Slope of the beta function at the fixed point of SU(2) gauge theory with six or eight flavors

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We consider measurement of the leading irrelevant scaling exponent $\gamma_g$, given by the slope of the beta function, at the fixed point of SU(2) gauge theory with six or eight flavors. We use the running coupling measured using the gradient flow method and perform the continuum extrapolation by interpolating the measured beta function. We study also the dependence of the results on different discretization of the flow. For the eight flavor theory we find $\gamma_g = 0.648(97)\pm0.16$, consistent with the earlier analysis.

I. INTRODUCTION

One of the basic goals of Beyond Standard Model lattice gauge theory is to establish the existence of infrared fixed points (IRFP) of gauge theories with sufficiently large number of flavors and to determine its properties. For recent reviews see [1–3].

A much studied case is SU(2) gauge theory with fermions in the fundamental representation [4–9]. While the upper edge of the conformal window is robust, as the asymptotic freedom is lost at $N_f = 11$, a consistent picture of the extent of the conformal window has only recently emerged: simulations of the 8 flavor theory have shown the existence of a fixed point [9] and similarly the 6 flavor case [10]. Theories with $N_f = 4$ and $N_f = 2$ are expected to break chiral symmetry [6]. Another benchmark case, where the existence of a fixed point has been established, is SU(2) gauge theory with two Dirac fermions in the adjoint representation [11–25].

In this paper we analyze further SU(2) gauge theory with eight or six flavors in the fundamental representation. We focus on the measurement of critical exponent $\gamma_g$, given by the slope of the $\beta$-function at the IRFP. For the first time we determine this scheme independent observable in the eight flavor theory, while the earlier results on the six flavor theory serve as a check on our methodology.

The slope of the $\beta$-function is directly measurable from the step scaling function of the coupling. We obtain $\gamma_g = 0.19(8)^{+0.21}_{-0.20}$ in the eight flavor theory, and similar analysis applied to the six flavor theory yields $\gamma_g = 0.648(97)\pm0.16$, consistent with the earlier analysis.

This paper is structured as follows: We first discuss briefly the lattice implementation: in Sec. II, and the measurement of the coupling in Sec. III. Then we present our results on the measurement of $\gamma_g$ for the six and eight flavor theories in Sec. IV. We end with conclusions and outlook in Sec. V.

II. LATTICE IMPLEMENTATION

We extend our analysis of the data generated in the studies [9, 10]. As the raw data and algorithmic details about the model are available in these papers, the discussion here will be in a form of brief summary.

The lattice formulations uses HEX-smeared [26] clover improved Wilson fermions together with gauge action that mixes the smeared and unsmeared gauge actions with mixing parameter $c_g = 0.5$:

$$S = (1 - c_g)S_G(U) + c_gS_G(V) + S_F(V) + c_{SW}\delta S_{SW}(V),$$

where the $V$ and $U$ are the smeared and unsmeared gauge fields respectively. This mixing of the smeared and unsmeared gauge actions helps to avoid the unphysical bulk phase transition within the interesting region of the parameter space [27] enabling simulations at larger couplings. In the fermion action, we set the Sheikholeslami-Wohlert coefficient to the tree-level value $c_{SW} = 1$, which is the standard choice for smeared clover fermions [20, 26, 28]. In earlier studies [6, 29] we have verified that this value is very close to the true non-perturbatively fixed $c_{SW}$ coefficient and cancels most of the $O(a)$ errors.

We use Dirichlet boundary conditions at the temporal boundaries $x_0 = 0, L$, as in the Schrödinger functional method [30–33], by setting fermion fields to zero and gauge link matrices to unity $U = V = 1$. The spatial boundaries are periodic. These boundary conditions allow us to tune the fermion mass to zero using the PCAC relation. In practice, the hopping parameter $\kappa_c(\beta_L)$ is tuned at lattices of size $24^4$, so that the PCAC fermion...
mass vanishes with accuracy $10^{-5}$. This critical hopping parameter $\kappa_c(\beta_L)$ is then used on all the lattice sizes.

