Arbitrary Lagrangian-Eulerian formulations for cutting pattern generation of tensile membrane structures

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\section*{ABSTRACT}
An application of Arbitrary Lagrangian-Eulerian (ALE) method is used to generate the cutting pattern shapes of membrane structures. The initial patterning panels are selected as the projection of three-dimensional (3D) membrane panels and are assumed as the reference configuration. While the Lagrangian displacements define the forced displacements from this configuration to the 3D configuration, the Eulerian terms describe the motion from this reference configuration to the current patterning panels. The cutting pattern panels can be determined by this process considering the effect of prescribed pre-stress and material properties. Several examples are performed to point out the effectiveness of the proposed method on simplifying the cutting pattern process and the reduction in the stress deviation from the intended pre-stress.

\section{1. Introduction}
Tensile membrane structures are mainly used in lightweight and long-span roofs. They may also be used in buildings such as sport facilities, warehousing and storage buildings, and exhibition venues. These structures are defined as a combination of tensioned fabric membrane and supporting elements such as rigid structural frame or flexible cables (Gale and Lewis 2016). The fabric membrane uses the pre-stress of tension and its shape to resist the external loading, while the supporting elements are compressed and bent by the tension in membrane. The design process of these structures consists of three steps including form-finding, load analysis and cutting pattern generation (Tabarrok and Qin 1992). The first step is aimed to find the shape of structures in which the prescribed pre-stress are in equilibrium with a given boundary condition (Veenendaal and Block 2012), while the second step investigates the behaviour of fabric membrane and supporting elements under service loads such as wind or snow loads. The shapes obtained from the form-finding process are usually in doubly curved surface, so they cannot be flattened into plane without being distorted (Bletzinger and Widhammer 2016). In addition, the fabric membrane itself is manufactured in plane panels of 1–5 m widths (Ishii 1999). As a result, a special design process, third step of cutting pattern, has to be conducted. The purpose of patterning process is to define the planar strips which are joined to form the desired 3D surface. In this cutting process, the stress distribution on the fabric 3D surface is intended to the one in form-finding.

The general process of computational cutting pattern generation can be divided into four stages. There are subdivision (1), flattening (2), stress reduction (3) and compensation (4) (Brew and Lewis 2013; Gale and Lewis 2016). In the first stage, the geodesic line, which will become a straight line upon the development of curved surface, is used (Moncrieff and Topping 1990; Ishii 1999). The cutting lines were then determined along geodesics lines and based on the width of membrane material. The implementation of general process is varied particularly in the last three stages. Ishii (Ishii 1999) incorporated the determination of geodesic lines during form-finding process, so after the form-finding shapes were obtained, the patterning strips were also determined along these geodesic lines. This method can eliminate the third stage, and the compensation was carried out in the standard of elastic-body theory. Tabarrok and Qin (1993) used a least-square minimization technique for flattening of second stage. They considered each side of triangular elements, which used in form-finding process, as link elements and used the least-squares approach to minimize the discrepancy between lengths of link elements in three-dimensional-curved surface and that of plane strips. However, the third and four stages were ignored in their research. Kim and Lee (2002) improved this work by adding the compensation stage, which reduced the stress deviation of the actual stress from...
the prescribed pre-stress. Tsubota and Yoshida (1989) used the optimization procedure for the compensation stage too, but the boundary points in patterning panels were selected as control variables in his work. Linhard, Wuchner, and Bletzinger (2008) and Maurin and Motro (1999) combined the last three stages to find the patterning panels. A form-finding via cutting pattern approach was represented, and the patterning panels were obtained with the consideration of stress reduction and compensation in the work of Linhard, Wuchner, and Bletzinger (2008). In contrast, Maurin and Motro (1999) used the stress composition method to optimize the assumed patterning panels. In this work, the initial patterning panels were assumed first, and the discrepancy between the prescribed pre-stress and the deformed stress was used to adjust these panels. Gale and Lewis (2016) suggested the un-roller method for flattening stage, and the panel assembly and re-meshing were used in stress reduction and compensation stages. In conclusion, the above-mentioned methods used the optimization technique, so each stage of cutting pattern process involved iterations. As a result, the computational cost and the complication of the process are high.

