EXTENSION OF LITTLEWOOD’S RULE TO THE 
MULTI-PERIOD STATIC REVENUE MANAGEMENT MODEL 
WITH STANDBY CUSTOMERS

HIDEAKI TAKAGI

Professor Emeritus, University of Tsukuba
Tsukuba Science City, Ibaraki 305-8573, Japan

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Abstract. Classical Littlewood’s rule (1972) for the two-period static revenue management of a single perishable resource is extended to a generic T-period model with monotonically increasing fixed fares, ending with standby customers with a special fare. The expected revenue in the entire period is expressed explicitly in terms of multiple definite integrals involving the distribution function of the demand in each period. The exact optimal protection level in each period is calculated successively, resulting in the maximized total expected revenue. The Brumelle-McGill’s theorem for the optimal booking limits in the T-period model is also extended to a similar model with standby customers. We show some numerical examples with comments on the effects of accepting standby customers on the optimal booking limits and the increase in the expected revenue.

1. Introduction. The mathematical theory for the revenue management of a single perishable resource was initiated with a seminal work by Littlewood [5], who studied a two-period model in the context of airline seat reservation in which low-yield passengers arrive in the early period wanting to reserve seats at a lower fare while high-yield passengers arrive in the late period with willingness to pay a higher fare. Given the probability distribution of the demand by high-yield passengers, he derived a simple formula (now called Littlewood’s rule in the literature) that determines the optimal value for the maximum number of seats allowed for reservation by low-yield passengers (called the optimal booking limit) such that it maximizes the expected total revenue over the two periods.

The two-period model is presented as the starter of mathematical revenue management in many book chapters and tutorials/surveys such as Müller-Bungart [6], Netessine and Shumsky [7], Phillips [8], Talluri [11], Talluri and van Ryzin [12], van Ryzin and Talluri [13], and Walczak et al. [14] using a variety of nomenclature for the low- and high-yield passengers as shown in Table 1.
Table 1. A variety of terms used for two classes of customers in the literature on the two-period static revenue management model.

| Literature                        | class 1 customers            | class 2 customers            |
|-----------------------------------|------------------------------|------------------------------|
| Littlewood [5]                    | high-yield passengers        | low-yield passengers         |
| Müller-Bungart [6, p. 55]         | high fare passengers         | low fare passengers          |
| Netessine and Shumsky [7]         | business customers           | leisure customers            |
| Phillips [8, p. 149]              | full-fare customers          | discount customers           |
| Talluri and van Ryzin [12]*       | class 1 demands              | class 2 demands              |
| Walczak et al. [14, p. 133]       | high fare demand             | low fare demand              |
*Also Talluri [11, p. 663] and van Ryzin and Talluri [13].

Littlewood’s two-period model has been extended to the multi-period version by a number of researchers in close succession. To name a few representative contributions, Belobaba [1] proposed a simple heuristic method called EMSR (expected marginal seat revenue). Curry [3] and Wollmer [15] proposed exact dynamic programming approaches based on Bellman’s optimality principle. Brumelle and McGill [2] formulated a Markov decision process. In particular, Robinson [9] considered a model in which fares do not necessarily monotonically increase as the time goes on. Li and Oum [4] noted that the approaches by Brumelle and McGill [2], Curry [3], and Wollmer [15] are mathematically equivalent. These models are called static as the fares and the probabilistic demands are fixed in each period in advance. Analyses of these models are presented in detail in the monograph by Talluri and van Ryzin [12].

The works cited above have triggered subsequent theoretical studies but, except for EMSR, they have never been broadly adopted and put into practice [14, p. 136]. In the dynamic programming approach, the optimal booking limits are calculated for each period, one by one, in the reversed order of progress in time. However, as far as the present author is aware from the internet literature search, there is no formulation which evaluates the expected revenue over the entire reservation period as a single function of multiple variables (a set of booking limits in each period) and finds the optimal set of the variables for maximizing the expected revenue by means of elementary calculus.

In this paper, we focus on a special case studied by Robinson [9, Section 4], in which the fares increase monotonically over multiple periods and the seats that are left unsold by the end of the last period are sold to standby customers at a discount fare. It is also assumed that there is no cancellation or no-show or overbooking and that customers whose request for reservation has been rejected once no longer come back for buy-up in any later period. These assumptions are in common with Brumelle and McGill [2], Curry [3], and Wollmer [15]. We express the expected revenue in the entire period explicitly in terms of multiple integrals involving the distribution function of the demand in each period. The exact optimal booking limit in each period is calculated successively, resulting in the maximized expected revenue. The Brumelle-McGill’s theorem for the optimal booking limits in a multi-period model with monotonically increasing fares is also extended to a similar model with standby customers for which a special fare is offered. We provide some numerical examples along with comments on the effects of accepting standby customers on the optimal booking limits and the increase in the expected revenue.
Our results complement the dynamic programming approaches published in the past and provide a new insight into the classical problems of revenue management.

![Diagram of a two-period static revenue management model with standby customers.](image-url)

**Figure 1.** Two-period static revenue management model with standby customers.

2. **Two-period model with standby customers.** We describe the model and notation in terms of seat reservation for an airplane. In this section, we show the optimization for the two-period model with standby customers as shown in Figure 1. Let us denote by $C$ the total number of seats in the airplane. We consider two consecutive periods which are numbered the 2nd and 1st periods in the direction of progress in time. Standby customers may be thought of as the customers in the 0th period.

The demands of seat reservation in each period are assumed to be mutually independent. We denote by $D_t$ the random demand in the $t$th period with distribution function $F_t(x) := P\{D_t \leq x\}$ and the density function $f_t(x) := dF_t(x)/dx$, where $F_t(0) = 0$ and $F_t(\infty) = 1$ for $t = 0, 1, 2$. The fare in the $t$th period is denoted by $r_t$ while $r_0$ is arbitrary. Let $b_t$ be the booking limit in the $t$th period, where $b_2 \leq b_1 \leq b_0 \equiv C$.

We assume the nested booking system, meaning that the seats not sold in the 2nd period (because less than $b_2$ customers arrived) can be sold to customers in the 1st period at a fare $r_1$. In these settings, the numbers of seat reservations in the 2nd and 1st periods and the number of seats sold to standby customers are given as follows:

$$S_2(b_2) = \min\{b_2, D_2\},$$

$$S_1(b_1, b_2) = \min\{b_1 - b_2 + \max\{0, b_2 - D_2\}, D_1\},$$

with seats left unsold in the 2nd period,

$$S_0(b_1, b_2) = \min\{C - b_1 + \max\{0, b_1 - b_2 + \max\{0, b_2 - D_2\} - D_1\}, D_0\},$$

with seats left unsold in the 1st period.

with expectation

$$E[S_2(b_2)] = \int_0^{b_2} [1 - F_2(x)]dx,$$
\[
E[S_1(b_1, b_2)] = \int_0^{b_1-b_2} [1 - F_1(x)]dx + \int_0^{b_2} F_2(x)[1 - F_1(b_1-x)]dx,
\]
\[
E[S_0(b_1, b_2)] = \int_0^{C-b_1} [1 - F_0(x)]dx + \int_0^{b_1-b_2} F_1(x)[1 - F_0(C-x-b_2)]dx
+ \int_0^{b_2} F_2(y)dy \int_0^{b_1-y} f_1(x)[1 - F_0(C-x-y)]dx.
\]

The expected revenue in the two periods and from standby customers is given by
\[
R(b_1, b_2) = r_2 E[S_2(b_2)] + r_1 E[S_1(b_1, b_2)] + r_0 E[S_0(b_1, b_2)].
\]

The values of \(\{b_1, b_2\}\) that maximize \(R(b_1, b_2)\), the optimal booking limits \(\{b_1^*, b_2^*\}\) are determined by the condition
\[
\frac{\partial R(b_1, b_2)}{\partial b_t} \bigg|_{b_1=b_1^*, b_2=b_2^*} = 0 \quad t = 1, 2.
\]
(1)

We treat the set of equations for \(\{b_1^*, b_2^*\}\) derived from Eq. (1) for the cases \(r_0 > r_1\) and \(r_0 \leq r_1\), separately.

