FREE FLAGS OVER LOCAL RINGS AND POWERING OF HIGH DIMENSIONAL EXPANDERS

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Abstract. Powering the adjacency matrix of an expander graph results in a better expander of higher degree. In this paper we seek an analogue operation for high-dimensional expanders. We show that the naive approach to powering does not preserve high-dimensional expansion, and define a new power operation, using geodesic walks on quotients of Bruhat-Tits buildings. Applying this operation results in high-dimensional expanders of higher degrees. The crux of the proof is a combinatorial study of flags of free modules over finite local rings. Their geometry describes links in the power complex, and showing that they are excellent expanders implies high dimensional expansion for the power-complex by Garland’s local-to-global technique. As an application, we use our power operation to obtain new efficient double samplers.

1. Introduction

A $k$-regular graph is called an expander if the nontrivial eigenvalues of its adjacency matrix are of small magnitude in comparison with $k$ (the trivial eigenvalues are, by definition, $\pm k$). The theory of expanders has long been a fruitful meeting point for combinatorics, algebra, number theory and computer science, see e.g. the surveys [22,34]. In recent years a theory of high-dimensional (HD) expanders has emerged, and is already seeing applications in mathematics and computer science, e.g. in PCPs [8], property testing [25], expansion in finite groups [3], quantum computation [13], counting problems in matroids [2], list decoding [7] and lattices [28]; we refer the reader to [36] for a recent survey.

A major obstacle in the study of HD expanders is that the means available for constructing simplicial complexes are much scarcer in dimension greater than one; for example, there are many well understood models of random graphs, whereas the theory of random complexes is still in its infancy. In this paper we focus on the operation of powering: given a graph $\mathcal{G}$ with adjacency matrix $A = A_{\mathcal{G}}$, one can regard the matrix $A^r$ as the adjacency matrix of a “power graph” $\mathcal{G}^r$. If $\mathcal{G}$ is $k$-regular with second largest eigenvalue (in absolute value) $\lambda$, then $\mathcal{G}^r$ is a $k^r$-regular graph with second largest eigenvalue $\lambda^r$. On average, distances between vertices become shorter in the power graph, and if the original graph was an expander, the power graph is a better one (namely, the ratio between first and second eigenvalues improves). Powering of graphs has been used, for example, in the proof of the PCP Theorem by Dinur [6] and in the Zig-Zag construction [47], and various researchers have independently raised the question whether there exists a power operation for high dimensional expanders.

Unlike the case of graphs, there are many non-equivalent definitions for HD expansion (e.g., [9,11,14-16,20,21,27,32,43-46,49]), reflecting the richness of high-dimensional combinatorics. A precursor of the modern theory of HD expanders is Garland’s seminal paper [17], which introduces a local-to-global approach: it shows that a $d$-dimensional simplicial complex, whose one-dimensional links are (very good) expander graphs, is cohomologically connected in all dimensions between zero and $d$. The complexes Garland was interested in are quotients of Bruhat-Tits buildings (see §2.2). The links of these complexes are flag complexes over finite fields, which are well known to be excellent expanders. Currently, we know that expansion in links implies several global expansion properties [4,21,30,43], and in this paper we define a HD expander as a complex whose links are good expanders (see Definition 2.1).

It turns out that the most natural approach to powering (which is described in §1.1) does not preserves high dimensional expansion, so that a more sophisticated one has to be taken. We devise
a powering method based on the notion of geodesic paths from [33]; as an example, if $X$ is a two-dimensional complex, its \textit{geodesic $r$-power} is the clique complex with the same vertices as $X$, and an edge between every two vertices which were connected by a monochromatic geodesic path of length $r$ in $X$. The rigorous definition appears in §2.3, after we recall the definitions of simplicial complexes (§2.1) and buildings (§2.2). An explicit example of a geodesic $r$-power Cayley complex appears in §2.4.

The central part of the paper is §3.1, which shows that when applied to $\tilde{A}$-complexes, namely, quotients of affine Bruhat-Tits buildings of type $\tilde{A}$, our geodesic power operation yields HD expanders. Let us explain where this expansion comes from: Each link in an $\tilde{A}$-complex is a spherical building over a finite field, and its cells correspond to flags in a finite vector space. The one-dimensional links are either complete bipartite graphs, or the projective plane $\mathbb{P}^2\mathbb{F}_q$, which is the incidence graph of lines and planes in $\mathbb{P}^3\mathbb{F}_q$. It is a classic exercise that the adjacency spectrum of $\mathbb{P}^2\mathbb{F}_q$ is $\{\pm(q+1), \pm\sqrt{q}\}$, making it an excellent expander, as $\frac{\sqrt{q}}{q+1}$ can be made arbitrarily small. This observation is the basis for Garland’s work, and our main task is to conduct a parallel study for links in our power-complex. In Proposition 3.6 we show that these links correspond to flags of free submodules of a fixed free module over a finite local ring $R$, such as $R = \mathbb{Z}/p\mathbb{Z}$. We call the complex which arises in this manner the \textit{free projective space over $R$} (see Definition 3.1), and we point out that this object may be of independent interest, outside the realm of buildings and HD expanders. Our main achievement is a complete analysis of the spectrum of one-dimensional links:

\textbf{Theorem} (Main theorem). For $d \geq 2$, the links of $(d-2)$-cells in the geodesic $r$-power of an $\tilde{A}$-complex of dimension $d$ are either complete bipartite graphs, or expanders with spectrum

$$\{\pm (q+1) q^{r-1}, \pm \sqrt{q^{2r-1}}, \pm \sqrt{q^{2r-2}}, \ldots, \pm \sqrt{q^2}, \pm \sqrt{q}\}.$$ 

Consequently, geodesic powers of $\tilde{A}$-complexes are high-dimensional expanders.

The computation of the spectrum of these links is carried out in Theorem 3.7, whose proof is long and technically challenging, in comparison with the elegance of the final result. This, together with the fact that they arise in a natural algebraic settings, suggest that a broader geometric theory of free flags over finite rings could perhaps be developed to give a more conceptual proof. Another interesting corollary of Theorem 3.7 is an isospectrality result (Corollary 3.8) for free projective planes over local rings with the same residue order (or equivalently, of links of geodesic powers of $\tilde{A}$-complexes of the same densities).

Having established local HD expansion in §3.1, we turn in §3.2 to demonstrating expansion between vertices in the power-complex (Theorem 3.9), and between vertices and geodesics (Proposition 3.10) – these are required for the applications which we present in §5. We stress that while the notion of geodesic walks comes from [33], the expansion types studied here and there are different, and no result from that paper is used in this one.

A special family of $\tilde{A}$-complexes which appears in the second half of the paper are \textit{Ramanujan complexes}: Ramanujan graphs, which were defined in [38], are $k$-regular graphs whose nontrivial eigenvalues belong to the $L^2$-spectrum of the $k$-regular tree. As regular trees are one dimensional buildings, Ramanujan complexes were defined in [31, 39] to be $\tilde{A}$-complexes whose spectral theory mimics that of the $\tilde{A}$-building (see Definition in §2.2). It turns out, however, that in dimension two and above all quotients of buildings have some expansion properties (by Garland or by Kazhdan’s property (T)); while in contrast, every regular graph is a quotient of a one dimensional building. Inspection reveals that many results on Ramanujan complexes actually apply to general quotients of HD buildings, e.g. [10, 12, 15, 19, 26] (some results which use the full power of the Ramanujan property appear in [3, 13, 33]). In this paper, the results of section §3.1 apply to all $\tilde{A}$-complexes, but in §3.2 we restrict ourselves to Ramanujan complexes in order to obtain a stronger result.

While we focus in this paper on $\tilde{A}$-complexes, our power operation makes sense for any colored complex (see Definition 2.2), and it is plausible that it yields HD expanders from other ones as well:
natural candidates are spherical buildings [37], the random ones constructed in [40, 41], and the ones constructed in [29].

1.1. Spheres and natural powering. When considering clique complexes, a natural power operation comes to mind: taking the clique complex afforded by the $r$-power of the one-skeleton of the original complex. In effect, this is not so simple, since the $r$-power of a graph is a “multigraph” with multiple edges and loops, and it is not clear how to define the clique complex in this case. Indeed, in the power graph $\mathcal{G}^r$ two vertices are neighbors if there is a path of length $r$ between them in $\mathcal{G}$, and backtracking paths always give rise to loops. One can replace $\mathcal{G}^r$ by the “non-backtracking $r$-power” $\mathcal{G}^{[r]}$, in which two vertices are neighbors if there is a non-backtracking $r$-path between them in $\mathcal{G}$. The expansion quality of $\mathcal{G}^{[r]}$ is even better than that of $\mathcal{G}^r$ [1], and if $\text{girth}(\mathcal{G}) > r$ then $\mathcal{G}^{[r]}$ is a simple graph (a graph with no multiedges and loops). However, for HD expanders this is still not useful, since the girth of the one-skeleton is only three, and more edges of $\mathcal{G}^r$ should be removed to obtain a simple graph. For $\mathcal{G}$ of high girth, the vertices of $\mathcal{G}^{[r]}$ correspond to the $r$-sphere in $\mathcal{G}$, which leads us to observe the $r$-sphere in our complex as a candidate for a “natural powering” process. In §4 we show the following:

**Proposition** (Prop. 4.1 and 4.2). *The $r$-spheres around a vertex in a two-dimensional $\tilde{\mathcal{A}}$-complex, and even the $r$-powers (as a graph) of the $r$-spheres, do not form a family of expanders.*

The consequence is that any powering scheme in which the resulting links are similar to the $r$-spheres, or to the $r$-powers of the $r$-spheres in the original graphs, does not give a family of HD expanders from $\tilde{\mathcal{A}}$-complexes (and in particular, from Ramanujan complexes). This proposition also relates to a conjecture of Benjamini, which states that there are no expander families in which the spheres of any radius form a family of expanders themselves. Being excellent local and global expanders, Ramanujan complexes are natural candidates for disproving Benjamini’s conjecture, but we show that they do not violate it.

