TRAVELING WAVES FOR NONLOCAL MODELS OF TRAFFIC FLOW

JOHANNA RIDDER AND WEN SHEN*

Mathematics Department, Pennsylvania State University
University Park, PA 16802, USA

(Communicated by Rinaldo M. Colombo)

Abstract. We consider several nonlocal models for traffic flow, including both microscopic ODE models and macroscopic PDE models. The ODE models describe the movement of individual cars, where each driver adjusts the speed according to the road condition over an interval in the front of the car. These models are known as the FtLs (Follow-the-Leaders) models. The corresponding PDE models, describing the evolution for the density of cars, are conservation laws with nonlocal flux functions. For both types of models, we study stationary traveling wave profiles and stationary discrete traveling wave profiles. (See definitions 1.1 and 1.2, respectively.) We derive delay differential equations satisfied by the profiles for the FtLs models, and delay integro-differential equations for the traveling waves of the nonlocal PDE models. The existence and uniqueness (up to horizontal shifts) of the stationary traveling wave profiles are established. Furthermore, we show that the traveling wave profiles are time asymptotic limits for the corresponding Cauchy problems, under mild assumptions on the smooth initial condition.

1. Introduction. Nonlocal ODE and PDE models describing traffic flow have attracted more research interests recently, and the field has observed an increasing growth of literature in recent year. In this paper we consider two nonlocal conservation laws, and the corresponding two nonlocal Follow-the-leaders models as their counter parts. In particular, our interests lie in the analysis of the stationary traveling waves and their associated properties.

For nonlocal conservation laws, we consider the following models for traffic flow,

\[
\rho_t(t, x) + \left[ \rho(t, x) \cdot v \left( \int_x^{x+h} \rho(t, y) w(y-x) \, dy \right) \right]_x = 0,
\]

(1.1)

and

\[
\rho_t(t, x) + \left[ \rho(t, x) \cdot \left( \int_x^{x+h} v(\rho(t, y)) w(y-x) \, dy \right) \right]_x = 0,
\]

(1.2)

where \(x, t \in \mathbb{R}\) and \(t \geq 0\). In both models, \(\rho : \mathbb{R} \to [0, 1]\) is the density function of cars, and \(h \in \mathbb{R}\) satisfies \(h > 0\). We have the following assumptions on the functions \(w\) and \(v\):

2010 Mathematics Subject Classification. Primary: 35L02, 35L65; Secondary: 34B99, 35Q99.
Key words and phrases. Traffic flow, nonlocal models, traveling waves, microscopic models, delay integro-differential equation, local stability.

* Corresponding author: Wen Shen.
(A1) The weight function $w$ is nonnegative and bounded. It is Lipschitz continuous on $(0, h)$, and satisfies
\[
\int_0^h w(x) \, dx = 1, \quad \text{and} \quad w(x) = 0 \quad \forall x < 0 \text{ and } x > h.
\] (1.3)

Note that $w$ can be discontinuous at $x = 0$ and $x = h$. Although in (1.1)-(1.2) $w$ is only used on the interval $[0, h]$, we define it on the whole real line for later use in the particle models.

(A2) The velocity function $v \in C^2$ satisfies
\[
v(0) = 1, \quad v(1) = 0, \quad \text{and} \quad v'(\rho) < 0 \quad \forall \rho \in [0, 1].
\] (1.4)

Assumptions (A2) are commonly used in traffic flow, indicating that for higher density the cars travel with lower speed. The function $v$ is defined on $[0, 1]$ since the car density satisfies $0 \leq \rho \leq 1$.

The conservation laws (1.1)-(1.2) are macroscopic models for traffic flow. They can be formally derived as the continuum limit of the corresponding particle models, commonly referred to also as microscopic models. Particle models consist of systems of ODEs that describe the time evolution of the position of each individual car.

**Particle model for (1.1).** In connection with the conservation law (1.1), we consider the following particle model. Assuming that all cars have the same length $\ell \in \mathbb{R}^+$, let $z_i(t)$ be the position of the $i$-th car at time $t$. We order the indices for the cars so that
\[
z_i(t) \leq z_{i+1}(t) - \ell \quad \forall t \geq 0, \quad \text{for every } i \in \mathbb{Z}.
\] (1.5)

For the $i$-th car, we define the local traffic density perceived by its driver, depending on the relative position of the car in front, namely
\[
\rho_i(t) := \frac{\ell}{z_{i+1}(t) - z_i(t)} \quad \forall t \geq 0, \quad \text{for every } i \in \mathbb{Z}.
\] (1.6)

Note that if $\rho_i = 1$, then the two cars with indices $i$ and $i+1$ are bumper-to-bumper. For the model to be meaningful, we therefore must have $0 \leq \rho_i(t) \leq 1$ for all $i \in \mathbb{Z}$ and $t \geq 0$.

We consider the “follow-the-leaders” (FtLs) model, defined as follows. The speed of the $i$-th car depends solely on a weighted average local density $\rho_i^*(t)$, where the average is taken over an interval of length $h$ in front of $z_i$. More precisely, denoting a time derivative with an upper dot, we assume
\[
\dot{z}_i(t) = v(\rho_i^*(t)), \quad \text{with} \quad \rho_i^*(t) := \sum_{k=0}^{+\infty} w_{i,k}(t) \rho_{i+k}(t).
\] (1.7)

For $k \geq 0$, by $w_{i,k}(t)$ we denote the weight assigned at time $t$ by the $i$-th driver to the car at $z_{i+k}(t)$. In connection with the weight function $w(\cdot)$ in (1.1)-(1.3), these weights are defined as
\[
w_{i,k}(t) := \int_{z_i(t)}^{z_{i+k+1}(t)} w(y - z_i(t)) \, dy, \quad k \geq 0, 1, 2, \ldots
\] (1.8)

Notice that the summation in (1.7) actually contains only finitely many non-zero terms. Indeed, if $(m+1)\ell \geq h$, then for every $k > m$ one has
\[
z_{i+k}(t) \geq z_i(t) + (m+1)\ell \geq z_i(t) + h.
\]
Hence, by (1.3), \( w_{i,k} = 0 \). With the above definition, from (1.3) it also follows
\[
\begin{align*}
\sum_{k=0}^{m} w_{i,k}(t) &= 1 \quad \forall i \in \mathbb{Z}, \\
w_{i,k}(t) &\geq 0 \quad \forall i \in \mathbb{Z}, k \in \{0, 1, \cdots , m\}, \ t \geq 0.
\end{align*}
\]

**Particle model for (1.2).** In this model, the speed of the car number \( i \) located at \( z_i \) is the weighted average of the function \( v(\cdot) \) over an interval in front of \( z_i \). This leads to the second FtLs model:
\[
\dot{z}_i(t) = v^*_i(\rho; t), \quad \text{where} \quad v^*_i(\rho; t) \doteq \sum_{k=0}^{m} w_{i,k}(t) v(\rho_{i+k}(t)).
\]

Here the weights \( w_{i,k}(t) \) are defined as in (1.8). As before, the sum may contain several zero terms.

The convergence of microscopic models to their macroscopic equivalents is of fundamental interest. In this paper, we study this issue by looking at traveling wave profiles. At the same time, the study of traveling waves and their related properties (existence, uniqueness, and stability) is of essential importance in the analysis of solutions for conservation laws, since stable traveling wave profiles often are time asymptotic solutions for the Cauchy problem.

Since the equations (1.1) and (1.2) are rather similar, as well as the systems (1.7) and (1.10), we analyze in detail (1.1) and (1.7). The analysis for (1.2) and (1.10) will only be presented briefly.

For (1.1) and (1.2), traveling wave solutions are special solutions of the Cauchy problem where a profile travels with a constant velocity. Below we give the precise definition.

**Definition 1.1.** We say that a continuously differentiable function \( Q : \mathbb{R} \mapsto [0, 1] \) is a “traveling wave profile” for (1.1) with speed \( \sigma \in \mathbb{R} \) if the function \( \rho(t,x) = Q(x - \sigma t) \) provides a solution to (1.1). In the special case where \( \sigma = 0 \), we call \( Q(\cdot) \) a “stationary profile”.

To simplify the discussion, throughout the sequel we seek stationary profiles \( Q(\cdot) \) for (1.1). Traveling waves with non-zero velocity can be transformed into a stationary profile using a coordinate translation (see Section 5.1). In Section 3 we derive the delay integro-differential equation (3.4) satisfied by a stationary profile \( Q(\cdot) \). The existence and uniqueness (up to horizontal shifts) of monotone profiles are established. The asymptotic stability of the traveling wave profiles is important in the study of the long-time behavior of solutions. We show that, under mild assumptions on the smooth initial condition, the profiles \( Q(\cdot) \) are the time asymptotic limits of solutions of (1.1), as \( t \to +\infty \).

In addition, we also seek “stationary discrete wave profiles” \( P(\cdot) \) for the FtLs model (1.7), defined as follows.

**Definition 1.2.** We say that a continuously differentiable function \( P : \mathbb{R} \mapsto [0, 1] \) is a “stationary discrete wave profile” for (1.7) if there exists a solution \( \{z_i(t); i \in \mathbb{Z}\} \) of (1.7), such that
\[
P(z_i(t)) = \rho_i(t) = \frac{\ell}{z_{i+1}(t) - z_i(t)}, \quad \forall t \geq 0, \ \forall i \in \mathbb{Z}.
\]
We derive a delay differential equation with a summation term, see (2.4), satisfied by $P(\cdot)$. In a similar way as for $Q(\cdot)$, we establish the existence and uniqueness (up to a horizontal shift) of the discrete profiles $P(\cdot)$. Furthermore, we show that these profiles provide attractors to the solutions of the FtLs model (1.7), for a wide family of initial data.

The profile $P(\cdot)$ depends on the length of the cars $\ell$. Taking the limit $\ell \to 0$, we prove the convergence of traveling wave solutions $P(\cdot)$ for the particle model to the profile $Q(\cdot)$ for the nonlocal conservation laws.

An entirely similar set of results is proved for the system (1.10) and the conservation law (1.2), with only small modifications in the analysis.

For the local follow-the-leader model, where the speed of each car is determined solely by the leader ahead, the traveling wave profiles have been studied in a recent work by Shen & Shikh-Khalil [38], where existence, uniqueness and stability of traveling waves were established. For the general Cauchy problem, convergence of the local FtL model to the corresponding local macroscopic PDE model

$$
\rho_t + f(\rho)_x = 0, \quad \text{where} \quad f(\rho) = \rho v(\rho),
$$

(1.12)

has been studied in various papers [19, 27, 28].

For the nonlocal model (1.1), existence of entropy weak solutions for the Cauchy problem was proved in [9] by utilizing the convergence of a finite difference scheme, and in [26] by means of a finite volume scheme. Well-posedness of the solutions for the Cauchy problem is also established in [9]. For model (1.2), existence and uniqueness of solutions are proved in [24], using a Godunov type scheme. A similar result was proved in [5] for kernel functions in $C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ instead of $C^1([0, \ell])$. Similar nonlocal conservation laws with symmetric kernel functions have been studied by [40], and [8] in the context of sedimentation modeling. Multi-dimensional versions and systems were studied as models for crowd dynamics [1, 12, 2, 13, 14, 17]. Other conservation laws with nonlocal flux functions include models for slow erosion of granular flow [39, 4], synchronization behavior [3], and materials with fading memory [10]. An overview over conservation laws with several other types of nonlocal flux functions can be found in [23, 15] and the references therein. See [29] for a recent result on uniqueness and regularity results on nonlocal balance laws and [30] for multi-dimensional nonlocal balance laws with damping. Further relevant references can be found in [20, 18, 11, 31]. For classical results on delay differential equations, we refer to [21, 22].

The paper is organized as follows. In Section 2 we study the nonlocal FtLs model (1.7), proving existence and uniqueness of the traveling wave profiles. We also show that these profiles are time asymptotic limits of more general solutions to the FtLs model. Similar results are proved for the nonlocal PDE model (1.1) in Section 3. In Section 4 we prove the convergence of the profiles of the FtLs model (1.7) to those of the PDE model (1.1), as the car length $\ell$ tends to 0. In Section 5 we discuss the case of travelling waves with non-zero velocity, and we consider a couple of examples where the profiles are unstable. In Section 6 we treat the alternative system (1.10) and the conservation law (1.2), and prove similar results. Finally, some concluding remarks are given in Section 7.

2. Nonlocal Follow-the-leaders models. We consider the nonlocal FtLs model in (1.7), i.e.
\[
\dot{z}_i(t) = v \left( \sum_{k=0}^{m} \int_{z_{i+k}(t)}^{z_{i+k+1}(t)} w(y - z_i(t)) \, dy \cdot \frac{\ell}{z_{i+k+1}(t) - z_{i+k}(t)} \right), \quad i \in \mathbb{Z}. \tag{2.1}
\]

Here \( m \) is chosen so that \((m + 1)\ell \geq h\). This family of countably many ODEs can be regarded as a nonlinear dynamical system on an infinite dimensional space. For example, we could set \( y_i(t) = z_i(t) - z_i(0) \) and write the system (2.1) as an evolution equation on the Banach space of bounded sequences of real numbers \( y = (y_i)_{i \in \mathbb{Z}} \), with norm \( \|y\| = \sup_i |y_i| \). For each \( i \in \mathbb{Z} \), the right hand side of (2.1) is Lipschitz continuous. For a given initial datum, the existence and uniqueness of solutions to this system follow from standard theory of evolution equations in Banach spaces, see for example [33, 35].

We now derive the equation satisfied by a stationary profile \( P(\cdot) \), considered in Definition 1.2. Note that (1.7) can be rewritten as a system of ODEs for the discrete density functions \( \rho_i(t) \), for \( i \in \mathbb{Z} \),
\[
\dot{\rho}_i(t) = -\frac{\ell}{\left( z_{i+1}(t) - z_i(t) \right)^2} \left[ v(\rho_i^*(t)) - v(\rho_{i+1}^*(t)) \right]. \tag{2.2}
\]

Differentiating both sides of (1.11) w.r.t. \( t \), and using (1.7) and (2.2), one obtains
\[
P'(z_i) = \frac{\dot{\rho}_i}{z_i} = \frac{\rho_i}{\ell} \rho_i^* \left[ v(\rho_i^*) - v(\rho_{i+1}^*) \right] = \frac{P^2(z_i)}{\ell} \left[ v(P^*(z_i)) - v(P^*(z_{i+1})) \right]. \tag{2.3}
\]
Here and in the sequel, a prime denotes a derivative w.r.t. the space variable \( x \). To shorten the notation, we do not explicitly write out the time dependence of \( z_i \) and \( \rho_i \). Furthermore, \( P^*(z_i) \) is the weighted average of \( P \) over an interval in front of \( z_i \), defined as
\[
P^*(z_i) \doteq \sum_{k=0}^{m} w_{i,k}P(z_{i+k}). \tag{2.4}
\]

To proceed with the analysis, we need to introduce some notations. Given a profile \( P \) with \( P(x) > 0 \) for every \( x \), we define an operator \( L^P(x) \) for the position of the leader of the car at \( x \),
\[
L^P(x) \doteq x + \frac{\ell}{P(x)}. \tag{2.5}
\]

We also use
\[
(L^P)^2 = L^P \circ L^P \quad \text{and} \quad (L^P)^k \doteq \underbrace{L^P \circ L^P \circ \cdots \circ L^P}_{k \text{ times}}
\]
to denote the composition of \( L^P \) with itself multiple times. We then have
\[
z_{i+1} = L^P(z_i),
\]
\[
z_{i+2} = L^P(z_{i+1}) = (L^P)^2(z_i) = z_i + \frac{\ell}{P(z_i)} + \frac{\ell}{P(z_{i+1})},
\]
and for a general index \( k \in \mathbb{N} \),
\[
z_{i+k} = (L^P)^k(z_i) = z_i + \sum_{j=0}^{k-1} \frac{\ell}{P(z_{i+j})}.
\]
We also define an averaging operator \( A^P \) as
\[
A^P(z_i) \doteq \sum_{k=0}^{m} w_{i,k} P \left( (L^P)^k(z_i) \right), \quad \forall i \in \mathbb{Z}. \tag{2.7}
\]

Since \( z_i \) is arbitrarily chosen, we now write \( x = z_i \). We have
\[
P'(x) = -\frac{P^2(x)}{\ell} \cdot v(A^P(x)) \left[ v(L^P(x)) - v(A^P(x)) \right]. \tag{2.8}
\]

We see that the profile \( P \) satisfies a delay differential equation, where the delays are introduced in the right-hand side of (2.8) by the operators \( L^P \) and \( A^P \). Since \( P(x) \in (0,1) \) for all \( x \), according to (2.6) the delay in (2.8) will always be larger than \( \ell \).

