A NON-LINEAR MONOTONICITY PRINCIPLE AND APPLICATIONS TO SCHRÖDINGER-TYPE PROBLEMS

JULIO BACKHOFF-VERAGUAS, MATHIAS BEIGLBÖCK, AND GIOVANNI CONFORTI

ABSTRACT. A basic idea in optimal transport is that optimizers can be characterized through a geometric property of their support sets called cyclical monotonicity. In recent years, similar monotonicity principles have found applications in other fields where infinite-dimensional linear optimization problems play an important role.

In this note, we observe how this approach can be transferred to non-linear optimization problems. Specifically we establish a monotonicity principle that is applicable to the Schrödinger problem and use it to characterize the structure of optimizers for target functionals beyond relative entropy. In contrast to classical convex duality approaches, a main novelty is that the monotonicity principle allows to deal also with non-convex functionals.

keywords: cyclical monotonicity, monotonicity principle, Schrödinger problem, $L^2$ divergence, non-linear optimization.

1. INTRODUCTION AND MAIN RESULTS

1.1. Motivation from optimal transport. Given probabilities $\mu$ and $\nu$ on Polish spaces $X$ and $Y$, and a cost function $c : X \times Y \to \mathbb{R}_+$, the Monge-Kantorovich problem is to find a cost-minimizing transport plan. More precisely, writing $\text{cpl}(\mu, \nu)$ for the set of all couplings (namely, measures) on $X \times Y$ with $X$-marginal $\mu$ and $Y$-marginal $\nu$, the problem is to find

$$\inf \left\{ \int c \, d\mathbb{P} : \mathbb{P} \in \text{cpl}(\mu, \nu) \right\}$$

(OT)

and to identify an optimal transport plan $\mathbb{P}^* \in \text{cpl}(\mu, \nu)$.

The notion of c-cyclical monotonicity leads to a geometric characterization of optimal couplings. Its relevance for (OT) has been highlighted by Gangbo and McCann [24], following earlier works of Knott and Smith [31] and Rüschendorf [41] among others.

We give here a slightly non-standard definition that is not inherently tied to the transport problem and serves our exposition more directly.

A set $\Gamma \subseteq X \times Y$ is c-cyclically monotone if any positive measure $\alpha$ that is finite and supported on finitely many points in $\Gamma$, is a cost-minimizing transport between its marginals. I.e., if $\alpha'$ has the same marginals as $\alpha$, then

$$\int c \, d\alpha \leq \int c \, d\alpha'.$$

JB and MB acknowledge the Austrian Science Fund (FWF) for its support via the project Y00782. GC acknowledges funding from the grant SPOT (ANR-20-CE40-0014).

1The arguments in [44] Exercise 2.21, p.79 can be used to prove the equivalence with the more familiar way of stating c-cyclical monotonicity of a set $\Gamma$ in the case of $c$ being the quadratic cost: usually one requires that for any $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$, $y_{n+1} = y_1$ it holds $\sum_{i=1}^n c(x_i, y_i) \leq \sum_{i=1}^n c(x_i, y_{i+1})$. The argument when $c$ is general carry over verbatim.
A transport plan $\gamma$ is called $c$-cyclically monotone if it is concentrated on such a set $\Gamma$, i.e. if there is such a $\Gamma$ with $\gamma(\Gamma) = 1$.

The equivalence of optimality and $c$-cyclical monotonicity has been established under progressively milder regularity assumption. Based on [1, 40, 43, 7, 14] the following ‘Monotonicity Principle’ holds true:

**Theorem 1.1.** Let $c : X \times Y \to [0, \infty)$ be measurable and assume that $\mathbb{P} \in \text{cpl}(\mu, \nu)$ is a transport plan with finite cost $\int c \, d\mathbb{P} \in \mathbb{R}_+$. Then $\mathbb{P}$ is optimal if and only if $\mathbb{P}$ is $c$-cyclically monotone.

The importance of this result stems from the observation that it is often an elementary and feasible task to see whether a transport behaves optimally on a finite number of points. But this would be a priori of no help for a problem where single points do not carry positive mass. Theorem 1.1 provides the required remedy to this obstacle as it establishes the connection to optimality on a “pointwise” level.

1.2. **Recent developments and aims of this article.** More recently, variants of this ‘monotonicity principle’ have been applied in transport problems for finitely or infinitely many marginals [38, 19, 27, 8, 45], the martingale version of the optimal transport problem [9, 36, 11], stochastic portfolio theory [37], the Skorokhod embedding problem [5, 28], the distribution constrained optimal stopping problem [6, 10] and the weak transport problem [20, 8, 4].

What all these articles have in common is that the original idea is applied to other infinite-dimensional linear optimization problems. In the present note, we advertise the idea that this optimality principle can be useful beyond linear problems and in fact to problems that are not susceptible to a convex duality approach. Given the versatile applicability of the idea in various linear optimization problems, the extension to non-linear problems appears highly promising.

In Section 1.3 we present the principal idea of what kind of structure such a monotonicity principle might take in applications to non-linear optimization problems. While the heuristic derivation in Section 1.3 is based on a purely formal linearization procedure, we rigorously establish this result in Section 1.4 for a large subclass of non-linear problems. We then further specify this rigorous monotonicity principle in the setup of a general and not necessarily convex version of the Schrödinger problem: In Theorem 1.4 we show how this non-linear monotonicity principle can be used to obtain necessary optimality conditions, which are shown to be also sufficient for convex problems such as the classical Schrödinger problem, see Theorem 1.6. Furthermore we derive novel variants of these conditions for more general entropy functionals in Theorem 1.5.

To illustrate the potential of our approach, we apply our results to obtain a shape theorem for the optimal solutions of a non-convex Schrödinger problem with congestion. Furthermore, we discuss briefly how a natural generalization of our findings, which we plan to address in future works, would allow to advance considerably the understanding of the recently introduced mean field Schrödinger problem [2].

1.3. **A ‘formal’ non-linear monotonicity principle.** In this section we introduce some notation and then state a non-linear monotonicity principle which is ‘formal’ in the sense that we do not give a rigorous proof or precise conditions under which it is expected to hold. In the next section we will then provide a rigorous version which is applicable to the Schrödinger problem and similar energy minimization problems.
Let $\Omega$ be a Polish space with $\mathcal{B}$ its Borel sigma-algebra. Consider $\mathcal{F}$ a family of real-valued functions on $\Omega$. We suppose either of the following:

1. $\mathcal{F}$ is a subset of $C_b(\Omega)$, the space of continuous bounded functions.
2. $\mathcal{F}$ is a countable sub-family of $B_b(\Omega)$, the space of Borel bounded functions.

We are given a functional

$$G : \mathcal{P}(\Omega) \to [0, +\infty],$$

and we are interested in the following problem

$$\inf \{ G(Q) : Q \in \mathcal{P}(\Omega), Q \in \text{Adm} \},$$

where

$$\text{Adm} := \text{Adm}_\mathcal{F} := \left\{ Q : \int f dQ = 0, \forall f \in \mathcal{F} \right\},$$

and $\mathcal{P}(\Omega)$ denotes the set of Borel probability measures on $\Omega$.