1.2. Applications. In §5 we demonstrate applications of the geodesic power operation of HD expanders. It is well known that walks on expander graphs sample the vertices well, and a natural question is whether longer walks along expanders can sample well short walks. We use the power complex to design a long walk which samples well short walks along geodesics. In this walk, two geodesics of length $r$ are considered neighbors if they border a common triangle in the geodesic $r$-power of the complex (see Figure 1.1). The fact that the power complex is a HD expander (§3.1) implies that this walk samples well the geodesics (Corollary 5.5).

![Figure 1.1. Four steps of the 3-walk on 3-geodesics in an \( \tilde{\mathcal{A}}_2 \)-complex.](image)

Combining this with the results of §3.2 yields an application for computer science: an explicit construction of a double sampler, as defined by Dinur and Kaufman for their work on de-randomization.
of direct product testing [8]. The advantage of our construction over that of [8] is that the power operation allows us to obtain arbitrary sampling quality for a fixed complex, in the same way that taking longer walks along an expander graph improves the sampling quality in a classical sampler. In contrast, the quality of the double sampler which appears in [8] is determined from the underlying complex, and a new complex has to be generated each time one seeks to obtain finer sampling quality.

Acknowledgement. The authors thank David Kazhdan for helpful discussions, and the anonymous referees for various improvements of the paper. Tali Kaufman was supported by ERC and BSF grants; Ori Parzanchevski was supported by ISF grant 1031/17.

2. Definitions

2.1. Simplicial complexes. A simplicial complex $X$ with vertex set $V$ is a collection of subsets of $V$, called faces or cells, which is closed under containment. We denote by $X(j)$ the cells of size $j + 1$, which are called $j$-dimensional (or $j$-cells), and by $d =$ dim $X$ the maximal dimension of a cell; $X$ is called pure if every cell is contained in a $d$-cell. The link of a cell $\tau \in X$, denoted $X_\tau$, is the complex obtained by taking all cells in $X$ that contain $\tau$ and removing $\tau$ from them. Thus, if dim $\tau = i$ then dim $X_\tau = d - i - 1$, and in particular $X_\emptyset = X$, and $X_\tau$ is a graph for $\tau \in X(d - 2)$. For any $\tau \in X(i)$ ($1 \leq i \leq d - 2$), the one-skeleton of $X_\tau$ is the graph obtained by taking only the vertices and edges in $X_\tau$, and the second largest eigenvalue of the normalized adjacency operator of this graph is denoted by $\mu_\tau$. If $X$ is maximal with respect to its underlying graph, i.e. every clique in the one-skeleton of $X$ is a cell, $X$ is called a clique complex.

Definition 2.1 (High dimensional expander). A pure $d$-dimensional complex $X$ is a $\lambda$-high dimensional expander if for every $-1 \leq i \leq d - 2$, and for every $\tau \in X(i)$, $\mu_\tau \leq \lambda$. In fact, by [43] it follows that if $X$ is connected and $\mu_\tau \leq \lambda < \frac{1}{2}$ for every $\tau \in X(d - 2)$, then $X$ is already a $\frac{\lambda}{\sqrt{d}}$-high dimensional expander.

2.2. $\tilde{A}$-complexes. We recall the definition of Ramanujan complexes, and more generally $\tilde{A}$-complexes, following [31, 39] (for further study of these complexes see [14, 15, 19, 26, 33, 35]). Let $F$ be a fixed non-archimedean local field with integer ring $\mathcal{O}$, uniformiser $\pi$ and residue field $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$ (for example, $(F, \mathcal{O}, \pi, q) = (\mathbb{Q}_p, \mathbb{Z}_p, p, p)$ with $p$ prime or $(\mathbb{F}_p((t)), \mathbb{F}_p[[t]], t, q)$ with $q$ a prime power). The affine Bruhat-Tits building $\mathcal{B} = \mathcal{B}_d(F)$ of type $\tilde{A}_{d-1}$ is an infinite $(d - 1)$-dimensional clique complex, whose vertices correspond to the cosets $G/K$, where $G = \text{PGL}_d(F)$ and $K = \text{PGL}_d(\mathcal{O})$. The map $gK \mapsto g\mathcal{O}^d$ constitutes a correspondence between $G/K$ and homothety-classes of $O$-lattices in $F^d$. Two such classes are neighbors in $\mathcal{B}$ when they have representatives $L, L'$ satisfying $\pi L < L' < L$, and $\mathcal{B}$ is the clique complex of the resulting graph. An $\tilde{A}_{d-1}$-complex is, by definition, a quotient of $\mathcal{B}$ by a cocompact lattice $\Gamma \subseteq G$ acting on $\mathcal{B}$ without fixed points (1), and we say that it is of density $q$. The $\tilde{A}_{d-1}$-complex $X = \Gamma \backslash \mathcal{B}$ is a finite simplicial complex with fundamental group $\Gamma$, and it is a $\frac{1}{\sqrt{d}}$-HD expander by [17]. The vertices of $\mathcal{B}$ are colored by

$$col: \mathcal{B}(0) \rightarrow \mathbb{Z}/d\mathbb{Z}, \quad col(gK) = \text{ord}_q \det g + d\mathbb{Z},$$

and we say that $X$ is $d$-partite if $\Gamma$ preserves vertex-colors, namely, col factors through $X(0) = \Gamma \backslash \mathcal{B}(0)$. The directed edges of $\mathcal{B}$, denoted $\mathcal{B}^\pm(1)$, are also colored:

$$col: \mathcal{B}^\pm(1) \rightarrow \mathbb{Z}/d\mathbb{Z}^X, \quad col(v \rightarrow w) = \text{col}(w) - \text{col}(v),$$

and $\Gamma$ always preserves the color of edges, so that any $\tilde{A}_{d-1}$-complex $X$ inherits a coloring of its directed edges, col: $X^\pm(1) \rightarrow \mathbb{Z}/d\mathbb{Z}^X$. We give this property a name:

Definition 2.2 (Colored complexes). A $d$-colored complex is a pure $(d - 1)$-dimensional complex whose directed edges are colored by $\mathbb{Z}/d\mathbb{Z}^X$, so that each $(d - 1)$-cell $\tau$ can be assigned vertex colors $\text{col}_\tau: \tau \rightarrow \mathbb{Z}/d\mathbb{Z}$ for which $\text{col}(v \rightarrow w) = \text{col}_\tau w - \text{col}_\tau v$ for any $v, w \in \tau$. We stress that $\text{col}_\tau$ and $\text{col}_{\tau'}$ need not agree on $\tau \cap \tau'$, so $X$ need not be vertex-colored.

(1) By a theorem of Tits, if $\Gamma$ is torsion-free then this condition is always satisfied.
We define the colored adjacency operator $A_j$ on $L^2 (X(0))$ for a $d$-colored complex $X$:

$$
(A_j f)(v) = \sum_{w \sim v, \text{col}(w-v)=j} f(w).
$$

In the case of $B_d$ and its quotients, $A_j$ is regular of degree $\left[\frac{d}{j}\right]_q$, namely $\# \{w | \text{col}(v-w)=j\} = \left[\frac{d}{j}\right]_q$ for every vertex $v$. Here $\left[\frac{d}{j}\right]_q$ is the Gaussian binomial coefficient, and eigenvalues of $A_j$ of this magnitude (which account for periodicity, by Perron-Frobenius theory) are said to be trivial.

**Definition 2.3** ($[31, 39]$). An $X_{d-1}$-complex $X$ is a Ramanujan complex if for $0 < j < d$ every eigenvalue of $A_j$ is either trivial or contained in the spectrum of $A_j$ acting on $L^2 (B_d(0))$.

**Remark 2.4.** Subsequent works [14, 33] suggest stronger definitions for Ramanujan complexes, but for the case $d = 3$, which we use in §5, it is shown in [23] that the various definitions agree.

**Example 2.5.** When $d = 3$, $B_d$ is a triangle complex with constant vertex degree $2 (q^2 + q + 1)$ and edge degree $q + 1$. The degrees of $A_1$ and $A_2$ are both $q^2 + q + 1$, so that a trivial eigenvalue satisfies $|\lambda| = q^2 + q + 1$. If (and only if) $X$ is Ramanujan, the nontrivial eigenvalues satisfy

$$
\lambda \in \text{Spec} \left( A_j |_{L^2(B_3)} \right) = \left\{ q (z_1 + z_2 + z_3) \mid |z_1| = |z_2| = |z_3| = 1, |z_1 \cdot z_2 \cdot z_3| = 1 \right\},
$$

and in particular $|\lambda| \leq 3q$.

