A wavenumber-frequency spectral model for atmospheric boundary layers

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Abstract. Motivated by the need to characterize power fluctuations in wind farms, we study spatio-temporal correlations of a neutral atmospheric boundary layer in terms of the joint wavenumber-frequency spectrum of the streamwise velocity fluctuations. To this end, we perform a theoretical analysis of a simple advection model featuring the advection of small-scale velocity fluctuations by the mean flow and large-scale velocity fluctuations. The model is compared to data from large-eddy simulations (LES). We find that the model captures the trends observed in LES, specifically a Doppler shift of frequencies due to the mean flow as well as a Doppler broadening due to random sweeping effects.

1. Introduction

Characterizing the space-time structure of atmospheric boundary layers is important from a geophysical perspective and has significant implications for understanding wind farm power output fluctuations. Turbines in a wind farm sample the turbulent wind field both in space and time, and the resulting power output signals of turbines contain complex spatio-temporal patterns and correlations from one turbine to another. These correlations can either be studied in real space, or equivalently, in the wavenumber-frequency domain.

Most models, like for example the Mann model for the spectral energy tensor [1, 2], focus on modeling spatial correlations of the velocity fields. In such models, time information is typically added by exploiting Taylor’s hypothesis. Here we discuss a recent model [3] that specifically aims at refining this temporal information by incorporating large-scale random sweeping effects in an analytically tractable form. The model is based on decomposing the streamwise velocity into small-scale turbulent fluctuations, mean flow velocity and random (large-scale) advection velocity. The mean flow advection leads to a Doppler shift in frequency space while the random advection of larger-scale eddies induces a Doppler broadening. For the application case of atmospheric boundary layers, we show that the Doppler shift and Doppler broadening...
can be parameterized in terms of the logarithmic laws for the mean velocity and the velocity fluctuations.

2. Large-Eddy Simulation of Atmospheric Boundary Layers
Here we use the results from large-eddy simulations (LES) of a neutral atmospheric boundary layer to test the model. LES is particularly well suited for such comparisons, because it provides access to the full space-time record of the flow. The LES considered here is run on a domain of $L_x \times L_y \times H = 4\pi \times 2\pi \times 1$ on a computational grid with $512 \times 256 \times 128$ grid points using periodic boundary conditions in the horizontal directions. In the simulations the filtered incompressible Navier-Stokes equations for neutral flows are solved. Because the Reynolds number is assumed to be very high, the viscous stresses are neglected while at the bottom surface a classic imposed wall stress boundary condition relates the wall stress to the velocity at the first grid point. This assumes that the first grid point resides in the logarithmic boundary layer region. The simulation has been performed with a wall roughness $z_0/H = 10^{-4}$ to satisfy this condition. In order to model the subgrid-scale stresses, the scale-dependent Lagrangian subgrid model, using a sharp spectral cutoff filter, is invoked [4]. In the horizontal directions a spectral scheme is used, while second-order finite differences are used to discretize the vertical direction. The time-stepping is performed with a second-order accurate Adams-Bashforth scheme. The flow is driven by a pressure gradient in the streamwise direction. In the statistically stationary state the pressure gradient is balanced by a linear shear stress profile, and the corresponding wall stress is $u_*^2$. The friction velocity $u_*$ and the domain height $H$ are used to non-dimensionalize the equations. All results are presented in these units. The flow evolution is solved for roughly 10 dimensionless time units (where the dimensionless time is in units of $H/u_*$) to make sure that the statistically stationary state is reached before data collection is started. Subsequently the time history of the flow field at preselected heights is collected every $5.5 \times 10^{-4}$ time units over a duration of 2.31 dimensionless time units.

3. Wavenumber-Frequency Spectrum from a Simple Advection Model & Comparison to LES Data

![Figure 1](image)

Figure 1. Sample space-time plot of the streamwise velocity fluctuations at height $z = 0.15H$ from LES. The fluctuations are advected by the mean velocity ($U = 19.2 u_*$) and random sweeping effects ($V = 2.09 u_*$). Additionally, they evolve in time.

Figure 1 shows a space-time plot of the streamwise velocity fluctuations at a height of $z = 0.15H$. It can be appreciated from the figure that the velocity fluctuations are predominantly advected by the mean velocity. If this was the only physical effect, as assumed in Taylor’s
frozen-eddy hypothesis [5], “world lines” of the velocity fluctuations would be perfectly straight. However, considerable deviations from this idealization can be seen, which originate from large-scale random sweeping effects [6, 7] and the temporal evolution of the small scales.

The main effects of mean flow advection and large-scale random sweeping can be captured in a simple model for the wavenumber-frequency spectrum [3]. The time-evolution of the small scales is neglected in the following. For simplicity, we consider the streamwise advection model. Spanwise and vertical random advection effects are hence neglected. Essentially, we are considering the linear advection equation $\frac{\partial}{\partial t} u + i[U + v] ku = 0$ for the Fourier coefficients of the streamwise velocity component, $u(k,t)$. Here, $k$ denotes the streamwise wavenumber. $U$ and $v$ are the mean velocity and large-scale random sweeping velocity, respectively. Both velocities are assumed constant in downstream direction and time compared to the velocity fluctuations by a scale-separation argument. Additionally, we assume that the large-scale random advection velocity obeys a Gaussian ensemble distribution. This problem now is simple enough to allow for a fully analytical derivation of the wavenumber-frequency spectrum, and we refer the reader to reference [3] for further details. The essence of the derivation is to solve the advection equation and calculate the two-time covariance of the Fourier coefficients, which is straightforwardly related to the two-time wavenumber spectrum. The wavenumber-frequency spectrum then is obtained by Fourier transform to frequency space. Here, we focus on the discussion of the main result for the wavenumber frequency spectrum, which takes the form

$$E(k, \omega, z) = \frac{E(k, z)}{\sqrt{2\pi k^2 V(z)^2}} \exp \left[ -\frac{(\omega - kU(z))^2}{2k^2 V(z)^2} \right].$$

