Abstract

Let $\prod_{i=1}^{d}(X - \alpha_i Y) \in \mathbb{C}[X,Y]$ be a binary form and let $\epsilon_1, \ldots, \epsilon_d$ be nonzero complex numbers. We consider the family of binary forms $\prod_{i=1}^{d}(X - \alpha_i \epsilon_i^a Y)$, $a \in \mathbb{Z}$, which we write as

$$X^d - U_1(a)X^{d-1}Y + \cdots + (-1)^{d-1}U_{d-1}(a)XY^{d-1} + (-1)^dU_d(a)Y^d.$$ 

In this paper we study these sequences $(U_h(a))_{a \in \mathbb{Z}}$ which turn out to be linear recurrence sequences.

Résumé

Soit $\prod_{i=1}^{d}(X - \alpha_i Y)$ une forme binaire de $\mathbb{C}[X,Y]$ et soit $\epsilon_1, \ldots, \epsilon_d$ des nombres complexes non nuls. Nous considérons la famille des formes binaires $\prod_{i=1}^{d}(X - \alpha_i \epsilon_i^a Y)$, $a \in \mathbb{Z}$, que nous écrivons sous la forme

$$X^d - U_1(a)X^{d-1}Y + \cdots + (-1)^{d-1}U_{d-1}(a)XY^{d-1} + (-1)^dU_d(a)Y^d.$$ 

Le but de cet article est d’étudier ces suites $(U_h(a))_{a \in \mathbb{Z}}$ qui s’avèrent être des suites récurrentes linéaires.

Keywords: Linear recurrence sequences; binary forms; units of algebraic number fields; families of Diophantine equations; exponential polynomials

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1 Introduction

Let us consider a binary form $F_0(X, Y) \in \mathbb{C}[X, Y]$ which satisfies $F_0(1, 0) = 1$. We write it as

$$F_0(X, Y) = X^d + a_1X^{d-1}Y + \cdots + a_dY^d = \prod_{i=1}^{d}(X - \alpha_iY).$$

Let $\epsilon_1, \ldots, \epsilon_d$ be $d$ nonzero complex numbers not necessarily distinct. Twisting $F_0$ by the powers $\epsilon_1^a, \ldots, \epsilon_d^a (a \in \mathbb{Z})$, we obtain the family of binary forms

$$F_a(X, Y) = \prod_{i=1}^{d}(X - \alpha_i^aY),$$

which we write as

$$F_a(X, Y) = X^d - U_1(a)X^{d-1}Y + \cdots + (-1)^{d-1}U_{d-1}(a)XY^{d-1} + (-1)^dU_d(a)Y^d. \tag{1}$$

Therefore

$$U_h(0) = (-1)^h a_h \quad (1 \leq h \leq d). \tag{2}$$

In [6] and [7], we consider some families of diophantine equations

$$F_a(x, y) = m$$

obtained in the same way from a given irreducible form $F(X, Y)$ with coefficients in $\mathbb{Z}$, when $\epsilon_1, \ldots, \epsilon_d$ are algebraic units and when the algebraic numbers $\alpha_1\epsilon_1, \ldots, \alpha_d\epsilon_d$ are Galois conjugates with $d \geq 3$. The results in [7] are effective, the results in [6] are more general but not effective. The next result follows from Theorem 3.3 of [6].

**Theorem 1.** Let $K$ be a number field of degree $d \geq 3$, $S$ a finite set of places of $K$ containing the places at infinity. Denote by $\mathcal{O}_S$ the ring of $S$-integers of $K$ and by $\mathcal{O}_S^\times$ the group of $S$-units of $K$. Assume $\alpha_1, \ldots, \alpha_d, \epsilon_1, \ldots, \epsilon_d$ belong to $K^\times$. Then there are only finitely many $(x, y, a)$ in $\mathcal{O}_S \times \mathcal{O}_S \times \mathbb{Z}$ satisfying

$$F_a(x, y) \in \mathcal{O}_S^\times, \quad xy \neq 0 \quad \text{and} \quad \text{Card}\{\alpha_1\epsilon_1^a, \ldots, \alpha_d\epsilon_d^a\} \geq 3.$$

Section 2 is an introduction to linear recurrence sequences. In Section 3 we observe that in the general case each of the sequences $(U_h(a))_{a \in \mathbb{Z}}$ coming from the coefficients of the relation (2) is a linear recurrence sequence.

2
2 Linear recurrence sequences

Let us recall some well known facts about linear recurrence sequences; (see for instance [10], Chapter C of [11], and also [1], [2], [4], [5], [9]). Then we apply these results to the families of binary forms given in (1) and (2).

