WHY PROPOSITIONAL QUANTIFICATION MAKES MODAL AND TEMPORAL LOGICS ON TREES ROBUSTLY HARD?

BARTOSZ BEDNARCZYK AND STÉPHANE DEMRI

Computational Logic Group, TU Dresden & University of Wroclaw
e-mail address: bartosz.bednarczyk@cs.uni.wroc.pl
Université Paris-Saclay, ENS Paris-Saclay, CNRS, LMF, 91190, Gif-sur-Yvette, France
e-mail address: demri@lsv.fr

ABSTRACT. Adding propositional quantification to the modal logics K, T or S4 is known to lead to undecidability but CTL with propositional quantification under the tree semantics (QCTL) admits a non-elementary Tower-complete satisfiability problem. We investigate the complexity of strict fragments of QCTL as well as of the modal logic K with propositional quantification under the tree semantics. More specifically, we show that QCTL restricted to the temporal operator EX is already Tower-hard, which is unexpected as EX can only enforce local properties. When QCTL restricted to EX is interpreted on N-bounded trees for some N ≥ 2, we prove that the satisfiability problem is AExp-pol-complete; AExp-pol-hardness is established by reduction from a recently introduced tiling problem, instrumental for studying the model-checking problem for interval temporal logics. As consequences of our proof method, we prove Tower-hardness of QCTL restricted to EF or to EXEF and of the well-known modal logics such as K, KD, GL, K4 and S4 with propositional quantification under a semantics based on classes of trees.

1. Introduction

Propositional quantification in modal and temporal logics. A natural way to design logics that dynamically update their models, consists in adding propositional quantification as done for example to define QBF from SAT. Propositional quantification is a very powerful feature to update models but this may have consequences in terms of computability. In the realm of modal logics [12], the paper [16] remains a quite early work adding propositional quantification and the undecidability of the propositional modal logic K (resp. T, K4 and S4) augmented with propositional quantification is first established in [25] and this is done thanks to a reduction from the second-order arithmetic. By contrast, the decidability of second-order versions of the modal logic S5 is first proved in [24, Chapter 3] (see also [25, 31]) but two S5 modalities and propositional quantification already lead to undecidability [1, 30].

Many subsequent works have dealt with second-order modal logics, see e.g. [30, 33, 56], but in the realm of temporal logics, LTL with propositional quantification (written QLTL) is

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introduced in Sistla’s PhD thesis [53] (see also [52]) and non-elementarity of the satisfiability problem is a consequence of [42]. So, QTLT satisfiability problem is decidable but with high complexity. By contrast, CTL with propositional quantification (written QCTL) already admits an undecidable satisfiability problem by [25] (as CTL captures the modal logic K) but its variant under the tree semantics (written QCTL^t) admits a non-elementary Tower-complete satisfiability problem [36, 19] (Tower is introduced in [49]). Having a tree semantics means that the formulae of QCTL^t are interpreted on computation trees obtained from the unfolding of finite (total) Kripke structures, which allows us to regain decidability (see a similar approach in [62] with a quantified version of the modal logic S4). This is a major observation from [36], partly motivated by the design of decision procedures for ATL with strategy contexts [37]. The tree semantics, as far as satisfiability is concerned, amounts to consider Kripke structures that are finite-branching trees in which all the maximal branches are infinite. This is an elegant way to regain decidability. More generally, decidability in the presence of propositional quantification can be regained when tree-like models are involved, see e.g., [4, 6, 62, 36], essentially by taking advantage of Rabin’s Theorem [46].

The modal logic K with propositional quantification from [25] is interpreted under the structure semantics, as classified in [36], but many variants of propositional quantification exist in the literature (see e.g., [44, 28, 14] and [9, 10] in the context of epistemic reasoning). Sometimes, propositional quantification is syntactically restricted in the temporal language, but its inclusion is motivated by a gain of expressive power while preserving the decidability of the reasoning tasks. By way of example, in [47], an extension of the modal μ-calculus with partial propositional quantification is introduced to perform control synthesis, whereas an extension for model-checking computer systems is also presented in [34].

Interestingly enough, propositional quantification can sometime have a more hidden presence. For instance, hybrid logics with the down-arrow operator ↓x, see e.g., [2, 3, 40, 61], can be understood as a form of propositional quantification since ↓x φ enforces that the propositional variable x holds true only at the current world (before evaluating the formula φ). In such logics, the companion formula @x ψ expresses that the unique world satisfying x also satisfied ψ; @x is a powerful operator to navigate in the structure [40] (obviously, it is related to the universal modality, see e.g., [29]).

**Our motivations.** As recalled above, the modal logic K augmented with propositional quantification is undecidable [25] and a fortiori, undecidability holds for fragments of CTL with propositional quantification. Actually, these results hold under the structure semantics but QCTL^t (tree semantics) admits Tower-complete satisfiability and model-checking problems [36]. As QCTL^t can express that every tree node has exactly one child, Tower-hardness for the satisfiability problem for QCTL^t, is a corollary of the Tower-hardness of the satisfiability problem for QLTL, see e.g., [42, 52, 20].

Given the central position of the modal logic K, surprisingly, the complexity of the satisfiability problem for K with propositional quantification under the tree semantics has never been investigated (closely related to QCTL^t & i.e. QCTL^t restricted to EX as EX corresponds to ◇ in K but with total models). QCTL^t for K is a natural and modest fragment of QCTL^t and we aim at characterising its complexity. Furthermore, the model-checking problem for QCTL^t is Tower-hard even with input Kripke structures having at most two successor worlds per world and Tower-hardness of QLTL holds with linear structures of length ω, see e.g., [52]. Thus, it is worth understanding what happens with the satisfiability
problem for $\text{QCTL}^t_X$ when the tree models are $N$-bounded (i.e. each node has at most $N$ children) for some fixed $N \geq 2$.

**Our contributions.** Given $\text{QCTL}^t$, the extension of $\text{CTL}$ with propositional quantification under the tree semantics (i.e. the models are finite-branching trees where all the maximal branches are infinite), let $\text{QCTL}^t_{\leq_N}$ be its variant in which the models are $N$-bounded, for some $N \geq 2$. We write $\text{QCTL}^t_X$ and $\text{QCTL}^t_{X \leq N}$, to denote respectively the restriction to the operator $\text{EX}$, and $\text{QCTL}^t_F$ to denote the restriction of $\text{QCTL}^t$ to $\text{EF}$.

- We prove that for all $N \geq 2$, the satisfiability problem for $\text{QCTL}^t_{X \leq N}$ is $\text{AExp}_{\text{pol}}$-complete (Section 3). $\text{AExp}_{\text{pol}}$ is the class of problems solvable by exponential-time alternating Turing machines with a polynomial number of alternations. By using a small model property and the complexity of model-checking (with upper bound $\text{AExp}_{\text{pol}}$), $\text{AExp}_{\text{pol}}$-easiness is established. As far as $\text{AExp}_{\text{pol}}$-hardness is concerned, the alternating multi-tiling problem introduced in [13] is instrumental to establish that the model-checking problem for the interval temporal logic $\text{BE}$ with regular expressions is $\text{AExp}_{\text{pol}}$-hard. As a corollary, we get that the modal logic $\mathcal{K}$ with propositional quantification interpreted on finite trees of branching degree bounded by some fixed $N \geq 2$ is also $\text{AExp}_{\text{pol}}$-complete.

- More generally and despite the modest scope of $\text{EX}$, the satisfiability problem for $\text{QCTL}^t_X$ is shown to be $\text{TOWER}$-hard (Theorem 4.8), by a uniform reduction from $k$-$\text{NExpTime}$-complete tiling problems (uniformity is with respect to $k$). The corresponding upper bound is known from [36] and it is worth noting that all the $\text{TOWER}$ upper bound results presented in this paper are based on translations into the satisfiability problem for $\text{QCTL}^t$ sometimes via intermediate decision problems, which eventually, uses Rabin’s Theorem [46] in some essential way. The hardness proof is one of the main results of the paper and amounts first to show that one can enforce that a node has a number of children equal to some tower of exponentials of height $k$ with a formula of size exponential in $k$. By contrast, checking the satisfiability status of $\text{CTL}^*$ formulae, requires only to consider tree models with branching degree bounded by the size of the formula, see e.g. [21, 20]. Once this complex construction enforcing a very high number of children is performed, the reduction from the tiling problems can be done with the help of other properties on the number of children. Hence, even though $\text{QCTL}^t_X$ under the structure semantics is undecidable [25] and the variant of $\text{QCTL}^t_X$ under the tree semantics ($\text{QCTL}^t$) is decidable by [36], the problem admits a high complexity despite the local range of $\text{EX}$.

- By adapting our proof method, we show that $\text{QCTL}^t_F$ and $\text{QCTL}^t_{XF}$ (a variant of $\text{QCTL}^t_F$ with the unique operator $\text{EXEF}$) are $\text{TOWER}$-hard too (consult Section 5.1).

- As $\text{EX}$, $\text{EF}$ and $\text{EXEF}$ correspond to the modality $\Diamond$ in several modal logics (e.g. $\text{EXEF}$ corresponds to $\Diamond$ in transitive frames), we are able to establish $\text{TOWER}$-completeness for standard modal logics with propositional quantification when interpreted on tree-like structures (see Section 5.2). For instance, as the provability logic $\mathcal{GL}$ (after Gödel and Löb) is complete for the class of finite transitive trees, i.e. the accessibility relation is the transitive closure of the child-relation in the tree, see e.g. [12], we also investigate the satisfiability problem under the finite tree semantics. We show that $\text{QCTL}^t_{XF}$ and $\text{QCTL}^t_{XF}$ (’$t$’ stands for ’finite tree semantics’) are $\text{TOWER}$-complete too. The satisfiability problem for $\mathcal{K}$ (resp. $\mathcal{GL}$) with propositional quantification under the finite (resp. transitive) tree
semantics is shown Tower-complete. Similar results are shown for KD, K4 and S4 with propositional quantification but interpreted on appropriate tree-like Kripke structures. See Section 5.2 for the details.

This paper is a revised and complete version of our conference paper [7].

2. Preliminaries

2.1. Kripke structures and computation trees. Below, we recall standard definitions about Kripke structures. Let \( AP = \{ p, q, x, y, \ldots \} \) be a countably infinite set of propositional variables. A Kripke structure \( \mathcal{K} \) is a triple \( (W, R, l) \), where \( W \) is a set of worlds, \( R \subseteq W \times W \) is a transition relation and \( l : W \rightarrow 2^{AP} \) is a labelling function. A Kripke structure \( \mathcal{K} \) is total whenever for all \( w \in W \), there is \( w' \in W \) such that \( (w, w') \in R \). Totality is a standard property for defining classes of models for temporal logics such as CTL. In the sequel, by a ‘Kripke structure’ we mean a structure according to the above definition, otherwise when arbitrary (or total, finite, etc.) Kripke structures need to be considered, we explicitly specify which classes of structures we have in mind. For instance, a tree model (resp. finite tree model) \( T = (V, E, l) \) is an arbitrary Kripke structure where \( (V, E) \) is a (resp. finite) rooted tree. Standard definitions about trees are omitted herein. Such structures play an important role in the paper, as the tree semantics involves specific tree models (those obtained as computation trees of Kripke structures).

Given a Kripke structure \( \mathcal{K} = (W, R, l) \) and a world \( w \in W \), a finite path \( \pi \) from \( w \) is a finite sequence \( w_0, w_1, \ldots, w_n \) such that \( w_0 = w \) and for all \( i \in [0, n-1] \), we have \( (w_i, w_{i+1}) \in R \). An infinite path from \( w \) is an infinite sequence \( w_0, w_1, \ldots \) such that \( w_0 = w \) and for all \( i \geq 0 \), we have \( (w_i, w_{i+1}) \in R \). With \( \Pi_{\mathcal{K}, w} \) we denote the set of all finite paths starting from a world \( w \in W \) on a Kripke structure \( \mathcal{K} \).

For a Kripke structure \( \mathcal{K} = (W, R, l) \) and \( w \in W \), the computation tree unfolding \( \mathcal{K} \) from \( w \) is the tree model \( \mathcal{T}_{\mathcal{K}, w} = (V, E, l') \) such that the following conditions are satisfied:

(a) \( V \triangleq \Pi_{\mathcal{K}, w} \),
(b) \( \pi E \pi' \triangleq \pi = w_0, \ldots, w_n, \pi' = \pi, w_{n+1} \text{ for some } w_{n+1} \in W \text{ and } (w_n, w_{n+1}) \in R \),
(c) for all \( \pi = w_0, \ldots, w_n \in V \), we have \( l'(\pi) \triangleq l(w_n) \).

Thus, when \( \mathcal{K} \) is finite and total, \( \mathcal{T}_{\mathcal{K}, w} \) is a finite-branching tree model in which all the maximal branches are infinite and it is also a total Kripke structure. Below, unless otherwise stated, the tree semantics involves such finite-branching trees with infinite maximal branches. In the sequel, tree-like Kripke structures are denoted by \( \mathcal{T} = (V, E, l) \) in order to emphasize the tree-like nature of such models.

2.2. The semantics for quantified temporal logics. Models for CTL or for its quantified extension QCTL are total Kripke structures (further refinements shall be operated along this section). The formulae of QCTL-like logics are defined from the grammar below by extending the set of formulae from the computation tree logic CTL [18] with propositional quantification:

\[
\phi ::= p \mid \neg \phi \mid \phi \land \phi \mid \mathbf{E} \phi \mid \mathbf{E} (\phi \mathbf{U} \phi) \mid \mathbf{A} (\phi \mathbf{U} \phi) \mid \exists p \phi,
\]

where \( p \in AP \). We use the standard abbreviations \( \forall, \rightarrow, \leftrightarrow, \bot, \top \), as well as other operators like \( \mathbf{A} \mathbf{X}, \mathbf{E} \mathbf{F}, \mathbf{A} \mathbf{G} \) and \( \mathbf{A} \mathbf{F} \):
• \( \text{AX}(\phi) \overset{\text{def}}{=} \neg \text{EX}(\neg \phi) \),
• \( \text{EF}(\phi) \overset{\text{def}}{=} \text{E}(\top \cup \phi) \), \( \text{AG}(\phi) \overset{\text{def}}{=} \neg \text{EF}(\neg \phi) \) and \( \text{AF}(\phi) \overset{\text{def}}{=} \text{A}(\top \cup \phi) \).
• The universal quantifier \( \forall \phi \) is used as \( \neg \exists p. \neg \phi \).

We denote by \(|\phi|\) the length of the formula \( \phi \) measured in a standard way, i.e. as the number of symbols used to write \( \phi \). The modal/temporal depth of a formula \( \phi \), written \( \text{md}(\phi) \), is the maximal number of nested temporal operators in \( \phi \). We stress that \( \text{md}(\phi) \) is linear in \(|\phi|\).

By restricting the set of allowed temporal operators in QCTL to the only temporal operator \( \text{EX} \) (resp. \( \text{EF} \)) we obtain the logic QCTL\( _X \) (resp. QCTL\( _F \)). Note that \( \text{AF} \) does not occur in QCTL\( _F \) but \( \text{AX} \) (resp. \( \text{AG} \)) is allowed in QCTL\( _X \) (resp. in QCTL\( _F \)) as it is the dual operator of \( \text{EX} \) (resp. \( \text{EF} \)). The main object of study in the paper is the logic QCTL\( _X \) under the tree semantics (below, written QCTL\( _X^t \)). Moreover, we write QCTL\( _{XF} \) to denote the restriction of QCTL to the (combined) temporal operator \( \text{EXEF} \), which provides a strict version of the future-time temporal operator \( \text{EF} \). In the rest of the paper, we refer to QCTL (or to some of its fragments) to denote a set of formulae and the notation QCTL\( ^* \) with a superscript ‘\( ^* \)’ refers to a logic based on QCTL (or on some of its fragments) under a specific semantics and for which the symbol ‘\( ^* \)’ is just a reminder of the semantics.

### 2.3. The logics QCTL\( ^s \), QCTL\( ^t \), QCTL\( ^ft \) and QCTL\( ^gt \).

The propositional quantifiers can be interpreted in various ways, see e.g. \([34, 27, 44, 47, 14]\). Below, we provide a satisfaction relation \( \models \) defined on total Kripke structures, providing the so-called structure semantics for QCTL (and leading to the logic QCTL\( ^s \)). However, the version of QCTL with formulae interpreted on computation trees obtained from the unfoldings of finite total Kripke structures (i.e. in that case the satisfaction relation operates on tree-like structures), is known as QCTL under the tree semantics (written QCTL\( ^t \)) and is extensively studied in \([36]\). To define formally the semantics for propositional quantifiers, we introduce the notion of \( X \)-equivalence between two Kripke structures, with \( X \) being a set of propositional variables. Intuitively, two Kripke structures are \( X \)-equivalent, whenever the only differences (if any) between them are restricted to the interpretation of the propositional variables not in \( X \). Formally, we say that two Kripke structures \( K = \langle W, R, l \rangle \) and \( K' = \langle W', R', l' \rangle \) are \( X \)-equivalent (written \( K \approx_X K' \)), iff \( W = W', R = R' \) and \( l(w) \cap X = l'(w) \cap X \) for all \( w \in W \).

Given a Kripke structure \( K = \langle W, R, l \rangle \), a world \( w \in W \) and a formula \( \phi \) in QCTL, the satisfaction relation \( K, w \models \phi \) is defined as follows:

\[
\begin{align*}
K, w \models p & \quad \text{iff } p \in l(w) \\
K, w \models \neg \phi & \quad \text{iff } K, w \not\models \phi \\
K, w \models \phi \land \psi & \quad \text{iff } K, w \models \phi \text{ and } K, w \models \psi \\
K, w \models \text{EX} \phi & \quad \text{iff there is } w' \text{ such that } (w, w') \in R \text{ and } K, w' \models \phi \\
K, w \models \text{E}(\phi \cup \psi) & \quad \text{iff there is a path } w_0, \ldots, w_n \text{ such that } w_0 = w, \\
& \quad \text{K, } w_i \models \psi \text{ and for all } i \in [0, n-1], \text{ we have } K, w_i \models \phi \\
K, w \models \text{A}(\phi \cup \psi) & \quad \text{iff for all infinite paths } w_0, \ldots, w_n, \ldots \text{ such that } w_0 = w, \text{ there is } \text{ j } \geq 0 \text{ such that } K, w_j \models \psi \text{ and for all } i \in [0, j-1], K, w_i \models \phi \\
K, w \models \exists p \phi & \quad \text{iff there is } K' \text{ such that } K \approx_{AP\{p\}} K' \text{ and } K', w \models \phi \\
\end{align*}
\]

Note that \( K, w \models \text{EF} \phi \) iff there is \( w' \in R^*(w) \) (i.e. \( R^* \) is the reflexive transitive closure of \( R \)) such that \( K, w' \models \phi \). Stating that there is a unique successor of \( w \) satisfying the propositional variable \( x \) can be expressed by the formula \( \text{EX} \ x \land \neg (\exists p \text{ EX}(x \land p) \land \text{EX}(x \land \neg p)) \), where \( p \) is distinct from \( x \).
The satisfiability problem for the logic \( QCTL^s \) (under the \textit{structure semantics}) takes as input a formula \( \phi \) in \( QCTL \) and asks whether there is a finite and total Kripke structure \( \mathcal{K} \) and a world \( w \) such that \( \mathcal{K}, w \models \phi \).

The \textit{tree semantics} introduced in [36] is obtained by considering as only admissible models the computation trees of finite and total Kripke structures. As noted in [36, Remark 5.7], an equivalent formulation can be provided: the satisfiability problem for the logic \( QCTL^t \) (under the \textit{tree semantics}) takes as input a formula in \( QCTL \) and asks whether there is a finite-branching tree model \( \Sigma \) in which all the maximal branches are infinite such that \( \Sigma, \varepsilon \models \phi \) and \( \varepsilon \) is the root of \( \Sigma \). The equivalence is mainly due to the fact that \( QCTL^t \) can be translated into Monadic Second-Order Logic (MSO) over tree models with arbitrary finite branching (getting decidability by Rabin’s Theorem [46]). Indeed, as MSO over tree models with arbitrary finite branching is decidable by Rabin’s Theorem [46], the satisfiability problem for \( QCTL^t \) is decidable too, by a standard translation internalising the tree semantics for \( QCTL^t \). As explained in [57, Section 6.3], the existence of a tree model implies also the existence of a \textit{regular tree model}, that can be precisely originated by a finite Kripke structure. Hence, \( QCTL^t \) has the finite model property. So, not only every finite and total Kripke structure can be unfolded as a finite-branching tree in which all the maximal branches are infinite (a direct consequence of the definition for computation trees) but the existence of a tree model with the above-mentioned properties and satisfying \( \phi \) implies the existence of the computation tree of a finite and total Kripke structure satisfying \( \phi \). Hence, satisfiability in the computation tree of a finite Kripke structure is equivalent to satisfiability in a finite-branching tree in which all the maximal branches are infinite, and in the sequel, we shall operate with the latter definition.

We write \( \text{SAT}(\mathcal{L}) \) to denote the satisfiability problem for the logic \( \mathcal{L} \). The distinction between the tree semantics and the structure semantics is crucial and affects the computational properties of the satisfiability problems.

Proposition 2.1. (I) The satisfiability problem for \( QCTL^s \) under the structure semantics is undecidable [36, Theorem 5.1]. (II) The satisfiability problem for \( QCTL^t \) under the tree semantics is decidable and Tower-complete [36, Theorem 5.3]. (III) The satisfiability problem for \( QCTL^t \) is Tower-hard even if restricted to \( \omega \)-sequences [42, 52].

All our forthcoming Tower upper bound results are based on translations into the satisfiability problem for \( QCTL^t \), sometimes via intermediate decision problems, which eventually invokes Rabin’s Theorem [46] in some essential way. This is not surprising, as considering tree-like models and propositional quantification naturally leads to invoking the decidability of \( \omega S [46] \) or its linear version, the second-order logic of one successor \( S1S [15] \). Decidability of other logics with propositional quantification is shown that way in [4, 6, 62]. Apart from \( \text{CTL} \)-like logics (see \textit{e.g.} [36] for a wealth of bibliographical references), other logics with propositional quantification have been shown decidable that way. Besides, in [4] a fragment of \( \text{GL} \) is shown decidable by translation into the weak monadic second-order logic of one successor \( WS1S [15] \), and a version of Gödel logic \( \text{LC} \) with propositional quantification is shown decidable by translation into \( S1S [15, 6] \), see also [62] solving partially an open problem from [32, §9].

Let us recapitulate the different versions of quantified \( \text{CTL} \) we have seen so far.

- \( QCTL^s \) is interpreted over finite and total Kripke structures and \( \text{SAT}(QCTL^s) \) is undecidable, see \textit{e.g.} [36].
• QCTL\(^t\) is interpreted over finite-branching trees in which all the maximal branches are infinite and \(\text{SAT}(\text{QCTL}\(^t\))\) is \text{TOWER}-complete, see \textit{e.g.} [36].

Let us introduce two additional versions that are closely related to modal logics with propositional quantification on tree-like models introduced in forthcoming Section 2.4.

• QCTL\(^ft\) is interpreted over finite trees, \(\text{SAT}(\text{QCTL}\(^ft\))\) can be shown in \text{TOWER} by a logspace reduction into \(\text{SAT}(\text{QCTL}\(^t\))\) and its restriction to EX will be shown to admit a \text{TOWER}-hard satisfiability problem.

• QCTL\(^gt\) is interpreted over finite-branching trees (maximal branches may be finite), \(\text{SAT}(\text{QCTL}\(^gt\))\) can be shown in \text{TOWER} by logspace reduction into \(\text{SAT}(\text{QCTL}\(^t\))\) and its restriction to \text{EXEF} will be shown to admit a \text{TOWER}-hard satisfiability problem.

