Family of mean-mixtures of multivariate normal distributions: properties, inference and assessment of multivariate skewness

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Abstract

In this paper, a new mixture family of multivariate normal distributions, formed by mixing multivariate normal distribution and a skewed distribution, is constructed. Some properties of this family, such as characteristic function, moment generating function, and the first four moments are derived. The distributions of affine transformations and canonical forms of the model are also derived. An EM-type algorithm is developed for the maximum likelihood estimation of model parameters. We have considered in detail, some special cases of the family, using standard gamma and standard exponential mixture distributions, denoted by MMNG and MMNE, respectively. For the proposed family of distributions, different multivariate measures of skewness are computed. In order to examine the performance of the developed estimation method, some simulation studies are carried out to show that the maximum likelihood estimates based on the EM-type algorithm do provide a good performance. For different choices of parameters of MMNE distribution, several multivariate measures of skewness are computed and compared. Because some measures of skewness are scalar and some are vectors, in order to evaluate them properly, we have carried out a simulation study to determine the power of tests, based on sample versions of skewness measures as test statistics to test the fit of the MMNE distribution. Finally, two real data sets are used to illustrate the usefulness of the proposed family of distributions and the associated inferential method.

Keywords: Canonical Form, EM Algorithm, Mean Mixtures of Normal Distribution, Moments, Multivariate Measures of Skewness.

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1 Introduction

The multivariate normal distribution plays a fundamental role in many statistical analyses and applications. One of the most basic properties of the normal distribution is the symmetry of its density function. However, in practice, data sets do not follow the normal distribution or even possess symmetry, and for this reason, researchers search for new distributions to fit data with different features allowing give flexibility in skewness, kurtosis, tails and multimodality;

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is a diagonal matrix formed by the standard deviations of the \(MMN\) distribution if, in the stochastic representation in (1),

\[
\mathbf{Y} \sim \mathcal{N}^p(\mathbf{U}, \mathbf{W})
\]

is denoted by \(\mathcal{N}(\mathbf{U}, \mathbf{W})\). The Hadamard product or entry-wise product of matrices \(H\) is an arbitrary positive random variable with CDF \(\Phi\), respectively. The parameters \(\mathbf{U}\) and covariance matrix \(\mathbf{W}\), and also \(\phi(.)\) and \(\Phi(.)\), to denote the PDF and CDF of the univariate standard normal distribution, respectively.

A \(p\)-dimensional random vector \(\mathbf{Y}\) follows a multivariate skew-normal distribution if it has the PDF

\[
f(y) = 2\phi_p(y; \xi, \Omega)\Phi\left(\frac{\delta^\top \mathbf{W}^{-1} \omega^{-1}(y - \xi)}{\sqrt{1 - \delta^\top \mathbf{W}^{-1} \delta}}\right),
\]

with stochastic representation

\[
\mathbf{Y} \overset{d}{=} \xi + \omega (\delta \mathbf{U} + \mathbf{Z}),
\]

where \(d\) stands for equality in distribution, \(\xi \in \mathbb{R}^p\), \(\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{W})\) and univariate random variable \(\mathbf{U}\) has a standard normal distribution within the truncated interval \((0, \infty)\), independently of \(\mathbf{Z}\). Truncated normal distribution in the interval \((0, \infty)\) with parameters \((a, b)\) is denoted by \(TN(a, b, (0, \infty))\). The vector \(\delta = (\delta_1, \ldots, \delta_p)^\top\) is the skewness parameter vector, such that \(-1 < \delta_i < 1\), for \(i = 1, 2, \ldots, p\). The matrix \(\omega = \text{diag}(\omega_1, \ldots, \omega_p) = (\Omega \odot \mathbf{I}_p)^{1/2} > 0\) is a diagonal matrix formed by the standard deviations of \(\mathbf{U}\) and \(\mathbf{W} = \omega^\top \omega\). In the stochastic representation in (1), positive definite matrices \(\xi\) and \(\mathbf{W}\) are covariance and correlation matrices, respectively. The parameters \(\xi, \omega\) and \(\delta\) are the location, scale and skewness parameters, respectively. The Hadamard product or entry-wise product of matrices \(\mathbf{A} = (a_{ij}): m \times n\) and \(\mathbf{B} = (b_{ij}): m \times n\) is given by \(m \times n\) matrix \(\mathbf{A} \odot \mathbf{B} = [a_{ij}b_{ij}]\).

Upon using the stochastic representation in (1), a general new family of mixture distributions of multivariate normal distribution can be introduced based on arbitrary random variable \(\mathbf{U}\). A \(p\)-dimensional random vector \(\mathbf{Y}\) follows a multivariate mean mixture of normal (MMN) distribution if, in the stochastic representation in (1), \(\mathbf{U}\) is an arbitrary random variable with CDF \(H(\cdot; \nu)\), independently of \(\mathbf{Z}\), indexed by the (possibly multivariate) parameter \(\nu = (\nu_1, \ldots, \nu_p)^\top\). Then, we say that \(\mathbf{Y}\) has a mean mixture of multivariate normal (MMN) distribution, and denote it by \(\mathbf{Y} \sim \text{MMN}_p(\xi, \Omega, \delta; H)\).

Negarstani et al. (2019) presented a new family of distributions as a mixture of normal distribution and studied its properties in the univariate and multivariate cases. These authors defined a \(p\)-dimensional random vector \(\mathbf{Y}\), to have a multivariate mean mixture of normal distribution if it has the stochastic representation \(\mathbf{Y} \overset{d}{=} \xi + \delta \mathbf{U} + \mathbf{Z}\), where \(\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{W})\) and \(\mathbf{U}\) is an arbitrary positive random variable with CDF \(H(\cdot; \nu)\), independently of \(\mathbf{Z}\), indexed by the parameter vectors \(\nu = (\nu_1, \ldots, \nu_p)^\top\) and \(\delta = (\delta_1, \ldots, \delta_p)^\top \in \mathbb{R}^p\). The stochastic representation used by Negarstani et al. (2019) is along the lines of the stochastic representation of the
restricted multivariate skew-normal distribution (see Azzalini 2005), but in this work, we use different stochastic representation in (1). Negarestani et al. (2019) examined some properties of this family in the univariate case for general $U$, and also two specific cases of the family. In the present work, we consider the multivariate form of this family and study its properties.

In the stochastic representation in (1), if the random variable $U$ is a skewed random variable, then the $p$-dimensional vector $Y$ will also be skewed. In the MMN family, skewness can be regulated through the parameter $\delta$. If in (1) $\delta = 0$, the MMN family is reduces to the multivariate normal distribution. The extended form of the skew-normal distribution obtains from (1) when $U$ is distributed as $\mathcal{N}(0,1)$ variable truncated below $-\tau$ instead of 0, for some constant $\tau$. The representation in (1) means that the MMN distribution is a mean mixture of the multivariate normal distribution when the mixing random variable is $U$. Specifically, we have the following hierarchical representation for MMN the distribution:

$$Y \mid U = u \sim \mathcal{N}_p(\xi + \omega \delta u, \Omega - \omega \delta \delta^\top \Omega), \quad U \sim H(\cdot; \nu).$$

According to the hierarchical representation in (2), in the MMN model, just the mean parameter is mixed with arbitrary random variable $U$, and so this class can not be obtained from the Normal Mean-Variance Mixture (NMVM) family. The family of multivariate NMVM distributions, originated by Barndorff-Nielsen et al. (1982), is another extension of multivariate normal distribution, with a skewness parameter $\delta \in \mathbb{R}^p$. A $p$-dimensional random vector $Y$ is said to have a multivariate NMVM distribution if it has the representation

$$Y = \xi + \delta U + \sqrt{U}Z,$$

where $Z \sim \mathcal{N}_p(0, \Omega)$ and $U$ is a positive random variable and the CDF of $U$, $H(\cdot; \nu)$, is the mean-variance mixing distribution.

Both families of distributions in (1) and (3) include the multivariate normal distribution as a special case and can be applied for data sets possessing skewness. In (3), both mean and variance are mixed with the same positive random variable $U$, but in (1) just the mean parameter is mixed with $U$. But, the class in (1) can not be obtained from the class in (3).

Skewness is a feature commonly found in the returns of some financial assets. For more information on applications of skewed distributions, in finance theory, one may refer to Adcock et al. (2015). In the presence of skewness in asset returns, the skew-normal and skew-t distributions have been found to be useful models in both theoretical and empirical work. Their parametrization is parsimonious, and they are mathematically tractable, and in financial applications, the distributions are interpretable in terms of the efficient market hypothesis. Furthermore, they lead to theoretical results that are useful for portfolio selection and asset pricing. In actuarial science, the presence of skewness in insurance claims data is the primary motivation for using skew-normal distribution and its extensions. In this regard, the MMN family, that is developed here will also prove useful in finance, insurance science, and other applied fields.

Simaan (1993) proposed that the $n$-dimensional vector of returns on financial assets should be represented as $X = U + \lambda V$. The $n$-dimensional vector $U$ is assumed to have a multivariate elliptically symmetric distribution, independently of the non-negative univariate random variable $V$, which has an unspecified skewed distribution. The vector $\lambda$, whose elements may take any real values, induces skewness in the return of individual assets. Adcock and Shutes (2012) have described multivariate versions of the normal-exponential and normal-gamma distributions. Both of them are specific cases of the model of Simaan (1993). Adcock (2014) and Adcock and Shutes (2012) used the representation of Simaan (1993), with specific choices of $U$ and $V$, such as skew-normal, extended skew-normal, skew-t, normal-exponential, and normal-gamma, are investigated the corresponding distributions and their applications in capital pricing, return on financial assets and portfolio selection.
In this paper, with arbitrary random variable $U$ for the MMN family with stochastic representation in $U$, basic distributional properties of the class such as the characteristic function (CF), the moment generating function (MGF), the first four moments of the model, distribution of linear and affine transformations, the canonical form of the family and the mode of the model are derived in general. Also, the maximum likelihood estimation of the parameters by using an EM-type algorithm is discussed, and then different measures of multivariate skewness are obtained.

The special cases when $U$ has standard gamma and standard exponential distributions, with the corresponding distributions denoted by MMNG and MMNE distributions, respectively, are studied in detail. For the MMNG distribution, in addition to all the above basic properties of the distribution the infinitely divisibility of the model is also studied. For the MMNE distribution, the basic properties of the distribution as well as log-concavity of the model are discussed. Moreover, various multivariate measures of skewness have been computed in a simulation study. Moreover, various multivariate measures of skewness have been computed and compared. Finally for two real data sets, the MMNE distribution is fitted and compared with the skew-normal and skew-t distributions in terms of log-likelihood value and AIC and BIC criteria.

## 2 Model and Properties

In this section, some basic properties of the model are studied. From $U$, if $U$ has a PDF $h(\cdot; \nu)$, an integral form of the PDF of $Y \sim MMN_p(\xi, \Omega, \delta; H)$ can be obtained as

$$f_{MMN_p}(y; \xi, \Omega, \delta, \nu) = \int_0^\infty \phi_p(y; \xi + \omega \delta u, \Omega - \omega \delta \delta^\top \omega) dH(u; \nu)$$

$$= \int_0^\infty \phi_p(y; \xi + \omega \delta u, \Omega - \omega \delta \delta^\top \omega) h(u; \nu) du.$$  \hspace{1cm} (4)

We now present some theorems and lemmas with regard to different properties of these distributions and their proofs are presented in Appendix A.

**Remark 1.** We can introduce the normalized MMN distribution through the transformation $X = \omega^{-1}(Y - \xi)$. It is immediate that the stochastic representation of $X = \delta U + Z$, has the following hierarchical representation:

$$X | U = u \sim N_p(\delta u, \overline{\Omega} - \delta \delta^\top), \quad U \sim H(\cdot; \nu).$$

Then, we say that $X$ has a normalized mean mixture of multivariate normal distributions, and denote it by $X \sim MMN_p(0, \overline{\Omega}, \delta; H)$.

**Lemma 2.** If $Y \sim MMN_p(\xi, \Omega, \delta; H)$, the CF and MGF of $Y$ are as follows:

$$C_Y(t) = e^{it^\top \xi + \frac{1}{2}t^\top \Sigma_Yt} C_U(it^\top \omega \delta; \nu), \quad M_Y(t) = e^{t^\top \xi + \frac{1}{2}t^\top \Sigma_Yt} M_U(t^\top \omega \delta; \nu),$$  \hspace{1cm} (5)

respectively, where $i = \sqrt{-1}$, $\Sigma_Y = \Omega - \omega \delta \delta^\top \omega$, and $C_U(\cdot; \nu) = C_U(\cdot)$ and $M_U(\cdot; \nu) = M_U(\cdot)$ are the CF and MGF of $U$, respectively.

Moreover, if $X \sim MMN_p(0, \overline{\Omega}, \delta; H)$, the CF and MGF of $X$ are

$$C_X(t) = e^{\frac{1}{2}t^\top \Sigma_Xt} C_U(it^\top \delta; \nu), \quad M_X(t) = e^{\frac{1}{2}t^\top \Sigma_Xt} M_U(t^\top \delta; \nu),$$  \hspace{1cm} (6)

respectively, where $\Sigma_X = \overline{\Omega} - \delta \delta^\top$. The first four moments of $X$ are presented in the following lemma, derived by using the partial derivatives of MGF of normalized MMN distribution, and these, in turn, can be used to obtain the first four moments of $Y$. 

