A FORMULA FOR THE BOUNDARY OF CHAOS IN THE LEXICOGRAPHICAL SCENARIO AND APPLICATIONS TO THE BIFURCATION DIAGRAM OF THE STANDARD TWO PARAMETER FAMILY OF QUADRATIC INCREASING-INCREASING LORENZ MAPS

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Abstract. The Geometric Lorenz Attractor has been a source of inspiration for many mathematical studies. Most of these studies deal with the two or one dimensional representation of its first return map. A one dimensional scenario (the increasing-increasing one’s) can be modeled by the standard two parameter family of contracting Lorenz maps. The dynamics of any member of the standard family can be modeled by a subshift in the Lexicographical model of two symbols. These subshifts can be considered as the maximal invariant set for the shift map in some interval, in the Lexicographical model. For all of these subshifts, the lower extreme of the interval is a minimal sequence and the upper extreme is a maximal sequence. The Lexicographical world (LW) is precisely the set of sequences (lower extreme, upper extreme) of all of these subshifts. In this scenario the topological entropy is a map from LW onto the interval $[0, \log 2]$. The boundary of chaos (that is the boundary of the set of $(a,b) \in \text{LW}$ such that $h_{\text{top}}(a,b) > 0$) is given by a map $b = \chi(a)$, which is defined by a recurrence formula. In the present paper we obtain an explicit formula for the value $\chi(a)$ for $a$ in a dense set contained in the set of minimal sequences. Moreover, we apply this computation to determine regions of positive topological entropy for the standard quadratic family of contracting increasing-increasing Lorenz maps.

1. Introduction. In a remarkable contribution, around 1963, the meteorologist E.N. Lorenz [23] exhibited numerical evidence for the existence of a strange attractor in a quadratic system of ordinary differential equations in three variables. Some time later, around 1974-1979, Afrajmovich, Bykov and Shilnikov ([1], [2]) and Guckenheimer, Williams ([10], [11], [31]) proposed the so called geometrical models for the...
behavior observed by Lorenz. An important feature of these models is the existence of a (partial) cross-section to the flow, as well as a smooth invariant foliation by curves. Using this foliation, one can reduce the dynamics of the flow to that of an interval transformation with a discontinuity. These transformations, generically, divide into two disjoint classes: the expanding ones (those whose derivative, from both sides, at the discontinuity is infinity) and the contracting ones (those whose derivative, from both sides, at the discontinuity is zero). As observed in [2] and [11], there exist uncountably many conjugacy classes of such transformations. In fact the moduli space is essentially 2-dimensional and can be parameterized by the admissible kneading sequences (forward itineraries of the discontinuity).

In view of these results, it is natural to look for a bifurcation theory of these transformations and flows using symbolic dynamics ([5, 15, 18, 19]). In this direction, de Melo and Martens ([7]) and Labarca and Moreira ([18]) showed the existence of parameterized families of contracting Lorenz flows that are topologically universal in the sense that given any geometric Lorenz flow, its dynamics is “essentially” the same as the dynamics of some element of the family.

In fact, the two parameter family of quadratic lexicographical Lorenz Maps \(F_{\mu,\nu} : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}\) given by
\[
F_{\mu,\nu}(x) = \begin{cases} 
-\mu + x^2, & x > 0, \\
-\nu - x^2, & x < 0.
\end{cases}
\]
is topologically universal. That is, for almost any continuous map \(g : (\mathbb{R} \setminus \{0\}) \to \mathbb{R}\) which is an increasing- increasing and such that \(g(0^-) \geq 0\) and \(g(0^+) \leq 0\) there is a parameter value \((\mu, \nu)\) such that the dynamics of \(F_{\mu,\nu}(x)\) is essentially the dynamics of \(g(x)\). Hence, up to topologically semi-conjugacy, the two parameter family of quadratic Lorenz Maps \(F_{\mu,\nu}(x)\) represents almost all the interesting dynamics of increasing-increasing one dimensional maps with one discontinuity.

Therefore, it is a very interesting problem to know the bifurcation theory associated to the family \(F_{\mu,\nu}(x)\) (form now and on we will call the family \(F_{\mu,\nu}(x)\) the standard family of quadratic increasing-increasing Lorenz Maps, or by short the standard family).

We recall that an isentrope of the topological entropy is a region of the parameter space \((\mu, \nu)\) such that \(h_{\text{top}}(F_{\mu,\nu}) = h_0 = \text{constant}\) (here \(h_{\text{top}}\) mean topological entropy). A way to understand the bifurcation theory associated to the standard family, is to understand the structure of the different isentropes of the family. As John Milnor ([26]) pointed out, it is not an easy task (see for instance [6] and [28]).

In the present work we advance a step further in these questions and in the programme established by Labarca and Moreira (see [18]).

To be more specific (for the necessary definitions see section 2): we obtain an explicit formula for the value \(\chi(c)\) for any \(c \in [a_-, b, a^n, a]\) for any \(a \in \mathcal{A}_\infty\) and we apply it for the computation of regions of positive entropy for the standard family.

2. Preliminaries. It is well known that one of the purposes of the topological theory of Dynamical Systems is to find universal models describing the topological dynamics of a large class of systems (see for instance [3], [8], [30]).

One of these universal models is the shift on \(n\)-symbols \(\sigma : \Sigma_n \to \Sigma_n\) where \(\Sigma_n\) is the set of sequences \(\{\theta : \mathbb{N}_0 \to \{0, 1, 2, \ldots, n - 1\}\}\) and \(\sigma\) is the shift map defined by \((\sigma(\theta))(i) = \theta(i + 1)\). Here \(\Sigma_n\) is endowed with a certain topology. This model has been introduced to study one dimensional dynamics, by Metropolis, Stein, Stein at [24] and [25] (actually, for \(n = 2\) with a different, said “naive” presentation) and
eventually stated formally by Milnor and Thurston at [27], where the notion of a
d signed order in the shift space \((\Sigma_n, \sigma)\) was also defined.

In fact, several signed orders can be defined in \(\Sigma_n\) in a different way (as Milnor
and Thurston did). Let us doing this here. Let \(0 = x_0 < x_1 < x_2 < \ldots < x_{2n-1} = 1\)
be \(2n\) points in the unit interval \([0, 1]\). Let \(I_j = [x_{2j}, x_{2j+1}]\) for \(j = 0, 1, 2, \ldots, n-1;\)
and \(T : \bigcup_{j=0}^{n-1} I_j \to [0, 1]\) be a map such that its restriction to \(I_j\) is linear and onto
\([0, 1]\), for any \(j = 0, 1, \ldots, n-1\). The restriction of the map \(T\) to any interval \(I_j\) can
be either orientation preserving or orientation reversing. Hence, we may define \(2^n\)

piecewise linear maps \(T : \bigcup_{j=0}^{n-1} I_j \to [0, 1]\) as before. Let \(Lin(n)\) denote the set formed
by these \(2^n\) maps. Associated to any \(T \in Lin(n)\) we have its maximal invariant
set \(\Lambda_T = \{x \in \bigcup_{j=0}^{n-1} I_j; T^n(x) \in \bigcup_{j=0}^{n-1} I_j, \text{ for all } i \in \mathbb{N}\}\). In this set, we consider the
topology induced by the euclidean topology of the interval \([0, 1]\).

It is not hard to see that the set \(\Lambda_T\) is bijective to \(\Sigma_n\). In fact, the itinerary map
\(I_T : \Lambda_T \to \Sigma_n\) defined by \(I_T(x)(i) = j\) if and only if \(T^i(x) \in I_j\) is bijective. Its
inverse map \(I_T^{-1} : \Sigma_n \to \Lambda_T\) is given by \(I_T^{-1}(\theta) = I_{\theta_0} \cap T^{-1}(I_{\theta_1}) \cap T^{-2}(I_{\theta_2}) \cap \ldots =
\bigcap_{j=0}^{\infty} T^{-j}(I_{\theta_j})\), where, from now and on, we denote \(\theta = (\theta_0, \theta_1, \theta_2, \ldots)\). Hence, by using
the bijective map \(I_T : \Lambda_T \to \Sigma_n\) we can induce in \(\Sigma_n:\)

\(\begin{align*}
(a) & \text{ A topology } \tau_T : U \subset \Sigma_n \text{ is open if and only if } I_T^{-1}(U) \subset \Lambda_T \text{ is open, and} \\
(b) & \text{ An order: } \theta \leq_T \beta \text{ in } \Sigma_n \text{ if and only if } I_T^{-1}(\theta) \leq I_T^{-1}(\beta) \text{ in } \Lambda_T.
\end{align*}\)

Let us denote by \(\Sigma_n(T)\) the ordered, compact topological space \((\Sigma_n, \tau_T, \leq_T)\). In
this way, we have introduced \(2^n\) of these ordered compact metric spaces.

These models has been extensively used to obtain a great amount of information
about maps defined in an interval (see for instance [3, 6, 8, 13, 15, 16, 17, 18, 27, 29]);
vector fields on three dimensional manifolds (see for instance [5, 11, 12, 14, 19, 20, 32]) among other kinds of dynamical systems.

In the special case of one dimensional dynamics, the shift of two symbols may be
used to study increasing (decreasing) maps with one discontinuity like the Lorenz
maps, unimodal maps like the quadratic family or increasing-decreasing (decreasing-
increasing) maps with one discontinuity. Namely, for \(n = 2\) the ordered metric
compact space \((\Sigma_2, \tau_T, \leq_T)\) corresponding to the increasing-increasing map \(T\) is
known as the lexicographical space which generates the lexicographical world (see for instance [15, 16, 17, 18, 21, 22]) which is denoted LW.

In the present work we deal with the lexicographical world. That is, here we
consider the set \(\Sigma_2\) with the topology induced by the map \(T : I_0 \cup I_1 \to [0, 1]\) such
that \(T|_{I_0}\) and \(T|_{I_1}\) are increasing maps.

Let \(\sigma : \Sigma_2 \to \Sigma_2\) be the shift map \(\sigma(\theta_0, \theta_1, \theta_2, \ldots) = (\theta_1, \theta_2, \ldots)\). Let
\(\Sigma_0\) and \(\Sigma_1\) denote the sets \(\{\theta \in \Sigma_2 : \theta_0 = 0\}\) and \(\{\theta \in \Sigma_2 : \theta_0 = 1\}\) respectively. It
is clear that \(\Sigma_2 = \Sigma_0 \cup \Sigma_1\).

In \(\Sigma_2\) the order induced by \(T, \leq_T\), is the **lexicographical order**: \(\theta < \alpha\) for any
\(\theta \in \Sigma_0\) and \(\alpha \in \Sigma_1\) or \(\theta < \alpha\) if there is \(n \in \mathbb{N}\) such that \(\theta_i = \alpha_i\) for \(i = 0, 1, 2, \ldots, n-1\)
and \(\theta_n = 0\) and \(\alpha_n = 1\).
For $a \leq b$ in $\Sigma_2$ let $[a, b]$ denote the interval $\{\theta \in \Sigma_2| a \leq \theta \leq b\}$. $\Sigma[a, b]$ will denote the invariant set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a, b])$.

Let $a$ denote the finite string $a = a_0a_1 ... a_n$ and $a$ be the infinite sequence $a = a_0a_1 ... a_n, a_0a_1 ... a_n, a_0a_1 ... a_n, ...$. For example, if $a = 0011$ then $\sigma(a) = 001$.

Let $\text{Max}_2 = \{\theta \in \Sigma_2; \sigma^i(\theta) \leq \theta, \forall i \in \mathbb{N} \cup \{0\}\}$ and $\text{Min}_2 = \{\alpha \in \Sigma_2; \alpha \leq \sigma^i(\alpha), \forall i \in \mathbb{N} \cup \{0\}\}$ denotes the sets of maximal and minimal sequences in the lexicographical order.

**Definition 2.1.** The set $LW = \{(a, b) \in \text{Min}_2 \times \text{Max}_2; \{a, b\} \subset \Sigma[a, b]\}$ will be called the lexicographical world.

**Remark 1.** 1.- If $a \in \text{Min}_2$, $a \neq 0$ then $b(a) = sup\{\sigma^i(a); i \in \mathbb{N}\} \in \text{Max}_2$. For $d \in \text{Max}_2$, $d \neq 1$ then $d(a) = inf\{\sigma^i(d); i \in \mathbb{N}\} \in \text{Min}_2$.

2.- If $b(a) = b_1b_2 ... b_n$ then we will denote $b(a) = b_1b_2 ... b_n$.

**Definition 2.2.** Let us define:

1.- If $a = a_0a_1 ... a_{n-1}1$ then $a_-$ will denote the string $a_- = a_0a_1 ... a_{n-1}0$ and for $b = 1b_1b_2 ... b_{n-1}0$ then $b_+$ will denote the string $b_+ = 1b_1b_2 ... b_{n-1}1$.

