From Rindler space to the electromagnetic energy-momentum tensor of a Casimir apparatus in a weak gravitational field

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This paper studies two perfectly conducting parallel plates in the weak gravitational field on the surface of the Earth. Since the appropriate line element, to first order in the constant gravity acceleration \( g \), is precisely of the Rindler type, we can exploit the formalism for studying Feynman Green functions in Rindler spacetime. Our analysis does not reduce the electromagnetic potential to the transverse part before quantization. It is instead fully covariant and well suited for obtaining all components of the regularized and renormalized energy-momentum tensor to arbitrary order in the gravity acceleration \( g \). The general structure of the calculation is therefore elucidated, and the components of the Maxwell energy-momentum tensor are evaluated up to second order in \( g \), improving a previous analysis by the authors and correcting their old first-order formula for the Casimir energy.

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I. INTRODUCTION

Although quantum field theory in curved spacetime is an hybrid framework, being based on the coupling of the classical Einstein tensor to a quantum concept like the vacuum expectation value of the regularized and renormalized energy-momentum tensor, it has led to exciting developments over many years \cite{1,2,3}. In particular, the theoretical discovery by Hawking of particle creation by black holes \cite{4} is a peculiar phenomenon of quantum field theory in curved spacetime, and all modern theories of quantum gravity face the task of evaluating and understanding black hole entropy and the ultimate fate of black holes.

Since the chief goal of quantum field theory in curved spacetime may be regarded as being the evaluation of the energy-momentum tensor on the right-hand side of the semiclassical Einstein equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \langle T_{\mu\nu} \rangle, \tag{1.1}
\]

it is very important even nowadays to look at problems where new physics (at least in principle) can be learned or tested while using Eq. (1.1). In particular, we are here concerned with a problem actively investigated over the last few years, i.e. the behavior of rigid Casimir cavities in a weak gravitational field \cite{5,6,7,8,9,10,11,12,13}. An intriguing theoretical prediction is then found to emerge, according to which Casimir energy obeys exactly the equivalence principle \cite{11,12,13}, and the Casimir apparatus should experience a tiny push (rather than being attracted) in the upwards direction. The formula for the push has been obtained in three different ways, i.e.,

(i) A heuristic summation over modes \cite{7};

(ii) A variational approach \cite{11};

(iii) An energy-momentum analysis \cite{10}.

While all approaches now agree about the push and the magnitude of the effect \cite{12}, the work in Ref. \cite{10}, despite its explicit analytic formulas for \( (T_{\mu\nu}) \), led to legitimate puzzlements, being accompanied by a theoretical prediction of nonvanishing trace anomaly. It has been therefore our aim to perform a more careful investigation of the energy-momentum tensor of our rigid Casimir apparatus. The nonvanishing trace will be shown to result from a calculational mistake, and a better understanding of the general calculation, possibly to all orders in \( g \), will be gained. Sections II, III and IV describe basic material on Rindler coordinates, scalar and photon Green functions. Ward identities are checked explicitly in Sec. V, while the various parts of the energy-momentum tensor are analyzed in Secs. VI and VII. Concluding remarks are presented in Sec. VIII, and relevant details are given in the appendices.

II. RINDLER COORDINATES

We work in natural units, in which \( \hbar = c = 1 \). In these units, the gravity acceleration has dimensions of an inverse length. Neglecting tidal forces, the weak gravitational field on the surface of the Earth is described by the line element

\[
ds^2 = -(1 + 2g z) dt^2 + dz^2 + dx_{\perp}^2, \tag{2.1}
\]

where \( g \) is the gravity acceleration, and \( x_{\perp} \equiv (x, y) \). We consider an ideal Casimir apparatus, consisting of two perfectly reflecting mirrors lying in the horizontal plane, and separated by an empty gap of width \( a \). We let the origin of the \( z \) coordinate coincide with the lower mirror, in such a way that the mirrors have coordinates \( z = 0 \) and \( z = a \), respectively. To first order in the small
quantity $g_\perp$, the line element in Eq. (2.1) coincides with the Rindler metric

$$ds^2 = -\left(\frac{\xi}{\xi_1}\right)^2 dt^2 + d\xi^2 + dx_\perp^2, \quad (2.2)$$

where

$$\xi \equiv \frac{1}{g} + z \equiv \xi_1 + z. \quad (2.3)$$

In the Rindler coordinates, the plates are located at

$$\xi_1 \equiv \frac{1}{g}, \quad \xi_2 \equiv \xi_1 + a. \quad (2.4)$$

The time coordinate $t$ in Eq. (2.2) therefore represents the proper time for an observer comoving with the mirror at $\xi_1$. In what follows, it shall be often convenient to work out exact formulas for a Casimir apparatus in the Rindler gravitational field, and to recover the corresponding formulas for the weak field in Eq. (2.1) by taking the large $\xi_1$ limit of the Rindler results. In

$$G^{(D/N)}(x, x') = \xi_1 \int \frac{d\omega}{2\pi} \exp[-i\omega(t-t')] \int \frac{d^2k}{(2\pi)^2} \exp[i\mathbf{k} \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)] \chi^{(D/N)}(\xi, \xi'|iv, k), \quad (3.4)$$

where $\mathbf{k} \equiv (k_x, k_y)$, $k \equiv \sqrt{k_x^2 + k_y^2}$ and $\nu \equiv \xi_1 \omega$. The functions $\chi^{(D/N)}(\xi, \xi'|iv, k)$ satisfy the equation

$$\left[\frac{d}{d\xi} \left(\xi \frac{d}{d\xi}\right) + (\nu^2 - \xi^2 k^2)\right] \chi^{(D/N)}(\xi, \xi'|iv, k) = -\xi \delta(\xi - \xi'), \quad (3.5)$$

together with the boundary conditions

$$\chi^{(D)}(\xi_1, \xi'|iv, k) = \frac{d}{d\xi} \chi^{(N)}(\xi_1, \xi'|iv, k) = 0. \quad (3.6)$$

