Comparison methods for a Keller–Segel-type model of pattern formations with density-suppressed motilities

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Abstract
This paper is concerned with global existence of classical solutions as well as occurrence of infinite-time blowups to the following fully parabolic system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta (\gamma(v) u) \\
\frac{\partial v}{\partial t} - \Delta v + v &= u
\end{aligned}
\] (1)

in a smooth bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 1 \) with no-flux boundary conditions. This model was recently proposed in Fu et al. (Phys Rev Lett 108:198102, 2012) and Liu et al. (Science 334:238, 2011) to describe the process of stripe pattern formations via the so-called self-trapping mechanism. The system features a signal-dependent motility function \( \gamma(\cdot) \), which is decreasing in \( v \) and will vanish as \( v \) tends to infinity. An essential difficulty in analysis comes from the possible degeneracy as \( v \to \infty \).

In this work we develop a novel comparison method to tackle the degeneracy issue, which greatly differs from the conventional energy method in literature. An explicit point-wise upper-bound estimate for \( v \) is obtained for the first time, which shows that \( v(x,t) \) grows point-wisely at most exponentially in time. An intrinsic mechanism is then unveiled that the finite-time degeneracy is prohibited in any spatial dimension with a generic decreasing \( \gamma(\cdot) \). With such new findings, we further study global existence of classical solutions when \( n \leq 3 \) and discuss uniform-in-time boundedness when \( \gamma(\cdot) \) decreases algebraically at large signal concentrations. Besides, a new critical-mass phenomenon in dimension two is observed if \( \gamma(v) = e^{-v} \). Indeed, we prove that the classical
solution always exists globally and remains uniformly-in-time bounded in the sub-critical case, while in the super-critical case a blowup may take place in infinite time rather than finite time.

Mathematics Subject Classification 35K51 · 35K65 · 35Q92

1 Introduction

Recently, Fu et al. [11] proposed a fully parabolic system to model the process of stripe pattern formation through the so-called self-trapping mechanism. Denote the density of cells and the concentration of signals by \( u(x,t) \) and \( v(x,t) \), respectively. The resulting system reads

\[
\begin{align*}
    u_t &= \Delta (\gamma(v) u) + \mu u (1 - u) \\
    \varepsilon v_t - \Delta v + v &= u,
\end{align*}
\]

where \( \mu, \varepsilon \geq 0 \) are given constants. Here, \( \gamma(\cdot) \) is a signal-dependent motility function decreasing in \( v \), which characterizes the repressive effect of signal concentration on cellular motility. As experimentally observed in [11,25], this model correctly captures the dynamics at the propagating front where new stripes are formed.

Note that \( \Delta (\gamma(v) u) = \nabla \cdot (\gamma(v) \nabla u) + \nabla \cdot (u \gamma'(v) \nabla v) \). The first equation of (2) has the following variant form

\[
    u_t - \nabla \cdot (\gamma(v) \nabla u) = \nabla \cdot (u \gamma'(v) \nabla v) + \mu u (1 - u).
\]

Since \( \gamma' \leq 0 \), system (2) can be regarded as a chemotaxis model of Keller–Segel type involving signal-dependent diffusion rates and chemo-sensitivities.

Apparently, the dependence of diffusion rate on \( v \) leads to a possible degeneracy as \( v \) becomes unbounded. Theoretical results concerning global solvability or existence of blowup are rather limited in the literature. In [35], Tao and Winkler considered the initial-boundary value problem of (2) with \( \mu = 0 \) and \( \varepsilon = 1 \). By assuming uniform lower and upper bounds of \( \gamma \) and \( \gamma' \), they obtained global existence and uniform-in-time boundedness of classical solutions in two dimensions and the existence of global weak solutions in higher dimensions. Global existence of classical solutions in the three-dimensional case was also examined under certain smallness assumptions on the initial data. Note the assumptions therein exclude the possibility of degeneracy.

If \( \gamma(v) \) vanishes as \( v \) tends to infinity, then the degeneracy becomes a serious issue in analysis. Therefore, the key problem lies in whether one can derive an upper bound estimate for \( v \). One classical way in literature is to establish a certain \( L^p \)-integrability of \( u \), since the \( L^\infty_r L^p_x \)-boundedness of \( u \) will give rise to an upper bound for \( v \) via the second equation with any \( p > \frac{n}{2} \). Along with this idea, Yoon and Kim [45] studied (2) with the specific motility function \( \gamma(v) = c_0 v^{-k} \), \( \varepsilon = 1 \) and \( \mu = 0 \). By introducing approximating step functions of the motility, they obtained global existence and uniform-in-time boundedness of classical solutions for all \( k > 0 \) under a smallness assumption on \( c_0 > 0 \).

On the other hand, the presence of logistic growth terms also helps to achieve the higher \( L^p \)-integrability of \( u \). In [23], the degeneracy issue was tackled with the aid of the logistic source where global existence and uniform-in-time boundedness of classical solutions were proved with any \( \mu > 0 \) when \( n = 2 \) and \( \varepsilon = 1 \). Note a crucial assumption therein is that \( \lim_{s \to \infty} \frac{\gamma'(s)}{\gamma(s)} \) exists, which excludes fast decay motilities like \( e^{-v^2} \) or \( e^{-e^v} \). More recently in
[38], making use of the approach developed by Winkler [39] in the study of Keller–Segel model with logistic sources together with the approximating idea in [45], global existence and uniform-in-time boundedness of classical solutions were shown when \( n \geq 3 \) with large \( \mu > 0 \) under an assumption of uniform boundedness of \(|\gamma'(\cdot)|\) on \([0, \infty)\). Some variant forms of model (2) with generalized logistic source terms or indirect signal productions were also investigated recently in [26–28].

From a mathematical point of view, the problem becomes even challenging when \( \mu = 0 \). To the best of our knowledge, global existence of classical solutions without any smallness assumption or logistic sources was only achieved in the simplified parabolic-elliptic case, i.e., \( \varepsilon = 0 \). With the specific motility \( \gamma(v) = v^{-k} \), global existence and uniform-in-time boundedness of classical solutions were established by delicate energy estimates in [1] when \( n \leq 2 \) for any \( k > 0 \) or \( n \geq 3 \) for \( k < \frac{2}{n-2} \). When \( \varepsilon = 1 \) and \( \gamma(v) = \frac{1}{c + v^k} \) with some \( c \geq 0 \), \( k > 0 \), existence of global weak solution was obtained for all \( k > 0 \) if \( n = 1 \), for \( 0 < k < 2 \) if \( n = 2 \), and for \( 0 < k < \frac{4}{3} \) if \( n = 3 \) in [9]. We also refer the readers to [44] where some doubly nonlinear diffusion operators, involving porous medium type degeneracies, are considered in similar frameworks.

In all work concerning (2) mentioned above, the upper bound of \( v \) was established via deriving the \( L^p \)-integrability of \( u \) with \( p > \frac{n}{2} \) by energy methods, where most calculations were carried out relied on the more familiar variant form (3). Such an indirect method only works for some specific choice of \( \gamma \), or under certain smallness assumptions. And moreover, this idea cannot help to identify blowups at time infinity. Thus, the global existence of classical solutions and the occurrence of blowups to system (1) with possible degenerate motilities are largely open before the present work.

In fact, the expansion in (3) breaks the original delicate structure and omits some significant information. Recently in [12], we considered the simplified parabolic-elliptic version of system (2) with generic motility functions satisfying

\[
(A0) : \gamma(\cdot) \in C^3[0, \infty), \quad \gamma(\cdot) > 0, \quad \gamma'(\cdot) \leq 0 \quad \text{on} \quad (0, \infty).
\]  

(4)

Keeping the integrity of \( \Delta(\gamma(v)u) \) in the first equation, we made a subtle observation of the nonlinear coupling structure. A new method based on the comparison principle for elliptic equations was introduced to derive directly the point-wise upper bounds of \( v \). Thus, the finite-time degeneracy cannot take place for all \( n \geq 1 \). Then we showed that the classical solution always exists globally in dimension two under the assumption (A0) with any \( \mu \geq 0 \). Moreover, the global solution was proven to be uniformly-in-time bounded if either \( \mu > 0 \) or \( 1/\gamma \) satisfies certain polynomial growth condition. More importantly, the occurrence of exploding solutions was examined for the first time for this signal-dependent model. In the case \( \gamma(v) = e^{-v} \) and \( \mu = 0 \), a novel critical-mass phenomenon in the two-dimensional setting was observed that with any sub-critical mass, the global solution is uniformly-in-time bounded while with a certain super-critical mass, the global solution will blow up at time infinity.
In this paper, we study the initial-boundary value problem for the original doubly parabolic degenerate system:

\[
\begin{cases}
  u_t = \Delta (\gamma(v)u) & x \in \Omega, \; t > 0 \\
  v_t - \Delta v + v = u & x \in \Omega, \; t > 0 \\
  \partial_n u = \partial_n v = 0 & x \in \partial \Omega, \; t > 0 \\
  u(x, 0) = u_0(x), \; v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\]  

(5)

where \( \Omega \subset \mathbb{R}^n \) with \( n \geq 1 \) is a smooth bounded domain.

Our motivation comes from the typical choice \( \gamma(v) = e^{-v} \) in (5). Recall that the first equation of (5) has a variant form (3), which allows us to regard system (5) as a Keller–Segel-type system with signal-dependent diffusion rates and chemo-sensitivities. Under the circumstance, our system reads

\[
\begin{cases}
  u_t = \Delta (ue^{-v}) = \nabla \cdot (e^{-v}(\nabla u - u \nabla v)) & x \in \Omega, \; t > 0 \\
  v_t - \Delta v + v = u & x \in \Omega, \; t > 0,
\end{cases}
\]  

(6)

which has certain important features in common with the classical/minimal fully parabolic Keller–Segel system:

\[
\begin{cases}
  u_t = \nabla \cdot (\nabla u - u \nabla v) \\
  v_t - \Delta v + v = u \\
  \partial_n u = \partial_n v = 0.
\end{cases}
\]  

(7)

Indeed, beyond the formal resemblance, they share the same set of equilibria, which consists of solutions to the following stationary problem:

\[
\begin{cases}
  -\Delta v + v = \Lambda e^v / \int_{\Omega} e^v \, dx & \text{in } \Omega \\
  u = \Lambda e^v / \int_{\Omega} e^v \, dx & \text{in } \Omega \\
  \partial_n v = 0 & \text{on } \partial \Omega
\end{cases}
\]  

(8)

with \( \Lambda = \|u_0\|_{L^1(\Omega)} > 0 \). In addition, they have the same Lyapunov functional. Define the Lyapunov functional by

\[
F(u, v) = \int_{\Omega} \left( u \log u + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - uv \right) \, dx.
\]

Then for any strictly positive smooth solution \((u, v)\) of classical Keller–Segel system (7), there holds

\[
\frac{d}{dt} F(u, v)(t) + \int_{\Omega} u |\nabla \log u - \nabla v|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0,
\]

while for our system (6), there holds

\[
\frac{d}{dt} F(u, v)(t) + \int_{\Omega} u e^{-v} |\nabla \log u - \nabla v|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0,
\]  

(9)

where an extra weighted function \( e^{-v} \) appears in the second dissipation term.

It is well-known that the classical solutions of the Keller–Segel system (7) may blow up when \( n \geq 2 \), i.e., there exists \( T_{\text{max}} \in (0, \infty) \) such that

\[
\limsup_{t \nearrow T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.
\]
In particular, when $n = 2$, the classical Keller–Segel system (7) has a critical-mass phenomenon. More precisely, there is a threshold number $\Lambda_c > 0$ such that if the conserved total mass is less than $\Lambda_c$, then the global classical solution exists and remains bounded for all time $[16,30]$; otherwise, it may blow up in finite or infinite time $[20,32]$. Recently, a finite-time blowup solution was constructed in $[29]$ and to our knowledge, an infinite-time blowup has not been examined yet for the classical fully parabolic Keller–Segel system (7) (see $[5,17]$ for an infinite-time blowup in Cauchy problems of the simplified parabolic-elliptic Keller–Segel system and see $[8,24,34]$ for an infinite-time blowup in initial-boundary value problems in different kinds of chemotaxis models; see also $[43]$ revealing that an infinite-time blowup is a generic feature for certain quasilinear Keller–Segel models considered in $[8]$ and see $[42]$ for a detection of certain slow infinite-time blowup phenomena in a quasilinear Keller–Segel system involving a density-dependent diffusion and a sensitivity of negative exponential type). In higher dimensions, on the one hand global classical solution exists with sufficiently small initial data in the scaling-invariant spaces $[7,39]$ while on the other hand, a finite-time blowup was observed for initial data with arbitrarily small mass $[41]$.

In view of the same steady states of the above two systems (6) and (7) as well as the slight difference in dissipations during the evolutionary process, one purpose of the present paper is to figure out whether their solutions have similar dynamical behaviors.

Now, we summarize the main results of problem (5) as follow.

(I) When $n = 2$, we prove global existence of classical solution for all motility functions that have a vanishing limit, i.e., $\lim_{s \to \infty} \gamma(s) = 0$ and satisfy (A0). Moreover, uniform-in-time boundedness is obtained provided that $1/\gamma$ grows at a polynomial rate at most; see Theorem 1.

(II) When $n = 3$, we show uniform-in-time boundedness of global classical solutions supposing additionally that $1/\gamma$ grows at most linearly in $v$; see Theorem 2.

(III) For the case $\gamma(v) = e^{-v}$ and $n = 2$, the classical solution always exists globally due to our first main result. Besides, we show that the solution is uniformly-in-time bounded if the total mass is less than some critical mass $\Lambda_c > 0$ while with a certain initial datum of super-critical mass, we verify the occurrence of infinite-time blowup; see Theorem 3.

