A New Path to Code-based Signatures via Identification Schemes with Restricted Errors

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Abstract

In this paper we introduce a variant of the Syndrome Decoding Problem (SDP), that we call Restricted SDP (R-SDP), in which the entries of the searched vector are defined over a subset of the underlying finite field. We prove the NP-completeness of R-SDP, via a reduction from the canonical SDP, and describe how information set decoding algorithms can be adapted to solve this new problem. We study the properties of random codes under this new decoding perspective (in the fashion of traditional coding theory results), in order to derive the conditions upon which R-SDP has a unique solution with overwhelming probability. As a concrete application, we describe how Zero-Knowledge Identification (ZK-ID) schemes based on SDP can be tweaked to rely on R-SDP, and show that this leads to compact public keys as well as significantly reduced communication costs. Thus, these schemes offer an improved basis for the construction of code-based digital signature schemes derived from identification schemes through the well-known Fiat-Shamir transformation.

1 Introduction

Public-key cryptography heavily relies on hard mathematical problems, that define the security target for the scheme by tying the system’s private key to the public one, or a plaintext to the corresponding ciphertext. Among the mathematical problems utilized in the context of public-key cryptography, one of the most studied is that of decoding random linear codes in the Hamming metric, which was proved to be NP-complete [3, 7] and boasts a vast literature of algorithms aimed at finding a
solution \([4, 8, 17, 21, 22, 25, 29, 32]\). Roughly speaking, this problem asks to find a vector of low Hamming weight, i.e., containing only a few non-null entries, such that its product with a given parity-check matrix returns a target vector, called syndrome. For this reason, the problem is known as Syndrome Decoding Problem (SDP).

For decades, SDP has represented the foundation of the area called code-based cryptography. Indeed, it allows building efficient and secure cryptosystems, which are largely inspired by the seminal work of McEliece [26]. However, the situation is not the same for signature schemes, and the vast majority of attempts to build a code-based signature scheme that is at the same time secure and efficient were unsuccessful. This is the case, for instance, of the CFS scheme [12] and its variants, which have tried to address the problem of decoding a random-like syndrome into a low-weight vector in several ways, but always yielding to either unpractical performance or even security breaches. The recent work of [10] highlights the fact that, over non-binary finite fields, the decoding problem remains hard also when the solution is required to have very high Hamming weight, as opposed to very low. Wave [15] is a signature scheme built upon this fact and, therefore, fills a gap, as it manages to build a secure CFS-like scheme with acceptable performance.

Constructing signatures via the Fiat-Shamir transform applied to a Zero-Knowledge Identification (ZK-ID) scheme is as a promising alternative. However, schemes such as Stern’s [33] feature non-trivial soundness errors and require many repetitions, leading to very large signature sizes, while attempts to translate the Schnorr-Lyubashevsky approach [23], such as [27], have been shown vulnerable to attacks based on statistical analysis [16, 30]. Nevertheless, the scheme proposed by Cayrel, Véron and El Yousfi Alaoui [11], denoted as CVE, has been shown suitable for obtaining very fast signature schemes with a reduction in the key size in the order of 25% over Stern’s [18].

Our contribution We introduce a variant of the SDP, in which the solution vector must take values over a restricted set of the finite field in which both the given code and the syndrome are defined. For this reason, we denote the corresponding decoding problem as Restricted SDP (R-SDP), and prove its NP-completeness through a reduction from the classical SDP in the Hamming metric, over the same finite field. We then focus on a special case of the R-SDP, in which the error vector lives in \(\{0, \pm1\}\), and use classical arguments from coding theory to derive conditions under which, for a random code, the solution of the problem is unique with overwhelming probability. Furthermore, we extend known approaches for solvers in the Hamming metric to our new problem, and derive tight estimates for their complexity. As a culminating development of our work, we revisit the CVE scheme with a new formulation based on restricted errors. Our results show that, using such an approach, we can obtain a noticeable performance improvement by significantly reducing the communication cost and the public key size, while preserving the security level.
Outline of the paper  The paper is organized as follows. In Section 2 we introduce the notation that will be used throughout the paper and recall some coding theory notions. In Section 3 we recall the CVE scheme, which we use as the starting point of our variant. In Section 4, we introduce the concept of restricted error vectors, studying their properties from a coding-theoretic point of view. In particular, we prove the NP-completeness of the restricted syndrome decoding problem and provide adaptations of known Information Set Decoding (ISD) algorithms to this case. In Section 5, we adapt the CVE scheme to our new framework, and compare its performance to that of schemes in the existing literature. Finally, we draw some concluding remarks in Section 6.

2  Preliminaries

In this section we introduce the notation that will be used throughout the paper, and provide some basic notions from coding theory.

2.1  Notation

We denote by $[a; b]$ the set of integers between $a$ and $b$ including $a$ and $b$. As usual, $F_q$ will denote the finite field with $q$ elements, where $q$ is a prime power, while $F_q^* = F_q \setminus \{0\}$ will denote the multiplicative group of $F_q$. We will use bold upper case (resp. lower case) letters to denote matrices (resp. vectors). For a matrix $A$, we refer to its entry in the $i$-th row and $j$-th column as $a_{i,j}$, and for a vector $a$ we denote its $i$-th entry by $a_i$. The $k \times k$ identity matrix is denoted by $I_k$. Let $S_n$ be the symmetric group on $n$ elements, where we will represent elements $\sigma$ of $S_n$ as bijections from the integer set $[0; n-1]$ to itself, and the action on a length $n$ vector is represented as

$$\sigma(a) = (a_{\sigma(0)}, a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}).$$

We will use $M_n$ to denote the set of monomial transformations, i.e., all linear transformations that can be represented through the action of a permutation and non-zero scaling factors. In other words, for each $\tau \in M_n$, there exist $\sigma \in S_n$ and $v \in (F_q^*)^n$ such that

$$\tau(a) = (v_{\sigma(0)} a_{\sigma(0)}, v_{\sigma(1)} a_{\sigma(1)}, \cdots, v_{\sigma(n-1)} a_{\sigma(n-1)}), \quad \forall a \in F_q^n.$$

We will use $\mathcal{U}(A)$ to denote the uniform distribution over a set $A$; for a random variable $a$, we will write $a \sim D$ if $a$ is distributed according to the distribution $D$, and $a \xleftarrow{\$} A$ if $a$ is sampled according to the uniform distribution over $A$, i.e., $a \sim \mathcal{U}(A)$.

The support of a vector $a \in F_q^n$ is defined as $\text{Supp}(a) = \{j \in [0; n-1] \mid a_j \neq 0\}$. For a set $J \subset [0; n-1]$ and a vector $a \in F_q^n$, we denote by $a_J$ the vector consisting of the entries of $a$ indexed by $J$. Analogously, for a set $J \subset [0; n-1]$ and a matrix $A \in F_q^{k \times n}$, we denote by $A_J$ the matrix consisting of the columns of $A$ indexed by $J$. 

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2.2 Coding theory preliminaries

Here we briefly recall some basic notions for codes in the Hamming metric.

**Definition 1.** An $[n, k]$ linear code $C$ over $\mathbb{F}_q$ is a linear subspace of $\mathbb{F}_q^n$ of dimension $k$.

Any linear code can be represented by a generator matrix, which has the code as image, or equivalently by a parity-check matrix, which has the code as kernel.

**Definition 2.** The **Hamming weight** of $x \in \mathbb{F}_q^n$ is equal to the size of its support, i.e.,

$$\text{wt}_H(x) := |\text{Supp}(x)| = |\{j \in [0; n-1] \mid x_j \neq 0\}|.$$

The **Hamming distance** between $x$ and $y \in \mathbb{F}_q^n$ is defined as the Hamming weight of their difference, i.e.,

$$d_H(x, y) := \text{wt}_H(x - y) = |\{i \in [0; n-1] \mid x_i \neq y_i\}|.$$

The minimum Hamming distance of a code is the minimum of all pairwise non-zero Hamming distances of the codewords.

The sphere of vectors of $\mathbb{F}_q^n$ with Hamming weight $\omega$ is denoted by $S_{H,n,q,\omega}$ and the ball of radius $\omega$ is denoted by $B_{H,n,q,\omega}$.

In the following proposition we recall the Gilbert-Varshamov bound in the Hamming metric.

**Proposition 3** (Theorem 13.73, [6]). Let $q$ be a prime power, $n$ and $d_H$ be positive integers. There exists a linear code $C$ over $\mathbb{F}_q$ of length $n$ and minimum Hamming distance $d_H$, such that

$$|C| \geq \frac{q^n}{\sum_{j=0}^{d_H-1} \binom{n}{j} (q-1)^j}.$$

**Definition 4.** Let $q$ be a prime power and $0 < k \leq n$ be positive integers. For a code over $\mathbb{F}_q$ of length $n$ and dimension $k$, the Gilbert-Varshamov distance is defined as follows

$$d_{GV} := \max \left\{ d_H \left| \sum_{i=0}^{d_H-1} \binom{n}{i} (q-1)^i < q^{n-k} \right. \right\}.$$

It is well known [24, Ch. 17, Problem (31)] that random codes in the Hamming metric asymptotically attain the Gilbert-Varshamov distance with overwhelming probability.

