Gravity and the Fermion Mass

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Abstract

It is shown that gravity generates mass for the fermion. It does so by coupling directly with the spinor field. The coupling term is invariant with respect to the electroweak gauge group $U(1) \otimes SU(2)_L$. It replaces the fermion mass term $m_\psi \bar{\psi} \psi$. 
1. Introduction.

The Lagrangian density for the spinor field is derived in appendix A

\[ L_f = \frac{i}{\hbar c} \left( \overline{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \overline{\psi}) \gamma^\mu \psi \right) + \frac{\hbar c}{4} \overline{\psi} \gamma_5 \gamma_\beta \gamma_\delta \psi e^{\delta \alpha \beta \gamma \lambda} \partial_\lambda e_\beta e_\alpha \]  \hspace{1cm} (1)

In this expression, \( \gamma^\mu(x) = e^{\alpha \mu}(x) \hat{\gamma}^\alpha \) where \( \hat{\gamma}^\alpha \) are the constant Dirac matrices. Gravitation is represented by the tetrad field \( e^{\alpha \mu}(x) \). Because of the coupling between tetrad and spinor, the two fields propagate together. It is similar to the coupling between electric and magnetic fields during the propagation of electromagnetic waves.

It is shown in appendix B that \( L_f \) yields an electroweak Lagrangian which is invariant under the gauge group \( U(1) \otimes SU(2)_L \). The conventional mass term \( m \overline{\psi} \psi \) has no such electroweak form. As a consequence, it cannot appear in the electroweak Lagrangian, and it does not contribute to the energy. What, then, is the source of fermion mass? We will find solutions of the coupling equation which also satisfy the Dirac equation. Therefore, these solutions describe particles of mass \( m \).

In this paper, the gauge fields will be ignored altogether. The gauge interactions play no part in the generation of fermion mass.
2. Field Equations. Energy Tensors.

The (non-linear) coupling equation follows from $L_f$ (1)

$$i\gamma^\mu \partial_\mu \psi + \frac{i}{2\sqrt{-g}}\partial_\mu (\sqrt{-g} \gamma^\mu) \psi + \frac{1}{4}\gamma_5 \gamma_8 \psi \epsilon^{\alpha\beta\gamma} e_\alpha e_\nu e_\gamma \partial_\gamma e_{\beta\nu} = 0$$

(2)

This, together with the conjugate equation, yields the conservation law

$$\partial_\mu \left( \sqrt{-g} \overline{\psi} \gamma^\mu \psi \right) = 0$$

(3)

If a plane wave solution exists

$$\psi = \frac{N'}{\sqrt{V}} \left( \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right) \exp(-ik_\mu x^\mu)$$

(4)

then $\partial_\mu \psi = -ik_\mu \psi$ and $\partial_\mu \overline{\psi} = ik_\mu \overline{\psi}$. It follows from the conservation law that $\partial_\mu (\sqrt{-g} \gamma^\mu) = 0$, and equation (2) simplifies to

$$i\gamma^\mu \partial_\mu \psi + \frac{1}{4}\gamma_5 \gamma_8 \psi \epsilon^{\alpha\beta\gamma} e_\alpha e_\nu e_\gamma \partial_\gamma e_{\beta\nu} = 0$$

(5)

The gravitational field equations are

$$\kappa \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + T^{(m)}_{\mu\nu} = 0$$

(6)

where $\kappa = c^4/8\pi G$ and the Ricci tensor is

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\lambda\nu}$$

(7)

The material energy tensor is that of the spinor [1]

$$T^{(m)}_{\mu\nu} = \frac{i}{4} \hbar c \left\{ \overline{\psi} \gamma_\mu \partial_\nu \psi + \overline{\psi} \gamma_\nu \partial_\mu \psi - (\partial_\mu \overline{\psi}) \gamma_\nu \psi - (\partial_\nu \overline{\psi}) \gamma_\mu \psi \right\}$$

$$+ \frac{\hbar c}{4} \psi \gamma_5 \gamma_8 \psi \epsilon^{\alpha\beta\gamma} e_\gamma \left\{ (e_\alpha \mu \partial_\lambda e_{\beta\nu} + e_\alpha e_\nu \partial_\lambda e_{\beta\mu}) - \frac{1}{2} (e_\alpha \mu \partial_\nu e_{\beta\lambda} + e_\alpha e_\nu \partial_\mu e_{\beta\lambda}) \right\}$$

(8)
The structure of space and time is expressed in terms of a scalar, 3-vector basis \( e_\mu = (e_0, e_i) \). The basis changes from point to point according to the formula

\[
\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu}
\]

(9)

which separates into scalar and 3-vector parts

\[
\begin{align*}
\nabla_\nu e_0 &= e_0 Q^0_{0\nu} \\
\nabla_\nu e_i &= e_j Q^j_{i\nu}
\end{align*}
\]

(10)

(11)

By definition \( Q^0_{j\nu} = Q^i_{0\nu} \equiv 0 \). All 28 independent coefficients \( Q^\mu_{\nu\lambda} \) are derivable from the scalar, three-vector metric \( g_{\mu\nu} = (g_{00}; g_{ij}) \). Explicitly,

\[
\begin{align*}
Q^0_{0\nu} &= \Gamma^0_{0\nu} = \frac{1}{2} g^{00} \partial_\nu g_{00} \\
Q^i_{j0} &= \Gamma^i_{j0} = \frac{1}{2} g^{in} \partial_0 g_{nj} \\
Q^i_{jk} &= \Gamma^i_{jk} = \frac{1}{2} g^{in} (\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{jk})
\end{align*}
\]

(12)

(13)

(14)

where

\[
\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} (\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda})
\]

