More About Trigonometric Series and Integration

1 Introduction

In a recent interesting paper by Gluchoff, [G], the development of the theory of integration was related to the needs of trigonometric series. The paper ended with the introduction of the Lebesgue integral at the beginning of the century. As the Lebesgue integral is the integral of everyday mathematics this was a natural place to stop; the needs of trigonometric series had produced a tool of first-rate importance for the whole of mathematical analysis.

However there is a question left over from [G]; it is the problem of determining the coefficients of an everywhere convergent trigonometric series in terms of its sum, the so called coefficient problem for convergent trigonometric series, that we will call shortly, the coefficient problem. Although the need for further developments in the theory of integration had disappeared, one of the basic problems of trigonometric series was not solved by the Lebesgue integral. As a result, right up to the present day further integrals, usually called trigonometric integrals, have been introduced in order to solve the coefficient problem. These integrals are more general than the Lebesgue integral.

Although such integrals have not found other uses it might be of some interest to continue the story started in [G] by describing some of this work.

As this article continues the discussion started in [G] reference will be made to the results and references there.

A knowledge of integrals more general than that of Lebesgue is important; see for instance [Bu2; Go; Ho; L; P]. The first reference was written to put this information, some of which is rather technical, in an easily available form. It is almost essential reading for the full understanding of the present topic, and use will be made of the results and references quoted there.

Finally certain terms that are defined in the the Appendix, Section 11, will be written as smooth.

2 The Coefficient Problem

The coefficient problem for convergent trigonometric series, or just the coefficient problem, is the following.

If for all $x, -\pi \leq x \leq \pi$, the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx,$$



(T)
converges, to \( f(x) \) say, then this representation of the function \( f \) is unique. This implies that the coefficients of the series, \( a_0, a_1, \ldots, b_1, \ldots \), are completely determined by the sum function, \( f \).

Calculate these coefficients from this function.

The fact that only one series of the form (T) can converge to the function \( f \) is a famous result due to Cantor; see [G I, IV], where references are given for proofs.

As explained in [G] if the series (T) converges uniformly then an argument that goes back to Fourier, and which is included in all first courses on Fourier analysis, shows that

\[
    a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad k = 0, 1, \ldots, \tag{F}
\]

\[
    b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k = 1, \ldots;
\]

see [G(I)].

In fact the argument that gives (F) only requires that the integrals in (F) have a meaning, and term-by-term integration of the series (T) is justified; that is the integral of \( f \) is obtained by integrating the series (T); see [G, pp.7-8]. In the case of uniform convergence of a series of continuous functions it is well known that a series can be integrated term-by-term.

Uniform convergence of the series (T) implies that the sum \( f \) is continuous and so the integrands in (F) are Riemann, or \( \mathcal{R} \)-, integrable. It is then natural to ask two questions.

(I) Do the formulæ (F) hold under the assumption that the sum \( f \) of the series (T) is \( \mathcal{R} \)-integrable?

(II) Is any \( f \) that is the sum of a series (T) \( \mathcal{R} \)-integrable?

The answer to (I) is yes: if the series (T) converges everywhere to an \( \mathcal{R} \)-integrable function then the coefficients of (T) are given by (F). This result is due to Du Bois-Reymond,[Ho, vol.II, p.659].

The answer to (II) is no: this was pointed out in [G IV, p.18]. Lebesgue has shown that the sum a function of a series (T) need not be \( \mathcal{R} \)-integrable; [G, ref.15].

3 Why The Lebesgue Integral Will Not Do

The next step is to repeat the above questions (I) and (II) with the Lebesgue, or \( \mathcal{L} \)-, integral instead of the \( \mathcal{R} \)-integral.

Again the answer to the question (I) is is yes; if the series (T) converges everywhere to a \( \mathcal{L} \)-integrable function then the coefficients of (T) are given by (F), where now the integrals are of course Lebesgue integrals. This is another result of Lebesgue; see [G IV, pp.16-17; G, ref.16].
Unfortunately the answer to the question (II) is again no. We will show why this is so, and at the same time see just how bad sum functions of trigonometric series can be.

A natural candidate for the integral of \( f \) is the sum of series you get on formally integrating (T) term-by-term:

\[
\Phi(x) = \frac{a_0}{2}x + \sum_{k=1}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{k}.
\]  

(1)

This series is seemingly better behaved than the original series (T), since the coefficients are essentially those of (T) divided by the terms of the sequence \( \{1/n, n = 1, 2, \ldots\} \). However this is is not true in general.

Consider the following example due to Fatou, where \( a_k = 0, k = 0,1,2,\ldots \) and \( b_k = 1/\log(k+1), k = 1,2,\ldots \). Then (T) and (1) become

\[
\sum_{k=1}^{\infty} \frac{\sin kx}{\log(k+1)},
\]

(2)

\[
\Xi(x) = -\sum_{k=1}^{\infty} \frac{\cos kx}{k \log(k+1)},
\]

(3)

respectively.

A simple application of the Dirichlet convergence test, \([K, pp.315, 316 Examples and Applications 5]\), shows that the series (2) converges for all \( x \), to \( \xi(x) \) say.

However the once integrated series in (3) does not converge when \( x = 0 \). In other words \( \Xi \), the “integral” of \( \xi \), is not defined at the origin. In particular it is not continuous there.

However all standard indefinite integrals are defined everywhere, and are continuous. So we must face the possibility that \( \xi \) is not integrable in any standard sense. In particular it cannot be \( L \)-integrable.

Of course the above argument is only heuristic, as (T) cannot necessarily be integrated term-by-term. However it can be made precise, and we will give a sketch of the proof; for details see \([D, vol.I, pp.42-43]\).

The series in (2), which as we saw is convergent for all \( x \), is uniformly convergent on any \([2k\pi + \epsilon, (2k+1)\pi - \epsilon]\), where \( 0 < \epsilon < \pi \), and \( k = 0, \pm 1, \pm 2, \ldots \); see \([K, pp. 347, 349 Examples and Applications 4]\).

The series obtained by the formally integrating (2), integrating term-by-term, is (3), which by a use of the Dirichlet test diverges for all \( x = 2k\pi, k = 0, \pm 1, \pm 2, \ldots \). Since (2) converges uniformly on any \([\alpha, \beta]\) that does not contain one of these points, that
is, on any \([\alpha, \beta]\) that lies in some \([2k\pi + \epsilon, (2k + 1)\pi - \epsilon]\), we get using term-by-term integration that

\[
\Xi(\beta) - \Xi(\alpha) = \int_{\alpha}^{\beta} \xi,
\]

where the integral is taken in the Riemann sense.