4
Lemma 3. Suppose \( X \sim MMN_p(0, \Omega, \delta; H) \). Then, the first four moments of \( X \) are as follows:

\[
\begin{align*}
M_1(X) &= M_1^X = E[U]\delta, \\
M_2(X) &= M_2^X = \Sigma_X + E[U^2](\delta \otimes \delta^T), \\
M_3(X) &= M_3^X = E[U] \left\{ \delta \otimes \Sigma_X + vec(\Sigma_X)\delta^T + (I_p \otimes \delta)\Sigma_X \right\} + E[U^3](I_p \otimes \delta)(\delta \otimes \delta^T), \\
M_4(X) &= M_4^X = (I_p \otimes U_{p,p})(\Sigma_X \otimes \Sigma_X) + vec(\Sigma_X)(vec(\Sigma_X))^T \\
&\quad + E[U^2]\delta \otimes \delta^T \otimes \Sigma_X + \delta \otimes \Sigma_X \otimes \delta^T + \Sigma_X \otimes \delta \otimes \delta^T + \delta^T \otimes \Sigma_X \otimes \delta \\
&\quad + \delta^T \otimes vec(\Sigma_X) \otimes \delta^T + (\delta \otimes \delta^T)(vec(\Sigma_X))^T \right\} + E[U^4]\delta^T \otimes \delta^T,
\end{align*}
\]

where \( E(U^k) = M_U^{(k)}(0) \), with \( M_U^{(k)}(.) \) being the k-th derivative of \( M_U(t) \) with respect to \( t \).

In the above, \( I_p \) is identity matrix of size \( p \). The Kronecker product of matrices \( A = (a_{ij}) : m \times n \) and \( B = (b_{ij}) : p \times q \) is a \( mp \times nq \) matrix \( A \otimes B = [a_{ij}b_{ij}] \). A matrix \( A = (a_1, \cdots, a_n) : m \times n \) with columns \( a_1, \cdots, a_n \) is sometimes written as a vector and called \( vec(A) \), defined by \( vec(A) = (a_1^T, \cdots, a_n^T)^T \). The matrix \( U_{p,p} \) is the permutation matrix (commutation matrix) associated with a \( p \times p \) matrix (its size is \( p^2 \times p^2 \)). For details about Kronecker product, permutation matrix and its properties, see Graham (1981) and Schott (2016). From Lemma 3, we readily obtain \( E(X) = E(U)\delta \) and \( \text{var}(X) = \Omega + (\text{var}(U) - 1)\delta^T\delta \).

We extend the results of Lemma 3 using stochastic representation in (11), to incorporate location and scale parameters, \( \xi \) and \( \omega \), through the transformation \( Y = \xi + \omega X \).

Theorem 4. If \( Y \sim MMN_p(\xi, \Omega, \delta; H) \), then its first four moments are as follows:

\[
\begin{align*}
M_1(Y) &= \xi + \omega M_1^X, \\
M_2(Y) &= \xi \otimes \xi^T + \xi \otimes (\omega M_1^X)^T + \omega M_1^X \otimes \xi^T + \omega M_2^X \omega, \\
M_3(Y) &= \xi \xi^T \otimes \xi + \xi \xi^T \otimes (\omega M_1^X) + \xi(\omega M_1^X) \otimes \xi + (\omega M_1^X) \otimes \xi^T + (\omega M_2^X) \omega \otimes \xi \\
&\quad + \xi \otimes (\omega M_2^X) \omega^T + (\omega \otimes \omega) vec(\omega M_2^X) \otimes \xi^T, \\
M_4(Y) &= \xi \xi^T \otimes \xi \xi^T + \xi \xi^T \otimes (\omega M_1^X) + \xi(\omega M_1^X) \xi^T + \xi \xi^T \otimes (\omega M_2^X) \omega \\
&\quad + \xi \xi^T \otimes \xi \xi^T + (\xi \otimes \xi)(vec(M_2^X))^T(\omega \otimes \omega) + \xi \otimes (\omega M_2^X) \omega \otimes \xi \\
&\quad + \xi \otimes (\omega M_2^X) \omega^T + (\omega \otimes \omega) vec(M_2^X) \otimes \xi^T + \xi^T \otimes (\omega M_2^X) \omega^T \\
&\quad + \xi \xi^T \otimes (\omega \otimes \omega) vec(M_2^X) \otimes \xi^T + \xi \xi^T \otimes (\omega \otimes \omega) M_2^X \omega \otimes \xi^T + (\omega \otimes \omega) M_2^X \omega \otimes \xi^T \\
&\quad + \omega(M_2^X)^T(\omega \otimes \omega) \otimes \xi + (\omega \otimes \omega) M_2^X \omega \otimes \xi^T + (\omega \otimes \omega) M_2^X \omega \otimes \xi^T.
\end{align*}
\]

From the above expressions, we can obtain mean vector and covariance matrix of \( MMN_p(\xi, \Omega, \delta; H) \) family as \( E(Y) = \xi + E(U)\omega \delta \) and \( \text{var}(Y) = \Omega + (\text{var}(U) - 1)\omega\delta^T\omega \).

Multiplication of \( M_X(t) \) by the MGF of the \( N_p(\mu, \Sigma) \) distribution, \( \exp(t^T \mu + \frac{1}{2}t^T \Sigma t) \), is still a function of type \( M_X(t) \), and we thus obtain the following result.

Theorem 5. If \( Y_1 \sim MMN_p(\xi, \Omega, \delta; H) \) and \( Y_2 \sim N_p(\mu, \Sigma) \) are independent variables, then

\[
Y = Y_1 + Y_2 \sim MMN_p(\xi_Y, \Omega_Y, \delta_Y; H),
\]

where \( \xi_Y = \xi + \mu, \Omega_Y = \Omega + \Sigma, \) and \( \delta_Y = \omega_Y^1 \omega \delta, \) with \( \omega_Y = (\Omega_Y \otimes I_p)^{1/2} \).

From the MGF’s \( M_X(t) \) and \( M_Y(t) \), it is clear that the family of MMN distributions is closed under affine transformations, as given in the following result.

Theorem 6. If \( X \sim MMN_p(0, \Omega, \delta; H) \) and \( A \) is a non-singular \( p \times p \) matrix such that \( \text{diag}(A^T \Omega A) = I_p \), that is, \( A^T \Omega A \) is a correlation matrix, then

\[
A^T X \sim MMN_p(0, A^T \Omega A, A^T \delta; H).
\]
Theorem 7. If \( Y \sim \text{MMN}_p(\xi, \Omega, \delta; H) \), \( A \) is a full-rank \( p \times h \) matrix, with \( h \leq p \), and \( c \in \mathbb{R}^h \), then

\[
T = c + A^T Y \sim \text{MMN}_h(\xi_T, \Omega_T, \delta_T; H),
\]

where \( \xi_T = c + A^T \xi \), \( \Omega_T = A^T \Omega A \), and \( \delta_T = \omega_T^{-1} A^T \omega \delta \), with \( \omega_T = (\Omega_T \circ I_h)^{1/2} \).

As in the case of skew-normal distribution in [see Azzalini and Capitanio (2014)], it can be shown that if the random vector \( Y \sim \text{MMN}_p(\xi, \Omega, \delta; H) \) is partitioned into a number of random vectors, the independence occurs between its components when at least one component follows the MMN distribution and the others have normal distribution, that is, the independence between components occurs when only one component of the skewness parameter \( \delta \) is non-zero and all others are zero. We can then state the following: if we partition \( Y \sim \text{MMN}_p(0, \Omega, \delta; H) \) into \( h \) blocks, so that \( Y^T = (Y_1^T, \ldots, Y_h^T) \), then the joint independence of these \( h \) blocks requires that the parameters have a structure of the following form (in an obvious notation):

\[
\Omega = \text{diag}(\Omega_1, \ldots, \Omega_h) \quad \text{and} \quad \delta = (0, \ldots, \delta_j, \ldots, 0)^T,
\]

so that the joint density in [3] can be factorized into a product with separate variables. Without loss of generality, from here on, it is assumed that the first element of \( \delta \) is non-zero. We now focus on a specific type of linear transformation of MMN variable, having special relevance for theoretical developments but also to some extent for practical reasons.

Theorem 8. For a given variable \( Y \sim \text{MMN}_p(\xi, \Omega, \delta; H) \), there exists a linear transformation \( Z' = A_\ast (Y - \xi) \) such that \( Z' \sim \text{MMN}_p(0, \mathbf{I}_p, \delta_{Z'}; H) \), where at most one component of \( \delta_{Z'} \) is not zero, and \( \delta_{Z'} = (\delta_\ast, 0, \ldots, 0)^T \) with \( \delta_\ast = (\delta^T \Omega^{-1} \delta)^{1/2} \).

The variable \( Z' \), which we shall sometimes refer to as a canonical variate, comprises \( p \) independent components. The joint density is given by the product of \( p - 1 \) standard normal densities and at most one non-Gaussian component \( \text{MMN}_1(0, 1, \delta_\ast; H) \); that is, the density of \( Z' \) is

\[
f_{Z'}(z) = f_{Z_1'}(z_1) \prod_{i=2}^{p} \phi(z_i),
\]

where \( Z_1' \sim \text{MMN}_1(0, 1, \delta_\ast; H) \) (for univariate MMN distribution, see Negarestani et al. (2019)).

Although Theorem 8 states that it is possible to obtain a canonical form, we should mention that in general there are many ways to achieve it, but it is not obvious how to achieve the canonical form in practice. To find the appropriate \( A_\ast \) in linear transformation \( Z' = A_\ast (Y - \xi) \), it is sufficient to find \( A_\ast \) with the following two conditions: \( A^\top_\ast \Omega A_\ast = I_p \) and \( A^\top_\ast \omega \delta = \delta_{Z'} = (\delta_\ast, 0, \ldots, 0)^T \), where \( \delta_\ast = (\delta^T \Omega^{-1} \delta)^{1/2} \). The canonical form facilitates the computation of the mode of the distribution and the multivariate coefficients of skewness.

Theorem 9. If \( Y \sim \text{MMN}_p(\xi, \Omega, \delta; H) \), the mode of \( Y \) is

\[
M_0 = \xi + m_0^\ast \omega \delta,
\]

where \( \delta_\ast = (\delta^T \Omega^{-1} \delta)^{1/2} \) and \( m_0^\ast \) is the mode of the univariate \( \text{MMN}_1(0, 1, \delta_\ast; H) \) distribution.

3 Likelihood Estimation through EM Algorithm

For obtaining the maximum likelihood estimates of all the parameters of \( \text{MMN}_p(\xi, \Omega, \delta; H) \), we propose an EM-type algorithm (Meng and Rubin; 1993). Let \( Y = (Y_1, \ldots, Y_n)^T \) be a
random sample of size $n$ from $MMN_p(\xi, \Omega, \delta; H)$ distribution. Consider the stochastic representation in (1) for $Y_i, i = 1, 2, \cdots, n$. Following the EM algorithm, let $(Y_i, U_i), i = 1, 2, \cdots, n$, be the complete data, where $Y_i$ is the observed data and $U_i$ is considered as missing data. Let $\theta = (\xi, \Omega, \delta, \nu)$. Using the representation in (1), we have that the distribution of $Y_i$, for $i = 1, 2, \cdots, n$, can be written hierarchically as

$$Y_i|U_i = u_i \sim N_p(\xi + \omega \delta u_i, \Sigma_Y), \quad U_i \sim H(\cdot; \nu),$$

where $\sim$ denotes independence of random variables and $\Sigma_Y = \Omega - \omega \delta \delta^T \omega$.

Let $y = (y_1^T, \cdots, y_n^T)$, where $y_i$ is a realization of $MMN_p(\xi, \Omega, \delta; H)$. Because

$$f(y_i, u_i) = f(y_i|u_i) h(u_i; \nu),$$

the complete data log-likelihood function, ignoring additive constants, is obtained from (17) as

$$\ell_c(\theta) = -\frac{n}{2} \log |\Sigma_Y| - \frac{1}{2} \sum_{i=1}^n (y_i - \xi)^T \Sigma_Y^{-1} (y_i - \xi) + \alpha^T \Sigma_Y^{-1} \sum_{i=1}^n u_i (y_i - \xi) - \frac{1}{2} \alpha^T \Sigma_Y^{-1} \alpha \sum_{i=1}^n u_i^2 + \sum_{i=1}^n \log h(u_i; \nu),$$

where $\alpha = \omega \delta$. Let us set

$$E_{i1}^{(k)} = E \left[ U_i|Y_i = y_i, \hat{\theta}^{(k)} \right], \quad E_{i2}^{(k)} = E \left[ U_i^2|Y_i = y_i, \hat{\theta}^{(k)} \right],$$

where $\hat{\theta}^{(k)} = (\hat{\xi}^{(k)}, \hat{\Omega}^{(k)}, \hat{\delta}^{(k)}, \hat{\nu}^{(k)})$. After some simple algebra and using (18), the expectation with respect to $U$ conditional on $Y$, of the complete log-likelihood function, has the form

$$Q(\theta|\hat{\theta}^{(k)}) = -\frac{n}{2} \log |\Sigma_Y^{-1}| - \frac{1}{2} \sum_{i=1}^n (y_i - \xi)^T \Sigma_Y^{-1} (y_i - \xi) + \sum_{i=1}^n \text{tr} \left[ \Sigma_Y^{-1} (y_i - \xi) \alpha^T \right] E_{i1}^{(k)} - \frac{1}{2} \text{tr} \left[ \Sigma_Y^{-1} \alpha \alpha^T \right] \sum_{i=1}^n E_{i2}^{(k)} + \sum_{i=1}^n E \left[ \log h(u_i; \nu)|Y_i = y_i, \hat{\theta}^{(k)} \right],$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ is the sample mean vector. The EM-type algorithm for the ML estimation of $\theta = (\xi, \Omega, \delta, \nu)$ then proceeds as follows:

**Algorithm 1.** Based on the initial value of $\theta^{(0)} = (\xi^{(0)}, \Omega^{(0)}, \delta^{(0)}, \nu^{(0)})$, the EM-type algorithm iterates between the following E-step and M-step:

**E-step:** Given the estimates of model parameters at the $k$-th iteration, say $\theta = \hat{\theta}^{(k)}$, compute $E_{i1}^{(k)}$ and $E_{i2}^{(k)}$, for $i = 1, 2, \cdots, n$;

**M-step 1:** Maximization of (19) over parameters $\xi$, $\alpha$ and $\Sigma_Y$ leads to the following closed-
form expressions:

\[
\hat{\alpha}^{(k+1)} = \frac{\sum_{i=1}^{n} y_i \hat{E}_{i1}^{(k)} - \bar{y} \sum_{i=1}^{n} \hat{E}_{i1}^{(k)}}{\sum_{i=1}^{n} \hat{E}_{i2}^{(k)} - \frac{1}{n} \left( \sum_{i=1}^{n} \hat{E}_{i1}^{(k)} \right)^2},
\]

\[
\hat{\xi}^{(k+1)} = \frac{\bar{y} - \hat{\alpha}^{(k+1)}}{n} \sum_{i=1}^{n} \hat{E}_{i1}^{(k)},
\]

\[
\hat{\Sigma}^{(k+1)}_Y = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\xi}^{(k+1)})(y_i - \hat{\xi}^{(k+1)})^\top - \frac{2}{n} \sum_{i=1}^{n} \hat{E}_{i1}^{(k)} (y_i - \hat{\xi}^{(k+1)}) \hat{\alpha}^{(k+1)} + \frac{1}{n} \hat{\alpha}^{(k+1)} \hat{\alpha}^{(k+1)}^\top \sum_{i=1}^{n} \hat{E}_{i2}^{(k)}.
\]

Therefore, we can compute \(\hat{\Omega}^{(k+1)} = \hat{\Sigma}^{(k+1)} + \hat{\alpha}^{(k+1)} \hat{\alpha}^{(k+1)}^\top\) and \(\hat{\theta}^{(k+1)} = \hat{\omega}^{(k+1)-1} \hat{\alpha}^{(k+1)}\), where \(\hat{\omega} = (\hat{\Omega} \otimes I_p)^{1/2}\).

**M-step 2**: The update of \(\hat{\nu}^{(k)}\) depends on the chosen distribution for \(U\), and is obtained as

\[
\hat{\nu}^{(k+1)} = \arg \max_{\nu} \sum_{i=1}^{n} E \left[ \log h(u_i; \nu) | Y_i = y_i, \hat{\theta}^{(k)} \right].
\]

Note that updating \(\hat{\nu}^{(k)}\) is strongly related to the form of \(h(u_i; \nu)\). If the conditional expectation \(E \left[ \log h(u_i; \nu) | Y_i = y_i, \hat{\theta}^{(k)} \right]\) is difficult to evaluate, one may resort to maximizing the restricted actual log-likelihood function, as follows:

**Modified M-step 2**: (Liu and Rubin; 1994) Update \(\hat{\nu}^{(k)}\) by

\[
\hat{\nu}^{(k+1)} = \arg \max_{\nu} \sum_{i=1}^{n} \log f_{MMN_p}(y_i; \hat{\xi}^{(k+1)}, \hat{\Omega}^{(k+1)}, \hat{\delta}^{(k+1)}, \nu).
\]

The above algorithm iterates between the E-step and M-step until a suitable convergence criterion is satisfied. Here, we adopt the distance involving two successive evaluations of the log-likelihood function \(\ell(\theta | y) = \sum_{i=1}^{n} \log f_{MMN_p}(y_i; \xi, \Omega, \delta, \nu)\), i.e., \(\left| \frac{\ell(\theta^{(k+1)} | y)}{\ell(\theta^{(k)} | y)} - 1 \right|\), as a convergence criterion.

## 4 Special Case of MMN Distribution

In this section, we study in detail a special case of the MMN family. In the stochastic representation in \(\Box\), if the random variable \(U\) follows the standard gamma distribution with corresponding PDF \(h(u; \nu) = u^{\nu-1} e^{-u/\Gamma(\nu)}, \nu > 0\), we denote \(Y \sim MMNG_p(\xi, \Omega, \delta, \nu)\). Then the PDF of \(Y\) can be obtained from \(\Box\) as follows:

\[
f_{MMNG_p}(y) = \frac{\sqrt{2\pi}}{\eta^p \Gamma(\nu)} \exp \left( \frac{A^2}{2} \right) \phi_p(y; \xi, \Sigma_Y) \int_{-\infty}^{+\infty} (z + A)^{\nu-1} \phi(z) dz, \quad y \in \mathbb{R}^p,
\]

where \(\eta = \sqrt{\delta^\top \omega \Sigma_Y^{-1} \omega \delta}, A = \eta^{-1} \left[ \delta^\top \omega \Sigma_Y^{-1} (y - \xi) - 1 \right]\) and \(\Sigma_Y = \Omega - \omega \delta \delta^\top \omega\). By using the MGF in \(\Box\), for \(Y \sim MMNG_p(\xi, \Omega, \delta, \nu)\), we obtain

\[
M_Y(t) = e^{t^\top \xi + \frac{1}{2} t^\top \Sigma_Y t (1 - t^\top \omega \delta)^{-\nu}}, \quad t^\top \omega \delta \neq 1, \quad \forall t.
\]
From the expressions in (7)-(14), and the fact that \( E(U^r) = \Gamma(v + r)/\Gamma(v) \), for positive constant \( r \), we can compute the first four moments of \( Y \) by substituting \( E(U) = v \), \( E(U^2) = v(v + 1) \), \( E(U^3) = v(v + 1)(v + 2) \), and \( E(U^4) = v(v + 1)(v + 2)(v + 3) \). Specifically, we find \( E(Y) = \xi + \nu \omega \delta \) and var \( (Y) = \Omega + (\nu - 1)\omega \delta^\top \omega \).

**Definition 10.** (Bose et al., 2002; Steutel and Van Harn, 2004) A random vector \( Y \) (or its distribution) is said to be infinitely divisible if, for each \( n \geq 1 \), there exist independent and identically distributed (iid) random vectors \( Y_1, \ldots, Y_n \) such that \( Y \overset{d}{=} Y_1 + \cdots + Y_n \).

**Theorem 11.** The MMNG distribution, in the multivariate case, is infinitely divisible.

**Proof.** Without loss of generality, suppose that \( X \sim MMNG_p(0, \Omega, \delta, \nu) \). Let \( X_i \overset{d}{=} \delta U_i + Z_i \), where \( U_i \sim Gamma(\alpha = \frac{v}{n}, \beta = 1) \) and \( Z_i \sim N_p(0, \frac{1}{n} \Omega - \delta \delta^\top) \) be independent random variables. It is easy to show that \( \sum_{i=1}^n U_i \sim Gamma(\nu, 1) \) and \( \sum_{i=1}^n Z_i \sim N_p(0, \Omega - \delta \delta^\top) \), and so we can write \( X \overset{d}{=} X_1 + \cdots + X_n \). Hence, the required result.
In the following, a particular case of the MMNG distribution with \( \nu = 1 \) is considered. Upon substituting \( \nu = 1 \), the mixing distribution of \( U \) follows the standard exponential distribution and the distribution of \( Y \) in this case is denoted by \( \text{MMNE}_p(\xi, \Omega, \delta) \). Then, the PDF of \( Y \) can be obtained as

\[
f_{\text{MMNE}_p}(y) = \frac{\sqrt{2\pi}}{\eta} \exp\left(\frac{A^2}{2}\right) \phi_p(y; \xi, \Sigma_Y) \Phi(A), \quad y \in \mathbb{R}^p.
\] (22)

Figure 1 presents the PDFs of the bivariate MMNE distribution for \( \Omega = (1,0;0,1) \) and \( \Omega = (1,1;1,1.5) \), and different choices of \( \delta \) for \( \xi = (0,0) \). Figure 1 shows that the MMNE distribution exhibits a wide variety of density shapes, in terms of skewness. The PDF of the MMNE distribution clearly depends on \( \Omega \) and \( \delta \).

The following theorem is useful in the implementation of the EM algorithm for the ML estimation of the parameters of the MMNE distribution.

**Theorem 12.** If \( Y \sim \text{MMNE}_p(\xi, \Omega, \delta) \) and the random variable \( U \) follows the standard exponential distribution, then \( U|Y = y \sim T N(\eta^{-1}A, \eta^{-2}, (0, \infty)) \). Furthermore,

\[
E[U|Y = y] = \eta^{-1} \left( A + \frac{\phi(A)}{\Phi(A)} \right),
\]

and for \( k = 2, 3, \ldots \),

\[
E[U^k|Y = y] = A\eta^{-1} E[U^{k-1}|Y = y] + (k-1)\eta^{-2} E[U^{k-2}|Y = y].
\]

**Proof.** The proof of the conditional distribution is completed easily by the use of Bayes rule. \( \square \)

Now, we can obtain the ML estimates of the parameters of MMNE distribution. By using Theorem 12 and letting

\[
\hat{E}_1^{(k)} = E[U_i|Y_i = y_i, \hat{\theta}^{(k)}] = \frac{1}{\hat{\eta}^{(k)}} \left( \hat{A}_i^{(k)} + \frac{\phi(\hat{A}^{(k)})}{\Phi(\hat{A}^{(k)})} \right),
\] (23)

\[
\hat{E}_2^{(k)} = E[U_i^2|Y_i = y_i, \hat{\theta}^{(k)}] = \frac{1}{\hat{\eta}^{(k)2}} \left[ \hat{A}_i^{(k)2} + \hat{A}_i^{(k)} \frac{\phi(\hat{A}^{(k)})}{\Phi(\hat{A}^{(k)})} + 1 \right],
\] (24)

in expression (18), the EM algorithm for the MMNE distribution can be performed. Here, we have \( \hat{\eta}^{(k)} = \sqrt{\hat{\alpha}^{(k)\top} \hat{\Sigma}_Y^{(k)-1} \hat{\alpha}^{(k)}}, \hat{A}^{(k)} = \hat{\eta}^{(k)-1} \left[ \hat{\alpha}^{(k)\top} \hat{\Sigma}_Y^{(k)-1} (y_i - \hat{\xi}^{(k)}) - 1 \right] \). Note that, in the case of MMNE distribution, the distribution of \( U \) does not have any parameter, ad so there is no need to estimate \( \nu \) in the EM algorithm and so M-step 2 must be skipped.

By using the fact that \( E(U^m) = m! \), for \( m = 1, 2, \ldots \), and by using expressions (3), (12)-(14), for \( Y \sim \text{MMNE}_p(\xi, \Omega, \delta) \), we have

\[
M_Y(t) = e^{t^\top \xi + \frac{1}{2} t^\top \Sigma_Y t} \frac{1}{1 - t^\top \Omega^\top \delta}, \quad t^\top \omega \delta \neq 1, \quad \forall t,
\] (25)
and

\[
M_1(Y) = \xi + \omega \delta, \\
M_2(Y) = \xi \otimes \xi^\top + \xi \otimes \delta^\top \omega + \omega \delta \otimes \xi^\top + (\Sigma_Y + 2\omega \delta \delta^\top \omega), \\
M_3(Y) = \xi \xi^\top \otimes \xi + \xi \xi^\top \otimes \omega \delta + \xi \delta^\top \omega \otimes \xi + \omega \delta \otimes \xi \xi^\top + (\Sigma_Y + 2\omega \delta \delta^\top \omega) + vec(\Sigma_Y + 2\omega \delta \delta^\top \omega) \otimes \xi^\top + \omega \delta \otimes \Sigma_Y + vec(\Sigma_Y) \delta^\top \omega + (I_p \otimes \omega \delta)^\top \left[ \Sigma_Y + 6\omega \delta \delta^\top \omega \right], \\
M_4(Y) = \xi \xi^\top \otimes \xi \xi^\top + \xi \xi^\top \otimes (\omega \delta) \xi^\top + \xi (\omega \delta) \xi^\top \otimes \xi \xi^\top + \xi \xi^\top \otimes (\Sigma_Y + 2\omega \delta \delta^\top \omega) + (\xi \otimes \xi)(vec(\Sigma_Y + 2\omega \delta \delta^\top \omega))^\top \\
+ \xi \otimes (\Sigma_Y + 2\omega \delta \delta^\top \omega) \otimes \xi^\top + \xi \otimes \omega(M_3^X)^\top (\omega \otimes \omega) + (\omega \delta) \xi^\top \otimes \xi \xi^\top + \xi^\top \otimes (\Sigma_Y + 2\omega \delta \delta^\top \omega) \otimes \xi^\top + \xi^\top \otimes (\omega \otimes \omega) M_3^X \omega + (\Sigma_Y + 2\omega \delta \delta^\top \omega) \otimes \xi \xi^\top + \omega (M_3^X)^\top (\omega \otimes \omega) \otimes \xi \\
+ (\omega \otimes \omega) M_3^X \omega \otimes \xi^\top + (\omega \otimes \omega) M_3^X \omega \otimes \omega, \\
\]

where

\[
M_3^X = \delta \otimes \Sigma_X + vec(\Sigma_X) \delta^\top + (I_p \otimes \delta) \Sigma_X + 6(I_p \otimes \delta)(\delta \otimes \delta^\top), \\
M_4^X = (I_{p^2} + U_{p,p})(\Sigma_X \otimes \Sigma_X) + vec(\Sigma_X)(vec(\Sigma_X))^\top \\
+ 2(\delta \otimes \delta^\top \otimes \Sigma_X + \delta \otimes \Sigma_X \otimes \delta^\top + \Sigma_X \otimes \delta \otimes \delta^\top + \delta \otimes \delta \otimes \Sigma_X \otimes \delta \\
+ \delta^\top \otimes vec(\Sigma_X) \otimes \delta^\top + (\delta \otimes \delta)(vec(\Sigma_X))^\top) + 24\delta \delta^\top \otimes \delta \delta^\top,
\]

and \( \Sigma_X = \Omega - \delta \delta^\top \). In particular, the mean vector and covariance matrix of \( Y \) are as follows:

\[
E(Y) = \xi + \omega \delta, \quad \text{var}(Y) = \Omega.
\]

**Theorem 13.** The MMNE distribution, in the multivariate case, is log-concave.

**Proof.** Because log-concavity is preserved by affine transformations, it is sufficient to prove this property for the canonical form \( Z^* \sim MMNE_p(0, I_p, \delta Z^*) \). From Prékopa (1973) and An (1996), if the elements of a random vector are independent, and each has a log-concave density function, then their joint density is log-concave. We know that in the canonical form with PDF in (15), the random variables \( Z_1, \ldots, Z_p \) are independent of each other. Log-concavity of MMNE distribution in the univariate case has been established in Proposition 3.1 of Negarestani et al. (2019), and the PDF of the univariate normal distribution is also known to be log-concave. Hence, the result.

