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M.S. Hashemi, M.C. Nucci

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Nonclassical Symmetries for a Class of Reaction-Diffusion Equations: 
the Method of Heir-Equations

M.S. Hashemi
Department of Mathematics
Imam Khomeini International University
Ghazvin 34149, Iran
hashemi@math396@yahoo.com

M.C. Nucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia & INFN Sezione di Perugia
06123 Perugia, Italy
nucci@unipg.it

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The nonclassical symmetries method is applied to a class of reaction-diffusion equations with nonlinear source, i.e. $u_t = u_{xx} + cu_x + R(u,x)$. Several cases are obtained by using suitable solutions of the heir-equations as described in [M.C. Nucci, Nonclassical symmetries as special solutions of heir-equations, J. Math. Anal. Appl. 279 (2003) 168–179].

Keywords: reaction-diffusion equations; Nonclassical symmetries; heir-equations

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1. Introduction

In a recent paper [5] a class of reaction-diffusion equations, i.e.

$$u_t = u_{xx} + cu_x + R(u,x),$$

with $R(u,x)$ arbitrary function of $u$ and $x$, was introduced as a model that incorporates climate shift, population dynamics, and migration for a population of individuals $u(t,x)$ that reproduce, disperse, and die within a patch of favorable habitat surrounded by unfavorable habitat. It is assumed that due to a shifting climate, the patch moves with a fixed speed $c > 0$ in a one-dimensional universe.

Motivated by this study here we look for nonclassical symmetries of equation (1.1) with the purpose of finding explicit expressions of the function $R(u,x)$ and deriving nonclassical symmetry solutions when feasible.

Nonclassical symmetries were introduced in 1969 in a seminal paper by Bluman and Cole [8]. After twenty years and few occasional papers, e.g. [50], [9], in the early Nineties there was a sudden spur of interest and several papers began to appear, e.g. [34], [20], [41], [40], [48], [52], [36], [26],

$^a$Actually equation (1.1) corresponds to the original model

$$u_t = u_{zz} + R(u, z - ct)$$

rewritten in terms of a moving coordinate system with $x = z - ct$ [5].
Since then the nonclassical symmetry method has been applied to various equations and systems in hundreds of published papers, e.g. [27], [37], [16], [28], [17], [54], [13], [11], [12], [15], [51], [4], the latest being [14], [31], [55], [10].

One should be aware that some authors call nonclassical symmetries as Q-conditional symmetries of the second type, e.g. [13], while others call them reduction operators, e.g. [51].

The nonclassical symmetry method can be viewed as a particular instance of the more general differential constraint method that, as stated by Kruglikov [33], dates back at least to the time of Lagrange... and was introduced into practice by Yanenko [57]. The method was set forth in details in Yanenko’s monograph [53] that was not published until after his death [21]. A more recent account and generalization of Yanenko’s work can be found in [39].

Among the papers dedicated to the application of the nonclassical symmetry method to diffusion-convection equation with source, we single out [11] where some nonclassical symmetries solutions were determined for the equation:

\[ u_t = u_{xx} + k(x)u^2(1-u). \]  

(1.2)

In particular nonclassical symmetries of the type \( V(t,x)\partial_x + \partial_t \) were found in the following three instances:

(i) \( k(x) = a^2 x^2 \),  
(ii) \( k(x) = a^2 \tanh^2 x \),  
(iii) \( k(x) = a^2 \tan^2 x \),

(1.3)

with \( a \) arbitrary constant.

In the next Section we recall the concept of heir-equations [43] and their link to nonclassical symmetries [46]. In Section 3 the nonclassical symmetries of equation (1.1) are reported, along with the corresponding reductions and solutions. The last Section contains some final remarks.

2. Heir-equations and nonclassical symmetries

Let us consider an evolution equation in two independent variables and one dependent variable of second order:

\[ u_t = H(t,x,u,u_x,u_{xx}) \]  

(2.1)

If

\[ \Gamma = V_1(t,x,u)\partial_t + V_2(t,x,u)\partial_x - F(t,x,u)\partial_u \]  

(2.2)

is a generator of a Lie point symmetry\(^d\) of equation (2.1) then the invariant surface condition is given by:

\[ V_1(t,x,u)u_t + V_2(t,x,u)u_x = F(t,x,u). \]  

(2.3)

\(^b\)Namely papers published within the first half of 2012.  
\(^c\)In [25] this name was introduced for the first time.  
\(^d\)The minus sign in front of \( F(t,x,u) \) was put there for the sake of simplicity: it could be replaced with a plus sign without affecting the following results.
Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (2.3) becomes:

$$u_x = G(t,x,u)$$  \hspace{1cm} (2.4)

Then, an equation for $G$ is easily obtained. We call this equation $G$-equation [42]. Its invariant surface condition is given by:

$$\xi_1(t,x,u,G)G_t + \xi_2(t,x,u,G)G_x + \xi_3(t,x,u,G)G_u = \eta(t,x,u,G)$$  \hspace{1cm} (2.5)

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (2.5) becomes:

$$G_x + GG_u = \eta(t,x,u,G)$$  \hspace{1cm} (2.6)

