Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping

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Abstract. This paper is concerned with the blowup phenomena for initial value problem of semilinear wave equation with critical space-dependent damping term

\[
\begin{align*}
\partial_t^2 u(x,t) - \Delta u(x,t) + V_0 |x|^{-1} \partial_t u(x,t) &= |u(x,t)|^p, \quad (x,t) \in \mathbb{R}^N \times (0,T), \\
u(x,0) &= \varepsilon f(x), \quad x \in \mathbb{R}^N, \\
\partial_t u(x,0) &= \varepsilon g(x), \quad x \in \mathbb{R}^N,
\end{align*}
\] (DW; V_0)

where \(N \geq 3\), \(V_0 \in [0, \frac{10}{10+1}]\), \(f\) and \(g\) are compactly supported smooth functions and \(\varepsilon > 0\) is a small parameter. The main result of the present paper is to give a solution of (DW; V_0) and to provide a sharp estimate for lifespan for such a solution when \(\frac{2}{p} < p \leq p_2(N + V_0)\), where \(p_2(N)\) is the Strauss exponent for (DW; 0). The main idea of the proof is due to the technique of test functions for (DW; 0) originated by Zhou–Han (2014, MR3169791). Moreover, we find a new threshold value \(V_0 = \frac{(N-1)^2}{N+1}\) for the coefficient of critical and singular damping \(|x|^{-1}\).

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1 Introduction

In this paper we consider the blowup phenomena for initial value problem of semilinear wave equation with scale-invariant damping term of space-dependent type as follows:

\[
\begin{align*}
\partial_t^2 u(x,t) - \Delta u(x,t) + a(x) \partial_t u(x,t) &= |u(x,t)|^p, \quad (x,t) \in \mathbb{R}^N \times (0,T), \\
u(x,0) &= \varepsilon f(x), \quad x \in \mathbb{R}^N, \\
\partial_t u(x,0) &= \varepsilon g(x), \quad x \in \mathbb{R}^N,
\end{align*}
\] (1.1)

where \(N \geq 3\), \(a(x) = V_0 |x|^{-1}\) \((V_0 \geq 0)\), \(\varepsilon > 0\) is a small parameter and \(f, g\) are smooth nonnegative functions satisfying \(g \equiv 0\) with

\[\text{supp}(f,g) \subset B(0,R_0) = \{x \in \mathbb{R}^N \mid |x| \leq R_0\}\]

for some \(R_0 > 0\). Note that by taking \(u_0(x,t) = \lambda^{-\frac{1}{2p}} \mu(\lambda x, \lambda t)\) with \(\lambda = R_0\), we can always assume \(R_0 = 1\) without loss of generality.

The study of blowup phenomena for (1.1) with \(N = 3\) and \(V_0 = 0\) was initially started by F. John in [5] for \(1 < p < 1 + \sqrt{2}\). Strauss conjectured in [9] that the number \(p_0(N)\) given by the positive root of the quadratic equation

\[(N-1)p^2 - (N+1)p - 2 = 0\]
is the threshold for dividing the following two situations: blowup phenomena at a finite time for arbitrary small initial data and global existence of small solutions. The conjecture of Strauss was completely solved until Yordanov–Zhang [11] and Zhou [12].

After that the lifespan of solutions to nonlinear wave equations (1.1) with small initial data has been considered by many authors. If \(1 < p < p_0(N)\), then by Sideris [8] and Di Pomponio–Georgiev [2] we have the two-sided estimates for lifespan of solution with small initial data as

\[
Ce^{\frac{2n(p-1)}{2n(p-n-1)p^2}} \leq \text{LifeSpan}(u) \leq Ce^{\frac{2n(p-1)}{2n(p-n-1)p^2}}
\]

with arbitrary \(\delta > 0\). For the critical case \(p = p_0(N)\), Takamura–Wakasa [10] succeeded in proving sharp upper bound of lifespan

\[
\exp[ce^{-p(p-1)}] \leq \text{LifeSpan}(u) \leq \exp[Ce^{-p(p-1)}]
\]

for remaining case \(N = 4\), and by [10] the study of the lifespan for blowup solutions to nonlinear wave equations with small data has been completed (for the other contributions see e.g. [10] and its references therein). In the connection to the previous paper, we have to remark that Zhou–Han [13] gave a short proof for verifying the sharp upper bound of lifespan by using an estimate established in [11] and a kind of test functions including the Gauss hypergeometric functions.

In this paper, we mainly deal with the problem (1.1) with \(N \geq 3\) and \(V_0 > 0\). Because of the strong singularity of damping term at the origin, the study of (1.1) has not been considered so far. Since the problem has a scaling-invariant structure, one can expect that some threshold for \(V_0\) appears.

The first purpose of this paper is to clarify the local wellposedness of (1.1) for \(1 < p < \frac{N-2}{N-4}\) in solutions in \(H^2(\mathbb{R}^N)\). The second is to show an upper bound of the lifespan of solutions to (1.1) with respect to small parameter \(\varepsilon > 0\) and to pose a threshold number for \(V_0\) dividing completely different situations.

The first assertion of this paper is for local wellposedness of (1.1).

**Proposition 1.1.** Let \(N \geq 3\), \(V_0 \geq 0\) and

\[
\begin{align*}
1 < p < \infty & \quad \text{if } N = 3, 4 \\
1 < p < \frac{N-2}{N-4} & \quad \text{if } N \geq 5.
\end{align*}
\]

For every \((f, g) \in H^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)\) and \(\varepsilon > 0\), there exist \(T = T(\|f\|_{H^2}, \|g\|_{H^1}, \varepsilon) > 0\) and a unique strong solution of (1.1) in the following class:

\[
\begin{align*}
u \in S_T = C^2([0, T]; L^2(\mathbb{R}^N)) \cap C^1([0, T]; H^1(\mathbb{R}^N)) \cap C([0, T]; H^2(\mathbb{R}^N))
\end{align*}
\]

Moreover, one has for every \(t \geq 0\),

\[
\text{supp } u(t) \subset B(0, R_0 + t).
\]

**Definition 1.1.** We denote \(\text{LifeSpan}(u)\) as the maximal existence time for solution of (1.1), that is,

\[
\text{LifeSpan}(u) = \sup \{T > 0 ; \ u \in S_T \ & \ u \text{ is a solution of (1.1) in } (0, T)\}
\]

**Definition 1.2.** We introduce the following quadratic polynomial

\[
\gamma(n; p) = 2 + (n + 1)p - (n - 1)p^2
\]
and denote \( p_0(n) \) as the positive root of the quadratic equation \( \gamma(n; p) = 0 \) as in Introduction. We also put

\[
V_\varepsilon = \frac{(N - 1)^2}{N + 1}
\]

and for areas for \((p, V_0)\) as follows:

\[
\Omega_0 = \{(p, V_0) : p = p_0(N + V_0), \quad 0 \leq V_0 < V_\varepsilon\} \quad (1.2)
\]

\[
\Omega_1 = \left\{ (p, V_0) : \max\left\{ \frac{p_0(N + 2 + V_0)}{N - 1 - V_0}, \frac{2}{N + 1} \right\} \leq p < p_0(N + V_0), \quad 0 \leq V_0 < V_\varepsilon \right\} \quad (1.3)
\]

\[
\Omega_2 = \left\{ (p, V_0) : \frac{2(N + 1)}{N + 1 + V_0} < p < \frac{2}{N - 1 - V_0}, \quad \frac{(N + 1)(N - 2)}{N + 2} < V_0 < V_\varepsilon \right\} \quad (1.4)
\]

\[
\Omega_3 = \left\{ (p, V_0) : \max\left\{ \frac{N}{N - 1}, \frac{N + 3 + V_0}{N + 1 + V_0} \right\} < p < \max\left\{ p_0(N + 2 + V_0), \frac{2(N + 1)}{N + 1 + V_0} \right\} \right\} \quad (1.5)
\]

![Figure 1: the regions \( \Omega_0, \Omega_1, \Omega_2 \) and \( \Omega_3 \)](image)

Now we are in a position to state our main result in this paper about upper bound of lifespan of solutions to (1.1).

