The maximal regularity and its application to a multi-dimensional non-conservative viscous compressible two-fluid model with capillarity effects in $L^p$-type framework

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Abstract The present paper is the continuation of work [32], devoted to extending it to a critical functional framework which is not related to the energy space. Employing the special dissipative structure of the non-conservative viscous compressible two-fluid model with capillarity effects, we first exploit the maximal regularity estimates for the corresponding linearized system in all frequencies which behaves like the heat equation. Then we construct the global well-posedness for the multi-dimensional model when the initial data are close to a stable equilibrium state in the sense of suitable $L^p$-type Besov norms. As a consequence, this allows us to work in the framework of Besov space with negative regularity indices and this fact is particularly important when the initial data are large highly oscillating in physical dimensions $N = 2, 3$. Furthermore, based on a refined time weighted inequalities in the Fourier spaces, we also establish optimal time decay rates for the constructed global solutions under a mild additional decay assumption involving only the low frequencies of the initial data.

Key words: non-conservative viscous compressible two-fluid model; global well-posedness; capillary effects; optimal time decay rates; $L^p$-type framework.

AMS subject classifications: 76T10, 76N10.

1 Introduction and Main Results

The models of multi-phase flows have a very broad applications of hydrodynamics in nature and industry, where the fluids under investigation contain more than one component. In nature, there is a variety of different multi-phase flow phenomena, such as sediment transport, geysers, volcanic eruptions, clouds, and rain (see [3]). On the other hand, it has been estimated that over half of anything which is produced in a modern industrial society depends, to some extent, on a multi-phase flow process for their optimum design and safe operations. In addition, multi-phase flows also naturally appear in many contexts within biology, ranging from tumor biology and anticancer therapies to developmental biology and plant physiology [25]. Recently, due to the

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*Research supported by the National Natural Science Foundation of China (11501332,11771043,51976112), the Natural Science Foundation of Shandong Province (ZR2021MA017,ZR2015AL007), and Young Scholars Research Fund of Shandong University of Technology.

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physical importance and mathematical challenges, the study of mathematical properties for the models becomes a significant and difficult topic.

In the present paper, we are going to study the following multiphase flows model, namely a non-conservative viscous compressible two-fluid system with capillarity effects in $\mathbb{R}^N (N \geq 2)$:

$$
\begin{cases}
\alpha^+ + \alpha^- = 1, \\
\partial_t(\alpha^+ \rho^+) + \text{div}(\alpha^+ \rho^+ u^+) = 0, \\
\partial_t(\alpha^- \rho^- u^-) + \text{div}(\alpha^- \rho^- u^- \otimes u^-) + \alpha^+ \nabla P^+(\rho^+) - \sigma^+ \alpha^+ \rho^+ \nabla \Delta(\alpha^+ \rho^+) = \text{div}(\alpha^+ \tau^+), \\
P^+(\rho^+) = A^+(\rho^+)\gamma^+ = P^-(\rho^-) = A^-(\rho^-)\gamma^-,
\end{cases}
$$

(1.1)

where the variable $0 \leq \alpha^+(x, t) \leq 1$ is the volume fraction of fluid $+$ in one of the two gases, and $0 \leq \alpha^-(x, t) \leq 1$ is the volume fraction of the other fluid $-$. Moreover, $\rho^+(x, t) \geq 0$, $u^+(x, t)$ and $P^\pm(\rho^\pm)$ are, respectively, the densities, the velocities, and the two pressure functions of the fluids. Here, for the sake of technical simplicity, we restricted ourselves to the study of the barotropic case, that is, $P^+(\rho^+) = A^+(\rho^+)\gamma^+$ where $\gamma^+ > 1$, $A^+ > 0$. In what follows, we set $A^+ = A^- = 1$ without loss of any generality. $\sigma^\pm > 0$ are the capillary coefficients. Also, $\tau^\pm$ are the viscous stress tensors

$$
\tau^\pm := 2\mu^\pm D(u^\pm) + \lambda^\pm \text{div} u^\pm \text{IN},
$$

(1.2)

where $D(u^\pm) \text{ def }= \frac{\nabla u^\pm + \nabla u^\pm^T}{2}$ stand for the deformation tensor, the constants $\mu^\pm$ and $\lambda^\pm$ are the (given) shear and bulk viscosity coefficients satisfying $\mu^\pm > 0$ and $\lambda^\pm + 2\mu^\pm > 0$. System (1.1) is known as a two-fluid flows model with algebraic closure, which is widely used in industrial applications, such as nuclear, power, oil-and-gas, micro-technology and so on, and we refer readers to Refs [5, 25–27] for more discussions about this model and related models.

From a mathematical point of view, model (1.1) is a highly nonlinear partial differential equation with the mixed hyperbolic-parabolic property. As a matter of fact, there is no diffusion on the mass conservation equations, whereas velocity evolves according to the parabolic equations due to the viscosity phenomena. In particular, the non-conservative pressure terms $\alpha^\pm \nabla P^\pm(\rho^\pm)$ typically prevent one from applying arguments used for compressible Navier-Stokes equations. Therefore, there are more challenges associated with this type of model when it comes to mathematical analysis (well-posedness and stability). In the last decade, many researchers have been devoted to studying system (1.1) and have made more progress. For example, Bresch et al. [3] first established the existence of global weak solutions to 3D model (1.1). Later, Bresch-Huang-Li [4] extended the result in [3] and proved the existence of global weak solutions to system (1.1) in one space dimension without capillarity terms. In 2016, Evje-Wang-Wen [16] proved the global existence of strong solutions to system (1.1) without capillary terms by the standard energy method under the condition that the initial data are close to the constant equilibrium state in $H^2(\mathbb{R}^3)$, and then constructed
the optimal time decay rates of the global strong solutions if the initial data belong to $L^1(\mathbb{R}^3)$ additionally. Lai-Wen-Yao [22] investigated the vanishing capillarity limit of smooth solutions to system (1.1) with unequal pressure functions if $\|(\alpha^+ \rho^+_0 - 1, \alpha^- \rho^-_0 - 1)\|_{H^4(\mathbb{R}^3)} + \|(u^+_0, u^-_0)\|_{H^3(\mathbb{R}^3)}$ are small enough. Recently, when $\mu^+ = \mu^-$ and $\lambda^+ = \lambda^-$, based on the complicated spectral analysis of Green’s function to the linearized system and the elaborate energy estimate to the nonlinear system, for system (1.1), authors [11] showed global solvability of smooth solutions close to an equilibrium state in $H^s(\mathbb{R}^3)(s \geq 3)$ and further got the time decay rates when the initial perturbation is bounded in $L^1(\mathbb{R}^3)$. More recently, Li et al. [23] constructed the global existence and optimal decay rates of system (1.1) with general constant viscosities and capillary coefficients when initial data $\|(\alpha^+ \rho^+_0 - 1, \alpha^- \rho^-_0 - 1)\|_{H^{l+1}(\mathbb{R}^3)} + \|(u^+_0, u^-_0)\|_{H^l(\mathbb{R}^3)}$ for an integer $l \geq 3$ are small enough. In $L^2$-type critical Besov spaces, Xu et al. [31] constructed the well-posedness and decay rates of strong solutions to a multi-dimensional system (1.1) without capillarity effects.

Later, Xu and Chi [32] further established the corresponding conclusions of system (1.1).

Here, it should be pointed out that the existing results mentioned above, including the global existence and time decay rates of strong solutions to system (1.1), are mainly based on $L^2$-framework, especially in critical Besov spaces. However, to our knowledge, so far there is no result on the global existence and time decay estimates to system (1.1) in $L^p$-framework. The main motivation of this paper is to give a positive answer to this question. In particular, we prove the global well-posedness to system (1.1) when the initial data are close to a stable equilibrium state in the sense of suitable $L^p$-type Besov norms, and establish optimal time decay rates for the constructed global solutions under a mild additional decay assumption involving only the low frequencies of the initial data. Let us emphasize that this framework allows us to construct global solutions of system (1.1) with highly oscillating initial velocities in larger spaces in physical dimensions $N = 2, 3$. However, due to the mixed hyperbolic-parabolic property of system (1.1), as in [12], the system has to be handled differently for the low and high frequencies. Roughly speaking, the first order terms predominate in low frequencies, so that system (1.1) has to be treated by means of hyperbolic energy methods, which implies that we must treat the low frequencies regime only in spaces constructed on $L^2$, as it is classical that hyperbolic systems are ill-posed in general $L^p$ spaces. In contrast, in the high frequencies, a $L^p$ approach may be used.

For the convenience of the reader, as in [3], we also show some derivations for another expression of the pressure gradient in terms of the gradients of $\alpha^+ \rho^+ \rho^-$ and $\alpha^- \rho^- \rho^-$ by using the pressure equilibrium assumption. Here, we only focus on the case that $\inf \rho^\pm > 0, 0 < \alpha^\pm < 1$ in our framework. The relation between the pressures of system (1.1) implies the following differential
identities
\[ dP^+ = s^2_+ d\rho^+, \quad dP^- = s^2_- d\rho^-, \quad \text{where} \ s_\pm := \sqrt{\frac{dP^\pm}{d\rho^\pm}(\rho^\pm)} = \sqrt{\frac{\rho^\pm}{\rho^\pm}}, \] (1.3)
where \( s_\pm \) denote the sound speed of each phase respectively.

Let
\[ R^\pm = \alpha^\pm \rho^\pm. \] (1.4)

Resorting to (1.1), we have
\[ d\rho^+ = \frac{1}{\alpha_+} (dR^+ - \rho^+ d\alpha^+), \quad d\rho^- = \frac{1}{\alpha_-} (dR^- + \rho^- d\alpha^+). \] (1.5)

Combining with (1.3) and (1.5), we conclude that
\[ d\alpha^+ = \frac{\alpha_- s^2_+}{\alpha_- \rho^+ s^2_+ + \alpha_+ \rho^- s^2_-} dR^+ - \frac{\alpha^+ s^2_-}{\alpha^- \rho^+ s^2_+ + \alpha^+ \rho^- s^2_-} dR^- . \]

Substituting the above equality into (1.5), we obtain
\[ d\rho^+ = \frac{s^2_+ - s^2_-}{\rho^+ - \rho^-} \left( \rho^- dR^+ + \rho^+ dR^- \right), \]
and
\[ d\rho^- = \frac{s^2_+ - s^2_-}{\rho^+ - \rho^-} \left( \rho^- dR^+ + \rho^+ dR^- \right), \]
which give, for the pressure differential \( dP^\pm \),
\[ dP^+ = C^2 (\rho^- dR^+ + \rho^+ dR^-), \]
and
\[ dP^- = C^2 (\rho^- dR^+ + \rho^+ dR^-), \]
where
\[ C^2 \overset{\text{def}}{=} \frac{s^2_- s^2_+}{\alpha^- \rho^+ s^2_+ + \alpha^+ \rho^- s^2_-}. \]

Recalling \( \alpha^+ + \alpha^- = 1 \), we get the following identity:
\[ \frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad \text{and therefore} \ \rho^- = \frac{R^- \rho^+}{\rho^+ - R^+}. \] (1.6)

Then it follows from the pressure relation (1.1) that
\[ \varphi(\rho^+) := P^+(\rho^+) - P^-(\frac{R^- \rho^+}{\rho^+ - R^+}) = 0. \] (1.7)

Differentiating \( \varphi \) with respect to \( \rho^+ \), we have
\[ \varphi'(\rho^+) = s^2_+ + s^2_- \frac{R^- R^+}{(\rho^+ - R^+)^2}. \]
By the definition of $R^+$, it is natural to look for $\rho^+$ which belongs to $(R^+, +\infty)$. Since $\varphi' > 0$ in $(R^+, +\infty)$ for any given $R^\pm > 0$, and $\varphi : (R^+, +\infty) \mapsto (-\infty, +\infty)$, this determines that 
$\rho^+ = \rho^+(R^+, R^-) \in (R^+, +\infty)$ is the unique solution of equation (1.7). Due to (1.5), (1.6) and (1.11), $\rho^-$ and $\alpha^\pm$ are defined as follows:

$$
\rho^-(R^+, R^-) = \frac{R^- \rho^+(R^+, R^-)}{\rho^+(R^+, R^-) - R^+},
\alpha^+(R^+, R^-) = \frac{R^+}{\rho^+(R^+, R^-)},
\alpha^-(R^+, R^-) = 1 - \frac{R^+}{\rho^+(R^+, R^-)} = \frac{R^-}{\rho^+(R^+, R^-)}.
$$

Based on the above analysis, system (1.1) is equivalent to the following form

$$
\begin{cases}
\partial_t R^\pm + \text{div}(R^\pm u^\pm) = 0, \\
\partial_t (R^+ u^+) + \text{div}(R^+ u^+ \otimes u^+) + \alpha^+ C^2 [\rho^- \nabla R^+ + \rho^+ \nabla R^-] - \sigma^+ R^+ \nabla \Delta R^+ = \text{div}(\alpha^+ [\mu^+(\nabla u^+ + \nabla^t u^+) + \lambda^+ \text{div} u^+ \text{Id}]), \\
\partial_t (R^- u^-) + \text{div}(R^- u^- \otimes u^-) + \alpha^- C^2 [\rho^- \nabla R^+ + \rho^+ \nabla R^-] - \sigma^- R^- \nabla \Delta R^- = \text{div}(\alpha^- [\mu^- (\nabla u^- + \nabla^t u^-) + \lambda^- \text{div} u^- \text{Id}]).
\end{cases}
$$

(1.8)

Here, we are concerned with the Cauchy problem of system (1.8) in $\mathbb{R}_+ \times \mathbb{R}^N$ subject to the initial data

$$
(R^+, u^+, R^-, u^-)(x, t)|_{t=0} = (R^+_0, u^+_0, R^-_0, u^-_0)(x), \quad x \in \mathbb{R}^N,
$$

(1.9)

and

$$
u^+(x, t) \to 0, \quad u^-(x, t) \to 0, \quad R^+ \to R^+_\infty > 0, \quad R^- \to R^-_\infty > 0, \text{ as } |x| \to \infty,$$

where $R^\pm_\infty$ denote the background doping profile, and in the present paper $R^\pm_\infty$ are taken as 1 without losing generality.

For simplicity, we take $\sigma^+ = \sigma^- = 1$. Set $c^\pm = R^\pm - 1$. Then, system (1.8) can be rewritten as

$$
\begin{cases}
\partial_t c^+ + \text{div} u^+ = H_1, \\
\partial_t u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu_1^+ \Delta u^+ - \nu_2^+ \text{div} u^+ - \nabla \Delta c^+ = H_2, \\
\partial_t c^- + \text{div} u^- = H_3, \\
\partial_t u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu_1^- \Delta u^- - \nu_2^- \text{div} u^- - \nabla \Delta c^- = H_4,
\end{cases}
$$

(1.10)

with initial data

$$
(c^+, u^+, c^-, u^-)(x, t)|_{t=0} = (c^+_0, u^+_0, c^-_0, u^-_0)(x),
$$

(1.11)

where $\beta_1 = \frac{C^2(1,1)\rho^-(1,1)}{\rho^+(1,1)}$, $\beta_2 = \beta_3 = C^2(1,1)$, $\beta_4 = \frac{C^2(1,1)\rho^+(1,1)}{\rho^-(1,1)}$, $\nu_1^+ = \frac{\mu^+}{\rho^+(1,1)}$, $\nu_2^+ = \frac{\mu^+ + \lambda^+}{\rho^+(1,1)}$ and the source terms are

$$
H_1 = H_1(c^+, u^+) = -\text{div}(c^+ u^+),
$$

(1.12)
\[ H_2 = H_2(c^+, u^+, c^-) = -g_+(c^+, c^-)\partial_t c^+ - \tilde{g}_+(c^+, c^-)\partial_t c^- - (u^+ \cdot \nabla)u^+_i \]
\[
+ \mu^+ h_+(c^+, c^-)\partial_j c^+\partial_j u^+_i + \mu^+ k_+(c^+, c^-)\partial_j c^-\partial_j u^+_i \\
+ \lambda^+ h_+(c^+, c^-)\partial_i c^+\partial_i u^+_i + \mu^+ k_+(c^+, c^-)\partial_i c^-\partial_i u^+_i \\
+ \lambda^+ l_+(c^+, c^-)\partial^2 u^+_i + (\mu^+ + \lambda^+)l_+(c^+, c^-)\partial_i \partial_j u^+_i, \quad i, j \in \{1, 2, \cdots N\},
\]
\[ H_3 = H_3(c^-, u^-) = -\text{div}(c^- u^-), \]
\[ H_4 = H_4(c^+, u^-, c^-) = -g_-(c^+, c^-)\partial_t c^- - \tilde{g}_-(c^+, c^-)\partial_t c^+ - (u^- \cdot \nabla)u^-_i \\
+ \mu^- h_-(c^+, c^-)\partial_j c^+\partial_j u^-_i + \mu^- k_-(c^+, c^-)\partial_j c^-\partial_j u^-_i \\
+ \lambda^- h_-(c^+, c^-)\partial_i c^+\partial_i u^-_i + \mu^- k_-(c^+, c^-)\partial_i c^-\partial_i u^-_i \\
+ \lambda^- l_-(c^+, c^-)\partial^2 u^-_i + (\mu^- + \lambda^-)l_-(c^+, c^-)\partial_i \partial_j u^-_i, \quad i, j \in \{1, 2, \cdots N\},
\]
where we define the nonlinear functions of \((c^+, c^-)\) by
\[
\begin{align*}
  g_+(c^+, c^-) &= \frac{(C^2 \rho^-)(c^+ + 1, c^- + 1)}{\rho^+(c^+ + 1, c^- + 1)} - \frac{(C^2 \rho^-)(1, 1)}{\rho^+(1, 1)}, \\
  g_-(c^+, c^-) &= \frac{(C^2 \rho^+)(c^+ + 1, c^- + 1)}{\rho^-(c^+ + 1, c^- + 1)} - \frac{(C^2 \rho^+)(1, 1)}{\rho^-(1, 1)}, \\
  h_+(c^+, c^-) &= \frac{(C^2 \alpha^-)(c^+ + 1, c^- + 1)}{[c^+ + 1]s^2_\alpha(c^+ + 1, c^- + 1)}, \\
  h_-(c^+, c^-) &= -\frac{C^2(c^+ + 1, c^- + 1)}{(\rho^- s^2_\alpha)(c^+ + 1, c^- + 1)}, \\
  k_+(c^+, c^-) &= \frac{-C^2(c^+ + 1, c^- + 1)}{[c^- + 1][s^2_\alpha \rho^+](c^+ + 1, c^- + 1)}, \\
  k_-(c^+, c^-) &= \frac{(\alpha^+ C^2)(c^+ + 1, c^- + 1)}{[c^- + 1][s^2_\alpha](c^+ + 1, c^- + 1)}, \\
  \tilde{g}_+(c^+, c^-) &= \tilde{g}_-(c^+, c^-) = C^2(c^+ + 1, c^- + 1) - C^2(1, 1), \\
  l_\pm(c^+, c^-) &= \frac{1}{\rho^\pm(c^+ + 1, c^- + 1)} - \frac{1}{\rho^\pm(1, 1)}.
\end{align*}
\]
then the Cauchy problem (1.8)-(1.9) admits a unique global-in-time solution \((c^+, u^+, c^-, u^-)\) in the space \(X\) defined by

\[
(c^+, u^+, c^-, u^-)^t \in \mathcal{C}(\mathbb{R}_+; \dot{B}^{\frac{N}{2}+1}_{2,1} ) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{2}+1}_{2,1} ),
\]

\[
(c^+, c^-)^h \in \mathcal{C}(\mathbb{R}_+; \dot{B}^{\frac{N}{p}}_{p,1} ) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+2}_{p,1} ),
\]

\[
(u^+, u^-)^h \in \mathcal{C}(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+1}_{p,1} ) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+1}_{p,1} ).
\]

Furthermore, we have for \(t \geq 0\),

\[
X(t) \lesssim X(0),
\]

where

\[
X(t) \overset{\text{def}}{=} \|(c^+, u^+, c^-, u^-)^t\|_{L^\infty_t(\dot{B}^{\frac{N}{2}+1}_{2,1} )} + \|(c^+, u^+, c^-, u^-)^t\|_{L^1_t(\dot{B}^{\frac{N}{2}+1}_{2,1} )} + \|(u^+, u^-)^h\|_{L^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1} )} + \|(c^+, c^-)^h\|_{L^1_t(\dot{B}^{\frac{N}{p}+2}_{p,1} )}.
\]

Our second main result on the optimal time decay rates of strong solutions states as follows.