3 Zero Knowledge Identification Schemes based on syndrome decoding

In this section we focus on the CVE scheme [11]. This scheme will be described in detail in Section 3.2, since it is the scheme to which we apply our new technique.
(see Section 5). For the sake of completeness, in the appendix we also provide a description of a scheme by Aguilar, Gaborit and Schrek [1], which we will denote by AGS and whose performance will be compared to that of the scheme we propose.

To begin, we briefly recall the operating principles of such schemes, highlighting their main features, and then, in the following sections, we show how switching to restricted errors leads to a strong boost in their performance.

3.1 General principles

Let \( R \) be a relation which is satisfied only by specific pairs of objects, such that checking whether a pair of elements satisfies the relation is efficient (i.e., it can be done in polynomial time). An identification scheme constructed upon \( R \) can be defined as a two-stage procedure, as follows:

- in the first stage, the prover randomly generates a pair \((sk, pk)\) satisfying \( R \);
- in the second stage, the prover exchanges messages with the verifier, which is only equipped with \( pk \), with the goal of demonstrating knowledge of \( sk \). At the end of the protocol, the verifier decides whether to accept the prover or not.

Usually, the key pair is such that \( pk \) represents an instance of a hard problem, with \( sk \) being a valid solution. Thus, the difficulty of finding, on input \( pk \), a value \( sk^* \) that satisfies the relation, without knowledge of \( sk \), is at the core of the scheme, since authentication is obtained through the proof of knowledge about the secret key.

An identification scheme is called zero-knowledge if no information about the secret key is revealed during the identification process. Other required properties for an identification scheme are completeness and soundness, the former meaning that an honest prover always gets accepted and the latter requiring that an impersonator has only a small probability of getting accepted. For a rigorous and complete description of these properties, we refer the interested reader to [20].

Reducing the communication cost  A crucial quantity to analyze in an identification scheme is the communication cost, i.e., the cost of a full interaction between the two parties, which is measured as the number of bits that are exchanged. Many identification schemes are constructed from Sigma protocols, i.e., three-pass proofs of knowledge for a certain relation. In this case, the scheme presents a soundness error, meaning that an adversary impersonating a prover (without access to the secret key) can “cheat” by pre-selecting a candidate response that works only for a subset of the challenge space, and hope that the chosen challenge is part of that subset. This implies that an impersonator is able to get accepted with a certain non-zero probability (e.g., 2/3 or 1/2), which depends on the scheme. It follows that, in order to achieve an acceptable level of authentication, the protocol is repeated several times, and a prover is accepted only if every instance was answered successfully. If the cheating probability is \( \eta \), executing \( N \) rounds of the protocol leads to an overall authentication level of \( \eta^N \), and \( N \) is chosen to achieve the desired
value (e.g., $2^{-128}$). To illustrate the process, we will analyze the case of the CVE scheme, that is the main focus of our work. In the scheme, each round is based on the following paradigm:

1. the prover prepares two commitments $c_0$, $c_1$, which are obtained on the base of some randomness;
2. the two commitments are sent to the verifier; after this exchange, some additional messages may be exchanged between the two parties;
3. the verifier randomly picks $b \in \{0, 1\}$, and sends it to the prover;
4. the prover provides information that only allows to verify $c_b$, but not $c_{b+1}$;
5. the verifier checks validity of $c_b$.

When the protocol is repeated for multiple rounds, it is possible to reduce the overall communication cost by exploiting the compression technique proposed in [1]. For the sake of completeness, we summarize this procedure as depicted in Figure 1. Before the 0-th round the prover generates the commitments for all the $N$ rounds, and then sends a unique hash value $c = \text{Hash}(c_0^0, c_1^0, \ldots, c_{N-1}^0, c_{N-1}^1)$ to the verifier. In the $i$-th round, after receiving the challenge $b$, the prover sets its response $f$ such that the verifier can compute $c_i^b$, and additionally includes $c_i^{b+1}$. At the end of each round, the verifier uses $f$ to compute $c_i^b$, and stores it together with $c_i^{b+1}$. After the final round only, the verifier is thus able to check validity of the initial commitment $c$, by computing the hash of all the stored $c_0^b$, $c_1^b$. This way, one hash is sent at the beginning of the protocol, and only one hash (instead of two) is transmitted in each round: this way, the number of exchanged hash values reduces from $2N$ to $N + 1$. For the sake of clarity, this compression technique will not be included in the description of the forthcoming schemes, but we remark that this can be applied, with slight modifications, to all the schemes we analyze in the rest of the paper.

3.2 The CVE scheme

The CVE scheme [11] is an improvement of Stern’s [33] and Véron’s [34] identification schemes, both based on the hardness of decoding a random binary code [7]. The scheme relies on non-binary codes over a large finite field. With this choice, the cheating probability for a single round is reduced to $2^{-\frac{1}{2q}}$, where $q$ is the finite field size. For the sake of completeness, the CVE scheme is summarized in Figure 2. We now briefly recall how the communication cost of this scheme is derived [11, Section 4.2]. We first note that, in order to represent a length-$n$ vector of weight $\omega$ over $\mathbb{F}_q$, we can either use the full vector, or just consider its support, together with the ordered non-zero entries. The first option requires $n \ceil{\log_2(q)}$ bits, while for the second one we need $\omega(\ceil{\log_2(n)} + \ceil{\log_2(q - 1)})$ bits. Depending on the particular parameters $n$, $\omega$ and $q$, we can then use $\psi(n, q, \omega) = \min\{n \ceil{\log_2(q)}, \omega(\ceil{\log_2(n)} + \ceil{\log_2(q - 1)})\}$ bits. Furthermore, objects that have been randomly generated (such as the monomial transformations) can be compactly represented by sending the seed that has been used as input of the pseudorandom generator. Taking all of this
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Generate $c_0^i$, $c_1^i$, for $i = 0, \cdots, N - 1$

Set $c = \text{Hash}(c_0^0, c_1^0, \ldots, c_0^{N-1}, c_1^{N-1})$

Repeat single round for $N$ times

Check validity of $c$

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**GENERIC $i$-th ROUND**

Exchange additional messages

Choose $b \leftarrow \{0, 1\}$.

Set $f := \text{information to compute } c_b^i$

Store $c_b^i$, compute and store $c_b^i$

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Figure 1: Description of the compression technique for $N$ rounds.

reasoning and the compression technique into account, and denoting with $l_{\text{Hash}}$ and $l_{\text{Seed}}$ the length of hash values and seeds, respectively, for $N$ rounds of the protocol we get the following average communication cost:

$$l_{\text{Hash}} + N\left(\lceil \log_2(q - 1) \rceil + n \lceil \log_2(q) \rceil + 1 + l_{\text{Hash}} + \frac{\psi(n, q, \omega) + l_{\text{Seed}}}{2}\right).$$

For the maximum communication cost, we take the maximum size of the response, and thus obtain

$$l_{\text{Hash}} + N\left(\lceil \log_2(q - 1) \rceil + n \lceil \log_2(q) \rceil + 1 + l_{\text{Hash}} + \max\{\psi(n, q, \omega), l_{\text{Seed}}\}\right).$$

To derive secure parameters for the CVE scheme, one can proceed as follows. Given $n$, $r$ and $q$, the Gilbert-Varshamov bound is used to estimate the minimum distance $d_H$ of the code whose parity-check matrix is $H$. The weight of the secret key can then be set as $\omega = \lceil d_H/2 \rceil$, since this guarantees that there is no other vector of weight smaller than or equal to $\omega$ with syndrome equal to the public key. To reach a security of $\lambda$ bits, $\omega$ must be sufficiently large, such that using ISD requires a number of operations that is not lower than $2^\lambda$. The authors of [11] have used the analysis due to Peters [28] to estimate the ISD complexity, and have proposed two parameters sets:

- $q = 256$, $n = 128$, $k = 64$, $\omega = 49$, for 87-bits security;
- $q = 256$, $n = 208$, $k = 104$, $\omega = 78$, for 128-bits security.
Public Data Parameters $q, n, r, \omega \in \mathbb{N}$, parity-check matrix $\mathbf{H} \in \mathbb{F}_q^{r \times n}$

Private Key $\mathbf{e} \in \mathbb{S}_n^q, \omega$

Public Key $\mathbf{s} = \mathbf{eH}^\top \in \mathbb{F}_q^r$

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| Choose $\mathbf{u} \overset{\$}{\leftarrow} \mathbb{F}_q^n$, $\tau \overset{\$}{\leftarrow} \mathbb{M}_n$ | $\mathbf{c}_0, \mathbf{c}_1 \overset{\$}{\leftarrow} \mathbb{F}_q$ |
| Set $\mathbf{c}_0 = \text{Hash}(\tau, \mathbf{uH}^\top)$ | Choose $z \overset{\$}{\leftarrow} \mathbb{F}_q^*$ |
| Set $\mathbf{c}_1 = \text{Hash}(\tau(\mathbf{u}), \tau(\mathbf{e}))$ | $\mathbf{y} \overset{\$}{\leftarrow} \mathbb{F}_q$ |
| If $b = 0$, set $f := \tau$ | Choose $b \overset{\$}{\leftarrow} \{0, 1\}$ |
| If $b = 1$, set $f := \mathbf{e}' = \tau(\mathbf{e})$ | $\overset{\$}{\leftarrow} f$ |
| $\overset{\$}{\leftarrow}$ If $b = 0$, accept if $\mathbf{c}_0 = \text{Hash}(\tau, \tau^{-1}(\mathbf{yH}^\top - zs))$ | If $b = 1$, accept if $\text{wt}_H(\mathbf{e}') = \omega$ and $\mathbf{c}_1 = \text{Hash}(\mathbf{y} - z\mathbf{e}', \mathbf{e}')$ |

Figure 2: The CVE scheme.