Then, an equation for $\eta$ is derived. We call this equation $\eta$-equation. Clearly:

$$G_x + GG_u \equiv u_{xx} \equiv \eta$$  \hspace{1cm} (2.7)

We could keep iterating to obtain the $\Omega$-equation, which corresponds to:

$$\eta_x + G\eta_u + \eta G_u \equiv u_{xxx} \equiv \Omega(t,x,u,G,\eta)$$  \hspace{1cm} (2.8)

the $\rho$-equation, which corresponds to:

$$\Omega_x + G\Omega_u + \eta \Omega_G + \Omega \eta_u \equiv u_{xxxx} \equiv \rho(t,x,u,G,\eta,\Omega)$$  \hspace{1cm} (2.9)

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the right prolongation: first prolongation for the $G$-equation, second prolongation for the $\eta$-equation, and so on. Therefore, these equations were named heir-equations in [43]. This implies that even in the case of few Lie point symmetries many more Lie symmetry reductions can be performed by using the invariant Lie point solution of any of the possible heir-equations, as it was shown in [43], [1], [38].

We recall that the heir-equations are just some of the many possible $n$-extended equations as defined by Guthrie in [30].

In [43] it was shown that this iterating method yields both partial symmetries as given by Vorobev in [56], and differential constraints as given by Olver [49].

Fokas and Liu [23] and Zhdanov [58] independently introduced the method of generalised conditional symmetries, i.e., conditional Lie-Bäcklund symmetries. In [44] it was shown that the heir-equations can retrieve all the conditional Lie-Bäcklund symmetries found by Zhdanov.

In [29] Goard has shown that Nucci’s method of constructing heir equations by iterating the nonclassical symmetries method is equivalent to the generalised conditional symmetries method.

The difficulty in applying the method of nonclassical symmetries consists in solving nonlinear determining equations in contrast with the linearity of the determining equations in the case of classical symmetries.

The concept of Gröbner basis has been used [19] for this purpose.

In [46] it was shown that one can find the nonclassical symmetries of any evolution equations of any order by using a suitable heir-equation and searching for a given particular solution among all its solutions, thus avoiding any complicated calculations. We recall the method as applicable to equation (2.1).

\*\*\*\*\*

\*We have replaced $F(t,x,u)$ with $G(t,x,u)$ in order to avoid any ambiguity in the following discussion.
We derive \( u_t \) from (2.1) and replace it into (2.3), with the condition \( V_1 = 1 \), i.e.:

\[
H(t, x, u, u_x, u_{xx}) + V_2(t, x, u)u_x = F(t, x, u)
\]  

(2.10)

Then, we generate the \( \eta \)-equation with \( \eta = \eta(x, t, u, G) \), and replace \( u_x = G, \ u_{xx} = \eta \) into (2.10), i.e.:

\[
H(t, x, u, G, \eta) = F(t, x, u) - V_2(t, x, u)G
\]  

(2.11)

For Dini’s theorem, we can isolate \( \eta \) in (2.11), e.g.:

\[
\eta = [h_1(t, x, u, G) + F(t, x, u) - V_2(t, x, u)G]h_2(t, x, u, G)
\]  

(2.12)

where \( h_i(t, x, u, G) (i = 1, 2) \) are known functions. Thus, we have obtained a particular solution of \( \eta \) which must yield an identity if replaced into the \( \eta \)-equation. The only unknowns are \( V_2 = V_2(t, x, u) \) and \( F = F(t, x, u) \). If any such solution is singular, i.e. does not form a group then we have found the nonclassical symmetries, otherwise one obtains the classical symmetries [46].

More recently in [7] Bilă and Niesen presented another method that reduces the partial differential equation (PDE) to an ordinary differential equation by using the invariant surface condition and then applies the Lie classical symmetry method in order to find nonclassical symmetries of the original PDE. We hope that an independent researcher will take up the task of comparing the two methods as it was done by Goard in [29] since we conjecture that Bilă and Niesen’s method, and its extension, as given in [12], are equivalent to Nucci’s method [46].

3. Nonclassical symmetries of (1.1)

We use a simple MAPLE program to derive the heir-equations. In particular the \( G \)-equation of (1.1) is:

\[
G_t + RG_u - G_{xx} - 2GG_{uu} - G^2G_{uu} - cG_x - R_uG - R_x = 0.
\]  

(3.1)

and the \( \eta \)-equation is

\[
\eta_t + R\eta_u + R_uG\eta_G - \eta_{xx} - 2G\eta_{uu} - 2\eta\eta_xG - G^2\eta_{uu} - 2G\eta\eta_{uG} - \eta^2\eta_{GG} - c\eta_x - R_{uu}G^2 - R_u\eta - 2GR_{uu} + R_xG - R_{xx} = 0.
\]  

(3.2)

The particular solution of the \( \eta \)-equation that we are looking for is

\[
\eta(t, x, u, G) = -R(u, x) - cG + F(t, x, u) - V_2(t, x, u)G,
\]  