2.- If $a_1, a_2$ are two sequences then we define the sequence $m(a_1, a_2)$ by $m(a_1, a_2) = a_1a_2$. For instance for $a_1 = 001$ and $a_2 = 01$ we have $m(a_1, a_2) = 00101$.

**Definition 2.3.** Let $m_0 < m_1$ be two finite words of 0’s and 1’s. Let $T_{m_0, m_1}: \Sigma_2 \to \Sigma_2$ be the renormalization map $T_{m_0, m_1}(\theta_0, \theta_1, ...) = (m_0\theta_0, m_1\theta_1, ...)$. This map is very well known (see for instance [9]); here we make the following considerations:

(1.-) $\Theta_1 \leq \Theta_2$ imply $T_{m_0, m_1}(\Theta_1) \leq T_{m_0, m_1}(\Theta_2)$.

(2.-) If $\Sigma_{m_0, m_1} = \{\Theta : \mathbb{N} \to \{m_0, m_1\}\}$ then $T_{m_0, m_1}(\Sigma_2) = \Sigma_{m_0, m_1}$ and it is an homeomorphism onto its image. The inverse map is constructed in the following way: let $\epsilon(m_0) = 0, \epsilon(m_1) = 1$ for $\alpha \in \Sigma_{m_0, m_1}$ we have $T_{m_0, m_1}^{-1}(\alpha) = (\epsilon(0\alpha), \epsilon(1\alpha), ...)$.

(3.-) An extension of the map $T_{m_0, m_1}^{-1}$ is given by the map $T_{m_0, m_1}^*: \Sigma_{m_0, m_1}$ defined, for $\alpha \leq m_1$, as $T_{m_0, m_1}^*(\alpha) = inf\{\beta \in \Sigma_2; T_{m_0, m_1}(\beta) \geq \alpha\}$.

(4.-) We define the map $\sigma_{m_0, m_1}: \Sigma_{m_0, m_1} \to \Sigma_{m_0, m_1}$, by

$$\sigma_{m_0, m_1}(T_{m_0, m_1}(a)) = T_{m_0, m_1}(\sigma(a)), \forall a \in \Sigma_2.$$  

3. Statement of the results.

3.1. The map $\chi$. We regard the definition of the maps $\varphi, \psi, \chi$ from [17].

**Definition 3.1.** We define the maps $\varphi, \psi, \chi: \Sigma_0 \to \Sigma_1$ by:

$\varphi(a) = inf\{b \in \Sigma_1: \Sigma[a, b] \neq \emptyset\}$

$\psi(a) = inf\{b \in \Sigma_1: \Sigma[a, b] \text{ contains } \infty\text{-elements}\}$

$\chi(a) = inf\{b \in \Sigma_1: \Sigma[a, b] \text{ is uncountable}\}$

For example $\varphi(0) = \psi(0) = \chi(0) = 10; \psi(01) = \chi(01) = 1$.

One of the main results in [17] is the following:

**Theorem 3.2.** [17] The function $\chi: \Sigma_0 \to \Sigma_1$ satisfies the following:

1. For $a < 001$, we have $\chi(a) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(a)$

2. For $001 \leq a < 00110$, we have $\chi(a) = 1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(a))$

3. For $00110 \leq a < 01$, we have $\chi(a) = 110$. 


4. For $01 \leq a \leq 01$, we have $\chi(a) = T_{10,1} \circ \chi \circ T_{10,1}^*(1a)$

Where $T_{x,y}$ is the renormalization map and $T_{x,y}^*$ is the inverse of the renormalization map, that we define in the section 2.

We also recall the following result from [17]

**Theorem 3.3.** The set $\{(a,b) \in \Sigma_0 \times \Sigma_1 : the \ topological \ entropy \ of \ the \ function \ (\sigma \mid_{\Sigma[a,b]} : \Sigma[a,b] \to \Sigma[a,b] \ is \ zero \}$ is equal to the set $\{(a,b) \in \Sigma_0 \times \Sigma_1 : b \leq \chi(a)\}$

3.2. The set $A_\infty$. Let us now define the set $A_\infty$

**Definition 3.4.** Let $A_0 = \{0^n1, 01^n \colon n, m \in \mathbb{N}\}$, that is:

$A_0 = \{\ldots, 00001, 0001, 001, 011, 0111, 01111\ldots\}$.

Let $A_1 = \{m(a_1, a_2); a_1 < a_2 \ are \ consecutive \ sequences \ in \ A_0 \} \cup A_0$ and $A_{n+1} = \{m(a_1, a_2); a_1 < a_2 \ are \ consecutive \ sequences \ in \ A_n \} \cup A_n$. So, we have:

$A_0 = \ldots, 00001, 0001, 001, 011, 0111, 01111\ldots$

$A_1 = \ldots, 0001, 0011, 00101, 011, 01011, 0011011, 0111\ldots$

$A_2 = \ldots, 001, 0010101, 00101, 001011, 011, 01011, 0101011, 011\ldots$

Let $A_\infty = \bigcup_{n=0}^{\infty} A_n$. For $a \in A_\infty$, let us define:

$A_0(a) = \{a_{-}(b(a))^{-1}b(a)+, a_{-}b(a)+a^{n-1} ; n, m \in \mathbb{N}\}$;

$A_1(a) = \{m(a_1, a_2); a_1 < a_2 \ are \ consecutive \ sequences \ in \ A_0(a) \} \cup A_0(a)$; and

$A_{n+1}(a) = \{m(a_1, a_2); a_1 < a_2 \ are \ consecutive \ sequences \ in \ A_n(a) \} \cup A_n(a)$ and let

$A_\infty(a) = \bigcup_{n=0}^{\infty} A_n(a)$.

**Example.** For $a = 01 \in A_\infty$, we have:

$A_0(a) = \{\ldots, 00101011, 001011, 00111, 0011011, 00111, 001101011, 0011011, 001101011, 011\ldots\}$

$A_1(a) = \{\ldots, 001011, 0011011011, 0011, 001101011, 0011011, 001101011, 0011011, 001101011, 011\ldots\}$.

**Definition 3.5.** Let us consider $A_\infty = A_\infty$ and $A_\infty = \bigcup_{a \in A_\infty} A_\infty(a) \cup A_\infty$.

Now, associated to any $a \in A_\infty$, we construct:

$A_0^2(a) = \{a_{-}(b(a))^{-1}b(a)+, a_{-}b(a)+a^{n-1} ; n, m \in \mathbb{N}\}$;

$A_{n+1}^2(a) = \{m(a_1, a_2); a_1 < a_2 \ are \ consecutive \ sequences \ in \ A_n^2(a) \} \cup A_n^2(a)$;

$A_\infty^2(a) = \bigcup_{n=0}^{\infty} A_n^2(a)$ and $A_\infty^2 = \bigcup_{a \in A_\infty} A_\infty(a) \cup A_\infty$.

Similarly, for any $n \geq 2$, we may define: $A_n^{n+1} = \bigcup_{a \in A_n} A_n^{n+1}$ and $A_\infty = \bigcup_{n=0}^{\infty} A_n^{n+1}$.

**Remark 2.** There are minimal sequences $a \in \text{Min}_2$ such that $a \notin A_\infty$.

For instance: $a = 001111, a = 0001111, a = 001011011$.

Our main results here are Theorems A and B:

**Theorem 3.6 (Theorem A).** 1. For any $a \in A_\infty$ and $c \in [a_{-}(b(a))+, a, a]$ we have that $\chi(c) = (b(a))+a$. 


2. For any \( a \in A_{\infty}^* \) we have that \( \chi(a - b(a)) = (b(a))_+ a - b(a) \)

**Remark 3.** This result was included as part of the PH.D. Thesis of S. Aranzubia ([4]) at the Departamento de Matemática y Ciencia de la Computación of the Universidad de Santiago de Chile.

Concerning the size of the set \( A_{\infty}^* \) in \( \Sigma_0 \) we have:

**Theorem 3.7.** (Theorem B) For any \( a \in A_{\infty}^* \) let us define \( I(a) = [a - b(a), a] \) and \( J(a) = [a - b(a)] + b, a \) we have that:

1. The set \( I_{\infty} = \bigcup_{a \in A_{\infty}} (I(a) \cap \text{Min}_2) \) is dense in \( \Sigma_0 \cap \text{Min}_2 \subset \Sigma_0 \);

2. The set \( J_{\infty} = \bigcup_{a \in A_{\infty}} (J(a) \cap \text{Min}_2) \) is dense in \( \Sigma_0 \cap \text{Min}_2 \).

**Remark 4.** These two results imply that we have an explicit formula for \( \chi(c) \), for \( c \) in a dense set of minimal sequences.

We want to acknowledge to an unknown referee who asked about the size of the set \( A_{\infty}^* \). In the way of answering this question we were forced to formalize the proof of the result in Theorem B.

### 3.3. Application of the Theorem A to the standard quadratic family \( F_{(\mu, \nu)} \)

As an application, of our main result, we will prove several results concerning the values of the topological entropy for the standard quadratic family. Initially let us define some special itineraries.

Let us consider the two parameter family \( F_{(\mu, \nu)} \). We denote by \( \Lambda(\mu, \nu) \) the maximal invariant set \( \cap_{j=0}^\infty F_{(\mu, \nu)}^{-j}([-\mu, \nu]) \). Associated to any \( x \in \Lambda(\mu, \nu) \) we can define its itinerary \( I_{(\mu, \nu)}(x) : \Sigma_2 \) by \( I_{(\mu, \nu)}(x)(i) = 0 \) if \( F_{(\mu, \nu)}^i(x) < 0 \) or \( I_{(\mu, \nu)}(x)(i) = 1 \) if \( F_{(\mu, \nu)}^i(x) > 0 \).

**Definition 3.8.** Associated with the points \( z = -\mu \) and \( z = \nu \) we define the kneading sequences \( I(-\mu) \) and \( I(\nu) \) by

\[
I(-\mu) = \lim_{x \downarrow -\mu, x \in \Lambda(\mu, \nu)} I_{(\mu, \nu)}(x) \quad \text{and} \quad I(\nu) = \lim_{x \uparrow \nu, x \in \Lambda(\mu, \nu)} I_{(\mu, \nu)}(x).
\]

**Remark 5.** In the sequel, sometimes we will denote \( I(-\mu) \) by \( a(\mu, \nu) \) and \( I(\nu) \) by \( b(\mu, \nu) \).

**Proposition 1.** For any \( (\mu, \nu) \) such that \( \mu = \frac{1 + \sqrt{1 + 4 \nu}}{2} \) or \( \nu = \frac{1 + \sqrt{1 + 4 \mu}}{2} \) the respective map \( F_{(\mu, \nu)} \) has positive entropy with the exception of the points \((1,0)\) and \((0,1)\) where the entropy is zero.

**Proposition 2.** For any \( \nu > 0 \) the map associated to the intersection of the curves \( \{(\mu(t), \nu)/\mu = t + \frac{1 + \sqrt{1 + 4 \nu}}{2}, t \geq 0\} \) and \( \{(\mu(t), \sqrt{t})/\mu(t) = t + \frac{1 + \sqrt{1 + 4 t}}{2}, t \geq 0\} \) has entropy zero.

**Corollary 1.** The map \( F_{(\mu, \nu)} \), for any point \( (\mu, \nu) \) in the curve \( \left(t + \frac{1 + \sqrt{1 + 4 \nu}}{2}, \nu\right) \) with \( 0 \leq t \leq \nu^2 \) has positive entropy, for any \( \nu > 0 \).
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Proposition 3. For any \( \mu > 0 \) the map associated to the intersection of the curves \( \{(\mu, \nu(t))/\nu = t + \frac{1 + \sqrt{1 + 4\mu}}{2}, t \geq 0\} \) and \( \{(\sqrt{t}, \nu(t))/\nu = t + \frac{1 + \sqrt{1 + 4\sqrt{t}}}{2}, t \geq 0\} \) has entropy zero.

Corollary 2. The map \( F_{\mu, \nu} \), for any point \((\mu, \nu)\) in the curve \( \left( \mu, t + \frac{1 + \sqrt{1 + 4\mu}}{2} \right) \) with \( 0 \leq t \leq \mu^2 \) has positive entropy, for any \( \mu > 0 \).

Remark 6. Taking together all of these results the figure 1 indicates the actual picture for the topological entropy

![Figure 1. Picture of the Topological Entropy](image1)

From now on, let denote by \( B(0, 1) \) or by \( B \) the set of parameters \((\mu, \nu)\) that belongs to the (compact) region whose boundary is given by the curves:

1.- \( \{(\mu, 0); 0 \leq \mu \leq 1\} \);
2.- \( \{(0, \nu); 0 \leq \nu \leq 1\} \);
3.- \( \left\{ \left( \mu, \frac{1 + \sqrt{1 + 4\mu}}{2} \right); 0 \leq \mu \leq 2 \right\} \) and
4.- \( \left\{ \left( \frac{1 + \sqrt{1 + 4\nu}}{2}, \nu \right); 0 \leq \nu \leq 2 \right\} \).