The functions $\chi^{(D/N)}(\xi, \xi'|iv, k)$ can be expressed in terms of the modified Bessel functions of imaginary order $I_\nu(k\xi)$ and $K_\nu(k\xi)$ that are solutions of the homogeneous equation corresponding to Eq. (3.3). We define the function

$$W_\nu(u, v) \equiv K_\nu(u)I_\nu(v) - I_\nu(u)K_\nu(v). \quad (3.7)$$

Thus we have

$$\chi^{(D)}(\xi, \xi'|iv, k) = -\frac{W_\nu(k\xi, k\xi_2)W_\nu(k\xi_1, k\xi_1)}{W_\nu(k\xi_1, k\xi_2)}, \quad (3.8)$$

and

$$G^{(0)}(x, x') = \frac{i\xi_1}{\pi} \int \frac{d\omega}{2\pi} \exp[-i\omega(t-t')] \int \frac{d^2k}{(2\pi)^2} \exp[i\mathbf{k} \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)] K_\nu(k\xi_2)K_\nu(e^{i\pi}k\xi_1), \quad (3.13)$$

III. SCALAR GREEN FUNCTIONS

We consider the Green functions $G^{(D)}(x, x')$ and $G^{(N)}(x, x')$ for a massless scalar field propagating in the Rindler metric and satisfying Dirichlet and Neumann boundary conditions, respectively, i.e.

$$G^{(D/N)}(x, x') = -g^{-1/2} \delta(x, x'), \quad (3.1)$$

$$G^{(D)}(x, x')|_{z=\xi_i} = 0 \quad i = 1, 2 \quad (3.2)$$

$$\partial_z G^{(N)}(x, x')|_{z=\xi_i} = 0 \quad i = 1, 2 \quad (3.3)$$

By virtue of translation invariance in the $t, x, y$ directions, they can be written as

$$\chi^{(D)}(\xi, \xi'|iv, k) = -\frac{(\partial_\nu W_\nu)(k\xi_2, k\xi_1)(\partial_\nu W_\nu)(k\xi_1, k\xi_1)}{(\partial_\nu \partial_\nu W_\nu)(k\xi_1, k\xi_1)}, \quad (3.9)$$

where $\xi_2 \equiv \max(\xi, \xi')$ and $\xi_1 \equiv \min(\xi, \xi')$. By using the identities

$$K_{\nu'}(e^{i\pi}\xi) = e^{-i\pi} K_{\nu'}(\xi) - i\pi I_{\nu'}(\xi), \quad (3.10)$$

$$K'_{\nu'}(e^{i\pi}\xi) = -e^{-i\pi} K'_{\nu'}(\xi) + i\pi I'_{\nu'}(\xi), \quad (3.11)$$

with $'$ denoting differentiation, to eliminate $I_{\nu'}$ and $I'_{\nu'}$ from Eqs. (3.8) and (3.9), the propagators can be expressed in the following form:

$$G^{(D/N)}(x, x') = G^{(0)}(x, x') + \tilde{G}^{(D/N)}(x, x'), \quad (3.12)$$

where $G^{(0)}(x, x')$ is the Feynman propagator for a massless scalar field in Minkowski space time [14], [15].
\[ \tilde{G}^{(D/N)}(x, x') = \xi_1 \int \frac{d\omega}{2\pi} \exp[-i\omega(t - t')] \int \frac{d^2k}{(2\pi)^2} \exp[i k \cdot (x_\perp - x'_\perp)] \tilde{\chi}^{(D/N)}(\xi, \xi' | iv, k), \] (3.14)

where
\[ \tilde{\chi}^{(D)} = \frac{i}{\pi} \frac{A^{(D)}(\xi, \xi' | iv, k)}{K_{iv}(e^{i\pi k_1})K_{iv}(k_2)} - K_{iv}(e^{i\pi k_1})K_{iv}(e^{i\pi k_2}) \]

(3.15)

and
\[ \tilde{\chi}^{(N)} = \frac{i}{\pi} \frac{A^{(N)}(\xi, \xi' | iv, k)}{K_{iv}(k_1)K_{iv}(e^{i\pi k_2}) - K_{iv}'(e^{i\pi k_1})K_{iv}'(k_2)} \]

(3.17)

As is clear from Eqs. (3.16) and (3.18), the quantities \( \tilde{\chi}^{(D/N)}(\xi, \xi') \) are symmetric functions of \( \xi \) and \( \xi' \), and are both regular at \( \xi = \xi' \). The integrands for \( \tilde{G}^{(D/N)}(x, x') \) in Eq. (3.14) have simple poles at the zeros of the quantities that occur in the denominators of the expressions for \( \tilde{\chi}^{(D/N)} \), Eqs. (3.15) and (3.17). These zeros are all located on the real \( \nu \) axis. The Feynman propagator is obtained by deforming the contour for the \( \nu \)-integration, in such a way that it passes below the poles on the negative \( \nu \)-axis and above those on the positive \( \nu \)-axis.