(3.3)

that replaced into (3.2) yields an overdetermined system in the unknowns \( F, V_2 \) and \( R(u, x) \). Since we obtain a polynomial of third degree in \( G \) then we let MAPLE evaluate the four coefficients that we call \( d_i, \ i = 0, 1, 2, 3 \) where \( i \) stands for the corresponding power of \( u \). We impose all of them to
be zero. From $d_3$, we obtain
\[ V_2(t,x,u) = ss_1(t,x)u + ss_2(t,x), \quad (3.4) \]
while $d_2$ yields
\[ F(t,x,u) = -\frac{1}{3} ss_1^2 u^3 + \frac{1}{2} \left( \frac{\partial ss_1}{\partial x} - 2css_1 - 2ss_1 ss_2 \right) + ss_3(t,x)u + ss_4(t,x), \quad (3.5) \]
with $ss_j(t,x), \ j = 1,\ldots,4$ arbitrary functions of $t$ and $x$. Then after differentiating $d[1]$ four times with respect to $u$ we obtain
\[ \frac{\partial^4 R(u,x)}{\partial u^4} = 0, \quad (3.6) \]
which implies that $R(u,x)$ must be a polynomial in $u$ of third degree at most, i.e.
\[ R(u,x) = -\frac{a_3^2(x)}{6} u^3 + \frac{a_2(x)}{2} u^2 + a_1(x)u + a_0(x), \quad (3.7) \]
where $a_i(x), \ i = 0,1,2,3$ are arbitrary functions of $x$. Since none of the remaining arbitrary functions depends on $u$, and $d_1$ has now become a polynomial of degree 3 in $u$, we have to annihilate all the four coefficients, i.e. $d_{1,i}, \ i = 0,1,2,3$. From $d_{1,3}$ we have that $ss_1(t,x)$ must be a constant, and two cases raise:

**Case 1.** $ss_1 = \pm \frac{\sqrt{3}}{2} a_3(x)$,

**Case 2.** $ss_1 = 0$.

We discuss the two cases\(^1\), separately. We remark that $a_3(x) = 0$ corresponds to a subcase of Case 2., and consequently in Case 1. we assume $a_3(x) \neq 0$.

Interestingly enough in Case 2. nonclassical symmetries exist for
\[ R(u,x) = f(u) k^2(x), \quad (3.8) \]
with $f(u)$ any arbitrary function of $u$, and $k(x)$ either of the following three particular functions of $x$, i.e.,
\[ k(x) = -\frac{cx + 2}{2x}, \quad k(x) = \frac{c}{e^{c(b_1-x)} - 1}, \quad k(x) = \frac{1}{b_1} \tan \left( \frac{x+b_2}{b_1} \right) - \frac{c}{2}. \quad (3.9) \]

\(^1\)In Case 1., one can choose either the plus or minus sign indifferently.
3.1. **Case I.** \( R(u,x) = -\frac{a_3'(x)}{6}u^3 + \frac{a_2'(x)}{2}u^2 + a_1'(x)u + a_0(x) \)

From coefficients \(d_{1,2}, d_{1,1}, d_{1,0}\) we obtain \( ss_2, ss_3, \) and \( ss_4, \) respectively. All of them are function of \( x \) only, e.g.

\[
s s_2 = -\frac{1}{2a_3'(x)} \left( -4a_3'(x) + \sqrt{3}a_2(x) + 2ca_3(x) \right),
\]

where ' denotes differentiation with respect to \( x \). Now the only remaining coefficient is \( d_0 \) which has become a linear polynomial in \( u \). Therefore we are left with two expressions to annihilate, namely the following underdetermined system of two equations that contain the derivative of \( a_1'(x) \) up to fifth order, and fourth order, respectively, and lower derivatives of the other three functions \( a_2'(x), a_1'(x), \) and \( a_0(x) \)

\[
-\alpha_3 a_3''(iv) - 4\alpha_2 a_2'a_1 - c^2 a_1^2 a_2' - 5\alpha_2^2 a_2 a_1' + 3\alpha_2 a_2 a_1'' + a_2^3 \sqrt{3}a_2'' - 36a_3 a_2' d_3''
\]

\[
-2a_2 a_3'' + 8\alpha_2 a_2 a_2'' + a_2^3 \sqrt{3}a_2 - 2\alpha_3 a_2 a_1' - 18a_3 a_2' \sqrt{3}a_2 + ca_3 a_1' + 4\alpha_2 a_3'' a_1'
\]

\[
+\alpha_3 \sqrt{3} a_2' - 5\alpha_3 a_2' \sqrt{3}a_2' - 6a_3 a_2' \sqrt{3}a_2' - 2a_2 \sqrt{3}a_3 a_2'' + 13\alpha_2 a_2' + a_2^3 \sqrt{3}a_2'
\]

\[
+12a_3 a_2' c + c^2 a_1 a_2^2 + 4a_1 a_2' a_2' + 3ca_2' \sqrt{3}a_2' - a_2^3 a_3 a_2' - 3a_3 a_2' \sqrt{3}a_2'
\]

\[
+24a_3'' - 14a_3 a_2 a_2' + a_2 a_2'' a_2' + 3a_2'' a_2' = 0,
\]

\[
\frac{6}{\sqrt{3}} a_3 d_3'' + 7c\sqrt{3} a_3 d_3 a_2' - c a_0 d_3 - 3a_2 a_0 + 192\sqrt{3}a_3 a_2'' a_3''
\]

