Quantum interpolating ensemble: Average entropies and orthogonal polynomials

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Abstract. The density matrix formalism is a fundamental tool in studying various problems in quantum information processing. In the space of density matrices, the most well-known and physically relevant measures are the Hilbert-Schmidt ensemble and the Bures-Hall ensemble. In this work, we propose a generalized ensemble of density matrices, termed quantum interpolating ensemble, which is able to interpolate between these two seemingly unrelated ensembles. As a first step to understand the proposed ensemble, we derive the exact mean formulas of entanglement entropies over such an ensemble generalizing several recent results in the literature. We also derive some key properties of the corresponding orthogonal polynomials relevant to obtaining other statistical information of the entropies. Numerical results demonstrate the usefulness of the proposed ensemble in estimating the degree of entanglement of quantum states.

Keywords: quantum entanglement, interpolating ensemble, entanglement entropy, orthogonal polynomials, special functions

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1. Introduction and quantum interpolating ensemble

Quantum information theory aims at understanding the theoretical underpinnings of quantum technologies including quantum computing and quantum communications. It is based on probabilistic interpretations of quantum states to explain various quantum effects. The density matrix formalism introduced by von Neumann [1] provides a natural framework to describe density matrices of quantum states. The density matrix is a fundamental object that encodes all the information of a quantum state. Among the different measures of density matrices, the most well-known and physically relevant ones [2] are the Hilbert-Schmidt measure and the Bures-Hall measure.

The Hilbert-Schmidt measure is formulated as follows. Consider a bipartite quantum system consisting of two subsystems $A$ and $B$ in the Hilbert space $\mathcal{H}_m$ and $\mathcal{H}_n$ (with $m \leq n$), respectively. A random pure state $|\psi\rangle$, defined as a linear combination of the complete basis of the subsystems, belongs to the composite Hilbert space $|\psi\rangle \in \mathcal{H}_m \otimes \mathcal{H}_n$. The reduced density matrix is obtained by partial tracing over the larger system of the full density matrix $\rho = |\psi\rangle \langle \psi|$ as $\rho_A = \text{tr}_B \rho$. The resulting density of eigenvalues of $\rho_A$ is the Hilbert-Schmidt measure [2]

$$f_{\text{HS}}(\lambda) \propto \delta(1 - \sum_{i=1}^m \lambda_i) \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)^2 \prod_{i=1}^m \lambda_i^{n-m},$$

where $\delta(\cdot)$ is the Dirac delta function. The joint density (1) is also referred to as the Hilbert-Schmidt (random matrix) ensemble, which is the eigenvalue density of a normalized Wishart matrix

$$\frac{GG^\dagger}{\text{tr}(GG^\dagger)}$$

with $G$ being an $m \times n$ complex Gaussian matrix. For the Bures-Hall measure, the random pure state is given by a superposition of that of the Hilbert-Schmidt measure as $|\varphi\rangle = |\psi\rangle + (U \otimes I_m)|\psi\rangle$, where $U$ is an $m \times m$ unitary matrix with the measure proportional to $\text{det}(I_m + U)^{2(n-m)}$. The resulting density of eigenvalues of the reduced density matrix $\rho_A = \text{tr}_B |\varphi\rangle \langle \varphi|$ is the (generalized) Bures-Hall measure [3, 4]

$$f_{\text{BH}}(\lambda) \propto \delta(1 - \sum_{i=1}^m \lambda_i) \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)^2 \prod_{i=1}^m \lambda_i^{n-m-\frac{1}{2}}.$$  

The joint density (3) is also known as the Bures-Hall ensemble in random matrix theory, which is the eigenvalue density of the normalized product of the matrix $I_m + U$ with a complex Gaussian matrix $G$,

$$\frac{(I_m + U)GG^\dagger(I_m + U^\dagger)}{\text{tr}((I_m + U)GG^\dagger(I_m + U^\dagger))}.$$ 

The Hilbert-Schmidt measure (1) and the Bures-Hall measure (3) are supported in the probability simplex

$$\Lambda = \left\{ 0 \leq \lambda_m < \ldots < \lambda_1 \leq 1, \sum_{i=1}^m \lambda_i = 1 \right\},$$
which reflects the constraint $\text{tr} \rho_A = 1$ of density matrices. Note also that the normalization constants in the densities (1) and (3) are omitted.

The study of the Hilbert-Schmidt measure has received substantial attention, see, for example, the results in [5–23]. These results include information-theoretic studies of different entanglement entropies [5–20] as well as applications to quantum information processing [21–23]. The relatively less-studied Bures-Hall ensemble [3, 4, 24–29] gains renewed interest very recently [30–32]. This is owing to the recent breakthrough in probability theory in understanding various aspects of the Bures-Hall ensemble [33–37]. Despite the distinct behavior of Hilbert-Schmidt measure and Bures-Hall measure, an interesting question is whether one could propose a measure that interpolates between the two measures. This question has also been motivated by the observation in [30] that the Bures-Hall measure tends to be more conservative than the Hilbert-Schmidt measure in estimating entanglement entropies. Namely, the Bures-Hall measure leads towards an estimate of less entangled states than the Hilbert-Schmidt measure does. In this context, one tries to control the appropriate amount of entanglement as a resource for quantum information processing by constructing new measures that interpolates between the two major measures.

In this work, we consider the following measure, supported in (5),

$$f(\lambda) = \frac{1}{C} \delta \left( 1 - \sum_{i=1}^{m} \lambda_i \right) \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \left( \lambda_i^\theta - \lambda_j^\theta \right) \prod_{i=1}^{m} \lambda_i^a$$

(6)

termed the quantum interpolating ensemble, where $\theta$ is assumed to be a positive real parameter and $a > -1$. Clearly, the proposed ensemble (6) reduces to the Hilbert-Schmidt ensemble (1) and the Bures-Hall ensemble (3) as special cases,

$$f(\lambda) = \begin{cases} f_{BH}(\lambda) & \text{for } \theta = 1, \ a = n - m - \frac{1}{2} \\ f_{HS}(\lambda) & \text{for } \theta = 2, \ a = n - m. \end{cases}$$

(7)

Namely, as $\theta$ varies from $\theta = 1$ to $\theta = 2$, the quantum interpolating ensemble interpolates between the Bures-Hall ensemble and the Hilbert-Schmidt ensemble. Due to Schur’s Pfaffian identity [38, 39]

$$\prod_{1 \leq i < j \leq 2m} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} = \text{Pf} \left( \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)_{1 \leq i, j \leq 2m},$$

(8)

the proposed ensemble (6) is described by a Pfaffian point process for any $\theta > 0$ except for the special value $\theta = 2$ when the ensemble becomes a determinantal point process. Therefore, in the interested interval $\theta \in [1, 2]$ the new ensemble (6) corresponds to the transition from a Pfaffian point process to a determinantal point process. It is also worth mentioning that besides the half integers values of $a$ for the Bures-Hall ensemble and the integers values of $a$ for the Hilbert-Schmidt ensemble, the proposed ensemble is valid for any $a > -1$. Therefore, in addition to $\theta$, the parameter $a$ is also considered as a deformation parameter that defines the interpolating ensemble.

With an interpolating ensemble being identified, a natural question is what will be the statistical behavior of entanglement entropies over such an ensemble? Will
the values of entropies also interpolate between those of the Hilbert-Schmidt ensemble and the Bures-Hall ensemble? In addition, what are the properties of the underlying orthogonal polynomial systems? We address these questions in this work. The rest of the paper is organized as follows. In section 2, we present the main results of the paper on the mean values of quantum purity and von Neumann entanglement entropy of the interpolating ensemble (section 2.1) and some basic properties of the corresponding orthogonal polynomials relevant to higher moment calculation (section 2.2). In section 3, we perform extensive numerical studies on the impact of the deformation parameters of the ensemble on the behavior of the entanglement. Some proofs of the results can be found in the two appendices.