As shown in Section 2, to compute the mode of the MMNE distribution, it is sufficient to obtain the mode of the distribution in its canonical form, and then to compute the mode of distribution using Theorem 3. To compute the mode of the distribution in its canonical form, we must calculate the value of the mode in the univariate case. Existence and uniqueness of the mode (log-concavity) of the MMNE distribution in the univariate case has been discussed in Proposition 3.1 of Negarestani et al. (2019). The mode of the univariate MNM distribution cannot be obtained in closed-form, and so one needs to use numerical methods. For this purpose, we recall the density function of univariate MMNE distribution [given by Negarestani et al. (2019)] as

\[
f_{Z_1}(z; \xi, \omega^2, \lambda) = \frac{\sqrt{1 + \lambda^2}}{\omega|\lambda|} e^{-\sqrt{1 + \lambda^2} \frac{z^2}{2\lambda^2} + \frac{1}{2\lambda^2} \Phi \left( \frac{\lambda \sqrt{1 + \lambda^2} z - 1}{|\lambda|} \right)},
\]
where \( z = \frac{y - \xi}{\omega} \), \( \lambda = \frac{\delta}{\sqrt{1 - \omega^2}} \neq 0 \), \( y \in \mathbb{R} \), \( \xi \in \mathbb{R} \) is a location parameter and \( \omega > 0 \) is a scale parameter. It is denoted by \( \text{MMNE}_1(\xi, \omega^2, \lambda) \). For obtaining the mode of \( \text{MMNE}_1 \), based on Theorem 4, we need to solve the following equation:

\[
\frac{\partial f_{Z_1}(z; \xi, \omega^2, \lambda)}{\partial z} = \frac{\sqrt{1 + \lambda^2}}{\omega |\lambda|} e^{\frac{-\lambda z + \frac{1}{2\lambda^2}}{\lambda}} \left[ -\frac{\sqrt{1 + \lambda^2}}{\lambda} \Phi\left( \frac{\lambda \sqrt{1 + \lambda^2} z - 1}{|\lambda|} \right) \right] + \frac{\lambda \sqrt{1 + \lambda^2}}{|\lambda|} \phi\left( \frac{\lambda \sqrt{1 + \lambda^2} z - 1}{|\lambda|} \right) = 0. \tag{32}
\]

The solution must be obtained by using numerical methods such as Newton-Raphson.

## 5 Multivariate Measures of Skewness

The skewed shape of the distribution is usually captured by multivariate skewness measures. The skewness is a measure of the asymmetry of a distribution about its mean and its value far from zero indicates stronger asymmetry of the underlying distribution than that with close to zero skewness value. In this work, multivariate measures of skewness by Mardia (1970),

\[
\begin{array}{|c|c|}
\hline
\text{Mardia} & \beta_{1,p} = (\gamma_1^p) \\
\text{Malkovich-Afifi} & \beta_1^p = (\gamma_1^p)^2 \\
\text{Srivastava} & \beta_2^p = \frac{1}{p} \sum_{i=1}^{p} E\left[ (\frac{Z_i^2}{\lambda_{1/2}})^3 \right] \\
\text{Móri-Rohatgi-Székely} & s = \sum_{i=1}^{p} E\left( Z_i^2 \right) = \left( \sum_{i=1}^{p} E\left( Z_i^2 \right) \right)^{\gamma_1^p} \\
\text{Kollo} & \mathbf{b} = E\left( \sum_{i,j} E\left( Z_i Z_j \right) \right) = \left( \sum_{i,j} E\left( Z_i Z_j \right) \right)^{\gamma_1^p} \\
\text{Balakrishnan-Brito-Quiroz} & T = \int_{\phi_p} u_{c_1}(u) d\Lambda(u), \quad Q^* = \mathbf{T}^T \mathbf{\Sigma}_Z^{-1} \mathbf{T} \\
\text{Isogai} & s_1 = \frac{\left[ \int_{\phi_p} u_{c_1}(u) d\Lambda(u) \right]^2}{\int_{\phi_p+\mathbb{I}} E(Z_i^2) \mathbf{T}^T \mathbf{\Sigma}_Z^{-1} \mathbf{T}} \\
\hline
\end{array}
\]

Malkovich and Afifi (1973), Srivastava (1984), Móri et al. (1993), Kollo (2008), Balakrishnan et al. (2007) and Isogai (1982) are studied for the MMN family. Table 1 presents these measures for the MMN family of distributions. The relevant derivations are given in Appendix B. In Table 4 \( \gamma_1^p \) is the skewness of \( Z_1^p \sim M\text{MN}\left( 0, 1, \delta; H \right) \) of the canonical form, respectively, where \( \delta = \left( \delta^T \Omega^{-1} \delta \right)^{1/2} \). Srivastava measures use principal components \( \mathbf{F} = \Gamma \mathbf{Y} \), where \( \Gamma = (\gamma_1, \ldots, \gamma_p) \) is the matrix of eigenvectors of the covariance matrix \( \Delta \), that is, an orthogonal matrix such that \( \Gamma^T \Delta \Gamma = \Lambda \), and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p) \) is diagonal matrix of corresponding eigenvalues.

Here, \( \mathbf{Z} = \mathbf{\Delta}^{-1/2}(\mathbf{Y} - \mathbf{\mu}) \) has the distribution \( \text{MMN}_p(\xi, \Omega, \delta; H) \), with its parameters as: \( \xi = \mathbf{\Delta}^{-1/2}(\mathbf{\mu} - \mathbf{\mu}) \), \( \Omega = \mathbf{\Delta}^{-1/2} \Omega \mathbf{\Delta}^{-1/2} \), \( \delta = \omega_\Delta^{-1/2} \mathbf{\Omega}^{-1/2} \), and \( \omega_\Delta = (\Omega \cdot I_p)^{1/2} \). Also, \( m_0^p \) is the mode of the scalar MMN distribution in the canonical form.

From Table 1 and using the moments in (23)–(24), we can obtain different measures of skewness for the \( \text{MMNE}_p(\xi, \Omega, \delta) \) distribution as follows:

- **Mardia and Malkovich-Afifi indices**: \( \beta_{1,p} = \beta_1^p = 4\delta^p \);
• **Srivastava index**: \( \beta_{3}^{2} = \frac{1}{p} \sum_{i=1}^{p} \left\{ \frac{E[(Y - \mu)^3]}{\lambda_i^3} \right\}^2 \), where \( \gamma_i \) and \( \lambda_i \) are eigenvectors and corresponding eigenvalues for covariance matrix \( \text{var}(Y) = \Omega \), when \( Y \sim MMNE_p(\xi, \Omega, \delta) \);

• **Móri-Rohatgi-Székely index**: If \( Y \sim MMNE_p(\xi, \Omega, \delta) \), then for the standardized variable \( Z = \Omega^{-1/2}(Y - \mu) \), and with \( A = \Omega^{-1/2} \), we have (see Appendix A)

\[
M_3(Z) = E[A^\top(Y - \mu)]^3 = (A^\top \otimes A^\top)M_3(Y)A - [A^\top M_2(Y)A] \otimes [A^\top E(Y)] - A^\top E(Y) \otimes [A^\top M_2(Y)A]E(Y)^\top A + 2[A^\top E(Y)E(Y)^\top A] \otimes [A^\top E(Y)].
\]

(33)

All the quantities in the Móri-Rohatgi-Székely measure of skewness are specific non-central moments of third order of \( Z \), where \( Z = \Omega^{-1/2}(Y - \mu) \sim MMNE_p(\xi_Z, \Omega_Z, \delta_Z) \), such that \( \xi_Z = -\Omega^{-1/2}\omega \delta, \Omega_Z = I_p, \delta_Z = \Omega^{-1/2}\omega \delta \);

• **Kollo index**: To get Kollo measures in Table 1 we use the elements of non-central moments of third order of \( Z \);

• **Balakrishnan-Brito-Quiroz index**: Upon substituting \( E(U^m) = m! \) for \( m = 1, 2, \ldots \), the elements of \( T \) in Table 1 for \( r = 1, 2, \ldots, p \), are

\[
T_r = \frac{3}{p(p+2)} \left( M_3^Z[(r-1)p+r, r] + \sum_{i \neq r} M_3^Z[(i-1)p+i, r] \right),
\]

where \( M_3^Z[.,.] \) denotes the elements of matrix \( M_3^Z \), third moments of \( MMNE_p(\xi_Z, \Omega_Z, \delta_Z) \) distribution, and we can then compute \( Q = T^\top \Sigma_T^{-1}T \) and \( Q^* = T^\top T \);

• **Isogai index**: By substituting \( E(U) = \text{var}(U) = 1 \) in Isogai measure of skewness in Table 1 we have \( S_I = (\delta - \bar{m}_0)^2 \), where \( \bar{m}_0 \) is the mode of the \( MMNE_1 \) distribution in the canonical form. This index is location and scale invariant. Vectorial measure, given by Balakrishnan and Scarpa (2012), is \( S_C = \left(1 - \frac{m_0}{\bar{m}_0}\right) \delta \). Therefore, the direction of \( \delta \) can be regarded as a measure of vectorial skewness for the MMNE distribution.

### 6 Simulation Study

#### 6.1 Model Fitting

This subsection presents the results of a Monte Carlo simulation study carried out to examine the performance of the proposed estimation method for the MMNE distribution in the trivariate case. We evaluate the estimates in terms of Bias and MSE (mean squared error). The results are based on 1000 simulated samples from the MMNE distribution with parameters \( \xi = (5, 10, 15)^\top, \Omega = \text{diag}(0.4, 0.6, 1.0), \delta = (0.3, 0.7, 0.4)^\top \) for different sample sizes \( n = 50, 100, 500, 1000 \). We computed the Bias and the MSE as

\[
\text{Bias} = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\theta}_j - \theta), \quad \text{MSE} = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\theta}_j - \theta)^2,
\]

where \( \theta \) is the true parameters (each of \( \xi = (\xi_1, \xi_2, \xi_3)^\top, \delta = (\delta_1, \delta_2, \delta_3)^\top \) and \( \Omega = \text{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33})^\top \)) and \( \hat{\theta}_j \) is the estimate from the \( j \)-th simulated sample. Table 2 presents the average values (Mean), the corresponding standard deviations (Std.), Bias and MSE of the EM estimates of
all the parameters of the MMNE model in 1000 simulated samples. It can be observed from Table 2 that the Bias and MSE decrease as $n$ increases, revealing the asymptotic unbiasedness and consistency of the ML estimates obtained through the EM algorithm.

