Emergent Gravity and Noncommutative Branes from Yang-Mills Matrix Models

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Abstract

The framework of emergent gravity arising from Yang-Mills matrix models is developed further, for general noncommutative branes embedded in $\mathbb{R}^D$. The effective metric on the brane turns out to have a universal form reminiscent of the open string metric, depending on the dynamical Poisson structure and the embedding metric in $\mathbb{R}^D$. A covariant form of the tree-level equations of motion is derived, and the Newtonian limit is discussed. This points to the necessity of branes in higher dimensions. The quantization is discussed qualitatively, which singles out the IKKT model as a prime candidate for a quantum theory of gravity coupled to matter. The Planck scale is then identified with the scale of $N = 4$ SUSY breaking. A mechanism for avoiding the cosmological constant problem is exhibited.

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1 Introduction

The notion of space-time which underlies the presently accepted models of fundamental matter and interactions goes back to Einstein. Space-time is modeled by a 4-dimensional manifold, whose geometry is determined by a metric with Lorentzian signature. This notion escaped the quantum revolution essentially unchanged, even though Quantum Mechanics combined with General Relativity strongly suggests a “foam-like” or quantum structure at the Planck scale. While some kind of quantum structure of space-time indeed arises e.g.
in string theory or loop quantum gravity, a satisfactory understanding is still missing.

A different approach to this problem has been pursued in recent years, starting with some explicit quantization of space-time and attempting to construct physical models on such a background. The classical space-time $\mathbb{R}^4$ is replaced by a quantized or “noncommutative” (NC) space, where the coordinate functions $x^\mu$ satisfy nontrivial commutation relations such as $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. This leads to non-commutative field theory, see e.g. [1,3]. At the semi-classical level, these commutation relations determine a Poisson structure $\theta^{\mu\nu}$ on space-time, which is fixed by construction. However, since quantized spaces are expected to arise from quantum gravity, it seems more appropriate to consider a dynamical Poisson structure at the semi-classical level. A straightforward generalization of General Relativity is then inappropriate; indeed any quantum structure of space-time rules out classical intuitive principles. Rather, one should look for simple models of dynamical noncommutative (or Poisson) spaces, with the hope that they will effectively incorporate gravity.

Such models are indeed available and known as Matrix-models of Yang-Mills type. They have the form $S = Tr[X^a, X^b][X^{a'}, X^{b'}]\delta_{aa'}\delta_{bb'} + ...$, where indices run from 1 to $D$. It is well known that these models admit noncommutative spaces (“NC branes”) as solutions, such as the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$; see e.g. [13–19]. However, most of the work up to now is focused on special NC branes with a high degree of symmetry. For generic NC spaces with non-constant $\theta^{\mu\nu}(x)$, it was shown in [4] that the kinetic term for any “field” coupled to the $D=4$ matrix model is governed by an effective metric $\tilde{G}_{ab}(x) = \rho \theta^{a\alpha'}(x)\theta^{b\beta'}(x)\delta_{\alpha\beta'}$, including nonabelian gauge fields. This nicely explains the observed relation in [5] between NC $U(1)$ gauge fields and gravitational degrees of freedom, see also [6–8] for related work. Since this effective metric is dynamical, these YM Matrix Models contain effectively some version of gravity, thus realizing the idea that gravity should emerge from NC gauge theory [10]. As argued in [1], an effective action for gravity is induced upon quantization, with the remarkable feature that the “would-be cosmological term” decouples from the model due to the constrained class of metrics. This makes the mechanism of induced gravity feasible at the quantum level, and suggests that the Newton constant resp. the Planck scale is related to an effective UV-cutoff of the model. A detailed analysis taking into account UV/IR mixing [10] and fermions [11] singles out the $N=4$ supersymmetric extensions of the model, where such a cutoff is given by the scale of $N=4$ SUSY breaking. This amounts to $D = 10$, which is nothing but the IKKT model [12], originally proposed as a nonperturbative definition of IIB string theory.

In the present paper, we develop the framework for emergent gravity on general NC branes with nontrivial embedding in $\mathbb{R}^D$. This works out very naturally, leading to a simple generalization of the effective metric which is strongly reminiscent of the open string

\footnote{As such, the presence of gravity in this model is expected and to some extent verified, cf. [12, 13, 20, 22, 24, 25]. However, what is usually considered are effects of D=10 (super)gravity, modeled by interactions of separated “D-objects”, represented by block-matrices. In contrast, emergent NC gravity describes interactions within (generic) NC branes in this model. Evidence for gravity on simple NC branes was obtained previously in [24, 26].}
metric \[27\], involving the general Poisson tensor and the embedding metric. We establish in Section 2 the relevant geometry, find the semi-classical form of the bare matrix-model action for general NC branes in \( \mathbb{R}^D \), and obtain covariant equations of motion. This generalizes the well-known case of flat or highly symmetric branes to the generic case, and shows how the would-be \( U(1) \) gauge field is absorbed in the effective metric on the brane. In Section 3, the Newtonian limit of emergent gravity is studied in detail. It turns out that even though it is possible to reproduce the Newtonian potential for general mass distributions, the relativistic corrections are in general not correctly reproduced in \( D = 4 \) matrix models. This provides one motivation to consider general branes embedded in higher-dimensional matrix models, which admit a much richer class of geometries and promise to overcome this problem. The compactification of higher-dimensional NC branes is described in Section 2.4 with the example of fuzzy spheres in extra dimensions.

Higher dimensions, more precisely \( D = 10 \) resp. \( N = 4 \) SUSY also appears to be required by consistency at the quantum level. From the point of view of emergent gravity, this condition arises as a result of UV/IR mixing in NC gauge theory. This is discussed qualitatively in Section 2.6 along the lines of \[4\], leading to an induced gravity action. In Section 2.3 some differences to General Relativity are discussed, most notably the presence of intrinsic scales and preferred coordinates, as well as the different role of the “would-be cosmological constant term”. As an illustration of the formalism, we also give a (unphysical) solution of the bare equations of motion in Section 4. Finally, a matrix version of a conserved energy-momentum tensor is derived.

The results of this paper provide a rich framework for the search of realistic solutions of emergent NC gravity. The main missing piece is the analog of the Schwarzschild solution, which is nontrivial because the quantum effective action at least at one loop must be taken into account. But in any case, it is clear that these models do contain a version of gravity in an intrinsically noncommutative way, and they have a good chance to be well-defined at the quantum level at least for the IKKT model. This certainly provides motivation for a thorough investigation.

## 2 The Matrix Model

Consider the matrix model with action

\[
S_{YM} = -Tr[X^\mu, X^\nu][X^{\mu'}, X^{\nu'}]g_{\mu\mu'}g_{\nu\nu'},
\]

(1)

for

\[
g_{\mu\mu'} = \delta_{\mu\mu'} \quad \text{or} \quad g_{\mu\mu'} = \eta_{\mu\mu'}
\]

(2)
in the Euclidean resp. Minkowski case. The ”covariant coordinates” \( X^\mu, \mu = 1, 2, 3, 4 \) are hermitian matrices or operators acting on some Hilbert space \( \mathcal{H} \). We will denote the commutator of 2 matrices as

\[
[X^\mu, X^\nu] = i\theta^{\mu\nu}
\]

(3)

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so that $\theta^{\mu\nu} \in L(\mathcal{H})$ are antihermitian matrices, which are not assumed to be proportional to $1_{\mathcal{H}}$. We focus here on configurations $X^\mu$ which can be interpreted as quantizations of coordinate functions $x^\mu$ on a Poisson manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$ with general Poisson structure $\theta^{\mu\nu}(x)$. This defines the geometrical background under consideration, and conversely essentially any Poisson manifold provides (locally) a possible background $X^\mu$ [28]. More formally, this means that there is an isomorphism of vector spaces

$$C(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})$$

$$f(x) \mapsto \hat{f}(X)$$

$$i \{ f, g \} \mapsto [\hat{f}, \hat{g}] + O(\theta^2)$$

(4)

Here $C(\mathcal{M})$ denotes some space of functions on $\mathcal{M}$, and $\mathcal{A}$ is interpreted as quantized algebra of functions on $\mathcal{M}$. This allows to replace $[\hat{f}(X), \hat{g}(X)] \rightarrow i\{ f(x), g(x) \}$ to leading order in $\theta$. In particular, we can then write

$$[X^\mu, f(X)] \sim i\theta^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} f(x)$$

(5)

which will be used throughout this paper, denoting with $\sim$ the leading contribution in a semi-classical expansion in powers of $\theta^{\mu\nu}$.

In order to derive the effective metric on $\mathcal{M}$, let us now consider a scalar field coupled to the matrix model (1). The only possibility to write down kinetic terms for matter fields is through commutators $[X^\mu, \Phi]$ using (5). Thus consider the action

$$S[\Phi] = -Tr g_{\mu\nu}[X^\mu, \Phi][X^{\mu'}, \Phi]$$

$$\sim \frac{1}{(2\pi)^2} \int d^4 x \rho(x) G^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \Phi(x) \frac{\partial}{\partial x^{\mu'}} \Phi(x).$$

(6)

Here

$$G^{\mu\nu}(x) = \theta^{\mu\nu'}(x) \theta^{\nu'\nu}(x) g_{\mu\nu}$$

is interpreted as metric on $\mathcal{M}$ in $x$ coordinates. We will assume in this paper that $\theta^{\mu\nu}(x)$ is nondegenerate. Then the symplectic measure on $(\mathcal{M}, \theta^{\mu\nu}(x))$ is given by the scalar density

$$\rho(x) \equiv |\theta^{-1}_{\mu\nu}(x)|^{1/2} = |G_{\mu\nu}(x)|^{1/4}|g_{\mu\nu}|^{1/4} \equiv \Lambda_{NC}^4(x),$$

(8)

which can be interpreted as “local” non-commutative scale $\Lambda_{NC}$. In the preferred $x$ coordinates characterized by (2), $\rho(x)$ coincides with the dimensionless scalar function

$$e^{-\sigma} = \frac{|G_{\mu\nu}(x)|^{1/4}}{|g_{\mu\nu}(x)|^{1/4}} = \frac{|\theta^{-1}_{\mu\nu}(x)|^{1/2}}{|g_{\mu\nu}|^{1/2}}.$$ 

(9)

\(^2\)in contrast to the conventions in [28]

\(^3\)Roughly speaking $\mathcal{A}$ is the algebra generated by $X^\mu$, but technically one usually considers some subalgebra corresponding to well-behaved functions.

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The action (6) can now be written in a covariant manner as
\[ S[\Phi] = \frac{1}{(2\pi)^2} \int d^4x \tilde{G}^{\mu\nu}(x) \partial_\mu \Phi(x) \partial_\nu \Phi(x) = \frac{1}{(2\pi)^2} \int d^4x \sqrt{|\tilde{G}_{\mu\nu}|} \Phi(x) \Delta \tilde{G} \Phi(x) , \] (10)

Here \( \Delta \tilde{G} \) is the Laplacian for the metric \( \tilde{G} \)

\[ \tilde{G}^{\mu\nu}(x) = |G_{\mu\nu}|^{1/4} G^{\mu\nu}(x) = e^{-\sigma} G^{\mu\nu}(x), \]
\[ |\tilde{G}^{\mu\nu}| = 1, \] (11)
which is unimodular in the preferred \( x^\mu \) coordinates. By definition, \( \tilde{G}^{\mu\nu}(x) \) is the effective metric for the scalar field. Because it enters in the kinetic term for any matter coupled to the matrix model, it plays the role of a gravitational metric
\[ ds^2 = \tilde{G}_{\mu\nu}(x) dx^\mu dx^\nu. \] (12)

Up to certain density factors, this also applies to nonabelian gauge fields as shown in [4] and for fermions [11]. Therefore the Poisson manifold under consideration naturally acquires a metric structure \( (M, \theta_{\mu\nu}(x), \tilde{G}^{\mu\nu}(x)) \), which is determined by the Poisson structure and the flat background metric \( g_{\mu\nu} \).

**Equations of motion.** The basic matrix model action (4) leads to the e.o.m. for \( X^\mu \)
\[ [X^\mu, [X^{\mu'}, X^{\nu'}]]g_{\mu\nu'} = 0. \] (13)
This can be written in the semi-classical limit as \( \theta^{\nu\gamma} \partial_\gamma \theta^{-1}_{\nu\mu'} g_{\mu\nu'} = 0 \), or
\[ G^{\gamma\eta}(x) \partial_\gamma \theta^{-1}_{\eta\nu} = 0. \] (14)
These equations are not covariant, they are valid only in the coordinates \( x^\mu \) where the “background metric” \( g_{\mu\nu} \) in the matrix model is either \( \delta_{\mu\nu} \) or \( \eta_{\mu\nu} \). As shown in Appendix B, these equations of motion can be written in a covariant manner as
\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma (e^\sigma \theta^{-1}_{\eta\nu}) = e^{-\sigma} \tilde{G}_{\mu\nu} \theta^{\mu\gamma} \partial_\gamma \eta(x) \] (15)
where
\[ \eta(x) = \frac{1}{4} G^{\mu\nu} g_{\mu\nu} = \frac{1}{4} G^{\mu\nu} G^{\mu'\nu'} \theta^{-1}_{\mu\nu} \theta^{-1}_{\mu'\nu'} \] (16)
and \( \tilde{\nabla} \) denotes the Levi-Civita connection with respect to the metric \( \tilde{G}^{\mu\nu} \). Note that the “background” metric \( g_{\mu\nu} \) is absorbed completely. (13) can be written as
\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma \theta^{-1}_{\eta\nu} = -\tilde{G}^{\gamma\eta}(x) \theta^{-1}_{\eta\nu} \partial_\gamma \sigma + e^{-2\sigma} \tilde{G}_{\mu\nu} \theta^{\mu\gamma} \partial_\gamma \eta(x), \] (17)
which has the form of covariant Maxwell equations with source. The obvious advantage of this covariant form of the equations of motion is that we can now use any adapted
coordinates, in particular rotation-symmetric ones etc. This should help to find solutions. Nevertheless, this should not obscure the fact that the underlying matrix model is not invariant under diffeomorphisms: the background metric $g_{\mu\nu}$ is constant, and there is no obvious way to transform it at the level of the matrix model. Only in the semi-classical limit we can allow general coordinates and rewrite things in a coordinate independent way, at the expense of introducing a flat background metric $g_{\mu\nu}$.

In principle of course, the equation of motion for $X^\mu$ is modified due to the presence of the scalar field. However for small coupling or energy, we can presumably neglect this back-reaction of matter on the geometry. It will be taken into account in section (2.1).

The equation of motion for the scalar field $\phi$ are

\[ 0 = [X^\mu, [X^\nu, \phi]] g_{\mu\nu} \sim \theta^{\mu\nu'} \partial_{\mu'} (\theta^{\nu\nu'} \partial_{\nu'} \phi) g_{\mu\nu}. \]  

As shown in Appendix A, this can be written as

\[ \Delta_{\tilde{G}} \phi = (\tilde{G}^{\mu\nu} \partial_\mu \partial_\nu - \tilde{\Gamma}^\mu \partial_\mu) \phi = 0 \]  

where $\tilde{\Gamma}^\mu = \tilde{G}^{\nu\eta} \tilde{\Gamma}_\mu^{\nu\eta}$ and $\tilde{\Gamma}_\mu^{\nu\eta}$ are the Christoffel symbols of $\tilde{G}_{\mu\nu}$. This follows also immediately from the covariant form (10) of the scalar action. We will show moreover in Appendix A that in the preferred $x^\mu$ coordinates defined by the matrix model, the equation of motion (14) for $X^\mu$ resp. $\theta^{-1}_{\mu\nu}$ is equivalent to the non-covariant equation (18) \[ \tilde{\Gamma}^\mu = 0. \]  

In these coordinates, the equation of motion for $\phi$ takes the simple form $\tilde{G}^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, for on-shell geometries.

The semi-classical form of the matrix model action (1) is

\[ S_{YM} = \frac{4}{(2\pi)^2} \int d^4 x \rho(x) \eta(x). \]  

The equation of motion (15) can be derived directly from this action. We will give this derivation in the next section, in the context of general branes embedded in $\mathbb{R}^D$.

