A New Look at Optimal Dual Problem Related to Fusion Frames

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Abstract. The purpose of this work is to examine the structure of optimal dual fusion frames and get more flexibility in the use of dual fusion frames for erasures of subspaces. We deal with optimal dual fusion frames with respect to different definitions of duality and compare the advantages of these approaches. In addition, we introduce a new concept so called partial optimal dual which involves less time and computation for detecting optimal dual for erasures in known locations. Then we study the relationship between local and global optimal duals by partial optimal duals which leads to some applicable results. In the sense that, we obtain an overcomplete frame and a family of associated optimal duals by a given Riesz fusion basis. We present some examples to exhibit the effect of error rate when dual fusion frames are applied in reconstruction.

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1. Introduction

Finite frame theory has been recently a major tool in many outstanding applications in engineering and applied mathematics such as filter bank theory, packet encoding, signal and image processing due to their resilience to additive noise and erasures [3, 6, 7, 8, 9, 10, 28]. Indeed, the redundancy property of frames reduces the errors of reconstructed signals when erasures fall out. Suppose a frame $F$ is preselected for encoding in a communication system in a finite dimensional Hilbert space. Optimal dual problem asks for finding the dual frames of $F$ that minimize the maximal reconstruction error when some coefficients in transmission have been erased or reshaped. In [23], the authors presented this problem and gave some sufficient conditions which the canonical dual is the unique optimal dual. Then the authors [21] obtained several results under which the canonical dual is optimal or not optimal for any $r$-erasures. Moreover, the characterization of extreme points
in the set of all 1-loss optimal duals and some discussions on conditions that an alternate dual frame is either optimal or not optimal dual can be found in [1]. Some generalizations of optimal dual problem have been done on reconstruction systems which are related to g-frames and fusion frames. See [8, 24]. Fusion frames are one of the most important generalization of frames which provides efficient frameworks for a wide range of applications that cannot be modeled by discrete frames [12, 13, 15, 19, 20]. So motivated by several applications in distributed processing, parallel processing and overall building efficient algorithms for reconstruction by fusion frames, Heineken et. al. generalized the optimal dual problem for fusion frames using \( Q \)-component preserving duals [18]. Some sufficient conditions which the canonical dual fusion frame is an optimal dual or not optimal as a \( Q \)-component preserving dual was presented in [25]. In this work, we are interested to set this idea in the context of dual fusion frames introduced by P. Găvruţa [16]. One of our aims is to compare the effect of this approach with \( Q \)-component preserving optimal duals. In addition, we would like to work more with alternate dual fusion frames.

The organization of this article is as follows. In Section 2, the basic definitions and notations of frame theory, fusion frame theory, optimal duals and some of their fundamental results will be given. In Section 3, we study the differences between optimal duals under two types of definitions of duality in this setting. This also shows our motivation for working with dual fusion frames defined by P. Găvruţa. Then, in Section 4 we give some sufficient conditions for determining the existence and robustness of optimal dual fusion frames. Moreover, we discuss the relation between local and global optimal duals, by a new notion called partial optimal duals. We present some examples to show the effect of the error rate when dual fusion frames are used to reconstruction, in Section 5.

2. Preliminaries and notations

Let \( \mathcal{H} \) be an \( n \)-dimensional Hilbert space and \( I_m = \{1, 2, ..., m\} \). A family \( F := \{f_i\}_{i \in I_m} \subseteq \mathcal{H} \) is called a frame for \( \mathcal{H} \) whenever \( \text{span}\{f_i\}_{i \in I_m} = \mathcal{H} \) so obviously \( m \geq n \). Three important operators associated with finite family \( F \) are defined as follows. The synthesis operator \( \theta_F : l^2(I_m) \to \mathcal{H} \) is defined by \( \theta_F\{c_i\} = \sum_{i \in I_m} c_i f_i \), the analysis operator that is the adjoint of \( \theta_F \), given by \( \theta_F^* f = \{\langle f, f_i \rangle\}_{i \in I_m} \), mapping \( \mathcal{H} \) into \( l^2(I_m) \). Also, the frame operator, given by \( S_F = \theta_F \theta_F^* \) or equivalently \( S_F f = \sum_{i \in I_m} \langle f, f_i \rangle f_i \). An easily argument shows that \( F \) is a frame if and only if the frame operator is invertible. The fact that \( \mathcal{H} \) is finite dimensional Hilbert space implies that the analysis and synthesis operators are continuous and specially the continuous function \( f \mapsto \|\theta_F^* f\| \) is non-zero on compact unit sphere in \( \mathcal{H} \). Thus, \( F = \{f_i\}_{i \in I_m} \) is a frame for \( \mathcal{H} \) if and only if there exist positive constants \( A, B \) such that

\[
A\|f\|^2 \leq \|\theta_F^* f\| = \sum_{i \in I_m} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).
\]  

The constants \( A \) and \( B \) are called the frame bounds. If \( A = B \), the frame \( F \) is called a tight frame, and in the case of \( A = B = 1 \) it is a Parseval frame. Applying a frame
F any elements of \( \mathcal{H} \) can be recovered from the frame coefficients \( \{\langle f, f_i \rangle\}_{i \in I_m} \)

\[
f = \sum_{i \in I_m} \langle f, f_i \rangle S_F^{-1} f_i = \sum_{i \in I_m} \langle f, S_F^{-1} f_i \rangle f_i = \sum_{i \in I_m} \langle f, S_F^{-1/2} f_i \rangle S_F^{-1/2} f_i.
\]

Hence, \( \{S_F^{-1/2} f_i\}_{i \in I_m} \) is a Parseval frame and \( \{S_F^{-1} f_i\}_{i \in I_m} \) is a frame that is called the canonical dual. A family \( G := \{g_i\}_{i \in I_m} \subseteq \mathcal{H} \) is called to be a dual for \( \{f_i\}_{i \in I_m} \) if \( \theta_G \theta_F^* = I_H \). It is well known that \( \{g_i\}_{i \in I_m} \) is a dual frame of \( \{f_i\}_{i \in I_m} \) if and only if \( g_i = S_F^{-1} f_i + u_i \), for all \( i \in I_m \) where \( \{u_i\}_{i \in I_m} \) satisfies

\[
\sum_{i \in I_m} \langle f, u_i \rangle f_i = 0, \quad (f \in \mathcal{H}). \tag{2.2}
\]

Every frame \( F \) with linearly independent vectors is called a Riesz basis and in case \( F \) is not a Riesz basis it is called overcomplete. We refer the reader to [14] for more information on frame theory. The optimal dual problem, one of the most important problems in frame theory, brings up the following problem: let \( F = \{f_i\}_{i \in I_m} \) be a frame for \( n \)-dimensional Hilbert space \( \mathcal{H} \), find a dual frame of \( F \) that minimize the reconstruction errors when erasures occur. If \( G = \{g_i\}_{i \in I_m} \) is a dual of \( F \) and \( \Lambda \subset I_m \), then the error operator \( E_\Lambda \) is defined by

\[
E_\Lambda = \sum_{i \in \Lambda} g_i \otimes f_i = \theta_G D \theta_F^*,
\]

where \( D \) is a \( m \times m \) diagonal matrix with \( d_{ii} = 1 \) for \( i \in \Lambda \) and 0 otherwise. Let

\[
d_r(F, G) = \max\{\|\theta_G D \theta_F^*\| : D \in \mathcal{D}_r\} = \max\{\|E_\Lambda\| : |\Lambda| = r\}, \tag{2.3}
\]

in which \( |\Lambda| \) is the cardinality of \( \Lambda \), the norm used in (2.3) is the operator norm, \( 1 \leq r < m \) is a natural number and \( \mathcal{D}_r \) is the set of all \( m \times m \) diagonal matrices with \( r \) ’s and \( n-r \) ’s. Then \( d_r(F, G) \) is the largest possible error when \( r \)-erasures occur. Indeed, \( G \) is called an optimal dual frame of \( F \) for 1-erasure or 1-loss optimal dual if

\[
d_1(F, G) = \min \{d_1(F, Y) : Y \text{ is a dual of } F\}. \tag{2.4}
\]

Inductively, for \( r > 1 \), a dual frame \( G \) is called an optimal dual of \( F \) for \( r \)-erasures (\( r \)-loss optimal dual) if it is optimal for \( (r-1) \)-erasures and

\[
d_r(F, G) = \min \{d_r(F, Y) : Y \text{ is a dual of } F\}.
\]

Also, some studies on optimal dual problem have been done in which the error operator was considered by different measurements instead of the operator norm in (2.3). See [2, 26]. This comes from the fact that dependent on applications using of different measurements for the error rate can simplify computations and be more suitable. In what follows, we prefer the Frobenius norm for fusion frame setting due to comparing our results with [18].

