Probability Thermodynamics and Probability Quantum Field

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Abstract: In this paper, we introduce probability thermodynamics and probability quantum fields. By probability we mean that there is an unknown operator, physical or non-physical, whose eigenvalues obey a certain statistical distribution. Eigenvalue spectra define spectral functions. Various thermodynamic quantities in thermodynamics and effective actions in quantum field theory are all spectral functions. In the scheme, eigenvalues obey a probability distribution, so a probability distribution determines a family of spectral functions in thermodynamics and in quantum field theory. This leads to probability thermodynamics and probability quantum fields determined by a probability distribution. In constructing spectral functions, we encounter a problem. The conventional definition of spectral functions applies only to lower bounded spectra. In our scheme, however, there are two types of spectra: lower bounded spectra, corresponding to the probability distribution with nonnegative random variables, and the lower unbounded spectra, corresponding to probability distributions with negative random variables. To take the lower unbounded spectra into account, we generalize the definition of spectral functions by analytical continuation. In some cases, we encounter divergences. We remove the divergence by a renormalization procedure. Moreover, in virtue of spectral theory in physics, we generalize some concepts in probability theory. For example, the moment generating function in probability theory does not always exist. We redefine the moment generating function as the generalized heat kernel introduced in this paper, which makes the concept definable when the definition in probability theory fails. As examples, we construct examples corresponding to some probability distributions. Thermodynamic quantities, vacuum amplitudes, one-loop effective actions, and vacuum energies for various probability distributions are presented.

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1 Introduction

A physical system is described by an operator. The eigenvalue of the operator contains only partial information of the system. This is the reason why one cannot hear the shape of a drum \([1]\). Nevertheless, the information embedded in eigenvalues, though partial, is of special importance. In physics, for example, in thermodynamics various thermodynamic quantities are determined by eigenvalues, and in quantum field theory the vacuum amplitude, the effective action, and the vacuum energy are determined by eigenvalues \([2]\). In mathematics, a famous example of spectral geometry is given by Kac that one can extract topological information, the Euler characteristics, from eigenvalues through the heat kernel or the spectral counting function \([1, 3]\). Quantities, such as thermodynamic quantities, effective actions, and spectral counting functions, are all spectral functions. The spectral function is determined by a spectrum \(\{\lambda_n\}\) and, on the other hand, all information of the spectrum is embedded in the spectral function \([4]\). Once an eigenvalue spectrum is given, even if the operator is unknown, we can also construct various spectral functions.

In the scheme, without knowing the operator, we preset an eigenvalue spectrum to construct spectral functions. The eigenvalue spectrum \(\{\lambda_n\}\) is supposed to obey a certain probability distribution. Spectral functions constructed in this way, therefore, are determined by the probability distribution. They are probability spectral functions. Technologically, in the scheme the state density of an eigenvalue spectrum is chosen as a probability density function, or equivalently, the spectral counting function of an eigenvalue spectrum is chosen as a cumulative probability function. Constructing the thermodynamic quality with a probability eigenvalue spectrum, we arrive at probability thermodynamics; constructing the vacuum amplitude and the effective action, etc. with a probability eigenvalue spectrum, we arrive at a probability quantum field.
In constructing probability spectral functions, we encounter a difficulty. The conventional definition of most spectral functions applies only to lower bounded spectra \( \{ \lambda_0, \lambda_1, \lambda_2, \cdots \} \) which has a finite lowest eigenvalue \( \lambda_0 \). In our scheme, eigenvalues obey a probability distribution. However, there are two types of probability distributions: one includes only nonnegative random variables, corresponding to the lower bounded spectrum, and the other includes negative random variables, corresponding to the lower unbounded spectrum. Often, for lower bounded cases, the spectral function can be constructed under the conventional definition, but for lower unbounded cases, we need to generalize the conventional definition of spectral functions.

Our starting point is the heat kernel. The local heat kernel \( K(t; \mathbf{r}, \mathbf{r}') \) of an operator \( D \) is the Green function of the initial-value problem of the heat-type equation:

\[
(\partial_t + D) K(t; \mathbf{r}, \mathbf{r}') = 0 \quad \text{with} \quad K(0; \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad [2].
\]

The global heat kernel is the trace of the local heat kernel \( K(t; \mathbf{r}, \mathbf{r}') \):

\[
K(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}.
\]  

(1.1)

Here \( \{ \lambda_n \} \) is the eigenvalue spectrum of the operator \( D \), determined by the eigenequation \( D\phi_n = \lambda_n \phi_n \). In the conventional definition, the lowest eigenvalue must be finite. In thermodynamics, the heat kernel is the partition function; in quantum field theory, the heat kernel is the vacuum amplitude. From the heat kernel, we can obtain all thermodynamic quantities in thermodynamics and effective actions and vacuum energies in quantum field theory.

As mentioned above, for a lower unbounded spectrum, there is no lowest eigenvalue, i.e., the lowest eigenvalues \( \lambda_0 \to -\infty \), and the sum in the definition (1.1) often diverges. In this paper, we first generalize the conventional definition of the heat kernel so as to make the definition of heat kernel applies to lower unbounded spectra. Then, we obtain other spectral functions from the heat kernel by the relation between spectral functions [2, 3, 5].

It should be emphasized that the eigenvalue spectrum of a real physical system must be lower bounded, or else the world would be unstable. While, the system considered here is a fictional physical system. Their eigenvalue spectra are not from a physical operator, but is required to satisfy a probability distribution artificially. The eigenvalue spectrum corresponding to a probability distribution with negative random variables leads to lower unbounded eigenvalue spectra. Therefore, unsurprisingly, some thermodynamic qualities may not be positive. That is, the operator corresponding to probability distributions is sometimes not a physically real operator.

Though the eigenvalue spectrum corresponding to probability distributions is sometimes not physically real, the probability eigenvalue spectrum has good behavior. This is because the probability distribution has good behavior, such as integrability. Therefore, for example, the vacuum energy though always diverges in quantum field theory, it converges for probability eigenvalue spectra.

The scheme suggested in the present paper also generalizes some concepts in probability theory. Take the moment generating function as an example. In probability theory, the moment generating function does not always exist. In this paper, the generalized heat
kernel is used to serve as a generalized moment generating function. In many cases, the new
definition is definable when the definition of the moment generating function in probability
theory fails.

When constructing quantum fields, in some cases, we encounter divergences. We remove
the divergence by a renormalization procedure.

As examples, we consider some probability distributions, including distributions without and with negative random variables. The corresponding thermodynamics and quantum fields are presented.

In section 2, a scheme for generalizing spectral functions to lower unbounded spectra
is suggested. In section 3, the probability spectral function is introduced. In sections 4
and 5, probability thermodynamics and probability quantum field theory are constructed.
In section 6, various kinds of probability thermodynamics and probability quantum field
theory corresponding to various probability distributions are constructed. The conclusions
and outlook are provided in section 7.

2 Generalizing spectral function to lower unbounded spectrum

Spectral functions are determined by a spectrum \( \{ \lambda_n \} \). In physics, the spectrum \( \{ \lambda_n \} \) are
always an eigenvalue spectrum of an lower bounded Hermitian operator \( D \). Nevertheless,
the eigenvalue spectrum we are interested in is given by a probability distribution, which
may be lower unbounded. For a lower bounded spectrum, there is a finite lowest eigenvalue
\( \lambda_{\text{min}} \). For a lower unbounded spectrum, however, the lowest eigenvalue tends to negative
infinity: \( \lambda_{\text{min}} \rightarrow -\infty \).

The conventional definition of spectral functions is valid for lower bounded spectra. One
of the main aim of the present paper is to generalize the definition of spectral functions to
lower unbounded spectra.

In this section, we generalize the definition of some essential spectral functions, the
heat kernel, the spectral counting function, and the spectral zeta function, to lower un-
bounded spectra. These spectral functions are essentially important in spectral geometry,
thermodynamics, and quantum field theory.

2.1 Generalized heat kernel

2.1.1 Generalized heat kernel: Wick rotation

In this section, we generalize the conventional definition of the global heat kernel, so as to
make it valid both for bounded and unbounded spectra.

The conventional definition of the global heat kernel for a lower bounded spectrum
\( \{ \lambda_0, \lambda_1, \lambda_1, \cdots \} \), by Eq. (1.1), is \( K(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \). For lower unbounded spectra,
\( \{ -\infty, \cdots, \lambda_{-1}, \lambda_0, \lambda_1, \cdots \} \), a natural and naive generalization of the heat kernel, following
the original intention of the conventional definition of the heat kernel, which is a sum of
\( e^{-\lambda_n t} \) over all eigenstates, is

\[
K(t) = \sum_{n=-\infty}^{\infty} e^{-\lambda_n t}. \tag{2.1}
\]
This is, however, ill defined, because the existence of the divergent lowest eigenvalue $\lambda_{-\infty} = -\infty$ brings a divergent term $e^{-\lambda_{-\infty}t}$ in the summation.

In order to seek a generalized heat kernel for taking lower unbounded spectra into account, we employ the Wick rotation.

Wick rotation performs the replacement $t = -i\tau$. (In physics, the Wick rotation usually takes $t = i\tau$. The reason why we take $t = -i\tau$ is that we want to apply the result to probability theory.) After the Wick rotation, the definition (2.1) becomes

$$K_{WR}(\tau) = \sum_{n=-\infty}^{\infty} e^{i\lambda_n \tau},$$

where $K_{WR}(\tau)$ denotes the Wick-rotated heat kernel.

After the Wick rotation, we may obtain a finite result $K_{WR}(\tau)$ for lower unbounded spectra \{-\infty, \cdots, \lambda_{-1}, \lambda_0, \lambda_1, \cdots\}. Nevertheless, the Wick-rotated heat kernel $K_{WR}(\tau)$ is, of course, not the heat kernel we want. To obtain a heat kernel rather than a Wick-rotated heat kernel, after working out the sum in Eq. (2.2), we perform an inverse Wick rotation $\tau = it$. After the inverse Wick rotation, the Wick-rotated heat kernel returns back to the heat kernel we want. The heat kernel obtained this way is finite.

In short, the generalized heat kernel for a lower unbounded spectrum \{-\infty, \cdots, \lambda_{-1}, \lambda_0, \lambda_1, \cdots\} is obtained by the following procedure:

$$K(t) = \sum_{n=-\infty}^{\infty} e^{i\lambda_n \tau} \bigg|_{\tau = it} = K_{WR}(it).$$

Such a procedure is essentially an analytic continuation, which allows us to achieve a finite result of a divergent series.

In this procedure, there are two steps to achieve the well-defined generalized heat kernel for lower unbounded spectra, Eq. (2.3). The first step, instead of the divergent summation $\sum_{n=-\infty}^{\infty} e^{-\lambda_n t}$ in the naive generalization of heat kernel (2.1), we turn to calculate a convergent Wick-rotated summation $\sum_{n=-\infty}^{\infty} e^{i\lambda_n \tau}$. The second step, we perform an inverse Wick rotation to the already worked out result of the Wick-rotated sum. Clearly, the key step in this procedure is to convert the divergent sum $\sum_{n=-\infty}^{\infty} e^{-\lambda_n \tau}$ to a convergent sum $\sum_{n=-\infty}^{\infty} e^{i\lambda_n \tau}$ by the Wick rotation.

Now we arrive at a definition (2.3) which defining a generalized heat kernel which is valid for lower unbounded spectra.

### 2.1.2 Generalized heat kernel: two-sided Laplace transformation and Fourier transformation

In the above, the generalized heat kernel is expressed in a summation form. By means of the state density

$$\rho(\lambda) = \sum_{\{\lambda_n\}} \delta(\lambda - \lambda_n),$$

we can rewrite the generalized heat kernel in an integral form. From the integral representation, we can see how the Wick rotation treatment works more intuitively.
Conventional heat kernel. First, let us consider the conventional definition of heat kernels, Eq. (1.1), which is only valid for lower bounded spectra.

By the state density (2.4), we can convert the sum in the conventional definition (1.1) into an integral,

\[ K(t) = \int_0^\infty d\lambda \rho(\lambda) e^{-\lambda t}. \]  

(2.5)

The equivalence between the summation form (1.1) and the integral form (2.5) can be checked directly by substituting the state density (2.4) into Eq. (2.5):

\[ K(t) = \sum_{\{\lambda_n\}} \int_0^\infty d\lambda \delta(\lambda - \lambda_n) e^{-\lambda t} = \sum_{n=0}^\infty e^{-\lambda_n t}. \]

For lower bounded spectra, the integral representation (2.5) of heat kernels shows that the relation between the heat kernel and the state density is a Laplace transformation.

Naive generalized heat kernel. Next, let us see the naive generalization of heat kernels, Eq. (2.1).

Converting the sum in Eq. (2.1) into an integral gives

\[ K(t) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{-\lambda t}. \]  

(2.6)

Notice that for lower unbounded spectra, \{\lambda_n\} runs from \(-\infty, +\infty\).

For lower unbounded spectra, by inspection of Eq. (2.6), we can see that the relation between the heat kernel and the state density is a two-sided Laplace transformation (also known as the bilateral Laplace transformation) [6] rather than a Laplace transformation. The relation between a two-sided Laplace transformation and Laplace transformation is

\[ \mathcal{BL}[f(\lambda); t] = \mathcal{L}[f(\lambda); t] + \mathcal{L}[f(-\lambda); -t], \]

where \(\mathcal{BL}[f(\lambda); t]\) denotes the two-sided Laplace transformation and \(\mathcal{L}[f(\lambda); t]\) denotes the Laplace transformation.

It is worth to point out here that the integral form of the naive generalized heat kernel (2.6) is also valid for a certain kind of lower unbounded spectra, though the sum form of the naive generalized heat kernel, Eq. (2.1), is invalid for all kinds of lower unbounded spectra. This is because in the summation form of the generalized heat kernel, Eq. (2.1), the term \(e^{-\lambda - \infty t}\) diverges when \(\lambda = -\infty\); while, in the integral form, Eq. (2.6), the validity of the definition relies on the integrability of the integral, \(\int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{-\lambda t}\). If the integral is integrable, then Eq. (2.6) can serve as a definition of heat kernels for lower unbounded spectra. The integrability condition, clearly, is that the state density, \(\rho(\lambda)\), must attenuate rapidly enough. In other words, the definition (2.6) is not valid for all kinds of lower unbounded spectra; it is only valid for \(\rho(\lambda)\) decreasing faster than \(e^{\lambda t}\).