To obtain the appropriate cutting patterns, which joined to desire the 3D surface and optimize the stress deviation between the final and prescribed stresses, the numerical methods such as meshless method (Rabczuk 2008, 2004), discrete singular convolution method (Erozy 2009; Civalek 2009), and Arbitrary Lagrangian-Eulerian method (Hughes, Liu, and Zimmermann 1981; Liu et al. 1991) are viable approaches. In terms of ALE approach, the implementation focused only on the contact between cable and membrane (Haber and Abel 1983; Noguchi and Kawashima 2004). To our best knowledge, there is no implementation of ALE method in patterning of tensile membrane structure. Therefore, this paper presents the application of an ALE method to obtain the cutting pattern. Firstly, the initial patterning panels are selected as the projection of three-dimensional (3D) membrane panels and are assumed as the reference configuration. Secondly, the Lagrangian displacements define the forced displacements from this configuration to the 3D configuration, while the Eulerian terms describe the motion from this reference configuration to the current patterning panels. The cutting pattern strips can be determined by this process considering the effect of prescribed pre-stress and material properties. The flattening, stress reduction and compensation stages are carried out simultaneously in the suggested method. In addition, the analytical data of form-finding can be reused and the Eulerian displacement conditions can be easily applied to control the patterning panels. Several examples are performed to point out the effectiveness of the proposed method on simplifying the cutting pattern process and the reduction in the stress deviation from the intended pre-stress.

2. ALE formulation for cutting pattern generation

The implementation of ALE method for cutting pattern generation is to find the planar configuration $\Omega_{2D}$ under unknown forced displacements to form the three-dimensional configuration $\Omega_{3D}$ of prescribed pre-stress $\mathbf{S}_0$. The kinematic description based on a combination of Eulerian and Lagrangian deformations is given in Section 2.1. Section 2.2 presents the equilibrium equation which is obtained by the principle of minimum potential energy. In addition, the numerical implementation of ALE method for the cutting pattern generation is provided in Section 2.3.

2.1. Kinematic description

Three configurations of deformed body, similar to that of Haber and Abel (1983), are shown in Figure 1. The initial configuration $^1\Omega_{2D}$ shows the 2D patterning panel, while the current configuration $^3\Omega_{2D}$ represents the 3D membrane panel. The position vectors of a material point in the initial and current configurations are denoted by $^1\mathbf{x}$ and $^3\mathbf{x}$, respectively. Both of them are treated as unknown variables and mapped by using position vector $\mathbf{x}$ belongs to 2D reference configuration $^0\Omega_{2D}$ as represented in Equation (2.1).

$$^1\mathbf{x} = \mathbf{X}(^0\mathbf{x})$$

$$^3\mathbf{x} = \mathbf{x}(^0\mathbf{x})$$

(2.1)

The deformation gradient $^0\mathbf{F}$ from $^1\Omega_{2D}$ to $^3\Omega_{3D}$ is decomposed into: (1) The Eulerian deformation $^1\tilde{\mathbf{F}}$ maps $^1\Omega_{2D}$ onto $^1\Omega_{3D}$, (2) The Lagrangian deformation $^3\tilde{\mathbf{F}}$ maps $^3\Omega_{2D}$ onto $^3\Omega_{3D}$. The total displacement of a material points under $^0\mathbf{F}$ is a function of reference state and is defined in Equation (2.2).

$$^0\mathbf{u} = \mathbf{u}(^0\mathbf{x}, t) = ^1\mathbf{x} - ^1\mathbf{X} = ^1\mathbf{x} - ^1\mathbf{x} + ^1\mathbf{x} - ^3\mathbf{x} = ^1\tilde{\mathbf{u}} + ^1\mathbf{a}$$

(2.2)

where $^1\tilde{\mathbf{u}}$ and $^1\mathbf{a}$ are the Lagrangian and Eulerian displacements, respectively. The strain $^1\mathbf{E}$, which measures the deformation $^1\mathbf{F}$, and its component are calculated by Equation (2.3). It should be noted that an Almanisi strain is used with the Eulerian displacements (Haber and Abel 1983).

$$^1\mathbf{E} = \frac{1}{2} \left( ^1\mathbf{F}^T ^1\mathbf{F} - ^1\mathbf{F} - ^1\mathbf{F}^T ^1\mathbf{F} \right) = \frac{1}{2} \left( ^3\mathbf{F}^T ^3\mathbf{F} - ^3\mathbf{F} + ^3\mathbf{F}^T ^3\mathbf{F} \right)$$

