NILPOTENT FUSION CATEGORIES

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Abstract. In this paper we extend categorically the notion of a finite nilpotent group to fusion categories. To this end, we first analyze the trivial component of the universal grading of a fusion category $C$, and then introduce the upper central series of $C$. For fusion categories with commutative Grothendieck rings (e.g., braided fusion categories) we also introduce the lower central series. We study arithmetic and structural properties of nilpotent fusion categories, and apply our theory to modular categories and to semisimple Hopf algebras. In particular, we show that in the modular case the two central series are centralizers of each other in the sense of M. Müger.

Dedicated to Leonid Vainerman on the occasion of his 60-th birthday

1. Introduction

The theory of fusion categories arises in many areas of mathematics such as representation theory, quantum groups, operator algebras and topology. The representation categories of semisimple (quasi-) Hopf algebras are important examples of fusion categories. Fusion categories have been studied extensively in the literature, and there exist many results concerning their structure and classification (see [ENO] and references therein). However, there are still many fundamental open questions about fusion categories, which are motivated by group theory and the theory of semisimple Hopf algebras, and it is desirable to continue to develop the theory of fusion categories along these lines.

The purpose of this paper is to extend categorically the notion of a finite nilpotent group to fusion categories. For this end, we analyze the trivial component of the universal grading of a fusion category $C$, and then introduce the upper central series of $C$. For fusion categories with commutative Grothendieck rings (e.g., braided fusion categories) we also introduce the lower central series. We study arithmetic and structural properties of nilpotent fusion categories, and apply our theory to modular categories and to semisimple Hopf algebras. In particular, we show that in the modular case the two central series are centralizers of each other in the sense of M. Müger [Mu].

The organization of the paper is as follows.

In Section 2 we recall necessary definitions and results about based rings and modules, fusion categories, and Frobenius-Perron dimensions, which are needed in the sequel.

In Section 3, we first define the notion of the adjoint subring $R_{ad}$ of a based ring $R$. Then we prove in Theorem 3.5 that $R$ is naturally graded by a group $U(R)$ (which we call the universal grading group of $R$) and that $R_{ad}$ is the trivial component of this grading. The adjective “universal” above is justified in Corollary
where it is proved that any faithful grading of $R$ arises from a quotient of $U(R)$. In case $R$ is the Grothendieck ring of a fusion category $C$ we prove that the character group of the maximal abelian quotient of $U(R)$ is isomorphic to the group of tensor automorphisms of the identity functor of $C$ (see Proposition 3.9). When $C = \text{Rep}(H)$, the representation category of a semisimple Hopf algebra $H$, we have $C_{\text{ad}} = \text{Rep}(H/HK^+)$ where $K = \text{Fun}(U(C))$ is the maximal central Hopf subalgebra of $H$ (see Theorem 3.8).

Next we consider based rings of integer Frobenius-Perron dimension. We show in Theorem 3.10 that any such based ring is graded by an elementary abelian 2-group, and that the objects of integer dimension form the trivial component of this grading. This in particular implies that if the Frobenius-Perron dimension of a fusion category $C$ is an odd integer then the dimension of any object is an integer, and hence $C$ is equivalent to the representation category of a semisimple quasi-Hopf algebra (see Corollary 3.11).

In Section 4 we use the construction of the adjoint based subring and adjoint fusion subcategory to define the notions of the upper central series, nilpotency and nilpotency class for based rings and fusion categories, generalizing the corresponding notions for groups (see Definitions 4.1, 4.2, 4.4). We say that a semisimple Hopf algebra $H$ is nilpotent if the fusion category $\text{Rep}(H)$ of representations of $H$ is nilpotent. When $H$ is the group Hopf algebra of a finite group this agrees with the classical notion of a nilpotent group. Other examples of nilpotent fusion categories are pointed fusion categories, Tambara-Yamagami categories, and $p$-fusion categories (fusion categories of dimension $p^n$, $p$ a prime). It follows from [ENO] that a nilpotent fusion category is pseudounitary and admits a canonical spherical structure.

Next, we define the notion of the commutator of a based subring of a commutative based ring and of a fusion subcategory of a fusion category with commutative Grothendieck ring (see Definition 4.8 and Definition 4.10). We then use it to define the lower central series of commutative based rings and fusion categories with commutative Grothendieck ring (e.g., braided fusion categories), generalizing the corresponding notion for groups (see Definitions 4.12, 4.13). Finally, in Theorem 4.16 we compare the two series and prove that the upper one converges to $Z1$ at step $n$ if and only if the lower one converges to $R$ at step $n$. This generalizes a classical result in group theory, cf. [H].

In Section 5 we consider indecomposable $R$-modules and prove that they decompose as $R_{\text{ad}}$-modules with index set being a transitive $U(R)$-set (see Proposition 5.1). We use this to prove that in a nilpotent fusion category $C$ the square of the Frobenius-Perron dimension of any simple object divides the Frobenius-Perron dimension of $C_{\text{ad}}$, see Theorem 5.2 and Corollary 5.3 (this generalizes a well-known property of nilpotent groups). When applied to semisimple Hopf algebras of a prime power dimension, Corollary 5.3 extends a result in [MW].

In Section 6 we focus on modular categories. We prove in Theorem 6.3 that for a modular category $C$ its universal grading group $U(C)$ is canonically isomorphic to the character group of the group of invertible objects of $C$. For a pseudounitary modular category $C$ we find out in Theorem 6.8 that the lower and upper series of $C$ are related via the operation of taking centralizers, as defined by M. M"uger in [Mu]. As consequences we obtain that $C_{\text{ad}}$ is the centralizer of the maximal pointed subcategory of $C$, and that $C$ contains a “large” symmetric fusion subcategory.
Based rings and modules. Let $R$ be a ring with identity which is a finite rank $\mathbb{Z}$-module. A $\mathbb{Z}_+$-basis of $R$ is a basis $B = \{X_i\}_{i \in I}$ such that $X_iX_j = \sum_{k \in I} c_{ij}^k X_k$, where $c_{ij}^k \in \mathbb{Z}_+$. An element of $B$ will be called basic.

Let us define a non-degenerate symmetric $\mathbb{Z}$-valued inner product on $R$ as follows. For all elements $X = \sum_{i \in I} a_i X_i$ and $Y = \sum_{i \in I} b_i X_i$ of $R$ we set
\[(X, Y) = \sum_{i \in I} a_i b_i.\]

**Definition 2.1.** A based ring (or multi-fusion ring) is a pair $(R, B)$ consisting of a ring $R$ with a $\mathbb{Z}_+$-basis $B = \{X_i\}_{i \in I}$ satisfying the following properties:

1. There exists a subset $I_0 \subset I$ such that $1 = \sum_{i \in I_0} X_i$.
2. There is an involution $i \mapsto i^*$ of $I$ such that the induced map $X = \sum_{i \in I} a_i X_i \mapsto X^* = \sum_{i \in I} a_{i^*} X_i$ satisfies
$$
(XY, Z) = (X, ZY^*) = (Y, X^* Z)
$$
for all $X, Y, Z \in R$.

In what follows we will usually denote a based ring $(R, B)$ simply by $R$ assuming that some basis $B = \{X_i\}_{i \in I}$ is fixed.

**Remark 2.2.** It follows from Definition 2.1 that $X \mapsto X^*$ is an anti-automorphism of the ring $R$. Also, for each $i \in I$ there is a unique $i_0 \in I_0$ such that the coefficient of $X_{i_0}$ in the linear decomposition of the product $X_i X_{i^*}$ is equal to 1 while the coefficient of $X_j$ is 0 for each $j \in I_0$ such that $j \neq i_0$.

A based ring is called unital (or fusion) if $|I_0| = 1$.

**Note 2.3.** In this paper all based rings will be assumed unital. All our definitions and results can be easily extended to the non-unital case.

2. Preliminaries

Throughout the paper we work with an algebraically closed field $k$ of characteristic 0. All categories considered in this paper are finite, abelian, semisimple, and $k$-linear. All rings and modules have finite $\mathbb{Z}$-rank.

2.1. Based rings and modules. Let $\mathbb{Z}_+$ be the semi-ring of non-negative integers. We will recall some definitions from [O]. Let $R$ be a ring with identity which is a finite rank $\mathbb{Z}$-module. A $\mathbb{Z}_+$-basis of $R$ is a basis $B = \{X_i\}_{i \in I}$ such that $X_iX_j = \sum_{k \in I} c_{ij}^k X_k$, where $c_{ij}^k \in \mathbb{Z}_+$. An element of $B$ will be called basic.