Wick-rotated heat kernel. Now, let us see the Wick rotated heat kernel, Eq. (2.2). Converting Eq. (2.2) into an integral gives

\[ K^{WR}(\tau) = \int_{-\infty}^{\infty} \rho(\lambda) e^{i\lambda \tau} d\lambda. \]  

(2.7)

The Wick rotated heat kernel \(K^{WR}(\tau)\) is obviously a Fourier transformation of the state density \(\rho(\lambda)\).

Now we can explain why the Wick rotated treatment is needed for lower unbounded spectra. Without the Wick rotation, the integral form of the naive generalization of the heat
kernel, Eq. (2.6), is a two-sided Laplace transformation, and the integrability condition is that the state density $\rho(\lambda)$ decreases faster than $e^{\lambda t}$. Nevertheless, the Wick rotated heat kernel, as shown in Eq. (2.7), is a Fourier transformation of the state density $\rho(\lambda)$. The integrability condition then becomes that $\rho(\lambda)$ is absolutely integrable, which is easier to be satisfied than that in the two-sided Laplace transformation.

**Generalized heat kernel.** The generalized heat kernel can be finally obtained by performing an inverse Wick rotation to the Wick rotated heat kernel $K_{WR}(\tau)$. From Eq. (2.7), we can see that the state density $\rho(\lambda)$ is an inverse Fourier transformation of the Wick rotated heat kernel $K_{WR}(\tau)$, i.e.,

$$\rho(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{WR}(\tau) e^{-i\lambda \tau} d\tau. \quad (2.8)$$

Substituting into Eq. (2.6) gives

$$K(t) = \int_{-\infty}^{\infty} d\tau K_{WR}(\tau) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda(\tau+it)} \right] = \int_{-\infty}^{\infty} d\tau K_{WR}(\tau) \delta(\tau - it). \quad (2.9)$$

Then we arrive at a relation between the heat kernel and the Wick rotated heat kernel:

$$K(t) = K_{WR}(it). \quad (2.10)$$

This is just the result given by the integral form of the definition of the generalized heat kernel, Eq. (2.3).

In a word, the key idea of introducing the generalized heat kernel for lower unbounded spectra is an analytic continuation treatment through a Wick rotation.

### 2.2 Generalized spectral counting function

The spectral counting function is the number of eigenstates whose eigenvalue is smaller than a given number $\lambda$ [3, 5],

$$N(\lambda) = \sum_{\lambda_0}^{\lambda_n \leq \lambda} 1, \quad (2.11)$$

where $\lambda_0$ denotes the minimum eigenvalue. The spectral counting function is an important spectral function, which is the starting point of the famous problem formulated by Kac “Can one hear the shape of a drum?” [1]. The spectral counting function $N(\lambda)$ has a direct relation with the global heat kernel $K(t)$, or, the partition function $Z(\beta)$ [3]. In probability theory, the spectral counting function corresponds the cumulative probability function.

For lower unbounded spectra, as that of heat kernels, the definition (2.11) needs to be generalized as

$$N(\lambda) = \sum_{-\infty}^{\lambda_n \leq \lambda} 1. \quad (2.12)$$

At the first sight, this seems to be a wrong definition: if the spectrum is not lower bounded, usually, there are infinite number of states below the state with eigenvalue $\lambda_n < \lambda$.

There are two ways to generalize the definition of the spectral counting function to lower unbounded spectra.
2.2.1 State density approach

Directly convert the sum in Eq. (2.12) into an integral is the most straightforward way to generalize the spectral counting function to lower unbounded spectra:

\[ N(\lambda) = \int_{-\infty}^{\lambda} \rho(\lambda) \, d\lambda. \] (2.13)

Similarly to heat kernels, the divergence encountered in the sum is avoided by the treatment of converting the sum into an integral. In this sense, the counting function is well-defined so long as the state density \( \rho(\lambda) \) is integrable.

2.2.2 Heat kernel approach

Alternatively, in Refs. [3, 4], we present a relation between heat kernels and counting functions for lower bounded spectra: \( K(t) = \int_{0}^{\infty} N(\lambda) e^{-\lambda t} \, d\lambda \), i.e., the counting function \( N(\lambda) \) is a Laplace transformation of \( K(t)/t \). For lower unbounded spectra, the lowest eigenvalue is \( -\infty \), so, as that of heat kernel, we can directly generalize the relation between \( K(t) \) and \( N(\lambda) \) as

\[ K(t)/t = \int_{-\infty}^{\infty} N(\lambda) e^{-\lambda t} \, d\lambda. \] (2.14)

Clearly, this generalized counting function for lower unbounded spectra is a two-sided Laplace transformation (bilateral Laplace transformation) of \( K(t)/t \) rather than a Laplace transformation.

Performing a Wick rotation to Eq. (2.14), \( t = -i\tau \), gives

\[ \frac{K(-i\tau)}{-i\tau} = \int_{-\infty}^{\infty} N(\lambda) e^{i\lambda \tau} \, d\lambda = \mathcal{F}[N(\lambda); \tau]. \] (2.15)

Then the two-sided Laplace transformation in Eq. (2.14) is converted to a Fourier transformation. Therefore, the counting function \( N(\lambda) \) can be immediately obtained by an inverse Fourier transformation:

\[ N(\lambda) = \mathcal{F}^{-1}\left[ \frac{K(-i\tau)}{-i\tau}; \lambda \right] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} K(\tau) \frac{e^{i\lambda \tau}}{\tau} \, d\tau. \] (2.16)

Eqs. (2.14) and (2.16) are relations between generalized heat kernels and generalized counting functions.

It should be noted that, at first sight, the counting function of a lower unbounded spectrum will diverge since there are infinite eigenvalues below the given number \( \lambda \). However, if instead the naive definition of counting function, \( N(\lambda) = \sum_{\lambda_n < \lambda} 1 \), by the definition (2.13), we may arrive at a finite result in the case of probability.
2.3 Generalized spectral zeta function

The spectral zeta function is important in quantum field theory, which is the basics in the calculation of the one-loop effective action and the vacuum energy, etc. \[2, 5\]. In probability, the spectral zeta function indeed corresponds to the logarithmic moment generating function.

The conventional definition of the spectral zeta function, which is valid only for lower bounded spectra, reads

\[
\zeta(s) = \sum_{n=0}^{\infty} \lambda_n^{-s}.
\] (2.17)

2.3.1 State density approach

To generalize the spectral zeta function to lower unbounded spectra, similarly as the generalized heat kernel, we convert the sum into an integral with the state density \(\rho(\lambda)\):

\[
\zeta(s) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \lambda^{-s}.
\] (2.18)

This definition is valid so long as state density \(\rho(\lambda) \lambda^{-s}\) is integrable.

2.3.2 Heat kernel approach

The relation between the conventional heat kernel and the conventional spectral zeta function is a Mellin transformation \[2, 5\],

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} K(t) \, dt
= \frac{1}{\Gamma(s)} \mathcal{M} [K(t) ; s],
\] (2.19)

where \(\mathcal{M} [K(t) ; s]\) denotes the Mellin transformation. It can be checked that such a relation also holds for generalized heat kernels and generalized spectral zeta functions. Substituting the generalized heat kernel (2.6) into Eq. (2.19) gives

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt t^{s-1} \left[ \int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{-t \lambda} \right]
= \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \lambda^{-s}.
\] (2.20)

2.3.3 Characteristic function approach

First introduce a spectral characteristic function defined as the Fourier transformation of the state density,

\[
f(k) = \int_{-\infty}^{\infty} \rho(\lambda) e^{ik\lambda} \, d\lambda.
\] (2.21)

Constructing a representation of \(\lambda^{-s}\),

\[
\lambda^{-s} = \frac{1}{i s \Gamma(s)} \int_{0}^{\infty} e^{ik\lambda} k^{s-1} \, dk = \frac{1}{i s \Gamma(s)} \mathcal{M} [e^{ik\lambda} ; s],
\] (2.22)
and substituting into Eq. (2.18) give

\[
\zeta(s) = \int_{-\infty}^{\infty} \rho(\lambda) \left[ \frac{1}{i^s \Gamma(s)} \int_{0}^{\infty} e^{ik\lambda} k^{s-1}dk \right] d\lambda
\]

\[
= \frac{1}{i^s \Gamma(s)} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} \rho(x) e^{ikx} dx \right] k^{s-1} dk.
\]

(2.23)

By the definition of the spectral characteristic function (2.21), we arrive at

\[
\zeta(s) = \frac{1}{i^s \Gamma(s)} \int_{0}^{\infty} f(k) k^{s-1} dk
\]

(2.24)

\[
= \frac{1}{i^s \Gamma(s)} \mathcal{M}[f(k); s].
\]

(2.25)

2.3.4 Mellin transformation approach

The spectral zeta can also be expressed as a Mellin transformation,

\[
\mathcal{M}[f(t); s] = \int_{0}^{\infty} f(t) t^{s-1} dt.
\]

(2.26)

Rewriting the expression of the spectral zeta function (2.18) as

\[
\zeta(s) = \int_{-\infty}^{0} p(x) x^{-s} dx + \int_{0}^{\infty} p(x) x^{-s} dx,
\]

(2.27)

we can representation the zeta by the Mellin transformation as

\[
\zeta(s) = (-1)^{-s-1} \mathcal{M}[-xp(-x); -s] + \mathcal{M}[xp(x); -s].
\]

(2.28)

3 Probability spectral function

The main aim of the present paper is to introduce probability spectral functions, which bridges spectral theory and probability theory.

The key idea to construct a probability spectral theory, including probability thermodynamics and probability quantum fields, is to regard random variables as eigenvalues.

In probability theory, there are two kinds of probability distributions:

1. the random variable ranges from 0 to \(\infty\);
2. the random variable ranges from \(-\infty\) to \(\infty\).

When regarding random variables as eigenvalue spectra, the distribution with random variables ranging from 0 to \(\infty\) corresponds to the conventional heat kernel which is appropriate for lower bounded spectra, and the distribution with random variables ranging from \(-\infty\) to \(\infty\) corresponds to the generalized heat kernel which is appropriate for lower unbounded spectra.

After bridging probability theory and the spectral theory by regarding random variables as eigenvalues, we can further construct a fictional physical system whose eigenvalues obey a probability distribution. Thermodynamics and quantum fields of such a system can be established.
Concretely, for a probability distribution, by regarding the eigenvalue $\lambda$ as the random variables $x$, regarding the state density $\rho(\lambda)$ as the probability distribution function $p(x)$, and regarding the counting function as the cumulative probability functions $P(x)$, etc., we arrive at a set of probability spectral functions. In probability theory, there are various probability distributions, such as the Gaussian distribution, the Laplace distribution, student’s $t$-distribution, etc. For each probability distribution, we can construct a family of spectral functions, including, e.g., the thermodynamic quantity and the effective action.

3.1 Spectral counting function and cumulative probability function

In this section, we show that the similarity between spectral counting functions and the cumulative probability functions is a bridge between spectral theory and probability theory.

The spectral counting function is the number of eigenstates with eigenvalues less than or equal to $\lambda$ [3, 4]. In section 2.2, we generalize the definition of the spectral counting function to lower-unbounded spectra. The generalized spectral counting function can be expressed as

$$N(\lambda) = \int_{\lambda_{\text{min}}}^{\lambda} \rho(\lambda) \, d\lambda,$$  \hspace{1cm} (3.1)

where $\rho(\lambda)$ is the state density and $\lambda_{\text{min}}$ tends to $-\infty$ for lower-unbounded spectra.

In probability theory, there is a cumulative distribution function —— the probability that a random variable will be found at a value less than or equal to $\lambda$ [7]. The cumulative probability function can be expressed as

$$P(x) = \int_{x_{\text{min}}}^{x} p(x) \, dx,$$  \hspace{1cm} (3.2)

where $p(x)$ is the probability density function. The random variable $x_{\text{min}}$ tends to $-\infty$ for probability distributions with negative infinite random variables.

Comparing the definition of the spectral counting function (3.1) and the cumulative probability functions (3.2), we can see that when regarding the random variable as an eigenvalue, the cumulative probability function serves as a spectral counting function.

Along this line of thought, furthermore, we can also find the similarity between the state density and the probability density function; the probability density function plays the same role as the state density.

The above observation builds a bridge between the probability theory and the spectral theory. In the following, by regarding the random variable as an eigenvalue, we transform the probability theory into a spectral theory.

3.2 Generalized heat kernel as generalized moment generating function

The moment generating function is an alternative description in addition to probability density functions and cumulative distribution functions. The moment generating function, however, unlike the characteristic function, does not always exist. In this section, we show that the generalized heat kernel can serve as a generalized moment generating function which is definable when the original definition of the moment generating function fails.
In probability theory, the moment generating function $M(t)$ is defined by a Riemann-Stieltjes integral\(^\text{[7]}\),

$$M(t) = \int_{-\infty}^{\infty} e^{tx} dP(x), \quad (3.3)$$

where $P(x)$ is the cumulative probability functions.

For continuous random variables the probability density function is $p(x)$. The moment generating function $M(-t)$ is then a two-sided Laplace transformation of $p(x)$,

$$M(-t) = \int_{-\infty}^{\infty} p(x) e^{-tx} dx. \quad (3.4)$$

Observing the relation between the generalized heat kernel and the state density, Eq. (2.6), we can see that the generalized heat kernel $K(t) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{-\lambda t}$ is just the moment generating function (3.4).

When $\rho(\lambda)$ decreases faster than $e^{-\lambda t}$, the moment generating function is ill-defined. Nevertheless, even for such a case, the generalized heat kernel is still well-defined after an analytic continuation treatment.