The simulations are run using the hybrid Monte Carlo algorithm with 2nd order Omelyan integrator [34, 35] and chronological initial values for the fermion matrix inversions [36]. We reach acceptance rate that is larger than 85%. For the analysis considered in this paper, we use lattices of volumes $(L/a)^4 = 10^4, 12^4, 16^4, 18^4, 20^4, 24^4, 30^4, 32^4$, where the $L = 18, 30$ are only used for $N_f = 6$ results and $L = 32$ is only available in the $N_f = 8$ analysis. The difference in available lattice sizes between the two cases is caused by the fact that we used step scaling step $s = 2$ for $N_f = 8$ in [9] and $s = 3/2$ for $N_f = 6$ in [10]. The bare couplings $\beta_L \equiv 4/g_0^2$ vary from 8 to 0.5 for $N_f = 6$ and to 0.4 for $N_f = 8$. For all combinations of $L/a$ and $\beta_L$, we generate between $(5 - 100) \times 10^3$ trajectories.

### III. GRADIENT FLOW COUPLING CONSTANT

The running coupling is defined by the Yang-Mills gradient flow method [37–39]. In the lattice flow equation the unsmeared lattice link variable $U$ is evolved using the tree-level improved Lüscher-Weisz pure gauge action.

The coupling at scale $\mu = 1/\sqrt{S t}$ [40] is defined via the energy measurement as

$$g^2_{GF}(\mu) = N^{-1} t^2 \langle E(t) \rangle |_{x_0 = L/2, t = 1/8 \mu^2},$$

where $a$ is the lattice spacing. The renormalization factor $N$ has been calculated in Ref. [41] for the Schrödinger functional boundary conditions so that $g^2_{GF}$ matches continuum $\overline{\text{MS}}$ coupling in the tree level of perturbation theory. Since the Schrödinger functional boundary conditions break the translation invariance in time direction, we measure the coupling only at central time slice $x_0 = L/2$. We measure the energy density $E(t)$ using both the clover and plaquette discretizations.

The flow time $t$ at which the gradient flow coupling is evaluated is arbitrary and defines the renormalization scheme. However, it is useful to link the lattice and renormalization scales with a dimensionless parameter $c_t$ so that the relation $\mu^{-1} = c_t L = \sqrt{S t}$ is satisfied [41, 42]. In Ref. [41] it is proposed, that for the Schrödinger functional boundary conditions the choice $c_t = 0.3–0.5$ yields reasonably small statistical variance and cutoff effects. In Ref. [10] we choose $c_t = 0.3$ for the six flavor theory, and in Ref. [9] we choose $c_t = 0.4$ for the eight flavor theory. In both studies, the $c_t$ was then varied to check the consistency of the results.

In the earlier studies [9, 10], we reduced the $\mathcal{O}(a^2)$ discretization effects by using the $\tau_0$-correction method [43], that modifies the Eq. (1) by measuring the energy density at flow time $E(t + \tau_0 a^2)$. The $\tau_0$ was tuned by hand to remove most order $\mathcal{O}(a^2)$ effects. Since the discretization effects grow with the coupling, we made the $\tau_0$ function of the gradient flow coupling $g^2_{GF}$; see [9, 10] for the details of this implementation.

In the present work we would like to investigate an alternative method to reduce the $\mathcal{O}(a^2)$ correction. In Ref. [44] it is noted that as the different discretizations have different $\mathcal{O}(a^2)$ behavior, it is possible to combine two discretizations so that the $\mathcal{O}(a^2)$ effects cancel each other. Combining the gradient flow measurement done with the plaquette and clover discretizations, we therefore get

$$g^2_{GF} = N^{-1} t^2 [(1 - X)\langle E_{\text{Clover}}(t) \rangle + X \langle E_{\text{Plaq}}(t) \rangle],$$

where mixing coefficient $X$ can in principle be chosen freely, but the perturbative results for periodic boundaries from Refs. [45, 46] suggest value of $X = 1.25$ for our choice of discretizations. We will investigate the dependence of the results on the value of the mixing parameter $X$.

Since the $\tau_0$-correction was optimized for the whole data and depended on the measured coupling, it naturally gives more fine tuned correction. On the other hand, in this paper we are interested only on the quantities at the IRFP so the data does not have to be perfectly improved at small couplings. Also, we will do bulk of our analysis with the unimproved $X = 0$ and then only use the parameter $X$ to study how the different discretizations affect the results.

