Groups and nonlinear dynamical systems

Chaotic dynamics on the $SU(2) \times SU(2)$ group

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Abstract: In our previous paper: K. Kowalski and J. Rembieliński, Groups and nonlinear dynamical systems. Dynamics on the $SU(2)$ group, Physica D 99, 237 (1996), we introduced an abstract Newton-like equation on a general Lie algebra such that submanifolds fixed by the second-order Casimir operator are attracting set. The corresponding group parameters satisfy the nonlinear dynamical system having an attractor coinciding with the submanifold. In this work we discuss the case with the $SU(2) \times SU(2)$ group. The resulting second-order system in $\mathbb{R}^6$ is demonstrated to exhibit chaotic behaviour.

Keywords: dynamical systems, ordinary differential systems, Lie groups, deterministic chaos, symmetric attractors

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One of the most important problems of the theory of dynamical systems relates to finding and classifying attracting limit sets or simply attractors. Another important problem is how to study the system on its invariant manifold. In our previous work [1] we introduced a method for the construction of $G$-invariant nonlinear dynamical systems having an attracting set coinciding with a submanifold generated by the second-order Casimir operator referring to a given Lie group $G$. The example of the $SU(2)$ group discussed therein led to the oscillatory dynamics on the orbit i.e. the sphere $S_2$. As suggested in [1] such regular dynamics need not be the case for groups with higher-dimensional submanifolds. In this paper we examine the case with the $SU(2) \times SU(2)$ group. Namely, following the general scheme described in [1] we introduce the nonlinear second-order system in $\mathbb{R}^6$ satisfied by the group parameters. We then show that the system exhibits chaotic behaviour on the sphere $S_5$ which is a submanifold fixed by the second-order Casimir operator corresponding to the $SU(2) \times SU(2)$ group.

2. THE NEWTON-LIKE EQUATION

In this section we recall the abstract Newton-like equation on a general Lie algebra such that submanifolds fixed by the second-order Casimir operator are attracting set [1]. This equation generates the nonlinear dynamical system satisfied by the group parameters, having an attractor coinciding with the submanifold. Consider the following second-order differential equation on a Lie algebra $\mathfrak{g}$:

$$\mu \ddot{X} + \nu \dot{X} + \rho X + \sigma Y = e^{iX}Ye^{-iX}, \quad X(0) = X_0, \quad \dot{X}(0) = \dot{X}_0, \quad (2.1)$$

where $X(t): \mathbb{R} \to \mathfrak{g}$ is a curve in $\mathfrak{g}$, $Y \in \mathfrak{g}$ is a fixed element, $\mu, \nu, \rho, \sigma \in \mathbb{R}$ and dot designates differentiation with respect to time.

Let us assume that $\mu \neq 0$. Clearly, we can set without loose of generality $\mu > 0$. On demanding that (2.1) admits the solution on the submanifold fixed by the second-order Casimir operator of the form

$$\text{Tr}X^2 = \text{const} \neq 0, \quad (2.2)$$

rescaling $t \to \sqrt{\mu} t$ and setting $\frac{\nu}{\sqrt{\mu}} = \beta$ we arrive at the following Newton-like equation:

$$\ddot{X} + \beta \dot{X} + \frac{\text{Tr}X^2}{\text{Tr}X^2}X = e^{iX}Ye^{-iX} - Y, \quad X(0) = X_0, \quad \dot{X}(0) = \dot{X}_0. \quad (2.3)$$
Using (2.3) we find
\[ \text{Tr} \nabla^2 = 2\text{Tr}_0 \dot{X}_0 \beta (1 - e^{-\beta t}) + \text{Tr}_0 \nabla^2. \] (2.4)

That is if $\beta > 0$ then the solution to (2.1) approaches the submanifold
\[ \text{Tr} X^2 = \frac{2}{\beta} \text{Tr}_0 \dot{X}_0 + \text{Tr}_0 X_0^2. \] (2.5)

It can be easily checked that for $\beta \leq 0$ there is no solution on the submanifold. The only exception are the initial data such that
\[ \text{Tr}_0 \dot{X}_0 = 0, \] (2.6)

The more detailed analysis of eq. (2.3) and (2.4) is provided in [1]. Notice that in the general case of an $n$-dimensional compact Lie algebra the manifold defined by (2.2) is simply the sphere $S_{n-1}$.