In a word, when replacing the moment generating function by the heat kernel, we in fact introduce a generalized moment generating function instead of the original definition.

The moment is fundamentally important in statistics: the first moment is the mean value, the second central moment is the variance, the third central moment is the skewness, and the fourth central moment is the kurtosis\(^\text{[8]}\). Moreover, the mean value of a quantity can be expressed as a series of the moments by expanding the quantity as a power series. Nevertheless, many statistical distributions have no moment, such as the Cauchy distribution and the intermediate distribution\(^\text{[9]}\).

By the generalized moment generating function, we can define moments for the statistical distributions which have no moments. In Ref. [9], for defining moments for statistical distributions, one introduces a weighted moment. When introducing the weighted moment, the moment depends not only on the distribution but also on the choice of weighted function. The moment defined in the present paper is more natural, which depends only the statistical distribution itself.

### 3.3 Generalized heat kernel and characteristic function

In this section, we present the relation between the generalized heat kernel in spectral theory and the characteristic function in probability theory.

In probability theory, the characteristic function is defined as the Fourier transformation of the distribution function $p(x)$\(^\text{[8]}\):

$$f(k) = \mathcal{F}[p(x); k] = \int_{-\infty}^{\infty} dx p(x) e^{ikx}, \quad (3.5)$$

where $\mathcal{F}[p(x); k]$ denotes the Fourier transformation of $p(x)$.

The relation between the generalized heat kernel and the characteristic function can be obtained by inspection of their definitions (2.6) and (3.5).
As discussed above, by regarding the statistical distribution function $p(x)$ as a state density $\rho(\lambda)$ of a spectrum, we have

$$K(t) = \int_{-\infty}^{\infty} dx p(x) e^{-xt}. \quad (3.6)$$

From the definition of the characteristic function (3.5), we can see that the statistical distribution function $p(x)$ is the inverse Fourier transformation of the characteristic function:

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(k) e^{-ikx}. \quad (3.7)$$

Substituting Eq. (3.7) into Eq. (3.6) gives

$$K(t) = \int_{-\infty}^{\infty} dk f(k) \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-i(k-it)x}$$

$$= \int_{-\infty}^{\infty} dk f(k) \delta(k-it). \quad (3.8)$$

Then we arrive at a relation between the generalized heat kernel and the characteristic function:

$$K(t) = f(it). \quad (3.9)$$

It can be directly seen from Eq. (2.10) that the characteristic function is just the Wick rotated heat kernel introduced in section 2.1.2.

### 3.4 Probability spectral zeta function

Similarly, we can construct the probability zeta function. From the generalized spectral zeta function (2.20), by regarding the eigenvalue $\lambda$ as the random variables $x$ and regarding the statistical distribution function $p(x)$ as the state density $\rho(\lambda)$, we arrive at a probability zeta function.

$$\zeta(s) = \int_{-\infty}^{\infty} dx p(x) x^{-s}. \quad (3.10)$$

Based on this probability zeta function, we can obtain various spectral functions in quantum field theory, e.g., the effective action.

### 4 Probability thermodynamics

All thermodynamic quantities are spectral functions. The thermodynamic behavior is determined by the spectrum. Starting from the partition function $Z(\beta)$, whether the conventional definition or the generalized definition, we arrive at thermodynamics. All thermodynamic quantities can be obtained from the partition function directly.

Regarding the probability density function as a state density of an eigenvalue spectrum, each probability distribution, such as the Gaussian distribution and the Cauchy distribution, defines a thermodynamic system. This allows us to establish various probability thermodynamics corresponding to various probability distributions.
4.1 Probability partition function

In statistical mechanics, the canonical partition function is defined by

\[ Z(\beta) = \sum_{\{\lambda_n\}} e^{-\beta \lambda_n}. \tag{4.1} \]

The canonical partition function is just the heat kernel with the replacement of \( t \) by \( \beta \).

By the same reason as in the definition of the heat kernel, the definition of the canonical partition function is only valid for lower bounded spectra. For lower unbounded spectra, as done for heat kernels, the canonical partition function can also be generalized.

According to the generalization of the heat kernel given above, by Eqs. (2.10) and (2.7), we immediately achieve a generalized partition function,

\[ Z(\beta) = \left. \int_{\lambda_{\text{min}}}^{\infty} \rho(\lambda) e^{-i\lambda \tau} d\lambda \right|_{\tau = -i\beta}; \tag{4.2} \]

or, by Eq. (2.6), we arrive at

\[ Z(\beta) = \int_{\lambda_{\text{min}}}^{\infty} d\lambda \rho(\lambda) e^{-\beta \lambda}. \tag{4.3} \]

For lower unbounded spectra, \( \lambda_{\text{min}} \rightarrow -\infty \).

It can be directly seen from Eq. (4.3) that, if regarding \( \rho(\lambda) e^{-\beta \lambda} \) as a density function, the partition function is the zero-order moment \( \langle \lambda^0 \rangle \).

It should be emphasized that the partition function given by Eq. (4.2) or (4.3) is a generalized partition function rather than the conventional partition function, and the corresponding thermodynamic quantities are also not the conventional thermodynamic quantities.

4.2 Probability thermodynamic quantity

Starting from the generalized canonical partition function, Eq. (4.2) or (4.3), we can construct whole thermodynamics regardless of the spectrum lower bounded or lower unbounded.

The internal energy is a statistical average of eigenvalues, formally defined by

\[ U(\beta) = \sum_{\{\lambda_n\}} \frac{\lambda_n e^{-\lambda_n \beta}}{\sum_{\{\lambda_n\}} e^{-\lambda_n \beta}}. \tag{4.4} \]

For lower unbounded spectra, such a definition becomes \( U(\beta) = \sum_{\lambda_n}^{\infty} \frac{\lambda_n e^{-\lambda_n \beta}}{\sum_{\lambda_n}^{\infty} e^{-\lambda_n \beta}} \). The divergence here is no longer a problem for it can be removed by the method discussed above. It is worthy to point out that the internal energy for lower unbounded spectra may be negative since part of the eigenvalues can take negative values.

Observing the definition of the internal energy (4.4) and the partition function (4.1), we can formally write down the relation of internal energy and partition function even for lower unbounded spectra

\[ U(\beta) = -\frac{\partial}{\partial \beta} \ln Z(\beta) = \frac{1}{Z(\beta)} \int_{\lambda_{\text{min}}}^{\infty} d\lambda \rho(\lambda) \lambda e^{-\beta \lambda}. \tag{4.5} \]
From Eq. (4.5) we can see that the internal energy is the first-order moment $\langle \lambda^1 \rangle$.

Furthermore, the specific heat can be calculated from the internal energy:

$$C_V = \frac{\partial U}{\partial T} = \beta^2 \left[ \int_{\lambda_{\text{min}}}^{\infty} d\lambda \rho(\lambda) \lambda^2 e^{-\lambda \beta} - \left( \int_{\lambda_{\text{min}}}^{\infty} d\lambda \rho(\lambda) \lambda e^{-\lambda \beta} \right)^2 \right]. $$ (4.6)

It can be seen that the specific heat is the difference between the second-order moment and the square of first-order moment $\langle \lambda^2 \rangle - \langle \lambda^1 \rangle^2$.

Moreover, from the generalized canonical partition function, we can obtain other thermodynamic quantities, e.g., the free energy

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta) $$ (4.7)

and the entropy

$$S = \ln Z(\beta) - \beta \frac{\partial}{\partial \beta} \ln Z(\beta) $$ (4.8)

for both lower bounded and lower unbounded spectra.

### 4.3 Probability thermodynamics: characteristic function approach

Starting from the relation between the heat kernel and the characteristic function, we can directly represent various thermodynamic quantities by the characteristic function. According to the correspondence between the generalized heat kernel and the canonical partition function (4.3) and the relation (3.9), we represent the canonical partition function as

$$Z(\beta) = f(i\beta). $$ (4.9)

Various thermodynamic quantities then can also be represented by the characteristic function, e.g., the internal energy

$$U(\beta) = -\frac{\partial}{\partial \beta} \ln f(i\beta) $$ (4.10)

and the specific heat capacity

$$C_V = \beta^2 \frac{\partial^2}{\partial \beta^2} \ln f(i\beta), $$ (4.11)

etc.

### 5 Probability quantum field theory

The vacuum amplitude in the Euclidean quantum field theory is

$$Z = \int {\cal D}\phi e^{-I[\phi]/\hbar}, $$ (5.1)

where $I[\phi]$ is the Euclidean action $I[\phi] = -\int d^3x d(it) \mathcal{L}$. The Euclidean vacuum amplitude in spectral representation is just the partition function, or, the global heat kernel (1.1).
The conventional definition of the vacuum amplitude (5.1) is, of course, only valid for lower bounded spectra. The procedure given in the present paper can also provide a generalized definition of vacuum amplitude for lower unbounded spectra. This allows us to construct a quantum field for lower unbounded operators.

By regarding the probability partition function as a probability vacuum amplitude, we arrive at a probability quantum field. Taking the one-loop effective action and the vacuum energy as examples, we show how to construct a probability quantum field.

5.1 One-loop effective action and vacuum energy as spectral function

The information of a mechanical system is embedded in an Hermitian operator $D$. Many important physical quantities are spectral functions constructed from the eigenvalues $\{\lambda_n\}$ of the operator $D$. Two important quantities in quantum field theory, the one-loop effective action and the vacuum energy, are both spectral functions: the one-loop effective action is the determinant of the operator, $\det D$, and the vacuum energy is the trace of the operator, $\text{tr} D$.

For an Hermitian operator, the determinant is the product of the eigenvalues:

$$\det D = \prod_n \lambda_n, \quad (5.2)$$

and the trace is the sum of the eigenvalues:

$$\text{tr} D = \sum_n \lambda_n. \quad (5.3)$$

They are, obviously, divergent.

In order to obtain a finite result, we need a renormalization treatment.

5.2 Probability one-loop effective action

The effective action for the operator $D$ can be expanded as

$$\Gamma [\phi] = I [\phi] + \hbar \frac{1}{2} \ln \det D + O (\hbar^2), \quad (5.4)$$

where

$$W = \frac{1}{2} \ln \det D = \frac{1}{2} \ln \prod_n \lambda_n = \frac{1}{2} \sum_n \ln \lambda_n \quad (5.5)$$

is the one-loop effective action which diverges.

In order to obtain a finite one-loop effective action, we analytically continue the sum in the one-loop effective action (5.5) by virtue of the spectral zeta function. By Eq. (2.17) we can see that the derivative of the spectral zeta function is $\zeta'(s) = -\sum_n \lambda_n^{-s} \ln \lambda_n$, so $\zeta'(0) = \sum_n \ln \lambda_n$ and then

$$W = -\frac{1}{2} \zeta'(0). \quad (5.6)$$

In practice, the one-loop effective action for continuous spectra can be rewritten as

$$W = -\frac{1}{2} \int_{-\infty}^{\infty} \rho(\lambda) \ln \lambda d\lambda, \quad (5.7)$$
by use of \( \zeta'(0) = \sum_n \ln \lambda_n = \int_{-\infty}^{\infty} \rho(\lambda) \ln \lambda d\lambda \).

Now, we show that, in probability quantum field theory, the expression (5.6) for one-loop effective action is often convergent, though it diverges in many cases of quantum field theory.

In quantum field theory, in order to remove the divergence, one introduces a regularized one-loop effective action \([2]\):

\[
W_s = -\frac{1}{2} \tilde{\mu}^2 \Gamma(s) \zeta(s).
\]  

(5.8)

where \( \tilde{\mu} \) is a constant of the dimension of mass introduced to keep proper dimension of the effective action. To remove the divergence in the one-loop effective action, we expand \( W_s \) around \( s = 0 \):

\[
W_s = -\frac{1}{2} \zeta(0) \frac{1}{s} - \frac{1}{2} \zeta'(0) + \left( \frac{1}{2} \gamma_E - \ln \tilde{\mu} \right) \zeta(0).
\]  

(5.9)

After the minimal subtraction which drops the divergent term, we arrive at a renormalized one-loop effective action,

\[
W_{\text{ren}} = -\frac{1}{2} \zeta'(0) + \left( \frac{1}{2} \gamma_E - \ln \tilde{\mu} \right) \zeta(0).
\]  

(5.10)

By the definition (2.17), \( \zeta(0) = \sum_n 1 \) is divergent. It is just the divergent \( \zeta(0) \) cancels the divergence in \( \zeta'(0) \).

In probability quantum field theory, the spectra satisfy probability distributions, so \( \zeta(0) = \sum_n 1 = \int_{-\infty}^{\infty} \rho(\lambda) d\lambda = 1 \) is the total probability which is of course finite and normalized. For finite \( \zeta(0) \), choosing the constant \( \tilde{\mu}^2 = e^{\gamma_E} \), we return to Eq. (5.6).

Moreover, it should be emphasized that in probability quantum field theory, \( \zeta'(0) \) still may diverge in some cases. For continuous spectra, \( \zeta'(0) = \sum_n \ln \lambda_n = \int_{-\infty}^{\infty} \rho(\lambda) \ln \lambda d\lambda \).

In probability theory, \( \rho(\lambda) \) is the probability distribution function. Obviously, for probability distribution possessing moments, \( \zeta'(0) \) is finite. For probability distribution without moments, we need to analyze the integrability case by case.

The one-loop effective action can also be calculated directly from the spectral characteristic function introduced in section 2.3.3.