### 2.3. Geodesic powering

The power operation which we define is based on geodesic paths in colored complexes:

**Definition 2.6** (Geodesic path). A sequence of vertices $v_0, \ldots, v_r$ in a $d$-colored complex $X$ is called an $r$-geodesic path of color one if it is:

1. A non-backtracking path: $\{v_i, v_{i+1}\} \in X(1)$ and $v_{i+2} \neq v_i$.
2. Geodesic: $\{v_i, v_{i+1}, v_{i+2}\} \notin X(2)$.
3. Of color one: $\text{col}(v_i \to v_{i+1}) = 1$.

Unless stated otherwise, by an $r$-geodesic we always mean an $r$-geodesic path of color one. Note that for $d \geq 3$, (1) follows from (3) since $\text{col} v_{i+2} \neq \text{col} v_i$.

The inverted path $v_r, \ldots, v_0$ is a “geodesic of color $d-1$”, satisfying (1), (2), and $\text{col}(v_r \to v_{r+1}) = d-1$. Geodesic paths of other colors can also be defined, but their geometric intuition is less obvious and originates from higher-dimensional geometry (1).

The geometric motivation for this definition is the following: when walking on graphs, every edge gives rise to a “trivial” local loop of length two, and the non-backtracking walk eliminates these loops. In complexes of higher dimension, even the non-backtracking walk has local loops formed by going around a triangle or a higher-dimensional cell. The (monochromatic) geodesic walks avoid such loops — indeed, on the building itself there are no closed geodesic paths at all, just as there are no closed non-backtracking paths on a tree (this follows from being “collision-free” — see [33]).

**Example 2.7.** With $F, \mathcal{O}, \pi, q$ and $B = B_d(F)$ as in §2.2, observe the vertices $v_i = \text{diag}(\pi^i, 1, \ldots, 1) K$ in $B(0)$ (which correspond to the homothety classes of the $O$-lattices $L_i = \pi^i \mathcal{O} \times \mathcal{O}^{d-1}$, respectively). The path $v_0 \to \ldots \to v_r$ is an $r$-geodesic in $B$, as $\pi L_i < L_{i+1} < L_i$, $[L_i : L_{i+1}] = q$ and no scaling $L'_{i+2}$ of $L_{i+2}$ satisfies $\pi L_i < L'_{i+2} < L_i$.

We can now define the geodesic power of a colored complex:

**Definition 2.8** (Geodesic powering). For a pure $d$-dimensional colored complex $X$, the geodesic $r$-power of $X$ is defined as follows: its vertices are the same as the vertices of $X$, and $d+1$ distinct vertices $v_0, \ldots, v_d$ form a $d$-cell iff, possibly after reordering them, there is an $r$-geodesic path (of color one) from each $v_i$ to $v_{i+1}$, and from $v_d$ to $v_0$. The cells of lower dimension are the subcells of the $d$-cells so defined.

(1) For an edge $v \to w$ of color $j$, a step in a geodesic walk of color $j$ can be performed by completing it to a $j$-geodesic path of color one $v = v_0, v_1, \ldots, v_i = w$ which is also a $j$-cell, performing $j$ steps of the $j$-dimensional flow described in [3, 33] on this (ordered) cell, and keeping the first and last vertex.
We do not assume here that $X$ is a clique complex, but the examples studied in this paper are. We also remark that the vertices contained in an edge in the power-complex are only the two endpoints of the corresponding $r$-geodesic in $X$ – as in graph powering, the interior vertices are "forgotten".

2.4. Explicit example. We give an explicit example of an $\tilde{A}$-complex and its geodesic $r$-powers, using the construction from [13]. Let $p, q$ be distinct primes equal to 1 modulo 4, let $p = \pi \equiv \mp 1$ be a decomposition of $p$ in $\mathbb{Z}[i]$, and define

$$S_p = \left\{ s \in M_3(\mathbb{Z}[i]) \mid s^*s = pI, \text{ord}_p (\det s) = 1, s \equiv \begin{pmatrix} 1 & * & * \\ \ast & 1 & \ast \\ \ast & \ast & 1 \end{pmatrix} \pmod{2+2i} \right\}.$$  

For example, taking $p = 5$ and $\pi = 2+i$ we have $S_5 = \left\{ \left( \begin{array}{ccc} 2i-1 & 0 & 0 \\ 0 & 2i-1 & 0 \\ 0 & 0 & -2i+1 \end{array} \right), \left( 2i-1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & 0 & -1 \end{array} \right), \left( 1+i & \ast & 1+i \\ \ast & 1 & \ast \\ 1+i & \ast & 1+i \end{array} \right) \pmod{2+2i} \right\}$ (see [13, Example 6.4] for the complete set). The set $S_p$ has $p^2 + p + 1$ elements, and the directed Cayley graph spanned by it in $PGL_3(\mathbb{Q}[i])$ is isomorphic to the graph of color-one edges in the $\tilde{A}_2$-building $B = B_3(\mathbb{Q}_{\mathfrak{p}})$ (the edges of color two are just the inverse edges). Fixing $\varepsilon = \sqrt{-1} \in \mathbb{F}_q$ and mapping $i \mapsto \varepsilon$ gives a ring homomorphism $\eta: \mathbb{Z}[i] \to \mathbb{F}_q$, and we denote by $S_{p,q}$ the set of matrices in $PGL_3(\mathbb{F}_q)$ obtained from $S_p$ by applying $\eta$. The generated group $G = \langle S_{p,q} \rangle$ equals either $PSL_2(\mathbb{F}_q)$ or $PGL_3(\mathbb{F}_q)$, and the directed Cayley graph $X^{P,q} = Cay(G,S_{p,q})$ is the graph of color-one edges in a finite $\tilde{A}_2$-complex of density $p$.

Returning to $B \cong Cay((S_{p,q}), S_p)$ (where $S_{p} \leq PGL_3(\mathbb{Q}[i])$), for any $s, s' \in S_p$ either $e \rightarrow s \rightarrow ss'$ or $e \rightarrow s \rightarrow ss'$ is a geodesic. For each $s$ there are $p+1$ choices of $s'$ for which the former occurs; they are the ones for which $ss's''$ is a scalar matrix for some $s'' \in S_p$.

Denoting the remaining $p^2$ choices of $s'$ by $\Sigma_s$, the $r$-geodesics starting at $e$ are precisely

$$e \rightarrow s_1 \rightarrow s_1 s_2 \rightarrow \cdots \rightarrow s_1 s_2 \cdots s_r \quad (s_1 \in S_p, s_i \in \Sigma_s, i = 2, \ldots, r).$$

In accordance, the $r$-power of $B$ coincides with the Cayley graph with generating set $S_{p,q}^r := \left\{ s_1 s_2 \cdots s_r \mid s_1 \in S_p, s_i \in \Sigma_s, i = 2, \ldots, r \right\} \subseteq PGL_3(\mathbb{Q}[i])$, and the $r$-power of $X^{P,q}$ is the Cayley graph of $G$ with the generators $\left\{ \eta(s) \mid s \in S_{p,q}^r \right\}$.

3. EXPANSION IN THE POWER-COMPLEX

3.1. Spectrum of links in the power-complex. Let $F, \mathcal{O}, \pi$ and $\mathbb{F}_q = \mathcal{O}/\pi \mathcal{O}$ be as in §2.2. The link of a vertex in $B = B_3(F)$ coincides with the spherical building of $PGL_3(\mathbb{F}_q)$, whose cells correspond to flags in the space $\mathbb{F}_q^d$. Our first goal is to give a similar description for the links in the power-complex of $B$. Recall that a module $M$ over a commutative ring $R$ is called free if it is isomorphic to $R^m$ for some $m$, which is denoted by rank $M$. The ring we are interested in is

$$\mathcal{O}_r = \mathcal{O}_r(F) := \mathcal{O}/\pi^r \mathcal{O},$$

and we remark that if $F$ is a completion of a global field $k$, then one can also realize $\mathcal{O}_r$ as a quotient of the integer ring of $k$; E.g., $\mathcal{O}_r(\mathbb{Q}_p) = \mathbb{Z}_p / \pi^r \mathbb{Z}_p \cong \mathbb{Z} / \pi^r \mathbb{Z}$, and $\mathcal{O}_r(\mathbb{F}_q(t)) \cong \mathbb{F}_q[t] / (t^r)$. We introduce the following definitions:

**Definition 3.1.** (1) A flag of $\mathcal{O}_r$-modules $F = \{0 < M_1 < \ldots < M_r < \mathcal{O}_r^d\}$ is called free if all $M_i$ are free $\mathcal{O}_r$-modules.

(2) The free projective $d$-space over $\mathcal{O}_r$, denoted $\mathbb{P}_r^d(\mathcal{O}_r)$, is the complex whose vertices correspond to free $\mathcal{O}_r$-submodules $0 < M < \mathcal{O}_r^{d+1}$, and whose cells are the free flags in $\mathcal{O}_r^{d+1}$.

