array}
\]

Up to geometric equivalence we can take here \( G = GL(n) \times SL(2) \times GL(m) \) acting on \( V = V_1 \oplus V_2 \) where \( V_1 = M_{n,2} \) and \( V_2 = M_{2,m} \) by

\[
(g_1, g_2, g_3)(u, v) = (g_1 u g_2^{-1}, g_2 v g_3^{-1}), \quad g_1 \in GL(n), \quad g_2 \in SL(2), \quad g_3 \in GL(m).
\]

a) Let us consider first the case where \( n = 2 \) and \( m > 2 \) (or equivalently \( m = 2 \) and \( n > 2 \)).

In this case the action of \( G \) on \( V_1 = M_{2,2} \) has a non-trivial relative invariant (the determinant), the (partial) generic isotropy of the matrix \( Id_2 \) is given by \( G_{Id_2} = \{ (g, g, g_3), g \in SL(2), g_3 \in GL(m) \} \), and the action of \( G_{Id_2} \) on \( V_2 = M_{2,m} \) is well known to have no non-trivial relative invariant. We deduce from Proposition 2.9 that this \( MF \) space is \( QD1 \). As the fundamental relative invariant which is given by \( f(u, v) = \det u \) depends only on the \( V_1 \) variable, it is not regular. It is case (7) in Table 3.

b) Consider now the case where \( n = m = 2 \). In this case there are obviously two fundamental relative invariants given by \( \det u \) and \( \det v \), \( u \in V_1, v \in V_2 \). Hence this \( MF \) space is not \( QD1 \).

c) Consider finally the case where \( n \geq m > 2 \) (or equivalently the case where \( m \geq n > 2 \)).

Define \( x_0 = \left[ \begin{array}{cc}
Id_2 & 0 \\
0 & 0 \\
\end{array} \right] \in M_{n,2} \) and \( y_0 = \left[ \begin{array}{cc}
Id_2 & 0 \\
0 & 0 \\
\end{array} \right] \in M_{2,m} \). The pair \( (x_0, y_0) \) is a generic element and the corresponding isotropy subgroup is given by

\[
G_{(x_0, y_0)} = \left\{ \left( \begin{array}{cc}
g_2 & \beta \\
0 & \delta \\
\end{array} \right), g_2, \left( \begin{array}{cc}
g_2 & 0 \\
c & d \\
\end{array} \right) \right\} \in G
\]

where \( g_2 \in SL(2), \delta \in GL(n-2), \beta \in M_{2,n-2}, d \in GL(m-2) \). It is then clear that the subgroup generated by the derived group \( SL(n) \times SL(2) \times SL(m) \) and \( G_{(x_0, y_0)} \) is equal to \( G = GL(m) \times SL(2) \times GL(m) \). Remark 2.1 implies that this space has no non-trivial relative invariant, and hence it is not \( QD1 \).

4.2.9. \( (SL(n) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes Sp(m)) \times \mathbb{C}^{*2}, (n, m \geq 2) \).

This space also is a \( PV \) of parabolic type corresponding to the diagram

\[
\begin{array}{cccccc}
\alpha_1 & \ldots & \alpha_n & \alpha_{n+2} & \ldots & \alpha_{n+m+2} \\
\end{array}
\]
Up to geometric equivalence we can take here \( G = GL(n) \times GL(2) \times Sp(m) \) acting on \( V = V_1 \oplus V_2 \) where \( V_1 = M_{n,2} \) and \( V_2 = M_{2m,2} \) by

\[
(g_1, g_2, g_3)(u, v) = (g_1 u^t g_2, g_3 v^t g_2), \quad g_1 \in GL(n), \ g_2 \in SL(2), \ g_3 \in Sp(m).
\]

a) Let us first consider the case where \( n > 2 \). The action of \( G \) on \( V_2 \) reduces to the action \( GL(2) \times Sp(m) \) on \( V_2 \) which is of parabolic type corresponding to the subdiagram

\[
\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \quad C_{m+2}
\]

This case has already been considered in 4.2.7. above. And we know from the calculation we did there that the generic isotropy subgroup of \( (GL(2) \times Sp(m), V_2) \) consists of certain pairs of the form \((g_2, g_3)\) where \( g_2 \) takes all values in \( SL(2) \). Therefore the generic isotropy subgroup of \((G, V_2)\) acting on \( V_1 \) is the representation \( (GL(n) \times SL(2), V_1) \) with \( n > 2 \). As this representation has no relative invariant we can apply Proposition 2.9, and obtain that this MF space is \( QD1 \). The fundamental relative invariant is given by \( f(u, v) = Pf(t^v Jv) \), where \( v \in M_{2m,2} \), and where \( J = \begin{bmatrix} 0 & Id_m \\ -Id_m & 0 \end{bmatrix} \). As this invariant does only depend on \( v \), the corresponding \( PV \) is not regular. It is case (8) in Table 3.

b) Consider now the case where \( n = 2 \). Here each of the two subspaces \((G, V_1)\) and \((G, V_2)\) has his own relative invariant (the determinant on \( V_1 \) and the preceding invariant \( f(u, v) = Pf(t^v Jv) \) on \( V_2 \)). Therefore this space is not \( QD1 \) if \( n = 2 \).