### IV. LEADING IRRELEVANT CRITICAL EXPONENT

#### A. Step scaling method

The leading irrelevant exponent of the coupling $\gamma_g$ is defined as the slope of the $\beta$-function. On the lattice the evolution of the coupling is measured with the step
scaling function
\[ \Sigma(g^2, L/a, s) = g_{GF}^2 (g_0^2, sL/a) |_{g_{GF}^2(g_0^2, L/a) = g^2}, \]  
(3)
\[ \sigma(g^2, s) = \lim_{L/a \to \infty} \Sigma(g^2, L/a, s). \]  
(4)

In the vicinity of the IRFP, where \( \beta \)-function is small, the step scaling function, \( \beta \)-function and \( \gamma_g^* \) can be related as follows:
\[ \beta(g) = \mu \frac{dg}{d\mu} \approx \gamma_g^*(g - g^*) \]  
(5)
\[ \approx \beta(g) = \frac{g}{2\ln(s)} \left( 1 - \frac{\sigma(g^2, s)}{g^2} \right). \]  
(6)

Here \( g^* \) is the coupling at the IRFP. In Ref. [10] we calculated the step scaling function for \( N_f = 6 \) by interpolating the measured couplings with 9th order polynomial, which led to continuum extrapolation shown in Fig. 1 for \( c_t = 0.3 \). In this case the final form of the function near the fixed point was smooth enough that we managed to measure the leading irrelevant exponent \( \gamma_g^* = 0.648(97)^{+0.16}_{-0.1} \), where the first set of errors implies the statistical errors with the parameters used in Ref. [10], and the second set of errors gives the variance between all measured discretizations. When the values of \( c_t \) were varied, the \( \gamma_g^* \) measurements remained consistent with each other, within the errors, indicating the scheme independence of this quantity.

However, we can also directly interpolate the finite volume \( \beta(g) \)-function (6) (where \( \sigma(g^2, s) \) is substituted with \( \Sigma(g^2, L/a, s) \)), instead of the measured couplings. Similar ideas have previously been implemented in Refs. [24, 25, 47]. Not only does this make the continuum limit smoother around the fixed point, but allows to limit the fit to a region near the IRFP. We show three different fits in Fig. 2 for \( c_t = 0.3 \) and \( X = 0 \), which corresponds to the unimproved clover measurements in [10]. We use three different polynomial fit functions: linear, quadratic, and quartic, and for each fit choose the number of points that minimizes the \( \chi^2 \)/d.o.f. While Fig. 2 shows the \( X = 0 \) -case, we do similar fits for \(-1.5 \leq X \leq 1 \). Depending on the value of \( X \) the \( \chi^2 \)/d.o.f.value varies between 0.5 and 2.5.

In Fig. 3 we show the continuum limit of \( \beta(g_*) \) at different values of the mixing parameter \( X \), assuming discretization errors \( \mathcal{O}(a/L)^2 \). The top and bottom panels correspond, respectively, to the values \( g_{GF} = 3 \) and \( g_{GF} = g^* = 3.8 \) of the coupling.

From the figure we can see that the continuum limit remains stable with respect to the variations of the parameter \( X \). At weaker coupling the values below \( X = -0.5 \) have reduced \( a^2 \)-effects, while near the fixed point the dependence on \( X \) becomes less pronounced.

In Fig. 4 we show the location of the IRFP as a function of the mixing parameter \( X \) and for different fits. The existence and the location of the IRFP in \( N_f = 6 \) theory agrees with our previous measurement [10], \( g^* = 14.5(3)^{+0.41}_{-1.38} \), within the error bounds indicating the variance between different discretizations shown with the dotted horizontal lines in the figure and corresponding to the second set of errors in the numerical result quoted above. For the \( \gamma_g^* \) measurement, we reproduce the value of \( \gamma^* \) obtained in the original analysis in [10]. Similarly, the results of \( \gamma_g^* \) measurements are shown in Fig. 5. Comparing the different discretizations, the quadratic fit seems to have most consistent behavior within the data. Therefore, we take the quadratic result with \( X = 0 \) as our main result and quote the variance between discretizations and choices of parameter \( X \) as systematic errors. We get \( \gamma_g^* = 0.66(4)^{+0.25}_{-0.13} \), which is in agreement with
the result $\gamma_g^* = 0.648(97)^{+0.16}_{-0.1}$ obtained earlier in [10].