The standing assumption on $G$ is that there exist directional derivatives with representation via functions, i.e. for any $Q$ in the domain $D(G) = \{ Q \in \mathcal{P}(\Omega) : G(Q) < \infty \}$ there exists $\delta G_Q : \Omega \to (-\infty, \infty]$ measurable such that

$$\forall Q \in D(G), \quad \lim_{\varepsilon \searrow 0} \frac{G(Q + \varepsilon[\bar{Q} - Q]) - G(Q)}{\varepsilon} = \int_{\Omega} \delta G_Q(\omega)[\bar{Q} - Q](d\omega),$$

where one implicitly assumed the limit to exist for all $Q, \bar{Q} \in D(G)$.

Positive finite measures $\alpha, \alpha'$ with equal mass and finite support are called competitors if

$$\int f \, d(\alpha - \alpha') = 0, \forall f \in \mathcal{F}.$$ 

We then expect the following:

**Formal Statement 1.2** (Non-Linear Monotonicity Principle, formal version). Suppose $Q^* \in \text{Adm} \cap D(G)$ is an optimizer for Problem (P). Then

1. $Q^*$ is a minimum of the linearized problem

$$\inf \left\{ \int_{\Omega} c(\omega) \, Q(d\omega) : Q \in \text{Adm} \cap D(G) \right\},$$

where $c(\omega) := \delta G_{Q^*}(\omega)$,

2. There exists a Borel set $\Gamma_{Q^*} \subseteq \Omega$ such that $Q^*(\Gamma_{Q^*}) = 1$ having the following property: given competitors $\alpha, \bar{\alpha}$, with $\text{supp} \, \alpha \subseteq \Gamma_{Q^*}$ we have

$$\int \delta G_{Q^*} \, d\alpha \leq \int \delta G_{Q^*} \, d\bar{\alpha}.$$ 

**Formal derivation.** By optimality of $Q^*$ and the fact that $\text{Adm}$ is convex, we easily obtain

$$\lim_{\varepsilon \searrow 0} \frac{G(Q^* + \varepsilon[\bar{Q} - Q^*]) - G(Q^*)}{\varepsilon} = \int_{\Omega} \delta G_{Q^*}(\omega)[\bar{Q} - Q^*](d\omega) \geq 0,$$

for all $\bar{Q} \in \text{Adm} \cap D(G)$, showing that $Q^*$ is a minimum of the linearized problem in (1). The monotonicity principle in [8, Theorem 1.4] applies, and we find exactly the desired condition in (2).
1.4. A Rigorous Non-Linear Monotonicity Principle. We consider throughout a continuous function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying at least:

\[
\text{h is differentiable on } (0, \infty) \text{ and the limit } h'(0) := \lim_{x \downarrow 0} h'(x) \text{ exists. (H)}
\]

Throughout we fix \( \mathcal{P} \in \mathcal{P}(\Omega) \) and consider

\[
G(\mathcal{Q}) := G_h(\mathcal{Q}) := \left\{ \int_\Omega h \left( \frac{d\mathcal{Q}}{d\mathcal{P}}(\omega) \right) \mathcal{P}(d\omega) \right\}, \text{ if } \mathcal{P}(\Omega) \gg \mathcal{Q} \ll \mathcal{P},
\]

and the associated minimization problem

\[
\inf \{ G_h(\mathcal{Q}) : \mathcal{Q} \in \mathcal{P}(\Omega), \mathcal{Q} \in \text{Adm} \}. \quad (\text{P}_h)
\]

**Lemma 1.3** (Non-Linear Monotonicity Principle). In addition to (H), suppose that \( h \) is twice differentiable on \( \mathbb{R}_+ \) with \( h'' \geq C \) everywhere for some \( C \in \mathbb{R} \) and that \( \lim_{x \to +\infty} h'(x) = +\infty \). Furthermore, assume that either \( h' \) is lower bounded or \( \lim_{x \downarrow 0} h'(x) = -\infty \) and let \( Q^* \) be an optimizer of Problem \((\text{P}_h)\). Then there exist sets \( \Gamma_{Q^*}, \Gamma_{\mathcal{P}} \) such that \( \mathcal{P}(\Gamma_{Q^*}) = Q^*(\Gamma_{Q^*}) = 1 \) and for all competitors \( \alpha, \alpha' \) with \( \text{supp}(\alpha) \subseteq \Gamma_{Q^*}, \text{supp}(\alpha') \subseteq \Gamma_{\mathcal{P}} \) we have

\[
\int h \left( \frac{d\alpha}{d\alpha'} \right) d\alpha \leq \int h \left( \frac{d\alpha}{d\alpha'} \right) d\alpha'. \quad (1.1)
\]

We defer the proof of the above lemma to Section 2.1. Typical examples of \( h \) satisfying the above conditions are \( h(x) = x \log x - x + 1 \) or \( h(x) = x^p \) with \( p > 1 \).

1.5. Schrödinger-type Problems. We specify the setting of Section 1.4. In this part we are interested in the case

\[
\Omega := X \times Y,
\]

for \( X, Y \) Polish spaces. As for the constraints set \( \mathcal{F} \), we are interested in

\[
\mathcal{F}_\mu := \{ \tilde{f}(x, y) = f(x) - \int_X f \, d\mu : f \in C_b(X) \} \quad \mathcal{F}_\nu := \{ \tilde{g}(x, y) = g(y) - \int_Y g \, d\nu : g \in C_b(Y) \},
\]

and

\[
\mathcal{F} := \mathcal{F}_\mu \cup \mathcal{F}_\nu,
\]

for given probability measures \( \mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y) \), satisfying

\[
\mu \ll \text{proj}^X(\mathcal{F}) \text{ and } \nu \ll \text{proj}^Y(\mathcal{F}).
\]

With these specifications, our minimization problem (Problem \((\text{P}_h)\)) clearly becomes:

\[
\inf \left\{ \int_{X \times Y} h \left( \frac{d\mathcal{Q}}{d\mathcal{P}}(x, y) \right) \mathcal{P}(dx, dy) : \mathcal{Q} \in \text{cpl}(\mu, \nu) \right\}. \quad (1.2)
\]

Notice that for the choice \( h(x) = x \log(x) \), Problem \((\text{P}_h)\) becomes the classical Schrödinger problem.\(^{2}\)

We now rigorously derive necessary optimality conditions for Problem \((1.2)\). The functions \( \varphi \) and \( \psi \) appearing in Theorem 1.4 can formally be seen as Lagrange multipliers and in the case \( h(x) = x \log(x) \) they are known as Schrödinger potentials, see [34, Sec 2.]. We remind the reader that \( \rho \sim \eta \) stands for equivalence of measures in the sense that \( \rho \ll \eta \text{ and } \eta \ll \rho \).

