Fife’s Theorem Revisited

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Abstract. We give another proof of a theorem of Fife — understood broadly as providing a finite automaton that gives a complete description of all infinite binary overlap-free words. Our proof is significantly simpler than those in the literature. As an application we give a complete characterization of the overlap-free words that are 2-automatic.

1 Introduction

Repetitions in words is a well-researched topic. Among the various themes studied, the binary overlap-free words play an important role, both historically and as an example exhibiting interesting structure. Here by an overlap we mean a word of the form $axaxa$, where $a$ is a single letter and $x$ is a (possibly empty) word.

It is easy to see that neither the finite nor the infinite binary overlap-free words form a regular language. Nevertheless, in 1980, Earl Fife [8] proved a theorem characterizing the infinite binary overlap-free words as encodings of paths in a finite automaton. His theorem was rather complicated to state and the proof was difficult. Berstel [3] later simplified the exposition, and both Carpi [6] and Cassaigne [7] gave an analogous analysis for the case of finite words. Also see [4].

In this note we show how to use the factorization theorem of Restivo and Salemi [11] to give an alternate (and, we hope, significantly simpler) proof of Fife’s theorem — here understood in the general sense of providing a finite automaton whose paths encode all infinite binary overlap-free words.

As a consequence we are able to disprove a conjecture on the fragility of overlap-free words.

2 Notation

Let $\Sigma$ be a finite alphabet. We let $\Sigma^*$ denote the set of all finite words over $\Sigma$ and $\Sigma^\omega$ denote the set of all (right-) infinite words over $\Sigma$. We say $y$ is a factor of a word $w$ if there exist words $x, z$ such that $w = xyz$.

If $x$ is a finite word, then $x^\omega$ represents the infinite word $xxx \cdots$.

As mentioned above, an overlap is a word of the form $axaxa$, where $a \in \Sigma$ and $x \in \Sigma^*$. An example of an overlap in English is the word alfalfa. A finite or infinite word is overlap-free if it contains no finite factor that is an overlap.
From now on we fix $\Sigma = \{0, 1\}$. The most famous infinite binary overlap-free word is $t$, the Thue-Morse word, defined as the fixed point, starting with 0, of the Thue-Morse morphism $\mu$, which maps 0 to 01 and 1 to 10. We have

$$t = t_0 t_1 t_2 \cdots = 0110100110010110 \cdots$$

The morphism $\mu$ has a second fixed point, $\overline{t} = \mu^\omega(1)$, which is obtained from $t$ by applying the complementation coding defined by $\overline{0} = 1$ and $\overline{1} = 0$.

We let $\mathcal{O}$ denote the set of (right-) infinite binary overlap-free words.

We now recall the infinite version of the factorization theorem of Restivo and Salemi [11] as stated in [1, Lemma 3].

**Theorem 1.** Let $x \in \mathcal{O}$, and let $P = \{p_0, p_1, p_2, p_3, p_4\}$, where $p_0 = \epsilon$, $p_1 = 0$, $p_2 = 00$, $p_3 = 1$, and $p_4 = 11$. Then there exists $y \in \mathcal{O}$ and $p \in P$ such that $x = p\mu(y)$. Furthermore, this factorization is unique, and $p$ is uniquely determined by inspecting the first 5 letters of $x$.

We can now iterate the factorization theorem to get

**Corollary 1.** Every infinite overlap-free word $x$ can be written uniquely in the form

$$x = p_{i_1} \mu(p_{i_2} \mu(p_{i_3} \mu(\cdots)))$$

with $i_j \in \{0, 1, 2, 3, 4\}$ for $j \geq 1$, subject to the understanding that if there exists $c$ such that $i_j = 0$ for $j \geq c$, then we also need to specify whether the “tail” of the expansion represents $\mu^\omega(0) = t$ or $\mu^\omega(1) = \overline{t}$. Furthermore, every truncated expansion

$$p_{i_1} \mu(p_{i_2} \mu(p_{i_3} \mu(\cdots p_{i_n-1} \mu(p_{i_n}) \cdots)))$$

is a prefix of $x$, with the understanding that if $i_n = 0$, then we need to replace 0 with either 1 (if the “tail” represents $t$) or 3 (if the “tail” represents $\overline{t}$).