**Theorem 1.2.** Let \( \frac{N}{N - 1} < p < \infty \) if \( N = 3, 4 \) and \( \frac{N}{N - 1} < p < \frac{N - 2}{N - 3} \) if \( N \geq 5 \). Fix \((f, g)\) satisfying \( f \geq 0, \ g \geq 0, \ g \neq 0 \) and \( \text{supp}(f, g) \subset B(0, 1) \). Let \( u_\varepsilon \) be the solution of (1.1) in Proposition 1.1 with the parameter \( \varepsilon > 0 \). If \((p, V_0) \in \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3\), then \( \text{LifeSpan}(u_\varepsilon) < \infty \). Moreover, one has

\[
\text{LifeSpan}(u_\varepsilon) \leq \begin{cases} 
\exp[C\varepsilon^{-(p-1)}] & \text{if } (p, V_0) \in \Omega_0, \\
C'_\delta \varepsilon^{-2(p-1)/\gamma(N+V_0;p)-\delta} & \text{if } (p, V_0) \in \Omega_1, \\
C''_\delta e^{-2\gamma(N+V_0;p)\delta} & \text{if } (p, V_0) \in \Omega_2, \\
C'''_\delta e^{-1-\delta} & \text{if } (p, V_0) \in \Omega_3,
\end{cases}
\]  

(1.6)
where \( \delta \) can be chosen arbitrary small and \( C_\delta, C'_\delta, C''_\delta \) and \( C'''_\delta \) are positive constants which depend on all parameters without \( \varepsilon \).

**Remark 1.1.** We emphasize the following two facts. The proof of \([13]\) depends on an estimate established by \([11]\) (for detail, see \([11]\) (2.5’)), however, our proof does not depend on that. The proof of Theorem \([13]\) can be applicable to weaker solutions of \([1.1]\) belonging to \( C([0, T]); H^1(\mathbb{R}^N) \cap L^p((0, T) \times \mathbb{R}^N) \).

**Remark 1.2.** Taking the threshold value \( V_0 = V_* \) formally, we have

\[
\gamma (N + V_*; p) = 2 (1 + Np) \left( 1 - \frac{N - 1}{N + 1} p \right)
\]

and therefore \( p_0(N + V_*) = \frac{N + 1}{N - 1} = 1 + \frac{2}{N - 1} \). On the one hand, critical exponent for the blowup phenomena for the problem

\[
\begin{align*}
\partial_t^2 u(x, t) - \Delta u(x, t) + (x)^{-\gamma} \partial_t u(x, t) &= |u(x, t)|^p, \quad (x, t) \in \mathbb{R}^N \times (0, T), \\
u(x, 0) &= \varepsilon f(x), \quad x \in \Omega, \\
\partial_t u(x, 0) &= \varepsilon g(x), \quad x \in \Omega,
\end{align*}
\]

(1.7)

is given by \( p_F(\alpha) = 1 + \frac{2}{N - \alpha}, \alpha \in [0, 1) \) which is so-called Fujita exponent (see e.g., Ikehata–Todorova–Yordanov \([4]\) and also Ikeda–Ogawa \([3]\)). We formally put again a threshold value \( \alpha = 1 \). Then one can find

\[ p_0 (N + V_0) = p_F(1). \]

The left-hand side comes from the blowup phenomena for nonlinear wave equation and the right-hand side comes from the one for nonlinear heat equation. In this connection, we would conjecture that if \( V_0 > V_* \), then the threshold of blowup phenomena is given by the Fujita exponent \( p_F(1) \).

**Remark 1.3.** If \( (p, V_0) \in \Omega_0 \cup \Omega_1 \), then Theorem \([12]\) seems to give a sharp lifespan of solutions to \([1.1]\) with small initial data. In the case \( (p, V_0) \in \Omega_3 \), we cannot derive the estimates for lifespan with \( e^{-\tau} \) with \( \tau \) less than one. So the estimate in \( \Omega_3 \) seems not to be sharp. For the case \( (p, V_0) \in \Omega_2 \) the effect of diffusion structure seems to appear in the estimate.

The present paper is organized as follows. In Section 2, we first give existence and uniqueness of local-in-time solutions to \([1.1]\) if \( p \leq \frac{N - 2}{N - 1} \) by using the standard semigroup properties. In Section 3, we construct special solutions of linear wave equation with anti-damping term \( -\frac{V_0}{N} \partial_t u \). In this point we use the idea due to \([13]\) (they only considered the case \( V_0 = 0 \)), which will be a test function for proving blowup phenomena. In Section 4, we prove blowup phenomena by dividing two cases \( p < p_0(N + V_0) \) and \( p = p_0(N + V_0) \).

## 2 Local solvability of nonlinear wave equation with singular damping

In this section we construct a solution of \([1.1]\) with initial data belonging to \( H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). To do so, we first treat the linear problem

\[
\begin{align*}
\partial_t^2 u(x, t) - \Delta u(x, t) + \alpha(x) \partial_t u(x, t) &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad \partial_t u(x, 0) = u_1(x), \quad x \in \mathbb{R}^N.
\end{align*}
\]

(2.1)
2.1 $C_0$-Semigroup for linear wave equation with singular damping

Now we start with the usual $N$-dimensional Laplacian

$$Au := -\Delta u, \quad D(A) = H^2(\mathbb{R}^N).$$

We note that $A$ is $m$-accretive in $L^2(\mathbb{R}^N)$, that is, $(I + A)D(A) = L^2(\mathbb{R}^N)$ and $(-\Delta u, u) \geq 0$ for every $u \in H^2(\mathbb{R}^N)$. Set $H = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix}, \quad D(\mathcal{A}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$$

and

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & |x|^{-1} \end{pmatrix}, \quad D(\mathcal{B}) = H^1(\mathbb{R}^N) \times \{ v \in L^2(\mathbb{R}^N) : |x|^{-1}v \in L^2(\mathbb{R}^N) \}$$

and put

$$\mathcal{A}_\kappa = \mathcal{A} + \kappa \mathcal{B}, \quad D(\mathcal{A}_\kappa) = D(\mathcal{A}) \cap D(\mathcal{B}).$$

Then in view of Hille–Yosida theorem, we have the following $C_0$-semigroup on $H$ (see e.g., Pazy [7]).

**Lemma 2.1.** Let $N \geq 3$. For every $V \geq 0$, $\frac{1}{2}I + \mathcal{A}_\kappa$ is $m$-accretive in $H$. Therefore $-\mathcal{A}_\kappa$ generates a $C_0$-semigroup $\{T_\kappa(t)\}_{t \geq 0}$ on $H$. Moreover, if supp$(u_0, u_1) \subset \overline{B(0, R)}$, then supp$[T_\kappa(t)(u_0, u_1)] \subset \overline{B(0, R + t)}$.

**Proof.** By Hardy’s inequality we have $D(\mathcal{A}) \subset D(\mathcal{B})$. This means that $D(\mathcal{A}_\kappa) = D(\mathcal{A}) \cap D(\mathcal{B}) = D(\mathcal{A})$.

**(Accretivity)** By integration by parts, we have

$$\langle \mathcal{A}(u), (v) \rangle_H = \left\langle \begin{pmatrix} -\Delta u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_H = \int_{\mathbb{R}^N} \left( -\nu u - \nabla v \cdot \nabla u - (\Delta u)v \right) dx = -\int_{\mathbb{R}^N} uv \, dx.$$

Since $\kappa \mathcal{B}$ is clearly accretive, we have the accretivity of $\mathcal{A}_\kappa$.