\[
+2c\sqrt{3} a_3 a_2'' a_2 + 264a_3 a_2' + 5a_3 a_3'' a_2 + 104\sqrt{3} a_2 a_2'' a_2'' - 18\sqrt{3} a_3 a_2'' a_3''
\]

\[
+2c^2 \sqrt{3} a_3 a_2'' a_2'' + 12c a_3 a_2'' a_2'' - 26c\sqrt{3} a_3 a_2'' a_2'' + 19c a_3 a_2'' a_2'' + 27ca_3 a_2'' a_3''
\]

\[
+6ca_3 a_2'' a_2'' + c^2 a_2^2 a_2'' + 4a_1 a_2''' a_2' - 4a_1 a_2 a_2'' a_2'' + 3a_3 a_2'' a_2''
\]

\[
+2a_2 a_2'' a_2'' a_2' - 6a_3 a_2'' a_2' a_2 + 464\sqrt{3} a_3 a_2'' a_2'' + 10\sqrt{3} a_3 a_2'' a_2'' + 178\sqrt{3} a_3 a_2'' a_3''
\]

\[
+22a_2 a_2'' a_2'' a_2' - 6a_3 a_2'' a_2' a_2 + 464\sqrt{3} a_3 a_2'' a_2'' + 10\sqrt{3} a_3 a_2'' a_2'' + 178\sqrt{3} a_3 a_2'' a_3''
\]

\[
+22a_2 a_2'' a_2'' a_2' - 6a_3 a_2'' a_2' a_2 + 464\sqrt{3} a_3 a_2'' a_2'' + 10\sqrt{3} a_3 a_2'' a_2'' + 178\sqrt{3} a_3 a_2'' a_3''
\]

\[
+22a_2 a_2'' a_2'' a_2' - 6a_3 a_2'' a_2' a_2 + 464\sqrt{3} a_3 a_2'' a_2'' + 10\sqrt{3} a_3 a_2'' a_2'' + 178\sqrt{3} a_3 a_2'' a_3''
\]

Since this system has infinite solutions we look for some particular solutions.
\[ R(u, x) = -\frac{1}{2}x^2u^3 + 3u^2 + \frac{1}{2}c^2u \]

If we assume \( a_3(x) = \sqrt{3}x \), and \( a_2(x), a_1(x), a_0(x) \) to be constants then from system (3.11)-(3.12) we obtain that

\[ R(u, x) = -\frac{1}{2}x^2u^3 + 3u^2 + \frac{1}{2}c^2u, \tag{3.13} \]

and

\[ ss_1(t, x) = \frac{3x}{2}, \quad ss_2(t, x) = -\frac{1 + cx}{x}, \quad ss_3(t, x) = c - \frac{2 + 3cx}{4x}, \quad ss_4(t, x) = 0. \tag{3.14} \]

Thus, (3.3) becomes

\[ \eta = \frac{x^3u^3 + 2cu - c^2ux + 6x^2uG - 4G}{4x}, \tag{3.15} \]

namely

\[ u_{xx} = -\frac{x^3u^3 + 2cu - c^2ux + 6x^2uux - 4ux}{4x}, \tag{3.16} \]

that can be solved in closed form, i.e.

\[ u(t, x) = \frac{c^2R_2(t)e^{\frac{2tu}{2}} - c^2(1 + cx)e^{-\frac{2tu}{2}}}{R_1(t) + (cx - 2)R_2(t)e^{\frac{2tu}{2}} + (10 + 5cx + c^2x^2)e^{-\frac{2tu}{2}}}, \tag{3.17} \]

with \( R_k(t), k = 1, 2 \) arbitrary functions of \( t \). Substituting (3.17) into (1.1) yields the following non-classical symmetry solution

\[ u(t, x) = \frac{c^2c_1e^{\frac{2tu}{2}} - c^2(1 + cx)e^{-\frac{2tu}{2}}}{c_2e^{\frac{2tu}{2}} + c_1(cx - 2)e^{\frac{2tu}{2}} + (10 + 5cx + c^2x^2)e^{-\frac{2tu}{2}}}, \tag{3.18} \]

with \( c_k, k = 1, 2 \) arbitrary constants. We observe that

\[ \lim_{t \to \infty} u(t, x) = \frac{c^2}{cx - 2}, \quad \lim_{x \to \pm \infty} u(t, x) = 0 \tag{3.19} \]

and that \( u(t, x) < 0 \) for \( t > 0, x < 0 \). This means that the solution (3.18) is not defined at \( x = 2/c \) and is positive\(^8\) if \( x \geq 0 \).

\[^8\text{It depends also on the values given to the arbitrary constants.}\]
3.1.2. \( R(u,x) = -\frac{1}{2}e^{\varepsilon x}u^3 + \frac{c^2}{4}u + e^{\frac{\eta}{2}} \)

