NEW HEAVY QUARK LIMIT SUM RULES INVOLVING ISGUR-WISE FUNCTIONS AND DECAY CONSTANTS

A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal
Laboratoire de Physique Théorique et Hautes Energies
Université de Paris XI, bâtiment 211, 91405 Orsay Cedex, France

Abstract.- We consider the dominant $c\bar{c}$ contribution to $\Delta \Gamma$ for the $B^0_s-\bar{B}^0_s$ system in the heavy quark limit for both $b$ and $c$ quarks. In analogy with the Bjorken-Isgur-Wise sum rule in semileptonic heavy hadron decay, we impose duality between the parton model calculation of $\Delta \Gamma$ and its estimation by a sum over heavy mesons. Varying the mass ratio $m_c/m_b$ and assuming factorization and saturation by narrow resonances ($N_c \to \infty$), we obtain new sum rules that involve the Isgur-Wise functions $\xi^{(n)}(w)$ and $\tau^{(n)}_{1/2}(w)$ and the decay constants $f^{(n)}$, $f^{(n)}_{1/2}$ ($n$ stands for any radial excitation). Alternatively, we deduce the sum rules with another method free of the factorization hypothesis, from the saturation of the expectation value of a product of two currents by heavy hadrons and by the corresponding free quarks. The sum rules read
\[ \sum_n \frac{f^{(n)}}{f^{(0)}} \xi^{(n)}(w) = 2 \sum_n \frac{f^{(n)}}{f^{(0)}} \tau^{(n)}_{1/2}(w) = 1, \] valid for all $w$. Moreover, we obtain, in the heavy quark limit, $f^{(n)}_{3/2} = 0$. As a consequence, unlike the BIW sum rule, the slope of the elastic function $\xi(w)$ is related to radial excitations alone. These are generalizations, rigorous for QCD in the heavy quark limit, of results that have an easy understanding in the non-relativistic quark model.

LPTHE Orsay 96-28
April 1996

---

1Laboratoire associé au Centre National de la Recherche Scientifique - URA D0063
email: oliver@qcd.th.u-psud.fr, pene@qcd.th.u-psud.fr
The Bjorken-Isgur-Wise (BIW) \cite{1,2} sum rule can be obtained from duality between the estimation of a heavy hadron semileptonic decay rate by the parton model or by a sum over exclusive heavy hadrons, or equivalently from the saturation of the expectation value of a product of currents by heavy hadrons or by the corresponding free quarks. We can call this the duality approach, used in the paper by Isgur and Wise \cite{2}, where they estimate the hadronic tensor involved in the total semileptonic decay rate 
\[ W_{\mu\nu} = \sum_i \langle B | J_{\mu}^{bc} | n \rangle \langle n | J_{\nu}^{cb} | B \rangle \]
either by the parton model, with the \( c \) quark considered as heavy, or by a sum over heavy hadron intermediate states. Indeed, one can expect duality owing to the heavy quark limit, the corrections being subleading in \( 1/m_Q \), as it has been demonstrated in a well-defined formalism \cite{3}. This approach is to be distinguished from the current algebra approach, used by Bjorken et al. \cite{1} that leads to the same result essentially because the two terms contributing to the commutator, the direct and \( Z \)-graph contributions, satisfy separately a sum rule of their own.

To fix our ideas, let us write down the heavy meson current matrix elements in terms of the corresponding Isgur-Wise (IW) functions. We have, for \( S \)-waves in terms of the function \( \xi(w) \):

\[ < D(v_f)|V_{\mu}|\bar{B}(v_i) > = N \xi(w) (v_i + v_f)_{\mu} \]
\[ < D^*(v_f,\bar{\epsilon})|V_{\mu}|\bar{B}(v_i) > = i N \xi(w) \varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{*\alpha} v_f^\beta v_i^\gamma \]
\[ < D^*(v_f,\bar{\epsilon})|A_{\mu}|\bar{B}(v_i) > = N \xi(w) [(v_i \cdot v_f + 1) \varepsilon_{\mu}^* - (\varepsilon^* \cdot v_i) v_f_{\mu}] \]

(1)

where \( N = \frac{1}{\sqrt{4v_0 B v_0 D}} \). For the transition from \( S \) to \( P \)-waves, there are two independent functions \( \tau_{1/2}(w), \tau_{3/2}(w) \), where \( j = \frac{1}{2}, \frac{3}{2} \) stand for the total angular momentum of the light degrees of freedom relative to the heavy quark. We list only the matrix elements that concern the specific purpose of this paper, the estimation of \( \Delta \Gamma \), as will become clear below :

\[ < D^{(1/2)_1^+}(v_f,\bar{\epsilon})|V_{\mu}|\bar{B}(v_i) > = N 2 [(w - 1)\varepsilon_{\mu}^* - (\varepsilon^* \cdot v_i) v_f_{\mu}] \tau_{1/2}(w) \]
\[ < D^{(1/2)_1^+}(v_f,\bar{\epsilon})|A_{\mu}|\bar{B}(v_i) > = N i \varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{*\alpha} (v_i + v_f)^\beta (v_i - v_f)^\gamma \tau_{1/2}(w) \]
\[ < D^{(1/2)_0^+}(v_f)|A_{\mu}|\bar{B}(v_i) > = N 2 (v_f - v_i)_{\mu} \tau_{1/2}(w) \] .