2.4. Quantified modal logics on trees. Numerous well-known modal logics are complete (a.k.a. determined) for classes of tree-like structures. For instance, the (propositional) modal logic \(K\) is complete for the class of finite trees [51, 12]. It is worth noting that a given modal logic can be complete for different classes of Kripke models (\textit{e.g.} \(K\) is complete for the class of all the Kripke models, but also complete for the class of finite Kripke models) and their extension to propositional quantification may lead to distinct logics. Typically, \(K\) with propositional quantification under the structure semantics is undecidable [25] whereas it is shown below to be \text{TOWER}-complete under the finite tree semantics (see Section 5.2).

Below, for the propositional modal logics \(L\) in \(K, KD, GL, K4\) and \(S4\) we define an extension \(QL\(^t\)\) with propositional quantification under a class of tree-like models that is complete for the logic \(L\). In Section 5, we shall establish that the satisfiability problem for each \(QL\(^t\)\) is \text{TOWER}-complete. To do so, in Section 5, we establish simple correspondences between each logic \(QL\(^t\)\) and QCTL\(^t\) (or its fragments and variants). Note also that we could complete the list of modal logics above but our intention is to illustrate how our results about QCTL\(^X\), QCTL\(^F\) or QCTL\(^XF\) could be applied to versions of standard modal logics with propositional quantification, without being necessarily exhaustive. In order to avoid too many notations, the modalities for each logic \(QL\(^t\)\) are EX and AX (instead of the more standard modal operators \(\Diamond\) and \(\Box\)) and therefore \(QL\(^t\)\) formulae are built from the grammar below: \(\phi := p \mid \neg \phi \mid \phi \land \phi \mid \text{EX} \phi \mid \text{AX} \phi \mid \exists p \phi\).

• The propositional modal logic \(K\) is complete for the class of finite trees and we define \(QK\(^t\)\) as the modal logic with propositional quantification over the class of finite trees.

• The propositional modal logic \(KD\) (\(K\) with seriality, a.k.a. totality) is known to be complete for the class of finite-branching trees for which all the maximal branches are infinite. Indeed, \(KD\) is complete for the class of finite total Kripke models, and therefore by using the unfolding construction completeness applies also for the class of finite-branching trees for which all the maximal branches are infinite that are exactly the models for QCTL\(^t\). Let \(QKD\(^t\)\) be the modal logic with propositional quantification over the class of finite-branching trees for which all the maximal branches are infinite. The satisfiability problem for \(QKD\(^t\)\) is exactly the problem for QCTL\(^t\).

• The modal logic \(GL\) is known to be complete for the class of finite transitive trees (\(GL\) is complete with respect to finite irreflexive transitive Kripke models [54]), \textit{i.e.} the class of Kripke structures \(\langle V, E^+, l \rangle\) such that \(\langle V, E, l \rangle\) is a finite tree model, see \textit{e.g.} [12]. Let \(QGL\(^t\)\) be the modal logic with propositional quantification over the class of finite transitive trees, which is precisely QCTL\(^{ft}\). It is worth noting that adding propositional quantification to
GL is studied in [4], where a fragment is shown decidable by translation into the weak monadic second-order logic of one successor WS1S [15].

- The modal logic K4 is complete for the class of Kripke structures \( \langle V, E^+, l \rangle \) such that \( \langle V, E, l \rangle \) is a finite-branching tree model (some branches may be infinite, some others not). Let QS4\(^t\) be the modal logic with propositional quantification over the class of finite-branching trees.

- The modal logic S4 is complete for the class of finite Kripke structures such that the accessibility relation is reflexive and transitive, and therefore complete for the class of structures \( \langle V, E^*, l \rangle \) such that \( \langle V, E, l \rangle \) is a finite-branching tree model in which all the branches are infinite (by unfolding). Let QS4\(^t\) be the modal logic with propositional quantification over the class of finite-branching trees in which all the branches are infinite (precisely the class of models for QCTL\(^t\)). The satisfiability problem for QS4\(^t\) happens to be exactly the problem for QCTL\(_F^t\), modulo the fact that EX in QS4\(^t\) corresponds to EF in QCTL\(_F^t\).

Section 5 shall provide TOWER-hardness results about the satisfiability problems for QK\(^t\), QKD\(^t\), QGL\(^t\), QK4\(^t\) and QS4\(^t\) respectively based on TOWER-hardness for SAT(QCTL\(_X\)), SAT(QCTL\(_F^t\)) and SAT(QCTL\(_X^t\)) in Sections 3.3 and 4.4.

2.5. Complexity classes and tiling problems. In this section, we introduce tiling problems that are mainly used in Sections 3.3 and 4.4.

Let \( t : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be a tetration function defined for integers \( n, k \geq 0 \), inductively as \( t(0, n) = n \) and \( t(k + 1, n) = 2^{t(k, n)} \). Intuitively the function \( t \) defines the tower of exponentials of height \( k \), e.g. we have \( t(1, n) = 2^n \), \( t(2, n) = 2^{2^n} \), and so on. By \( k\text{-NExpTime} \) we denote the class of all problems decidable with a nondeterministic Turing machines of working time in \( O(t(k, p(n))) \) for some polynomial \( p(\cdot) \), on each input of length \( n \). We define TOWER as the class of all problems of time complexity bounded by a tower of exponentials, whose height is an elementary function [49]. Thus, to show TOWER-hardness (using elementary reductions [49]), it is sufficient to prove \( k\text{-NExpTime} \)-hardness for all \( k \) using uniform reductions [49].

For proving hardness results, we make extensive use of tiling problems, see e.g. [59].

**Definition 2.2.** The tiling problem TILING\(_k\) takes as inputs a triple \( \langle T, \mathcal{H}, \mathcal{V} \rangle \) and \( c \in T^n \) for some \( n \geq 1 \) such that \( T \) is a finite set of tile types, \( \mathcal{H} \subseteq T \times T \) (resp. \( \mathcal{V} \subseteq T \times T \)) represents the horizontal (resp. vertical) matching relation, and \( c = t_0, t_1, \ldots, t_{n-1} \in T^n \) is the initial condition. A solution for the instance \( \langle T, \mathcal{H}, \mathcal{V} \rangle, c \) is a mapping \( \tau : [0, t(k, n) - 1] \times [0, t(k, n) - 1] \to T \) such that:

- **(init):** For all \( i \in [0, n - 1] \), \( \tau(0, i) = t_i \).
- **(hori):** For all \( i, j \in [0, t(k, n) - 1] \), \( i < t(k, n) - 1 \) implies \( (\tau(i, j), \tau(i+1, j)) \in \mathcal{H} \).
- **(verti):** For all \( i, j \in [0, t(k, n) - 1] \), \( j < t(k, n) - 1 \) implies \( (\tau(i, j), \tau(i, j+1)) \in \mathcal{V} \).

A mapping \( \tau \) satisfying (hori) and (verti) is called a tiling.

The problem of checking if an instance of TILING\(_k\) has a solution (note that \( k \) does not appear in the instance and it governs the size of the grid) is \( k\text{-NExpTime} \)-complete [59].

Given \( N \geq 2 \), let us consider the satisfiability problem for QCTL\(_X_{\leq N}\) in which the structures are tree models where all the maximal branches are infinite but each node has at most \( N \) children (and at least one child). To characterise the complexity of SAT(QCTL\(_X_{\leq N}\)),
Given an initial condition $c(e)$ and for all worlds $v$, there is a solution $(\forall w I)$, where $f(n)$ refers to the restriction on the Space, $g(n)$ refers to the restriction on the Time, and $h(n)$ refers to the restriction on the number of Alternations. The complexity of several logical problems has been captured by the class $\text{AExp}_\text{pol}$, see e.g. [23, 14, 13].

For proving $\text{AExp}_\text{pol}$-hardness, we use an elegant modification of $\text{Tiling}_1$, introduced in [13, 43]. The extension amounts to consider a stack of $n$ tilings, with a matching relation between two consecutive tile types on the same position of the grid, and quantifications over the tile types on the first row (initial conditions). Details follow below.

**Definition 3.1.** The alternating multi-tiling problem $\text{AMTP}$ takes as inputs an even number $n$ (in unary), $\langle T, H, V \rangle$ (as for defining $\text{Tiling}_1$), $T_0 \subseteq T$, $T_{\text{acc}} \subseteq T$ and $T_{\text{multi}} \subseteq T \times T$. Given an initial condition $c = (w_1, \ldots, w_n) \in (T_0^{2n})^n$, a solution for $c$ is a multi-tiling $(\tau_1, \ldots, \tau_n)$ on the grid $[0, 2^n - 1] \times [0, 2^n - 1]$ such that:

- (m-init): For all $\alpha \in [1, n]$, and all $j \in [0, 2^n - 1]$, $\tau_\alpha(0, j) = w_\alpha(j)$ (i.e. the first row of $\tau_\alpha$ is $w_\alpha$).

- (m-tiling): For $\alpha \in [1, n]$, $\tau_\alpha$ satisfies (hori) and (verti).

- (m-multi): For $\alpha \in [1, n - 1]$, and all $i, j \in [0, 2^n - 1]$, $(\tau_\alpha(i + 1, j), \tau_\alpha(i, j)) \in T_{\text{multi}}$.

- (m-accept): For some $j \in [0, 2^n - 1]$, $\tau(n, 2^n - 1, j) \in T_{\text{acc}}$.

An instance $I$ for $\text{AMTP}$ made of $n$, $\langle T, H, V \rangle$, $T_0, T_{\text{acc}} \subseteq T$, $T_{\text{multi}} \subseteq T \times T$ is positive iff for all $w_1 \in T_0^{2n}$, there is $w_2 \in T_0^{2n}$ such that ... for all $w_{n-1} \in T_0^{2n}$, there is $w_n \in T_0^{2n}$ such that there is a solution $(\tau_1, \ldots, \tau_n)$ for $(w_1, \ldots, w_n)$.

The alternating multi-tiling problem is shown $\text{AExp}_\text{pol}$-complete in [13, 43].

### 3. What happens when trees are bounded?

In this section, we study the satisfiability problem for $\text{QCTL}^X_{\leq N}$ with $N \geq 1$, i.e. $\text{QCTL}^X$ over trees, where the degree of each node is bounded by a fixed natural number $N \geq 1$.

#### 3.1. A toolkit for introducing local nominals.

Below, we introduce formulae to simulate partially the use of nominals from hybrid modal logics [3]. A nominal $x$ is usually understood as a propositional variable true at exactly one world of the model (a global property). In $\text{QCTL}^X$, such a property cannot be enforced but it can be done with respect to nodes at a bounded depth from the evaluation node, whence the adjective ‘local’ for the nominals.

The use of local nominals is essential in all our hardness proofs, as it allows us to simulate first-order quantification on a given set of nodes of bounded depth.

**Definition 3.1.** Given a tree model $\mathfrak{X}$ and a node $v$, we say that the propositional variable $x$ is a nominal for the depth $k \geq 1$ from $v$ iff there is $v' \in V$ with $\text{EX}^k v' \text{ such that } \mathfrak{X}, v' \models x$, and for all worlds $v'' \neq v'$ s.t. $\text{EX}^k v''$, we have $\mathfrak{X}, v'' \not\models x$ ($\text{EX}^k$ is the $k$-fold composition of $E$).

The formula $\text{nom}(x, k)$ defined as $\text{EX}^k x \land \neg \exists p (\text{EX}^k(x \land p) \land \text{EX}^k(x \land \neg p))$, where $p$ is distinct from $x$, states that $x$ is a nominal for the depth $k$ ($\text{EX}^k$ denotes $k$ copies of $\text{EX}$).

**Lemma 3.2.** $x$ is a nominal for the depth $k \geq 1$ from $v$ iff $\mathfrak{X}, v \models \text{nom}(x, k)$. 


Proof. First, suppose that \( x \) is a nominal for the depth \( k \geq 0 \) from \( v \). By definition, this means that there is \( v' \in V \) satisfying \( vE^kv' \) such that \( \mathfrak{T}, v' \models x \) and for all \( v'' \neq v' \) satisfying \( vE^kv'' \) we have \( \mathfrak{T}, v'' \models \neg x \). Obviously, we have \( \mathfrak{T}, v \models \text{EX}^k x \). Ad absurdum, suppose that \( \mathfrak{T}, v \models \exists p (\text{EX}^k(x \land p) \land \text{EX}^k(x \land \neg p)) \) with \( p \) being a fresh propositional variable. So, there is a \((\text{AP} \setminus \{p\})\)-variant \( \mathfrak{T}' \) of \( \mathfrak{T} \) (i.e., \( \mathfrak{T} \approx_{\text{AP} \setminus \{p\}} \mathfrak{T}' \)) such that \( \mathfrak{T}', v \models \text{EX}^k(x \land p) \land \text{EX}^k(x \land \neg p) \). Therefore, there is \( v' \) such that \( vE^kv' \) and \( \mathfrak{T}', v' \models x \land p \). Similarly, there is \( v'' \) such that \( vE^kv'' \) and \( \mathfrak{T}', v'' \models x \land \neg p \). Because of the constraint on the satisfaction of the propositional variable \( p \), the nodes \( v' \) and \( v'' \) are distinct, which leads to a contradiction.

Conversely, suppose that \( \mathfrak{T}, v \models \text{nom}(x, k) \). As \( \mathfrak{T}, v \models \text{EX}^k x \), there exists \( v' \) such that \( vE^kv' \) and \( \mathfrak{T}, v' \models x \). The uniqueness of \( v' \) can be concluded from the satisfaction \( \neg \exists p (\text{EX}^k(x \land p) \land \text{EX}^k(x \land \neg p)) \) (as above), since that formula characterises exactly the property that there are no two distinct nodes reachable in \( k \) steps from \( v \) satisfying \( x \).

Let us next define \( @^k_x \phi \) as the formula \( \text{EX}^k(x \land \phi) \) (usually assuming that \( \text{nom}(x, k) \) holds).

Lemma 3.3. Assuming that \( x \) is a nominal for the depth \( k \geq 0 \) from \( v \) such that \( vE^kv' \) and \( \mathfrak{T}, v' \models x \), we have \( \mathfrak{T}, v \models @^k_x \phi \) iff \( \mathfrak{T}, v' \models \phi \).

Proof. By assumption, there is a node \( v' \in V \) satisfying \( vE^kv' \) such that \( \mathfrak{T}, v' \models x \), and for all \( v'' \neq v' \) satisfying \( vE^kv'' \), we have \( \mathfrak{T}, v'' \models \neg x \). First, suppose that \( \mathfrak{T}, v \models @^k_x \phi \), i.e., \( \mathfrak{T}, v \models \text{EX}^k(x \land \phi) \). Hence, there exists \( v'' \) such that \( vE^kv'' \) and \( \mathfrak{T}, v'' \models x \land \phi \). As \( \mathfrak{T}, v'' \models x \), the node \( v'' \) is necessarily equal to \( v' \) and therefore \( \mathfrak{T}, v' \models \phi \). For the opposite direction, suppose that \( \mathfrak{T}, v' \models \phi \). As \( vE^kv' \) and \( \mathfrak{T}, v' \models x \), by the definition of the satisfaction relation \( \models \), we conclude that \( \mathfrak{T}, v \models \text{EX}^k(x \land \phi) \) and \( \text{EX}^k(x \land \phi) \) is equal to \( @^k_x \phi \).
Every formula \( k \)-depth iff there are \( \alpha \). The (easy) proof is a slight variant of the proof of Lemma 3.2 and is therefore omitted herein.

Thus, a formula \( \phi \) has an upper bound for SAT(\( N \)) more importantly, Section 3.3 proves AExp-pol-hardness of SAT(QCTL\( X_{\leq N} \)) as soon as \( N \geq 2 \). Hence, the goal of this section is to establish an AExp-pol upper bound. The tight upper bound for SAT(QCTL\( X_{\leq N} \)) relies on the following ingredients.

(i) Every formula \( \phi \) of QCTL\( X \) is logically equivalent to a QCTL\( X \) formula \( \phi' \) in prenex normal form (PNF) such that \( \phi' \) can be computed in polynomial-time in \( |\phi| \). Formulae in PNF are of the form \( \exists \psi \exists \psi \psi \) where \( \{ Q_1, \ldots, Q_\beta \} \subseteq \{ \exists, \forall \} \) and \( \psi \) is quantifier-free.
(ii) Existence of an $N$-bounded tree model for $\phi$ is equivalent to the existence of an $N$-bounded finite tree structure such that all branches are of length $md(\phi)$. Then, we simply guess a finite tree of a small (exponential) size with a help of shallow model property - such a tree will be later unravelled to become an infinite tree model.

(iii) Checking whether $\exists \varepsilon \mid \varphi_1 \cdots \varphi_\beta \varphi_3 \psi$ (involving an $N$-bounded finite tree with branches of length $md(\psi)$ and the input formula in PNF) can be done with an alternating Turing machine in time $O((|\varphi| + \beta)|\Sigma|)$ and with at most $\beta$ alternations.

To establish (i), we cannot rely directly on PNF for QCTL from [36, Prop. 3.1] as the translation in [36, Prop. 3.1] involves temporal operators beyond the language of $\mathsf{QCTL_X}$. Fortunately, on tree models, the following formulae are tantologies, assuming that $p$ does not occur in $\psi'$.

\[ \mathsf{EX} \varphi \psi \leftrightarrow \mathsf{EX} \varphi' \psi' \leftrightarrow \mathsf{EX} \varphi \psi \] 

\[ \mathsf{EX} \varphi \psi \leftrightarrow \mathsf{EX} \varphi' \psi' \leftrightarrow \mathsf{EX} \varphi \psi \] 

\[ \mathsf{EX} \varphi \psi \leftrightarrow \mathsf{EX} \varphi' \psi' \leftrightarrow \mathsf{EX} \varphi \psi \] 

\[ \mathsf{AX} \varphi \psi \leftrightarrow \mathsf{AX} \varphi' \psi' \leftrightarrow \mathsf{AX} \varphi \psi \] 

\[ \mathsf{AX} \varphi \psi \leftrightarrow \mathsf{AX} \varphi' \psi' \leftrightarrow \mathsf{AX} \varphi \psi \] 

Thus, the equivalent formula in PNF remains equivalent in $\mathsf{QCTL_X}$.  

\textbf{Lemma 3.6.} For every formula $\phi$ in $\mathsf{QCTL_X}$, one can compute in polynomial-time (Q \in \{\exists, \forall\}) a logically equivalent formula in PNF $\mathsf{Q_1} \varphi_1 \cdots \mathsf{Q_\beta} \varphi_\beta \psi$ with $\beta \leq |\phi|$.

\textbf{Proof.} By way of example, we prove that $\mathsf{EX} \forall \varphi \psi \leftrightarrow \forall \mathsf{EX} \psi$ is valid. (Other formulae, e.g. $\mathsf{EX} \exists \varphi \psi \leftrightarrow \exists \mathsf{EX} \psi$ can be proved valid in an even simpler way). Before doing so, note that then the valid equivalences below provide a rewriting system (by reading the equivalences from left to right) that pushes the propositional quantification outside (in the usual way), leading to formulae in PNF in polynomial-time ($\mathsf{Q \in \{\exists, \forall\}}$).

\[ \mathsf{EX} \varphi \psi \leftrightarrow \mathsf{EX} \varphi' \psi' \leftrightarrow \mathsf{EX} \varphi \psi \] 

assuming that $p$ does not occur in $\psi'$ (otherwise, rename the quantified variable).

First, assume that $\Sigma, v \models \mathsf{EX} \forall \varphi \psi$ with $\Sigma = \langle V, E, l \rangle$. Thus, there exists $v'$ such that $vEv'$ and $\Sigma, v' \models \forall \varphi \psi$. Hence, for all $\Sigma' \approx_{\mathsf{AP}}(p) \Sigma$, we have $\Sigma', v' \models \psi$. As $v'$ remains a child of $v$ for all such variants $\Sigma'$, we have that for all $\Sigma' \approx_{\mathsf{AP}}(p) \Sigma$, $vE'v'$ and $\Sigma', v' \models \psi$. Thus, for all $\Sigma' \approx_{\mathsf{AP}}(p) \Sigma$ we have $\Sigma', v \models \mathsf{EX} \psi$. We conclude $\Sigma, v \models \forall \mathsf{EX} \psi$. Conversely (and this is the place where the tree structure is essential), assume $\Sigma, v \models \forall \mathsf{EX} \psi$. It means that for all $\Sigma' \approx_{\mathsf{AP}}(p) \Sigma$, we have $\Sigma', v \models \mathsf{EX} \psi$. Ad absurdum, suppose $\Sigma, v \not\models \mathsf{EX} \forall \varphi \psi$. Equivalently, we have $\Sigma, v \models \mathsf{AX} \exists \varphi \psi$. Thus, for all $v'$ such that $vEv'$, there is $\Sigma_{v'} \approx_{\mathsf{AP}}(p) \Sigma$ (say $\Sigma_{v'} = \langle V, E, l_{v'} \rangle$) such that $\Sigma_{v'}, v' \not\models \psi$. Let $\Sigma_{v'} = \langle V_{v'}, E_{v'}, l_{v'} \rangle$ be the restriction of $\Sigma_{v'}$ to the subtree whose root is $v'$. Obviously $\Sigma_{v'}, v' \not\models \psi$ too. Two distinct children $v_1$ and $v_2$ of $v$ in $\Sigma$, lead to tree models $\Sigma_{v_1} = \langle V_{v_1}, E_{v_1}, l_{v_1} \rangle$ and $\Sigma_{v_2} = \langle V_{v_2}, E_{v_2}, l_{v_2} \rangle$ with $V_{v_1} \cap V_{v_2} = \emptyset$. Indeed, $\Sigma$ is a tree. Let $\Sigma^*$ be defined as $\langle V, E, (\cup_{v' \in E(v)} l_{v'}) \cup \{ v \mapsto l(v) \} \rangle$, where $\cup$ denotes the disjoint sum. One can show that $\Sigma^* \approx_{\mathsf{AP}}(p) \Sigma$ and for all $v'$ such that $vEv'$, $\Sigma^*, v' \not\models \psi$, which leads to a contradiction. \hfill \Box

Now we proceed with the second property. Let us be a bit more precise. Given a tree model $\Sigma$, we write $\Sigma|_{n}$ to denote its subtree obtained by taking only nodes on the depth at most $n$ from the root. A completion of a finite tree $\Sigma'$ of maximal depth $n$ is an infinite tree $\Sigma$ (finite-branching and all the maximal branches are infinite) such that $\Sigma = \Sigma'|_{n}$. By the naive completion of $\Sigma'$ of maximal depth $n$ we refer to the unique completion achieved by replacing each node $v$ at depth $n$ from $\Sigma$ by an infinite chain of copies of itself.

A shallow model property, states that what matters for a $\mathsf{QCTL_X}$ formula $\phi$ in its infinite tree model is a relatively small finite part, with paths bounded by the modal depth of $\phi$.  


Lemma 3.7 (Shallow Model Property). Let $\mathcal{T}, \varepsilon$ be a model for the QCTL$^4_X$-formula $\phi$. Then, any completion of $\mathcal{T}|_{md(\phi)}, \varepsilon$ is also a model for $\phi$.

The construction is standard and goes in exactly the same way as in the literature, e.g. [12, Theorem 2.3]. Since we are interested in the satisfiability problem over $N$-bounded trees, the overall size of a structure $\mathcal{T}|_{md(\phi)}$ can be easily bounded. A simple estimation can be obtained by counting the number of nodes with a certain distance from the root, namely:

$$|\mathcal{T}|_{md(\phi)}| \leq 1 + N + N^2 + \ldots + N^{md(\phi)} < |\phi| \cdot N^{|\phi|}$$

As a direct consequence of the above estimation and Lemma 3.7, we obtain:

Lemma 3.8. For any formula $\phi$, $\phi$ is satisfiable for QCTL$^4_X_{\leq N}$ iff $\phi$ is satisfiable in a finite $N$-bounded tree structure of size bounded by $|\phi|N^{|\phi|}$ and each branch is of length $md(\phi)$.