If we assume \( a_3(x) = \sqrt{3}e^{\varepsilon x}, a_2(x) = 0 \), and \( a_1(x) = b_1, a_0(x) = b_0 \), i.e. constants, then from system (3.11)-(3.12) we obtain that

\[
R(u,x) = -\frac{1}{2}e^{\varepsilon x}u^3 + b_1u + b_0e^{\frac{\eta}{2}}, \quad [b_1,b_0 = \text{const.}]
\]

and thus \( \eta \) becomes

\[
\eta = -\frac{1}{8}\left(2e^{\varepsilon x}u^3 + (3c^2 - 4b_1)u + 8cG + 6e^{\frac{\eta}{2}}cu^2 + 12e^{\frac{\eta}{2}}uG - 4b_0e^{-\frac{\eta}{4}}\right),
\]

namely

\[
u_{xx} = -\frac{1}{8}\left(2e^{\varepsilon x}u^3 + (3c^2 - 4b_1)u + 8cu_x + 6e^{\frac{\eta}{2}}cu^2 + 12e^{\frac{\eta}{2}}uu_x - 4b_0e^{-\frac{\eta}{4}}\right),
\]

that can be solved in closed form, although the solution is very lengthy. If we assume

\[
b_1 = \frac{c^2}{4}, \quad b_0 = 1,
\]

then the solution of (3.22) becomes:

\[
u(t,x) = \frac{\sqrt{2}}{2}\left(R_1(t)e^{\frac{\eta}{4}x} - R_2(t)e^{-\frac{\eta}{4}x}\sin \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right) + e^{\frac{\eta}{4}x}\cos \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right)e^{\frac{\eta}{4}}\right)
\times
\]

\[
\times \left[2R_1(t)e^{\frac{\eta}{4}x} + R_2(t)e^{-\frac{\eta}{4}x}\left(\sin \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right) - \sqrt{3}\cos \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right)\right)\right]
\]

\[
- e^{-\frac{\eta}{4}x}\left(\sqrt{3}\sin \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right) + \cos \left(\frac{\sqrt{2}\sqrt{3x}}{4}\right)\right)\right]
\]

which if replaced into (1.1) yields

\[
R_1(t) = 0, \quad R_2(t) = -\tan \left(\frac{3\sqrt{3}\sqrt{4}}{8}(t + c_1)\right).
\]

This solution oscillates between negative and positive values. Consequently it is not a valid solution for the biological model set in [5]. However, equation

\[
u_t = \nu_{xx} + cu_x - \frac{1}{2}e^{\varepsilon x}u^3 + \frac{c^2}{4}u + e^{\frac{\eta}{2}}
\]

maybe of interest for other biological or physical models.
3.1.3. \( R(u, x) = \frac{-u^3}{6} - \frac{\sqrt{3}u^2}{x} + \frac{c^2u}{6} + \frac{\sqrt{3}c^2}{3x} \)

If we assume \( a_3(x) = b_3, a_1(x) = b_1 \) and substitute them into system (3.11)-(3.12), after some further simplifications such \( b_1 = c^2 / 6 \) and having to impose that \(^b b_3 = -1 \), then we obtain

\[
R(u, x) = \frac{-u^3}{6} - \frac{\sqrt{3}u^2}{x} + \frac{c^2u}{6} + \frac{\sqrt{3}c^2}{3x}.
\]  

(3.27)

Thus (3.3) becomes

\[
\eta = \frac{36xG - 36u + 6\sqrt{3}Gu_2 - u^3x^2 - 6\sqrt{3}u^2x + c^2ux^2 + 12\sqrt{3}c + 2\sqrt{3}c^2x}{12x^2},
\]

namely

\[
u_{xx} = \frac{36xu_2 - 36u + 6\sqrt{3}uu_2 - u^3x^2 - 6\sqrt{3}u^2x + c^2ux^2 + 12\sqrt{3}c + 2\sqrt{3}c^2x}{12x^2}.
\]

(3.29)

If we assume \( c = 0 \) then we find that its solution is

\[
u(t, x) = \frac{2\sqrt{3}(4R_2(t)x^3 + 2x)}{R_1(t) + R_2(t)x^4 + x^2}\]

(3.30)

that substituted into (1.1) yields the following solution

\[
u(t, x) = \frac{4\sqrt{3}(2c^2 + c_1 + 12t)}{6c_1t + 36t^2 - c_2 - x^2 - x^2c_1 - 12tx^2}.
\]

(3.31)

Although this solution is not valid for the biological problem set in [5] since \( c = 0 \), we are reporting it since equation

\[
u_t = u_{xx} + \frac{u^2}{6x}(xu + 6\sqrt{3})
\]

(3.32)

may be of interest for other biological or physical problems. We observe that solution (3.31) is such that

\[
\lim_{t \to \infty} u(t, x) = 0, \quad \lim_{x \to \pm \infty} u(t, x) = 0
\]

(3.33)

Moreover (3.31) is not defined for the following set of values of \( x \) and \( t \):

\[
\begin{align*}
&\{ x = \frac{1}{2} \sqrt{-2c_1 - 24t + 2 \sqrt{c_1^2 + 48c_1t + 288t^2 - 4c_2}, \forall t} \}, \\
&\{ x = -\frac{1}{2} \sqrt{-2c_1 - 24t + 2 \sqrt{c_1^2 + 48c_1t + 288t^2 - 4c_2}, \forall t} \}, \quad (3.34) \\
&\{ x = \frac{1}{2} \sqrt{-2c_1 - 24t - 2 \sqrt{c_1^2 + 48c_1t + 288t^2 - 4c_2}, \forall t} \}, \quad (3.35) \\
&\{ x = -\frac{1}{2} \sqrt{-2c_1 - 24t - 2 \sqrt{c_1^2 + 48c_1t + 288t^2 - 4c_2}, \forall t} \}. \quad (3.36)
\end{align*}
\]