We state now a few useful facts which follow from the theory of modules over local principal ideal rings (for example from the existence of a Smith Normal Form over $\mathcal{O}_r$, see e.g. [24]):

**Fact 3.2.** Every submodule $M$ of $\mathcal{O}_r^d$ is equivalent under $GL_d(\mathcal{O}_r)$ to $\text{diag}(m_1, \ldots, m_d) \mathcal{O}_r^d$ for a unique choice of $r \geq m_1 \geq \ldots \geq m_d \geq 0$. For these $m_i$, $\mathcal{O}_r^d / M \cong \text{diag}(m_1 - m_d, \ldots, m_1 - m_1)$ $\mathcal{O}_r^d$, and $M \leq \mathcal{O}_r^d$ is free iff all $m_i$ are either 0 or $r$. When $M$ is free, the free submodules in $\mathcal{O}_r^d / M$ are in correspondence with free submodules in $\mathcal{O}_r^d$ which contain $M$. 

An important consequence is that a maximal free flag $F = \{M_i\}$ in $O_d$ has a unique refinement to a maximal flag, since each quotient $M_{i+1}/M_i$ is isomorphic to the local ring $O_r$, which has a unique composition series.

**Example 3.3.** For $F = \mathbb{Q}_p, r = 2, d = 3$ we have $O_d \cong \mathbb{Z}/p^2\mathbb{Z}$, and $0 < \left(\begin{array}{c} 2 \\ 0 \end{array}\right) < \left(\begin{array}{c} 2 \\ \phi \end{array}\right) < \left(\begin{array}{c} 2 \\ 2 \end{array}\right)$ is a maximal free flag; its maximal refinement is $0 < \left(\begin{array}{c} p^2 \\ 0 \end{array}\right) < \left(\begin{array}{c} p^2 \\ \phi \end{array}\right) < \left(\begin{array}{c} p^2 \\ 2 \end{array}\right)$.

To see how $O_d$ relates to the building $B$, let $L_0 \to \ldots \to L_m$ be a closed path of color one in $B$. By some abuse of notation we use $L_i$ to refer to a specific choice of lattice in the homothety class $L_i$, that was chosen so that $\pi L_i < L_{i+1} < L_i$; it follows from col$(L_i \to L_{i+1}) = 1$ that $[L_i : L_{i+1}] = q$. By col $L_0 = \text{col} L_m$ one has $m = rd$ for some $r$, and from $L_m = \pi^r L_0$ (as they are homothetic) it follows that $t = r$, so that $L_m < L_{m-1} < \ldots < L_0$ projects to a maximal flag in $L_0/\pi^r L_0 \cong O_d$. On the other hand, each maximal flag in $L_0/\pi^r L_0$ lifts to a distinct path, since if $L_{i+1} \neq L_{i+1}$ and both are of index $q$ in $L_i$ then they cannot be homothetic. We conclude that color-one cycles of length $rd$ around $L_0$ are in correspondence with maximal flags in $O_d$. In addition, as $B$ is $d$-colored any color-one path $L_0 \to \ldots \to L_r$ can be completed to a color-one cycle of length $rd$, so that color-one paths of length $r$ starting from a given vertex correspond to flags $M_r < \ldots < M_0 = O_d$ such that $[M_i : M_{i+1}] = q$. We can now prove two useful Lemmas:

**Lemma 3.4.** The group $G = \text{PGL}_d(F)$ acts transitively on all $r$-geodesics in $B$.

**Proof.** We show by induction on $r$ that any $r$-geodesic (of color one) $L_0 \dashrightarrow L_r$ can be translated by $G$ to the “standard” geodesic $L_0' \dashrightarrow L_r'$, where $L_0' := \pi^r O \times O^{d-1}$ as in Example 2.7. For $r = 0$ this holds by transitivity of $G$ on $B(0) = G/K$, and for $r > 0$ since $K = \text{Stab}(O)$ acts transitively on the edges of color one leaving $O_d$. For $r \geq 2$, we can assume by induction that $L_i = L_i'$ for $0 \leq i \leq r - 1$, and we observe that the color one edges leaving $L_i' \dashrightarrow \pi^{r-1} O \times O^{d-1}$ enter either

$$G_{\pi a_i} = \left(\begin{array}{c} \pi^r \\ 0 \\ \ldots \end{array}\right)$$

and take $L_i'$ to $G_{\pi a_i} L_i' = \pi^{r-1} O \times O^{d-1}$, which completes the proof.

**Lemma 3.5.** For a color-one path $\gamma = L_0 \dashrightarrow L_r$ with $[L_i : L_{i+1}] = q$, the following are equivalent:

1. $L_r$ is a free submodule of $L_0/\pi^r L_0 \cong O_d$.
2. $\gamma$ is the unique color-one path of length $r$ from $L_0$ to $L_r$.
3. $\gamma$ is geodesic.

**Proof.** (1)$\Rightarrow$(2): By Fact 3.2, if $L_r \leq O_d$ is free so is $O_d/L_r$, and by index considerations, it is isomorphic to $O_r$ which has a unique composition series. (2)$\Rightarrow$(3): If $\gamma$ is not geodesic, then $\pi L_i \leq L_{i+2} \leq L_i$ for some $i$, and $[L_i : L_{i+2}] = q^2$. As $L_{i+2}$ corresponds in $L_i/\pi L_i \cong O_d$ to a subspace of codimension two, there are $(q+1)$ color-one paths $L_i \dashrightarrow \ldots \dashrightarrow L_{i+2}$ (including $\ast = L_{i+1}$), hence $\gamma$ is not unique. (3)$\Rightarrow$(1): By Lemma 3.4, some $g \in G$ takes $L_0 \dashrightarrow L_r$ to $O_d \dashrightarrow \pi^r O \times O^{d-1}$, and $\pi^r O \times O^{d-1}$ corresponds to a free submodule in $O_d$.

**Proposition 3.6.** The link of a vertex in the $r$-power-complex of $B = B_d$ is isomorphic to $\mathbb{P}^{d-1}_r(O_r)$, and the link of a cell of codimension two is either a complete bipartite graph, or isomorphic to the graph $\mathbb{P}^{d-1}_r(O_r)$.

**Proof.** By definition, the $r$-power of $B$ is defined by its $(d-1)$-cells, which correspond to cycles of $O$-lattices $L_0, L_1, \ldots, L_{d-1}, L_0 \in B(0)$ such that each $L_i, L_{i+1} \text{(mod } d\text{)}$ are connected by an $r$-geodesic. Such a cycle can be rotated to start in any of its vertices, hence by Lemma 3.5 each $(d-1)$-cell containing $L_0$ in the power-complex corresponds to a unique color-one $rd$-cycle $\{v_i\}_{i=0}^{rd}$ in $B$ such that $v_i = L_i$. By the discussion following Example 3.3, this $rd$-cycle corresponds to a maximal
flag \{M_i\} in L_0/\pi^r L_0 \cong O^r_i which satisfies that M_{ir}/M_{i+1}r \cong O_r for each i, hence the sub-flag \{M_{ir}\}_{i=0}^d is a maximal free flag in O^d_i. On the other hand, each maximal free flag in L_0/\pi^r L_0 \cong O^d_r refines to a unique maximal flag, which corresponds to a cycle consisting of d geodesics of length \( r \), yielding a \((d-1)\)-cell containing \( L_0 \) in the power-complex. As the cells in \( \mathbb{P}_{2r}^{-1}(O_r) \) are all the subsets of maximal free flags in O^d_r, this establishes the stated isomorphism.

If \( \tau \) is a \((d-3)\)-cell in the \( r \)-power of \( \mathcal{B} \), it corresponds to a free flag \{M_i\} of length \( d - 2 \) in \( O^d_r \), and two cases arise: either there are two different \( i \) such that \( M_{i+1}/M_i \cong O^d_r \), or there is a single \( i \) for which \( M_{i+1}/M_i \cong O^d_r \). In the former case, the link of \( \tau \) is a complete bipartite graph, since the choices of the (free) refinements in the two places with \( M_{i+1}/M_i \cong O^d_r \) are independent. In the latter case, the possible refinements correspond precisely to maximal free flags in \( O^d_r \), resulting in a complex isomorphic to \( \mathbb{P}_{2r}^0(O_r) \).

We can now prove that one-dimensional links in the power-complexes are excellent expanders:

**Theorem 3.7.** The graph \( \mathbb{P}_{2r}^1(O_r) \) is a \((q^2 + q + 1)q^{2(r-1)}\)-regular connected bipartite graph on \( 2(q^2 + q + 1)q^{2(r-1)} \) vertices, with adjacency spectrum

\[
\text{Spec } (\mathbb{P}_{2r}^1(O_r)) = \left\{ \pm (q + 1)q^{-1}, \pm q^{2r-1}, \pm q^{2r-2}, \ldots, \pm q^{1+r}, \pm q^{r} \right\}.
\]

In particular, its second-largest normalized eigenvalue equals \( \frac{\sqrt{q}}{\sqrt{q+1}} \) independently of \( r \).