Suppose \(\xi\) were \(\mathcal{R}\)-integrable and that \(0 < \alpha < \beta < \pi\), then

\[
\int_{0}^{\beta} \xi = \lim_{\alpha \to 0} \int_{\alpha}^{\beta} \xi = \Xi(\beta) - \lim_{\alpha \to 0} \Xi(\alpha).
\]

However it can be shown that for some function \(m(x)\), with \(xm(x)\) bounded away from zero,

\[
\lim_{x \to 0} \left( \Xi(x) + \sum_{k=1}^{m(x)} \frac{1}{k \log(k+1)} \right) = 0.
\]

So, noting again that \(\sum_{k=1}^{\infty} \frac{1}{k \log(k+1)}\) diverges, \(\lim_{\alpha \to 0} \Xi(\alpha)\) does not exist.

Hence \(\xi\) is not \(\mathcal{R}\)-integrable.

It cannot be \(\mathcal{L}\)-integrable either, since the above argument could be repeated using the Lebesgue integral.

More generally \(\xi\) cannot be integrable in any sense that extends the \(\mathcal{R}\)-integral, and has a continuous indefinite integral; see, for instance, the \(\mathcal{D}^*\)-integral in the next section.

The above argument applies to any series

\[
\sum_{k=1}^{\infty} b_k \sin kx,
\]

subject to the conditions

\[
b_1 > b_2 > \cdots; \quad \lim_{k \to \infty} b_k = 0; \quad \sum_{k=1}^{\infty} \frac{b_k}{k} = \infty;
\]

as the references to \([K]\) show; see \([D, \text{vol.I, pp. 42–43}; Z, \text{vol.I, pp.185–186}].

4 Why No Classical Integral Will Do

There are various extensions of the Lebesgue integral and simplest of these extensions goes under various names that indicate the method of approach; the narrow Denjoy, or \(\mathcal{D}^*\)-, integral, the Perron, or \(\mathcal{P}\)-, integral and the Henstock-Kurzweil, or \(\mathcal{HK}\)-, integral; when we do not care about the actual form we will refer to this
integral as the $D^*$-integral. This integral extends the $L$-integral, in that if a function is $L$-integrable then it is $D^*$-integrable and the integrals are equal. In addition if a function is bounded, or even just bounded below, and is $D^*$-integrable then it is $L$-integrable. Finally the $D^*$-integral, unlike the $L$-integral, is not an absolute integral, it is a non-absolute integral; that is if $f$ is $D^*$-integrable it does not follow that $|f|$ is. Finally the $D^*$-integral solves the classical primitive problem, see Section 5 and [Bu2], and the indefinite $D^*$-integral, the $D^*$-primitive, is continuous. The various approaches to this integral are described in [B2] and these techniques will be used to define various trigonometric integrals.

From the last remarks, and the comments in the previous section, the sum of (2) is not $D^*$-integrable. So this integral, that is more general than the $L$-integral, does not solve the coefficient problem either; that is the answer to question (II) for of Section 2 for this integral is again no.

However if the integral in the question (I) of Section 2 is taken to be this more general $D^*$-integral the answer is yes; that is if the sum of (T) is $D^*$-integrable, the coefficients are given by (F), where the integral is if course the $D^*$-integral; this result is due to Nalli, see [N; Z, vol.II, pp. 83–86].

In [Bu2, 6.3] a definition of an integral that is slightly more general than the $D^*$-integral was given, the $DH$-integral. In this case the primitive is differentiable almost everywhere and is $ACG$, while the $D^*$-primitive is differentiable almost everywhere and is $ACG^*$; these terms are defined in [Bu2, 6.3].

Sklyarenko, [S2; Th3, p.401], has given an example of a $DH$-integrable function, $f$ say, that is the sum of an everywhere convergent series (T), but for which the coefficients are not given by (F), using the $DH$-integral. In particular

$$a_0 \neq \frac{1}{\pi} DH\int_{-\pi}^{\pi} f.$$

So with this slight extension of the $D^*$-integral definition, the answer to question (I) becomes no. Of course the answer to question (II) is also no, as the $DH$-primitive is continuous.

However, it is correct to say that $\Phi$, the sum of the series in (1) and often called the Lebesgue function of (T), is the natural primitive of $f$, the sum of the series (T).

This suggests that in order to have (F) hold in general a rather strange integral will be needed; one whose primitive need not be continuous, or may even fail to be defined at some points.

The possibility of having an integration process with a primitive that is not continuous was first considered by W.H.Young, [Y]. It would appear that his discussion

---

1 Denjoy-Hinčin; Hinčin, Хинчин, is also transliterated as Khintchine; as a result this integral is sometimes called the DK-integral.
was motivated by theoretical considerations only, but it is not completely impossi-
ble, given the important contributions that Young made to the theory trigonometric
series, that he was thinking of the need for such integrals in the coefficient problem.

5 An Approach To The Coefficient Problem
As the above discussion shows the coefficient problem can be solved if an extra
assumption is made—uniform convergence of the series (T), the $\mathcal{R}$-, $\mathcal{L}$-, or $\mathcal{D}^*$-
integrability of the sum, $f$, of the series (T). Such an assumption enables the problem
to be solved but is unnecessary, since, as we have seen, the series (T) when it
converges need not be uniformly convergent, and its sum function need not be
integrable in any of these senses..

What is needed is an integral, call it using a conceit due to Hardy, [H, p.7], the
Pickwickian integral, or shortly the $\Pi$-integral, for which the answers to both of
the questions (I) and (II) will be yes. That is:

(A) all sum functions of everywhere convergent series (T) will be $\Pi$-integrable;
(B) the coefficients of the series (T) are given by (F), where the integrals are $\Pi$-
integrals.

Given a function $f$ for which the integrals in (F) exist in some sense, the series in
(T) can be written down. It is called the Fourier series of $f$, whether the series
converges to $f$ or not. The discussion of two questions (I), and (II) above shows
that if the series in (T) converges to a $\mathcal{R}$-, $\mathcal{L}$-, or $\mathcal{D}^*$-integrable function $f$ then
this series is the Fourier series of $f$, more precisely it is a Fourier-Riemann series, a
Fourier-Lebesgue series, or a Fourier-$\mathcal{D}^*$ series of $f$. The series (T) of the example
above of Sklyarenko is not a Fourier-$\mathcal{D}^*$ series; the sum function is $\mathcal{D}^*$-
integrable but the coefficients calculated by (F) using this integral are not the coefficients of
(T).

If we can define the $\Pi$-integral having the properties (A) and (B), then (T) is the
Fourier-Pickwickian series of its sum, and the coefficient problem will have been
solved.

As we have noted the Pickwickian integral will be unusual as its primitive will not
be continuous, may even fail to exist at some points. The example of Fatou given
above that has one bad point can, by the usual classical technique of condensation
of singularities, [Ho, vol.II, pp. 389–399], be generalized to having an uncountable
zero measure set of discontinuities, or where it is not defined; [Bu2; D, vol.I, pp. 42;
vol.IVb, pp.497–503].