Table 2: Bias and MSE of the EM estimates over 1000 samples from the MMNE distribution.

| $n$ | $\xi_1$   | $\xi_2$   | $\xi_3$   | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
|-----|-----------|-----------|-----------|-------------|-------------|-------------|---------------|---------------|---------------|
| 50  | 5.0779    | 10.1124   | 15.1344   | 0.1803      | 0.5455      | 0.2596      | 0.3960        | 0.5928        | 0.9844        |
|     | 0.2533    | 0.2931    | 0.4074    | 0.3744      | 0.3642      | 0.3883      | 0.8081        | 0.1523        | 0.2088        |
|     | 0.0779    | 0.1124    | 0.1344    | -0.1197     | -0.1545     | -0.1404     | -0.0040       | -0.0072       | -0.0156       |
|     | 0.0702    | 0.0985    | 0.1839    | 0.1544      | 0.1564      | 0.1703      | 0.0064        | 0.0232        | 0.0438        |
| 100 | Mean      | 5.0376    | 10.0637   | 15.0686     | 0.2393      | 0.6162      | 0.3233        | 0.3959        | 0.9895        |
|     | Std       | 0.1751    | 0.2038    | 0.2832      | 0.2652      | 0.2427      | 0.2658        | 0.0562        | 0.1048        |
|     | Bias      | 0.0376    | 0.0630    | 0.0686      | -0.0607     | -0.0838     | -0.0767       | -0.0041       | -0.0081       |
|     | MSE       | 0.0321    | 0.0454    | 0.0848      | 0.0740      | 0.0659      | 0.0765        | 0.0032        | 0.0110        |
| 500 | Mean      | 4.9992    | 10.0022   | 15.0004     | 0.3001      | 0.6980      | 0.4001        | 0.3994        | 0.9970        |
|     | Std       | 0.0478    | 0.0508    | 0.0721      | 0.0621      | 0.0466      | 0.0588        | 0.0250        | 0.0488        |
|     | Bias      | -0.0008   | 0.0022    | 0.0004      | 0.0001      | -0.0020     | 0.0001        | -0.0006       | -0.0004       |
|     | MSE       | 0.0023    | 0.0026    | 0.0052      | 0.0039      | 0.0022      | 0.0035        | 0.0006        | 0.0024        |
| 1000| Mean      | 4.0010    | 10.0009   | 15.0025     | 0.2987      | 0.6078      | 0.2971        | 0.3992        | 0.9991        |
|     | Std       | 0.0352    | 0.0353    | 0.0545      | 0.0441      | 0.0321      | 0.0432        | 0.0182        | 0.0332        |
|     | Bias      | 0.0010    | 0.0009    | 0.0025      | -0.0013     | -0.0022     | -0.0029       | -0.0008       | -0.0009       |
|     | MSE       | 0.0012    | 0.0012    | 0.0030      | 0.0019      | 0.0010      | 0.0019        | 0.0003        | 0.0011        |

6.2 Assessment of Skewness

To study and compare different multivariate measures of skewness for the MMN distributions, we consider the MMNE distribution. We compute the values of all the skewness measures for different choices of the parameters of the bivariate and trivariate MMNE distributions; Tables 3 and 4 present the values of all the skewness measures. It should be noted that all the measures are location and scale invariant, a desirable property indeed for any measure of skewness. For similar work on skewness comparisons for skew-normal distribution, one may refer to Balakrishnan and Scarpa (2012), and also to Kim and Zhao (2018) for similar work on scale mixtures of skew-normal distributions.

From Table 3, we find that in all cases with scalar measures of skewness, Mardia’s measure have the highest value and Srivastava’s measure is the next largest.

Just as in the case of skew-normal distribution, for the bivariate MMNE distribution, the vectorial measures yield very similar results in terms of skewness directions, especially when the distribution is highly asymmetric (Balakrishnan and Scarpa, 2012). It is important to note that Cases 9 and 10 deal with reflected distributions; and in these cases, all the measures are the same and the vectorial ones are reflected as well.

Table 4 presents the values of all the measures for the trivariate MMNE distribution. In this case, differences among the measures become much more pronounced. From Table 4 we find that in all cases, among the vectorial measures of skewness, Mardia’s measure has the highest value. Of course, the magnitude of the measures alone does not say much; one has to know how significant the values are!

6.3 Comparison and Performance of Different Skewness Measures

The measures studied in Section 5 and preceding subsection are not directly comparable with each other. So, for comparing them, we should have measures obtained on the same scale. To get such a set of comparable indices, we study the sample version for each of the skewness measures considered as test statistics for the hypothesis of normal distribution against MMNE distribution upon using the power of test based on different test statistics. If $Y_1, Y_2, \cdots, Y_n$ denote a sample of $p \times 1$ observations from any $p$-dimensional distribution. A sample version
of all the skewness measures described can be obtained by replacing $\xi$, $\Omega$, and $\delta$ with the maximum likelihood estimates of these quantities (Balakrishnan and Scarpa, 2012).

As seen in the previous sections, the Mardia and Malkovich-Affifi measure $\beta_{1,p}$, Srivastava measure $\beta_{1,p}^2$, Balakrishnan-Brito-Quiroz measure $Q^*$, Isogai measure $s_T$ are scalar indices and the Mőri-Rohatgi-Székely measure $s$, Kollo measure $b$, Balakrishnan-Brito-Quiroz measure $T$, and Isogai measure $s_C$ are vectorial indices. Here, we study different statistics for testing the null hypothesis and powers for each of these tests to quantify the power of each skewness measure to identify the specific asymmetry present in the MMNE distribution. The power of the test is a probability, and its use lets us compare different statistics, no matter what the original scales of them were. To obtain a single test statistic for the vectorial measures, we propose two different metrics, namely, the sum and the maximum (see Balakrishnan and Scarpa, 2012, pages 82-83). For the Mőri-Rohatgi-Székely measure we compute, $s_{\text{sum}} = \sum_{r=1}^p s_r$, and $s_{\text{max}} = \max_{r \in \{1, \ldots, p\}} s_r$, for the Kollo measure $b_{\text{sum}} = \sum_{r=1}^p b_r$ and $b_{\text{max}} = \max_{r \in \{1, \ldots, p\}} b_r$, for the Balakrishnan-Brito-Quiroz measure $T_{\text{sum}} = \sum_{r=1}^p T_r$ and $T_{\text{max}} = \max_{r \in \{1, \ldots, p\}} T_r$, for the Isogai’s measure $s_{C_{\text{sum}}} = \sum_{r=1}^p s_{Cr}$ and $s_{C_{\text{max}}} = \max_{r \in \{1, \ldots, p\}} s_{Cr}$.

The distributions of sample versions of measures, $\beta_{1,p}$, $\beta_{1,p}^2$, $s_{\text{sum}}$, $s_{\text{max}}$, $b_{\text{sum}}$, $b_{\text{max}}$, $Q^*$, $T_{\text{sum}}$, $T_{\text{max}}$, $s_T$, $s_{C_{\text{sum}}}$, and $s_{C_{\text{max}}}$ are not analytically computable easily, and so we may determine the critical values of these tests through Monte Carlo simulations. Two sets of the critical values obtained by a Monte Carlo simulation, based on 10 000 samples from the standard multivariate normal distribution are tabulated in Tables 5 and 6 for dimensions $p = 2, \ldots, 8$. To get the
values of critical values, first, we simulate 10,000 samples of size \( n = 100 \) from the standard multivariate normal distribution with dimensions \( p = 2, \cdots, 8 \). We estimate the parameters and then find the values of test statistics. Then we arrange the obtained values in increasing order and then picked up the 2.5 and 5 lower and upper percentage points as the critical values.

Table 5: Upper and lower 2.5% critical levels for all tests for \( n = 100 \) and \( p = 2, \cdots, 8 \), obtained from 10,000 simulated samples of standard multivariate normal distribution.

| Test Statistics | Percentile | \( p = 2 \) | \( p = 3 \) | \( p = 4 \) | \( p = 5 \) | \( p = 6 \) | \( p = 7 \) | \( p = 8 \) |
|-----------------|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \beta_{1,p} \) | 0.025 | 0.0000 | 0.0000 | 0.0068 | 0.1252 | 0.2304 | 0.3170 | 0.4388 |
| \( \beta_{2,p} \) | 0.025 | 0.0000 | 0.0000 | 0.0025 | 0.0038 | 0.0041 | 0.0054 | 0.0042 |
| \( s_{\text{max}} \) | 0.025 | -0.3562 | -0.1626 | -0.0975 | -0.0244 | 0.0160 | 0.1061 | 0.1452 |
| \( s_{\text{sum}} \) | 0.025 | -0.9945 | -1.3716 | -1.6814 | -1.9739 | -2.2822 | -2.5103 | -2.7708 |
| \( b_{\text{max}} \) | 0.025 | -0.6804 | -0.3890 | -0.2855 | -0.0707 | 0.0000 | 0.0002 | 0.0018 |
| \( b_{\text{sum}} \) | 0.025 | -1.7299 | -2.8419 | -3.9202 | -5.5741 | -6.4345 | -7.3387 | -8.0308 |
| \( Q^* \) | 0.025 | 0.0000 | 0.0000 | 0.0011 | 0.0009 | 0.0009 | 0.0007 | 0.0006 |
| \( T_{\text{max}} \) | 0.025 | -0.1336 | -0.0325 | -0.0122 | -0.0021 | 0.0019 | 0.0051 | 0.0054 |
| \( T_{\text{sum}} \) | 0.025 | -0.3729 | -0.2743 | -0.2102 | -0.1692 | -0.1426 | -0.1195 | -0.1039 |
| \( s_T \) | 0.025 | 0.0000 | 0.0000 | 0.0008 | 0.0133 | 0.0229 | 0.0304 | 0.0407 |
| \( s_{C,\text{max}} \) | 0.025 | -0.1130 | -0.0536 | -0.0147 | -0.0098 | 0.0109 | 0.0311 | 0.0482 |
| \( s_{C,\text{sum}} \) | 0.025 | -0.2920 | -0.1146 | -0.0547 | -0.0276 | -0.0363 | -0.0894 | -1.0657 |

For computing the powers of the different tests, based on the above test statistics, we
simulate 1000 samples from MMNE distribution of size $n = 100$ for different choices of the parameters of the MMNE distribution and estimate the test statistics by replacing the ML estimates of parameters evaluated by EM algorithm. Then, we compute the proportion of samples falling in the same rejection region.

For test statistics $\beta_{1p}, \beta^2_{1p}, Q^*, s_I$, we consider the sample versions exceeding the critical values, as critical regions, in the form $CR = \{Q_0 > q_0\}$, and for all other test statistics, the rejection regions will be the two-sided areas in the form $CR = \{Q_0 < q_1 - \alpha/2 \text{ or } Q_0 > q_0 + \alpha/2\}$, where $Q_0$ is test statistics under null hypothesis and $q_0$ is upper $\alpha$ percentile of distribution of test statistics.

In the simulation study, we use $\xi = 0$, the parameters $\Omega$ and $\delta$ are given in the tables. Tables 4, 8 and 9 present the power of the proposed tests for bivariate, trivariate and seven dimensional normal distribution against MMNE distribution, respectively. The comparison of the different measures may be directly performed by considering Tables 7-9. These results show clearly which are the poorer indices of skewness among those considered. Based on our empirical study, by considering different cases of the MMNE distributions in two, three and seven dimensions, we obtain the following points.

For all cases with small skewness, as expected, the power of the tests are lower for distributions more similar to the normal, and test statistics $\beta_{1p}, \beta^2_{1p}, Q^*$ and $s_I$ have better performance. From Tables 7-9 as expected, for increasing values of the elements of the skewness parameters, the power of the tests increases. For the large elements close to 1 or -1, for skewness parameters, the power of the tests are higher and almost have the same values for different test statistics.

The behaviour of test statistics $\beta_{1p}, Q^*$ and $s_I$, are very close to each other and have the same power. For small values of the skewness parameter, this test statistics have poorer performance. The power of the test for $s_{\text{max}}$, and $T_{\text{max}}$ statistics are the same, and the test statistics $s_{\text{sum}}$ and $T_{\text{sum}}$ often have similar behaviour. For large and middle values of skewness parameters, $b_{\text{sum}}$ and $b_{\text{max}}$ statistics have the lowest test power and have the worst performance compared with other test statistics. For the bivariate case in Table 7 when one element of the skewness parameter is large, and one is small, the statistics $\beta^2_{1p}, T_{\text{max}}$ and $s_{\text{max}}$ perform well, but $b_{\text{sum}}$ and $b_{\text{max}}$ statistics have the lowest test power.

For the trivariate case in Table 5 when one element of the skewness parameter is large, and two elements is small, the statistics $\beta^2_{1p}, T_{\text{max}}$ and $s_{\text{max}}$ have better performance.

