Resonance features of coupled Josephson junctions: radiation and shunting

Yu. M. Shukrinov\textsuperscript{a,b,c}, P. Seidel\textsuperscript{d}, E. Il’ichev\textsuperscript{e}, W. Nawrocki\textsuperscript{f}, M. Grajc\textsuperscript{a} P. A. Plecenik\textsuperscript{g}, I. R. Rahmonov\textsuperscript{a,h}, K. Kulikov\textsuperscript{a,c}

\textsuperscript{a} BLTP, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia
\textsuperscript{b} Max-Planck-Institute for the Physics of Complex Systems, 01187 Dresden, Germany
\textsuperscript{c} Department of Theoretical Physics, International University of Dubna, Dubna, 141980, Russia
\textsuperscript{d} Institut für Festkörperphysik, Jena, D-07743 Jena, Germany
\textsuperscript{e} Institute of Photonic Technology, P.O. Box 100239, D-07702 Jena, Germany
\textsuperscript{f} Poznan University of Technology, Poznan, Poland
\textsuperscript{g} Department of Experimental Physics, Comenius University, Bratislava, Slovakia
\textsuperscript{h} Umarov Physical Technical Institute, TAS, Dushanbe, 734063 Tajikistan

E-mail: shukrinv@theor.jinr.ru

Abstract. We study the phase dynamics and the resonance features of coupled Josephson junctions in layered superconductors and their manifestations in the current-voltage characteristics and temporal dependence of the electric charge in the superconducting layers. Results on the effect of the external radiation and shunting of the stack of Josephson junctions by \textit{LC}-elements are presented. We discuss the ideas concerning the experimental observation of these resonances.

1. Introduction

Strongly anisotropic high-\textit{T}_c superconductor (HTSC) forms a natural stack of intrinsic Josephson junctions (JJ) and shows the intrinsic Josephson effect \cite{1}. A model for describing the physical properties of intrinsic JJs in HTSCs, including nonlinear effects and various nonequilibrium phenomena, is offered by a system of coupled JJs \cite{2, 3}. It should be emphasized that the system of coupled JJs is a promising object in HTSC electronics that has been extensively studied in recent years \cite{4, 5, 6, 7}. In particular, broad possibilities for various applications are offered by the recently discovered coherent electromagnetic radiation of noticeable power in a terahertz frequency range, which is generated by the intrinsic JJ system \cite{8}. An interesting aspect of this phenomenon is that the emission is related to a certain region on the current-voltage characteristic (CVC), namely, that it corresponds to parametric resonance (PR) in the JJ system \cite{9, 10}. Numerical modeling the CVC predicts important properties of the system of coupled JJs and describes its phase dynamics in the branching (hysteresis) region of interest \cite{11}.

2. Model and method

A system of \(N+1\) superconducting layers in an anisotropic HTSC, which is characterized by the order parameter \(\Delta_l(t) = |\Delta| \exp(i\theta_l(t))\) with the time dependent phase \(\theta_l(t)\), comprises \(N\)
Josephson junctions [1]. Figure 1(a) shows a schematic diagram of this layered system.

Superconducting layers, which are numbered by \( l \) running from 0 to \( N \) and characterized by the time-dependent order parameters with moduli \( \Delta_l \) and phases \( \theta_l \), form a system of JJs with phase differences \( \varphi_l = \theta_l - \theta_{l-1} \); \( d_s \) and \( d \) are the thicknesses of the superconducting and dielectric layers, respectively.

**Figure 1.** (a) Schematic diagram of a system of intrinsic JJs in HTSCs (see text); (b) Total CVC of the stack with 10 JJs under radiation at \( \omega_R = 2 \) and amplitude \( A = 0.5 \). The part of the outermost branch of CVC marked by circle shown enlarged in (c) together with charge-time dependence, demonstrating the “bump” structure in CVC and charging of \( S \)-layers above Shapiro step.