Now, let us sketch the idea of our comparison method in deriving the point-wise upper-bound estimate for $v$, which is the main novelty of the present contribution. First, inspired by our previous work $[12]$, we introduce a non-negative auxiliary function $w(x,t)$, which is the solution of the following elliptic Helmholtz equation:

$$\begin{cases}
-\Delta w + w = u & x \in \Omega, \; t > 0 \\
\partial_n w = 0 & x \in \partial \Omega, \; t > 0.
\end{cases}$$

We can formally write $w(x,t) = (I - \Delta)^{-1}[u](x,t)$ and we denote $w_0(x) = (I - \Delta)^{-1}[u_0]$. One notes that in the parabolic-elliptic case, i.e., $\varepsilon = 0$ in (2), $w$ is identical to $v$. However, in the present doubly parabolic case, from the second equation we formally have

$$v = w - (I - \Delta)^{-1}[v_t].$$

Thus, it suffices to derive upper bounds for both terms on the right-hand side of (11).

To this aim, we begin with deducing an upper bound for the auxiliary function $w$, which is nontrivial since we only have the $L^1$-boundedness of $u$ due to the conservation of mass. This goal is achieved by a subtle observation of the nonlinear coupling structure and an application of the comparison principle for elliptic equations. In the same manner as we have
previously done in [12], taking \((I - \Delta)^{-1}\) on both sides of the first equation of (5), we obtain
the following key identity:

\[
\partial_t w(x, t) + u \gamma(v) = (I - \Delta)^{-1}[u \gamma(v)](x, t),
\]

which captures the intrinsic mechanism of the system. Indeed, making use of the decreasing property of \(\gamma\), thanks to the comparison principle of elliptic equations together with Gronwall’s inequality, one can deduce from (12) that

\[
w(x, t) \leq w_0(x)e^{Ct}, \quad \text{for all } x \in \Omega \text{ and } t \geq 0
\]

with some \(C > 0\) depending only on \(\gamma, \Omega\) and the initial data.

The second step is to obtain an upper bound of \(v - w = -(I - \Delta)^{-1}[v_t]\), where the comparison principle for heat equations now plays a crucial role. Denote \(L[g] = g_t - \Delta g + g\) for any smooth function \(g(x, t)\) satisfying the homogeneous Neumann boundary conditions. Thanks to the key identity (12) again and via delicate calculations, we are able to construct an auxiliary function \(\Gamma(\cdot)\) such that

\[
L[v - w] \leq L[\Gamma(v) + K], \quad \text{for all } x \in \Omega \text{ and } t \geq 0,
\]

with some sufficiently large constant \(K > 0\) satisfying \(v_0(x) - w_0(x) \leq \Gamma(v_0(x)) + K\) for all \(x \in \Omega\). Making use of the property that \(\gamma\) is decreasing and has a vanishing limit, we may further show that

\[
\Gamma(v) \leq \varepsilon_0 v, \quad \text{for all } v > 0
\]

with some \(0 < \varepsilon_0 < 1\). Then it follows directly from the comparison principle of heat equations that

\[
v(x, t) \leq \frac{w(x, t) + K}{1 - \varepsilon_0}
\]

for all \(x \in \Omega\) and \(t \geq 0\).

Our method relies on comparison principles, which greatly differs from the energy methods used in all previous work. By our approach, an explicit point-wise upper-bound estimate for \(v\) is obtained for the first time showing that \(v(x, t)\) grows point-wisely at most exponentially in time. As a consequence, an insight information of the nonlinear structure is unveiled that the finite-time degeneracy is prohibited in any spatial dimension in this system. The main strategy of our approach lies in the idea to compare the solution \(v\) of the heat equation with the auxiliary function \(w\), which is the solution of the Helmholtz elliptic equation. To our knowledge, such an idea is used for the first time in related research and our approach makes fully use of the nonlinear coupling structure together with the decreasing property of \(\gamma\), but does not need any \(L^p\)-integrability of \(u\).

We remark that the aim of a study on system (5) with the specific choice \(\gamma(v) = e^{-v}\) is beyond just warranting the accessibility of our techniques. Based on the existing results for the classical Keller–Segel model, the dynamics of solutions is closely related to a balance between the diffusion rate and the chemotactic effect. Such a viewpoint is justified for a large variety of Keller–Segel type models when the diffusion and the chemo-sensitivity depend on \(u\) [4]. However, it lacks affirmative evidences when the system is solely signal-dependent. In our system, the decay rate of \(\gamma\) is seemingly playing a similar role. Indeed in 2D, we first prove that the classical solution always exists globally with generic \(\gamma\). However, the bounds of solution may depend on time, i.e., it is a growing-up solution in general. Thus a natural question is when is the solution uniformly-in-time bounded and do we really have unbounded
global solutions? Since when \( \gamma \) decreases at any polynomial rate, the solution is shown to be uniformly-in-time bounded (Theorem 1), a further attempt next is to consider the exponential rate. Our study justifies a new critical-mass phenomenon asserting an infinite-time blowup in the super-critical case, which partially supports our conjecture on the role of the decay rate of \( \gamma \) played in the dynamics of solutions.

Finally, we would like to stress that our results on global existence as well as infinite-time blowups are both new for the fully parabolic system (5) since this problem has not been tackled before without any smallness assumptions or the presence of source terms. The adaptation of some methods from previous works on Keller–Segel systems, is valid for our system only when the upper bound of \( v \) is obtained in advance. The latter issue as mentioned above is the major difficulty of the problem, which is completely solved via our comparison method in the current work. Even in the discussion on blowups, the conventional approach borrowed from the precedent work [20] can only assert that the solution may become unbounded in finite or infinite time. By our first global existence result stated in Theorem 1 given by comparison methods, we successfully exclude the possibility of finite-time blowup. Moreover, our new method also enables us to give new descriptions on blowup behaviors of \( v \) and \( w = (I - \Delta)^{-1}u \) besides of the \( L^\infty \)-unboundedness of \( u \) (see Theorem 3).

The rest of the paper is organized as follows. In Sect. 2, we state our main results on problem (5). In Sect. 3, we provide some preliminary results and recall some useful lemmas. Then in Sect. 4 we use our comparison argument to derive the upper bounds of \( v \). Uniform-in-time upper bounds of \( v \) are also established under certain growth conditions on \( 1/\gamma \). Thanks to the upper bound of \( v \), we are able to study global existence of classical solutions in Sect. 5. The last section is devoted to the case \( \gamma(v) = e^{-v} \), where the critical-mass phenomenon is proved in the two-dimensional setting.

## 2 Main results

In this section, we state the main results concerning global existence as well as an infinite-blowup of problem (5). To begin with, we introduce some notations and basic assumptions. Throughout this paper we assume that

\[
(u_0, v_0) \in (W^{1,p_0}(\Omega))^2 \quad \text{with some } p_0 > n, \quad u_0 \geq 0, \quad v_0 \geq 0 \quad \text{in } \Omega, \quad u_0 \neq 0
\]  
(14)

and for \( \gamma \) we require

\[
(A0): \gamma(\cdot) \in C^3[0, \infty), \quad \gamma(\cdot) > 0, \quad \gamma'(\cdot) \leq 0 \quad \text{on } (0, \infty),
\]  
(15)

and the following asymptotically vanishing property:

\[
(A1): \lim_{s \to \infty} \gamma(s) = 0.
\]  
(16)

Throughout this paper, \( C, M > 0 \) will stand for generic positive constants and for reader’s convenience, we will label different constants by \( C_i, M_i \) \((i = 1, 2, 3 \ldots)\) or \( C', C'', M', M'', \ldots \) inside proofs. Special dependence will be pointed out if necessary.

Now we state our first result on global existence of classical solutions in dimension two.

**Theorem 1** Assume \( n = 2 \) with \( \gamma(\cdot) \) satisfying (A0) and (A1). For any given initial data \((u_0, v_0)\) satisfying (14), system (5) permits a unique global classical solution \((u, v) \in (C^0([0, \infty); W^{1,p_0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)))^2\).
In addition, if $1/\gamma$ satisfies the following growth condition:

\[(A2) : \text{there is } k > 0 \text{ such that } \lim_{s \to \infty} s^k \gamma(s) = \infty, \tag{17}\]

then the global solution is uniformly-in-time bounded.

**Remark 1** The above result still holds true if one replaces assumption (A1) by the following

\[(A1') : \lim_{s \to \infty} \gamma(s) = \gamma_\infty < 1. \tag{18}\]

**Remark 2** If $v_0 > 0$ in $\overline{\Omega}$, thanks to the positive time-independent lower bound $v_\kappa$ of $v$ for $(x, t) \in \overline{\Omega} \times [0, \infty)$ given in Lemma 2 in the next section, our global existence and boundedness results also hold true if $\gamma(s)$ has singularities at $s = 0$, for example $\gamma(s) = s^{-k}$ with $k > 0$. In such cases, we can simply replace $\gamma(s)$ by a new motility function $\tilde{\gamma}(s)$ which satisfies (A0) and coincides with $\gamma(s)$ for $s \geq v_\infty^2$.

**Remark 3** Our result generalizes the corresponding boundedness result in [1] established for the simplified parabolic-elliptic system with the special motility $v^{-k}$ with any $k > 0$ or more general functions satisfying (A0), (A1) and (A2), for example, $\gamma(v) = \frac{1}{v^k \log(1+v)}$ with any $k > 0$.

**Remark 4** Theorem 1 is independent of the choice of coefficients in the system. In particular, if the second equation of (5) is replaced with

$$\varepsilon v_t = \Delta v - u$$

with $\varepsilon > 0$, Theorem 1 is still valid for any $\varepsilon > 0$. See Remark 9, Remark 11 and Remark 13.

**Remark 5** In the case $\gamma(v) = v^{-k}$ with $k > 0$, the variant form reads

$$u_t = \nabla \cdot \left[ \gamma(v) (\nabla u - ku \nabla \log v) \right], \tag{19}$$

which resembles the classical Keller–Segel model with a logarithmic chemo-sensitivity:

\[
\begin{aligned}
    u_t &= \nabla \cdot (\nabla u - ku \nabla \log v), \\
    \varepsilon v_t &= \Delta v - u + u. 
\end{aligned} \tag{20}
\]

Indeed, they have the same stationary problem. As to the two dimensional Keller–Segel model with a logarithmic chemo-sensitivity, global existence and uniform-in-time boundedness of solutions were established for sufficiently small or sufficiently large $\varepsilon > 0$ in [13,14]. Even global existence of solutions for any $\varepsilon > 0$ is still open. In contrast, Remark 4 claims global existence and uniform-in-time boundedness of solutions to (5) for any $\varepsilon > 0$.

In the three-dimensional setting, we obtain global existence and uniform-in-time boundedness of classical solutions with a stronger growth condition on $1/\gamma$.

**Theorem 2** Assume $n = 3$ and $\gamma(\cdot)$ satisfies (A0), (A1) and additionally

\[(A3) : 2|\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \forall s > 0. \tag{21}\]

For any given initial data $(u_0, v_0)$ satisfying (14), system (5) permits a unique global classical solution $(u, v) \in (C^0([0, \infty); W^{1,p_0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)))^2$, which is uniformly-in-time bounded.
Remark 6 Note that (A3) is a more restrictive growth condition than (A2). Under assumptions (A0), (A1) and (A3), $1/\gamma(s)$ can grow at most linearly in $s$; see Lemma 13.

In fact when $n = 3$, we can establish a uniform-in-time boundedness of $v$ with $\gamma(\cdot)$ satisfying (A0), (A1) and (A2) with some $0 < k < 2$. However, for technical reasons, we can now only achieve uniform-in-time bounds of $u$ with the help of assumption (A3); see Sect. 5.3 for more details.

Remark 7 When $n = 3$ and $\gamma(v) = v^{-k}$ with $k > 0$, (A3) is equivalent to a constraint $0 < k \leq 1$. Comparing with the Keller–Segel model with a logarithmic chemo-sensitivity (20), the condition (A3) reduces to a restriction on the chemo-sensitivity coefficient $k$. We remark that global existence of (20) is still open for large $k$ when $n \geq 3$ and we refer the readers to [4, 14] for reviews of related topics.

Last, we verify the following critical mass phenomenon for the case $\gamma(v) = e^{-v}$.

Theorem 3 Assume $n = 2$, $\gamma(v) = e^{-v}$ and $(u_0, v_0)$ satisfies (14). Let

$$A_c = \begin{cases} 
8\pi & \text{if } \Omega = B_R(0) \triangleq \{ x \in \mathbb{R}^2; |x| < R \} \text{ with } R > 0 \\
4\pi & \text{otherwise.}
\end{cases}$$

Then if $\Lambda \triangleq \int_{\Omega} u_0 dx < A_c$, the global classical solution of (5) is uniformly-in-time bounded. Moreover, the solution converges to an equilibrium as time goes to infinity, i.e., there is a solution $(u_s, v_s)$ to the stationary problem (8), such that

$$\lim_{t \to \infty} (u(t), v(t)) = (u_s, v_s) \text{ in } C^2(\Omega).$$

On the other hand, there exists non-negative initial datum $(u_0, v_0)$ satisfying (14) with $\Lambda \in (8\pi, \infty) \setminus 4\pi\mathbb{N}$ such that the corresponding global classical solution blows up at time infinity. More precisely,

$$\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \limsup_{t \to \infty} \|(I - \Delta)^{-1}[u](\cdot, t)\|_{L^\infty(\Omega)} = \limsup_{t \to \infty} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Remark 8 In [20], the authors considered a transformed problem of the minimal Keller–Segel system, where non-radial unbounded solutions were constructed when $\Lambda \in (4\pi, \infty) \setminus 4\pi\mathbb{N}$. However since their unknowns are the density and the reduced signal concentration, their initial datum for the reduced concentration does not preserve non-negativity.

In our problem, it is still unknown whether their construction of unbounded solution can be modified suitably to show that for arbitrary two-dimensional domains, there exists a “non-negative” initial datum such that $\Lambda \in (4\pi, 8\pi)$ inducing an unbounded solution; see Remark 17.