**IV. THE PHOTON PROPAGATOR**

We quantize the classical solutions of the field equations
\[ \nabla_\mu \nabla^\mu A_\nu(x) = 0, \quad \xi_1 \leq \xi \leq \xi_2 \] (4.1)

which are obtained on choosing the Lorenz gauge \[16\], subject to the boundary conditions
\[ A_r(\xi_1) = A_j(\xi_1) = 0, \quad \partial_\xi(\xi A_\xi)(\xi_1) = 0. \] (4.2)

Equation (4.2) expresses the mixed boundary conditions on the potential corresponding to the choice of perfect conductor boundary conditions \[17\]. They are preserved under gauge transformations provided that the Faddeev–Popov ghost fields \( \chi \) and \( \psi \) obey homogeneous Dirichlet conditions on the plates \[17\]. The modes are normalized via the following Klein–Gordon inner product:
\[ (w, v) = i \int d^2x \int \xi_1 d\xi \frac{\xi_1}{\xi} w^{\ast} \nabla_\nu v_{\nu}. \] (4.3)

Note that the above inner product is not positive definite, by virtue of the Lorentz signature of the metric. A convenient basis of gauge fields \( A_\nu \) can be obtained in terms of the normalized modes for the Dirichlet and Neumann scalar problems, \( \phi^{(D)}_{r \nu}(x) \) and \( \phi^{(N)}_{r \nu}(x) \) respectively:
\[ \phi^{(D/N)}_{r \nu}(x) = \exp[-i \omega^{(D/N)}]t + i k \cdot x_\perp \] \( \phi^{(D/N)}_{r \nu}(\xi) \) (4.4)

These modes obey the differential equation (cf. Eq. (3.5))
\[ \left[ \frac{\xi}{\xi_1} \frac{d}{d\xi} \left( \frac{\xi}{\xi_1} \frac{d}{d\xi} \right) + \left( \frac{\xi}{\xi_1} \right)^2 \right] \tilde{\phi}^{(D/N)}_{r \nu}(\xi) = 0, \] (4.5)

the boundary conditions
\[ \tilde{\phi}^{(D)}_{r \nu}(\xi_1) = \frac{d}{d\xi} \tilde{\phi}^{(N)}_{r \nu}(\xi_1) = 0, \] (4.6)

and the orthogonality relation
\[ \int d^2x \int \xi_1 d\xi \frac{\xi_1}{\xi} \phi^{(D/N)}_{r \nu}(x) \phi^{(D/N)\ast}_{r \nu}(x) = \frac{1}{2 \omega^{(D/N)}_{r \nu}} \delta(k - k') \] (4.7)

We obtain
\[ A_\mu = \sum_{r=1}^{\infty} \int \frac{d^2k}{k(2\pi)^2} \sum_{\lambda=0}^{3} \left[ A^{(\lambda)}_{r \mu}(x)a_{r \lambda}(k) + A^{(\lambda)\ast}_{r \mu}(x)a_{r \lambda}^\ast(k) \right], \] (4.8)

where
\[ A^{(0)}_{r \mu}(x) = (\nabla_\mu, 0) \phi^{D}_{r \nu}(x), \] (4.9)

\[ A^{(1)}_{r \mu}(x) = (p_\mu, 0) \phi^{N}_{r \nu}(x), \] (4.10)
\[ A^{(2)}_{\nu k}(x) = (0, p_{\nu}) \phi^D_{\nu k}(x), \quad (4.11) \]
\[ A^{(3)}_{\nu k}(x) = (0, \nabla_i) \phi^D_{\nu k}(x), \quad (4.12) \]
where \( p_a = \epsilon_{ab} \nabla^b, \ p_i = \epsilon_{ij} \nabla^j, \) with
\[ \epsilon_{ab} = \frac{1}{\xi_1} \left( \begin{array}{cc} 0 & \xi \\ -\xi & 0 \end{array} \right) \quad (4.13) \]
and
\[ \epsilon_{ij} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right). \quad (4.14) \]
The above modes satisfy the orthogonality relations
\[ (A^{(\lambda)}_{\nu k}, A^{(\lambda')}_{\nu' k'}) = \eta^{\lambda\lambda'} \delta_{\nu\nu'} (2\pi)^2 \delta(k - k'), \quad (4.15) \]
\[ (A^{(\lambda)}_{\nu k}, A^{(\lambda')}_{\nu' k'}) = 0, \quad (4.16) \]
where \( \eta^{\lambda\lambda'} = \eta_{\lambda\lambda'} = \text{diag}(-1, 1, 1, 1). \)

It should be stressed that, despite some formal analogies with the work in Ref. \[15\], we are not reducing the theory to its physical degrees of freedom before quantization. In quantum theory the amplitudes \( a_{\nu \lambda}(k) \) are replaced by operators satisfying the commutation relations
\[ \left[ a_{\nu \lambda}(k), a^*_{\nu' \lambda'}(k') \right] = \eta^{\lambda\lambda'} \delta_{\nu\nu'} (2\pi)^2 \delta(k - k'), \quad (4.17) \]
with all other commutators vanishing. The Feynman propagator can be now obtained by taking the time-ordered product of gauge fields, i.e. (recall that \( A_\nu(x') \equiv A_{\nu'} \)),
\[ G_{\nu\nu'} = i \langle 0 | T A_\nu(x) A_{\nu'}(x') | 0 \rangle. \quad (4.18) \]

