METRIC CHARACTERIZATION OF CONNECTEDNESS FOR TOPOLOGICAL SPACES

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Abstract. Connectedness, path connectedness, and uniform connectedness are well-known concepts. In the traditional presentation of these concepts there is a substantial difference between connectedness and the other two notions, namely connectedness is defined as the absence of disconnectedness, while path connectedness and uniform connectedness are defined in terms of connecting paths and connecting walks, respectively. In compact metric spaces uniform connectedness and connectedness are well-known to coincide, thus the apparent conceptual difference between the two notions disappears. Connectedness in topological spaces can also be defined in terms of chains governed by open coverings in a manner that is more reminiscent of path connectedness. We present a metric formalism for connectedness which unifies all of the mentioned approaches to connectedness. The resulting connectedness criterion is valid for all topological spaces.

1. Introduction

The notions of connectedness and path connectedness for topological spaces are, of course, well-known and, while the concepts are related, their standard formulations are almost diametrically opposite each other, in the following sense. Path connectedness is a positive condition in that if a space is path connected, then for all pairs of points in it there exists a path between the two points. Connectedness is a negative condition in that if a space is connected, then no non-trivial clopen sets exist. An equivalent definition of connectedness in a non-empty topological space \( X \) is the following Čech-type formalism. Say that the points \( x \) and \( y \) are connected if for every open covering of \( X \) there exists a chain from \( x \) to \( y \), namely a finite sequence \( x_1, \ldots, x_n \), with \( x_1 = x \) and \( x_n = y \), such that for all \( 1 \leq i < n \) both \( x_i \) and \( x_{i+1} \) lie in the same open set of the covering. Then \( X \) is connected if any two points in it are connected (the same criterion can be given in a point-free fashion as well).

If \( X \) is a metric space, then open coverings of \( X \) correspond (roughly bijectively) to functions \( R: X \to (0, \infty) \) (every such function yields the

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covering \( \{ B_R(x) \mid x \in X \} \), and any open covering \( \{ U_a \} \) yields a function \( R \) by choosing, for each \( x \in X \), a radius \( R(x) > 0 \) such that \( B_R(x) \) is contained in some member of the covering. The above description of connectedness can then be given in terms of distances as follows. Say that two points \( x \) and \( y \) are connected if for any such function \( R \) there exists a walk from \( x \) to \( y \), i.e., a finite sequence \( x_1, \ldots, x_n \), with \( x_1 = x \) and \( x_n = y \), and such that, for all \( 1 \leq i < n \), either \( x_i \in B_R(x_{i+1}) \) or \( x_{i+1} \in B_R(x_i) \). Then \( X \) is connected if any two points in it are connected.

In the presence of a metric (or a uniform structure) a weaker notion of connectedness, namely that of uniform connectedness (or Cantor connectedness) is well-known. A metric space \( X \) is said to be uniformly connected if for all \( \varepsilon > 0 \) and for all \( x, y \in X \) there exists an \( \varepsilon \)-walk from \( x \) to \( y \), that is a sequence \( x_1, \ldots, x_n \in X \) such that \( x_1 = x \), \( x_n = y \), and \( d(x_{i+1}, x_i) \leq \varepsilon \), for all \( 1 \leq i < n \). It is immediate that if \( X \) is connected, then it is uniformly connected, and that for compact spaces, uniform connectedness implies connectedness. More generally, it is known that if \( X \) is a compact topological space with a compatible uniformity, then \( X \) is connected if, and only if, \( X \) as a uniform space is uniformly connected, i.e., the only uniformly continuous functions from \( X \) to a non-trivial discrete space are the constants functions.

Below, the above observations are strengthened to include non-compact spaces and a suitable generalization of metric spaces (that of Flagg’s continuity spaces) which encompass all topological spaces.

**Remark 1.1.** Notions of connectedness were investigated early on in the development of topology, e.g., for uniform, syntopogenous, and bitopological spaces in, respectively, [8, 9, 6], and surveyed in general in [5, 7]. The current article can be seen as a companion article to the former three.

**Remark 1.2.** Flagg’s [3] investigates continuity spaces in a broad context. However, it appears that most further work utilizing continuity spaces is largely in the context of domain theory. Recent work, namely [10, 11, 2] investigates the applicability of continuity spaces in the context of topology. This article is another result in this direction.

Section 2 briefly introduces continuity spaces, which we call \( V \)-valued metric spaces, and recounts the results used in Section 3 where the metric connectedness criterion is introduced and investigated.