**(Maximality)** Let $F = (f, g) \in H$. Then $\lambda u + \mathcal{A} u + \kappa \mathcal{B} u = F$ is equivalent to the system

$$\lambda u - v = f, \quad \lambda v - \Delta u + \kappa |x|^{-1} v = g.$$

Substituting $v = \lambda u - f$, we see that

$$\lambda^2 u - \Delta u + \lambda \kappa |x|^{-1} u = g + \lambda f + \kappa |x|^{-1} f = f_{\lambda \kappa}.$$

Taking $\tilde{u}(y) = u(\lambda^{-1}y)$ and $\tilde{f}_{\lambda \kappa}(y) = f_{\lambda \kappa}(\lambda^{-1}y)$ yields that

$$\tilde{u} - \Delta \tilde{u} + \kappa |y|^{-1} \tilde{u} = \lambda^{-2} \tilde{f}_{\lambda \kappa}.$$

This is nothing but the resolvent problem of the Schrödinger operator with positive Coulomb potentials. Therefore there exists $\tilde{u}_{\lambda \kappa} \in H^2(\mathbb{R}^N)$ such that

$$\tilde{u}_{\lambda \kappa} - \Delta \tilde{u}_{\lambda \kappa} + \kappa |y|^{-1} \tilde{u}_{\lambda \kappa} = \lambda^{-2} \tilde{f}_{\lambda \kappa}.$$

Putting $u_{\lambda \kappa}(x) = \lambda \tilde{u}_{\lambda \kappa}(\lambda^2 x) \in H^2(\mathbb{R}^N)$, we obtain

$$\lambda^2 u_{\lambda \kappa} - \Delta u_{\lambda \kappa} + \lambda \kappa |x|^{-1} u_{\lambda \kappa} = f_{\lambda \kappa} = g + \lambda f + \kappa |x|^{-1} f.$$

Finally setting $v_{\lambda \kappa} = \lambda u_{\lambda \kappa} - f \in H^1(\mathbb{R}^N)$, we obtain $(\lambda u + \mathcal{A} + \kappa \mathcal{B})(u_{\lambda \kappa}, v_{\lambda \kappa}) = (f, g)$.

The finite propagation property follows from the standard argument for wave equation with regular damping term. The proof is complete. \qed
2.2 Local solvability of nonlinear problem

We consider \( u(0), \partial_t u(0) \in U_0 = (u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) which is equivalent to the following problem

\[
\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \mathcal{A}_c \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{N}(u(t), v(t)) \end{pmatrix}
\]

with \( \mathcal{N}(u, v) = (0, |u|^p) \). Here we construct the corresponding mild solution in \( H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) given by

\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T_s(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^s T_s(t-s) \begin{pmatrix} 0 \\ \mathcal{N}(u(s), v(s)) \end{pmatrix} \, ds,
\]

where \( \{T_s(t)\}_{t \geq 0} \) is determined in Lemma 2.3.

**Lemma 2.2.** (i) The following metric space

\[
X_T = \left\{ U \in C([0, T]; \mathcal{H}) \cap L^\infty([0, T]; D(\mathcal{A}_c)) : \sup_{0 < t < T} \|U(t)\|_{D(\mathcal{A}_c)} \leq M \right\}, \quad M := 2(\|U_0\|_{D(\mathcal{A}_c)} + 1)
\]

with the distance

\[
d(U_1, U_2) := \max_{0 < t < T} \|U_1(t) - U_2(t)\|_{\mathcal{H}}, \quad U_1, U_2 \in X_T.
\]

is complete.

(ii) If \( \text{supp}(u_0, u_1) \subset B(0, R) \), then the following metric space

\[
Y_T = \left\{ U \in C([0, T]; \mathcal{H}) \cap L^\infty([0, T]; D(\mathcal{A}_c)) : \sup_{0 < t < T} \|U(t)\|_{D(\mathcal{A}_c)} \leq M, \quad \text{supp}(u(t), v(t)) \subset B(0, R + t) \right\}
\]

with the same distance \( d \) is also complete.

**Proof.** Take a Cauchy sequence \( \{U_n\}_{n \in \mathbb{N}} \) in \( X_T \). The completeness of \( C([0, T]; \mathcal{H}) \) yields that there exists \( U_\infty \in C([0, T]; \mathcal{H}) \) such that

\[
U_n \to U_\infty \quad \text{strongly in } C([0, T]; \mathcal{H}) \quad \text{as } n \to \infty.
\]

Moreover, we can subtract a subsequence \( U_{n_j} \) and \( \tilde{U} \in L^\infty([0, T]; D(\mathcal{A}_c)) \) such that

\[
\sup_{0 < t < T} \|\tilde{U}(t)\|_{D(\mathcal{A}_c)} \leq M
\]

and

\[
U_{n_j} \to \tilde{U} \quad \text{*-weakly in } L^\infty([0, T]; D(\mathcal{A}_c)) \quad \text{as } j \to \infty.
\]

Therefore we have \( U_\infty = \tilde{U} \). Therefore \( X_T \) is a complete metric space. If \( \text{supp} U_n(t) \subset \{x \in \mathbb{R}^N : |x| \leq R + t\} \), then by strong convergence we have \( \text{supp} U_\infty(t) \subset \{x \in \mathbb{R}^N : |x| \leq R + t\} \). This means that \( Y_T \) is also complete. \( \square \)

**Lemma 2.3.** There exists \( T_0 \) such that \( \Psi : X_{T_0} \to X_{T_0} \) and \( \Psi : Y_{T_0} \to Y_{T_0} \) are both well-defined, and \( \Psi \) is contractive in \( X_{T_0} \) and in \( Y_{T_0} \).

**Proof.** First observe that by finite propagation property in Lemma 2.1, we can deduce \( \text{supp} \Psi U(t) \subset B(0, R + t) \) when \( \text{supp} U_0 \subset B(0, R) \). Since \( X_T \) and \( Y_T \) are endowed with the same distance, it suffices to prove the assertion for \( X_T \).

We recall that the norms in \( D(\mathcal{A}_c) \) and in \( H^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) are equivalent.
(well-defined) If $U = (u, v) \in X_T$, then

$$
\|N(U(t))\|_{D(\mathcal{A}_u)}^2 \leq C^2_e \|u(t)\|_{H^2}^2
$$

$$
= C^2_e \left\| u(t) \right\|_{H^2}^2
$$

$$
= C^2_e \int_{\mathbb{R}^N} \left( |u(t)|^p + |\nabla (|u(t)|^p)|^2 \right) dx
$$

$$
= C^2_e \left( \int_{\mathbb{R}^N} |u(t)|^{2p} dx + p^2 \int_{\mathbb{R}^N} |u(t)|^{2(p-1)} |\nabla u(t)|^2 \right)
$$

$$
\leq C^2_e \left( \int_{\mathbb{R}^N} |u(t)|^{2(p-1)} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|\nabla u(t)|^2} dx \right)^{\frac{1}{2}}
$$

$$
+ p^2 C^2_e \left( \int_{\mathbb{R}^N} |u(t)|^{2(p-1)} dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|\nabla u(t)|^2} dx \right)^{1-\frac{1}{2}}.
$$

Since $p \leq \frac{N-2}{N-4}$, we have $N(p-1) \leq \left( \frac{1}{2} - \frac{2}{N} \right)$ and therefore

$$
\|N(U(t))\|_{D(\mathcal{A}_u)} \leq C'_e \|u(t)\|_{H^2}
$$

$$
\leq C''_e \|u(t)\|_{H^2} \times H^1
$$

$$
\leq C''_e \|u(t)\|_{D(\mathcal{A}_u)}
$$

$$
\leq C''_e M.
$$

Therefore we have for $0 \leq t \leq T \leq \log 2$,

$$
\|\psi U(t)\|_{D(\mathcal{A}_u)} \leq \|T_e(t)U_0\|_{D(\mathcal{A}_u)} + \int_0^t \|T_e(t-s)[N(U(s))]\|_{D(\mathcal{A}_u)} ds
$$

$$
\leq e^{t/2} \|U_0\|_{D(\mathcal{A}_u)} + \int_0^t e^{(t-s)/2} \|N(U(s))\|_{D(\mathcal{A}_u)} ds
$$

$$
\leq \sqrt{2} |U_0|_{D(\mathcal{A}_u)} + \sqrt{2} C''_e Mt.
$$

Therefore there exists $T_1 \in (0, \log 2)$ such that sup$_{0 \leq t \leq T_1} \|\psi U(t)\|_{D(\mathcal{A}_u)} \leq M$.