\(^{2}\)See Léonard’s survey [34] on classical results around the Schrödinger problem and its probabilistic meaning. Recently this problem has seen a surge in interest owing to the overture to machine learning by Cuturi [20].
Theorem 1.4. Assume that Problem (1.2) is finite and $Q^*$ is an optimizer thereof. Importantly we also assume that $P \sim \mu \otimes \nu$. Let $h : [0, \infty) \to (-\infty, \infty)$ be twice continuously differentiable, \(\lim_{x \to 0} h'(x) = -\infty, \lim_{x \to +\infty} h'(x) = +\infty\) and \(\inf_{\mathbb{R}_+} h'' > -\infty\). Then $Q^* \sim P$ and there exist measurable functions $\varphi : X \to [-\infty, +\infty)$ and $\psi : Y \to [-\infty, +\infty)$ such that
\[
h' \circ dQ^*/dx (x, y) = \varphi(x) + \psi(y), \quad P-a.s. \tag{1.3}
\]

It is worth remarking that the above theorem applies to $h(x) = x \log(x)$ (where $h'(x) = 1 + \log(x)$) but not to $h(x) = x^2$ (where $h'(x) = 2x$). This latter case (and similar ones) is covered by the following complementary theorem:

Theorem 1.5. Assume that Problem (1.2) is finite and $Q^*$ is an optimizer thereof. Assume that $P \sim \mu \otimes \nu$. Let $h : [0, \infty) \to (-\infty, \infty)$ be strictly increasing, continuously differentiable, \(\lim_{x \to 0} h'(x) = 0, \lim_{x \to +\infty} h'(x) = +\infty, \inf_{\mathbb{R}_+} h'' > -\infty\). Then there exist measurable functions $\varphi : X \to [-\infty, +\infty)$ and $\psi : Y \to [-\infty, +\infty)$ such that
\[
h' \circ dQ^*/dy (x, y) = (\varphi(x) + \psi(y))_+, \quad P-a.s. \tag{1.4}
\]

We remark that uniqueness of an optimizer to Problem (1.2) is guaranteed if $h$ is strictly convex. On the other hand, Conditions (1.3)–(1.4) do not characterize optimizers even when these are unique (e.g. when $h'$ is not one-to-one).

Comparison with the existing literature. Minimization problems of the form (1.2) have been studied for a long time, the most notable example being the Schrödinger problem. Indeed, analogues of Theorem 1.4 for the case where $h(x) = x \log x$ have been obtained in seminal works of Fortet and Beurling [23, 13]. In more recent works, Borwein and Lewis [15] and Borwein, Lewis and Nussbaum [16] proposed an approach to entropy minimization that combines fixed point-arguments and convex optimization techniques. We refer to Gigli and Tamanini’s article [25] for adaptations of these results to the setting of RCD spaces. Convex duality is also at the heart of the proof strategy of Pennanen and Perkkiö [39]. A different viewpoint is adopted by Rüschendorf and Thomsen [42]: therein the shape of the optimal measure is found as a consequence of the closedness property of sum spaces of integrable functions. We also refer to Carlier and Laborde [17] for multidimensional generalizations. A large part of the above mentioned results is surveyed by Léonard in [34]. This author has also proven shape theorems for the Schrödinger problem analogous to Theorem 1.4 in [32, 33]. Cattiaux and Gamboa [18] treat the more general case when $h$ is the log-Laplace transform of a probability measure: this condition implies that $h$ is convex. However, it is not assumed there (unlike what we do here) that $P \sim \mu \otimes \nu$, but only $P \ll \mu \otimes \nu$ is needed. Their proofs rely essentially on ideas and tools coming from large deviations and on the earlier findings of [42]. To the best of our knowledge, the case when $h$ is not convex has not been treated before the present article. As for Lemma 1.2 a more explicit version in the particular case of the classical Schrödinger Problem has been obtained in parallel by Bernton, Ghosal and Nutz in [12], where it is furthermore leveraged to obtain stability and large deviations estimates.

We now study the converse direction: how structure of a measure implies optimality. Here we do need to assume convexity.
Theorem 1.6. Let $h : [0, \infty) \to (-\infty, \infty)$ be strictly convex, lower-bounded, and continuously differentiable, $\lim_{x \to 0} h'(x) = 0$, $\lim_{x \to +\infty} h'(x) = +\infty$, and $h(2x) \leq ah(x) + bx + c$ for constants $a, b, c$. Suppose that $Q^* \in \text{cpl}(\mu, \nu)$ is absolutely continuous with respect to $\mathbb{P}$, with

$$h' \circ \frac{dQ}{d\mathbb{P}}(x, y) = (\varphi(x) + \psi(y))_+, \quad \mathbb{P} - \text{a.s.}$$

for measurable $\varphi : X \to [-\infty, +\infty)$ and $\psi : Y \to [-\infty, +\infty)$. Then $Q^*$ is optimal for (1.2).

With the same techniques used to prove Theorem 1.6, variants of this result can be established if $h'(0) \in (-\infty, \infty)$. This covers in particular the Schrödinger problem, and relatives thereof, for which the converse direction is contained in Theorem 1.4. As a side remark, we also want to stress that Theorem 1.5 can be plainly adapted to cover the case $h'(0) \in (-\infty, \infty)$.

A Toy Example: Schrödinger Problem with Congestion. Let us say $x \in X$ and $y \in Y$ denote respectively origins and destinations for car users in a city. Hence an origin-destination pair $(x, y)$ can stand for the route that a car has to travel from $x$ to $y$. Experts have determined that $\mathbb{P} \in \mathcal{P}(X \times Y)$ is the optimal use of the road network (here $\mathbb{P}(dx, dy)$ is the infinitesimal proportion of cars taking route $(x, y)$) in the stationary case. However, the actual proportion of car trip origins and car trip destinations are described by $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ respectively, rather than $\text{proj}^X(\mathbb{P})$ and $\text{proj}^Y(\mathbb{P})$. In the vanilla version of the Schrödinger Problem we aim to determine a minimizer $Q^*$ of the relative entropy $\int \frac{dQ}{d\mathbb{P}} \log \left(\frac{dQ}{d\mathbb{P}}\right) d\mathbb{P}$ over $Q \in \text{cpl}(\mu, \nu)$, $Q \ll \mathbb{P}$, amounting to the distribution of car trips compatible with the experts’ guess $\mathbb{P}$ and the marginal information $\mu$ and $\nu$. However, we may also want to consider congestion effects, codified by an added term $f(\frac{dQ}{d\mathbb{P}})$ with $f(\cdot)$ increasing, the idea being that adding traffic above the experts’ recommendation should be more costly than the opposite. This way we arrive at the non-convex Schrödinger-type problem of minimizing $\int \left[\frac{dQ}{d\mathbb{P}} \log \left(\frac{dQ}{d\mathbb{P}}\right) + f(\frac{dQ}{d\mathbb{P}})\right] d\mathbb{P}$ under the same constraints. The optimality condition in Theorem 1.4 now reads:

$$(\log + f')\left(\frac{dQ^*}{d\mathbb{P}}\right) = \varphi(x) + \psi(y),$$

from which $Q^*$ can even be determined depending on the choice of $f$.

