Explicit Zero Density Estimate for the Riemann Zeta-Function Near the Critical Line

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Abstract. In 1946, A. Selberg proved $N(\sigma, T) \ll T^{1 - \frac{1}{4} (\sigma - \frac{1}{2})} \log T$ where $N(\sigma, T)$ is the number of nontrivial zeros $\rho$ of the Riemann zeta-function with $\Re{\rho} > \sigma$ and $0 < \Im{\rho} \leq T$. We provide an explicit version of this estimate, together with an explicit approximate functional equation and an explicit upper bound for the second power moment of the zeta-function on the critical line.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function and denote by $\rho = \beta + i \gamma$ a nontrivial zero of $\zeta(s)$ in the critical strip $0 \leq \Re{s} \leq 1$. Denote by $N(T)$ the number of zeros $\rho$ with $\gamma \in (0, T]$, and let $N(\sigma, T)$ be the number of those zeros with $\beta > \sigma \geq 1/2$. Trivially, $N(\sigma, T) \leq \frac{1}{2} N(T)$, where

$$|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8}| \leq 0.11 \log T + 0.29 \log \log T + 2.29 + \frac{1}{5T}. \quad (1)$$

$T \geq e$, is an explicit version of the Riemann–von Mangoldt formula, see [PT15, Corollary 1] and [Tru14, Corollary 1]. The Riemann Hypothesis is equivalent to $N(1/2, T) = 0$ for every $T > 0$. It has been rigorously verified for all nontrivial zeros with $|\Im{\rho}| \leq H_0$, where

$$H_0 := 3.0610046 \cdot 10^{10},$$

the result due to Platt [Pla17]. Non-trivial upper bounds for $N(\sigma, T)$ are called zero density estimates. There exist many such estimates in the literature, for instance Ingham’s theorem

$$N(\sigma, T) \ll T^{\frac{3}{2}(1-\sigma)} \log^5 T. \quad (2)$$

There are other zero density estimates which are better than (2) in smaller regions of the critical strip. Possible applications strongly depend on the position of such a region, e.g., to the distribution of prime numbers if $\sigma$ is close to 1, see [PT19], and to problems connected to the function $S(t)$ and to the pair correlation conjecture when $\sigma$ is close to 1/2. We refer the reader to [KLN18] and references therein for zero density estimates near the one-line. Much less work was done in the latter case. Selberg proved in [Sel46, Theorem 1] that

$$N(\sigma, T) \ll T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T, \quad (3)$$

which supersedes (2) for $\sigma - 1/2 \ll \log \log T / \log T$. In fact, he provided a bound for $N(\sigma, T + H) - N(\sigma, T)$ where $H \in [T^a, T]$ and $a \in (1/2, 1]$, such that (3) is a special case of $H = T$ after a dyadic partition. Later Jutila improved in [Jut83] the

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\[^1\text{Platt and Trudgian will soon announce that } H_0 \text{ can be replaced by } 2.5 \cdot 10^{12}, \text{ see [PT19, Lemma 4].}\]
constant 1/4 to 1 − ε and Conrey announced in [Con89] a further improvement to 8/7 − ε, where the implied constant in [3] depends on ε > 0. Observe that if one could prove [3] with 2 instead of 1/4, this would imply the Density Conjecture.

There exist only few explicit zero density estimates, e.g., in recent papers [Kad13] and [KLN18] where they improve older results by Cheng and Ramaré, and could prove (3) with 2 instead of 1/

Lumley for calculating with a are believed to be new, and the latter inequality greatly improves the recently announced estimate [DHZA19] Theorem 4.3].

2Note that the first column in Table 1 in [KLN18] should have σ in place of σ0. It seems that A(σ) and B(σ) are increasing and decreasing functions, respectively. The author thanks Allysia Lumley for calculating A(0.638) = 2.789 . . . and B(0.638) = 5.312 . . . .

\[ N(\sigma, T) \leq A(\sigma) \cdot T^{\frac{1}{4}(1 - \sigma)} \log^{\frac{5}{2} - 2\varepsilon} T + B(\sigma) \log^2 T, \]  

valid for \( \sigma \in [3/5, 1) \) and \( T \geq H_0, \) see [KLN18], where \( A(\sigma) \) and \( B(\sigma) \) are positive and calculable functions, e.g., \( A(37/58) \leq 2.9 \) and \( B(37/58) \leq 5.6. \) However, \( \frac{1}{2} \) produces non-trivial bound for \( \sigma > 5/8. \) It seems that the only explicit result of Selberg-type zero density estimate was done by Karatsuba and Korolëv in [KK06] Theorem 1. They proved

\[ N(\sigma, T + H) - N(\sigma, T - H) \leq 13HT^{(1 - 2\varepsilon)/10} \log T \]  

for \( 0 < \varepsilon < 0.001, T \geq T_0(\varepsilon) > 0 \) and \( H = T^{27/82 + \varepsilon}. \) Unfortunately, \( T_0(\varepsilon) \) is not explicitly known.

The main result of this paper is the following explicit version of (3).

**Theorem 1.** Let \( T \geq H_0 \) and \( \sigma \in [1/2, 0.831]. \) Then we have

\[ N(\sigma, 2T) - N(\sigma, T) \leq aT^{1 - \frac{1}{2}(\sigma - \frac{1}{2})} \log T + b \log^2 T + c \log T \log \log T + d \log T, \]

with \( a = 10395.2, b = 1.104, c = 0.173 \) and \( d = 0.51. \)

Under the assumptions of Theorem 1 an immediate corollary is

\[ N(\sigma, T) \leq \frac{10395.21}{2^{1 - \frac{1}{4}(\sigma - \frac{1}{2})}} T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T \]

for \( T \geq 2H_0. \) In virtue of (4), this bound is of interest only for \( \sigma \in (1/2, 5/8]. \) Nevertheless, for \( \sigma = 37/58, \) when exponents of \( T \) in both inequalities are equal, it is better than (3). For larger \( H_0 \) we can obtain smaller values for the leading constant, e.g., for \( T \geq 10^{50} \) and \( \sigma \in [1/2, 0.569] \) we have

\[ N(\sigma, 2T) - N(\sigma, T) \leq 5.357 \cdot T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T + 1.11 \cdot \log^2 T. \]  

But with the method presented here we cannot get a smaller constant than 3.259.

Our approach to Theorem 1 strongly relies on Selberg’s original proof with the simplification \( \varepsilon = T. \) The main idea is using the approximate functional equation (Theorem 3) to prove the second power moment of \( \zeta(s) \) with a special weight (Theorem 4), which is then used to estimate the main term in Littlewood’s zero-counting lemma for Selberg’s mollifier (Proposition 2). These three crucial steps constitute Sections 2, 3, and 4 respectively. Beside the proof of Theorem 1 which is presented in Section 1.5, we also provide three additional results which might be interesting on their own, namely explicit versions of the approximate functional equations for \( \zeta(s) \) and \( \zeta^2(s), \) see Theorem 4 and Corollaries 1 and 3 and an explicit upper bound for the second power moment of \( \zeta(s) \) on the critical line

\[ \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \leq T \log T - \left( 1 + \log 2\pi - 2\gamma \right) T + 70.26 \cdot T^{\frac{3}{2}} \log \frac{T}{2\pi}, \]

valid for \( T \geq 2000, \) see Corollary 5 for a more precise statement. All results are believed to be new, and the latter inequality greatly improves the recently announced estimate [DHZA19] Theorem 4.3].
2. Explicit approximate functional equation

We can approximate $\zeta(s)$ with Dirichlet polynomials to arbitrary precision on every compact set in $\Re\{s\} > 1$. Hardy and Littlewood showed in [HL21 Lemma 2] that this is also possible to some extent in the critical strip.

**Theorem 2.** Let $s = \sigma + it$ where $\sigma \in (0, 1]$ and $s \neq 1$. Also assume that $x \geq 1$ and $|t| < 2\pi x$. Then

$$\zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + R(s;x), \quad (6)$$

where $R(s;x) = O(x^{-\sigma})$ uniformly.

In many cases the sum in (6) has too many terms to be useful. Remember that the functional equation for the Riemann zeta-function is $\zeta(s) = \chi(s)\zeta(1-s)$ where

$$\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \quad (7)$$

Hardy and Littlewood proved in [HL23 Theorem A] the following refinement of (6) which is known as the approximate functional equation.

**Theorem 3.** Let $s = \sigma + it$ where $\sigma \in [0, 1]$ and $|t| \geq 2\pi$. Also assume that $2\pi xy = |t|$ for $x, y \geq 1$. Then

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \chi(s)\sum_{n \leq y} n^{s-1} + R_1(s;x,y), \quad (8)$$

where $R_1(s;x,y) = O\left(x^{-\sigma} + y^{\sigma-1}|t|^{1/2-\sigma}\right)$ uniformly.

Equation (8) first appeared in [HL21], but with a factor $\log |t|$ in the remainder. The proof of this “imperfect” approximate functional equation exploits the Poisson summation formula while their approach to Theorem 3 was complete analytic in the sense that they used contour integration; it is sketched in [Tit86, p. 81] where also Theorem 2 is proved in such a way. Later they provided in [HL29] a proof along the similar lines as in [HL21]. However, the more common proof, see [Tit86 pp. 82–84] or [Ivi03, pp. 99–104], has roots in the celebrated paper of Siegel [Sie32] where he developed Riemann’s ideas on the zeta function and derived

$$\zeta(s) = \mathcal{R}(s) + \chi(s)\overline{\mathcal{R}(1-s)}, \quad (9)$$

where $\mathcal{R}(s)$ is some function given as the contour integral. A more useful expression for this function is

$$\mathcal{R}(s) = \sum_{n \leq x} n^{-s} + \left(\frac{|t|}{2\pi}\right)^{-\frac{s}{2}} E_L(s), \quad (10)$$

where $E_L(s)$ has a known asymptotic expansion in powers of $|t|^{-1/2}$, see [AdR11 Theorem 3.1]. Equation (10), now called the Riemann–Siegel formula, was first proposed by Lehmer in [Leh59] for values on the critical line. Equations (9) and (10) imply (8) in the symmetric case $x = y = \sqrt{|t|/(2\pi)}$.

The Riemann–Siegel formula can be used to calculate values of $\zeta(s)$ relatively fast, e.g., through the Odlyzko–Schönhage algorithm which is suitable for large scale computations, and thus it replaced the previous method based on the Euler–Maclaurin summation formula or on its simpler version (6). For high precision calculations we still need to know explicit bounds. Titchmarsh [Tit35] carried out a complete analysis of the error terms which comes from Siegel’s method. Since his estimates are most suitable only for sufficiently large values of $t$, Turing developed a

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3See also [Tit86] pp. 79–80] and [KV92 Chapter III].
different method, see [Tur43]. Gabcke provided in [Gab79] good bounds for $E_L(s)$ in case of $\sigma = 1/2$ and Arias de Reyna [AdR11, Section 4] for all values in the critical strip. There also exist generalisations of (10) to $L$-functions and to specially designed smooth functions, see [Hia16]. In the context of the Knopp–Hasse–Sondow formula for $\zeta(s)$, it is possible to obtain even better error term, see [Jer19].

None of the previously mentioned authors considered explicit versions of equation (8) in the non-symmetrical case. The main result of this section is an explicit form of the approximate functional equation (Theorem 6) which comes from the standard proof of Theorem 3. The main advantage of this is having a uniform bound on constants in the $O$-estimate of $R_1$, independent of $x$ and $y$. In Section 2.4 we prove the following.

**Theorem 4.** Let $s = \sigma + it$ where $\sigma \in [1/2, 1]$ and $|t| \geq 2\pi$. Also assume that $2\pi xy = |t|$ for $x, y \geq 1$. If $R_1(s; x, y)$ is defined by equation (8), then

$$|R_1(s; x, y)| \leq E \cdot x^{-\sigma} + F \cdot \left(\frac{|t|}{2\pi}\right)^{\frac{\sigma}{2}} y^{\sigma-1},$$

where $E$ and $F$ are non-negative real numbers, whose values are given by Table 1 for $|t| \geq 2\pi$, Table 2 for $|t| \geq 10^3$ and Table 3 for $|t| \geq 10^{10}$.

We used bounds from [AdR11] to give constants in Tables 1 and 2 in the symmetrical case. While these are expected to be better than those obtained by the classical method, they are not so large at all.

\[
\begin{array}{c|c|c|c}
| & x \leq y & x > y & x = y \\
\hline
E & 36.094 & 0 & 4.257 \\
F & 0 & 127.126 & 0 \\
\hline
\tilde{E} & 36.214 & 0 & 4.376 \\
\tilde{F} & 0 & 127.245 & 0 \\
\end{array}
\]

**Table 1.** Bounds for $|t| \geq 2\pi$.

\[
\begin{array}{c|c|c|c}
| & x \leq y & x > y & x = y \\
\hline
E & 10.983 & 0 & 1.195 \\
F & 0 & 15.726 & 0 \\
\hline
\tilde{E} & 10.992 & 0 & 1.205 \\
\tilde{F} & 0 & 15.726 & 0 \\
\end{array}
\]

**Table 2.** Bounds for $|t| \geq 10^3$.

\[
\begin{array}{c|c|c|c}
| & x \leq y & x > y & x = y \\
\hline
E, \tilde{E} & 10.7502 & 0 & 1.00007 \\
F, \tilde{F} & 0 & 15.203 & 0 \\
\end{array}
\]

**Table 3.** Bounds for $|t| \geq 10^{10}$.

Sometimes it is more convenient to have (8) in the form

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \tilde{\chi}(s) \sum_{n \leq y} n^{s-1} + \tilde{R}_1(s; x, y)$$

(11)

where

$$\tilde{\chi}((\sigma+it)) := \left(\frac{2\pi}{|t|}\right)^{\frac{\sigma}{2}} \left(\frac{|t|}{2\pi e}\right)^{-it} e^{\text{sgn}(t)\frac{\pi}{4}}.$$  

(12)

Note that a consequence of Stirling’s formula is $\chi(\sigma + it) \sim \tilde{\chi}(\sigma + it)$ for $t \to \infty$ where $\chi(s)$ is defined by (7). In Section 2.2 we will provide an explicit version of this asymptotic relation, see Proposition 1. This will enable us to prove the following corollary of Theorem 3.
Corollary 1. Let \( s = \sigma + it \) where \( \sigma \in [1/2, 1] \) and \(|t| \geq 2\pi\). Also assume that \( 2\pi xy = |t| \) for \( x, y \geq 1 \). If \( \tilde{R}_x(s; x, y) \) is defined by equation (14), then

\[
\left| \tilde{R}_x(s; x, y) \right| \leq \tilde{E} \cdot x^{-\sigma} + \tilde{F} \cdot \left( \frac{|t|}{2\pi} \right)^{\frac{1}{2}} y^{\sigma-1},
\]

(13)

where \( \tilde{E} \) and \( \tilde{F} \) are non-negative real numbers, whose values are given by Table 7 for \(|t| \geq 2\pi\), Table 8 for \(|t| \geq 10^3\) and Table 9 for \(|t| \geq 10^{10}\).

2.1. Some estimates for \( R(s; x) \). It seems that an explicit version of Theorem 2 first appeared in [Che99, Proposition 1]. Cheng’s result was considerably improved by Kadiri in [Kad13, Theorem 1.2]. Following the proof outlined there, we can obtain an explicit bound for \( R(s; x) \) which also slightly improves Kadiri’s bound.

Theorem 5. With assumptions and notations as in Theorem 3 we have

\[
|R(s; x)| \leq x^{-\sigma} \left( \frac{1}{2} + \frac{3x}{|t|} \right) \sqrt{1 + \left( \frac{\sigma}{t} \right)^2} \left( 1 - \frac{t}{2x} \cot \frac{t}{2x} \right)
\]

(14)

for \( t \neq 0 \).

Proof. Our proof is basically the same as the proof in [Kad13], except that we use closed expression for the sum in (16).

Let \( N \geq 2 \). We start with the classical summation formula

\[
\sum_{x < n \leq N} n^{-s} = N^{1-s} \frac{x^{1-s}}{1-s} + \frac{1}{2N^s} + s \int_x^N \frac{\langle u \rangle}{u^{s+1}} du
\]

where \( \langle u \rangle := |u| - x + 1/2 \), see [Tit86, Equation 2.1.2]. Then

\[
R(s; x) = \frac{\langle u \rangle}{x^s} + s \int_x^\infty \frac{\langle u \rangle}{u^{s+1}} du + s \int_x^N \frac{\langle u \rangle}{u^{s+1}} du
\]

and from this it follows that

\[
|R(s; x)| \leq \frac{|s|}{2\pi N^s} + \frac{1}{2x^s} + |t| \sqrt{1 + \left( \frac{\sigma}{t} \right)^2} \left| \int_x^N \frac{\langle u \rangle}{u^{s+1}} du \right|.
\]

(15)

Writing \( \langle u \rangle \) in form of the Fourier series and applying the second mean value theorem, we have

\[
\int_x^N \frac{\langle u \rangle}{u^{s+1}} du = \sum_{n=1}^\infty \frac{I(n) - I(-n)}{n} \leq 3 \frac{\pi x^{-\sigma}}{2\pi 2\pi x n \mp t}.
\]

For details of this derivation see [Kad13, pp. 189–190]. Then

\[
\left| \int_x^N \frac{\langle u \rangle}{u^{s+1}} du \right| \leq \frac{6x^{1-s}}{(2\pi x)^2} \sum_{n=1}^\infty \left( n^2 - \left( \frac{t}{2\pi x} \right)^2 \right)^{s-1} = \frac{3x^{1-s}}{t^2} \left( 1 - \frac{t}{2x} \cot \frac{t}{2} \right)
\]

(16)

and (14) clearly follows from (15) and (13) after taking \( N \to \infty \). Equality in (16) is established by a well-known identity

\[
\sum_{n=1}^\infty \frac{1}{n^2 - a^2} = \frac{1 - (\pi a) \cot (\pi a)}{2a^2},
\]

see [GR13, Eq. 1.421 3].