(2.3)

where $^1\mathbf{F}$, which maps $^1\Omega_{2D}$ onto $^1\Omega_{3D}$, is the inverse Eulerian deformation $^1\tilde{\mathbf{F}}$. 
2.2. Equilibrium equation

A linear elastic material is assumed, and the potential energy in the initial configuration $\Pi$ is expressed in Equation (2.4). The body and external forces are ignored as regards the cutting pattern generation problem.

$$\Pi = \frac{1}{2} \int_{\Omega_2D} \varepsilon^T \sigma \varepsilon \, d\Omega_2D$$

(2.4)

where $\varepsilon$ is the second Piola-Kirchhoff stress for time $t$ referred to the initial configuration. Equation (2.4) can be rewritten in the reference configuration.

$$\Pi = \frac{1}{2} \int_{\Omega_2D} \varepsilon^T \sigma \varepsilon' \, d\Omega_2D$$

(2.5)

where $\varepsilon'$ is determinant of the deformation gradient $F$ which maps from $\Omega_2D$ onto $\Omega_2D$. All $\varepsilon$, $\sigma$, $E$, and $\varepsilon'$ are functions of Eulerian and Lagrangian displacements. The equilibrium equation of the structure can be obtained by taking the variation of Equation (2.5) with respect to Eulerian and Lagrangian displacements.

$$\delta\Pi = \frac{1}{2} \int_{\Omega_2D} \delta\varepsilon^T \sigma \varepsilon' \, d\Omega_2D + \frac{1}{2} \int_{\Omega_2D} \varepsilon^T \delta\varepsilon' \sigma' \, d\Omega_2D$$

$$+ \frac{1}{2} \int_{\Omega_2D} \varepsilon^T \sigma \varepsilon' \, d\Omega_2D$$

$$= 0$$

(2.6)

2.3. Numerical implementation

This section presents the numerical implementation of the ALE method to obtain the patterning panels. The triangular membrane element, which is a typical element in form-finding process (Tabarrok and Qin 1992), is selected to derive the formulations. This element has three nodes numbered as 1, 2 and 3. There are total eighteen degrees of freedom (DOF) per element as nine Eulerian and nine Lagrangian displacements, which are denoted by $\bar{u}$ and $\bar{u}$, respectively. The numerical solution of patterning problem is obtained in steps by incrementally applying Lagrangian displacement boundary conditions. The discrete set of Equation (2.6) is formed at element level and solved for each Lagrangian displacement step denoted by $\bar{t}(t=1, 2, \ldots, T)$. The next formulations are established to obtain the incremental nodal Eulerian displacements from step $t$ to step $t+1$.

The Eulerian and Lagrangian displacements at step $t$ of nodes per element are denoted by $\bar{u}$ and $\bar{u}$, respectively, as shown in Equation (2.7).

$$\bar{u} = \begin{b matrix} u_1 & v_1 & w_1 \ u_2 & v_2 & w_2 \ u_3 & v_3 & w_3 \ end{b matrix}$$

$$\bar{u} = \begin{b matrix} u_1' & v_1' & w_1' \ u_2' & v_2' & w_2' \ u_3' & v_3' & w_3' \ end{b matrix}$$

(2.7)

The right subscript $i$ indicates the displacements at node $i$. It should be noted that the displacements $\bar{w}_i$ are always zeros, but we still include them in Equation (2.7) for general using. When the patterning is carried out, the fixed Eulerian conditions can be easily applied for those displacements. Expressing the Eulerian and Lagrangian displacements linearly over the element in the reference configuration as:

$$\bar{u} = a_1 + a_2x + a_3y$$

$$\bar{v} = a_4 + a_5x + a_6y$$

$$\bar{w} = a_7 + a_8x + a_9y$$

(2.8)

$$\bar{u} = b_1 + b_2x + b_3y$$

$$\bar{v} = b_4 + b_5x + b_6y$$
\[ \dot{\mathbf{w}} = b_7 + b_8 x + b_9 y \]  
(2.9)

where \( x \) and \( y \) are the coordinate axes of the reference configuration. The values of \( a_i \) and \( b_i \) are obtained by evaluating the displacements at three nodes of element as:

\[ a_2 = (y_{23} \dot{u}_1 + y_{31} \dot{u}_2 + y_{12} \dot{u}_3) / 2S, \]

