New conformal map for the trapezoidal formula for infinite integrals of unilateral rapidly decreasing functions

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Abstract

While the trapezoidal formula can attain exponential convergence when applied to infinite integrals of bilateral rapidly decreasing functions, it is not capable of this in the case of unilateral rapidly decreasing functions. To address this issue, Stenger proposed the application of a conformal map to the integrand such that it transforms into bilateral rapidly decreasing functions. Okayama and Hanada modified the conformal map and provided a rigorous error bound for the modified formula. This paper proposes a further improved conformal map, with two rigorous error bounds provided for the improved formula. Numerical examples comparing the proposed and existing formulas are also given.

Keywords: trapezoidal formula, Conformal map, Computable error bound
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1. Introduction and summary

In this paper, we are concerned with the trapezoidal formula for the infinite integral, expressed as

$$\int_{-\infty}^{\infty} f(x)dx \approx h \sum_{k=-\infty}^{\infty} f(kh),$$

where $h$ is a mesh size. This approximation formula is fairly accurate if the integrand $f(x)$ is analytic, which has been known since several decades ago [3, 7]. For example, the approximation

$$\int_{-\infty}^{\infty} e^{-x^2} dx \approx h \sum_{k=-\infty}^{\infty} e^{-(kh)^2}$$

gives the correct answer in double-precision with $h = 1/2$, and the approximation

$$\int_{-\infty}^{\infty} \frac{1}{4 + x^2} dx \approx h \sum_{k=-\infty}^{\infty} \frac{1}{4 + (kh)^2}$$

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Furthermore, they theoretically showed that the error of the modified formula, say $E_n$, is bounded by

$$|E_n| \leq C \cdot e^{-\sqrt{2\pi} \mu'},$$

where $\mu \geq \mu'$, and $C$ is explicitly given in a computable form. This inequality not only shows that the modified formula $E_n$ can attain faster convergence than $E_n$, but it also indicates that the error can be rigorously estimated by the right-hand side. This is useful for verified numerical integration.
The present work improves upon their results. Rather than the conformal map \( x = \psi(t) \) or \( x = \tilde{\psi}(t) \), we propose a new conformal map

\[
x = \phi(t) = 2 \sinh(\log(1 + e^t)).
\]

The principle of this conformal map is derived from the fact that the convergence rate is improved by replacing \( \text{arcsinh}(e^t) \) with \( \log(1 + e^t) \) in some fields \([1, 3, 5]\). Consequently, the following approximation formula is derived:

\[
\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(\phi(t))\phi'(t)dt = h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh).
\]

Furthermore, as the main contribution of this work, we provide two (general and special) theoretical error bounds in the same form as (3), where \( \mu_1 > 0 \). Consequently, the following approximation

\[
\text{holds for all } h > 0 \text{ and } (\mu_1, \beta_1), \text{ and}
\]

\[
\text{holds for all } h > 0, \text{ and } (\mu_2, \beta_2).
\]

2. Summary of existing and new results

Sections 2.1 and 2.2 describe the existing results, and Sections 2.3 and 2.4 describe the new results. First, the relevant notations are introduced. Let \( D \) be a strip domain defined by \( D = \{ \zeta \in \mathbb{C} : |\text{Im} \zeta| < d \} \) for \( d > 0 \). Furthermore, let \( D_\mu = \{ \zeta \in D : \text{Re} \zeta < 0 \} \) and \( D_\beta = \{ \zeta \in D : \text{Re} \zeta \geq 0 \} \).

2.1. Error analysis of Stenger’s formula

An error analysis for Stenger’s formula \([1]\) can be expressed as the following theorem, which is a restatement of an existing theorem \([9, \text{Theorem 1.5.16}]\).

**Theorem 2.1 (Okayama–Hanada \([2, \text{Theorem 2.1}]\)).** Assume that \( f \) is analytic in \( \psi(D) \) with \( 0 < \beta < \pi/2 \), and that there exist positive constants \( K, \alpha, \beta \) such that

\[
|f(z)| \leq K|z|^{-\beta}
\]

holds for all \( z \in \psi(D) \), and

\[
|f(z)| \leq K \frac{1}{|z|^{\alpha+1}}
\]

holds for all \( z \in \psi(D) \). Let \( \mu = \min\{\alpha, \beta\} \), let \( M \) and \( N \) be defined as

\[
\begin{cases}
M = n, & N = \lceil an/\beta \rceil \quad (\text{if } \mu = \alpha), \\
N = n, & M = \lceil \beta n/a \rceil \quad (\text{if } \mu = \beta),
\end{cases}
\]

and let \( h \) be defined as

\[
h = \sqrt{\frac{2\pi d}{\mu n}}.
\]

Then, there exists a constant \( C \) independent of \( n \), such that

\[
\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\psi(kh))\psi'(kh) \right| \leq C e^{-\sqrt{2\pi d \mu n}}.
\]
2.2. Error bound for the formula by Okayama and Hanada

Okayama and Hanada [3] proposed the replacement of $\psi$ with $\tilde{\psi}$ in Stenger’s formula (1). They also provided the following theoretical error bound for the modified formula (2).

**Theorem 2.2 (Okayama–Hanada [2, Theorem 2.2]).** Assume that $f$ is analytic in $\tilde{\psi}(D_d)$ with $0 < d < \pi/2$, and that there exist positive constants $K, \alpha, \beta$ such that

$$|f(z)| \leq Ke^{-\beta z}$$

holds for all $z \in \tilde{\psi}(D_d)$, and

$$|f(z)| \leq K \frac{1}{|4 + z^2(\alpha+1)/2|}$$

holds for all $z \in \tilde{\psi}(D_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $M$ and $N$ be defined as (7), and let $h$ be defined as (8). Then, it holds that

$$\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\tilde{\psi}(kh))\tilde{\psi}'(kh) \right| \leq K \left( \frac{2C_1}{1 - e^{-\sqrt{2}\mu}} + C_2 \right) e^{-\sqrt{2}\mu},$$

where $C_1$ and $C_2$ are constants defined as

$$C_1 = \gamma_d \frac{\gamma_d}{\alpha \arctan(\gamma_d)} \left( \frac{\gamma_d}{2} \left( 1 + \frac{1}{\sin^2 1} \right) \right) + \frac{(1 + \alpha^2) \sqrt{2\pi}}{\beta} \left( \sqrt{2} e^\alpha \cos(d/2) \right)^{\beta},$$

$$C_2 = \frac{1}{\alpha} \left( \frac{1}{2} \left( 1 + \frac{1}{\sin^2 1} \right) \right) + \frac{1 + \alpha^2}{\beta} \left( e^\alpha / 2 \right)^{\beta},$$

where $\gamma_d = 1/\cos(d)$ and $\sigma = 1/\arcsinh(1)$.

In Theorem 2.2, the condition (5) is modified to (9), and the condition (6) is modified to (10). The former constitutes the most significant difference, because $\beta$ in Theorem 2.2 can be two times greater than that in Theorem 2.1, while $\alpha$ remains unchanged. Owing to the difference, $\mu$ in Theorem 2.2 may be greater than that in Theorem 2.1, which affects the convergence rate $O(e^{-\sqrt{2}\mu})$.

Another difference between Theorems 2.1 and 2.2 lies in the constants on the right-hand side of the inequalities. All the constants in Theorem 2.2 are explicitly revealed, and the right-hand side can be computed to provide an error bound. This paper provides two error bounds for the improved formula (4) in the same manner as Theorem 2.2.

2.3. General error bound for the proposed formula

As a general case, we present the following error bound for the improved formula (4). The proof is given in Section 4.

**Theorem 2.3.** Assume that $f$ is analytic in $\phi(D_d)$ with $0 < d < \pi$, and that there exist positive constants $K, \alpha, \beta$ such that (9) holds for all $z \in \phi(D_d)$, and (10) holds for all $z \in \phi(D_d)$. Let $\mu = \min\{\alpha, \beta\}$, let $M$ and $N$ be defined as (7), and let $h$ be defined as (8). Then, it holds that

$$\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh) \right| \leq K \left( \frac{2C_3}{1 - e^{-\sqrt{2}\mu}} + C_4 \right) e^{-\sqrt{2}\mu},$$
where $C_3$ and $C_4$ are constants defined as

\begin{align*}
C_3 &= \left( \frac{1}{\alpha + 1} + \frac{1}{\alpha} \right) \left( \frac{e^{c_d}}{(1 - \log 2)(e - 1)} \right)^{\alpha + 1} \frac{1 + (\log(2 + c_d))^2}{(1 + c_d)^2} + \frac{(1 + \lambda^2) c_d}{\beta} (e^{\lambda} c_d)^\beta, \\
C_4 &= \frac{e^{1/n^3}}{\alpha(1 - \log 2)^{\alpha + 1}} + \frac{1 + \lambda^2}{\beta} (e^{\lambda})^\beta,
\end{align*}

where $c_d = 1/\cos(d/2)$ and $\lambda = 1/\log 2$.