The figure 2 shows \( B(0, 1) \)

![Figure 2. Bubble B(0, 1)](image2)
Remark 7. Using the notation of de Melo and Martens, in [7], the bubble $B(0, 1)$ is an archipelago.

Proposition 4. Let $B_1(0, 0010, 1101, 1) = \{(\mu, \nu) \in B; 0 \leq I(-\mu) \leq 0010; 1101 \leq I(\nu) \leq 1\}$ and $\Lambda(\mu, \nu) = \bigcap_{j=0}^{\infty} F_{\mu, \nu}^{-j}([-\mu, \nu])$. For any $(\mu, \nu) \in B_1(0010, 1101, 0, 1)$, we have that $h_{top}(F_{\mu, \nu} |_{\Lambda(\mu, \nu)}) > 0$

![Figure 3. Region $B_1(0, 0010, 1101, 1)$](image)

Proposition 5. Let $B_2(0010, 01, 1101, 1) = \{(\mu, \nu) \in B; 0010 \leq I(-\mu) \leq 01, 1101 < I(\nu) \leq 1\}$ and $\Lambda(\mu, \nu) = \bigcap_{j=0}^{\infty} F_{\mu, \nu}^{-j}([-\mu, \nu])$. For any $(\mu, \nu) \in B_2(0010, 01, 1101, 1)$ we have that $h_{top}(F_{\mu, \nu} |_{\Lambda(\mu, \nu)}) > 0$

![Figure 4. Region $B_2(0010, 01, 1101, 1)$](image)

Remark 8. We observe that if $(\mu, \nu)$ satisfy $b(\mu, \nu) = 1101$ and $0010 \leq a(\mu, \nu) = a < 001101$, then $h_{top}(F_{\mu, \nu} |_{\Lambda(\mu, \nu)}) > 0$. In fact, $\chi(001101) = 1101$ and $001101$ is the lower value with this property, therefore $\chi(a(\mu, \nu)) < 1101$ and hence $h_{top}(F_{\mu, \nu} |_{\Lambda(\mu, \nu)}) > 0$
Proposition 6. Let $B_3(0,0010,10,1101) = \{(\mu, \nu) \in B; 0 \leq I(-\mu) < 0010, 10 \leq I(\nu) \leq 1101\}$. For $(\mu, \nu) \in B_3(0,0010,10,1101)$ we have $h_{top}(F_{\mu,\nu} |_{\Lambda(\mu,\nu)}) > 0$.

Remark 9. We note that if $(\mu, \nu)$ satisfy $a(\mu, \nu) = 0010$ and $110010 < b(\mu, \nu) \leq 1101$, then $h_{top}(F_{\mu,\nu} |_{\Lambda(\mu,\nu)}) > 0$. In fact, $\chi(0010) = 110010$ therefore for $110010 < b(\mu, \nu)$ we have that $h_{top}(F_{\mu,\nu} |_{\Lambda(\mu,\nu)}) > 0$ for $a(\mu, \nu) = 0010$.

Now, let us consider $a \in A_\infty$ and $b = b(a) = \sup\{\sigma^j(a); j \in \mathbb{N}\}$.

Let $B_1(0, a, -a, b, 0, 1) = \{(\mu, \nu) \in B; 0 \leq I(-\mu) \leq a - a, b \leq I(\nu) \leq 1\} = B_1$; $B_2(a, b, a, b, 0, 1) = \{(\mu, \nu) \in B; a - b \leq I(-\mu) \leq a, b \leq I(\nu) \leq 1\} = B_2$ and $B_3(0, a, -a, b, 0, 1) = \{(\mu, \nu) \in B; 0 \leq I(-\mu) < a - b, b \leq I(\nu) \leq b + a\} = B_3$.

For $(\mu, \nu) \in B_1 \cup B_2 \cup B_3$ let $\Lambda(\mu, \nu) = \bigcap_{j=0}^{\infty} F_{\mu,\nu}^{-j}([-\mu, \nu])$.

Remark 10.

1. The set $B_2(a, b, a, b, 0, 1)$ include the region $L_a = \{(\mu, \nu); I(-\mu) = a, b \leq I(\nu) \leq 1\}$ which has a non-empty interior (see [18]).

2. The set $B_3(0, a, -a, b, 0, 1)$ include the region $R_a = \{(\mu, \nu); I(\nu) = b; 0 \leq I(-\mu) < a - b\}$ which has a non-empty interior.

Proposition 7. For $(\mu, \nu) \in B_1 \cup B_2 \cup B_3$ we have that $h_{top}(F_{\mu,\nu} |_{\Lambda(\mu,\nu)}) > 0$.

For $a \in A_\infty$ let $C_1(a, b, a) = \{(\mu, \nu) \in B; I(-\mu) = a, I(\nu) = b + a\} = C_1$; $C_2(0, b, \infty) = \{(\mu, \nu) \in B; I(-\mu) = a, I(\nu) = b\} = C_2$; $C_3(a, b) = \{(\mu, \nu) \in B; I(-\mu) = a - b, I(\nu) = b\} = C_3$. 

Figure 5. Region $B_3(0,0010,10,1101)$

Figure 6. Region $B_1 \cup B_2 \cup B_3$
Proposition 8. For any \((\mu, \nu) \in C_1 \cup C_2 \cup C_3\), we have that \(h_{\text{top}}(F_{\mu, \nu} |_{\Lambda(\mu, \nu)}) = 0\)

Remark 11. 1. We observe that the map \((\mu, \nu) \rightarrow (I(-\mu), I(\nu))\) from \(B(0, 1)\) into \(\text{Min}_2 \times \text{Max}_2\) is not continuous. In fact, let \((\mu_n, \nu_n)\) be such that \(I(-\mu_n) = 001(01)_n\) and \(I(\nu_n) = 110010\). We have \(I(-\mu_n) \rightarrow 00110 = 0010\), \(a = 01, b = 10\), and \(I(\nu_n) = 110010\).

Therefore \((I(-\mu_n), I(\nu_n)) \rightarrow (0010, 110010)\).

From the other side, for \((\pi, \nu) = \lim (\mu_n, \nu_n)\) we have \((I(-\pi), I(\nu)) = (0010, 10)\).

Therefore \(\lim_{n \to \infty} (I(-\mu_n), I(\nu_n)) \neq (I(\pi), I(\nu))\) and \((\pi, \nu) = \lim_{n \to \infty} (\mu_n, \nu_n)\).

2. There are a countable number of discontinuity points for the map \((\mu, \nu) \rightarrow (I(-\mu), I(\nu))\). In fact, it is enough to consider \(a \in A_\infty^1\) and \((\mu_n, \nu_n)\) such that \(a(\mu_n, \nu_n) = a \cdot b\), \(b(\mu_n, \nu_n) = b + a \cdot b^n\), with \(b = b(a)\). We may verify that \((\mu_n, \nu_n) \rightarrow (\pi, \nu)\) with \(a(\pi, \nu) = a \cdot b\) and \(b(\pi, \nu) = b\). Since \(\lim a(\mu_n, \nu_n) = a \cdot b\) and \(\lim b(\mu_n, \nu_n) = b, a \cdot b\), we obtain \((I(\pi), I(\nu)) = \lim (I(\mu_n), I(\nu_n)))\).

3. For any \(a \in A_\infty^1\) let denote by \(B(a)\) the bounded region of the \((\mu, \nu)\) plane whose boundary is given by the curves:

(a) \(I(-\mu) = a, I(\nu) = b + a\), denoted \(\gamma_{a,b,a} \subset C_1(a, b, a)\);

(b) \(I(-\mu) = a - b, I(\nu) = b\), denoted \(\gamma_{a-b,b} \subset C_2(a-b, b)\);

(c) \(I(-\mu) = a - b, I(\nu) = b\), denoted \(\gamma_{a-b,b} \subset C_3(a-b, b)\);

(d) \(I(\nu) = b + a\), denoted \(\gamma_{b,a}^2\).

Accordingly to [7], the region \(B(a)\) is an island, and consequently an archipelago. Hence: we conclude that there are infinitely many archipelagos and islands in the standard family.

4. Proof of the Theorem A. Let us now prove:

Theorem 4.1 (Theorem A). 1. For any \(a \in A_\infty^1, c \in [a \cdot b(a)] \cdot a, b(a)]\) we have that \(\chi(c) = (b(a)) \cdot a\).

2. For any \(a \in A_\infty^1\) we have that \(\chi(a \cdot b(a)) = (b(a)) \cdot a \cdot b(a)\).

Proof. The map \(\chi\) satisfy:

1. If \(a < 001\): \(\chi(a) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^* (a)\)
Let us prove, initially, the result for \( a \in A_0 \).

If \( a = 01 \), then \( 001101 \leq c \leq 01 \) and \( \chi(c) = 1101 = b_2 \alpha \).

If \( a = 001 \), then for \( 000101001 \leq c \leq 001 \), and we obtain \( T_{0,01}^* (000101001) = T_{0,01}^*(001) \leq T_{0,01}^*(0001) \) therefore, \( 001101 \leq T_{0,01}^*(c) \leq 01 \) and, as a consequence \( \chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c) \); therefore \( \chi(c) = \sigma \circ T_{0,01}(1101) = \sigma(0101001) = 101001 = b_2 \alpha \).

Let us now assume that the result is true for \( a_1 = 0_{n-1}1 \). Let \( a = 0n_11 \), then for 

\[
0_{n+1}10n_{n-1}10n_{n-1}^1 \leq c \leq 0n_11
\]

we have \( T_{0,01}^*(0_{n+1}10n_{n-1}10n_{n-1}^1) \leq T_{0,01}^*(c) \leq T_{0,01}^*(0n_11) \) therefore, \( 0n_{n+1}10n_{n-1}10n_{n-1}^1 \leq T_{0,01}^*(c) \leq 0_{n-1}1 \) and we obtain \( \chi(T_{0,01}^*(c)) = 10_{n-2}1 \) and hence, the result is true for \( a \in A_0 \).

If \( a = 0n_11 \) and \( a \in A_1 \), we have \( a \in A_0 \) or \( a = m(a_1, a_2); a_1, a_2 \in A_0 \). If \( a \in A_0 \) then the result was actually proved. Let us assume that \( a = m(a_1, a_2); a_1, a_2 \in A_0 \).

Initially, let us assume that \( a = 00101 \). In this case: for \( 001001010100101 \leq c \leq 00101 \), we have \( T_{0,01}^*(001001010100101) \leq T_{0,01}^*(00101) \leq T_{0,01}^*(0001) \) and, consequently: \( 010111011 \leq T_{0,01}^*(c) \leq 0111 \), therefore \( \chi(T_{0,01}^*(c)) = 11101 \). So:

\[
\chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c)
\]

\[
\sigma \circ T_{0,01}^*(11101)
\]

\[
\sigma(0101001001) = 1010001001
\]

Let us now assume that the result is true for \( a_1 = 0_{n+1}10n_{n-1}^1 \) and let us prove the result for \( a = 0_{n+1}10n_{n-1}^1 \).
In this case for any \( c \) such that \( 0_{n+1}10_{n+1}10_n10_n10_{n+1}10_n1 \leq c \leq 0_{n+1}10_n1 \) we have: \( T_{0,01}^* \left( 0_{n+1}10_{n+1}10_n10_n10_{n+1}10_n1 \right) \leq T_{0,01}^* (c) \leq T_{0,01}^* \left( 0_{n+1}10_n1 \right) \), therefore \( 0_n10_n10_{n+1}10_n10_{n+1}10_n1 \leq T_{0,01}^* (c) \leq 0_n10_{n+1}10_n10_{n+1}10_n1 \) and consequently: \( \chi(T_{0,01}^*(c)) = 10_n10_{n+1}10_n10_{n+1}10_n1 \). Hence:

\[
\chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c)
\]

\[
= \sigma \circ T_{0,01} \left( 10_n10_{n+1}10_n10_{n+1}10_n1 \right)
\]

\[
= \sigma \left( 010_n10_{n+1}010_n10_{n+1}10_n1 \right)
\]

\[
= 10_n10_n10_{n+1}10_n1
\]

\[
= b_{+1}
\]

The proof is similar for \( a = 01_n01_{n+1} \). We conclude that the result is true for any \( a \in A_1 \).

Let us now assume that the result is true for any \( a \in A_n \) and let us prove the result for \( a \in A_{n+1} \).

For any \( a \in A_{n+1} \), we have that \( a \in A_n \) or \( a = (a_1)^{n+1}a_2 \) or \( a = (a_1)^n a_2 (a_1)^n a_2 (a_1)^n a_2 \) or \( a = (a_1)^n a_2 (a_1)^n a_2 (a_1)^n a_2 \), and so on; where \( a_1 \) and \( a_2 \) are two consecutive periodic sequences in \( A_0 \).