2. Entropies and orthogonal polynomials

2.1. Average entropies over interpolating ensemble

The degree of entanglement of the subsystems A and B is estimated by entanglement entropies, which are functions of the eigenvalues (entanglement spectrum) of a given ensemble. Any function that satisfies a list of axioms can be considered as an entanglement entropy. In particular, an entropy should monotonically change from the separable state

$$\lambda_1 = 1, \ \lambda_2 = \ldots = \lambda_m = 0$$

(9)
to the maximally-entangled state

$$\lambda_1 = \lambda_2 = \ldots \lambda_m = 1/m.$$  

(10)

A standard one we consider here is quantum purity [2]

$$S_P = \sum_{i=1}^{m} \lambda_i^2,$$  

(11)
supported in $S_P \in [1/m, 1]$, which attains the separable state and maximally-entangled state when $S_P = 1$ and when $S_P = 1/m$, respectively. Quantum purity (11) is an example of polynomial entropies, whereas a well-known non-polynomial entropy that we also consider here is von Neumann entropy [2]

$$S_{vN} = -\sum_{i=1}^{m} \lambda_i \ln \lambda_i.$$  

(12)

The von Neumann entropy (12) is supported in $S_{vN} \in [0, \ln m]$ that achieves the separable state and maximally-entangled state when $S_{vN} = 0$ and when $S_{vN} = \ln m$, respectively.

Statistical information of entanglement entropies is encoded through their moments: the first moment (average value) implies the typical behavior of entanglement, the second moment (variance) specifies the fluctuation around the typical value, and the higher order moments (such as skewness and kurtosis) describe the tails of the distributions. In this work, we focus on the first moments of purity (11) and von Neumann entropy (12).
over the quantum interpolating ensemble (6). Moment computation over an ensemble with the probability constraint \( \delta (1 - \sum_{i=1}^{m} \lambda_i) \) is typically performed over an ensemble without the constraint \([7, 9, 11, 12, 30–32]\). As will be seen, the unconstrained ensemble that corresponds to (6) is given by

\[
h(x) = \frac{1}{C} \prod_{1 \leq i < j \leq m} \frac{x_i - x_j}{x_i + x_j} (x_i^\theta - x_j^\theta) \prod_{i=1}^{m} x_i^\theta e^{-x_i},
\]

where \( x_i \in [0, \infty), i = 1, \ldots, m \). This ensemble has been recently studied in [40] in connection to a \( \theta \)-deformed Cauchy-Laguerre two-matrix model. In the case when \( \theta = 1 \), the corresponding ensembles have been studied in [33–37]. Among other results in [40], the \( k \)-point \((1 \leq k \leq m)\) correlation function of the unconstrained ensemble is shown to follow a Pfaffian point process

\[
\rho_k(x_1, \ldots, x_k) \propto \text{Pf} \left( \begin{pmatrix} \Delta K_{11}(x_i, x_j) & \Sigma K_{01}(x_i, x_j) \\ -\Sigma K_{01}(x_j, x_i) & \Delta K_{00}(x_i, x_j) \end{pmatrix} \right)_{1 \leq i, j \leq k},
\]

where we denote

\[
\Delta K_{00}(x, y) = K_{00}(x, y) - K_{00}(y, x) \quad (15a)
\]

\[
\Sigma K_{01}(x, y) = K_{01}(x, y) + K_{10}(y, x) \quad (15b)
\]

\[
\Delta K_{11}(x, y) = K_{11}(x, y) - K_{11}(y, x). \quad (15c)
\]

The above kernel functions can be expressed via the following Fox H-functions [40]

\[
H_\mu(x) = H_{2,3}^{1,1} \left( \begin{pmatrix} (-\alpha - m, 1); (m, 1) \\ (0, 1); (-q, \theta), (-\alpha, 1) \end{pmatrix} \left| tx^\theta \right. \right) \quad (16a)
\]

\[
G_\mu(x) = H_{2,3}^{2,1} \left( \begin{pmatrix} (-\alpha - m, 1); (m, 1) \\ (0, 1); (-q, \theta), (-\alpha, 1) \end{pmatrix} \left| tx^\theta \right. \right) \quad (16b)
\]

as

\[
K_{00}(x, y) = \theta \int_0^1 t^\alpha H_a(x) H_{a+1}(y) \, dt \quad (17a)
\]

\[
K_{01}(x, y) = \theta x^{2a+1} \int_0^1 t^\alpha H_a(y) G_{a+1}(x) \, dt \quad (17b)
\]

\[
K_{10}(x, y) = \theta y^{2a+1} \int_0^1 t^\alpha H_{a+1}(x) G_a(y) \, dt \quad (17c)
\]

\[
K_{11}(x, y) = \theta (xy)^{2a+1} \int_0^1 t^\alpha G_{a+1}(x) G_a(y) \, dt - \frac{x^a y^{a+1}}{x + y}, \quad (17d)
\]

where we denote

\[
\alpha = \frac{2(a + 1)}{\theta} - 1. \quad (18)
\]

In general, the Fox H-function is defined through the following contour integral [41]

\[
H_{p,q}^{m,n} \left( \begin{pmatrix} (a_1, A_1), \ldots, (a_n, A_n); (a_{n+1}, A_{n+1}), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_m, B_m); (b_{m+1}, B_{m+1}), \ldots, (b_q, B_q) \end{pmatrix} \left| x \right. \right) = \frac{1}{2\pi i} \int_L \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - A_j - a_j s) \prod_{j=m+1}^{p} \Gamma(1 - b_j - B_j s) x^{-s} \, ds, \quad (19)
\]
where the contour $\mathcal{L}$ separates the poles of $\Gamma(b_j + B_j s)$ from the poles of $\Gamma(1 - a_j - A_j s)$. In the special case $A_1 = \ldots = A_p = 1$ and $B_1 = \ldots = B_q = 1$, the Fox H-function reduces to the Meijer G-function [41]. The integral forms of the kernel functions (17a)–(17d) are useful for the computation of mean entropies, whereas several other distinct representations of the kernel functions exist [32, 35–37, 40]. In particular, we present the following biorthogonal polynomial forms [40] useful for a later discussion

\begin{align}
K_{00}(x, y) &= \sum_{k=0}^{m-1} p_k(x^\theta) q_k(y^\theta) \tag{20a} \\
K_{01}(x, y) &= x^a e^{-x} \int_0^\infty \frac{y^a+1 e^{-v} K_{00}(y,v)}{x + v} dv \tag{20b} \\
K_{10}(x, y) &= y^a e^{-y} \int_0^\infty \frac{x^a e^{-w} K_{00}(w,x)}{y + w} dw \tag{20c} \\
K_{11}(x, y) &= x^a y^{a+1} e^{-x-y} \int_0^\infty \int_0^\infty \frac{v^a e^{-v} w^{a+1} e^{-w} K_{00}(v,w) dv dw}{y + v x + w} - W(x,y), \tag{20d}
\end{align}

where the biorthogonal polynomials

\begin{align}
p_j(x^\theta) &= \sum_{k=0}^{j} \frac{\sqrt{2}(-1)^k j! \Gamma(k + j + \alpha + 1)x^{\theta k}}{\Gamma(\theta k + a + 1)\Gamma(k + \alpha + 1)(j - k)! k!} \tag{21a} \\
q_j(y^\theta) &= \sum_{k=0}^{j} \frac{\sqrt{2}(-1)^k j! \Gamma(k + j + \alpha + 1)y^{\theta k}}{\Gamma(\theta k + a + 2)\Gamma(k + \alpha + 1)(j - k)! k!} \tag{21b}
\end{align}

are orthogonal with respect to the weight function

\begin{equation}
W(x,y) = \frac{x^a y^{a+1} e^{-x-y}}{x + y} \tag{22}
\end{equation}

as

\begin{equation}
\int_0^\infty \int_0^\infty p_k(x^\theta) q_l(y^\theta) W(x,y) dx dy = \delta_{kl}. \tag{23}
\end{equation}

