Integral bases and invariant vectors for Weil representations

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Abstract

We construct, for the Weil representation associated with any discriminant form, an explicit basis in which the action of the representation involves algebraic integers over its field of definition. The action of a general element of $\text{SL}_2(\mathbb{Z})$ on many parts of these bases is simple and explicit, a fact that we use for determining the dimension of the space of invariants for some families of discriminant forms.

Keywords: Weil Representations, Representations over Rings, Invariant Subspaces

Introduction

Let $D$ be a discriminant form, also known as finite quadratic module. Then there is a representation $\rho_D$ of $\text{SL}_2(\mathbb{Z})$ (or sometimes its double cover), called the Weil representation associated with $D$ on the space $\mathbb{C}[D]$. It is defined on the generators of that group by Eq. (2). These representations are essentially the case of the finite group $D$ in the general theory of Weil representations, initiated in [14].

These Weil representations now form an important technical tool in the theory of modular forms. Indeed, when the discriminant form $D$ comes from an even lattice $L$, the most natural way to present the theta function of $L$ is as a vector-valued modular form of level 1 and the Weil representation associated with $D$. This allows for many modularity proofs to be reduced to the case of level 1, where the group is $\text{SL}_2(\mathbb{Z})$ (or its double cover), with well-known generators and relations.

Given a discriminant form $D$, the subspace of $\mathbb{C}[D]$ that is invariant under $\rho_D$ is of particular importance. Indeed, it is the space of holomorphic modular forms of weight 0 (constants) with representation $\rho_D$. As one example of the role it plays, assume that $D$ is the discriminant form of a lattice $L$ of signature $(2, 2)$. Then a weakly holomorphic modular form of weight 0 is determined by its singular part precisely up to this subspace. Thus knowing it resolves certain technical questions involving the corresponding theta lifts.

We shall not give a comprehensive list of references involving these questions. The latter arose, for example, in [3], a reference which cites many papers discussing various forms of theta lifts, and Section 5 of [1] presents the theta lifts of signature $(2, 2)$ arising from these invariant vectors in some important cases. However, we do mention another result, illustrating the importance of Weil representations: The paper [8] shows that every irre-
ducible representation of $\mathrm{SL}_2(\mathbb{Z})$ that factors through a congruence quotient is contained inside some Weil representation of this sort.

A seemingly unrelated question involving Weil representations is the following one. Equation (2) below easily implies that in the natural basis for $\mathbb{C}[D]$, the Weil representation $\rho_D$ is defined over the cyclotomic number field $\mathbb{Q}(\zeta_N)$, where $N$ is the level of $D$. However, the coefficients include denominators. In the 33rd Automorphic Forms Workshop, L. Candelori presented the question of finding an explicit basis for $\mathbb{C}[D]$ in which the coefficients of the Weil representation are algebraic integers. The fact that $\rho_D$ factors through a finite quotient easily implies that this can be done abstractly over a field extension, and some extra observations allows one to prove that for Weil representations no field extension is required (see Remark 4.8). Thus, the precise formulation of our question is: Can one explicitly write a basis for $\mathbb{C}[D]$ such that the action of $\rho_D$ in that basis involves only algebraic integral coefficients from $\mathbb{Z}[\zeta_N]$? Moreover, can one do it in such a way that the action of $\rho_D$ has a simple description?

The main result of this paper (Theorem 4.7 below) solves this question entirely. Moreover, away from some small local representations, a typically large sub-representation of $\rho_D$ is supported on basis vectors such that the action of every group element takes one basis vector to another basis vector multiplied by a root of unity (or, more precisely, a power of $\zeta_N$). The only parts where the representations become more complicated, and less explicit in terms of the action of arbitrary elements of $\mathrm{SL}_2(\mathbb{Z})$, are those essentially coming from prime discriminants.

For explaining the basic idea, note that the inversion element $S \in \mathrm{SL}_2(\mathbb{Z})$ takes, under $\rho_D$, a natural basis element $\epsilon_\gamma$ of $\mathbb{C}[D]$ (with $\gamma \in D$) to a certain sum over all the $\epsilon_\gamma$’s, with coefficients that are based on $\gamma$. By taking a subgroup $H$ of $D$, one defines in Eq. (3) intermediate vectors between these two types of vectors. For the correct choice of $H$, these vectors produce bases with the properties that we seek.

As an example, assume that $H$ is a self-dual isotropic subgroup of $D$. In this case Theorem 2.2 shows that any element of $\mathrm{SL}_2(\mathbb{Z})$ takes such a vector associated with $H$ to another such vector associated with $H$, times an explicit root of unity. Moreover, the action on the indices of these vectors is just the action of $\mathrm{SL}_2(\mathbb{Z})$ on vectors of length 2 in $D$. For a particular case where we can give the basis and formulae explicitly, assume that there is a subgroup $\hat{H}$ of $D$ that maps surjectively onto $D/H$. Then the vectors $a_{\eta, \lambda}^H$ from Eq. (3), with $\eta$ and $\lambda$ from $\hat{H}$, form a basis for $\mathbb{C}[D]$, and in this basis the action of a matrix $M = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z})$ takes $a_{\eta, \lambda}^H$ to $e(Q(M, \nu))a_{\eta, \lambda}^H$, where $Q(M, \nu)$ is the expression defined in Eq. (4) for our matrix $M$ and for $\nu := \eta \lambda^t$.

Assume now that $H$ is isotropic, yielding a quotient of exponent $p$, and let $J \subseteq H$ be of size $p$, with quotient $B$ (of size $|D|/p^2$). In this case $\mathbb{C}[B]$ embeds into $\mathbb{C}[D]$ as a sub-representation via the arrow operator from [2] and others, and Theorem 3.10 shows (via Remark 3.11) that its orthogonal complement inside $\mathbb{C}[D]$ also admits a basis with a similar action (but up to some 8th root of unity that are harder to determine). This reduces the question to anisotropic discriminant forms, whose $p$-parts are given in, e.g., [18], and solving these cases yields the final result. We remark that in the prime discriminant case we do not obtain the formula for a general element in this basis.

Finally, the fact that in large enough parts of the representations we have a closed formula for the action of a general element allows one to use the classical formula of Frobenius to determine the dimension of the space of invariants in $\mathbb{C}[D]$ discussed above.
Note that some properties of this space are known in general: In relation to our integrality question, the result from [5] proves that $\mathbb{C}[D]^{\text{inv}}$ is defined over $\mathbb{Z}$. Moreover, any self-dual isotropic subgroup yields a 1-dimensional subspace of $\mathbb{C}[D]$, and a theorem of [7] (also proven in [1]) shows that when such subgroups exist, these spaces generate $\mathbb{C}[D]^{\text{inv}}$. While determining this dimension in this method might still present difficulties in general, we obtain the formula for $\dim \mathbb{C}[D]^{\text{inv}}$ for generalized hyperbolic planes in Theorem 5.4, and for discriminants of prime level in Theorem 5.6. The generators, in the particular case of the latter discriminants in which the space of invariants is 1-dimensional, are used (among some additional variants) in the recent pre-print [6] for generating the invariants in any discriminant form, in the spirit of the result of [7] and [1] but when no self-dual isotropic subgroups exist.

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The paper is divided into 5 sections. Section 1 introduces the Weil representations and the vectors that we later use for our bases. Section 2 proves the formula for self-dual isotropic and quasi-isotropic subgroups, and Sect. 3 establishes the result involving quotients of prime level. Then Sect. 4 produces the construction of integral bases in general, and finally Sect. 5 shows how to apply these formulae to the determination of the dimension of the space of invariant vectors.

1 Weil representations and subgroups

The group $\text{SL}_2(\mathbb{Z})$ is known to be generated by the elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, whose only relations are $S^2 = (ST)^3 = Z$ and $Z^2 = I$ (the matrix $Z$ is minus the identity matrix). It admits a non-trivial double cover, denoted by $M_p^2(\mathbb{Z})$, which is generated by appropriate lifts of $T$ and $S$, satisfying the first relation but in which $Z$ now has order 4 (and $Z^2$ generates the kernel of the projection onto $\text{SL}_2(\mathbb{Z})$, of order 2).

Let $D$ be a discriminant form, also called a finite quadratic module, namely a finite group with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form, which we write as $\gamma \mapsto \frac{\gamma^2}{2}$. It induces the $\mathbb{Q}/\mathbb{Z}$-valued bilinear form $(\gamma, \delta) = \frac{(\gamma + \delta)^2}{2} - \frac{\gamma^2}{2} - \frac{\delta^2}{2}$, which we assume to be non-degenerate. This identified $D$ with its $\mathbb{Q}/\mathbb{Z}$-dual, and there is a signature $\text{sgn} D \in \mathbb{Z}/8\mathbb{Z}$ which can be defined using Milgram’s formula

$$\sum_{\gamma \in D} e\left(\frac{\gamma^2}{2}\right) = e(\text{sgn} D/8) \cdot \sqrt{|D|}.$$ (1)

To $D$ we associate the Weil representation $\rho_D$ of $M_p^2(\mathbb{Z})$ onto the space $\mathbb{C}[D]$, with the natural basis $\{e_\gamma\}_{\gamma \in D}$, via the formulae

$$\rho_D(T)e_\gamma = e\left(\frac{\gamma^2}{2}\right)e_\gamma \quad \text{and} \quad \rho_D(S)e_\gamma = \frac{e(-\text{sgn} D/8)}{\sqrt{|D|}} \sum_{\delta \in D} e\left(-\frac{(\gamma, \delta)}{2}\right)e_\delta,$$ (2)
where \( e(x) \) stands for \( e^{2\pi ix} \). The fact that it is a representation can be proved using Milgram's formula, and it factors through a representation of \( SL_2(\mathbb{Z}) \) if and only if \( \text{sgn } D \) is even (otherwise \( \mathbb{Z}^2 \) acts as \( -1 \)). We endow \( \mathbb{C}[D] \) with the inner product in which the basis \( \{ e_\gamma \}_{\gamma \in D} \) is orthonormal, and then \( \rho_D \) becomes a unitary representation.

Recall that the level of \( D \) is the minimal integer \( N \) such that \( N \frac{\gamma^2}{2} = 0 \) in \( \mathbb{Q}/\mathbb{Z} \) for every \( \gamma \in D \), and set \( \zeta_N \) to be a primitive \( N \)th root of unity (e.g., \( \zeta_N = e^{i\pi/N} \)). We get the following simple result.

**Proposition 1.1** Using the canonical basis, the representation \( \rho_D \) is defined over the cyclotomic field \( \mathbb{Q}(\zeta_N) \).

**Proof** The definition of the level \( N \) implies that for every \( \gamma \) and \( \delta \) in \( D \), the coefficients \( e\left(\frac{\gamma^2}{2}\right) \) and \( e(-\langle \gamma, \delta \rangle) \) from Eq. (2) lie in \( \mathbb{Q}(\zeta_N) \). For the coefficient \( e(-\text{sgn } D/8)/\sqrt{|D|} \) in \( \rho_D(S) \) there, this can be verified locally, but it also follows from the reciprocal of this number appearing on the right hand side of Eq. (1). This proves the proposition. \( \square \)

Given any subgroup \( H \) of \( D \), the subgroup

\[
H^\perp = \{ \gamma \in D : \langle \gamma, \delta \rangle = 0 \ \forall \delta \in H \}
\]

has index \( |H| \) in \( D \), and the \( \mathbb{Q}/\mathbb{Z} \)-dual of \( H \) is \( D/H \perp \). For such \( H \) and elements \( \eta \) and \( \lambda \) in \( D \) we define

\[
a_{H,\lambda}^\eta := \frac{1}{\sqrt{|H|}} \sum_{\gamma \in H} e(\langle \gamma, \eta \rangle) e_{\lambda + \gamma}.
\]  

(3)

The first properties of these vectors are given in the following lemma.

**Lemma 1.2** The vector from Eq. (3) depends only on the image of \( \eta \) modulo \( H^\perp \), and adding \( \delta \) in \( H \) to \( \lambda \) multiplies \( a_{H,\lambda}^\eta \) by \( e(-\langle \delta, \eta \rangle) \). Choosing a set \( \mathcal{R} \) of representatives for \( D/H \) in \( D \), the set \( \{ a_{H,\lambda}^\eta \}_{\eta \in D/H, \lambda \in \mathcal{R}} \) is an orthonormal basis for \( \mathbb{C}[D] \).

**Proof** The dependence on \( \eta \) in Eq. (3) is by pairings with \( H \), and adding \( \delta \in H \) to \( \lambda \) results in a summation index change. The pairing of \( a_{H,\lambda}^\eta \) with \( a_{K,\nu}^\lambda \) for \( \nu \neq \lambda \in \mathcal{R} \) (i.e., \( \nu \neq \lambda + H \)) is based on disjoint subsets of \( \{ e_\gamma \}_{\gamma} \), and if \( \nu = \lambda \) the pairing is \( \frac{1}{|H|} \sum_{\gamma \in H} e(-\langle \nu, \gamma \rangle) \). Since this equals 1 when \( \nu \in H \) and vanishes otherwise, the orthonormality follows. This proves the lemma. \( \square \)

In particular, if \( \eta \in H^\perp \) then \( a_{H,\lambda}^\eta \) is invariant under replacing \( \lambda \) by \( \lambda + \delta \) with \( \delta \in H \).

**Remark 1.3** Each basis vector \( e_\gamma \) can be written as \( a_{\gamma,0}^{(0)} \) via Eq. (3), where the independence of \( \eta \in D \) and the orthonormality are generalized in Lemma 1.2. Moreover, we can rewrite Eq. (2) as stating that \( \rho_D(T) \) and \( \rho_D(S) \) take \( e_\gamma = a_{\gamma,0}^{(0)} \) to \( e(\langle \lambda^2/2 \rangle a_{\gamma,0}^{(0)} \) and \( e(-\text{sgn } D/8)a_{\gamma,0}^{(0)} \) respectively. We soon generalize these formulae in Lemma 1.7 below.

A subgroup \( H \) of \( D \) is called quasi-isotropic if \( \langle \gamma, \delta \rangle = 0 \) for every \( \gamma \) and \( \delta \) in \( H \), namely, if \( H \subseteq H^\perp \), and isotropic when \( \langle \gamma^2/2 \rangle = 0 \) for every \( \gamma \in H \). Any isotropic subgroup is quasi-isotropic, and the isotropic condition is equivalent to the quotient \( A := H^\perp/H \) inheriting from \( D \) a natural structure of a discriminant form. This discriminant form is non-degenerate, and appropriately gathering elements in Milgram’s formula from Eq. (1) shows that it has the same signature as \( D \). The following lemma is an immediate consequence of the definition.
**Lemma 1.4** The subgroup $H$ is quasi-isotropic if and only if $\gamma \mapsto \frac{\chi^2}{\gamma}$ is linear on $H$, namely there is $\xi_H \in D$ such that $\frac{\chi^2}{\gamma} = (\gamma, \xi_H)$ for every $\gamma \in H$. The element $\xi_H$ is unique modulo $H^\perp$, and its image in $D/H^\perp$ is trivial if $H$ is isotropic and has order 2 otherwise.