## 2.1 Noncommutative branes and extra dimensions

Let us discuss scalar matter from the point of view of extra dimensions. Recall that e.g. the action for a scalar field is given by additional terms of the type

\[ Tr[X^\mu, \phi][X^\nu, \phi] \eta_{\mu\nu}. \]  

The combined action can be interpreted as matrix model with extra dimensions, where one coordinate denoted as $\phi$ is a function of the other 4 coordinates. Therefore we consider more generally

\[ S_{YM} = -Tr[X^a, X^b][X^{a'}, X^{b'}] \eta_{a\alpha} \eta_{b\beta}, \]  

5
for hermitian matrices or operators $X^a$, $a = 1, \ldots, D$ acting on some Hilbert space $\mathcal{H}$. To avoid a proliferation of symbols we fix the background to have the Minkowski metric; the Euclidean case is completely parallel, replacing $\eta_{ab}$ with $\delta_{ab}$. A scalar field can therefore be interpreted as defining an embedding of a 4-dimensional manifold (a “3-brane") in a higher-dimensional space. This naturally suggests to consider a higher-dimensional version of the Yang-Mills matrix model, such as the IKKT model in 10 dimensions.

We want to consider general $2n$-dimensional noncommutative spaces $\mathcal{M}_\theta^{2n} \subset \mathbb{R}^D$ (a $2n - 1$ brane) in $D$ dimensions. We correspondingly split the matrices as

$$X^a = (X^\mu, \phi^i), \quad \mu = 1, \ldots, 2n, \quad i = 1, \ldots, D - 2n.$$  \hspace{1cm} (24)

The basic example is a flat embedding of a 4-dimensional NC background with

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4,$$

$$\phi^i = 0, \quad i = 1, \ldots, D - 4$$  \hspace{1cm} (25)

where $X^\mu$ generates a 4-dimensional NC brane $\mathcal{M}_\theta^4$. Then the discussion of the previous section applies, and fluctuations of $\phi^i(x)$ can be interpreted as scalar fields on $\mathcal{M}_\theta^4$. More generally, we can interpret $\phi^i(x)$ as defining the embedding of a $2n$-dimensional submanifold $\mathcal{M}_{2n} \subset \mathbb{R}^D$, equipped with a nontrivial induced metric. The support (“D-dimensional spectrum") of $X^a \sim x^a$ will then be concentrated on $\mathcal{M}_{2n} \subset \mathbb{R}^D$ in the semi-classical limit.

Expressing the $\phi^i$ in terms of $X^\mu$, we obtain

$$[\phi^i, f(X^\mu)] \sim i\theta^{\mu\nu} \partial_\mu \phi^i \partial_\nu f = i\epsilon^\mu(f) \partial_\mu \phi^i$$  \hspace{1cm} (26)

in the semi-classical limit. This involves only the components $\mu = 1, \ldots, 2n$ of the antisymmetric tensor $[X^a, X^b] \sim i\theta^{ab}(x)$, which has rank $2n$ in this case. Here

$$\epsilon^\mu := -i[X^\mu, \ldots] \sim \theta^{\mu\nu} \partial_\nu$$  \hspace{1cm} (27)

are derivations, which span the tangent space of $\mathcal{M}_{2n} \subset \mathbb{R}^D$. They will define a preferred frame below. We can then interpret

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x)$$  \hspace{1cm} (28)

as Poisson structure on $\mathcal{M}_{2n}$ (assuming that it is non-degenerate), noting that the Jacobi identity is trivially satisfied. This is the Poisson structure on $\mathcal{M}_{2n}$ whose quantization is given by the matrices $X^\mu$, $\mu = 1, \ldots, 2n$, interpreted as quantization of the coordinate functions $x^\mu$ on $\mathcal{M}_{2n}$. Conversely, any $\theta^{\mu\nu}(x)$ (28) can be (locally) quantized, and provides together with arbitrary $\phi^i(x)$ a quantization of $\mathcal{M}_{2n} \subset \mathbb{R}^D$ as described above. Note that this Poisson structure is defined intrinsically by the configurations of the matrix model, independent of the choice made in (24). Assuming that $\theta^{\mu\nu}(x)$ is non-degenerate, we

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4For generic embeddings, this separation (24) is arbitrary, and we are free to choose different $2n$ components among the $\{X^a\}$ as generators of tangential vector fields. This is a particular change of coordinates on $\mathcal{M}_{2n}$, which from the field theory point of view corresponds to a remarkable transformation exchanging fields with coordinates, reminiscent of T-duality in string theory. In any case, note that $i\theta^{\mu\nu}$ is not naturally a pull-back of some non-degenerate Poisson or symplectic structure on $\mathbb{R}^D$. 

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denote its inverse matrix with
\[ \theta_{\mu\nu}^{-1}(x), \] (29)
which defines a symplectic form on \( M^{2n} \). Finally, the trace is again given semi-classically by the volume of the symplectic form,
\[ (2\pi)^n Tr f \sim \int d^{2n}x \rho(x) f \] (30)
with \( \rho = (\det \theta_{\mu\nu}^{-1})^{1/2} \) generalizing (8).

We are now in a position to extract the semi-classical limit of the matrix model and its physical interpretation. To understand the effective geometry on \( M^{2n} \), consider again a (test-) particle on \( M^{2n} \), modeled by some additional scalar field \( \varphi \) (this could be e.g. \( su(k) \) components of \( \phi^{i} \)). The kinetic term due to the matrix model must have the form
\[ S[\varphi] = -Tr[X^{a}, \varphi][X^{b}, \varphi]\eta_{ab} = -Tr ([X^{\mu}, \varphi][X^{\nu}, \varphi]\eta_{\mu\nu} + [\phi^{i}, \varphi][\phi^{j}, \varphi]\delta_{ij}) \]
\[ \sim Tr \left( \theta^{\mu\nu} \theta^{\nu\sigma} \partial_{\mu} \varphi \partial_{\nu} \varphi \eta_{\mu\sigma} + \theta^{\mu\nu} \theta^{\nu\sigma} \partial_{\nu} \varphi \partial_{\mu} \varphi \partial_{\nu} \varphi \delta_{ij} \right) \]
\[ = Tr \theta^{\mu\nu} \theta^{\nu\sigma} \left( \eta_{\mu\sigma} + \partial_{\mu} \phi^{i} \partial_{\nu} \phi^{j} \delta_{ij} \right) \partial_{\nu} \varphi \partial_{\mu} \varphi \]
\[ \sim \frac{1}{(2\pi)^{n}} \int d^{2n}x \rho(x) G^{\mu\nu}(x) \partial_{\mu} \varphi \partial_{\nu} \varphi \] (31)
where
\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_{\mu} \phi^{i} \partial_{\nu} \phi^{j} \delta_{ij} = \partial_{\mu} x^{a} \partial_{\nu} x^{b} \eta_{ab} \] (32)
\[ G^{\mu\nu}(x) = \theta^{\mu\nu}(x) \theta^{\nu\sigma}(x) g_{\mu\nu}(x) \] (33)
\[ \rho(x) = |\theta_{\mu\nu}^{-1}|^{1/2} = |G_{\mu\nu}|^{1/4} |g_{\mu\nu}(x)|^{1/4}. \] (34)
Here \( g_{\mu\nu}(x) \) is the metric induced on \( M^{2n} \subset \mathbb{R}^{D} \) via pull-back of \( \eta_{ab} \) on \( \mathbb{R}^{D} \). Now \( g_{\mu\nu}(x) \) is no longer flat in general. So far, the kinetic term does not quite have the correct covariant form. This can be achieved by a suitable rescaling of \( G^{\mu\nu}(x) \): generalizing the corresponding quantities in (3) and (11), we define
\[ \tilde{G}^{\mu\nu}(x) = e^{-\sigma} G^{\mu\nu}(x) \]
\[ \rho \tilde{G}^{\mu\nu} = |\tilde{G}_{\mu\nu}|^{1/2} \tilde{G}^{\mu\nu}(x) \]
\[ e^{-(n-1)\sigma} = |G_{\mu\nu}|^{1/4} |g_{\mu\nu}(x)|^{-\frac{1}{2}} = \rho |g_{\mu\nu}(x)|^{-\frac{1}{2}} \]
\[ |\tilde{G}_{\mu\nu}| = |\theta_{\mu\nu}^{-1}|^{\frac{n-2}{n-1}} |g_{\mu\nu}(x)|^{\frac{1}{n-1}}. \] (35)

Then the action (31) has the correct covariant form
\[ S[\varphi] = \frac{1}{(2\pi)^{n}} \int d^{2n}x |\tilde{G}_{\mu\nu}|^{1/2} \tilde{G}^{\mu\nu}(x) \partial_{\mu} \varphi \partial_{\nu} \varphi. \] (37)

Therefore the kinetic term on \( M^{2n}_{\theta} \) is governed by the metric \( \tilde{G}^{\mu\nu}(x) \), which has almost the same form as (11) except that the constant background metric \( g_{\mu\nu} \) is now replaced by the
induced metric \( g_{\mu \nu}(x) \) on \( \mathcal{M}^{2n} \subset \mathbb{R}^D \). The matrix model action (23) can be written in the semi-classical limit as

\[
S_{YM} = -\text{Tr}[X^a, X^b][X^a', X^b'] \eta_{aa'} \eta_{bb'} \sim \frac{4}{(2\pi)^n} \int d^{2n}x \rho(x) \eta(x),
\]

where

\[
4\eta(y) = G^{\mu \nu}(x)g_{\mu \nu}(x) = (\eta_{\mu \nu} + \partial_\mu \phi_i^j \partial_\nu \phi_j^i \delta_{ij}) \theta^{\mu \nu'}(\eta_{\mu' \nu'} + \partial_{\mu'} \phi_i^{\prime j} \partial_{\nu'} \phi_j^{\prime i} \delta_{ij}') \]

\[
= \left( \theta^{\mu \mu'} \theta^{\nu \nu'} \eta_{\mu, \nu} + 2\theta^{\mu \mu'} \theta^{\nu \nu'} \eta_{\mu', \nu} \partial_\mu \phi_i^{\prime j} \partial_\nu \phi_j^{\prime i} \delta_{ij} + \theta^{\mu \nu} \partial_\mu \phi_i^j \partial_\nu \phi_j^i \theta^{\mu' \nu'} \partial_{\mu'} \phi_i^{\prime j} \partial_{\nu'} \phi_j^{\prime i} \delta_{ij} \right)
\]

\[
\sim -[X^a, X^b][X^a', X^b'] \eta_{aa'} \eta_{bb'}
\]

generalizes (14).

There are 2 interesting special cases. For 4-dimensional NC spaces, we have

\[
|\tilde{G}_{\mu \nu}(x)| = |g_{\mu \nu}(x)|, \quad 2n = 4
\]

which means that the Poisson tensor \( \theta^{\mu \nu} \) does not enter the Riemannian volume at all. This provides a very interesting mechanism for “stabilizing flat space”, and may hold the key for the cosmological constant problem as discussed below. In the case of 2-dimensional NC spaces, (35) has no solution\(^5\), so that the action cannot be written in standard form at all. This will be discussed elsewhere.

The emergence of such noncommutative vacua is very compelling in closely related (Euclidean) matrix models admitting compact NC branes as vacua \([25, 30]\), and supported by a considerable body of analytical and numerical work at least in 2 dimensions, including \([19, 31–33]\) and references therein. In higher dimensions, it may be necessary to consider supersymmetric matrix models as discussed below, cf. \([34]\).

**Relation with string theory.** In string theory, a somewhat related situation occurs in the context of D-branes in a nontrivial B-field background. This leads to an effective description in terms of NC Yang-Mills theory on a noncommutative D-brane with Poisson structure \( \theta^{\mu \nu} \) inherited from the B field, see e.g. \([27]\) and references therein. This effective gauge theory is governed by the open string metric \([27]\) which is strongly reminiscent of \( \tilde{G}_{\mu \nu}(x) \) (apart from the density factor), while \( g_{\mu \nu}(x) \) corresponds to the closed string metric (more precisely its pull-back on the brane). Most of these results are restricted to the case of constant \( \theta^{\mu \nu} \) and slowly varying fields, while the case of general NC curved branes has received only limited attention, notably \([35, 36]\).

However, the results of the present paper should be compared more properly with previous work on string-theoretical matrix models such as the IKKT model \([12]\). NC branes have indeed been studied in considerable detail in this context, and it is well-known that the

\(^5\)I would like to thank A. Much for related discussions
matrix models can be interpreted as NC gauge theory on the brane. However, this has been worked out only for NC branes with a high degree of symmetry, such as fuzzy spaces (see e.g. [14, 17, 32, 37]), or other special branes satisfying a BPS condition [13, 18, 23, 38]. The role of the effective metric \( \tilde{G}^{\mu\nu}(x) \) is well-known in these cases, and evidence for the existence of gravitons on the branes has been obtained [26]. For generic NC branes in matrix models, the effective metric \( \tilde{G}^{\mu\nu}(x) \) and its role in the effective field theory on branes has not been elaborated previously, to the best knowledge of the author. Moreover, it is essential to note that the would-be \( U(1) \) gauge field on the brane is absorbed in \( \tilde{G}^{\mu\nu}(x) \), leading to a dynamical emergent gravity. Therefore the present approach could be seen as a novel way of obtaining gravity from string-theoretical matrix models, avoiding the conventional picture of string compactification.

In this context, it is worth recalling the relation between the semi-classical action (38) and the Dirac-Born-Infeld action for \( \theta^{-1}_{\mu\nu} := B_{\mu\nu} + F_{\mu\nu} \) which governs the dynamics of branes in string theory [27]. The action (21) arises from the DBI action at leading “nontrivial” order,

\[
\sqrt{\det(g_{\mu\nu} + \theta^{-1}_{\mu\nu})} \sim \rho(x) (1 + 2 \eta(x) + ...) \tag{41}
\]

omitting all constants, cf. [39].

**Equation of motion for test particle \( \phi \).** The covariant e.o.m. for \( \phi \) obtained from the semi-classical action (37) is

\[
\Delta_{\tilde{G}} \phi = (\tilde{G}^{\mu\nu} \partial_\mu \partial_\nu - \tilde{\Gamma}^{\mu} \partial_\mu) \phi = 0 \tag{42}
\]

On the other hand, starting from the matrix model (31) we obtain the e.o.m. for the same scalar field \( \phi \) as

\[
0 = [X^a, [X^b, \phi_j]] \eta_{ab} = [X^\mu, [X^\nu, \phi]] \eta_{\mu\nu} + [\phi^i, [\phi^j, \phi]] \delta_{ij}
\]

\[
= i[X^\mu, \theta^{\rho\eta} \partial_\eta \phi] \eta_{\mu\nu} + i[\phi^i, \theta^{\rho\mu} \partial_\rho \phi^j \partial_\nu \phi^j] \delta_{ij}
\]

\[
= -\theta^{\rho\mu} \partial_\rho (\theta^{\nu\eta} \partial_\eta \phi) \eta_{\mu\nu} - \theta^{\rho\sigma} \partial_\rho \phi^i \partial_\sigma (\theta^{\nu\eta} \partial_\eta \phi^i) \delta_{ij}
\]

\[
= - \left( \eta_{\mu\nu} \theta^{\rho\sigma} \partial_\rho \phi^i \partial_\sigma (\theta^{\nu\eta} \partial_\eta \phi^i) \delta_{ij} \right) \partial_\eta \phi
\]

\[
- \theta^{\rho\sigma} \theta^{\nu\eta} (\eta_{\mu\nu} + \delta g_{\mu\nu}) \partial_\rho \partial_\sigma \phi
\]

\[
eq_{\text{e.o.m.}} -G^{\rho\eta} \partial_\rho \partial_\eta \phi \tag{43}
\]

The last equality holds for on-shell geometries defined by (48), and \( \delta g_{\mu\nu} \equiv \partial_\mu \phi^i \partial_\nu \phi^j \delta_{ij} \). Comparing with the covariant form (42), it follows that

\[
\tilde{G}^{\mu\nu} \partial_\mu \partial_\nu \phi = 0 = \Delta_{\tilde{G}} \phi \tag{44}
\]

for on-shell geometries, which implies the “harmonic gauge”

\[
\tilde{\Gamma}^{\mu} \overset{\text{e.o.m.}}{=} 0 \tag{45}
\]

This holds only in the preferred \( x^\mu \) coordinates defined by the matrix model; a direct derivation based on (48) is given in Appendix A.
Equation of motion for $X^a$. The same argument as above gives the equations of motion for the embedding functions $\phi^i$ in the matrix model \[ \Delta \tilde{G} \phi^i = 0 \] (46) and similarly for $x^\mu \sim X^\mu$, \[ \Delta \tilde{G} x^\mu = 0. \] (47) This reflects the freedom of choosing the separation of $X^a = (X^\mu, \phi^i)$ into coordinates and scalar fields. In particular, on-shell geometries (48) imply harmonic coordinates, which in General Relativity \[40\] would be interpreted as gauge condition. We will now derive an equivalent but more useful form of (47) in terms of the “tangential” $\theta^{-1}_{\mu\nu}(x)$:

Equation of motion for $\theta^{-1}_{\mu\nu}(x)$. Reconsider the e.o.m. for the tangential components $X^\mu$ from the matrix model \[23\]:

\[ 0 = [X^b, [X^\nu, X^\nu]] \eta_{b\nu} = [X^\mu, [X^\nu, X^\nu]] \eta_{\mu\nu} + [\phi^i, [X^\nu, X^\nu]] \delta_{ij} \]

\[ = -\theta^{\mu\nu} \partial_\mu \theta^{\nu\rho} \eta_{\rho\nu} - \theta^{\mu\rho} \partial_\mu \phi^i \partial_\nu (\theta^{\nu\rho} \delta_{ij}) \]

\[ = -\theta^{\mu\nu} \partial_\mu \theta^{\nu\eta} (\eta_{\mu\eta} + \delta g_{\mu\eta}) - \theta^{\nu\eta} \theta^{\mu\rho} \partial_\nu \delta g_{\mu\eta} \]

\[ = -\theta^{\mu\nu} G^{\eta\mu}(x) \partial_\nu \theta^{-1}_{\nu\eta} - \theta^{\nu\eta} \theta^{\mu\rho} \partial_\nu g_{\mu\eta} \] (48)

since $\partial_\nu \delta g_{\mu\eta}(x) = \partial_\mu g_{\mu\eta}(x)$, i.e.

\[ G^{\eta\mu}(x) \partial_\nu \theta^{-1}_{\nu\eta} = \theta^{\mu\rho} \partial_\nu g_{\mu\eta}(x) \equiv J_\nu. \] (49)

These are essentially Maxwell equations coupled to an external current $J_\nu$, which depends on the matter field $\phi$. As shown in Appendix B, this can be written in covariant form as

\[ \tilde{G}^\gamma_{\nu\mu}(x) \tilde{\nabla}_\gamma (e^\sigma \theta^{-1}_{\nu\mu}) = e^{-\sigma} \tilde{G}^\gamma_{\mu\nu} \theta^{\rho\gamma} \partial_\rho \eta(x) \] (50)

Here $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to the effective metric $\tilde{G}_{\mu\nu}$ \[36\], which is no longer unimodular in general. This has the same form as \[13\], and can be rewritten as

\[ \tilde{G}^\gamma_{\nu\mu}(x) \tilde{\nabla}_\gamma \theta^{-1}_{\nu\mu} = -\tilde{G}^\gamma_{\nu\mu}(x) \theta^{-1}_{\nu\mu} \partial_\gamma \sigma + e^{-2\sigma} \tilde{G}^\gamma_{\mu\nu} \theta^{\rho\gamma} \partial_\rho \eta(x) \] (51)

The derivation in Appendix B assumes that the embedding functions $\phi^i$ also satisfy their e.o.m. \[46\]. It can also be derived directly from the semi-classical action \[38\]:

Semi-classical derivation of e.o.m. for $\theta^{-1}_{\mu\nu}(x)$. Starting from \[38\], we can derive the covariant e.o.m. of the matrix model using

\[ \delta \theta^{-1}_{\mu\nu} = \tilde{\nabla}_\mu \delta A_\nu - \tilde{\nabla}_\nu \delta A_\mu. \] (52)
This gives
\[ \delta S_{YM} = 2 \int d^{2n}x \left( \delta \eta(x) \sqrt{\det \theta^{-1}} + \eta(x) \frac{1}{2 \sqrt{\det \theta^{-1}}} \det \theta^{-1}(\theta^{\mu\nu} \delta \theta^{-1}) \right) \]
\[ = \int d^{2n}x \rho \left( g_{\mu\nu} \theta^{\mu\nu} \delta \eta + g_{\mu\nu} \theta^{\mu\nu} \delta \eta + \eta(x) \theta^{\mu\nu} \delta \theta^{-1} \right) \]
\[ = \int d^{2n}x \rho \left( G^{\mu\nu} \theta_{\mu\nu} - G^{\mu\nu} \delta \theta_{\rho\eta} + G^{\mu\nu} \delta g_{\mu\nu} + \eta(x) \theta^{\mu\nu} \delta \theta^{-1} \right) \]
\[ = 2 \int d^{2n}x \sqrt{|\tilde{G}|} \left( G^{\mu\nu} \delta g_{\mu\nu} - e^{-\sigma} \eta \theta^{\rho\eta} \tilde{\nabla}_\rho \delta A_\eta \right) \]
\[ + 2 \int d^{2n}x \sqrt{|\tilde{G}|} \tilde{\nabla}_\mu \delta A_\mu \]
\[ (53) \]
using
\[ \rho = |\tilde{G}|^{1/2} e^{-\sigma} \]
\[ (54) \]
which follows from (33). Noting that
\[ \int d^{2n}x \sqrt{|\tilde{G}|} \tilde{\nabla}_\mu V^\mu = 0 \]
\[ (55) \]
and \( \tilde{\nabla} \tilde{G} = 0 \) we obtain
\[ \delta S = -2 \int d^{2n}x \sqrt{|\tilde{G}|} \delta A_\eta \left( \tilde{G}^{\mu\rho} \tilde{\nabla}_\rho (e^{\sigma} \theta^{-1}) - \tilde{\nabla}_\rho (e^{-\sigma} \eta \theta^{\rho\eta}) \right) + \delta \phi^i \delta_{ij} \partial_i \left( \sqrt{|\tilde{G}|} \tilde{G}^{\mu\nu} \tilde{\partial}_\mu \phi^j \right) \]
\[ = -2 \int d^{2n}x \sqrt{|\tilde{G}|} \left( \delta A_\eta \left( \tilde{G}^{\mu\rho} \tilde{\nabla}_\rho (e^{\sigma} \theta^{-1}) - |\tilde{G}|^{-1/2} \partial_\rho (|\tilde{G}|^{1/2} e^{-\sigma} \eta \theta^{\rho\eta}) \right) + \delta \phi^i \delta_{ij} \Delta G \phi^j \right) \]
\[ = -2 \int d^{2n}x \sqrt{|\tilde{G}|} \left( \delta A_\eta \left( \tilde{G}^{\mu\rho} \tilde{\nabla}_\rho (e^{\sigma} \theta^{-1}) - e^{-\sigma} \theta^{\rho\eta} \partial_\rho \eta \right) + \delta \phi^i \delta_{ij} \Delta G \phi^j \right) \]
using (180) and (54) in the last steps. This gives precisely the equations of motion (50) and (46).

**Formal considerations.** From a more formal point of view, we have the following structures: The submanifold \( \mathcal{M}^{2n} \subset \mathbb{R}^D \) carries an embedding metric \( g \), and a preferred frame \( e^\mu = \theta^{\mu\nu} \partial_\nu \sim [X^\mu,] \) which encodes the noncommutative structure. The effective metric \( G^{\mu\nu} \) on \( T^* \mathcal{M} \) is defined by
\[ (\beta, \beta')_G := (\star \beta, \star \beta')_g, \quad \beta, \beta' \in T^* \mathcal{M}, \]
\[ (56) \]
where \( (\partial_\mu, \partial_\nu)_g = g_{\mu\nu}(x) \) and \( \star : T^* \mathcal{M} \to T^* \mathcal{M} \) is the canonical map defined by the Poisson structure \( \theta^{\mu\nu} \).

Notice the unusual role of the indices. This makes sense here, because the frame \( e^\mu \) is given in terms of the antisymmetric Poisson structure \( \theta^{\mu\nu} \) in the preferred coordinates \( x^\mu \).
There is no distinction between "Lorentz" and "covariant" indices here, and neither local Lorentz nor general coordinate transformations are allowed a priori. One could proceed to introduce differential forms in terms of one-forms $\theta^a$, $a = 1, ..., D$ through $[\theta^a, X^b] = 0$, $\theta^a \theta^b = -\theta^b \theta^a$. The exterior differential of functions is then defined in terms of a “special” one-form $\theta$,

$$df = [\theta, f], \quad \theta = X^a \eta_{ab} \theta^b.$$  \hfill (57)

This is similar to the formalism in [41, 42]; however the calculus is $D$-dimensional, similar to the case of the fuzzy sphere [43]. The scalar action can then be written as

$$S[\varphi] \sim \int d^{2n} x \rho \langle d\phi, d\phi \rangle$$  \hfill (58)

where $\langle \theta^a, \theta^b \rangle = \eta^{ab}$. Similarly, the semi-classical form of the matrix model is

$$S_{YM} \sim \frac{1}{(2\pi)^n} \int d^{2n} x \rho \langle \theta \wedge \theta, \theta \wedge \theta \rangle.$$  \hfill (59)

We can also write

$$df = [\theta, f] = e^\mu(f) \tilde{\theta}_\mu,$$

$$\tilde{\theta}_\mu = \eta_{\mu\nu} \theta^\nu + \partial_\mu \phi^i_j \delta^{2n+j}$$  \hfill (60)

where $\tilde{\theta}_\mu$ is in some sense dual to $e^\mu$. This should illuminate the relation and difference to [41]. These considerations will be pursued further elsewhere.

### 2.2 Nonabelian gauge fields

Now consider backgrounds of the form

$$Y^a = X^a \otimes 1_n + A^a_\alpha \otimes \lambda^\alpha$$  \hfill (61)

where $\lambda^\alpha$ are generators of $su(n)$. According to [1], the $U(1)$ sector (i.e. the components proportional to $1_n$) is absorbed in the geometrical degrees of freedom defined by $X^a$, and the discussion of the previous sections applies without change. On the other hand, the $su(n)$ components $A^\alpha_\mu$ behave as nonabelian gauge fields, and similarly the transversal $su(n)$ components $\phi^i_\alpha$ in

$$\phi^i = \tilde{\phi}^i \otimes 1_n + \phi^i_\alpha \otimes \lambda^\alpha$$  \hfill (62)

are nonabelian scalars from the brane point of view. The $\phi^i_\alpha$ then propagate in the background geometry $\tilde{G}^{\mu\nu}$ as discussed above. If some of the $\phi^i_\alpha$ develop a nontrivial vev, they might be viewed as part of the geometry.

\footnote{However the frame and metric here have a specific form in terms of $\theta^\mu$, unlike in [41].}
It was shown in [4] that the effective action for nonabelian gauge fields $A_\mu^\alpha$ due to the 4-dimensional matrix model (1) in the semi-classical limit is

$$S_{YM}[A] \sim \int d^4x \rho(x) tr \left( G^{\mu\nu} G^{\nu\rho} F_{\mu\rho} F_{\mu\rho}' \right) + 2 \int \eta(x) tr F \wedge F$$

This is the Yang-Mills action for a nonabelian gauge fields coupled to the effective metric $\tilde{G}_{\mu\nu}$, apart from the “would-be topological term” and the density factor $e^\sigma$. The latter could be interpreted as varying bare gauge coupling “constant”

$$g_{YM}^2 = g^2 e^{-\sigma}$$

introducing an overall coupling constant $\frac{1}{g^2}$ to the matrix model (4). In order to be physically acceptable, it is probably required that $\sigma$ is slowly varying. Indeed, a kinetic term for the “dilaton” $\rho$ resp. $\sigma$ is induced in the quantum effective action [11], except in the case of unbroken $N = 4$ supersymmetry. This might ensure that $\sigma$ is nearly constant.

Due to the strong constraints of gauge invariance, we expect that (63) applies without change to the case of non-trivially embedded 4-dimensional branes in $\mathbb{R}^D$; however this remains to be shown. Note that $\eta \sim e^\sigma$ due to (68), hence the two terms in (63) have roughly the same coefficients. This changes for higher-dimensional branes, where the “would-be topological term” $\int \eta(x) tr F \wedge F$ will be replaced by a different term which could be determined along the lines in [4]. Before relating this e.g. to the strong CP problem one would first have to identify more realistic models, elaborate the symmetry breaking etc..

2.3 Fermions

Then the most obvious (perhaps the only reasonable) action for a spinor which can be written down in the matrix model framework is

$$S = (2\pi)^2 Tr \overline{\Psi} \gamma_a [X^a, \Psi] \sim \int d^{2n}x \rho(x) \overline{\Psi} i(\gamma_\mu + \gamma_{2n+i} \partial_\mu \phi^i) \theta^{\mu\nu}(x) \partial_\nu \Psi$$

$$= \int d^{2n}x \rho(x) \overline{\Psi} \tilde{\gamma}_\mu \theta^{\mu\nu}(x) \partial_\nu \Psi$$

where $\gamma_a$ defines the D-dimensional Euclidean Clifford algebra, and

$$\tilde{\gamma}_\mu = \gamma_\mu + \gamma_{2n+i} \partial_\mu \phi^i$$

satisfies the Clifford algebra associated with the embedding metric $g_{\mu\nu}(x)$ on $\mathcal{M}$,

$$\{ \tilde{\gamma}_\mu, \tilde{\gamma}_\nu \} = 2\eta_{\mu\nu} + 2 \partial_\mu \phi^j \partial_\nu \phi^j \delta_{ij} = 2g_{\mu\nu}(x).$$

In particular, fermions should also be in the adjoint, otherwise they cannot acquire a kinetic term. This does not rule out its applicability in particle physics, see e.g. [4].
This is indeed the appropriate coupling of a spinor to the background geometry with metric $G_{\mu\nu}$ (up to rescaling), albeit with a non-standard spin connection which vanishes in the $x^\mu$ coordinates. This nicely generalizes at the classical level the analysis in [11], where this action was shown to provide a reasonable coupling of fermions to emergent gravity for flat $g_{\mu\nu}$. At the quantum level, it was shown in [11] that the Einstein-Hilbert term is indeed induced (along with a Dilaton-like term for $\sigma$), for flat $g_{\mu\nu}$ and on-shell geometries. It remains to be verified whether this generalizes to the case of non-trivially embedded branes. This is expected to be the case since it does give the correct Dirac operator e.g. in the case of the fuzzy sphere [45] or for $S_N^2 \times S_N^2$ [29].

Given the above Dirac operator, one could also consider the associated spectral action in the sense of [16]. It is an open problem in that context how to quantize gravity, more precisely how to integrate over the various geometries. The present framework suggests a simple answer: The Dirac operator should have the form as given in (63), and the integral over the geometries should be realized as integral over the matrices $X^a$ with measure defined by the bosonic matrix model, $d\mu(X^a) = e^{-S_{YM}[X^a]}$ (23). Nevertheless, this is not entirely equivalent to the present matrix model framework: The spectral action is based on the dependence of the spectrum as a function of the cutoff, while in the $N = 4$ case as considered here such a cutoff should not be required.

### 2.4 Compactification of branes

Consider a 2n–dimensional NC brane $M_{\theta}^{2n} \subset \mathbb{R}^{10}$. In order to obtain a 4-dimensional space at low energies, we assume that this higher-dimensional brane has compact extra dimensions, for example

$$M_{\theta}^{2n} \sim M_{\theta}^{4} \times K_{\theta}.$$ (68)

If $K$ is “small” enough, this looks like $M_{\theta}^{4}$ at low energies, as in standard compactification scenarios. Particularly natural examples would be $M_{\theta}^{6} \sim M_{\theta}^{4} \times S_N^2$ or $M_{\theta}^{8} \sim M_{\theta}^{4} \times S_N^2 \times S_N^2$, where $S_N^2$ denotes the fuzzy sphere. Such extra-dimensional fuzzy spaces can indeed be embedded naturally in the matrix models considered here (possibly upon adding soft SUSY-breaking terms) [13,31], or alternatively they can arise spontaneously from the scalar fields from the 4-dimensional point of view [14,47]. These 2 points of view are essentially equivalent.

Let us count degrees of freedom for the effective metric. For a 2n - dimensional NC brane, $\theta^{\mu\nu}$ resp. $\theta^{-1} = dA$ has $2n - 2$ physical (on-shell) plus one off-shell degrees of freedom, after gauge fixing. Upon compactification on $K_{\theta}$, the components $A_i$ tangential to $K_{\theta}$ become massive, leaving only 2 massless d.o.f. from a 4-dimensional point of view. The embedding of $M^{2n} \subset \mathbb{R}^{10}$ defined by $\phi^i$ provides $10 - 2n$ additional degrees of freedom. They are absorbed in the effective metric and governed by the quantum effective action. From the 4-dimensional point of view, this will lead to an effective “brane tension” on $M_{\theta}^{4}$ depending on the moduli of the compactification (e.g. the radius) as indicated below. Those are likely
to become “off-shell” d.o.f. which enlarge the class of effective 4-dimensional metrics, as desired. Therefore one can expect to recover most of the 2 on-shell plus 4 off-shell d.o.f. of the 4-dimensional metric in General Relativity. All this requires a more detailed analysis.