For a given \( \ell > 0 \), we seek monotone and continuously differentiable profiles \( P(\cdot) \) that satisfy (2.8) with given asymptotic conditions at \( x \to \pm \infty \). In the analysis below we also study the initial value problem, where the solutions might be non-differentiable at the initial point \( x_0 \in \mathbb{R} \). In that case, the derivative \( P'(x_0) \) on the lefthand side of (2.8) indicates \( P'(x_0^-) \).

2.1. Technical lemmas. We assume that \( w(\cdot) \) satisfies (1.3) and \( v(\cdot) \) satisfies (1.4). Let \( \hat{\rho} \) denote the unique stagnation point of the local conservation law (1.12), i.e., the point where, for \( f(\hat{\rho}) = \rho v(\hat{\rho}) \), we have
\[
0 < \hat{\rho} < 1, \quad f'(\hat{\rho}) = 0, \quad \text{and} \quad \begin{cases} f'(\rho) > 0 & \text{for } \rho < \hat{\rho}, \\ f'(\rho) < 0 & \text{for } \rho > \hat{\rho}. \end{cases} \tag{2.9}
\]

The existence of solutions of (2.8) will be established in Section 2.2 for initial value problem and Section 2.3 for asymptotic value problems. Assuming that solutions exist, in this section we establish several technical lemmas. We start with a definition.

**Definition 2.1.** Let a continuously differentiable function \( P : \mathbb{R} \to (0,1) \) be given, and let \( \ell \in \mathbb{R}^+ \) be the length of each car. Assume that
\[
\ell P'(x) < P^2(x) \quad \forall x \in \mathbb{R}. \tag{2.10}
\]

We call a sequence of car positions \( \{z_i; \ i \in \mathbb{Z}\} \) “a distribution generated by \( P(\cdot) \)” if
\[
z_{i+1} - z_i = \frac{\ell}{P(z_i)}, \quad \forall i \in \mathbb{Z}. \tag{2.11}
\]

The assumption (2.10) ensures that for each car position \( z_i \), there exists a unique follower \( z_{i-1} \) such that \( z_{i-1} + \ell/P(z_{i-1}) = z_i \). Furthermore, it implies that
\[
x + \ell/P(x) > y + \ell/P(y) \quad \text{for every } x > y.
\]

Note that if \( P(\cdot) \) satisfies the equation (2.8), then (2.10) holds, because
\[
P'(x) = \frac{P^2(x)}{\ell} \cdot v(A^P(L^P(x))) \left[ 1 - \frac{v(A^P(L^P(x)))}{v(A^P(x))} \right] \leq \frac{P^2(x)}{\ell}. \tag{2.12}
\]

We further note that there exist infinitely many car distributions generated by \( P \) for any given profile \( P \). However, if we fix the position of one car, say \( z_0 \), then the distribution is unique.

In the next Lemma we establish properties for the asymptotic limits as \( x \to \pm \infty \).
Lemma 2.2 (Asymptotic limits). Assume that $P(\cdot)$ is a continuously differentiable solution of (2.8) whose asymptotic limits satisfy

$$
\lim_{x \to -\infty} P(x) = \rho^-, \quad \lim_{x \to +\infty} P(x) = \rho^+, \quad \lim_{x \to \pm\infty} P'(x) = 0,
$$

where $\rho^-, \rho^+ \in \mathbb{R}$ and $\rho^-, \rho^+ \in (0, 1)$. Then, the following holds.

- As $x \to +\infty$, $P(x)$ approaches $\rho^+$ with an exponential rate $\lambda^+ \in \mathbb{R}^+$ if and only if $\rho^+ > \hat{\rho}$, where $\hat{\rho}$ satisfies (2.9). The rate $\lambda^+$ satisfies the estimate

$$\lambda^+ > \frac{b}{bh + a}, \quad \text{where} \quad a = \frac{\ell}{\rho^+}, \quad b = -\frac{\rho^+v'(\rho^+)}{v(\rho^+)}. \tag{2.13}$$

- As $x \to -\infty$, $P(x)$ approaches $\rho^-$ with an exponential rate $\lambda^- \in \mathbb{R}^+$ if and only if $\rho^- < \hat{\rho}$, where $\hat{\rho}$ satisfies (2.9). The rate $\lambda^-$ satisfies the estimate

$$\lambda^- > \frac{b'}{b'h + a'}, \quad \text{where} \quad a' = \frac{\ell}{\rho^-}, \quad b' = -\frac{\rho^-v'(\rho^-)}{v(\rho^-)}. \tag{2.14}$$

Proof. We consider the limit $x \to +\infty$, and linearize (2.8) at $\rho^+$. We write

$$P(x) = \rho^+ + \eta(x),$$

where $\eta(\cdot)$ is a first order perturbation. Below we use “$\approx$” to denote the first order approximation. Let $\{z_i : i \in \mathbb{Z}\}$ be a car distribution generated by $P(\cdot)$, we have,

$$L^P(x) \approx x + \frac{\ell}{\rho^+}, \quad z_{i+k+1} - z_{i+k} \approx \frac{\ell}{\rho^+}, \quad (L^P)^k(x) \approx x + \frac{k\ell}{\rho^+}. \tag{2.15}$$

For the weights $w_{i,k}$ we have

$$w_{i,k} \approx \int_{z_{i+k}}^{z_{i+k}+\ell/\rho^+} w(y - z_i) \, dy \approx \int_{k\ell/\rho^+}^{(k+1)\ell/\rho^+} w(s) \, ds \approx \hat{w}_k. \tag{2.16}$$

Note that the approximated weights $\hat{w}_k$ are independent of the index $i$. We have

$$\hat{w}_k \geq 0 \quad (0 \leq k \leq m), \quad \sum_{k=0}^{m} \hat{w}_k = 1. \tag{2.17}$$

We compute

$$A^P(x) \approx \sum_{k=0}^{m} \hat{w}_k \left( \rho^+ + \eta \left( x + \frac{k\ell}{\rho^+} \right) \right) \approx \rho^+ + \sum_{k=0}^{m} \hat{w}_k \cdot \eta \left( x + \frac{k\ell}{\rho^+} \right), \tag{2.18}$$

$$A^P(L^P(x)) \approx \rho^+ + \sum_{k=0}^{m} \hat{w}_k \cdot \eta \left( x + \frac{(k+1)\ell}{\rho^+} \right), \tag{2.19}$$

$$v(A^P(x)) \approx v(\rho^+) + v'(\rho^+) \cdot \sum_{k=0}^{m} \hat{w}_k \cdot \eta \left( x + \frac{k\ell}{\rho^+} \right), \tag{2.20}$$

$$v(A^P(L^P(x))) \approx v(\rho^+) + v'(\rho^+) \cdot \sum_{k=0}^{m} \hat{w}_k \cdot \eta \left( x + \frac{(k+1)\ell}{\rho^+} \right). \tag{2.21}$$

Plugging all these approximations into (2.8), we obtain

$$\eta'(x) \approx -v'(\rho^+) \left( \frac{\rho^+}{\ell \cdot v(\rho^+)} \right)^2 \sum_{k=0}^{m} \hat{w}_k \cdot \left[ \eta \left( x + \frac{(k+1)\ell}{\rho^+} \right) - \eta \left( x + \frac{k\ell}{\rho^+} \right) \right]. \tag{2.22}$$
Using the positive coefficients $a, b$ defined in (2.13), we can write the linearization of (2.8) as

$$
\eta'(x) = \frac{b}{a} \sum_{k=0}^{m} \hat{w}_k \left[ \eta(x + (k + 1)a) - \eta(x + ka) \right],
$$

(2.16)

where the linearized weights $\hat{w}_k$ are given in (2.15).

Note that (2.16) is a linear delay differential equation, which can be solved explicitly using its characteristic equation. Seeking solutions of the form

$$
\eta(x) = Me^{-\lambda x}, \quad \lambda \in \mathbb{R},
$$

where $M$ is an arbitrary constant (positive or negative), we have the characteristic equation

$$
-\lambda = \frac{b}{a} \sum_{k=0}^{m} \hat{w}_k e^{-ka\lambda},
$$

Thus, the rate $\lambda$ must satisfy the equation $\mathcal{L}(\lambda) = \mathcal{R}(\lambda)$ where

$$
\begin{cases}
\mathcal{L}(\lambda) = \frac{b}{a} \sum_{k=0}^{m} \hat{w}_k e^{-ka\lambda}, \\
\mathcal{R}(\lambda) = \frac{a\lambda}{1 - e^{-a\lambda}}, \quad \text{and} \quad \mathcal{R}(0) = \lim_{\lambda \to 0} \mathcal{R}(\lambda) = 1.
\end{cases}
$$

(2.17)

Furthermore, we define the derivative

$$
\mathcal{R}'(0) = \lim_{\lambda \to 0} \mathcal{R}'(\lambda) = \lim_{\lambda \to 0} \frac{a(1 - e^{-a\lambda}) - a^2 \lambda e^{-a\lambda}}{(1 - e^{-a\lambda})^2} = \frac{1}{2a},
$$

where the last equality is obtained by applying twice the L'Hôpital's rule. The functions $\mathcal{L}(\cdot)$ and $\mathcal{R}(\cdot)$ are Lipschitz continuous and satisfy the properties

$$
\mathcal{L}(0) = b, \quad \lim_{\lambda \to \infty} \mathcal{L}(\lambda) = 0, \quad \lim_{\lambda \to -\infty} \mathcal{L}(\lambda) = \infty, \quad \text{and} \quad \mathcal{L}'(\lambda) < 0 \quad \forall \lambda \in \mathbb{R},
$$

$$
\mathcal{R}(0) = 1, \quad \lim_{\lambda \to \infty} \mathcal{R}(\lambda) = \infty, \quad \lim_{\lambda \to -\infty} \mathcal{R}(\lambda) = 0, \quad \text{and} \quad \mathcal{R}'(\lambda) > 0 \quad \forall \lambda \in \mathbb{R}.
$$

We see that $\mathcal{L}(\cdot)$ is monotonically decreasing and $\mathcal{R}(\cdot)$ is monotonically increasing, and the range of both functions is $(0, \infty)$. We conclude that there exist exactly one solution $\lambda$ for (2.17). Furthermore, we observe that:

- If $b > 1$, the solution $\lambda$ is positive, which we denote by $\lambda^\ell > 0$;
- if $b = 1$, the solution is $\lambda = 0$, which leads to the trivial solution $P(x) \equiv \rho^+$;
- if $b < 1$, the solution $\lambda$ is negative, which leads to an unstable asymptote.

Thus, we obtain a stable asymptote at $x \to +\infty$ if and only if $b > 1$. We have

$$
b > 1 \iff -v'(\rho^+) \frac{(\rho^+)}{v(\rho^+)} > 1 \iff f'(\rho^+) = v(\rho^+) + \rho^+ v'(\rho^+) < 0 \iff \rho^+ > \hat{\rho},
$$

where $f(\rho) = \rho v(\rho)$, and $\hat{\rho}$ satisfies (2.9).

To get an estimate on $\lambda^\ell$, we define the monotone function $H(\lambda) \doteq \mathcal{L}(\lambda) - \mathcal{R}(\lambda)$. Since $b > 1$, we have

$$
H(0) = b - 1 > 0, \quad H'(\lambda) < 0 \quad \forall \lambda \in \mathbb{R},
$$

therefore $H$ has a unique zero which is positive. Using the inequality

$$
\frac{x}{1 - e^{-x}} < 1 + x \quad \forall x > 0,
$$
we obtain
\[
-\mathcal{R}(\lambda) > -(1 + a\lambda) \quad \text{for } \lambda > 0.
\]  
(2.18)

Moreover, since \( \hat{w}_k = 0 \) if \( ka \geq h \), then for \( \lambda > 0 \) we have
\[
\mathcal{L}(\lambda) > b \sum_{k=0}^{m} \hat{w}_k e^{-ka\lambda} = be^{-h\lambda}.
\]  
(2.19)

Combining (2.18)-(2.19), we have
\[
H(\lambda) > H'(\lambda) = be^{-h\lambda} - 1 - a\lambda \quad \forall \lambda > 0.
\]  
(2.20)

The function \( H(\cdot) \) has the properties
\[
H(0) = b - 1, \quad H'(\lambda) = -bhe^{-h\lambda} - a < 0, \quad H''(\lambda) = bh^2 e^{-h\lambda} > 0 \quad \forall \lambda > 0.
\]

Therefore \( H \) has a unique positive root \( \lambda^b \), satisfying the rough estimate
\[
\lambda^b > \frac{H(0)}{H'(0)} = \frac{b - 1}{bh + a}.
\]

Using the fact that \( \lambda^f_+ > \lambda^b \), we obtain the estimate (2.13).

A completely similar computation can be carried out for the limit \( x \to -\infty \), replacing \( \rho^+ \) with \( \rho^- \). We omit the details.

We now remark a few trivial cases.

**Remark 2.3.** We see that, if \( \rho^+ = \hat{\rho} \), then \( b = 1 \) and so \( \lambda^f_+ = 0 \). If \( \rho^+ \to 1 \), then \( b \to \infty \), so (2.17) implies
\[
\sum_{k=0}^{m} \hat{w}_k e^{-ka\lambda^f_+} \to 0 \quad \Rightarrow \quad \lambda^f_+ \to \infty.
\]

A similar argument shows that if \( \rho^- \to 0 \), then \( \lambda^- \to \infty \). Therefore, we obtain two trivial cases for the profile \( P(\cdot) \):

- If \( \rho^- = \hat{\rho} = \rho^+ \), then \( P(x) \equiv \hat{\rho} \) for all \( x \in \mathbb{R} \);
- If \( \rho^- = 0 \) and \( \rho^+ = 1 \), then \( P \) is a unit step function, with a jump at some \( x_0 \in \mathbb{R} \).

Thanks to Remark 2.3, in the sequel we consider only the nontrivial cases:
\[
0 < P(x) < 1 \quad \forall x \in \mathbb{R}, \quad \text{and} \quad 0 < \rho^- < \hat{\rho} < \rho^+ < 1. \quad (2.21)
\]

**Definition 2.4.** Let \( \{z_i(t) : i \in \mathbb{Z}\} \) be the solution of (1.7) with initial condition \( \{z_i(0) : i \in \mathbb{Z}\} \). We say that \( \{z_i(t) : i \in \mathbb{Z}\} \) is **periodic** if there exists a constant \( t_p \in \mathbb{R}^+ \), independent of \( i \) and \( t \), such that
\[
z_i(t + t_p) = z_{i+1}(t), \quad \forall i \in \mathbb{Z}, \forall t \geq 0. \quad (2.22)
\]

We refer to \( t_p \) as the **period**.

Definition 2.4 indicates that, in a periodic solution \( \{z_i(t) : i \in \mathbb{Z}\} \), after a time period of \( t_p \), each car takes over the position of its leader. Intuitively, if \( P \) is a stationary profile, and \( \{z_i(t)\} \) satisfies (1.11), then \( \{z_i(t)\} \) must be periodic. Indeed, in next Lemma we show that these two situations are equivalent.
Lemma 2.5. Let \( P : \mathbb{R} \to (0,1) \) be a continuously differentiable function. Then, \( P(\cdot) \) satisfies (2.8) if and only if
\[
\int_0^{x+\ell/P(x)} \frac{1}{v(A^p(z))} \, dz = t_p, \quad \forall x \in \mathbb{R} \tag{2.23}
\]
for some constant \( t_p \in \mathbb{R}^+ \). Moreover we have
\[
\lim_{x \to \infty} P(x) = \rho^+ \quad \implies \quad t_p = \frac{\ell}{\bar{f}}, \text{ where } \bar{f} = f(\rho^+), \tag{2.24}
\]
\[
\lim_{x \to -\infty} P(x) = \rho^- \quad \implies \quad t_p = \frac{\ell}{\bar{f}}, \text{ where } \bar{f} = f(\rho^-). \tag{2.25}
\]

Furthermore, let \( \{ z_i(t) : i \in \mathbb{Z} \} \) be a solution of (1.7). Then, \( \{ z_i(t) \} \) satisfies (1.11) if and only if \( \{ z_i(t) \} \) is periodic, with period \( t_p \) given in (2.23).

