Cosmic acceleration and the change of the Hubble parameter

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A new model of accelerating expansion of the universe is presented. A universal vacuum de Sitter solution is obtained. A new explanation of the acceleration of the cosmic expansion is given. It is proved that the changing of the expansion from decelerating to accelerating is an intrinsic property of the universe without need of an exotic dark energy. The cosmological constant problem, the coincidence problem and the problem of phantom divide line crossing are naturally solved.

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I. Introduction

Very recently, there have been proposals for constructing generalizations of teleparallel gravity in Refs. [1], [2] and [3] which followed the spirit of $f(R)$ gravity (see [4] for a review) as a generalization of general relativity. That is, the Lagrangians of the theories were generalized to the form $f(T)$, where $f$ is some suitably differentiable function and $T$ is the Lagrangian of teleparallel gravity [5]. The interest in these theories was aroused by the claim that their dynamics differ from those of general relativity but their equations are still second order in derivatives and, therefore, they might be able to account for the accelerated expansion of the universe and remain free of pathologies. It has been shown, however, that this last expectation was unfounded: these theories are not locally Lorentz invariant and appear to harbour extra degrees of freedom [6]. Even if one decides to give up teleparallelism, such actions would not make sense as descriptions of the dynamics of gravity if local Lorentz symmetry was restored [7]. Based on the definition given in these theories, the Weitzenbock connection, the torsion tensor and the contortion tensor are not local Lorentz scalars (i.e. they do not remain invariant under a local Lorentz transformation in tangent space). This is the root of the lack of Lorentz invariance in generalized teleparallel theories of gravity. The $f(T)$ models behave very differently from the $\Lambda$CDM model on large scales, and are, therefore, very unlikely to be suitable alternatives to it [8]. Since this theory is not invariant under local Lorentz transformations, the choice of tetrad plays a crucial role in such models. Different tetrads will lead to different field equations which in turn have different solutions [9].

It is known that the Lagrangian $T$ of teleparallel gravity only differs from the Riemann curvature scalar $R$ by a boundary term [10]. Recall that in general relativity the Einstein field equation can be derived from a reduced Lagrangian $\Gamma = \{\sigma^\mu\nu\} \{\sigma^\nu\rho\} - \{\nu^\rho\} \{\sigma^\mu\nu\} g^{\rho\sigma}$ which only differs from the Riemann curvature scalar $R$ of the Christoffel symbol $\{\sigma^\mu\nu\}$ by a boundary term $\nabla_\mu w^\mu$ with $w^\mu = g^{\rho\lambda} \{\nu^\rho\} \{\nu^\lambda\} - g^{\rho\nu} \{\nu^\lambda\}$ [10]. Following the spirit of $f(R)$ gravity we can propose a modified theory of gravity called reduced $f(R)$ gravity or $f(\Gamma)$ gravity given by

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a Lagrangian which is a function of $\Gamma$: $L_G = f(\Gamma)$. It will be seen in this letter that this theory is identified as a metrical formulation of $f(T)$ gravity. In contrast with $f(T)$ gravity, the new theory, $f(\Gamma)$ gravity respects local Lorentz symmetry and harbour no extra degrees of freedom, since the dynamical variable is the metric instead of the tetrad. At the same time it has the advantage over $f(R)$ gravity that its field equations are second-order instead of fourth-order and then is free of pathologies. The disadvantage of $f(\Gamma)$ gravity is that its field equations do not respect general covariance. The significant pay-off that will be suggested in this letter is that $f(\Gamma)$ gravity may provide an satisfactory alternative to conventional dark energy in general relativistic cosmology and a new explanation of the acceleration of the cosmic expansion. The changing of the expansion from decelerating to accelerating is an intrinsic property of the universe without need of an exotic dark energy. The cosmological constant problem, the coincidence problem and the problem of phantom divide line crossing are naturally solved.

II. Field equations

We start from the reduced action

$$S = \frac{1}{2\kappa^2} \int d\Omega \left( \sqrt{-g} f(\Gamma) + L_m \right),$$

where $\kappa^2 = 8\pi G_N$ with the bare gravitational constant $G_N$,

$$\Gamma = \{\sigma_{\mu \nu}\} \{\nu_{\mu} - \{\mu_{\nu}\} \{\sigma_{\mu}\}\} g^{\rho \sigma},$$

is the reduced gravitational Lagrangian of general relativity \[10\] with the Christoffel symbol $\{\nu_{\mu \sigma}\}$. $\Gamma$ is a quadratic form of the gravitational strength $\{\nu_{\mu \sigma}\}$ and identified with the kinetic energy of the gravitational potential $g_{\mu \nu}$. The variational principle yields the field equations for the metric $g_{\mu \nu}$:

$$f_{\Gamma}\left(R_{\rho \sigma} - \frac{1}{2} R g_{\rho \sigma}\right) + \frac{1}{2} g_{\rho \sigma}\left(f_{\Gamma}\Gamma - f(\Gamma)\right) - f_{\Gamma\Gamma\Gamma,\mu} \frac{\partial \Gamma}{\partial g_{\rho \sigma},\mu} = \kappa^2 T_{\rho \sigma},$$

where

$$f_{\Gamma} = \frac{\partial f}{\partial \Gamma}, f_{\Gamma\Gamma} = \frac{\partial^2 f}{\partial \Gamma^2},$$

$$R_{\rho \sigma}$$ is the Ricci curvature tensor of $\{\nu_{\mu \sigma}\}$ and

$$T_{\rho \sigma} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\rho \sigma}},$$

is the energy-momentum of the matter fields. These equations can be re-arranged in the Einstein-like form

$$\left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}\right) = \frac{1}{2 f_{\Gamma}} (f - f_{\Gamma}) g_{\mu \nu} + \frac{f_{\Gamma\Gamma}}{f_{\Gamma}} \Gamma,_{\lambda} \frac{\partial \Gamma}{\partial g^{\mu \nu},\lambda} + \kappa^2 T_{\mu \nu}.$$

III. Cosmological model

We now investigate the cosmological dynamics for the models based on $f(\Gamma)$ gravity. In order to derive conditions for the cosmological viability of $f(\Gamma)$ models we shall carry out a general analysis without specifying the form of $f(\Gamma)$ at first. We consider a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background with the metric

$$g_{\mu \nu} = \text{diag} \left(-1, a(t)^2, a(t)^2, a(t)^2\right),$$

$$a(t) = \int d\xi \sqrt{1 - \frac{\kappa^2}{4\pi}}.$$
where $a(t)$ is a scale factor. The non-vanishing components of the Christoffel symbol are

\begin{align*}
\{0^0_0\} &= 0, \{0^0_i\} = \{i^0_0\} = 0, \{0^i_j\} = a \dot{a} \delta_{ij}, \\
\{0^i_0\} &= 0, \{i^0_0\} = \{i^0_j\} = H \delta^2_{ij}, \{i^i_k\} = 0, i, j, k, \ldots = 1, 2, 3.
\end{align*}

and

\[ \Gamma = (\{\sigma^\rho_{\mu \nu}\} \{\rho^\nu_{\mu \sigma}\} - \{\mu^\nu_{\rho \nu}\} \{\sigma^\rho_{\mu \nu}\}) g^{\rho \sigma} = -6H^2. \]

Here $H \equiv \dot{a} / a$ is the Hubble parameter and a dot represents a derivative with respect to the cosmic time $t$. One can find that $\Gamma$ is identified with the gravitational Lagrangian of the cosmological model of $f(T)$ gravity given by Bengochea and Ferraro [1]. The the field equations (6) take the forms

\begin{align*}
3H^2 &= -\frac{1}{2f}\left(f + 6H^2 f_{\Gamma}\right) - 18H^2 \frac{\dot{f}_{\Gamma}}{f_{\Gamma}} + \frac{\kappa^2}{f_{\Gamma}} \rho, \\
-2 \dot{H} - 3H^2 &= \frac{1}{2f_{\Gamma}} \left(f + 6H^2 f_{\Gamma}\right) - 18H^2 \frac{\dot{f}_{\Gamma}}{f_{\Gamma}} + \frac{\kappa^2}{f_{\Gamma}} p,
\end{align*}

which lead to

\[ \dot{H} = -\frac{\kappa^2}{f_{\Gamma}} \left(\rho + p - \frac{36}{\kappa^2} H^2 \frac{\dot{f}_{\Gamma}}{f_{\Gamma}} \right). \]

It is easy to see that, in $f(\Gamma)$ gravity, the gravitational constant $\kappa^2$ is replaced by an effective (time dependent) $\kappa^2_{\text{eff}} = \kappa^2/\dot{f}_{\Gamma}(\Gamma)$. On the other hand, it is reasonable to assume that the present day value of $\kappa^2_{\text{eff}}$ is the same as the $\kappa^2$ so that we get the simple constraint:

\[ \kappa^2_{\text{eff}}(z = 0) = \kappa^2 \rightarrow f_{\Gamma}(\Gamma_0) = 1, \]

where $z$ is the redshift.