With the notation of Ref. \[15\], one obtains from Eqs. (4.9)–(4.12) and (4.18)
\[ G_{\mu\nu} = \left( \begin{array}{cc} G_{ab} & 0 \\ 0 & G_{ij} \end{array} \right), \quad (4.19) \]
where
\[ G_{ab} = -\frac{p_a p_b}{\nabla^2} G^{(N)}(x, x') + \frac{\nabla_a \nabla_b}{\nabla^2} G^{(D)}(x, x'), \quad (4.20) \]
\[ G_{ij} = \delta_{ij} G^{(D)}(x, x'). \quad (4.21) \]

If one follows instead a differential equation approach, one can verify that the vanishing of off-diagonal blocks in Eq. (4.19) is also obtainable by finding the kernel of the operator
\[ \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \left( \frac{\nu^2}{\xi^2} - k^2 \right) \]
and of the operator matrix
\[ M \equiv \left( \begin{array}{cc} \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\xi^2}{\xi^4} - k^2 \\ -\frac{2\nu}{\xi^2} \end{array} \right), \quad (4.22) \]
when the boundary conditions (4.2) are imposed. On setting \( \nu = i \mu, \mu \in \mathbb{R}, \) one finds no real roots of the resulting equations, which involve modified Bessel functions \( I_\mu, K_\mu \) with \( \rho = \mu - 1, \mu, \mu + 1. \) For example, no real roots exist of the equation
\[ \frac{I_\mu(k_1^\xi)}{I_\mu(k_2^\xi)} - \frac{K_\mu(k_1^\xi)}{K_\mu(k_2^\xi)} = 0, \quad \rho = \mu - 1, \mu, \mu + 1, \quad (4.23) \]

\[ \left[ \frac{I_\mu(k_1^\xi) + kI'_\mu(k_1^\xi)}{I_\mu(k_2^\xi) + kI'_\mu(k_2^\xi)} \right] = \left[ \frac{K_\mu(k_1^\xi) + kK'_\mu(k_1^\xi)}{K_\mu(k_2^\xi) + kK'_\mu(k_2^\xi)} \right] = 0. \quad (4.24) \]

Hereafter, \( \nabla^2 \equiv \nabla_i \nabla^i. \) The action of the operator \( 1/\nabla^2 \) in Eq. (4.20) is easily defined, since we shall require it to act only on functions that have Fourier integral representation. The ghost Green function is defined
by
\[ G(x, x') \equiv i \langle 0 | T \chi(x) \psi(x') | 0 \rangle , \] (4.25)
and is required to obey homogeneous Dirichlet conditions as we said before, i.e.,
\[ G(x, x') = G^{(D)}(x, x') . \] (4.26)

V. WARD IDENTITIES

We now verify that the following Ward identities hold:
\[ G_{\nu', \mu}^\mu + G_{\nu}^{\nu'} = 0 , \] (5.1)
\[ G_{\nu'}^{\mu} + G^{\mu} = 0 . \] (5.2)
To prove these identities, use is made of the following properties:
1) The order of covariant derivatives \( \nabla_\mu \) can be freely interchanged because the metric is flat, i.e.,
\[ \nabla_\mu \nabla_\nu = \nabla_\nu \nabla_\mu . \] (5.3)
2) The identity holds
\[ \nabla_a \nabla^a G^{(D/N)}(x, x') = - \nabla^2 G^{(D/N)}(x, x') \quad \text{for } x \neq x' \] (5.4)
which easily follows from the Klein–Gordon equation.
3) Translation invariance in the \((x, y)\) directions implies
\[ \nabla_x G^{(D/N)}(x, x') = - \nabla_{x'} G^{(D/N)}(x, x') . \] (5.5)
4) since \( \epsilon_{ab} \) is antisymmetric and covariantly constant, 
\[ \nabla_a \epsilon_{bc} = 0 , \] it follows that
\[ \nabla_a p_a = \nabla^a \epsilon_{ab} \nabla^b = \epsilon_{ab} \nabla^a \nabla^b = 0 . \] (5.6)
By using the above ingredients, we can easily prove Eq. (5.1). Take first \( \nu' = b' \)
\[ G_{b', a}^a + G_{a, b'} = \left( \nabla_a \nabla_{b'} - \nabla_{b'} \nabla_a \right) G^{(D)}(x, x') = 0 . \] (5.7)
For \( \nu' = j' \), we get
\[ G_{j', i}^i + G_{i, j'} = \left( \nabla_j + \nabla_{j'} \right) G^{(D)}(x, x') = 0 . \] (5.8)
By following analogous steps one proves also Eq. (5.2).

