1 Proof that Yau-Hausdorff distance is a metric

1.1 Lemma 1:

Let $A$ and $B$ be two sets of finite points in $\mathbb{R}^d$, $d(a, b) = |a - b|$ is the Euclidean distance. For $a \in A$, we define $d(a, B) = \min_{b \in B} d(a, b)$. Similarly, we define $d(b, A) = \min_{a \in A} d(b, a)$. Then define $d(A, B) = \max_{a \in A} d(a, B)$, $d(B, A) = \max_{b \in B} d(b, A)$ and $h(A, B) = \max\{d(A, B), d(B, A)\}$. Then $h$ is a metric.

Proof:

1. Obviously $h \geq 0$.

If $h(A, B) = 0$, then $d(A, B) = d(B, A) = 0$. $\max_{a \in A} d(a, B) = 0$, implies for each $a \in A$ $d(a, B) = 0$. We have $\min_{b \in B} d(a, b) = 0$ for any $a \in A$. Because $B$ is a finite set, we can find $b \in B$, s.t. $b = a$.

This gives us $A \subset B$, similarly we have $B \subset A$. Hence $A = B$.

On the other hand if $A = B$, we have $h(A, B) = 0$ from definition, so $h(A, B) = 0$ if and only if $A = B$.

2. $h(A, B) = \max\{d(A, B), d(B, A)\} = \max\{d(B, A), d(A, B)\} = h(B, A)$.

3. We take three sets finite point sets $A, B, C$ in $\mathbb{R}^d$ and show that $h(A, B) \leq h(A, C) + h(C, B)$, i.e.

$$\max\{d(A, B), d(B, A)\} \leq \max\{d(A, C), d(C, A)\} + \max\{d(C, B), d(B, C)\}$$  \hspace{1cm} (1)

First we show that

$$d(a, B) \leq d(a, C) + d(C, B)$$  \hspace{1cm} (2)

for each $a \in A$. 


Assume

\[ d(a, C) = \min_{c \in C} d(a, c) = d(a, c_0), c_0 \in C \]  
(3)

\[ d(c_0, B) = \min_{b \in B} d(c_0, b) = d(c_0, b_0), b_0 \in B \]  
(4)

It follows that

\[ d(a, B) \leq d(a, b_0) \]  
(5)

\[ \leq d(a, c_0) + d(c_0, b_0) \]  
(6)

\[ = d(a, C) + d(c_0, B) \]  
(7)

\[ \leq d(a, C) + d(C, B) \]  
(8)

and equation (2) holds. Hence

\[ d(a, B) \leq d(a, C) + d(C, B) \]  
(9)

\[ \leq d(A, C) + d(C, B) \]  
(10)

\[ \leq \max\{d(A, C), d(C, A)\} + \max\{d(C, B), d(B, C)\} \]  
(11)

\[ = h(A, C) + h(C, B) \]  
(12)

for each fixed \( a \in A \).

Take the maximum of the left hand of this inequality,

\[ d(A, B) = \max_{a \in A} d(a, B) \leq h(A, C) + h(C, B) \]  
(13)

Similarly we can get

\[ d(B, A) \leq h(A, C) + h(C, B) \]  
(14)

\[ h(A, B) = \max\{d(A, B), d(B, A)\} \leq h(A, C) + h(C, B) \]  
(15)

The triangle inequality holds.
We have proven that $h$ is a metric.

1.2 Lemma 2:

Let $A$ and $B$ be two sets of finite points in $\mathbb{R}^d$. For a translation vector $t \in \mathbb{R}^d$, we define $A + t = \{a + t | a \in A\}$. For a rotation $\theta$, we define $A^\theta$ to be the set $A$ rotated around the origin by $\theta$. Let $H^d(A, B) = \inf_{t, \theta} h(A^\theta + t, B)$, then $H^d$ is a metric, and is called minimum $d$-dimensional Hausdorff metric.

Proof:

1. $$H^d(A, B) = \inf_{t, \theta} h(A^\theta + t, B) \geq 0$$ (16)

If $H^d(A, B) = 0$, then we can find $t_0$ and $\theta_0$, such that $h(A^\theta_0 + t_0, B) = 0$.