(i) Case of \(r_0 > r_1\). If we introduce the optimal protection levels \(y_0^*, y_1^*\) in place of optimal booking limits \(b_1^*, b_2^*\) by the relation
\[
y_0^* := C - b_1^* \quad ; \quad y_1^* := C - b_2^* \quad (b_1^* - b_2^* = y_1^* - y_0^*),
\]

we get from Eq. (1) the following:

\[
r_1 = r_0[1 - F_0(C - b_1^*)] = r_0 P\{D_0 > y_0^*\}, \quad \text{ (Littlewood's rule)}
\]

\[
r_2 = r_1[1 - F_1(b_1^* - b_2^*)] + r_0 \int_0^{b_1^*-b_2^*} f_1(x)[1 - F_0(C-x-b_2^*)]dx
= r_1 P\{D_1 > b_1^* - b_2^*\} + r_0 P\{D_0 + D_1 > C - b_2^*, D_1 \leq b_1^* - b_2^*\}
= r_0 P\{D_0 > y_0^*, D_1 > y_1^*\} + P\{D_0 + D_1 > y_1^*, D_1 \leq y_1^* - y_0^*\}
= r_0 P\{D_0 > y_0^*, D_0 + D_1 > y_1^*\}. \quad \text{ (2)}
\]

See Figure 2 for the manipulation of these probabilities. Given \(\{r_2, r_1, r_0\}\), we determine \(\{b_1^*, b_2^*\}\) from Eq. (2), which are used to calculate the maximized expected revenue \(R(b_1^*, b_2^*)\).

(ii) Case of \(r_0 \leq r_1\). Since there is no point in protecting any seats for standby customers in the 1st period, we set the booking limit \(b_1 = C\). Then, the expected revenue in the two periods and from standby customers is given by
\[
R(C, b_2) = r_2 E[S_2(b_2)] + r_1 E[S_1(C, b_2)] + r_0 E[S_0(C, b_2)],
\]
which is a function of a single variable \(b_2\). Therefore, from the condition \(dR(C, b_2)/db_2|_{b_2=b_2^*} = 0\), we get the following equation for \(b_2^*\):

\[
r_2 = r_1[1 - F_1(C - b_2^*)] + r_0 \int_0^{C-b_2^*} f_1(x)[1 - F_0(C-x-b_2^*)]dx
= r_1 P\{D_1 > C - b_2^*\} + r_0 P\{D_0 + D_1 > C - b_2^*, D_1 \leq C - b_2^*\}
= r_1 P\{D_1 > y_1^*\} + r_0 P\{D_0 + D_1 > y_1^*, D_1 \leq y_1^*\}
= r_1 P\{D_1 > y_1^*\} + r_0 (P\{D_0 + D_1 > y_1^*\} - P\{D_1 > y_1^*\})
= (r_1 - r_0) P\{D_1 > y_1^*\} + r_0 P\{D_0 + D_1 > y_1^*\}. \quad \text{ (3)}
\]
Given \( \{r_2, r_1, r_0\} \), we determine \( b_2^* \) from Eq. (3) and calculate the maximized expected revenue \( R(C, b_2^*) \). In the case of \( r_0 = 0 \), no revenue incurs from standby customers. Then we recover Littlewood’s rule \( r_2 = r_1 P\{D_1 > y_1^*\} \).

We note in both Eqs. (2) and (3) that the optimal protection level \( y_1^* \) for class 1 customers does not depend on the demand \( D_2 \) of class 2 customers.

We can proceed to treat the 3-, 4-, \ldots period models with standby customers in a similar fashion and obtain the set of formulas such as Eqs. (2) and 3 for calculating the optimal booking limit in each period. Specifically, for the 3-period model with standby customers, the numbers of seats reserved in the 3rd, 2nd, and 1st periods and sold to standby customers are respectively given by

\[
\begin{align*}
S_3(b_3) &= \min\{b_3, D_3\}, \\
S_2(b_2, b_3) &= \min\{b_2 - b_3 + \max\{0, b_3 - D_3\} - D_2\}, D_2\}, \\
S_1(b_1, b_2, b_3) &= \min\{b_1 - b_2 + \max\{0, b_2 - b_3 + \max\{0, b_3 - D_3\} - D_2\} - D_1\}, D_1\}, \\
S_0(b_1, b_2, b_3) &= \min\{C - b_1 + \max\{0, b_1 - b_2 + \max\{0, b_2 - b_3 + \max\{0, b_3 - D_3\} - D_2\} - D_1\} - D_0\}, D_0\}.
\end{align*}
\]

For the 4-period model with standby customers, the numbers of seats reserved in the 4th, 3rd, 2nd, and 1st periods and sold to standby customers are respectively
We extend the analysis to a

period.

The nested booking system is used such that the seats not reserved up to the

are arbitrary:

The nested booking system is used such that the seats not reserved up to the

tth period can be sold to customers in the t-1st period. Brumelle and McGill [2] derived the following necessary and sufficient condition that \( y_1^*, y_2^*, \ldots, y_T^* \) are optimal protection levels in the T-period model without standby customers:

\[
\begin{align*}
& r_{t+1} = r_1 P\{D_1 > y_1^*, D_1 + D_2 > y_2^*, D_1 + D_2 + D_3 > y_3^*, \ldots, \\
& \quad \quad \quad \quad \quad D_1 + D_2 + \cdots + D_t > y_t^* \} \quad 0 \leq t \leq T - 1,
\end{align*}
\]

(4)

where \( y_t^* \) is related with the optimal booking limit \( b_{t+1}^* \) via

\[
y_t^* := C - b_{t+1}^* \quad 0 \leq t \leq T - 1.
\]

(5)

This theorem is referred to as Brumelle-McGill’s theorem throughout this paper. It implies that the optimal protection level \( y_t^* \) in the tth period depends on the

demands \( \{D_1, D_2, \ldots, D_{t-1}\} \) as well as the optimal protection levels \( \{y_1^*, y_2^*, \ldots, y_{t-1}^*\} \) in the subsequent periods but that it does not depend on the demand \( D_t \) in the tth period.
3.1. **Expected number of reserved seats.** For the $T$-period model with standby customers, we assume the same property for the optimal protection levels $\{y_0^*, y_1^*, y_2^*, \ldots, y_{T-1}^*\}$ as mentioned above for the Brumelle-McGill’s theorem. Thus, in terms of the booking limits $\{b_1, b_2, \ldots, b_T\}$, the number of reserved seats in the $T$th period and its expectation are given by

$$S_T(b_T) = \min\{b_T, D_T\} \quad \text{and} \quad E[S_T(b_T)] = \int_0^{b_T} [1 - F_T(x)]dx.$$  