(15)

are the Christoffel coefficients. The symbols \( \Gamma^\mu_{\nu\lambda} \) are symmetric in \( \nu\lambda \), while the \( Q^\mu_{\nu\lambda} \) are not. The following formula holds good

\[
Q^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + g^{\mu\rho} g_{\lambda\eta} Q^\eta_{[\nu\rho]}
\]

(16)

where

\[
Q^\mu_{[\nu\lambda]} \equiv Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu}
\]

(17)
The gravitational energy tensor is given by

\[ T_{\mu\nu}^{(g)} = \kappa \{ Q_{\lambda\mu}^\rho Q_{\rho\nu}^\lambda + Q_\mu Q_\nu - \frac{1}{2} g_{\mu\nu} g^{\rho\tau} (Q_{\lambda\rho\nu}^\rho Q_{\rho\tau}^\lambda + Q^\rho Q_\tau) \} \]  

(18)

where \( Q_\mu = Q_{\nu\mu}^\rho \). There are 9 independent components \( Q_{\mu\lambda\nu}^\rho \), namely

\[ Q_{0i}^0 = \frac{1}{2} g^{00} \partial_i g_{00} \quad \text{and} \quad Q_{j0}^i = \frac{1}{2} g^{ijn} \partial_i g_{nj} \]  

(19)

For a static, Newtonian potential \( \psi \)

\[ g_{00} = 1 + \frac{2}{c^2} \psi \]  

(20)

so that

\[ Q_{0i}^0 = \frac{1}{c^2} \partial_i \psi \quad \text{and} \quad Q_{j0}^i = 0 \]  

(21)

It follows that

\[ T_{00}^{(g)} = \frac{1}{8\pi G} (\nabla \psi)^2 \]  

(22)

\[ T_{0i}^{(g)} = 0 \]  

(23)

\[ T_{ij}^{(g)} = \frac{1}{4\pi G} \left\{ \partial_i \psi \partial_j \psi - \frac{1}{2} \delta_{ij} (\nabla \psi)^2 \right\} \]  

(24)

which is the Newtonian stress-energy tensor. The topic of energy, momentum, and stress is taken up in section 6.

In the calculations to follow, frequent use is made of the conditions

\[ e_1^1 = e_2^2 = e_3^3 \quad e_1^2 = e_2^3 = e_3^1 = e_3^1 \]  

(25)

Moreover, the focus is upon oscillating gravitational fields of very small amplitude. The coordinate system is taken to be nearly rectangular

\[ e^\alpha_{\mu} = \delta^\alpha_{\mu} + \xi^\alpha_{\mu} \]  

(26)

\[ e_{\alpha}^\mu = \delta_{\alpha}^\mu - \xi_{\alpha}^\mu \]  

(27)

\[ e^\alpha_{\mu} e^\mu_{\alpha} = \delta_\alpha^\nu (\delta^\mu_{\mu} + \xi^\mu_{\mu})(\delta^\nu_{\nu} - \xi^\nu_{\nu}) \]  

\[ = \delta_\alpha^\nu + \delta^\nu_{\nu} e^\alpha_{\mu} - \delta^\alpha_{\mu} \xi^\nu_{\nu} + O(\xi^2) \]  

(28)

1The components \( e^\alpha_{\iota} \) are projections of the basis vectors \( e_\iota \) onto an orthonormal frame. The \( e^\alpha_{\iota} \) can be made symmetric, at any point, by suitably rotating the frame. The three rotation parameters yield the three conditions (25). These frame rotations do not affect the coordinate system \( \{ x^\iota \} \).
Therefore,

\[ \xi^\nu_{\mu} = \xi^\nu_{\mu} + O(\xi^2) \]  

(\xi^\alpha_{\mu} \text{ and } \xi^\alpha_{\mu} \text{ are small compared with unity.}) Only the largest terms will be retained in the Ricci tensor

\[ R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} \]  

Expansion of the metric 

\[ g_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_{\mu} e^\beta_{\nu} = \eta_{\alpha\beta} (\delta^\alpha_{\mu} + \xi^\alpha_{\mu}) (\delta^\beta_{\nu} + \xi^\beta_{\nu}) \]  

yields

\[ R_{\mu\nu} = \eta^{\lambda\rho} \partial_\lambda \partial_\rho \xi_{\mu\nu} + \partial_\mu \partial_\nu \xi^\lambda_{\lambda} - \partial_\mu \partial_\lambda \xi^\lambda_{\nu} - \partial_\nu \partial_\lambda \xi^\lambda_{\mu} \]  

and

\[ R = g^{\mu\nu} R_{\mu\nu} = 2\eta^{\mu\nu} (\partial_\mu \partial_\nu \xi^\lambda_{\lambda} - \partial_\mu \partial_\lambda \xi^\lambda_{\nu}) \]
3. The Coupling Equation.

Define the pseudovector
\[ \varepsilon^\delta = -\frac{1}{4} \epsilon^{\delta\alpha\beta\gamma} e_\alpha^\nu e_\gamma^\lambda \partial_\lambda e_{\beta\nu} \] (34)

so that the coupling equation (5) becomes
\[ i\gamma^\mu \partial_\mu \psi - \gamma_5 \hat{\gamma}^\delta \varepsilon^\delta \psi = 0 \] (35)