(2)

Identifying the parton model \( B \) meson total semileptonic decay rate to its estimation by the sum over the possible meson final states listed above, one finds the BIW sum
rule:
\[
\sum_n w + \frac{1}{2} \left\{ \left[ \xi^{(n)}(w) \right]^2 + (w - 1) \left[ 2 \left[ \tau^{(n)}_{1/2}(w) \right]^2 + (w + 1) \left[ \tau^{(n)}_{3/2}(w) \right]^2 \right] + \ldots \right\} = 1.
\] (3)

Where \(n\) stands for radial excitations and the dots indicate a possible continuum. For the transition to a radial excitation one has \(\xi^{(n)}(w) \sim (w - 1)\) \((n \neq 1)\) due to the orthogonality of the wave functions at zero recoil. Notice that for these transitions we have changed the notation relatively to Isgur and Wise [2], as in their paper they define \(\xi^{(n)}(w)\) factorizing out \((w - 1)\). It is more convenient for us to keep an homogeneous definition for any \(S\)-wave meson in the final state. It is well known that from (3) follows the sum rule for the slope of the ground state IW function, and Bjorken’s lower bound for it:

\[\rho^2 = \frac{1}{4} + \sum_n \left\{ \left[ \tau^{(n)}_{1/2}(1) \right]^2 + 2 \left[ \tau^{(n)}_{3/2}(1) \right]^2 \right\} , \quad \rho^2 \geq \frac{1}{4}\] (4)

We want now to apply duality to a non-leptonic process, \(\Delta \Gamma\), the decay rate difference between the two CP eigenstates of the system \(B_s^0 - \bar{B}_s^0\) (we assume CP to be conserved). Our aim is to obtain, in the factorization approximation, sum rules involving both annihilation constants and form factors.

Before proceeding to the calculation, we need to make a number of important remarks on questions of principle.

First, there is an essential difference between the total semileptonic rate \(B \to X_c \ell \nu\) and \(\Delta \Gamma\). In the former, by varying \(q^2\), the (mass)\(^2\) of the lepton system, one has equivalently a variable \(w\) for the hadronic matrix element \(< X_c | J_\mu | B >\). On the contrary, \(\Delta \Gamma\) is a non-leptonic quantity, and, for a given intermediate state, \(q^2\) is fixed. To set our ideas, let us remind that there are two contributions to \(\Delta \Gamma\), corresponding to two ways of cutting the box diagram: the exchange contribution (Fig. 1) and the spectator contribution (Fig. 2), as pointed out by Hagelin in his pioneering work within the parton model [4] and in ref. [7], where we discussed \(\Delta \Gamma\) from the point of view of ground state exclusive modes. As will be justified later, we will only consider the spectator contribution.

For the spectator contribution, one has two types of color singlet groupings, \((c\bar{s})_1(\bar{c}s)_1(D_s \bar{D}_s \ldots)\) and \((c\bar{c})_1(s\bar{s})_1(\psi \phi \ldots)\). As we will argue below, we will only consider the former. We draw a typical contribution of this type in Fig. 3. It is clear that any \(B \to D\) form factor \(F^{DB}(q^2)\) \((D\) is any meson with charm quantum number\) must be taken at \(q^2 \approx m_c^2\) because there is emission of another \(D\) meson, and the charmed quark is heavy. In the
heavy quark limit, a form factor $F^{DB}(q^2) \to \eta(w)$ (up to kinematic factors), where $\eta(w)$ is a generic IW function. At $q^2 = m_c^2$ the variable $w = \frac{m_b^2 + m_c^2 - q^2}{2m_b m_c}$ will take a fixed value

$$q^2 = m_c^2 \to w = \frac{m_b}{2m_c}$$  \hspace{1cm} (5)

However, without any assumption on the value of $m_c$, $m_b$ (except to be heavy), by varying the mass ratio $\frac{m_b}{m_c}$ one can express $\Delta \Gamma$ in terms of $w$, both in the parton model and in the exclusive calculation as well. The parton model rate difference will be given in terms of some kinematic function of $w$, while the exclusive calculation will be expressed in terms of the IW functions $\xi^{(n)}(w)$, $\tau^{(n)}_{1/2}(w)$ and $\tau^{(n)}_{3/2}(w)$.

The exclusive calculation of $\Delta \Gamma$ involves also other quantities, namely, within the factorization approximation, the following decay constants:

$$<D_s(0^-)(v)|A^{sc}_\mu|0> = N \ f_D \sqrt{m_D} \ v_\mu$$

$$<D_s(1^-)(v,\varepsilon)|V^{sc}_\mu|0> = N \ g_D \sqrt{m_D} \ \varepsilon^*_\mu$$  \hspace{1cm} (6)

for $S$ states, where $N = \frac{1}{\sqrt{2}v_D}$, and

$$<D_s(1/2^+)(v)|V^{sc}_\mu|0> = N \ f_D^{(1/2)} \ \sqrt{m_D} \ v_\mu$$

$$<D_s(1/2^+)(v,\varepsilon)|A^{sc}_\mu|0> = N \ g_D^{(1/2)} \ \sqrt{m_D} \ \varepsilon^*_\mu$$

$$<D_s(3/2^+)(v,\varepsilon)|A^{sc}_\mu|0> = N \ g_D^{(3/2)} \ \sqrt{m_D} \ \varepsilon^*_\mu$$  \hspace{1cm} (7)

for $P$ states. The annihilation constant $f^{(1/2)}$ is non-zero because the vector current is not conserved. The annihilation constants are defined in such a way that they are flavor-independent in the heavy quark limit.