### 3. DYNAMICS ON THE $SU(2) \times SU(2)$ GROUP

Our aim now is to discuss the nonlinear dynamical system implied by the abstract Newton-like equation in the case of the $SU(2) \times SU(2)$ group. We first observe that due to the product structure of the considered group, the general elements of the Lie algebra $X(t)$ and $Y$ can be written in the form
\[ X(t) = X_+(t) + X_-(t), \]
\[ Y = Y_+ + Y_-, \] (3.1a)

where
\[ [X_+, X_-] = 0, \quad [X_+, Y_-] = [X_-, Y_+] = 0, \]
\[ \text{Tr} X_+ X_- = \text{Tr} \dot{X}_+ \dot{X}_- = 0. \] (3.1b)

Notice that in view of (3.1) the manifolds (2.2) take the form
\[ \text{Tr}(X_+^2 + X_-^2) = \text{const.} \] (3.2)

We now return to (2.3). From (2.3) and (3.1) we get
\[ \dot{X}_+ + \beta \dot{X}_+ + \frac{\text{Tr}(\dot{X}_+^2 + \dot{X}_-^2)}{\text{Tr}(X_+^2 + X_-^2)} X_+ = e^{ix_+} Y_+ e^{-ix_+} - Y_+, \]
\[ \dot{X}_- + \beta \dot{X}_- + \frac{\text{Tr}(\dot{X}_+^2 + \dot{X}_-^2)}{\text{Tr}(X_+^2 + X_-^2)} X_- = e^{ix_-} Y_- e^{-ix_-} - Y_-, \] (3.3)
\[ X_+(0) = X_{+0}, \quad X_-(0) = X_{-0}, \quad \dot{X}_+(0) = \dot{X}_{+0}, \quad \dot{X}_-(0) = \dot{X}_{-0}. \]
Thus whenever the initial data satisfy the inequality

$$\frac{1}{2}\sigma_1 \geq 0, \quad \frac{1}{2}\sigma_2 \leq 0, \quad \frac{1}{2}\sigma_3 \leq 0,$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices. Clearly, we can write the general elements of the Lie algebra $X_{\pm}(t)$ and $Y_{\pm}$ of the Lie algebra as

$$X_{\pm}(t) = x_{\pm}(t) \cdot J_{\pm}, \quad Y_{\pm} = a_{\pm} \cdot J_{\pm}, \quad a_{\pm} = \frac{1}{2}(1 \pm e^{i\frac{1}{2}\sigma}).$$

(3.4)

where $x_{\pm}(t) : \mathbb{R} \to \mathbb{R}^3$, $a_{\pm}$ is a constant vector of $\mathbb{R}^3$ and the dot designates the inner product. Substituting (3.5) into (3.3) we obtain the following nonlinear system of second-order equations:

$$\begin{align*}
\ddot{x}_+ + \beta \dot{x}_+ + \frac{x^2_+}{x^2_+ + x^2_-} x_+ = (\cos |x_+| - 1) a_+ + \frac{|x_+|}{x_+} \sin |x_+| a_+ \times x_+ + (1 - \cos |x_+|) \frac{(a_+ \cdot x_+) x_+}{x^2_+}, \\
\ddot{x}_- + \beta \dot{x}_- + \frac{x^2_-}{x^2_+ + x^2_-} x_- = (\cos |x_-| - 1) a_- + \frac{|x_-|}{x_-} \sin |x_-| a_- \times x_- + (1 - \cos |x_-|) \frac{(a_- \cdot x_-) x_-}{x^2_-}, \\
x_+(0) = x_{+0}, \quad x_-(0) = x_{-0}, \quad \dot{x}_+(0) = \dot{x}_{+0}, \quad \dot{x}_-(0) = \dot{x}_{-0},
\end{align*}$$