4 Decoding random codes with restricted errors

In this section we introduce a variant of the decoding problem, in which the error vector is constrained to take values in a subset of the finite field; for this reason, we speak of restricted errors. In particular, we show that the associated decoding problem is NP-complete, and adapt the Gilbert-Varshamov bound to this case. We also study the complexity of solving the decoding problem with restricted errors, through the adaptation of ISD algorithms for the Hamming case.

4.1 NP-completeness of R-SDP

In this section we introduce a new variant of the decoding problem, by choosing a set of vectors $\mathbf{B}$ over $\mathbb{F}_q$, with $q$ being an odd prime power, whose Hamming weight is below some threshold value and, as an additional restriction, takes values in a specific
subset of the finite field. For the elements of the field, we adopt a representation \( F^* = \{ x_1 = 1, x_2, \cdots, x_{q-1} \} \), such that \( x_i + x_{q-1} = 0 \), for \( i \in [0; \frac{q-1}{2}] \). We will call this representation for the finite field elements “symmetric”; it is evident that, for the same finite field, many symmetric representations may exist. Note that, when \( q \) is a prime, the canonical symmetric representation is \( F^*_q = \{ 1, 2, \cdots, q-1 \} \).

For a positive integer \( a \leq \frac{q-1}{2} \), that we will call the restriction parameter, we define the restricted Hamming ball of radius \( t \) and parameter \( a \) as

\[
E^{(a)}_{n,q,t} := \{ e \in F_q^n | \text{wt}_H(e) \leq t, \ e \in \{ 0, \pm x_1, \cdots, \pm x_a \}^n \}.
\]

Hence, we are always interested in a subset of \( F_q \) of size \( 2a + 1 \) where, for each element, the set also contains its additive inverse. We formally define the syndrome decoding problem for a restricted Hamming ball over an arbitrary finite field as follows.

**Problem 1. Restricted Syndrome Decoding Problem (R-SDP)** Let \( q = p^m \), with \( p \neq 2 \) being a prime and \( m \in \mathbb{N} \), and denote with \( F_q \) the corresponding finite field with \( q \) elements, described through a symmetric representation. On input \( H \in F_r \times F^n_q \), \( s \in F_q^r \) and \( t \in \mathbb{N} \), decide whether there exists an \( e \in E^{(a)}_{n,q,t} \), such that \( eH^\top = s \).

The above problem is obtained by applying an additional restriction to the classical SDP over a finite field \( F_q \). It may thus seem intuitive that, like SDP, R-SDP belongs to the hierarchical class of NP-complete problems. A formal proof of this property is provided in the following theorem. Note that, in our proof, the restriction parameter \( a \) is not treated as an input to R-SDP, since we can prove the NP-completeness for any fixed value of \( a \).

**Theorem 5.** R-SDP is NP-complete.

**Proof.** We provide a reduction from the classical SDP in the Hamming metric, defined over \( F_q \), which has been proved to be NP-complete in [3]. Formally, this problem is defined as follows:

Given \( H \in F_q^{r \times n} \), \( s \in F_q^r \) and \( t \in \mathbb{N} \), decide whether there exists \( e \in F_q^n \), with Hamming weight smaller than or equal to \( t \), such that \( eH^\top = s \).

Clearly, the finite field representation does not interfere with the definition of the problem; thus, we directly consider a symmetric representation for the field. From now on, for the sake of simplicity, we will denote \( F^{(a)}_q = \{ 0, \pm x_1, \cdots, \pm x_a \} \).

We denote by \( \{ H, s, t \} \) an arbitrary instance of SDP, and map it into an R-SDP instance, which we denote by \( \{ H', s, t \} \). If \( q = 3 \), then we can set \( H' = H \); otherwise, we construct \( H' \) according to the following procedure. We first select a set \( U \subseteq F_q^* \), such that every element of \( F_q^{(a)} \) can be obtained as the product of one element from \( F_q^{(a)} \) and one from \( U \), that is

\[
\forall b \in F_q^* \ \exists u \in U, \ x \in F_q^{(a)} \text{ such that } -xu = b \text{ or } xu = b.
\]
It is easily seen that, for all possible sets $F_q^{(a)}$, a choice for $U$ always exists and for its cardinality, which we denote with $v$, it is straightforward to show that $\frac{q-1}{2a} \leq v \leq q - 1$. It may also happen that, for the same element of $F_q^*$, more than one pair of factors from $U$ and $F_q^{(a)}$ exists but, for our purposes, this is not an issue. Let $u = (u_0, \ldots, u_{v-1})$ be a vector formed by the elements of $U$: we finally obtain $H'$ as

$$H' = \left[ u \otimes h_0, \ u \otimes h_1, \ldots, \ u \otimes h_{n'-1} \right],$$

where $\otimes$ denotes the Kronecker product and $h_i$ is the $i$-th column of $H$. This way, $H'$ has $r$ rows and $n' = nv$ columns, given by

$$h_{iv+j} = u_j h_i, \ i \in [0; n-1], \ j \in [0; v-1].$$

We first show that, to each $e' \in (F_q^{(a)})^{n'}$ such that $e' H^T = s$, we can associate a vector $e \in F_q^n$ such that $e H^T = s$. In fact, consider that, $\forall i \in [0; n-1]$

$$\sum_{j=0}^{v-1} e'_{iv+j} h'_{iv+j} = \left( \sum_{j=0}^{v-1} e'_{iv+j} u_j \right) h_i = \beta_i h_i,$$

where $\beta_i \in F_q$. On the other hand, it is easily seen that, if $e'_{iv+j} = 0$ for all $j \in [0; v-1]$, then $\beta_i = 0$ as well. For the other cases, it might happen that $\beta_i \neq 0$. The number of coefficients $\beta_i$ that are non-zero is surely smaller than or equal to $\text{wt}_H(e')$ and, if $s \neq \mathbf{0}_r$, then at least one of the $\beta_i$ must be zero. Let $e \in F_q^n$, such that $e_i = \beta_i \neq 0$: it is then immediately seen that $e H^T = s$, and $\text{wt}_H(e) \leq \text{wt}_H(e') \leq t$. If $e' \in E_{n',q,t}^{(a)}$, then $e$ has weight smaller than or equal to $t$: existence of an $e'$ satisfying the constructed R-SDP instance, then, implies existence of an $e$ satisfying the initial SDP instance.

Finally, we show that $e' \in E_{n',q,t}^{(a)}$ exists if and only if a desired $e$ exists, i.e., if the initial SDP instance is a “yes” instance. In fact, as a consequence of the requirements on $U$, for each $\beta_i$ we can always write

$$\beta_i = \lambda_i u_{\ell_i} x_{y_i},$$

for proper indices $\ell_i \in [0; v-1]$, $y_i \in [1; a]$ and $\lambda_i \in \{ \pm 1 \}$. Then, we can always build a vector $e' \in E_{n',q,t}^{(a)}$ having at most one entry among the ones in positions $\{ iv, \cdots, iv + v - 1 \}$, whose elements are defined as

$$e'_{iv+j} = \begin{cases} 
\lambda_i x_{y_i} & \text{if } j = \ell_i, \\
0 & \text{elsewhere}.
\end{cases}$$

This vector has the same weight as $e$, and is such that $e H^T = e' H^T$. Given the bijection between $e$ and $e'$, it becomes clear that solving the given R-SDP instance means solving the initial SDP instance. It follows that R-SDP is NP-complete. □
We have a two-fold motivation to study R-SDP, arising from our interest in identification schemes based on syndrome decoding. First, in these schemes, the proof of knowledge is provided by publishing a masked version of the secret key, which is a low-weight vector over $\mathbb{F}_q$. As already stressed in the previous sections, a crucial quantity to study the performance of an identification scheme is the communication cost per round. Vectors with restricted entries can clearly be represented with a lower number of bits, compared to vectors over the full field $\mathbb{F}_q$. Thus, we expect the performance of these schemes to benefit from the use of restricted vectors. For this reason, from now on we limit our attention to the case of $a = 1$, which leads to the lowest communication cost, since we impose the maximal non-trivial restriction to the subset $B$.

Secondly, these schemes are parametrized by choosing the weight of the secret error vector as high as possible, but not larger than half the minimum distance of the code. This guarantees the uniqueness of the solution to SDP, for which the currently-known best solvers are ISD algorithms. For a random code, the minimum Hamming distance is estimated via the Gilbert-Varshamov bound, which only depends on the code length, dimension and on the finite field size. We generalize all these concepts to the case of restricted vectors, and derive conditions to guarantee that, for a given random code, only a unique solution to the R-SDP exists. Our results show that, for random codes with the same parameters, we can achieve higher security levels by relying on R-SDP instead of SDP. As we show more extensively in Section 5, our results lead to strong improvements in the performances of already existing identification schemes.