From Table 9, for case 3, the statistics $s_{C_{\text{max}}}$ have the best performance and $T_{\text{max}}$ and $s_{\text{max}}$ have the lower power close to 0.05. From Table 9 for case 4, the statistic $b_{\text{max}}$ has the best performance, but $T_{\text{max}}$ and $s_{\text{max}}$ have the lower power close to 0.05. A result the we can find from 7,9 is that the test statistic $s_{C_{\text{max}}}$ performs better than others in many cases.
Table 7: Simulated values of power for test for bivariate normal distribution against MMNE distribution.

| #  | Parameters | $\Omega_p$ | $\Omega_1$ | $\delta$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ | $s_9$ | $s_{10}$ | $s_{11}$ | $s_{12}$ | $s_{13}$ | $s_{14}$ |
|----|------------|-------------|-------------|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1  | $\Omega = \begin{bmatrix} 1 & 0 \ 0 & 2.5 \end{bmatrix}$ | $\delta = 0.1$ | 0.030 | 0.041 | 0.034 | 0.040 | 0.038 | 0.035 | 0.030 | 0.034 | 0.040 | 0.030 | 0.030 | 0.034 | 0.040 | 0.030 | 0.044 |
| 2  | $\delta = 0.5$ | 0.283 | 0.121 | 0.172 | 0.467 | 0.492 | 0.497 | 0.283 | 0.172 | 0.467 | 0.283 | 0.167 | 0.459 |
| 3  | $\delta = 0.1$ | 0.659 | 0.748 | 0.711 | 0.688 | 0.631 | 0.349 | 0.659 | 0.711 | 0.668 | 0.659 | 0.690 | 0.634 |
| 4  | $\delta = 0.8$ | 0.682 | 0.724 | 0.729 | 0.700 | 0.661 | 0.383 | 0.682 | 0.729 | 0.700 | 0.682 | 0.711 | 0.682 |
| 5  | $\delta = 0.1$ | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 | 0.994 |
| 6  | $\delta = 0.8$ | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 | 0.982 |
| 7  | $\Omega = \begin{bmatrix} 1 & 1 \ 1 & 2.5 \end{bmatrix}$ | $\delta = 0.1$ | 0.032 | 0.035 | 0.043 | 0.057 | 0.066 | 0.059 | 0.032 | 0.043 | 0.057 | 0.032 | 0.087 | 0.140 |
| 8  | $\delta = 0.5$ | 0.074 | 0.079 | 0.062 | 0.159 | 0.189 | 0.186 | 0.074 | 0.062 | 0.159 | 0.074 | 0.088 | 0.315 |
| 9  | $\delta = 0.1$ | 0.979 | 0.929 | 0.980 | 0.668 | 0.105 | 0.004 | 0.979 | 0.980 | 0.668 | 0.979 | 0.974 | 0.956 |
| 10 | $\delta = 0.8$ | 0.982 | 0.978 | 0.984 | 0.941 | 0.639 | 0.084 | 0.982 | 0.984 | 0.941 | 0.982 | 0.977 | 0.963 |
| 11 | $\delta = 0.1$ | 0.956 | 0.958 | 0.959 | 0.816 | 0.076 | 0.002 | 0.956 | 0.959 | 0.816 | 0.956 | 0.960 | 0.938 |
| 12 | $\delta = 0.8$ | 0.953 | 0.955 | 0.023 | 0.832 | 0.003 | 0.005 | 0.953 | 0.023 | 0.832 | 0.953 | 0.024 | 0.936 |
| 13 | $\delta = 0.1$ | 0.968 | 0.963 | 0.004 | 0.921 | 0.004 | 0.088 | 0.968 | 0.004 | 0.921 | 0.968 | 0.248 | 0.944 |
| 14 | $\delta = 0.8$ | 0.929 | 0.929 | 0.815 | 0.977 | 0.979 | 0.980 | 0.929 | 0.815 | 0.977 | 0.929 | 0.918 | 0.987 |
Table 8: Simulated values of power for test for trivariate normal distribution against MMNE distribution.

| #  | Parameters | $\beta_{1, \rho}$ | $\beta_{2, \rho}$ | $\rho_{\text{min}}$ | $\rho_{\text{sum}}$ | $b_{\text{min}}$ | $b_{\text{sum}}$ | $Q^2$ | $T_{\text{min}}$ | $T_{\text{sum}}$ | $\gamma$ | $\gamma_{\text{min}}$ | $\gamma_{\text{sum}}$ |
|----|------------|-------------------|-------------------|-------------------|-------------------|----------------|----------------|-----|----------------|----------------|-------|----------------|----------------|
| 1  | $\Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$, $\delta = 0.1$ | 0.056 | 0.058 | 0.047 | 0.036 | 0.042 | 0.035 | 0.056 | 0.047 | 0.036 | 0.056 | 0.053 | 0.038 |
| 2  | $\delta = \begin{bmatrix} -0.1 \\ -0.1 \\ -0.1 \end{bmatrix}$ | 0.057 | 0.061 | 0.057 | 0.035 | 0.048 | 0.052 | 0.057 | 0.057 | 0.035 | 0.057 | 0.056 | 0.039 |
| 3  | $\Omega = \begin{bmatrix} 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$, $\delta = 0.7$ | 0.601 | 0.192 | 0.637 | 0.166 | 0.082 | 0.031 | 0.601 | 0.192 | 0.637 | 0.166 | 0.082 | 0.031 |
| 4  | $\delta = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ | 0.272 | 0.324 | 0.309 | 0.277 | 0.262 | 0.168 | 0.272 | 0.309 | 0.277 | 0.272 | 0.308 | 0.287 |
| 5  | $\Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$, $\delta = 0.3$ | 0.899 | 0.513 | 0.816 | 0.902 | 0.901 | 0.902 | 0.899 | 0.816 | 0.902 | 0.899 | 0.815 | 0.901 |
| 6  | $\delta = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.3 \end{bmatrix}$ | 0.745 | 0.674 | 0.750 | 0.501 | 0.277 | 0.114 | 0.745 | 0.750 | 0.501 | 0.745 | 0.638 | 0.709 |
| 7  | $\Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\delta = 0.7$ | 0.572 | 0.349 | 0.461 | 0.726 | 0.729 | 0.703 | 0.572 | 0.461 | 0.726 | 0.572 | 0.476 | 0.711 |
| 8  | $\delta = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{bmatrix}$ | 0.381 | 0.228 | 0.025 | 0.025 | 0.015 | 0.015 | 0.380 | 0.025 | 0.025 | 0.380 | 0.029 | 0.029 |
| 9  | $\delta = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ | 0.415 | 0.248 | 0.480 | 0.061 | 0.034 | 0.013 | 0.415 | 0.401 | 0.061 | 0.415 | 0.414 | 0.081 |
| 10 | $\Omega = \begin{bmatrix} 2.5 & 1 \\ 1 & 1 \end{bmatrix}$, $\delta = 0.7$ | 0.394 | 0.240 | 0.378 | 0.344 | 0.259 | 0.152 | 0.392 | 0.378 | 0.344 | 0.394 | 0.385 | 0.362 |
| 11 | $\delta = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{bmatrix}$ | 0.404 | 0.206 | 0.073 | 0.304 | 0.059 | 0.146 | 0.404 | 0.073 | 0.304 | 0.404 | 0.072 | 0.317 |
| 12 | $\delta = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{bmatrix}$ | 0.415 | 0.222 | 0.034 | 0.050 | 0.015 | 0.022 | 0.415 | 0.034 | 0.050 | 0.415 | 0.044 | 0.069 |
| 13 | $\delta = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ | 0.393 | 0.257 | 0.356 | 0.031 | 0.020 | 0.015 | 0.393 | 0.356 | 0.031 | 0.393 | 0.370 | 0.037 |
| 14 | $\delta = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ | 0.996 | 0.965 | 0.997 | 0.993 | 0.993 | 0.993 | 0.996 | 0.997 | 0.993 | 0.996 | 0.997 | 0.993 |
Table 9: Simulated values of power for test for seven dimensional normal distribution against MMNE distribution when $\Omega = I_7$.

| # | Parameter | $s_T$ | $s_{T_{1\alpha}}$ | $s_{x_{1\alpha}}$ | $s_{x_{2\alpha}}$ | $s_{x_{3\alpha}}$ | $s_{x_{4\alpha}}$ | $s_{x_{5\alpha}}$ | $s_{x_{6\alpha}}$ | $s_{x_{7\alpha}}$ | $s_{x_{1\alpha}}$ | $s_{x_{2\alpha}}$ | $s_{x_{3\alpha}}$ | $s_{x_{4\alpha}}$ | $s_{x_{5\alpha}}$ | $s_{x_{6\alpha}}$ | $s_{x_{7\alpha}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\delta = 0.1$ | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| 2 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 3 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 4 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 5 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 6 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 7 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |
| 8 | $\delta = 0.7$ | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 |

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7 Illustrative Examples

In this section, we fit the MMNE model for two real data sets to illustrate the flexibility of the model. It is also compared with skew-normal and skew-t distributions in terms of some measures of fit.

7.1 AIS data

The first example considers the Australian Institute of Sport (AIS) data (Cook and Weisberg 1994), containing 11 biomedical measurements on 202 Australian athletes (100 female and 102 male). Here we focus solely on the first 100, and the trivariate case corresponding to BMI, SSF, and Bfat variables, where the three acronyms denote Body Mass Index, Sum of Skin Folds, and Body Fat percentage, respectively. These data are available in the R software, sn package. Figure 2 presents the histograms for the three variables. Upon using the EM algorithm, we obtained the maximum likelihood estimates of parameters of the model. Table 10 presents the estimates of parameters (\(\xi, \Omega, \delta\)). Table 11 presents values of all skewness measures upon substituting the estimates of parameters, given in Table 10.

![Histogram of BMI](image)

![Histogram of SSF](image)

![Histogram of Bfat](image)

Figure 2: The marginal histograms for the three selected variables of the AIS data set.

Table 10: Parameter estimates of MMNE model for the AIS data set.

|   | \(\xi\)  | \(\Omega\)  | \(\delta\) |
|---|---------|------------|-----------|
| 1 | 20.1099 | 7.2870     | 66.3650   | 10.2745   | 0.6063    |
| 2 | 56.1969 | 66.3650    | 1238.1858 | 191.2748  | 0.8747    |
| 3 | 13.6666 | 10.2745    | 191.2748  | 31.3535   | 0.7471    |

Table 11: Values of skewness measures for the AIS data set.

| \(\beta_1\) | \(\beta_2\) | \(s\) | \(b\) | \(Q^*\) | \(T\) | \(s_I\) | \(s_C\) |
|------------|------------|-------|-------|--------|------|--------|--------|
| 3.5539     | 0.5973     | 0.3182| 0.2080| 0.1422 | 0.0636| 0.4800 | 0.4920 |
| 3.5539     | 0.5973     | 0.1703| 1.1571| 0.3541 | 0.3541| 0.6181 | 0.5279 |
| 3.5539     | 0.5973     | -0.5644| -0.3680| -0.1129| 0.4800| 0.4920 | 0.4920 |

The relative difference in the fit of a number of candidate models can be compared by using the log-likelihood values \(\ell(\hat{\theta}|y)\), the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The AIC and BIC indices are defined as \(AIC = 2k - \ell(\hat{\theta}|y)\) and

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\( BIC = k \ln n - 2\ell(\hat{\theta}|y) \), where \( k \) is the number of model parameters and \( \ell(\hat{\theta}|y) \) is the log-likelihood value of a fitted model. The larger value of \( \ell(\hat{\theta}|y) \) and the smaller value of AIC or BIC, indicates a better fit of the model to the data. Table 12 summarizes the fitting performance of MMNE model, as compared to the skew-normal and skew-t distributions. From Table 12, it is seen that the MMNE model provides the best fit overall as it provides the highest \( \ell(\hat{\theta}|y) \) value and the lowest AIC and BIC scores. Figure 3 shows the scatter plots of pairs of the three variables BMI, SSF, Bfat, along with the marginal contour plots for the fitted MMNE, skew-normal and skew-t distributions.

Table 12: comparison of fitting measures for skew-normal, skew-t and MMNE distributions for the AIS dataset.

| Distribution | \( \ell(\hat{\theta}|y) \) | AIC       | BIC       |
|--------------|----------------------------|-----------|-----------|
| Skew-normal  | -866.2725                  | 1756.545  | 1787.807  |
| Skew-t       | -852.1354                  | 1730.271  | 1764.138  |
| MMNE         | -850.7388                  | 1725.478  | 1756.740  |

Figure 3: Scatter plots of pairs of three selected variables for the AIS data set along with the marginal contour plots for the fitted MMNE, skew-normal and skew-t distributions.
7.2 Italian olive oil data

As a second example, we consider the well-known data on the percentage composition of eight fatty acids found by lipid fraction of 572 Italian olive oils. This data come from three areas; within each area there are a number of constituent regions, 9 in total. The data set includes a data frame with 572 observations and 10 columns. The first column gives the area (one of Southern Italy, Sardinia, and Northern Italy), the second gives the region, and the remaining 8 columns give the variables. Southern Italy comprises the North Apulia, Calabria, South Apulia, and Sicily regions, Sardinia is divided into Inland Sardinia, and Coastal Sardinia and Northern Italy comprises the Umbria, East Liguria, and West Liguria regions. These data are available in the R software, pgmm package.

![Histogram of Linolenic](image1)

![Histogram of Arachidic](image2)

Figure 4: The marginal histograms of two selected variables of olive oil data set.

For the purpose of illustration, we consider 323 cases from Southern Italy, and columns (8, 9), Linolenic and Arachidic fatty acids, respectively, so as to consider the bivariate case. Figure 4 shows the marginal histograms of the two selected variables, while Table 13 presents the estimates of parameters and Table 14 presents the values of skewness measures.

Table 13: Parameter estimates of the MMNE model for the olive oil data set.

|   | ξ   | Ω   | δ   |
|---|-----|-----|-----|
|   | 36.8344 | 63.3962 | 40.9481 | 0.1546 |

Table 14: Values of skewness measures for the olive oil data set.

|   | β1,p | β2,p | s   | b   | Q | T   | sI | sC |
|---|------|------|-----|-----|---|-----|----|----|
|   | 0.5707 | 0.1218 | −0.0392 | 0.7538 | −0.0428 | 0.6557 | 0.0802 | −0.0185 | 0.0517 | 0.0486 | 0.2195 |

Table 15 provides the fit of MMNE model, as compared to those of skew-normal and skew-t distributions for the considered data. From Table 15, it is clear that the MMNE model provides the best overall fit as it offers the largest $\ell(\hat{\theta}|y)$ value and the lowest AIC and BIC scores. Figure 5 shows the scatter plot of the data and the marginal contour plots of the fitted MMNE, skew-normal and skew-t distributions.
Table 15: Measures of fit of skew-normal, skew-t and MMNE distributions for the olive oil data set.

| Distribution | \( \ell(\theta|y) \) | AIC        | BIC        |
|--------------|----------------|------------|------------|
| Skew-normal  | -2320.039      | 4654.079   | 4680.522   |
| Skew-t       | -2316.320      | 4648.640   | 4678.861   |
| MMNE         | -2314.604      | 4643.207   | 4669.651   |

Figure 5: Scatter plots of olive oil data, and the marginal contour plots of the fitted MMNE, skew-normal and skew-t distributions.