**Proof.** Denoting by \( \mathcal{F}_r^i \) the set of free modules of rank \( i \) in \( O^d_r \), the graph \( \mathbb{P}_{2r}^1(O_r) \) is the bipartite graph with vertices \( \mathcal{F}_r^i \cup \mathcal{F}_r^j \) and edges given by inclusion. By Fact 3.2, \( GL_2(O_r) \) acts transitively on each \( \mathcal{F}_r^i \), and the stabilizer of \( O^d_r \times 0 \times 0 \in \mathcal{F}_r^1 \) is \( \{g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in GL_2(O_r) \} \), hence

\[
|\mathcal{F}_r^i| = |GL_2(O_r)| / \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = \frac{(q^3 - 1)(q^3 - q)(q^3 - q^2)q^{i(r-1)}}{(q - 1)^2(q^2 - 1)(q^2 - q)q^{2(r-1)}} = (q^2 + q + 1)q^{2(r-1)},
\]

and \( |\mathcal{F}_r^j| = |\mathcal{F}_r^i| \) by a similar computation or by duality (Fact 3.2), and to compute its degree we observe that the neighbors of a fixed vertex in \( \mathcal{F}_r^2 \) correspond to rank-one free submodules in \( O^2_r \), of which there are

\[
|GL_2(O_r)| / \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = \frac{(q^2 - 1)(q^2 - q)q^{4(r-1)}}{(q - 1)^2(q - 1)} = (q + 1)q^{4(r-1)}.
\]

For any regular bipartite graph \( \mathcal{G} \) on vertices \( L \cup R \), the spectrum of \( A = \text{Adj}_L \) satisfies \( \text{Spec } A = \{ \pm \sqrt{\delta} \mid \delta \in \text{Spec } A^2 \} \). We denote \( A = \text{Adj}(\mathbb{P}_{2r}^1(O_r)) \), and let \( Q = A^2 \mid \mathcal{F}_r^1 \). Recalling that \( \mathcal{F}_r^i \) corresponds to the endpoint of \( r \)-geodesics leaving \( v_0 \in \mathcal{B} \), we observe the subgraph of \( \mathcal{B} \) formed by these geodesics (including their inner vertices). This is a rooted tree of height \( r \), with the vertices \( \mathcal{F}_r^1 \) as leaves, root degree \( (q^2 + q + 1) \), and all inner nodes having \( q^2 \) descendants\(^{(1)}\). For \( v, w \in \mathcal{F}_r^1 \), denote by \( \Delta(v, w) \) the shortest distance from \( v \) and \( w \) to a common ancestor in this tree. This gives an ultrametric distance function on \( \mathcal{F}_r^1 \), and the \( (v, w) \)-entries of all polynomials in \( Q \) only depend on \( \Delta(v, w) \). For the rest of the proof all matrices will be indexed by \( \mathcal{F}_r^1 \), and by \( \delta \) we mean the value of \( M_{v,w} \) for any \( v, w \) with \( \Delta(v, w) = \delta \). Defining \( B^{(\ell)} = \prod_{v \in \mathcal{F}_r^1}(Q - q^\ell) \), we will show by induction that \( B^{(\ell)} \) is constant for \( 0 \leq \delta \leq \ell \). In particular, \( B^{(r)} = \prod_{v \in \mathcal{F}_r^1}(Q - q^r) \) is a constant multiple of the all-one matrix, and this constant is not zero as \( Q^2 = (q + 1)q^{2(r-1)}Q \). This implies that \( \mathbb{P}_{2r}^1(O_r) \) is connected, and that the spectrum of \( Q \) is \( \{ (q + 1)q^{2(r-1)}, q^{2r-1}, q^{2r-2}, \ldots, q^r \} \), which yields the theorem. In fact, one can show by induction the following:

\[
B^{(\ell)}_{\delta-1} - B^{(\ell)}_\delta = \begin{cases} q^{\ell-\delta+r} \left( \begin{smallmatrix} (q^2 - 1) \prod_{j=\delta-1}^{\ell-1} (q^j - 1) \\ 0 \end{smallmatrix} \right) & 0 \leq \delta \leq \ell \\
0 & 1 \leq \delta \leq \ell \end{cases}
\]

(it turns out that this is not needed for the proof, but we record this observation here for the benefit of future research). For two matrices indexed by \( \Delta \)-values, the appropriate multiplication

\(^{(1)}\)Any color one edge has \( q^2 \) extensions to a geodesic of length two - this follows from (3.1), (3.2).
rule is \((AB)_\delta = \sum_{e<\zeta} N^\delta_{e,\zeta} A_e B_\zeta\), where \(N^\delta_{e,\zeta}\) is the number of vertices \(u\) satisfying \(\Delta(v,u) = \varepsilon, \Delta(u,w) = \zeta\), for (any pair of) \(v,w\) with \(\Delta(v,w) = \delta\). It is not hard to see that \(N^\delta_{e,\zeta} = N^\delta_{\zeta,e}\), so from now on we always assume \(\varepsilon \leq \zeta\), writing
\[
(AB)_\delta = \sum_{\varepsilon<\zeta} N^\delta_{\varepsilon,\zeta} (A_\varepsilon B_\zeta + A_\zeta B_\varepsilon) + \sum_{\varepsilon} N^\delta_{\varepsilon,\varepsilon} A_\varepsilon B_\varepsilon.
\]
By the ultrametric triangle inequality for \(\Delta\), if \(N^\delta_{e,\zeta} \neq 0\) then either \(\delta < \varepsilon = \zeta\) or \(\varepsilon \leq \zeta = \delta\), so that we can further write
\[
(AB)_\delta = \sum_{\varepsilon=0}^{\delta-1} N^\delta_{\varepsilon,\delta} (A_\varepsilon B_\delta + A_\delta B_\varepsilon) + \sum_{\varepsilon=\delta}^r N^\delta_{\varepsilon,\varepsilon} A_\varepsilon B_\varepsilon.
\]

Careful counting reveals that whenever \(N^\delta_{e,\zeta} \neq 0\), it is given by
\[
N^\delta_{e,\zeta} = \begin{cases} 1 & \text{if } \varepsilon = 0, 0 < \varepsilon < r, \varepsilon = r \\ 0 & \text{if } \varepsilon = \delta \neq 0 \end{cases}
\]

In particular, whenever \(\varepsilon < \delta\) or \(\delta < \varepsilon < r\), the value of \(N^\delta_{e,\zeta}\) does not depend on \(\delta\). This leads to many simplifications for the differences between entries of \(AB\), resulting in:

\[
(AB)_{\delta-1} - (AB)_\delta = (A_{\delta-1} - A_\delta) \sum_{\varepsilon<\delta-1} N^\delta_{\varepsilon,\delta} B_\varepsilon + (B_{\delta-1} - B_\delta) \sum_{\varepsilon<\delta} N^\delta_{\varepsilon,\delta} A_\varepsilon
\]

\ outlets \((AB)_{\delta-1} - (AB)_\delta = (M B)_\delta - (M \delta)\) does not depend on \(\delta\). We finally compute \(Q\) for \(L_1, L'_1 \in F^2\r
\) with \(\Delta(L_1, L'_1) = \delta\), the entry \(Q_\delta\) is the number of \(L_2 \in F^2\) which complete both of them to a free flag. This corresponds to \(L_2 \leq \Omega^2/\ell (L_1 + L'_1)\) with \(\Omega^2/\ell \cong \Omega, r\), giving
\[
Q_\delta = \begin{cases} (q+1)^{q-1} & \text{if } 1 \leq \delta < r \\ q^{r-\delta} & \text{if } \delta = r \\ 0 & \text{if } \delta = 0 \end{cases}
\]

We see that it is easier to work with \(M := Q - q^{-\delta}\), which is simply \(M_\delta = q^{r-\delta}\), so that taking \(A = M - q^{r+\delta} + q^{r-1}\) and \(B = B^\delta\) we obtain \(B^{\delta+1} = AB\), and using (3.4) we have
\[
B^{\delta+1} - B^\delta = \left\{(q^{r-\delta} - q^{r-1}) \sum_{\varepsilon<\delta-1} N^\delta_{e,\delta} B_e + (B_{\delta-1} - B_\delta) \sum_{\varepsilon<\delta} N^\delta_{e,\delta} q^{r-\varepsilon}\right\}
\]
\[
+ N^\delta_{\delta-1,\delta} (A_{\delta-1} B_{\delta-1} - N^\delta_{\delta-1,\delta} (A_{\delta-1} B_\delta + A_\delta B_{\delta-1}) + q^{2(\delta-1)} A_\delta B_\delta
\]
\]

For \(\delta \leq \ell\), we have by the induction hypothesis \(B_0 = \ldots = B_\delta\), and thus (3.5) simplifies to
\[
q^{r-\delta} B_\delta \left\{(q-1) \sum_{\varepsilon<\delta-1} N^\delta_{e,\delta} + N^\delta_{\delta-1,\delta-1} q - N^\delta_{\delta-1,\delta} (q+1) + q^2(\delta-1)\right\}.
\]

For \(1 \leq \delta \leq \ell\), plugging (3.3) into (3.6) gives
\[
q^{r-\delta} B_\delta \left\{(q-1) \left(1 + (q^2 - 1) \sum_{\varepsilon=1}^{\delta-2} q^{2(\varepsilon-1)}\right) + (q^2 - 2)q^{2\delta-3} - (q^2 - 1)q^{2\delta-4}(q+1) + q^2(\delta-1)\right\} = 0,
\]
and for $\delta = 1$, it gives $q^{r-1} B_0 [q - (q + 1) + 1] = 0$ as well. Finally, taking $\delta = \ell + 1$ in (3.5) yields

$$B_0 \left[ (q^{r-\ell} - q^{r-\ell-1}) (q^{2^\ell - 2} - q^{r+\ell} + q^{r+\ell-1} + q^{r+\ell-2} + q^{r+\ell-1}) \right] + B_{\ell+1} \left[ q^{r-\ell} - q^{r-\ell-1} - q^{r+\ell-2} - (q^2 - 1) q^{2\ell - 2} \cdot q^{r-\ell} + q^{r+\ell-1} \right] = B_0 \cdot 0 + B_{\ell+1} \cdot 0 = 0,$$

which establishes the induction. \qed

In particular, we conclude that the link spectrum only depends on the residue field:

**Corollary 3.8.** If $F, F'$ are non-archimedean local fields with the same residue order (e.g. $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$), then the free projective planes $\mathbb{P}_F^2(O_r)$ and $\mathbb{P}_{F'}^2(O_r)$ are isospectral graphs.

