Worldtube conservation laws for the null-timelike evolution problem

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I treat the worldtube constraints which arise in the null-timelike initial-boundary value problem for the Bondi-Sachs formulation of Einstein’s equations. Boundary data on a worldtube and initial data on an outgoing null hypersurface determine the exterior spacetime by integration along the outgoing null geodesics. The worldtube constraints are a set of conservation laws which impose conditions on the integration constants. I show how these constraints lead to a well-posed initial value problem governing the extrinsic curvature of the worldtube, whose components are related to the integration constants. Possible applications to gravitational waveform extraction and to the well-posedness of the null-timelike initial-boundary value problem are discussed.

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I. INTRODUCTION

It is extremely gratifying to contribute this article in appreciation of Josh Goldberg’s friendship and guidance, especially because this opportunity to recall Josh’s early work has led me to an interesting approach to a current problem. Josh and I first overlapped in 1963 when he came to Syracuse University as a new Professor. At that time I was very busy finishing my PhD thesis and our interaction was fortuitous in more ways than one. Not only did we share the same research interests but the next year, when I was looking for my first position, Josh recommended me to the Aerospace Research Laboratory at Wright Patterson Air Force Base, where in his prior position he had organized a general relativity research group which included such budding young relativists as Roy Kerr. It was curious to me that the Air Force sponsored research in a topic with no apparent military relevance. Several years later, in a cost-cutting measure, Congress also came to this curious realization and the general relativity group was disbanded. However, during the intervening years the lab was a Camelot for basic research, much to the credit of Josh’s legacy. It was there that I came to meet and work with the organizers of this volume, David Robinson and Ed Glass, two PhD students of Josh who came to the Lab on National Academy of Science postdoctoral fellowships.

Much of Josh’s research centered around conservation laws and null hypersurfaces \cite{1,2,3}. In the 1960’s, these two topics were the focus of some of the most exciting results in gravitational theory. They came together in Bondi’s \cite{4} and Sachs’s \cite{5} treatment of Einstein’s equations via the characteristic initial value problem, which led to the formulation of the Bondi mass $M_B$ and news function $N$ as the quantities of prime physical importance for an isolated system. The conservation law

$$
\frac{dM_B}{du} = - \oint |N|^2 \sin \theta d\theta d\phi,
$$

which relates the retarded time derivative of the Bondi mass to the integral of the news function over the sphere at future null infinity, was the conclusive theoretical evidence that ended any serious debate over whether gravitational waves carry energy from a system. The conformal compactification of future null infinity into the boundary $\mathcal{I}^+$ of an asymptotically flat spacetime put this mass loss relation into a well-defined geometrical setting \cite{6}. One of Josh’s little known contributions to these results was the grant support of Bondi’s King College group, which he arranged through the Lab.

The Bondi mass loss equation is a result of certain constraints that arise in the characteristic formulation of Einstein’s equations. The conservation law (1.1) is obtained by applying these constraints at $\mathcal{I}^+$. In this paper, I will discuss the content of these constraints when applied on a timelike worldtube of finite size but surrounding the matter sources of the gravitational field, as arises in the null-timelike initial-boundary value problem for the gravitational field. There is much overlap between my results presented here and Josh’s work \cite{2} which supplies the basic ideas for interpreting these constraints as conservation laws. Josh applied these conservation laws to the theory of equations of motion for isolated systems being developed by Newman and his students \cite{7,8}. Here I apply these conservation laws to a problem in numerical relativity that did not exist at that time. It is fitting, when I had a chance to talk to...
Josh at a recent meeting on mathematical relativity, that he commented (essentially) “these are the same problems that we tried to solve in the 1960’s”. That is true. The importance of those problems was recognized back then and only recently have many of them been elevated to a high level status by the mathematicians; and their importance to numerical simulation has also been recognized. But they are the same problems!

In the Cauchy problem, initial data on a spacelike hypersurface \( S_0 \) are extended to a solution in the domain of dependence \( D(S_0) \) (which consists of those points whose past directed characteristics all intersect \( S_0 \)). In the initial-boundary value problem (IBVP), data on a timelike boundary \( T \) transverse to \( S_0 \) is used to further extend the solution to the domain of dependence \( D(S_0 \cup T) \).

The IBVP for Einstein’s equations only recently received widespread attention due to its importance to numerical relativity \([4, 5]\), where the introduction of a finite artificial outer boundary is standard practice. It is essential for numerical evolution that the underlying analytic problem be well-posed, i.e. that the solution depend continuously on the data so that it is not destabilized by numerical error. The first well-posed IBVP was achieved for a formulation based upon a tetrad, connection and curvature as evolution fields \([10]\) and subsequently for the harmonic formulation of Einstein’s equations \([11]\). However, much of the work in numerical relativity is based upon other formulations where the well-posedness of the IBVP remains an unresolved issue, including the null-timelike formulation. The properties of the worldtube constraints treated here is important for a clearer understanding of this characteristic IBVP.

I begin with a short review of the null-timelike IBVP in Sec. \( \text{II} \). In the traditional approach to the Cauchy problem for Einstein’s equations, initial data on a spacelike hypersurface with unit timelike normal \( n_\mu \) are formulated in a purely 3-dimensional form in terms of the intrinsic metric \( h_{\mu\nu} \) and extrinsic curvature \( k_{\mu\nu} \) of the initial Cauchy hypersurface. The components \( G^{\mu\nu} n_\mu \) of the Einstein tensor contain only first time-derivatives of the metric so that they constrain this initial data, i.e. the Hamiltonian and momentum constraints

\[
0 = (3)R + (k_\mu^2 - k_{\mu\nu}k^{\mu\nu}) = 2G_{\mu\nu}n^\mu n^\nu \tag{1.2}
\]

and

\[
0 = (3)\nabla_\mu(k_\nu^\mu - \delta_\nu^\mu k_\rho^\rho) = h_{\nu\rho}G_{\mu\rho}n^\mu, \tag{1.3}
\]

where \( (3)\nabla_\mu \) is the covariant derivative and \( (3)R \) is the curvature scalar associated with \( h_{\mu\nu} \). Subject to these constraints, the Cauchy data determine a solution of Einstein’s equations which is unique up to a diffeomorphism (cf. \([12]\)).