2. Continuity spaces

Continuity spaces, introduced by Flagg ([3]) following on ideas of Kopperman ([1]), are a generalization of metric spaces where the codomain of
the metric function is a value quantale, rather than the particular value quantale $[0, \infty]$ of non-negative reals. A value quantale is a complete lattice $V$, with the bottom element denoted by 0 and meet denoted by $\land$, together with an associative and commutative binary operation $+$ such that the conditions

- $a + \land S = \land(a + S)$, for all $a \in V$ and $S \subseteq V$ (where $a + S = \{a + s \mid s \in S\}$)
- $a + 0 = a$ for all $a \in V$
- $a = \land\{b \in V \mid b \succ a\}$
- $a \land b \succ 0$ for all $a, b \in V$ with $a, b \succ 0$

hold. Here the meaning of $a \prec b$, or $b \succ a$, is that $b$ is well above $a$, i.e., that for all $S \subseteq V$, if $a \geq \land S$, then there exists $s \in S$ with $b \geq s$. It is straightforward that

- if $a \prec b$, then $a \leq b$
- if $a \leq b \prec c$ or $a \prec b \leq c$, then $a \prec c$
- $0 = \land\{\varepsilon \in V \mid \varepsilon \succ 0\}$
- if $a \succ 0$, then there exists $\varepsilon \succ 0$ with $a \not\prec \varepsilon$.

For the proofs of the following two properties in a value quantale refer, respectively, to [3, Theorem 1.6, Theorem 2.9].

- if $a \prec \varepsilon$, then there exists $b \in V$ with $a \prec b \prec c$
- if $\varepsilon \succ 0$, then there exists $\delta \succ 0$ with $2\delta \leq \varepsilon$ (where $2\delta = \delta + \delta$).

A continuity space or a $V$-valued metric space is a triple $(V, X, d)$ where $V$ is a value quantale, $X$ is a set, and $d: X \times X \to V$ is a function satisfying $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$. With every $V$-valued metric space $X$ one may associate the open ball topology, the one generated by the open balls $B_\varepsilon(x) = \{y \in X \mid d(x, y) \prec \varepsilon\}$, where $x \in X$ and $\varepsilon \succ 0$. Every topological space is metrizable in the sense that if $(X, \tau)$ is a topological space, then there exists a value quantale $V$ and a $V$-valued metric structure on $X$, such that the induced open ball topology is precisely $\tau$ ([3, Theorem 4.15]).

In [10] it is shown that this construction extends functorially to an equivalence $\text{Met} \to \text{Top}$, as follows. Given value quantales $V$ and $W$ and metric spaces $(V, X, d)$ and $(W, Y, d)$, declare a function $f: X \to Y$ to be continuous if for all $x \in X$ and $\varepsilon \succ 0_W$, there exists $\delta \succ 0_V$ such that $d(fx, fy) \leq \varepsilon$ for all $y \in Y$ with $d(x, y) \leq \delta$. $\text{Met}$ is then the category whose objects are $(V, X, d)$ where $V$ ranges over all value quantales and $X$ is a $V$-valued metric space. The morphisms in $\text{Met}$ are the continuous functions. Flagg’s construction of the open ball topology extends functorially to $\mathcal{O}: \text{Met} \to \text{Top}$,
with \( O(V, X, d) \) sent to \( X \) with the open ball topology, and \( O(f) = f \) for all continuous functions \( f \). It then holds that \( O \) is an equivalence of categories. In light of this equivalence \( V \)-valued metric spaces with their continuous functions can be taken as models for topology. We thus make no distinction between a topological space and a \( V \)-valued metric space. Every \( V \)-valued metric space is silently endowed with the open ball topology, and every topological space is automatically assumed to come with a \( V \)-valued metric structure inducing its topology.

The following useful property is \([3, \text{Theorem 4.6}]\). Given \( x \in X \) and \( S \subseteq X \) in a \( V \)-valued metric space \( X \), let \( d(x, S) = \bigwedge_{s \in S} d(x, s) \). With respect to the open ball topology, a set \( C \subseteq X \) is closed if, and only if, \( d(x, C) = 0 \) implies \( x \in C \), for all \( x \in X \).

3. Metric connectedness

Given a \( V \)-valued metric space \( X \), consider a function \( R: X \to V \) with the only restriction being that \( R(x) > 0 \) for all \( x \in X \) (which we abbreviate to \( R > 0 \)). An \( R \)-step (or simply a step, if \( R \) is understood) is an ordered pair \((x, y) \in X \times X\) such that either one of the conditions

- \( d(x, y) < R(x) \)
- \( d(y, x) < R(y) \)

holds. An \( R \)-walk (or simply a walk) is a finite sequence \( x_1, \ldots, x_n \) of points in \( X \) such that \((x_i, x_{i+1})\) is a step for all \( 1 \leq i < n \). Such a walk is said to \( R \)-connect (or simply to connect) \( x_1 \) and \( x_n \), which are then said to be \( R \)-connected, and we write \( x_1 \sim_R x_n \), clearly an equivalence relation. \( X \) is said to be \( R \)-connected if any two of its points are \( R \)-connected. We say that two points \( x, y \in X \) are connected if \( x \sim_R y \) for all \( R: X \to V \) with \( R > 0 \), and we then write \( x \sim y \), again an equivalence relation (indeed, \( \sim = \bigcap_{R>0} \sim_R \)).