We now study the moment relations of entanglement entropies between the proposed ensemble (6) and its unconstrained version (13). Firstly, the density $g_d(r)$ of the trace

\begin{equation}
r = \sum_{i=1}^{m} x_i, \quad r \in [0, \infty) \tag{24}
\end{equation}

of the unconstrained ensemble (13) is obtained as

\begin{align}
g_d(r) &= \int_x h(x) \delta \left( r - \sum_{i=1}^{m} x_i \right) \prod_{i=1}^{m} dx_i \tag{25} \\
&= \frac{C}{Cr} r^{-d-1} \int_{\lambda} f(\lambda) \prod_{i=1}^{m} d\lambda_i \tag{26} \\
&= \frac{1}{\Gamma(d)} e^{-r d-1}, \tag{27}
\end{align}
where we have used the change of variables
\[ x_i = r \lambda_i, \quad i = 1, \ldots, m \] (28)
and the resulting Jacobian calculation leads to the normalization \( \Gamma(d) \) with
\[ d = \frac{m}{2} (m \theta - \theta + 2a + 2). \] (29)
The above calculation implies that the density \( h(x) \) can be factored as
\[ h(x) \prod_{i=1}^{m} dx_i = f(\lambda) g_d(r) dr \prod_{i=1}^{m} d\lambda_i, \] (30)
i.e., the random variable \( r \) is independent of each \( \lambda_i \) (hence independent of \( S_P \) and \( S_{vN} \)). Similar factorizations also exist for the Hilbert-Schmidt ensemble [7] and the Bures-Hall ensemble [30, 32]. Introducing the corresponding quantum purity of the unconstrained ensemble
\[ T_P = \sum_{i=1}^{m} x_i^2, \] (31)
the \( k \)-th moment of quantum purity \( S_P \) is represented as
\[ \mathbb{E}_f [S_P^k] = \int_{\lambda} S_P^k f(\lambda) \prod_{i=1}^{m} d\lambda_i \] (32)
\[ = \int_{\lambda} \frac{T_P^k}{r^{2k}} f(\lambda) \prod_{i=1}^{m} d\lambda_i \int_r g_{d+2k}(r) dr \] (33)
\[ = \frac{\Gamma(d)}{\Gamma(d+2k)} \int_{\lambda} \int_r T_P^k f(\lambda) g_d(r) dr \prod_{i=1}^{m} d\lambda_i \] (34)
\[ = \frac{\Gamma(d)}{\Gamma(d+2k)} \mathbb{E}_h [T_P^k], \] (35)
where we have used the change of variables (28) and the independence property (30). Therefore, computing the \( k \)-th moment of \( S_P \) can be converted to computing the \( k \)-th moment of \( T_P \). In particular, the first moments are related by
\[ \mathbb{E}_f [S_P] = \frac{1}{d(d+1)} \mathbb{E}_h [T_P]. \] (36)
We now introduce von Neumann entropy of the unconstrained ensemble
\[ T_{vN} = \sum_{i=1}^{m} x_i \ln x_i, \] (37)
that leads to the identity
\[ S_{vN} = \ln r - r^{-1} T_{vN}, \] (38)
then the first moment relation is similarly obtained as
\[ \mathbb{E}_f [S_{vN}] = \int_{\lambda} S_{vN} f(\lambda) \prod_{i=1}^{m} d\lambda_i \int_r g_{d+1}(r) dr \] (39)
\[
\psi_0(d + 1) = \psi_0(d + 1) = \frac{1}{d}E_h[T_{\nu N}],
\]

where we have also used
\[
\int_0^\infty e^{-r^a r^{-1}} \ln r \, dr = \Gamma(a)\psi_0(a), \quad \Re(a) > 0
\]

with \(\psi_0(x) = d \ln \Gamma(x)/dx\) denoting the digamma function [42]. For a positive integer \(l\), the digamma function admits the following useful identities
\[
\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \quad \psi_0\left(l + \frac{1}{2}\right) = -\gamma - 2 \ln 2 + 2 \sum_{k=0}^{l-1} \frac{1}{2k + 1},
\]

where \(\gamma \approx 0.5772\) is the Euler’s constant.

With the above preparation, we now state the results of average entropies over the interpolating ensemble.

**Proposition 1** The average value of quantum purity (11) under the quantum interpolating ensemble (6), for any \(\theta > 0\) and \(a > -1\), is given by
\[
E_f[S_P] = \sum_{k=0}^{m-1} \frac{(-1)^{k+m-1} \theta^{k+a+2}^2}{2d(d+1)(m-1-k)!k!} A_\theta(k),
\]

where
\[
A_\theta(k) = \frac{\Gamma\left(k + 2(a+2)\theta\right)\Gamma\left(k + m + 2(a+1)\theta\right)}{\Gamma\left(k + 2(a+1)\theta\right)\Gamma\left(k + m + 2(a+2)\theta\right)}.
\]

Before proving proposition 1, two remarks on its special cases are in order.

**Remark 1** In the special case \(\theta = 1\), \(a = n - m - \frac{1}{2}\) that corresponds to the Bures-Hall ensemble (3), by recognizing \(A_1(m-2)\) and \(A_1(m-1)\) as the only non-vanishing terms in (44) with
\[
A_1(k) = \frac{(k + 2n - 2m + 2)(k + 2n - 2m + 1)\Gamma(k + 3)}{(k + 2n - m + 2)(k + 2n - m + 1)\Gamma(k + 3 - m)},
\]

the expression (43) simplifies to
\[
E_f[S_P] = \sum_{k=m-2}^{m-1} \frac{2(-1)^{k+m-1}(k + n - m + 3/2)^2}{m(2n-m)(2mn-m^2+2)(m-1-k)!k!} A_1(k)
\]
\[
= \frac{2n(2n+m) - m^2 + 1}{2n(2mn-m^2+2)}.
\]

This recovers the mean purity formula of the Bures-Hall ensemble recently reported in [30, 31].
Remark 2  In the special case $\theta = 2$, $a = n - m$ that corresponds to the Hilbert-Schmidt ensemble (1), by recognizing $A_2(m - 1)$ as the only non-vanishing term in (44) with

$$A_2(k) = \frac{(k+n-m+1)\Gamma(k+2)}{(k+n+1)\Gamma(k+2-m)},$$

the expression (43) simplifies to

$$\mathbb{E}_f[S_P] = \sum_{k=m-1}^{m-1} \frac{(-1)^{k+m-1}(2k+n-m+2)^2}{mn(mn+1)(m-1-k)!k!} A_2(k)$$

$$= \frac{m+n}{mn+1}.\quad (49)$$

We recover the mean purity formula of the Hilbert-Schmidt ensemble obtained in [5].

We now prove the proposition 1.