These constraints are elliptic partial differential equations which couple their solution with data on the boundary of the initial Cauchy hypersurface. Once these constraints have been satisfied initially, the dynamical equations used in a consistent hyperbolic reduction of Einstein’s equations ensure that they continue to be satisfied in time in the domain of dependence of the Cauchy problem. Boundary conditions must then be formulated to preserve these constraints in the larger domain of dependence of the IBVP problem. That is the strategy which lies behind the constraint-free Cauchy evolution schemes used in numerical relativity. In characteristic versions of the IBVP, data are given on an initial null hypersurface \( u = \text{const} \). This problem has the peculiar feature that the normal co-vector \( \nabla_\mu u \) is tangent to the hypersurface when interpreted as a vector \( k_\nu = g^{\mu\nu}\nabla_\mu u \) by the standard technique of “raising indices” with the metric. As a result, the corresponding Hamiltonian and momentum constraints reduce to propagation laws consisting of ordinary differential equations (ODE’s) along the characteristics of the null hypersurface. In the null-timelike version of the characteristic IBVP the worldtube integration constants for these ODE’s uniquely determine the exterior spacetime. In Sec. \( \text{II} \) we review these ODE’s and their integration constants.

**II. THE WORLDTUBE-NULLCONE PROBLEM**

**A. The null cone formalism**

We use coordinates based upon a family of outgoing null hypersurfaces. We let \( u \) label these hypersurfaces, \( x^A \) \((A = 2, 3)\) be labels for the null rays and \( r \) be a surface area coordinate. In the resulting \( x^\alpha = (u, r, x^A) \) coordinates, the metric takes the Bondi-Sachs form

\[
ds^2 = -(e^{2\beta} \frac{V}{r} - r^2 h_{AB} U^A U^B) \, du^2 - 2e^{2\beta} \, du \, dr - 2r^2 h_{AB} U^A \, du \, dx^A + r^2 h_{AB} \, dx^A \, dx^B, \tag{2.1}
\]

where \( \det(h_{AB}) = \det(q_{AB}) = q \), with \( q_{AB} \) a unit sphere metric. The contravariant components are

\[
g^{rr} = e^{-2\beta} \frac{V}{r}. \tag{2.2}
\]
to the components \( m \), which points in the angular direction with components \( h \), where

\[
g^A_r = -\epsilon^{-2\beta} U^A \tag{2.3}
g^u = -\epsilon^{-2\beta} \tag{2.4}
g^{AB} = \epsilon^{-2} h^{AB}, \tag{2.5}
\]

where \( h^{AB} h_{BC} = \delta^A_C \).

The Einstein equations \( G_{\mu\nu} = 0 \) decompose into hypersurface equations, evolution equations and conservation laws. We express these equations following the formalism in [13, 14]. The hypersurface equations correspond to the components \( G^\mu_{\nu} \nabla_{\nu} u = 0 \) and take the specific form

\[
\beta_r = \frac{1}{16} r h^{AC} h^{BD} h_{AB,r} h_{CD,r} \tag{2.6}
\]

\[
(\epsilon^4 \epsilon^{-2\beta} h_{AB} U^B_{,r})_r = 2(\epsilon^{-2\beta} A)_r - \epsilon^2 h^{BC} D C h_{AB,r} \tag{2.7}
\]

\[
2\epsilon^{-2\beta} V_r = \mathcal{R} - 2 D^A D_A \beta - 2 D^A \beta D_A \beta + \epsilon^{-2\beta} D_A (\epsilon^4 U_A) - \frac{1}{2} \epsilon^4 \epsilon^{-2\beta} h_{AB} U_A U^B_r, \tag{2.8}
\]

where \( D_A \) is the covariant derivative and \( \mathcal{R} \) the curvature scalar of the 2-metric \( h_{AB} \).

The evolution equations can be picked out by introducing a complex polarization dyad \( m^\mu \) satisfying \( m^\mu \nabla_\mu u = 0 \) which points in the angular direction with components \( m^\mu = (0, 0, m^A) \) satisfying \( m^A \bar{m}^B = h^{AB} \), so that \( h_{AB} m^A \bar{m}^B = 2 \), which determines \( m^A \) up to the phase freedom \( m^A \rightarrow e^{i\pi m^A} \). The evolution equations correspond to the components \( m^\mu m^\nu G_{\mu\nu} = 0 \) and take the form

\[
m^A m^B \left\{ r (r h_{AB,u})_r - \frac{1}{2} (r V h_{AB})_r - 2\epsilon^2 (D_A D_B \beta + D_A \beta D_B \beta) + h_{CD}(A B) (r^2 U^C)_r - \frac{1}{2} \epsilon^4 \epsilon^{-2\beta} h_{AB} h_{CD} U^C U^D_{,r} + \frac{1}{2} r^2 h_{AB,r} D C U^C + r^2 U^C D C h_{AB,r} - D^C U_{(B A) C,r} + D_{(B C) h_{A} C,r} \right\} = 0. \tag{2.9}
\]

If we introduce the auxiliary variable \( Q_A = r^2 \epsilon^{-2\beta} h_{AB} U^B_{,r} \), the system of hypersurface and evolution equations can be cast into a hierarchy of first order ODE’s,

\[
\beta_r = \mathcal{N}_\beta(h_{CD}) \tag{2.10}
\]

\[
(\epsilon^2 Q_A)_r = \mathcal{N}_Q(h_{CD}, \beta) \tag{2.11}
\]

\[
U^A_r = \mathcal{N}_U(h_{CD}, \beta, Q_C) \tag{2.12}
\]

\[
V_r = \mathcal{N}_V(h_{CD}, \beta, Q_C, U^C) \tag{2.13}
\]

\[
m^A m^B (r h_{AB,u})_r = \mathcal{N}_h(h_{CD}, \beta, Q_C, U^C, V), \tag{2.14}
\]

where the \( \mathcal{N} \)-terms on the right hand side can be calculated from the values of their arguments on a given \( u = \text{const.} \) null hypersurface. Each \( \mathcal{N} \)-term only depends upon previous members of the hierarchy in the order \( \mathcal{N}_\beta, \mathcal{N}_Q, \mathcal{N}_U, \mathcal{N}_V \). Thus, given \( h_{AB} \) on the null hypersurface \( u = \text{const.} \), these equations can be integrated radially, in sequential order, to determine \( \beta, Q_A, U^A, V \) and \( m^A m^B h_{AB,u} \) on the hypersurface in terms of integration constants on an inner worldtube. For an inner worldtube given by \( r = R(u, x^A) \), the necessary integration constants are

\[
\beta \big|_R, \quad Q_A \big|_R, \quad U^A \big|_R, \quad V \big|_R, \quad m^A m^B h_{AB,u} \big|_R. \tag{2.15}
\]

In addition, the location of the worldtube specified by \( R(u, x^A) \) is another essential part of the data.