Representing the state density \( \rho(\lambda) \) by the inverse Fourier transformation of the spectral characteristic function, by Eq. (3.7),

\[
\rho(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{ik\lambda} dk
\]  

(5.11)

and substituting into the expression of the one-loop effective action, Eq. (5.7), we arrive at

\[
W = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{ik\lambda} dk \right] \ln \lambda d\lambda
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) \left[ \int_{-\infty}^{\infty} e^{ik\lambda} \ln \lambda d\lambda \right] dk.
\]  

(5.12)

The integral in Eq. (5.12) diverges. This integral, however, is a Fourier transformation of \( \ln \lambda \). After analytic continuation, we achieve a finite result:

\[
\int_{-\infty}^{\infty} e^{ik\lambda} \ln \lambda d\lambda = \pi \left[ -2\gamma_E \delta(k) + i\pi \delta(k) - \frac{\text{sgn}(k) - 1}{k} \right].
\]  

(5.13)
Then we have

$$W = \left( -\gamma_E + \frac{i\pi}{2} \right) f(0) - \frac{1}{2} \int_{-\infty}^{\infty} f(k) \frac{\text{sgn}(k) - 1}{k} dk. \quad (5.14)$$

Therefore,

$$W = \left( -\gamma_E + \frac{i\pi}{2} \right) f(0) + \int_{0}^{\infty} f(k) \frac{k}{k} dk. \quad (5.15)$$

This expression is useful when only the characteristic function of a probability distribution is known.

In some cases, the integral term in Eq. (5.15) still diverges. However, the divergence can be removed by a renormalization procedure [10]. We will illustrate such a renormalization procedure by taking the Cauchy distribution as an example in section 6.2.5.

### 5.3 Probability vacuum energy

The vacuum energy is essentially a sum of all eigenvalues, or, the trace of the operator $D$:

$$E_0 = \sum_n \lambda_n. \quad (5.16)$$

Physical operators are not upper bounded, so the vacuum energy diverges. In section 2.3, we suggest an approach to obtain an finite result of the trace of the operator.

The trace of the operator $D$, Eq. (5.3), is a Dirichlet series defined by $\sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^s}$ which converges absolutely on the right open half-plane such that $\text{Re} \, s > 1$. A Dirichlet series can be analytically continued as a spectral zeta function.

In our problem, the spectrum may be not lower bounded. That is, the trace of the operator encountered sometimes is a generalized Dirichlet series. By the procedure developed in section 2.3, we can analytically continue such a generalized Dirichlet series as a generalized spectral zeta function.

Concretely, the vacuum energy is just a spectral zeta function:

$$E_0 = \text{tr} \, D = \sum_n \left. \frac{1}{\lambda_n^s} \right|_{s=-1} = \zeta(s)\big|_{s=-1} = \zeta(-1). \quad (5.17)$$

The trace, after being analytically continued as a spectral zeta function, $\zeta_D(-1)$, is already finite. That is, it is renormalized.

Similarly, in quantum field theory, the vacuum energy always diverges. However, it recovers $\zeta(-1)$ after a renormalization treatment.

The vacuum energy can also be calculated from the characteristic function.

The vacuum energy can be expressed by the state density,

$$E_0 = \int_{-\infty}^{\infty} \rho(\lambda) \lambda d\lambda. \quad (5.18)$$
The state density can be expressed by the inverse Fourier transformation of the characteristic function, so by Eq. (3.7),

\[
E_0 = (-i) \frac{d}{dk} \left[ \int_{-\infty}^{\infty} \rho(\lambda) e^{ik\lambda} d\lambda \right]_{k=0} = (-i) \frac{df(k)}{dk} \bigg|_{k=0},
\]

(5.19)

6 Probability thermodynamics and probability quantum field: various statistical distributions

Following the approach introduced above, we can construct probability thermodynamics and probability quantum fields for various statistical distributions.

There are two kinds of probability distributions: lower bounded and lower unbounded.

For distributions with nonnegative random variables, such as the gamma distribution, the exponential distribution, the beta distribution, the chi-square distribution, the Pareto distribution, the generalized inverse Gaussian distribution, the tempered stable distribution, and the log-normal distribution, we can use the conventional definition of the spectral function.

For distributions with negative random variables, such as the normal distribution, the skew Gaussian distribution, Student’s t-distribution, the Laplace distribution, the Cauchy distribution, the q-Gaussian distribution, the intermediate distribution, we must use the generalized definition of the spectral function.

6.1 Probability thermodynamics and probability quantum field: distribution with nonnegative random variable

For distributions with nonnegative random variables, probability thermodynamics and probability quantum fields can be constructed by the conventional definition of spectral functions.

6.1.1 Gamma distribution

The Gamma distribution is used to describe nonstationary earthquake activity, the drop size in sprays [11], and maximum entropy approach to H-theory [12], etc.

The gamma distribution [13]

\[
p(x) = \frac{1}{\Gamma(a)} b^{-a} e^{-x/b} x^{a-1},
\]

(6.1)

recovers many distributions when taking the shape parameter \(a\) and the inverse scale parameter \(b\) as some special values. For example, the exponential distribution is recovered when \(a = 1\) and \(b = 1/\lambda\) and the normal distribution is recovered when \(a \to \infty\).

The cumulative distribution function, by Eq. (3.2), reads

\[
P(x) = Q \left( a, 0, \frac{x}{b} \right),
\]

(6.2)

where \(Q(a, z_0, z_1) = \Gamma(a, z_0, z_1) / \Gamma(a)\) is the generalized regularized incomplete gamma function with \(\Gamma(a, z_0, z_1)\) the generalized incomplete gamma function [14].
Thermodynamics In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = (1 - ibk)^{-a}.
\]  

(6.3)

The heat kernel, by performing the Wick rotation \( k \rightarrow it \) to the characteristic function, reads

\[
K(t) = (1 + bt)^{-a}.
\]  

(6.4)

The thermodynamics corresponding to the gamma distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = (1 + b\beta)^{-a}.
\]  

(6.5)

Then the free energy

\[
F = -\frac{1}{\beta} \ln(1 + b\beta)^{-a},
\]  

(6.6)

the internal energy

\[
U = \frac{ba}{1 + b\beta},
\]  

(6.7)

the entropy

\[
S = \frac{ba\beta}{1 + b\beta} - a \ln(1 + b\beta),
\]  

(6.8)

and the specific heat

\[
C_V = \frac{b^2 \beta^2 a}{(1 + b\beta)^2}.
\]  

(6.9)

Quantum field The spectral zeta function, by Eq. (3.10), reads

\[
\zeta(s) = \frac{1}{\Gamma(a)} b^{-s} \Gamma(a - s).
\]  

(6.10)

The one-loop effective action by Eq. (5.6) reads

\[
W = \frac{1}{2} \left( \psi^{(0)}(a) + \ln b \right),
\]  

(6.11)

where \( \psi^{(n)}(z) \) is the \( n \)-th derivative of the digamma function.

The vacuum energy can be obtained by Eq. (5.17):

\[
E_0 = ab.
\]  

(6.12)

6.1.2 Exponential distribution

The exponential distribution is used to describe the central spin problem [15], dislocation network [16], and the Anderson-like localization transition in \( SU(3) \) gauge theory [17].

The probability density function of the exponential distribution is [13, 18]

\[
p(x) = \lambda e^{-\lambda x},
\]  

(6.13)

where \( \lambda > 0 \) is the rate parameter.

The cumulative distribution function, by Eq. (3.2), reads

\[
P(x) = 1 - e^{-\lambda x}.
\]  

(6.14)
**Thermodynamics**  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = (1 - ibk)^{-a}. \tag{6.15}
\]

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[
K(t) = \frac{\lambda}{t + \lambda}. \tag{6.16}
\]

The thermodynamics corresponding to the exponential distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = \frac{\lambda}{\beta + \lambda}. \tag{6.17}
\]

Then the free energy

\[
F = - \frac{1}{\beta} \ln \frac{\lambda}{\beta + \lambda}. \tag{6.18}
\]

the internal energy

\[
U = \frac{1}{\beta + \lambda}, \tag{6.19}
\]

the entropy

\[
S = \frac{\beta}{\beta + \lambda} + \ln \frac{\lambda}{\beta + \lambda}. \tag{6.20}
\]

and the specific heat

\[
C_V = \frac{\beta^2}{(\beta + \lambda)^2}. \tag{6.21}
\]

**Quantum field**  The spectral zeta function, by Eq. (3.10), reads

\[
\zeta(s) = \lambda^s \Gamma(1 - s). \tag{6.22}
\]

The one-loop effective action by Eq. (5.6) reads

\[
W = - \ln \frac{\lambda}{2} - \frac{\gamma_E}{2}, \tag{6.23}
\]

where \( \gamma_E \approx 0.577216 \) is Euler’s constant.

The vacuum energy can be obtained by Eq. (5.17):

\[
E_0 = \frac{1}{\lambda}. \tag{6.24}
\]
6.1.3 Beta distribution

The beta distribution fits the data in the measurement of monoenergetic muon neutrino charged current interactions [19] and describes the collective behavior of active particles locally aligning with neighbors [20].

The beta distribution [21]

\[ p(x) = \frac{(1 - x)^{b-1}x^{a-1}}{B(a, b)}, \] (6.25)

is defined on the interval \([0, 1]\), where \(a\) and \(b\) are two positive shape parameters [18].

The cumulative distribution function, by Eq. (3.2), is

\[ P(x) = I_x(a, b), \] (6.26)

where \(I_x(a, b) = B(x, a, b)/B(a, b)\) is the regularized incomplete beta function with \(B(z, a, b)\) the incomplete beta function [14].

**Thermodynamics**  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \(p(x)\),

\[ f(k) = {}_1F_1(a; a + b; ik), \] (6.27)

where \( {}_1F_1(a; b; z) \) is the Kummer confluent hypergeometric function.

The heat kernel, by performing the Wick rotation \(k \rightarrow i\tau\) to the characteristic function, reads

\[ K(\tau) = {}_1F_1(a; a + b; -\tau). \] (6.28)

The thermodynamics corresponding to the beta distribution can be then constructed. The partition function, by replacing \(t\) with \(\beta\) in the heat kernel, reads

\[ Z(\beta) = {}_1F_1(a; a + b; -\beta). \] (6.29)

Then the free energy

\[ F = -\frac{1}{\beta} \ln {}_1F_1(a; b + a; -\beta), \] (6.30)

the internal energy

\[ U = \frac{a {}_1F_1(a + 1; b + a + 1; -\beta)}{(a + b) {}_1F_1(a; b + a; -\beta)}, \] (6.31)

the entropy

\[ S = \frac{a\beta {}_1F_1(a + 1; b + a + 1; -\beta)}{(a + b) {}_1F_1(a; b + a; -\beta)} + \ln {}_1F_1(a; b + a; -\beta), \] (6.32)

and the specific heat

\[ C_V = \frac{a\beta^2}{(a + b)} \left[ \frac{(a + 1) {}_1F_1(a + 2; b + a + 2; -\beta)}{{}_1F_1(a; b + a; -\beta) (a + b + 1)} - \frac{a {}_1F_1(a + 1; b + a + 1; -\beta)^2}{(a + b) {}_1F_1(a; b + a; -\beta)^2} \right]. \] (6.33)
Quantum field  The spectral zeta function, by Eq. (3.10), reads

\[ \zeta(s) = \frac{\Gamma(b)\Gamma(a-s)}{B(a,b)\Gamma(b-s+a)}. \]  

(6.34)

The one-loop effective action by Eq. (5.6) reads

\[ W = \frac{1}{2} \left[ \psi^{(0)}(a) - \psi^{(0)}(b+a) \right], \]  

(6.35)

where \( \psi^{(n)}(z) \) is the \( n \)-th derivative of the digamma function.

The vacuum energy can be obtained by Eq. (5.17):

\[ E_0 = \frac{a}{a+b}. \]  

(6.36)

6.1.4 Chi-square distribution

The chi-squared distribution (\( \chi^2 \)-distribution) is the distribution of a sum of the squares of standard normal random variables. The chi-square distribution appears in neutrino experiments [22], measurements of microwave background [23], and statistical inference [18, 24].

The probability density function of the chi-squared distribution with \( \nu \) degrees of freedom is [18]

\[ p(x) = \frac{1}{\Gamma(\nu/2)^{2\nu/2}} e^{-x^2/2} x^{\nu-1}. \]  

(6.37)

The cumulative distribution function, by Eq. (3.2), reads

\[ P(x) = Q(\nu/2, 0, x/2), \]  

(6.38)

where \( Q(a, z_0, z_1) \) is the generalized regularized incomplete gamma function, defined as \( \Gamma(a, z_0, z_1)/\Gamma(a) \) with \( \Gamma(a, z_0, z_1) \) the generalized incomplete gamma function [25].

Thermodynamics  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[ f(k) = {}_1F_1 \left( \frac{\nu}{2} : \frac{1}{2}; -\frac{k^2}{2} \right) + \sqrt{\frac{\pi}{\nu}} \Gamma(\nu/2) \Gamma \left( \frac{\nu}{2} + 1 \right) \frac{ik}{2} {}_1F_1 \left( \frac{\nu}{2} + 1, 2, -\frac{k^2}{2} \right). \]  

(6.39)

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[ K(t) = {}_1F_1 \left( \frac{\nu}{2} : \frac{1}{2}; \frac{t^2}{2} \right) - t \sqrt{\frac{\pi}{\nu \Gamma(\nu/2)}} \frac{\Gamma((\nu + 1)/2)}{\Gamma(\nu/2)} {}_1F_1 \left( \frac{\nu + 1}{2}, 2, \frac{3 \cdot t^2}{2} \right). \]  

(6.40)

The thermodynamics corresponding to the chi-squared distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[ Z(\beta) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{\nu + 1}{2} \right) U \left( \frac{\nu}{2}, \frac{1}{2}, \frac{\beta^2}{2} \right), \]  

(6.41)

where \( U(a, b, z) \) is the confluent hypergeometric function.