We shall also use the following consequence of Lemma 1.4.

**Corollary 1.5** Every quasi-isotropic subgroup $H$ of $D$ contains a unique maximal isotropic subgroup $H_0$. It equals $H$ when $H$ is isotropic, and has index 2 there otherwise. The associated subgroup $H_0^\perp$ is the union of $H^\perp \cup (\xi_H + H^\perp)$.

Indeed, the subgroup $H_0$ from Corollary 1.5 is just the kernel of the pairing with $\xi_H$ from Lemma 1.4.

The fact that a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form is linear if and only if the associated bilinear form vanishes, and thus the quadratic form takes values in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, extends Lemma 1.4 as follows.

**Lemma 1.6** If $H$ is a subgroup of $D$ and $l \in \mathbb{Z}$ is such that $(\gamma, \delta) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ for every $\gamma$ and $\delta$ in $H$ then there exists an element $\xi_{lH} \in D$, unique modulo $H^\perp$, such that $l\chi^2 = (\gamma, \xi_{lH})$ for all $\gamma \in H$.

The element $\xi_{lH}$ from Lemma 1.6 is closely related to the element denoted by $x_c$ in [11], [12], and [17], with $c = l$.

Most of our calculations will be based on evaluating $\rho_D$ on the vectors from Eq. (3). On the generators we get the following result.

**Lemma 1.7** For every $\lambda$ and $\eta$ we have the equality

$$\rho_D(S)a_{\eta, \lambda}^H = e(-\text{sgn } D/8)e(- (\lambda, \eta))a_{\eta, \lambda}^H.$$  

If $l \in \mathbb{Z}$ is as in Lemma 1.6 and $\xi_{lH}$ is the resulting element, then we get

$$\rho_D(T^l)a_{\eta, \lambda}^H = e(l\chi^2_2)_{\eta, \lambda, +\xi_{lH}}.$$

**Proof** Unfolding the definitions from Eqs. (2) and (3) and recalling that $|H^\perp| = \frac{|D|}{|H|}$ expresses $\rho_D(S)a_{\eta, \lambda}^H$ as $e(-\text{sgn } D/8)/\sqrt{|H^\perp|}$ times

$$\sum_{\gamma \in H} \sum_{\beta \in D} e((\gamma, \eta) - (\lambda + \gamma, \beta)) = \sum_{\beta \in D} e(- (\lambda, \beta))e_{\eta, \lambda + H^\perp} \epsilon_{\beta}.$$

Then writing $\beta$ as $\eta + \sigma$ with $\sigma \in H^\perp$ and applying Eq. (3) (with $H^\perp$) yields the first expression. For the second one, the multiplier $e(l\chi^2_2/2)$ in front of $e_{\eta, \lambda}$ in $\rho_D(T^l)a_{\eta, \lambda}^H$ expanded as in Eq. (3) equals $e(l\chi^2_2)$ times $e((l\lambda + \xi_{lH}, \gamma))$, and the result follows. This proves the lemma.

Note that the two factors in the expression for $\rho_D(S)a_{\eta, \lambda}^H$ in Lemma 1.7 may depend on $\eta \in D$, but Lemma 1.2 shows that their product is well-defined for $\eta \in D/H^\perp$. Moreover, the case $H = \{0\}$ of Lemma 1.7 reproduces the presentation from Remark 1.3.

**Remark 1.8** Lemma 1.7 suggests that a better indexing for the vector $a_{\eta, \lambda}^H$ from Eq. (3) is using the vector $v := \binom{\eta}{\lambda} \in D^2 = \mathbb{Z}^2 \otimes D$, on which matrices in $\text{SL}_2(\mathbb{Z})$ have a natural action. Then the first formula there reads $\rho_D(S)a_v^H = e(-\text{sgn } D/8)e(- (\lambda, \eta))a_v^H$, and when the element $\xi_{lH}$ from Lemma 1.6 vanishes, the second formula there becomes
\( \rho_D(T^I) a^H_{\nu} = e\left(\frac{\xi_{I,H}}{2}\right) a^H_{T^I \nu} \). With a general \( \xi_{I,H} \), we can set \( T^I \ast \nu := T^I \nu + \left(\frac{\xi_{I,H}}{0}\right) \), and get \( \rho_D(T^I) a^H_{\nu} = e\left(\frac{\xi_{I,H}}{2}\right) a^H_{T^I \nu} \).

The case of isotropic \( H \) with \( \lambda \) and \( \eta \) in \( H^\perp \) in Lemma 1.7 reproduces the following operator, from [2] and others, in which we denote again the discriminant form \( H^\perp / H \) by \( A \).

**Corollary 1.9** The map \( \uparrow_H : \mathbb{C}[A] \to \mathbb{C}[D] \) that is defined by the linear extension of

\[ \uparrow_H \epsilon_{\mu} := \frac{1}{|H|} \sum_{\gamma \in H^+, \gamma + H = \mu} \epsilon_\gamma \]

is an isometric map of representations, embedding \( \rho_A \) into \( \rho_D \).

**Proof** The vector \( \uparrow_H \epsilon_{\mu} \) is, in the notation of Eq. (3), just \( a^H_{\mu,\lambda} \), where \( \eta \) and \( \lambda \) in \( H^\perp \) with \( \lambda + H = \mu \). These vectors, for \( \mu \in A \), are orthonormal vectors that are independent of the representatives (by Lemma 1.2). The result now follows from the formulae from Lemma 1.7, together with the fact that \( \text{sgn} A = \text{sgn} D \) and the vector \( a^H_{\lambda,\eta} \) can be viewed, as in Remark 1.3, as \( \frac{1}{\sqrt{|A|}} \sum_{\tau \in H^+/H} a^H_{\tau,\eta,\tau} = \frac{1}{\sqrt{|A|}} \sum_{\tau \in H^+/H} \uparrow_H \epsilon_\tau \). This proves the corollary. \( \square \)

**Remark 1.10** Let \( J \) be an isotropic subgroup of \( D \), with the associated discriminant form \( B := J^\perp / J \), and take \( \kappa \) and \( \nu \) in \( B \). It is then clear from Eq. (3) and Corollary 1.9 that for every subgroup \( H \) of \( D \) that contains \( J \) we have \( \uparrow_J a^H_{\kappa,\nu} = a^H_{\eta,\lambda} \) for \( \eta \) and \( \lambda \) in \( J^\perp \) with respective \( B \)-image \( \kappa \) and \( \nu \). In particular this gives the relation \( \uparrow_H = \uparrow_J \circ \uparrow_{H/J} \) of the operators from Corollary 1.9.

### 2 Discriminant forms with trivial quotients

A quasi-isotropic subgroup \( H \) of \( D \) is called **self-dual** if \( H^\perp = H \). Thus for a self-dual isotropic subgroup \( H \), the associated discriminant form \( A \) is trivial. Note that “self-dual” here does not mean that \( H \) is identified with its dual, but rather that when \( D \) is the discriminant of a lattice, \( H \) corresponds to an over-lattice that is self-dual in the usual sense ([1] uses the same term).

If \( H \) is a self-dual isotropic subgroup of \( D \), then the quotient \( A = H^\perp / H \) is a trivial discriminant form. In this case we have the equality \( \text{sgn} D = \text{sgn} A = 0 \), and \( \rho_D \) is a representation of \( \text{SL}_2(\mathbb{Z}) \). Then the simple formulae from Lemma 1.7 are enough for establishing a simple formulae for \( \rho_D(M) \) for every \( M \in \text{SL}_2(\mathbb{Z}) \) using the vectors from Eq. (3).

To do so, write \( \nu := I_{\nu}^H \) for \( \eta \) and \( \lambda \) in \( D \) as in Remark 1.8, and for \( M = (a' \ b' \ c' \ d') \in \text{SL}_2(\mathbb{Z}) \) and such \( \nu \) we set

\[ Q(M, \nu) := ac \frac{\nu^2}{2} + bd \frac{\nu^2}{2} + bc(\lambda, \eta) \tag{4} \]

(such expressions arise naturally in the general theory of Weil representations, by appropriate substitutions in the general expressions from [14]). This expression has the following cocycle property.

**Lemma 2.1** For \( M \) and \( N \) in \( \text{SL}_2(\mathbb{Z}) \) and \( \nu \in D^2 \), the expression from Eq. (4) satisfies the equality \( Q(MN, \nu) = Q(N, \nu) + Q(M, N\nu) \).
Theorem 2.2
Assume that $H$ is a self-dual quasi-isotropic subgroup. Then for $M \in \text{SL}_2(\mathbb{Z})$ and a vector $v \in D^2$ the operator $\rho_D(M)$ sends the vector $a^M_v$ to $\psi(Q(M,v))a^M_{Mv}$.

Proof
The fact that the operation on the vectors is a group action combines with Lemma 2.1 to show that if the formula holds for two matrices $M$ and $N$ and every $v$ then it is valid for the product $MN$. But since $\text{sgn} \ D = 0$, Lemma 1.7 and Remark 1.8 produce the desired formula for $T$, $T^{-1}$, and $S$, which generate $\text{SL}_2(\mathbb{Z})$ multiplicatively, and for every vector $v$. This proves the theorem. 

The following well-known lemma allows us to extend Theorem 2.2 to the case of a self-dual quasi-isotropic subgroup $H$, which is not necessarily isotropic.

Lemma 2.3
The elements $T^2$ and $S$ of the double cover $\text{Mp}_2(\mathbb{Z})$ of $\text{SL}_2(\mathbb{Z})$ generate a subgroup $\Gamma_{\text{odd}}$ of index 3, which is the semi-direct product in which $\langle S \rangle$ acts by conjugation on the free group generated by $T^2$ and $ST^2S^{-1}$. The non-trivial cosets of $\Gamma_{\text{odd}}$ in $\text{Mp}_2(\mathbb{Z})$ are represented by $T$ and $ST$.

Consider now a character $\chi$ of $\Gamma_{\text{odd}}$ that is trivial on the free subgroup from Lemma 2.3, and is thus determined by the 8th root of unity $\chi(S)$. We extend $\chi$ to $\text{Mp}_2(\mathbb{Z})$ by taking elements $TM$ and $STM$ with $M \in \Gamma_{\text{odd}}$ to $\chi(M)$ and $\chi(S)\chi(M)$ respectively, and we define a twisted operation of $\text{Mp}_2(\mathbb{Z})$ on $D^2$ by

$$M \ast v := \begin{cases} Mv, & \text{when } M \in \Gamma_{\text{odd}}; \\ Mv + (\xi_H), & \text{in case } M \in T\Gamma_{\text{odd}}; \\ Mv + (0), & \text{if } M \in ST\Gamma_{\text{odd}}, \end{cases}$$

where $\xi_H$ is the element from Lemma 1.4. We also modify the cocycle from Lemma 2.1, and define, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $v$ as above, the expression

$$\tilde{Q}_H(M, v) := \begin{cases} Q(M, v), & \text{if } M \in \Gamma_{\text{odd}} \cup T\Gamma_{\text{odd}}; \\ Q(M, v) + (\xi_H, a\eta + b\lambda), & \text{when } M \in ST\Gamma_{\text{odd}}. \end{cases}$$

Using these expressions we obtain the following generalization of Theorem 2.2.

Theorem 2.4
Let $H$ be a self-dual quasi-isotropic subgroup of $D$, and consider the map $\chi: \text{Mp}_2(\mathbb{Z}) \to \mathbb{C}^\times$ extending the character of $\Gamma_{\text{odd}}$ that sends $S$ to $e(-\text{sgn} \ D/8)$. Then we have the equality $\rho_D(M)a^H_{Mv} = \chi(M)e(\tilde{Q}_H(M, v))a^H_{Mv}$ for every $M \in \text{Mp}_2(\mathbb{Z})$ and $v \in D^2$, with $M \ast v$ and $\tilde{Q}_H(M, v)$ from Eqs. (5) and (6) respectively.
Remark 2.5 Note that while the expression $M \ast v$ for $M \neq \Gamma_{\text{odd}}$ and $\hat{Q}_H(M, v)$ when $M \in ST\Gamma_{\text{odd}}$ depend on the choice of $\xi_H$, Lemma 1.2 implies that $a^H_{M,v}$, with $M \in T \Gamma_{\text{odd}}$ and the combination $e(\hat{Q}_H(M, v))a^H_{M,v}$ for $M \in ST\Gamma_{\text{odd}}$ only depend on the image of $\xi_H$ in $D/H$, which is canonical by Lemma 1.4. Moreover, if $H$ is isotropic then $\chi$ is trivial and Eqs. (5) and (6) reduce to $M \ast v = Mv$ and Eq. (4) respectively for all $M \in \text{Mp}_2(\mathbb{Z})$, so that Theorem 2.4 reproduces Theorem 2.2 in this case. See Remark 2.6 below for the other cases.

Proof Lemma 1.6 implies that $\xi_{l,H}$ vanishes for even $l$ and equals $\xi_H$ when $l$ is odd. Thus Lemma 1.7 and Remark 1.8 extend the proof of Theorem 2.2 to the case of $M$ from $\Gamma_{\text{odd}} = \langle S, T^2 \rangle$, up to scalar multiples coming from the fact that $\rho_D(S)$ has the additional multiplier $e(-\text{sgn} D/8) = \chi(S)$. As $\rho(S^{-1})$ comes with the inverse multiplier, we deduce that the formula for $M = ST^2 S^{-1}$ involves no additional factors, so that the formula from Theorem 2.2 is valid for elements of the free subgroup from Lemma 2.3. It is now clear that the asserted formula holds for every $M \in \Gamma_{\text{odd}}$ (with our character $\chi$), and note that Remark 1.8 with $l = 1$ and $\xi_{1,H} = \xi_H$ and Eq. (5) give $(TN) \ast v = T \ast Nv$ and $(STN) \ast v = S((TN) \ast v)$ for $N \in \Gamma_{\text{odd}}$. This establishes the desired result when $M = TN \in T \Gamma_{\text{odd}}$, and since the proof of Lemma 2.1 combines with Eqs. (5) and (6) to show that $Q(TN, v) + Q(S, TN \ast v) = \hat{Q}_H(STN, v)$ for such $N$, we obtain the formula for $M = STN \in ST\Gamma_{\text{odd}}$ as well. This completes the proof of the theorem. \qed

Remark 2.6 When $H$ is quasi-isotropic and self-dual, Corollary 1.5 yields the subgroup $H_0$, and the associated discriminant form $A_0 := H_0^+ / H_0$. The latter is trivial when $H$ is isotropic, but otherwise it has order 4. It is therefore either cyclic, where $\chi(S) = e(-\text{sgn} A_0/8)$ is of order 8 and $H_0$ is the unique subgroup of order 2, or isomorphic to the Klein 4-group. In the latter case we need the element $\tau \in A_0$ generating $H/H_0$ to satisfy $\frac{\tau^2}{2} = \frac{1}{2} + \mathbb{Z}$, so that using the notation from [11], [12], [17], and others, $A_0$ is either isomorphic to $2 \mathbb{Z}^2$ with a unique choice of $\tau$ and $\chi(S)$ of order 4, or to $2 \mathbb{H}^2$ with a unique $\tau$ and $\chi(S) = 1$, or to $2 \mathbb{H}_H^2$, where $\chi(S) = -1$ and $\tau$ can be any non-trivial element.