We will identify $\frac{m_b}{2m_c} \equiv w$ in the parton model and in the exclusive calculation. By imposing duality, we will then obtain sum rules involving IW functions and decay constants. We will begin by considering the physical V-A case (the ground state contribution to $\Delta \Gamma$ and duality as $2m_c \to m_b$ were studied in [7]), but we must keep in mind that the heavy quark limit concerns QCD alone and is independent of the Lorentz structure of the current. As we will see below, we will obtain various sum rules by varying the chirality structure of the electroweak part of the theory. Indeed, we shall use $\Delta \Gamma$ as an intermediary step to get a sum rule for annihilation constants and form factors. For this purpose we can choose a fictitious but convenient four fermion interaction. Concerning short distance QCD coefficients we can choose in particular $c_+ = c_- = 1$. 

4
In the following, we will assume that the excitation energies verify the hierarchy
\[ \Delta E_n \ll p = \left( \frac{1}{4}m_b^2 - m_c^2 \right)^{1/2}, \] and \[ \Delta E_n \ll m_c, m_b. \] Strictly speaking, the excitation energies range up to infinity and the latter hierarchy cannot be valid for all states. In practice we assume that these higher states do not contribute significantly in the sum rules. One could introduce a cut-off \( \mu \) as in [2], but we preferred to avoid any unessential complication. On the other hand, we will neglect hard gluons, because these appear as corrections to the heavy quark limit.

To be able to get sum rules involving IW functions and decay constants (semileptonic quantities) from a non-leptonic quantity like \( \Delta \Gamma \) it would seem that we need to be in the limit in which factorization (i.e. vacuum insertion) holds for the non-leptonic decays, i.e. the \( N_c \to \infty \) limit.

Taking \( c_+ = c_- = 1 \), the amplitude to decay into the color grouping \((c\bar{c})_1(s\bar{s})_1\) is proportional to \( 1/N_c \). As pointed out by Shifman [3], this type of final state violates duality at \( O(1/N_c) \). However, we have shown that the sum of both color configurations \((cs)_1(\bar{c}s)_1\) and \((c\bar{c})_1(s\bar{s})_1\) satisfy duality if the final state interaction (FSI) is taken into account [3]. But since we want here to relate semileptonic quantities and we therefore adopt \( N_c \to \infty \) to avoid violations to factorization, we take, to summarize: \( c_+ = c_- = 1, N_c \to \infty \). Therefore, factorization holds, the color configurations \((c\bar{c})_1(s\bar{s})_1\) vanish and, moreover, the width is saturated by narrow resonances.

The estimation of the exchange contribution to \( \Delta \Gamma \) in the parton model is straightforward [4]. However, the calculation using exclusive channels would involve form factors \( F^{DD}(m_b^2) \), quantities that are very different [7] from the ones we are interested in the case of the spectator contributions, namely \( F^{DB}(m_c^2) \). Therefore, we need to compare the contributions \( \Delta \Gamma_{\text{partons}}^{\text{spectator}} \) to \( \Delta \Gamma_{\text{exclusive}}^{\text{spectator}} \) alone.

However, let us proceed with care and make explicit the \( N_c \) dependence. Let us substract from the parton model result
\[
\Delta \Gamma_{\text{partons}} = \frac{G^2}{2\pi} f_B^2 |V_{cb}^* V_{cs}|^2 p
\]
\[
\left\{ 2 \left( 1 - \frac{1}{2N_c} \right) 2m_b^2 \left[ \frac{1}{2} - \frac{1}{3} \left( \frac{3}{4} + \frac{p^2}{m_b^2} \right) \right] - 2 \left( 1 + \frac{1}{N_c} \right) \left[ m_c^2 - \frac{1}{2} m_b^2 + \frac{1}{3} m_b^2 \left( \frac{3}{4} + \frac{p^2}{m_b^2} \right) \right] \right\}
\]
(the two terms correspond respectively to the two operators contributing to the mixing
\[\bar{c} \gamma_{\mu} (1 - \gamma_5) b \bar{c} \gamma_{\mu} (1 - \gamma_5) b\] \[\text{and}\ \bar{c} (1 + \gamma_5) b \bar{c} (1 + \gamma_5) b\) the exchange contribution:

\[\Delta \Gamma_{\text{exchange}}^{\text{partons}} = -\frac{G^2}{2\pi} \frac{2}{N_c} f^2_B m_c^2 |V_{cb}^* V_{cs}|^2 p\]

where \(p = \left(\frac{1}{4} m_b^2 - m_c^2\right)^{1/2}\) is the three-momentum. We get the simple result

\[\Delta \Gamma_{\text{spectator}}^{\text{partons}} = \frac{G^2}{2\pi} f^2_B m_b^2 |V_{cb}^* V_{cs}|^2 p .\] (10)

We need now to compute the sum over exclusive modes. Let us see what are the constraints on the annihilation constants defined in (7). Let us first realize that, on grounds of Lorentz covariance and parity,

\[<D_s\left(\frac{1}{2} 0^+\right)(v) |A^{sc}_\mu|0> = <D_s\left(\frac{3}{2} 2^+\right)(v, \varepsilon) |A^{sc}_\mu|0> = <D_s\left(\frac{3}{2} 2^+\right)(v, \varepsilon) |V^{sc}_\mu|0> \equiv 0 .\] (11)

On the other hand, using (for the longitudinal polarizations of \(1^+, 2^+\)),

\[S^3_c |D_s(\frac{1}{2} 1^+)(0, 0) > = \frac{1}{2} |D_s(\frac{1}{2} 0^+)(0) >\]

\[S^3_c |D_s(\frac{3}{2} 1^+)(0, 0) > = \frac{1}{2} |D_s(\frac{3}{2} 2^+)(0, 0) >\] (12)

and the commutation relations [7]

\[\left[ S^3_c, A^{sc}_3 \right] = -\frac{1}{2} V_0^{sc}\]

\[\left[ S^3_c, V^{sc}_0 \right] = -\frac{1}{2} A^{sc}_3\] (13)

together with (11), we obtain:

\[f^{(1/2)} = g^{(1/2)}\]

\[g^{(3/2)} = 0 \quad \text{(heavy mass limit)} .\] (14)