The thickness of superconducting layers (about 3 Å) in an HTSC is comparable with the Debye length \( r_D \) of electric charge screening. Therefore, there is no complete screening of the charge in the separate layers, and the electric field induced in each JJ penetrates into the adjacent junctions. Thus, the electric neutrality of superconducting layers is dynamically broken and, in the case of the alternating current (ac) Josephson effect, a capacitive coupling appears between the adjacent junctions [12]. The absence of complete screening of charge in the superconducting layer leads to the formation of a generalized scalar potential \( \Phi_l \) of the layer, which is defined in terms of the scalar potential \( \phi_l \) and the derivative of phase \( \theta_l \) of the superconducting order parameter as follows: \( \Phi_l(t) = \phi_l - V_0 \frac{d\phi_l}{dt} \), where \( V_0 = h\omega_p/(2e) \), \( \omega_p = \sqrt{2eI_c/\hbar C_j} \) is the plasma frequency, \( I_c \) is the critical current, and \( C_j \) is the capacitance of the JJ. The generalized scalar potential is related to the charge density \( Q_l \) on the superconducting layer \( Q_l = -\frac{1}{4\pi r_D^2} \Phi_l \) [12, 13].

The existence of a relationship between the electric charge \( Q_l \) of the \( l \)th layer and the generalized scalar potential \( \Phi_l \) of this layer reflects a nonequilibrium nature of the ac Josephson effect in layered HTSCs [13].

When an external current passes through the stack of coupled Josephson junctions, the system appears under nonequilibrium conditions [13]. The Josephson relation is generalized in this case. Also the diffusion contribution to the quasiparticle current arises due to the generalized scalar potential difference

\[
I_{\text{diff}} = \frac{\Phi_l - \Phi_{l-1}}{I_s R_j} = \frac{(Q_l - Q_{l-1})}{4\pi r_D^2 I_s R_j} = \frac{(Q_l - Q_{l-1})}{2e^2 N(0) I_s R_j} = \beta \dot{\phi}_l - \beta V_l = -\alpha \beta (V_{l+1} + V_{l-1} - 2V_l) \tag{1}
\]

where \( R_j \) is the resistance of the Josephson junction in the resistive state and \( N(0) \) denotes the density of states on the Fermi level. The structure of the CVC in the capacitively coupled Josephson junction model with diffusion current (CCJJ+DC model) is equidistant [14] in agreement with the experiments [15]. In Ref. [16] was demonstrated that the diffusion current plays an important role in the formation of the CVC, particularly, determining the width of the hysteresis region.
To investigate the phase dynamics of an intrinsic JJ we use the one-dimensional CCJJ+DC model with the gauge-invariant phase differences $\varphi_l(t)$ between S-layers $l$ and $l+1$ in the presence of electromagnetic irradiation described by the system of equations:

$$\begin{cases}
\frac{\partial \varphi_l}{\partial t} = V_l - \alpha(V_{l+1} + V_{l-1} - 2V_l) \\
\frac{\partial V_l}{\partial t} = I - \sin \varphi_l - \beta \frac{\partial \varphi_l}{\partial t} + A \sin \omega_R t + I_{\text{noise}}
\end{cases}$$

(2)

where $t$ is dimensionless time, normalized to the inverse plasma frequency $\omega_p^{-1}$, $\beta = \frac{1}{R_j} \sqrt{\frac{\hbar}{2eIC_j}} = \frac{1}{\sqrt{\beta_c}}$, $\beta_c$ is McCumber parameter, $\alpha$ gives the coupling between junctions [12], $A$ is the amplitude of the radiation. To find the CVC of the stack of intrinsic JJs we solve the system of nonlinear second-order differential equations (1) using the fourth order Runge-Kutta method. In our simulations we measure the voltage in units of $V_0$, the frequency in units of $\omega_p$, the bias current $I$ and the amplitude of radiation $A$ in units of $I_c$. We note that different kind of couplings between junctions, like inductive coupling in the presence of magnetic field [17], capacitive [12, 18], charge-imbalance [19] and phonon [20, 21] couplings determine a variety of CVCs observed in HTSC. The influence of these couplings on the parametric resonance in the system is still an open problem.