**Example 1: The fuzzy sphere $S^2_N$.** The fuzzy sphere $S^2_N$ \[13\] is a natural realization of this framework, being realized in terms of an embedding $S^2 \subset \mathbb{R}^3$. Consider our matrix model in $D = 3$, with the configuration

$$X^a = \frac{r}{c_N} \lambda^{(N), a}, \quad a = 1, 2, 3$$

$$[X^a, X^b] = i \theta_N \varepsilon^{abc} X^c, \quad X^a X^{a'} \delta_{aa'} = \frac{r^2}{c_N^2} \frac{1}{4} (N^2 - 1) = r^2. \quad (69)$$

Here $r$ is an arbitrary radius, and $\lambda^{(N), a}$ denotes the generators of the $N$-dimensional irreducible representation of $SU(2)$,

$$c_N^2 = \frac{1}{4} (N^2 - 1) \quad (70)$$

$$\theta_N = \frac{r^2}{c_N} \quad (71)$$

Even though \[13\] is not a solution of the basic matrix model \[1\], it makes nevertheless sense to consider such configurations since the induced gravity action is not yet taken into account. Moreover, it becomes a solution once a mass term and/or a cubic term is added to the action, e.g.

$$S_{YM} + S_{corr} = (2\pi) T r \left( [X^a, X^b] [X^{a'}, X^{b'}] \delta_{aa'} \delta_{bb'} + m^2 X^a X^{a'} \delta_{aa'} + \gamma X^a X^b X^c \varepsilon_{abc} \right). \quad (72)$$

Such terms might be induced in the quantum effective action, possibly after SSB. The general discussion of Section 2.1 applies as follows: consider e.g. some neighborhood of the north pole $x^3 \approx r, x^1 \approx x^2 \ll r$ of $S^2$. Then we separate the coordinates as in (24) in 2 tangential ones and one scalar “embedding” function,

$$X^a = (X^\mu, \phi), \quad \mu = 1, 2 \quad (73)$$

where $\phi = \phi(X^\mu) \approx 0$ for small $X^1, X^2$. Indeed it is not difficult to write $\phi = X^3$ in (69) as a function of $X^1, X^2$, in a suitable domain. Hence $S^2_N$ is a NC brane embedded in $\mathbb{R}^3$. The Poisson tensor e.g. at the north pole is $\theta^{12} = \theta_N$, and $S^2_N$ is a quantization of $S^2$ with the symplectic structure

$$\omega_{S^2} = \theta_N^{-1} \frac{1}{r} \varepsilon_{abc} x^a \, dx^b \, dx^c, \quad (74)$$

where $x^a$ is the semi-classical limit of $X^a$. It satisfies the semi-classical quantization condition

$$2\pi N = \int_{S^2} \omega_{S^2} = \int_{S^2} d^2 x \, \rho = \theta_N^{-1} 4\pi r^2 = 4\pi c_N \quad (75)$$
consistent with (70), where \( \rho = \theta^{-1}_N \). Therefore \( S^2_N \) can be considered as a compactification of the \( D = 2 \) Moyal-Weyl plane. The embedding metric \( g_{\mu\nu}(x) \) is the round metric for a sphere \( S^2 \) with radius \( r \), and \( G^{\mu\nu} = \theta^2_N g^{\mu\nu} \).

**Example 2**: \( \mathbb{R}^4_\theta \times S^2_N \). Now consider a configuration \( M^6_\theta = \mathbb{R}^4_\theta \times S^2_N \). This can be realized in the \( D \)-dimensional matrix model for \( D \geq 7 \)

\[
X^\mu = \bar{X}^\mu, \quad \mu = 0, 1, 2, 3 \\
\phi^i = \frac{r}{c_N} \lambda^{(N),i}, \quad i = 1, 2, 3
\]

where \( \bar{X}^\mu \) are the generators of \( \mathbb{R}^4_\theta \). This should be interpreted as a 6-dimensional NC space, which for small \( r \) looks like \( \mathbb{R}^4_\theta \). Such configurations can lead to interesting low-energy gauge groups and zero modes in the nonabelian case, as discussed in [44,47]. Similar configurations were discussed previously in the IKKT model [31], see also [48]. The radius of the fuzzy spheres will be dynamical \( r = r(x) \sim r(x^\mu) \), determined by the effective action. Inserting this configuration in the action and recalling (38), we obtain

\[
S_{YM} = (2\pi)^3 Tr[X^a, X^b][X^{a'}, X^{b'}] \delta_{aa'} \delta_{bb'} \sim 4 \int_{\mathbb{R}^4 \times S^2} d^4x \omega S^2 \rho^{(4)}(x) \eta^{(6)}(x)
\]

using (75) and

\[
\eta^{(6)}(x) \sim [X^a, X^b][X^{a'}, X^{b'}] \delta_{aa'} \delta_{bb'} \sim \eta^{(4)}(x) + 2G^{\mu\nu} \partial_{\mu}r(x) \partial_{\nu}r(x) + 2 \frac{r(x)^4}{c_N^2},
\]

where \( \rho(x) \) and \( g_{\mu\nu}(x) \) involve only the 4-dimensional metric. This leads to an effective potential \( V(r) \) for the radius \( r(x) \), which will receive additional contributions from further terms such as (72) and from the induced gravitational action. For example, consider the 6-dimensional “would-be cosmological constant” term in (76), which using (38) can be written as

\[
\int d^6x \sqrt{|\tilde{G}|} \Lambda^6 \sim \int d^6x |\theta^{-1}_\mu|^{1/4} |g_{\mu\nu}|^{1/4} \Lambda^6
\]

\[
= 4\pi c_N^{1/2} \int d^4x r(x) \rho(x)^{1/2} |g_{\mu\nu}|^{1/4} \Lambda^6
\]

where \( \rho(x) \) and \( g_{\mu\nu}(x) \) in the last line are 4-dimensional quantities. Somewhat surprisingly, this term now depends on \( \rho(x) \), unlike in the case of 4D branes. However, this expression should be taken with much caution, because the IR condition (90) for the applicability of the semi-classical expressions (96) may not be appropriate for the compact dimensions. In that case, it might be more appropriate to use a 4-dimensional description rather than the 6-dimensional metric. In any case, this will contribute to the effective potential for \( V(r) \), but a more detailed analysis is required.
Example 3: \( \mathbb{R}^4_g \times S^2_{N_L} \times S^2_{N_R} \). A generalization of the above configuration which can be realized in the 10-dimensional IKKT matrix model is

\[
X^\mu = \bar{X}^\mu, \quad \mu = 0, 1, 2, 3
\]
\[
\phi^i = \frac{r_L}{c_L} \lambda^{(N_L),i}, \quad i = 1, 2, 3,
\]
\[
\phi^i = \frac{r_R}{c_R} \lambda^{(N_R),i}, \quad i = 4, 5, 6
\]

(81)

which should be interpreted as a 8-dimensional NC space. The effective 4-dimensional action now involves 2 parameters \( r_L, r_R \), which will be governed by an effective potential \( V(r_L, r_R) \). This should provide sufficient structure to obtain interesting solutions from the particle physics point of view; see also e.g. [29, 31] and references therein for related work.

2.5 Departures from General Relativity: preferred scales and coordinates

There are several features of the model under consideration which differ radically from the conventional picture of General Relativity. We focus on the case of 4-dimensional NC branes for simplicity.

First, recall that there are preferred coordinates in the model, given by the covariant coordinates \( x^\mu \). In those coordinates, the background metric is explicitly constant, \( g_{\mu\nu} = \delta_{\mu\nu} \) resp. \( g_{\mu\nu} = \eta_{\mu\nu} \) in the \( D = 4 \) case, and the preferred frame is given by the antisymmetric Poisson tensor, \( e^\mu = \theta_{\mu\nu} \partial_\nu \). This is physically not very significant a priori, but it simplifies the issue of gauge fixing. We recall that in the preferred \( x^\mu \) coordinates, the on-shell condition for \( X^\mu \) amounts to

\[
\tilde{\Gamma}^\mu = 0,
\]

(82)

which would be interpreted as gauge choice in General Relativity.

A more significant feature of the matrix model is the presence of the scalar density

\[
\rho = (\det \theta^{-1})^{1/2}
\]

(14), which defines the scale of noncommutativity

\[
\rho = \Lambda_{NC}^4 = L_{NC}^{-4}
\]

(83)

and provides the symplectic measure \( (2\pi)^2 \text{Tr} f \sim \int dx \rho f \). Such a structure does not exist in the commutative framework. This leads to an analog of the Bohr-Sommerfeld quantization condition,

\[
\text{Vol}_\theta = (2\pi)^2 \mathcal{N}
\]

(84)

where \( \text{Vol}_\theta \) denotes the volume measured in units of \( L_{NC} \), and \( \mathcal{N} \) the dimension of the corresponding Hilbert (sub)space. This means that the volume is quantized in integer multiples of \( L_{NC}^4 \), so that NC branes are automatically “large” for large \( \mathcal{N} \). This is already
a hint that NC spaces like to be flat, which is very interesting in connection with the cosmological constant problem.

There is another scale in the model determined by the embedding metric \( g_{\mu \nu} \) resp. the effective metric \( \tilde{G}_{\mu \nu} \),

\[
L_g^4 = \Lambda_g^{-4} = |\tilde{G}^{\mu \nu}|^{1/2}
\]

(85)

which we could set to 1 thereby fixing the units; recall that |\( \tilde{G}_{\mu \nu} \)| \( \equiv |g_{\mu \nu}| \) \( (10) \) for general 4-dimensional branes in \( \mathbb{R}^D \). The ratio of these scales defines the dimensionless scalar function

\[
e^{-\sigma} = \frac{|\theta_{\mu \nu}|^{1/2}}{|G_{\mu \nu}|^{1/2}} = \frac{\Lambda_{NC}^4}{\Lambda_g^4}
\]

(86)

using (36). We can relate this with the Riemannian volume of \((\mathcal{M}_\theta^4, \tilde{G}_{\mu \nu})\) measured by \(\tilde{G}_{\mu \nu}\),

\[
\text{Vol}_{\tilde{G}} = (2\pi)^2 N e^\sigma.
\]

(87)

The “dilaton” \(e^\sigma\) will be determined dynamically by the model resp. the background under consideration. For example, in matrix models for fuzzy spheres it depends on the coefficient of additional (soft SUSY-breaking) terms such as \(\text{Tr} \varepsilon_{abc} X^a X^b X^c\), see also the related discussion in [47]. Note that \(e^{-\sigma}\) also gives the scale of \(\eta\),

\[
\eta \approx \frac{|\tilde{G}_{\mu \nu}|^{1/2}}{|\theta_{\mu \nu}|^{1/2}} = e^\sigma,
\]

(88)

at least for simple 4-dimensional configurations. This may be a significant large dimensionless number.

In the context of quantization, we will encounter 2 additional scales \(\Lambda_4 \gg \Lambda_1\) in the 10-dimensional version of the model, where \(\Lambda_4\) is the scale of \(N = 4\) SUSY breaking which is argued to coincide with the Planck scale \(\Lambda_{Pl}\) below, and \(\Lambda_1\) is the scale of \(N = 1\) SUSY breaking. These should also be dynamical scales. We will furthermore argue that \(\Lambda_{NC} > \Lambda_4\) simplifies the semi-classical analysis, however this is not essential; it seems actually plausible that \(N = 4\) SUSY is broken by the NC background, so that \(\Lambda_{NC} = \Lambda_4\). In summary, we expect 3 a priori distinct physical scales

\[
\Lambda_{NC} \geq \Lambda_4 = l_{Pl}^{-1} \gg \Lambda_1
\]

(89)

in addition to the dimensionless number \(e^\sigma\) in the model.

### 2.6 Quantization and induced gravity

Now consider the quantization of our matrix model, which can contain scalar fields (such as e.g. arising from extra dimensions), fermions, the “would-be \(U(1)\) gauge field” which is absorbed in \(\theta^{\mu \nu}(x)\), and possibly nonabelian gauge fields. In principle, the quantization
is defined in terms of an integral over all matrices. This is expected to be well-defined at least in the case of the IKKT model, which leads to $N = 4$ SUSY on $\mathbb{R}^4$. Some modifications such as soft SUSY breaking terms may also be allowed. Note that this quantization implies an integration over all geometries of the NC branes embedded in $\mathbb{R}^{10}$, via the quantization of the embedding functions $\phi^i$ as well as $X^\mu$ resp. $\theta^{\mu\nu}(x)$. In particular, (emergent) gravity is also quantized.

To obtain a qualitative understanding of the model at the quantum level, we can take advantage of the above semi-classical form of the action in terms of conventional field theory coupled to $\tilde{G}^{\mu\nu}$. Then the low-energy effective action at one loop can be extracted from standard results of ordinary quantum field theory on curves spaces. As shown in [10, 11], this is indeed justified (based on a comparison with a fully NC computation and UV/IR mixing) provided there exists an effective UV-cutoff $\Lambda$, and the following IR regime [10] is respected

$$p \Lambda < \Lambda_{NC}^2 \quad \text{and} \quad \Lambda < \Lambda_{NC}.$$  

These conditions ensure that the effects of noncommutativity are mild even in the loops, so that the phase factors in non-planar diagrams are small and are well approximated by the Poisson structure. This reflects the fact that emergent NC gravity is an IR phenomenon. A violation of e.g. $\Lambda < \Lambda_{NC}$ is acceptable, but implies corrections\(^8\) to the effective action (90) given below, some of which have been discussed in [10]. Such a cutoff is realized in the $N = 4$ supersymmetric version of the model, assuming that $N = 4$ SUSY is broken at $\Lambda = \Lambda_4$ from now on. This is essential, because no bare term in the action is available which could cancel the induced (gravitational) action discussed below. We will furthermore assume that some smaller supersymmetry survives down to a much lower energy scale $\Lambda_1$, below which no supersymmetry survives. These are reasonable assumptions, which appear to be necessary for the proposed framework to be physically viable. Note that these scales are measured using the physical metric $\tilde{G}_{\mu\nu}$.

The results of the one-loop computation of fields coupled to the background metric $\tilde{G}_{\mu\nu}$ can be obtained conveniently using the Seeley- de Witt coefficients of the corresponding heat kernel. The essential features are illustrated by the quantization of scalar fields. Hence consider the effective action obtained by integrating out the scalars, which in the Euclidean case is

$$e^{-\Gamma_{\phi}[\tilde{G}]} = \int d\Phi e^{-S[\Phi]}.$$  

(91)

Since we are mainly interested in the induced gravitational action here, it is sufficient to consider the case of non-interacting scalar fields coupled to the metric $\tilde{G}_{\mu\nu}$, where

$$\Gamma_{\phi}[\tilde{G}] = \frac{1}{2} \text{Tr} \log \frac{1}{2} \Delta_{\tilde{G}}$$  

(92)

\(^8\) if this condition is violated, a more refined analysis of NC corrections is required, cf. [10]. It turns out that the apparently-quartic divergent term $\int \Lambda_4^4 \sqrt{G}$ actually becomes milder, being a difference between quadratically-divergent planar and non-planar diagrams. A similar comment applies to the $\int \Lambda_4^2 R[\tilde{G}]$ term.
assuming Euclidean signature for simplicity. Here $\Delta_{\tilde{G}}$ is the Laplacian of a scalar field on the Riemannian manifold $(M, \tilde{G}^{ab}(y))$ with action (10). The UV cutoff $\Lambda$ is incorporated using the Schwinger parametrization

$$\text{Tr} \left( \frac{1}{2} \log \Delta_{\tilde{G}} - \frac{1}{2} \log \Delta_0 \right) \sim -\text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \frac{1}{2} \Delta_{\tilde{G}}} - e^{-\alpha \frac{1}{2} \Delta_0} \right) \equiv -\text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \frac{1}{2} \Delta_{\tilde{G}}} - e^{-\alpha \frac{1}{2} \Delta_0} \right) e^{-\frac{1}{\alpha^2} \Lambda^2}. \tag{93}$$

Now we can use the heat kernel expansion,

$$\text{Tr} e^{-\frac{1}{2} \alpha \Delta_{\tilde{G}}} \sim \sum_{m \geq 0} \left( \frac{\alpha}{2} \right)^{m-n} \int_M d^2n x \sqrt{|\tilde{G}_{\mu\nu}|} \ a_{2m}(x, \Delta_{\tilde{G}}). \tag{94}$$

The $a_m(x, \Delta_{\tilde{G}})$ are known as Seeley-de Witt (or Duhamel) coefficients, which for scalar fields with action (10) are given by (95)

$$\begin{align*}
a_0(x) &= \frac{1}{(4\pi)^n}, \\
a_2(x) &= \frac{1}{(4\pi)^n} \left( \frac{1}{6} R(\tilde{G}) \right), \\
a_4(x) &= \frac{1}{(4\pi)^n} \left( \frac{1}{360} \left( 12 R_{\mu\nu}^\mu + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right). \tag{95}
\end{align*}$$

The full effective action is complicated by the fact that there are several types of fields in the model including scalars, fermions, and gauge fields. Because they all couple to the same effective metric up to possibly density factors, they will all induce essentially the same type of gravitational action, with an additional kinetic term for the “dilaton” $\sigma$ due to the fermions and gauge fields resp. gravitons. Therefore one obtains the following type of induced gravitational action:

$$\Gamma_{1-\text{loop}}[\tilde{G}] = \frac{1}{(4\pi)^n} \int d^2n x \sqrt{|\tilde{G}_{\mu\nu}|} \left( -c_0 \Lambda_1^{2n} - c_2 R(\tilde{G}) \Lambda_4^{2n-2} + \ldots \right) \tag{96}$$

where $c_m$ are model-dependent constants, omitting dilaton-like terms. This allows already to draw some qualitative conclusions, focusing on $2n = 4$ -dimensional NC branes. More detailed computations should be performed elsewhere.