This result of the heavy quark limit (one heavy quark and one light quark) is quite different from the usual intuition of the case of mesons made of quark-antiquark of equal masses, in which the state \(1^+\) is decoupled of the axial current, and the state \(0^{++}\) is decoupled from the vector current, owing to current conservation. These decouplings are completely general in QCD for mesons with equal mass valence quarks. In the language of the quark model, in the equal mass limit, from the change of basis

\[|1 P_1> = \sqrt{\frac{T}{3}} |1/2 1^+ > + \sqrt{\frac{T}{3}} |3/2 1^+ >\]

\[|3 P_1> = -\sqrt{\frac{T}{3}} |1/2 1^+ > + \sqrt{\frac{T}{3}} |3/2 1^+ >\] (15)
(we neglect the \(L.S\) coupling) one gets indeed
\[
g^{(1/2)} + \sqrt{2} g^{(3/2)} = 0 \quad f^{(1/2)} = 0 \quad \text{(equal mass limit)} \quad .
\] (16)

One can understand intuitively the transition between these two regimes by using the non-relativistic quark model with unequal masses. In the lowest order in the quark velocities, one has, in terms of the relative momentum within the bound state:

\[
V_{0}^{sc} \rightarrow u_{c}^{+}v_{s} \rightarrow \frac{1}{2\sqrt{2}} \left( \frac{1}{m_{c}} - \frac{1}{m_{s}} \right) \chi_{c}^{+} \alpha \cdot p \chi_{s}
\] (17)

\[
A_{3}^{sc} \rightarrow u_{c}^{+} \sigma_{3}v_{s} \rightarrow \frac{1}{2\sqrt{2}} \left( \frac{1}{m_{c}} - \frac{1}{m_{s}} \right) \chi_{c}^{+} p_{z} \chi_{s} + \frac{1}{2\sqrt{2}} \left( \frac{1}{m_{c}} + \frac{1}{m_{s}} \right) \chi_{c}^{+} i(\alpha \times p)_{z} \chi_{s} .
\]

After some angular momentum calculations, one obtains (14) in the \(m_{c} \rightarrow \infty\) limit, and (16) in the equal mass limit.

An important consequence of (14) is that, in the problem that we are considering of the estimation of \(\Delta \Gamma\), the emission of \(3/2, 1^{+}\) states is forbidden in the hypothesis of factorization. \(\Delta \Gamma\) will read, in the estimation through exclusive modes:

\[
\Delta \Gamma^{\text{exclusive}}_{\text{spectator}} = 2\Gamma_{12} = \frac{G^{2}}{4\pi} m_{B} |V_{cb}^{*}V_{cs}|^{2} p
\]

\[
\left\{ \sum_{n,n'} < 0 |V_{\mu}^{sc} - A_{\mu}^{sc}|D_{n}(v_{f}) > < B^{0}(v_{i})|V_{\mu}^{cb} - A_{\mu}^{cb}|D_{n'}(v'_{f}) > \right. \\
\times < \bar{D}_{n'}(v'_{f})|V_{\alpha}^{sc} - A_{\alpha}^{sc}|0 > < D_{n}(v_{f})|V_{\alpha}^{cb} - A_{\alpha}^{cb}|\bar{B}^{0}(v_{i}) > \}
\] (18)

where the factor \(p\) is the three-momentum that also appears in the parton model expression (10). For the sake of clarity, it is convenient to split the different contributions into three types (we call from now on the exclusive spectator contribution simply \(\Delta \Gamma\)):

\[
\Delta \Gamma = \Delta \Gamma^{(-)} + \Delta \Gamma^{(-+)} + \Delta \Gamma^{(++)}
\] (19)

where \(\Delta \Gamma^{(-)}\), \(\Delta \Gamma^{(-+)}\) and \(\Delta \Gamma^{(++)}\) denote respectively two \(S\)-wave mesons of parity - (the ground state and its radial excitations), one \(S\)-wave meson with parity - and one \(P\)-wave meson with parity +, and two \(P\)-wave mesons of parity + in the final state [8].

Since each transition matrix element is contracted with a creation matrix element out of the vacuum, we need a number of current matrix elements contracted with four-vectors. Recall the identification \(\frac{m_{b}}{2m_{c}} \equiv w\).

1) Transition matrix element to \(S\)-wave meson contracted with pseudoscalar or scalar emission (since the velocity of the emitted meson is \((m_{B}v_{i} - m_{D}v_{f})/m_{D})\) :
\[
\frac{1}{m_D} < D(v_f) | (m_B v_i - m_D v_f) \cdot V | \bar{B}(v_i) >= N \xi(w) (2w - 1)(w + 1)
\]
\[
\frac{1}{m_D} < D^*(v_f, \varepsilon) | (m_B v_i - m_D v_f) \cdot V | \bar{B}(v_i) >= i N \xi(w) \frac{1}{m_D} \varepsilon_{\alpha \beta \gamma} (m_B v_i - m_D v_f)^\mu \varepsilon^{*\alpha} v^\beta_f v^\gamma_i \equiv 0
\]
\[
\frac{1}{m_D} < D^*(v_f, \varepsilon) | (m_B v_i - m_D v_f) \cdot A | \bar{B}(v_i) >= N \xi(w) \sqrt{w^2 - 1}(2w + 1) \quad (20)
\]