Fusion frame theory is a fundamental mathematical theory introduced in [11] to model sensor networks perfectly. Although, recent studies show that fusion frames provide effective frameworks not only for modeling of sensor networks but also for a variety of applications that cannot be modeled by discrete frames. In the following, we review basic definitions of fusion frames.
Let \( \{W_i\}_{i \in I_m} \) be a family of closed subspaces of \( \mathcal{H} \) and \( \{\omega_i\}_{i \in I_m} \) a family of weights, i.e. \( \omega_i > 0, i \in I_m \). Then \( \{(W_i, \omega_i)\}_{i \in I_m} \) is called a \textit{fusion frame} for \( \mathcal{H} \) if 
\[
\text{span}\{W_i\}_{i \in I_m} = \mathcal{H},
\]
equivalently, there exist constants \( 0 < A \leq B < \infty \) such that
\[
A\|f\|^2 \leq \sum_{i \in I_m} \omega_i^2\|\pi_{W_i}f\|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}),
\]
where \( \pi_{W_i} \) denotes the orthogonal projection from Hilbert space \( \mathcal{H} \) onto a closed subspace \( W_i \). The constants \( A \) and \( B \) are called the \textit{frame bounds}. Also, a fusion frame is called \textit{\( A \)-tight}, if \( A = B \), Parseval if \( A = B = 1 \) in (2.5), \( \omega \)-\textit{uniform} if \( \omega_i = \omega \) for all \( i \in I_m \) and we abbreviate 1-\textit{uniform} fusion frames as \( \{W_i\}_{i \in I_m} \).

A family of closed subspaces \( \{W_i\}_{i \in I_m} \) is called an orthonormal basis for \( \mathcal{H} \) when \( \bigoplus_{i \in I_m} W_i = \mathcal{H} \). Furthermore, the sequence \( \{(W_i, \omega_i)\}_{i \in I_m} \) is called a Riesz fusion basis whenever it is a complete family in \( \mathcal{H} \) and there exist positive constants \( A, B \) so that for every finite subset \( J \subset I_m \) and arbitrary vector \( f_i \in W_i \), we have
\[
A \sum_{i \in J} \|f_i\|^2 \leq \left\| \sum_{i \in J} f_i \right\|^2 \leq B \sum_{i \in J} \|f_i\|^2.
\]

Let \( \{(W_i, \omega_i)\}_{i \in I_m} \) be a sequence of subspaces and consider the Hilbert space 
\[
\mathcal{W} := \sum_{i \in I_m} \oplus W_i = \left\{ \{f_i\}_{i \in I} : f_i \in W_i \right\}.
\]
The \textit{synthesis operator} \( T_W \in \mathcal{B}(\mathcal{W}, \mathcal{H}) \) given by
\[
T_W (\{f_i\}_{i \in I_m}) = \sum_{i \in I_m} \omega_i f_i, \quad (\{f_i\}_{i \in I_m} \in \mathcal{W}).
\]
Its adjoint \( T_W^* \in \mathcal{B}(\mathcal{H}, \mathcal{W}) \), which is called the \textit{analysis operator}, is obtained by 
\[
T_W^*(f) = \{\omega_i \pi_{W_i}(f)\}_{i \in I_m}, \quad (f \in \mathcal{H}),
\]
and the \textit{fusion frame operator} \( S_W \in \mathcal{B}(\mathcal{H}) \) is defined by 
\[
S_W f = \sum_{i \in I_m} \omega_i^2 \pi_{W_i} f,
\]
which is a bounded, invertible and positive operator [11].

Throughout this paper, we use \( (W, w) \) to denote a fusion frame \( \{(W_i, \omega_i)\}_{i \in I_m} \) in a finite dimensional Hilbert space \( \mathcal{H} \), if there exists \( i \in I_m \) so that \( W_i \neq \mathcal{H} \) we call \( (W, w) \) a non-trivial fusion frame and in this paper, we consider non-trivial case for all fusion frames. Also, if \( (W, w) \) is a fusion frame and \( F_i = \{f_{i,j}\}_{j \in J_i} \) is a frame for \( W_i \) for \( i = 1, 2, ..., m \). Then \( (W, w, F) \) is called a fusion frame system, where \( F = \{F_i\}_{i \in I_m} \). Let \( \mathcal{H}, \mathcal{K} \) be two Hilbert spaces, we use of \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) for denoting of all bounded linear operators of \( \mathcal{H} \) into \( \mathcal{K} \) and we abbreviate \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \) by \( \mathcal{B}(\mathcal{H}) \).

Finally, we use of \( \mathcal{L}(\mathcal{H}) \), for the set of all left inverses of \( T_W^* \), \( \|\cdot\|_F \), for the Frobenius norm and \( \|\cdot\| \), for the operator norm.

Let \( (W, w) \) be a fusion frame for Hilbert space \( \mathcal{H} \) then for reconstruction of the elements of \( \mathcal{H} \) there are some approaches towards definition of dual fusion frames, one approach was presented in [17, 18].

**Definition 2.1.** Let \( (W, w) \) and \( (V, v) \) be two fusion frames for \( \mathcal{H} \). Then \( (V, v) \) is called a \( Q \)-dual of \( (W, w) \) if there exists a linear operator \( Q \in \mathcal{B}(\mathcal{W}, \mathcal{V}) \) such that 
\[
T_V QT_W^* = I_{\mathcal{H}}.
\]
Consider the operator $M_{j,W} : W \rightarrow W$, as $M_{j,W}\{f_j\}_{j \in I_m} = \{\chi_j(f_j)\}_{j \in I_m}$, that $\chi_j$ is the characteristic function and so clearly $M_{j,W}$ is a self-adjoint operator. We simply write $M_j$ if it is clear to which $W$ we refer to. Also, we abbreviate $M(j,j)_W = M_j W$ and $M(j) = M_j$.

**Definition 2.2.** Let $Q \in L(W,V)$

1. If $QM_{j,V} W \subseteq M_{j,V} V$ for all $j \in I_m$ then $Q$ is called block-diagonal.
2. If $QM_{j,W} W = M_{j,V} V$ for all $j \in I_m$ then $Q$ is called component preserving.

In Definition 2.1, we say that $(V,v)$ is a $Q$-block-diagonal (component preserving) dual fusion frame of $(W,w)$ if $Q$ is block-diagonal (component preserving). To characterize component preserving dual fusion frames, the authors in [18] considered some notations as follows. Let $A \in B(W,H)$, and $v$ be a collection of weights, $V_i = AM_i W$, for all $i \in I_m$ and consider the linear transformation

$$Q_{A,v} : W \rightarrow V, \quad Q_{A,v}\{f_j\}_{j \in I_m} = \left\{ \frac{1}{\nu_i} AM_i\{f_j\}_{j \in I_m} \right\}_{i \in I_m}.$$ (2.7)

Then by these notations for a given fusion frame we have a complete characterization of $Q$-component preserving dual fusion frames.