**Theorem 1.2** Let the data \((c_0^+, u_0^+, c_0^-, u_0^-)\) satisfy the assumptions of Theorem 1.1. Let \(N \geq 2\) and \(p\) satisfy Theorem 1.1. Denote \((t) \overset{\text{def}}{=} (1+t)\) and \(\alpha \overset{\text{def}}{=} \frac{N}{p} + \frac{1}{2} - \varepsilon\) with \(\varepsilon > 0\) arbitrarily small. There exists a positive constant \(c\) such that if in addition

\[
D_0 \overset{\text{def}}{=} \|(c_0^+, u_0^+, c_0^-, u_0^-)\|_{L^\infty_t(\dot{B}^{\frac{N}{2}+1}_{2,1} )} \leq c \quad \text{with } s_0 \overset{\text{def}}{=} \frac{2N}{p} - \frac{N}{2}
\]

then the global solution \((c^+, u^+, c^-, u^-)\) given by Theorem 1.1 satisfies for all \(t \geq 0\),

\[
D(t) \lesssim \left( D_0 + \|\nabla c_0^+, u_0^+, \nabla c_0^-, u_0^-\|_{L^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1} )} \right)
\]

with

\[
D(t) \overset{\text{def}}{=} \sup_{s \in [s_0, \frac{N}{2}+1]} \|\langle \tau \rangle^{\frac{N}{2}+1} (c^+, u^+, c^-, u^-)^t\|_{L^\infty_t(\dot{B}^{\frac{N}{2}+1}_{2,1} )} + \|\tau^\alpha (\nabla c^+, u^+, \nabla c^-, u^-)^h\|_{L^\infty_t(\dot{B}^{\frac{N}{p}+1}_{p,1} )}.
\]

We would like to give some comments on our main results.

**Remark 1.3** In Theorem 1.1, the regularity indices for the high frequency parts of \(u_0^\pm\) may be negative. Especially, this allows us to obtain the global well-posedness of system (1.1) for the highly oscillating initial velocities \(u_0^\pm\). For example, let

\[
u_0^+(x) = \sin \left( \frac{x_1}{\varepsilon_1} \right) \phi(x), \quad u_0^-(x) = \sin \left( \frac{x_1}{\varepsilon_2} \right) \phi(x), \quad \phi(x) \in \mathcal{S}(\mathbb{R}^N).
\]
Thus for any $\varepsilon_i > 0 (i = 1, 2)$

$$\|u_0^+\|_{h^{N/p}_{B_{p,1}^1}} \leq C\varepsilon_1^{1-N/p}, \quad \|u_0^-\|_{h^{N/p}_{B_{p,1}^1}} \leq C\varepsilon_2^{1-N/p}$$

for $p > N$.

Hence such data with small enough $\varepsilon_i > 0 (i = 1, 2)$ generate global unique solutions in dimensions $N = 2, 3$.

**Remark 1.4** Compared with [11, 23, 29, 30], in Theorems 1.1 and 1.2, we obtain the global well-posedness and optimal time decay rates for multi-dimensional non-conservative viscous compressible two-fluid system (1.1) in critical $L^p$-framework respectively. Additionally, in Theorem 1.2, the regularity index $s$ can take both negative and nonnegative values, rather than only nonnegative integers, which improves the classical decay results in high Sobolev regularity, such as [11, 23, 29, 30]. Moreover, our results cover the case $N = 2$.

**Remark 1.5** Compared with [11], in this paper, we consider the case of general constant viscosities and relax the special choice for viscosities in [11].

As a consequence of Theorem 1.2, we can show the following decay rates of $L^p$ norm of solution $(R^+ - 1, u^+, R^- - 1, u^-)$.

**Corollary 1.6** The solution $(R^+ - 1, u^+, R^- - 1, u^-)$ constructed in Theorem 1.1 satisfies

$$\|\Lambda^s(\xi^+ - 1, R^- - 1)\|_{L^p} \lesssim \left(D_0 + \|\langle \nabla R_0^+ u_0^+, \nabla R_0^- u_0^-\rangle\|_{h^{N/p}_{B_{p,1}^1}}^1\right) \langle t \rangle^{-\frac{s+s_0}{2}} \quad \text{if } -s_0 < s \leq N/p, \quad (1.27)$$

$$\|\Lambda^s(u^+, u^-)\|_{L^p} \lesssim \left(D_0 + \|\langle \nabla R_0^+ u_0^+, \nabla R_0^- u_0^-\rangle\|_{h^{N/p}_{B_{p,1}^1}}^1\right) \langle t \rangle^{-\frac{s+s_0}{2}} \quad \text{if } -s_0 < s \leq N/p - 1, \quad (1.28)$$

where the fractional derivative operator $\Lambda^s$ is defined by $\Lambda^s f = F^{-1}(|\cdot|^s F f)$.

**Remark 1.7** In Corollary 1.6, taking $p = 2$ (hence $s_0 = N/2$), $s = 0$ leads back to the standard optimal $L^1$-$L^2$ time decay rates which is a consistent with the optimal time decay rates from a single phase flow model in [29, 30].

Before going into the heart of the proof of our main results, we make a brief interpretation of the main difficulties and techniques involved in the proof. Due to the mixed hyperbolic-parabolic property of system (1.10), the system has to be handled differently in the low and high frequencies respectively. Roughly speaking, the first order terms predominate in low frequencies, so that system (1.10) has to be treated by means of hyperbolic energy methods, which implies that we must treat the low frequencies regime only in spaces constructed on $L^2$, as it is classical that hyperbolic systems are ill-posed in general $L^p$ spaces. In contrast, in the high frequencies, a
$L^p$ approach may be used. On the other hand, various important mathematical difficulties occur when we want to generalize well-known results of the compressible Navier-Stokes equations to the two-phase system (1.10) since the corresponding model is non-conservative.

First, in the low frequencies, we need deal with system (1.10) in $L^2$-framework by means of hyperbolic energy methods. In general, the proof consists of spectral analysis of Green’s function for the corresponding linearized system and energy estimates of the solutions to the nonlinear system, refer for instance to [6, 10, 19]. However, we encounter a fundamental obstacle that Green’s function of the viscous compressible two-fluid model (1.10) is an 8-order matrix and is not self-adjoint so that we can not make some complicate analysis of Green’s function. To get around this difficulty, we will follow the main ideas of [32] (see Lemma 3.1 for details) to exploit the maximal regularity estimates for the corresponding linearized system in $L^2$-type critical Besov spaces by employing the energy argument of Godunov [18] for partially dissipative first-order symmetric systems (further developed by [17]) and Fourier-Plancherel theorem.

Second, we need to cover more general values of the integration parameter $p$ in the high frequencies. Generally speaking, in order to solve this problem, there are two fundamental effective methods concerning the standard barotropic compressible Navier-Stokes equations. The first is some suitable effective velocity field (named viscous effective flux in Hoff’s work [14]) in order to kill the relation of coupling between the velocity and the pressure from [19]. The second is the study of the paralinearized system combined with a Lagrangian change of coordinates (in the spirit of that introduced by Hmidi in [20] for the convection-diffusion equation) from [6, 10] so as to counter the loss of regularity coming from the convection terms and almost completely avoid the undesired coupling. Unfortunately, these two methods fail to work for the current system (1.10) due to the non-conservation. To explain this problem, one study the following non-conservative coupled hyperbolic-parabolic model

$$
\begin{align*}
&\partial_t c^+ + \text{div} Q u^+ = 0, \\
&\partial_t Q u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu^+ \Delta Q u^+ - \nabla \Delta c^+ = 0, \\
&\partial_t c^- + \text{div} Q u^- = 0, \\
&\partial_t Q u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu^- \Delta Q u^- - \nabla \Delta c^- = 0,
\end{align*}
$$

where $Q$ is potential vector fields. For simplicity, here we only consider the above non-conservative coupled hyperbolic-parabolic model without capillarity effects. Following from Haspot’s idea in [19], we introduce two new auxiliary functions $W_1 \overset{\text{def}}{=} Q u^+ + \frac{\gamma_1}{\nu^+} (-\Delta)^{-1} \nabla c^+$ and $W_2 \overset{\text{def}}{=} Q u^- + \frac{\gamma_4}{\nu^-} (-\Delta)^{-1} \nabla c^-$, then we get

$$
\begin{align*}
\partial_t W_1 - \nu^+ \Delta W_1 &= \frac{\gamma_1^2}{\nu^+} W_1 - \frac{\gamma_3^2}{(\nu^+)^2} (-\Delta)^{-1} \nabla c^+ - \gamma_2 \nabla c^- , \\
\partial_t W_2 - \nu^- \Delta W_2 &= \frac{\gamma_2^2}{\nu^-} W_2 - \frac{\gamma_4^2}{(\nu^-)^2} (-\Delta)^{-1} \nabla c^- - \gamma_3 \nabla c^+ ,
\end{align*}
$$
and
\[
\begin{align*}
\partial_t c^+ + \frac{\gamma_2}{\nu} c^+ + \frac{1}{\sqrt{\beta_1}} u^+ \cdot \nabla c^+ &= -\frac{1}{\sqrt{\beta_1}} c^+ \text{div} u^+ - \gamma_1 \text{div} W_1, \\
\partial_t c^- + \frac{\gamma_2}{\nu} c^- + \frac{1}{\sqrt{\beta_4}} u^- \cdot \nabla c^- &= -\frac{1}{\sqrt{\beta_4}} c^- \text{div} u^- - \gamma_4 \text{div} W_2.
\end{align*}
\]

By simple calculation, we have
\[
\begin{align*}
\|W_1\|_{L_t^\infty(B_{p,1}^{\infty})}^h + \nu^+ \|W_1\|_{L_t^1(B_{p,1}^{\infty+1})}^h &
\leq C_1 \left( \|W_{10}\|_{B_{p,1}^{\infty+1}}^h + \frac{\gamma_2}{\nu} \|W_1\|_{L_t^1(B_{p,1}^{\infty+1})}^h + \frac{\gamma_3}{(\nu)^2} \|c^+\|_{L_t^1(B_{p,1}^{\infty+2})}^h + \gamma_2 \|c^-\|_{L_t^1(B_{p,1}^{\infty+1})}^h \right), \\
\|W_2\|_{L_t^\infty(B_{p,1}^{\infty+1})}^h + \nu^+ \|W_2\|_{L_t^1(B_{p,1}^{\infty+1})}^h &
\leq C_2 \left( \|W_{20}\|_{B_{p,1}^{\infty+1}}^h + \frac{\gamma_2}{\nu} \|W_2\|_{L_t^1(B_{p,1}^{\infty+1})}^h + \frac{\gamma_3}{(\nu)^2} \|c^-\|_{L_t^1(B_{p,1}^{\infty+2})}^h + \gamma_3 \|c^+\|_{L_t^1(B_{p,1}^{\infty+1})}^h \right), \\
\|c^+\|_{L_t^\infty(B_{p,1}^{\infty})}^h + \frac{\gamma_2}{\nu} \|c^+\|_{L_t^1(B_{p,1}^{\infty})}^h &
\leq C_3 \left( \|c_0^+\|_{B_{p,1}^{\infty}}^h + \gamma_1 \|\text{div} W_1\|_{B_{p,1}^{\infty}}^h + \frac{1}{\sqrt{\beta_1}} \int_0^t \|\nabla u^+\|_{B_{p,1}^{\infty}}^h \|c^+\|_{B_{p,1}^{\infty}}^h \, dt \right), \\
\|c^-\|_{L_t^\infty(B_{p,1}^{\infty})}^h + \frac{\gamma_2}{\nu} \|c^-\|_{L_t^1(B_{p,1}^{\infty})}^h &
\leq C_4 \left( \|c_0^-\|_{B_{p,1}^{\infty}}^h + \gamma_4 \|\text{div} W_2\|_{B_{p,1}^{\infty}}^h + \frac{1}{\sqrt{\beta_4}} \int_0^t \|\nabla u^-\|_{B_{p,1}^{\infty}}^h \|c^-\|_{B_{p,1}^{\infty}}^h \, dt \right).
\end{align*}
\]

Different from the barotropic compressible Navier-Stokes equations [19] and the compressible Navier-Stokes system with capillarity [7,21], there are four terms \(\|\text{div} W_1\|_{L_t^1(B_{p,1}^{\infty})}^h\), \(\|\text{div} W_2\|_{L_t^1(B_{p,1}^{\infty})}^h\), \(\|c^+\|_{L_t^1(B_{p,1}^{\infty})}^h\) and \(\|c^-\|_{L_t^1(B_{p,1}^{\infty})}^h\) appearing in the righthand sides of the above estimates. Obviously, it seems impossible to obtain the desired estimates of \((c_1, W_1, c_2, W_2)\) when the four terms are simultaneously treated as source terms in the high frequencies. Moreover, the method from [6,10] has also similar difficulty for system (1.10). To overcome this essential difficulty, the main idea here is that in view of a crucial observation according to the nice mathematical structures of system (1.10). More precisely, applying operator \(\Delta\) to the above equations of \(c^\pm\) and taking operator \(\text{div}\) to the equations of \(Q u^\pm\), we immediately notice that \(\text{div} Q u^\pm\) should have the same regularity as \(\Delta c^\pm\). Moreover, we also find that \((\Delta c^\pm, \text{div} Q u^\pm)\) satisfy two similar linearized coupling systems which likely exhibit parabolic properties in the high frequencies due to the presence of the capillary terms by the spectral analysis (see (3.14) and Lemma 3.4 for details). More importantly, based on the observation, the perturbations \(\beta_1 \Delta c^+, \beta_2 \Delta c^-, \beta_3 \Delta c^+\) and \(\beta_4 \Delta c^-\) can be treated as harmless source terms in the high frequencies, which induces us to get the desired optimal \(a\ priori\) estimates of \((c^+, u^+, c^-, u^-)\) in critical \(L^p\)-framework.
Third, we also investigate how global strong solutions constructed above look like for large time. In this part, our main ideas are based on a refined time-weighted energy inequalities in the Fourier spaces and the benefit of low-frequency and high-frequency decomposition. In the low frequencies, making good use of Fourier localization analysis to a linearized parabolic-hyperbolic system and Parseval’s equality in order to obtain smoothing effects of Green’s function and avoid some complicate spectral analysis as in [11]. Consequently, it is possible to adapt the standard Duhamel’s principle handling those nonlinear terms. With the aid of the nonclassical product estimates in Besov spaces, one can obtain the desired estimates. In the high frequencies, in order to close the estimate from time-weighted energy functional, using the similar method of the proof for global existence in Section three together with elaborate nonlinear estimates, we further exploit some decay estimates with gain of regularity of $(\nabla c^+, u^+, \nabla c^-, u^-)$.

Finally, we also have to deal with some difficulties caused by much more complicate nonlinear terms by using the usual product estimates in Besov spaces, Bony’s decomposition and the low-high frequency decomposition, and some new composition of the binary functions from harmonic analysis.

The rest of the paper unfolds as follows. In the next section, we recall some basic facts about Littlewood-Paley decomposition, Besov spaces and some useful lemmas. Section 3 is devoted to the proof of the global well-posedness for initial data near equilibrium in critical Besov spaces. In Section 4, we present the optimal time decay rates of the global strong solutions. Some material concerning paradifferential calculus and product estimates in Besov spaces is recalled in Appendix.

**Notations.** We assume $C$ be a positive generic constant throughout this paper that may vary at different places and denote $A \leq CB$ by $A \lesssim B$. We shall also use the following notations

$$z^\ell = \sum_{j \leq k_0} \hat{\Delta}_j z \quad \text{and} \quad z^h = z - z^\ell,$$

for some $k_0$.

$$\|z\|_{B^{s}_{2,1}}^\ell = \sum_{j \leq k_0} 2^{js} \|\hat{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{B^{s}_{2,1}}^h = \sum_{j \geq k_0} 2^{js} \|\hat{\Delta}_j z\|_{L^2},$$

for some $k_0$.

Noting the small overlap between low and high frequencies, we have

$$\|z^\ell\|_{B^{s}_{2,1}} \lesssim \|z\|_{B^{s}_{2,1}}^\ell \quad \text{and} \quad \|z^h\|_{B^{s}_{2,1}} \lesssim \|z\|_{B^{s}_{2,1}}^h.$$

## 2 Littlewood-Paley Theory and Some Useful Lemmas

Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{3}{2}\}$ such that

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1 \quad \text{for all} \ \xi \neq 0.$$
The homogeneous frequency localization operators $\hat{\Delta}_q$ and $\hat{S}_q$ are defined by
\[
\hat{\Delta}_q f = \varphi(2^{-q}D)f, \quad \hat{S}_q f = \sum_{k \leq q-1} \hat{\Delta}_k f \quad \text{for} \quad q \in \mathbb{Z}.
\]
With our choice of $\varphi$, one can easily verify that
\[
\hat{\Delta}_q \hat{\Delta}_k f = 0 \quad \text{if} \quad |q - k| \geq 2 \quad \text{and} \quad \hat{\Delta}_q(\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if} \quad |q - k| \geq 5.
\]
We denote the space $Z'(\mathbb{R}^N)$ by the dual space of $Z(\mathbb{R}^N) = \{ f \in S(\mathbb{R}^N); D^\alpha f(0) = 0; \forall \alpha \in \mathbb{N}^N \text{ multi-index} \}$. It also can be identified by the quotient space of $S'(\mathbb{R}^N)/P$ with the polynomials space $P$. The formal equality
\[
f = \sum_{q \in \mathbb{Z}} \hat{\Delta}_q f
\]
holds true for $f \in Z'(\mathbb{R}^N)$ and is called the homogeneous Littlewood-Paley decomposition.

Let us recall the definition of homogeneous Besov spaces (see [11]).