2) Transition matrix element to S-wave meson contracted with vector or axial emission (\(\lambda\) indicates the polarization):
\[
< D(v_f) | \varepsilon^{*\mu} \cdot V | \bar{B}(v_i) >= -N \xi(w)(2w + 1)\sqrt{w^2 - 1}
\]
\[
< D^*(v_f, \varepsilon) | \varepsilon^{*\mu} \cdot V | \bar{B}(v_i) >= \pm i N \xi(w)\sqrt{w^2 - 1} \quad (\lambda = \pm 1, \lambda' = \mp 1)
\]
\[
< D^*(v_f, \varepsilon) | \varepsilon^{*\mu} \cdot A | \bar{B}(v_i) >= -N \xi(w)(w + 1)(2w - 1) \quad (\lambda = 0)
\]
\[
< D^*(v_f, \varepsilon) | \varepsilon^{*\mu} \cdot A | \bar{B}(v_i) >= -N \xi(w)(w + 1) \quad (\lambda = \pm 1, \lambda' = \pm 1) \quad (21)
\]

3) Transition matrix element to P-wave meson contracted with pseudoscalar or scalar emission:
\[
\frac{1}{m_D} < D^{(1/2) +} (v_f, \varepsilon) | (m_B v_i - m_D v_f) \cdot V | \bar{B}(v_i) >= -N 2 \sqrt{w^2 - 1}(2w - 1) \tau_{1/2}(w)
\]
\[
\frac{1}{m_D} < D^{(1/2) +} (v_f, \varepsilon) | (m_B v_i - m_D v_f) \cdot A | \bar{B}(v_i) >= 0
\]
\[
\frac{1}{m_D} < D^{(1/2) 0+} (v_f) | (m_B v_i - m_D v_f) \cdot A | \bar{B}(v_i) >= N 2 (2w + 1)(w - 1) \tau_{1/2}(w) \quad (22)
\]

4) Transition matrix element to P-wave meson contracted with vector or axial emission:
\[
< D^{(1/2) 1+} (v_f, \varepsilon) | \varepsilon^{*\mu} \cdot V | \bar{B}(v_i) >= N 2(w - 1)(2w + 1) \tau_{1/2}(w) \quad (\lambda = 0)
\]
\[
< D^{(1/2) 1+} (v_f, \varepsilon) | \varepsilon^{*\mu} \cdot V | \bar{B}(v_i) >= -N 2(w - 1) \tau_{1/2}(w) \quad (\lambda = \pm 1, \lambda' = \pm 1)
\]
\[
< D^{(1/2) 1+} (v_f, \varepsilon) | \varepsilon^{*\mu} \cdot A | \bar{B}(v_i) >= \pm i N 2 \sqrt{w^2 - 1} \tau_{1/2}(w) \quad (\lambda = \pm 1, \lambda' = \mp 1)
\]
\[
< D^{(1/2) 0+} (v_f) | \varepsilon^{*\mu} \cdot A | \bar{B}(v_i) >= -N 2(2w - 1)\sqrt{w^2 - 1} \tau_{1/2}(w) \quad (\lambda = 0) \quad . \quad (23)
\]

When summing the different contributions one must take special care in their relative sign. Let us detail, as an example, the calculation of the sum over the ground state mesons. From \((20)\) and \((21)\) we get, for the ground state contributions:
\[ \Delta \Gamma^{\text{ground state}} = \frac{G^2}{8\pi} f_B^2 m_B^2 |V_{cb}^* V_{cs}|^2 p [\xi(w)]^2 \frac{1}{4w^2} \]  
\{ (2w - 1)^2 (w + 1)^2 + 2(w^2 - 1)(2w + 1)^2 + (w + 1)^2 (2w - 1)^2 + 2(w + 1)^2 - 2(w^2 - 1) \}

where the terms correspond respectively to \( PP, PV, VV(p.v.L), VV(p.v.T), VV(p.c.T) \) (\( P, V \) stand for pseudoscalar, vector; \( p.v., p.c. \) for parity violating and parity conserving; \( L, T \) for longitudinal, transverse). We recover the same signs for the different states that we found in ref. [7], in particular for the non-trivial case of the \( PV \) contribution. We get therefore:

\[ \Delta \Gamma^{\text{ground state}} = \frac{G^2}{2\pi} f_B^2 m_B^2 |V_{cb}^* V_{cs}|^2 p \frac{(2w - 1)^2 (w + 1)^2}{4w^2} [\xi(w)]^2 . \]  

Notice that in (25) we have adopted the heavy quark limit relation:

\[ f_B^2 m_B = f_D^2 m_D \]  

in order to have the same overall factors as in the parton model expression (10).

The sum over all excitations is simple in its final expression. To this aim, let us define the scale invariant sums extended over all radial excitations:

\[ X(w) = \sum_n \frac{f^{(n)}}{f^{(0)}} \xi^{(n)}(w) \]
\[ T_{1/2}(w) = \sum_n \frac{f_{1/2}^{(n)}}{f^{(0)}} \tau_{1/2}^{(n)}(w) \]  

where \( f^{(0)} \) denotes the ground state annihilation constant. Obviously, one has

\[ X(1) = 1 \]  

From (24) we can write down the sum over all \( S \)-wave radial excitations, as defined in (14):

\[ \Delta \Gamma^{(-)} = \frac{G^2}{2\pi} f_B^2 m_B^2 |V_{cb}^* V_{cs}|^2 p \frac{(2w - 1)^2 (w + 1)^2}{4w^2} [X(w)]^2 . \]  