**Theorem 2.3.** [18] Let $(W,w)$ be a fusion frame for $H$. Then $(V,v)$ is a $Q$-component preserving dual fusion frame of $(W,w)$ if and only if $V_i = AM_i W$ for all $i \in I_m$ and $Q = Q_{A,v}$ for some $A \in \mathcal{L}_{T'_V}$. Moreover, any element of $\mathcal{L}_{T'_W}$ is of the form $T_V Q$ where $(V,v)$ is some $Q$-component preserving dual fusion frame of $(W,w)$.

Now let $(W,w)$ be a fusion frame for $H$ and $(V,v)$ a $Q$-dual fusion frame of $(W,w)$. Then every $f \in H$ can be reconstructed by $T_V Q T'_V f = f$. Suppose that $J \subseteq I_m$ and the data vectors corresponding to the subspace $\{V_i\}_{i \in J}$ are erased. Then the reconstruction gives $T_V Q M_{I_m \setminus J} T'_W f$. In [18], the authors presented some approaches for finding those dual fusion frames of $(W,w)$ which are optimal for these situations. In the following we state their method in summary.

Consider a fix $r \in I_m$ and take $P_{r}^{\infty} := \{J \subseteq I_m : |J| = r\}$, also notice that $M_J = I_W \setminus M_{I_m \setminus J}$, where $W = \bigoplus_{i \in I_m} W_i$. In this case, the worst case error is

$$\|e(r,W,V)\|_\infty = \max_{J \in P_{r}^{\infty}} \|T_V Q M_J T'_W\|_F.$$ (2.8)

The set of all 1-loss optimal dual fusion frames under this measure, $D_1^{\infty}(W,w)$, is defined as the set of all $Q$-duals $(V,v)$ of $(W,w)$ such that $e_1^{\infty}(W,w) := \|e(1,W,V)\|_\infty = \inf \{\|e(1,W,Z)\|_\infty : (Z,z) \text{ is a } Q\text{-dual of } (W,w)\}$ and inductively, the set of all $r$-loss optimal $Q$-dual fusion frames is defined as

$$D_r^{\infty}(W,w) = \{(V,v) \in D_{r-1}^{\infty}(W,w) : \|e(r,W,V)\|_\infty = e_r^{\infty}(W,w)\}.$$ In [18], the authors considered this concept for $Q$-component preserving dual fusion frames and showed that if $(W,w)$ is a fusion frame for $H$, then

$$\mathfrak{A} = \{A \in \mathcal{L}_{T'_W} : \max_{i \in I_m} \|AM_i T'_W\|_F = \min_{B \in \mathcal{L}_{T'_W}} \max_{i \in I_m} \|BM_i T'_W\|_F\},$$
is a non empty, compact and convex set. Then by this fact and using Theorem 2.3 they could present some results under which the canonical dual is the unique optimal as a $Q$-component preserving dual fusion frame. These results in [24] was extended for the case that the canonical dual, as a $Q$-component preserving dual, is optimal or not optimal for probabilistic erasures.

The other approach to define dual fusion frames was defined by P. Găvruţa in [16]. In the sense that, a sequence $(V, v)$ of subspaces is called a dual fusion frame of $(W, w)$ if

$$T_V \phi_{vw} T_W^* = \sum_{i \in I_m} \omega_i \nu_i \pi V_i S_W^{-1} \pi W_i = I_H,$$

where $\phi_{vw} : W \to V$ is defined as

$$\phi_{vw} \{f_i\}_{i \in I} = \{\pi V_i S_W^{-1} f_i\}_{i \in I_m}, \quad (\{f_i\}_{i \in I} \in W).$$

The family $\{(S_W^{-1} W_i, \omega_i)\}_{i \in I_m}$ is called the canonical dual of $(W, w)$.

In this work, we are going to continue optimal dual problem specially for alternate dual fusion frames in this concept.

Remark 2.4. It is worth to note that, these dual fusion frames can be considered as a $Q$-block diagonal dual. Indeed, using (2.9) we observe that $\phi_{vw} M_j W \subseteq M_j V$, for all $j \in I_m$ so $\phi_{vw}$ is block diagonal. However, Example 5.1 shows that $\phi_{vw}$ is not necessarily component preserving.

3. Opposite relations among optimal dual and $P$-optimal dual fusion frames

In this section, we survey similarities and specially differences between optimal duals and $P$-optimal duals. Throughout the paper we will work with the worst case of error as in (2.8), so we define our consideration of optimality as follows;

Definition 3.1. Let $(W, w)$ be a fusion frame for $H$ and $(V, v) \in D_W$. We say that $(V, v)$ is 1-loss optimal dual of $(W, w)$ whenever

$$\max_{i \in I_m} \|\omega_i \nu_i \pi V_i S_W^{-1} \pi W_i\|_F = \inf \left\{ \max_{i \in I_m} \|\omega_i \nu_i \pi V_i S_W^{-1} \pi W_i\|_F : (Z, z) \in D_W \right\}.$$  

Inductively, we can extend the above definition for any $r$-erasures. More precisely, $(V, v)$ is called an optimal dual of $(W, w)$ for any $r$-erasures, whenever it is an optimal dual of $(W, w)$ for any $(r - 1)$-erasures and

$$\max_{J \in P_m} \|T_V \phi_{vw} M_j T_W^*\|_F = \inf \left\{ \max_{J \in P_m} \|T_Z \phi_{zw} M_j T_W^*\|_F : (Z, z) \in D_W \right\}. \quad (3.1)$$

As we mentioned in Remark 2.4 $\phi_{vw}$ is block diagonal but not necessarily component preserving. In this respect, if a dual is optimal in the set of all $Q$-component preserving dual fusion frames we call it $P$-optimal dual and it is called an optimal dual if it is optimal in the notion (3.1). To simplify the notations, we use of $D_W$ and $OD_W$ ($OD_W$) to denote the set of all duals of $W$ and optimal duals of $W$.
under $r$-erasures (1-erasure) under dual definition as in \([2,9]\), respectively. The following lemmas are useful for later use.

**Lemma 3.2.** [16] Let $\mathcal{H}$ be a Hilbert spaces and $U \in B(\mathcal{H})$. Also, let $V$ be a closed subspace of $\mathcal{H}$. Then $\pi_Y U^* = \pi_Y U^* \pi_Y$. Moreover, the following are equivalent

(i) $U \pi_Y = \pi_Y U$.

(ii) $U^* UV \subseteq V$.

In the following lemma, we suspect that the set $\mathfrak{D}$ is compact. Indeed it is obviously bounded but maybe it is not closed in general. However, the compactness is a secondary problem, we only want to show that $OD_W$ is non-empty.

**Lemma 3.3.** Let $(W, w)$ be a fusion frame for finite dimensional Hilbert space $\mathcal{H}$. Then $OD_W^r$ is non-empty under any $r$-erasures.