(3.37)

\(^b\)It is also possible to have \( b_3 = 1 \) although it leads to very lengthy calculations.
3.1.4. \( R(u, x) = -u^3 + 6 \frac{u}{x^2} + 6 \frac{\sqrt{2}}{x^3}, \ [c = 0] \)

If we impose \( a_3(x) = b_3, a_2(x) = 0, \) and \( c = 0, \) then we obtain

\[
R(u, x) = -u^3 + 6 \frac{u}{x^2} + 6 \frac{\sqrt{2}}{x^3}. \tag{3.38}
\]

Thus (3.3) becomes

\[
\eta = -\frac{6\sqrt{2} - 6xu + 3\sqrt{2}x^3 uG + x^3 u^3}{2x^3}, \tag{3.39}
\]

namely

\[
u_{xx} = -\frac{6\sqrt{2} - 6xu + 3\sqrt{2}x^3 uu_x + x^3 u^3}{2x^3}. \tag{3.40}
\]

We find its solution, i.e.

\[
u(t, x) = \frac{\sqrt{2}(-R_1(t) + 3R_2(t)x^4 + x^2)}{x(R_1(t) + R_2(t)x^4 + x^2)} \tag{3.41}
\]

that substituted into (1.1) yields the following solution

\[
u(t, x) = -\frac{3\sqrt{2}(12c_2^2 + 24c_2 + 4c_1 + 12c_1 + 12c_1 + 12c_1 + 36t^2 - 12tx^2 + x^4)}{x(36c_2 + 72c_2t - 12c_2x^2 + 12c_1 + 36t^2 - 12tx^2 + x^4)}. \tag{3.42}
\]

Although this solution is not valid for the biological problem set in [5] since \( c = 0, \) we report it here because equation

\[
u_t = u_{xx} - u^3 + 6 \frac{u}{x^2} + 6 \frac{\sqrt{2}}{x^3} \tag{3.43}
\]

maybe of interest for other biological or physical problems. We observe that solution (3.42) is such that

\[
\lim_{t \to \infty} u(t, x) = -\frac{\sqrt{2}}{x}, \quad \lim_{x \to \pm\infty} u(t, x) = 0 \quad \text{(3.44)}
\]
and that \(u(t,x) < 0\) for \(t > 0, x < 0\). Moreover (3.42) is not defined for the following set of values of \(x\) and \(t\):

\[
\begin{align*}
\{ & x = 0, & (3.45) \\
& x = \sqrt{6t - 6c_2 + 2\sqrt{18t^2 + 36tc_2 + 18c_2^2 + 3c_1}}, \forall t, & (3.46) \\
& x = -\sqrt{6t - 6c_2 + 2\sqrt{18t^2 + 36tc_2 + 18c_2^2 + 3c_1}}, \forall t, & (3.47) \\
& x = \sqrt{6t - 6c_2 - 2\sqrt{18t^2 + 36tc_2 + 18c_2^2 + 3c_1}}, \forall t, & (3.48) \\
& x = -\sqrt{6t - 6c_2 - 2\sqrt{18t^2 + 36tc_2 + 18c_2^2 + 3c_1}}, \forall t. & (3.49)
\end{align*}
\]

\[
\begin{align*}
& \forall x, t = \frac{1}{6}x^2 - c_2 + \frac{1}{6}\sqrt{2x^4 - 12c_1}, & (3.50) \\
& \forall x, t = \frac{1}{6}x^2 - c_2 - \frac{1}{6}\sqrt{2x^4 - 12c_1}. & (3.51)
\end{align*}
\]

3.2. **Case 2.** \(R(u,x) = \frac{f(u)}{k^2(x)}\)

If we assume \(ss_1 = 0\) then \(V_2(t,x,u) = ss_2(t,x)\) and \(d_0\) yields that

\[
R(u,x) = \frac{f(u)}{k^2(x)}, \quad ss_2 = k(x). \quad (3.52)
\]

Following [11] we impose \(F(t,x,u) = 0\), thus the annihilation of \(d_1\) imposes that \(k(x)\) is either one of the three particular functions of \(x\) in (3.9) and their nonclassical symmetry operators are

\[
\partial_t - \frac{cx + 2}{2x} \partial_x \quad \partial_t + \frac{c}{e^{t(b_0-x)}} - 1 \partial_x, \quad \partial_t + \left( \frac{1}{b_1} \tan \left( \frac{x + b_2}{b_1} \right) - \frac{c}{2} \right) \partial_x, \quad (3.53)
\]

respectively. In each case we can solve the corresponding invariant surface condition (2.3) and reduce the original diffusion equation (1.1) to an ordinary differential equation that involves an arbitrary function of the unknown due to the arbitrariness of \(f(u)\). We consider some instances where \(f(u)\) has a given expression in order to derive the nonclassical symmetry solution of equation (1.1).