In order to establish (iii), the details are omitted but we apply the naive model-checking algorithm for MSO with an ATM: existential (resp. universal) quantification $\exists \varphi$ (resp. $\forall \varphi$) requires time $O(|\mathcal{T}|)$ and the machine enters in a sequence of existential (resp. universal) states. The quantifier-free formula $\psi$ is evaluated as a first-order formula by the standard translation for modal logic. Note also that checking $\mathcal{T}', v \models \psi$ can be done in polynomial time in $|\psi| + |\mathcal{T}'|$ (see [18, 50]). By combining (i)-(iii) we establish an improved upper bound.

Theorem 3.9. For any $N \geq 1$, the satisfiability problem for QCTL$^4_X_{\leq N}$ is in $AExp_{pol}$.

When $N = 1$, the upper bound can be improved as the number of alternations is linear and the size of the finite witness “tree” is polynomial in $|\phi|$, and therefore the whole procedure can be implemented with a polynomial-time alternating Turing machine (thus in PSPACE [17]). The matching lower bound is inherited from quantified propositional logic QBF, see e.g. [41].

Corollary 3.10. The satisfiability problem for QCTL$^4_X_{\leq 1}$ is PSPACE-complete.

### 3.3. A reduction from AMTP (with fixed $N \geq 2$). Let $N \geq 2$ and let us consider the satisfiability problem for QCTL$^4_X_{\leq N}$ in which the structures are tree models where all the maximal branches are infinite and each node has at most $N$ children (and at least one child). In order to show that the problem is $AExp_{pol}$-hard, we define below a reduction from the alternating multi-line tiling problem AMTP presented in Section 2 and introduced in [13].

To define a grid $[0, 2^n - 1] \times [0, 2^n - 1]$, a major part in the solution of an instance of AMTP, we specify a tree such that every node at a distance less than $2n$ from the root $\varepsilon$ has exactly two children, implying that there are exactly $2^{2n}$ nodes at a distance $2n$ from $\varepsilon$. Moreover, each node at a distance $2n$ encodes a position $(\mathcal{S}, \mathcal{W})$ in $[0, 2^n - 1] \times [0, 2^n - 1]$, by employing the propositional variables $h_{n-1}, \ldots, h_0$ and $v_{n-1}, \ldots, v_0$ responsible, respectively, for the horizontal and vertical axes. The $i$-th bit of $\mathcal{S}$ (resp. $\mathcal{W}$) is taken care of by the truth value of $h_i$ (resp. $v_i$) and by convention the least significant bit is encoded by $h_0$ (resp. $v_0$).

The forthcoming formula $\text{grid}(2n)$ is dedicated to encode such a grid.

\[
\text{grid}(2n) \overset{\text{def}}{=} \left( \bigwedge_{i \in [0, 2n-1]} \text{AX}^i \text{EX}_{\geq 2} \top \right) \land \\
\forall x, y \text{ diff-nom}(x, y, 2n) \rightarrow \left( \bigvee_{j \in [0, n-1]} \neg((\oplus_x^{2n} h_j) \leftrightarrow (\oplus_y^{2n} h_j)) \lor \neg((\oplus_x^{2n} v_j) \leftrightarrow (\oplus_y^{2n} v_j)) \right),
\]

[12, Theorem 2.3] Since we are interested in the satisfiability problem over $N$-bounded trees, the overall size of a structure $\mathcal{T}|_{md(\phi)}$ can be easily bounded. A simple estimation can be obtained by counting the number of nodes with a certain distance from the root, namely:

$$|\mathcal{T}|_{md(\phi)}| \leq 1 + N + N^2 + \ldots + N^{md(\phi)} < |\phi| \cdot N^{|\phi|}$$

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Theorem 3.9. For any $N \geq 1$, the satisfiability problem for QCTL$^4_X_{\leq N}$ is in $AExp_{pol}$.

When $N = 1$, the upper bound can be improved as the number of alternations is linear and the size of the finite witness “tree” is polynomial in $|\phi|$, and therefore the whole procedure can be implemented with a polynomial-time alternating Turing machine (thus in PSPACE [17]). The matching lower bound is inherited from quantified propositional logic QBF, see e.g. [41].

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The forthcoming formula $\text{grid}(2n)$ is dedicated to encode such a grid.

\[
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\forall x, y \text{ diff-nom}(x, y, 2n) \rightarrow \left( \bigvee_{j \in [0, n-1]} \neg((\oplus_x^{2n} h_j) \leftrightarrow (\oplus_y^{2n} h_j)) \lor \neg((\oplus_x^{2n} v_j) \leftrightarrow (\oplus_y^{2n} v_j)) \right),
\]
where \( \text{EX}_{x_2 T} \overset{\text{def}}{=} \exists x_1, x_2 \text{ diff-nom}(x_1, x_2, 1) \land \text{AX}(x_1 \lor x_2) \) states that there are exactly two children. Moreover, \( \text{AX}^0 \psi \overset{\text{def}}{=} \psi \) and \( \text{AX}^{i+1} \psi \overset{\text{def}}{=} \text{AXAX}^i \psi \).

Note that the upper part of grid\( (2n) \) enforces that there are exactly \( 2^{2n} \) descendants at a distance \( 2n \) from the root, while the lower part imposes that any two such descendants differ by at least one propositional variable from \( h_{n-1}, \ldots, h_0, v_{n-1}, \ldots, v_0 \). Hence, the full grid \([0, 2^n - 1] \times [0, 2^n - 1] \) is encoded with grid\( (2n) \). The correctness of grid\( (2n) \) follows from Lemma 3.3 and Lemma 3.5.

**Corollary 3.11.** \( \Sigma, v \models \text{grid}(2n) \) iff \( \Sigma|_{2n} \) is a binary tree in which there are exactly \( 2^{2n} \) nodes \( v' \) satisfying \( E^{2n}(v, v') \) and each of such nodes \( v', v'' \) is labelled by a different subset of atomic propositions from \( \{v_0, v_1, \ldots, v_{n-1}, h_0, h_1, \ldots, h_{n-1}\} \).

Let \((T, H, V)\) be a triple from an instance of AMTP and let \( j \in \mathbb{N} \). Each tile type \( t \in T \) will be represented by a fresh propositional variable \( t' \). Hence, \( \{t' : t \in T\} \) (written below \( T' \)) is a set of propositional variables used to provide a tile type for each position of the grid \([0, 2^n - 1] \times [0, 2^n - 1] \), while the superscript \( 'j' \) is handy to remember that this concerns the \( j \)-th tiling (as several tilings are involved in AMTP instances).

We first define the formulae \( \phi^j_{\text{cov}}, \phi^j_H \) and \( \phi^j_V \), whose conjunction states that every position of the grid has a unique tile type in \( T' \), and the horizontal and vertical matching conditions are satisfied. Hence, we have a valid tiling of the grid made from \( T' \).

The formula \( \phi^j_{\text{cov}} \) expresses that every position of the grid has a unique tile type:

\[
\phi^j_{\text{cov}} \overset{\text{def}}{=} \forall x \text{ nom}(x, 2n) \rightarrow @_{x}^{2n} \left( \bigvee_{t \in T} t^j \land \bigwedge_{t \neq t' \in T} \neg(t^j \land t'^j) \right).
\]

For the horizontal matching constraints, we need to express when two nodes at a distance \( 2n \) interpreted respectively by \( x \) and \( y \) and representing respectively the position \((h_i, \mathcal{V})\) and \((h'_i, \mathcal{V}')\), satisfy \( \mathcal{V} = \mathcal{V}' \) and \( h'_i = h_i + 1 \). The formula \( \text{HN}(x, y) \) (‘HN’ for ‘horizontal neighbours’) does the job using a standard arithmetical reasoning on numbers encoded in binary.

The intuition is that we treat \( h_i \) propositions as bits and to verify that the number encoded on \( y \) is equal to the number encoded on \( x \) plus 1, we need to (i) find an index \( i \) on which the \( i \)-th bit is switched on for \( y \) but switched off for \( x \), (ii) check that all bits on more significant positions after \( i \) for \( x \) and \( y \) are equal and (iii) ensure that all less significant bits are switched on for \( x \) while switched off for \( y \). This idea is formalised with the formula below.

\[
\text{HN}(x, y) \overset{\text{def}}{=} \left( \bigwedge_{\alpha \in [0, n-1]} @_{x}^{2n} v_\alpha \leftrightarrow @_{y}^{2n} v_\alpha \right) \land \bigvee_{i \in [0, n-1]} \left( @_{x}^{2n} h_i \land @_{y}^{2n} h_i \land \bigwedge_{\alpha \in [0, i-1]}\left( @_{x}^{2n} h_\alpha \land @_{y}^{2n} h_\alpha \land \bigvee_{\alpha \in [i+1, n]} @_{x}^{2n} h_\alpha \leftrightarrow @_{y}^{2n} h_\alpha \right) \right).
\]

Employing \( \text{HN}(x, y) \) we can provide a formula \( \phi^j_H \) that encodes horizontal matching constraints:

\[
\phi^j_H \overset{\text{def}}{=} \forall x, y \ (\text{nom}(x, 2n) \land \text{nom}(y, 2n) \land \text{HN}(x, y)) \rightarrow \bigwedge_{(t, t') \in H} ( @_{x}^{2n} t^j \land @_{y}^{2n} t'^j ).
\]
Let VN(x, y) (where ‘VN’ stands for ‘vertical neighbours’) be the formula obtained from HN(x, y) by replacing each occurrence of hα (resp. vα) by vα (resp. hα).

The following formula \( \phi^j_V \) encodes the vertical matching constraints:

\[
\phi^j_V \overset{\text{def}}{=} \forall x, y \ (\nom(x, 2n) \land \nom(y, 2n) \land \text{VN}(x, y)) \rightarrow \bigvee_{(t, t') \in V} (\oplus^2_{x} t^j \land \oplus^2_{y} t'^j).
\]

To state that a root satisfying grid(2n) encodes a tiling with respect to \( T^j \), we consider the formula

\[
\phi^{j}_{\text{tiling}} \overset{\text{def}}{=} \phi^{j}_{\text{cov}} \land \phi^{j}_{H} \land \phi^{j}_{V}.
\]

**Lemma 3.12.** Assume that \( \mathfrak{S}, v \models \text{grid}(2n) \) holds. Then:

- If \( \mathfrak{S}, v \models \phi^{j}_{\text{tiling}} \) then \( \tau : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow T^j \), defined as \( \tau(x, y) = t^j \) for \( \mathfrak{S}, v_{(x, y)} \models t^j \), where \( v_{(x, y)} \) is the unique encoding of the position \( (x, y) \) satisfying \( E^{2n}(v, v_{(x, y)}) \), is a tiling.
- If \( \tau : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow T^j \) is a tiling, then there exists a tree \( \mathfrak{S}' \), \( v' \) satisfying \( \phi^{j}_{\text{tiling}} \land \text{grid}(2n) \) and being a \( T^j \)-variant of \( \mathfrak{S} \).

**Proof.** We start from proving the first item of the lemma. The definition of \( \tau \) is correct since:

- for each position \( (x, y) \in [0, 2^n - 1] \times [0, 2^n - 1] \) there is a unique node \( v' \) in the distance \( 2n \) from \( v \) encoding the position \( (x, y) \) (follows from Corollary 3.11),
- for each \( v' \), as defined above, there is a unique tile proposition \( t^j \) such that \( \mathfrak{S}, v' \models t^j \) holds (it is a direct consequence of the satisfaction of \( \phi^{j}_{\text{cov}} \) at \( v \)).

The satisfaction of the conditions (hori) and (verti) follows from the satisfaction of \( \phi^{j}_{H} \land \phi^{j}_{V} \). Indeed, let us discuss the condition (hori) only, since (verti) is analogous. Take any two consecutive positions \( (a, b) \) and \( (a, b + 1) \) such that \( \tau(a, b) = t \) and \( \tau(a, b + 1) = t' \). Then, let \( v_t, v_{t'} \) be nodes at the distance \( 2n \) from \( v \), representing the positions \( (a, b) \) and \( (a, b + 1) \). Set the local nominals \( x \) and \( y \) at \( v_t \) and \( v_{t'} \). Then, note that the formula HN(x, y) is satisfied at \( v \) by elementary operations on binary encodings of numbers. Hence, by the right-hand side of the implication in \( \phi^{j}_{H} \), we conclude that \( (t, t') \in \mathcal{H} \).

To show the second item, we take \( \mathfrak{S}' = \langle V, E, l' \rangle \) obtained from \( \mathfrak{S} = \langle V, E, l \rangle \) by setting \( l'(v_{(a,b)}) = (l(v_{(a,b)}) \setminus T^j) \cup \{ \tau(a, b) \} \) for the unique node \( v_{(a,b)} \) at the distance \( 2n \) from \( v \) corresponding to \( (a, b) \) in the grid. Otherwise \( l \) and \( l' \) coincide. Since the formula grid(2n) does not employ propositions from \( T^j \) we conclude \( \mathfrak{S}', v \models \text{grid}(2n) \). By the definition of \( \tau \) we know that each node at the distance \( 2n \) from \( v \) is labelled with exactly one tile proposition from \( T^j \) and hence, \( \mathfrak{S}', v \models \phi^{j}_{\text{cov}} \). Checking that \( \mathfrak{S}', v \) satisfies \( \phi^{j}_{H} \land \phi^{j}_{V} \) is routine and follows from the fact that \( \tau \) is a tiling (so it satisfies (hori) and (verti)).

In order to encode an instance of AMTP, there are still properties that need to be expressed. Let us assume that the root node \( \varepsilon \) satisfies grid(2n).

- Given the set of initial tile types \( T_0 \subseteq T \), let us express that for each position of the first row of the grid, exactly one tile type in \( T_0^\varepsilon \) holds.

\[
\phi^{j}_{\text{init}} \overset{\text{def}}{=} \forall x \left( \nom(x, 2n) \land \oplus^2_{x} \bigwedge_{\alpha \in [0, n-1]} \neg h_{\alpha} \right) \rightarrow \oplus^2_{x} \left( \bigvee_{t \in T_0} t^{j} \land \bigwedge_{t \neq t' \in T_0} \neg (t^{j} \land t'^{j}) \right).
\]
• Assuming that \( \varepsilon \) satisfies \( \phi_{\text{tiling}}^j \land \phi_{\text{init}}^j \), we express that for each position of the first row of the grid, the tile type in \( T_0^j \) coincides with the tile type in \( T_0^{j'} \) (corresponding to \( (m-\text{init}) \)):

\[
\phi_{\text{coinci}}^{j,j'} \overset{\text{def}}{=} \forall x \left( \left( \bigwedge_{\alpha \in \{0, n-1\}} \neg h_{\alpha} \right) \land \left( \bigvee_{t \in T_0} (t^j \land t^{j'}) \right) \right).
\]

• Given the set of accepting tile types \( T_{\text{acc}} \subseteq T \) and assuming that \( \varepsilon \) satisfies \( \phi_{\text{tiling}}^j \), we state that there is a position on the last row with a tile type in \( T_{\text{acc}} \) (satisfying \( (m-\text{accept}) \)):

\[
\phi_{\text{acc}}^j \overset{\text{def}}{=} \exists x \left( \bigwedge_{\alpha \in \{0, n-1\}} h_{\alpha} \land \left( \bigvee_{t \in T_{\text{acc}}} t^j \right) \right).
\]

• Given the multi-matching tiling relation \( T_{\text{multi}} \subseteq T \times T \), and assuming \( \varepsilon \) satisfies \( \phi_{\text{tiling}}^j \land \phi_{\text{tiling}}^{j+1} \), on every position, the tile type in \( T^j \) and the tile type in \( T^{j+1} \) are in the relation \( T_{\text{multi}} \) (fulfilling the requirements of \( (m-\text{multi}) \)):

\[
\phi_{\text{multi}}^j \overset{\text{def}}{=} \forall x \left( \bigwedge_{\alpha \in \{0, n-1\}} h_{\alpha} \land \left( \bigvee_{(t,t') \in T_{\text{multi}}} (t^j \land t^{j+1}) \right) \right).
\]

It is time to wrap-up. Given a finite set of propositional variables \( X = \{r_1, \ldots, r_\beta\} \), we write \( \exists X \psi \) to denote the formula \( \exists r_1 \exists r_2 \cdots \exists r_\beta \psi \). \( \forall X \psi \) is defined similarly. Given an instance \( \mathcal{I} \) of \( \text{AMTP} \) made of \( n, \langle T, H, V \rangle, T_0, T_{\text{acc}}, T_{\text{multi}} \), let us define the formula \( \phi_{\mathcal{I}} \) below:

\[
\phi_{\mathcal{I}} \overset{\text{def}}{=} \left( \bigwedge_{j \in [1, n]} \phi_{\text{init}}^j \right) \land \left( \bigwedge_{j \in [n+1, 2n]} \phi_{\text{coinci}}^j \right) \land \left( \bigwedge_{j \in [n+1, 2n]} \phi_{\text{multi}}^j \right) \land \left( \bigwedge_{j \in [n+1, 2n]} \phi_{\text{acc}}^j \right).
\]

Now, we can state the correctness of the reduction. The proof is a bit tedious but has no serious difficulties, as all the conditions for being a solution of \( \mathcal{I} \) can be easily expressed, as soon as the grid \( [0, 2^n - 1] \times [0, 2^n - 1] \) is encoded. Moreover, the quantifications involved in \( \text{AMTP} \) are straightforwardly taking care of in \( \text{QCTL}^X_{\leq N} \) thanks to the presence of propositional quantification.

**Lemma 3.13.** \( \mathcal{I} \) is a positive instance of \( \text{AMTP} \) iff \( \phi_{\mathcal{I}} \) is satisfiable in \( \text{QCTL}^X_{\leq N} \).

**Proof.** Let \( \mathcal{I} \) be the instance \( n, \langle T, H, V \rangle, T_0, T_{\text{acc}}, T_{\text{multi}} \) of \( \text{AMTP} \). Before showing that the reduction is correct, we need to state preliminary properties. Moreover, in the proof below, we repeat several formula definitions to follow more smoothly the technical developments.

**GRID** We recall that \( \operatorname{grid}(2n) \) is defined as the formula below

\[
\left( \bigwedge_{i \in [0, 2n-1]} \left( \bigwedge_{x,y \in \{0,\ldots,2n-1\}} \left( \bigwedge_{t \in T_0} (t \land (x \lor y)) \land \bigwedge_{t \in T} (t \land \neg (x \lor y)) \right) \right) \right) \land \left( \bigvee_{j \in [0, n-1]} \left( \neg \left( \left( \bigwedge_{x \in \{0,\ldots,2n-1\}} h_{x} \leftrightarrow \bigwedge_{y \in \{0,\ldots,2n-1\}} h_{y} \right) \lor \left( \bigwedge_{x \in \{0,\ldots,2n-1\}} v_{x} \leftrightarrow \bigwedge_{y \in \{0,\ldots,2n-1\}} v_{y} \right) \right) \right).
\]
Given a tree model $\mathfrak{T}$ and a root node $\varepsilon$, one can show that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$ iff the properties below hold:

(a) For all $j \in [0, 2n - 1]$, we have that $\varepsilon E^j v$ implies that $v$ has exactly two children.

(b) For all distinct nodes $v, v'$ such that $\varepsilon E^{2n} v$ and $\varepsilon E^{2n} v'$, there is some propositional variable $r$ in $\{h_{n-1}, \ldots, h_0, v_{n-1}, \ldots, v_0\}$, such that $v$ satisfies $r$ iff $v'$ does not satisfy $r$.

Satisfaction of (a) is essentially due to the fact that $\mathbf{EX}_{=2} \top$ is defined as

$$\exists x_1, x_2 \ \text{diff-nom}(x_1, x_2, 1) \land \text{AX}(x_1 \lor x_2)$$

and one can check that it holds true on nodes having exactly two children. For the satisfaction of (b), we need to invoke Lemma 3.2, Lemma 3.3 and Lemma 3.5. Assuming that $\text{diff-nom}(x, y, 2n)$ holds, is equivalent to have two distinct nodes that are the respective interpretations of the nominals $x$ and $y$ for the depth $2n$. More generally, Lemma 3.12 states the main properties that are used about $\text{grid}(2n)$.

(TILING) Let us state a few properties about the conjunction $\phi^j_{\text{cov}} \land \phi^j_{\mathcal{H}} \land \phi^j_v$, assuming that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$ (and therefore the set of nodes of distance $2n$ from the root encodes the grid $[0, 2^n - 1] \times [0, 2^n - 1]$). The formula $\phi^j_{\text{cov}}$ defined as

$$\forall x \ \text{nom}(x, 2n) \rightarrow \@^2_x (\bigvee_{t \in T} t^j \land \bigwedge_{t \neq t' \in T} \neg(t^j \land t'^j))$$

states that all the nodes at distance $2n$ from the root satisfy exactly one tile type from $T^j$. This is again a consequence of Lemma 3.2 and Lemma 3.3. The formula $\phi^j_{\mathcal{H}}$ is defined as:

$$\forall x, y \ (\text{nom}(x, 2n) \land \text{nom}(y, 2n) \land \text{HN}(x, y)) \rightarrow \bigvee_{(t, t') \in \mathcal{H}} \@^2_x t^j \land \@^2_y t'^j,$$

where $\text{HN}(x, y)$ is the formula below:

$$\text{HN}(x, y) \overset{\text{def}}{=} \left( \bigwedge_{\alpha \in [0, n-1]} \@^2_x v_\alpha \leftrightarrow \@^2_y v_\alpha \right) \land \bigvee_{i \in [0, n-1]} \left( \@^2_x h_i \land \@^2_y h_i \land \bigwedge_{\alpha \in [i+1, n]} \left( \@^2_x h_\alpha \land \@^2_y h_\alpha \right) \right).$$

The formula $\text{HN}(x, y)$ expresses that, assuming that $x$ and $y$ for the depth $2n$, the two nodes at distance $2n$ interpreted respectively by $x$ and $y$ and representing respectively the positions $(\mathfrak{S}, \mathfrak{V})$ and $(\mathfrak{S}', \mathfrak{V}')$ of the grid, satisfies $\mathfrak{V} = \mathfrak{V}'$ and $\mathfrak{S}' = \mathfrak{S} + 1$. So, $\phi^j_{\mathcal{H}}$ encodes the horizontal constraints for the set of tile types $T^j$. Similarly, $\phi^j_v$ encodes the vertical constraints for the set of tile types $T^j$. Indeed, in $\text{VN}(x, y)$ we swap the variable $h_{n}$ with the variable $v_{\alpha}$ (with respect to the definition for $\text{HN}(x, y)$), which amounts to state that assuming that $x$ and $y$ are nominals for the depth $2n$ from $\varepsilon$, the two nodes at distance $2n$ interpreted respectively by $x$ and $y$ and representing respectively the positions $(\mathfrak{S}, \mathfrak{V})$ and $(\mathfrak{S}', \mathfrak{V}')$ of the grid, satisfies $\mathfrak{V}' = \mathfrak{V} + 1$ and $\mathfrak{S}' = \mathfrak{S}$. So, assuming that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$, the formula $\phi^j_{\text{cov}} \land \phi^j_{\mathcal{H}} \land \phi^j_v$ expresses that the way the tile types from $T^j$ holds on the nodes at distance $2n$ from the root defines a proper tiling.

(INIT) Let $\phi^j_{\text{init}}$ be the formula below:

$$\forall x \ (\text{nom}(x, 2n) \land \@^2_x \left( \bigwedge_{\alpha \in [0, n-1]} \neg h_\alpha \right)) \rightarrow \@^2_x \left( \bigvee_{t \in T_0} t^j \land \bigwedge_{t \neq t' \in T_0} \neg(t^j \land t'^j) \right).$$
Note that \((\bigwedge_{\alpha \in [0,n-1]} -h_{\alpha})\) states on a node \(v\) at distance \(2n\) from the root, that all the propositional variables \(h_{\alpha}\) are false and therefore this is a node on the row zero of the grid. \(\phi_{\text{init}}^{j}\), therefore states (using again Lemma 3.2 and Lemma 3.3) that all the nodes of the row zero have a unique tile type from \(T_0\).

