THE POLYNOMIAL METHOD FOR LIST-COLOURING EXTENDABILITY OF OUTERPLANAR GRAPHS

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ABSTRACT. We restate theorems of Hutchinson [5] on list-colouring extendability for outerplanar graphs in terms of non-vanishing monomials in a graph polynomial, this way obtaining Alon-Tarsi equivalent for her work. This allows to simplify the proofs as well as obtain more general results.

1. Introduction

In his famous paper [6] Thomassen proved that every planar graph is 5-choosable. Actually, to proceed with inductive argument, he proved the stronger result.

Theorem 1.1 ([6]). Let \( G \) be any plane near-triangulation (every face except the outer one is a triangle) with outer cycle \( C \). Let \( x, y \) be two consecutive vertices on \( C \). Then \( G \) can be coloured from any list of colours such that the length of lists assigned to \( x, y \), any other vertex on \( C \) and any inner vertex is 1, 2, 3, and 5, respectively.

In other words vertices \( x \) and \( y \) can be precoloured in different colours. Basically, this theorem implies that any outerplanar graph is 3-choosable. Moreover, lists of any two neighbouring vertices can have a deficiency. To formalise this fact we say that a triple \((G, x, y)\), where \( G \) is outerplanar graph, \( x, y \in V(G) \) are neighbouring vertices is \((1, 2)\)-extendable in sense that \( G \) is colourable from any lists whose length is 1, 2 and 3 for vertex \( x \), \( y \) and any other vertex, respectively.

Hutchinson [5] analysed extendability of outerplanar graphs, in case when the selected vertices are not adjacent, showing that for any two vertices \( x, y \) of outerplanar graph \( G \) every triple \((G, x, y)\) is \((2, 2)\)-extendable. Of course, it is enough to prove this for outerplane 2-connected near-triangulation only as each outerplane graph can be extended to it just by adding some edges. It is folklore that 2-connected outerplane near-triangulations have unique proper 3-colouring, up to permutation of colours. The main theorem was the following.

Theorem 1.2 ([5]). Let \( G \) be outerplane 2-connected near-triangulation and \( x, y \in V(G) \), \( x \neq y \). Let \( c : V(G) \to \{1, 2, 3\} \) be any proper 3-colouring of \( G \). Then

(i) \((G, x, y)\) is not \((1, 1)\)-extendable;

(ii) \((G, x, y)\) is \((1, 2)\)-extendable if and only if \( c(x) \neq c(y) \);

(iii) \((G, x, y)\) is \((2, 2)\)-extendable.

Indeed, it is enough to prove the above theorem for near-triangulations with exactly 2 vertices of degree 2 and to let \( x \) and \( y \) be these degree 2 vertices. Hutchinson called such configurations fundamental subgraphs. Such a configuration can be obtained by successively shrinking outerplane near-triangulation along some chord.
(inner edge) that separates the component of the graph not containing vertices \(x\) and \(y\) (in case when \(xy \in E(G)\) this reduces to an edge \(xy\)). The general result follows now by successive colouring of shrinked parts using Theorem 1.1 — the chord is an outer edge of the shrinked component and its endpoints (already coloured) are these 2 precoloured vertices. Details are in [5]. Also in [5], Hutchinson provided further results about extendability of general outerplanar graphs, for which the conditions are more relaxed than those of Theorem 1.2, allowing for \((1,1)\)-extendability.

Recently, Zhu [8] strengthened the theorem of Thomassen in the language of graph polynomials showing that Alon-Tarsi number of any planar graph \(G\) satisfies \(AT(G) \leq 5\). His approach utilizes a certain polynomial arising directly from the structure of the graph. This graph polynomial is defined as:

\[
P(G) = \prod_{uv \in E(G), u < v} (u - v),
\]

where the relation \(<\) is an arbitrary ordering of the vertices of \(G\). Here we understand \(u\) and \(v\) both as the vertices of \(G\) and variables of \(P(G)\), depending on the context.

We refer the reader to [1, 2, 3] for the connection between list colourings and graph polynomials. The approach of Zhu may be described in the following form, analogous to Theorem 1.1.

**Theorem 1.3 ([8]).** Let \(G\) be any plane near-triangulation, let \(e = xy\) any its boundary edge. Denote other boundary vertices by \(v_1, \ldots, v_k\) and inner vertices by \(u_1, \ldots, u_m\). Then the graph polynomial of \(G - e\) contains a non-vanishing monomial of the form \(\eta x^{\beta} y^{\gamma} \prod_{i=1}^{n} v_i^{\alpha_i} u_i^{\beta_i} \ldots u_m^{\beta_m}\) with \(\alpha_i \leq 2, \beta_j \leq 4\) for \(i \leq k, j \leq m\).

The main tool connecting graph polynomials with list colourings is Combinatorial Nullstellensatz [1]. It implies that for every non-vanishing monomial of \(P(G)\), if we assign to each vertex of \(G\) a list of length greater than the exponent of corresponding variable in that monomial, then such list assignment admits a proper colouring.

We note, that this approach can be continued, allowing to obtain stronger equivalents of already known results for list-colouring. Moreover, an example of [4] where is proven that every planar graph \(G\) contains a matching \(M\) such that \(AT(G - M) \leq 4\), shows that with this approach it is possible to get results not known (or hard to prove) for ordinal list colouring.

In this paper we provide a graph polynomial analogue to the result of Hutchinson, obtaining a characterisation of polynomial extendability for outerplanar graphs, which may be presented in the form of the following theorem.