3 Preliminaries

In this section, we recall some useful lemmas. First, local existence and uniqueness of classical solutions to system (5) can be established by an application of Amann’s theory [2, 3].
Theorem 4  Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$. Suppose that $\gamma(\cdot)$ satisfies (A0) and $(u_0, v_0)$ satisfies (14). Then there exists $T_{\max} \in (0, \infty]$ such that problem (5) permits a unique classical solution

$$(u, v) \in (C^0([0, T_{\max}); W^{1,p_0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2$$

such that $(u, v) > 0$ on $\overline{\Omega} \times (0, T_{\max})$. Moreover, the following mass conservation holds

$$\int_{\Omega} u(\cdot, t) \, dx = \int_{\Omega} u_0 \, dx \quad \text{for all } t \in (0, T_{\max}).$$

If $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.$$

Proof The proof for local existence and uniqueness of a non-negative classical solution can be carried out in the same manner as done in [19, Theorem 1] applying the abstract theory for parabolic systems in [2, Theorem] and [3, Theorems 14.4, 14.6, 15.5]. Hence we omit the details for brevity here. Moreover, under the assumption (14), the positivity follows from the strong maximum principle for classical solution to single parabolic equations.  

Next, we recall the following lemma given in [1,6] about estimates for the solution of Helmholtz equations. Let $I_+ = \max(I, 0)$. Then we have

Lemma 1  Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$ and let $f \in C(\overline{\Omega})$ be a non-negative function such that $\int_{\Omega} f \, dx > 0$. For any $1 \leq q < \frac{n}{(n-2)_+}$, there exists a positive constant $C = C(n, q, \Omega)$ such that for any $z \in C^2(\overline{\Omega})$ satisfying

$$
\begin{cases}
-\Delta z + z = f, & x \in \Omega, \\
\frac{\partial z}{\partial \nu} = 0 & x \in \partial \Omega,
\end{cases}
$$

there holds

$$\|z\|_{L^q(\Omega)} \leq C\|f\|_{L^1(\Omega)}. \quad (23)$$

A strictly positive uniform-in-time lower bound for $v$ was given in [13, Lemma 2.1] provided that $v_0$ is strictly positive in $\overline{\Omega}$.

Lemma 2  Assume that $(u_0, v_0)$ satisfies (14) and moreover $v_0 > 0$ in $\overline{\Omega}$. If $(u, v)$ is the solution of (5) in $\Omega \times (0, T)$, then there exists some $v_* > 0$ such that

$$\inf_{x \in \Omega} v(x, t) \geq v_* > 0 \quad \text{for all } t \in (0, T).$$

Here the constant $v_*$ is independent of $T > 0$.

Then, we recall the following lemma given in [13, Lemma 2.4].

Lemma 3  Let $n = 2$ and $p \in (1, 2)$. There exists $K_{\text{Sob}} > 0$ such that for all $s > 1$ and for all $t \in [0, T_{\max}),$

$$\int_{\Omega} u^{p+1} \, dx \leq \frac{K_{\text{Sob}}(p + 1)^2}{\log s} \int_{\Omega} (u \log u + e^{-1}) \, dx \int_{\Omega} u^{p-2} |\nabla u|^2 \, dx$$

$$+ 6s^{p+1} |\Omega| + 4K_{\text{Sob}}^2 |\Omega|^{2-p} \|u_0\|_{L^1(\Omega)}^{p+1}. $$
In addition, we need the following uniform Gronwall inequality [37, Chapter III, Lemma 1.1] to deduce uniform-in-time estimates for the solutions.

**Lemma 4** Let \( g, h, \gamma \) be three positive locally integrable functions on \((t_0, \infty)\) such that \( \gamma' \) is locally integrable on \((t_0, \infty)\) and the following inequalities are satisfied:

\[
\gamma'(t) \leq g(t)\gamma(t) + h(t) \quad \text{for a.e. } t \geq t_0,
\]

\[
\int_t^{t+r} g(s) \, ds \leq a_1, \quad \int_t^{t+r} h(s) \, ds \leq a_2, \quad \int_t^{t+r} \gamma(s) \, ds \leq a_3, \quad \text{for a.e. } t \geq t_0
\]

where \( r, a_i, (i = 1, 2, 3) \) are positive constants. Then

\[
\gamma(t + r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \quad \text{for a.e. } t \geq t_0.
\]

### 4 The comparison method and the upper bound of \( v \)

In this section, we establish the upper bounds of \( v \) by our comparison method as illustrated in the Introduction. To begin with, we define an auxiliary variable \( w(x,t) \), which is the unique non-negative solution of the following Helmholtz equation:

\[
\begin{align*}
-\Delta w + w &= u, \quad x \in \Omega, \quad t > 0, \\
\partial_\nu w &= 0, \quad x \in \partial \Omega, \quad t > 0.
\end{align*}
\]

Then we derive the key identity and establish a point-wise upper bound for \( w \) as follow. Here and in the sequel, \( v_* = 0 \) if \( v_0 \geq 0 \) and \( v_* > 0 \) if \( v_0 > 0 \) in \( \tilde{\Omega} \) due to Lemma 2.

**Lemma 5** Assume \( n \geq 1 \). For any \( 0 < t < T_{\text{max}} \), there holds

\[
w_t + \gamma(v)u = (I - \Delta)^{-1}[\gamma(v)u].
\]

Moreover, for any \( x \in \Omega \) and \( t \in [0, T_{\text{max}}) \), we have

\[
w(x, t) \leq w_0(x)e^{\gamma(v_*)t}.
\]

**Proof** The proof was already given in our previous paper [12]. For the completeness of the present work, we report in detail here. First, the key identity (24) follows by taking \((I - \Delta)^{-1}\) on both sides of the first equation in (5). Here, \( \Delta \) is the Laplacian operator with homogeneous Neumann boundary conditions.

Note that \( v \) is non-negative due to the maximum principle of heat equations. Since \( \gamma \) is non-increasing in \( v \), there holds \( \gamma(v) \leq \gamma(v_*) \) for all \((x, t) \in \Omega \times [0, T_{\text{max}})\). As a result, we infer by comparison principle of elliptic equations that for any \((x, t) \in \Omega \times [0, T_{\text{max}})\),

\[
(I - \Delta)^{-1}[\gamma(v)u] \leq (I - \Delta)^{-1}[\gamma(v_*)u] = \gamma(v_*)w
\]

and it follows from (24) that

\[
w_t + \gamma(v)u \leq \gamma(v_*)w.
\]

Since \( \gamma(v)u \geq 0 \), an application of Gronwall’s inequality together with (26) gives rise to

\[
w(x, t) \leq w_0(x)e^{\gamma(v_*)t},
\]

which completes the proof. \( \square \)
Next, we aim to compare $v$ with the bounded auxiliary function $w$. Observing that $\lim_{s \to \infty} \gamma(s) = 0$, we can fix some $\rho > 0$ such that $0 < \gamma(\rho) < 1$ and for any $s \geq 0$ we define

$$\Gamma(s) = \int_{\rho}^{s} \gamma(\eta) \, d\eta.$$  

(27)

Then, one can easily verify the following relation between $\gamma$ and $\Gamma$.

**Lemma 6** Under the assumption of (A0) and (A1), for any $s_0 \in [0, \rho)$, there is $C_\rho(s_0) > 0$ depending on $\rho$ and $s_0$ such that

$$s \gamma(s) - C_\rho(s_0) \leq \Gamma(s) \leq \gamma(\rho) s, \quad \forall \ s \geq s_0.$$  

(28)

**Proof** Consider first the case $s \geq \rho$. Since $\gamma$ is decreasing,

$$(s - \rho) \gamma(s) \leq \Gamma(s) \leq (s - \rho) \gamma(\rho) \quad \text{for } s \geq \rho.$$  

By using $\gamma(s) \leq \gamma(\rho)$ and $\rho \gamma(\rho) > 0$, we have

$$s \gamma(s) - \rho \gamma(\rho) \leq \Gamma(s) < \gamma(\rho) s \quad \text{for } s \geq \rho.$$  

Then in order to establish (28), it remains to check the case $s_0 \leq s \leq \rho$. The most right-hand side is trivial since $\Gamma(s) \leq 0$ by definition when $s_0 \leq s \leq \rho$. On the other hand when $s_0 \leq s \leq \rho$, using the decreasing property of $\gamma$ again, there holds

$$s \gamma(s) - \Gamma(s) = s \gamma(s) + \int_{\rho}^{s} \gamma(\eta) \, d\eta$$

$$\leq \rho \gamma(s_0) + \gamma(s_0)(\rho - s_0)$$

$$\leq 2 \rho \gamma(s_0),$$

(29)

which completes the proof. \qed

Now, we are ready to apply the comparison principle of parabolic equations to obtain the following result.

**Lemma 7** Under the assumption of (A0) and (A1), there is $K > 0$ depending on $\rho$ and the initial data such that for all $(x, t) \in \Omega \times [0, T_{\text{max}})$,

$$v(x, t) \leq \frac{1}{1 - \gamma(\rho)} \left( w(x, t) + K \right).$$  

(30)

**Proof** Recall that $w - \Delta w = u$. Substituting the key identity (24) into the second equation of (5), we observe that

$$v_t - \Delta v + v = w - \Delta w$$

$$= w - \Delta w + w_t - w_t$$

$$= w_t - \Delta w + w + \gamma(v)u - (I - \Delta)^{-1}[\gamma(v)u].$$  

(31)

Using the second equation of (5) again, we observe that

$$\gamma(v)u = \gamma(v)(v_t - \Delta v + v)$$

$$= \left( \partial_t \Gamma(v) - \Delta \Gamma(v) + \Gamma(v) \right) + \gamma'(v)|\nabla v|^2 + v \gamma(v) - \Gamma(v).$$  

(32)
Then plugging (32) into (31) yields that

\[ v_t - \Delta v + v + (I - \Delta)^{-1}[\gamma(v)u - \gamma'(v)|\nabla v|^2 = \left( \partial_t (w + \Gamma(v)) - \Delta (w + \Gamma(v)) + (w + \Gamma(v)) \right) + (v\gamma(v) - \Gamma(v)). \]  

(33)

According to Lemma 6, there is \( C(v_\ast) > 0 \) depending on \( \rho \) and \( v_\ast \) such that for all \((x, t) \in \Omega \times [0, T_{\text{max}})\)

\[ v\gamma(v) - \Gamma(v) \leq C(v_\ast). \]  

(34)

In addition, since \((I - \Delta)^{-1}[\gamma(v)u] \) and \(-\gamma'(v)|\nabla v|^2 \) are both non-negative, it follows from (33) that for all \((x, t) \in \Omega \times [0, T_{\text{max}}),\)

\[ v_t - \Delta v + v \leq \left( \partial_t (w + \Gamma(v)) - \Delta (w + \Gamma(v)) + (w + \Gamma(v)) \right) + C(v_\ast). \]  

(35)

Now, in view of our assumption (14) on the initial data, we may choose a positive constant \( K \geq C(v_\ast) \) such that \( v_0 \leq w_0 + \Gamma(v_0) + K \) for all \( x \in \Omega \). Then we deduce by comparison principle for heat equations that

\[ v(x, t) \leq w(x, t) + \Gamma(v(x, t)) + K, \quad \forall (x, t) \in \Omega \times [0, T_{\text{max}}). \]  

(36)

Finally, we may conclude the proof with the fact that \( \Gamma(v(x, t)) \leq \gamma(\rho)v(x, t) \) due to Lemma 6 again. \( \square \)

**Remark 9** The similar result of Lemma 7 still holds true if we replace the second equation of (5) by

\[ \varepsilon v_t = \Delta v - v + u \]

with a constant \( \varepsilon > 0 \). Indeed, one can give a suitable modification as follows. For fixed \( \varepsilon > 0 \), we can choose some \( \rho > 0 \) such that \( 0 < \gamma(\rho) < \frac{1}{\varepsilon} \) due to the assumption \( \lim_{s \to \infty} \gamma(s) = 0 \).

With the function \( \Gamma \) which is defined by the above \( \rho > 0 \), we proceed the similar lines as

\[ \varepsilon v_t - \Delta v + v = w - \Delta w \]

\[ = \varepsilon w_t - \Delta w + w + \varepsilon \left( \gamma(v)u - (I - \Delta)^{-1}[\gamma(v)u] \right), \]

and

\[ \varepsilon \gamma(v)u = \varepsilon \gamma(v)(\varepsilon v_t - \Delta v + v) \]

\[ = \left( \varepsilon \partial_t (\varepsilon \Gamma(v)) - \Delta (\varepsilon \Gamma(v)) + (\varepsilon \Gamma(v)) \right) \]

\[ + \varepsilon \gamma'(v)|\nabla v|^2 + \varepsilon v\gamma(v) - \varepsilon \Gamma(v), \]

thus we derive

\[ \varepsilon v_t - \Delta v + v + \varepsilon (I - \Delta)^{-1}[\gamma(v)u] - \gamma'(v)|\nabla v|^2 \]

\[ = \left( \varepsilon \partial_t (w + \varepsilon \Gamma(v)) - \Delta (w + \varepsilon \Gamma(v)) + (w + \varepsilon \Gamma(v)) \right) + \varepsilon (v\gamma(v) - \Gamma(v)). \]

By the same discussion, for any \((x, t) \in \Omega \times [0, T_{\text{max}})\) we have

\[ v(x, t) \leq w(x, t) + \varepsilon \Gamma(v(x, t)) + K \leq w(x, t) + \varepsilon \gamma(\rho)v(x, t) + K, \]

\[ \vdots \]
which implies

\[ v(x, t) \leq \frac{1}{1 - \varepsilon \gamma(\rho)} \left( w(x, t) + K \right). \]

Next, we establish uniform-in-time boundedness of \( v \) with the growth condition (A2) on \( 1/\gamma \).