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A based ring is called unital (or fusion) if $|I_0| = 1$.

**Note 2.3.** In this paper all based rings will be assumed unital. All our definitions and results can be easily extended to the non-unital case.
We will say that \( X \) is a partition of \( V, U \) and set each other. We will call an element of the basis of a based module basic by \( Z \) or \( R \).

Similarly to (1), for all elements \( V = \sum_{j \in J} m_j V_j \) and \( U = \sum_{j \in J} n_j V_j \) of \( M \) we set

\[
(V, U) = \sum_{i \in I} m_i n_i.
\]

A based submodule of a based \( R \)-module is defined in an obvious way. Based modules enjoy a complete reducibility [O] Lemma 1; i.e., every based \( R \)-submodule of a based \( R \)-module \( M \) has a direct complement which is also a based \( R \)-submodule of \( M \).

A based ring is pointed if all its basic elements are invertible. The invertible elements of \( R \) generate a maximal pointed based subring of \( R \) which we will denote by \( R_p \). A typical example of a pointed based ring is \((ZG, G)\), where \( G \) is a finite group.

Let \( R \) be a based ring and let \( R_+ \) consist of all \( Z_+ \)-linear combinations of \( \{X_1\}_{i \in I} \). We will say that \( X \in R_+ \) contains \( Y \in R_+ \) if \( X - Y \in R_+ \). A similar terminology will be used for based modules.

Let \( G \) be a finite group. We say that a based ring \((R, B)\) is graded by \( G \) if there is a partition \( B = \bigcup_{g \in G} B_g \) such that \( R = \bigoplus_{g \in G} R_g \), where \( R_g \) is a \( Z \)-submodule of \( R \) generated by \( B_g \) and \( R_g R_h \subset R_{gh}, R_g = R_{g^{-1}} \) for all \( g, h \in G \). Such a grading is faithful if \( R_g \neq 0 \) for all \( g \in G \).

2.2. Frobenius-Perron dimensions. Let us recall the definition and basic properties of Frobenius-Perron dimensions in based rings and their based modules from [ENO] Section 8.1. We note that Frobenius-Perron dimensions for commutative based rings were defined and used in the book [FK].

Let \( R \) be a unital based ring with basis \( B = \{X_1\}_{i \in I} \). Let \( X \) be a non-zero element of \( R_+ \) and let \( N_X \) be the matrix of multiplication by \( X \) in the basis \( B \). This matrix has nonnegative entries and is not nilpotent. The Frobenius-Perron theorem (see [G]) implies that \( N_X \) has a positive eigenvalue. The largest such eigenvalue is called the Frobenius-Perron dimension of \( X \) and is denoted by \( \text{FPdim}(X) \).

The assignment \( X \mapsto \text{FPdim}(X) \) extends to a homomorphism from \( R \) to \( \mathbb{R} \). This is the unique such homomorphism with the property that it sends basic elements to positive numbers. Clearly, Frobenius-Perron dimensions of elements of \( R \) are algebraic integers. The Frobenius-Perron dimension of \( R \) is defined to be \( \text{FPdim}(R) = \sum_{i \in I} \text{FPdim}(X_i)^2 \).

Let \( R = \bigoplus_{g \in G} R_g \) be a faithfully graded based ring with basis \( B = \bigcup_{g \in G} B_g \). Define \( \text{FPdim}(R_g) = \sum_{X_i \in B_g} \text{FPdim}(X_i)^2 \). Then \( \text{FPdim}(R_g) = \text{FPdim}(R)/|G| \) for all \( g \in G \) [ENO] Proposition 8.20]. In particular, \( |G| \) divides \( \text{FPdim}(R) \) in the ring of algebraic integers.

Let \( M \) be an indecomposable based module over a based ring \( R \) with a basis \( \{V_j\}_{j \in J} \). For every \( X \in R \) let \( L_X \) denote the matrix of left multiplication by \( X \) in this basis. There exists a unique (up to a positive scalar multiple) common eigenvector \( Q = \sum_{j \in J} \mu_j V_j \in M \otimes \mathbb{R} \) of the matrices \( L_X, X \in R \), such that \( \mu_j > 0 \) for all \( j \in J \) [ENO] Proposition 8.5]. The corresponding eigenvalue of \( L_X \) is \( \text{FPdim}(X) \). Define the Frobenius-Perron dimension of the basic element \( V_j \in M \) by \( \text{FPdim}(V_j) = \mu_j \). Unlike the Frobenius-Perron dimensions of elements of \( R \),
Remark 3.1. (1) Here the index \( K \) corresponding elements of \( I \) of basic elements contained in quantum groups and affine Lie algebras \([BK]\) and from subfactor theory. Hopf and quasi-Hopf algebras as well as semisimple tensor categories coming from \( C \) refer the reader to \([ENO]\) for a general theory of such categories. We will use this fact in Section 6.

The adjoint subcategory.

\[ \text{ad} \]

Examples of fusion categories include representation categories of semisimple Hopf and quasi-Hopf algebras as well as semisimple tensor categories coming from quantum groups and affine Lie algebras \([BK]\) and from subfactor theory.

Let \( C \) be a fusion category. Its Grothendieck ring \( K_0(C) \) is the free \( \mathbb{Z} \)-module generated by the isomorphism classes of simple objects of \( C \) with the multiplication coming from the tensor product in \( C \). The Grothendieck ring of a fusion category is a based unital ring. The Frobenius-Perron dimensions of objects in \( C \) (respectively, \( \text{FPdim}(C) \)) are defined as the Frobenius-Perron dimensions of their images in the based ring \( K_0(C) \) (respectively, as \( \text{FPdim}(K_0(C)) \)). For a semisimple quasi-Hopf algebra \( H \) one has \( \text{FPdim}(X) = \dim_k(X) \) for all \( X \in \text{Rep}(H) \), and so \( \text{FPdim}(\text{Rep}(H)) = \dim_k(H) \).

By a fusion subcategory of a fusion category \( C \) we understand a full tensor subcategory of \( C \). An example of a fusion subcategory is the maximal pointed subcategory \( C_{pt} \) generated by the invertible objects of \( C \).

A fusion category \( C \) is pseudounitary if its categorical dimension \( \dim(C) \) coincides with its Frobenius-Perron dimension, see \([ENO]\) for details. In this case \( C \) admits a canonical spherical structure (a tensor isomorphism between the identity functor of \( C \) and the second duality functor) with respect to which categorical dimensions of objects coincide with their Frobenius-Perron dimensions \([ENO, \text{Proposition 8.23}]\). We will use this fact in Section 6.

If we identify (the isomorphism classes of ) objects \( X, Y \in C \) with the corresponding elements of \( K_0(C) \) then \( (X, Y) = \dim_k \text{Hom}_C(X,Y) \).

A fusion category \( C \) is graded by a finite group \( G \) if the based ring \( K_0(C) \) is \( G \)-graded. That is, \( C \) decomposes into a direct sum of full abelian subcategories \( C = \bigoplus_{g \in G} C_g \) such that \( C_g^* = C_{g^{-1}} \) and the tensor product maps \( C_g \times C_h \) to \( C_{gh} \) for all \( g, h \in G \).

3. Graded based rings and fusion categories

3.1. The adjoint subcategory. Let \( R \) be a based ring with a \( \mathbb{Z}_+ \)-basis \( B = \{ X_i \}_{i \in I} \). Let \( R_{ad} \) be the minimal based subring of \( R \) with the property that \( X_i X_i^* \) belongs to \( R_{ad} \) for all \( i \in I \); i.e., \( R_{ad} \) is generated by all basic elements of \( R \) contained in \( X_i X_i^* \), \( i \in I \).

Equivalently, let us define \( I(1) := \sum_{i \in I} X_i X_i^* \). Then \( R_{ad} \) is the \( \mathbb{Z} \)-linear span of basic elements contained in \( I(1)^n \), \( n = 1, 2, \ldots \). Note that \( 1 \in R_{ad} \).

