A generalization of the boundedness of certain integral operators in variable Lebesgue spaces

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Abstract

Let $A_1, \ldots, A_m$ be a $n \times n$ invertible matrices. Let $0 \leq \alpha < n$ and $0 < \alpha_i < n$ such that $\alpha_1 + \ldots + \alpha_m = n - \alpha$. We define

$$T_{\alpha} f(x) = \int \frac{1}{|x - A_1 y|^\alpha_1 \ldots |x - A_m y|^\alpha_m} f(y) dy.$$ 

In [8] we obtained the boundedness of this operator from $L^{p(.)}(\mathbb{R}^n)$ into $L^{q(.)}(\mathbb{R}^n)$ for $\frac{1}{q(.)} = \frac{1}{p(.)} - \frac{\alpha}{n}$, in the case that $A_i$ is a power of certain fixed matrix $A$ and for exponent functions $p$ satisfying log-Hölder conditions and $p(Ay) = p(y), \ y \in \mathbb{R}^n$. We will show now that the hypothesis on $p$, in certain cases, is necessary for the boundedness of $T_{\alpha}$ and we also prove the result for more general matrices $A_i$.

1 Introduction

Given a measurable function $p(.) : \mathbb{R}^n \to [1, \infty)$, let $L^{p(.)}(\mathbb{R}^n)$ be the Banach space of measurable functions $f$ on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \frac{dx}{\lambda} < \infty,$$

with norm

$$\|f\|_{p(.)} = \inf \left\{ \lambda > 0 : \int \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \frac{dx}{\lambda} \leq 1 \right\}.$$ 

These spaces are known as variable exponent spaces and are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. They have been widely studied lately. See

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for example [1], [3] and [4]. The first step was to determine sufficient conditions on $p(.)$ for the boundedness on $L^{p(.)}$ of the Hardy Littlewood maximal operator

$$Mf(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ containing $x$. Let $p_- = \text{ess inf} \, p(x)$ and let $p_+ = \text{ess sup} \, p(x)$. In [3], D. Cruz Uribe, A. Fiorenza and C. J. Neugebauer proved the following result.

**Theorem 1** Let $p(.) : \mathbb{R}^n \to [1, \infty)$ be such that $1 < p_- \leq p_+ < \infty$. Suppose further that $p(.)$ satisfies

$$|p(x) - p(y)| \leq \frac{c}{-\log |x-y|}, \quad |x-y| < \frac{1}{2}, \quad (1)$$

and

$$|p(x) - p(y)| \leq \frac{c}{\log (e + |x|)}, \quad |y| \geq |x|, \quad (2)$$

Then the Hardy Littlewood maximal operator is bounded on $L^{p(.)} (\mathbb{R}^n)$.

We recall that a weight $\omega$ is a locally integrable and non negative function. The Muckenhoupt class $A_p$, $1 < p < \infty$, is defined as the class of weights $\omega$ such that

$$\sup_Q \left[ \left( \frac{1}{|Q|} \int_Q \omega \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} \right] < \infty,$$

where $Q$ is a cube in $\mathbb{R}^n$.

For $p = 1$, $A_1$ is the class of weights $\omega$ satisfying that there exists $c > 0$ such that

$$\mathcal{M}\omega(x) \leq c\omega(x) \text{ a.e. } x \in \mathbb{R}^n.$$

We denote $[\omega]_{A_1}$ the infimum of the constant $c$ such that $\omega$ satisfies the above inequation.

In [3], B. Muckenhoupt y R.L. Wheeden define $A(p, q)$, $1 < p < \infty$ and $1 < q < \infty$, as the class of weights $\omega$ such that

$$\sup_Q \left[ \left( \frac{1}{|Q|} \int_Q \omega(x)^q \, dx \right)^\frac{1}{q} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p'} \, dx \right)^{\frac{1}{p'}} \right] < \infty.$$  

When $p = 1$, $\omega \in A(1, q)$ if only if

$$\sup_Q \left[ \|\omega^{-1} \chi_Q\|_\infty \left( \frac{1}{|Q|} \int_Q \omega(x)^q \, dx \right)^\frac{1}{q} \right] < \infty.$$  

Let $0 \leq \alpha < n$. For $1 \leq i \leq m$, let $0 < \alpha_i < n$, be such that

$$\alpha_1 + ... + \alpha_m = n - \alpha.$$  

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Let $T_\alpha$ be the integral operator given by

$$T_\alpha f(x) = \int k(x, y) f(y) \, dy$$

where

$$k(x, y) = \frac{1}{|x - A_1 y|^{\alpha_1}} \cdots \frac{1}{|x - A_m y|^{\alpha_m}},$$

and where the matrices $A_i$ are certain invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$.