**Proposition 1** Assume \( n = 2, 3 \). Then under the assumptions (A0), (A1) and (A2) with \( 0 < k < \frac{2}{(n-2)_+} \), there exists \( C > 0 \) depending only on \( \gamma, \Omega \) and the initial data such that for all \((x, t) \in \Omega \times [0, T_{\text{max}})\),

\[ v(x, t) \leq C. \]

The proof of the preceding result consists of two steps. First, we prove the following lemma.

**Lemma 8** Under the same assumptions of Proposition 1, there is \( C > 0 \) depending only on \( \gamma, \Omega \) and the initial data such that

\[
\sup_{0 \leq t < T_{\text{max}}} \left( \| \nabla w \|_{L^2(\Omega)} + \| w \|_{L^2(\Omega)} \right) \leq C, \tag{37}
\]

and for any \( t \in (0, T_{\text{max}} - \tau) \) with \( \tau = \min\{1, \frac{1}{2}T_{\text{max}}\}, \)

\[
\int_{t}^{t+\tau} \int_{\Omega} \gamma(v) u^2 \, dx \, ds \leq C. \tag{38}
\]

**Proof** By the assumption (A0) and the non-negativity of \( v \), it follows that for all \((x, t) \in \Omega \times [0, T_{\text{max}})\),

\[
\gamma(v(x, t)) \leq \gamma(0). \tag{39}
\]

Multiplying the first equation of (5) by \( w = (I - \Delta)^{-1}[u] \) and integrating over \( \Omega \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} (\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2) + \int_{\Omega} \gamma(v) u^2 \, dx = \int_{\Omega} \gamma(v) u w \, dx. \tag{40}
\]

On the other hand, observe that

\[
\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 = \int_{\Omega} w u \, dx. \tag{41}
\]

We combine (40) and (41), then by (39) and Young’s inequality to infer that

\[
\frac{d}{dt} (\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2) + \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 + 2 \int_{\Omega} \gamma(v) u^2 \, dx
\]

\[
\leq (1 + 2\gamma(0)) \int_{\Omega} w u \, dx
\]

\[
\leq \int_{\Omega} \gamma(v) u^2 \, dx + \frac{(1 + 2\gamma(0))^2}{4} \int_{\Omega} \frac{w^2}{\gamma(v)} \, dx. \tag{42}
\]

In view of our assumption (A2), we infer that there exist \( k \in (0, \frac{2}{(n-2)_+}) \), \( b > 0 \) and \( s_b > 0 \) such that for all \( s \geq s_b \)

\[
\frac{1}{\gamma(s)} \leq b s^k
\]
and on the other hand, since $\gamma(\cdot)$ is decreasing,

$$\frac{1}{\gamma(s)} \leq \frac{1}{\gamma(s_b)}$$

for all $0 \leq s < s_b$. Therefore, for all $s \geq 0$, there holds

$$\frac{1}{\gamma(s)} \leq bs^k + \frac{1}{\gamma(s_b)}. \quad (43)$$

Therefore, we deduce from (43) and Lemma 7 that

$$\int_{\Omega} \frac{1}{\gamma(v)} w^2 \, dx \leq \int_{\Omega} (bv^k + \frac{1}{\gamma(s_b)}) w^2 \, dx$$

$$\leq \int_{\Omega} \left( b \left( \frac{1}{1 - \gamma(\rho)} (w + K) \right)^k + \frac{1}{\gamma(s_b)} \right) w^2 \, dx$$

$$\leq C_1 \int_{\Omega} w^{k+2} \, dx + C_1 \quad (44)$$

with $C_1 > 0$ depending only on the initial data, $\gamma$ and $\Omega$. Now, from (42) and (44) we arrive at

$$\frac{d}{dt} (\|w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) \leq \|\nabla w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2 + \gamma(v) u^2 \, dx$$

$$\leq C_1 \frac{(1 + 2\gamma(0))^2}{4} \int_{\Omega} w^{k+2} \, dx + C_1 \frac{(1 + 2\gamma(0))^2}{4} \quad (45)$$

for all $0 \leq t < T_{\text{max}}$.

When $n = 2$, by Lemma 1, for all $k > 0$ there exists $M > 0$ such that

$$\int_{\Omega} w^{k+2} \, dx \leq M, \quad (46)$$

where $M$ depends on $k$, $\Omega$ and $\|u_0\|_{L^1(\Omega)}$.

When $n = 3$, we first deduce similarly from Lemma 1 that for all $0 < k < 1$ there exists $M' = M'(k, \Omega, \|u_0\|_{L^1(\Omega)}) > 0$ satisfying

$$\int_{\Omega} w^{k+2} \, dx \leq M'. \quad (47)$$

For the case $1 \leq k < 2$, one easily checks that $3 \leq k + 2 < 6$ and $\frac{3k}{2} \in (1, 3)$. Now we may fix $q \in (\frac{3k}{2}, 3)$ and make use of interpolation and the Sobolev embedding $H^1 \hookrightarrow L^6$ to infer that there exists some $C_{\text{sob}} > 0$ depending on $\Omega$ such that

$$\int_{\Omega} w^{k+2} \, dx \leq \|w\|_{L^6(\Omega)}^{\beta'(k+2)} \|w\|_{L^4(\Omega)}^{(1-\beta')(k+2)} \leq C_{\text{sob}}^{\beta'(k+2)} \|w\|_{H^1}^{\beta'(k+2)} \|w\|_{L^4(\Omega)}^{(1-\beta')(k+2)}$$

with

$$\beta' = \left( \frac{1}{q} - \frac{1}{k+2} \right) / \left( \frac{1}{q} - \frac{1}{6} \right).$$

Since $q < 3$, in view of Lemma 1 again, there exists some $C_2 > 0$ depending on $q$, $\Omega$ and the initial data such that

$$\int_{\Omega} w^{k+2} \, dx \leq C_{\text{sob}}^{\beta'(k+2)} C_2^{(1-\beta')(k+2)} \|w\|_{H^1}^{\beta'(k+2)}. \quad (48)$$
Thanks to \( q \in \left( \frac{3k}{2}, 3 \right) \), we observe that
\[
0 < \beta'(k + 2) < 2.
\]

Now we may invoke Young’s inequality to get some \( C_3 > 0 \) fulfilling
\[
C_1 \left( \frac{1}{4} + 2\gamma(0)^2 \right) \int_{\Omega} w^{k+2} \, dx \leq C_1 \| w \|_{H^1}^{\beta(k+2)} C_2 \left( \frac{1}{4} + 2\gamma(0)^2 \right) \| w \|_{H^1}^{\beta'(k+2)} \leq \frac{1}{2} \| w \|_{H^1}^2 + C_3, \tag{48}
\]
where \( C_3 > 0 \) depends on \( \Omega, k, \gamma \) and the initial data.

In conclusion, combining (45), (46), (47) and (48), for \( n = 2, 3 \) and \( k \in (0, \frac{2}{2-n}) \), we obtain some \( C_4 > 0 \) depending on \( \Omega, k, \gamma \) and the initial data such that for all \( 0 \leq t < T_{\max} \)
\[
\frac{d}{dt} (\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2) + \frac{1}{2} \left( \| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \int_{\Omega} \gamma(v) u^2 \, dx \leq C_4 \tag{49}
\]
which by means of ODE analysis yields that there exists some \( C > 0 \) satisfying
\[
\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2 \leq C \tag{50}
\]
for all \( 0 \leq t < T_{\max} \). Here we use the fact that \( w \) is a continuous \( H^1(\Omega) \)-valued function on \( [0, T_{\max}] \) due to the continuity of \( u \) given by Theorem 4. Moreover, it follows from (49) and (50) that there exists some \( C' > 0 \) such that for any \( t \in (0, T_{\max} - \tau) \) with \( \tau = \min\{1, \frac{1}{2} T_{\max}\} \),
\[
\int^t_{t+\tau} \int_{\Omega} \gamma(v) u^2 \, dx \, ds \leq C'.
\]
This completes the proof. \( \square \)

With above uniform-in-time estimates, we can establish the uniform-in-time upper bounds of \( v \) relying on an application of the uniform Gronwall inequality and the Sobolev embedding \( H^2 \hookrightarrow L^\infty \) when \( n \leq 3 \).

**Lemma 9** Under the same assumptions of Proposition 1, there is \( C > 0 \) depending only on \( \Omega, \gamma \) and the initial data such that
\[
\sup_{0 \leq t < T_{\max}} (\| w \|_{L^\infty(\Omega)} + \| v \|_{L^\infty(\Omega)}) \leq C. \tag{51}
\]

**Proof** For any \( \frac{3}{2} < p < 2 \), due to the Sobolev embedding theorem and Hölder’s inequality, we have some \( C_{sob} > 0 \) satisfying
\[
\| w \|_{L^\infty(\Omega)} \leq C_{sob} \| u \|_{L^p(\Omega)} \leq C_{sob} \left( \int_{\Omega} \gamma(v) u^2 \, dx \right)^{1/2} \left( \int_{\Omega} \gamma^{-\frac{p}{2-p}}(v) \, dx \right)^{2-p} \tag{52}
\]
where $C_{sob} > 0$ depends on $p$ and $\Omega$. Recalling (43) and Lemma 7, in the same manner as before, we infer that there exists some $C_1 > 0$ such that

$$\int_{\Omega} \gamma^{-\frac{p}{2-p}}(v) \, dx \leq \int_{\Omega} \left( b v^k + \frac{1}{\gamma(s_b)} \right)^{\frac{p}{2-p}} \, dx$$

$$\leq \int_{\Omega} \left( b \left( \frac{1}{1 - \gamma(p)} (w + K) \right)^k + \frac{1}{\gamma(s_b)} \right)^{\frac{p}{2-p}} \, dx$$

$$\leq C_1 \int_{\Omega} w^{\frac{pk}{2-p}} \, dx + C_1,$$  \hspace{1cm} (53)

where $C_1 > 0$ depends only on the initial data, $p, k, \gamma$ and $\Omega$.

Next, we divide our argument into two cases. First, when $n = 2$, recalling that $w = (I - \Delta)^{-1}[u]$ and thanks to Lemma 1, we have

$$\int_{\Omega} \gamma^{-\frac{p}{2-p}}(v) \, dx \leq C_1 \int_{\Omega} w^{\frac{pk}{2-p}} \, dx + C_1 \leq C_2$$  \hspace{1cm} (54)

with some $C_2 > 0$ depending only on the initial data, $p, k, \gamma$ and $\Omega$.

When $n = 3$, for any fixed $0 < k < 2$, we can always choose $p = p(k) \in \left( \frac{3}{2}, 2 \right)$ such that $\frac{pk}{2-p} \leq 6$. Then if $0 \leq \frac{pk}{2-p} < 3$, we infer by Lemma 1 and a possible application of Young’s inequality (in the case $\frac{pk}{2-p} < 1$) that

$$\int_{\Omega} \gamma^{-\frac{p}{2-p}}(v) \, dx \leq C_1 \int_{\Omega} w^{\frac{pk}{2-p}} \, dx + C_1 \leq C_2'$$  \hspace{1cm} (55)

with some $C_2' > 0$ depending only on the initial data, $p, k, \gamma$ and $\Omega$.

For the case $\frac{pk}{2-p} \in [3, 6]$, by (53) and the three-dimensional Sobolev embeddings, there exists $C'_{sob} > 0$ such that

$$\left( \int_{\Omega} \left( \gamma(v) \right)^{-\frac{p}{2-p}} \, dx \right)^{-\frac{2-p}{2p}} \leq \left( C_1 \int_{\Omega} w^{\frac{pk}{2-p}} \, dx + C_1 \right)^{-\frac{2-p}{2p}}$$

$$\leq \left( C_1 C'_{sob} \|w\|_{H^2(\Omega)} + C_1 \right)^{-\frac{2-p}{2p}},$$

where $C'_{sob} > 0$ depends on $p, k, \Omega$. Then in view of Lemma 8 there exists some $C_2'' > 0$ such that

$$\left( \int_{\Omega} \left( \gamma(v) \right)^{-\frac{p}{2-p}} \, dx \right)^{-\frac{2-p}{2p}} \leq C_2''$$  \hspace{1cm} (56)

where $C_2'' > 0$ depends on the initial data, $p, k, \gamma$ and $\Omega$.

In summary, combining (52), (53), (54), (55) and (56), when $n = 2, 3$ and $0 < k < \frac{2}{(n-2)^+}$, we obtain that there exists some $C_3 > 0$ such that

$$\|w\|_{L^\infty(\Omega)} \leq C_3 \left( \int_{\Omega} u^2 \gamma(v) \, dx \right)^{1/2} \leq C_3 \int_{\Omega} u^2 \gamma(v) \, dx + C_3 \frac{3}{4},$$

where $C_3 > 0$ depends on $p, k, \Omega$ and the initial data. Invoking Lemma 8 again, we have some $C_4 > 0$ such that for any $t \in (0, T_{\text{max}} - \tau)$ with $\tau = \min\{1, \frac{1}{2} T_{\text{max}}\},$

$$\int_0^{t+\tau} \|w\|_{L^\infty(\Omega)} \, ds \leq C_3 \int_t^{t+\tau} \gamma(v) u^2 \, dx \, ds + C_3 \frac{3}{4} \leq C_4.$$  \hspace{1cm} (57)
which in particular indicates that for any fixed \( x \in \Omega \) and any \( t \in (0, T_{\text{max}} - \tau) \) with \( \tau = \min\{1, \frac{1}{2}T_{\text{max}}\} \),

\[
\int_{t}^{t+\tau} w(x, s) \, ds \leq \int_{t}^{t+\tau} \|w\|_{L^\infty(\Omega)} \, ds \leq C_4.
\] (58)

Then, we recall that

\[
w_t \leq w_t + \gamma(v) u = (I - \Delta)^{-1} [\gamma(v) u] \leq \gamma(0) w.
\]

With the aid of the uniform Gronwall inequality Lemma 4, we infer that for any \( x \in \Omega \) and \( t \in (\tau, T_{\text{max}}) \)

\[
w(x, t) \leq C
\] (59)

with some \( C > 0 \) independent of \( x, t \) and \( T_{\text{max}} \), which together with Lemma 5 for \( t \leq \tau \) gives rise to the uniform-in-time boundedness of \( w \) such that for all \((x, t) \in \Omega \times [0, T_{\text{max}})\),

\[
w(x, t) \leq C'
\] (60)

with some \( C' > 0 \). This concludes the proof due to Lemma 7 since

\[
v(x, t) \leq \frac{1}{1 - \gamma(\rho)} \left( w(x, t) + K \right).
\]

\[\square\]

**Remark 10** The results of Lemma 6, Lemma 7 and Proposition 1 still hold true if one replaces the assumption \((A1)\) by the following

\[(A1') : \lim_{s \to \infty} \gamma(s) = \gamma_\infty < 1.\] (61)

**Remark 11** In light of Remark 9, the result of Proposition 1 still holds if we replace the second equation of (5) by

\[
\varepsilon v_t = \Delta v - v + u
\]

with a constant \( \varepsilon > 0 \).