The important information concerning the resonance features of intrinsic JJ in HTSC can be obtained by detailed investigation of the charge dynamics of superconducting layers. To study the time dependence of the electric charge in the S-layers, we use the Maxwell equation $\text{div}(\varepsilon \varepsilon_0 E) = Q$, where $\varepsilon$ and $\varepsilon_0$ are relative dielectric and electric constants, respectively. The charge density $Q_l$ in the S-layer $l$ is proportional to the difference between the voltages $V_l$ and $V_{l+1}$ in the neighbor insulating layers $Q_l = Q_0\alpha(V_{l+1} - V_l)$, where $Q_0 = \varepsilon \varepsilon_0 V_0/\sqrt{2}\beta$. The details of the simulation procedure are presented in Refs. [10, 22].

3. Results and discussions

An explanation of coherent radiation from the stack of intrinsic JJ in BSCCO at zero magnetic field [8] is one of the important unsolved problem today. Association of the emission with conventional Fiske steps [23, 24, 25] is problematic because their amplitude is zero at zero magnetic field [26]. Different mechanisms of emission are discussed in literature [23, 26, 27]. A question if the parametric resonance play any role in the radiation physics was not investigated yet.

In this paper we discuss the resonance features of coupled Josephson junctions related to the effects of the external radiation and shunting the system of junctions by $LC$-elements. First, we demonstrate the effect of microwave’s amplitude variation on the CVC of this system and time dependence (temporal oscillations) of the electric charge in the superconducting layers.

In Fig. 1(b) we show the results of simulation of the total CVC for the stack with 10 JJ under external radiation with frequency $\omega_R = 2$ and amplitude $A = 0.5$. The simulation was done at coupling parameter $\alpha = 1$ and the dissipation parameter $\beta = 0.2$. The CVC demonstrates the first Shapiro step on each branch. Results are in qualitative agreement with the experimental ones presented in Ref. [28]. Circle marks a part of the outermost branch demonstrating an interesting behavior of the above steps. Its enlarged part shown in Fig. 1(c).

To clarify this feature, we study a charging of superconducting layers in this interval of bias current. The charge-time dependence together with CVC are shown in Fig. 1(c), too. We see that an irradiation changes the character of the charge-time dependence essentially and brings about a “bump” structure on the outermost branch of CVC, as shown in Fig. 1(c). So, we observe here an additional parametric resonance. We call this resonance as radiation related
The results of simulations at smaller amplitude show some “excess” current in CVC in this region [30, 31]. The “bump” structure and “excess current” structure were recently observed experimentally [4, 15, 32, 33]. We stress the importance of these features and necessity of the additional experimental and theoretical studies of “excess current” and “bump” structures. The question if the charging of the $S$-layers appears at other types of resonances in coupled JJs [24, 25, 26] has not yet been investigated.

Another important problem on the way of using the intrinsic Josephson junctions in HTSC as terahertz electromagnetic waves sources [34] is a synchronization of all junctions in a stack to increase a power of radiation. Intensive attempts to solve this problem are based on using $LC$-shunting which leads to such synchronization [34, 35, 36].

Let us consider the system, presented in Fig. 2(a). In normalized units the system of equations, describing this electric scheme, can be written in the form [37]

$$
\begin{align*}
\frac{\partial \varphi_l}{\partial t} &= V_l - \alpha(V_{l+1} + V_{l-1} - 2V_l) \\
\frac{\partial V_l}{\partial t} &= I - \sin \varphi_l - \beta \frac{\partial \varphi_l}{\partial t} - C \frac{\partial u_c}{\partial t} \\
\frac{\partial^2 u_c}{\partial t^2} &= \frac{1}{LC} \left( \sum_{l=1}^{N} V_l - u_c \right)
\end{align*}
$$

Here $u_c$ is the voltage at the capacitance, which is normalized to $V_0$. The bias current $I$ is normalized to the critical current $I_c$ of JJ, time - to the inverse plasma frequency $\omega_p$; shunt capacitance $C_{sh}$ - to the capacitance of the JJ $C_j$ and denote as $C$, and shunt inductance $L_{sh}$ - to $(C_j \omega_p^2)^{-1}$ and denote as $L$.