8 Concluding Remarks

In this paper, we have discussed the mean mixture of multivariate normal distribution (MMN), which includes the normal, skew-normal, and extended skew-normal distributions as particular cases. We have studied several features of this family of distributions, including the first four moments, the distribution of affine transformations and canonical forms, estimation of parameters by using an EM-type algorithm with closed-form expressions, and different measures of multivariate skewness. Two special cases of the MMN family, with standard gamma and standard exponential distributions as mixing distributions, denoted by MMNG and MMNE distributions, have been studied in detail. A simulation study has been performed to evaluate the performance of the MLEs of parameters of the MMNE distribution. For the AIS and olive oil data sets, the MMNE distribution provides a better fit than the skew-normal and skew-t distributions. Different multivariate measures of skewness have been derived for the MMNE distribution, and the evaluation of test based on these measures is carried out in terms of powers of tests.

There are several possible directions for future research. For example, the study of finite mixtures and scale mixtures of MMN family will be of great interest. In the stochastic representation in (11), if the skewness parameter is a matrix, with representation \( Y \sim \xi + \omega (\Delta U + Z) \), then \( Y \) has the unified skew normal (SUN) distribution (see Arellano-valle and Azzalini, 2006), wherein elements of \( U \) have the standard half-normal distribution. In this connection, consideration of a general distribution for \( U \) would be of interest.
Appendix A. Proofs

Proof of Lemma 3. By using (9), we can calculate the partial derivatives of $M_X(t)$, the MGF of normalized MMN distribution, which are directly related to the moments of the MMN random vectors. Suppose $X \sim MMN_p(\mathbf{0}, \Omega; \delta; H)$. Then, some derivatives of $M_X(t)$ in (3) are as follows:

$$
\frac{\partial M_X(t)}{\partial t} = e^{\frac{1}{2}t^\top \Sigma_X t} \left[ \Sigma_X t M_U(t^\top \delta) + \delta M_U^{(1)}(t^\top \delta) \right],
$$

$$
\frac{\partial^2 M_X(t)}{\partial t^2} = e^{\frac{1}{2}t^\top \Sigma_X t} \left\{ M_U(t^\top \delta) \left[ \Sigma_X + (\Sigma_X t) \otimes (\Sigma_X t)^\top \right] + M_U^{(1)}(t^\top \delta) \left[ (\Sigma_X t) \otimes \delta^\top + \delta \otimes (\Sigma_X t)^\top \right] + M_U^{(2)}(t^\top \delta) \delta \otimes \delta^\top \right\},
$$

$$
\frac{\partial^3 M_X(t)}{\partial t^3} = e^{\frac{1}{2}t^\top \Sigma_X t} \left\{ M_U(t^\top \delta) \left[ (\Sigma_X t) \otimes \Sigma_X + vec(\Sigma_X)(\Sigma_X t)^\top + (I_p \otimes (\Sigma_X t))(\Sigma_X + (\Sigma_X t) \otimes (\Sigma_X t)^\top) \right] \right.
\left. + M_U^{(1)}(t^\top \delta) \left[ \delta \otimes \Sigma_X + vec(\Sigma_X)\delta^\top + (I_p \otimes (\Sigma_X t))\delta \otimes (\Sigma_X t)^\top + (\Sigma_X t) \otimes \delta^\top \right] \right.
\left. + M_U^{(2)}(t^\top \delta) \left[ (I_p \otimes (\Sigma_X t))\delta \otimes \delta^\top + (I_p \otimes \delta) (\delta \otimes (\Sigma_X t)^\top + (\Sigma_X t) \otimes \delta^\top) \right] \right.
\left. + M_U^{(3)}(t^\top \delta) \left[ (I_p \otimes \delta)(\delta \otimes \delta^\top) \right] \right\},
$$

where $M_U^{(i)}(t^\top \delta) = \frac{\partial^i M_U(t^\top \delta)}{\partial t^i}$ and $M_U^{(3)}(t^\top \delta) = \frac{\partial^3 M_U(t^\top \delta)}{\partial t^3}$. Setting $t = 0$, as in Genton et al. (2001), we obtain the first three moments of the MMN family. To find the fourth moment, since we only need the value of fourth partial derivative of $M_X(t)$ at $t = 0$, say $M_4(X) = \frac{\partial^4 M_X(t)}{\partial t^4}|_{t=0}$, we do not need to compute the whole expression. Instead, we can simply single out all the terms in $\frac{\partial^4 M_X(t)}{\partial t^4}$ that do not contain the factor $t$ or $t^3$.

Note 1: The stochastic representation $Y \overset{d}{=} \mathbf{\xi} + \omega(\delta U + Z)$ can be used directly as a way to obtain the first four moments of $Y$ in the following formulas:

$$
M_1(Y) = E(Y),
M_2(Y) = E(Y \otimes Y^\top) = E(YY^\top),
M_3(Y) = E(Y \otimes Y^\top \otimes Y) = E[(Y \otimes Y^\top)Y^\top],
M_4(Y) = E(Y \otimes Y^\top \otimes Y \otimes Y^\top) = E[(YY^\top) \otimes [(YY^\top)^\top]].
$$

The corresponding central moments of $Y$ are then

$$
\overline{M}_1(Y) = 0,
\overline{M}_2(Y) = E \left\{ [Y - E(Y)] \otimes [Y - E(Y)]^\top \right\} = \text{var}(Y),
\overline{M}_3(Y) = E \left\{ [Y - E(Y)] \otimes [Y - E(Y)]^\top \otimes [Y - E(Y)] \right\},
\overline{M}_4(Y) = E \left\{ (|Y - E(Y)| \otimes |Y - E(Y)|^\top) \otimes (|Y - E(Y)| \otimes |Y - E(Y)|^\top) \right\}.
$$

Note 2: We know that for any multivariate random vector $Y$, the central moments of third and fourth orders are related to the non-central moments by the following relationships (see,
Proof of Theorem 6.

The moment generating function of \( \Omega \) for example, Kollo and Srivastava, 2004; Kollo and von Rosen, 2005):

\[
\begin{align*}
\overline{M}_3(Y) &= M_3(Y) - M_2(Y) \otimes E(Y) - E(Y) \otimes M_2(Y) - vec(M_2(Y))E(Y)^T \\
+& \quad 2E(Y)E(Y)^T \otimes E(Y), \\
\overline{M}_4(Y) &= M_4(Y) - (M_3(Y)^T \otimes E(Y) - M_3(Y) \otimes E(Y)^T - E(Y) \otimes (M_3(Y))^T \\
-& \quad E(Y)^T \otimes M_2(Y) + M_2(Y) \otimes E(Y)E(Y)^T + (E(Y) \otimes E(Y))(vec(M_2(Y)))^T \\
+& \quad E(Y) \otimes M_2(Y) \otimes E(Y)^T + E(Y)^T \otimes M_2(Y) \otimes E(Y) \\
-& \quad 3E(Y)E(Y)^T \otimes E(Y)E(Y)^T.
\end{align*}
\]

(34)

Upon using the relations for affine transformations of moments, we then obtain

\[
\begin{align*}
M_1(AY) &= E(AY) = AE(Y), \\
M_2(AY) &= E(AY \otimes (AY)^T) = AE(Y \otimes Y^T)A^T, \\
M_3(AY) &= E(AY \otimes (AY)^T \otimes AY) = E[(AY \otimes AY)(AY)^T] \\
&= E\left\{ vec(AY(AY)^T)(AY)^T \right\} = (A \otimes A)M_3(Y)A^T, \\
M_4(AY) &= E(AY(AY)^T \otimes AY(AY)^T) = (A \otimes A)M_4(Y)(A \otimes A)^T.
\end{align*}
\]

(35)

The moment generating function of \( A^T X \) can be written as

\[
M_{A^T X}(t) = E[e^{t^T A^T X}] = E[e^{(At)^T X}] = M_X(At)
\]

\[
= e^{\frac{1}{2}t^T (\mathbf{A}^T \mathbf{\Omega} - \delta \delta^T) A t} M_U(t^T A^T \delta; \nu)
\]

Upon using the uniqueness property of the moment generating function, the required result is obtained.

Proof of Theorem 8. The moment generating function of \( X = c + A^T Y \) can be written as

\[
M_X(t) = E[e^{t^T (c + A^T Y)}] = e^{t^T c} M_Y(At)
\]

\[
= e^{t^T c} e^{t^T A^T \xi + \frac{1}{2} t^T A^T (\mathbf{\Omega} - \omega \delta^T \omega) A t} M_U(t^T A^T \omega \delta; \nu)
\]

\[
= e^{t^T c} \xi + \frac{1}{2} t^T (\mathbf{\Omega} - \omega \delta^T \omega) \nu t M_U(t^T \omega \delta \xi; \nu),
\]

which completes the proof.

Proof of Theorem 8. We have introduced the MMN distribution by assuming \( \Omega > 0 \) through the factorization \( \Omega = \omega \mathbf{\Pi} \omega \). The matrix \( \mathbf{\Pi} \) is a positive definite non-singular matrix if and only if there exists some invertible(non-singular) matrix \( \mathbf{C} \) such that \( \mathbf{\Pi} = \mathbf{C}^T \mathbf{C} \). If \( \delta \neq 0 \), there exists an orthogonal matrix \( \mathbf{P} \) with the first column proportional to \( \mathbf{C}^{-1} \delta \), while for \( \delta = 0 \) we set \( \mathbf{P} = I_p \). Finally, define \( A_\ast = (\mathbf{C}^{-1} \mathbf{P})^T \omega^{-1} \). By using Theorem 7 we see that \( Z^\ast = A_\ast (Y - \xi) \) has the stated distribution with \( \delta Z^\ast = (\delta_\ast, 0, \cdots, 0)^T \).

Proof of Theorem 9. First, consider the mode of the corresponding canonical variable \( Z^\ast \sim MMN_p(0, I_p, \delta Z^\ast; H) \). We find this mode by solving the following equations with respect to \( z_1, z_2, \cdots, z_p \):

\[
\frac{\partial f_{Z_1}(z_1)}{\partial z_1} = 0, \quad z_i f_{Z_1}(z_1) = 0, \quad \text{for } i = 2, 3, \cdots, p.
\]
The last \( p - 1 \) equations are fulfilled when \( z_i = 0 \), while the root of the first equation corresponds to the mode, \( m_0^* \) say, of the \( MMN_1(0, 1, \delta_*; H) \) distribution. Therefore, the mode of \( Z^* \) is \( M_0^* = (m_0^*, 0, \ldots, 0)^\top = \frac{m_0^*}{\delta_*} \delta Z^* \). From Theorem \( \text{B}_3 \) we can write \( Y = \xi + \omega C^\top P Z^* \) and \( \delta Z^* = P^\top C \Omega^{-1} \delta \). As the mode is equivariant with respect to affine transformations, the mode of \( Y \) is

\[
M_0 = \xi + \frac{m_0^*}{\delta_*} \omega C^\top P \delta Z^* = \xi + \frac{m_0^*}{\delta_*} \omega C^\top P P^\top C \Omega^{-1} \delta = \xi + \frac{m_0^*}{\delta_*} \omega \delta.
\]

Hence, the result.

**Appendix B. Computation of Different Measures of Skewness**

**B1. Mardia Measure of Skewness**

Mardia (1970, 1974) presented a multivariate measure of skewness of an arbitrary \( p \)-dimensional distribution \( F \) with mean vector \( \mu \) and covariance matrix \( \Delta \). Let \( X \) and \( Y \) be two independent and identically distributed random vectors from distribution \( F \). Then, the measure of skewness is

\[
\beta_{1,p} = E \left\{ \frac{1}{3} (X - \mu)^\top \Delta^{-1} (Y - \mu) \right\}^3, \tag{40}
\]

where \( \mu = E(X) \) and \( \Delta = \text{var}(X) \). Mardia measure of skewness is location and scale invariant (see Mardia 1970). From Theorems \( \text{B}_7 \) and \( \text{B}_8 \) the MMN family is closed under affine transformations and have a canonical form. If \( X \sim MMN_p(\xi, \Omega, \delta; H) \), there exists a linear transformation \( Z^* = A_*(Y - \xi) \) such that \( Z^* \sim MMN_p(0, I_p, \delta Z^*; H) \) where at most one component of \( \delta Z^* \) is not zero. Without loss of any generality, we take the first component of \( Z^* \) to be skewed and denote it by \( Z^*_1 \), and so for computing the measure in \( \text{B}_7 \), we can use the canonical form of the MMN family. Let \( X^* \) and \( Y^* \) be two independent and identically distributed random vectors from \( MMN_p(0, I_p, \delta Z^*; H) \). Now, by using \( \mu^* = E(X^*) = E(Y^*) = E(U) \delta Z^* \) and \( \Delta^* = \text{var}(X^*) = \text{var}(Y^*) = I_p + (\text{var}(U) - 1) \delta Z^* \delta Z^* \), in \( \text{B}_7 \), the Mardia measure of skewness can be expressed as

\[
\beta_{1,p} = E \left\{ \frac{1}{3} (X^* - \mu^*)^\top [\Delta^*]^{-1} (Y^* - \mu^*) \right\} = \left( E \left\{ \frac{Z^*_1 - \delta_*}{\sqrt{\text{var}(Z^*_1)}} \right\}^3 \right)^2 = (\gamma_1^*)^2, \tag{41}
\]

where \( \gamma_1^* \) is the univariate skewness of \( Z^*_1 \sim MMN_1(0, 1, \delta_*; H) \) of the canonical form (see Theorem \( \text{B}_8 \)). An explicit formula of \( \gamma_1^* \) can be found in Negarestani et al. (2019) for the univariate case.