Some perspectives on the mean field Schrödinger problem. In the recent article [2] a mean field version of the Schrödinger problem has been introduced. A simplified discrete-time version of it consists in finding the most likely evolution conditionally to observations at initial and terminal times of the particle system $(X_i^t)_{t=0,1,2}$ where $(X_0^1, \ldots, X_0^N)$ are i.i.d. samples from a probability measure $\mu$ on $\mathbb{R}^d$ and

$$X_{t+1}^i - X_t^i = -\sum_{j \in N} \nabla W(X_t^i - X_t^j) + \xi_t^i, \quad i = 1, \ldots, N, \quad t = 0, 1. \quad (1.5)$$

Here the random variables $(\xi_t^i)_{t=1,\ldots,N; i=1,2}$ are i.i.d. standard Gaussians. The large deviations rate function for the empirical distribution of the particle system (1.5) in the regime $N \to +\infty$ is known explicitly (see [21] for a general result in continuous time and [22] for the analysis of the toy model (1.5)) and leads to the following problem formulation

$$\inf \left\{ \int h \left( \frac{dQ}{dR(Q)}(x_0, x_1, x_2) \right) R(Q)(dx_0, dx_1, dx_2) : Q \in \text{cpl}(\mu, \nu) \right\}. \quad (1.6)$$
In the above we denoted \( h(x) = x \log x \) and, adapting the convention used throughout this paper, we denoted by \( \text{cpl}(\mu, \nu) \) the subset of \( \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) \) whose first marginal \((t = 0)\) is \( \mu \) and whose last \((t = 2)\) marginal is \( \nu \). Finally, for a given \( Q, R(\cdot) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) \) is defined as the law of the controlled discrete stochastic differential equation

\[
\begin{cases}
Z_{t+1} = Z_t - \int \nabla W(Z_t - x)Q(dx_0, dx_1, dx_2) + \xi_t, & t = 0, 1, \\
Z_0 \sim \mu,
\end{cases}
\]

where \((\xi_0, \xi_1, \xi_2)\) are i.i.d. standard Gaussians. Despite several analogies with (1.2), including the fact that the function \( R(\cdot) \) naturally introduces non-convexity into the problem, the analysis of (1.6) is outside the reach of this work, essentially because the “reference” measure \( R(\cdot) \) depends on \( Q \). However, the heuristics put forward in the introduction based on the linearization procedure still apply and leads to natural conjectures on the kind of monotonicity principle and shape theorem for optimizers to be expected in this situation. For this reason, the present work resides in the fact that a shape theorem for the mean field Schrödinger problems yields existence of solutions for the coupled Fokker Planck-Hamilton Jacobi Bellman system describing the dynamics of mean field Schrödinger bridges. We redirect the interested reader to [2, Sec 1.3] for the precise form of such PDE system as well as for more explanations.

2. Proofs

2.1. Proof of the Non-linear Monotonicity Principle: Lemma [1.3]. The proof requires two preliminary results. The first is a lemma telling essentially that, if \( G = G_h \) directional derivatives can be computed with

\[
\delta G_Q(\omega) = h'(\frac{dQ}{dP})(\omega).
\]

More precisely, we will need this in the form of the following lemma:

**Lemma 2.1.** Let \( h \) satisfy the hypotheses of Lemma [1.3] Consider now a probability measure \( Q \) and positive measures \( \theta, \theta' \) satisfying

(i) \( \theta(\Omega) = \theta'(\Omega) \),

(ii) \( \int h(\frac{dQ}{dP}) \, dP \) exists and is finite.

(iii) \( \theta \leq Q, \theta' \leq P \).

(iv) There is a constant \( l \in \mathbb{R} \) such that \( -l \leq h'(\frac{dQ}{dP}) \leq l \) hold \( \theta + \theta' \)-a.s.

(v) \( \int h'(\frac{dQ}{dP}) \, d(\theta' - \theta) < 0 \).

Setting \( Q_\varepsilon := Q + \varepsilon(\theta' - \theta) \) we then find that, for all \( \varepsilon > 0 \) small enough, \( \int h(\frac{dQ}{dP}) \, dP \) exists and

\[\int h(\frac{dQ}{dP}) \, dP < \int h(\frac{dQ}{dP}) \, dP.\]

**Proof.** If \( 0 \leq \varepsilon \leq 1 \) then \( Q_\varepsilon \) is by (i) and (iii) a probability measure. By hypothesis \( h'' \geq C \) we have

\[
h(\frac{dQ}{dP}) \geq h(\frac{dQ}{dP}) + \varepsilon h'(\frac{dQ}{dP}) \frac{d(\theta' - \theta)}{dP} - \frac{\varepsilon^2 C}{2} \left( \frac{d(\theta' - \theta)}{dP} \right)^2,
\]

Combining (iii) and (iv) we get
\[
\sup_{x,y} \left| \frac{d(f - \theta)}{dx} \right| \leq \sup_{\text{supp}(\theta)} \frac{d(f - \theta)}{dx} + \frac{d\theta}{dx} \\
\leq 1 + \sup_{\text{supp}(\theta)} \frac{d\theta}{dx} \leq 1 + \sup (h' - 1)([0,1]) < +\infty,
\]

where to obtain the last inequality we used that \(\lim_{x \to +\infty} h'(x) = +\infty\). Using this result in (2.1) shows that \(\int h\left(\frac{d\gamma}{dx}\right) d\mathbb{P}\) exists and belongs to \((-\infty, +\infty]\). Similarly,
\[
h\left(\frac{d\gamma}{dx}\right) - h\left(\frac{d\theta}{dx}\right) \geq \varepsilon h'\left(\frac{d\theta}{dx}\right) \frac{d(\theta - \gamma)}{dx} - \frac{\varepsilon^2 C}{2} \left(h' - 1\right)^2.
\]

Next, we observe that if we can prove that for \(\gamma = \theta, \theta'\) we have
\[
\int h'\left(\frac{d\gamma}{dx}\right) \, dy \to \int h'\left(\frac{d\theta}{dx}\right) \, dy,
\]
then we obtain the conclusion dividing by \(\varepsilon\) on both sides in (2.2), integrating in \(d\mathbb{P}\) and letting \(\varepsilon \to 0\). We only argue in the case when \(\lim_{x \to +\infty} h'(x) = -\infty\), the other case being simpler. In this case, condition (iv) implies that \(\gamma\text{-a.s. } d\gamma/d\mathbb{P}\) takes values in a compact set of \((0, +\infty)\). Using this last observation and (iii) we deduce that \(\gamma\text{-a.s. } d\gamma/d\mathbb{P}\), viewed as a function of \(x, y\) and \(\varepsilon\), takes its values in a compact set of \((0, +\infty)\) provided \(\varepsilon\) is small enough. The desired conclusion follows by dominated convergence. \(\square\)

The second ingredient, towards the proof of Lemma 1.3, is the following result from [7], which is a consequence of a duality result by Kellerer [30]. We recall that if \(\alpha, \beta\) are two measures, we write \(\alpha \leq \beta\) if \(\alpha(A) \leq \beta(A)\) for all \(A\) measurable sets. In the following, we denote by \(p_i\) the projection onto the \(i\)-th coordinate of a product space, so that if \(\eta\) is a measure on such product then \(p_i(\eta)\) denotes its \(i\)-th marginal.

Lemma 2.2 ([7, Proposition 2.1]). Let \((E_i, m_i), i \leq k\) be Polish probability spaces, and \(M\) an analytic subset of \(E_1 \times \ldots \times E_k\), then one of the following holds true:

(i) there exist \(m_i\)-null sets \(M_i \subseteq E_i\) such that \(M \subseteq \bigcup_{i=1}^k p_i^{-1}(M_i)\), or

(ii) there is a measure \(\eta\) on \(E_1 \times \ldots \times E_k\) such that \(\eta(M) > 0\) and \(p_i(\eta) \leq m_i\) for \(i = 1, \ldots, k\).