We conclude this section by modifying the vectors from Eq. (3) in order to respect the actions of automorphisms. Let $D$ be any discriminant form, and consider a subgroup $H$ of $D$, a group $G$ of automorphisms of $D$ (that preserve the quadratic form) that fixes $H$ (and thus also $H^\perp$), and a character $\psi : G \to \mathbb{C}^\times$. For a pair of elements $\eta$ and $\lambda$ in $D$, denote by $G^H_{\eta,\lambda}$ the subgroup of $G$ that stabilizes the cosets $\eta + H^\perp \in D/H^\perp$ and $\lambda + H \in D/H$. Then we define

$$a^H_{\eta,\lambda} = \frac{1}{|G| \cdot |G^H_{\eta,\lambda}|} \sum_{\psi \in G} \psi^{-1}(\phi) a^H_{\psi(\eta),\psi(\lambda)},$$

(7)

Note that for $\phi \in G^H_{\eta,\lambda}$, Lemma 1.2 presents $a^H_{\psi(\eta),\psi(\lambda)}$ as $a^H_{\psi(\eta),\psi(\lambda)}$ times a factor that depends only on $\phi$, so that the vector from Eq. (7) vanishes unless this multiple is $\psi(\phi)$ (and then it can be presented as $\frac{1}{|G^H_{\eta,\lambda}|} \cdot |G| / |G^H_{\eta,\lambda}|$ times a sum over $G / G^H_{\eta,\lambda}$).

A special case of interest is where $G = \{ \pm \text{Id}_D \}$, which fixes every $H$, and the character $\psi$ is simply a sign $\pm 1$. This case is related to the action of the central element $Z$ of $\text{Mp}_2(\mathbb{Z})$ decomposing the representation space $\mathbb{C}[D]$ into the symmetric and anti-symmetric elements. In the case where $2\eta \notin H^\perp$ or $2\lambda \notin H$, i.e., when $G^H_{\eta,\lambda}$ is trivial, Eq. (7) simplifies
to
\[
\xi^H_{n,\lambda} = \frac{a^H_{n,\lambda}}{\sqrt{2}} \pm \frac{a^H_{-n,-\lambda}}{\sqrt{2}},
\]
when \(2\lambda \notin H\),
\[
\xi^H_{n,\lambda} = \frac{\sum_{\gamma \in \Gamma} e((\gamma, \eta)) (-2\lambda - \gamma)}{\sqrt{2|\Gamma|}} + \frac{\sum_{\gamma \in \Gamma} e((\gamma, \eta)) \pm e(-((2\lambda + \gamma, \eta))}{\sqrt{2|\Gamma|}} \xi_{\lambda + \gamma},
\]
if \(2\lambda \in H, 2\eta \notin H^\perp\) (8)
(the second expression comes from Lemma 1.2). When \(2\lambda \in H\) and \(2\eta \in H^\perp\), i.e., the case where \(G^H_{n,\lambda} = G\), the number \(e((2\lambda, \eta))\) is a sign, and Eq. (7) collapses \(a^H_{n,\lambda}\) to simply \(a^H_{n,\lambda}\) if \(e((2\lambda, \eta)) \equiv \pm 1\) and to 0 when it is the opposite sign.

Many results from before extend to these more general vectors.

**Proposition 2.7** Fix \(G\) and \(\psi\). Then Lemma 1.2 is valid for the vectors from Eq. (7), with the condition for the non-orthogonality of \(a^H_{n,\lambda}\) and \(a^H_{v,\psi}\) being that the action of \(G\) does not relate \((\eta + H^\perp, \lambda + H)\) to \((\kappa + H^\perp, \nu + H)\). The formulae from Lemma 1.7 also remain valid when the superscript \(\psi\) added throughout. Moreover, if \(H\) is a self-dual isotropic subgroup, then Theorems 2.2 and 2.4 hold for the vectors with \(\psi\) as well.

**Proof** The first and second assertions follow from the pairings and quadratic form values being invariant under \(G\), as well as the fact that \(2\xi_{\lambda H} \in H^\perp\) and \(l\lambda \in H^\perp\) for \(\lambda \in H\) in the situation from Lemma 1.6. This also implies that the expression from Eq. (4) is invariant under \(G\), and since the action of \(G\) commutes with that of \(\text{Mp}_2(\mathbb{Z})\), this establishes the last assertion in the isotropic case, as well as in the quasi-isotropic one for \(M \in \Gamma_{odd}\).

For \(M \in \Gamma_{odd}\) the same argument combines with the fact that \(2\xi_{H} \in H^\perp = H\) to give the desired result, and when \(M \in \text{ST} \Gamma_{odd}\), Remark 2.5 implies that replacing \(\xi_H\) by \(\varphi(\xi_H)\) (which lies in the same coset modulo \(H\) by Lemma 1.4) in the summand associated with \(\varphi\) does not change the result. Thus the assertion is true also in the remaining cases. This proves the proposition.

Such characters can be used to determine the complete decomposition of some Weil representations into irreducible components. For example, assume that \(D\) is a cyclic discriminant form, of size \(N\), and fix a generator \(\gamma\) of \(D\). Then \(\xi^2\) equals \(\frac{1}{N} + \frac{1}{N}\) when \(N\) is odd and \(\frac{1}{2N} + \frac{1}{2N}\) in case \(N\) is even, where \(t\) is prime to \(N\). The automorphism group \(G\) of \(D\) is a product of \(\{\pm 1\}\)’s, one for each prime dividing \(N\), except for \(p = 2\) when \(N\) is even but not divisible by 4. A subgroup of \(D\) is determined by its cardinality, which is a divisor \(M\) of \(N\), and the subgroup \(H_M\) of cardinality \(M\) is isotropic if and only if \(M^2\) divides \(N\) and \(\frac{N}{M^2}\), which is the size of the quotient \(A_M := H_M^\perp / H_M^\perp\), has the same parity of \(N\). We say that a character \(\psi\) of \(G\) is admissible for \(M\) if it is trivial on every factor of \(G\) that is associated with a prime \(p\) that does not divide \(\frac{N}{M^2}\), as well as with \(p = 2\) in case 4 divides \(N\) but does not divide \(\frac{N}{M^2}\).

For such a discriminant form we obtain the following decomposition.

**Theorem 2.8** The set of irreducible representations of \(\rho_D\) is in one-to-one correspondence with pairs \((M, \psi)\) where \(M\) is such a divisor of \(N\) and \(\psi\) is a character of \(G\) that is admissible for \(M\). The sub-representation associated with such a pair \(M\) and \(\psi\) consists of those elements of \(\mathbb{C}[D]\) on which \(G\) operates via \(\psi\), which are in the image of \(\uparrow_{H_M} \mathbb{C}[A_M]\), and which are perpendicular to \(\uparrow_{H_L} \mathbb{C}[A_L]\) for any divisor \(L\) of \(N\) which is properly divisible by \(M\).

Indeed, we have a surjective map from \(G\) onto the automorphism group of \(A_M\) for each \(M\), and the kernel of this map consists precisely of those \(\{\pm 1\}\)’s that are associated with
the primes in the definition of admissibility. Hence Remark 1.10 restricts the proof of Theorem 2.8 to verifying the irreducibility of the representations associated with $M = 1$, where it is clear that vectors on which $G$ operates via $\psi$ exist for every $\psi$ (check the vectors obtained from a generator). Then one can verify that the space in question admits a basis consisting of eigenvectors for $\rho_D(T)$ with different eigenvalues, and applying $\rho_D(S)$ to each one of them gives a linear combination involving all the different eigenvalues, and the irreducibility easily follows. It is clear that neither the description of the subgroups of $D$, nor the result of Theorem 2.8, hold when $D$ is not cyclic.

3 Quotients of prime exponent

Some discriminant forms $D$ do not contain self-dual quasi-isotropic subgroups, and for a subgroup $H$ that is not quasi-isotropic and self-dual, Lemma 1.7 indicates that bases in which the action of $Mp_2(\mathbb{Z})$ look particularly simple may need to involve more than one group (e.g., $H$ and $H^\perp$). For doing so we shall need some additional formulae.

Lemma 3.1 Assume that $H$ is contained in another subgroup $K$ of $D$, and let $\mathcal{R}$ be a set of representatives for $K/H$ in $K$. Then for $\eta$ and $\lambda$ in $D$ we have

$$a^K_{\eta, \lambda} = \frac{1}{\sqrt{|K/H|}} \sum_{\tau \in \mathcal{R}} e((\tau, \eta))a^H_{\eta + \tau, \lambda}.$$  

If, in addition, we take $l$ and $\xi_{l,H}$ as in Lemma 1.6, then we have the equality

$$\rho_D(T^l)a^K_{\eta, \lambda} = \frac{1}{\sqrt{|K/H|}} e(l^2) \sum_{\tau \in \mathcal{R}} e((\tau, \eta + l\lambda) + l^2) a^H_{\eta + \tau, \lambda + l\xi_{l,H} + l\tau}.$$  

The summands in these formulae are independent of the choice of $\mathcal{R}$, and in case $K \subseteq H^\perp$ we can omit $l\tau$ from the first index in the latter expression.

Proof The first formula follows from re-ordering the sum in Eq. (3), the second one follows easily from the first via Lemma 1.7, and the remaining ones are now simple consequences of Lemma 1.2 and the definition of $\xi_{l,H}$ in Lemma 1.6. This proves the lemma.

We shall need these formulae when $H$ is quasi-isotropic and $K = H^\perp$, for powers $l$ satisfying some co-primality conditions. But first we shall need the following extension of Milgram’s formula. Let $H \subseteq D$ be quasi-isotropic, with the isotropic subgroup $H_0$ from Corollary 1.5 and the associated discriminant form $A_0$. Take an integer $l$ that is prime to $|H^\perp/H|$, and then if $H$ is isotropic or $l$ is odd then we write $A_0(l)$ for the re-scaling of $A_0$ by $l$. When $H$ is not isotropic and $l$ is even, so that $|H^\perp/H|$ is odd and thus $H^\perp/H_0$ is the direct sum of $H/H_0$ (whose non-trivial element $\beta$ satisfies $\beta^2 = \frac{1}{2}$) and the lift $2H^\perp/H_0$ of $H^\perp/H$, multiplying the quadratic form on $H^\perp$ by $l$ transforms $A$ to a discriminant form, and we allow the abuse of notation of writing $A_0(l)$ for the resulting discriminant form there as well.

Let $\xi_{l,H}$ be the element from Lemma 1.6, and choose $k \in \mathbb{Z}$ with the following properties. In case $l$ is odd or $H$ is isotropic, we require that $kl$ is congruent to 1 modulo the denominators of $(\gamma, \xi_{l,H})$ and $\frac{k^2}{2}$ for $\gamma \in H^\perp$, as well as that of $\frac{k^2}{4}$. When $H$ is not isotropic and $l$ is even, so that $\xi_{l,H} \in H^\perp$ and $|H^\perp/H|$ is odd, we impose these congruences only modulo the odd parts of these denominators, and demand that $k$ be even.
Lemma 3.2  Given such $H$ and $l$, take $\xi_{lH}$ and $k$ as defined above. Then each term in the sum $\sum_{\sigma \in H^{-1}/H} e\left(l_{lH}^2 \left( - (\sigma, \xi_{lH}) \right) \right)$ is well-defined, and the value of the entire sum is $e\left( -k \frac{\xi_{lH}^2}{2} \right)e\left( \text{sgn} A_0(l)/8 \right) \cdot \sqrt{|H^{-1}/H|}$.

Proof  The invariance under changing $\sigma \in H^{-1}$ by an element of $H$ follows directly from Lemma 1.6, making each summand well-defined.

When $H$ is isotropic or $l$ is even we have $\xi_{lH} \in H^{-1}$, and then writing the variable $\sigma$ as $\gamma + k\xi_{lH}$ with $\gamma \in A := H^{-1}/H$ gives $e\left( -k \frac{\xi_{lH}^2}{2} \right)$ times the left hand side of Eq. (1), with $D$ replaced by $A_0(l)$ (this is $A(l)$ if $H$ is isotropic). The result thus follows from Milgram’s formula in this case.

If $H$ is quasi-isotropic and $l$ is odd, then we have seen that $\frac{\xi_{lH}^2}{2} = \frac{1}{2} + \mathbb{Z}$ for any $\beta \in H \setminus H_0$, so by fixing such an element we have $H = H_0 \cup (\beta + H_0)$. Since Corollary 1.5 gives $H_0^+ = H^+ \cup (\xi_H + H^+)$, $k$ must be odd as well, and the non-trivial coset can be written as $k\xi_H + H^+$. The left hand side of Milgram’s formula for $A_0(l)$ can thus be written as the sum of $\sum_{\sigma \in H^+/H_0} e\left(l_{lH}^2 \right)$ and $\sum_{\sigma \in H^+/H_0} e\left(l_{\sigma(k\xi_H)^2}^2 \right)$. Now, replacing $\sigma$ by $\sigma + \beta$ in the former sum inverts all the summands, so that this sum vanishes. Thus Eq. (1) compares

$e\left( \frac{\text{sgn} A_0(l)/8 \cdot \sqrt{|H^+/H_0|}}{e\left( k \frac{\xi_{lH}^2}{2} \right)} \right)$ times $\sum_{\sigma \in H^+/H_0} e\left(l_{lH}^2 - (\sigma, \xi_{lH}) \right)$.

But the well-definedness of the required sum means that it equals half of the latter sum, which gives the desired result by the index 2 property in Corollary 1.5 and the fact that $\xi_{lH} = \xi_H$ in this setting. This proves the lemma.  

In this case the formulae from Lemmas 1.7 and 3.1 are complemented by the following evaluation.

Proposition 3.3  Let $H$, $l$, $A_0(l)$, $\xi_{lH}$, and $k$ be as in Lemma 3.2, and take $\eta$ and $\lambda$ in $D$. Then $\rho_D(ST^l)\alpha_{\eta, \lambda}^{H^+}$ equals $e\left( \frac{\text{sgn} A_0(l) - \text{sgn} D}{8} \right)$ times

$e\left[ (kl - 1)(l_{lH}^2 + (\lambda, \eta + \xi_{lH})) + k \frac{\eta^2}{2} + k(\eta, \xi_{lH}) \right] \rho_D(T^{-k})\alpha_{(kl - 1)\lambda + k\eta, \eta + \lambda + \xi_{lH}}^{H^+}$.