\[ a_3 = (x_{23} \dot{u}_1 + x_{13} \dot{u}_2 + x_{21} \dot{u}_3) / 2S, \]

\[ a_5 = (y_{23} \dot{v}_1 + y_{31} \dot{v}_2 + y_{12} \dot{v}_3) / 2S, \]

\[ a_6 = (x_{23} \dot{v}_1 + x_{13} \dot{v}_2 + x_{21} \dot{v}_3) / 2S, \]

\[ a_8 = (y_{23} \dot{w}_1 + y_{31} \dot{w}_2 + y_{12} \dot{w}_3) / 2S, \]

\[ a_9 = (x_{23} \dot{w}_1 + x_{13} \dot{w}_2 + x_{21} \dot{w}_3) / 2S, \]  
(2.10)

\[ b_2 = (y_{23} \dot{u}_1 + y_{31} \dot{u}_2 + y_{12} \dot{u}_3) / 2S, \]

\[ b_3 = (x_{23} \dot{u}_1 + x_{13} \dot{u}_2 + x_{21} \dot{u}_3) / 2S, \]

\[ b_5 = (y_{23} \dot{v}_1 + y_{31} \dot{v}_2 + y_{12} \dot{v}_3) / 2S, \]

\[ b_6 = (x_{23} \dot{v}_1 + x_{13} \dot{v}_2 + x_{21} \dot{v}_3) / 2S, \]

\[ b_8 = (y_{23} \dot{w}_1 + y_{31} \dot{w}_2 + y_{12} \dot{w}_3) / 2S, \]

\[ b_9 = (x_{23} \dot{w}_1 + x_{13} \dot{w}_2 + x_{21} \dot{w}_3) / 2S, \]  
(2.11)

where \( S_t \) is the area of the membrane element in the reference configuration; \( \mathbf{x}_j = x_i - x_j; \mathbf{y}_j = y_i - y_j; \mathbf{r}_x \) and \( \mathbf{r}_y \) are reference coordinates at node \( i \). Expressing Equation (2.3) with respect to Eulerian and Lagrangian displacements, the strain components from initial configuration \( \Omega_{3D} \) to the current configuration \( \Omega_{3D} \) at step \( t \) can be obtained by Equation (2.12).

\[ \dot{E}_{xx} = \frac{\partial^2 \dot{u}}{\partial x^2} + \frac{\partial^2 \dot{u}}{\partial y^2} + \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

\[ - \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

\[ \dot{E}_{yy} = \frac{\partial^2 \dot{v}}{\partial y^2} + \frac{\partial^2 \dot{v}}{\partial y^2} + \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

\[ - \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

\[ \dot{E}_{xy} = \frac{\partial^2 \dot{w}}{\partial x \partial y} + \frac{\partial^2 \dot{w}}{\partial y^2} + \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

\[ - \frac{1}{2} \left[ \left( \frac{\partial^2 \dot{u}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{v}}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 \dot{w}}{\partial x \partial y} \right)^2 \right] \]

Substituting Equation (2.8), Equation (2.9), Equation (2.10) and Equation (2.11) into Equation (2.12), Equation (2.13) can be obtained.

\[ \dot{E}_{xx} = a_2 + b_2 + \frac{1}{2} \left[ b_2^2 + b_5^2 + b_8^2 \right] \]

\[ - \frac{1}{2} \left[ a_2^2 + a_5^2 + a_8^2 \right] \]

\[ \dot{E}_{yy} = a_6 + b_6 + \frac{1}{2} \left[ b_3^2 + b_6^2 + b_9^2 \right] \]

\[ - \frac{1}{2} \left[ a_3^2 + a_6^2 + a_9^2 \right] \]

\[ \dot{E}_{xy} = a_3 + a_5 + b_3 + b_5 + \left[ b_2 b_3 + b_5 b_6 + b_8 b_9 \right] \]

\[ - \left[ a_2 a_3 + a_5 a_6 + a_8 a_9 \right] \]  
(2.13)

The variations of these strains are shown in Equation (2.14).

