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Bounds on $F$-index of tricyclic graphs with fixed pendant vertices

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Abstract: The $F$-index $F(G)$ of a graph $G$ is obtained by the sum of cubes of the degrees of all the vertices in $G$. It is defined in the same paper of 1972 where the first and second Zagreb indices are introduced to study the structure-depency of total $\pi$-electron energy. Recently, Furtula and Gutman [J. Math. Chem. 53 (2015), no. 4, 1184–1190] reinvestigated $F$-index and proved its various properties. A connected graph with order $n$ and size $m$, such that $m = n + 2$, is called a tricyclic graph. In this paper, we characterize the extremal graphs and prove the ordering among the different subfamilies of graphs with respect to $F$-index in $\Omega^a_n$, where $\Omega^a_n$ is a complete class of tricyclic graphs with three, four, six and seven cycles, such that each graph has $a \geq 1$ pendant vertices and $n \geq 16 + a$ order. Mainly, we prove the bounds (lower and upper) of $F(G)$, i.e.

$$8n + 12a + 76 \leq F(G) \leq 8(n - 1) - 7a + (a + 6)^3$$

for each $G \in \Omega^a_n$.

Keywords: extremal graphs, tricyclic graphs, $F$-index

MSC 2010: 05C12, 05C50, 05C35

1 Introduction and preliminaries

A representative number of a molecular graph that expresses the various features of the involved organic molecules, usually known as a topological index (TI). It plays an important role to study the certain changes in the molecular structures which may be physical or chemical. Moreover, Cheminformatics studies quantitative structural activity and property relationships that are used to examine the bioactivities and chemical reactivities of the chemical compounds in a molecular graph on the bases of obtained computational results for the different topological indices (TI’s), see [1]. Most importantly, all the TI’s are invariants under the parameter of graphs-isomorphism. For a connected graph, there are many TI’s in literature. These are classified into three main classes degree-based TI’s, distance-based TI’s and polynomial-based TI’s. The TI’s depending upon degrees are more familiar than the others, see [2].

Wiener (1947) defined the first distance based TI, when he was working on paraffin, see [3]. Later on, it was called by Wiener index and much more work has been done on it. Recently, Furtula and Gutman (2015) [4] reinvestigated a degree-based TI and named it forgotten index ($F$-index). They also proposed its basic properties in the same paper and reported that it can enhance the physico-chemical capability of the molecules. The $F$-index and its co-index of the different graphs are studied by De et al. [5], Milovanovic et

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al. [6] and Basavanagoud et al. [7]. Khaksari and Ghorbani [8] studied the certain product of graphs with the same index. The extremal graphs with respect to F-index among the unicyclic and bicyclic graphs are studied in [9, 10]. For more studies, we refer to [11] and [12-25].

In this paper, we prove the existence of extremal graphs with respect to F-index in the class of tricyclic graphs with three, four, six and seven cycles under the condition of certain pendant vertices. We also investigate the ordering and compute the bounds (lower and upper) of the F-index in the same class of graphs.

Throughout the paper, $G(V(G), E(G))$ for vertex-set $V(G)$ and edge-set $E(G)$ is considered as simple (no loops and parallel edges), finite and undirected graph. For $r \in V(G)$, $d(r)$ shows its degree (number of incident edges on $r$). For more theoretic terminologies, we refer [26]. Now, some important TI’s are defined as follows:

**Definition 1.1.** For a (molecular) graph $G$, the first and second Zagreb indices are

$$M_1(G) = \sum_{rs \in E(G)} [d(r) + d(s)] \quad \text{and} \quad M_2(G) = \sum_{rs \in E(G)} [d(r) \times d(s)].$$

**Definition 1.2.** For a (molecular) graph $G$, the general Randić index ($R_\alpha(G)$) is

$$R_\alpha(G) = \sum_{rs \in E(G)} [d(r) \times d(s)]^\alpha.$$

For $\alpha = -\frac{1}{2}, \alpha = \frac{1}{2}$ and $\alpha = 1$, we obtain Randić, reciprocal Randić and second Zagreb indices respectively.

**Definition 1.3.** For a (molecular) graph $G$ the forgotten index ($F$-index) is defined as follow:

$$F(G) = \sum_{s \in V(G)} [d(s)]^3.$$

For more studies, we refer to [4, 11, 27-29]. Following lemma is frequently used in the main results.

**Lemma 1.1.** [9] For $1 \leq i \leq n$ and $1 \leq j \leq 2$, assume that $< d_1^i, d_2^i, d_3^i, ..., d_n^i >$ and $< d_1^j, d_2^j, d_3^j, ..., d_n^j >$ are degree sequences with the condition of $d_i^1 = d_i^2$, where $d_i^j$ is degree of the vertex $v_i^j \in V(G_j)$ and $n = |V(G_1)| = |V(G_2)|$. Then, $F(G_1) = F(G_2)$.

## 2 Computational results of $F$-index

A connected graph with order $n$ and size $m$ such that $m = n - 1 + c$ is called a $c$-cyclic graph. In particular, if $c = 0, c = 1, c = 2$ or $c = 3$ then it is a tree, unicyclic, bicyclic or tricyclic graph respectively. A tricyclic graph contains at least three and at most seven cycles except of exactly five cycles. There are seven possibilities for a tricyclic graph with three cycles as shown in Figure 1. Moreover, the possibilities for the tricyclic graphs with four, six and seven cycles are four, three and one respectively, see Figure 2. Now, we define some more tricyclic graphs with respect to the attachment of $k \geq 1$ pendant vertices to the $l$ vertices of the graphs which are defined in Figure 1. To choose $l$ vertices, we have the following choices:

(i) cycle-vertex of degree 2,
(ii) tree-vertex of degree 2,
(iii) cycle-vertex of degree greater or equal to 2,
(iv) cycle-vertex and tree-vertex of degree exactly 2,
(v) cycle-vertex of degree greater or equal to 2 and tree-vertex of degree exactly 2.