**Definition 2.1** Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by
\[
\dot{B}^s_{p,r} = \left\{ f \in Z'(\mathbb{R}^N) : \|f\|_{\dot{B}^s_{p,r}} < +\infty \right\},
\]
where
\[
\|f\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \left\|2^{qs}\|\hat{\Delta}_q f(t)\|_{\mathcal{L}^p}\right\|_{L^r}.\]

**Remark 2.2** Some properties about the Besov spaces are as follows

- **Derivation**: 
  \[
  \|f\|_{\dot{B}^s_{2,1}} \approx \|\nabla f\|_{\dot{B}^{s-1}_{2,1}};
  \]
- **Algebraic properties**: for $s > 0$, $\dot{B}^s_{2,1} \cap L^\infty$ is an algebra;
- **Interpolation**: for $s_1, s_2 \in \mathbb{R}$ and $\theta \in [0, 1]$, we have
  \[
  \|f\|_{\dot{B}^{s_2 + (1-\theta)s_2}_{2,1}} \leq \|f\|^{\theta}_{\dot{B}^{s_1}_{2,1}} \|f\|^{1-\theta}_{\dot{B}^{s_2}_{2,1}}.
  \]

**Definition 2.3** Let $s \in \mathbb{R}, (\rho, p) \in [1, +\infty]^3$ and $T \in (0, +\infty]$. We say then that $f \in L_T^\rho(\dot{B}^s_{p,r})$, if
\[
\|f\|_{L_T^\rho(\dot{B}^s_{p,r})} \overset{\text{def}}{=} \left\|\left\|2^{qs}\|\hat{\Delta}_q f\|_{L^p}\right\|_{L^\rho_T}\right\|_{L^\rho_T} < +\infty.
\]

We next introduce the Besov-Chemin-Lerner space $\dot{L}^q_T(\dot{B}^s_{p,r})$ which is initiated in [9].
Definition 2.4 Let $s \leq \frac{N}{p}$ (respectively $s \in \mathbb{R}$), $(r, \rho, p) \in [1, +\infty]^3$ and $T \in (0, +\infty)$. We define $\tilde{L}^p_T(B^s_{p,r})$ as the completion of $C([0, T]; S^r_{h})$ by the norm

$$
\|f\|_{\tilde{L}^p_T(B^s_{p,r})} \overset{\text{def}}{=} \left\|2^{js}\|\hat{\Delta}^j f(t)\|_{L^p(0, T; L^p)}\right\|_{\ell^r} < \infty,
$$

with the usual change if $r = \infty$.

Obviously, $\tilde{L}^1_T(B^s_{p,1}) = L^1_T(B^s_{p,1})$. By a direct application of Minkowski’s inequality, we have the following relations between these spaces

$$
L^p_T(B^s_{p,r}) \hookrightarrow \tilde{L}^p_T(B^s_{p,r}), \text{ if } r \geq \rho,
$$

$$
\tilde{L}^p_T(B^s_{p,r}) \hookrightarrow L^p_T(B^s_{p,r}), \text{ if } \rho \geq r.
$$

The following Bernstein’s inequalities will be frequently used.

Lemma 2.5 Let $1 \leq p_1 \leq p_2 \leq +\infty$. Assume that $f \in L^{p_1}(\mathbb{R}^N)$, then for any $\gamma \in (\mathbb{N} \cup \{0\})^N$, there exist constants $C_1, C_2$ independent of $f, q$ such that

$$
\text{supp } \hat{f} \subseteq \{\|\xi\| \leq A_0 2^q\} \Rightarrow \|\partial^\gamma f\|_{p_2} \leq C_1 2^{q\|\gamma\| + qN(\frac{1}{p_1} - \frac{1}{p_2})}\|f\|_{p_1},
$$

$$
\text{supp } \hat{f} \subseteq \{A_1 2^q \leq \|\xi\| \leq A_2 2^q\} \Rightarrow \|f\|_{p_1} \leq C_2 2^{-d\|\gamma\|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_{p_1}.
$$

We here recall basic nonlinear estimates in Besov spaces which will be used repeatedly in our proof.

Proposition 2.6 For all $1 \leq r, p, p_1, p_2 \leq +\infty$, there exists a positive universal constant such that

$$
\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}, \text{ if } s > 0;
$$

$$
\|fg\|_{\dot{B}^{s_1+s_2}_{p,r}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,r}}\|g\|_{\dot{B}^{s_2}_{p,r}}, \text{ if } s_1, s_2 < \frac{N}{p}, \text{ and } s_1 + s_2 > 0;
$$

$$
\|fg\|_{\dot{B}^s_{p,\infty}} \lesssim \|f\|_{\dot{B}^s_{p,r}}\|g\|_{\dot{B}^0_{p,\infty} \cap L^\infty}, \text{ if } |s| < \frac{N}{p};
$$

$$
\|fg\|_{\tilde{B}^s_{2,1}} \lesssim \|f\|_{\tilde{B}^s_{2,1}}\|g\|_{\tilde{B}^s_{2,1}}, \text{ if } s \in (-N/2, N/2].
$$

The basic tool of the paradifferential calculus is Bony’s decomposition. Formally, the product of two tempered distributions $u$ and $v$ may be decomposed into

$$
uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v)
$$

with

$$
\hat{T}_u v = \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{\Delta}_j v, \quad \hat{\Delta}_j v = \sum_{|j-j'| \leq 1} \hat{\Delta}_j v.
$$

As a consequence, the estimates for the paraproduct and remainder operators can be given by
**Proposition 2.7** \[13\] Let \( N \geq 2 \), \( s \in \mathbb{R} \) and \( 2 \leq p \leq \min(4, \frac{2N}{N-2}) \), we have
\[
||Tfg||_{L^1_t(B^{s-1+\frac{N}{p}}_{2,1})} \leq C||f||_{L^p_{x,t}(B^{s}_{p,1})}||g||_{L^1_t(B^{s}_{p,1})}.
\] \tag{2.1}

In particular, for \( s \in \mathbb{R} \), \( m \geq 0 \), we also have
\[
||(Tfg)^t||_{L^1_t(B^{s-1+\frac{N}{p}}_{2,1})} \leq C||f||_{L^p_{x,t}(B^{s}_{p,1})}||g||_{L^1_t(B^{s-m}_{p,1})}.
\] \tag{2.2}

**Proposition 2.8** \[13\] Let \( N \geq 2 \), \( s > 1 - \min\left(\frac{N}{p}, \frac{N}{p'}\right) \) and \( 1 \leq p \leq 4 \), we have
\[
||R(f,g)||_{B^{s}_{2,1}} \leq C||f||_{B^{s}_{p,1}}||g||_{B^{s}_{p,1}}.
\]

**Proposition 2.9** \[13\] Let the real numbers \( \sigma_1, \sigma_2, p_1 \) and \( p_2 \) satisfy
\[
\sigma_1 + \sigma_2 > 0, \quad \sigma_1 \leq \frac{N}{p_1}, \quad \sigma_2 \leq \frac{N}{p_2}, \quad \sigma_1 \geq \sigma_2, \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1
\]

then
\[
||fg||_{B^{\sigma_2}_{q,1}} \leq ||f||_{B^{\sigma_1}_{p_1,1}}||g||_{B^{\sigma_2}_{p_2,1}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma_1}{N}.
\]

Let the exponents \( \sigma > 0 \) and \( 1 \leq p_1, p_2, q \leq \infty \) satisfy
\[
\frac{N}{p_1} + \frac{N}{p_2} - d \leq \sigma \leq \min\left(\frac{N}{p_1}, \frac{N}{p_2}\right) \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma}{N},
\]

then
\[
||fg||_{B^{\sigma}_{q,\infty}} \leq ||f||_{B^{\sigma}_{p_1,1}}||g||_{B^{\sigma}_{p_2,\infty}}.
\]

**Corollary 2.10** Let \( N \geq 2 \) and \( p \) satisfy \( 2 \leq p \leq \min(4, \frac{2N}{N-2}) \) and, additionally, \( p \neq 4 \) if \( N = 2 \), then
\[
||fg||_{B^{-s_0}_{2,\infty}} \leq ||f||_{B^{1}_{p_1,1}}||g||_{B^{-s_0}_{p_2,1}}.
\] \tag{2.3}

**Corollary 2.11** Let \( N \geq 2 \) and \( p \) satisfy \( 2 \leq p \leq N \), then
\[
||fg^h||_{B^{-s_0}_{2,\infty}} \leq ||f||_{B^{1}_{p_1,1}}||g^h||_{B^{-s_0}_{p_2,1}}.
\] \tag{2.4}

**Proposition 2.12** \[13\] Let \( q_0 \in \mathbb{Z}_+ \), and denote \( \hat{S}_{q_0} u \triangleq u^t \) and for any \( s \in \mathbb{R} \), there exists a universal integer \( N_0 \) such that for any \( 2 \leq p \leq 4 \) and \( \sigma > 0 \), then
\[
||u^h||_{B^{s}_{p,\infty}} \leq C(||u||_{B^s_{p,1}} + ||\hat{S}_{q_0+N_0} u||_{L^p})||u^h||_{B^{-s}_{p,\infty}}
\] \tag{2.5}

and
\[
||u^h v||_{B^{-s_0}_{2,\infty}} \leq C(||u^h||_{B^{s}_{p,1}} + ||\hat{S}_{q_0+N_0} u^h||_{L^p})||v||_{B^{-s_0}_{p,\infty}}
\] \tag{2.6}

with \( s_0 := \frac{2N}{p} - \frac{N}{2} \) and \( \frac{1}{p'} := \frac{1}{2} - \frac{1}{p} \), and \( C \) depending only on \( k_0, N \) and \( \sigma \).
Then for all $T > 0$, we can conclude (2.9).

Employing Taylor’s expansion, we have
\[
\|F(f, g)\|_{\tilde{L}^s_t(B_{p,r})} \leq C \left(1 + \|f\|_{L^{p,\infty}(\mathbb{R}^n)} + \|g\|_{L^{p,\infty}(\mathbb{R}^n)}\right) \left(\|f\|_{L^{s,\infty}(B_{p,r})} + \|g\|_{L^{s,\infty}(B_{p,r})}\right)
\]

(i) If $(f_1, g_1) \in \tilde{L}^\infty_t(B_{p,r})$ and $(f_2, g_2) \in \tilde{L}^\infty_t(B_{p,r})$, then $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^n) \times W^{[s]+2,\infty}_{loc}(\mathbb{R}^n)$ with $\partial_t F(0,0) = 0$ and $\partial_y F(0,0) = 0$. Then, there exists a positive constant $C$ depending only on $s, p, N$ and $F$ such that
\[
\|F(f_2, g_2) - F(f_1, g_1)\|_{\tilde{L}^s_t(B_{p,r})} \leq C \left(\|f_1\|_{L^{s,\infty}(B_{p,r})} + \|g_1\|_{L^{s,\infty}(B_{p,r})}\right) \left(\|f_2\|_{L^{s,\infty}(B_{p,r})} + \|g_2\|_{L^{s,\infty}(B_{p,r})}\right)
\]

In $L^p$-framework, we need deal with the composition of the binary functions the case $s < 0$.

Proposition 2.14 Let $I$ be an open interval of $\mathbb{R}$ containing 0, $s > -\min\{\frac{N}{p}, \frac{N}{p}\}$, $t \geq 0$, $1 \leq p, q, r \leq \infty$ and $(f, g) \in \tilde{L}^\infty_t(B_{p,r}) \times \tilde{L}^\infty_t(B_{p,r})$. If $F \in W^{[s]+2,\infty}_{loc}(I) \times W^{[s]+2,\infty}_{loc}(I)$ with $F(0,0) = 0$, then $F(f, g) \in \tilde{L}^s_t(B_{p,r})$. Moreover, there exists a positive constant $C$ depending only on $s, p, N$ and $F$ such that
\[
\|F(f, g)\|_{\tilde{L}^s_t(B_{p,r})} \leq \frac{C}{(1 + \|f\|_{L^{s,\infty}(B_{p,r})} + \|g\|_{L^{s,\infty}(B_{p,r})})} \left(\|f\|_{L^{s,\infty}(B_{p,r})} + \|g\|_{L^{s,\infty}(B_{p,r})}\right)
\]

Proof. Employing Taylor’s expansion, we have
\[
F(f, g) = f F'_1(0,0) + g F'_2(0,0) + f \tilde{F}_1(f, g) + g \tilde{F}_2(f, g),
\]

where $\tilde{F}_i : I \times I \to \mathbb{R} (i = 1, 2)$ are smooth and vanish at $(0,0)$. Furthermore, using Propositions 2.6 and 2.13, we can conclude (2.9).

We also recall some maximal regularity properties for the heat equation.

Proposition 2.15 Let $\mu > 0, \sigma \in \mathbb{R}, (p, r) \in [1, \infty]^2$ and $1 \leq \rho_2 \leq \rho_1 \leq \infty$. Let $u$ satisfy
\[
\mu \frac{1}{\rho_1} \|u\|_{\tilde{L}^\infty_t(B_{p,r})} \leq C \left(\|u_0\|_{\tilde{L}^{\rho_2}_t(B_{p,r})} + \mu^{\frac{1}{\rho_2}} \|f\|_{\tilde{L}^{\rho_2}_t(B_{p,r})}\right).
\]

Then for all $T > 0$ the following a priori estimate is fulfilled
\[
\frac{1}{\rho_1} \|u\|_{\tilde{L}^\infty_t(B_{p,r})} \leq C \left(\|u_0\|_{\tilde{L}^{\rho_2}_t(B_{p,r})} + \mu^{\frac{1}{\rho_2}} \|f\|_{\tilde{L}^{\rho_2}_t(B_{p,r})}\right).
\]
We finish this subsection by listing an elementary but useful inequality.

**Lemma 2.16 [24]** Let $r_1, r_2 > 0$ satisfy $\max\{r_1, r_2\} > 1$. Then

$$
\int_0^t (1 + t - \tau)^{-r_1} (1 + \tau)^{-r_2} d\tau \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2\}}.
$$

(2.13)

**Lemma 2.17 [25]** Let $0 \leq r_1 \leq r_2$ with $r_2 > 1$. Then

$$
\int_0^t (1 + t - \tau)^{-r_1} \tau^{-\theta} (1 + \tau)^{\theta - r_2} d\tau \leq C(r_1, r_2)(1 + t)^{1 - r_1} \quad \text{for} \quad 0 \leq \theta < 1.
$$

(2.14)

3 The proof of Theorem 1.1

In this section, we shall exhibit the proof of Theorem 1.1. We divide it into the following three parts.

3.1 Maximal regularity estimates in the low frequencies

Here, we shall establish the following a priori estimates based on the $L^2$-framework of the Cauchy problem (1.10)-(1.11).

**Proposition 3.1** Let $T \geq 0$, $N \geq 2$, $p$ satisfy $2 \leq p \leq \min(4, \frac{2N}{N-2})$ and, additionally, $p \neq 4$ if $N = 2$. Assume that $(c^+, u^+, c^-, u^-)$ is a solution to the Cauchy problem (1.10)-(1.11) on $[0, T] \times \mathbb{R}^N$, then

$$
\|(c^+, u^+, c^-, u^-)\|_{L^\infty_t(B^{\frac{N}{2}+1}_{2,1})} + \|(c^+, u^+, c^-, u^-)\|_{L^1_t(B^{\frac{N}{2}+1}_{2,1})} \lesssim X(0) + X^2(t) + (1 + X^2(t))^{\frac{N}{2}+1} \left( X^2(t) + X^3(t) \right) \quad \text{for all} \quad t \in [0, T].
$$

(3.1)

**Proof.** Employing the energy argument of Godunov [18] for partially dissipative first-order symmetric systems (further developed by [17]), by a similar derivation in Lemma 3.1 in [32], we conclude that

$$
\|(\hat{c}^+, \hat{u}^+, \hat{c}^-, \hat{u}^-)\| \leq Ce^{-c_0|\xi|^{\frac{2}{N}}} \|(c^+, u^+, c^-, u^-)(0)\| \quad \text{for} \quad \xi \leq \xi_0,
$$

(3.2)

which together with Fourier-Plancherel theorem and Duhamel’s formula, implies that

$$
\|(c^+, u^+, c^-, u^-)\|_{L^\infty_t(B^{\frac{N}{2}+1}_{2,1})} + \|(c^+, u^+, c^-, u^-)\|_{L^1_t(B^{\frac{N}{2}+1}_{2,1})} \lesssim \|(c^+_0, u^+_0, c^-_0, u^-_0)\|_{B^{\frac{N}{2}+1}_{2,1}} + \|(H_1, H_2, H_3, H_4)\|_{L^1_t(B^{\frac{N}{2}+1}_{2,1})}.
$$

(3.3)

In what follows, we derive some estimates for the nonlinear terms $\|(H_1, H_2, H_3, H_4)\|_{L^1_t(B^{\frac{N}{2}+1}_{2,1})}$.

For the sake of simplicity, we first show the following five important estimates of the paradifferential calculus from Propositions 2.7 and 2.8 respectively which will be frequently used in our process later.

$$
T : \dot{B}^{\frac{N}{p}-1}_{p,1} \times \dot{B}^{\frac{N}{p}+1}_{p,1} \rightarrow \dot{B}^N_{2,1} \quad \text{for} \quad 2 \leq p \leq \min(4, \frac{2N}{N-2}),
$$

(3.4)
For the term $H_1$, using Bony’s decomposition we see that

\[(c^+ u^+)^\ell = (T_{c^+ u^+})^\ell + (R(c^+ u^+))^\ell + (T_{u^+ c^+})^\ell.\]

Employing (3.3) and the embedding relation $\dot{B}_{2,1}^{\frac{N}{2} + s} \times \dot{B}_{2,1}^{\frac{N}{2} + s}$ for $s \in \mathbb{R}$, $p \geq 2$ yields that

\[
\|(T_{c^+ u^+})^\ell\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \lesssim \|c^+\|_{L^\infty_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \\
\lesssim \left(\|c^+\|_{L^\infty_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \left(\|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \\
\lesssim X^2(t),
\]

and

\[
\|(T_{u^+ c^+})^\ell\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \lesssim \|u^+\|_{L^\infty_t(B_{2,1}^{\frac{N}{2}})} \|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \\
\lesssim \left(\|u^+\|_{L^\infty_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \left(\|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \\
\lesssim X^2(t).
\]

For the remainder term, using (3.5) and the embedding $\dot{B}_{2,1}^{\frac{N}{2} + s} \times \dot{B}_{2,1}^{\frac{N}{2} + s}$ for $s \in \mathbb{R}$, $p \geq 2$ gives rise to

\[
\|(R(c^+ u^+))^\ell\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \\
\lesssim \|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \\
\lesssim \left(\|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \left(\|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \|u^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right) \\
\lesssim X^2(t),
\]

where we have used the following interpolation inequalities (which will be frequently used later),

\[
\|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \lesssim \left(\|c^+\|_{L^\infty_t(B_{2,1}^{\frac{N}{2}})} \right)^{\frac{1}{2}} \left(\|c^+\|_{L^1_t(B_{2,1}^{\frac{N}{2}})} \right)^{\frac{1}{2}},
\]
Hence we handle the third term as follows

$$\|c^+\|_{L^2(B_{\frac{N}{p},1}^+)}^h \lesssim \left( \|c^+\|_{L^\infty(B_{\frac{N}{p},1}^+)}^h \right)^{\frac{1}{2}} \left( \|c^+\|_{L^1(B_{\frac{N}{p},1}^+)}^h \right)^{\frac{1}{2}},$$

$$\|u^+\|_{L^2(B_{\frac{N}{p},1}^+)}^\ell \lesssim \left( \|u^+\|_{L^\infty(B_{\frac{N}{p},1}^+)}^\ell \right)^{\frac{1}{2}} \left( \|u^+\|_{L^1(B_{\frac{N}{p},1}^+)}^\ell \right)^{\frac{1}{2}},$$

and

$$\|u^+\|_{L^2(B_{\frac{N}{p},1}^+)}^h \lesssim \left( \|u^+\|_{L^\infty(B_{\frac{N}{p},1}^+)}^h \right)^{\frac{1}{2}} \left( \|u^+\|_{L^1(B_{\frac{N}{p},1}^+)}^h \right)^{\frac{1}{2}}.$$ 

Hence

$$\|H_1\|_{L^1([0,t];B_{\frac{N}{p},1}^+)}^\ell \lesssim \|(c^+ u^+)^\ell\|_{L^1([0,t];B_{\frac{N}{p},1}^+)} \lesssim X^2(t).$$

The term $H_3$ may be treated along the same lines, and we omit it.