When $r = 1$, both graphs coincide with the projective plane over the residue field $\mathbb{F}_q$, and are isomorphic. However, we conjecture that for $r \geq 2$ these are non-isospectral graphs when $F \neq F'$ (we have verified this for some small values of $r$ and $q$).

### 3.2. Expansion between vertices and $r$-geodesics

In this section we establish some global expansion results for geodesic powers of Ramanujan complexes. These will be useful for the applications we present in §5. Let $F$, $G$, and $q$ be as in §3.1, $G = \text{PGL}_3(F)$, $\Gamma < G$ a cocompact torsion-free lattice, and $X = \Gamma \backslash \mathcal{B}(F)$ a non-tripartite Ramanujan complex. Denote by $G^{(r)} = G^{(r)}(X)$ the bipartite graph formed by $r$-geodesics in $X$ on one side, and the vertices $X(0)$ on the other one, with $v \in X(0)$ connected to a geodesic $\gamma$ if it appears along it. In order to study expansion in $G^{(r)}$ we first compute the expansion across $m$-geodesics in $X$, for all $0 \leq m \leq r$. Let $S_m(\mathbb{R})$ be all endpoints of $m$-geodesics (of color one) starting at $v$, and $S_m(\mathbb{V})$ the set of $w$ with an $m$-geodesic from $w$ to $v$ (or equivalently, a color-two $m$-geodesic from $v$ to $w$). Denote

$$(\mathcal{A}_m f)(v) := \sum_{w \in S_1(\mathbb{R}) \cup S_2(\mathbb{V})} f(w) \quad (f \in L^2(X(0))),$$

and observe that $A_1 = A_1 + A_2$, the colored adjacency operators from (2.1). We call an eigenvalue of $\mathcal{A}_m$ trivial if its eigenfunction is an eigenfunction of $A_1, A_2$ with eigenvalue of magnitude $q^2 - q + 1$, and denote by $\lambda^{(m)}$ the largest nontrivial eigenvalue of $\mathcal{A}_m$ (in absolute value).

**Theorem 3.9.** Let $X$ be a Ramanujan $\check{A}_2$-complex of density $q$, and $m \geq 1$. The degree of $\mathcal{A}_m$ on $X$ is $k = 2 (q^2 + q + 1) q^{2(m-1)}$, and

$$\lambda^{(m)} \leq (m^2 + 3m + 2) q^m - 2 (m^2 - 1) q^{m-1} + (m^2 - 3m + 2) q^{m-2},$$

so that the largest normalized nontrivial eigenvalue of $\mathcal{A}_m$ is $\frac{\lambda^{(m)}}{k} \leq \frac{(m^2 + 3m + 2)}{q^m}$. The trivial spectrum of $\mathcal{A}_m$ (including multiplicities) is

$$\text{trivial spectrum}(\mathcal{A}_m) = \begin{cases} \{k\} & X \text{ is non-tripartite} \\ \{k, -\frac{k}{2}, -\frac{k}{2}\} & X \text{ is tripartite and } 3 \nmid m \\ \{k, k, k\} & X \text{ is tripartite and } 3 \mid m. \end{cases}$$

**Proof.** The degree of $\mathcal{A}_m$ was already computed in the proof of Theorem 3.7. We recall the notion of a Hecke operator on $\mathcal{B}$, which is a $G$-invariant operator $A$ on $L^2(B(0))$ such that $A(I_v)$ has compact support for (any) $v \in V$. Such an operator induces an action on quotients of $\mathcal{B}$, and we denote $A$ acting on $X = \Gamma \backslash \mathcal{B}$ by $A(X)$. As $\Gamma$ is cocompact, $L^2(\Gamma \backslash \mathcal{G})$ decomposes as a Hilbert sum of irreducible $G$-representations, $L^2(\Gamma \backslash \mathcal{G}) = \bigoplus V_i$. Letting $v_0 = K \in B(0)$, $\Gamma g \mapsto \Gamma g v_0$ gives an identification $X(0) \cong \Gamma \backslash \mathcal{G}/K$ and thus $L^2(X(0)) \cong L^2(\Gamma \backslash \mathcal{G})^K = \bigoplus V_K^i$, where $V_K$ is the space of $K$-fixed vectors in $V$. It is well known (see e.g. [39, 48] or [42, §IV]) that for every $i$ either $V_K^i = 0$ or $V_K^i = \mathbb{C} f_i$, where $f_i$ is a common eigenfunction of all Hecke operators on $L^2(X(0))$. By [31, 39], $X$ is a Ramanujan complex if each $V_i$ with $V_K^i \neq 0$ is either finite-dimensional or tempered, namely, the matrix coefficient $\varphi_0(g) = \int_{\Gamma \backslash \mathcal{G}} f_i(gx) \frac{d\ell}{\ell} dx$ satisfies $\varphi_0 \in \cap_{r \geq 0} L^{2+2r}(G)$. The finite-dimensional representations account for the trivial eigenvalues, so as $X$ is Ramanujan every nontrivial eigenvalue $\lambda_i$ of $\mathcal{A}_m$ comes from a tempered $V_i$. As $\varphi_i$ is a bi-$K$-spherical function on $G$, it can be interpreted as a $K$-spherical function on $B(0)$, which is also a $\lambda_i$-eigenfunction for $\mathcal{A}_m$. In the language of
and the bound for $\mu_i$ can be deduced as before. The parameters: $P_{m,0,0}$, $S_3^{m,0,0}$, and $\lambda_i$ are obtained by specialization of the Hall-Littlewood polynomial corresponding to the partition $\mu$. This is the spherical function with Satake parameters $\varphi_i$. Direct computation then gives

$$P_{m,0,0}(1,1,1;\varphi_i(0)) = \sum_{\sigma \in S_3^{m,0,0}} |\varphi_i(\sigma(0))| = \sum_{\varphi \in \mathcal{H}} \varphi(\sigma(0))$$

and the bound for $\lambda_i$ follows. The finite-dimensional representations of $\mathcal{H}$ are $\rho_{ij} g = \omega^i \omega^j \det(g)$, with $\rho_0$ being the trivial representation (which appears once in $L^2(\Gamma \backslash G)$, and $\rho_1, \rho_2$ appearing in $L^2(\Gamma \backslash G)$ each once) if $X$ is tri-partite. The eigenvalue of $\mathcal{H}$ on $\mathcal{H}$ can be computed by graph theory (observing that $\mathcal{H}$ shifts colors by $m$), or using its Satake parameters: $P_{m,0,0}(\varphi_i, \omega^j; \omega^q; \varphi_i(0)) = \omega^{jm} (q^2 + q + 1) q^{2(m-1)}$, and the trivial eigenvalues of $\mathcal{H}$ are deduced as before.

We return to the expansion of $G^{(r)}$:

**Proposition 3.10.** If $X$ is a non-tripartite Ramanujan $\tilde{A}_2$-complex of density $q$ then the Perron-Frobenius eigenvalue of $G^{(r)}(X)$ satisfies $\lambda_1(A_{G^{(r)}}) = rq^r (1 + o(1))$ (as $q, r \to \infty$), and its second eigenvalue satisfies $\lambda_2(A_{G^{(r)}}) \leq \sqrt{rq} (1 + o(1))$.