More precisely, we define that $l$ vertices are either of degree exactly 2 or, greater or equal to 2. By joining $k \geq 1$ pendant vertices to $l$ vertices of degree 2, and the vertices of degree greater or equal to 2 of the graph $G_1$ in Figure 1, the tricyclic graphs $A_{i,k,1}^{m,r} = A_1$ and $A_{i,k,2}^{m,r} = A_2$ are obtained respectively. In $G_1$, vertices of degree 3 are four and of degree 2 are $m_1 + m_2 + m_3 + r$ such that $m = m_1 + m_2 + m_3$ are cycle-vertex and $r$ are tree-vertex. Table 1 shows the vertex-partition with respect to degrees of vertices of graphs $A_1$ and $A_2$. 
respectively. In Figure 1, and obtained, respectively. The Table 2 presents the vertex-partitions of graphs vertices of degree greater or equal to \( k \), and greater or equal to \( m \). Moreover for \( i \in \{1, 2\} \), we note that (i) \( |V(A_i)| = |V(B_i)| = |V(C_i)| = |V(D_i)| = |V(E_i)| = m_1 + m_2 + m_3 + l + r + 4 = m + l + k + r + 4 \) and (ii) the graphs \( G_6 \) and \( G_7 \) have the same degree sequences as of \( G_1 \) and \( G_2 \) respectively. For more explanation, \( B_1, B_2, E_1 \) and \( E_2 \) are given in Figure 2 with certain value of the parameters \( l, m, k \) and \( r \).

Now, \( A_1^2 \) from \( A_1 \) are obtained by deleting \( k \) pendant vertices from a vertex of degree \( k + 2 \) and joining these vertices to another vertex of degree \( k + 2 \). Similarly, \( A_1^2 \) is derived from \( A_1^1 \) by deleting \( 2k \) pendant vertices.

Table 1: Vertex-partitions of the tricyclic graphs \( A_1 \) and \( A_2 \).

| \( d(v) \), for \( v \in V(A_1) \) | 1 | 2 | 3 | \( k + 2 \) |
| \( |d(v)| \) | \( lk \) | \( m + r - l \) | 4 | \( l \) |
| \( d(v) \), for \( v \in V(A_2) \) | 1 | 2 | \( k + 2 \) | \( k + 3 \) |
| \( |d(v)| \) | \( lk \) | \( m + r + 4 - l \) | \( l - 4 \) | \( 4 \) |

In \( G_2 \) (Figure 1), the vertices of degrees 4, 3, and 2 are 1, 2 and \( m_1 + m_2 + m_3 + r + 1 \) such that \( m = m_1 + m_2 + m_3 \) are cycle-vertex and \( r + 1 \) are tree-vertex. By joining \( k \geq 1 \) pendant vertices to \( l \) vertices of degree 2, and the vertices of degree greater or equal to 2 of \( G_2 \) in Figure 1, the tricyclic graphs \( B_{l,k,1}^{m,r} = B_1 \) and \( B_{l,k,2}^{m,r} = B_2 \), are obtained, respectively. The Table 2 presents the vertex-partitions of graphs \( B_1 \) and \( B_2 \).

The graph \( G_3 \) (Figure 1) has 2 and \( m_1 + m_2 + m_3 + r + 2 \) vertices of degrees 4 and 2, respectively such that \( m = m_1 + m_2 + m_3 \) and \( r + 2 \) are cycle-vertex. The tricyclic graphs \( C_{l,k,1}^{m,r} = C_1 \) and \( C_{l,k,2}^{m,r} = C_2 \) are obtained by joining \( k \geq 1 \) pendant vertices to \( l \) vertices of degree 2, and degree greater or equal to 2 of the graph \( G_3 \) in Figure 1, respectively. The Table 3 presents the vertex-partitions with respect to the degrees of vertices of the graphs \( C_1 \) and \( C_2 \).

Similarly, we obtain the tricyclic graphs \( D_{l,k,1}^{m,r} = D_1 \), \( D_{l,k,2}^{m,r} = D_2 \), \( E_{l,k,1}^{m,r} = E_1 \) and \( E_{l,k,2}^{m,r} = E_2 \), by joining \( k \geq 1 \) pendant vertices to \( l \) vertices of degree 2, and greater or equal to 2 of the graphs \( G_4 \) and \( G_5 \) in Figure 1, respectively. In Figure 1, \( G_6 \) has \( m \) cycle-vertex and \( r + 2 \) tree-vertex of degrees 2 and \( G_7 \) has \( m \) cycle-vertex and \( r + 3 \) tree-vertex of degrees 2. The vertex-partitions of these derived tricyclic graphs are shown in Table 4 and Table 5.

Moreover for \( i \in \{1, 2\} \), we note that (i) \( |V(A_i)| = |V(B_i)| = |V(C_i)| = |V(D_i)| = |V(E_i)| = m_1 + m_2 + m_3 + l k + r + 4 = m + l k r + 4 \) and (ii) the graphs \( G_6 \) and \( G_7 \) have the same degree sequences as of \( G_1 \) and \( G_2 \) respectively. For more explanation, \( B_1, B_2, E_1 \) and \( E_2 \) are given in Figure 2 with certain value of the parameters \( l, m, k \) and \( r \).

Now, \( A_1^2 \) from \( A_1 \) are obtained by deleting \( k \) pendant vertices from a vertex of degree \( k + 2 \) and joining these vertices to another vertex of degree \( k + 2 \). Similarly, \( A_1^2 \) is derived from \( A_1^1 \) by deleting \( 2k \) pendant vertices.

Table 2: Vertex-partitions of the tricyclic graphs \( B_1 \) and \( B_2 \).

| \( d(v) \), for \( v \in V(B_1) \) | 1 | 2 | 3 | 4 | \( k + 2 \) |
| \( |d(v)| \) | \( lk \) | \( m + r + 1 - l \) | 2 | 1 | \( l \) |
| \( d(v) \), for \( v \in V(B_2) \) | 1 | 2 | \( k + 2 \) | \( k + 3 \) | \( k + 4 \) |
| \( |d(v)| \) | \( lk \) | \( m + r + 4 - l \) | \( l - 3 \) | 2 | \( 1 \) |
Figure 2: Tricyclic graphs with four cycles ($H_1$, $H_2$, $H_3$, and $H_4$), six cycles ($L_1$, $L_2$, and $L_3$) and seven cycles ($K$).