Proof  The essential task is to compute $\mathbb{E}_h[T_P]$, which, after inserting into the moment relation (36), will establish the proposition 1. The required single eigenvalue density $h_1(x)$ of the unconstrained ensemble (13) can be read off from the correlation function (14) that corresponds to a Pfaffian of a $2 \times 2$ matrix as

$$h_1(x) = \frac{1}{m} \rho_1(x) = \frac{1}{2m} (K_{01}(x,x) + K_{10}(x,x)).\quad (51)$$

The computation now boils down to computing two integrals

$$\mathbb{E}_h[T_P] = m \int_0^{\infty} x^2 h_1(x) \, dx$$

$$= \frac{1}{2} \int_0^{\infty} x^2 K_{01}(x,x) \, dx + \frac{1}{2} \int_0^{\infty} x^2 K_{10}(x,x) \, dx.\quad (53)$$

The starting point to calculate the above integrals is the fact that the contour form (19) of the Fox H-function (16a) admits a finite number of single poles, which by residue calculation gives a finite sum

$$H_q(x) = \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k+\alpha+m+1) (tx^\theta)^k}{\Gamma(k+\alpha+1) \Gamma(\theta k + q + 1)(m-1-k)!k!}.\quad (54)$$

Therefore, we have

$$\int_0^{\infty} x^2 K_{01}(x,x) \, dx$$

$$= \int_0^{\infty} \int_0^{1} t^\alpha H_a(y) G_{a+1}(x) \, dt \, dx$$

$$= \int_0^{\infty} x^{2a+3} \int_0^{1} t^\alpha H_a(y) G_{a+1}(x) \, dt \, dx$$

$$= \sum_{k=0}^{m-1} \frac{\theta(-1)^k \Gamma(k+\alpha+m+1)}{\Gamma(k+\alpha+1) \Gamma(\theta k + q + 1)(m-1-k)!k!} \int_0^{\infty} x^{\theta k + 2a+3} \int_0^{1} t^{\alpha+k} G_{a+1}(x) \, dt \, dx$$

$$= \sum_{k=0}^{m-1} \frac{(-1)^{k+m-1} \theta(\theta k + a + 1)(\theta k + a + 2)}{2(m-1-k)!k!} A_\theta(k).\quad (58)$$
where the integrals over $t$ and $x$ in (57) are evaluated respectively by the identity [41]

$$
\int_0^1 x^{t-1} H^{m,n}_{p,q} \left( \begin{array}{c} (a_1, A_1), \ldots, (a_n, A_n); (a_{n+1}, A_{n+1}), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_m, B_m); (b_{m+1}, B_{m+1}), \ldots, (b_q, B_q) \end{array} \right) | \eta x \rangle \ dx
$$

$$
= H^{m,n+1}_{p+1,q+1} \left( \begin{array}{c} (1 - \rho, 1), (a_1, A_1), \ldots, (a_n, A_n); (a_{n+1}, A_{n+1}), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_m, B_m); (b_{m+1}, B_{m+1}), \ldots, (b_q, B_q), (-\rho, 1) \end{array} \right) \langle \eta \rangle
$$

(59)

and the Mellin transform of the Fox H-function [41], cf. (19),

$$
\int_0^\infty x^{s-1} H^{m,n}_{p,q} \left( \begin{array}{c} (a_1, A_1), \ldots, (a_n, A_n); (a_{n+1}, A_{n+1}), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_m, B_m); (b_{m+1}, B_{m+1}), \ldots, (b_q, B_q) \end{array} \right) | \eta x \rangle \ dx
$$

$$
= \frac{\eta^{-s} \prod_{j=1}^m \Gamma (b_j + B_j s) \prod_{j=1}^n \Gamma (1 - a_j - A_j s)}{\prod_{j=n+1}^p \Gamma (a_j + A_j s) \prod_{j=m+1}^q \Gamma (1 - b_j - B_j s)}.
$$

(60)

In the same manner, the second integral in (53) is evaluated to

$$
\int_0^\infty x^2 K_{10}(x, x) \ dx = \sum_{k=0}^{m-1} \frac{(-1)^{k+m-1}}{2(m-1-k)!} \frac{\theta(\theta k + a + 2)(\theta k + a + 3)}{d(m-1-k)!} A_\theta(k).
$$

(61)

Putting together the results (36), (53), (58), and (61), we complete the proof of proposition 1. □

The next result on the average von Neumann entropy is summarized in the following proposition.

**Proposition 2** The average value of von Neumann entropy (12) under the quantum interpolating ensemble (6), for any $\theta > 0$ and $a > -1$, is given by

$$
\mathbb{E}_f[S_{\text{VN}}] = \psi_0(d+1) - \sum_{k=0}^{m-1} \frac{(-1)^{k+m-1} (\theta k + a + 3/2)}{d(m-1-k)!} \frac{\theta(\theta k + a + 3/2)}{\theta k + a + 3/2} B_\theta(k),
$$

(62)

where

$$
B_\theta(k) = \frac{\Gamma \left( k + \frac{2a+3}{\theta} \right) \Gamma \left( k + m + \frac{2a+3}{\theta} \right) \Gamma \left( k + 1 + \frac{1}{\theta} - m \right)}{\Gamma \left( k + \frac{2a+3}{\theta} \right) \Gamma \left( k + m + \frac{2a+3}{\theta} \right) \Gamma \left( k + 1 + \frac{1}{\theta} - m \right)} \left( \psi_0 \left( k + \frac{2a+3}{\theta} \right) \right.
\left. + \psi_0 \left( k + 1 + \frac{1}{\theta} \right) \right)
$$

$$
+ \psi_0 \left( k + m + \frac{2a+3}{\theta} \right) - \psi_0 \left( k + 1 + \frac{1}{\theta} - m \right) - \psi_0 \left( k + m + \frac{2a+3}{\theta} \right) - \psi_0 \left( k + 1 + \frac{1}{\theta} - m \right)
$$

$$
+ \theta \left( \psi_0(\theta k + a + 2) - \frac{\theta k + a + 1}{\theta k + a + 3/2} \right).
$$

(63)

**Proof** The main task is to compute the average $\mathbb{E}_h[T_{\text{VN}}]$, which, after inserting into the moment relation (40), establishes the proposition 2. By employing the one arbitrary eigenvalue density (51), the task boils down to computing two integrals

$$
\mathbb{E}_h[T_{\text{VN}}] = \frac{1}{2} \int_0^\infty x \ln x K_{01}(x, x) \ dx + \frac{1}{2} \int_0^\infty x \ln x K_{10}(x, x) \ dx.
$$

(64)

We first compute the integral

$$
\int_0^\infty x^\beta K_{01}(x, x) \ dx, \quad \beta > 0
$$

(65)
by using the results (54), (59), and (60) as
\[
\int_{0}^{\infty} x^\beta K_{01}(x, x) \, dx = \sum_{k=0}^{m-1} \frac{(-1)^{k+m} \Gamma(k + \alpha + m + 1)}{2\theta(m - 1 - k)!k!\Gamma(k + \alpha + 1)\Gamma(\theta k + a + 1)} \\
\times \frac{\Gamma(s)\Gamma(s - \alpha)\Gamma(k + \alpha - s + 1)\Gamma(\theta s - a - 1)}{\Gamma(s + m)\Gamma(s - \alpha - m)\Gamma(k + \alpha - s + 2)}, \tag{66}
\]
where \(\alpha\) is given by (18) and we denote
\[
s = \beta - 1 + k + \frac{2a + 3}{\theta}. \tag{67}
\]
Similarly, we also obtain
\[
\int_{0}^{\infty} x^\beta K_{10}(x, x) \, dx = \sum_{k=0}^{m-1} \frac{(-1)^{k+m} \Gamma(k + \alpha + m + 1)}{2\theta(m - 1 - k)!k!\Gamma(k + \alpha + 1)\Gamma(\theta k + a + 2)} \\
\times \frac{\Gamma(s)\Gamma(s - \alpha)\Gamma(k + \alpha - s + 1)\Gamma(\theta s - a)}{\Gamma(s + m)\Gamma(s - \alpha - m)\Gamma(k + \alpha - s + 2)}, \tag{68}
\]
Taking the derivative of (66) and (68) with respect to \(\beta\) before setting \(\beta \to 1\) leads to the desired expression of (64), which upon inserting into the moment relation (40) completes the proof of proposition 2. \(\square\)