**(COINCI)** Let \(\phi_{\text{coinci}}^{j,j'}\) be the formula below (with \(j, j' \in \mathbb{N}\)):

\[
\forall x \ (\text{nom}(x, 2n) \land \bigoplus_x^{2n} (\bigwedge_{\alpha \in [0,n-1]} -h_{\alpha})) \rightarrow \bigoplus_x^{2n} (i^j \land t^{j'}).
\]

Assuming that \(\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^{j} \land \phi_{\text{init}}^{j}\), the formula \(\phi_{\text{coinci}}^{j,j'}\) states that for every node of the row zero, the tile type from \(T_0\) is the same as the tile type from \(T_0^{j'}\). The reasoning is exactly the same as for (INIT).

**(ACCEPT)** Let \(\phi_{\text{acc}}^{j}\) be the formula below:

\[
\exists x \ \text{nom}(x, 2n) \land \bigoplus_x^{2n} ((\bigwedge_{\alpha \in [0,n-1]} h_{\alpha}) \land \bigvee_{t \in T_{\text{acc}}} t^j).
\]

Note that \((\bigwedge_{\alpha \in [0,n-1]} h_{\alpha})\) states on a node \(v\) at distance \(2n\) from the root, that all the propositional variables \(h_{\alpha}\) are true and therefore this is a node on the (last) row \(2^n - 1\) of the grid.

Assuming that \(\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^{j}\), the formula \(\phi_{\text{acc}}^{j}\) therefore states that there is a node encoding a position of the grid on the last row such that the tile type is in \(T_{\text{acc}}^{j}\).

**(MULTI)** Finally, let \(\phi_{\text{multi}}^{j}\) be the formula below:

\[
\forall x \ \text{nom}(x, 2n) \rightarrow \bigoplus_x^{2n} (\bigvee_{(t,t') \in T_{\text{multi}}} t^j \land t^{j+1}).
\]

As done in (INIT), \(\phi_{\text{multi}}^{j}\) (assuming that \(\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^{j} \land \phi_{\text{init}}^{j+1}\)) states that for all nodes at distance \(2n\) from the root, the tile type from \(T^{j}\) and the tile type from \(T^{j+1}\) are in the relation \(T_{\text{multi}}\). Consequently,

\[
\mathfrak{T}, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [1,n]} \phi_{\text{init}}^{j}) \land (\bigwedge_{j \in [n+1,2n]} \phi_{\text{tiling}}^{j}) \land (\bigwedge_{j \in [n+1,2n]} \phi_{\text{coinci}}^{j}) \land (\bigwedge_{j \in [n+1,2n-1]} \phi_{\text{multi}}^{j} \land \phi_{\text{acc}}^{2n})
\]

if and only if the initial condition induced from the satisfaction of \((\bigwedge_{j \in [1,n]} \phi_{\text{init}}^{j})\) (see the definition of AMTP in Section 2.5) of the form \((w_1, \ldots, w_n) \in (T_0)_{T_0}^n\), and the multi-tiling \((\tau_1, \ldots, \tau_n)\) induced by the satisfaction of \((\bigwedge_{j \in [n+1,2n]} \phi_{\text{tiling}}^{j})\), entails that \((\tau_1, \ldots, \tau_n)\) is a solution and satisfies the condition (m-init), (m-tiling), (m-multi) and (m-accept). Indeed, satisfying \(\mathfrak{T}, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [n+1,2n]} \phi_{\text{tiling}}^{j})\) defines a multi-tiling \((\tau_1, \ldots, \tau_n)\) (and reciprocally). Similarly, satisfying \(\mathfrak{T}, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [1,n]} \phi_{\text{init}}^{j})\) defines an initial condition \((w_1, \ldots, w_n) \in (T_0)_{T_0}^n\) (and reciprocally). The details are omitted but it does not pose any difficulty. So given an initial condition \(c = (w_1, \ldots, w_n) \in (T_0)_{T_0}^n\), we write \(\mathfrak{T}_c\) to denote a tree model such that \(\mathfrak{T}_c, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [1,n]} \phi_{\text{init}}^{j})\) and on the grid induced by \(\mathfrak{T}_c\), we have precisely the initial condition \(c\). Similarly, given a multi-tiling \(M = (\tau_1, \ldots, \tau_n)\), we write \(\mathfrak{T}_M\) to denote a tree model such that \(\mathfrak{T}_M, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [n+1,2n]} \phi_{\text{tiling}}^{j})\) and on the grid induced by \(\mathfrak{T}_M\), we have precisely the multi-tiling \((\tau_1, \ldots, \tau_n)\). More generally, given \(c\) and \(M\), we write \(\mathfrak{T}_{c,M}\) to denote a tree model such that \(\mathfrak{T}_{c,M}, \varepsilon \models \text{grid}(2n) \land (\bigwedge_{j \in [1,n]} \phi_{\text{init}}^{j})\) and on the grid induced by \(\mathfrak{T}_{c,M}\), we have precisely the initial condition \(c\) and the multi-tiling \((\tau_1, \ldots, \tau_n)\).
grid(2n) \land (\bigwedge_{j \in [1,n]} \phi^j_{\text{init}}) \land (\bigwedge_{j \in [n+1,2n]} \phi^j_{\text{tiling}}) \; \text{and on the grid induced by } \Sigma_{c,M}, \text{ we have precisely the initial condition } c \text{ and the multi-tiling } M. \text{ Reciprocally, assuming a tree model } \mathfrak{I} \text{ such that } \Sigma, \varepsilon, \models \text{grid}(2n) \land (\bigwedge_{j \in [1,n]} \phi^j_{\text{init}}), \text{ we write } c_{\mathfrak{I}} = (w_1, \ldots, w_n) \text{ to denote the initial condition from the grid defined by } \mathfrak{I}. \text{ Similarly, assuming a tree model } \mathfrak{I} \text{ such that } \Sigma, \varepsilon, \models \text{grid}(2n) \land (\bigwedge_{j \in [n+1,2n]} \phi^j_{\text{tiling}}), \text{ we write } M_{\mathfrak{I}} = (1, \ldots, n) \text{ to denote the multi-tiling from the grid defined by } \mathfrak{I}.

It remains to conclude by explaining how to handle the quantifications over the tuples \((w_1, \ldots, w_n) \in (\bigwedge_{j \in [1,n]} \phi^j_{\text{init}})^n\). The instance \mathfrak{I} is positive iff

- (by definition) for all \(w_1 \in \mathcal{T}_0^2 \cap \mathcal{T}_0^n\), there is \(w_2 \in \mathcal{T}_0^2 \cap \mathcal{T}_0^n\) such that \(\ldots\) for all \(w_{n-1} \in \mathcal{T}_0^2 \cap \mathcal{T}_0^n\), there is \(w_n \in \mathcal{T}_0^2 \cap \mathcal{T}_0^n\) such that there is a solution \((1, \ldots, n)\) for \((w_1, \ldots, w_n)\), iff

- (by the above correspondences) there is a tree model \(\mathfrak{I}^0\) such that \(\mathfrak{I}^0, \varepsilon, \models \text{grid}(2n)\) such that for all \(\mathfrak{I}^1\) such that \((\mathfrak{I}^1 \equiv_{\text{AP} \setminus \mathcal{T}_0} \mathfrak{I}^0), \mathfrak{I}^1, \varepsilon \models \phi^1_{\text{init}}\) and \(w_1^{\mathfrak{I}^1} = w_1\) (where \(w_1^{\mathfrak{I}^1}\) is the new word induced by \(\mathfrak{I}^1\)) there is \(\mathfrak{I}^2\) such that \((\mathfrak{I}^2 \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^0} \mathfrak{I}^1), \mathfrak{I}^2, \varepsilon \models \phi^2_{\text{init}}\) and \(w_2^{\mathfrak{I}^2} = w_2\) ... for all \(\mathfrak{I}^{n-1}\) such that \((\mathfrak{I}^{n-1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-2}), \mathfrak{I}^{n-1}, \varepsilon \models \phi^{n-1}_{\text{init}}\) and \(w_{n-1}^{\mathfrak{I}^{n-1}} = w_{n-1}\) there is \(\mathfrak{I}^n\) such that \((\mathfrak{I}^n \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-1}), \mathfrak{I}^n, \varepsilon \models \phi^n_{\text{init}}\) and \(w_{n}^{\mathfrak{I}^n} = w_n\), there is \(\mathfrak{I}^{n+1}\) such that \((\mathfrak{I}^{n+1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n+1}} \mathfrak{I}^{n+1}), \mathfrak{I}^{n+1}, \varepsilon \models \phi_{\text{init}}\) and \(M_{\mathfrak{I}^{n+1}}\) is a solution for \(c_{\mathfrak{I}^{n+1}}\), iff

- (by first-order reasoning) there is a tree model \(\mathfrak{I}^0\) such that \(\mathfrak{I}^0, \varepsilon, \models \text{grid}(2n)\) such that for all \(\mathfrak{I}^1\) such that \((\mathfrak{I}^1 \equiv_{\text{AP} \setminus \mathcal{T}_0} \mathfrak{I}^0)\) there is \(\mathfrak{I}^2\) such that \((\mathfrak{I}^2 \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^0} \mathfrak{I}^1)\) ... for all \(\mathfrak{I}^{n-1}\) such that \((\mathfrak{I}^{n-1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-2})\) there is \(\mathfrak{I}^n\) such that \((\mathfrak{I}^n \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-1})\), if \(\mathfrak{I}^n, \varepsilon \models (\bigwedge_{j \in [1,n]} \phi^j_{\text{init}})\), then there is \(\mathfrak{I}^{n+1}\) such that \((\mathfrak{I}^{n+1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n+1}} \mathfrak{I}^{n+1}), \mathfrak{I}^{n+1}, \varepsilon \models \phi_{\text{init}}\) and \(M_{\mathfrak{I}^{n+1}}\) is a solution for \(c_{\mathfrak{I}^{n+1}}\), iff

- (by the encodings of \(\text{m-init}, \text{m-coinci}, \text{m-multi}, \text{m-accept}\)) there is a tree model \(\mathfrak{I}^0\) such that \(\mathfrak{I}^0, \varepsilon, \models \text{grid}(2n)\) such that for all \(\mathfrak{I}^1\) such that \((\mathfrak{I}^1 \equiv_{\text{AP} \setminus \mathcal{T}_0} \mathfrak{I}^0)\) there is \(\mathfrak{I}^2\) such that \((\mathfrak{I}^2 \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^0} \mathfrak{I}^1)\) ... for all \(\mathfrak{I}^{n-1}\) such that \((\mathfrak{I}^{n-1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-2})\) there is \(\mathfrak{I}^n\) such that \((\mathfrak{I}^n \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n-1}} \mathfrak{I}^{n-1})\), if \(\mathfrak{I}^n, \varepsilon \models (\bigwedge_{j \in [1,n]} \phi^j_{\text{init}})\), then there is \(\mathfrak{I}^{n+1}\) such that \((\mathfrak{I}^{n+1} \equiv_{\text{AP} \setminus \hat{\mathfrak{I}}^{n+1}} \mathfrak{I}^{n+1}), \mathfrak{I}^{n+1}, \varepsilon \models \phi_{\text{init}}\) and \(M_{\mathfrak{I}^{n+1}}\) is a solution for \(c_{\mathfrak{I}^{n+1}}\), iff

- (by definition of \(\models\)) the formula below is satisfiable:

\[
\text{grid}(2n) \land \forall \mathcal{T}_1^2 \exists \mathcal{T}_0^2 \land \mathcal{T}_0^3 \ldots \exists \mathcal{T}_0^n \forall j \in [1,n] \phi^j_{\text{init}} \rightarrow \exists \{i^j : t \in \mathcal{T}, j \in [n + 1, 2n]\} (\bigwedge_{j \in [n + 1, 2n]} \phi^j_{\text{tiling}} \land \phi^j_{\text{coinci}}) \land (\bigwedge_{j \in [n + 1, 2n - 1]} \phi^j_{\text{multi}}) \land \phi^2_{\text{acc}}.
\]

The above lemma leads us to one of the main results of the paper.

**Theorem 3.14.** For all \(N \geq 2\), the satisfiability problem for \(\text{QCTL}^1_{X, \leq N}\) is \(\text{AEXP}_{\text{pol.}}\)-hard.

As a consequence, for all \(N \geq 2\), the satisfiability problem for \(\text{QK}^2_{X, \leq N}\) is \(\text{AEXP}_{\text{pol.}}\)-complete too where \(\text{QK}^2_{X, \leq N}\) is interpreted on finite trees whose branching degree is at most \(N\). The \(\text{AEXP}_{\text{pol.}}\) lower bound can be obtained using the same reduction as for \(\text{QCTL}^1_{X, \leq N}\), whereas the \(\text{AEXP}_{\text{pol.}}\) upper bound uses also the same arguments as for \(\text{QCTL}^1_{X, \leq N}\).
4. Tower-hardness of the satisfiability problem \(\text{QCTL}^k_X\)

We are back to the (general) satisfiability problem for \(\text{QCTL}^k_X\), i.e. with no further restrictions on the number of children per node.

4.1. Overview of the method. In order to show Tower-hardness, we shall reduce the \(k\)-NExpTime-complete tiling problem \(\text{Tiling}^{k}_{\mathbb{E}}\) introduced in Section 2.5 to \(\text{SAT}(\text{QCTL}^k_X)\) and this should be done in a uniform way so that Tower-hardness can be concluded (see the discussion in [49, Section 3.1.2]). Hence, we need to encode concisely a grid \(t(k,n) \times t(k,n)\) and to do so, the main task consists in enforcing that a node has \(t(k,n)\) children with a formula of elementary size in \(k + n\) (bounded by a tower of exponentials of fixed height).

Actually, our method produces a formula of exponential size in \(k + n\). Of course, this is not the end of the story as we need to encode the grid \(t(k,n) \times t(k,n)\) and to express on it constraints about the tiling \(\tau\). First, let us explain how to enforce that a node has exactly \(t(k,n)\) children, by partly taking advantage of the proof technique of local nominals (see Section 3.1).

We recall that \(t(1,n)=2^n\) and \(t(k+1,n)=2^{t(k,n)}\) for \(k > 0\). For the forthcoming subsections, we assume that \(n\) is fixed. Below, we classify the nodes of a tree model by their type (a value in \(\mathbb{N}\)) such that any node is of type 0, and if a node is of type \(k > 0\), then it has exactly \(t(k,n)\) children and all the children are of type \(k - 1\). To be more precise, a node may have two types (as one of them is always zero). So, a node of type 1 has exactly \(2^n\) children, a node of type 2 has exactly \(2^{2n}\) children, etc. As every node is of type 0 and for \(k > 0\), being of type \(k\) entails that the total number of children equal to \(t(k,n)\), if a node is of type \(k > 0\), the value \(k\) is unique. Additional conditions apply for being of type \(k > 0\) but we can already notice that a node \(v\) of type \(k\) implicitly defines a balanced subtree of depth \(k\) with root \(v\).

In order to enforce that a node is of type \(k \geq 1\), (this is a trivial property for \(k = 0\)), and therefore has exactly \(t(k,n)\) children, with each node \(v\) of type \(k \geq 0\) is associated a number in \([0, t(k+1,n)−1]\). Such a number is written \(\text{nb}_\mathcal{T}(v)\). The subscript \(\mathcal{T}\) may be omitted when the context is clear. Similarly, when \(\mathcal{T}, v_0 \models \text{nom}(x_1, 1) \land \bigwedge_{i \in [2,d]} \alpha_{x_1}^{1} \circ \alpha_{x_2}^{1} \circ \cdots \circ \alpha_{x_{i-1}}^{1} \text{nom}(x_i, 1)\) and the nodes \(v_1, \ldots, v_d\) are associated with \(x_1, \ldots, x_d\), we write \(\text{nb}_{\mathcal{T}}(x_i)\) instead of \(\text{nb}_{\mathcal{T}}(v_i)\) for all \(i \in [1,d]\) (in general, by a slight abuse of notation, we may refer to a node by its local nominal when it exists).

When the type of the node \(v\) is zero, its number is defined as the unique \(m\) such that the number represented by the truth values of \(p_{n−1}, p_{n−2}, \ldots, p_0\) is equal to \(m\). As usual, the propositional variable \(p_i\) is responsible for the \(i\)th bit of the number \(m\) and by convention, the least significant bit is encoded by the truth value of \(p_0\). We illustrate the encoding below:
Otherwise, when the type of $v$ is equal to some $k > 0$, its number is represented by the binary encoding of the propositional variable $val$ on its children assuming that there are $t(k, n)$ children whose respective (bit) numbers span all over $[0, t(k, n) − 1]$ and therefore all the children are implicitly ordered. This principle makes sense conceptually but it remains to express it in $\text{QCTL}_X$, similarly to the Tower-hardness proof in [45] for the fluted fragment in which counters with high values has to be enforced within a restricted language (see also [55]). That is why, in Table 1, we present a list of formulae to be defined. All of them are interpreted on a node $v_0$ of level $k \geq 0$, $1 \leq d \leq k$, and $\vec{x}, \vec{y}$ denotes the formula

$$\text{nom}(x_1, 1) \land \text{nom}(y_1, 1) \land \bigwedge_{i \in [2, d]} (\@_{x_1} \@_{x_2} \cdots \@_{x_{i-1}} \text{nom}(x_i, 1) \land \@_{y_1} \@_{y_2} \cdots \@_{y_{i-1}} \text{nom}(y_i, 1)).$$

Assuming that the nodes $v_1, \ldots, v_d$ are associated with $\vec{x} = x_1, \ldots, x_d$ (resp. $v_1', \ldots, v_d'$ are associated with $\vec{y} = y_1, \ldots, y_d$), $v_0, v_1, \ldots, v_d$ and $v_0, v_1', \ldots, v_d'$ can be understood as two branches rooted at $v_0$ ending at the node $v_d$ and at the node $v_d'$ respectively. The formula $\vec{x}, \vec{y}$ uses subformulae introduced in Section 3.1 and the wide hat symbol in $\vec{x}, \vec{y}$ above $\vec{x}$ and $\vec{y}$ is a graphical reminder of these two branches. By contrast, the specific formula $\vec{x}, \vec{x}$ states the existence of a single branch with nodes $v_0, \ldots, v_d$.

| Formulae to be defined | Intuitive meaning |
|------------------------|-------------------|
| $\text{type}(k)$       | $v_0$ is of type $k$ |
| $\text{first}(k)$      | $\text{nb}(v_0) = 0$ |
| $\text{last}(k)$       | $\text{nb}(v_0) = t(k + 1, n) − 1$ |
| $\text{uniq}(k)$       | $\forall v, v' ((v_0Ev) \land (v_0Ev') \land v \neq v') \rightarrow \text{nb}(v) \neq \text{nb}(v')$ |
| $\text{compl}(k)$      | $\forall v ((v_0Ev) \land \text{nb}(v) < t(k, n) − 1) \rightarrow \exists v' (v_0Ev') \land \text{nb}(v') = \text{nb}(v) + 1$ |
| $\text{nb}(\vec{x}) = k \text{nb}(\vec{y})$ | $\text{nb}(v_d) = \text{nb}(v_d')$ |
| $\text{nb}(\vec{y}) = k \text{nb}(\vec{x}) + 1$ | $\text{nb}(v_d') = 1 + \text{nb}(v_d)$ |
| $\text{nb}(\vec{x}) < k \text{nb}(\vec{y})$ | $\text{nb}(v_d) < \text{nb}(v_d')$ |

Table 1: Family of auxiliary formulae.

In the last 3 lines of the table, the subscript ‘$k$’ below ‘$=$’ and ‘$<$’ allows us to remember that the formula is evaluated at a node of type $k$; as $\vec{x}$ and $\vec{y}$ are of length $d$, the number comparison is done on nodes of type $k − d$ and the numbers can take values in $[0, t(k − d + 1, n) − 1]$. Though most of the intuiting meanings are straightforward, let us notice that $\text{uniq}(k)$ is intended to express that two distinct children of $v_0$ have distinct numbers. Similarly, $\text{compl}(k)$ is intended to express that if a child of $v_0$ has a number $n$ strictly less than $t(k, n) − 1$, then $v_0$ has also another child with number equal to $n + 1$ (‘compl’ in $\text{compl}(k)$ stands for ‘complete’).
In order to define type\((k)\) \((k \geq 1)\), we specify that every child is of type \(k - 1\), there is a child with number equal to zero, and if a child has number \(m < t(k, n) - 1\), then there is a child with number equal to \(m + 1\). Moreover, two distinct children have distinct numbers in \([0, t(k, n) - 1]\). Satisfying these conditions guarantees that the number for the children span all over \([0, t(k, n) - 1]\). The formula type\((k)\) \((k \geq 1)\) is defined as

\[
\text{type}(k) \overset{\text{def}}{=} \text{AX}(\text{type}(k-1)) \land \text{EX}(\text{first}(k-1)) \land \text{uniq}(k) \land \text{compl}(k).
\]

Note that the above formula is intended to be built over the propositional variables \(p_0, \ldots, p_{n-1}, \text{val}\) (only).

Let us first explain how we proceed to define all the mentioned formulae. For successive values \(N \in \mathbb{N}\), we define inductively the formulae:

- \(\text{type}(N)\), \(\text{first}(N)\) and \(\text{last}(N)\),
- \(\text{nb}(x_1, \ldots, x_d) =_k \text{nb}(y_1, \ldots, y_d)\),
- \(\text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d)\),
- \(\text{nb}(y_1, \ldots, y_d) =_k \text{nb}(x_1, \ldots, x_d) + 1\)

for all \(k - d = N - 1\). For \(N = 0\), only the formulae \(\text{type}(0)\), \(\text{first}(0)\) and \(\text{last}(0)\) make sense. The case \(N = 1\) is not yet an instance of the general case. We first treat the cases for \(N \in \{0, 1\}\) and then we proceed with the general case \(N \geq 2\).

### 4.2. Formulae for levels zero and one.

When \(k = 0\) \((i.e., k = N\) and \(N = 0\)) only the intended properties for the formulae type\((0)\), \(\text{first}(0)\) and \(\text{last}(0)\) are meaningful. Let us define them, in the simplest way.

\[
\begin{align*}
\text{type}(0) & \overset{\text{def}}{=} \top, \\
\text{first}(0) & \overset{\text{def}}{=} \neg p_{n-1} \land \cdots \land \neg p_0, \\
\text{last}(0) & \overset{\text{def}}{=} p_{n-1} \land \cdots \land p_0.
\end{align*}
\]

It can be rapidly checked that \(\Sigma, v \models \text{type}(0)\) iff \(v\) is of type 0. Moreover, assuming that \(v\) is understood as a node of type 0, we have \(\Sigma, v \models \text{first}(0)\) iff \(\text{nb}_v(v) = 0\) and \(\Sigma, v \models \text{last}(0)\) iff \(\text{nb}_v(v) = t(1, n) - 1 = 2^n - 1\).

We next focus on the case when \(N = 1\) and we define the formulae type\((k)\), \(\text{first}(k)\) and \(\text{last}(k)\) with \(k = 1\) \((i.e., k = N = 1)\) as well as \(\text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d)\) and \(\text{nb}(y_1, \ldots, y_d) =_k \text{nb}(x_1, \ldots, x_d) + 1\) with \(k - d = 0\) (that is when \(k - d = N - 1\) with \(N = 1\)). We stress that \(k\) and \(d\) can be arbitrarily large as soon as \(k = d\).