(3.6)

where $a_{\pm} \times x_{\pm}$ designates the vector product of vectors $a_{\pm}$ and $x_{\pm}$, and $|x_{\pm}| = \sqrt{x^2_{\pm}}$ is the norm of the vector $x_{\pm}$.

Notice that the manifolds given by (3.2) and (3.5) are the five-dimensional spheres

$$x^2_+ + x^2_- = \text{const.}$$

(3.7)

We now examine the asymptotic behaviour of the system (3.6). First we observe that (3.6) implies

$$\frac{\partial}{\partial x_+} \dot{x}_+ + \frac{\partial}{\partial x_-} \dot{x}_- = -6\beta - 2 \frac{x_+ \cdot \dot{x}_+ + x_- \cdot \dot{x}_-}{x^2_+ + x^2_-}.$$  

(3.8)

Thus as the solution to (3.6) when $\beta > 0$ approaches the sphere (3.7), the system (3.6) becomes dissipative one with exponential contraction of a volume element. Furthermore, it can be easily checked that the $SU(2) \times SU(2)$ realization of (2.4) takes the form

$$x^2_+ + x^2_- = 2(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0}) \frac{1}{\beta}(1 - e^{-\beta t}) + x^2_{+0} + x^2_{-0}.$$  

(3.9)

Thus whenever the initial data satisfy the inequality

$$\frac{2}{\beta}(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0}) + x^2_{+0} + x^2_{-0} > 0,$$

(3.10)
where a restricted to the interval \([0, \alpha]\) and the transformation

\[
\frac{2}{\beta}(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0}) + x_{+0}^2 + x_{-0}^2 = 0, \quad x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0} \neq 0, \tag{3.13}
\]

then the solutions to (3.6) tend asymptotically to the singular point \(x_+ = 0, x_- = 0\). Furthermore, if \(\beta > 0\) and

\[
\frac{2}{\beta}(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0}) + x_{+0}^2 + x_{-0}^2 < 0, \tag{3.14}
\]

then the singular point is approached after a finite period of time

\[
t_* = -\frac{1}{\beta} \ln \left(1 + \frac{\beta(x_{+0}^2 + x_{-0}^2)}{2(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0})}\right). \tag{3.15}
\]

Finally, for \(\beta \leq 0\) and \(x_{+0} \cdot \dot{x}_{+0} + x_{-0} \cdot \dot{x}_{-0} \neq 0\), the trajectories go to infinity.

We now return to (3.6). Notice that in view of the covariance of (3.3) with respect to the group transformations we can set in (3.6) without loose of generality

\[
a_+ = (0, 0, a \cos \alpha), \quad a_+ = (0, 0, a \sin \alpha). \tag{3.16}
\]