VI. ENERGY-MOMENTUM TENSORS

Since in what follows we always consider pairs of space-time points \((x, x')\) with space-like separations, we do not have to worry about operator ordering, and as a result we can replace in all formulas the Hadamard function by twice the imaginary part of the Feynman propagator. The Maxwell energy-momentum tensor \( T_A^{\mu \nu} \) reads as
\[ T_A^{\mu \nu} = F_\beta^\rho F^{\nu \beta} - \frac{1}{4} \eta^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} . \] (6.1)
The gauge and ghost parts of the energy-momentum tensor are
\[ T_{\text{gauge}}^{\mu \nu} = - A_{\alpha}^{\mu} A_{\nu}^\alpha - A_{\alpha}^{\mu} A_\alpha^\nu + \left[ A_{\alpha}^{\mu} A_\beta^\nu + \frac{1}{2} \left( A_\alpha^\alpha \right)^2 \right] g^{\mu \nu} , \] (6.2)
\[ T_{\text{ghost}}^{\mu \nu} = - \chi^{\mu} \psi^\nu - \chi^\nu \psi^\mu + g^{\mu \nu} \chi_{\alpha} \psi^\alpha . \] (6.3)
By adopting the point-split regularization we define
\[ \langle 0 | T_A^{\mu \nu} | 0 \rangle \equiv \lim_{x' \rightarrow x} T_A^{\mu \nu}(x, x'), \] (6.4)
\[ \langle 0 | T_{\text{gauge}}^{\mu \nu} | 0 \rangle \equiv \lim_{x' \rightarrow x} T_{\text{gauge}}^{\mu \nu}(x, x'), \] (6.5)
\[ \langle 0 | T_{\text{ghost}}^{\mu \nu} | 0 \rangle \equiv \lim_{x' \rightarrow x} T_{\text{ghost}}^{\mu \nu}(x, x'), \] (6.6)
where, on denoting by \( g_{\nu'}^{\mu} \) the parallel displacement bivector \( [18] \),
\[ T_A^{\mu \nu}(x, x') = \frac{1}{2} g_{\rho \beta} \left( g^{\mu \rho} g^{\nu \beta} - \frac{1}{4} g^{\mu \nu} g^{\rho \tau} \right) \]
\[ \times \langle 0 | \left\{ F_{\tau \alpha} g_{\beta}^{\rho} g_{\rho \beta}^{\nu} F_{\tau \alpha}^{\nu} + F_{\rho \beta} g_{\sigma}^{\nu} g_{\alpha}^{\rho} F_{\tau \alpha}^{\nu} \right\} | 0 \rangle , \] (6.7)
\[ T_{\text{gauge}}^{\mu \nu}(x, x') = \frac{1}{2} \langle 0 | \left\{ - A_{\alpha}^{\mu} g_{\nu}^{\rho} A_{\nu}^{\alpha} - A_{\nu}^{\mu} g_{\nu}^{\rho} A_{\alpha}^{\rho} - A_{\alpha}^{\nu} g_{\nu}^{\rho} A_{\alpha}^{\rho} - A_{\alpha}^{\nu} g_{\nu}^{\rho} A_{\alpha}^{\rho} + g^{\mu \nu} \left[ A_{\alpha}^{\alpha} g_{\beta}^{\beta} A_{\beta}^{\alpha} \right] 
\quad + A_{\beta}^{\rho} g_{\beta}^{\mu} A_{\alpha}^{\nu} - A_{\alpha}^{\rho} g_{\beta}^{\mu} A_{\beta}^{\alpha} \right\} | 0 \rangle , \] (6.8)
and
\[ T_{\text{ghost}}^{\mu \nu}(x, x') = \frac{1}{2} \langle 0 | \left\{ - \chi^{\mu} \psi^\nu - \chi^\nu \psi^{\mu} - \chi^{\mu} g_{\nu}^{\rho} \psi^{\rho} - \chi^\nu g_{\mu}^{\rho} \psi^{\rho} + g^{\mu \nu} \left[ \chi_{\alpha} \psi^\alpha + g_{\nu}^{\rho} \chi^\alpha \psi^{\rho} \right] \right\} | 0 \rangle \] (6.9)
Note that $T^{\mu\nu}_{\text{Maxwell}}(x,x')$, $T^{\mu\nu}_{\text{gauge}}(x,x')$ and $T^{\mu\nu}_{\text{ghost}}(x,x')$ all transform as tensors at $x$ and as scalars at $x'$, and therefore Eqs. (6.4,6.6) are well defined. By using the Ward identities Eq. (5.1) and Eq. (5.2) it is easy to prove that

$$T^{\mu\nu}_{\text{gauge}}(x,x') + T^{\mu\nu}_{\text{ghost}}(x,x') = 0. \quad (6.10)$$

Indeed, upon expressing $T^{\mu\nu}_{\text{gauge}}(x,x')$ and $T^{\mu\nu}_{\text{ghost}}(x,x')$ in terms, respectively, of the photon and ghost propagators $G_{\mu\nu'}$ and $G$, one can show that the l.h.s of Eq. (6.10) is equal to

$$\tilde{G}_{ij'} = \delta_{ij} \tilde{G}^{(D)}(x,x'). \quad (7.5)$$

The important thing to notice is that the singularities of the photon propagator are all included in the $G^{(0)}_{\mu\nu'}$ piece, while $\tilde{G}_{\mu\nu'}$ is perfectly regular in the coincidence limit $x' \to x$. The Maxwell tensor admits a representation analogous to Eq. (7.1), i.e.,

$$T^{\mu\nu}_A(x,x') = T^{(0)\mu\nu}_A(x,x') + T^{\mu\nu}_A(x,x'). \quad (6.16)$$