To start with the formula \(\text{nb}(x_1, \ldots, x_d) =_k \text{nb}(y_1, \ldots, y_d)\), it can be easily defined in terms of \(\text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d)\) as:

\[
\neg (\text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d)) \land \neg (\text{nb}(y_1, \ldots, y_d) <_k \text{nb}(x_1, \ldots, x_d)).
\]

Second, we turn our attention to the formula type\((1)\). It states that there is a child with number equal to zero, if a child has number \(m < 2^n - 1\), then there is a child with number equal to \(m + 1\), all the children are of type 0, and two distinct children have distinct numbers in \([0, 2^n - 1]\). Remember that the number of each child (of type 0) is computed from the propositional variables in \(\{p_{n-1}, \ldots, p_0\}\). The arithmetical reasoning between children, leading to the fact that there are exactly \(2^n\) children whose numbers span all over \([0, 2^n - 1]\) takes advantage of standard arithmetical properties on numbers encoded by \(n\) bits. Here is the formula type\((1)\):

\[
\text{type}(1) \overset{\text{def}}{=} \text{AX}(\text{type}(0)) \land \text{EX}(\text{first}(0)) \land \text{uniq}(1) \land \text{compl}(1).
\]
It remains to specify what exactly are the formulae uniq(1) and compl(1). In order to define uniq(1), responsible for enforcing the uniqueness among the children’s numbering, we simply state that there are no two distinct children (of type 0) having the same number:

\[ \forall x, y \; \text{diff-nom}(x, y, 1) \to \neg (\text{nb}(x) =_1 \text{nb}(y)). \]

Note that the formula \text{diff-nom}(x, y, 1) guarantees that we pick two distinct children and the nominals \( x \) and \( y \) allow us to access to them (and check the values of the propositional variables in \( \{p_{n-1}, \ldots, p_0\} \)). Consult Lemma 3.5 for the correctness.

The formula compl(1) below states that for each child \( v \) (of type 0) that is not the last one (i.e. does not have the highest possible number among all other nodes), there is also a child \( v' \) (also of type 0) such that \( \text{nb}_T(v') = \text{nb}_T(v) + 1 \). Here is the formula compl(1):

\[ \forall x \; (\text{nom}(x, 1) \land \neg \text{last}(0)) \to \exists y \; \text{nom}(y, 1) \land \text{nb}(y) =_1 \text{nb}(x) + 1. \]

Finally, it remains to define the formulae \( \text{nb}(x) =_1 \text{nb}(y) \) and \( \text{nb}(y) =_1 \text{nb}(x) + 1 \) used respectively in uniq(1) and compl(1). Below, we treat the more general situation with \( k = d \) (\( k \) is not necessarily equal to 1), and \( \text{nb}(x) =_1 \text{nb}(y) \) and \( \text{nb}(y) =_1 \text{nb}(x) + 1 \) are specific instances with \( k = d = 1 \). Let assume that \( \hat{x} = x_1, \ldots, x_k \) and \( \hat{y} = y_1, \ldots, y_k \) (thus \( d = k \)). The forthcoming definitions are standard and rely on elementary operations on binary encoding of natural numbers with \( n \) bits (again, the least significant bit is represented by the truth value of \( p_0 \)).

- \( \text{nb}(y_1, \ldots, y_k) =_k \text{nb}(x_1, \ldots, x_k) + 1 \) is defined as

\[
\bigg( \bigvee_{i=0}^{n-1} \left( \bigwedge_{j=0}^{i-1} -p_j \land \bigvee_{j=0}^{i-1} p_j \right) \bigg) \land \bigg( \bigwedge_{j=0}^{i-1} -p_j \land \bigvee_{j=0}^{i-1} p_j \bigg) \land \bigg( \bigwedge_{j=i+1}^{n-1} \bigwedge_{j=0}^{i} p_j \Leftrightarrow \bigwedge_{j=0}^{i} y_j p_j \bigg)
\]

- \( \text{nb}(x_1, \ldots, x_k) <_k \text{nb}(y_1, \ldots, y_k) \) is defined as

\[
\bigg( \bigvee_{i=0}^{n-1} \left( \bigwedge_{j=0}^{i-1} p_j \land \bigwedge_{j=0}^{i-1} -p_j \land \bigvee_{j=i+1}^{n-1} \bigwedge_{j=0}^{i} p_j \Leftrightarrow \bigwedge_{j=0}^{i} y_j p_j \right) \bigg)
\]

For the sake of completeness, we define first(1) \( \overset{\text{def}}{=} \text{AX}(\neg \text{val}) \) and last(1) \( \overset{\text{def}}{=} \text{AX}(\text{val}) \).

Here is a preliminary lemma that states that we have properly proceeded for the binary encoding of numbers with the propositional variables in \( p_{n-1}, \ldots, p_0 \) (and Lemma 3.4–3.5 need to be used).

**Lemma 4.1.** Let \( \mathcal{T} \) be a tree model and \( v \) be one of its nodes such that \( v \) satisfies \( \hat{x}, \hat{y} \) (\( \hat{x} \) and \( \hat{y} \) are both of length \( k \)) and, \( v_k \) and \( v'_k \) are understood as nodes of type 0.

(I) : \( \mathcal{T}, v \models \text{nb}(\hat{y}) =_k \text{nb}(\hat{x}) + 1 \) iff \( \text{nb}_T(v'_k) = 1 + \text{nb}_T(v_k) \).

(II): \( \mathcal{T}, v \models \text{nb}(\hat{x}) <_k \text{nb}(\hat{y}) \) iff \( \text{nb}_T(v_k) < \text{nb}_T(v'_k) \).

(III): \( \mathcal{T}, v \models \text{nb}(\hat{x}) =_k \text{nb}(\hat{y}) \) iff \( \text{nb}_T(v_k) = \text{nb}_T(v'_k) \).

**Proof.** The property (III) is a direct consequence of (II). Since the proof of (II) is similar to (I), we focus on showing (I) only. In order to define \( \text{nb}(y_1, \ldots, y_k) =_k \text{nb}(x_1, \ldots, x_k) + 1 \),
and therefore to express that $\text{nb}_\Sigma(v'_k) = \text{nb}_\Sigma(v_k) + 1$ with numbers computed with the propositional variables $p_{n-1}, \ldots, p_0$, we perform a standard comparison of the respective truth values of $p_{n-1}, \ldots, p_0$ for the node $v_k$ (the interpretation of $x_k$) and for the node $v'_k$ (the interpretation of $y_k$). Typically, $\text{nb}_\Sigma(v'_k) = \text{nb}_\Sigma(v_k) + 1$ iff there is $i \in [0, n-1]$, such that

- for every $j \in [i+1, n-1]$, $v_k$ and $v'_k$ agree on $p_j$ ($\text{nb}_\Sigma(v'_k)$ and $\text{nb}_\Sigma(v_k)$ agree on their $j$th bit),
- $v_k$ does not satisfy $p_i$ and $v'_k$ satisfies $p_i$ ($\text{nb}_\Sigma(v'_k)$ and $\text{nb}_\Sigma(v_k)$ disagree on their $i$th bit and the $i$th bit of $\text{nb}_\Sigma(v_k)$ is equal to zero),
- for every $j \in [0, i-1]$, $v_k$ satisfies $p_j$ and $v'_k$ does not satisfy $p_j$.

The formula $\text{nb}(y_1, \ldots, y_k) = k \cdot (x_1, \ldots, x_k) + 1$ indeed quantifies existentially on $i$ via a generalised disjunction and the three conditions are checked by three conjuncts in the standard manner.

We conclude by presenting the main lemma that gathers all established formulæ.

**Lemma 4.2.** Let $\Sigma$ be a tree model and let $v$ be any of its nodes. The following properties hold true:

(I) : $\Sigma, v \models \text{type}(1) \text{ iff } v \text{ is of type } 1$,

(II): Assuming $v$ satisfies type(1), we have $\Sigma, v \models \text{first}(1) \text{ iff } \text{nb}_\Sigma(v) = 0$.

(III): Assuming $v$ satisfies type(1), we have $\Sigma, v \models \text{last}(1) \text{ iff } \text{nb}_\Sigma(v) = t(2, n) - 1$.

**Proof.** The properties (II)-(III) follow immediately from the way we encode numbers. Let us focus then on proving the property (I).

A node $v$ is of type 1 iff it has exactly $2^n$ children, say $v_0, \ldots, v_{2^n-1}$, and for all $j \in [0, 2^n - 1]$, the number associated to $v_j$ is precisely $j$ when encoded with the truth values of the propositional variables $p_{n-1}, \ldots, p_0$. The latter proposition amounts to have the following conditions:

(a) The node $v$ has a child whose number is zero.
(b) If $v'$ is a child of $v$ with number $m < 2^n - 1$, then $v$ has also a child with number $m + 1$.
(c) Two distinct children of $v$ have distinct numbers.

Let us recall that $\text{type}(1) = \text{AX}(\text{type}(0)) \land \text{EX}(\text{first}(0)) \land \text{uniq}(1) \land \text{compl}(1)$. Obviously, $\text{AX}(\text{type}(0))$ always holds and $\text{EX}(\text{first}(0))$ expresses exactly the condition (a). It remains to show that $\text{uniq}(1)$ (resp. $\text{compl}(1)$) characterises the conditions (c) (resp. the condition (b)). The formula $\text{uniq}(1)$ is equal to $\forall x, y \ \text{diff-nom}(x, y, 1) \rightarrow \neg(\text{nb}(x) = 1 \land \text{nb}(y))$. By Lemma 4.1(III), and by Lemma 3.5, when interpreted on a node of type 1, $\text{uniq}(1)$ states that for any two distinct children, their respective numbers are different, which is precisely the condition (c). Finally, let us recall the definition of the formula $\text{compl}(1)$:

$$\forall x \ (\text{nom}(x, 1) \land @x \neg(\text{last}(0))) \rightarrow \exists y \ \text{nom}(y, 1) \land \text{nb}(y) = 1 \land \text{nb}(x) + 1.$$  

By Lemma 4.1(I), by the fact that last(0) already characterises the nodes of type 0 whose number is $2^n - 1$. By Lemma 3.2 and Lemma 3.3, the formula $\text{compl}(1)$, states that for all children whose number $m$ is different from $2^n - 1$, there is a child with number $m + 1$, which is precisely the condition (b). This ends the proof as all the formulæ $\text{AX}(\text{type}(0))$, $\text{EX}(\text{first}(0))$, $\text{uniq}(1)$, $\text{compl}(1)$ capture exactly the properties specified above. \qed
4.3. Formulae with arbitrary \( N \geq 2 \). Let us consider the arbitrary case \( N \geq 2 \). Below, we define the formulae \( \text{type}(N) \), \( \text{first}(N) \) and \( \text{last}(N) \) as well as \( \text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d) \), and \( \text{nb}(y_1, \ldots, y_d) =_k \text{nb}(x_1, \ldots, x_d) + 1 \) with \( k - d = N - 1 \). The formula \( \text{nb}(x_1, \ldots, x_d) =_k \text{nb}(y_1, \ldots, y_d) \) is also defined but does not require much developments as by definition, it is equal to

\[
-(\text{nb}(x_1, \ldots, x_d) <_k \text{nb}(y_1, \ldots, y_d)) \land -(\text{nb}(y_1, \ldots, y_d) <_k \text{nb}(x_1, \ldots, x_d)).
\]

We assume that for all \( k < N \), the formulae \( \text{type}(k) \), \( \text{last}(k) \) and \( \text{first}(k) \) are already defined and for \( k - d \leq N - 2 \), the formulae \( \text{nb}(\bar{x}) <_k \text{nb}(\bar{y}) \), \( \text{nb}(\bar{x}) =_k \text{nb}(\bar{y}) \) and \( \text{nb}(\bar{y}) =_k \text{nb}(\bar{x}) + 1 \) are already defined too (\( \bar{x} \) and \( \bar{y} \) are of length \( d \)). This can be understood as an implicit induction hypothesis when proving the correctness of the formulae built for \( N \).

As for the case \( N = 1 \), the formula \( \text{type}(N) \) follows the general schema: it states that there is a child with number equal to zero, if a child has number \( m < t(N, n) - 1 \), then there is a child with number equal to \( m + 1 \), and two distinct children have distinct numbers in \( [0, t(N, n) - 1] \). Of course, all the children are enforced to be of type \( N - 1 \). We present the claimed formula \( \text{type}(N) \) below.

\[
\text{type}(N) \overset{\text{def}}{=} \text{AX}(\text{type}(N - 1)) \land \text{EX}(\text{first}(N - 1)) \land \text{uniq}(N) \land \text{compl}(N).
\]

Again, it remains to specify what are \( \text{uniq}(N) \) and \( \text{compl}(N) \). In order to define \( \text{uniq}(N) \), we simply state that there are no two distinct children (of type \( N - 1 \)) with the same number:

\[
\text{uniq}(N) \overset{\text{def}}{=} \forall x, y \iff \text{diff-nom}(x, y, 1) \rightarrow -(\text{nb}(x) = \text{nb}(y)).
\]

Again, the formula \( \text{diff-nom}(x, y, 1) \) allows us to select two distinct children (of type \( N - 1 \)). The formula \( \text{compl}(N) \) below states that for each child \( v \) (of type \( N - 1 \)) that is not the last one, there is also a child \( v' \) (of type \( N - 1 \) too) such that \( \text{nb}_T(v') = \text{nb}_T(v) + 1 \). Here is the formula \( \text{compl}(N) \):

\[
\text{compl}(N) \overset{\text{def}}{=} \forall x (\text{nom}(x, 1) \land @^{1}_{2}(-\text{last}(N - 1))) \rightarrow \exists y \text{nom}(y, 1) \land \text{nb}(y) = \text{nb}(x) + 1.
\]

It remains to define \( \text{nb}(x) = N \text{nb}(y) \) and \( \text{nb}(y) = N \text{nb}(x) + 1 \) used respectively in \( \text{uniq}(N) \) and in \( \text{compl}(N) \). This time, this requires much lengthier developments, apart from using the properties of the formulae constructed for \( N - 1 \) and for smaller values (implicit induction hypothesis). Below, we treat the more general situation with \( k - d = N - 1 \), and \( \text{nb}(x) =_N \text{nb}(y) \) and \( \text{nb}(y) =_N \text{nb}(x) + 1 \) are just particular instances with \( k = N \) and \( d = 1 \). Thus, let \( \bar{x} = x_1, \ldots, x_d \) and \( \bar{y} = y_1, \ldots, y_d \).

For defining \( \text{nb}(y_1, \ldots, y_k) =_k \text{nb}(x_1, \ldots, x_k) + 1 \) (see Section 4.2), we have compared the respective truth values of the propositional variables \( p_{n - 1}, \ldots, p_0 \) for the node \( v_k \) (the interpretation of \( x_k \)) and for the node \( v_k' \) (the interpretation of \( y_k \)). The same principle applied for defining \( \text{nb}(x_1, \ldots, x_k) =_k \text{nb}(y_1, \ldots, y_k) \). Typically, \( \text{nb}(y_1, \ldots, y_k) =_k \text{nb}(x_1, \ldots, x_k) + 1 \) holds iff there is an index \( i \in [0, n - 1] \), such that

- for every \( j \in [i + 1, n - 1] \), \( v_k \) and \( v_k' \) agree on \( p_j \),
- \( v_k \) does not satisfy \( p_i \) and \( v_k' \) satisfies \( p_i \),
- for \( j \in [0, i - 1] \), \( v_k \) satisfies \( p_j \), \( v_k' \) does not satisfy \( p_j \).

Thus, we needed to define a partition \( \{[i + 1, n - 1], \{i\}, [0, i - 1]\} \) of \([0, n - 1]\) (understood as the set of bit numbers to encode a value in \([0, 2^n - 1]\)). The same principle applies when the bit numbers are among \([0, t(k - d, n) - 1]\) to encode a value in \([0, t(k - d + 1, n) - 1]\). This needs to be done concisely as we cannot go through all the \( t(k - d, n) \) bit numbers because the whole reduction has to be of elementary complexity.
Now, when attempting to define \( \text{nb}(y_1, \ldots, y_d) =_k \text{nb}(x_1, \ldots, x_d) + 1 \), the nodes \( v_d \) and \( v'_d \) are of type \( k - d > 0 \) with \( t(k - d, n) \) children each. The truth values of \( \text{val} \) on their respective children determine precisely the numbers \( \text{nb}(v_d) \) and \( \text{nb}(v'_d) \). Below, we describe how look like \( v_d \)'s children.

Let \( u_{i(k-d,n)-1}, \ldots, u_0 \) be the children of \( v_d \) such that \( \text{nb}(u_j) = j \) for all \( j \). Similarly, let \( u'_{i(k-d,n)-1}, \ldots, u'_{0} \) be the children of \( v'_d \) such that \( \text{nb}(u'_j) = j \) for all \( j \). Hence, \( \text{nb}(y_1, \ldots, y_d) =_k \text{nb}(x_1, \ldots, x_d) + 1 \) holds iff there is a position \( i \in [0, t(k - d, n) - 1] \) satisfying

- for \( j \in [i + 1, t(k - d, n) - 1] \), \( u_j \) and \( u'_j \) agree on \( \text{val} \),
- \( u_i \) does not satisfy \( \text{val} \) and \( u'_i \) satisfies \( \text{val} \),
- for every \( j \in [0, i - 1] \), \( u_j \) satisfies \( \text{val} \) and \( u'_j \) does not satisfy \( \text{val} \).

We have to define a partition \( \{u_{i+1}, \ldots, u_{i(k-d,n)-1}\}, \{u_i\}, \{u_0, \ldots, u_{i-1}\} \) of \( \{u_{i(k-d,n)-1}, \ldots, u_0\} \) (and similarly for the children of \( v'_d \)). To do so, we employ the existential quantification on the fresh propositional variables \( l \) (left), \( s \) (selected bit), \( r \) (right) such that

(a) for every child of \( v_d \) (resp. \( v'_d \)), exactly one propositional variable among \( \{l, s, r\} \) holds true,

(b) exactly one child of \( v_d \) (resp. \( v'_d \)) satisfies \( s \),

(c) if \( v \) is a child of \( v_d \) satisfying \( l \) (resp. \( s \)) and \( v' \) is child of \( v_d \) satisfying \( s \) (resp. \( r \)), then \( \text{nb}^l(v) < \text{nb}^l(v') \). The same condition holds with \( v'_d \).

Below, we illustrate how are distributed the propositional variables \( l, s \) and \( r \).

Additional arithmetical constraints are needed to relate the partition of \( \bar{x} \) with the partition of \( \bar{y} \) (see below the details) but in a way, it is independent of the partition itself. For instance, the unique child of \( v_d \) satisfying \( s \) and the unique child of \( v'_d \) satisfying \( s \) should have the same (bit) number. Nevertheless, it is clear that we need, at least, to be able to state in \( \text{QCTL}_X^f \) the existence of a partition satisfying the conditions (a), (b) and (c). In the sequel, such partitions are called \( \text{lsr-partitions} \). The forthcoming formula \( \text{LSR}_d(k) \) does the job for \( v_d \) (then use \( \text{LSR}_d(k) \) for \( v'_d \)).

Take \( \bar{x} = x_1, \ldots, x_d \). In the context of the definition of \( \text{LSR}_d(k) \), we allow the limit case \( d = 0 \), with empty sequence \( \epsilon \), assuming that \( \epsilon \psi \) and \( \epsilon \rho \). Below \( 0 \leq d < k \) and the formula \( \text{LSR}_d(k) \) is defined as the conjunction \( \text{LSR}_d^k(k) \wedge \text{LSR}_d^2(k) \wedge \text{LSR}_d^3(k) \) and is
interpreted on a node $v_0$ of type $k$ satisfying $\bar{x}, \bar{x}$, and therefore this satisfaction is witnessed by the branch $v_0, \ldots, v_d$ (notations for developments below).

First, $\text{LSR}^1_k(k) \overset{\text{def}}{=} \@_{\bar{x}}(\text{EX}_{=1}(s))$, with $\text{EX}_{=1}(\psi)$ defined as $\text{EX}\psi \land \neg \exists p (\text{EX}(\psi \land p) \land \text{EX}(\psi \land \neg p))$ for a fresh $p$. Note that the formula $\text{LSR}^1_k(k)$ simply states that there is a unique child of $v_d$ satisfying $s$. Next, let $\text{LSR}^2_k(k)$ be defined below, stating that for every child of $v_d$, exactly one propositional variable among $\{l, s, r\}$ holds true:

$$\text{LSR}^1_k(k) \overset{\text{def}}{=} \@_{\bar{x}}(\text{AX}((s \lor l \lor r) \land \neg(s \land l) \land \neg(s \land r) \land \neg(l \land r))).$$

Finally, $\text{LSR}^3_k(k)$ is defined as follows.

$\@_{\bar{x}}(\forall w \forall w' \text{diff-nom}(w, w', 1) \land ((\@^1_{w}(s) \land \@^1_{w'}(r)) \lor (\@^1_{w}(l) \land \@^1_{w'}(s))) \rightarrow_nb(w') <_{k-d} nb(w)).$

The formula $\text{LSR}^3_k(k)$ states if $v$ is a child of $v_d$ satisfying $l$ (resp. $s$) and $v'$ is another child of $v_d$ satisfying $s$ (resp. $r$), then $nb_{\bar{x}}(v) < nb_{\bar{x}}(v')$. The nodes $v$ and $v'$ are obviously of type $k - d - 1$ and their respective numbers belong to $[0, t(k - d, n) - 1]$. It is important to observe that $nb(w') <_{k-d} nb(w)$ is well-defined recursively as soon as $k - d \leq N - 2$.

**Lemma 4.3.** Let $\mathfrak{T}$ be a tree model, $v_0$ be a node of type $k \geq 0$, and $\bar{x}$ be a (possibly empty) sequence of nominals $x_1, \ldots, x_d$ for some $d \in [0, k - 1]$ such that $k - d \leq N - 1$, $\mathfrak{T}, v_0 = \bar{x}, \bar{x}$, and its witness branch is $v_0, \ldots, v_d$. Then $\mathfrak{T}, v_0 \models \text{LSR}_k(k)$ iff the conditions below hold:

(a) For every child of $v_d$, exactly one propositional variable among $\{l, s, r\}$ holds true.

(b) Exactly one child of $v_d$ satisfies $s$.

(c) If $v$ is a child of $v_d$ satisfying $l$ (resp. $s$) and $v'$ is a child of $v_d$ satisfying $s$ (resp. $r$), then $nb_{\bar{x}}(v) < nb_{\bar{x}}(v')$.

**Proof.** Assume that $v_0$ is of type $k$, $\mathfrak{T}, v_0 \models \bar{x}, \bar{x}$ and the witness branch is $v_0, \ldots, v_d$.

First, suppose that $\mathfrak{T}, v \models \text{LSR}_k(k)$. As $\mathfrak{T}, v_0 \models \@_{\bar{x}}(\text{EX}_{=1}(s))$, by Lemma 3.4, $\mathfrak{T}, v_d \models \text{EX}_{=1}(s)$. It is easy to show that $\text{EX}_{=1}(\psi)$ holds whenever there is a unique child satisfying $\psi$. Consequently, there is a unique child of $v_d$ satisfying $s$, which corresponds to the satisfaction of (b). As $\mathfrak{T}, v_0 \models \@_{\bar{x}}(\text{AX}((s \lor l \lor r) \land \neg(s \land l) \land \neg(s \land r) \land \neg(l \land r)))$, by Lemma 3.4, we have $\mathfrak{T}, v_d \models \text{AX}((s \lor l \lor r) \land \neg(s \land l) \land \neg(s \land r) \land \neg(l \land r))$ and therefore for all children $v'$ of $v_d$, we have $\mathfrak{T}, v' \models (s \lor l \lor r) \land \neg(s \land l) \land \neg(s \land r) \land \neg(l \land r)$. As the formula $(s \lor l \lor r) \land \neg(s \land l) \land \neg(s \land r) \land \neg(l \land r)$ precisely states that exactly one propositional variable among $\{l, s, r\}$ holds true, we can conclude that (a) is satisfied. Moreover, $\mathfrak{T}, v_0 \models \text{LSR}_2(k)$ and by Lemma 3.4, we have that $\mathfrak{T}, v_d \models \forall w \forall w' \text{diff-nom}(w, w', 1) \land ((\@^1_{w}(s) \land \@^1_{w'}(r)) \lor (\@^1_{w}(l) \land \@^1_{w'}(s))) \rightarrow_nb(w') <_{k-d} nb(w)$.