Note that we associated different scales $\Lambda_1$ resp. $\Lambda_4$ to the different terms in (96), which arise as follows. It is well-known that the coefficient of the leading “would-be cosmological constant” term $\int d^4x \sqrt{|G_{\mu\nu}|}$ is determined by the scale $\Lambda_1$ for $N = 1$ SUSY breaking. In contrast, the coefficient of the induced Einstein-Hilbert term in emergent NC gravity is

\footnote{This was shown in \cite{4} for gauge fields and in \cite{11} for fermions. Similar results are expected for the gravitons (i.e. the would-be $U(1)$ gauge field) due to supersymmetry, at least in the case of $N = 4$ SUSY.}
determined by the scale \( \Lambda_4 \) where \( N = 4 \) SUSY is broken. This reflects the well-known fact that UV/IR mixing in NC gauge theory persists even in SUSY gauge theory \[50, 51\], except in the \( N = 4 \) case. Since \( \Gamma_{1-loop}[\tilde{G}] \) is nothing but a re-interpretation of the UV/IR mixing terms in NC gauge theory \[4,10\], it follows that the induced term \( R[\tilde{G}] \) has a cutoff given by the scale of \( N = 4 \) SUSY breaking; this is discussed in \[11\] from the point of view of gravity. Because there is no bare gravity action, it follows that the effective Newton constant resp. the Planck scale in emergent gravity is given by

\[
l^2_{Pl} = \frac{1}{G} \sim \Lambda^2_4. \tag{97}
\]

This also suggests what happens in models without a finite effective cutoff \( \Lambda_4 \): Then \( G \to 0 \), hence the induced gravitational action becomes a constraint and there is no more back-reaction of matter to the geometry. However there might still be interesting scaling limits.

### 2.7 Effective action and the (ir)relevance of the cosmological constant term

Consider again the case of 4-dimensional NC branes embedded in \( \mathbb{R}^D \). The full semi-classical effective action of the matrix model at one loop is given by

\[
S_{eff} = S_{YM} + S_{1-loop} \tag{98}
\]

where \( S_{1-loop} = \Gamma_{1-loop}[\tilde{G}] + \ldots \), and

\[
S_{YM} = \frac{1}{(2\pi)^2 g^2} \int d^4x \sqrt{|\tilde{G}_{\mu\nu}|} \left( e^\sigma \tilde{G}^{\mu\nu} \tilde{G}^{\mu\nu} tr F_{\mu\nu} F_{\mu'\nu'} + \tilde{G}^{\mu\nu} tr \partial_\mu \phi^i \partial_\nu \phi^j \delta_{ij} \right) + \frac{1}{(2\pi)^2 g^2} \int d^4x \left( 4\rho + 2\eta(x) tr F \wedge F \right) \tag{99}
\]

including nonabelian gauge fields and the nonabelian components \( \phi^i = \phi^i_\alpha \lambda^\alpha \) of the scalar fields. Fermionic terms are omitted. We introduced an explicit coupling constant \( g \), which does not enter the induced gravitational action since it can be absorbed in the fields. The bare YM coupling constant is given by \( g_{YM}^2 = g^2 e^{-\sigma} \) \[64\], which receives the standard quantum corrections, and might play a role similar to a GUT coupling. The one-loop induced gravitational term \( \Gamma_{1-loop}[\tilde{G}] \) for 4-dimensional NC branes is \[30\]

\[
\Gamma_{1-loop}[\tilde{G}] \sim \int d^4x \sqrt{|\tilde{G}_{\mu\nu}|} \left( \Lambda_4^2 + R[\tilde{G}] \Lambda_4^2 \right) \tag{100}
\]

where \( |\tilde{G}_{\mu\nu}| = |g_{\mu\nu}| \) using \[40\]. The first term is therefore simply the invariant volume of the embedding metric.

Consider the geometric equations of motion. It is well-known that

\[
\delta \int d^4x \sqrt{|\tilde{G}|} R[\tilde{G}] = \int d^4x \sqrt{|\tilde{G}|} \left( \frac{1}{2} R[\tilde{G}] \tilde{G}^{\mu\nu} - R^{\mu\nu}[\tilde{G}] \right) \delta \tilde{G}_{\mu\nu} \tag{101}
\]
while the variation of the “would-be cosmological term” is

$$\delta \int d^4x \sqrt{|\tilde{G}|} = \frac{1}{2} \int d^4x \sqrt{|\tilde{G}|} \tilde{G}_{\mu\nu} \delta \tilde{G}_{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} \quad ,$$

(102)

using (100) in the case of 4-dimensional NC branes. This vanishes identically in the case $D = 4$ due to (11). The variation of the bare gravitational action $\delta \int d^4x 4 \rho \eta$ was worked out in (54).

In General Relativity, (101) and (102) imply the Einstein equations for vacuum. The essential difference here is that the metric $\tilde{G}_{\mu\nu}$ is constrained, and the fluctuations do not span the space of symmetric $4 \times 4$ matrices. Therefore we do not simply obtain the Einstein equations; this is seen most strikingly for the cosmological constant term as discussed below. However, note that Ricci-flat spaces which can be realized in terms of our $\tilde{G}_{\mu\nu}$ certainly satisfy $\delta \int d^4x \sqrt{|G|} R[\tilde{G}] = 0$. This supports the physical viability of this framework. The correct e.o.m. which follow from the effective action are complicated here by the presence of possible dilaton-like terms at one loop, and will be derived elsewhere. They will in particular modify (50), which takes into account the bare action only.

Let us briefly discuss the geometrical degrees of freedom. We can decompose the variations $\delta X^a$ of the basic matrices into tangential and normal fluctuations w.r.t. the background brane $\mathcal{M}$. Using an orthogonal transformation if necessary, we can assume that $\delta X^a \in T\mathcal{M}$ are tangential, and $\delta \phi^i \in T\mathcal{M}^\perp$ are normal to the brane at some given point. Consider first the tangential variations $\delta X^a = \mathcal{A}^a$. They lead to variations of the Poisson tensor $\theta^{\mu\nu}$ on the brane (which can be interpreted as diffeomorphism), but they do not change the embedding metric, which is fixed in the matrix model (recall that e.g. $g_{\mu\nu} \equiv \eta_{\mu\nu}$ in the simplest case $D = 4$): $\delta_\mathcal{A} g_{\mu\nu} = 0$. Therefore these tangential fluctuations imply nontrivialootnote{except for gauge transformations resp. symplectomorphisms $\mathcal{A}^\mu = [f, X^\mu]$.} physical fluctuations of the effective metric $\delta_\mathcal{A} G_{\mu\nu} \sim (G \theta^F + F \theta G)_{\mu\nu}$ (113) corresponding to the 2 on-shell graviton helicities plus one off-shell deformation; cf. the discussion in Section 3. In particular, the term $\int d^4x \sqrt{|\tilde{G}|} = \int d^4x \sqrt{|g|}$ in (96) is independent of these tangential degrees of freedom. This provides (part of) a mechanism for avoiding the cosmological constant problem.

Now consider the normal fluctuations $\delta \phi^i$ of the brane embedding, which in general imply nontrivial physical fluctuations of the effective metric. The corresponding variation of the “would-be cosmological term” is

$$\delta \int d^4x \sqrt{|g_{\mu\nu}|} = \int d^4x \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} = 2 \int d^4x \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \delta \phi^j \delta_{ij}$$

$$= 2 \int d^4x \sqrt{|g|} \partial_\mu \phi^i \delta \phi^j \Delta_g \delta_{ij} \quad (103)$$

using partial integration, where $\Delta_g$ is the covariant Laplacian corresponding to the metric

22
This vanishes if the $\phi^i$ satisfy the constraint
\[ \Delta g \phi^i = 0, \tag{104} \]
which is similar to (106) except that the metric is now $g_{\mu\nu}$. Flat embeddings do satisfy this condition. Therefore flat space is a solution at the quantum level, even in the presence of this “would-be cosmological constant” term; the same applies to any surfaces embedded in $\mathbb{R}^D$ which satisfies (104). This can also be seen from the fact that there is no tadpole contribution at one loop in NC gauge theory [10]. This is in stark contrast to General Relativity, where the term $\int d^4x \sqrt{G} \Lambda^4$ corresponds to a huge cosmological constant, requiring unreasonable fine-tuning. Together with the observations of the previous paragraph, we obtain strong evidence here that the cosmological constant problem is resolved or at least much milder in the present context. This is a robust mechanism, rooted in the fact that the metric is not a fundamental degree of freedom but emerges from the matrix model.

To obtain a more complete understanding of the cosmological constant issue in emergent NC gravity, a more complete analysis is required, as for related claims in the literature [24]. At present, the only known solution for the full effective action (98) is flat Moyal-Weyl space. Nevertheless, the fact that this is a solution without fine-tuning a cosmological constant is very remarkable. Moreover, since the Einstein-Hilbert term contains two explicit derivatives, the bare action together with the “would-be cosmological constant” term will govern the extreme IR (cosmological) scale which should indeed be flat, while the induced E-H action will determine the gravitational fields due to localized (point) masses. This would be a very satisfactory picture.

3 Linearized metric and gravitational waves

Moyal-Weyl case. A particular solution of the e.o.m. (13) is given by the 4D Moyal-Weyl quantum plane. Its generators $\bar{X}^\mu$ satisfy
\[ [\bar{X}^\mu, \bar{X}^\nu] = i\bar{\theta}^{\mu\nu} \mathbb{I}, \tag{105} \]
where $\bar{\theta}^{\mu\nu}$ is a constant antisymmetric tensor. The effective geometry (7) for the Moyal-Weyl plane is indeed flat, given by
\[ \bar{g}^{\mu\nu} = \bar{\theta}^{\mu\nu}', \bar{\theta}^{\nu\mu}, \]
\[ \bar{g}^{\mu\nu} = \bar{\rho} \bar{g}^{\mu\nu}, \]
\[ \bar{\rho} = |\bar{g}_{\mu\nu}|^{1/4} = |\bar{\theta}^{-1}_{\mu\nu}|^{1/2} \equiv \Lambda^{4}_{NC}. \tag{106} \]
In this section, lower-case $\bar{g}^{\mu\nu}$ resp. $\bar{g}^{\mu\nu}$ will denote the flat effective Moyal-Weyl metric rather than the embedding metric, and we will raise and lower indices using $\bar{g}_{\mu\nu}$. First, we can choose coordinates where $\bar{g}_{\mu\nu} = (-1, 1, 1, 1)$, so that $x^0 = ct$ corresponds to the time.
One can use the remaining $SO(3, 1)$ (resp. $SO(4)$ in the Euclidean case) to bring $\tilde{\theta}^{\mu\nu}$ into canonical form

$$\tilde{\theta}^{\mu\nu} = \theta \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (107)$$

The NC scale is then

$$\tilde{\rho} = \theta^{-2} \alpha^{-1} = \Lambda_{NC}^4,$$  \hspace{1cm} (108)

and the original flat background metric is

$$\eta_{\mu\nu} = \bar{\theta}^{-1} \eta_{\mu\nu} \bar{\theta}^{-1} \bar{g}^{\mu\nu} = \tilde{\rho}^{-1} \theta^{-2} \text{diag}(1, \alpha^{-2}, \alpha^{-2}, -1) = \alpha \text{diag}(1, \alpha^{-2}, \alpha^{-2}, -1). \quad (109)$$

The bare action for this flat Moyal-Weyl background is

$$S_{YM} = Tr \tilde{\theta}^{\mu\nu} \tilde{\theta}^{\nu\sigma} \eta_{\mu\nu} \eta_{\nu\sigma} = Tr \eta_{\mu\nu} \bar{g}^{\mu\nu} = \int d^4 x \eta_{\mu\nu} \bar{g}^{\mu\nu} = 2 \int d^4 x \alpha (\alpha^{-2} - 1) \quad (110)$$

which vanishes in the case $\alpha = 1$ where $\tilde{\theta}^{\mu\nu}$ admits an enhanced $SO(2, 1) \times U(1)$ symmetry. In the Euclidean case, the action is positive definite. From now on, we assume that

$$\alpha = 1$$

for simplicity. It is also worth pointing out that we are not in the case of “space-like” noncommutativity, since $\tilde{\theta}^{\mu\nu}$ is non-degenerate. However, the problems of unitarity etc. discussed e.g. in [53] are expected to be benign in the present context due to the assumed $N = 4$ supersymmetry at the Planck scale.

**Deformations of the flat Moyal-Weyl plane.** Consider small deformations of the flat Moyal-Weyl plane,

$$X^\mu = \bar{X}^\mu - \tilde{\theta}^{\mu\nu} A_\nu(x), \quad (111)$$

where $A_\nu$ are hermitian and can be interpreted as $U(1)$ gauge fields on $\mathbb{R}^4$ with field strength $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. The linearized metric $G^{\mu\nu}$ is

$$G^{\mu\nu} = (\bar{\theta}^{\mu\nu} - \bar{\theta}^{\mu\eta} \bar{\theta}^{\nu\rho} \bar{F}_{\eta\rho})(\bar{\theta}^{\nu\sigma} - \bar{\theta}^{\nu\kappa} \bar{\theta}^{\sigma\kappa} \bar{F}_{\sigma\kappa})(\eta_{\mu\nu} + \delta g_{\mu\nu}) \quad (112)$$

where

$$h^{\mu\nu} = -\bar{g}^{\mu\rho} \bar{F}_{\mu\eta} \bar{g}^{\nu\rho} - \bar{g}^{\mu\rho} \bar{F}_{\nu\rho} \bar{g}^{\eta\rho} - \bar{g}^{\mu\rho} \bar{g}^{\eta\rho} \delta g_{\mu\eta} + O(A^2). \quad (113)$$

where $\delta g_{\mu\nu} = \partial_\mu \phi^i \partial_\nu \phi^j \delta_{ij}$. Correspondingly, the inverse metric is

$$G_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \ldots, \quad (114)$$

with

$$h_{\mu\nu} \equiv \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} h^{\rho\sigma} = -\bar{g}_{\nu\rho} \bar{\theta}^{\rho\sigma} \bar{F}_{\rho\mu} - \bar{g}_{\mu\rho} \bar{\theta}^{\rho\sigma} \bar{F}_{\rho\nu} - \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} \bar{\theta}^{\rho\sigma} \delta g_{\rho\sigma}. \quad (115)$$
This gives
\[ h = h_{\mu\nu} \bar{\gamma}^{\mu\nu} = 2 \bar{\gamma}^{\mu\nu} F_{\mu\nu} - \eta^{\mu\nu} \delta g_{\mu\nu}. \] (116)

We focus on the case of flat embeddings \( \delta g_{\mu\nu} = 0 \). Then \( h^{\mu\nu} = -\bar{g}^{\mu\nu} F_{\mu\nu} - \bar{\theta}^{\mu\nu} \delta g_{\mu\nu} + O(A^2) \) gives the linearized fluctuation resp. graviton in terms of the \( U(1) \) degrees of freedom. The linearized Ricci tensor for the unimodular metric \( \tilde{G}_{\mu\nu} \) resp. the traceless graviton \( \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4} \tilde{g}_{\mu\nu} h \) is given by
\[ R_{\mu\nu}[\tilde{G}] = \frac{1}{2} \left( \bar{\theta}_{\mu} \eta \partial^{\rho} \partial_{\eta} F_{\rho\nu} + \bar{\theta}_{\nu} \eta \partial^{\rho} \partial_{\eta} F_{\rho\mu} + \frac{1}{2} \tilde{g}_{\mu\nu} \partial^{\rho} \partial_{\rho} F_{\eta\sigma} \bar{\theta}^{\eta\sigma} \right) \] (117)
in agreement with results of \([5]\), and
\[ R[\tilde{G}] = \frac{1}{2} \partial^{\rho} \partial_{\rho} \bar{\theta}^{\eta\sigma} F_{\eta\sigma}. \] (118)

Now consider the equations of motion for the bare action \((14)\), which in the present context amount to \( \partial^{\rho} F_{\mu\nu} = 0 = \partial^{\rho} \partial_{\rho} F_{\mu\nu} \) up to possibly corrections of order \( \theta \), i.e. the vacuum Maxwell equations for the flat metric \( \bar{g}_{\mu\nu} \). As pointed out in \([3]\), this implies that the vacuum geometries are Ricci-flat to leading nontrivial order,
\[ R_{ab}[\tilde{G}] = 0 + O(\theta^2), \] (119)
while the general curvature tensor \( R_{\mu\nu\rho\sigma} \) is first order in \( \theta \) and does not vanish\(^{11}\). This shows that the effective metric does contain the 2 physical degrees of freedom (helicities) of gravitational waves. It is quite remarkable that \((13)\) is obtained from the bare action, without invoking the mechanism of induced gravity in section \(2.5\). Note that the cosmological constant vanishes to this order. A generalization to non-trivially embedded branes remains to be elaborated.