At the end of this section, it will be shown that tetrad solutions \( e^\alpha_\mu(x) \) exist such that \( \varepsilon^\delta \) is constant in space and time. Anticipating this, substitute the trial solution (4)
\[ (k_\alpha \hat{\gamma}^\alpha - \varepsilon_\alpha \gamma_5 \hat{\gamma}^\alpha) \psi = 0 \] (36)

where \( k_\alpha = e_\alpha^\mu k_\mu \). Make use of the matrix representation
\[ \hat{\gamma}^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad \hat{\gamma}^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (37)

and then introduce \( u = N \begin{pmatrix} \phi \\ \chi \end{pmatrix} \) to find
\[ \begin{pmatrix} (k^0 - \sigma_a \varepsilon^a) & (\varepsilon^0 - \sigma_a k^a) \\ -\varepsilon^0 + \sigma_a k^a & -(k^0 - \sigma_a \varepsilon^a) \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \] (38)

or
\[ (k^0 - \sigma_a \varepsilon^a)\phi + (\varepsilon^0 - \sigma_a k^a)\chi = 0 \] (39)
\[ (\varepsilon^0 - \sigma_a k^a)\phi + (k^0 - \sigma_a \varepsilon^a)\chi = 0 \] (40)

Add and subtract, in order to arrive at the uncoupled equations
\[ \{(k^0 + \varepsilon^0) - \sigma_a (k^a + \varepsilon^a)\}(\phi + \chi) \quad = 0 \] (41)
\[ \{(k^0 - \varepsilon^0) + \sigma_a (k^a - \varepsilon^a)\}(\phi - \chi) \quad = 0 \] (42)

For each equation, the determinant of the coefficient must vanish. In the first case,
\[ \left| \begin{array}{cc} (k^0 + \varepsilon^0) - (k^3 + \varepsilon^3) & -(k^0 + \varepsilon^0) + (k^3 + \varepsilon^3) \\ -(k^0 + \varepsilon^0) + (k^3 + \varepsilon^3) & (k^0 + \varepsilon^0) - (k^3 + \varepsilon^3) \end{array} \right| = 0 \] (43)
where \( k_\pm = k^1 \pm i k^2 \) and \( \varepsilon_\pm = \varepsilon^1 \pm i \varepsilon^2 \). This gives

\[
k_\mu k^\mu + \varepsilon_\mu \varepsilon^{\mu} + 2k_\mu \varepsilon^\mu = 0 \tag{44}
\]

while the second equation gives

\[
k_\mu k^\mu + \varepsilon_\mu \varepsilon^{\mu} - 2k_\mu \varepsilon^\mu = 0 \tag{45}
\]

Therefore, the constants \( k^\mu \) and \( \varepsilon^\mu \) are related by

\[
k^\mu \varepsilon^\mu = 0 \quad \text{and} \quad k_\mu k^\mu + \varepsilon_\mu \varepsilon^{\mu} = 0 \tag{46}
\]

In the expansion of \( \varepsilon^\delta \), all first order terms vanish

\[
\varepsilon^\delta = -\frac{1}{4} \delta^{\alpha\beta\gamma} e_\alpha^\nu e_\gamma^\lambda \partial_\lambda e_\beta^\nu = -\frac{1}{4} \delta^{\alpha\lambda\nu} e_\alpha^\nu e_\lambda^\gamma \partial_\gamma e_\nu^\lambda
\]

\[
= \frac{1}{4} \delta^{\alpha\beta\gamma} e_\alpha^k e_\gamma^l \partial_k e_{\beta l} \tag{47}
\]

This expression contains no derivatives of \( e^0 \). Thus, it plays no role in the spinor coupling, \( e^0 = 1 \). With the substitution of the trial solution

\[
\xi^i_\mu = a^i_\mu \cos(-k^\mu x^\mu) + b^i_\mu \sin(-k^\mu x^\mu) \tag{48}
\]

\( \varepsilon^\delta \) becomes constant in space and time (as was to be shown)

\[
\varepsilon^\delta = -\frac{1}{4} \delta^{\alpha\mu\nu} K_\gamma a_\alpha b_\beta^\gamma \tag{49}
\]

For later use, the components are

\[
\varepsilon^c = -\frac{1}{4} \epsilon^{cab} K_0 a^{\alpha b}_c = \frac{1}{4} \epsilon^{0abc} K_0 a^{\alpha b}_0 \tag{50}
\]

\[
\varepsilon^0 = -\frac{1}{4} \epsilon^{abc} K_0 a^{\alpha b}_c = \frac{K_c}{K_0} \varepsilon^c \tag{51}
\]

or \( K_\mu \varepsilon^\mu = 0 \).
4. The Dirac Equation. Helicity.

Fermions are described by solutions of the Dirac equation \[^2\]

\[ i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad (52) \]

[In this section, \(\hbar = c = 1\). The replacement \(m \rightarrow mc/\hbar\) gives the explicit formula.] Substitute the plane wave (4) with \(u = N \begin{pmatrix} \phi \\ \chi \end{pmatrix}\) to find

\[ \chi = \frac{\sigma_a k^a}{k^0 + m} \phi \quad \text{and} \quad k_\mu k^\mu = m^2 \quad (53) \]

The two component spinor is normalized to unity, \(\phi^\dagger \phi = 1\), while \(u\) is normalized to \(k^0/m\)

\[ u^\dagger u = N^2 \frac{2k^0}{k^0 + m} = \frac{k^0}{m} \quad N = \sqrt{\frac{k^0 + m}{2m}} \quad (54) \]

In the following section, it will be shown that functions (48) satisfy the gravitational field equations, if the spinor is in a state of definite helicity. Helicity states are defined by the eigenvalue equation [3, 4, 5]

\[ \frac{\sigma_a k^a}{k} \phi_\pm = \pm \phi_\pm \quad (55) \]

where \(k = [(k^1)^2 + (k^2)^2 + (k^3)^2]^{1/2}\). For positive helicity,

\[ \begin{pmatrix} (k^3 - k) \\ k_+ \end{pmatrix} \begin{pmatrix} k_- \\ -(k^3 + k) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \quad (56) \]

which yields \(\phi_2 = \frac{k_+}{k_+ + k^3} \phi_1\). Normalize \(\phi^\dagger \phi = 1\) to find \(\phi_1 = \sqrt{\frac{k + k^3}{2k}}\) and

\[ \phi_+ = \frac{1}{\sqrt{2k(k + k^3)}} \begin{pmatrix} k + k^3 \\ k_+ \end{pmatrix} \quad (57) \]