The calculation of the \( P \)-wave meson contributions \( \Delta \Gamma^{(-)} \) and \( \Delta \Gamma^{(++)} \) can be done using formulae (20)-(23) and is rather tedious. One must be careful with the relative signs. Using the notation (27) we find

\[ \Delta \Gamma^{(-)} = -\frac{G^2}{2\pi} f_B^2 m_B^2 |V_{cb}^* V_{cs}|^2 p \frac{(4w^2 - 1)(w^2 - 1)}{4w^2} 2[X(w)] 2T_{1/2}(w) \]
\[ \Delta \Gamma^{(++)} = \frac{G^2}{2\pi} f_B^2 m_B^2 |V_{cb}^* V_{cs}|^2 p \frac{(2w + 1)^2 (w - 1)^2}{4w^2} [2T_{1/2}(w)]^2 . \]  

9
Therefore, we obtain for the total sum:

\[
\Delta \Gamma = \frac{G^2}{2\pi} f_B^2 m_B^2 V_{cb}^* V_{cs}^2 p \\
\frac{1}{4w^2} \left\{ (2w^2 - 1) \left[ X(w) - 2T_{1/2}(w) \right] + w \left[ X(w) + 2T_{1/2}(w) \right] \right\}^2 .
\] (31)

Identifying with the parton model calculation (10):

\[
\frac{1}{4w^2} \left\{ (2w - 1)(w + 1)X(w) - 2(2w + 1)(w - 1)T_{1/2}(w) \right\}^2 = 1 .
\] (32)

This equation is not the only one that one can obtain from duality arguments for \( \Delta \Gamma \) in the heavy quark limit. As pointed out above, the heavy quark symmetry limit concerns QCD and is independent of the structure of the electroweak theory. We can obtain other relations of the type (32) by just varying the chirality structure of the theory.

For example if instead of having the V-A structure \( \gamma_\mu L \) one would have a pure vector coupling \( \gamma_\mu \), one would select just a few states in the exclusive calculation. Only production and annihilation by a vector current can occur, and then only the parity conserving vector-vector final state, i.e. \( D^* \bar{D}^* \) in the relative \( P \) wave (and its radial excitations) contribute. In particular, there is no contribution of the \( P \) states! The calculation follows along similar lines as the case of the V-A current. We find, in the \( N_c \rightarrow \infty \) limit:

\[
\Delta \Gamma_{\text{partons spectator (pure V)}} = -\frac{G^2}{4\pi} f_B^2 m_b^2 |V_{cb}^* V_{cs}|^2 p \frac{w^2 - 1}{2w^2}
\] (33)

\[
\Delta \Gamma_{\text{exclusive spectator (pure V)}} = -\frac{G^2}{4\pi} f_B^2 m_b^2 |V_{cb}^* V_{cs}|^2 p [X(w)]^2 \frac{w^2 - 1}{2w^2}
\] (34)

Duality implies, in this case:

\[
[X(w)]^2 = 1 .
\] (35)

One could consider also a pure axial coupling \( \gamma_\mu \gamma_5 \). Then, only production and annihilation by an axial current can occur, and only the parity conserving axial-axial states \( D^{**}(1^+) \bar{D}^{**}(1^+) \) in the relative \( P \) wave (and its radial excitations) contribute. In particular, there is no contribution of the ground state. We find, in the \( N_c \rightarrow \infty \) limit, the same result for the parton model calculation:

\[
\Delta \Gamma_{\text{partons spectator (pure A)}} = -\frac{G^2}{4\pi} f_B^2 m_b^2 |V_{cb}^* V_{cs}|^2 p \frac{w^2 - 1}{2w^2}
\] (36)
and, for the exclusive calculation:

$$\Delta \Gamma_{\text{exclusive}}^{\text{pure } A} = -\frac{G^2}{4\pi} f_B^2 m_b^2 |V_{cb}^* V_{cs}|^2 \left[2T_{1/2}(w)\right]^2 \frac{w^2 - 1}{2w^2}.$$  \hspace{1cm} (37)

And duality implies:

$$\left[2T_{1/2}(w)\right]^2 = 1.$$ \hspace{1cm} (38)

From (32), (35) and (38), owing to the fact that $X(w)$ and $T_{1/2}(w)$ are real and we have the normalization condition (28), we obtain the simple sum rules, the main result of this paper:

$$X(w) = 2T_{1/2}(w) = 1.$$ \hspace{1cm} (39)

This solution is very constrained, since $X(w)$ and $T_{1/2}(w)$ are functions of $w$.

Let us now use another method to obtain this result, namely duality applied to a product of currents $J^{sc}(x) J^{cb}(y)$, as indicated in Fig. 4. This method is simpler as it gives a linear equation involving $X(w)$ and $T_{1/2}(w)$ that leads precisely to (39). Moreover, the method does not rely on the hypothesis of factorization when we have used $\Delta \Gamma (N_c \to \infty)$. It seems to have a higher degree of generality. Let us consider the currents:

$$J^{sc}(x) = \bar{s}(x) \Gamma^{sc} c(x) \quad J^{cb}(x) = \bar{c}(x) \Gamma^{cb} b(x).$$ \hspace{1cm} (40)

We will estimate the matrix element

$$<0|J^{sc}(0) \tilde{J}^{cb}(q)|\bar{s}_B>(v_B)$$ \hspace{1cm} (41)

within the parton model approach in the heavy mass limit for both $c$ and $b$ quarks and, on the other hand, by taking heavy hadrons as intermediate states. We will first compute the product of currents $J^{sc}(0) \tilde{J}^{cb}(q)$ acting on a $b$ quark state $|b, p_b, s_b>$ where $\tilde{J}^{cb}(q)$ is the Fourier transform:

$$\tilde{J}^{cb}(q) = \int d\mathbf{x} \ J^{cb}(x, 0) e^{-iq \cdot x}.$$ \hspace{1cm} (42)