Next we prove continuity of $\psi U$ on $[0, T]$. For $0 \leq t_1 < t_2 \leq T$,

$$
\|\psi U(t_1) - \psi U(t_2)\|_H \leq \|T_e(t_1) - T_e(t_2)U_0\|_H
$$

$$
+ \left\| \int_{t_1}^{t_2} T(t_1 - s)N(U(s)) ds - \int_{t_1}^{t_2} T(t_2 - s)N(U(s')) ds' \right\|_H
$$

$$
\leq |t_1 - t_2| \|\mathcal{A}_u U_0\|_H
$$

$$
+ \left\| \int_{t_1}^{t_2} [I - T(t_2 - t_1)]T(t_1 - s)N(U(s)) ds \right\|_H
$$

$$
+ \left\| \int_{t_1}^{t_2} T(t_2 - s)N(U(s')) ds' \right\|_H
$$

$$
\leq |t_1 - t_2| \|\mathcal{A}_u U_0\|_H
$$

$$
+ e^{T/2} |t_2 - t_1| \int_0^{t_1} \|\mathcal{A}_u N(U(s))\|_H ds + e^{T/2} \int_{t_1}^{t_2} \|N(U(s'))\|_H ds'
$$

$$
\leq M' \|t_1 - t_2\|.$
(contractivity) If \( U_1 = (u_1, v_1), U_2 = (u_2, v_2) \in X_T \), then
\[
\|\mathcal{N}(U_1(t)) - \mathcal{N}(U_2(t))\|^2_{\mathcal{H}} = \left\| \left( \begin{array}{c} 0 \\ |u_1(t)|^p - |u_2(t)|^p \end{array} \right) \right\|^2_{\mathcal{H}} \\
\leq p^2 \int_{\mathbb{R}^N} (|u_1(t)| + |u_2(t)|)^{(p-1)}|u_1(t) - u_2(t)|^2 \, dx \\
\leq p^2\|u_1(t)| + |u_2(t)|\|^{2(p-1)}_{L^\infty}|u_1(t) - u_2(t)|^2_{L^\infty} \\
\leq C\|\|u_1(t)| + |u_2(t)|\|_{L^\infty}^{2(p-1)}\|u_1(t) - u_2(t)\|^2_{\mathcal{H}}.
\]
This implies that
\[
\|\Psi U_1(t) - \Psi U_2(t)\|_{\mathcal{H}} \leq \int_0^t \|T_x(t - s)\|_x\mathcal{N}(U_1(s)) - \mathcal{N}(U_2(s))\|_{\mathcal{H}} \, ds \\
\leq C'M^{p-1}e^{T/2} \int_0^t \|U_1(s) - U_2(s)\|_{\mathcal{H}} \, ds.
\]
Consequently, taking \( T_0 \in (0, T_1) \) satisfying
\[
2C'M^{p-1}Te^{T/2} \leq 1,
\]
we obtain \( d(\Psi U_1, \Psi U_2) \leq \frac{1}{M}d(U_1, U_2) \), that is, \( \Psi \) is contractive in \( X_{T_0} \) and also in \( Y_{T_0} \).

**Proof of Proposition 1.1** By Lemma 2.3 we can find a unique fixed point \( U_\infty \) of \( \Psi \) in \( Y_{T_0} \). Moreover, combining the previous arguments implies
\[
\|\mathcal{N}(U_\infty(t_1)) - \mathcal{N}(U_\infty(t_2))\|_{\mathcal{H}} \leq C'M^{p-1}\|U_\infty(t_1) - U_\infty(t_2)\|_{\mathcal{H}} \\
\leq C'M^{p}|t_1 - t_2|.
\]
Thus \( \mathcal{N}(U_\infty(\cdot)) \) is Lipschitz continuous on \([0, T]\). By [7, Corollary 4.2.11(p.109)], we verify that \( U_t + \mathcal{A}_x U = F(U_\infty) \) has a unique strong solution \( U_\infty^* \) given by
\[
U_\infty^*(t) = T_x(t)U_0 + \int_0^t T_x(t - s)\mathcal{N}(U_\infty(s)) \, ds, \quad t \in [0, T_0].
\]
Since \( U_\infty \) is a fixed point of \( \Psi \), we obtain \( U_\infty^*(t) = U_\infty(t) \) for \( t \in [0, T_0] \). Since \( U_\infty \) is a strong solution, we have \( \partial_t U_\infty = -\mathcal{A}_x U_\infty + \mathcal{N}(U_\infty) \in L^\infty(0, T; \mathcal{H}) \) a.e. on \([0, T]\). This gives us that
\[
\partial_t u = v \in L^\infty(0, T; H^2(\mathbb{R}^N)) \cap W^{1,\infty}(0, T; H^1(\mathbb{R}^N)) \cap C([0, T]; H^1(\mathbb{R}^N)),
\]
and
\[
\partial_t v = \Delta u - \frac{K}{|x|} v + |u|^p \in L^\infty(0, T; H^1(\mathbb{R}^N)) \cap C([0, T]; L^2).
\]
Hence \( u \in C^2([0, T]; L^2(\mathbb{R}^N)) \) is nothing but a strong solution of
\[
\partial_t^2 u - \Delta u + \frac{K}{|x|} \partial_t u = |u|^p
\]
on \([0, T]\). Uniqueness of local solutions is due to a proof similar to the contractivity of \( \Psi \) and the finite propagation property follows from the use of \( Y_{T_0} \).
3 Special solutions of linear damped wave equation

In this section we construct special solutions of linear damped wave equation which will be test functions for proving blowup properties.

The following function plays an essential role in the proof of upper bound of lifespan of solutions to (1.1). Similar test functions appear in Zhou–Han [13].

Definition 3.1. For $\beta > 0$, set

$$
\Phi_\beta(x, t) = (|x| + t)^{-\beta} F \left( \frac{N - 1 + V_0}{2}, N - 1; \frac{2|x|}{2 + t + |x|} \right),
$$

where $F(a, b, c; z)$ is the Gauss hypergeometric function given by

$$
F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}
$$

with $(d)_0 = 1$ and $(d)_n = \prod_{k=1}^{n} (d + k - 1)$ for $n \in \mathbb{N}$. (For further properties of $F(\cdot, \cdot, \cdot; z)$, see e.g., Chapter 8 in Beals–Wong [11]).

For the reader’s convenience we would give a derivation the Gauss hypergeometric function from the wave equation.

Lemma 3.1. For $\beta > 0$, $\Phi_\beta$ satisfies the wave equation with the anti-damping term

$$
\partial_t^2 \Phi_\beta - \Delta \Phi_\beta - \frac{V_0}{|x|} \partial_t \Phi_\beta = 0, \quad \text{in } Q = \{(x, t) \in \mathbb{R}^N \times (0, \infty) ; |x| < 2 + t\}.
$$

Proof. We can put $\Phi(x, t) = \Phi_\beta(x, t - 2)$ for $t > 0$. We start with the desired equation

$$
\partial_t^2 \Phi(x, t) - \Delta \Phi(x, t) - \frac{V_0}{|x|} \partial_t \Phi(x, t) = 0, \quad \text{in } \{(x, t) \in \mathbb{R}^N \times (0, \infty) ; |x| < t\}. \quad (3.1)
$$

Put

$$
u(x, t) = \frac{2|x|}{|x| + t} = 2 - \frac{2t}{|x| + t},$$

and therefore

$$
u(x, t) = (2t)^{-\beta}(2 - z)^{-\beta} \varphi(z).
$$

Observing that

$$
\frac{\partial z}{\partial t} = \frac{2|x|}{(|x| + t)^2} = \frac{z(2 - z)}{2t},
$$

we have

$$
\partial_t u = -2\beta(2t)^{-\beta-1}(2 - z)^{-\beta} \varphi(z) + (2t)^{-\beta}[\beta(2 - z)^{\beta-1} \varphi(z) + (2 - z)^{\beta} \varphi'(z)] \frac{\partial z}{\partial t}
$$

$$
= -2\beta(2t)^{-\beta-1}(2 - z)^{-\beta} \varphi(z) - (2t)^{-\beta-1}[\beta(2 - z)^{\beta-1} \varphi(z) + (2 - z)^{\beta} \varphi'(z)]z(2 - z).
$$

$$
= -(2t)^{-\beta-1}(2 - z)^{\beta+1} \left[ \beta \varphi(z) + z \varphi'(z) \right]
$$

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and also
\[ \partial_t^2 u = (2t)^{-\beta-2}(2-z)^{\beta+2}
\left[(\beta + 1)(\beta \varphi(z) + z \varphi'(z)) + z(\beta \varphi(z) + z \varphi'(z))'\right]
\]
\[ = (2t)^{-\beta-2}(2-z)^{\beta+2}\left[\beta(\beta + 1)\varphi(z) + (2\beta + 1)z \varphi'(z) + z^2 \varphi''(z)\right].\]