Thus we finally arrive at the system

\[
\begin{align*}
\ddot{x_{\pm1}} + \beta \dot{x_{\pm1}} + \frac{\dot{x}_{\pm1}^2 + \dot{x}_{\pm2}^2}{x_{\pm1}^2 + x_{\pm2}^2} x_{\pm1} &= -\frac{\sin |x_{\pm1}|}{x_{\pm1}} a_{\pm3} x_{\pm2} + (1 - \cos |x_{\pm1}|) \frac{a_{\pm3} x_{\pm1} x_{\pm3}}{x_{\pm2}^2}, \\
\ddot{x_{\pm2}} + \beta \dot{x_{\pm2}} + \frac{\dot{x}_{\pm2}^2 + \dot{x}_{\pm3}^2}{x_{\pm2}^2 + x_{\pm3}^2} x_{\pm2} &= \frac{\sin |x_{\pm2}|}{x_{\pm2}} a_{\pm3} x_{\pm1} + (1 - \cos |x_{\pm2}|) \frac{a_{\pm3} x_{\pm2} x_{\pm3}}{x_{\pm1}^2}, \\
\ddot{x_{\pm3}} + \beta \dot{x_{\pm3}} + \frac{\dot{x}_{\pm3}^2 + \dot{x}_{\pm1}^2}{x_{\pm3}^2 + x_{\pm1}^2} x_{\pm3} &= (\cos |x_{\pm1}| - 1) a_{\pm3} + (1 - \cos |x_{\pm3}|) \frac{a_{\pm3} x_{\pm3} x_{\pm1}}{x_{\pm2}^2}, \\
x_{\pm}(0) &= x_{\pm0}, \quad x_{\pm}(0) = \dot{x}_{\pm0}, \tag{3.17}
\end{align*}
\]

where \(a_{\pm3} = a \cos \alpha\) and \(a_{-3} = a \sin \alpha\). Notice that the parameter \(\alpha\) can be restricted to the interval \([0, \pi/2]\). Indeed, taking into account (3.16) we find that the transformation \(\alpha \rightarrow \alpha + \pi/2\) leads to

\[
a_{\pm3} \rightarrow -a_{-3}, \quad a_{-3} \rightarrow a_{\pm3}, \tag{3.18}
\]

or in view of (3.17)

\[
x_- \rightarrow x_+, \quad x_{+1} \rightarrow -x_{-1}, \quad x_{+2} \rightarrow x_{-2}, \quad x_{+3} \rightarrow -x_{-3}. \tag{3.19}
\]
Further, we observe that for $\alpha = \pi/4$ the system (3.17) is symmetric in $x^+\pm$ and $x^-\pm$ variables. For an easy illustration of this observation see Fig. 2. It should also be noted that by virtue of the following transformation law of the equation (2.3) under the scaling $t \rightarrow \lambda t$:

$$\ddot{X} + \lambda \beta \dot{X} + \frac{\text{Tr} \dot{X}^2}{\text{Tr} X^2} X = e^{iX} \lambda^2 Y e^{-iX} - \lambda^2 Y,$$  

(3.20)

we have actually two bifurcation parameters: $\beta^2/a$, where $a \neq 0$, and $\alpha$.

In order to discuss the symmetries of (3.17) consider the original abstract equation (2.3). Evidently, (2.3) is invariant under the transformations referring to the stability group of $Y$, that is transformations leaving $Y$ unchanged. Hence, making use of (3.3) and (3.5) we find that solutions to (3.17) have the form invariant under rotations about $x^+3$ and $x^-3$ axes such that

$$x'_\pm = \cos \varphi \pm x_{\pm1} + \sin \varphi \pm x_{\pm2},$$
$$x'_\pm = -\sin \varphi \pm x_{\pm1} + \cos \varphi \pm x_{\pm2},$$
$$x'_\pm = x_{\pm3}, \quad \varphi \pm \in [0, 2\pi).$$  

(3.21)

This suggests that whenever the initial condition $x^+_0, x^-_0, \dot{x}^+_0$ and $\dot{x}^-_0$ corresponds to an attractor $A$ then the basin of attraction of $A$ contains the points $x'_+0, x'_-0, \dot{x}'_+0$ and $\dot{x}'_-0$ of the form

$$x'_{\pm10} = \cos \varphi \pm x_{\pm10} + \sin \varphi \pm x_{\pm20},$$
$$x'_{\pm20} = -\sin \varphi \pm x_{\pm10} + \cos \varphi \pm x_{\pm20},$$
$$x'_{\pm30} = x_{\pm30},$$
$$\dot{x}'_{\pm10} = \cos \varphi \pm \dot{x}_{\pm10} + \sin \varphi \pm \dot{x}_{\pm20},$$
$$\dot{x}'_{\pm20} = -\sin \varphi \pm \dot{x}_{\pm10} + \cos \varphi \pm \dot{x}_{\pm20},$$
$$\dot{x}'_{\pm30} = \dot{x}_{\pm30}, \quad \varphi \pm \in [0, 2\pi).$$  