For the term $(g_+(c^+, c^-)\partial_t c^+)^\ell$ in $H_2^i$, employing Bony’s decomposition and splitting $c^+$ into $c^+^\ell + c^+^h$ we have

$$(g_+(c^+, c^-)\partial_t c^+)^\ell = (T_{\partial_t c^+} g_+(c^+, c^-))^\ell + (R(g_+(c^+, c^-), \partial_t c^+))^\ell + (T_{g_+(c^+, c^-)} \partial_t c^+)^\ell + (T_{g_+(c^+, c^-)} \partial_t c^+)^h)^\ell.$$ 

We bound the above items one by one. To bound the first two terms, it suffices to notice that (3.6) and (3.7) and use Proposition 2.13(i). Hence

$$\|(T_{\partial_t c^+} g_+(c^+, c^-))^\ell + (R(g_+(c^+, c^-), \partial_t c^+))^\ell\|_{L^1([0,t];B_{\frac{N}{p},1}^+)} \lesssim \left(1 + X^2(t)\right) \|(c^+, c^-)\|_{L^2(B_{\frac{N}{p},1}^+)}^{\frac{N}{p}+1} X^2(t).$$

Employing $\|T_f g\|_{B_{p,r}^\infty} \lesssim \|f\|_{L^\infty} \|g\|_{B_{p,r}^\infty}$, imbedding relation $B_{\frac{N}{p},1}^\infty \hookrightarrow L^\infty$ and Proposition 2.13(i), we handle the third term as follows

$$\|(T_{g_+(c^+, c^-)} \partial_t c^+)^\ell\|_{L^1([0,t];B_{\frac{N}{p},1}^+)} \lesssim \|\nabla c^+\|_{L^2(B_{\frac{N}{p},1}^+)} \|g_+(c^+, c^-)\|_{L^2(L^\infty)} \lesssim \|c^+\|_{L^2(B_{\frac{N}{p},1}^+)} \|g_+(c^+, c^-)\|_{L^2(B_{\frac{N}{p},1}^+)} \lesssim \left(1 + X^2(t)\right)^{\frac{N}{p}+1} X^2(t).$$

The last term may be bounded thanks to (3.6). Moreover, we just have to use Proposition 2.14.
as it may happen that $\frac{N}{p} - 1 < 0$. Thus
\[
\| (T_{u_i}(c^+, c^-) \partial_i c^+) \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| g^+(c^+, c^-) \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla c^+ \|^{\ell_{L_1^N(B_{2,1}^N)}} 
\lesssim \left( \| (c^+, c^-) \|^\ell_{L_1^N(B_{2,1}^N)} + (1 + \| (c^+, c^-) \|^{2} \right)_{L_1^N(B_{2,1}^N)} \| (c^+, c^-) \|^{\ell_{L_1^N(B_{2,1}^N)}} \| (c^+, c^-) \|^{\ell_{L_1^N(B_{2,1}^N)}} 
\times \| c^+ \|^{h}_{L_1^N(B_{2,1}^N)} 
\lesssim X^2(t) + \left( 1 + X^2(t) \right) \| c^+ \|^{\ell_{L_1^N(B_{2,1}^N)}+1} X^3(t). 
\]

Hence
\[
\| (g^+(c^+, c^-) \partial_i c^+) \|^\ell_{L_1^N([0,t];B_{2,1}^N)} \lesssim X^2(t) + \left( 1 + X^2(t) \right) \| c^+ \|^{\ell_{L_1^N(B_{2,1}^N)}+1} X^3(t). 
\]

Bounding the term $(\tilde{g}^+(c^+, c^-) \partial_i c^-)^\ell$ is totally similar, we omit it.

For the term with $((u^+ \cdot \nabla) u_i^+)\ell$ in $H_2^1$, employing Bony’s decomposition implies that
\[
((u^+ \cdot \nabla) u_i^+)\ell = (T_{\nabla u_i^+} u^+)\ell + \left( \sum_{j=1}^N (R(u_i^+, \partial_j u_i^+)) \right)^\ell + (T_{u^+ \nabla u_i^+})^\ell.
\]

Then it follows from (3.6) and (3.7) that
\[
\| (T_{\nabla u_i^+} u^+)\ell + (R(u_i^+, \partial_j u_i^+)) \|^\ell_{L_1^N(B_{2,1}^N)} + \| (T_{u^+ \nabla u_i^+})\ell \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| \nabla u^+ \|^\ell_{L_1^N(B_{2,1}^N)} \| u^+ \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| u^+ \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla u^+ \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| u^+ \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla u^+ \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim X^2(t).
\]

Hence
\[
\| ((u^+ \cdot \nabla) u_i^+)\ell \|^\ell_{L_1^N([0,t];B_{2,1}^N)} \lesssim X^2(t).
\]

For the term $(h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+)^\ell$ in $H_2^1$, we decompose it into
\[
(h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+)^\ell = (T_{h_+(c^+, c^-) \partial_j c^+} \partial_j u_i^+)^\ell + (R(h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+))^\ell 
+ (T_{\partial_j u_i^+} h_+(c^+, c^-) \partial_j c^+)^\ell + (T_{\partial_j u_i^+} h_+(c^+, c^-) \partial_j c^+)^\ell.
\]

Thanks to (3.6) and (3.7), Proposition 2.13 (i), we deduce that
\[
\| (T_{h_+(c^+, c^-) \partial_j c^+} \partial_j u_i^+)^\ell + (R(h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+))^\ell \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| h_+(c^+, c^-) \partial_j c^+ \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla u^+ \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \| h_+(c^+, c^-) \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla c^+ \|^\ell_{L_1^N(B_{2,1}^N)} \| \nabla u^+ \|^\ell_{L_1^N(B_{2,1}^N)} 
\lesssim \left( 1 + \| (c^+, c^-) \|^{2} \right)_{L_1^N(B_{2,1}^N)} \| (c^+, c^-) \|^{\ell_{L_1^N(B_{2,1}^N)}} \| c^+ \|^{\ell_{L_1^N(B_{2,1}^N)}} \| u^+ \|^{\ell_{L_1^N(B_{2,1}^N)}} 
\lesssim X^2(t) + \left( 1 + X^2(t) \right) \| c^+ \|^{\ell_{L_1^N(B_{2,1}^N)}+1} X^3(t),
\]
\[ \| (T \partial_j u_1^+ h_+ (c^+, c^-) \partial_j c^+) \|_{L^1_t(B_{2,1}^\infty)} \leq \| \nabla u^+ \|_{L^\infty_t(B_{p,1}^{N\infty})} \| h_+ (c^+, c^-) \nabla c^+ \|_{L^1_t(B_{p,1}^N)} \]
\[ \leq \| \nabla u^+ \|_{L^\infty_t(B_{p,1}^{N\infty})} \left( 1 + (1 + \| (c^+, c^-) \|_2^2 L^\infty_t(B_{p,1}^N)) \right) \| (c^+, c^-) \|_{L^1_t(B_{p,1}^N)} \| \nabla c^+ \|_{L^1_t(B_{p,1}^N)} \]
\[ \lesssim X^2(t) + \left( 1 + X^2(t) \right)^{\frac{N+1}{p}} X^3(t), \]
and
\[ \| (T h_+ (c^+, c^-) \partial_j c^+ \partial_j c^+) \|_{L^1_t(B_{2,1}^\infty)} \leq \| h_+ (c^+, c^-) \partial_j c^+ \|_{L^1_t(B_{p,1}^N)} \| \partial_j c^+ \|_{L^\infty_t(B_{p,1}^N)} \]
\[ \leq \left( 1 + (1 + \| (c^+, c^-) \|_2^2 L^\infty_t(B_{p,1}^N)) \right) \| \nabla u^+ \|_{L^1_t(B_{p,1}^N)} \| \partial_j c^+ \|_{L^\infty_t(B_{p,1}^N)} \]
\[ \lesssim X^2(t) + \left( 1 + X^2(t) \right)^{\frac{N+1}{p}} X^3(t). \]

Hence
\[ \| (h_+ (c^+, c^-) \partial_j c^+ \partial_j u_1^+) \|_{L^1([0, t] ; B_{2,1}^\infty)} \lesssim X^2(t) + \left( 1 + X^2(t) \right)^{\frac{N+1}{p}} X^3(t). \]

Similarly, we also obtain the corresponding estimates of other terms \((\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u_1^+)\), \((\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_1^+)\), \((\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u_1^+)\), \((\lambda^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_1^+)\) and \((\lambda^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u_1^+)\). Here, we omit the details.

Finally, for the term \((l_+ (c^+, c^-) \partial^2_j u_i^+)\) in \(H^2_1\), we may decompose it into
\[ (l_+ (c^+, c^-) \partial^2_j u_i^+) = (T \partial^2_j u_1^+ l_+ (c^+, c^-)) + (R (l_+ (c^+, c^-), \partial^2_j u_i^+)) + (T l_+ (c^+, c^-) \partial^2_j u_i^+) + (T_+ (c^+, c^-) \partial^2_j u_i^+) \]
To handle the first two terms, by virtue of (3.6) and (3.7), Proposition (2.13(i), we get
\[ \| (T \partial^2_j u_1^+ l_+ (c^+, c^-)) + (R (l_+ (c^+, c^-), \partial^2_j u_i^+)) \|_{L^1_t(B_{2,1}^{N\infty})} \]
\[ \lesssim \| \nabla^2 u^+ \|_{L^1_t(B_{p,1}^{N\infty})} \| l_+ (c^+, c^-) \|_{L^\infty_t(B_{p,1}^N)} \]
\[ \lesssim \| u^+ \|_{L^1_t(B_{p,1}^{N+1})} \left( 1 + \| (c^+, c^-) \|_2^2 L^\infty_t(B_{p,1}^N) \right) \| (c^+, c^-) \|_{L^1_t(B_{p,1}^N)} \]
\[ \lesssim X^2(t) + \left( 1 + X^2(t) \right)^{\frac{N+1}{p}} X^3(t). \]

To bound the third term, according to \(\| T g \|_{B_{p,r}} \lesssim \| f \|L^\infty \| g \|_{B_{p,r}}\), imbedding relation \(B_{p,1}^\infty \hookrightarrow L^\infty\),
and Proposition 2.13(i), we infer that
\[
\left\| (T_{t+}(c^+, c^-) \partial^2 \mathcal{L}_t^h u^+_{l_1}) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \\
\lesssim \|I_4(c^+, c^-)\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 2)} \\
\lesssim \|I_4(c^+, c^-)\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \|\nabla^2 u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \\
\lesssim (1 + \|\nabla^2 u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)})^{\frac{N}{p} - 1} \|\nabla^2 u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \|u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \\
\lesssim X^2(t) + \left(1 + X^2(t)\right)^{\frac{N}{p} - 1} X^3(t).
\]

Using (3.8) and Proposition 2.14, we bound the last term as follows
\[
\left\| (T_{t+}(c^+, c^-) \partial^2 \mathcal{L}_t^h u^+_{l_1}) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \\
\lesssim \|I_4(c^+, c^-)\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \|\nabla^2 u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)} \\
\lesssim \left(1 + \|\nabla^2 u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}\right) \|u^+\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}^{\frac{N}{p} - 1} X^3(t).
\]

Hence
\[
\left\| H_2 \right\|_{L^1([0, t]; L^1_t(B_{\mathcal{L}_t}^{\infty} - 1))} \lesssim X^2(t) + \left(1 + X^2(t)\right)^{\frac{N}{p} - 1} X^3(t).
\]

The term $H_4$ may be treated along the same lines, we omit it.

Putting together all the above estimates for the terms of $H_1-H_4$, we deduce that (3.1). This completes the proof of Proposition 3.1.

\[\Box\]

### 3.2 Maximal regularity estimates in the high frequencies

In the following proposition, we shall exploit the parabolic properties of the Cauchy problem (1.10)-(1.11) in the high frequencies and construct a priori estimates based on the general $L^p$-framework.

**Proposition 3.2** Let $T \geq 0$, $N \geq 2$, $p$ satisfy $2 \leq p \leq \min(4, \frac{2N}{N-2})$ and, additionally, $p \neq 4$ if $N = 2$. Assume that $(c^+, u^+, c^-, u^-)$ is a solution to the Cauchy problem (1.10)-(1.11) on $[0, T] \times \mathbb{R}^N$, then
\[
\left\| (u^+, u^-) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}^{h} + \left\| (u^+, u^-) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}^{h} + \left\| (c^+, c^-) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}^{h} + \left\| (c^+, c^-) \right\|_{L^1_t(B_{\mathcal{L}_t}^{\infty} - 1)}^{h} \\
\lesssim X(0) + X^2(t) + \left(1 + X^2(t)\right)^{\frac{N}{p} - 1} X^3(t) \quad \text{for all} \quad t \in [0, T].
\]

(3.9)
In order to prove Proposition 3.2, we introduce the orthogonal projectors over divergence-free \( \mathcal{P} \) and potential vector fields \( \mathcal{Q} \) satisfying the identity \( I = \mathcal{P} + \mathcal{Q} \). Then, applying the orthogonal projectors \( \mathcal{P} \) and \( \mathcal{Q} \) over divergence-free and potential vector-fields, respectively, to (1.10) and (1.11), and setting \( \nu^\pm \equiv \nu_1^\pm + \nu_2^\pm \) yield that
\[
\begin{align*}
&\partial_t \mathcal{P} u^+ - \nu^+ \Delta \mathcal{P} u^+ = \mathcal{P} H_2, \\
&\partial_t \mathcal{P} u^- - \nu^- \Delta \mathcal{P} u^- = \mathcal{P} H_4,
\end{align*}
\]
and
\[
\begin{align*}
&\partial_t c^+ + \text{div} \mathcal{Q} u^+ = H_1, \\
&\partial_t \mathcal{Q} u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu^+ \Delta \mathcal{Q} u^+ - \nabla \Delta c^+ = QH_2, \\
&\partial_t c^- + \text{div} \mathcal{Q} u^- = H_3, \\
&\partial_t \mathcal{Q} u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu^- \Delta \mathcal{Q} u^- - \nabla \Delta c^- = QH_4.
\end{align*}
\]
For system (3.10), according to Proposition 2.15 (restricted to the high frequencies), we have
\[
\| \mathcal{P} u^+ \|_{L_0^h(\hat{B}_{p,1}^N)}^h + \| \mathcal{P} u^- \|_{L_0^h(\hat{B}_{p,1}^N)}^h \lesssim \left( \| \mathcal{P} u^+ \|_{B_{p,1}^N}^h + \| \mathcal{P} H_2, \mathcal{P} H_4 \|_{L_0^h(\hat{B}_{p,1}^N)}^h \right).
\]
Next, to handle the coupling system (3.11) in the high frequencies, applying operator \( \Delta \) to (3.11)_1 and (3.11)_3, and taking operator div to (3.11)_2 and (3.11)_4, respectively, we get
\[
\begin{align*}
&\partial_t \mathcal{P} u^+ - \nu^+ \Delta \mathcal{P} u^+ = \Delta H_1, \\
&\partial_t \text{div} \mathcal{Q} u^+ - \nu^+ \Delta \text{div} \mathcal{Q} u^+ - \Delta \Delta c^+ = \text{div} \mathcal{Q} H_2 - \beta_1 \Delta c^+ - \beta_2 \Delta c^-,
\end{align*}
\]
and
\[
\begin{align*}
&\partial_t \mathcal{P} u^- - \nu^- \Delta \mathcal{P} u^- = \Delta H_3, \\
&\partial_t \text{div} \mathcal{Q} u^- - \nu^- \Delta \text{div} \mathcal{Q} u^- - \Delta \Delta c^- = \text{div} \mathcal{Q} H_4 - \beta_3 \Delta c^+ - \beta_4 \Delta c^-.
\end{align*}
\]
Obviously, \( (\Delta c^\pm, \text{div} \mathcal{Q} u^\pm) \) in system (3.13) satisfies the linearized coupling system
\[
\begin{align*}
&\partial_t a + \Delta v = F, \\
&\partial_t v - \nu \Delta v - \Delta a = G.
\end{align*}
\]
In what follows, we will prove system (3.14) has the parabolic properties in \( L^p \)-framework in the high frequencies, which implies that the terms \( \beta_1 \Delta c^+ + \beta_2 \Delta c^- \) and \( \beta_3 \Delta c^+ + \beta_4 \Delta c^- \) on the right hands of system (3.13) can be treated as harmless perturbations in the high frequencies. We first make some analysis for Green’s matrix \( \mathcal{G}(x,t) \) of the following linearized system without outer forces, namely
\[
\begin{align*}
&\partial_t a + \Delta v = 0, \\
&\partial_t v - \nu \Delta v - \Delta a = 0.
\end{align*}
\]
Lemma 3.3 Let $\mathcal{G}$ be Green's matrix of system \((3.15)\). Then we have the following explicit expression for $\hat{\mathcal{G}}$:

(i) when $\nu^2 \neq 4$,

\[
\hat{\mathcal{G}}(\xi, t) = \left( \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} |\xi|^2 \right) \left( \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} |\xi|^2 \right),
\]

where

\[
\lambda_\pm = -\frac{\nu}{2} |\xi|^2 \pm |\xi|^2 \sqrt{\frac{\nu^2}{4} - 1}.
\]

(ii) when $\nu^2 = 4$,

\[
\hat{\mathcal{G}}(\xi, t) = e^{-\frac{\nu}{2} |\xi|^2 t} \left( 1 + \frac{\nu}{2} |\xi|^2 t - \frac{1}{2} \frac{|\xi|^2 t}{1 - \frac{\nu}{2} |\xi|^2 t} \right),
\]

where

\[
\lambda_\pm = -\frac{\nu}{2} |\xi|^2.
\]

Moreover, there is a positive constant $\theta$ such that for $\xi \in \mathbb{R}^N$,

\[
|\hat{\mathcal{G}}(\xi, t)| \leq C e^{-\theta |\xi|^2 t}.
\]

Proof. Taking Fourier transforms to system \((3.15)\) yields that

\[
\begin{aligned}
\partial_t \hat{a} - |\xi|^2 \hat{v} &= 0, \\
\partial_t \hat{v} + \nu |\xi|^2 \hat{v} + |\xi|^2 \hat{a} &= 0.
\end{aligned}
\]

Differentiating with respect to the time variable $t$ of \((3.19)\) yields that

\[
\partial_t \hat{v} + \nu |\xi|^2 \hat{v} + |\xi|^2 \hat{a} = 0.
\]

Plugging \((3.19)_1\) into \((3.20)\) gives rise to

\[
\begin{aligned}
\partial_t \hat{v} + \nu |\xi|^2 \hat{v} + |\xi|^2 \hat{v} &= 0, \\
\hat{v}(\xi, 0) &= \hat{v}_0(\xi), \hat{v}_t(\xi, 0) = -\nu |\xi|^2 \hat{v}_0 - |\xi|^2 \hat{a}_0.
\end{aligned}
\]

It is easy to check that

\[
\lambda_\pm = -\frac{\nu}{2} |\xi|^2 \pm |\xi|^2 \sqrt{\frac{\nu^2}{4} - 1}
\]

are two roots of the corresponding characteristic equation of \((3.21)\).