### III. THE BOUNDARY CONSTRAINTS AS CONSERVATION LAWS

The components of Einstein’s equations independent of the hypersurface and evolution equations are the conservation conditions (called supplementary conditions by Bondi and Sachs)

\[
h^{AB} G_{AB} = 0 \tag{3.1}
\]

\[
G^r_r = 0 \tag{3.2}
\]

\[
G^u = 0. \tag{3.3}
\]
As was shown by Bondi and Sachs, the Bianchi identity
\[ \nabla_\mu G^{\mu}_\nu = \frac{1}{\sqrt{-g}} (\sqrt{-g} G^{\mu}_\nu)_{,\mu} + \frac{1}{2} g^{\rho\sigma} G_{\rho\sigma} = 0 \] (3.4)
implies that these equations need only be satisfied on a worldtube \( r = R(u, x^A) \). When the hypersurface and evolution equations are satisfied, the Bianchi identity for \( \nu = r \) reduces to \( h^{AB} G_{AB} = 0 \) so that (3.4) becomes trivially satisfied. (Here it is necessary that the worldtube have nonvanishing expansion so that the areal radius \( r \) is a non-singular coordinate.) The Bianchi identity for \( \nu = A \) then reduces to
\[ (r^2 G^r_A)_{,r} = 0, \] (3.5)
so that \( G^r_A = 0 \) if it is set to zero at \( r = R(u, x^A) \). When that is the case, the Bianchi identity for \( \nu = u \) then reduces to
\[ (r^2 G^r_u)_{,r} = 0, \] (3.6)
so that \( G^r_u = 0 \) also vanishes if it vanishes for \( r = R(u, x^A) \).

As a result, the conservation conditions can be replaced by the condition that the Einstein tensor satisfy
\[ \xi^\mu G^{\mu}_\nu \nabla_\nu r = 0, \] (3.7)
where \( \xi^\mu \) is any vector field tangent to the worldtube. This allows these conditions to be interpreted as flux conservation laws for the \( \xi \)-momentum contained in the worldtube [15]. The unit normal to the worldtube \( N_\mu \) lies in the direction \( \nabla_\mu (r - R(u, x^A)) \), i.e.
\[ N_\mu = \eta \nabla_\mu (r - R(u, x^A)) \] (3.8)
where \( \eta \) is a normalization constant. Since we are assuming that the hypersurface equations are satisfied, we can replace (3.7) by
\[ \xi^\mu G^{\mu}_\nu N_\nu = 0. \] (3.9)
These are the boundary analogue of the momentum constraints (1.3) for the Cauchy problem. In a treatment of a timelike boundary for the Cauchy problem, it has been pointed out [9] that the Cauchy momentum constraints (1.3) must be enforced on the boundary. Here, in the characteristic initial-boundary value problem, it is the boundary-momentum constraints (3.9) which must be enforced.

Since \( \xi^\mu N_\mu = 0 \), we can further replace (3.9) by the condition on the Ricci tensor
\[ \xi^\mu R^{\mu}_\nu N_\nu = 0. \] (3.10)
The Ricci identity
\[ \xi^\mu R^{\mu}_\nu = \nabla_\mu \nabla^{(\nu \xi^\mu)} + \nabla_\mu \nabla^{[\nu \xi^\mu]} - \nabla^{\nu} \nabla_\mu \xi^\mu \] (3.11)
then gives rise to the strict Komar conservation law
\[ \nabla_\mu \nabla^{(\nu \xi^\mu)} = 0 \] (3.12)
when \( \xi^\mu \) is a Killing vector corresponding to an exact symmetry. More generally, (3.11) gives rise to the flux conservation law
\[ P_\xi (u_2) - P_\xi (u_1) = \int_{u_1}^{u_2} dS_\nu \{ \nabla^{\nu} \nabla_\mu \xi^\mu - \nabla^{\mu} \nabla_\nu \xi^\mu \} \] (3.13)
where
\[ P_\xi = \oint dS_{\mu\nu} \nabla^{[\nu \xi^\mu]} \] (3.14)
and \( dS_{\mu\nu} \) and \( dS_\nu \) are, respectively, the appropriate surface and 3-volume elements on the worldtube. For the limiting case when \( R \to \infty \), these flux conservation laws govern the energy-momentum, angular momentum and
supermomentum corresponding to the generators of the Bondi-Metzner-Sachs asymptotic symmetry group \([15]\). For an asymptotic time translation, they give rise to the Bondi mass loss relation \([11]\).

Josh applied these conservation laws to a new treatment of equations of motions in general relativity \([3]\). Here I pursue a different application to the mathematical basis of the worldtube constraints for the null-timelike IBVP. For this purpose it is useful to rewrite the conservation laws \((3.10)\) in terms of the intrinsic metric and extrinsic curvature of the worldtube.

The intrinsic metric of a worldtube embedded in the spacetime with unit spacelike normal \(N_{\mu}\) is

\[
H_{\mu\nu} = g_{\mu\nu} - N_{\mu}N_{\nu} \tag{3.15}
\]

and its extrinsic curvature is

\[
K_{\mu\nu} = H_{\mu}^{\rho}\nabla_{\rho}N_{\nu}. \tag{3.16}
\]

We then have the worldtube analogue of \((1.3)\) for the Cauchy problem,

\[
0 = D_\mu(K^\rho_{\mu} - 2\delta^\rho_{\mu}R^\rho_{\nu}) = H^\rho_{\mu}G_{\mu\nu}N^\nu, \tag{3.17}
\]

where \(D_\mu\) is the covariant derivative associated with \(H_{\mu\nu}\). These are equivalent to the conservation conditions \((3.10)\) and allow the conserved quantities to be expressed in terms of the extrinsic curvature of the boundary. For any vector field \(\xi^\mu\) tangent to the worldtube, \((3.17)\) implies

\[
D_\mu(\xi^\nu K^\rho_{\nu} - \xi^\nu R^\rho_{\nu}) = D^\mu(\xi^\nu)(K^\rho_{\nu} - H_{\mu\nu}R^\rho_{\nu}). \tag{3.18}
\]

In particular, this show that \(\xi^\mu\) need only be a Killing vector for the 3-metric \(H_{\mu\nu}\) to obtain a strict conservation law on the boundary.

\section{IV. THE WELL-POSEDNESS OF THE BOUNDARY CONSTRAINT PROBLEM}

The conservation conditions \((3.17)\) constrain the boundary data for the nullcone-worldtube problem. I now show that these constraints can be formulated in terms of a well-posed initial value problem intrinsic to the worldtube. The traditional 3 + 1 decomposition of spacetime used in the Cauchy formalism is not applicable to the nullcone-worldtube problem since a foliation by hypersurfaces with a null normal gives rise to a degenerate 3-metric. However, an analogous 2 + 1 decomposition can be made on a timelike worldtube. Let vector and tensor fields intrinsic to the worldtube be denoted by \(v^a\), etc.