Table 3: Vertex-partitions of the tricyclic graphs $C_1$ and $C_2$.

| $d(v)$, for $v \in V(C_1)$ | 1 | 2 | 4 | $k + 2$ |
|-----------------------------|---|---|---|--------|
| $|d(v)|$                     | $lk$ | $m + r + 2 - l$ | 2 | 1       |

| $d(v)$, for $v \in V(C_2)$ | 1 | 2 | $k + 2$ | $k + 4$ |
|-----------------------------|---|---|--------|--------|
| $|d(v)|$                     | $lk$ | $m + r + 4 - l$ | $l - 2$ | 2      |

Table 4: Vertex-partitions of the tricyclic graphs $D_1$ and $D_2$.

| $d(v)$, for $v \in V(D_1)$ | 1 | 2 | 3 | 5 | $k + 2$ |
|-----------------------------|---|---|---|---|--------|
| $|d(v)|$                     | $lk$ | $m + r + 2 - l$ | 1 | 1 | $l$    |

| $d(v)$, for $v \in V(D_2)$ | 1 | 2 | $k + 2$ | $k + 3$ | $k + 5$ |
|-----------------------------|---|---|--------|--------|--------|
| $|d(v)|$                     | $lk$ | $m + r + 4 - l$ | $l - 2$ | 1 | 1      |

Table 5: Vertex-partitions of the tricyclic graphs $E_1$ and $E_2$.

| $d(v)$, for $v \in V(E_1)$ | 1 | 2 | 6 | $k + 2$ |
|-----------------------------|---|---|---|--------|
| $|d(v)|$                     | $lk$ | $m + r + 3 - l$ | 1 | $l$    |

| $d(v)$, for $v \in V(E_2)$ | 1 | 2 | $k + 2$ | $k + 6$ |
|-----------------------------|---|---|--------|--------|
| $|d(v)|$                     | $lk$ | $m + r + 4 - l$ | $l - 1$ | 1      |
Joining \( k \) vertices to \( l \) vertices of degree \( \geq 2 \) respectively.

### Table 6

| Base Graphs (BG) | \( H_1 \) | \( H_2 \) | \( H_3 \) |
|------------------|-----------|-----------|-----------|
| Joining \( k \) vertices to \( l \) vertices of degree \( \geq 2 \) | \( R_{lk,1}^{m,r} = R_{1} \) | \( S_{lk,1}^{m,r} = S_{1} \) | \( T_{lk,1}^{m,r} = T_{1} \) |
| Classes of tricyclic graphs generated from BG | \( \xi_{1} \) | \( \xi_{2} \) | \( \xi_{3} \) |

### Table 7

| Base Graphs (BG) | \( L_{1} \) | \( L_{2} \) | \( L_{3} \) |
|------------------|-----------|-----------|-----------|
| Joining \( k \) vertices to \( l \) vertices of degree \( \geq 2 \) | \( X_{lk,1}^{m,r} = X_{1} \) | \( Y_{lk,1}^{m,r} = Y_{1} \) | \( Z_{lk,1}^{m,r} = Z_{1} \) |
| Classes of tricyclic graphs generated from BG | \( \xi_{1} \) | \( \xi_{2} \) | \( \xi_{3} \) |

from the vertex of degree \( 2k+2 \) and joining these vertices to the vertex of degree \( k+2 \). After \( l-1 \) iteration, we obtain \( A_{1}^{l-1} \) from \( A_{1}^{l} \) by deleting \((l-1)k \) pendant vertices from a vertex of degree \((l-1)k+2 \) and joining these vertices to the last vertex of degree \( k+2 \), where \( 2 \leq l \leq m+r \). Using the same transformation, we obtain \( A_{2}^{j} \) from \( A_{2}^{j-1} \) for \( 1 \leq i \leq l-5 \) by the deletion of \( ik \) pendant vertices from a vertex of degree \( ik+2 \) and joining these vertices to the vertex of degree \( k+2 \) plus two. Moreover, we obtain \( A_{3}^{j} \) from \( A_{3}^{j-1} \) for \( l-4 \leq i \leq l-1 \) by the deletion of \( ik \) pendant vertices from a vertex of degree \( ik+3 \) and joining these vertices to the last vertex of degree \( k+3 \), where \( A_{0}^{0} = A_{2} \). Similarly, for \( 1 \leq i \leq l-1 \), we obtain \( B_{2}^{j}, C_{2}^{j}, D_{2}^{j}, E_{2}^{j} \) from \( B_{2}^{j-1}, C_{2}^{j-1}, D_{2}^{j-1}, E_{2}^{j-1} \) respectively.

Assume that \( u_{1}, u_{2}, u_{3}, u_{4} \) and \( u_{5} \), are classes of the tricyclic graphs obtained from \( G_{1}, G_{2}, G_{3}, G_{4} \) and \( G_{5} \) (shown in Figure 1) respectively such that the order of each graph is \( m+ik+r+4 \) with \( lk \) pendant vertices. Let \( u_{i}^{ik} \) be a class of all the tricyclic graphs with three cycles such that each graph has order \( n \) and pendant vertices \( lk \), where \( k \geq 1, n \geq 16 \) and \( 1 \leq l \leq n \). Similarly, tricyclic graphs with four and six cycles obtained from the base graphs presented in Figure 2 are given in Table 6 and Table 7. Moreover, \( \xi_{1}^{ik} \) and \( \xi_{2}^{ik} \) are classes of all the tricyclic graphs with four and six cycles respectively that include each graph of order \( n \) and pendant vertices \( lk \). Finally, we obtain the tricyclic graphs with seven cycles \( (K_{i,k,1}^{m,r} = K_{1}) \) and \( (K_{i,k,2}^{m,r} = K_{2}) \) from the base graph \( K \) (see, Figure 2) and \( u_{i}^{ik} \) be a class of all the tricyclic graphs with seven cycles such that each graph has order \( n \) and pendant vertices \( lk \). Now, by the deletion and addition of pendant vertices, we have \( R_{i}^{j}, S_{j}^{i}, T_{j}^{i}, U_{j}^{i}, V_{j}^{i}, W_{j}^{i} \) and \( K_{j} \) from \( R_{1}^{j-1}, S_{j}^{i-1}, T_{j}^{i-1}, U_{j}^{i-1}, V_{j}^{i-1}, W_{j}^{i-1} \) and \( K_{i-1}^{j} \) respectively, where \( 1 \leq i \leq l-1 \) and \( 1 \leq j \leq 2 \).

Now, we present some important lemmas which are frequently used in the next section of main results.

**Lemma 2.1.** For \( u, v, a, b \geq 1 \), the functions (i) \( f_{1}(u) = -au(u+b) \), (ii) \( f_{2}(u) = -au(u+b) \), (iii) \( f_{3}(u, v) = -3u^{3}(v^{2} + 3v + 2) - 12u^{2}(v+1) \), and (iv) \( f_{4}(u, v) = -au(u+v+b) - c \) are strictly decreasing functions.

**Proof:** Since, (i) \( \frac{df_{1}(u)}{du} = -a(2u+b) < 0 \), (ii) \( \frac{df_{2}(u)}{du} = -a(2u+b) < 0 \), (iii) \( \frac{df_{3}(u,v)}{du} = -9u^{2}(v^{2} + 3v + 2) - 24u(v+1) < 0 \) and \( \frac{df_{3}(u,v)}{dv} = -3u^{3}(2v+3) - 12u^{2} < 0 \), (iv) \( \frac{df_{4}(u,v)}{du} = -av(2u+v+b) < 0 \) and \( \frac{df_{4}(u,v)}{dv} = -au(2u+v+b) < 0 \) for \( u, v, a, b \geq 1 \). Therefore, \( f_{1}(u), f_{2}(u), f_{3}(u, v) \) and \( f_{4}(u, v) \) are strictly decreasing functions.