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Then the free energy

\[ F = -\frac{1}{\beta} \ln \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{\nu + 1}{2} \right) U \left( \frac{\nu}{2}, \frac{1}{2}, \frac{\beta^2}{2} \right) \right), \quad (6.42) \]

the internal energy

\[
U = \frac{2^{\nu-1}}{3\Gamma(\nu)U(\nu/2,1/2,\beta^2/2)} \left\{ \sqrt{2\Gamma \left( \frac{\nu + 1}{2} \right)} \left[ (\nu + 1)\beta^2 \ _1F_1 \left( \frac{\nu + 3}{2}; \frac{5}{2}; \frac{\beta^2}{2} \right) \right] + 3 \ _1F_1 \left( \frac{\nu + 1}{2}; \frac{3}{2}; \frac{\beta^2}{2} \right) \right\}, \quad (6.43)
\]

the entropy

\[
S = \frac{\beta}{12\Gamma(\nu/2)\Gamma(\nu)U(\nu/2,1/2,\beta^2/2)} \times \left\{ \sqrt{2\Gamma(\nu)} \left[ 4\sqrt{\pi} \left\{ \beta^2(\nu + 1) \ _1F_1 \left( \frac{\nu + 3}{2}; \frac{5}{2}; \frac{\beta^2}{2} \right) \right] + 3 \ _1F_1 \left( \frac{\nu + 1}{2}; \frac{3}{2}; \frac{\beta^2}{2} \right) \right] \right\}
- 3\beta \Gamma \left( \frac{\nu}{2} \right) U \left( \frac{\nu}{2}; \frac{1}{2}, \frac{\beta^2}{2} \right) \left[ \ln \pi - 2\ln \left( \Gamma \left( \frac{\nu + 1}{2} \right) U \left( \frac{\nu}{2}, \frac{1}{2}, \frac{\beta^2}{2} \right) \right) \right]
- 3\beta^2 \nu + 3 \Gamma \left( \frac{\nu}{2} \right) \Gamma(\nu/2) \ _1F_1 \left( \frac{\nu}{2}; \frac{1}{2}, \frac{3}{2}, \frac{\beta^2}{2} \right) \right\}, \quad (6.44)
\]

and the specific heat

\[
C_V = -\frac{\beta^2}{45 \left[ \Gamma(\nu/2) \ _1F_1 \left( \nu/2,1/2,\beta^2/2 \right) - \sqrt{2}\beta \Gamma \left( (\nu + 1)/2 \right) \ _1F_1 \left( (\nu + 1)/2;3/2;\beta^2/2 \right) \right]^2}
\times \left\{ 5 \left\{ \sqrt{2}\Gamma \left( \frac{\nu + 1}{2} \right) \left[ \beta^2(\nu + 1) \ _1F_1 \left( \frac{\nu + 3}{2}; \frac{5}{2}; \frac{\beta^2}{2} \right) \right] + 3 \ _1F_1 \left( \frac{\nu + 1}{2}; \frac{3}{2}; \frac{\beta^2}{2} \right) \right\} \right\}
- 3\beta \nu \Gamma \left( \frac{\nu}{2} \right) \ _1F_1 \left( \frac{\nu}{2}; \frac{1}{2}, \frac{3}{2}, \frac{\beta^2}{2} \right) \right\} \times \left\{ \sqrt{2}\beta(\nu + 1) \Gamma \left( \frac{\nu + 1}{2} \right) \left[ \beta^2(\nu + 3) \ _1F_1 \left( \frac{\nu + 5}{2}; \frac{7}{2}; \frac{\beta^2}{2} \right) \right] + 15 \ _1F_1 \left( \frac{\nu + 3}{2}; \frac{5}{2}, \frac{\beta^2}{2} \right) \right\}
- 5\nu \Gamma \left( \frac{\nu}{2} \right) \left[ \beta^2(\nu + 2) \ _1F_1 \left( \frac{\nu}{2}; \frac{1}{2}, \frac{5}{2}, \frac{\beta^2}{2} \right) \right] + 3 \ _1F_1 \left( \frac{\nu}{2}; \frac{1}{2}, \frac{3}{2}, \frac{\beta^2}{2} \right) \right\} \right\}, \quad (6.45)
\]

**Quantum field**  The spectral zeta function, by Eq. (3.10), reads

\[
\zeta \left( s \right) = \frac{\Gamma \left( (\nu - s)/2 \right)}{2^{s/2}\Gamma \left( \nu/2 \right).} \quad (6.46)
\]

The one-loop effective action by Eq. (5.6) reads

\[
W = \frac{1}{2} \left( \psi^{(0)} \left( \frac{\nu}{2} \right) + \ln 2 \right), \quad (6.47)
\]

where \( \psi^{(n)} (z) \) is the \( n \)-th derivative of the digamma function.

The vacuum energy can be obtained by Eq. (5.17):

\[
E_0 = \nu. \quad (6.48)
\]
6.1.5 Pareto distribution

The Pareto distribution was introduced by Pareto in social science [26]. The Pareto distribution appears in Calabi-Yau moduli spaces [27], the Anderson localization with disordered medium [28], and the phenomenon of cumulative inertia in intracellular transport [29].

The probability density function of the Pareto distribution is [18]

\[
p(x) = b^{\alpha} x^{-1-\alpha}, \quad (6.49)
\]

where \( b \) is the minimum value parameter and \( \alpha \) is the shape parameter.

The cumulative distribution function, by Eq. (3.2), reads

\[
P(x) = 1 - \left( \frac{b}{x} \right)^{\alpha}. \quad (6.50)
\]

**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = \alpha E_{\alpha+1}(-ibk), \quad (6.51)
\]

where \( E_n(z) \) is the exponential integral function.

The heat kernel, by performing the Wick rotation \( k \rightarrow it \) to the characteristic function, reads

\[
K(t) = \alpha E_{\alpha+1}(bt). \quad (6.52)
\]

The thermodynamics corresponding to the Pareto distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = \alpha E_{\alpha+1}(b\beta). \quad (6.53)
\]

Then the free energy

\[
F = -\frac{1}{\beta} \ln (\alpha E_{\alpha+1}(b\beta)), \quad (6.54)
\]

the internal energy

\[
U = \frac{bE_\alpha(b\beta)}{E_{\alpha+1}(b\beta)}, \quad (6.55)
\]

the entropy

\[
S = \frac{b\beta E_\alpha(b\beta)}{E_{\alpha+1}(b\beta)} + \ln (\alpha E_{\alpha+1}(b\beta)), \quad (6.56)
\]

and the specific heat

\[
C_V = \frac{b^2 \beta^2 \left[ E_{\alpha-1}(b\beta)E_{\alpha+1}(b\beta) - E_\alpha(b\beta)^2 \right]}{E_{\alpha+1}(b\beta)^2}. \quad (6.57)
\]
Quantum field  The spectral zeta function, by Eq. (3.10), reads
\[ \zeta(s) = \frac{\alpha}{s + \alpha} b^{-s}. \] (6.58)

The one-loop effective action by Eq. (5.6) reads
\[ W = \frac{1}{2} \left( \frac{1}{\alpha} + \ln b \right), \] (6.59)
where \( \psi^{(n)}(z) \) is the \( n \)-th derivative of the digamma function.

The vacuum energy can be obtained by Eq. (5.17):
\[ E_0 = \frac{\alpha}{\alpha - 1} b. \] (6.60)

6.1.6 Generalized inverse Gaussian distribution

The generalized inverse Gaussian distribution is proposed to study the population frequencies of species [30], which is also called the Halphen type A distribution [31]. Sichel uses the distribution to construct the mixture of the Poisson distribution [32, 33]. Barndorff-Nielsen points out that the mixture of the normal distribution and the generalized inverse Gaussian distribution is just the generalized hyperbolic distribution [34, 35]. The generalized inverse Gaussian distribution appears in for example fractional Brownian motion [36] and inverse Ising problem [37].

The probability density function of the generalized inverse Gaussian distribution is [18]
\[ p(x) = \frac{\mu^{-\theta} x^{\theta-1}}{2K_\theta(\lambda/\mu)} \exp\left(-\frac{\lambda}{2x} \left(\frac{x^2}{\mu^2} + 1\right)\right), \] (6.61)
where \( \mu > 0 \) is the mean, \( \lambda > 0 \) is the shape parameter, and \( \theta \) is a real number. \( K_n(z) \) is the modified Bessel function of the second kind [14].

Thermodynamics  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),
\[ f(k) = \left(1 - \frac{2ik\mu^2}{\lambda}\right)^{-\theta/2} K_\theta\left(\frac{\lambda}{\mu} \sqrt{1 - \frac{2ik\mu^2}{\lambda}}\right). \] (6.62)

The heat kernel, by performing the Wick rotation \( k \rightarrow it \) to the characteristic function, reads
\[ K(t) = \left(\frac{2\mu^2 t}{\lambda} + 1\right)^{-\theta/2} K_\theta\left(\sqrt{\lambda(2t\mu^2 + \lambda)/\mu}\right). \] (6.63)

The thermodynamics corresponding to the generalized inverse Gaussian distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads
\[ Z(\beta) = \frac{1}{K_\theta(\lambda/\mu)} \left(\frac{\lambda}{2\beta\mu^2 + \lambda}\right)^{\theta/2} K_\theta(\sigma). \] (6.64)
Then the free energy
\[ F = -\frac{1}{\beta} \ln \left( \frac{\lambda}{2\beta \mu^2 + \lambda} \frac{K_\theta (\sigma)}{K_\theta (\lambda/\mu)} \right), \] (6.65)
the internal energy
\[ U = \frac{\mu}{K_\theta (\sigma)} \sqrt{\frac{\lambda}{2\beta \mu^2 + \lambda}} K_{\theta + 1} (\sigma), \] (6.66)
the entropy
\[ S = \frac{\beta \lambda K_{\theta + 1} (\sigma)}{\sigma K_\theta (\sigma)} + \ln \left( \frac{\lambda}{2\beta \mu^2 + \lambda} \frac{K_\theta (\sigma)}{K_\theta (\lambda/\mu)} \right), \] (6.67)
and the specific heat
\[ C_V = \frac{\beta^2 \lambda^2}{4\sigma^4} \left[ 2\sigma \frac{K_{\theta + 1} (\sigma)}{K_\theta (\sigma)} + \frac{(2\beta \mu^2 - \lambda) K_{\theta + 1} (\sigma)^2}{\mu^2 K_\theta (\sigma)^2} + \sigma^2 \frac{K_{\theta + 2} (\sigma)}{K_\theta (\sigma)} - \sigma^2 \frac{K_{\theta - 1} (\sigma)^2}{K_\theta (\sigma)^2} \right. \\
+ 2\sigma \frac{K_{\theta - 1} (\sigma)}{K_\theta (\sigma)} + \sigma^2 \frac{K_{\theta - 2} (\sigma)}{K_\theta (\sigma)} - 2\sigma^2 \frac{K_{\theta - 1} (\sigma) K_{\theta + 1} (\sigma)}{K_\theta (\sigma)^2} + \frac{2 \lambda^2}{\mu^2} + 8\theta + 4\beta \lambda \right], \] (6.68)
where \( \sigma = \sqrt{\lambda (2\beta \mu^2 + \lambda)/\mu} \).

Note that the generalized inverse Gaussian distribution recovers the inverse Gaussian distribution when \( \theta = -1/2 \).

Quantum field
The spectral zeta function, by Eq. (3.10), reads
\[ \zeta (s) = \frac{\mu^{-s} K_{\theta - s} (\lambda/\mu)}{K_\theta (\lambda/\mu)}. \] (6.69)

The one-loop effective action by Eq. (5.6) reads
\[ W = \frac{1}{2} \left( \ln \mu + \frac{1}{K_\theta (\lambda/\mu)} \frac{\partial K_{\theta - s} (\lambda/\mu)}{\partial \theta} \right), \] (6.70)
where \( K_n (z) \) gives the modified Bessel function of the second kind.

The vacuum energy can be obtained by Eq. (5.17):
\[ E_0 = \mu \frac{K_{\theta + 1} (\lambda/\mu)}{K_\theta (\lambda/\mu)}. \] (6.71)

6.1.7 Tempered stable distribution
The tempered stable distribution is introduced in Ref. [38]. In literature, the distribution is mentioned as the truncated Levy flight [39], and the KoBoL distribution [40]. Barndorff-Nielsen and Shephard generalize the tempered stable distribution to the modified stable distribution [41]. The tempered stable distribution appears in for example continuous-time random walks [42].

The probability density function of the tempered stable distribution is [43]
\[ p(x; \kappa, \alpha, b) = \frac{e^{\alpha b} - \alpha^{4/k}}{2\pi \alpha^{1/k}} \sum_{k=1}^{\infty} (-1)^{k-1} \sin (k\pi \kappa) \frac{\Gamma (k\kappa + 1)}{k!} 2^{k\kappa+1} \left( \frac{x}{\alpha^{1/k}} \right)^{-k\kappa-1}, \] (6.72)
where \( \alpha > 0, b \geq 0, \) and \( 0 < \kappa < 1. \)
**Thermodynamics**  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = \exp \left( ab - a \left( b^{1/\kappa} - 2ik \right)^{\kappa} \right).
\]  

(6.73)

For \( \kappa = 1/2 \) the tempered stable distribution reduces to the inverse Gaussian distribution. For the limiting case \( \kappa \to 0 \), we obtain the Gamma distribution.

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[
K(t) = \exp \left( ab - a \left( b^{1/\kappa} + 2t \right)^{\kappa} \right).
\]  

(6.74)

The thermodynamics corresponding to the tempered stable distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = \exp \left( ab - a \left( b^{1/\kappa} + 2\beta \right)^{\kappa} \right).
\]  

(6.75)

Then the free energy

\[
F = -a \frac{\beta}{\beta} \left[ b - \left( b^{1/\kappa} + 2\beta \right)^{\kappa} \right],
\]

(6.76)

the internal energy

\[
U = 2a \left( b^{1/\kappa} + 2\beta \right)^{-1+\kappa} \kappa,
\]

(6.77)

the entropy

\[
S = 2a\beta \left( b^{1/\kappa} + 2\beta \right)^{-1+\kappa} \kappa + a \left[ b - \left( b^{1/\kappa} + 2\beta \right)^{\kappa} \right],
\]

(6.78)

and the specific heat

\[
C_V = -4a\beta^2 \left( b^{1/\kappa} + 2\beta \right)^{-2+\kappa} (\kappa - 1)\kappa.
\]

(6.79)

**Quantum field**  The spectral zeta function, by Eq. (3.10), reads

\[
\zeta(s) = -2^{-s} e^{ab} \sin \kappa \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k \kappa + 1) \Gamma(-s - k \kappa)}{(k - 1)!} a^k b^{k+s/\kappa}.
\]  

(6.80)

The one-loop effective action by Eq. (5.6) reads

\[
W = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\pi (-a)^k b^k e^{ab}}{\Gamma(k)} \left[ \ln b \sin(\csc(\pi k \kappa)) - \ln 2 \sin \kappa \csc(\pi k \kappa) - \sin \kappa \csc(\pi k \kappa) \psi^{(0)} (-k \kappa) \right].
\]  

(6.81)

The vacuum energy can be obtained by Eq. (5.19):

\[
E_0 = 2ab b^{(\kappa-1)/\kappa}.
\]

(6.82)
6.1.8 Log-normal distribution

The log-normal distribution is a continuous probability distribution whose logarithm is normally distributed. The Log-normal distribution appears in for example the global ocean models [44], particle physics [45], and the problem of dark matter annihilation [46].