Here, $T^{(0)\mu\nu}_A(x,x')$ is the contribution arising from $G^{(0)}_{\mu\nu'}$, while $\tilde{T}^{\mu\nu}_A(x,x')$ is the contribution involving $\tilde{G}_{\mu\nu'}$. The quantity $T^{(0)\mu\nu}_A(x,x')$ coincides with the point-split expression for the Maxwell tensor in Minkowski space-time, transformed to Rindler coordinates, and it diverges in the limit $x' \to x$. Being independent of the plates' separation $a$, we shall simply disregard it. On the contrary, the expression $T^{\mu\nu}_A(x,x')$ is perfectly well defined in the coincidence limit. In this limit, its explicit expression is

$$\langle 0| \tilde{T}^\mu_A | 0 \rangle = \text{diag}(-\gamma + \delta, \gamma, \gamma, -\gamma - \delta), \quad (7.7)$$

where

$$\delta = \frac{i}{2} \left( \frac{x^2}{\xi \xi'} \left( \frac{\partial^2}{\partial \xi \partial \xi'} + \frac{\partial^2}{\partial \xi' \partial \xi} \right) \right)_{\xi' = \xi} \tilde{G}^{(N)} + \tilde{G}^{(D)}, \quad (7.8)$$

$$\gamma = \frac{i}{2} \nabla^2 \tilde{G}^{(N)} + \tilde{G}^{(D)} \quad (7.9)$$

It is clear from the above formulas that $\langle 0| \tilde{T}^\mu_A | 0 \rangle$ is traceless, i.e.,

$$\langle 0| \tilde{T}^\mu_A | 0 \rangle = 0. \quad (7.10)$$

It can also be verified that $\gamma$ and $\delta$ satisfy the relation

$$-\gamma + \delta = -\frac{d}{d\xi}[\xi(\gamma + \delta)], \quad (7.11)$$

which represents the condition for $\tilde{T}^\mu_A$ to be covariantly conserved. The expressions for $\gamma$ and $\delta$ can be obtained...
by inserting Eq. (3.14) into Eqs. (7.8) and (7.9), i.e.,

$$\delta = \frac{ia}{2} \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \left( \frac{\xi_i^2}{\xi^j} + \frac{\partial^2}{\partial \xi_i \partial \xi_j} \right) \tilde{\psi}(\xi, \xi_i', i\xi_1, k) \bigg|_{\xi_i' = \xi},$$

(7.12)

$$\gamma = -\frac{a}{2} \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} k^2 \tilde{\psi}(\xi, i\xi_1, k),$$

(7.13)

where we have defined

$$\tilde{\psi}(\xi, \xi_i', i\xi_1, k) \equiv \frac{\xi_i}{a}(\tilde{\chi}^{(D)} + \tilde{\chi}^{(N)})(\xi, \xi_i', i\xi_1, k)$$

(7.14)

It is convenient to rotate the contour of the $\omega$-integration away from the singularities to the positive imaginary axis. Since $\tilde{\psi}$ is an even function of $\omega$, and is invariant under rotations in the $(x, y)$ plane, upon setting $\omega \equiv i\eta$ and then performing the change of variables $ak \equiv q\sqrt{1 - s^2}$, $\eta \equiv sq/a$, the integrals for $\delta$ and $\gamma$ become

$$\delta = \frac{1}{4\pi^2a^2} \int \frac{dq}{q} \int_0^1 ds \left( s^2q^2 \frac{\xi_i^2}{\xi_i'} - \frac{\partial^2}{\partial \xi_i \partial \xi_i'} \right) \tilde{\psi}(\xi, \xi_i') \bigg|_{\xi_i' = \xi},$$

(7.15)

$$\tilde{\psi}(1) = \frac{\{e^{q(\tilde{z} - \tilde{z}')} - e^{q(2 - \tilde{z} - \tilde{z}')}} - 2q(\tilde{z} + \tilde{z}')(\cosh q(\tilde{z} - \tilde{z}'))\} (1 - s^2) - 2s^2q^2(\tilde{z}^2 - \tilde{z}'^2) \sinh q(\tilde{z} - \tilde{z}') + \frac{\cosh q(\tilde{z} - \tilde{z}')}{2\sinh^2(q)s^2}.$$

(7.19)

The expression for $\tilde{\psi}(2)$ is exceedingly lengthy and will not be reported here.

Evaluation of the integrals then gives the result

$$\delta \sim \frac{\pi^2}{360a^4} + \frac{g^2}{a^3} \left( \frac{\pi^2}{450} (1 - 2\tilde{z}) + \frac{\pi}{60 \sin^3(\pi \tilde{z})} \right) + \frac{g^2}{a^2} \left[ \pi^2(1 - 104\tilde{z} + 160\tilde{z}^2) - 160 \frac{25200}{240} - \pi^2(\tilde{z} - 1) - \frac{160}{240} \pi^2(\tilde{z} - 1) - 8 \frac{240}{280} \sin^3(\pi \tilde{z}) \right] + O(g^3),$$

(7.20)

$$\gamma \sim \frac{\pi^2}{720a^4} + \frac{g^2}{a^3} \left( \frac{\pi^2}{1800} (1 - 2\tilde{z}) - \frac{\pi}{60 \sin^3(\pi \tilde{z})} \right) + \frac{g^2}{a^2} \left[ \pi^2(44\tilde{z}^2 - 16\tilde{z} - 9) - 100 \frac{50400}{420} + \pi^2(\tilde{z} - 1) - \frac{100}{420} \pi^2(\tilde{z} - 1) - 1 \frac{280}{240} \sin^3(\pi \tilde{z}) \right] + O(g^3).$$

(7.21)

It can be verified that the above expressions for $\gamma$ and $\delta$ satisfy the fundamental conservation condition Eq. (7.11). Moreover, on inserting these values into Eq. (7.17),

$$\gamma = \frac{1}{4\pi^2a^4} \int_0^\infty dq \int_0^1 ds (1 - s^2) \tilde{\psi}(\xi, \xi),$$