For \( a \in A_n \) the result was already proved. Assume that we have: \( a = (a_1)^{n+1}a_2 \).

Let us assume initially that \( a = (001)^{n+1}01 \). To prove the result for this \( a \), let us observe the following: for \( a_0 = (01)^{n+1}1 = (01)^n01 \in A_n \) and any \( c \) such that \( (01)^{n+1}011(01)^{n+1}01 \leq c \leq (01)^{n+1}1 \) we have \( \chi(c) = 11(01)^n1(01)^{n+1}1 \).

For \( a = (001)^{n+1}01 \), and any \( c \) such that \( (001)^{n+1}0010(100)^{n+1}1 = (001)^n1(01)^{n+1}01 \leq c \leq (001)^{n+1}01 \), we have \( T_{0,01}^* \left( (001)^{n+1}0010(100)^{n+1}1 \right) \leq T_{0,01}^* (c) \leq T_{0,01}^* \left( (001)^{n+1}01 \right) \), and consequently \( (01)^{n+1}011(01)^n1(01)^{n+1}01 \leq \chi(T_{0,01}^*(c)) \leq (01)^{n+1}1 \).

Therefore: \( \chi(T_{0,01}^*(c)) = 11(01)^n(01)^{n+1}1 \). Hence:

\[
\chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c)
\]

\[
= \sigma \circ T_{0,01} \left( 11(01)^n01(01)^{n+1}1 \right)
\]

\[
= \sigma \left( 0101001001001^{n+1}01 \right)
\]

\[
= 101(001)^n01001^{n+1}01
\]

\[
= b_{+1}
\]

as we announced.

Let us now assume that the result is true for \( (0_m1)^{n+1}(0_m0_m1)^{n+1} \). Now, for \( a = (0_m1)^{n+1}(0_m1) \), and any \( c \) such that \( (0_m1)^{n+1}0_m1(0_m1)^{n+1}0_m1 \leq c \leq (0_m1)^{n+1}0_m0_m1 \), we have that \( T_{0,01}^* \left( (0_m1)^{n+1}0_m1(0_m1)^{n+1}0_m1 \right) \leq T_{0,01}^* (c) \leq T_{0,01}^* \left( (0_m1)^{n+1}0_m0_m1 \right) \), and consequently \( \chi(T_{0,01}^*(c)) = 10_m1(0_m1)^n0_m0_m1(0_m1)^{n+1}0_m0_m1 \). Therefore:
\[ \chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c) \]
\[ = \sigma \circ T_{0,01} \left( 10_{m-1}1(0_m1)^a0_{m-1}1(0_m1)^a10_{m-1}1 \right) \]
\[ = \sigma \left( 010_{m-1}1(0_m01)^a0_{m-1}10_{m-1}1(0_m01)^a10_{m-1}1 \right) \]
\[ = 10_m1(0_{m+1})^a0_{m-1}1(0_{m+1})^a10_{m-1}1 \]
\[ = b_+a \]

Similarly we prove the result for \( a = (01_m)^{n+1}01_{m+1} \). Let us now assume that \( a = (a_1)^n a_2 (a_1)^{n-1} a_2 \).

Let us assume initially that: \( a = (001)^{n+1}(001)^{n-1}01, \) In this case for any \( c \) such that: \((001)^{n+1}(001)^{n-1}001(001)^{n-1}01(001)^{n-1}01 \leq c \leq (001)^{n+1}(001)^{n-1}01(001)^{n-1}01 \) we have that \( T_{0,01}^* \left( (001)^{n+1}(001)^{n-1}001(001)^{n-1}01 \right) \leq T_{0,01}^* \left( (001)^n \right) \leq T_{0,01}^* \left( (001)^{n+1}(001)^{n-1}01 \right) \). Therefore \( (001)^{n+1}(001)^{n-1}001(001)^{n-1}01 \leq (001)^n \). Consequently \( (001)^{n+1}(001)^{n-1}001(001)^{n-1}01 \leq \chi(T_{0,01}^*(c)) = (001)^{n+1}(001)^{n-1}01 \), and \( (001)^{n+1}(001)^{n-1}001(001)^{n-1}01 \leq \chi(T_{0,01}^*(c)) = (001)^{n+1}(001)^{n-1}01 \).

Hence:
\[ \chi(c) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(c) \]
\[ = \sigma \circ T_{0,01} \left( 10(001)^{n-1}(001)^{n-1}01(001)^{n-1}01 \right) \]
\[ = \sigma \left( 010(001)^{n-1}(001)^{n-1}01(001)^{n-1}01 \right) \]
\[ = 10(001)^{n-1}(001)^{n-1}01(001)^{n-1}01 \]
\[ = b_+a \]

Inductively, let us assume that: \( a = (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_{m+1} \). In this case, for any \( c \) such that \((0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1 \leq c \leq (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1 \) we have:
\[ T_{0,01}^* \left( (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1 \right) \leq T_{0,01}^* \left( (0_m1)^n \right) \leq T_{0,01}^* \left( (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1 \right) \).

And, consequently:
\[ (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1(0_{m+1})^{n-1}0_m1 \leq T_{0,01}^*(c) \leq (0_m1)^n \leq T_{0,01}^* \left( (0_m1)^{n+1}0_m1(0_{m+1})^{n-1}0_m1 \right) \)
\[ = b_+a \]

We proceed in a similar way for \( a = (01_m)^{n+1}(01_m)^{n-1}01_{m+1} \). Successively, we prove the result for any \( a \in A_{n+1} \).

Therefore, for any \( a \in A_{\infty} \) and any \( c \) such that \( c \in [a, b_+a, a] \) we have: \( \chi(c) = b_+a \).

Let us now prove the result for elements in \( A_{\frac{1}{2}} \). Let \( a_0 = 01 \) and \( a = a_+-b_+ = 0011 \).
For any $c$ such that $001011010011 \leq c \leq 0011$ we have: $01011010011 \leq \sigma(c) \leq 0110; T_{01,10}^*(01011010011) \leq T_{01,10}^*(\sigma(c)) \leq T_{01,10}^*(0110)$ and $001101 \leq T_{01,10}^*(\sigma(c)) \leq 011$. Hence:

\[
\chi(c) = \begin{cases} 
1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(c)) \\
1T_{01,10} (1101) \\
= 110100110 \\
= 110100111 \\
= (b(001))_+ + 0011 \\
= b.a.
\end{cases}
\]

Let us now consider $a = (a_0)_- = (a_0)_+ = 001011$. In this case, for any $c$ such that $00101011001101 \leq c \leq 001011$ we have: $0101011001101 \leq \sigma(c) \leq 0110110; T_{01,10}^*(0101011001101) \leq T_{01,10}^*(\sigma(c)) \leq T_{01,10}^*(011101), and $001011 \leq T_{01,10}^*(\sigma(c)) \leq 001111$. Hence:

\[
\chi(c) = \begin{cases} 
1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(c)) \\
1T_{01,10} (101001) \\
= 1100110010110 \\
= 110011001111 \\
= b.a.
\end{cases}
\]

Let us now consider $a = (a_0)_- = (b_0)_+ = 00(10)$. In this case, for any $c$ such that $00(10)^{n-1}1100(10)^{n-1}1100(10)^n11 \leq c \leq 00(10)^n11$, we have that $0(10)^{n-1}1100(10)^{n-1}1100(10)^n11 \leq \sigma(c) \leq 0(10)^{n}110$ and $T_{01,10}^*(010(10)^{n}101100(10)^{n-1}1100(10)^n11) \leq T_{01,10}^*(\sigma(c)) \leq T_{01,10}^*(0(10)^{n}110)$. So $0^{n+2}10^n10^{n+1}1 \leq T_{01,10}^*(\sigma(c)) \leq 0^{n+1}1$ and we obtain:

\[
\chi(c) = \begin{cases} 
1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(c)) \\
1T_{01,10} (10^n0^{n+1}1) \\
= 10(01)^n10(01)^n10^{n-1} \\
= 1100(10)^{n-1}1100(10)^n11 \\
= b.a.
\end{cases}
\]

Let us assume that the result is true for $(a_1)_- = (b_1)_+ = 1$, where $a_1 = 0_{n-1}$ and let us prove the result for $a = (a_0)_- = (b_0)_+ = 0_{n-1}$. In this case, for any $c$ such that $0_{n+1} + 1(10) = 0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10)$, we have: $T_{01,10}^*(0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10)) \leq T_{01,10}^*(c) \leq T_{01,10}^*(0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10))$, and consequently $0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) \leq T_{01,10}^*(c) \leq 0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10)$ and we obtain:

\[
\chi(c) = \begin{cases} 
\sigma \circ T_{01,10} \circ \chi \circ T_{01,10}^*(c) \\
\sigma \circ T_{01,10} \circ (10_{n-2}10_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10)) \\
= 0_{n+1}10_{n-1}10_{n+1}(10) + 110_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) + 10_{n-1}10_{n+1}(10) \\
= b.a.
\end{cases}
\]

Clearly, we can continue in an inductive way for elements in $A^\infty$. The proof of the second part of this result can be done in a similar way as we did for the first part.
Remark 12. We note that if \( d \in \text{Max}_2 \) satisfies \( d > b_+a \) and \( c \) is a sequence such that \( c \in [a_-, b_+ a, a] \) then \( h_{top}(c, d) = h_{top}\left( \sigma \mid_{\gamma_{j=0}^{\infty} \sigma^{-j}([c, d])} \right) > 0. \)

Also, for any \( c \) such that \( c < a_-, b_+ a \), we may find a natural number \( n \) such that \( c < c_n = a_+ b_+ a_n < a_+ b_+ a \) and \( \chi(c) \leq \chi(a_+ b_+ a_n) = \chi(a_n) = (b_+ a_n + 1) a_+ b_+ a_n < b_+ a = \chi(a_+ b_+ a). \) In this case: \( \Lambda[a_+ b_+ a_n, \chi(a_+ b_+ a_n)] \subset \text{int}(\Lambda[c, b_+ a]) \) and, as a consequence: \( h_{top}(c, b_+ a) = h_{top}\left( \sigma \mid_{\gamma_{j=0}^{\infty} \sigma^{-j}([c, b_+ a])} \right) > 0. \)

In particular, for \( c \) and \( d \) such that: \( b_+ a < d \) and \( c < a_+ b_+ a \) we have \( h_{top}(c, d) > 0. \)

In a similar way, for any \( c \) and \( d \) such that \( b \leq d \leq b_+ a_+ b \) and \( c < a_+ b \) we have that \( h_{top}(c, d) > 0. \) Also, for any \( d \) such that \( b_+ a_+ b \leq d \) we have \( h_{top}(a_+ b_+, d) > 0. \)

Moreover, for \( a_+ b < a_+ b_+ a \), we have that \( h_{top}(a_+ b, b_+ a) > 0 \) for any \( a \in \mathcal{A}_\infty^\infty. \)

In fact [4] proved that
\[
\ h_{top}(a_+ b, b_+ a) = \frac{\log 2}{\#(a)}.
\]

Where \( \#(a) = \text{period of the sequence } a. \)

5. Proof of the Theorem B. The topology, in the Lexicographical space, is equivalent to the topology induced by the metric \( d : \Sigma_2 \times \Sigma_2 \to [0, 2] \) given by: \( d(\alpha, \beta) = \sum_{i=0}^{\infty} \delta(\alpha_i, \beta_i) \); where \( \delta(\alpha_i, \beta_i) = 0, i_f \alpha_i = \beta_i; \) and \( \delta(\alpha_i, \beta_i) = 1, i_f \alpha_i \neq \beta_i. \)

With this topology the Lexicographical space is compact and complete.

Moreover, for a sequence of intervals \( I_n = [\alpha_n, \beta_n] \subset \Sigma_2 \) such that
1. \( \lim_{n \to \infty} d(\alpha_n, \beta_n) = 0 \) and
2. \( I_{n+1} \subset I_n \)

Then we have that \( \bigcap_{n=0}^{\infty} I_n \) is a unique point.

Now, for any \( a \in \mathcal{A}_\infty \), let us define \( I(a) = [a_-, b_+(\overline{a}), a] \) for \( \theta \in \Sigma_2; a_+ b(a) \leq \theta \leq a \) and we have the following:

Proposition 9. \( I_\infty = \bigcup_{a \in \mathcal{A}_\infty} (I(a) \cap \text{Min}_2) \) is a dense set in \( \Sigma_0 \cap \text{Min}_2 \subset \Sigma_0 \subset \Sigma_2. \)

Proof. Initially, let us assume that \( \alpha = 0 \) or \( \alpha = 1 \). In this case for \( a_n = 0_n 1 \) and \( \alpha_n = 01_n \), we have that \( d(a_n, 0) \to 0, n \to \infty \) and \( d(\alpha_n, 01) \to 0, n \to \infty \). Therefore \( 0 \in I_\infty \) and \( 01 \in I_\infty. \)

Let \( \alpha \in \Sigma_0 \cap \text{Min}_2 \) and assume that \( \alpha_n < \alpha < \alpha_n 1 \) or \( \alpha_n 1 < \alpha < \alpha_n 1 n \), some \( n \in \mathbb{N}. \)

Let us assume that \( 0_n 1 < \alpha < 0_n 1 \) is the case.