Corollary 2. Let \( s = \sigma + it \), \( \sigma \in [1/2, 1] \), \(|t| \geq t_0 > 0 \) and \( c > 1/(2\pi) \). Then

\[
\left| \zeta(s) - \sum_{n < c \cdot |t|} n^{-s} \right| \leq (tc)^{-\sigma} \left( c + \frac{1}{2} + \frac{3c}{t_0} \sqrt{1 + t_0^2} \left( 1 - \frac{1}{2c} \cot \frac{1}{2c} \right) \right).
\]
In particular, if $c = 1$ and $t_0 = \gamma_1$ where $\gamma_1 \approx 14.1347$ is the imaginary part of the first non-trivial zero of $\zeta(s)$, then
\[
\left| \zeta(s) - \sum_{n=1}^{t} n^{-s} \right| \leq 1.755 \cdot t^{-\sigma}.
\]

This improves Kadiri’s constant 2.1946, see [Kad13, Corollary 1.3]. It was shown in [DHZA19, Lemma 2.10] that the Euler–Maclaurin summation formula implies that (3) is true with $|R(s;x)| \leq 5/6 \cdot x^{-\sigma}$ for $\sigma \in (0,1]$ and $|t| \leq x$. Numerical calculations show that for $|t| \geq 1.18$ inequality (13) always provides the better bound.

Taking $t \to 0$ in (13), we obtain $|R(\sigma;x)| \leq x^{-\sigma} (1/2 + \sigma/(4x))$. However, it is possible to prove in quite elementary way that $|R(\sigma;x)| \leq x^{-\sigma}/2$ for all $\sigma \in (0,\infty) \setminus \{1\}$, see [DHZA19, Lemma 2.9]. We will use this estimate in the proof of Theorem 7, see Section 3.6.

2.2. Explicit Stirling approximation of $\chi(s)$. In the proof we make use of the following upper and lower bounds of $\arctan x$ which are asymptotically sharp. We note that the second inequality in (17) can be found in [AM15, Corollary V.14].

**Lemma 1.** For $x \geq 0$ we have
\[
\frac{\pi}{2} x \geq \arctan x \geq \frac{\pi}{2} x + \sqrt{x^2 + \left(\frac{\pi}{2} - \frac{1}{2}\right)^2}.
\]  

**Proof.** Denote by $\Delta_1(x)$ the difference between the upper bound and arctan $x$, and by $\Delta_2(x)$ the difference between arctan $x$ and the lower bound. For $x \geq 0$ these functions are smooth, and we have $\Delta_1(0) = \Delta_2(0) = 0$ and $\lim_{x \to \infty} \Delta_1(x) = \lim_{x \to \infty} \Delta_2(x) = 0$. Numerical verification reveals that both functions are positive for $x = 1$. Equations $\Delta_1'(x) = 0$ and $\Delta_2'(x) = 0$ can be reduced to a linear and a quadratic equation, respectively. After simple calculations we can conclude that both functions have only one stationary point on the interval $(0, \infty)$. Hence they cannot have any zeros for $x > 0$ due to zero limits at infinity. This implies that both functions are positive throughout this region. $\blacksquare$

**Proposition 1.** Let $\sigma \in (1/2,1]$ and $|t| \geq t_0 \geq 1/\pi$. Then
\[
\chi(\sigma + it) = \tilde{\chi}(\sigma + it) \left(1 + \frac{C(\sigma,t,t_0)}{|t|}\right)
\]
where
\[
|C(\sigma,t,t_0)| \leq C_1(\sigma,t) \left(1 + \frac{t_0 e^{-\pi t_0}}{|t|}\right) C_2(t) + C_4(t,t_0)
\]
with
\[
C_1(\sigma,t) := (1-\sigma)^2 \left(\frac{1}{2} + \frac{2}{\pi}\right) + (1-\sigma) \left(\sigma - \frac{1}{2}\right) \left(\frac{\pi}{2}\right)^2 + \frac{1-\sigma}{2|t|},
\]
\[
C_2(t) := \exp\left(\frac{1}{12|t|} + \frac{1}{90|t|^3}\right),
\]
\[
C_4(t,t_0) := \frac{C_2(t_0)}{\log C_2(t_0)} \left(\frac{1}{12} + \frac{1}{90t_0}\right) + t_0 e^{-\pi t_0} C_2(t).
\]

**Proof.** It is enough to prove the case when $t$ is positive since $\chi(s) = \chi(\bar{s})$ and $\tilde{\chi}(s) = \tilde{\chi}(\bar{s})$. We use Stieltjes’ explicit version of the Stirling formula for $\Gamma(z)$ where $\Re\{z\} > 0$, see [Dit74]:
\[
\Gamma(z) = \sqrt{2\pi e^{-z} z^{z-1/2}} e^{-R(z)}, \quad |R(z)| \leq \frac{1}{12|z|} + \frac{1}{90|z|^3}.
\]
From the explicit expressions for $\chi(s)$ and $\Gamma(z)$ we obtain

\[ \frac{\chi}{\chi}(\sigma + it) - 1 = ((a(\sigma, t) - 1)e^{i\varphi(\sigma,t)} + e^{i\varphi(\sigma,t)} - 1) (1 + \varepsilon(\sigma, t)) + \varepsilon(\sigma, t) \]

where

\[ a(\sigma, t) := \left( \frac{1}{1 + \left( \frac{1 - \sigma}{t} \right)^2} \right)^{\frac{1}{2}} e^{r(\sigma, t)}, \]

\[ r(\sigma, t) := \frac{\pi}{2} t - t \arctan \frac{t}{1 - \sigma} + \sigma - 1, \]

\[ \varphi(\sigma, t) := \left( \frac{1}{2} - \sigma \right) \left( \frac{\pi}{2} - \arctan \frac{t}{1 - \sigma} \right) - \frac{t}{2} \log \left( 1 + \left( \frac{1 - \sigma}{t} \right)^2 \right), \]

\[ \varepsilon(\sigma, t) := e^{R(1 - \sigma - it)} - 1 - e^{-\pi t + \pi i + R(1 - \sigma - it)}. \]

Then

\[ \left| \frac{\chi}{\chi}(\sigma + it) - 1 \right| \leq (|a(\sigma, t) - 1| + |\varphi(\sigma, t)|) \cdot |1 + \varepsilon(\sigma, t)| + |\varepsilon(\sigma, t)|. \]  \hspace{1cm} (22)

Using Stieltjes’ error term (20) and noting that $|e^z - 1| \leq e^{|z|} - 1$, and that $(e^x - 1)x^{-1}$ and $e^{-\pi t}$ are strictly decreasing functions for $x > 0$ and $t \geq t_0 \geq 1/\pi$ respectively, we get

\[ |1 + \varepsilon(\sigma, t)| \leq \left( 1 + \frac{t_0 e^{-\pi t_0}}{t} \right) C_2(t), \quad |\varepsilon(\sigma, t)| \leq \frac{C_3(t, t_0)}{t}. \]  \hspace{1cm} (23)

The second inequality in (17) gives us

\[ \left| \frac{\pi}{2} - \arctan \frac{t}{1 - \sigma} \right| \leq \frac{\pi}{2} \left( 1 - \frac{1 - \sigma}{\frac{\pi}{2} - \arctan \frac{t}{1 - \sigma} + \sigma - 1} \right)^2 \leq \left( \frac{\pi}{2} \right)^2 \frac{1 - \sigma}{t}, \]  \hspace{1cm} (24)

which implies

\[ |\varphi(\sigma, t)| \leq \frac{1 - \sigma}{t} \left( \frac{\pi}{2} \left( \sigma - \frac{1}{2} \right) + \frac{1 - \sigma}{2} \right). \]  \hspace{1cm} (25)

By (24) we have

\[ \frac{\partial}{\partial t} r(\sigma, t) = \frac{\pi}{2} - \arctan \frac{t}{1 - \sigma} - \frac{\sigma}{1 + \left( \frac{t}{1 - \sigma} \right)} > 0. \]

Together with $r(\sigma, 0) < 0$ and $\lim_{t \to \infty} r(\sigma, t) = 0$ this implies $r(\sigma, t) < 0$. Next,

\[ |a(\sigma, t) - 1| \leq \left( 1 - \left( \frac{1}{1 + \left( \frac{1 - \sigma}{t} \right)^2} \right)^{\frac{1}{2}} e^{r(\sigma, t)} + |e^{r(\sigma, t)} - 1| \right) \]

\[ \leq \left( 1 - \exp \left( - \frac{1}{2} \left( \sigma - \frac{1}{2} \left( \frac{1 - \sigma}{t} \right)^2 \right) \right) \right) e^{r(\sigma, t)} + |e^{r(\sigma, t)} - 1|. \]

Applying the first inequality in (17), we get

\[ |r(\sigma, t)| \leq \frac{2(1 - \sigma)^2}{\pi t}. \]

Using the inequalities $e^{-x} \leq 1$ and $1 - e^{-x} \leq x$, both valid for $x \geq 0$, we obtain

\[ |a(\sigma, t) - 1| \leq \frac{(1 - \sigma)^2}{t} \left( \frac{1}{2t} \left( \sigma - \frac{1}{2} \right) + \frac{2}{\pi} \right). \]  \hspace{1cm} (26)

Inserting (26), (25) and (23) into (22), we finally obtain (18). \hspace{1cm} \blacksquare
Corollary 3. Let $s = \sigma + it$ where $\sigma \in (0, 1)$ and $t \geq 1$. Then
\[ \frac{1}{2\pi} |e^{-i\pi} \Gamma(1 - s)| \leq \frac{C_2(t)}{2\pi \sqrt{\pi}} e^{\frac{x}{2}} e^{\frac{t}{2} + R(1 - \sigma - it)} \]
where $C_2(t)$ is defined by \[19\] .

Proof. We have
\[ |e^{-i\pi} \Gamma(1 - s)| = \sqrt{2\pi} |1 - \sigma - it|^{\frac{1}{2} - \sigma} e^{\frac{x}{2} + R(1 - \sigma - it)} \]
where $r(\sigma, t)$ and $R(z)$ are defined by \[21\] and \[20\], respectively. Because $t \leq 1 - \sigma - it \leq 2t$, we have $|1 - \sigma - it|^{\frac{1}{2} - \sigma} \leq t^{\frac{1}{2} - \sigma}$ and $|1 - \sigma - it|^{\frac{1}{2} - \sigma} \leq (2t)^{\frac{1}{2} - \sigma}$ for $\sigma \in [1/2, 1)$ and $\sigma \in (0, 1/2]$, respectively. The result now follows since $r(\sigma, t)$ is always negative.

Assume that $t \geq t_0 \geq 2\pi$. Observe that $C_1(\sigma, t) \leq C_1(\sigma, 2\pi)$, $C_2(t) \leq C_2(2\pi)$ and $C_3(t, t_0) \leq C_3(2\pi, 2\pi)$. This implies
\[ |C(\sigma, t, t_0)| \leq C_1(\sigma, 2\pi) \left(1 + e^{-2\pi^2}\right) C_2(2\pi) + C_3(2\pi, 2\pi) < 0.3746 \]
(27)
since the function in the middle has the maximum at $\sigma \approx 0.54162$.

Let $\sigma \in [0, 1/2)$ and $t \geq t_0 \geq 2\pi$. Because $\chi(s)\chi(1 - s) = 1$, we have
\[ |\chi(\sigma + it)| = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2} - \sigma} \left|1 + \frac{C(1 - \sigma, t, t_0)}{|t|}\right|^{-1} \]
\[ \leq \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2} - \sigma} \left|\frac{|t|}{|t| - |C(1 - \sigma, t, t_0)|}\right|. \]
(28)

2.3. Explicit estimate for $R_1(s; x, y)$. In this section we will provide an explicit upper bound for the remainder in \[5\]. Our proof requires a bound of $|e^z - 1|$ for $z = re^{i\phi}$ with $r > 0$ and $\phi \in [0, 2\pi]$. We would like to obtain non-zero and $\phi$-independent lower bound. Observe that the trivial estimate $|e^z - 1| \geq |e^{r\cos \phi} - 1|$ is not good since it is zero for $\phi \in \{\pi/2, 3\pi/2\}$.

Lemma 2. Let $D_r := \bigcup_{k \in \mathbb{Z}} \{z \in \mathbb{C} : |z - 2k\pi| < r\}$ where $r \in (0, \pi/\sqrt{2}]$. For $z \in \mathbb{C} \setminus D_r$ we have
\[ |e^z - 1| \geq 1 - e^{-\sqrt{r}}. \]
(29)

Proof. Firstly, observe that $|e^{x+iy} - 1| \geq e^x - 1$ for $x > 0$ and $|e^{x+iy} - 1| \geq 1 - e^x$ for $x < 0$. This means that $|e^{x+iy} - 1| \geq 1 - e^{-h}$ for $|x| \geq h > 0$.

Let
\[ S_r := \bigcup_{k \in \mathbb{Z}} \{x + iy \in \mathbb{C} : |x| < \frac{r}{\sqrt{2}}, |y - 2k\pi| < \frac{r}{\sqrt{2}}\} \]
be a set of squares inscribed in $D_r$. For $|x| \geq r/\sqrt{2}$ we have $|e^{x+iy} - 1| \geq 1 - e^{-r/\sqrt{2}}$ while for $|y| = r/\sqrt{2}$ we have $|e^{x+iy} - 1| \geq \sin \left(r/\sqrt{2}\right)$. This gives us
\[ \min_{z \in \partial S_r} |e^z - 1| \geq \min \left\{\sin \left(\frac{r}{\sqrt{2}}\right) - 1, e^{-\sqrt{r}}\right\} = 1 - e^{-\sqrt{r}}. \]

Take large $k \in \mathbb{N}$ and let $S'_k$ be a two-dimensional closed square with vertices $-k\pm 2k\pi$ and $k\pm 2k\pi$. Define $\Omega(r, k) := (\mathbb{C} \setminus S'_r) \cap S'_k$. Then $\min_{z \in \partial \Omega(r, k)} |e^z - 1| \geq 1 - e^{-\sqrt{r}}$. Because the set $\Omega(r, k)$ is bounded and $e^z - 1$ is holomorphic in the interior, the minimum principle implies $|e^z - 1| \geq 1 - e^{-\sqrt{r}}$ for every $z \in \Omega(r, k)$. Lemma 2 now follows because for every $z \in \mathbb{C} \setminus D_r$ there exists $k$ such that $z \in \Omega(r, k).$
Numerical calculations suggest that the minimum value of $|e^z - 1|$ on the set $\{z \in \mathbb{C}; |z| = r\}$ occurs at $z = -r$, thus giving lower bound $1 - e^{-r}$ in (29). But the author is unable to prove this claim.

We are now in a position to prove Theorem 6. We follow the proof presented in [Tit86] but with flexible parameters that have exactly prescribed domains of validity. This allows some optimisation when trying to get the best possible uniform bound for the remainder.

**Theorem 6.** Let $s = \sigma + it$ where $\sigma \in [0, 1]$ and $|t| > t_0 > 2\pi$. Also assume that $2\pi xy = |t|$ for $x, y \geq 1$ where $x \leq y$. In addition, let $r_0, c, \lambda_0$ and $d$ be four real numbers satisfying the following conditions:

(a) $0 < r_0 \leq \pi/\sqrt{2}$,
(b) $\frac{r_0}{|t|} \leq c \leq \frac{10 + 3\sqrt{2}}{10 + 3\sqrt{2}}$,
(c) $r_0 \leq \lambda_0 \leq \frac{|t|}{x}$,
(d) $d \geq \frac{\pi}{\sqrt{2}x}$.

Define four functions $E_1(\sigma, t, c, d, x)$, $E_2(\sigma, t, r_0, c, \lambda_0, x, y)$, $E_3(\sigma, t, c, x)$ and $E_4(\sigma, t, r_0, c, x, y)$ in the following way:

$$E_1 := E_1(c) := 2e^c + 2c + 1, \quad E_2(t, c, d, x) := d - c + \frac{x}{|t|} \log \frac{1 - e^{-\frac{|t|}{2}}}{1 - e^{-\frac{2c}{x}}},$$

$$E_3(\sigma, t, c, x) := \frac{x}{\sqrt{\pi t}} E_2 e^{-|t|\Phi_1} + E_3 e^{-|t|\Phi_2},$$

where

$$E_1(c) := 2e^c + 2c + 1, \quad E_2(t, c, d, x) := d - c + \frac{x}{|t|} \log \frac{1 - e^{-\frac{|t|}{2}}}{1 - e^{-\frac{2c}{x}}},$$

$$\Phi_1(c) := c - \arctan \frac{c}{1 + c}, \quad \Phi_2(d, x) := d \frac{|x|}{x} - \frac{\pi}{2},$$

$$E_2 := \sqrt{\frac{2}{\pi}} E_4 \left( \frac{1}{1 - e^{-\frac{c}{x}}} + \frac{1}{e^{\lambda_0} - 1} \right) e^{\varepsilon_5} + \frac{r_0}{1 - e^{-\frac{c}{x}}} \sqrt{\frac{2}{\pi}} (1 + \frac{1}{y})^{-1} e^{\varepsilon_5},$$

where

$$E_4(\sigma, t, c) := 1 - \frac{(1 - \sigma + |t|)2\sqrt{2}c}{3|t| \left(1 - c\sqrt{2}\right)},$$

$$\varepsilon_5(\sigma, t, c, \lambda_0, x) := \lambda_0 (x - |x|) + \frac{(1 - \sigma)^2}{4|t| E_4(\sigma, |t|, c)};$$

$$E_6(\sigma, t, r_0, x, y) := \frac{(1 - \sigma)r_0}{2\pi |y|} + \frac{r_0 \left(1 - \sigma\right) |x| \left(1 - \sigma\right)}{|y|}$$

$$+ \frac{r_0^2 x y \left(1 + \frac{1 - \sigma}{y}ight) \left(1 + \frac{r_0}{2(y + |x| - r_0)} \right)}{2\pi |y|^2};$$

$$E_3 := e^{\sigma - 1} \frac{2 - c + \frac{\pi x}{|t|}}{1 - e^{-\frac{c}{x}}} \sqrt{\frac{|t|}{\pi}} e^{-|t|\Phi_1(-c)},$$

and

$$E_4 := \frac{x \left(1 - \frac{r_0}{|t|}\right)^{\sigma - 1}}{|x| \left(1 - e^{-\frac{c}{x}\sqrt{2}}\right) \sqrt{\pi |t|}} e^{-|t|\Phi_2}.$$
Proof. Firstly, we will show how to obtain (31) from (30). Changing $x, y$ equation with reversed role of $\chi$ in (8) and multiplying both sides by inequality (28). This implies assertions for $\sigma$.