Remark 3.1. (1) Here the index \( ad \) stands for “adjoint” which is justified by the following observation. If \( H \) is a semisimple Hopf algebra and \( R = \)
\[ K_0(\text{Rep}(H)) \] is the Grothendieck ring of the representation category of \( H \) then \( R_{ad} \) is the subring generated by the subrepresentations of the adjoint representation of \( H \).

(2) Recall from [ENO] Section 5.8 that for a fusion category \( C \) with simple objects \( \{X_i\}_{i \in I} \) the induction functor \( I : C \to Z(C) \) to the center of \( C \) is defined as the left adjoint of the forgetful functor \( F : Z(C) \to C \). One has \( F(I(1)) \cong \sum_{i \in I} X_i \otimes X_i^* \), which explains our notation.

For a fusion category \( C \) let \( C_{ad} \) be the full tensor subcategory of \( C \) generated by all subobjects of \( X \otimes X^* \) where \( X \) runs through simple objects of \( C \). We have \( K_0(C_{ad}) = K_0(C)_{ad} \).

**Example 3.2.** Let \( G \) be a finite group and let \( C = \text{Rep}(G) \) be the representation category of \( G \). Then \( C_{ad} = \text{Rep}(G/Z(G)) \), where \( Z(G) \) is the center of \( G \).

**Proposition 3.3.**

1. The element \( I(1) \in R_{ad} \) is central in \( R \).
2. A based \( R_{ad} \)-subbimodule \( M \) of \( R \) is indecomposable if and only if it is indecomposable as a left (or right) \( R_{ad} \)-module.

**Proof.** To prove (1) we first observe the equality

\[ \sum_{i \in I} X_k X_i \otimes Z X_i^* X_k = \sum_{i \in I} X_i \otimes Z X_i^* X_k, \]

for all \( k \in \mathbb{I} \),

which can be obtained by evaluating the pairing \( (, ) \) of both sides with \( X_j \otimes Z X_l, j, l \in \mathbb{I}, \) and using Definition 2.1. Applying the multiplication of \( R \) to both sides of (3) we obtain \( X_k I(1) = I(1) X_k \) for all \( k \in \mathbb{I} \).

Since every basic element of \( R_{ad} \) is contained in some power of \( I(1) \), every one-sided based \( R_{ad} \)-submodule of \( M \) is automatically an \( R_{ad} \)-submodule. Indeed, if \( XY \in M \) for all \( X \in R_{ad} \) and \( Y \in M \) then \( I(1)^n Y = YT^n(1) \in M \) for all non-negative integers \( n \), and hence \( YX \in M \).

### 3.2. The universal grading of a based ring and of a fusion category.

Let \( R \) be a based ring. We can view \( R \) as a based \( R_{ad} \)-bimodule. As such, it decomposes into a direct sum of indecomposable based \( R_{ad} \)-bimodules: \( R = \bigoplus_{a \in A} R_a \), where \( A \) is the index set. This decomposition is unique up to a permutation of \( A \). We may assume that there is an element \( 1 \in A \) such that \( R_1 = R_{ad} \). Note that \( (R_a)^* = \{ X^* \mid X \in R_a \} \), \( a \in A \), is an indecomposable based \( R_{ad} \)-submodule of \( R \) and hence \( (R_a)^* = R_{a^*} \) for some \( a^* \in A \).

**Lemma 3.4.** For all \( X_a, Y_a \in R_a, a \in A \), we have \( X_a Y_a^* \in R_{ad} \).

**Proof.** Recall that each \( X \in R_{ad} \) is contained in some power of \( I(1) \). Observe that \( M = \{ X \in R_a \mid X \text{ is contained in } I(1)^n Y_a \text{ for some } n \} \) is a \( R_{ad} \)-submodule of \( R_a \). Since \( R_a \) is an indecomposable based \( R_{ad} \)-bimodule, it follows from Proposition 3.3(2) that \( M = R_a \) and therefore \( X_a \) is contained in \( I(1)^n Y_a \) for some \( n \). Hence, \( X_a Y_a^* \) is contained in \( I(1)^n Y_a Y_a^* \in R_{ad} \), as required.

**Theorem 3.5.** There is a canonical group structure on the index set \( A \) with the multiplication defined by the following property:

1. \( ab = c \) if and only if \( X_a X_b \in R_c \), for all \( X_a \in R_a, X_b \in R_b, a, b, c \in A \).

The identity of \( A \) is 1 and the inverse of \( a \in A \) is \( a^* \).
Proof: We need to check that the binary operation in (4) is well defined. Let \( a, b \in A \) and let \( X_a, Y_a \in R_a, X_b, Y_b \in R_b \) be basic elements of \( R \).

Suppose that the product \( X_aX_b \) contains a basic element \( x \in R_c \), and the product \( Y_aY_b \) contains a basic element \( y \in R_d \) for \( c \neq d \). By Lemma 3.4 there is a positive integer \( n \) such that the element \( X_aI(1)^nY_a^* \in R_1 \) contains \( X_aX_bY_b^*Y_a^* = (X_aX_b)(Y_aY_b)^* \) and, therefore, \( Y = X_aY_a^* \) is in \( R_1 \). Multiplying both sides of the last equality by \( Y_c \) on the right we conclude that \( YY_c \) is in \( R_c \cap R_d \), a contradiction.

Thus, \( X_aX_b \) and \( Y_aY_b \) both belong to the same component \( R_e \) and so the binary operation (4) is well-defined. It is easy to see that it defines a group structure on \( A \). □

Definition 3.6. We will call the grading \( R = \oplus_{a \in A} R_a \) constructed in Theorem 3.5 the universal grading of \( R \). The group \( A \) will be called the universal grading group of \( R \) and denoted by \( U(R) \).

Corollary 3.7. Every based ring \( R \) has a canonical faithful grading by the group \( U(R) \). Any other faithful grading of \( R \) by a group \( G \) is determined by a surjective group homomorphism \( \pi : U(R) \to G \).

Proof. Let \( R = \oplus_{g \in G} R^g \) be a grading of \( R \). Since for every basic \( X \in R \) we have \( XX^* \in R^1 \) it follows that \( R^1 \) contains \( R_{ad} \) as a based subring. Hence, each \( R^g \) is a based \( R_{ad} \)-submodule of \( R \). This means that every component \( R_a, a \in U(R) \), of the universal grading \( R = \oplus_{a \in U(R)} R_a \) of \( R \) belongs to some \( R^\pi(a) \) for some well-defined \( \pi(a) \in G \). Clearly, the map \( a \mapsto \pi(a) \) is a surjective homomorphism. □

For a fusion category \( C \) let \( U(C) = U(K(\mathcal{C})) \). We will call \( U(C) \) the universal grading group of \( C \).

Theorem 3.8. Let \( H \) be a semisimple Hopf algebra and let \( \mathcal{C} = \text{Rep}(H) \). There exists a unique Hopf subalgebra \( K \) of \( H \) which is contained in the center of \( H \) and is maximal with respect to this property. Let \( H_{ad} := H/HK^+ \) be the quotient of \( H \) by \( K \) (see [MG]). Then \( \mathcal{C}_{ad} = \text{Rep}(H_{ad}) \). Furthermore, \( K = \text{Fun}(U(C)) \).

Proof. Clearly such \( K \) exists. If \( K_1, K_2 \) are Hopf subalgebras of \( H \) contained in its center, then so is the Hopf subalgebra \( K_1K_2 \), so that \( K \) is unique. Let \( \Delta \) and \( S \) denote the comultiplication and antipode of \( H \). Every grading of \( \text{Rep}(H) \) comes from a decomposition \( H = \bigoplus_{a \in A} H_a \) where each \( H_a \) is an ideal of \( H \) such that \( \Delta(H_a) \subseteq \bigoplus_{xy = a} H_x \otimes H_y \) and \( S(H_a) = H_a^{-1} \). Let \( p_a \in H \) be the central idempotents such that \( p_aH = H_a, a \in A \). Then \( \Delta(p_a) = \sum_{xy = a} p_x \otimes p_y \), i.e., the linear span of \( p_a, a \in A \), is a Hopf subalgebra of \( H \) (isomorphic to \( \text{Fun}(A) \)) and is contained in the center of \( H \). Clearly, the universal grading of \( \text{Rep}(H) \) corresponds to a maximal such subalgebra. The trivial component of \( H \) is precisely the quotient \( H/\mathcal{C} \), so that \( \mathcal{C}_{ad} = \text{Rep}(H_1) = \text{Rep}(H/\mathcal{C}) \).