In this situation, for \( a = 0_n 1 \), we may assume that \( 0_n 1 < \alpha < a_+ b(a) = 0_n 1 10_n = 0_n 1 10_n 1 = 0_n 1 1a_+. \) In fact, otherwise \( a_+ b(a) = 0_n 1 10_n \leq \alpha < a \) and \( \alpha \in I(a) \), so the result is proved.

Remark 13. We observe that for \( a_m = 0_n m 1 n(m) \in \mathcal{A}_m \subset \mathcal{A}_\infty \) we have that \( d(a_m, 0_n 1 n(m)) \to 0 \) as \( m \to \infty \). That is \( 0_n 1 n 1 \in I_\infty. \)

So, let us consider \( 0_n 1 < \alpha < 0_n 1 n 1 \). This imply that \( \alpha = 0_n 1 n 1 \alpha_n 2 \alpha_2 2n 3 \ldots \)
Let us consider the periodic sequence $0_{n+1}10_n1 \in A_1$. We must have $0_{n+1}1 < \alpha < 0_{n+1}10_n1$ or $0_{n+1}10_n1 < \alpha < 0_{n+1}10_n1$ or $\alpha = 0_{n+1}10_n1$.

In the case $\alpha = 0_{n+1}10_n1$, we have $\alpha \in I_\infty$ and we are done.

Let us assume that $0_{n+1}1 < \alpha < 0_{n+1}10_n1$ is the case. In this situation for $\rho = 0_{n+1}10_n1$, we must have $0_{n+1}1 < \alpha < \rho$ or $\rho \leq \alpha < \rho$.

In the case $\rho \rho < \alpha \leq \rho$, we have $\alpha \in I(\rho), \rho \in I_\infty$ and we are done.

So, let us assume that $0_{n+1}1 < \alpha < \rho$ is the case. That is $0_{n+1}1 < \alpha < 0_{n+1}10_n10_n10_n10_n1 = 0_{n+1}110_n110_n110_n110_n1$.

In this situation we must have $\alpha = 0_{n+1}10_n10_n10_n110_n10_n110_n10_n1$.

**Remark 14.** We observe that $a_m = 0_{n+1}(0_{n+1}10_n1)_m \in A_{m+1}$ and $d(a_m, 0_{n+1}10_n1) \to 0$ as $m \to \infty$. That is $0_{n+1}10_n10_n110_n1 \in I_\infty$.

Let us assume that $0_{n+1}10_n1 < \alpha < 0_{n+1}10_n1$ is the case. In this situation let us consider the sequence $0_{n+1}10_n10_n1$. We have $0_{n+1}10_n1 < 0_{n+1}10_n1 \in 0_{n+1}10_n1$.

We may have: $0_{n+1}10_n1 < \alpha < 0_{n+1}10_n10_n1$ or $0_{n+1}10_n10_n1 < \alpha < 0_{n+1}10_n10_n1$.

In the case $\alpha = 0_{n+1}10_n10_n1$, we have $\alpha \in I_\infty$ and we are done.

Let us assume that $0_{n+1}10_n10_n1 < \alpha < 0_{n+1}10_n10_n1$ is the case.

In this situation for $\rho = 0_{n+1}10_n10_n10_n1$ we must have $0_{n+1}10_n10_n1 < \alpha < \rho$ or $\rho \leq \alpha < \rho$.

So, let us assume that $0_{n+1}10_n10_n1 < \alpha < \rho$, is the case.

**Remark 15.** We observe that for $a_m = 0_{n+1}10_n1(0_{n+1}10_n10_n1)_m$ we have that $a_m \in A_{m+2}$ and $d(a_m, 0_{n+1}10_n10_n10_n10_n10_n1) \to 0$ as $m \to \infty$.

Hence $0_{n+1}10_n10_n10_n10_n1 \in A_\infty$.

In this situation we have: $0_{n+1}10_n1 < \alpha < 0_{n+1}10_n10_n10_n10_n10_n1$. We have $0_{n+1}10_n10_n1 < \alpha$ is the case. Since $0_{n+1}10_n10_n10_n1 = 0_{n+1}10_n10_n10_n10_n1$.

Let us consider $\varepsilon_0 = 0_{n+1}, a_0 = 0_{n+1}$. We notice that we started with $\varepsilon_0 < \alpha < (a_0) - _b(a_0)$, with

$$d(\varepsilon_0, (a_0) - _b(a_0)) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((\varepsilon_0)i, ((a_0) - _b(a_0))i)$$

$$= \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((\varepsilon_0)i, ((a_0) - _b(a_0))i)$$

and $(a_0) - _b(a_0) \in I_\infty, \varepsilon_0 \in I_\infty$.

We can consider $\rho_1 = 0_{n+1}10_n1$. We may assume $\varepsilon_0 < \alpha < (\rho_1) - _b(\rho_1)$ or $\rho_1 < \alpha < (a_0) - _b(a_0)$. Let denote $\varepsilon_1 = \varepsilon_0$ or $\varepsilon_1 = \rho_1$ and $a_1 = (\rho_1) - _b(\rho_1)$ or $a_1 = (a_0) - _b(a_0)$ we have $\varepsilon_1, a_1 \in I_\infty$ and $d(\varepsilon_1, a_1) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((\varepsilon_1)i, (a_1)i) \leq \sum_{i=5n+3}^{\infty} \frac{1}{2^{i+1}} \delta((\varepsilon_1)i, (a_1)i) \leq \frac{1}{2} d(\varepsilon_0, a_0)$.

Successively we construct sequences $(\varepsilon_n), (a_n)$ such that $\varepsilon_n < \varepsilon_{n+1} < a_{n+1} \leq a_n$ and $d(a_{n+1}, \varepsilon_{n+1}) \leq \frac{1}{2} d(a_n, \varepsilon_n)$ and $\varepsilon_{n+1} < \alpha < a_{n+1}$ with $(\varepsilon_n), (a_n) \subset I_\infty$.

Therefore we obtain $\alpha \in I_\infty$ as we announced.
In the case $01_n < \alpha < 01_{n+1}$, we proceed in a similar way.
This complete the proof of the proposition 5.1.

Let us now consider for any $a \in A_\infty^\omega$ the interval $J(a) = [a_{-b+a}, a]$.

**Proposition 10.** The set $J_\infty = \bigcup_{a \in A_\infty^\omega} (J(a) \cap \text{Min}_2)$ is a dense set in $\Sigma_0 \cap \text{Min}_2 \subseteq \Sigma_0 \subset \Sigma_2$.

**Proof.** By the previous proposition, it is enough to prove that for any $a \in A_\infty^\omega$, $J_\infty \cap I(a)$ is a dense set in $I(a) \cap \text{Min}_2$.

So, let us start initially with $a = 01$ and let us consider $I(01) = [0010, 01]$. For $a = 01$ we have $J(01) = \{00110010, 00110101\}$ and since $0010 < 001101$ we have $J(01) \subset I(01)$.

Let us assume that $\alpha \in I(01) \cap \text{Min}_2$ satisfies $0011 < \alpha < 001101$. Associated to $01$ we have the set $A(01) = \{00(01)^{n-1}11, 0011(01)^{n-1}, n \in \mathbb{N}\} \subset A_\infty^\omega$.

For these sequences we have: $0010 < \ldots < 00(01)^{n-1}11 < 00(01)^{n-2}11 < \ldots < 00111 < 0011101 < 001101101 < \ldots < 0011101$ and $0010 = \lim_{\infty \to \infty} 00(01)^{n+1}11, 001101 = \lim_{n \to \infty} 0011(01)^n$.

Therefore $0010 \in J_\infty$ and $001101 \in J_\infty$.

So, let us assume that there is $n \in \mathbb{N}$ such that $00(01)^{n+1}11 \leq \alpha < 00(01)^{n+1}11$ or $0011(01)^n < \alpha < 0011(01)^n$. Without loss, let us assume that $001001 < \alpha < 0011$ or $0011101 < \alpha < 001101$.

In the first case, $001011 < \alpha < 0011$ we must have $001011 < \alpha < (0011)_{-1}(100) + 0011 = a_{-b+a}$, for $a = 0011$. In this situation we must have: $001011 < \alpha \leq 00101011 = a_{-b}$ or $00101100 < \alpha < (0011)_{-1}(100), 0011$.

Let us assume that $00101100 < \alpha < (0011)_{-1}(1100), 0011$. And we have proved that $001101 < \alpha < 00110011$.

In this situation let us consider $a_0 = 01, a_1 = 0011$. Initially, we have started with $\alpha \in I(a_0) \cap \text{Min}_2$ and we conclude that $\alpha \in I(a_1) \cap \text{Min}_2$. Now, the length of the interval $I(a_0)$ satisfies: $|I(a_0)| = d(01, (01)_{-1}10) = d(01, 001) = \sum_{n=3}^{\infty} \frac{1}{2^{n+1}} \delta((01), (01))$.

and $|I(a_1)| = d(0011, 00101100) = \sum_{n=3}^{\infty} \frac{1}{2^{n+1}} \delta((0011), (00101100))$ satisfies: $|I(a_1)| \leq \frac{1}{2} |I(a_0)|$.

Let us now assume that $001011 < \alpha < 00101100 = a_{-b}$ for $a = 0011$. In this situation, we may have: $001011 < \alpha < 0010110011$ or $0010110011 < \alpha < 00101100$.

Let us assume that $00101100 < \alpha < 0010110011$ is the case.

In this situation, let $a_1 = 0011, a_2 = 00110, a_3 = 0010110011$.

We have started with $\alpha \in [a_2, a_1] \cap \text{Min}_2$ and we have proved that $\alpha \in [a_2, a_3, a_1] \cap \text{Min}_2$ with $|a_3, a_1| \leq |a_2, a_1|$ and $|a_2, a_1| = d(a_2, a_1) = \sum_{i=2}^{\infty} \frac{1}{2^{i+1}} \delta((a_2), (a_1))$ and $|a_3, a_1| = d(a_3, a_1) = \sum_{i=2}^{\infty} \frac{1}{2^{i+1}} \delta((a_3), (a_1))$. That is $||a_3, a_1| \leq \frac{1}{2} |a_2, a_1|$.

Let us assume that $0010110011 < \alpha < 00101100 = a_{-b}$ is the case. In this situation, let $a_0 = 0011, a_1 = 0010110011$. We have started with $\alpha \in [a_1, 001011000] \cap \text{Min}_2$ and we have proved that $\alpha \in [0010110011, 00101100] \cap \text{Min}_2$ with $|0011$, 

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\[ \text{[00101100]} = \sum_{i=3}^{\infty} \frac{1}{2^{i+1}} \delta(\text{(0011)\,i},(\text{00101100)}), \quad [\text{00101100111}, \text{00101100]} = \]
\[ \sum_{i=3}^{13} \frac{1}{2^{i+1}} \delta(\text{(00101100111)}, \text{(00101100)}), \text{ and } [\text{00101100111}, \text{00101100}] \leq \frac{1}{2}[\text{0011}]. \]

So, in any case we have started with \( \alpha \in I_1 \cap \text{Min}_2 \) and we have proved that \( \alpha \in I_2 \cap \text{Min}_2 \) for intervals \( I_1, I_2 \) that contains \( a_1, a_2 \in \mathcal{J}_\infty \), respectively, with \( I_2 \subset I_1 \) and \( |I_2| \leq \frac{1}{2}|I_1| \). Successively, we construct sequence of intervals \( I_1, I_2, \ldots I_{n+1} \) such that \( (i)I_{n+1} \subset I_n \) and \( (ii)|I_{n+1}| \leq \frac{1}{2}|I_n| \) and elements \( a_1, a_2, \ldots a_{n+1} \) in \( \mathcal{J}_\infty \) such that \( \alpha \in I_j \cap \text{Min}_2; j = 1, \ldots, n + 1 \). So, we conclude that \( \alpha \in \mathcal{J}_\infty \).

We proceed in a similar way for \( \text{0011} < \alpha \subset \text{001101} \).

Then, we proved the result for the case \( a = 01 \).

Let us now prove the result in the general case.

Let us take any \( \alpha \in \mathcal{A}_\infty \) and let us prove that \( \mathcal{J}_\infty \cap I(\alpha) \) is a dense set in \( I(\alpha) \cap \text{Min}_2 \).

We have \( I(\alpha) = [a_-, a_+] \) and \( J(\alpha) = [a_-, a_+] \) that is \( a_- < a_+ \). Hence, let us assume \( \alpha \in I(\alpha) \cap \text{Min}_2 \). Without loss we may assume that \( a_- < \alpha < a_+ \).

