On the Approximation of Convex Bodies by Ellipses with Respect to the Symmetric Difference Metric

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Abstract Given a centrally symmetric convex body $K \subset \mathbb{R}^d$ and a positive number $\lambda$, we consider, among all ellipsoids $E \subset \mathbb{R}^d$ of volume $\lambda$, those that best approximate $K$ with respect to the symmetric difference metric, or equivalently that maximize the volume of $E \cap K$: these are the maximal intersection (MI) ellipsoids introduced by Artstein-Avidan and Katzin. The question of uniqueness of MI ellipsoids (under the obviously necessary assumption that $\lambda$ is between the volumes of the John and the Loewner ellipsoids of $K$) is open in general. We provide a positive answer to this question in dimension $d = 2$. Therefore we obtain a continuous 1-parameter family of ellipses interpolating between the John and the Loewner ellipses of $K$. In order to prove uniqueness, we show that the area $I_K(E)$ of the intersection $K \cap E$ is a strictly quasiconcave function of the ellipse $E$, with respect to the natural affine structure on the set of ellipses of area $\lambda$. The proof relies on smoothing $K$, putting it in general position, and obtaining uniform estimates for certain derivatives of the function $I_K(\cdot)$. Finally, we provide a characterization of maximal intersection positions, that is, the situation where the MI ellipse of $K$ is the unit disk, under the assumption that the two boundaries are transverse.

Keywords Convex bodies · Ellipsoids · Symmetric difference metric · Approximation

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1 Introduction

1.1 Convex Bodies and Approximation Problems

The euclidean distance induces the well-known Hausdorff metric on the set $S^d$ of nonempty compact subsets of $\mathbb{R}^d$. Namely, $d_{\text{Haus}}(K_1, K_2)$ is defined as the least $\varepsilon \geq 0$ such that every point in one of the sets $K_i$ is within euclidian distance at most $\varepsilon$ from some point in the other set. By the Blaschke selection theorem [8, p. 37], bounded subsets of $S^d$ are compact; in particular the metric space $(S^d, d_{\text{Haus}})$ is complete and locally compact.

We are interested in the space $K^d$ of convex bodies (i.e., compact convex sets with nonempty interior), which is a locally closed subset of $S^d$. There are other natural metrics on $K^d$ that also induce the Hausdorff topology: see [19]. Among these, we highlight the symmetric difference metric and the normalized symmetric difference metric:

$$d_{\text{sym}}(K_1, K_2) := |K_1 \triangle K_2|, \quad d_{\text{nsym}}(K_1, K_2) := \frac{|K_1 \triangle K_2|}{|K_1 \cup K_2|},$$

(1.1)

where $|\cdot|$ denotes volume (Lebesgue measure) in $\mathbb{R}^d$. These two metrics make sense in broader classes of sets and are known in Measure Theory as the Fréchet–Nikodym and the Marczewski–Steinhaus pseudometrics, respectively. Note that the metric $d_{\text{Haus}}$ is preserved by the action euclidian isometries of $\mathbb{R}^d$, while $d_{\text{sym}}$ is preserved by volume-preserving affine transformations, and $d_{\text{nsym}}$ is preserved by all affine transformations. In this paper we focus on the symmetric difference metric $d_{\text{sym}}$.

There is a large body of literature on approximation of convex bodies by simpler ones, as e.g. polyhedra: see the survey articles [5,10]. Let us mention a few of the most classical results. Given a plane convex body $K \in K^2$, for each $n \geq 3$, let $P_n^{(1)}$ be an inscribed $n$-gon of maximal area, let $P_n^{(3)}$ be a circumscribed $n$-gon of minimal area, and let $P_n^{(2)}$ be a convex $n$-gon that best approximates $K$ with respect to the symmetric difference metric. The approximation errors $\varepsilon_n^{(i)} := d_{\text{sym}}(K, P_n^{(i)})$ obviously tend to zero. Dowker [6] (see also [9, §II.3]) proved that the sequence $(\varepsilon_n^{(1)})$ is concave and the sequence $(\varepsilon_n^{(3)})$ is convex, and Eggleston [7] proved that the sequence $(\varepsilon_n^{(2)})$ is also convex. On the other hand, L. Fejes Tóth stated in his famous book [9, p. 43] that if $\partial K$ is sufficiently differentiable and positively curved then each of these three sequences is asymptotic to $c_i n^{-2}$, for some explicitly defined constant $c_i = c_i(K) > 0$; curiously, $(c_1, c_2, c_3)$ is proportional to $(1, 3/4, 2)$. These formulas were later proved by McClure and Vitale [17] for $i = 1$ and 3, and by Ludwig [16] for $i = 2$. For higher-dimensional versions of these results, see [10,11,16].

Another class of “simple” convex bodies consists on ellipsoids. Let us note that ellipsoids are the convex bodies that are worst approximable by polytopes: see [16, Rem. 2].
It is well-known that every convex body $K$ admits a unique inscribed ellipsoid $J_K$ of maximal volume and a unique circumscribed ellipsoid $L_K$ of minimal volume; they are called respectively the John ellipsoid and the Loewner ellipsoid of $K$. Moreover, if $K$ is centrally symmetric in the sense that $K = -K$, then so are the ellipsoids $J_K$ and $L_K$. See [3, Lecture 3] for proofs, [13] for historical information, and [18, §10.12] for other types of ellipsoids associated to a convex body.

Our original motivation comes from the following approximation problem posed by W. Kuperberg [15]:

**Question 1.1** If $K$ is a plane convex body of area 1, and if $E$ is an ellipse of area 1 that minimizes $d_{\text{sym}}(K, E)$ among all such ellipses, is $E$ necessarily unique?