4.2.10. \((Sp(n) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes Sp(m)) \times \mathbb{C}^* \), \((n, m \geq 2)\).

Up to geometric equivalence we can take \( G = Sp(n) \times GL(2) \times Sp(m) \times \mathbb{C}^* \) acting on \( V = V_1 \oplus V_2 \) where \( V_1 = M_{2n,2} \) and \( V_2 = M_{2m,2} \) by

\[
(g_1, g_2, g_3, \lambda)(X, Y) = (g_1 X^t g_2, \lambda g_3 Y^t g_2),
\]

where \( g_1 \in Sp(n), g_2 \in GL(2), g_3 \in Sp(m), \lambda \in \mathbb{C}^*, X \in M_{2n,2}, Y \in M_{2m,2} \).

According to the diagrammatical rules in Remark 2.3 d) this space is not of parabolic type.

As each of the representations \((G, V_1)\) and \((G, V_2)\) is of the type seen in 4.1.7. above, they have each their own fundamental relative invariant. Therefore this MF space is not \( QD1 \).

4.2.11. \( Sp(n) \oplus_{Sp(n)} Sp(n) \times \mathbb{C}^* \), \((n \geq 2)\).

Here \( G = Sp(n) \times \mathbb{C}^* \) acts on \( V = M_{2n,1} \oplus M_{2n,1} \) by

\[
(g, \lambda, \mu)(u, v) = (\lambda g u, \mu g v), \ g \in Sp(n), \lambda, \mu \in \mathbb{C}, \ u, v \in M_{2n,1}.
\]

At the infinitesimal level the Lie algebra \( g = sp(n) \times \mathbb{C}^2 \) acts on \( V \) by

\[
(x, \lambda, \mu)(u, v) = (\lambda u + x u, \mu v + x v), \ x \in sp(n), \lambda, \mu \in \mathbb{C}, \ u, v \in M_{2n,1}.
\]
First of all let us remark that there is at least one fundamental relative invariant, namely \( f(u,v) = tuJV\) where \( J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \).

Consider the element \( X_0 = (e_1, e_{n+1}) \in M_{2n,1} \oplus M_{2n,1} \) where \( e_j \) is the \( j \)-th vector of the canonical base of \( M_{2n,1} \cong \mathbb{C}^{2n} \). An easy calculation shows that the isotropy subalgebra \( g_{X_0} \) of \( X_0 \) is given by:

\[
g_{X_0} = \left\{ \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda \right\}
\]

where \( A \in \mathfrak{gl}(n-1), B, C \in \text{Sym}(n-1), \lambda \in \mathbb{C}^* \). As \( \dim g - \dim g_{X_0} = \dim V \), the point \( X_0 \) is generic. As \( g_{X_0} \) is the Lie algebra of a reductive subgroup, this PV is regular. Let \( \tilde{g} \) be the Lie algebra generated by \( [g,g] = \mathfrak{sp}(n) \times \{0\} \times \{0\} \) and \( g_{X_0} \). We have \( \dim(\tilde{g}/g) = 1 \). Then according to Remark 2.1, the polynomial \( f(u,v) = tuJV \) is the only fundamental relative invariant. Therefore this space is \( QD1 \).

According to Remark 2.3 d), it is not of parabolic type. It is case (9) in Table 3.

4.2.12. \( \text{Spin}(8) \oplus \text{Spin}(8) \) \( \text{SO}(8) \times \mathbb{C}^* \).

Let \( \rho \) be one of the Spin representations of \( \text{Spin}(8) \). Here \( G = \text{Spin}(8) \times \mathbb{C}^* \) acts on \( V = \mathbb{C}^8 \oplus \mathbb{C}^8 \) by

\[
(g, \lambda, \mu)(u,v) = (\lambda gu, \mu \rho(g)v), g \in \text{Spin}(8), \lambda, \mu \in \mathbb{C}^*, u,v \in \mathbb{C}^8.
\]

This is a parabolic \( PV \) in \( E_6 \) corresponding to the diagram:

As each of the two summands of this representation has his own fundamental relative invariant (a quadratic form), this space is not \( QD1 \).