Remark 3 In the special case \(\theta = 1\), \(a = n - m - \frac{1}{2}\) that corresponds to the Bures-Hall ensemble (3), the result (62) in proposition 2 reduces to
\[
\mathbb{E}_f[|S_{mN}|] = \psi_0 \left( mn - \frac{m^2}{2} + 1 \right) - \psi_0 \left( n + \frac{1}{2} \right) + \psi_0(2n + 1) - \psi_0(2n - m + 1) + \psi_0(1) \\
- \psi_0(m + 1) + \frac{2n - 1}{2n} + \sum_{k=0}^{m-2} \frac{2(k + 1)(k + n - m + 1)(k + 2n - 2m + 1)}{m(2n - m)(m - 1 - k)(k + 2n - m + 1)} \\
= \psi_0 \left( mn - \frac{m^2}{2} + 1 \right) - \psi_0 \left( n + \frac{1}{2} \right). \tag{69}
\]
This recovers the mean formula of von Neumann entropy under the Bures-Hall ensemble recently obtained in [30, 31].

Remark 4 In the special case \(\theta = 2\), \(a = n - m\) that corresponds to the Hilbert-Schmidt ensemble (1), the mean formula of von Neumann entropy is well-known
\[
\mathbb{E}_f[|S_{mN}|] = \psi_0(mn + 1) - \psi_0(n) - \frac{m + 1}{2n}, \tag{70}
\]
which was conjectured by Page [7] and later proved in [8, 9]. For this special case, to show that the proposition 2 can be reduced to the above result requires showing the identity
\[
\frac{1}{mn} \sum_{k=0}^{m-1} \frac{(-1)^{k+m-1}(2k + n - m + \frac{3}{2})\Gamma(k + \frac{3}{2})\Gamma(k + n + 1)\Gamma(k + n - m + \frac{3}{2})}{(m - 1 - k)!k!\Gamma(k - m + \frac{3}{2})\Gamma(k + n + \frac{3}{2})\Gamma(k + n - m + 1)} \\
\times \left( \psi_0 \left( k + \frac{3}{2} \right) + \psi_0 \left( k + n - m + \frac{3}{2} \right) - \psi_0 \left( k + n + \frac{3}{2} \right) - \psi_0 \left( k - m + \frac{3}{2} \right) \right) + 2\psi_0(2k + n - m + 2) - \frac{2(2k + n - m + 1)}{2k + n - m + 3/2} = \psi_0(n) + \frac{m + 1}{2n}, \tag{71}
\]
which, despite strong numerical evidence, remains unproven.
2.2. Some properties of orthogonal polynomials

Before illustrating the behavior of average entropies via numerical simulations, we present some results on the $\theta$-deformed orthogonal polynomials (21a), (21b). Our first result is the recurrence relation of $p_j (x^\theta)$ for $\theta = 2$, which will be useful in computing the higher order moments of entanglement entropies.

**Proposition 3** The $\theta$-deformed orthogonal polynomials $p_j (x^\theta)$ in (21a) for $\theta = 2$ satisfy a five-term recurrence relation as

$$
x^2 (a_2 p_{j+2} (x^2) + a_1 p_{j+1} (x^2) + a_0 p_j (x^2)) =
$$

$$r_3 p_{j+3} (x^2) + r_2 p_{j+2} (x^2) + r_1 p_{j+1} (x^2) + r_0 p_j (x^2) + r_{-1} p_{j-1} (x^2),
$$

(72)

where the coefficients are explicitly given by

$$a_2 = \frac{2aj + 3a + 2j^2 + 6j + 5}{a + 2j + 4}$$

(73a)

$$a_1 = \frac{2(a + 2j + 3) (2aj + 3a + 2j^2 + 6j + 5)}{(a + 2j + 2)(a + 2j + 4)}$$

(73b)

$$a_0 = \frac{2aj + 3a + 2j^2 + 6j + 5}{a + 2j + 2}$$

(73c)

$$r_3 = \frac{(j + 3)(a + j + 3) (2aj + 3a + 2j^2 + 6j + 5)}{a + 2j + 4}$$

(73d)

$$r_2 = \frac{a^3 + 6a^2j + 12a^2 + 12aj^2 + 46aj + 41a + 8j^3 + 46j^2 + 82j + 42}{(2aj + 3a + 2j^2 + 6j + 5)^{-1} (a + 2j + 2)(a + 2j + 4)}$$

(73e)

$$r_1 = \frac{(a + 2j + 3) (2aj + 3a + 2j^2 + 6j + 5) (2a^2 + 6aj + 9a + 6j^2 + 18j + 10)}{(a + 2j + 2)(a + 2j + 4)}$$

(73f)

$$r_0 = \frac{a^3 + 6a^2j + 6a^2 + 12aj^2 + 26aj + 11a + 8j^3 + 26j^2 + 22j + 6}{(2aj + 3a + 2j^2 + 6j + 5)^{-1} (a + 2j + 2)(a + 2j + 4)}$$

(73g)

$$r_{-1} = \frac{j(a + j) (2aj + 3a + 2j^2 + 6j + 5)}{a + 2j + 2}$$

(73h)

The proof of proposition 3 is in Appendix A.

**Remark 5** The corresponding recurrence relation of the dual polynomial $q_j (y^\theta)$ can be also similarly derived. We defer to a future publication a more complete study of the orthogonal polynomials including the recurrence relations for an arbitrary integer $\theta$ and the resulting Christoffel-Darboux formulas. Note that the recurrence relations for the undeformed case $\theta = 1$ has been established in [34, 37] in terms of a four-term recurrence.

The second result is a factorization property of the kernels in the following lemma. This property is useful in simplifying the $k$-point densities for the higher moment calculations as demonstrated in [32] for the case $\theta = 1$.
Lemma 1 For any $\theta > 0$, the correlation kernels (20a)–(20d) can be factorized as

\begin{align}
K_{00}(x, y) + K_{00}(y, x) &= w(x)w(y) \\
K_{01}(x, y) - K_{10}(y, x) &= v(x)w(y) \\
K_{11}(x, y) + K_{11}(y, x) &= -v(x)v(y),
\end{align}

where

\begin{align}
w(x) &= \theta \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k + \alpha + m + 1)x_\theta^k}{\Gamma(\theta k + a + 2)\Gamma(k + \alpha + 1)\Gamma(m - k)k!} \\
v(x) &= e^{-x^a} - \theta x_\theta^{2a+1} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k + \alpha + m + 1)(-\theta k - a, x)x_\theta^k}{(\theta k + a + 1)\Gamma(k + \alpha + 1)\Gamma(m - k)k!}
\end{align}

with $\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}dt$ denoting the incomplete Gamma function.

The proof of lemma 1 is in Appendix B. Note that the corresponding factorization property of the special case $\theta = 1$ has been established in [36].