For $1 \leq t \leq T - 1$, the number of reserved seats in the $t$th period is given by

$$S_t(b_t, b_{t+1}, \ldots, b_{T-1}, b_T) := \min\{b_t - b_{t+1} + \max\{0, b_{t+1} - b_{t+2} + \max\{0, b_{t+2} - b_{t+3} + \cdots + \max\{0, b_{T-1} - b_T + \max\{0, b_T - D_T \} - D_{T-1} \} - \cdots - D_{t+2} \} - D_{t+1} \}, D_t \}. \quad (6)$$

Then the expectation of this quantity is expressed in terms of multiple definite integrals as follows:

$$E[S_t(b_t, b_{t+1}, \ldots, b_{T-1}, b_T)]$$

$$= \int_0^{b_t-b_{t+1}} [1 - F_t(x)]dx_t$$

$$+ \int_0^{b_{t+1}-b_{t+2}} F_{t+1}(x_{t+1})[1 - F_t(b_t - x_{t+1} - b_{t+2})]dx_{t+1}$$

$$+ \int_0^{b_{t+2}-b_{t+3}} F_{t+2}(x_{t+2})dx_{t+2}$$

$$+ \int_0^{b_{t+3}-b_{t+4}} f_{t+1}(x_{t+1})[1 - F_t(b_t - x_{t+1} - x_{t+2} - b_{t+3})]dx_{t+1}$$

$$+ \int_0^{b_{t+4}-b_{t+5}} F_{t+3}(x_{t+3})dx_{t+3} \int_0^{b_{t+2}-x_{t+3}-b_{t+4}} f_{t+2}(x_{t+2})dx_{t+2}$$

$$+ \int_0^{b_{t+5}-b_{t+6}} f_{t+1}(x_{t+1})[1 - F_t(b_t - x_{t+1} - x_{t+2} - x_{t+3} - b_{t+4})]dx_{t+1}$$

$$+ \int_0^{b_{t+6}-b_{t+7}} F_{t+4}(x_{t+4})dx_{t+4} \int_0^{b_{t+3}-x_{t+4}-b_{t+5}} f_{t+3}(x_{t+3})dx_{t+3}$$

$$+ \int_0^{b_{t+7}-b_{t+8}} f_{t+2}(x_{t+2})dx_{t+2}$$

$$+ \int_0^{b_{t+8}-b_{t+9}} f_{t+1}(x_{t+1})$$

$$\times [1 - F_t(b_t - x_{t+1} - x_{t+2} - x_{t+3} - x_{t+4} - b_{t+5})]dx_{t+1}$$

$$+ \cdots$$
Therefore, the expected number of seats reserved during the $T$ periods is given by

$$
\sum_{t=1}^{T} E[S_t(b_t, b_{t+1}, b_{t+2}, \ldots, b_{T-1}, b_T)]
$$

\begin{align*}
&= b_1 - \int_{0}^{b_1-b_2} F_1(x_1)dx_1 - \int_{0}^{b_2-b_3} F_2(x_2)F_1(b_1 - x_2 - b_3)dx_2 \\
&\quad - \int_{0}^{b_3-b_4} F_3(x_3)dx_3 \int_{0}^{b_2-x_3-b_4} f_2(x_2)F_1(b_1 - x_2 - x_3 - b_4)dx_2 \\
&\quad - \int_{0}^{b_4-b_5} F_4(x_4)dx_4 \int_{0}^{b_3-x_4-b_5} f_3(x_3)dx_3 \\
&\quad - \int_{0}^{b_2-x_4-b_5} f_2(x_2)F_1(b_1 - x_2 - x_3 - x_4 - b_5)dx_2 \\
&\quad \cdots \\
&\quad - \int_{0}^{b_{T-1}-b_T} F_{T-1}(x_{T-1})dx_{T-1} \int_{0}^{b_T-x_{T-1}-b_T} f_{T-2}(x_{T-2})dx_{T-2} \\
&\quad \int_{0}^{b_3-x_5-\cdots-x_{T-1}-b_T} f_3(x_3)dx_3 \\
&\quad \int_{0}^{b_2-x_3-\cdots-x_{T-1}-b_T} f_2(x_2)F_1(b_1 - x_2 - x_3 - \cdots - x_{T-1} - b_T)dx_2
\end{align*}
The expectation of this quantity is expressed as follows:

\[
E[S_0(b_1, b_2, \ldots, b_{T-1}, b_T)] = 
\int_0^{b_T} F_T(x_T) dx_T \int_0^{b_{T-1}-x_T} f_{T-1}(x_{T-1}) dx_{T-1} 
\int_0^{b_{T-2}-x_{T-1}-x_T} f_{T-2}(x_{T-2}) dx_{T-2} 
\int_0^{b_{T-3}-x_{T-2}-x_{T-1}-x_T} f_{T-3}(x_{T-3}) dx_{T-3} \cdots 
\int_0^{b_3-x_3-x_4-\cdots-x_{T-1}-x_T} f_3(x_3) dx_3 
\int_0^{b_2-x_2-x_3-\cdots-x_{T-1}-x_T} f_2(x_2) f_1(b_1-x_2-x_3-\cdots-x_{T-1}-x_T) dx_2.
\]

The number of seats sold to standby customers is given by

\[
S_0(b_1, b_2, \ldots, b_{T-1}, b_T) = \min\{\max\{0, C - b_1 + \max\{0, b_1 - b_2 + \max\{0, b_2 - b_3 + \cdots 
+ \max\{0, b_{T-1} - b_T + \max\{0, b_T - D_T - D_{T-1} - \cdots - D_2 - D_1\}, D_0\}\}, 0\}. 
\]

(7)

The expectation of this quantity is expressed as follows:

\[
E[S_0(b_1, b_2, \ldots, b_{T-1}, b_T)] = 
\int_0^{C-b_1} [1 - F_0(x_0)] dx_0 + \int_0^{b_1-b_2} F_1(x_1)[1 - F_0(C - x_1 - b_2)] dx_1 
+ \int_0^{b_2-b_3} F_2(x_2) dx_2 \int_0^{b_1-x_2-b_3} f_1(x_1)[1 - F_0(C - x_1 - x_2 - b_3)] dx_1 
+ \int_0^{b_3-b_4} F_3(x_3) dx_3 \int_0^{b_2-x_3-b_4} f_2(x_2) dx_2 
\int_0^{b_1-x_3-b_4} f_1(x_1)[1 - F_0(C - x_1 - x_2 - x_3 - b_4)] dx_1 
+ \int_0^{b_4-b_5} F_4(x_4) dx_4 \int_0^{b_3-x_4-b_5} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-b_5} f_2(x_2) dx_2 
\int_0^{b_1-x_2-x_3-x_4-b_5} f_1(x_1)[1 - F_0(C - x_1 - x_2 - x_3 - x_4 - b_5)] dx_1 
+ \cdots 
+ \int_0^{b_{T-1}-b_T} F_{T-1}(x_{T-1}) dx_{T-1} \int_0^{b_{T-2}-x_{T-1}-b_T} f_{T-2}(x_{T-2}) dx_{T-2} \cdots 
\int_0^{b_3-x_3-x_4-\cdots-x_{T-1}-b_T} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-\cdots-x_{T-1}-b_T} f_2(x_2) dx_2 
\int_0^{b_1-x_2-x_3-\cdots-x_{T-1}-b_T} f_1(x_1)[1 - F_0(C - x_1 - x_2 - \cdots - x_{T-1} - b_T)] dx_1.
\]
\[
\begin{align*}
&\int_0^{b_T} F_T(x_T) dx_T \int_0^{b_T-1-x_T} f_{T-1}(x_{T-1}) dx_{T-1} \\
&\int_0^{b_T-2-x_{T-1}-x_T} f_{T-2}(x_{T-2}) dx_{T-2} \\
&\int_0^{b_T-3-x_{T-2}-x_{T-1}-x_T} f_{T-3}(x_{T-3}) dx_{T-3} \cdots \\
&\int_0^{b_3-x_4-x_5-\cdots-x_{T-1}-x_T} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-\cdots-x_T-1-x_T} f_2(x_2) dx_2 \\
&\int_0^{b_1-x_2-x_3-\cdots-x_T-1-x_T} f_1(x_1) [1 - F_0(C - x_1 - x_2 - \cdots - x_{T-1} - x_T)] dx_1.
\end{align*}
\]