In the paper [7] the authors studied this kind of integral operators and they obtained weighted $(p, q)$ estimates, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, for weights $w \in A(p, q)$ such that $w(A_i x) \leq cw(x)$. In [8] we use extrapolation techniques to obtain $p(-) - q(-)$ and weak type estimates, in the case where $A_i = A^j$, and $A^N = I$, for some $N \in \mathbb{N}$. This technique allows us to replace the log-Hölder conditions about the exponent $p(-)$ by a more general hypothesis concerning the boundedness of the maximal function $M$. We obtain the following results

**Theorem 2** Let $A$ be an invertible matrix such that $A^N = I$, for some $N \in \mathbb{N}$, let $T_\alpha$ be the integral operator given by (3), where $A_i = A^j$ and such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p : \mathbb{R}^n \to [1, \infty)$ be such that $1 < p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(Ax) = p(x)$ a.e. $x \in \mathbb{R}^n$. Let $q(-)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{n}{\alpha}$. If the maximal operator $M$ is bounded on $L^{\left(\frac{n - \alpha}{p} - q(-)\right)}(\mathbb{R}^n)$ then $T$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

**Theorem 3** Let $A$ be an invertible matrix such that $A^N = I$, for some $N \in \mathbb{N}$, let $T_\alpha$ be the integral operator given by (3), where $A_i = A^j$ and such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p : \mathbb{R}^n \to [1, \infty)$ be such that $1 \leq p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(Ax) = p(x)$ a.e. $x \in \mathbb{R}^n$. Let $q(-)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{n}{\alpha}$. If the maximal operator $M$ is bounded on $L^{\left(\frac{n - \alpha}{p} - q(-)\right)}(\mathbb{R}^n)$ then there exists $c > 0$ such that

$$\left\|t\chi_{\{x: T_\alpha f(x) > t\}}\right\|_{q(-)} \leq c \|f\|_{p(-)}.$$

We also showed that this technique applies in the case when each of the matrices $A_i$ is either a power of an orthogonal matrix $A$ or a power of $A^{-1}$.

In this paper we will prove that these theorems generalize to any invertible matrices $A_1, \ldots, A_m$ such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. We will also show, in some cases, that the condition $p(A_i x) = p(x)$, $x \in \mathbb{R}^n$ is necessary to obtain $p(-) - q(-)$ boundedness.

## 2 Necessary conditions on $p$

Let $A$ be a $n \times n$ invertible matrix and let $0 < \alpha < n$. We define

$$T_A f(x) = \int \frac{1}{|x - Ay|^{n-\alpha}} f(y) dy.$$


Proposition 4 Let $A$ be a $n \times n$ invertible matrix. Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function such that $p$ is continuous at $y_0$ and at $A y_0$ for some $y_0 \in \mathbb{R}^n$. If $p(A y_0) > p(y_0)$ then there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $T_A f \not\in L^{q(\cdot)}(\mathbb{R}^n)$ for 
\[
\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}.
\]