We will have, in the heavy mass limit for the $c$ and $b$ quarks, as one replaces the heavy quark propagator by the free propagator:

$$J^{sc}(0) \tilde{J}^{cb}(q) |b, p_b, s_b> =$$

$$\bar{s}(0) \int d\mathbf{x} \ e^{-iq \cdot x} [<0|c(0)\bar{c}(x, 0) |0>]|_{E>0} b(x, 0) |b, p_b, s_b>$$ \hspace{1cm} (43)
where
\[ \langle 0|c(0)\bar{c}(x,0)|0 \rangle_{E>0} = \int \frac{d^3p}{(2\pi)^3} \frac{\gamma^\mu_p + 1}{2v^\mu_p} e^{-ip\cdot x} \]
\[ b(x,0)|b, p_b, s_b > = e^{-ip\cdot x} b(0)|b, p_b, s_b > , \]  
(44)
with \( v^\mu_p = \frac{p^\mu}{m_c} \). Finally,
\[ J^{sc}(0) \bar{J}^{cb}(q)|b, p_b, s_b > = \bar{s}(0) \left[ \Gamma_{sc} \frac{\gamma^\mu_c + 1}{2v^\mu_c} \Gamma^{cb} \right] b(0) \]  
(45)
where \( p_c = p_b - q \). Particularizing to the product of currents \( A^0V^0 \), i.e. \( \Gamma^{sc} = \gamma^0\gamma_5 \) and \( \Gamma^{cb} = \gamma^0 \) we obtain:
\[ \gamma^0\gamma_5 \gamma^\mu_c + \frac{1}{2v^\mu_c} \gamma^\alpha = \frac{\bar{v}^\mu_c \gamma^0\gamma_5 - \gamma_5 + \bar{v}^\mu_c \cdot \hat{v} \gamma_5}{2v^\mu_c} \]  
(46)
Using \( < 0|\bar{s}\gamma^\mu\gamma_5 b|B(v_B) > = -v^\mu_B < 0|\bar{s}\gamma_5 b|B(v_B) > = \frac{f_B \sqrt{m_B} v^\mu_c}{\sqrt{2v^\mu_c}} \) we obtain, in the limit \( v_c = v_D, v_b = v_B \):
\[ < 0|J^{sc}(0) \bar{J}^{cb}(q)|B(v_B) > = < 0|\bar{s}(0)\gamma^0\gamma_5 \gamma^\mu_c + \frac{1}{2v^\mu_c} \gamma^0 b(0)|B(v_B) > = \frac{f_B \sqrt{m_B}}{2v^\mu_D \sqrt{2v^\mu_B}} \left( v^\mu_B v^0_D + 1 + v_D \cdot v_B \right) \]  
(47)
We must now compute the matrix element (44) by a sum over exclusive heavy hadrons \(|n > \) as intermediate states. Moreover, we will assume dominance of the sum by intermediate narrow resonances. Using the formulae above for transition and annihilation matrix elements, we find:
\[ < 0|J^{sc}(0) \bar{J}^{cb}(q)|B(v_B) > = \frac{f_B \sqrt{m_B}}{2v^\mu_D \sqrt{2v^\mu_B}} \left[ X(w) v^0_D (v^0_D + v^0_B) + 2T_{1/2}(w)v_D \cdot (v_B - v_D) \right] \]  
(48)
Identifying with the parton model result (47), and choosing \( v_D, v_B, v^2_D \) and \( w \) as independent variables (only \( w \) is covariant), we find the identity:
\[ (v_D \cdot v_B) \left[ X(w) + 2T_{1/2}(w) - 2 \right] + v^2_D \left[ X(w) - 2T_{1/2}(w) \right] + (w+1) [X(w) - 1] = 0 \]  
(49)
that implies the solution (39).

By the way, notice that formula (45) leads to a simple computation of the r.h.s. of (44). Particularizing (45) to \( (V^\mu - A^\mu)(V^\nu - A^\nu) \), i.e. \( \Gamma^{sc} = \gamma^\mu(1 - \gamma_5), \Gamma^{cb} = \gamma^\nu(1 - \gamma_5) \), we get
\[ <0|J^{sc}(0)\tilde{J}^{cb}(q)|B(v_B)> = \frac{f_B\sqrt{m_B}}{\nu_D\sqrt{2\nu_B^0}} \]

\[ [v_D^\mu v_B^\nu + v_D^\nu v_B^\mu - g_{\mu\nu}v_D \cdot v_B - i\varepsilon^{\mu\nu\rho\sigma}v_D^\rho v_B^\sigma] \]  

(50)

Contracting with the same tensor except for \( v_c \rightarrow v_c' = \frac{m_b v_b - m_c v_c}{m_c} \) and using \( v_b \cdot v_c = v_b \cdot v_c' = w, v_c \cdot v_c' = 2w^2 - 1 \) we obtain \( \frac{f_{BmB}}{2(\nu_D^0)^2\nu_B^0} 4w^2 = 2f_B^2 m_B \) in the \( B \) rest frame. From (18) we then obtain the result (10).