**Theorem 1.4.** Let \(G\) be any outerplane graph with \(V(G) = \{x, y, v_1, \ldots, v_n\}\). Then in \(P(G)\) there is non-vanishing monomial of the form \(\eta x^{\beta} y^{\gamma} \prod_{i=1}^{n} v_i^{\alpha_i}\) with \(\alpha_i \leq 2, \beta, \gamma \leq 1\) satisfying:

(i) \(\beta = \gamma = 1\) when every proper 3-colouring \(C\) of \(G\) forces \(C(x) = C(y)\);
(ii) \(\beta + \gamma = 1\) when every proper 3-colouring \(C\) of \(G\) forces \(C(x) \neq C(y)\);
(iii) \(\beta = \gamma = 0\) otherwise.

We note that our proofs are simpler than the ones of Hutchinson, which show the strength of the graph polynomial method for graph colouring problems. All considered graphs are simple, undirected, and finite. For background in graph theory see [7].
2. Outerplane near-triangulations

In this section we provide a graph polynomial analogue to Theorem 1.2. The main tool is the following theorem.

**Theorem 2.1.** Let $G$ be any 2-connected, outerplane near-triangulation with $V(G) = \{a, b, x, v_1, \ldots, v_n\}$, where $\deg(x) = \deg(a) = 2$, $\deg(v_i) \geq 3$ for $i \in \{1, \ldots, n\}$ and $a, b \neq x$. Then

$$P(G) = Q(G) + \eta_1 x v_1^2 \ldots v_n^2 a^0 b^2 + \eta_2 x v_1^2 \ldots v_n^2 a^1 b^1 + \eta_3 x v_1^2 \ldots v_n^2 a^2 b^0,$$

where $\{\eta_1, \eta_2, \eta_3\} = \{-1, 0, 1\}$, while $Q(G)$ is a sum of monomials of the form $\eta x^{\alpha_x} v_1^{\alpha_1} \ldots v_n^{\alpha_n} a^{\alpha_a} b^{\alpha_b}$, $\eta \neq 0$, with $(\alpha_x, \alpha_1, \ldots, \alpha_n) \neq (1, 2, \ldots, 2)$.

**Proof.** The proof is done by induction. For the base step, let $G$ be a triangle on vertices $\{a, b, x\}$. It is easy to check, that:

$$P(G) = (x - a)(a - b)(b - x) = x^2 a^0 b^1 - x^2 a^1 b^0 - x^0 a^2 b^1 + x^0 a^1 b^2 + x^1 a^2 b^0 - x^1 a^0 b^2 = Q(G) + x^1 a^2 b^0 - x^1 a^0 b^2,$$

hence we have $\eta_2 = 0$ and $\{\eta_1, \eta_3\} = \{-1, 1\}$, and with $Q(G)$ having necessary form, $G$ is concordant with the thesis.

We now proceed with the induction. Suppose $G$ is such that it fulfills the conditions of the theorem. Now we extend the graph by one vertex, without loss of generality we can adjoin it to vertices $a$ and $b$. Call this new vertex $y$, and the new graph $G'$. There are three possible cases:

1. $\eta_1 = 0$. As $\eta_1 = 0$ and $\{\eta_2, \eta_3\} = \{-1, 1\}$, we know that:

$$P(G) = Q(G) + \eta_2 x v_1^2 \ldots v_n^2 a^1 b^1 + \eta_3 x v_1^2 \ldots v_n^2 a^2 b^0 =$$

$$Q(G) + \eta_2 x v_1^2 \ldots v_n^2 a^1 b^1 - \eta_2 x v_1^2 \ldots v_n^2 a^2 b^0 =$$

$$Q(G) + \eta_2 x v_1^2 \ldots v_n^2 (a^1 b^1 - a^2 b^0).$$

Now $P(G') = P(G)(a - y)(b - y) = P(G)(ab - ay - by + y^2)$, thus:

$$P(G') = (Q(G) + \eta_2 x v_1^2 \ldots v_n^2 (a^1 b^1 - a^2 b^0))(ab - ay - by + y^2) =$$

$$Q(G)(ab - ay - by + y^2) + \eta_2 x v_1^2 \ldots v_n^2 (a^2 b^2 y^0 - a^2 b^1 y^1 - a^1 b^2 y^1 + a^1 b^1 y^2 - a^3 b^1 + a^3 y^1 + a^2 b^1 y^1 - a^2 b^0 y^2) = Q'(G') + \eta_2 x v_1^2 \ldots v_n^2 (a^2 b^2 y^0 - a^1 b^2 y^1 - a^2 b^0 y^2).$$

Now $y$ assumes role of $a$, and $a$ and $b$ can assume roles of $b$ and $v_{n+1}$, respectively, or the inverse may occur. In the first case, we have:

$$P(G') = Q'(G') + \eta_2 x v_1^2 \ldots v_n^2 (v_{n+1}^2 a^0 b^2 - v_{n+1}^2 a^1 b^1 - v_{n+1}^2 a^2 b^2),$$

thus $\{\eta_1, \eta_2\} = \{-1, 1\}$ and $\eta_3 = 0$, with the last monomial going into $Q'(G')$. With analogous calculations, in the second case we have $\{\eta_1, \eta_3\} = \{-1, 1\}$ and $\eta_2 = 0$. As $Q'(G')$ obviously contains only monomials of the form $\eta x^{\alpha_x} v_1^{\alpha_1} \ldots v_n^{\alpha_n+1} a^{\alpha_a} b^{\alpha_b}$, $\eta \neq 0$, $(\alpha_x, \alpha_1, \ldots, \alpha_n+1) \neq (1, 2, \ldots, 2)$, it can assume role of $Q(G)$, and the case is finished.