**Remark 4.1.** There is an alternate way to check the \( QD1 \) condition. Let \( (G,V) \) be a \( MF \) space of rank \( r \), \( B \subset G \) a Borel subgroup and \( h_1, \ldots, h_r \in \mathbb{C}[V] \) fundamental relative invariants of \( (B,V) \). If \( \Delta \in \mathbb{C}[V] \) is a non-constant irreducible \( G' \)-invariant polynomial then \( \Delta \) is necessarily a constant multiple of one of the \( h_j \)'s. Thus \( (G,V) \) is \( QD1 \) if and only if exactly one of \( h_1, \ldots, h_r \) is \( G' \)-invariant.

This condition can be checked using formulas for the \( h_j \)'s. These can be found in [6] for the irreducible \( MF \) spaces and in [3] for the non-irreducible cases.
Tables of indecomposable, saturated, multiplicity free representations
with one dimensional quotient

Table 2: Irreducible representations
(Notations for representations as in [2], see section 4.1.)

| Representation, rank | Weighted Dynkin diagram (if parabolic type) | Regular | Fundamental invariant |
|---------------------|--------------------------------------------|---------|----------------------|
| (1) \(SO(n) \times \mathbb{C}^*\) \((n \geq 3)\), rank=2 | \(n = 2p + 1\) \[\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \\
\end{array}\] \(B_{p+1}\) | Yes | Non degenerate quadratic form |
|                    | \(n = 2p\) \[\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \\
\end{array}\] \(D_{p+1}\) |         |                      |
|                    | Commutative Parabolic (both) | | |
| (2) \(S^2(SL(n)) \times \mathbb{C}^*\) \((n \geq 2)\), rank= n | \[\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \\
\end{array}\] \(C_n\) | Yes | Determinant on symmetric matrices |
|                    | Commutative Parabolic | | |
| (3) \(\Lambda^2(SL(n)) \times \mathbb{C}^*\) \((n \geq 4)\) and \(n = 2p\) rank=p | \[\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \\
\end{array}\] \(D_2p\) | Yes | pfaffian on skew symmetric matrices |
|                    | Commutative Parabolic | | |
| (4) \((SL(n)^* \otimes SL(n)) \times \mathbb{C}^*\) \((n \geq 2)\), rank=n | \[\begin{array}{c}
\alpha_1 \bullet \bullet \bullet \bullet \bullet \alpha_n \bullet \bullet \bullet \bullet \bullet \alpha_{2n-1} \end{array}\] \(A_{2p-1}\) | Yes | Determinant on full matrix space |
|                    | Commutative Parabolic | | |
| (5) \(E_6 \times \mathbb{C}^*\) (dim=27) rank=3 | \[\begin{array}{c}
\bullet \bullet \bullet \bullet \\
\end{array}\] \(E_7\) | Yes | Freudenthal cubic |
|                    | Commutative Parabolic | | |
| (6) \((SL(2) \otimes Sp(n)) \times \mathbb{C}^*\) \((n \geq 2)\), rank=3 | \[\begin{array}{c}
\bullet \bullet \bullet \bullet \\
\end{array}\] \(C_{n+2}\) | Yes | \(Pf(\{XJX\})\) \(X \in M(2n, 2)\) \(Pf=pfaffian of 2 \times 2\) matrices |
| (7) \(SL(4) \times Sp(2) \times \mathbb{C}^*\) rank=0 | \[\begin{array}{c}
\bullet \bullet \bullet \bullet \\
\end{array}\] \(C_6\) | Yes | \(Det(X), X \in M(4)\) |
| (8) \(Spin(7) \times \mathbb{C}^*\) rank=2 | \[\begin{array}{c}
\bullet \bullet \bullet \\
\end{array}\] \(F_4\) | Yes | Non degenerate quadratic form \((Spin(7) \hookrightarrow SO(8))\) |
| (9) \(Spin(9) \times \mathbb{C}^*\) rank=3 | Non parabolic | Yes | Non degenerate quadratic form |
| (10) \(G_2 \times \mathbb{C}^*\) (dim = 7) rank=2 | Non parabolic | Yes | Non degenerate quadratic form \((G_2 \hookrightarrow SO(7))\) |
### Table 3: Non Irreducible representations

(Notations for representations as in [2], see section 4.2.)