However it cannot get worse than this. Consider the following facts.
If (T) converges everywhere then $a_k, b_k \to 0$, see [G IV, p.18], and so by a simple
application of the comparison test $\sum (a_k^2 + b_k^2)/k^2$ converges. Hence by the Riesz-
Fischer Theorem, see [Z, vol. I, pp.127, 321], if $\Phi$ is given by (1) then $\Phi^2$ is $\mathcal{L}$-
integrable; and this implies $\Phi$ is finite almost everywhere; equivalently the series (1) converges almost everywhere, that is $\Phi$ can only fail to be defined on a set of measure zero. Although the above remarks concentrate on the pathology of $\Phi$ they also indicate a way to approach our problem.

Suppose that we could obtain $\Phi$ from $f$, only knowing $f$ to be the sum of (T). Then by the above remarks $\Phi$ is $\mathcal{L}$-integrable, and the series in (1) is its Fourier-$\mathcal{L}$ series. Hence the coefficients in (1) are given by (F), with $\Phi$ replacing $f$, the integrals being in the Lebesgue sense of course. A comparison of (T) and (1) shows that their coefficients are simply related, so having the coefficients in (1) we have those of (T). Since $\Phi$ can be considered as an integral of $f$, we can also consider $f$ as a derivative of $\Phi$, in the Pickwickian sense say: $D_{P}i \Phi = f$. Then the problem of obtaining $\Phi$ from $f$ becomes the problem of (a) defining what we mean by the $P_i$-derivative, (b) showing that $D_{P}i \Phi = f$ and finally, (c) obtaining a method for inverting the $P_i$-derivative.

In other words the coefficient problem can be viewed as a generalization of the classical primitive problem.

The classical primitive problem is as follows.

If on $[a, b]$, $G' = g, G(a) = 0$ find $G$, or if $G' = g$, calculate $G(b) - G(a)$, from $g$.

A full discussion of this problem, together with the various solutions that have been given can be found in several places; see in particular [Bu2] where there are further references. The various solutions, $\mathcal{D}^*$- integral, the Perron, or $P$, integral and the Henstock-Kurzweil, or $\mathcal{H}K$, integral are all equivalent. The names indicate different approaches to the definition of the integral. As all approaches—totalization for the $\mathcal{D}^*$- integral, [Bu2, 6.1], major and minor functions for the $P$- integral, [Bu2, 6.2], and Riemann sums for the $\mathcal{H}K$, integral, [Bu2, 6.4], — are used in the coefficient problem it is important to know about these methods. In addition two very important classes of functions are introduced when discussing these integrals, the classes of continuous $ACG$ and continuous $ACG^*$ functions; [Bu2, 6.3].

The various solutions of the coefficient problem that have been proposed are obtained by replacing the ordinary derivative of the classical primitive problem by various Pickwickian derivatives and then solving the primitive problem for these derivatives.

It is clear that if we have inverted the Pickwickian derivative with a Pickwickian integral, that is

$$D_{P}i G = g \iff G = Pi \int g;$$

and if we also have, for $f$ the sum of (T), and $\Phi$ given by (1), that $D_{P}i \Phi = f$, then
\[ Pi - \int_{\alpha}^{\alpha + 2\pi} f = \Phi(\alpha + 2\pi) - \Phi(\alpha) = \frac{a_0}{2} x \bigg|_{\alpha}^{\alpha + 2\pi} = a_0 \pi, \]

giving the first formula in (F),

\[ a_0 = \frac{1}{\pi} Pi - \int_{\alpha}^{\alpha + 2\pi} f. \]  

(4)

Since \( \Phi \) may only be defined almost everywhere, \( \alpha \) is chosen so that the limits of integration are points where \( \Phi \) is defined.

Once we have defined the correct \( Pi \)-integral of course (4) will hold and so if \( f \) is Sklyarenko’s function it will be integrable in both senses and

\[ \mathcal{DH} - \int_{-\pi}^{\pi} f \neq \mathcal{Pi} - \int_{-\pi}^{\pi} f. \]

This gives another indication of how awkward the \( Pi \)-integral has to be. The class of \( Pi \)-integrable functions includes, by Nalli’s result quoted section 4, all \( \mathcal{D}^* \)-integrable function. So in the class of \( Pi \)-primitives includes all continuous \( ACG^* \) functions, see [Bu, 6.3], but not, by Sklyarenko’s example, all continuous \( ACG \) functions. In fact the sum of the Sklyarenko series has a \( \mathcal{DH} \)-primitive that is not its \( Pi \)-primitive, so the \( Pi \)-primitive is not an \( ACG \) function.

This example also shows that the difficulties the \( Pi \)-integral presents are inherent in the coefficient problem.

Given that the \( Pi \)-integral calculates \( a_0 \), how do we calculate the rest of the coefficients of (T)?

One method is based on the theory of formal multiplication of trigonometric series; [Z, vol. I, pp.335–344; Ho. vol. II, pp.585–587]. In this method the functions \( f(x) \sin nx, f(x) \cos nx \) are shown to be the sums of trigonometric series with constant terms \( a_n/2, b_n/2 \) respectively. Then the above particular case (4) applies.

Another method is to prove an integration by parts formula for the \( Pi \)-integral. This formula then plays two roles; firstly it ensures that the functions \( f(x) \sin nx, f(x) \cos nx \) are \( Pi \)-integrable, and secondly it gives the required formulæ. For instance:

\[ Pi - \int_{\alpha}^{\alpha + 2\pi} f(x) \sin nx \, dx = \Phi(x) \sin nx \bigg|_{\alpha}^{\alpha + \pi} - \int_{\alpha}^{\alpha + 2\pi} \Phi(x) n \cos nx \, dx \]

\[ = \pi b_n, \]

where the right a Lebesgue integral.

Both methods are useful as in some cases an integration by parts formula is not easily obtained.
6 The $\mathcal{P}_i$-derivative Will Be Symmetric Derivative

To see what role symmetry plays in the coefficient problem consider the formula, given in [G II, p.10], for the partial sums of the series in (T) regarded as the Fourier series of $f$:

$$s_n(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x + 2t) \frac{\sin(2n + 1)t}{\sin t} \, dt.$$  \hspace{1cm} (5)

A simple change of variable in the integral (5) results in the alternative form

$$s_n(x) = \frac{2}{\pi} \int_0^{\pi/2} \left( \frac{f(x + 2t) + f(x - 2t)}{2} \right) \frac{\sin(2n + 1)t}{\sin t} \, dt.$$  \hspace{1cm} (6)

The implication of (6) is that these partial sums evaluated at $x$ depend not on the value of $f$ at $x$ but on $f_e^x(t)$, the **even part** of $f$ at $x$. In fact the heuristic argument in [G II, pp.10–11] can be used to show that the series in (T) converges to $\lim_{t \to 0} f_e^x(t) = (f(x+) + f(x-))/2$ under the assumption that $f$ is monotonic—the **Dirichlet-Jordan test** for Fourier series; [Z, vol. I, p.57].