### 3.1 Newtonian Limit and relativistic corrections

The Newtonian limit of General Relativity corresponds to static metric perturbations of the form
\[ ds^2 = -c^2 dt^2 \left( 1 + \frac{2U}{c^2} \right) + dx^2 \left( 1 + O\left( \frac{1}{c^2} \right) \right) \] (120)
where \( \Delta_{(3)} U(x) = 4\pi G m(x) \) and \( m(x) \) denotes the mass density. Including the leading relativistic corrections, this takes the form
\[ ds^2 = -c^2 dt^2 \left( 1 + \frac{2U}{c^2} \right) + dx^2 \left( 1 - \frac{2U}{c^2} \right). \] (121)

\(^{11}\)while this is true generically, there may be particular momenta \( k^\mu \) determined by \( \theta^{\mu\nu} \) for which the corresponding “graviton” is pure gauge and hence \( R_{\mu\nu\rho\sigma} \) vanishes. This should be studied in more detail elsewhere.
It follows from the results of the previous section that (121) is reproduced correctly in the case without matter, where the vacuum equations of motion amount to $\partial x F_{cb} = 0$ resp. $R_{ab} = 0$. In the presence of matter, it was essentially shown in [4] that one can indeed obtain metrics of the form $\eta_{\mu\nu}$ for arbitrary static $m(x)$. We will re-analyze this issue in more detail here. It will turn out that even though one can find a metric $\tilde{G}_{\mu\nu}$ corresponding to (120) in the case $D = 4$ without nontrivial embeddings, the relativistic corrections of General Relativity in (121) are not correctly reproduced in the presence of matter. In particular, it appears that the Schwarzschild solution is not correctly reproduced in this minimal framework. This is not a real problem for emergent NC gravity, since we concluded on different grounds above that the model with $D = 10$ and $N = 4$ SUSY is required for a consistent model at the quantum level. Therefore realistic solutions for point masses should be realized by nontrivially embedded branes.

In order to reproduce (120) with $\tilde{G}_{\mu\nu}$, we have to find $U(1)$ gauge fields $A_\mu$ on the Moyal-Weyl quantum plane with the desired $h_{\mu\nu}$, which is (115)

$$h_{\mu\nu} = \bar{\vartheta}\eta_{\mu\nu} F_{\mu\nu} - \bar{\vartheta} \eta_{\mu\nu} \vartheta^\nu F_{\mu\nu}. \quad (122)$$

Choose coordinates where $\bar{g}_{\mu\nu} = (-1, 1, 1, 1)$ as discussed above, so that $x^0 = ct$ corresponds to the time, and $\bar{\vartheta}_{\mu\nu}$ has the form (107). Then

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad (123)$$

gives

$$h_{\mu\nu} = \bar{\rho} \vartheta \begin{pmatrix} -2E_3 & B_2 - E_2 & -B_1 + E_1 & 0 \\ B_2 - E_2 & -2B_3 & 0 & B_1 + E_1 \\ -B_1 + E_1 & 0 & -2B_3 & B_2 + E_2 \\ 0 & B_1 + E_1 & B_2 + E_2 & 2E_3 \end{pmatrix} \quad (124)$$

which is the most general metric fluctuation available. Let us denote its trace with

$$h(x) = \bar{g}^{\mu\nu} h_{\mu\nu}(x) = 4\theta(E_3 - B_3). \quad (125)$$

The physical graviton is the traceless version,

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4} g_{\mu\nu} h = \bar{\rho} \bar{\theta} \begin{pmatrix} -(B_3 + E_3) & B_2 - E_2 & -B_1 + E_1 & 0 \\ B_2 - E_2 & -(B_3 + E_3) & 0 & B_1 + E_1 \\ -B_1 + E_1 & 0 & -(B_3 + E_3) & B_2 + E_2 \\ 0 & B_1 + E_1 & B_2 + E_2 & B_3 + E_3 \end{pmatrix}. \quad (126)$$

As shown above, $R_{\mu\nu} [\tilde{G}] = 0$ holds if $\tilde{E}$ and $\tilde{B}$ satisfy the Maxwell equations without source. We would like this to be the case where the mass density $m(x)$ vanishes.
3.1.1 Static point charges

To gain some intuition, we first consider a static point charges with electric and magnetic charge at the origin, and determine the corresponding effective geometry. Thus consider electromagnetic fields given by

\[ E_i = q_E \frac{1}{r^3} x_i, \quad B_i = q_M \frac{1}{r^3} x_i. \]  

(127)

Then the metric fluctuation (126) is

\[ \tilde{h}_{\mu\nu} = \frac{1}{r^3} \bar{\rho}\bar{\theta} \begin{pmatrix}
-(q_M + q_E)x_3 & (q_M - q_E)x_2 & -(q_M - q_E)x_1 & 0 \\
(q_M - q_E)x_2 & -(q_E + q_M)x_3 & 0 & (q_M + q_E)x_1 \\
-(q_M - q_E)x_1 & 0 & -(q_E + q_M)x_3 & (q_M + q_E)x_2 \\
0 & (q_M + q_E)x_1 & (q_M + q_E)x_2 & (q_M + q_E)x_3
\end{pmatrix}, \]  

(128)

which in the “extremal” case \( q_M = q_E =: q \) is

\[ \tilde{h}_{\mu\nu} = \frac{2q\bar{\rho}\bar{\theta}}{r^3} \begin{pmatrix}
-x_3 & 0 & 0 & 0 \\
0 & -x_3 & 0 & x_1 \\
0 & 0 & -x_3 & x_2 \\
0 & x_1 & x_2 & x_3
\end{pmatrix}. \]  

(129)

This can be brought into diagonal form using the diffeomorphism \( \xi^\mu = 2q\bar{\rho}\bar{\theta}(0, 0, 0, \frac{1}{r}) \), which gives the linearized metric

\[ \tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu \xi_b + \partial_\nu \xi_a = \frac{2q\bar{\rho}\bar{\theta}}{r^3} \begin{pmatrix}
-x_3 & 0 & 0 & 0 \\
0 & -x_3 & 0 & 0 \\
0 & 0 & -x_3 & 0 \\
0 & 0 & 0 & -x_3
\end{pmatrix}. \]  

(130)

This has indeed the form of (121) of a Ricci-flat metric for \( \vec{x} \neq 0 \), with Newtonian potential \( U = \frac{q\bar{\rho}\bar{\theta}}{r} x_3 \sim \partial_3 \frac{1}{r} \) which is harmonic away from the origin. However, the corresponding “mass” distribution

\[ m(x) \sim \Delta U \sim \partial_3 \delta^{(3)}(\vec{x}) \]  

(131)

is not positive. This is not what we want; it corresponds to an unphysical gravitational dipole rather than a point mass. Note however that there is no charged field in the model for this \( U(1) \), hence this is only a toy configuration which is not expected to play any physical role. Moreover, it is not expected to be a solution of the e.o.m. at the quantum level.

This result is easy to understand: Since the electromagnetic field of a localized charge distribution decays as \( \frac{1}{r^2} \), the corresponding gravitational field also decays like \( \frac{1}{r^4} \) at the linearized level. The correct \( U(r) \sim \frac{1}{r} \) gravitational potential for a point mass can be
recovered either at the cost of violating relativistic corrections as elaborated next, or – presumably – by a nontrivial deformation of the brane embedding due to the point mass, governed by the induced gravitational action. The trace-$U(1)$ modes under consideration here are to be interpreted as gravitational waves.

3.1.2 General mass distributions

Now consider more generally the static case

\[ \tilde{h}_{0i} = 0, \quad \partial_0 \tilde{h}_{\mu\nu} = 0, \]  

(132)

which amounts to

\[ B_2 = E_2, \quad B_1 = E_1. \]  

(133)

The metric fluctuation (126) is then

\[ \tilde{h}_{\mu\nu} = \bar{\rho} \theta \begin{pmatrix} -E_3 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 2E_1 \\ 0 & 2E_1 & B_2 & 0 \\ 0 & 0 & 0 & B_3 \end{pmatrix}. \]  

(134)

To determine the covariant coordinates \( x^\mu = \tilde{x}^\mu - \tilde{\theta}^\mu_\nu A_\nu \) (111), we have to fix a gauge. A natural gauge choice in the present context is the “static gauge” \( \partial_0 A_\mu = 0 \), so that

\[ \vec{E} = -\partial_i A_0(\vec{x}), \quad \vec{B} = \vec{\nabla} \times \vec{A}(\vec{x}). \]  

(135)

The metric can be brought into diagonal form using the diffeomorphism \( x^{\mu'} = x^{\mu} + \xi^{\mu}(x) \) with \( \xi^{\mu}(x) = 2\rho\theta(0, 0, 0, A_0(x)) \), which gives

\[ \tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \bar{\rho} \theta \begin{pmatrix} -(B_3 + E_3) & 0 & 0 & 0 \\ 0 & -B_3 & 0 & 0 \\ 0 & 0 & -B_3 & 0 \\ 0 & 0 & 0 & B_3 - 3E_3 \end{pmatrix}, \]  

(136)

where we used (125) to write \( E_3 = B_3 + \frac{1}{4\theta} h \), hence

\[ \vec{B} = \vec{E} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{4\theta} h(x) \end{pmatrix}. \]  

(137)

\(^{12}\)Alternatively one can also impose e.g. \( A_0 = 0 \), but then the \( A_i \) become \( x^0 \)-dependent \[1\].
For $h(x) = 0$, this has precisely the form of the metric (121) with Newtonian potential

$$2U = \bar{\rho}\theta (B_3 + E_3) = -\bar{\rho}(2\theta \partial_3 A_0 + \frac{1}{4}h)$$

(138)

including leading relativistic corrections, while for $h \neq 0$ it agrees with the Newtonian limit (120) but the last term in (136) violates the relativistic corrections.

To determine $A_\mu$ explicitly for given $U(x)$, we act with $\partial_3$ on (138) and combine the result with the Bianchi identity for $\vec{B}$

$$0 = \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{E} - \frac{1}{4\theta} \partial_3 h(x) = -\Delta A_0(x) - \frac{1}{4\theta} \partial_3 h(x).$$

(139)

This gives

$$\partial_3 \frac{2U(x)}{\bar{\rho}\theta} = -2\partial_3^2 A_0 + \Delta A_0(x) = (-\partial_3^2 + \partial_1^2 + \partial_2^2)A_0,$$

(140)

which can be solved for $A_0$ as

$$A_0(x) = \frac{2}{\bar{\rho}\theta} \int d^3y \, G(x-y) \frac{\partial}{\partial y^3} U(y).$$

(141)

Here $G(x-y)$ is a 3-dimensional propagator, $(-\partial_3^2 + \partial_1^2 + \partial_2^2)G(x-y) = \delta^{(3)}(x-y)$. The (static) Bianchi identity for $\vec{E}$

$$0 = \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{B} + \frac{1}{4\theta} (\partial_2 h(x), -\partial_1 h(x), 0)$$

(142)

then determines the conserved current

$$\vec{J} \equiv \vec{\nabla} \times \vec{B} = -\frac{1}{4\theta} (\partial_2 h, -\partial_1 h, 0), \quad \vec{\nabla} \cdot \vec{J} = 0,$$

(143)

so that $\vec{B} = \vec{\nabla} \times \vec{A}$ can be solved for $\vec{A}$. Therefore for an arbitrary given potential $U(\vec{x})$, we can indeed find $A_\mu$ corresponding to a static metric fluctuation $h_{\mu\nu}$ which reproduces the Newtonian potential $U(\vec{x})$. There is some freedom in the solution of $A_0$ (140), and $h(x)$ is (almost) determined by (139). Note that even though a preferred direction $x^3$ is singled out through $\theta^{\mu\nu}$ and $x^0$, this merely amounts to preferred coordinates $x^\mu$ for the desired geometry.

Let us consider the vacuum case $\Delta U(x) = 0$ in more detail. By integrating (138), we can obtain a Ricci-flat solution with $A_i = 0$, $h(x) = 0,$

$$A_0(x) = \frac{1}{\theta} \int_0^{x^3} ds \, U(x^1, x^2, s) + H(x^1, x^2)$$

(144)

which solves $\Delta A_0 = 0$ provided $(\partial_1^2 + \partial_2^2)H = -\frac{1}{\theta} \frac{\partial}{\partial x^3} U(x)|_{x^3=0}.$
Now consider the case with non-vanishing mass distribution $\Delta U(x) = 4\pi G m(x) \neq 0$ in a region of space near the origin. Then the presence of a (hyperbolic!) propagator in (141) implies that $A_0$ will not be harmonic even in regions where the mass density $m(x)$ vanishes. This in turn implies e.g. through (139) that $h(x) \neq 0$, and the leading relativistic corrections to Newtonian gravity are not correctly reproduced. In other words, while it is possible to reproduce e.g. the Newtonian potential $U(r) \sim \frac{1}{r}$ for a point mass, it implies in the electromagnetic picture a nontrivial charge density which is not localized at the origin, leading to a violation of Ricci flatness. This is in accord with the results of Section 3.1.1.

We conclude that the consideration of nontrivially embedded branes in matrix models with extra dimensions is required in order to obtain a gravity theory which reproduces the leading relativistic corrections of General Relativity in the presence of masses. This is in accord with the results of Section 2.6 that $N = 4$ SUSY is required at the quantum level, leading to the $D = 10$ IKKT model and hence to embedded branes. Since the embedding degrees of freedom can be viewed as scalar fields, their quantization is straightforward, and expected to be well-behaved.

4 Solution with spherically symmetric Poisson structure

In this section, we discuss an exact but unphysical solution of the tree-level e.o.m. (50), in order to illustrate nontrivial geometries and the covariant formulation. The solution is unphysical, because the induced gravity action is not taken into account; this will be explored elsewhere. We start from the covariant e.o.m. (51) for the Poisson structure $\theta^{\mu\nu}$

$$\tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma \theta^{-1}_{\rho\sigma} = -\tilde{G}^{\gamma\eta}(x) \theta^{-1}_{\rho\sigma} \partial_\gamma \sigma + e^{-2\sigma} \tilde{G}^{\mu\nu} \theta^{\mu\gamma} \partial_\gamma \eta(x)$$

(145)

and look for a solution $\theta^{-1}_{\mu\nu}$ which is static and spherically symmetric. This Ansatz is actually not appropriate in order to look e.g. for a Schwarzschild-like solution; for that purpose one should presumably look for a deformation of the flat Moyal-Weyl solution, where $\theta^{\mu\nu}$ breaks rotational invariance. Nevertheless, finding a nontrivial exact solution of (143) is certainly instructive.