Proof  Taking the second formula from Lemma 3.1 with $K = H^{-1}$ and applying $\rho_D(S)$ presents, via Lemma 1.7, the vector $\rho_D(ST^l)\alpha_{\eta, \lambda}^{H^+}$ as

$e\left( -\frac{\text{sgn} D}{8} \right) \sqrt{|H^{-1}/H|}$ times $e\left( -l_{lH}^2 - (\lambda, \eta + \xi_{lH}) \right)$ times $\rho_D(T^{-k})\eta, \lambda - (\tau, \eta + \lambda + \xi_{lH}) \right] \alpha_{(kl - 1)\lambda + k\eta, \eta + \lambda + \xi_{lH}}^{H^+}$.

Using the first formula from Lemma 3.1, the sum over $\tau$ becomes

$\frac{1}{\sqrt{|H^{-1}/H|}} \sum_{\tau \in \mathfrak{R}} e\left( - (\lambda, \rho) \right) \sum_{\tau \in \mathfrak{R}} e\left( l_{lH}^2 - (\tau, \rho + \xi_{lH}) \right) \alpha_{(kl - 1)\lambda + \xi_{lH} + \xi_{lH}}^{H^+}$

where Lemma 1.2 allowed us to ignore $\tau \in H^+$ in the first index before interchanging the summation order.

Now, we may replace $\mathfrak{R}$ by $H^{-1}/H$ in the internal sum, and writing $\tau$ as $\sigma + k\rho$ with $\sigma \in H^+/H$ transforms the latter sum into $e\left( -k \frac{\xi_{lH}^2}{2} - k(\rho, \xi_{lH}) \right)$ times the expression from Lemma 3.2. The considerations from the proof of that lemma thus present $\rho_D(ST^l)\alpha_{\eta, \lambda}^{H^+}$ as $e\left( \frac{\text{sgn} A_0(l) - \text{sgn} D}{8} \right)$ times

$e\left( -l_{lH}^2 - (\lambda, \eta + \xi_{lH}) - k \frac{\xi_{lH}^2}{2} \right)$ times $\sum_{\rho \in \mathfrak{R}} e\left( - (\rho, \lambda + k\xi_{lH}) - k \frac{\xi_{lH}^2}{2} \right) \alpha_{(kl - 1)\lambda + \xi_{lH} + \xi_{lH} + \rho}^{H^+}$.
Lemma 1.2 allows us to add \((1 - k)\xi_{l,H} \in H^\perp\) to \(-\lambda\) (note that \((1 - k)\xi_{l,H} \in H^\perp\) also when \(\xi_{l,H} \notin H^\perp\) since then \(k\) is odd), and since the resulting expression equals \(\beta - k\alpha + \xi_{l,H}\) for \(\beta := (kl - 1)\lambda + k\eta\) and \(\alpha := \eta + \lambda + \xi_{l,H}\), and \(-\lambda - k\xi_{l,H}\) equals \(\beta - k\alpha\), Lemma 3.1 shows that the sum over \(\mathfrak{N}\) here equals \(\sqrt{|H^\perp/H|} e(k \frac{\alpha^2}{2})\) times \(\rho_D(T^{-k}) a_{\rho,\alpha}^H\). Substituting the value of \(\alpha\) and simple algebra now produces the desired result. \(\Box\)

Remark 3.4 Proposition 2.7 can be extended, with a similar proof, to show that the formulae from Proposition 3.3 are also valid for the vectors from Eqs. (7) and (8).

We shall be using these expressions in the case where \(H\) is a quasi-isotropic subgroup of \(D\) such that the quotient \(H^\perp/H\) has prime exponent \(p\). We shall need the following technical lemma.

Lemma 3.5 Let \(H_0\) be the subgroup from Corollary 1.5. If \(H^\perp/H\) has prime exponent \(p\) then there exists a subgroup \(H^\perp\) of \(H^\perp\) such that \(H^\perp/H_0\) is a complement of \(H/H_0\) inside the finer quotient \(H^\perp/H_0\).

Proof If \(p\) is odd then the fact that \(H/H_0\) has order dividing 2 determines \(H^\perp/H_0\) as the subgroup containing those elements of \(H^\perp/H_0\) whose order is odd. For \(p = 2\) we note that \(2(\gamma, \delta)\) vanishes for every \(\gamma\) and \(\delta\) in \(H^\perp\) (since \(2\mathfrak{H} \in H\)), implying that \(H^\perp\) and 2 satisfy the condition from Lemma 1.6. But this means that \(\frac{\gamma^2}{2} \in \frac{1}{4}\mathbb{Z}/\mathbb{Z}\) for every \(\gamma \in H^\perp\), and therefore \(2\gamma\) lies in the subgroup \(H_0\) from Corollary 1.5. Hence \(H^\perp/H_0\) also has exponent 2, and every subgroup in it has a complement. This proves the lemma. \(\Box\)

For easing the notation, we define \(H\) to be such that \(H^\perp\) carries the usual meaning, and obtain the following consequence.

Corollary 3.6 There exists an element \(\xi_{H^\perp,H} \in H_0^\perp\) which pairs trivially with \(H^\perp/H_0\), but not trivially with \(H/H_0\) in case the latter group is non-trivial. This element is uniquely determined modulo \(H\), has order at most 2 in \(D/H\), and the order of \(\frac{\xi_{H^\perp,H}^2}{2}\) in \(\mathbb{Q}/\mathbb{Z}\) divides 8.

Proof The first statement follows directly from Lemma 3.5, and the second one as in Lemma 1.6. Next we note that \(2\xi_{H^\perp,H}\) is perpendicular to \(H^\perp\) hence lies in \(H\), and the remaining assertions follows. This proves the corollary. \(\Box\)

It follows immediately from Corollary 3.6 that the element \(\xi_{H^\perp,H}\) can serve as a representative for \(\xi_{H}\) from Corollary 1.5. Moreover, given \(l \in \mathbb{Z}\), we recall from that Lemma that \(\xi_{l,H}\) vanishes if \(l\) is even and equals \(\xi_{H}\) for odd \(l\). Using the third assertion in Corollary 3.6, we therefore set \(\xi_{l,H^\perp,H}\) to be 0 in case \(l\) is even and \(\xi_{H^\perp,H}\) when \(l\) is odd. We remark that if \(p\) is odd then \(\xi_{l,H^\perp,H}\) is simply \(\xi_{p,H^\perp}\) from Lemma 1.4, but this is not true when \(p = 2\).

Assume thus that \(H\) is such a subgroup, fix \(H^\perp\) as in Lemma 3.5, and assume that \(J\) is subgroup of \(H\), of order \(p\). Then the \(\mathbb{Q}/\mathbb{Z}\)-dual \(D/J^\perp\) of \(J\) is also cyclic of order \(p\), and the image of any element of \(D \setminus J^\perp\) generates it. It follows that given two elements \(\eta\) and \(\lambda\) in \(D\), if \(\lambda \notin J^\perp\), then there exists \(l \in \mathbb{Z}\) such that \(\eta - l\lambda \in J^\perp\). The subgroup \(H^\perp\) thus produces the element \(\xi_{l,H^\perp,H}\). Denote \(A = H^\perp/H\) (also when \(H\) is not isotropic), and we set

\[
\xi_{l,H^\perp,H} = \begin{cases} 
\frac{1}{\sqrt{|A|}} \sum_{\tau \in A} e((\tau, \eta - \xi_{l,H^\perp,H}) + l^2/2) a_{\eta,\lambda + \tau}^H, & \text{if } \lambda \notin J^\perp, \\
0, & \text{for } \lambda \in J^\perp,
\end{cases}
\]
where we could take \( \tau \) in \( A \) rather than in a representing set by Lemmas 1.2 and 1.6. Note that when \( \lambda \notin J^\perp \), the property that \( \eta - \lambda \notin J^\perp \) determines \( l \) modulo \( p \), but a priori the definition of \( b_{\eta,\lambda}^{HJ} \) may differ when \( l \) is changed by a multiple of \( p \). Now, it is clear that for fixed \( l \) the expression from Eq. (9) depends only on the class of \( \xi_{l,H,H} \) in \( D/H \), and for showing that it is well-defined as a formula of \( \eta \) and \( \lambda \) alone, we prove the following lemma.

**Lemma 3.7** Given two elements \( \eta \in D \) and \( \lambda \in D \setminus J^\perp \), the vector \( b_{\eta,\lambda}^{HJ} \) from Eq. (9) equals \( e\left(-\frac{l^2}{2}\right)\rho_D(T)\xi_{\eta,\lambda}^{H} + \xi_{l,H,H}^{\perp} \), and thus depends only on \( l \) modulo \( p \).

**Proof** The expression for \( b_{\eta,\lambda}^{HJ} \) is a consequence of Lemma 3.1 and Eq. (9). The invariance under changing \( l \) by a multiple of \( p \) follows from Remark 1.8 and the fact that the difference between \( \xi_{l,H,H} \) and \( \xi_{l+np,H,H} \) is just \( \xi_{np,H,H} \) from Corollary 3.6. This proves the lemma. \( \square \)

From Lemma 3.7 we also obtain the following analogue of Lemma 1.2.

**Corollary 3.8** For \( \lambda \in J^\perp \), the vector \( b_{\eta,\lambda}^{HJ} \) from Eq. (9) depends only on the image of \( \eta \) modulo \( H^\perp \), while adding \( \delta \in H \subseteq J^\perp \) to \( \lambda \) multiplies it by \( e(-\delta,\eta) \). On the other hand, if \( \lambda \notin J^\perp \), then this vector depends on the image of \( \eta \) in \( D/H \), and for \( \delta \in H^\perp \) we have \( b_{\eta,\lambda}^{HJ} = e\left(-\frac{l^2}{2},(\delta,\eta)\xi_{l,H,H}^\perp\right)\xi_{l,H,H}^{\perp} \), with \( l \) as in Eq. (9). Two vectors \( b_{\eta,\lambda}^{HJ} \) and \( b_{\nu,\kappa}^{HJ} \) with indices that are not related by these transformations are orthogonal, and each \( b_{\eta,\lambda}^{HJ} \) has norm 1.

The actions of the generators \( S \) and \( T \) on the vectors from Eq. (9) take the following form.

**Proposition 3.9** For every such \( H, J, \eta, \) and \( \lambda \) we have the equality

\[
\rho_D(T)b_{\eta,\lambda}^{HJ} = e\left(-\frac{l^2}{2}\right)\xi_{\eta,\lambda}^{H} + \xi_{l,H,H}^{\perp},
\]

If either \( \eta \) or \( \lambda \) lies outside of \( J^\perp \), then there exists an 8th root of unity \( \varepsilon_j(S,v) \), for \( v := (v_1^\nu) \) as in Remark 1.8, such that

\[
\rho_D(S)b_{\eta,\lambda}^{HJ} = \varepsilon_j(S,v)e\left(-\lambda,\eta\right)\xi_{l,H,H}^{\perp}.
\]

**Proof** If \( \lambda \in J^\perp \) then both formulae follow from Lemma 1.7 (note that for \( S \) we assume \( \eta \notin J^\perp \)), and we can apply the formula from Lemma 3.7, where in the second one we have \( \varepsilon_j(S,v) = e(-\text{sgn}D/8) \). When \( \lambda \notin J^\perp \), the expression for \( T \) is a consequence of Lemma 3.7, since the difference between \( \xi_{l,H,H} \) and \( \xi_{l+1,H,H} \) is \( \xi_{H,H}^\perp \), which has order 2 in \( D/H \) by Corollary 3.6.

For evaluating the action of \( \rho_D(S) \) when \( \lambda \notin J^\perp \), we take \( l \) as in Eq. (9), and write \( b_{\eta,\lambda}^{HJ} \) as in Lemma 3.7 once again. If \( \eta \notin J^\perp \), then \( l \) can be taken to be 0, and the desired formula is again obtained from Lemma 1.7, again with \( \varepsilon_j(S,v) = e(-\text{sgn}D/8) \). On the other hand, when \( \eta \notin J^\perp \), the index \( l \) is prime to the \( p \)-power \( |A| \), we choose \( k \) as in Lemma 3.2, and then Proposition 3.3 presents \( \rho_D(S)b_{\eta,\lambda}^{HJ} \), after some cancelations, as

\[
e\left(\frac{\text{sgn}A(-l) - \text{sgn}D}{8}\right) - (\lambda,\eta) - k\xi_{H,H}^{\perp} + k\xi_{H,H}^{\perp} + k\xi_{H,H}^{\perp}.
\]
Another application of Lemma 3.7 compares this with the desired result, in which $\varepsilon_f(S, v) = e^{\left(\text{sgn} A_0(l) - \text{sgn} D - \frac{k^2}{2}\right)}$, which is indeed an 8th root of unity by Corollary 3.6 again. This proves the proposition. □

Using Proposition 3.9, the proofs of Theorems 2.2 and 2.4 establish the following result.

**Theorem 3.10** Let $H, J, \eta, \lambda$ be as in Proposition 3.9, and assume that $J$ is isotropic and that not both of $\eta$ and $\lambda$ are in $\lambda$. Take $M \in \text{Mp}_2(\mathbb{Z})$, set $\nu = (\nu)$, and write the associated vector $v_{H, J}^\nu$ from Eq. (9) as $v_{\nu}^H$. Then the action of $\rho_D(M)$ takes $v_{\nu}^H$ to $\varepsilon_f(M, v) e(\tilde{Q}_H(M, v)) v_{\nu}^H$, where $M = v$ and $\tilde{Q}_H(M, v)$ are defined in Eqs. (5) and (6) respectively (with $\xi_{H, J, v}$ replacing $\xi_H$), and $\varepsilon_f(M, v)$ is an 8th root of unity.

**Remark 3.11** We need that $J$ be isotropic in Theorem 3.10, in order for the element $\xi_{H, J, v}$ to be in $\lambda$, so that the operations from that theorem preserve the property that not both $\lambda$ and $\eta$ are in $\lambda$. We can then set $B := \lambda / \lambda$, and get a description of the orthogonal complement, inside $\mathbb{C}[D]$, of the sub-representation $\lambda \uparrow \mathbb{C}[B]$ from Corollary 1.9. Moreover, using Corollary 3.6 we can easily describe this orthogonal complement using an orthonormal basis.

**Remark 3.12** Note that unlike the situation from Remark 2.6, the parameter $\varepsilon_f(M, v)$ from Theorem 3.10 is no longer a character of $\text{Mp}_2(\mathbb{Z})$. This is because of its dependence on $v$, as seen in the proof of Proposition 3.9. Moreover, since $\varepsilon_f(T, v) = 1$ for every $v$ (see Proposition 3.9), the root of unity $\varepsilon_f(M, v)$ depends only on the lower row of $M$ (as well as on the metaplectic sign when $\text{sgn} D$ is odd). For odd $p$ and non-isotropic $H$, where as in Remark 2.6, every signature may appear, the two presentations of $\varepsilon_f(S, v)$ when $p$ does not divide $l$ look different: One with even $l$ where the term involving $k$ disappears, and one with odd $k$. Note that in this case $2\xi_{H, J, v}$ lies in $H$ (see the proofs of Lemma 3.5 and Corollary 3.6), and there are two possibilities: Either $2\xi_{H, J, v} \in H_0$, $\frac{\xi_{H, J, v}}{2}$ has order dividing 4, $A_0$ is the direct sum of a lift of $A$ and a Klein 4-group, and the signatures of $D$ and $A_0$ are even; Or $2\xi_{H, J, v} \notin H_0$, its image in $A_0$ is of order 4 and generates the complement of the lift of $A$, $\frac{\xi_{H, J, v}}{2}$ has order 8, and the signatures of $D$ and $A_0$ are odd. Recalling that $\text{sgn} A_0(l)$ is even for even $l$ (since $|A_0(l)| = |A|$ is odd), this explains how the two expressions for $\varepsilon_f(S, v)$ are indeed equal.

