Chekanov–Eliashberg dg-algebras for singular Legendrians

Johan Asplund

Uppsala University

April 2, 2021

Based on joint work with Tobias Ekholm (arXiv:2102.04858)
Setup and main results

The Chekanov–Eliashberg dg-algebra

Computations and examples

Proof of the pushout diagrams

Setup and main results
Setup

Let $X$ be a $2n$-dimensional Weinstein manifold with ideal contact boundary $\partial X$.

\[ \Lambda \subset \partial X \quad \text{smooth Legendrian} \quad \sim \quad CE^*(\Lambda) \quad \text{Chekanov–Eliashberg dg-algebra} \]

Singular Legendrians

Let $(V, \lambda)$ be a $(2n - 2)$-dimensional Weinstein domain, together with a handle decomposition $h$. Assume there is an embedding of $V$ in $\partial X$ such that it extends to a (strict) contact embedding

\[ F: (V \times (-\varepsilon, \varepsilon), dz + \lambda) \longrightarrow (\partial X, \alpha) \]

We call $F$ a Legendrian embedding of $V$ in $\partial X$. 
Setup

Singular Legendrians
In particular, the union of the top dimensional strata of $\text{Skel} V$ is Legendrian, and we will refer to $\text{Skel} V$ as a "singular Legendrian" in $\partial X$.

$(V, h) \subset \partial X$
Legendrian embedding

$\leadsto CE^*((V, h); X)$
Setup

Stopped Weinstein manifolds

We consider *stops* using a surgery description.

\[ C = \text{union of co-core disks of top handles of } V \times D^*_\varepsilon[-1, 1] \]
Main results

Theorem A (A.–Ekholm)

There is a surgery isomorphism of $A_\infty$-algebras

$$\Phi: C W^*(C; X_V) \rightarrow CE^*((V, h); X)$$

Let $\Lambda \subset \partial X$ be a smooth Legendrian and let $(V(\Lambda), h(\Lambda))$ denote a small disk cotangent neighborhood of $\Lambda$ with a handle decomposition with a single top handle.

Theorem B (A.–Ekholm)

There is a quasi-isomorphism of dg-algebras

$$\Psi: CE^*((V(\Lambda), h(\Lambda)); X) \rightarrow CE^*(\Lambda, C_{-*}(\Omega \Lambda); X)$$

Theorem A and B together prove a conjecture by Ekholm–Lekili and independently by Sylvan.
Main results

Now assume $V$ is Legendrian embedded in the ideal contact boundary of $X$ and $X'$. We can join $X$ and $X'$ together via $V$.

$$V \times D_\varepsilon^*[-1, 1]$$

$C_\#$ = union of co-core disks of top handles of $V \times D_\varepsilon^*[-1, 1]$. $\Sigma_\#$ := union of attaching spheres dual to $C_\#$. 
Main results

**Theorem C (A.–Ekholm)**

_Below, the front face is a pushout. After passing to cohomology, the diagram commutes and the back face is a pushout._

\[
\begin{align*}
&\text{CW}^*(c; V) \quad \text{CE}^*(\partial l; V_0) \\
&\downarrow \quad \downarrow \\
&\text{CE}^*(\partial l; V_0) \quad \text{CE}^*((V, h); X') \\
&\uparrow \quad \uparrow \\
&\text{CE}^*((V, h); X) \quad \text{CE}^*(\Sigma_{\#}; X \#_0 X') \\
&\downarrow \quad \downarrow \\
&\text{CW}^*(C; X_V) \quad \text{CW}^*(C'; X'_V) \quad \text{CW}^*(C_{\#}; X_{\#} V X')
\end{align*}
\]
The Chekanov–Eliashberg dg-algebra
Setup

Let $X$ be a $2n$-dimensional Weinstein manifold with ideal contact boundary $\partial X$. ($c_1(X) = 0$)

Let $\Lambda \subset \partial X$ be a smooth Legendrian with vanishing Maslov class.

- $\alpha$ contact form on $\partial X$
- $R_\alpha$ Reeb vector field, defined by

$$\begin{cases} 
  d\alpha(R_\alpha, -) = 0 \\
  \alpha(R_\alpha) = 1
\end{cases}$$

Consider $R = \{\text{Reeb chords of } \Lambda\}$ and let $\Lambda = \bigsqcup_{i=1}^n \Lambda_i$. Then $R_{ij} \subset R$ is the set of Reeb chords from $\Lambda_i$ to $\Lambda_j$.