**Proof.** The form (1) is unique, since each $p_i$ is uniquely determined by the first 5 characters of the associated word.

Thus, we can associate each infinite binary overlap-free word $x$ with the essentially unique infinite sequence of indices $i := (i_j)_{j \geq 0}$ coding elements in $P$, as specified by (1). If $i$ ends in $0^\omega$, then we need an additional element (either 1 or 3) to disambiguate between $t$ and $\overline{t}$ as the “tail”. In our notation, we separate this additional element with a semicolon so that, for example, the string $000 \cdots ; 1$ represents $t$ and $000 \cdots ; 3$ represents $\overline{t}$.

Other sequences of interest include $203000 \cdots ; 1$, which codes $001001\overline{t}$, the lexicographically least infinite word, and $2(31)^\omega$, which codes the word having, in the $i$‘th position, the number of 0’s in the binary expansion of $i$.

Of course, not every possible sequence of $(i_j)_{j \geq 1}$ of indices corresponds to an infinite overlap-free word. For example, every infinite word coded by $21 \cdots$ represents $00\mu(0\mu(\cdots))$ and hence begins with 000 and has an overlap. Our goal is to characterize precisely, using a finite automaton, those infinite sequences corresponding to overlap-free words.

We recall some basic facts about overlap-free words.
Lemma 1. Let $a \in \Sigma$. Then

(a) $x \in \mathcal{O} \iff \mu(x) \in \mathcal{O}$;
(b) $a \mu(x) \in \mathcal{O} \iff \overline{\sigma}x \in \mathcal{O}$;
(c) $aa \mu(x) \in \mathcal{O} \iff \overline{\sigma}x \in \mathcal{O}$ and $x$ begins with $\overline{aa}$.

Proof. See, for example, [1].

We now define 11 subsets of $\mathcal{O}$:

\[ A = \mathcal{O} \]
\[ B = \{x \in \Sigma^\omega : 1x \in \mathcal{O}\} \]
\[ C = \{x \in \Sigma^\omega : 1x \in \mathcal{O}$ and $x$ begins with $101$} \]
\[ D = \{x \in \Sigma^\omega : 0x \in \mathcal{O}\} \]
\[ E = \{x \in \Sigma^\omega : 0x \in \mathcal{O}$ and $x$ begins with $010$} \]
\[ F = \{x \in \Sigma^\omega : 0x \in \mathcal{O}$ and $x$ begins with $11$} \]
\[ G = \{x \in \Sigma^\omega : 0x \in \mathcal{O}$ and $x$ begins with $1$} \]
\[ H = \{x \in \Sigma^\omega : 1x \in \mathcal{O}$ and $x$ begins with $1$} \]
\[ I = \{x \in \Sigma^\omega : 1x \in \mathcal{O}$ and $x$ begins with $00$} \]
\[ J = \{x \in \Sigma^\omega : 1x \in \mathcal{O}$ and $x$ begins with $0$} \]
\[ K = \{x \in \Sigma^\omega : 0x \in \mathcal{O}$ and $x$ begins with $0$} \]

Next, we describe the relationships between these classes:
Lemma 2. Let $x$ be an infinite binary word. Then