For our main result, namely Theorem 4.7 below, we shall need the following observation.

**Proposition 3.13** All the coefficients in Theorems 2.2 and 2.4 are powers of $\zeta_N$, where $N$ is the level of the respective discriminant form $D$. These theorems thus produce bases for the corresponding representations in which the coefficients are from $\mathbb{Z}[\zeta_N]$. The same holds for the coefficients appearing in Theorem 3.10, provided that we replace each basis vector $v_{\eta, \lambda}^H$ by $e^{(\text{sgn} D / 8)} a_{\eta, \lambda}^H$.

**Proof** It is clear from Eqs. (4) and (6) that the result it true for the roots of unity $e(Q(M, v))$ and $e(\tilde{Q}_H(M, v))$ from Theorems 2.2 and 2.4. The fact that Lemma 1.2 allows one to reduce the set of vectors $a_{\eta}^H$ from these theorems to a subset that constitutes a basis, and the additional vectors are the same ones multiplied by powers of $\zeta_N$, establishes the result for Theorem 2.2. As for the expression $\chi(M)$ in Theorem 2.4, it is a power of
\( \chi(S) = e(-\text{sgn} D/8). \) But the latter number is \( e(-\text{sgn} A_0/8) \) in the notation of Remark 2.6, and the considerations of that lemma shows that in all the cases considered there, \( e(-\text{sgn} A_0/8) \) lies in the cyclotomic field associated with the level of \( A_0. \) As \( A_0 = H_0^\perp / H_0 \) for an isotropic subgroup \( H_0 \subseteq D, \) its level divides \( N, \) and our result for Theorem 2.4 follows.

For the coefficients in the representation from Theorem 3.10 (with the modified basis), we note that the coefficients in Corollary 3.8 are also powers of \( \zeta_N, \) and \( \rho_D(T) \) does not interchange the part of the basis that is affected by the modification with the one that is not. By generation and the value \( \varepsilon_J(T,v) = 1 \) for all \( v, \) we only need to consider the roots of unity arising from the coefficients \( \varepsilon_J(S,v), \) given explicitly in the proof of Proposition 3.9, and the modification of the basis.

Now, this root of unity is 1 when \( \lambda \in J_\perp, \) and is \( e(-\text{sgn} D/4) \) in case \( \eta \in J_\perp. \) The latter expression is \( \pm 1 \) for even \( \text{sgn} D, \) and the well-known fact that if \( \text{sgn} D \) is odd then 4 divides \( N \) yields the desired property in these cases. Finally, when neither \( \lambda \) nor \( \eta \) are in \( J_\perp \) and \( l \) from Eq. (9) is not divisible by \( p \) and \( k \) as in Lemma 3.2, recall that the part arising from \( \varepsilon_{H_0^\perp} \) is a power of \( \zeta_N. \) Now, when \( H \) is isotropic or \( l \) is odd, \( \text{sgn} D = \text{sgn} A_0 \) and \( \text{sgn} A_0(l) \) have the same parity, which yields the desired result in case 4 divides \( N. \) As when \( N \) is not divisible by 4 the difference between \( e(\text{sgn} A_0(l)/8) \) and \( e(\text{sgn} D/8) = e(\text{sgn} A_0/8) \) is only a sign, the result follows in this case. Finally, when \( H \) is not isotropic and \( p \) is odd, Remark 3.12 shows that on one hand, if \( 2\varepsilon_{H_0^\perp} \notin H_0 \) then the denominator of \( \frac{\varepsilon_{H_0^\perp}}{2}, \) which divides \( N, \) is 8 and we are done since \( \varepsilon_J(S,v) \) is an 8th root of unity. On the other hand, when \( 2\varepsilon_{H_0^\perp} \in H_0 \) we take \( l \) to be odd and argue as before. This proves the proposition. \( \square \)

In fact, the proof of Proposition 3.13 can be slightly adapted to show that when \( D \) has even signature, \( |A| \) is an even power of \( p, \) and either \( p \) is odd or \( A \) satisfies some signature condition, all the coefficients coming from \( \varepsilon_J(S,v) \) disappear in that basis, and we get an action like in Theorem 2.2.

We conclude by remarking that the formula from Eq. (7), and its properties given in Proposition 2.7, can be extended to the vectors from Eq. (9) and their properties, once extra assumptions are made on \( G. \) Specifically we need \( G \) to preserve \( f \) as well (and then the parameter \( l \) from Eq. (7) remains unaffected), but also the group \( \tilde{H}^\perp, \) whose definition in Lemma 3.5 involved a choice in some cases, must be preserved. These assumptions are clearly satisfied in the cyclic case considered in Theorem 2.8, but not in general.

4 Integral bases for discriminant forms

In this section we establish the first main goal of this paper, namely providing an explicit basis for the Weil representation \( \rho_D \) associated with any discriminant form \( D \) using which all the matrices arising from \( \rho_D \) have integral entries from the ring \( \mathbb{Z}[\zeta_N] \) of algebraic integers in the field \( \mathbb{Q}(\zeta_N) \) from Proposition 1.1.

We begin with the case where \( D \) is the cyclic discriminant form \( p^{\pm 1}, \) where \( p \) is some odd prime. The even part of this case was essentially dealt with in [13], but the odd part requires some technicalities.

Lemma 4.1 Given positive integers \( h \) and \( m, \) assume that we are given an analytic function \( \varphi_{m,h} \) of the variable \( \zeta, \) with the following properties:
1. The Taylor expansion of \( \varphi_{m,h} \) at \( \zeta = 1 \) is \( \sum_{n=0}^{\infty} P_{h,n}(m)(\zeta - 1)^n \), where \( P_{h,n} \) is an odd polynomial of degree \( 2n + 2h - 1 \).

2. The function \( \varphi_{m,h} \) is the constant \( \delta_{m,h} \) when \( m \leq h \).

Fix a third integer \( r \geq 0 \), and define functions \( f_{m,h}^{(r)} \) of \( \xi \) as follows: For \( h = 0 \) we set \( f_{m,0}^{(r)} \) to be the constant \((\frac{m+r}{2r+1})\), and for \( h \geq 1 \) the function \( f_{m,h}^{(r)} \) is defined inductively as \( f_{m,h-1}^{(r)}(\xi) = \varphi_{m,h}(\xi)f_{h-1}^{(r)}(\xi) \). Then the function \( f_{m,h}^{(r)} \) is the constant \((\frac{m+r}{2r+1})\) wherever \( h \leq r \), and vanishes to order at least \( h - r \) at \( \zeta = 1 \) in case \( h \geq r \).

**Proof** The binomial coefficients \((\frac{m+r}{2r+1})\) with \( j \geq 0 \) form a basis for the space of odd polynomials in \( m \), where the \( j \)th such expression has degree \( 2j + 1 \), it vanishes for \( m \leq j \), and it attains 1 on \( m = j + 1 \). The degree bound means that \( P_{h,n}(m) \) is spanned by \((\frac{m+r}{2r+1})\) for \( 0 \leq j \leq n + h - 1 \), and the values for small \( m \) mean that the terms with \( j \leq h - \delta_{m,0} \) do not appear in \( P_{h,n}(m) \), and \( P_{h,0}(m) = (\frac{m+r}{2r+1}) \) with the coefficients \( \delta_{m,0} \) do not appear in \( P_{h,n}(m) \), and \( P_{h,0}(m) = (\frac{m+r}{2r+1}) \) with the coefficient 1. We can thus write

\[
\varphi_{m,h}(\xi) = \left( \frac{m+h-1}{2h-1} \right) - \sum_{n=1}^{\infty} \sum_{j=h}^{n+h-1} \alpha_{nj}^{(h)} \left( \frac{m+j}{2j+1} \right)(\xi - 1)^n
\]

for some constants \( \alpha_{nj}^{(h)} \) for any \( n \geq 1, h \geq 1 \), and \( h \leq j \leq n + h - 1 \).

Now, the constant \( f_{m,0}^{(r)} \) is the constant \((\frac{m+r}{2r+1})\) for some \( 1 \leq h \leq r \) (which is given when \( h = 1 \)), then the vanishing of \( f_{m,h-1}^{(r)} \) in the definition of \( f_{m,h}^{(r)} \) implies that the latter equals the same constant as well. This proves the first assertion.

We now claim that for any \( h \geq r \), the function \( f_{m,h}^{(r)} \) expands at \( \zeta = 1 \) as \( \sum_{s=1+r}^{\infty} x_{s,h}^{(r)}(m+s)^{(|(m+s)/2r+1)|}(\zeta - 1)^s \) for some constants \( x_{s,h}^{(r)} \), which will clearly establish the second assertion. The claim is evident for \( h = r \), with the coefficients \( x_{s,h}^{(r)} = \delta_{s,0}\delta_{s,r} \).

Assuming that the claim holds for \( h - 1 \) for some \( h > r \), the fact that for \( j \geq h - 1 \) the expression \((\frac{h+1}{2r+1})\) equals \( \delta_{j,h-1} \) reduces \( f_{m,h}^{(r)}(\xi) \) to \( \sum_{s=h-1-r}^{\infty} x_{s,h}^{(r)}(m+s)^{(|(m+s)/2r+1)|}(\zeta - 1)^s \). Substituting these expressions into the formula \( f_{m,h-1}^{(r)}(\xi) = \varphi_{m,h}(\xi)f_{h-1}^{(r)}(\xi) \), the summands with \( j = h - 1 \) cancel with the part coming from the constant term of \( \varphi_{m,h} \), and our expression for \( f_{m,h}^{(r)}(\xi) \) becomes the sum of \( \sum_{s=h-r}^{\infty} x_{s,h}^{(r)}(m+s)^{(|(m+s)/2r+1)|}(\zeta - 1)^s \) and

\[
\sum_{l=h-1-r}^{\infty} x_{l,h-1}^{(r)}(\xi - 1)^l \times \sum_{n=1}^{n+h-1} \sum_{j=h}^{n+h-1} \alpha_{nj}^{(h)} \left( \frac{m+j}{2j+1} \right)(\zeta - 1)^n
\]

But with \( s = n + l \geq h - r \) and \( n = s - l \leq n - h + 1 + r \) we get the inequalities \( h \leq j \leq n + h - 1 \leq s + r \), and our claim (with the second assertion) follows, with \( x_{s,h}^{(r)} = x_{s,h}^{(r)} + \sum_{n=1-h}^{s+h-1} x_{n,h}^{(r)} \). This proves the lemma. \( \Box \)

We shall also need a decomposition of a Vandermonde matrix.

**Lemma 4.2** Let \( M \) be the Vandermonde matrix of some parameters \( \{x_i\}_{i=1}^{n} \), in the convention in which the first column of \( M \) consists of 1’s. Then in the presentation of \( M \) as \( LL^{\dagger} \), where \( L \) is lower triangular and \( U \) is upper triangular and unipotent, the entries of \( U \) are polynomials in \( \{x_i\}_{i=1}^{n} \), and \( L \) decomposes further as the following product. Let \( N_h \) be the lower triangular unipotent matrix with \( ij \) entry 1 in case \( i = h \) and \( \delta_{ij} \) otherwise, and set \( D_h \) to be the diagonal matrix with \( ij \)th diagonal entry 1 if \( i \leq h \) and \( x_i - x_h \) if \( i > h \).

Then \( L \) is the product \( N_1D_1N_2D_2...N_{n-1}D_{n-1} \). 

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Proposition 4.4  Let $h_k$ denote the complete homogeneous symmetric polynomial of degree $k$ (with $h_0 = 1$ and $h_k = 0$ for $k < 0$). Then a classical result (see, e.g., Theorem 2 of [15] in the alternative convention, though it was known much earlier) implies that in this decomposition of $M$, the $ij$th entry of $U$ is $h_{j-i}(x_1, \ldots, x_i)$, while the $ij$th entry of $L$ is $\prod_{m=1}^{i-1}(x_i - x_m)$ (indeed vanishing when $i < j$), with the empty product 1 when $j = 1$. This proves the first assertion. The second one follows by induction, once one verifies that $L = N_1 D_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ where $L$ is the $L$-matrix of the Vandermonde matrix of $\{x_m\}_{m=2}^n$. This proves the lemma.

Next we establish a certain explicit formula.

Lemma 4.3  Take some parameter $\zeta$ and some $k \geq 1$, set $\epsilon_m := \frac{\zeta^{2km} - \zeta^{2km}}{\zeta^{2km} - \zeta^{-2km}}$, and consider them to be the entries of a column vector $\epsilon$. Let $N_h$ and $D_h$ be the matrices from Lemma 4.2, where $x_m = \zeta^{m^2}$. For every $m$ and $h$ we define the function $\psi_{m,h}(\zeta) = \frac{\zeta^{2m} - \zeta^{-2m}}{\zeta^{2m} - \zeta^{-2m}} \prod_{j=1}^{h-1} \left( \frac{\zeta^{m^2} - \zeta^{2j}}{\zeta^{j^2} - \zeta^{2j}} \right)$, and define $f_{m,h}(\zeta)$ as in Lemma 4.1. Then for $m > h$ the $m$th entry of $D_h^{-1} N_h^{-1} \cdots D_1^{-1} N_1^{-1} \epsilon$ is

$$\frac{(\zeta^{2k} - \zeta^{-2k}) \sum_{r=0}^{m-1} f_{m,h}(\zeta) \eta_k^{2r}}{(\zeta^{2m} - \zeta^{-2m}) \prod_{j=1}^{h} (\zeta^{m^2} - \zeta^{j^2})},$$

where $\eta_k := \zeta^k - \zeta^{-k}$.

Moreover, the $m$th entry of $L^{-1} \epsilon$ is given by the same expression, with $h = m - 1$.

