Anomalies, entropy and boundaries

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Abstract

A relation between the conformal anomaly and the logarithmic term in the entanglement entropy is known to exist for CFT’s in even dimensions. In odd dimensions the local anomaly and the logarithmic term in the entropy are absent. As was observed recently, there exists a non-trivial integrated anomaly if an odd-dimensional spacetime has boundaries. We show that, similarly, there exists a logarithmic term in the entanglement entropy when the entangling surface crosses the boundary of spacetime. The relation of the entanglement entropy to the integrated conformal anomaly is elaborated for three-dimensional theories. Distributional properties of intrinsic and extrinsic geometries of the boundary in the presence of conical singularities in the bulk are established. This allows one to find contributions to the entropy that depend on the relative angle between the boundary and the entangling surface.
1 Introduction

It is by now well known that for conformal field theories there is a relation between the conformal anomalies and the logarithmic term in the entanglement entropy computed for an entangling surface $\Sigma$. This relation was first found in [1] in the case when $\Sigma$ is a black hole horizon. The general structure of this relation, see [3], can be obtained by applying the replica method in which the entropy is derived from the action by introducing a small angle deficit at surface $\Sigma$ and then differentiating with respect to the deficit. The geometrical aspects of this technique were developed in [4]. Black hole horizons, static or stationary, are characterized by vanishing extrinsic curvature. For a generic surface there appears an extra contribution, see [5], to the logarithmic term in the entropy that is due to the extrinsic curvature of the surface. Geometrically, these contributions originate from a squashed conical singularity as was shown in [6].

In odd dimensions, the local conformal anomaly is absent since there is no invariant of an odd dimension constructed from the Riemann curvature and its derivatives. Similarly, the logarithmic term in the entropy for any CFT in odd dimensions is absent. This is due to the fact that one cannot construct a geometric invariant of odd dimension from the bulk curvature and the extrinsic curvature of the entangling surface. Since the invariant in question should not depend on the choice of the vectors normal to the surface one always needs an even number of extrinsic curvatures to construct an invariant.

As became clear from the recent studies [7], [8], [9] the situation is different if the space-time in question has boundaries. First of all, the integrated conformal anomaly in even dimensions is modified by certain boundary terms as shows the explicit calculation [7] in $d = 4$ dimensions. Then, quite surprisingly, the anomaly in odd dimensions occurs to be non-trivial solely due to the boundary terms, as was suggested in [8]. The reason for this is simple: the boundary in this case is even-dimensional and the required conformal invariants of even dimension can be easily constructed from the Riemann curvature and the extrinsic curvature of the boundary.

In this paper we build on these findings and suggest that, similarly, the logarithmic term in entanglement entropy is non-trivial in odd dimensions provided the entangling surface crosses the boundary of spacetime. The earlier study of the boundary entanglement is given in [10]. For simplicity, we mainly focus on the $d = 3$ case although the general argument is applicable to the entanglement entropy calculation in any odd-dimensional space-time and we comment on the possible structure of the logarithmic term in higher dimensions.

Our results can be summarized as follows. The integrated anomaly in a conformal field theory in three dimensions is solely determined by two boundary conformal invariants $\chi_2$, the Euler number of the boundary, and $j$, a quadratic combination of the boundary extrinsic curvature,

$$A = -a\chi_2 + qj,$$  \hspace{1cm} (1.1)

and by two boundary charges $a$ and $q$ specific for the theory. The logarithmic term in the entanglement entropy which appears under partition of the space-time by an entangling surface (curve) $\Sigma$ crossing the boundary can be defined via the anomaly of the effective

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1The logarithmic term in $d = 4$ entanglement entropy was discovered in [2] for the Schwarzschild black hole.
action by following [3]. We denote this term $s_{\text{anom}}$ and find

$$s_{\text{anom}} = -\alpha \mathcal{N} + q \sum_k f(\alpha_k)$$

(1.2)

where $\mathcal{N}$ is the number of points, where $\Sigma$ crosses the boundary, and $\alpha_k$ are angles (at each of such points) between a normal vector to the boundary and the tangent vector to $\Sigma$.

Derivation of (1.2) from (1.1) requires one to consider (1.1) on manifolds with conical singularities when the singular surface crosses the boundary. One of our new results is that quadratic combinations of the boundary extrinsic curvature tensor behave as delta-functions in a way similar to components of the Riemann tensor [4]. The angle function $f(\alpha)$ can be calculated explicitly, see (6.3).

We also study the logarithmic term in the genuine entanglement entropy derived in a standard way from the divergent part of the effective action on a manifold with conical singularities. This term is denoted as $s$. By analysing the case when $\Sigma$ is orthogonal to the boundary, $\alpha_k = 0$, we conclude that $s$ and $s_{\text{anom}}$ do not coincide, in general. The differences appear due to the presence of the non-minimal coupling. This is a novel feature specific for 3-dimensional entropies. This property does not appear in 4 dimensions [3], [5].

The paper is organized as follows. In Sec. 2 we compute the integrated conformal anomaly in scalar and spinor CFT’s in three dimensions. The Renyi entanglement entropy in these theories for a 3D Euclidean space with two parallel boundary planes and $\Sigma$ orthogonal to the boundary planes is considered in Sec. 3. A relation between this entropy and the integrated anomaly is studied in Sec. 4. Distributional properties of the boundary geometry due to conical singularities when $\Sigma$ crosses the boundary plane under an arbitrary angle are analysed in Sections 5 and 6. In particular we present there new results on distributional properties of extrinsic curvatures. Formula (1.2) and a relation of $s_{\text{anom}}$ to $s$ are considered in Sec. 7. A brief discussion of the logarithmic terms in the entropy in higher dimensions and conclusions are in Sections 8 and 9.