The sum of the expected number of seats reserved in each of the \( T \) periods and the expected number of seats sold to standby customers is

\[ S(b_1, b_2, \ldots, b_{T-1}, b_T) := \sum_{t=1}^{T} E[S_t(b_1, \ldots, b_{T-1}, b_T)] + E[S_0(b_1, b_2, \ldots, b_{T-1}, b_T)], \]

which can be written explicitly for the expected number of \textit{vacant seats} at the time of airplane’s departure as

\[
C - S(b_1, b_2, \ldots, b_{T-1}, b_T)
\]

\[= \int_0^{C-b_1} F_0(x_0) dx_0 + \int_0^{b_1-b_2} F_1(x_1) F_0(C - x_1 - b_2) dx_1
+ \int_0^{b_2-b_3} F_2(x_2) dx_2 \int_0^{b_1-x_2-b_3} f_1(x_1) F_0(C - x_1 - x_2 - b_3) dx_1
+ \int_0^{b_3-b_4} F_3(x_3) dx_3 \int_0^{b_2-x_3-b_4} f_2(x_2) dx_2
+ \int_0^{b_4-b_5} F_4(x_4) dx_4 \int_0^{b_3-x_4-b_5} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-b_5} f_2(x_2) dx_2
+ \int_0^{b_5} F_5(x_5) dx_5 \int_0^{b_4-x_5-b_5} f_4(x_4) dx_4 \int_0^{b_3-x_4-x_5-b_5} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-x_5-b_5} f_2(x_2) dx_2
+ \cdots
\]

\[+ \int_0^{b_T-1-b_T} F_{T-1}(x_{T-1}) dx_{T-1} \int_0^{b_T-2-x_{T-2}-b_T} f_{T-2}(x_{T-2}) dx_{T-2} \cdots
+ \int_0^{b_3-x_4-x_5-\cdots-x_{T-1}-b_T} f_3(x_3) dx_3 \int_0^{b_2-x_3-x_4-\cdots-x_{T-1}-b_T} f_2(x_2) dx_2
+ \int_0^{b_2-x_3-x_4-\cdots-x_T-b_T} f_1(x_1) F_0(C - x_1 - x_2 - x_3 - \cdots - x_{T-1} - b_T) dx_1.\]
The expected revenue from the reservations during $T$ periods and from the sales to standby customers amounts to

$$R(b_1, b_2, \ldots, b_{T-1}, b_T) = \sum_{t=1}^{T} r_t E[S_t(b_t, \ldots, b_{T-1}, b_T)] + r_0 E[S_0(b_1, b_2, \ldots, b_{T-1}, b_T)].$$

The values of $\{b_1, b_2, \ldots, b_{T-1}, b_T\}$, denoted by $\{b_1^*, b_2^*, \ldots, b_{T-1}^*, b_T^*\}$, that maximize this quantity can be obtained from the condition that the first partial derivatives vanish at these points as follows:

$$\frac{\partial R(b_1, b_2, \ldots, b_{T-1}, b_T)}{\partial b_t} \bigg|_{b_1=b_1^*, b_2=b_2^*, \ldots, b_{T-1}=b_{T-1}^*, b_T=b_T^*} = 0 \quad 1 \leq t \leq T. \quad (8)$$

We treat the set of equations for $\{b_1^*, b_2^*, \ldots, b_{T-1}^*, b_T^*\}$ derived from Eq. (8) for the cases $r_0 > r_1$ and $r_0 \leq r_3$ separately.

(i) Case of $r_0 > r_1$, in which the fare for standby customers is set higher than that for the fare in the 1st period. In this case, we impose the booking limit $b_1 < C$ on the reservation in the 1st period as well and get the following set of equations from Eq. (8):

$$r_{t+1} = r_t \left[1 - F_t(b_t^* - b_{t+1}^*)\right]$$

$$+ r_{t-1} \int_0^{b_t^* - b_{t+1}^*} f_t(x_t) \left[1 - F_{t-1}(b_{t-1}^* - x_t - b_{t+1}^*)\right] dx_t$$

$$+ r_{t-2} \int_0^{b_t^* - b_{t+1}^*} f_t(x_t) dx_t$$

$$+ \int_0^{b_t^* - b_{t+1}^*} f_{t-1}(x_{t-1}) \left[1 - F_{t-2}(b_{t-2}^* - x_{t-1} - x_t - b_{t+1}^*)\right] dx_{t-1}$$

$$+ r_{t-3} \int_0^{b_t^* - b_{t+1}^*} f_t(x_t) dx_t \int_0^{b_{t-1}^* - b_{t+1}^*} f_{t-1}(x_{t-1}) dx_{t-1}$$

$$+ \int_0^{b_t^* - b_{t+1}^*} f_{t-2}(x_{t-2}) \left[1 - F_{t-3}(b_{t-3}^* - x_{t-2} - x_{t-1} - x_t - b_{t+1}^*)\right] dx_{t-2}$$

$$+ \cdots$$
\[
+ r_2 \int_0^{b_{t+1}^* - b_{t+1}^-} f_t(x_t) \, dx_t \int_0^{b_{t-1}^* - x_{t-1} - b_{t+1}^-} f_{t-1}(x_{t-1}) \, dx_{t-1} \\
+ \int_0^{b_{t-2}^* - x_{t-2} - b_{t+1}^-} f_{t-2}(x_{t-2}) \, dx_{t-2} \\
+ \int_0^{b_{t-3}^* - x_{t-3} - x_{t-1} - b_{t+1}^-} f_{t-3}(x_{t-3}) \, dx_{t-3} \cdots \\
+ \int_0^{b_{t-4}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_4(x_4) \, dx_4 \int_0^{b_{t-2}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_2(x_2) \, dx_2 \\
\times [1 - F_2(b_2^* - x_3 - x_4 - x_5 - \cdots - x_t - b_{t+1}^-)] \, dx_3 \\
+ r_1 \int_0^{b_{t+1}^* - b_{t+1}^-} f_t(x_t) \, dx_t \int_0^{b_{t-1}^* - x_{t-1} - b_{t+1}^-} f_{t-1}(x_{t-1}) \, dx_{t-1} \\
+ \int_0^{b_{t-2}^* - x_{t-2} - x_{t-1} - b_{t+1}^-} f_{t-2}(x_{t-2}) \, dx_{t-2} \\
+ \int_0^{b_{t-3}^* - x_{t-3} - x_{t-1} - b_{t+1}^-} f_{t-3}(x_{t-3}) \, dx_{t-3} \cdots \\
+ \int_0^{b_{t-4}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_4(x_4) \, dx_4 \int_0^{b_{t-2}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_2(x_2) \, dx_2 \\
\times [1 - F_1(b_1^* - x_2 - x_3 - x_4 - \cdots - x_t - b_{t+1}^-)] \, dx_2 \\
+ r_0 \int_0^{b_{t+1}^* - b_{t+1}^-} f_t(x_t) \, dx_t \int_0^{b_{t-1}^* - x_{t-1} - b_{t+1}^-} f_{t-1}(x_{t-1}) \, dx_{t-1} \\
+ \int_0^{b_{t-2}^* - x_{t-2} - b_{t+1}^-} f_{t-2}(x_{t-2}) \, dx_{t-2} \\
+ \int_0^{b_{t-3}^* - x_{t-3} - x_{t-1} - b_{t+1}^-} f_{t-3}(x_{t-3}) \, dx_{t-3} \cdots \\
+ \int_0^{b_{t-4}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_4(x_4) \, dx_4 \int_0^{b_{t-2}^* - x_4 - \cdots - x_2 - b_{t+1}^-} f_2(x_2) \, dx_2 \\
\times [1 - F_0(C - x_1 - x_2 - x_3 - \cdots - x_t - b_{t+1}^-)] \, dx_1 \\
0 \leq t \leq T - 1.
\]