2. $\eta_2 = 0$. As $\eta_2 = 0$ and $\{\eta_1, \eta_3\} = \{-1, 1\}$, we know that:

$$P(G) = Q(G) + \eta_1 x v_1^2 \ldots v_n^2 a^0 b^2 + \eta_2 x v_1^2 \ldots v_n^2 a^2 b^0 =$$

$$Q(G) + \eta_1 x v_1^2 \ldots v_n^2 a^0 b^2 - \eta_1 x v_1^2 \ldots v_n^2 a^2 b^0 =$$

$$Q(G) + \eta_1 x v_1^2 \ldots v_n^2 (a^0 b^2 - a^2 b^0).$$

And then:
Continuing as in case 1, with one concluding the proof. □

As in each case we proved that either of the newly created edges can assume role of in the original graph, any 2-connected, outerplane near-triangulation with exactly two degree 2 vertices can be constructed in the way described above, thus concluding the proof.

Recall that by Combinatorial Nullstellensatz, \((i, j)\)-extendability of \((G, x, y)\) can be expressed as the fact that there is a non-vanishing monomial in \(P(G)\) where exponents of \(x\) and \(y\) are \(i - 1\) and \(j - 1\), respectively, and every other exponent is less than 3. We obtain an analogue to Theorem 1.2 as the following

**Corollary 2.2.** Let \(G\) be any 2-connected, outerplane near-triangulation with \(V(G) = \{x, y, v_1, \ldots, v_n\}\). Let \(c: V(G) \to \{1, 2, 3\}\) be any proper 3-colouring of \(G\). Then in the graph polynomial \(P(G)\)

(i) there is no monomial of the form \(\eta x^0 y^0 \prod_{i=1}^{n} v_i^{\alpha_i}\) with \(\alpha_i \leq 2\);
(ii) the monomial of the form \(\eta x^1 y^0 \prod_{i=1}^{n} v_i^{\alpha_i}\) with \(\alpha_i \leq 2\) does not vanish if and only if \(c(x) \neq c(y)\);
(iii) there is non-vanishing monomial of the form \(\eta x^\beta y^\gamma \prod_{i=1}^{n} v_i^{\alpha_i}\) with \(\alpha_i \leq 2, \beta, \gamma \leq 1\).

**Proof.** For the first point, simply note that outerplane near-triangulation on \(n + 2\) vertices has \(2n + 1\) edges, while the sum of the exponents of the given monomial is 2n.

For the second one: when \(x\) and \(y\) are adjacent one may apply Theorem 1.3 directly; otherwise at first notice that under assumption of Theorem 2.1 there is \(\eta_1 = 0\) if and only if \(c(x) = c(a)\) and similarly \(\eta_3 = 0\) if and only if \(c(x) = c(b)\). The inductive proof is very simple, as on each triangle must appear all 3 colours.

For the rest of the proof of the second point as well as to prove the third one it is enough now to repeat Hutchinson’s shrinking argument. Then for each shrunked component applying Theorem 1.3 one may find a proper non-vanishing monomial. The resulting monomial is a product of all these monomials and the one obtained by Theorem 2.1.

3. Poly-extendability of general outerplanar graphs

The results of previous section can be of course applied to any outerplanar graph, not necessarily triangulated. This, however, leads to loss of information, as usually there is more than one way to triangulate the graph, and different triangulations may lead to different types of extendability. Moreover, in the case of non-triangulated graphs, as well as those that are not 2-connected, the counting argument behind point (i) of Corollary 2.2 does not work any more. Hence, it is possible for general
outerplanar graph to be \((1,1)\)-extendable. At first, a formal definition of fundamental subgraphs is provided, followed by three instrumental lemmas.

**Definition 3.1.** Let \(G\) be a 2-connected outerplanar graph, \(x, y \in V(G)\) and let \(T(G)\) be the weak dual of \(G\). The fundamental \(x - y\) subgraph of \(G\) is the subgraph of \(G\) induced by the vertices belonging to faces that have vertices representing them in \(T(G)\) lying on the shortest path between vertices representing faces on which \(x\) and \(y\) lie. If \(xy \in E(G)\), then the fundamental subgraph reduces to an edge \(xy\).

**Definition 3.2.** Let \(G\) be a connected outerplanar graph with cutvertices, and let \(BC(G)\) be the block-cutvertex graph of \(G\). Let \(x, y \in V(G)\) be vertices lying in two different blocks of \(G\). The fundamental \(x - y\) subgraph of \(G\) consists of all blocks that have vertices representing them in \(BC(G)\) lying on the shortest path between vertices representing blocks containing \(x\) and \(y\), and each of those blocks is restricted to the fundamental \(a - b\) subgraph, where \(a, b \in V(G)\) are the two cutvertices belonging to given block and to the shortest path between blocks containing \(x\) and \(y\) in \(BC(G)\).

**Definition 3.3.** An outerplanar graph \(G\) with \(x, y \in V(G)\) is \(xy\)-fundamental if its fundamental \(x - y\) subgraph is equal to \(G\).

**Lemma 3.4.** Let \(G\) be a 2-connected \(xy\)-fundamental near-triangulation, such that \(C(x) = C(y)\). Let \(v_0\) be the vertex of \(G\) that has degree 2 in \(G - y\), and \(v_1, \ldots, v_n\) be the remaining vertices. Then in \(P(G)\) there is a non-vanishing monomial of the form \(\eta x^0y^2v_1^2v_2^2 \ldots v_n^2\), with \(\eta \in \{-1, 1\}\).