**Proof.** We first show that $OD_W$ is a non-empty set. To this end, it is sufficient to prove that

$$\mathcal{O} = \left\{ T_Y \phi_{yw} : (Y, y) \in D_W, \max_{i \in I_m} \|T_Y \phi_{yw} M_i T_{W_i}^* \|_F = \inf_{(Z, z) \in D_W} \max_{i \in I_m} \|T_Z \phi_{zw} M_i T_{W_i}^* \|_F \right\}$$

is a non-empty set. The mapping $\| \|_{w_1} : B(W, \mathcal{H}) \to \mathbb{R}^+$ defined as $\|A\|_{w_1} = \max_{J \in P_{w_1}} \|AM_J T_{W_i}^* \|_F$ is a norm on $B(W, \mathcal{H})$ by Theorem 3.1 of [25]. On the other hand,

$$\mathfrak{D} = \left\{ T_Y \phi_{yw} : (Y, y) \in D_W, \|T_Y \phi_{yw}\|_{w_1} \leq \|S^{-1}_W T_W\|_{w_1} \right\}$$

is a non-empty and compact subset of $B(W, \mathcal{H})$. Hence, $\| \|_{w_1}$ attains its infimum on $\mathfrak{D}$, i.e., there exists a dual fusion frame $(V, v)$ of $(W, w)$ so that

$$\|T_V \phi_{yw}\|_{w_1} = \inf_{T_Y \phi_{yw} \in \mathfrak{D}} \|T_Y \phi_{yw}\|_{w_1}.$$ 

Since $OD_W \subseteq \{(Y, y) : T_Y \phi_{yw} \in \mathfrak{D}\}$, therefore $(V, v) \in OD_W$, i.e., $\mathcal{O} \neq \emptyset$. The rest of the proof is done by a simple induction. Let the set of all optimal dual fusion frames for any $(r-1)$-erasures, $OD_W^{r-1}$, is non-empty. Then a similar argument as in the above shows that the set

$$\left\{ T_Y \phi_{yw} : (Y, y) \in OD_W^{r-1}, \|T_Y \phi_{yw}\|_{w_r} = \inf_{(Z, z) \in D_W} \|T_Z \phi_{zw}\|_{w_r} \right\}$$

is non-empty and consequently $OD_W^r$ is non-empty.

We can associate to every $Q$-block diagonal dual fusion frame $(V, v)$ of $(W, w)$ a $Q$-preserving dual fusion frame. See Remark 3.6 of [18]. In the following, we state this result for dual fusion frames as mutual relation.

**Lemma 3.4.** Let $(W, w)$ and $(V, v)$ be two fusion frames for $\mathcal{H}$. Then $(V, v)$ is a dual fusion frame of $(W, w)$ if and only if $X := \{(\pi_V S_{W_i}^{-1} W_i, \nu_i)\}_{i \in I_m}$ is a $\phi_{vw}$-component preserving dual of $(W, w)$.

**Proof.** Suppose $(V, v)$ is a dual fusion frame of $(W, w)$. So $\mathcal{A} = T_V \phi_{vw}$ is a left inverse of $T_{W_i}, Q_{\mathcal{A}, v} = \phi_{vw}$ and $\mathcal{A} \mathcal{W} = \pi_V S_{W_i}^{-1} W_i$, for all $i \in I_m$. Thus, by Remark 3.6 of [18] we imply that $X$ is a $\phi_{vw}$-component preserving dual of $(W, w)$.
Conversely, if \( X \) is a \( \phi_{vw} \)-component preserving dual of \((W, w)\) then \( T_X \phi_{vw} \in \mathcal{L}_{T_W} \) and
\[
T_X \{ \pi_V, S_W^{-1} f_i \}_{I_m} = \sum_{i \in I_m} \nu_i \pi_V, S_W^{-1} f_i = T_V \{ \pi_V, S_W^{-1} f_i \}_{I_m}, \quad \{ \{ f_i \}_{i \in I_m} \in W, \}
\]
Thus, \( T_V \phi_{vw} T_{W}^* = T_X \phi_{vw} T_{W}^* = I_\mathcal{H} \) and this completes the proof. \( \square \)

Applying the above lemma, if the canonical dual is a \( P \)-optimal dual then it is an optimal dual fusion frame. Indeed, for every dual fusion frame \((V, v)\) of \((W, w)\)
\[
\max_{i \in I_m} \| T_V \phi_{vw} M_i T_{W}^* \|_F = \max_{i \in I_m} \| T_X \phi_{vw} M_i T_{W}^* \|_F \geq \max_{i \in I_m} \| \omega_i^2 S_W^{-1} \pi_W, \|_F, \quad (3.2)
\]
where \( X = \{ (\pi_V, S_W^{-1} W_i, \nu_i) \}_{i \in I_m} \). Similarly, if \((V, v)\) is a \( p \)-optimal dual, as a \( \phi_{vw} \)-preserving dual, of \((W, w)\) then \((V, v) \in OD_W\). However, there are several essential differences between optimal dual and \( P \)-optimal dual fusion frames. These differences are due to the fact that a component preserving dual is not necessarily a dual fusion frame and vice versa. For a simple example we consider a special case of Example 6.3 in [18]. Let \( \mathcal{H} = \mathbb{R}^3, W_1 = (1, 0, 0)^\perp, W_2 = (0, 1, 0)^\perp \) and \( \omega_1 = \omega_2 = 1 \). Then \( W := \{ (W_i, \omega_i) \}_{i=1} \) is a fusion frame for \( \mathcal{H} \). Put \( V_1 = \text{span}\{(0, 1, 0), (1, 2, -1/2)\} \), \( V_2 = \text{span}\{(1, 0, 0), (-1, -2, 3/2)\} \) and \( \nu_1 = \nu_2 = 1 \). Then the mapping \( A : \sum_{i=1}^2 \oplus W_i \to \mathcal{H} \) given by
\[
A((0, x_2, x_3), (y_1, 0, y_3)) = (x_3 + y_1 - y_3, x_2 + 2x_3 - 2y_3, -1/2x_3 + 3/2y_3)
\]
is a left inverse of \( T_{W}^* \) and \( V := \{ (V_i, \nu_i) \}_{i=1}^2 \) is a \( Q, \lambda, v \)-preserving dual of \( W \). However, a straightforward computation shows that \( V \) is not a dual fusion frame of \( W \). More precisely, \( S_W^{-1}(a, b, c) = (a, b, c/2) \), for all \((a, b, c) \in \mathbb{R}^3 \) and so
\[
\sum_{i=1}^2 \pi_{V_i} S_W^{-1} \pi_{V_i}, (a, b, c) = \left( a - \frac{c}{5}, b - \frac{6c}{25}, \frac{7c}{25} \right)
\]
Using these discussions one implies that if a dual fusion frame is optimal dual then it is not necessarily a \( P \)-optimal dual and vice versa. As mentioned above, only under a limited and special condition if a dual fusion frame is \( P \)-optimal dual then it is also an optimal dual. However, even in this case, the uniqueness cannot be preserved in general. See Example 5.1. In the next remark we observe more cases of a fusion frame \( W \) where \( |OD_W| \geq 2 \).

**Remark 3.5.** Assume that \((W, w)\) is a fusion frame for \( \mathcal{H} \). Applying Lemma 3.3 \( OD_W \neq \emptyset \) for any \( r \)-erasures. So, let \((V, v) \in OD_W\). If there exists \( i \in I_m \) so that \( 0 \neq (S_W^{-1} W_i)^\perp \subseteq V_i \) then we confront two cases. If \((S_W^{-1} W_i)^\perp = V_i \) we take \( Z_i = \{0\} \) and \( Z_j = V_j \) for \( j \neq i \). Then obviously \((Z, v) \in OD_W\). Also, in case \((S_W^{-1} W_i)^\perp \subset V_i \) we have that \( V_i = (S_W^{-1} W_i)^\perp \oplus (S_W^{-1} W_i \cap V_i) \). Consider \( Z_i = S_W^{-1} W_i \cap V_i \) and \( Z_j = V_j \) for \( j \neq i \), then \((Z, v) \in OD_W\).