Case 1. For $\nu^2 \neq 4$. We assume that the solution of \((3.21)\) has the following form

\[
\hat{v}(\xi, t) = A(\xi) e^{\lambda_-(\xi) t} + B(\xi) e^{\lambda_+(\xi) t}.
\]

Using the initial conditions, we get

\[
A(\xi) = \frac{-(\lambda_+ + \nu |\xi|^2) \hat{v}_0 - |\xi|^2 \hat{a}_0}{\lambda_- - \lambda_+},
\]

\[
B(\xi) = \frac{(\lambda_- + \nu |\xi|^2) \hat{v}_0 + |\xi|^2 \hat{a}_0}{\lambda_- - \lambda_+},
\]

\[
\lambda_\pm = -\frac{\nu}{2} |\xi|^2 \pm |\xi|^2 \sqrt{\frac{\nu^2}{4} - 1}
\]
which implies that

\[
\hat{v}(\xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} |\xi|^2 \hat{a}_0(\xi) + \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{v}_0(\xi). \tag{3.25}
\]

On the other hand, from (3.19), we obtain

\[
\hat{a}(\xi, t) = \hat{a}(\xi, 0) + |\xi|^2 \int_0^t \hat{v}(\xi, \tau) d\tau. \tag{3.26}
\]

Putting (3.25) into the above equality and using the following relations

\[
\lambda_+ + \lambda_+ = -\nu |\xi|^2; \lambda_+ = |\xi|^4,
\]

we finally get

\[
\hat{a}(\xi, t) = \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{a}_0(\xi) + \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-} |\xi|^2 \hat{v}_0(\xi). \tag{3.28}
\]

Thus, we get an explicit derivation of the Fourier transform \(\hat{G}(\xi, t)\) of Green’s matrix corresponding linearized system (3.15) as follows

\[
\hat{G}(\xi, t) = \left(\begin{array}{cc}
\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-} |\xi|^2
\end{array}\right).
\tag{3.29}
\]

For \(\nu^2 < 4\), we denote \(h = \sqrt{1 - \frac{\nu^2}{4}}\), thus \(h > 0\) and \(\lambda_\pm = -\frac{\nu}{2} |\xi|^2 \pm ih |\xi|^2\). Employing Euler’s formula, we have

\[
\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{\sin(h |\xi|^2 t)}{h |\xi|^2} e^{-\frac{\nu}{2} |\xi|^2 t},
\]

\[
\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \left[ \cos(h |\xi|^2 t) + \frac{\nu}{2h} \sin(h |\xi|^2 t) \right] e^{-\frac{\nu}{2} |\xi|^2 t},
\]

\[
\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \left[ \cos(h |\xi|^2 t) - \frac{\nu}{2h} \sin(h |\xi|^2 t) \right] e^{-\frac{\nu}{2} |\xi|^2 t}.
\]

Thus, we can easily verify that there is a positive constant \(\theta\) such that for \(\xi \in \mathbb{R}^N\),

\[
|\hat{G}(\xi, t)| \leq Ce^{-\theta |\xi|^2 t}. \tag{3.30}
\]

For \(\nu^2 > 4\), we denote \(\nu' = \sqrt{\frac{\nu^2}{4} - 1}\), thus \(\nu' > 0\) and \(\lambda_\pm = -\frac{\nu}{2} |\xi|^2 \pm \nu' |\xi|^2\). Then,

\[
\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{\sinh(\nu' |\xi|^2 t)}{\nu' |\xi|^2} e^{-\frac{\nu}{2} |\xi|^2 t},
\]

\[
\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{\nu}{2\nu'} \sinh(\nu |\xi|^2 t) \cosh(\nu' |\xi|^2 t) e^{-\frac{\nu}{2} |\xi|^2 t},
\]

\[
\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{-\nu}{2\nu'} \sinh(\nu |\xi|^2 t) \cosh(\nu' |\xi|^2 t) e^{-\frac{\nu}{2} |\xi|^2 t},
\]

which implies that (3.18) holds.

Case 2. For \(\nu^2 = 4\). We assume that the solution of (3.21) has the following form

\[
\hat{v}(\xi, t) = (A(\xi) + B(\xi) t) e^{-\frac{\nu}{2} |\xi|^2 t}. \tag{3.31}
\]
In terms of Green’s matrix and Duhamel’s principle, the solution of system (3.14) can be expressed as
\[
A(\xi) = \hat{v}_0(\xi),
\]
\[
B(\xi) = -\nu |\xi|^2 \hat{v}_0 - |\xi|^2 \hat{a}_0 - \frac{\nu}{2} |\xi|^2 \hat{v}_0,
\]
which give rise to
\[
\hat{v}(\xi, t) = -|\xi|^2 t e^{-\frac{\nu}{2} |\xi|^2 t} \hat{a}_0 + \left(1 - \frac{\nu}{2} |\xi|^2 t\right) e^{-\frac{\nu}{2} |\xi|^2 t} \hat{v}_0.  \tag{3.33}
\]
Furthermore, from (3.19), we obtain
\[
\hat{a}(\xi, t) = \hat{a}(\xi, 0) + |\xi|^2 \int_0^t \hat{v}(\xi, \tau) d\tau.  \tag{3.34}
\]
Plugging (3.25) into the above equality, we finally conclude that
\[
\hat{a}(\xi, t) = (1 + \frac{\nu}{2} |\xi|^2 t) e^{-\frac{\nu}{2} |\xi|^2 t} \hat{a}_0(\xi) + (|\xi|^2 t - \frac{\nu}{2}) e^{-\frac{\nu}{2} |\xi|^2 t} \hat{v}_0(\xi).  \tag{3.35}
\]

Then
\[
\hat{\mathcal{G}}(\xi, t) = e^{-\frac{\nu}{2} |\xi|^2 t} \left(1 + \frac{\nu}{2} |\xi|^2 t \right) \left(\begin{array}{c}
|\xi|^2 t - \frac{\nu}{2} \\
1 - \frac{\nu}{2} |\xi|^2 t
\end{array}\right).  \tag{3.36}
\]
By a simple computation, we also conclude that (3.18) holds. The proof of Lemma 3.3 is complete.

With Lemma 3.3 at hand, we can exploit the following parabolic properties of system (3.14) in $L^p$-framework in the high frequencies.

**Lemma 3.4** Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $1 \leq \rho_2 \leq \rho_1 \leq \infty$, $\nu > 0$, and $T \in (0, +\infty]$. Suppose that $(a_0, v_0) \in (B^p_{\rho_1})^2$ and $(F, G) \in (\tilde{L}^p_T (B^{s-2+\frac{2}{p}}_{\rho_1}))^2$. Then, system (3.14) has a unique solution $(a, v)$ in $(\tilde{C}_T (B^s_{\rho_1}) \cap \tilde{L}^p_T (B^{s-2+\frac{2}{p}}_{\rho_1}))^2$. Moreover, there exists $C > 0$ depending only on $\nu, \rho, \rho_1$ such that
\[
\| (a, v) \|_{\tilde{L}^p_T (B^s_{\rho_1})} \leq C \left(\| (a_0, v_0) \|_{B^p_{\rho_1}} + \| (F, G) \|_{\tilde{L}^p_T (B^{s-2+\frac{2}{p}}_{\rho_1})} \right).  \tag{3.37}
\]

**Proof.** In terms of Green’s matrix and Duhamel’s principle, the solution of system (3.14) can be expressed as
\[
\begin{pmatrix}
a \\
v
\end{pmatrix} = \mathcal{G}(x, t) * \begin{pmatrix}
a_0 \\
v_0
\end{pmatrix} + \int_0^t \mathcal{G}(x, t - \tau) * \begin{pmatrix}
F \\
G
\end{pmatrix} d\tau.  \tag{3.38}
\]
Applying homogeneous frequency localization operators $\hat{\Delta}_j$ on both sides of (3.38), we get
\[
\begin{pmatrix}
\hat{\Delta}_j a \\
\hat{\Delta}_j v
\end{pmatrix} = \mathcal{G}(x, t) * \begin{pmatrix}
\hat{\Delta}_j a_0 \\
\hat{\Delta}_j v_0
\end{pmatrix} + \int_0^t \mathcal{G}(x, t - \tau) * \begin{pmatrix}
\hat{\Delta}_j F \\
\hat{\Delta}_j G
\end{pmatrix} d\tau.
\]
From (3.18) and Young’s inequality, we infer that
\[
\| \Delta_j a(t) \|_{L^p} + \| \Delta_j v(t) \|_{L^p} \leq C e^{-c_2 t} \left(\| \Delta_j a_0 \|_{L^p} + \| \Delta_j v_0 \|_{L^p} \right)
+ C \int_0^t e^{-c_2 (t - \tau)} \left(\| \Delta_j F(\tau) \|_{L^p} + \| \Delta_j G(\tau) \|_{L^p} \right) d\tau.
\]
Taking $L^p$ norm with respect to $t$, and using convolution inequality with $1 + \frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p}$ yield that
\[
\|\tilde{\Delta}u\|_{L^p_t L^p_x} + \|\tilde{\Delta}v\|_{L^p_t L^p_x} \\
\leq C2^{\frac{2j}{p_2}} \left( \|\Delta u_0\|_{L^p_x} + \|\Delta v_0\|_{L^p_x} + 2^{-2j} \left( \|\tilde{\Delta}F\|_{L^p_t L^p_x} + \|\tilde{\Delta}G\|_{L^p_t L^p_x} \right) \right). 
\] (3.39)

Multiplying $2^{2j}$ on both sides of (3.39), and summing up over $j \geq j_0$, where $j_0 \in \mathbb{Z}$, we conclude that (3.37). The proof of Lemma 3.4 is complete.

**Proof of Proposition 3.2** Applying Lemma 3.4 to system (3.13), we have
\[
\|((\Delta c^+, \text{div}Q^+, \Delta c^+, \text{div}Q^-))\|_{L^\infty_t(B^p_{\infty,1})} + \|((\Delta c^+, \text{div}Q^+, \Delta c^-, \text{div}Q^-))\|_{L^\infty_t(B^p_{\infty,1})} \\
\lesssim \|((\Delta c^+_0, \text{div}Q^+_0, \Delta c^-_0, \text{div}Q^-_0))\|_{B^p_{\infty,-2}} + \|((\text{div}QH_2, \text{div}QH_4))\|_{L^1(B^p_{\infty,-2})} \\
+ \|((\Delta c^+, \Delta c^-))\|_{L^1(B^p_{\infty,-2})} + \|((\Delta H_1, \Delta H_3))\|_{L^1(B^p_{\infty,-2})}.
\] (3.40)

Thanks to the high-frequency cutoff, we get
\[
\|((\Delta c^+, \Delta c^-))\|_{L^1(B^p_{\infty,-2})} \lesssim 2^{-2j} \|((\Delta c^+, \Delta c^-))\|_{L^1(B^p_{\infty,-2})} \lesssim 2^{-2j_0} \|((\Delta c^+, \Delta c^-))\|_{L^1(B^p_{\infty,-2})}, \text{ for } j \geq j_0.
\]

Hence, taking $j_0$ large enough, it follows from (3.40) that
\[
\|((\Delta c^+, \text{div}Q^+, \Delta c^-, \text{div}Q^-))\|_{L^\infty_t(B^p_{\infty,1})} + \|((\Delta c^+, \text{div}Q^+, \Delta c^-, \text{div}Q^-))\|_{L^\infty_t(B^p_{\infty,1})} \\
\lesssim \|((\Delta c^+_0, \text{div}Q^+_0, \Delta c^-_0, \text{div}Q^-_0))\|_{B^p_{\infty,-2}} + \|((\text{div}QH_2, \text{div}QH_4))\|_{L^1(B^p_{\infty,-2})} \\
+ \|((H_1, H_3))\|_{L^1(B^p_{\infty,-1})}.
\] (3.41)

Noticing that $u^+ = \mathcal{P}u^+ + Qa^+$, and then combining with (3.12) and (3.41) yields that
\[
\|((u^+, u^-))\|_{L^\infty_t(B^p_{\infty,1})} + \|((u^+, u^-))\|_{L^1_t(B^p_{\infty,1})} + \|((c^+, c^-))\|_{L^\infty_t(B^p_{\infty,1})} + \|((c^+, c^-))\|_{L^1_t(B^p_{\infty,1})} \\
\lesssim X(0) + \|((H_1, H_3))\|_{L^1(B^p_{\infty,1})} + \|((H_2, H_4))\|_{L^1(B^p_{\infty,1})}.
\] (3.42)

where we have used $\mathcal{P}$ and $\mathcal{Q}$ are continuous on $B^p_{\infty,1}$ (being 0 order multipliers).

In what follows, we shall bound the nonlinear terms on the right-side of (3.42). First, due to $H_1 = -\text{div}(c^+ u^+) = -c^+ \text{div}u^+ - u^+ \cdot \nabla c^+$, $H_3 = -\text{div}(c^- u^-) = -c^- \text{div}u^- - u^- \cdot \nabla c^-$, employing Proposition 2.6 yields that
\[
\|((H_1, H_3))\|_{L^1_t(B^p_{\infty,1})} \\
\lesssim \int_0^t \|\nabla u^+\|_{B^p_{\infty,1}} \|c^+\|_{B^\infty_{\infty,1}} d\tau + \int_0^t \|\nabla u^+\|_{B^p_{\infty,1}} \|c^-\|_{B^\infty_{\infty,1}} d\tau \\
+ \int_0^t \|\nabla c^+\|_{B^\infty_{\infty,1}} \|u^+\|_{B^\infty_{\infty,1}} d\tau + \int_0^t \|\nabla c^-\|_{B^\infty_{\infty,1}} \|u^-\|_{B^\infty_{\infty,1}} d\tau \\
\lesssim \|c^+\|_{L^\infty_t(B^p_{\infty,1})} \|u^+\|_{L^1_t(B^p_{\infty,1})} + \|c^-\|_{L^\infty_t(B^p_{\infty,1})} \|u^-\|_{L^1_t(B^p_{\infty,1})} \\
+ \|c^+\|_{L^2_t(B^p_{\infty,1})} \|u^+\|_{L^2_t(B^p_{\infty,1})} + \|c^-\|_{L^2_t(B^p_{\infty,1})} \|u^-\|_{L^2_t(B^p_{\infty,1})} \\
\lesssim X^2(t).
\]
For the term \( \|(H_2, H_4)\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \), omitting some positive constants, it suffices to bound \( \|(H_2, H_4)\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \).

For the term \( H_2^j \), thanks to Proposition 2.10 and Proposition 2.13(i), we get
\[
\|g_+ (c^+, c^-) \partial_t c^+ \|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \lesssim \|g_+ (c^+, c^-)\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \|\nabla c^+\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \left(1 + X^2(t)\right)^{\left[\frac{N}{p}\right]+1} \|(c^+, c^-)\|_{L_t^2(\mathcal{B}_{p, 1}^{N^{-1}})}^2 \\
\lesssim \left(1 + X^2(t)\right)^{\left[\frac{N}{p}\right]+1} X^2(t), \\
\|u^t \cdot \nabla u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \lesssim \|u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \|\nabla u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \|u^t\|_{L_t^2(\mathcal{B}_{p, 1}^{N^{-1}})}^2 \\
\lesssim X^2(t),
\]
\[
\|h_+ (c^+, c^-) \partial_j c^+ \partial_j u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \\
\lesssim \|h_+ (c^+, c^-) \partial_j c^+ \|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|\partial_j u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \|h_+ (c^+, c^-)\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|\nabla c^+\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \|\nabla u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \left(1 + (1 + \|(c^+, c^-)\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})}^2)^{\left[\frac{N}{p}\right]+1} \|(c^+, c^-)\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|c^+\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \|u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim X^2(t) + \left(1 + X^2(t)\right)^{\left[\frac{N}{p}\right]+1} X^3(t),
\]
and
\[
\|l_+ (c^+, c^-) \partial_j^2 u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})}^h \lesssim \|l_+ (c^+, c^-)\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|\partial_j^2 u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \|(c^+, c^-)\|_{L_t^\infty(\mathcal{B}_{p, 1}^{N^{-1}})} \|u^t\|_{L_t^1(\mathcal{B}_{p, 1}^{N^{-1}})} \\
\lesssim \left(1 + X^2(t)\right)^{\left[\frac{N}{p}\right]+1} X^2(t).
\]

Similarly, we also obtain the corresponding estimates of other terms \((\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u^t_j)^h, (\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u^t_j)^h, (\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u^t_j)^h, (\lambda^+ h_+ (c^+, c^-) \partial_t c^+ \partial_j u^t_j)^h, (\lambda^+ k_+ (c^+, c^-) \partial_t c^- \partial_j u^t_j)^h, ((\mu^+ + \lambda^+ l_+ (c^+, c^-) \partial_t c^- \partial_j u^t_j)^h \). Here, we omit the details.

Hence
\[
\|H_2\|_{L_t^1([0,T]; \mathcal{B}_{p, 1}^{N^{-1}})}^h \lesssim X^2(t) + \left(1 + X^2(t)\right)^{\left[\frac{N}{p}\right]+1} \left(X^2(t) + X^3(t)\right).
\]
The term \( H_4 \) may be treated along the same lines, and we omit it.

Plugging all the above nonlinear estimates of the terms \( H_1 - H_4 \) into (3.42), we conclude that (3.9). This completes the proof of Proposition 3.2.

3.3 The Unique Global Solvability

Combining with Propositions 3.1 and 3.2 we have the following \textit{a priori} estimates in all frequencies.
Proposition 3.5 Let $T \geq 0, N \geq 2, p$ satisfy $2 \leq p \leq \min(4, \frac{2N}{N-2})$ and, additionally, $p \neq 4$ if $N = 2$. Assume that $(c^+, u^+, c^-, u^-)$ is a solution to the Cauchy problem (1.10)-(1.11) on $[0, T] \times \mathbb{R}^N$, then

$$X(t) \lesssim \left(X(0) + (1 + X^2(t))^{\frac{N}{2}}(X^2(t) + X^3(t))\right) \text{ for all } t \in [0, T]. \quad (3.43)$$

From (3.43), it is not difficult to work out a fixed point argument as in [32], we finally conclude the global well-posedness of the Cauchy problem (1.10)-(1.11). This completes the proof of Theorem 1.1.