3. Numerical results

In this section, we perform numerical studies of the main results on the average entanglement entropies of the interpolating ensemble. We first focus on the result of
average purity in proposition 1. In figure 1, we plot the numerical values of average purity (43) as a function of the subsystem dimensions. For different values of $\theta$, we consider both the $\theta$-deformed Hilbert-Schmidt ensemble assuming $a = 0$ and the $\theta$-deformed Bures-Hall ensemble assuming $a = -1/2$ as shown in the left subfigure and the right subfigure, respectively. Note that the choices of $a$ imply equal subsystem dimensions $m = n$ in both cases. The solid curves in figure 1 describe the behavior of the standard ensembles (7) with no deformations, whereas the other curves represent the corresponding $\theta$-deformed ones. It is observed that as the deformation parameter $\theta$ increases, the values of average purity decrease monotonically resulting in estimations of entanglement towards more entangled states. The observation suggests that the proposed interpolating ensemble (6) is indeed able to continuously interpolate among the possible values of purity by varying the $\theta$ parameter. It is also observed in figure 1 that for a given $\theta$ the average purity under the Bures-Hall ensemble tends to an estimate of more separable state (i.e., a larger purity value) than that of the Hilbert-Schmidt ensemble. On the other hand, the differences are diminishing as the dimension increases. This behavior has also been recently observed in [30].

Since the parameter $a$ of the proposed ensemble (6) can be also considered as a deformation variable, we wish to understand its impact on the quantum purity. In figure 2, we plot the average purity (43) as a function of the parameter $a$ for different values of $\theta$, where the dimension of subsystem is assumed to be $m = 8$. The data
Figure 3. Average von Neumann entropy (62) as a function of subsystem dimensions: the impact of \( \theta \)-deformation. The two solid curves represent the cases of the standard ensembles (7) with no deformations and the other curves represent the cases of the deformed ensemble (6).

points marked by diamond shape for \( \theta = 1 \) and square shape for \( \theta = 2 \) corresponds to the special case of Bures-Hall ensemble and Hilbert-Schmidt ensemble, respectively. It is observed in figure 2 that as \( a \) increases, the values of average purity decrease monotonically indicating more entangled states. In particular, for the cases \( \theta = 1 \) and \( \theta = 2 \), the average purity is seen to interpolate continuously among the permissible values (7) of the parameter \( a \).

We now turn to the numerical study of the von Neumann entropy in proposition 2. In figure 3, we plot the average von Neumann entropy (62) as a function of the subsystem dimensions for different values of \( \theta \). We consider both the \( \theta \)-deformed Hilbert-Schmidt and Bures-Hall ensembles with the same values of \( a \) as in figure 1. It is seen that as the deformation parameter \( \theta \) increases, the average von Neumann entropy increases monotonically, which also results in estimations of entanglement towards more entangled states as in figure 1. In particular, the proposed interpolating ensemble (6) continuously interpolates among the possible values of the von Neumann entropy. Similar to figure 1, we also observe in figure 3 that the average von Neumann entropy under the Bures-Hall ensemble tends to an estimate of more separable state (i.e., a smaller value of von Neumann entropy) than that of the Hilbert-Schmidt ensemble. The differences, however, diminish as the dimension increases, which is in line with the recent observation [30].

To understand the impact of parameter \( a \), we plot in figure 4 the average von Neumann
entropy (62) as a function of the parameter $a$ for different values of $\theta$. The dimension of subsystem is also assumed to be $m = 8$. Similarly as observed in figure 2, the values of average von Neumann entropy increase monotonically indicating more entangled states as $a$ increases. Finally, we point out that various other numerical simulations have been performed, where the same relative behavior as discussed in above four figures persists.

4. Conclusion and outlook

In this work, we proposed and studied a generalized measure that interpolates between the two major measures of density matrices - the Hilbert-Schmidt ensemble and the Bures-Hall ensemble. In particular, we derived the mean quantum purity and the mean von Neumann entropy over the proposed ensemble as summarized in proposition 1 and proposition 2, respectively. We also studied the corresponding orthogonal polynomial system including its recurrence relation in proposition 2 and a kernel factorization property in lemma 1. Extensive numerical simulations show that the proposed ensemble provides additional power in estimating the degree of entanglement by varying the deformation parameters. Future work includes further study of the statistical information of the ensemble such as higher order moments of entropies, fidelity, and volumes as well as further study of the underlying orthogonal polynomials.
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Appendix A. Proof of proposition 3

Proof We give a proof of the recurrence relation for the general bi-orthogonal system \( \{p_j, q_j\}_{j=0}^\infty \) for all positive integer \( \theta \in \mathbb{N} \) with respect to the generalized weight function

\[
\tilde{W}(x, y) = \frac{x^a y^b e^{-x-y}}{x+y}, \quad (A.1)
\]

and all \( \mathbb{R}(a, b) > -1 \). This reduces to the weight function (22) when \( b = a + 1 \). The essence of our proof is a generalization of that used in [34] from the rank one shift condition to a rank-\( \theta \) condition. Towards the end of proof we will specialize to \( \theta = 2 \) only when evaluating the explicit recurrence coefficients, without loss of generality.

To begin we define the parameter

\[
\beta = \frac{a + b + 1}{\theta}, \quad (A.2)
\]

which reduces to \( \alpha + 1 \) in (18) when \( b = a + 1 \). We will see that the simplest form of the recurrence will be expressed in terms of the bi-orthogonal polynomials

\[
\tilde{p}_j(x) = \frac{(-1)^j}{\sqrt{2}} p_j(x) \quad (A.3)
\]

of argument \( x \), which we call hybrid polynomials, rather than those defined in (21) of argument \( x^\theta \). The normalized bi-orthogonal polynomials are denoted by \( \{P_j, Q_j\}_{j=0}^\infty \) and the monic system by \( \{p_j, q_j\}_{j=0}^\infty \) with the inter-relations

\[
P_j(x) = \frac{1}{\sqrt{h_j}} p_j(x) \quad (A.4)
\]

\[
p_j(x) = (-1)^j j! \frac{\Gamma(a + 1 + \theta j) \Gamma(j + \beta)}{\Gamma(2j + \beta)} \tilde{p}_j(x) \quad (A.5)
\]

along with the squared norm

\[
h_j = \theta^{-1} \frac{(j!)^2 \Gamma^2(j + \beta)}{\Gamma(2j + \beta) \Gamma(2j + \beta + 1)} \Gamma(a + 1 + \theta j) \Gamma(b + 1 + \theta j). \quad (A.6)
\]

The bi-moments are defined and evaluated as

\[
I_{k,l} = \int_{\mathbb{R}_+^2} e^{-x-y} x^{a+\theta k} y^{b+\theta l} \, dx \, dy = \frac{\Gamma(a + 1 + \theta k) \Gamma(b + 1 + \theta l)}{a + b + 1 + \theta(k + l)}, \quad k, l = 0, 1, \ldots \quad (A.7)
\]

Now a key observation is that \( x + y \) divides \( x^\theta - e^{\pi i \theta} y^\theta \) without remainder and therefore

\[
I_{k+1,l} - e^{\pi i \theta} I_{k,l+1} = \sum_{s=0}^{\theta-1} (-1)^s \Gamma(a + \theta - s + \theta k) \Gamma(b + 1 + s + \theta l). \quad (A.8)
\]
Employing semi-infinite matrices $I = (I_{k,l})_{k,l \geq 0}$ and the shift matrix $\Lambda = (\delta_{k+1,l})_{k,l \geq 0}$, this is written as the rank-$\theta$ decomposition

$$\Lambda I - e^{\pi i \theta} \Lambda^T = \sum_{s=0}^{\theta-1} (-1)^s \alpha_s \beta_s^T \tag{A.9}$$

with semi-infinite column vectors $\alpha_s = (\Gamma(a+\theta-s+\theta k))_{k \geq 0}$, $\beta_s = (\Gamma(b+1+s+\theta k))_{k \geq 0}$. Now denote the vectors of monomials $x = (x^k)_{k \geq 0}$, $y = (y^k)_{k \geq 0}$ and the vector of normalised bi-orthogonal polynomials $P = (P_k(x))_{k \geq 0}$, $Q = (Q_k(y))_{k \geq 0}$. These basis vectors are related by lower triangular matrices $S_P$ and $S_Q$ as