![Figure 2. (a) Schema of the JJs stack with $LC$ shunting elements; (b) CVC of the single JJ at different values of shunting parameters $L$ and $C$.](image)

Now we study the phase dynamics of this system based on the model (3) and discuss the influence of LC shunting on CVC and parametric resonance in the system. We note that the stack of JJs together with $LC$-elements form a resonance circuit with its eigenfrequency

$$
\omega_{rc} = \sqrt{\frac{1 + NC}{LC}}
$$

which depends on the number of junctions in the stack $N$. To simplify analysis we consider first a case of single junction instead of stack of JJ shown in Fig. 1(a). We see that the $LC$ shunting leads to a step-like structure in the oneloop CVC, when the value of Josephson frequency approaches to the eigenfrequency of the $LC$ resonance circuit [38, 39]. The location of the step depends on parameters of this $LC$ circuit. For hysteretic JJ these steps are $rc$-branches, and they might be obtained completely by changing sweeping current direction after jump on it in the current decreasing procedure [37].
In Fig. 2 we present CVC of a single JJ shunted by the LC-circuit at different values of L and C. The simulation was done in the framework of the RCSJ model, $\beta = 0.2$. It means that the calculations were done by Eqs. (3), but at $\alpha = 1$ and $N = 1$. When Josephson frequency $\omega_J$ coincides with the resonance circuit frequency $\omega_{rc}$, the CVC demonstrates a step in CVC at corresponding voltage value. We see such step at $\omega_{rc} = 3.0$ in Fig. 2(a). An increase in $\omega_{rc}$ brings this step up on the voltage scale. We see that in Fig. 2(b). Here we observe a jump to this branch at $I = I_c$ as well. So, this $rc$-branch is separated for two parts in one loop CVC simulation procedure. Then, in some frequency interval we observe a jump at $I = I_c$ to the total $rc$-branch, as shown at $\omega_{rc} = 4.899$ in Fig. 2c.

Let us now consider the influence of the external electromagnetic radiation on the CVC of the investigated system. In the pioneering work of Werthamer and Shapiro [40] was shown that a JJ in a cavity will show additional self-induced resonant steps which lead to subharmonics and some additional structures also in combination with those steps induced by an external microwave frequency. In some cases besides subharmonics also chaotic dynamics was observed [41, 42, 43]. The situation becomes much more complex if an external microwave is taking into account and can result in situations where the chaotic dynamics can be controlled and synchronization of JJ can be improved [44, 45, 46]. Thus we restrict ourself to the question: what would happen with the CVC at double resonance conditions: $\omega_J = \omega_R = \omega_{rc}$. To make it clear we show first CVC of coupled JJs under radiation, but without shunting. In Fig. 3a we present such CVC of ten coupled JJ without shunting under external radiation with frequency $\omega_R = 3.045$ and amplitude $A = 0.5$. As we see, this CVC has a Shapiro step at $V = 30.45$, indicating the external frequency $\omega_R = V/10 = 3.045$. Fig. 3b enlarges this step.