The probability density function of the log-normal distribution is [18]

\[ p(x) = \frac{1}{\sqrt{2\pi x\sigma}} \exp\left(\frac{-(\mu + \ln x)^2}{2\sigma^2}\right). \] (6.83)

The logarithm of a log-normal distribution \( X \) with parameters \( \mu \) and is normally distributed. \( \ln X \) has a mean \( \mu \) and standard deviation . Then we can write \( X \) as

\[ X = \exp(\mu + \sigma Z) \] (6.84)

with \( Z \) a standard normal variable.

The cumulative distribution function, by Eq. (3.2), reads

\[ P(x) = \frac{1}{2} \text{erfc}\left(\frac{\mu - \ln x}{\sqrt{2\sigma}}\right), \] (6.85)

where erfc \( (x) \) is the complementary error function given by erfc \( (z) = 1 - \text{erf}(z) \) with \( \text{erf}(z) \) the integral of the Gaussian distribution: \( \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt \).

**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[ f(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right). \] (6.86)

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[ K(t) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right). \] (6.87)

The thermodynamics corresponding to the log-normal distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[ Z(\beta) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right). \] (6.88)

Then the free energy

\[ F = -\frac{1}{\beta} \ln \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right), \] (6.89)

the internal energy

\[ U = -\frac{\sum_{n=0}^{\infty} \frac{n(-\beta)^{n-1} + n^2}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)}{\sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)}, \] (6.90)
the entropy
\[
S = \ln \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \exp \left( n\mu + \frac{n^2 \sigma^2}{2} \right) - \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \exp \left( n\mu + \frac{n^2 \sigma^2}{2} \right),
\]
(6.91)
and the specific heat
\[
C_V = \frac{\beta^2}{\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n}{n!} \exp \left( n\mu + \frac{n^2 \sigma^2}{2} \right)}
\times \left\{\sum_{n=0}^{\infty} \frac{(-\beta)^{n-2}}{(n-2)!} e^{n^2 \sigma^2/2+\mu n} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} e^{n^2 \sigma^2/2+\mu n} - \sum_{n=0}^{\infty} \frac{(-\beta)^{n-1}}{(n-1)!} e^{n^2 \sigma^2/2+\mu n} \right\}^2.
\]
(6.92)

Quantum field The spectral zeta function, by Eq. (3.10), reads
\[
\zeta (s) = \exp \left( \frac{1}{2} s \left( s \sigma^2 - 2 \mu \right) \right).
\]
(6.93)
The one-loop effective action by Eq. (5.6) reads
\[
W = \frac{\mu}{2}.
\]
(6.94)
The vacuum energy can be obtained by Eq. (5.17):
\[
E_0 = \exp \left( \mu + \frac{\sigma^2}{2} \right).
\]
(6.95)

6.2 Probability thermodynamics and probability quantum field: distribution with negative random variable

In the above, we construct probability thermodynamics and probability quantum fields for probability distributions with nonnegative random variables. For constructing probability spectral functions with nonnegative random variables, we can use the conventional definition of spectral functions. For the distribution with negative random variables, however, the conventional definition is invalid, and we need to use the generalized definition given in the present paper.

6.2.1 Normal distribution

The normal distribution is also known as the bell curve or the Gaussian distribution. Normal distribution is the most important distribution in statistics. The normal distribution appears in almost every areas.

The probability density function of the normal distribution is [13, 18]
\[
p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right),
\]
(6.96)
where \( \mu \) is the mean of the distribution and \( \sigma \) is the standard deviation.

The cumulative distribution function, by Eq. (3.2), reads
\[
P(x) = \frac{1}{2} \operatorname{erfc} \left( \frac{\mu - x}{\sqrt{2} \sigma} \right).
\]
(6.97)
**Thermodynamics**  In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = \exp \left( ik\mu - \frac{k^2\sigma^2}{2} \right).
\]

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[
K(t) = \exp \left( \frac{1}{2} t (t\sigma^2 - 2\mu) \right). \tag{6.98}
\]

The thermodynamics corresponding to the normal distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = \exp \left( \frac{1}{2} \beta (\beta\sigma^2 - 2\mu) \right). \tag{6.99}
\]

Then the free energy

\[
F = \mu - \frac{1}{2} \beta\sigma^2, \tag{6.100}
\]

the internal energy

\[
U = \mu - \beta\sigma^2, \tag{6.101}
\]

the entropy

\[
S = -\frac{1}{2} \beta^2\sigma^2, \tag{6.102}
\]

and the specific heat

\[
C_V = \beta^2\sigma^2. \tag{6.103}
\]

**Quantum field**  The spectral zeta function, by Eq. (3.10), reads

\[
\zeta(s) = \frac{(-1)^s}{2^{s/2+1}\sqrt{\pi}\sigma^s} \left\{ [1 + (-1)^s] \Gamma \left( \frac{1}{2} - \frac{s}{2} \right) \frac{1}{\Gamma \left( \frac{1}{2} - \frac{s}{2} \right)} _1F_1 \left( \frac{s+1}{2}; \frac{1}{2}; -\frac{\mu^2}{2\sigma^2} \right) \right. \\
+ \frac{\sqrt{2\mu} (-1)^s (1 - (-1)^s)}{\sigma} \Gamma \left( 1 - \frac{s}{2} \right) _1F_1 \left( \frac{s+1}{2}; \frac{3}{2}; -\frac{\mu^2}{2\sigma^2} \right) \right\}, \tag{6.104}
\]

where \( \Gamma(z) \) is the Euler gamma function and \(_1F_1(\alpha; b; z)\) is the confluent hypergeometric function. When \( s \) is an integer,

\[
\zeta(s) = \frac{1}{2^{s/2+1}\sqrt{\pi}\sigma^s} \left\{ [1 + (-1)^s] \Gamma \left( \frac{1-s}{2} \right) \frac{1}{\Gamma \left( \frac{1-s}{2} \right)} _1F_1 \left( \frac{s+1}{2}; \frac{1}{2}; -\frac{\mu^2}{2\sigma^2} \right) \right. \\
+ \frac{\sqrt{2\mu} (-1)^s [1 - (-1)^s]}{\sigma} \Gamma \left( 1 - \frac{s}{2} \right) _1F_1 \left( \frac{s+1}{2}; \frac{3}{2}; -\frac{\mu^2}{2\sigma^2} \right) \right\}. \tag{6.105}
\]

When \( s \) is an even number

\[
\zeta(s) = \frac{1}{2^{s/2}\sqrt{\pi}\sigma^s} \Gamma \left( \frac{1-s}{2} \right) _1F_1 \left( \frac{s+1}{2}; \frac{1}{2}; -\frac{\mu^2}{2\sigma^2} \right). \tag{6.106}
\]
and when $s$ is an odd number

$$\zeta(s) = -\frac{2^{(1-s)/2}\mu}{\sqrt{\pi\sigma^2+1}}\Gamma\left(1 - \frac{s}{2}\right) \left. {}_1F_1\left(\frac{s + 1}{2}; \frac{3}{2}; -\frac{\mu^2}{2\sigma^2}\right) \right|_{a=0}. \quad (6.107)$$

The one-loop effective action by Eq. (5.6) reads

$$W = \frac{1}{4} \left( \partial_z {}_1F_1\left(a, 1/2, -\mu^2/2\sigma^2\right) \right) \left|_{a=0} + i\pi \text{erfc}\left(\frac{\mu}{\sqrt{2}\sigma}\right) + \ln\frac{\sigma^2}{2} - \gamma_E \right). \quad (6.108)$$

The vacuum energy can be obtained by Eq. (5.17):

$$E_0 = \mu. \quad (6.109)$$

6.2.2 Skewed normal distribution

The skewed normal distribution is a generalization of the normal distribution with a shape parameter which accounts for skewness. Azzalini introduces the density function of the skewed normal distribution [47]. In physics, the skewed normal distribution appears in gravitational wave problems [48] and in black hole problems [49].

The probability density function of the Skewed Gaussian distribution is [47]

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{erf}\left(-\frac{\alpha(x-\mu)}{\sqrt{2}\sigma}\right), \quad (6.110)$$

where $\alpha$ is the shape parameter, $\mu$ is the location parameter, and $\sigma$ is the scale parameter. The Gaussian distribution is recovered when $\alpha = 0$.

The cumulative distribution function, by Eq. (3.2), reads

$$P(x) = \frac{1}{2} \text{erf}\left(-\frac{x-\mu}{\sqrt{2}\sigma}\right) - 2T\left(\frac{x-\mu}{\sigma}, \alpha\right), \quad (6.111)$$

where $\text{erfi}(z)$ is the imaginary error function given by $\text{erfi}(iz) = -i\text{erf}(iz)$, and $T[x,\alpha]$ is Owen’s $T$ function given by $T[x,\alpha] = \frac{1}{2\pi} \int_0^\alpha \frac{-x^2(1+t^2)/2}{1+t^2} dt$ for real $\alpha$ [50].

**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function $p(x)$,

$$f(k) = \exp\left(-\frac{k^2\sigma^2}{2} + ik\mu\right) \left[ 1 + i\text{erfi}\left(\frac{\alpha k\sigma}{\sqrt{2}\sqrt{\alpha^2+1}}\right) \right]. \quad (6.112)$$

The heat kernel, by performing the Wick rotation $k \rightarrow it$ to the characteristic function, reads

$$K(t) = e^{\frac{1}{2}(\sigma^2t - 2\mu)} \text{erfc}\left(\frac{\alpha\sigma t}{\sqrt{2}\sqrt{\alpha^2+1}}\right). \quad (6.113)$$

The thermodynamics corresponding to the skewed normal distribution can be then constructed. The partition function, by replacing $t$ with $\beta$ in the heat kernel, reads

$$Z(\beta) = e^{\frac{1}{2}\beta(\beta\sigma^2 - 2\mu)} \text{erfc}\left(\frac{\alpha\sigma^2}{\sqrt{2}\sqrt{\alpha^2+2}}\right). \quad (6.114)$$
Then the free energy
\[ F = -\frac{1}{2} (\beta \sigma^2 - 2\mu) + \frac{1}{\beta} \ln \text{erfc} \left( \frac{\alpha \beta}{\sqrt{2 \alpha^2 + 2}} \right), \] (6.115)
the internal energy
\[ U = \mu - \beta \sigma^2 + \frac{2\sigma \alpha}{\sqrt{\pi \sqrt{2 \alpha^2 + 2}}} \exp \left( -\frac{\alpha^2 \sigma^2 \beta^2}{2(2\alpha^2 + 2)} \right) \text{erfc} \left( \frac{\beta \sigma \alpha}{\sqrt{2 \alpha^2 + 2}} \right), \] (6.116)
the entropy
\[ S = \frac{2\alpha \beta \sigma}{\sqrt{\pi \sqrt{2 \alpha^2 + 2}}} \text{erfc} \left( \frac{\beta \sigma \alpha}{\sqrt{2 \alpha^2 + 2}} \right) + \ln \left( \frac{\text{erfc} \left( \frac{\beta \sigma \alpha}{\sqrt{2 \alpha^2 + 2}} \right)}{\beta \sigma} \right) - \frac{1}{2} \beta^2 \sigma^2, \] (6.117)
and the specific heat
\[ C_V = \frac{\alpha^2 \beta^2 \sigma^2 e^{-\frac{\alpha^2 \beta^2 \sigma^2}{2(\alpha^2 + 2)}} \text{erfc} \left( \frac{\beta \sigma \alpha}{\sqrt{2 \alpha^2 + 2}} \right) - 2\sqrt{\alpha^2 + 1}}{\pi (\alpha^2 + 1)^{3/2} \text{erfc} \left( \frac{\beta \sigma \alpha}{\sqrt{2 \alpha^2 + 2}} \right)^2} + \beta^2 \sigma^2. \] (6.118)

**Quantum field**  The spectral zeta function, by Eq. (3.10), reads
\[ \zeta(s) = \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{erfc} \left( -\frac{\alpha(x-\mu)}{\sqrt{2\sigma}} \right) x^{-s} dx. \] (6.119)
The one-loop effective action by Eq. (5.7) reads
\[ W = \frac{1}{2\sqrt{2\pi \sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{erfc} \left( -\frac{\alpha(x-\mu)}{\sqrt{2\sigma}} \right) \ln x dx. \] (6.120)
The vacuum energy can be obtained by Eq. (5.19):
\[ E_0 = \mu + \sqrt{\frac{2}{\pi}} \frac{\alpha \sigma}{\sqrt{\alpha^2 + 1}}. \] (6.121)

When $\mu = 0$, the integral in Eqs. (6.119) and (6.120) can be carried out,
\[ \zeta(s) = \frac{1}{\sqrt{\pi \sigma^s}} \left\{ 2^{s/2-1} [(-1)^{s-1} + 1] \alpha^{s-1} \Gamma(1-s) \frac{\Gamma \left( 1 - \frac{s}{2} \right)}{\Gamma \left( \frac{s}{2} \right) 2 \Gamma \left( \frac{3-s}{2} \right) \Gamma \left( \frac{1}{2} - \frac{s}{2} \right) \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)}} \right\}. \] (6.122)
and
\[ W = \frac{1}{4} \left( 2\cot^{-1}(\alpha + \ln 2\sigma^2 + \psi^{(0)} \left( \frac{1}{2} \right) \right)). \] (6.123)
The vacuum energy can be obtained by Eq. (5.19):
\[ E_0 = \frac{2}{\pi} \frac{\alpha \sigma}{\sqrt{\alpha^2 + 1}}. \] (6.124)
6.2.3 Student’s t-distribution

Student’s t-distribution is a symmetric and bell shaped distribution, which rises to estimate the mean of a normally distributed population whose sample size is small and population standard deviation is unknown. Student’s t-distribution has heavier tails which means that it more likely to produce values that fall far from the mean. This is the main difference between the normal distribution and Student’s t-distribution.