Moreover, if $c, d, \lambda_0$ and $r_0$ are fixed, then $E$ is bounded and the parts $E_1, E_3, E_4$ are decreasing to zero while $t \to \infty$.

Proof. Firstly, we will show how to obtain (31) from (30). Changing $s$ to $1 - s$ in (30) and multiplying both sides by $\chi(s)$, we obtain the approximate functional equation with reversed role of $x, y$ and $R_1(s; x, y) = R_1(1 - \sigma - it; x, y)\chi(\sigma + it)$. This implies $|R_1(s; x, y)| \leq E(1 - \sigma)|\chi(\sigma + it)|x^{\sigma-1}$. Since $|\chi(1/2 + it)| \equiv 1$ our assertions for $\sigma \in [1/2, 1]$ follow directly from Proposition 1 and for $\sigma \in [0, 1/2)$ by inequality (28).

The main equation in the analytical proof of (8) is

$$
\zeta(s) = \sum_{n=1}^{x} n^{-s} + \chi(s) \sum_{n=1}^{y} n^{s-1} + \frac{e^{-i\pi x}(1-s)}{2\pi i} \int_{C} z^{s-1}e^{-|z|z} \frac{dz}{e^{z} - 1} \tag{32}
$$

where $C$ is a positively oriented contour $C$ which goes from $+\infty$, encircles zeros $\pm 2\pi i$ of $e^{z} - 1$ with $t \in \{0, 1, 2, \ldots, \{y\}\}$, and returns back to $+\infty$, see [1v03 pp. 99–100] for a detailed derivation of (32).

Let $[a, b]$ be a line segment in the complex plane with endpoints $a$ and $b$. Define $\eta := 2\pi y = t/x$, $z_1 := cy + i(1 + c)$, $z_2 := -cy + i(1 - c)$ and $z_3 := -cy - i(2/|y| + 1)$. Also define $q$ as $|y|$ if $\{y\} \leq 1/2$ and $|y| + 1$ otherwise. The reader is advised to consult Figure 1. Because of the condition (a), the set $J := [z_1, z_2] \cap \partial D_{r_0}$, where $D_r$ is defined in Lemma 2, is empty or contains exactly two elements, say $w_1$ and $w_2$. Without loss of generality we can assign $w_1$ to the point closer to 1. In the latter case, these two points are on the same circle with radius $r_0$ and center at $2\pi i q$, unless $r_0 = \pi / \sqrt{2}$ and $\eta = \pi (2l + 1)$.

Deform $C$ into four curves. Let $C_1 := [\infty + i(1 + c), z_1]$, $C_3 := [z_2, z_3]$ and $C_4 := [z_3, -\infty - i(2/|y| + 1)]$. Let $C_2 := [z_1, z_2]$ if $J = \emptyset$, and $[z_1, w_1] \cup [w_1, w_2] \cup [w_2, z_1]$ otherwise where $w_1w_2$ is a smaller arc on circle if both points belong to the same circle. If this is not the case, we take the segment $[w_1, w_2]$ instead of the arc. Anyway, such contour always lies in $C \setminus D_{r_0}$.

Write $z = u + i r e^{i\theta}$ where $r > 0$. Then $|z^{s-1}| = r^{\sigma-1} e^{-\sigma t}$ and $|e^{-m|z|}| = e^{-mu}$. Denote by $I_k$ the integral in (32) which goes along $C_k$. In the next paragraphs we will derive explicit bounds for each $I_k$ which will, together with Corollary 3, give the final bounds.

Consider integration along $C_1$. We have

$$
\frac{|z^{s-1}e^{-|z|\eta}|}{e^{z} - 1} \leq \eta^\sigma \mathcal{E}_1(c) \frac{\eta^{x+1} e^{-|\eta|u}}{\eta e^\eta - 1}, \tag{33}
$$
and also
\[
\left| \frac{z^{s-1}e^{-|x|z}}{e^z - 1} \right| \leq \eta^s E_1(c) \frac{z^{s-1}e^{-|x|z}}{e^z - 1} \leq \frac{\eta^s E_1(c)}{e^z - 1} \frac{e^u}{\eta^s e^z - 1}.
\] (34)

Note that \( \Phi_1(c) \) is strictly increasing, thus \( \Phi_1(c) > 0 \). The last inequality is true because
\[
-t \arctan \frac{\eta(1+c)}{u} - (|x| + 1)u \leq -t \arctan \frac{\eta(1+c)}{u} - \frac{tu}{\eta} \leq -\frac{\pi}{2} t - t \Phi_1(c)
\]
since the function in the middle is strictly decreasing in the variable \( u \) and \( \arctan \alpha + \arctan 1/\alpha = \pi/2 \) for \( \alpha > 0 \). Let \( d \) satisfies the condition (d). Then \( d > c \) and
\[
|I_1| \leq \left( \int_{c+d}^{\infty} + \int_{d_0}^{\infty} \right) \left| \frac{z^{s-1}e^{-|x|z}}{e^z - 1} \right| du
\leq \eta^s E_1(c) \frac{z^{s-1}e^{-|x|z}}{e^z - 1} \left( e^{-\frac{\pi}{2} t - t \Phi_1(c)} \frac{1}{\eta} \log \frac{e^{dz}}{e^{c\eta} - 1} + \frac{e^{-|x|dz}}{|x| \eta (e^{dz} - 1)} \right)
\]
where we use (34) for the first integral and (33) for the second one. This implies that
\[
\left| \frac{e^{-i\pi s} \Gamma(1-s)}{2\pi i} I_1 \right| \leq \frac{C_2(t)}{2^\sigma} E_1 x^{-\sigma}.
\]

Note that \( \Phi_1(c) > 0 \) and condition (d) imply that \( E_1 \to 0 \) while \( t \to \infty \) if \( c \) and \( d \) are fixed.

Consider integration along \( C_2 \). The main idea is to apply the bound from Lemma 2 on a part of \( C_2 \) which goes through \( \{ z \in \mathbb{C} : |\Re\{z\}| \leq \lambda_0 \} \setminus \mathcal{D}_{r_0} \), where \( \lambda_0 \) satisfies the condition (c). This set is represented by the grey colour in Figure 1. Firstly, observe that for \( |z| < 1 \) we can write \( \log(1+z) = z + f_1(z) = z - z^2/2 + z^3 f_2(z) \) with
\[
|f_1(z)| \leq |z|^2 \left( \frac{1}{2} + \frac{|z|}{3(1-|z|)} \right), \quad |f_2(z)| \leq \frac{1}{3(1-|z|)}.
\]
Let \( z(\lambda) = i\eta + \lambda \sqrt{2} e^{i \pi/4}, \lambda \in [-c\eta, c\eta], \) be a parametrisation of the line \([z_1, z_2]\).
Then we have
\[
\log z(\lambda)^{s-1} - \log \left(e^{(s-1)\lambda i} \eta^{s-1}\right) = (s - 1) \log \left(1 + \frac{\lambda \sqrt{2}}{\eta} e^{\frac{-\eta}{s}}\right)
\]
\[
= (s - 1) \left(\frac{\lambda \sqrt{2}}{\eta} e^{\frac{-\eta}{s}} - \frac{\lambda^2}{\eta^2} e^{\frac{-\eta}{s}} + \frac{\lambda^3 2 \sqrt{2}}{\eta^3} e^{\frac{-\eta}{s}} o\right)
\]
where \(|o| \leq \left(3 \left(1 - c\sqrt{2}\right)\right)^{-1}.\)
The above equation is valid if \(|\lambda| < \eta/\sqrt{2},\) and this is true because \(c < 1/\sqrt{2}\) due to the condition (b).
From this we obtain
\[
|z(\lambda)^{s-1}| \leq \eta^{s-1} \exp \left(t \left(\frac{\eta}{2} + \lambda - E_4(\sigma, t, c) \frac{\lambda^2}{\eta^2} + \frac{(\sigma - 1)\lambda}{t}\right)\right).
\]
Note that \( E_4 > 0. \)
Writing \( e^{-|x|z} = e^{z(|-x|)} e^{-xx} \) and noticing that \( \cosh (ax) \geq \cosh ((1 - a)x) \) for \( a \geq 0, \) we have
\[
\frac{e^{-|x|z(\lambda)}}{|z(\lambda) - 1|} \leq \frac{e^{\lambda_0 (z(|-x|))}}{e^{\lambda_0 - 1}}
\]
for \(|\lambda| \geq \lambda_0.\)
Denote the integration along segments \([z_1, z(\lambda_0)]\) and \([z(-\lambda_0), z_2],\)
and \([z(\lambda_0), w_1]\) and \([w_2, z(-\lambda_0)]\) by \( I_{21} \) and \( I_{22},\) respectively.
Because
\[
I(a, b) := \int_{-\infty}^{\infty} e^{-(a\lambda)^2 + b\lambda} d\lambda = \frac{\sqrt{\pi}}{|a|} \exp \left(\frac{b^2}{4a^2}\right)
\]
for real numbers \( a \) and \( b,\) see [GR15, Eq. 3.323 2], it follows
\[
|I_{21}| \leq \sqrt{2}\eta^{\sigma-1} \frac{e^{\lambda_0 (z(|-x|))}}{e^{\lambda_0 - 1}} e^{-\frac{\eta}{s} I} \left(\frac{\sqrt{E_4}}{\eta} \frac{\sigma - 1}{\eta}\right)
\]
\[
\leq t^{\sigma-\frac{\eta}{s}} x^{-\sigma} \sqrt{\frac{2\pi}{E_4}} \frac{1}{e^{\lambda_0 - 1}} \exp \left(-\frac{\pi}{2} t + E_5\right).
\]
The bound for \( I_{22} \) is the same except that we must replace \( e^{\lambda_0} - 1 \) by \( 1 - e^{-\lambda_0} \) in the above inequality.
Let \( z = 2\pi qi + r_0 e^{i\varphi} \) be a parametrisation of the circle with center at \( 2\pi qi \) and radius \( r_0.\)
Denote the integration along the arc \( w_1 w_2 \) by \( I_{23}.\)
Since
\[
(s - 1) \log \left(1 + \frac{r_0 e^{i\varphi}}{2\pi qi}\right) - r_0 |x| e^{i\varphi} = \frac{(\sigma - 1) r_0 e^{i\varphi}}{2\pi qi} + (\sigma - 1 + i)t f_1 \left(\frac{r_0 e^{i\varphi}}{2\pi qi}\right)
\]
\[
+ \left(\frac{t}{2\pi q} - |x|\right) r_0 e^{i\varphi},
\]
we have
\[
|I_{23}| \leq \frac{r_0 \pi}{1 - e^{-\frac{\sqrt{2}}{\eta}}} e^{\frac{-\eta}{s} x^{-\sigma}} \sqrt{\frac{e}{2\pi y}} \left(1 + \frac{1}{y}\right)^{\sigma-1} e^{-\frac{\eta}{s} t + E_6}.
\]
Because \( I_2 = I_{21} + I_{22} + I_{23}\), we finally obtain
\[
\left|e^{-i\sigma s} \Gamma(1 - s)\right| I_2 \leq \frac{C_2(t)}{2^\sigma} E_2 x^{-\sigma}.
\]
Note that \( E_2,\) although bounded for fixed \( c, \lambda_0 \) and \( r_0,\) does not tend to zero while \( t \to \infty\) due to a contribution from parts \( I_{21} \) and \( I_{22}.\)
Consider integration along \( C_3.\)
Because
\[
\left|\frac{z^{s-1} e^{-|x|z}}{e^z - 1}\right| \leq \left(\frac{\sigma - 1}{1 - e^{-\eta}}\right) e^{-\frac{\eta}{s} t + E_1(-c)},
\]
we have
\[ |I_3| \leq \frac{e^{\sigma-1}(2 - c + \pi / \eta)}{1 - e^{-\eta c}} \eta^\sigma e^{-\frac{\pi}{2} t \Phi_1(-c)} \]

since \( \eta(1-c) + (2 [y] + 1) \pi \leq \eta(2-c+\pi/\eta) \). Note that \( \Phi_1(-c) \) is strictly increasing, thus \( \Phi_1(-c) > 0 \). From this we obtain
\[ \left| \frac{e^{-is\pi} \Gamma(1 - s)}{2^\pi} I_3 \right| \leq \frac{C_2(t)}{2^\sigma} E_3 x^{-\sigma}. \]

Note that \( \Phi_1(-c) > 0 \) implies that \( E_3 \to 0 \) while \( t \to \infty \) if \( c \) is fixed.

Consider integration along \( C_4 \). Because \( (2 [y] + 1) \pi > \eta - \pi, \) we have
\[ |z^{s-1} e^{-[x]z}| \leq \frac{1}{\eta} \left( 1 - \frac{\pi}{\eta} \right)^{\sigma-1} \eta^\sigma \exp \left( -\frac{\pi}{2} t \Phi_3 - \frac{|x|}{x} \right). \]

Then
\[ |I_4| \leq \frac{x}{1 - e^{-\eta c}} \left( 1 - \frac{\pi x}{t} \right)^{\sigma-1} t^{\sigma-1} x^{-\sigma} e^{-\frac{\pi}{2} t (\int_{-c\eta}^\infty e^{-[x]u} du)} e^{-t \Phi_3 - \frac{|x|}{x}} \]

which gives
\[ \left| \frac{e^{-is\pi} \Gamma(1 - s)}{2^\pi} I_4 \right| \leq \frac{C_2(t)}{2^\sigma} E_4 x^{-\sigma}. \]

Note that \( E_4 \to 0 \) while \( t \to \infty \) since \( c < \pi/2 \leq \pi x / (2 [x]) \).

2.4. Numerical analysis of the error term. Let \( 0 \leq \sigma_0 \leq \sigma \leq 1 \). Among all four terms in \( E \), the \( E_2 \) is the only one which does not go asymptotically to zero, also because of term \( E_5 \). This suggests we choose \( \lambda_0 \) as small as possible according to the condition (c) of Theorem 5 therefore \( \lambda_0 = r_0 \). Because \( r_0 x / |t| \leq r_0 / (2 \pi) \leq 1 / (2 \sqrt{2}) \), the choice \( c = r_0 / (2 \pi) \) satisfies the condition (b). Putting these two parameters into \( E_2 \), we can obtain
\[
E_2 \leq \sqrt{\frac{6 (\pi \sqrt{2} - r_0)}{3 \pi \sqrt{2} - r_0 \left( 5 + \frac{2(1-\sigma_0)}{|t|} \right)}} \left( \frac{1}{1 - e^{-\eta c}} + \frac{1}{e^{\eta c} - 1} \right) \]
\[
\cdot \exp \left( r_0 + \frac{(1-\sigma_0)^2 (6 \pi - 3 r_0 \sqrt{2})}{4|t| (6 \pi - 5 r_0 \sqrt{2}) - 8 (1 - \sigma_0) r_0 \sqrt{2}} \right) + \frac{r_0 \sqrt{2}}{1 - e^{-\eta c}} \]
\[
\cdot \exp \left( \frac{r_0^2 (|t| + 1 - \sigma_0)}{2|t|} \right) \left( \frac{1}{2} + \frac{r_0}{3 \left( 2 \pi \left( \sqrt{\frac{2 \pi}{r_0}} - r_0 \right) \right)} \right) \left( 1 + \frac{\sqrt{|t|}}{2 \pi} \right)^{-1} \]
\[
+ r_0 \left( 2 + \frac{1}{3 \pi} \right) \left( \frac{1}{2 \pi} \right)^{-1} \left( \sqrt{\frac{|t|}{2 \pi}} \right)^{-1} \right) .
\]

Taking \( |t| \to \infty \) in the above expression, we get
\[
\sqrt{\frac{6 (\pi \sqrt{2} - r_0)}{3 \pi \sqrt{2} - 5 r_0}} \left( \frac{1}{1 - e^{-\eta c}} + \frac{1}{e^{\eta c} - 1} \right) e^{r_0 \sqrt{2}} + \frac{r_0 \sqrt{2}}{1 - e^{-\eta c}} \exp \left( \frac{r_0^2}{4 \pi} + 2 r_0 \right).
\]