Let $\mathbb{F}$ be a field. Let $\{e_i\}_{i=1}^n$ be such that

- $e_i^2 = e_i$
- $e_i e_j = 0$ if $i \neq j$
**CE** for smooth Legendrians

Graded algebra

Define $k := \bigoplus_{i=1}^{n} \mathbb{F} e_i$. Then $\mathcal{R}$ is a $k$-$k$-bimodule via

\[
e_i \cdot c = \begin{cases} 
  c, & \text{if } c \in \mathcal{R}_{ji} \\
  0, & \text{otherwise}
\end{cases}
\]

\[
c \cdot e_i = \begin{cases} 
  c, & \text{if } c \in \mathcal{R}_{ij} \\
  0, & \text{otherwise}
\end{cases}
\]

Then define

\[
CE^*(\Lambda) := k \langle \mathcal{R} \rangle.
\]

Grading is given by

\[
|c| = -CZ(c) + 1.
\]
**CE* for smooth Legendrians**

**Differential**

\[ \partial: CE^*(\Lambda) \longrightarrow CE^*(\Lambda) \] counts (anchored) rigid \( J \)-holomorphic disks in \( \mathbb{R} \times \partial X \) with boundary on \( \mathbb{R} \times \Lambda \) with 1 positive puncture, and several negative punctures.

A curve giving the term \( \partial c = b_1 b_2 b_3 + \cdots \).
Assume $V^{2n-2}$ is a Weinstein domain which is Legendrian embedded in $\partial X$ with handle decomposition $h$ and $c_1(V) = 0$. Let $V_0$ denote its subcritical part.

Let

$$ l := \bigcup_{j=1}^{m} l_j = \text{union of core disks of top handles} $$

$$ \partial l := \bigcup_{j=1}^{m} \partial l_j = \text{union of the attaching spheres of top handles} $$
CE* for singular Legendrians

Now attach $V_0 \times D_{\varepsilon}^*[-1, 1]$ to $V_0 \times (-\varepsilon, \varepsilon) \subset \partial X$ to construct $X_{V_0}$.

Define

$$\Sigma(h) := l \sqcup_{\partial l \times \{-1\}} (\partial l \times [-1, 1]) \sqcup_{\partial l \times \{1\}} l$$
Definition.

We define the Chekanov–Eliashberg dg-algebra of a Legendrian embedding of \((V, h)\) in \(\partial X\) as

\[
CE^*((V, h); X) := CE^*(\Sigma(h); X_{V_0}).
\]

Theorem A.

There is a surgery isomorphism of \(A_\infty\)-algebras

\[
\Phi: CW^*(C; X_V) \longrightarrow CE^*((V, h); X)
\]
Proof of the surgery formula

Proof of Theorem A.
Follows immediately from the definition together with the Bourgeois–Ekholm–Eliashberg surgery formula.

\[ CW^*(C; X_V) \cong CE^*(\Sigma(h); X_{V_0}) = CE^*((V, h); X) \]
Description of generators

Lemma

For any $a > 0$, there is some $\varepsilon > 0$ small enough (size of the stop) so that we have the following one-to-one correspondence

\[
\left\{ \text{Reeb chords of } \Sigma(h) \subset \partial X V_0 \text{ of action } < \alpha \right\} \\
\left\{ \text{Reeb chords of } l \subset \partial X \text{ of action } < \alpha \right\} \cup \\
\left\{ \text{Reeb chords of } \partial l \subset \partial V_0 \text{ of action } < \alpha \right\}
\]

1:1

Lemma

There is a dg-subalgebra of $CE^*((V, h); X)$ which is freely generated by Reeb chords of $\partial l \subset \partial V_0$ and canonically isomorphic to $CE^*(\partial l; V_0)$. 
Computations and examples
Special case: $\partial X = P \times \mathbb{R}$

Assume $V \subset P \times \mathbb{R}$ is a Legendrian embedding so that $\pi(V_0) \subset P$ is embedded. Consider

$$P^\circ := (P \setminus \pi(V_0)) \sqcup_{\pi(\partial V_0)} ((-\infty, 0] \times \pi(\partial V_0))$$
Special case: \( \partial X = P \times \mathbb{R} \)

Then we can consider \( CE^*(l; P^\circ \times \mathbb{R}) \), where \( l \) is the Legendrian lift of \( \pi(l) \subset P^\circ \).

**Proposition**

There is an isomorphism of dg-algebras

\[
CE^*(l; P^\circ \times \mathbb{R}) \cong CE^*((V, h); \mathbb{R} \times (P \times \mathbb{R}))
\]

**Upshot**

Can compute \( CE^*(l; P^\circ \times \mathbb{R}) \) and hence \( CE^*((V, h); \mathbb{R} \times (P \times \mathbb{R})) \) by projecting \( l \) and holomorphic curves to \( P^\circ \).

(cf. An–Bae)
Computations

Example \((n\) points in the circle, \(I_n\))

Let \(X = \mathbb{R}^2\) and \(\Lambda = n\) pts \(\subset\) \(\partial X = S^1\).

Let \(V = T^*\Lambda \subset S^1\). The only generators of \(CE^*((V, h); \mathbb{R}^2)\) are Reeb chords in \(S^1\) of the top handles \(l = \Lambda\)

- \(c^{0}_{ij}\) for \(1 \leq i < j \leq n\)
- \(c^{p}_{ij}\) for \(1 \leq i, j \leq n\)

The differential \(\partial\) is given by

\[
\partial(c^{0}_{ij}) = (-1)^* \sum_{k=1}^{n} c^{0}_{k}c^{0}_{i k},
\]

\[
\partial(c^{1}_{ij}) = \delta_{ij} + (-1)^* \sum_{k=1}^{n} c^{1}_{k}c^{0}_{i k} + (-1)^* \sum_{k=1}^{n} c^{0}_{k}c^{1}_{i k}.
\]

\(\ast\)

\(\ast\)

\(\ast\)
Computations

Example (Link of Lagrangian arboreal $A_2$-singularity)

Let $X = \mathbb{R}^4$ and $\Lambda \subset S^3$. Then $V = T^*\Lambda$ has 0-handles $x$ and $y$ and 1-handles $l_1$, $l_2$ and $l_3$.