It should be remarked that on integrating a function, the even, respectively odd, part of that function yields the odd, respectively even part of its primitive.

It is natural to look at the Pickwickian derivatives that use the odd part of a function; that is, for example, a derivative of $\Phi$ of the kind defined in (19).

For a more detailed discussion of the role of symmetry see [Th1,2], and the very important book [Th3].

7 The Integrals of Denjoy and of James

7.1 Denjoy’s $(T_{2,s})_o$-totalization. The first solution of the coefficient problem, given by Denjoy, avoided the use of the primitive $\Phi$ because of the difficulties expected for the Pickwickian integral that have been discussed above. Denjoy wanted an integral with a continuous primitive defined everywhere.

Denjoy published his result in a series of very short notes around 1920, but a full version was only given in the fifties as a result of a series of lectures that Denjoy gave at Harvard on the invitation of McShane, [D].

The basic idea behind Denjoy’s solution is to by-pass the first integral, $\Phi$, and go to the second integral

$$F(x) = \frac{a_0 x^2}{4} - \sum_{k=1}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{k^2};$$  \hspace{1cm} (7)

that is the series obtained by formally integrating (T) twice.
As we remarked in Section 5 the coefficients of an everywhere convergent series \((T)\) tend to zero; this implies, by a simple application of the comparison test, that the series in \((7)\) is uniformly, and absolutely, convergent. Its sum \(F\) is then continuous, and the series in \((7)\) is the Fourier-R series of \(F\). Hence the coefficients in this series, that are very simply related to those in \((T)\), are given by \((F)\), using the Riemann integral, and with suitable changes in the integrand, using \(F\) instead of \(f\).

This would then calculate the coefficients of \((T)\), except that we do not yet know how \(F\) is related to \(f\) except in this formal way—it is a sort of second integral. Equivalently \(f\) is a sort of second order derivative of \(F\). It follows from our discussion in Section 6 that we should look at a derivative of the even part of \(F\), and a careful reading of the discussion in \([G\ IV]\) will show in fact that the coefficient problem is equivalent to a primitive problem in which the Pickwickian derivative is the \textbf{Schwarz derivative}; see \((20)\).

There is a famous result of Riemann discussed in \([G\ IV]\) that tells us how \(f\), the sum of \((T)\) is related to \(F\), the sum in \((7)\): namely, \(D^2_s F = f\). A proof can be found in \([Z,\ vol.I,\ p.319]\).

In our situation we know \(f\) and wish to find \(F\), the function of which it is the second symmetric derivative. This is a the generalization of the classical primitive problem used by Denjoy in his solution of the coefficient problem.

It is known that if \(g = D^2_s G\) on \([a, b]\) for some continuous function \(G\), then \(G\) is unique up to a linear function; this follows from the various results on trigonometric series given in \([G]\), but a direct proof is preferable and was given by Schwarz; see \([Th3,\ p.12]\).

Hence if \(G\) is any continuous function such that \(D^2_s G = g\), the unique function that has the same property and is zero at both \(a\) and \(b\) is

\[
\tilde{G}(x) = G(x) - G(a) - \frac{x-a}{b-a} (G(b) - G(a)) = (x-a)(x-b)[a, x, b; G],
\]

where \([a, x, b; G]\) is the second \textbf{divided difference} of \(G\) at \(a, x, b\).

In trying to solve the coefficient problem \(F\), the sum of the series in of \((7)\), is then regarded as a second primitive of \(f\), in an integration process that inverts the Schwarz derivative.

The process that Denjoy used was a extremely complicated extension of his totalization process, see \([Bu2,\ 6.1]\) where other references are given. It defined what he called the \((T_{2, s})_o\)-integral and we write

\[
T_{(2, s)_o} \int_{(a, b)}^x g = \tilde{G}(x) = (x-a)(x-b)[a, x, b; G];
\]

the various subscripts are readily explained—2 for second order, \(s\) for symmetric, and \(o\) for ordinary. This last needs bit of explanation; in Section 9 we will introduce an approximate derivative, here however we are using ordinary derivative.
Formula (8) can be compared with
\[ \int_a^x g = (x - a)[a, x; \Gamma] \]
that gives the first primitive in terms of the first divided difference.
If \( g \) is Lebesgue integrable then and so:
\[ T_{(2, s)_0} \int_{(a, b)}^x g = \frac{1}{b - a} \left\{ (x - b) \int_a^x (t - a)g(t) \, dt + (x - a) \int_x^b (t - b)g(t) \, dt \right\} \] (9).
Formula (9) is easily shown if we note that
\[ G(x) - G(a) = \int_a^x \int_a^t f(u) \, du \, dt = (x - a)\Gamma(x) - \int_a^x (t - a)g(t) \, dt, \]
and another similar formula for \( G(b) - G(x) \).
While the sum \( f \) of the series (T) is, as we have seen, \( T_{(2, s)_0} \)-integrable on \([-\pi, \pi]\), it is not necessarily \( L \)-integrable. However, as in the case of the classical primitive problem, \( f \) is \( L \)-integrable on lots of subsets of this interval. On these sets \( f \) can be integrated twice to give contributions to the value of \( T_{(2, s)_0} \int_{(-\pi, \pi)} f \) that in the case of intervals are of the form (9). The difficulty is to put these contributions together correctly to get the value of the second order primitive \( T_{(2, s)_0} \int_{(-\pi, \pi)} f \).
The totalization process used here is much more difficult than the totalization used for classical primitive problem. Much deeper knowledge of the fine properties of sets is required, as well as some very deep properties of the Schwarz derivative.
As a further evidence of the extra complications Denjoy distinguishes nine distinct types of limit calculations that are needed to complete the calculation, each used infinitely often in general—and he gives examples to show that all are needed; for the classical primitive problem he only distinguishes three, and all are very simple. These difficulties were overcome by Denjoy who went on to show how to complete the calculation and gave formulæ for the coefficients, like those in (F), but now more complicated since we are using a second order integral, rather than a first order integral; see (14) below.
To see the order of these difficulties consider the simplest situation: we have computed the total on two adjoining intervals \([a, b]\) and \([b, c]\) how do we compute the value on \([a, c]\)?
In the case of the classical primitive problem this means that if
\[ \int_a^x g = \Gamma_1(x) - \Gamma_1(a), \ a \leq x \leq b, \quad \text{and} \quad \int_b^x g = \Gamma_2(x) - \Gamma_2(b), \ b \leq x \leq c, \]
then, as is easy to see,

\[ \int_a^x g = \Gamma(x) - \Gamma(a), \quad a \leq x \leq c, \]

where

\[
\Gamma(x) = \begin{cases} 
\Gamma_1(x), & \text{if } a \leq x \leq b, \\
\Gamma_2(x) + \Gamma_1(b) - \Gamma_2(b), & \text{if } b \leq x \leq c.
\end{cases}
\]

However the situation for the \((T_{2,s})_o\)-integral is much more complicated; [D, vol. III, pp.278–279].