## 5 Existence and boundedness of classical solutions

In this section, we prove Theorems 1 and 2 via the classical energy method.

### 5.1 Energy estimates

Let \( A \) denote the self-adjoint realization of \(-\Delta\) under homogeneous Neumann boundary condition in the Hilbert space \( L^2_\perp(\Omega) := \{ \psi \in H^2(\Omega) \mid \int_\Omega \psi = 0 \} \) with domain \( D(A) := \{ \psi \in H^2(\Omega) \cap L^2_\perp(\Omega) \mid \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial \Omega \} \). Moreover we denote the bounded self-adjoint fractional powers \( \Delta^{-\alpha} \) with any \( \alpha > 0 \). Then in the same spirit of [35, Lemma 3.1], we first derive the following duality estimates.
Lemma 10 Assume \( n \geq 1 \). Then for any \( t \in [0, T_{\text{max}}) \), there holds
\[
\| A^{-\frac{1}{2}} (u - \overline{u}_0) \|_{L^2(\Omega)}^2 + 2 \int_0^t \int_\Omega \gamma(v) u^2 \, dx \, ds \\
\leq \| A^{-\frac{1}{2}} (u_0 - \overline{u}_0) \|_{L^2(\Omega)}^2 + 2 \gamma(0) \overline{u}_0^2 |\Omega| t,
\] (62)
where \( \overline{\varphi} \triangleq \frac{1}{|\Omega|} \int_\Omega \varphi \, dx \) for any \( \varphi \in L^1(\Omega) \).

Proof Multiplying the first equation of (5) by \( A^{-1}(u - \overline{u}_0) \) and integrating over \( \Omega \), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \| A^{-\frac{1}{2}} (u - \overline{u}_0) \|_{L^2(\Omega)}^2 + \int_\Omega \gamma(v) u^2 \, dx = \overline{u}_0 \int_\Omega \gamma(v) u \, dx.
\]
Since \( \gamma(v) \leq \gamma(0) \), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \| A^{-\frac{1}{2}} (u - \overline{u}_0) \|_{L^2(\Omega)}^2 + \int_\Omega \gamma(v) u^2 \, dx \leq \gamma(0) \overline{u}_0^2 |\Omega|,
\]
which concludes the proof by a direct integration with respect to time. \( \square \)

Remark 12 With the energy estimates in Lemma 10 and in the same manner as done in [12, Lemma 3.5], we may show that the upper bounds of \( w \) and hence of \( v \) grow at most linearly in time if \( n \leq 3 \).

Lemma 11 Assume \( n \leq 3 \) and \( (u, v) \) is a classical solution of system (5) on \( \Omega \times (0, T) \). Then there exists \( C(T) > 0 \) depending on \( \Omega, T \) and the initial data such that
\[
\sup_{0 < t < T} \int_\Omega u(t) \log u(t) \, dx + \int_0^T \int_\Omega |\nabla u|^2 \, u \, dx \, ds \leq C(T).
\]

Proof First, we recall that due to Lemma 5 and Lemma 7 there holds
\[
0 \leq v(x, t) \leq v^* \quad \text{for all } (x, t) \in \Omega \times [0, T),
\] (63)
where
\[
0 < v^* = v^*(T) \triangleq \frac{e^{\gamma(v_0) T} \| w_0 \|_{L^\infty} + K}{1 - \gamma(\rho)}
\] (64)
depends on \( \Omega, \gamma, T \) and the initial data. Recall that \( \rho \) here is the constant appearing in the definition of \( \Gamma \) in (27), which is chosen sufficiently large in Lemma 7 such that \( \gamma(\rho) < 1 \).

Multiplying the first equation of (5) by \( \log u \), integrating by parts and applying Young’s inequality, we obtain that
\[
\frac{d}{dt} \int_\Omega u \log u \, dx + \int_\Omega \gamma(v) \frac{|\nabla u|^2}{u} \, dx = - \int_\Omega \gamma'(v) \nabla v \cdot \nabla u \, dx
\]
\[
\leq \frac{1}{2} \int_\Omega \gamma(v) \frac{|\nabla u|^2}{u} \, dx + \int_\Omega \frac{\gamma'(v)^2}{\gamma(v)} u |\nabla v|^2 \, dx
\]
\[
\leq \frac{1}{2} \int_\Omega \gamma(v) \frac{|\nabla u|^2}{u} \, dx + \int_\Omega \gamma(v) u^2 \, dx + \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)^3} |\nabla v|^4 \, dx.
\]
Thus it follows
\[
\frac{d}{dt} \int_{\Omega} u \log u \, dx + \frac{1}{2} \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx \\
\leq \int_{\Omega} \gamma(v) u^2 \, dx + \int_{\Omega} \frac{|\gamma'(v)|^4}{\gamma(v)^3} |\nabla v|^4 \, dx.
\] (65)

In view of (63) and our assumption (A0) on \( \gamma \), there is \( C_1(T) > 0 \) depending on the initial data and \( \Omega, \gamma, T \) such that for all \( (x, t) \in \Omega \times (0, T) \),
\[
\frac{|\gamma'(v)|^4}{\gamma(v)^3} (x, t) \leq C_1(T).
\]

Therefore, with the aid of the Gagliardo–Nirenberg inequality ([36, Exercise 1, pp. 12], or [46])
\[
||\nabla v||_{L^4(\Omega)} \leq C_{GN} ||v||_{H^2(\Omega)}^{1/2} ||v||_{L^\infty(\Omega)}^{1/2} + C_{GN} ||v||_{L^\infty(\Omega)}
\]
with some \( C_{GN} > 0 \), we infer that
\[
\int_{\Omega} \frac{|\gamma'(v)|^4}{\gamma(v)^3} |\nabla v|^4 \, dx \leq C_1(T) \int_{\Omega} |\nabla v|^4 \, dx \\
\leq C_1(T) \left( C_{GN} ||v||_{H^2(\Omega)} ||v||_{L^\infty(\Omega)} + C_{GN} ||v||_{L^\infty(\Omega)} \right)^4 \\
\leq C_1(T) \left( C_{GN} ||v||_{H^2(\Omega)} (v^*(T))^{1/2} + C_{GN} v^*(T) \right)^4,
\]
and thus there exists \( C_2(T) > 0 \) such that
\[
\int_{\Omega} \frac{|\gamma'(v)|^4}{\gamma(v)^3} |\nabla v|^4 \, dx \leq C_2(T) ||v||_{H^2(\Omega)}^2 + C_2(T),
\]
where \( C_2(T) > 0 \) depends on \( \Omega, \gamma, T \) and the initial data. Then we observe from the maximal regularity estimate of heat equations (see [18])
\[
\int_0^T \int_{\Omega} \frac{|\gamma'(v)|^4}{\gamma(v)^3} |\nabla v|^4 \, dx \, ds \\
\leq C_2(T) \int_0^T ||v||_{H^2(\Omega)}^2 \, ds + TC_2(T) \\
\leq C_2(T) C_{MR} \left( ||v_0||_{H^1(\Omega)}^2 + \int_0^T ||u||_{L^2(\Omega)}^2 \, ds \right) + TC_2(T),
\]
with some \( C_{MR} > 0 \). Finally, by integrating (65) and using (63) together with Lemma 10, we deduce that
\[
\int_{\Omega} u \log u \, dx + \int_0^T \int_{\Omega} \gamma(v) \frac{|\nabla u|^2}{u} \, dx \, ds \leq C(T),
\]
which completes the proof. \( \square \)

5.2 Classical solution in dimension two

In this part, we deal with the case \( n = 2 \) by a similar argument as done for the classical Keller–Segel models (cf. [14]). First, we have
Lemma 12 Assume $n = 2$ and let $(u, v)$ be a classical solution of system (5) on $\Omega \times (0, T)$. Then there exist $p \in (1, 2)$ and some $C(T) > 0$ such that

$$\|u(t)\|_{L^p(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T).$$

Proof Multiplying the first equation of (5) by $u^{p-1}$ with some $p \in (1, 2)$ we have

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \, dx = \int_{\Omega} u^{p-1} u_t \, dx = \int_{\Omega} u^{p-1} \nabla \cdot (\gamma(v) \nabla u + u \gamma'(v) \nabla v) \, dx,$$

and by integration by parts, it follows that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \, dx + (p - 1) \int_{\Omega} u^{p-2} \gamma(v) |\nabla u|^2 \, dx = -(p - 1) \int_{\Omega} u^{p-1} \gamma'(v) \nabla u \cdot \nabla v \, dx.$$

By the Cauchy–Schwarz inequality we have

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \, dx + \frac{p - 1}{2} \int_{\Omega} u^{p-2} \gamma'(v) |\nabla u|^2 \, dx \leq \frac{p - 1}{2} \int_{\Omega} \frac{u^p |\gamma'(v)|^2}{\gamma(v)} \, dx \leq pM_{\gamma}(T) \int_{\Omega} u^p |\nabla v|^2 \, dx,$$

where we set

$$M_{\gamma}(T) = \sup_{s \in [0, v^*(T)]} \frac{|\gamma'(s)|^2}{\gamma(s)}$$

with $v^*(T)$ given in (64). Using Young’s inequality we obtain that

$$\int_{\Omega} u^p |\nabla v|^2 \, dx \leq \frac{p}{p + 1} \int_{\Omega} u^{p+1} \, dx + \frac{1}{p + 1} \int_{\Omega} |\nabla u|^{2(p+1)} \, dx,$$

and in view of (63), we obtain

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \, dx + \frac{(p - 1)\gamma(v^*(T))}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 \, dx \leq \frac{p^2 M_{\gamma}(T)}{p + 1} \int_{\Omega} u^{p+1} \, dx + \frac{pM_{\gamma}(T)}{p + 1} \int_{\Omega} |\nabla v|^{2(p+1)} \, dx. \quad (66)$$

On the other hand, by the Sobolev embedding theorem and the elliptic regularity theory, we deduce that

$$\|\nabla v\|_{L^{2(p+1)}(\Omega)} \leq C_{sob}\|v\|_{W^{2\frac{p+1}{2},\frac{p+1}{2}}(\Omega)} \leq C_{sob} C_{elli}\|(-\Delta + 1)v\|_{L^{2\frac{p+1}{2},\frac{p+1}{2}}(\Omega)}$$
with positive constants $C_{sob}$ and $C_{elli}$. By applying the maximal regularity argument \[18\] we estimate that for some fixed $\tau_0 \in (0, \frac{1}{2} T_{\text{max}})$ and any $t \in (\tau_0, T)$,

\[
\int_{\tau_0}^{t} \int_{\Omega} |\nabla v|^{2(p+1)} \, dx \, ds \\
\leq (C_{sob}C_{elli})^{2(p+1)} \int_{\tau_0}^{t} \|(-\Delta + 1)v\|_{L^{\frac{2(p+1)}{p+2}}(\Omega)}^{2(p+1)} \, ds \\
\leq (C_{sob}C_{elli})^{2(p+1)} C_{MR} \left( \|v(\tau_0)\|_{W^{2, \frac{2(p+1)}{p+2}}(\Omega)}^{2(p+1)} + \int_{\tau_0}^{t} \|u\|_{L^{\frac{2(p+1)}{p+2}}(\Omega)}^{2(p+1)} \, ds \right) \\
\leq C_1 \int_{\tau_0}^{t} \int_{\Omega} u^{p+1} \, dx \, ds + C'_1 \|v(\tau_0)\|_{W^{2, \frac{2(p+1)(p+2)}{p+2}}(\Omega)}^{2(p+1)} , \tag{67}
\]

where $C_1 := (C_{sob}C_{elli})^{2(p+1)} C_{MR}\|u_0\|_{L^1(\Omega)}^{p+1}$ and $C'_1 := (C_{sob}C_{elli})^{2(p+1)} C_{MR}$, and here we used the relation

\[
\|u\|_{L^{\frac{2(p+1)}{p+2}}(\Omega)}^{2(p+1)} \leq \|u\|_{L^1(\Omega)}^{p+1} \int_{\Omega} u^{p+1} \, dx .
\]

Therefore by (66) and (67) we have that any $t \in (\tau_0, T)$,

\[
\frac{1}{p} \int_{\Omega} u^p(t) \, dx + \frac{(p-1)\gamma(v^*(T))}{2} \int_{\tau_0}^{t} \int_{\Omega} |\nabla u|^2 \, dx \, ds \\
\leq \frac{1}{p} \int_{\Omega} u^p(\tau_0) \, dx + C'_1 \frac{p M_{\gamma}(T)}{p+1} \|v(\tau_0)\|_{W^{2, \frac{2(p+1)(p+2)}{p+2}}(\Omega)}^{2(p+1)} \\
+ \left( \frac{p^2 M_{\gamma}(T)}{p+1} + C_1 \frac{p M_{\gamma}(T)}{p+1} \right) \int_{\tau_0}^{t} \int_{\Omega} u^{p+1} \, dx \, ds .
\]

Finally by picking $s > 0$ sufficiently large in Lemma 3 and recalling Lemma 11, it follows that there exists some $C(T) > 0$ such that any $t \in (\tau_0, T)$,

\[
\int_{\Omega} u^p(t) \, dx \leq C(T) ,
\]

which completes the proof together with the local existence result Theorem 4. \qed

**Proof of Theorem 1** After the above preparation, we may use the Moser type iteration (see [33, Lemma A.1]) to prove that there exists some $C(T) > 0$ such that

\[
\sup_{0 < t < T} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(T)
\]

for any $T < T_{\text{max}}$ and hence by Theorem 4, we deduce that $T_{\text{max}} = \infty$. Therefore, we prove global existence of classical solutions of problem (5) when $n = 2$ if (14), (A0) and (A1) or (A1') are satisfied.