Generators are Reeb chords of $l$: $a$ and $b$, and generators of $\partial l \subset \partial V_0$: $\{x_{ij}^p\}$ and $\{y_{ij}^p\}$.

\[\text{Diagram of the link with Reeb chords and handles.}\]
Computations

Example (Link of Lagrangian arboreal $A_2$-singularity)

The dg-subalgebra $CE^*(\partial l; V_0)$ consists of two copies of $I_3$. The differential of $a$ and $b$ is as follows

$$\partial a = e_1 + y_{31}^1 b x_{12}^0 + y_{31}^1 x_{12}^0 - y_{21}^1 x_{12}^0, \quad \partial b = x_{23}^0 - y_{23}^0$$
Computations

Example (Singular torus)

Let $X = \mathbb{R}^6$ and $\Lambda \subset S^5$ is given by the following front.

$\begin{align*}
\begin{array}{c}
\includegraphics[width=\textwidth]{example.png}
\end{array}
\end{align*}$

The intersection $l \cap \partial h^0$ is a standard Hopf link in $S^3$.

The dg-subalgebra $CE^*(\partial l; V_0)$ is generated by the generators of the Hopf link together with a copy of $I_2$.

Suitable augmentation of $CE^*(\partial l; V_0)$ gives Chekanov–Eliashberg dg-algebra of nearby smooth tori obtained by smoothing.
Proof of the pushout diagrams
Joining Weinstein manifolds along $V$

Recall the construction of $X \#_V X'$. Assume $V$ is Legendrian embedded in the ideal contact boundary of $X$ and $X'$. We can join $X$ and $X'$ together via $V$. 
Joining Weinstein manifolds along $V$

**Theorem C (A.–Ekholm)**

Below, the \textit{front face} is a pushout. After passing to cohomology, the diagram commutes and the \textit{back face} is a pushout.

\[
\begin{align*}
&\text{CW}^*(c; V) \quad \text{\textit{\scriptsize [BEE]}} \quad \text{\textit{\scriptsize [BEE]}} \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Proof of the pushout diagram for $\text{CE}^*$

**Proof of Theorem C.**

Consider $X \#_{V_0} X'$, and $\Sigma_\#(h) \subset \partial (X \#_{V_0} X')$ the attaching spheres obtained by joining $l$ on either side by $\partial l \times [-1,1]$ through the handle.
**Proof of the pushout diagram for $CE^*$**

**Proof of Theorem C.**

By the description of the generators we obtain

$$CE^*(\Sigma\#(h); X\#V_0X') \cong CE^*((V, h); X) \ast CE^*(\partial l; V_0) CE^*((V, h); X')$$

which means that the diagram

$$\begin{array}{ccc}
CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V, h); X') \\
\downarrow\text{incl.} & & \downarrow\text{incl.} \\
CE^*((V, h); X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma\#(h); X\#V_0X')
\end{array}$$

is a pushout.

Key observation: $CE^*((V, h); X) \subset CE^*(\Sigma\#(h); X\#V_0X')$ since curves cannot “cross” the handle.
Stop removal

Corollary (Stop removal)

Let $X' := V \times D_1^*[-1, 1]$ equipped with the Liouville vector field $Z_V + x\partial_x + y\partial_y$. Then $CW^*(C_#; X_V^# X')$ has trivial cohomology.

Proof.

The key is to observe that after rounding corners $V \times \{(-1, 0)\} \subset \partial (V \times D_1^*[-1, 1])$ is loose (meaning that each core disk $l_j$ of every top handle of $V$ admits a loose chart).

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]
Stop removal

Proof.

Since we can create loose charts it means that there is at least one generator $b \in CE^*((V, h); X')$ such that $\partial b = 1$. Use

$$
\begin{array}{c}
CE^*(\partial l; V_0) \\ \text{incl.}
\end{array}
\xrightarrow{\text{incl.}}
\begin{array}{c}
CE^*((V, h); X') \\ \text{incl.}
\end{array}
\xrightarrow{\text{incl.}}
\begin{array}{c}
CE^*((V, h); X) \\ \text{incl.}
\end{array}
\xrightarrow{\text{incl.}}
\begin{array}{c}
CE^*(\Sigma\#(h); X\#V_0X') \\
\end{array}
$$

To conclude that the same is true for $CE^*(\Sigma\#(h); X\#V_0X')$. By surgery we therefore have

$$CW^*(C\#; X\#VX') \cong CE^*(\Sigma\#(h); X\#V_0X') \cong 0$$
Thank you!