As $k - d - 1 \leq N - 2$ by assumption, the satisfaction of the formula above on $v_d$ straightforwardly states that for all children $v$ and $v'$ of $v_d$ such that ($v$ satisfies $s$ and $v'$ satisfies $r$) or ($v$ satisfies $l$ and $v'$ satisfies $s$), we have $nb_{\bar{x}}(v) < nb_{\bar{x}}(v')$ (here we use the induction hypothesis), which corresponds precisely to the satisfaction of (c).

The proof in the other direction is quite similar as there are equivalences between the formulae $\text{LSR}_1^1(k), \text{LSR}_2^1(k)$, and $\text{LSR}_1^2(k)$, and the conditions (a), (b) and (c).}

We come back to the question of defining formulae expressing number comparisons. The formula $\text{nb}(\bar{y}) =_k nb(\bar{x}) + 1$ (remember $k - d = N - 1$) is defined as the expression

$$\exists l, s, r \text{LSR}_k(k) \land \text{LSR}_y(k) \land \psi_{\text{left}}(k) \land \psi_{\text{select}}(k) \land \psi_{\text{right}}(k).$$

The conjunction $\psi_{\text{left}}(k) \land \psi_{\text{select}}(k) \land \psi_{\text{right}}(k)$ takes care of the arithmetical constraints. The formula $\psi_{\text{select}}(k)$ states that the unique $v_d$'s child satisfying $s$ (whose number is the
pivōt bit $i$ does not satisfy $val$, and the unique $v'_d$'s child satisfying $s$ (whose number is also the pivot bit $i$) satisfies $val$:

$$\phi_{select}(k) \overset{\text{def}}{=} @_x (AX(s \to \neg val)) \land @_y (AX(s \to val)).$$

The formula $\phi_{right}(k)$ states that for every children of $v_d$ satisfying $r$ (and therefore with bit number strictly smaller than $i$), the bit value is 1, and for all the children of $v'_d$ satisfying $r$ (and therefore with bit number strictly smaller than $i$), the bit value is 0.

$$\phi_{right}(k) \overset{\text{def}}{=} @_x (AX(r \to val)) \land @_y (AX(r \to \neg val)).$$

The formula $\phi_{left}(k)$ states that the children of $v_d$ satisfying $l$ induce a set of bit numbers equal to the set of bit numbers induced by the children of $v'_d$ satisfying $l$. This entails also that the unique respective children of $v_d$ and $v'_d$ satisfying $s$ have the same (bit) number. Moreover, we require that children with the same bit number satisfying $l$ (taken from $v_d$ and from $v'_d$) have the same bit value witnessed by the truth value of $val$. The formula $\phi_{left}(k)$ is equal to $\phi_{left}(k) \land \phi_{left}(k)$ with $\phi_{left}(k)$ defined below:

$$\phi_{left}(k) \overset{\text{def}}{=} \forall w \ @_x (nom(w, 1) \land @_w (l)) \to (\exists w' @_y (nom(w', 1) \land @^{w'}_w (l))) \land nb(x, w) = k \land nb(y, w') = k \land (\forall @_{x,w}, val \leftrightarrow @_{y,w}, val)).$$

Note that $nb(x, w) = k \land nb(y, w') = k$ is well-defined as $k - (d + 1) \leq N - 2$. Below, we define the formula $nb(x) < k \land nb(y)$ with $k = d = N - 1, x = x_1, \ldots, x_d$ and $y = y_1, \ldots, y_d$. Based on previous developments and on standard arithmetical properties of numbers encoded in binary with $k - d$ bits, we define the formula $nb(x) < k \land nb(y)$ as the expression

$$nb(x) < k \land nb(y) \overset{\text{def}}{=} \exists s, l, r LSR_x(k) \land LSR_y(k) \land \phi_{right}(k) \land \phi_{select}(k).$$

As previously, the formula $nb(x) = k \land nb(y)$ is defined as follows:

$$nb(x) = k \land nb(y) \overset{\text{def}}{=} \neg(nb(x) < k \land nb(y)) \land \neg(nb(y) < k \land nb(x)).$$

**Lemma 4.4.** Let $\Sigma$ be a tree model, $v$ be a node satisfying $\widehat{x}, \widehat{y}$ and, $v_d, v'_d$ are of type $k - d$.

**I:** We have $\Sigma, v \models nb(y) = k \land nb(x) + 1 \iff nb\Sigma(v_d) = 1 + nb\Sigma(v_d)$.

**II:** We have $\Sigma, v \models nb(x) < k \land nb(y) \iff nb\Sigma(v_d) < nb\Sigma(v'_d)$.

**III:** We have $\Sigma, v \models nb(x) = k \land nb(y) \iff nb\Sigma(v_d) = nb\Sigma(v'_d)$.

**Proof.** Let $\Sigma$ be a tree model and $v$ be such that $v$ satisfies $\widehat{x}, \widehat{y}, k - d = N - 1$ and $v_d$ and $v'_d$ are of type $k - d$. We will focus on the proof of (I) only. The proof of (II) is similar to the proof of (I) and (III) is a direct consequence of (II). Hence, we omit the details.

As $v_d$ and $v'_d$ are of type $k - d$, both have $t(k - d, n)$ children. Those children are ordered, let $u_0, \ldots, u_{n(k - d, n) - 1}$ be the children of $v_k$ such that $nb(u_j) = j$ for all $j$. Similarly, let $u'_0, \ldots, u'_{n(k - d, n) - 1}$ be the children of $v'_k$ such that $nb(u'_j) = j$ for all $j$. By arithmetical reasoning, we have $nb\Sigma(v_d) = nb\Sigma(v_d) + 1$ iff there is $i \in [0, t(k - d, n) - 1]$, such that

(A) for every $j \in [i + 1, t(k - d, n) - 1], nb\Sigma(v'_d)$ and $nb\Sigma(v_d)$ agree on the $j$th bit, which is equivalent to $u_j$ and $u'_j$ agree on $val$,

(B) the $i$th bit of $nb\Sigma(v_d)$ is equal to 0 and the $i$th bit of $nb\Sigma(v'_d)$ is equal to 1, which is equivalent to $u_i$ does not satisfy $val$ and $u'_i$ satisfies $val$,

(C) for every $j \in [0, i - 1]$, the $j$th bit of $nb\Sigma(v'_d)$ is equal to 1 and the $j$th bit of $nb\Sigma(v'_d)$ is equal to 0, which is equivalent to $u_j$ satisfies $val$ and $u'_j$ does not satisfy $val$. 
By using Lemma 3.4, it is easy to check that the condition (A) (resp. (B), (C)) is taken care by $\phi_{left}(k)$ (resp. $\phi_{select}(k)$, $\phi_{right}(k)$). This is quite immediate for $\phi_{select}(k)$ and $\phi_{right}(k)$, as no induction hypothesis is used, in particular no comparison between numbers is performed. Concerning the satisfaction of $\phi_{left}(k)$, we have to be more careful. Let us recall the definition of the formula $\phi_{left}(k)$ below:

$$\forall w \; @x (\text{nom}(w, 1) \land @w(1)) \rightarrow$$

$$(\exists w' @y (\text{nom}(w', 1) \land @w'(1)) \land \text{nb}(x, w) = k \; \text{nb}(y, w') \land (@x, w \ \Leftrightarrow \ @y, w')).$$

We have defined $\phi_{left}(k)$ as $\phi_{left}^x(y)(k) \land \phi_{left}^y(x)(k)$. By Lemma 3.4, and by the induction hypothesis (as $(k - (d + 1)) \leq N - 2$), $\phi_{left}^x(y)(k)$ enforces that for all children $v$ of $v_d$ satisfying $l$, there is a child $v'$ of $v_d'$ satisfying $l$, such that $v$ and $v'$ agree on $val$. Moreover, $\phi_{left}^y(x)(k)$ enforces that for all children $v$ of $v_d'$ satisfying $l$, there is a child $v'$ of $v_d$ satisfying $l$, such that $v$ and $v'$ agree on $val$. So, the set of numbers of the children of $v_d$ satisfying $l$ is equal to the set of numbers of the children of $v_d'$ satisfying $l$. This implies also that this property applies for the children satisfying $r$, and the number of the unique child of $v_d$ satisfying $s$ is equal to the number of the unique child of $v_d'$. This is a direct consequence of the properties of $\text{lsr}$-partitions (see Lemma 4.3). Consequently, $\text{nb}_T(v_d') = \text{nb}_T(v_d) + 1$ implies $\text{nb}(y) = k \; \text{nb}(x) + 1$. The proof for the other direction is similar, as $\phi_{left}(k)$ is equivalent to (A), $\phi_{select}(k)$ is equivalent to (B) and $\phi_{right}(k)$ is equivalent to (C). Moreover, the existential quantification over $i$ corresponds to the existential quantification leading to an $\text{lsr}$-partition.

Mainly, this allows us to prove Lemma 4.5(I).

**Lemma 4.5.** Let $N \geq 2$, $\Sigma$ be a tree model and $v$ be one of its nodes.

(I) $\Sigma, v \models \text{type}(N)$ iff $v$ is of type $N$,

(II): Assuming that $v$ satisfies $\text{type}(N)$, we have $\Sigma, v \models \text{first}(N)$ iff $\text{nb}_T(v) = 0$.

(III): Assuming that $v$ satisfies $\text{type}(N)$, we have $\Sigma, v \models \text{last}(N)$ iff $\text{nb}_T(v) = (t(N, n) + 1) - 1$.

**Proof.** The properties (II) and (III) are easy to verify, hence let us focus on (I). The proof is actually very similar to the proof of Lemma 4.2. A node $v$ is of type $N$ for some $N \geq 2$, iff the conditions below hold:

(a) $v$ has exactly $t(N, n)$ children, (b) all its children are of type $N - 1$ and (c) $\{\text{nb}_T(v') : vE v'\} = [0, t(N, n) - 1]$. The mentioned conditions can be reformulated as follows so, by induction, $(a') + (b') + (c')$ is equivalent to $(a) + (b) + (c)$:

(a') The node $v$ has a child whose number is zero.

(b') If $v'$ is a child of $v$ with number $m < t(N, n) - 1$, then $v$ has also a child with number $m + 1$.

(c') Two distinct children of $v$ have distinct numbers.

Let us recall below the definition of the formula $\text{type}(N)$:

$$\text{AX}(\text{type}(N - 1)) \land \text{EX}(\text{first}(N - 1)) \land \text{uniq}(N) \land \text{compl}(N).$$

Obviously, the formula $\text{AX}(\text{type}(N - 1))$ expresses exactly the condition (a), assuming that type($N - 1$) already characterised the nodes of type $N - 1$. Similarly, the formula $\text{EX}(\text{first}(N - 1))$ expresses exactly the condition (a'), assuming that first($N - 1$) already characterises the nodes of type $N - 1$ whose number is zero. It remains to show that uniq($N$) (resp. compl($N$)) characterises the conditions (c') (resp. the condition (b')).

Let us recall below the definition of the formula $\text{uniq}(N)$:

$$\forall x, y \; \text{diff-nom}(x, y, 1) \rightarrow \neg(\text{nb}(x) = N \; \text{nb}(y)).$$
By Lemma 4.4(III), and by the fact that $\text{diff-nom}(x, y, 1)$ enforces that $x$ and $y$ are interpreted by two distinct children (see Lemma 3.5), when interpreted on a node of type $N$, the formula $\text{uniq}(N)$ states that for any two distinct children, their respective numbers are different, which is precisely the condition (c'). Finally, let us recall the formula $\text{compl}(N)$:

$$\forall x \ (\text{nom}(x, 1) \land \@_{x}(\neg \text{last}(N - 1))) \rightarrow \exists y \ \text{nom}(y, 1) \land \text{nb}(y) = N \ \text{nb}(x) + 1.$$ 

By Lemma 4.4(I), by the fact that $\text{last}(N - 1)$ already characterises the nodes of type $N - 1$ whose number is $t(N, n) - 1$ and by the properties of the subformulae $\text{nom}(x, 1)$ and $\text{nom}(y, 1)$ enforcing local nominals $x$ and $y$, the formula $\text{compl}(N)$, when interpreted on a node of type $N$, states that for all children whose number $m$ is different from $t(N, n) - 1$, there is a child with number $m + 1$, which is precisely the condition (b'). This ends the proof as the formulae $\text{AX}(\text{type}(N - 1))$, $\text{EX}(\text{first}(N - 1))$, $\text{uniq}(N)$, $\text{compl}(N)$ capture $(a') + (b') + (c')$. $
$  
Consequently, for all $k \geq 0$, $\text{type}(k)$, first($k$) and last($k$) characterise exactly the discussed properties, and similarly for the formulae of form $\text{nb}(\bar{y}) = k \ \text{nb}(\bar{x}) + 1$, $\text{nb}(\bar{x}) < k \ \text{nb}(\bar{y})$, $\text{nb}(\bar{x}) = k \ \text{nb}(\bar{y})$ where $\bar{x}$ and $\bar{y}$ are of length $d$ in $[1, k]$ and $k \geq 1$. It is natural to wonder what is the size of type($k$), using a reasonably succinct encoding for formulae. As the definition of type($k$) requires the subformulae $\text{type}(k - 1)$, $\text{nb}(\bar{x}) = k \ \text{nb}(\bar{y})$ and $\text{nb}(\bar{y}) = k \ \text{nb}(\bar{x}) + 1$ (the other subformulae are of constant size), and the formula type(1) is quadratic in $n$, one can show that type($k$) is of size $2^{O(k+n)}$.

4.4. Uniform reduction leading to Tower-hardness. Let $\mathcal{P} = (T, \mathcal{H}, \mathcal{V})$, $c = t_0, t_1, \ldots, t_{n-1}$ be an instance of Tiling$_{k}$ known to be $k$-NExpTime-complete (see also [20, Chapter 11]).

Let us reduce the existence of a tiling $\tau : [0, t(k, n) - 1] \times [0, t(k, n) - 1] \rightarrow T$, respecting the initial condition and the horizontal and vertical matching conditions, to the satisfiability of a formula $\phi$ in QCTL$^*_X$.

To encode the grid $[0, t(k, n) - 1] \times [0, t(k, n) - 1]$, we consider a root node $\varepsilon$ of type $k + 1$, and we distinguish $t(k, n)$ children among all of its $t(k + 1, n)$ children. Each child of $\varepsilon$ has exactly $t(k, n)$ children as it is a node of type $k$. In order to identify the $t(k, n)$ first children of $\varepsilon$, we use an lsr-partition so that the unique child satisfying $s$ has precisely the number $t(k, n)$. This guarantees that exactly the children of $\varepsilon$ whose numbers are in $[0, t(k, n) - 1]$ satisfy $r$. So, the lsr-partition is used in a new context.

Below, we define the new formula $\text{nb} = k \ t(k, n)$ that expresses that a node of type $k$ has number $t(k, n)$ ($k \geq 1$). We recall that a node of type $k$ takes its values in $[0, t(k + 1, n) - 1]$. Let us provide an inductive definition for $\text{nb} = k \ t(k, n)$ with the base case $k = 1$.

- $\text{nb} =_0 t(0, n) \overset{\text{def}}{=} \neg p_{n-1} \land \cdots \land \neg p_0$.
- $\text{nb} =_1 t(1, n) \overset{\text{def}}{=} \text{AX}(\text{val} \leftrightarrow X_n = n)$, where $X_n = n$ is an abbreviation for the formula stating that the truth values for $p_{n-1}, \ldots, p_0$ encode $n$ in binary.
- For $k \geq 2$, $\text{nb} =_k t(k, n) \overset{\text{def}}{=} \text{AX}(\text{val} \leftrightarrow (\text{nb} =_{k-1} t(k - 1, n)))$.

Lemma 4.6. Assume that $\mathcal{F}, v \models \text{type}(k)$. Then $\text{nb}_{\mathcal{F}}(v) = t(k, n)$ iff $\mathcal{F}, v \models \text{nb} =_k t(k, n)$.

Proof. The base case is $k = 1$. Assuming that $\mathcal{F}, v \models \text{type}(1)$, the node $v$ is of type 1, and therefore $v$ has exactly $t(1, n) = 2^n$ children. The number $\text{nb}_{\mathcal{F}}(v)$ is determined by the truth values of $\text{val}$ on its children, and the children have themselves (bit) numbers spanning all over $[0, 2^n - 1]$ and the numbers are encoded by the truth values of $p_{n-1}, \ldots, p_0$. Consequently, $\text{nb}_{\mathcal{F}}(v) = t(1, n) = 2^n$ iff the unique child $v'$ of $v$ such that $\text{nb}_{\mathcal{F}}(v') = n$ satisfies $\text{val}$ and all the other children do not satisfy $\text{val}$. If the value $n$ encoded with $n$ bits is represented by
the sequence of bits $b_{n-1}b_{n-2} \cdots b_0$, then $X_n = n$ is a shortcut for $\bigwedge_{i=0}^{n-1} \ell_i$ where $\ell_i = p_i$ if $b_i = 1$, otherwise $\ell_i = -p_i$. Hence, we get $\mathsf{nb}_T(v) = t(k,n)$ iff $\mathcal{S}, v \models \mathsf{AX}(\mathsf{val} \leftrightarrow X_n = n)$.

For the induction step, we reason in a similar way. Assume that for all $1 \leq k' < k$, if $\mathcal{S}', v' \models \mathsf{type}(k')$, we have $\mathsf{nb}_T(v') = t(k',n)$ iff $\mathcal{S}, v \models \mathsf{nb} = k' \iff t(k',n)$. Now, assume that $\mathcal{S}, v \models \mathsf{type}(k)$, the node $v$ is of type $k$, and therefore $v$ has exactly $t(k,n)$ children. The number $\mathsf{nb}_T(v)$ is determined by the truth values of $\mathsf{val}$ on its children, and the children have themselves (bit) numbers spanning all over $[0, t(k,n) - 1]$. Consequently, $\mathsf{nb}_T(v) = t(k,n)$ iff the unique child $v'$ of $v$ such that $\mathsf{nb}_T(v') = t(k-1,n)$ satisfies $\mathsf{val}$ and all the other children do not satisfy $\mathsf{val}$. Indeed, if only the $t(k-1,n)$th bit is equal to 1, $\mathsf{nb}_T(v)$ is equal to $2^{t(k-1,n)}$, which is precisely $t(k,n)$. Now, checking whether a node of type $k-1$, has value $t(k-1,n)$ can be expressed by $\mathsf{nb} = k-1 \iff t(k-1,n)$ invoking the induction hypothesis. Putting all together, we get $\mathsf{nb}_T(v) = t(k,n)$ iff $\mathcal{S}, v \models \mathsf{AX}(\mathsf{val} \leftrightarrow (\mathsf{nb} = k-1 \iff t(k-1,n)))$.  

Let $\phi_P$ be the formula built from the instance $(\mathcal{P} = (T, \mathcal{H}, \mathcal{V}), c)$ as follows:

$$\mathsf{type}(k+1) \land \exists l, s, r \mathsf{LSR}, (k+1) \land \mathsf{EX}(s \land \mathsf{nb} = k \iff t(k,n)) \land \mathsf{cov} \land \mathsf{init} \land \mathsf{val} \land \mathsf{phi}. $$

Definitions and explanations for $\mathsf{phi}, \mathsf{init}, \mathsf{val}$ and $\mathsf{phi}$ follow but observe that an $\mathsf{LSR}$-partition is performed for a node of type $k + 1$ and exactly $t(k,n)$ children satisfy $\mathsf{val}$ thanks to the satisfaction of the subformula $\mathsf{EX}(s \land \mathsf{nb} = k \iff t(k,n))$. The formula $\mathsf{phi}$ states that every position in $[0, t(k,n) - 1] \times [0, t(k,n) - 1]$ has a unique tile:

$$\forall x, y \mathsf{nom}(x,1) \land \mathsf{at}^1_r \land \mathsf{at}^1_x \mathsf{nom}(y,1) \rightarrow \mathsf{at}_{x,y}^1 \left( \bigwedge_{t \in T} t \land \bigwedge_{t \neq t' \in T} \neg(t \land t') \right).$$

The tile types in $T$ are understood as propositional variables. In order to access from the root node $\varepsilon$ to a node encoding a position of the grid, one needs first to access to a child $v$ of $\varepsilon$ satisfying $r$ (and this is done with the help of the local nominal $x$) and then to access to any child $v'$ of $v$ (done with the local nominal $y$). Then, to reason propositionally on $v'$, it is sufficient to consider subformulas of the form $\mathsf{at}_{x,y}^1 \psi$. This principle is applied to all the formulae below. The formula $\mathsf{phi}$ defined below encodes the horizontal matching constraints.

$$\forall x, x', y, y' \mathsf{nom}(x,1) \land \mathsf{at}^1_r \land \mathsf{nom}(x',1) \land \mathsf{at}^1_x \land \mathsf{at}^1_x \mathsf{nom}(y,1) \land \mathsf{at}^{1,y} \mathsf{nom}(y',1) \land$$

$$\mathsf{nb}(x') = k+1 \mathsf{nb}(x) + 1 \land \mathsf{nb}(x,y) = k+1 \mathsf{nb}(x,y) ) \rightarrow \bigvee_{(t,t') \in T} \mathsf{at}_{x,y}^1 t \land \mathsf{at}_{x',y'}^1 t' $$

Similarly, the following formula $\mathsf{phi'}$ encodes the vertical matching constraints.

$$\forall x, x', y, y' \mathsf{nom}(x,1) \land \mathsf{at}^1_r \land \mathsf{nom}(x',1) \land \mathsf{at}^1_x \land \mathsf{at}^1_x \mathsf{nom}(y,1) \land \mathsf{at}^{1,y} \mathsf{nom}(y',1) \land$$

$$\mathsf{nb}(x) = k+1 \mathsf{nb}(x') \land \mathsf{nb}(x,y') = k+1 \mathsf{nb}(x,y) + 1 \rightarrow \bigvee_{(t,t') \in V} \mathsf{at}_{x,y}^1 t \land \mathsf{at}_{x',y'}^1 t' $$

It remains to express the initial conditions. It is sufficient to identify the $n$ first children of the first child of $\varepsilon$ (identified by the satisfaction of first($k$)). For example, to express that the $j$th child of the first child of $\varepsilon$ (say $v$ is this first child of $\varepsilon$) satisfies $t_j$, perform an $\mathsf{LSR}$-partition on $v$, enforce that the unique child satisfying $s$ also satisfies $t_j$ and express that there are exactly $j - 1$ children of $v$ satisfying $r$. This is a condition from graded modal logic that is easy to express. Let $\mathsf{EX}_{\geq 1} \psi$ be the formula below stating that exactly $i \geq 1$ children satisfy $\psi$:

$$\exists q_1, \ldots, q_i \mathsf{diff-nom}(q_1, \ldots, q_i, 1) \land \mathsf{AX}((q_1 \lor \cdots \lor q_i) \leftrightarrow \psi),$$
where \( q_1, \ldots, q_t \) are fresh propositional variables. By convention \( \mathbf{EX}_{=0} \psi \) is defined as \( \mathbf{AX} \neg \psi \).

The formula \( \phi_{\text{init}} \) is defined as
\[
\forall x \ (\text{nom}(x, 1) \land @^1_x (\text{first}(k))) \rightarrow @^1_x (\bigwedge_{i \in [0, n-1]} \exists l, s, r \ LSR_v(k) \land \mathbf{EX}_{=i} r \land \mathbf{EX}(s \land t_j)).
\]

The correctness of the reduction is stated below.

**Lemma 4.7.** \( \mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V}) \), \( c = t_0, t_1, \ldots, t_{n-1} \) is a positive instance of \( \text{Tiling}_k \) iff \( \phi_{\mathcal{P}} \) is satisfiable in \( \text{QCTL}_k^\mathcal{X} \).