**B2. Malkovich-Afifi Measure of Skewness**

Malkovich and Afifi (1973) proposed a measure of multivariate skewness as a different type of generalization of the univariate measure. By denoting the unit \( p \)-dimensional sphere by \( \phi_p = \{ u \in \mathbb{R}^p; ||u|| = 1 \} \), for \( u \in \phi_p \), the usual univariate measure of skewness in the \( u \)-direction is

\[
\beta_1(u) = \frac{\left[ E \left( u^\top (Y - E(Y)) \right) \right]^2}{\left[ \text{var}(u^\top Y) \right]^3}, \tag{42}
\]

27
and so the Malkovich-Afifi multivariate extension of it is defined as

$$\beta_1^* = \sup_{u \in \phi_p} \beta_1(u). \tag{43}$$

Malkovich-Afifi measure of multivariate skewness is also location and scale invariant. Malkovich and Afifi (1973) then defined the measures in (12) and (13) and showed that if $Z$ is the standardized variable $Z = \Delta^{-1/2}(Y - \mu)$, an equivalent version of $\beta_1^*$ is

$$\beta_1^* = \sup_{u \in \phi_p} \left( E \left[ (u^T Z)^3 \right] \right)^2. \tag{44}$$

For obtaining this measure for the MMN family, it is convenient to use the canonical form. If $Y \sim MMN_p(\xi, \Omega, \delta; H)$, there exists a linear transformation $Z^* = \Lambda \gamma (Y - \mu)$ such that $Z^* \sim MMN_p(0, I_p, \delta Z^*; H)$, where at most one component of $\delta Z^*$ is not zero. This means that the Malkovich-Afifi index, which is the maximum of the univariate skewness measure among all the directions of the unit sphere, will be, for $Z^*$, the index of asymmetry in the only (if there is) skew direction (without loss of any generality, we take the first component of $Z^*$ to be skewed and denote it by $Z_1^*$):

$$\beta_1^* = \beta_1^*(u) = \sup_{u \in \phi_p} \left[ \frac{E \left[ (u^T (Y - E(Y))^3 \right]}{\left[ \text{var}(u^T Y) \right]^3} \right]^2 = \frac{[E (Z_1^* - E(Z_1^*))^3]^2}{[\text{var}(Z_1^*)]^3} = \left( \frac{E (Z_1^* - E(Z_1^*))^3}{[\text{var}(Z_1^*)]^3/2} \right)^2 = (\gamma_1^*)^2. \tag{45}$$

As in the case of Mardia index, we have used $\gamma_1^*$ to denote the univariate skewness measure of the unique (if any) skewed component of the canonical form $Z^*$. Since this measure is location and scale invariant, it is invariant for linear transforms and consequently (45) is also the Malkovich-Afifi measure for $Y$, and thus it is the same as the Mardia index in (11).

**B3. Srivastava Measure of Skewness**

Using principal components $F = \Gamma Y$, Srivastava (1984) developed a measure of skewness for the multivariate vector $Y$, where $\Gamma = (\gamma_1, \ldots, \gamma_p)$ is the matrix of eigenvectors of the covariance matrix $\Delta$, that is, an orthogonal matrix such that $\Gamma^T \Delta \Gamma = \Lambda$, and $\lambda_1, \ldots, \lambda_p$ are the corresponding eigenvalues. Srivastava’s measure of skewness for $Y$ may then be presented as

$$\beta_{1p}^2 = \frac{1}{p} \sum_{i=1}^p \left\{ \frac{E(F_i - \theta_i)^3}{\lambda_i^{3/2}} \right\}^2 \leq \frac{1}{p} \sum_{i=1}^p \left\{ \frac{E[\gamma_i^T(Y - \mu)]^3}{\lambda_i^{3/2}} \right\}^2, \tag{46}$$

where $F_i = \gamma_i^T Y$ and $\theta_i = \gamma_i^T \mu$. The measure in (46) is based on central moments of third order $E[\gamma_i^T(Y - \mu)]^3$. For obtaining this measure for the MMN distribution, we only need to obtain the non-central moments up to third order. Upon using the relations in (31)-(38), we can obtain the third central moment to be

$$E[\gamma_i^T(Y - \mu)]^3 = M_3(F) = (\gamma_i^T \otimes \gamma_i^T) M_3(Y) \gamma_i - [\gamma_i^T M_2(Y) \gamma_i] \otimes [\gamma_i^T E(Y)] - \gamma_i^T E(Y) \otimes [\gamma_i^T M_2(Y) \gamma_i] - vee(\gamma_i^T M_2(Y) \gamma_i) E(Y)^T \gamma_i + 2[\gamma_i^T E(Y)^T \gamma_i] \otimes [\gamma_i^T E(Y)], \tag{47}$$

where $M_1(Y) = E(Y)$, $M_2(Y) = E(Y^2)$ and $M_3(Y) = E(Y \otimes Y^T \otimes Y) = E \{ (Y \otimes Y) Y^T \}$. These expressions are as given in Theorem 4.
B4. Móri-Rohatgi-Székely Measure of Skewness

Móri et al. (1993) suggested a vectorial measure of skewness as a p-dimensional vector. If \( Z = \Delta^{-1/2}(Y - \mu) = (Z_1, \cdots, Z_p)^\top \) is the standardized vector, this measure can be written in terms of coordinates of \( Z \) as

\[
s(Y) = E(||Z||^2Z) = E\left((Z^\top Z)Z\right) = \sum_{i=1}^{p} E\left(Z_i^2Z\right) = \left(\sum_{i=1}^{p} E\left(Z_i^2Z_1\right), \cdots, \sum_{i=1}^{p} E\left(Z_i^2Z_p\right)\right)^\top . \tag{48}
\]

All the quantities involved in (48) are specific non-central moments of third order of \( Z \). When \( Y \) has a multivariate MN distribution, \( Z \) is still MN distribution, and so we can use once again the expressions in Theorem 4. Now, \( Z = \Delta^{-1/2}(Y - \mu) = (Z_1, \cdots, Z_p)^\top \) has the distribution \( \text{MMN}_p(\xi, \Omega, \delta; H) \), whose parameters are \( \xi = \Delta^{-1/2}(\xi - \mu) \), \( \Omega = \Delta^{-1/2}\Omega\Delta^{-1/2} \) and \( \delta = \omega Z^\top \Delta^{-1/2}\omega \), with \( \omega Z = (\Omega Z \circ I_p)^{1/2} \).

Furthermore, upon replacing \( \gamma_i \) by matrix \( \Lambda = \Delta^{-1/2} \) in the third central moment in (47), using (33)-(38), and the moments in Theorem 4, we can compute \( s(Y) \) in (48).

B5. Kollo Measure of Skewness

Kollo (2008) noticed that Móri-Rohatgi-Székely skewness measure \( s(Y) \) does not include all third-order mixed moments. To include all mixed moments of the third order, he defined a skewness vector of \( Y \) as:

\[
b(Y) = E\left(\sum_{i,j} (Z_iZ_j)Z\right) = \left(\sum_{i,j} E\left((Z_iZ_j)Z_1\right), \cdots, \sum_{i,j} E\left((Z_iZ_j)Z_p\right)\right)^\top . \tag{49}
\]

The required moments can be obtained from Theorem 4 and the corresponding measure in (49) can then be computed.

B6. Balakrishnan-Brito-Quiroz Measure of Skewness

When reporting the skewness of a univariate distribution, it is customary to indicate skewness direction by referring to skewness ‘to the left’ (negative) or ‘to the right’ (positive). It seems natural that, in the multivariate setting, one would also like to indicate a direction for the skewness of a distribution.

Both Mardia and Malkovich-Afifi measures give an overall view of skewness measures without any specific reference to the direction of skewness. For this reason, Balakrishnan et al. (2007) modified the Malkovich-Afifi measure to produce an overall vectorial measure of skewness as

\[
T = \int_{\phi_p} u c_1(u)d\lambda(u), \tag{50}
\]

where \( c_1(u) = E\left([u^\top Z]^3\right) \) is a signed measure of skewness of the standardized variable \( Z = \Delta^{-1/2}(Y - \mu) \) in the direction of \( u \), and \( \lambda \) denotes the rotationally invariant probability measure on the unit p-dimensional sphere \( \phi_p = \{ u \in \mathbb{R}^p; ||u|| = 1 \} \).

From Balakrishnan et al. (2007) and Balakrishnan and Scarpa (2012), it turns out that the computation of \( T \) is straightforward and, when the distribution of \( Y \) is absolutely continuous with respect to Lebesgue measure and symmetric (in the broad sense specified below) it has,
under some moment assumptions, a Gaussian asymptotic distribution with a limiting covariance
matrix, $\Sigma_T$, that can be consistently estimated from the $Z_i$ sample.

If $c_1(u)$ is negative, it indicates skewness in the direction of $-u$, while $uc_1(u)$ provides a
vectorial index of skewness in the $u$ (or $-u$) direction. Summation of these vectors over $u$ (in
the form of an integral) will then yield an overall vectorial measure of skewness presented earlier
in [50].

For obtaining a single measure, Balakrishnan et al. (2007) proposed the quantity $Q = T^\top \Sigma_T^{-1}T$, where $T$ is as in (50) and $\Sigma_T$ is the covariance matrix of $T$. However, the covariance
matrix $\Sigma_T$ depends on the moments of sixth order. Sixth order moments in this family are not
in explicit form, and so as done in Balakrishnan and Scarpa (2012), by replacing $\Sigma_T$ by $\Sigma_Z$, we obtain $Q^* = T^\top \Sigma_Z^{-1}T$, to provide a reasonable measure of overall skewness.

In the following, evaluation of $T$ using the integrals of some monomials over the unit sphere
$\Phi_p$ are required. From Balakrishnan et al. (2007), let $u_j$ be the $j$-th coordinate of a point
$u \in \Phi_p$. Then, the values of the integrals

$$J_4 = \int_{\Phi_p} u_i^4 d\lambda(u) = \frac{3}{p(p+2)} \quad \text{and} \quad J_{2,2} = \int_{\Phi_p} u_i^2 u_j^2 d\lambda(u) = \frac{1}{p(p+2)},$$  \quad (51)

for $j \neq i, 1 \leq j, i \leq p$, are obtained using Theorem 3.3 of Fang et al. (1990). We see that
the above integrals do not depend on the particular choices of $j$ and $i$. Therefore, the $r$-th
coordinate of $T$ is simply

$$T_r = J_4 E(Z_i^3) + 3 \sum_{i \neq r} J_{2,2} E(Z_i^2 Z_r).$$  \quad (52)

So, we must obtain the moments $E(Z_i^3)$ and $E(Z_i^2 Z_r)$. From (47), by replacing $\gamma_i$ by matrix $A = \Delta^{-1/2}$, the required moments can be obtained as

$$E(Z_i^3) = M_3[Z][i] + i, \quad E(Z_i^2 Z_j) = M_3[Z][i-j],$$

where $M_3[Z][i-j]$ denotes the elements of $M_3[Z]$, third moment of the $MMN_p(\xi_Z, \Omega_Z, \delta_Z; H)$
distribution, such that $\xi_Z = \Delta^{-1/2}(\xi - \mu)$, $\Omega_Z = \Delta^{-1/2}\Omega\Delta^{-1/2}$, $\delta_Z = \omega^{-1}_Z \Delta^{-1/2} \omega\delta$, with

$$\omega_Z = (\Omega_Z \odot I_p)^{1/2}.$$  \quad (53)

Upon using the above moments, we can obtain the elements of $T$ as follows:

$$T_r = \frac{3}{p(p+2)} E(Z_i^3) + 3 \sum_{i \neq r} \frac{1}{p(p+2)} E(Z_i^2 Z_r).$$  \quad (53)

B7. Isogai Measure of Skewness

Isogai (1982) considered an overall extension of Pearson measure of skewness to a multivari-
ate case in the form

$$S_I = (\mu - M_0)^\top g^{-1}(\Delta)(\mu - M_0),$$  \quad (54)

where $\mu$, $M_0$ and $\Delta$ are the mean, mode and the covariance matrix of $Y$, respectively. The
function $g(\Delta)$ is an “appropriate” function of the covariance matrix. To derive this measure
of skewness, we need to obtain the mode of the MMN distribution. But, the uniqueness of the
mode for the family of mean mixture of normal distributions is an open problem. For obtaining
this measure, we choose $g(.)$ to be the identity function. This measure is location and scale
invariant, and so by using canonical form of MMN distribution, we get

$$S_I = \frac{[\delta E(U) - m_0]^2}{1 + \delta_0^2[\text{var}(U) - 1]},$$  \quad (55)
where $\delta_* = \left(\delta^\top \Omega^{-1} \delta\right)^{1/2}$ and $m_0^*$ is the mode of the single scalar MMN distribution in the canonical form. This index is essentially the Mahalanobis distance between the null vector and the vector $E(Y) - M_0$, and it is location and scale invariant.

Another vectorial measure has been given by Balakrishnan and Scarpa (2012) as $S_C = \omega^{-1}(\mu - M_0)$, which is a natural choice to characterize the direction of the asymmetry of the multivariate skew-normal distribution. Using the same reasoning for the MMN distribution, we can consider $S_C = \left(E(U) - \frac{m_0^*}{\delta_*}\right) \delta$, and so, the direction of $\delta$ can be regarded as a measure of vectorial skewness for the MMN distribution.

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