2.1 Generalities

Let $\mathbb{K}$ be a field of characteristic 0. The sequences $(u(a))_{a \in \mathbb{Z}}$, with values in $\mathbb{K}$ and indexed by $\mathbb{Z}$, form a vector space $\mathbb{K}\mathbb{Z}$ over $\mathbb{K}$. Let $c = (c_1, \ldots, c_d) \in \mathbb{K}^d$ with $c_d \neq 0$. The sequences, satisfying the linear recurrence relation of order $d$ given by

$$u(a + d) = c_1 u(a + d - 1) + \cdots + c_d u(a),$$

form a $\mathbb{K}$-vector subspace $E_c$ of $\mathbb{K}\mathbb{Z}$ of dimension $d$, a natural canonical basis being given by the $d$ sequences $u_0, \ldots, u_{d-1}$ defined by the initial conditions $u_j(a) = \delta_{ja}$ $(0 \leq j, a \leq d - 1)$, $\delta_{ja}$ being the Kronecker symbol

$$\delta_{ja} = \begin{cases} 1 & \text{if } j = a, \\ 0 & \text{if } j \neq a. \end{cases}$$

For $u \in E_c$, we have

$$u = u(0)u_0 + u(1)u_1 + \cdots + u(d - 1)u_{d-1}.$$

By definition, the characteristic polynomial of the linear recurrence relation (3) is

$$P(T) = T^d - c_1 T^{d-1} - \cdots - c_{d-1} T - c_d \in \mathbb{K}[T],$$

where $P(0) = -c_d \neq 0$.

A sequence $u \in \mathbb{K}\mathbb{Z}$ satisfies a linear recurrence relation of order $\leq d$ if and only if the sequences

$$(u(a + j))_{a \in \mathbb{Z}} \quad (j = 0, 1, 2, \ldots)$$

generate a vector space over $\mathbb{K}$ of dimension $\leq d$. Remark that a linear recurrence relation of order $d$ may be viewed as a linear recurrence relation of order $d + s$ for any $s \geq 1$. The dimension $d_0$ of this vector space is the
minimal order of the linear recurrence relation satisfied by $u$. The linear
recurrence relation of order $d_0$ satisfied by $u$ is unique; the characteristic
polynomial of this relation generates an ideal of $\mathbb{K}[T]$ and the characteristic
polynomials of these linear recurrence relations satisfied by $u$ are the monic
polynomials of this ideal.

2.2 Decomposed characteristic polynomial

As a preliminary step, let us assume that the polynomial $P(T)$ of degree $d$
 splits completely in $\mathbb{K}[T]$ as a product of linear factors:

$$P(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j}$$

with $t_j \geq 1$, $t_1 + \cdots + t_\ell = d$ and with nonvanishing pairwise distinct ele-
ments $\gamma_1, \ldots, \gamma_\ell$. Let us prove that a basis of $E_c$ is given by the $d$
sequences $\left( a^i \gamma_j^a \right)_{a \in \mathbb{Z}}$, $1 \leq j \leq \ell$, $0 \leq i \leq t_j - 1$.

Firstly, we will show that these $d$ sequences belong to the vector space $E_c$
(this part was omitted in [5]). Next, we will prove that they form a linearly
independent subset of $E_c$.

By hypothesis, for $1 \leq j \leq \ell$ and $0 \leq i \leq t_j - 1$, the derivative of order
$i$ of the polynomial $P(T)$ is vanishing at the point $\gamma_j$. Let us recall that the
characteristic of $\mathbb{K}$ is 0. Instead of using the operator $d/dT$, we will use the
operator $T d/dT$ which has the property

$$\left(T \frac{d}{dT}\right)^i (T^a P)(\gamma_j) = 0$$

for $i \geq 0$ and $h \geq 0$; we stipulate that $h^i = 1$ for $i = h = 0$. For $a \in \mathbb{Z}$,
$1 \leq j \leq \ell$ and $0 \leq i \leq t_j - 1$, the equation

$$\left(T \frac{d}{dT}\right)^i (T^a P)(\gamma_j) = 0$$

can be written as

$$(a + d)^i \gamma_j^{a+d} = \sum_{k=1}^{d} (a + d - k)^i c_k \gamma_j^{a+d-k} \quad (a \in \mathbb{Z}),$$
with the convention that for \( k = a + d \), the term \((a + d - k)^i\) takes the value 1 for \( i = 0 \) and the value 0 for \( i \geq 1 \). Therefore the sequence \((a^i \gamma_j^a)_{a \in \mathbb{Z}}\) belongs to the vector space \( E_c \) for \( 1 \leq j \leq \ell \) and \( 0 \leq i \leq t_j - 1 \).

**Remark.** In the literature, there are at least two further classical proofs of this fact. One is to write the linear recurrence relation in a matrix form

\[
U(a + 1) = CU(a)
\]

with

\[
U(a) = \begin{pmatrix}
u(a) \\
u(a + 1) \\
\vdots \\
u(a + d - 1)
\end{pmatrix}, \quad C = \begin{pmatrix}0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
c_d & c_{d-1} & c_{d-2} & \cdots & c_1
\end{pmatrix}.
\]

The determinant of \( I_d T - C \) (the characteristic polynomial of \( C \)) is nothing but \( P(T) \). To obtain the result, one writes the matrix \( C \) in its Jordan normal form.