All in all, we can prove Lemma 1.3 now:

Proof of Lemma 1.3 Set \(d := \frac{d\gamma}{dx}\) and \(c := h' \circ d\).

We want to find finitely minimal sets \(\Gamma_{\theta'}, \Gamma_{\theta}\) supporting \(\theta', \theta\). To obtain this, it is sufficient to show that for each \(l \in \mathbb{N}\) there are sets \(\Gamma_{\theta'} \subset \Gamma_{\theta}\) of full \(\theta'\) measure such that: for any finite measure \(\alpha\) concentrated on at most \(l\) points in \(\Gamma_{\theta'}\) and satisfying \(\alpha(\Omega) \leq 1\) as well as \(c_{\text{supp} \alpha} \leq l\), there is no \(c\)-better competitor \(\alpha'\) on at most \(l\) points in \(\Gamma_{\theta}\) and satisfying \(c_{\text{supp} \alpha'} \leq l\). If we achieve this, we can just take the intersection over countably many such \(\Gamma_{\theta'}\).

Hence, fix \(l\) and define \(M\) the subset of \(\Omega' \times \Omega'\) through
\[
M = \{(z_1, \ldots, z_l), (z'_1, \ldots, z'_l)\} \in \Omega' \times \Omega' : \exists \text{ a measure } \alpha \text{ on } \Omega, \alpha(\Omega) \leq 1, \supp \alpha \subseteq \{z_1, \ldots, z_l\}, -l \leq c_{\text{supp} \alpha} \leq l, \\
\text{s.t. there is a } c\text{-better competitor } \alpha', \supp \alpha' \subseteq \{z'_1, \ldots, z'_l\}, -l \leq c_{\text{supp} \alpha'} \leq l.
\]
Note that $M$ is a projection of the set

$$
\hat{M} = \{(z_1, \ldots, z_l, \alpha_1, \ldots, \alpha_l, z'_1, \ldots, z'_l, \alpha'_1, \ldots, \alpha'_l) \in \Omega' \times \mathbb{R}_+^l \times \Omega' \times \mathbb{R}_+^l : \\
\sum \alpha_i \leq 1, \sum \alpha_i = \sum \alpha'_i, -l \leq c_{\text{supp } \alpha, \text{supp } \alpha'} \leq l \text{ where } \alpha := \sum \alpha_i \delta_{z_i}, \alpha' := \sum \alpha'_i \delta_{z'_i}, \\
\sum \alpha_i f(z_i) = \sum \alpha'_i f(z'_i) \text{ for all } f \in \mathcal{F}, \sum \alpha_i c(z_i) > \sum \alpha'_i c(z'_i)\}.
$$

The set $\hat{M}$ is Borel; this is immediate if $\mathcal{F}$ is countable, and otherwise follows from the well-known argument that $\mathcal{F} \subseteq C_c(\Omega)$ contains a separating sequence. Hence $M$ is an analytic set.

We apply Lemma 2.2 to the $l$ copies of the spaces $(\Omega, Q^*)$, $(\Omega, \mathbb{P})$ and the set $M$. To be precise, we take $E_i = \Omega$, $i = 1, \ldots, 2l$, $m_i = Q^*$ if $i \leq l$ and $m_i = \mathbb{P}$ otherwise. By Lemma 2.2, if (i) holds, then there are sets $N_1, N_2$ with $Q^*(N_i) = \mathbb{P}(N_2) = 0$ such that $M \subseteq N_1' \times \Omega' \cup \Omega' \times N_2'$.

Indeed, noticing that the set $M$ must be symmetric in its first $l$ coordinates, and also on the remaining $l$ ones, we get that if a point is in $M$, then at least one of its first $l$ coordinates are in a given $Q^*$-null set $N_1$, or one of the remaining $l$ coordinates are in a given $\mathbb{P}$-null set $N_2$. We set $\Gamma_{Q^*} := \Omega \setminus N_1, \Gamma_{\mathbb{P}} := \Omega \setminus N_2$, which have full $Q^*/\mathbb{P}$ measure respectively. From the definition of $M$ it can be directly seen that $\Gamma_{Q^*}, \Gamma_{\mathbb{P}}$ are as needed.

If (i) does not hold, (ii) has to. Hence, let us derive a contradiction from it.

For $j \leq 2, i \leq l$, write $p_i^j$ for the projection of an element of $\Omega' \times \Omega'$ onto its $((j-1) \times l + i)$-th component. We may assume that the measure $\eta$ given by Point (ii) in Lemma 2.2 is concentrated on $M$, and also fulfills $p_i^j(\eta) \leq \frac{1}{4} Q^*, p_i^j(\eta) \leq \frac{1}{4} \mathbb{P}$ for $i = 1, \ldots, l$.

We now apply Jankow – von Neumann uniformization [29, Theorem 18.1] to the set $\hat{M}$ to define a mapping

$$
M \to \hat{M}
$$

$$(z_1, \ldots, z_l, z'_1, \ldots, z'_l) \mapsto (z_1, \ldots, z_l, \alpha_1(z, z'), \ldots, \alpha_\ell(z, z'), z'_1, \ldots, z'_l, \alpha'_1(z, z'), \ldots, \alpha'_\ell(z, z'))
$$

which is measurable with respect to the $\sigma$-algebra generated by the analytic subsets of $\Omega' \times \Omega'$ in the domain and the Borel $\sigma$-algebra of $\Omega' \times \mathbb{R}_+^l \times \Omega' \times \mathbb{R}_+^l$ in the range. In the above, we denoted $z_i$ resp. $z'_i$ the i-th coordinate of $z \in \Omega'$ resp. $z' \in \Omega'$. Setting

$$
\alpha_{(z, z')} := \sum_{i=1}^l \alpha_i(z, z') \delta_{z_i}, \alpha'_{(z, z')} := \sum_{i=1}^l \alpha'_i(z, z') \delta_{z'_i},
$$

we thus obtain kernels $(z, z') \mapsto \alpha_{(z, z')}$, $(z, z') \mapsto \alpha'_{(z, z')}$ from $\Omega' \times \Omega'$ with the $\sigma$-algebra generated by its analytic subsets to $\mathcal{P}(\Omega)$ with its Borel sets. We use these kernels to define measures $\theta, \theta'$ on the Borel sets of $\Omega$ through

$$
\theta(B) = \int \alpha_{(z, z')} (B) \, \text{d} \eta(z, z'), \quad \theta'(B) = \int \alpha'_{(z, z')} (B) \, \text{d} \eta(z, z').
$$

By construction $\theta \leq Q^*$. Indeed we have,

$$
\theta(B) \leq \sum_{i=1}^l \int \delta_{z_i}(B) \, \text{d} \eta(z, z') = \sum_{i=1}^l p_i^j(B) \leq \frac{1}{4} Q^*(B).
$$

Arguing similarly we obtain $\theta' \leq \mathbb{P}$. Moreover $\theta'$ is a $c$-better competitor of $\theta$. To see this, we first observe that for each $f \in \mathcal{F}$ we have

$$
\int_{\Omega} f(\bar{z}) \, \text{d} \theta(\bar{z}) = \int_{\Omega} f(\bar{z}) \, \text{d} \alpha_{(z, z')} \eta(z, z') = \int_{\Omega} f(\bar{z}) \, \text{d} \alpha'_{(z, z')} \eta(z, z') = \int_{\Omega} f(\bar{z}) \, \text{d} \theta(\bar{z}),
$$