To illustrate the case of nontrivially embedded branes, consider a 4-dimensional brane $\mathcal{M}^4 \subset \mathbb{R}^5$ in the matrix model (23), with Cartesian coordinates $x^a = (x^\mu, \phi)$ given by the semi-classical limit of the matrices $X^a$. We also use the radial variable $r^2 = x_1^2 + x_2^2 + x_3^2$ and the Euclidean time $\tau = x_4$. This leads to the following spherically symmetric closed 2-form

$$\theta^{-1} = \omega^{(2)} + f(r, \tau) dr \wedge d\tau$$

(146)

where $\omega^{(2)} = \sin(\theta) d\theta \wedge d\phi$ can be interpreted as field of a magnetic monopole on $S^2$, which is singular at the origin. This will define a spherically symmetric metric $G_{\mu\nu}(x)$ if the
induced metric $g_{\mu\nu}(x)$ on $M^4$ is spherically symmetric. Hence we consider an embedding function $\phi = \phi(r)$, so that

$$
g_{\mu\nu}(x) = \delta_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi$$

$$
ds_g^2 = r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) + (1 + \phi'^2(r))dr^2 + d\tau^2$$  \hspace{1cm} (147)

or

$$
g_{rr} = 1 + \phi'^2(r), \quad g_{\tau\tau} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2(\theta).$$  \hspace{1cm} (148)

Since $\theta^{-1} = \sin(\theta)$, this gives

$$
ds^2_G = r^{-2}(d\theta^2 + \sin^2(\theta)d\varphi^2) + f(r, \tau)^2 \left((1 + \phi'^2(r))^{-1}d\tau^2 + dr^2\right).$$  \hspace{1cm} (149)

The effective metric $\tilde{G}_{\mu\nu} = e^\sigma G_{\mu\nu}$ is

$$
ds^2_{\tilde{G}} = \frac{\sqrt{1 + \phi'^2(r)}}{f(r, \tau)} (d\theta^2 + \sin^2(\theta)d\varphi^2) + r^2 f(r, \tau)^2 \left(\frac{1}{\sqrt{1 + \phi'^2(r)}} d\tau^2 + \sqrt{1 + \phi'^2(r)} dr^2\right)$$  \hspace{1cm} (150)

where

$$e^{-\sigma} = \frac{|g_{ab}|^{1/2}}{|g_{ab}|^{1/2}} = \frac{f(r, \tau)}{r^2 \sqrt{1 + \phi'^2(r)}}$$  \hspace{1cm} (151)

and

$$\eta = \frac{1}{4} G^{ab} g_{ab} = \frac{1}{2} \left(\frac{1 + \phi'^2(r)}{f^2(r, \tau)} + r^4\right).$$  \hspace{1cm} (152)

We could now define

$$f(r, \tau) = r^{-2},$$

$$\sqrt{1 + \phi'^2(r)} = \left(1 - \frac{R_s}{R}\right)^{-1},$$

$$R^2 \left(1 - \frac{R_s}{R}\right) = r^2$$  \hspace{1cm} (153)

which reproduces the (Euclidean) Schwarzschild metric. However, we are not free to choose the embedding function $\phi(r)$, which must satisfy an e.o.m. which for the bare matrix model is given by $\Delta_{\tilde{G}} \phi = 0$ (140), modified by the quantum effective action, cf. (104). Therefore the above metric is only an illustration how nontrivial geometries may be realized. We leave this for future work, and proceed to give an illustrative solution only for the case of flat embedding $\phi = 0$ resp. $D = 4$.

\[\text{A seemingly more general } \phi(r, \tau) \text{ could be reduced to the above through a redefinition of } \tau\]
Flat embedding $\phi = 0$, or $D = 4$. Now consider the purely 4D case with $\phi = 0$. The effective metric (150) then becomes

$$\tilde{G}_{ab} = \frac{1}{f(r, \tau)} \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) + r^2 f(r, \tau) \left( d\tau^2 + dr^2 \right)$$

(154)

where

$$e^{-\sigma} = \frac{f(r, \tau)}{r^2}, \quad \eta = \frac{1}{2} \left( \frac{1}{f^2(r, \tau)} + r^4 \right).$$

(155)

Computing the Christoffel symbols for the metric (154) gives

$$\tilde{\Gamma}_{rr} = r^{-1} + \frac{1}{2} f(r, \tau)^{-1} \partial_r f(r, \tau) = -\tilde{\Gamma}_{rr} = \tilde{\Gamma}_{rr}$$

$$\tilde{\Gamma}_{\tau\tau} = \frac{1}{2} f(r, \tau)^{-1} \partial_r f(r, \tau) = -\tilde{\Gamma}_{\tau\tau} = -\tilde{\Gamma}_{\tau\tau}$$

$$\tilde{\Gamma}_r = -\tilde{G}^{rr} f \partial_r f^{-1}, \quad \tilde{\Gamma}_\tau = -\tilde{G}^{rr} f \partial_\tau f^{-1}.$$ (156)

The covariant Maxwell equations (145) for the $\tau$ component is then

$$\tilde{G}^{rr} \partial_r f - \tilde{\Gamma}^r f + \tilde{G}^{\tau r} \tilde{\Gamma}_r f - \tilde{G}^{\tau r} \tilde{\Gamma}_\tau f = e^{-2\sigma} \tilde{G}_{\tau\tau} \theta^{r r} \partial_\tau \eta(x) - \tilde{G}^{rr} \theta^{-1}_r \partial_\tau \sigma$$

(157)

which gives

$$f(r, \tau)^{-1} \partial_r f(r, \tau) = -2 f^2 r^3$$

(158)

using (155). Similarly, the $r$ component of (145)

$$-\tilde{G}^{rr} \partial_r f + \tilde{\Gamma}^r f - \tilde{G}^{\tau r} \tilde{\Gamma}_r f + \tilde{G}^{\tau r} \tilde{\Gamma}_\tau f = -e^{-2\sigma} \tilde{G}_{rr} f(r, \tau)^{-1} \partial_r \eta(x) + \tilde{G}^{rr} f(r, \tau) \partial_\tau \sigma$$

(159)

gives

$$f(r, \tau)^{-1} \partial_r f(r, \tau) = -e^{-2\sigma} r^4 \partial_r \eta(x) + \partial_\tau \sigma = 0$$

(160)

which implies $f = f(r)$. Together with (158) we obtain

$$f(r) = \frac{1}{\sqrt{r^4 + c}}.$$ (161)

To make contact with the standard notation, denote

$$R^2 = f^{-1} = \sqrt{r^4 + c},$$

$$r^2 f = \frac{r^2}{R^2} = \sqrt{1 - \frac{c}{R^4}}.$$ (162)

Then $R^3 dR = r^3 dr$, hence

$$\frac{dr}{dR} = \frac{R^3}{r^3}$$

(163)

and we obtain

$$r^2 f dr^2 = \frac{r^2}{R^2} \left( \frac{dr}{dR} \right)^2 dR^2 = \frac{1}{1 - \frac{c}{R^4}} dR^2$$

(164)
Therefore the effective metric \((154)\) becomes
\[
ds^2_G = R^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) + \sqrt{1 - \frac{c}{R^4}} d\tau^2 + \frac{1}{1 - \frac{c}{R^4}} dR^2
\] (165)
which is flat for \(c = 0\). This metric, in particular the radial dependence is quite strange. However this is not surprising, because it is a solution of the "bare" equations of motion only, without taking into account the induced gravitational action. Therefore this serves merely as an illustration how nontrivial solutions can arise. Since the \(\theta^{-1}_{\mu\nu}\) we used is far from the Moyal-Weyl plane, the results of Section 3 do not apply, and there is no contradiction with the fact that \((163)\) is not Ricci-flat. Indeed, note that
\[
e^{-\sigma} = \frac{1}{r^2 R^2} \sim \frac{1}{R^4}
\] (166)
which is far from the Moyal-Weyl case where \(e^{-\sigma} = \text{const}\). In particular, even the solution \(f = r^{-2}\) with flat \(G_{\mu\nu}\) is very different from the Moyal-Weyl plane. This shows how the same geometry may be realized in different ways. These different realizations however will be distinguished once the induced gravitational action is taken into account, which includes in particular an action for the dilaton-like field \(\sigma\) \([11]\).

**Cartesian coordinates.** To clarify the above solution, reconsider the spherically symmetric symplectic form \((146)\) for \(\phi = 0\). The most general rotationally invariant antisymmetric tensor in 3+1 (Euclidean) dimensions has the form
\[
\theta_{0i}^{-1} = x_i f(r, \tau), \quad \theta^{-1}_{ij} = \varepsilon_{ijk} x_k g(r, \tau)
\] (167)
which is actually also invariant under \(SO(3)\). The corresponding 2-form
\[
\theta^{-1} = f(r, \tau) x_i d\tau d x_i + g(r, \tau) \varepsilon_{ijk} x_k d x_i d x_j
\] (168)
is closed if and only if
\[
g(r) = r^{-3},
\] (169)
recovering \((146)\)
\[
\theta^{-1} = f(r, \tau) x_i d\tau d x_i + r^{-3} \varepsilon_{ijk} x_k d x_i d x_j
\] (170)
for an aritrary function \(f(r, \tau)\). The corresponding effective metric is
\[
G_{00} = \theta^{-1}_{0i} \delta^{ij} \theta^{-1}_{0j} = r^2 f^2
\]
\[
G_{ii} = \theta^{-1}_{ik} \delta^{kl} \theta^{-1}_{il} + \theta^{-1}_{i0} \delta^{ij0} \theta^{-1}_{i0} = r^{-6} \delta_{ii} r^2 + x_i x_i (f^2 - r^{-6}).
\] (171)
For \(f = r^{-3}\), we obtain
\[
G_{\mu\nu} = \frac{1}{r^4} \begin{pmatrix}
1 & 0 \\
0 & \delta_{ii}
\end{pmatrix},
\] (172)
so that $\tilde{G}_{\mu\nu}$ reproduces the flat solution found above (165) for $c = 0$. Note again that the corresponding $\theta^{-1}_{\mu\nu}$ is very different from the Moyal-Weyl case where $\theta^{-1}_{\mu\nu} = \text{const.}$

We conclude that in order to obtain realistic metrics such as the Schwarzschild-metric, different nontrivial embeddings must be used, solving the equations of motion derived from the combined bare action plus induced gravitational action.

5 Symmetries and conservation laws

The basic matrix model (23) is invariant under the D-dimensional Poincaré group, consisting of translations

$$X^a \rightarrow X^a + c^a, \quad c^a \in \mathbb{R}$$

and rotations resp. Lorentz transformations

$$X^a \rightarrow \Lambda^a_b X^b, \quad \Lambda^a_b \in SO(D - 1, 1).$$

These symmetries lead to conservation laws according to Noether's theorem, which are elaborated below for the case of translations; see also [2] for a related discussion. Adapting a standard trick, we consider the following non-constant infinitesimal transformation $X^a \rightarrow X^a + \delta X^a$ for

$$\delta X^a = \{X^b, [X^a, \varepsilon^b']\} g_{ab}$$

where $\varepsilon^b$ is an arbitrary matrix, and $g_{ab} = \delta_{ab}$ or $g_{ab} = \eta_{ab}$. As elaborated in Appendix C, this leads to

$$\frac{1}{4} \delta S_{\text{YM}} = -T r \varepsilon^c [X^a, \tilde{T}^{a'c}] g_{a'a'} g_{cc'}$$

for arbitrary $\varepsilon^a$, where

$$\tilde{T}^{ab} = [X^a, X^c][X^b, X^{c'}] g_{cc'} + [X^b, X^c][X^a, X^{c'}] g_{cc'} - \frac{1}{2} g^{ab}[X^d, X^c][X^{d'}, X^{c'}] g_{dd'} g_{cc'}$$

is the matrix - “energy-momentum tensor”. Since (176) vanishes on-shell, the conservation law

$$[X^a, \tilde{T}^{a'c}] g_{a'a'} = 0$$

follows. This can of course also be checked directly using $[X^a, [X^b, X^c]] g_{a'a'} = 0$. Moreover, since it is a consequence of a symmetry of the action, this will survive quantization in the form of a Ward identity. Indeed it is easy to check that (173) defines a measure-preserving vector field on the space of matrices $X^a$, so that (178) also holds under the matrix path integral i.e. upon quantization; there will be additional terms in the presence of matter or in correlators. Note that the indices of the “tensor” $\tilde{T}^{ab}$ range from 1 to $D$, including transversal components. A covariant form of these conservation laws and their physical meaning in the context of gravity remains to be elaborated.
A very similar conserved energy-momentum tensor was obtained previously in \cite{54,55} in the context of NC gauge theory on the Moyal-Weyl quantum plane. In that case, it was possible to find a suitable gauge invariant version of $\tilde{T}^{ab}$ which satisfies a standard conservation law \cite{54}. The present result is somewhat different since (178) is obtained for NC spaces with general $\theta^{\mu\nu}(x)$, involving also components which are transversal to the brane. Moreover, the meaning of gauge invariance versus locality is somewhat different (and not entirely clear) in the present context; for example, $U(1)$ gauge transformations are now interpreted as symplectomorphisms. In any case, a similar “local” version of (177) involving Wilson lines might help to clarify its interpretation. An analogous energy-momentum tensor in the context of the BFSS matrix model was also found in \cite{56}.

6 Discussion and outlook

We present in this paper a general framework for studying emergent gravity in the context of Yang-Mills type matrix models, on generic noncommutative branes embedded in $\mathbb{R}^P$. The basic message is that the dynamics of fields on the brane is governed by an effective metric in the semi-classical limit, which depends both on the embedding and the Poisson or noncommutative structure on the brane. The resulting geometry is dynamical, governed by the matrix model and its induced effective action which includes in particular the Einstein-Hilbert term. Therefore Yang-Mills matrix models contain some type of gravity theory. The results of \cite{4} are thus generalized to a richer class of geometries, setting the stage for a systematic exploration of the physical properties of the models. This necessity to consider nontrivially embedded branes in higher dimensions is shown by a detailed analysis of the Newtonian limit of the $D = 4$ model, which does not correctly reproduce the relativistic corrections to the Newtonian limit.

Matrix models such as the IKKT model therefore provide a simple and transparent mechanism for gravity, which arises from fluctuations of the basic matrix degrees of freedom, along with nonabelian gauge fields. While the IKKT model was proposed originally as non-perturbative description of IIB string theory \cite{12,57}, the progress in this and related works shifts the emphasis towards the consideration of general noncommutative branes and geometries, which promise to provide the physically relevant backgrounds. They appear to be simpler and more natural in this context than classical spaces and geometries, the essential difference being the effective metric which involves the noncommutative resp. Poisson structure. Similar considerations should apply also to time-dependent matrix models such as the BFSS model \cite{21}.

There are some important differences to General Relativity. The essential point is that the metric is not a fundamental degree of freedom, but arises effectively as described above. This leads to important simplifications for the quantization: first, the issue of gauge fixing is much simpler, involving degrees of freedom which can be viewed as scalar and gauge fields in a NC background. Second, it is not the Einstein-Hilbert action which is quantized,
rather the matrix model action, which is similar to a Yang-Mills action. This allows to compute e.g. the one-loop effective action in a straightforward way, which boils down to computations in a NC gauge theory or the use of standard heat-kernel expansions under certain conditions. Most remarkably, in the case of maximal supersymmetry (i.e. the IKKT model in $D = 10$) the model can be expected to be finite, leading to the identification of the Planck scale with the scale of $N = 4$ SUSY breaking. This suggests that the IKKT model may provide a well-defined quantum theory of fundamental interactions including gravity.

Remarkably, emergent NC gravity appears to provide a mechanism for avoiding the cosmological constant problem, which is explained in the case of 4-dimensional branes. Again, the full significance can only be judged once near-realistic solutions are found and understood. Here the compactification of higher-dimensional NC branes as indicated may turn out to be important, which is motivated also from particle physics, providing a mechanism for gauge symmetry breaking and fermionic zero modes. A full discussion of emergent gravity in such cases is a challenging subject for future work.

This paper contains only semi-classical considerations. These are the leading terms in a systematic expansion in $\theta^{\mu\nu}$, which should be elaborated eventually. This can be achieved using the Seiberg-Witten map [27], which allows to systematically re-write a noncommutative (gauge) theory in terms of a commutative one. While this was used in [4] to obtain the semi-classical limit of the nonabelian gauge fields in emergent gravity, it is not part of the definition of the model: it is simply – by definition – a natural way to extract the physical content of a NC model. In principle, the quantization should be done on the level of the matrix model, and its effective action can then be interpreted in a commutative language. For example, the issue of UV/IR mixing is resolved here not through the Seiberg-Witten map but through its proper interpretation in terms of gravity [4, 11]. At least in the case of (softly broken) $N = 4$ SUSY, one may hope to resolve similarly the issues of unitarity and Wick rotation. All this clearly requires much more work.

Let us summarize the main arguments supporting emergent NC gravity as described by $D$-dimensional matrix models:

- The models do describe some gravity theory on 4-dimensional NC branes, since matter couples to a universal metric (up to possibly density factors). Gauge fields and gravity are naturally unified.
- The class of geometries is rather rich in the case of models with $D > 4$.
- The geometry is dynamical, governed by an effective action which includes the Einstein-Hilbert term at the quantum level. The quantization is likely to be well-defined, at least for the IKKT model.
- Flat space is a solution even at the quantum level, without fine-tuning
- The models are extremely simple, without any classical-geometric prerequisites.
This certainly describes a very promising theory of gravity, the main missing item being the analog of the Schwarzschild solution. This requires to consider nontrivial embedding as shown here, and is complicated by the fact that the quantum effective action is required at least at one loop.

