HAMILTONIAN CIRCLE ACTIONS ON EIGHT DIMENSIONAL MANIFOLDS WITH MINIMAL FIXED SETS

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Abstract. Consider a Hamiltonian circle action on a closed 8-dimensional symplectic manifold $M$ with exactly five fixed points, which is the smallest possible fixed set. In their paper [GS], L. Godinho and S. Sabatini show that if $M$ satisfies an extra “positivity condition” then the isotropy weights at the fixed points of $M$ agree with those of some linear action on $\mathbb{C}P^4$. Therefore, the (equivariant) cohomology rings and the (equivariant) Chern classes of $M$ and $\mathbb{C}P^4$ agree; in particular, $H^*(M; \mathbb{Z}) \cong \mathbb{Z}[y]/y^5$ and $c(TM) = (1 + y)^5$. In this paper, we prove that this positivity condition always holds for these manifolds. This completes the proof of the “symplectic Petrie conjecture” for Hamiltonian circle actions on 8-dimensional closed symplectic manifolds with minimal fixed sets.

The classification of Hamiltonian $S^1$-actions is an important subject in symplectic geometry. Although in general the problem is probably too hard to be tractable, progress has been made in special cases. In particular, let $(M, \omega)$ be a closed $2n$-dimensional symplectic manifold with a Hamiltonian circle action such that $H^{2i}(M; \mathbb{R}) = H^{2i}(\mathbb{C}P^n; \mathbb{R})$ for all $i$. Since $[\omega]^i \neq 0$ for $0 \leq i \leq n$, this means that $M$ has the smallest possible even Betti numbers. In analogy with the Petrie conjecture [P], one can reasonably hope to classify all the possible (equivariant) cohomology rings and (equivariant) Chern classes in this case. Indeed, these have been successfully classified when the fixed set is discrete and the first Chern class is large, or when $\dim M \leq 6$, etc ([A], [GS], [H1], [H2], [H3], [J], [LH], [LT], [M]).

In this paper, we consider the case that all the fixed points are isolated. Because the family of almost complex structures $J$ on $M$ compatible with $\omega$ is contractible, this implies that there is a multiset of $n$ non-zero integers $w_{p1}, \ldots, w_{pn}$, called isotropy weights, associated to each fixed point $p \in M^{S^1}$. Moreover, the moment map $\phi : M \to \mathbb{R}$ is a perfect Morse function, the critical points are the fixed points, and the index of each critical point is twice the number of negative

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weights (counted with multiplicity). Therefore, there are at least $n + 1$ fixed points, and $H^2_i(M; \mathbb{R}) = H^2_i(\mathbb{CP}^n; \mathbb{R})$ for all $i$ if and only if there are exactly $n + 1$ fixed points – one fixed point of index $2i$ for each $i \in \{0, \ldots, n\}$. In this case, by [11], the isotropy weights at the fixed points completely determine the ordinary cohomology ring $H^*(M; \mathbb{Z})$, the equivariant cohomology ring $H^*_S(M; \mathbb{Z})$, its restriction to the equivariant cohomology of the fixed set $H^*_S(M^{S^1}; \mathbb{Z})$, and the (equivariant) Chern classes. Hence, it is enough to classify the possible isotropy weights.

In their paper [GS], L. Godinho and S. Sabatini show that if $M$ is 8 dimensional and satisfies an extra “positivity condition” $(\mathcal{P})^{+}_0$, then the isotropy weights at the fixed points of $M$ agree with those of some linear action on $\mathbb{CP}^4$; see Definitions 2.1, 2.2, and 2.3, and Theorem 2.4. Here, a linear action on $\mathbb{CP}^4$ is the action induced by an embedding $S^1 \hookrightarrow U(5)$, where $U(5)$ acts on $(\mathbb{C}^5$ and hence) $\mathbb{CP}^4$ by the defining representation. In Theorem 2.7, we show that the positivity condition $(\mathcal{P})^{+}_0$ is always satisfied for these manifolds. Thus, we prove the following theorem:

**Theorem 0.1.** Consider a Hamiltonian circle action on a 8-dimensional closed symplectic manifold $(M, \omega)$ with exactly 5 fixed points. Then the isotropy weights at the fixed points of $M$ agree with those of some linear action on $\mathbb{CP}^4$. Therefore, the (equivariant) cohomology and (equivariant) Chern classes of $M$ and $\mathbb{CP}^4$ agree. In particular, $H^*(M; \mathbb{Z}) = \mathbb{Z}[y]/(y^5)$ and $c(TM) = (1 + y)^5$.

Moreover, the claims in Lemma 2.5 and its corollary reduce the number of cases one needs to consider to prove Theorem 2.4. More generally, in higher dimensions, they can be used to simplify the classification; cf. [GS].

Note that the theorem follows from [11] if $M$ is homotopic to $\mathbb{CP}^4$.

Unfortunately, the proof of Theorem 2.7 does not work in higher dimensions. However, we do not know any counter-examples to the following much more general question:

**Question 0.2.** Let the circle act on a closed symplectic manifold $(M, \omega)$ with isolated fixed points and with moment map $\phi: M \to \mathbb{R}$. Does there exist a labelled graph describing $M$ so that $\phi(i(e)) < \phi(t(e))$ for every edge $e$?

In contrast, condition $(\mathcal{P})^{+}_0$ is not satisfied for general Hamiltonian actions; see [GS] Remark 5.7]. Nevertheless, an affirmative answer to

\footnote{See Definitions 2.1 and 2.2}
the above question would imply that \((\mathcal{P})_0^+\) holds whenever the fixed set is minimal.

1. The main technical tool

In this section, we prove our main technical tool, which gives restrictions on the smallest positive weight for actions on almost complex manifolds (and therefore symplectic manifolds).

Let the circle act on a closed \(2n\)-dimensional almost complex manifold \((M, J)\) with isolated fixed points. As before, there is a multiset \(w_{p1}, \ldots, w_{pn}\) of isotropy weights at each fixed point \(p\). In analogy with the Hamiltonian case, define the index of \(p\) to be \(2\lambda_p\), where \(\lambda_p\) is the number of negative isotropy weights at \(p\). Let \(N_k\) be the number of fixed points of index \(2k\), and let \(\sigma_k\) denote the elementary symmetric polynomial of degree \(k\) in \(n\) variables. In [LP], P. Li proves the following:

**Theorem 1.1.** Let the circle act on a closed \(2n\)-dimensional almost complex manifold \((M, J)\) with isolated fixed points. Then for all \(0 \leq k \leq n\),

\[
\sum_{p \in M^{S^1}} \frac{\sigma_k(t^{w_{p1}}, \ldots, t^{w_{pn}})}{\prod_{j=1}^n (1 - t^{w_{pj}})} = (-1)^k N^k.
\]

Kosniowski proved a claim analogous to Proposition 1.2 below for holomorphic vector fields on complex manifolds with simple isolated zeroes [K]. Closely following his idea, we use Theorem 1.1 to prove the following:

**Proposition 1.2.** Let the circle act on a closed \(2n\)-dimensional almost complex manifold \((M, J)\) with isolated fixed points. Let \(a\) be the smallest positive isotropy weight that occurs at any fixed point. Given any \(k \in \{0, 1, \ldots, n-1\}\), the number of times the isotropy weight \(-a\) occurs at fixed points of index \(2k + 2\) is equal to the number of times the isotropy weight \(+a\) occurs at fixed points of index \(2k\).