By the use of Definition 1.3 and the generalization of Table 1/Table 5 for the \( i \)th iteration of the deletion of \( ik \) pendant vertices from the vertex of degree \( ik+2 \) and joining them to a vertex of degree \( k+2 \), we obtain the \( F \)-index of the tricyclic graphs \( A_{1}^{i}, B_{1}^{i}, C_{1}^{i}, D_{1}^{i} \) and \( E_{1}^{i} \) with three cycles and \( lk \) pendant vertices for \( 0 \leq i \leq l-1 \) in the following lemma.

**Lemma 2.2.** For \( m = m_{1} + m_{2} + m_{3} \geq 9, k \geq 1, r \geq 2, 2 \leq l \leq m+r, 0 \leq i \leq l-1 \) and \( m_{j} \geq 3 \) with \( j = 1, 2, 3 \), the \( F \)-index of the tricyclic graphs defined above are

(i) \( F(A_{1}^{i}) = lk+(l-i-1)(k+2)^{3} + 8(m+r-l+i)+(i+1)k+2)^{3} + 108 \),

(ii) \( F(B_{1}^{i}) = lk+(l-i-1)(k+2)^{3} + 8(m+r-l+i)+(i+1)k+2)^{3} + 108 \),

(iii) \( F(C_{1}^{i}) = lk+(l-i-1)(k+2)^{3} + 8(m+r-l+i)+(i+1)k+2)^{3} + 108 \),

(iv) \( F(D_{1}^{i}) = lk+(l-i-1)(k+2)^{3} + 8(m+r-l+i)+(i+1)k+2)^{3} + 108 \),

(v) \( F(E_{1}^{i}) = lk+(l-i-1)(k+2)^{3} + 8(m+r-l+i)+(i+1)k+2)^{3} + 108 \).
Lemma 2.3. \( F \) for the tricyclic graphs \( A^i_1, B^i_2, C^i_2, D^i_2 \) and \( E^i_2 \) are

(i) \( F(A^i_2) = \begin{cases} 
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 8(m+r+1-l+i) + [(i+1)k+3]^3 + 118, \\
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 8(m+r-l+2+i) + [(i+1)k+3]^3 + 128, \\
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 8(m+r-l+2+i) + [(i+1)k+3]^3 + 152, \\
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 8(m+r-l+3+i) + [(i+1)k+3]^3 + 216.
\end{cases} \)

(ii) \( F(B^i_2) = \begin{cases} 
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 4(k+3)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 5, \\
\frac{1}{i} + \frac{1}{i-1}(k+3)^3 + 8(m+r) + 27(i-l+4) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 1, \\
\frac{1}{i} + \frac{1}{i-1}(k+4)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 4,
\end{cases} \)

(iii) \( F(C^i_2) = \begin{cases} 
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + 2(k+3)^3 + (k+4)^3 + 8(m+r+l-i) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 3, \\
\frac{1}{i} + \frac{1}{i-1}(k+4)^3 + 8(m+r+2) + (l-k+3)^3; \quad & \text{for } i = l - 2, \\
\frac{1}{i} + \frac{1}{i-1}(k+4)^3 + 8(m+r+2) + 64; \quad & \text{for } i = l - 1,
\end{cases} \)

(iv) \( F(D^i_2) = \begin{cases} 
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + (k+3)^3 + (k+5)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 3, \\
\frac{1}{i} + \frac{1}{i-1}(k+5)^3 + 8(m+r+2) + (l-k+3)^3; \quad & \text{for } i = l - 2, \\
\frac{1}{i} + \frac{1}{i-1}(k+5)^3 + 8(m+r+2) + 27; \quad & \text{for } i = l - 1,
\end{cases} \)

(v) \( F(E^i_2) = \begin{cases} 
\frac{1}{i} + \frac{1}{i-1}(k+2)^3 + (k+6)^3 + 8(m+r+4-l+i) + [(i+1)k+3]^3; \quad & \text{for } 1 \leq i \leq l - 2, \\
\frac{1}{i} + \frac{1}{i-1}(k+6)^3 + 8(m+r+3); \quad & \text{for } i = l - 1.
\end{cases} \)

Lemma 2.4. For \( m = m_1 + m_2 + m_3 \geq 9, k \geq 1, r \geq 2, 2 \leq l \leq m + r, 0 \leq i \leq l - 1, 1 \leq j \leq 2 \) and \( m_p \geq 3 \) with \( p = 1, 2, 3 \), we have
Proof. (a) Since the degree sequences of the base graphs of the tricyclic graph with four cycles \( T_4 \) (see, \( H_3 \) in Figure 2) and tricyclic graph with three cycles \( A'_1 \) (see, \( G_1 \) in Figure 1) are equal. Therefore, the degree sequences of the graphs \( T_4 \) and \( A'_1 \) having \( k \geq 1 \) pendant vertices attached with \( l \) vertices of degree (i) exactly 2 for \( j = 1 \) and (ii) greater or equal 2 for \( j = 2 \) are equal. Consequently, by Lemma 1.1, \( F(T_4) = F(A'_1) \). Similarly, the degree sequences of the tricyclic graphs with four cycles \( S'_j \) and \( R'_j \) are equal to the degree sequences of the tricyclic graphs with three cycles \( B'_j \) and \( D'_j \) respectively. Thus, by Lemma 1.1, we have \( F(S'_j) = F(B'_j) \) and \( F(R'_j) = F(D'_j) \). (b) Proof is same as of part (a). (c) Proof is same as of part (a).

3 Extremal graphs with respect to \( F \)-index

The results of extremal graphs in the complete class of tricyclic graphs with fixed pendant vertices are obtained in this section.

Lemma 3.1. For \( m = m_1 + m_2 + m_3 \geq 9, k \geq 1, r \geq 2, 2 \leq l \leq m + r \) and \( m_j \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(A_1) \leq F(A_2), F(B_1) \leq F(B_2), F(C_1) \leq F(C_2), F(D_1) \leq F(D_2) \) and \( F(E_1) \leq F(E_2) \).

Proof. Consider:

Case 1: Using Lemma 2.2(i) and Lemma 2.3(i) for \( i = 0 \), we have \( F(A_1) - F(A_2) = F(A_{m,r}^{m_1,m_2}) - F(A_{m,r}^{m_1,m_2}) = -12(k + 5) \). By Lemma 2.1(i), it follows that the tricyclic graph \( A_1 \) has \( F \)-index less than of \( A_2 \).