$$P = S_P x, \quad Q = S_Q y. \tag{A.10}$$

Thus, we have the L-U decomposition of the bi-moment matrix

$$I = S_P^{-1} (S_Q^{-1})^T. \tag{A.11}$$

The monomial bases are the right and left eigenvectors of the shift matrix and its transpose

$$\Lambda x = x x, \quad y^T \Lambda^T = y y^T. \tag{A.12}$$

Thinking of the action of a multiplication operator on the normalized polynomial bases, it is clear that it can be written generally as

$$x P = X P, \quad y^T Q^T = Q^T Y^T \tag{A.13}$$

for some lower Hessenberg matrices $X$ and $Y$. These can be explicitly calculated although they are not needed and so we will not exhibit them here. These multiplication matrices now have the decomposition

$$X = S_P \Lambda S_P^{-1}, \quad Y = S_Q \Lambda S_Q^{-1}, \tag{A.14}$$

and by premultiplying by $S_P$ and postmultiplying (A.9) by $S_Q^T$ we deduce

$$X - e^{\pi i \theta} Y^T = \sum_{s=0}^{\theta-1} (-1)^s \pi_s \eta_s^T, \tag{A.15}$$

where $\pi_s = S_P \alpha_s$ and $\eta_s = S_Q \beta_s$.

In order to proceed further it is useful to construct rank-$\theta$ annihilators, along the lines that was done in [34] for $\theta = 1$, for the right-hand side of (A.15). We will do this recursively in $\theta$ steps, but will show the details for $\theta = 2$ only. For any vector $\pi$ let us construct the semi-infinite diagonal matrix $D_{\pi}$ so that $\pi = D_{\pi} \mathbf{1}$. From the knowledge that the unit vector $\mathbf{1}$ (or any constant vector) is left-annihilated by $\Lambda - \text{Id}$ we can left-annihilate the $s = 0$ term on the right-hand side of (A.15) by premultiplying with $(\Lambda - \text{Id}) D_{\pi_0}^{-1}$. To annihilate the remaining $s = 1$ term we need to calculate $\psi = (\Lambda - \text{Id}) D_{\pi_0}^{-1} D_{\pi_1} \mathbf{1}$. A simple calculation gives the components of $\psi$ as

$$\psi_j = \frac{\pi_{1,j}}{\pi_{0,j}} - \frac{\pi_{1,j+1}}{\pi_{0,j+1}}, \quad j = 0, 1, \ldots, \tag{A.16}$$
assuming $\pi_{0,j} \neq 0$. Our required second left-annihilator is therefore $(\Lambda - Id) \ D_{\psi}^{-1}$ and so the composite operator is the second order difference operator

$$(\Lambda - Id) \ D_{\psi}^{-1} (\Lambda - Id) \ D_{\pi_0}^{-1}. \quad (A.17)$$

This recursion can be repeated up to $\theta$ levels leaving us with a $\theta$-order difference operator in the general case, modulo the non-vanishing condition given above. Before we compute this our final step in deriving the recurrence relation for $P_j$ is to put some of these pieces together. Let us act on the first equation of (A.13) with this operator - doing so on the left-hand side gives

$$x (\Lambda - Id) \ D_{\psi}^{-1} (\Lambda - Id) \ D_{\pi_0}^{-1} P, \quad (A.18)$$

whereas acting on the right-hand side gives

$$(\Lambda - Id) \ D_{\psi}^{-1} (\Lambda - Id) \ D_{\pi_0}^{-1} XP = AP \quad (A.19)$$

that defines a banded matrix $A$. This banded matrix has non-zero elements only for $\theta + 1$ super-diagonals above the diagonal - the $\theta$-order difference operator adds $\theta$ super-diagonals to the initial single one of the lower Hessenberg $X$ - and a single sub-diagonal - this operator does not add any additional sub-diagonals to the upper Hessenberg matrix $Y^T$. For $\theta = 2$ the second order difference operator acting on $P$ has $j$-th component

$$\frac{\pi_{0,j+1}}{\pi_{0,j+2} \pi_{1,j+1} - \pi_{0,j+1} \pi_{1,j+2}} P_{j+2} + \frac{\pi_{0,j+1}}{\pi_{0,j+1} \pi_{1,j} - \pi_{0,j} \pi_{1,j+1}} P_j + \frac{\pi_{0,j+1}(\pi_{0,j} \pi_{1,j+1} - \pi_{0,j} \pi_{1,j+2})}{(\pi_{0,j+1} \pi_{1,j} - \pi_{0,j} \pi_{1,j+1})} P_{j+1}. \quad (A.20)$$

In the case at hand we compute the components of $\pi_0$ and $\pi_1$ to be

$$\pi_{0,j} = \frac{1}{\sqrt{h_j}} j! \Gamma(a + 1 + 2j) \Gamma(\beta + j) \Gamma(\beta + 2j) (a + 1 + 2j(j + \beta)) \quad (A.21)$$

$$\pi_{1,j} = \frac{1}{\sqrt{h_j}} j! \Gamma(a + 1 + 2j) \Gamma(\beta + j) \Gamma(\beta + 2j). \quad (A.22)$$

Thus, we arrive at a structure for the recurrence for our hybrid system

$$x (a_2 \tilde{p}_{j+2}(x) + a_1 \tilde{p}_{j+1}(x) + a_0 \tilde{p}_j(x)) =$$

$$r_3 \tilde{p}_{j+3}(x) + r_2 \tilde{p}_{j+2}(x) + r_1 \tilde{p}_{j+1}(x) + r_0 \tilde{p}_j(x) + r_{-1} \tilde{p}_{j-1}(x), \quad (A.23)$$

along with

$$a_2 = \frac{(a + 1 + 2(j + 1)(j + 1 + \beta))}{2j + 3 + \beta} \quad (A.24)$$

$$a_1 = \frac{2(a + 1 + 2(j + 1)(j + 1 + \beta))(2j + 2 + \beta)}{(2j + 1 + \beta)(2j + 3 + \beta)} \quad (A.25)$$

$$a_0 = \frac{(a + 1 + 2(j + 1)(j + 1 + \beta))}{2j + 1 + \beta}. \quad (A.26)$$

On the other hand, the $r$ coefficients can be deduced in a number of ways, such as employing the explicit series form and peeling off the leading terms from the highest degree $(j + 3)$ down in four successive iterations. Either way we find

$$r_3 = \frac{-(a + 1 + 2(j + 1)(j + 1 + \beta))(j + 3)(a + 2j + 5)(a + 2j + 6)(\beta + j + 2)}{(\beta + 2j + 3)(\beta + 2j + 4)(\beta + 2j + 5)} \quad (A.27)$$
where the last step is obtained by lemma 4.1 in [35]. We then have
\[ K = \frac{1}{\beta + j + 5} \]

and specialize (A.3) prove the proposition 3.