Using the optimal protection levels \(\{y_0^*, y_1^*, \ldots, y_{T-1}^*\}\) defined in Eq. (5), we obtain

\[
r_{t+1} = r_t P\{D_t > y_t^* - y_{t-1}^*\} \\
+ r_{t-1} P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t > y_t^* - y_{t-2}^*\} \\
+ r_{t-2} P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t > y_t^* - y_{t-3}^*\}
\]
calculate the maximized expected revenue by
than that for the fare in the 1st period. Then there is no point in protecting

t\to T
\text{of} \quad T
\text{ing with Littlewood's rule}
D
\text{Equation (9) can be proved by mathematical induction with respect to } t
\text{ing for standby customers is set equal to or lower than that for the fare in the 1st period. Then there is no point in protecting}
any seats for standby customers in the 1st period. In other words, we set the booking limit $b_1 = C$. The set of equations by letting $b_1 = C$ in (i) becomes

$$r_{t+1} = r_t [1 - F_t(b^*_t - b^*_{t+1})]$$

$$+ r_{t-1} \int_0^{b^*_t - b^*_{t+1}} f_t(x_t) [1 - F_{t-1}(b^*_{t-1} - x_t - b^*_t)] dx_t$$

$$+ r_{t-2} \int_0^{b^*_t - b^*_{t+1}} f_t(x_t) dx_t$$

$$\int_0^{b^*_{t-1} - x_t - b^*_t} f_{t-1}(x_{t-1}) [1 - F_{t-2}(b^*_{t-2} - x_{t-1} - x_t - b^*_t)] dx_{t-1}$$

$$+ r_{t-3} \int_0^{b^*_t - b^*_t} f_t(x_t) dx_t \int_0^{b^*_{t-1} - x_t - b^*_t} f_{t-1}(x_{t-1}) dx_{t-1}$$

$$\int_0^{b^*_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-2}(x_{t-2}) dx_{t-2}$$

$$\times [1 - F_{t-3}(b^*_t - x_{t-2} - x_{t-1} - x_t - b^*_{t+1})] dx_{t-2}$$

$$+ \cdots$$

$$+ r_2 \int_0^{b^*_t - b^*_t} f_t(x_t) dx_t \int_0^{b^*_{t-1} - x_t - b^*_t} f_{t-1}(x_{t-1}) dx_{t-1}$$

$$\int_0^{b^*_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-2}(x_{t-2}) dx_{t-2}$$

$$\int_0^{b^*_{t-3} - x_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-3}(x_{t-3}) dx_{t-3} \cdots$$

$$\int_0^{b^*_t - x_5 - x_6 - \cdots - x_t - b^*_{t+1}} f_4(x_4) dx_4 \int_0^{b^*_t - x_4 - x_5 - \cdots - x_t - b^*_{t+1}} f_3(x_3)$$

$$\times [1 - F_2(b^*_t - x_3 - x_4 - x_5 - \cdots - x_t - b^*_{t+1})] dx_3$$

$$+ r_1 \int_0^{b^*_t - b^*_t} f_t(x_t) dx_t \int_0^{b^*_{t-1} - x_t - b^*_t} f_{t-1}(x_{t-1}) dx_{t-1}$$

$$\int_0^{b^*_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-2}(x_{t-2}) dx_{t-2}$$

$$\int_0^{b^*_{t-3} - x_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-3}(x_{t-3}) dx_{t-3} \cdots$$

$$\int_0^{b^*_t - x_5 - x_6 - \cdots - x_t - b^*_{t+1}} f_4(x_4) dx_4 \int_0^{b^*_t - x_4 - x_5 - \cdots - x_t - b^*_{t+1}} f_3(x_3) dx_3$$

$$\int_0^{b^*_t - x_3 - x_4 - \cdots - x_t - b^*_{t+1}} f_2(x_2)$$

$$\times [1 - F_1(C - x_2 - x_3 - x_4 - \cdots - x_t - b^*_{t+1})] dx_2$$

$$+ r_0 \int_0^{b^*_t - b^*_t} f_t(x_t) dx_t \int_0^{b^*_{t-1} - x_t - b^*_t} f_{t-1}(x_{t-1}) dx_{t-1}$$

$$\int_0^{b^*_{t-2} - x_{t-1} - x_t - b^*_t} f_{t-2}(x_{t-2}) dx_{t-2}$$
In terms of optimal protection levels with $y_0^* = 0$, we get

$$r_{t+1} = r_t P\{D_t > y_t^* - y_{t-1}^*\} + r_{t-1} P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t > y_t^* - y_{t-2}^*\} + r_{t-2} P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t > y_t^* - y_{t-3}^*\} + r_{t-3} P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-3}^*, D_{t-3} + D_{t-2} + D_{t-1} + D_t > y_t^* - y_{t-4}^*\} + \cdots$$

$$+ r_2 P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-3}^*, D_{t-3} + D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-4}^*, \ldots, D_4 + D_5 + D_6 + \cdots + D_t \leq y_t^* - y_3^*, \ldots, D_3 + D_4 + D_5 + \cdots + D_t \leq y_t^* - y_2^*, D_2 + D_3 + D_4 + D_5 + \cdots + D_t > y_t^* - y_1^*\} + r_1 P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-3}^*, D_{t-3} + D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-4}^*, \ldots, D_4 + D_5 + D_6 + \cdots + D_t \leq y_t^* - y_3^*, D_3 + D_4 + D_5 + \cdots + D_t \leq y_t^* - y_2^*, D_2 + D_3 + D_4 + \cdots + D_t \leq y_t^* - y_1^*, D_1 + D_2 + D_3 + D_4 + \cdots + D_t > y_t^*\} + r_0 P\{D_t \leq y_t^* - y_{t-1}^*, D_{t-1} + D_t \leq y_t^* - y_{t-2}^*, D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-3}^*, D_{t-3} + D_{t-2} + D_{t-1} + D_t \leq y_t^* - y_{t-4}^*, \ldots, D_4 + D_5 + D_6 + \cdots + D_t \leq y_t^* - y_3^*, D_3 + D_4 + D_5 + \cdots + D_t \leq y_t^* - y_2^*, D_2 + D_3 + D_4 + \cdots + D_t \leq y_t^* - y_1^*, D_1 + D_2 + D_3 + \cdots + D_t \leq y_t^*, D_0 + D_1 + D_2 + D_3 + \cdots + D_t > y_t^*\}$$
Carlo simulation for large values of $T$ the convention (equated by software package such as Mathematica for small values of $T$) functions for the demand in each period. Then, such integrals can be readily evaluated in terms of multiple definite integrals of the distribution probabilities appearing in these formulas. Therefore, it would be useful to provide McGill’s theorem in Eq. (3.3). Expressions of the joint probabilities in the extended Brumelle-McGill’s theorem. The results in Eqs. (3.3) and (3.4) are derived by mathematical induction with respect to $t$ starting with $r_2$ given in Eq. (3.1). Solving Eq. (3.3) for $t = 1, 2, \ldots$ (using the expressions in terms of multiple integrals shown above) successively, we obtain the optimal booking limits $\{b_2^*, \ldots, b_T^*\}$ and calculate the maximized expected revenue by