Because this function is positive and continuous for \( 0 < r_0 \leq \pi / \sqrt{2} \) with a pole at \( r_0 = 0 \), it must have a minimum value on this interval. Let \( R_0 \) be the upper bound of the set where the minimum value is attained. Numerical calculations show that there is only one stationary point at \( R_0 \approx 0.52777 \) and the minimum value is \( \approx 15.2029 \).
Choosing \( d = \pi x / (2 \lfloor x \rfloor) < \pi \), and using inequalities \( \pi x / |t| \leq \sqrt{\pi / (2|t|)} \) and \( c|t|/x \geq \pi \sqrt{\pi / (2|t|)} \), we can estimate

\[
E_1 \leq \sqrt{|t| / \pi} \exp \left( -|t| \left( \frac{r_0}{2 \pi} - \arctan \frac{r_0}{2\pi + r_0} \right) \right)
\cdot \left( \pi - \frac{r_0}{2 \pi} - \frac{1}{\sqrt{2\pi |t|}} \log \left( 1 - e^{-r_0 \sqrt{\pi / |t|}} \right) + \frac{2}{\sqrt{\pi |t|}} \left( e^{2\pi \sqrt{\pi |t|}} - 1 \right) \right).
\]

Furthermore, we also have

\[
E_3 \leq \left( \frac{r_0}{2 \pi} \right)^{\sigma_0 - 1} \frac{2 - r_0}{\pi} \sqrt{2|t| / \pi} + 1 \exp \left( -|t| \left( - \frac{r_0}{2 \pi} + \arctan \frac{r_0}{2\pi - r_0} \right) \right),
\]

\[
E_4 \leq \left( 1 - \sqrt{2|t| / \pi} \right)^{\sigma_0 - 1} \frac{2}{\sqrt{1 - e^{-r_0 \sqrt{\pi / |t|}}}} \pi |t| \exp \left( -|t| \left( \frac{\pi}{2} - \frac{r_0}{2 \pi} + \arctan \frac{2\pi}{r_0} \right) \right).
\]

From [AdR11] Theorems 4.1 and 4.2 we can deduce

\[
|E_L(s)| \leq \frac{1}{2} + \frac{9^\sigma}{2\sqrt{t}} + \left( \frac{11}{10} \right)^{2\pi / (4\tau)} 2\pi 2^{2\pi}
\]

for \( \sigma \in (0,1] \) by taking the first two terms in \( E_L(s) \). Together with Proposition 1 this implies

\[
R_1 \left( s; \sqrt{|t| / 2\pi} \sqrt{|t| / 2\pi} \right) \leq \left( \frac{|t|}{2\pi} \right)^{-\frac{3}{2}} \left( |E_L(1)| + |E_L(1 - \sigma_0)| \left( 1 + \frac{0.3746}{|t|} \right) \right).
\]

Taking \( r_0 = 0.52777 \) and \( \sigma_0 = 1/2 \) in the above inequalities, we easily obtain bounds from Theorem 4. After applying Proposition 1 to Theorem 5 we obtain (13) with

\[
|E| \leq \frac{0.3746}{\sigma \sqrt{2\pi |t|}} + E, \quad |F| \leq \frac{0.3746}{\sigma |t|} + F
\]

since \( \sum_{n \leq x} n^{\sigma - 1} \leq x^\sigma / \sigma \) is valid for \( \sigma \in (0,1] \). This implies inequalities from Corollary 1.

### 2.5. Application to the approximate functional equation for \( \zeta^2(s) \)

Hardy and Littlewood proved in [HL29] that

\[
\zeta^2(s) = \sum_{n \leq x} \frac{d(n)}{n^s} + \chi^2(s) \sum_{n \leq y} \frac{d(n)}{n^{1-s}} + R_2(s; x, y)
\]

with \( 4\pi^2 xy = t^2 \), where \( R_2(s; x, y) \ll x^{1/2-\sigma} (x + y)/|t|)^{1/4} \log |t| \). Here \( d(n) \) is the divisor function, and it is well-known that \( \sum_{n \leq X} d(n) = X \log X + (2\gamma - 1)X + \Delta(X) \) where \( \Delta(X) \ll X \). Later Titchmarsh provided a different proof of (35) with \( R_2(s; x, y) \ll x^{1/2-\sigma} \log |t| \), see also [Ivi03] pp. 104–121. Both proofs are quite elaborate. In the symmetric case \( x = y = |t|/(2\pi) \), Motohashi [Mot83] found a simple connection between (35) and (8) by means of Dirichlet’s hyperbola method.
He obtained

\[ R_2(s) := R_2 \left( s; \frac{|t|}{2\pi}, \frac{|t|}{2\pi} \right) = 2\chi(s) \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} \frac{1}{n^s} + 2 \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} R_1 \left( s; \frac{|t|}{2\pi n}, n \right) + 2\chi^2(s) \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} R_1 \left( 1 - s; \frac{|t|}{2\pi n}, n \right) + R_1^2 \left( s; \sqrt{\frac{|t|}{2\pi}}, \sqrt{\frac{|t|}{2\pi}} \right), \]

where \( R_1(s; x, y) \) is the error term in the approximate functional equation. Theorem 3 enables us to obtain an explicit version of \( R_2(s) \) and thus of (35) in the symmetric case.

**Corollary 4.** Let \( s = \sigma + it \) where \( \sigma \in [1/2, 1] \) and \( |t| \geq 10^3 \). Then we have

\[ \zeta^2(s) = \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} \frac{d(n)}{n^s} + \text{sgn}(t) i \left( \frac{|t|}{2\pi} \right)^{1-2\sigma} \left( \frac{|t|}{2\pi e} \right)^{-2it} \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} \frac{d(n)}{n^{1-s}} + \tilde{R}_2(s), \quad (36) \]

where

\[ \left| \tilde{R}_2(s) \right| \leq 34.765 \left( \frac{|t|}{2\pi} \right)^{-\sigma} \log \frac{|t|}{2\pi}, \]

and also

\[ \left| \tilde{R}_2 \left( \frac{1}{2} + it \right) \right| \leq 28.621 \log \frac{|t|}{2\pi}. \]

**Proof.** By symmetry, we can assume that \( t \geq 10^3 \). By Proposition 1 (12) and (27) we have \( \left| \tilde{R}_2(s) \right| \leq |R_2(s)| + r_2(s) \), where

\[ r_2(s) := \left( \frac{t}{2\pi} \right)^{1-2\sigma} 0.75 \sum_{n \leq \sqrt{\frac{2\pi}{|t|}}} \frac{d(n)}{n^s}. \]

Partial summation assures that

\[ \sum_{n \leq X} \frac{d(n)}{n^{1-s}} = \frac{1}{\sigma} X^\sigma \log X + \frac{2\gamma \sigma - 1}{\sigma^2} X^\sigma + \frac{(2\gamma - 1)\sigma^2 - 2\gamma \sigma + 1}{\sigma^2} + (1 - \sigma) \int_1^X \frac{\Delta(u)}{u^{2-\sigma}} du + X^{\sigma-1} \Delta(X) \]

for \( X \geq 1 \). Now we consider two cases: \( \sigma = 1/2 \) and \( \sigma \in (1/2, 1] \). Using the elementary bound \( \Delta(X) \leq 3\sqrt{X} \), we obtain

\[ r_2 \left( \frac{1}{2} + it \right) \leq 0.106, \quad r_2(s) \leq 0.265 \left( \frac{t}{2\pi} \right)^{1-2\sigma} \log \frac{t}{2\pi}. \]

There exist much better estimates for \( \Delta(X) \), see [BBR12] Theorem 1.1, but this bound is good enough for our purposes.

Now we need to bound \( R_2(s) \). Because \( n \leq \sqrt{t/(2\pi)} \), this implies \( t/(2\pi n) \geq n \). By Theorem 4 and Theorem 6 we thus have

\[ \left| R_1 \left( s; \frac{t}{2\pi n}, n \right) \right| \leq 15.726 \left( \frac{t}{2\pi} \right)^{1-\sigma} n^{\sigma-1}, \]

\[ \left| R_1 \left( 1 - s; \frac{t}{2\pi n}, n \right) \right| \leq 10.983 \left( \frac{t}{2\pi} \right)^{\sigma-\frac{1}{2}} n^{-\sigma}, \]

\[ \left| R_1 \left( \frac{1}{2} + it; \frac{t}{2\pi n}, n \right) \right| \leq \frac{10.983}{\sqrt{n}}. \]
Using also $|\chi(s)| \leq 1.00038 \left(\frac{t}{(2\pi)}\right)^{1/2-\sigma}$, $|\chi(1/2 + it)| = 1$, and the inequality
$$\sum_{n \leq X} \frac{1}{n} \leq \log X + \gamma + \frac{1}{2X},$$
see [DHZA19] Lemma 2.8, we obtain
$$|R_2(s)| \leq 34.5 \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} \log \frac{t}{2\pi}.$$ 
These bounds give the desired estimates from Corollary 4.

3. Explicit second power moment of the Riemann zeta-function

The main analytic tool used by Selberg in his proof of the zero density estimate is a weighted second power moment of $\zeta$, see [Sel46] Lemma 6. The main idea is to use the approximate functional equation in the form (11), together with (6) for real values. In the forthcoming subsections we will provide a proof of the following explicit version of Selberg’s lemma with $H = T$.

**Theorem 7.** Let $\sigma \in (1/2, \sigma_0]$, $\sigma_0 \in (1/2, 1)$, and $T \geq T_0 \geq 2\pi$. Furthermore, let $1 \leq \mu_1 \leq \mu_2 \leq T/(2\pi)$ be a positive coprime integers, and denote $z := (\sigma, T, \mu_1, \mu_2)$. Define
$$S(z) := \int_{T}^{2T} |\zeta(s + it)|^2 \left(\frac{\mu_1}{\mu_2}\right)^{it} dt$$
and
$$\mathcal{S} (z) := \zeta(2\sigma) (\frac{\mu_2}{\mu_1})^{\sigma} T^{1-\sigma} \frac{(2\pi)^{2\sigma-1} (4^{1-\sigma} - 1) \zeta(2-2\sigma) T^{2(1-\sigma)}}{2(1-\sigma)(\mu_1 \mu_2)^{1-\sigma}.} \quad (37)$$
Then
$$|S - \mathcal{S}| \leq \mathcal{S}_1(z) \left(\frac{\mu_2}{\mu_1}\right)^{\sigma} T^{1-\sigma} \sqrt{\log \frac{T \mu_2}{\mu_1},} + \mathcal{S}_2(z) \mu_1 \mu_2 T^{1-\sigma} \log \frac{T \mu_1 \mu_2}{\pi},$$
where
$$\mathcal{S}_1(z) := \sqrt{\log \frac{T_0}{\pi}} \left(\frac{B_5(z) + B_6(z)}{B_5(z) + B_6(z)} + B_7(z) + B_8(z), \right)$$
$$\mathcal{S}_2(z) := B_1(\sigma_0, T_0) + \sqrt{\frac{\pi}{2} B_3(z) + B_4(z) + \frac{B_9(\sigma_0)}{\log \frac{T}{\sigma}}},$$
and positive functions $B_1, B_3, \ldots, B_9$, defined by equations (51), (51), (51), (51), (51), (51), (51), (51), and (51), respectively, are bounded for fixed $\sigma_0$ and $T_0$. Additionally, they are continuous for $\sigma \in [1/2, \sigma_0]$ and $\sigma_0 \in [1/2, 1]$.

Although $\mathcal{S}(z)$ is not defined for $\sigma = 1/2$, the limit $\sigma \to 1/2$ exists. This enables us to obtain an explicit upper bound for second power moment of $\zeta$ on the critical line, see Corollary 5. It turns out that we get an explicit version of Littlewood’s bound
$$\int_{0}^{T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = T \log T - (1 + \log 2\pi - 2\gamma) T + \mathcal{E}(T)$$
with $\mathcal{E}(T) = O(T^{4+\varepsilon})$, announced incorrectly and without proof in 1922. This estimate was the first improvement of the fact that the integral is asymptotically
equal to $T \log T$, a result due to Hardy and Littlewood. Their second proof uses the approximate functional equation, see [11868 Theorem 7.3]. New turn in the mean square theory was Atkinson’s formula for $\delta(T)$ and its various generalisations, e.g., Matsumoto–Meurman formulas. They enabled to prove that $\delta(T) = O(T^{3/4 + \epsilon})$ and it is plausible to believe that $\delta(T) = O(T^{4/5 + \epsilon})$ is true since $\delta(T) = \Omega(T^{1/2})$, see [11803 Chapter 15] for proofs and techniques, and [Mat00] for an overview of the mean square theory.

**Corollary 5.** Let $T \geq 2\pi$. Then

$$\delta(T) \leq 13.803 T^{3/4} \sqrt{\frac{\log T}{2\pi}} + 83.964 \sqrt{T} \log \frac{T}{2\pi} + 2 \cdot 10^3 \log T + 3691.24. \quad (38)$$

**Proof.** Let $T_0 = 10^3$ and $T \geq 2T_0$. Define $S(\sigma, T) := S(\sigma, T, 1, 1)$, $\mathcal{F}(\sigma, T) := \mathcal{F}(\sigma, T, 1, 1)$, $\mathcal{F}_1(\sigma, T) := \mathcal{F}_1(\sigma, T, 1, 1)$ and $\mathcal{F}_2(\sigma, T) := \mathcal{F}_2(\sigma, T, 1, 1)$. Take an arbitrary $\sigma_0 \in (1/2, 1)$ and let $\sigma \in (1/2, \sigma_0]$. By Theorem 7 there exist a continuous functions $\mathcal{F}_1(\sigma_0, T_0)$ and $\mathcal{F}_2(\sigma_0, T_0)$ for $\sigma_0 \in [1/2, 1)$ such that $\mathcal{F}_1(\sigma, T) \leq \mathcal{F}_1(\sigma_0, T_0)$ and $\mathcal{F}_2(\sigma, T) \leq \mathcal{F}_2(\sigma_0, T_0)$. Also $\mathcal{F}_1(1/2, T_0) \leq 9.4104$ and $\mathcal{F}_2(1/2, T) \leq 34.779$. We thus have

$$\int_{2T_0}^{T} |\zeta(\sigma + it)|^2 \, dt \leq \sum_{n=1}^{n_0} S\left(\sigma, \frac{T}{2^n}\right) \leq \mathcal{F}(\sigma) + \mathcal{F}_1(\sigma_0, T_0) T^{1-\frac{\sigma}{2}} \log \frac{T}{2\pi} + \mathcal{F}_2(\sigma_0, T_0) T^{1-\sigma} \log \frac{T}{2\pi},$$

where $n_0 := [\log_2 (T/T_0)]$ and

$$\mathcal{F}(\sigma) := \sum_{n=1}^{n_0} \mathcal{F}\left(\sigma, \frac{T}{2^n}\right).$$

A simple calculation shows that

$$\mathcal{F}(\sigma) = \zeta(2\sigma) \left(1 - 2^{-n_0}\right) T + f(\sigma) \zeta(2 - 2\sigma),$$

where

$$f(\sigma) := \frac{(2\pi)^{2\sigma-1} \left(1 - 4^{-1(1-\sigma)n_0}\right) T^{2(1-\sigma)}}{2(1-\sigma)}.$$  

Remember that the Laurent series of $\zeta(s)$ around $s = 1$ is $\zeta(s) = (s-1)^{-1} + \gamma + g(s)$ for some holomorphic function $g(s)$ with $g(1) = 0$, and $\gamma$ is the Euler–Mascheroni constant. Then

$$\mathcal{F}(\sigma) = \frac{(1 - 2^{-n_0}) T - f(\sigma)}{2\sigma - 1} + \left(\gamma + g(2\sigma)\right) \left((1 - 2^{-n_0}) T + f(\sigma)\right).$$

Since $\lim_{\sigma \to 1/2} f(\sigma) = (1 - 2^{-n_0}) T$, we have

$$\lim_{\sigma \to \frac{1}{2}} \mathcal{F}(\sigma) = -\frac{1}{2} f\left(\frac{1}{2}\right) + 2\gamma (1 - 2^{-n_0}) T = (1 - 2^{-n_0}) (\log T + 2\gamma) T$$

$$- (1 + \log 2\pi) T + 2^{n_0} (1 + n_0 \log 2 + \log 2\pi) T$$

$$\leq T \log T - (1 + \log 2\pi - 2\gamma) T + 2T_0 \log \frac{2\pi e^{1/2}}{T_0}.$$  

Take $\sigma_0 \to 1/2$. For $T \geq 2T_0$, the main inequality now easily follows from this since we can numerically verify that $\int_0^{2T_0} |\zeta(1/2 + it)|^2 \, dt \leq 11831$, and this also implies that it is true for $T \geq 35$. Finally $\int_0^{35} |\zeta(1/2 + it)|^2 \, dt \leq 67$, which concludes the proof. ■
Evaluation of the above integrals was performed in Mathematica, using the built-in function \( \text{RiemannSiegelZ}[t] \) and integration method \( \text{NIntegrate} \). In principle, it is possible to improve the constants in the first two terms in (38) because \( \lim_{T \to \infty} I_1(1/2, T, 1, 1) \approx 8.953 \) and \( \lim_{T \to \infty} I_2(1/2, T, 1, 1) \approx 22.6 \), but unfortunately \( T_0 \), and consequently the last two constants in (38), grow too rapidly to be numerically useful. Note that our estimate is for \( T \geq 1545 \) better than the recent explicit bound in [DHZA19, Theorem 4.3].

