Traveling Dark Solitons in Superfluid Fermi Gases

Renyuan Liao and Joachim Brand
New Zealand Institute for Advanced Study and Centre for Theoretical Chemistry and Physics, Massey University, Private Bag 102904 NSMC, Auckland 0745, New Zealand
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Families of dark solitons exist in superfluid Fermi gases. The energy-velocity dispersion and number of depleted particles completely determines the dynamics of dark solitons on a slowly-varying background density. For the unitary Fermi gas we determine these relations from general scaling arguments and conservation of local particle number. We find solitons to oscillate sinusoidally at the trap frequency reduced by a factor of $1/\sqrt{3}$. Numerical integration of the time-dependent Bogoliubov-de Gennes equation determines spatial profiles and soliton dispersion relations across the BEC-BCS crossover and proves consistent with the scaling relations at unitarity.

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Dark solitons are elementary nonlinear excitations that play a key role in understanding complex dynamics of superfluids [1,2]. Superfluid Fermi gases have only recently become accessible experimentally and their nonlinear wave dynamics are largely unexplored [3,4]. These systems offer the intriguing possibility to tune between the perturbatively accessible regimes of Bose-Einstein condensation (BEC) of preformed pairs and Bardeen-Cooper-Schrieffer (BCS) superfluidity and a strongly correlated regime of unitarity-limited interactions. While the existence and properties of dark solitons in the BEC regime can be inferred from the solutions of Gross-Pitaevskii (GP) mean-field theory and experiments with atomic BECs, it is an outstanding question what happens outside this regime. So far, only numerical solutions for stationary dark solitons within Bogoliubov-de Gennes (BdG) mean-field theory have been available [5].

In this work, we report theoretical results supporting the existence and detailing the properties of a family of traveling (grey) solitons that are parameterized by their velocity of propagation $v_s$. We are aware of parallel efforts to understand soliton dynamics in trapped Fermi gases [6] and to determine grey soliton profiles [7]. For the unitary gas, we find a closed analytic form of the energy-velocity dispersion relation that is fully determined from a set of general assumptions: (a) Upon adiabatic change of the environment, the soliton can adjust its dynamical state and consistently conserve locally both energy and particle number. (b) Energy and particle number vanish as $v_s$ approaches the speed of sound. (c) The superfluid order parameter has a well defined phase step across the soliton that also vanishes under the conditions of (b).

As a dynamical consequence we are able to predict oscillations of dark solitons in a harmonically trapped Fermi gas. While for BECs, the oscillation frequency was predicted accurately [9] and observed [10,11] to be reduced from the trapping frequency $\omega_t$ by a factor of $1/\sqrt{2}$ $\approx 0.707$, we find the oscillation frequency further reduced across the BEC-BCS crossover. The BdG calculation yields $\omega/\omega_t=0.480$, 0.572, and 0.687 for $\eta = -0.5$ (BCS regime), 0 (unitary) and 1 (BC regime), respectively, where $\eta = 1/(k_F a)$, the Fermi wave number $k_F = (3\pi^2 n)^{1/3}$ parametrizes the density $n$, and $a$ is the s-wave scattering length. Our analytic theory for the unitary case of $\eta = 0$ predicts $\omega/\omega_t = 1/\sqrt{3} \approx 0.577$, which is in excellent agreement with the numerical data.

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Let us consider a superfluid Fermi gas with a soliton that is localized along the z direction in a region small compared to the system length $L$ on a homogeneous background. We can extract the scaling of the system energy with density using the inverse Fermi wave number $k_F^{-1}$ and the Fermi energy $E_F = \hbar^2 k_F^2/(2m)$ as units of length and energy, re-
consider pure one-dimensional motion of the soliton, treating it as a quasiparticle, there is at most only a single independent constant of the motion. The condition that $N_s$ and $E_s$ have identical contours in phase space leads to the condition

$$\frac{\partial N_s}{\partial \mu} \frac{\partial E_s}{\partial v_s} = \frac{\partial N_s}{\partial v_s} \frac{\partial E_s}{\partial \mu},$$  

(3)

where $\mu$ and $v_s$ represent the quasiparticle’s coordinate and momentum, respectively. It follows from Eqs. (1) and (2) that the third derivative of $E$ vanishes identically and that $E(\hat{v}^2)$ can be parameterized by

$$E(\hat{v}^2) = \hat{e}(\hat{v}^2 - \bar{v}^2)^2,$$  

(4)

where $\hat{e}$ and $\bar{v}$ are yet undetermined parameters. The functional form (4) already has important implications for soliton oscillations in a trapped gas:

Requiring $dE_s/dt = 0$, we find the Newtonian equation of motion for the soliton

$$\frac{d\mu}{dz} = -\frac{m(1 + \beta)}{\bar{v}^2} \hat{v}_s = 0.$$  

(5)

In the case of harmonic trapping and under validity of the Thomas Fermi approximation, we can write $\mu(z) = \mu_0 - m\omega_s^2 z^2/2$ and Eq. (5) reduces to a harmonic oscillator. The frequency $\omega/\omega_1 = \hat{v}/\sqrt{1 + \beta}$ is independent of amplitude! The parameter $\hat{v}$ can be determined from assumption (b): From Eq. (4) we find that the energy and particle number vanish when the dimensionless velocity $\hat{v}$ reaches the critical value $\bar{v}$. We expect this to happen at the speed of sound, which takes the value $c = \sqrt{(1 + \beta)/\omega_F}$. This leads to $\bar{v} = \sqrt{(1 + \beta)/\beta}$ and yields the oscillation frequency $\omega/\omega_1 = 1/\sqrt{3}$. We have thus derived the oscillation frequency of a dark soliton in a harmonically trapped unitary gas from the assumptions (a) and (b).

The remaining coefficient $\hat{e}$ can be determined from the relation between the physical momentum of the soliton $p_s = mN_s v_s$ and the canonical momentum $p_{\phi}$, which is defined by $\partial E_s/\partial p_{\phi} = v_s$. The difference between the two quantities accounts for the counterflow that would have to occur in a toroidal system to compensate for the phase difference $\delta \phi$ in the superfluid order parameter across the soliton [12]. For the superfluid Fermi gas the counterflow term was recently found by Pitaevskii [3].

$$p_s - p_e = \hbar n_1 (\pi - \delta \phi)/2,$$  

(6)

where $n_1 = nA = k_F^2 A/(3\pi^2)$ is the one-dimensional density. From Eqs. (1) and (4) we evaluate the difference using $p_e = \int v_s^{-1} \partial E_s/\partial v_s \, dv_s = -\hbar n_1 \delta \phi \hat{v} (\bar{v}^2 - \hat{v}^2)/3$ to yield

$$p_s - p_e = \hbar n_1 \pi \hat{v} \sqrt{\hat{v}^2 - \bar{v}^2}.$$  

(7)

Comparing Eqs. (7) and (6), we find that the phase difference varies linearly with velocity in contrast to the GP soliton, where $\cos(\delta \phi/2) = v_s/\hat{c}^{\phi}$. Fixing the remaining
constant $\hat{e}$ by requiring the phase step to vanish at the speed of sound [assumption (c)], we find

$$\hat{e} = \frac{\hat{v}}{2\pi} + \delta \phi = \pi(1 - v_s/c).$$

(8)

Thus, the energy and particle number dispersion (shown as full lines in Figs. 1b and 2d, respectively) as well as the phase step for the family of dark solitons in the unitary gas (shown in Fig. 2b) are obtained without any free parameters. The success of this derivation shows, that the assumption (a) of particle-number conservation under quasiparticle motion is consistent with the universal scaling relations of the unitary Fermi gas.

We have not yet proven that dark solitons exist. Within the realm of mean-field theory, this can be done by finding self-consistent solutions of the BdG equations. In addition to testing the stated assumptions against a physical theory, this allows us to determine spatial profiles as well as dispersion relations outside the unitary regime.

We now more generally consider a Fermi gas with equal density for two spin components at zero temperature. The time-dependent BdG equations provide a convenient mean-field theory of the BEC-BCS crossover [4]

$$i\hbar \partial_t (v_u(r, t)) = \left( \frac{\hbar}{\Delta} (r, t) \right) (v_u(r, t)),$$

(9)

where $\Delta = \frac{\hbar^2}{2m} \nabla^2 - \mu$ and $u$ and $v$ are space- and time-dependent quasi-particle amplitudes satisfying $\hbar^2 \delta^4 \{ u^*_v(r, t) u_u(r, t) + v^*_u(r, t) v_v(r, t) \} = \delta_{uv}$. The problem simplifies to a time-independent eigenvalue problem when we seek soliton solutions of the superfluid order parameter of the form $\Delta(z, t) = \Delta(z - v_s t) = \Delta(\xi)$ and write $v_u(r, t) = (LA)^{-1/2} \epsilon^{\mu}(p, x + p, y) - i E_{p,n} \hat{v}_p(n)(\xi)$ and likewise for $u$. The energies $E_{p,n}$ are the eigenvalues of the resulting time-independent BdG equation, which contain the soliton velocity $v_s$ as a parameter. The transverse momentum $p$ is discretized according to the transverse area $A$ of the computational box. The above equations must be solved together with the equation for the order parameter $\Delta(\xi) = g \sum_{p,n} u_p(n)(\xi)^* v_{p,n}(\xi)$ in a self-consistent way. The density is then given by $\hat{\rho} = \rho(1 - v_s/c)$.