$$
R(C, b_2^*, \ldots, b_T^*) = \sum_{t=2}^{T} r_t E[S_t(b_1^*, \ldots, b_{t-1}^*, b_t^*)] + r_1 E[S_1(C, b_2^*, \ldots, b_{T-1}^*, b_T^*)] + r_0 E[S_0(b_2^*, \ldots, b_{T-1}^*, b_T^*)].
$$

In particular, if there are an infinite number of standby customers ($D_0 \to \infty$), we get

$$
r_{t+1} = (r_1 - r_0)P\{D_1 > y_1^*, D_1 + D_2 > y_2^*, \ldots, D_1 + D_2 + \cdots + D_t > y_t^*\}
$$

which is recursively reduced to

$$
r_{t+1} = (r_1 - r_0)P\{D_1 > y_1^*, D_1 + D_2 > y_2^*, \ldots, D_1 + D_2 + \cdots + D_t > y_t^*\}
$$

Equation (10) can be proved by mathematical induction with respect to $t$ starting with $r_2$ given in Eq. (3.1). Solving Eq. (10) for $t = 1, 2, \ldots$ (using the expressions in terms of multiple integrals shown above) successively, we obtain the optimal booking limits $\{b_2^*, \ldots, b_T^*\}$ and calculate the maximized expected revenue by

$$
r_{t+1} = (r_1 - r_0)P\{D_1 > y_1^*, D_1 + D_2 > y_2^*, \ldots, D_1 + D_2 + \cdots + D_t > y_t^*\}
$$

This formula can be obtained by applying the Brumelle-McGill’s theorem shown in Eq. (4) to the $T$-period model without standby customers after the fare in each period is subtracted by $r_0$. In the case of $r_0 = 0$, there is no revenue from standby customers. Then, Eqs. (10) and (11) are reduced to the Brumelle-McGill’s theorem. The results in Eqs. (10) and (11) are derived by the author for the first time.

3.3. Expressions of the joint probabilities in the extended Brumelle-McGill theorem with multiple definite integrals. Although the Brumelle-McGill’s theorem in Eq. (4) or (9) and our extension in Eq. (10) for the model with standby customers are neat, it is not obvious how to evaluate numerically the joint probabilities appearing in these formulas. Therefore, it would be useful to provide their explicit expressions in terms of multiple definite integrals of the distribution functions for the demand in each period. Then, such integrals can be readily evaluated by software package such as Mathematica for small values of $T$ or by Monte Carlo simulation for large values of $T$ as suggested by Robinson [9]. Below, we use the convention $(x)^+ := \max\{0, x\}$. 

For \( r_0 > r_1 \), we have Eq. (9) with
\[
P\{D_0 > y_0^*, D_0 + D_1 > y_1^*, D_0 + D_1 + D_2 > y_2^*,
D_0 + D_1 + D_2 + D_3 > y_3^*, \ldots,
D_0 + D_1 + D_2 + \cdots + D_{t-3} + D_{t-2} > y_{t-2}^*,
D_0 + D_1 + D_2 + \cdots + D_{t-2} + D_{t-1} > y_{t-1}^*,
D_0 + D_1 + D_2 + \cdots + D_{t-1} + D_t > y_t^*\}
\[
= \int_{y_0^*}^{\infty} f_0(x_0) dx_0 \int_{y_1^* - x_0}^{\infty} f_1(x_1) dx_1 \int_{y_2^* - x_0 - x_1}^{\infty} f_2(x_2) dx_2
\int_{y_3^* - x_0 - x_1 - x_2}^{\infty} f_3(x_3) dx_3 \cdots
\int_{y_{t-2}^* - x_0 - x_1 - \cdots - x_{t-3}}^{\infty} f_{t-2}(x_{t-2}) dx_{t-2}
\int_{y_{t-1}^* - x_0 - x_1 - \cdots - x_{t-2}}^{\infty} f_{t-1}(x_{t-1})
\times [1 - F_t(y_t^* - x_0 - x_1 - x_2 \cdots - x_{t-1})] dx_{t-1}
\]
\[
= 1 - F_0(y_0^*) - \int_{y_0^*}^{y_1^*} f_0(x_0) F_1(y_1^* - x_0) dx_0
- \int_{y_0^*}^{y_2^*} f_0(x_0) \int_{y_1^* - x_0}^{y_2^* - x_0} f_1(x_1) F_2(y_2^* - x_0 - x_1) dx_1 dx_0
- \int_{y_0^*}^{y_3^*} f_0(x_0) \int_{y_1^* - x_0}^{y_2^* - x_0} f_1(x_1) dx_1
\int_{y_3^* - x_0 - x_1}^{y_3^* - x_0 - x_1} f_2(x_2) F_3(y_3^* - x_0 - x_1 - x_2) dx_2
\cdots
- \int_{y_0^*}^{y_t^*} f_0(x_0) \int_{y_1^* - x_0}^{y_2^* - x_0} \int_{y_3^* - x_0 - x_1}^{y_3^* - x_0 - x_1} \cdots \int_{y_{t-2}^* - x_0 - x_1 - \cdots - x_{t-3}}^{y_{t-2}^* - x_0 - x_1 - \cdots - x_{t-3}} f_{t-1}(x_{t-1})
\times [1 - F_t(y_t^* - x_0 - x_1 - x_2 \cdots - x_{t-1})] dx_{t-1}.
\]