Figure 3. Demonstration of the effect of radiation on shunted coupled JJ at double resonance conditions: (a, b) CVC under radiation, but without shunting; (c, d) CVC with shunting by $L = 55$ and $C = 0.002$, but without radiation; (e) CVC at both effects simultaneously.
Then we show in Fig. 3c the CVC of ten coupled JJ with shunting by $L = 55$ and $C = 0.002$, but without external radiation. At this parameters the eigenfrequency of the resonance circuit according to the formula $\omega_{rc} = \sqrt{\frac{1+NC}{LC}}$ is equal to 3.045, so CVC has a corresponding $rc$-branch. Fig. 3d enlarges this $rc$-branch. Finally we demonstrate both effects simultaneously. In Fig. 3e we see a crucial changes in CVC when shunting and radiation are both taken into account. Resonance of Josephson and circuit oscillations triggers the appearance of the Shapiro step harmonics and subharmonics. At this resonance the width of Shapiro steps and its harmonics are sharply increased because of the changes in the CVC induced by the $rc$-branch structure. An interesting question concerns the influence of parametric resonance on this feature which will be investigated separately [47].

Figure 4. (a) Schema of parallel connections of two stacks with $N_1$ and $N_2$ coupled JJ; CVC for the different cases: (b) $N_1 = 1$, $N_2 = 1$; (c) $N_1 = 1$, $N_2 = 2$; (d) $N_1 = 1$, $N_2 = 3$; (e) $N_1 = 1$, $N_2 = 9$; (f) $N_1 = 1$, $N_2 = 10$.

For some applications it is interesting to consider the circuits which consist of two stacks of coupled JJ in parallel connection shown in Fig. 4(a). In the simplest case $N1=N2=1$ this is the well-known superconducting quantum interference device (DC-SQUID). For DC-SQUIDs the appearance of resonance features corresponding to the inductance $L$ and capacitance $C$ of the circuit like separate $rc$-branches were reported [48, 49, 50]. There is an additional strong influence of the external magnetic field leading to so-called “beating branches”. Two junction SQUIDs can be extended to symmetric as well as asymmetric multi-junction loops, see [35, 51, 52] and references therein. Such circuits have a very complex dynamics. Multi-junction SQUIDs based on intrinsic Josephson junctions were investigated e.g. by [53, 54, 55]. In Ref. [55] Shapiro steps in a DC-SQUID with multiple identical junctions in each arm were investigated.

Here we shortly discuss the cases $N_1 = 1$ with different $N_2$ ($N_2=1, 2, 3, 9, 10$). System of equations which describes the phase dynamics in this circuit can be written in the form
\[
\begin{align*}
\dot{\psi} &= U \\
\dot{\phi}_l &= V_l - \alpha (V_{l+1} + V_{l-1} - 2V_l) \\
\dot{V}_l &= \sum_{k=1}^{N_2} B_{lk}^{-1} \left( I - I_c \sin \psi - \frac{\beta}{R} U - \sin \phi_l - \beta \dot{\phi}_l \right) \\
U &= \sum_{l=1}^{N_2} V_l
\end{align*}
\]

where $\psi$ and $U$ is the phase difference and voltage in the single JJ, relatively. We use the same notations for JJ in the stack and notations with index 1 for single JJ: $C_1$-capacitance, $I_{c1}$-critical current, $R_1$-resistance for single JJ, and $C_{st1}$-capacitance, $I_{c, st1}$-critical current, $R_{st1}$-resistance for JJ in the stack.

We introduced dimensionless parameters $C = \frac{C_1}{C_j}$, $I_c = \frac{I_{c1}}{I_c}$ and $R = \frac{R_1}{R_j}$. Matrix $B$ is determined by

\[
\hat{B} = \begin{pmatrix}
C + 1 & C & \cdots & C \\
C & C + 1 & \cdots & C \\
\vdots & \vdots & \ddots & \vdots \\
C & \cdots & C & C + 1
\end{pmatrix}
\]

We solve the system of nonlinear second-order differential equations (5) using the fourth order Runge-Kutta method. Results for the following cases (b) $N_2 = 1$; (c) $N_2 = 2$; (d) $N_2 = 3$; (e) $N_2 = 9$; (f) $N_2 = 10$. We see strengthening of branching with increase in $N_2$, see Fig. 4(c) to (f). Detailed analysis of these CVC will be published elsewhere.