From Lemma 1 we have $A^\theta_0 + t_0 = B$ in $\mathbb{R}^d$, so $A \triangleq B$. (Here $A \triangleq B$ means that $A$ and $B$ are of the same shape, i.e. we can find translation $t$ and rotation $\theta$, such that $A^\theta + t = B$).

On the other hand, if $A \triangleq B$, then we can find $t_0$ and $\theta_0$, s.t. $A^\theta_0 + t_0 = B$.

Then $h(A^\theta_0 + t_0, B) = 0$ and $H^d(A, B) = \inf_{t, \theta} h(A^\theta + t, B) = 0$.

$H^d(A, B) = 0$ if and only if $A \triangleq B$. 


2. 

\[
\begin{align*}
H^d(A, B) & \quad (17) \\
= & \inf_{t, \theta} h(A^\theta + t, B) \quad (18) \\
= & \inf_{t, \theta} h(A, (B - t)^{-\theta}) \quad (19) \\
= & \inf_{t, \theta} h((B - t)^{-\theta}, A) \quad (20) \\
= & \inf_{t, \theta} h(B^{-\theta} - t, A) \quad (21) \\
= & \inf_{t', \theta'} h(B^{\theta'} + t', A) \quad (22) \\
= & H^d(B, A) \quad (23)
\end{align*}
\]

3. Take three finite point sets \( A, B, C \) in \( \mathbb{R}^d \) and show that \( H^d(A, B) \leq H^d(A, C) + H^d(B, C) \). This is equivalent to

\[
\inf_{t, \theta} h(A^\theta + t, B) \leq \inf_{t, \theta} h(A^\theta + t, C) + \inf_{t, \theta} h(B^\theta + t, C) \quad (24)
\]

Since the rotation group is compact and we only need to consider the translation in a compact region, we can find \( \theta_1, t_1, \theta_2, t_2 \), s.t.

\[
\begin{align*}
 h(A^{\theta_1} + t_1, C) & = \inf_{t, \theta} h(A^\theta + t, C) \quad (25) \\
 h(B^{\theta_2} + t_2, C) & = \inf_{t, \theta} h(B^\theta + t, C) \quad (26)
\end{align*}
\]
That gives us

\[ H^d(A, C) + H^d(B, C) \]  \hspace{1cm} (27)
\[ = \inf_{t, \theta} h(A^\theta + t, C) + \inf_{t, \theta} h(B^\theta + t, C) \]  \hspace{1cm} (28)
\[ = h(A^{\theta_1} + t_1, C) + h(B^{\theta_2} + t_2, C) \]  \hspace{1cm} (29)
\[ \geq h(A^{\theta_1} + t_1, B^{\theta_2} + t_2) \]  \hspace{1cm} (30)
\[ = h(A^{\theta_1} + t_1 - t_2, B^{\theta_2}) \]  \hspace{1cm} (31)
\[ = h((A^{\theta_1} + t_1 - t_2)^{-\theta_2}, B) \]  \hspace{1cm} (32)
\[ = h(A^{\theta_1} - \theta_2 + t_1 - t_2, B) \]  \hspace{1cm} (33)
\[ \geq \inf_{t, \theta} h(A^\theta + t, B) \]  \hspace{1cm} (34)
\[ = H^d(A, B) \]  \hspace{1cm} (35)

The triangle inequality holds.

We have proven that \( H^d \) is a metric.

1.3 Theorem:

Let \( A \) and \( B \) be two point sets of finite points in \( \mathbb{R}^2 \). For a rotation \( \theta \), we define \( P_x(A^\theta) \) to be the x-axis projection of \( A^\theta \).

\[ D(A, B) = \max \{ \sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^\varphi)) \} \]  \hspace{1cm} (36)

Here \( H^1 \) is the minimum one-dimensional Hausdorff distance,

\[ H^1(A, B) = \inf_{t \in \mathbb{R}} \max \min_{a \in A + t} |a - b|, \max_{b \in B} \min_{a \in A + t} |b - a| \]  \hspace{1cm} (37)

then \( D \) is a metric.