The other method consists in introducing the formal power series

\[
U(z) = \sum_{a \geq 0} u(a) z^a.
\]

One has

\[
\left(1 - \sum_{i=1}^{d} c_i z^i\right) U(z) = \sum_{j=0}^{d-1} \left(u(j) - \sum_{i=1}^{j} c_i u(j - i)\right) z^j.
\]

Hence \( U(z) \) is a rational fraction, with denominator

\[
1 - \sum_{i=1}^{d} c_i z^i = z^d P(1/z) = \prod_{j=1}^{\ell} (1 - \gamma_j z)^{t_j},
\]

while the numerator is of degree \(< d\). This rational fraction can be rewritten using a partial fraction decomposition:

\[
U(z) = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j - 1} \frac{q_{ij}}{(1 - \gamma_j z)^{i+1}}.
\]
For $1 \leq j \leq \ell$, one develops $(1 - \gamma_j z)^{-i-1}$ as a power series expansion to get
\[
\frac{1}{(1 - \gamma_j z)^{i+1}} = \frac{1}{i! \gamma_j} \left( \frac{d}{dz} \right)^i \frac{1}{1 - \gamma_j z} = \sum_{a \geq 0} \frac{(a + 1)(a + 2) \cdots (a + i)}{i!} \gamma_j^a z^a.
\]
This allows to write $u(a)$ as a linear combination of the elements $\gamma_j^a$ with coefficients being polynomials of degree $< t_j$ evaluated at $a$.

Proving the linear independence of the set of the $d$ sequences
\[(a^i \gamma_j^a)_{a \in \mathbb{Z}}, \quad \text{with } 1 \leq j \leq \ell \text{ and } 0 \leq i \leq t_j - 1,
\]
boils down to showing that the determinant of the matrix
\[
A = \begin{pmatrix}
1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^k & \cdots & \gamma_1^{d-1} \\
0 & 1 & 2\gamma_1 & \cdots & k\gamma_1^{k-1} & \cdots & (d-1)\gamma_1^{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \left(\frac{k}{t_1-1}\right)\gamma_1^{k-t_1+1} & \cdots & \left(\frac{d-1}{t_1-1}\right)\gamma_1^{d-t_1} \\
1 & \gamma_2 & \gamma_2^2 & \cdots & \gamma_2^k & \cdots & \gamma_2^{d-1} \\
0 & 1 & 2\gamma_2 & \cdots & k\gamma_2^{k-1} & \cdots & (d-1)\gamma_2^{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \left(\frac{k}{t_2-1}\right)\gamma_2^{k-t_2+1} & \cdots & \left(\frac{d-1}{t_2-1}\right)\gamma_2^{d-t_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \gamma_\ell & \gamma_\ell^2 & \cdots & \gamma_\ell^k & \cdots & \gamma_\ell^{d-1} \\
0 & 1 & 2\gamma_\ell & \cdots & k\gamma_\ell^{k-1} & \cdots & (d-1)\gamma_\ell^{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \left(\frac{k}{t_\ell-1}\right)\gamma_\ell^{k-t_\ell+1} & \cdots & \left(\frac{d-1}{t_\ell-1}\right)\gamma_\ell^{d-t_\ell}
\end{pmatrix}
\]
is different from 0. Note that $\binom{r}{k} = 0$ for $r < k$. Let us define $s_j$ to be
\[
s_j = t_1 + \cdots + t_{j-1} \quad \text{for } 1 \leq j \leq \ell \text{ with } s_1 = 0.
\]
For $1 \leq j \leq \ell$, $0 \leq i \leq t_j - 1$, $0 \leq k \leq d - 1$, the $(s_j + i, k)$ entry of the matrix $A$ is
\[
\frac{1}{i!} \left( \frac{d}{dT} \right)^i T^k \bigg|_{T = \gamma_j} = \binom{k}{i} \gamma_j^{k-i}.
\]
As a matter of fact, $A$ is best described as being made of $\ell$ vertical blocks $A_1, A_2, \ldots, A_\ell$ where for $1 \leq j \leq \ell$, $A_j$ is the $t_j \times d$ matrix

$$A_j = \begin{pmatrix}
1 & \gamma_j & \gamma_j^2 & \ldots & \gamma_j^{t_j-1} & \gamma_j^{t_j} & \ldots & \gamma_j^{d-1} \\
0 & 1 & \binom{2}{1} \gamma_j & \ldots & (t_j-1) \gamma_j^{t_j-2} & \binom{t_j}{1} \gamma_j^{t_j-1} & \ldots & (d-1) \gamma_j^{d-2} \\
0 & 0 & 1 & \ldots & (t_j-1) \gamma_j^{t_j-3} & \binom{t_j}{2} \gamma_j^{t_j-2} & \ldots & (d-1) \gamma_j^{d-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & (t_j-1) \gamma_j & \ldots & (d-1) \gamma_j^{d-t_j}
\end{pmatrix}. \quad (5)$$

Denote by $C_0, \ldots, C_{d-1}$ the $d$ columns of $A$. Let $b_0, \ldots, b_{d-1}$ be complex numbers such that

$$b_0 C_0 + \cdots + b_{d-1} C_{d-1} = 0.$$ 

The left side of this equality is an element of $\mathbb{K}^d$, the $d$ components of which are all 0, and these $d$ relations mean that the polynomial

$$b_0 + b_1 T + \cdots + b_{d-1} T^{d-1}$$

vanishes at the point $\gamma_j$ with multiplicity at least $t_j$ for $1 \leq j \leq \ell$. Since $t_1 + \cdots + t_\ell = d$, we deduce that $b_0 = \cdots = b_{d-1} = 0$.