**Proof.** As \(C(x) = C(y)\), then \(C(x) \neq C(v_0)\). Hence by Theorem 2.1, there is a non-vanishing monomial \(\eta x^0v_1^2v_2^2 \ldots v_n^2\), with \(\eta \in \{-1, 1\}\) in \(P(G - y)\). Adding \(y\) back, thus multiplying \(P(G - y)\) by \((y - v_0)(y - v_n) = y^2 - yv_0 - yv_n + v_0v_n\), we get the monomial specified in thesis, and as it is the only way to obtain it, it is non-vanishing.

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**Figure 1.** Top: a connected, outerplanar graph \(G\); Bottom: a fundamental \(x - y\) subgraph of \(G\).
Lemma 3.5. Let $G, G'$ be two connected, outerplanar near-triangulations, $V(G) = \{x, v_1, \ldots, v_n\}$, $V(G') = \{x', u_1, \ldots, u_m\}$. Let $G''$ be the graph obtained from $G$ and $G'$ by identifying $x$ with $x'$, thus creating vertex $x''$, and carrying neighbour-
ulating relations from $G, G'$. Suppose there are non-vanishing monomials $\eta x^\alpha v_i^{\alpha_i}$ and $\eta' x'^{\alpha'} u_j^{\beta_j}$ in $P(G)$ and $P(G')$ respectively. Then in $P(G'')$ there is a non-vanishing monomial $A(G'') = \eta \eta' x''^{\alpha + \beta} v_i^{\alpha_i} u_j^{\beta_j}$.

Proof. As both $\eta$ and $\eta'$ are non-zero, then the only way $A(G'')$ would vanish is that there were a monomial $A'(G'') = \nu \nu' x''^{\alpha'} v_i^{\alpha_i} u_j^{\beta_j}$, where $\nu \nu' = -\eta \eta'$ and $\alpha' + \beta' = \alpha + \beta$. But as the sum of exponents in every monomial in a polynomial of given graph is fixed, hence $\alpha = \alpha'$ and $\beta = \beta'$, a contradiction. Thus $A(G'')$ is non-vanishing. □

Lemma 3.6. Let $G$ be a path of length $n$, $n \geq 2$, where $x, y$ are the endpoints and $v_1, \ldots, v_{n-1}$ are the inner vertices of $G$. Then in $P(G)$ there is a non-vanishing monomial of the form $\eta x^0 y^0 v_1^1 v_2^1 \ldots v_{n-1}^1$, where $\eta \in \{-1, 1\}$.

Proof. Suppose at first that $n = 2$. Then $P(G) = (x-v_1)(y-v_1) = xy-xv_1-xy_1+v_1^2$, and the last monomial is the one fulfilling the thesis. Now suppose that the theorem holds for $n = k-1$. Then in $P(G)$, where $G$ is a path $x v_1 \ldots v_{k-1}$, there is a monomial $\eta x^0 y^0 v_1^1 v_2^1 \ldots v_{k-2}^1 v_{k-1}^0$. Now adjoining $v_k$ to $v_{k-1}$, thus multiplying $P(G)$ by $(v_{k-1} - v_k)$ we obtain a monomial $\eta x^0 y^0 v_1^1 v_2^1 \ldots v_{k-2}^1 v_{k-1}^1 v_k^0$ for path of length $k$, hence completing the induction. □

3.1. Near-triangulations with cutvertices.

The following theorem is a polynomial analogue of [5, Theorem 5.3] that characterizes extendability of outerplane near-triangulations with cutvertices.

Theorem 3.7. Let $G$ be a fundamental $x-y$ subgraph with cutvertices $\{v_1, \ldots, v_{j-1}\}$, $CV(G) = (x, v_1, \ldots, v_{j-1}, y)$, and $u_{i,k}$ being the remaining vertices in $i$-th block. Then in $P(G)$:
(i) there is a non-vanishing monomial of the form \( \eta_1 x^1 y^1 \Pi^{\alpha_m} u^\beta_{i,k} , \alpha_m, \beta_{i,k} \leq 2 \) if every vertex from \( CV(G) \) is in the same colour class;
(ii) there is a non-vanishing monomial of the form \( \eta_2 x^0 y^1 \Pi^{\alpha_m} u^\beta_{i,k} , \alpha_m, \beta_{i,k} \leq 2 \) if there is a single pair of successive vertices in \( CV(G) \) that are in different colour classes;
(iii) there is a non-vanishing monomial of the form \( \eta_3 x^0 y^0 \Pi^{\alpha_m} u^\beta_{i,k} , \alpha_m, \beta_{i,k} \leq 2 \) if there are at least two pairs of successive vertices in \( CV(G) \) that are in different colour classes;

![Figure 3. An example of labelling described in Theorem 3.7](image)

**Proof.** Start with partitioning \( G \) by its cutvertices into separate, 2-connected, \( v_{i-1}, v_i \)-fundamental outerplane near-triangulations \( B_1, \ldots, B_j \). To each of these graphs, Theorem 2.1 applies, and \( P(G) = P(B_1) \ldots P(B_j) \). If in each of those blocks colour class of degree 2 vertices is the same, then in each of their polynomials there is a non-vanishing monomial such that exponents of degree 2 vertices are equal to 1, with other exponents no larger than 2. Thus case 1 is just a repeated use of lemma 3.5.

In the second case, let \( B_i \) be the block with degree 2 vertices in different colour classes. If \( i = 1 \), then in \( P(B_1) \) there is a non-vanishing monomial of the form \( \eta_0 x^0 v_1^1 \Pi^{2}_{u_{i,k}} \). Hence again by lemma 3.5 we get the desired monomial. If \( i > 1 \), then we apply lemma 3.4 to each block \( B_1 \) to \( B_{i-1} \), thus by lemma 3.5 obtaining monomial with \( x^0 \) and \( v_{i-1}^1 \). As \( C(v_{i-1}) \neq C(v_i) \), \( P(B_i) \) contains a non-vanishing monomial \( \eta_i x^0 v_1^1 \Pi^{2}_{u_{i,k}} \), hence through lemma 3.5 we finish the case.