**Proof.** Fix an integer \(k \in \{0, 1, \ldots, n-1\}\). By Theorem 1.1

\[
(1.3) \quad N^k = \sum_{p \in M^{S^1}} (-1)^k \frac{\sigma_k(t^{w_{p1}}, \ldots, t^{w_{pn}})}{\prod_{j=1}^n (1 - t^{w_{pj}})}
= \sum_{p \in M^{S^1}} (-1)^{\lambda_p+k} \frac{\sigma_k(t^{w_{p1}}, \ldots, t^{w_{pn}})}{\prod_{j=1}^n (1 - t^{w_{pj}})} \prod_{w_{pj} < 0} t^{-w_{pj}}.
\]

\(^2\)We assume the action preserves the almost complex structure.
Moreover, \( a \) is the smallest positive isotropy weight that occurs at any fixed point. Thus, as \( t \to 0 \), at any fixed point \( p \) we have

\[
\sigma_k(t^{w_{p_1}}, \ldots, t^{w_{p_n}}) \prod_{w_{p_j} < 0} t^{-w_{p_j}} = \begin{cases} 
N_p(a)t^a + O(t^{a+1}) & \text{if } \lambda_p = k - 1 \\
1 + O(t^{2a}) & \text{if } \lambda_p = k \\
N_p(-a)t^a + O(t^{a+1}) & \text{if } \lambda_p = k + 1 \\
O(t^{2a}) & \text{otherwise}
\end{cases}
\]

where \( N_p(\pm a) \) is the number of times the weight \( \pm a \) occurs at \( p \). Similarly,

\[
\prod_j (1 - t^{|w_{p_j}|}) = 1 - (N_p(a) + N_p(-a))t^a + O(t^{a+1}).
\]

Hence, replacing \( N^k \) by \( \sum_{\lambda_p=k} \prod_j \frac{(1-t^{|w_{p_j}|})}{\prod_j (1-t^{|w_{p_j}|})} \) in (1.3), combining like terms, and then multiplying each term by \( -\prod_{p,j} (1 - t^{|w_{p_j}|}) = -1 + O(t^a) \), we have

\[
\sum_{\lambda_p=k} (N_p(-a) + N_p(a))t^a + O(t^{a+1}) = \sum_{\lambda_p=k-1} N_p(a)t^a + \sum_{\lambda_p=k+1} N_p(-a)t^a + O(t^{a+1}),
\]

where each sum is over all fixed points with the given number of negative weights. Therefore,

\[
\sum_{\lambda_p=k} (N_p(-a) + N_p(a)) = \sum_{\lambda_p=k-1} N_p(a) + \sum_{\lambda_p=k+1} N_p(-a).
\]

Since, by definition, there are no fixed point with negative index, and points of index 0 have no negative weights, the claim is now obvious for \( k = 0 \). The general case follows by induction. \( \square \)

2. Consequences for multigraphs

We are now ready to prove Theorem [2.7] once we have reviewed some material which we adapt from [GS].

**Definition 2.1.** A labelled (directed) multigraph is a set \( V \) of vertices, a set \( E \) of edges, maps \( i: E \to V \) and \( t: E \to V \) giving the initial and terminal vertices of each edge, and a map \( w \) from \( E \) to the positive integers.

The next two definitions correspond to [GS Definition 4.9] and [GS Definitions 4.12], respectively.
Definition 2.2. Let the circle act on a closed almost-complex manifold $(M, J)$ with isolated fixed points. We say that a labelled multigraph with vertex set $M^T$ describes $M$ if the following hold:

1. The isotropy weights are $\{w(e) \mid i(e) = p\} \cup \{-w(e) \mid t(e) = p\}$ at any $p \in M^{S^1}$; and
2. the two endpoints $i(e)$ and $t(e)$ are in the same component of the isotropy submanifold $M^Z/(w(e))$ for each edge $e$.

Definition 2.3. We say that $M$ satisfies the positivity property $P_0^+$ if it can be described by a labelled multigraph so that

$$\sum_{i=1}^{n} w_{pi} \geq \sum_{i=1}^{n} w_{qi}$$

if there is any edge from $p$ to $q$.

The following theorem is an immediate consequence of [GS, Theorem 1.3].