3.1. Setting of the proof. Assume the conditions of Theorem 7. Define

\[
x(t) := \sqrt{\frac{t \mu_1}{2 \pi \mu_2}} \quad \text{and} \quad y(t) := \sqrt{\frac{t \mu_2}{2 \pi \mu_1}}.
\]

Then \( 1 \leq x(t) \leq y(t) \) and \( 2 \pi x(t) y(t) = t \). Using Corollary 11 we obtain

\[
\zeta(\sigma + it) = \sum_{n \leq x(t)} n^{-\sigma - it} + \tilde{\chi}(\sigma + it) \sum_{n \leq y(t)} n^{\sigma - 1 + it} + \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t))
\]

with \( |\tilde{E}(s; x(t), y(t))| \leq \tilde{E} \). Changing roles of \( x(t) \) and \( y(t) \), Corollary 11 also implies

\[
\zeta(\sigma - it) = \sum_{m \leq y(t)} m^{-\sigma + it} + \tilde{\chi}(\sigma - it) \sum_{m \leq x(t)} m^{\sigma - 1 - it} + \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t))
\]

with \( |\tilde{E}(s; x(t), y(t))| \leq \tilde{E} \). If we multiply these two equations, we get an expression for \( |\zeta(s)|^2 \) consisting of nine terms and arranged into five groups:

\[
A_1 := \sum_{n \leq x(t)} \sum_{m \leq y(t)} (nm)^{-\sigma} \left( \frac{m}{n} \right)^{it},
\]

\[
A_2 := \left( \frac{t}{2\pi} \right)^{1-\sigma} \sum_{n \leq y(t)} \sum_{m \leq x(t)} (nm)^{\sigma-1} \left( \frac{n}{m} \right)^{it};
\]

\[
A_3 := \tilde{\chi}(\sigma + it) \sum_{n,m \leq y(t)} n^{\sigma-1} m^{-\sigma} (nm)^{it},
\]

\[
A_4 := \tilde{\chi}(\sigma - it) \sum_{n,m \leq x(t)} n^{\sigma-1} m^{-\sigma} (nm)^{-it};
\]

\[
A_5 := \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t)) \sum_{n \leq x(t)} n^{-\sigma - it},
\]

\[
A_6 := \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t)) \sum_{n \leq y(t)} n^{-\sigma + it};
\]

\[
A_7 := \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t)) \tilde{\chi}(\sigma + it) \sum_{n \leq y(t)} n^{\sigma-1 + it},
\]

\[
A_8 := \left( \frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} \left( \frac{\mu_1}{\mu_2} \right)^{\frac{\sigma-1}{2}} \tilde{E}(s; x(t), y(t)) \tilde{\chi}(\sigma - it) \sum_{n \leq x(t)} n^{\sigma-1 - it};
\]
$A_9 := \left(\frac{1}{2\pi}\right)^{-\sigma} \sqrt{\frac{\mu_2}{\mu_1}} E(s; x(t), y(t)) \widetilde{F}(s; x(t), y(t))$.

Therefore,

$$S(\sigma, T; \mu_1, \mu_2) = \sum_{j=1}^9 \int_{T}^{2T} A_j \cdot \left(\frac{\mu_1}{\mu_2}\right)^{it} dt.$$ 

Denote by $B_i$ the $i$th summand in the above equation. In the following subsections we provide an explicit bounds on each $B_i$. Before doing this we firstly collect some lemmas which are used in the forthcoming subsections.

### 3.2. Some lemmas

The first lemma is a rule for changing integration and summation when the range in the sum depends on the integration variable.

**Lemma 3.** Let $f(n, t)$ be an integrable function in variable $t \in [T_1, T_2]$ where $T_1 \geq 1$, and let $g(t)$ be a strictly increasing differentiable function with $g(T_1) \geq 1$. Then

$$\int_{T_1}^{T_2} \sum_{n \leq g(t)} f(n, t) dt = \sum_{n \leq g(T_2)} \int_{\max(T_1, g^{-1}(n))}^{T_2} f(n, t) dt.$$ 

**Proof.** We first prove the special case when $g(t) = t$. We can assume that $T_2 - T_1 \geq 2$ since otherwise the lemma is obviously true. Then

$$\int_{T_1}^{T_2} \sum_{n \leq t} f(n, t) dt = \left(\sum_{n \leq [T_1]} \int_{T_1}^{[T_1] + 1} + \sum_{j=1}^{[T_2] - 1} \sum_{n \leq j} \int_{j}^{j+1} + \sum_{n \leq [T_2]} \int_{T_2}ight) f(n, t) dt.$$ 

The second integral equals to

$$\sum_{n \leq [T_2] - 1} \int_{j}^{j+1} f(n, t) dt = \sum_{n \leq [T_2] - 1} \int_{\max([T_1] + 1, n)}^{[T_2]} f(n, t) dt.$$ 

This implies

$$\int_{T_1}^{T_2} \sum_{n \leq t} f(n, t) dt = \left(\sum_{n \leq [T_1]} \int_{T_1}^{[T_1] + 1} + \sum_{n \leq [T_2] - 1} \int_{\max([T_1] + 1, n)}^{[T_2]} \right) f(n, t) dt$$

and consequently the lemma in this special case.

Write $g(t) = u$. Because $g(t)$ is strictly increasing differentiable function, there exists its inverse $t = g^{-1}(u)$ and $dt = (g^{-1}(u))' du$. We have

$$\int_{T_1}^{T_2} \sum_{n \leq g(t)} f(n, t) dt = \int_{g(T_1)}^{g(T_2)} \sum_{n \leq u} f(n, g^{-1}(u)) (g^{-1}(u))' du$$

$$= \sum_{n \leq g(T_2)} \int_{\max(g(T_1), n)}^{g(T_2)} f(n, g^{-1}(u)) (g^{-1}(u))' du$$

$$= \sum_{n \leq g(T_2)} \int_{\max(g(T_1), n)}^{T_2} f(n, t) dt,$$

where the second equality follows from the first part of the proof. Clearly, the last integral equals to the second integral from the lemma.

From Lemma 3 it easily follows that

$$\int_{T_1}^{T_2} \sum_{n \leq g_1(t)} \sum_{m \leq g_2(t)} f(n, m, t) dt = \sum_{n \leq g_1(T_2)} \sum_{m \leq g_2(T_2)} \int_{\mathcal{M}(n, m)}^{T_2} f(n, m, t) dt.$$  \hspace{1cm} (39)
Then bound the first part, and Lemma 5 for the second part. What we obtain is Lemma 4.

Let \( T \) summation variables \( n \) and \( m \) of Preissmann’s inequality integration of Dirichlet polynomials. The first one is a slightly modified corollary of Lemma 3.

\[
\left| \sum_{n,m \leq X} \frac{u_n u_m}{x_n - x_m} \right| \leq \pi m_0 \sum_{n \leq X} \frac{|u_n|^2}{\min_{n \neq m}|x_n - x_m|},
\]  

(40)

where \( X \geq 2 \), \( \{x_n\}_{n \leq X} \) are distinct real numbers, \( \{u_n\}_{n \leq X} \) are complex numbers and \( m_0 := \sqrt{1 + \frac{2 + \sqrt{3}}{5}} \), see [Pre84].

**Lemma 4.** Let \( X \geq 2 \), \( \{a_n\}_{n \leq X} \) be a sequence of complex numbers and \( Y \in \mathbb{R} \). Then

\[
\left| \sum_{n,m \leq X} \frac{a_n u_m}{\log (n/m)} \left( \frac{n}{m} \right)^Y \right| \leq \pi m_0 \sum_{n \leq X} |a_n|^2 \left( \frac{1}{2} + n \right),
\]

where \( m_0 := \sqrt{1 + \frac{2 + \sqrt{3}}{5}} \).

**Proof.** Use (40) for \( x_j = \log j \) and \( u_j = a_j j^Y \), and observe that \( |\log (n/m)| \geq (n + 1/2)^{-1} \) for all distinct integers \( n \) and \( m \).

**Lemma 5.** Let \( X \geq 2 \) and \( |a| < 1 \). Then

\[
\sum_{n,m \leq X} \frac{(nm)^a}{|\log (n/m)|} \leq \left( \sum_{n \leq X} n^a \right)^2 - \sum_{n \leq X} n^{2a} + 2 \sum_{n \leq X} n^{1+2a} \sum_{n \leq X} \frac{1}{n}.
\]

**Proof.** Firstly, observe that \( 1/|\log \lambda| \leq 1 + \lambda^{1+a}/(|\lambda - 1|) \) is true for \( \lambda > 1 \). Then

\[
\sum_{n,m \leq X} \frac{(nm)^a}{|\log (n/m)|} \leq \sum_{n,m \leq X} (nm)^a + 2 \sum_{n \leq m \leq X} \frac{n^{1+2a}}{m - n}
\]

from which the main inequality follows.

The idea is to combine Lemmas 4 and 5 to bound the following double sum

\[
D(a, T_1, T_2; X) := \sum_{n,m \leq X} \frac{(nm)^a}{|\log (n/m)|} \left( \left( \frac{n}{m} \right)^{iT_2} - \left( \frac{n}{m} \right)^{iT_1} \right).
\]

In the most subsequent applications, \( T_2 \) is independent while \( T_1 \) depends on the summation variables \( n \) and \( m \). Thus we will use Lemma 4 for \( a_n = n^a \), \( Y = T_2 \) to bound the first part, and Lemma 5 for the second part. What we obtain is

\[
|D(a, T_1, T_2; X)| \leq \left( \pi m_0 + 2 \sum_{n \leq X} \frac{1}{n} \right) \sum_{n \leq X} n^{1+2a}
\]

\[
+ \left( \sum_{n \leq X} n^a \right)^2 + \left( \frac{\pi m_0}{2} - 1 \right) \sum_{n \leq X} n^{2a}.
\]

(41)
Observe that $\pi m_\theta/2 - 1 > 0$. We will need (11) only for $a \in \{-\sigma, \sigma - 1\}$. Particular sums are estimated by

\[
\sum_{n \leq X} n^{-2\sigma} \leq \log X + \gamma + \frac{1}{2X}, \tag{42}
\]

\[
\sum_{n \leq X} n^{2(\sigma - 1)} \leq X^{2\sigma - 1} \left( \log X + \gamma + \frac{1}{2X} \right), \tag{43}
\]

\[
\sum_{n \leq X} n^{1-2\sigma} \leq \frac{X^{2(1-\sigma)}}{2(1 - \sigma)}, \tag{44}
\]

\[
\sum_{n \leq X} n^{1-\sigma} \leq \frac{X^{1-\sigma}}{1 - \sigma}, \tag{45}
\]

\[
\sum_{n \leq X} n^{2\sigma - 1} \leq \frac{X^{2\sigma}}{2\sigma} \left( 1 + \frac{2\sigma}{X} - \frac{1}{X^{2\sigma}} \right). \tag{46}
\]

These bounds are good also for $\sigma = 1/2$. Inequalities (11), (16) and (19) follow simply from integration.

The next two lemmas are explicit versions of Selberg’s Lemmas 2 and 3, with the same proof in principle. The first one is needed to estimate $B_2$ while the second one is useful to obtain bounds for $B_3$ and $B_4$.

**Lemma 6.** Let $\sigma \geq 1/2$, $\lambda \neq 0$ and $T_1 \leq T_2$. Then

\[
\left| \int_{T_1}^{T_2} t^{1-2\sigma} e^{\lambda t} dt \right| \leq \frac{2}{|\lambda|} T_1^{1-2\sigma}. \tag{47}
\]

**Proof.** The stated inequality is clearly true in case of $\sigma = 1/2$. If we assume that $\sigma > 1/2$, it is not hard to see that integration by parts implies the stated bound. ■

**Lemma 7.** Let $\sigma \geq 1/2$, $\xi \in (0, T_1]$ and $2 \leq T_1 < T_1 + \sqrt{T_1} \leq T_2$. Then

\[
\left| \int_{T_1}^{T_2} t^{\frac{1}{2}-\sigma} \left( \frac{t}{\xi} \right)^{\pm i t} \frac{dt}{\log \xi} \right| \leq \frac{8 T_1^{\frac{1}{2}-\sigma}}{\log T_1}. \tag{48}
\]

**Proof.** Denote by $I$ the above integral and assume $\xi \neq T_1$. Separating real and imaginary part of the exponential function, we obtain

\[
I = \int_{T_1}^{T_2} \frac{t^{\frac{1}{2}-\sigma}}{\log \xi} \cos \left( t \log \frac{t}{c_\xi} \right) \log \frac{t}{\xi} dt \pm i \int_{T_1}^{T_2} \frac{t^{\frac{1}{2}-\sigma}}{\log \xi} \sin \left( t \log \frac{t}{c_\xi} \right) \log \frac{t}{\xi} dt
\]

\[
= \frac{T_1^{\frac{1}{2}-\sigma}}{\log \xi} \left( \int_{u(T_1)}^{u(T_1 + \eta_1(T_2 - T_1))} \cos u du \right) \pm i \int_{u(T_1)}^{u(T_1 + \eta_2(T_2 - T_1))} \sin u du
\]

for some $\eta_1, \eta_2 \in [0, 1]$. The second equality follows from the second mean value theorem and after making substitution $u(t) := t \log (t/\xi)$ with $u'(t) = \log (t/\xi) > 0$. This implies that

\[
|I| \leq \frac{4 T_1^{\frac{1}{2}-\sigma}}{\log \frac{T_1}{\xi}}. \tag{49}
\]

Define

\[
f(\xi) := \frac{\log \frac{T_1 + \sqrt{T_1}}{\xi}}{\log \frac{T_1}{\xi}}, \quad g(\xi) := \frac{\sqrt{T_1}}{4} \log \frac{T_1 + \sqrt{T_1}}{\xi}.
\]
The first function is strictly increasing while the second one is strictly decreasing. For $\xi \leq T_1 - \sqrt{T_1}$ we thus have

$$1 \leq \frac{f(T_1 - \sqrt{T_1} - \xi)}{\log \frac{T_1 - \sqrt{T_1} + \xi}{\xi}} \leq 2 \frac{2}{\log \frac{T_1 + \sqrt{T_1}}{\xi}}$$

since $\lim_{T_1 \to \infty} f(T_1 - \sqrt{T_1}) = 2$. In this case we obtain the desired inequality. Now, let $T_1 - \sqrt{T_1} \leq \xi \leq T_1$. By the already known inequality (37) we have

$$|I| \leq \int_{T_1}^{T_1 + \sqrt{T_1}} t^{1-\sigma} dt + \int_{T_1 + \sqrt{T_1}}^{T_1 + \sqrt{T_1}} t^{1-\sigma} \left(\frac{t}{\epsilon \xi}\right)^{\pm i t} dt$$

$$\leq T_1^{1-\sigma} + \frac{4T_1^{1-\sigma}}{\log \frac{T_1 + \sqrt{T_1}}{\xi}} \leq \frac{4T_1^{1-\sigma}}{\log \frac{T_1 + \sqrt{T_1}}{\xi}} \left(1 + g(T_1 - \sqrt{T_1})\right).$$

This also proves the main bound since $g(T_1 - \sqrt{T_1}) < 1$. The proof of Lemma 7 is thus complete.

We are now in position to obtain desired bounds for integrals $B_1$. We will do this in pairs of indices, namely for $\{5,6\}, \{7,8\}, \{1,2\}$ and $\{3,4\}$, but starting with the most simple one $B_3$. Derivation of bounds for one part of a pair give bounds for the other part when changing the roles of parameters $\mu_1$ and $\mu_2$. Note that the order of appearance of parameters $\mu_1$ and $\mu_2$ is crucial when obtaining bounds which depend only on $\sigma_0$ and $T_0$. Here we use inequalities $\mu_1/\mu_2 \leq 1$ and $\pi \mu_2 / (T \mu_1) \leq 1/2$.

3.3. Bound on $B_0$. A straightforward calculation shows that

$$|B_0| \leq T_1^{1-\sigma} \sqrt{\frac{\mu_2}{\mu_1}} B_0(\sigma_0),$$

where

$$B_0(\sigma_0) := \pi^{\sigma_0} \frac{2-2\sigma_0}{1-\sigma_0} E \cdot F. \tag{48}$$

Here we used the fact that $(2 - 2x) / (1 - x)$ is a strictly increasing function on $[1/2,1)$.

3.4. Bounds on $B_5$ and $B_6$. Firstly, we will consider $B_5$. Define

$$S_x(t)(s) := \sum_{n \leq \pi(t)} \frac{1}{n^{s+it}}.$$

Hölder’s inequality implies

$$\int_{T}^{2T} \left(\frac{t}{2\pi}\right)^{-\frac{2}{p}} |S_x(t)(s)| \, dt \leq T^{\frac{1-\sigma}{p}} (2\pi)^{\frac{1}{p}} \sqrt{\frac{2^{1-\sigma} - 1}{1 - \sigma}} I_x(t)(s),$$

where

$$I_x(t)(s) := \int_{T}^{2T} \left|S_x(t)(s)\right|^2 \, dt.$$

The same inequality is also true for $y(t)$ and $\sigma = \sigma - it$. The problem is thus reduced to bounding the second integral. Separation of the diagonal and off-diagonal terms which appear after multiplication gives, together with equality (39),

$$I_x(t)(s) = \int_{T}^{2T} S_x(t)(2\sigma) \, dt + i D (-\sigma, -M_1, -2T; x(2T)) \tag{49}$$
where $M_1 := \max\{T, x^{-1}(n), x^{-1}(m)\}$. Using (12), we can deduce by straightforward integration that

$$\int_T^{2T} S_x(t)(2\sigma)dt \leq B_{5,1}(T, \mu_1, \mu_2) T \log T,$$

where

$$B_{5,1}(T, \mu_1, \mu_2) := \frac{1}{2} + \frac{\gamma - 1}{2} + \log \left(\frac{\pi \mu_1 \mu_2}{\pi \mu_2} + \left(\sqrt{2} - 1\right) \sqrt{\frac{\pi \mu_1 \mu_2}{\pi \mu_2}} \right).$$

After changing roles of $\mu_1$ and $\mu_2$, the resulting bound is also true for $y(t)$ in place of $x(t)$ since (49) is also true in this case with $M_2 := \max\{T, y^{-1}(n), y^{-1}(m)\}$ in place of $M_1$. We shall see that $B_{5,1}$ contributes the most in (49).