The determinant of $A$ was calculated in\[5\]:

$$\det A = \prod_{1 \leq i < j \leq \ell} (\gamma_j - \gamma_i)^{t_j t_i}.$$ 

2.3 Interpolation.

The matrix $A$ is associated with the linear system of $d$ equations in $d$ unknowns which amounts to finding a polynomial $f \in \mathbb{K}[z]$ of degree $< d$ for which the $d$ numbers

$$\frac{d^i f}{dz^i}(\gamma_j), \quad (1 \leq j \leq \ell, \ 0 \leq i \leq t_j - 1)$$

take prescribed values. Sharp estimates related with this linear system are provided by Lemma 3.1 of\[8\].

Before stating and proving the next proposition, we introduce the following notation.
Let \( g \in \mathbb{K}(z) \), let \( z_0 \in \mathbb{K} \) and let \( t \geq 1 \). Assume \( z_0 \) is not a pole of \( g \). We set
\[
T_{g,z_0,t}(z) = \sum_{i=0}^{t-1} \frac{d^i g}{d z^i}(z_0) \frac{(z - z_0)^i}{i!}.
\]
In other words, \( T_{g,z_0,t} \) is the unique polynomial in \( \mathbb{K}[z] \) of degree \(< t \) such that there exists \( r(z) \in \mathbb{K}(z) \) having no pole at \( z_0 \) with
\[
g(z) = T_{g,z_0,t}(z) + (z - z_0)^{t} r(z).
\]
Notice that if \( g \) is a polynomial of degree \(< t \), then \( g = T_{g,z_0,t} \) for any \( z_0 \in \mathbb{K} \).

**Proposition 1.** Let \( \gamma_j \) (\( 1 \leq j \leq \ell \)) be distinct elements in \( \mathbb{K} \), \( t_j \) (\( 1 \leq j \leq \ell \)) be positive integers, \( \eta_{ij} \) (\( 1 \leq j \leq \ell, 0 \leq i \leq t_j - 1 \)) be elements in \( \mathbb{K} \). Set \( d = t_1 + \cdots + t_\ell \). There exists a unique polynomial \( f \in \mathbb{K}[z] \) of degree \(< d \) satisfying
\[
\frac{d^i f}{dz^i}(\gamma_j) = \eta_{ij}, \quad (1 \leq j \leq \ell, 0 \leq i \leq t_j - 1).
\]
For \( j = 1, \ldots, \ell \), define
\[
h_j(z) = \prod_{1 \leq k \leq \ell \atop k \neq j} \left( \frac{z - \gamma_k}{\gamma_j - \gamma_k} \right)^{t_k} \quad \text{and} \quad p_j(z) = \sum_{i=0}^{t_j-1} \frac{\eta_{ij} (z - \gamma_j)^i}{i!}.
\]
Then the solution \( f \) of the interpolation problem (6) is given by
\[
f = \sum_{j=1}^{\ell} h_j T_{p_j, h_j, \gamma_j, t_j}.
\]

**Proof.** The conditions (6) can be written
\[
T_{f, \gamma_k, t_k} = p_k \quad \text{for} \quad k = 1, \ldots, \ell.
\]
The unicity is clear: the difference between two solutions is a polynomial of degree \(< d \) which vanishes at \( d \) points (including multiplicity), hence is the zero polynomial.

Since \( h_j(\gamma_j) = 1 \), the quantity \( q_j = T_{p_j, h_j, \gamma_j, t_j} \) is well defined and is a polynomial of degree \(< t_j \). Since \( h_j \) is a polynomial of degree \( d - t_j \), the polynomial \( f \) in (7), namely
\[
f = h_1 q_1 + \cdots + h_\ell q_\ell,
\]
is a polynomial of degree \(< d\). Let us prove that this polynomial \(f\) verifies the equalities in (6). For \(1 \leq k \neq j \leq \ell\) and \(0 \leq i \leq t_k - 1\), we have

\[
\frac{d^i h_j}{dz^i}(\gamma_k) = 0,
\]
and therefore also

\[
\frac{d^i (h_j q_j)}{dz^i}(\gamma_k) = 0.
\]
Hence, for the function \(f\) given by (7) and for \(1 \leq k \leq \ell\), \(0 \leq i \leq t_k - 1\), we have

\[
\frac{d^i f}{dz^i}(\gamma_k) = \frac{d^i (h_k q_k)}{dz^i}(\gamma_k).
\]
In other words, for \(1 \leq k \leq \ell\), we have

\[
T_{f,\gamma_k,t_k} = T_{h_k q_k,\gamma_k,t_k}.
\]
By definition of \(T\), the function \(q_k - \frac{p_k}{h_k}\) has a zero of multiplicity \(\geq t_k\) at \(\gamma_k\), hence the same is true for the function \(h_k q_k - p_k\). Therefore, for any \(k \in \{1, \ldots, \ell\}\), we have

\[
T_{h_k q_k,\gamma_k,t_k} = p_k,
\]
whereupon, \(T_{f,\gamma_k,t_k} = p_k\). This completes the proof. \(\square\)

The Lagrange–Hermite interpolation formula \([3]\) deals with this question when \(K = \mathbb{C}\) and when the values \(\eta_{ij}\) are of the form

\[
\eta_{ij} = \frac{d^i F}{dz^i}(\gamma_j) \quad (1 \leq j \leq \ell, \ 0 \leq i \leq t_j - 1)
\]
for a function \(F\) which is analytic in a domain containing the points \(\gamma_1, \ldots, \gamma_\ell\).