(2.3)

and similarly, since $c \leq l$, $\theta + \theta'$-a.s. we obtain

$$
\int_{\Omega} c(\bar{z}) \, \text{d} \theta(\bar{z}) = \int_{\Omega} c(\bar{z}) \, \text{d} \alpha_{(z, z')} \eta(z, z') < \int_{\Omega} c(\bar{z}) \, \text{d} \alpha'_{(z, z')} \eta(z, z') = \int_{\Omega} c(\bar{z}) \, \text{d} \theta(\bar{z}).
$$
Therefore, since \( \int c \, d(\theta' - \theta) < 0 \) we obtain from Lemma 2.1 that if we set \( Q_x^* = Q^* + \varepsilon(\theta' - \theta) \), then

\[
\int h\left(\frac{dQ}{d\theta}\right) \, d\mathbb{P} < \int h\left(\frac{dQ}{d\theta}\right) \, d\mathbb{P}
\]

for \( \varepsilon \) small enough. Since (2.3) makes sure that \( Q_x^* \in \text{Adm} \), we have derived a contradiction to the optimality of \( Q^* \). \( \square \)

2.2. Proof of Necessity: Theorems 1.4 and 1.5. In the coming proofs the assumption \( \mathbb{P} \sim \mu \otimes \nu \) is used in the following form: we use \( \mu \otimes \nu \ll \mathbb{P} \) to apply \cite[Lemma 4.3]{7} and \( \mathbb{P} \ll \mu \otimes \nu \) to guarantee w.l.o.g. that, as we trim down certain sets in the product space \( X \times Y \), their \( X \)- and \( Y \)-projections remain unaffected.

Proof of Theorem 1.4. Let \( \Gamma_{Q^*}, \Gamma_\mathbb{P} \) be as in Lemma 1.3. Passing to subsets if necessary we may assume that \( \text{proj}_X \Gamma_{Q^*} = X, \text{proj}_Y \Gamma_{Q^*} = Y, \Gamma_{Q^*} \subseteq \Gamma_\mathbb{P} \). Apparently \( Q^* \ll \mathbb{P} \). Hence, shrinking \( \Gamma_{Q^*} \) by an irrelevant \( \mathbb{P} \)-null set, we may assume that \( \Gamma_{Q^*} \subseteq \{d(x, y) > 0\} = \{h'(d(x, y)) > -\infty\} \), with \( d' := dQ^*/d\mathbb{P} \). In the present transport case the finitistic optimality property (1.1) boils down to cyclical monotonicity, i.e. we find that for \( (x_i, y_i) \in \Gamma_{Q^*}, i \leq N, x_{N+1} = x_0, (x_{i+1}, y_i) \in \Gamma_\mathbb{P} \) we have

\[
\sum_{i \leq N} h' \circ d(x_i, y_i) \leq \sum_{i \leq N} h' \circ d(x_{i+1}, y_i).
\]

(2.4)

We say that \( x_i, y_i, i \leq N \) form a \((\Gamma_{Q^*}, \Gamma_\mathbb{P})\)-path if \( (x_i, y_i) \in \Gamma_{Q^*} \) for \( i \leq N, (x_{i+1}, y_i) \in \Gamma_\mathbb{P} \) for \( i \leq N - 1 \).

Based on the assumption \( \mathbb{P} \sim \mu \otimes \nu \) we can apply \cite[Lemma 4.3]{7} with the cost function \( c := 0 \) on \( \Gamma_\mathbb{P} \) and \( c := +\infty \) otherwise, to obtain that there exist subsets \( \bar{X} \subseteq X \) and \( \bar{Y} \subseteq Y \) with respectively full measure under \( \mu \) and \( \nu \), so \( \bar{\Gamma}_{Q^*} := \Gamma_{Q^*} \cap (\bar{X} \times \bar{Y}) \) has \( Q^* \)-full measure, and such that crucially for any points \( (x, y), (\bar{x}, \bar{y}) \in \bar{\Gamma}_{Q^*} \) there exists a \((\bar{\Gamma}_{Q^*}, \bar{\Gamma}_\mathbb{P})\)-path satisfying \( (x_0, y_0) = (x, y) \) and \( (x_N, y_N) = (\bar{x}, \bar{y}) \). Passing to subsets if necessary, we can w.l.o.g. assume that \( \bar{X} = X, \bar{Y} = Y, \bar{\Gamma}_{Q^*} = \Gamma_{Q^*} \). We use this to establish

\[
d(x, y) > 0 \quad \text{for all} \quad (x, y) \in \Gamma_\mathbb{P}.
\]

(2.5)

To see this, pick an arbitrary point \((x, y) \in \Gamma_\mathbb{P}\) and points \(\bar{x}, \bar{y}\) such that \((\bar{x}, \bar{y}), (x, y) \in \Gamma_{Q^*}\) and a \((\Gamma_{Q^*}, \Gamma_\mathbb{P})\)-path \((x_0, y_0) := (x, y), (x_1, y_1), \ldots, (x_N, y_N) := (\bar{x}, \bar{y})\) which connects these points. By (2.4) we then have (with \( x_{N+1} = x_0 \))

\[
\sum_{i \leq N} h' \circ d(x_i, y_i) \leq \sum_{i \leq N} h' \circ d(x_{i+1}, y_i)
\]

\[
\Leftrightarrow \sum_{i \leq N} h' \circ d(x_i, y_i) \leq \sum_{i \leq N-1} h' \circ d(x_{i+1}, y_i) + h' \circ d(x_N, y_N).
\]

Since the left-hand side is finitely valued and \( h' \circ d(x_{i+1}, y_i) < \infty \) for \( i \leq N - 1 \) we obtain indeed \( h' \circ d(x, \bar{y}) > -\infty \). This establishes (2.5). It follows that \( \mathbb{P} \sim Q^* \) and, by passing to subsets if needed, we can assume without loss of generality that \( \Gamma_{Q^*} = \Gamma_\mathbb{P} \). Next, we say that \((x_i, y_i), i \leq N\) form a \(\Gamma_{Q^*}\)-loop if \((x_i, y_i), (x_{i+1}, y_i) \in \Gamma_{Q^*}\) for \(i \leq N\), where \(x_{N+1} := x_0\). Note that for any \(\Gamma_{Q^*}\)-loop we have

\[
\sum_{i \leq N} h' \circ d(x_i, y_i) = \sum_{i \leq N} h' \circ d(x_{i+1}, y_i);
\]

(2.6)

to see this, apply (2.4) twice, i.e. to the loop in the usual direction as well as to running the loop in the ‘reverse’ direction. By \cite[Prop. 1]{33}, Condition 2.6 is necessary and sufficient to obtain functions \( \varphi, \psi \) satisfying

\[
h' \circ d(x, y) = \varphi(x) + \psi(y),
\]

