THE SOFTWARE PACKAGE SpectralSequences

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ABSTRACT. We describe the computer algebra software package SpectralSequences for the computer algebra system Macaulay2. This package implements many data types, objects and algorithms which pertain to, among other things, filtered complexes, spectral sequence pages and maps therein. We illustrate some of the syntax and capabilities of SpectralSequences by way of several examples.

1. Introduction

Spectral Sequences play an important role in many areas of mathematics, including Algebraic Topology, Algebraic Geometry and Commutative Algebra. Here, we describe the software package SpectralSequences for the computer algebra system Macaulay2. This package provides tools for effective computation of the pages and differentials in spectral sequences obtained from many kinds of filtered chain complexes. We refer the reader to the package’s documentation for precise details regarding the kinds of filtered chain complexes, and the spectral sequences they determine, which we consider. Further, we mention that SpectralSequences version 1.0 is designed to run on Macaulay2 version 1.9.2 and is available at: https://github.com/Macaulay2/M2/blob/master/M2/Macaulay2/packages/SpectralSequences.m2.

Spectral sequences, while often very useful, are notoriously cumbersome and difficult to compute. Indeed, as evidenced by their recursive nature, making explicit computations by hand is often impossible except when they degenerate quickly. At the same time, the mere existence of a given spectral sequence can be enough to prove many interesting and important results.

Before illustrating our package, we mention that we were motivated in part by phrases in the literature, along the lines of:

• “There is a spectral sequence for Koszul cohomology which abuts to zero and [provides this formula for syzygies],” [3];
• “Thus the spectral sequence degenerates ... [and] φ is a differential from the mth page and other maps are differentials from the first page,” [1].

We also point out that, in spite of the many classes of spectral sequences which the package SpectralSequences can compute, there are still several, including some considered for instance in [3] and [1], which we would like to compute but which remain out of reach using present techniques and computational power.

In writing this package, our goal was to create a solid foundation and language in Macaulay2 for working with filtered chain complexes and spectral sequences. In the class of spectral sequences we can compute, not only can we compute the modules, but we can compute all of the maps as well. We hope that this package develops further and, as it develops, more examples will be incorporated which will facilitate computations and intuition. Computation and experimentation in algebra in recent decades have led to countless conjectures, examples
and theorems. It is our hope that this package will allow for experimentation previously not possible.

As a first easy example, and also to help illustrate one aspect in developing our package, consider the simplicial complex $\Delta$ on the vertex set $\{x, y, z, w\}$, $x < y < z < w$, with facet description given by $\Delta := \{x y z, w z\}$. Further, put $F_2 \Delta := \Delta$, and define simplicial subcomplexes by $F_1 \Delta := \{x y, w\}$ and $F_0 \Delta := \{x, w\}$. By considering the reduced chain complexes of the simplicial complexes $F_\bullet \Delta$, over a given field $k$, we obtain a filtered chain complex

$$F_\bullet \mathring{C}_\bullet : 0 = F_{-1} \mathring{C}_\bullet \subseteq F_0 \mathring{C}_\bullet \subseteq F_1 \mathring{C}_\bullet \subseteq F_2 \mathring{C}_\bullet = \mathring{C}_\bullet.$$ 

As it turns out, the spectral sequence $E := E(F_\bullet \mathring{C}_\bullet)$ determined by this filtered complex $F_\bullet \mathring{C}_\bullet$ has the property that the map $d_{2, -1}^2 : E_{2, -1}^2 \to E_{0, 0}^2$ is an isomorphism of one dimensional $k$-vector spaces. In fact, it is not difficult to establish this fact directly by hand. On the other hand, an important first step in developing our package was for it to successfully compute the map $d_{2, -1}^2$ as well as other similar kinds of examples.

Using the package `SpectralSequences`, we can compute the map $d_{2, -1}^2$ as follows:

```
11 : needsPackage "SpectralSequences"; A = QQ[x,y,z,w];
13 : F2D = simplicialComplex {x*y*z, w*z}; F1D = simplicialComplex {x*y, w}; F0D = simplicialComplex {x,w};
16 : K = filteredComplex{F2D, F1D, F0D}; E = prune spectralSequence K;
18 : E^2 .dd_{2,-1}
o8 = | -1 |
 |   1
o8 : Matrix QQ <--- QQ
```

In the sections that follow, we briefly describe the structure of our package; we also illustrate it with two examples. We remark that the package documentation contains many more examples—examples which illustrate how the package can be used to compute, among other things, spectral sequences arising from filtrations of simplicial complexes, triangulations of Hopf fibrations, non-Koszul syzygies and change of rings maps.