### 2 Integrated conformal anomaly in $d = 3$

Boundary terms in conformal anomalies of effective actions of different conformal field theories have been studied recently in four dimensions [7], [8]. The aim of these works was to establish the universal structure of boundary terms and relation of boundary and bulk charges.

We consider an integral conformal anomaly of the effective action $W$

$$\mathcal{A} \equiv \partial_\sigma W [e^{2\sigma} g_{\mu
u}]_{\sigma=0} = \int_{\mathcal{M}_d} \sqrt{g} d^d x \left< T^\mu_{\phantom{\mu} \mu} \right>,$$

(2.1)

which is defined under scaling with a constant factor $\sigma$. The right hand side of (2.1) relates $\mathcal{A}$ to the trace anomaly and holds on a closed manifold. In odd dimensions ($d = 2k + 1$) the local trace anomaly is zero. The integral anomaly, however, may be non-trivial in the case of boundaries.

We calculate the integral anomaly for a number of models for $d = 3$ and comment on its generalizations to higher dimensions ($d = 5$). For $d = 3$ the anomaly is a pure
Table 1: Charges in the anomaly of the effective action

| Theory          | a   | q  | boundary condition |
|-----------------|-----|----|-------------------|
| real scalar     | $\frac{1}{96}$ | $\frac{1}{64}$ | Dirichlet          |
| real scalar     | $-\frac{1}{96}$ | $\frac{1}{64}$ | Robin             |
| Dirac spinor    | 0   | $\frac{1}{32}$ | mixed             |

boundary term \((1.1)\) composed of two conformal invariants set on the boundary\(^2\)

\[
\chi_2 = \frac{1}{4\pi} \int_{\partial M} \sqrt{H} d^2 x \hat{R}, \tag{2.2}
\]

\[
j = \frac{1}{4\pi} \int_{\partial M} \sqrt{H} d^2 x \text{Tr} \hat{K}^2. \tag{2.3}
\]

Here \(\chi_2\) is the Euler number of \(\partial M\), \(\hat{R}\) is the curvature on the boundary, \(a\) and \(q\) are boundary charges which depend on the model under consideration. Conformal invariance of \(j\) in \((2.3)\) follows from the fact that \(\hat{K}_{\mu\nu}\) transforms homogeneously under conformal transformations.

We use the following notations: \(g_{\mu\nu}\) is the metric of the background manifold \(M\), the metric induced on the boundary \(\partial M\) of \(M\) is \(H_{\mu\nu} = g_{\mu\nu} - N_\mu N_\nu\), \(N^\nu\) is a unit outward pointing normal vector to \(\partial M\), \(\hat{K}_{\mu\nu} = K_{\mu\nu} - H_{\mu\nu}K/2\) is a traceless part of \(K_{\mu\nu}\). Also we put \(\text{Tr}K^m = K_{\mu_1}^{\mu_2}...K_{\mu_p}^{\mu_p}\), \(K = \text{Tr}K\).

In three dimensions we have 3 non-interacting conformal field theories: massless scalar fields with a conformal coupling (either with the Dirichlet or with the conformal Neumann condition) and a massless spin 1/2 field with mixed boundary conditions. The boundary charges for these models are listed in Table 1.

Boundary charge \(a\) is not positive definite: it changes the sign depending on the type of the boundary condition. On the other hand, the boundary charge \(q\) appears to be positive and independent of the boundary conditions. In a renormalization in which for a scalar field the value of this charge is \(q = 1\) it takes value \(q = 2\) for a Dirac fermion. This behavior suggests that \(q\) may represent the number of degrees of freedom in the quantum field theory. The charges for the spinor theory are just a sum of the charges of two scalar CFT’s with the Dirichlet and Neumann boundary conditions.

In the scalar case \(a\) and \(q\) have been computed in [8]. For completeness we comment on computations of charges in Table 1.

We use the relation between the anomaly and the heat coefficient of a Laplace operator \(\Delta = -\nabla^2 + X\) for the corresponding \(d\)-dimensional conformal theory,

\[
\mathcal{A} = \eta A_d , \tag{2.4}
\]

\(^2\)See also [11] for a related discussion in the presence of defects.
where \( \eta = +1 \), for Bosons, and \( \eta = -1 \), for Fermions. The heat coefficients for the asymptotic expansion of the heat kernel of \( \Delta \) are defined as

\[
K(\Delta; t) = \text{Tr} e^{-t\Delta} \simeq \sum_{p=0} A_p(\Delta) t^{(p-d)/2} , \quad t \to 0 .
\]

(2.5)

If the classical theory is scale invariant the heat coefficient \( A_d \) is a conformal invariant, see e.g. [13]. Therefore \( A_d \) can be represented as a linear combination of conformal invariants constructed of the geometrical characteristics of \( \mathcal{M} \), \( \partial \mathcal{M} \) and embedding of \( \partial \mathcal{M} \) in \( \mathcal{M} \).

We are interested in \( d = 3 \) case. The heat coefficient in this case is a pure boundary term. On dimensional grounds \( A_3 \) can be written in terms of 2 invariants \( \chi^2 \). Thus, one comes to (1.1). Explicit values of the charges can be found by using results for the heat kernel coefficients collected in [12]. The boundary conditions are

\[
(\nabla_N - S)\Pi_+ \phi = 0 , \quad \Pi_- \phi = 0 ,
\]

(2.6)

where \( \nabla_N = N^\mu \nabla_\mu \), \( \Pi_\pm \) are corresponding projectors, \( \Pi_+ + \Pi_- = 1 \), definition of \( S \) coincides with [12]. Some general formulas for the heat kernel coefficients (see [12]) in dimension \( d \) are

\[
A_1 = \frac{1}{4(4\pi)^{d/2}} \int_{\partial \mathcal{M}} \sqrt{H} d^{d-1}x \text{Tr} \chi ,
\]

\[
A_3 = \frac{1}{384(4\pi)^{d/2}} \int_{\partial \mathcal{M}} \sqrt{H} d^{d-1}x \text{Tr} \left[ -96\chi X + 16\chi R - 8\chi R_{\mu\nu} N^\mu N^\nu 
\right.
\]

\[
+(13\Pi_+ - 7\Pi_-) K^2 + (2\Pi_+ + 10\Pi_-) \text{Tr} K^2 + 96SK + 192S^2 - 12\chi_{a\chi a} \right] .
\]

(2.7)

Here we use 'flat' indices \( a, b \) in the tangent space to the boundary, \( \chi = \Pi_+ - \Pi_- \).