The crucial difference between Theorems 2.2 and 2.3 is the upper bound of $d$: $d < \pi/2$ in Theorem 2.2 whereas $d < \pi$ in Theorem 2.3. This implies that in the new approximation (4), $d$ may be greater than $d$ in the previous approximation (2). In this case, the convergence rate $O(e^{-2\sqrt{\mu}d})$ is improved (note that $\mu$ is not changed between the two theorems).

This difference in the range of $d$ originates from the conformal maps $\tilde{\psi}$ and $\phi$. By observing the derivatives of the functions

\begin{align*}
\tilde{\psi}'(\zeta) &= \frac{1 + \arcsinh^2(e^\zeta)}{\sqrt{1 + e^{-2\zeta}} \arcsinh^2(e^\zeta)}, \\
\phi'(\zeta) &= \frac{1 + (\log(1 + e^\zeta))^2}{(1 + e^{-\zeta})(\log(1 + e^\zeta))^2},
\end{align*}

we see that $\tilde{\psi}'(\zeta)$ is not analytic at $\zeta = \pm i(\pi/2)$, and $\phi'(\zeta)$ is not analytic at $\zeta = \pm i \pi$. Accordingly, $f(\tilde{\psi}'(\zeta))\tilde{\psi}'(\zeta)$ is analytic at most $\mathcal{P}_{\pi/2}$, and $f(\phi'(\zeta))\phi'(\zeta)$ is analytic at most $\mathcal{P}_\pi$. Therefore, the range of $d$ is $0 < d < \pi/2$ in Theorem 2.2 and $0 < d < \pi$ in Theorem 2.3.

### 2.4. Special error bound for the proposed formula

As a special case, restricting the range of $d$ to $0 < d < (1 + \pi)/2$, we present the following error bound for the improved formula (4). The proof is given in Section 5.

**Theorem 2.4.** Assume that $f$ is analytic in $\phi(\mathcal{P}_d)$ with $0 < d < (1 + \pi)/2$, and that there exist positive constants $K$, $\alpha$, and $\beta$ such that (9) holds for all $z \in \phi(\mathcal{P}_d)$, and

\begin{equation}
|f(z)| \leq K \frac{1}{|4 + 2z|^{1/2}|z|^\alpha}
\end{equation}

holds for all $z \in \phi(\mathcal{P}_d^-)$. Let $\mu = \min\{\alpha, \beta\}$, let $M$ and $N$ be defined as (7), and let $h$ be defined as (8). Then, it holds that

\begin{equation}
\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh) \right| \leq K \left( \frac{2C_5}{1 - e^{-2\sqrt{\mu}d}} + C_6 \right) e^{-\sqrt{2\mu}d},
\end{equation}

where $C_5$ and $C_6$ are constants defined as

\begin{align*}
C_5 &= \frac{1}{\alpha} \left( \frac{e^{c_d}}{(1 - \log 2)(e - 1)} \right)^{a} \frac{1 + c_d}{\log(2 + c_d)} + \frac{(1 + \lambda^2) c_d}{\beta} (e^{\lambda} c_d)^\beta, \\
C_6 &= \frac{1}{\alpha(1 - \log 2)^{\alpha + 1}} + \frac{1 + \lambda^2}{\beta} (e^{\lambda})^\beta,
\end{align*}

where $c_d = 1/\cos(d/2)$ and $\lambda = 1/\log 2$.

In this theorem, the upper bound of $d$ is $(1 + \pi)/2$, which is smaller than that in Theorem 2.3. This is because the condition (6) is changed to (13), where $4 + (\phi(\zeta))^2$ (put $z = \phi(\zeta)$) has zero points at $\zeta = \log(2\sin(1/2)) \pm i(1 + \pi)/2$. However, the constants $C_5$ and $C_6$ are considerably smaller than $C_3$ and $C_4$, respectively (comparing the first term). Therefore, Theorem 2.4 is useful for attaining a sharp error bound rather than a large upper bound of $d$. It must be noted here that $(1 + \pi)/2$ is still greater than $\pi/2$ in Theorems 2.1 and 2.2.
3. Numerical examples

This section presents the numerical results obtained in this study. All the programs were written in C language with double-precision floating-point arithmetic. The following three integrals are considered:

\[
\int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{1 + (x/2)^2}} + \frac{1}{4 + x^2} \right) \exp \left( -\frac{x}{2} - \sqrt{1 + \left( \frac{x}{2} \right)^2} \right) \, dx = 3 - 4eE_1(1), \tag{16}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{4 + x^2} \exp \left( -\frac{x}{2} - \sqrt{1 + \left( \frac{x}{2} \right)^2} \right) \, dx = \text{Ci}(1)\sin 1 - \text{si}(1)\cos 1, \tag{17}
\]

\[
\frac{1}{2} \left( 1 + \frac{x}{\sqrt{4 + x^2}} \right) \frac{1}{1 + e^{(x/2)^2}} \, dx = 1.13677446810281077257 \cdots, \tag{18}
\]

where \( E_1(x) \) is the exponential integral defined by \( E_1(x) = \int_{1}^{\infty} (e^{-tx} / t) \, dt \), \( \text{Ci}(x) \) is the cosine integral defined by \( \text{Ci}(x) = -\int_{1}^{\infty} (\cos t / t) \, dt \), and \( \text{si}(x) \) is the sine integral defined by \( \text{si}(x) = -\int_{1}^{\infty} (\sin t / t) \, dt \). The third integral (18) is taken from the previous study [2].

| Table 1: Parameters for the integral (16). |
|------------------|---|---|---|---|
| Theorem 2.1 | 1 | 1/2 | 3/2 | K |
| Theorem 2.2 | 1 | 1 | 3/2 | 1 |
| Theorem 2.3 | 1 | 1 | 3 | 78 |
| Theorem 2.4 | 1 | 1 | 2 | 6/5 |

| Table 2: Parameters for the integral (17). |
|------------------|---|---|---|---|
| Theorem 2.1 | 1 | 1/2 | 3/2 | K |
| Theorem 2.2 | 1 | 1 | 3/2 | 16/9 |
| Theorem 2.3 | 1 | 1 | 2 | 215 |
| Theorem 2.4 | 1 | 1 | 2 | 39 |

| Table 3: Parameters for the integral (18). |
|------------------|---|---|---|---|
| Theorem 2.1 | 1 | π/4 | 3/2 | K |
| Theorem 2.2 | 1 | π/2 | 3/2 | 12 |
| Theorem 2.3 | 1 | π/2 | 3/2 | 9 |
| Theorem 2.4 | 1 | π/2 | 3/2 | 9/2 |

The integrand in (16) satisfies the assumptions of Theorems 2.1, 2.2, 2.3, and 2.4 with the parameters shown in Table 1. In Theorem 2.1, \( K \) is not investigated since \( K \) is not used for computation. In Theorems 2.1 and 2.2, \( d \) is taken as \( d = 3/2 \) since \( d < \pi/2 \). In Theorem 2.3, \( d \) is taken as \( d = 3 \) since \( d < \pi \). In Theorem 2.4, \( d \) is taken as \( d = 2 \) since \( d < (1 + \pi)/2 \). The results are shown in Figure 1. As seen in the graph, the proposed formula with \( d = 3 \) shows the fastest convergence as compared to the others. However, the corresponding error bound by Theorem 2.3 is relatively large, because the constant \( C_3 \) in (11) is large. In contrast, Theorem 2.4 produces a sharp error for the proposed formula with \( d = 2 \), although the convergence rate is slightly worse than that from Theorem 2.3.
The integrand in (17) satisfies the assumptions of Theorems 2.1, 2.2, 2.3, and 2.4 with the parameters shown in Table 2. In this case, \( d \) must satisfy \( d < (1 + \pi)/2 \) in Theorem 2.3 due to the singular points of \( 1/(4 + x^2) \). The results are shown in Figure 2. As seen in the graph, the proposed formula with \( d = 2 \) shows the fastest convergence as compared to the others. Note that in this example, Theorem 2.3 and Theorem 2.4 have the same \( d \) values, and thus, their approximation formulas are exactly the same. As for the error bound, Theorem 2.4 produces a sharper error than Theorem 2.3 in this case as well.