Similarly,

\[ \phi_- = \frac{1}{\sqrt{2k(k + k^3)}} \begin{pmatrix} -k_- \\ k + k^3 \end{pmatrix} \quad (58) \]

If the motion is along \(x^3\), then these eigenstates coincide with those of spin, \(\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\).
For the remainder of the paper, we focus upon the positive frequency, positive helicity solutions of the Dirac equation

\[ \psi_+ = \sqrt{\frac{m}{k^0 V}} u_+ \exp(-ik_\mu x^\mu) \]  

(59)

where

\[ u_+ = N \left( \frac{\phi_+}{\sqrt{k^0 + m}} \right) \]  

(60)

The solution \( \psi_+ \) must also satisfy the coupling equation (35). This will be the case, if

\[ \gamma_5 \gamma_\delta \varepsilon^\delta u_+ = mu_+ \]  

(61)

The corresponding two-component systems are

\[ \begin{aligned}
\left\{ \sigma_a \varepsilon^a - m - \frac{\varepsilon^0 k}{k^0 + m} \right\} \begin{pmatrix} k + k^3 \\ k_+ \end{pmatrix} &= 0 \\
\left\{ \varepsilon^0 - \frac{k(\sigma_a \varepsilon^a + m)}{k^0 + m} \right\} \begin{pmatrix} k + k^3 \\ k_+ \end{pmatrix} &= 0
\end{aligned} \]  

(62-63)

After some algebra, these equations yield the expression

\[ \varepsilon^a = \frac{k^a}{k} \left( \frac{\varepsilon^0 k}{k^0 + m} + m \right) \]  

(64)

Make use of \( k_\mu \varepsilon^\mu = 0 \) (46) in order to obtain

\[ \varepsilon^0 = k \quad \text{and} \quad \varepsilon^a = \frac{k^a k^0}{k} \quad a = 1, 2, 3 \]  

(65)

This solution, together with (49), relates the propagation vectors \( k_\mu \) and \( K_\mu \). In appendix C, it is shown that \( k_\mu = \alpha K_\mu \), where \( \alpha \) is a very small constant, second order in the amplitudes. Therefore, the spinor and tetrad propagate with the same phase velocity \( k^0/k^i = K^0/K^i \).
5. Gravity.

It remains to be determined whether the gravitational field equations are satisfied. Setting \( \xi^0_0 = 0 \) in (32) leaves

\[
R_{00} = \partial_0 \partial_0 \xi_n^n \\
R_{ij} = \eta^{\lambda \rho} \partial_\lambda \partial_\rho \xi_{ij} + \partial_i \partial_j \xi_n^n - \partial_i \partial_n \xi_j^j - \partial_j \partial_n \xi_i^i
\]

(66) (67)

Conditions \( \partial_\mu (\sqrt{-g} e_\mu^\nu) \hat{\gamma}^\alpha = 0 \) (section 2) are \( \partial_\lambda \xi_\lambda^\lambda + \partial_\alpha \xi_\lambda^\alpha = 0 \) to first order. It follows that

\[
\xi_n^n = 0 \quad \text{and} \quad \partial_n \xi_i^i = 0 \quad i = 1, 2, 3
\]

(68)

Therefore, the solution is both traceless and transverse; also \( R_{00} = R = 0 \) and

\[
R_{ij} = \eta^{\lambda \rho} \partial_\lambda \partial_\rho \xi_{ij}
\]

(69)

Since \( R_{ij} \) is first order in \( \xi_{ij} \), only first order terms will be retained in \( T_{\mu \nu}^{(m)} \) (8). The derivative \( \partial_\mu \psi \) is proportional to \( k_\mu \), which was shown to be second order in \( \xi_{ij} \). Expand \( T_{\mu \nu}^{(m)} \) to find \( T_{00}^{(m)} = 0 \) and

\[
T_{ij}^{(m)} = \frac{\hbar c}{4} \gamma^5 \gamma^\delta \psi \epsilon^{\delta \alpha \beta \gamma} (\eta_{ai} \partial_\gamma \xi_{bj} + \eta_{aj} \partial_\gamma \xi_{bi})
\]

(70)

The factor \( \gamma^5 \gamma^\delta \psi \) is evaluated by means of the helicity state

\[
\gamma^5 \gamma^\delta \psi^+_+ = \frac{mc}{\hbar k^0 V} \gamma^5 \gamma^\delta u^+_+ = \frac{\epsilon \delta}{k^0 V}
\]

(71)

therefore

\[
T_{ij}^{(m)} = -\frac{\hbar c}{4k^0 V} \epsilon^{\delta \alpha \beta \gamma} \epsilon_\delta (\eta_{ai} \partial_\gamma \xi_{bj} + \eta_{aj} \partial_\gamma \xi_{bi})
\]

(72)

Substitute the trial solution (48) and simplify, making use of \( k_\mu = \alpha K_\mu \), in order to obtain

\[
T_{ij}^{(m)} = \frac{\hbar c}{4k^0 V} (K_0^2 - K^2) \epsilon^{\alpha \beta \gamma \delta} \frac{K_\epsilon}{K} \left\{ \eta_{ai} a_{bj} \sin(-K_\mu x^\mu) - \eta_{ai} b_{bj} \cos(-K_\mu x^\mu) \right\}
\]