(7.16)

where we have set $\xi \equiv \xi/a$, $\xi_i \equiv \xi_i/a$, $i = 1, 2$. The weak-field limit is obtained by taking $\xi_i \rightarrow \infty$ in the previous formulas, for fixed $s$ and $q$. By using the large-order uniform asymptotic expansions of the modified Bessel functions, quoted in Appendix A, we have obtained the asymptotic expansion for $\tilde{\psi}$, to second order in $ga$ (hereafter $\tilde{z} \equiv z/a$, $\tilde{z}' \equiv z'/a$):

$$\tilde{\psi}(\tilde{z}, \tilde{z}') \sim \tilde{\psi}(0) + ga \tilde{\psi}(1) + (ga)^2 \tilde{\psi}(2) + O((ga)^3),$$

(7.17)

where

$$\tilde{\psi}(0) = e^{q(\tilde{z} - \tilde{z}') + e^{q(\tilde{z}' - \tilde{z})}} q(e^{2q} - 1)$$

(7.18)
we obtain

\[
\langle 0 | \tilde{T}_{\mu \nu}^{\hat{A}} | 0 \rangle \sim \frac{\pi^2}{720 a^4} + \frac{g}{a^2} \left( \frac{\pi^2}{600} (1 - 2 \hat{z}) + \frac{\pi}{30} \cos(\pi \hat{z}) \right) + \frac{g^2}{a^2} \left[ -\frac{11}{2520} + \frac{\pi^2}{50400} (11 - 192 \hat{z} + 276 \hat{z}^2) 
+ \frac{9 - 2\pi^2 \hat{z}(\hat{z} - 1)}{420 \sin^2(\pi \hat{z})} - \frac{\pi(20 \hat{z} - 3) \cos(\pi \hat{z})}{420 \sin^2(\pi \hat{z})} - \frac{\pi^2 (1 - \hat{z})}{140 \sin^2(\pi \hat{z})} \right] + O(g^3),
\]

(7.22)

\[
\langle 0 | \tilde{T}_{\hat{A} \hat{z}} | 0 \rangle \sim -\frac{\pi^2}{240 a^4} - \frac{\pi}{360} (1 - 2 \hat{z}) + \frac{g^2}{a^2} \left( \frac{1}{120} + \frac{\pi^2}{7200} [1 + 4 \hat{z}(8 - 13 \hat{z})] - \frac{1}{60 \sin^2(\pi \hat{z})} \right) + O(g^3),
\]

(7.23)

while of course

\[
\langle 0 | \tilde{T}_{\hat{A} \hat{z}} | 0 \rangle = \langle 0 | \tilde{T}_{\hat{A} \hat{y}} | 0 \rangle = \gamma.
\]

(7.24)

We note that the quantities \( \gamma \) and \( \delta \) both diverge as \( \hat{z} \) approaches the locations of the plates at \( \hat{z} = 0 \) and \( \hat{z} = a \). In particular, for \( \hat{z} \to 0 \), from Eqs. (7.22) and (7.24) we find

\[
\langle 0 | \tilde{T}_{\hat{A} \hat{t}}^{\hat{A}} | 0 \rangle \sim \frac{g}{30 \pi \hat{z}^3} + O(z^{-2}),
\]

(7.25)

\[
\langle 0 | \tilde{T}_{\hat{A} \hat{t}}^{\hat{z}} | 0 \rangle \sim -\frac{g^2}{60 \pi^2 \hat{z}^3} + O(z^{-1}),
\]

(7.26)

\[
\langle 0 | \tilde{T}_{\hat{A} \hat{t}}^{\hat{y}} | 0 \rangle = \langle 0 | \tilde{T}_{\hat{A} \hat{y}}^{\hat{y}} | 0 \rangle \sim -\frac{g}{60 \pi^2 \hat{z}^3} + O(z^{-2}).
\]

(7.27)

These behaviors are in full agreement with the results derived in Ref. [13], for the case of a single mirror [26]. The valuable work in Ref. [19], devoted to the scalar and electromagnetic Casimir effects in the Fulle–Rindler vacuum, can also be shown to agree with our energy-momentum formulas.

**VIII. CONCLUDING REMARKS**

Our analysis has made it possible to put on completely firm ground the set of formulas for the vacuum expectation value of the regularized and renormalized energy-momentum tensor for an electromagnetic Casimir apparatus in a weak gravitational field. In particular, the term of first order in \( g \) in Eq. (7.22) corrects an unfortunate mistake in Eq. (4.4) of Ref. [10] (see Ref. [20]). Using our original Eqs. (7.15) and (7.16) we have been able to evaluate second-order corrections (with respect to the expansion parameter \( ga/c^2 \)) to \( \langle T_{\mu \nu} \rangle \), which represent one new result of the present paper. The physical interpretation that can be attributed to these corrections is doubtful, in view of the divergencies they exhibit on approaching the plates. The existence of these divergencies is well known in the literature [22], and it is usually attributed to the pathological character of perfect-conductor boundary conditions. Indeed, divergencies arise already in first order corrections to some components of \( T_{\mu \nu} \), but they constitute somewhat less of a problem, because while on the one hand no divergence is found in \( T_{zz} \), which provides the Casimir pressure, the nonintegrable divergencies in \( T_{tt} \) are of such a nature that one can still obtain a finite value for the total mass-energy of the Casimir apparatus (per unit area of the plates), by taking the principal-value integral of \( T_t^t \) over the volume of the cavity [10]. Neither of these fortunate circumstances occurs at second order, since on the one hand \( T_{zz} \) is now found to diverge on approaching the plates, so that no definite meaning can be given to the gravitational correction to the Casimir pressure, and on the other hand the divergencies in \( T_t^t \) are such that the resulting correction to the total mass-energy of the cavity is infinite, even on taking the principal-value integral of \( T_t^t \).