The last case is starts analogously to the second one, with \( B_i, B_j, i < j \) being two blocks with endpoints in different colour classes. Let \( G' \) be the \( v_{i-1}, v_j \) fundamental subgraph of \( G \). By Theorem 2.1 there is a non-vanishing monomial in \( P(B_i) \) with \( v_{i-1}^0 \) and \( v_i^1 \) and a monomial in \( P(B_j) \) with \( v_{j-1}^1 \) and \( v_j^0 \). As every block between \( B_i \) and \( B_j \) has a monomial with endpoints in power 1, by lemmas 3.4 and 3.5 there is a monomial in \( P(G') \) with both \( v_{i-1} \) and \( v_j \) in power 0. Again by lemmas 3.4 and 3.5 we can now adjoin remaining parts of \( G \) to \( G' \), with their suitable monomials creating a desired monomial in \( P(G) \). □
3.2. 2-connected outerplane graphs with non-triangular faces.

The following three theorems are jointly analogous to [5, Theorem 4.3].

**Theorem 3.8.** Let $G$ be a 2-connected $xy$-fundamental graph with exactly one non-triangular interior face, and that face contains $x$ and does not contain $y$. Let $V(G) = \{x, y, a, b, v_1, \ldots, v_n\}$, where $a, b$ are the two vertices of non-triangular face belonging to the neighbouring interior face. Let $C(v)$ be the colour class of vertex $v$ in the 3-colouring of the graph induced by all of the triangular faces. Then in $P(G)$:

(i) there is a non-vanishing monomial of the form $\eta_1 x^0 y^1 a^{\alpha} b^{\alpha_i} \prod_{v_i}^{\alpha_i}, \alpha_k \leq 2$ if $d(x, a) = 1$ and $C(a) = C(y)$ OR $d(x, b) = 1$ and $C(b) = C(y)$;

(ii) there is a non-vanishing monomial of the form $\eta_2 x^0 y^0 a^{\alpha} b^{\alpha_i} \prod_{v_i}^{\alpha_i}, \alpha_k \leq 2$ otherwise;

![Diagram of Theorem 3.8](image)

**Figure 4.** Examples of labelling as in Theorem 3.8. Left: example to point (i); Right: example to point (ii).

**Proof.** Suppose that $d(x, a) = 1$ and $C(a) = C(y)$. Let $G'$ be the subgraph of $G$ created by deleting all the vertices on the non-triangular face except for $a$ and $b$. As $G'$ is an outerplanar near-triangulation Theorem 2.1 applies, and as $C(a) = C(y)$, then in $P(G')$ there is a non-vanishing monomial with $a^1$ and $y^1$. If we now adjoin vertex $x$ to $a$, creating graph $G''$, then it $P(G'')$ there is a non-vanishing monomial with $x_0$, $a^2$ and $y^1$. Now adding a path between $x$ and $b$, thus reconstructing $G$ (notice that the length of this path is at least 2, as the face is not a triangle), by lemma 3.6 we obtain a desired monomial. The case when $d(x, b)$ and $C(b) = C(y)$ is handled analogously.

If this is not the case, then either $d(x, a) \geq 1$ and $d(x, b) \geq 1$, or $d(x, a) = 1$ and $C(a) \neq C(y)$ (or analogously $d(x, b) = 1$ and $C(b) \neq C(y)$). In the first case, then by Theorem 2.1 and lemma 3.4 in $P(G')$ (with $G'$ defined as previously) there is a non-vanishing monomial with $y^0$ and all other powers less than 3. Now as we join $x$ with $a$ and $b$ with previously deleted paths, lemma 3.6 gives us a monomial with $x^0, y^0$ and all other powers less than 3. In the second case, as $C(a) \neq C(y)$, by 2.1 there is a monomial in $P(G')$ where $y$ has power 0 and $a$ has power 1. Adjoining $x$ to $a$, we obtain a monomial with $x^0, y^0$ and $a^2$, and as we join $x$ with $b$ by a path, lemma 3.6 gives us a desired monomial. Case when $d(x, b) = 1$ and $C(b) \neq C(y)$ is again analogous to the last one.  

□
Theorem 3.9. Let $G$ be a 2-connected $xy$-fundamental graph with exactly one non-triangular interior face, and that face does not contain $x$ nor $y$. Let $V(G) = \{x, y, a, b, c, v_1, \ldots, v_n\}$, where $a, b$ and $a, c$ are the two pairs of vertices of non-triangular face belonging to the neighbouring interior faces, and let $C(v)$ be the colour class of vertex $v$ in the 3-colouring of the subgraph of $G$ created by deleting the path $bc$. Then in $P(G)$:

(i) there is a non-vanishing monomial of the form $\eta_1 x^1 y^1 a^{\alpha_2} b^{\alpha_3} c^{\alpha_4} \Pi_{v_i}^{\alpha_i}$, $\alpha_k \leq 2$, if $C(x) = C(a) = C(y)$;
(ii) there is a non-vanishing monomial of the form $\eta_2 x^0 y^1 a^{\alpha_2} b^{\alpha_3} c^{\alpha_4} \Pi_{v_i}^{\alpha_i}$, $\alpha_k \leq 2$, if $C(x) \neq C(a) = C(y)$ or $C(x) = C(a) \neq C(y)$;
(iii) there is a non-vanishing monomial of the form $\eta_3 x^0 y^0 a^{\alpha_2} b^{\alpha_3} c^{\alpha_4} \Pi_{v_i}^{\alpha_i}$, $\alpha_k \leq 2$, if $C(x) \neq C(a) \neq C(y)$;