Let

\[
T_{(2,s)_o} - \int_{(a,b)}^x g = (x - a)(x - b)[a, x, b; G_1], \quad a \leq x \leq b;
\]

and

\[
T_{(2,s)_o} - \int_{(b,c)}^x g = (x - b)(x - c)[b, x, c; G_2], \quad b \leq x \leq c.
\]

There is an \(G\) such that \(D_s^2 G = g\) on \([a, c]\), which differs by a linear function from \(G_1\) on \([a, b]\), and by a linear function from \(G_2\) on \([b, c]\). Assume for simplicity that \(a \leq x < b\), (the case \(b < x \leq x\) can be treated similarly); then by (17) below

\[
T_{(2,s)_o} - \int_{(a,c)}^x g = (x - a)(x - c)[a, x, c; G] + (b - c)(x - a)[a, b, c; G] + \Delta^2 G(x; h) h.
\]

It remains to see how we can compute the second divided difference in first term in the last line, knowing only \(G_1\) and \(G_2\).

Let \(a < b - h < b < b + h < c\) and then

\[
[a, b, c; G] = (c - a) \left( (b - h - a)[a, b - h, b; G] + (c - b + h)[b, b + h, c; G] + \frac{\Delta^2 G(x; h)}{h} \right)
\]

Since \(D_s^2 G\) is finite \(G\) is smooth, and so the last term on the right tends to zero with \(h\). Hence we get that:

\[
[a, b, c; G] = (c - a) \lim_{h \to 0} \left( (b - h - a)[a, b - h, b; G_1] + (c - b + h)[b, b + h, c; G_2] \right); \quad (11)
\]
of course this gives the value of $T_{(2,s)} \int_{(a,c)}^{b} f$.
This is a difficulty occurs at the very first step of the calculations and there are many more difficulties along the road.

7.2 The James $P^2$-integral

Given the simplicity of the Perron approach to the classical primitive problem, see [Bu, 6.2], it would seem natural to ask whether a similar idea could not be used for the solution of the of the coefficient problem. However it was not until the final appearance of Denjoy’s work in the fifties that R.D. James, on the suggestion of A. Zygmund, used the Perron method on the Schwarz derivative; see [J1; J-G; Z, vol. II, p. 86]. The extension of the Perron method to this situation is not immediate. The derivative being used is a second order derivative and the corresponding integral, like Denjoy’s in section 7.1, is a second order integral.

The classes of major and minor functions defined in [Bu2, 6.2] are generalized in a natural way. Call $M, m$ major and minor functions for $f$ on $[a, b]$ if

$$M(a) = m(a) = M(b) = m(b) = 0 \quad D^2_s M(x) \geq f(x) \geq D^2_s m(x), \ a < x < b;$$

and then the symmetric second Perron, or (James) $P^2$-, integral of $f$ is

$$P^2 - \int_{(a,b)}^{x} f = \sup M(x) = \inf m(x), \ a \leq x \leq b \quad (12)$$

when these two families of functions have a common sup and inf.

Any function, $F$, that differs from the right-hand side of (12) by a linear function will be called a (James) $P^2$- integral of $f$; and so by (8),

$$P^2 - \int_{(a,b)}^{x} f = (x - a)(x - b)[a, x, b; F] \quad (13)$$

A $P^2$-integral $F$, of $f$, can be proved to be continuous, $ACG$, smooth, differentiable almost everywhere, and $D^2 F = f$ almost everywhere; see [J1; S1]. However a $P^2$-integral need not be differentiable everywhere; as the following example, [J1], shows:

$$H(x) = \begin{cases} x \cos x^{-1}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases} \quad h(x) = \begin{cases} -x^{-3} \cos x^{-1}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $H$ is continuous, smooth and $D^2 H = h$ and so its $P^2$-integral is given by (13), with $H, h$ replacing $f, F$ respectively. However $H'(0)$ does not exist, as is easily seen.
In addition the $P^2$-integral does not have nice properties with respect to additivity. That is a function can be integrable both on $[a,b]$ and on $[b,c]$ but not on $[a,c]$. Consider the following example due Skvortsov, [Sk];

$$g(x) = \begin{cases} 0, & \text{if } -2/\pi \leq x \leq 0, \\ -x^{-3}\cos x^{-1}, & \text{if } 0 < x \leq 2/\pi. \end{cases}$$

Obviously $g$ is integrable on $[-2/\pi, 0]$, with integral 0, and by the previous example it is integrable on $[0, 2/\pi]$, with integral given by the function $H$ of that example. Suppose then that it was integrable on the complete interval $[-2/\pi, 2/\pi]$ with integral $G$. Then $G$ would be linear on the left half of the interval, and differ from $H$ by a linear function on the second half of the interval. This implies that $G$ has a left derivative at the origin, but that it does not have a right derivative at the origin, and so is not smooth there. As a result we deduce that $g$ is not integrable on the complete interval.

Skvortsov gave a necessary and sufficient condition for the additivity to hold; it is a little complicated to state; see [Sk] and [C2]. A simple corollary is the sufficient condition: if $g$ is James integrable on $[a,b]$, and also on $[b,c]$, with integrals $G_1$ and $G_2$ respectively, then $g$ is integrable on $[a,c]$ if $G_1$ has a left derivative at $b$, and $G_2$ has a right derivative at $b$.

It is worth remarking that if the integral exists on $[a,b]$, $[b,c]$ and on $[a,c]$ then the argument used in (10) and (11) can be repeated to connect the values of the three integrals; the function $F$ used there is now the integral $[a,c]$ which is smooth so we can get (10) from (11) as was done for the $T_{(2,s),a}$-integral.