4.2 Restricted minimum distance and properties of random codes

In this section we study the properties of random linear codes regarding the weight of codewords with restricted entries. As mentioned before, we will focus on the case $a = 1$ and, to ease the notation, we will simply use $E_{n,q,t} := E_{n,q,t}^{(1)}$ to denote the ball of vectors with entries in $\{0, \pm 1\}$ and with Hamming weight not larger than $t$. Our goal is to derive conditions upon which, for a given code over $\mathbb{F}_q$, with $q \geq 5$ being an odd prime power, R-SDP has at most a unique solution with overwhelming probability.

Definition 6. Let $q \geq 5$ be an odd prime power, and denote with $\mathbb{F}_q$ the corresponding finite field with $q$ elements, described with a symmetric representation. For $a \in \mathbb{F}_q^n$, we define the restricted pre-weight of $a$ as

$$\tilde{\text{wt}}(a) := \begin{cases} \#_1(a) + 2 \cdot \#_2(a) & \text{if } a \in \{0, \pm 1, \pm 2\}^n, \\ \infty & \text{otherwise}, \end{cases}$$

where $\#_1(a)$ is the number of entries of $a$ equal to $\pm 1$ and $\#_2(a)$ is the number of entries equal to $\pm 2$.

Note that the restricted pre-weight is not a weight, since it does not verify the triangular inequality. On the other hand note that, since $q$ is a prime, the restricted pre-weight of a vector over $\{0, \pm 1, \pm 2\}$ corresponds to its Lee weight.
Definition 7. Let \( q \geq 5 \) be an odd prime power, and denote with \( \mathbb{F}_q \) the corresponding finite field with \( q \) elements, described with a symmetric representation. Let \( C \) be a linear code with length \( n \) over \( \mathbb{F}_q \). We define its restricted minimum distance as

\[
\tilde{d} := \min \left\{ \tilde{wt}(c) \mid c \in C \cap \{0, \pm1, \pm2\}^n \setminus \{0_n\} \right\}.
\]

If \( C \cap \{0, \pm1, \pm2\}^n = \{0_n\} \), then we set \( \tilde{d} = \infty \).

Note that the restricted minimum distance of \( C \) is equal to the minimum Lee distance of \( C \cap \{0, \pm1, \pm2\}^n \). Its importance is stated in the following theorem.

Theorem 8. Let \( q \geq 5 \) be an odd prime power, and denote with \( \mathbb{F}_q \) the corresponding finite field with \( q \) elements, described with a symmetric representation. Let \( C \) be a code over \( \mathbb{F}_q \), with length \( n \) and restricted minimum distance \( \tilde{d} \). For any parity-check matrix \( H \) for \( C \), and for all \( t < \tilde{d}/2 \), there cannot exist two distinct vectors \( e, e' \in E_{n,q,t} \) such that \( eH^\top = e'H^\top \).

Proof. If \( eH^\top = e'H^\top \), then necessarily \( \tilde{e} = e - e' \in C \). Furthermore, given that \( e, e' \in \{0, \pm1\}^n \), we have \( \tilde{e} \in \{0, \pm1, \pm2\}^n \) and hence \( \tilde{wt}(\tilde{e}) \leq 2t \). This contradicts the fact that the restricted minimum distance of \( C \) is \( \tilde{d} > 2t \).

Note that, if a code has no codewords (apart the zero one) living in \( \{0, \pm1, \pm2\}^n \), then its restricted minimum distance is infinite and all vectors over \( \{0, \pm1\} \) correspond to distinct syndromes.

Analogously to the Hamming metric, we define a ball of radius \( t \) and center \( a \in \mathbb{F}_q^n \) as the set of all vectors whose difference with \( a \) has restricted pre-weight smaller than or equal to \( t \), that is

\[
\widetilde{B}(a, t, q, n) := \left\{ x \in \mathbb{F}_q^n \mid \tilde{wt}(x - a) \leq t \right\}.
\]

The volume of each such ball does not depend on its center but only on its radius, hence we get the following.

Proposition 9. Let \( q \geq 5 \) be an odd prime power, and \( n \) and \( t \leq 2n \) be positive integers. Then the size of a restricted ball in \( \mathbb{F}_q^n \) of radius \( t \), as defined in (1), is given by

\[
\widetilde{V}(n, t) := \sum_{i=0}^{t} \sum_{j=\max\{0, i-n\}}^{\lfloor i/2 \rfloor} \binom{n}{j} \binom{n-j}{i-2j} 2^{i-j}.
\]

Note that this is the same as the size of the Lee ball of radius \( w \) in \( \mathbb{F}_q^n \), which can be found in [36, Proposition 8, Corollary 9].

In the following theorem we derive a bound which, in the same fashion of the Gilbert-Varshamov bound, states the minimal maximum dimension for a code achieving a given restricted minimum distance.
**Theorem 10.** For a given finite restricted minimum distance $\tilde{d}$ and length $n$, there exists a code in $\mathbb{F}_q^n$ of dimension $\tilde{k}$, where

$$\tilde{k} \geq n - 1 - \log_q \left( \tilde{V}(n, \tilde{d} - 1) \right)$$

$$= n - 1 - \log_q \left( \sum_{i=0}^{\tilde{d}-1} \sum_{j=\max\{0,i-n\}}^{\lfloor i/2 \rfloor} \binom{n}{j} \binom{n-j}{i-2j} 2^{i-j} \right).$$

**Proof.** Let $C \subseteq \mathbb{F}_q^n$ with restricted minimum distance $\tilde{d}$ and maximal dimension $\tilde{k}$. In the following we show that for every vector $x \in \mathbb{F}_q^n$, there must be at least an $\alpha \cdot x \in \mathbb{F}_q^*$ and a codeword $c$ in $C$ such that $\tilde{w}_t(\alpha \cdot x - c) \leq \tilde{d} - 1$ or, analogously, $\alpha \cdot x - c \in \tilde{B}(c, \tilde{d} - 1, q, n)$. In other words, for each $x \in \mathbb{F}_q^n$, at least one of its scalar multiples is contained in the ball of radius $\tilde{d} - 1$ with center in a codeword of $C$. Note that, if $x \in C$, then this is trivially true since it is enough to consider a ball with center in $x$; to show that this holds for all vectors in the space, we consider $x \not\in C$. If there is no pair $(\alpha \cdot x, c)$ satisfying the above requirement, then we can consider a new code $C'$, defined as

$$C' = \{ \beta \cdot x + c \mid \beta \in \mathbb{F}_q^*, \ c \in C \}.$$

Such a code will be linear and will have dimension $\tilde{k} + 1$. Furthermore, its restricted minimum distance will not be lower than that of $C$, because by hypothesis all “new” codewords $\beta \cdot x + c$, with $\beta \neq 0$, are outside of balls of radius $\tilde{d} - 1$ centered in codewords of $C$. However, the existence of $C'$ contradicts the fact that, by hypothesis, $C$ has maximum dimension: thus, it must be

$$\forall x \in \mathbb{F}_q^n \exists \alpha \cdot x \in \mathbb{F}_q^*, \ c \in C \ s.t. \ \alpha \cdot x \in \tilde{B}(c, \tilde{d} - 1, q, n).$$

Let $A$ be the set of all valid vectors $\alpha \cdot x$: for a given $x$, we include all the scalar multiples such that the above condition is verified. It is easily demonstrated that

$$q^{n-1} \leq |A| \leq q^n.$$

In fact, $A \subseteq \mathbb{F}_q^n$ proves the upper bound, while the lower bound is obtained by assuming that for each $x$, only one scalar multiple satisfies the condition. Given that $A = \bigcup_{c \in C} \tilde{B}(c, \tilde{d} - 1, q, n)$, we consider the following chain of inequalities

$$q^{n-1} \leq |A| = \left| \bigcup_{c \in C} \tilde{B}(c, \tilde{d} - 1, q, n) \right| \leq \sum_{c \in C} |\tilde{B}(c, \tilde{d} - 1, q, n)| = q^\tilde{k} \tilde{V}(n, \tilde{d} - 1).$$

Simple further computations yield the claimed inequality. \hfill \square

Starting from the previous theorem, in the same fashion of the commonly called Gilbert-Varshamov distance for the Hamming metric, we define the restricted Gilbert-Varshamov minimum distance.
Definition 11. For a code with length $n$ and dimension $k$ over $\mathbb{F}_q$, with $q \geq 5$ being an odd prime power, we define the restricted Gilbert-Varshamov distance as follows

$$\tilde{d}_{GV} := \begin{cases} \infty & \text{if } \tilde{V}(n, 2n) < q^{n-k-1}, \\ \max \{ \phi > 0 \mid \tilde{V}(n, \phi) \leq q^{n-k-1} \} & \text{otherwise.} \end{cases}$$

Analogously to other formulations (such as the classical Gilbert-Varshamov distance), the restricted Gilbert-Varshamov distance tells us the maximum distance that a code of fixed length and rate can achieve. Note that $\tilde{V}(n, 2n) = 5^n$, and hence for large values of $q$ and/or $n$ (if $q > 5$), the restricted Gilbert-Varshamov distance is equal to $\infty$, as long as $k < n \left(1 - \log_q(5) - \frac{1}{2}\right)$. Thus, it makes sense to study the probability with which a random code achieves this restricted minimum distance; in the next theorem we show that this probability is asymptotically equal to $1$.