![Figure 5. An example of labelling described in Theorem 3.9](image)

Proof. Let $G'$ be the subgraph of $G$ obtained by deleting path $bc$ from $G$. Obviously $G'$ is an outerplanar near-triangulation with a single cutvertex $a$, hence Theorem 3.7 applies to it. Notice moreover, that the first case of the above theorem leads to the first case of Theorem 3.9, and the second and third case also relate similarly. As Theorem 3.7 gives us suitable monomials, when we add back the path we previously deleted, application of lemma 3.6 finishes the proof. \(\square\)

Theorem 3.10. Let $G$ be a 2-connected $xy$-fundamental graph with exactly one non-triangular interior face, and that face does not contain $x$ nor $y$. Let $V(G) = \{x, y, a, b, c, d, v_1, \ldots, v_n\}$, where $a, b$ and $c, d$ are the two pairs of vertices of non-triangular face belonging to the neighbouring interior faces with $ab \in E(G)$ and $cd \in E(G)$, and let $C(v)$ be the colour class of vertex $v$ in the 3-colouring of the subgraphs of $G$ created by deleting the paths $ac$ and $bd$. Then in $P(G)$:

(i) there is a non-vanishing monomial of the form $\eta_1 x^0 y^1 a^{\alpha_2} b^{\alpha_3} c^{\alpha_4} d^{\alpha_5} \Pi_{v_i}^{\alpha_i}$, $\alpha_k \leq 2$, if $d(a, c) = 1$, $C(x) = C(a)$ and $C(y) = C(c)$ OR $d(b, d) = 1$, $C(x) = C(b)$ and $C(y) = C(d)$;
(ii) there is a non-vanishing monomial of the form $\eta_2 x^0 y^0 a^{\alpha_2} b^{\alpha_3} c^{\alpha_4} d^{\alpha_5} \Pi_{v_i}^{\alpha_i}$, $\alpha_k \leq 2$ otherwise;

Proof. Suppose at first that $C(x) = C(a)$ and $C(y) = C(c)$. We can connect vertex $a$ with $d$, and if $d(b, d) > 1$, also with every interior vertex on the path $bd$, thus
obtaining an $xy$-fundamental 2-connected near triangulation $G'$. If $d(a,c) = 1$, then $C(a) \neq C(c)$, thus $C(x) \neq C(y)$, and by Corollary 2.2 $P(G')$ contains a non-vanishing monomial with $x^0$, $y^1$ and every other exponent equal to 2. As neither $x$ nor $y$ were affected by addition of edges to $G$, $P(G)$ contains a non-vanishing monomial of the form $\eta_1 x^0 y^1 a^{\alpha_0} b^{\alpha_0} c^{\alpha_0} d^{\alpha_0} \Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$. If $d(a,c) > 1$, then $G'$ fulfills the conditions of Theorem 3.9, with $d$ serving as vertex $a$ in the statement of that theorem. Moreover, as $C(x) = C(a)$ and $C(y) = C(c)$, and $d$ neighbours both $a$ and $c$ in $G'$, then in colouring of $G'$ $C(x) \neq C(d)$ and $C(y) \neq C(d)$. Hence by Theorem 3.9 $P(G')$ contains a non-vanishing monomial with $x^0$, $y^1$ and every other exponent no larger than 2, and this again implies that there is a non-vanishing monomial of the form $\eta_2 x^0 y^1 a^{\alpha_0} b^{\alpha_0} c^{\alpha_0} d^{\alpha_0} \Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$ in $P(G)$. Case when $C(x) = C(b)$ and $C(y) = C(d)$ is analogous.

Suppose now that $C(x) \neq C(a)$ and $C(y) = C(c)$. Start by removing paths $ac$ and $bd$ from $G$. This leaves us with two separate, 2-connected near triangulations $G'$ and $G''$ with $\{x,a,b\} \in V(G')$ and $\{y,c,d\} \in V(G'')$. As $C(y) = C(c)$, then $C(y) \neq C(d)$, and by Corollary 2.2 in $P(G'')$ there is a non-vanishing monomial of the form $\eta_3 y^0 d^2 \Pi v_i^2$. Now as $C(x) \neq C(a)$, there exists a non-vanishing monomial $\eta_4 x^0 a^1 b^2 \Pi v_i^2$ in $P(G')$, as the polynomial of $xa$-fundamental subgraph of $G'$ contains a non-vanishing monomial with $x^0$ and $a^1$, and as $G'$ is a 2-connected near triangulation, every other exponent must be then equal to 2. Now add back paths $ac$ and $bd$. Each of them contains in its polynomial a non-vanishing monomial with every exponent equal to 1, except for one of its endpoints, which is in power 0. We will call that monomial oriented towards the endpoint with non-zero exponent. Add paths $ac$ and $bd$ to $G'$ and $G''$, and by multiplication of the monomials described above we obtain a monomial of the form $\eta_2 x^0 y^1 a^{\alpha_0} b^{\alpha_0} c^{\alpha_0} d^{\alpha_0} \Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$ in $P(G)$, where exponent of each of the vertices $a,b,c,d$ is equal to 2. This monomial does not vanish, as the only other way to get this monomial would require to orient both of the paths in the opposite direction, but this would imply that there were a non-vanishing monomial $\eta_1 y^0 d^2 \Pi v_i^2$ in $P(G'')$, which is not the case as $C(y) = C(c)$. Cases where $C(x) = C(a)$ and $C(y) \neq C(c)$, $C(x) \neq C(b)$ and $C(y) = C(d)$ or $C(x) = C(b)$ and $C(y) \neq C(d)$ are sorted out in the same manner.