Proof. 1. Differentiating (2.23) in \( x \) on both sides, we obtain
\[
\left( 1 - \frac{\ell}{P^2(x)} P'(x) \right) \frac{1}{v(A^p(L^p(x)))} - \frac{1}{v(A^p(x))} = 0,
\]
which is equivalent to (2.8).

2. Furthermore, since (2.23) holds for all \( x \in \mathbb{R} \), we take the limit \( x \to \infty \) and get
\[
t_p = \lim_{x \to \infty} \int_x^{x+\ell/P(x)} \frac{1}{v(A^p(z))} \, dz = \frac{\ell}{\rho^+}, \quad \frac{1}{v(A^p(x))} = \frac{\ell}{\bar{f}(\rho^+)},
\]
proving (2.24). A completely similar argument leads to (2.25).

3. Let \( \{ z_i(t) : i \in \mathbb{Z} \} \) be a solution of (1.7). Assume that \( \{ z_i(t) \} \) satisfies (1.11). Fix an index \( i \in \mathbb{Z} \) and a time \( \hat{t} \geq 0 \). We have
\[
\frac{dz_i}{dt} = v(A^p(z_i)) \quad \implies \quad \frac{dz_i}{v(A^p(z_i))} = dt,
\]
a separable equation which can be solved implicitly. The time it takes for the car at \( z_i \) to reach \( z_{i+1} = z_i + \ell/P(z_i) \) is
\[
t_{p,i} = \int_{\hat{t}}^{\hat{t}+t_{p,i}} dt = \int_{z_i}^{z_i+\ell/P(z_i)} \frac{1}{v(A^p(z))} \, dz.
\]
Thanks to (2.23), we conclude \( t_{p,i} = t_p \) for all \( i \in \mathbb{Z} \), therefore \( \{ z_i(t) \} \) is periodic.

On the other hand, assume that \( \{ z_i(t) \} \) is periodic, and let \( t_p \) denote its period. Fix a time \( \hat{t} \in \mathbb{R}^+ \), and consider the interval \( t \in [\hat{t}, \hat{t} + t_p] \). On each space interval \( x \in [z_i(\hat{t}), z_{i+1}(\hat{t})] \), we define the function \( P \) as
\[
P(z_i(t)) = \rho_i(t) \quad t \in [\hat{t}, \hat{t} + t_p].
\]
Since \( \{ z_i(t) \} \) is periodic, we have \( P(z_i(\hat{t} + t_p)) = P(z_{i+1}(\hat{t})) \), and \( P \) is continuous.
The period, i.e., the time it takes for car at \( z_i(\hat{t}) \) to reach \( z_{i+1}(\hat{t}) \), is
\[
t_p = \int_{z_i}^{z_i+\ell/P(z_i)} \frac{1}{v(A^p(z))} \, dz \quad \forall i.
\]
Since \( \hat{t} \) is arbitrarily chosen, we conclude (1.11). \( \square \)
2.2. Initial value problems. We assume that the assumptions (1.3)-(1.4) hold. Fix any point \( x_0 \in \mathbb{R} \), and let \( \Psi : [x_0, +\infty] \to (0, 1) \) be a smooth increasing function, satisfying
\[
0 < \Psi(x) < 1, \quad \Psi'(x) > 0, \quad \forall x \geq x_0. \tag{2.26}
\]

We let the “initial condition” be given on \( x \geq x_0 \), i.e.,
\[
P(x) = \Psi(x), \quad \text{for } x \geq x_0. \tag{2.27}
\]

Then, the delay differential equation (2.8) can be solved backwards in \( x \), for \( x < x_0 \). We refer to this as the “initial value problem” for (2.8), and we seek continuously differentiable solutions \( P(\cdot) \) on \( x < x_0 \). Note that, since \( \Psi(x) \) might not satisfy (2.8) on \( x \geq x_0 \), the derivative \( P' \) might not be continuous at \( x_0 \). Therefore, it is assumed that the derivative \( P'(x_0) \) at the initial point in (2.8) denotes the left derivative \( P'(x_0^-) \).

We remark that for the delay differential equation (2.8) the minimum delay is \( \ell \) and the maximum delay is bounded by \( h/\ell + \ell/P_0 \) where \( P_0 > 0 \) is a lower bound for \( P \). If \( P_0 \) is given, then the initial condition is only needed on an interval \([x_0, x_0 + d]\) where \( d = (h+\ell/P_0) - \ell \). However we do not have an a priori bound for \( P_0 \), therefore we provide the initial condition on the whole half line \( x \geq x_0 \).

**Lemma 2.6** (Monotonicity and positivity). Let \( P(\cdot) \) be a solution of the initial value problem for (2.8), with initial condition (2.27), satisfying (2.26). Then, we have
\[
0 < P(x) < \Psi(x_0), \quad P'(x) > 0, \quad \forall x < x_0. \tag{2.28}
\]

**Proof.** We first prove the monotonicity. Observe that, since \( \Psi \) is monotone increasing on \( x \geq x_0 \), by (2.8) we have \( P'(x_0^-) > 0 \). We now proceed with contradiction. Assume that \( P(\cdot) \) fails to be monotone on \( x < x_0 \). Then there exists a point \( \hat{x} < x_0 \) such that
\[
P'(\hat{x}) = 0, \quad P'(x) > 0 \quad \forall x > \hat{x}. \tag{2.29}
\]
Since \( P(x) \) is monotone on \( x > \hat{x} \), and \( \hat{A}P \) is an averaging operator, we have
\[
\hat{A}P(\hat{x}) < \hat{A}P \left(L^P(\hat{x})\right), \quad v \left(\hat{A}P(\hat{x})\right) > v \left(\hat{A}P(L^P(\hat{x}))\right).
\]
By (2.8) we get \( P'(\hat{x}) > 0 \), contradicting (2.29).

The positivity of \( P(\cdot) \) follows from the fact that equation (2.8) is “autonomous” and \( P = 0 \) is a critical point.

The next theorem states the existence and uniqueness for the initial value problem.

**Theorem 2.7.** Consider the initial value problem for (2.8) with initial condition (2.27), satisfying (2.26). Then, there exists a unique continuous solution \( P(\cdot) \) on \((-\infty, x_0]\) with the following properties.

- \( P(\cdot) \) is monotone and continuously differentiable on \( x < x_0 \), with
  \[
P'(x) \leq \ell^{-1} P^2(x) \leq \ell^{-1} \Psi^2(x_0) < \ell^{-1}. \tag{2.30}
  \]

- \( P(\cdot) \) has the asymptotic value
  \[
  \lim_{x \to -\infty} P(x) = \rho_0^-,
  \]
  where \( \rho_0^- \in \mathbb{R} \) satisfies
  \[
  0 < \rho_0^- < \hat{\rho}, \quad \frac{\ell}{f(\rho_0^-)} = \int_{x_0}^{x_0 + \ell/\Psi(x_0)} \frac{1}{v(A^z(z))} \, dz. \tag{2.31}
  \]
Here \( \hat{\rho} \) is defined in (2.9), and the operator \( A^\Psi \) is defined in (2.7), replacing \( P \) with \( \Psi \).

- Let \( L_v \) be the Lipschitz constant for the map \( \rho \mapsto (1/v(\rho)) \) for \( 0 < \rho \leq \Psi(x_0) < 1 \), we have

\[
P \left( x + \frac{\ell}{P(x)} + h \right) - P(x) > \frac{P(x)}{L_v} \cdot \left( \frac{1}{f(\rho_0)} - \frac{1}{f(P(x))} \right)
\]

(2.32)

for all \( x \leq x_0 \).

Proof. (1). Since (2.8) is a delay differential equation with delay of at least \( \ell \), the existence and uniqueness can be established by the standard method of steps, see [21]. We define the intervals

\[
I_k = [x_0 - (k + 1)\ell, x_0 - k\ell], \quad k \in \mathbb{Z}^+.
\]

Consider first the interval \( I_0 = [x_0 - \ell, x_0] \). For \( x \in I_0 \), the right hand side of (2.8) involve only values with \( x \geq x_0 \), which are given as initial condition. The existence and uniqueness of solutions follow from standard theory for scalar ODEs. Furthermore, since \( \Psi \) is monotone and positive, by Lemma 2.6 the solution \( P(\cdot) \) is monotone and positive on \( I_0 \). Finally, thanks to (2.12) we have

\[
P' = \ell - \frac{1}{P^2} \leq \ell - \frac{1}{\Psi^2(x_0)},
\]

proving (2.30).

(2). The argument can be repeated on all subsequent intervals \( I_k \) for \( k \in \mathbb{Z}^+ \), leading to the existence and uniqueness of Lipschitz solution on \( x \leq x_0 \).

(3). Furthermore, by the periodic property in Lemma 2.5, we have

\[
\int_x^{x + \ell/P(x)} \frac{1}{v(A^P(z))} \, dz = t_p, \quad \forall x \leq x_0.
\]

Setting \( x = x_0 \), we have

\[
t_p = \int_{x_0}^{x_0 + \ell/\Psi(x_0)} \frac{1}{v(A^\Psi(z))} \, dz,
\]

and together with Lemma 2.5, we obtain (2.31).

(4). By the periodic property in step (3), we have, for every \( x < x_0 \),

\[
f / f_0 = \frac{\ell}{f(P(x)) \cdot v(P(x))} = \int_x^{x + \ell/P(x)} \frac{1}{v(A^P(z))} \, dz, \quad \tilde{f}_0 = f(\rho_0).
\]

Subtracting from it the identity

\[
\frac{\ell}{f(P(x))} = \frac{\ell}{P(x) \cdot v(P(x))} = \int_x^{x + \ell/P(x)} \frac{1}{v(A^P(z))} \, dz,
\]

we get

\[
\frac{\ell}{f_0} - \frac{\ell}{f(P(x))} = \int_x^{x + \ell/P(x)} \left[ \frac{1}{v(A^P(z))} - \frac{1}{v(P(z))} \right] \, dz.
\]

(2.33)

Since \( P \) is monotone increasing, we have, for any \( z \in [x, x + \ell/P(x)] \),

\[A^P(z) < A^P(x + \ell/P(x)) < P(x + \ell/P(x) + h).\]
Combining this with (2.33), we get

\[
\frac{\ell}{f_0} - \frac{\ell}{f(P(x))} < \frac{\ell}{P(x)} \left[ \frac{1}{v(P(x + \ell/P(x) + h))} - \frac{1}{v(P(x))} \right] < \frac{\ell}{P(x)} \cdot L_v \cdot [P(x + \ell/P(x) + h) - P(x)],
\]

where \(L_v\) is the Lipschitz constant for the map \((1/v)\). This proves (2.32). \(\square\)

2.3. Asymptotic value problems.

**Theorem 2.8** (Asymptotic value problem). Let \(w(\cdot)\) satisfy (1.3) and let \(v(\cdot)\) satisfy (1.4). Consider the asymptotic value problem for (2.8), with asymptotic conditions

\[
\lim_{x \to -\infty} P(x) = \rho^-, \quad \lim_{x \to \infty} P(x) = \rho^+.
\]

If \(\rho^-, \rho^+ \in \mathbb{R}\) satisfy

\[
0 < \rho^- < \hat{\rho} < \rho^+ < 1, \quad f(\rho^-) = f(\rho^+) = \bar{f},
\]

then there exists a monotone increasing, Lipschitz continuous solution \(P(\cdot)\), defined for all \(x \in \mathbb{R}\).

Furthermore, the solutions are unique up to horizontal shifts, in the following sense. If \(P_1\) and \(P_2\) are two solutions of the same asymptotic value problem, then there exists a constant \(c \in \mathbb{R}\) such that \(P_1(x) = P_2(x + c)\) for all \(x \in \mathbb{R}\).

**Proof.** Existence of solutions. The proof for the existence of solutions takes several steps.

1. A solution will be constructed by taking the limit of a sequence of approximations. Let \(\lambda^x_{\ell}\) be the exponential rate given in Lemma 2.2 for the asymptotic condition \(\lim_{x \to -\infty} P(x) = \rho^+\), and let

\[
\Psi(x) = \rho^+ - e^{-\lambda^x_{\ell}x}.
\]

Let the sequence \(\{x_n \in \mathbb{R} : n \in \mathbb{N}\}\) satisfy \(x_n < x_{n+1}\) for all \(n\), and \(\lim_{n \to \infty} x_n = \infty\), and let \(P^{(n)}(\cdot)\) be the unique solution for the initial value problem of (2.8) with initial condition \(P^{(n)}(x) = \Psi(x)\) for \(x \geq x_n\), established in Theorem 2.7. Then \(P^{(n)}(\cdot)\) is Lipschitz, positive and monotone increasing for \(x \in ]-\infty, x_n]\). Denoting

\[
\rho^+_n \doteq \lim_{x \to -\infty} P^{(n)}(x),
\]

by Lemma 2.2 we have \(\rho^+_n < \hat{\rho}\). We further claim that

\[
\lim_{n \to \infty} f(\rho^-_n) = f(\rho^+) = f(\rho^-).
\]

Indeed, denoting

\[
\bar{f}_n \doteq \ell \cdot \left[ \int_{x_n}^{x_n+\ell/\Psi(x_n)} \frac{1}{v(A^\Psi(z))} dz \right]^{-1},
\]

by Theorem 2.7, we have

\[
\frac{\ell}{f(\rho^-_n)} = \frac{\ell}{\bar{f}_n}.
\]
Let $\varepsilon > 0$. There exists an $N$, sufficiently large, such that $e^{-\lambda_{\varepsilon}x_n} < \varepsilon$ for all $n > N$. Using (2.36), we compute, for $n > N$,

\[
\frac{\ell}{f(\bar{\rho}_n)} < \int_{x_n}^{x_n + \ell/(\rho^+ - \varepsilon)} \frac{1}{v(\rho^+)} \, dz = \frac{\ell}{(\rho^+ - \varepsilon)v(\rho^+)},
\]

\[
\frac{\ell}{f(\bar{\rho}_n)} > \int_{x_n}^{x_n + \ell/\rho^+} \frac{1}{v(\rho^+ - \varepsilon)} \, dz = \frac{\ell}{\rho^+v(\rho^+ - \varepsilon)}.
\]

Since $\varepsilon > 0$ is arbitrary, we conclude that

\[
\lim_{n \to \infty} \frac{\ell}{\bar{f}_n} = \lim_{n \to \infty} \frac{\ell}{f(\bar{\rho}_n)} = \frac{\ell}{\rho^+v(\rho^+ - \varepsilon)},
\]

which proves (2.37). This further implies that

\[
\lim_{n \to \infty} \rho_n = \rho^-, \quad \lim_{n \to \infty} \bar{f}_n = \bar{f} = f(\rho^-) = f(\rho^+).
\]

2. By Theorem 2.7, $P^{(n)}$ is Lipschitz continuous, with Lipschitz constant $\ell^{-1}$. Since we are assuming $\rho^- < \hat{\rho} < \rho^+$, by (2.38) and (2.30) it follows that, for every $n$ large enough, there exists a value $\xi_n$ such that $P^{(n)}(\xi_n) = \hat{\rho}$. We can thus consider the sequence of shifted profiles

\[
\bar{P}^{(n)}(x) = P^{(n)}(x - \xi_n).
\]

This guarantees that $\bar{P}^{(n)}(0) = \hat{\rho}$, for every $n$ large enough.

3. Since the functions $\bar{P}^{(n)}$ are uniformly bounded, monotone increasing and uniformly Lipschitz continuous, by the Arzelà-Ascoli theorem $\bar{P}^{(n)}$ converges to a limit function $P$, where the convergence is uniform on every compact subset of $\mathbb{R}$.

From the properties of all $\bar{P}^{(n)}$ it immediately follows that $P$ is nondecreasing and uniformly Lipschitz continuous. Moreover

\[
P(0) = \lim_{n \to \infty} \bar{P}^{(n)}(0) = \hat{\rho}.
\]

By Lemma 2.5, the differential equation (2.8) can be written in the integral form for $\bar{P}^{(n)}$,

\[
\int_x^{x + \ell/\bar{P}^{(n)}(x)} v^{-1}(A\bar{P}^{(n)}(z)) \, dz = \bar{f}_n.
\]

Recalling that the convergence $\bar{P}^{(n)}(x) \to P(x)$ is uniform on bounded sets, we conclude that the limit function $P$ satisfies the integral equation

\[
\int_x^{x + \ell/P(x)} v^{-1}(AP(z)) \, dz = \bar{f}.
\]

By Lemma 2.5, $P$ provides a solution to (2.8).