**Proposition 2.** Let \(D\) be a domain in \(\mathbb{C}\), \(F\) an analytic function in \(D\), \(\gamma_1, \ldots, \gamma_\ell\) distinct points in \(D\) and \(\Gamma\) a simple curve inside which the points \(\gamma_1, \ldots, \gamma_\ell\) are located. Then the unique polynomial \(f \in \mathbb{C}[z]\) of degree \(< d\) satisfying

\[
\frac{d^i f}{dz^i}(\gamma_j) = \frac{d^i F}{dz^i}(\gamma_j), \quad (1 \leq j \leq \ell, \ 0 \leq i \leq t_j - 1)
\]
is given, for \(z\) inside \(\Gamma\), by

\[
f(z) = F(z) + \frac{1}{2i\pi} \int_\Gamma \Phi(\zeta) d\zeta
\]

...
with

\[ \Phi(\zeta) = \frac{F(\zeta)}{z-\zeta} \prod_{j=1}^{\ell} \left( \frac{z-\gamma_j}{\zeta-\gamma_j} \right)^{t_j}. \]

Proof. The residue at \( \zeta = z \) of \( \Phi(\zeta) \) is \(-F(z)\). Under the assumptions of Proposition 2 and with the notations of Proposition 1, we have

\[ p_j = T_{F, \gamma_j, t_j}. \]

It remains to show that for \( 1 \leq j \leq \ell \), the residue at \( \zeta = \gamma_j \) of \( \Phi(\zeta) \) is

\[ h_j(z)T_{p_j \gamma_j, \gamma_j, t_j}(z). \]

We first notice that for \( m \in \mathbb{Z} \) and \( t \in \mathbb{Z} \) with \( t \geq 0 \), the residue at \( \zeta = 0 \) of

\[ \zeta^m \left( \frac{z}{\zeta} \right)^t \frac{1}{z-\zeta} \]

is \( z^m \) for \( m \leq t - 1 \) and \( z \neq 0 \), and is 0 otherwise, namely for \( z = 0 \) as well as for \( m \geq t \). Therefore, when \( \varphi(\zeta) \) is analytic at \( \zeta = \gamma \), the residue at \( \zeta = \gamma \) of

\[ \varphi(\zeta) \left( \frac{z-\gamma}{\zeta-\gamma} \right)^t \frac{1}{z-\zeta} \]

is \( T_{\varphi, \gamma, t}(z) \). Since

\[ \Phi(\zeta) = \frac{F(\zeta)}{z-\zeta} \left( \frac{z-\gamma_j}{\zeta-\gamma_j} \right)^{t_j} \frac{h_j(z)}{h_j(\zeta)}, \]

and since \( h_j(\gamma_j) \neq 0 \), the residue at \( \zeta = \gamma_j \) of \( \Phi(\zeta) \) is

\[ h_j(z)T_{\varphi, \gamma_j, t_j}(z). \]

Finally, we notice that when \( \varphi_1 \) and \( \varphi_2 \) are analytic at \( \gamma \), then \( T_{\varphi_1 \varphi_2, \gamma, t} = T_{\tilde{\varphi}_1 \tilde{\varphi}_2, \gamma, t} \) with \( \tilde{\varphi}_1 = T_{\varphi_1, \gamma, t} \). This final remark with \( \gamma = \gamma_j, t = t_j, \varphi_1 = F, \tilde{\varphi}_1 = p_j, \varphi_2 = 1/h_j \) completes the proof.

There are other formulae for the solution to the interpolation problem [6]. For instance, writing \( t_j \) times each \( \gamma_j \), one gets a sequence \( z_1, \ldots, z_d \), and the so-called Newton's divided differences interpolation polynomials give formulae for the coefficients \( c_0, \ldots, c_{d-1} \) in

\[ f(z) = c_0 + c_1(z-z_1) + c_2(z-z_1)(z-z_2) + \cdots + c_{d-1}(z-z_1)(z-z_2) \cdots (z-z_{d-1}). \]
2.4 Polynomial combinations of powers.

From the preceding sections, we deduce that the linear recurrence sequences over an algebraically closed field of characteristic 0 are in bijection with the linear combinations of the powers $\gamma_j^a$ ($1 \leq j \leq \ell$) with polynomial coefficients of the form

$$u(a) = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j-1} v_{ij} a^i \gamma_j^a \quad (a \in \mathbb{Z}). \quad (8)$$

The piece of data $c = (c_1, \ldots, c_d) \in \mathbb{K}^d$ is equivalent to being given $\ell$ distinct nonzero complex numbers $\gamma_1, \ldots, \gamma_\ell$ and $\ell$ positive integers $t_1, \ldots, t_\ell$ together with the property that

$$T^d - c_1 T^{d-1} - \cdots - c_{d-1} T - c_d = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j}$$

with $d = t_1 + \cdots + t_\ell$.