\( + \ i \leftrightarrow j \)

(73)
At this point, the gravitational field equations are

$$- \kappa R_{ij} = T_{ij}^{(m)}$$  \hspace{1cm} (74)

The integration volume is introduced on the left-hand side by means of a length parameter \( \lambda \): \( \kappa \rightarrow \kappa \lambda^3 / V \). Substitute the trial solution to find

\[
- \kappa R_{ij} = \frac{\kappa \lambda^3}{V} (K_0^2 - K^2) \xi_{ij} = \frac{\kappa \lambda^3}{V} (K_0^2 - K^2) \left\{ a_{ij} \cos(-K_{\mu} x^\mu) + b_{ij} \sin(-K_{\mu} x^\mu) \right\} \hspace{1cm} (75)
\]

Equate coefficients of \( \cos(-K_{\mu} x^\mu) \) and \( \sin(-K_{\mu} x^\mu) \) in (73) and (75) to find that solutions exist only if

$$\kappa \lambda^3 = \frac{\hbar c}{2K^0}$$  \hspace{1cm} (76)

In this case,

\[
a_{ij} = -\frac{1}{2} \epsilon^{abc} K_c \left\{ \eta_{ai} b_{bj} + \eta_{aj} b_{bi} \right\} \hspace{1cm} (77)
\]

\[
b_{ij} = \frac{1}{2} \epsilon^{abc} K_c \left\{ \eta_{ai} a_{bj} + \eta_{aj} a_{bi} \right\} \hspace{1cm} (78)
\]

Therefore, the gravitational field equations are satisfied. An explicit solution is given in appendix D. The following are two special cases:

(a) Motion along the \( x^3 \)-axis; \( k^1 = k^2 = K_1 = K_2 = 0 \).

In this case, \( K = K_3 = -K_3 \). From appendix D, \( a^{3 \ i} = b^{3 \ i} = 0 \). The solution is

\[
\xi_1^1 = a_1^1 \cos(-K_{\mu} x^\mu) + b_1^1 \sin(-K_{\mu} x^\mu) \hspace{1cm} (79)
\]

\[
\xi_2^2 = -\xi_1^1 \hspace{1cm} (80)
\]

\[
\xi_1^2 = -b_1^1 \cos(-K_{\mu} x^\mu) + a_1^1 \sin(-K_{\mu} x^\mu) \hspace{1cm} (81)
\]

If the phase is such that \( b_1^1 = 0 \), then this is identical to the solution found in [1].
(b) Motion in the \((x^2, x^3)\) plane; \(k^1 = K^1 = 0\). In this case, \((K)^2 = K^2_2 + K^2_3\). The solution is

\[
\begin{align*}
\xi^1_1 &= a^1_1 \cos(-K_\mu x^\mu) + b^1_1 \sin(-K_\mu x^\mu) \tag{82} \\
\xi^2_2 &= -\left(\frac{K_3}{K}\right)^2 \left\{ a^1_1 \cos(-K_\mu x^\mu) + b^1_1 \sin(-K_\mu x^\mu) \right\} \tag{83} \\
\xi^3_3 &= -\left(\frac{K_2}{K}\right)^2 \left\{ a^1_1 \cos(-K_\mu x^\mu) + b^1_1 \sin(-K_\mu x^\mu) \right\} \tag{84} \\
\xi^1_2 &= \frac{K_3}{K} \left\{ b^1_1 \cos(-K_\mu x^\mu) - a^1_1 \sin(-K_\mu x^\mu) \right\} \tag{85} \\
\xi^2_3 &= \frac{K_2 K_3}{(K)^2} \left\{ a^1_1 \cos(-K_\mu x^\mu) + b^1_1 \sin(-K_\mu x^\mu) \right\} \tag{86} \\
\xi^3_1 &= -\frac{K_2}{K} \left\{ b^1_1 \cos(-K_\mu x^\mu) - a^1_1 \sin(-K_\mu x^\mu) \right\} \tag{87}
\end{align*}
\]
6. Energy, Momentum, Stress.

The total density of energy, momentum, and stress is given by

\[ T_{\mu\nu} = T^{(g)}_{\mu\nu} + T^{(m)}_{\mu\nu} \]  

(88)

The gravitational energy tensor (18) is simplified in the present case, because

\[ Q^0_0 = \partial_0 \xi^0 = 0 \quad \text{and} \quad Q^n_0 = \partial_0 \xi^n = 0, \]

so that \( Q_\mu = 0 \). This leaves

\[ T^{(g)}_{00} = \frac{\hbar c}{4K^0V} Q^l_{m0} Q^m_{l0} \]  

(89)

\[ T^{(g)}_{0i} = 0 \]  

(90)

\[ T^{(g)}_{ij} = \frac{\hbar c}{4K^0V} Q^l_{m0} Q^m_{l0} \delta_{ij} \]  

(91)

Make use of (77, 78) to find

\[ a^{lm} a_{lm} = b^{lm} b_{lm} = -\epsilon^{0abc} \frac{K_c}{K} a_{an} b^n_b \]  

(92)

while \( a^{lm} b_{lm} = 0 \). It follows that

\[ Q^l_{m0} Q^m_{l0} = \partial_0 \xi^l \partial_0 \xi^m = (K_0)^2 a^{lm} a_{lm} \]  

(93)

Compare (92) with (51), then make use of \( \epsilon^0 = k \) and \( k_\mu = \alpha K_\mu \) in order to obtain \( a^{lm} a_{lm} = 4\alpha \). Therefore, \( Q^l_{m0} Q^m_{l0} = 4k_0 K_0 \) and

\[ T^{(g)}_{00} = \frac{\hbar c k_0}{V} = \frac{\hbar \omega}{V} \]  

(94)

\[ T^{(g)}_{0i} = 0 \]  

(95)

\[ T^{(g)}_{ij} = \frac{\hbar \omega}{V} \delta_{ij} \]  

(96)

The stresses are constant and purely compressive.