4. It remains to prove the asymptotic limits

\[
\lim_{x \to -\infty} P(x) = \rho^-, \quad \lim_{x \to +\infty} P(x) = \rho^+.
\]

Since $P(\cdot)$ is nondecreasing and bounded, it is clear that these two limits exist. Assume that

\[
P^+ \doteq \lim_{x \to +\infty} P(x), \quad P^+ < \rho^+.
\]
From Lemma 2.2 we have $P^+ > \hat{\rho}$. Let $\epsilon > 0$. There exist $M \in \mathbb{R}$ sufficiently large, such that $P^+ - P(x) < \epsilon$ for every $x > M$. This implies

$$P(x + \ell/P(x) + h) - P(x) < \epsilon \quad \forall x > M.$$ 

However, using (2.32) we have

$$P(x + \ell/P(x) + h) - P(x) > \frac{P^+ - \epsilon}{L_v} \left[ \frac{1}{f(\rho^+)} - \frac{1}{f(P^+)} \right],$$

a contradiction. We conclude that $P^+ = \rho^+$. A similar analysis yields the asymptotic limit at $x \to -\infty$. This proves the existence of the asymptotic value problem.

**Uniqueness.** Assume that there exist two solutions $P_1$ and $P_2$ that are distinct after any horizontal shift. Then, we can consider some shifted versions of $P_1, P_2$ such that their graphs cross each other. Let $\hat{x}$ be the rightmost point where they cross, and assume

$$P_1(\hat{x}) = P_2(\hat{x}), \quad P_1(x) > P_2(x) \quad \forall x > \hat{x}. \quad (2.42)$$

Denote by $A^{P_1}, A^{P_2}$ and $L^{P_1}, L^{P_2}$ the averaging operators and the leader operators corresponding to $P_1, P_2$, respectively. By the assumptions (2.42), we have, for all $x \geq \hat{x}$,

$$A^{P_1}(x) > A^{P_2}(x), \quad \text{therefore} \quad \frac{1}{v(A^{P_1}(x))} > \frac{1}{v(A^{P_2}(x))}. \quad (2.43)$$

Observe that we have

$$L^{P_1}(\hat{x}) = \hat{x} + \frac{\ell}{P_1(\hat{x})} = \hat{x} + \frac{\ell}{P_2(\hat{x})} = L^{P_2}(\hat{x}).$$

Since both profiles admit the same period, we have

$$\int_{\hat{x}}^{L^{P_1}(\hat{x})} \frac{1}{v(A^{P_1}(z))} \, dz = \int_{\hat{x}}^{L^{P_2}(\hat{x})} \frac{1}{v(A^{P_2}(z))} \, dz,$$

a contradiction to (2.43). Thus, we conclude that the solutions of the asymptotic value problem are unique, up to horizontal shifts.

Sample profiles. Sample profiles for $P(\cdot)$ with various $(\rho^-, \rho^+)$ values and $w(\cdot)$ functions are given in Figure 1. The profiles are generated using the approximate solutions described in Theorem 2.8.

2.4. **Stability of the traveling waves.** It is natural to assume that the road condition right in front of the driver is more important than the condition further ahead. This leads to the additional assumption

$$w'(x) \leq 0 \quad \forall x \in (0, h). \quad (2.44)$$

Furthermore, we also assume

$$v''(\rho) \leq 0 \quad \forall \rho \in [0, 1]. \quad (2.45)$$

This assumption gives $f''(\rho) < 0$ for all $\rho \in [0, 1]$, where $f(\rho) = \rho v(\rho)$. With these additional assumptions, the traveling wave profiles turn out to be local attractors for solutions of the FtLs model (1.7). Specifically, with mild assumptions on the initial condition, the traveling wave profiles provide time asymptotic limits for the solutions of the FtLs model (1.7).
Figure 1. Typical profiles \( P(\cdot) \) with \( v(\rho) = 1 - \rho, h = 0.2, \ell = 0.01 \) and various \([\rho^-, \rho^+]\) values given in the legends. In the left we use the weight function \( w(x) = \frac{2}{h} - \frac{2x^2}{h^2} \) on \((0, h)\) where \( w' < 0 \), while in right plot we use \( w(x) = \frac{2x}{h} \) on \((0, h)\) where \( w' > 0 \).

Theorem 2.9. Let \( w(\cdot) \) satisfy (1.3) and (2.44), and \( v(\cdot) \) satisfy (1.4) and (2.45), and let \( \rho^-, \rho^+ \in \mathbb{R} \) be given with

\[
0 < \rho^- < \hat{\rho} < \rho^+ < 1, \quad f(\rho^-) = f(\rho^+) = \bar{f}.
\]

Let \( P(\cdot) \) be the solution for the asymptotic value problem, satisfying (2.8), the asymptotic conditions (2.34), and the additional condition \( P(0) = \hat{\rho} \). Let \( \{z_i(t) : i \in \mathbb{Z}\} \) be the solution of (1.7) with initial condition \( \{z_i(0) : i \in \mathbb{Z}\} \), and let \( \{\rho_i(t) : i \in \mathbb{Z}\} \) be the corresponding discrete densities, defined in (1.6).

Assume that there exist two constants \( c_1, c_2 \in \mathbb{R} \), such that the initial condition satisfies

\[
P(z_i + c_1) \geq \rho_0(0) \geq P(z_i + c_2), \quad \forall i \in \mathbb{Z},
\]

Then, there exists a constant \( \hat{c} \in \mathbb{R} \), such that

\[
\begin{align*}
\lim_{t \to \infty} [\rho_i(t) - P(z_i(t) + \hat{c})] &= 0, \quad \forall i \in \mathbb{Z}.
\end{align*}
\]

Proof. Step 1. We first observe that assumption (2.46) implies

\[
\lim_{i \to \pm \infty} \rho_0(0) = \rho^\pm.
\]

Fix a time \( t \geq 0 \), and let \( \{z_i(t) : i \in \mathbb{Z}\} \) be the solution of the FtLs model and \( \{\rho_i(t) : i \in \mathbb{Z}\} \) the corresponding discrete densities. Denote by \( \hat{P}(\cdot) \) the profile that satisfies

\[
\begin{align*}
\hat{P}(x) &= P(x + \hat{c}) \quad \forall x \in \mathbb{R}, \text{ for some } \hat{c} \in \mathbb{R}, \\
\hat{P}(z_i(t)) &\geq \rho_i(t) \quad \forall i \in \mathbb{Z}.
\end{align*}
\]

Let \( k \) be an index such that

\[
\hat{P}(z_k(t)) = \rho_k(t), \quad \text{and } \hat{P}(z_i(t)) > \rho_i(t) \quad \forall i > k,
\]

and hence

\[
L^\hat{P}(z_k) = z_{k+1} = z_k + \frac{\ell}{\hat{P}(z_k)}.
\]

We claim that

\[
\frac{\hat{P}(z_k)}{z_k} < \hat{P}'(z_k),
\]
Indeed, (2.48) and \( v' \leq 0 \) imply
\[
A^P(z_k) > \rho^*_k, \quad A^P \left( L^P(z_k) \right) > \rho^*_{k+1}.
\] (2.50)
Furthermore, using \( v' \leq 0 \) we get
\[
v(A^P(z_k)) \leq v(\rho^*_k).
\] (2.51)
Equation (2.8) can be written as
\[
\dot{P}(z_k) = A_1 B_1 C_1,
\]
where
\[
A_1 = \frac{\dot{P}(z_k)}{v(A^P(z_k))},
\]
\[
B_1 = \frac{v(A^P(z_k)) - v(A^P(L^P(z_k)))}{A^P(L^P(z_k)) - A^P(z_k)},
\]
\[
C_1 = \frac{A^P(L^P(z_k)) - A^P(z_k)}{L^P(z_k) - z_k}.
\]
On the other hand, equations (1.7) and (2.2) lead to
\[
\frac{\dot{\rho}_k}{z_k} = A_2 B_2 C_2,
\]
where
\[
A_2 = \frac{\rho_k}{v(\rho^*_k)}, \quad B_2 = \frac{v(\rho^*_k) - v(\rho^*_{k+1})}{\rho^*_{k+1} - \rho^*_k}, \quad C_2 = \frac{\rho^*_{k+1} - \rho^*_k}{z_{k+1} - z_k}.
\]
By (2.48) and (2.50), we have \( A_2 < A_1 \). Since \( v'' \leq 0 \), by (2.50) we have \( B_2 \leq B_1 \). Finally, to compare \( C_1 \) and \( C_2 \), let \( \{y_i\} \) be the car distribution generated by the profile \( \dot{P}(\cdot) \) with \( y_i = z_k \). We also have \( y_{k+1} = z_{k+1} \). Define the piecewise constant functions \( \dot{P}^\ell(\cdot) \) and \( \rho^\ell(\cdot, \cdot) \) as
\[
\dot{P}^\ell(x) = \dot{P}(y_i) \quad \text{for} \quad x \in [y_i, y_{i+1}),
\]
\[
\rho^\ell(x, t) = \rho_i(t) \quad \text{for} \quad x \in [z_i(t), z_{i+1}(t)).
\]
We have
\[
\dot{P}^\ell(x) > \rho^\ell(x, t) \quad \forall x > z_{k+1}.
\] (2.52)
Moreover,
\[
\rho^*_{k+1} - \rho^*_k = - \int_{z_k}^{z_{k+1}} \rho_k w(y - z_k) \, dy + \int_{z_{k+1}}^{\infty} \rho^\ell(y, t) [w(y - z_{k+1}) - w(y - z_k)] \, dy,
\]
and
\[
A^P \left( L^P(z_k) \right) - A^P(z_k) = - \int_{z_k}^{z_{k+1}} \dot{P}(z_k) w(y - z_k) \, dy + \int_{z_{k+1}}^{\infty} \dot{P}^\ell(y) [w(y - z_{k+1}) - w(y - z_k)] \, dy.
\]
Since \( w' \leq 0 \), we have \( w(y - z_{k+1}) - w(y - z_k) \geq 0 \). Using (2.48) and (2.52), we conclude that \( C_2 \leq C_1 \). This proves (2.49).
On the other hand, let \( \tilde{P}(\cdot) \) be a profile such that
\[
\tilde{P}(x) = P(x + \hat{c}) \quad \forall x \in \mathbb{R},
\]
for some \( \hat{c} \in \mathbb{R} \), and \( \tilde{P}(z_i(t)) \leq \rho_i(t) \), \( \forall i \).
Let \( k \) be an index such that
\[
\tilde{P}(z_k(t)) = \rho_k(t), \quad \text{and} \quad \tilde{P}(z_i(t)) < \rho_i(t) \quad \forall i > k.
\]
(2.53)

Then, by a totally similar argument one concludes
\[
\frac{\dot{\rho}_k}{z_k} > \tilde{P}'(z_k).
\]
(2.54)

**Step 2.** The stability of the stationary profiles is a consequence of (2.49) and (2.54). Let \( P(\cdot) \) denote the profile with \( P(0) = \hat{\rho} \), where \( \hat{\rho} \) is defined in (2.9). Since any horizontal shift of \( P(\cdot) \) is also a profile, we have a family of non-intersecting profiles generated by horizontal shifts of \( P(\cdot) \). Then, in the \((x, P)\)-plane, any point \((x, \rho)\) with \( \rho^- < \rho < \rho^+ \) must lie on a unique profile. This motivates the introduction of the following mapping
\[
\Phi(x, \rho) = P(0), \quad \text{where } P(\cdot) \text{ is a profile such that } P(x) = \rho.
\]
(2.55)

Let \( \{z_i(t) : i \in \mathbb{Z}\} \) be the solution of (1.7) and \( \{\rho_i(t) : i \in \mathbb{Z}\} \) the corresponding discrete densities, as in the setting of the theorem. Define the functions
\[
\phi_i(t) = \Phi(z_i(t), \rho_i(t)), \quad i \in \mathbb{Z}.
\]
Fix a time \( t \geq 0 \). Let \( k_{\min} \) and \( k_{\max} \) be the indices where \( \{\phi_i(t) : i \in \mathbb{Z}\} \) attains its minimum and maximum values, respectively, such that
\[
\phi_i(t) \geq \phi_{k_{\min}}(t) \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \phi_i(t) > \phi_{k_{\min}}(t) \quad \forall i > k_{\min},
\]
\[
\phi_i(t) \leq \phi_{k_{\max}}(t) \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \phi_i(t) < \phi_{k_{\max}}(t) \quad \forall i > k_{\max}.
\]

By the results in Step 1, we now have
\[
\frac{d}{dt} \phi_{k_{\min}}(t) > 0, \quad \frac{d}{dt} \phi_{k_{\max}}(t) < 0.
\]
This further implies that
\[
\lim_{t \to \infty} [\phi_{k_{\max}}(t) - \phi_{k_{\min}}(t)] = 0,
\]
therefore
\[
\lim_{t \to \infty} \phi_i(t) = \hat{\phi} = \text{constant} \quad \forall i \in \mathbb{Z}.
\]
This proves (2.47), where \( \hat{c} \) satisfies \( P(-\hat{c}) = \hat{\rho} \).

**Numerical simulations.** We consider an initial condition \( \{z_i(0), \rho_i(0)\} \) which satisfies
\[
\rho_i(0) = \begin{cases} 
0.2, & z_i(0) \leq -0.3, \\
0.5 - 0.3 \times \sin(5\pi z_i(0)), & -0.3 < z_i(0) < 0.3, \\
0.8, & z_i(0) \geq 0.3.
\end{cases}
\]
(2.56)

see the left plots in Figure 2. Typical solutions of the FtLs model \( (z_i(t), \rho_i(t)) \) at various time \( t \) are given in the same figure, for two different weight functions \( w(\cdot) \).
We observe that if \( w' < 0 \), as in the top plots, the oscillations damp out quickly as \( t \) grows, and the solution approaches some profile \( P(\cdot) \). On the other hand, when \( w' > 0 \), as in the bottom plots, the solution becomes more oscillatory as \( t \) grows, indicating the instability of the profiles.
Figure 2. Typical solutions of the FtLs model \((z_i(t), \rho_i(t))\) at various time \(t\), with oscillatory initial condition. Above: \(w(x) = \frac{2}{h^2} - \frac{2x}{h^2}\) and the solution approaches some profile as \(t\) grows. Below: \(w(x) = \frac{2x}{h^2}\) and the solution oscillates more as \(t\) grows.

Remark 2.10. We remark that similar oscillatory behaviors with \(w' > 0\) were observed in other literatures, e.g. [9], for the solution of the nonlocal conservation law (1.1), where the maximum principle fails. It would be of interest to carry out rigorous analysis for the increase in the oscillation as time grows.

Remark 2.11. If the initial condition satisfies the assumptions in Theorem 2.9, then the solution approaches a stationary profile as \(t \to \infty\), by Theorem 2.9. The above numerical simulation suggests possible improvements for Theorem 2.9. Indeed, note that for the profile \(P(\cdot)\) with \(\lim_{x \to -\infty} P(x) = 0.2\) and \(\lim_{x \to \infty} P(x) = 0.8\), one has that \(0.2 < P(x) < 0.8\) for every bounded \(x\). We remark that the initial condition (2.56) does not satisfy the assumptions in Theorem 2.9, since \(\rho(x) = 0.8\) for \(x \geq 0.3\). Nevertheless, we observe stability in the simulation. This indicates that the basin of attraction is probably larger than the assumptions in Theorem 2.9.

3. The nonlocal conservation law. In this section we consider the stationary traveling wave profile \(Q(\cdot)\) for (1.1). We denote by \(A\) the continuous averaging operator

\[
A(Q; x) \doteq \int_x^{x+h} Q(y) w(y - x) \, dy = \int_0^h Q(x + s)w(s) \, ds, \tag{3.1}
\]

and with a slight abuse of notation, we denote the operator also for a function of two variables,

\[
A(\rho; t, x) \doteq \int_x^{x+h} \rho(t, y)w(y - x) \, dy = \int_0^h \rho(t, x + s)w(s) \, ds. \tag{3.2}
\]

A stationary profile for (1.1) satisfies

\[
Q(x) \cdot v(A(Q; x)) \equiv \bar{f} = \text{constant}. \tag{3.3}
\]

In the case where

\[
\lim_{x \to -\infty} Q(x) = \rho^-, \quad \text{and/or} \quad \lim_{x \to +\infty} Q(x) = \rho^+,
\]
the constant \( \bar{f} \) satisfies (respectively)
\[
\bar{f} = \lim_{x \to -\infty} Q(x) \cdot v(A(Q; x)) = f(\rho^-), \quad \text{and/or}
\]
\[
\bar{f} = \lim_{x \to +\infty} Q(x) \cdot v(A(Q; x)) = f(\rho^+).
\]

We seek continuously differentiable solutions \( Q(\cdot) \) for the integral equation (3.3). Differentiating (3.3) in \( x \), we can rewrite it equivalently as a differential equation:
\[
Q'(x) = -\frac{Q(x)v'(A(Q; x))}{v(A(Q; x))} \int_0^h Q'(x + s)w(s) \, ds. \tag{3.4}
\]
Note that (3.4) is a delay integro-differential equation. We remark that, for smooth solutions of \( Q(\cdot) \), (3.3) and (3.4) are equivalent. In the sequel we consider the initial value problems in Section 3.2, where the solution for \( Q(\cdot) \) is not differentiable at the initial point \( x_0 \), and the derivative \( Q'(x_0) \) in (3.4) is assumed to be the left derivative \( Q'(x_0^-) \).