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Appendix A: Some identities

The following is an important identity for Poisson tensors:

\[ \partial_\mu \theta^{\mu \nu} = -\theta^{\mu \nu'} \partial_\mu \theta^{-1}_{\mu' \nu} \theta^{\nu \nu'} = \theta^{\mu \nu'} \theta^{\nu \nu'} (\partial_\mu \theta^{-1}_{\nu' \mu} + \partial_\nu \theta^{-1}_{\mu' \mu}) = -\theta^{\mu \nu'} \partial_\mu \theta^{-1}_{\nu' \nu} \theta^{\nu \nu'} - \theta^{\nu \nu'} \partial_\nu \theta^{\mu \nu'} \theta^{-1}_{\mu' \mu} = -\partial_\mu \theta^{\mu \nu} - 2 \theta^{\nu \nu'} \rho^{-1} \partial_\nu \rho \]  

noting that \( 2 \rho^{-1} \partial_\nu \rho = \partial_\nu \theta^{\mu \nu'} \theta^{-1}_{\mu' \mu} \), hence

\[ \partial_\mu (\rho \theta^{\mu \nu}) \equiv 0. \]  

On-shell vanishing of \( \tilde{\Gamma}^{\mu} \): For our restricted class of metrics, the above identity \((180)\) together with \(|\tilde{G}^{\mu \nu}|^{1/2} = \rho e^\sigma \) \((54)\) implies

\[ \tilde{\Gamma}^{\mu} = -|\tilde{G}^{\mu \sigma}|^{-1/2} \partial_\nu (\tilde{G}^{\nu \mu} |\tilde{G}^{\sigma \nu}|^{1/2}) = -\frac{1}{\rho} e^{-\sigma} \partial_\nu (G^{\mu \nu}) = \frac{1}{\rho} e^{-\sigma} \partial_\nu (\rho \theta^{\nu \nu'} \theta^{\mu \nu'} g_{\mu' \nu'}(x)) = -e^{-\sigma} \theta^{\nu \nu'} \partial_\nu (\theta^{\mu \nu'} g_{\mu' \nu'}(x)) \text{ e.o.m.} \equiv 0 \]  

using the e.o.m. \((48)\) for \( X^\mu \). This can also be seen from \((44)\). Therefore the equations of motion for \( X^\mu \) are equivalent to \( \tilde{\Gamma}^{\mu} = 0 \). From the point of view of General Relativity, this would be interpreted rather as a gauge-fixing condition. This is not the case here due to the constrained class of metrics.

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Appendix B: Derivation of the covariant e.o.m.

Consider

\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_{\gamma} \theta^{-1}_{\eta\nu} = \tilde{G}^{\gamma\eta}(x) \left( \partial_{\gamma} \theta^{-1}_{\eta\nu} - \tilde{\Gamma}^{\rho}_{\gamma\nu} \theta^{-1}_{\rho\mu} - \tilde{\Gamma}^{\rho}_{\gamma\nu} \theta^{-1}_{\rho\eta} \right) \]

\[ = \tilde{G}^{\gamma\eta} \partial_{\gamma} \theta^{-1}_{\eta\nu} - \tilde{\Gamma}^{\rho}_{\gamma\nu} \theta^{-1}_{\rho\mu} - \tilde{G}^{\gamma\eta} \tilde{\Gamma}^{\rho}_{\gamma\nu} \theta^{-1}_{\rho\eta} \]

(182)

where

\[ \tilde{\Gamma}^{\gamma}_{\gamma\nu} = \tilde{G}^{ab}_{\gamma} \tilde{G}^{\gamma}_{ab} = -\frac{1}{\sqrt{\tilde{G}_{ab}}} \partial_{\rho}(\tilde{G}^{\gamma\rho} \sqrt{\tilde{G}_{ab}}). \]

(183)

Using (B6)

\[ \tilde{G}^{\mu\nu} = e^{-\sigma} G^{\mu\nu} = \theta^\mu \theta^\nu \theta'(x) \theta^{\mu'} \theta^{\nu'}(x) \]

where

\[ \tilde{g}_{\mu' \nu'}(x) \equiv e^{-\sigma} g_{\mu' \nu'}(x), \]

we can write

\[ \tilde{G}^{\gamma\eta} \tilde{\Gamma}^{\delta\eta}_{\gamma\nu} \theta_{\eta\delta}^{-1} = \frac{1}{2} \tilde{G}^{\gamma\eta} \tilde{G}^{\delta\xi} \theta_{\eta\delta}^{-1} \left( \partial_{\gamma} \tilde{G}_{\rho\xi} + \partial_{\rho} \tilde{G}_{\gamma\xi} - \partial_{\xi} \tilde{G}_{\gamma\rho} \right) \]

\[ = \frac{1}{2} \tilde{\theta}^{\gamma\rho} \left( \partial_{\gamma} \tilde{G}_{\rho\xi} + \partial_{\rho} \tilde{G}_{\gamma\xi} - \partial_{\xi} \tilde{G}_{\gamma\rho} \right) \]

\[ = \tilde{\theta}^{\gamma\rho} \partial_{\gamma} \tilde{G}_{\rho\xi} = \tilde{G}^{\gamma\eta} \tilde{G}^{\rho\xi} \theta_{\eta\rho}^{-1} \partial_{\gamma} \tilde{G}_{\rho\xi} \]

\[ = -\tilde{G}^{\gamma\eta} \theta_{\eta\rho}^{-1} \tilde{G}_{\rho\xi} \partial_{\gamma}(\theta^{\rho\xi} \tilde{g}_{\mu' \nu'}) \theta^{\delta\rho} + \theta^{\rho\xi} \tilde{g}_{\mu' \nu'} \partial_{\gamma} \theta^{\delta\rho} \]

\[ = -\tilde{G}^{\gamma\eta} \tilde{G}_{\rho\xi} \tilde{g}_{\mu' \xi} \partial_{\gamma} \theta^{\rho\xi} - \tilde{G}^{\gamma\eta} \theta^{\rho\xi} \tilde{G}_{\rho\xi} \tilde{g}_{\mu' \xi} + \tilde{G}^{\gamma\eta} \theta_{\eta\rho}^{-1} \]

(186)

since \( \tilde{\theta}^{\gamma\rho} := \tilde{G}^{\gamma\eta} \tilde{G}^{\rho\delta} \theta_{\eta\delta}^{-1} \) is antisymmetric. The Jacobi identity gives

\[ \tilde{G}^{\gamma\eta} \tilde{g}_{\mu' \xi} \partial_{\gamma} \theta^{\rho\xi} = \theta^{\gamma\rho} \tilde{g}_{\gamma' \eta} \tilde{g}_{\mu' \xi} \theta^{\gamma' \rho} \theta_{\gamma' \rho} \]

\[ \theta^{\gamma\rho} \tilde{g}_{\gamma' \eta} \tilde{g}_{\mu' \xi} \theta_{\gamma' \rho} \theta_{\gamma' \rho} = \theta^{\gamma\rho} \tilde{g}_{\gamma' \eta} \tilde{g}_{\mu' \xi} \theta_{\gamma' \rho} \theta_{\gamma' \rho} \]

\[ = \theta^{\gamma\rho} \tilde{g}_{\gamma' \eta} \tilde{g}_{\mu' \xi} \theta_{\gamma' \rho} \theta_{\gamma' \rho} - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu' \xi} \partial_{\gamma} \theta^{\rho\xi} \]

(187)

hence

\[ \tilde{G}^{\gamma\eta} \tilde{g}_{\mu' \xi} \partial_{\gamma} \theta^{\rho\xi} = \frac{1}{2} \theta^{\gamma\rho} \tilde{g}_{\gamma' \eta} \theta^{\gamma' \rho} \theta_{\gamma' \rho}. \]

(188)

Finally, observe that

\[ G^{\gamma\eta} \partial_{\gamma} g_{\mu\eta} - G^{\gamma\eta} \partial_{\mu} \delta g_{\gamma\eta} = G^{\gamma\eta} \partial_{\gamma} (\partial_{\mu} \delta g_{\eta\phi}) - G^{\gamma\eta} \partial_{\mu} (\partial_{\gamma} \delta g_{\eta\phi}) \]

\[ = \partial_{\mu} \delta g_{\gamma\eta} G^{\gamma\eta} + G^{\gamma\eta} \partial_{\gamma} \partial_{\mu} \delta g_{\eta\phi} - G^{\gamma\eta} \partial_{\mu} \partial_{\gamma} \delta g_{\eta\phi} - G^{\gamma\eta} \partial_{\gamma} \partial_{\mu} \delta g_{\eta\phi} \]

\[ = \partial_{\mu} \delta g_{\gamma\eta} G^{\gamma\eta} \partial_{\gamma} \partial_{\mu} \delta g_{\eta\phi} - \frac{1}{2} G^{\gamma\eta} \partial_{\mu} \delta g_{\gamma\eta} \]

(189)
hence
\[ G^{\gamma\eta} \partial_\gamma \delta g_{\mu\eta} = \frac{1}{2} G^{\gamma\eta} \partial_\mu \delta g_{\gamma\eta} + \partial_\mu \phi G^{\gamma\eta} \partial_\gamma \partial_\eta \phi \] (190)

and
\[ \tilde{G}^{\gamma\eta} \partial_\gamma g_{\mu\eta} = \frac{1}{2} \tilde{G}^{\gamma\eta} \partial_\mu g_{\gamma\eta} + \partial_\mu \phi \tilde{G}^{\gamma\eta} \partial_\gamma \partial_\eta \phi . \] (191)

Therefore
\[ \tilde{G}^{\gamma\eta} \partial_\gamma \tilde{g}_{\mu\eta} = \frac{1}{2} \tilde{G}^{\gamma\eta} \partial_\mu \tilde{g}_{\gamma\eta} + e^{-\sigma} \partial_\mu \phi \tilde{G}^{\gamma\eta} \partial_\gamma \partial_\eta \phi - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \sigma + \frac{1}{2} \tilde{G}^{\gamma\eta} \tilde{g}_{\gamma\eta} \partial_\mu \sigma \] (192)

and we obtain
\[ \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \theta^{\rho\mu} + \tilde{G}^{\gamma\eta} \theta^{\rho\mu} \partial_\gamma \tilde{g}_{\mu\eta} = \frac{1}{2} \theta^{\rho\gamma} \tilde{g}_{\gamma\eta} \theta^{\rho\eta} \tilde{g}_{\eta\rho} \partial_\gamma \theta^{\rho\mu} + \frac{1}{2} \theta^{\rho\gamma} \tilde{G}^{\mu\eta} \partial_\gamma \tilde{g}_{\mu\eta} \\
+ \theta^{\rho\mu} (e^{-\sigma} \partial_\mu \phi \tilde{G}^{\gamma\eta} \partial_\gamma \partial_\eta \phi - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \sigma + \frac{1}{2} \tilde{G}^{\gamma\eta} \tilde{g}_{\gamma\eta} \partial_\mu \sigma) \]
\[ = \theta^{\rho\mu} (\partial_\mu \tilde{\eta}(x) + e^{-\sigma} \partial_\mu \phi \tilde{G}^{\gamma\eta} \partial_\gamma \partial_\eta \phi - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \sigma + 2\tilde{\eta}(x) \partial_\mu \sigma) \]

using the scalar function
\[ \tilde{\eta}(x) = \frac{1}{4} \tilde{G}^{\mu\nu} g_{\mu\nu} = \frac{1}{4} \theta^{\mu\nu} \tilde{g}_{\mu\nu} \theta^{\rho\nu} g_{\mu\nu} \] (193)

which satisfies
\[ \partial_\gamma \tilde{\eta}(x) = \frac{1}{2} \partial_\gamma \theta^{\rho\mu} \tilde{g}_{\mu\nu} \theta^{\rho\nu} g_{\mu\nu} + \frac{1}{2} \tilde{G}^{\mu\nu} \partial_\gamma \tilde{g}_{\mu\nu} . \] (194)

Putting all this together, we obtain
\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma \theta_{\eta^{-1}} = \tilde{G}^{\gamma\eta} \partial_\gamma \theta_{\eta^{-1}} - \tilde{G}^{\gamma\eta} \tilde{\Gamma}^{\mu}_{\nu} \theta_{\eta^{-1}} - \tilde{\Gamma}^{\mu}_{\nu} \theta_{\eta^{-1}} = \tilde{G}_{\rho\nu} \left( \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \theta^{\rho\mu} + \tilde{G}^{\gamma\eta} \theta^{\rho\mu} \partial_\gamma \tilde{g}_{\mu\eta} \right) - \tilde{\Gamma}^{\mu}_{\nu} \theta_{\eta^{-1}} = \tilde{G}_{\rho\nu} \left( \theta^{\rho\mu} (\partial_\mu \tilde{\eta}(x) + e^{-\sigma} \partial_\mu \phi \tilde{G}^{\gamma\eta} \partial_\gamma \partial_\eta \phi - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \sigma + 2\tilde{\eta}(x) \partial_\mu \sigma) \right) - \tilde{\Gamma}^{\mu}_{\nu} \theta_{\eta^{-1}} . \]

This holds identically, i.e. it characterizes the constraint of the metric.

Now we take into account the equations of motion $\tilde{\Gamma}^{\mu} = 0$ (181) and $\Delta_{\tilde{G}} \phi = \tilde{G}^{\mu\nu} \partial_\nu \partial_\rho \phi = 0$ (16), which hold in the special coordinates $x^\mu$ defined by the dynamical matrices (this is why the above is non-covariant). We can then rewrite this as
\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma \theta_{\eta^{-1}} = \tilde{G}_{\rho\nu} \theta^{\rho\mu} \left( \partial_\mu \tilde{\eta}(x) - \tilde{G}^{\gamma\eta} \tilde{g}_{\mu\eta} \partial_\gamma \sigma + 2\tilde{\eta}(x) \partial_\mu \sigma \right) \]
\[ = \tilde{G}_{\rho\nu} \theta^{\rho\mu} \left( \partial_\mu \tilde{\eta}(x) + 2\tilde{\eta}(x) \partial_\mu \sigma \right) - \tilde{G}^{\gamma\eta} \theta_{\eta^{-1}} \partial_\gamma \sigma . \] (195)

Using $\eta = e^{2\sigma} \tilde{\eta}(x) = \frac{1}{4} \theta^{\mu\nu} \tilde{g}_{\mu\nu} \theta^{\rho\nu} g_{\mu\nu}$ (183), we obtain
\[ \tilde{G}^{\gamma\eta}(x) \tilde{\nabla}_\gamma (e^{\sigma} \theta_{\eta^{-1}}) = e^{\sigma} \tilde{G}_{\rho\nu} \theta^{\rho\gamma} \left( \partial_\gamma \tilde{\eta}(x) + 2\tilde{\eta}(x) \partial_\gamma \sigma \right) = e^{-\sigma} \tilde{G}_{\rho\nu} \theta^{\rho\gamma} \partial_\gamma \eta(x) \] (196)

which is (12) resp. (70). This is the covariant form of the equation of motion, independent of the choice of coordinates.
Appendix C: Derivation of (178)

We use here a short-hand notation where double upper indices are understood to be contracted with $\delta_{ab}$ or $\eta_{ab}$. Then

$$0 = -\frac{1}{4} \delta S_{YM} = Tr[\delta X^a, X^b][X^a, X^b]$$

$$= Tr\{[X^c, [X^a, \varepsilon^c]], X^b][X^a, X^b]$$

$$= Tr[X^c[X^a, \varepsilon^c] + [X^a, \varepsilon^c]X^c, X^b][X^a, X^b]$$

$$= Tr\left(X^c[[X^a, \varepsilon^c], X^b][X^a, X^b] + [[X^a, \varepsilon^c], X^b]X^c[X^a, X^b]$$

$$+ [X^a, \varepsilon^c][X^c, X^b][X^a, X^b] + [X^c, X^b][X^a, \varepsilon^c][X^a, X^b] \right)$$

$$= Tr\left(\{X^c, [[X^a, \varepsilon^c], X^b]]\}X^a, X^b\right) + \{X^a, \varepsilon^c\}[[X^c, X^b], [X^a, X^b]]\right)$$

$$= Tr\left(\frac{1}{2}\{X^c, [\varepsilon^c, X^b, [X^a, X^b]]\}X^a, X^b\right) + \{X^a, \varepsilon^c\}[[X^c, X^b], [X^a, X^b]]\right)$$

(197)

for arbitrary $\varepsilon^a$. Using $Tr\{[A, [B, C]]\}C = Tr([A, B]C^2)$ this can be written as

$$0 = Tr\left(\frac{1}{2}\varepsilon_c[X^c, [X^a, \varepsilon^c, X^b]]X^a, X^b\right) + \{X^a, \varepsilon^c\}[[X^c, X^b], [X^a, X^b]]\right)$$

$$= Tr\frac{1}{2}\varepsilon_c[X^c, [X^a, X^b]]X^a, X^b - \varepsilon_c[X^a, \{[X^c, X^b], [X^a, X^b]\}]$$

$$= Tr\varepsilon_c[X^a, \tilde{T}^{ac}] .$$

(198)

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