\[
\begin{align*}
    x \in A & \iff \mu(x) \in A \quad (2) \\
    x \in B & \iff 0\mu(x) \in A \quad (3) \\
    x \in C & \iff 00\mu(x) \in A \quad (4) \\
    x \in D & \iff 1\mu(x) \in A \quad (5) \\
    x \in E & \iff 11\mu(x) \in A \quad (6) \\
    x \in B & \iff \mu(x) \in B \quad (7) \\
    x \in E & \iff 1\mu(x) \in B \quad (8) \\
    x \in B & \iff \mu(x) \in D \quad (9) \\
    x \in D & \iff 1\mu(x) \in D \quad (10) \\
    x \in I & \iff \mu(x) \in E \quad (11) \\
    x \in C & \iff 0\mu(x) \in E \quad (12) \\
    x \in F & \iff \mu(x) \in C \quad (13) \\
    x \in E & \iff 1\mu(x) \in C \quad (14) \\
    x \in J & \iff 0\mu(x) \in I \quad (15) \\
    x \in G & \iff 1\mu(x) \in F \quad (16) \\
    x \in K & \iff \mu(x) \in J \quad (17) \\
    x \in J & \iff \mu(x) \in K \quad (18) \\
    x \in B & \iff 0\mu(x) \in J \quad (19) \\
    x \in C & \iff 0\mu(x) \in K \quad (20) \\
    x \in H & \iff \mu(x) \in G \quad (21) \\
    x \in G & \iff \mu(x) \in H \quad (22) \\
    x \in D & \iff 1\mu(x) \in G \quad (23) \\
    x \in E & \iff 1\mu(x) \in H \quad (24)
\end{align*}
\]

Proof.

- (2): Follows immediately from Lemma 1 (a).
- (3), (4), (7), (10): Follow immediately from Lemma 1 (b).
- (5), (6), (9), (12): Follow immediately from Lemma 1 (c).
- (8): \[0\mu(x) \in B \iff 10\mu(x) = \mu(1x) \in O \iff 1x \in O.\]
- (11): Just like (8).
\(\mu(x) \in E \iff (0\mu(x) \in \mathcal{O} \text{ and } \mu(x) \text{ begins with 010}) \iff (1x \in \mathcal{O} \text{ and } x \text{ begins with 00}).\)

15: Just like 13.

14: \(0\mu(x) \in E \iff (00\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with 010}) \iff (1x \in \mathcal{O} \text{ and } x \text{ begins with 101}).\)

16: Just like 14.

17: \(0\mu(x) \in I \iff (10\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with 00}) \iff (\mu(1x) \in \mathcal{O} \text{ and } x \text{ begins with 0}) \iff (1x \in \mathcal{O} \text{ and } x \text{ begins with 0}).\)

18: Just like 17.

19: \(\mu(x) \in J \iff (1\mu(x) \in \mathcal{O} \text{ and } \mu(x) \text{ begins with 0}) \iff (0x \in \mathcal{O} \text{ and } x \text{ begins with 0}).\)

20, 21: Just like 19.

21: \(0\mu(x) \in J \iff (10\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with 00}) \iff (\mu(1x) \in \mathcal{O} \iff 1x \in \mathcal{O}).\)

22: Just like 21.

23: \(0\mu(x) \in K \iff (00\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with 00}) \iff (1x \in \mathcal{O} \text{ and } x \text{ begins with 101}).\)

24: Just like 22.

We can now use the result of the previous lemma to create an 11-state automaton that accepts all infinite sequences \((i_j)_{j \geq 1}\) over \(\Delta := \{0, 1, 2, 3, 4\}\) such that \(p_{i_1}\mu(p_{i_2}\mu(p_{i_3}\mu(\cdots)))\) is overlap-free. Each state represents one of the sets \(A, B, \ldots, K\) defined above, and the transitions are given by Lemma 2.