Proof:
1.  
\[ D(A, B) = \max \{ \sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^\varphi)), \frac{\sup_{\varphi} \inf_{\theta} H^1(P_x(A^\theta), P_x(B^\varphi))}{\} \} \]  
\( (38) \)

\[ D(B, A) = \max \{ \sup_{\theta} \inf_{\varphi} H^1(P_x(B^\theta), P_x(A^\varphi)), \frac{\sup_{\varphi} \inf_{\theta} H^1(P_x(B^\theta), P_x(A^\varphi))}{\} \} \]  
\( (39) \)

Since \( H^1(A, B) = H^1(B, A) \), we have

\[ \frac{\sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^\varphi))}{\} = \frac{\sup_{\varphi} \inf_{\theta} H^1(P_x(B^\theta), P_x(A^\varphi))}{\} \]  
\( (40) \)

which gives us \( D(A, B) = D(B, A) \).

2. We take three sets \( A, B, C \) of finite points in \( \mathbb{R}^2 \) and show that \( D(A, B) \leq D(A, C) + D(C, B) \). First we prove that

\[ \inf_{\varphi} H^1(P_x(A^{\theta_0}), P_x(B^\varphi)) \leq D(A, C) + D(C, B) \]  
\( (41) \)

for each fixed \( \theta_0 \). Assume \( \alpha_0 \) is a rotation, s.t.

\[ H^1(P_x(A^{\theta_0}), P_x(C^{\alpha_0})) = \inf_{\alpha} H^1(P_x(A^{\theta_0}), P_x(C^{\alpha})) \]  
\( (42) \)

\( \varphi_0 \) is a rotation, s.t.

\[ H^1(P_x(C^{\alpha_0}), P_x(B^{\varphi_0})) = \inf_{\varphi} H^1(P_x(C^{\alpha_0}), P_x(B^{\varphi})) \]  
\( (43) \)
So

$$\inf_{\varphi} H^1(P_x(A^{\theta_0}), P_x(B^\varphi)) \leq H^1(P_x(A^{\theta_0}), P_x(B^{\varphi_0})) \leq H^1(P_x(A^{\theta_0}), P_x(C^{\alpha_0})) + H^1(P_x(C^{\alpha_0}), P_x(B^{\varphi_0}))$$

$$= \inf_{\alpha} H^1(P_x(A^{\theta_0}), P_x(C^{\alpha_0})) + \inf_{\varphi} H^1(P_x(C^{\alpha_0}), P_x(B^{\varphi}))$$

$$\leq \sup_{\theta} \inf_{\alpha} H^1(P_x(A^{\theta}), P_x(C^{\alpha})) + \sup_{\alpha} \inf_{\varphi} H^1(P_x(C^{\alpha}), P_x(B^{\varphi}))$$

$$\leq D(A, C) + D(C, B)$$

for each fixed rotation $\theta_0$.

We take the maximum of all rotation $\theta$ in the left hand, and we get

$$\sup_{\theta} \inf_{\varphi} H^1(P_x(A^{\theta}), P_x(B^\varphi)) \leq D(A, C) + D(C, B)$$

Similarly, we can get

$$\sup_{\varphi} \inf_{\theta} H^1(P_x(A^{\theta}), P_x(B^\varphi)) \leq D(A, C) + D(C, B)$$

So

$$D(A, B) = \max\{\sup_{\theta} \inf_{\varphi} H^1(P_x(A^{\theta}), P_x(B^\varphi)), \sup_{\varphi} \inf_{\theta} H^1(P_x(A^{\theta}), P_x(B^\varphi))\}$$

$$\leq D(A, C) + D(C, B)$$

The triangle inequality holds.

3. Obviously $D(A, B) \geq 0$ for any two point sets $A, B$.

We need to prove that $A \cong B$ if and only if $D(A, B) = 0$.

If $A \cong B$, then $D(A, B) = 0$.

Conversely, if $D(A, B) = 0$, we need to show that $A \cong B$. 

7
Assume that there are \( m \) points in set \( A \) and \( n \) points in set \( B \). We assume that \( m > n \).

We can find a rotation \( \theta_0 \), s.t. the number of points in \( P_x(A^{\theta_0}) \) has \( m \) different points, but the number of points in \( P_x(B^{\varphi}) \) is no more than \( n \), so

\[
\begin{align*}
P_x(A^{\theta_0}) & \neq P_x(B^{\varphi}) \\
\implies \inf_{\varphi} H^1(P_x(A^{\theta_0}), P_x(B^{\varphi})) & > 0 \\
\implies D(A, B) & > 0
\end{align*}
\]

Contradiction! So we must have \( m \leq n \). Similarly we can get \( n \leq m \). So \( m = n \). The number of points of the two sets must be the same.