Otherwise \( \alpha \in J(\alpha) \) and we complete the proof.

In this case the interval \( I_1(\alpha) = [a_-, a_+] \) satisfies \( |I_1(\alpha)| = d(a_-, a_+) \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((001\,i), (a_-\,a_+)). \)

Let us assume that \( \rho(\alpha) = p \) then \( a_- = 0a_1 \ldots a_{2p-2}0b_0b_1 \ldots b_{p-2}0b_0b_1 \ldots b_{p-2} \ldots a_+ \).

That is \( (a_-)_{i} = (a_+)_{i} \) for \( i = 0, 1, \ldots, 2p - 2 \), so \( \delta((a_-\,i), (a_+\,i)) = 0 \) for \( i = 0, 1, \ldots, 2p - 2 \), therefore \( |I_1(\alpha)| \leq \sum_{i=2p-2}^{\infty} \frac{1}{2^{i+1}} \delta((a_-\,i), (a_+\,i)). \)

It is clear that if \( \alpha \in \mathcal{J}_\infty \), we are done. Also if there is \( \rho \in \mathcal{A}_\infty \) such that \( \alpha \in J(\rho) \) we are done.

So, let assume that \( \alpha \notin J(\rho) \) for any \( \rho \in \mathcal{A}_\infty \). In this case let us construct a sequence of intervals \( (I_j(\alpha))_{n=1}^{\infty} \) such that \( |I_{j+1}(\alpha)| \leq \frac{1}{2}|I_j(\alpha)| \) and one of the end points of the interval \( I_j(\alpha) = [a^j_1, a^j_2] \) satisfies \( a^j_n \in \mathcal{J}_\infty \). In this case if \( \varepsilon_j(\alpha) = a^j_n \), the respective \( a^j_n \in \mathcal{J}_\infty \) then we have \( \lim_{j \to \infty} d(\varepsilon_j(\alpha), \alpha) = 0 \) and the result is proved.

Let us consider \( A_0(\alpha) = \{a_-b^{m-1}b_+, a_+b_+a^n; n, m \in N \} \subset \mathcal{A}_\infty \).

We have \( a_- < \ldots < a_-b^n < a_-b^{n-1}b_+ < \ldots < a_-b_+ < a_- < \ldots < a_+ \).

We note \( a_-b_+ = \lim_{n \to \infty} a_-b^n b_+ \) and \( a_-b_+ = \lim_{n \to \infty} a_-b^n a^n \), therefore \( a_-b_+ \in \mathcal{J}_\infty \) and \( a_+ \in \mathcal{J}_\infty \).

So there is \( n \in N \) such that \( a_-b^n b_+ < \alpha < a_-b^{n-1}b_+ \) or \( a_+b_+a^n < \alpha < a_+b_+a^n \).

Without loss let us assume that \( a_-b^n b_+ < \alpha < a_-b_+ \) or \( a_+b_+a^n < \alpha < a_-b_+a^n \) is the case.

In the case \( a_-b^n b_+ < \alpha < a_-b_+ \) we must have \( a_-b^n b_+ < \alpha < (a_-b_+) \).

In this case let \( I_2(\alpha) = [a_-b^n b_+, a_+b_+a^n] \) we have \( I_2(\alpha) \subset \mathcal{J}_1(\alpha) \) and
In the case $a_-b_+ < \alpha < a_-b_+a$, we must have $a_-b_+ < \alpha < (a_-b_+a) - (b_+a-a-) + a_-b_+$.

In this case let $I_2(a) = [a_-b_+, (a_-b_+a) - (b_+a-a-) + a_-b_+]$. We have $I_2(a) \subset I_1(a)$ and

$$|I_2(a)| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((a_-b_+)_+, (a_-b_+a_+)_+)$$

$$\leq \sum_{i=4p-2}^{\infty} \frac{1}{2^{i+1}} \delta((a_-b_+)_+, (a_-b_+a_+)_+)$$

$$\leq \frac{1}{2} |I_1(a)|$$

We observe that $(a_-b_+)_-(b_+a-a-) + a_-b_+ = \lim_{n \to \infty} (a_-b_+)_-(b_+a-a-) + a_-b_+^n$ and for all $n \in \mathbb{N}$, $(a_-b_+)_-(b_+a-a-) + a_-b_+^n \in \mathcal{A}_\infty$.

Also we observe that: $(a_-b_+)_-(b_+a-a-) + a_-b_+ = \lim_{n \to \infty} (a_-b_+)_-(b_+a-a-) + a_-b_+^n$ and for all $n \in \mathbb{N}$, $(a_-b_+)_-(b_+a-a-) + a_-b_+^n \in \mathcal{A}_\infty$. This complete the second step of our inductive construction.

Let us proceed, for the sake of completeness, with the third step for our inductive construction.

Let us consider the case $I_1(a) = [a_-b_+, a_-b_+a]$ and $I_2(a) = [a_-b_+, a_-b_+a_+a_-]$. We may have $a_-b_+ < \alpha < (a_-b_+a_+a_-)$ or $a_-b_+ < \alpha < a_-b_+a_+a_-$. Let us assume that $a_-b_+ < \alpha < (a_-b_+a_+a_-)$ is the case.

In this situation let us consider the sequence $a_-b_+a_+a_- \in \mathcal{A}_\infty$. We have $a_-b_+ < a_-b_+a_+a_- < (a_-b_+a_+a_-)$. We may have $a_-b_+ < \alpha < a_-b_+a_+a_-$. Assume that $a_-b_+ < \alpha < a_-b_+a_+a_-$. In this situation we must have $a_-b_+ < \alpha < (a_-b_+a_+a_-)$. We note that $(a_-b_+a_+a_-)_-(b_+a_+a_-)_+ a_-b_+a_+a_- = \lim_{n \to \infty} (a_-b_+a_+a_-)_-(b_+a_+a_-)_+ a_-b_+a_+a_-^n \in \mathcal{A}_\infty$. So let $I_3(a) = [a_-b_+, (a_-b_+a_+a_-)_-(b_+a_+a_-)_+ a_-b_+a_+a_-^n]$, we have $I_3(a) \subset I_2(a)$ and

$$|I_3(a)| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((a_-b_+)_+, ((a_-b_+a_+a_-)_-(b_+a_+a_-)_+ a_-b_+a_+a_-)_+)$$

$$\leq \sum_{i=10p-2}^{\infty} \frac{1}{2^{i+1}} \delta((a_-b_+)_+, ((a_-b_+a_+a_-)_-(b_+a_+a_-)_+ a_-b_+a_+a_-)_+)$$

$$\leq \frac{1}{2} |I_2(a)|$$

Assume $a_-b_+a_+a_- < \alpha < (a_-b_+a_+a_-)$ is the case.
In this situation let us consider \( I_3(a) = [a\_bb\_a\_a\_b\_+](a\_b\_+\_b\_+\_a\_a\_b\_+) \). We have
\[
|I_3(a)| \leq \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \delta((a\_b\_+\_a\_b\_+)_i, ((a\_b\_+)\_b\_+\_a\_a\_b\_+)_i)
\leq \sum_{i=p-2}^{\infty} \frac{1}{2^{i+1}} \delta((a\_b\_+\_a\_b\_+)_i, ((a\_b\_+)\_b\_+\_a\_a\_b\_+)_i)
\leq \frac{1}{2}|I_2(a)|
\]
Let us assume that \((a\_b\_+)\_b\_+\_a\_a\_b\_+ < \alpha < a\_bb\_a\_a\_b\_+\)

In this situation let us consider
\[
A_1(a) = \{(a\_b\_+\_b\_+\_a\_a\_b\_+)_{n-1}(b\_a\_a\_b\_+)_{n-1}(a\_b\_+)_{n-1}(a\_b\_+)_{n-1}, n, m \in \mathbb{N}\} \subset \mathcal{A}_\infty^\infty
\]
We have
\[
(a\_b\_+)\_b\_+\_a\_a\_b\_+ = \lim_{n \to \infty} (a\_b\_+)\_b\_+\_a\_a\_b\_+ \text{ and } a\_bb\_a\_a\_b\_+
\]
Hence \((a\_b\_+)\_b\_+\_a\_a\_b\_+ \in \mathcal{J}_\infty\) and \(a\_bb\_a\_a\_b\_+ \in \mathcal{J}_\infty\)

Without loss, let us assume that \(n = 1\) and that we have
\[
(a\_b\_+)\_b\_+\_a\_a\_b\_+ < \alpha < (a\_b\_+)\_b\_+\_a\_a\_b\_+ \text{ or } (a\_b\_+)\_b\_+\_a\_a\_b\_+
\]
Let us assume that \((a\_b\_+)\_b\_+\_a\_a\_b\_+ < \alpha < (a\_b\_+)\_b\_+\_a\_a\_b\_+\)

In this situation we must have
\[
a\_bb\_a\_a\_b\_+ < \alpha < (a\_bb\_a\_a\_b\_+)\_b\_+\_a\_a\_b\_+
\]
Hence we must have either
\[
a\_bb\_a\_a\_b\_+ < \alpha < (a\_b\_+)\_b\_+\_a\_a\_b\_+ = a\_bb\_a\_a\_b\_+\text{ or } a\_bb\_a\_a\_b\_+
\]
Assume \(a\_bb\_a\_a\_b\_+ < \alpha < a\_bb\_a\_a\_b\_+)

In this situation let us consider
\[
(a\_b\_+)\_b\_+\_a\_a\_b\_+ = a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+ \in \mathcal{A}_\infty^\infty
\]
We have
\[
a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+ < a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+ < a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+
\]
Hence we must have
\[
a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+ < a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+
\]
Let us assume that \(a\_bb\_a\_a\_b\_+ < \alpha < a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+\)

In this situation we must have
\[
a\_bb\_a\_a\_b\_+ < \alpha < a\_bb\_a\_a\_b\_+ a\_bb\_a\_a\_b\_+
\]
Let us assume now that
\[
|I_4(a)| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \delta((A_1)_i, (A_2)_i) \leq \sum_{i=12p-2}^{\infty} \frac{1}{2^{i+1}} \delta((A_1)_i, (A_2)_i) \leq \frac{1}{2}|I_3(a)|
\]
\[ a - bb_+a - b_+aa - bb_+a < \alpha < a - bb_+a - b_+aa - b \] is the case.

In this situation let
\[ I_1(a) = \frac{a - bb_+a - b_+aa - bb_+a, a - bb_+a - b_+aa - bb_+aa - bb_+a}{a - bb_+a - b_+aa - b} \]
\[ A_1 = a - bb_+a - b_+aa - bb_+a, A_2 = a - bb_+a - b_+aa - b \]

We have
\[ |I_4(a)| \leq \sum_{i=0}^{\infty} \frac{1}{2} |A_1, (A_2)_{i+1} \leq \sum_{i=14p-2}^{\infty} \frac{1}{2} |I_5(a)| \]

This complete the inductive step in the case \( a - bb_+a - b_+a < \alpha < a - bb_+a - b_+aa - b \).

Let us finally, consider the case
\[ a - bb_+a - b_+aa - b < \alpha < (a - bb_+a)(b_+aa - b)_{a - bb_+a} \]

In this situation let us consider the set
\[ A_2(a) = \{(a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a}\} \]
\[ n, m \in \mathbb{N}. \]

We have:
\[ a - bb_+a - b_+aa - b = \lim_{n \rightarrow \infty} (a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a} \] and
\[ a - bb_+a - b_+aa - b_{a - bb_+a} = \lim_{n \rightarrow \infty} (a - bb_+a)(b_+aa - b)_{a - bb_+a} \]

Hence
\[ a - bb_+a - b_+aa - b < \ldots < (a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a} < \]
\[ < (a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a} < \ldots < (a - bb_+a)(b_+aa - b)_{a - bb_+a} < \]
\[ < (a - bb_+a)(b_+aa - b)_{a - bb_+a} < \ldots < (a - bb_+a)(b_+aa - b)_{a - bb_+a} < \]
\[ < (a - bb_+a)(b_+aa - b)_{a - bb_+a}. \]

So there is \( n \in \mathbb{N} \) such that
\[ (a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a} < \alpha < (a - bb_+a)(b_+aa - b)^{n-1}(b_+aa - b)_{a - bb_+a} \] or
\[ (a - bb_+a)(b_+aa - b)^{n-1} < \alpha < (a - bb_+a)(b_+aa - b)_{a - bb_+a} \]

Without loss, let us assume that \( n = 1 \) and that we have
\[ (a - bb_+a)(b_+aa - b)^{n-1} < \alpha < (a - bb_+a)(b_+aa - b)_{a - bb_+a} \] or
\[ (a - bb_+a)(b_+aa - b)_{a - bb_+a} < \alpha < (a - bb_+a)(b_+aa - b)_{a - bb_+a} \]

From now on we continue as in the previous case.