□

Let \( \mathcal{C} \) be a fusion category and let \( \text{Aut}(\mathcal{C}) \) denote the group of tensor auto-
morphisms of the identity functor of \( \mathcal{C} \). Any \( \Phi \in \text{Aut}(\mathcal{C}) \) is determined by a collection of non-zero scalars \( \{ \Phi(x) \} \) \( x \), one for every simple object \( X \) in \( \mathcal{C} \), such that \( \Phi(X_1)\Phi(X_2) = \Phi(Y) \) whenever \( Y \) is contained in \( X_1 \otimes X_2 \). Clearly, \( \text{Aut}(\mathcal{C}) \) is an abelian group.

For every abelian group \( A \) let \( \hat{A} \) denote the group of characters of \( A \).
Proposition 3.9. Let \( C \) be a fusion category and let \( G = U(C) \) be its universal grading group. Let \( G_{ab} \) be the maximal abelian quotient group of \( G \). There is an isomorphism between \( \hat{G}_{ab} \) and the group \( \text{Aut}_\otimes(\text{id}_C) \).

Proof. Define a grading of \( C \) by \( \hat{\text{Aut}_\otimes(\text{id}_C)} \) by setting
\[
C_\chi = \{ X \in C \mid \Phi_X = \chi(\Phi)\text{id}_X \text{ for all } \Phi \in \hat{\text{Aut}_\otimes(\text{id}_C)} \},
\]
where \( \chi \in \hat{\text{Aut}_\otimes(\text{id}_C)} \). Since \( G_{ab} \) is the universal abelian grading group of \( C \) it follows that there is a surjective group homomorphism \( G_{ab} \to \hat{\text{Aut}_\otimes(\text{id}_C)} \).

Conversely, let \( C = \bigoplus_{a \in G_{ab}} C_a \) be the grading of \( C \) by \( G_{ab} \) constructed in Corollary 3.7. For each \( \chi \in \hat{\text{Aut}_\otimes(\text{id}_C)} \) define \( \Phi_\chi \in \text{Aut}_\otimes(\text{id}_C) \) by \( (\Phi_\chi)_X = \chi(a)\text{id}_X \) for all \( X \in C_a \). The map \( \chi \mapsto \Phi_\chi \) is an injective group homomorphism, and hence it establishes the required isomorphism. \( \square \)

3.3. Based rings of integer Frobenius-Perron dimension. It was shown in \([ENO, \text{Section 8.5}]\) that a fusion category of integer Frobenius-Perron dimension is pseudounitary (and hence, admits a pivotal structure with respect to which the categorical dimensions of all simple objects coincide with their Frobenius-Perron dimensions).

Let us analyze the structure of based rings of integer Frobenius-Perron dimension.

Theorem 3.10. Let \( R \) be a based ring such that \( \text{FPdim}(R) \in \mathbb{Z} \). Then there is an elementary abelian \( 2 \)-group \( E \), a set of distinct square free positive integers \( n_x, x \in E \), with \( n_0 = 1 \), and a faithful grading \( R = \bigoplus_{x \in E} R(n_x) \) such that \( \text{FPdim}(X) \in \mathbb{Z}\sqrt{n_x} \) for each \( X \in R(n_x) \).

Proof. By \([ENO, \text{Proposition 8.27}]\) every basic element of \( R \) has dimension \( \sqrt{N} \) for some \( N \in \mathbb{Z} \). Let \( R(1) \subset R \) be the based subring of \( R \) generated by all basic elements of integer dimension. Observe that \( R_{ad} \subset R(1) \) and for each square free \( n \in \mathbb{Z} \) the basic elements of \( R \) whose dimension is in \( \mathbb{Z}\sqrt{n} \) generate an \( R(1) \)-subbimodule \( R(n) \) of \( R \). Let
\[
E = \{ n \text{ is square free } \mid \exists X \in R_+, X \neq 0, \text{ such that } \text{FPdim}(X) \in \mathbb{Z}\sqrt{n} \}.
\]
It is clear that for \( X \in R(n) \) and \( Y \in R(m) \) their product \( XY \) is in \( R((nm)') \) where \( l' \) denotes the square free part of \( l \). This defines a commutative group operation on \( E \) and a grading on \( R \). Since the order of every \( e \in E \) is at most two, \( E \) is an elementary abelian \( 2 \)-group. \( \square \)

Corollary 3.11. Let \( C \) be a fusion category of odd Frobenius-Perron dimension. Then the Frobenius-Perron dimension of every object of \( C \) is an integer (and hence, \( C \) is the category of representations of some semisimple quasi-Hopf algebra, see \([ENO]\)).

4. The central series and nilpotency for based rings and fusion categories

4.1. The upper central series. Let \( R \) be a based ring. Let \( R^{(0)} = R, R^{(1)} = R_{ad}, \) and \( R^{(n)} = (R^{(n-1)})_{ad} \) for every integer \( n \geq 1 \).
Definition 4.1. The non-increasing sequence of based subrings of $R$
\begin{equation}
R = R^{(0)} \supseteq R^{(1)} \supseteq \cdots \supseteq R^{(n)} \supseteq \cdots
\end{equation}
will be called the upper central series of $R$.

Similarly, for a fusion category $C$ we define $C^{(0)} = C$, $C^{(1)} = C_{ad}$, and $C^{(n)} = (C^{(n-1)})_{ad}$ for every integer $n \geq 1$.

Definition 4.2. The non-increasing sequence of fusion subcategories of $C$
\begin{equation}
C = C^{(0)} \supseteq C^{(1)} \supseteq \cdots \supseteq C^{(n)} \supseteq \cdots
\end{equation}
will be called the upper central series of $C$.

Example 4.3. For every group $H$ let $Z(H)$ denote its center. Let $G$ be a finite group and $C = \text{Rep}(G)$. Let
\begin{equation}
\{1\} = C^0(G) \subseteq C^1(G) \subseteq \cdots \subseteq C^n(G) \subseteq \cdots
\end{equation}
be the upper central series of $G$; i.e., $C^0(G) := \{1\}$, $C^1(G) := Z(G)$ and for $n \geq 1$ the subgroup $C^n(G)$ is defined by $C^n(G) = Z(G/C^{n-1}(G))$. Then $C^{(n)}(G) = \text{Rep}(G/C^n(G))$, so that our definition of the upper central series agrees with the classical one.

Definition 4.4. A based ring $R$ is nilpotent if its upper central series converges to $Z1$; i.e., $R^{(n)} = Z1$ for some $n$. The smallest number $n$ for which this happens is called the nilpotency class of $R$.

A fusion category $C$ is nilpotent if its upper central series converges to $\text{Vec}$; i.e., $C^{(n)} = \text{Vec}$ for some $n$. The smallest such $n$ is called the nilpotency class of $C$.

A semisimple Hopf algebra (or a quasi-Hopf algebra, or a weak Hopf algebra) is nilpotent if its representation category is nilpotent.

Example 4.5. (1) By [ENO, Theorem 8.28], every fusion category whose dimension is a prime power (i.e., a $p$-fusion category) is graded by $\mathbb{Z}/p\mathbb{Z}$, and hence is nilpotent. In particular, every semisimple Hopf or quasi-Hopf algebra of a prime power dimension is nilpotent.

(2) Semisimple Hopf algebras of dimension $2n^2$, $n \geq 2$, constructed by G. Kac and V. Paljutkin in [KP] (see also [Sc]) are nilpotent. These Hopf algebras have algebra structure $k^{n^2} \oplus M_n(k)$ and representation categories of Tambara-Yamagami type [TY]. In particular, a semisimple nilpotent Hopf algebra is not necessarily a tensor product of Hopf algebras of prime power dimension, which is in contrast with a classical result in group theory.

Proposition 4.6. Let $C$ be a nilpotent fusion category. Every fusion subcategory $E \subset C$ is nilpotent. If $F : C \to D$ is a surjective tensor functor, then $D$ is nilpotent.

Proof. We have $E^{(n)} \subset C^{(n)}$ and $D^{(n)} \subset F(C^{(n)})$ for all $n$. □

Remark 4.7. (1) Let $G$ be a finite group and $C = \text{Rep}(G)$. Then $C$ is nilpotent if and only if $G$ is nilpotent.
(2) By Theorem 3.5, a based ring \( R \) (respectively, a fusion category \( \mathcal{C} \)) is nilpotent if and only if every non-trivial based subring of \( R \) (respectively, a non-trivial fusion subcategory of \( \mathcal{C} \)) has a non-trivial grading.