We consider a new question. If we know all the x-axis projections of set \( A \) with different rotation \( \theta \), can we reconstruct set \( A \) in the x,y-plane? This question is equivalent to the original question because all the projection of set \( A \) and set \( B \) are the same if \( D(A, B) = 0 \), and we are about to show that the answer of this new question is yes.

First we consider a simple situation. There are only three different points in set \( A \). Without loss of generality we fix a point at the origin. Then we rotate the set \( A \) three times so that each time a line that connects two points of \( A \) parallels the x-axis. So we can know the distance of any two points in set \( A \) from the information of projections, then the shape of set \( A \) is fixed.

If there are \( n \) different points in set \( A \), again we fix a point at the origin \( O \). Similarly we can determine the shape of the triangle \( \triangle OA_1A_2 \) with three rotations.

For the next point \( A_3 \), we can know the distance between \( A_3, O \), the distance between \( A_3, A_1 \) and the distance between \( A_3, A_2 \) by three rotations. So the location of \( A_3 \) is fixed. The other points are fixed in the same way.
For each point, we need three other rotations. So with \(3 + 3(n-3) = 3n-6\) rotations, the shape of \(A\) is fixed.

It means that we can reconstruct the set \(A\) in a plane from the information of \(P_\theta(A^\theta)\) for all \(\theta\). If \(D(A, B) = 0\), the projections of \(A\) and \(B\) with all the rotations are the same. \(A \triangleq B.\)

With symmetry, triangle inequality, non-negativity and identity of indiscernibles as shown above, we have proven that \(D\) is a metric. Q.E.D.

**Remark:** This theorem has a more general version. \(D(A, B)\) defined in Euclidean space \(\mathbb{R}^d\) is a metric, for all \(d \geq 2\).

Proof: Symmetry, triangle inequality an non-negativity can be proven the same way above. We only need to prove identity of indiscernibles.

Again, we only need to show that we can reconstruct set \(A\) up to rigid motion in \(\mathbb{R}^d\) with all the x-axis projections of \(A\) with different rotation \(\theta\). For \(d=2\), we have shown that \(3n - 6\) rotations is enough to reconstruct \(A\). There is a similar formula for arbitrary \(d\). Once we reconstruct \(A\) in \(\mathbb{R}^d\) with all the x-axis projections of \(A\) with different rotation \(\theta\), \(D(A, B) = 0\) gives us \(A \triangleq B.\)

We have proven that \(D\) is a metric.

## 2 Proof that \(H^2(A, B) \geq D(A, B)\)

### 2.1 Lemma

Let \(A = \{a_1, a_2, ..., a_n\} \subset \mathbb{R}^2\), \(B = \{b_1, b_2, ..., b_m\} \subset \mathbb{R}^2\). Let

\[
\begin{align*}
  d(A, B) &= \max_{1 \leq i \leq n} \min_{1 \leq j \leq m} d(a_i, b_j) \quad (57) \\
  d(B, A) &= \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} d(b_j, a_i) \quad (58) \\
  h(A, B) &= \max\{d(A, B), d(B, A)\} \quad (59)
\end{align*}
\]
then
\[ h(A^\theta, B^\varphi) \geq H^1(P_x(A^\theta), P_x(B^\varphi)) \] (60)

for any rotation \( \theta \) and \( \varphi \). Here \( H^1 \) is the minimum one-dimensional Hausdorff distance.

Proof: Assume \( A^\theta = \{a_{1\theta}, a_{2\theta}, ..., a_{n\theta}\} \), \( B^\varphi = \{b_{1\varphi}, b_{2\varphi}, ..., b_{m\varphi}\} \), \( P_x(A^\theta) = \{x_{1\theta}, x_{2\theta}, ..., x_{n\theta}\} \), \( P_x(B^\varphi) = \{y_{1\varphi}, y_{2\varphi}, ..., y_{m\varphi}\} \). \( x_{i\theta} \) is the x-projection of \( a_{i\theta}, 1 \leq i \leq n \) and \( y_{j\varphi} \) is the x-projection of \( b_{j\varphi}, 1 \leq j \leq m \).