Theorem 12. Let $q > 5$ be an odd prime power and $k \leq n \left(1 - \log_q(5) - \epsilon\right)$, for $0 < \epsilon < 1 - \log_q(5)$. Let $G \stackrel{\$}{\leftarrow} \mathbb{F}_q^{k \times n}$ with rank $k$. Then the code generated by $G$ has restricted minimum distance $\tilde{d} = \infty$ with probability at least $1 - q^{-\epsilon n}$.

Proof. First, the requirements on $k$ and $\epsilon$ assure that $0 < k < n$. The code generated by $G$ will have restricted minimum distance $\tilde{d} = \infty$ if all of its codewords (apart from the zero one) do not live in $\{0, \pm 1, \pm 2\}^n$. We first focus on a single codeword: since $G$ is random over $\mathbb{F}_q$, each linear combination of its rows is a random length-$n$ vector over $\mathbb{F}_q$, as well. Thus, the probability that this codeword has a finite restricted pre-weight (i.e., it takes values in $\{0, \pm 1, \pm 2\}$) is

$$\frac{|\{0, \pm 1, \pm 2\}^n \setminus 0_n|}{|\mathbb{F}_q^n \setminus 0_n|} = \frac{5^n - 1}{q^n - 1} < \frac{5^n}{q^n} = q^{-n\left(1 - \log_q(5)\right)}.$$ 

Since there are $q^k$ codewords, from a union bound argument we get that the probability that the code contains at least a codeword of restricted pre-weight smaller than or equal to $2n$ is at most

$$q^k q^{-n\left(1 - \log_q(5)\right)} \leq q^n \left(1 - \log_q(5) - \epsilon\right) q^{-n\left(1 - \log_q(5)\right)} = q^{-\epsilon n}.$$ 

Then, $q^{-\epsilon n}$ is an upper bound on the probability that the code generated by $G$ contains at least a codeword with finite pre-weight; taking its complement, we obtain a lower bound on the desired probability.

4.3 Solving R-SDP

To solve the R-SDP, one can rely on known ISD algorithms, with slight modifications if necessary; as in the previous section, we focus on the case $a = 1$. Remember that
we want to solve the following problem: given the parity-check matrix $H \in \mathbb{F}_q^{(n-k)\times n}$ of a linear code of length $n$ and dimension $k$, $s \in \mathbb{F}_q^{n-k}$ and $t \in \mathbb{N}$, find $e \in \{0, \pm 1\}^n$ such that $\text{wt}_H(e) \leq t$ and $eH^\top = s$. Therefore, we assume $q \geq 3$. Looking at the practical application we will consider in the next section, we here derive a complexity estimate for the case in which

$$s \xleftarrow{\$} \left\{ eH^\top \mid e \in \{0, \pm 1\}^n, \text{wt}_H(e) = t \right\},$$

where $t < \tilde{d}/2$ and $\tilde{d}$ is the restricted minimum distance of the code described by $H$.

In other words, we consider the particular case in which there is exactly one vector, with known weight $t$, whose syndrome through $H$ is $s$.

Since the error vector is restricted to $\{0, \pm 1\}^n$, the ISD algorithms can be adapted from the ternary case, where the Hamming and the Lee metric coincide. We consider an adaption of Lee-Brickell’s algorithm [21] to the restricted case. Note that we could also use an adaptation of [9], where the authors make a similar partitioning of the error vector and, instead of sampling all possible vectors and checking for collisions, they solve the smaller instances as a subset sum problem via Wagner’s algorithm [35]. An important observation in [9] is that SDP in the ternary case becomes harder for large weights (see [9, Figure 1]) which is also the case in our proposal.

Now, the vectors we are interested in have large Hamming weight, i.e., larger than $n - k$; this means that we cannot adapt the ISD algorithm of Prange, which relies on the assumption that there are no errors in the information set. Also, a zero window like in Stern’s algorithm, where there are no errors, is not useful in the extreme case, where $t = n$. Note that if one assumes that the zero window is of size zero, then Stern’s ISD algorithm and Lee-Brickell’s ISD algorithm coincide.

Thus, we want to adapt Lee-Brickell’s ISD algorithm. As a setup we have $H \in \mathbb{F}_q^{(n-k)\times n}$, $s \in \mathbb{F}_q^{n-k}$ and the positive integers $0 \leq t \leq n, k$ and $0 \leq v \leq \min\{k, t\}$. We want to find $e \in \{0, \pm 1\}^n$ with $\text{wt}_H(e) = t$. We assume that, within an information set $I$, the error vector $e$ has Hamming weight $v$ and outside $I$ the remaining Hamming weight $t - v$.

As a first step we bring the parity-check matrix into systematic form, by multiplying with some invertible matrix $U$. For simplicity, but without loss of generality, we assume that the information set $I$ is given in the first $k$ positions and we denote its complement by $J$. Then we get the following situation:

$$eH^\top U^\top = \begin{pmatrix} e_I & e_J \end{pmatrix} \begin{pmatrix} A \\ 1_{n-k} \end{pmatrix} = sU^\top,$$

for some $A \in \mathbb{F}_q^{k \times (n-k)}$. From this we get the following condition

$$e_J A + e_J = sU^\top.$$

Hence we will go through all vectors $e_J \in \{0, \pm 1\}^k$ having Hamming weight $v$ and define $e_J$ such that the above condition is verified. Then the algorithm will check
whether each entry of $e_J$ is in the restricted subset $\{0, \pm 1\}$ and if $e_J$ has the remaining Hamming weight $t - v$.

In Algorithm 1 we provide the formal ISD algorithm by Lee-Brickell adapted to the case of restricted error vectors.

**Algorithm 1:** Lee-Brickell’s Algorithm over $\mathbb{F}_q$ with restricted error vector

We now provide the cost of the adapted algorithm by Lee-Brickell.

**Theorem 13.** Algorithm 1 requires

$$\binom{k}{v}^{-1} \binom{n-k}{t-v}^{-1} \binom{n}{t} \left( (n-k)(n+1)[\log_2(q)]^2 + \binom{k}{v} \frac{2 q}{2} (t-v+1)[\log_2(q)] \right)$$

binary operations.

**Proof.** As a first step we compute $\begin{pmatrix} H^\top \\ s \end{pmatrix} U^\top$, which requires

$$(n-k)(n+1)[\log_2(q)]^2$$

binary operations. As a second step we go through all $e_I \in \{0, \pm 1\}^k$ having Hamming weight $v$, which are $\binom{k}{v} 2^v$ many, and compute $s' - e_I A$. The computation of $e_I A$, i.e., adding or subtracting $v$ rows of $A$, requires $(v-1)(n-k)$ additions. By subtracting $s'$ we get $(n-k)$ more additions.

Since the algorithm only proceeds if the resulting vector is in $\{0, \pm 1\}^{n-k}$ and if it has Hamming weight $t - v$, we can use the concept of *early abort*. According to this concept, we compute one entry of the resulting vector and check if the conditions are verified; if not, we abort the computation. We may assume that each entry of the resulting vector is uniformly distributed over $\mathbb{F}_q$; for this, note that $s$, as well as $e_I A$, are uniformly distributed as they are obtained by adding and subtracting rows of $H$ and $A$ which are assumed to be completely random. Hence, the probability of
one entry to be in \( \{ \pm 1 \} \) is \( \frac{2}{q} \). Thus we expect to abort after \( \frac{2}{q}(t - v + 1) \) computations of one entry, since then we should have exceeded the target Hamming weight \( t - v \).

In total, one iteration requires a number of binary operations equal to

\[
(n - k)^2(n + 1)[\log_2(q)]^2 + \binom{n}{v}2^v q \left( \frac{2}{q}(t - v + 1)v[\log_2(q)] \right).
\]

The success probability is the same as for Lee-Brickell’s algorithm over the Hamming metric, i.e.,

\[
\binom{k}{v} \binom{n - k}{t - v} \frac{1}{(n - k) !}.
\]

Then, the estimated overall cost of Lee-Brickell’s ISD algorithm is given as in the claim.

In the particular case in which \( \text{wt}(e) = n \), that is, when \( \tilde{\text{wt}}(e) = n \), each chosen information set contains exactly \( k \) non-zero entries; in such a case, one iteration of the algorithm with parameter \( v = k \) will always return the valid error vector, with an overall complexity given by

\[
C = (n - k)^2(n + 1)[\log_2(q)]^2 + 2^k q \left( \frac{2}{q}(n - k + 1)k[\log_2(q)] \right).
\]

### 4.4 R-SDP with multiple solutions

In the previous section we have described the complexity of solving R-SDP, in the regime in which a unique solution exists. On the contrary, when multiple solutions exist, the computational effort to solve R-SDP presumably gets reduced; at a first glance, similarly to the analysis in [31], one expects that if \( M \) solutions exist, then the complexity gets reduced by a factor of \( M \), with respect to the case in which there is a unique solution. However, unless \( M \) does not grow exponentially, the expected gain in solving R-SDP is quite limited. Roughly speaking, the existence of multiple solutions does not imply that the problem becomes easy, because even finding one of such solutions may be hard; note that the security of WAVE [15] is based on an analogous consideration.