Proof  We establish the result by induction, where for $h = 0$ we need to express $\epsilon_m$ in a more convenient manner. Recalling that $\frac{\chi_{m+1}}{\chi_{m+1}}$ is given by $U_m(\frac{x + x^{-1}}{2})$ where $U_m(x) = \sum_{s=0}^{(n/2)} (-1)^s (n-s) y^{n-2s}$ is the Chebyshev polynomial of the second kind (with $[x]$ being the lower integral function), we can write $\epsilon_m$ as $\frac{\zeta^{2m} - \zeta^{2k}}{\zeta^{2m} - \zeta^{-2m}} U_m(\frac{\zeta^{2m} + \zeta^{2k}}{2})$. As the argument of $U_m$ here is $\frac{\zeta^{2m} + \zeta^{2k}}{2} + 1$, the formula $\sum_{r=0}^{m} \binom{n+r}{2r+1} (2y - 2y')$ for $U_n(y)$ now allows us to express $\epsilon_m$ as

$$\frac{\zeta^{2k} - \zeta^{-2k}}{\zeta^{2m} - \zeta^{-2m}} \sum_{r=0}^{m-1} (m+r)(m+r+1) \eta_k^{2r},$$

which is the assertion for $h = 0$ by the definition of $f_{m,0}$ in Lemma 4.1 and its vanishing for $m \leq r$.

Take now $h \geq 1$, and assume that the $m$th entry of $D_{h-1}^{-1} N_{h-1}^{-1} \cdots D_1^{-1} N_1^{-1} \epsilon$ is expressed, for $m > h$, by our formula with $h - 1$. The action of $N_h^{-1}$, which is defined like $N_h$, but with $hj$-entry $-1$ for $j > h$, subtracts from it the same term with $m = h$. As $\psi_{m,h}(\zeta)$ is the quotient of the denominators, we indeed obtain the recursive definition of $f_{m,h}(\zeta)$ from Lemma 4.1, and the action of $D_h^{-1}$ divides by the remaining expression in the denominator.

This establishes the formula with $h$, and the for $L^{-1} \epsilon$, with $L$ decomposed as in Lemma 4.2, we just note that the $m$th entry is not affected by the matrices with index $m$ and larger, and thus preserves the value that it attains for $h = m - 1$. This proves the lemma.

The application that we shall need is the following one.

Proposition 4.4  Let $\zeta$ be a non-trivial root of unity of prime order $p$, and take some $1 \leq k \leq p - 1$. Let $c_l$ with $0 \leq l \leq \frac{p-3}{2}$ be the elements of $\mathbb{Q}(\zeta)$ such that the equality $\sum_{l=0}^{p-3/2} (\zeta^{2m} - \zeta^{-2m}) c_l m^2 = \zeta^{2km} - \zeta^{-2km}$ holds for every $1 \leq m \leq \frac{p-1}{2}$. Then $c_l$ lies in $\mathbb{Z}[\zeta]$ for every $0 \leq l \leq \frac{p-3}{2}$.

Proof  Simple division shows that the vector $c$ of our elements is related to $\epsilon$ from Lemma 4.3 via $Mc = \epsilon$, where $M$ is the Vandermonde matrix of $\zeta^{m^2}$, $1 \leq m \leq \frac{p-1}{2}$, in the convention from Lemma 4.2. We thus need the integrality of the entries of
\( M^{-1} e = U^{-1} L^{-1} e \), and since the first assertion of that lemma implies that both \( U \) and \( U^{-1} \) have entries from \( \mathbb{Z}[\zeta] \), the desired integrality is equivalent to that of \( L^{-1} e \). We thus need the integrality of the expression from Lemma 4.3 with our \( \zeta \).

Now, it is clear that the formula in question consists of quotients of differences of the form \( \zeta^a - \zeta^b \). Recall that every non-zero such difference generates the same prime ideal \( p \) in \( \mathbb{Z}[\zeta] \), and quotients between such differences are units in \( \mathbb{Z}[\zeta] \). Moreover, the value obtained by substituting \( \zeta = 1 \) in such a unit is non-zero. It therefore follows that the power of \( p \) dividing any polynomial in \( \zeta \) is at least the order of that expression at \( \zeta = 1 \) (this is not an equality because the are relations among powers of \( \zeta \)), but when this polynomial is a product of non-vanishing differences, the power of \( p \) dividing it is precisely the number of factors. In particular, an expression like that from Lemma 4.3 is integral in case it becomes finite at \( \zeta = 1 \) as a function of \( \zeta \).

Next we verify that the functions \( \varphi_{m,h} \) from that proposition satisfy the conditions from Lemma 4.1. The equality \( \varphi_{m,h} (\zeta) = \delta_{m,h} \) for \( m \leq h \) is clear. Moreover, the \( t \)th term in the Taylor expansion of each normalized multiplier \( \frac{\zeta^m}{\zeta^h - 1} \) at \( \zeta = 1 \) is an even polynomial of degree at most \( t + 1 \), while for \( \frac{\zeta^{2m} - \zeta^{-2m}}{\zeta - 1} \) it is odd of degree \( 2t + 1 \). Since the Taylor expansions of the corresponding terms in the denominator of \( \varphi_{m,h}(\zeta) \) are independent of \( m \), we deduce that the \( n \)th term in the Taylor expansion of \( \varphi_{m,h}(\zeta) \) is indeed an odd polynomial in \( m \), of degree \( 2n + 2h - 1 \).

Finally, the denominator in Lemma 4.3 has order \( h + 1 \) at \( \zeta = 1 \). On the other hand, for every \( r \) we have an order of \( 2r \) from \( \eta_{2r} \), Lemma 4.1 implies that \( f_{m,h}(\zeta) \) has order at least \( h - r \) (this is clear also if \( r > h \)), and the external multiplier contributes another 1 to the order. Therefore this summand has order at least \( h + r + 1 \geq h + 1 \) at \( \zeta = 1 \), meaning that the total quotient is finite at \( \zeta = 1 \), hence lies in \( \mathbb{Z}[\zeta] \), for every \( m \) and \( h \) as desired. This proves the proposition.

We can now prove the basic case \( D = p^{\pm 1} \) of our main result. For the even part we reproduce the proof from [13] (which becomes shorter using our results), but the proof for the odd part seems to be new.

**Lemma 4.5** Let \( D \) be the prime discriminant form \( p^{\pm 1} \), and take \( 0 \neq \eta \in D \). Then the action of \( \rho_D \) on the basis \( \{ \rho_D(T^a)a_{0,0}^{D,0}, 1 \leq a \leq p^{\pm 1/2} \} \cup \{ \rho_D(T^a)a_{n,0}^{D,0}, 1 \leq a \leq p^{-(p-1)/2} \} \) involves only integral elements from \( \mathbb{Z}[\zeta_D] \).

**Proof** As the choice of basis suggests, we decompose \( \rho_D \) into the even and odd parts, and consider each one separately. As elements of \( D \) are \( m \eta \) for \( m \in \mathbb{F}_p^{\times} \), we simplify the notation by writing just \( m \) for every index \( m \eta \), and set \( \zeta := \epsilon(p^{1/2}) \epsilon(p^{1/2}) \) for some \( t \in \mathbb{F}_p^{\times} \) with \( (\frac{t}{p}) = \pm 1 \) is the sign for which \( D = p^{\pm 1} \), so that \( \mathbb{Z}[\zeta_D] = \mathbb{Z}[\zeta] \).

For the even part, consider the vector \( a_{0,0}^{D,0} = a_{0,0}^D \). Recall from the proof of Theorem 2.8 that the \( \pm \) part of \( \mathbb{C}[D] \) consists of a basis of eigenvectors for \( \rho_D(T) \) with different values, and all of these appear in \( a_{0,0}^{D,0} \). Since there are \( \frac{p+1}{2} \) such eigenvalues, which are of the form \( \zeta^j \) for \( t \in \mathbb{F}_p^{\times}/(\mathbb{F}_p^{\times})^2 \) determined by the sign \( \pm \) of \( D \) and \( 0 \leq j \leq \frac{p-1}{2} \), we deduce that the first asserted set, namely \( \{ \rho_D(T^a)a_{0,0}^{D,0}, 1 \leq a \leq p^{1/2} \} \), forms a basis for the corresponding sub-representation of \( \rho_D \). We prove that this basis has the desired property.

Indeed, it suffices to verify this property for \( T \) and \( S \). For \( T \), each vector but the last is taken to the next one, and the statement is clear. For the last vector, the previous paragraph implies that the presentation of its \( \rho_D(T) \)-image in our basis is based on the coefficients of
the characteristic polynomial of $\rho_D(T)$. But we have seen that the roots of this polynomial are powers of $\zeta$, so that these coefficients are in $\mathbb{Z}[\zeta]$ as desired.

For the action of $S$, Remark 3.4 allows us to apply Proposition 3.3, with the parameters $\eta = \lambda = \ell_2$, for evaluating the action on the vectors with $l > 0$. This determines the corresponding $\rho_D(S)$-image as $e\left(\frac{\text{sgn}(D(l)) - \text{sgn} D}{8}\right)$, which is just a sign (in fact, the results of [11, 12, 17], and others determine this coefficient to be just the Legendre symbol $(\frac{\ell}{p})$), times $\rho_D(T^{-k})a_{0,0}^{D,+}$, where we can take $-p < k < 0$. This vector thus has coefficients from $\mathbb{Z}[\zeta]$ in our basis, by which we know about powers of $T$.

It remains to verify that $\rho_D(S)a_{0,0}^{D,+} = e(-\text{sgn} D/8)\epsilon_0$ is spanned in our basis with coefficients from $\mathbb{Z}[\zeta]$. Writing the latter vector as $\sum c_l^{(p-1)/2} \rho(D(T))a_{0,0}^{D,+}$, and checking the coefficients of $a_{0,0}^{D,+}$, we find that $\sum c_l^{(p-1)/2} \zeta^{2l} = 0$ for every $1 \leq j \leq \frac{p-1}{2}$, so that the polynomial $\sum c_l^{(p-1)/2} \zeta^{2l}$ is a multiple of $\prod_{j=1}^{(p-1)/2}(\zeta^{2j})$ (the multiplier is $c := c_{p-1/2}$). It suffices to check that this multiplier is in $\mathbb{Z}[\zeta]$, for which we recall that $e(-\text{sgn} D/8)$ is $\pm 1$ when $p \equiv 1 \pmod{4}$ and $\pm i$ in case $p \equiv 1 \pmod{3}$, and comparing the coefficients of $\epsilon_0$ on both sides yields $1 = \sum c_l^{(p-1)/2} = \frac{\epsilon_0}{\sqrt{p}} \prod_{j=1}^{(p-1)/2}(1 - \zeta^{2j})$, with $\pm = \left(\frac{-1}{p}\right)$. But each multiplier generates the prime ideal $p$ from the proof of Proposition 4.4, and it is known that $\sqrt{\pm p}$ lies in $\mathbb{Z}[\zeta]$ and generates the ideal $p^{(p-1)/2}$ there. It follows that the product and $\sqrt{\pm p}$ generate the same ideal, and thus $c \in \mathbb{Z}[\zeta]$ (and is a unit there). This establishes the desired property of our basis of the even part.

In the odd part there is no vector $a_{0,0}^{D,-}$, and indeed our basis consists of images of $a_{1,0}^{D,-}$ instead. These images indeed form a basis for this sub-representation, since the eigenvalues of $\rho_D(T)$ there are $\zeta^p$ with $1 \leq j \leq \frac{p-1}{2}$, and the action of $T$ involves coefficients from $\mathbb{Z}[\zeta]$ by the same argument.

Next, we claim that any other possible generator, namely $a_{k,0}^{D,-}$ for some non-zero $k \in \mathbb{F}_p$, is generated by our basis over $\mathbb{Z}[\zeta]$. Indeed, using that basis for the anti-symmetric part of $\rho_D$, there are coefficients $c_l$ in $\mathbb{C}$ (and, in fact, in $\mathbb{Q}(\zeta)$ by Proposition 1.1) with $0 \leq l \leq \frac{p-3}{2}$, such that $a_{k,0}^{D,-} = \sum_{l=0}^{(p-3)/2} c_l \rho(D(T))a_{1,0}^{D,-}$. But Eq. (8) gives

$$a_{1,0}^{D,-} = \sum_{m=0}^{p-1} \frac{\zeta^{2m} - \zeta^{-2m}}{\sqrt{2p}} \epsilon_m = \sum_{m=1}^{(p-1)/2} \frac{\zeta^{2m} - \zeta^{-2m}}{\sqrt{p}} a_{0,m}^{D,-},$$

$\rho_D(T)$ multiplies the $m$th summand by $\zeta^{bm^2}$, and in $a_{0,m}^{D,-}$ we replace $\zeta^m$ by $\zeta^{\pm km}$. Therefore our coefficients are the ones from Proposition 4.4, yielding their integrality.

Combining with the action of $T$, it thus follows that $\rho_D(T^{-k})a_{k,0}^{D,-}$ is also spanned by our basis over $\mathbb{Z}[\zeta]$, and since for $l > 0$ a similar application of Proposition 3.3 (using Remark 3.4 again) identifies $\rho_D(S)\rho_D(T^l)a_{k,0}^{D,-}$ with this vector (for the appropriate $k$) up to $\pm 1$ and a power of $\zeta$, the action of $S$ on these vectors has coefficients in $\mathbb{Z}[\zeta]$ in this basis.

Finally, for $\rho_D(S)a_{1,0}^{D,-}$, which equals $e(-\text{sgn} D/8)a_{0,1}^{D,-}$ via Proposition 2.7, writing it again as $\sum_{l=0}^{(p-3)/2} c_l \rho(D(T))a_{1,0}^{D,-}$ and expanding using the $a_{0,m}^{D,-}$'s, the coefficients become those of $c \prod_{j=1}^{(p-1)/2}(\zeta^{2j})$ (now with $c := c_{p-3/2}$), and since $c$ now satisfies the equality $1 = \frac{\epsilon_0}{\sqrt{p}} \prod_{j=1}^{(p-1)/2}(1 - \zeta^{2j})$, we deduce that $c$ is in $\mathbb{Z}[\zeta]$ and (and now generates $p$), and the action of $S$ in our basis is defined over $\mathbb{Z}[\zeta]$ as well. This proves the lemma. 

In fact, one can prove Lemma 4.1 with even polynomials and $h \geq 0$, with the initial condition where $f_{0,0}^{(r)}$ equals 2 for $r = 0$ and $m_r^{(m-1+r)}$ when $r > 0$. Then one can
Theorem 4.7 Let $D$ be any discriminant form, and set $N$ to be the level of $D$. Then there is an explicit basis for the space $\mathbb{C}[D]$ on which the associated Weil representation $\rho_D$ operates with coefficients that lie in $\mathbb{Z}([\zeta_N])$. Moreover, away from the anisotropic part of the $p$-parts of $D$, the action is given via permutation matrices with non-vanishing entries that are powers of $\zeta_N$.

Proof Decomposing $\rho_D$ as the tensor product of the Weil representations arising from its $p$-adic parts, and recalling that if $N = \prod_{p} p^{\nu_p}$ then the level of the $p$-adic part is $p^{\nu_p}$ and $\mathbb{Z}([\zeta_N])$ is the compositum ring of the $\mathbb{Z}([\zeta_{p^\nu}])$'s, it suffices to consider the case where $D$ has prime power level. In this case we apply induction on the cardinality of the maximal
isotropic subgroup of $D$, where the case in which the maximal isotropic subgroup is trivial is proved in Proposition 4.6. Consider now the case where $D$ contains isotropic elements, and let $H$ be a maximal quasi-isotropic subgroup of $D$. Observing that the anisotropic discriminant forms $2^{−2}$, $4^{±1}$, and $2^{±1} 4^{±1}$ from Proposition 4.6 contain quasi-isotropic subgroups, we deduce that the quotient $A := H^⊥/H$ is either trivial or of prime exponent $p$.

