On the existence of min-max minimal torus

Xin Zhou

Abstract

In this paper, we will study the existence problem of minmax minimal torus. We use classical conformal invariant geometric variational methods. We prove a theorem about the existence of minmax minimal torus in Theorem 5.1. Firstly we prove a strong uniformization result (Proposition 3.1) using method of [1]. Then we use this proposition to choose good parametrization for our minmax sequences. We prove a compactification result (Lemma 4.1) similar to that of Colding and Minicozzi [2], and then give bubbling convergence results similar to that of Ding, Li and Liu [7]. In fact, we get an approximating result similar to the classical deformation lemma (Theorem 1.1).

1 Introduction

The existence problem of minimal surfaces is always an interesting topic. We know the existence of minimizing minimal disk, i.e. the classical Plateau problem (see Chapter 4 of [3]) since 1931. There are many results from that time. In general, a minimal surface is a harmonic conformal branched immersion from a Riemann surface to a compact Riemannian manifold. Most results only consider existence of area minimizing minimal surfaces in a given homotopy class. In particular, the existence of area-minimizing surfaces has been proved for all genus in a suitable sense (cf. [11], [12], [5] etc.).

Besides minimizing minimal surfaces, we naturally ask whether there exist min-max minimal surfaces. Here min-max means the area of the minimal surfaces is just the min-max critical point of the area functional in a homotopy class. In general, suppose $A$ is a functional on a Banach manifold $M$, $\Omega = \{ \nu(t) : [0, 1] \to M, \nu \in C^0([0, 1], M) \}$ the path space in $M$ with $\sigma \in \Omega$. Then $W_A = \inf_{\rho \in [\sigma]} \max_{t \in [0, 1]} A(\rho(t))$ is the min-max critical value in the homotopy class of $\rho$. It will be more complicated when considering min-max minimal surfaces than the minimizing case. From the point of view of variational method, the approximation sequences will be one parameter families of mappings, which makes it difficult to do compactification. J. Jost gave such an approach in his book [8]. Recently Colding and Minicozzi [2] also gave such an approach in the case of sphere.
using geometric variational methods. They all used the bubble convergence of almost harmonic mappings from closed surfaces given by Sacks and Uhlenbeck [11]. Colding and Minicozzi also found a good approximation sequence which plays an important role in their proof of finite time extinction of the Ricci flow.

We will extend Colding and Minicozzi’ approach to the case of torus, i.e. the existence of min-max minimal torus. In fact, we give a stronger approximation for a special minimizing sequence. Using notations in Section 2.1, the main result is:

**Theorem 1.1** For any homotopically nontrivial path \( \beta \in \Omega \), if \( W > 0 \), there exists a sequence \( (\rho_n, \tau_n) \in [\beta] \), with \( \max_{t \in [0,1]} E(\rho_n(t), \tau_n(t)) \to W \), and \( \forall \epsilon > 0 \), there exist \( N \) and \( \delta > 0 \) such that if \( n > N \), then for any \( t \in (0,1) \) satisfying:

\[
E(\rho_n(t), \tau_n(t)) > W - \delta,
\]

there are possibly a conformal harmonic torus \( u_0 : T^2_{\tau_0} \to N \) and finitely many harmonic sphere \( u_i : S^2 \to N \), such that:

\[
d_V(\rho_n(t), \bigcup_i u_i) \leq \epsilon.
\]

Here \( d_V \) means varifold distance as in Appendix A in [2]. It is a corollary of Theorem 5.1 and Appendix A in [2]. It is a stronger approximation result than Theorem 1.14 of [2]. We use the energy condition inequality \( \| \) for the special sequence \( \rho_n \), while Theorem 1.14 of [2] use area condition.

In the case of torus, we have to include the variation of conformal structures as discussed in [12] and [13]. The analysis of singularity in the bubble convergence will be more complicated than in the case of sphere. We will give existence results similar to that of Ding, Li and Liu [7]. In the following, we will first give our notations, and then give the sketch of this paper at the end of Section 2.2.

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2 Sketch of the variational methods for min-max minimal torus

In the paper [2], Colding and Minicozzi used variational methods to give the existence of min-max minimal spheres. Let’s firstly sketch their idea. Let \((N, h)\) be the ambient space. \(\Omega = \left\{ \gamma(t) \in C^0([0,1], C^0 \cap W^{1,2}(S^2, N)) \right\}\) is the path space. Here for all \(\gamma(t) \in \Omega\), \(\gamma(0), \gamma(1)\) are constant mappings. We call all such one parameter family of mappings \(\gamma(t) \in \Omega\) paths in the following. For \(\beta \in \Omega\), let \([\beta]\) be the homotopy class of \(\beta\) in \(\Omega\). The min-max critical value is \(W = \inf_{\rho \in [\beta]} \max_{t \in [0,1]} \text{Area}(\rho(t))\). They want to learn the behavior of critical points corresponding to \(W\). They firstly chose an arbitrary minimizing sequence \(\tilde{\gamma}_n(t) \in [\beta]\), such that \(\lim_{n \to \infty} \max_{t \in [0,1]} \text{Area}(\tilde{\gamma}_n(t)) = W\). Then they did almost conformal reparametrization for these paths to get \(\gamma_n(t) \in [\beta]\) which are almost conformal, i.e. \(E(\gamma_n(t)) - \text{Area}(\gamma_n(t)) \to 0\). Finally they perturbed \(\gamma_n(t)\) to \(\rho_n(t)\) by local harmonic replacement so that the new paths \(\rho_n(t)\) have certain compactness. The existence of min-max minimal spheres follows from this construction and Sacks and Uhlenbeck’s bubbling compactness [11].

We want to extend the min-max variational method given by Colding and Minicozzi to the case of torus \(T^2\). The difference between sphere and torus is that torus has more than one conformal structures, while the conformal structure of sphere is unique. Generally speaking, the pull-back metrics of the mappings on the area minimizing sequence of paths will correspond to different conformal structures. It is natural to include the variance of the conformal structures in the min-max construction. In fact, we need to consider the Teichmüller space of torus in order to maintain the homotopy class of the paths as discussed in [13]. It will be difficult to do both conformal reparametrization and compactification, and we must also consider whether the corresponding conformal structures converge. Fortunately, the Teichmüller space of \(T^2\) is easy to manipulate, and the singularity arising from the absence of compactness of conformal structures has been given in [7] by Ding, Li and Liu.

2.1 Teichmüller space of torus and the notations

We know that any flat torus \(T^2\) can be viewed as the quotient space of \(\mathbb{C}\) modulated by a lattice generated by bases \(\{\omega_1, \omega_2\}\). After some conformal linear transformation, we can assume \(\omega_1 = 1\), and \(\omega_2 = \tau = \frac{w_2}{w_1}\), where \(\tau\) lies in the upper half plane \(\mathbb{H}\). In fact the Teichmüller spaces of torus \(T_1\), is just the upper half plane \(\mathbb{H}\). We call each element \(\tau \in T_1\) a mark, and denote \(\tau\) by a marked torus \((T^2, \tau)\) as in Definition 2.7.2 of [9], which means a torus by gluing edges of the lattice \(\{1, \tau\}\) with the plane metric \(dzd\tau\).
Denoting \( \tau = \tau_1 + \sqrt{-1}\tau_2 \), we have another normalization such that the area of the corresponding torus \( \text{Area}(\{\omega_1, \omega_2\}) = 1 \), i.e. by letting \( \omega_1 = \frac{1}{\sqrt{T_2}}, \omega_2 = \frac{1}{\sqrt{T_2}} + \sqrt{-1}\sqrt{T_2} \). Let \( T_0^2 \) be the marked torus \( (T^2, \sqrt{-1}) \), then there is a natural diffeomorphism \( i_\tau \) from \( (T^2, \tau) \) to \( (T^2, \sqrt{-1}) \), which is the quotient map of the linear map of \( \mathbb{C} \) keeping 1 and sending \( \tau \) to \( \sqrt{-1} \). So we can also denote \( \tau \in T_1 \) as \( (T^2_\tau, i_\tau) \) as in page 78 of [9]. We will show that every metric on \( T_0^2 \) is conformal to a marked torus \( (T^2, \tau) \), while keeping the conformal homeomorphism in the homotopy class of \( i_\tau^{-1} \).

**Definition 2.1** Let \( \tilde{\Omega} = \left\{ (\gamma(t), \tau(t)) ; \gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}((T^2, \tau(t)), \mathbb{T})) , \tau(t) \in C^0([0, 1], T_1) \right\} \), and \( \Omega = \left\{ (\gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(T_0^2, N)) \right\} \). We assume \( \gamma(0), \gamma(1) \) are constant mapping or map the torus to some circles in \( N \). And \( \tau(0), \tau(1) = \sqrt{-1} \), if mappings on the endpoints are constant mappings, and not restrained if not.

We use varying domains \( (T^2, \tau(t)) \) in the definition of \( \tilde{\Omega} \), and there are two ways to understand this: we can pull back all \( \gamma(t) \) to \( T_0^2 \) by \( i_\tau^{-1} \) and the continuity is defined w.r.t the same domain \( T_0^2 \); we can also consider \( \gamma(t) \) as defined on a large ball of \( \mathbb{C} \) containing all parallelograms generated by \( \{1, \tau(t)\} \), and continuity is defined w.r.t. the plane ball. Since \( \tau(t) \) is continuous, the two definitions are equivalent. Here \( \tilde{\Omega} \) and \( \Omega \) are our variational spaces.

For the area functional, we only need to consider variational problem in the space \( \tilde{\Omega} \), since changing domain metrics will not change the area. But for energy functional, different conformal structures may lead to different energy, so we have to consider variational problem in the space \( \tilde{\Omega} \). Fix a homotopically nontrial path \( \beta \in \Omega \), \( (\beta(t), \tau_0(t)) \in \tilde{\Omega} \). Let \( [\beta] \) be the homotopy class of \( \beta \) in \( \Omega \). Since path \( \gamma(t) \in \tilde{\Omega} \) may have different domains \( T_0^2(\tau(t)) \), the homotopy equivalence \( \alpha \sim \beta \) of \( \alpha(t) : T_0^2(\tau(t)) \to N \) and \( \beta(t) : T_0^2(\tau(t)) \to N \) is defined as follows. We can identify \( T_0^2(\tau(t)) \) and \( T_0^2(\tau(t)) \) to \( T_0^2 \) by \( i_\tau(t) \) and \( i_{\tau(t)} \), then we can view \( \alpha(t) \) and \( \beta(t) \) as mappings defined on the same domain \( T_0^2 \) and hence define their homotopy equivalence.

**Definition 2.2** Let \( \mathcal{W} = \inf_{\rho \in [\beta]} \max_{t \in [0, 1]} \text{Area}(\rho(t)) \). Considering the energy, similarly define \( \mathcal{W}_E = \inf_{(\rho, \tau) \in [\beta, \tau_0]} \max_{t \in [0, 1]} \mathcal{E}(\rho(t), \tau(t)) \).

In fact, we will show that \( \mathcal{W} = \mathcal{W}_E \) in Remark 3.2. What we are interested is the case when \( \mathcal{W} > 0 \). So we assume that \( \mathcal{W} > 0 \) in the following.

1Here \( \tau_0(t) \equiv \sqrt{-1} \).

2The Teichmüller space \( T_1 \) is simply connected, so we do not need to consider the homotopy class of conformal structures, i.e.\( ([\rho, \tau]) \) is the same as \( [\rho] \).
2.2 Sketch of the variational approach

**Question:** Whether one can find a minimal torus or a minimal torus together with several minimal spheres with total area equal \(W\)? Here we will follow the method of Colding and Minicozzi. We want to reduce the variational problem for the area functional to that of the energy functional, i.e. to change a variational problem in \(\Omega\) to one in \(\tilde{\Omega}\). **Firstly** choose a sequence \(\tilde{\gamma}_n(t)\in[\beta]\), such that \(\lim_{n\to\infty}\max_{t\in[0,1]}\text{Area}(\tilde{\gamma}_n(t)) = W\).

By a smoothing argument, we can assume \(\tilde{\gamma}_n(t)\) varies in the \(C^2\) class w.r.t \(t\), i.e. \(\tilde{\gamma}_n(t)\in C^0([0,1],C^2(T_0^2,N))\). Pull back the ambient metric \(\tilde{g}_n(t) = \tilde{\gamma}_n(t)^*h\). We want to show that \(\tilde{g}_n(t)\), which may be degenerate, determine a family of marks \(\tau_n\), such that there exist almost conformal parametrizations \(h_n(t): T^2_{\tau_n(t)} \to T^2_{\tilde{g}_n(t)}\) isotopic to \(i_{\tau_n(t)}\). Hence the reparametrization \((\gamma_n(t),\tau_n(t)) = (\tilde{\gamma}_n(h_n(t),t),\tau_n(t)) \in [(\tilde{\gamma}_n(t),\tau_0)]\) have energy close to area, i.e. \(E(\gamma_n(t),\tau_n(t)) - \text{Area}(\gamma_n(t)) \to 0\). **Next** we want to perturb \(\gamma_n(t)\) to \(\rho_n(t)\) to get bubble compactness. Clearly, we can not globally change the mappings on each path to harmonic or almost harmonic ones like in the Plateau Problem. Local harmonic replacement is a good choice here, and this is just what Colding and Minicozzi did. **Finally** we will study what we will get when the the corresponding marks \(\{\tau_n\} \subset \mathcal{T}_1\) converge or degenerate. If the marks \(\tau_n\) being considered will not degenerate, we will get a good solution to this variational problem. In fact, we will show that \((\rho_n(t),\tau_n(t))\) are almost conformal when their energy are closed to the min-max value \(W_E\).

We will give details of the above approach in the following sections.

3 Conformal parametrization

We will do almost conformal reparametrization for the minimizing sequence of paths \(\tilde{\gamma}_n(t)\), and we can assume that \(\tilde{\gamma}_n(t)\) have some regularity.