If $f(\Gamma)$ has the form

\[ f(\Gamma) = \Gamma + \Phi(\Gamma), \]

then the equations (10), (11) and (12) become

\begin{align*}
3H^2 &= \kappa^2 (\rho + \rho_{de}), \\
-2 \dot{H} - 3H^2 &= \kappa^2 (p + p_{de}),
\end{align*}

and

\[ \frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} ((\rho + \rho_{de}) + 3 (p + p_{de})), \]

where

\[ \rho_{de} = -\frac{1}{2\kappa^2} \rho - \frac{6}{\kappa^2} H^2 \Phi_{\Gamma} - \frac{18}{\kappa^2} H^2 \frac{\dot{f}_{\Gamma}}{f_{\Gamma}} \Phi_{\Gamma}, \]
and

$$p_{de} = \frac{1}{2\kappa^2} \dot{\Phi} + \frac{6}{\kappa^2} H^2 \Phi + \frac{2}{\kappa^2} \dot{H} \left( \Phi_{\Gamma} - 9H^2 \Phi_{\Gamma\Gamma} \right),$$  \quad (19)

are the density and the pressure of the "dark energy" $\Phi (\Gamma)$, respectively. The state equation of the "dark energy" is then

$$w_{de} = \frac{p_{de}}{\rho_{de}} = -1 - \frac{2}{\frac{1}{2} \dot{H} \left( \Phi_{\Gamma} - 18H^2 \Phi_{\Gamma\Gamma} \right)} \frac{\frac{1}{2} \dot{\Phi} + 6H^2 \Phi_{\Gamma} + 18H^2 H \Phi_{\Gamma\Gamma}}{\frac{1}{2} \dot{\Phi} + 6H^2 \Phi_{\Gamma} + 18H^2 H \Phi_{\Gamma\Gamma}}. \quad (20)$$

Let $N = \ln a$. For any function $\Phi(a)$ we have $\Phi' = \frac{d\Phi}{dN} = H \dot{\Phi}$. The the equation (20) becomes

$$w_{de} = -1 + \frac{1}{3} \frac{\Phi'_{\Gamma} \Phi_{\Gamma} + 3\Phi_{\Gamma\Gamma}}{\Phi_{\Gamma} / \Gamma - 2\Phi_{\Gamma} + \frac{1}{2} \Phi'_{\Gamma\Gamma}}. \quad (21)$$

The equations (15) and (16) become

$$H^2 = \frac{\kappa^2}{3} \rho - \frac{1}{6} \Phi + \frac{1}{3} \Gamma \Phi_{\Gamma} + \Gamma H \Phi_{\Gamma\Gamma}, \quad (22)$$

$$\left(H^2\right)' = -2\kappa^2 \rho + \Gamma + \Phi - 2\Gamma \Phi_{\Gamma} \frac{3 \Gamma \Phi_{\Gamma\Gamma} + 2 \Phi_{\Gamma} + 2}{3 \Gamma \Phi_{\Gamma\Gamma} + 2 \Phi_{\Gamma} + 2}. \quad (23)$$

We see that a constant $\Phi$ acts just like a cosmological constant (dark energy), and $\Phi$ linear in $\Gamma$ (i.e. $\Phi_{\Gamma} =$constant) is simply a redefinition of gravitational constant $\kappa^2$. The the equation (23) can be written as

$$\frac{1}{6} \left( 1 + \Phi_{\Gamma} - \frac{3}{2} \Gamma \Phi_{\Gamma\Gamma} \right) \Gamma' = -\frac{1}{2} \Gamma + \frac{1}{2} \Phi - \Gamma \Phi_{\Gamma} + \kappa^2 \rho. \quad (24)$$

Taking a universe with only dust matter so

$$p = 0, \quad (25)$$

we have

$$-\frac{1 + \Phi_{\Gamma} - \frac{3}{2} \Gamma \Phi_{\Gamma\Gamma}}{3 \Gamma (1 - \Phi / \Gamma + 2 \Phi_{\Gamma})} d\Gamma = dN, \quad (26)$$

and then we find the solution $\Gamma(a)$ in closed form:

$$a(\Gamma) = \exp \left\{ \frac{1}{3} \int_{-6H_0^2}^{\Gamma} \frac{dx}{x} \left( 1 + \Phi_x (x) - \frac{3}{2} x \Phi_{xx} (x) \right) \right\}. \quad (27)$$

The equations (22), (23) and (27) take the same forms as the ones for $f(T)$ gravity given by Linder [2]. In a sense our model can be considered as a metric formulation of the $f(T)$ model.