Proof of the proposition 4. Since $p$ is continuous at $y_0$, there exists ball $B = B(y_0, r)$ such that $p(y) \sim p(y_0)$ for $x \in B$. We suppose $p(y_0) < p(A y_0)$. In this case we take 
\[
f(y) = \frac{\chi_B(y)}{|y - y_0|^\beta},
\]
for certain $\beta < \frac{n}{p(y_0)}$ that will be chosen later. We will show that, for certain $\beta$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ but $T_A f \not\in L^{q(\cdot)}(\mathbb{R}^n)$. Indeed,
\[
T_A f(x) = \int \frac{1}{|x - A y|^{n-\alpha}} f(y) dy = \int_B \frac{1}{|x - A y|^{n-\alpha} |y - y_0|^{\beta}} dy,
\]
so 
\[
\int (T_A f(x))^{q(x)} dx = \int \left( \int_B \frac{1}{|x - A y|^{n-\alpha} |y - y_0|^{\beta}} dy \right)^{q(x)} dx 
\geq \int_{B(A y_0, \varepsilon)} \left( \int_B \frac{1}{|x - A y|^{n-\alpha} |y - y_0|^{\beta}} dy \right)^{q(x)} dx
\]
\[
\geq \int_{B(A y_0, \varepsilon)} \left( \int_{B \cap \{ |y - A y_0| < |A y_0 - x| \}} \frac{1}{|x - A y|^{n-\alpha} |y - y_0|^{\beta}} dy \right)^{q(x)} dx
\]
Now, we denote by $M = \|A\| = \sup_{\|y\|=1} |Ay|$. Now for $\varepsilon < M r$ and $x \in B(A y_0, \varepsilon)$,
\[
B(y_0, \frac{1}{M} |A y_0 - x|) \subset B \cap \{ |y - A y_0| < |A y_0 - x| \}.
\]
Indeed, $|y - y_0| \leq \frac{1}{M} |A y_0 - x| \leq \frac{1}{M} \varepsilon \leq r$ and $|A y - A y_0| \leq M |y - y_0| \leq |A y_0 - x|$, so 
\[
\geq \int_{B(A y_0, \varepsilon)} \left( \int_{B(y_0, \frac{1}{M} |A y_0 - x|)} \frac{1}{|x - A y|^{n-\alpha} |y - y_0|^{\beta}} dy \right)^{q(x)} dx,
\]
also, for $y \in B(y_0, \frac{1}{M} |A y_0 - x|)$ 
\[
|x - A y| \leq |x - A y_0| + |A y_0 - A y| \leq |x - A y_0| + M |y_0 - y| \leq 2 |x - A y_0|,
\]
so
\[
\geq \int_{B(A y_0, \varepsilon)} \left( \frac{1}{2^{n-\alpha} |x - A y_0|^{n-\alpha}} \right)^{q(x)} \left( \int_{B(y_0, \frac{1}{M} |A y_0 - x|)} \frac{1}{|y - y_0|^{\beta}} dy \right)^{q(x)} dx
\]
\[
= \int_{B(A y_0, \varepsilon)} \left( \frac{1}{2^{n-\alpha} |x - A y_0|^{n-\alpha}} \right)^{q(x)} (c |A y_0 - x|^{\beta})^{q(x)} dx
\]
= \int_{B(y_0, \varepsilon)} \left( \frac{c}{2^{\beta-\alpha} |x - Ay_0|^{\beta-\alpha}} \right) \gamma(x) \, dx.

Now, since \( q(Ay_0) > q(y_0) \), \( q(Ay_0) - \gamma > q(y_0) \) for \( \gamma = \frac{q(Ay_0) - q(y_0)}{2} \). We observe that if \( \frac{1}{q(y_0)} = \frac{1}{p(y_0)} - \frac{\alpha}{n} \) for \( \beta_0 = n p(y_0) \), \((\beta_0 - \alpha) q(y_0) = \left( \frac{n}{p(y_0)} - \alpha \right) q(y_0) = n \), so since \( q(Ay_0) - \gamma > q(y_0) \), we obtain that \( (\frac{n}{p(y_0)} - \alpha) (q(Ay_0) - \gamma) > n \) and still \( (\beta - \alpha) (q(Ay_0) - \gamma) > n \) for \( \beta = n p(y_0) - \frac{1}{2} \left( \frac{n}{p(y_0)} - \left( \alpha + \frac{n}{q(q(Ay_0) - \gamma)} \right) \right) \). So \( \beta = \frac{n}{p(y_0)} (1 - \delta) \) for some \( \delta > 0 \). Since \( q \) is continuos, we chose \( \varepsilon \) so that, for \( x \in B(Ay_0, \varepsilon) \), \( q(x) > q(Ay_0) - \gamma \). and \( \frac{c}{2^{\beta-\alpha} |x - Ay_0|^{\beta-\alpha}} > 1 \) so this last integral is bounded from below by

\[ \frac{c}{|x - Ay_0|^{\beta-\alpha}} \right) \gamma(x) \, dx = \infty. \]

For this \( \beta \) we chose \( r \) to obtain that the ball \( B = B(y_0, r) \subset \{ y : p(y) < \frac{n}{\alpha} \} \). In this way we obtain that \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) but \( T_A f \notin L^{p(\cdot)}(\mathbb{R}^n) \).

\[ \square \]

**Corollary 5** If \( A^N = I \) for some \( N \in \mathbb{N} \), \( p \) is continuos and \( T_A \) bounded is from \( L^{p(\cdot)} \) into \( L^{q(\cdot)} \), then \( p(Ay) = p(y) \) for all \( y \in \mathbb{R}^n \).