\[ d(a_{i\theta}, b_{j\varphi}) \geq d(x_{i\theta}, y_{j\varphi}) \] for any i,j. Take the minimum of \( j = 1, 2, ..., m \) in this inequality, and we get

\[ \min_{1 \leq j \leq m} d(a_{i\theta}, b_{j\varphi}) \geq \min_{1 \leq j \leq m} d(x_{i\theta}, y_{j\varphi}) \] (61)

Take the max of \( i = 1, 2, ..., n \) in this inequality, and we get

\[ \max_{1 \leq i \leq n} \min_{1 \leq j \leq m} d(a_{i\theta}, b_{j\varphi}) \geq \max_{1 \leq i \leq n} \min_{1 \leq j \leq m} d(x_{i\theta}, y_{j\varphi}) \] (62)

This means

\[ d(A^\theta, B^\varphi) \geq d(P_x(A^\theta), P_x(B^\varphi)) \] (63)

Similarly we have

\[ d(B^\varphi, A^\theta) \geq d(P_x(B^\varphi), P_x(A^\theta)) \] (64)

\[ h(A^\theta, B^\varphi) \]
\[ = \max\{d(A^\theta, B^\varphi), d(B^\varphi, A^\theta)\} \] (65)
\[ \geq \max\{d(P_x(A^\theta), P_x(B^\varphi)), d(P_x(B^\varphi), P_x(A^\theta))\} \] (66)
\[ = h(P_x(A^\theta), P_x(B^\varphi)) \] (67)
\[ \geq \inf_{t \in \mathbb{R}} h(P_x(A^\theta) + t, P_x(B^\varphi)) \] (68)
\[ = H^1(P_x(A^\theta), P_x(B^\varphi)) \] (69)

Q.E.D.
2.2 Theorem

Let \( A = \{a_1, a_2, ..., a_n\} \subset \mathbb{R}^2 \), \( B = \{b_1, b_2, ..., b_m\} \subset \mathbb{R}^2 \). \( H^2(A, B) \) is the minimum two-dimensional Hausdorff distance of \( A \) and \( B \), i.e.

\[
H^2(A, B) = \inf_{t \in \mathbb{R}^2} \inf_\theta h(A^\theta + t, B)
\]

\( D(A, B) = \max\{\sup_\phi \inf_\varphi H^1(P_x(A^\varphi), P_x(B^\varphi)), \sup_\varphi \inf_\theta H^1(P_x(A^\theta), P_x(B^\varphi))\} \).

Then \( H^2(A, B) \geq D(A, B) \).

Proof: Assume

\[
d(A, B) = \max_{1 \leq i \leq n} \min_{1 \leq j \leq m} d(a_i, b_j) \quad (71)
\]

\[
d(B, A) = \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} d(b_j, a_i) \quad (72)
\]

\[
h(A, B) = \max\{d(A, B), d(B, A)\} \quad (73)
\]

\[
H^2(A, B) = \inf_{t \in \mathbb{R}^2} \inf_\theta h(A^\theta + t, B) \quad (74)
\]

First we prove that \( h(A^{\theta_1} + t_1, B) \geq D(A, B) \) for any fixed \( \theta_1 \) and \( t_1 \). We only need to show that \( h(A^{\theta_1} + t_1, B) \geq \sup_\varphi \inf_\theta H^1(P_x(A^\varphi), P_x(B^\varphi)) \).

Fix \( \theta = \theta_2 \),

\[
h(A^{\theta_1} + t_1, B) \quad (75)
\]

\[
h(A^{\theta_1}, B - t_1) \quad (76)
\]

\[
h(A, (B - t_1)^{-\theta_1}) \quad (77)
\]

\[
h(A^{\theta_2}, (B - t_1)^{-\theta_1 + \theta_2}) \quad (78)
\]

\[
h(A^{\theta_2}, B^{-\theta_1 + \theta_2} - t_1) \quad (79)
\]

\[
\geq H^1(P_x(A^{\theta_2}), P_x(B^{-\theta_1 + \theta_2} - t_1)) \quad (80)
\]

\[
= H^1(P_x(A^{\theta_2}), P_x(B^{-\theta_1 + \theta_2})) \quad (81)
\]

\[
\geq \inf_\varphi H^1(P_x(A^{\theta_2}), P_x(B^\varphi)) \quad (82)
\]
Equation (80) above is from the lemma. That gives us

\[ h(A^{\theta_1} + t_1, B) \geq \inf_{\varphi} H^1(P_x(A^{\theta_2}), P_x(B^\varphi)) \]  