In particular, we consider here the case of vectors with restricted pre-weight equal to \( n \) and derive an approximation of the probability that, given \( s = eH^\top \) with \( \tilde{\text{wt}}(e) = n \), there is no other vector \( e' \neq e \) of restricted pre-weight equal to \( n \), such that \( e'H^\top = s \).

**Proposition 14.** Let \( q \geq 3 \) and \( C \subseteq \mathbb{F}_q^n \) be a random code of length \( n \), dimension \( k \), with parity-check matrix \( H \in \mathbb{F}_q^{(n - k) \times n} \). Let \( s = eH^\top \), for \( e \in \{ \pm 1 \}^n \); if \( 2^q(n - k) \ll 1 \), the probability that there is no other vector in \( \{ \pm 1 \}^n \) and syndrome equal to \( s \) can be approximated as

\[
1 - 2^{-n((1 - R) \log_2(q) - 1)},
\]

where \( R = k/n \) is the code rate.
Proof. The set of vectors \( \{ \pm 1 \}^n \) contains \( 2^n - 1 \) elements different from \( e \). Since \( H \) is random, we can model each product \( e' H^\top \), for \( e' \in \{ \pm 1 \}^n \), as a random vector of length \( n - k \) over \( \mathbb{F}_q \). Thus, the probability that \( e' \neq e \) does not have syndrome \( s \) can be obtained as

\[
1 - \frac{1}{q^{n-k}}.
\]

Assuming that, for all such vectors \( e' \), the syndromes are well modeled as random and uncorrelated vectors from \( \mathbb{F}_q^{n-k} \), the probability that the syndrome of \( e \) is unique is obtained as

\[
\left(1 - q^{-(n-k)}\right)^{2^n-1} \approx \left(1 - q^{-(n-k)}\right)^{2^n}.
\]

If \( 2^n q^{-(n-k)} \ll 1 \), we can approximate this probability with

\[
1 - 2^n q^{-(n-k)} = 1 - 2^{-n(1-R) \log_2(q)-1}.
\]

\( \square \)

Note that even for codes such as those not fulfilling the conditions of Theorem 12, we may still expect uniqueness of the solution to R-SDP with high probability, according to the previous proposition.

5 Identification schemes based on R-SDP

In this section we show how to redesign existing ZK-ID schemes based on the hardness of syndrome decoding, in order to rely on R-SDP. In particular, we focus on the CVE scheme, and rewrite it using the decoding problem we have introduced; our proposed protocol is reported in Figure 3. For the sake of clarity, we have reported the procedure for the case of a single round of communication, but we remark that it is always possible to take advantage of the compression technique described in detail in Figure 1, when multiple rounds are considered.

In the protocol we use again \( E_{n,q,t} \) to denote the sphere of vectors in \( \{0, \pm 1\}^n \) with restricted pre-weight \( t \), and we denote by \( \mathcal{M}_n \subseteq \mathcal{M}_n \) the set of monomial transformations whose scaling factors are only \( \pm 1 \). Finally, we choose \( q, n, k, t \) such that Theorems 8 and 12, or Proposition 14 assure uniqueness of the solution to the R-SDP with high probability. Note that the only modifications, with respect to the CVE scheme, are in the fact that we are restricting the secret key and the monomial transformations. By doing this, we base the security of the protocol on the hardness of the R-SDP, which we have proven to be NP-complete in Section 4.1. A formal security analysis of the proposed scheme is provided next.

5.1 Security

We now show that our protocol satisfies all properties required for a zero-knowledge identification scheme. Note that our proofs are very similar to those in [11], from which our scheme is adapted.
Public Data Parameters $q, n, r, t \in \mathbb{N}$, parity-check matrix $H \in \mathbb{F}_q^{r \times n}$

Private Key $e \in E_{n,q,t}$

Public Key $s = eH^\top \in \mathbb{F}_q^r$

PROVER

Choose $u \leftarrow \mathbb{F}_q^n$, $\tau \leftarrow \mathbb{M}_{n}$

Set $c_0 = \text{Hash}(\tau, uH^\top)$

Set $c_1 = \text{Hash}(\tau(u), \tau(e))$

$\overset{(c_0,c_1)}{\longrightarrow}$

Choose $z \leftarrow \mathbb{F}_q^*$

$\overset{z}{\longleftarrow}$

Set $y = \tau(u + ze)$

$\overset{y}{\longrightarrow}$

Choose $b \leftarrow \{0,1\}$

$\overset{b}{\longleftarrow}$

If $b = 0$, set $f := \tau$

If $b = 1$, set $f := e' = \tau(e)$

$\overset{f}{\longrightarrow}$

If $b = 0$, accept if $c_0 = \text{Hash}(\tau, \tau^{-1}(y)H^\top - zs)$

If $b = 1$, accept if $\tilde{w}(e') = t$

and $c_1 = \text{Hash}(y - ze', e')$

VERIFIER

Choose $\tau \leftarrow \mathbb{M}_{n}$

Set $c_0 = \text{Hash}(\tau, uH^\top)$

Set $c_1 = \text{Hash}(\tau(u), \tau(e))$

$\overset{(c_0,c_1)}{\longrightarrow}$

Choose $z \leftarrow \mathbb{F}_q^*$

$\overset{z}{\longleftarrow}$

Set $y = \tau(u + ze)$

$\overset{y}{\longrightarrow}$

Choose $b \leftarrow \{0,1\}$

$\overset{b}{\longleftarrow}$

If $b = 0$, accept if $c_0 = \text{Hash}(\tau, \tau^{-1}(y)H^\top - zs)$

If $b = 1$, accept if $\tilde{w}(e') = t$

and $c_1 = \text{Hash}(y - ze', e')$

Figure 3: Our adaptation of the CVE scheme to restricted error vectors.

Completeness It is easy to show that an honest prover is always successfully verified. In fact, if $b = 0$, we have

$$\tau^{-1}(y)H^\top - zs = (u + ze)H^\top - zs = uH^\top,$$

and therefore

$$\text{Hash}(\tau, \tau^{-1}(y)H^\top - zs)$$

matches the commitment $c_0$. Similarly, if $b = 1$, we have that

$$\tilde{w}(e') = \tilde{w}(\tau(e)) = \tilde{w}(e) = t$$

and

$$y - ze' = \tau(u) + z\tau(e) - ze' = \tau(u).$$

It follows that $\text{Hash}(y - ze', e')$ matches the commitment $c_1$, and therefore both conditions are verified.
Zero-Knowledge To prove this property, we construct a simulator $S$, modelled as a probabilistic polynomial-time algorithm, that uses a dishonest verifier $A$ as a subroutine. The goal of such a simulator is to produce a communication record that is indistinguishable from one which would be obtained through an honest execution of the protocol. Note that, since our scheme is a 5-pass protocol, $A$ has two strategies for his attack, corresponding to the two interactions with the prover. In the first strategy, which we call $ST_0$, $A$ takes as input the prover’s commitments $c_0, c_1$ and produces a value $z \in \mathbb{F}_q^*$. In the second strategy, which we call $ST_1$, $A$ takes as input both the commitments $c_0, c_1$ and the first response $y$, and generates a challenge $b \in \{0, 1\}$.

The simulator is constructed as follows. First, it picks a random challenge $b \xleftarrow{} \{0, 1\}$; then:

- if $b = 0$, choose uniformly at random $u$ and $\tau$, then find a vector $\hat{e}$ such that $\hat{e}H^\top = s$. No limitation is placed on the restricted value of $\hat{e}$; so, this can be accomplished by simple linear algebra. Generate the commitments by setting $c_0 = \text{Hash}(\tau, uH^\top)$ and picking $c_1$ as a random string of the proper length. Call on $A$ with input $c_0, c_1$; $A$ will apply $ST_0$ and return a value $z \in \mathbb{F}_q^*$. Compute $y = \tau(u + z\hat{e})$ and call on $A$ again with input $c_0, c_1, y$; this time, $A$ will apply $ST_1$ and respond with a bit $b$.

- if $b = 1$, choose again $u$ and $\tau$ uniformly at random, then pick a random vector $\hat{e}$ of the correct restricted value $t$. Generate the commitments by picking $c_0$ as a random string of the proper length, and setting $c_1 = \text{Hash}(\tau(u), \tau(\hat{e}))$. As before, call on $A$ with input $c_0, c_1$; $A$ will apply $ST_0$ and return $z \in \mathbb{F}_q^*$. Compute again $y = \tau(u + z\hat{e})$ and call on $A$ with input $c_0, c_1, y$ to obtain the bit $b$.

At this point, the simulator has two options. If $b = b$, the simulator halts and produces the communication consisting of $c_0, c_1, z, y, b$ and $f$; otherwise, it restarts the procedure. Note that all the objects comprising the record are distributed uniformly at random. Therefore, on an average of $2N$ rounds, the record produced by $S$ is indistinguishable from one which would be produced in an honest execution over $N$ rounds, as we conjectured.