(3) A fusion category \( \mathcal{C} \) is nilpotent if and only if \( K_0(\mathcal{C}) \) is nilpotent. The nilpotency class of \( \mathcal{C} \) is equal to the nilpotency class of \( K_0(\mathcal{C}) \).

(4) Pointed based rings (respectively, pointed fusion categories) are precisely the nilpotent based rings (respectively, fusion categories) of nilpotency class 1. Fusion categories constructed by Tambara and Yamagami \([TY]\) have nilpotency class 2.

(5) A nilpotent semisimple Hopf algebra is lower solvable in the sense of \([MW]\). (The special case of a twisted group algebra of a nilpotent group is discussed in \([GN]\).)

(6) It follows from Theorem 3.5 that a nilpotent fusion category comes from a sequence of gradings, in particular it has an integer Frobenius-Perron dimension. It follows from results of \([ENO]\) that a nilpotent fusion category \( \mathcal{C} \) is pseudounitary, i.e., its global dimension coincides with \( \text{FPdim}(\mathcal{C}) \), and that \( \mathcal{C} \) admits a pivotal (in fact, spherical) structure.

4.2. Lower central series for commutative based rings. For a commutative based ring \( R \) we can define the notion of a lower central series.

**Definition 4.8.** Let \( S \) be a based subring of a commutative based ring \( R \). We define the commutator \( S^{co} \) of \( S \) in \( R \) to be the based subring of \( R \) generated by all basic elements \( Y \in R \) such that \( Y Y^* \in S \).

**Remark 4.9.** Note that the \( \mathbb{Z} \)-span of basic elements \( Y \in R \) with the property \( Y Y^* \in S \) is already a based subring of \( R \) and so is equal to \( S^{co} \). Indeed, if \( Y_1 Y_1^*, Y_2 Y_2^* \in S \) and \( Y \) is contained in \( Y_1 Y_2 \) then \( Y Y^* \) is contained in \( (Y_1 Y_1^*)(Y_2 Y_2^*) \in S \).

**Definition 4.10.** Let \( \mathcal{C} \) be a fusion category such that \( K_0(\mathcal{C}) \) is commutative (e.g., \( \mathcal{C} \) is braided). Let \( \mathcal{K} \) be a fusion subcategory of \( \mathcal{C} \). Define the commutator \( \mathcal{K}^{co} \) of \( \mathcal{K} \) to be the fusion subcategory of \( \mathcal{C} \) generated by all simple objects \( Y \) of \( \mathcal{C} \) such that \( Y \otimes Y^* \in \mathcal{K} \). One has \( K_0(\mathcal{K}^{co}) = K_0(\mathcal{K})^{co} \).

**Example 4.11.** Let \( G \) be a finite group and let \( \mathcal{C} = \text{Rep}(G) \). Any fusion subcategory of \( \mathcal{C} \) is of the form \( \text{Rep}(G/N) \) for some normal subgroup \( N \) of \( G \). The simple objects of the category \( \text{Rep}(G/N)^{co} \) are irreducible representations \( Y \) of \( G \) for which \( Y \otimes Y^* \) restricts to the trivial representation of \( N \). Equivalently, each \( x \in N \) acts on \( Y \) by a scalar. This happens if and only if the commutator group \( [G, N] := \langle gxyg^{-1}x^{-1} \mid g \in G, x \in N \rangle \) acts trivially on \( Y \). Thus, \( \text{Rep}(G/N)^{co} = \text{Rep}(G/[G, N]) \).

For a commutative based ring \( R \) we define \( R_{(0)} = \mathbb{Z}1, R_{(1)} = R_{pt} \) (the based subring of \( R \) spanned by the invertible basic objects of \( R \)), and \( R_{(n)} = (R_{(n-1)})^{co} \) for every integer \( n \geq 1 \).

**Definition 4.12.** Let \( R \) be a commutative based ring. The non-decreasing sequence of based subrings of \( R \)

\[ \mathbb{Z}1 = R_{(0)} \subseteq R_{(1)} \subseteq \cdots \subseteq R_{(n)} \subseteq \cdots \]

will be called the lower central series of \( R \).
Similarly, for a fusion category $\mathcal{C}$ we define $\mathcal{C}(0) = \text{Vec}$, $\mathcal{C}(1) = \mathcal{C}_{pt}$ (the maximal pointed subcategory of $\mathcal{C}$) and $\mathcal{C}(n) = (\mathcal{C}(n-1))^{co}$ for every integer $n \geq 1$.

**Definition 4.13.** The non-decreasing sequence of fusion subcategories of $\mathcal{C}$

$$\text{Vec} = \mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \cdots \subseteq \mathcal{C}(n) \subseteq \cdots$$

will be called the lower central series of $\mathcal{C}$.

**Example 4.14.** Let $G$ be a finite group and $C = \text{Rep}(G)$. Let

$$G = C_0(G) \supseteq C_1(G) \supseteq \cdots \supseteq C_n(G) \supseteq \cdots$$

be the lower central series of $G$; i.e., $C_n(G) = [G, C_{n-1}(G)]$ for all $n \geq 1$. Then $\mathcal{C}(n) = \text{Rep}(G/C_n(G))$, so that our definition of the lower central series agrees with the classical one.

For a commutative based ring $R$ we can prove an analogue of the classical result in group theory saying that the upper central series of a group $G$ converges to $G$ if and only if its lower central series converges to $\{1\}$, which gives an alternative criterion of nilpotency $\square$.

**Lemma 4.15.** Let $R$ be a commutative based ring and let $S \subseteq R$ be a based subring. Then $(S^{co})_{ad} \subseteq S \subseteq (S_{ad})^{co}$.

**Proof.** Both inclusions are immediate from the definitions of the adjoint subring and commutator. $\square$

**Theorem 4.16.** Let $R$ be a commutative based ring and let

$$R^{(0)} \supseteq R^{(1)} \supseteq \cdots \supseteq R^{(n)} \supseteq \cdots \tag{10}$$

$$Z1 = R^{(0)} \subseteq R^{(1)} \subseteq \cdots \subseteq R^{(n)} \subseteq \cdots \tag{11}$$

be the upper and lower central series of $R$. Then $R^{(n)} = Z1$ if and only if $R^{(n)} = R$, and when this is the case one has $R^{(k)} \subseteq R^{(n-k)}$ for all $k = 0, \ldots, n$.

**Proof.** Suppose $R^{(n)} = R$; i.e., $R^{(0)} = R^{(n)}$. If $R^{(k)} \subseteq R^{(n-k)}$ for some $k$ ($0 \leq k < n$) then, using Lemma 4.15, we obtain $R^{(k+1)} = (R^{(k)})_{ad} \subseteq (R^{(n-k)})_{ad} = (R^{(n-k)})^{co}_{ad} \subseteq R^{(n-k-1)}$. Hence, $R^{(n)} = Z1$ by induction.

Conversely, suppose $R^{(n)} = Z1$; i.e., $R^{(n)} = R^{(0)}$. If $R^{(n-k)} \subseteq R^{(k)}$ for some $k$ ($n \geq k > 0$) then $R^{(n-k-1)} \subseteq (R^{(n-k-1)})^{co} = (R^{(n-k)})^{co} \subseteq (R^{(k)})^{co} = R^{(k+1)}$. Hence, $R^{(n)} = R$.

This means that neither central series can be longer than the other, as required. $\square$

5. **Divisibility properties of Frobenius-Perron dimensions of basic elements in a nilpotent based ring**

Let $(R, \{X_i\}_{i \in I})$ be a based ring with the universal grading $R = \oplus_{a \in U(R)} R_a$, and let $(M, \{V_j\}_{j \in J})$ be an indecomposable based $R$-module. Then $M$ is also a left based $R_{ad}$-module and therefore decomposes into a direct sum of indecomposable based $R_{ad}$-submodules: $M = \oplus_{x \in S} M_x$. This decomposition is unique up to a permutation of $S$.