For \( r_0 \leq r_1 \), we have Eq. (10) with
\[
P\{D_1 > y_1^*, D_1 + D_2 > y_2^*, \ldots, D_1 + D_2 + D_3 > y_3^*, \ldots,
D_1 + D_2 + \cdots + D_{t-3} + D_{t-2} > y_{t-2}^*,
D_1 + D_2 + \cdots + D_{t-2} + D_{t-1} > y_{t-1}^*,
D_1 + D_2 + \cdots + D_{t-1} + D_t > y_t^*\}
\]
\[
\begin{align*}
&= \int_{y_1^*}^{\infty} f_1(x_1)dx_1 \int_{y_2^* - x_1}^{\infty} f_2(x_2)dx_2 \int_{y_3^* - x_1 - x_2}^{\infty} f_3(x_3)dx_3 \cdots \\
&\quad \int_{y_{t-2}^* - x_1 - x_2 - \cdots - x_{t-3}}^{\infty} f_{t-2}(x_{t-2})dx_{t-2} \\
&\quad \int_{y_{t-1}^* - x_1 - x_2 - \cdots - x_{t-2}}^{\infty} f_{t-1}(x_{t-1}) \\
&\quad \times [1 - F_t(y_t^* - x_1 - x_2 - \cdots - x_{t-1})]dx_{t-1} \\
&= 1 - \int_{0}^{y_1^*} f_1(x_1)F_2(y_2^* - x_1)dx_1 \\
&\quad - \int_{y_1^*}^{y_2^*} f_1(x_1)\int_{y_2^* - x_1}^{\infty} f_2(x_2)F_3(y_3^* - x_1 - x_2)dx_2 \\
&\quad - \cdots \\
&\quad - \int_{y_1^*}^{y_2^*} f_1(x_1)\int_{y_2^* - x_1}^{\infty} f_2(x_2)\int_{y_3^* - x_1 - x_2}^{\infty} f_3(x_3)dx_3 \cdots \\
&\quad \int_{y_{t-2}^* - x_1 - x_2 - \cdots - x_{t-3}}^{\infty} f_{t-2}(x_{t-2})dx_{t-2} \\
&\quad \int_{y_{t-1}^* - x_1 - x_2 - \cdots - x_{t-2}}^{\infty} f_{t-1}(x_{t-1}) \\
&\quad \times F_t(y_t^* - x_1 - x_2 - \cdots - x_{t-1})dx_{t-1}
\end{align*}
\]

and

\[
P\{D_0 + D_1 > y_1^*, D_0 + D_1 + D_2 > y_2^*, D_0 + D_1 + D_2 + D_3 > y_3^*, \ldots, \\
D_0 + D_1 + D_2 + \cdots + D_{t-3} + D_{t-2} > y_{t-2}^*, \\
D_0 + D_1 + D_2 + \cdots + D_{t-2} + D_{t-1} > y_{t-1}^*, \\
D_0 + D_1 + D_2 + \cdots + D_{t-1} + D_t > y_t^* \}
\]

\[
= \int_{0}^{\infty} f_0(x_0)dx_0 \int_{y_1^* - x_0}^{\infty} f_1(x_1)dx_1 \int_{y_2^* - x_0 - x_1}^{\infty} f_2(x_2)dx_2 \\
\quad \int_{y_3^* - x_0 - x_1 - x_2}^{\infty} f_3(x_3)dx_3 \cdots \\
\quad \int_{y_{t-3}^* - x_0 - x_1 - x_2 - \cdots - x_{t-3}}^{\infty} f_{t-2}(x_{t-2})dx_{t-2} \\
\quad \int_{y_{t-2}^* - x_0 - x_1 - x_2 - \cdots - x_{t-2}}^{\infty} f_{t-1}(x_{t-1}) \\
\quad \times [1 - F_t(y_t^* - x_0 - x_1 - x_2 - \cdots - x_{t-1})]dx_{t-1} \\
= 1 - \int_{0}^{y_1^*} f_0(x_0)F_1(y_1^* - x_0)dx_0 \\
- \int_{0}^{y_2^*} f_0(x_0)\int_{y_2^* - x_0}^{\infty} f_1(x_1)F_2(y_2^* - x_0 - x_1)dx_1
\]
In the numerical examples in this paper, we use the values of the demand in each period in the four-period model with standby customers. These are given by where the density and distribution functions for the standard normal distribution deviate from the respective. However the deviation is negligible if $\sigma_t/\mu_t \ll 1$.

In Table 2, we show an example of the fares and parameters for the distribution of the demand in each period in the four-period model with standby customers. These values have been taken from Belobaba [1, Section 5.6] and Robinson [9, Section 4]. Let the total number of seats be $C = 107$. In Table 3, we show numerical values for the optimal booking limits and the resultant maximized expected revenue in the 2-, 3- and 4-period models with standby customers. In addition, we show the expected total number of seats booked before the departure of the airplane. We display excessive numbers of digits in order to demonstrate the subtlety of optimal values.

In Table 3, we may observe the following.

- The partial set of optimal booking limits in 1st through $t (< T)$th periods in the $T$-period model is identical with the full set of optimal booking limits in the $T$-period model, which complies with Bellman’s optimality principle.
- If any standby customers are forecast, one should try to accept them at the highest fare by lowering the booking limits over the preceding periods in order to maximize the expected revenue. However, we must note that such a policy is

4. Numerical example. In the numerical examples in this paper, we use the following distribution function $F_t(x)$ and the density function $f_t(x)$ for the demand $D_t$ in the $t$th period by modifying those of the normal distribution with mean $\mu_t$ and standard deviation $\sigma_t$ so that $F_t(0) = 0$ and $F_t(\infty) = 1$:

$$
F_t(x) := \left[ \Phi \left( \frac{x - \mu_t}{\sigma_t} \right) - \Phi \left( \frac{-\mu_t}{\sigma_t} \right) \right] / \left[ 1 - \Phi \left( \frac{-\mu_t}{\sigma_t} \right) \right],
$$

$$
f_t(x) := \frac{1}{\sigma_t} \phi \left( \frac{x - \mu_t}{\sigma_t} \right) / \left[ 1 - \Phi \left( \frac{-\mu_t}{\sigma_t} \right) \right] \quad 0 \leq x < \infty,
$$

where the density and distribution functions for the standard normal distribution are given by

$$
\phi(x) := \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right); \quad \Phi(x) := \int_{-\infty}^{x} \phi(y)dy = \frac{1}{2} \left[ 1 + \text{Erf} \left( \frac{x}{\sqrt{2}} \right) \right].
$$

By this modification, the mean and standard deviation for $D_t$ deviate from the values of $\mu_t$ and $\sigma_t$, respectively. However the deviation is negligible if $\sigma_t/\mu_t \ll 1$.

In Table 2, we show an example of the fares and parameters for the distribution of the demand in each period in the four-period model with standby customers. These values have been taken from Belobaba [1, Section 5.6] and Robinson [9, Section 4]. Let the total number of seats be $C = 107$. In Table 3, we show numerical values for the optimal booking limits and the resultant maximized expected revenue in the 2-, 3- and 4-period models with standby customers. In addition, we show the expected total number of seats booked before the departure of the airplane. We display excessive numbers of digits in order to demonstrate the subtlety of optimal values.

In Table 3, we may observe the following.

- The partial set of optimal booking limits in 1st through $t (< T)$th periods in the $T$-period model is identical with the full set of optimal booking limits in the $T$-period model, which complies with Bellman’s optimality principle.
- If any standby customers are forecast, one should try to accept them at the highest fare by lowering the booking limits over the preceding periods in order to maximize the expected revenue. However, we must note that such a policy is
Table 2. The fare and parameters of demand in each period used in the numerical example.

| th period | Fare | Mean | Standard deviation | \( \sigma_t / \mu_t \) |
|-----------|------|------|--------------------|--------------------------|
| 0         | Variable | 10.0 | 10.0000 | 2.0 | 2.0000 | 0.2000 |
| 1         | 105  | 20.3 | 20.5135 | 8.6 | 8.3414 | 0.4236 |
| 2         | 83   | 33.4 | 33.9289 | 15.1 | 14.9436 | 0.4521 |
| 3         | 57   | 19.3 | 19.7139 | 9.2 | 8.7453 | 0.4767 |
| 4         | 39   | 29.7 | 30.1047 | 13.1 | 12.6264 | 0.4411 |

Based on the assumption of the static model in which the fare and the demand distribution of standby customers are respectively set fixed in advance so that the demand does not change dynamically depending on the fare seen by them on the spot.