The equations (22) and (23) i.e.

$$3H^2 = -\frac{1}{2} \Phi - 6H^2 \Phi_{\Gamma} - 18H^2 H \Phi_{\Gamma\Gamma} + \kappa^2 \rho. \quad (28)$$

$$-2 \dot{H} - 3H^2 = \frac{1}{2} \Phi + 6H^2 \Phi_{\Gamma} + 2 \dot{H} \left( \Phi_{\Gamma} - 9H^2 \Phi_{\Gamma\Gamma} \right) + \kappa^2 \rho. \quad (29)$$

yield

$$\dot{H} = -\frac{\kappa^2}{2(1 + \Phi_{\Gamma} - 18H^2 \Phi_{\Gamma\Gamma})} (\rho + p). \quad (30)$$
So, \( \frac{1}{1 + \Phi - 9H^2\Phi} \) is simply a redefinition of gravitational constant \( \kappa^2 \).

In the vacuum (30) gives a **universal** de Sitter solution

\[
H = 0,
\]

for *any function* \( \Phi (\Gamma) \). Then (20) gives

\[
w_{de} = -1,
\]

which means that the function \( \Phi (\Gamma) \) plays the role of the cosmological constant or dark energy.

The equation (29) gives

\[
\dot{H} = - \frac{\kappa^2 p + 3H^2 + \frac{1}{2} \Phi + 6H^2\Phi}{2 (1 + \Phi - 9H^2\Phi)},
\]

Substituting (33) into (20) yields

\[
w_{de} = - \frac{\Phi + 6H^2\Phi + 54H^4\Phi - 2\kappa^2 (9H^2\Phi - \Phi) p}{(1 + \Phi - 12H^2\Phi - 18H^2\Phi + 3H^2 + \Phi) - 18H^2\Phi^{2}\kappa^2 p}.
\]

In the case

\[
\Phi (\Gamma) = \alpha (-\Gamma)^n = \alpha 6^n H^{2n},
\]

(33) and (34) become

\[
\dot{H} = - \frac{\kappa^2 p + 3H^2 + \frac{1}{2} \alpha 6^n H^{2n}}{2 (1 + 3n (3n - 5) \alpha 6^n - 2 H^{2n - 2})},
\]

and

\[
w_{de} = - \frac{3 (3n - 2) (n - 1) H^2 + n (3n - 1) \kappa^2 p}{-3 (n + 1) (3n - 2) H^2 + 6^n n (2n - 1) (3n - 2) \alpha H^{2n} - 3n (n - 1) \kappa^2 p},
\]

For dust matter

\[p = 0,\]

we have

\[
\dot{H} = - \frac{3H^2 + \frac{1}{2} (n - 3) \alpha 6^n H^{2n}}{2 (1 + 3n (3n - 5) \alpha 6^n - 2 H^{2n - 2})},
\]

\[
w_{de} = - \frac{3 (3n - 2) (n - 1) H^2}{-3 (n + 1) (3n - 2) H^2 + 6^n n (2n - 1) (3n - 2) \alpha H^{2n}}.
\]

When

\[n = 2,\]

i.e. for the gravitational Lagrangian

\[
L_g = \sqrt{-g} (\Gamma + \alpha \Gamma^2),
\]
(39) gives

\[ w_{\text{de}} = -\frac{1}{3 (24\alpha H^2 - 1)}. \]  

(41)

Letting

\[ w_{\text{de}} = -1 \]

and

\[ H_0 = 74 \text{ km/sec/Mpc} \approx 2.4 \times 10^{-18} \text{ sec}^{-1} \]

one can compute

\[ \alpha = 1.0145 \times 10^{-5} (\text{km/sec/Mpc})^{-2} = 9.6451 \times 10^{33} \text{ sec}^2. \]  

(42)

Then when

\[ w_{\text{de}} = -\frac{1}{3}, \]

(43)

we obtain

\[ H = 90.631 \text{ km/sec/Mpc}. \]  

(44)

Observations of type Ia supernovae at moderately large redshifts \((z \sim 0.5 \text{ to } 1)\) have led to the conclusion that the Hubble expansion of the universe is accelerating [11]. This is consistent also with microwave background measurements [12]. According to the result of [13], \(H = 90.631 \text{ km/sec/Mpc}\) corresponds to

\[ z \sim 0.88, \]

which is consistent with the observations.