\[ \delta \mathbf{E} = \begin{bmatrix} \delta E_{xx} \\ \delta E_{yy} \end{bmatrix} = \begin{bmatrix} B_0 + \dot{B}_e \end{bmatrix} \begin{bmatrix} \delta \dot{u} \end{bmatrix} \]  
(2.14)

where,

\[ B_0 = \frac{1}{2S} \begin{bmatrix} y_{23} & 0 & 0 & y_{31} & 0 & 0 & y_{12} & 0 & 0 \\ 0 & y_{12} & 0 & 0 & x_{13} & 0 & 0 & x_{21} & 0 \\ x_{23} & 0 & y_{23} & 0 & x_{31} & 0 & 0 & x_{21} & y_{12} \end{bmatrix} \]

\[ \dot{B}_e = \frac{1}{2S} \begin{bmatrix} b_2 y_{23} & b_3 y_{23} & b_8 y_{23} \\ b_3 x_{23} & b_6 x_{23} & b_9 x_{23} \\ b_3 y_{23} + b_6 x_{23} & b_3 x_{23} + b_9 x_{23} \end{bmatrix} \]
\[ T\mathbf{B}_L = \frac{1}{2S_y} \begin{bmatrix} a_1y_{23} & a_1y_{23} & a_1y_{23} \\ a_1x_{32} & a_1x_{32} & a_1x_{32} \\ a_1'y_{23} + a_1'x_{32} & a_1'y_{23} + a_1'x_{32} & a_1'y_{23} + a_1'x_{32} \end{bmatrix} \]

where \( T\mathbf{K}_j \) is defined as

\[ T\mathbf{K}_j = \begin{bmatrix} (a_6 - 1)y_{23} - a_5'x_{32} \\ (a_2 - 1)x_{32} - a_3'y_{23} \\ 0 \\ (a_6 - 1)y_{31} - a_5'x_{13} \\ (a_2 - 1)x_{13} - a_3'y_{31} \\ 0 \\ (a_6 - 1)y_{12} - a_5'x_{21} \\ (a_2 - 1)x_{21} - a_3'y_{12} \\ 0 \end{bmatrix}^{T} \]

Equation (2.17) is nonlinear with Lagrangian and Eulerian strain components, so the Newton-Raphson scheme is used to obtain the results of the incremental displacements from step t to step \( t + 1 \). This process is standard and therefore skipped here.

3. Analytical samples

This section shows the comparison between the proposed method and other published methods in three well known samples. The first and second are Chinese hat and HP shapes, while the third is a cone shape.

3.1. Chinese hat type membrane structures

The same geometry and material used in Linhard, Wuchner, and Bletzinger (2008) and Gale and Lewis (2016) were applied in this sample. The Chinese hat membrane structure comprises two fixed rings. The diameters of upper and lower rings are 1.2 m and 5.6 m, respectively. The height difference of these two fixed rings is 1.0 m. The isotropic ETFE foil of Young’s modulus \( E = 880 \) kN/m, Poisson’s ratio \( \mu = 0.4 \) and thickness \( th=0.25 \times 10^{-3} \) m was chosen. The isotropic prescrted pre-stress of 2 kN/m is used for form-finding and cutting pattern generation. It should be noted that the unit of pre-stress is used in the standard of tensile membrane structures. The SI unit can be obtained by dividing this stress to the thickness of membrane. The whole surface was divided into 12 equal segments in similar pattern to that of previous works, and one selected segment which was patterned by using the proposed method is shown in Figure 2(a).

In this sample, the 2D reference configuration was chosen as the projection of 3D selected segment and was shown by solid line in Figure 2(b). All nodes were fixed in Lagrangian displacement and were free in Eulerian one except for corner node. The results of 2D patterning panel were presented by dash-dot line in Figure 2(b).

Figure 3(a) shows the deviations of resulting stresses from the prescribed pre-stress of 2 kN/m in warp and weft directions, while Figure 3(b) represents the stress distribution of 192 elements. It should be noted that 192 elements are enough to give the converged results. The greatest stress deviations where stresses are higher or lower than the prescribed pre-stress of 2
kN/m are +1.1288 kN/m and −0.6847 kN/m, respectively. The comparison of stress deviations with Linhard, Wuchner, and Bletzinger (2008) and Gale and Lewis (2016) are listed in Table 1. The maximum stress deviation in presented work is considerably smaller than that of these two references, while the minimum value is in middle of the previous results. The effectiveness of the present work is confirmed as the final stress distribution (Figure 3) is closer to the prescribed pre-stress than the previous works.