Moreover, if \( V_i^\perp \cap (S_W^{-1} W_i)^\perp \neq \{0\} \) for some \( i \in I_m \). Then we can choose a non zero element \( u \in V_i^\perp \cap (S_W^{-1} W_i)^\perp \). Take \( Z_i = V_i \oplus \text{span}\{u\} \) and \( Z_j = V_j \) for \( j \neq i \). Then \((Z, v) \in D_W\). Moreover,
\[
\max_{i \in J} \| T_V \phi_{vw} M_j T_{W}^* \|_F = \max_{i \in J} \| T_Z \phi_{zw} M_j T_{W}^* \|_F,
\]
for every $J \in P^m_r$. Thus $(Z, v) \in OD^r_\mathcal{W}$ and is different from $(V, v)$.

By the above explanations we obtain the following results that gives sufficient conditions for finding 1-loss optimal dual fusion frames.

**Theorem 3.6.** Let $(W, w)$ be a fusion frame for $\mathcal{H}$. Consider

$$c = \max\{\omega^2_i \|S^{-1}_W \pi_{W_i}\|_F, i \in I_m\}$$

$$\Lambda_1 = \{i \in I_m : \omega^2_i \|S^{-1}_W \pi_{W_i}\|_F = c\}, \quad \Lambda_2 = I_m \setminus \Lambda_1$$

and $\mathcal{H}_j = \operatorname{span} \bigcup_{i \in \Lambda_j} W_i$, $j = 1, 2$. If $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\{(W_i, \omega_i)\}_{i \in \Lambda_2}$ is a Riesz fusion basis for $\mathcal{H}_2$, then $(S^{-1}_W W, w) \in OD_\mathcal{W}$, but not the unique optimal one.

**Proof.** Using Theorem 3.3 of [25] the canonical dual is the unique $P$-optimal dual of $(W, w)$ and so it is an optimal dual fusion frame, by the assertions after Lemma 3.4 Thus, one can easily deduce the result by Remark 3.5.

**Corollary 3.7.** Let $(W, w)$ be an $\alpha$-tight fusion frame for $\mathcal{H}$ so that $\omega^2_i \sqrt{\dim W_i} = c$, for all $i \in I_m$. Then $(V, v) \in OD_\mathcal{W}$, for every $(V, v) \in D_\mathcal{W}$ where $\max_{i \in I_m} \frac{\nu_i}{\omega_i} \leq 1$.

**Proof.** We first note that the canonical dual is an optimal dual of $(W, w)$ by Theorem 3.6. Assume $(V, v) \in D_\mathcal{W}$ so that $\max_{i \in I_m} \frac{\nu_i}{\omega_i} \leq 1$ then

$$\max_{i \in I_m} \|\omega_i \nu_i \pi_V \pi^{-1}_W \pi_{W_i}\|_F = \frac{1}{\alpha} \max_{i \in I_m} \omega_i \nu_i \sqrt{\text{tr} \left(\pi_W \pi_V \pi_W \pi_{W_i}\right)}$$

$$= \frac{1}{\alpha} \max_{i \in I_m} \omega_i \nu_i \sqrt{\text{tr} \left(\pi_V \pi_W \pi_{W_i}\right)}$$

$$\leq \frac{1}{\alpha} \max_{i \in I_m} \omega_i \nu_i \sqrt{\text{tr} \left(\pi_W \pi_{W_i}\right)}$$

$$= \frac{1}{\alpha} \max_{i \in I_m} \frac{\nu_i}{\omega_i} \sqrt{\dim W_i}$$

$$\leq c/\alpha \max_{i \in I_m} \frac{\nu_i}{\omega_i}$$

$$\leq c/\alpha = \max_{i \in I_m} \omega_i^2 \|S^{-1}_W \pi_{W_i}\|_F.$$
4. Optimal and partial optimal dual fusion frames

According to the differences between optimal and $P$-optimal duals mentioned in Section 2, in the sequel, we survey more on optimal dual fusion frames and their constructions. We present some sufficient condition, for building optimal dual frames. Specially, we introduce a new concept called partial optimal dual fusion frame and using that we study the relation among local and global optimal duals. Also, we get an overcomplete frame with a family of optimal duals by a given Riesz fusion basis. Finally, we examine optimal dual fusion frames under operator perturbations.

Motivating the notations in Theorem 3.6 we define the following symbols to get sufficient conditions in order that a dual fusion frame is optimal. So, let $(W, w)$ be a fusion frame of $\mathcal{H}$ and $(V, v) \in D_W$. Denote

$$c_v = \max\{\omega_i \nu_i \|\pi_{V_i} S_W^{-1} \pi_{W_i}\|_F, i \in I_m\},$$

$\Lambda_{1,v} = \{i \in I_m : \omega_i \nu_i \|\pi_{V_i} S_W^{-1} \pi_{W_i}\|_F = c_v\}, \Lambda_{2,v} = I_m \setminus \Lambda_{1,v}$ and $\mathcal{H}_{j,v} = \text{span} \{U_{i} \in \Lambda_{j,v}, W_{i}\}, j = 1, 2$.

**Proposition 4.1.** Let $(W, w)$ be a fusion frame of $\mathcal{H}$ and $(V, v) \in D_W$ so that $\{(W_i, \omega_i)\}_{i \in \Lambda_{1,v}}$ is a Riesz fusion basis for $\mathcal{H}_{1,v}$ and $\mathcal{H}_{1,v} \cap \mathcal{H}_{2,v} = \{0\}$. Then $(V, v)$ is a 1-loss optimal dual of $(W, w)$.

**Proof.** Suppose that $(Z, z) \in D_W$, then

$$TW M_{\Lambda_{1,v}} (T_Z \phi_z w - T_V \phi_{vw})^* + TW M_{\Lambda_{2,v}} (T_Z \phi_z w - T_V \phi_{vw})^*$$

$$= TW (T_Z \phi_z w - T_V \phi_{vw})^*$$

$$= TW \phi_z^* M_{i}^* - TW \phi_{vw}^* T_V = 0.$$

So, the hypothesis $\mathcal{H}_{1,v} \cap \mathcal{H}_{2,v} = \{0\}$ assures that $TW M_{\Lambda_{1,v}} (T_Z \phi_z w - T_V \phi_{vw})^* = 0$, and consequently, $T_V \phi_{vw} M_i = T_Z \phi_z w M_i$, for all $i \in \Lambda_{1,v}$. Hence,

$$\max_{i \in I_m} \|T_Z \phi_z w M_i T_W^*\|_F \geq \max_{i \in \Lambda_{1,v}} \|T_Z \phi_z w M_i T_W^*\|_F$$

$$\geq \max_{i \in \Lambda_{1,v}} \|T_V \phi_{vw} M_i T_W^*\|_F$$

$$= \max_{i \in I_m} \|T_V \phi_{vw} M_i T_W\|_F.$$

Thus, $(V, v) \in OD_W$ as required. \hfill $\Box$

A version of the above proposition is a sufficient condition under which the canonical dual is optimal dual for probabilistic erasures [25]. Probability model first of all was introduced in [22] by J. Leng et al. for finding optimal dual frames when probabilistic erasures occur. In fact, they considered different weights for the coefficients of $\theta^*_f f$ according to their degree of loss possibility. Then, by some examples illustrated the advantages of using the probability optimal dual frames over the optimal dual frames when the coefficients with large erasure probability are lost.

Herein, we consider the other problem: given a finite frame $F$ so that the probability of elimination of $r$ coefficients of $\theta^*_f f$ is near to 1 (or certainly we know that which $r$ coefficients have been lost). In the other words, we suppose
that the receiver knows which coefficients have been received. Then we would like to find optimal dual frames of $F$ just for elimination of these $r$ elements. This scheme avoids checking the existence of optimal dual frames for all erasures when partial erasures occur.