4. Summary

We studied the resonance features of coupled Josephson junctions and demonstrated a strong effect of the external radiation on the CVC in the parametric resonance region. A crucial changes in CVC found at the double resonance condition when radiation frequency coincides with Josephson and resonance circuit frequencies.

5. Acknowledgments

The work was supported by Heisenberg-Landau and Bogoliubov-Infeld Programs, Collaboration between JINR, Dubna and Slovakia, by the Slovak Research and Development Agency under the contract No. APVV-0494-11, Russian Fund for Basic Research (RFBR) under grant 12-02-90901-mob-sng-st., and JINR under grant 12-302-03. E.I. acknowledges a support by the German Ministry for Education and Science under Grant No. 13N9307 (project THz-Videocam).

6. References

[1] Kleiner R, Steinmeyer F, Kunkel G and Müller P 1992 Phys. Rev. Lett. 68 2394
[2] Krasnov V M 2011, Phys. Rev. B 83 174517
[3] Kurter C, Zhuravel A P, Ustinov A V and Anlage S M 2011 Phys. Rev. B 83 104515
[4] Benseman T M, Koshelev A E, Gray K E, Kwok W -K, Welp U, Kadowaki K, Tachiki M and Yamamoto T 2011 Phys. Rev. B 84 064523
[5] Koshelev A E 2010 Phys. Rev. B 82 174512
[6] Pfeiffer J, Abdumalikov Jr A A, Schuster M and Ustinov A V 2008 Phys. Rev. B 77 024511
[7] Yurgens A A 2000 Supercond. Sci. Technol. 13 R85
[8] Ozyuzer L et al. 2007 Science 318 1291
[9] Shukrinov Yu M and Mahfouzi F 2007 Phys. Rev. Lett. 98 157001
[10] Shukrinov Yu M, Mahfouzi F and Suzuki M 2008 Phys. Rev. B 78 134521
[11] Shukrinov Yu M and Rahmanov I R 2010 JETP Lett. 92 327; Pis'ma v ZhETF 92 364
[12] Koyama T and Tachiki M 1996 Phys. Rev. B 54 16183
[13] Ryndyk D A 1998 Phys. Rev. Lett. 80 3376
[14] Shukrinov Yu M, Mahfouzi F and Seidel P 2006 Physica C 449 62
[15] Irie A, Shukrinov Yu M and Oya G 2008 Appl. Phys. Lett. 93 157001
[16] Shukrinov Yu M, Mahfouzi F and Suzuki M 2008 Phys. Rev. B 78 134521
[17] Machida M, Koyama T and Tachiki M 1999 Phys. Rev. Lett. 83 4816
[18] Helm Ch, Preis Ch, Schmalzl K, Keller J, Kleiner R and Muller P 2001 Physica C 362 51
[19] Shukrinov Yu M, Nasrulaev Kh, Sargolzaei M, Oya G and Irie A 2002 Supercond. Sci. Technol. 15 178
[20] Darula M, Seidel P, Busse F and Benacka S 1993 J. Appl. Phys. Lett. 65 1618
[21] Shirahata N, Sengupta D C and Edoh K D 2001 IEEE Trans Circuits Systems 48 990
[22] Dana S K, Roy P K, Sethia G C, Sen A and Sengupta D C 2006 IEEE Proc.-CircuitsDevices Syst. 153 453
[23] Shukrinov Yu M, Seidel P, Illich E, Rahmanov I R and Kulikov K unpublished
[24] Voss R F, Laibowitz R B, Broers A N, Raeder S I, Knoedler C M and Viggiano J M 1981 IEEE Trans. Magnetics 17 395
[25] Schmidt W -D, Seidel P and Heinemann S 1985 Phys. Stat. Sol. (a) 91 K155
[26] Grib A N, Seidel P, Busse F and Benacka S 1993 J. Appl. Phys. 74 2674
[27] Irie A and Oya G 2005 IEEE Trans Appl Supercond 15 813
[28] De Luca R and Romeo F 2005 J Appl Phys 98 073904