(1)

The joint wavenumber-frequency spectrum turns out to be the product of the wavenumber spectrum $E(k, z)$ and a Gaussian distribution of the frequency $\omega$, characterized by the height-dependent mean velocity $U(z)$ and large-scale fluctuations $V(z) = \sqrt{\langle v^2 \rangle(z)}$. This result demonstrates that the mean velocity induces a Doppler shift of frequencies, whereas the large-scale random advection effects lead to a Doppler broadening. Both effects depend on wavenumber $k$ and become more pronounced at the small scales, i.e. at high wavenumbers. Note that application of Taylor’s hypothesis would imply a sharp frequency distribution $\sim \delta(\omega - kU(z))$.

The linear advection equation does not contain the physics reflected in the wavenumber spectrum of the small-scale velocity fluctuations, which we need to model explicitly. For an atmospheric boundary layer of height $H$, the streamwise energy spectrum exhibits two distinct power law regimes and can be modeled as [8]

$$E(k, z) = \begin{cases} \frac{C_1}{\kappa^{2/3}} u_*^2 H & k < 1/H \\ \frac{C_1}{\kappa^{2/3}} u_*^2 k^{-1} & 1/H < k < 1/z \\ C_1 \left( \frac{u_*^2}{\kappa} \right)^{2/3} k^{-5/3} & k > 1/z. \end{cases}$$

(2)

Here $C_1 \approx 0.49$ is the Kolmogorov constant for the streamwise energy spectrum, and $\kappa \approx 0.4$ is the von-Kármán constant. The mean vertical velocity profile is captured by the log-law [9, 10, 11]

$$U(z) = \frac{u_*}{\kappa} \log \left( \frac{z}{z_0} \right)$$

(3)

with $z_0$ being the roughness length. Also the fluctuations have been observed to display a logarithmic behavior of the form [12, 13, 8]

$$V(z)^2 = u_*^2 \left[ B - A \log \left( \frac{z}{H} \right) \right],$$

(4)
Figure 2. Wavenumber-frequency spectrum from LES (left) compared to model spectrum (right) evaluated at \( z = 0.15H \). Both spectra exhibit the characteristic Doppler shift and Doppler broadening induced by the mean flow velocity and random sweeping effects, respectively. The most significant deviations occur at low wavenumbers, see discussion in the main text. The dashed line indicates a wavenumber corresponding to the typical length scale of a wind turbine.

where \( A \) is the “Perry-Townsend” constant and \( B \) depends on the specific flow conditions. For the current simulation we find parameter values of \( A \approx 1.15 \) and \( B \approx 2.17 \), consistent with prior findings [14, 15].

This concludes the specification of our model spectrum, which we can compare to the wavenumber-frequency spectrum from LES. Figure 2 shows such a comparison for a height of \( z = 0.15H \). The LES spectrum is clearly tilted due to a Doppler shift of frequencies, and the Doppler broadening increases with increasing wavenumbers. The model spectrum captures these main trends, especially for high wavenumbers. A typical length scale associated with wind energy applications is the rotor diameter of a wind turbine. If we assume that the domain height \( H \) represents a physical height of 1 km, a reasonable estimate for the rotor diameter is \( D = 0.1H \). This corresponds to a wavenumber of \( k = \pi/D \approx 31.5/H \) (associating \( D \) with a half-wavelength of wavenumber \( k \)); \( z = 0.15H \) thus roughly corresponds to the tip height of a wind turbine. Given its simplicity, the model performs rather well in this range of scales.

For low wavenumbers, the model spectrum underestimates the scatter of frequencies. This appears plausible in appreciation of the fact that we consider a one-dimensional model in which the time-evolution of the small scales has been neglected. This excludes both aliasing effects from transverse and vertical wavenumbers, and Doppler broadening due to the intrinsic evolution of the velocity fluctuations. Further discrepancies may be rooted in the fact that scale separation of the large-scale random sweeping effects and the velocity fluctuations has been assumed, which clearly does not hold for the low-wavenumber part of the spectrum.

Figure 3 shows the frequency distribution at wavenumber \( k = 31.5/H \) compared to the Gaussian prediction of the advection model. The model captures the essence of the Doppler shift and Doppler broadening. The latter effect is entirely absent when Taylor’s hypothesis is applied as indicated by the sharp frequency distribution in figure 3. Especially the logarithmic plot shows that the frequency distribution in LES is not perfectly Gaussian at this length scale.
Further tests, not shown here, indicate that the frequency distributions become more Gaussian with increasing wavenumber.

4. Summary
We have motivated a model for the joint wavenumber frequency spectrum of streamwise velocity fluctuations for atmospheric boundary layers. As a consequence of assuming a Gaussian large-scale random advection, the model predicts a Gaussian frequency distribution, which exhibits a Doppler shift due to the mean velocity and a Doppler broadening due to a large-scale random sweeping velocity. Both velocity profiles are known to follow logarithmic laws in certain heights of atmospheric boundary layers, which was used to parameterize the two effects. In comparison to LES results, the model was found to capture the main trends, especially in the intermediate to high wavenumber region.

It will be interesting to extend this analysis further into the direction of wind energy conversion. Apart from estimating spatio-temporal correlations of power fluctuations and fatigue loads, it will be interesting see how the presence of a wind farm alters the space-time structure of the correlations.

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