On the other hand, for radial derivative, we see from \( \frac{\partial}{\partial r} = \frac{2t}{\beta} \) that
\[ \partial_r u = (2t)^{-\beta} \left[-\beta(2-z)^{\beta-1}\varphi(z) + (2-z)^{\beta}\varphi'(z)\right] \frac{\partial z}{\partial r}
\[ = (2t)^{-\beta-1}(2-z)^{\beta+1}\left[-\beta \varphi(z) + (2-z) \varphi'(z)\right]\]
and
\[ \partial_r^2 u = (2t)^{-\beta-2}(2-z)^{\beta+2}\left[(\beta + 1)\beta \varphi(z) - 2(\beta + 1)(2-z) \varphi'(z) + (2-z)^2 \varphi''(z)\right].\]

Combining these equalities and \( \frac{1}{2}(2-z)^{-1} = \frac{1}{z} \), we obtain
\[ 0 = \left(\partial_t^2 u - \partial_r^2 u - \frac{N-1}{r} \partial_r u + \frac{V_0}{r} \partial_r u\right)(2t)^{-\beta-2}(2-z)^{-2}
\[ = \beta(\beta + 1)\varphi(z) + 2(\beta + 1)z \varphi'(z) + z^2 \varphi''(z) - \beta \varphi(z) + (2-z) \varphi'(z) + (2-z)^2 \varphi''(z)
\[ - \frac{N-1}{r}(2t)(2-z)^{-1}\left[-\beta \varphi(z) + (2-z) \varphi'(z)\right] - \frac{V_0}{r}(2t)(2-z)^{-1}\left[-\beta \varphi(z) + (2-z) \varphi'(z)\right]
\[ = -4(1-z)\varphi''(z) + 4(\beta + 1)\varphi'(z) + \frac{2\beta(N-1 + V_0)}{z} \varphi(z) + \frac{2(N-1)}{z}\varphi'(z) + (2-z) \varphi'(z) - 2V_0 \varphi'(z)
\[ = -4\left[(1-z)\varphi''(z) + \left[N-1 - \left(1 + \beta + \frac{N-1 + V_0}{z}\right)\right] \varphi'(z) - \frac{\beta(N-1 + V_0)}{2} \varphi(z)\right].\]

This is nothing but the Gauss hypergeometric differential equation
\[ z(1-z)\varphi''(z) + (c - (1+a+b)z)\varphi'(z) - ab \varphi(z) = 0\]
with
\[ (a, b, c) = \left(\beta, \frac{N-1 + V_0}{2}, N-1\right).\]

This implies that \( \varphi(z) = F(\beta, \frac{N-1 + V_0}{2}, N-1; z) \). \( \square \)

**Lemma 3.2.** (i) For every \( \beta > 0 \) and \( (x, t) \in \mathcal{Q} \),
\[ \partial_t \Phi_\beta(x, t) = -\beta \Phi_{\beta+1}(x, t).\]
(ii) If \( 0 < \beta < \frac{N-1 - V_0}{t} \), then there exists a constant \( c_\beta > 0 \) such that for every \( (x, t) \in \mathcal{Q} \),
\[ c_\beta(2 + t)^{-\beta} \leq \Phi_\beta(x, t) \leq c_\beta^{-1}(2 + t)^{-\beta}.\]
(iii) If \( \beta > \frac{N-1 - V_0}{2} \), then there exists a constant \( c'_\beta > 0 \) such that for every \( (x, t) \in \mathcal{Q} \),
\[ c_\beta(2 + t)^{-\beta} \left(1 - \frac{|x|}{t + 2}\right)^{\frac{N-1 - V_0}{2} - \beta} \leq \Phi_\beta(x, t) \leq c_\beta^{-1}(2 + t)^{-\beta} \left(1 - \frac{|x|}{t + 2}\right)^{\frac{N-1 - V_0}{2} + \beta}.\]

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Proof. (i) In view of the proof of Lemma \[3.1\] we have

$$\partial_t \Phi_\beta(x, t) = -(2t)^{-\beta-1}(2 - z)^{\beta+1}[\beta \varphi(z) + z \varphi'(z)]$$

with $s = \frac{2|d|}{2 + 1 + |\beta|}$. It suffices to show that

$$\beta \varphi(z) + z \varphi'(z) = \beta F\left(\beta + 1, \frac{N - 1 + V_0}{2}, N - 1; z\right), \quad z \in (0, 1). \quad (3.2)$$

Put $\psi(z) = \beta \varphi(z) + z \varphi'(z)$ for $z \in [0, 1]$. Then by the definition of $F(\cdot, \cdot; z)$, we have $\psi(0) = \beta$. On the other hand, we see from the Gauss hypergeometric equation with $a = \beta, b = \frac{N - 1 + V_0}{2}$ and $c = N - 1$ that

$$(1 - z)\psi'(z) = (1 - z)(\beta + 1)\varphi'(z) + z \varphi''(z)$$

$$(1 - z)\psi'(z) = (\beta + 1)(1 - z)\varphi'(z) + z(1 - z)\varphi''(z)$$

$$= (a + 1 - c)\varphi'(z) + b z \varphi'(z) + a b \varphi(z)$$

and therefore $(1 - z)\psi'(z) - b \psi(z) = (a + 1 - c)\varphi'(z)$. The definition of $\psi$ yields

$$z(1 - z)\psi'(z) - b z \varphi(z) = (a + 1 - c) z \varphi'(z)$$

$$= (a + 1 - c) \varphi(z) - (a + 1 - c) a \varphi(z)$$

Differentiating the above equality, we have

$$z(1 - z)\varphi''(z) + (1 - (2 + b)z)\psi'(z) - b \psi(z) = (a + 1 - c) \varphi'(z) - (a + 1 - c) a \varphi'(z)$$

$$= (a + 1 - c) \varphi'(z) - d \left(1 - z\right)\psi'(z) - b \psi(z)\right).$$

Hence we have $z(1 - z)\varphi''(z) + (c - (2 + a + b)z)\psi'(z) + (a + 1) b \psi(z) = 0$. Since $N \geq 2$, all bound solutions of this equation near $0$ can be written by $\psi(z) = h F(a + 1, b, c; z)$ with $h \in \mathbb{R}$. Combining the initial value $\psi(0) = \beta$, we obtain \[3.2\].

The remaining assertions (ii) and (iii) are a direct consequence of the integral representation formula

$$F(a, b, c, z) = \frac{1}{B(c, c - a)} \int_0^1 s^{a-1}(1 - s)^{c-a-1}(1 - Zs)^{-b} ds, \quad 0 \leq z < 1$$

when $c > 0$ and $c - a > 0$. The proof is complete. \[ \square \]

4 Proof of blowup phenomena

In this section we prove upper bound of the lifespan of solutions to \[1.1\] and its dependence of $\varepsilon$ under the condition $0 \leq V_0 < V_*$ with $V_* = \frac{(N-1)^2}{N+1}$. \[11\]
4.1 Preliminaries for showing blowup phenomena

We first state a criterion for derivation of upper bound for lifespan.

**Lemma 4.1.** Let $H \in C^2([\sigma_0, \infty)$ be nonnegative function.

(i) Assume that there exists positive constants $c, C, C'$ such that

$$C[H(\sigma)]^p \leq H''(\sigma) + c \frac{H'(\sigma)}{\sigma}$$

with $H(\sigma) \geq \varepsilon^p C_0^2$ and $H'(\sigma) \geq \varepsilon^p C\sigma$. Then $H$ blows up before $\sigma = C'\varepsilon^{-\frac{1}{p-1}}$ for some $C' > 0$.

(ii) Assume that there exists positive constants $c, C, C'$ such that

$$C\sigma^{1-p}[H(\sigma)]^p \leq H''(\sigma) + 2H'(\sigma)$$

with $H(\sigma) \geq \varepsilon^p C\sigma$ and $H'(\sigma) \geq \varepsilon^p C$. Then $H$ blows up before $\sigma = C'\varepsilon^{-\frac{p-1}{p}}$ for some $C' > 0$.

**Proof.** The assertion follows from [13, Lemma 2.1] with the argument in [13, Section 3]. □

We focus our eyes to the following functionals.