Let \(t^a\) be an evolution vector field on the worldtube, i.e. the flow of \(t^a\) carries an initial spacelike cross section \(t = 0\) into a \(t\)-foliation of the worldtube, with \(L_a t = 1\). Coordinates \(y^A\) of the initial slice then induce adapted coordinates \(y^a = (t, y^A)\) on the worldtube by requiring \(L_{t^a}y^A = 0\). Thus the choice of \(t^a\) and initial coordinates \(y^A\) fix the gauge freedom on the worldtube.

The intrinsic 3-metric \(H_{ab}\) of the worldtube has the further 2 + 1 decomposition

\[
H_{ab} = -T_a T_b + R^2 h_{ab}, \quad H^{ab} = -T^a T^b + R^{-2} h^{ab}. \tag{4.1}
\]

where \(T^a\) is the future timelike unit normal to the \(t\)-foliation and \(R^2 h_{ab}\) is the intrinsic 2-metric of the \(t = \text{const}\) slices, with \(h^{AB}h_{BC} = \delta^A_C\). Here, for later application to the characteristic problem, We have introduced the surface area factor \(R(t, y^A)\), so that \(\det(h_{AB}) = \det(Q_{AB}) = Q\), where \(Q_{AB}\) is a unit sphere metric. For convenience, we chose stereographic coordinates \(y^A = (q, p)\) for the unit sphere metric for which \(Q_{AB}dq^A dq^B = Q^{1/2}(dq^2 + dp^2)\) with \(Q^{1/2} = 4/(1 + q^2 + p^2)^2\). Since \(h^{AB}\) has \((++)\) signature, it can be put in the matrix form

\[
h^{AB} = Q^{-1/2} \begin{pmatrix} 
  e^{-2\gamma} \cosh 2\alpha & -\sinh 2\alpha \\
  -\sinh 2\alpha & e^{2\gamma} \cosh 2\alpha
\end{pmatrix}, \tag{4.2}
\]

where \(\gamma\) and \(\alpha\) represent the two degrees of freedom. A specific choice of polarization dyad associated with this representation is

\[
m^A = Q^{-1/4} \begin{pmatrix}
  e^{-\gamma}(\cosh \alpha - i \sinh \alpha), \ & i e^{\gamma}(\cosh \alpha + i \sinh \alpha) 
\end{pmatrix}. \tag{4.3}
\]

The operator

\[
h^a_b = \delta^a_b + T^a T_b \tag{4.4}
\]
which the lapse $A$ and shift $B^a$ on the worldtube analogous to the standard 3+1 decomposition of the Cauchy problem. In the adapted coordinates, $H_{AB} = R^2 h_{AB}$ is the metric of the 2-surfaces of constant $t$ on the worldtube,

$$A^2 = -H_{tt} + H_{AB} B^A B^B$$

is the square of the lapse function and

$$B^A = R^{-2} h_{AB} H_{tB}$$

is the shift vector. We have $T_a = -A \partial_a t$, the contravariant components of the worldtube metric are

$$H^{tt} = -A^{-2}$$
$$H^{tA} = A^{-2} B^A$$
$$H^{AB} = R^{-2} h_{AB} - A^{-2} B^A B^B.$$}

and we again have the dyad decomposition $h^{ab} = m^a v^b$ with $m^a = (0, m^A)$.

The mathematical structure of the conservation conditions (3.17) is simplest to analyze in a worldtube gauge in which the lapse $A = 1$ and the shift $B^A = 0$. This corresponds to the introduction of Gaussian normal coordinates on the worldtube, which is always possible locally. In this Gaussian gauge, $H_{tt} = H^{tt} = -1, H_{tA} = H^{tA} = 0, H^{AB} = R^{-2} h_{AB}$ and $T_a = -\partial_a t$.

Let $D_a$ denote the 3-connection on the worldtube associated with $H_{ab}$. Then in the Gaussian gauge, the conservation conditions (3.17) reduce to

$$D_b (K^b_a - \delta^b_a K) = \frac{1}{R^2} \partial_b \left( R^2 \sqrt{Q} (K^b_a - \delta^b_a K) \right) + \frac{1}{2} H^{BC}_{,a} (K_{BC} - H_{BC} K) = 0 \quad (4.9)$$

with components

$$\frac{1}{R^2} \partial_b (h^{BC} K_{BC}) - D_B K^B_t - \frac{1}{2} H^{BC}_{,t} (K_{BC} - H_{BC} K) = 0 \quad (4.10)$$
$$\frac{1}{R^2} \partial_b (R^2 K_{tA}) - D_B K^B_A + \mathcal{D}_A K = 0 \quad (4.11)$$

These equations can be re-expressed in terms of the 2-connection $d_A$ on the slices of the worldtube associated with $h_{AB}$ as

$$\partial_t (h^{CD} K_{CD}) - d_A (h^{AB} K_{tB}) = \frac{1}{2} h^{BC}_{,t} (K_{BC} - \frac{1}{2} h_{BC} h^{DE} K_{DE}) - \frac{1}{R} R_{tA} (h^{BC} K_{BC} - 2 R^2 K) \quad (4.12)$$
$$\partial_b (h^{AB} K_{tB}) - \frac{1}{2} H^{AB} d_B (h^{CD} K_{CD}) = R^2 h_{AB} K_{tB} + h^{BC} d_B (K^A_{,C} - \frac{1}{2} h^{AB} K_{C}) - h^{AB} d_B K + \frac{2}{R} R_{,B} (h^{BC} K^A_{,C} - h^{AB} K^C_{,C}) \quad (4.13)$$

We interpret (4.12) - (4.13) as a system of equations for $h^{BC} K_{BC}$ and $K_{tA}$ given their right hand sides. As such, they take the form of a symmetric hyperbolic system (cf. [16]). This requires that, up to lower differential terms, the left hand sides have the form

$$\partial_t V^\alpha - M^{\alpha \beta}_\gamma A^\beta \partial A V^\beta$$

where, for each $A$-component, $\mathcal{H}_{\gamma A} M^{\alpha \beta}_\gamma$ is symmetric in $(\gamma, \beta)$ for some symmetric, positive-definite matrix $\mathcal{H}_{\gamma A}$. In the present case, we have $V^\alpha = (V^t, V^q, V^p) = (h^{CD} K_{CD}, h^{tB} K_{tB}, h^{pB} K_{tB})$. The component $M^{\alpha q}_\beta$ is given (up to lower order terms) by

$$M^{\alpha q}_\beta = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ H^{q q} & 0 & 0 \\ H^{p p} & 0 & 0 \end{pmatrix}$$

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$$M^{\alpha q}_\beta = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ H^{q q} & 0 & 0 \\ H^{p p} & 0 & 0 \end{pmatrix}$$
Referring to (4.12), the symmetrizer is

\[
\mathcal{H}_{\gamma \alpha} = \begin{pmatrix}
(2R^2 \cosh 2\alpha)^{-1} & 0 & 0 \\
0 & e^{2\gamma} \tanh 2\alpha & 0 \\
0 & \tanh 2\alpha & e^{-2\gamma}
\end{pmatrix}. \tag{4.16}
\]

The component \( M^{ab}_{\beta} \) has the same symmetrizer.