The last case is when $C(x) \neq C(a)$ and $C(y) \neq C(c)$. Observe at first, that we can also assume that $C(x) \neq C(b)$ and $C(y) \neq C(d)$, as all the other cases were already solved in previous arguments due to symmetries. Let $G'$ and $G''$ be as in previous

![Figure 6. An example of labelling described in Theorem 3.10.](image-url)
case. As $C(b) \neq C(x) \neq C(a)$, then in $P(G')$ there are non-vanishing monomials $\eta_1x^a a^2b^2\Pi^2_i$ and $-\eta_1 x^a a^2b^2\Pi^2_i$. Similarly, there are non-vanishing monomials $\eta_2y^b c^2d^2\Pi^2_i$ and $-\eta_2 y^b c^2d^2\Pi^2_i$ in $P(G'')$. Now reconstruct $G$ as previously, orienting $ac$ towards $a$ and $bd$ towards $d$. To comply with requirements of the thesis, we have to use the first and fourth monomial from those specified above, thus in $P(G)$ we have a monomial $-\eta_1\eta_2 x^a y^b a^2b^2c^2d^2\Pi^2_i$. The only other way to reach this set of exponents is be to use second and third monomial, and orient paths in opposite directions, but as simultaneous switch of orientations preserves sign, we again obtain $-\eta_1\eta_2 x^a y^b a^2b^2c^2d^2\Pi^2_i$, so those monomials do not annihilate each other, but rather double the coefficient. As all cases are now addressed, the proof is concluded. \qed

### 3.3. General outerplane graphs.

The three theorems above can be combined with Theorem 5.10 to obtain a general characterisation of $(i, j)$-extendability of outerplane graphs. We will start with some technicalities.

**Definition 3.11.** Let $G$ be an outerplane graph. A non-triangular inner face of $G$ will be called **type 0** if it is as defined in Theorem 3.8 (with possibly $y$ belonging to that face instead of $x$), **type 1** if it is as defined in Theorem 3.9 and **type 2** if it is as defined in Theorem 3.10. In case of type 1 faces, the vertex belonging to the two neighbouring faces will be called an apex of that face.

**Lemma 3.12.** Let $G$ be a connected outerplane graph with $V(G) = \{x, y, v_1, \ldots, v_i\}$ and let $G'$ be a supergraph of $G$ obtained by adding a path of the length 2 to $G$ in a way that preserves outerplanarity. Then the monomial $x^\alpha y^\alpha \Pi^\alpha_i$ does not vanish in $P(G)$ if and only if the monomial $x^\alpha y^\alpha \Pi^\alpha_i z^2$ does not vanish in $P(G')$, where $z$ is the middle point of the added path.

**Proof.** The implication from $P(G)$ to $P(G')$ is obvious and was shown to be true and utilized multiple times in this paper. Suppose there is a non-vanishing monomial $x^\alpha y^\alpha \Pi^\alpha_i z^2$ in $P(G')$. As $P(G') = P(G)(ab - az - bz + z^2)$, where $a, b$ are the endpoints of the added path, and none of the monomials from $P(G)$ contains $z$ due to the fact that $z \notin V(G)$, then the only way to obtain the monomial above is by multiplying $x^\alpha y^\alpha \Pi^\alpha_i$ by $z^2$, thus the former must occur in $P(G)$. \qed

**Definition 3.13.** Let $G$ be a 1-connected outerplane graph. For every cutvertex of $G$ add a path to length 2, connecting it to the neighbours of that cutvertex without disrupting outerplanarity, thus creating a non-triangular face of type 0. Then for every bridge (or chain of bridges) of $G$ add a path of the length 2 to the neighbours of the endpoints of that bridge (or to the endpoint if it has degree 1) in a way that preserves outerplanarity, creating a face of type 2 (or type 0). The resulting supergraph of $G$ will be called a **2-connection** of $G$. The 2-connection of 2-connected graph would be the graph itself.

The following remark is a direct consequence of lemma 3.12.

**Remark 3.14.** Let $G$ be a connected outerplane graph, $V(G) = \{x, y, v_1, \ldots, v_i\}$ and let $G'$ be its 2-connection, $V(G') = \{x, y, v_1, \ldots, v_i, u_1, \ldots, u_j\}$. There is a non-vanishing monomial $x^\alpha y^\alpha \Pi^\alpha_i$ in $P(G)$ if and only if there is a non-vanishing monomial $x^\alpha y^\alpha \Pi^\alpha_i \Pi^\alpha_2$ in $P(G')$. 
We can now present the final theorem.

**Theorem 3.15.** Let $G$ be a connected $xy$-fundamental outerplane graph, $V(G) = \{x, y, v_1, \ldots, v_i\}$, and let $G'$ be a 2-connection of $G$. Then in $P(G)$:

(i) there is a non-vanishing monomial of the form $x^1y^0\Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$ if $G$ is a 2-connected near-triangulation with $C(x) = C(y)$ OR $G$ is as in point 1 of Theorem 3.7 OR every non-triangular face of $G'$ is of type 1 and every apex, $x$ and $y$ have the same colour in every 3-colouring of $G$.