Case 2: Using Lemma 2.2(ii) and Lemma 2.3(ii) for \( i = 0 \), we have \( F(B_1) - F(B_2) = F(B_{m,r}^{m_1,m_2}) - F(B_{m,r}^{m_1,m_2}) = -6k(2k + 11) \). By Lemma 2.1(i), it follows that \( F \)-index of the tricyclic graph \( B_1 \) is less than the \( F \)-index of the tricyclic graph \( B_2 \).

Case 3: Using Lemma 2.2(iii) and Lemma 2.3(iii) for \( i = 0 \), we have \( F(C_1) - F(C_2) = F(C_{m,r}^{m_1,m_2}) - F(C_{m,r}^{m_1,m_2}) = -12(k + 6) \). By Lemma 2.1(i), it follows that \( F \)-index of the tricyclic graph \( C_1 \) is less than the \( F \)-index of the tricyclic graph \( C_2 \).

Case 4: Using Lemma 2.2(iv) and Lemma 2.3(iv) for \( i = 0 \), we have \( F(D_1) - F(D_2) = F(D_{m,r}^{m_1,m_2}) - F(D_{m,r}^{m_1,m_2}) = -12(k + 6) \). By Lemma 2.1(i), it follows that \( F \)-index of the tricyclic graph \( D_1 \) is less than the \( F \)-index of the tricyclic graph \( D_2 \).

Case 5: Using Lemma 2.2(v) and Lemma 2.3(v) for \( i = 0 \), we have \( F(E_1) - F(E_2) = F(E_{m,r}^{m_1,m_2}) - F(E_{m,r}^{m_1,m_2}) = -12(k + 8) \). By Lemma 2.1(i), it follows that \( F \)-index of the tricyclic graph \( E_1 \) is less than the \( F \)-index of \( E_2 \).

Consequently, from all the cases, we have \( F(A_1) \leq F(A_2), F(B_1) \leq F(B_2), F(C_1) \leq F(C_2), F(D_1) \leq F(D_2) \) and \( F(E_1) \leq F(E_2) \).

Lemma 3.2. For \( m = m_1 + m_2 + m_3 \geq 9, k \geq 1, r \geq 2, 2 \leq l \leq m + r \) and \( m_j \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(A_1) \leq F(B_1) \leq F(C_1) \leq F(D_1) \leq F(E_1) \).

Proof. Consider:

Case 1: By Lemma 2.2(ii) and (iii) for \( i = 0 \), we have \( F(A_1) - F(B_1) = F(A_{m,r}^{m_1,m_2}) - F(B_{m,r}^{m_1,m_2}) = -18 < 0 \). It follows that \( F(A_1) < F(B_1) \).

Case 2: Using Lemma 2.2(ii) and (iii) for \( i = 0 \), we have \( F(B_1) - F(C_1) = F(B_{m,r}^{m_1,m_2}) - F(C_{m,r}^{m_1,m_2}) = -36 < 0 \). Which implies that \( F(B_1) < F(C_1) \).
Case 3: By Lemma 2.2((iii) and (iv)) for $i = 0$, we have $F(C_1) - F(D_1) = F(H_{t,k,l}^m) - F(D_{t,k,l}) = -24 < 0$. It follows that $F(C_1) < F(D_1)$.

Case 4: By Lemma 2.2((iv) and (v)) for $i = 0$, we have $F(D_1) - F(E_1) = F(H_{t,k,l}^m) - F(D_{t,k,l}) = -72 < 0$. It implies that $F(D_1) < F(E_1)$.

From all the cases, we conclude that $F(A_1) \leq F(B_1) \leq F(C_1) \leq F(D_1) \leq F(E_1)$.

Theorem 3.3. If $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A_1) \leq F(G), F(B_1) \leq F(G), F(C_1) \leq F(G), F(D_1) \leq F(G)$ and $F(E_1) \leq F(G)$ for each $G \in \mathbb{U}_1$, $G \in \mathbb{U}_2$, $G \in \mathbb{U}_3$, $G \in \mathbb{U}_4$ and $G \in \mathbb{U}_5$ respectively. Moreover, equalities hold if $G \cong A_1, G \cong B_1, G \cong C_1, G \cong D_1$ and $G \cong E_1$ respectively.

Proof. We consider the following cases:

Case 1: If $G = A_1$ then by Lemma 2.2(i) (for $i = 0$ and $i = 1$) $F(A_1) - F(A_1) = -6k^2(k+2)$. Moreover using Lemma 2.1(i), we have $F(A_1) < F(A_1)$. Similarly, by Lemma 2.2((ii)-(iii)) and Lemma 2.1(ii), we have $F(B_1) - F(B_1) = F(C_1) - F(C_1)^{i} = F(D_1) - F(D_1)^{i} = F(E_1) - F(E_1)^{i} = -6k^2(k+2) < 0$ which implies that $F(B_1) < F(B_1), F(C_1) < F(C_1)^{i}, F(D_1) < F(D_1)^{i}$ and $F(E_1) < F(E_1)^{i}$. Consequently, if $G = A_1, G = B_1, G = C_1, G = D_1$ and $G = E_1$ then $F(A_1) < F(G), F(B_1) < F(G), F(C_1) < F(G), F(D_1) < F(G)$ and $F(E_1) < F(G)$ for each $G \in \mathbb{U}_1, G \in \mathbb{U}_2, G \in \mathbb{U}_3, G \in \mathbb{U}_4$ and $G \in \mathbb{U}_5$ respectively.

Case 2: If $G = A_1$ for $1 \leq i \leq l - 2$, by Lemma 2.2(i) $F(A_1) - F(A_1) = -8 + (k + 2)^3 + (i + k + 2)^3 - (i + k + 2)^3 - 3k^3(i^2 + 1) - 12k^2(i + 1)$. By Lemma 2.1(iii), $F(A_1) < F(A_1)$. Using $i = 1, 2, 3, ..., l - 2$, we have $F(A_1) < F(A_1), F(A_1) < F(A_1)$, ..., $F(A_1) < F(A_1)$ By Combining these inequalities $F(A_1) < F(A_1) < F(A_1) < ... < F(A_1)$.