5. Concluding remarks. In this paper we have presented a method of calculating the optimal booking limits in the multiple-period static model with standby customers as extension of the classical Littlewood’s rule and Brumelle-McGill’s theorem. Our method complements the traditional approaches based on Bellman’s dynamic programming. Given the distribution function for the demand in each period, our explicit formulas include multiple definite integrals for calculating the optimal booking limits as well as the resulting maximized expected revenue, which can be easily evaluated and quickly computed by off-the-shelf software such as Mathematica or by Monte Carlo simulation.

One of the merits of our approach is that we can treat the successive buy-up behavior of customers [1] in a multi-period model with monotonically increasing fares in an exact manner, which is said to be difficult by Talluri and van Ryzin [12, p. 62], [13]. In fact, a static model with buy-up cannot be formulated by dynamic programming because the optimal booking limit in each period does depend on the demands and the booking limits in the past periods. A result of study for this model will be presented elsewhere.

REFERENCES

[1] P. P. Belobaba, Air Travel Demand and Airline Seat Inventory Management, Ph.D thesis, Massachusetts Institute of Technology, 1987.
[2] S. L. Brumelle and J. I. McGill, Airline seat allocation with multiple nested fare classes, Operations Research, 41 (1993), 127–137.
[3] R. E. Curry, Optimal airline seat allocation with fare classes nested by origins and destinations, Transportation Science, 24 (1990), 169–243.
[4] M. Z. F. Li and T. H. Oum, A note on the single leg, multifare seat allocation problem, Transportation Science, 36 (2002), 271–354.
[5] K. Littlewood, Forecasting and control of passenger bookings, J. Revenue Pricing Manag., 4 (2005), 111–123.
[6] M. Müller-Bungart, Revenue Management with Flexible Products: Models and Methods for the Broadcasting Industry, Springer-Verlag Berlin Heidelberg, 2007.
[7] S. Netessine and R. Shumsky, Introduction to the theory and practice of yield management, INFORMS Transactions on Education, 3 (2002), 34–44.
[8] R. Phillips, Pricing and Revenue Optimization, Stanford University Press, 2005.
[9] L. W. Robinson, Optimal and approximate control policies for airline booking with sequential nonmonotonic fare classes, Operations Research, 43 (1995), 252–263.
Table 3. Optimization of booking limits in the 2-, 3-, and 4-period static revenue management models with standby customers. \( C = 107 \) is the total number of seats.

(a) 2-period model with standby customers (expected total demand = 64.442).

| \( r_0 \) | \( b_1^* \) | \( b_2^* \) | \( R(b_1^*, b_2^*) \) | \( S(b_1^*, b_2^*) \) |
|---|---|---|---|---|
| 150 | 98.04880 | 82.53349 | 6465.337 | 64.40167 |
| 120 | 99.30070 | 83.08922 | 6165.701 | 64.40301 |
| 106 | 101.69624 | 83.55498 | 6025.950 | 64.40352 |
| 105 | 107 | 83.61577 | 6015.975 | 64.40354 |
| 90 | 107 | 85.47684 | 5866.468 | 64.40366 |
| 83 | 107 | 86.34620 | 5796.699 | 64.40374 |
| 50 | 107 | 91.58374 | 5268.461 | 64.40375 |
| 30 | 107 | 93.43602 | 4969.460 | 64.40375 |
| 0 | 107 | 93.43602 | 4969.460 | 64.40375 |

(b) 3-period model with standby customers (expected total demand = 84.156).

| \( r_0 \) | \( b_1^* \) | \( b_2^* \) | \( b_3^* \) | \( R(b_1^*, b_2^*, b_3^*) \) | \( S(b_1^*, b_2^*, b_3^*) \) |
|---|---|---|---|---|---|
| 150 | 98.049  | 82.533  | 46.720  | 7468.847 | 82.9636 |
| 120 | 99.301  | 83.089  | 47.235  | 7174.891 | 83.0073 |
| 106 | 101.696 | 83.555  | 47.639  | 7039.305 | 83.0082 |
| 105 | 107  | 83.616  | 47.687  | 7029.778 | 83.0082 |
| 90 | 107  | 85.477  | 49.525  | 6824.975 | 83.0211 |
| 83 | 107  | 86.346  | 49.525  | 6825.326 | 83.0250 |
| 50 | 107  | 91.584  | 52.730  | 6249.932 | 83.0332 |
| 30 | 107  | 93.436  | 57.734  | 6057.888 | 83.0352 |
| 0 | 107  | 93.436  | 57.734  | 6057.888 | 83.0352 |

(c) 4-period model with standby customers (expected total demand = 114.261).

| \( r_0 \) | \( b_1^* \) | \( b_2^* \) | \( b_3^* \) | \( b_4^* \) | \( R(b_1^*, b_2^*, b_3^*, b_4^*) \) | \( S(b_1^*, b_2^*, b_3^*, b_4^*) \) |
|---|---|---|---|---|---|---|
| 150 | 98.049  | 82.533  | 46.720  | 18.503  | 7864.765 | 95.6543 |
| 120 | 99.301  | 83.089  | 47.235  | 19.009  | 7438.307 | 96.0016 |
| 106 | 101.696 | 83.555  | 47.640  | 19.404  | 7312.431 | 96.2417 |
| 105 | 107  | 83.616  | 47.687  | 19.449  | 7304.006 | 96.2670 |
| 90 | 107  | 85.477  | 49.525  | 20.577  | 7191.889 | 96.8510 |
| 83 | 107  | 86.346  | 49.525  | 21.152  | 7141.103 | 97.1230 |
| 50 | 107  | 91.584  | 52.731  | 24.219  | 6914.435 | 98.3425 |
| 30 | 107  | 93.436  | 57.735  | 29.540  | 6606.416 | 99.7556 |
| 0 | 107  | 93.436  | 57.735  | 29.540  | 6606.416 | 99.7556 |

[10] H. Takagi, Explicit calculation of optimal booking limits for the static revenue management with standby customers, in Conference Proceedings, Joint International Conference of Service Science and Innovation and Serviceology, ICSSI2018 and ICServe2018, Taichung, Taiwan, (2018), 119–126.

[11] K. T. Talluri, Revenue management, in The Oxford Handbook of Pricing Management (eds. Ö. Özer and R. Phillips), Oxford University Press, (2012), 655–678.
[12] K. T. Talluri and G. J. van Ryzin, *The Theory and Practice of Revenue Management*, International Series in Operations Research & Management Science, vol. 68, Kluwer Academic Publishers, Boston, MA, 2004.

[13] G. J. van Ryzin and K. T. Talluri, *An introduction to revenue management*, Tutorials in Operations Research, (2005), 142–194.

[14] D. Walczak, E. A. Boyd and R. Cramer, *Revenue management*, in *Quantitative Problem Solving Methods in the Airline Industry* (eds. C. Barnhart and B. Smith), Springer, Boston, MA, (2012), 101–161.

[15] R. D. Wollmer, *An airline seat management model for a single leg route when lower fare classes book first*, Operations Research, 40 (1992), 26–37.

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E-mail address: takagi@sk.tsukuba.ac.jp