4 The proof of Theorem 1.2

In this section, our central task is to prove Theorem 1.2 taking for granted the global-in-time existence result of Theorem 1.1. The proof is divided into two parts, according to the two terms of the time-weighted functional $D(t)$ (see (1.26)). In what follows, we shall use frequently elementary fact that the global solution $(c^+, u^+, c^-, u^-)$ provided by Theorem 1.1 fulfills

$$\|\{c^+, c^-\}\|_{L^\infty_t(B_{p,1}^N)} \leq c \ll 1 \text{ for all } t \geq 0. \quad (4.1)$$

4.1 In the low frequencies

Denoting by $A(D)$ the semi-group associated to system (1.10) with $H_1 \equiv H_2 \equiv H_3 \equiv H_4 \equiv 0$ for $U = (c^+, u^+, c^-, u^-)$, and using Parseval’s equality, (3.2) and the definition of $\hat{\Delta}_q$, we get for all $q \leq q_0$

$$\|e^{tA(D)}\hat{\Delta}_q U\|_{L^2} \lesssim e^{-\sigma_0 q^2 t} \|\hat{\Delta}_q U\|_{L^2}.$$

Hence, multiplying by $t^{rac{s+q_0}{2}} 2^{qs}$ and summing up on $q$, we readily have

$$t^{rac{s+q_0}{2}} \sum_{q \leq q_0} 2^{qs} \|e^{tA(D)}\Delta_q U\|_{L^2} \lesssim \sum_{q \leq q_0} 2^{qs} e^{-\sigma_0 q^2 t} \|\Delta_q U\|_{L^2} t^{rac{s+q_0}{2}} \lesssim \sum_{q \leq q_0} 2^{q(s+q_0)} e^{-\sigma_0 q^2 t} \|\Delta_q U\|_{L^2} 2^{q(s-q_0)} t^{rac{s+q_0}{2}}$$

$$\lesssim \|U\|_{\ell^2_{\frac{r}{2}, \infty}} \sum_{q \leq q_0} 2^{q(s+q_0)} e^{-\sigma_0 q^2 t} t^{rac{s+q_0}{2}}$$

$$\lesssim \|U\|_{\ell^2_{\frac{r}{2}, \infty}} \sum_{q \leq q_0} 2^{q(s+q_0)} e^{-\sigma_0 q^2 t} t^{rac{s+q_0}{2}}. \quad (4.2)$$

As for any $\sigma > 0$ there exists a constant $C_\sigma$ so that

$$\sup_{t \geq 0} \sum_{q \in \mathbb{Z}} \frac{\sigma}{t^2} 2^{qs} e^{-\sigma_0 q^2 t} \leq C_\sigma. \quad (4.3)$$

We get from (4.2) and (4.3) that for $s + q_0 > 0$,

$$\sup_{t \geq 0} t^{\frac{s+q_0}{2}} \|e^{tA(D)} U\|_{\ell^2_{\frac{r}{2}, \infty}} \lesssim \|U\|_{\ell^2_{\frac{r}{2}, \infty}}.$$
Furthermore, it is obvious that for $s + s_0 > 0$,
\[
\|e^{tA(D)}U\|_{B_{2,1}^{s}}^{\ell} \lesssim \|U\|_{B_{2,\infty}^{-s_0} \cap \mathbb{S}} \sum_{q \leq q_0} 2^q \|s + s_0\| \lesssim \|U\|_{B_{2,\infty}^{-s_0} \cap \mathbb{S}}.
\]
Hence, setting $\langle t \rangle \overset{\text{def}}{=} (1 + t)$, we get
\[
\sup_{t \geq 0} \langle t \rangle^{-\frac{s + s_0}{2}} \|e^{tA(D)}U\|_{B_{2,1}^{s}}^{\ell} \lesssim \|U\|_{B_{2,\infty}^{-s_0} \cap \mathbb{S}}.
\]
(4.4)

Thus, from (4.4) and Duhamel’s formula, we have
\[
\|e^{tA(D)}U\|_{B_{2,1}^{s}}^{\ell} \lesssim \sup_{t \geq 0} \langle t \rangle^{-\frac{s + s_0}{2}} \|e^{tA(D)}(c^+, u^+, c^-, u^-)\|_{B_{2,\infty}^{-s_0}}^{\ell}
\]
\[
+ \int_0^t \langle t - \tau \rangle^{-\frac{s + s_0}{2}} \|\dot{H}_1, H_2, H_3, H_4\|_{B_{2,\infty}^{-s_0}}^{\ell} d\tau.
\]
(4.5)

We claim that for all $s \in [\varepsilon - s_0, \frac{N}{2} + 1]$ and $t \geq 0$, then
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + s_0}{2}} \|\dot{H}_1, H_2, H_3, H_4\|_{B_{2,\infty}^{-s_0}}^{\ell} d\tau \lesssim \langle t \rangle^{-\frac{s + s_0}{2}} \left(X^2(t) + D^2(t)\right),
\]
(4.6)

where $X(t)$ and $D(t)$ have been defined in (1.23) and (1.26), respectively.

In order to prove our claim, we first present the following some important inequalities which will be frequently used in our process later.

\[
\|e^{tA(D)}U\|_{B_{2,1}^{s}}^{\ell} \lesssim \sup_{t \geq 0} \langle t \rangle^{-\frac{s + s_0}{2}} \|e^{tA(D)}(c^+, u^+, c^-, u^-)\|_{B_{2,\infty}^{-s_0} \cap \mathbb{S}}^{\ell}
\]
\[
+ \|\dot{H}_1, H_2, H_3, H_4\|_{B_{2,\infty}^{-s_0}}^{\ell} \lesssim \langle t \rangle^{-\frac{s + s_0}{2}} \left(X^2(t) + D^2(t)\right),
\]
(4.7)

To bound the term $H_1$, using low-high frequency decomposition we see that
\[
H_1 = u^+ \cdot \nabla c^{\ell} + u^+ \cdot \nabla c^{h} + c^+ \cdot \text{div}(u^+)^{\ell} + c^+ \cdot \text{div}(u^+)^{h}.
\]

We now bound the above items one by one. For the term $u^+ \cdot \nabla c^{\ell}$, employing (2.3) yields that
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + s_0}{2}} \|u^+ \cdot \nabla c^{\ell}\|_{B_{2,\infty}^{-s_0} \cap \mathbb{S}}^{\ell} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + s_0}{2}} \|u^+\|_{B_{p,1}^{-\frac{N}{2p}} \cap \mathbb{S}}^{\ell} \|\nabla c^{\ell}\|_{B_{2,1}^{s}}^{\ell} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + s_0}{2}} \left(\|u^+\|_{B_{p,1}^{-\frac{N}{2p}} \cap \mathbb{S}}^{\ell} + \|u^+\|_{B_{p,1}^{-\frac{N}{2p}} \cap \mathbb{S}}^{\ell} \right) \|\nabla c^{\ell}\|_{B_{2,1}^{s}}^{\ell} d\tau.
\]
(4.12)
We are going to bound \( \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \) in (4.12). Here, we shall proceed differently depending on whether \( 2 \leq p \leq N \) or \( N < p < 2N \). If \( 2 \leq p \leq N \) then \( 1 - \frac{N}{p} \leq \frac{N}{p} - 1 \), then we conclude, from (4.7), that

\[
\|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \lesssim \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \lesssim \langle \tau \rangle^{-\alpha} \langle \tau \rangle^{\alpha} \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \lesssim \langle \tau \rangle^{-\alpha} \left(D(\tau) + X(\tau)\right).
\]

(4.13)

Plugging (4.13) and (4.9) into (4.12) implies that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \|(u^+ \cdot \nabla c^+)(\tau)\|_{L^{\infty}} d\tau \\
\lesssim \left(D^2(t) + X^2(t)\right) \int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \left(\langle \tau \rangle^{-\frac{3}{2}} + \langle \tau \rangle^{\frac{a}{2}}\right) d\tau
\]

(4.14)

\[
\lesssim \langle t \rangle^{-\frac{a+s}{2}} \left(D^2(t) + X^2(t)\right).
\]

where we have used Lemma 2.16 and the fact \( \frac{a+s}{2} \leq \min\left\{1, \frac{N}{p} \right\} \) with \( \alpha = \frac{N}{p} + \frac{1}{2} - \varepsilon \) for all \( s \leq \frac{N}{2} + 1 \). If \( N < p < 2N \) then \( 1 - \frac{N}{p} \leq \frac{N}{p} \). Employing interpolation inequality and (4.7) yields that

\[
\|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \lesssim \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \lesssim \left(\|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}}\right)^{\frac{1}{2}} \lesssim \left(\langle \tau \rangle^{-\alpha} \langle \tau \rangle^{\alpha} \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}} \|u^+h\|_{B^{\frac{N}{p},1}_{p,1}}\right)^{\frac{1}{2}} \lesssim \langle \tau \rangle^{-\frac{a+s}{2}} \left(X(\tau) + D(\tau)\right).
\]

(4.15)

Putting (4.15) and (4.9) into (4.12), noticing \( \frac{a}{2} = \frac{N}{2p} + \frac{1}{2} - \varepsilon < 1 \), \( \frac{a+s}{2} \leq \alpha + \frac{N}{p} \) for all \( s \leq \frac{N}{2} + 1 \), and \( \alpha + \frac{N}{p} > 1 \), and then using Lemmas 2.16, 2.17 give rise to

\[
\int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \|(u^+ \cdot \nabla c^+)(\tau)\|_{L^{\infty}} d\tau \\
\lesssim \left(D^2(t) + X^2(t)\right) \int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \langle \tau \rangle^{-\frac{3}{2}} + \langle \tau \rangle^{\frac{a}{2}} d\tau
\]

\[
\lesssim \langle t \rangle^{-\frac{a+s}{2}} \left(D^2(t) + X^2(t)\right).
\]

For the term \( c^+ \cdot \text{div}(u^+) \), using (2.3), (4.8), (4.9) and (4.10) implies that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \|(c^+ \cdot \text{div}(u^+))\|_{L^{\infty}} d\tau
\]

\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \|c^+\|_{L^{\infty}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \langle \min\left\{1, \frac{a+s}{2} + \alpha, \frac{a+s}{2} + N\right\} \rangle d\tau
\]

(4.16)

\[
\lesssim \left(D^2(t) + X^2(t)\right) \int_0^t \langle t - \tau \rangle^{-\frac{a+s}{2}} \langle \min\left\{1, \frac{a+s}{2} + \alpha, \frac{a+s}{2} + N\right\} \rangle d\tau
\]

\[
\lesssim \langle t \rangle^{-\frac{a+s}{2}} \left(D^2(t) + X^2(t)\right).
\]
where we have used Lemma 2.16 and the fact \( \frac{2q+8}{2} \leq \min\{ \frac{1}{2} + \frac{N}{p}, \alpha + \frac{N}{p} \} \) with \( \alpha = \frac{N}{p} + \frac{1}{2} - \varepsilon \) for all \( s \leq \frac{N}{2} + 1 \).

For the term \( u^+ \cdot \nabla c^+ h \), we shall also proceed differently depending on whether \( 2 \leq p \leq N \) or \( N < p < 2N \). If \( 2 \leq p \leq N \), we observe that applying (4.1) with \( \sigma = \frac{N}{p} - 1 \) yields

\[
\|f g^h\|_{B_{2,\infty}^{\ell, q_0}} \lesssim \|f\|_{B_{p,1}^{\frac{N}{p}-\frac{N}{2}}} \left( \|g^h\|_{B_{p,1}^{\frac{N}{p}}} + \|\tilde{S}_{q_0+N_0} g^h\|_{L^p} \right) \lesssim \|f\|_{B_{p,1}^{\frac{N}{p}-1}} \|g^h\|_{B_{p,1}^{\frac{N}{p}-1}},
\]

(4.17)

where we used Bernstein's inequality (recall that \( p^* = \frac{2p}{p-2} \geq p \)) and the fact that only middle frequencies of \( g \) are involved in \( \tilde{S}_{q_0+N_0} g^h \). Employing (1.8), (1.13), (1.10), noticing \( \frac{2q+8}{2} \leq \min\{ \frac{1}{2} + \alpha, 2\alpha \} \) with \( \alpha = \frac{N}{p} + \frac{1}{2} - \varepsilon \) for all \( s \leq \frac{N}{2} + 1 \), and then using Lemma 2.16 we get

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \|u^+ \cdot \nabla c^+ h(\tau)\|_{B_{2,\infty}^{\ell, q_0}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \|u^+\|_{B_{p,1}^{\frac{N}{p}}} \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \|u^+\|_{B_{p,1}^{\frac{N}{p}}} + \|u^+\|_{B_{p,1}^{\frac{N}{p}}} \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} \right) \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} d\tau \\
\lesssim \left( D^2(t) + X^2(t) \right) \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \langle \tau \rangle^{-(\frac{1}{4} + \alpha)} + \langle \tau \rangle^{-2\alpha} \right) d\tau \\
\lesssim \left( t \right)^{-\frac{s+q_0}{2}} \left( D^2(t) + X^2(t) \right),
\]

If \( N < p < 2N \), applying (2.5) with \( \sigma = 1 - \frac{N}{p} \) yields

\[
\|f g^h\|_{B_{2,\infty}^{\ell, q_0}} \lesssim \left( \|f\|_{B_{p,1}^{\frac{N}{p}}} + \|\tilde{S}_{q_0+N_0} f\|_{L^p} \right) \|g^h\|_{B_{p,1}^{\frac{N}{p}}} \lesssim \left( \|f\|_{B_{p,1}^{\frac{N}{p}}} + \|f\|_{B_{p,1}^{\frac{N}{p}}} \right) \|g^h\|_{B_{p,1}^{\frac{N}{p}}} \lesssim \left( \|f\|_{B_{p,1}^{\frac{N}{p}}} + \|f\|_{B_{p,1}^{\frac{N}{p}}} \right) \|g^h\|_{B_{p,1}^{\frac{N}{p}}},
\]

(4.18)

where \( p^* = \frac{2p}{p-2} \). Then it follows from (4.8), (4.15), (4.11) and (4.18) that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \|u^+ \cdot \nabla c^+ h(\tau)\|_{B_{2,\infty}^{\ell, q_0}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \|u^+\|_{B_{p,1}^{\frac{N}{p}}} + \|u^+\|_{B_{p,1}^{\frac{N}{p}}} \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} \right) \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \|u^+\|_{B_{p,1}^{\frac{N}{p}}} + \|u^+\|_{B_{p,1}^{\frac{N}{p}}} \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} \right) \|\nabla c^+ h\|_{B_{p,1}^{\frac{N}{p}}} d\tau \\
\lesssim \left( D^2(t) + X^2(t) \right) \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \langle \tau \rangle^{\min\{ \frac{1}{2} + \alpha, \frac{N}{2p} - \frac{N}{2} + \alpha \} + \langle \tau \rangle^{-2\alpha} \right) d\tau \\
\lesssim \left( t \right)^{-\frac{s+q_0}{2}} \left( D^2(t) + X^2(t) \right),
\]

where we have used Lemmas 2.16 and 2.17 and the fact \( \frac{\alpha}{2} = \frac{N}{2p} + \frac{1}{4} - \frac{\varepsilon}{2} < 1, \frac{s+q_0}{2} \leq \min\{ \frac{1}{2} + \alpha, \frac{3N}{2p} - \frac{N}{4} + \alpha, 2\alpha \} \) for all \( s \leq \frac{N}{2} + 1 \), and \( 2\alpha > 1 \).

For the term \( c^+ \text{div}(u^+)^h \), we shall also proceed differently depending on whether \( 2 \leq p \leq N \) or \( N < p < 2N \). If \( 2 \leq p \leq N \), applying (2.4) yields that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \|c^+ \text{div}(u^+)^h(\tau)\|_{B_{2,\infty}^{\ell, q_0}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \|c^+\|_{B_{p,1}^{\frac{N}{p}}} \|\text{div}u^+\|_{B_{p,1}^{\frac{N}{p}}} d\tau.
\]
In what follows, we split the integral on $[0, t]$ into integrals on $[0, 2]$ and $[2, t]$, respectively. The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and we infer that
\[
\int_0^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \| c^+ \div (u^+)^h(\tau) \|_{B_{2,\infty}^0} d\tau
\]
\[
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \| c^+ \|_{B_{p,1}^\infty} + \| c^+ \|_{B_{p,1}^0} \right) \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \| c^+ \|_{B_{p,1}^\infty} + \| c^+ \|_{B_{p,1}^0} \right) \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\lesssim \langle t \rangle^{-\frac{\sigma + q_0}{2}} X^2(t).
\]

On the other hand, if $t \geq 2$, we infer that
\[
\int_0^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \| c^+ \div (u^+)^h(\tau) \|_{B_{2,\infty}^0} d\tau
\]
\[
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \| c^+ \|_{B_{p,1}^\infty} \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
+ \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \| c^+ \|_{B_{p,1}^\infty} \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\overset{\text{def}}{=} I_1 + I_2.
\]

Remembering the definition of $X(t)$ and $D(t)$, we obtain
\[
I_1 \lesssim \langle t \rangle^{-\frac{q_0}{2} - \frac{s}{2}} X^2(1).
\]

To bound the term $I_2$, according to (4.10), and using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we conclude that
\[
I_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \| c^+ \|_{B_{2,1}^\infty} \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \| c^+ \|_{B_{p,1}^\infty} + \| c^+ \|_{B_{p,1}^0} \right) \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \| c^+ \|_{B_{p,1}^\infty} + \| c^+ \|_{B_{p,1}^0} \right) \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \| c^+ \|_{B_{p,1}^\infty} + \| c^+ \|_{B_{p,1}^0} \right) \| u^+ \|_{B_{p,1}^0}^h d\tau
\]
\[
\langle D^2(t) + X^2(t) \rangle \left( \int_1^t \langle t - \tau \rangle^{-\frac{\sigma + q_0}{2}} \left( \langle \tau \rangle^{-\left(\frac{q_0}{2} + \frac{N}{p} - \frac{1}{2} + \alpha\right)} + \langle \tau \rangle^{-2\alpha} \right) d\tau \right)
\]
\[
\lesssim \langle t \rangle^{-\frac{s}{2} - \frac{\sigma}{2}} \left( D^2(t) + X^2(t) \right),
\]

where we have used Lemma 2.10 and the fact $\frac{s}{2} + \frac{\sigma}{2} \leq \min\{\frac{q_0}{2} + \frac{N}{p} - \frac{1}{2} + \alpha, 2\alpha\}$ with $\alpha = \frac{N}{p} + \frac{1}{2} - \varepsilon$ for all $s \leq \frac{N}{p} + 1$. If $N < p < 2N$, thanks to (4.8), (4.13), (4.10), (4.11) and (4.18), we deduce
Lemma 2.16, we conclude that

\[ \left\| \langle \tau \rangle^{-\frac{s+\alpha}{2}} u^+ \right\|_{B^{s}_{2,\infty}} \]

where

\[ g \]

For the term \( H \)

Then we infer that

\[ N \leq \min \left\{ \frac{1}{2} + \alpha, 2\alpha, \frac{3N}{2p} - \frac{N}{4} + \alpha \right\} \]

where we have used Lemma 2.17 and the fact \( \frac{a}{2} < 1 \), \( \min \left\{ \frac{1}{2} + \alpha, 2\alpha, \frac{3N}{2p} - \frac{N}{4} + \alpha \right\} > 1 \) for

\[ N \leq p \leq \min(4, \frac{2N}{p}), \quad \frac{N}{2p} + \frac{a}{2} \leq \min \left\{ \frac{1}{2} + \alpha, 2\alpha, \frac{3N}{2p} - \frac{N}{4} + \alpha \right\} \]

with \( \alpha = \frac{N}{p} + \frac{1}{2} - \varepsilon \) for all \( s \leq \frac{N}{2} + 1 \). Then we infer that

\[ \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left\| H_1(\tau) \right\|_{B^{s}_{2,\infty}}^\ell d\tau \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} \left( D^2(t) + X^2(t) \right). \tag{4.19} \]

The term \( H_3 \) may be treated along the same lines, and we obtain

\[ \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left\| H_3(\tau) \right\|_{B^{s}_{2,\infty}}^\ell d\tau \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} \left( D^2(t) + X^2(t) \right). \tag{4.20} \]

Next, we bound the term \( H_2 \). To handle the first term \( g_+(c^+, c^-) \partial_t c^+ \) in \( H_2 \), we decompose it into

\[ g_+(c^+, c^-) \partial_t c^+ = g_+(c^+, c^-) \partial_t c^{+\ell} + g_+(c^+, c^-) \partial_t c^{+h}, \]

where \( g_+ \) stands for some smooth function vanishing at 0.