Soundness We now analyze the cheating probability of an adversary, in this case a dishonest prover $A$. We show that such an adversary has a cheating probability that is asymptotically (in $q$) close to $1/2$. To this end, we show that $A$ can behave in one of two ways, depending on what is the expected challenge value. In the first case, which we call $ST_0$, assume without loss of generality that $A$ is preparing to receive the challenge $b = 0$. Then $A$ will choose $u$ and $\tau$ uniformly at random, and find a vector $\hat{e}$ such that $\hat{e}H^\top = s$, without any limitation on the restricted value. Commitments are generated by setting $c_0 = \text{Hash}(\tau, uH^\top)$ and picking $c_1$ as a random string of the proper length. Thus, $A$ is able to successfully answer the challenge $b = 0$, regardless of the value $z$ chosen by the verifier. In fact, the value $y = \tau(u + z\hat{e})$ and the response $f = \tau$ computed by $A$ are enough to pass the
verification, since the restricted value of $\hat{e}$ is not checked.

In the second case, which we call $ST_1$, $A$ is instead prepared to receive the challenge $b = 1$. In this case, $A$ will choose again $u$ and $\tau$ uniformly at random, then he will pick a random vector $\hat{e}$ of the correct restricted weight $t$. Commitments are generated by picking $c_0$ as a random string of the proper length, and setting $c_1 = \text{Hash}(\tau(u), \tau(\hat{e}))$. Thus, $A$ is able to successfully answer the challenge $b = 1$, regardless of the value $z$ chosen by the verifier. In fact, the value $y = \tau(u + z\hat{e})$ and the response $f = \tau(\hat{e})$ computed by $A$ are enough to pass the verification, since the same vector $\hat{e}$ is used to calculate both objects, and $\hat{e}$ has the correct restricted value.

Note that the adversary’s strategy can be improved in both cases, by taking a guess $\hat{z}$ on the value $z$ chosen by the verifier, so that $A$ is able to answer not only the challenge $b = 0$ regardless of $z$, but also the challenge $b = 1$ if $z$ was guessed correctly – or vice versa. With this improvement, we can calculate the probability of success of the adversary as follows, where we model the values $b$ and $z$ as random variables:

$$\Pr[A \text{ is accepted}] = \sum_{i=0}^{1} \Pr[ST = ST_i](\Pr[b = i] + \Pr[b = 1 - i]\Pr[z = \hat{z}])$$

$$= \frac{q}{2(q-1)}.$$

To conclude this section, we state the following theorem, relating the cheating probability to the security of the hash function and finding the secret key in the scheme.

**Theorem 15.** Let $V$ be an honest verifier, running $N$ rounds of the protocol in Figure 3 with a dishonest prover $A$. If $A$ is accepted with probability $(\frac{q}{2(q-1)})^N + 1$, where $\epsilon > 0$, then it is possible to devise an extractor algorithm $E$ that is able to either recover the secret $e$, or to find a collision for $\text{Hash}$.

The proof of Theorem 15 proceeds along the lines of that given in [11, Theorem 2], and therefore we do not repeat it here for the sake of brevity.

### 5.2 Communication cost

In our scheme, the public matrix is the parity-check matrix of a linear code over $F_q$, with length $n$ and dimension $k$. To reduce the computational complexity of the protocol, we can rely on the systematic form of such a matrix, i.e., we can choose $H = [I, P]$, where $P \in F_{q \times k}$. Note that, since the code is chosen uniformly at random, its full representation is provided by the associated seed.

The public key is the syndrome of the secret key through $H$, thus it is a vector of length $r$ over $F_q$; its size equals $r \lceil \log_2(q) \rceil$.

To properly calculate the communication cost of the scheme, we make the following considerations:
- The vector $y$ is random over $\mathbb{F}_q^n$ and thus, it is represented through $n \lceil \log_2(q) \rceil$ bits.

- When $b = 0$, the monomial transformation can be represented through the associated seed.

- When $b = 1$, $e'$ is a random vector over $\{0, \pm 1\}^n$ with restricted pre-weight, or equivalently Hamming weight, $t$. In particular, we will consider the worst case of $t = n$, in which $e'$ can be efficiently represented by a binary string of length $n$.

Given these considerations, and assuming $N$ rounds are performed with the compression technique which we have illustrated in Figure 1, the average communication cost of our proposed scheme is derived as

$$l_{\text{Hash}} + N \left( \lceil \log_2(q-1) \rceil + n \lceil \log_2(q) \rceil + 1 + \frac{n + l_{\text{Seed}}}{2} \right).$$

For the maximum communication cost, we instead have

$$l_{\text{Hash}} + N \left( \lceil \log_2(q-1) \rceil + n \lceil \log_2(q) \rceil + 1 + l_{\text{Hash}} + \max\{n, l_{\text{Seed}}\} \right).$$

To reach a cheating probability equal to $2^{-t}$, the number of rounds is obtained as

$$N = \left\lceil \frac{-t}{\log_2 \left( \frac{q}{2(q-1)} \right)} \right\rceil.$$

Note that, in practice, this means that number of rounds is approximately equal to $t$.

### 5.3 Practical instances

In this section we propose practical instances of our scheme, and compare them with other code-based identification schemes based on the Hamming metric, at the same security level. We first describe how secure parameters for the scheme can be designed, by recalling the analysis in Section 4. We will consider parameters $q, n, k$ for the public code (i.e., the one having the public $H$ as parity-check matrix) such that solving R-SDP for an error vector of pre-weight $n$ requires at least $2^\lambda$ operations, where $\lambda$ is the security level expressed in bits. We will focus on the case in which the secret key is a vector $e \leftarrow E_{n,q,n} = \{\pm 1\}^n$, and use (2) to estimate the complexity of an attack aiming at recovering the secret key. To guarantee uniqueness of the solution to the R-SDP, we will consider two different scenarios:

(A) the parameters of the public code are such that its restricted minimum distance is infinite with probability at least $1 - 2^{-\lambda}$, according to Theorem 8;
the parameters of the public code are such that the public code has, with non
negligible probability, a finite restricted minimum distance but the probability
of having a unique solution to the R-SDP is at least $1 - 2^{-\lambda}$, according to
Proposition 14.

Note that, in scenario (B), the security assumptions are more relaxed than those
of scenario (A), thus we expect it to provide more competitive parameters.

We initially focus on the security level $\lambda = 87$ and target a cheating probability
equal to $2^{-16}$, as is common in literature. For the original CVE, we have considered
the parameters that were originally given in [11], assuming seeds and hashes of re-
spective lengths 128 and 160. To provide a fair comparison, in the appendix we have
designed updated parameters for the AGS scheme, to target the same security level;
for the cheating probability of a single round, we have conservatively approximated
it to $1/2$. For our adaptation of CVE, we have the following parameters:

(A) $n = 163$, $k = 69$, $q = 31$, resulting in $N = 17$;
(B) $n = 110$, $k = 70$, $q = 31$, resulting in $N = 17$.

Table 1 compares the performances of these three schemes, also taking into
account the compression technique.

|                     | CVE | AGS | Rest. CVE (A) | Rest. CVE (B) |
|---------------------|-----|-----|---------------|---------------|
| Number of rounds    | 17  | 16  | 17            | 17            |
| Public key size (bits) | 512 | 1094 | 470          | 200          |
| Average comm. cost (kB) | 3.472 | 3.463 | 2.414        | 1.794         |
| Max comm. cost (kB) | 4.117 | 4.894 | 2.451        | 1.814         |

As we see from Table 1, our proposed scheme leads to significant improvements in
the communication cost for both scenarios (A) and (B), which is the most important
feature when analyzing these schemes. As another important advantage, the size of
the public key is strongly reduced as well.

**Signatures**

Signature schemes can be obtained, in the Random Oracle Model, by applying the
very famous Fiat-Shamir transform [19] to any ZK-ID. The transform is very intu-
itive for 3-pass schemes, in which the protocol is made non-interactive by generating
the challenge bits as the hash output of the commitment and the message. The idea
can easily be generalized to 5-pass schemes such as ours, as illustrated in [13]; in this
The communication cost roughly corresponds to the size of a signature. Some minor optimizations are possible, but, especially for the case of 5-pass schemes, lead only to a very limited improvement. Thus, to keep the analysis as simple as possible, we do not consider such optimizations in our analysis. Note that, for a signature scheme to be of practical interest, the requirements are higher in terms of security. As a consequence, we provide below parameters for $\lambda = 128$, corresponding to an authentication level of $2^{-128}$, and we update the lengths of both seeds and hash digests, setting $l_{\text{Seed}} = l_{\text{Hash}} = 256$.

The new instances that we propose for our scheme are as follows:

- (A) $n = 213$, $k = 108$, $q = 61$, resulting in $N = 132$;
- (B) $n = 171$, $k = 110$, $q = 31$, resulting in $N = 135$.

In Table 2 the features of our scheme are compared, again, with those of CVE and AGS.