**Definition 3.1.** A non-empty \( V \)-valued metric space \( X \) is connected if \( X \) is \( R \)-connected for all \( R > 0 \). Equivalently, \( X \) is connected if any two of its points are connected.

**Remark 3.2.** If a step is instead defined to be a pair \((x, y)\) such that one of

- \( d(x, y) \leq R(x) \)
- \( d(y, x) \leq R(y) \)

holds, with the rest of the concepts above unchanged, then the final notion of connectedness is the same as above. To see that, note first that \( a < b \)
follows that any \( R \) \( \epsilon \) walk under the new definition. For the converse, note that for any \( \epsilon > 0 \) there is a \( \delta \) with \( 0 < \delta < \epsilon \), and then if \( a \leq \delta \), then \( a \leq \epsilon \). Given \( R: X \to V \) with \( R > 0 \) let \( R': X \to V \) with \( 0 < R'(x) < R(x) \), for all \( x \in X \). It then follows that any \( R' \)-walk under the new definition is an \( R \)-walk under the old one. We will choose whichever alternative to use based on convenience.

Given \( R: X \to V \) with \( R > 0 \) and \( z \in X \), let \( C_z^R = \{ x \in X \mid z \sim_R x \} \), the \( R \)-connected component of \( z \). Obviously, \( z \in C_z^R \).

**Proposition 3.3.** In a \( V \)-valued metric space \( X \), given any \( R: X \to V \) with \( R > 0 \), and \( z \in X \), the \( R \)-connected component \( C_z^R \) is clopen.

**Proof.** Using the first definition of step, the first step condition implies that if \( x \in C_z^R \), then \( B_{R(x)}(x) \subseteq C_z^R \), and thus \( C_z^R \) is open. By the second step condition, if \( y \in X \setminus C_z^R \), then \( B_{R(y)}(y) \subseteq X \setminus C_z^R \), and thus \( C_z^R \) is closed.

**Theorem 3.4.** Let \((V, X, d)\) be a non-empty \( V \)-valued metric space and \( O \) its associated open ball topology. Then \((V, X, d)\) is connected if, and only if, \((X, O)\) is connected.

**Proof.** Assume that \((X, O)\) is connected and let \( R: X \to V \) with \( R > 0 \) be given. For an arbitrary \( x \in X \) the \( R \)-connected component \( C_x^R \) is clopen and thus \( C_x^R = X \). In other words, \( x \sim y \) for all \( x, y \in X \).

Assume now that \((X, O)\) is not connected and let \( C \subseteq X \) be a non-trivial clopen subset. For every \( x \in C \) there exists \( \epsilon_x > 0 \) with \( B_{\epsilon_x}(x) \subseteq C \). Let \( R(x) = \epsilon_x \). For every \( y \in X \setminus C \) we have that \( d(y, C) > 0 \), and thus there exists \( \delta_y > 0 \) with \( d(y, C) \leq \delta_y \). Let \( R(y) = \delta_y \). Fix \( x_0 \in C \) and \( y_0 \in X \setminus C \), and suppose that \( x_0 \sim_R y_0 \). There is then a walk from \( x_0 \) to \( y_0 \), and consider in it a step \((x, y)\) with \( x \in C \) and \( y \in X \setminus C \) (clearly such a step must exist). We must then have either \( d(x, y) < R(x) \) or \( d(y, x) < R(y) \). However, \( d(x, y) < R(x) = \epsilon_x \) implies \( y \in B_{\epsilon_x}(x) \subseteq C \), a contradiction. Similarly \( d(y, x) < R(y) \) implies \( d(y, x) \leq \delta_y \), and thus \( d(y, C) \leq d(y, x) \leq \delta_y \), another contradiction.

**Definition 3.5.** Let \( X \) be a \( V \)-valued metric space and let \( z \in X \). The **connected component** of \( x \) is the set \( C_z = \{ x \in X \mid z \sim x \} \).