For the term \( g_+(c^+, c^-) \partial_t c^{+\ell} \), employing (2.8), (4.8), (4.9), (4.10), (4.11), Proposition 2.11 and Lemma 2.16, we conclude that

\[ \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left\| g_+(c^+, c^-) \partial_t c^{+\ell}(\tau) \right\|_{B^{s}_{2,\infty}} d\tau \]

\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left\| g_+(c^+, c^-) \right\|_{B^{1-\frac{N}{2}}_{p,1}} \left\| c^{+\ell} \right\|_{B^{\frac{N}{4}}_{2,1}} d\tau \]

\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left( \left\| (c^+, c^-)^{+\ell} \right\|_{B^{1-\frac{N}{2}}_{p,1}} + \left\| (c^+, c^-)^{+h} \right\|_{B^{1-\frac{N}{2}}_{p,1}} \right) \left\| c^{+\ell} \right\|_{B^{\frac{N}{4}}_{2,1}} d\tau \]

\[ \lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left( \langle \tau \rangle^{-\min\left\{ \frac{1}{2}, \frac{N}{p}, \frac{N}{2p} + \frac{a}{2} \right\}} \right) d\tau \]

\[ \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} D^2(t). \]

For the term \( g_+(c^+, c^-) \partial_t c^{+h} \), if \( 2 \leq p \leq N \), it follows from (4.8), (4.10), (4.17), (4.11), Proposition
Similarly, and Lemma 2.16 that
\[
\int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| g_+(c^+, c^-) \partial_t c^{+h}(\tau) \|_{B_{2,\infty}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| (c^+, c^-) \|_{B_{\frac{n}{p}, 1}}^{\frac{\alpha}{p}} \| \nabla c^{+h} \|_{B_{\frac{p-1}{p}, 1}}^{\frac{\alpha}{p}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \langle \tau \rangle^{-\frac{1}{2} + \alpha} \, d\tau \\
\lesssim \langle t \rangle^{\frac{s+\alpha}{2}} D^2(\tau).
\]

Thus
\[
\int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| g_+(c^+, c^-) \partial_t c^{+h}(\tau) \|_{B_{2,\infty}} \, d\tau \lesssim \langle t \rangle^{\frac{s+\alpha}{2}} D^2(\tau).
\]

Similarly,
\[
\int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| g_+(c^+, c^-) \partial_t c^{-}(\tau) \|_{B_{2,\infty}} \, d\tau \lesssim \langle t \rangle^{\frac{s+\alpha}{2}} D^2(t).
\]

To bound the term with \((u^+ \cdot \nabla)u_i^+\) in \(H^1_2\), we employ the following decomposition:
\[
(u^+ \cdot \nabla)u_i^+ = (u^+ \cdot \nabla)(u_i^+) + (u^+ \cdot \nabla)(u_i^+)^h.
\]

For the term \((u^+ \cdot \nabla)(u_i^+)\), if \(2 \leq p \leq N\), it follows from (2.13), (4.8), (4.9), and (4.13), that
\[
\int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| (u^+ \cdot \nabla)(u_i^+) \|_{B_{2,\infty}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \| u^+(\tau) \|_{B_{\frac{n}{p}, 1}}^{\frac{\alpha}{p}} \| \nabla u_i^+ \|_{B_{2,1}}^{\frac{\alpha}{p}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \left( \| u^+(\tau) \|_{B_{\frac{p-1}{p}, 1}}^{\frac{\alpha}{p}} + \| u^+(\tau) \|_{B_{\frac{p-1}{p}, 1}}^{\frac{\alpha}{p}} \right) \| \nabla u_i^+ \|_{B_{2,1}}^{\frac{\alpha}{p}} \, d\tau \\
\lesssim \left( D^2(t) + X^2(t) \right) \int_0^t \langle t - \tau \rangle^{\frac{s+\alpha}{2}} \langle \tau \rangle^{-\frac{1}{2} + \alpha} \, d\tau \\
\lesssim \langle t \rangle^{\frac{s+\alpha}{2}} \left( D^2(t) + X^2(t) \right).
\]
If \( N < p \leq 2N \), using (2.3), (4.8), (4.9), and (4.15) yields that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| \nabla u_1^+ (\tau) \|^h_B \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| u^+ (\tau) \|^h_B \, d\tau.
\]

To deal with the term \((u^+ \cdot \nabla)(u_1^+)^h\), let us first consider the case \( 2 \leq p \leq N \). Applying (2.4) implies that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| (u^+ \cdot \nabla)(u_1^+)^h (\tau) \|^h_B \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau.
\]

In what follows, we divide the integral on \([0, t]\) into integrals on \([0, 2]\) and \([2, t]\), respectively. When \( t \leq 2 \), thus \( \langle t \rangle \approx 1 \) and \( \langle t - \tau \rangle \approx 1 \) for \( 0 \leq \tau \leq t \leq 2 \). We have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| (u^+ \cdot \nabla)(u_1^+)^h (\tau) \|^h_B \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau \\
\approx \int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| (u^+ (\tau))^{h, B} + \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| (u^+ (\tau))^{h, B} + \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau \\
\lesssim \langle t \rangle^{-\frac{N + p - N}{2}} X^2 (t).
\]

When \( t \geq 2 \), we deduce that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| (u^+ \cdot \nabla)(u_1^+)^h (\tau) \|^h_B \, d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau \\
\quad + \int_1^t \langle t - \tau \rangle^{-\frac{N + p - N}{2}} \| u^+ (\tau) \|^h_B \| \nabla u^+ (\tau) \|^h_B \, d\tau \\
\quad \overset{\text{def}}{=} II_1 + II_2.
\]

Using the definition of \( X(t) \) and \( D(t) \), we get

\[
II_1 \lesssim \langle t \rangle^{-\frac{N + p - N}{2}} X^2 (1).
\]
For the term $I_2$, according to (4.10), and using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we conclude, from Lemma 2.16 that

$$
I_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} ||u^+(\tau)||_{B_{p,1}^{N/2}} ||\nabla u^+(\tau)||_{B_{p,1}^{N/2}} d\tau
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \left( ||u^+||_{B_{p,1}^{N/2}} + ||u^+||_{B_{p,1}^{N/2}}^{N/2} \right) ||u^+||_{B_{p,1}^{N/2}}^{N/2} d\tau
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \left( ||u^+||_{B_{p,1}^{N/2}} + ||u^+||_{B_{p,1}^{N/2}}^{N/2} \right) ||u^+||_{B_{p,1}^{N/2}}^{N/2} d\tau
$$

$$
\lesssim \left( D^2(t) + X^2(t) \right) \int_1^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \langle \tau \rangle^{-\frac{3N}{2} + \frac{N}{2} - \frac{3\alpha}{2}} d\tau
$$

$$
\lesssim \langle t \rangle^{-\frac{p + \alpha}{2}} \left( D^2(t) + X^2(t) \right).
$$

If $N < p \leq 2N$, according to (4.9), (4.15), (4.10), (4.11) and (4.18), we deduce, from Lemma 2.16 that

$$
\int_0^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} ||(u^+ \cdot \nabla)(u_i^+)h(\tau)||_{B_{2,\infty}^{-N/2}} d\tau
$$

$$
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \left( ||u^+||_{B_{p,1}^{N/2}} + ||u^+||_{B_{p,1}^{N/2}}^{N/2} \right) ||\text{div} u^+||_{B_{p,1}^{N/2}}^{N/2} d\tau
$$

$$
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \left( ||u^+||_{B_{p,1}^{N/2}} + ||u^+||_{B_{p,1}^{N/2}}^{N/2} \right) ||u^+||_{B_{p,1}^{N/2}}^{N/2} d\tau
$$

$$
\lesssim \left( D^2(t) + X^2(t) \right) \int_0^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} \langle \tau \rangle^{-\frac{3N}{2} + \frac{N}{2} - \frac{3\alpha}{2}} d\tau
$$

$$
\lesssim \langle t \rangle^{-\frac{p + \alpha}{2}} \left( D^2(t) + X^2(t) \right).
$$

Therefore

$$
\int_0^t \langle t - \tau \rangle^{-\frac{p + \alpha}{2}} ||(u^+ \cdot \nabla)u_i^+(\tau)||_{B_{2,\infty}^{-N/2}} d\tau \lesssim \langle t \rangle^{-\frac{p + \alpha}{2}} D^2(\tau).
$$

To bound the term with $\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+$ in $H_2^T$, employing the following low-high frequency decomposition yields that

$$
\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ = \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ + \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+. \quad (4.10)
$$

For the term $\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+$, it follows from $\alpha \geq \frac{N}{p}$, (4.9), (4.10), (4.11), Proposition 2.14.
and Lemma 2.16 that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ (\tau) \|_{B_2^{-s_0}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ (\tau) \|_{B_2^{-\frac{N}{p}}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| h_+ (c^+, c^-) \nabla c^+ \|_{B_{p,1}^{N-1}} \| \nabla u^+ \|_{B_{p,1}^{1-\frac{N}{p}}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| c^+ \|_{B_{p,1}^{N}} \| \nabla u^+ \|_{B_{p,1}^{1-\frac{N}{p}}} \, d\tau \\
\lesssim \left( X^2 (t) + D^2 (t) \right) \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \langle \tau \rangle^{-(1+\frac{N}{p}+\frac{N}{2p})} + \langle \tau \rangle^{-(1+\alpha+\frac{N}{p}+\frac{N}{2p})} \right) \, d\tau \\
\lesssim \langle t \rangle^{-\frac{s+q_0}{2}} \left( X^2 (t) + D^2 (t) \right),
\]

where we have used the embedding $B_{p,1}^{\frac{N}{p}} \hookrightarrow B_{2,\infty}^{s_0}$ as $s_0 \leq \frac{N}{p}$ (for $p \geq 2$), $\| f \|_{B_{p,1}^{\frac{N}{p}}} \lesssim \| f \|_{B_{2,\infty}^{s_0}} \| g \|_{B_{2,\infty}^{1-\frac{N}{p}}}$ and the fact $\frac{a}{2} + \frac{s}{2} \leq \min \{ 1 + \alpha + \frac{N}{p} - \frac{N}{1}, 1 - \frac{N}{p} + \frac{3N}{2p} \}$ for all $s \leq \frac{N}{p} + 1$, $p \leq \frac{2N}{N-2}$.

For the term $\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ h$, let us first consider the case $2 \leq p \leq N$. It follows from (4.1), (2.4) and Proposition 2.13 that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ h (\tau) \|_{B_{2,\infty}^{s_0}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \nabla c^+ (\tau) \|_{B_{2,\infty}^{\frac{N}{p}}} \| \nabla u^+ (\tau) \|_{B_{2,\infty}^{\frac{N}{p}}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| c^+ (\tau) \|_{B_{p,1}^{\frac{N}{p}+1}} \| \nabla u^+ (\tau) \|_{B_{p,1}^{\frac{N}{p}+1}} \, d\tau.
\]

Due to $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, we deduce, if $t \leq 2$, that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ h (\tau) \|_{B_{2,\infty}^{s_0}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} + \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} \right) \| \nabla u^+ \|_{B_{p,1}^{\frac{N}{p}+1}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \left( \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} + \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} \right) \| \nabla u^+ \|_{B_{p,1}^{\frac{N}{p}+1}} \, d\tau \\
\lesssim \langle t \rangle^{-\frac{s+q_0}{2}} X^2 (t).
\]

If $t \geq 2$, we conclude that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u_i^+ h (\tau) \|_{B_{2,\infty}^{s_0}} \, d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} \| \nabla u^+ \|_{B_{p,1}^{\frac{N}{p}+1}} \, d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-\frac{s+q_0}{2}} \| c^+ \|_{B_{p,1}^{\frac{N}{p}}} \| \nabla u^+ \|_{B_{p,1}^{\frac{N}{p}+1}} \, d\tau \\
\overset{\text{def}}{=} III_1 + III_2.
\]
From the definition of $X(t)$ and $D(t)$, we get

$$III_1 \lesssim \langle t \rangle^{-\frac{N}{2}} X^2(1).$$

To handle the term $II_2$, according to (4.10), Lemma 2.16 and using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we infer that

$$II_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{N}{2}} \| c^+ \|_{B^0_{p,1}} \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} d\tau$$

$$\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{N}{2}} \left( \| c^+ \|_{L^0_{B^0_{p,1}}} + \| c^+ \|_{L^p_{B^0_{p,1}}} \right) \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} d\tau$$

$$\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{N}{2}} \left( \| c^+ \|_{L^0_{B^0_{p,1}}} + \| c^+ \|_{L^p_{B^0_{p,1}}} \right) \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} d\tau$$

$$\lesssim \left( D^2(t) + X^2(t) \right) \int_1^t \langle t - \tau \rangle^{-\frac{N}{2}} \left( \langle \tau \rangle^{-1} \right) d\tau$$

$$\lesssim \langle t \rangle^{-\frac{N}{2}} \left( D^2(t) + X^2(t) \right).$$

Let us consider the case $N \leq p \leq 2N$. Applying (4.4) with $\sigma = 1 - \frac{N}{p}$ yields that

$$\| h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+ \|_{L^p_{B^0_{p,1}}} \lesssim \left( \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} + \sum_{k=k_0}^{k_0+N-1} \| \Delta_k \nabla u^+ \|_{L^p_{B^0_{p,1}}} \right) \| h_+(c^+, c^-) \nabla c^+ \|_{L^p_{B^0_{p,1}}}$$

$$\lesssim \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} \| h_+(c^+, c^-) \nabla c^+ \|_{L^p_{B^0_{p,1}}},$$

where we have used $1 - \frac{N}{p} > 0$ and Bernstein inequality $\| \Delta_k \nabla u^+ \|_{L^p_{B^0_{p,1}}} \lesssim \| \Delta_k \nabla u^+ \|_{L^p}$ for $k_0 \leq k \leq k_0 + N_0$ as $p \geq p$. Further, due to the smooth function $h_+(c^+, c^-)$ vanishing at $(0,0)$ and $1 - \frac{N}{p} \leq \frac{N}{p}$, employing (4.4) and Proposition 2.13 we thus get

$$\int_0^t \langle t - \tau \rangle^{-\frac{N}{2}} \| \mu^+ h_+(c^+, c^-) \partial_j c^+ \partial_j u_i^+ (\tau) \|_{L^p_{B^0_{p,1}}} d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N}{2}} \| c^+ \|_{L^p_{B^0_{p,1}}} \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{N}{2}} \| \nabla u^+ \|_{L^p_{B^0_{p,1}}} d\tau.$$
On the other hand, when \( t \geq 2 \), we have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| \mu^+ \Delta_h^+ (c^+, c^-) \partial_j c^+ \partial_j u^+_i (\tau) \right\|_{\dot{B}_{2,\infty}^{-s_0}} d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| \partial_j c^+ \right\|_{B_{p,1}^N} \left\| \nabla u^+_i \right\|_{B_{p,1}^N}^h d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| \partial_j c^+ \right\|_{B_{p,1}^N} \left\| \nabla u^+_i \right\|_{B_{p,1}^N}^h d\tau \\
\overset{\text{def}}{=} IV_1 + IV_2.
\]

Remembering the definition of \( X(t) \) and \( D(t) \) implies that

\[
IV_1 \lesssim \langle t \rangle^{-\frac{2n}{2} - \frac{n}{4}} X^2(1).
\]

To handle the term \( IV_2 \), thanks to (4.10) and the fact that \( \langle \tau \rangle \approx \tau \) when \( \tau \geq 1 \), we deduce that

\[
IV_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left( \left\| \partial_j c^+ \right\|_{B_{p,1}^N}^\ell + \left\| \partial_j c^+ \right\|_{B_{p,1}^N}^h \right) \left\| \nabla u^+_i \right\|_{B_{p,1}^N}^h d\tau \\
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left( \left\| c^+ \right\|_{B_{p,1}^N}^\ell + \left\| c^+ \right\|_{B_{p,1}^N}^h \right) \left\| \nabla u^+_i \right\|_{B_{p,1}^N}^h d\tau \\
\lesssim \left( D^2(t) + X^2(t) \right) \int_1^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left( \langle \tau \rangle^{-\left( \frac{n}{2} + \frac{N}{4} - \frac{n}{4} + \alpha \right) + \langle \tau \rangle^{-2\alpha}} d\tau \\
\lesssim \langle t \rangle^{-\frac{2n}{2} - \frac{n}{4}} \left( D^2(t) + X^2(t) \right).
\]

Thus

\[
\int_0^t \left\| \mu^+ \Delta_h^+ (c^+, c^-) \partial_j c^+ \partial_j u^+_i (\tau) \right\|_{\dot{B}_{2,\infty}^{-s_0}} d\tau \lesssim \langle t \rangle^{-\frac{s + 4}{2}} \left( D^2(t) + X^2(t) \right).
\]

Similarly, we also obtain the corresponding estimates of other terms \( \mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_i \), \( \mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_i \), \( \mu^+ k_+(c^+, c^-) \partial_j c^- \partial_j u^+_i \), \( \lambda^+ h_+(c^+, c^-) \partial_i c^+ \partial_j u^+_i \) and \( \lambda^+ k_+(c^+, c^-) \partial_i c^- \partial_j u^+_i \). Here, we omit them.

To bound the term \( \mu^+ l_+(c^+, c^-) \partial_j^2 u^+_i \) in \( H^1 \). We decompose it into

\[
\mu^+ l_+(c^+, c^-) \partial_j^2 u^+_i = \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i) + \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i)^h,
\]

where \( l_+ \) stands for some smooth function vanishing at 0. The term \( \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i)^h \) may be treated as \( c^+ \text{ div} (u^+_i)^h \) in (4.10), that is,

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i)^h \right\|_{\dot{B}_{2,\infty}^{-s_0}} d\tau \lesssim \langle t \rangle^{-\frac{s + 4}{2}} \left( D^2(t) + X^2(t) \right).
\]

To handle the term \( \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i)^h \), if \( 2 \leq p \leq N \), it follows from (4.11), (2.4) and Proposition 2.11 that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| \mu^+ l_+(c^+, c^-) \partial_j^2 (u^+_i)^h (\tau) \right\|_{\dot{B}_{2,\infty}^{-s_0}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + 4}{2}} \left\| (c^+, c^-)(\tau) \right\|_{B_{p,1}^N} \left\| \nabla^2 u^+(\tau) \right\|_{B_{p,1}^N} d\tau.
\]
When \( t \leq 2 \), then \( \langle t \rangle \approx 1 \) and \( \langle t - \tau \rangle \approx 1 \) for \( 0 \leq \tau \leq t \leq 2 \). We have

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \| \mu^+ l_+ (c^+, c^-) \partial^2_j (u^+_i)^h(\tau) \|_{\dot B_{2,\infty}^{-\alpha}} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{p,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| u^+(\tau) \|^{h}_{B_{p,1}^{\lambda}} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{2,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| u^+(\tau) \|^{h}_{B_{p,1}^{\lambda}} \ d\tau \\
\lesssim \langle t \rangle^{-\frac{\gamma + \alpha}{2}} X^2(t).
\]

When \( t \geq 2 \),

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \| \mu^+ l_+ (c^+, c^-) \partial^2_j (u^+_i)^h(\tau) \|_{\dot B_{2,\infty}^{-\alpha}} \ d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{p,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| \nabla^2 u^+ h(\tau) \|_{B_{p,1}^{\lambda}} \ d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{2,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| \nabla^2 u^+ h(\tau) \|_{B_{p,1}^{\lambda}} \ d\tau \\
def \equiv V_1 + V_2.
\]

Obviously,

\[ V_1 \lesssim \langle t \rangle^{-\frac{\gamma}{2} - \frac{s}{2}} X^2(1). \]

To deal with the term \( V_2 \), based on the fact that \( \langle \tau \rangle \approx \tau \) when \( \tau \geq 1 \), we conclude, according to (4.10), that

\[
V_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \| (c^+, c^-)(\tau) \|^{\ell}_{B_{p,1}^{\lambda}} \| \nabla^2 u^+ h(\tau) \|_{B_{p,1}^{\lambda}} \ d\tau \\
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{p,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| \nabla u^+(\tau) \|^{h}_{B_{p,1}^{\lambda}} \ d\tau \\
\lesssim \int_1^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{2,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{p,1}^{\lambda}} \right) \| \nabla u^+(\tau) \|^{h}_{B_{p,1}^{\lambda}} \ d\tau \\
\lesssim \left( D^2(t) + X^2(t) \right) \int_1^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \langle \tau \rangle^{-\frac{s+\alpha}{2} + \frac{\gamma}{2} - \frac{\alpha}{2} + \frac{\gamma}{2}} + \langle \tau \rangle^{-\frac{s}{2}} \right) \ d\tau \\
\lesssim \langle t \rangle^{-\frac{s+\alpha}{2} - \frac{\gamma}{2}} \left( D^2(t) + X^2(t) \right).
\]