The probability density function of Student’s t-distribution is [21]

\[
p(x) = \frac{1}{\sqrt{\nu} \sigma B \left( \frac{\nu}{2}, \frac{1}{2} \right)} \left( \frac{\nu}{\nu + (x - \mu)^2 / \sigma^2} \right)^{(\nu+1)/2},
\]

where \(\mu\) is location parameter, \(\sigma\) scale parameter, and \(\nu\) degrees of freedom. For \(\nu \to \infty\), Student’s t-distribution goes to Gaussian distribution. For \(\nu = 1\), Student’s t-distribution goes to Cauchy distribution.

The cumulative distribution function, by Eq. (3.2), reads

\[
P(x) = \begin{cases} 
\frac{1}{2} I_{\frac{(x-\mu)^2}{\nu + (x-\mu)^2 / \sigma^2}} \left( \frac{\nu}{2}, \frac{1}{2} \right), & x \leq \mu, \\
\frac{1}{2} \left[ I_{\frac{(x-\mu)^2}{\nu + (x-\mu)^2 / \sigma^2}} \left( \frac{\nu}{2}, \frac{1}{2} \right) + 1 \right], & x > \mu, 
\end{cases}
\]

where \(B(a,b)\) is the Euler beta function, \(K_\nu(z)\) is the modified Bessel function of the second kind, and \(I_z(a,b)\) is regularized incomplete beta function given by \(I_z(a,b) = B_z(a,b) / B(a,b)\) [14].

**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \(p(x)\),

\[
f(k) = \frac{2^{1-\nu/2} \nu^{\nu/4} \sigma^{\nu/2}}{\Gamma(\nu/2)} e^{ik\mu |k|^{\nu/2}} K_{\nu/2} \left( \sqrt{\nu} |k| \right).
\]

The heat kernel, by performing the Wick rotation \(k \to it\) to the characteristic function, reads

\[
K(t) = \frac{2^{1-\nu/2} \nu^{\nu/4}}{\Gamma(\nu/2)} e^{-t \mu (\sigma t)^{\nu/2}} K_{\nu/2} \left( \sqrt{\nu} |t| \right).
\]

The thermodynamics corresponding to Student’s t-distribution can be then constructed. The partition function, by replacing \(t\) with \(\beta\) in the heat kernel, reads

\[
Z(\beta) = \frac{2^{1-\nu/2} \nu^{\nu/4}}{\Gamma(\nu/2)} e^{-\beta \mu (\beta \sigma)^{\nu/2}} K_{\nu/2} \left( \beta \sqrt{\nu} \sigma \right).
\]

Then the free energy

\[
F = -\frac{1}{\beta} \ln \frac{2^{1-\nu/2} \nu^{\nu/4} e^{-\beta \mu (\beta \sigma)^{\nu/2}} K_{\nu/2} \left( \beta \sqrt{\nu} \sigma \right)}{\Gamma(\nu/2)}
\]

the internal energy

\[
U = \frac{1}{2} \left\{ \sqrt{\nu} \sigma \frac{K_{\nu/2 - 1} \left( \beta \sqrt{\nu} \sigma \right) + K_{\nu/2 + 1} \left( \beta \sqrt{\nu} \sigma \right)}{K_{\nu/2} \left( \beta \sqrt{\nu} \sigma \right)} - \frac{\nu}{\beta} + 2 \mu \right\},
\]
the entropy
\[ S = \frac{1}{2} \left[ \ln \frac{4\nu^\nu e^{-2\beta\mu}(\beta\nu)^\nu K_{\nu}/(\beta\nu)}{\Gamma(\nu/2)^2} + 2\beta\mu + \frac{2\beta\sqrt{\nu}\sigma K_{\nu-2}(\beta\sqrt{\nu}\sigma)}{K_{\nu}/(\beta\nu)} - \nu \ln 2 \right], \]
and the specific heat
\[ C_V = \frac{-\beta^2\nu \sigma^2}{4K_{\nu/2}(\beta\sqrt{\nu}\sigma)^2} \left\{ \left[ K_{\nu/2-1}(\beta\sqrt{\nu}\sigma) + K_{\nu/2+1}(\beta\sqrt{\nu}\sigma) \right]^2 
- K_{\nu/2}(\beta\sqrt{\nu}\sigma) \left[ K_{\nu/2-2}(\beta\sqrt{\nu}\sigma) + 2K_{\nu/2}(\beta\sqrt{\nu}\sigma) + K_{\nu/2+2}(\beta\sqrt{\nu}\sigma) \right] \right\} - \frac{\nu}{2}. \]

**Quantum field** The spectral zeta function, by Eq. (2.25), reads
\[ \zeta(s) = \sqrt{\pi}(\frac{-1}{2})^{s/4} e^{\pi(s+1)/2} \left( \frac{i}{\mu} \right)^s \Gamma(s+\nu) \frac{\Gamma(s/2+1/2)}{\Gamma(\nu/2)} 2\tilde{F}_1\left(\frac{s}{2},\frac{s+1}{2};\frac{1}{2};(2s+\nu+1);\frac{\nu \sigma^2}{\mu^2}+1\right), \]
where \( 2\tilde{F}_1(a,b;c;d) \) is the regularized hypergeometric function and \( 2\tilde{F}_1(a,b;c;d) = 2F_1(a,b;c;d)/\Gamma(c) \).

The one-loop effective action by Eq. (5.6) reads
\[ W = \frac{(-1)^\nu}{4} \left[ \ln 4\mu^2 - 2\psi(0)(\nu) - 2i\pi - \Gamma\left(\frac{\nu+1}{2}\right) \left( 2\tilde{F}_1(0,0,1,0) \left(0,\frac{1}{2},\frac{\nu+1}{2},\frac{\nu \sigma^2}{\mu^2}+1\right) \right) \right. \]
\[ \times + \left. \left. 2\tilde{F}_1(1,0,0,0) \left(0,\frac{1}{2},\frac{\nu+1}{2},\frac{\nu \sigma^2}{\mu^2}+1\right) \right) \right] . \]

The vacuum energy can be obtained by Eq. (5.17):
\[ E_0 = \mu. \]

**6.2.4 Laplace distribution**

The Laplace distribution is known as the double exponential distribution that can be considered as two exponential distributions jointed together back-to-back. The Laplace distribution is also a symmetric distribution. The Laplace distribution has fatter tails than the normal distribution. The Laplace distribution appears in for example neuronal growth [51], complex networks [52], and complex biological systems [53].

The probability density function of the Laplace distribution is [21, 54]
\[ p(x) = \frac{1}{2} e^{-\lambda|x-\mu|}, \]
where \( \mu \) is the mean and \( \lambda \) is the scale parameter.

The cumulative distribution function, by Eq. (3.2), reads
\[ P(x) = \begin{cases} 1 - \frac{1}{2} e^{-\lambda(x-\mu)}, & x \geq \mu \\ \frac{1}{2} e^{\lambda(x-\mu)}, & x < \mu \end{cases}. \]
**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[
f(k) = \frac{\lambda^2}{k^2 + \lambda^2} e^{ik\mu}. \quad (6.139)
\]

The heat kernel, by performing the Wick rotation \( k \to it \) to the characteristic function, reads

\[
K(t) = \frac{\lambda^2}{\lambda^2 - t^2} e^{-t\mu}. \quad (6.140)
\]

The thermodynamics corresponding to the Laplace distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[
Z(\beta) = \frac{\lambda^2}{\lambda^2 - \beta^2} e^{-\beta\mu}. \quad (6.141)
\]

Then the free energy

\[
F = -\frac{1}{\beta} \ln \frac{\lambda^2}{\lambda^2 - \beta^2} e^{-\beta\mu}, \quad (6.142)
\]

the internal energy

\[
U = \frac{2\beta}{\beta^2 - \lambda^2} + \mu, \quad (6.143)
\]

the entropy

\[
S = \left( \beta\mu + 1 - \frac{2\beta^2}{\lambda^2 - \beta^2} \right) \left( -\beta\mu + \ln \frac{\lambda^2}{\lambda^2 - \beta^2} \right), \quad (6.144)
\]

and the specific heat

\[
C_V = \frac{2\beta^2 (\beta^2 + \lambda^2)}{(\beta^2 - \lambda^2)^2} \quad (6.145)
\]

\[
C_V = \frac{2\beta^2 (\beta^2 + \lambda^2)}{(\beta^2 - \lambda^2)^2} \quad (6.146)
\]

**Quantum field** The spectral zeta function, by Eq. (3.10), reads

\[
\zeta(s) = \frac{1}{2} e^{-\lambda\mu} \mu^{-s} \left( -\lambda\mu E_s(-\lambda\mu) + e^{2\lambda\mu}(\lambda\mu)^s \Gamma(1 - s, \lambda\mu) - 2i \sin(\pi s) \Gamma(1 - s)(\lambda\mu)^s \right), \quad (6.147)
\]

where \( E_n(z) \) gives the exponential integral function

The one-loop effective action by Eq. (5.6) reads

\[
W = \frac{1}{4} e^{-\lambda\mu} \left( 2 e^{\lambda\mu} \ln \mu + \Gamma(0, -\lambda\mu) + e^{2\lambda\mu} \Gamma(0, \lambda\mu) + 2i \pi \right). \quad (6.148)
\]

The vacuum energy can be obtained by Eq. (5.17):

\[
E_0 = \mu. \quad (6.149)
\]
6.2.5 Cauchy distribution

In statistics the Cauchy distribution is often used as an example of a pathological distribution, for it does not have finite moments of order greater than one, but only has fractional absolute moments [18]. The Cauchy distribution appears in for example holographic dual problems [55] and lasers [56].

The probability density function of the Cauchy distribution is [18]

\[ p(x) = \frac{\sigma}{\pi ((x - \mu)^2 + \sigma^2)}, \tag{6.150} \]

where \( \mu \) is location parameter and \( \sigma \) is scale parameter.

The cumulative distribution function, by Eq. (3.2), reads

\[ P(x) = \frac{1}{\pi} \tan^{-1} \left( \frac{x - \mu}{\sigma} \right) + \frac{1}{2}. \tag{6.151} \]

**Thermodynamics** In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \( p(x) \),

\[ f(k) = e^{ik\mu - \sigma |k|}. \tag{6.152} \]

The heat kernel, by performing the Wick rotation \( k \rightarrow it \) to the characteristic function, reads

\[ K(t) = e^{-i\mu - \sigma |t|}. \tag{6.153} \]

The thermodynamics corresponding to the Cauchy distribution can be then constructed. The partition function, by replacing \( t \) with \( \beta \) in the heat kernel, reads

\[ Z(\beta) = e^{-\beta(\mu + \sigma)}. \tag{6.154} \]

Then the free energy

\[ F = \mu + \sigma \tag{6.155} \]

and the internal energy

\[ U = \mu + \sigma. \tag{6.156} \]

**Quantum field** The spectral zeta function, by Eq. (2.25), reads

\[ \zeta(s) = (\mu + i\sigma)^{-s}. \tag{6.157} \]

The one-loop effective action by Eq. (5.6) reads

\[ W = \frac{1}{2} \ln(\mu + i\sigma). \tag{6.158} \]

The vacuum energy can be obtained by Eq. (5.17):

\[ E_0 = \mu + i\sigma. \tag{6.159} \]
Renormalization  In this section, we show that the renormalization procedure can remove the divergence appears in the one-loop effective action and the result coincides with the renormalized one-loop effective action obtain by the standard quantum field method.

Consider a standard Cauchy distribution which is the distribution (6.150) with \( \mu = 0 \) and \( \sigma = 1 \),

\[
p(x) = \frac{1}{\pi (x^2 + 1)},
\]

(6.160)

By Eq. (5.7), directly calculation gives

\[
W = \int_{-\infty}^{\infty} \frac{1}{\pi (\lambda^2 + 1)} \ln \lambda d\lambda
\]

\[
= \frac{i\pi}{2},
\]

(6.161)

The one-loop effective action can also be obtained by the characteristic function. The result can be obtained by Eq. (5.15) with a renormalization treatment.

Substituting the distribution (6.160) into Eq. (5.15) gives

\[
W = \left( -\gamma_E + \frac{i\pi}{2} \right) f(0) + \int_{-\infty}^{0} \frac{f(k)dk}{k}.
\]

(6.162)

The characteristic function of the standard Cauchy distribution is

\[
f(k) = e^{-|k|},
\]

(6.163)

so

\[
W = -\gamma_E + \frac{i\pi}{2} + \int_{-\infty}^{0} \frac{e^k}{k}dk.
\]

(6.164)

The integral diverges since \( k = 0 \) is singular. Cutting off the upper limit of the integral gives

\[
\int_{-\infty}^{\varepsilon} \frac{e^k}{k}dk = \text{Ei} (\varepsilon),
\]

(6.165)

where \( \text{Ei} (z) \) is the exponential integral function. Expanding at \( \varepsilon \to 0 \), we arrive at

\[
\int_{-\infty}^{\varepsilon} \frac{e^k}{k}dk = \text{Ei} (\varepsilon) = \gamma_E + \ln (-\varepsilon).
\]

(6.166)

Dropping the divergent term \( \ln (-\varepsilon) \), we have

\[
\int_{-\infty}^{0} \frac{e^k}{k}dk \bigg|_{\text{reg}} = \gamma_E.
\]

(6.167)

Then we obtain a renormalized one-loop effective action

\[
W|_{\text{reg}} = -\gamma_E + \frac{i\pi}{2} + \int_{-\infty}^{0} \frac{e^k}{k}dk \bigg|_{\text{reg}}
\]

\[
= \frac{i\pi}{2},
\]

(6.168)

which agrees with Eq. (6.161).