Such symmetric hyperbolic systems have a well-posed Cauchy problem, i.e. there exists a unique solution which depends continuously on the values of \( h^{BC}K_{BC} \) and \( K_{IA} \) on the initial slice of the worldtube. In addition to lower order terms in \( V^a \), the right hand sides of (4.12) and (4.13) depend upon \( K_B^A - (1/2)h_B^AK_C^C, K, h_{AB} \) and \( R \). Here \( K_B^A - (1/2)h_B^AK_C^C \) is determined by \( m^a m^b K_{ab} \) or, in tensorial form, by \( (h_0^2 h_0^b - 1/2 h_0^b h_0^c)K_{ab}^c \). Thus, in the Gaussian gauge, we have established the following Worldtube Theorem:

Given \( H_{ab}, m^a m^b K_{ab} \) and \( K \), the worldtube constraints constitute a well-posed initial-value problem which determines the remaining components of the extrinsic curvature \( K_{ab} \).

This theorem extends to any gauge. More generally, the conservation conditions (3.17) take the form

\[
T^b \partial_b (h_0^2 K_{ab}^c) - d_B(h_0^B K_{ad}^d) = S_t, \tag{4.17}
\]

\[
T^b \partial_b (h_0^c K_{ad}^d T_d) - \frac{1}{2} d_A(h_0^2 K_{ab}^c) = S_A, \tag{4.18}
\]

where the source terms \( S_t \) and \( S_A \) are determined by \( H_{ab} \), \( m^a m^b K_{ab} \), \( K \) and lower order terms. The system (4.17) - (4.18) can again be symmetrized.

It is important to note that whether or not \( H_{ab}, m^a m^b K_{ab} \) and \( K \) can be chosen independently depends upon the choice of gauge conditions and formulation of Einstein’s equations. In the next Section, we discuss this issue in the context of the Bondi-Sachs formulation.

V. CONSERVATION LAWS AND THE CHARACTERISTIC INTEGRATION CONSTANTS

The worldtube theorem established in Sec. [IV] formulates the boundary constraints as a well-posed initial value problem given the required source terms. We now explore the implications for the integration constants necessary to integrate the characteristic hypersurface and evolution equations. The integration constants (2.15) constitute 8 real functions on the worldtube. The areal radius of the worldtube \( R(u, x^A) \) is a 9th function necessary to determine a unique solution. Thus no more than 9 pieces of worldtube data can be freely specified. This matches the number of functions assumed in the worldtube theorem, i.e. \( H_{ab}, m^a m^b K_{ab} \) and \( K \). However, the boundary constraints introduce 3 relations between the 9 pieces of worldtube data.

We denote by \( y^a = (t, y^A) \) the coordinates intrinsic to a foliation of the worldtube by topological spheres \( t = \text{const} \), with angular coordinates \( y^A \). The propagation of these coordinates along the null geodesics of the outgoing \( u = \text{const} \) null hypersurfaces, which are uniquely determined by this foliation, induces Bondi-Sachs coordinates \( x^a = (u, r, x^A) \) in the exterior spacetime with \( x^a = (u, x^A) = y^a \) on the worldtube. When convenient we will switch from 4-dimensional notation with coordinates \( x^a \) to 3-dimensional notation with coordinates \( y^a \). We have

\[
\frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^a} + R_{a\alpha} \frac{\partial}{\partial r} \tag{5.1}
\]

where we continue to denote partial derivatives by commas when it does not lead to ambiguity. In addition, in this Section, we denote \( \partial/\partial y^a = \partial_a \).

The metric on the worldtube \( r = R(u, x^A) \) which is induced by the Bondi-Sachs metric (2.1) is

\[
H_{ab} dy^a dy^b = -e^{2\beta} \frac{V}{R} dt^2 - 2e^{2\beta} R_{\alpha} dt dy^a + R^2 h_{AB}(dy^A - U^A dt)(dy^B - U^B dt). \tag{5.2}
\]

Here \( V(t, y^A) \) is related to the Bondi-Sachs variable \( V(u, r, x^A) \) by \( V(t, y^A) = V(t, R(t, y^A), y^A) \). Similarly, \( h_{AB}(t, y^A) = h_{AB}(t, R(t, y^A), y^A) \), etc. We again denote by \( d_A \) the 2-connection on the slices of the worldtube associated with \( h_{AB} \). It is important to distinguish between \( d_A \) and the 2-dimensional covariant derivative \( D_A \) associated with \( h_{AB} \) on the \( (r = \text{const}, u = \text{const}) \) Bondi-Sachs spheres. In the tangent space of the worldtube, we again have the dyad decomposition \( h^{ab} = m^a m^b \) with \( m^a = (0, m^A) \). It is also important to distinguish between
In Bondi-Sachs coordinates, \( K \) so that \( \eta \) acceleration components \( T^a \) and the 2-dimensional extrinsic curvature of the \( t \) and \( r \) vectors tangent to the worldtube and vectors tangent to the worldtube and vectors tangent to the spacetime.

In order to interpret the implication of the boundary constraints for the nullcone-worldtube problem we now relate the intrinsic metric and extrinsic curvature of the worldtube to the integration constants (2.15) for the hypersurface by

\[ A^2 = e^{2\beta} \left( \frac{V}{R} + 2R_1 \right) + R^2 h_{AB}(B^A + U^A)(B^A - U^A) \] (5.3)

and

\[ B^A = -R^{-2}e^{2\beta}h^{AB}R_{,B} - U^A. \] (5.4)

The geometric properties associated with the \( t \)-foliation of the boundary, with normal \( T_a = -\partial_\eta t \), are the 3-acceleration

\[ a^b = T^a D_a T^b \] (5.5)

and the 2-dimensional extrinsic curvature of the \( t \)-foliation

\[ \kappa_{ab} = h^c_{\alpha} D_c T_b. \] (5.6)

The independent components of \( a^b \) and \( \kappa_{ab} \) are

\[ m^b_{ab} = m^B d_B \log A \] (5.7)

\[ h^{ab}\kappa_{ab} = -A^{-1} \left( d_C(R^2 B^C) - \partial_t(R^2) \right) \] (5.8)

and

\[ m^a m^b \kappa_{ab} = \frac{R^2}{2A} m^A m^B \partial_t h_{AB} - \frac{R^2}{A} m^C m^D d_C B^D. \] (5.9)