(83)

for any fixed \( \theta_2 \), which means

\[ h(A^{\theta_1} + t_1, B) \geq \sup_{\theta} \inf_{\varphi} H^1(P_x(A^{\theta}), P_x(B^\varphi)) \]  

(84)

Similarly we can get

\[ h(A^{\theta_1} + t_1, B) \geq \sup_{\varphi} \inf_{\theta} H^1(P_x(A^{\theta}), P_x(B^\varphi)) \]  

(85)

Equations (84) and (85) give us

\[
\begin{align*}
& h(A^{\theta_1} + t_1, B) \\
\geq & \max \{ \sup_{\theta} \inf_{\varphi} H^1(P_x(A^{\theta}), P_x(B^\varphi)), \sup_{\varphi} \inf_{\theta} H^1(P_x(A^{\theta}), P_x(B^\varphi)) \} \\
=& D(A, B)
\end{align*}

(86)

(87)

(88)

for any \( \theta_1 \) and \( t_1 \). Take minimum of the left hand, and we have

\[
\begin{align*}
& \inf_{t \in \mathbb{R}^2} \inf_{\theta} h(A^\theta + t, B) \geq D(A, B) \\
\implies & H^2(A, B) \geq D(A, B)
\end{align*}

(89)

(90)

Q.E.D.

3 A simple example

Let \( A = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2 \), \( B = \{(0, 0), (0, 1), (1, 1)\} \subset \mathbb{R}^2 \). We will show that \( H^2(A, B) = \frac{1}{2} > \frac{\sqrt{5}}{10} = D(A, B) \)
3.1 Compute $H^2(A, B)$

First we prove that $h(A, B^\theta + t) \geq \frac{1}{2}$ for all fixed $\theta$ and $t$.

Draw 4 disks of radius $\frac{1}{2}$ centered at $O(0,0), M(0,1), P(1,0), N(1,1)$. Because there are three points in $B^\theta + t$, there must be a disk that does not contain any point of $B^\theta + t$. We denote the four disks $C_O, C_M, C_N, C_P$ and assume that there is no point of $B^\theta + t$ in $C_O$.

So

$$\min_{b_j \in B^\theta + t} d(O, b_j) \geq \frac{1}{2} \quad (91)$$

$$\implies d(O, B^\theta + t) \geq \frac{1}{2} \quad (92)$$

which gives us

$$d(A, B^\theta + t) = \max_{a_i \in A} d(a_i, B^\theta + t) \geq \frac{1}{2} \quad (93)$$

$$h(A, B^\theta + t) = \max\{d(A, B^\theta + t), d(B^\theta + t, A)\} \geq \frac{1}{2} \quad (94)$$

Take minimum of the left hand of equation (94), we have

$$\inf_{t \in \mathbb{R}^2} \inf_{\theta} h(A, B^\theta + t) \geq \frac{1}{2} \quad (95)$$

We then show that $\inf_{t \in \mathbb{R}^2} \inf_{\theta} h(A, B^\theta + t) = \frac{1}{2}$.

Take a rigid motion from $B$ to $B' = \{(\frac{1}{2}, 0), (\frac{3}{2}, 0), (\frac{1}{2}, 1)\}$.

$$d(A, B') = \max_{a_i \in A} \min_{b_j \in B'} d(a_i, b_j) = \frac{1}{2} \quad (96)$$

$$d(B', A) = \max_{b_j \in B'} \min_{a_i \in A} d(b_j, a_i) = \frac{1}{2} \quad (97)$$

$$h(A, B') = \max\{d(A, B'), d(B', A)\} = \frac{1}{2} \quad (98)$$
So

\[ H^2(A, B) = H^2(B, A) \]  \hspace{1cm} (99)

\[ = \inf_{t \in \mathbb{R}^2} \inf_{\theta} h(B^\theta + t, A) \]  \hspace{1cm} (100)

\[ = \inf_{t \in \mathbb{R}^2} \inf_{\theta} h(A, B^\theta + t) \]  \hspace{1cm} (101)

\[ = \frac{1}{2} \]  \hspace{1cm} (102)

3.2 Compute D(A,B)

First we compute \( \sup_{\theta} \inf_{\phi} H_1(P_x(A^\theta), P_x(B^\phi)) \).