The validity of the above renormalization procedure is discussed in Ref. [10].
6.2.6 \textit{q-Gaussian distribution}

The \textit{q-Gaussian} distribution is a generalization of the Gaussian distribution as well as that the Tsallis entropy is a generalization of the standard Boltzmann-Gibbs entropy or the Shannon entropy \cite{57}. The \textit{q-Gaussian} distribution is an example of the Tsallis distribution arises from the maximization of the Tsallis entropy under some appropriate constraints.

The probability density function of the \textit{q-Gaussian} distribution is \cite{58}

\begin{equation}
p(x) = \frac{\sqrt{b}}{C_q} e_q \left(-b x^2\right),
\end{equation}

where \(e_q(x) = [(1-q)x + 1]^\frac{1}{q-1}\) is the \textit{q}-exponential, \(q\) is deformation parameter, \(b\) is positive real number, and the normalization factor \(C_q\) is given by

\begin{equation}
C_q = \begin{cases}
\frac{2\sqrt{\pi} \Gamma \left(\frac{1}{1-q}\right)}{(3-q) \sqrt{1-q} \Gamma \left(\frac{3-q}{2(1-q)}\right)}, & -\infty < q < 1, \\
\frac{\sqrt{\pi} \Gamma \left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1} \Gamma \left(\frac{1}{q-1}\right)}, & 1 < q < 3,
\end{cases}
\end{equation}

where \(x \in (-\infty, +\infty)\) for \(1 < q < 3\) and \(x \in \left(-\frac{1}{\sqrt{b(1-q)}}, \frac{1}{\sqrt{b(1-q)}}\right)\) for \(q < 1\) with \(b\) the scale parameter and \(q\) the deformation parameter. For \(q = 1\), the \textit{q-Gaussian} distribution reduces to the Gaussian distribution. For \(q = 2\) and \(\sigma = \sqrt{\frac{1}{b}}\), the \textit{q-Gaussian} distribution reduces to the Cauchy distribution.

The cumulative distribution function, taking \(1 < q < 3\) as an example, by Eq. (3.2), reads

\begin{equation}
P(x) = \begin{cases}
x \sqrt{b^{(q-1)} \Gamma \left(\frac{1}{q-1}\right)} \frac{2F_1 \left(\frac{1}{2}; q-1; 3; -b(q-1)x^2\right)}{\sqrt{\pi} \Gamma \left(\frac{1}{q-1} - \frac{1}{2}\right)} + \frac{1}{2}, & 1 < q < 3, \\
x \sqrt{b-bq \Gamma \left(\frac{3}{2} + \frac{1}{1-q}\right)} \frac{2F_1 \left(\frac{1}{2}; q-1; 3; -b(q-1)x^2\right)}{\sqrt{\pi} \Gamma \left(1 + \frac{1}{1-q}\right)} + \frac{1}{2}, & q < 1,
\end{cases}
\end{equation}

where \(2F_1(a; b; c; z)\) is the hypergeometric function and \(\text{erf}(z)\) is the error function.

\textbf{Thermodynamics} In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \(p(x)\):

\begin{equation}
f(k) = \frac{\Gamma \left(-\frac{3}{2(q-1)}\right)}{\Gamma \left(-\frac{q-3}{2(q-1)}\right)} K_{\frac{q-3}{2(q-1)}} \left(\frac{|k|}{\sqrt{b(q-1)}}\right).
\end{equation}
The heat kernel, by performing the Wick rotation $k \rightarrow it$ to the characteristic function, reads

$$K(t) = \frac{2^{1+\frac{3}{2}} [b(q-1)/t^2]^{\frac{q-3}{4(q-1)}}}{\Gamma \left( \frac{1}{q-1} - \frac{3}{2} \right) K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)}.$$  \hspace{1cm} (6.172)

The thermodynamics corresponding to the $q$-Gaussian distribution can be then constructed. The partition function, by replacing $t$ with $\beta$ in the heat kernel, reads

$$Z(\beta) = 2^{1+\frac{3}{2}} \left[ \frac{b(q-1)}{\beta^2} \right]^{\frac{q-3}{4(q-1)}} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \frac{1}{\Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)}.$$  \hspace{1cm} (6.173)

Then the free energy

$$F = -\frac{1}{\beta} \ln \left( \frac{2^{1+\frac{3}{2}} \left[ \frac{b(q-1)}{\beta^2} \right]^{\frac{q-3}{4(q-1)}} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)}{2\sqrt{b(q-1)} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)} \right),$$  \hspace{1cm} (6.174)

the internal energy

$$U = \frac{K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) + K_{\frac{3}{2} + \frac{1}{2-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)}{2\sqrt{b(q-1)} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)} + \frac{q-3}{2\beta(q-1)},$$  \hspace{1cm} (6.175)

the entropy

$$S(\beta) = \frac{1}{4\sqrt{b(q-1)^{3/2} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)}} \left\{ \sqrt{b(q-1)} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \right\} \times \left\{ 4(q-1) \ln \left( \frac{2^{1+\frac{3}{2}} \left[ \frac{b(q-1)}{\beta^2} \right]^{\frac{q-3}{4(q-1)}} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)}{2\sqrt{b(q-1)} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)} \right) + q-3 \right\} + 2\beta(q-1) \left[ K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) + K_{\frac{3}{2} + \frac{1}{2-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \right] + \beta(q-3) \sqrt{b(q-1)} K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right),$$  \hspace{1cm} (6.176)

and the specific heat

$$C_V = \frac{q-3}{2(q-1)} + \frac{\beta^2}{2b(q-1)} + \frac{\beta^2 \left[ K_{\frac{1}{1-q} - \frac{1}{2}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) + K_{\frac{1}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \right]}{4b(q-1) K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)} - \frac{\beta^2 \left[ K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) + K_{\frac{3}{2} + \frac{1}{2-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right) \right]^2}{4b(q-1) K_{\frac{3}{2} + \frac{1}{1-q}} \left( \frac{\beta}{\sqrt{b(q-1)}} \right)^2},$$  \hspace{1cm} (6.177)
Quantum field The spectral zeta function, by Eq. (3.10), reads
\[
\zeta(s) = \frac{[1 + (-1)^s] \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) [b(q - 1)]^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{q-1} - \frac{1}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}. \tag{6.178}
\]
When \(s\) is a integer
\[
\zeta(s) = \frac{[1 + (-1)^s] \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) [b(q - 1)]^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{q-1} - \frac{1}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}; \tag{6.179}
\]
when \(s\) is even
\[
\zeta(s) = \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right) [b(q - 1)]^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{q-1} - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}; \tag{6.180}
\]
when \(s\) is odd
\[
\zeta(s) = \frac{[1 + (-1)^s] \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) [b(q - 1)]^{s/2} \Gamma\left(\frac{s}{2} + \frac{1}{q-1} - \frac{1}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)} = 0. \tag{6.181}
\]
The one-loop effective action by Eq. (5.6) reads
\[
W = \frac{1}{4} \left[ - \ln(4b(q - 1)) - H_{\frac{1}{q-1}-\frac{3}{2}} + \pi \right], \tag{6.182}
\]
where \(H_n\) gives the \(n\)th harmonic number.

The vacuum energy can be obtained by Eq. (5.17):
\[
E_0 = 0. \tag{6.183}
\]

6.2.7 Intermediate distribution

The intermediate distribution is a distribution linking the Gaussian and the Cauchy distribution [9], which reduces to the Gaussian and the Cauchy distribution at some special values of the parameter. The intermediate distribution can be applied to spectral line broadening in laser theory and the stock market return in quantitative finance [9].

The probability density function of the intermediate distribution is [9]
\[
p(x, \mu, \sigma, \nu) = \frac{1}{\nu \pi} e^{\left(1/\nu - 1\right) \sigma^2} \int_0^1 \cos\left(\frac{x - \mu}{\nu} \ln t\right) t^{\sigma^2 - 1} \exp\left(\left(1 - \frac{1}{\nu^2}\right) \sigma^2 t\right) dt. \tag{6.184}
\]

Thermodynamics In order to obtain the thermodynamic quantity, we first calculate the characteristic function by Eq. (3.5), which is a Fourier transformation of the density function \(p(x)\),
\[
f(k) = \exp\left(ik\mu + (-1 + e^{-\nu|k|}) \left(1 - \frac{1}{\nu}\right) \sigma^2 - \frac{\sigma^2 |k|}{\nu}\right). \tag{6.185}
\]
The heat kernel, by performing the Wick rotation $k \rightarrow it$ to the characteristic function, reads
\[ K(t) = \exp \left( -t\mu + \frac{(-1 + e^{-\nu|t|})}{\nu} (-1 + \nu)\sigma^2 - \frac{\sigma^2|t|}{\nu} \right). \] (6.186)

The thermodynamics corresponding to the intermediate distribution can be then constructed. The partition function, by replacing $t$ with $\beta$ in the heat kernel, reads
\[ Z(\beta) = \exp \left( -\beta \left( \mu + \frac{\sigma^2}{\nu} \right) + \frac{\sigma^2}{\nu} (\nu - 1) \left( e^{-\beta\nu} - 1 \right) \right). \] (6.187)

Then the free energy
\[ F = \mu + \frac{\sigma^2}{\nu} - \frac{(\nu - 1)\sigma^2}{\beta\nu} \left( e^{-\beta\nu} - 1 \right), \] (6.188)
the internal energy
\[ U = \mu + \left[ e^{-\beta\nu}(\nu - 1) + \frac{1}{\nu} \right] \sigma^2, \] (6.189)
the entropy
\[ S = e^{-\beta\nu}(\nu - 1)\beta\sigma^2 + \frac{(\nu - 1)\sigma^2}{\nu} \left( e^{-\beta\nu} - 1 \right), \] (6.190)
and the specific heat
\[ C_V = e^{-\beta\nu} \beta^2 (\nu - 1)\nu\sigma^2. \] (6.191)

Quantum field The spectral zeta function, by Eq. (3.10), reads
\[ \zeta(s) = \frac{1}{\pi} \nu^{-s} \sin(\pi s) \Gamma(1-s) \exp \left( -i\pi s \frac{1}{2} + \frac{\sigma^2(1 - \nu^2)}{\nu^2} \right) \int_0^1 \exp \left( t \frac{\sigma^2(\nu^2 - 1)}{\nu^2} \right) t^{\frac{\mu}{\nu} + \frac{\sigma^2}{\nu^2} - 1} (\ln t)^{s-1} dt. \] (6.192)
\[ \zeta(s) = \frac{1}{\pi} \nu^{-s} \sin(\pi s) \Gamma(1-s) \exp \left( -i\pi s \frac{1}{2} + \frac{\sigma^2(1 - \nu^2)}{\nu^2} \right) \int_0^1 \exp \left( t \frac{\sigma^2(\nu^2 - 1)}{\nu^2} \right) t^{\frac{\mu}{\nu} + \frac{\sigma^2}{\nu^2} - 1} (\ln t)^{s-1} dt. \] (6.193)

The one-loop effective action by Eq. (5.7) reads
\[ W = \frac{1}{2} e^{-\sigma^2(1-1/\nu)} \int_0^1 \exp \left( t \frac{\sigma^2(\nu^2 - 1)}{\nu^2} + \left( \frac{i\mu}{\nu} + \frac{\sigma^2}{\nu^2} - 1 \right) \ln t \right) \frac{1}{\ln t} dt. \] (6.194)

The vacuum energy can be obtained by Eq. (5.19):
\[ E_0 = \mu - i\sigma^2 \left( 1 - \frac{1}{\nu} - \nu \right). \] (6.195)

7 Conclusions and outlook

In this paper, we suggest a scheme for constructing probability thermodynamics and quantum fields. In the scheme, we suppose that there is a fictional system whose eigenvalues obey a probability distribution. Most characteristic quantities in thermodynamics and quantum
field theory are spectral functions, such as various thermodynamics qualities and effective actions.

Our starting point is to analytically continue the heat kernel which is the partition function in thermodynamics and the vacuum amplitude in quantum field theory. We note here that the Wick-rotated heat kernel is a special case of the analytic continuation of the heat kernel,

\[ K(z) = \sum_{n=-\infty}^{\infty} e^{\lambda_n z}, \]  

which is defined in the complex plane with a complex number \( z \). By Eq. (7.1), we analytically continue the heat kernel to the \( z \)-complex plane. When \( z \) is a pure imaginary number, the complex variable heat kernel, \( K(z) \), returns to Eq. (2.2). By the complex variable heat kernel, \( K(z) \), we can discuss the analyticity property of the heat kernel on the complex \( z \)-plane.

In the scheme, we suppose that the eigenvalue obeys a probability distribution without giving the operator. In further work, we can consider what an operator corresponds a eigenvalue spectra obeying a certain given probability distribution and find operators for various probability distributions. As a rough example, the Gaussian distribution, \( p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \), corresponds to a Laplace operator \( \Delta \). The negative Laplace operator \( -\Delta \) is lower bounded, corresponding to nonnegative random variables and the positive Laplace operator \( +\Delta \) is lower unbounded, corresponding to negative random variables.

The work of this paper bridges two fundamental theories: probability theory and heat kernel theory. The heat kernel theory is important in both physics and mathematics. In physics, the heat kernel theory plays an important role in quantum field theory \([2, 59–64]\). Scattering spectral theory is an important theory in quantum field theory \([65–68]\). The relation between heat kernels and scattering phases bridges the heat kernel theory, the scattering spectral theory, and scattering theory \([67, 69–71]\). This inspires us to relate the probability theory and the scattering theory which is an important issue in mathematical physics \([72]\).

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