With regard to the matter tensor \( T^{(m)}_{\mu\nu} \), the stress components \( T^{(m)}_{ij} \) were given in (73). They are first order and propagate along with the tetrad. Components \( T^{(m)}_{00} \) and \( T^{(m)}_{0i} \) are calculated from (8) by means of

\[ \overline{\psi_+} \gamma_\delta \gamma_\delta \psi_+ = \frac{k_\delta}{k^0 V} \quad \text{and} \quad \overline{\psi_+} \gamma_5 \gamma_\delta \gamma_\delta 

(97)

\[ \psi_+ = - \frac{\varepsilon \delta}{k^0 V} \]

The material energy density is
$T^{(m)}_{00} = \frac{\hbar c k_0}{V} - \frac{\hbar c}{4k_0 V} \varepsilon_a \varepsilon^{abc} \xi_b \partial_0 \xi_{cn} = \frac{\hbar c k_0}{V} + \frac{\hbar c}{k_0 V} \varepsilon_a e^a = 0 \quad (98)$

where (49) and (65) have been used. The density of momentum is

$T^{(m)}_{0i} = \frac{\hbar c k_i}{V} - \frac{\hbar c}{4k_0 V} \varepsilon_a \varepsilon^{abc} e^b \partial_i e_{ci} + \frac{\hbar c}{8k_0 V} \varepsilon_a \xi^a \left\{ \varepsilon_i \partial_i \xi_{bn} - \varepsilon_c \partial_i \xi_{bn} \right\} \quad (99)$

Substitute the solution (48) to find that the last term is zero. The second term is a divergence $e^a \partial_0 e_{ci} = \partial_n (e^b n e_{ci})$, since $\partial_n e^b n = 0$. Therefore,

$T^{(m)}_{0i} = \frac{\hbar c k_i}{V} + \text{a divergence} \quad (100)$

In sum, the fermion’s energy is given by the gravitational component $T^{(g)}_{00}$, while its momentum is given by $T^{(m)}_{0i}$. The total energy $\hbar \omega$ is conserved, as is the total momentum $\hbar k^i$. 

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7. Concluding Remarks.

Despite the equality (61), the coupling equation and the Dirac equation are not equivalent. They differ in their matrix algebra. The coupling equation conserves not only the vector current (3), but the axial vector current as well

\[ \partial_{\mu} \left( \sqrt{-g} \bar{\psi} \gamma_5 \gamma^\mu \psi \right) = 0 \]  

(101)

The Dirac equation does not conserve the axial current

\[ \partial_{\mu} \left( \sqrt{-g} \bar{\psi} \gamma_5 \gamma^\mu \psi \right) = -2im\sqrt{-g} \bar{\psi} \gamma_5 \psi \]  

(102)

This is clearly due to the mass term \( m \bar{\psi} \psi \). The electroweak interaction contains conserved vector and axial vector parts. Thus, the mass term cannot appear in the Lagrangian.
Appendix A: Spinor Lagrangian.

The covariant spinor derivative is [6, 7]

$$D_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi$$  \hspace{1cm} (103)

where

$$\Gamma_\mu = \frac{1}{8} \left( \gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha \right) \omega_{\alpha\beta\mu} = \frac{1}{4} \gamma^{[\alpha} \gamma^{\beta]} \omega_{\alpha\beta\mu}$$  \hspace{1cm} (104)

The conjugate expression is

$$D_\mu \overline{\psi} = \partial_\mu \overline{\psi} - \overline{\psi} \Gamma_\mu$$  \hspace{1cm} (105)

The spinor Lagrangian density is given by

$$L_f = \frac{i}{2 \hbar c} \{ \overline{\psi} \gamma^\mu D_\mu \psi - (D_\mu \overline{\psi}) \gamma^\mu \psi \}$$

$$= \frac{i}{2 \hbar c} \{ \overline{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \overline{\psi}) \gamma^\mu \psi \} + \frac{i}{2 \hbar c} \overline{\psi} \gamma^\mu (\gamma_\mu + \Gamma_\mu \gamma^\mu) \psi$$  \hspace{1cm} (106)

The coupling term may be reduced, by first expanding

$$\gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = \frac{1}{4} \gamma^\mu \left( \gamma^\gamma \gamma^{[\alpha} \gamma^{\beta]} + \gamma^{[\alpha} \gamma^{\beta]} \gamma^\gamma \right) \omega_{\alpha\beta\gamma}$$

$$= \frac{1}{2} \gamma^{[\alpha} \gamma^{\beta]} \gamma^\gamma \omega_{\alpha\beta\gamma}$$  \hspace{1cm} (107)

where

$$\gamma^{[\gamma \gamma^{[\alpha} \gamma^{\beta]} = \frac{1}{2} \left( \gamma^\gamma \gamma^{[\alpha} \gamma^{\beta]} + \gamma^{[\alpha} \gamma^{\beta]} \gamma^\gamma \right)$$  \hspace{1cm} (108)

The identity [7]

$$\gamma^{[\alpha} \gamma^{\beta]} \equiv -i \gamma_5 \gamma_5 \epsilon^{\alpha\beta\gamma}$$  \hspace{1cm} (109)

then gives

$$\gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = -\frac{i}{2} \gamma_5 \gamma_5 \epsilon^{\delta\alpha\beta\gamma} \omega_{\alpha\beta\gamma}$$

$$= -\frac{i}{2} \gamma_5 \gamma_5 \epsilon^{\delta\alpha\beta\gamma} \omega_{\alpha\beta\gamma}$$  \hspace{1cm} (110)
since $\epsilon^{\delta\alpha\beta\gamma}$ is totally anti-symmetric.