**Lemma 3.1** (Lemma D.1 of [2]) Suppose \(\tilde{\gamma}_n(t)\) are chosen as a minimizing sequence of paths as above, we can perturb them to get a new minimizing sequence in the same homotopy class \([\beta]\). If denoting them still as \(\tilde{\gamma}_n(t)\), we have \(\tilde{\gamma}_n(t) \in C^0([0,1],C^2(T_0^2,N))\).

3.1 Uniformization for torus

We need the following uniformization result. For a marked torus \(T^2_\tau\), we have a standard covering map \(\pi_\tau: \mathbb{C} \to T^2_\tau\), which is just the map quotient by the lattices generated by \(\{1, \tau\}\). We denote \(\pi_0 = \pi_{\sqrt{-1}}\).
Proposition 3.1 Let $g$ be a $C^1$ metric on $T^2_0$. We can view $g$ as a metric on the complex plane $\mathbb{C}$, with double periods. Then there is a unique mark $\tau \in \mathcal{T}_1$, and a unique orientation preserving $C^{1,\frac{1}{2}}$ conformal diffeomorphism $h : T^2_\tau \to T^2_g$, such that $h$ is isotopic to $i_\tau$, with normalization that if pulling the map back to $\mathbb{C}$ by $\pi_\tau$ and $\pi_0$, it maps 0 to 0, 1 to 1 and $\tau$ to $\sqrt{-1}$. Furthermore, if $g(t)$ is a family of $C^1$ metrics on $T^2_0$ which varies continuously in the $C^1$ class, i.e. $g(t) \in C^1([0,1], C^1)$ metrics, and $g(t) \geq \epsilon g_0$ for some uniform $\epsilon > 0$, let $\tau(t)$, $h(t)$ be the corresponding marks and normalized conformal diffeomorphisms, then $\tau(t)$ varies continuously in $\mathcal{T}_1$ and $h(t)$ varies continuously in $C^0 \cap W^{1,2}(T^2_\tau(t), T^2_0)$.

Remark 3.1 Here the space $C^0 \cap W^{1,2}(T^2_\tau(t), T^2_0)$ have different domain spaces $T^2_\tau(t)$, and the continuity is defined as the Section 2.

Proof: The existence of a lattice $\{1, \tau\}$ and the conformal homeomorphism $h : T^2_\tau \to T^2_g$ follows from Theorem 3.3.2 of [8] by variational methods.

We firstly give the existence of a conformal homeomorphism satisfying the above normalization. Let $f : T^2_\hat{g} \to T^2_g$ be the inverse mapping of the conformal homeomorphism $h$ given by the variational methods. Pulling back $T^2_\hat{g}$ to $\mathbb{C}$ by $\pi_0$, $g$ can be viewed as double periodic metrics $(g_{ij})$. By Lemma 6.1 we can write $g = \lambda |dz + \mu d\bar{z}|^2$, with $|\mu| \leq k < 1$. Let $\tilde{f}$ be the lifting of $f$ to the covering space $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by $\pi_0$ and $\pi_\tau$. After possibly composing with a conformal diffeomorphism of $T^2$, we can assume $\tilde{f}(1) = 1$. By the uniqueness of $\mu$-conformal homeomorphisms which fix $(0,1,\infty)$ as described in section 6.1, we know that $\tilde{f}$ is just the map $w^\mu$ given by Ahlfors and Bers in [11]. Since $\tilde{f}$ is orientation preserving, $\tilde{f}(\sqrt{-1}) \in \mathbb{H}$. Denoting $\tau' = \tilde{f}(\sqrt{-1})$, since $f_\#$ is homeomorphism between $\pi_1(T^2_\hat{g})$ and $\pi_1(T^2_\tau)$, we know $\{1, \tau'\}$ is another generator of the lattice generalized by $\{1, \tau\}$. After pulling down $\tilde{f}$ by $\pi_0$ and $\pi_{\tau'}$, we get $f'$. In fact $f'$ differs from $f$ by an automorphism $\pi_{\tau'} \circ \pi_0^{-1}$ of $T^2_0$. $f'$ maps $T^2_\hat{g}$ conformally and homeomorphicly to $T^2_{\tau'}$. Since $\tilde{f}$ maps 1 to 1 and $\sqrt{-1}$ to $\tau'$, we know that $f'$ is homotopic to $i_{\tau'}^{-1}$ by Lemma 2.7.1 of [9]. So $f'$ and $\tau'$ are our unique conformal homeomorphism and mark, and we will denote them by $f$ and $\tau$. Let $h = f^{-1} : T^2_\tau \to T^2_g$ be our unique conformal homeomorphism, then $h$ is isotopic to $i_\tau$.

The uniqueness under the above normalization and the continuous dependence of the conformal homeomorphisms and the marks on the variance of the metric follow from Appendix 6. For a family of metrics $g(t)$, $g(t) = \lambda(z)|dz + \mu(t) d\bar{z}|^2$, with $|\mu(t)| \leq k(\epsilon) < 1$. Here $\mu(t)$ are double periodic functions on $\mathbb{C}$ with periods generalized by $\{1, \tau_0\}$, and $\mu(t) = \mu_\tau$ change continuously in the $C^1$ class w.r.t $t$ by Lemma 6.1 and the following Remark 6.1. Let $f(t)$ be the inverse of $h(t)$, with $\tilde{f}(t)$ and $\tilde{h}(t)$ being pulled back by $\pi_0$ and $\pi_{\tau(t)}$. Hence $\tilde{f}(t) = w^{\mu_t}$ are just the maps given by Ahlfors and Bers described in
Appendix [6]

We will show that $\tau(t)$ vary continuously w.r.t $t$. We know that $\tau(t) = w^\mu t(\sqrt{-1})$, and then $w^\mu t(\sqrt{-1}) \rightarrow w^\mu t_0(\sqrt{-1})$ as $t \rightarrow t_0$. This is because we have convergence under sphere distance in Lemma 6.2, i.e. $d_\mathbb{Z}(w^\mu t(\sqrt{-1}), w^\mu t_0(\sqrt{-1})) \rightarrow 0$. And we know from the variational methods that $w^\mu t_0(\sqrt{-1}) = \tau_{t_0}$ is away from $\infty$, so all $\tau_t = w^\mu t(\sqrt{-1})$ are away from $\infty$. Since the sphere distance is equivalent to plane distance of $\mathbb{C}$, we know $|w^\mu t(\sqrt{-1}) - w^\mu t_0(\sqrt{-1})| \rightarrow 0$, i.e. $\tau(t) \rightarrow \tau(t_0)$ in $T_1$.

We will give the continuous dependence of $h_t = f_t^{-1}$ on $t$. The lifting are $\mu(t)$-conformal $\tilde{\tau}_t : \mathbb{C}_{dw \mu t} \rightarrow \mathbb{C}_{[dz + \mu(t)dr]^2}$. Here, we only need to consider $\tilde{\tau}_t$ as mappings defined on a large ball $B_R$, which contains all the parallelograms of $\{1, \tau_t\}$. This is because $\tau_t$ vary continuously, so they will lie on a large ball $B_R$ for all $t \in [0, 1]$. Here $\tilde{\tau}(t)$ are the conformal homeomorphism solutions of Lemma 6.2. We know the convergence under sphere distance in Lemma 6.2, i.e. $d_\mathbb{Z}$.

We will give the continuous dependence of $h_t = f_t^{-1}$ on $t$. The lifting are $\mu(t)$-conformal $\tilde{\tau}_t : \mathbb{C}_{dw \mu t} \rightarrow \mathbb{C}_{[dz + \mu(t)dr]^2}$. Here, we only need to consider $\tilde{\tau}_t$ as mappings defined on a large ball $B_R$, which contains all the parallelograms of $\{1, \tau_t\}$. This is because $\tau_t$ vary continuously, so they will lie on a large ball $B_R$ for all $t \in [0, 1]$. Here $\tilde{\tau}(t)$ are the conformal homeomorphism solutions of Lemma 6.2. We know the convergence under sphere distance, i.e. equation 64. The image $\tilde{\tau}(t)(B_R)$ are restrained to a neighborhood of $[0, 1] \times [0, 1]$, since $\tilde{\tau}(t)$ have uniform Hölder continuity and map parallelograms $\{1, \tau_t\}$ homeomorphically to $T_0^2$. So $\|\tilde{\tau}_t - \tilde{\tau}_{t_0}\|_{L^\infty(B_R)} \rightarrow 0$, as $t \rightarrow t_0$, and hence:

$$\|\tilde{\tau}_t - \tilde{\tau}_{t_0}\|_{C^0(T_0^2, T_0^2)} \rightarrow 0. \quad (3)$$

From the second convergence in Lemma 6.3, we know $\|((\tilde{\tau}_t - \tilde{\tau}_{t_0})_w\|_{L^\infty(B_R)} \rightarrow 0$, as $t \rightarrow t_0$, so $\|((\tilde{\tau}_t - \tilde{\tau}_{t_0})_w\|_{L^p(T_0^2, T_0^2)} \rightarrow 0$, and hence:

$$\|\tilde{\tau}_t - \tilde{\tau}_{t_0}\|_{W^{1, 2}(T_0^2, T_0^2)} \rightarrow 0. \quad (4) \square$$

### 3.2 Construction of the conformal reparametrization

As above, we consider $\tilde{\gamma}_n(t) = \tilde{\gamma}_n(t) h$, which vary continuously in the $C^1$ class. Since there may be degenerations, we let $\gamma_n(t) = \tilde{\gamma}_n(t) + \delta_n g_0$, where $g_0$ is the standard metric of $T_0^2$, and $\delta_n$ arbitrarily small. The corresponding marks in $T_1$ and conformal diffeomorphisms are $\tau_n(t)$ and $h_n(t)$ given by Proposition 3.1. We have the following result.

**Theorem 3.1** Using the above notion, we have reparametrizations $(\gamma_n(t), \tau_n(t)) \in \hat{\Omega}$ for $\tilde{\gamma}_n(t)$, i.e. $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t)$, such that $\gamma_n(t) \in [\tilde{\gamma}_n]$. And

$$E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \rightarrow 0,$$

as $\delta_n \rightarrow 0$. 

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Proof: We know that $h_n(t) : T^2_{\tau_n(t)} \to T^2_{g_n(t)}$ are conformal diffeomorphisms. Let $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t) : T^2_{\tau_n(t)} \to N$ be the composition of our test path with the almost conformal parametrization, we know $\gamma_n(t) \in \Omega$. The continuity of $t \to \gamma_n(t)$ from $[0, 1]$ to $C^0 \cap W^{1,2}(T^2_{\tau_n(t)}, N)$ follows from the continuity of $t \to \tilde{\gamma}_n(t)$ in $C^2$ by Lemma 3.1 and $t \to h_n(t)$ in $C^0 \cap W^{1,2}$ by Proposition 3.1. We will show that $\gamma_n(t) \in [\tilde{\gamma}_n]$. From our discussion of homotopy equivalence of mappings defined on different domains in Section 2, we view $\gamma_n(t)$ as mappings defined on $T^2_0$ by composing with $i^{-1}_{\tau_n(t)} : T^2_0 \to T^2_{\tau_n(t)}$ and compare it to $\tilde{\gamma}_n(t)$. Since $h_n$ are homotopic equivalent to $i_{\tau_n(t)}$ by Proposition 3.1, $h_n(t) \circ i^{-1}_{\tau_n(t)}$ is homotopic equivalent to identity map of $T^2_0$. While $\gamma_n$ are composition of $\tilde{\gamma}_n$ with $h_n(t)$, $\gamma_n \circ i^{-1}_{\tau_n}$ is homotopic equivalent to $\tilde{\gamma}_n$, hence $\gamma_n \sim \tilde{\gamma}_n$.

We can get estimates as in Appendix D of [2]:

$$E(\gamma_n(t), \tau_n(t)) = E(h_n(t) : T^2_{\tau_n(t)} \to T^2_{g_n(t)}) \leq E(h_n(t) : T^2_{\tau_n(t)} \to T^2_{g_n(t)})$$

$$= \text{Area}(h_n(t) : T^2_{\tau_n(t)} \to T^2_{g_n(t)})$$

$$= \text{Area}(T^2_{g_n(t)}) = \int_{T^2_0} [\text{det}(g_n(t))]^{\frac{1}{2}} dvol_0$$

$$= \int_{T^2_0} [\text{det}(\tilde{g}_n(t))] + \delta_n Tr_{\tilde{g}_n} \tilde{g}_n(t) + C(\tilde{g}_n(t))\delta_n^{\frac{1}{2}} dvol_0$$

$$\leq \text{Area}(T^2_{g_n(t)}) + C(\tilde{g}_n(t))\sqrt{\delta_n}$$

$$= \text{Area}(\gamma_n(t) : T^2_0 \to N) + C(\tilde{\gamma}_n)(\sqrt{\delta_n}).$$

The first and last equality follow from the definition of energy and area integral, and the second inequality is due to the fact $\tilde{g}_n(t) \leq g_n(t)$. Hence we have equation 5 as $\delta_n \to 0$.

Remark 3.2 We point out that the above Lemma implies that $W = W_E$. Since we always have that $\text{Area}(u) \leq E(u, \tau)$, we get $W \leq W_E$. We will be done if we know $W_E \leq W$. By definition $W_E \leq \max_{t \in [0, 1]} E(\gamma_n(t), \tau_n(t))$. Since $W = \lim_{n \to \infty} \max_{t \in [0, 1]} \text{Area}(\gamma_n(t))$, we have $W_E \leq \lim_{n \to \infty} \max_{t \in [0, 1]} \text{Area}(\gamma_n(t)) = W$.

Now we have reduced the problem in $\Omega$ to that in $\tilde{\Omega}$ as we discussed above, and it is now easy to deal with energy $E$ by analytical methods.