**Definition 4.2.** Suppose that $(W, w)$ is a fusion frame of $\mathcal{H}$. Then we say $(V, v)$ is a partial optimal dual fusion frame of $(W, w)$ for $r$-erasures if $(V, v)$ is an optimal dual fusion frame of $(W, w)$ for the elimination of some $r$-elements of $(W, w)$. Equivalently, $(V, v)$ is a partial optimal dual of $(W, w)$ for $r$-erasures if there exists $J \in P_r$ so that

$$\|Tv\phi_{vw}M_JT_W^*\|_F = \inf \{\|Tz\phi_{zw}M_JT_W^*\|_F : (Z, z) \in D_W\}.$$ 

Also, the notion of partial optimal duality can be considered for discrete frames as a new concept. More precisely, let $F$ be a fusion frame for $V, v$ and $I \in \mathbb{N}$, be a frame for $n$-dimensional Hilbert space $\mathcal{H}$ and $G \in D_F$ then $G$ is called a partial optimal dual of $F$ for $r$-erasures if there exists a $k \times k$ diagonal matrix $D$ with $r$ 1’s and $n-r$ 0’s so that

$$\|\theta_G D \theta_F^*\|_F = \inf \{\|\theta_X D \theta_F^*\|_F : X \in D_F\}.$$ 

In the next theorem, using this notion we present a connection between local and global optimal duals.

**Theorem 4.3.** Let $(W, w)$ be a fusion frame for $n$-dimensional Hilbert space $\mathcal{H}$ and $\{e_j\}_{j \in I_n}$ be an orthonormal basis of $\mathcal{H}$. Then the following assertions hold;

1. If $(V, v)$ is a fusion frame so that $G := \{\nu_i \pi_{V_e} e_j\}_{j \in I_n, i \in I_m}$ is a partial optimal dual frame for $n$-erasures as $\{\omega_i \pi_{W_e} S_W^{-1} e_j\}_{j \in I_n, i \in I_m}$ of the frame $F := \{\omega_i \pi_{W_e} S_W^{-1} e_j\}_{j \in I_n, i \in I_m}$, then $(V, v)$ is a 1-loss optimal dual fusion frame of $(W, w)$

2. If $(W, w)$ is a Riesz fusion basis then all dual frames of $F$ can be considered as a partial optimal dual frame of $F$ for $n$-erasures.

**Proof.** Using Proposition 3.3 of [5] follows that $(V, v)$ is a dual fusion frame of $(W, w)$ if and only if $G = \{\nu_i \pi_{V_e} e_j\}_{j \in I_n, i \in I_m}$ is a dual of $F = \{\omega_i \pi_{W_e} S_W^{-1} e_j\}_{j \in I_n, i \in I_m}$. Assume $G$ is a partial optimal dual frame for $n$-erasures as $\{\omega_i \pi_{W_e} S_W^{-1} e_j\}_{j \in I_n, i \in I_m}$ of $F$. Then for every $(Z, z) \in D_W$ the sequence $Z = \{z_i \pi_{Z_e} e_j\}_{j \in I_n, i \in I_m}$ is a dual frame of $F$. Consider $\Lambda_i = \{(i, j) : j \in I_n\}$ and the operator $D_{\Lambda_i} : \mathbb{C}^{nm} \rightarrow \mathbb{C}^{nm}$ defined by $D_{\Lambda_i} \{c_{j,k}\}_{k \in I_m, j \in I_n} = \{\chi_{\Lambda_i}(k, j)c_{j,k}\}_{k \in I_m, j \in I_n}$, for all $i \in I_m$. Then we have

$$\max_{i \in I_m} \|Tv\phi_{vw} M_i T_W^*\|_F = \max_{i \in I_m} \|\theta_G D_{\Lambda_i} \theta_F^*\|_F \leq \max_{i \in I_m} \|\theta_Z D_{\Lambda_i} \theta_F^*\|_F = \max_{i \in I_m} \|Tz\phi_{zw} M_i T_W^*\|_F.$$ 

Hence, $(V, v)$ is a 1-loss optimal dual fusion frame of $(W, w)$. For proving (2), we first obtain the canonical dual of $F$. To this end, let $S_F$ denotes the frame operator.
of $F$. Then 

$$S_F = \sum_{i \in I_m} \omega_i^2 \pi_{W_i} S^{-2}_{W_i} \pi_{W_i},$$

and consequently 

$$S_F \omega_i \pi_{S_{W_i}^{-1} W_i} e_j = \sum_{k \in I_m} \omega_i \omega_k^2 \pi_{W_k} S^{-2}_{W_k} \pi_{W_k} \pi_{S_{W_i}^{-1} W_i} e_j$$

$$= \omega_i^3 \pi_{W_i} S^{-2}_{W_i} \pi_{W_i} \pi_{S_{W_i}^{-1} W_i} e_j$$

$$= \omega_i \pi_{W_i} S^{-2}_{W_i} \sum_{k \in I_m} \omega_k^2 \pi_{W_k} \pi_{S_{W_i}^{-1} W_i} e_j$$

$$= \omega_i \pi_{W_i} S^{-2}_{W_i} \pi_{S_{W_i}^{-1} W_i} e_j$$

$$= \omega_i \pi_{W_i} \pi_{S_{W_i}^{-1} W_i} e_j$$

for every $i \in I_m, j \in I_n$, where the second equality is due to the assumption that $W$ is a Riesz fusion basis and is orthogonal with its canonical dual, see [11, 27], and the last equality is obtained by Lemma 8.2. Therefore the canonical dual of $F$ is obtained as follows

$$S^{-1}_F F = \{S^{-1}_F \omega_i \pi_{W_i} S^{-1}_{W_i} e_j\}_{j \in I_n, i \in I_m} = \{\omega_i \pi_{S_{W_i}^{-1} W_i} e_j\}_{j \in I_n, i \in I_m}.$$ 

So, for every dual frame $G$ of $F$ we can write $G = \{\omega_i \pi_{S_{W_i}^{-1} W_i} e_j + u_{i,j}\}_{j \in I_n, i \in I_m}$ where

$$\sum_{j \in I_n, i \in I_m} \langle \cdot, u_{i,j} \rangle \omega_i \pi_{W_i} S^{-1}_{W_i} e_j = 0.$$ 

The fact that $(W, w)$ is a Riesz fusion basis, implies that

$$\pi_{W_i} \sum_{j \in I_n} \langle \cdot, u_{i,j} \rangle S^{-1}_{W_i} e_j = 0, \quad (i \in I_m). \quad (4.1)$$ 

Therefore, we can write

$$\|\theta_G D_{\Lambda}, \theta^*_F\|_F = \left\| \sum_{j \in I_n} \langle \cdot, \omega_i \pi_{W_i} S^{-1}_{W_i} e_j \rangle (\omega_i \pi_{S_{W_i}^{-1} W_i} e_j + u_{i,j}) \right\|_F$$

$$= \left\| \sum_{j \in I_n} \langle \cdot, \omega_i \pi_{W_i} S^{-1}_{W_i} e_j \rangle \omega_i \pi_{S_{W_i}^{-1} W_i} e_j \right\|_F$$

$$= \|\theta^{-1}_{S^{-1}_F} D_{\Lambda}, \theta^*_F\|_F,$$

for all $i \in I_m$. The above computations show that all dual frames of $F$ have the same error rate and so are optimal for any $n$-erasures as $\{\omega_i \pi_{W_i} S^{-1}_{W_i} e_j\}_{j \in I_n, i \in I_m}$. This follows the desired result. \hfill \Box

An immediate result of the above theorem is as follows;
Corollary 4.4. Suppose $(W, w)$ is a fusion frame of Hilbert space $\mathcal{H}$ and $\{e_j\}_{j \in I_n}$ is an orthonormal basis of $\mathcal{H}$. If $(V, v)$ is a fusion frame so that $G := \{\nu_i \pi_{V_i} e_j\}_{j \in I_n, i \in I_m}$ is an optimal dual frame of $F := \{\omega_i \pi_{W_i} S_W^{-1} e_j\}_{j = 1, i \in I_m}$, for any $n$-erasures then $(V, v)$ is a 1-loss optimal dual fusion frame of $(W, w)$.