The integrand in (18) satisfies the assumptions of Theorems 2.1, 2.2, 2.3, and 2.4 with the parameters shown in Table 3. In this case, \( d \) must satisfy \( d < \pi/2 \) in both Theorems 2.3 and 2.4 due to the singular points of \( 1/(1 + e^{(\pi/2)x}) \). The results are shown in Figure 3. As seen in the graph, all formulas show a similar convergence rate, mainly because all formulas use the same value of \( d \). Approximation formulas of Theorem 2.3 and Theorem 2.4 are exactly the same, but Theorem 2.4 produces a sharper error than Theorem 2.3 in this case as well.

4. Proofs for Theorem 2.3

This section presents the proof of Theorem 2.3. It is organized as follows. In Section 4.1 the task is decomposed into two lemmas: Lemmas 4.2 and 4.3. To prove these lemmas, useful inequalities are presented in Sections 4.2, 4.3, 4.4, and 4.5. Following this, Lemma 4.2 is proved in Section 4.6 and Lemma 4.3 is proved in Section 4.7.
Figure 2: Numerical results for (17).

Figure 3: Numerical results for (18).
4.1. Sketch of the proof

Let \( F(t) = f(\phi(t))\phi'(t) \). The main strategy in the proof of Theorem 2.3 is to split the error into two terms as follows:

\[
\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh) \right| = \left| \int_{-\infty}^{\infty} F(t)dt - h \sum_{k=-M}^{N} F(kh) \right| \\
\leq \left| \int_{-\infty}^{\infty} F(t)dt - h \sum_{k=-\infty}^{\infty} F(kh) \right| + h \sum_{k=-\infty}^{\infty} F(kh) + h \sum_{k=N+1}^{\infty} F(kh). \tag{19}
\]

The first and second terms are called the discretization error and truncation error, respectively. The following function space is important for bounding the discretization error.

**Definition 4.1.** Let \( \mathcal{D}_d(e) \) be a rectangular domain defined for \( 0 < e < 1 \) by

\[ \mathcal{D}_d(e) = \{ \zeta \in \mathbb{C} : |\Re \zeta| < 1/e, |\Im \zeta| < d(1-e) \}. \]

Then, \( H^1(\mathcal{D}_d) \) denotes the family of all functions \( F \) that are analytic in \( \mathcal{D}_d \) such that the norm \( N_1(F, d) \) is finite, where

\[ N_1(F, d) = \lim_{e \to 0} \int_{\partial \mathcal{D}_d(e)} |F(\zeta)|d\zeta. \]

For functions belonging to this function space, the discretization error is estimated as follows.

**Theorem 4.1 (Stenger [8, Theorem 3.2.1]).** Let \( F \in H^1(\mathcal{D}_d) \). Then,

\[
\left| \int_{-\infty}^{\infty} F(x)dx - h \sum_{k=-\infty}^{\infty} F(kh) \right| \leq \frac{N_1(F, d)}{1 - e^{-2nd/h}} e^{-2nd/h}.
\]

In this paper, we show the following lemma, which completes estimation of the discretization error. The proof is given in Section 4.6.

**Lemma 4.2.** Let the assumptions made in Theorem 2.3 be fulfilled. Then, the function \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta) \) belongs to \( H^1(\mathcal{D}_d) \), and \( N_1(F, d) \) is bounded as

\[ N_1(F, d) \leq 2KC_3, \]

where \( C_3 \) is a constant defined as \( [1] \).

In addition, we bound the truncation error as follows. The proof is given in Section 4.7.

**Lemma 4.3.** Let the assumptions made in Theorem 2.3 be fulfilled. Then, setting \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta) \), we have

\[
\left| h \sum_{k=-\infty}^{N-1} F(kh) + h \sum_{k=N+1}^{\infty} F(kh) \right| \leq KC_4 e^{-\mu nh},
\]

where \( C_4 \) is a constant defined as \( [12] \).

Setting \( h \) as \( [8] \), the above estimates (Theorem 4.1, Lemmas 4.2 and 4.3) yield the desired result as

\[
\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh) \right| \leq \frac{2KC_3}{1 - e^{-2nd/h}} e^{-2nd/h} + KC_4 e^{-\mu nh}
\]

\[
= K \left( \frac{2C_3}{1 - e^{-\sqrt{2nd/\mu}}} + C_4 \right) e^{-\sqrt{2nd/\mu}}
\]

\[
\leq K \left( \frac{2C_3}{1 - e^{-\sqrt{2nd/\mu}}} + C_4 \right) e^{-\sqrt{2nd/\mu}}.
\]

This completes the proof of Theorem 2.3.
4.2. Useful inequalities on $\mathbb{R}$

We prepare two lemmas here.

Lemma 4.4 (Okayama et al. [5, Lemma 4.7]). We have

$$\left| \frac{\log(1 + e^x)}{1 + \log(1 + e^x)} \cdot \frac{1 + e^x}{e^x} \right| \leq 1 \quad (x \in \mathbb{R}). \quad (20)$$

Lemma 4.5. We have

$$\arccos\left( \frac{1}{2} \right) \geq \sqrt{2 - t} \quad (0 \leq t \leq 2). \quad (21)$$

Proof. Integrating both sides of the obvious inequality

$$2 - 2 \cos w \geq 0 \quad (w \geq 0),$$

we have

$$\int_0^u 2(1 - \cos w)dw = 2v - 2 \sin v \geq 0 \quad (v \geq 0).$$

In the same manner, integrating both sides of the above inequality, we have

$$\int_0^u 2(v - \sin v)dv = u^2 + 2 \cos u - 2 \geq 0 \quad (u \geq 0).$$

Here, putting $u = \arccos(t/2)$, we rewrite the inequality as

$$\arccos^2\left( \frac{t}{2} \right) \geq 2 - t \quad (0 \leq t \leq 2),$$

which is equivalent to the desired inequality (21).

4.3. Useful inequalities on $\mathbb{D}^+_d$

We prepare three lemmas here. Note that $\overline{\mathbb{D}}$ denotes the closure of $\mathbb{D}$.

Lemma 4.6. It holds for all $\zeta \in \overline{\mathbb{D}}^+$ that

$$\left| \frac{1}{\log(1 + e^\zeta)} \right| \leq \frac{1}{\log 2}. \quad (22)$$

Proof. Let $\zeta = x + iy$ where $x$ and $y$ are real numbers with $x \geq 0$ and $|y| \leq \pi$. By the definition of $\log z$, it holds that

$$\left| \frac{1}{\log(1 + e^\zeta)} \right|^2 = \left| \frac{1}{\log |1 + e^\zeta| + i \arg(1 + e^\zeta)} \right|^2 = \frac{1}{\left\{ \log |1 + e^{x+iy}| \right\}^2 + \left\{ \arg(1 + e^{x+iy}) \right\}^2}.$$

Since $|1 + e^{x+iy}|$ and $|\arg(1 + e^{x+iy})|$ monotonically increase with respect to $x$, we have

$$\frac{1}{\left\{ \log |1 + e^{x+iy}| \right\}^2 + \left\{ \arg(1 + e^{x+iy}) \right\}^2} \leq \frac{1}{\left\{ \log |1 + e^{0+iy}| \right\}^2 + \left\{ \arg(1 + e^{0+iy}) \right\}^2}.$$

Furthermore, using

$$\log |1 + e^y| = \log \sqrt{(1 + \cos y)^2 + \sin^2 y} = \log \sqrt{4 \cos^2 \left( \frac{y}{2} \right)} = \log \left( 2 \cos \frac{y}{2} \right),$$

$$\arg(1 + e^y) = \arctan \left( \frac{\sin y}{1 + \cos y} \right) = \arctan \left( \tan \frac{y}{2} \right) = \frac{y}{2},$$

we have

$$\left| \frac{1}{\log(1 + e^\zeta)} \right| \leq \frac{1}{\log 2}. \quad (22)$$
and putting \( t = 2 \cos(y/2) \), we have
\[
\frac{1}{\{ \log |1 + e^{iy}| \}^2 + \{ \arg(1 + e^{iy}) \}^2} = \frac{1}{\{ \log (2 \cos(y/2)) \}^2 + (y/2)^2} = \frac{1}{(\log t)^2 + \arccos^2(t/2)}.
\]