For \( d = 3 \) in the case of a conformal scalar field \( X = R/8 \). For the Dirichlet condition \( \Pi_+ = 0 \), \( \chi = -1 \). Conformally invariant scalar Robin condition requires \( S = -K/4 \), \( \Pi_+ = 1 \), \( \chi = 1 \).

In case of a massless Dirac field \( \psi \) the operator is \( \Delta^{(1/2)} = (i\gamma^\mu \nabla_\mu)^2 \). The boundary conditions are mixed ones,

\[
\Pi_- \psi \big|_{\partial \mathcal{M}} = 0 , \quad (\nabla_N + K/2)\Pi_+ \psi \big|_{\partial \mathcal{M}} = 0 ,
\]

(2.9)

where \( \Pi_\pm = \frac{1}{2}(1 \pm i\gamma_s N^\mu \gamma_\mu) \), and \( \gamma_s \) is a chirality gamma matrix. Therefore, \( X = R/4 \), \( S = -\Pi_+ K/2 \).

Results of Table 1 easily follow from (2.8) if we use the Gauss-Codazzi identities and relations, \( \text{Tr} \Pi_\pm = 1 \), \( \chi_{a} = 2\Pi_+ a = i\gamma^b \gamma_s K_{ba} \), \( \text{Tr} \chi_{a\chi a} = 2\text{Tr} K^2 \).

### 3 Entanglement Renyi entropy in \( d = 3 \)

We consider the entropy of entanglement and, more generally, entanglement Renyi entropy \( S^{(n)} \). The order of the entropy \( n \) is a natural number, entanglement entropy corresponds to value \( n = 1 \). The entropy is assumed to appear under partition of the system into
different parts by an entangling surface Σ. In $d = 3$ Σ is a curve. The Renyi entropy in three dimensions is expected to depend on the UV cutoff parameter as follows:

$$S^{(n)} \simeq c(n)L\Lambda + \ln(\mu\Lambda) \sum_{P} s(n, \alpha_k),$$

(3.1)

where $L$ is the length of Σ, $\Lambda$ is the UV cutoff and $\mu$ is a typical scale of the system, $c(n)$ is some polynomial of $n$. The aim of our analysis is the logarithmic term of the entropy in (3.1). This term, as we shall see, comes from the points where the entangling surface Σ crosses the boundary. We denote this set of points as $P = \Sigma \cap \partial M = \bigcup \! \! \! \! p_k$. The sum in (3.1) goes over all such points $p_k$. Coefficient functions $s(n, \alpha_k)$ are polynomials of $n$. Their conformal invariance in a CFT allows dependence on the angles $\alpha_k$ between $N$ and tangent vectors to Σ at those boundary points.

The question which we address in this paper is the relation between the integrated conformal anomaly of the effective action and the logarithmic term in (3.1). We study the models introduced in Sec. 2.

The logarithmic part of the entropy in (3.1) can be derived from the ultraviolet part of the effective action by setting the action on manifolds with conical singularities. The method is described, e.g. in [14], and yields the formula:

$$s(n) = \eta \frac{nA_3(1) - A_3(n)}{n - 1}$$

(3.2)

(the zero modes in (3.2) are ignored). Here $A_3(n)$ is the heat coefficient of the corresponding Laplacian on a manifold $M_n$ which appears in the replica method of the entropy computations. $M_n$ is constructed from $n$ copies of the spacetime manifold $M$ by a `cut and glue' procedure. The details of this procedure are not relevant for our purposes.

The heat coefficients $A_3(n)$ when Σ meets the boundary under an arbitrary angle are unknown. So we derive them for configurations when Σ and the boundary are orthogonal. We suppose that the spacetime $M$ has the structure $R^2 \times L$, where $L$ is an interval with 2 end points. The space $M$ is flat and the operators reduce to simple Laplacians.

The boundary $\partial M$ consists of two planes (located at end points of $L$). The entangling surface Σ is an interval (identical to $L$). Σ touches $\partial M$ at its two end points, $p_1$ and $p_2$. The replica method introduces manifolds $M_n = C_n \times L$, where $C_n$ is a two-dimensional cone with an opening angle $2\pi n$. The heat kernel coefficient $A_3(n)$ on $M_n$ is a product

$$A_3(n) = A_2(C_n) \times A_1(L),$$

(3.3)

where

$$A_2(C_n) = \eta \frac{1}{12n^2} (1 - n^2)$$

(3.4)

is the heat kernel coefficient for a Laplace operator for the considered models on a two-dimensional cone, and

$$A_1(L) = \frac{1}{4} \sum_{P_k} \text{Tr} \chi$$

(3.5)

is the heat coefficient on one-dimensional interval $L$, with two ends $p_1$ and $p_2$, see Eq.(2.7).
The result for the entropy is:

\[ s(n) = -4a \frac{n+1}{n}, \quad s = -8a, \quad (3.6) \]

where \( a \) are defined in Table 1. Here \( s = s(1) \) is the logarithmic term in the entanglement entropy.