The years to come will hopefully tell us whether the push predicted and confirmed by theory is amenable to experimental verification [8]. It also remains to be seen whether the experience gained in the detailed evaluation of the energy-momentum tensor in Ref. [10] and in the present paper can be used to obtain a better understanding of the intriguing relation between Casimir effect and Hawking radiation found in Ref. [21].

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\[\text{\scriptsize{8}}\]
**APPENDIX A: ASYMPTOTIC FORMULAS**

For large orders $\nu$, the modified Bessel functions $I_\nu(w)$, $K_\nu(w)$ and their first derivatives admit the following asymptotic expansions, which hold uniformly with respect to $w$ in the half-plane $|\arg w| \leq \pi/2 - \varepsilon$, for $\varepsilon$ in the open interval $]0, \pi/2[$ [23, 24]:

\[
I_\nu(w) \sim \frac{1}{\sqrt{2\pi w}} e^{w} \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right), \quad (A1)
\]

\[
K_\nu(w) \sim \frac{1}{\sqrt{2\pi w}} e^{-w} \left( 1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k} \right), \quad (A2)
\]

\[
I_\nu'(w) \sim \frac{1}{\sqrt{2\pi \nu w}} (1 + w^2)^{1/4} e^{w} \left( 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} \right), \quad (A3)
\]

\[
K_\nu'(w) \sim -\frac{1}{\sqrt{2\pi \nu w}} (1 + w^2)^{1/4} e^{-w} \left( 1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{\nu^k} \right), \quad (A4)
\]

where

\[
t \equiv 1/\sqrt{1 + w^2} \quad (A5)
\]

\[
\rho \equiv \sqrt{1 + w^2} + \log \frac{w}{1 + \sqrt{1 + w^2}}. \quad (A6)
\]

and, for $k = 0, 1, 2$, one has

\[
v_0 = 1, \quad (A7)
\]

\[
v_1 = (-9t + 7t^3)/24, \quad (A8)
\]

\[
v_2 = (-135t^2 + 594t^4 - 455t^6)/1152; \quad (A9)
\]

\[
u_0 = 1, \quad (A10)
\]

\[
u_1 = (3t - 5t^3)/24, \quad (A11)
\]

\[
u_2 = (81t^2 - 462t^4 + 385t^6)/1152, \quad (A12)
\]

The generating formulas of these Olver polynomials [23] are

\[
u_{k+1}(t) = \frac{t^2}{2} (1 - t^2) \frac{du_k}{dt} + \frac{1}{8} \int_0^t (1 - 5\beta^2) u_k(\beta) d\beta, \quad (A13)
\]

\[
v_k(t) = u_k(t) - t(1 - t^2) \left( \frac{1}{2} u_{k-1} + \frac{du_{k-1}}{dt} \right), \quad (A14)
\]

**APPENDIX B: THE BIVECTOR OF PARALLEL DISPLACEMENT**

The bivector of parallel displacement $\delta_{R'}^\nu$ [18] in the Rindler spacetime is easily evaluated by exploiting the coordinate transformation

\[
\bar{t} = \xi \sinh \tau, \quad \bar{z} = \xi \cosh \tau, \quad \bar{\bar{z}} = x, \quad \bar{\bar{y}} = y, \quad (B1)
\]

where $\tau = t/\xi_1$, that turns the Rindler metric in Eq. (2.2) into the Minkowski metric

\[
ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2. \quad (B2)
\]

In the Minkowski coordinates we obviously have $\bar{g}_{R'}^\nu = \delta_{R'}^\nu$. Therefore

\[
g_{R'}^\nu = \bar{g}_{R'}^\nu \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{x}^\mu}{\partial \bar{x}^\nu} = \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{x}^\mu}{\partial \bar{x}^\nu}, \quad (B3)
\]

We then obtain

\[
g_{R'}^\nu = \left( \begin{array}{cc}
\delta_{R'}^\nu & 0 \\
0 & \delta_j^\nu
\end{array} \right), \quad (B4)
\]

where

\[
g_{R'}^\nu = g_{R'}^\nu g_{\nu \sigma} \sigma_{R'}^\nu = \left( \begin{array}{cc}
g_{R'}^\nu & 0 \\
0 & \delta_j^\nu
\end{array} \right), \quad (B6)
\]

Similarly, one finds

\[
g_{R'}^\nu = g_{R'}^\nu g_{\nu \sigma} \sigma_{R'}^\nu = \left( \begin{array}{cc}
g_{R'}^\nu & 0 \\
0 & \delta_j^\nu
\end{array} \right), \quad (B6)
\]
\[
\begin{aligned}
g_{\nu}^\mu &= \left( \begin{array}{cc}
g_{\tau}\tau & g_{\tau}\xi' \\
g_{\xi'}\tau & g_{\xi'}\xi'
\end{array} \right) = \left( \begin{array}{cc}
\xi'(\cosh \tau' \cosh \tau - \sinh \tau' \sinh \tau) & -\frac{1}{2}(\cosh \tau' \sinh \tau - \sinh \tau' \cosh \tau) \\
-\xi'(\cosh \tau' \cosh \tau - \sinh \tau' \sinh \tau) & \cosh \tau' \cosh \tau - \sinh \tau' \sinh \tau
\end{array} \right) .
\end{aligned}
\]
(B7)

It can be checked that
\[
g_{\nu}^\mu g_{\mu}^\rho = \delta_{\nu}^\rho . \quad \text{(B8)}
\]