In Ref. [9] we measured the running coupling for $N_f = 8$ by interpolating the couplings with rational ansatz, where the numerator was 7th order polynomial and the denominator a 1st order polynomial, and then taking the continuum limit. This continuum limit, together with the raw $\tau_0$-corrected data, is shown in the Fig. 6. The interpolation function was chosen by extensive statistical tests to give best fit to the whole data. While we were sure to check the existence of the IRFP within the reported errors, the chosen fit function develops a curvature at the fixed point, which renders a reliable measurement of $\gamma_g^*$ impossible.

On the basis of the results in six flavor theory, we now directly interpolate the raw $\beta$-function instead of the raw couplings also in the $N_f = 8$ case. Again we perform the linear, quadratic, and quartic fits for the regions of data where these fit ansatz give the best $\chi^2/d.o.f$. These fits, together with their continuum limits, are shown in Fig. 7. Similar as with the six flavor theory we choose the quadratic fit with $X = 0$ to be our main result and report the variance in fit function and parameter $X$ as the systematic errors.
Similarly as in the six flavor case studied above, in Fig. 8 we show the continuum limit of $\beta(g_\star)$ at different mixing parameters $X$, assuming discretization errors $O((a/L)^2)$ in the eight flavor theory. From the figure we can see that the continuum limit remains stable with respect to the variations of the parameter $X$, but clearly values around $X \sim -0.5$ have reduced $a^2$-effects as the slope is small. On the other hand, we do not observe good scaling with the value $X = 1.25$ as suggested by perturbation theory [46].

In Fig. 9 we show the location of the IRFP as a function of the mixing parameter $X$ and for different fits. Compared with the earlier analysis [9], $g^2 = 8.24(59)^{+0.97}_{-1.04}$; the existence and location of the IRFP does not change when the methods discussed in this paper are implemented: the results are consistent within the bounds given by the variance between different discretizations and shown by the dashed lines in the figure and corresponding to the second set of errors in the numerical result quoted above.

Similarly, the results for the $\gamma^*_g$ measurement are shown in Fig 10. The upper panel shows the results for different fits at $c_t = 0.4$. The theoretical scheme independent result $\gamma^*_g = 0.25$ [48, 49] is shown by the dashed black line. We measure $\gamma^*_g = 0.19(8)^{+0.21}_{-0.09}$. The lower panel shows the dependence of the result on the value of the scheme parameter $c_t$ for the quadratic fit. We measure $\gamma^*_g = 0.16(4)$ for the $c_t = 0.45$ case and $\gamma^*_g = 0.2(1)$ for the $c_t = 0.35$ case. Overall, all the fits when the $X$ is between -0.5 and 0.5 are in agreement with each other and the scheme independent result.

**B. Finite-size scaling method**

The results obtained in the previous subsection rely on the $a^2$-scaling of the lattice observables. Near the IRFP this scaling can be modified by nontrivial scaling...
exponents. If these scaling exponents remain small near the infrared fixed point, we can assume that the power counting argument holds and cutoff effects dominated by dimension 6 operators decrease with the power of lattice spacing $a$. This can modify the $a^2$-scaling, which relies on Symanzik improvements around the Gaussian UV fixed point. Since we checked our continuum limit with multiple different $a^2$ scalings (by varying $X$), and since the results hold between discretizations and varying $c_t$, we argue that the scaling violation is small and the continuum limit is robust.

In order to check the consistency of the $a^2$-scaling in the continuum limit, we also measure the leading irrelevant exponent using a finite-size scaling method developed in Refs. [50–53]. In the close proximity of the IRFP, by integrating the $\beta$-function, we obtain a scaling relation between lattices of size $L_{ref}$ and $L$ [52]:

$$g^2_{\text{GF}}(\beta, L) - g^2_\ast = [g^2_{\text{GF}}(\beta, L_{ref}) - g^2_\ast] \left(\frac{L_{ref}}{L}\right)^{\nu_0}. \quad (7)$$

Again, we start with the analysis of $N_f = 6$ theory. In Fig. 11 we show the value of $\gamma_0^\ast$ obtained by Eq. (7) in six flavor theory for two different values of $L_{ref}/a = 18$ and 20, and vary the lattice size between $L_{ref}$ and 30. We use a polynomial interpolation to the $\tau_0$-corrected measurements and choose the IRFP to be at the measured value $g^2_\ast = 14.5(3)^{+0.41}_{-1.38}$. The dashed lines around the shaded bands show the variation of the result when $g^2_\ast$ is varied within its statistical errors. This analysis was carried out already in [10].