However the James integral does integrate second symmetric derivatives, and so the sum function of (T), $f$ say, is integrable in this sense, and the sum of (7), $F$, is a James integral of $f$. Hence using the remarks in 11.1 about second divided differences

$$a_0 = -\frac{1}{\pi^2}P^2\int_{(-2\pi,2\pi)} f.$$ 

Further, using the method of formal multiplication, each of $f(x) \sin kx$, $f(x) \cos kx$, $k = 1, 2, \ldots$ are also integrable in this sense and

$$a_k = -\frac{1}{\pi^2}P^2\int_{(-2\pi,2\pi)} f(y) \cos ky \, dy, \quad k = 0, 1, \ldots,$$

$$b_k = -\frac{1}{\pi^2}P^2\int_{(-2\pi,2\pi)} f(y) \sin ky \, dy, \quad k = 1, \ldots.$$  

(A comment on the signs in these formulae might be in order. If $f > 0$ we would expect $a_0 > 0$, but then $D^2F = f > 0$, so $F$ would be convex and so $[-2\pi, 0, 2\pi; F] > 0$, see 11.1, and $-4\pi^2[-2\pi, 0, 2\pi; F] = P^2\int_{(-2\pi,2\pi)} f < 0.$)
8 Two Other Integrals of Perron-type

8.1 The Integral of Marcinkiewicz & Zygmund

The first trigonometric integral to solve the coefficient problem, after the work of Denjoy was announced in the early twenties, was given by Marcinkiewicz & Zygmund, [M-Z]. They took on the difficulties of the first primitive, getting around the lack of continuity and the fact that $\Phi$, (1), was defined only almost everywhere in an ingenious way. As we have seen $\Phi$ is Lebesgue integrable, and to integrate it we do not need it to be defined everywhere. Using a symmetric derivative called the symmetric Borel derivative that is based on smoothing the function to be differentiated by first integrating; [Th3 pp.15–16], they were able to solve the modification of the classical primitive problem for this derivative and to show that the integral obtained, the so called $T$-integral, solved the coefficient problem. The details of the construction of the $T$-integral are very similar to those of the better known integral due to Burkill to be described below. Formal multiplication was used to obtain (F).

8.2 The Burkill Integral

A trigonometric integral that is related to the James integral was defined by J.C. Burkill, see [BJ]. His approach has the advantages of that of the previous section as it results in a first order integral, but the tools used are standard results from the theory of trigonometric series, which makes this integral more approachable than the $T$-integral.

Consider the following: the sum of the series (7), $F$, is the indefinite integral of the function $\Phi$ in (1) hence

$${\frac{F(x + h) + F(x - h) - 2F(x)}{h^2}} = \frac{1}{h^2} \int_x^{x+h} \Phi(y) \, dy - \frac{1}{h^2} \int_x^{x-h} \Phi(y) \, dy$$

$$= \frac{1}{h^2} \int_0^h (\Phi(x + u) - \Phi(x - u)) \, du.$$  \hspace{1cm} (15)

So that

$$D_x^2 F(x) = \lim_{h \to 0} \frac{1}{h^2} \int_0^h (\Phi(x + u) - \Phi(x - u)) \, du.$$  \hspace{1cm} (15)

The right-hand side of (15) can be considered as firstly, by integrating, smoothing out the bad function $\Phi$, and then differentiating, giving a generalized first order derivative of $\Phi$, called the symmetric Cesàro derivative of $\Phi$, at $x$; see 11.3.

Burkill’s idea was to use this symmetric Cesàro derivative to define an integral, using the Perron approach, and to show that it solves the coefficient problem. In this way the use the second integrated series, (7), is avoided.

As we saw above the first integrated series does not necessarily converge everywhere and so, like the $T$-integral, the integral defined by Burkill, the symmetric Cesàro
-Perron, or SCP-, integral has a primitive that is only defined almost everywhere. This makes the details of the construction, the modifications needed to the definitions of the major and minor functions in this case, very intricate and so will not be given here.

Using this integral the formulæ (F) were obtained by an integration by parts formula proved by Burkill. The proof of this was found, by Skvorcov, to contain a flaw that was only corrected much later, by Sklyarenko; [S3; C-Th ]. In the meantime the formulæ (F) were reproved using the method of formal multiplication; [BH]

If the correct formulation of the definition of the major and minor functions is made a function is SCP-integrable if and only if it is \( \mathcal{P}^2 \)-integrable, but the discussion of this point is very technical; see [V, p. 681].

9 The Story Continues

As we have seen in the \( \mathcal{T} \)- and the SCP-integrals the difficulties with \( \Phi \) are avoided by the use of symmetric differentiation and by using integration to smooth out bad functions. However there is a much more direct modification of the classical primitive problem that can be used.

Associated with \( \Phi \) there is a symmetric derivative that plays a role analogous to that played by the Schwarz derivative for the function \( F \).

The first order symmetric derivative of \( \Phi \), see (19), is not quite right since although it does not need \( \Phi \) to be defined everywhere it does need it to be defined in some interval around each point; and this may not be the case. This difficulty is avoided by generalizing the derivative in (19) even further.

The limit process used in defining continuity and derivatives can be modified in many ways. The most useful modification in this connection is the so-called approximate limit. In this we only require that \( x+h \to x \) through a set that has density 1 at \( x \).

That is \( x+h \) is restricted to a set \( E_x \) with

\[
\lim_{h \to 0} \frac{|E_x \cap [x-h, x+h]|}{2h} = 1,
\]

(where by \( |A| \) we mean the measure of \( A \).) The theory of such limits, approximate limits, can be developed and gives these limits all the properties of usual limits; see for instance [G, p.223; Z I, p.23]. The correct derivative for use here is one in which the limit in (19) is taken in this approximate sense giving the approximate symmetric derivative of \( \Phi \) at \( x \);

\[
D_{s, ap} \Phi(x) = \text{ap-} \lim_{h \to 0} \frac{\Phi(x+h) - \Phi(x-h)}{2h}, \tag{16}
\]

where of course \( \text{ap-} \lim_{h \to 0} \) means the approximate limit as \( h \to 0 \). It is the correct derivative because of the following result of Rajchman & Zygmund, [Th3, pp.17–18; Z, vol.I, p.324].
If $f$ is the sum of the series $(T)$ and if $\Phi$ is as in (1) then

$$D_{s,ap}\Phi = f.$$  

This result was proved in 1926 and it is reasonable to ask why it could not have been used to give an exceedingly simple generalization of the Perron integral, one with major and minor functions defined by

$$M(a) = m(a) = 0 \quad D_{s,ap}M \geq g \geq D_{s,ap}m$$

to get a trigonometric integral that would solve the coefficient problem?

A critical part of the Perron theory is the fact that for each major function $M$, and each minor function $m$, the difference $M - m$ is an increasing function.

In the classical Perron case this derives from a fairly elementary result:

if $Dh \geq 0$ then $h$ is increasing.

This implies that in the classical case $D(M - m) \geq 0$, and so $M - m$ is increasing.

The same result for the approximate symmetric derivative turns out to be much more difficult to prove, was not known until very recently when Freiling & Rinnie, [F-R], showed that:

if $D_{s,ap}h \geq 0$ then $h$ is increasing.

This allowed Preiss & Thomson to give another solution to the coefficient problem along the lines suggested above; a Perron integral that inverts the approximate symmetric derivative; [P-Th2].