Now, when $A$ is trivial the result follows directly from Theorems 2.2 and 2.4, combined with Proposition 3.13. Assuming now that $A$ is non-trivial, and so is $H$, we take an isotropic subgroup $J$ of $H$, of cardinality $p$, set $B := J^⊥/J$, and recall that Corollary 1.9 embeds the representation $\mathbb{C}[B]$ into $\mathbb{C}[D]$ via $\uparrow_J$. Then Remark 3.11 explains, via the same argument and the last assertion of Proposition 3.13, how Theorem 3.10 yields a basis for the orthogonal complement of $\uparrow_J \mathbb{C}[B]$ inside $\mathbb{C}[D]$ in which the coefficients are from $\mathbb{Z}[\zeta_N]$, and it remains to obtain such a basis for $\uparrow_J \mathbb{C}[B]$ itself. But as Corollary 1.9 shows that $\uparrow_J$ is an embedding of representations, the level of $B$ divides that of $D$, and a maximal isotropic subgroup in $B$ is $H/J$ which is of smaller cardinality, this part is covered by the induction hypothesis. This proves the theorem.  

$\blacksquare$

Remark 4.8 The Weil representation $\rho_D$ is essentially a representation of a finite group over a number field $\mathbb{K}$. Then the sum of the images of any lattice under the group action yields an invariant lattice (see, e.g., Theorem 2.3 of [9]), which has, as a module, a Steinitz class in the class group of $\mathbb{K}$. By extending scalars to a field in which this class becomes trivial (e.g., the Hilbert class field of $\mathbb{K}$), one immediately obtains the existence of some basis in which the action is integral, perhaps over some field extension (see also Theorem 23.18 of [4]). In fact, the local nature of $\rho_D$, which we already used in the proof of Theorem 4.7, allows us to establish this result without the need to extend the field (this is not possible for every representation—for an example see, e.g., [10]): The representation associated with the $p$-part of $D$ is defined over $\mathbb{Q}[\zeta_{pe}]$ for some $e \geq 1$, with ring of integers $\mathbb{Z}[\zeta_{pe}]$, and the denominators in the natural basis only involve the unique prime $P$ lying over $p\mathbb{Z}$. Therefore the module considered in [9] contains the initial free module with finite index, and the quotient admits a composition series all of whose simple quotients are isomorphic to $\mathbb{Z}[\zeta_{pe}]/P$. But since $P = \mathbb{Z}[\zeta_{pe}](1 - \zeta_{pe})$ is principal, such quotients do not change the ideal class of the module. Thus the Steinitz class in question is trivial, the module from [9] is free and admits a basis over $\mathbb{Q}[\zeta_{pe}]$ in which the action is integral, and by tensoring over all primes $p$ we obtain the result for the full representation $\rho_D$. Thus, the strength of Theorem 4.7 is not so much in the proof of existence, but rather in the explicit construction of that basis, where in (generally many) parts of that basis the action is also particularly simple.

We conclude this section by remarking that while Theorems 2.2 and 2.4 make the formulae for the action of every element in that basis very explicit, we can no longer do that in the general case considered in Theorem 4.7. Indeed, when one $p$-part involves a non-trivial anisotropic quotient, the formulae resulting from Lemma 4.5 and Proposition 4.6 become complicated when one tries to consider a general element of $\text{Mp}_2(\mathbb{Z})$, and Remark 3.12 shows that the extra parameter from Theorem 3.10, in the corresponding complement, can also be hard to evaluate.
5 Invariant vectors in Weil representations

Let $D$ be an arbitrary discriminant form, with the associated Weil representation $\rho_D$ of $\text{Mp}_2(\mathbb{Z})$ on the space $\mathbb{C}[D]$. A natural question would be to determine the subspace $\mathbb{C}[D]^{\text{inv}}$ of $\mathbb{C}[D]$ on which $\rho_D$ operates trivially, and investigate its properties. The paper [5] established one property, which is related to the integrality questions considered in this paper, by showing that $\mathbb{C}[D]^{\text{inv}}$ can always be defined over $\mathbb{Z}$.

Two well-known and basic conditions that make $\mathbb{C}[D]^{\text{inv}}$ trivial are as follows.

Lemma 5.1 If $\text{sgn} D$ is odd then $\mathbb{C}[D]^{\text{inv}} = \{0\}$. In addition, let $D_{\text{iso}}$ denote the set of isotropic elements of $D$. Then if $D \setminus D_{\text{iso}}$ surjects onto the quotient $D/D_{\text{iso}}^\perp$ then $\mathbb{C}[D]^{\text{inv}} = \{0\}$ as well.

Proof The first assertion follows from the fact that $Z^2 \in \text{Mp}_2(\mathbb{Z})$ always acts as the scalar $(-1)^{\text{sgn} D}$. For the second one, the action of $\rho_D(T)$ implies that $\mathbb{C}[D]^{\text{inv}} \subseteq \bigoplus_{\gamma \in D_{\text{iso}}^\perp} \mathbb{C}e_\gamma$, which we embed into $\bigoplus_{\gamma \in D_{\text{iso}}^\perp} \mathbb{C}e_\gamma$ where $D_{\text{iso}}^\perp$ denotes the subgroup of $D$ that is generated by $D_{\text{iso}}$. Since $\rho_D(S)$ must preserve $\mathbb{C}[D]^{\text{inv}}$, we deduce that $\mathbb{C}[D]^{\text{inv}}$ is contained in the subspace of $\bigoplus_{\gamma \in D_{\text{iso}}^\perp} \mathbb{C}e_\gamma$ that is taken to $\bigoplus_{\gamma \in D_{\text{iso}}^\perp} \mathbb{C}e_\gamma$ by $\rho_D(S)$.

The formula for $\rho_D(S)$ from Eq. (2) thus shows that if $\sum_{\gamma \in D_{\text{iso}}^\perp} c_\gamma e_\gamma$ is in the latter space then $\sum_{\gamma \in D_{\text{iso}}^\perp} c_\gamma e_\gamma (-(\gamma, \delta)) = 0$ for every $\delta \notin D_{\text{iso}}$. But elements in the same coset modulo $D_{\text{iso}}^\perp$ give the same relation, while a collection of different cosets yields linearly independent relations. Thus the surjectivity condition amounts to the elements of $D/D_{\text{iso}}^\perp$ producing all the $|D/D_{\text{iso}}^\perp| = |D_{\text{iso}}^\perp|$ linearly independent conditions, meaning that the space in question is $\{0\}$, and thus so is $\mathbb{C}[D]^{\text{inv}}$. This proves the lemma.

Remark 5.2 If $D_{\text{iso}}$ is a subgroup of $D$ such that the inclusion in $D_{\text{iso}}^\perp$ is strict (e.g., when $D$ is anisotropic and non-trivial, or cyclic of non-square order), then the surjectivity condition from Lemma 5.1 holds, and we get $\mathbb{C}[D]^{\text{inv}} = \{0\}$. Moreover, the signature condition implies that wherever $\mathbb{C}[D]^{\text{inv}} \neq \{0\}$, the Weil representation $\rho_D$ is a representation of $\text{SL}_2(\mathbb{Z})$, without double covers.

In contrast to Lemma 5.1 and Remark 5.2, when $D$ contains a self-dual isotropic subgroup $H$, the representation $\mathbb{C}[A]$ for $A = H^\perp / H$ is trivial, and its image under the operator from Corollary 1.9 lies in $\mathbb{C}[D]^{\text{inv}}$. Moreover, it is known that in this case $\mathbb{C}[D]^{\text{inv}}$ is spanned by these images for the various self-dual isotropic subgroups of $D$. This was proved as Theorem 5.5.7 of [7] in the language of codes and mentioned as Theorem 1 of [5], and the proof was translated to our terminology in Theorem 4.4 of [1]. Note that some discriminant forms need not contain self-dual isotropic subgroups at all, but still admit invariant vectors—see, e.g., Theorem 5.6 below.

These results, however, do not say much about the dimension of $\mathbb{C}[D]^{\text{inv}}$ (or alternatively the number of such subgroups $H$ and the dimension of relations between their images). We now demonstrate how the explicit formulae from Theorems 2.2, 2.4, and 3.10 can be used for determining the dimension of $\mathbb{C}[D]^{\text{inv}}$ in many cases. We only present two families, one involving self-dual isotropic subgroups in a strong way, and the other not necessarily having ones at all.

For defining the first family, let $G$ be any finite Abelian group, with dual $G^*$ (which is thus isomorphic to $G$ as an Abelian group, although not canonically). Then the generalized hyperbolic plane associated with $G$ is the discriminant form $U_G := G \oplus G^*$, where given $\gamma \in G$ and $\phi \in G^* = \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$, the element $\gamma + \phi \in U_G$ satisfies $((\gamma + \phi)^2) = \phi(\gamma)$. For defining the first family, let $G$ be any finite Abelian group, with dual $G^*$ (which is thus isomorphic to $G$ as an Abelian group, although not canonically). Then the generalized hyperbolic plane associated with $G$ is the discriminant form $U_G := G \oplus G^*$, where given $\gamma \in G$ and $\phi \in G^* = \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$, the element $\gamma + \phi \in U_G$ satisfies $((\gamma + \phi)^2) = \phi(\gamma)$.
For obtaining the formula for \( \dim \mathbb{C}[U_G]^{\text{inv}} \) we shall use the following simple lemma. We denote Euler’s totient function as usual by \( \varphi(n) := n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \).

**Lemma 5.3** Take two numbers \( n \) and \( m \), with \( m \) dividing \( n \) consider the Abelian group \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \), and let \( X_{n,m} \) be the set of pairs of elements in that group that generate it. Then the group \( \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) \) acts on \( X_{n,m} \) with \( \varphi(m) \) orbits.

**Proof** We write our group as a quotient of \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \). Then \( X_{n,m} \) is the quotient of \( X_{n,n} \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \), on which \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \) acts transitively. Now, \( X_{n,n} \) can be presented as \( \bigcup_{h \in (\mathbb{Z}/n\mathbb{Z})^*} \text{SL}_2(\mathbb{Z}/n\mathbb{Z})\left(\begin{smallmatrix} 1 & 0 \\ 0 & h \end{smallmatrix}\right) \), and we project this onto \( X_{n,m} \). This gives \( \varphi(m) \) orbits with \( h \in (\mathbb{Z}/m\mathbb{Z})^* \), and since they are clearly distinct after projecting onto \( X_{m,m} \) and working modulo \( m \), they are distinct in \( X_{n,m} \). This proves the lemma.

The dimension of \( \mathbb{C}[D]^{\text{inv}} \) for \( D = U_G \) for such \( G \) can now be determined.

**Theorem 5.4** Take \( D \) to be the generalized hyperbolic plane \( U_G \) that is associated with the finite Abelian group \( G \). For every pair of positive integers \( m \) and \( n \), denote by \( S_{m,n}(G) \) the number of subgroups of \( G \) that are isomorphic to \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \). Then we have

\[
\dim \mathbb{C}[D]^{\text{inv}} = \sum_{0 < m \mid n} S_{n,m}(G) \varphi(m).
\]

**Proof** Since \( \text{sgn} D = 0 \) (because \( G \) is a self-dual isotropic subgroup of \( D := U_G \), and so is \( G^* \)), and \( D \) has level \( N \) which equals the exponent of \( G \), it is known (see, e.g., [12] or [17]) that \( \rho_D \) is essentially a representation of the finite group \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \), whose cardinality is \( \Delta_N := N^3 \prod_{p \mid N} \left(1 - \frac{1}{p}\right) \). Relating \( \rho_D \) with the trivial representation of this finite group via the classical formula of Frobenius thus expresses \( \dim \mathbb{C}[D]^{\text{inv}} \) as

\[
\frac{1}{\Delta_N} \sum_{M \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})} \text{Tr} \rho_D(M).
\]

We take a basis for \( \mathbb{C}[D] \) as in Lemma 1.2, where for the set of representatives \( \mathcal{R} \) for \( D/H \) (also for \( \eta \)) we can take \( G^* \). Theorem 2.2 expresses the action of any element of \( M \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) in this basis, where for every such \( \nu \) in the given basis, \( M \nu \) also lies there because both parameters are from the subgroup \( G^* \). Moreover, as \( G^* \) is also isotropic, the expression from Eq. (4) is trivial for every \( M \) and \( \nu \). Thus the contribution of the vector \( a^\nu \) to \( \text{Tr} \rho_D(M) \) is 1 when \( M \nu = \nu \), and 0 otherwise. This allows us to write

\[
\dim \mathbb{C}[D]^{\text{inv}} = \sum_{\nu \in (G^*)^2} \frac{\text{St}(\nu)}{\Delta_N},
\]

where \( \text{St}(\nu) \) is the stabilizer of \( \nu \) under the action of \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) induced from Remark 1.8.

We partition the latter sum according to the orbits, in \( (G^*)^2 \), of the action of \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \).

Since all the elements in the same orbit have the same stabilizer, they give the same contribution. But as the Orbit-Stabilizer Theorem determines the size of the orbit of \( \nu \) to be \( \frac{\Delta_N}{\text{St}(\nu)} \), we deduce that \( \dim \mathbb{C}[D]^{\text{inv}} \) is just the number of these orbits.

Now, given \( \nu = (\eta \lambda) \) with \( \eta \) and \( \lambda \) in \( G^* \), it is easy to check that for every element of the orbit of \( \nu \), the corresponding pair of elements of \( G^* \) generates the same subgroup as \( \eta \) and \( \lambda \). We thus partition our orbits according to these subgroups, and since only groups that can be generated by two elements are involved, each such group is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \) for \( n \) and \( m \) with \( m \mid n \) (note that \( m \), and also \( n \), can be 1 when this group is cyclic). On an orbit associated with such a subgroup of \( G \), the group \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) acts through its quotient \( \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) \) (as \( n \mid N \)).

But the fact that the action is on sets of generators yields, via Lemma 5.3, that each such subgroup contributes \( \varphi(m) \) orbits. Since for such \( n \) and \( m \) there are \( S_{n,m}(G) \) such subgroups, the total number of orbits is given by the asserted formula. This proves the theorem. □
As a special case, we deduce a quick proof of Lemma 3.2 of [16] (see also Corollary 4.16 of [1]). We denote by $\sigma_0(N)$ the number of positive divisors of the integer $N$.