4 Compactification for mappings

In this case, we can view $\gamma_n(t)$ as double periodic mappings on $C$, with periods generated by lattices $\{1, \tau_n(t)\}$. So all the mappings have the same domain, but with different periods, with periods varying continuously. We can do similar perturbation procedure as what Colding and Minicozzi did in the case of sphere in [2].
Lemma 4.1 Let $[\beta]$ and $\mathcal{W}_E$ be as in section 2. For any $(\gamma(t), \tau(t)) \in [\beta] \subset \tilde{\Omega}$ with \[ \max_{t \in [0,1]} E(\gamma(t), \tau(t)) - \mathcal{W}_E \ll 1, \] if $(\gamma(t), \tau(t))$ is not harmonic unless $\gamma(t)$ is a constant map, we can perturb $\gamma(t)$ to $\rho(t)$, such that $\rho(t) \in [\gamma]$ and $E(\rho(t), \tau(t)) \leq E(\gamma(t), \tau(t))$, and for any $t$ such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2} \mathcal{W}_E$, $\rho(t)$ satisfies:

\[ (* ) \text{ For any finite collection of disjoint balls } \bigcup_i B_i \text{ on } T^2_{\tau(t)}, \text{ which can also be viewed as disjoint balls on the parallelogram generated by } \{1, \tau(t)\} \subset \mathbb{C}, \text{ such that } E(\rho(t), \bigcup_i B_i) \leq \epsilon_0, \] if we let $v$ be the energy minimizing harmonic map with the same boundary value as $\rho(t)$ on $\frac{1}{2} \bigcup_i B_i$, then we have:

\[ \int_{\frac{1}{2} \bigcup_i B_i} |\nabla \rho(t) - \nabla v|^2 \leq \Psi \left( E(\gamma(t), \tau(t)) - E(\rho(t), \tau(t)) \right). \quad (7) \]

Here $\epsilon_0$ is some small constant, and $\Psi$ is a positive continuous function with $\Psi(0) = 0$.

Remark 4.1 In the paper [2] of Colding and Minicozzi, all the results about harmonic maps on disks are still valid here. The other two most important ingredients are continuity of local maps and comparison of energy of local harmonic replacements. For the first one, since all the balls $\bigcup_i B_i$ can be viewed as balls on $\mathbb{C}$, and $\gamma(t)$ are continuous as mappings on $\mathbb{C}$, so continuity of $\gamma(t)$ restricted to local balls is valid. The comparison results are just for a fixed mapping $\gamma(t)$, and when $t$ is fixed, all the comparison results can be viewed as on the plane, so we can show that they are still valid here.

We will give the proof by combining results in the following sections by following the proof of Theorem 2.1 of [2]. To do such compactification, we use repeated local harmonic replacements, which means that we replace the map $u$ on a ball $B$ by the energy-minimizing map $H(u)$ with the same boundary value as $u$.

4.1 Harmonic replacement on disks

In this section, we will list some results about harmonic replacement on disks with small energy as given in Section 3 of [2]. Firstly we recall that for small energy harmonic map, energy gap can control the difference of $W^{1,2}$-norm. Here $B_1 \in \mathbb{R}^2$ is the unit disk, and $N$ is the ambient manifold.

Theorem 4.1 (Theorem 3.1 of [2]) There exists a small constant $\epsilon_1$ (depending on $N$) such that for all maps $u, v \in W^{1,2}(B_1, N)$, if $v$ is weakly harmonic with the same boundary value as $u$, and $v$ has energy less than $\epsilon_1$, then we have:

\[ \int_{B_1} |\nabla u|^2 - \int_{B_1} |\nabla v|^2 \geq \frac{1}{2} \int_{B_1} |\nabla u - \nabla v|^2. \quad (8) \]
Remark 4.2 This theorem tells us that for small energy harmonic map, we can use the gap of energy to control the difference of $W^{1,2}$ norm. Hence we will focus on the energy gaps when we do harmonic replacement. It also implies the uniqueness of small energy weakly harmonic map among maps with the same boundary values (Corollary 3.3 of \[2\]).

Using this theorem and boundary regularity of harmonic maps (i.e. \[10\]), we have the following continuity property of harmonic replacements.

Corollary 4.1 (Corollary 3.4 of \[2\]) Let $\epsilon_1$ be as in the previous theorem. Suppose $u \in C^0(\overline{B}_1) \cap W^{1,2}(B_1)$ with energy $E(u) \leq \epsilon_1$, then there exists a unique energy minimizing harmonic map $v \in C^0(\overline{B}_1) \cap W^{1,2}(B_1)$ with the same boundary value as $u$. Set $\mathcal{M} = \{u \in C^0(\overline{B}_1) \cap W^{1,2}(B_1), E(u) \leq \epsilon_1\}$. \(\exists C\) (depending on $N$), \(\forall u_1, u_2 \in \mathcal{M}\), let $w_1, w_2$ be the corresponding energy minimizing maps, and let $E = E(u_1) + E(u_2)$, then we have:

\[
|E(w_1) - E(w_2)| \leq C\|u_1 - u_2\|_{C^0(\overline{B}_1)} E + C\|\nabla u_1 - \nabla u_2\|_{L^2(B_1)} E^{\frac{1}{2}}. \tag{9}
\]

If we denote $v$ by $H(u)$, the mapping $H: \mathcal{M} \to \mathcal{M}$ is continuous w.r.t the norm on $C^0(\overline{B}_1) \cap W^{1,2}(B_1)$. Here the norm is the sum of $C^0(\overline{B}_1)$-norm and $W^{1,2}(B_1)$-norm.

We will need the following extension of the above result:

Corollary 4.2 Suppose $u_i, u$ are defined on a ball $B_{1+\epsilon}$ with energy less than $\epsilon_1$. Suppose $u_i \to u$ in $C^0(\overline{B}_{1+\epsilon}) \cap W^{1,2}(B_{1+\epsilon})$. Choose a sequence $r_i \to 1$, and let $w_i, w$ be the mappings which coincide with $u_i, u$ outside $r_i B_1$ and $B_1$ and are energy minimizing inside $r_i B_1$ and $B_1$ respectively. We have $w_i \to w$ in $C^0(\overline{B}_{1+\epsilon}) \cap W^{1,2}(B_{1+\epsilon})$.

Proof: Firstly we show the following claim:

Claim: Let $\tilde{w}_i$ be the energy minimizing map with the same boundary value as $u$ on $r_i B_1$, then we have: $\tilde{w}_i \to w$ in $C^0(\overline{B}_{1+\epsilon}) \cap W^{1,2}(B_{1+\epsilon})$.

Since $E(u, B_{1+\epsilon}) \leq \epsilon_1 < \epsilon_{SU}$, with $\epsilon_{SU}$ the constant given in \[11\], we know that $\tilde{w}_i$ have uniform inner $C^{2,\alpha}$ bounds on $B_1$, so $\forall r < 1$, $\tilde{w}_i \to w'$ in $C^{2,\alpha}(B_r)$, and $w'$ is a harmonic map on $B_1$. By scaling argument, we can show that there are no energy concentration near the boundary of $B_1$. So $\tilde{w}_i \to w'$ in $W^{1,2}(B_{1+\epsilon})$. We also know from \[10\], as indicated by the proof of Corollary 3.4 of \[2\] that $\tilde{w}_i$ are equi-continuous near $\partial(r_i B_1)$ and hence equi-continuous near $\partial B_1$ since $r_i \to 1$. So $\tilde{w}_i \to w'$ in $C^0(\overline{B}_{1+\epsilon})$. By the uniqueness of small energy harmonic map of Corollary 3.3 of \[2\], we know $w' = w$. So the claim holds.

Let $v_i = \Pi(\tilde{w}_i + u_i - u)$ which have the same boundary value as $u_i$ and $w_i$ on $\partial(r_i B_1)$. Here $\Pi: N_\delta \to N$ is the nearest point projection defined on a tubular neighborhood $N_\delta$. When \(\delta\) is small enough, we have $|d\Pi| \leq 2$. So $\|v_i - \tilde{w}_i\|_{W^{1,2}(B_{1+\epsilon})} \to 0$, hence...
\[ \|v_i - w\|_{W^{1,2}(B_{1+\epsilon})} \to 0 \] by our Claim. By Corollary 4.1, \( |E(w_i) - E(\bar{w}_i)| \to 0 \), hence \( |E(w_i) - E(v_i)| \to 0 \). By Theorem 4.1, \( \|w_i - v_i\|_{W^{1,2}(\tilde{r}, B_1)} \to 0 \). So:

\[
\int_{B_{1+\epsilon}} |\nabla w_i - \nabla w|^2 = \int_{r, B_1} |\nabla w_i - \nabla w|^2 + \int_{B_{1+\epsilon}\setminus r, B_1} |\nabla w_i - \nabla w|^2 \to 0. \tag{10}
\]

The convergence to 0 of the second part of the last term in the above is due to \( u_i \to u \) and \( w = u \) outside \( B_1 \). Hence \( w_i \to w \) in \( W^{1,2}(B_{1+\epsilon}) \).

To show the \( C^0(B_{1+\epsilon}) \) convergence, we know from similar argument as in the proof of the claim, that \( w_i \) are equi-continuous near \( \partial B_1 \) by the equi-continuity of \( u_i \). Recall that \( \ref{corollary:equi} \) gives uniform inner \( C^{2,\alpha} \) for \( w_i \) on \( B_1 \). We have that every subsequence of \( w_i \) must have \( w_i \to w \) in \( C^0(\bar{B}_{1+\epsilon}) \) possibly after taking a further subsequence. So we get \( C^0(\bar{B}_{1+\epsilon}) \) continuity.

\[ \square \]

\textbf{Remark 4.3} Since we always work on path of mappings, and we will do harmonic replacement on balls with continuously varying radii, this result tells us that harmonic replacements will give us another continuous path if we do harmonic replacement continuously on the initial path. We can continuously shrink the radii of the disks on which we do harmonic replacement to 0, so the new path given by harmonic replacement can be continuously deformed to the original one, i.e. they lie in the same homotopy class.

\section*{4.2 A comparison result for repeated harmonic replacement}

In this section, we will extend the comparison result of local harmonic replacements given in Lemma 3.11 of \cite{2} to the case of torus. We will use \( B \) to denote a finite collection of disjoint balls on the complex plane \( \mathbb{C} \). If \( \mu \in [0, 1] \), we denote \( \mu B \) by a finite collection of balls with the same centers as \( B \), but the radii \( \mu \) timing those of \( B \). If \( u \) is a \( C^0 \cap W^{1,2} \) mapping on the complex plane with small energy on a collection \( B \), let \( H(u, B) \) be the mapping which coincides with \( u \) outside \( B \), and is the energy minimizing inside \( B \). If \( B_1, B_2 \) are two such collections, we denote \( H(u, B_1, B_2) \) to be \( H(H(u, B_1), B_2) \). We will give the relationship between the energy gaps of \( u, H(u, B_1) \) and \( H(u, B_1, B_2) \).

\textbf{Lemma 4.2} Fix a torus \( T_{\tau}^2 \) with mark \( \tau \in \mathcal{T}_1 \), and \( u \in C^0 \cap W^{1,2}(T_{\tau}^2, N) \). Let \( B_1, B_2 \) be two finite collection of disjoint balls on \( T_{\tau}^2 \), which can also be viewed as collections of disjoint balls on \( \mathbb{C} \). If \( E(u, B_i) \leq \frac{1}{3}\epsilon_1 \), with \( \epsilon_1 \) as in Theorem 4.1 for \( i = 1, 2 \), then there exists a constant \( k \) depending on \( N \), such that:

\[
E(u) - E[H(u, B_1, B_2)] \geq k \left( E(u) - E[H(u, \frac{1}{2}B_2)] \right)^2, \tag{11}
\]
and for any $\mu \in [\frac{1}{2}, \frac{1}{2}]$,
\[
\frac{1}{k}(E(u) - E[H(u, B_1)]) + E(u) - E[H(u, 2\mu B_2)] \geq E[H(u, B_1)] - E[H(u, B_1, \mu B_2)]. \tag{12}
\]

**Remark 4.4** We know from the energy minimizing property of small energy harmonic maps that the following estimates hold:
\[
E(u) - E[H(u, B_1, B_2)] \geq E(u) - E[H(u, \frac{1}{2} B_1)]. \tag{13}
\]
So the above three inequalities tell us the relationship of energy improvement between any two successive harmonic replacements.

We will give the proof by constructing comparison mappings. We will use the following Lemma in our construction. Let $B_R$ be the ball of radius $R$ and center 0 in $\mathbb{C}$, and $N$ the ambient manifold.

**Lemma 4.3** (Lemma 3.14 of [2]) There exists a $\delta$ and a large constant $C$ depending on $N$, such that for any $f, g \in C^0 \cap W^{1,2}(\partial B_R, N)$, if $f, g$ are equal at some point on $\partial B_R$, and:
\[
R \int_{\partial B_R} |f' - g'|^2 \leq \delta^2, \tag{14}
\]
we can find some $\rho \in (0, \frac{1}{2} R]$, and a mapping $w \in C^0 \cap W^{1,2}(B_R \setminus B_{R-\rho}, N)$ with $w|_{B_R} = f$, $w|_{B_{R-\rho}} = g$, which satisfies estimates:
\[
\int_{B_R \setminus B_{R-\rho}} |\nabla w|^2 \leq C \left( R \int_{\partial B_R} |f'|^2 + |g'|^2 \right)^{\frac{3}{2}} \left( R \int_{\partial B_R} |f' - g'|^2 \right)^{\frac{1}{2}}. \tag{15}
\]

**Remark 4.5** The condition and result of this Lemma are all scaling invariant, so we can apply it to balls of any radius $R$.

**Proof:** (of Lemma 4.2) We know that both $u$ and $H(u, B_1)$ have energy less than $\frac{2}{4} \epsilon_2$ on $B_1 \cup B_2$, so Theorem 4.1 shows that energy gaps can control $W^{1,2}$-norm gaps in this case. Denote balls in $B_1$ by $B_1^\alpha$, and balls in $B_2$ by $B_2^\beta$.