**Definition 4.1.** For $\beta \in (0, \frac{N-1-V_0}{2})$, define the following three functions

$$G_\beta(t) := \int_{\mathbb{R}^N} |u(x, t)|^p \Phi_\beta(x, t) \, dx, \quad t \geq 0,$$

$$H_\beta(t) := \int_0^t (t-s)(2+s)G_\beta(t) \, ds, \quad t \geq 0,$$

$$J_\beta(t) := \int_0^t (2+s)^{-3}H_\beta(t) \, ds, \quad t \geq 0.$$

Note that we can see from Lemma [5.2(ii)] that $G_\beta(t) \approx (2+t)^{-\beta}\|u(t)\|^p_{L^p(\mathbb{R}^N)}$.

**Lemma 4.2.** If $u_\varepsilon$ is a solution of (1.1) in Proposition [7.1] with parameter $\varepsilon > 0$, then $J_\beta(t)$ does not blow up until $\text{LifeSpan}(u_\varepsilon)$.

**Proof.** It follows from the embedding $H^2(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ (given by Gagliardo-Nirenberg-Sobolev inequalities) that $\|u_\varepsilon(t)\|_{L^p}$ is continuous on $[0, \text{LifeSpan}(u_\varepsilon))$ and also $G_\beta(t)$. This means that $J_\beta(t)$ is finite for all $t \in [0, \text{LifeSpan}(u_\varepsilon))$. □

**Lemma 4.3.** For every $\beta > 0$ and $t \geq 0$,

$$(2+t)^2 J_\beta(t) = \frac{1}{2} \int_0^t (t-s)^2G_\beta(s) \, ds.$$

**Proof.** This can be verified by integration by parts twice, by noting that

$$\frac{d}{ds}((t-s)^2(1+s)^{-1}) = \frac{2(1+t)^2}{(1+s)^3}.$$

□

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Lemma 4.4. Let \( u \) be a solution of (1.1). Then for every \( \beta > 0 \) and \( t \geq 0 \),

\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} + \int_0^t (t-s)G_{\beta}(s) \, ds = \int_{\mathbb{R}^N} u(x,t)\Phi_{\beta}(x,t) \, dx + 2\beta \int_0^t \int_{\mathbb{R}^N} u(x,s)\Phi_{\beta+1}(x,s) \, dx \, ds + V_0 \int_0^t \int_{\mathbb{R}^N} \frac{1}{|x|} u(x,t)\Phi_{\beta}(x,t) \, dx \, ds,
\]

where

\[
E_{\beta,0} = \int_{\mathbb{R}^N} f(x)\Phi_{\beta}(x,0) \, dx > 0.
\]

\[
E_{\beta,1} = \int_{\mathbb{R}^N} g(x)\Phi_{\beta}(x,0) \, dx + \beta \int_{\mathbb{R}^N} f(x)\Phi_{\beta+1}(x,0) \, dx + V_0 \int_{\mathbb{R}^N} \frac{1}{|x|} f(x)\Phi_{\beta}(x) \, dx > 0.
\]

**Proof.** By the equation in (1.1) we see from integration by parts that

\[
G_{\beta}(t) = \int_{\mathbb{R}^N} \left( \partial_t^2 u(t) - \Delta u(t) + \frac{V_0}{|x|} \partial_t u(t) \right) \Phi_{\beta}(t) \, dx
= \int_{\mathbb{R}^N} \left( \partial_t^2 u(t) - \Delta u(t) - \frac{V_0}{|x|} \partial_t u(t) \right) \Phi_{\beta}(t) \, dx + \int_{\mathbb{R}^N} u(t)(\Delta \Phi_{\beta}(t)) \, dx.
\]

Using Lemma 3.1 we have

\[
G_{\beta}(t) = \frac{d}{dt} \left[ \int_{\mathbb{R}^N} \left( \partial_t u(t) \Phi_{\beta}(t) - u(t)\partial_t \Phi_{\beta}(t) \right) \, dx \right] + V_0 \int_{\mathbb{R}^N} \frac{1}{|x|} u(t)\Phi_{\beta}(t) \, dx
= \frac{d}{dt} \left[ \int_{\mathbb{R}^N} \left( \partial_t u(t) \Phi_{\beta}(t) - u(t)\partial_t \Phi_{\beta}(t) \right) \, dx \right] + 2\beta \int_{\mathbb{R}^N} u(t)\Phi_{\beta+1}(t) \, dx + V_0 \int_{\mathbb{R}^N} \frac{1}{|x|} u(t)\Phi_{\beta}(t) \, dx.
\]

Integrating it again, we obtain (4.1). \( \square \)

The following lemma makes sense when \( \frac{2}{N-1-V_0} < p \leq \frac{(N-1)^2}{N+1} \) which is equivalent to \( 0 \leq V_0 < \frac{(N-1)^2}{N+1} \).

**Lemma 4.5.** Assume \( \frac{N}{N-1} < p < \infty \) and \( 0 \leq V_0 < \frac{(N-1)^2}{N+1} \). (i) Let \( q > 1 \) satisfy \( \max[p, \frac{2}{N-1-V_0}] < q < \infty \) and put

\[
\beta = \frac{N - 1 - V_0}{2} - \frac{1}{q} \in \left( 0, \frac{N - 1 - V_0}{2} \right).
\]

Then there exists a positive constant \( C_1 > 0 \) such that

\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} + \int_0^t (t-s)G_{\beta}(s) \, ds \leq C_1 \left[ ||u(t)||_{L^p(2+t)^{\frac{N}{p'-\beta}}} + \int_0^t ||u(s)||_{L^p(2+s)^{\frac{N-1-V_0}{2}}} (\log(2+s))^{\frac{1}{p'}} \, ds \right].
\]

(ii) If \( p > \frac{2}{N-1-V_0} \), then setting \( \beta_0 = \frac{N-1-V_0}{2} - \frac{1}{p} \in \left( 0, \frac{N - 1 - V_0}{2} \right) \), one has

\[
\int_0^t (t-s)G_{\beta}(s) \, ds \leq C_1 \left[ ||u(t)||_{L^p(2+t)^{\frac{N}{p'-\beta}}} + \int_0^t ||u(s)||_{L^p(2+s)^{\frac{N-1-V_0}{2}}} (\log(2+s))^{\frac{1}{p'}} \, ds \right].
\]

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Proof. By Lemma 3.4 with finite propagation property, we have
\[ \varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s)G_{\beta}(s)\,ds = I_{\beta,1}(t) + 2\beta I_{\beta,2}(t) + V_0 I_{\beta,3}(t). \]
where
\[ I_{\beta,1}(t) = \int_{B(0,1+t)} u(x,t)\Phi_{\beta}(x,t)\,dx \]
\[ I_{\beta,2}(t) = \int_0^t \left( \int_{B(0,1+t)} u(x,s)\Phi_{\beta+1}(x,s)\,dx \right)\,ds \]
\[ I_{\beta,3}(t) = \int_0^t \left( \int_{B(0,1+t)} \frac{1}{|x|} u(x,s)\Phi_{\beta}(x,s)\,dx \right)\,ds. \]
Using Lemma 3.2(ii), we have
\[ I_{\beta,1}(t) \leq \left( \int_{B(0,1+t)} |u(x,t)|^p\,dx \right)^{\frac{1}{p}} \left( \int_{B(0,1+t)} \Phi_{\beta}(x,t)^{p'}\,dx \right)^{\frac{1}{p'}} \leq c_\beta^{-1} N^{-\frac{1}{p'}} |S^{N-1}|^\frac{1}{p'} \|u(t)\|_{L^p(2+t)^\frac{N}{p}} - \beta. \]
and
\[ I_{\beta,3}(t) \leq \left( \int_{B(0,1+t)} |u(x,t)|^p\,dx \right)^{\frac{1}{p}} \left( \int_{B(0,1+t)} \frac{1}{|x|^p} \Phi_{\beta}(x,t)^{p'}\,dx \right)^{\frac{1}{p'}} \leq c_\beta^{-1} (N - p')^{-\frac{1}{p'}} |S^{N-1}|^\frac{1}{p'} \int_0^t \|u(t)\|_{L^p(2+t)^\frac{N}{p}} - \beta - 1\,ds. \]
Noting that \( \beta + 1 = \frac{N-1}{2} - \frac{1}{q'} \), we see from Lemma 3.2(iii) that
\[ I_{\beta,2}(t) \leq \left( \int_{B(0,1+t)} |u(x,t)|^p\,dx \right)^{\frac{1}{p}} \left( \int_{B(0,1+t)} \Phi_{\beta+1}(x,t)^{p'}\,dx \right)^{\frac{1}{p'}} \leq (c_\beta')^{-1} \|u(t)\|_{L^p(2+t)}^{N-\beta-1} \left( \int_{B(0,1+t)} \left( 1 - \frac{|x|}{t+2} \right)^{\frac{N-1}{2} - \beta - 1} \,dx \right)^{\frac{1}{p'}} \]
\[ = (c_\beta')^{-1} |S^{N-1}|^\frac{1}{p'} \|u(t)\|_{L^p(2+t)}^{N-\beta-1} \left( \int_0^r \left( 1 - \frac{r}{t+2} \right)^{\frac{N-1}{2} - \beta - 1} \,dr \right)^{\frac{1}{p'}} \]
\[ = (c_\beta')^{-1} |S^{N-1}|^\frac{1}{p'} \|u(t)\|_{L^p(2+t)}^{N-\beta-1} \left( \int_0^1 \rho^{-\frac{N}{2}} (1 - \rho)^{N-1} \,d\rho \right)^{\frac{1}{p'}} \]
\[ \leq (c_\beta')^{-1} |S^{N-1}|^\frac{1}{p'} \frac{N}{q'} \|u(t)\|_{L^p(2+t)}^{\frac{N}{q'} - \frac{1}{q'}}. \]
Thus we have
\[ \varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s)G_{\beta}(s)\,ds \leq C_1 \left[ \|u(t)\|_{L^p(2+t)}^{\frac{N}{q'} - \beta} + \int_0^t \|u(s)\|_{L^p(2+s)}^{\frac{N}{q'} - \beta - 1} \,ds \right]. \]
By the definition of \( \beta \) we have the first desired inequality. The second is verified by noticing \( q'/p' = 1 \) in the previous proof.
4.2 Proof of Theorem [1.2] for subcritical case \( \max \{ N - 1 + V_0, N + 3 + V_0 \} < p < p_0(N + V_0) \)