3.1. Technical lemma. The existence of solutions for (3.4) will be established in Section 3.2 for initial value problems, and in Section 3.3 for asymptotic value problems. Assuming that the solutions exist, we establish some technical lemma.

**Lemma 3.1** (Asymptotic limits). Assume that \( Q(\cdot) \) is a solution of (3.4) which satisfies
\[
\lim_{x \to -\infty} Q(x) = \rho^- \quad \text{and} \quad \lim_{x \to +\infty} Q(x) = \rho^+.
\]
Then, the following holds.

i) As \( x \to +\infty \), \( Q(x) \) approaches \( \rho^+ \) with an exponential rate if and only if \( \rho^+ > \hat{\rho} \), where \( \hat{\rho} \) satisfies (2.9). The rate \( \lambda_+ \) satisfies the estimate
\[
\lambda_+ > \frac{1}{h} \ln \left( -\frac{\rho^+ v'(\rho^+)}{v(\rho^+)} \right). \tag{3.6}
\]

ii) As \( x \to -\infty \), \( Q(x) \) approaches \( \rho^- \) with an exponential rate if and only if \( \rho^- < \hat{\rho} \), where \( \hat{\rho} \) satisfies (2.9). The rate \( \lambda_- \) satisfies the estimate
\[
\lambda_- > \frac{1}{h} \ln \left( -\frac{\rho^- v'(\rho^-)}{v(\rho^-)} \right). \tag{3.7}
\]

**Proof.** We consider the limit \( x \to +\infty \), and assume that \( Q(x) \to \rho^+ \) in the limit. We linearize (3.3) at \( \rho^+ \) and write
\[
Q(x) = \rho^+ + \epsilon(x),
\]
where \( \epsilon(x) \) is a small perturbation. Keeping only the first order terms of \( \epsilon \) and using the notation \( \approx \), we compute,
\[
A(Q; x) = \int_x^{x+h} (\rho^+ + \epsilon(y))w(y - x) \, dy = \rho^+ + \int_x^{x+h} \epsilon(y)w(y - x) \, dy,
\]
\[
v(A(Q; x)) \approx v(\rho^+) + v'(\rho^+) \cdot \int_x^{x+h} \epsilon(y)w(y - x) \, dy.
\]
Putting these in (3.3), one gets
\[
(\rho^+ + \epsilon(x)) \cdot \left[ v(\rho^+) + v'(\rho^+) \cdot \int_x^{x+h} \epsilon(y)w(y - x) \, dy \right] \approx \bar{f} = \rho^+ v(\rho^+).
\]
We obtain the following linear integral equation for the perturbation $\epsilon(x)$:

$$
- \beta \int_x^{x+h} \epsilon(y)w(y-x) \, dy + \epsilon(x) = 0,
$$

where $\beta = - \frac{\rho^+ v'(\rho^+)}{v(\rho^+)}$. \hspace{1cm} (3.8)

Note that we have $\beta = b$, where $b$ is defined in (2.13). We can solve (3.8) using the characteristic equation. We seek solutions of the form

$$
\epsilon(x) = Me^{-\lambda x},
$$

where $M$ is an arbitrary constant (positive or negative), and $\lambda \in \mathbb{R}$ is the exponential rate. Plugging this into (3.8), we get

$$
G(\lambda) = \int_0^h e^{-\lambda s}w(s) \, ds - \frac{1}{\beta} = 0.
$$

The function $G(\cdot)$ has the properties

$$
G(0) = 1 - \frac{1}{\beta}, \quad \lim_{\lambda \to +\infty} G(\lambda) = - \frac{1}{\beta} < 0,
$$

$$
G'(\lambda) = - \int_0^h se^{-\lambda s}w(s) \, ds < 0 \quad \forall \lambda.
$$

Thus, $G(\lambda) = 0$ has a unique positive solution if and only if $G(0) > 0$, i.e.,

$$
\frac{1}{\beta} < 1 \iff - \frac{v(\rho^+)}{\rho^+ v'(\rho^+)} < 1 \iff \rho^+ > \hat{\rho}.
$$

We denote this solution by $\lambda_+$. For any given $\rho^+$, $h$, and $w(\cdot)$, an estimate for $\lambda_+$ can be obtained by observing

$$
e^{-\lambda_+ h} = \int_0^h e^{-\lambda_+ s}w(s) \, ds < \frac{1}{\beta},
$$

which implies (3.6). A completely symmetric argument leads to the result in ii) for the limit $x \to -\infty$.

**Remark 3.2.** We observe two trivial cases.

1. If $\rho^+ = \hat{\rho}$, then $\beta = 1$, and we have $\lambda_+ = 0$. Similarly, if $\rho^- = \hat{\rho}$ then $\lambda_- = 0$.

Thus, if $\rho^- = \rho^+ = \hat{\rho}$, the only profile is the constant function $Q(x) \equiv \hat{\rho}$.

2. On the other hand, as $\rho^+ \to 1$, we have that $\beta^{-1} \to 0$, therefore $\lambda_+ \to \infty$. Similarly, as $\rho^- \to 0$, then $\lambda^- \to \infty$ as well. Thus, the only stationary traveling wave profile connecting $\rho^- = 0, \rho^+ = 1$ is the unit step function, taking the step at an arbitrary point.

In the sequel we consider the nontrivial cases where

$$
0 < Q(x) < 1 \quad \forall x \in \mathbb{R}, \quad 0 < \rho^- < \hat{\rho} < \rho^+ < 1. \quad (3.10)
$$

3.2. Initial value problems. Assume that $w(\cdot)$ satisfies (1.3) and $v(\cdot)$ satisfies (1.4). Fixing a point $x_0 \in \mathbb{R}$, we consider the initial value problem of (3.3), with initial condition

$$
Q(x) = \Phi(x) \quad \text{for} \quad x \geq x_0, \quad (3.11)
$$

where $\Phi(\cdot)$ satisfies

$$
0 < \Phi(x) < 1, \quad \Phi'(x) > 0, \quad \text{for} \quad x \geq x_0. \quad (3.12)
$$

We seek continuously differentiable solutions $Q(\cdot)$ for (3.3), solved backward in $x$ on the interval $x \leq x_0$. Since $\Phi$ might not satisfy the equation (3.3), the solution
We consider the closed set \( U \) on \( \rho \) where \( \kappa \) is monotone increasing and Lipschitz continuous, with Lipschitz constant \( \bar{Q} \).

Then there exists a unique solution of the initial value problem of (3.3) with initial condition (3.11), satisfying (3.12). Then for all \( x < x_0 \), we have

(i) \( Q \) is positive, bounded and monotone

\[ 0 < Q(x) < \Psi(x_0) < 1, \quad Q'(x) > 0. \tag{3.13} \]

(ii) We have

\[ Q(x)\nu(A(Q; x)) = \bar{f}_0, \quad \text{where} \quad \bar{f}_0 = \Phi(x_0) \cdot \nu(A(\Phi; x_0)). \tag{3.14} \]

(iii) We have the asymptotic limit

\[ \lim_{x \to -\infty} Q(x) = \rho^-_0, \quad \text{where} \quad 0 < \rho^-_0 < \hat{\rho}, \quad f(\rho^-_0) = \bar{f}_0. \tag{3.15} \]

**Proof.**

(i) We first observe that \( Q'(x_0^-) > 0 \). Indeed, this follows immediately from the assumptions (3.12) on \( \Phi() \) and the equation (3.4).

We now assume that \( Q() \) is not monotone increasing on \( x < x_0 \). Let \( \hat{y} < x_0 \) be the local minimum such that \( Q'(\hat{y}) = 0 \) and \( Q'(x) > 0 \) for all \( x \geq \hat{y} \). By (3.4), this implies \( Q'(\hat{y}) > 0 \), a contradiction. Thus, we conclude that \( Q'(x) > 0 \) for all \( x < x_0 \).

The positivity of the solution follows from the fact that (3.3) is autonomous and \( Q = 0 \) is a critical value.

(ii) By (3.3), we immediately have (3.14).

(iii) By Lemma 3.1, the asymptotic value satisfies \( \rho^-_0 < \hat{\rho} \). Taking the limit \( x \to -\infty \) in (3.14) we obtain (3.15). \( \square \)

**Theorem 3.4.** Assume that \( w() \) satisfies (1.3) and \( v() \) satisfies (1.4). Consider the initial value problem of (3.3) with initial condition (3.11), satisfying (3.12). Then there exists a unique solution \( Q \) on the interval \( x \leq x_0 \). The solution \( Q \) is monotone increasing and Lipschitz continuous, with Lipschitz constant \( \bar{f}_0 L_v \kappa \), where \( \kappa = \| w() \|_\infty \) and \( L_v \) is the Lipschitz constant for the map \( \rho \mapsto (1/v(\rho)) \) on \( \rho \in [0, \Phi(x_0)] \).

**Proof.**

**Existence.** For delay differential equations with strictly positive delays, the existence and uniqueness of solutions can be proved by the standard method of steps, cf. [22, 21]. Unfortunately, for (3.4) the delay is arbitrarily small, and the method of steps does not apply. Instead, we apply a fixed point argument.

Let \( \Phi \) be the initial condition on \( x \geq x_0 \), and let \( \bar{f}_0 \) and \( \rho^-_0 \) be defined as in (3.14)-(3.15) in Lemma 3.3. Let \( L_v \) be the Lipschitz constant for the map \( \rho \mapsto (1/v(\rho)) \) on \( \rho \in [0, \Phi(x_0)] \), and consider the constants

\[ \kappa = \| w() \|_\infty, \quad L_Q = \bar{f}_0 L_v \kappa, \quad \gamma = 2\bar{f}_0 L_v \kappa. \tag{3.16} \]

We consider the closed set \( \mathcal{U} \) of functions, defined on \( x \leq x_0 \), as

\[ \mathcal{U} = \left\{ u : (-\infty, x_0] \mapsto [\rho^-_0, \Phi(x_0)] : \text{u is continuously differentiable} \right. \]

\[ \left. \text{with Lipschitz constant} \ L_Q, \ u(x_0) = \Phi(x_0), \right. \]

\[ \lim_{x \to -\infty} u(x) = \rho^-_0 > 0, \quad u'(x) \geq 0 \ \forall x \leq x_0 \} \tag{3.17} \]

Let \( u \in \mathcal{U} \). We define a Picard operator on \( \mathcal{U} \) as

\[ (\mathcal{P} u)(x) = \bar{f}_0 \frac{1}{v(A(u; x))}, \quad \forall x \geq x_0. \tag{3.18} \]
Note that a fixed point for $P$ is a solution for (3.3).

We first claim that the Picard operator $P$ maps $\mathcal{U}$ into itself, i.e.

$$ (Pu) \in \mathcal{U} \quad \text{if } u \in \mathcal{U}. $$

(3.19)

Indeed, from the construction we have

$$ (Pu)(x_0) = \frac{\bar{f}_0}{v(\mathcal{A}(\Phi; x_0))} = \Phi(x_0). $$

Moreover, since $v$ is decreasing, we conclude that $(1/v)$ is increasing. Then, since $u \in \mathcal{U}$ is increasing, so is the averaged function $\mathcal{A}(u; x)$. Therefore $(Pu)$ is also increasing. Furthermore, for the asymptotic value, we have

$$ \lim_{x \to -\infty} (Pu)(x) = \lim_{x \to -\infty} \frac{\bar{f}_0}{v(\rho^-_0)} = \rho^-_0. $$

Finally, since $u$ is Lipschitz, so is $(Pu)$. To obtain the Lipschitz constant, we compute

$$ \mathcal{A}(u; x) = \int_0^h u'(x + s)w(s) \, ds \leq \kappa \int_0^h u'(x + s) \, ds $$

$$ = \kappa [u(x + h) - u(x)] \leq \kappa, $$

therefore

$$ (Pu)'(x) = \bar{f}_0 \cdot \left( \frac{1}{v(\mathcal{A}(u; x))} \right)_x \leq \bar{f}_0 L_v A(u; x) \leq \bar{f}_0 L_v \kappa = L_Q. $$

We conclude that $(Pu) \in \mathcal{U}$, proving the claim (3.19).

We further claim that the Picard operator $P$ is a strict contraction w.r.t. the norm

$$ \|u\|_{\gamma} = \sup_{x \leq x_0} e^{\gamma x}|u(x)|, $$

(3.20)

where $\gamma$ is defined in (3.16). Note that the norm (3.20) controls the uniform norm on bounded set on $x \leq x_0$. Furthermore, since $\lim_{x \to -\infty} u(x) = \rho^-_0$ for any $u \in \mathcal{U}$, the asymptotic value at $x \to -\infty$ is fixed.

We now prove that $P$ is a strict contraction with the norm defined in (3.20). Let $u_1, u_2 \in \mathcal{U}$. Assume

$$ \|u_1 - u_2\|_{\gamma} = \delta, \quad \text{i.e. } |u_1(x) - u_2(x)| \leq \delta e^{-\gamma x} \quad \forall x \leq x_0. $$

Then, for any $x \leq x_0$ we have

$$ \left| (Pu_1)(x) - (Pu_2)(x) \right| \leq \bar{f}_0 L_v \cdot \int_x^{x+h} |u_1(y) - u_2(y)| w(y - x) \, dy $$

$$ \leq \bar{f}_0 L_v \kappa \cdot \int_x^{x+h} \delta e^{-\gamma y} \, dy $$

$$ \leq \bar{f}_0 L_v \kappa \delta \frac{1}{\gamma} \left( e^{-\gamma x} - e^{-\gamma(x+h)} \right) $$

$$ < \frac{\delta}{2} e^{-\gamma x}. $$

Hence

$$ \left\| (Pu_1) - (Pu_2) \right\|_{\gamma} \leq \sup_{x \leq x_0} e^{\gamma x} \left| (Pu_1)(x) - (Pu_2)(x) \right| \leq \frac{\delta}{2} = \frac{1}{2} \|u_1 - u_2\|_{\gamma}. $$
This shows that the Picard operator is a strict contraction from $\mathcal{U}$ to itself, hence it has a unique fixed point in $\mathcal{U}$. The fixed point iterations converge pointwise on bounded sets. Furthermore, since all functions in $\mathcal{U}$ have a fixed asymptotic limit as $x \to -\infty$, we conclude the pointwise convergence for all $x \leq x_0$. This establishes the existence of solutions for the initial value problem.

**Uniqueness.** The uniqueness of solutions can be proved by contradiction. Let $Q(\cdot)$ and $\hat{Q}(\cdot)$ be two distinct solutions of the initial value problem, with the same initial condition (3.11). Without loss of generality, we assume that for some $\bar{x} \leq x_0$ we have $Q(x) = \hat{Q}(x)$ on $x \geq \bar{x}$ and $Q(x) > \hat{Q}(x)$ on some non-empty interval $[\bar{x} - c, \bar{x}]$ where $c \in \mathbb{R}^+$. Since the solutions are monotone, there exist $x_1, x_2$, with $\bar{x} - c < x_1 < x_2 < \bar{x} < x_1 + h$ and

$$Q(x_1) = \hat{Q}(x_2), \quad Q(x) > \hat{Q}(x), \quad Q'(x) < \hat{Q}'(x) \quad \forall x \in [x_1, \bar{x}], \quad (3.21)$$

see Figure 3 for an illustration. Note that, since $Q(\cdot)$ and $\hat{Q}(\cdot)$ are smooth functions, by continuity the assumptions (3.21) hold for some $x_1, x_2$ sufficiently close to $\bar{x}$. This implies

$$Q(x_1 + s) < \hat{Q}(x_2 + s), \quad \forall s \in (0, h].$$

![Figure 3](image)

**Figure 3.** Left: Graphs of $Q(x)$ and $\hat{Q}(x)$ on $[x_1, x_1 + h]$. Right: Graphs of shifted functions $Q(s + x_1)$ and $\hat{Q}(s + x_2)$.