**Step 1** (inequality 11): Since if the second harmonic replacements are done on balls which are disjoint with the balls of the first step, the comparison is easy. So we divide the second class of balls into two disjoint subcollections $\mathcal{B}_2 = \mathcal{B}_2^+ \cup \mathcal{B}_2^-$, where $\mathcal{B}_2^+ = \{B_2^\beta : \frac{1}{2} B_2^\beta \subset B_1^\alpha \text{ or } \frac{1}{2} B_2^\beta \cap B_1 = \emptyset\}$ for some $B_1^\alpha \in \mathcal{B}_1$. We know that:
\[
E(u) - E[H(u, \frac{1}{2} B_2)] = E(u) - E[H(u, \frac{1}{2} B_2^+)] + E(u) - E[H(u, \frac{1}{2} B_2^-)]. \tag{16}
\]
We will deal with $\mathcal{B}_2^+$ and $\mathcal{B}_2^-$ separately.
For $B_{2+}$, we have:

$$
E(u) - E[H(u, \frac{1}{2}B_{2+})] = \sum_{\{\frac{1}{2}B_i^2 \cap B_1 = \emptyset\}} (E(u) - E[H(u, \frac{1}{2}B_i^2)])
+ \sum_{\{\frac{1}{2}B_i^2 \subset B_1\}} (E(u) - E[H(u, \frac{1}{2}B_i^2)]).
$$

(17)

For balls $\frac{1}{2}B_i^2 \cap B_1 = \emptyset$, we get from the minimizing property of small energy harmonic maps that:

$$
E(u) - E[H(u, \frac{1}{2}B_i^2)] = E[H(u, B_1)] - E[H(u, B_i^2) \subset \frac{1}{2}B_i^2, B_1 = \emptyset] \leq E[H(u, B_1)] - E[H(u, B_i^2, B_2)].
$$

(18)

So, we have:

$$
\sum_{\{\frac{1}{2}B_i^2 \cap B_1 = \emptyset\}} (E(u) - E[H(u, \frac{1}{2}B_i^2)])
\leq \sum_{\{\frac{1}{2}B_i^2 \cap B_1 = \emptyset\}} E[H(u, B_1)] - E[H(u, B_i^2, B_2)]
\leq E[H(u, B_1)] - E[H(u, B_i^2, B_2) \cap B_1 = \emptyset] \leq E(u) - E[H(u, B_1, B_2+)].
$$

(19)

For balls $\frac{1}{2}B_i^2 \subset B_{2+}^1$, we have $H(u, B_1, \frac{1}{2}B_i^2) = H(u, B_1)$, so

$$
\int_{B_i^2} |\nabla H(u, B_1, B_i^2)|^2 \leq \int_{B_i^2} |\nabla H(u, B_1, \frac{1}{2}B_i^2)|^2 = \int_{B_i^2} |\nabla H(u, B_1)|^2
\leq \int_{B_i^2} |\nabla H(u, \frac{1}{2}B_i^2)|^2.
$$

(20)

Hence:

$$
\int_{B_i^2} |\nabla u|^2 - \int_{B_i^2} |\nabla H(u, \frac{1}{2}B_i^2)|^2 \leq \int_{B_i^2} |\nabla u|^2 - \int_{B_i^2} |\nabla H(u, B_1, B_i^2)|^2.
$$

(21)

Summarizing all the results of this case, we have,

$$
\int_{\cup_{B_i^2 \subset B_1^1} B_i^2} |\nabla u|^2 - |\nabla H(u, \frac{1}{2}B_i^2)|^2 \leq \int_{\cup_{B_i^2 \subset B_1^1} B_i^2} |\nabla u|^2 - |\nabla H(u, B_1, B_i^2)|^2
\leq \int |\nabla u|^2 - |\nabla u_1|^2 + \int_{\cup_{B_i^2 \subset B_1^1} B_i^2} |\nabla u_1|^2 - |\nabla H(u, B_1, B_i^2)|^2.
$$

(22)
For the first term, by Theorem 4.1, we have $\int |\nabla u|^2 - |\nabla u_1|^2 \leq 4 \left( E(u) - E(u_1) \right)$. For the second term, we have $E(u_1) - E[H(u,B_1, \cup B_j^2)] \leq E(u) - E[H(u,B_1, \cup B_j^2)]$. Combining them together,

$$E(u) - E[H(u,\frac{1}{2}B_2^+)] \leq C \left( E(u) - E[H(u,B_1,B_2^+)] \right). \quad (23)$$

For the collection $B_{2-}$, we should consider balls separately. Specify a ball $B_j^2$ such that $B_j^2 \cap B_1^a \neq \emptyset$ for some $B_1^a \in B_1$. Denote $B_j^2$ by $B_R$, and $u_1 = H(u,B_1)$. We will compare $E[H(u,\frac{1}{2}B_R)]$ with $E[H(u_1,B_R)]$. Using simple measure theory or the Courant-Lebesgue Lemma (Lemma 3.1.1 of [8]), we can find a subset of $[\frac{3}{4}R, R]$ with measure $\frac{1}{3}R$, such that for any $r$ in this subset, we have:

$$\int_{\partial B_r} |\nabla u_1 - \nabla u|^2 \leq \frac{9}{R} \int_{\frac{1}{4}R}^{R} \int_{\partial B_s} |\nabla u_1 - \nabla u|^2 \leq \frac{9}{r} \int_{B_R} |\nabla u_1 - \nabla u|^2, \quad (24)$$

$$\int_{\partial B_r} |\nabla u_1|^2 + |\nabla u|^2 \leq \frac{9}{R} \int_{\frac{1}{4}R}^{R} \int_{\partial B_s} |\nabla u_1|^2 + |\nabla u|^2 \leq \frac{9}{r} \int_{B_R} |\nabla u_1|^2 + |\nabla u|^2. \quad (25)$$

By choosing $\epsilon_1$ small enough, we can get $r \int_{\partial B_r} |\nabla u_1|^2 + |\nabla u|^2 \leq \delta^2$ and $r \int_{\partial B_r} |\nabla u_1 - \nabla u|^2 \leq \delta^2$ with $\delta$ as in the above Lemma 4.3. Since $\frac{1}{2}B_R \cap B_1^a \neq \emptyset$, but $\frac{1}{2}B_R \not\subseteq B_1^a$, $u$ and $u_1$ must be equal at some point on $\partial B_r$. So from Lemma 4.3, we can find a $\rho \in (0, \frac{1}{2}r]$ and a mapping $w \in C^0 \cap W^{1,2}(B_r \setminus B_{r-\rho})$ with $w|_{\partial B_r} = u_1$, $w|_{\partial B_{r-\rho}} = u$, and:

$$\int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 \leq C (r \int_{\partial B_r} |\nabla u_1 - \nabla u|^2)^{\frac{1}{2}} (r \int_{\partial B_r} |\nabla u_1|^2 + |\nabla u|^2)^{\frac{1}{2}}$$

$$\leq C \left( \int_{B_R} |\nabla u_1 - \nabla u|^2 \right)^{\frac{1}{2}} (\int_{B_R} |\nabla u_1|^2 + |\nabla u|^2)^{\frac{1}{2}}. \quad (26)$$

Define a comparison map $v$ on $B_R$ such that:

$$v = \begin{cases} 
  u_1 & \text{on } B_R \setminus B_r \\
  w & \text{on } B_r \setminus B_{r-\rho} \\
  H(u,B_r)(\frac{r}{r-\rho} \cdot) & \text{on } B_{r-\rho}.
\end{cases}$$

We know $E[H(u_1,B_R)] \leq E(v)$ since $H(u_1,B_R)$ is energy minimizing among all maps with the same boundary value on $B_R$. So we have:

$$\int_{B_R} |\nabla H(u_1,B_R)|^2 \leq \int_{B_R} |\nabla v|^2$$

$$= \int_{B_R \setminus B_r} |\nabla u_1|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_{r-\rho}} |\nabla H(u,B_r)(\frac{r}{r-\rho} \cdot)|^2$$

$$= \int_{B_R \setminus B_r} |\nabla u_1|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_r} |\nabla H(u,B_r)|^2. \quad (27)$$
The second equation is due to conformal invariance of the Dirichlet integral. Hence

\[
\int_{B_R} |\nabla u|^2 - \int_{B_R} |\nabla H(u, \frac{1}{2} B_R)|^2 \leq \int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla H(u, B_r)|^2 \\
\leq \int_{B_r} |\nabla u|^2 - \int_{B_R} |\nabla H(u_1, B_R)|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_R \setminus B_r} |\nabla u_1|^2 \\
\leq \int_{B_R} |\nabla u_1|^2 - \int_{B_R} |\nabla H(u_1, B_R)|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_R \setminus B_r} |\nabla u_1|^2
\]

(28)

By argument similar to the above, we know \( \int |\nabla u|^2 - |\nabla u_1|^2 \leq 4 \left( E(u) - E(u_1) \right) \). Put the estimates 26 into the above inequality, and sum over \( B_j^2 \in B_{2-} \):

\[
E(u) - E[H(u, \frac{1}{2} B_{2-}^2)] \leq E(u_1) - E[H(u_1, B_{2-}^2)] \\
+ C \left( E(u) - E(u_1) \right)^\frac{1}{2} + E(u) - E(u_1) \\
= E(u) - E[H(u_1, B_{2-}^2)] + C \left( E(u) - E(u_1) \right)^\frac{1}{2} \\
\leq E(u) - E[H(u, B_1, B_2)] + C \left( E(u) - E[H(u, B_1, B_2)] \right)^\frac{1}{2}.
\]

(29)

Using the fact that all the maps have energy less than \( \frac{1}{2} \varepsilon_1 \), we have:

\[
E(u) - E[H(u, \frac{1}{2} B_{2-}^2)] \leq C' \left( E(u) - E[H(u, B_1, B_2)] \right)^\frac{1}{2}.
\]

(30)

Combining results on \( B_{2+} \) and \( B_{2-} \), we have:

\[
E(u) - E[H(u, \frac{1}{2} B_{2-}^2)] \leq C \left( E(u) - E[H(u, B_1, B_2)] \right)^\frac{1}{2},
\]

(31)

i.e. the first inequality 11.

**Step 2** (inequality 12): In this step, we also divide \( B_2 \) into two classes with \( B_{2+} = \{ B_j^2 : \mu B_j^2 \subset B_1^1 \text{ or } \mu B_j^2 \cap B_1 = \emptyset \} \). For \( \mu B_j^2 \subset B_1^1 \), we have \( H(u, B_1) = H(u, B_1, \mu B_j^2) \), so we need not to consider such ball. For \( \mu B_j^2 \cap B_1 = \emptyset \), we have:

\[
E[H(u, B_1)] - E[H(u, B_1, \mu B_j^2)] = E(u) - E[H(u, \mu B_j^2)] \leq E(u) - E[H(u, 2 \mu B_j^2)].
\]

(32)

So summing all the balls in \( B_{2+} \), we have:

\[
E[H(u, B_1)] - E[H(u, B_1, \mu B_{2+}^2)] \leq E(u) - E[H(u, 2 \mu B_{2+}^2)].
\]

(33)
For the class $B_{2-}$, we use similar method as above. The difference are that $B_R = 2\mu B_j^2$, and in the definition of $v$, the role of $u$, $u_1$ changed:

$$v = \begin{cases} 
  u & \text{on } B_R \setminus B_r \\
  w & \text{on } B_r \setminus B_{r-\rho} \\
  H(u_1, B_r)(\frac{r}{r-\rho} x) & \text{on } B_{r-\rho}
\end{cases}$$

So we have:

$$\int_{B_R} |\nabla H(u, B_R)|^2 \leq \int_{B_R \setminus B_r} |\nabla u|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_r} |\nabla H(u_1, B_r)|^2. \tag{34}$$

And

$$\int_{B_R} |\nabla u_1|^2 - \int_{\frac{1}{2}B_R} |\nabla H(u_1, B_R)|^2 \leq \int_{B_r} |\nabla u_1|^2 - \int_{B_r} |\nabla H(u_1, B_R)|^2$$

$$\leq \int_{B_r} |\nabla u_1|^2 - \int_{B_r} |\nabla H(u, B_R)|^2 + \int_{B_r \setminus B_{r-\rho}} |\nabla w|^2 + \int_{B_r} |\nabla u_1|^2 - |\nabla u|^2. \tag{35}$$

Here we use our argument $\int |\nabla u|^2 - |\nabla u_1|^2 \leq 4 \left( E(u) - E(u_1) \right)$ again. Use estimates again observing that $u, u_1$ have local energy less than $\frac{1}{2}e_1$, and sum over $B_j^2 \in B_{2-}$:

$$E(u_1) - E[H(u_1, \mu B_{2-}^2)] \leq E(u) - E[H(u, 2\mu B_{2-}^2)] + C \left( E(u) - E(u_1) \right)^\frac{1}{2}. \tag{36}$$

Combining results on $B_{2+}$ and $B_{2-}$, we will get inequality $\text{[12]}$.

\[\square\]

### 4.3 Construction of the perturbation

To construct a perturbation satisfying condition $(\ast)$ in Lemma [4.1] we can reduce to control the energy gaps instead of $W^{1,2}$-norm. Since we only focus on balls with small energy, there must be a maximal possible energy decrease for a fixed map on certain such balls. If we firstly do harmonic replacement on such balls, we can then control the energy decrease for harmonic replacement on other small energy balls by the comparison Lemma [4.2]. For a path $(\sigma(t), \tau(t)) \in \tilde{\Omega}, \epsilon \in (0, \epsilon_1]$, define: $e_{\epsilon, \sigma(t)} = \sup_{\mathcal{B}} \{ E(\sigma(t), \tau(t)) - E[H(\sigma(t), \frac{1}{2}\mathcal{B}), \tau(t)] \}$. Here $\mathcal{B}$ are chosen as any finite collection of disjoint balls on $T^2_{\tau_1}$, satisfying: $E(\sigma(t), \mathcal{B}) \leq \epsilon$. We know $e_{\epsilon, \sigma(t)} > 0$ if $(\sigma(t), \tau(t))$ is not harmonic. $e_{\epsilon, \sigma}$ has some continuity as follows:

**Lemma 4.4** Use notations as above, $\forall t \in (0, 1)$, if $\sigma(t)$ is not harmonic, we can find a neighborhood $I^t \subset (0, 1)$ of $t$ depending on $t, \epsilon$ and the path $\sigma$, such that

$$e_{\frac{1}{2}t, \sigma(s)} \leq 2e_{\epsilon, \sigma(t)}, \tag{37}$$

for $s \in 2I^t$. 