Using Case 1 and above inequality, $F(A_1) < F(A_1)$ for $1 \leq i \leq l - 1$ which implies that $F(A_1) < F(G)$. Similarly, by Lemma 2.2((ii)-(v)) $F(B_1) - F(B_1) = F(C_1) - F(C_1)^{i} = F(D_1) - F(D_1)^{i} = F(E_1) - F(E_1)^{i} = -8 + (k + 2)^3 + (i + k + 2)^3 - (i + k + 2)^3 - 3k^3(i^2 + 1) - 12k^2(i + 1)$. By Lemma 2.1(iii), $F(B_1) < F(B_1), F(C_1) < F(C_1)^{i}$, $F(D_1) < F(D_1)^{i}$ and $F(E_1) < F(E_1)^{i}$, where $1 \leq i \leq l - 2$. Using Case 1 and above inequalities, we have $F(B_1) < F(B_1), F(C_1) < F(C_1)^{i}, F(D_1) < F(D_1)^{i}$ and $F(E_1) < F(E_1)^{i}$, for $1 \leq i \leq l - 1$. Consequently, if $G = A_1, G = B_1, G = C_1, G = D_1$ and $G = E_1$ for $1 \leq i \leq l - 1$ then $F(A_1) < F(G), F(B_1) < F(G), F(C_1) < F(G), F(D_1) < F(G)$ and $F(E_1) < F(G)$ for each $G \in \mathbb{U}_1, G \in \mathbb{U}_2, G \in \mathbb{U}_3, G \in \mathbb{U}_4$ and $G \in \mathbb{U}_5$ respectively.

Case 3: If $G = A_1, G = B_2, G = C_2, G = D_2$ and $G = E_2$, using the same way as of Case 2, we can prove that $F(A_1) < F(G), F(B_2) < F(G), F(C_2) < F(G), F(D_2) < F(G)$ and $F(E_2) < F(G)$ respectively, where $1 \leq i \leq l - 1$. Moreover, by Lemma 3.1 $F(A_1) < F(A_2), F(B_2) < F(B_2), F(C_2) < F(C_2), F(D_2) < F(D_2)$ and $F(E_2) < F(E_2)$. Consequently, $F(A_1) < F(G), F(B_2) < F(G), F(C_2) < F(G), F(D_2) < F(G)$ and $F(E_2) < F(G)$ for each $G \in \mathbb{U}_1, G \in \mathbb{U}_2, G \in \mathbb{U}_3, G \in \mathbb{U}_4$ and $G \in \mathbb{U}_5$ respectively.

Case 4: If $G \in \mathbb{U}_1 - \{A_1, A_2, B_2\}, G \in \mathbb{U}_2 - \{B_1, B_2\}, G \in \mathbb{U}_3 - \{C_1, C_2\}, G \in \mathbb{U}_4 - \{D_1, D_2\}$ and $G \in \mathbb{U}_5 - \{E_1, E_2\}$. Using transformation of the deletion of pendant vertices and joining them with degree greater or equal to two, we obtain $A_1$ or $A_1, B_1$ or $B_2, C_1$ or $C_1, D_1$ or $D_2$, and $E_1$ or $E_2$ respectively. Thus, we follow the Case 2 or Case 3.

Thus, from all the cases, we have $F(A_1) < F(G)$ for each $G \in \mathbb{U}_1, F(B_1) < F(G)$ for each $G \in \mathbb{U}_2, F(C_1) < F(G)$ for each $G \in \mathbb{U}_3, F(D_1) < F(G)$ for each $G \in \mathbb{U}_4$ and $F(E_1) < F(G)$ for each $G \in \mathbb{U}_5$, where equalities hold if $G \cong A_1, G \cong B_1, G \cong C_1, G \cong D_1$ and $G \cong E_1$ respectively.

Theorem 3.4. If $k \geq 1, n \geq 16 + lk$ and $1 \leq l \leq n - lk$. Then $F(A_1) < F(G)$ for each $G \in \mathbb{U}_1$, where $\mathbb{U}_1$ is a class of all the tricyclic graphs with three cycles such that each graph has order $n$ and pendant vertices $lk$. Moreover, equality holds if $G \cong A_1$. 
Proof. We consider the following cases:

Case 1: If $G \in U_1$, by Theorem 3.3 $F(A_1) \leq F(G)$ for each $G \in U_1$.

Case 2: If $G \in U_2$, by Theorem 3.3 $F(B_1) \leq F(G)$ for each $G \in U_2$. Also, by Lemma 3.6 $F(A_1) \leq F(B_1)$ which implies that $F(A_1) \leq F(G)$ for each $G \in U_2$.

Case 3: Assume that $G \in U_3$. Using Theorem 3.3, we have $F(C_1) \leq F(G)$ for each $G \in U_3$. Now, by Lemma 3.2 $F(A_1) \leq F(C_1)$. Consequently, $F(A_1) \leq F(G)$ for each $G \in U_3$.

Case 4: If $G \in U_4$ then by Theorem 3.3 $F(D_1) \leq F(G)$ for each $G \in U_4$. Moreover, by Lemma 3.2 $F(A_1) \leq F(D_1)$ which implies that $F(A_1) \leq F(G)$ for each $G \in U_4$.

Case 5: If $G \in U_5$, by Theorem 3.3 $F(C_1) \leq F(G)$ for each $G \in U_5$ and by Lemma 3.2 $F(A_1) \leq F(E_1)$. Consequently, $F(A_1) \leq F(G)$ for each $G \in U_5$.

So, from all the cases, we conclude that $F(A_1) \leq F(G)$ for each $G \in U^{lk}_n$ and equality holds if $G \cong A_1$.

Lemma 3.5. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq n - lk$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A^{l-1}_1) \leq F(A^{l-1}_2)$, $F(B^{l-1}_1) \leq F(B^{l-1}_2)$, $F(C^{l-1}_1) \leq F(C^{l-1}_2)$, $F(D^{l-1}_1) \leq F(D^{l-1}_2)$ and $F(E^{l-1}_1) \leq F(E^{l-1}_2)$.

Proof. Consider:

Case 1: By Lemma 2.2(i) and Lemma 2.3(i) for $i = l - 1$, we have $F(A^{l-1}_1) - F(A^{l-1}_2) = -3lk(lk + 5)$. By Lemma 2.1(iv), $F(A^{l-1}_1) < F(A^{l-1}_2)$.

Case 2: Using Lemma 2.2(ii) and Lemma 2.3(ii) for $i = l - 1$, we have $F(B^{l-1}_1) - F(B^{l-1}_2) = -6lk(lk + 6)$. Using Lemma 2.1(iv), we have $F(B^{l-1}_1) \leq F(B^{l-1}_2)$.

Case 3: By Lemma 2.2(iii) and Lemma 2.3(iii) for $i = l - 1$, we have $F(C^{l-1}_1) - F(C^{l-1}_2) = -6lk(lk + 6)$. By Lemma 2.1(iv), $F(C^{l-1}_1) \leq F(C^{l-1}_2)$.