| Representation | Weighted Dynkin diagram (if parabolic type) | Regular | Fundamental invariant |
|----------------|---------------------------------------------|---------|----------------------|
| (1) \( (SL(n)^* \oplus_{SL(n)} SL(n)) \times (\mathbb{C}^*)^2 \) \( n \geq 2 \) rank=3 | ![Dynkin diagram for A_{n+1}]() | Yes | \( f(u, v) = uv \) on \( M(1, n) \oplus M(n, 1) \) |
| (2)(a) \( (SL(n) \oplus_{SL(n)} SL(n)) \times (\mathbb{C}^*)^2 \) \( n \geq 4, n = 2p \) even rank=2p | ![Dynkin diagram for E_7, n=6]() | No | Pfaffian on skew symmetric matrices (on 2nd component) |
| (2)(b) \( (SL(n) \oplus_{SL(n)} SL(n)) \times (\mathbb{C}^*)^2 \) \( n \geq 4, n = 2p + 1 \) odd rank=2p+1 | ![Dynkin diagram for E_6, n=5, E_7, n=7]() | Yes | \( f(v, x) = Pf\left( \begin{array}{c} x \\ -tv \\ 0 \end{array} \right) \) \( v \in \mathbb{C}^n \) \( x \in AS(2p+1) \) |
| (3) \( (SL(n)^* \oplus_{SL(n)} SL(n)) \times (\mathbb{C}^*)^2 \) \( n \geq 4, n = 2p \) even rank=2n | ![Dynkin diagram for D_{2p}]() | No | Pfaffian (on 2nd component) |
| (4)(a) \( SL(n) \oplus_{SL(n)} (SL(n) \otimes SL(n)) \times (\mathbb{C}^*)^2, n \geq 2 \) rank=2n | ![Dynkin diagram for A_{2n}]() | No | Determinant (on 2nd component) |
| (4)(b) \( (SL(n) \oplus_{SL(n)} SL(n) \otimes SL(n-1)) \times (\mathbb{C}^*)^2, n \geq 3 \) rank=2n+1 | ![Dynkin diagram for E_7, n=4]() | Yes | \( \det(x; v) \) \( v \in M_{n,1}, x \in M_{n,n-1} \) |
| (5) \( SL(n)^* \oplus_{SL(n)} (SL(n) \otimes SL(n)) \times (\mathbb{C}^*)^2, n \geq 3 \) rank=2n | ![Dynkin diagram for A_{2n}]() | No | Determinant (on 2nd component) |

Continued next page.
### Table 3 (continued): Non Irreducible representations

(Notations for representations as in [2], see section 4.2)

| Representation, rank | Weighted Dynkin diagram (if parabolic type) | Regular | Fundamental invariant |
|---------------------|---------------------------------------------|---------|-----------------------|
| (6) \[(SL(2) \oplus_{SL(2)} (SL(2) \otimes Sp(n))) \times (\mathbb{C}^*)^2\] \[n \geq 2\] \[\text{rank}=5\] | \[\begin{array}{cccccccccccccccc}
\bullet & \bullet & \bullet & \ldots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\] \[C_{n+3}\] | No | \(Pf'(XJX)\) \[X \in M(2n, 2)\] \(Pf = pfaffian\) \(\text{(on 2nd component)}\) |
| (7) \[(SL(2) \otimes SL(2)) \oplus_{z_{SL(2)}} (SL(2) \otimes SL(n)) \times (\mathbb{C}^*)^2\] \[n \geq 3\] \[\text{rank}=5\] | \[\begin{array}{cccccccccccccccc}
\alpha_1 & \alpha_2 & \alpha_4 & \ldots & \alpha_{n+3}
\end{array}\] \[A_{n+3}\] | No | \(Det(X), X \in M(2, 2)\) \(\text{(on 1st component)}\) |
| (8) \[(SL(n) \otimes SL(2)) \oplus_{z_{SL(2)}} (SL(2) \otimes Sp(m)) \times (\mathbb{C}^*)^2\] \[n \geq 3, m \geq 2\] \[\text{rank}=6\] | \[\begin{array}{cccccccccccccccc}
\alpha_1 & \alpha_2 & \alpha_n & \alpha_{n+2} & \alpha_{n+m+2}
\end{array}\] \[C_{n+m+2}\] | No | \(Pf'(XJX)\) \[X \in M(2n, 2)\] \(Pf = pfaffian\) \(\text{(on 2nd component)}\) |
| (9) \[(Sp(n) \oplus_{Sp(n)} Sp(n)) \times (\mathbb{C}^*)^2, n \geq 2\] \[\text{rank}=4\] | Non parabolic | Yes | \(f(u, v) = t_uJv\) \(\text{on } M(1, 2n) \oplus M(1, 2n)\) |

### Acknowledgement

I would like to thank the referee for her/his careful reading of the manuscript and her/his valuable remarks and suggestions.
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Hubert Rubenthaler
Institut de Recherche Mathématique Avancée
Université de Strasbourg et CNRS
7 rue René Descartes
67084 Strasbourg Cedex, France
rubenth@math.unistra.fr

Received May 2, 2012
and in final form September 17, 2012