No logarithmic terms in the entropy appear for spin 1/2 fields, since \( \text{Tr} \chi = 0 \). For scalar fields each point of the boundary, where it meets the entangling surface, yields a contribution to the logarithmic term \( (3.6) \) equal \( -4a \). \( a = \pm 1/96 \). The extra factor 2 in \( (3.6) \) follows from the fact that there are 2 such points. In general, if there are \( N \) such points, the logarithmic term is \( s = -4Na \).

### 4 Entropy from the integrated conformal anomaly, and non-minimal coupling

The logarithmic terms in entropy of entanglement can be alternatively derived from the anomaly of the effective action, as has been shown in [3]. This definition for the integrated anomaly looks as follows:

\[ s_{\text{anom}} = \lim_{n \to 1} \frac{nA - A(n)}{n - 1}. \quad (4.1) \]

Here \( A(n) \) is the integrated anomaly \( (1.1) \) taken on the corresponding replicated manifolds \( M_n \) glued from \( n \) copies of \( M \). Formula \( (4.1) \) requires definitions of invariants \( \chi \) and \( j \) on the boundary \( \partial M_n \) in case when \( \Sigma \) crosses \( \partial M_n \).

We show that \( (4.1) \) results in \( (1.2) \). To simplify the analysis in this Section we again consider the case of the entangling surface orthogonal to the boundary. Conical singularities result in delta-function contributions in the scalar curvature of \( \partial M_n \). Thus, we expect that \( \chi \) should produce a non-trivial contribution in \( (4.1) \), since the Euler characteristics in two dimension is determined by the curvature scalar. As for another invariant \( j \), Eq. \( (2.3) \), we demonstrate in next Section that \( \text{Tr} \hat{K}^2 \) can be non-trivial only when \( \Sigma \) is not orthogonal to the boundary.

Let us see what happens with the integrated anomaly \( (1.1) \) by using example of the previous Section, \( M_n = C_n \times L \). Since invariant \( q \) does not contribute at the conical singularity then

\[ A(n) = -a\chi_2[\partial M_n] = -2a(1 - n). \quad (4.2) \]

Coefficient 2 in the r.h.s. of \( (4.2) \) corresponds to two components of the boundary. For the configuration in question we put \( \chi_2[\partial M_n] = 2(1 - n) \), where we took into account that each conical singularity adds an extra \( 4\pi(1 - n) \) to the integral of the curvature. The logarithmic term in the entropy obtained via \( (4.1) \) is

\[ s_{\text{anom}} = -2a. \quad (4.3) \]

Formula \( (4.3) \) is a particular case of \( (1.2) \) for \( N = 2 \) and \( \alpha_k = 0 \).

In case of scalar theory \( (4.3) \) contradicts the straightforward derivation of the logarithmic term, \( s = -8a \), see \( (3.6) \). The explanation of this contradiction is as follows. In a scalar CFT the corresponding Laplace operator \( \Delta = -\nabla^2 + X \) includes the non-minimal
coupling with the curvature, since the conformal invariance requires that \( X = R/8 \). Computation of \( s_{\text{anom}} \) implies that all curvatures including those in the non-minimal couplings should acquire delta-function terms. Opposite to that, calculation of \( s \) treats \( X \) as a non-singular term in the Laplacian.

Contribution of the non-minimal coupling to the heat coefficient is, see (2.8),

\[
A_{3}^{nc} = \frac{1}{384(4\pi)} \int_{\partial\mathcal{M}} \sqrt{H} d^{2}x \: \text{Tr} \left[ -12 \chi R \right]. 
\] (4.4)

If conical singularities are allowed to contribute in (4.4) one finds

\[
A_{3}^{nc}(n) = nA_{3}^{nc}(1) - \frac{1 - n}{32} \sum_{k} \text{Tr} \chi_k. 
\] (4.5)

Notice that in (4.4) the bulk curvature \( R \) appears in the integral over the boundary. For the orthogonal configuration this makes no problem since the delta-function in \( R \) is in fact the delta-function normalized to unity with respect to the integration over the boundary. Formula (7.5) results in the following addition to the entropy:

\[
s_{nc} = \lim_{n \to 1} \frac{nA_{3}^{nc}(1) - A_{3}^{nc}(n)}{n - 1} = - \frac{1}{32} \sum_{k} \text{Tr} \chi_k = 6a. 
\] (4.6)

Now one can check that

\[
s_{\text{anom}} = s + s_{nc}. 
\] (4.7)

The effect of the non-minimal couplings is a new feature in three dimensional theories. This feature is not known in four dimensions. There, despite the presence of the non-minimal coupling in the scalar conformal operator, there is a complete agreement between the direct entropy calculation and the calculation via the conformal anomaly.

5 Conical singularity and induced geometry on the boundary

So far our analysis has been restricted to cases where the boundary and the entangling surface are orthogonal. In this section we study distributional properties of the boundary geometry provided the singular surface crosses the boundary at an arbitrary angle. To this aim it is useful to consider a simplest three-dimensional set up in which the 3-dimensional space-time \( \mathcal{M} \), its boundary and the entangling surface are flat. We choose Cartesian coordinates \((x^0, x^1, x^2)\) on \( \mathcal{M} \) where the metric is

\[
ds^2 = dx_0^2 + dx_1^2 + dx_2^2.
\] (5.1)

The entangling surface \( \Sigma \) is supposed to be a plane \( x_1 = 0 \), while the boundary \( \partial \mathcal{M} \) is an arbitrary plane parallel to \( x^0 \)

\[
x_2 = ax_1, \quad a = \tan \alpha, \quad \cos \alpha = N^\mu l_\mu,
\] (5.2)

where \( N^\mu \) is vector normal to the boundary \( \partial \mathcal{M} \) and \( l^\mu \) is vector tangent to the entangling surface \( \Sigma \). The boundary \( \partial \mathcal{M} \) and the entangling surface intersect at point \( P \). One sees
that \((\pi/2-\alpha)\) is the angle with which \(\Sigma\) intersects the boundary \(\partial M\) on the hypersurface of constant time \(x_0\).