FIG. 3. (Color online) The spatial structure of the soliton order parameter $\Delta$ at different velocities by its imaginary part (a) and magnitude (b) in the BCS regime at $\eta = -0.5$ (1a,b), unitarity limit $\eta = 0$ (2a,b), and BEC regime $\eta = 1$ (3a,b).

While the length scale for the BEC soliton is $\ell = \hbar/(m \sqrt{\epsilon^2 - v_s^2})$ from GP theory, there is no clear evidence in our data for a velocity dependence of the length scale in the unitarity limit. There we expect on general grounds that the only length scale is $k_F^{-1}$. In the BCS regime, we expect small scale Friedel oscillations with size $2k_F^{-1}$, and a second length scale in the Cooper pair size $\xi_C = h v_F/\Delta_0$, which evaluates to about $5k_F^{-1}$ for $\eta = -0.5$ [5]. From Fig. 3 panel (1a) and our experience with boundary effects, it appears that there is an additional velocity dependence and the total size grows with increasing velocity.

Relevant velocity scales for the problem are the speed of sound $c = \sqrt{n(\partial \mu/\partial n)/m}$ and the pair breaking velocity $mv^2_{sp} = \sqrt{\mu^2 + \Delta_0^2 - \mu}$. We consistently found it difficult to converge to self-consistent solutions approaching these velocities from below and thus assume that soliton solutions exist only below $v_c = \min(c, v_{sp})$, which is also the critical velocity for dissipationless motion of infinitesimal impurities [3, 10]. $v_c$ takes a maximum around unitarity, where $c \approx v_{sp}$. For $\eta \gtrsim 0$ (BCG to unitarity), $c < v_{sp}$ and thus solitons are limited by the speed of sound. In the BCS regime, pair breaking dominates and $v_{sp}$ limits soliton propagation.

The dimensionless soliton energy as a function of the velocity is shown in Fig. 1. In all three regimes the energy is positive with negative curvature, which supports the understanding of a dark soliton as a quasiparticle with negative ef-
effective mass. In the unitarity limit, the numerical data fits beautifully with the analytical result \( \frac{\omega}{\omega_c} = 0.480, 0.572, \) and 0.687 for \( \eta = -0.5, 0, \) and 1 respectively.

The oscillation frequency of solitons thus decreases significantly from the BEC towards the BCS regime in agreement with time-dependent simulations \[6\]. In the unitarity regime we were able to determine the dispersion relation in closed form starting from a small number of global assumptions. In particular we find that the oscillation frequency does not depend on the many-body parameter \( \beta \) in the equation of state or other details of the soliton solutions. Dark solitons thus offer the opportunity to explore complimentary properties of the unitary gas to previous studies that pinpointed the equation of state \[3\]. The important questions of stability of dark solitons against strong quantum fluctuations and the consistency of local energy and particle number conservation deserve to be studied beyond mean-field theory and thus make an excellent subject for future experimental investigation.

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\[\text{FIG. 4. (Color online) The density profile at different velocities for (a) } \eta = -0.5 \text{ (BCS), (b) } \eta = 0 \text{ (unitarity) and (c) } \eta = 1 \text{ (BEC).}\]

\[\text{Note that both } N_s \text{ is a constant of the motion by directly checking Eq. (5), which is fulfilled within our expected numerical errors.}\]

Finally, we consider small amplitude soliton oscillations in a harmonically trapped Fermi gas beyond the unitarity regime. From the Thomas-Fermi and local density approximation we have \( E_s(v_s, \mu_0 - m\omega^2 z_s^2 / 2) = \text{const} \) and taking a time derivative obtain
\[\frac{1}{v_s} \frac{\partial E_s}{\partial z_s} z_s + N_s m \omega^2 z_s = 0. \quad (10)\]

Noting that both \( E_s \) and \( N_s \) should be even functions of velocity, Eq. (10) describes a harmonic oscillator. Defining the effective mass \( M_s = v_s^{-1} \frac{\partial E_s}{\partial z_s} |_{v_s=0} \), we find the frequency of small oscillations
\[\frac{\omega^2}{\omega_c^2} = \sqrt{\frac{m N_s(0)}{M_s}}, \quad (11)\]

which is nicely interpreted as the ratio between the physical mass \( mN_s \) and effective mass \( M_s \). Both \( N_s(0) \) and \( M_s \) are negative and can be easily extracted from our numerical data. We find the oscillation frequencies \( \omega / \omega_c = 0.480, 0.572, \) and 0.687 for \( \eta = -0.5, 0, \) and 1 respectively.