Of course, we also need to verify that transitions not shown correspond to the empty set of infinite words. For example, a transition out of \(B\) on the symbol 2 would correspond to the set \(\{x : 100\mu(x) \in \mathcal{O}\}\). But if \(x\) begins with 0, then \(100\mu(x) = 10001\cdots\) contains the overlap 000 as a factor, whereas if \(x\) begins with 10, then \(100\mu(x) = 1001001\cdots\) contains the overlap 1001001 as a factor, and if \(x\) begins with 11, then \(100\mu(x) = 1001010\cdots\) contains 01010 as a factor. Similarly, we can (somewhat tediously) verify that all other transitions not given in Figure 1 correspond to the empty set:
\[ \delta(B, 4) = \{ x \in \Sigma^\omega : 111\mu(x) \in \mathcal{O} \} = \emptyset \]
\[ \delta(D, 2) = \{ x \in \Sigma^\omega : 000\mu(x) \in \mathcal{O} \} = \emptyset \]
\[ \delta(D, 4) = \{ x \in \Sigma^\omega : 011\mu(x) \in \mathcal{O} \} = \emptyset \]
\[ \delta(C, 1) = \{ x \in \Sigma^\omega : 10\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with } 101 \} = \emptyset \]
\[ \delta(C, 2) = \{ x \in \Sigma^\omega : 100\mu(x) \in \mathcal{O} \text{ and } 00\mu(x) \text{ begins with } 101 \} = \emptyset \]
\[ \delta(C, 4) = \{ x \in \Sigma^\omega : 111\mu(x) \in \mathcal{O} \text{ and } 11\mu(x) \text{ begins with } 101 \} = \emptyset \]
\[ \delta(E, 2) = \{ x \in \Sigma^\omega : 00\mu(x) \in \mathcal{O} \text{ and } 00\mu(x) \text{ begins with } 010 \} = \emptyset \]
\[ \delta(E, 3) = \{ x \in \Sigma^\omega : 01\mu(x) \in \mathcal{O} \text{ and } 1\mu(x) \text{ begins with } 010 \} = \emptyset \]
\[ \delta(E, 4) = \{ x \in \Sigma^\omega : 011\mu(x) \in \mathcal{O} \text{ and } 11\mu(x) \text{ begins with } 010 \} = \emptyset \]
\[ \delta(F, 0) = \{ x \in \Sigma^\omega : 0\mu(x) \in \mathcal{O} \text{ and } \mu(x) \text{ begins with } 11 \} = \emptyset \]
\[ \delta(F, 1) = \{ x \in \Sigma^\omega : 00\mu(x) \in \mathcal{O} \text{ and } 0\mu(x) \text{ begins with } 11 \} = \emptyset \]
\[ \delta(F, 2) = \{ x \in \Sigma^\omega : 00\mu(x) \in \mathcal{O} \text{ and } 00\mu(x) \text{ begins with } 11 \} = \emptyset \]
\[ \delta(F, 4) = \{ x \in \Sigma^\omega : 011\mu(x) \in \mathcal{O} \text{ and } 11\mu(x) \text{ begins with } 11 \} = \emptyset \]
\[ \delta(J, 2) = \{ x \in \Sigma^\omega : 100\mu(x) \in \mathcal{O} \text{ and } 00\mu(x) \text{ begins with } 0 \} = \emptyset \]
\[ \delta(J, 3) = \{ x \in \Sigma^\omega : 11\mu(x) \in \mathcal{O} \text{ and } 1\mu(x) \text{ begins with } 0 \} = \emptyset \]
\[ \delta(J, 4) = \{ x \in \Sigma^\omega : 111\mu(x) \in \mathcal{O} \text{ and } 11\mu(x) \text{ begins with } 0 \} = \emptyset \]
\[ \delta(K, 2) = \{ x \in \Sigma^\omega : 00\mu(x) \in \mathcal{O} \text{ and } 00\mu(x) \text{ begins with } 0 \} = \emptyset \]
\[ \delta(K, 3) = \{ x \in \Sigma^\omega : 01\mu(x) \in \mathcal{O} \text{ and } 1\mu(x) \text{ begins with } 0 \} = \emptyset \]
\[ \delta(K, 4) = \{ x \in \Sigma^\omega : 011\mu(x) \in \mathcal{O} \text{ and } 11\mu(x) \text{ begins with } 0 \} = \emptyset \]

The proof of most of these is immediate. (We have not listed \( \delta(I, a) \) for \( a \in \{0, 2, 3, 4\} \), nor \( \delta(G, a) \) for \( a \in \{1, 2, 4\} \), nor \( \delta(H, a) \) for \( a \in \{1, 2, 4\} \), as these are symmetric with other cases.) The only one that requires some thought is \( \delta(F, 4) \):