Last, in light of the time-independent upper bound of $v$ in Proposition 1, we can proceed along the same lines in [35] to show the uniform-in-time boundedness of the classical solutions under assumption (A2). This completes the proof of Theorem 1. \qed

**Remark 13** In light of Remark 11, the above discussion still holds true if we replace the second equation of (5) by

\[
\varepsilon v_t = \Delta v - v + u
\]

with a constant $\varepsilon > 0$. \hfill \(\textcopyright\) Springer
5.3 Classical solutions in dimension three

In this part, we study global existence of classical solution when \( n = 3 \). First of all, we show that (A3) is a stronger condition than (A2).

**Lemma 13** A function satisfying (A0), (A1) and (A3) must fulfil assumption (A2) with any \( k > 1 \).

**Proof** First, we point out that under the assumptions (A0), (A1) and (A3), \( \gamma'(s) < 0 \) on \([0, \infty)\). In fact, due to (A0) and (A3), we have \( \gamma''(s) \geq 0 \) for all \( s \geq 0 \). Then if there is \( s_1 \geq 0 \) such that \( \gamma'(s_1) = 0 \), it must hold that \( 0 = \gamma'(s_1) \leq \gamma'(s) \leq 0 \) for all \( s \geq s_1 \) which contradicts to our assumptions (A0) and (A1).

Now, we may divide (21) by \(- \gamma(s) \gamma'(s)\) to obtain that

\[
- \frac{2\gamma'(s)}{\gamma(s)} \leq - \frac{\gamma''(s)}{\gamma'(s)}, \quad \forall s > 0,
\]

which indicates that

\[
(\log(-\gamma^{-2} \gamma'))' \leq 0.
\]

An integration of above ODI from \( v_* \) to \( s \) yields that

\[
- \gamma^{-2}(s) \gamma'(s) \leq - \gamma^{-2}(v_*) \gamma'(v_*) \triangleq d > 0,
\]

which further implies that

\[
\left( \frac{1}{\gamma(s)} \right)' \leq d
\]

Thus for any \( s \geq v_* \), there holds

\[
\frac{1}{\gamma(s)} \leq d(s - v_*) + \frac{1}{\gamma(v_*)}.
\]

As a result, for any \( k > 1 \), we have

\[
\frac{1}{s^k \gamma(s)} \leq \frac{d(s - v_*)}{s^k} + \frac{1}{s^k \gamma(v_*)} \to 0, \quad \text{as } s \to \infty.
\]

This completes the proof. \( \square \)

As an immediate conclusion of Lemma 13 and Proposition 1, we have the following

**Corollary 1** Assume that \( n = 3 \) and \( \gamma(\cdot) \) satisfies (A0), (A1) and (A3). Then \( v \) has a uniform-in-time upper bound in \( \Omega \times [0, T_{\text{max}}) \), i.e., there is a positive constant \( v^{**} \) which may depend on \( \Omega, \gamma \) and the initial data but is independent of \( t \) and \( T_{\text{max}} \) such that for all \( (x, t) \in \Omega \times [0, T_{\text{max}}) \),

\[
v(x, t) \leq v^{**}.
\]

Next, we derive the following energy estimates.

**Lemma 14** Assume \( n = 3 \). Suppose that \( \gamma(\cdot) \) satisfies (A0), (A1), and (A3). Then there is \( C > 0 \) depending only on the initial data and \( \Omega \) such that

\[
\sup_{0 \leq t < T_{\text{max}}} \int_{\Omega} u^2 \, dx \leq C.
\]
Proof. Multiplying the first equation of (5) by 2 and integrating by parts, we obtain that
\[ \frac{d}{dt} \int_{\Omega} u^2 \, dx + 2 \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx = -2 \int_{\Omega} \gamma'(v) u \nabla u \cdot \nabla v \, dx. \tag{71} \]

On the other hand, we multiply the second equation by \(-u^2 \gamma'(v)\) to obtain that
\[ -\int_{\Omega} v_t u^2 \gamma'(v) \, dx - 2 \int_{\Omega} u \gamma'(v) \nabla u \cdot \nabla v \, dx - \int_{\Omega} u^2 \gamma''(v) |\nabla v|^2 \, dx \]
\[ - \int_{\Omega} u^2 \gamma'(v) v \, dx = - \int_{\Omega} u^3 \gamma'(v) \, dx, \]
where we observe that
\[ -\int_{\Omega} v_t u^2 \gamma'(v) \, dx = - \frac{d}{dt} \int_{\Omega} \gamma(v) u^2 \, dx + 2 \int_{\Omega} u u_t \gamma(v) \, dx \]
\[ = - \frac{d}{dt} \int_{\Omega} \gamma(v) u^2 \, dx + 2 \int_{\Omega} u \gamma(v) \Delta (u \gamma(v)) \, dx \]
\[ = - \frac{d}{dt} \int_{\Omega} \gamma(v) u^2 \, dx - 2 \int_{\Omega} |\nabla (u \gamma(v))|^2 \, dx. \]

Therefore, we have
\[ \frac{d}{dt} \int_{\Omega} \gamma(v) u^2 \, dx + 2 \int_{\Omega} |\nabla (u \gamma(v))|^2 \, dx + \int_{\Omega} u^2 \gamma''(v) |\nabla v|^2 \, dx - \int_{\Omega} u^3 \gamma'(v) \, dx \]
\[ = - \int_{\Omega} u^2 \gamma'(v) v \, dx - 2 \int_{\Omega} u \gamma'(v) \nabla u \cdot \nabla v \, dx. \tag{72} \]

Now, multiplying (72) by \(\lambda > 0\) (we will choose \(\lambda = 1\) below. But in order show that assumption (A3) is independent of the choice of coefficients in the PDE-system, we keep this \(\lambda\) for convenience; see Remark 14) and adding the resultant to (71), we obtain that
\[ \frac{d}{dt} \int_{\Omega} (1 + \lambda \gamma(v)) u^2 \, dx + 2 \lambda \int_{\Omega} |\nabla (u \gamma(v))|^2 \, dx + 2 \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx \]
\[ + \lambda \int_{\Omega} \gamma''(v) u^2 |\nabla v|^2 \, dx - \lambda \int_{\Omega} u^3 \gamma'(v) \, dx \]
\[ = - \int_{\Omega} (2 + 2 \lambda) u \gamma'(v) \nabla u \cdot \nabla v \, dx - \lambda \int_{\Omega} u^2 \gamma'(v) v \, dx. \tag{73} \]

Invoking the Young inequality, we infer that
\[ \left| \int_{\Omega} (2 + 2 \lambda) u \gamma'(v) \nabla u \cdot \nabla v \, dx \right| \]
\[ \leq 2 \int_{\Omega} \gamma(v) |\nabla u|^2 \, dx + \int_{\Omega} \frac{(1 + \lambda)^2 |\gamma'(v)|^2}{2 \gamma(v)} u^2 |\nabla v|^2 \, dx. \]

Under the assumption
\[ 2 |\gamma'(v)|^2 \leq \gamma(v) \gamma''(v), \]
one finds that \(\lambda = 1\) fulfills
\[ \frac{(1 + \lambda)^2 |\gamma'(v)|^2}{2 \gamma(v)} \leq \lambda \gamma''(v). \tag{74} \]
As a result, we obtain from above that
\[
\frac{d}{dt} \int_{\Omega} \left(1 + \gamma(v)\right) u^2 \, dx + 2 \int_{\Omega} |\nabla (u \gamma(v))|^2 \, dx + \int_{\Omega} u^3 |\gamma'(v)| \, dx \\
\leq - \int_{\Omega} u^2 \gamma'(v) v \, dx.
\] (75)

Thanks to Corollary 1, there exists some \( C_1 > 0 \) depending on \( \Omega, \gamma \) and initial data such that
\[
\left| \int_{\Omega} u^2 \gamma'(v) v \, dx \right| \leq C_1 \int_{\Omega} u^2 \, dx.
\]

Thus, we obtain that
\[
\frac{d}{dt} \int_{\Omega} \left(1 + \gamma(v)\right) u^2 \, dx + 2 \int_{\Omega} |\nabla (u \gamma(v))|^2 \, dx + \int_{\Omega} u^3 |\gamma'(v)| \, dx \leq C_1 \int_{\Omega} u^2 \, dx.
\] (76)

On the other hand, since now \( v \) is bounded from above and below, there is \( \gamma_* = \gamma(v^{**}) > 0 \) with \( v^{**} \) being the time-independent upper bound given in Corollary 1 such that \( \gamma_* \leq \gamma(v) \leq \gamma(0) \) and it follows from (38) that there exists some \( C_2 > 0 \) depending on \( \Omega, \gamma \) and initial data such that
\[
\int_{t}^{t+\tau} \int_{\Omega} \left(1 + \gamma(v)\right) u^2 \, dx \, ds \leq C_2.
\] (77)

Now we may apply the uniform Gronwall inequality together with the local existence result to conclude that
\[
\int_{\Omega} \left(1 + \gamma(v)\right) u^2 \, dx \leq C, \quad \forall \ t \in [0, T_{\text{max}}),
\]
with some \( C > 0 \). This completes the proof. \( \square \)

**Remark 14** Our assumption \((A_3)\) is independent of the coefficients of the system. If we replace the second equation of system (5) by \( v_t - \alpha \Delta v + \beta v = \theta u \) with some \( \alpha, \beta, \theta > 0 \), one easily checks that condition (74) becomes
\[
\frac{(1 + \alpha \lambda)^2 |\gamma'(v)|^2}{2 \gamma(v)} \leq \alpha \lambda \gamma''(v),
\] (78)
which holds with \( \lambda = 1/\alpha \) under assumption \((A_3)\).

**Proof of Theorem 2** With the aid of Lemma 14, we may further use the standard bootstrap argument to prove that
\[
\sup_{0 < t < T} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C
\]
for any \( T < T_{\text{max}} \). Since similar argument is given in detail in [1,33], we omit the proof here. Finally, by Theorem 4, we deduce that \( T_{\text{max}} = \infty \) and Theorem 2 is proved. \( \square \)
6 The critical mass phenomenon with $\gamma(v) = e^{-v}$

This section is devoted to the special case $\gamma(v) = e^{-v}$. Namely, we consider the following initial Neumann boundary value problem:

$$
\begin{cases}
    u_t = \Delta(ue^{-v}) & x \in \Omega, 
    t > 0, \\
    v_t - \Delta v + v = u & x \in \Omega, 
    t > 0, \\
    \partial_\nu u = \partial_\nu v = 0, & x \in \partial \Omega, 
    t > 0, \\
    u(x, 0) = u_0(x), 
    v(x, 0) = v_0(x) & x \in \Omega,
\end{cases}
$$

(79)

with $\Omega \subset \mathbb{R}^2$.

6.1 Uniform-in-time boundedness with sub-critical mass

In this part, we first prove the following uniform-in-time boundedness of the classical solutions with sub-critical mass.

**Proposition 2** Assume $n = 2$ and let

$$
\Lambda_c = \begin{cases}
    8\pi & \text{if } \Omega = \{x \in \mathbb{R}^2; |x| < R\} \text{ and } (u_0, v_0) \text{ is radial in } x, \\
    4\pi & \text{otherwise}.
\end{cases}
$$

If $\Lambda \triangleq \int_\Omega u_0 dx < \Lambda_c$, then the global classical solution $(u, v)$ to system (79) is uniformly-in-time bounded in the sense that

$$
\sup_{t \in (0, \infty)} \left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \right) < \infty.
$$

First, system (79) is a dissipative dynamical system.

**Lemma 15** There holds

$$
\frac{d}{dt} F(u, v)(t) + \int_\Omega ue^{-v} |\nabla \log u - \nabla v|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0,
$$

(80)

where the functional $F(\cdot, \cdot)$ is defined by

$$
F(u, v) = \int_\Omega \left( u \log u + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - uv \right) \, dx.
$$

**Proof** Multiplying the first equation of (79) by $\log u - v$, the second equation of (79) by $v_t$ and integrating by parts, then adding the resultants together, we get

$$
\frac{d}{dt} \int_\Omega \left( u \log u + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - uv \right) \, dx + \int_\Omega ue^{-v} |\nabla \log u - \nabla v|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0.
$$

This completes the proof. \( \square \)

Since the energy $F(\cdot, \cdot)$ is the same as that of the classical Keller–Segel model, we may recall [30, Lemma 3.4] (see also [16, Lemma 4.9]) stated as follows.

**Lemma 16** If $\Lambda < \Lambda_c$, there exists a positive constant $C$ independent of $t$ such that

$$
\|v\|_{H^1(\Omega)} \leq C, \quad \int_\Omega uv \, dx \leq C \quad \text{and} \quad |F(u(t), v(t))| \leq C, \quad \forall \ t \geq 0.
$$
Next, we aim to derive a time-independent upper bound of $v$ with sub-critical mass. For this purpose, we need the following uniform-in-time estimates.