**Theorem 2.4.** Consider a Hamiltonian circle action on an 8-dimensional closed symplectic manifold $M$ with exactly five fixed points. If $M$ satisfies the positivity property $P_0^+$, then the isotropy weights at the fixed points of $M$ agree with those of some linear action on $\mathbb{C}P^4$.

With this theorem as motivation, we explore the consequences of Proposition 1.2 on the labelled graphs that describe actions on almost complex manifolds. Given a directed multigraph, a self-loop is an edge $e$ that connects a vertex to itself, that is, $i(e) = t(e)$.

**Lemma 2.5.** Let the circle act on a closed almost-complex manifold $(M, J)$ with isolated fixed points. There exists a graph describing $M$ such that, for each edge $e$, the index of $i(e)$ in the isotropy submanifold $M^Z/(w(e))$ is two less than the index of $t(e)$ in $M^Z/(w(e))$. In particular, the graph has no self-loops.

**Proof.** For any $l \in \mathbb{N}$, the smallest positive weight in the isotropy submanifold $M^Z/(l)$ is $l$ itself. Therefore, by applying Proposition 1.2 above to each component $Z \subset M^Z/(l)$, we see that the number of times the weight $-l$ occurs in points of index $2k$ in $Z$ is equal to the number of times the weight $+l$ occurs in points of index $2k - 2$ in $Z$. Both claims follow immediately.

This lemma has the following consequence for Hamiltonian actions.

**Corollary 2.6.** Let the circle act on a $2n$-dimensional closed symplectic manifold $(M, \omega)$ with isolated fixed points and with moment map
\( \phi: M \to \mathbb{R} \). There exists a graph describing \( M \) so that, if \( e \) is an edge and the index of \( t(e) \) is 2 (or the index of \( i(e) = 2n - 2 \), then \( \phi(i(e)) < \phi(t(e)) \).

Proof. Let \( e \) be an edge such that the index of \( t(e) \) is 2. By the lemma above, the index of \( i(e) \) in the isotropy submanifold \( M^{Z/(w(e))} \) is 0. Hence, \( i(e) \) is the point in the component of \( M^{Z/(w(e))} \) that contains \( t(e) \) where \( \phi \) achieves its minimum. A fortiori, \( \phi(i(e)) < \phi(t(e)) \). The remaining case is similar.

\( \square \)

**Theorem 2.7.** Consider a Hamiltonian circle action on an 8-dimensional closed symplectic manifold \((M, \omega)\) with exactly 5 fixed points. Then \( M \) satisfies the positivity property \( P^+_0 \).

Proof. As we saw in the introduction, we can label the fixed points \( p_0, \ldots, p_4 \) so that \( p_i \) has index \( 2i \). By Lemma 2.5 and its corollary, \( M \) is described by a labelled graph with no self-loops such that the edge \( e' \) with \( t(e') = p_1 \) and the edge \( e'' \) with \( i(e'') = p_3 \) satisfy \( \phi(i(e'')) < \phi(t(e')) \) and \( \phi(i(e'')) \leq \phi(t(e')) \). By Proposition 3.4 in [I], \( \phi(p_i) < \phi(p_j) \) exactly if \( i < j \). Therefore, we must have \( i(e') = p_0 \) and \( t(e'') = p_4 \). By considering the remaining istropy weights, it is clear that for any other edge \( e \), the index of \( i(e) \) is at most 4 and the index of \( t(e) \) is at least 4. Since there is only one point of index four and no self-loops, this implies that the index of \( i(e) \) is less than the index of \( t(e) \) for every edge \( e \). Therefore, \( \phi(i(e)) < \phi(t(e)) \) for every edge \( e \). Since there exists a positive number \( a \) and a real number \( b \) so that \( \sum_i w_{pi} = a \phi(p) + b \) for all fixed points \( p \), (see Lemma 3.23 in [I]) the claim follows immediately.

\( \square \)

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