Noting that \( C_z = \bigcap_{R>0} C_z^R \), and seeing that \( C_z^R \) is clopen, it follows immediately that \( C_z \) is closed (but not necessarily open). Moreover, \( C_z \) is obviously a maximal connected subset of \( X \), and every maximal connected subset of \( X \) is the connected component of any one of its members.
Remark 3.6. It is interesting to note the similarly between $R$-connectedness and path-connectedness, and to note that some of the qualitative differences between $R$-connectedness and connectedness (e.g., that $R$-connected components are clopen while connected components are closed) are attributed to the extra quantification over $R > 0$ required for connectedness. Numerous (and perhaps all) of the fundamental facts about connectedness can be proven using the metric formalism presented above. Sometimes the proof is shorter, other times it may be a bit longer, but always the proof employs arguments strongly reminiscent of path connectedness arguments, and the proof is, arguably, closer to one’s natural intuition of connectedness. The reader is invited to contemplate the perspective offered by the above on connectedness at the fundamental level of introductory topology.

As stated in the introduction, for ordinary metric spaces, i.e., for symmetric $[0, \infty]$-valued metric spaces, connectedness implies uniform connectedness, and under compactness the two notions coincide. We now establish the analogous result for general $V$-valued metric spaces, and thus for all topological spaces.

Given $\varepsilon > 0$ and a $V$-valued metric space $X$, one may consider the constant function $R_{\varepsilon} : X \to V$, $x \mapsto \varepsilon$. We say that $X$ is $\varepsilon$-connected if it is $R_{\varepsilon}$-connected, and we say that $X$ is uniformly connected if $X$ is $\varepsilon$-connected for all $\varepsilon > 0$.

**Theorem 3.7.** Let $X$ be a non-empty $V$-valued metric space. If $X$ is connected, then it is uniformly connected, and the converse holds if $X$ is compact.

*Proof.* If $X$ is connected, then it is $R$-connected for all $R > 0$, certainly then also for all $R_{\varepsilon}$, $\varepsilon > 0$. Assume now that $X$ is compact and uniformly connected. Let $R > 0$ be given. For each $x \in X$ let $R'(x) > 0$ satisfying $2R'(x) \leq R(x)$, and consider the open covering $\{B_{R'(x)}(x) \mid x \in X\}$, from which we may extract a finite subcovering, say with centers $x_1, \ldots, x_n$, and let $\varepsilon = R'(x_1) \land \cdots \land R'(x_k)$. Suppose now that $d(x, y) \leq \varepsilon$ (cf. Remark 3.2). There is then $1 \leq k \leq n$ with $x \in B_{R'(x_k)}(x_k)$. We then have $d(x_k, x) \prec R'(x_k)$, and thus $d(x_k, y) \leq d(x_k, x) + d(x, y) \leq R'(x_k) + R'(x_k) \leq R(x_k)$. It now follows that $(x, x_k, y)$ is an $R$-walk. A similar observation holds if $d(y, x) \leq \varepsilon$. We thus showed that any $\varepsilon$-step $(x, y)$ may be split into a 2-step $R$-walk, and thus any $\varepsilon$-walk can be refined to an $R$-walk without altering the end points. As $X$ is assumed uniformly connected, it follows that it is connected. □
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References

[1] J. Bruno and I. Weiss. Metrix axioms: a structural study. Topology proceedings (in press).
[2] A. Chand and I. Weiss. Completion of continuity spaces with uniformly vanishing asymmetry. Topology and its Applications (submitted).
[3] R. C. Flagg. Quantales and continuity spaces. Algebra Universalis, 37(3):257–276, 1997.
[4] R. Kopperman. All topologies come from generalized metrics. Amer. Math. Monthly, 95(2):89–97, 1988.
[5] H. Lord. Connectednesses and disconnectednesses. In Papers on general topology and applications (Slippery Rock, PA, 1993), volume 767 of Ann. New York Acad. Sci., pages 115–139. New York Acad. Sci., New York, 1995.
[6] H. Lord. Connectedness and disconnectedness in bitopology. In Papers on general topology and applications (Gorham, ME, 1995), volume 806 of Ann. New York Acad. Sci., pages 265–278. New York Acad. Sci., New York, 1996.
[7] H. Lord. Connectednesses and disconnectednesses. II. In Proceedings of the Eighth Prague Topological Symposium (1996), pages 216–250 (electronic). Topol. Atlas, North Bay, ON, 1997.
[8] S. G. Mrówka and W. J. Pervin. On uniform connectedness. Proc. Amer. Math. Soc., 15:446–449, 1964.
[9] J. L. Sieber and W. J. Pervin. Connectedness in syntopogenous spaces. Proc. Amer. Math. Soc., 15:590–595, 1964.
[10] I. Weiss. A note on the metrizability of spaces. Algebra Universalis (in press).

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