A change of basis for $\mathbb{K}^d$, involving the transition matrix

$$(a^i \gamma_j^a)_{\substack{0 \leq a \leq d-1 \quad 0 \leq i \leq t_j-1 \quad 1 \leq j \leq \ell}},$$

allows to switch from the initial conditions $u(a)$ for $0 \leq a \leq d-1$ to the $d$ coefficients $v_{ij}$ of (8).

Since

$$\frac{1}{1 - \gamma_j z} = \sum_{a \geq 0} (\gamma_j z)^a$$

and

$$\left( z \frac{d}{dz} \right)^i (\gamma_j z)^a = a^i (\gamma_j z)^a,$$

the generating function of the sequence $(u(a))_{a \in \mathbb{Z}}$ given by (8) is

$$U(z) = \sum_{a \geq 0} u(a) z^a = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j-1} v_{ij} \left( z \frac{d}{dz} \right)^i \left( \frac{1}{1 - \gamma_j z} \right),$$

which is a rational fraction with denominator $\prod_{j=1}^{\ell} (1 - \gamma_j z)^{t_j}$, as expected.
2.5 The ring of linear recurrence sequences.

A sum and a product of two polynomial combinations of powers is still a polynomial combination of powers. If \( U_1 \) and \( U_2 \) are two linear recurrence sequences of characteristic polynomials \( P_1 \) and \( P_2 \) respectively, then \( U_1 + U_2 \) satisfies the linear recurrence, the characteristic polynomial of which is

\[
\frac{P_1 P_2}{\gcd(P_1, P_2)}.
\]

Consequently, the union of all vector spaces \( E_c \), with \( c \) running through the set of \( d \)-tuples \( (c_1, \ldots, c_d) \in \mathbb{K}^d \) subject to \( c_d \neq 0 \), and \( d \) running through the set of integers \( \geq 1 \), is still a vector subspace of \( \mathbb{K}^Z \).

Moreover, if the characteristic polynomials of the two linear recurrence sequences \( U_1 \) and \( U_2 \) are respectively

\[
P_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad P_2(T) = \prod_{k=1}^{\ell'} (T - \gamma_k')^{t_k'},
\]

then \( U_1U_2 \) satisfies the linear recurrence, the characteristic polynomial of which is

\[
\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma_k')^{t_j + t_k' - 1}.
\]

As a consequence, the linear recurrence sequences form a ring.

2.6 Non homogeneous linear recurrence sequences

Let us suppose now that a factorisation of the characteristic polynomial \( P(T) \) of a linear recurrence relation is of the form \( P = QR \), with \( R \) completely decomposed in \( \mathbb{K}[T] \). Let us write

\[
P(T) = T^d - \sum_{i=1}^{d} c_i T^{d-i}, \quad Q(T) = T^m - \sum_{i=1}^{m} b_i T^{m-i}, \quad R(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j}.
\]

Hence \( d = m + t_1 + \cdots + t_\ell \). Then the elements of \( E_c \) are the sequences \( (u(a))_{a \in \mathbb{Z}} \) for which there exist \( d - m \) elements

\[
\lambda_{ij} \quad (1 \leq j \leq \ell, \quad 0 \leq i \leq t_j - 1)
\]

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In \( K \) such that
\[
    u(a + m) = b_1 u(a + m - 1) + \cdots + b_m u(a) + \sum_{j=1}^{t} \sum_{i=0}^{t_j-1} \lambda_{ij} a^i \gamma_{j}^a.
\] (9)

In order to define an element \( (u(a))_{a \in \mathbb{Z}} \) of \( E_c \) by using the homogenous recurrence relation in (3), we have to give \( d \) initial values, for instance \( u(0), \ldots, u(d - 1) \). In order to define this sequence by using the non homogeneous recurrence relation (9), it is sufficient to have \( m \) initial conditions, say \( u(0), \ldots, u(m - 1) \), but we also have to know the elements \( \lambda_{ij} \) for \( 1 \leq j \leq \ell \) and \( 0 \leq i \leq t_j - 1 \) (which altogether are \( d \) conditions, as is required in a vector space of dimension \( d \)).

Consider the transition matrix associated to the change of basis, allowing to switch from the initial conditions
\[
u(a) \quad \text{for} \quad 0 \leq a \leq d - 1
\]
to the initial conditions
\[
u(a) \quad \text{for} \quad 0 \leq a \leq m - 1 \quad \text{and} \quad \lambda_{ij} \quad \text{for} \quad 1 \leq j \leq \ell \quad \text{and} \quad 0 \leq i \leq t_j - 1.
\]

It is a matrix which has only a diagonal of two blocks,
\[
    \begin{pmatrix}
        I_m & 0 \\
        0 & A
    \end{pmatrix}

\text{with}
A = \begin{pmatrix}
    A_1 \\
    \vdots \\
    A_\ell
\end{pmatrix}.
\]

The first block \( I_m \) is the \( m \times m \) identity matrix. The second block \( A \) is a generalized Vandermonde matrix similar to the matrix in (4) made of the blocks \( A_1, \ldots, A_\ell \) described in (5).