**Proof of the corollary 5.** We suposse that \( p(Ay_0) < p(y_0) \). Since \( p \) is continuos in \( y_0 \), by the last proposition,

\[ p(Ay_0) < p(y_0) = p(A^N y_0) \leq p(A^{N-1} y_0) \leq \ldots \leq p(Ay_0) = p(Ay_0) \]

which is a contradiction. \( \square \)

## 3 The main results

Given \( 0 \leq \alpha < n \), we recall that we are studying fractional type integral operators of the form

\[ T_\alpha f(x) = \int k(x, y) f(y) \, dy, \tag{4} \]

with a kernel

\[ k(x, y) = \frac{1}{|x - A_1 y|^\alpha_1} \frac{1}{|x - A_m y|^\alpha_m} = \frac{1}{|x - A_1 y|^{\alpha_1 - \alpha} |x - A_m y|^{\alpha_m - \alpha}}. \]

\( \alpha_1 + \ldots + \alpha_m = n - \alpha, 0 < \alpha_i < n \).

**Theorem 6** Let \( m \in \mathbb{N} \), let \( A_1, \ldots A_m \) be invertibles matrices such that \( A_i - A_j \) is invertible for \( i \neq j, 1 \leq i, j \leq m \). Let \( T_\alpha \) be the integral operator given by (3), let \( p : \mathbb{R}^n \rightarrow [1, \infty) \) be such that \( 1 < p_- \leq p_+ < \frac{n}{\alpha} \) and such that \( p(A, x) = p(x) \)
a.e. $x \in \mathbb{R}^n$, $1 \leq i \leq m$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} \frac{1}{q(x)} = \frac{a}{n}$. If the maximal operator $M$ is bounded on $L^{\left(\frac{n-a}{np-\alpha q(\cdot)}\right)}$ then $T_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

**Theorem 7** Let $m \in \mathbb{N}$, let $A_1, ..., A_m$ be invertibles matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $T_\alpha$ be the integral operator given by (3), let $p : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 \leq p_- \leq p_+ < \infty$ and such that $p(A_i x) = p(x)$ a.e. $x \in \mathbb{R}^n$, $1 \leq i \leq m$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} \frac{1}{q(x)} = \frac{a}{n}$. If the maximal operator $M$ is bounded on $L^{\left(\frac{n-a}{np-\alpha q(\cdot)}\right)}$ then there exists $c > 0$ such that

$$\|t\chi_{\{x: T_\alpha f(x) > t\}}\|_{q(i)} \leq c \|f\|_{p(i)} ;$$

**Lemma 8** If $f \in L^1_{loc}(\mathbb{R}^n)$ and $A$ an invertible $n \times n$ matrix then

$$M(f \circ A)(x) \leq c(M(f) \circ A)(x).$$

**Proof of the Lemma 8.** Indeed, $M(f \circ A) = \sup_B \frac{1}{|B|} \int_B |(f \circ A)(y)| dy$, where the supremum is taken over all balls $B$ containing $x$. By a change of variable we see that,

$$\frac{1}{|B|} \int_B |(f \circ A)(y)| dy = |det(A^{-1})| \frac{1}{|B|} \int_{A(B)} |f(z)| dz,$$

where $A(B) = \{Ay : y \in B\}$. Now, if $y \in B = B(x_0, r)$ then $|Ay - Ax_0| \leq M|x - x_0| \leq Mr$, where $M = \|A\|$. That is $Ay \in \tilde{B} = B(Ax_0, Mr)$. So

$$\leq \frac{M^n |det(A^{-1})|}{|B|} \int_{\tilde{B}} f(z) dz$$

$$\leq M^n |det(A^{-1})| Mf(Ax).$$

Therefore we obtain that,

$$M(f \circ A) \leq c(M(f) \circ A),$$

with $c = M^n |det(A^{-1})|$. ■

**4 Proofs of the main results**

**Proof of theorem 6.** In the paper [7] the authors obtain an estimate of the form

$$\int (T_\alpha f)^p (x) w(x) dx \leq c \sum_{j=1}^{m} (M_{\alpha} f)^p (x) w(A_j x) dx,$$