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Proof: \( \sigma(t) \in C^0 \cap W^{1,2}(T^2_{\tau}) \) can be viewed as defined on a uniform domain \( B_R \subset \mathbb{C} \) with \( \{1, \tau(t)\} \subset B_R \) for all \( t \in [0,1] \), i.e. \( \sigma \in C^0([0,1], C^0 \cap W^{1,2}(B_R, N)) \). Since \( e_{\epsilon, \sigma(t)} > 0 \), we can find a neighborhood \( \tilde{I} \) of \( t \) such that for all \( s \in \tilde{I} \), and for any \( B \subset B_R \), we have

\[
\frac{1}{2} \int_B |\nabla \sigma(s) - \nabla \sigma(t)|^2 \leq \min\{\frac{1}{4} e_{\epsilon, \sigma(t)}, \frac{1}{2}\epsilon\}.
\]

(38)

For fixed \( s \in \tilde{I} \), we can find a finite collection of balls \( B \subset B_R \), such that \( \mathcal{E}(\sigma(s), B) \leq \frac{1}{2} \epsilon \) and \( \mathcal{E}(\sigma(s)) - \mathcal{E}[H(\sigma(s), \frac{1}{2}B)] \geq \frac{3}{4} e_{\epsilon, \sigma(s)} \) by the definition of \( e_{\epsilon, \sigma(s)} \). Hence \( E(\sigma(t), B) \leq E(\sigma(s), B) + \frac{1}{2} \epsilon \leq \epsilon \), so we have \( E(\sigma(t)) - E[H(\sigma(t), \frac{1}{2}B)] \leq e_{\epsilon, \sigma(t)} \).

Thus:

\[
E(\sigma(s)) - E[H(\sigma(s), \frac{1}{2}B)] \leq |E(\sigma(t)) - E(\sigma(s))| + E(\sigma(t)) - E[H(\sigma(t), \frac{1}{2}B)]
\]

\[
+ |E[H(\sigma(t), \frac{1}{2}B)] - E[H(\sigma(s), \frac{1}{2}B)]|.
\]

(39)

By Corollary 4.4, after possibly shrinking the neighborhood \( \tilde{I} \) to a smaller one \( I \), we will have \( |E(\sigma(t)) - E(\sigma(s))| \leq \frac{1}{2} e_{\epsilon, \sigma(t)} \) and \( |E[H(\sigma(t), \frac{1}{2}B)] - E[H(\sigma(s), \frac{1}{2}B)]| \leq \frac{1}{2} e_{\epsilon, \sigma(t)} \).

So we know \( E(\sigma(s)) - E[H(\sigma(s), \frac{1}{2}B)] \leq \frac{3}{4} e_{\epsilon, \sigma(t)} \), and hence \( e_{\epsilon, \sigma(s)} \leq 2e_{\epsilon, \sigma(t)} \).

\( \square \)

Now we will find a good family of coverings of the time parameter on which we do harmonic replacement for fixed \( \gamma(t) \). In fact, there will be at most two overlaps for these coverings for a fixed time \( t \).

Lemma 4.5 Let \( (\gamma(t), \tau(t)) \) be as in Lemma 4.4, \( B_R \supset \{1, \tau(t)\} \) as above. There exist \( m \) collection of disjoint balls \( B_1, \ldots, B_m \subset B_R \), which are disjoint balls on \( T^2_{\tau(t)} \) after quotient by \( \{1, \tau(t)\} \), and continuous functions \( r_j : [0,1] \to [0,1], \; j = 1, \ldots, m \), satisfying:

1°. At most two \( r_j \) are positive for a fixed \( t \), and \( E(\gamma(t), r_j(t)B_j) \leq \frac{1}{3} \epsilon_1 \);

2°. If \( t \in [0,1] \), such that \( E(\gamma(t), \tau(t)) \geq \frac{1}{2} W \), there exists a \( j \), such that \( E(\gamma(t)) - E[H(\gamma(t), \frac{1}{2}B_j)] \geq \frac{1}{8} \epsilon_1 \).

Proof: By continuity, \( I = \{ t \in [0,1] : E(\gamma(t), \tau(t)) \geq \frac{1}{2} W \} \) is a compact subset of \( (0,1) \), since the boundary maps \( \gamma(0), \gamma(1) \) have energy almost 0 by our almost conformal parametrization. Since \( \gamma(t) \) has no nonconstant harmonic slices, \( \forall t \in I \), we can find a finite collection of disjoint balls \( B_t \), such that, \( E(\gamma(t), B_t) \leq \frac{1}{4} \epsilon_1 \), and:

\[
E(\gamma(t)) - E[H(\gamma(t), \frac{1}{2}B_t)] \geq \frac{1}{2} e_{\epsilon_1, \gamma(t)} > 0.
\]

(40)

By Lemma 4.4 and continuity of \( \gamma \), we can find a neighborhood \( I^t \ni t \), such that:

\( e_{\epsilon_1, \gamma(s)} \leq 2e_{\epsilon_1, \gamma(t)} \), and \( E(\gamma(s), B_t) \leq \frac{1}{3} \epsilon_1 \) for \( s \in 2I^t \). By the continuity of harmonic
replacement Corollary 4.1, after possibly shrinking $I^t$, we can get for $s \in 2I^t$:

$$\left|\lbrace E(\gamma(t)) - E[H(\gamma(t), \frac{1}{2}B_i)]\rbrace - \lbrace E(\gamma(s)) - E[H(\gamma(s), \frac{1}{2}B_i)]\rbrace\right| \leq \frac{1}{4}\epsilon_1, \gamma(t).$$ (41)

So we have $E(\gamma(s)) - E[H(\gamma(s), \frac{1}{2}B_i)] \geq \frac{1}{4}\epsilon_1, \gamma(t) \geq \frac{1}{8}\epsilon_1, \gamma(s)$, for $s \in 2I^t$. By the compactness of $I$, we can find a finite covering $\{I^t\}$ of $I$, and we can shrink $I^t$ such that each $I^t$ intersects at most two $I^{t_k}$, and these two intervals do not intersect with each other. Choose $B_j = B_{i_j}$, and choose $r_j$ which are equal to 1 on $I^{t_j}$, and 0 outside $2I^{t_j}$. We also urge that $r_j(t) = 0$, if $t$ lies in other interval $I^{t_k}$ which does not intersect with $I^{t_j}$. It is easy to see these $B_j$ and $r_j$ satisfy the Lemma.

\[\square\]

**Proof:** (of Lemma 4.1) Choose the covering $B_j$ and functions $r_j$ as the above Lemma. Let $\gamma^0(t) = \gamma(t)$, and $\gamma^k(t) = H(\gamma^{k-1}(t), r_k(t)B_k)$, for $k = 1, \cdots, m$. and let $\rho(t) = \gamma^m(t)$. We will show that $\rho \in \gamma$. By Corollary 4.2 we know $t \to \gamma^k(t)$ is continuous from $[0, 1]$ to $C^0 \cap W^{1,2}$, so $\rho \in \Omega$. Since we can continuously shrink $r_j$ to 0, and again Corollary 4.2 and the Remark 4.3 show that we can hence continuously deform $\rho$ to $\gamma$ in $\Omega$. So $\rho \in \gamma$. Clearly we have $E(\rho(t)) \leq E(\gamma(t))$.

Now we show property (*). Property 1° of the above Lemma shows that there are at most two steps of harmonic replacements from $\gamma$ to $\rho$, and for fixed $t$ with $E(\gamma(t)) \geq \frac{1}{2}W$ we denote the possible middle nontrivial harmonic replacement by $\gamma^k(t)$. For any finite collection of disjoint balls $B = \cup B_i$ with $E(\rho(t), B) \leq \frac{1}{12}\epsilon_1$, we can assume that $\gamma(t), \gamma^k(t)$ have energy at least $\frac{1}{8}\epsilon_1$ on $B$, or we have a lower bound of $E(\gamma(t)) - E(\rho(t))$, hence inequality 7 holds. By property 2° of the above Lemma, the energy decrease from $\gamma(t)$ to $\gamma^k(t)$ or from $\gamma^k(t)$ to $\rho(t)$ is at least $\frac{1}{8}\epsilon_1, \gamma(t)$. We have estimates at worst by Lemma 4.2

$$E(\gamma(t)) - E(\rho(t)) \geq k\left(\frac{1}{8}\epsilon_1, \gamma(t)\right)^2.$$ (42)

Now using inequality 12 of Lemma 4.2 with $\mu = \frac{1}{8}, \frac{1}{4}$ twice in the case that two $r_j(t) > 0$, we have:

$$E(\rho(t)) - E[H(\rho(t), \frac{1}{8}B)] \leq E(\gamma^k(t)) - E[H(\gamma^k(t), \frac{1}{4}B)] + \frac{1}{k}[E(\gamma^k(t)) - E(\rho(t))]^{\frac{1}{2}}$$

$$\leq E(\gamma(t)) - E[H(\gamma(t), \frac{1}{2}B)] + \frac{1}{k}[E(\gamma(t)) - E(\gamma^k(t))]^{\frac{1}{2}}$$

$$+ \frac{1}{k}[E(\gamma(t)) - E(\rho(t))]^{\frac{1}{2}}$$

$$\leq e^{\frac{1}{8}\epsilon_1, \gamma(t)} + C[E(\gamma(t)) - E(\rho(t))]^{\frac{1}{2}}$$

$$\leq C[E(\gamma(t)) - E(\rho(t))]^{\frac{1}{2}}.$$ (43)

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It is easy to get similar estimates in the case only one \( r_j(t) > 0 \). If we choose \( \epsilon_0 = \frac{1}{2}\epsilon_1 \) and \( \Psi \) a square root function, together with Theorem 4.1, we will get property (\( * \)).

\[ \square \]

**Remark 4.6** Before going on, we have to give some restrictions on the area minimizing sequence \( \tilde{\gamma}_n(t) \). In fact, we can assume that \( \tilde{\gamma}_n(t) \) have no non-constant harmonic slices, i.e. \( (\tilde{\gamma}_n(t), T^2_0) \) is not harmonic unless it is a constant map. We can do this by a reparametrization on \( T^2_0 \) as on page 10 of [2]. In fact, we can assume \( \tilde{\gamma}_n(t) \) is a constant map on a small region on \( T^2_0 \) by small perturbation. Since \( \gamma_n(t) \) differ from \( \tilde{\gamma}_n(t) \) by a diffeomorphism \( T^2_{\tau_n(t)} \) to \( T^2_0 \), \( \gamma_n(t) \) is also a constant map on a small region of \( T^2_{\tau_n(t)} \). Hence \( \gamma_n(t) \) is not harmonic unless it is a constant map by the unique continuation of harmonic maps (Corollary 2.6.1 of [3]). So we can apply Lemma 4.1 to \( (\gamma_n(t), \tau_n(t)) \). Hence there always exist a min-max sequence \( (\rho_n(t_n), \tau_n(t_n)) \), such that \( E(\rho_n(t_n), \tau_n(t_n)) \rightarrow W \) satisfying property (\( * \)) of Lemma 4.1, which will imply bubbling convergence of \( \{\rho_n(t_n), \tau_n(t_n)\} \). But we have to remember that we do not know the behavior of \( \tau_n(t_n) \), so we will discuss two cases in the next section.

## 5 Convergence results

In the paper [7] of Ding, Li and Liu, they discussed bubbling convergence results of almost harmonic maps from tori with conformal structures converging or diverging. If the conformal structures converge, the sequence of almost harmonic maps will bubbling converge to a minimal torus together with possibly several minimal spheres. Here convergence of conformal structures will possibly ensure existence of a nontrivial minimal torus. If the conformal structures diverge to infinity, the bubbling limits only contain several minimal spheres, with the body map from torus degenerate. We will have similar results for our minimizing sequences \( (\rho_n(t_n), \tau_n(t_n)) \). In fact, our sequence are almost conformal.

**Lemma 5.1** If \( E(\rho_n(t_n), \tau_n(t_n)) \rightarrow W \), we have \( E(\rho_n(t_n), \tau_n(t_n)) - \text{Area}(\rho_n(t_n)) \rightarrow 0 \).

**Remark 5.1** Although after the perturbation is Section 4, \( (\rho_n(t), \tau_n(t)) \) may be far from conformal for some \( t \in [0, 1] \), this result tells us that it will still be almost conformal for the mappings with energy closed to \( W \).