This method assumes vanishing discretization artifacts, and thus it can be used only if the lattices are close to the continuum (i.e. large $L$). Also, the method can not be applied at the fixed point. However, from Fig. 11 we see that in the vicinity of the fixed point $g^2_{\ast} \approx g^2_\ast \approx 14$, the method gives value of $\gamma_0^\ast$ consistent with the result obtained directly from the measurement of the $\beta$-function.

The dependence on the mixing parameter $X$ is shown in Fig. 12. Here the open symbols show the maximum value of $\gamma_0^\ast$ at each $X$ obtained with the finite size scaling method. This method seems to be very sensitive to the $a^2$-effects. Not only does the measurement of $\gamma_0^\ast$ depend on the value of $X$, but we cannot even get a non-zero result with unimproved data $X > 0.1$. The measurements in the region $X < -0.5$, with small $a^2$-scaling, we get results that agree with the $\tau_0$-corrected result in Fig. 11. Overall, for $X < -0.5$ this analysis yields results consistent with the earlier analysis [10] and also with the theoretical scheme independent result [49].

Finally, we show similar results for the $N_f = 8$ theory. In Fig. 13 we show the fit to Eq. (7) using the results from [9] with $\tau_0$-correction, $c_t = 0.4$, $L_{ref}/a = 16$ and $\beta$ between 0.4 and 0.7. We see that also for the eight flavor theory this method produces results consistent with the one from the direct measurement of the $\beta$-function shown by the horizontal lines in the figure.

In Fig. 14 we show the value of $\gamma_0^\ast$ obtained by the finite size scaling method in the eight flavor theory. In the figure the horizontal red lines correspond to the value $\gamma_0^\ast = 0.28(12)$ obtained from the slope of the $\beta$-function at the fixed point and the black line corresponds to the
scheme independent result $\gamma^*_{\bar{g}} = 0.25$. To obtain the finite size scaling result we use $L_{\text{ref}} = 16$ or 20 and vary the lattice sizes between $L_{\text{ref}}$ and 32. Rational interpolation of the $\tau_0$-corrected measurements is used and the measured fixed point value $g^2_{\text{FP}} = 8.24(59)^{+0.97}_{-1.64}$.

In Fig. 15 we show the $X$-dependence of $\gamma^*_{\bar{g}}$. The circles show the maximum value of $\gamma^*_{\bar{g}}$ obtained by the finite size scaling method. The red and black horizontal lines are the same as in Fig. 14.

V. CONCLUSIONS AND OUTLOOK

We have studied the properties of the IRFP of SU(2) lattice gauge theory with 6 or 8 fermions in the fundamental representation. The existence of IRFP in these theories has been established in earlier work [9, 10], and in this paper we focused on determination of the leading irrelevant critical exponent, $\gamma^*_g$ in these two theories. For the first time, we obtain in the eight flavor theory the result $\gamma^*_g = 0.28(12)$, and the black line shows the scheme invariant estimate. The result is compatible with the scheme independent analytical results 0.25 obtained in [48, 49]. We also studied the robustness of the results with respect to different interpolations used in the analysis.

Furthermore, we showed that the methodology we have applied here is consistent with the earlier analysis of the six-flavor theory. In this paper we obtained the result $\gamma^*_g = 0.66(4)^{+0.25}_{-0.13}$ using a quadratic fit to the $\beta$-function near the IRFP. This result was shown to be very stable with respect to the discretization mixing parameter in the definition of the gradient flow. The result is also consistent with the earlier result $\gamma^*_g = 0.648(97)^{+0.16}_{-0.1}$ in [10] and hence also with the analytic results of [48, 49].
Our results indicate the emergence of a consistent picture of strong coupling features of SU(2) gauge theory inside the conformal window.

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