Using (41), we get

$$|iD (-\sigma, -M_1, -2T; x(2T))| \leq \left(\frac{\mu_1}{\pi \mu_2}\right)^{1-\sigma} B_{5,2}(z) T \log T,$$

where

$$B_{5,2}(z) := \frac{T^{-\sigma}}{2(1-\sigma)} + \frac{4 + 2(1-\sigma)(2\gamma + \log \frac{\mu_1}{\mu_2} + \sqrt{\frac{\pi \mu_1}{\pi \mu_2}} + \pi m_0)}{4(1-\sigma)^2 T^\sigma \log T} + \left(\frac{\pi \mu_2}{\mu_1 T}\right)^{1-\sigma} \pi m_0 - 2 \pi m_0 \left(1 + \frac{2\gamma + \log \frac{\mu_1}{\mu_2} + \sqrt{\frac{\pi \mu_1}{\pi \mu_2}}}{\log T}\right).$$

This bound is also true for $y(T)$ after changing roles of $\mu_1$ and $\mu_2$. Define

$$\overline{B}_5(z) := B_{5,1}(T, \mu_1, \mu_2) + \left(\frac{\mu_1}{\pi \mu_2}\right)^{1-\sigma} B_{5,2}(z)$$

$$\leq \frac{1}{2} + \frac{0.266}{\log T_0} + \frac{\pi^{\sigma_0-1}}{2(1-\sigma_0) \sqrt{T_0}} \left(1 + \frac{4.43}{(1-\sigma_0) \log T_0} + 0.534 \cdot 2^{\sigma_0-1} \left(1 + \frac{0.717}{\log T_0}\right)\right),$$

$$\overline{B}_6(z) := \left(\frac{\pi \mu_1}{\mu_2}\right)^{1-\sigma} B_{5,1}(T, \mu_2, \mu_1) + B_{5,2}(\sigma, T, \mu_2, \mu_1)$$

$$\leq \sqrt{\pi} + \frac{1}{(1-\sigma_0) \sqrt{T_0}} \left(1 + \frac{1.577 + \frac{4}{\pi \mu_1 \mu_2} \sqrt{\frac{\pi \mu_1 \mu_2}{\pi \mu_2}} + 0.534}{(1-\sigma_0) \log T_0} \right).$$

In derivation of the second inequality we used $B_{5,1}(T, \mu_2, \mu_1) < 1$. Both functions and their bounds are continuous for $\sigma \in [1/2, \sigma_0]$ and $\sigma_0 \in [1/2, 1]$. Observe also that $B_{5,1}(T, 1, 1) < 1/2$ for $T \geq 50$. This gives

$$\overline{B}_5\left(\frac{1}{2} T, 1, 1\right) \leq \frac{1}{2} + 2.3 \sqrt{\frac{1}{\pi T_0}}, \quad \overline{B}_6\left(\frac{1}{2} T, 1, 1\right) \leq \sqrt{\pi} + 2.3 \sqrt{\frac{1}{T_0}},$$

for $T_0 \geq 50$. In case $\mu_1 = \mu_2 = 1$ and $\sigma = 1/2$ we will use these bounds instead of (50) and (51). We have

$$|B_5| \leq B_6(y) \left(\frac{\mu_2}{\mu_1}\right)^{1-\sigma} T^{1-\frac{1}{2}} \sqrt{\log T}, \quad |B_6| \leq B_6(y) \left(\frac{\mu_2}{\mu_1}\right)^{1-\sigma} T^{1-\frac{1}{2}} \sqrt{\log T},$$

where

$$B_6(y) := \overline{B}_6\left(\frac{1}{2} T, 1, 1\right) \leq \sqrt{\pi} + 2.3 \sqrt{\frac{1}{T_0}}.$$
where

\[ B_5(z) := \mathcal{E}_{\pi \sigma_0} \sqrt{\frac{2 - 2\sigma_0}{1 - \sigma_0}} B_5(z), \quad (52) \]

\[ B_6(z) := \mathcal{E}_{\pi \sigma_0} \sqrt{\frac{2 - 2\sigma_0}{1 - \sigma_0}} B_6(z). \quad (53) \]

In the general case we will use (50) and (51) to bound \( B_5(z) \) and \( B_6(z) \). This implies that both functions are bounded for fixed \( \sigma_0 \) and \( T_0 \).

3.5. **Bounds on** \( B_7 \) **and** \( B_8 \). The strategy here is the same as in Section 3.4.

Hölder’s inequality implies

\[
\int_T^{2T} (\frac{1}{2\pi})^{\frac{1-3\sigma}{2}} |S_{y(t)}(1-s)| \, dt \leq T^{\frac{\lambda_{\sigma}}{\pi}} (2\pi)^{\frac{3\sigma}{4}} \sqrt{\frac{2^{\sigma} - 3\sigma - 1}{2 - 3\sigma}} I_{y(t)}(1-s),
\]

and we have

\[
I_{y(t)}(1-s) = \int_T^{2T} S_{y(t)}(2(1-s)) \, dt + iD(\sigma - 1, -M_2, -2T; y(2T)). \quad (54)
\]

Using inequality (43), we can estimate by straightforward integration that

\[
\int_T^{2T} S_{y(t)}(2(1-s)) \, dt \leq B_{T,1}(z) T^{\frac{1}{2} + \sigma} (\frac{\mu_2}{\pi \mu_1})^\sigma \log \frac{T \mu_2}{\pi \mu_1},
\]

where

\[
B_{T,1}(z) := (1 + 2\sigma)2^{\frac{1}{2} + \sigma} - \frac{2^{\frac{1}{2} - \sigma}}{1 + 2\sigma} + \frac{1}{\log \frac{T \mu_2}{\pi \mu_1}} \left( \frac{2^{\frac{1}{2} - \sigma} (2^{\frac{1}{2} + \sigma} - 2)}{(1 + 2\sigma)^2} \right. \\
+ \left. \frac{4 - 2^{\frac{1}{2} - \sigma}}{1 + 2\sigma} \gamma + 2^{\frac{1}{2} - \sigma} \log 2 \right) + \frac{1 - 2^{\sigma}}{\sigma} \sqrt{T \mu_1} \frac{\mu_1}{T \mu_2}.
\]

By inequality (41) we also have

\[
|iD(\sigma - 1, -M_2, -2T; y(2T))| \leq B_{T,2}(z) (\frac{\mu_2}{\pi \mu_1})^\sigma T^{\frac{1}{2} + \sigma} \log \frac{T \mu_2}{\pi \mu_1},
\]

where

\[
B_{T,2}(z) := \frac{1 + 2\sigma \sqrt{T \mu_2}}{2\sigma \sqrt{T}} \left( \frac{\mu_1}{\pi \mu_2} \right)^\sigma \log \frac{T \mu_2}{\pi \mu_1} \\
+ \frac{\pi m_0 - 2}{4 \sqrt{T}} \sqrt{T \mu_1} \frac{\mu_1}{\mu_2 T} \left( 1 + \frac{2 \gamma + \sqrt{T \mu_2}}{\log \frac{T \mu_2}{\pi \mu_1}} \right).
\]
After changing roles of \( \mu_1 \) and \( \mu_2 \), both bounds are also true for \( x(t) \) and \( M_1 \) in place of \( y(t) \) and \( T \), respectively. Define

\[
\bar{B}_7 (z) := \sqrt{\frac{\mu_1}{\mu_2}} B_{7,1} (z) + B_{7,2} (z)
\]

\[
\leq \sqrt{\pi} \left( 1 + 2 \sigma_0 \right) 2^{\frac{1}{2} + \sigma_0} - \frac{2^{\frac{1}{2} - \sigma_0}}{1 + 2 \sigma_0} \left( 4 - 2^{\frac{1}{2} - \sigma_0} \right) \gamma + \log 2 \cdot \left( \frac{1 + 2 \sigma_0}{(1 + 2 \sigma_0)^2} \right) + 2 \left( 1 - 2^{-\sigma_0} \right) \frac{\sqrt{2}}{\sqrt{\pi T_0}} \right) \right) \]

\[
+ \frac{1}{\log \frac{2}{\pi}} \left( 1 + \frac{9.287 + \sqrt{\pi T_0}}{\log \frac{2}{\pi}} \right) + 0.534 \sqrt{\pi} \left( 1 + \frac{2 \gamma + \sqrt{\pi T_0}}{\log \frac{2}{\pi}} \right),
\]

\[
B_8 (z) := B_{7,1} (\sigma, T, \mu_2, \mu_1) + \sqrt{\frac{\mu_1}{\pi \mu_2}} B_{7,2} (\sigma, T, \mu_2, \mu_1)
\]

\[
\leq \frac{1}{\log 2} \left( 4 - 2^{\frac{1}{2} - \sigma_0} \right) \gamma + \log 2 \cdot \left( \frac{1 + 2 \sigma_0}{(1 + 2 \sigma_0)^2} \right) + 27.7101 \frac{\sqrt{2}}{\sqrt{\pi T_0}}.
\]

Both functions and their bounds are continuous for \( \sigma \in [1/2, \sigma_0] \) and \( \sigma_0 \in [1/2, 1] \). In case \( \mu_1 = \mu_2 = 1 \) and \( \sigma = 1/2 \) we will use

\[
\bar{B}_7 \left( \frac{1}{2}, T, 1, 1 \right) \leq \sqrt{\frac{\mu_1}{\pi \mu_2}} B_{7,1} \left( \frac{1}{2}, T_0, 1, 1 \right) + B_{7,2} \left( \frac{1}{2}, T_0, 1, 1 \right)
\]

\[
\bar{B}_8 \left( \frac{1}{2}, T, 1, 1 \right) \leq B_{7,1} \left( \frac{1}{2}, T_0, 1, 1 \right) + \sqrt{\frac{\mu_1}{\pi \mu_2}} B_{7,2} \left( \frac{1}{2}, T_0, 1, 1 \right)
\]

instead of (55) and (56). Then

\[
|B_7| \leq B_7 (z) \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2}} T^{\frac{1}{2} - \sigma} \sqrt{\log \frac{\mu_1}{\pi \mu_2}}, \quad |B_8| \leq B_8 (z) \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2} + \frac{1}{2}} T^{\frac{1}{2} - \sigma} \sqrt{\log \frac{\mu_1}{\pi \mu_2}}.
\]

where

\[
B_7 (z) := \overline{F} \cdot \pi^{\sigma_0 - \frac{1}{2}} \frac{\lambda_1 (\sigma_0)}{\pi^{\frac{1}{2}}} \bar{B}_7 (z),
\]

\[
B_8 (z) := \overline{E} \cdot \pi^{\sigma_0 - \frac{1}{2}} \frac{\lambda_1 (\sigma_0)}{\pi^{\frac{1}{2}}} \bar{B}_8 (z),
\]

and \( \lambda_1(x) \) is a continuous function on \( \mathbb{R} \), defined as

\[
\lambda_1 (x) := \begin{cases} \frac{2^{x - 2^{x/2}}}{2 \log 2} & x \neq 2/3, \\ \frac{2^{x - 2^{x/2}}}{2 \log 2} & x = 2/3.
\end{cases}
\]

Observe that \( \lambda_1(x) \) is a strictly increasing function on \([1/2, 1]\). In the general case we use (55) and (56) to bound \( B_7 (z) \) and \( B_8 (z) \), which means that both functions are bounded for fixed \( \sigma_0 \) and \( T_0 \).

3.6. Bounds on \( B_1 \) and \( B_2 \). Separation of the diagonal (here we need coprimality of \( \mu_1 \) and \( \mu_2 \)) and off-diagonal terms in \( A_1 \) and \( A_2 \), together with equality (59) gives

\[
B_1 = (\mu_1 \mu_2)^{-\sigma} \int_T^{2T} \sum_{n \leq (t/\mu_1)} n^{-2\sigma} dt + \sum_{n \leq (2T)} \sum_{m \neq (m/\mu_2)} \int_{M_n}^{2T} (nm)^{-\sigma} \left( \frac{m \mu_1}{n \mu_2} \right)^{it} dt.
\]
where \( M_4 := \max \{ T, x^{-1}(n), y^{-1}(m) \} \), and

\[
B_2 = (\mu_1 \mu_2)^{\sigma-1} \int_T^{2T} \left( \frac{2\pi}{l} \right)^{2\sigma-1} \sum_{n \leq y(l/\mu_2^2)} n^{2\sigma-2} dt \\
+ \sum_{n \leq y(2T)} \sum_{n \neq \mu_1 \mu_2} \int_{M_4}^T \left( \frac{2\pi}{l} \right)^{2\sigma-1} (nm)^{\sigma-1} \left( \frac{n \mu_1}{\mu_2} \right)^{it} dt.
\]

where \( M_4 := \max \{ T, y^{-1}(n), x^{-1}(m) \} \). Denote by \( B_{1,1} \) and \( B_{1,2} \) the first integral and the double sum in \( B_1 \), respectively. In the same vein define also \( B_{2,1} \) and \( B_{2,2} \). If we apply Theorem 2 on the sums in \( B_{1,1} \) and \( B_{2,1} \), and knowing that \(|R(\sigma; z)| \leq (1/2)x^{-\sigma} \), then we obtain

\[
B_{1,1} + B_{2,1} = \mathcal{J}(z) - \frac{(2\pi)^\sigma (2^{1-\sigma} - 1)}{1 - \sigma} T^{1-\sigma},
\]

where \( \mathcal{J}(z) \) is defined by (67). Writing \( n \mu_1 = M \) and \( n \mu_2 = N \), we get

\[
|B_{1,2}| \leq 2(\mu_1 \mu_2)^\sigma \sum_{N} \sum_{M \leq 2N} \frac{(NM)^{-\sigma}}{\log (N/M)} \leq \mu_1 \mu_2 \bar{B}_{1,2}(\sigma_0, T_0) T^{1-\sigma} \log \frac{T \mu_1 \mu_2}{\pi},
\]

where

\[
\bar{B}_{1,2}(\sigma_0, T_0) := \pi^{\sigma_0-1} \left( 1 + \frac{2\gamma + \frac{2}{\log \frac{t}{\pi}}}{\log \frac{t}{\pi}} \right)
\]

Lemma 6 implies

\[
|B_{2,2}| \leq 2(2\pi)^{2\sigma-1} (\mu_1 \mu_2)^{1-\sigma} T^{1-2\sigma} \sum_{N,M \leq 2N/\mu_2} \frac{(NM)^{-\sigma-1}}{\log (N/M)}
\]

\[
\leq \mu_1 \mu_2 \bar{B}_{2,2}(\sigma_0, T_0) T^{1-\sigma} \log \frac{T \mu_1 \mu_2}{\pi},
\]

where

\[
\bar{B}_{2,2}(\sigma_0, T_0) := 2^{2\sigma} \pi^{\sigma_0-1} \left( 1 + 2\sigma_0 \sqrt{\frac{\pi}{T_0}} \left( 4 + 2\gamma + (1 + 4\gamma \sigma_0) \sqrt{\frac{\pi}{T_0}} + \frac{2\sigma_0}{T_0} \right) \right).
\]

In both cases we have used the inequality from Lemma 5 without the term with the minus sign. We get

\[
|B_1 + B_2 - \mathcal{J}(z)| \leq B_1(\sigma_0, T_0) \mu_1 \mu_2 T^{1-\sigma} \log \frac{T \mu_1 \mu_2}{\pi},
\]

where

\[
B_1(\sigma_0, T_0) := \bar{B}_{1,2}(\sigma_0, T_0) + \bar{B}_{2,2}(\sigma_0, T_0) + \frac{\pi^{\sigma_0} (2 - 2\sigma_0)}{(1 - \sigma_0) \log \frac{T_0}{\pi}}.
\]

This function is clearly bounded for fixed \( \sigma_0 \) and \( T_0 \), and is also continuous for \( \sigma_0 \in [1/2, 1) \).
3.7. Bounds on $B_3$ and $B_4$. Using (12) and (39), we obtain

$$B_3 = e^{\tilde{t}} (2\pi)^{\sigma - \frac{1}{2}} \sum_{n,m \leq y(2T)} n^{\sigma - 1} m^{-\sigma} \int_{M_2(n,m)} t^{\frac{\sigma}{2}} \left( \frac{1}{2\pi\log \left( T \mu_1 \right)} \right)^{\sigma} dt.$$ 

The equation for $B_4$ is the same except that we need to replace $e^{i\pi/4}$ by $e^{-i\pi/4}$, $y(2T)$ by $x(2T)$, and $-i\pi$ by $i\pi$. Lemma 7 and separation of diagonal and off-diagonal terms imply

$$|B_3| \leq 8(2\pi)^{\sigma - \frac{1}{2}} T^{\frac{1}{2} - \sigma} (B_{3,1} + B_{3,2}),$$

where

$$B_{3,1} := \sum_{n \leq y(2T)} \left( n \log \frac{M_2(n,n) + \sqrt{M_2(n,n)}}{2\pi n^2 \mu_2} \right)^{-1},$$

$$B_{3,2} := \sum_{n,m \leq y(2T) \atop n \neq m} n^{2\sigma - 1} \left( (nm)^{\sigma} \log \frac{M_2(n,m) + \sqrt{M_2(n,m)}}{2\pi nm \mu_2} \right)^{-1}.$$ 