We now have

$$A(Q; x_1) < A(\hat{Q}; x_2). \quad (3.22)$$

Since both $Q(\cdot), \hat{Q}(\cdot)$ are solutions of (3.3), we have

$$Q(x_1)v(A(Q; x_1)) = \hat{Q}(x_2)v(A(\hat{Q}; x_2)),$$

and hence

$$v(A(Q; x_1)) = v(A(\hat{Q}; x_2)),$$

a contradiction to (3.22). Thus, we conclude that $Q(x) \equiv \hat{Q}(x)$ for all $x \leq x_0$, proving the uniqueness of the solutions for the initial value problem.

A numerical algorithm and an alternative proof for existence. To generate profiles of $Q$, the fixed point iterations for the Picard operator $\mathcal{P}$ is not most convenient. Alternatively, as a more direct construction, one may adopt the following numerical scheme. Fix a step size $\Delta x$, and discretize the space by

$$x_i = x_0 + i\Delta x, \quad i \in \mathbb{Z}^-.$$
We construct a continuous and piecewise affine approximate solution $Q^\Delta(\cdot)$. Denoting the approximate value at the grid points as

$$Q_0 \doteq \Phi(x_0) < 1, \quad \text{and} \quad Q_i \doteq Q^\Delta(x_i), \quad i \in \mathbb{Z}^-,$$

then we interpolate as

$$Q^\Delta(x) = Q_{i-1} \frac{x-x_i}{\Delta x} + Q_i \frac{x-x_{i-1}}{\Delta x} \quad \text{for} \ x \in [x_{i-1}, x_i], \quad \forall i \in \mathbb{Z}^-.$$

Fix an $i \in \mathbb{Z}^-$, and assume that $Q^\Delta(x)$ is given for all $x \geq x_i$. The value $Q_{i-1}$ is generated by solving the nonlinear equation

$$G(Q_{i-1}) \doteq Q_{i-1} \cdot v(A(Q^\Delta; x_{i-1})) - Q_i \cdot v(A(Q^\Delta; x_i)) = 0. \quad (3.23)$$

Numerically, (3.23) can be computed efficiently using Newton iterations, with $Q_i$ as the initial guess. We remark that (3.23) can be viewed as a finite difference approximation for (3.4),

$$\frac{Q_i - Q_{i-1}}{\Delta x} \cdot v(A(Q^\Delta; x_{i-1})) + Q_i \cdot \frac{v(A(Q^\Delta; x_i)) - v(A(Q^\Delta; x_{i-1}))}{\Delta x} = 0. \quad (3.24)$$

The algorithm (3.24) is somewhat similar to the symplectic method for systems of ODEs.

We compute

$$G(Q_i) = Q_i \cdot \left[ v(A(Q^\Delta; x_{i-1})) - v(A(Q^\Delta; x_i)) \right] > 0, \quad (3.25)$$

$$G(0) = -Q_i \cdot v(A(Q^\Delta; x_i)) < 0, \quad (3.26)$$

and

$$\frac{\partial}{\partial Q_{i-1}} A(Q^\Delta; x_{i-1})$$

$$= \frac{\partial}{\partial Q_{i-1}} \int_{x_{i-1}}^{x_i} \left[ Q_{i-1} \frac{x_i-y}{\Delta x} + Q_i \frac{y-x_{i-1}}{\Delta x} \right] w(y-x_{i-1}) \, dy$$

$$= \int_{x_{i-1}}^{x_i} \frac{x_i-y}{\Delta x} w(y-x_{i-1}) \, dy$$

$$= \int_0^{\Delta x} \frac{\Delta x - s}{\Delta x} w(s) \, ds.$$ 

Then, for $\Delta x$ sufficiently small, the above term is arbitrarily small, and we have

$$G'(Q_{i-1}) = v(A(Q^\Delta; x_{i-1})) + Q_{i-1} v'(A(Q^\Delta; x_{i-1})) \cdot \int_0^{\Delta x} \frac{\Delta x - s}{\Delta x} w(s) \, ds$$

$$> 0. \quad (3.27)$$

By (3.25)-(3.27) we conclude that there exists a unique solution $Q_{i-1}$ of (3.23), satisfying $0 < Q_{i-1} < Q_i$.

Iterating the above step for $i \in \mathbb{Z}^-$, we generate the values $\{Q_i : i \in \mathbb{Z}^-\}$, satisfying

$$0 < Q_{i-1} < Q_i < Q_0, \quad Q(x_i)v(A(Q^\Delta; x_i)) = Q(x_0)v(A(Q^\Delta; x_0)) = f_0. \quad (3.28)$$
We now establish an upper bound on the gradient of $Q^\Delta$, on $x < x_0$. Recall the constants $L_v, \kappa, L_Q$ defined in (3.16). Using the scheme (3.24), we compute

$$\frac{Q_i - Q_{i-1}}{\Delta x} = Q_{i-1}v(A(Q^\Delta;x_{i-1})) \left[ \frac{1}{v(A(Q^\Delta;x_i))} - \frac{1}{v(A(Q^\Delta;x_{i-1}))} \right] \leq \bar{f}_0 L_v [A(Q^\Delta;x_i) - A(Q^\Delta;x_{i-1})] \leq \bar{f}_0 L_v \kappa = L_Q.$$ 

Since $Q^\Delta(\cdot)$ is piecewise affine on $x < x_0$, $(Q^\Delta)'$ is piecewise constant. We conclude that

$$\|(Q^\Delta)'(\cdot)\|_{L^\infty(-\infty,x_0]} \leq L_Q.$$ 

Convergence of the sequence $\{Q^\Delta(x)\}$ as $\Delta x \to 0$ follows from Helly’s compactness theorem. This in turn offers an alternative proof for the existence of solutions for the initial value problem.

3.3. Asymptotic value problems.

**Theorem 3.5** (Asymptotic value problem). Assume that $w(\cdot)$ satisfies (1.3) and $v(\cdot)$ satisfies (1.4). Let $\rho^- \in \mathbb{R}$ and $\rho^+ \in \mathbb{R}$ be given which satisfy

$$0 < \rho^- < \hat{\rho} < \rho^+ < 1 \quad \text{and} \quad f(\rho^-) = f(\rho^+).$$

There exist monotone and continuously differentiable solutions $Q$ for (3.3) with the asymptotic values

$$\lim_{x \to +\infty} Q(x) = \rho^+, \quad \lim_{x \to -\infty} Q(x) = \rho^-.$$  

(3.29)

Furthermore, the solutions are unique up to horizontal shifts, in the following sense. Let $Q_1(\cdot)$ and $Q_2(\cdot)$ be two solutions of the asymptotic value problem with the same asymptotic conditions (3.29), then there exists a constant $c \in \mathbb{R}$ such that $Q_1(x) = Q_2(x + c)$ for all $x \in \mathbb{R}$.

**Proof. Existence.** The existence of solutions to the asymptotic value problem is established through convergence of approximate solutions, similar to the approach used for the proof of Theorem 2.8. Let $\lambda_+ > 0$ be the exponential rate given in Lemma 3.1, and let $\{x_n : n \in \mathbb{N}\}$ be an increasing sequence of real numbers such that $\lim_{n \to \infty} x_n = +\infty$. For each given $n \in \mathbb{N}$, let $Q^{(n)}(\cdot)$ be the solution of the initial value problem of (3.3), defined on $x \leq x_n$, with initial condition

$$Q^{(n)}(x) = \Phi(x) \equiv \rho^+ - e^{-\lambda_+ x}, \quad \text{on} \quad x \geq x_n.$$ 

By Theorem 3.4, $Q^{(n)}(\cdot)$ exists and is unique, and it satisfies

$$Q^{(n)}(x) \cdot v(A(Q^{(n)}; x)) = \bar{f}_n$$

where

$$\bar{f}_n = \Phi(x_n) \cdot v \left( \int_{x_n}^{x_n+h} \Phi(y) w(y - x_n) \, dy \right).$$

By Lemma 3.3, we have

$$\lim_{x \to -\infty} Q^{(n)}(x) = \rho^- \quad \text{where} \quad f(\rho^-) = \bar{f}_n, \quad \rho^- < \hat{\rho}.$$
Using the exact expression of \( \Phi(x) \), we compute
\[
\lim_{n \to \infty} \bar{f}_n = \lim_{n \to \infty} \left( \rho^+ - e^{-\lambda x_n} \right) v \left( \int_{x_n}^{x_n+h} \left( \rho^+ - e^{-\lambda y} \right) w(y-x_n) \, dy \right).
\]
\[
= \rho^+ \cdot v \left( \rho^+ \int_0^h w(s) \, ds \right) = \rho^+ v(\rho^+) = f(\rho^+) = f(\rho^-).
\]
This implies that
\[
\lim_{n \to \infty} \rho_n^- = \rho^-.
\]
Furthermore, replacing \( Q \) by \( Q^{(n)} \) in (3.3) and subtracting it from the identity
\[
\mathcal{A} \left( Q^{(n)}; x \right) \cdot v \left( \mathcal{A}(Q^{(n)}; x) \right) = f \left( \mathcal{A}(Q^{(n)}; x) \right),
\]
we get the estimate
\[
\frac{f(\mathcal{A}(Q^{(n)}; x)) - \bar{f}}{v(\mathcal{A}(Q^{(n)}; x))} = \mathcal{A}(Q^{(n)}; x) - Q^{(n)}(x) < Q^{(n)}(x + h) - Q^{(n)}(x).
\]
Finally, let \( \tilde{Q}^{(n)} \) be a horizontally shifted function of \( Q^{(n)} \), such that for some \( c_n \in \mathbb{R} \),
\[
\tilde{Q}^{(n)}(x) = Q^{(n)}(x + c_n) \quad \forall x, \quad \text{and} \quad \tilde{Q}^{(n)}(0) = \tilde{\rho}.
\]
Then, \( \tilde{Q}^{(n)} \) is bounded, Lipschitz continuous, and monotonically increasing. By Helly’s compactness Theorem, as \( n \to \infty \), there exists a subsequence of functions \( \{\tilde{Q}^{(n)}\} \) that converges uniformly on bounded set to a limit function \( Q \). The limit function is bounded, Lipschitz continuous, and monotone increasing, satisfying the integral equation (3.3) and the estimate
\[
\frac{f(\mathcal{A}(Q; x)) - \bar{f}}{v(\mathcal{A}(Q; x))} < Q(x + h) - Q(x). \tag{3.30}
\]
It remains to establish the asymptotic values of the limit function \( Q \). Since \( Q \) is monotone and bounded, the limits as \( x \to \pm \infty \) exist. Let \( Q^+ = \lim_{x \to +\infty} Q(x) \) and assume that \( Q^+ \neq \rho^+ \). From Lemma 3.1 it holds \( Q^+ > \tilde{\rho} \), and from the construction \( Q^+ < \rho^+ \). Therefore we have \( \bar{f} > f(Q^+) \). Let \( \epsilon > 0 \). There exists an \( M \in \mathbb{R} \), sufficiently large, such that \( Q^+ - Q(x) < \epsilon \) for all \( x > M \). In particular, we have
\[
Q(x + h) - Q(x) < \epsilon \quad \forall x > M.
\]
However, from (3.30) we get, for any \( x > M \),
\[
Q(x + h) - Q(x) > f(Q^+) - \bar{f} > 0
\]
a contradiction. We conclude that \( Q^+ = \rho^+ \). A completely similar argument gives the limit as \( x \to -\infty \), proving the asymptotic values (3.29). This establishes the existence of solutions for the asymptotic value problems.

**Uniqueness.** The uniqueness of solutions is proved by a contradiction argument. Let \( Q_1 \) and \( Q_2 \) be two distinct solutions for the asymptotic value problem with the same asymptotic values (3.29). We may horizontally shift the profiles, such that the graphs of \( Q_1 \) and \( Q_2 \) intersect. Let \( y \in \mathbb{R} \) be the rightmost intersection point, such that
\[
Q_1(y) = Q_2(y), \quad \text{and} \quad Q_1(x) > Q_2(x) \quad \forall x > y.
\]
Then, we have \( \mathcal{A}(Q_1; y) > \mathcal{A}(Q_2; y) \), so
\[
Q_1(y)v(\mathcal{A}(Q_1; y)) < Q_2(y)v(\mathcal{A}(Q_2; y)). \tag{3.31}
\]
On the other hand, by (3.3) we must have
\[ Q_1(y)v(A(Q_1; y)) = Q_2(y)v(A(Q_2; y)) = \bar{f} = f(\rho^\pm), \]
which leads to a contradiction to (3.31). This proves that solutions to the asymptotic value problems are unique up to horizontal shifts.

Sample profiles. Sample profiles for \( Q(\cdot) \) with various \([\rho^-, \rho^+]\) values and two different weight functions \( w(\cdot) \) are given in Figure 4. The profiles are generated using the numerical algorithm given after the proof of Theorem 3.5.

**Figure 4.** Typical profiles \( Q(\cdot) \) with \( v(\rho) = 1 - \rho, h = 0.2 \), and various asymptotic values \([\rho^-, \rho^+]\) given in the legends. For the left plot we use \( w(x) = \frac{2}{h} - \frac{2x}{hx^2} \) for \( x \in (0, h) \), and for right plot we use \( w(x) = \frac{2x}{hx^2} \) for \( x \in (0, h) \).

**Remark 3.6.** We observe that the profiles for \( Q \) look very similar to those in Figure 1 for \( P(\cdot) \). A side-by-side comparison of the profiles \( P \) and \( Q \) with the same weight function \( w \) and same asymptotic values \([\rho^-, \rho^+]\) shows that the profile for \( Q \) has steeper gradient at the points where \( P \) and \( Q \) share the same function values. This indicates that the particle model (1.7) with \( \ell > 0 \) has more diffusive stationary profiles.

### 3.4. Stability of the traveling waves.
For the Cauchy problem of (1.1), the existence and uniqueness of weak solution is established in [9]. In particular, if the initial condition \( \rho(0, \cdot) \) is smooth and the weight \( w \) is Lipschitz continuous on its support, then the solution \( \rho(t, \cdot) \) remains smooth for all \( t \geq 0 \).

Under the additional assumptions (2.44) and (2.45), we now show that the stationary profiles are the stable time asymptotic limit for the solutions of the Cauchy problem of the nonlocal conservation law, under suitable assumptions on smooth initial condition.

**Theorem 3.7 (Stability).** Let \( w(\cdot) \) satisfy (1.3) and (2.44), and let \( v(\cdot) \) satisfy (1.4) and (2.45). Let \( \rho^-, \rho^+ \in \mathbb{R} \) satisfy
\[ f(\rho^-) = f(\rho^+) = \bar{f}, \quad 0 < \rho^- < \check{\rho} < \rho^+ < 1. \]
Let \( Q(\cdot) \) be the unique stationary profile with asymptotic conditions (3.29) and \( Q(0) = \check{\rho} \).
Let $\rho(0, \cdot)$ be a smooth function, and assume that there exist constants $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}$ such that
\begin{equation}
Q(x + c_1) \leq \rho(0, x) \leq Q(x + c_2), \quad \forall x \in \mathbb{R}. \tag{3.32}
\end{equation}
For $t \geq 0$, let $\rho(t, \cdot)$ be the solution of the Cauchy problem for (1.1), with initial condition $\rho(0, \cdot)$. Then, there exists a constant $\bar{c} \in \mathbb{R}$ such that
\begin{equation}
\lim_{t \to \infty} [\rho(t, x) - Q(x + \bar{c})] = 0, \quad \forall x \in \mathbb{R}. \tag{3.33}
\end{equation}

**Proof. Step 1.** Since the initial condition $\rho(0, \cdot)$ is smooth, the solution $\rho(t, \cdot)$ for the nonlocal conservation law (1.1) is smooth for all $t > 0$.