From \( 0 \leq t \leq 2 \) and (21), we have
\[
\frac{1}{(\log t)^2 + \arccos^2(t/2)} \leq \frac{1}{(\log t)^2 + \{ \sqrt{2} - t \}^2} = q(t),
\]
where
\[
q(t) = \frac{1}{(\log t)^2 + 2 - t}.
\]

Since \( \log t \leq t - 1 \), it holds that
\[
q'(t) = \frac{t - 2 \log t}{t \{ (\log t)^2 + 2 - t \}^2} \geq \frac{t - 2(t - 1)}{t \{ (\log t)^2 + 2 - t \}^2} = \frac{2 - t}{t \{ (\log t)^2 + 2 - t \}^2} \geq 0.
\]
Therefore, \( q(t) \) monotonically increases, from which we have \( q(t) \leq q(2) = 1/(\log 2)^2 \). This completes the proof.

**Lemma 4.7.** It holds for all \( \zeta \in D^- \) that
\[
|e^{1/\log(1 + e^\zeta)}| \leq e^{1/\log 2}. \tag{23}
\]

**Proof.** Using (22), we have
\[
|e^{1/\log(1 + e^\zeta)}| \leq e^{1/\log(1 + e^\zeta)} \leq e^{1/\log 2}.
\]
This completes the proof.

**Lemma 4.8.** It holds for all \( \zeta \in D^- \) that
\[
\left| \frac{1 + \{ \log(1 + e^\zeta) \}^2}{\{ \log(1 + e^\zeta) \}^2} \right| \leq 1 + \frac{1}{(\log 2)^2}. \tag{24}
\]

**Proof.** Using (22), we have
\[
\left| \frac{1 + \{ \log(1 + e^\zeta) \}^2}{\{ \log(1 + e^\zeta) \}^2} \right| = \left| 1 + \frac{1}{\{ \log(1 + e^\zeta) \}^2} \right| \leq 1 + \left| \frac{1}{\{ \log(1 + e^\zeta) \}^2} \right| \leq 1 + \frac{1}{(\log 2)^2}.
\]
This completes the proof.

**4.4. Useful inequality on \( D^- \)**

We prepare the following lemma here.

**Lemma 4.9.** It holds for all \( \zeta \in D^- \) that
\[
\frac{1}{|1 + \log(1 + e^\zeta)|} \leq \frac{1}{1 - \log 2}. \tag{25}
\]
Proof. By the definition of log z, it holds that
\[ \frac{1}{| -1 + \log(1 + e^z) |} = \frac{1}{| -1 + \log |1 + e^z| + i \arg(1 + e^z) |} \leq \frac{1}{| -1 + \log |1 + e^z| + 0 |} \]

Let \( \zeta = x + iy \) where \( x \) and \( y \) are real numbers with \( x < 0 \) and \( |y| \leq \pi \). Then, we have
\[ |1 + e^z| \leq 1 + |e^z| = 1 + |e^{x+iy}| = 1 + e^x < 1 + e^0 < e, \]
from which we have \( \log |1 + e^z| < 1 \). Therefore, it holds that
\[ \frac{1}{| -1 + \log |1 + e^z| |} = \frac{1}{1 - \log |1 + e^z|} \]
which is further bounded as
\[ \frac{1}{1 - \log |1 + e^z|} \leq \frac{1}{1 - \log(1 + |e^z|)} = \frac{1}{1 - \log(1 + e^x)} \leq \frac{1}{1 - \log(1 + e^0)} = \frac{1}{1 - \log 2}. \]

This completes the proof.

4.5. Useful inequalities on \( \mathcal{D}_d \)

We prepare four lemmas here.

Lemma 4.10 (Okayama et al. [5, Lemma 4.6]). It holds for all \( \zeta \in \mathcal{D}_d \) that
\[ \left| \frac{\log(1 + e^z)}{1 + \log(1 + e^z)} \cdot \frac{e^{-i} + e^z}{e^z} \right| \leq 1, \tag{26} \]
where \( l = \log(e/(e-1)) \).

Lemma 4.11 (Okayama et al. [4, Lemma 2.21]). For all \( x \in \mathbb{R} \) and \( y \in (-\pi, \pi) \), putting \( \zeta = x + iy \), we have
\[ \frac{1}{|1 + e^z|} \leq \frac{1}{(1 + e^x) \cos(y/2)} \tag{27} \]
\[ \frac{1}{|1 + e^{-z}|} \leq \frac{1}{(1 + e^{-x}) \cos(y/2)} \tag{28} \]

Lemma 4.12 (Three lines lemma, cf. [7, p. 133]). Let \( g \) be analytic and bounded in \( \mathcal{D}_d \) and continuous on \( \overline{\mathcal{D}_d} \). Let \( M_g(y) = \sup_{x \in \mathbb{R}} |g(x + iy)|. \) Then, we have
\[ \{M_g(y)\}^{2d} \leq \{M_g(-d)\}^{d-y} \{M_g(d)\}^{y+d} \quad (-d \leq y \leq d). \]

Lemma 4.13. Let \( d \) be a constant satisfying \( 0 < d < \pi \). For all \( \zeta \in \mathcal{D}_d \) and \( x \in \mathbb{R} \), we have
\[ \left| \frac{1 + \{\log(1 + e^z)\}^2}{(1 + e^{-z})^2 \{\log(1 + e^z)\}^2} \right| \leq \frac{1 + \{\log(2 + c_d)\}^2}{\{\log(2 + c_d)\}^2} \frac{1 + c_d}{(1 + c_d)^2}, \tag{29} \]
\[ \left| \frac{1 + \{\log(1 + e^z)\}^2}{(1 + e^{-z})^2 \{\log(1 + e^z)\}^2} \right| \leq e^{1/\pi^3}, \tag{30} \]
where \( c_d = 1/\cos(d/2) \).
Proof. First, consider (30), which is proved by showing
\[ p(t) = \frac{1 + t^2}{t^2} (1 - e^{-t})^2 \leq e^{1/3} \]
for all \( t > 0 \) (put \( t = \log(1 + e^t) \)). The derivative of \( p(t) \) is expressed as
\[ p'(t) = \frac{2(e^t - 1)(e^t - t^3 - t - 1)}{t^3 e^{2t}}. \]

Let \( \kappa \) be a value that satisfies \( p'(\kappa) = 0 \) and \( \log(108) < \kappa < \log(109) \), i.e., \( p(t) \) has its maximum at \( t = \kappa \). Using \( e^\kappa = \kappa^3 + \kappa + 1 \), we have
\[ p(\kappa) = \frac{1 + \kappa^2}{\kappa^2} \left( \frac{(\kappa^3 + \kappa + 1) - 1}{e^\kappa} \right)^2 = \frac{(1 + \kappa^2)^3}{e^{2\kappa}}. \]

Since the function \( q(x) = (1 + x^2)^3/e^{2x} \) monotonically decreases for \( x \geq (3 + \sqrt{5})/2 \), \( q(\log(109)) < q(\kappa) < q(\log(108)) \) holds (note that \( \log(108) > (3 + \sqrt{5})/2 \)). Thus, it holds that
\[ 1 < q(\log(109)) < p(\kappa) = q(\kappa) < q(\log(108)) < e^{1/3}. \]

Next, we show (29). Let
\[ g(\zeta) = \frac{1 + \log(1 + e^\zeta)}{(1 + e^{-\zeta})^2 \log(1 + e^\zeta)}^2. \]

Since the function \( g(\zeta) \) is analytic and bounded in \( \mathcal{D}_d \) and continuous on \( \overline{\mathcal{D}_d} \), by Lemma 4.12 we obtain (29) if we show the following two inequalities:
\[ M_\delta'(d) \leq \frac{1 + \log(2 + c_d)^2}{\log(2 + c_d)^2} (1 + c_d)^2, \quad M_\delta'(-d) \leq \frac{1 + \log(2 + c_d)^2}{\log(2 + c_d)^2} (1 + c_d)^2, \]
where \( M_\delta(y) = \sup_{x \in \mathbb{R}} |g(x + iy)| \). We show only the first one, because the second one is also shown in the same way. Putting \( \xi = \log(1 + e^{\xi + di}) \), \( g(x + id) = p(\xi) \) holds, and thus, in what follows we prove
\[ |p(\xi)| \leq \frac{1 + \log(2 + c_d)^2}{\log(2 + c_d)^2} (1 + c_d)^2. \]