**Corollary 5.5** Let $U_N$ denote the discriminant form associated with the lattice $\Gamma_{1,1}(N)$ for some integer $N$, where $e$ and $f$ are the $U_N$-images of the natural, isotropic generators of $\Gamma_{1,1}(N)^*$. For every $d|N$, set $a_d := \sum_{d|t} \sum_{N/t} \sum_{0 < a < N/t} e^{i\pi af^2}$. Then $\{a_d\}_{d|N}$ form a basis for $\mathbb{C}[U_N]^{\text{inv}}$.

**Proof** The discriminant form $D := U_N$ is the generalized hyperbolic plane associated with a cyclic group $G$ of order $N$. We shall denote by $e$ a generator of $G$, and by $f$ a generator of $G^*$. Applying Theorem 5.4, we have $S_{n,m} = 1$ when $m = 1$ and $n|N$ and 0 otherwise, implying that $\dim \mathbb{C}[D]^{\text{inv}} = \sigma_0(N)$.

Now, Corollary 1.9 shows that for every self-dual isotropic subgroup $H$ of $D$, which is of size $N$, the image of the trivial representation under $\uparrow_H$ gives an 1-dimensional subspace of $\mathbb{C}[D]^{\text{inv}}$. For $d|N$ we denote by $H_d$ the subgroup generated by $de$ and $N/f$, and it is easy to verify that $(H_d)_{d|N}$ are precisely the self-dual isotropic subgroup of $D$. Moreover, It is clear that $a_d = \sqrt{\frac{N}{d}}$ times $a_{H_d}$, which thus generates the image of $\uparrow_{H_d}$, and there are $\sigma_0(N)$ such vectors.

But given a divisor $D|N$, the coefficient in front of $e_{de}$ in a linear combination of the sort $\sum_{d|N} c_d a_d$ is just $\sum_{d|D} c_d$, so that if $d$ is the minimal divisor of $N$ with $c_d \neq 0$ then $e_{de}$ appears with a non-zero coefficient. It follows that $\{a_d\}_{d|N}$ are $\sigma_0(N)$ linearly independent vectors in a $\sigma_0(N)$-dimensional space, which thus form a basis. This proves the corollary.

\[ \square \]

In contrast with Corollary 5.5, when $G$ is not cyclic, the images arising from self-dual are no longer linearly independent. Section 4.3 of [1] examines the case where $G$ is isomorphic to $G_{N,M} := (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/M\mathbb{Z})$ where $M$ divides $N$ (then $U_G$ is $D_{N,M}$ in the notation of [1]). To illustrate the type of expressions resulting in this case, we define

$$
\psi(n) := n \prod_{p|n} (1 + \frac{1}{p}).
$$

Then, since $S_{k,1}(G_{k,d})$ equals $\frac{\psi(d)}{\psi(k/d)}$ wherever $d$ divides $k$, and a subgroup counted in $X_{n,m}(G_{N,M})$ is contained in $G[n]$ (the subgroup of $G$ annihilated by $n$) but must contain $G[m]$, we deduce that $S_{n,m}(G_{N,M}) = \frac{\psi(n/m)}{\psi(n)/\gcd(n,M)}$. Thus Theorem 5.4 determines $\dim \mathbb{C}[D_{N,M}]$ as

$$
\sum_{m|M} \sum_{d|M} \frac{\psi(kd)}{\psi(k)} = \sum_{t|M} \sum_{k|\frac{N}{t}} \frac{\psi(kd)}{\psi(k)}.
$$

In the prime power case, with $N = p^r$ and $M = p^s$ for $r \geq s$, the latter number becomes

$$
(r + 1 - s)(s + 1)p^s - (r - 1 - s)sp^{r-1}.
$$

On the other hand, if $M = p$ is prime and $N_p$ is the maximal divisor of $N$ that is prime to $p$ then it equals $(2p - 1)\sigma_0(N/p) + 2\sigma_0(N_p)$ (it also produces $\sigma_0(N)$ when $M = 1$, as in Corollary 5.5).

Section 4.3 of [1] also indicates a few constructions, which may serve to count the number of self-dual isotropic subgroups of $D_{N,M}$. We only remark that for $M = p$ a prime, there are $\sigma_0(N/p)$ isotropic subgroups $H$ of $D_N$ with the property that $K^\perp/K \cong D_p$, and each one of them is the kernel of a surjective projection onto $D_p$ of $2p - 2$ self-dual isotropic subgroups of $D_{N,p}$. Since there are also $2\sigma_0(N) = 2\sigma_0(N_p) + 2\sigma_0(N/p)$ products of a self-dual isotropic subgroup of $D_N$ with one of $D_p$, the total number of such subgroups
of $D_{N,p}$ is $2p\sigma_0(N/p) + 2\sigma_0(N_p)$. The $\sigma_0(N/p)$ resulting linear relations are in one-to-one correspondence with the groups $K$ mentioned above (the case $N = M = p$, with one relation among $2p + 2$ groups, appears in Proposition 4.25 of [1]). It would be interesting to investigate the combinatorics of the relations arising from more complicated groups $G$.

We now determine the dimension of the space of invariants for another family of discriminant forms. This result also appears as Theorem 4.3 of [6], but with a different proof.

**Theorem 5.6** Assume that $D$ is a vector space over $\mathbb{F}_p$, and that if $p = 2$ then its index is II. Let $p^d$ be the cardinality of a maximal isotropic subgroup $H$ of $D$, and denote the size of the anisotropic discriminant form $A := H^\perp / H$ by $p^r$. Then the dimension of $\mathbb{C}[D]^{\text{inv}}$ is

$$p^r \frac{(p^d-1)(p^{d-2}-1)}{p^2-1} + \frac{p^d-1}{p-1} + \delta_{r,0}.$$ 

Proof Lemma 5.1 (or Remark 5.2) deals with the case where $d = 0$ and $r > 0$, and if $d = r = 0$ then $D$ is trivial and so is $\rho_D$. We have thus established the induction basis for working by induction on $d$.

If $d > 0$ then $H$ is non-trivial, so we can take any cyclic subgroup $J \subseteq H$, and set $B := J^\perp / J$. Then we have an orthogonal decomposition of $\mathbb{C}[D]$ as the direct sum of the sub-representation $\uparrow_J \mathbb{C}[B]$ from Corollary 1.9 and its orthogonal complement. The former contributes $\dim \mathbb{C}[B]^{\text{inv}}$, which by the induction hypothesis (with $d - 1$ and the same $r$) is $p^r \frac{(p^{d-1}-1)(p^{d-2}-1)}{p^2-1} + \frac{p^{d-1}-1}{p-1} + \delta_{r,0}$, and for the complement we argue as in the proof of Theorem 5.4.

Indeed, the signature is even and the level is $p$ (this is why we need the index to be II when $p = 2$), so we view $\rho_D$ as a representation of $\text{SL}_2(\mathbb{F}_p)$, of cardinality $p^3 - p$. Take now a representing set $\mathcal{Q}$ for the vectors $v \in D^2 \setminus (J^\perp)^2$ modulo the relations from Corollary 3.8, and we allow ourselves the abuse of terminology by saying that $M \in \text{SL}_2(\mathbb{F}_p)$ stabilizes a vector $v \in \mathcal{Q}$ if $b^H_{Mv}$ is a multiple of $b^H_v$ (or pairs non-trivially with it, which is the same by this corollary). By setting $\delta_{HJ}^{Mv} = \delta_{HJ}(M,v)b^H_v$, the argument from the proof of Theorem 5.4, but now using Theorem 3.10 and Remark 3.11, yields

$$\dim \left(\uparrow_J \mathbb{C}[B]\right)^{\text{inv}} = \sum_{v \in \mathcal{Q}} \sum_{M \in \text{St}(v)} \frac{\delta_{HJ}^{Mv}(M,v)\mathcal{e}(Q(M,v))}{p^3 - p},$$

with our modified notion of the stabilizer.

Moreover, a conjugation argument implies that if $M \in \text{St}(v)$, $\alpha$ is the constant such that $\rho_D(M)\alpha b^H_v = \alpha b^H_{Mv}$, and $N \in \text{SL}_2(\mathbb{F}_p)$, then the element $N M N^{-1}$ stabilizes $N v$ and we have $\rho_D(N M N^{-1})b^H_{Nv} = \alpha b^H_{Nv}$ with the same constant $\alpha$. Therefore we can once again replace $\mathcal{Q}$ by a subset $\mathcal{O}$ consisting of one representative for each orbit of $\text{SL}_2(\mathbb{F}_p)$ on $\mathcal{Q}$ modulo the relations from Corollary 3.8, and obtain

$$\dim \left(\uparrow_J \mathbb{C}[B]\right)^{\text{inv}} = \sum_{v \in \mathcal{O}} \sum_{M \in \text{St}(v)} \frac{\delta_{HJ}^{Mv}(M,v)\mathcal{e}(Q(M,v))}{|\text{St}(v)|}.$$  

(10)

Now, the images in $D/H^\perp$ of the entries $\eta$ and $\lambda$ of a vector $v \in \mathcal{O} \subseteq \mathcal{Q}$ span a subspace that is not contained in $J^\perp / H^\perp$, and can therefore be of dimension 1 or 2. Moreover, this subspace remains invariant under the action of $\text{SL}_2(\mathbb{F}_p)$, and thus for each such subspace we can consider the set of orbits associated with it. Via the action of $\text{SL}_2(\mathbb{F}_p)$ we may assume that in any element $v = \left(\begin{smallmatrix} \eta \\ \lambda \end{smallmatrix}\right) \in \mathcal{O}$ we have $\lambda \in J^\perp$ and $\eta \notin J^\perp$, so that $b^H_v = \alpha^H_v$.  

Thus Corollary 3.8 (or Lemma 1.2) implies that for a representative associated with a subspace of \( D/H^\perp \), the \( \eta \)-coordinate only appears in the basis via its \( D/H^\perp \)-image, but there are \( p' \) different \( \lambda'_s \) in the basis with the same given \( D/H^\perp \)-image.

Given a representative \( \nu \in \mathcal{O} \) that is associated with a 2-dimensional subspace of \( D/H^\perp \), the group \( \text{St}(\nu) \) is trivial, and thus such an orbit contributes 1 to the right hand side of Eq. (10). Moreover, as in Lemma 5.3, the group \( \text{SL}_2(\mathbb{F}_p) \) acts on the set of bases for such a space with \( \varphi(p) = p - 1 \) orbits. Since there are \( \frac{(p^d - 1)(p^d - 2)}{p(p - 1)} \) 2-dimensional subspaces of \( D/H^\perp \), out of which \( \frac{(p^d - 1)(p^d - 2)}{p(p - 1)} \) are contained in \( J^\perp \) and must be excluded, and each subspace is associated with \( p' \) \( (p - 1) \) orbits, these combine to a total of \( p' \left( \frac{(p^d - 1)(p^d - 2)}{p(p - 1)} \right) \) (this can be simplified to \( p^{d+r-2}(p^d - 1) \), but the expanded form is better for merging with the induction hypothesis).

On the other hand, for \( \nu \in \mathcal{O} \) for which the subspace of \( D/H^\perp \) is 1-dimensional, we have \( \lambda \in H^\perp \) and therefore the powers of \( T \) stabilize \( \nu \), with \( \delta_{H^\perp}(T^l, \nu) = \epsilon_l(T^l, \nu) = 1 \) (see Proposition 3.9 and Corollary 3.8 or Lemma 1.2 once again), and \( Q(T^l, \nu) = l_{\mathcal{O}}^2 \) from Eq. (4). It follows that the contribution of such an orbit is \( \frac{1}{p} \sum_{l \in \mathbb{F}_p} e(l_{\mathcal{O}}^2) \), which is 1 if \( \lambda \) is isotropic and 0 otherwise. But since we assumed that \( H \) was a maximal isotropic subgroup of \( D \), the only contributing elements are those with \( \lambda \in H \). Therefore for each one of the \( \frac{p^d - 1}{p - 1} \) 1-dimensional subspaces of \( D/H^\perp \), except for the \( \frac{p^d - 1}{p - 1} \) subspaces that are contained in \( J^\perp /H^\perp \), we have a contribution of 1 (this difference reduces to \( p^{d-1} \), but once again we leave it in the expanded form).

Adding these terms to the formula for \( \dim \mathbb{C}[D]_{\text{inv}} \) from the induction hypothesis gives the desired result. This proves the theorem. \( \square \)

Note that the case \( r = 0 \) in Theorem 5.6 is also covered by Theorem 5.4, essentially with \( G = D/H^\perp = D/H \). As the calculations from the proof show, the three terms in Theorem 5.6 are \( S_{p+p}(G)(p - 1) \), \( S_{p,1}(G) \), and \( S_{1,1}(G) \) respectively. On the other hand, with \( r > 0 \) we only have \( 2^r \left( \frac{2d+1}{2} \right) \) with \( r = 2 \) when \( p = 2 \), and for odd \( p \) the possibilities are \( p^{\pm(2d+1)} \) with \( r = 1 \) or \( p^{2(2d+2)} \) with signature 4 and \( r = 2 \).

Finally, note that when \( d = 1 \) and \( r > 0 \) Theorem 5.6 gives \( \dim \mathbb{C}[D]_{\text{inv}} = 1 \), so it is interesting to see what form does a generator of this space takes. It turns out that for \( r = 2 \), i.e., when \( D \) is \( p^{-2} \) or \( 2^{-4} \), this vector has a simple form \( \sum_{0 \neq \gamma \in \mathcal{O}_{(a)} \mathcal{T}_{\gamma} - (p - 1)l_0} \). For odd primes \( p \) and \( r = 1 \), one can show that while the special orthogonal group of \( D \) as a quadratic space over \( \mathbb{F}_p \) operates transitively on the non-zero isotropic vectors in \( D \), its connected component (defined by trivial spinor norm) operates with two orbits. Moreover, for such an isotropic element \( \gamma \), a multiple \( l_{\gamma} \mathcal{T}_{\gamma} \) lie in the same orbit as \( \gamma \) if and only if \( \left( \frac{l}{p} \right) = +1 \). Writing these orbits as \( \mathcal{O}_{\pm} \) (with an arbitrary choice of signs), the expression \( \sum_{\gamma \in \mathcal{O}_{\pm}} \mathcal{T}_{\gamma} - \sum_{\gamma \in \mathcal{O}_{-}} \mathcal{T}_{\gamma} \) spans \( \mathbb{C}[D]_{\text{inv}} \) (the invariance under \( \rho_{D}(S) \) and the form of intersections with isotropic lines is closely related to the classical Gauss sum for \( p \)). The lifts of these vectors by the corresponding arrow operators (or, in general, their tensor products over different primes) were recently shown in [6] to generate the invariants of the Weil representation of every discriminant form \( D \) (necessarily of even signature—see Lemma 5.1).

We conclude by posing a related question, which is to determine, for some discriminant forms \( D \), the subspace of \( \mathbb{C}[D] \) on which \( \rho_{D} \) operates via a given non-trivial character of \( \text{Mp}_2(\mathbb{Z}) \). The method proving Theorems 5.4 and 5.6, and possibly also twists of the maps from [6], can surely be applied to shed light on this question as well in some instances.
Data availability  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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