(2.7)

for all \((x, y) \in \Gamma_{Q^*}\).
Fix \( x_0 \in X \), and observe that (2.7) yields
\[
\sum_{i \leq M} h' \circ d(x_{i+1}, y_i) - h' \circ d(x_i, y_i) = \varphi(x) - \varphi(x_0),
\]
whenever \((x_i, y_i), (x_{i+1}, y_i) \in \Gamma_{Q^*}\) for \( i \leq M \in \mathbb{N} \) is such that \( x_{M+1} = x \). In particular we have
\[
\varphi(x) = \inf \left\{ \sum_{i \leq M} h' \circ d(x_{i+1}, y_i) - h' \circ d(x_i, y_i) + \varphi(x_0) : (x_i, y_i), (x_{i+1}, y_i) \in \Gamma_{Q^*} \right\}.
\]
(2.8)

The right-hand side of (2.8) is upper semi-analytic, hence \( \varphi \) is upper semi-analytic. Indeed, if \( M \) is fixed in the r.h.s. of (2.8), then we would have the partial infimum of a jointly Borel function, which must be upper semi-analytic; this is also the case as we let \( M \in \mathbb{N} \). Of course (2.8) pertains if we replace the inf with a sup, hence \( \varphi \) is also lower semi-analytic. Putting the two together, we find that \( \varphi \) is Borel. (For the same reason that a set is Borel iff its complement are analytic (Suslin theorem) we have that \( \varphi \) must be Borel.)

**Proof of Theorem 1.5** The start of the proof is the same as the one of Theorem 1.4: Let \( \Gamma_{Q^*}, \Gamma_{P^*} \) be as in Lemma 1.3. Apparently \( Q^* \ll P \) and we may assume that \( \Gamma_{Q^*} \subseteq \Gamma_{P^*} \). Redefining \( d := \frac{dQ^*}{dP} \) on an irrelevant \( P \)-null set we may assume that \( d = +\infty \) exactly on \((X \times Y) \setminus \Gamma_{P^*} \).

As above, the finitistic optimality property amounts to cyclical monotonicity, i.e. we find that for \((x_i, y_i) \in \Gamma_{Q^*}, i \leq N, x_{N+1} = x_0 \) we have
\[
\sum_{i \leq N} h' \circ d(x_i, y_i) \leq \sum_{i \leq N} h' \circ d(x_{i+1}, y_i).
\]
(2.9)

Note that we do not have to assume \((x_i, y_{i+1}) \in \Gamma_{P^*} \) since \( h' \circ d(x_i, y_{i+1}) = +\infty \) whenever \((x_i, y_{i+1}) \notin \Gamma_{P^*} \). Passing to subsets if necessary we may assume that \( \text{proj}_X \Gamma_{Q^*} = X, \text{proj}_Y \Gamma_{Q^*} = Y \).

We say that \( x_i, y_i, i \leq N \) form a \((x, y)-\text{path} \) on \( \Gamma \) if \((x_i, y_i) \in \Gamma \) for \( i \leq N, d(x_{i+1}, y_i) < \infty \) for \( i \leq N - 1 \).

Based on the assumption \( P \sim \mu \otimes \nu \) we can apply [7] Lemma 4.3] (with the cost function \( c = 0 \) on \( \Gamma_{P^*} \) and \( +\infty \) else) to obtain the following:

There exist respectively full \( \mu, \nu \)-measure subsets \( X_0 \subseteq X, Y_0 \subseteq Y \), so \( \Gamma_0 := \Gamma_{Q^*} \cap (X_0 \times Y_0) \) has \( Q^* \)-full measure, such that for any points \((x, y), (\bar{x}, \bar{y}) \in \Gamma_0 \) there exists a \((\bar{0}, \bar{c})\)-path satisfying \((x_0, y_0) = (x, y) \) and \((x_N, y_N) = (\bar{x}, \bar{y}) \). Of course, we immediately assume w.l.o.g. that \( X_0 = X, Y_0 = Y, \Gamma_0 = \Gamma_{Q^*} \).

In the terms of [7] we would say that \((\Gamma_{Q^*}, h' \circ d) \) is connecting. It then follows from [7] Proposition 3.2 that there exist Borel functions \( \varphi : X \to [-\infty, \infty), \psi : Y \to [-\infty, \infty) \) such that for all \( x \in X, y \in Y \)
\[
\varphi(x) + \psi(y) \leq h' \circ d(x, y)
\]
with equality holding \( Q^* \)-a.s. Hence we also have
\[
h' \circ \frac{dQ^*}{dP}(x, y) = (\varphi(x) + \psi(y)) 1_{\frac{dQ^*}{dP}(x, y) > 0} = (\varphi(x) + \psi(y))_+, \quad P \text{-a.s.}
\]

**Proof of Sufficiency: Theorem 1.6**

**Proof of Theorem 1.6** If Problem (1.2) has value \(+\infty \) then there is nothing to prove. Hence, let \( Z \in \text{cpl}(\mu, \nu) \) with \( I := \int h(Z) dP < \infty \).

Denote \( h'(y) = \sup_{x \geq 0} \{ xy - h(x) \} \), and notice that under the assumptions on \( h \) we have \( h'(y) = (h')^{-1}(y_+) y - h \circ (h')^{-1}(y_+) \).
Introduce $\varphi_n(x) = (-n) \vee \varphi(x) \wedge n$, $\psi_n(x) = (-n) \vee \psi(x) \wedge n$. Clearly (c.f. [43, Lemma 3]), on $[\varphi + \psi \geq 0]$ we have $0 \leq \varphi_n + \psi_n \geq \varphi + \psi$, while on $[\varphi + \psi \leq 0]$ we have $0 \geq \varphi_n + \psi_n \geq \varphi + \psi$. Since $h(Z) \geq (\varphi_n + \psi_n)Z - h^*(\varphi_n + \psi_n)$, we find

$$
I \geq \int (\varphi_n + \psi_n)Zd\mu - \int h^*(\varphi_n + \psi_n)d\mu
= \int \varphi_n d\mu + \int \psi_n d\nu - \int h^*(\varphi_n + \psi_n)d\mu
= \int (\varphi_n + \psi_n)Zd\nu - \int h^*(\varphi_n + \psi_n)d\mu.
$$

Since $h^*(\cdot)$ is increasing, we have by monotone convergence

$$
\int_{\varphi + \psi \geq 0} h^*(\varphi_n + \psi_n)d\mu \to \int_{\varphi + \psi \geq 0} h^*(\varphi + \psi)d\mu,
$$

and in particular

$$
\int_{\varphi + \psi \geq 0} (\varphi_n + \psi_n)Zd\mu \to \int_{\varphi + \psi \geq 0} (\varphi + \psi)Zd\mu.
$$

Collecting integrals we have found

$$
\int (\varphi_n + \psi_n)Zd\mu - \int h^*(\varphi_n + \psi_n)d\mu \to \int (\varphi + \psi)Zd\mu - \int h^*(\varphi + \psi)d\mu
= \int \left[ (\varphi + \psi)^{d\nu} - h^*(\varphi + \psi) \right]d\mu
= \int \left[ (\varphi + \psi)(h')^{-1} ((\varphi(x) + \psi(y))_+) - h^*(\varphi + \psi) \right]d\mu
= \int h \circ (h')^{-1} ((\varphi(x) + \psi(y))_+)d\mu
= \int h \left( \frac{d\nu}{d\mu} \right) d\mu.
$$

We conclude that $I \geq \int h(dQ^*/d\mu)d\mu$. □

References

[1] L. Ambrosio and A. Pratelli. Existence and stability results in the $L^1$ theory of optimal transportation. In *Optimal transportation and applications* (Martina Franca, 2001), volume 1813 of *Lecture Notes in Math.*, pages 123–160. Springer, Berlin, 2003.