(3.22a)

Thus the basin of attraction contains two circles given by (3.22a) such that

$$\mathbf{x}'^2_+0 = \mathbf{x}'^2_-0, \quad \mathbf{x}'^2_-0 = \mathbf{x}'^2_+0,$$  

(3.23)

and two vector fields (3.22b) obtained from $\dot{x}_\pm0$ by rotating this about $x^+3$ and $x^-3$ axis, respectively, by the angle referring to the position of the corresponding point of the circle.

In figures 1 and 2 we show examples of strange attractors from numerical integration of the system (3.17). As expected these attractors are symmetric under rotations about $x^+3$ and $x^-3$ axes.
It follows from the computer simulations illustrated in Fig. 3 that in the parameter space of the system (3.17) in a neighbourhood of the attractor from Fig. 1 there exists a quasiperiodic trajectory.

A look at Fig. 3 is enough to conclude that in the case with the attractor from Fig. 1 we deal with the quasiperiodicity to chaos transition like in the Ruelle-Takens-Newhouse scenario [2,3]. In the case of the attractor from Fig. 2 we have most probably the new scenario.

Namely, it turns out that there exist in the parameter space of the system (3.17) two nearby quasiperiodic orbits shown in Fig. 4. The computer simulations suggest that these quasiperiodic orbits are separated by an infinitesimal perturbation of the bifurcation parameter $\beta$. As a consequence of the infinitesimal nature of the perturbation the irregular transitions occur between the chaotic attractors arising from the perturbation of the quasiperiodic orbits from Fig. 4a and Fig. 4b, respectively. These attractors are shown separately in Fig. 6. We remark that the attractors have the same form as those shown in Fig. 5 arising in the transient chaos before reaching the quasiperiodic state. The authors did not find such scenario of transitions between chaotic attractors in the literature. As the bifurcation parameter decreases and approaches the value corresponding to the attractor from Fig. 2 the frequency of transitions between the two attractors increases. On the other hand, the decay of the bifurcation parameter leads to the deformation of the attractors. As a result of these two combined processes we arrive at the attractor illustrated in Fig. 2. In this sense that attractor arise from the perturbation of two quasiperiodic states.

4. CONCLUSION

In the present work we have studied the particular $SU(2) \times SU(2)$ realization of the abstract Newton-like equation having attractors coinciding with the submanifolds fixed by the second-order Casimir operator referring to a given Lie group. The resulting second-order system in $\mathbb{R}^6$ has been shown to exhibit chaotic behaviour. The concrete examples of the chaotic attractors with the $SU(2) \times SU(2)$ symmetry have been provided. One of them (see Fig. 2 and discussion below) is related to the most probably new scenario of transitions between chaotic attractors and effectively forming one by perturbation of two quasiperiodic states. We point out that the case with the $SU(2) \times SU(2)$ real-
The investigation of the abstract Newton-like equation discussed herein seems to be the simplest one leading to chaotic dynamics. As far as we are aware, the investigated system (3.17) is the first example of the chaotic dynamical system on the manifold determined by the continuous Lie-group structure. We remark that the importance of such an example was indicated in [4–6]. Furthermore, the knowledge of chaotic attractors with fixed symmetry and known the invariant measure like those introduced in this work would be useful for testing methods of detecting symmetry of chaotic attractors such as for example the method of detectives [7]. Whenever the theory of groups appears to lead to some insight into the nature of chaos then it seems that the observations introduced herein would be an important point of departure in solving numerous problems.