**Proof.** Given an instance \( \mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V}) \), \( c = t_0, t_1, \ldots, t_{n-1} \) of the tiling problem \( \text{Tiling}_k \), let \( \tau : [0, t(k, n) - 1] \times [0, t(k, n) - 1] \rightarrow \mathcal{T} \) be a tiling respecting all the constraints (init), (hori) and (verti). Below, we build a tree model \( \Xi = (V, E, l) \) such that \( \Xi, \varepsilon \models \phi_{\mathcal{P}} \) where \( \varepsilon \) is the root node of \( \Xi \). Let \( V \) be the following subset of \( \mathbb{N}^* \) (set of finite sequences over \( \mathbb{N} \)):

- \( \varepsilon \) is the empty string and it belongs to \( V \),
- For all \( j \in [1, k + 1] \), \( [0, t(k, n) - 1] \times \cdots \times [0, t(j, n) - 1] \subseteq V \),
- \( [0, t(k+1, n) - 1] \times \cdots \times [0, t(1, n) - 1] \times 0^+ \subseteq V \).

The binary relation \( E \) is simply defined as: \( v E v' \overset{\text{def}}{=} v \) is a prefix of \( v' \), and \( v \cdot \alpha = v' \) for some \( \alpha \in \mathbb{N} \). So, \( (V, E) \) is a finite-branching tree such that all the maximal branches are infinite.

The labelling map \( l \) is defined in a way so that \( \Xi, \varepsilon \models \text{type}(k + 1) \). For instance, any node \( m_{k+1}, \ldots, m_2 \in V \cap \mathbb{N}^{k-1} \) has \( 2^n \) children, and their numbers should span over \([0, 2^n - 1]\). This is easy to realise by setting properly the truth values for \( p_{n-1}, \ldots, p_0 \). Similarly, any node \( m_{k+1}, \ldots, m_3 \in V \cap \mathbb{N}^{k-2} \) has \( t(2, n) \) children, and their numbers should span over \([0, t(2, n) - 1]\). Again, this is easy to realise by setting properly the truth values of \( \text{val} \) on the nodes in \([0, t(k+1, n) - 1] \times \cdots \times [0, t(1, n) - 1]\). So, it remains to take care of the propositional variables dedicated to the tile types.

- For all \( i, j \in [1, k + 1] \), \( [0, t(k, n) - 1] \times [0, t(k, n) - 1] \subseteq V \), there is exactly one tile type (understood as a propositional variable) satisfied by \( (i, j) \).
- For all \( v \in V \setminus [0, t(k, n) - 1] \times [0, t(k, n) - 1] \), the value of the set \( l(v) \cap \mathcal{T} \) is irrelevant.

It remains to check that \( \Xi, \varepsilon \models \text{type}(k + 1) \land \exists \ l, s, r \ LSR_e(k+1) \land \mathbf{EX}(s \land \text{nb} = k t(k, n)) \land \phi_{\text{cov}} \land \phi_{\text{init}} \land \phi_{\mathcal{H}} \land \phi_{\mathcal{V}} \).

We have seen that the satisfaction of \( \text{type}(k + 1) \) is guaranteed by the way propositional variables hold on the nodes. Observe that for the nodes in \([0, t(k+1, n) - 1] \times \cdots \times [0, t(1, n) - 1] \times 0^+ \), the truth values of the propositional variables is irrelevant. Moreover, for all \( j \in [1, k + 1] \), for all \( m_{k+1}, \ldots, m_j \in [0, t(k+1, n) - 1] \times \cdots \times [0, t(j, n) - 1] \), we have \( \text{nb}(m_{k+1}, \ldots, m_j) = m_j \). In order to check the satisfaction of the existentially quantified subformula, we consider the labelling \( l' \) variant of \( l \) only for the propositional variables \( l \), \( s \), and \( r \) such that \( l \) holds on \([t(k, n) + 1, t(k, n) + 1] \), \( s \) holds on \([t(k, n), t(k, n)] \), and \( r \) holds on \([0, t(k, n) - 1] \). By Lemma 4.3, \( (V, E, l'), \varepsilon \models LSR_e(k+1) \) and by Lemma 4.6, we have \( (V, E, l'), \varepsilon \models \mathbf{EX}(s \land \text{nb} = k t(k, n)) \). The satisfaction of the formulae \( \phi_{\text{cov}}, \phi_{\mathcal{H}} \) and \( \phi_{\mathcal{V}} \) is inherited from the fact that for all \( (i, j) \in [0, t(k, n) - 1] \times [0, t(k, n) - 1] \), there is exactly one tile type satisfied by \( (i, j) \) and the mapping \( \tau \) satisfies horizontal and vertical matching conditions. Here, we use the properties of the formulae of the form \( \text{nom}(x, 1), @^1_x \psi, \text{nb}(x) = k+1 \text{nb}(x') \) and \( \text{nb}(x, y') = k+1 \text{nb}(x, y) + 1 \) (see e.g. Lemma 4.4), apart from the fact that \( r \) holds exactly on the nodes in \([0, t(k, n) - 1]\). Concerning the satisfaction of \( \phi_{\text{init}} \), first
observe that $\text{EX}_{x,j} \psi$ holds true exactly when there are $j$ children of the node satisfying the formula $\psi$. Hence, the formula below

$$\forall x (\text{nom}(x, 1) \land \exists x\,(\text{first}(k))) \rightarrow \exists x\,(\bigwedge_{j \in [0, n-1]} \exists l, r \, \text{LSR}_x(k) \land \text{EX}_{x,j} \, r \land \text{EX}(s \land t_j))$$

states for all $(i, j) \in \{0\} \times [0, n-1]$, $t_j$ holds on it. In order to access to the $j$th child of $\{0\}$, an $\text{lsr}$-partition on the children of $\{0\}$ is performed and $s$ holds true exactly on $(i, j)$ by counting how many children satisfies $r$. The proof for the other direction uses similar principles and is omitted herein. The main idea is to build $\tau$ so that assuming that $\exists, v \models \phi_p$, for all $v, v'$ such that $\exists E v E v'$, and $(\text{nb}(v), \text{nb}(v')) \in [0, t(k, n) - 1] \times [0, t(k, n) - 1]$, $\tau(\text{nb}(v), \text{nb}(v'))$ takes the value of the unique $t$ in $T$ satisfied on the node $v'$.

We are ready to conclude the main theorem of this paper.

**Theorem 4.8.** $\text{SAT}(\text{QCTL}^f_X)$ is Tower-complete.

Theorem 4.8 significantly improves the Tower lower bound from [36, Cor. 5.6] by considering as only temporal operator, the (local) modality $\text{EX}$. Tower-hardness can be also obtained with arbitrary countable trees. In Section 5 below, we show that this entails more Tower-hardness results for other fragments of $\text{QCTL}^f_X$ and for modal logics with propositional quantification under appropriate tree semantics.

5. A HArvest of Tower-complete modal and Temporal logics

Now, we capitalise on the Tower-hardness of the satisfiability problem for $\text{QCTL}^f_X$, by showing Tower-hardness of other fragments of $\text{QCTL}^f$ that involve only $\text{EF}$ or its strict variant $\text{EXEF}$ (Section 5.1). Tower-hardness is obtained by reduction from $\text{SAT}(\text{QCTL}^f_X)$ by introducing propositional variables that enforce layers from the root in the tree model and therefore this allows us to simulate $\text{EX}$. In Section 5.2, we consider well-known modal logics that are complete for classes of tree-like Kripke structures (formal definitions are provided in Section 2.4), and we show that their extension with propositional quantification for such classes of tree-like Kripke structures is decidable in Tower, but more importantly Tower-hard. Some of such classes involve finite trees and therefore, we take also the opportunity to study $\text{QCTL}^f_X$ and $\text{QCTL}^f_{\text{XF}}$ that happen, for instance, to be closely related to the modal logics $\text{QK}^f$ and $\text{QGL}^f$.

5.1. Satisfiability problem for $\text{QCTL}^f_X$ and $\text{QCTL}^f_{\text{XF}}$ are Tower-hard! The fragment $\text{QCTL}^f_X$ of $\text{QCTL}^f$ is defined according to the following grammar

$$\phi ::= p \mid \neg \phi \mid \phi \land \phi \mid \text{EF} \phi \mid \exists p \phi.$$ 

We recall the standard semantics for $\text{EF}$-formulae: $\exists, v \models \text{EF} \phi$ if and only if there is $j \geq 0$ such that $v E^j v'$ and $\exists, v' \models \phi$, and as usual, $\text{AG} \phi$ if and only if $\neg \text{EF} \neg \phi$.

In order to show that $\text{SAT}(\text{QCTL}^f_X)$ is Tower-hard, we design a logarithmic-space many-one reduction from $\text{SAT}(\text{QCTL}^f_X)$. We believe that the reduction we proposed in [7] and for which we provide more details herein, is simple enough as it shows how propositional quantification and $\text{EF}$ can simulate a strict discipline to access to nodes, allowing us to reduce $\text{SAT}(\text{QCTL}^f_X)$. A more sophisticated analysis is also possible to establish Tower-hardness for even smaller fragments, see the recent work [38].
Let $\phi$ be a formula in $\QCTL^t_X$ with modal depth $md(\phi) = k$. Without loss of generality, we assume that $\phi$ may contain occurrences of $\EX$ and no occurrences of $\AX$. Let us define the formula $\phi' = \text{trans}(k, \phi) \land \text{shape}(k)$ in $\QCTL^t_F$, where the formula $\text{shape}(k)$ enforces a discipline for layers (explained below) and $\text{trans}(k, \phi)$ admits a recursive definition, by relativising the occurrences of $\EX$. We consider the set of propositional variables $Y_k = \{\text{layer}_{-1}, \text{layer}_0, \ldots, \text{layer}_k\}$ with the intended meaning that a node satisfying $\text{layer}_i$ is of “layer $i$”, the root node being of layer $k$. Indeed, there is a need for such propositional variables, as unlike with the formulae in $\QCTL^t_X$, we have to enforce that moving with $\EF$ leads to a lower layer.

The formula $\text{shape}(k)$ is defined as the conjunction of the following formulae.

- Every node satisfies exactly one propositional variable from $Y_k$ (layer unicity):

$$\AG\left(\left(\text{layer}_{-1} \lor \text{layer}_0 \lor \cdots \lor \text{layer}_k\right) \land \bigwedge_{-1 \leq i \neq j \leq k} \neg(\text{layer}_i \land \text{layer}_j)\right)$$

- When a node satisfies $\text{layer}_i$ with $i \leq k$, none of its descendants satisfies $\text{layer}_j$ with $j > i$ (monotonicity of layer numbers):

$$\bigwedge_{-1 \leq i \leq k} \AG(\text{layer}_i \rightarrow \AG(\text{layer}_{-1} \lor \text{layer}_0 \lor \cdots \lor \text{layer}_i))$$

- When a node satisfies $\text{layer}_i$ with $0 \leq i \leq k$, there is a descendant satisfying $\text{layer}_{i-1}$ (weak progress):

$$\bigwedge_{0 \leq i \leq k} \AG(\text{layer}_i \rightarrow \EF \text{layer}_{i-1})$$

- When a node satisfies $\text{layer}_i$ with $0 \leq i \leq k$, it has no (strict) descendant satisfying $\text{layer}_i$ (no stuttering). This type of constraints does not apply to $\text{layer}_{-1}$.

$$\bigwedge_{0 \leq i \leq k} \AG(\text{layer}_i \rightarrow \neg\exists p (p \land \EF(\text{layer}_i \land \neg p)))$$

- The root node is at layer $k$: $\text{layer}_k$.

A tree model $\mathfrak{T} = \langle V, E, l \rangle$ for $\QCTL^t$ with root node $\varepsilon$ is $k$-layered iff the conditions below hold:

(a) For every node $v \in V$, $\text{card}(l(v) \cap Y_k) = 1$.

(b) For all $v \in V$ such that $\text{layer}_j \in l(v)$ for some $j \in [-1, k]$,

- if $j \geq 0$, then there is $v'$ such that $vEv'$ and $\text{layer}_{j-1} \in l(v')$ and,
- for all $v'$ such that $vE^+v'$ and $\text{layer}_{j'} \in l(v')$, we have $j' \leq j$.

(c) For all $j \in [0, k]$, there are no distinct nodes $v$ and $v'$ such that $vE^+v'$ and $\text{layer}_j \in l(v) \cap l(v')$.

(d) $\text{layer}_k \in l(\varepsilon)$, where $\varepsilon$ is the root of the tree model.

This means that the only propositional variable from $Y_k$ satisfied by a node reachable in $j \in [1, k]$ steps from $\varepsilon$ is $\text{layer}_m$ for some $m \leq k-j$, and the only propositional variable from $Y_k$ satisfied by a node reachable in strictly more than $k$ steps from $\varepsilon$ is $\text{layer}_{-1}$. Moreover, once $\text{layer}_{-1}$ holds true, it holds for all its descendants. Actually, the formula $\text{shape}(k)$ characterises $k$-layered structures.

**Lemma 5.1.** Let $\mathfrak{T} = \langle V, E, l \rangle$ be a tree model for $\QCTL^t$ with the root node $\varepsilon$. We have $\mathfrak{T}, \varepsilon \models \text{shape}(k)$ holds if and only if $\mathfrak{T}$ is $k$-layered.
Proof. First, let us suppose that $T$ is $k$-layered and we show that $T, \varepsilon \models \text{shape}(k)$.

- By the condition (a), we have that:
  \[
  T, \varepsilon \models \text{AG} \left( (\text{layer}_1 \lor \text{layer}_0 \lor \cdots \lor \text{layer}_k) \land \bigwedge_{1 \leq i \neq j \leq k} \neg(\text{layer}_i \land \text{layer}_j) \right)
  \]
as the formula inside $\text{AG}$ states that there is exactly one proposition from $Y_k$ holds.

- By the condition (b), namely its second point, for all $i \in [0, k]$, we have
  \[
  T, \varepsilon \models \bigwedge_{i \leq k} \text{AG}(\text{layer}_i \rightarrow \text{AG}(\text{layer}_1 \lor \text{layer}_0 \lor \cdots \lor \text{layer}_i)).
  \]

- By the first point of the condition (b), for all $i \in [0, k]$, we have $T, \varepsilon \models \text{AG}(\text{layer}_i \rightarrow \text{EF layer}_{i-1})$. Actually, when $T, v \models \text{layer}_i$, the witness descendant satisfying $\text{layer}_{i-1}$ is a child of $v$ by (b).

- By the condition (d), we have $T, \varepsilon \models \text{layer}_k$.

- By the condition (c), for every node $v$ satisfying $\text{layer}_j$ for some $j \in [0, k]$, there is no proper descendant of $v$ satisfying $\text{layer}_j$. Observe that the formula $\neg \exists p (p \land \text{EF}(\text{layer}_j \land \neg p))$ holds exactly on the nodes such that there is no proper descendant satisfying $\text{layer}_j$. Hence
  \[
  T, \varepsilon \models \bigwedge_{0 \leq i \leq k} \text{AG}(\text{layer}_i \rightarrow \neg \exists p (p \land \text{EF}(\text{layer}_i \land \neg p))).
  \]

Conversely, suppose that $T, \varepsilon \models \text{shape}(k)$ holds. The satisfaction of (a), (c) and (d) holds thanks to the corresponding formulae in $\text{shape}(k)$ (see above). Let us check that (c) holds true. As
  \[
  T, \varepsilon \models \bigwedge_{0 \leq i \leq k} \text{AG}(\text{layer}_i \rightarrow \neg \exists p (p \land \text{EF}(\text{layer}_i \land \neg p)))
  \]
holds and for all $i \in [0, k]$, we have
  \[
  T, \varepsilon \models \text{AG}(\text{layer}_i \rightarrow \text{EF} \text{ layer}_{i-1}),
  \]
on the same branch two distinct nodes cannot satisfy $\text{layer}_i$ for some $i \in [0, k]$. Moreover, the satisfaction of $\text{layer}_i$ implies that some (proper) descendant satisfies $\text{layer}_{i-1}$. Due to the monotonicity of the layer numbers and no stuttering, $\text{layer}_i$ implies that a child satisfies $\text{layer}_{i-1}$, which corresponds to the first point of (b). The second point of (b) is a consequence of the monotonicity of the layer numbers. \qed

To define $\text{trans}(k, \phi)$, we define inductively $\text{trans}(i, \psi)$ where $\psi$ is a subformula of $\phi$ and $md(\psi) \leq i$.

- $\text{trans}(i, p) \overset{\text{def}}{=} p$ for all propositional variables $p$,
- $\text{trans}$ is homomorphic for Boolean connectives and $\text{trans}(i, \exists p \psi) \overset{\text{def}}{=} \exists p \text{trans}(i, \psi)$,
- $\text{trans}(i, \text{EX} \psi) \overset{\text{def}}{=} \text{EF} (\text{layer}_{i-1} \land \text{trans}(i-1, \psi))$.

Note that $\text{trans}(k, \phi)$ has no occurrence of $\text{layer}_{-1}$ since $md(\phi) = k$ and that translating an $\text{EX}$-formula decreases the index of the layer by exactly one. The correctness of the reduction can be now stated as follows.

**Lemma 5.2.** A formula $\phi$ is satisfiable for $\text{QCTL}_X^k$ iff $\text{trans}(k, \phi) \land \text{shape}(k)$ is satisfiable for $\text{QCTL}_F^k$. 

Proof. First, let us assume that $\phi$ $(md(\phi) = k \geq 0)$ is satisfiable for $\text{QCTL}^i_F$, i.e. $\mathfrak{T}, \varepsilon \models \phi$ holds with the tree model $\mathfrak{T} = \langle V, E, l \rangle$. Let $\mathfrak{T}' = \langle V, E, l' \rangle$ be the tree model obtained from $\mathfrak{T}$ by providing truth values for the propositional variables in $Y_k$. More precisely, for all $v \in V$ with $\varepsilon E^j v$, we have that $l'(v) \defeq (l(v) \setminus Y_k) \cup \{\text{layer}_{\max(-1,k-j)}\}$. As $\mathfrak{T}$ is a tree model, $\varepsilon E^j v$ implies that $j$ is the unique number of steps to reach $v$ from $\varepsilon$. Obviously, $\mathfrak{T}'$ is $k$-layered and hence, by Lemma 5.1, $\mathfrak{T}', \varepsilon \models \text{shape}(k)$ holds. Moreover, by structural induction, one can show that for all $j \in [0, k]$, for all $v \in V$ with $\varepsilon E^j v$, and for all subformulas $\psi$ of $\phi$ of modal depth less than $j$, $\mathfrak{T}, v \models \psi$ if and only if $\mathfrak{T}', v \models \text{trans}(j, \psi)$. This leads to the satisfaction of $\mathfrak{T}', \varepsilon \models \text{trans}(k, \phi)$.

- For the base case $j = 0$, for all formulae $\psi$ of modal degree 0, we have $\mathfrak{T}, v \models \psi$ iff $\mathfrak{T}', v \models \text{trans}(0, \psi)$ due to the fact that $\text{trans}(0, \psi) = \psi$ and, that $\mathfrak{T}$ and $\mathfrak{T}'$ agree on the propositional variables occurring in $\phi$.

- For the induction step, the proof for the cases with Boolean connectives is immediate.

- We now consider the case with propositional quantification. Suppose that $\mathfrak{T}, v \models \exists p \psi$. Hence, there is $\mathfrak{T}^* = \langle V, E, l^* \rangle$ such that $\mathfrak{T}^* \approx_{AP \setminus \{p\}} \mathfrak{T}$ and $\mathfrak{T}^*, v \models \psi$. Let $\mathfrak{T}^{**} = \langle V, E, l^{**} \rangle$ be the variant obtained from $\mathfrak{T}^*$ such that for all $v' \in V$ with $\varepsilon E^{j'} v'$, we have $l^{**}(v') \defeq (l'(v') \setminus Y_k) \cup \{\text{layer}_{\max(-1,k-j)}\}$. By the induction hypothesis, $\mathfrak{T}^{**}, v \models \text{trans}(j, \psi)$. It is easy to check that $\mathfrak{T}^{**} \approx_{AP \setminus \{p\}} \mathfrak{T}'$ and therefore $\mathfrak{T}', v \models \exists p \text{trans}(j, \psi)$. The proof for the other direction is analogous.

- Finally, we consider the case with $\text{EX}$. First, let us suppose that $\mathfrak{T}, v \models \text{EX} \psi$ with $\varepsilon E^{(k-j)} v$ and the modal depth of $\psi$ is less than $j$. Hence, there is $v'$ such that $v E v'$ and $\mathfrak{T}, v' \models \psi$. Thus, $\varepsilon E^{(k-(j-1))} v'$ and therefore $v'$ satisfies $\text{layer}_{j-1}$ in $\mathfrak{T}'$. By the induction hypothesis ($\psi$ is also of modal depth less than $j - 1$), we conclude $\mathfrak{T}', v' \models \text{layer}_{j-1} \land \text{trans}(j-1, \psi)$. As $v'$ is also a child of $v$ in $\mathfrak{T}'$ (and therefore a descendant), we obtain $\mathfrak{T}', v \models \text{EF}(\text{layer}_{j-1} \land \text{trans}(j-1, \psi))$. Conversely, suppose that $\mathfrak{T}', v \models \text{EF}(\text{layer}_{j-1} \land \text{trans}(j-1, \psi))$. Thus, there is a descendant $v''$ such that $v E v''$ and $\mathfrak{T}', v'' \models \text{layer}_{j-1} \land \text{trans}(j-1, \psi)$. By definition of $l'$, we have $\varepsilon E^{k-j-1} v''$ and therefore $v E v''$. By the induction hypothesis, we obtain $\mathfrak{T}, v'' \models \psi$ (again, $\psi$ is also of modal depth less than $j - 1$), which implies $\mathfrak{T}, v \models \text{EX} \psi$.

For the other implication, we assume that $\text{trans}(k, \phi) \land \text{shape}(k)$ is satisfiable for $\text{QCTL}^i_F$, that is $\mathfrak{T}, \varepsilon \models \text{trans}(k, \phi) \land \text{shape}(k)$ holds with the tree model $\mathfrak{T} = \langle V, E, l \rangle$ and $md(\phi) = k \geq 0$. By Lemma 5.1, the tree model $\mathfrak{T}$ is $k$-layered and therefore satisfying $\text{layer}_i$ and jumping to a node with the help of $\text{EF}(\text{layer}_{i-1} \land \ldots)$ leads to a child node (assuming that $i \in [0, k]$). Let $\mathfrak{T}' = \langle V', E', l' \rangle$ be the tree model defined as follows:

- $V'$ is the least subset of $V$ satisfying the conditions below:
  - $\varepsilon \in V'$,
  - if $v \in V'$ and $\text{layer}_j \in l(v)$ for some $j \in [0, k]$, then for all $v' \in V$ such that $\text{layer}_{j-1} \in l(v')$ and $v E v'$, then $v' \in V'$. The children of $v$ that do not satisfy $\text{layer}_{j-1}$ are ignored in $\mathfrak{T}'$.
- $l'$ is the restriction of $l$ to $V'$.
- For all $v, v' \in V'$, $v E v'$ $\Longleftrightarrow$ one the conditions below holds:
  - $\text{layer}_{-1} \in l(v) \cap l(v')$ and $v E v'$.
  - For some $j \in [0, k]$, $v E v'$, $\text{layer}_j \in l(v)$ and $\text{layer}_{j-1} \in l(v')$. 
It is not difficult to check that $\mathcal{T}'$ is a tree model (finite-branching tree and all the maximal branches are infinite), as $\mathcal{T}$ satisfies the formula below (due to the satisfaction of $\text{shape}(k)$):

$$\bigwedge_{i \in [0,k]} \text{AG}(\text{layer}_i \rightarrow \text{EF} \text{ layer}_{i-1}).$$

Similarly to what we did above, by structural induction, one can show that for all $j \in [0,k]$, for all $v \in V'$ such that $\text{layer}_j \in l(v)$, and for all subformulae $\psi$ of $\phi$ of modal depth less than $j$, we have $\mathcal{T}, v \models \text{trans}(j, \psi)$ iff $\mathcal{T}', v \models \psi$. This leads to the satisfaction of $\mathcal{T}', \varepsilon \models \phi$ as $\mathcal{T}, \varepsilon \models \text{layer}_k$ holds by the satisfaction of $\text{shape}(k)$.