**Lemma 17** If $\Lambda < \Lambda_c$, then there holds

$$\sup_{t \geq 0} \int_t^{t+1} \int_\Omega e^{-v} u^2 \, dx \, ds \leq C,$$

where $C > 0$ depends on $\Omega$ and the initial data only.

**Proof** Multiplying the first equation of (79) by $w$ and integrating over $\Omega$, we obtain that

$$\int_\Omega u_t w \, dx = \int_\Omega e^{-v} u \Delta w \, dx.$$  

Recalling that $w - \Delta w = u$, the above equality implies that

$$\int_\Omega (-\Delta w_t + w_t) w \, dx + \int_\Omega e^{-v} u^2 \, dx = \int_\Omega e^{-v} u w \, dx.$$  

Hence, we have

$$\frac{1}{2} \frac{d}{dt} (\| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)}) + \int_\Omega e^{-v} u^2 \, dx$$

$$= \int_\Omega e^{-v} u w \, dx$$

$$\leq \frac{1}{2} \int_\Omega e^{-v} u^2 \, dx + \frac{1}{2} \int_\Omega e^{-v} w^2 \, dx.$$  

In view of Lemma 1 and the non-negativity of $v$, there exists some $C_1 > 0$ such that

$$\int_\Omega e^{-v} w^2 \, dx \leq \int_\Omega w^2 \, dx \leq C_1 \| u \|^2_{L^1(\Omega)} = C_1 \Lambda^2.$$  

On the other hand, by integration by parts and Young’s inequality, we infer that

$$\| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)} = \int_\Omega w u \, dx$$

$$\leq \frac{1}{2} \int_\Omega e^{-v} u^2 \, dx + \frac{1}{2} \int_\Omega e^v w^2 \, dx.$$  

Thanks to Hölder’s inequality and Lemma 1, we infer that

$$\int_\Omega e^v w^2 \, dx \leq \left( \int_\Omega e^{2v} \, dx \right)^{1/2} \left( \int_\Omega w^4 \, dx \right)^{1/2} \leq C_2$$  

with $C_2 > 0$ depending only on the initial data and $\Omega$, where we also used the 2D Trudinger–Moser inequality [30, Theorem 2.2] together with Lemma 16 to infer that

$$\int_\Omega e^{2v} \, dx \leq C_{TM} e^{C_{TM} (\| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)})} \leq C_3$$  

with $C_{TM}, C_{TM}' > 0$ depending only on $\Omega$ and $C_3 > 0$ depending only on $\Omega$ and initial data. Therefore, by (81), (82), (83) and (84) we deduce from above that

$$\frac{d}{dt} (\| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)}) + \frac{1}{2} \int_\Omega e^{-v} u^2 \, dx + (\| \nabla w \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)})$$

$$\leq C_1 \Lambda^2 + \frac{1}{2} C_2.$$  

\( \diamond \) Springer
Then we may conclude by a direct integration that
\[
\sup_{t \geq 0} (\| \nabla w \|_{L^2(\Omega)}^2 + \| w \|_{L^2(\Omega)}^2) \leq C
\]
with \( C > 0 \) depending only on the initial data and \( \Omega \). Here we use the fact that \( w \) is a continuous \( H^1 \)-valued function again. Moreover, an integration of (85) with respect to time from \( t \) to \( t + 1 \) together with the fact \( \sup_{t \geq 0} \| w \|_{H^1} \leq C \) will finally yields to our assertion. This completes the proof.

\[\square\]

**Remark 15** If \( w \) or \( v \) has a uniform-in-time upper bound, then one has
\[
\sup_{t \geq 0} \left( \| v \|_{H^1(\Omega)} + \int_{\Omega} u v \, dx + |\mathcal{F}(u(t), v(t))| + \int_t^{t+1} \int_{\Omega} e^{-v} u^2 \, dx \, ds \right) \leq C,
\]
where \( C > 0 \) depends on \( \Omega \) and the initial data only.

**Proof** If \( w \) is uniformly-in-time bounded, then it follows from Lemma 7 that \( \sup_{t \geq 0} \| v(t, \cdot) \|_{L^\infty(\Omega)} \leq C \) by some \( C > 0 \) independent of \( t \). As a result, we infer that
\[
\int_{\Omega} \left( u \log u + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 \right) \, dx = \mathcal{F}(u, v) + \int_{\Omega} u v \, dx \leq \mathcal{F}(u, v) + \| v \|_{L^\infty(\Omega)} \int_{\Omega} u \, dx \leq \mathcal{F}(u_0, v_0) + C \Lambda,
\]
which indicates that there exists some \( C' > 0 \) such that
\[
\sup_{t \geq 0} \left( \| v \|_{H^1(\Omega)} + \int_{\Omega} u v \, dx + |\mathcal{F}(u(t), v(t))| \right) \leq C'.
\]
Then we may concludes the proof in the same manner as in Lemma 17. \[\square\]

**Lemma 18** If \( \Lambda < \Lambda_c \), then there exists \( C > 0 \) depending on \( \Omega \) and the initial data such that for all \( x \in \Omega \)
\[
\sup_{t \geq 0} v(x, t) \leq C.
\]

**Proof** First, we apply the Sobolev embedding theorem, the elliptic regularity theorem and Hölder’s inequality to infer that there exist positive constants \( C_{sob}, C_{elli} \) depending on \( \Omega \) such that
\[
\| w \|_{L^\infty(\Omega)} \leq C_{sob} \| w \|_{W^{2, \frac{3}{2}}(\Omega)} \leq C_{sob} C_{elli} \| u \|_{L^\frac{3}{2}(\Omega)} \leq C_{sob} C_{elli} \left( \int_{\Omega} u^2 e^{-v} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} e^{3v} \, dx \right)^{\frac{1}{6}}.
\]
By the 2D Trudinger–Moser inequality [30, Theorem 2.2] together with Lemma 16, we can deduce that
\[
\int_{\Omega} e^{3v} \, dx \leq C_{TM} e^{C_{TM} \left( \| w \|_{L^2(\Omega)}^2 + \| v \|_{L^2(\Omega)}^2 \right)} \leq C_1
\]
\[\square\]
with positive constants $C_{TM}$ and $C'_{TM}$ depending only on $\Omega$, and $C_1$ depending only on $\Omega$ and initial data. Thus, by Lemma 17, for any $t \geq 0$, there exists some $C_2 > 0$ depending on $\Omega$ and initial data such that
\[
\int_t^{t+1} \|w\|_{L^\infty(\Omega)}^2 ds \leq (C_{sob} C_{elli})^2 (C_1)^{\frac{3}{2}} \int_t^{t+1} \int_\Omega u^2 e^{-v} dx ds \leq C_2,
\]
which due to Young’s inequality indicates that
\[
\int_t^{t+1} \|w\|_{L^\infty(\Omega)} ds \leq \int_t^{t+1} \|w\|_{L^\infty(\Omega)}^2 ds + \frac{1}{4} \leq C_2 + \frac{1}{4}.
\]
Hence, for any $x \in \Omega$ and $t \geq 0$, we obtain that
\[
\int_t^{t+1} w(x, s) ds \leq \int_t^{t+1} \|w\|_{L^\infty(\Omega)} ds \leq C_2 + \frac{1}{4}. \tag{88}
\]
Observing that
\[
w_t \leq w_t + u e^{-v} = (I - \Delta)^{-1} [u e^{-v}] \leq (I - \Delta)^{-1} [u] = w,
\]
we may fix $x \in \Omega$ and apply the uniform Gronwall inequality Lemma 4 to deduce that there exists some $C > 0$ such that
\[
w(x, t) \leq C \quad \text{for all } t \geq 1.
\]
Since $C > 0$ above is independent of $x$ and
\[
w(x, t) \leq w_0(x) e^{-v_0} \leq e w_0(x) \quad \text{for any } x \in \Omega \text{ and } t \in [0, 1]
\]
we conclude that there exists some $C' > 0$ such that
\[
\sup_{t \geq 0} w(x, t) \leq C'.
\]
As a result, $v$ is uniformly-in-time bounded as well according to Lemma 7. This completes the proof. $\square$

**Proof of Proposition 2** Proceeding along the same lines as in [35], we can invoke the time-independent upper bound of $v$ to show the uniform-in-time boundedness of the classical solutions, which concludes the proof. $\square$

### 6.2 Unboundedness with super-critical mass

In this part we construct blowup solutions in infinite time. Since the system (79) has the similar energy structure and the same stationary problem as the Keller–Segel system, we may verify existence of blowup solutions following the idea in [20,32].

Stationary solutions $(u_s, v_s)$ to (79) satisfy that
\[
\begin{cases}
0 = \nabla \cdot u_s e^{-v} \nabla (\log u_s - v_s) & \text{in } \Omega, \\
0 = \Delta v_s - v_s + u_s & \text{in } \Omega, \\
u_s > 0, \ v_s > 0 & \text{in } \Omega, \\
\frac{\partial u_s}{\partial v} = \frac{\partial v_s}{\partial v} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
In view of the boundary condition, the set of equilibria consists of solutions to the following problem:

\[
\begin{cases}
  v_s - \Delta v_s = \frac{\Lambda}{\int_\Omega e^{v_s} \, dx} e^{v_s} & \text{in } \Omega, \\
  u_s = \frac{\Lambda}{\int_\Omega e^{v_s} \, dx} e^{v_s} & \text{in } \Omega, \\
  \frac{\partial v_s}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(89)

for some $\Lambda > 0$.

Proceeding the same way as in [40, Lemma 3.1], we have the following result.

**Proposition 3** Let $(u, v)$ be a classical non-negative solution to (79) in $\Omega \times (0, \infty)$. If the solution is uniformly-in-time bounded, there exist a sequence of time $\{t_k\} \subset (0, \infty)$ and a solution $(u_s, v_s)$ to (89) with $\Lambda = \|u_0\|_{L^1(\Omega)}$ such that $\lim_{k \to \infty} t_k = \infty$ and that

\[
\lim_{k \to \infty} (u(t_k), v(t_k)) = (u_s, v_s) \quad \text{in } C^2(\overline{\Omega}).
\]

as well as

\[
\mathcal{F}(u_s, v_s) \leq \mathcal{F}(u_0, v_0).
\]

**Remark 16** Observe that for any solution $(u_s, v_s)$ to (89), $u_s$ is strictly positive on $\Omega$ (see, e.g., [10, Sect. 2]). Assume for any $j \geq 1$, there is $t_j > 0$ and $x_j \in \Omega$ such that $u(t_j, x_j) < 1/j$. Then by a similar compactness argument as in [40, Lemma 3.1], one may extract a time subsequence, still denoted by $t_j$, such that $u(t_j)$ converges to some $u_s$ in $C^2(\overline{\Omega})$, which leads to a contradiction since $u_s$ is strictly positive. Thus, we infer that for any uniformly-in-time bounded solution $(u, v)$, $u$ is strictly positive for $(t_0, \infty) \times \Omega$ with some sufficiently large $t_0$ and we can now apply the non-smooth Lojasiewicz–Simon inequality established in [10] (see, also [21, 22]) to deduce that

\[
\lim_{t \to \infty} (u(t), v(t)) = (u_s, v_s) \quad \text{in } C^2(\overline{\Omega}).
\]

For $\Lambda > 0$ put

\[
S(\Lambda) \triangleq \{ (u_s, v_s) \in C^2(\overline{\Omega}) : (u_s, v_s) \text{ is a solution to (89)} \}.
\]

Here we recall the quantization property of solutions to (89). By [20, Lemma 3.5] and [31, Theorem 1], for $\Lambda \notin 4\pi \mathbb{N}$ there exists some $C > 0$ such that

\[
\sup\{\|(u_s, v_s)\|_{L^\infty(\Omega)} : (u, v) \in S(\Lambda)\} \leq C
\]

and

\[
\mathcal{F}_*(\Lambda) \triangleq \inf\{\mathcal{F}(u_s, v_s) : (u_s, v_s) \in S(\Lambda)\} \geq -C.
\]

Thus taking account of Proposition 3, for a pair of non-negative functions $(u_0, v_0)$ satisfying

\[
\begin{cases}
  \|u_0\|_{L^1(\Omega)} = \Lambda \notin 4\pi \mathbb{N}, \\
  \mathcal{F}(u_0, v_0) < \mathcal{F}_*(\Lambda),
\end{cases}
\]

the corresponding global solution must blow up in infinite time.

From now on we will construct an example satisfying the above condition based on calculations in [15]. A straightforward calculation leads us to the following lemma.
Lemma 19  For any $\lambda > 0$ the following functions

$$u_{\lambda}(x) := \frac{8\lambda^2}{(1 + \lambda^2 |x|^2)^2}, \quad v_{\lambda}(x) := 2 \log \frac{\lambda}{1 + \lambda^2 |x|^2} + \log 8$$

for all $x \in \mathbb{R}^2$, satisfy

$$e^{v_{\lambda}} = u_{\lambda}, \quad 0 = \Delta v_{\lambda} + u_{\lambda}, \quad \int_{\mathbb{R}^2} u_{\lambda} \, dx = 8\pi.$$ 

We modify the above function as: for any $\lambda \geq 1$ and $r \in (0, 1)$,

$$v_{\lambda, r}(x) := 2 \log \frac{1 + \lambda^2 r^2}{1 + \lambda^2 |x|^2} + \log 8,$$

and by simple calculations it follows that

$$u_{\lambda}(x) \leq 8\lambda^2, \quad v_{\lambda, r}(x) > \log 8 > 0 \text{ in } B(0, r).$$

Hereafter we fix $r \in (0, 1)$ and $q \in \Omega$ such that $B(q, 2r) \subset \Omega$. By translation, we may assume that $q = 0$. We also fix $r_1 \in (0, r)$ and then we can construct a smooth and radially symmetric function $\phi = \phi_{r, r_1}$ satisfying

$$\phi(B(0, r_1)) = 1, \quad 0 \leq \phi \leq 1, \quad \phi(\mathbb{R}^2 \setminus B(0, r)) = 0, \quad x \cdot \nabla \phi(x) \leq 0.$$ 

Noting that

$$f(\lambda) := 1 - \frac{1}{1 + (\lambda r_1)^2} \to 1 \text{ as } \lambda \to \infty,$$

and that

$$f'(\lambda) = \frac{2\lambda r_1^2}{(1 + (\lambda r_1)^2)^2} > 0 \text{ for } \lambda > 0,$$

we have that $1 > f(\lambda) \geq f(1)$ for all $\lambda \geq 1$.