(ii) there is a non-vanishing monomial of the form $x^0y^1\Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$ if $G$ is a 2-connected near-triangulation with $C(x) \neq C(y)$ OR $G$ is as in point 2 of Theorem 3.7 OR $G'$ is as in point 1 of Theorem 3.8 OR $G'$ is as in point 1 of Theorem 3.10 OR every non-triangular face of $G'$ is of type 1 and in every 3-colouring of $G'$ there is exactly one pair of consecutive apexes (or either $x$ or $y$ with the closest apex) with different colours OR only one of the non-triangular faces of $G'$ is of the type 1 and conditions of point 1 of Theorem 3.10 are fulfilled on that face.

(iii) there is a non-vanishing monomial of the form $x^0y^0\Pi v_i^{\alpha_i}$, $\alpha_k \leq 2$ otherwise.

**Proof of Theorem 3.15.** We will omit every case that is covered already by previous theorems, leaving us only with the cases when there are multiple non-triangular faces. Suppose all of those are of type 1. It is easy to see (with some help of lemma 3.6) that for every such face removal of all vertices belonging only to this (and outer) face produces a cutvertex, simultaneously changing nothing in terms of extendability-relevant monomials. Hence Theorem 3.7 with each apex acting as a cutvertex.

Suppose now there is a face of type either 0 or 2 in $G'$. Theorems 3.8 and 3.10 show that the only cases where there is no monomial in $P(G')$ (and thus in $P(G)$) with both $x$ and $y$ in power 0 is when 3-colouring $G'$ we cannot avoid a situation described in point 1 of either of these theorems on any of such faces, and in those cases there is a non-vanishing monomial with $x^0$ and $y^1$. Observe that this is not the case when there are at least two faces of type 0 or 2, as we can avoid this situation by either permuting the colours, or by changing them on vertices of degree 2 (as in...
case of type 0 faces at least one such vertex other than \(x\) and \(y\) definitely exists). So there are only two cases when we cannot avoid that. The first is when in \(G'\) there is only one face of type 2, no faces of type 0, there is a pair of neighbouring vertices belonging to this face such that the only other face of \(G'\) they belong simultaneously is the outer face, and in any 3-colouring of \(G\) (and thus also \(G'\)) each of those vertices has the same colour as \(x\) or \(y\), depending on which of those vertices lies on the same 'side' of that face. Label the vertex from this pair lying closer to \(x\) as \(v_x\), and the one being closer to \(y\) as \(v_y\). The case of \(C(x) = C(v_x)\) can occur either when on one side there are only triangular faces between \(x\) and \(v_x\), with the structure of that triangulation forcing the same colour of those vertices, or when for every type 1 face between those vertices, the triangular structure between neighbouring faces or between \(x/v_x\) and the nearest such face forces the same colour on each of those vertices. The same is true for \(y\) and \(v_y\), with the restriction that the former situation cannot occur for both of those pairs. The second case is when there is exactly one face of type 0 in \(G'\) (without loss of generality we can assume that \(x\) lies on that face), no faces of type 2, \(x\) has a neighbour (\(v_0\)) that lies also on adjacent inner face, and the colour of that vertex is the same as colour of \(y\) in every 3-colouring of \(G'\). This can be only caused by the fact that the apex of every type 1 face is forced to have the same colour as the others, as well as \(y\) and \(v_0\). □

To see that Theorem 3.15 implies Theorem 1.4 we have to consider consequences of each of the situations described in the statement of the former in terms of 3-colourings. Start with point (i). If \(G\) is a 2-connected near-triangulation with \(C(x) = C(y)\), then obviously every 3-colouring of \(G\) forces \(C(x) = C(y)\), as the 3-colouring of every 2-connected near-triangulation is unique up to permutation of colours. The remaining two conditions in that point force \(C(x) = C(y)\) directly in similar way.

Moving to the second point, the first condition again directly state that \(C(x) \neq C(y)\). If \(G\) is as in point 2 of Theorem 3.7 or every non-triangular face of \(G'\) is of type 1 and in every 3-colouring of \(G'\) there is exactly one pair of consecutive apexes (or either \(x\) or \(y\) with the closest apex) with different colours, as the colour class change only once on the cutvertices/apexes, then obviously classes of terminal vertices \(x\) and \(y\) have to be different. If \(G'\) is as in point 1 of Theorem 3.8 then it is directly stated that the colour of one of terminal vertices is the same as the colour of one of the neighbours of the other terminal vertex, thus the colours of terminal vertices have to be different. Finally, if \(G'\) is as in point 1 of Theorem 3.10 or only one of the non-triangular faces of \(G'\) is not of the type 1 and conditions of point 1 of Theorem 3.10 are fulfilled on that face, the vertices \(x\) and \(y\) are in the same colour class as vertices \(a\) and \(c\) (or \(b\) and \(d\)), respectively, and those vertices are adjacent, hence their colours cannot possibly be the same.

### 4. Further work

In [9] and [10] Postle and Thomas provided extendability results for triangulated planar graphs. Namely, their results can be stressed in the following theorem.

**Theorem 4.1.** Let \(G = (V, E)\) be any plane graph, let \(C \subseteq V\) be the set of vertices on the outer face, \(x, y \in C, x \neq y\). Then

(i) \((G, x, y)\) is \((1, 2)\)-extendable if and only if there exists a proper colouring \(c: C \rightarrow \{1, 2, 3\}\) such that \(c(x) \neq c(y)\);
(ii) \((G, x, y)\) is \((2, 2)\)-extendable.

Extendability is naturally transformed into plane graphs by allowing interior vertices to have a list of colours of length 5. One may ask, whether it is possible to restate the above theorem in the terms of a graph polynomial, i.e. to extend, at least partially Theorem 1.4 to planar graphs. Our partial results suggest that it is possible.

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