If \( N < p \leq 2N \), employing (4.11), (4.18) and Proposition 2.14 yields that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \| \mu^+ l_+ (c^+, c^-) \partial^2_j (u^+_i)^h(\tau) \|_{\dot B_{2,\infty}^{-\alpha}} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\gamma + \alpha}{2}} \left( \| (c^+, c^-)(\tau) \|^{\ell}_{B_{p,1}^{\lambda}} + \| (c^+, c^-)(\tau) \|^{h}_{B_{2,1}^{\lambda}} \right) \| \nabla^2 u^+ h(\tau) \|_{B_{p,1}^{\lambda}} \ d\tau.
\]
When \( t \leq 2 \), then \( \langle t \rangle \approx \langle t - \tau \rangle \approx 1 \) for \( 0 \leq \tau \leq t \leq 2 \). We get

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \| \mu^+ l_+ (c^+, c^-) \partial_j^2 (u^+_j)^h (\tau) \|_{B_{2, \infty}^s} \, d\tau \leq \int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \left( \| (c^+, c^-) \|^s_{B_{p,1}^{1-\frac{N}{p}}} + \| (c^+, c^-) \|^h_{B_{p,1}^N} + \| (c^+, c^-) (\tau) \|^\ell_{B_{p,1}^N} \right) \| u^+ \|^h_{B_{p,1}^N} \, d\tau
\]

\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \left( \| (c^+, c^-) \|^s_{B_{p,1}^{1-\frac{N}{p}}} + \| (c^+, c^-) (\tau) \|^\ell_{B_{p,1}^N} \right) \| \nabla^2 u^+ (\tau) \|^h_{B_{p,1}^N} \, d\tau + \int_1^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \left( \| (c^+, c^-) \|^s_{B_{p,1}^{1-\frac{N}{p}}} + \| (c^+, c^-) (\tau) \|^\ell_{B_{p,1}^N} \right) \| \nabla^2 u^+ (\tau) \|^h_{B_{p,1}^N} \, d\tau
\]

\[
\text{def } VI_1 + VI_2.
\]

It follows from the definition of \( X(t) \) and \( D(t) \), that

\[
VI_1 \lesssim \langle t \rangle^{-\frac{N}{p} - \frac{s}{2}} \left( D^2 (1) + X^2 (1) \right).
\]

To bound the term \( VI_2 \), according to (4.8), (4.10) and (4.11), and using the fact that \( \langle \tau \rangle \approx \tau \) when \( \tau \geq 1 \), we conclude that

\[
VI_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \left( \| (c^+, c^-) \|^s_{B_{p,1}^{1-\frac{N}{p}}} + \| (c^+, c^-) \|^h_{B_{p,1}^N} + \| (c^+, c^-) (\tau) \|^\ell_{B_{p,1}^N} \right) \| u^+ \|^h_{B_{p,1}^N} \, d\tau
\]

\[
\lesssim \left( D^2 (t) + X^2 (t) \right) \int_0^t \langle t - \tau \rangle^{-\min \left( \frac{1}{2} + a, 2a, \frac{3N}{p - N} - \frac{N}{4} + \alpha \right)} \, d\tau
\]

\[
\lesssim \langle t \rangle^{-\frac{N}{p} - \frac{s}{2}} \left( D^2 (t) + X^2 (t) \right).
\]

Hence

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \| \mu^+ l_+ (c^+, c^-) \partial_j^2 (u^+_j)^h \|_{B_{2, \infty}^s} \, d\tau \lesssim \langle t \rangle^{-\frac{s + \alpha}{2}} \left( X^2 (t) + D^2 (t) \right).
\]

Similarly,

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \| (\mu^+ + \lambda^+) l_+ (c^+, c^-) \partial_i \partial_j u^+_j \|_{B_{2, \infty}^s} \, d\tau \lesssim \langle t \rangle^{-\frac{s + \alpha}{2}} \left( X^2 (t) + D^2 (t) \right)
\]

We finally conclude that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{s + \alpha}{2}} \| H_2 (\tau) \|_{B_{2, \infty}^s} \, d\tau \lesssim \langle t \rangle^{-\frac{s + \alpha}{2}} \left( X^2 (t) + D^2 (t) \right). \tag{4.21}
\]
The term $H_4$ may be treated along the same lines, and we have

$$
\int_0^t (t - \tau) \frac{\tau^{s + \alpha}}{2} \| H_4(\tau) \|^2_{B^{\frac{N}{2} - 1}_{2,1}} d\tau \lesssim (t - \tau)^{s + \alpha} \left( X^2(t) + D^2(t) \right).
$$

(4.22)

Thus, putting together inequalities (4.19), (4.21), (4.20) and (4.22), we complete the proof of (4.6). Further, combining with (4.4) and (4.6), we conclude that for all $t \geq 0$ and $s \in (\varepsilon - \frac{N}{2}, \frac{N}{2} + 1)$,

$$
\langle t \rangle^{s + \alpha} \| (c^+, u^+, c^-, u^-) \|^2_{B^{\frac{N}{2} - 1}_{2,1}} \lesssim D_0 + X^2(t) + D^2(t).
$$

(4.23)

### 4.2 In the high frequencies

This part is devoted to bounding the last term of $D(t)$. We first introduce the following system in terms of the weighted unknowns the term $(t^\alpha c^+, t^\alpha u^+, t^\alpha c^-, t^\alpha u^-)$

$$
\begin{align*}
\partial_t (t^\alpha c^+) + \text{div}(t^\alpha u^+) &= \alpha t^{\alpha - 1} c^+ + t^\alpha H_1, \\
\partial_t (t^\alpha u^+) + \beta_1 \nabla(t^\alpha c^+) + \beta_2 \nabla(t^\alpha c^-) - \nu_1^+ \Delta(t^\alpha u^+) - \nu_2^+ \nabla \text{div}(t^\alpha u^+) - \nabla \Delta(t^\alpha c^+) &= \alpha t^{\alpha - 1} u^+ + t^\alpha H_2, \\
\partial_t (t^\alpha c^-) + \text{div}(t^\alpha u^-) &= \alpha t^{\alpha - 1} c^- + t^\alpha H_3, \\
\partial_t (t^\alpha u^-) + \beta_1 \nabla(t^\alpha c^-) + \beta_2 \nabla(t^\alpha c^+) - \nu_1^- \Delta(t^\alpha u^-) - \nu_2^- \nabla \text{div}(t^\alpha u^-) - \nabla \Delta(t^\alpha c^-) &= \alpha t^{\alpha - 1} u^- + t^\alpha H_4.
\end{align*}
$$

(4.24)

Applying Lemma 3.4 and Proposition 2.15 to system (4.24), by a similar derivation process of (3.32), we also have

$$
\| \tau^\alpha (\nabla c^+, u^+, \nabla c^-, u^-) \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})} \lesssim \| \tau^\alpha (c^+, u^+, c^-, u^-) \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})} + \| \tau^\alpha (H_1, H_2, H_3, H_4) \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})}.
$$

(4.25)

We now handle the lower order linear terms on the right hand-side of the above inequality. When $v \in \{c^+, u^+, c^-, u^-\}$, for $0 \leq \tau \leq t \leq 2$, we have

$$
\| \alpha \tau^\alpha v \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})} \lesssim \| v \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})} \lesssim X(t).
$$

When $t \geq 2$, for $0 \leq \tau \leq 1$, we get

$$
\| \alpha \tau^\alpha v \|^2_{L^\infty_t([0,1];B_{p,1}^{\frac{N}{2} - 1})} \lesssim \| v \|^2_{L^\infty_t(B_{p,1}^{\frac{N}{2} - 1})} \lesssim X(t).
$$

When $t \geq 2$, for $1 \leq \tau \leq t$, we have

$$
\| \alpha \tau^\alpha v \|^2_{L^\infty_t([1,t];B_{p,1}^{\frac{N}{2} - 1})} = \alpha \sum_{j \geq j_0} 2^{j(N+1)} 2^{-2j} \| \tau^\alpha \hat{\Delta} j v \|^2_{L^\infty_t([1,t];L^p)} \lesssim \alpha 2^{-2j_0} \sum_{j \geq j_0} 2^{j(N+1)} \| \tau^\alpha \hat{\Delta} j v \|^2_{L^\infty_t([1,t];L^p)} \lesssim \alpha 2^{-2j_0} \| \tau^\alpha v \|^2_{L^\infty_t([1,t];B_{p,1}^{\frac{N}{2} + 1})}.
$$

(4.26)
Choosing \( j_0 \) large enough such that
\[
C \alpha 2^{-2j_0} \leq \frac{1}{4},
\]
which implies that \((4.26)\) may be absorbed by the left hand-side of \((4.25)\). Thus
\[
\| \tau^\alpha (\nabla c^+, u^+, \nabla c^-, u^-) \|_{L_1^\infty(\Omega)} \lesssim X(t) + \| \tau^\alpha (H_1, H_2, H_3, H_4) \|_{L_1^\infty(\Omega)}.
\]
(4.27)

It now comes down to estimating the above nonlinear terms. We first show the following inequalities which are repeatedly used later.
\[
\| \tau^\alpha \nabla (c^+, c^-) \|_{L_1^\infty(\Omega)} \lesssim \| \tau^\alpha \nabla (c^+, c^-) \|_{L_1^\infty(\Omega)} + \| \tau^\alpha \nabla (c^+, c^-) \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim \| \tau^\alpha (c^+, c^-) \|_{L_1^\infty(\Omega)} + \| \tau^\alpha (c^+, c^-) \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim D(t),
\]
\[
\| \tau^\alpha (u^+, u^-) \|_{L_1^\infty(\Omega)} \lesssim \| \tau^\alpha (u^+, u^-) \|_{L_1^\infty(\Omega)} + \| \tau^\alpha (u^+, u^-) \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim \| \tau^\alpha (u^+, u^-) \|_{L_1^\infty(\Omega)} + \| \tau^\alpha (u^+, u^-) \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim D(t).
\]
(4.28)

For \( \| \tau^\alpha H_1 \|_{L_1^\infty(\Omega)} \), from \((4.28), (4.29)\) and Proposition 2.6, we have
\[
\| \tau^\alpha H_1 \|_{L_1^\infty(\Omega)} \lesssim \| \tau^\alpha c^+ \|_{L_1^\infty(\Omega)} + \| \tau^\alpha u^+ \cdot \nabla c^+ \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim \| \tau^\alpha c^+ \|_{L_1^\infty(\Omega)} + \| \tau^\alpha u^+ \cdot \nabla c^+ \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim \| u^+ \|_{L_1^\infty(\Omega)} \| \tau^\alpha \nabla c^+ \|_{L_1^\infty(\Omega)} + \| u^+ \|_{L_1^\infty(\Omega)} \| \tau^\alpha \nabla c^+ \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim X(t)D(t).
\]
(4.30)

Similarly,
\[
\| \tau^\alpha H_3 \|_{L_1^\infty(\Omega)} \lesssim X(t)D(t).
\]
(4.31)

In what follows, we bound the term \( \| \tau^\alpha H_2 \|_{L_1^\infty(\Omega)} \). To bound the first part of \( H_2^i \), employing \((4.28),\) Proposition 2.6 Proposition 2.14 and (4.1), we infer that
\[
\| \tau^\alpha g_+ (c^+, c^-) \partial_t c^+ \|_{L_1^\infty(\Omega)} \lesssim \| g_+ (c^+, c^-) \|_{L_1^\infty(\Omega)} \| \tau^\alpha \partial_t c^+ \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim \| (c^+, c^-) \|_{L_1^\infty(\Omega)} \| \tau^\alpha \nabla c^+ \|_{L_1^\infty(\Omega)}
\]
\[
\lesssim X(t)D(t).
\]
(4.32)

Similarly,
\[
\| \tau^\alpha \tilde{g}_+ (c^+, c^-) \partial_t c^- \|_{L_1^\infty(\Omega)} \lesssim X(t)D(t).
\]
(4.33)
To bound the term \((u^+ \cdot \nabla) u^+_i\), from Proposition 2.6 and (4.29), we get
\[
\|\tau^a (u^+ \cdot \nabla) u^+_i \|^h_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \lesssim \|u^+\|_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \|\tau^a \nabla u^+\|_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \lesssim X(t) D(t). \tag{4.34}
\]

Using (4.29), Proposition 2.6, Proposition 2.14 and (4.1) yields that
\[
\|\tau^a \mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u^+_i \|^h_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \lesssim \|\nabla c^+\|_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \|\tau^a \nabla u^+\|_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \lesssim X(t) D(t). \tag{4.35}
\]
Similarly, we also obtain the corresponding estimates of other terms \(\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u^+_i\), \(\mu^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u^+_i\), \(\mu^+ l_+ (c^+, c^-) \partial_j u^+_i\), \((\mu^+ + \lambda^+) l_+ (c^+, c^-) \partial_j \partial_j u^+_i\), \(\mu^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u^+_i\), \(\lambda^+ h_+ (c^+, c^-) \partial_j c^+ \partial_j u^+_i\) and \(\lambda^+ k_+ (c^+, c^-) \partial_j c^- \partial_j u^+_i\). Here, we omit the details.

Then
\[
\|\tau^a (\nabla c^+, u^+, \nabla c^-, u^-) \|^h_{L_t^\infty (B_{p,1}^{\frac{N}{N-1}})} \lesssim X(t) + X(t) D(t), \tag{4.36}
\]
which together with (4.38) for all \(t \geq 0\), yields that
\[
D(t) \lesssim D_0 + X(t) + X^2(t) + D^2(t).
\]
As Theorem 1.1 ensures that \(X(t) \lesssim X(0)\) with \(X(0)\) being small, and \(X(0) = \|(c_0^+, u_0^+, c_0^-, u_0^-)\|_{B_{p,1}^{\frac{N}{N-1}}} \lesssim \|(c_0^+, u_0^+, c_0^-, u_0^-)\|_{B_{2,\infty}^{2-\frac{N}2}}\), one can conclude that (1.25) is fulfilled for all time if \(\|(\nabla R_0^+, u_0^-, \nabla R_0^-, u_0^-)\|_{B_{p,1}^{\frac{N}{N-1}}} \lesssim \)
and \(D_0\) are small enough. This completes the proof of Theorem 1.2.

References

1. H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften, No. 343. Springer, Heidelberg, (2011).
2. J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Annales Scientifiques de l’École Normale Supérieure, 14 (1981), 209–246.
3. D. Bresch, B. Desjardins, J.M. Ghidaglia and E. Grenier, *Global weak solutions to a generic two-fluid model*, Arch. Rational Mech. Anal. 196 (2009), 599–629.
4. D. Bresch, X. Huang and J. Li, *Global weak solutions to one-dimensional nonconservative viscous compressible two-phase system*, Commun. Math. Phys. 309 (2012), 737–755.
[5] D. Bresch, B. Desjardins, J.-M. Ghidaglia, E. Grenier, and M. Hillairet, *Multi-fluid Models Including Compressible Fluids*. In: Giga, Y., Novotny, A. (eds.) *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*. Springer International Publishing, Cham (2016).

[6] F. Charve and R. Danchin, *A global existence result for the compressible Navier-Stokes equations in the critical $L^p$ framework*, Arch. Rational Mech. Anal. **198** (2010), 233–271.

[7] F. Charve, R. Danchin and J. Xu, *Gevrey analyticity and decay for the compressible Navier-Stokes system with capillarity*, arXiv:1805.01764v1, 2018, to appear in Indiana Univ. Math. J. (2020).

[8] J.-Y. Chemin, *Perfect incompressible fluids*, Oxford University Press, New York, (1998).

[9] J.-Y. Chemin and N. Lerner, *Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes*, J. Differential Equations, **121** (1992), 314–328.

[10] Q. Chen, C. Miao and Z. Zhang, *Global well-posedness for compressible Navier-Stokes equations with highly oscillating initial velocity*, Comm. Pure Appl. Math. **63** (2010), 1173–1224.

[11] H. Cui, W. Wang, L. Yao and C. Zhu, *Decay rates for a nonconservative compressible generic two-fluid model*, SIAM J. Math. Anal. **48** (2016), 470–512.

[12] R. Danchin, *Global existence in critical spaces for compressible Navier-Stokes equations*, Invent. Math. **141** (2000), 579–614.

[13] B. Haspot, *Existence of global strong solutions in critical spaces for barotropic viscous fluids*, Arch. Ration. Mech. Anal. **202** (2011), 427–460.

[14] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids*, Arch. Ration. Mech. Anal. **139** (1997), 303–354.

[15] R. Danchin and J. Xu, *Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical $L^p$ framework*, Arch. Rational Mech. Anal. **224** (2017), 53–90.

[16] S. Evje, W. Wang and H. Wen, *Global well-posedness and decay rates of strong solutions to a non-conservative compressible two-fluid model*, Arch. Rational Mech. Anal. **221** (2016), 1285–1316.

[17] K. Friedrichs and P. Lax, *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 1686–1688.

[18] S. Godunov, *An interesting class of quasi-linear systems*, Dokl. Akad. Nauk. SSSR. **139** (1961), 521–523 (Russian).
[19] B. Haspot, *Existence of global strong solutions in critical spaces for barotropic viscous fluids*, Arch. Rational Mech. Anal. **202** (2011), 427–460.

[20] T. Hmidi, *Régularité höldérienne des poches de tourbillon visqueuses*, J. Math. Pure Appl. **84** (2005), 1455–1495.

[21] S. Kawashima, Y. Shibata and J. Xu, *The L^p energy methods and decay for the compressible Navier-Stokes equations with capillarity*, J. Math. Pure Appl. **154** (2021), 146–184.

[22] J. Lai, H. Wen and L. Yao, *Vanishing capillarity limit of the non-conservative compressible two-fluid model*, Discrete Continuous Dyn. Syst. Ser. B, **22** (2017), 1361–1392.

[23] Y. Li, H. Wang, G. Wu and Y. Zhang, *Global existence and decay rates for a generic compressible two-fluid model*, arXiv:2108.06973.

[24] A. Matsumura and T. Nishida, *The initial value problems for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ. **20** (1980), 67–104.

[25] M. Ishii and T. Hibiki, *Thermo-fluid Dynamics of Two-Phase Flow*, Springer-Verlag, New York, 2006.

[26] N. Kolev, *Multiphase Flow Dynamics. Vol. 1. Fundamentals*, Springer-Verlag, Berlin, 2005.

[27] N. Kolev, *Multiphase Flow Dynamics. Vol. 2. Thermal and Mechanical Interactions*, Springer-Verlag, Berlin, 2005.

[28] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3 (Berlin: Walter de Gruyter, 1996).

[29] Z. Tan, H. Wang and J. Xu, *Global existence and optimal L^2 decay rate for the strong solutions to the compressible fluid models of Korteweg type*, J. Math. Anal. Appl. **390** (2012), 181–187.

[30] Y. Wang and Z. Tan, *Optimal decay rates for the compressible fluid models of Korteweg type*, J. Math. Anal. Appl. **379** (2011), 256–271.

[31] F. Xu, M. Chi, L. Liu and Y. Wu, *On the well-posedness and decay rates of strong solutions to a multi-dimensional non-conservative viscous compressible two-fluid system*, Discrete and Continuous Dynamical Systems, **40** (2020), 2515–2559.

[32] F. Xu and M. Chi, *The unique global solvability and optimal time decay rates for a multi-dimensional compressible generic two-fluid model with capillarity effects*, Nonlinearity, **34** (2021), 164–204.