Let us fix a Frobenius-Perron eigenvector $Q = \sum_{j \in J} \text{FPdim}(V_j)V_j \in M \otimes_\mathbb{R} \mathbb{R}$. For each $x \in S$ let $\text{FPdim}(M_x) = \sum_{V_j \in M_x} \text{FPdim}(V_j)^2$. The vector $Q$ is unique
Proposition 5.1. There is a canonical structure of a transitive $U(R)$-set on the index set $S$ defined by the following property:

$$(12)\quad ax = y \quad \text{if and only if} \quad X_aV_x \in M_y, \quad \text{for all} \quad X_a \in R_a, V_x \in M_x, \quad a \in U(R), \quad x, y \in S.$$ 

Furthermore, $\text{FPdim}(M_x)/\text{FPdim}(M_y) = 1$ for all $x, y \in S$.

Proof. We need to check that the action in (12) is well defined. Let $X_a, Y_a \in R_a, a \in U(R)$, and let $V_x, U_x$ be basic elements in $M_x, x \in S$. Suppose that the product $X_aV_x$ contains a non-zero element $V_y \in M_y$ and the product $Y_aU_x$ contains a non-zero element $U_z \in M_z$ for some $y \neq z$ in $S$. Let $I(1) = \sum_{i \in \mathbb{Z}} X_iX_i^* \in R_{ad}$. Since $M_x$ is an indecomposable based $R_{ad}$-module, for some $n \in \mathbb{Z}_+$ the element $U_x$ is contained in $I(1)^nV_x$. Therefore, we have

$$(V_x, X_a^*V_y) = (X_aV_x, V_y) > 0$$

and

$$(V_x, I(1)^nY_a^*U_z) = (Y_aI(1)^nV_x, U_z) \geq (Y_aU_x, U_z) > 0.$$ 

Combining these inequalities we get $(Y_aX_a^*V_y, I(1)^nU_z) > 0$. Since $Y_aX_a^* \in R_{ad}$, we obtain a contradiction: $(V_x', U_z') > 0$ for $V_x' = Y_aX_a^*V_y \in M_y$ and $U_z' = I(1)^nU_z \in M_z$, while $y \neq z$. This proves that the action in question is well-defined.

To prove the statement about Frobenius-Perron dimensions let

$$E = \sum_{a \in U(R)} E_a \in R \otimes \mathbb{R}$$

be the uniquely determined virtual regular element of $R$, where

$$E_a = \sum_{X_i \in R_a} \text{FPdim}(X_i)X_i,$$

and let $Q = \sum_{x \in S} Q_x \in M \otimes \mathbb{R}$, where

$$Q_x = \sum_{V_j \in M_x} \text{FPdim}(V_j)V_j \in M_x \otimes \mathbb{R}$$

for each $x \in S$.

The positive number $\text{FPdim}(E_a), a \in U(R)$, does not depend on $a$ by [ENO, Proposition 8.20]. Let us denote it by $d$. One has $E_aQ = dQ$, whence $E_aQ_x = dQ_{ax}$. Since $\text{FPdim} : M \to \mathbb{R}$ is an $R$-module homomorphism, taking $\text{FPdim}$ of both sides of the last equality we obtain $\text{FPdim}(Q_x) = \text{FPdim}(Q_{ax})$. Hence, $\text{FPdim}(M_x) = \text{FPdim}(M_{ax})$. Since the action of $U(R)$ on $S$ is transitive, the statement follows. □

Theorem 5.2. Let $R$ be a nilpotent based ring, let $M$ be an indecomposable based $R$-module, and let $V \in M$ be a basic element. Then $\text{FPdim}(M)/\text{FPdim}(V)^2 \in \mathbb{Z}$. In particular, $\text{FPdim}(R_{ad})/\text{FPdim}(X)^2 \in \mathbb{Z}$ for every basic $X$ in $R$.

Proof. Note that the ratio $\text{FPdim}(M)/\text{FPdim}(V)^2$ is well defined.

We use induction on the nilpotency class of $R$. For pointed based rings (i.e., those of nilpotency class 1) the statement is clear, since any indecomposable based module $M$ over $\mathbb{Z}G$, where $G$ is a finite group, comes from a transitive $G$-set and one can choose $\text{FPdim}(V) = 1$ for any basic $V \in M$. 

up to a scalar, hence the ratios $\text{FPdim}(M_x)/\text{FPdim}(M_y)$ are well-defined for all $x, y \in S$. 


Let $R$ be nilpotent of class $n > 1$, then $R_{ad}$ is nilpotent of class $n - 1$. Suppose the statement is true for all indecomposable $R_{ad}$-modules. Let $M$ be an indecomposable $R$-module and let $M_x, x \in S$, be its indecomposable $R_{ad}$-submodules.

Let $V$ be a basic object of $M$, then $V \in M_x$ for some $x \in S$. Applying Proposition 5.1 and using the inductive assumption, we have

\[ \frac{\text{FPdim}(M)}{\text{FPdim}(V)^2} = \frac{|S|\text{FPdim}(M_x)}{\text{FPdim}(V)^2} \in \mathbb{Z}, \]

as required. The second statement follows since each component $R_a$ in the universal grading $R = \oplus_{a \in U(R)} R_a$ is an indecomposable based $R_{ad}$-module and $\text{FPdim}(R_a) = \text{FPdim}(R_{ad})$ for all $a \in A$. \hfill \Box

**Corollary 5.3.** Let $\mathcal{C}$ be a nilpotent fusion category. Then for each simple object $X$ in $\mathcal{C}$, $\text{FPdim}(X)^2$ divides $\text{FPdim}(\mathcal{C}_{ad})$.

**Remark 5.4.**

1. In the case when $\mathcal{C}$ is the representation category of a nilpotent group, we recover a well-known fact about group representations: the square of the degree of an irreducible representation of a finite nilpotent group divides the index of its center.

2. It follows from [MW] that for an irreducible representation $V$ of an upper or lower semi-solvable Hopf algebra $H$ one has $\text{dim}_k(V) | \text{dim}_k(H)$. Corollary 5.3 generalizes this result in the special case when $H$ is nilpotent (e.g., has dimension $p^n, p$ a prime).

### 6. The Lower and Upper Central Series of Modular Categories

Let $\mathcal{C}$ be a modular category [BK] with simple objects $X_i, i \in \mathcal{I}$, the braiding $c_{X,Y} : X \otimes Y \cong Y \otimes X$ for all $X,Y \in \mathcal{C}$, the $S$-matrix $(s_{ij})_{i,j \in \mathcal{I}}$, where $s_{ij} = \text{tr}_q(c_{X_i,X_j} \circ c_{X_j,X_i})$, and the $T$-matrix $(\delta_{ij} \theta_i)_{i,j \in \mathcal{I}}$. Each dimension $d_i := d(X_i) = s_{0i}$ is a real number and each $\theta_i, i \in \mathcal{I}$, is a root of unity. The entries of the $S$-matrix are known to satisfy $s_{ij} = s_{ji} = s_{i^*j^*}$ and $\sum_j s_{ij} s_{j^*} = \delta_{i^*1} D$, where $i^* \in \mathcal{I}$ is an index such that $X_{i^*} = X_i^*$ and $D = \sum_{i \in \mathcal{I}} d_i^2$ is the categorical dimension of $\mathcal{C}$.

A theorem due to J. de Boere, J. Goeree, A. Coste and T. Gannon states that the entries of the $S$-matrix of a semisimple modular category lie in a cyclotomic field, see [BG] [CG]. That is, there exists a root of unity $\xi$ such that $s_{ij} \in \mathbb{Q}(\xi) \subset \mathbb{C}$. So we will always assume that the entries of the $S$-matrix are complex numbers.

It is known that $S/\sqrt{D}$ is a unitary matrix; i.e., $s_{ij} = \overline{s_{ij}}$ for all $i,j \in \mathcal{I}$ [ENO] Proposition 2.12]. It follows that $D = \sum_j s_{ij} s_{i^*j^*} = \sum_i |s_{ij}|^2$ for each $j \in \mathcal{I}$.