3.2. **HP type membrane structures**

In the second sample, two HP type membrane structures with four fixed corners, which were used in Kim and Lee (2002), are analysed. The diagonal distance between two corner nodes at the same height is 10.0 m. The height differences are 1.0 m and 4.0 m for Model 1 and Model 2, respectively. An orthotropic elastic material is used, and its properties taken from (Kim and Lee 2002) are shown in Table 2. It is assumed that the prescribed pre-stress of 5 kg/cm is used for both warp and weft directions in form finding and cutting pattern generation. The whole surface was divided into 4 segments in similar pattern to the previous work, and Figure 4 shows the analytical models. Because of symmetry, two segments which named Part 1 and Part 2 as shown in Figure 4(a) were patterned by using the proposed method.

| Analysis | Maximum stress deviation | Minimum stress deviation |
|----------|--------------------------|--------------------------|
|          | (kN/m)                   | (%)                      | (kN/m)                   | (%)                      |
| Linhard, Wuchner, and Bletzinger (2008) | +2.08                     | +104%                    | −0.83                    | −41.5%                   |
| Gale and Lewis (2016) (with shear)   | +2.06                     | +103%                    | −0.395                   | −19.8%                   |
| Present work                          | +1.1288                   | +56.4%                   | −0.6847                  | −34.2%                   |
In this sample, the 2D reference configuration was chosen as the projection of 3D segments and was shown by solid lines in Figure 5. All nodes were fixed in Lagrangian displacement and were free in Eulerian displacement except for the solid nodes (Figure 5). The results of 2D patterning panel were presented by dash-dot line in Figure 5. It should be noted that a half of Part 1 and Part 2 are shown because of symmetry.

Figures 6(a) and 7(a) show the deviations of resulting stresses from the prescribed pre-stress of 5 kg/cm in the warp and weft directions for Model 1 and Model 2, respectively. The stress distributions of 100 elements are represented in Figures 6(b) and 7(b) for Model 1 and Model 2, respectively. The number of 100 elements are used to have a fair comparison with previous work. The same observations with Kim and Lee (2002) are seen as: (1) stresses in the warp direction are more dramatically affected by area changing than those of the weft direction; (2) Model 2 which is higher curvature shows a larger stress deviation than Model 1. Table 3 presents the comparison of maximum and minimum stress deviation in present and previous works. The results of the proposed method show good agreement with Kim and Lee (2002), except in the warp direction of Model 2. In this direction, the greatest stress deviations vary from −64.23(%) to +158.28 (%) in compared with −23.20 (%) to +36.60 (%) of previous work. However, it can be seen from Figure 7 that the maximum stress deviation only occurs in few elements near the cutting lines. These errors are acceptable. The stress coefficients defined in Equation (3.1) are used to extend the comparisons.

### Table 2. Material properties for both Model 1 and Model 2.

| Thickness (th cm) | Young's modulus $E_x$ (kg/cm²) | Young's modulus $E_y$ (kg/cm²) | Shear modulus $G_{xy}$ (kg/cm²) | Poisson's ratio $\nu_{yx}$ | Poisson's ratio $\nu_{xy}$ |
|------------------|---------------------------------|---------------------------------|---------------------------------|--------------------------|--------------------------|
| 0.08             | 2725                            | 8225                            | 712.5                           | 0.29                     | 0.87                     |

![Figure 4. The form finding shape and cutting pattern plan for HP type.](image)

![Figure 5. Cutting pattern shapes for Model 1 and Model 2.](image)
Table 4 shows a good agreement in those coefficients between our present work and Kim and Lee (2002).

\[
S_{xAVE} = \frac{\sum S_x}{\sum S_x0}
\]

\[
S_{yAVE} = \frac{\sum S_y}{\sum S_y0}
\]
\[ S_{x\text{DEV}0} = \frac{\sum |S_{x0} - S_x|}{\sum S_{x0}} \]

\[ S_{y\text{DEV}0} = \frac{\sum |S_{y0} - S_y|}{\sum S_{y0}} \]  

(3.1)

where \( S_{\text{AVE}x}, S_{\text{AVE}y} \) are the average stresses for weft and warp directions; \( S_{x\text{DEV}0}, S_{y\text{DEV}0} \) are the average stresses deviation for weft and warp directions; \( S_x, S_y \) are the resulting stresses for weft and warp directions; and \( S_{x0}, S_{y0} \) are the prescribed pre-stresses for weft and warp directions.