**Proof:** We know \( \max_tE(\gamma_n(t), \tau_n(t)) \rightarrow W \), and \( E(\gamma_n(t), \tau_n(t)) \geq E(\rho_n(t), \tau_n(t)) \). So we have \( E(\gamma_n(t), \tau_n(t)) - E(\rho_n(t), \tau_n(t)) \rightarrow 0 \). As we know from the construction from \( \gamma_n(t) \) to \( \rho_n(t) \), \( \rho_n(t) \) is gotten by at most twice harmonic replacements from \( \gamma_n(t) \) on balls where \( \gamma_n(t) \) have energy less than \( \epsilon_1 \). We denote the possible middle harmonic
replacement by $\gamma_n^k(t)$ as in the proof of Lemma 4.1. From Theorem 4.1, we know that $\|\nabla \gamma_n(t_n) - \nabla \gamma_n^k(t_n)\|_{L^2} \leq 4 |E(\gamma_n(t_n), \tau_n(t_n)) - E(\gamma_n^k(t_n), \tau_n(t_n))| \to 0$, and $\|\nabla \gamma_n(t_n) - \nabla \rho_n(t_n)\|_{L^2} \leq 4 |E(\gamma_n(t_n), \tau_n(t_n)) - E(\rho_n(t_n), \tau_n(t_n))| \to 0$. Since all the energy of $\gamma_n(t)$, $\rho_n(t)$ are bounded, we know that $|\text{Area}(\gamma_n(t_n)) - \text{Area}(\rho_n(t_n))| \leq |\text{Area}(\gamma_n(t_n)) - \text{Area}(\gamma_n^k(t_n))| + |\text{Area}(\gamma_n^k(t_n)) - \text{Area}(\rho_n(t_n))| \leq C\{\|\nabla \gamma_n(t_n) - \nabla \gamma_n^k(t_n)\|_{L^2} + \|\nabla \gamma_n^k(t_n) - \nabla \rho_n(t_n)\|_{L^2}\} \to 0$. As $E(\gamma_n(t_n), \tau_n(t_n)) - \text{Area}(\gamma_n(t_n)) \to 0$, we have $E(\rho_n(t_n), \tau_n(t_n)) - \text{Area}(\rho_n(t_n)) \to 0$.

□

To discuss bubble convergence for $(\rho_n(t_n), \tau_n(t_n))$, we firstly talk about the convergence of the metrics given by $\tau_n(t_n) \in T_1$. In fact, two metrics $\tau$ and $\tau'$ are conformally equivalent, if they lie in the same orbit of $PSL(2, \mathbb{Z})$. Denote $\mathcal{M}_1 = \{z \in \mathbb{C}, |z| \geq 1, \Im z > 0, -\frac{1}{2} < \Re z \leq \frac{1}{2}, \text{ if } |z| = 1, \Re z \geq 0\}$ to be the fundamental region of $PSL(2, \mathbb{Z})$, which is also the moduli space of all conformal structures on $T^2$. So every such metric in $T_1$ is conformally equivalent to an element in $\mathcal{M}_1$ after a $PSL(2, \mathbb{Z})$-action. We say a sequence $\{\tau_n\}$ converge to $\tau_0 \in \mathcal{M}_1$ if after being conformally translated to $\{\tau_n'\} \subseteq \mathcal{M}_1$ by actions in $PSL(2, \mathbb{Z})$, $\tau'_n \to \tau_0$. Since area and energy are all conformally invariant, we can always consider bubble convergence after conformally changing the domain metrics to the moduli space $\mathcal{M}_1$.

There is a criterion for convergence of conformal structures on Riemann surfaces with genus $g$ given by Mumford, i.e. Lemma 3.3.2 in [8], or Section 4 in [7]. If the lengths of the shortest closed geodesics on a family of genus $g$ surfaces have a positive lower bound, then the conformal structures on these surfaces will converge after possibly taking a subsequence. In the case of torus $T^2$, this criterion is relatively simple. Denote $\tau = \tau_1 + \sqrt{-1}\tau_2$ to be the conformal structure on a marked torus, and we use the second normalization as discussed above. So $T^2_\tau = \{\frac{1}{\sqrt{\tau_2}}, \frac{\tau_1}{\sqrt{\tau_2}} + \sqrt{\tau_2}\}$. That the conformal structure $\tau$ degenerate means $\tau_2 \to \infty$. The length of the shortest closed geodesic on $T^2_\tau$ has the same order as $\frac{1}{\sqrt{\tau_2}}$. So the criterion is obvious.

**Theorem 5.1** Using the above notations, let $(\rho_n(t), \tau_n(t))$ be what we get in the last section by perturbation of $(\gamma_n(t), \tau_n(t))$ as in Lemma 4.1 and Remark 4.6, then all subsequences $\rho_n(t_n)$ with $E(\rho_n(t_n), \tau_n(t_n)) \to \mathcal{W}_E$, satisfy:

(4) For any finite collection of disjoint balls $\bigcup B_i$ on $T^2_{\tau_n(t_n)}$ such that $E(\rho_n(t_n), \bigcup B_i) \leq \epsilon_0$, let $v$ be the harmonic replacement of $\rho_n(t_n)$ on $\frac{1}{8} \bigcup B_i$. We have:

\[ \int_{\frac{1}{8} \bigcup B_i} |\nabla \rho_n(t_n) - \nabla v|^2 \to 0. \]

\[^3\text{Area}(\omega_1, \omega_2) = 1\]
Here $\epsilon_0$ is the small constant given in Lemma 4.1. We have the following two possible cases for $\{\rho_n(t_n), \tau_n(t_n)\}$:

(1). If $\tau_n(t_n) \to \tau_\infty$ in the above sense, then there exist a conformal harmonic map $u : (T^2, \tau_\infty) \to N$, and harmonic spheres $\{u_i\}$, such that $\rho_n(t_n)$ bubble converge to $(u,u_1,\ldots,u_l)$, with:

$$\lim_{n \to \infty} E(\rho_n(t_n), \tau_n(t_n)) = E(u, \tau_\infty) + \sum_i E(u_i).$$

(45)

(2). If $\tau_n(t_n)$ diverge, then there exist only several harmonic spheres $\{u_i\}$, such that $\rho_n(t_n)$ bubble converge to $(u_1,\ldots,u_l)$, with body map degenerated, and

$$\lim_{n \to \infty} E(\rho_n(t_n), \tau_n(t_n)) = \sum_i E(u_i).$$

(46)

**Remark 5.2** We point out here that property ($\ast$) is invariant when we do recaling in the bubble process. Property ($\ast$) also holds when we conformally change the metrics $\tau_n(t_n)$ to $M_1$. These two invariance properties ensure us to use property ($\ast$) in all our proof. For case (1), we can use the bubbling convergence given by Sacks and Uhlenbeck in [11]. Since the area and energy of this sequence will converge to the same value $W = W_E$, the energy identity holds. The bubbling limits are the solution of this variational problem.

For case (2), the length of the shortest closed geodesics will converge to 0. So by the argument given by Ding, Li and Liu in Section 4 of [7], we can tear the torus to a long cylinder. After some conformal scaling, we can assume the radii of the cylinders equal 1. So the sequence of almost harmonic mappings on long cylinders will converge to a set of harmonic spheres by an argument given in an unpublished note [6] of Ding. Similar argument as case (1) ensures the energy identity.

We will need the following Proposition when we prove identities 45 and 46. We denote $C_{r_1,r_2}$ as a part of the cylinder $S^1 \times \mathbb{R}$ with radial coordinates between $r_1$ and $r_2$. Clearly $C_{r_1,r_2}$ is conformally equivalent to the annulus $B_{e^{-r_2}} \setminus B_{e^{-r_1}}$. Here we have to recall the concept of almost harmonic maps defined by [2]. Let $N$ be the ambient manifold. For $\nu > 0$, we call $u \in W^{1,2}(C_{r_1,r_2}, N)$ a $\nu$-almost harmonic map (Definition B.27 in [2]) if for any finite collection of disjoint balls $B$ in the conformally equivalent annulus $B_{e^{-r_2}} \setminus B_{e^{-r_1}}$ of $C_{r_1,r_2}$, there is an energy minimizing map $v : \cup_{B} B \to N$ with the same boundary value as $u$ such that:

$$\int_B |\nabla u - \nabla v|^2 \leq \nu \int_{C_{r_1,r_2}} |\nabla u|^2.$$  

(47)

**Proposition 5.1** (Proposition B.29 of [2]) Let $\delta > 0$, there exist small constants $\nu > 0$, $\epsilon_2 > 0$ and large constant $l \geq 1$ (depending on $\delta$ and $N$), such that for any integer $m$, if
$u$ is a $\nu$-almost harmonic map as defined above on $C_{-(m+3)\ell,3\ell}$ with $E(u) \leq \epsilon_2$, then:

$$
\int_{C_{-(m+3)\ell,3\ell}} |u_\theta|^2 \leq 7\delta \int_{C_{-(m+3)\ell,3\ell}} |\nabla u|^2.
$$

(48)

Here we use $(\theta,t)$ as coordinates on $S^1 \times \mathbb{R}$, and $u_\theta$ means the differentiation w.r.t $\theta$.

**Proof:** (of Theorem 5.1) **Case (1):** We denote $\rho_n = \rho_n(t_n)$, and let $\tau_n \in \mathcal{M}_1$ be the corresponding conformal structure of $\tau_n(t_n)$. We divide the bubbling convergence into several steps, and we will then focus on the neck parts.

**Step 1.** Since $\tau_n \to \tau_\infty$, we can identify a point $x \in T^2_{\tau_\infty}$ as on $T^2_{\tau_n}$ by viewing it as on the fundamental regions of lattices $\{1,\tau_\infty\}$ and $\{1,\tau_n\}$ of corresponding conformal structures. So for any $x \in T^2_{\tau_\infty}$, for a fixed small constant $\epsilon_1 < \epsilon_0$, we can consider a sequence of energy concentration radii $r_n(x)$ defined as follows:

$$
r_n(x) = \sup \{ r > 0, E(\rho_n, B(x, r)) \leq \epsilon_1 \}.
$$

(49)

Such $r_n(x)$ exist and are positive. Now we say $x$ is an energy concentration point if $\lim_{n \to \infty} r_n(x) \to 0$. If $x$ is an energy concentration point, we have that:

$$
\inf_{r > 0} \{ \lim_{n \to \infty} E(\rho_n, B(x, r)) \} \geq \epsilon_1.
$$

(50)

Since our sequence $\rho_n$ have uniform bounded energy $2W$, we know the number of the energy concentration points are bounded by $2W/\epsilon_1$. Denote these points by $\{x_1, \cdots, x_m\}$. If $x \in T^2_{\tau_\infty} \setminus \{x_1, \cdots, x_m\}$, we can find a $r(x) > 0$ such that $E(\rho_n, B(x, r(x))) \leq \epsilon_1$ for all $n$. and by condition (*), there exist $\nu_n$ which are the energy minimizing harmonic maps defined on $\frac{1}{\delta}B(x, r(x))$ with the same boundary value as $\rho_n$, such that $\|\rho_n - \nu_n\|_{W^{1,2}(\frac{1}{\delta}B(x, r(x)))} \to 0$. Since $E(\nu_n, \frac{1}{\delta}B(x, r(x))) \leq \epsilon_1 < \epsilon_{SU}$, we know from [11] that $\nu_n$ have uniform interior $C^{2,\alpha}$-estimates on $\frac{1}{2}B(x, r(x))$, and hence converge to a harmonic map $u$ on $\frac{1}{2}B(x, r(x))$ in $C^{2,\alpha}$ after taking a subsequence. Hence $\rho_n \to u$ in $W^{1,2}(\frac{1}{2}B(x, r(x)))$. So for any compact subset $K \subset T^2_{\tau_\infty} \setminus \{x_1, \cdots, x_m\}$, we can cover them by finite many balls $\frac{1}{2}B(x, r(x))$, and hence $\rho_n \to u$ in $W^{1,2}(K)$ after taking a subsequence. Here $u$ is a harmonic map defined on $K$. After exhausting $T^2_{\tau_\infty} \setminus \{x_1, \cdots, x_m\}$ by a sequence of compact sets $K_i$, and a diagonal argument, we know $u$ is a harmonic map on $T^2_{\tau_\infty} \setminus \{x_1, \cdots, x_m\}$, and by the Theorem 3.6 of removable singularity in [11], we know $u$ extends to a harmonic map on $T^2_{\tau_\infty}$.

**Step 2.** We now see what happens near the energy concentration points. Fix an energy concentration point $x_i$, and denote $r_{n,i} = r_n(x_i)$. Find a small $r > 0$, such that $E(u, B(x_i, r)) \leq \frac{1}{2}\epsilon_1$. We rescale $\rho_n$ on $B(x_i, r_{n,i})$. Define $u_{n,i} = \rho_n(x_i + r_{n,i}(x - x_i))$.
So \( B(x_i, r_{n,i}) \) are now rescaled to \( B_1 \), and \( B(x_i, r) \) to \( B(0, r/r_{n,i}) \). \( u_{n,i} \) can be viewed as defined on balls \( B(0, r/r_{n,i}) \) with radii converging to infinity. Since the domains converge to the whole complex plane \( \mathbb{C} \), which is conformal equivalent to the sphere \( S^2 \) without the south pole, we can think \( u_{n,i} \) as defined on any compact subsets of \( S^2 \) away from the south pole for \( n \) large enough. Since the property \((*)\) is conformal invariant, we can do the first step to \( u_{n,i} \). We can find finitely many energy concentration points \( \{x_{i,1}, \cdots, x_{i,m_i}\} \subset S^2 \setminus \text{south pole} \), such that \( u_{n,i} \) converge to a harmonic map \( u_i \) defined on \( S^2 \setminus \text{south pole} \) in the sense of the above step, and hence \( u_i \) is a harmonic sphere defined on \( S^2 \) by the Theorem of removable singularity. From our definition, we know that \( E(u_{n,i}, B_1) = \epsilon_1 < \epsilon_{SU} \). So \( x_{i,j} \in S^2 \setminus B_1^4 \). A key point is that the total energy of \( u_{n,i} \) on \( S^2 \setminus \{ \text{south pole} \cup B_1 \} \) is decreased by a finite amount \( \epsilon_1 \) compared to the original \( u_{n,i} \), as \( u_{n,i}|_{B_1} \) taking the energy. We call such rescaling and convergence procedure bubbling convergence.

Step 3. We can repeat the bubbling convergence given in step 2 for \( u_{n,i} \) on balls centered at \( x_{i,j} \). We point out here that there are only finite many such steps, and then the bubbling convergence stops. Each time, we come from a sequence of maps \( u_n \) defined on a small ball \( B_r \), and we rescale them to exhaust the whole complex plane. Each time \( u_n|_{B_1} \) take a finite amount of energy after recaling. So after several steps, the total energy of \( u_n \) will be less than \( \epsilon_1 < \epsilon_{SU} \), and there will be no energy concentration points. The bubbling convergence stops.