Finally, upon the substitution \( x \to x^2 \) and specialization \( b = a + 1 \), \( \beta = \alpha + 1 \) while keeping in mind the relation (A.3) prove the proposition 3.

\[ \square \]

Appendix B. Proof of lemma 1

Proof The starting point of the proof is the orthogonal polynomial forms of the kernel functions (20a)–(20d). To show (74a), we first represent (20a) via (21a) and (21b) as

\[ r_2 = \frac{(a + 1 + 2(j + 1)(j + 1 + \beta))}{(\beta + 2j + 1)(\beta + 2j + 3)} \left( - (j + 2)(a + 2j + 3)(a + 2j + 4)(\beta + j + 1) + \frac{(j + 3)(a + 2j + 5)(a + 2j + 6)(\beta + j + 2)(\beta + 2j + 1)}{\beta + j + 5} \right) \]  

\[ (A.28) \]

\[ r_1 = - (\beta + 2j + 2)(a + 1 + 2(j + 1)(j + 1 + \beta)) \left( \frac{1}{8} \beta(\beta - 2)(a - \beta + 1)(a - \beta + 2) \times \left( \frac{1}{\beta + 2j + 4} - \frac{1}{\beta + 2j} \right) + \frac{1}{4} \beta^4 - (2a + 5)^3 + (a(a + 7) + 8)\beta^2 - 2a(a + 2)\beta - a(a + 7) + \beta - 9 \right) \left( \frac{1}{\beta + 2j + 1} - \frac{1}{\beta + 2j + 3} \right) + \frac{3}{2} \]  

\[ (A.29) \]

\[ r_0 = (a + 1 + 2(j + 1)(j + 1 + \beta)) \left( \frac{4\beta - 2a - 3}{2} + \frac{(\beta - 1)^2(a - \beta)(a - \beta + 1)}{8(\beta + 2j - 1)} - \beta^4 - (2a + 5)^3 + (a(a + 7) + 8)\beta^2 - 2a(a + 2)\beta - a(a + 7) + \beta - 9 \right) \]  

\[ + \frac{(\beta - 3)(\beta + 1)(a - \beta + 2)(a - \beta + 3)}{8(\beta + 2j + 3)} + 2j \]  

\[ (A.30) \]

\[ r_{-1} = - \frac{j(\beta + j - 1)(a - 2\beta - 2j + 1)(a - 2(\beta + j - 1))}{(\beta + 2j - 1)(\beta + 2j)(\beta + 2j + 1)} \times (a + 1 + 2(j + 1)(j + 1 + \beta)) \]  

\[ (A.31) \]
with which in general admits \(4\) as
\[
\sum_{k=0}^{m-1} \frac{\theta(-1)^k \Gamma(k + \alpha + m + 1)x^\theta_k}{\Gamma(\theta k + a + 2)\Gamma(k + \alpha + 1)(m - k)!} = w(x)w(y),
\]
which establishes (74a). To show (74b), we insert (20a) into (20b) and (20c) that gives
\[
K_{01}(x, y) - K_{10}(y, x) = \int_{0}^{\infty} \frac{x^a v^a e^{-x-\theta}}{x + v} (vK_{00}(y, v) - xK_{00}(v, y)) \, dv
\]
\[
= \sum_{i=0}^{m-1} \frac{\theta(-1)^i k \Gamma(i + \alpha + m + 1)\Gamma(k + \alpha + m + 1)\Gamma(m - i)!\Gamma(m - k)!^{-1} y^\theta_i}{(i + k + \alpha + 1)\Gamma(\theta i + a + 2)\Gamma(\theta k + a + 2)\Gamma(i + \alpha + 1)\Gamma(k + \alpha + 1)}
\]
\[
\times \int_{0}^{\infty} \frac{x^a v^a e^{-x-\theta}}{x + v} \left( (\theta i + a + 1)v - x(\theta k + a + 1) \right) \, dv
\]
\[
= \sum_{i=0}^{m-1} \frac{\theta(-1)^i k \Gamma(i + \alpha + m + 1)e^{-x} x^a y^\theta_i}{(i + k + \alpha + 1)\Gamma(\theta i + a + 2)\Gamma(\theta k + a + 2)\Gamma(i + \alpha + 1)\Gamma(k + \alpha + 1)}
\]
\[
\times \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k + \alpha + m + 1)\Gamma(m - k)!^{-1} y^\theta_k}{(\theta k + a + 1)\Gamma(k + \alpha + 1)\Gamma(m - k)!}
\]
\[
\left( e^{-x} x^a - \theta x^{2a+1} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k + \alpha + m + 1)\Gamma(-\theta k - a, x)x^\theta_k}{(\theta k + a + 1)\Gamma(k + \alpha + 1)\Gamma(m - k)!} \right)
\]
\[
\times \sum_{i=0}^{m-1} \frac{\theta(-1)^i \Gamma(i + \alpha + m + 1)y^\theta_i}{(\theta i + a + 2)\Gamma(i + \alpha + 1)\Gamma(m - i)!} = v(x)w(y),
\]
where the second to last equality is obtained by the identity
\[
\sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k + \alpha + m + 1)}{(i + k + \alpha + 1)(\theta k + a + 1)\Gamma(k + \alpha + 1)\Gamma(m - k)!} = \frac{1}{\theta i + a + 1}.
\]
This identity is established by the fact that the sum can be written in terms of a unit argument terminating hypergeometric function of Saalschützian type [42] as
\[
_{4}F_{3} \left( 1 - m, \frac{\alpha + 1}{2}, \alpha + m + 1, i + \alpha + 1; \frac{\alpha + 3}{2}, \alpha + 1, i + \alpha + 2; 1 \right),
\]
which in general admits [42]
\[
_{4}F_{3}(-n, b, c, d; b + 1, c - l, d - m; 1) = \frac{n!(b - c + 1)l(b - d + 1)m}{(b + 1)q(1 - c)(1 - d)m}, \quad 0 \leq l + m \leq n
\]
with
\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}
\]
denoting the Pochhammer’s symbol. This completes the proof of (74b). To show (74c), we insert (20a) into (20d) that leads to

\[
K_{11}(x, y) + K_{11}(y, x) = x^a y^ae^{-x-y} \left( \int_0^\infty \int_0^\infty \frac{v^a w^a e^{-v-w}}{(x+w)(y+v)} (ywK_{00}(v, w) + xvK_{00}(w, v) \, dw \, dv) - 1 \right)
\]

\[
= \sum_{k=0}^{m-1} \sum_{i=0}^{m-1} \frac{\theta(-1)^i k \Gamma(i + \alpha + m + 1) \Gamma(-\theta k - a, x)x^{\theta k}y^{\theta k+2a+1}e^{-x}}{\Gamma(k + \alpha + 1) \Gamma(m - k)k!} \\
\times x^a y^a e^{-x-y} \int_0^\infty \int_0^\infty \frac{v^a w^a e^{-v-w} (yw^{\theta k}w^{\theta i+1} + xv^{\theta k}v^{\theta i+1})}{(x+w)(y+v)} \, dw \, dv - x^a y^a e^{-x-y}
\]

\[
= \sum_{k=0}^{m-1} \frac{\theta(-1)^k \Gamma(k + \alpha + m + 1) \Gamma(-\theta k - a, x)x^{\theta k}y^{\theta k+2a+1}e^{-x}}{\Gamma(k + \alpha + 1) \Gamma(m - k)k!} \\
\times \sum_{i=0}^{m-1} \frac{(i + k + \alpha + 1) \Gamma(-\theta k - a, y)y^{\theta k+2a+1}x^{\alpha}e^{-x}}{\Gamma(k + \alpha + 1) \Gamma(m - k)k!} \\
\times \sum_{i=0}^{m-1} \frac{(i + k + \alpha + 1) \Gamma(i + \alpha + 1) \Gamma(-\theta i - a, y)y^{\theta i}x^{\alpha+1}}{\Gamma(i + \alpha + 1) \Gamma(m - i)k!} \\
\times \sum_{i=0}^{m-1} \frac{(i + k + \alpha + 1) \Gamma(i + \alpha + 1) \Gamma(-\theta i - a, y)y^{\theta i}x^{\alpha+1}}{\Gamma(i + \alpha + 1) \Gamma(m - i)k!} \\
\times v(x) = v(y),
\]

where the second to last equality is obtained by applying twice the identity (B.7). This completes the proof of lemma 1.

\[\square\]

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