(5)
for any \( w \in A_\infty \) and \( 0 < p < \infty \) (See the last lines of page 454 in [7]). We denote \( \tilde{q}(.) = \frac{q(1)}{q_0} \), we define an iteration algorithm on \( L^{\tilde{q}(.)'} \) by

\[
R h(x) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h(x)}{2^k ||\mathcal{M}||^k_{\tilde{q}(.)'}},
\]

where, for \( k \geq 1 \), \( \mathcal{M}^k \) denotes \( k \) iteration of the maximal operator \( \mathcal{M} \) and \( \mathcal{M}^0 (h) = |h| \). We will check that

a) For all \( x \in \mathbb{R}^n \), \( |h(x)| \leq R h(x) \),

b) For all \( j : 1, ..., m \), \( ||R h \circ A_j||_{\tilde{q}(.)'} \leq c ||h||_{\tilde{q}(.)'} \),

c) For all \( j : 1, ..., m \), \( (R h \circ A_j) \in A(p_-, q_0) \).

Indeed, a) is evident from the definition b) is verified by the following,

\[
||R h \circ A_j||_{\tilde{q}(.)'} \leq \sum_{k=0}^{\infty} \frac{||\mathcal{M}^k h \circ A_j||_{\tilde{q}(.)'}}{2^k ||\mathcal{M}||^k_{\tilde{q}(.)'}}
\]

and

\[
||\mathcal{M}^k h \circ A_j||_{\tilde{q}(.)'} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(A_j x)}{\lambda} \right)^{\tilde{q}(x)'} dx \leq 1 \right\}
\]

But, by a change of variable and using the hypothesis on the exponent,

\[
\int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(A_j x)}{\lambda} \right)^{\tilde{q}(x)'} dx = |det(A_j^{-1})| \int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(A_j^{-1} y)'} dy,
\]

put \( D = \max \{|det(A_j^{-1})| \}, j = 1...m \),

\[
\leq D \int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(y)'} dy
\]

(7)

If \( D \leq 1 \),

\[
||\mathcal{M}^k h \circ A_j||_{\tilde{q}(.)'} \leq ||\mathcal{M}^k h||_{\tilde{q}(.)'}
\]

So,

\[
||R h \circ A_j||_{\tilde{q}(.)'} \leq \sum_{k=0}^{\infty} \frac{||\mathcal{M}^k h(x)||_{\tilde{q}(.)'}}{2^k ||\mathcal{M}||^k_{\tilde{q}(.)'}} \leq ||h||_{\tilde{q}(.)'} \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 ||h||_{\tilde{q}(.)'}
\]

If \( D > 1 \) then from (7) it is follows that

\[
D \int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(y)'} dy = \int_{\mathbb{R}^n} \left( \frac{\mathcal{M}^k h(y)}{\lambda C^{\tilde{q}(.)'}} \right)^{\tilde{q}(.)'} dy
\]

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and $D = \frac{1}{\nu}$ where $C = \min\{|\det(A_j)|, j = 1 \ldots m\}$. So,

$$\leq \int_{\mathbb{R}^n} \left( \frac{M^k h(y)}{\lambda C^{(q)}_{-}} \right) \tilde{q}(\cdot)' \, dy$$

That is,

$$\int_{\mathbb{R}^n} \left( \frac{M^k h(A_j x)}{\lambda} \right) \tilde{q}(x)' \, dx \leq \int_{\mathbb{R}^n} \left( \frac{M^k h(x)}{\lambda C^{(q)}_{-}} \right) \tilde{q}(\cdot)' \, dx.$$

From this last inequality it follows that

$$\|M^k h \circ A_j\|_{\tilde{q}(\cdot)'} \leq D^{q^-} \|M^k h\|_{\tilde{q}(\cdot)'}$$

and so b) is verified with $c = 2D^{q^-}$.

To see c), by Lemma 8,

$$\mathcal{M}(\mathcal{R} \frac{1}{\nu} \circ A_j)(x) \leq c \mathcal{M}(\mathcal{R} \frac{1}{\nu})(A_j x)$$

$\mathcal{R} h \in A_1$ (see [2]) implies that $\mathcal{R} \frac{1}{\nu} \in A_1$ and so,

$$\leq c \mathcal{R} \frac{1}{\nu}(A_j x) = c(\mathcal{R} \frac{1}{\nu} \circ A_j)(x).$$