To introduce \(M_n\) we switch to polar coordinates in the metric of \(M\):

\[
x_0 = \rho \sin n\phi \quad x_1 = \rho \cos n\phi
\]

and assume that \(\phi\) has period \(2\pi\). \(M_n\) has a conical singularity at \(\rho = 0\) with the angle surplus \(2\pi(1-n)\).

Since the singular surface crosses the boundary we expect some sort of conical singularities on the boundary. We first study intrinsic geometry on the boundary by using two independent methods in the limit \(n \to 1\) (by assuming, as usual, that such a limit can be done in final integral expressions).

It is instructive to start with the analysis of the distributional properties of the scalar curvature of the boundary by using a regularization procedure [4]. The subtle point is that we want to regularize the boundary geometry by regularizing the bulk geometry. The regularized bulk metric takes the form

\[
ds^2 = f_n(\rho) d\rho^2 + n^2 \rho^2 d\phi^2 + dx_2^2, \quad f_n(\rho) = \frac{\rho^2 + n^2 b^2}{\rho^2 + b^2},
\]

where \(b\) is a regularization parameter to be taken to zero at the end. The regularity of (5.4) at \(\rho = 0\) can be seen in coordinates \(y_1 = \rho \cos \phi, y_2 = \rho \sin \phi\) which near \(\rho = 0\) behave as Cartesian coordinates.

To find regularized geometry induced on the boundary \(\partial M\) one should take (5.4) on some embedding equation which does not spoil analyticity at \(\rho = 0\). Equation (5.2) in the polar coordinates takes the form:

\[
x_2 = a \rho \cos n\phi, \quad a = \tan \alpha
\]

and cannot be used since it does not have the analytic behaviour due to the presence of \(\cos n\phi\). The problem is similar to manifolds with squashed conical singularities considered in [6] and it is cured by changing embedding equation (5.5) to

\[
x_2 = c^{1-n} \rho^n \cos n\phi, \quad n > 1
\]

where \(c\) is an arbitrary parameter which has the dimension of a length. Coordinate \(x_2\) in (5.6) is an analytic function with respect to Cartesian-like coordinates \(y_1, y_2\).

One can now compute the integral of the scalar curvature \(\hat{R}\) of the regularized boundary geometry over some domain around \(\rho = 0\), say a disk of a finite radius \(\rho_0\) with the center at \(\rho = 0\). Calculations can be done in two steps. First, one computes the integral of \(\hat{R}\) along \(\rho\) with the condition that \(f(\rho = 0) = n^2\) and \(f(\rho_0) \to 1\), when \(b \to 0\). With these conditions the integral does not depend on the concrete form of \(f(\rho)\).

The integration over angle \(\phi\) is complicated. To perform it we relax the condition that \(n\) is a natural number and consider the limit \(n \to 1\), which is actually what we are interested in. By decomposing the integral in powers of \((n-1)\) we obtain

\[
\int_0^{2\pi} d\phi \int_0^{\rho_0} d\rho \sqrt{H} \hat{R} = \frac{4\pi}{\sqrt{1 + a^2}} (1-n) + O(1-n)^2,
\]

(5.7)
where the higher order terms are functions of $a$ (vanishing identically if $a = 1$). This means that $\hat{R}$ has a delta-function behaviour at point $\rho = 0$

$$\hat{R} \simeq 4\pi \cos \alpha (1 - n) \hat{\delta}_P, \ n \to 1,$$

(5.8)

where $\hat{\delta}_P$ is normalized with respect to the integration measure on $\partial M$.

If the entangling surface and the boundary are orthogonal, $\alpha = 0$, the boundary conical singularity at $P$ results in the usual addition $4\pi(1 - n)$ to the curvature integral. For arbitrary $\alpha$ we find an extra factor $\cos(\alpha)$. If the entangling surface touches the boundary at zero angle ($\alpha = \pi/2$) the conical singularity on the boundary effectively disappears.

We now obtain (5.8) by an alternative method based on some simple geometrical arguments. Note that in the considered geometrical setup (Eqs. (5.1), (5.2)) there is a rotational symmetry around the conical singularity. Instead of studying $n$ replicas of $M$ with conical singularities on the entangling line $x_0 = x_1 = 0$ (and the surplus of the conical angle) we can consider a manifold $M_\beta$ with conical singularities on $x_0 = x_1 = 0$ and the length of a unit radius circle around the singularity equal $2\pi \beta$. As a result of the rotational symmetry, $\beta$ can be an arbitrary positive parameter. We choose $0 < \beta < 1$ which corresponds to the conical angle deficit.

In this case the orbifold $M_\beta$ can be obtained by cutting a part of the 3D Euclidean space between two planes which intersect at the line $x_0 = x_1 = 0$ under angle $\delta = 2\pi(1 - \beta)$, $\delta$ being the angle deficit of the cone. The boundary $\partial M_\beta$ of $M_\beta$ is a 2-plane with a conical singularity. When the two planes intersect the boundary they produce two lines under the angle $\delta_b = 2\pi(1 - \beta_b)$, which can be determined from the relation

$$\tan(\delta_b) = \cos \alpha \tan(\delta).$$

(5.9)

We see that the angle deficit on the boundary is different from the one in the bulk. At small angle deficits ($\beta \to 1$) the relation (5.9) takes a simple form

$$2\pi(1 - \beta_b) \simeq 2\pi(1 - \beta) \cos \alpha.$$

(5.10)

The boundary metric on $\partial M$ can be written as

$$dl^2 = dx_0^2 + (1 + a^2)dx_1^2 = dx_0^2 + \frac{dx_1^2}{\cos^2 \alpha} = dx_0^2 + d\bar{x}_1^2.$$