So this complete the inductive step, and the proof of the result. \( \square \)

6. Proof of the results on topological entropy for the standard quadratic family of Lorenz maps. In this section we will prove the results established in section 3.3.

Proposition 11. For any \((\mu, \nu)\) such that \( \mu = \frac{1 + \sqrt{1 + 4\nu}}{2} \) or \( \nu = \frac{1 + \sqrt{1 + 4\mu}}{2} \) the respective map \( F_{\mu, \nu} \) has positive entropy, with the exception of the points \((1, 0)\) and \((0, 1)\), where the entropy is zero.

The figure 2 represents the graph of the curves in proposition 11.

Proof. Let us remind that the quadratic family is defined by:
\[ F_{\mu, \nu}(x) = \begin{cases} -\mu + x^2 & \text{if } x > 0 \\ \nu - x^2 & \text{if } x < 0 \end{cases} \]

Figure 9 shows the graph of the map \( F_{\mu, \nu}(x) \).

The left hand fixed point satisfies: \( \nu - x^2 = x \); that is \( y(\nu) = -\frac{1 + \sqrt{1 + 4\nu}}{2} \).
The curve \(-\mu = y(\nu)\) (or \(-\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}\)) satisfies that \(I(-\mu) = 0\) (here \(I(x)\) is the itinerary of the point \(x\)). Figure 10 shows the map satisfying \(-\mu = y(\nu)\).

So, for \((\mu, \nu)\) such that \(\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}\) we have that \(I(-\mu) = 0\).

The right hand fixed point satisfies \(-\mu + x^2 = x\); that is: \(x^2 - x - \mu = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 4\mu}}{2}\). Hence, we get: \(x(\mu) = \frac{1 + \sqrt{1 + 4\mu}}{2}\).

The curve \(\nu = x(\mu)\) (or \(\nu = \frac{1 + \sqrt{1 + 4\mu}}{2}\)) satisfies: \(I(\nu) = 1\). Figure 11 shows the map satisfying \(\nu = x(\mu)\).

For \(\mu = \nu = 2\), we obtain \(\mu(2) = \frac{1 + \sqrt{3}}{2} = \frac{1 + 3}{2} = 2, \nu(2) = 2\). That is: the curves \(\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}\) and \(\nu = x(\mu)\) o \(\nu = \frac{1 + \sqrt{1 + 4\mu}}{2}\) transversally intersects at \((2, 2)\). See figure 12.

Now, for \(\mu = \nu = 2\) we have that \(F_{\mu, \nu}(x) = \begin{cases} -2 + x^2 & \text{if } x > 0 \\ 2 - x^2 & \text{if } x < 0 \end{cases}\)

Figure 13 shows the graph of the map \(F_{(2,2)}\).
Figure 11. Graph of the equation $\nu = x(\mu)$.

Figure 12. Transversal intersection at $(2, 2)$

Figure 13. Quadratic family for $\mu = \nu = 2$

Hence, it is clear that: $h_{top}(F_{(2,2)}) = \log(2)$. Here, we compute the entropy of the map $F_{(2,2)}$ restricted to the set $([-2, 2] \setminus \{0\})$.

From the equation $-\mu + x^2 = 0$, we obtain: $x = \pm \sqrt{\mu}$; we consider $x = \sqrt{\mu}$

Clearly: $I(\sqrt{\mu}) = (1, 0; 1, 0; 1, 0; 1, 0; \ldots)$.
Now, for $\nu = \sqrt{\mu}$ and $\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}$ figure 14 shows the graph of the map $F_{\mu, \nu}$.

\[
\text{Figure 14. } F_{\mu, \nu} \text{ for } \nu = \sqrt{\mu} \text{ and } \mu = \frac{1 + \sqrt{1 + 4\nu}}{2}
\]

In this case an easy computation shows that: $h_{\text{top}}(F_{\mu, \nu}) = \log \left( \frac{1 + \sqrt{5}}{2} \right)$. Figure 15 shows, in the plane $(\mu, \nu)$ the point of intersection of the curves $\nu = \sqrt{\mu}$ and $\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}$.

\[
\text{Figure 15. Intersection of the curves } \nu = \sqrt{\mu} \text{ and } \mu = \frac{1 + \sqrt{1 + 4\nu}}{2}
\]

Second preimage of zero.

Now, for the zero preimage of the left hand side we have: $\nu - x^2 = 0$ or $x(\nu) = -\sqrt{\nu}$.

Now, let us find $x > 0$ such that $-\mu + x^2 = -\sqrt{\nu}$. we have: $x(\nu) = \sqrt{u + \sqrt{\nu}}$.

The equality $\nu = x(\nu)$ or $\nu = \sqrt{\mu - \sqrt{\nu}}$ or $\mu = \nu^2 + \sqrt{\nu}$ determines the $\nu$-value that satisfies: $I(\nu) = 100$. Therefore, the intersection of the curves : $\mu = \nu^2 + \sqrt{\nu}$ and $\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}$ determine values $(\mu_1, \nu_1)$ such that: $I(-\mu_1) = 0$, $I(\nu_1) = 100$.

Figure 16 shows the graph of the map $F_{\mu_1, \nu_1}$.

In this case: $h_{\text{top}}(F_{\mu_1, \nu_1}) = \log(x_1)$, where $x_1$ is the greatest real root of the equation: $-x^3 + x^2 + 1 = 0$. An easy computation shows that $x_1 \approx 1.465571232$.

Let us now compute $y_2(\nu)$ such that $\nu^2 - y_2(\nu) = -\sqrt{\nu}$. we have: $y_2(\nu) = -\sqrt{\nu^2 + \sqrt{\nu}}$.

Now, let $y_3(\nu)$ be such that $\nu^2 - y_3(\nu) = y_2(\nu)$, that is: $y_3(\nu) = -\sqrt{\nu^2 - y_2(\nu)}$ and, inductively, $y_n(\nu)$ such that $\nu^2 - y_n(\nu) = y_{n-1}(\nu)$. That is: $y_n(\nu) = -\sqrt{\nu^2 - y_{n-1}(\nu)}$. 

Let us now consider \( x_n(\mu, \nu) > 0 \) defined by \(-\mu + x_n^2 = y_n(\nu)\). we obtain:
\[
x_n(\mu, \nu) = \sqrt{\mu + y_n(\nu)}.
\]

Is not hard to verify that the curves \( \nu = x_n(\mu, \nu) \) and \( \mu = 1 + \sqrt{1 + 4\nu^2} \) transversally intersects at the point \((\mu_n, \nu_n)\) such that \( I(\mu_n) = 0 \) and \( I(\nu_n) = 10n \). The figure 17 shows the graph of the map \( F_{(\mu, \nu_n)} \).

For \( F_{(\mu_n, \nu_n)} \) we have that \( h_{top}(F_{(\mu_n, \nu_n)}) = \log(x_n) \), where \( x_n \) is the greatest real root of the polynomial: \((-1)^{n+1}(x_n^{n+1} - x_n^{n-1} - 1)\), we note that: \( x_1 > x_2 > x_3 > \ldots \).

Since \( x_n^{n+1} - x_n^{n-1} - 1 = 0 \), we have: \( 1 - \frac{1}{x_n} - \frac{1}{x_n^{n+1}} = 0 \) and, as a consequence:
\[
\lim_{n \to \infty} \left( 1 - \frac{1}{x_n} - \frac{1}{x_n^{n+1}} \right) = 0.
\]

Let \( \pi = \lim x_n \). If \( \pi > 1 \) then we have \( \lim_{n \to \infty} \left( 1 - \frac{1}{x_n} \right) = \lim_{n \to \infty} \frac{1}{x_n^{n+1}} = 0 \) then \( 1 = \frac{1}{\pi} \) so \( \pi = 1 \), which is a contradiction with the assumption about the value of the point \( \pi \). Therefore \( \pi = 1 \).

So, we conclude that the sequence of parameter values \((\mu_n, \nu_n)\) satisfies:

1. \( \mu_n = \frac{1 + \sqrt{1 + 4\nu_n}}{2} \)
2. \( I(-\mu_n) = 0, I(\nu_n) = 10n \)
Therefore: \( h_{\text{top}} \left( F_{\mu_n, \nu_n} \mid \cap_{j=0}^{\infty} F_{\mu_n, \nu_n}(\mu_n, \nu_n) \right) = \log(x_n) > 0 \) and \( \lim_{n \to \infty} x_n = 1 \)

Now, let us assume that \((\mu, \nu)\) satisfies: \( \mu = \frac{1 + \sqrt{1 + 4\nu}}{2}, 0 < \nu < 2 \). It is clear that there is \( n \) such that \( \mu_{n+1} < \mu < \mu_n \). In this situation: \( \log(x_{n+1}) \leq h_{\text{top}}(F|_{[-\mu, \nu]}) \leq \log(x_n) \), and we conclude that the value \( h_{\text{top}}(F|_{[-\mu, \nu]}) \) is positive.

Let us now prove the following:

**Lemma 6.1.** Let \((\mu, \nu)\) be parameter values such that \( \mu = \frac{1 + t + \sqrt{1 + 4\nu}}{2} \) with \( t > 0 \) then \( I(-\mu) = 0 \)

**Proof.** The curve \( \mu = \frac{1 + t + \sqrt{1 + 4\nu}}{2}; t > 0 \), is a right hand translation of the curve \( \mu = \frac{1 + \sqrt{1 + 4\nu}}{2} \). To prove that \( I(-\mu) \) is 0 we have to prove that:

\[
-\mu = F(0^+) < F \left( -\frac{1 - \sqrt{1 + 4\nu}}{2} \right) = \frac{-1 - \sqrt{1 + 4\nu}}{2}
\]

Since \( t > 0 \), we have that: \( -\frac{t}{2} - \frac{\sqrt{1 + 4\nu}}{2} \leq -\frac{\sqrt{1 + 4\nu}}{2} \), hence: \( -\frac{1 + t}{2} - \frac{\sqrt{1 + 4\nu}}{2} \leq -\frac{1}{2} - \frac{\sqrt{1 + 4\nu}}{2} \)

That is: \( -\frac{1 + t + \sqrt{1 + 4\nu}}{2} \leq -\frac{1}{2} - \frac{\sqrt{1 + 4\nu}}{2} \)

In this way: \( -\mu < F \left( -\frac{1 - \sqrt{1 + 4\nu}}{2} \right) \)

So \( F(0^+) < F \left( -\frac{1 - \sqrt{1 + 4\nu}}{2} \right) \) and we conclude that \( \mu = \frac{1 + t + \sqrt{1 + 4\nu}}{2} \) satisfy \( I(-\mu) = 0 \), as announced.
The figure 18 shows the graph of the map \( F(\mu,\nu) \) for \(-\mu < -\frac{1 + \sqrt{1 + 4\nu}}{2}\). \( \square \)

We also have the following:

**Lemma 6.2.** For parameter values \((\mu,\nu)\) such that \(\nu = \frac{1 + t + \sqrt{1 + 4\mu}}{2}\) with \(t > 0\)

we have: \(I(\nu) = \frac{1}{2}\)

*Proof.* Similar to the proof of the lemma 6.1. \( \square \)

**Lemma 6.3.** For any \(\nu > 0\) the map associated to the intersection of the curves

\(\{(\mu(t),\nu)/\mu = t + \frac{1 + \sqrt{1 + 4\nu}}{2}, t \geq 0\}\) and \(\{(\mu(t),\sqrt{t})/\mu(t) = t + \frac{1 + \sqrt{1 + 4\sqrt{t}}}{2}, t \geq 0\}\)

has entropy zero.

*Proof.* Let us consider the curve \(\mu = t + \frac{1 + \sqrt{1 + 4\nu}}{2}, t \geq 0\), which is a translation of the curve \(\mu = \frac{1 + \sqrt{1 + 4\nu}}{2}\). Let us consider the associated map:

\[
F_{\mu(t),\nu}(x) = \begin{cases} 
\nu - x^2 & \text{if } x < 0 \\
- \left(t + \frac{1 + \sqrt{1 + 4\nu}}{2}\right) + x^2 & \text{if } x > 0
\end{cases}
\]

Figure 19 shows the graph of the map \(F_{\mu(t),\nu}(x)\)

![Figure 19. Graph of the map \(F_{\mu(t),\nu}(x)\)](image)

We observe that for any \(t\) such that \(0 \leq t < \nu^2\) the corresponding map \(F_{\mu(t),\nu}\) has a graph like in the figure 20.

In this case if we solve the equation: \(- \left(t + \frac{1 + \sqrt{1 + 4\nu}}{2}\right) + x^2 = \frac{1 + \sqrt{1 + 4\nu}}{2}\)

we obtain: \(x = \sqrt{t}\).

In this situation the map: \(F_{(\mu,\nu)}|_{[-\frac{t + \sqrt{1 + 4\nu}}{2},\nu] \setminus \{0\}} : [-\frac{1 + \sqrt{1 + 4\nu}}{2},\nu] \setminus \{0\} \rightarrow [-\frac{1 + \sqrt{1 + 4\nu}}{2},\nu]\) has positive topological entropy.