- For the base case $j = 0$, for all formulae $\psi$ of modal degree 0, we have $\mathcal{T}, v \models \text{trans}(0, \psi)$ iff $\mathcal{T}', v \models \psi$ due to the fact that $\text{trans}(0, \psi) = \psi$ and, $\mathcal{T}$ and $\mathcal{T}'$ agree on the propositional variables occurring in $\phi$.

- For the induction step, the proof for the cases with Boolean connectives and propositional quantification is by easy verification (cf. the proof in the other direction).

- Let us treat in depth the case with $\text{EX} \psi$, with $\text{layer}_j \in l(v)$ and $\text{EX} \psi$ is of modal depth less than $j$. Suppose that $\mathcal{T}, v \models \text{trans}(j, \text{EX} \psi)$. So, this means that $\mathcal{T}, v \models \text{EF}(\text{layer}_{j-1} \land \text{trans}(j-1, \psi))$. There is $v' \in V$ such that $vE^*v'$ and $\mathcal{T}, v' \models \text{layer}_{j-1} \land \text{trans}(j-1, \psi)$. As $\mathcal{T}$ is a $k$-layered tree model, necessarily $vEv'$ (otherwise there are two distinct nodes on the branch from $v$ such that both satisfy $\text{layer}_j$ or both satisfy $\text{layer}_{j-1}$, which leads to a contradiction). By definition of $E'$, we get $vE'v'$ and by the induction hypothesis ($\psi$ is also of modal depth less than $j-1$), we get $\mathcal{T}', v' \models \psi$. Hence, we obtain $\mathcal{T}', v \models \text{EX} \psi$.

Conversely, assume that $\mathcal{T}', v \models \text{EX} \psi$. Thus, there exists $v' \in V'$ such that $vE'v'$ and $\mathcal{T}', v' \models \psi$. By definition of $E'$, $\mathcal{T}, v' \models \text{layer}_{j-1}$ and hence, by the induction hypothesis, we get $\mathcal{T}, v' \models \text{layer}_{j-1} \land \text{trans}(j-1, \psi)$. As one can check that $E' \subseteq E$ and hence, we conclude that $\mathcal{T}, v \models \text{EF}(\text{layer}_{j-1} \land \text{trans}(j-1, \psi))$.

Hence we conclude yet another important result.

**Theorem 5.3.** The satisfiability problem for $\text{QCTL}^t_\mathcal{F}$ is Tower-complete.

The Tower upper bound is established for the full logic $\text{QCTL}^t$ in [36] and in particular for $\text{QCTL}^t_F$. Theorem 5.3 also admits a variant in which we only allow to move to proper descendants. It amounts to replace $\text{EF}$ by $\text{EXEF}$ in $\text{QCTL}^t_F$, leading to the variant $\text{QCTL}^t_{\text{XF}}$, with formulae obtained from

$$\phi ::= p \mid \neg \phi \mid \phi \land \phi \mid \text{EXEF} \phi \mid \exists p \phi$$

As usual, we write $\text{AXAG} \psi$ as an abbreviation of $\neg \text{EXEF} \neg \phi$.

**Theorem 5.4.** The satisfiability problem for $\text{QCTL}^t_{\text{XF}}$ is Tower-complete.

As above, the Tower upper bound for SAT($\text{QCTL}^t_{\text{XF}}$) is inherited from SAT($\text{QCTL}^t$) [36]. Note that in $\text{QCTL}^t$, the formula $\text{EF} p$ is logically equivalent to $p \lor \text{EXEF} p$. Thus, we get a reduction from SAT($\text{QCTL}^t_F$) to SAT($\text{QCTL}^t_{\text{XF}}$), whence the Tower-hardness of SAT($\text{QCTL}^t_{\text{XF}}$).

### 5.2. Modal logics with propositional quantification on trees

In this section, we show that the satisfiability problem for the logics $\text{QK}^t$, $\text{QKD}^t$, $\text{QGL}^t$, $\text{QK4}^t$ and $\text{QS4}^t$ introduced in Section 2.4 and whose models are tree-like Kripke structures is Tower-complete. For instance, $\text{QK}^t$ corresponds to the modal logic K interpreted on finite trees with propositional quantification, which is precisely $\text{QCTL}^{ft}_{\mathcal{X}}$, i.e. $\text{QCTL}^{ft}$ restricted to the $\text{EX}$ operator.
Theorem 5.5. SAT(QK\textsuperscript{f}) is Tower-complete.

Proof. As far as Tower-hardness is concerned, in order to enforce finite tree models, it is sufficient to consider the reduction defined for QCTL\textsuperscript{f} in Section 4 but to modify the definition of the formula type(0) so that type(0) is now equal to ¬\textit{EX}\top. In that way, the finite grids of the form \([0, t(k, n) - 1] \times [0, t(k, n) - 1]\) can still be encoded but with finite tree models.

In order to get the Tower upper bound, let us define a reduction to the satisfiability problem for QCTL\textsuperscript{f} by simply identifying finite trees within tree models for QCTL\textsuperscript{f} (known to be in Tower by [36]). Let \(\phi\) be a formula in QK\textsuperscript{f}. Without loss of generality, we assume that \(\phi\) may contain occurrences of \textit{EX} and no occurrences of \textit{AX}. We introduce the formula \(\text{trans}(\phi) \land \phi_{\text{fin}}\) in QCTL\textsuperscript{f}, where \(\phi_{\text{fin}}\) enforces that the fresh propositional variable \(in\) holds true only finitely on each branch and \(\text{trans}(\phi)\) admits a recursive definition, by relativising the occurrences of \textit{EX} with respect to \(in\). Let \(\phi_{\text{fin}}\) be the formula \(in \land \text{AF AG } \neg in \land \text{AG } (\neg in \rightarrow \text{AG } \neg in)\). The satisfiability of \(\phi_{\text{fin}}\) at the root node \(\varepsilon\) implies \(in\) holds exactly on a subtree from \(\varepsilon\) where all the branches are finite. It remains to define \(\text{trans}(\phi)\):

- \(\text{trans}(p) \overset{\text{def}}{=} p\) for all propositional variables \(p\), and \(\text{trans}\) is homomorphic for Boolean connectives and propositional quantification,
- \(\text{trans}(\text{EX}\psi) \overset{\text{def}}{=} \text{EX}(in \land \text{trans}(\psi))\).

We recall that the satisfaction of \(\phi_{\text{fin}}\) at the root node \(\varepsilon\) implies \(in\) holds exactly on a subtree from \(\varepsilon\) where all the branches are finite.

One can show that \(\phi\) is satisfiable for QK\textsuperscript{f} iff \(\text{trans}(\phi) \land \phi_{\text{fin}}\) is satisfiable in QCTL\textsuperscript{f} (note that \(in\) does not need to be part of the conjunction as the root is always part of the model). Moreover, as the models for QCTL\textsuperscript{f} are finite-reaching trees (see e.g. [36, Remark 5.7]), \(\phi\) is satisfiable in a finite tree model iff \(\text{trans}(\phi) \land \phi_{\text{fin}}\) is satisfiable in a finite-reaching tree model, which leads to the desired upper bound Tower. \(\Box\)

In Section 2.4, we have seen that QKD\textsuperscript{f} is actually defined as QCTL\textsuperscript{f} (KD is characterised by total and finite Kripke structures whose unfoldings generate finite-reaching trees in which all the branches are infinite). Consequently,

Theorem 5.6. SAT(QKD\textsuperscript{f}) is Tower-complete.

Similarly, the modal logic GL is known to be complete for the class of finite transitive trees, i.e. the class of Kripke structures \(\langle V, E^+, l \rangle\) such that \(\langle V, E, l \rangle\) is a finite tree model, see e.g. [12]. In Section 2.4, QGL\textsuperscript{f} is defined as the modal logic with propositional quantification over the class of finite transitive trees, which can be shown to be precisely QCTL\textsuperscript{ff}_XF when \textit{EX} is translated into \textit{EXEF}. Theorem 5.7 below provides a new decidability result by reduction into SAT(QCTL\textsuperscript{f}) and a complexity characterisation with a remarkable lower bound.

Theorem 5.7. SAT(QGL\textsuperscript{f}) is Tower-complete.

Proof. First, let us show that SAT(QGL\textsuperscript{f}) and SAT(QCTL\textsuperscript{ff}_XF) are identical problems modulo the rewriting of \textit{EX} into \textit{EXEF}. Herein, QCTL\textsuperscript{ff}_XF is defined as QCTL\textsuperscript{ff} restricted to the combined temporal operator \textit{EXEF}. According to Section 2, the models for QCTL\textsuperscript{ff} are finite trees. In a second part of the proof, we show that SAT(QCTL\textsuperscript{ff}_XF) is Tower-complete.

(I) Let \(t\) be the map from QGL\textsuperscript{f} formulae into QCTL\textsuperscript{ff}_XF formulae such that \(t(\phi)\) is defined from \(\phi\) by replacing every occurrence of \textit{EX} by \textit{EXEF}. Similarly, the inverse map
$t^{-1}$ is defined so that $t^{-1}(\psi)$ with a $\text{QCTL}^{ft}_{XF}$ formula $\psi$, is defined from $\psi$ by replacing every occurrence of $\text{EXEF}$ by $\text{EX}$.

Let us show that (a) $\phi$ is $\text{QGL}^l$ satisfiable iff $t(\phi)$ is $\text{QCTL}^{ft}_{XF}$ satisfiable and (b) $\psi$ is $\text{QCTL}^{ft}_{XF}$ satisfiable iff $t^{-1}(\psi)$ is $\text{QGL}^l$ satisfiable. Since $t$ is bijective, it is sufficient to show (a). Let $K = \langle W, R, l \rangle$ be a $\text{QGL}^l$ model and $w \in W$ such that $K, w \models \phi$. As $K$ is a $\text{QGL}^l$ model, $(W, R) = (W, E^+)$ for some finite tree $(W, E)$ with root $w$. Consequently for all $w_1, w_2 \in W$, we have $(w_1, w_2) \in R \text{ iff } (w_1, w_2)$ belongs to the transitive closure of $E$. By structural induction, one can easily show that for all $w' \in W$ and for all subformulae $\phi'$ of $\phi$, we have $K, w' \models \phi'$ iff $\langle W, E, l \rangle, w' \models t(\phi')$. Consequently, $\phi$ is $\text{QGL}^l$ satisfiable implies $t(\phi)$ is $\text{QCTL}^{ft}_{XF}$ satisfiable as $(W, E, l)$ is a model for $\text{QCTL}^{ft}_{XF}$ since it is a finite tree.

Conversely, let $\exists = \langle V, E, l \rangle$ be a $\text{QCTL}^{ft}_{XF}$ model (finite tree) with root $\varepsilon$ such that $\exists, \varepsilon \models t(\phi)$. Let $K$ be the Kripke structure $\langle V, E^+, l \rangle$ defined from $\exists$ and by definition, $K$ is an $\text{QGL}^l$ model. Again, by structural induction, one can easily show that for all $w' \in V$ and for all subformulae $\phi'$ of $\phi$, we have $\langle V, E, l \rangle, w' \models t(\phi')$ iff $K, w' \models \phi'$. Consequently, $\phi$ is $\text{QGL}^l$ satisfiable as we have assumed that $\exists, \varepsilon \models t(\phi)$ and therefore $K, \varepsilon \models \phi$.

(II) Let us show that the satisfiability problem for $\text{QCTL}^{ft}_{XF}$ is Tower-complete. We need to take care of both the lower bound and of the upper bound. As far as Tower-hardness is concerned, we define a reduction from the satisfiability for $\text{QCTL}^{ft}_{X}$ (see Theorem 5.5 as $\text{QCTL}^{ft}_{X}$ and $\text{QKL}^l$ are identical), and the proof is very similar to the one for $\text{QCTL}^{ft}_{XF}$ (actually, it is a bit simpler). The main steps are summarised below. Let $\phi$ be a formula in $\text{QCTL}^{ft}_{X}$ of the modal depth $md(\phi) = k$. Similarly to what was done before, let us consider the set of fresh propositional variables $Y_k = \{layer_0, \ldots, layer_k\}$ with the intended meaning that a node satisfying $\text{layer}_i$ is of “layer $i$”, the root node being of layer $k$. Let us define the formula $\text{trans}(k, \phi) \land \text{shape}(k)$ in $\text{QCTL}^{ft}_{XF}$, where $\text{shape}(k)$ is the conjunction of the following formulæ:

- Every node satisfies exactly one propositional variable from $Y_k$ (layer unicity) and the root satisfies $\text{layer}_k$.

\[ (\text{layer}_k \land \bigwedge_{i \neq k} \neg \text{layer}_i) \land \text{AXAG}(\text{layer}_0 \lor \cdots \lor \text{layer}_k) \land \bigwedge_{0 \leq i < j \leq k} \neg (\text{layer}_i \land \text{layer}_j) \].

- When a node satisfies $\text{layer}_i$ with $i \leq k$, none of its descendants satisfies some $\text{layer}_j$ with $j > i$ (monotonicity of layer numbers).

\[ \text{AXAG}(\text{layer}_0 \lor \cdots \lor \text{layer}_{k-1}) \land \bigwedge_{i \leq k-1} \text{AXAG}(\text{layer}_i \rightarrow \text{AXAG}(\text{layer}_0 \lor \cdots \lor \text{layer}_{i-1})) \].

- When a node satisfies $\text{layer}_i$ with $1 \leq i \leq k-1$, there is a descendant satisfying $\text{layer}_{i-1}$ (weak progress).

\[ \text{EXEF} \text{layer}_{k-1} \land \text{AXAG}(\text{layer}_i \rightarrow \text{EXEF} \text{layer}_{i-1}) \].

- The nodes satisfying $\text{layer}_0$ have no successor: $\text{AXAG}(\text{layer}_0 \rightarrow \neg \text{EXEF} \top)$.

The formula $\text{trans}(k, \phi)$ is defined as in the reduction from $\text{QCTL}^{ft}_{X}$ to $\text{QCTL}^{ft}_{XF}$ (see Section 5.1). One can show that $\phi$ is satisfiable in $\text{QCTL}^{ft}_{X}$ iff $\text{trans}(k, \phi) \land \text{shape}(k)$ is satisfiable in $\text{QCTL}^{ft}_{XF}$. 
To get the Tower upper bound, let us define a reduction to the satisfiability problem for $\text{QCTL}^t$ (known to be in Tower by [36]). Let $\phi$ be a formula in $\text{QCTL}^t_{\text{XF}}$. We introduce the formula $\text{trans}(\phi) \land \phi_{\text{fin}}$ in $\text{QCTL}^t$, where $\text{trans}(\phi)$ is recursively defined as follows:

- $\text{trans}(p) \overset{\text{def}}{=} p$ for all propositional variables $p$, and $\text{trans}$ is homomorphic for Boolean connectives and propositional quantification,

- $\text{trans}(\text{EXEF}\psi) \overset{\text{def}}{=} \text{EXEF}(\text{in} \land \text{trans}(\psi))$.

As in the proof of Theorem 5.5, $\phi_{\text{fin}}$ is equal to $\text{in} \land \text{AF AG \neg in} \land \text{AG(\neg in} \rightarrow \text{AG \neg in)}$. One can show that $\phi$ is satisfiable in a tree model without infinite branches iff $\text{trans}(\phi) \land \phi_{\text{fin}}$ is satisfiable in $\text{QCTL}^t$. Again, as the models for $\text{QCTL}^t$ are finite-branching, $\phi$ is satisfiable in a finite tree model iff $\text{trans}(\phi) \land \phi_{\text{fin}}$ is satisfiable in a finite-branching tree models, which leads to Tower-easiness.

In Section 2.4, $\text{QK4}^t$ is defined as the modal logic with propositional quantification over the class of finite-branching trees as the modal logic $K4$ is complete for the class of Kripke structures $(V, E^+, l)$ such that $(V, E, l)$ is a finite-branching tree model (some branches may be infinite, some others not).

**Theorem 5.8.** $\text{SAT}(\text{QK4}^t)$ is Tower-complete.

**Proof.** First, let us show that $\text{SAT}(\text{QK4}^t)$ and $\text{SAT}(\text{QCTL}^t_{\text{XF}})$ are identical problems modulo the rewriting of EX into EXEF. Herein, $\text{QCTL}^t_{\text{XF}}$ is defined as $\text{QCTL}^t$ restricted to the temporal operator EXEF. According to Section 2, the models for $\text{QCTL}^t$ are finite-branching trees. In a second part of the proof, we show that $\text{SAT}(\text{QCTL}^t_{\text{XF}})$ is Tower-complete.

(I) Let $t$ be the map from $\text{QK4}^t$ formulae into $\text{QCTL}^t_{\text{XF}}$ formulae such that $t(\phi)$ is defined from $\phi$ by replacing every occurrence of EX by EXEF. Similarly, the inverse map $t^{-1}$ is defined so that $t^{-1}(\psi)$ with a $\text{QCTL}^t_{\text{XF}}$ formula $\psi$, is defined from $\psi$ by replacing every occurrence of EXEF by EX. This is similar to what is done in the proof of Theorem 5.7.

Let us show that (a) $\phi$ is $\text{QK4}^t$ satisfiable iff $t(\phi)$ is $\text{QCTL}^t_{\text{XF}}$ satisfiable and (b) $\psi$ is $\text{QCTL}^t_{\text{XF}}$ satisfiable iff $t^{-1}(\psi)$ is $\text{QK4}^t$ satisfiable. It is sufficient to show (a). Let $\mathcal{K} = (W, R, l)$ be an $\text{QK4}^t$ model and $w \in W$ such that $\mathcal{K}, w \models \phi$. As $\mathcal{K}$ is an $\text{QK4}^t$ model $(W, R) = (W, E^+)$ for some finite-branching tree $(W, E)$ with root $w$. Consequently for all $w_1, w_2 \in W$, we have $(w_1, w_2) \in R$ iff $(w_1, w_2)$ belongs to the transitive closure of $E$. By structural induction, one can easily show that for all $w' \in W$ and for all subformulae $\phi'$ of $\phi$, we have $\mathcal{K}, w' \models \phi'$ iff $(W, E, l), w' \models t(\phi')$. Consequently, $\phi$ is $\text{QK4}^t$ satisfiable implies $t(\phi)$ is $\text{QCTL}^t_{\text{XF}}$ satisfiable as $(W, E, l)$ is a model for $\text{QCTL}^t_{\text{XF}}$.

Conversely, let $\mathcal{X} = (V, E, l)$ be a $\text{QCTL}^t_{\text{XF}}$ model (finite-branching tree) with root $\varepsilon$ such that $\mathcal{X}, \varepsilon \models t(\phi)$. Let $\mathcal{K}$ be the Kripke structure $(V, E^+, l)$ defined from $\mathcal{X}$ and by definition, $\mathcal{K}$ is an $\text{QK4}^t$ model. Again, by structural induction, one can easily show that $\mathcal{K}, \varepsilon \models \phi$.

(II) In order to establish the upper bound Tower, let us provide a reduction from the satisfiability for $\text{QCTL}^t_{\text{XF}}$ to the satisfiability problem for $\text{QCTL}^t_{\text{XF}}$. Let $\phi$ be a formula in $\text{QCTL}^t_{\text{XF}}$. We introduce the formula $\text{trans}(\phi) \land \phi_{\text{fin}}'$ in $\text{QCTL}^t_{\text{XF}}$, where $\phi_{\text{fin}}'$ enforces that the propositional variable $\text{in}$ holds false on all descendants, as soon as it does not hold on a node and that $\text{trans}(\phi)$ admits a recursive definition, by relativising the occurrences of EXEF with respect to $\text{in}$. Let $\phi_{\text{fin}}$ be the formula $\text{in} \land \text{AXAG}(\neg \text{in} \rightarrow \text{AXAG} \neg \text{in})$.

It remains to define $\text{trans}(\phi)$:
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- \(\text{trans}(p) \overset{\text{def}}{=} p\) for all propositional variables \(p\), and \(\text{trans}\) is homomorphic for Boolean connectives and propositional quantification,
- \(\text{trans}(\text{EXEF}\psi) \overset{\text{def}}{=} \text{EXEF}(\text{in} \land \text{trans}(\psi))\).

It is easy to see that \(\phi\) is satisfiable for \(\text{QCTL}^{qf}_{XF}\) iff \(\text{trans}(\phi) \land \phi'_{\text{fin}}\) is satisfiable for \(\text{QCTL}^{t}_{XF}\).

As far as Tower-hardness is concerned, for any formula \(\phi\) in \(\text{QCTL}^{t}_{XF}\), one can show that \(\phi\) is satisfiable for \(\text{QCTL}^{t}_{XF}\) iff \(\phi \land \text{EX EF} \top \land \text{AX AG EXEF} \top\) is satisfiable in \(\text{QCTL}^{qf}_{XF}\). The two last conjuncts simply state that from any node, there is a child, which enforces that all the maximal branches are infinite. As the satisfiability problem for \(\text{QCTL}^{t}_{XF}\) is Tower-hard (Theorem 5.4), this concludes the proof.

Finally, by noting that \(\text{QS4}^{t}\) is equal to \(\text{QCTL}^{t}_{F}\) modulo that \(\text{EX}\) is rewritten into \(\text{EF}\), using Theorem 5.3, we get the following complexity characterisation.

**Theorem 5.9.** \(\text{SAT}(\text{QS4}^{t})\) is Tower-complete.

6. Conclusion

In the paper, we have developed a relatively simple proof method to show that the satisfiability problems for \(\text{QCTL}^{t}_{X}, \text{QCTL}^{t}_{F}\) and \(\text{QCTL}^{t}_{XF}\) are Tower-complete, see also similar methods in [55, 45]. Our contribution is essentially to establish Tower-hardness, which could be also shown for several modal logics with propositional quantification whose respective classes of models are tree-like structures. Moreover, in the case of fixed degree, we have shown that for all \(N \geq 2\), the satisfiability problem for the variant \(\text{QCTL}^{t}_{X,\leq N}\) is \(\text{AExp}_{\text{pol}}\)-complete. Whereas \(\text{AExp}_{\text{pol}}\)-hardness is established by reducing the alternating multi-tiling problem recently introduced in [13], the Tower-hardness of \(\text{SAT}(\text{QCTL}^{t}_{X})\) is essentially based on the fact that one can enforce concisely that a node has a number of children equal to some tower of exponentials.

Section 5 deals with the Tower-completeness of \(\text{SAT}(\text{QCTL}^{t}_{X})\) and \(\text{SAT}(\text{QCTL}^{t}_{XF})\), as well as Tower-completeness for the well-known modal logics \(K, \text{KD}, \text{GL}, \text{K4}\) and \(\text{S4}\) extended with propositional quantification but with adequate classes of tree-like structures. Though the Tower upper bound for decision problems on trees should not come as a real surprise, all our Tower-hardness results significantly improve the current state-of-the-art regarding the fragments of \(\text{QCTL}^{t}\) and for the above-mentioned modal logics. In particular, our proof technique for Tower-hardness of \(\text{SAT}(\text{QCTL}^{t}_{X})\) (and therefore for \(\text{QK}^{t}\) on finite trees) is simple enough so that it could be further reused or adapted, see e.g. a recent refinement of the proof in [8].

This work can be continued in several directions. For instance, Tower-hardness of \(\text{SAT}(\text{QCTL}^{t}_{F})\) is recently refined in [39, 38] by establishing that already \(\text{QCTL}^{p}_{F}\) restricted to formulae of modal/temporal depth two is also Tower-hard. Among the several directions, one of them would be to characterise the expressiveness of \(\text{QCTL}^{t}_{X}\) or \(\text{QCTL}^{t}_{F}\) along the lines of [19] or [35], see also [5]. More generally, we believe that standard modal logics with propositional quantification, but under the tree semantics, deserve to be much better understood.
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