In the following result we get a family of optimal dual frame pairs with operator norm $[2,3]$ by a given Riesz fusion basis.

Theorem 4.5. Let $(W, 1)$ be a Riesz fusion basis of $\mathcal{H}$. Then $F = \{\pi_{S_W^{-1/2} W_i} e_j\}_{j \in I_n, i \in I_m}$ is a Parseval frame with a family of optimal dual frames as $\{\pi_{V_i} e_j\}_{j \in I_n, i \in I_m}$, for some orthonormal basis $\{e_j\}_{j \in I_n}$ of $\mathcal{H}$ and all sequences $(V, 1)$ of subspaces which satisfy $S_W^{-1/2} W_i \subseteq V_i$, $i \in I_m$.

Proof. Since $(W, 1)$ is a Riesz fusion basis, the family $\{(S_W^{-1/2} W_i, 1)\}_{i \in I_m}$ is an orthonormal fusion basis and so $F$ is a Parseval frame for $\mathcal{H}$. Consider an element $\alpha_j \in S_W^{-1/2} W_j$ for all $j \in I_m$ and put $e_j = \alpha_j / \|\alpha_j\|$. Then $\{e_j\}_{j \in I_m}$ is an orthonormal subset of $\mathcal{H}$ and so it can be extended to an orthonormal basis $\{e_j\}_{j \in I_n}$ for $\mathcal{H}$. Thus, $\max_{i \in I_m, j \in I_n} \|\pi_{S_W^{-1/2} W_i} e_j\|^2 = 1$. Hence,

$$\Lambda_1 = \{(i, j) : i \in I_m, j \in I_n, \|\pi_{S_W^{-1/2} W_i} e_j\|^2 = 1\}$$

$$\Lambda_2 = \Lambda_1 \setminus \Lambda_1$$

The fact that $\Lambda_2$ is a Parseval frame with a family of optimal dual frames as $\{\pi_{V_i} e_j\}_{j \in I_n, i \in I_m}$ is also a 1-loss optimal dual of $F$. Moreover, for every family $(V, 1)$, $S_W^{-1/2} W_i \subseteq V_i$, $i \in I_m$ we obtain $\max_{i \in I_m, j \in I_n} \|\pi_{V_i} e_j\| \|\pi_{S_W^{-1/2} W_i} e_j\| = 1$, i.e., $G$ is also a 1-loss optimal dual of $F$. \hfill $\square$

4.1. Robustness of optimal dual fusion frames

In what follows, we survey the robustness of optimal dual fusion frames under operator perturbations. First we recall that, if $U \in B(\mathcal{H})$ is an invertible operator and $(W, w)$ is a fusion frame of $\mathcal{H}$ then the family $\{(UW_i, w)\}$ is also a fusion frame for $\mathcal{H}$, see [16]. Moreover, in [14] the stability of dual fusion frames under operator perturbations was considered, although that result needs an extra condition. In the next lemma we present and improve that result.

Lemma 4.6. Let $(W, w)$ be a fusion frame of $\mathcal{H}$ and $(V, v)$ be a family of closed subspaces along with a family of weights. Also, let $U \in B(\mathcal{H})$ be an invertible operator such that $U^* UW_i \subseteq W_i$ and $U^* UV_i \subseteq V_i$ for every $i \in I_m$. Then $(V, v)$
is a dual fusion frame of \((W, w)\) if and only if \((UV, v)\) is a dual fusion frame of \((UW, w)\).

**Proof.** The family \((UW, w)\) is a fusion frame with the frame operator \(US_WU^{-1}\), \([13]\). Therefore, applying Lemma 3.2 we obtain

\[
\sum_{i \in I_m} \omega_i v_i \pi_{UV} S_{UV}^{-1} \pi_{UW} f = U \sum_{i \in I_m} \omega_i v_i \pi_{V_i} S_{W_i}^{-1} \pi_{W_i} U^{-1} f
\]

for each \(f \in \mathcal{H}\). This implies the desired result. \(\square\)

**Theorem 4.7.** Let \((W, w)\) be a fusion frame for \(\mathcal{H}\) and \(U \in B(\mathcal{H})\) be a unitary operator. Then \((V, v)\) is a 1-loss optimal dual fusion frame of \((W, w)\) if and only if \(\{(UV_i, v_i)\}_{i \in I_m}\) is a 1-loss optimal dual fusion frame of \(\{(UW_i, w_i)\}_{i \in I_m}\).

**Proof.** Since \(U\) is unitary \(UW := \{(UW_i, w_i)\}_{i \in I_m}\) is also a fusion frame for \(\mathcal{H}\), with the frame operator \(S_{UW} = US_WU^*\), see \([16]\). Moreover, if \((V, v)\) is a dual fusion frame of \((W, w)\) then \(\{(UV_i, v_i)\}_{i \in I_m}\) is a dual fusion frame of \(\{(UW_i, w_i)\}_{i \in I_m}\) by Lemma 4.6. Moreover, the set of all dual fusion frames of \(UW\) is as follows

\[D_{UW} = \{(UV, v) : (V, v) \in D_W\}.
\]

To prove the robustness of optimal dual under this operator, let \((Z, z)\) be a dual fusion frame of \((W, w)\) then

\[
\max_{i \in I_m} ||T_{UV} \phi_{uv, uu} M_i T_{UW}^*||_F = \max_{i \in I_m} ||\omega_i v_i \pi_{UV} S_{UV}^{-1} \pi_{UW}||_F
\]

\[
= \max_{i \in I_m} ||\omega_i v_i U \pi_{V_i} S_{W_i}^{-1} \pi_{W_i} U^*||_F
\]

\[
= \max_{i \in I_m} ||\omega_i v_i \pi_{V_i} S_{W_i}^{-1} \pi_{W_i}||_F
\]

\[
\leq \max_{i \in I_m} ||\omega_i z_i \pi_{Z_i} S_{W_i}^{-1} \pi_{W_i}||_F
\]

\[
= \max_{i \in I_m} ||\omega_i z_i U \pi_{Z_i} S_{W_i}^{-1} \pi_{W_i} U^*||_F
\]

\[
= \max_{i \in I_m} ||\omega_i z_i \pi_{Z_i} S_{W_i}^{-1} \pi_{UW}||_F
\]

\[
= \max_{i \in I_m} ||T_{UZ} \phi_{uz, uu} M_i T_{UW}^*||_F.
\]

Similarly, let \((UV, v)\) be a 1-loss optimal dual fusion frame of \((UW, w)\). Then for every dual fusion frame \((Z, z)\) of \((W, w)\) we can write

\[
\max_{i \in I_m} ||T_V \phi_{vw} M_i T_W^*||_F = \max_{i \in I_m} ||T_{UV} \phi_{uv, uu} M_i T_{UW}^*||_F
\]

\[
\leq \max_{i \in I_m} ||T_{UZ} \phi_{uz, uu} M_i T_{UW}^*||_F = \max_{i \in I_m} ||T_Z \phi_{zw} M_i T_W^*||_F.
\]

This completes the proof. \(\square\)

5. Examples

In this section we give some examples related to the previous sections. The first example determines the advantage of using optimal dual over \(P\)-optimal dual fusion frames.
Example 5.1. (A non-Riesz fusion frame which has several optimal duals but the unique $P$-optimal dual.)