Case 4: By Lemma 2.2(iv) and Lemma 2.3(iv) for $i = l - 1$, we have $F(D^{l-1}_1) - F(D^{l-1}_2) = -9lk(lk + 7)$. By Lemma 2.1(iv), $F(D^{l-1}_1) < F(D^{l-1}_2)$.

Case 5: By Lemma 2.2(v) and Lemma 2.3(v) for $i = l - 1$, we have $F(E^{l-1}_1) - F(E^{l-1}_2) = -12lk(lk + 8)$. Using Lemma 2.1(iv), $F(E^{l-1}_1) < F(E^{l-1}_2)$.

From all the cases, we conclude that $F(A^{l-1}_1) \leq F(A^{l-1}_2)$, $F(B^{l-1}_1) \leq F(B^{l-1}_2)$, $F(C^{l-1}_1) \leq F(C^{l-1}_2)$, $F(D^{l-1}_1) \leq F(D^{l-1}_2)$ and $F(E^{l-1}_1) \leq F(E^{l-1}_2)$.

Lemma 3.6. For $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(A^{l-1}_2) \leq F(B^{l-1}_2)$, $F(C^{l-1}_2) \leq F(D^{l-1}_2) \leq F(E^{l-1}_2)$.

Proof. Consider:

Case 1: By Lemma 2.3(i) and (ii) for $i = l - 1$, we have $F(A^{l-1}_2) - F(B^{l-1}_2) = -3l^2k^2 - 21lk - 28$. Using Lemma 3.1(iv), we have $F(A^{l-1}_2) < F(B^{l-1}_2)$.

Case 2: Using Lemma 2.3(iii) and (iii) for $i = l - 1$, we have $F(B^{l-1}_2) - F(C^{l-1}_2) = -18 < 0$. By Lemma 3.1(iv), we have $F(B^{l-1}_2) \leq F(C^{l-1}_2)$.

Case 3: By Lemma 2.3(iv) and (iv) for $i = l - 1$, we have $F(C^{l-1}_2) - F(D^{l-1}_2) = -3l^2k^2 - 27lk - 24$. Using Lemma 3.1(iv), we have $F(C^{l-1}_2) < F(D^{l-1}_2)$.

Case 4: By Lemma 2.3(iv) and (v) for $i = l - 1$, we have $F(D^{l-1}_2) - F(E^{l-1}_2) = -3l^2k^2 - 33lk - 72$. By Lemma 3.1, we have $F(D^{l-1}_2) < F(E^{l-1}_2)$.

From all the cases, we conclude that $F(A^{l-1}_2) \leq F(B^{l-1}_2)$, $F(C^{l-1}_2) \leq F(D^{l-1}_2) \leq F(E^{l-1}_2)$.

Theorem 3.7. If $m = m_1 + m_2 + m_3 \geq 9$, $k \geq 1$, $r \geq 2$, $2 \leq l \leq m + r + 4$ and $m_j \geq 3$ with $j = 1, 2, 3$. Then, $F(G) \leq F(A^{l-1}_2)$, $F(G) \leq F(B^{l-1}_2)$, $F(G) \leq F(C^{l-1}_2)$, $F(G) \leq F(D^{l-1}_2)$ and $F(G) \leq F(E^{l-1}_2)$ for each $G \in U_4$, $G \in U_2$. 


\( G \in \mathcal{U}_3 \), \( G \in \mathcal{U}_4 \) and \( G \in \mathcal{U}_5 \) respectively. Moreover, equalities hold if \( G \cong A_2^{l_1} \), \( G \cong B_2^{l_1} \), \( G \cong C_2^{l_1} \), \( G \cong D_2^{l_1} \) and \( G \cong E_2^{l_1} \) respectively.

**Proof.** Proof is same as of Theorem 3.3 with the help of Lemma 2.3, Lemma 3.5 and Lemma 3.6.

**Theorem 3.8.** If \( k \geq 1 \), \( n \geq 16 + lk \) and \( 1 \leq l \leq n - lk \). Then \( F(G) \leq F(E_2^{l_1}) \) for each \( G \in \mathcal{U}_n^{\mathcal{L}_k} \), where \( \mathcal{U}_n^{\mathcal{L}_k} \) is a class of all the tricyclic graphs with three cycles such that each graph has order \( n \) and pendant vertices \( lk \). Moreover, equality holds if \( G \cong E_2^{l_1} \).

**Proof.** Proof follows by Theorem 3.4 with the help of Theorem 3.3, Lemma 3.5 & Lemma 3.6.

By the similar arguments as of the tricyclic graphs with three cycles, we obtain the following result for the tricyclic graphs with four, six and seven cycles.

**Theorem 3.9.** If \( k \geq 1 \), \( n \geq 16 + lk \) and \( 1 \leq l \leq n - lk \). Then
(a) \( F(T_1) \leq F(G) \leq F(R_2^{l_1}) \) for each \( G \in \xi_{3k}^{\mathcal{L}_k} \), where lower and upper bounds holds for \( G \cong T_1 \) and \( G \cong R_2^{l_1} \) respectively.
(b) \( F(Z_1) \leq F(G) \leq F(X_2^{l_1}) \) for each \( G \in \xi_{3k}^{\mathcal{L}_k} \), where lower and upper bounds holds for \( G \cong Z_1 \) and \( G \cong X_2^{l_1} \) respectively.
(c) \( F(L_1) \leq F(G) \leq F(L_2^{l_1}) \) for each \( G \in \mu_{3k}^{\mathcal{L}_k} \), where lower and upper bounds holds for \( G \cong L_1 \) and \( G \cong L_2^{l_1} \) respectively.