We consider the following two cases: (a) \( |\xi| \leq \log(2 + c_d) \) and (b) \( |\xi| > \log(2 + c_d) \). In case (a), we have
\[ |p(\xi)| \leq \frac{1 + \xi^2}{\xi^2} \left( \sum_{k=1}^\infty (-\xi)^k / k! \right)^2 \leq (1 + |\xi|^2) \left( \sum_{n=1}^\infty |\xi|^n / n! \right)^2 \leq \frac{1 + |\xi|^2}{|\xi|^2} \left( e^{\log(1 + \xi)} - 1 \right)^2. \]

Here, if we put \( q(x) = (1 + x^2)^3(e^{x^2 - x})^2/x^2 \), then we have \( q'(x) = 2(e^x - 1)r(x)/x^3 \), where \( r(x) = 1 + e^x(x^3 + x - 1) \). Since \( r(x) = x e^x((x + 1)^2 + x) \geq 0 \) for \( x \geq 0 \), \( r(x) \) monotonically increases for \( x \geq 0 \). Therefore, \( r(x) \geq r(0) = 0 \) holds, from which we have \( q'(x) \geq 0 \) for \( x \geq 0 \), i.e., \( q(x) \) monotonically increases for \( x \geq 0 \). Thus, from \( |\xi| \leq \log(2 + c_d) \), we have (31)
\[ |p(\xi)| \leq \frac{1 + \log(2 + c_d)^2}{\log(2 + c_d)^2} (e^{\log(2 + c_d)} - 1)^2 = \frac{1 + \log(2 + c_d)^2}{\log(2 + c_d)^2} (1 + c_d)^2. \]

In case (b), from (27), it holds that
\[ \text{Re}(\xi) = \text{Re}(\log(1 + e^{\xi + di})) = \log|1 + e^{\xi + di}| \geq \log(|1 + e^\zeta| \cos(d/2)) \geq \log(\cos(d/2)). \]

Using this, we have
\[ |p(\xi)| \leq \frac{1 + |\xi|^2}{|\xi|^2} (1 + |e^{-\zeta}|)^2 \leq \frac{1 + |\xi|^2}{|\xi|^2} (1 + e^{-\text{Re}(\xi)})^2 \leq \frac{1 + |\xi|^2}{|\xi|^2} (1 + e^{-\log(\cos(d/2))})^2 = \frac{1 + |\xi|^2}{|\xi|^2} (1 + c_d)^2. \]

Furthermore, since \((1 + x^2)/x^2\) decreases monotonically for \( x > 0 \), we have (31). This completes the proof.
4.6. Estimation of the discretization error (proof of Lemma 4.2)

Lemma 4.2 is shown as follows.

Proof. Let \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta) \). Since \( f \) is analytic in \( \phi(D_d) \), \( f(\phi(i)) \) is analytic in \( D_d \). In addition, since \( \phi' \) is analytic in \( D_n \), \( F \) is analytic in \( D_d \) (note that \( d < \pi \)). Therefore, the remaining task is to show \( N_1(F, d) \leq 2KC_3 \).

From (9), by using (23) and (24), it holds for all \( \zeta \in D_d^\ast \) that

\[
|F(\zeta)| \leq K |e^{-\phi(\zeta)}| |\phi'(\zeta)| \leq K \left| e^{(1/\log(1+e^\zeta)) \beta} \frac{1}{1 + e^\zeta} \right|^\beta |1 + \{\log(1 + e^\zeta)^2\}| \leq K \left( e^{(1/\log 2)} \beta \right)^\beta \left| \frac{1}{1 + e^\zeta} \right|^\beta \left| 1 + \frac{1}{\log(1/e^2)} \right|.
\]

Furthermore, from (10), by using (25), (26), and (29), it holds for all \( \zeta \in D_d^\ast \) that

\[
|F(\zeta)| \leq K \frac{1}{|\phi'(\zeta)|^{\beta+1}} |\phi'(\zeta)| \leq K \left| \frac{\log(1 + e^\zeta)}{1 + \log(1 + e^\zeta)} \right|^{\beta+1} \frac{|1 + e^{-\zeta}|}{1 + \log(1 + \zeta^2)} |1 + \{\log(1 + e^\zeta)^2\}| \leq K \frac{1}{|\log(2 + c_d)|^{\beta+1} (1 + \log 2)^{\beta+1}} |1 + \frac{1}{(1 + e^{-\zeta})(1 + e^{-\zeta})} | \leq \frac{1}{c_d} \left( 1 + \log 2 \right)^{\beta+1} (1 + e^{-\zeta})^{\beta+1},
\]

where \( c_d = 1/\cos(d/2) \) and \( l = \log(e/(e - 1)) \). By definition, \( N_1(F, d) \) is expressed as

\[
N_1(F, d) = \lim_{e \to 0} \left[ \int_{-1/e}^{1/e} |F(x - i \cdot d)| dx + \int_{-d(1-e)}^{d(1-e)} |F(1/e + i \cdot y)| dy + \int_{-1/e}^{1/e} |F(x + i \cdot d)| dx + \int_{-d(1-e)}^{d(1-e)} |F(-1/e + i \cdot y)| dy \right].
\]

Using (27), (28), and (33), we can bound the second term as

\[
\int_{-d(1-e)}^{d(1-e)} |F(1/e + i \cdot y)| dy \leq K \left( e^{(1/\log 2)} \beta \right)^\beta \int_{-d(1-e)}^{d(1-e)} \left| \frac{1}{1 + e^{1/e+iy}} \right|^\beta \frac{1}{(1 + e^{1/e+iy})^{\beta+1}} dy \leq \frac{K \beta}{(1 + e^{1/e})^{\beta+1} (1 + e^{-1/e})^{\beta+1}} \int_{-d(1-e)}^{d(1-e)} \frac{1}{\cos^\beta(y/2) \cos(y/2)} dy,
\]

from which we have

\[
\lim_{e \to 0} \int_{-d(1-e)}^{d(1-e)} |F(1/e + i \cdot y)| dy = 0.
\]

In the same manner, with regard to the fourth term of (33), using (27), (28), and (34), we have

\[
\lim_{e \to 0} \int_{-d(1-e)}^{d(1-e)} |F(-1/e + i \cdot y)| dy = 0.
\]

Therefore, \( N_1(F, d) \) is expressed as

\[
N_1(F, d) = \int_{-\infty}^{\infty} |F(x - i \cdot d)| dx + \int_{-\infty}^{\infty} |F(x + i \cdot d)| dx
\]