3.2.1. \(k(x) = -\frac{cx + 2}{2x}\)

We solve the invariant surface equation (2.3), i.e.

\[
u_t - \frac{cx + 2}{2x} u_x = 0 \quad (3.54)
\]

and derive its complete solution as

\[
u(t,x) = H(\xi), \quad \xi = \frac{4\log(cx + 2) - 2cx - c^2t}{c^2} \quad (3.55)
\]
where $H(\xi)$ is an arbitrary function of $\xi$. After substituting this solution into equation (1.1), i.e.

$$u_t = u_{xx} + cu_x + \frac{4x^2}{(cx+2)}f(u),$$

we obtain the following ordinary differential equation

$$4\frac{d^2 H}{d\xi^2} - c^2 \frac{dH}{d\xi} + 4f(H) = 0.$$  (3.57)

Let us consider $f(u) = 1/u$. In this instance equation (3.57) admits a two-dimensional nonabelian transitive Lie point symmetry algebra generated by

$$\partial_\xi, \quad e^{\frac{c^2}{4} \xi^4} \left(4\partial_\xi + c^2 H \partial_H\right),$$

and equation (3.57) can be integrated by quadrature. In fact taking a canonical representation of the generators of the two-dimensional Lie point symmetry algebra, i.e.

$$4e^{\frac{c^2}{4} \xi^4} \left(4\partial_\xi + c^2 H \partial_H\right), \quad -\frac{4}{c^2} \partial_\xi + 4e^{\frac{c^2}{4} \xi^4} \left(4\partial_\xi + c^2 H \partial_H\right),$$

we can derive the corresponding canonical variables, i.e.

$$\tilde{\xi} = He^{\frac{c^2}{4} \xi^4}, \quad \tilde{H} = 1 + e^{\frac{c^2}{4} \xi^4} \left(-\frac{1}{4c^2} + H\right).$$

(3.60)

These variables transform equation (3.57) into its canonical form, i.e.

$$\frac{d^2 \tilde{H}}{d\tilde{\xi}^2} = \frac{1}{\tilde{\xi}} \left(256 \left(\frac{d\tilde{H}}{d\tilde{\xi}}\right)^3 - 3 \left(\frac{d\tilde{H}}{d\tilde{\xi}}\right)^2 + 3 \frac{d\tilde{H}}{d\tilde{\xi}} - 1\right)$$

(3.61)

that can be solved by two quadratures and thus its general solution is

$$\tilde{H} = \tilde{\xi} + c_2 \pm 16 \int \frac{d\tilde{\xi}}{\sqrt{2c_1 - 2\log(\tilde{\xi})}}.$$  (3.62)

Unfortunately the last integral cannot be expressed in finite terms.

If we assume $c = 0$ then (3.56) becomes

$$u_t = u_{xx} + x^2 f(u).$$  (3.63)

In [11] the same nonclassical symmetry was determined if $f(u) = u^2(1 - u) - (i) \, \text{in}(1.3) -$. We found that this is true for any $f(u)$.

---

1We recall that the classification of real two-dimensional Lie symmetry algebra and derivation of corresponding canonical variables were done by Lie himself [35], retold in Bianchi’s 1918-textbook [6] and also in more recent textbooks, e.g. [32].
3.2.2. \( k(x) = \frac{c}{e^{c(b_0-x)} - 1} \)

We solve the invariant surface equation (2.3), i.e.

\[
\frac{c}{e^{c(b_0-x)} - 1} u_t + c e^{c(b_0-x)} - 1 u_x = 0
\]  

(3.64)

and derive its complete solution as

\[
u(t, x) = H(\gamma), \quad \gamma = -\frac{c x + c^2 t + e^{c(b_0-x)}}{c^2}, \]

(3.65)

where \( H(\gamma) \) is an arbitrary function of \( \gamma \). After substituting this solution into equation (1.1), i.e.

\[
u_t = u_{xx} + cu_x + \frac{(e^{c(b_0-x)} - 1)^2}{c^2} f(u),
\]

(3.66)

we obtain the following ordinary differential equation

\[
\frac{d^2 H}{d \gamma^2} + f(H) = 0.
\]

(3.67)

Its general solution in implicit form is

\[
\pm \int \frac{dH}{\sqrt{c_1 - 2 f(H) dH}} - \gamma - c_2 = 0.
\]

(3.68)

Let us consider some instances:

(A) \( f(u) = u^2 \implies u(t, x) = -6\text{WeierstrassP}(\gamma + c_1, 0, c_2), \)

where \( \text{WeierstrassP} \) represents Weierstrass elliptic function.

(B) \( f(u) = u^3 \implies u(t, x) = c_2 \text{JacobiSN} \left( \left( \frac{\gamma}{\sqrt{2}} + c_1 \right) c_2, \sqrt{-1} \right) \)

where \( \text{JacobiSN}(z, k) = \sin(\text{JacobiAM}(z, k)) \) and \( \text{JacobiAM} \) represents the Jacobi amplitude function am.

(C) \( f(u) = u^2(1-u) \implies \int \frac{6dH}{\sqrt{18H^4 - 24H^3 + 36c_1}} - \gamma - c_2 = 0 \)

Let us consider two particular values of \( c_1 \).