At the same time they also gave an equivalent Riemann definition of an integral based on the Kurzweil-Henstock methods mentioned in [Bu2].. To do this they had to consider symmetric partitions of an interval; that is partitions in which we would want that $x_{i-1} = y_i - h_i$, $x_i = y_i + h_i$, $h_i < \delta(y_i)$; this extension is far from trivial and is due to Thomson; [Th2], [P-Th1]. In addition they had to consider approximate partitions that would have the $x_i$ lie in some set $E_{y_i}$ of density 1 at $y_i$. Such partitions were first considered by Henstock, but in the case of symmetric partitions they are much more difficult to handle; [P-Th2]. The details as is always the case with the coefficient problem are complicated. Finally they also gave a variational form for this integral. A very detailed discussion of all these integrals can be found in [Th3].

10 Final Remarks
Firstly we note that all the integrals introduced extend the $\mathcal{D}^*$-integral although relationships with other general classical integral is more complicated; see [V].
Do these solutions to the classical coefficient problem end of the story? No, there are at several other areas of further interest.

An obvious question is, are all the solutions above equivalent, as the various solutions to the classical primitive problems are. In general the answer to this is unknown. Using the classical definitions the $P^2$-integral is more general than the $SCP$-integral but as we have pointed out these integrals are essentially equivalent if some changes are made in the definitions; although even here there are technical problems. A trigonometric integral that lies between the $SCP$-integral and the $P^2$-integral has recently been given by Mukhopadhyay, [M]. Just whether any of the various integrals are equivalent to the original $T(2,s)_{0}$-integral is unknown. A very detailed examination of this problem can be found in the recent masterly paper by Cross & Thomson, [C-Th].

Then there is no end to the class of integrals that can be investigated by considering other methods of convergence for (T). Equality in (T) has so far meant everywhere convergence of the series to the function, but we could instead consider some summability method, as is very common in the theory of trigonometric series. The use of Cesàro summability leads to a family of integrals that generalize the James integral described above, [J2–4]; also, less successfully the Burkill integral has been extended to a class of integrals that does a similar job, [Bu-L]. A mention should also be made of an attempt to give a Riemann form to these higher order integrals, [C2]. A full discussion of all of these integrals has not been given and the technical problems seem to be daunting.

In addition an integral to be used when Abel summability is considered has been defined by Taylor; this integral is more general than the $P^2$-integral; see [C1; T].

By far the best reference for this material is the excellent book by Brian S. Thomson, [Th3], where all the facts are sorted out and given a very lucid exposition. However it not a book for the light-hearted. The article by James, [J3], is an easy introduction to the field although a little dated; the same can be said of the lecture by R.L. Jeffery, [Je]. A short but excellent discussion can also be found in the article by Vinogradova & Skvorcov, [V], as well as in the book of Zygmund, where it is preceded by an equally short review of the classical primitive problem; [Z II, pp.83–90].

11 Appendix

11.1 Divided Differences. The first divided difference of $G$ at $a, x$ is written $[a, x; G]$, where

$$[a, x; G] = \frac{G(x) - G(a)}{x - a};$$
The second divided difference of $G$ at $a, x, b$ is written $[a, x, b; G]$, where

$$
[a, x, b; G] = \frac{1}{b-a} \left\{ \frac{G(b) - G(x)}{b-x} - \frac{G(x) - G(a)}{x-a} \right\} = \frac{G(a)}{(a-x)(a-b)} + \frac{G(x)}{(x-b)(x-a)} + \frac{G(b)}{(b-a)(b-x)}.
$$

A few very simple facts about the second divided difference are worth noting.

(i) If $h$ is a polynomial of degree at most two then for all $a, x, b$,

$$
[a, x, b; h] = \frac{1}{2} h''.
$$

In particular if $h(x) = Ax^2$, $[a, x, b; h] = A$.

(ii) If $h(x) = \sin nx$ or $\cos nx$, $n = 0, 1, 2, \ldots$ then $[-2\pi, 0, 2\pi; h] = 0$.

(iii) If $a, b, c, x$ are any four distinct points then

$$
(c-x)[a, x, c; G] = (c-b)[a, b, c; G] + (b-x)[a, x, b; G].
$$

(iv) A function $G$ is convex on $[a, b]$ if and only if $[x, y, z; G] \geq 0$ for all distinct $x, y, z$ in $[a, b]$.

11.2 Odd and Even Parts of Function

Any function $G$ can be written in terms of its odd and even parts at $x$

$$
G(x+t) = \frac{G(x+t) + G(x-t)}{2} + \frac{G(x+t) - G(x-t)}{2} = G^e_t(x) + G^o_t(x)
$$

and the derivatives of these two parts at $t = 0$ depend on

$$
G^e_t(x) - G^e_t(0) = \frac{G(x+t) - G(x-t)}{2};
$$

$$
G^e_t(x) - G^e_t(0) = \frac{G(x+t) + G(x-t) - 2G(x)}{2} = \frac{1}{2} \Delta^2 f(x; t).
$$

So the derivative at the origin of the odd part of a function exists precisely when the function has a first order symmetric derivative at $x$:

$$
D_s G(x) = \lim_{h \to 0} \frac{G(x+h) - G(x-h)}{2h}.
$$

This derivative certainly exists when $G'(x)$ exists but not conversely as taking $G(x) = |x|$ shows; more, the value of $G$ at $x$ is not used in (19) so $D_s G(x)$ can exist even if $G$ is not continuous at $x$, or even if it not defined there; consider for instance $G(x) = \cos 1/x, x \neq 0$, when $D_s G(0) = 0$. See [Th3, pp.5–7].
The second of the quantities in (18) was introduced in [G IV, p.17] when defining the Schwarz, or second symmetric derivative of a function $G$ at $x$,

$$D_s^2 GF(x) = \lim_{h \to 0} \frac{G(x + h) + G(x - h) - 2G(x)}{h^2},$$

(20)

The derivative at the origin of the even part of a function exists precisely when the function has the property

$$\lim_{h \to 0} \frac{G(x + h) + G(x - h) - 2G(x)}{h} = 0;$$

the function $G$ is then said to be smooth at $x$. If $G$ is smooth at $x$ and if $G'_+(x)$ exists then so does $G'_-(x)$ and $G'_-(x) = G'_+(x)$; of course a function can be smooth without being continuous, let alone differentiable. The concept goes back to Riemann and was given the name by Zygmund; [Th3, p.159; Z, vol.I, p. 43].