\[
= \int_{-\infty}^{0} |F(x - i \cdot d)| dx + \int_{0}^{\infty} |F(x - i \cdot d)| dx + \int_{-\infty}^{0} |F(x + i \cdot d)| dx + \int_{0}^{\infty} |F(x + i \cdot d)| dx.
\]
With regard to the first term, using (27), (28), and (34), we have
\[
\int_{-\infty}^{0} |F(x - i d)| dx \leq \frac{K}{(1 - \log 2)^{\alpha+1}} \frac{1 + \{\log(2 + c_d)\}^2}{(1 + c_d)^2} \int_{-\infty}^{0} \frac{1 + |e^{-x+id}|}{|1 + e^{-x+id}|^{\alpha+1}} dx
\]
\[
\leq \frac{K}{(1 - \log 2)^{\alpha+1}} \frac{1 + \{\log(2 + c_d)\}^2}{(1 + c_d)^2} \int_{-\infty}^{0} \frac{1 + |e^{-x+id}|}{(1 + e^{-x+id})^{\alpha+1} \cos^{\alpha+1}(d/2)} dx
\]
\[
= \frac{K_{d}^{\alpha+1}}{(1 - \log 2)^{\alpha+1}} \frac{1 + \{\log(2 + c_d)\}^2}{(1 + c_d)^2} \int_{-\infty}^{0} \frac{1 + e^{-x}}{(1 + e^{-x})^{\alpha+1}} dx.
\]
The integral is further bounded as
\[
\int_{-\infty}^{0} \frac{1 + e^{-x}}{(1 + e^{-x})^{\alpha+1}} dx = \int_{-\infty}^{0} \left( \frac{e^{x(\alpha+1)}}{(e^{x} + e^{-x})^{\alpha+1}} + \frac{e^{x}}{(e^{x} + e^{-x})^{\alpha+1}} \right) dx
\]
\[
\leq \int_{-\infty}^{0} \left( \frac{e^{x(\alpha+1)}}{0 + e^{-x})^{\alpha+1}} + \frac{e^{x}}{(0 + e^{-x})^{\alpha+1}} \right) dx
\]
\[
= \left( \frac{e}{e-1} \right)^{\alpha+1} \left( \frac{1}{\alpha+1} + \frac{1}{\alpha} \right).
\]
In the same manner, the third term is bounded as
\[
\int_{-\infty}^{0} |F(x + i d)| dx \leq \frac{K_{d}^{\alpha+1}}{(1 - \log 2)^{\alpha+1}} \frac{1 + \{\log(2 + c_d)\}^2}{(1 + c_d)^2} \left( \frac{1}{\alpha+1} + \frac{1}{\alpha} \right).
\]
With regard to the second term, using (27), (28), and (33), we have
\[
\int_{0}^{\infty} |F(x - i d)| dx \leq K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) \int_{0}^{\infty} \frac{1}{|1 + e^{-x+id}|^{\beta+1} |1 + e^{-x+id}|} dx
\]
\[
\leq K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) \int_{0}^{\infty} \frac{1}{(1 + e^{x})^{\beta}(1 + e^{-x})^{\beta+1} \cos^{\beta+1}(d/2)} dx
\]
\[
= K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) c_{d}^{\beta+1} \int_{0}^{\infty} \frac{e^{-\beta x}}{(1 + e^{-x})^{\beta+1}} dx
\]
\[
\leq K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) c_{d}^{\beta+1} \int_{0}^{\infty} \frac{e^{-\beta x}}{(1 + 0)^{\beta+1}} dx
\]
\[
= K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) c_{d}^{\beta+1} \beta.
\]
In the same manner, the fourth term is bounded as
\[
\int_{0}^{\infty} |F(x + i d)| dx \leq K e^{b/\log 2} \left( 1 + \frac{1}{(\log 2)^2} \right) c_{d}^{\beta+1} \beta.
\]
Thus, we have \( \mathcal{N}_{1}(F, d) \leq 2KC_{3} \).

4.7. Estimation of the truncation error (proof of Lemma 4.3)

Lemma 4.3 is shown as follows.
Proof. Let $F(t) = f(\phi(t))\phi'(t)$. From (9), by using (23) and (24), it holds for all $t \geq 0$ that

$$|F(t)| \leq K \left( e^{-\phi(t)} \right)^{\beta} \phi'(t)$$

$$= K \left( e^{1/\log(1+e^t)} \right)^{\beta} \left( \frac{e^{-t}}{1+e^{-t}} \right)^{\beta} \frac{1 + \{ \log(1 + e^t) \}^2}{(1 + e^{-t}) \{ \log(1 + e^t) \}^2}$$

$$\leq K e^{\delta/\log^2} \frac{e^{-\beta t}}{(1 + 0)^{\beta + 1}} \left( 1 + \frac{1}{(\log 2)^2} \right).$$

Using this estimate, we have

$$\left| h \sum_{k=N+1}^{\infty} F(kh) \right| \leq h \sum_{k=N+1}^{\infty} |F(kh)|$$

$$\leq K e^{\delta/\log^2} \left( 1 + \frac{1}{(\log 2)^2} \right) h \sum_{k=N+1}^{\infty} e^{-\beta k h}$$

$$\leq K e^{\delta/\log^2} \left( 1 + \frac{1}{(\log 2)^2} \right) \int_{Nh}^{\infty} e^{-\beta x} \, dx$$

$$= K e^{\delta/\log^2} \left( 1 + \frac{1}{(\log 2)^2} \right) \frac{e^{-\beta Nh}}{\beta}.$$

Next, from (6), using (20), (25), and (30), it holds for all $t \leq 0$ that

$$|F(t)| \leq K \frac{1}{|\phi(t)|^{\alpha+1}} \phi'(t)$$

$$= K \left| \frac{\log(1 + e^t)}{1 + \log(1 + e^t)} \right|^{\alpha+1} \frac{1 + e^t}{e^t - 1 + \log(1 + e^t)} \left( \frac{1 + e^t}{1 + e^{-t}} \right)^{\frac{1}{\alpha+1}} \frac{1 + \{ \log(1 + e^t) \}^2}{(1 + e^{-t}) \{ \log(1 + e^t) \}^2}$$

$$\leq K \left( \frac{e^t}{1 + e^t} \right)^{\alpha+1} \frac{1 + e^t}{e^t (1 - \log 2)^{\alpha+1}} e^{1/\alpha+1}$$

$$\leq K \frac{e^{\delta t}}{(1 + 0)^{\alpha} (1 - \log 2)^{\alpha+1}} e^{1/\alpha+1}.$$

Using this estimate, we have

$$\left| h \sum_{k=-\infty}^{-M-1} F(kh) \right| \leq h \sum_{k=-\infty}^{-M-1} |F(kh)|$$

$$\leq K \frac{e^{1/\alpha+1}}{(1 - \log 2)^{\alpha+1}} h \sum_{k=-\infty}^{-M-1} e^{kh}$$

$$\leq K \frac{e^{1/\alpha+1}}{(1 - \log 2)^{\alpha+1}} \int_{-\infty}^{-Mh} e^{\alpha x} \, dx$$

$$= K \frac{e^{1/\alpha+1}}{(1 - \log 2)^{\alpha+1}} \frac{e^{-\alpha M h}}{\alpha}.$$
Thus, using (7), we have
\[
\left| h \sum_{k=-\infty}^{-M-1} F(kh) + h \sum_{k=N+1}^{\infty} F(kh) \right| \leq \frac{Ke^{1/\pi^3}}{\alpha(1 - \log 2)^{\mu+1}} e^{-\alpha Nh} + \frac{Ke^{\delta/\log 2}}{\beta} \left\{ 1 + \frac{1}{(\log 2)^2} \right\} e^{-\delta Nh}
\]
\[
\leq \frac{Ke^{1/\pi^3}}{\alpha(1 - \log 2)^{\mu+1}} e^{-\mu Nh} + \frac{Ke^{\delta/\log 2}}{\beta} \left\{ 1 + \frac{1}{(\log 2)^2} \right\} e^{-\mu Nh},
\]
which is the desired estimate.

5. Proofs for Theorem 2.4

This section presents the proof of Theorem 2.4. It is organized as follows. In Section 5.1, the task is decomposed into two lemmas: Lemmas 5.1 and 5.2. To prove these lemmas, a useful inequality is presented in Section 5.2. Then, Lemma 5.1 is proved in Section 5.3, and Lemma 5.2 is proved in Section 5.4.

5.1. Sketch of the proof

The main strategy in the proof of Theorem 2.4 is identical to that of Theorem 2.3, that is, splitting the error into the discretization error and the truncation error as (19). For the discretization error, we show the following lemma. The proof is given in Section 5.3.

**Lemma 5.1.** Let the assumptions made in Theorem 2.4 be fulfilled. Then, the function \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta) \) belongs to \( H^1(\mathcal{D}_d) \), and \( \mathcal{N}_f(F, d) \) is bounded as
\[
\mathcal{N}_f(F, d) \leq 2KC_5,
\]
where \( C_5 \) is a constant defined as (14).

In addition, we bound the truncation error as follows. The proof is given in Section 5.4.

**Lemma 5.2.** Let the assumptions made in Theorem 2.4 be fulfilled. Then, setting \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta) \), we have
\[
\left| h \sum_{k=-\infty}^{-M-1} F(kh) + h \sum_{k=N+1}^{\infty} F(kh) \right| \leq KC_6 e^{-\mu Nh},
\]
where \( C_6 \) is a constant defined as (15).