Step 4. We will discuss energy identity \((53)\) now. We can decompose \( T_{r_n}^2 \) into the bubble part \( \cup B(x_i, r) \) and the body part \( T_{r_n}^2 \setminus \cup B(x_i, r) \). So the total energy has decomposition \( E(\rho_n, T_{r_n}^2) = E(\rho_n, T_{r_n}^2 \setminus \cup B(x_i, r)) + \sum_i E(\rho_n, B(x_i, r)) \). Now we can calculate the energy of the first limit map \( u_0 \) as follows:

\[
E(u_0) = \lim_{r \to 0} E(u_0, T_{r_n}^2 \setminus \cup B(x_i, r)) = \lim_{r \to 0} \lim_{n \to \infty} E(\rho_n, T_{r_n}^2 \setminus \cup B(x_i, r)).
\] (51)

So we only need to show that \( \lim_{r \to 0} \lim_{n \to \infty} \sum_i E(\rho_n, B(x_i, r)) = \sum_i E(u_i) \). Here \( u_i \) are the bubble maps. As in the second step, we know that \( u_i \) are limits of \( u_{n,i} \) on any compact set of \( \mathbb{C} \), so we can calculate the energy of the first bubble map \( u_i \) as follows:

\[
E(u_i) = \lim_{R \to \infty} E(u_i, B(R)) = \lim_{R \to \infty} \lim_{n \to \infty} E(u_{n,i}, B(R)).
\] (52)

By the conformal invariance of energy, \( E(u_{n,i}, B(R)) = E(\rho_n, B(x_i, r_{n,i}R)) \). So we only need to show that:

\[
\lim_{r \to 0, R \to \infty} \lim_{n \to \infty} E(\rho_n, B(x_i, r) \setminus B(x_i, r_{n,i}R)) = 0.
\] (53)

\(^4\)Here \( B_1 \) is a unit ball centered at the north pole.
We denote the annulus \( A(x_i, r, r_n, R) = B(x_i, r) \setminus B(x_i, r_n, R) \). Since \( A(x_i, r, r_n, R) \) is conformally equivalent to a long cylinder \( C_{r_1, r_2} \), with \( r_1 = -\ln(r_n, R), r_2 = -\ln(r) \), we call such annuli or such cylinders **necks**. So what left is to show that there will be no energy concentration on necks.

**Step 5.** We use Proposition \([5.1]\) to show that necks support no energy in our case. We will use step 1 as an example, and others follow in the same way. Suppose there is a lower bound for \( E(\rho_n, C_{r_1, r_2}) \). Since \( \rho_n \) will converge to \( u_0 \) on any small annulus centered at \( x_i \), and \( u_{n,i} \) will converge to \( u_i \) on any large annulus centered at \( 0 \), for fixed \( L > 0 \) we know that there can be no energy concentration on \( A(x_i, re^{-L}, r) \) and \( A(x_i, r_n, Re^{-L}, r_n, R) \) for \( r \to 0 \) and \( R \to \infty \). Changing to the cylinder, we know there will be no energy concentration on a region with fixed length towards boundary of \( C_{r_1, r_2} \). Now fix a \( \delta = \frac{1}{10} \), and let \( \nu, \epsilon_2 \) and \( l \) be as in Proposition \([5.1]\). We can find a sub-cylinder \( C_{r_1', r_2'} \) with the distance between boundaries of them converging to \( \infty \), i.e. \( d(\partial C_{r_1, r_2}, \partial C_{r_1', r_2'}) \to \infty \), such that \( E(\rho_n, C_{r_1', r_2'}) = \frac{1}{2} \epsilon_2 \). We want to show that \( \rho_n \) is \( \nu \)-almost harmonic on \( C_{r_1', r_2'} \) for \( n \) large. In fact for any finite collection of disjoint balls \( B \) on the annulus, \( E(\rho_n, B) \leq E(\rho_n, C_{r_1', r_2'}) \leq \epsilon_2 \). We can assume \( \epsilon_2 \leq \epsilon_1 \), so \( \rho_n \) satisfy property \((*)\), i.e. \( \int_{\{B\}} |\nabla \rho_n - v|^2 \to 0 \), with \( v \) the energy minimizing map. Since \( E(\rho_n, C_{r_1', r_2'}) \) have uniform lower bound, \( \int_{\{B\}} |\nabla \rho_n - v|^2 \leq \nu \int_{C_{r_1', r_2'}} |\nabla \rho_n|^2 \) hold for \( n \) large enough. We can assume we first do the above on a cylinder a little bit larger than \( C_{r_1', r_2'} \), then by Proposition \([5.1]\) we have:

\[
\int_{C_{r_1', r_2'}} |(\rho_n)_{\partial t}|^2 \leq \frac{1}{10} \int_{C_{r_1', r_2'}} |\nabla \rho_n|^2. \tag{54}
\]

Hence we have a lower bound on the gap between energy and area.

\[
E(\rho_n, C_{r_1', r_2'}) - \text{Area}(\rho_n, C_{r_1', r_2'}) = \frac{1}{2} \int_{C_{r_1', r_2'}} |(\rho_n)_{\partial t}|^2 + |(\rho_n)_{\partial \theta}|^2 - 2 |(\rho_n)_{\partial t} \times (\rho_n)_{\partial \theta}| \\
\geq \frac{1}{8} \int_{C_{r_1', r_2'}} |(\rho_n)_{\partial t}|^2 - |(\rho_n)_{\partial \theta}|^2. \tag{55}
\]

So \( E(\rho_n, C_{r_1', r_2'}) - \text{Area}(\rho_n, C_{r_1', r_2'}) \) have a lower bound by the above estimates. It is a contradiction to \( E(\rho_n(t), r_n(t)) - \text{Area}(\rho_n(t)) \to 0 \) given in Lemma \([5.1]\).

**Case (2).** We use \((t, \theta)\) as parameters on \( T_{r_n}^2 \). In fact, we assume \( \arg(r_n) = \theta_n \), and let \( z' = t + \sqrt{-1} \theta = e^{-\sqrt{-1} \left( \frac{1}{2} \pi - \theta_n \right)} z \) be another conformal parameter system on \( T_{r_n}^2 \). We conformally expand the torus such that the length of the circle of parameter \( \theta \) is 1, and the length of parameter \( t \) is denoted by \( 2l_n \). Then we divide the torus \( T_{r_n}^2 \) into sections with length 1 in the parameter \( t \), i.e. \( T_{r_n}^2 = \bigcup_i S^1 \times [t_i, t_{i+1}] \).

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We claim that there exists a large $L > 0$, such that for $n$ large, there exist $t_{n,0}$, such that $E(\rho_n, S^1 \times [t_{n,0} - L, t_{n,0} + L]) > \epsilon_2$. If the claim fails, $\forall L > 0$, we can find a subsequence of $n \rightarrow \infty$, such that $\forall t_{n,i}$, $E(\rho_n, S^1 \times [t_{n,i} - L, t_{n,i} + L]) \leq \epsilon_2$. After possibly extending some $[t_{n,i} - L, t_{n,i} + L]$, we have $E(\rho_n, S^1 \times [t_{n,i} - L, t_{n,i} + L]) = \epsilon_2$. So $\rho_n|_{[t_{n,i} - L, t_{n,i} + L]}$ satisfy condition of Proposition 5.1 and hence is a contradiction to Lemma 5.1 as argued in step 5 of case 1.

Now consider $\rho_n : S^1 \times [t_{n,0} - l, t_{n,0} + l) \rightarrow N$. There may be bubbles near $t_{n,0}$. Argument as in case 1 shows that $\rho_n$ converge to a harmonic map $u_1$ defined on $S^1 \times \mathbb{R}$ besides some energy concentration points. $u_1$ is nontrivial since $E(\rho_n, S^1 \times [t_{n,0} - L, t_{n,0} + L]) > \epsilon_2$. As $S^1 \times \mathbb{R}$ is conformally equivalent to $S^2 \setminus \text{north and south pole}$, we can extend $u_1$ to a harmonic map on $S^2$. We can rescale $\rho_n$ near the energy concentration points, and the rescaled map will converge as we discussed in Case 1 to several bubble maps $\{u_{1,1}, \cdots, u_{1,l_1}\}$. Energy identity during these bubbles will follow as in the last step of Case 1 on each long cylinder. Now we calculate the total energy:

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} E(\rho_n, S^1 \times [t_{n,0} - l, t_{n,0} + l]) = \lim_{l \rightarrow \infty} E(u_1, S^1 \times [-l, l]) + \sum_i E(u_{1,i})$$

$$= E(u_1, S^2) + \sum_i E(u_{1,i}).$$

(56)

So if $\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} E(\rho_n, S^2 \times [-l, -l] \cup [l, l]) = 0$, there will be no other bubbles except for $\{u_1, u_{1,1}, \cdots, u_{1,l_1}\}$, and $\lim_{n \rightarrow \infty} E(\rho_n) = E(u_1) + \sum_i E(u_{1,i})$, i.e energy identity (56) holds. If $\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} E(\rho_n, S^2 \times [-l, -l] \cup [l, l]) > 0$, we can consider maps on the other part of the rescaled torus, i.e. we can find another base point denoted by $t_{n,1}$, such that $|t_{n,1} - t_{n,0}| \rightarrow \infty$ and $E(\rho_n, S^1 \times [t_{n,1} - L, t_{n,1} + L]) > \epsilon_2$. Consider $\rho_n : S^1 \times [t_{n,1} - l, t_{n,1} + l] \rightarrow N$. We can repeat the above step and get another set of harmonic spheres $\{u_2, u_{2,1}, \cdots, u_{2,l_2}\}$. Since each bubble is a harmonic sphere and must take a finite mount of energy by (11), there are only finitely many such steps. We will get all these harmonic spheres $u_i$ and energy identity (56) by summing over all the steps.

\[\square\]

What is left? The aim of this method is to find a min-max minimal torus, but only when the conformal structures do not degenerate can we get a nontrivial minimal torus. So we do want to know under what condition does there exist a subsequence $\{\rho_n(t_n), \tau_n(t_n)\}$ satisfying condition (1) in the above theorem.

6 Appendix 1–a uniformization result

In this section, we discuss a general uniformization theorem on the complex plane. We will focus on the continuous dependence of the conformal diffeomorphisms on the
variance of general metrics. Let \( g \) be a Riemannian metric on the complex plane \( \mathbb{C} \).

**Lemma 6.1** In the complex coordinates \( \{ z, \overline{z} \} \), we can write \( g = \lambda(z) |dz + \mu(z) d\overline{z}|^2 \). Here \( \lambda(z) > 0 \), and \( \mu(z) \) is complex function on the complex plane with \( |\mu| < 1 \). If \( g \geq \epsilon |dz d\overline{z}| \), there exists a \( k = k(\epsilon) < 1 \), such that \( |\mu| \leq k \).

**Remark 6.1** The proof is just simple calculation. Hence we can always identify a plane non-degenerate metric with \( |dz + \mu(z) d\overline{z}|^2 \) conformally. In fact, \( \mu \) is a rational function of the components \( g_{ij}(z) \), so if a family \( g(t) \) vary continuously in the \( C^1 \) class, the corresponding \( \mu(t) \) also vary continuously in the \( C^1 \) class.

### 6.1 Results in \cite{1}

Let us discuss what Ahlfors and Bers did in \cite{1}. They gave the existence and uniqueness of conformal diffeomorphism \( w^\mu : \mathbb{C}_{|dz + \mu d\overline{z}|^2} \to \mathbb{C}_{dwd\overline{w}} \) fixing three points \( (0, 1, \infty) \) for any \( L^\infty \) function \( \mu \) with \( |\mu| \leq k < 1 \). Such maps must satisfy the following equation:

\[
\frac{w^\mu}{z} = \mu(z) \frac{w^\mu}{z}.
\] (57)

Define function space \( B_p(\mathbb{C}) = C^{1 - \frac{2}{p}} \cap W^{1, p}_{loc}(\mathbb{C}) \), where \( p > 2 \) depends on the bound \( k \) of \( \mu \). They showed that \( w^\mu \) are uniformly bounded in \( B_p(\mathbb{C}) \) for a uniform bound \( k \), and that \( w^\mu \) vary continuously in \( B_p(\mathbb{C}) \) while \( \mu \) varying continuously in \( L^\infty(\mathbb{C}) \). Suppose \( \mu, \nu \in L^\infty(\mathbb{C}) \), and \( |\mu|, |\nu| \leq k \), with \( k < 1 \). Let \( w^\mu, w^{\nu} \) be the corresponding conformal homeomorphisms, then:

**Lemma 6.2** (Lemma 16, Theorem 7, Lemma 17, Theorem 8 of \cite{1})

\[
d_{S^2}(w^\mu(z_1), w^{\nu}(z_2)) \leq c d_{S^2}(z_1, z_2)^{\alpha},
\] (58)

\[
\|w^\mu_z\|_{L^p(B_R)} \leq c(R),
\] (59)

\[
d_{S^2}(w^\mu(z), w^{\nu}(z)) \leq C \|\mu - \nu\|_{\infty},
\] (60)

\[
\|(w^\mu - w^{\nu})_z\|_{L^p(B_R)} \leq C(R) \|\mu - \nu\|_{\infty}.
\] (61)

Here \( d_{S^2} \) is the sphere distance, which is equivalent to the plane distance of \( \mathbb{C} \) on compact sets. \( \alpha = 1 - \frac{2}{p} \). All constants are uniformly bounded depending on \( k < 1 \).