11.3 Derivatives Using Integrals The idea of using integrals to smooth a function before taking the derivative goes back to Borel, [Th3, pp.15–16]: his mean value derivative is

$$D_B f(x) = \lim_{h \to 0} \frac{1}{h} \int_0^h \frac{f(x + t) - f(x)}{t} \, dt,$$

and the symmetric Borel derivative, the one used for the $\mathcal{T}$-integral in 8.1, is

$$D_{s,B} f(x) = \lim_{h \to 0} \frac{1}{h} \int_0^h \frac{f(x + t) - f(x - t)}{2t} \, dt,$$

Burkill introduced his Cesàro derivative,

$$CD f(x) = \lim_{h \to 0} \frac{2}{h^2} \int_0^h (f(x + t) - f(x)) \, dt.$$

and used the symmetric version, (15), for the $\mathcal{SCP}$-integral.

Of course given the variety of integrals at our disposal it is natural to ask which integral should be used in these definitions. It turns out to make some difference. Burkill used the very natural $D^*$-integral, but by using the $D\mathcal{H}$-integral instead Sklyarenko was able to show that the more general $\mathcal{SCP}$-integral thus obtained was equivalent to the $\mathcal{P}^2$-integral; [S1].

12 References
[Bu] P S Bullen The search for the primitive, submitted.
[Bu-L] P S Bullen & C.M.Lee The $SC_nP$-integral and the $P^{n+1}$-integral, *Canad. J. Math.*, 25 (1973), 1274 –1284.
[BH] H Burkill Fourier series of SCP-integrable functions, *J. Math. Anal. Appl.*, 57 (1977), 587–609.
[BJ] J C Burkill Integrals and trigonometric series, *Proc. London Math. Soc.*, (3) 1 (1951), 46–57. Corrigendum: *Proc. London Math. Soc.*, (3) 47 (1983), 192.
[C1] G E Cross On the generality of the $AP$-integral, *Canad. J. Math.*, 23 (1971), 557-561.
[C2] G E Cross Generalized integrals as limits of Riemann-like sums, *Real Anal. Exchange*, 13 (1987–1988), 390–403.
[C-Th] G E Cross & B S Thomson Symmetric integrals and trigonometric series, *Diss. Math.*, CCCXIX, 1992.
[D] A Denjoy *Leçons sur le Calcul de Coefficients d’une Série Trigonométrique, I–IVa,b*, Gauthier-Villars, Paris 1941, 1949.
[F-R] C Freiling & D Rinnie A symmetric density property; monotonicity and the approximate symmetric derivative, *Proc. Amer. Math. Soc.*, 104 (1988), 1098 –1102.
[G] A D Gluchoff Trigonometric series and theories of integration, *Math. Mag.*, 67 (1994), 3–20.
[Go] R A Gordon *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Amer. Math. Soc Memoir, 1994.
[H] G H Hardy *Divergent Series*, Oxford University Press, 1949.
[Ho] E W Hobson *The Theory of Functions of a Real Variable and the Theory of Fourier’s Series I, II*, Cambridge University Press, 1926.
[J1] R D James A generalized integral II, *Canad. J. Math.*, 2 (1950), 297–306.
[J2] R D James Generalized $nth$ primitives, *Trans. Amer. Math. Soc.*, 76 (1954), 149–176.
[J3] R D James Integrals and summable trigonometric series, *Bull. Amer. Math. Soc.*, 61 (1955), 1–15.
[J4] R D James Summable trigonometric series *Pacific J. Math.*, 6 (1956), 99–110.
[J-G] R D James & W H Gage A generalized integral, *Trans. Roy. Soc. Canada, III*, (3) 40 (1946), 25–35.
[Je] R L Jeffery Trigonometric Series, *Canad. Math. Congress Lecture Ser.*, # 2, Toronto, 1956.
[K] K Knopp *Theory and Application of Infinite Series*, Blackie & Son Ltd., London, 1948.
[L] H Lebesgue *Leçons sur l’Intégration et la Recherche des Fonctions Primitives,*
Gauthiers–Villars, Paris; Ist Ed. 1904; 2nd Ed. 1928.

[M-Z] J Marcinkiewicz & A Zygmund On the differentiability of functions and the summability of series, Fund. Math., 26 (1936), 1–43.

[M] S N Mukhopadhyay An extension of the SCP-integral with a relaxed integration by parts formula, Analysis Math., 25 (1999), 103–132.

[N] P Nalli Sulle serie di Fourier delle funzione non assolutamente integrabili, Rend. Circ. Math. Palermo, 40 (1915), 33–37. New York, 1964

[P] I N Pesin Razvite Ponyatiya Integrala2, Moscow, 1966. Engl. transl.: Classical and Modern Integration Theory,3 New York, 1970.

[P-Th1] D Preiss & B S Thomson A symmetric covering theorem, Real Anal. Exchange, 14 (1988–1989), 253–254.

[P-Th2] D Preiss & B S Thomson An approximate symmetric integral, Canad. J. Math., 41 (1989), 508–555.

[S1] V A Sklyarenko4 Nekotorie svoistva $P^2$-primitivnoi, Mat. Zametki, 12 (1972), 693–700. Eng. transl.: Math. Notes, 12 (1972), 856–860, (1973).

[S2] V A Sklyarenko On Denjoy integrable sums of everywhere convergent trigonometric series, Dokl. Akad. Nauk SSSR, 210 (1973), 533–536. Eng. transl.: Soviet Math. Dokl., 14 (1973), 771–775.

[S3] V A Sklyarenko Ob integrirovani po chastam v SCP-integrale Burkillya, Mat. Sb., (154) 112 , (1980), 630–646. Engl. transl. Math. USSR Sbornik, 40 (1981), 567–583.

[S4] V A Sklyarenko On a property of the Burkill SCP-integral, Mat. Zamet., 65(1999), 599–604.

[Sk] V A Skvorstov5 Po povody opredelenii $P^2$-i SCP-integralov, Vestnik Moscov. Univ.; Ser.I Mat. Meh., 21 (1966), 12–19.

[T] S J Taylor An integral of Perron’s type defined with the help of trigonometric series, Quart. J. Math. Oxford, (2) 6 (1955), 255–274.

[Th1] B S Thomson Symmetric derivatives and symmetric integrals, Real Anal. Exchange, 15 (1989–1990), 49–61.

[Th2] B S Thomson Some symmetric covering lemmas, Real Anal. Exchange, 15 (1989–1990), 346–383.

[Th3] B S Thomson Symmetry Properties of Real Functions, Marcel Dekker Inc., 1994.

[V] I A Vinogradova & V A Skvorcov6 Generalized Fourier integrals and series,

---

2 И Н Песин, Разные Понятия Интеграла.
3 Care using this translation, several terms are used in their historical, rather than their modern senses.
4 В.А.Скляренко
5 В.А.Скворцов. Also transliterated as Skvorcov.
6 И.А.Виноградова
J. Soviet Math., 1 (1973), 677–703.

[Y] W H Young  On non-absolutely convergent, not necessarily continuous, integrals, Proc. London Math. Soc., (2) 16 (1917), 175–218.

[Z] A Zygmund  Trigonometric Series I, II, Cambridge, 1959.