In particular, the equation \( X(w) = 1 \) implies a formula for the slope of the elastic IW function. Deriving this equation at \( w = 1 \), we get :

\[ \rho^2 = \sum_{n \neq 0} \frac{f^{(n)}}{f^{(0)}} \xi^{(n)'}(1) \]  

(51)

We get a new formula for the slope in terms of radial excitations alone. This is to be compared to BIW formula (4) that expressed the slope in terms of ground state to \( P \)-wave matrix elements. We must add that (51) does not have the positivity properties of (4) and does not imply any lower bound for \( \rho^2 \). Notice that the sum rules (39), (51) imply drastic cancellations among the ground state and the excited states. To have a feeling of what happens, we can compute \( f^{(n)} \) and \( \xi^{(n)}(w) \) in the non-relativistic harmonic oscillator model. One gets for \( \omega \cong 1 \)

\[ \xi(w) \cong 1 + \frac{R^2 m^2}{2\sqrt{2}} (w - 1) \]

\[ \frac{f^{(n)}}{f^{(0)}} = (-1)^n \left[ \frac{(2n + 1)!!}{2^n n!} \right]^{1/2} \]

\[ \xi^{(n)}(w) \cong \frac{(-1)^n}{n!} \left[ \frac{n!}{2^n(2n + 1)!!} \right]^{1/2} \left( \frac{R^2 m^2}{\sqrt{2}} \right)^n (w - 1)^n \xi(w) \]  

(52)

where \( R \) is the radius of a light \( q\bar{q} \) meson and \( m \) is the constituent \( q \) mass. One sees that (51) is satisfied by the contribution of the first radial excitation, since higher excitations vanish as \( (w - 1)^n \) \( (n \geq 2) \).

The sum rules that we have found in the heavy quark limit have a simple, explicit and easy proof in the non-relativistic limit (i.e. if the light quark were also non-relativistic). Indeed, we find, for a non-relativistic two-body system :

\[ \sum_n \psi_n(0) < n|r^2|0> = 0 \]  

(53)
which is the non-relativistic version of (51). The demonstration follows from
\[ \sum_n \psi_n(0) \int dr \, \psi_n^*(r) \, r^2 \, \psi_0(r) = \int dr \, r^2 \, \delta(r) \, \psi_0(r) = 0 \] (54)
since
\[ \sum_n \psi_n(r) \, \psi_n^*(r') = \delta(r - r') \] (55)
Similarly, one can generalize for any value of the recoil \( q \):
\[ \sum_n \psi_n(0) \, | <n | e^{iq \cdot r} | 0 > = \psi_0(0) \] (56)
and one can deduce, similarly:
\[ \sum_n \nabla \psi_n(0) \cdot | <n | r \, e^{iq \cdot r} | 0 > = 3 \psi_0(0) \] (57)
Equations (56) and (57) are the non-relativistic limit version of our sum rules in the heavy quark limit (39). In (56) and (57) only respectively the \( S \) and \( P \) waves contribute to the sum over the spectrum. In (56), the wave functions at the origin \( \psi_n(0) \) are related to \( f_n \) and one recovers \( X(w) = 1 \) (39) for \( w \approx 1 \) (non-relativistic approximation). In the Pauli approximation (first relativistic correction) the gradient at the origin \( \nabla \psi_n(0) \) is, introducing spin, related to \( f_{1/2}^{(n)} \) and \( f_{3/2}^{(n)} \). In the limit in which one of the quarks has infinite mass, \( f_{3/2}^{(n)} = 0 \) and one recovers the first equation (39), \( 2T_{1/2}(w) = 1 \), for \( w \approx 1 \).

In conclusion, from duality arguments applied to the non-leptonic quantity \( \Delta \Gamma \), or to the matrix element of the product of two currents between the vacuum and a pseudoscalar heavy meson, we have found new sum rules that involve the Isgur-Wise functions \( \xi_n(w) \) and \( \tau_{1/2}^{(n)}(w) \) and the decay constants \( f_n \), \( f_{1/2}^{(n)} \) (\( n \) denotes any radial excitation). These sum rules give a new insight on the role of the excitations of the spectrum in the heavy quark theory, and on their relation to the slope of the elastic Isgur-Wise function.

**Acknowledgements**

This work was supported in part by the CEC Science Project SC1-CT91-0729 and by the Human Capital and Mobility Program, contract CHRX-CT93-0132.
References

[1] J. D. Bjorken, invited talk at Les Rencontres de Physique de la Vallée d’Aoste, La Thuile (SLAC Report No. SLAC-PUB-5278, 1990 (unpublished) ; J.D. Bjorken, I. Dunietz and J. Taron, Nucl. Phys. B371, 111 (1992).

[2] N. Isgur and M. B. Wise, Phys. Rev. D43, 819 (1991).

[3] M. A. Shifman, in Proceedings of the International Symposium on Production and Decay of Heavy Hadrons, Heidelberg (1986) ; I.I. Bigi, M.A. Shifman, N.G. Uraltsev, A.L. Vainstein, Int. J. of Mod. Phys. A9, 2467 (1994) ; B. Blok, L. Koyraehk, M.A. Shifman and A.L. Vainshtein, Phys. Rev. D49, 3356 (1994).

[4] S. Hagelin, Nucl. Phys. B193, 123 (1981).

[5] M. A. Shifman, Nucl. Phys. B388, 346 (1992).

[6] A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Rev. D52, 2813 (1995).

[7] R. Aleksan, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Lett. B316, 567 (1993).

[8] We will use throughout the paper the quark model notation : $S$-wave mesons mean $J^P = 0^-, 1^-$ states and $P$-wave, $J^P = 0^+, 1^+, 2^+$. 

Figure Captions

Fig. 1. The cut of the box diagram corresponding to the exchange diagram.

Fig. 2. The cut of the box diagram corresponding to the spectator diagram.

Fig. 3. Generic final states $D_s \bar{D}_s$, common to $B_s$ and $\bar{B}_s$, from the color allowed spectator diagram.

Fig. 4. The matrix element between $\bar{B}_s$ and vacuum of the product of currents $A^{sc}_0(0) \bar{V}^{cb}_0(q)$, estimated within the parton model or by a sum over bound states in the heavy quark limit.