We observe that assumption (3.32) implies
\[
\lim_{x \to \infty} \rho(0, x) = \rho^+ \quad \text{and} \quad \lim_{x \to -\infty} \rho(0, x) = \rho^-.
\]
Fix a time $t \geq 0$. Let $\hat{Q}(x) = Q(x + \hat{c})$ for some $\hat{c} \in \mathbb{R}$ be a profile such that
\[
\hat{Q}(x) \geq \rho(t, x) \quad \forall x,
\]
and the graph of $\hat{Q}$ touches the graph of $\rho(t, \cdot)$ tangentially from above. In other words, the profile $\hat{Q}$ is an “upper envelop” for $\rho(t, \cdot)$. Let $\hat{x}$ be the largest value such that
\[
\hat{Q}(\hat{x}) = \rho(t, \hat{x}), \quad \hat{Q}'(\hat{x}) = \rho_x(t, \hat{x}), \quad \text{and} \quad \hat{Q}(x) > \rho(t, x) \quad \forall x > \hat{x}. \tag{3.34}
\]
We claim that the upper envelope is decreasing in time, i.e.,
\[
\rho_x(t, \hat{x}) < 0, \quad \text{i.e.,} \quad [\rho(t, \hat{x})v(A(\rho; t, \hat{x})]\] > 0. \tag{3.35}
\]
Indeed, we have the estimate
\begin{equation}
A(\hat{Q}; \hat{x}) - A(\rho; t, \hat{x}) = \int_0^h [\hat{Q}(\hat{x} + s) - \rho(\hat{x} + s)] w(s) \, ds > 0. \tag{3.36}
\end{equation}
Since $v' < 0$ and $v'' \leq 0$, we have
\begin{equation}
v(A(\hat{Q}; \hat{x})) < v(A(\rho; t, \hat{x})) \quad \text{and} \quad v'(A(\hat{Q}; \hat{x})) \leq v'(A(\rho; t, \hat{x})). \tag{3.37}
\end{equation}
Finally, since
\begin{equation}
A(\rho; t, \hat{x}) = \rho(t, \hat{x} + h)w(h) - \rho(t, \hat{x})w(0) + \int_{\hat{x}}^ {\hat{x} + h} -\rho(t, y)w'(y - x) \, ds,
\end{equation}
\begin{equation}
A(\hat{Q}; \hat{x}) = \hat{Q}(\hat{x} + h)w(h) - \hat{Q}(\hat{x})w(0) + \int_{\hat{x}}^ {\hat{x} + h} -\hat{Q}(y)w'(y - x) \, ds,
\end{equation}
and using $w'(x) \leq 0$, we get
\begin{equation}
A(\rho; t, \hat{x}) - A(\hat{Q}; \hat{x}) = \frac{[\rho(t, \hat{x} + h) - \hat{Q}(t, \hat{x} + h)]w(h) - \int_{\hat{x}}^ {\hat{x} + h} [\rho(t, y) - \hat{Q}(y)]w'(y - x) \, ds}{0}.
\tag{3.38}
\end{equation}
By using (3.3), we compute, at $(t, \hat{x})$,
\begin{equation}
[\rho(t, \hat{x})v(A(\rho; t, \hat{x}))] = [\rho(t, \hat{x})v(A(\rho; t, \hat{x}))] - [\hat{Q}(\hat{x})v(A(\hat{Q}; \hat{x}))] + \hat{Q}'(\hat{x})v(A(\rho)) - v(A(\hat{Q})) + v'(A(\hat{Q})) = Q(\hat{x})A(\hat{Q})[v'(A(\rho)) - v'(A(\hat{Q}))]. \tag{3.39}
\end{equation}
Since the profile $\hat{Q}(x)$ is monotone increasing, we have $A(\hat{Q})_x > 0$. By further using the properties (3.36)–(3.38), all three terms on the righthand side of (3.39) are positive. We conclude (3.35).

The corresponding property for the “lower envelop” can be established in a similar way. Let $\tilde{Q}(x) = Q(x + \tilde{c})$ for some $\tilde{c} \in \mathbb{R}$ be a profile such that $\tilde{Q}(x) \leq \rho(t, x)$ for all $x$, and the graph of $\tilde{Q}$ touches the graph of $\rho(t, \cdot)$ tangentially from below. Let $\hat{x}$ be the largest $x$ value such that

$$\tilde{Q}(\hat{x}) = \rho(\hat{x}), \quad \tilde{Q}'(\hat{x}) = \rho_x(t, \hat{x}), \quad \text{and} \quad \tilde{Q}(x) < \rho(t, x) \quad \forall x > \hat{x}.$$ 

By a completely symmetric argument we conclude that the lower envelope is increasing in time, i.e., $\rho(t, \hat{x}) > 0$. We omit the details.

**Step 2.** Since the upper envelope is decreasing in time while the lower envelope is increasing in time, the time asymptotic stability for the solution $\rho(t, \cdot)$ follows, with similar arguments as in Step 2 of the proof of Theorem 2.9.

**Numerical simulations.** Solutions for the nonlocal conservation law (1.1) using oscillatory initial condition

$$\rho(0, x) = \begin{cases} 0.2, & x \leq -0.3, \\ 0.5 - 0.3 \sin(5\pi x), & -0.3 < x < 0.3, \\ 0.8, & x \geq 0.3. \end{cases} \quad \text{(3.40)}$$

are computed with a finite difference method, at various time $t$. See Figure 5, where two cases of the weight functions $w(\cdot)$ are treated. In the plots in the top row we have $w' < 0$. Here, we observe that the oscillations are quickly damped and that the solution approaches some profile as $t$ grows. On the other hand, in the plots in the bottom row we have $w' > 0$, and we observe that the solution becomes more oscillatory as $t$ grows, indicating instability of the profiles. This behavior is analyzed in some detail in Section 5.

![Figure 5](image-url)
4. Micro-macro limit of traveling waves.

**Theorem 4.1** (Micro-macro limit of traveling waves). Fix a weight function $w(\cdot)$ that satisfies the assumption (A1). Let $\{\ell_n\}$ be a sequence of car length such that $\lim_{n \to \infty} \ell_n = 0$. Let $P^{(n)}(\cdot)$ be the discrete stationary profile for the FDLs model, with car length $\ell_n$, such that

$$\lim_{x \to \pm \infty} P^{(n)}(x) = \rho^\pm, \quad P^{(n)}(0) = \hat{\rho}.$$ 

Then, as $n \to \infty$, the sequence of functions $P^{(n)}(\cdot)$ converges to a unique limit function $Q(\cdot)$, where $Q(\cdot)$ is the stationary profile for the nonlocal conservation law (1.1), satisfying the conditions

$$\lim_{x \to \pm \infty} Q(x) = \rho^\pm, \quad Q(0) = \hat{\rho}. \quad (4.1)$$

**Proof.** Step 1. Let $\lambda^\ell_+ > 0$ be the exponential rate for the profile $P(\cdot)$ as $x \to \infty$, derived in Lemma 2.2, where $\lambda^\ell_+$ is the unique solution of (2.17). Let $\lambda_+$ be the exponential rate for the profile $Q(\cdot)$ derived in Lemma 3.1, where $\lambda_+$ is the unique solution of (3.9). Recalling (2.17), we define the function

$$H(a, \lambda) = b \sum_{k=0}^m \hat{w}_k e^{-ka\lambda} - \frac{a\lambda}{1 - e^{-a\lambda}}. \quad (4.2)$$

Recall that $a = \ell/\rho^\ell_+$, and $\hat{w}_k$ is given in (2.15). We observe that the first term on the right-hand side of (4.2) is an approximate Riemann sum for the integral $\int_0^h e^{-\lambda s} w(s) \, ds$. We compute, using (2.15), and get

$$H(0, \lambda) = \lim_{a \to 0} H(a, \lambda) = b \sum_{k=0}^m \hat{w}_k e^{-ka\lambda} - \frac{a\lambda}{1 - e^{-a\lambda}} - \frac{a\lambda}{1 - e^{-a\lambda}} = b \int_0^h e^{-\lambda s} w(s) \, ds - 1. \quad (4.2)$$

By (2.17) and (3.9), we have

$$H(0, \lambda_+) = 0, \quad H(a, \lambda_+) = 0.$$

For $0 < \lambda \leq \lambda^\ell_+ \leq \max\{\lambda^\ell_+, \lambda_+\}$, we have

$$\left| \frac{\partial}{\partial a} H(a, \lambda) \right| = \left| -b \sum_{k=0}^m \hat{w}_k k^a e^{-ka\lambda} - \frac{\lambda(1 - e^{-a\lambda}) - a\lambda^2 e^{-a\lambda}}{(1 - e^{-a\lambda})^2} \right| \leq M_1,$$

and

$$\left| \frac{\partial}{\partial \lambda} H(0, \lambda) \right| = \int_0^h -se^{-\lambda s} w(s) \, ds \leq -e^{-\lambda h} \int_0^h ws(s) \, ds \leq -M_2 < 0,$$

so

$$\left| \frac{\partial}{\partial \lambda} H(0, \lambda) \right| \geq M_2.$$
Since $\lambda \mapsto H(0, \lambda)$ is strictly decreasing, we have $H(0, \lambda^+_{\ell}) \neq 0$. Then, it holds

$$|\lambda_+ - \lambda^+_{\ell}| = \left| \frac{\lambda_+ - \lambda^+_{\ell}}{H(0, \lambda_+ - H(0, \lambda^+_{\ell}))} \right| \cdot \left| \frac{H(0, \lambda^+_{\ell}) - H(a, \lambda^+_{\ell})}{a} \right| \cdot a \leq \frac{\ell}{\rho^+} \cdot M_1 \cdot M_2,$$

therefore, we have

$$\lim_{\ell \to 0^+} \lambda^+_{\ell} = \lambda_+.$$

A completely similar proof yields

$$\lim_{\ell \to 0^+} \lambda^-_{\ell} = \lambda_-.$$

**Step 2.** Fix a small $\ell > 0$, and let $P^\ell(\cdot)$ be a profile that satisfies (2.8) with asymptotic conditions $\lim_{x \to \pm \infty} P^\ell(x) = \rho^{\pm}$. Then $P^\ell(\cdot)$ is monotone and Lipschitz continuous. Taking the limit $\ell \to 0$, by Helly’s compactness theorem there exists a subsequence of $\{P^\ell\}$ that converges to a limit function $Q(\cdot)$ uniformly on bounded set. Thanks to the asymptotic conditions $\lim_{x \to \pm \infty} P^\ell(x) = \rho^{\pm}$ for all $\ell > 0$ and the result in step 1 on the exponential rates, the convergence $P^\ell \to Q$ is uniform for all $x \in \mathbb{R}$. Moreover, the limit function $Q$ is monotone, Lipschitz continuous, and satisfies

$$Q(0) = \hat{\rho}, \quad \lim_{x \to -\infty} Q(x) = \rho^-, \quad \lim_{x \to \infty} Q(x) = \rho^+.$$

It remains to show that $Q$ satisfies the integral equation (3.3). Indeed, recalling the definition of the operator $A_{P^\ell}$ in (2.7), we write

$$A_{P^\ell}(x) = \sum_{k=0}^{\infty} \int_{(L_{P^\ell})^k(x)}^x P^\ell \left( (L_{P^\ell})^k(x) \right) w(y - x) \, dy.$$

Since the convergence $P^\ell \to Q$ is uniform for all $x \in \mathbb{R}$, and the weight function $w$ is continuous on its support $x \in [0, h]$, we have

$$\lim_{\ell \to 0^+} A_{P^\ell}(x) = \int_x^{x+h} Q(y) w(y - x) \, dy = A(Q; x) \quad \forall x \in \mathbb{R}. \quad (4.3)$$

By the periodic property, $P^\ell$ satisfies the integral equation

$$\int_x^{x+\ell/P^\ell(x)} \frac{1}{v(A_{P^\ell}(z))} \, dz = \frac{\ell}{\bar{f}} \quad \forall x \in \mathbb{R}.$$

This can be written as

$$\frac{1}{\ell/P^\ell(x)} \int_x^{x+\ell/P^\ell(x)} \frac{1}{v(A_{P^\ell}(z))} \, dz = \frac{P^\ell(x)}{\bar{f}} \quad \forall x \in \mathbb{R}, \quad (4.4)$$

where the left-hand side is the average value of $v(A_{P^\ell}(z))^{-1}$ over the interval $[x, x + \ell/P^\ell(x)]$. Taking the limit $\ell \to 0^+$ on both sides of (4.4), we obtain

$$\frac{1}{v(A(Q; x))} = \frac{Q(x)}{\bar{f}} \quad \Rightarrow \quad Q(x)v(A(Q; x)) = \bar{f}, \quad \forall x \in \mathbb{R}.$$

Thus, we conclude that $Q$ satisfies (3.3), completing the proof. \qed
5. Further discussions.

5.1. Traveling waves with non-zero speed. So far in this paper we considered stationary profiles. In the case where a traveling wave has non-zero speed, a coordinate translate can be used. Let $\sigma$ be the velocity of the travelling waves. Assume that the function $f(\rho) = \rho v(\rho)$ is concave with $f'' < 0$. Let $\rho^- \in \mathbb{R}, \rho^+ \in \mathbb{R}$ satisfy

$$0 < \rho^- < \hat{\rho}_\sigma < \rho^+ < 1, \quad \sigma = \frac{f(\rho^-) - f(\rho^+)}{\rho^- - \rho^+}, \quad \text{where} \quad f'(\hat{\rho}_\sigma) = \sigma.$$ 

We consider the nonlocal conservation law (1.1). Let $\xi = x - \sigma t$. We seek profile $Q(\cdot)$ such that $\rho(t,x) = Q(\xi) = Q(x - \sigma t)$ is a solution for (1.1). The profile $Q(\cdot)$ satisfies the ODE

$$-\sigma Q_{\xi} + (Q \cdot v(A(Q; \xi)))_{\xi} = 0,$$

where $A$ is the averaging operator defined in (3.1). This leads to the integro-equation

$$Q(\xi) [v(A(Q; \xi)) - \sigma] \equiv \bar{f}_\sigma,$$  \hspace{1cm} (5.1)

where

$$\bar{f}_\sigma \doteq f(\rho^-) - \sigma \rho^- = f(\rho^+) - \sigma \rho^+.$$ 

The analysis in Section 3 can be applied to (5.1) in a similar way, achieving similar results.

On the other hand, for the FtLs model we seek a profile $P(\cdot)$ such that

$$P(z_i(t) - \sigma t) = \rho_i(t).$$

Differentiating this in $t$, and after direct computation we arrive at

$$P'(\xi) = \frac{P^2(\xi)}{\ell [v(A(\xi)) - \sigma]} \left[ v(A^P(L^p(\xi))) - v(A^P(\xi)) \right].$$  \hspace{1cm} (5.2)

Here $L^p$ is the operator in (2.6) and $A^P$ is defined in (2.7). Note the similarity between (5.2) and (2.8). Again, similar results are achieved by applying the same approach as in Section 2.

5.2. Unstable profiles with $w' > 0$. We discuss a case where the traveling wave profiles are not local attractors for the solution of the Cauchy problem for the nonlocal conservation law. Assume that the weight function $w(\cdot)$ satisfies

$$w'(x) > 0, \quad x \in [0,h].$$

This implies that, on the interval $[0,h]$ ahead of a driver, the situation further ahead is more important than the one closer to the driver. Of course, from a practical point of view, this assumption is rather obscure, and we expect the mathematical models to exhibit erroneous behavior. Indeed, as we have observed in the numerical simulations (see Figure 5), when $w'(x) > 0$ on $x \in (0,h)$, the maximum principle fails for the solution of the Cauchy problem for (1.1), and the solution does not approach any profile $Q(\cdot)$ as $t$ grows. Here we offer a supplementary argument.

To fix ideas, assume that $w(0) = 0$ and $w'(h) \geq c_o > 0$ on $(0,h)$. We revisit the proof of Theorem 3.7 and observe that (3.39) and (3.36) give

$$\rho(t,\hat{x}) = -[\rho(t,\hat{x})v(A(\rho; t, \hat{x}))]_x$$

$$= -Q'(\hat{x})[v(A(\rho)) - v(A(Q))] - Qv'(A(\rho))[A(\rho)x - A(Q)x]$$
Since the profile \( Q \)

\[
- Q(A(Q)_x [v'(A(\rho)) - v'(A(Q))] = \int_0^h \left[ Q'(\hat{x})v'(p_1) w(s) + Q(\hat{x})A(Q; \hat{x})_x v''(p_2) w(s) - Q(\hat{x})v'(A(\rho; t, \hat{x}))(p) \right] ds,
\]

where

\[
v'(p_1) = \frac{v(A(\rho)) - v(A(Q))}{A(\rho) - A(Q)}, \quad v''(p_2) = \frac{v'(A(\rho)) - v'(A(Q))}{A(\rho) - A(Q)}.
\]

Since the profile \( Q(x) \) approaches \( \rho^+ \) as \( x \to +\infty \), therefore, for \( \hat{x} \) sufficiently large, \( Q'(\hat{x}) \) and \( A(Q; \hat{x})_x \) become arbitrarily small, and we have

\[
\rho_\ell(t, \hat{x}) > 0.
\]

This shows that the solution \( \rho(t, x) \) can never settle around \( Q(x) \) for \( x \) large, therefore the profiles \( Q \) are not time asymptotic limits in the sense of Theorem 3.7.