**Proof.** Since $G^{(r)}$ is bipartite, its nonzero spectrum is obtained as $\{ \pm \sqrt{X} \}$ where $\lambda$ runs over the nonzero eigenvalues of $A_{G^{(r)}}^2$ restricted to either side of $G^{(r)}$. For any vertex $v$, we have

$$\lambda_1(A_{G^{(r)}}^2) = \# \left\{ r\text{-geodesics containing } v \right\} \cdot \# \left\{ \text{vertices contained in an } r\text{-geodesic} \right\} = (r+1) (q^2 + q + 1) q^{2(r-1)} \cdot (r+1) = r^2 q^{2r} (1 + o(1)).$$


Next, we observe that
\[ N^{(r)}_m := \# \left\{ r\text{-geodesics containing } v, w \text{ whenever } w \in S^1_m(v) \right\} = \begin{cases} (r + 1) \left(q^2 + q + 1\right) q^{2(r-1)} & m = 0 \\ (r - m + 1) q^{2(r-m)} & 0 < m \leq r, \end{cases} \]
and that on the vertex side \( A^2 \) can be described using \( A_{m} \):
\[ A^2_{G(v)} \big|_{\text{vertex}} = \sum_{m=0}^{r} N^{(r)}_m A_{m}. \]
We observe that all \( A_{m} \) have a unique trivial eigenvalue, obtained on the constant functions. This shows another way to compute \( \lambda_1 \left( A^2_{G(v)} \right) \), as \( \sum_{m=0}^{r} N^{(r)}_m \deg(A_{m}). \) More importantly, since all \( A_{m} \) are self-adjoint this gives \( \lambda_2(A^2_{G(v)}) \leq \sum_{m=0}^{r} N^{(r)}_m \lambda_2(A_{m}), \) and we recall that \( \lambda_2(A_{m}) \leq m^2 q^m (1 + o(1)) \) for \( m \geq 1 \) by Theorem 3.9. In addition \( \lambda_2(A_0) = \lambda_2(I) = 1, \) so that
\[ \lambda_2(A^2_{G(v)}) \leq (r + 1) \left(q^2 + q + 1\right) q^{2(r-1)} (1 + o(1)) + \sum_{m=1}^{r} (r - m + 1) q^{2(r-m)} \cdot m^2 q^m (1 + o(1)) = rq^{2r} (1 + o(1)). \]

4. Spheres in \( \tilde{A} \)-complexes

In this section we show that \( r \)-spheres in \( \tilde{A} \)-complexes (of a fixed degree) do not form a family of expanders, and neither do their \( r \)-powers. This shows that any power operation on \( \tilde{A} \)-complexes whose links are similar to these spheres or to \( r \)-paths in them, do not form a family of high-dimensional expanders. We carry out the analysis for dimension two, but it is evident that similar phenomena occur in general dimension.

Proposition 4.1. The \( r \)-th spheres around a vertex in \( B = B_3(F) \) are not a family of expanders.

Proof. This can be deduced from the spectral analysis in Proposition 4.2, but we prefer to show how the geometry of the building gives an explicit sparse cut in the \( r \)-sphere \( S_r \) around a vertex. Let \( F, O, \pi, q, G, K \) be as in §3.2, and denote
\[ T = \{ \text{diag}(\pi^a, \pi^b, \pi^c) \mid a, b, c \in \mathbb{N}, \min(a, b, c) = 0 \}. \]
The subcomplex induced by the vertices \( \{ t v_0 \mid t \in T \} \) (where \( v_0 = K \in B(0) \)) is a triangular tiling of the Euclidean plane, called the fundamental apartment of \( B \). There is a simplicial retraction from \( B \) to this apartment, which corresponds to a decomposition \( G = \bigsqcup_{t \in T} B t K \); here \( B \) is the Iwahori group in \( PGL_3(F) \), which is the subgroup of elements in \( K \) with subdiagonal entries in \( \pi O \). In particular, each vertex in \( B \) lies in \( X_{a,b,c} := B t v_0 \) for a unique \( t = \text{diag}(\pi^a, \pi^b, \pi^c) \in T \). The \( r \)-sphere around \( v_0 \) is the preimage of the \( r \)-sphere in the fundamental apartment, which is a Euclidean hexagon: \( S_r = \bigsqcup_{\max(a,b,c)=r} X_{a,b,c} \) (see [13, §3.2]).

The size of \( X_{a,b,c} \) can be determined by computing Weyl lengths [18, §6.2]:
\[ |X_{a,b,c}| = \begin{cases} q^{2 \max(a,b,c)} & a \geq b \geq c \\ q^{2 \max(a,b,c)-1} & a \geq c > b \text{ or } b \geq a \geq c \\ q^{2 \max(a,b,c)-2} & b \geq c > a \text{ or } c \geq a \geq b \\ q^{2 \max(a,b,c)-3} & c > b > a, \end{cases} \]
so that for \( r \geq 1 \)
\[ |S_r| = q^{2r-3} (q^r + q + r - 1) (q^2 + q + 1) \approx (r + 1) q^{2r}. \]
Finally, the degrees of vertices in \( S_r \) (for \( r \geq 1 \)) are
\[ \deg(v \in X_{a,b,c}) = \begin{cases} q + 1 & |\{a, b, c\}| = 2 \\ 2q & |\{a, b, c\}| = 3. \end{cases} \]
Assume for simplicity that \( r \) is odd and larger than one (the computations are similar in the even case), and let \( A \subseteq S_r \) be the half sphere
\[
A = \bigcup \left\{X_{a,b,c} \mid \text{max}(a,b,c) = r \text{ and } \begin{cases} a \geq b \geq \frac{r+1}{2}, & b > a \geq c, \\ b > c > a, & \text{or } c > b \geq \frac{r+1}{2} \end{cases} \right\}.
\]
All edges crossing from \( A \) to \( S_r \setminus A \) connect either \( X_{r,\frac{r+1}{2}, 0} \) with \( X_{r,\frac{r-1}{2}, 0} \), or \( X_{0,\frac{r+1}{2}, r} \) with \( X_{0,\frac{r-1}{2}, r} \). Each vertex in \( X_{r,\frac{r+1}{2}, 0} \) has \( q \) neighbors in \( X_{r,\frac{r-1}{2}, 0} \) and \( q \) neighbors in \( X_{r,\frac{r+1}{2}, r} \), and similarly in the other case, giving
\[
\phi(S_r) \leq \frac{|E(A, S_r \setminus A)|}{\sum_{v \in A} \deg v} = \frac{q \left( |X_{r,\frac{r+1}{2}, 0}| + |X_{0,\frac{r+1}{2}, r}| \right)}{r (q^2 + q + 1) (q + 1) q^{2r-2}} = \frac{q^2 - q + 1}{q^2 + q + 1} \cdot \frac{1}{r} < \frac{1}{r}
\]
where \( \phi \) is the graph conductance (also known as the normalized Cheeger constant).

While \( S_r \) do not form an expander family as \( r \to \infty \), it is more interesting to ask whether the \( r \)-th power of \( S_r \) (as a graph) form together such a family, since when we take the \( r \)-sphere as an \( r \)-link, we should also expect edges in this link to correspond to \( r \)-paths. Denoting by \( \lambda_{(r)} \) the second normalized eigenvalue of \( S_r \), we have from the computation above and the discrete Cheeger inequality that \( \lambda_{(r)} \geq 1 - 2\phi(S_r) > 1 - \frac{2}{r} \), so that potentially we might have \( \lambda_{(r)}^{\rightarrow r\infty} e^{-2} < 1 \). With a finer analysis we can rule out this possibility:

**Proposition 4.2.** The normalized second eigenvalue \( \lambda_{(r)} \) of the \( r \)-th sphere \( S_r \subseteq \mathcal{B}_3 \) satisfies
\[
\lambda_{(r)} \geq \cos \left( \frac{2\pi}{r} \right) = 1 - \frac{2\pi^2}{r^2} + O \left( \frac{1}{r^4} \right).
\]
In particular, \( \lambda_{(r)}^{\rightarrow r\infty} 1 \), so the \( r \)-power graphs of the \( r \)-spheres in \( \mathcal{B}_3 \) are not expanders.

**Proof.** Let \( A \) be the adjacency operator on \( S_r \), and \( M \) its symmetric normalization \( M = D^{-1/2}AD^{-1/2} \) (where \( D \) is the diagonal operator of degrees in \( S_r \)). Let \( f : S_r \to \mathbb{R} \) be the function
\[
f(v) = \begin{cases} \sin \left( \frac{2\pi}{r} j \right) & v \in X_{r,j,0} \text{ with } 0 \leq j \leq r \\ 0 & \text{otherwise,} \end{cases}
\]
for which
\[
\langle f, f \rangle = \sum_{j=1}^{r-1} |X_{r,j,0}| \sin \left( \frac{2\pi}{r} j \right)^2 = q^2 \sum_{j=1}^{r-1} \sin \left( \frac{2\pi}{r} j \right)^2 = \frac{rq^{2r}}{2}.
\]
Since for \( 0 < j < r \) any \( x \in X_{r,j,0} \) has degree \( 2q \) with \( q \) neighbors in each of \( X_{r,j-1,0} \) and \( X_{r,j+1,0} \), and \( f \) vanishes elsewhere, we have \( D^{-1/2}f = \frac{f}{\sqrt{2q}} \), and
\[
\langle Mf, f \rangle = \langle AD^{-1/2}f, D^{-1/2}f \rangle = \frac{1}{2q} \langle Af, f \rangle
\]
\[
= \frac{1}{2q} \sum_{j=1}^{r-1} |X_{r,j,0}| \left( q \sin \left( \frac{2\pi(j-1)}{r} \right) + q \sin \left( \frac{2\pi(j+1)}{r} \right) \right) \sin \left( \frac{2\pi j}{r} \right) = \frac{rq^{2r}}{2} \cos \left( \frac{2\pi}{r} \right).
\]
The involution \( \tau : g \mapsto \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \) of \( \text{PGL}_3 \) induces an automorphism \( \tau \) of \( \mathcal{B}_3 \) which restricts to \( S_r \) and interchanges \( X_{r,j,0} \) and \( X_{r,j-1,0} \). Since \( f \) is \( \tau \)-antisymmetric (by construction) and the Perron-Frobenius eigenvector of \( M \) is \( \tau \)-symmetric (by connectedness of \( S_r \)), they are orthogonal, hence \( \lambda_{(r)} \geq \langle Mf, f \rangle = \cos \left( \frac{2\pi}{r} \right) \).