Moreover, for the parameter value \(\nu = \sqrt{t}\), the corresponding map \(F_{(\mu(t),\sqrt{t})}\) has zero topological entropy when restricted to the interval \((y(\nu),\nu) \setminus \{0\}\). Figure 21 shows the graph of the map \(F_{(\mu(t),\sqrt{t})}\).
Figure 20. Graph of map $F_{\mu(t),\nu}$, $0 \leq t < \nu^2$

Figure 21. Graph of map $F_{(\mu(t),\sqrt{t})}$.

We observe that any $x$ such that $0 < x < \sqrt{t}$ satisfy: $\lim_{n \to \infty} F^n_{\mu,\nu}(x) = -\infty$. 

Let us now establish the following:

**Corollary 3.** The map $F_{\mu,\nu}$, for parameter values $(\mu, \nu)$ in the curve

$\left(t + \frac{1 + \sqrt{1 + 4\nu}}{2}, \nu\right)$

with $\nu > \sqrt{t}$ has positive topological entropy.

**Proof.** We known that $\chi(0) = 10$. For the given parameter values $\mu$ and $\nu$ we have that $I(-\mu) = 0$ and $I(\nu) = 10, 1, \ldots$, hence we obtain $I(\nu) > 10 = \chi(0)$. Accordingly with the observation 12 we have that:

$$h_{top}\left(F_{\mu,\nu} \mid \bigcap_{j=0}^{\infty} F^{-j}_{\mu,\nu}([-\mu, \nu])\right) > 0.$$ 

Now, we establishes the following:
Lemma 6.4. For any \( \mu > 0 \) the map associated to the intersection of the curves
\[
\{ (\mu, \nu(t))/\nu = t + \frac{1 + \sqrt{1 + 4\mu}}{2}, t \geq 0 \} \quad \text{and} \quad \{ (|\sqrt{1 + 4\mu}|(t))/\nu(t) = t + \frac{1 + \sqrt{1 + 4\nu}}{2}, t \geq 0 \}
\] has entropy zero.

Proof. The proof is similar to that of lemma 6.3.

\( \square \)

Corollary 4. For parameter values \((\mu, \nu)\) in the curve \(\mu, t + \frac{1 + \sqrt{1 + 4\mu}}{2}\) with \(\mu > \sqrt{1}\) we have that the corresponding map \(F_{(\mu, \nu)}\) has positive entropy.

Proof. Similar to that of the corollary 3

\( \square \)

Remark 16. The figure 1 shows the picture that we have, up to now, with respect to the topological entropy.

In what follows let us denote by \(B(0, 1)\) or simply \(B\) the region of the parameter values \((\mu, \nu)\) such that \((\mu, \nu)\) belongs to the region bounded by the curves:

1. \(\{ (\mu, 0); 0 \leq \mu \leq 1 \}\)
2. \(\{ (0, \nu); 0 \leq \nu \leq 1 \}\)
3. \(\{ \left( \mu, \frac{1 + \sqrt{1 + 4\mu}}{2} \right); 0 \leq \mu \leq 2 \}\)
4. \(\{ \left( \frac{1 + \sqrt{1 + 4\nu}}{2}, \nu \right); 0 \leq \nu \leq 2 \}\).

The figure 2 represents the region \(B(0, 1)\). We observe that this region contains the interesting part of the set \(H_0 = \{ (\mu, \nu) : h_{\text{top}}(\mu, \nu) = 0 \}\). Moreover we can connect the bubble with the part of \(H_0\) which is outside of the bubble through the points \((0, 1)\) and \((1, 0)\).

Lemma 6.5. Let \(B_1(0, 0010, 1101, 1) = \{ (\mu, \nu) \in B; 0 \leq I(-\mu) \leq 0010; 1101 \leq I(\nu) \leq 1 \}\) and \(\Lambda(\mu, \nu) = \bigcap_{j=0}^{\infty} F_{\mu, \nu}^{-j}([-\mu, \nu])\) for \((\mu, \nu) \in B_1(0010, 1101, 0, 1)\), we have that: \(h_{\text{top}}(F_{\mu, \nu} \mid_{\Lambda(\mu, \nu)}) > 0\)

The figure 3 shows region \(B_1(0, 0010, 1101, 1)\)

Proof. We have that: \(\chi(0010) = 110010\), so \(\chi(0010) < 1101\). Hence, if we have: \(0 \leq I(-\mu) \leq 0010\) then: \(110 \leq \chi(I(-\mu)) \leq 110010 < 1101 \leq I(\nu)\).

Therefore, \(h_{\text{top}}(F_{\mu, \nu} \mid_{\Lambda(\mu, \nu)}) \geq h_{\text{top}}(\sigma \mid_{\cap_{j=0}^{\infty} \sigma^{-j}([-\mu, \nu])}) > 0\).

Now, let us establishes the following:

Lemma 6.6. Let \(B_2(0010, 01, 1101, 1) = \{ (\mu, \nu) \in B; 0010 \leq I(-\mu) \leq 01, 1101 < I(\nu) \leq 1 \}\) and \(\Lambda(\mu, \nu) = \bigcap_{j=0}^{\infty} F_{\mu, \nu}^{-j}([-\mu, \nu])\). Then, for any parameter values \((\mu, \nu) \in B_2(0010, 01, 1101, 1)\) we have that: \(h_{\text{top}}(F_{\mu, \nu} \mid_{\Lambda(\mu, \nu)}) > 0\).

Proof. Similar to the proof of the lemma 6.5

The figure 4 shows region \(B_2(0010, 01, 1101, 1)\)
Remark 17. We observe that for parameter values \((\mu, \nu)\) such that \(b(\mu, \nu) = 1101\) and \(0010 \leq a(\mu, \nu) = a < 001101\), we have \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\).

In fact, we have that \(\chi(001101) = 1101\) and the minimal sequence 001101 is the lower minimal sequence with this property, therefore \(\chi(a(\mu, \nu)) < 1101\) and \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\).

Let us now establishes the following result:

**Lemma 6.7.** Let \(B_3(0, 0010, 10, 1101) = \{(\mu, \nu) \in B; 0 \leq I(\mu) < 0010, 10 \leq I(\nu) \leq 1101\}. For any parameter value \((\mu, \nu) \in B_3(0, 0010, 10, 1101)\) we have that \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\).

**Proof.** Similar to the proof of the lemma 6.5.

The figure 5 shows region \(B_3(0, 0010, 10, 1101)\).

**Remark 18.** We observe that if \((\mu, \nu)\) is a parameter value such that \(a(\mu, \nu) = 0010\) and \(110010 < b(\mu, \nu) \leq 1101\), then \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\). In fact, \(\chi(0010) = 110010\), then for any \(b(\mu, \nu)\) such that \(110010 < b(\mu, \nu)\) we have that \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\) in the case that \(a(\mu, \nu) = 0010\).

Let us now consider \(a \in A^\infty\) and \(b = b(a) = \sup\{\sigma^j(a); j \in \mathbb{N}\}\).

Let \(B_1(0, a, b, a, a, 1) = \{(\mu, \nu) \in B; 0 \leq I(\mu) < a - b, b + a \leq I(\nu) \leq 1\} = B_1\),

\(B_2(a, b, a, a, a, 1) = \{(\mu, \nu) \in B; a - b \leq I(\mu) < a, b + a \leq I(\nu) \leq 1\} = B_2\),

\(B_3(0, a, b, a, b, a) = \{(\mu, \nu) \in B; 0 \leq I(\mu) < a - b, b \leq I(\nu) \leq b + a\} = B_3\).

For a parameter value \((\mu, \nu) \in B_1 \cup B_2 \cup B_3\) let \(\Lambda(\mu, \nu) = \bigcap_{j=0}^{j=n} F_{\mu, \nu}^{-j}([-\mu, \nu])\).

Let us now establishes the following:

**Proposition 12.** For any parameter value \((\mu, \nu) \in B_1 \cup B_2 \cup B_3\) we have that: \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) > 0\).

**Proof.** Similar to the proof of the lemma 6.5, considering the observation 12.

See figure 6 shows the region \(B_1 \cup B_2 \cup B_3\).

**Remark 19.**

1. We observe that the region \(B_2(a - b, a, b, a, 1)\) include the region \(L_2 = \{(\mu, \nu); I(\mu) = a, I(\nu) < b + a\}\) which has non-empty interior.

2. The region \(B_3(0, a, b, a, b, a)\) include the region \(R_2 = \{(\mu, \nu); I(\nu) = b, 0 \leq I(\mu) < a - b\}\) which has non-empty interior.

For any \(a \in A^\infty\) let us define:

\(C_1(a, b, a) = \{(\mu, \nu) \in B; I(\mu) = a, I(\nu) = b + a\} = C_1\);

\(C_2(a, b) = \{(\mu, \nu) \in B; I(\mu) = a, I(\nu) = b\} = C_2\) and

\(C_3(a, b, a) = \{(\mu, \nu) \in B; I(\mu) = a - b, I(\nu) = b\} = C_3\).

We have the following:

**Proposition 13.** For any \((\mu, \nu) \in C_1 \cup C_2 \cup C_3\), we have that: \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) = 0\).

**Proof.** Since \(\chi(a) = b + a\) then \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) = 0\) for any \((\mu, \nu) \in C_1 \cup C_2\).

Now, as \(\chi(a - b) = b + a - b\), then \(h_{\text{top}}(F_{\mu, \nu} | \Lambda(\mu, \nu)) = 0\) if \((\mu, \nu) \in C_3\), see figure 7.

**Remark 20.**

1. Now observe that the map \((\mu, \nu) \rightarrow (I(\mu), I(\nu))\) defined from \(B(0, 1)\) into \(\text{Min}_2 \times \text{Max}_2\) is not continuous. In fact, let \((\mu_n, \nu_n)\) be a parameter value such that \(I(-\mu_n) = 001(01)\) and \(I(\nu_n) = 110010\). We have:
\[ I(-\mu_n) \to 00101 = 0010 = a \cdot \underbar{b}, \quad a = 01, \quad b = 10, \quad I(\nu_n) = 110010 \]

hence \((I(-\mu_n), I(\nu_n)) \to (0010, 110010)\). Now, for \((\overline{\mu}, \overline{\nu}) = \lim_{n \to \infty} (\mu_n, \nu_n)\) we have that \((I(-\mu), I(\nu)) = (0010, 10)\). We conclude that: \(\lim_{n \to \infty} (I(-\mu_n), I(\nu_n)) \neq (I(\overline{\mu}), I(\overline{\nu}))\) and \((\overline{\mu}, \overline{\nu}) = \lim_{n \to \infty} (\mu_n, \nu_n)\).

2. There are a countable number of parameter values \((\mu, \nu)\) such that the map \((\mu, \nu) \to (I(-\mu), I(\nu))\) is not continuous. In fact, it is enough to consider \(a \in A_{\infty}^1\) and \((\mu_n, \nu_n)\) such that 
\(a(\mu_n, \nu_n) = a \cdot b; b(\mu_n, \nu_n) = b + a \cdot b^n\), with 
\(b = b(a)\). Now, it is possible to verify that \((\mu_n, \nu_n) \to (\overline{\mu}, \overline{\nu})\) where 
\(a(\overline{\mu}, \overline{\nu}) = a \cdot \underbar{b}\) and \(b(\overline{\mu}, \overline{\nu}) = \underbar{b}\). Since 
\(\lim a(\mu_n, \nu_n) = a \cdot \underbar{b}\) and \(\lim b(\mu_n, \nu_n) = b + a \cdot \underbar{b}\), we have that: \((I(\overline{\mu}), I(\overline{\nu})) \neq \lim(I(\mu_n), I(\nu_n))\).

3. For any \(a \in A_{\infty}^1\) let denote by \(B(a)\) the bounded region of the \((\mu, \nu)\) plane whose boundary is given by the curves:
(a) \(I(-\mu) = a, I(\nu) = b + a\); denoted \(\gamma_{a, b + a} \subset C_3(a, b + a)\);
(b) \(I(-\mu) = a \cdot \underbar{b}, I(\nu) = \underbar{b}\); denoted \(\gamma_{a \cdot \underbar{b}, \underbar{b}} \subset C_3(a \cdot \underbar{b}, \underbar{b})\);
(c) \(I(-\mu) = a \cdot \underbar{b}, I(\nu) = b, \underbar{a}\); denoted \(\gamma_{a \cdot \underbar{b}, b, \underbar{a}}\)
(d) \(I(\nu) = b + a, \underbar{a}\); denoted \(\gamma_{b + a, \underbar{a}}\).

Accordingly with [7], the region \(B(a)\) is an island, and consequently an archipelago. Hence: we conclude that there are infinitely many archipelagos and island in the standard family (see figure 8).

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