Setting \( h \) as (8), the above estimates (Theorem 4.1, Lemmas 5.1 and 5.2) yield the desired result as
\[
\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(\phi(kh))\phi'(kh) \right| \leq \frac{2KC_5}{1 - e^{-2\pi d/h}} e^{-2\pi d/h} + KC_6 e^{-\mu Nh}
\]
\[
= K \left( \frac{2C_5}{1 - e^{-\sqrt{2\pi d}}/2} + C_6 \right) e^{-\sqrt{2\pi d}u_n}
\]
\[
\leq K \left( \frac{2C_5}{1 - e^{-\sqrt{2\pi d}}/2} + C_6 \right) e^{-\sqrt{2\pi d}u_n}.
\]

This completes the proof of Theorem 2.4.
5.2. Useful inequality on \(\mathcal{D}_d\)

We prepare the following lemma here.

**Lemma 5.3.** Let \(d\) be a constant satisfying \(0 < d < \pi\). For all \(\zeta \in \mathcal{D}_d\) and \(x \in \mathbb{R}\), we have

\[
\frac{1}{(1 + e^{-\zeta}) \log(1 + e^\zeta)} \leq \frac{1 + c_d}{\log(2 + c_d)},
\]

(36)

\[
\frac{1}{(1 + e^{-x}) \log(1 + e^x)} \leq 1,
\]

(37)

where \(c_d = 1/\cos(d/2)\).

**Proof.** First, consider (37), which is proved by showing

\[
p(t) = \frac{1 - e^{-t}}{t} \leq 1
\]

for all \(t > 0\) (put \(t = \log(1 + e^\zeta)\)). Differentiating \(p(x)\), we have

\[
p'(t) = -\frac{e^t - (1 + t)}{e^t t^2} \leq 0,
\]

since \(e^t \geq 1 + t\) holds. Therefore, \(p(t)\) decreases monotonically, and thus, it holds that \(p(t) \leq \lim_{t \to 0} p(t) = 1\).

Next, we show (36). Let \(g(\zeta) = 1/((1 + e^{-\zeta}) \log(1 + e^\zeta))\). Since the function \(g(\zeta)\) is analytic and bounded in \(\mathcal{D}_d\) and continuous on \(\mathcal{D}_d\), by Lemma 4.12, we obtain (36) if we show the following two inequalities:

\[
M_g(d) \leq \frac{1 + c_d}{\log(2 + c_d)},
\]

\[
M_g(-d) \leq \frac{1 + c_d}{\log(2 + c_d)},
\]

where \(M_g(y) = \sup_{x \in \mathbb{R}} |g(x + iy)|\). We show only the first one, because the second one is also shown in the same way. Putting \(\zeta = \log(1 + e^{\pi i d})\), \(g(x + id) = p(\xi)\) holds, and thus, in what follows we prove

\[
|p(\xi)| \leq \frac{1 + c_d}{\log(2 + c_d)}.
\]

(38)

We consider the following two cases: (a) \(|\xi| \leq \log(2 + c_d)\) and (b) \(|\xi| > \log(2 + c_d)\). In case (a), we have

\[
|p(\xi)| = \sum_{k=1}^{\infty} \frac{(-\xi)^{k-1}}{k!} \leq \sum_{k=1}^{\infty} \frac{|\xi|^{k-1}}{k!} = \frac{e^{\xi} - 1}{|\xi|}.
\]

Here, if we put \(q(x) = (e^x - 1)/x\), then we have \(q'(x) = r(x)/x^2\), where \(r(x) = 1 + (x - 1) e^x\). Since \(r'(x) = x e^x \geq 0\) for \(x \geq 0\), \(r(x)\) monotonically increases for \(x \geq 0\). Therefore, \(r(x) \geq r(0) = 0\) holds, from which we have \(q'(x) \geq 0\) for \(x \geq 0\), i.e., \(q(x)\) monotonically increases for \(x \geq 0\). Thus, from \(|\xi| \leq \log(2 + c_d)\), we have (38) as

\[
|p(\xi)| \leq q(|\xi|) \leq \frac{e^{\log(2 + c_d)} - 1}{\log(2 + c_d)} = \frac{1 + c_d}{\log(2 + c_d)}.
\]

In case (b), using (32), we have

\[
|p(\xi)| \leq \frac{1 + |e^{-\zeta}|}{|\xi|} = \frac{1 + e^{-\Re \zeta}}{|\xi|} \leq \frac{1 + e^{-\log(\cos(d/2))}}{|\xi|} = \frac{1 + c_d}{|\xi|}.
\]

Furthermore, since \(1/x\) decreases monotonically for \(x > 0\), we have (38). This completes the proof.
5.3. Estimation of the discretization error (proof of Lemma 5.1)

Lemma 5.1 is essentially shown by the following lemma, which holds for $0 < \delta < \pi$ (not only $0 < d < (1 + \pi)/2$).

**Lemma 5.4.** Assume that $F$ is analytic in $D_\delta$ with $0 < \delta < \pi$, and that there exist positive constants $K_+$, $K_-$, $\alpha$, and $\beta$ such that

$$|F(\zeta)| \leq K_+ \left| \frac{e^{1/\log(1+e^\zeta)}}{1 + e^\zeta} \right|^{\beta}$$

holds for all $\zeta \in D_\delta^+$, and

$$|F(\zeta)| \leq K_- \frac{\log(1 + e^\zeta)}{\{1 + \log(1 + e^\zeta)\{-1 + \log(1 + e^\zeta)\}^a}$$

holds for all $\zeta \in D_\delta^-$. Then, $F$ belongs to $H^1(D_\delta)$, and $N_1(F, \delta)$ is bounded as

$$N_1(F, \delta) \leq \frac{2K_+}{\alpha} \left( \frac{e^c_0}{(1-\log 2)(e-1)} \right)^a + \frac{2K_+}{\beta} \left( e^{1/\log^2 c_0} \right)^\beta,$$

where $c_0 = 1/\cos(\delta/2)$.

**Proof.** Since $F$ is analytic on $D_\delta$, the remaining task is to show (41). From (39), by using (23), it holds for all $\zeta \in D_\delta^+$ that

$$|F(\zeta)| \leq K_+ \left| \frac{e^{1/\log(1+e^\zeta)}}{1 + e^\zeta} \right|^{\beta} \left| \frac{1}{1 + e^\zeta} \right| \leq K_+ \left( e^{1/\log 2} \right)^\beta \left| \frac{1}{1 + e^\zeta} \right|^{\beta}.$$  (42)

Furthermore, from (40), by using (25) and (26), it holds for all $\zeta \in D_\delta^-$ that

$$|F(\zeta)| \leq K_- \frac{\log(1 + e^\zeta)}{1 + \log(1 + e^\zeta)} \frac{1}{1 - \log(1 + e^\zeta)^a} \leq K_- \frac{1}{1 + e^{-\zeta} - l^a} \frac{1}{1 - \log 2^a},$$  (43)

where $l = \log(e/(e-1))$. As described earlier, $N_1(F, d)$ is expressed as (35). Using (27) and (42), we have

$$\int_{-\beta(1-e)}^\beta |F(1/e + iy)| dy \leq K_+ \left( e^{1/\log 2} \right)^\beta \int_{-\beta(1-e)}^\beta \frac{1}{1 + e^{1/e}y} |y|^\beta dy \leq K_+ \left( e^{1/\log 2} \right)^\beta \int_{-\beta(1-e)}^\beta \frac{1}{\cos^\beta(y/2)} dy,$$

from which we have

$$\lim_{e \to 0} \int_{-\beta(1-e)}^\beta |F(1/e + iy)| dy = 0.$$  

In the same manner, using (25) and (43), we have

$$\lim_{e \to 0} \int_{-\beta(1-e)}^\beta |F(-1/e + iy)| dy = 0.$$
Therefore, $\mathcal{N}_1(F, \delta)$ is expressed as

$$
\mathcal{N}_1(F, \delta) = \int_{-\infty}^{\infty} |F(x - i \delta)|dx + \int_{-\infty}^{\infty} |F(x + i \delta)|dx
$$

$$
= \int_{-\infty}^{0} |F(x - i \delta)|dx + \int_{0}^{\infty} |F(x - i \delta)|dx + \int_{-\infty}^{0} |F(x + i \delta)|dx + \int_{0}^{\infty} |F(x + i \delta)|dx.
$$