### 5.3. A symmetric kernel \( w \)

We now consider the case when a driver consider the situation both in front and behind the car. Although in reality one only adjusts the speed according to the leaders, there has been an interest in nonlocal models where the integral kernel has support both in front and behind particle. This leads to the nonlocal conservation law

\[
\rho_t + \left[ \rho(t, x) \cdot v \left( \int_{x-h}^{x+h} \rho(t, y) w(y-x) \, dy \right) \right] = 0, \tag{5.3}
\]

where the weight function \( w(\cdot) \) has support on \([{-h_1}, h_2] \). We remark that, for (5.3) global existence and uniqueness of the solutions is not yet available in the literature, and a maximum principle is lacking as well.

One motivation for (5.3) stems from possible extensions of the one-dimensional conservation law into several space dimensions, for applications such as pedestrian flow and flock flow. A radially symmetric kernel is commonly used, i.e. \( w(\vec{x}) = w(r) \) where \( r = |\vec{x}| \). Then, the corresponding one-dimensional kernel \( w(x) \) is necessarily an even function. We now consider (5.3) with \( h_1 = h_2 = h \) and a weight function \( w(\cdot) \) that satisfies

\[
\begin{cases}
  w(s) = 0 & \forall |s| \geq h, \\
  w(-s) = w(s) & \forall s \in \mathbb{R}, \\
  \int_0^h w(s) \, ds = 0.5.
\end{cases} \tag{5.4}
\]

We seek stationary smooth and monotone profiles \( \bar{Q}(\cdot) \) such that

\[
\lim_{x \to \pm \infty} \bar{Q}(x) = \rho^\pm, \quad f(\rho^-) = f(\rho^+) = \bar{f}.
\]

The profile \( \bar{Q}(\cdot) \) satisfies the integral equation

\[
\bar{Q}(x) \cdot v \left( \int_{x-h}^{x+h} \bar{Q}(y) w(y-x) \, dy \right) \equiv \bar{f}.
\]

Carrying out a similar asymptotic analysis as in Lemma 3.1, for the limit \( x \to \infty \), equation (3.9) is modified to

\[
\int_{-h}^h e^{-\lambda s} w(s) \, ds = \frac{1}{\beta}. \tag{5.7}
\]
We seek positive roots \( \lambda > 0 \). By using \( w(-s) = w(s) \), we get

\[
\frac{1}{\beta} = \int_{-h}^{0} e^{-\lambda s} w(s) \, ds + \int_{0}^{h} \frac{e^{-\lambda h}}{2} w(s) \, ds
\]

\[
= \int_{0}^{h} \left( e^{-\lambda s} + e^{\lambda s} \right) w(s) \, ds
\]

\[
= \int_{0}^{h} 2 \cosh(\lambda s) w(s) \, ds > 2 \int_{0}^{h} w(s) \, ds = 1.
\]

Therefore, a positive root \( \lambda \) exists if and only if \( \beta < \frac{1}{2} \), i.e., \( \rho^+ < \hat{\rho} \). Thus, as \( x \to +\infty \), the profile \( \tilde{Q} \) converges to \( \rho^+ \) exponentially if and only if \( \rho^+ < \hat{\rho} \) as in Lemma 3.1. In the limit as \( \ell \to 0^+ \), these profiles converge to a downward jump with \( \rho^- > \rho^+ \).

6. Another nonlocal model: Averaging the velocity. In this section we consider the conservation law (1.2) and the corresponding FtLs model (1.10). Note that (1.10) now gives

\[
\dot{\rho}_i(t) = -\frac{\ell}{(z_{i+1} - z_i)^2} \left( \frac{\rho^2_i}{z_{i+1} - z_i} \cdot (v^*_i(\rho; t) - v^*_{i+1}(\rho; t)) \right), \quad i \in \mathbb{Z}. \tag{6.1}
\]

The same results on stationary profiles as in Sections 3 and 4 apply to these models. The proofs are very similar, with only mild modifications. Below we go through the analysis briefly, focusing mainly on the differences.

6.1. The FtLs model. We seek “discrete stationary traveling wave profiles” \( \mathcal{P}(\cdot) \) such that

\[
\mathcal{P}(z_i(t)) = \rho_i(t), \quad \forall t \geq 0, \quad \forall i \in \mathbb{Z}. \tag{6.2}
\]

We define the operators

\[
L^\mathcal{P}(x) = x + \frac{\ell}{\mathcal{P}(z_i)},
\]

\[
A(v(\mathcal{P}(z_i))) = \sum_{k=0}^{m} w_{i,k} v(\mathcal{P}((L^\mathcal{P})^k(z_i))).
\]

After a similar derivation, we find that \( \mathcal{P}(x) \) satisfies a delay differential equation,

\[
\mathcal{P}'(x) = -\frac{\mathcal{P}^2(x)}{\ell \cdot A(v(\mathcal{P}(\mathcal{P}(x))))} \left[ A(v(\mathcal{P}(L^\mathcal{P}(x)))) - A(v(\mathcal{P}(x))) \right]. \tag{6.3}
\]

The asymptotic limits at \( x \to \pm \infty \), the periodic behavior, the existence and uniqueness of solutions of the initial value problems, and the existence and uniqueness of two-point asymptotic value problem all follow in almost the same way as those in Section 3.

Stability. The analysis for the stability of the profiles is slightly different. In the same setting as in the proof of Theorem 2.9, we let \( k \) be the index such that

\[
\mathcal{P}(z_k(t)) = \rho_k(t), \quad \mathcal{P}(z_i(t)) > \rho_i(t) \quad \forall i > k,
\]
and
\[ \hat{P}(z_i(t)) \geq \rho_i(t) \quad \forall i \in \mathbb{Z}, \]
and claim that
\[ \frac{\hat{p}_k}{z_k} < \hat{P}'(z_k). \quad (6.4) \]
Indeed, using \( L^\hat{P} (z_k) = z_k+1 \), we have
\[
\hat{P}'(z_k) = \frac{\hat{P}(z_k)}{A(v(\hat{P}(z_k)))} \cdot \frac{A(v(\hat{P}(z_{k+1}))) - A(v(\hat{P}(z_k)))}{v(\hat{P}(z_{k+1})) - v(\rho_k)} \cdot \frac{v(\hat{P}(z_{k+1})) - v(\rho_k)}{z_k - z_{k+1}}.
\]
Then, using \( v' < 0 \), we have
\[
A(v(\hat{P}(z_k))) < v_k' \implies A_2 < A_1,
\]
\[
\hat{P}(L^\hat{P}(z_k)) > \rho_{k+1} \implies v(\hat{P}(L^\hat{P}(z_k))) < v(\rho_{k+1}) \implies C_2 < C_1.
\]
It remains to show that \( B_2 \leq B_1 \). Indeed, we observe that
\[
v(\rho_{k+1}) - v(\rho_k) > v(\hat{P}(z_{k+1})) - v(\rho_k).
\]
Now let \( V_1(\cdot), V_2(\cdot) \) denote the piecewise constant functions defined as
\[
V_1(x) = v(\hat{P}((L^\hat{P})^j(z_k))), \quad \text{for} \quad (L^\hat{P})^j(z_k) < x < (L^\hat{P})^{j+1}(z_k),
\]
\[
V_2(x) = \rho_{k+j}, \quad \text{for} \quad z_{k+j} < x < z_{k+j+1}.
\]
In this setting we have
\[
V_1(x) = V_2(x), \quad x \in (z_k, z_{k+1}); \quad \text{and} \quad V_1(x) < V_2(x), \quad x > z_{k+1}.
\]
Now, we can write
\[
A(v(\hat{P}(z_{k+1}))) - A(v(\hat{P}(z_k))) = -\rho_k \int_{z_k}^{z_{k+1}} w(y - z_k) \, dy + \int_{z_{k+1}}^{\infty} (w(y - z_{k+1}) - w(y - z_k))V_1(y) \, dy,
\]
\[
v^*_k = v^*_{k+1} - v^*_k = -\rho_k \int_{z_k}^{z_{k+1}} w(y - z_k) \, dy + \int_{z_{k+1}}^{\infty} (w(y - z_{k+1}) - w(y - z_k))V_2(y) \, dy.
\]
Now, since \( w' \leq 0 \), we have \( w(y - z_{k+1}) - w(y - z_k) \geq 0 \). We conclude
\[
v^*_{k+1} - v^*_k \leq A(v(\hat{P}(z_{k+1}))) - A(v(\hat{P}(z_k))).
\]
This implies \( B_2 \leq B_1 \), and therefore proves (6.4).

**Remark 6.1.** Note that in the above proof we do not use the assumption \( v'' \leq 0 \). Recall that this assumption is needed in Theorem 2.9, for the stability of traveling waves for the first FtLs model (1.7), where the averaging operator is taken over the discrete density.
6.2. The nonlocal conservation law. Let $Q(x)$ denote the stationary wave profile for (1.2). Introduce the operator $A$ such that

$$A(v(p); t, x) = \int_{x}^{x+h} v(p(t, y))w(y - x)\,dy = \int_{0}^{h} v(p(t, x + s))w(s)\,ds,$$

$$A(v(Q); x) = \int_{x}^{x+h} v(Q(y))w(y - x)\,dy = \int_{0}^{h} v(Q(x + s))w(s)\,ds.$$ 

A stationary solution $Q(x)$ for (1.2) satisfies the equation

$$Q(x) \cdot A(v(Q); x) \equiv \bar{f} = \text{constant} = f(\rho^{\pm}).$$

(6.5)

This can also be written in the form of a delay integro-differential equation,

$$Q'(x) = -\frac{Q(x)}{A(v(Q); x)} \cdot \int_{0}^{h} v'(Q(x + s))Q'(x + s)w(s)\,ds.$$ 

The asymptotic limits are analyzed in the same way as for Section 3.

Approximate solutions for initial value problem. The approximate solutions for the initial value problem are generated using a similar algorithm as for Theorem 3.4, with a few different details. Fix a $k \in \mathbb{Z}$, let $Q_i$ for $i \geq k$ be given. We compute $Q_{k-1}$ by solving the nonlinear equation

$$G(Q_{k-1}) = Q_{k-1}A(v(Q); x_{k-1}) - \bar{f} = 0,$$

where on $x \in [x_{k-1}, x_k]$ the function $Q(x)$ is reconstructed by linear interpolation. The above nonlinear equation has a unique zero, if it is monotone. We claim that, for $\Delta x$ sufficiently small,

$$\partial_{Q_{k-1}} G(Q) = A(v(Q); x_{k-1}) + Q_{k-1}\partial_{Q_{k-1}} A(v(Q); x_{k-1}) > 0.$$ 

Indeed, we have

$$\partial_{Q_{k-1}} A(v(Q); x_{k-1}) = \int_{x_{k-1}}^{x_k} \partial_{Q_{k-1}} v \left( Q_{k-1} \frac{x_k - x}{x_k - x_{k-1}} + Q_k \frac{x - x_{k-1}}{x_k - x_{k-1}} \right)w(y - x)\,dy$$

$$= -\mathcal{O}(1) \cdot \int_{x_{k-1}}^{x_k} \frac{x_k - x}{x_k - x_{k-1}} w(y - x)\,dy$$

$$= -\mathcal{O}(1) \cdot \Delta x,$$

proving the claim.

The existence and uniqueness of the initial value problem and the two-point asymptotic value problem follow in a very similar way as those in Section 4.

Stability. The existence and uniqueness of weak solutions for the Cauchy problem of (1.2) is recently established in [24]. Similar to the equation (1.1), if the initial condition $\rho(0, \cdot)$ is smooth, the solution for the Cauchy problem of (1.2) remains smooth for $t \geq 0$.

Fix a time $t \geq 0$. Similar to the proof of Theorem 3.7, let $\hat{Q}(x)$ be a profile such that $\hat{Q}(x) \geq \rho(t, x)$, and let $\hat{x}$ be a point satisfying

$$\hat{Q}(\hat{x}) = \rho(t, \hat{x}), \quad \hat{Q}_{x}(\hat{x}) = \rho_{x}(t, \hat{x}), \quad \hat{Q}(x) > \rho(t, x) \quad \forall x > \hat{x}. \quad (6.6)$$

We claim that

$$\rho_{t}(t, \hat{x}) < 0, \quad \text{i.e.} \quad [\rho(t, \hat{x})A(v(p); t, \hat{x})]_x > 0. \quad (6.7)$$
Indeed, we compute
\[
\begin{align*}
\left[ \rho(t, \hat{x}) A(v(\rho); t, \hat{x}) \right]_x &= \left[ \rho(t, \hat{x}) A(v(\rho); t, \hat{x}) \right]_x - \left[ \hat{Q}(\hat{x}) A(v(\hat{Q}); \hat{x}) \right]_x \\
&= \hat{Q}_x \left[ A(v(\rho); t, \hat{x}) - A(v(\hat{Q}); \hat{x}) \right] \\
&+ \hat{Q}(\hat{x}) \left[ A(v(\rho); t, \hat{x})_x - A(v(\hat{Q}); \hat{x})_x \right].
\end{align*}
\tag{6.8}
\]

Since \( \hat{Q}(x) \) is monotone increasing, we have \( \hat{Q}_x > 0 \). And, by (6.6) we have
\[
v(\rho; t, \hat{x}) > v(\hat{Q}; \hat{x}) \quad (x > \hat{x}), \quad \text{so} \quad A(v(\rho); t, \hat{x}) > A(v(\hat{Q}); \hat{x}).
\]
Thus, the first term on the righthand side of (6.8) is positive. To estimate the second term, using
\[
\begin{align*}
A(v(\rho); t, \hat{x})_x &= v(\rho(t, \hat{x} + h))w(h) - v(\rho(t, \hat{x}))w(0) \\
&\quad - \int_0^h v(\rho(t, \hat{x} + s))w'(s) \, ds, \\
A(v(\hat{Q}); \hat{x})_x &= v(\hat{Q}(\hat{x} + h))w(h) - v(\hat{Q}(\hat{x}))w(0) \\
&\quad - \int_0^h v(\hat{Q}(\hat{x} + s))w'(s) \, ds,
\end{align*}
\]
and that \( w' \leq 0, v' < 0 \), we get
\[
A(v(\rho); t, \hat{x}) - A(v(\hat{Q}); \hat{x}) > 0,
\]
proving the claim.

**Remark 6.2.** Note again that we do not need the assumption \( v'' \leq 0 \) in the above proof. Recall that the assumption is need in Theorem 3.7, for (1.1) where the averaging operator is taken over the density \( \rho \).

Finally, as \( \ell \to 0 \), the profiles \( P^\ell(\cdot) \) converges to \( Q(\cdot) \), following the same argument as in the proof of Theorem 4.1. We omit the details.

### 7. Concluding remarks

In this paper we analyze existence, uniqueness and stability of stationary traveling wave profiles for several nonlocal models for traffic flow, for both particle models and PDE models. Furthermore, we prove the convergence of the traveling waves of the FtLs models to those of the corresponding nonlocal conservation laws. However, the convergence of solutions of the nonlocal microscopic model to the macroscopic model remains open. We recall that, for the local models, the micro-macro limits are well treated in the literature, see [16, 19, 27, 28]. Existence of solutions for the Cauchy problem of the nonlocal conservation laws is also well studied, cf. [9]. We speculate that an adaptation of the approach in [27] combined with the results in [9] could yield the micro-macro limit. Details may come in a future work.

It is also interesting to study stationary profiles for the case where the road condition is discontinuous, for example where the speed limit has a jump at \( x = 0 \). Preliminary results in [37] show that the profiles for the models (1.1) and (1.2) are very different. In both cases, some profiles are non-monotone, some are non-unique, and some are also unstable, (similar to the results in [36]), portraying a much more complex picture.
Codes for the numerical simulations used in this paper can be found: http://www.personal.psu.edu/wxs27/TrafficNL

Acknowledgment. We thank the anonymous reviewers for their careful readings and useful comments which led to an improvement of our manuscript.

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Received August 2018; 1st revision November 2018; 2nd revision December 2018.

E-mail address: jur436@psu.edu
E-mail address: wxs27@psu.edu