Proof. Fix \( q > p \) as the following way:

\[
\frac{1}{q} \in \left( 0, \frac{N - 1 - V_0}{2} \right) \cap \left( \left( \frac{N - 1 + V}{2}, \frac{N + V_0 p - (N + 3 + V_0)}{2} \right) \right).
\]

The above set is not empty when \((p, V_0) \in \Omega_1 \cup \Omega_2 \cup \Omega_3\); note that for respective cases we can take

\[
\frac{1}{q} = \begin{cases} 
\frac{1}{2} - \delta & \text{if } (p, V_0) \in \Omega_1, \\
\frac{2}{N - 1 - V_0} - \delta & \text{if } (p, V_0) \in \Omega_2, \\
\frac{2}{(N + 1 + V_0) p - (N + 3 + V_0)} - \delta & \text{if } (p, V_0) \in \Omega_3
\end{cases}
\]

with arbitrary small \( \delta > 0 \). Moreover, this condition is equivalent to

\[
q > p, \quad \beta = \frac{N - 1 - V_0}{2} - \frac{1}{q} > 0, \quad \& \quad \lambda = \frac{\gamma(N + V_0 p)}{2p} - \frac{1}{p} + \frac{1}{q} \in (0, p - 1).
\]

Then we see by Lemma 4.5(i) that

\[
\varepsilon E_{\beta,0} + \varepsilon E_{\beta,1} t + \int_0^t (t - s)G_\beta(s) \, ds \leq C_1 \left[ G_\beta(t)^{\frac{1}{p}} (2 + t)^{\frac{N - \beta}{p}} + \int_0^t G_\beta(s)^{\frac{1}{p}} (2 + s)^{\frac{N - \beta}{p} - \frac{\lambda}{p}} \, ds \right]. \tag{4.2}
\]

Observe that

\[
\frac{N - \beta}{p} - \frac{1}{q} - \frac{1}{p} = \frac{1}{p} (p - 1 - \lambda) > 0,
\]

Integrating (4.2) over \([0, t] \), we deduce

\[
\varepsilon E_{\beta,0} t + \varepsilon E_{\beta,1} t^2 + \frac{1}{2} \int_0^t (t - s)^2 G_\beta(s) \, ds
\]

\[
\leq C_1 \left[ \int_0^t G_\beta(s)^{\frac{1}{p}} (2 + s)^{\frac{N - \beta}{p}} \, ds + \int_0^t (t - s)G_\beta(s)^{\frac{1}{p}} (2 + s)^{\frac{N - \beta}{p} - \frac{\lambda}{p}} \, ds \right]
\]

\[
\leq C_1 \left( \int_0^t (2 + s)G_\beta(s) \, ds \right)^{\frac{1}{p}} \left[ \left( \int_0^t (2 + s)^{\frac{N - \beta}{p} - \frac{\lambda}{p}} \, ds \right)^{\frac{1}{p}} + \left( \int_0^t (t - s)^{\frac{\lambda}{p}} (2 + s)^{\frac{N - \beta}{p} - \frac{1}{p}} \, ds \right)^{\frac{1}{p}} \right]
\]

\[
\leq C_2 \left( \int_0^t (2 + s)G_\beta(s) \, ds \right)^{\frac{1}{p}} \left[ (2 + t)^{\frac{N - \beta}{p} - \frac{1}{p} + \frac{\lambda}{p}} + (2 + t)^{1 + \frac{N - \beta}{p} - \frac{1}{p}} \right]
\]

\[
\leq 2C_2 \left( \int_0^t (2 + s)G_\beta(s) \, ds \right)^{\frac{1}{p}} (2 + t)^{1 + \frac{N - \beta}{p}}.
\]

We see from the definition of \( H_\beta \) that

\[
(2C_2)^{-p} (2 + t)^{1 + \lambda - 2p} \left( \varepsilon E_{\beta,0} t + \varepsilon E_{\beta,1} t^2 \right)^p \leq H_\beta'(t).
\]

Hence

\[
H_\beta'(t) \geq C_4 \varepsilon^p (2 + t)^{1 + \lambda}, \quad t \geq 1.
\]
Integrating it over $[0, t]$, we have for $t \geq 2$,
\[ H_\beta(t) = \int_0^t H_\beta'(s) \, ds \geq \int_1^t H_\beta'(s) \, ds \geq C_4 e^p \int_1^t (2 + s)^4 \, ds \geq \frac{C_4 e^p}{4(2 + \lambda)} (2 + t)^{2+\lambda}. \]

We see from the definition of $I_\beta$ that for $t \geq 2$,
\[ J_\beta(t) = (2 + s)^{-3} H_\beta(t) \geq \frac{C_4 e^p}{4(2 + \lambda)} (2 + t)^{-1+\lambda}. \]

and for $t \geq 4$,
\[ J_\beta(t) = \int_2^t J_\beta'(s) \, ds \geq \frac{C_4 e^p}{8(2 + \lambda)} (2 + t)^{\lambda}. \]

On the other hand, we see from Lemma 4.3 that
\[ (2C_2)^{-p}[J_\beta(t)]^p \leq (2 + r)^{2-\lambda} J_\beta''(t) + 3(2 + t)^{1-\lambda} J_\beta'(t). \]

Moreover, setting $J_\beta(t) = \tilde{J}_\beta(\sigma)$, $\sigma = \frac{t}{2}(2 + t)^{\frac{\lambda}{2}}$, we see
\[ (2 + r)^{1-\frac{\lambda}{2}} J_\beta'(t) = \tilde{J}_\beta'(\sigma), \quad (2 + t)^{2-\lambda} J_\beta''(t) + \frac{2 - \lambda}{2} (2 + t)^{1+\lambda} J_\beta'(t) = \tilde{J}_\beta''(\sigma). \]

Then
\[ C_5^{-p}[\tilde{J}_\beta(\sigma)]^p \leq \tilde{J}_\beta''(\sigma) + \frac{4 + \lambda}{\lambda} \sigma^{-1} \tilde{J}_\beta'(\sigma), \quad \sigma \geq \sigma_0 = \frac{2}{\lambda} 2^{\frac{\lambda}{2}}, \]
\[ \tilde{J}_\beta(\sigma) \geq C_6 e^p \sigma^2, \quad \sigma \geq \sigma_1 = \frac{2}{\lambda} 4^{\frac{\lambda}{2}}, \]
\[ \tilde{J}_\beta(\sigma) \geq C_6 e^p \sigma^2, \quad \sigma \geq \sigma_2 = \frac{2}{\lambda} 6^{\frac{\lambda}{2}}. \]

Consequently, by Lemma 4.1 (i) we see that $\tilde{J}_\beta$ blows up before $C_7 e^{-\frac{\lambda}{p+\delta}}$ and then, $J_\beta$ blows up before $C_7 e^{-\frac{\lambda}{p+\delta}}$. By virtue of Lemma 4.2, we have LifeSpan($u_\epsilon$) $\leq C_7 e^{-\frac{\lambda}{p+\delta}}$.