(5.11)

The same metric can be used on $\partial M_\beta$ if we introduce a set of polar coordinates

$$x_0 = \bar{\rho} \sin(\bar{\phi} \beta_b) \quad x_1 = \bar{\rho} \cos(\bar{\phi} \beta_b)$$

(5.12)

where $\bar{\phi}$ ranges from 0 to $2\pi$. The scalar curvature for the metric induced on the boundary, therefore, acquires the following contribution at small angle deficits:

$$\hat{R} = 4\pi(1 - \beta_b) \hat{\delta}_P \simeq 4\pi \cos \alpha (1 - \beta) \hat{\delta}_P.$$

(5.13)

This results agrees with Eq. (5.7) obtained by using the regularization procedure. We use these results in next Sections.
Extrinsic geometry of conical singularity on the boundary

One of the results of this paper is that not only intrinsic geometry but also geometrical characteristics related to its embedding in a higher dimensional space may have distributional properties. Let us demonstrate this for invariants quadratic in extrinsic curvatures, $K^2$, $\text{Tr}K^2$, by using the model introduced in the previous section.

Regularization (5.4), (5.6) makes the extrinsic curvature tensor regular at $\rho = 0$ and integrable provided $n > 1$. If instead of regularized embedding equation (5.6) one used equation (5.5) the components of the extrinsic curvature tensor had a power law singularity in $\rho = 0$.

Since for $n = 1$ the extrinsic curvature vanishes one would expect that integrals of $K^2$, $\text{Tr}K^2$ yield results which are at least quadratic in $(n - 1)$. A careful analysis shows that it is not so. The integration over a disk of radius $\rho_0$ produces terms $\rho_0^{2n-2}/2(n-1) = 1/2(n-1) + O(1)$. So that in the limit $n \to 1$ this pole cancels one power of $(n - 1)$ and transforms $(n - 1)^2$ term into $(n - 1)$ term. This is exactly the same mechanism which happens in the case of the squashed cones considered in [6]. With these remarks in the limit of small angle deficit ($n \to 1$) one obtains the following expressions:

\[
\int_{\partial \mathcal{M}} K^2 \simeq \int_{\partial \mathcal{M}} \text{Tr} K^2 \simeq 8\pi(1 - n)f(\alpha), \quad (6.1)
\]

\[
j \simeq (1 - n)f(\alpha), \quad (6.2)
\]

\[
f(\alpha) = -\frac{1}{32\cos^2 \alpha}\left(1 + 2\cos^2 \alpha + 5\cos^4 \alpha\right). \quad (6.3)
\]

Invariant $j$ is defined in (2.3).

Our next step is to study the effect of conical singularities in quadratic combinations of $K_{\mu \nu}$ of a general form and check the consistence of expressions (6.1). We continue with a 3D manifold $\mathcal{M}_\beta$ with a rotational symmetry around conical singularities and return to an arbitrary angle $\beta$ around the singular points. Components of the Riemann tensors on $\mathcal{M}_\beta$ and its boundary $\partial \mathcal{M}_\beta$, respectively, are

\[
R_{\mu \nu \lambda \rho} = 2\pi(1 - \beta) \left(G_{\mu \lambda}G_{\nu \rho} - G_{\mu \rho}G_{\nu \lambda}\right) \delta_P, \quad (6.4)
\]

\[
\hat{R}_{\mu \nu \lambda \rho} = 2\pi(1 - \beta_0) \left(H_{\mu \lambda}H_{\nu \rho} - H_{\mu \rho}H_{\nu \lambda}\right) \hat{\delta}_P. \quad (6.5)
\]

Here $G_{\mu \nu} = (p_i)_\mu(p_i)_\nu$ and $p_i$ is a pair of orthonormalized vectors orthogonal to the entangling line in $\mathcal{M}$, while $H_{\mu \nu} = (\hat{p}_i)_\mu(\hat{p}_i)_\nu$, where $\hat{p}_i$ is a pair of orthonormalized vectors at the conical singularity in the tangent space of $\partial \mathcal{M}$. In fact, $H_{\mu \nu}$ is an induced metric on $\partial \mathcal{M}$. In (6.4), (6.5) we also denote $\delta_P$ and $\hat{\delta}_P$ normalized delta-functions on $\mathcal{M}_\beta$ and $\partial \mathcal{M}_\beta$, respectively. The delta-functions in the limit $\beta \to 1$ are related as

\[
\hat{\delta}_P = \delta(x_0)\delta(x_1) = \cos \alpha \delta(x_0)\delta(x_1) = \cos \alpha \delta_P, \quad (6.6)
\]
where we used the fact that \( \bar{x}_1 = x_1 / \cos \alpha \).

One can suggest the following general structure:

\[
K_{\mu\nu}K_{\lambda\rho} = 2\pi(1 - \beta_b) (a_1 H_{\mu\nu}H_{\lambda\rho} + a_2 (H_{\mu\lambda}H_{\nu\rho} + H_{\mu\rho}H_{\nu\lambda})) \hat{\delta}_P ,
\]

(6.7)

where \( a_1 \) and \( a_2 \) are unknown functions of \( \alpha \). Eq. (6.7) reflects required symmetries under permutations of indices. Another property of Eq. (6.7) is that it takes into account \( O(2) \) symmetry associated with the rotation of the pair of vectors \( \hat{p}_i \). This property can be seen if one rewrites (6.7) in the basis \( \hat{p}_i \)

\[
K_{ij}K_{mn} = 2\pi(1 - \beta_b) (a_1 \delta_{ij}\delta_{mn} + a_2 (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})) \hat{\delta}_P ,
\]

(6.8)

where \( K_{ij} = \hat{p}^\mu_i \hat{p}^\nu_j K_{\mu\nu} \).