- If \( x \) begins 00, then \( 011\mu(x) = 0110101 \cdots \), which has 10101 as a factor.
- If \( x \) begins 01, then \( 011\mu(x) = 0110110 \cdots \), which has 0110110 as a factor.
- If \( x \) begins 1, then \( 011\mu(x) = 01110 \cdots \), which has 11 as a factor.
From Lemma 2 and the results above, we get

**Theorem 2.** Every infinite binary overlap-free word $x$ is encoded by an infinite path, starting in $A$, through the automaton in Figure 1.

Every infinite path through the automaton not ending in $0^\omega$ codes a unique infinite binary overlap-free word $x$. If a path $i$ ends in $0^\omega$ and this suffix corresponds to a cycle on state $A$ or a cycle between states $B$ and $D$, then $x$ is coded by either $i;1$ or $i;3$. If a path $i$ ends in $0^\omega$ and this suffix corresponds to a cycle between states $J$ and $K$, then $x$ is coded by $i;1$. If a path $i$ ends in $0^\omega$ and this suffix corresponds to a cycle between states $G$ and $H$, then $x$ is coded by $i;3$.

**Corollary 2.** Each of the 11 sets $A, B, \ldots, K$ is uncountable.

**Proof.** We prove this for $K$, with the proof for the other sets being similar. Elements in the set $K$ correspond to those infinite paths leaving the state $K$ in Figure 1. It therefore suffices to produce uncountably many distinct paths leaving $K$. One way to do this, for example, is by $\{13010, 1301000\}^\omega$.

### 3 The lexicographically least overlap-free word

We now recover a theorem of [1]:

![Fig. 1. Automaton coding infinite binary overlap-free words](image-url)
Theorem 3. The lexicographically least infinite binary overlap-free word is 001001$^\infty$.

Proof. Let $x$ be the lexicographically least infinite word, and let $y$ be its code. Then $y[1]$ must be 2, since any other choice codes a word that starts with 01 or something lexicographically greater. Once $y[1] = 2$ is chosen, the next two symbols must be $y[2..3] = 03$. Now we are in state $G$. We argue that the lexicographically least string that follows causes us to alternate between states $G$ and $H$ on 0, producing 100· · ·. For otherwise our only choices are 30, 31, or (if we are in $G$) 33 as the next two symbols, and all of these code a word lexicographically greater than 100. Hence $y = 2030^\infty$; 1 is the code for the lexicographically least sequence, and this codes 001001$^\infty$.

4 Automatic infinite binary overlap-free words

As a consequence of Theorem 2, we can give a complete description of the infinite binary overlap-free words that are 2-automatic [2]. Recall that an infinite word $(a_n)_{n \geq 0}$ is $k$-automatic if there exists a deterministic finite automaton with output that, on input $n$ expressed in base $k$, produces an output associated with the state last visited that is equal to $a_n$.

Theorem 4. An infinite binary overlap-free word is 2-automatic if and only if its code is both specified by the DFA given above in Figure 1, and is ultimately periodic.

First, we need two lemmas:

Lemma 3. An infinite binary word $x = a_0a_1a_2 \cdots$ is 2-automatic if and only if $\mu(x)$ is 2-automatic.

Proof. For one direction, we use the fact that the class of $k$-automatic sequences is closed under uniform morphisms ([2, Theorem 6.8.3]). So if $x$ is 2-automatic, so is $\mu(x)$.

For the other, we use the well-known characterization of automatic sequences in terms of the $k$-kernel ([2, Theorem 6.6.2]: a sequence $(c_n)_{n \geq 0}$ is $k$-automatic if and only if its $k$-kernel defined by

$\{(c_{k^e n + i})_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i < k^e\}$

is finite. Furthermore, each sequence in the $k$-kernel is $k$-automatic.