The same inequality holds also for $B_4$. Using the fact that $M_2(n,n) \geq 2\pi n^2 \mu_1 / \mu_2$, $\sqrt{x} \log (1 + 1/\sqrt{x}) \geq \sqrt{T} / (1 + \sqrt{T})$ for $x \geq T$, and $\sqrt{M_2(n,n)} \leq \sqrt{2T}$, we get

$$B_{3,1} \leq \frac{T \mu_2}{\pi \mu_1} \sqrt{T} \log \frac{T \mu_2}{\pi \mu_1},$$

where

$$\tilde{B}_{3,1} (T, \mu_1, \mu_2) := \sqrt{\frac{\pi \mu_1}{2 \mu_2}} \left( 1 + \frac{2\gamma + \sqrt{T} \mu_2}{\log \frac{T \mu_2}{\pi \mu_1}} \left( 1 + \frac{1}{\sqrt{T}} \right) + \frac{1}{\sqrt{T}} \right),$$

$$\leq \sqrt{\frac{\pi}{2}} \left( 1 + \frac{2\gamma + \sqrt{T} \mu_2}{\log \frac{\pi}{\sqrt{2}}} \left( 1 + \frac{1}{\sqrt{T_0}} \right) + \frac{1}{\sqrt{T_0}} \right).$$

Because

$$\frac{M_2(n,m) + \sqrt{M_2(n,m)}}{2\pi nm \mu_2} \geq \frac{M_2(n,m)}{2\pi nm \mu_2} = \max \left\{ \frac{T \mu_2}{2\pi nm \mu_1}, \frac{n \cdot m \cdot n}{m \cdot n} \right\},$$

it follows by Lemma 5 that

$$B_{3,2} \leq \left( \frac{T \mu_2}{\pi \mu_1} \right)^{\sigma - \frac{1}{2}} \sum_{n,m \leq y(2T) \atop n \neq m} \frac{(nm)^{-\sigma} \log (n/m)}{2\pi \log \left( T \mu_1 \right)} \leq \tilde{B}_{3,2}(z) \sqrt{\frac{T \mu_2}{\pi \mu_1}} \sqrt{T} \log \frac{T \mu_2}{\pi \mu_1},$$

where

$$\tilde{B}_{3,2}(z) := \frac{1}{2(1 - \sigma)} \left( 1 + \frac{1}{\log \frac{T \mu_2}{\pi \mu_1}} \left( 2 \frac{1}{1 - \sigma} + 2\gamma + \frac{\pi \mu_1}{T \mu_2} \right) \right),$$

$$\leq \frac{1}{2(1 - \sigma_0)} \left( 1 + \frac{1}{\log \frac{T}{\pi}} \left( 2 \frac{1}{1 - \sigma_0} + 2\gamma + \frac{\pi}{T_0} \right) \right).$$

Define also

$$\tilde{B}_{4,1} (T, \mu_1, \mu_2) := \frac{2\gamma + \sqrt{T} \mu_2}{2 \sqrt{2} \log \frac{T \mu_2}{\pi \mu_2}} \left( 1 + \frac{1}{\sqrt{T}} \right) + \frac{1}{\sqrt{2T}}$$

$$\leq 2.607 \left( 1 + \frac{1}{\sqrt{T_0}} \right).$$

(62)
\[ \tilde{B}_{4,2} (z) := \sqrt{\frac{\mu_1}{\pi \mu_2}} \tilde{B}_{3,2} (\sigma, T, \mu_2, \mu_1) \leq \frac{1}{2\sqrt{\pi}} \left( \frac{2.9 + 2.886}{1 - \sigma_0} \right). \] (63)

Functions \( \tilde{B}_{3,2} \) and \( \tilde{B}_{4,2} \), and their bounds are continuous for \( \sigma \in [1/2, \sigma_0] \) and \( \sigma_0 \in [1/2, 1) \). In case \( \mu_1 = \mu_2 = 1 \) and \( \sigma = 1/2 \) it is better to use:

\[ \tilde{B}_{4,1} (T, 1, 1) \leq \tilde{B}_{4,1} (T_0, 1, 1), \quad \tilde{B}_{4,2} \left( \frac{1}{2}, T, 1, 1 \right) \leq \sqrt{\frac{T}{\pi}} \tilde{B}_{3,2} \left( \frac{1}{2}, T_0, 1, 1 \right). \]

Putting all together finally gives:

\[ |B_3| \leq B_3(z) \sqrt{\frac{\mu_2}{\pi \mu_1}} T^{1-\sigma} \log \frac{T \mu_2}{\pi \mu_1}, \quad |B_4| \leq B_4(z) T^{1-\sigma} \log \frac{T \mu_1}{\pi \mu_2} \]

where:

\[ B_3(z) := 8(2\pi)^{\sigma - \frac{1}{2}} \left( \tilde{B}_{3,1} (T, \mu_1, \mu_2) + \tilde{B}_{3,2} (z) \right), \] (64)

\[ B_4(z) := 8(2\pi)^{\sigma - \frac{1}{2}} \left( \tilde{B}_{4,1} (T, \mu_1, \mu_2) + \tilde{B}_{4,2} (z) \right). \] (65)

In the general case we will use (60), (61), (62) and (63) to bound \( B_3(z) \) and \( B_4(z) \). This also implies that both functions are bounded for fixed \( \sigma_0 \) and \( T_0 \).

3.8. Proof of Theorem 7. The statement of Theorem 7 now easily follows by using bounds for \( B_i \) developed in Sections 3.4 and 3.5, which give \( \mathcal{R}_1(z) \), and Sections 3.6, 3.7 and 3.8 which give \( \mathcal{R}_2(z) \).

4. Explicit Selberg’s zero density result

4.1. The mollifier. Let \( s = \sigma + it \) with \( \sigma \geq 1/2 \) and \( X \geq 1 \). Selberg introduced:

\[ S_X(s) := \sum_{n \leq X} \lambda_X(n)n^{-s} \]

where:

\[ \lambda_X(n) := n^{2\sigma} \left( \sum_{m \leq X} \frac{\mu^2(m)}{\varphi_{2\sigma}(m)} \right)^{-1} \sum_{m \leq X/n} \frac{\mu(nm)\mu(m)}{\varphi_{2\sigma}(nm)} \]

and:

\[ \varphi_X(n) := n^x \sum_{d|n} \frac{\mu(d)}{d^x} = n^x \prod_{p|n} (1 - p^{-x}) \]

for \( x \in \mathbb{R} \). Observe that \( \varphi_1(n) \) is the ordinary Euler totient function \( \varphi(n) \), and also that \( \lambda_X(1) = 1 \). Because \( \mu(n) \) and \( \varphi_{2\sigma}(n) \) are multiplicative functions, and \( \mu(nm) = 0 \) if \( (m, n) \neq 1 \), it follows:

\[ \lambda_X(n) = \frac{\mu(n)n^{2\sigma}}{\varphi_{2\sigma}(n)} \left( \sum_{m \leq X} \frac{\mu^2(m)}{\varphi_{2\sigma}(m)} \right)^{-1} \sum_{m \leq X/n} \frac{\mu^2(m)}{\varphi_{2\sigma}(m)}. \]

This implies:

\[ |\lambda_X(n)| \leq \frac{n^{2\sigma}}{\varphi_{2\sigma}(n)} = \prod_{p|n} \frac{1}{1 - p^{-2\sigma}}. \]

If \( \sigma > 1/2 \), then \( |\lambda_X(n)| \leq \zeta(2\sigma) \). Let \( 1 < n \leq X \). Then the above product is not greater than the same product for \( p \leq X \) and \( \sigma = 1/2 \). Therefore,

\[ |\lambda_X(n)| \leq \tilde{\lambda}(X) \log X \] (66)

for some bounded function \( \tilde{\lambda}(X) \leq 2.2 \) where \( X \geq 8 \), see [RS62, Corollary 1].
Lemma 8. We have
\[
\sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)}{(nm)^{2\sigma}} (n, m)^{2\sigma} = \left( \sum_{k \leq X} \frac{\mu^2(k)}{\varphi_{2\sigma}(k)} \right)^{-1},
\]
\[
\sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)}{nm} (n, m)^{2-2\sigma} > 0.
\]

Proof. This is quite straightforward to prove if we notice that
\[(n, m)^x = \sum_{d | (n, m)^x} \varphi_x(d) .\]
For details see [Sel46, pp. 15–16] or [KK06, p. 401].

Lemma 9. Let \( \sigma \geq 1/2 + 1/ \log X \). Then
\[
\zeta(2\sigma) \sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)}{(nm)^{2\sigma}} (n, m)^{2\sigma} \leq 1 + \frac{1 + 2\sigma - 1}{1 - e^{-2}} X^{1-2\sigma}.
\]

Proof. By Lemma 8 we have
\[
\zeta(2\sigma) \sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)}{(nm)^{2\sigma}} (n, m)^{2\sigma} = \zeta(2\sigma) \left( \sum_{k \leq X} \frac{\mu^2(k)}{\varphi_{2\sigma}(k)} \right)^{-1} \leq \frac{\zeta(2\sigma)}{\sum_{k \leq X} k^{-2\sigma}} = 1 + \frac{\sum_{k \geq X} k^{-2\sigma}}{\sum_{k \leq X} k^{-2\sigma}}.
\]

Because
\[
\sum_{k > X} k^{-2\sigma} \leq \frac{X^{1-2\sigma}}{2\sigma - 1} \left( 1 + \frac{2\sigma - 1}{X} \right), \quad \sum_{k \leq X} k^{-2\sigma} \geq \frac{1 - e^{-2}}{2\sigma - 1},
\]
the stated inequality follows.

4.2. Littlewood’s lemma. Let \( f(s) \) be a holomorphic function on some domain in the complex plane which includes a rectangle with vertices \( \sigma + iT, a + iT, a + i2T \) and \( \sigma + i2T \), where \( 1/2 \leq \sigma < 1 < a \). Denote by \( n_f(\tau) \) the number of zeros of \( f(s) \) in the set \( \{ s \in \mathbb{C} : \sigma < \tau < \Re{s} < a, T < \Im{s} < 2T \} \), and assume that no zeros are on the boundary of the rectangle. Then Littlewood’s lemma asserts that
\[
2\pi \int_{\sigma}^{a} n_f(\tau) \, d\tau = \int_{-T}^{2T} \log |f(\sigma + it)| - \log |f(a + it)| \, dt \quad + \int_{\sigma}^{a} \arg f(\tau + i2T) - \arg f(\tau + iT) \, d\tau.
\]

Define \( \Phi_X(s) := \zeta(s)S_X(s) \). Then \( \Phi_X(s) \) is holomorphic in \( \mathbb{C} \setminus \{1\} \) and \( N(\tau, 2T) - N(\tau, T) = n(\tau) \leq n_{\Phi}(\tau) \). We need some trivial estimates on \( \Phi_X(s) \) in order to apply Littlewood’s lemma.

Lemma 10. Let \( \sigma \geq \sigma_1 \geq 2.4 \) and define
\[
\begin{align*}
  h(\sigma) := & \left( 2\zeta(\sigma) + \left( \frac{3}{2} \right)^{-\sigma} - 1 \right) \left( 1 + \zeta(\sigma) \zeta(2\sigma) \right). 
\end{align*}
\]  (67)

Then \( |\Phi_X(s) - 1| \leq 2^{-\sigma} h(\sigma_1) \) and
\[
|\arg \Phi_X(s)| \leq 2^{-\sigma} \frac{h(\sigma_1)}{1 - 2^{-\sigma} h(\sigma_1)} .
\]
Proof. By definition of $\Phi_X(s)$, we have for $\sigma > 1$ the following:

$$\Phi_X(s) = 1 + \sum_{n=2}^{\infty} n^{-s} + \sum_{n=2}^{\infty} \frac{|\lambda_X(n)|}{n^s} + \sum_{m=2}^{\infty} \sum_{n=2}^{m} \frac{|\lambda_X(m)|}{(nm)^s}.$$ 

Then the first inequality easily follows since $|\lambda_X(n)| \leq \zeta(2\sigma)$,

$$\left|\sum_{n=2}^{\infty} n^{-s}\right| \leq \zeta(\sigma) - 1 \leq 2^{-\sigma} \left(1 + \left(\frac{3}{2}\right)^{-\sigma} + \sum_{n=4}^{\infty} \left(\frac{3}{2}\right)^{-\sigma}\right) \leq \frac{2^{-\sigma} h(\sigma)}{1 + \zeta(\sigma)\zeta(2\sigma)},$$

and $h(\sigma)$ is decreasing function. Similarly, we also have $|3 \{\Phi_X(s)\}| \leq 2^{-\sigma} h(\sigma_1)$ and $\Re \{\Phi_X(s)\} \geq 1 - 2^{-\sigma_1} h(\sigma_1) > 0$. This implies the bound on the argument of $\Phi_X(s)$.

Lemma \[\text{Lemma 10}\] assures that $\lim_{a \to \infty} \Phi_X(a + it) = 1$, and also that $n_{\Phi}(\tau) = 0$ for $\tau \geq 1$. Therefore, Littlewood’s lemma implies

$$\int_{\tau}^{1} N(\tau, 2\tau) - N(\tau, T) d\tau \leq \frac{1}{2\pi} \int_{T}^{2T} \log|\Phi_X(\sigma + it)| d\tau$$

$$+ \frac{1}{2\pi} \int_{\tau}^{\sigma_1} \left|\arg \Phi_X(\tau + i2T)\right| + \left|\arg \Phi_X(\tau + iT)\right| d\tau$$

$$+ \frac{2^{-\sigma_1} h(\sigma_1)}{(1 - 2^{-\sigma_1} h(\sigma_1)) \pi \log 2}$$

for $\sigma_1 \geq 2.4$. In the following two subsections we will provide explicit estimates of the above integrals.

4.3. Explicit upper bound for $\int_{T}^{2T} \log|\Phi_X(\sigma + it)| d\tau$. We will use Theorem \[\text{Theorem 7}\] together with Lemmas \[\text{Lemma 8}\] and \[\text{Lemma 9}\] in order to estimate

$$\int_{T}^{2T} |\Phi_X(\sigma + it)|^2 d\tau = \sum_{n,m \leq X} \frac{\lambda_X(n)\lambda_X(m)S(\sigma, T, n, m)}{(nm)^{\sigma}}.$$ 

The following proposition is an explicit version of \[\text{[So14] Lemma 7}\] for $H = T$. It is important to note that in this approach it is crucial to keep $\sigma$ far enough from 1/2.

**Proposition 2.** Let $\sigma_0 \in (1/2, 1)$, $T \geq T_0 > e^{\frac{\sigma_0}{\sigma_0 - 1/2}}$, $X = T^\frac{\sigma}{\sigma_0} - \varepsilon$, and

$$0 < \varepsilon \leq \frac{1}{6} - \frac{1}{(\frac{\sigma_0}{\sigma_0 - 1/2}) \log T_0}. \quad (68)$$

Then

$$\int_{T}^{2T} |\Phi_X(\sigma + it)|^2 d\tau \leq T + \phi(\sigma_0, T_0, \varepsilon, T) T^{1 - 2(\frac{\sigma}{\sigma_0})(\sigma - \frac{1}{2})}$$

for

$$\sigma \in \left[\frac{1}{2} + \frac{1}{(\frac{\sigma_0}{\sigma_0 - 1/2}) \log T_0}, \sigma_0\right], \quad (69)$$

where

$$\phi(\sigma_0, T_0, \varepsilon, T) := \frac{1 + (2\sigma_0 - 1) T_0^{\frac{1}{2}}}{1 - e^{-2}} + \phi_1 \frac{\log \frac{2}{\sigma_0} T}{T^{\frac{3}{2}}} + \phi_2 \frac{\log^3 T}{T^{3\varepsilon}},$$

and

$$\phi_1 := 2\sqrt{2} \left(\frac{1}{6} - \varepsilon\right)^{\frac{3}{2}} \mathcal{J}_1(\sigma_0, T_0) \tilde{\lambda}(X)^2 \left(1 + \frac{\gamma}{(\frac{6}{\sigma_0 - 1}) \log T_0} + \frac{1}{2T_0^{\frac{3}{2} - \varepsilon}}\right),$$

$$\phi_2 := 6 \left(\frac{1}{6} - \varepsilon\right)^2 \mathcal{J}_2(\sigma_0, T_0) \tilde{\lambda}(X)^2,$$
with \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) as in Theorem 7 and \( \hat{\lambda}(X) \) from (66).

Proof. Let \( n \) and \( m \) be positive integers not greater than \( T/(2\pi) \). Define \( z := (\sigma, T, n, m) \) and \( \hat{z} := (\sigma, T, n/(n, m), m/(n, m)) \). Because \( S(z) = S(\hat{z}) \), by Theorem 7 we have

\[
|S(z) - \mathcal{S}(\hat{z})| \leq \frac{2J_2(\sigma, T_0) n m}{(n, m)^2} T^{1-\sigma} \log \frac{nm}{\pi(n, m)^2} + \mathcal{S}_1(\sigma, T_0) T^{1-\frac{2}{\sigma}} \left( \frac{m}{n} \right)^{\sigma} \sqrt{\log \frac{nm}{\pi n}} + \frac{m}{n} \left( \frac{n}{m} \right)^{\sigma} \sqrt{\log \frac{nm}{\pi m}}.
\]

Let \( X := T^\alpha \) for some \( \alpha \in (0, 1 - \log (2\pi)/\log T) \). Then

\[
\left| \sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)(S(z) - \mathcal{S}(\hat{z}))}{(nm)^\sigma} \right| \leq 2\sqrt{2} \alpha^3 \mathcal{S}_1 \hat{\lambda}(T)^2 \left( 1 + \frac{\gamma}{\alpha \log T_0} + \frac{1}{2T_0^\alpha} \right) T^{1-\frac{2}{\sigma}} \log \frac{T}{\pi T} + 6\alpha^2 J_2(\sigma) T^{1+\alpha(1-2\sigma)+3\alpha-\sigma} \log^3 T.
\]

Now let \( \sigma \geq 1/2 + 1/\log X \). By Lemmas 8 and 9 we have

\[
\left| \sum_{n, m \leq X} \frac{\lambda_X(n)\lambda_X(m)\mathcal{S}(\hat{z})}{(nm)^\sigma} \right| \leq T + \frac{1 + \frac{2}{\sigma-1}}{1 - e^{-2T}} T^{1+\alpha(1-2\sigma)}
\]

since \( \zeta(2-2\sigma) \) is negative for \( \sigma \in (1/2, 1) \). Take \( \sigma_0 \in (0, 1) \) and define \( \alpha := 1/6 - \varepsilon \) with \( \varepsilon \) satisfying (68). Then (69) is a well-defined set and the bound for the integral now clearly follows.