Without loss of generality we may assume \( 0 \leq \theta \leq \frac{\pi}{2} \).

Let the projection of M,N,P after rotation \( \theta \) be \( M',N',P' \) (Fig.12).

Let \( a = OM' = \sin \theta, b = M'P' \), then \( P'N' = \sin \theta = a \).

Next we prove that \( \inf_{\phi} H_1(P_x(A^\theta), P_x(B^\phi)) = \frac{1}{2} \min\{a, b\} \).

Assume \( a \leq b \), draw four disks of radius \( \frac{1}{2}a \) centered at \( O, M', N', P' \), denoted as \( C_0, C_{M'}, C_{N'}, C_{P'} \).

Because there are no more than three points in the projection of \( B^\varphi \), there must be a disk that does not contain any point of \( P_x(B^\varphi) \). So \( H_1(P_x(A^\theta), P_x(B^\varphi)) \geq \frac{1}{2}a \), for any rotation \( \varphi \).

We then take a rigid motion \( \varphi_0 \), s.t. \( P_x(B^{\varphi_0}) = \{O', M', N'\} \).

Take \( t = -\frac{1}{2}a \), and translate \( P_x(B^{\varphi_0}) \) by \( t \).

Assume \( P_x(B^{\varphi_0}) - \frac{1}{2}a = \{O'', M'', N''\} \). We can see that the Hausdorff distance

Figure 12: Diagram for computing the Yau-Hausdorff distance
between $P_x(A^\theta)$ and $P_x(B^{\varphi_0}) - \frac{1}{2}a$ is $\frac{1}{2}a$. So

$$H^1(P_x(A^\theta), P_x(B^{\varphi_0})) = \frac{1}{2}a$$

(104)

$$\inf_{\varphi} H^1(P_x(A^\theta), P_x(B^{\varphi})) = \frac{1}{2}a = \frac{1}{2}\min\{a, b\}$$

(105)

Assume $b \leq a$, we can prove equation (105) in the same way.

Now we compute $\sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^{\varphi}))$, it is equal to $\frac{1}{2}\sup_{\theta}\min\{a, b\}$. We can see that $\min\{a, b\}$ achieves the maximum for $\theta$ if and only if $a = b$, because if one of the values of $\{a, b\}$ increases, the other will decrease. Assume that the rotation of $A$ is $\theta_0$, s.t. $a=b$.

So

$$a = OM \sin \theta_0 = \sin \theta_0, \angle OM M' = \theta_0$$

(106)

$$\angle M'MP = \angle OMP - \angle OM M' = \frac{\pi}{4} - \theta_0$$

(107)

$$b = MP \sin \angle M'MP$$

(108)

$$= \sqrt{2} \sin(\frac{\pi}{4} - \theta_0)$$

(109)

$$= \sqrt{2}(\frac{\sqrt{2}}{2} \cos \theta_0 - \frac{\sqrt{2}}{2} \sin \theta_0)$$

(110)

$$= \cos \theta_0 - \sin \theta_0$$

(111)

$$a = b$$

(112)

$$\Rightarrow \sin \theta_0 = \cos \theta_0 - \sin \theta_0$$

(113)

$$\Rightarrow \cos \theta_0 = 2 \sin \theta_0$$

(114)

$$\Rightarrow \sin \theta_0 = \frac{\sqrt{5}}{5}, \cos \theta_0 = \frac{2\sqrt{5}}{5}$$

(115)
So \( a = b = \sin \theta_0 = \frac{\sqrt{5}}{5} \).

\[
\sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^\varphi)) = \inf_{\varphi} H^1(P_x(A^0), P_x(B^\varphi)) = \frac{1}{2} \min\{\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\} = \frac{\sqrt{5}}{10} \tag{116}
\]

Similarly we can prove that

\[
\sup_{\varphi} \inf_{\theta} H^1(P_x(A^\theta), P_x(B^\varphi)) < \frac{\sqrt{5}}{10} \tag{120}
\]

\[
D(A, B) = \max\{\sup_{\theta} \inf_{\varphi} H^1(P_x(A^\theta), P_x(B^\varphi)), \sup_{\varphi} \inf_{\theta} H^1(P_x(A^\theta), P_x(B^\varphi))\} \tag{121}
\]

\[
= \frac{\sqrt{5}}{10} \tag{123}
\]