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Fig. 1. The system (3.17) with $\beta = 0.1$, $a = 0.5$, $\alpha = 0.2$, $x_{i0} = (1, 1, 1.5)$, $x_{-0} = (1, 1, 1.5)$, $\dot{x}_{i0} = (0, 0, 0, 0, 0, 0)$ and $x_{-0} = (0, 1, 0, 1, 0, 1)$. Left: the projection of the Poincaré section of the attractor on the $(x_{i1}, x_{i3})$ plane. This section and the following ones in the case with the projection on the $(x_{-i1}, x_{-i3})$ plane are defined by the hyperplane $n(x - x(0)) = 0$, where $n$ is the normal vector and $x(0)$ is a point of the hyperplane, $n$, $x(0)$, $x \in \mathbb{R}^{12}$, $n = (0, 0, 0, \ldots, 0)$ and $x(0) = 0$. Right: the projection of the Poincaré section of the attractor on the $(x_{-1}, x_{-3})$ plane. This section and the following ones in the case with the projection on the $(x_{-1}, x_{-3})$ plane are defined by the hyperplane $n(x - x(0)) = 0$, where $n = (0, 0, 0, 1, 0, \ldots, 0)$ and $x(0) = 0$. The Lyapunov exponents are: $\lambda_1 = 0.02$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \pm 0.00$, $\lambda_7 = -0.01$, $\lambda_8 = -0.06$, $\lambda_9 = -0.08$, $\lambda_{10} = -0.09$, $\lambda_{11} = -0.10$, $\lambda_{12} = -0.12$ and $\lambda_{13} = -0.16$. The Lyapunov dimension is 6.1.

Fig. 2. The system (3.17) with $\beta = 0.1$, $a = 1$, $\alpha = \pi/4$, $x_{i0} = (2, 2, 2)$, $x_{-0} = (2, 2, 2)$, $\dot{x}_{i0} = (0.1, 0.1, -1)$ and $x_{-0} = (1, 1, 1)$. Left: the projection of the Poincaré section of the attractor on the $(x_{i1}, x_{i3})$ plane. Right: the projection of the Poincaré section of the attractor on the $(x_{-1}, x_{-3})$ plane. The Lyapunov exponents are: $\lambda_1 = 0.06$, $\lambda_2 = 0.01$, $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \pm 0.00$, $\lambda_7 = -0.04$, $\lambda_8 = -0.07$, $\lambda_9 = -0.10$, $\lambda_{10} = -0.11$, $\lambda_{11} = -0.14$ and $\lambda_{12} = -0.21$. The Lyapunov dimension is 7.4.

Fig. 3. The system (3.17) with $\beta = 0.117$. The remaining parameters and the initial condition are the same as in Fig. 1. Left: the projection of the Poincaré section of the quasiperiodic attractor on the $(x_{i1}, x_{i3})$ plane. Right: the projection of the Poincaré section of the quasiperiodic attractor on the $(x_{-1}, x_{-3})$ plane.

Fig. 4. The system (3.17) with a) $\beta = 0.1102766$ and b) $\beta = 0.1102767$. The remaining parameters and the initial condition are the same as in Fig. 2. The projection of the Poincaré section of the quasiperiodic attractors on the $(x_{i1}, x_{i3})$ plane.

Fig. 5. The transient chaos in the system (3.17) with a) $\beta = 0.1102766$ and b) $\beta = 0.1102767$ before reaching the quasiperiodic states from Fig. 4. The remaining parameters and the initial condition are the same as in Fig. 2. The projection of the Poincaré section of the attractors on the $(x_{i1}, x_{i3})$ plane.

Fig. 6. The system (3.17) with $\beta = 0.1099$. The remaining parameters and the initial condition are the same as in Fig. 2. For such data the irregular transitions appear between the attractor from the left and the attractor from the right. In both figures the projection is shown of the Poincaré section of
the attractor on the \((x+1, x+3)\) plane.