[2] J. Backhoff, G. Conforti, I. Gentil, and C. Léonard. The mean field Schrödinger problem: ergodic behavior, entropy estimates and functional inequalities. *Probability Theory and Related Fields*, 178(1):475–530, 2020.

[3] J. Backhoff-Veraguas, M. Beiglböck, M. Huesmann, and S. Källblad. Martingale bamou–brenier: a probabilistic perspective. *Annals of Probability*, 48(5):2258–2289, 2020.
[4] J. Backhoff-Veraguas, M. Beiglböck, and G. Pammer. Existence, duality, and cyclical monotonicity for weak transport costs. *Calculus of Variations and Partial Differential Equations*, 58(6):203, 2019.

[5] M. Beiglböck, A. Cox, and M. Huesmann. Optimal transport and Skorokhod embedding. *Invent. Math.*, 208(2):327–400, 2017.

[6] M. Beiglböck, M. Eder, C. Elgert, and U. Schmock. Geometry of distribution-constrained optimal stopping problems. *Probability theory and related fields*, 172(1-2):71–101, 2018.

[7] M. Beiglböck, M. Goldstern, G. Maresch, and W. Schachermayer. Optimal and better transport plans. *J. Funct. Anal.*, 256(6):1907–1927, 2019.

[8] M. Beiglböck and C. Griessler. A land of monotone plenty. *Annali della SNS*, Vol. XIX, issue 1, Apr. 2019.

[9] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.*, 44(1):42–106, 2016.

[10] M. Beiglböck, M. Nutz, and F. Stebegg. Fine properties of the optimal skorokhod embedding problem. *Journal of the European Mathematical Society*, 2021.

[11] M. Beiglböck, M. Nutz, and N. Touzi. Complete duality for martingale optimal transport on the line. *The Annals of Probability*, 45(5):3038–3074, 2017.

[12] E. Bernton, P. Ghosal, and M. Nutz. Entropic optimal transport: geometry and large deviations. *arXiv preprint arXiv:2102.04397*, 2021.

[13] A. Beurling. An automorphism of product measures. *Annals of Mathematics*, pages 189–200, 1960.

[14] S. Bianchini and L. Caravenna. On optimality of c-cyclically monotone transference plans. *C. R. Math. Acad. Sci. Paris*, 348(11-12):613–618, 2010.

[15] J. M. Borwein and A. S. Lewis. Decomposition of multivariate functions. *Canadian journal of mathematics*, 44(3):463–482, 1992.

[16] J. M. Borwein, A. S. Lewis, and R. D. Nussbaum. Entropy minimization, dad problems, and doubly stochastic kernels. *Journal of Functional Analysis*, 123(2):264–307, 1994.

[17] G. Carlier and M. Laborde. A differential approach to the multi-marginal Schroedinger system. *SIAM Journal on Mathematical Analysis*, 52(1):709–717, 2020.

[18] P. Cattiaux, F. Gamboa, et al. Large deviations and variational theorems for marginal problems. *Bernoulli*, 5(1):81–108, 1999.

[19] M. Colombo, L. De Pascale, and S. Di Marino. Multimarginal optimal transport maps for one-dimensional repulsive costs. *Canad. J. Math.*, 67(2):350–368, 2015.

[20] M. Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. *Advances in neural information processing systems*, 26:2292–2300, 2013.

[21] D. Dawson and J. Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. *Stochastics: An International Journal of Probability and Stochastic Processes*, 20(4):247–308, 1987.

[22] M. Fischer. On the form of the large deviation rate function for the empirical measures of weakly interacting systems. *Bernoulli*, 20(4):1765–1801, 2014.

[23] R. Fortet. Résolution d’un système d’équations de M. Schrödinger. *J. Math. Pure Appl. IX*, 1:83–105, 1940.

[24] W. Gangbo and R. McCann. The geometry of optimal transportation. *Acta Math.*, 177(2):113–161, 1996.

[25] N. Gigli and L. Tamanini. Second order differentiation formula on RCD(k,N) spaces. *arXiv preprint arXiv:1802.02463*, 2018.

[26] N. Gozlan and N. Juillet. On a mixture of Brenier and Strassen theorems. *Proceedings of the London Mathematical Society*, 120(3):434–463, 2020.

[27] C. Griessler. c-cyclical monotonicity as a sufficient criterion for optimality in the multimarginal Monge-Kantorovich problem. *Proceedings of the American Mathematical Society*, 146(11):4735–4740, 2018.

[28] G. Guo, X. Tan, and N. Touzi. On the monotonicity principle of optimal Skorokhod embedding problem. *SIAM Journal on Control and Optimization*, 54(5):2478–2489, 2016.

[29] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[30] H. Kellerer. Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete*, 67(4):399–432, 1984.

[31] M. Knott and C. Smith. On Hoeffding-Fréchet bounds and cyclic monotone relations. *J. Multivariate Anal.*, 40(2):328–334, 1992.

[32] C. Léonard. Minimizers of energy functionals. *Acta Mathematica Hungarica*, 93(4):281–325, 2001.
[33] C. Léonard. Entropic projections and dominating points. *ESAIM: Probability and Statistics*, 14:343–381, 2010.

[34] C. Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst.*, 34(4):1533–1574, 2014.

[35] B. D. Miller. Coordinatewise decomposition of group-valued Borel functions. *Fund. Math.*, 196(2):119–126, 2007.

[36] M. Nutz and F. Stebegg. Canonical supermartingale couplings. *The Annals of Probability*, 46(6):3351–3398, 2018.

[37] S. Pal and T.-K. L. Wong. The geometry of relative arbitrage. *Math. Financ. Econ.*, 10(3):263–293, 2016.

[38] B. Pass. On the local structure of optimal measures in the multi-marginal optimal transportation problem. *Calc. Var. Partial Differential Equations*, 43(3-4):529–536, 2012.

[39] T. Pennanen and A.-P. Perkkiö. Convex duality in nonlinear optimal transport. *Journal of Functional Analysis*, 277(4):1029–1060, 2019.

[40] A. Pratelli. On the sufficiency of c-cyclical monotonicity for optimality of transport plans. *Mathematische Zeitschrift*, 258(3):677–690, 2008.

[41] L. Rüschendorf. On c-optimal random variables. *Statist. Probab. Lett.*, 27(3):267–270, 1996.

[42] L. Rüschendorf and W. Thomsen. Note on the Schrödinger equation and i-projections. *Statistics and probability letters*, 17(5):369–375, 1993.

[43] W. Schachermayer and J. Teichmann. Characterization of optimal transport plans for the Monge-Kantorovich problem. *Proc. Amer. Math. Soc.*, 137(2):519–529, 2009.

[44] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[45] D. A. Zaev. On the Monge–Kantorovich problem with additional linear constraints. *Mathematical Notes*, 98(5-6):725–741, 2015.