The covariant derivative of the tetrad is zero

$$\partial_{\nu}e_{\alpha}^{\mu} - e_{\beta}^{\mu}\omega_{\alpha\nu}^{\beta} + e_{\lambda}^{\lambda}Q_{\lambda\nu}^{\mu} = 0 \quad (111)$$

so that

$$Q_{\lambda\nu}^{\mu} = e_{\alpha}^{\mu}\partial_{\nu}e_{\alpha}^{\lambda} + \omega_{\lambda\nu}^{\mu} \quad (112)$$

Beginning with formula (16), it is readily shown that the anti-symmetric tensor

$$Q_{[\mu\nu\lambda]} = g_{\mu\rho}Q_{[\nu\lambda]}^{\rho} + g_{\nu\rho}Q_{[\lambda\mu]}^{\rho} + g_{\lambda\rho}Q_{[\mu\nu]}^{\rho} = 0 \quad (113)$$

Therefore,

$$0 = e_{[\mu}^{\alpha}\partial_{\nu}e_{\alpha\mu]} + \omega_{[\mu\nu]} \quad (114)$$

or

$$\omega_{[\mu\nu\lambda]} = e_{[\mu}^{\alpha}\partial_{\nu}e_{\alpha\lambda]} \quad (115)$$

Substitute this expression into (110) and (106), in order to obtain

$$L_f = \frac{i}{2}\hbar c\left\{\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right\} + \frac{\hbar c}{4}\psi\gamma_{5}\gamma_{5}\psi e^{\delta\alpha\beta\gamma}e_{\alpha}^{\gamma}\partial_{\lambda}e_{\beta\nu} \quad (116)$$
Appendix B. $U(1) \otimes SU(2)_L$ Gauge Invariance.

The gravitational coupling term in $L_f$ (1) contains the factor $\bar{\psi} \gamma_5 \tilde{\gamma}_5 \gamma_5 \psi$. This factor does not mix right- and left-handed spinor components, $\psi_R$ and $\psi_L$.

In order to prove this, set

$$\psi = \psi_R + \psi_L = \frac{1 + \gamma_5}{2} \psi + \frac{1 - \gamma_5}{2} \psi$$  \hspace{1cm} (117)

where $(\gamma_5)^2 = 1$ and $\gamma_5 \gamma_6 = -\gamma_6 \gamma_5$. Also, $\bar{\psi}_R = \bar{\psi} \frac{1 - \gamma_5}{2}$ and $\bar{\psi}_L = \bar{\psi} \frac{1 + \gamma_5}{2}$.

In the expansion

$$\bar{\psi} \gamma_5 \tilde{\gamma}_5 \gamma_5 \psi = (\bar{\psi}_R + \bar{\psi}_L) \gamma_5 \gamma_6 (\psi_R + \psi_L)$$

$$= \bar{\psi}_R \gamma_5 \gamma_6 \psi_R + \bar{\psi}_L \gamma_5 \gamma_6 \psi_L + \bar{\psi}_R \gamma_5 \gamma_6 \psi_L + \bar{\psi}_L \gamma_5 \gamma_6 \psi_R$$  \hspace{1cm} (118)

the mixed terms are identically zero. For example,

$$\bar{\psi}_R \gamma_5 \tilde{\gamma}_5 \gamma_6 \psi_L = \bar{\psi} \frac{1 - \gamma_5}{2} \gamma_5 \gamma_6 \frac{1 - \gamma_5}{2} \psi = \bar{\psi} \gamma_5 \gamma_6 \psi = 0$$  \hspace{1cm} (119)

Therefore,

$$\bar{\psi} \gamma_5 \tilde{\gamma}_5 \gamma_5 \psi = \bar{\psi}_R \gamma_5 \gamma_6 \psi_R + \bar{\psi}_L \gamma_5 \gamma_6 \psi_L$$  \hspace{1cm} (120)

An expression of this type will be invariant under $U(1) \otimes SU(2)_L$ gauge transformations. \footnote{The Dirac mass term $m \bar{\psi} \psi = m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$ mixes right- and left-handed spinors and cannot appear in the electroweak Lagrangian.}

Introduce the right-handed singlet $\psi_R = e_R$ and left-handed doublet $\psi_L = \left( \begin{array}{c} \nu_L \\ e_L \end{array} \right)$ in order to form the Lagrangian

$$L_{e-w} = \frac{i}{2} \hbar c \left\{ \bar{\psi}_R \gamma_\mu \partial_\mu \psi_R + \bar{\psi}_L \gamma_\mu \partial_\mu \psi_L \right\} + \text{h.c.}$$

$$+ \frac{\hbar c}{4} \left\{ \bar{\psi}_R \gamma_5 \tilde{\gamma}_5 \psi_R + \bar{\psi}_L \gamma_5 \tilde{\gamma}_5 \psi_L \right\} \epsilon^{\delta \alpha \beta \gamma} e_\alpha \lambda e_\gamma \nu \partial_\beta e_\delta + L_{int}$$  \hspace{1cm} (121)

(plus expressions for the muon and tau). $L_{int}$ contains the electroweak interaction as well as kinetic terms for $A_\mu, W^\pm_\mu$, and $Z^0_\mu$. 

Appendix C. Proof of $k^\mu = \alpha K^\mu$.