Finally, we remark that if $(p, V_0) \in \Omega_\lambda$, then we can take $1/q = 1/p - \delta$ for arbitrary small $\delta > 0$ and then $\lambda = \gamma(N + V_0; p)/(2p) - \frac{1}{p} + \frac{1}{q} = \gamma(N + V_0; p)/(2p) - \delta$. This implies that
\[ \text{LifeSpan } u \leq C_7 e^{-\frac{2p}{8p+6p+\delta'q}}. \]

for arbitrary small $\delta' > 0$. The proof is complete. $\square$

### 4.3 Proof of Theorem 1.2 for critical case $p = p_0(N + V_0)$

**Proof.** In this case we set
\[ \beta_\delta = \frac{N - 1 - V_0}{2} - \frac{1}{p + \delta} \in \left(0, \frac{N - 1 - V_0}{2}\right). \]

Then by Lemma 4.5 (ii) with $\beta = \beta_\delta$,
\[ \epsilon E_{\beta_\delta, 0} + \epsilon E_{\beta_\delta, 2} t \leq C_1 \left[ ||u(t)||_{L^r} (2 + t)^{\frac{\lambda}{p} - \beta} + \int_0^t ||u(s)||_{L^r} (2 + s)^{\frac{\lambda}{p} - \beta_\delta - 1} \, ds \right] \]
\[ \leq K_1 \left[ (G_{\beta_\delta}(t))^\frac{1}{2} (2 + t)^{\frac{\lambda}{p} - \beta_\delta + (\beta_\delta - \beta)} + \int_0^t \left( G_{\beta_\delta}(t) \right)^{\frac{1}{2}} (2 + s)^{\frac{\lambda}{p} - \beta_\delta - 1} \, ds \right]. \]
Noting that $\frac{N-\beta_0}{p} = 1 + \frac{1}{p}$ and integrating it over $[0, t]$, we have

$$
\varepsilon E_{\beta_0,0}t + \frac{E_{\beta_0,1}}{2}t^2 \leq K_1 \left[ \int_0^t (G_{\beta_0}(s)^\frac{1}{p} (2 + s)^{1 + \frac{1}{p} + (\beta_0 - \beta_0)} ds + \int_0^t (t - s)(G_{\beta_0}(t)^\frac{1}{p} (2 + s)^{\frac{1}{p}} ds \right]

\leq K_1 \left( \int_0^t G_{\beta_0}(s)(1 + s) ds \right) \frac{1}{p} \left[ \left( \int_0^t (2 + s)^{\frac{1}{p} + (\beta_0 - \beta_0)} p' ds \right)^\frac{1}{p} + \left( \int_0^t (t - s)^{p'} ds \right)^\frac{1}{p} \right]

\leq K_2 \left( \int_0^t G_{\beta_0}(s)(1 + s) ds \right) \frac{1}{p} (2 + t)^{1 + \frac{1}{p}}.

By the definition of $H_{\beta_0}$, we have for $t \geq 1$,

$$
H'_{\beta_0}(t) \geq K_2^2 \varepsilon^p \left( E_{\beta_0,0}t + \frac{E_{\beta_0,1}}{2}t^2 \right)^p (2 + t)^{-2p} \geq K_3 \varepsilon^p (2 + t)^2
$$

and then for $t \geq 2$,

$$
H_{\beta_0}(t) \geq \int_1^t C_{\beta_0} s ds \geq K_4 \varepsilon^p (2 + t)^2.
$$

On the other hand, by Lemma 4.5(ii) we have

$$
\int_0^t (t - s)G_{\beta_0}(s) ds \leq C_1 \left[ \|u(t)\|_{L^p}(2 + t)^{\frac{N-\beta_0}{p}} + \int_0^t \|u(s)\|_{L^p}(2 + s)^{\frac{N-\beta_0}{p}} (\log(2 + s))^{\frac{N-\beta_0}{p}} ds \right].
$$

Noting $\frac{N-\beta_0}{p} = 1 + \frac{1}{p}$ again and integrating it over $[0, t]$, we have

$$
\frac{1}{2} \int_0^t (t - s)G_{\beta_0}(s) ds \leq K_4' \left[ \int_0^t G_{\beta_0}(s)^{\frac{1}{p} (2 + s)^{\frac{\beta_0}{p}}} ds + \int_0^t (t - s)G_{\beta_0}(s)^{\frac{1}{p} (2 + s)^{\frac{\beta_0}{p}} (\log(2 + s))^{\frac{N-\beta_0}{p}}} ds \right]

\leq K_4' H'_{\beta_0}(t)^{\frac{1}{p}} \left[ \int_0^t (2 + s)^{p'} ds + \int_0^t (t - s)^{p'} \log(2 + s) ds \right]

\leq K_4' H'_{\beta_0}(t)^{\frac{1}{p}} (2 + t)^{1 + \frac{1}{p} (\log(2 + t))^{\frac{1}{p}}}.
$$

As in the proof of subcritical case, we deduce

$$
(K_2')^{-p} (\log(2 + t))^{1-p} J_{\beta_0}(t)^p \leq H'_{\beta_0}(t)(1 + t)^{-1}

\leq (2 + t)^2 J'_{\beta_0}(t) + 3(2 + t)J_{\beta_0}(t).
$$

Here we take $J_{\beta_0}(t) = \tilde{J}_{\beta_0}(\sigma)$ with $\sigma = \log(2 + t)$. Since

$$
(2 + t)J_{\beta_0}'(t) = \tilde{J}_{\beta_0}'(\sigma), \quad (2 + t)^2 J_{\beta_0}''(t) + (2 + t)J_{\beta_0}'(t) = \tilde{J}_{\beta_0}''(\sigma),
$$

we obtain for $\sigma \geq \sigma_0 := \log 2$,

$$
(K_2')^{-p} \sigma^{1-p} \tilde{J}_{\beta_0}(\sigma)^p \leq \tilde{J}_{\beta_0}''(\sigma) + 2\tilde{J}_{\beta_0}''(\sigma).
$$
Moreover, we have for $\sigma \geq \sigma_1 = \log 4$,
\[
\tilde{J}_{\beta_0}(\sigma) = (2 + t) J'_{\beta_0}(t)
= (2 + t)^{-2} H_{\beta_0}(t)
\geq K_4 e^{\rho t}
\]
and therefore for $\sigma \geq \sigma_2 = 2 \log 4$,
\[
\tilde{J}_{\beta_0}(\sigma) \geq \frac{K_4}{2} e^{\rho \sigma}.
\]
Applying Lemma 4.1(ii) we deduce that $\tilde{J}_{\beta_0}$ blows up before $\sigma = K_5 e^{-p(\rho-1)}$. Then by definition $J_{\beta_0}$ blows up before $\exp[K_5 e^{-p(\rho-1)}]$. Consequently, using Lemma 4.2 we obtain
\[
\text{LifeSpan } u \leq \exp[K_5 e^{-p(\rho-1)}].
\]
The proof is complete. \(\Box\)

Remark 4.1. In particular, in the proof of Theorem 1.2 with $p = p(N + V_0)$, we have used two kind of auxiliary parameters $1/q = \frac{N-1-V_0}{2} - \frac{1}{p}$ and $1/q = \frac{N-1-V_0}{2} - \frac{1}{p+\delta}$. The first choice is for deriving lower bound of the functional $J_{\beta_0}$ and the second is for deriving differential inequality for $J_{\beta_0}$. The first choice is essentially different from the idea of Yordanov–Zhang [11] to prove the lower bound of a functional.

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