## 2. The structure of the package `SpectralSequences`

The package `SpectralSequences` is able, at least in principle, to compute all aspects of the spectral sequence obtained from a bounded filtered chain complex of finitely generated modules over a finite type $k$-algebra. The actual implementation is achieved by first defining a number of auxiliary data structures combined with constructors and other methods to work with these data types.

To use the package `SpectralSequences`, the user must first create a filtered chain complex. Such filtered chain complexes are represented by the data type `FilteredComplexes`. The most basic constructor for this type has as input a collection of chain complex maps, whose images determine the given filtered chain complex. We also provide other methods for creating filtered complexes. For instance, the natural filtration induced by truncation can be inputted by the command `filteredComplex`. For those familiar with the filtered complexes coming from a double complex, we have implemented these in the cases of $\text{Hom}(C_\bullet, C'_\bullet)$ and $C_\bullet \otimes C'_\bullet$. Finally, given such a filtered chain complex, represented as an instance of the type `FilteredComplexes`,
one uses a constructor associated to the type `SpectralSequence` to create the spectral sequence determined by the given filtered chain complex.

In fact, upon initializing a new spectral sequence, using the type `SpectralSequence`, no calculations are actually performed by the computer. Rather, calculations are performed using a sort of “lazy evaluation”. There are a number of methods associated to the type `SpectralSequence`. In brief, for each of the aspects of the spectral sequence, there exists a method which takes, as input, the spectral sequence represented as an instance of the type `SpectralSequence`. The output of such methods are either modules, or maps between modules, depending on what is asked by the user.

Finally, the package `SpectralSequences` contains methods which take as input a given spectral sequence represented as type `SpectralSequence` together with a non-negative integer and has as output the modules of the resulting spectral sequence page or the maps between such modules, depending on what is desired by the user. Such outputs are represented by respective data types `SpectralSequencePage` and `SpectralSequencePageMap`.

3. Spectral sequences and hypercohomology calculations

If \( F \) is a coherent sheaf on a smooth complete toric variety \( X \), then multigraded commutative algebra can be used to compute the cohomology groups \( H^i(X, F) \). Indeed, if \( B \subseteq R \) is the irrelevant ideal of \( X \), then the cohomology group \( H^i(X, F) \) can be realized as the degree zero piece of the multigraded module \( \text{Ext}^i_R(B[\ell], F) \) for sufficiently large \( \ell \); here \( B[\ell] \) denotes the \( \ell \)th Frobenius power of \( B \) and \( F \) is any multigraded module whose corresponding sheaf on \( X \) is \( F \). Given the fan of \( X \) and \( F \), a sufficiently large power of \( \ell \) can be determined effectively. We refer to sections 2 and 3 of [2] for more details.

Here we consider the case that \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( F = \mathcal{O}_C(1, 0) \), where \( C \) is a general divisor of type (3, 3) on \( X \). In this setting, \( H^0(C, F) \) and \( H^1(C, F) \) are both 2-dimensional vector spaces. We can compute these cohomology groups using a spectral sequence associated to a Hom complex.

We first make the multigraded coordinate ring of \( \mathbb{P}^1 \times \mathbb{P}^1 \), the irrelevant ideal, and a sufficiently high Frobenius power of the irrelevant ideal needed for our calculations. Also the complex \( G \) below is a resolution of the irrelevant ideal.

```plaintext
i1 : needsPackage"SpectralSequences";

i2 : R = ZZ/101[a_0..b_1, Degrees=>{2:{1,0},2:{0,1}}]; -- PP^1 x PP^1

i3 : B = intersect(ideal(a_0,a_1),ideal(b_0,b_1)); -- irrelevant ideal

i4 : B = B_*/(x -> x^5)//ideal; -- Sufficiently high Frobenius power

i5 : G = res image gens B;

We next make the ideal, denoted by \( I \) below, of a general divisor of type (3, 3) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Also the chain complex \( F \) below is a resolution of this ideal.

```
To use hypercohomology to compute the cohomology groups of the line bundle $\mathcal{O}_C(1,0)$ on $C$ we twist the complex $F$ above by a line of ruling and then make a filtered complex whose associated spectral sequence abuts to the desired cohomology groups. This is the complex $K$ below.

$$K = \text{Hom}(G, \text{filteredComplex}(F \star R^{-\{1,0\}}))$$

$$E = \text{prune spectralSequence}(K)$$

The cohomology groups we want are obtained as follows.