Consider restrictions imposed by the Gauss-Codazzi equations on Eq. (6.7). For smooth geometries one has

\[
K_{\mu\rho}K_{\nu\lambda} - K_{\mu\lambda}K_{\nu\rho} = R_{\mu\nu\lambda\rho}^\parallel - \hat{R}_{\mu\nu\lambda\rho} ,
\]

(6.9)

where \( R_{\mu\nu\lambda\rho}^\parallel \) are the components of the Riemann tensor in the bulk projected on the tangent space to \( \partial M \). Let us study the right hand side of (6.9) by using expressions (6.4), (6.5) for singular parts. In the limit of a small angle deficit

\[
R_{\mu\nu\lambda\rho}^\parallel - \hat{R}_{\mu\nu\lambda\rho} \simeq
\]

\[
2\pi \frac{(1 - \beta)}{\cos \alpha} \left( G_{\mu\lambda}^\parallel G_{\nu\rho}^\parallel - G_{\mu\rho}^\parallel G_{\nu\lambda}^\parallel - \cos^2 \alpha (H_{\mu\lambda}H_{\nu\rho} - H_{\mu\rho}H_{\nu\lambda}) \right) \hat{\delta}_P
\]

(6.10)

where we took into account (6.6). We need to calculate \( G_{\mu\nu}^\parallel = H^{\lambda}_{\mu}H^\nu_{\nu}G_{\lambda\rho} \). Let \( l \) be a unit vector tangent to the entangling line. One has

\[
G_{\mu\nu} = \delta_{\mu\nu} - l_\mu l_\nu .
\]

(6.11)

Therefore,

\[
G_{\mu\nu}^\parallel = H_{\mu\nu} - l_\mu l_\nu^\parallel ,
\]

(6.12)

where \( l_\mu^\parallel = H^{\lambda}_{\mu}l_\lambda \).

Let us choose the basis \( \hat{p}_i \) such that \( \hat{p}_2 \) is directed along the \( x^0 \) axis, while \( \hat{p}_1 \) is directed along the \( x^1 \). Then vector \( l_\mu^\parallel \) is directed along \( \hat{p}_1 \), and one can easily see that \( l_\mu^\parallel = \sin \alpha \hat{p}_1 \). With these definitions,

\[
G_{\mu\lambda}^\parallel G_{\nu\rho}^\parallel - G_{\mu\rho}^\parallel G_{\nu\lambda}^\parallel - \cos^2 \alpha (H_{\mu\lambda}H_{\nu\rho} - H_{\mu\rho}H_{\nu\lambda}) = \sin^2 \alpha (H_{\mu\lambda}H_{\nu\rho} - H_{\mu\rho}H_{\nu\lambda}) +
\]

\[
H_{\mu\rho}(\hat{p}_1)_{(\nu}(\hat{p}_1)_{\lambda)} + H_{\nu\lambda}(\hat{p}_1)_{(\mu}(\hat{p}_1)_{\rho)} - H_{\mu\lambda}(\hat{p}_1)_{(\nu}(\hat{p}_1)_{(\rho)} - H_{\nu\rho}(\hat{p}_1)_{(\mu}(\hat{p}_1)_{\lambda)} \equiv 0 ,
\]

(6.13)

Therefore, in the limit of a small angle deficit we have the following condition for the singular parts:

\[
K_{\mu\rho}K_{\lambda\nu} - K_{\mu\lambda}K_{\nu\rho} = 0 ,
\]

(6.14)

which should hold up to terms quadratic in \( \beta - 1 \). It follows immediately that expressions (6.11) obtained via the prescribed regularization procedure are consistent with (6.14). Another consequence is that \( a_1 = a_2 \) in (6.7), and in the limit \( \beta \to 1 \) one can write

\[
K_{\mu\nu}K_{\lambda\rho} \simeq \pi(1 - \beta) f (H_{\mu\nu}H_{\lambda\rho} + H_{\mu\lambda}H_{\nu\rho} + H_{\mu\rho}H_{\nu\lambda}) \hat{\delta}_P ,
\]

(6.15)

where \( f = f(\alpha) \) is defined in (6.3).
7 A general expression for the entropy

We can now provide arguments in support of formula (1.2) for a general geometric configuration. We use definition (4.1) for entropy $s_{anom}$ derived from the integrated conformal anomaly (1.1) and definitions of invariants (2.2), (2.3).

The expression at $q$-charge in (1.2) comes out immediately from (6.2). The term associated to $a$-charge requires one to know topological characteristic $\chi$ in (4.1). Despite the fact that we find a dependence on angle $\alpha$ in the formula for the boundary curvature (5.8) this dependence disappears in the Euler characteristics of the boundary. This is clearly the case for the Euler number of a disk containing a conical singularity. Indeed, the regularization as in Section 5 does not change the Euler number. Therefore, in the limiting procedure the number for a disk remains to be 1 and thus does not depend neither on the number of replicas $n$ nor on the angle $\alpha$. To see what happens for compact boundaries one can use again a simple set up. Suppose that $M$ is a flat Euclidean three dimensional space. Choose $\partial M$ as a cylinder with the axis along the entangling surface $\Sigma$, which is, say, $x^2$ axis, as in the examples considered before. We suppose that the cylinder has the length $L$ and some radius. The size of cylinder is restricted by two parallel planes going along $x^0$ and having an angle $\alpha$ with the axis $x^2$. The topology of the complete boundary is that of a sphere $S^2$. Hence $\chi[\partial M] = 2$. Since $\Sigma$ is along $x^2$ axis, $M_n$ are obtained from $n$ copies of $M$ when cuts are made by a half plane which end on $x^2$. By this construction $\partial M_n$ has the same topology as $\partial M$ regardless the value of $\alpha$. Therefore,

$$\frac{n\chi[\partial M] - \chi[\partial M_n]}{n - 1} = 2$$ ,

and one comes to (1.2) with $N = 2$. One can notice that each boundary end point of $\Sigma$ in this example yields addition $-a$ to the $a$-term in (1.2). This can be extended to an arbitrary number $N$ of boundary points.