The equation (38) now becomes

\[ \frac{\dot{H}}{H} = \frac{3H^2 (18\alpha H^2 - 1)}{2 (6\alpha H^2 + 1)}. \]  

(45)

Using the formula

\[ H = \frac{\dot{a}}{a} = -\frac{1}{1+z} \frac{\dot{z}}{z}, \]

(46)

(45) can be rewritten as

\[ -\frac{2 (6\alpha H^2 + 1) \frac{dH}{H}}{3H (18\alpha H^2 - 1)} = \frac{dz}{(1+z)}, \]

(47)

and integrated

\[ z = \left( \frac{H^2}{H_0^2} \right)^{1/3} \left( \frac{18\alpha H_0^2 - 1}{18\alpha H^2 - 1} \right)^{4/9} - 1. \]  

(48)

This function is illustrated in Fig. 1, which is roughly consistent with the observations [13].
The equation (15) indicates that during the evolution of the universe $H^2$ decreases owing to decreasing of the matter density $\rho$. This makes $w_{de}$ descend during the evolution of the universe. The evolution of the function $w_{de} = w_{de}(\alpha H^2)$ given by (41) is illustrated in Fig. 2. According to (41) when $H^2 = \frac{1}{12\alpha}$, $w_{de} = -\frac{1}{3}$, $\Phi(\Gamma)$ changes from "visible" to dark as indicated by (17). If $H^2 > \frac{1}{12\alpha}$, it decelerates the expansion, if $H^2 < \frac{1}{12\alpha}$, it accelerates the expansion. When $H^2 = \frac{1}{18\alpha}$, $w_{de}$ crosses the phantom divide line $-1$. In other words, the expansion of the universe naturally includes a decelerating and an accelerating phase. $\alpha$ given by (42) can be seen as a new constant describing the evolution of the universe.

![FIG. 1: The function of $z = z(H)$.](image1)

![FIG. 2: The evolution of $w_{de}$.](image2)

[1] G. R. Bengochea and R. Ferraro, Phys. Rev. D79, 124019 (2009);
[2] E. V. Linder, Phys. Rev. D81, 127301 (2010);
[3] P. Wu and H. Yu, Phys. Lett. B 692, 176 (2010); P. Wu and H. Yu, Phys. Lett. B 693, 415 (2010); P. Wu and H. Yu, Eur. Phys. J. C 71, 1552 (2011); K. Bamba, C. Q. Geng and C. C. Lee, JCAP, 08, 021 (2010); S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis (2010), Phys. Rev. D 83, 023508 (2011); G. R. Bengochea, Phys. Lett. B 695, 405 (2011); R. J. Yang, Europhys. Lett. 93, 60001 (2011); R. Zheng and Q. -G. Huang (2010), J. Cosmol. Astropart. Phys. 03 (2011) 002; K. Bamba, C. Q. Geng, C. C. Lee and L. -W. Luo, J. Cosmol. Astropart. Phys. 01 (2011) 021.
[4] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010)
[5] R. Aldrovandi and J. G. Pereira, An Introduction to Teleparallel Gravity, Instituto de Fisica Teorica, UNSEP, Sao Paulo (http://www.ift.unesp.br/gcg/tele.pdf).
[6] B. Li, T. P. Sotiriou and J. D. Barrow, Phys. Rev. D 83, 064035 (2011)
[7] T. P. Sotiriou, B. Li and J. D. Barrow, Phys. Rev. D 83, 104030 (2011).
[8] B. Li, T. P. Sotiriou, and J. D. Barrow, Phys. Rev. D 83, 104017 (2011).
[9] N. Tamanini and C. G. Boehmer Phys. Rev. D 86: 044009 (2012).
[10] L. D. Landau and E. M. Lifshitz, The classical theory of fields, Addison-Wesley, Reading, Massachusetts, and Pergamon, London, 1971.
[11] A. G. Riess et. al., Astron. J. 116, 1009 (1998); Astron. J. 607 665 (2004); S. Perlmutter et. al., Astrophys. J. 517 565 (1999); J. L. Tonry et. al., Astrophys. J. 594, 1 (2003).
[12] C. L. Bennett et. al, WMAP team, Astrophys. J. 583, 1 (2003); e-print arXiv: 1001.4758 (2010).
[13] J. Simon et al, 2005 Phys. Rev. D 71, 123001. astro-ph/0412269; A. G. Riess et al., Astrophys.J. 699, 539 (2009) arXiv:0905.0605; D. Stern, R. Jimenez, L. Verde, M. Kamionkowski and S. A. Stanford, [arXiv:astro-ph/0907.3149].