The 3-dimensional extrinsic curvature of the worldtube is \( K_{\mu\nu} = H^a_{\mu} \nabla_\alpha N_\nu \) where the unit outward normal is given by

\[ N_\mu = \eta \nabla_\mu (r - R). \] (5.10)

The normalization condition expressed in Bondi-Sachs coordinates gives

\[ \eta^{-2} = \frac{V}{R}e^{-2\beta} + 2e^{-2\beta} R_{,u} + 2e^{-2\beta} U^A R_{,A} + R^{-2} h^{AB} R_{,A} R_{,B} = e^{-4\beta} A^2, \] (5.11)

so that \( \eta = e^{2\beta} A^{-1} \). The outgoing null vector normal to the \( t \)-foliation of the worldtube is

\[ K^\mu = T^\mu + N^\mu. \]

In Bondi-Sachs coordinates, \( K^\mu \partial_\mu \) is proportional to \( \partial_r \). The proportionality constant is determined by the normalization condition \( K^\mu N_\mu = 1 \), which evaluated on the worldtube gives

\[ K^\mu \partial_\mu = \eta^{-1} \partial_r = e^{-2\beta} A \partial_r. \] (5.12)

The computation of \( K_{\mu\nu} \) in terms of Bondi-Sachs coordinates can be simplified by the construction

\[ K_{\mu\nu} = H^a_{\mu} \nabla_\alpha N_\nu = H^a_{(\mu} H^\beta_{\nu)} \nabla_\alpha N_\beta \] (5.13)

\[ = H^a_{(\mu} H^\beta_{\nu)} \nabla_\alpha K_\beta - H^a_{(\mu} H^\beta_{\nu)} D_\alpha T_\beta \] (5.14)

\[ = \frac{1}{2} H^a_{(\mu} H^\beta_{\nu)} \mathcal{L}_K g_{\alpha\beta} - h^a_{(\mu} h^\beta_{\nu)} D_\alpha T_\beta + h^a_{(\mu} T_{\nu)} T^\beta D_\beta T_\alpha \] (5.15)
so that
\[ K_{\mu \nu} = \frac{1}{2} H^\alpha_{\mu} H^\beta_{\nu} \mathcal{L}_{K} g_{\alpha \beta} - \kappa_{\mu \nu} + a_{(\mu} T_{\nu)}. \] (5.16)

Expressions for the components of \( \mathcal{L}_{K} g_{\alpha \beta} \) in terms of the Bondi-Sachs variables, which are valid for an arbitrary gauge, are given in Appendix A. Collecting the pieces of (5.16), the components of the extrinsic curvature are

\[ h^{ab} K_{ab} = 2 A R e^{-2\beta} + A^{-1} \left( d_C (R^2 B^C) - \partial_t (R^2) \right) \] (5.17)

\[ m^a m^b K_{ab} = \frac{1}{2} A R^2 e^{-2\beta} m^A m^B h_{AB,r} - \frac{R^2}{2A} m^A m^B \partial_t h_{AB} + \frac{R^2}{A} m^C m_D d_C B^D \] (5.18)

\[ m^a T^b K_{ab} = - \frac{1}{2} m^A Q_A - m^A R^B m^B h_{AB,r} + \frac{1}{4} m^A m^B h_{AB,r} m^C R_C - \frac{1}{2} m^C \partial_C \log(e^{-2\beta} A^2) \] (5.19)

and

\[ T^a T^b K_{ab} = - \frac{1}{A} \left( \frac{V}{R} + R_t - B^A R_A \right) \beta_r + \frac{V}{2 A R^2} - \frac{1}{2 A R} V_r - \frac{1}{A R^2} e^{2\beta} h^{AB} R_A R_B Q_A \]

\[ - \frac{1}{2 A R^2} e^{2\beta} h^{AB} R_A R_B + \frac{1}{2 A R^3} e^{2\beta} h^{AB} R_A R_B - T^a \partial_a \log(e^{-2\beta} A). \] (5.20)

The interplay between the boundary constraints on \( K_{ab} \) and the characteristic integration constants is complicated by the choice of gauge and the choice of free data. We consider two complementary scenarios which simplify the discussion of the underlying problems.

A. Waveform extraction

In the first scenario, the Bondi-Sachs integration constants \((\beta, V, U^A, Q_A, h_{AB}, R)\) are obtained from the metric in the neighborhood of the worldtube which is supplied by the numerical results of a \(3+1\) Cauchy evolution. This provides the inner boundary data for a numerical characteristic evolution on a Penrose compactified grid, which yields the waveform at \(I^+\). This approach, called Cauchy-characteristic extraction [17, 18], is used in numerical relativity to provide \((\beta, V, U^A, Q_A, h_{AB}, R)\) on the boundary. The worldtube theorem then determines the integration constant for \(C_{\beta} \), which is then determined from \(Q_A \) using the hypersurface equation (2.6). The integration constant for \(Q^A \) is then determined from \(m^a T^b K_{ab} \), as given by (5.18).

We have thus shown that the worldtube theorem leads to the following Extraction Corollary:

*Given \(H_{ab}, m^a m^b K_{ab} \) and \(K\), the worldtube constraints constitute a well-posed initial-value problem which determine the Bondi-Sachs integration constants \((\beta, V, U^A, Q_A, h_{AB}, R)\).*
B. The initial-boundary value problem

In the foregoing waveform extraction scenario, the nine integration constants \((\beta, V, U^A, Q_A, h_{AB}, R)\), which are required to integrate the Bondi-Sachs equations were supplied by a Cauchy evolution in a manner consistent with Einstein’s equations. The data consisted of the boundary metric \(H_{ab}\), (6 functions), \(m^m m^n K_{ab}\) (2 functions) and \(K\) (1 function). The constraints were then enforced via the worldtube theorem to determine the remaining components of \(K_{ab}\), which in turn supplied the integration constants.

In an initial-boundary value problem, boundary data consistent with the constraints must be prescribed \textit{apriori}, i.e. before the evolution is carried out. Enforcement of the boundary constraints has been the major difficulty in attempting to show that the various formulations of the gravitational initial-boundary value problem are well-posed. In this regard, the characteristic formulation is no exception. The coupling between the Bondi-Sachs evolution system and the boundary constraint system is complicated. The details depend upon the choice of free boundary data and the choice of gauge conditions adopted on the boundary. We illustrate this problem with two examples.