With regard to the first term, using (28) and (43), we have

$$
\int_{-\infty}^{0} |F(x - i \delta)|dx \leq \frac{K_1}{(1 - \log 2)^{\alpha}} \int_{-\infty}^{0} \frac{1}{|1 + e^{-x(i+1/2)}|^\alpha} dx
$$

$$
\leq \frac{K_1}{(1 - \log 2)^{\alpha}} \int_{-\infty}^{0} \frac{1}{(1 + e^{-x})^\alpha \cos^2(\delta/2)} dx
$$

$$
= \frac{K_1 e_0^{\beta}}{(1 - \log 2)^{\alpha}} \int_{-\infty}^{0} \frac{1}{(1 + e^{-x})^\alpha} dx.
$$

The integral is further bounded as

$$
\int_{-\infty}^{0} \frac{1}{(1 + e^{-x})^\alpha} dx = \int_{-\infty}^{0} \frac{e^x}{(e^x + e^{-x})^\alpha} dx
$$

$$
\leq \int_{-\infty}^{0} \frac{e^x}{(0 + e^{-x})^\alpha} dx
$$

$$
= \left( \frac{e}{e-1} \right)^\alpha \frac{1}{\alpha}.
$$

In the same manner, the third term is bounded as

$$
\int_{-\infty}^{0} |F(x + i \delta)|dx \leq \frac{K_1 e_0^{\beta}}{\alpha(1 - \log 2)^{\alpha}} \left( \frac{e}{e-1} \right)^\alpha.
$$

With regard to the second term, using (27) and (42), we have

$$
\int_{0}^{\infty} |F(x - i \delta)|dx \leq K_2 e^{\beta/\log 2} \int_{0}^{\infty} \frac{1}{|1 + e^{-x(i)^\beta}|} dx
$$

$$
\leq K_2 e^{\beta/\log 2} \int_{0}^{\infty} \frac{1}{(1 + e^{-x})^\beta \cos^2(\delta/2)} dx
$$

$$
= K_2 e^{\beta/\log 2} e_0^{\beta} \int_{0}^{\infty} \frac{e^{-x}}{(1 + e^{-x})^\beta} dx
$$

$$
\leq K_2 e^{\beta/\log 2} e_0^{\beta} \int_{0}^{\infty} \frac{e^{-x}}{(1 + 0)^\beta} dx
$$

$$
= K_2 e^{\beta/\log 2} \frac{e_0^{\beta}}{\beta}.
$$

In the same manner, the fourth term is bounded as

$$
\int_{0}^{\infty} |F(x + i \delta)|dx \leq K_2 e^{\beta/\log 2} \frac{e_0^{\beta}}{\beta}.
$$

Thus, we obtain (43).
Using this lemma, Lemma 5.4 is shown as follows.

Proof. Let $F(\zeta) = f(\phi(\zeta))\phi'(\zeta)$. Since $f$ is analytic in $\phi(\mathbb{D})$, $f(\phi(\cdot))$ is analytic in $\mathbb{D}$. In addition, since $\phi'$ is analytic in $\mathbb{D}$, $F$ is analytic in $\mathbb{D}$ (note that $d < \pi$). Therefore, the remaining task is to show $N_1(F, d) \leq 2KC_5$. Using (28), we have

$$\frac{1}{|1 + e^{-\zeta}|} \leq \frac{1}{(1 + e^{-\Re \zeta}) \cos((\Im \zeta)/2)} \leq \frac{1}{(1 + 0) \cos(d/2)}$$

for all $\zeta \in \mathbb{D}_d$. Therefore, from (9), by using (24), it holds for all $\zeta \in \mathbb{D}_d^+$ that

$$|F(\zeta)| \leq K |e^{-\phi(\zeta)}|^{\beta} |\phi'(\zeta)|$$

for all $\zeta \in \mathbb{D}_d^+$. From which we have $N_1(F, d) \leq 2KC_5$. Thus, the assumptions of Lemma 5.4 are fulfilled with $\delta = d$ and

$$K_+ = KC_d \left(1 + \frac{1}{\log(2)}\right),$$

$$K_- = K \frac{1 + c_d}{\log(2 + c_d)},$$

from which we have $N_1(F, d) \leq 2KC_5$.

5.4. Estimation of the truncation error (proof of Lemma 5.2)

Lemma 5.2 is essentially shown by the following lemma.

Lemma 5.5. Assume that there exist positive constants $K_+, K_-, \alpha,$ and $\beta$ such that

$$|F(x)| \leq K_+ \frac{e^{1/\log(1 + e^x)}}{1 + e^x}$$

holds for all $x \geq 0$, and

$$|F(x)| \leq K_- \frac{\log(1 + e^x)}{(1 + \log(1 + e^x)) \{-1 + \log(1 + e^x)}$$

holds for all $x < 0$. Let $\mu = \min\{\alpha, \beta\}$, and let $M$ and $N$ be defined as (7). Then, we have

$$h \sum_{k=-\infty}^{-M} |F(kh)| + h \sum_{k=N+1}^{\infty} |F(kh)| \leq \left(\frac{K_-}{\alpha (1 - \log 2)^\alpha} + \frac{K_+}{\beta} \left(e^{1/\log 2}\right)^\beta\right) e^{-\mu h}.$$
Proof. From (44), by using (25), it holds for all \( x \geq 0 \) that
\[
|F(x)| \leq K_+ \left( e^{1/(\log(1 + e^x))} \right)^{\beta} \left( \frac{e^{-x}}{1 + e^{-x}} \right)^{\beta} \leq K_+ e^{\beta/\log 2} \frac{e^{-\beta x}}{(1 + 0)^{\beta}}.
\]
Using this estimate, we have
\[
h \sum_{k=N+1}^{\infty} |F(kh)| \leq K_+ e^{\beta/\log 2} h \sum_{k=N+1}^{\infty} e^{-\beta kh} \leq K_+ e^{\beta/\log 2} \int_{Nh}^{\infty} e^{-\beta x} \, dx = K_+ e^{\beta/\log 2} \frac{e^{-\beta Nh}}{\beta}.
\]
Next, from (49), using (20) and (25), it holds for all \( x < 0 \) that
\[
|F(x)| \leq K_- \left| \frac{\log(1 + e^x)}{1 + \log(1 + e^x)} \right|^a \left| \frac{1}{1 - 1 + \log(1 + e^x)} \right|^{\beta_1} \leq K_- \left( \frac{e^x}{1 + e^x} \right)^a \left( \frac{1}{(1 - \log 2)^\alpha} \right) \leq K_- e^{a x} \left( \frac{1}{(1 - \log 2)^\alpha} \right).
\]
Using this estimate, we have
\[
h \sum_{k=-\infty}^{-M-1} |F(kh)| \leq K_- \frac{1}{(1 - \log 2)^\alpha} h \sum_{k=-\infty}^{-M-1} e^{\alpha k h} \leq K_- \frac{1}{(1 - \log 2)^\alpha} \int_{-\infty}^{-Mh} e^{\alpha x} \, dx = \frac{K_- e^{-\alpha Mh}}{(1 - \log 2)^\alpha}.
\]
Thus, using (7), we have (46).

Using this lemma, Lemma 5.2 is shown as follows.

Proof. Let \( F(x) = f(\phi(x))\phi'(x) \). From (9), by using (24), it holds for all \( x \geq 0 \) that
\[
|F(x)| \leq K |e^{-\phi(x)}|^\beta |\phi'(x)|
= K \left| e^{1/(\log(1 + e^x))} \right|^{\beta} \left| \frac{1 + \{\log(1 + e^x)\}^2}{1 + e^x} \right| \frac{1}{\log(1 + e^x)^2} \leq K \left( \frac{1}{1 + e^x} \right)^{\beta} \left( \frac{1}{1 + 0 + \log(1 + e^x)} \right)^a \left( \frac{1}{(1 - \log 2)^2} \right)^{\alpha}.
\]
Next, from (13), using (37), it holds for all \( x < 0 \) that
\[
|F(x)| \leq K \left| \frac{1}{\log(1 + e^x)} \right|^a \left| \frac{\log(1 + e^x)}{1 + \log(1 + e^x)} \right|^{1/2} \left| \frac{1}{\log(1 + e^x)} \right| \left| \frac{1}{(1 + e^x) \log(1 + e^x)} \right| \cdot 1.
\]
Thus, the assumptions of Lemma 5.5 are fulfilled with
\[
K_+ = K \left( 1 + \frac{1}{\log(2)^2} \right),
K_- = K,
\]
which completes the proof.
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