**Remark 6.2** This Lemma comes from estimates of equation \cite{57}. Here we use sphere distance because what we concern is just local properties.
6.2 Similar results

If we write our metrics conformally on \( \mathbb{C} \), what we concern in our case are the conformal homeomorphisms \( h^\mu : \mathbb{C}_{dw} \rightarrow \mathbb{C}_{|dz+\mu dz|^2} \) fixing three points \((0, 1, \infty)\), which are just the inverse mappings of those of Ahlfors and Bers. We also concern the continuous dependence of \( h^\mu \) in \( C^0 \cap W^{1,2}_{loc}(\mathbb{C}, \mathbb{C}) \) on the variance of \( \mu \) in \( C^1(\mathbb{C}) \). In fact:

\[
h^\mu(w) = (w^\mu)^{-1}(w),
\]

and our mappings satisfy:

\[
h^\mu_{\overline{w}} = -\mu(h^\mu(w))h^\mu_{\overline{w}}.
\]

If \( \mu_n \) are a sequence of metric coefficients as above, such that \( \| \mu_n - \mu \|_{C^1} \rightarrow 0 \), and \( h^{\mu_n} \) as above, we want to have results similar to the above:

**Lemma 6.3**

\[
d_{S^2}(h^{\mu_n}, h^\mu) \rightarrow 0,
\]

\[
\|(h^{\mu_n} - h^\mu)_{w}\|_{L^p(B_R)} \rightarrow 0.
\]

Here because the equation (63) is quasi-linear, we may not get the linear control as Lemma 6.2. We will give a self contained proof of this result by argument similar to those of Ahlfors and Bers. We will use their notions. In fact we will proof the following two claims:

**Claim 1**

\[
d_{S^2}(h^{\mu_n}, h^\mu) \rightarrow 0.
\]

**Proof:** Let \( w^\mu \) be the conformal diffeomorphism described above, so we have uniform Hölder estimates \( d_{S^2}(w^\mu(z_1), w^\mu(z_2)) \leq c d_{S^2}(z_1, z_2)^\alpha \). Here the constant \( c \) is uniform for fixed \( k < 1 \), when all \( \| \mu \| \leq k \). Let \( h^\mu = (w^\mu)^{-1} \), we have:

\[
h^\mu_{\overline{w}} = \nu(w)h^\mu_{\overline{w}},
\]

here \( \nu(w) = ( - \mu w^\mu_{\overline{w}} ) \circ h \). Since \( \| \nu \|_{L^\infty} = \| \mu \|_{L^\infty} \), we have similar Hölder estimates \( d_{S^2}(h^\mu(w_1), h^\mu(w_2)) \leq c' d_{S^2}(w_1, w_2)^\alpha \).

We use contradiction arguments. Suppose \((w^{\mu_n})^{-1}\) do not converge to \((w^\mu)^{-1}\) in \( L^\infty(S^2, S^2) \), then there exists an \( \epsilon > 0 \) and a sequence \( x_n \in S^2 \) such that \( d_{S^2}((w^{\mu_n})^{-1}(x_n), (w^\mu)^{-1}(x_n)) > \epsilon \). By the compactness of \( S^2 \), we can assume \( x_n \rightarrow x_0 \), and \( (w^{\mu_n})^{-1}(x_n) \rightarrow z_1 \), \( (w^\mu)^{-1}(x_n) \rightarrow z_0 \). Clearly \( d_{S^2}(z_0, z_1) \geq \epsilon \). But \( w^\mu(z_0) = w^\mu(z_1) = x_0 \), which forms a contradiction since \( w^\mu \) is a homeomorphism. This is because of the following.
Denoting $z_n = (w^{\mu_n})^{-1}(x_n)$ and $z'_n = (w^\mu)^{-1}(x_n)$, we have the following:

\[
d_{S^2}(w^{\mu_n}(z_n), w^\mu(z_1)) \leq d_{S^2}(w^{\mu_n}(z_n), w^{\mu_n}(z_1)) + d_{S^2}(w^{\mu_n}(z_1), w^\mu(z_1)) \to 0, \quad (68)
\]

The convergence of the first term is because $w^{\mu_n}$ have uniform Hölder norm. So $w^\mu(z_1) = x_0$. And

\[
w^\mu(z_0) = \lim_{n \to \infty} w^\mu(z'_n) = \lim_{n \to \infty} x_n = x_0 \quad (69)
\]

So we have $w^\mu(z_0) = w^\mu(z_1)$.

\[\square\]

**Claim 2**  
The conformal diffeomorphism solution $h : \mathbb{C} \to \mathbb{C}$ fixing $(0, 1, \infty)$ of the equation:

\[
hw = \alpha(w)hw,
\]

have estimates:

\[
\| (h^\alpha - h^\beta)w \|_{L^p(B_R)} \leq C(R)\| \alpha - \beta \|_{L^\infty}^{2\alpha}. \quad (71)
\]

Here constants depend only on bound $k$ of $|\alpha| \leq k < 1$. $\alpha = 1 - \frac{2}{p}$ as in Lemma 6.2 and $p$ depends only on $k$.

**Proof:** We show this in five steps, and we may use $w$ to denote $h$.

**Step 1.** We consider the following non-homogeneous equation:

\[
w_\sigma = \mu \bar{w_\sigma} + \sigma. \quad (72)
\]

We want to find solutions satisfying: $w(0) = 0$, $w_\sigma \in L^p(\mathbb{C})$, and we denote such solution by $w^{\mu,\sigma}$. We firstly consider the following preliminary equation:

\[
q = T(\mu \bar{\sigma} + \sigma). \quad (73)
\]

Here $T$, $P$ denote the operators defined in Section 1.2 of [1]. By the fixed point theorem, we know there is a unique solution $q \in L^p(\mathbb{C})$ when $p$ is appropriate. Let $w = P(\mu \bar{\sigma} + \sigma)$. We have $w(0) = 0$, $w_\sigma = T(\mu \bar{\sigma} + \sigma) = q$, and $w_\sigma = \mu \bar{\sigma} + \sigma$ by properties of operators $T$ and $P$ given in Lemma 3 in [1]. So $w_\sigma = \mu \bar{w_\sigma} + \sigma$, and $w$ satisfy our restriction. So $w$ is our solution. We can know that such $w$ is unique by estimating corresponding homogenous equation similar to that of Lemma 1 in [1]. From properties of operators $T$ and $P$ given in Lemma 3 in [1], we have estimates for $w^{\mu,\sigma}$:

\[
\| w^{\mu,\sigma} \|_{L^p} = \| q \|_{L^p} \leq c(p)\| \sigma \|_{L^p}, \quad (74)
\]

\[
| w^{\mu,\sigma}(z_1) - w^{\mu,\sigma}(z_2) | \leq c|z_1 - z_2|^{\alpha} \quad (75)
\]

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Here \( \alpha = 1 - \frac{2}{p} \). In fact, by the properties of \( P \), we have:
\[
|w^{\mu,\sigma}(z_1) - w^{\mu,\sigma}(z_2)| \leq c\|\mu\bar{q} + \sigma\|_{L^p}|z_1 - z_2|^{1 - \frac{2}{p}} \leq c' \|\sigma\|_{p}|z_1 - z_2|^{1 - \frac{2}{p}}. \tag{76}
\]

This mean our solution also have uniform Hölder norm.

**Step 2.** \( w^{\mu,\sigma} \) varies continuously in \( L^\infty \) and \( L^p \) as \( \mu, \sigma \) vary continuously. Let \( w = w^{\mu,\sigma} \), \( w' = w^\mu,\rho \). We have:
\[
(w - w')_z = \mu(w - w')_z + \lambda, \tag{77}
\]
where \( \lambda = (\mu - \nu)w^\nu_z + (\sigma - \rho) \). By the above results, we have estimate:
\[
\| (w - w')_z \|_{L^p} \leq c \|\lambda\|_{L^p} \leq c\|\mu - \nu\|_{L^\infty} + \|\sigma - \rho\|_{L^p}. \tag{78}
\]
Similarly, we also have estimates of Hölder norm for \( w - w' \).

**Step 3.** Suppose \( \mu \) is compactly supported. We want to have homeomorphism \( w^\mu : C \to \overline{C} \) satisfying: \( w^\mu_\mu = \mu w^\mu_z \), with normalization \( w^\mu(0) = 0 \), and \( w^\mu_z - 1 \in L^p(C) \). In fact, let \( w^\mu = z + w^{\mu,\mu}(z) \), with \( w^{\mu,\mu}(z) \) as in the above step, we have:
\[
w^\mu_\mu = (w^{\mu,\mu})_\mu = \mu(w^\mu_z) + \mu = \mu(w^\mu_z + 1) = \mu w^\mu_z. \tag{79}
\]
Clearly, \( w^\mu(0) = 0 \), and \( w^\mu_z + 1 = w^{\mu,\mu}_z \in L^p(C) \). From argument similar to Section 3.3 of [1], we know \( w^\mu \) is homeomorphism. So \( w^\mu \) is our solution. In this case, to have a solution fixing \( (0,1,\infty) \), we only need to divide \( w^\mu \) by \( w^\mu(1) \). We also have results similar to Lemma 15 in [1] that \( c(R)^{-1} \leq |w^\mu(1)| \leq c(R) \) when \( \mu \) has compact support in \( B_R \). We will also denote \( w^\mu/w^\mu(1) \) by \( w^\mu \) in the following.

**Step 4.** Let \( \alpha, \beta \) be the two coefficients with \( |\alpha|, |\beta| \leq k < 1 \), we give a decomposition formula:
\[
w^\alpha = w^\beta \circ w^\gamma, \tag{80}
\]
here \( \gamma = \frac{\alpha - \beta}{1 - \alpha \beta} \beta \left( \frac{w^\beta}{w^\beta - 1} \right)_z \circ w^\alpha \). Hence \( \|\gamma\|_{L^\infty} \leq C\|\alpha - \beta\|_{L^\infty} \). The proof is just simple calculation. Using sphere distance, we have the following estimates:
\[
d_s(z, w^\beta(z)) = d_s(z, w^\beta \circ w^\gamma(z)) \leq cd_s(w^\gamma(z), z)^\alpha. \tag{81}
\]
Decompose \( \gamma = \gamma_1 + \gamma_2 \), with \( \gamma_1 \) and \( \gamma_2 \) supported near 0 and \( \infty \) separately, we have:
\[
d_s(z, w^\gamma(z)) \leq d_s(w^\gamma(z), w^{\gamma_1}(z)) + d_s(w^{\gamma_1}(z), z). \tag{82}
\]
In the case \( \gamma \) having compact support, for \( |z| \leq R \) we have:
\[
d_s(z, w^\gamma(z)) \leq c(R)\|w^\gamma, \gamma\|_{L^\infty(B_R)} = c(R)\|w^\gamma(0) - w^{\gamma, \gamma}(0)\|_{L^\infty(B_R)} \leq C(R)\|\gamma\|_{L^\infty}. \tag{83}
\]
By arguments as Section 5.1 of [1], for $|z| \geq R$, we have $d_{S^2}(w^\gamma(z), z) \leq c(R)\|\gamma\|_{L^\infty}$.

Combining all the above together, we have:

$$d_{S^2}(w^\alpha(z), w^\beta(z)) \leq C(R)\|\alpha - \beta\|_{L^\infty}^{2\alpha}.$$  \hfill (84)

Here sphere distance is equivalent to the ordinary plane distance when restricted to a compact set on $\mathbb{C}$.

Step 5. Choose cutoff function $\eta$ supported in $B_{2R}$, with $\eta \equiv 1$ on $B_R$, $\eta \leq 1$. Then we have:

$$\left(\eta(w^\alpha - w^\beta)\right)_z = \alpha \left(\eta(w^\alpha - w^\beta)\right)_z + \lambda,$$  \hfill (85)

here $\lambda = \eta(\alpha - \beta)w^\beta_z + \eta\gamma((w^\alpha - w^\beta) - \alpha(w^\alpha - w^\beta))$. And $\|\lambda\|_{L^p} \leq C(R)\|\alpha - \beta\|_{L^\infty(B_{2R})} + C'(R)\|w^\alpha - w^\beta\|_{L^\infty(B_{2R})}$. So the results in the step 1 and step 4 give:

$$\|\left(w^\alpha - w^\beta\right)_z\|_{L^p(B_{2R})} \leq \|\left(\eta(w^\alpha - w^\beta)\right)_z\|_{L^p}$$
$$\leq C(R)\|\lambda\|_{L^p}$$
$$\leq C(R)\|\alpha - \beta\|_{L^\infty(B_{2R})} + C'(R)\|w^\alpha - w^\beta\|_{L^\infty(B_{2R})},$$  \hfill (86)

Here we abuse the use of notion, and if we change $w^\alpha$ to $h^\alpha$, and $z$ to $w$, we will get the result.

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**Proof:** (of Lemma 6.3) The first convergence 64 follows from the first claim. For the second convergence 65 since $h^{\mu_n} \to h^\mu$ in $L^\infty(S^2, S^2)$, $h^{\mu_n}(B_{2R})$ must be restrained in a uniform finite ball $B_{2R}$. As $\mu_n \to \mu$ in $C^1(\mathbb{C})$, we know $\mu_n(h^{\mu_n}(w)) \to \mu(h^\mu(w))$ in $L^\infty$ on any bounded balls $B_{2R}$ for fixed $R < \infty$. We know from the proof of the second claim that:

$$\|\left(h^{\mu_n} - h^\mu\right)_w\|_{L^p(B_{2R})} \leq C(R)\|\mu_n(h^{\mu_n}(w)) - \mu(h^\mu(w))\|_{L^\infty(B_{2R})}$$
$$+ C'(R)\|h^{\mu_n} - h^\mu\|_{L^\infty(B_{2R})}. $$  \hfill (87)

Since the sphere distance is equivalent to the plane distance on compact sets, the first convergence result shows that $\|h^{\mu_n} - h^\mu\|_{L^\infty(B_{2R})} \to 0$. So the second convergence result 65 holds.

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Department of Mathematics, Stanford University, building 380, Stanford, California 94305.
E-mail: xzhou08@math.stanford.edu