$$\text{basis}(\{0,0\}, E^{-2}_{0,0}) \quad == \quad \text{HH}^0 \mathcal{O}_C(1,0)$$

$$\text{basis}(\{0,0\}, E^{-2}_{1,-2}) \quad == \quad \text{HH}^1 \mathcal{O}_C(1,0)$$
We now consider an example which is similar to the ones contained in the documentation node **Seeing Cancellations**. Let $S := k[x_1, \ldots, x_n]$, $I := \langle x_1, \ldots, x_n \rangle^2$ and put $R := S/I$. By abuse of notation, we write $x_i \in R$ also for the residue class of $x_i \in S$ modulo $I$. If $K_\bullet$ is the Koszul complex over $R$ on the sequence $(x_1, \ldots, x_n)$ and $F_\bullet$ is a free resolution of $k$ over $R$ then we can form the tensor product complex $D_\bullet = F_\bullet \otimes K_\bullet$. We now describe these two filtrations and their associated spectral sequences.

\begin{figure}[h]
\centering
\begin{align*}
\text{(A) A double complex} & \quad \text{(B) } E^1 \text{ has degenerated}
\end{align*}
\end{figure}

The filtrations $'F_\bullet(D_\bullet)$ and $''F_\bullet(D_\bullet)$: The vertical columns in Figure 1A determine a filtration, whose corresponding spectral sequence $'E = E('F_\bullet(D_\bullet))$ has zeroth page given by $'E^0_{p,q} = F_p \otimes K_q$ and the maps $'d^0_{p,q} : F_p \otimes K_q \to F_p \otimes K_{q-1}$ are induced by those of $K_\bullet$. Analogously, one can consider the filtration $''F_\bullet(D_\bullet)$ determined by the horizontal maps and its spectral sequence $''E$. 

\*E degenerates at the first page.\* Since \( F_\bullet \) is a resolution, the spectral sequence associated to the horizontal maps degenerates at the first page (see Figure 1). We have

\[
\\*E_\infty = \*E_1^{p,q} = \begin{cases} k(-q)^n & \text{if } p = 0 \\ 0, & \text{otherwise} \end{cases}.
\]

The second spectral sequence results in an \( E_1 \) page which will be the Koszul homology of \( R \) tensored with \( F_\bullet \). The modules are given by: \( \*E_1^{p,q} = F_p \otimes H^K_q(R) \) where \( H^K_q(R) \cong \text{Tor}_q^S(R,k) = \bigoplus_j k(-j)^{\beta_{ij}(R)} \). Since the \( E_\infty \) page consists only of \( k(-q) \)'s, eventually the \( q \)th diagonal of \( \*E_i \) must reduce to only terms generated in degree \( q \).

We can see this in our package \texttt{SpectralSequences} in any particular example. For instance, if \( S = k[x,y] \) and \( I = (x^2, xy, y^2) \), then \( \*E_1^{p,q} \) is the following complex of \( k \)-vector spaces.

\begin{align*}
& k \otimes k^2(-3) \longrightarrow k^2(-1) \otimes k^2(-3) \longrightarrow k^4(-2) \otimes k^2(-3) \longrightarrow k^8(-3) \otimes k^2(-3) \longrightarrow \\
& k \otimes k^3(-2) \longrightarrow k^2(-1) \otimes k^3(-2) \longrightarrow k^4(-2) \otimes k^3(-2) \longrightarrow k^8(-3) \otimes k^3(-2) \longrightarrow \\
& k \otimes k \longrightarrow k^2(-1) \otimes k \longrightarrow k^4(-2) \otimes k \longrightarrow k^8(-3) \otimes k \longrightarrow \\
\end{align*}

(A) \( \*E_1 \)

\begin{align*}
& k^3(-3) \longrightarrow k^4(-4) \longrightarrow k^5(-5) \longrightarrow k^{16}(-6) \longrightarrow \\
& k^2(-2) \longrightarrow k^2(-3) \longrightarrow k^4(-4) \longrightarrow k^{16}(-5) \longrightarrow \\
& k \longrightarrow k^2(-1) \longrightarrow k(-2) \otimes k(-2) \longrightarrow k^8(-3) \longrightarrow \\
\end{align*}

(B) \( \*E^2 \)

We have separated out the \( E_\infty \) terms in blue. Notice that all the remaining (black) terms must disappear at some later page. In this case, the maps from the \( \*E_{3,0} \) position are both surjective, which we illustrate below:

\begin{align*}
& i9 : S = \mathbb{Z}/101[x,y]; \\
& i10 : I = \text{ideal}(x^2,x*y,y^2); \hspace{1cm} R = S/I; \\
& i11 : kR = \text{coker vars R}; \hspace{1cm} kS = \text{coker vars S}; \\
& i12 : K = (\text{res kS})**R; \hspace{1cm} F = \text{res(kR,LengthLimit=>6)}; \\
& i13 : E = \text{prune spectralSequence} (K ** \text{filteredComplex} F); \\
& i14 : \text{length image} ((E_2)_{dd\{3,0\}}) \\
& o14 : 6 \\
& i15 : \text{length image} (E_3)_{dd\{3,0\}} \\
& o15 : 2
\end{align*}
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