A particular case is the trivial one when \( P = Q, m = d \) and \( R = 1 \). Another one is when \( P = R, Q = 1 \) and \( m = 0 \), which corresponds to the case studied in Section 2.2.

**Example.** Let us consider
\[
P(T) = (T - \gamma)^2, \quad Q(T) = R(T) = T - \gamma.
\]

There are three ways of defining an element \( (u(a))_{a \in \mathbb{Z}} \) of the vector space \( E_c \) when \( c = (2, -1) \). The first one is to mention that the sequence satisfies the binary linear recurrence relation
\[
u(a + 2) = 2u(a + 1) - u(a) \quad (a \in \mathbb{Z})
\]

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and give two initial values, for, say \( u(0) \) and \( u(1) \). The second one is to write
\[
u(a) = (\lambda_1 + \lambda_2 a) \gamma^a \quad (a \in \mathbb{Z})
\]
and give the values of \( \lambda_1 \) and \( \lambda_2 \). The third one is in-between the previous ones; one writes that the sequence satisfies
\[
u(a + 1) = \gamma \nu(a) + \lambda \gamma^a \quad (a \in \mathbb{Z})
\]
while providing an initial value, for, say \( u(0) \), and the value of \( \lambda \).

### 2.7 Exponential polynomials

The sequence of derivatives of an exponential polynomial evaluated at one point satisfies a linear recurrence relation. This allows us to deduce the following well known result (Ch. I, §7 of [12]).

**Lemma 1.** Let \( a_1(z), \ldots, a_\ell(z) \) be nonzero polynomials of \( \mathbb{C}[z] \) of degrees smaller than \( t_1, \ldots, t_\ell \) respectively. Let \( \gamma_1, \ldots, \gamma_\ell \) be distinct complex numbers. Let us suppose that the function
\[
F(z) = a_1(z)e^{\gamma_1 z} + \cdots + a_\ell(z)e^{\gamma_\ell z}
\]
is not identically 0. Then its vanishing order at a point \( z_0 \) is smaller than or equal to \( t_1 + \cdots + t_\ell - 1 \).

**Proof.** Define \( d = t_1 + \cdots + t_\ell \). We give two proofs of Lemma 1. A short one by induction on \( d \) is as follows. For \( d = 1 \) we have \( \ell = 1 \) and \( F \) has no zero. Assume \( \ell \geq 2 \). Without loss of generality we may assume \( \gamma_1 = 0 \). If \( F \) has a zero of multiplicity \( \geq T_0 \) at \( z_0 \), then \( F(z) - a_1(z) \) has a zero of multiplicity \( \geq T_0 - t_1 \) at \( z_0 \). The result follows.

Our second proof relates Lemma 1 with linear recurrence sequences. We now assume \( \gamma_1, \ldots, \gamma_\ell \) all nonzero, as we may without loss of generality. Write the Taylor expansion of \( F(z + z_0) \) at \( z = 0 \):
\[
F(z + z_0) = \sum_{a \geq 0} \frac{u(a)}{a!} z^a.
\]
Let us show that the sequence \((u(0), u(1), \ldots, u(a), \ldots)\) satisfies a linear recurrence relation of order \( \leq d \). Define \( a_{ij} \in \mathbb{C} \) by
\[
a_j(z + z_0)e^{\gamma_j z_0} = \sum_{i=0}^{t_j-1} a_{ij} z^i \quad (1 \leq j \leq \ell),
\]
so that
\[ F(z + z_0) = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j-1} a_{ij} z^i e^{\gamma_j z}. \]

Since \( \gamma_j \neq 0 \) for \( j = 1, \ldots, \ell \),
\[ u(a) = \sum_{j=1}^{\ell} \sum_{i=0}^{t_j-1} a_{ij} a(a - 1) \cdots (a - i + 1) \gamma_j^{a-i} \]
has the same form as in (8). Therefore the sequence \( \{u(a)\}_{a \in \mathbb{Z}} \) satisfies a linear recurrence relation of order \( \leq d \). It follows that the conditions
\[ u(0) = \cdots = u(d-1) = 0 \]
imply \( u(a) = 0 \) for any \( a \geq 0 \).

We can state this lemma in the following way: When the complex numbers \( \gamma_j \) are distinct, the determinant
\[ \begin{vmatrix} \left( \frac{d}{dz} \right)^a (z^i e^{\gamma_j z})_{z=0} \end{vmatrix}_{0 \leq i \leq t_j-1, 1 \leq j \leq \ell, 0 \leq a \leq d-1} \]
is different from 0. This is no surprise that we come across the determinant of the matrix (4).