Let $X_i \otimes X_j \cong \bigoplus_{k \in \mathcal{I}} N_{ij}^k X_k, i,j \in \mathcal{I}$, be the fusion rules of $\mathcal{C}$. The Verlinde formula (see e.g., [BK] 3.1]) relates the fusion coefficients and entries of the $S$-matrix:

\[ N_{ij}^k = \sum_{k \in \mathcal{I}} s_{ik} s_{jk} s_{i^*j^*k} / d_k. \]

It follows from this formula that every homomorphism from $K_0(\mathcal{C})$ to $\mathbb{C}$ has the form

\[ h_j : X_i \mapsto \frac{s_{ij}}{s_{0j}}, \quad j \in \mathcal{I}. \]
The entries of the $S$- and $T$-matrices are related by the following formula [BK 3.1.2]:

\begin{equation}
    s_{ij} = \theta_i^{-1}\theta_j^{-1} \sum_{k\in I} N_{ij}^k \theta_k d_k.
\end{equation}

6.1. The universal grading group of a modular category.

**Lemma 6.1.** Let $j \in I$. We have $|s_{ij}| = |d_id_j|$ for all $i \in I$ if and only if $X_j$ is invertible.

**Proof.** Suppose $X_j$ is invertible. Then $d_j = \pm 1$ and for each $i \in I$ there is a unique $k \in I$ such that $N_{ij}^k \neq 0$ (in fact, $N_{ij}^k = 1$). Hence, $s_{ij} = \theta_i^{-1}\theta_j^{-1} \theta_k d_k$. Since every $\theta_i$, $i \in I$, is a root of unity and $d_k = d_id_j$ we have $|s_{ij}| = |d_id_j|$.

Conversely, if $j \in I$ is such that $|s_{ij}| = |d_id_j|$ for all $i \in I$ then $D = \sum |s_{ij}|^2 = d_i^2 D$ and hence $d_j = \pm 1$. Therefore, $s_{ij} = \xi_{ij} d_i$ for some $\xi_{ij}$ such that $|\xi_{ij}| = 1$. But then it follows from (14) that the image of $X_j$ under any ring homomorphism $K_0(C) \to \mathbb{C}$ has absolute value 1. Hence, FPdim($X_j$) = 1 and $X_j$ is invertible. □

Let $U(C)$ be the universal grading group of $C$ and let $G(C)$ be the group of invertible objects of $C$. Let $X(C)$ denote the set of ring homomorphisms $K_0(C) \to \mathbb{C}$. Formula (14) gives a bijection between $X(C)$ and the set $I$ of isomorphism classes of simple objects of $C$.

**Lemma 6.2.** The group $\text{Aut}_\otimes(id_C)$ acts on $X(C)$ by $\Phi h(X) := \Phi(X)h(X)$ for all $\Phi \in \text{Aut}_\otimes(id_C)$, $h \in X(C)$, and simple $X \in C$.

**Proof.** This is a direct consequence of the definition of a tensor automorphism. □

**Theorem 6.3.** There is a canonical isomorphism $U(C) \cong \hat{G(C)}$.

**Proof.** Since the group $U(C)$ is abelian, it suffices to establish an isomorphism $G(C) \cong \text{Aut}_\otimes(id_C)$, thanks to Proposition 3.9.

For any $\Phi \in \text{Aut}_\otimes(id_C)$ the function $X_i \mapsto \Phi(X_i)d_i$ extends to a homomorphism $K_0(C) \to \mathbb{C}$ and hence there is a unique $j_\Phi \in I$ such that $h_{j_\Phi}(X_i) = \Phi(X_i)d_i$. That is,

\begin{equation}
    s_{ij_\Phi} = \Phi(X_i)d_id_{j_\Phi}, \quad \text{for all } i \in I.
\end{equation}

Since $\Phi(X)$ is a root of unity we have $|s_{ij_\Phi}| = |d_id_{j_\Phi}|$. By Lemma 6.1, $X_{j_\Phi}$ is an invertible object of $C$.

Conversely, for any invertible $X_j \in C$ let $\Phi_j(X_i) = \frac{s_{ij}}{d_id_j}$, $i \in I$. To show that $\Phi_j$ defines a tensor automorphism of $id_C$ we need to check that

\begin{equation}
    \frac{s_{kj}}{d_i d_j} = \frac{s_{ij}}{d_i d_j} \frac{s_{kj}}{d_i d_j}, \quad \text{for all } k, i, l \in I \text{ such that } X_k \text{ is contained in } X_i \otimes X_l.
\end{equation}

for all $k, i, l \in I$ such that $X_k$ is contained in $X_i \otimes X_l$. Let $\gamma_i$, $i \in I$, be the scalar such that $c_{X_i} X_i \circ c_{X_i} X_j = \gamma_i id_{X_i \otimes X_j}$. Clearly, $\gamma_i = \frac{1}{d_id_j}$. Using the hexagon identity, we compute

\[
c_{X_i \otimes X_l} X_j \circ c_{X_i \otimes X_l} X_j = c_{X_i \otimes X_l} X_j \circ c_{X_i \otimes X_l} X_j = \gamma_i id_{X_i \otimes X_j \otimes X_l},
\]

where we omit identity morphisms and associativity constraints. Taking projection of both sides on $X_k \otimes X_j$ and computing the trace we obtain the identity (17).

It is clear that the above maps between $G(C)$ and $\text{Aut}_\otimes(id_C)$ are inverses of each other. That they are group homomorphisms follows from the following observation:
if $X_j, X_{j_1}, X_{j_2}$ are invertible objects of $\mathcal{C}$ such that $X_j \cong X_{j_1} \otimes X_{j_2}$ then $s_{j_1} s_{j_2} = d_i s_{j_1}$ for all $i \in \mathcal{I}$ (this is also a consequence of the hexagon identity, see [Mu Lemma 2.4(i)]).

**Remark 6.4.** For the representation category of a modular Hopf algebra $H$ the result of Theorem 6.3 was proved in [S3 Theorem 2.3(b)]. Namely, it was shown that the groups $G(H^\ast)$ and $G(H) \cap Z(H)$ are isomorphic.

### 6.2. The lower and upper central series of a pseudounitary modular category.

Let $\mathcal{K}$ be a full tensor subcategory of a braided tensor category $\mathcal{C}$. In [Mu] M. Müger introduced the *centralizer* $\mathcal{K}'$ of $\mathcal{K}$, which is a full tensor subcategory of $\mathcal{C}$. In the case when $\mathcal{C}$ is pseudounitary and modular, $\mathcal{K}'$ is the fusion category generated by all simple objects $X_i$ of $\mathcal{C}$ such that $i$

\[
\begin{equation}
\label{eq:18}
s_{ij} = d_i d_j
\end{equation}
\]

for all simple $X_j \in \mathcal{K}$. If (18) holds we will say that $X_i$ and $X_j$ *centralize* each other.

It was also shown in [Mu] that $\mathcal{K}' = \mathcal{K}$ and $\text{dim}(\mathcal{K}) \text{dim}(\mathcal{K}') = \text{dim}(\mathcal{C})$. A fusion subcategory $\mathcal{K} \subseteq \mathcal{C}$ is symmetric if and only if $\mathcal{K} \subseteq \mathcal{K}'$ and is modular if and only if $\mathcal{K} \cap \mathcal{K}' = \text{Vec}$.

From now on we will assume that $\mathcal{C}$ is a pseudounitary modular category. By [ENO] Proposition 8.23, there is a canonical spherical structure on $\mathcal{C}$ with respect to which the categorical dimensions of objects coincide with their Frobenius-Perron dimensions: $s_i = \text{FPdim}(X_i), i \in \mathcal{I}$. In particular, each $d_i$ is a positive real number.

We will use the notion of a centralizer to relate the upper and lower central series of a pseudounitary modular category.

**Lemma 6.5.**

1. For all $i, j \in \mathcal{I}$ we have $|s_{ij}| \leq d_i d_j$.
2. The real part of $d_i d_j - s_{ij}$ is non-negative and is equal to 0 if and only if $s_{ij} = d_i d_j$.

**Proof.** These results are essentially contained in [Mu]; we give proofs for the reader’s convenience. The first inequality follows from (18) and the Cauchy inequality:

\[
|s_{ij}| = |\theta_i^{-1} \theta_j^{-1} \sum_{k \in \mathcal{I}} N^k_{ij} \theta_k d_k| \leq \sum_{k \in \mathcal{I}} N^k_{ij} d_k = d_i d_j.
\]

Subtracting (18) from $d_i d_j = \sum_{k \in \mathcal{I}} N^k_{ij} d_k$ we obtain

\[
d_i d_j - s_{ij} = \sum_{k \in \mathcal{I}} N^k_{ij} d_k (1 - \theta_i^{-1} \theta_j^{-1} \theta_k).
\]

It is clear that the real part of every summand in the last sum is non-negative. Hence, the real part of $d_i d_j - s_{ij}$ equals 0 if and only if $\theta_k = \theta_i \theta_j$ for all $k \in \mathcal{I}$ such that $N^k_{ij} \neq 0$. By (18), this is equivalent to $s_{ij} = d_i d_j$.