Observe that Proposition 2 implies

\[
\int_T^{2T} |\Phi_X(\sigma + it)|^2 dt \leq T + O \left( T^{1-2(\hat{z} - \varepsilon)(\sigma - \hat{z})} \right),
\]

uniformly for \( \sigma \) on the set (69) while \( \sigma_0 \) and \( \varepsilon \in (0, 1/6) \) are fixed. This is a slight generalisation of Selberg’s result for \( H = T \) since his bound follows for \( \varepsilon = 1/24 \).

Corollary 6. With assumptions and notations as in Proposition 4 we have

\[
\frac{1}{2\pi} \int_T^{2T} \log |\Phi_X(\sigma + it)| dt \leq \frac{1}{4\pi} \phi(\sigma_0, T_0, \varepsilon, T) T^{1-2(\hat{z} - \varepsilon)(\sigma - \hat{z})}.
\]

Proof. The first inequality follows from Proposition 2 and because log \((1 + x) \leq x \) and

\[
\int_a^b \log f(u) du \leq (b - a) \log \left( \frac{1}{b-a} \int_a^b f(u) du \right)
\]

for \( x \geq 0 \), and positive continuous functions \( f(u) \) on \([a, b] \subset \mathbb{R}\).

4.4. Explicit upper bound for \( \int_\sigma^{\sigma_1} |\arg \Phi_X(\tau + it)| \, d\tau \). Let \( \sigma_1 \) be as in Lemma 10 and let \( w = \sigma_1 + (2\sigma_1 - 1)e^{i\varphi} + iU \) where \( \varphi \in [\pi/2, 3\pi/2] \) and \( U \) is not the ordinate of a zero of \( \Phi(z) \). Assume that there is a function \( \hat{\Phi}(\sigma_1, \varphi, U, X) \) such that \( |\Phi_X(w)| \leq \hat{\Phi}(\sigma_1, \varphi, U, X) \). According to Proposition 4.10 in [KLN18], for \( \sigma \in (0, \sigma_1] \) we have

\[
|\arg \Phi_X(\sigma + iU)| \leq \frac{1}{2\log 2} \int_{\varphi}^{2\varphi} \log \hat{\Phi}(\sigma_1, \varphi, U) d\varphi + \frac{\pi}{2\log 2} \log \frac{1 + 2^{1-\sigma_1} h(\sigma_1)}{(1 - 2^{-\sigma_1} h(\sigma_1))^2} + \frac{\pi}{2}.
\]
Trivially,
\[ |S_X(w)| \leq \hat{\Phi}_1(\sigma_1, X) := \frac{11}{5} X^{\sigma_1} \log X. \tag{71} \]

For the second part we will use the following convexity result.

**Lemma 11.** Let \( s = \sigma + it \) with \( \sigma \in [1 - \sigma_1, \sigma_1] \) where \( \sigma_1 \geq 2.4 \), and \( s \neq 1 \). Then
\[ |\zeta(s)| \leq b_1(\sigma_1) \frac{\sigma - \frac{1}{2}}{\frac{3}{2} - \sigma_1 - 1} \frac{\zeta(\sigma_1)}{|1 - s|}, \]
where
\[ b_1(\sigma_1) := \sqrt{2 \pi (2\pi)^{1-\sigma_1} e^{-\frac{\sigma_1}{2}} \frac{\Gamma(\frac{1}{2})}{\zeta(1/2)}}. \]

**Proof.** By the functional equation for \( \zeta(s) \) and the Stirling formula \( \text{(20)} \), we have
\[ |\zeta(1 - \sigma_1 + it)| \leq b_1(\sigma_1) |\sigma_1 + it|^{\sigma_1 - \frac{1}{2}}. \]

Applying the Phragmén–Lindelöf theorem, see [Rad60, Theorem 2], on the function \( (1 - s)\zeta(s) \) in the strip \( \{ s \in \mathbb{C}: 1 - \sigma_1 \leq \sigma \leq \sigma_1 \} \) with \( Q = 2\sigma_1 - 1 \), we obtain the main inequality.

**Proposition 3.** Let \( U \geq T_0 > 2\sigma_1 - 1 \geq 3.8 \) and \( U \) is not the ordinate of a zero of \( \Phi_X(s) \). Then
\[ |\arg \Phi_X(\sigma + iU)| \leq \frac{\pi (3 + \sigma_1)}{2 \log 2} \log U + \frac{\pi \sigma_1}{2 \log 2} \log X + \frac{\pi}{2 \log 2} \log \log X + b_3 \]
for \( \sigma \in (0, \sigma_1) \), where
\[ b_3(\sigma_1, T_0) := \frac{\pi}{2 \log 2} \log \zeta(\sigma_1) + \frac{\log b_1(\sigma_1)}{\log 2} + \frac{\pi}{4 \log 2} \frac{\log b_2(\sigma_1, T_0)}{2} \left( \frac{1}{2} + \sigma_1 \right) \]
\[ + \frac{\pi}{2 \log 2} \frac{11 (1 + 2^{-\sigma_1} h(\sigma_1))}{5 (T_0 - 2\sigma_1 + 1) (1 - 2^{-\sigma_1} h(\sigma_1))^{\frac{3}{2}}} + \frac{\pi}{2} \]
and \( h(\sigma_1) \) is defined by \( \text{(67)} \).

**Proof.** We can take \( \hat{\Phi} = \hat{\Phi}_1(\sigma_1, X) \hat{\Phi}_2(\sigma_1, \varphi, T_0, U) \), where \( \hat{\Phi}_1 \) and \( \hat{\Phi}_2 \) are defined by \( \text{(71)} \) and \( \text{(72)} \), respectively. Now the result simply follows from \( \text{(70)} \).

**Corollary 7.** Let \( T \geq T_0 > 2\sigma_1 - 1 \geq 3.8, 1 > \sigma \geq 1/2, X = T^{2\sigma}, \) and \( T \) or \( 2T \) is not the ordinate of a zero of \( \Phi_X(s) \). Then
\[ \frac{1}{2\pi} \int_{\sigma}^{\sigma_1} [\arg \Phi_X(\tau + i2T)] + [\arg \Phi_X(\tau + iT)] \, d\tau \]
\[ \leq \frac{1}{2\pi} \left( \sigma - \frac{1}{2} \right) \left( \frac{\pi (9\sigma_1 + 4)}{8 \log 2} \log T + \frac{\pi}{2 \log 2} \log \log T - \frac{\pi (3 + \sigma_1)}{2} + 2b_3 \right). \]
4.5. Proof of Theorem 1. Firstly, we will provide some general bounds. Let 
$T \geq T_0 \geq e^{4} \approx 2.65 \cdot 10^{10}$ and $\sigma \in [1/2 + 8/\log T, 1/2 + 8/\log T_0]$. We can assume that $\Phi_X(s)$ does not have any zeros with imaginary parts equal to $T$ or $2T$ since the following inequalities can be extended by continuity principle also to these cases. Applying Corollaries 3 and 4 with $\varepsilon = 1/24$, we obtain

$$\int_{\sigma}^{1} N(\tau, 2T) - N(\tau, T)d\tau \leq \alpha T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} + \beta \log T + \gamma \log \log T + \delta,$$  \hspace{1cm} (73)

where

$$\alpha (T_0) := \frac{1}{4\pi} \left( \frac{1}{2} + \frac{8}{\log T_0} \right) \cdot \frac{1}{T_0},$$

$$\beta (\sigma_1) := \frac{(\sigma_1 - \frac{1}{2})(9\sigma_1 + 16)}{16 \log 2},$$

$$\gamma (\sigma_1) := \frac{\sigma_1 - \frac{1}{2}}{4 \log 2},$$

$$\delta (\sigma_1, T_0) := \left( \sigma_1 - \frac{1}{2} \right) \left( \frac{2\sigma_1 - 11}{8} + \frac{b_4 (\sigma_1, T_0)}{\pi} \right) + \frac{2^{-\sigma_1} h (\sigma_1)}{(1 - 2^{-\sigma_1} h (\sigma_1)) \pi \log 2},$$

and $\sigma_1 \geq 2.4$. By (1) we have

$$N(2T) - N(T) \leq \frac{T}{2\pi} \log T + 0.22 \log T + 0.6 \log \log T + 5.$$  \hspace{1cm} (74)

Let $\sigma \in [1/2, 1/2 + 8/\log T]$. Because $N(\tau, 2T) - N(\tau, T) \leq (N(2T) - N(T))/2$ and $T \leq e^{2T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})}}$, we have

$$\int_{\sigma}^{1} N(\tau, 2T) - N(\tau, T)d\tau \leq \frac{1}{2} \int_{\frac{1}{2}}^{1} N(2T) - N(T)d\tau$$

$$+ \int_{\frac{1}{2}}^{1} N(\tau, 2T) - N(\tau, T)d\tau$$

$$\leq \left( \frac{2e^{2} + \alpha}{\pi} \right) T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} + \beta \log T + \gamma \log \log T$$

$$+ \delta + 0.88 + 2.4 \frac{\log \log T_0}{\log T_0} + \frac{20}{\log T_0}. \hspace{1cm} (75)$$

Because the right-hand side of (73) is always smaller than the right-hand side of (74), the latter inequality is true for all $\sigma \in [1/2, 1/2 + 8/\log T_0]$.

With help of (42) we are ready to estimate $N(\sigma, 2T) - N(\sigma, T)$. Let $\sigma \in [1/2 + 4/\log T, 1/2 + 8/\log T_0]$. Then

$$N(\sigma, 2T) - N(\sigma, T) \leq \frac{\log T}{4} \int_{\sigma}^{1} N(\tau, 2T) - N(\tau, T)d\tau$$

$$\leq \frac{e}{4} \left( \frac{2e^{2} + \alpha}{\pi} \right) T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T$$

$$+ \frac{\beta}{4} \log^{2} T + \frac{\gamma}{4} \log T \cdot \log \log T + \left( \frac{\delta}{4} + 0.51 \right) \log T. \hspace{1cm} (75)$$

For $\sigma \in [1/2, 1/2 + 4/\log T]$ we have

$$N(\sigma, 2T) - N(\sigma, T) \leq \frac{e}{4\pi} T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T + \frac{1}{4} \log T.$$  \hspace{1cm} (76)

The right-hand side of the latter inequality is obviously smaller than the right-hand side of (75), therefore this inequality is true for all $\sigma \in [1/2, 1/2 + 8/\log T_0]$.

**Proof of Theorem 1.** Take $T_0 = H_0$ and $\sigma_1 = 2.40764$. Since $\mathcal{S}_1 \leq 219.618$ and $\mathcal{S}_2 \leq 611.578$ by Theorem 7, we have $\alpha (H_0) < 15291.986$, $\beta (\sigma_1) < 4.416$, $\gamma (\sigma_1) < 0.6881$ and $\delta (\sigma_1, H_0) < 0$. Then the constants in Theorem 1 follows from (75).
Observe that
\[ e^{\frac{2e^2}{\pi} + \alpha(T_0)} > \frac{e^3}{2\pi} \left( 1 + \frac{1}{8(e^2 - 1)} \right) > 3.259, \]
where the minimum is attained in the limit \( T_0 \to \infty \). This means that the leading term in Theorem \( \text{III} \) can be significantly improved if we take larger values for \( T_0 \), but its value could not be below 3.259. For instance, if \( T_0 = 10^{50} \), then \( \alpha(T_0) < 3.18 \).

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References

[AdR11] J. Arias de Reyna, *High precision computation of Riemann’s zeta function by the Riemann-Siegel formula, I*, Math. Comp. 80 (2011), no. 274, 995–1009.

[AM15] G. Alirezaei, R. Mathar, *Analytical bounds on the average error probability for Nakagami fading channels*, 2015 Information Theory and Applications Workshop (ITA), San Diego, CA, 2015, pp. 54–63.

[BBR12] D. Berkane, O. Bordellès, and O. Ramaré, *Explicit upper bounds for the remainder term in the divisor problem*, Math. Comp. 81 (2012), no. 278, 1025–1051.

[Che99] Y. Cheng, *An explicit upper bound for the Riemann zeta-function near the line \( \sigma = 1 \)*, Rocky Mountain J. Math. 29 (1999), no. 1, 115–140.

[Con89] J. B. Conrey, *At least two-fifths of the zeros of the Riemann zeta function are on the critical line*, Bull. Amer. Math. Soc. (N.S.) 20 (1989), no. 1, 79–81.

[DHZA19] D. Dona, H. A. Helfgott, and S. Zuniga Altermann, *Explicit \( L^2 \) bounds for the Riemann \( \zeta \) function*, preprint available at arXiv:1906.01097v5.

[GR15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 8th ed., Elsevier/Academic Press, Amsterdam, 2015.

[Hia16] G. A. Hiary, *An alternative to Riemann-Siegel type formulas*, Math. Comp. 85 (2016), no. 298, 1017–1032.

[HL21] G. H. Hardy and J. E. Littlewood, *The zeros of Riemann’s zeta-function on the critical line*, Math. Z. 10 (1921), no. 3-4, 283–317.

[HL23] G. H. Hardy and J. E. Littlewood, *The Approximate Functional Equation in the Theory of the Zeta-Function, with Applications to the Divisor-Problem of Dirichlet and Piltz*, Proc. London Math. Soc. (2) 21 (1923), 39–74.

[HL29] G. H. Hardy and J. E. Littlewood, *The Approximate Functional Equations for \( \zeta(s) \) and \( \zeta' (s) \)*, Proc. London Math. Soc. (2) 29 (1929), no. 2, 81–97.

[Ivi03] A. Ivic, *The Riemann Zeta-Function*, Dover Publications, Inc., Mineola, NY, 2003.

[Jut83] M. Jutila, *Zeros of the zeta-function near the critical line*, Studies in pure mathematics, Birkaüser, Basel, 1983, pp. 385–394.

[Kad13] H. Kadiri, *A zero density result for the Riemann zeta function*, Acta Arith. 160 (2013), no. 2, 185–200.

[KK06] A. A. Karatsuba and M. A. Korolëv, *The behavior of the argument of the Riemann zeta function on the critical line*, Russian Math. Surveys 61 (2006), no. 3(369), 389–482.

[KLN18] H. Kadiri, A. Lumley, and N. Ng, *Explicit zero density for the Riemann zeta function*, J. Math. Anal. Appl. 465 (2018), no. 1, 22–46.

[KV92] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, De Gruyter Expositions in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 1992.

[Leh56] D. H. Lehmer, *Extended computation of the Riemann zeta-function*, Mathematika 3 (1956), 102–108.
Explicit zero density estimate

[Mat00] K. Matsumoto, Recent developments in the mean square theory of the Riemann zeta and other zeta-functions, Number theory, Trends Math., Birkhäuser, Basel, 2000, pp. 241–286.

[Mot83] Y. Motohashi, A note on the approximate functional equation for $\zeta^2(s)$, Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), no. 8, 393–396.

[Olv74] F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.

[Pla17] D. J. Platt, Isolating some non-trivial zeros of zeta, Math. Comp. 86 (2017), no. 307, 2449–2467.

[Pre84] E. Preissmann, Sur une inégalité de Montgomery-Vaughan, Enseign. Math. (2) 30 (1984), no. 1-2, 95–113.

[PT15] D. J. Platt and T. S. Trudgian, An improved explicit bound on $|\zeta(\frac{1}{2} + it)|$, J. Number Theory 147 (2015), 842–851.

[PT19] D. J. Platt and T. S. Trudgian, The error term in the prime number theorem, preprint available at [arXiv:1809.03134].

[Rad60] H. Rademacher, On the Phragmén-Lindelöf theorem and some applications, Math. Z. 72 (1959/1960), 192–204.

[RS62] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), 64–94.

[Sel46] A. Selberg, Contributions to the theory of the Riemann zeta-function, Arch. Math. Naturvid. 48 (1946), no. 5, 89–155.

[Sie32] C. L. Siegel, Über Riemanns Nachlaß zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik 2 (1932), 45–80.

[Tit35] E. C. Titchmarsh, The zeros of the Riemann zeta-function, Proc. Roy. Soc. London 151 (1935), no. 873, 234–255.

[Tit86] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., The Clarendon Press, Oxford University Press, New York, 1986.

[Tru11] T. Trudgian, Improvements to Turing’s method, Math. Comp. 80 (2011), no. 276, 2259–2279.

[Tru14] T. S. Trudgian, An improved upper bound for the argument of the Riemann zeta-function on the critical line II, J. Number Theory 134 (2014), 280–292.

[Tru16] T. Trudgian, Improvements to Turing’s method II, Rocky Mountain J. Math. 46 (2016), no. 1, 325–332.

[Tur43] A. M. Turing, A method for the calculation of the zeta-function, Proc. London Math. Soc. (2) 48 (1943), 180–197.

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