Table 2: Comparison between ZK-ID schemes for a cheating probability $2^{-128}$, considering seeds and hashes of 256 bits.

|                  | CVE  | AGS  | Rest. CVE (A) | Rest. CVE (B) |
|------------------|------|------|---------------|---------------|
| Number of rounds | 129  | 128  | 132           | 135           |
| Public key size (bits) | 832  | 1574 | 630           | 305           |
| Average sig. size (kB) | 43.263 | 41.040 | 29.328      | 23.201         |
| Max sig. size (kB)  | 51.261 | 56.992 | 29.328      | 22.484         |

Once again, our proposal leads to significant reductions in all the considered sizes.

To complete the picture, we comment about some schemes appeared recently in literature. First, we consider the work of [5], that is an adaptation of Veron’s scheme [34] to the rank metric. For the instance denoted as cRVDC-125 in the paper, which reaches a security of 125 bits, the average signature size is estimated as 22.482 kB and the public key is 1212 bits long. Note that, despite a slightly larger security level, under scenario (B) our scheme leads to signatures of essentially the same length as those of cRVDC-125, while under scenario (A) we obtain minimally larger signatures; yet, in both cases our scheme has a much more compact public key. Next, we consider Durandal [2], which is again obtained via Fiat-Shamir and also uses rank metric, but is based on a different ZK-ID that is an adaptation of the Schnorr-Lyubashevsky approach. The authors propose two sets of parameters: for the smallest of the two, the public key is 121,961 bits and the signature 32,514 bits, corresponding to approximately 15 kB and 4 kB, respectively. It is immediate to notice that the main benefit of this approach is the very short signature size, due to the absence of soundness error, meaning that no repetitions of the protocol are necessary. However, this comes at the cost of a considerably larger public key. Finally, Wave [14] uses an entirely different paradigm (hash-and-sign), which is not
based on ZK-ID at all. It follows that the differences in performance with our scheme are even more stark. In fact, the protocol uses random linear codes in the Hamming metric, leading to a public key of $O(n^2)$ bits and a signature of $O(n^2)$ bits, where $n$ is the length of the chosen linear code. The authors set this parameter at $n = 8492$, which leads to roughly 3.2 MB of public key, and about 1.6 kB of signature.

5.4 Implementation aspects

Given that the prover and the verifier perform, besides hash function computation, only basic linear algebra operations (i.e., sums, multiplications and monomial transformations), we expect our protocol to be at least as fast as the other ZK-ID schemes we have considered. In particular, with respect to the standard CVE, it is very likely that our proposed solution can lead to strong improvements from the implementation side. In fact, our scheme uses codes with essentially the same length and dimension, but in a finite field of smaller size: given that sums and multiplications in $\mathbb{F}_q$ essentially cost $O\left(\lceil \log_2(q) \rceil\right)$ and $O\left(\lceil \log_2(q) \rceil^2 \right)$, respectively, linear algebra operations in $\mathbb{F}_{61}$ or $\mathbb{F}_{31}$ should be considerably faster than those performed in $\mathbb{F}_{256}$. Furthermore, using restricted monomial transformations should be easier to handle, with respect to the general case of monomial transformations over $\mathbb{F}_q$. In fact, multiplying by $\pm 1$ either corresponds to doing nothing, or simply applying a change of sign. Roughly speaking, it seems plausible that scaling according to a restricted transformation costs as much as $n$ sums, instead of $n$ multiplications. Finally, note that computing the inverse of a restricted monomial transformation is also easier: indeed, the inverse of $\pm 1$ is equal to itself (so, no actual inverse needs to be computed). Given all of these considerations, we believe that an optimized, ad-hoc implementation can achieve particularly favourable running times.

To provide a proof of concept, we have implemented our scheme using Sagemath; the corresponding code is open source and available online\(^1\). We have performed experiments on an Intel(R) Core(TM) i7-8565U CPY, running at 1.80 GHz, for the instances reaching 128 bits of security. For both scenarios, we have measured the average running time of a single round verification. We have averaged timings over 1000 runs: for scenarios (A) and (B), the average values of the running time has been, respectively, 3.05 ms and 2.39 ms. For the sake of completeness, we also provide an implementation for the case of multiple rounds, in which we have used the compression technique to reduce the communication cost. We remark that these implementations may be strongly optimized and there is large room for improvements.

6 Conclusion

In this paper we have studied generalizations of decoding problems, and their application to zero-knowledge identification schemes. In particular, we have introduced

\(^1\)The proof-of-concept implementation of our scheme is available at https://re-zkid.github.io/.
R-SDP, a new decoding problem in which the searched error, corresponding to the
given syndrome, must have entries living in a restricted subset of the finite field. We
have shown that the decisional version of this new problem is NP-complete, via a
reduction from the Hamming version of SDP, and have adapted classical arguments
about random codes (such as the Gilbert-Varshamov bound) to take into account
error vectors with this particular structure. Thus, we have determined conditions
guaranteeing the uniqueness of the solution to R-SDP and, in this regime, we have
determined the complexity of solving the problem. We have adapted the ISD algo-
rithm of Lee-Brickell to the case of restricted error vectors and analyzed its cost.
We have provided the adaption of the CVE scheme to the case of restricted error
vectors and compared this proposal to the original CVE scheme and to the AGS
scheme. Finally, we have observed that using restricted error vectors we can achieve
a reduction in the communication cost of more than 38%, which coincides with the
achievable reduction in the signature size when these schemes are used as the basis
for digital signature schemes obtained through the Fiat-Shamir transform.

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Appendix - The AGS scheme

The AGS scheme is constructed upon quasi-cyclic codes over $\mathbb{F}_2$. Before describing the scheme, we need the following definition. For a vector $a \in \mathbb{F}_2^{jk}$, such that

$$a = [a_0^{(0)}, \ldots, a_{k-1}^{(0)} | \ldots | a_0^{(j-1)}, \ldots, a_{k-1}^{(j-1)}],$$

we use $\rho_i^{(k)}$ to denote the shift of the entries of each block by $i$ positions towards right, i.e.,

$$\rho_i^{(k)}(a) = [a_{-i \mod k}^{(0)}, \ldots, a_{k-1-i \mod k}^{(0)} | \ldots | a_{-i \mod k}^{(j-1)}, \ldots, a_{k-1-i \mod k}^{(j-1)}].$$

Public Data Parameters $k, n, \omega \in \mathbb{N}$, hash function $\text{Hash}$, generator matrix $G \in \mathbb{F}_2^{k \times n}$

Private Key $m \in \mathbb{F}_2^k, e \in S_n^{n, \omega}$

Public Key $x = mG + e \in \mathbb{F}_2^n$

---

| PROVER | VERIFIER |
|--------|----------|
| Choose $u \leftarrow \mathbb{F}_2^k, \sigma \leftarrow S_n$ | Choose $z \leftarrow [0; k - 1]$ |
| Set $c_0 = \text{Hash}(\sigma)$ | $\leftarrow z$ |
| Set $c_1 = \text{Hash}(\sigma(uG))$ | $\leftarrow c_0, c_1$ |
| Set $c_2 = \text{Hash}(\sigma(uG + e'))$ | $\leftarrow c_2$ |
| If $b = 0$, set $f := \{\sigma, u + \rho_i^{(k)}(m)\}$ | Choose $b \leftarrow \{0, 1\}$ |
| If $b = 1$, set $f := \{\sigma(uG), \sigma(e')\}$ | $\leftarrow b$ |
| If $b = 0$, accept if $c_0 = \text{Hash}(\sigma)$ and $c_2 = (u + \rho_i^{(k)}(m))G + \rho_i^{(k)}(x)$ | If $b = 1$, accept if $\text{wt}_H(e') = \omega$ and $c_1 = \text{Hash}(\sigma(uG))$ and $c_2 = \text{Hash}(\sigma(uG + \sigma(e')))$ |

Figure 4: The AGS scheme.
The AGS scheme is presented in Fig. 4. In this scheme, the cheating probability asymptotically tends to $\frac{1}{2}$; since in the original paper a straightforward formula to compute the actual cheating probability is not provided, we will conservatively assume that its value is $\frac{1}{2}$, which is optimal. When performing $N$ rounds, the average communication cost is

$$l_{\text{Hash}} + N\left(\lceil \log_2(k) \rceil + 1 + 2l_{\text{Hash}} + \frac{l_{\text{Seed}} + k + n + \psi(n, \omega, 2)}{2}\right),$$

while the maximum communication cost is

$$l_{\text{Hash}} + N\left(\lceil \log_2(k) \rceil + 1 + 2l_{\text{Hash}} + \min\{l_{\text{Seed}} + k, n + \psi(n, \omega, 2)\}\right).$$

In the original paper, three parameters sets are proposed:

- $n = 698, k = 349, \omega = 70$, for 81-bits security;
- $n = 1094, k = 547, \omega = 109$, for 128-bits security.

Given the modern progress in binary ISD techniques [4], and taking into account the polynomial gain due to the quasi-cyclicity structure [31], the security that this scheme can reach is well estimated as $\omega - \frac{1}{2}\log_2(k)$ bits. It is immediately seen that, for all proposed parameters sets, the actual security is below the claimed level. So, we have updated the scheme parameters, to reach security levels of practical interest, which can directly be compared with the ones of the CVE scheme proposed in [11]:

- $n = 1094, k = 547, \omega = 92$, for 87-bits security;
- $n = 1574, k = 787, \omega = 133$, for 128-bits security.