**Lemma 6.6.** Let $X_i, X_k$ be simple objects in $\mathcal{C}$. Then $s_{kj} = d_k d_j$ for each $j \in \mathcal{I}$ such that $X_j$ is contained in $X_i \otimes X_k^\ast$ if and only if $|s_{ik}| = d_i d_k$.

**Proof.** Let us write the Verlinde formula (13) in the following equivalent form (cf. [Mu Lemma 2.4(iii)]):

\[
\begin{equation}
\label{eq:19}
\frac{1}{d_k} s_{ik} s_{kl} = \sum_{j \in \mathcal{I}} N^j_{il} s_{kj} \quad \text{for all } i, k, l \in \mathcal{I}.
\end{equation}
\]
In particular, we have
\[
\frac{1}{d_k}|s_{ik}|^2 = \frac{1}{d_k}s_{ik}s_{i,k} = \sum_{j \in \mathcal{I}} N_{ik}^j s_{kj},
\]
If \( s_{kj} = d_k d_j \) whenever \( X_j \) is contained in \( X_i \otimes X_i^* \) then
\[
d_k d_i^2 = \sum_{j \in \mathcal{I}} N_{ij}^j s_{kj} = \frac{1}{d_k}|s_{ik}|^2,
\]
whence \( |s_{ik}| = d_i d_k \).
Conversely, suppose \( |s_{ik}| = d_i d_k \). Then letting \( l = i^* \) in (14) we compute
\[
d_k d_i^2 = \sum_{j \in \mathcal{I}} N_{ij}^j s_{kj}.
\]
By Lemma 6.6.2) the last equality is possible if and only if \( s_{kj} = d_k d_j \) for all \( j \) such that \( N_{ij}^j \neq 0 \), so the proof is complete. \hfill \Box

**Proposition 6.7.** Let \( \mathcal{K} \) be a fusion subcategory of a pseudounitary modular category \( \mathcal{C} \). Let \( \mathcal{L} \) be the fusion subcategory generated by simple objects \( X_k \in \mathcal{C} \) such that \( |s_{ik}| = d_i d_k \) whenever \( X_i \in \mathcal{K} \). Then \( (\mathcal{K}_{ad})' = \mathcal{L} = (\mathcal{K}')^c \).

Proof. By definition, \( (\mathcal{K}')^c \) is generated, as an abelian category, by all simple \( X_i \in \mathcal{C} \) such that every subobject of \( X_i \otimes X_i^* \) centralizes \( \mathcal{K} \). By Lemma 6.6 the last condition is equivalent to \( |s_{ik}| = d_i d_k \) for all \( X_k \in \mathcal{K} \). Therefore, \( (\mathcal{K}')^c = \mathcal{L} \).

Let \( X_k \) be a simple object of \( (\mathcal{K}_{ad})' \). Then \( X_k \) centralizes every simple subobject \( X_j \) of \( X_i \otimes X_i^* \) whenever \( X_i \) is in \( \mathcal{K} \). It follows from Lemma 6.6 that \( |s_{ki}| = d_k d_i \); i.e., \( X_k \in \mathcal{L} \) and \( (\mathcal{K}_{ad})' \subseteq \mathcal{L} \).

Let \( X_k \) be a simple object of \( \mathcal{L} \), let \( X_i \) be a simple object of \( \mathcal{K} \), and let \( X_j \) be a simple subobject of \( X_i \otimes X_i^* \), \( k, j, i \in \mathcal{I} \). Then \( s_{kj} = d_k d_j \) by Lemma 6.6 and hence \( X_j \) belongs to \( \mathcal{L}' \). This implies that \( \mathcal{K}_{ad} \) is generated by some of the simple objects of \( \mathcal{L}' \), hence \( \mathcal{K}_{ad} \subseteq \mathcal{L}' \).

By Müger’s double centralizer theorem [Mu], \( \mathcal{L} = \mathcal{L}'' \), therefore the above inclusions imply \( \mathcal{L} = (\mathcal{K}_{ad})' \). \hfill \Box

**Theorem 6.8.** Let \( \mathcal{C} \) be a pseudounitary modular category and let
\[
\begin{align*}
\mathcal{C} &= \mathcal{C}^{(0)} \supseteq \mathcal{C}^{(1)} \supseteq \cdots \supseteq \mathcal{C}^{(n)} \supseteq \cdots \\
\text{Vec} &= \mathcal{C}^{(0)} \subseteq \mathcal{C}^{(1)} \subseteq \cdots \subseteq \mathcal{C}^{(n)} \subseteq \cdots
\end{align*}
\]
be the upper and lower central series of \( \mathcal{C} \). Then \( (\mathcal{C}^{(n)})' = \mathcal{C}^{(n)} \) for all \( n \geq 0 \).

Proof. We have \( (\mathcal{C}^{(0)})' = \mathcal{C}' = \text{Vec} = \mathcal{C}^{(0)} \). If \( (\mathcal{C}^{(n)})' = \mathcal{C}^{(n)} \) for some \( n \geq 0 \) then, using Proposition 6.7 we obtain \( (\mathcal{C}^{(n+1)})' = ((\mathcal{C}^{(n)})_{ad})' = ((\mathcal{C}^{(n)})')' = (\mathcal{C}^{(n)})^{co} = (\mathcal{C}^{(n)})' = \mathcal{C}^{(n+1)} \), and the statement follows by induction. \hfill \Box

Recall that \( \mathcal{C}_{pt} \) denotes the maximal pointed fusion subcategory of \( \mathcal{C} \); i.e., the subcategory generated by all invertible objects of \( \mathcal{C} \).

**Corollary 6.9.** Let \( \mathcal{C} \) be a pseudounitary modular category. Then \( \mathcal{C}_{ad} = (\mathcal{C}_{pt})' \).

**Corollary 6.10.** Let \( \mathcal{C} \) be a nilpotent modular category of nilpotency class \( c \). Then for each \( n \geq \frac{c}{2} \) the fusion subcategory \( \mathcal{C}^{(n)} \subseteq \mathcal{C} \) is symmetric.
Theorem 6.11. Let \( c \) be a braided fusion category. Then \( C \) is nilpotent if and only if its center \( Z(C) \) is nilpotent. Moreover, if the nilpotency class of \( C \) is \( c \) then the nilpotency class of \( Z(C) \) is at most \( 2c \).

Proof. By Proposition 4.6, if \( Z(C) \) is nilpotent, then so is \( C \).

Conversely, suppose \( C \) is nilpotent and identify it with \( C_+ \subseteq Z(C) \). Let
\[
C = C^{(0)} \supseteq C^{(1)} \supseteq \cdots \supseteq C^{(c)} = \text{Vec}
\]
be the upper central series of \( C \). Let \( \mathcal{E}_k, k = 0, \ldots, c \), be the centralizer of \( C^{(k)} \) in \( Z(C) \). By Proposition 6.7, we have
\[
\mathcal{E}_k = (C^{(k)})' = ((C^{(k-1)})_{ad})' = ((C^{(k-1)})')^0 = (\mathcal{E}_{k-1})^0.
\]
Combining this with Lemma 4.13, we obtain \( (\mathcal{E}_k)_{ad} = ((\mathcal{E}_{k-1})^0)_{ad} \subseteq \mathcal{E}_{k-1} \) for all \( k = 0, \ldots, c \). Note that \( \mathcal{E}_c = (C^{(c)})' = Z(C) \). Assume that \( Z(C)^{(k)} \subseteq \mathcal{E}_{c-k} \) for some \( k (0 \leq k < c) \). Then \( Z(C)^{(k+1)} = (Z(C)^{(k)})_{ad} \subseteq (\mathcal{E}_{c-k})_{ad} \subseteq \mathcal{E}_{c-k-1} \). Hence, \( Z(C)^{(c)} \subseteq \mathcal{E}_0 = C' = C_- \) by induction. Since \( C_- \) is nilpotent of class \( c \), the result follows.

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