Now we define the pair $(u_0, v_0) \triangleq (a u_{\lambda}, \phi, a v_{\lambda, r}, \phi)$ with some $a > 1$. Then, we prove that

Lemma 20  Let $\Lambda \in (8\pi, \infty) \setminus 4\pi \mathbb{N}$. For $\lambda > 1$ there is

$$a = a(\lambda) \in \left[ \frac{\Lambda}{8\pi}, \frac{\Lambda}{8\pi f(1)} \right]$$

such that

$$\int_{\Omega} u_0 \, dx = \Lambda.$$
Proof Firstly by changing variables, we see that

\[
\int_{B(0,\ell)} u_\lambda \, dx = 8 \int_{B(0,\ell)} \frac{\lambda^2}{(1 + \lambda^2 |x|^2)^2} \, dx
\]
\[
= 8 \int_{B(0,\lambda \ell)} dy \frac{\lambda^2}{(1 + |y|^2)^2}
\]
\[
= 16\pi \int_0^{\lambda \ell} \frac{s}{(1 + s^2)^2} \, ds
\]
\[
= 8\pi \int_0^{(\lambda \ell)^2} \frac{d\tau}{(1 + \tau)^2}
\]
\[
= 8\pi \cdot \left( 1 - \frac{1}{1 + (\lambda \ell)^2} \right) \quad \text{for } \ell > 0,
\]
and that

\[
8\pi \cdot \left( 1 - \frac{1}{1 + (\lambda \ell)^2} \right) < \int_{\Omega} u_\lambda \phi \, dx < 8\pi \cdot \left( 1 - \frac{1}{1 + (\lambda \ell)^2} \right).
\]

Then there is a unique constant \( a = a(\lambda) \) satisfying

\[
\frac{\Lambda}{8\pi} \leq a \leq \frac{\Lambda}{8\pi f(1)}
\]
and (91). \( \square \)

Next, we want to show that \( \mathcal{F}(u_0, v_0) \) can be sufficiently negative as \( \lambda \to \infty \). First, we note that

Lemma 21 There is \( C > 0 \) such that for all \( \lambda > 1 \),

\[
\int_{\Omega} u_0 \log u_0 \, dx \leq 16a\pi \cdot \log \lambda + C.
\]

Proof Since \( s \log s \leq t \log t + \frac{1}{e} \) for \( s \leq t \),

\[
\int_{\Omega} u_0 \log u_0 \, dx \leq a \int_{\Omega} \bar{u}_\lambda \log \bar{u}_\lambda \, dx + a \log a \int_{\Omega} \bar{u}_\lambda \, dx + \frac{|\Omega|}{e}.
\]

Since \( \log \bar{u}_\lambda \leq \log(8\lambda^2) = 2 \log \lambda + \log 8 \) and \( \int_{\Omega} \bar{u}_\lambda \leq \int_{\mathbb{R}^2} \bar{u}_\lambda = 8\pi \),

\[
\int_{\Omega} u_0 \log u_0 \, dx \leq 2a \cdot 8\pi \cdot \log \lambda + C,
\]
where we remark that the constant \( C \) is independent of \( a \) in view of (90). \( \square \)

Lemma 22 There exists \( C > 0 \) such that for all \( \lambda > 1 \),

\[
\int_{\Omega} u_0 v_0 \, dx \geq 32a^2\pi \log \lambda - C,
\]
and for any \( \varepsilon_1 > 0 \), there is \( C(\varepsilon_1) > 0 \) such that

\[
\frac{1}{2} \int_{\Omega} (v_0^2 + |\nabla v_0|^2) \, dx \leq 16(1 + \varepsilon_1)a^2\pi \log \lambda + C(\varepsilon_1).
\]
**Proof** Using $\overline{v}_{\lambda,r} > 0$ in $B(0, r)$ and $u_0 = 0$ outside $B(0, r)$, we see that

$$
\int_{\Omega} u_0 v_0 \, dx \geq a^2 \int_{B(0,r_1)} \overline{u}_\lambda \overline{v}_{\lambda,r} \, dx.
$$

Since

$$
\overline{v}_{\lambda,r}(x) > 2 \log \frac{1 + \lambda^2 r^2}{1 + \lambda^2 |x|^2} \quad \text{for } x \in B(0, r_1),
$$

then we have that

$$
\int_{\Omega} u_0 v_0 \, dx \geq a^2 \int_{B(0,r_1)} \overline{u}_\lambda \cdot 2 \log \frac{1 + \lambda^2 r^2}{1 + \lambda^2 |x|^2} \, dx
$$

and that

$$
\int_{B(0,r_1)} \overline{u}_\lambda \log(1 + \lambda^2 |x|^2) \, dx = 8 \int_{B(0,r_1)} \frac{\lambda^2 \log(1 + \lambda^2 |x|^2)}{(1 + \lambda^2 |x|^2)^2} \, dx
$$

$$
= 16\pi \int_{0}^{r_1} s \log(1 + s^2) \, ds
$$

$$
< 8\pi \int_{0}^{\infty} \frac{\log(1 + \xi)}{(1 + \xi)^2} \, d\xi < \infty.
$$

Combining these with (92), we obtain that

$$
\int_{\Omega} u_0 v_0 \, dx \geq 4a^2 \log(\lambda r) \cdot 8\pi \left(1 - \frac{1}{1 + (\lambda r_1)^2}\right) - C
$$

$$
\geq 32\pi a^2 \log \lambda - C'
$$

for $\lambda > 1$ with some positive constants $C, C'$. We remark that the constant $C'$ is independent of $a$ due to (90).

On the other hand, since

$$
\frac{1 + \lambda^2 r^2}{1 + \lambda^2 |x|^2} \leq \left(\frac{1 + \lambda r}{\lambda |x|}\right)^2,
$$

we see that for $\lambda \geq 1$

$$
|\overline{v}_{\lambda,r}(x)| \leq 4 \log \frac{1 + r}{|x|} + \log 8 \quad \text{in } B(0, r).
$$

Hence it follows from straightforward calculations that there is a positive constant $C$ satisfying

$$
\frac{1}{2} \int_{\Omega} v_0^2 \, dx \leq a^2 \int_{B(0,r)} \left(4 \log \frac{1 + r}{|x|} + \log 8\right)^2 \, dx
$$

$$
\leq C,
$$

(97)

where the constant $C$ is independent of $a$ due to (90). Moreover by the direct calculations,

$$
|\nabla \overline{v}_{\lambda,r}(x)| = \frac{4\lambda^2 |x|}{1 + \lambda^2 |x|^2} \quad \text{in } B(0, r).$$
By Young’s inequality, for any $\varepsilon_1 > 0$ there holds
\[
|\nabla v_0|^2 = a^2|\phi \nabla v_\lambda, r + \nabla \phi v_\lambda, r|^2 \\
\leq a^2 (1 + \varepsilon_1)\phi^2|\nabla v_\lambda, r|^2 + C(\varepsilon_1)a^2|\nabla \phi|^2|\nabla v_\lambda, r|^2.
\]
In view of (97), we have
\[
a^2 \int_\Omega |\nabla \phi|^2|\nabla v_\lambda, r|^2 d x \leq C
\]
with some $C$ independent of $\lambda$ and $a$ in view of (90). We infer that
\[
\int_\Omega |\nabla v_0|^2 d x \leq a^2 (1 + \varepsilon_1) \int_\Omega \phi^2|\nabla v_\lambda, r|^2 d x + C(\varepsilon_1)a^2 \int_\Omega |\nabla \phi|^2|\nabla v_\lambda, r|^2 d x \\
\leq 16(1 + \varepsilon_1)a^2 \int_{B(0,r)} \frac{\lambda^4 |x|^2}{(1 + \lambda^2 |x|^2)^2} d x + C'(\varepsilon_1) \\
= 16(1 + \varepsilon_1)a^2 \int_{B(0,\lambda \varepsilon)} \frac{|y|^2}{(1 + |y|^2)^2} d y + C'(\varepsilon_1) \\
= 32(1 + \varepsilon_1)\pi a^2 \int_0^{\lambda \varepsilon} \frac{s \cdot s^2}{(1 + s^2)^2} d s + C'(\varepsilon_1) \\
= 16(1 + \varepsilon_1)\pi a^2 \int_0^{(\lambda \varepsilon)^2} \frac{\tau}{(1 + \tau^2)^2} d \tau + C'(\varepsilon_1) \\
\leq 16(1 + \varepsilon_1)\pi a^2 \int_0^{(\lambda \varepsilon)^2} \frac{1}{1 + \tau} d \tau + C'(\varepsilon_1) \\
= 16(1 + \varepsilon_1)\pi a^2 \cdot \log (1 + (\lambda \varepsilon)^2) + C'(\varepsilon_1)
\]
thus
\[
\frac{1}{2} \int_\Omega |\nabla v_0|^2 d x \leq 16(1 + \varepsilon_1)\pi a^2 \log \lambda + C''(\varepsilon_1),
\]
where we again remark that the constant $C''(\varepsilon_1)$ is independent of $a$ and $\lambda$ due to (90). \qed

With above remark, we are now ready to conclude the proof for Theorem 3.

**Proof of Theorem 3** By Proposition 2, Proposition 3 and Remark 16, the proof for the first part in Theorem 3 concerning bounded solutions was complete. It remains to show the unboundedness part.

Collecting (94), (95) and (96), we infer that for any $\varepsilon_1 \in (0, 1)$, there exists some $C(\varepsilon_1) > 0$ such that
\[
\mathcal{F}(u_0, v_0) \leq 16a\pi a \log \lambda - 32\pi a^2 \log \lambda + 16(1 + \varepsilon_1)\pi a^2 \log \lambda + C(\varepsilon_1) \\
= -16\pi a[(1 - \varepsilon_1)a - 1] \log \lambda + C(\varepsilon_1)
\]
Recalling that $a \geq \frac{A}{8\pi} > 1$ due to (90), we now fix some small $\varepsilon_1$ independent of $a$ and $\lambda$ such that
\[
(1 - \varepsilon_1)a \geq (1 - \varepsilon_1)\frac{A}{8\pi} > 1
\]
and hence due to (90) again we infer that
\[
a[(1 - \varepsilon_1)a - 1] \geq \frac{A}{8\pi} \left(\frac{A(1 - \varepsilon_1)}{8\pi} - 1\right) > 0.
\]
As a result,
\[
F(u_0, v_0) \leq -2A \left( \frac{A(1 - \varepsilon_1)}{8\pi} - 1 \right) \log \lambda + C(\varepsilon_1) \to -\infty \text{ as } \lambda \to \infty. \tag{98}
\]

In the last step, we construct a suitable initial data based on the above discussion. For \( \Lambda \in (8\pi, \infty) \setminus 4\pi\mathbb{N} \), in view of (98) we can choose some \( \lambda > 1 \) such that
\[
-2\Lambda \left( \frac{A(1 - \varepsilon_1)}{8\pi} - 1 \right) \log \lambda + C < F_*(\Lambda),
\]
where \( C > 0 \) is the constant in (98). Finally we choose \( a \) satisfying (90) and (91). Therefore by the above discussion \((u_0, v_0)\) also satisfies
\[
F(u_0, v_0) < F_*(\Lambda).
\]

Thus let \((u, v)\) be the solution to (79) with the non-negative initial function \((u_0, v_0)\). If the solution is globally bounded in time, Proposition 3 guarantees that there are a subsequence \( \{t_k\} \subset (0, \infty) \) and a stationary solution \((u_s, v_s)\) satisfying that
\[
\lim_{t_k \to \infty} (u(t_k), v(t_k)) = (u_s, v_s) \text{ in } C^1(\Omega)
\]
and that
\[
F(u_s, v_s) < F_*(\Lambda),
\]
which contradicts to the definition of \( F_*(\Lambda) \). Thus we can conclude that
\[
\limsup_{t \nearrow \infty} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) = \infty. \tag{99}
\]

Furthermore, we notice that if \( \sup_{t > 0} \|u(t)\|_{L^\infty(\Omega)} < \infty \), by the second equation of (79) the semigroup estimate immediately implies \( \sup_{t > 0} \|v(t)\|_{L^\infty(\Omega)} < \infty \), which contradicts to (99). Thus we deduce
\[
\limsup_{t \nearrow \infty} \|u(t)\|_{L^\infty(\Omega)} = \infty.
\]

Besides, in view of Remark 15 and Proof of Proposition 2, if \( \|v\|_{L^\infty(\Omega)} \) or \( \|(I - \Delta)^{-1}[u]\|_{L^\infty(\Omega)} \) is uniformly-in-time bounded, \( \|u\|_{L^\infty(\Omega)} \) should be bounded as well, which also contradicts to (99). Hence, we have
\[
\limsup_{t \nearrow \infty} \|(I - \Delta)^{-1}[u](\cdot, t)\|_{L^\infty(\Omega)} = \limsup_{t \nearrow \infty} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]

We complete the proof. \( \square \)

**Remark 17** We remark that the example used in [20] cannot be directly applied in our problem to show unboundedness. Indeed, their construction of initial datum for the (reduced-)signal concentration has a zero mean integral and thus cannot be non-negative.

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