We now turn to the entropy $s$ which is defined by (3.2) in the limit $n \to 1$. Formula (3.2) can be used if heat coefficients $A_3(n)$ are known for an arbitrary angle between the singular surface and the boundary. Since straightforward derivations of $A_3(n)$ is this case are absent one can apply results of Sections 5 and 6 to formula (2.8) by assuming this can be done at small angle deficits. For spin 1/2 field it yields $s = s_{anom}$. For the scalar theory $s \neq s_{anom}$ since there are non-minimal couplings. To calculate $s$ we can proceed as in Sec. 4 and write, by taking into account Eqs. (4.6),(4.7),

$$s = s_{anom} - s_{nc}$$ ,

$$s_{nc} = \lim_{n \to 1} \frac{nA_3^{nc}(1) - A_3^{nc}(n)}{n - 1}$$ ,

$$A_3^{nc} = \frac{1}{384(4\pi)} \int_{\partial M} \sqrt{H} d^2x \ Tr \ [-12\chi R]$$ ,

where conical singularities are allowed to contribute in (7.4). With the help of (6.6) one finds

$$A_3^{nc}(n) = nA_3^{nc}(1) - \frac{1 - n}{32 \cos \alpha} \sum_k \Tr \chi_k$$ .

This yields

$$s_{nc}(\alpha) = \frac{6a}{\cos \alpha}$$ ,

13
which reduces to (4.6) at $\alpha = 0$.

Both $f(\alpha)$ and $s_{nc}(\alpha)$ are singular at $\alpha \rightarrow \pi/2$. The case $\alpha = \pi/2$ is degenerate since the partition of the system onto two parts cannot be defined. Note that $f(\alpha)$ is monotonically decreasing function and $f(0) = 0$.

8 Comments on higher dimensions

Let us make some comments on what happens in higher dimensions. We are interested in the case when the entangling surface $\Sigma$ intersects the boundary $\partial M$ so that $P = \Sigma \cap \partial M$. We then expect that in any dimension $d$ there is a contribution to the logarithmic term in entanglement entropy due to $P$. We shall denote this contribution as $s_0(P)$. The concrete form of $s_0(P)$ depends on how many conformal invariants may contribute to this term in dimension $d$. Let us list the possible contributions. First of all we notice that $\dim P = d - 3$ so that $s_0(P)$ is given by integral over $P$ of a geometric quantity of dimension $(d - 3)$. This dimension is odd if $d$ is odd and even if $d$ is even. Being considered as a subset in the boundary $\partial M$ the intersection $P$ is a co-dimension 2 surface and therefore there are 2 vectors $p^a_\mu$, $a = 1, 2$ normal to $P$ in $\partial M$. The respective extrinsic curvature is $k^a_{\mu\nu}$, $a = 1, 2$. By projecting the extrinsic curvature $K_{\mu\nu}$ of the boundary $\partial M$ on these orthogonal vectors we construct $\tilde{K}_{ab} = p^a_\mu p^b_\nu K_{\mu\nu}$, where $\tilde{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{d-1} R_{\mu\nu}K$ is trace-free extrinsic curvature. Then, the possible contributions to $s_0(P)$ are the bulk Weyl tensor both projected on normal vector $N^\mu$ and the normal vectors $p^a_\mu$, $W_{a\mu
u}$ and $W_{abcd}$, the conformal invariants constructed from the extrinsic curvature of the boundary, $\tilde{K}_{\mu\nu}$, and its projections on vectors $p^\mu_a$, $\tilde{K}_{ab}$, and also the trace free extrinsic curvature of $P$, $k^a_{\mu\nu} = k^a_{\mu\nu} - \frac{1}{d-3} \gamma_{\mu\nu} k^\alpha_a$, $\gamma_{\mu\nu}$ is the induced metric on $P$. We will not attempt here give a general form for the possible contributions to $s_0(P)$ since their number is rapidly growing with spacetime dimension. The possible terms in $s_0(P)$ in $d = 4$ were suggested in [10].

9 Concluding remarks

In this paper we have demonstrated that there is a non-trivial logarithmic term in entanglement entropy of a conformal field theory in $d = 3$ dimensions provided the entangling surface intersects the boundary of the spacetime. This feature is not typical only for three dimensions and it appears in any higher dimension $d$ when there is a non-trivial intersection $P$, see [10]. In odd dimensions, when the standard contribution is vanishing, there always exists a non-trivial logarithmic term due to $P$.

In dimension $d = 3$ we have found that there is a certain relation between the logarithmic term in the entropy and the conformal charges in the integral trace anomaly. However, this relation is less straightforward than in dimension $d = 4$. We have demonstrated that the non-minimal coupling in the conformal field operator should be properly taken into account.

We have analyzed the conical geometry induced on the boundary by a conical singularity in the bulk. This geometry, both intrinsic and extrinsic, appears to be rather non-trivial in the case when the entangling surface $\Sigma$ intersects the boundary under angle different from $\pi/2$. These findings indicate that the conical geometry still has some surprises for us and perhaps more interesting features will be revealed in the future.
Regarding the interplay of anomalies, entropy and the boundaries which is the main focus of the present paper we can mention at least two interesting and important open questions:

1) How the boundary charges which appear in the integrated Weyl anomaly can be derived from the \( n \)-point correlation functions of the CFT stress-energy tensor?

2) What is the holographic description of the boundary terms in the Weyl anomaly and in the entanglement entropy?

The work towards the understanding the answer on the second question is on-going and we hope to report the progress in the nearest future.

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