Now if $y = \mu(x) = b_0b_1b_2 \cdots$, then $b_{2n} = a_n$. So one of the sequences in the 2-kernel of $y$ is $x$, and if $y$ is 2-automatic, then so is $x$.

Now we can prove Theorem 4.

Proof. Suppose the code of $x$ is ultimately periodic. Then we can write its code as $yz^\omega$ for some finite words $y$ and $z$. Since the class of 2-automatic sequences is closed under appending a finite prefix ([2, Corollary 6.8.5], by Lemma 3 it suffices to show that the word coded by $z^\omega$ is 2-automatic.
The word $z\omega$ codes an overlap-free word $w$ satisfying $w = t\varphi(w)$, where $t$ is a finite word and $\varphi$ is a power of $\mu$. If $t$ is empty the result is clear. Otherwise, by iteration, we get that
\[ w = t\varphi(t)\varphi^2(t) \cdots. \] (27)

The 2-kernel of a sequence is obtained by repeated 2-decimation, that is, recursively splitting a sequence into its even- and odd-indexed terms. When we apply 2-decimation to $\mu^k(t)$, where $t$ is a finite word, we get $\mu^{k-1}(t)$ and $\mu^{k-1}(\overline{t})$. These words are both of even length, provided $k$ is at least 1. Hence iteratively applying 2-decimation to $w$, as given in (27), shows that if $\varphi = \mu^k$, then the 2-kernel of $w$ is contained in
\[ S := \{ u\mu^i(v)\mu^{i+k}(v)\mu^{i+2k}(v) \cdots : |u| \leq 2|t| \text{ and } v \in \{t, \overline{t}\} \text{ and } 1 \leq i \leq k \}, \]
which is a finite set.

On the other hand, suppose the code for $x$ is not ultimately periodic. Then we show that the 2-kernel is infinite. To see this, note that the code for $x$ contains a 2 or 4 only at the beginning, so we can assume without loss of generality that the code for $x$ contains only the letters 0, 1, 3. Now it is easy to see that if the code for $x$ is $ay$ for some letter $a \in \{0, 1, 3\}$ and infinite string $y \in \{0, 1, 3\}^\omega$, then one of the sequences in the 2-kernel (obtained by taking either the odd- or even-indexed terms) is either coded by $y$ or its complement is coded by $y$. Since the code for $x$ is not ultimately periodic, there are infinitely many distinct sequences in the orbit of the code for $x$, under the shift. (By the orbit of $y$ we mean the set of sequences of the form $y[i..\infty]$ for $i \geq 1$.) Now infinitely many of these sequences correspond to a sequence in the 2-kernel, or its complement. Hence $x$ is not 2-automatic.

5 A fragility conjecture disproved

Brown, Rampersad, Shallit, and Vasiga showed that the Thue-Morse word $t$ is fragile in the following sense: if any finite nonempty set of positions is chosen, and the bits in those positions are simultaneously flipped to the complement of their original values, the result has an overlap [5].

It is natural to wonder if a similar result holds more generally for all overlap-free words. However, the statement must be modified in this more general setting, as (for example) both $0t$ and $1t$ are overlap-free.

The author made the following conjecture at the Oberwolfach meeting in 2010:

Conjecture 1. For each infinite binary overlap-free word $w$ there exists a constant $C$ (depending on $w$) such that if the bits at any finite nonempty set of positions > $C$ are flipped, then the result has an overlap.

Using our result we can disprove this conjecture. For consider the infinite words coded by $1\{113011, 313011\}^\omega$. By examining the automaton, each such word is easily seen to be a valid code for an overlap-free word. These words have
blocks that line up exactly at the same positions, but each 6th block can be replaced by the appropriate power of \( \mu \) evaluated at either 0 or 1, and each such choice gives a distinct overlap-free word.

6 Remarks

According to a theorem of Karhumäki and the author [9], there is a similar factorization theorem for all exponents \( \alpha \) with \( 2 < \alpha \leq \frac{7}{3} \). Recently we have proven similar results for \( \alpha = \frac{7}{3} \) [10].

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