**Proof.** (a) By Lemma 2.4, \( F(T_1) = F(A_1), F(T_2) = F(A_2), F(S_1) = F(B_1), F(S_2) = F(B_2), F(R_1) = F(D_1) \) and \( F(R_2) = F(D_2) \). Now, by Lemma 3.1 \( F(A_1) \leq F(A_2) \), \( F(B_1) \leq F(B_2) \) and \( F(D_1) \leq F(D_2) \). Consequently, \( F(T_1) \leq F(T_2) \), \( F(S_1) \leq F(S_2) \) and \( F(R_1) \leq F(R_2) \). Moreover, by Lemma 3.2 \( F(A_1) \leq F(B_1) \leq F(D_1) \) which implies that \( F(T_1) \leq F(S_1) \leq F(R_1) \). Finally, by Theorem 3.3 and Theorem 3.4, we have \( F(A_1) \leq F(G) \) for each \( G \in \mathcal{U}_n^{\mathcal{L}_k} \), where \( \mathcal{U}_n^{\mathcal{L}_k} \) is a class of all the tricyclic graphs with three cycles such that each graph has order \( n \) and pendant vertices \( lk \). Consequently, \( F(T_1) \leq F(G) \) for each \( G \in \xi_{3k}^{\mathcal{L}_k} \), where lower bound holds for \( G \cong T_1 \). Similarly, by Theorem 3.5, Lemma 3.6, Theorem 3.7 and Theorem 3.8, we have \( F(G) \leq F(E_2^{l_1}) \) for each \( G \in \xi_{3k}^{\mathcal{L}_k} \), where \( \xi_{3k}^{\mathcal{L}_k} \) is a class of all the tricyclic graphs with four cycles such that each graph has order \( n \) and pendant vertices \( lk \). Thus, \( F(T_1) \leq F(G) \leq F(R_2^{l_1}) \) for each \( G \in \xi_{3k}^{\mathcal{L}_k} \), where lower and upper bounds holds for \( G \cong T_1 \) and \( G \cong R_2^{l_1} \) respectively. (b) Proof is same as of part (a). (c) Proof is same as of part (a).

**Theorem 3.10.** If \( k \geq 1 \), \( n \geq 16 + \alpha \) and \( 1 \leq l \leq n - \alpha \). Then \( F(A_1) \leq F(G) \leq F(E_2^{\alpha_1}) \) for each \( G \in \Omega_\alpha^{\mathcal{L}_k} \), where \( \Omega_\alpha^{\mathcal{L}_k} = \{ \mathcal{L}_k, \xi_{3k}^{\mathcal{L}_k}, \mu_{3k}^{\mathcal{L}_k} \} \), \( \alpha \) is number of pendant vertices and bounds (lower and upper) holds for \( G \cong T_1 \) and \( G \cong R_2^{\alpha_1} \) respectively.

**Proof.** Using Lemma 2.4, Theorem 3.4, Theorem 3.8 and Theorem 3.9.

## 4 Lower and upper bounds

The ordering and investigate of bounds (lower and upper) of the \( F \)-index in the complete class of tricyclic graphs of three, four, six or seven cycles with fixed pendant vertices is given in this section.

**Theorem 4.1.** If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_j \geq 3 \) with \( j = 1, 2, 3 \). Then, (a) \( F(l_{14}) < F(l_{12}) < F(l_{13}) < F(l_{14}) < F(l_{15}) < F(\xi_3) < F(\xi_2) < F(\xi_1) \) and (c) \( F(\xi_1) < F(\xi_2) < F(\xi_1) \).

**Proof.** (a) Firstly, we prove that \( F(l_{14}) < F(l_{12}) \). For the purpose, we show that for each \( G \in \mathcal{U}_4 \) there exists \( G^* \in \mathcal{U}_2 \) such that \( F(G) < F(G^*) \), where \( n = m + r + 4 + lk \) is order of both \( G \) and \( G^* \) with \( lk \) pendant vertices in each. We assume that \( G = A_1^1 \) and \( G^* = B_1^1 \) for \( 1 \leq i \leq l - 1 \). By Lemma 2.2, \( F(G) - F(G^*) = -10 < 0 \) which implies that \( F(G) < F(G^*) \). Similarly, by Lemma 2.3 it can be proved that if \( G = A_2^1 \) for \( 1 \leq i \leq l - 1 \) then there exists \( G^* = B_2^1 \) such that \( F(G) < F(G^*) \). In addition, if \( G \in \mathcal{U}_4 - \{ A_1^1, A_2^1 \} \), using transformation of delation and
joining the pendant vertices. Then, \( G = A^1 \) or \( G = A^2 \). Thus, we get \( G^* \in \mathbb{L}_2 \) such that \( F(G) < F(G^*) \). So, we conclude that \( F(l_1) < F(l_2) \). Similarly, we can prove that \( F(l_2) < F(l_3) \), \( F(l_3) < F(l_4) \), and \( F(l_4) < F(l_5) \). Consequently, we have \( F(l_1) < F(l_2) < F(l_3) < F(l_4) < F(l_5) \). Proves of (b) and (c) are same as of (a).

**Theorem 4.2.** If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \). If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \). If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \). If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \). If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \). If \( m = m_1 + m_2 + m_3 \geq 9 \), \( k \geq 1 \), \( r \geq 2 \), \( 2 \leq l \leq m + r + 4 \) and \( m_1 \geq 3 \) with \( j = 1, 2, 3 \). Then, \( F(l_1) < F(l_2) < F(l_3) \).

**Proof.** Proof is obvious using Lemma 2.4, Theorem 4.1 (a), (b) and (c).

**Theorem 4.3.** Let \( G \in \mathcal{O}_{3+}^n \) be a tricyclic graph of order \( n \geq 16 + \alpha \) with three, four, six or seven cycles and \( \alpha \geq 1 \) pendant vertices. Then, \( 8n + 12\alpha + 76 \leq F(G) \leq 8(n - 1) - 7\alpha + (\alpha + 6)^3 \), where the lower and upper bounds are achieved if and only if \( G \cong A_1 \) with \( k = 1 \) and \( G \cong E_2^{1-1} \) respectively.

**Proof.** Assuming \( l = \alpha \) and \( k = 1 \) in Lemma 2.2 (i) for \( i = 0 \) and Lemma 2.3(v) for \( i = l - 1 \), we have \( F(A_1) = 8n + 12\alpha + 76 \) and \( F(E_2^{-1}) = 8(n - 1) - 7\alpha + (\alpha + 6)^3 \). Moreover, by Theorem 3.4 and Theorem 3.8 \( F(A_1) < F(G) \) and \( F(G) \leq F(E_2^{1-1}) \) for each \( G \) being a tricyclic graph of order \( n \geq 16 \) with three cycles and \( \alpha \geq 1 \) pendant vertices. Thus, we have \( 8n + 12\alpha + 76 \leq F(G) \leq 8(n - 1) - 7\alpha + (\alpha + 6)^3 \), where lower and upper bounds are achieved if and only if \( G \cong A_1 \) with \( k = 1 \) and \( G \cong E_2^{1-1} \) respectively.

## 5 Conclusion

In this paper, we studied the complete class of tricyclic graphs consisting on three, four, six and seven cycles for certain number of pendant vertices with respect to F-index. We proved the existence of the extremal graphs and construct the ordering of graphs with respect to F-index. Mainly, we computed the bounds (lower and upper) of F-index for the same family of graphs.

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