1. Constant \(R\) boundary data

Consider first the case in which the the 5 worldtube integration constants for \((\beta, U^A, h_{AB})\) are prescribed freely along with a constant value of the areal radius \(R\). We then attempt to prescribe the remaining 3 integration constants for \((V, Q_A)\) via the 3 boundary constraints (4.17) and (4.18). For simplicity, assume the boundary data are \(\beta = 0\), \(U^A = 0\), \(h_{AB} = Q_{AB}\) (where \(Q_{AB}\) is the unit sphere metric), along with \(R = \text{const}\). The lapse (5.3) and shift (5.4) corresponding to this data reduce to

\[ A^2 = V/R, \quad B^A = 0. \quad (5.21) \]

Along with the initial data at \(t = 0\),

\[ h_{AB}(0, r, x^C) = Q_{AB}, \quad V(0, R, x^C) = R - 2M, \quad Q_A(0, R, x^C) = 0, \quad (5.22) \]

this worldtube data determine a mass \(M\) Schwarzschild spacetime in spherically symmetric Bondi coordinates.

Now, with the same boundary data, let the initial data \(h_{AB}(0, r, x^C)\) consist of a pulse whose support is isolated from the worldtube. This evolves to produce ingoing radiation so that the spacetime in the neighborhood of the boundary is no longer Schwarzschild in the domain of dependence of the initial pulse. The boundary constraints for this problem reduce to

\[ \partial_t K_B^B - \frac{1}{A} d_B(AK_i^B) = 0 \quad (5.23) \]

and

\[ \partial_t \left( \frac{1}{A} K_B^B \right) - \frac{1}{2} d_B(AK_C^C) + \frac{1}{2} d_B(AK_D^D) = \frac{1}{2} \delta_C^D K_{BC}^D \quad (5.24) \]

where the components of the extrinsic curvature are

\[ h^{BC} K_{BC} = 2AR \quad (5.25) \]

\[ m^C T_b K_{Cb} = -m^C \left( \frac{1}{2} Q_C + \frac{1}{A} \partial_C A \right) \quad (5.26) \]

\[ m^B m^C K_{BC} = \frac{AR^2}{2} m^B m^C h_{BC,r} \quad (5.27) \]

and

\[ K = \frac{3A}{2R} + A \beta_r + \frac{1}{2AR} V_r + \frac{1}{A^2} \partial_r A. \quad (5.28) \]

Before the pulse hits the worldtube, \(V(u, R, x^A) = R - 2M\) and \(A^2 = 1 - 2M/R\), but afterward the worldtube values of \(h_{BC,r}\) and \(Q_A\) become time dependent.
If these boundary constraints were to constitute a well-posed problem for $K_B^B$ and $K_B^R$ then they would supply the boundary values of $A$ and $Q_A$ necessary to determine the remaining Bondi-Sachs integration constants. However, two problems arise from the right hand side of (5.24). First, the term $K^C_A - \frac{1}{2} \delta^C_A K_B^B$ is determined by the optical shear $\sigma$ at the boundary. But this shear is not known until the evolution is carried out and it depends upon the detailed shape of the initial radiation pulse. Thus the boundary system [5.23 - 5.24] is coupled to the evolution system.

The second problem is more serious. For this system the evaluation of the hypersurface equations on the boundary implies

$$\beta, r = -\frac{R}{16} h^{AB} h_{AB, r} = \frac{R}{2} \sigma \bar{\sigma}$$  \hspace{1cm} (5.29)$$

and

$$V, r = 1 + \frac{1}{2} h^{CD} dC Q_D - \frac{1}{4} h^{CD} Q_{CD, r}$$  \hspace{1cm} (5.30)$$

so that

$$K = \frac{3A}{2R} + \frac{AR}{2} \sigma \bar{\sigma} + \frac{1}{2AR} (1 + \frac{1}{2} h^{CD} dC Q_D - \frac{1}{4} h^{CD} Q_{CD, r}) + \frac{1}{A^2} \partial_t A.$$  \hspace{1cm} (5.31)$$

As a result, in addition to coupling the boundary and evolution systems, the term $d_B(AK)$ on the right hand side of [5.21] introduces a $h^{CD} dB dC Q_D$ term which alters the principle part of the boundary system so that it is no longer guaranteed to determine $A$ and $Q_A$ in a well-posed manner.

2. Trace $K$ boundary data

The problem in the preceding example arises because the boundary data for $R$ and $K$ (the trace of the extrinsic curvature of the boundary) cannot in general be given independently, as is required for application of the worldtube theorem. Specification of $R = \text{const}$ boundary data simplifies the boundary system since [5.1] then reduces to $\partial/\partial y^i = \partial/\partial x^\alpha$. However, this simplification is offset by the complicated way in which $K$ changes the principle part of the boundary system and affects the well-posedness through the $d_B(AK)$ term in [5.21].

The only way this complication can be avoided is to prescribe $K(t, y^A)$ as explicit boundary data, so that it does not affect the principle part of the boundary system. In Bondi-Sachs coordinates, $R$ and $K$ are related by

$$K = H^{\mu \nu} \nabla_\mu N_\nu = \eta H^{\mu \nu} \partial_\mu (r - R) - H^{\mu \nu} \Gamma^p_{\mu \nu} N_p = -\eta H^{ab} \partial_a \partial_b R - H^{\mu \nu} \Gamma^p_{\mu \nu} N_p.$$  \hspace{1cm} (5.32)$$

Here the timelike nature of the worldtube ensures that $H^{ab} \partial_a \partial_b$ is a wave operator. Thus, given $K(t, y^A)$, the lapse $A$, the shift $B^A$, the conformal 2-metric $h^{ab}$ and the Bondi-Sachs Christoffel symbols $H^{\mu \nu} \Gamma^p_{\mu \nu} N_p$ on the worldtube, $R$ can be determined from its initial data by a well-posed quasilinear wave problem based upon [5.32]. This relationship between the function locating the worldtube and the extrinsic curvature scalar of the worldtube was first pointed out in the Friedrich-Nagy [10] treatment of the initial-boundary value problem.

Although this might at first sound like a promising approach, it leads to serious difficulties. Foremost, if $R_A \neq 0$ then the boundary constraint [4.18] couples the evolution system with the boundary system in a way which makes the formulation of a well-posed evolution-boundary system appear to be intractable. This arises from the terms

$$-m^A R_A \beta, r + \frac{1}{4} m^A m^B h_{AB, r} m^C R_{AC}$$

in the expression [5.19] for $m^a T^b K_{ab}$. Here $\beta, r$ and $h_{AB, r}$ cannot be determined without knowledge of the evolution. Consequently, the boundary constraint governing $m^a T^b K_{ab}$ cannot be used to determine the Bondi-Sachs integration constant for $Q_A$ independently of the evolution.