If we assume \( c_1 = \frac{1}{6} \) then we obtain

\[
u(t, x) = \frac{(1-H)\sqrt{18H^2 + 12H + 6}}{\sqrt{18H^4 - 24H^3 + 36c_1}} \text{arctanh} \left( \frac{2(1+2H)}{\sqrt{18H^2 + 12H + 6}} \right) - \gamma - c_2 = 0,
\]

although still an implicit solution of (3.66);

instead \( c_1 = 0 \) yields

\[
u(t, x) = -\frac{12}{4c_2 \gamma + 2c_2^2 + 2\gamma^2 - 9,}
\]
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a nonclassical symmetry solution of equation (3.66), i.e.

\[
  u_t = u_{xx} + cu_x + \frac{(e^{c(b_0-x)} - 1)^2}{c^2} u^2 (1-u). \tag{3.69}
\]

This solution tends to zero when \( t \) (or \( x \)) goes to infinity. Also it blows up in finite time if

\[
  t = \frac{1}{2c^2} \left( 2c_2c^2 - 2c_2x - 2e^{c(b_0-x)} \pm c^2 \sqrt{18} \right), \quad \forall x.
\]

3.2.3. \( k(x) = \frac{1}{b_1} \tan \left( \frac{x+b_2}{b_1} \right) - \frac{c}{2} \)

We solve the invariant surface equation (2.3), i.e.

\[
  u_t + \left( \frac{1}{b_1} \tan \left( \frac{x+b_2}{b_1} \right) - \frac{c}{2} \right) u_x = 0 \tag{3.70}
\]

and derive its complete solution as

\[
  u(t,x) = H(\rho), \quad \rho = t + \frac{2b_1}{4 + b_1^2 c^2} \left[ c(x+b_2) + \log \left( 1 + \tan^2 \left( \frac{x+b_2}{b_1} \right) \right) \\
  - 2\log \left( 2\tan \left( \frac{x+b_2}{b_1} \right) - b_1c \right) \right], \tag{3.71}
\]

where \( H(\rho) \) is an arbitrary function of \( \rho \). After substituting this solution into equation (1.1), i.e.

\[
  u_t = u_{xx} + c u_x + \frac{4b_1^2}{\left( 2\tan \left( \frac{x+b_2}{b_1} \right) - b_1c \right)^2} f(u), \tag{3.72}
\]

we obtain the following ordinary differential equation

\[
  4b_1^2 \frac{d^2H}{d\rho^2} + (4 + b_1^2 c^2) \frac{dH}{d\rho} + 4b_1^2 f(H) = 0. \tag{3.73}
\]

If \( f(u) = u \), namely if equation (3.72) is linear then we obtain that the general solution is

\[
  u(t,x) = c_1 e^{-\frac{4+b_1^2c^2 - \rho \sqrt{(b_1^2 c^2 + 4)^2 - 64b_1^4}}{8b_1^2}} + c_2 e^{-\frac{4+b_1^2c^2 + \rho \sqrt{(b_1^2 c^2 + 4)^2 - 64b_1^4}}{8b_1^2}}. \tag{3.74}
\]

4. Final remarks

The application of the nonclassical symmetry method to equation (1.1) yields different possible expressions of \( R(u,x) \) and in several instances even a class of solutions in finite form. These solutions may not be all suitable for the problem set in [5] since they may take negative values, and also have singularities for finite values of \( t \) and \( x \).
However, we have found that nonclassical symmetries exist for the following reaction-diffusion equations, i.e.

\[ u_t = u_{xx} + cu_x - \frac{a_3^2(x)}{6} u^3 + \frac{a_2(x)}{2} u^2 + a_1(x) u + a_0(x), \]  

(4.1)

with \( a_0, a_1, a_2, a_3 \) satisfying system (3.11)-(3.12). Particular instances have been obtained, i.e.

\[ u_t = u_{xx} + cu_x - \frac{1}{2} x^2 u^3 + 3u^2 + \frac{1}{2} c^2 u, \]  

(4.2)

\[ u_t = u_{xx} + cu_x - \frac{1}{2} e^{c^2} u^3 + \frac{c^2}{4} u + e^{\frac{c^2}{2}}, \]  

(4.3)

\[ u_t = u_{xx} + cu_x - \frac{u^3}{6} - \frac{\sqrt{3} u^2}{x} + \frac{c^2 u}{6} + \frac{\sqrt{3} c^2}{3x}, \]  

(4.4)

\[ u_t = u_{xx} - u^3 + \frac{6 u}{x^2} + \frac{6 \sqrt{2}}{x^3}. \]  

(4.5)

Moreover we have found that nonclassical symmetries exist for the following three families in the class of equation (1.1), i.e.

\[ u_t = u_{xx} + cu_x + \frac{4 x^2}{(cx+2)} f(u), \quad \forall f(u) \]  

(4.6)

\[ u_t = u_{xx} + cu_x + \frac{e^{c(b_0-x)} - 1}{c^2} f(u), \quad \forall f(u) \]  

(4.7)

\[ u_t = u_{xx} + cu_x + \frac{4 b_1^2}{(2 \tan \left( \frac{x+b_1}{b_1} \right) - b_1 c)} f(u), \quad \forall f(u). \]  

(4.8)

We conclude by observing that the method of heir-equations [46] very much facilitates the search for nonclassical symmetries.

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