**Remark 4.3.** (1) If one can show that the bound in Proposition 4.2 is asymptotically tight, this would show that the \( r^2 \)-powers of the \( r \)-spheres in \( \mathcal{B} \) form a family of expanders (but with growing degrees).
(2) For small values of $r$, the exact values of $\lambda(r)$ are:

| $r$  | 1     | 2     | 3     |
|------|-------|-------|-------|
| $\lambda(r)$ | $\frac{\sqrt{7}}{q+1}$ | $\sqrt{\frac{1}{2} + \frac{\sqrt{7}}{2(q+1)}} \cdot \frac{(q+1)(2\sqrt{q^3+q^2+q-q^2-1})^{1/3}+q+1}{2(q+1)^{2/3}(2\sqrt{q^3+q^2+q-q^2-1})^{1/6}}$ |

$$\lim_{q \to \infty} \lambda(r) = 0 \quad \sqrt{\frac{1}{2}} \quad \sqrt{\frac{3}{4}}$$

Finding $\lambda(r)$ for general $r$ seems to be hard, but determining $\lim_{q \to \infty} \lambda(r)$ could be a nice problem.

5. Walks on geodesics and double samplers

In this section we describe a mixing random walk on the space of geodesics in a Ramanujan complex, and use it to construct double samplers. We first recall the definition of a sampler:

**Definition 5.1** (Sampler). A connected bipartite incidence graph $G(L \sqcup R, E)$ with $L = [n], R \subseteq \binom{[n]}{k}$ (and $E = \{(\ell, r) | \ell \in r\}$) is called an $f(\varepsilon, \alpha)$-**sampler** if for any $S \subseteq L$ and $\varepsilon > 0$

$$\frac{1}{|R|} \left\{ r \in R : \frac{|r \cap S|}{k} - \frac{|S|}{|L|} \geq \varepsilon \right\} \leq \frac{1}{f(\varepsilon, |S|/|L|)}.$$

Namely, a random element in $R$ samples well any “property” $S$ which may be assigned to the elements of $L$. It is a classical result that random walks on expanders sample well the vertices: Indeed, taking $L$ to be the vertex set of a regular $\lambda$-expander (expander with normalized nontrivial eigenvalues bounded by $\lambda$), and $R$ the set of all paths of length $k$ in it, one obtains an $f(\varepsilon, \alpha)$-sampler with

$$f(\varepsilon, \alpha) = e^{\pi^2 k (1-\lambda)/60}$$

(for a proof take [50, Thm. 3.2] with $f(v) = 1_{L(v) = |S|/|L|}$). A crucial point is that given a fixed expander, one can improve the sampling precision by taking longer and longer walks.

Double samplers were defined in [8], where they are used for studying PCP agreement tests and for a strong de-randomization of direct products tests. Roughly, a double sampler gives a way to sample well a set, and at the same time sample well the sampling sets themselves. It it not known whether this can be achieved from expander graphs - for example, whether longer walks on an expander graph (say of length $k^2$) sample short walks well (say, of length $k$).

**Definition 5.2** (Double sampler). A tripartite incidence graph $G(L \sqcup R \sqcup W, E_1 \sqcup E_2)$ with $L = [n], R \subseteq \binom{[n]}{k}, W \subseteq \binom{[n]}{w}$ (where $k \leq K, E_1 = \{(\ell, r) | \ell \in r\}$ and $E_2 = \{(r, w) | r \subseteq w\}$) is called a $\{

f(\varepsilon, \alpha), f'(\varepsilon, \alpha)\}$-double-sampler if $G(L \sqcup R, E_1)$ is an $f(\varepsilon, \alpha)$-sampler, and $G(R \sqcup W, E_2)$ is an $f'(\varepsilon, \alpha)$-sampler in the sense that for any $T \subseteq R$

$$\frac{1}{|W|} \left\{ w \in W : \frac{|\{r \in T | r \subseteq w\}|}{|T|/|R|} \geq \varepsilon \right\} \leq \frac{1}{f(\varepsilon, |T|/|R|)}.$$

Double samplers were constructed in [8] by taking $L$ to be the vertex set of an HD expander of dimension $K-1$, $R$ to be the cells of dimension $d-1$, and $W$ the cells of dimension $K-1$. The downside of this construction is that the sampling quality of $L$ depends on the dimension of the complex, and cannot be improved by taking longer walks as in the classic sampler construction.

We propose here a new approach for the double sampling problem, by designing a special walk on the space of geodesics $(d-1)$-cells in an $A_d$-complex. The upshot of our approach is that the quality of the sampler depends on the length of the walks performed and not on the dimension of the complex (which remains two dimensional). First we introduce a walk which is interesting in its own right:

**Definition 5.3.** The $r$-walk on an $A_d$-complex $X$ is the simple random walk on the set of $(d-1)$-cells of the geodesic $r$-power of $X$, where two cells are neighbors if they bound a joint $d$-cell (in the power complex).
Using the local-to-global technique, we obtain:

**Proposition 5.4.** The adjacency operator of the $r$-walk on an $\tilde{A}_d$-complex of density $q$ has normalized nontrivial eigenvalues bounded by $\frac{d}{\sqrt{r}} + \frac{2}{\sqrt{q}}$.

**Proof.** By Proposition 3.6 and Theorem 3.7, the links of codimension two in the power complex are either complete bipartite graphs or $\sqrt{q}$-expanders, and the claim follows from [30].

This shows that the $r$-walk on an $\tilde{A}_d$-complex of density $q > d^2(d + 1)^2$ samples well the geodesics $(d - 1)$-cells in it. The case which we will use for the double sampler construction is that of $d = 2$. There, the $r$-walk is carried on the (monochromatic) geodesics of length $r$ in a two-dimensional complex $X$, and two geodesics are neighbors if they share a joint triangle in the geodesic $r$-power of $X$ (see Figure 1.1). In this way, $K/k$ steps of the $k$-walk yield a long walk (of length $K$) which samples well the short walks (of length $k$) along geodesics. Indeed, applying the classical results on expander samplers (5.1) we obtain:

**Corollary 5.5.** Let $X$ be an $\tilde{A}_2$-complex of density $q \geq 37$. The incidence graph where $L$ are the $k$-geodesics of $X$ and $R$ are the $k$-walks of length $K/k$ in $X$ is a $\exp\left(e^2\left(\frac{1}{3} - \frac{2}{\sqrt{q}}\right)\frac{K}{\text{box}}\right)$-sampler.

Combining this with the results of §3.2 we arrive at a double sampler:

**Theorem 5.6.** Let $X$ be a non-tripartite Ramanujan $\tilde{A}_2$-complex of density $q \geq 37$. Taking $L$ to be the vertices of $X$, $R$ to be all $k$-geodesics in $X$, and $W$ to be all $k$-walks of length $K/k$ in $X$, yields a $\left(\frac{2\sqrt{q}}{\alpha}, \exp\left(e^2\left(\frac{1}{3} - \frac{2}{\sqrt{q}}\right)\frac{K}{\text{box}}\right)\right)$-double-sampler.

**Proof.** Observe that $L \cup R$ is the graph $G(k)$ of §3.2, hence $\lambda_1(G(k)) \approx kq^k$, $\lambda_2(G(k)) \approx \sqrt{k}q^k$ (where $\approx$ stands for a multiplicative error of $(1 + o(1))$ as $k, q \to \infty$). Let $S \subset L$ be of size $|S| \geq \alpha |L|$ and let $T = \{r \in R : |E(S, T)| \geq \alpha |L| \}$. Using $|R| \approx q^{2|L|}$ and the expander mixing lemma we obtain

$$|T|r(\alpha + \varepsilon) \leq |E(S, T)| \leq \lambda_1 |S||T| + \lambda_2 \sqrt{|S||T|} \approx k|S| |\alpha + \sqrt{k}q^k \alpha |L||T|,$$

so that $|T|r \leq \frac{|T|}{\alpha}$ as claimed. The expansion quality of $RLW$ is addressed in Corollary 5.5, with the difference that there the incidence relation is of membership, and here it is of containment. However, if $w = w_0, \ldots, w_K$ is a $k$-walk of length $K/k$ (so that each $w_{kj}, \ldots, w_{k(j+1)}$ is a $k$-geodesic), it follows from the definition of the $k$-walk that for each $1 \leq j \leq K/k - 1$ the vertices $w_{kj-1}, w_{kj}, w_{kj+1}$ form a triangle. Thus, $w$ contains no other $k$-geodesics, and the two relations agree (and in particular, $|\{r \in R : r \subseteq w\}| = K/k$).

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