One possible way to circumvent this problem would be to pick a gauge for the boundary in which $R_A = 0$, i.e. $R = R(t)$. Since $R$ is a scalar density defined by the determinant of the 2-metric of the boundary slices, this can be achieved via the Jacobian of an appropriately chosen angular transformation. However, this raises the new problem of how to pick explicit data for $K$ which would be consistent with $R_A = 0$.

VI. DISCUSSION

We have shown how the boundary constraints for the Bondi-Sachs equations can be posed as a symmetric hyperbolic system governing the evolution of certain components of the extrinsic curvature of the worldtube, as described by the
In Bondi-Sachs coordinates, the tangents to the worldtube foliation have components

Thus

wordtube theorem in Sec. [X]. In Sec. [V] we described how these extrinsic curvature components were related to the integration constants for the Bondi-Sachs system. The application of the wordtube theorem requires knowledge of the intrinsic metric of the wordtube and the remaining components of the extrinsic curvature. We considered two different versions.

The first application was to waveform extraction. In that case, the data $(H_{ab}, m^a m^b K_{ab}, K)$ necessary to apply the wordtube theorem are supplied by the numerical results of a $3 + 1$ Cauchy evolution. The remaining components of the extrinsic curvature can then be determined by means of a well-posed initial value problem on the boundary. The integration constants $(\beta, V, U^A, Q_A, h_{AB}, R)$, for the Bondi-Sachs equations are then determined. This approach can be used to enforce the constraints in the numerical computation of waveforms at $I^+$ by means of Cauchy-characteristic extraction.

In the second application, we considered the initial-boundary value problem, for which boundary data consistent with the constraints must be prescribed a priori, i.e. independent of the evolution. The object was to obtain a well-posed version of the characteristic initial-boundary value problem. However, the complicated coupling between the Bondi-Sachs evolution system and the boundary constraint system prevented any definitive results. Two choices of free boundary data and boundary gauge conditions were explored. In both cases, the Bondi-Sachs choice of areal coordinate $r$ complicated the analysis. This results from the way that the coordinates and geometry are mixed, i.e. $r$ cannot be assigned freely on the boundary without specifying its area. It is possible that other formulations of the characteristic initial-boundary value problem might be more amenable. Bartnik [19] previously explored a quasi-spherical version, in which the 2-metric $h_{AB}$ is transformed into a conformally unit sphere form. He found similar complications in trying to establish well-posedness as in the Bondi-Sachs case. A formal computational algorithm for the evolution-constraint system was possible, but the well-posedness of the corresponding initial-boundary value problem, which is necessary for numerical stability, was not clear.

Another possibility is to choose a gauge in which the areal coordinate $r$ is replaced by an affine parameter $\lambda$, so that the affine freedom allows the specification $\lambda = 0$ on the boundary, independently of its geometry. Rendall [20] has shown that such a characteristic initial-boundary value is well-posed in the double null case where the boundary is also a null hypersurface. However, Rendall’s approach cannot be applied to the corresponding null-timelike problem. Although the full treatment of the null-timelike problem lies outside the scope of this paper, it is clear that the worldtube conservations laws must enter in an essential way.

Appendix A

We use (5.12),

$$K^\nu \partial_\mu = e^{-2\beta} A \partial_r, \tag{A1}$$

to simplify the calculation of $\mathcal{L}_K g_{\alpha\beta}$ in Bondi coordinates. We have

$$\mathcal{L}_K g_{\alpha\beta} = e^{-2\beta} A g_{\alpha\beta,r} + e^{2\beta} A^{-1} \left( K_\alpha \partial_\beta (e^{-2\beta} A) + K_\beta \partial_\alpha (e^{-2\beta} A) \right). \tag{A2}$$

Thus

$$H^\alpha_\mu H^\beta_\nu \mathcal{L}_K g_{\alpha\beta} = e^{-2\beta} A H^\alpha_\mu H^\beta_\nu g_{\alpha\beta,r} + e^{2\beta} A^{-1} H^\alpha_\mu H^\beta_\nu \left( K_\alpha \partial_\beta (e^{-2\beta} A) + K_\beta \partial_\alpha (e^{-2\beta} A) \right)$$

$$= e^{-2\beta} A H^\alpha_\mu H^\beta_\nu g_{\alpha\beta,r} + T_\mu D_\nu \log (e^{-2\beta} A) + T_\nu D_\mu \log (e^{-2\beta} A). \tag{A3}$$

In Bondi-Sachs coordinates, the tangents to the worldtube foliation have components $m^\mu = (0, m^B R, B, m^A)$ and $T^\mu = A^{-1} (1, R_A - B^C R_C, -B^A)$. This leads to the components

$$m^\mu m^\nu H^\alpha_\mu H^\beta_\nu \mathcal{L}_K g_{\alpha\beta} = 4 A R e^{-2\beta} \tag{A4}$$

$$m^\mu m^\nu H^\alpha_\mu H^\beta_\nu \mathcal{L}_K g_{\alpha\beta} = A R^2 e^{-2\beta} m^A m^B h_{AB,r} \tag{A5}$$

$$m^\mu T^\nu H^\alpha_\mu H^\beta_\nu \mathcal{L}_K g_{\alpha\beta} = -m^A Q_A - 2 m^A R_A \beta_r - e^{-2\beta} m^B g_{AB,r} (U^A + B^A) - m^A d_A \log (e^{-2\beta} A)$$

$$= -m^A Q_A - 2 m^A R_A (\beta_r - \frac{1}{R}) + \frac{1}{2} m^A m^B h_{AB,r} m^C R_C - m^A \partial_A \log (e^{-2\beta} A) \tag{A6}$$
\[ T^\mu T^\nu H^\alpha H^\beta \mathcal{L}_K g_{\alpha \beta} = -\frac{2}{A} \left( \frac{V}{R} + 2R_t - 2B^A R_A \right) \beta \partial_r + \frac{V}{AR^2} - \frac{1}{AR} V_r - \frac{2}{A} (U^A + B^A) Q_A \]
\[ + \frac{1}{A} e^{-2\beta} g_{AB,r} (U^A + B^A) (U^C + B^C) - 2T^a \partial_a \log(e^{-2\beta} A) \]
\[ = -\frac{2}{A} \left( \frac{V}{R} + 2R_t - 2B^A R_A \right) \beta \partial_r + \frac{V}{AR^2} - \frac{1}{AR} V_r - \frac{2}{AR^2} e^{2\beta} h^{AB} R_B Q_A \]
\[ - \frac{1}{AR^2} e^{2\beta} h^{AB}_r R_A R_B + \frac{2}{AR^3} e^{2\beta} h^{AB} R_A R_B - 2T^a \partial_a \log(e^{-2\beta} A). \] (A7)

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