Then c) follows since a weight $\omega \in A_1$ implies that $\omega \in A(p_{-}, q_0)$. We now take a bounded function $f$ with compact support. We will check later that $\|T_{\alpha}f\|_{\tilde{q}(\cdot)} < \infty$, so as in Theorem 5.24 in [2],

$$\|T_{\alpha}f\|_{\tilde{q}(\cdot)} = \|(T_{\alpha}f)^{q_0}\|_{\tilde{q}(\cdot)} = C \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} \int (T_{\alpha}f)^{q_0}(x)h(x)dx$$

$$\leq C \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} \int (T_{\alpha}f)^{q_0}(x)\mathcal{R}h(x)dx \leq C \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} \sum_{j=1}^{m} \int (M_{\alpha}f)^{q_0}(x)\mathcal{R}h(A_j x)dx,$$

$$\leq C \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} \sum_{j=1}^{m} \left( \int |f(x)|^{p_{-}} \mathcal{R}h_{\frac{p_{-}}{q_0}}(A_j x)dx \right)^{\frac{q_0}{p_{-}}}$$

where the last inequality follows since $\mathcal{R} \frac{1}{\nu} \circ A_j$ are weights in $A(p_{-}, q_0)$ (by c). We denote by $\bar{\nu}(\cdot) = \frac{1}{\nu_{\alpha}}$. Holder’s inequality, 2) and Proposition 2.18 in [2] and again the hypothesis about $A_1$ and $p$ give

$$\|T_{\alpha}f\|_{q_0}^{q_0} \leq C \|f^{p_{-}}\|_{\bar{\nu}(\cdot)} \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} \sum_{j=1}^{m} \left( \mathcal{R}h_{\frac{p_{-}}{q_0}} \circ A_j \right)_{\tilde{q}(\cdot)}^{\frac{q_0}{p_{-}}}$$

$$\leq \sup_{\|h\|_{\tilde{q}(\cdot)} = 1} C m \|f^{q_0}\|_{q_0} \|h\|_{q_0} \cdot \leq C \|f\|_{p_{-}}^{q_0}.$$
Now we show that \( \|T_\alpha f\|_{q(\cdot)} < \infty \). By Prop. 2.12, p.19 in [2], it is enough to check that \( \rho_{\bar{q}(\cdot)}(T_\alpha f) < \infty \).

\[
|T_\alpha f(x)|^{q(x)} \leq |T_\alpha f(x)|^{q^+} \chi(x:T_\alpha f(x) > 1) + |T_\alpha f(x)|^{q^-} \chi(x:T_\alpha f(x) \leq 1),
\]

now \( f \) is bounded and with compact support, so \( T_\alpha f \in L^s(\mathbb{R}^n) \) for \( \frac{1}{s} < \frac{1}{n} \), (see Lemma 2.2 in [7]) thus \( \int |T_\alpha f(x)|^{q(x)} \, dx < \infty \). The theorem follows since bounded functions with compact support are dense in \( L^{p(\cdot)}(\mathbb{R}^n) \) (See Corollary 2.73 in [2]).

**Proof of theorem 7.** We observe that it is enough to check (7) for \( f \in L^\infty_c(\mathbb{R}^n) \). In [7] (See page 459) the authors prove that there exists \( c > 0 \) such that,

\[
\sup_{\lambda > 0} \lambda \left( \omega^{q_0} \{ x : |T_\alpha f(x) | > \lambda \} \right)^{\frac{1}{q_0}} \leq \sup_{\lambda > 0} \left( \omega^{q_0} \{ x : \sum_{i=1}^m A_i f(A_i^{-1} x) > c\lambda \} \right)^{\frac{1}{q_0}}
\]

for all \( \omega \in \mathcal{A}_\infty \) and \( f \in L^\infty_c(\mathbb{R}^n) \).

Let \( F_\lambda = \lambda^{q_0} \chi(\{ x : |T_\alpha f(x) | > \lambda \} \) the last inequality implies that,

\[
\int_{\mathbb{R}^n} F_\lambda(x) \omega(x)^{q_0} \, dx \leq \sup_{\lambda > 0} \int_{\mathbb{R}^n} \lambda^{q_0} \chi(\{ x : \sum_{i=1}^m A_i f(A_i^{-1} x) > c\lambda \} \omega(x)^{q_0} \, dx
\]

for some \( c > 0 \) and for all \( \omega \in \mathcal{A}_\infty \). Now by proposition 2.18 in [2], if \( \bar{q}(\cdot) = \frac{q(\cdot)}{q_0} \),

\[
\| \lambda \chi(\{ x : |T_\alpha f(x) | > \lambda \} \|_{\bar{q}(\cdot)} = \| \lambda^{q_0} \chi(\{ x : \sum_{i=1}^m A_i f(A_i^{-1} x) > c\lambda \} \|_{\bar{q}(\cdot)}
\]

\[
= \| F_\lambda \|_{\bar{q}(\cdot)} \leq c \sup_{\| h \|_{\bar{q}(\cdot)} = 1} \int_{\mathbb{R}^n} F_\lambda(x) h(x) \, dx,
\]