Let $\{e_i\}_{i \in I_4}$ be the standard orthonormal basis of $\mathbb{R}^4$ and put

$$W_1 = \text{span}\{e_1, e_2\}, \quad W_2 = \text{span}\{e_2, e_3\}, \quad W_3 = \text{span}\{e_4\},$$

and $\omega := \omega_i = 1$, $i \in I_3$. Then $W = \{(W_i, \omega_i)\}_{i \in I_3}$ is a fusion frame for $\mathbb{R}^4$ and

$$S_W^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

Hence

$$S_W^{-1}W_i = W_i, \quad (i \in I_4).$$

Also,

$$S_W^{-1}\pi W_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad S_W^{-1}\pi W_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

$$S_W^{-1}\pi W_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

Thus, $\max_{i \in I_3} \|S_W^{-1}\pi W_i\|_F = \sqrt{5}/4$ and $\Lambda_1 = \{1, 2\}$. Moreover, $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\{(W_i, \omega_i)\}_{i \in \Lambda_2}$ is a Riesz fusion basis for $\mathcal{H}_2$. By Theorem 3.3 of [25], the canonical dual is the unique $P$-optimal dual of $(W, w)$ and consequently it is also an optimal dual, by (3.2). However, the canonical dual is not the unique optimal one, and $(W, w)$ has many optimal dual fusion frame. For example, take

$$V_1 = \text{span}\{e_1, e_2, (0, 0, \xi_1, \xi_2)\},$$

$$V_2 = \text{span}\{e_2, e_3, (\xi_3, 0, 0, \xi_4)\},$$

$$V_3 = \text{span}\{e_4, (\xi_5, \xi_6, \xi_7, 0)\},$$

for every $\xi_i \in \mathbb{R}$, $i \in I_7$. Then $V = \{(V_i, w)\}_{i=1}^3$ is a dual fusion frame of $(W, w)$. Moreover, for every $\xi_i \in \mathbb{R}$, $i \in I_7$, the sequence $V$ constitutes a 1-loss optimal dual fusion frame of $(W, w)$. In fact,

$$\max_{i \in I_3} \|\pi V_i S_W^{-1}\pi W_i\|_F = \max_{i \in I_3} \|S_W^{-1}\pi W_i\|_F = \sqrt{5}/4.$$

Note that, in this example $\phi_{vw}$ associated with many dual fusion frames $V$ are not component preserving. For instance, put $V_1 = \text{span}\{e_1, e_2, ce_4\}, c \neq 0$. Then $\pi V_i S_W^{-1}\pi W_i \subset V_1$, indeed $ce_4$ is not in $\pi V_i S_W^{-1}\pi W_i$ and so $\phi_{vw} \mathcal{M}_1 \mathcal{W} = \mathcal{M}_1 \mathcal{V}$.

Example 5.2. (construction an overcomplete frame with a family of optimal duals by using an orthonormal fusion basis)
Suppose that \( \{e_j\}_{j \in I_3} \) is the standard orthonormal basis of \( \mathbb{R}^3 \) and
\[
W_1 = \text{span}\{ (1, 0, 1) \}, \quad W_2 = \text{span}\{ (-1, 0, 1), (0, 1, 0) \}.
\]
Then \( W = \{(W_i, 1)\}_{i=1}^2 \) is an orthonormal fusion basis for \( \mathbb{R}^3 \). A straightforward computation shows that
\[
\{\pi W_i e_j\}_{j \in I_3, i \in I_2} = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}.
\]
So, \( F := \{\pi W_i e_j\}_{j \in I_3, i \in I_2} \) is an overcomplete Parseval frame and \( \{\pi V_i e_j\}_{j \in I_3, i \in I_2} \)
is an optimal dual frame of \( F \) for every sequence \( \{V_i\}_{i=1}^2 \) so that of \( W_i \subseteq V_i \), by Theorem \( \text{4.5} \). For example \( F \in OD_F \) and also by putting
\[
V_1 = \text{span}\{(1, 0, 1), (1, 0, -1)\}, V_2 = W_2,
\]
we obtain an optimal alternate dual frame of \( F \) as follows
\[
G = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \right\}.
\]
Finally, we present an example of a non-Riesz fusion frame which provides an overcomplete frame that its canonical dual is optimal dual for 1-erasure, however it is not partial optimal for \( r \)-erasures, for all \( r > 1 \).

**Example 5.3.** Let \( \mathcal{H} = \mathbb{R}^3 \) with the standard orthonormal basis \( \{e_j\}_{j \in I_3} \) and take
\[
W_1 = \text{span}\{e_1, e_2\}, \quad W_2 = \text{span}\{e_2\}, \quad W_3 = \text{span}\{e_3, e_1 - e_2\},
\]
and \( \omega_i = 1, i \in I_3 \). Then \( W = \{(W_i, \omega_i)\}_{i \in I_3} \) is a fusion frame for \( \mathcal{H} \) and
\[
S_W^{-1} = \begin{bmatrix}
\frac{5}{7} & \frac{1}{7} \\
\frac{1}{7} & \frac{3}{7} \\
0 & 0
\end{bmatrix}.
\]
Hence, we compute \( F = \{\pi W_i S_W^{-1} e_j\}_{i,j} \) as follows
\[
F = \begin{bmatrix}
\begin{bmatrix} 5/7 \\ 1/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/7 \\ 3/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/7 \\ -2/7 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/7 \\ 1/7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\]
and
\[
S_F^{-1}F = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ -1/10 & 3/10 \\ 10/10 & -2/5 \\ 15/10 & 15/10 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},
\]

Then we obtain \( \max_{i \in I_7} \|S_F^{-1}f_i\| = \|S_F^{-1}f_7\| = 1 \). By taking a sequence \( \{u_i\}_{i \in I_7} \) satisfies \((2.2)\) one derives the following relations
\[
\begin{align*}
5u_1 + u_2 + 2u_5 - u_6 &= 0, \\
u_1 + 3u_2 + u_3 + 3u_4 - 2u_5 + u_6 &= 0, \\
u_7 &= 0.
\end{align*}
\]

Due to \((5.3)\) for every dual frame \( \{g_i\}_{i \in I_7} = \{S_F^{-1}f_i + u_i\}_{i \in I_7} \) of \( F \) we have that
\[
\max_{i \in I_7} \|S_F^{-1}f_i\| = \|g_7\| = 1 \leq \max_{i \in I_7} \|g_i\| = \|f_i\|,
\]
i.e., \( S_F^{-1}F \in OD_F \). We show that the canonical dual cannot be considered as a partial optimal dual for 2-erasures. To this end, let \( i \)th and \( j \)th component are lost.

we set \( u_i = \frac{-1}{2}S_F^{-1}f_i \) and \( u_j = \frac{-1}{2}S_F^{-1}f_j \) then one obtain \( u_k, \ k \in I_7, \ k \neq i, j \) by \((5.1), (5.2) \) and \((5.3)\). So \( \{g_i\}_{i \in I_7} = \{S_F^{-1}f_i + u_i\}_{i \in I_7} \) is a dual frame of \( F \). Suppose \( D \) is a 7 \times 7 diagonal matrix with \( d_{ii} = d_{jj} = 1 \) and other matrix elements 0. Then
\[
\left\| T_{S_F^{-1}F}DT_F^* \right\|_F = \left\| \sum_{k=i,j} S_F^{-1}f_k \right\|_F
\]
\[
= \frac{1}{2} \left\| \sum_{k=i,j} S_F^{-1}f_k \right\|_F
\]
\[
= \left\| \sum_{k=i,j} g_k \right\|_F
\]
\[
= \left\| T_GDT_F^* \right\|_F.
\]

Thus the canonical dual is not a partial optimal dual for 2-erasures and so for any \( r \)-erasures, \( r \geq 1 \).

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