Begin with equation (49) and substitute the transversality conditions (68) in the form

\[
\begin{align*}
    a_2^1 &= \frac{K_3}{2} \left\{ -\frac{K_1}{K_2 K_3} a_1^1 - \frac{K_2}{K_3 K_1} a_2^2 + \frac{K_3}{K_1 K_2} a_3^3 \right\} \\
    a_3^2 &= \frac{K_1}{2} \left\{ -\frac{K_1}{K_2 K_3} a_1^1 - \frac{K_2}{K_3 K_1} a_2^2 - \frac{K_3}{K_1 K_2} a_3^3 \right\} \\
    a_1^3 &= \frac{K_2}{2} \left\{ -\frac{K_1}{K_2 K_3} a_1^1 + \frac{K_2}{K_3 K_1} a_2^2 - \frac{K_3}{K_1 K_2} a_3^3 \right\}
\end{align*}
\]

(122) (123) (124)

This yields

\[
\begin{align*}
    \varepsilon^i &= -\frac{K_0 K^i}{8K_1 K_2 K_3} \left\{ (K_2^2 + K_3^2)(a_2^2 b_3^3 - a_3^3 b_2^2) \\
    &\quad + (K_3^2 + K_1^2)(a_3^3 b_1^1 - a_1^1 b_3^3) + (K_1^2 + K_2^2)(a_1^1 b_2^2 - a_2^2 b_1^1) \right\}
\end{align*}
\]

(125)

Comparison with $\varepsilon^\mu = (k, k^i k^0/k)$ shows that $k^i = \alpha K^i$. Next, make use of $K_\mu \varepsilon^\mu = 0$ to find

\[
\begin{align*}
    \varepsilon^0 &= \frac{(K)^2}{8K_1 K_2 K_3} \left\{ (K_2^2 + K_3^2)(a_2^2 b_3^3 - a_3^3 b_2^2) \\
    &\quad + (K_3^2 + K_1^2)(a_3^3 b_1^1 - a_1^1 b_3^3) + (K_1^2 + K_2^2)(a_1^1 b_2^2 - a_2^2 b_1^1) \right\}
\end{align*}
\]

(126)

and $k^0 = \alpha K^0$. 

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Appendix D. Gravitational Solution.

The transverse, traceless conditions (68) are imposed upon the solution (48). In the equation \( \partial_{\eta} \xi_{\eta} = 0 \), the coefficients of \( \cos(-K_{\mu} x^\mu) \) and \( \sin(-K_{\mu} x^\mu) \) must equal zero

\[
K_n a_i^n = 0 \quad \text{and} \quad K_n b_i^n = 0 \quad i = 1, 2, 3 \quad (127)
\]

Similarly, the trace condition \( \xi_{\eta} = 0 \) requires

\[
a_n^n = 0 \quad \text{and} \quad b_n^n = 0 \quad (128)
\]

These four conditions leave two independent amplitudes \( a^i_j \) as well as two \( b^i_j \). [8]

In order to solve the three-dimensional problem, we first choose the independent amplitudes to be \( a^1_1, a^1_2, b^1_1, \) and \( b^1_2 \). The transverse, traceless conditions are then used to eliminate all other amplitudes. Finally, equations (77, 78) are used to eliminate \( a^1_2 \) and \( b^1_2 \). The solution is

\[
\xi_{ij} = a^i_j \cos(-K_{\mu} x^\mu) + b^i_j \sin(-K_{\mu} x^\mu) \quad (129)
\]

with

\[
a^2_2 = \frac{1}{(K_2^2 + K_3^2)^2} \left\{ (K_1^2 K_2^2 - K_3^2 K_2^2) a_1^1 - 2K_1 K_2 K_3 K b_1^1 \right\} \quad (130)
\]

\[
a^3_3 = \frac{1}{(K_2^2 + K_3^2)^2} \left\{ (K_1^2 K_3^2 - K_2^2 K_2^2) a_1^1 + 2K_1 K_2 K_3 K b_1^1 \right\} \quad (131)
\]

\[
a^1_2 = \frac{1}{K_2^2 + K_3^2} \left\{ -K_1 K_2 a_1^1 + K_3 K b_1^1 \right\} \quad (132)
\]

\[
a^2_3 = \frac{1}{(K_2^2 + K_3^2)^2} \left\{ K_2 K_3 (K^2 + K_1^2) a_1^1 + K_1 K (K_2^2 - K_3^2) b_1^1 \right\} \quad (133)
\]

\[
a^3_1 = -\frac{1}{K_2^2 + K_3^2} \left\{ K_1 K_3 a_1^1 + K_2 K b_1^1 \right\} \quad (134)
\]

and

\[
b^2_2 = \frac{1}{(K_2^2 + K_3^2)^2} \left\{ 2K_1 K_2 K_3 K a_1^1 + (K_1^2 K_2^2 - K_3^2 K_2^2) b_1^1 \right\} \quad (135)
\]

\[
b^3_3 = -\frac{1}{(K_2^2 + K_3^2)^2} \left\{ 2K_1 K_2 K_3 K a_1^1 - (K_1^2 K_3^2 - K_3^2 K_2^2) b_1^1 \right\} \quad (136)
\]
\[ b^1_2 = -\frac{1}{K_2^2 + K_3^2} \left\{ K_3 K a^1_1 + K_1 K_2 b^1_1 \right\} \] (137)

\[ b^2_3 = -\frac{1}{(K_2^2 + K_3^2)^2} \left\{ K_1 K (K_2^2 - K_3^2) a^1_1 - K_2 K_3 (K_1^2 + K_2^2) b^1_1 \right\} \] (138)

\[ b^3_1 = \frac{1}{K_2^2 + K_3^2} \left\{ K_2 K a^1_1 - K_1 K_3 b^1_1 \right\} \] (139)
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