Let \( \mathcal{R}h \) be defined by (6). We can verify that,

a) \( |h(x)| \leq \mathcal{R}h(x) \) \( x \in \mathbb{R}^n \);

b) For all \( j : 1, \ldots, m, \| \mathcal{R}h \circ A_j \|_{\bar{q}(\cdot)'} \leq c \| h \|_{\bar{q}(\cdot)'} ;

\]

c) For all \( j : 1, \ldots, m, \mathcal{R}h^{\frac{1}{q_0}} \circ A_j \in \mathcal{A}(p_-, q_0) \)

and so,

\[
\leq c \sup_{\| h \|_{\bar{q}(\cdot)} = 1} \int_{\mathbb{R}^n} F_\lambda(x) \mathcal{R}h(x) \, dx = c \sup_{\| h \|_{\bar{q}(\cdot)} = 1} \int_{\mathbb{R}^n} F_\lambda(x) (\mathcal{R} h^{\frac{1}{q_0}}(x))^{q_0} \, dx,
\]

and by (8), since \( \mathcal{R}h^{\frac{1}{q_0}} \in \mathcal{A}(p_-, q_0) \) and \( \mathcal{R}h \in \mathcal{A}_1 \subset \mathcal{A}_\infty \),

\[
\leq c \sup_{\| h \|_{\bar{q}(\cdot)} = 1} \sup_{\lambda > 0} \int_{\mathbb{R}^n} \lambda^{q_0} \chi(\{ x : \sum_{i=1}^m A_i f(A_i^{-1} x) > c\lambda \} (\mathcal{R} h^{\frac{1}{q_0}}(x))^{q_0} \, dx
\]
Since,
\[ \left\{ x : \sum_{i=1}^{m} M_\alpha f(A_i^{-1}x) > c\lambda \right\} \subseteq \bigcup_{i=1}^{m} \left\{ x : M_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m} \right\} \]
then,
\[ \chi\left\{ x : \sum_{i=1}^{m} M_\alpha f(A_i^{-1}x) > c\lambda \right\} \leq m \sum_{i=1}^{m} \chi\left\{ x : M_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m} \right\}, \]
so
\[ \leq c \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \int_{\mathbb{R}^n} \lambda^{q_0} \chi\left( \{ x : M_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m} \} \right) (R h \frac{1}{\psi'}(x))^{q_0} dx \]
\[ = \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \int_{\{ x : M_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m} \}} \lambda^{q_0} (R h \frac{1}{\psi'}(x))^{q_0} dx \]
\[ = \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \int_{\{ y : M_\alpha f(A_i^{-1}y) > \frac{c\lambda}{m} \}} \lambda^{q_0} (R h \frac{1}{\psi'}(A_i^0 y))^{q_0} dy, \]
\[ \leq \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \int_{\{ y : M_\alpha f(y) > \frac{c\lambda}{m} \}} \lambda^{q_0} (R h \frac{1}{\psi'}(A_i^0 y))^{q_0} dy, \]
\[ \leq \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \left( \int_{\mathbb{R}^n} |f(y)|^{p-} (R h \frac{p^{-}}{\psi'}(A_i^0 y)) dy \right)^{q_0}, \]
\[ = \sup_{\|h\|_{\psi'(\cdot)}} \sum_{i=1}^{m} \left( \int_{\mathbb{R}^n} |f(y)|^{p-} (R h \frac{p^{-}}{\psi'}(A_i^0 y)) dy \right)^{q_0}, \]
where the last inequality follows since \( R h \frac{1}{\psi'} \circ A_i \in A(p-, q_0) \) for all \( i = 1 \ldots m \).

Now we follow as in the proof of Theorem 6 to obtain
\[ \leq c \|f\|_{\psi_p(\cdot)}^{q_0}. \]

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