The effect of prescribed pre-stress in Equation (2.16) on cutting pattern is also investigated. Figure 8 shows the patterning of Model 1 and Model 2 with and without prescribed pre-stress. The solid line represents the patterning when pre-stress is not considered, while the dash-and-dot and dash lines show the patterning results of prescribed stresses of 5 kg/cm and 20 kg/cm, respectively. The cutting pattern shapes are shrunk when the prescribed pre-stress accounted in Equation (2.16) increases. The compensation stage can be combined in the proposed method effectively.

### 3.3. Cone type membrane structures

A more complicated shape of membrane structures is analysed in this Section. This structure comprises fabric membrane panels and cables which are indicated by thin and thick lines in Figure 9(c). The dimensions of the structure taken from Kim and Lee (2002) are shown in Figure 9(a). All dimensions are in millimetre. The membrane material is described in detail in Section 3.2, while the axial stiffness of the cables, which was not given explicitly in (Kim and Lee 2002), is assumed as \( 1.95 \times 10^4 \) kg to obtain the appropriate form-finding shape with the previous work. The pre-stress of 5 kg/cm is assumed for both warp and weft directions in form-finding and cutting pattern process. The whole surface is divided into 14 segments in similar pattern to the previous work, and four segments named as (a), (b), (c) and (d) (Figure 9(a)) are selected to pattern because of symmetry.

![Figure 8. Cutting pattern shapes with and without prescribed pre-stress.](image)

![Figure 9. Form finding shape and cutting pattern plan for cone type. Unit of length: mm.](image)
Figure 10. The results of cutting pattern generation for cone type.

| Table 5. Stress coefficients for cone type membrane structures. |
|---------------------------------------------------------------|
|                                                              |
| Present work Kim and Lee (2002) Present work Kim and Lee (2002) |
| S_{AVE} Seg. (a) 0.987 1.010 Seg. (b) 0.987 0.999 |
| S_{AVF} 0.990 1.005 |
| S_{DEFO} 0.027 0.046 |
| S_{AVE} Seg. (c) 0.986 1.001 Seg. (d) 0.989 0.975 |
| S_{AVF} 0.988 0.999 |
| S_{DEFO} 0.044 0.164 |
| S_{AVF} 0.030 0.158 |

Figure 10(a) shows the patterning results of four selected segments. The solid line shows the 2D reference configuration, while the dash-dot line represents the patterning results. It should be noted that all Lagrangian displacements are fixed, while Eulerian displacements are free except for the solid points in Figure 10(a). The warp and weft directions for segment (a) are assigned as shown in Figure 10(a) to have a fair comparison with the previous work. The stress deviations from the intended pre-stress are shown in Figure 10(b). It can be seen that the final stress distribution is close to the prescribed pre-stress as the deviations are nearly zeros. The stress coefficients calculated by Equation (3.1) are shown in Table 5 in comparison to the previous work. The results of the proposed method are better in this sample than that of Kim and Lee (2002) as the average stresses deviation coefficients are close to zeros, especially in segment (c). The previous work based on the optimization techniques, so the stress deviations at the parts of steeper curvature were large (Kim and Lee 2002). The choice of reference configuration in the proposed method could solve this problem. In this analytical sample, the reference configuration was selected as the projection of 3D curved surface, and the obtained results were better than previous ones.

4. Conclusions

This paper presents the implementation of ALE methods for cutting patterning generation of tensile membrane structures. The process of proposed method is:
(1) The 2D initial patterning is selected as the projection of 3D form-finding panel; (2) All nodes are fixed in Lagrangian displacements and free in Eulerian displacements except given boundary conditions; (3) The Eulerian motion or 2D patterning shapes can be obtained with and without the prescribed pre-stress by the proposed formulations. Examples of patterning applied to Chinese hat, HP and cone types of membrane structures show the applicability and accuracy of the proposed cutting pattern method. The stress results are in high agreement with previous ones, although the flattening, stress reduction and compensation stages are carried out simultaneously in the proposed method. However, the theorem of minimum potential energy was employed to derive the equilibrium equation, so the material is in the elastic region. The future work needs to focus on the modification of the proposed formulations to include the nonlinear properties of material.

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