### 3 Families of binary forms

The equations (1) and (2) give, for \( 1 \leq h \leq d \) and \( a \in \mathbb{Z} \),
\[ U_h(a) = \sum_{1 \leq i_1 < \cdots < i_h \leq d} \alpha_{i_1} \cdots \alpha_{i_h} (\epsilon_{i_1} \cdots \epsilon_{i_h})^a. \] (10)

For example, for \( a \in \mathbb{Z} \),
\[ U_1(a) = \sum_{i=1}^{d} \alpha_i \epsilon_i^a, \quad U_d(a) = \prod_{i=1}^{d} \alpha_i \epsilon_i^a. \]

The relations (10) show that for \( 1 \leq h \leq d \), the sequence \( \{U_h(a)\}_{a \in \mathbb{Z}} \) is a linear combination of the sequences
\[ ((\epsilon_{i_1} \cdots \epsilon_{i_h})^a)_{a \in \mathbb{Z}}, \quad (1 \leq i_1 < \cdots < i_h \leq d). \]
For $1 \leq h \leq d$, consider the set
\[ \mathcal{E}_h = \{ \epsilon_{i_1} \cdots \epsilon_{i_h} \mid 1 \leq i_1 < \cdots < i_h \leq d \} \]
and note $m_h$ its cardinality. The elements of $\mathcal{E}_h$ are values of monomials in $m_1$ variables of degree $h$. The map from $\mathcal{E}_h$ to $\mathcal{E}_{d-h}$ defined by
\[ \eta \mapsto \epsilon_1 \cdots \epsilon_{d-h} \eta^{-1} \]
is a bijection and we have
\[ m_h = m_{d-h} \leq \min \left\{ \left( \frac{d}{h} \right), \left( \frac{m_1 + h - 1}{h} \right), \left( \frac{m_1 + d - h - 1}{d - h} \right) \right\}. \]

The sequence $(U_h(a))_{a \in \mathbb{Z}}$ satisfies the linear recurrence relation of order $m_h$ with the characteristic polynomial
\[ \prod_{\eta \in \mathcal{E}_h} (T - \eta). \]
This polynomial is also written as
\[ \prod_{\eta \in \mathcal{E}_{d-h}} (T - \epsilon_1 \cdots \epsilon_d \eta^{-1}), \]
which is matching (10) via
\[ U_h(a) = U_d(a) \sum_{1 \leq j_1 < \cdots < j_{d-h} \leq d} (\alpha_{j_1} \cdots \alpha_{j_{d-h}})^{-1}(\epsilon_{j_1} \cdots \epsilon_{j_{d-h}})^{-a}. \]

For example, the sequence $(U_{d-1}(a))_{a \in \mathbb{Z}}$ satisfies the linear recurrence relation of order $d$, the characteristic polynomial of which is
\[ \prod_{i=1}^{d}(T - \epsilon_1 \cdots \epsilon_d \epsilon_i^{-1}) = (T - \epsilon_2 \cdots \epsilon_d)(T - \epsilon_1 \epsilon_3 \cdots \epsilon_d) \cdots (T - \epsilon_1 \cdots \epsilon_{d-1}). \]

The case $\epsilon_1 = \ldots = \epsilon_d$ is trivial: we have
\[ U_h(a) = \epsilon_1^a U_h(0) = (-1)^h a_h \epsilon_1^a, \]
and each of the sequences $(U_h(a))_{a \in \mathbb{Z}}$ satisfies
\[ U_h(a + 1) = \epsilon_1 U_h(a). \]
Let us consider the example

\[ \epsilon_1 = \ldots = \epsilon_\ell = \epsilon, \quad \epsilon_{\ell+1} = \ldots = \epsilon_d = \eta, \]

with \( \epsilon \) and \( \eta \) being two distinct complex numbers. We have

\[ \mathcal{E}_1 = \{ \epsilon, \eta \}, \quad \mathcal{E}_{d-1} = \{ \epsilon^{\ell-1}\eta^{d-\ell}, \epsilon^{\ell}\eta^{d-\ell-1} \} \]

and

\[ \mathcal{E}_2 = \{ \epsilon^2, \epsilon\eta, \eta^2 \}, \quad \mathcal{E}_{d-2} = \{ \epsilon^{\ell-2}\eta^{d-\ell}, \epsilon^{\ell-1}\eta^{d-\ell-1}, \epsilon^{\ell}\eta^{d-\ell-2} \}. \]

The sequence \( (U_1(a))_{a \in \mathbb{Z}} \) satisfies the binary recurrence relation, the characteristic polynomial of which is

\[ (T - \epsilon)(T - \eta); \]

the sequence \( (U_{d-1}(a))_{a \in \mathbb{Z}} \) satisfies the binary recurrence relation, the characteristic polynomial of which is

\[ (T - \epsilon^{\ell-1}\eta^{d-\ell})(T - \epsilon^{\ell}\eta^{d-\ell-1}); \]

while the sequence \( (U_2(a))_{a \in \mathbb{Z}} \) satisfies the ternary recurrence relation, the characteristic polynomial of which is

\[ (T - \epsilon^2)(T - \eta^2)(T - \epsilon\eta). \]

In particular, if one writes

\[ (T - \epsilon^2)(T - \eta^2) = T^2 - AT - B, \]

then there exists a constant \( C \in \mathbb{C} \) such that, for any \( a \in \mathbb{Z} \), one has

\[ U_2(a + 2) = AU_2(a + 1) + BU_2(a) + C(\epsilon\eta)^a. \]

Finally, the sequence \( (U_{d-2}(a))_{a \in \mathbb{Z}} \) satisfies the ternary recurrence relation, the characteristic polynomial of which is

\[ (T - \epsilon^{\ell-2}\eta^{d-\ell})(T - \epsilon^{\ell-1}\eta^{d-\ell-1})(T - \epsilon^{\ell}\eta^{d-\ell-2}). \]

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