In this paper, we answer this question positively under the assumption that $K$ is centrally symmetric. Actually, we prove uniqueness of a family of a certain ellipses that includes $E$ and interpolates between the John and the Loewner ellipses, as explained below.

### 1.2 Maximal Intersection Ellipsoids

Let $C^d \subset K^d$ denote the set of centrally symmetric $d$-dimensional convex bodies, where $d \geq 2$. Following Artstein-Avidan and Katzin [1], we say that an ellipsoid $E \subset \mathbb{R}^d$ is a maximal intersection (MI) ellipsoid for $K \in C^d$ if among all ellipsoids with the same volume as $E$, it maximizes the volume of $E \cap K$. In view of the relation

$$d_{\text{sym}}(K, E) = |K| + |E| - 2|K \cap E|,$$  \hspace{1cm} (1.2)

it is equivalent to say that $E$ is an optimal approximation for $K$ with respect to the symmetric difference metric, among all ellipsoids of a fixed volume.

Immediate examples of MI ellipsoids are the John ellipsoid $J_K$ and the Loewner ellipsoid $L_K$. Furthermore, there are no other MI ellipsoids with volume $|J_K|$ or $|L_K|$. On other hand, if $\lambda > 0$ is either smaller than $|J_K|$ or bigger than $|L_K|$ then $K$ obviously admits infinitely many MI ellipsoids of volume $\lambda$. Artstein-Avidan and Katzin [1] ask whether uniqueness of MI ellipsoids holds when $\lambda$ is in the interesting range $|J_K| < \lambda < |L_K|$. We provide a positive answer for this question in dimension $d = 2$:

**Theorem 1.2** Let $K \subset \mathbb{R}^2$ be a centrally symmetric convex body, and let $\lambda$ be a number in the range $|J_K| \leq \lambda \leq |L_K|$. Then there exists a unique MI ellipse $M_K(\lambda)$ of area $\lambda$, and it is centrally symmetric.

In particular, taking $\lambda = |K|$, we obtain the announced positive answer for Question 1.1 in the centrally symmetric case.

As a simple consequence of uniqueness (using the Blaschke selection theorem), the MI ellipse $M_K(\lambda)$ provided by Theorem 1.2 depends continuously on both $K$ and $\lambda$, provided that $|J_K| \leq \lambda \leq |L_K|$. In particular, these MI ellipses continuously interpolate between the John and Loewner ellipses.
As remarked in [1], every centrally symmetric convex body in $\mathbb{R}^d$ admits MI ellipsoids of any prescribed volume $\lambda > 0$ that are centrally symmetric. In dimension 2, as an ingredient of the proof of Theorem 1.2, we need to establish the following:

**Lemma 1.3** Let $K \subset \mathbb{R}^2$ be a centrally symmetric convex body, and let $\lambda$ be a number in the range $|\mathcal{J}_K| < \lambda < |\mathcal{L}_K|$. Then every MI ellipse of area $\lambda$ for $K$ is centrally symmetric.

### 1.3 Quasiconcavity of the Area Function

Theorem 1.2 follows from a sharper result. In order to state it, let us introduce some notation.

Given $\lambda > 0$, let $\mathcal{C}^2_\lambda$ be the set of centrally symmetric bodies $K \in \mathcal{C}^2$ that satisfy $|\mathcal{J}_K| < \lambda < |\mathcal{L}_K|$. Note that $\mathcal{C}^2_\lambda$ is an open subset of $\mathcal{C}^2$. Consider the family of ellipses of area $\pi$ whose axes are the $x$ and $y$ axes (together with the unit disk), which we parameterize by $t \in \mathbb{R}$ as follows:

$$E_t := \{ (x, y) \in \mathbb{R}^2 ; e^t x^2 + e^{-t} y^2 \leq 1 \}. \quad (1.3)$$

For any $K \in \mathcal{C}^2$, its **intersection function** is the function $I_K : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$I_K(t) := |E_t \cap K|. \quad (1.4)$$

Recall that a real function $f$ defined on an interval $J \subseteq \mathbb{R}$ is called **quasiconcave** if for every $s, t, u \in J$,

$$t_0 < t_1 < t_2 \Rightarrow f(t_1) \geq \min\{ f(t_0), f(t_2) \},$$

and is called **strictly quasiconcave** if the inequality on the right is always strict.

Our crucial technical result, whose proof occupies the bulk of this paper, is the following:

**Theorem 1.4** For every $K \in \mathcal{C}^2_\pi$, the associated intersection function $I_K$ is strictly quasiconcave.

If Lemma 1.3 and Theorem 1.4 are assumed, we can immediately deduce our uniqueness result:

**Proof of Theorem 1.2** Let $K \in \mathcal{C}^2$. Since the John and Loewner ellipses are known to be unique and centrally symmetric, it is sufficient to consider $|\mathcal{J}_K| < \lambda < |\mathcal{L}_K|$. Applying an homothecy if necessary, we can assume that $\lambda = \pi$, so $K \in \mathcal{C}^2_\pi$. As remarked before, $K$ admits at least one MI ellipse of area $\pi$. Suppose there are two, say $E \neq E'$. By Lemma 1.3, these ellipses must be centrally symmetric. Applying an appropriate element of $\text{SL}(2, \mathbb{R})$ (i.e., a linear map of determinant 1), we can assume that $E$ and $E'$ are elements of the family (1.3). So the intersection function $I_K$ attains its maximum at two distinct points, contradicting Theorem 1.4. This proves uniqueness of the MI ellipse of area $\pi$. \qed
Remark 1.5 The space $\mathcal{H}$ of centrally symmetric ellipses of area $\pi$ has a natural affine structure, which is in fact equivalent to the affine structure of the hyperbolic plane: see [4]. As a reformulation of Theorem 1.4, for every $K \in C^2_\pi$, the function $E \in \mathcal{H} \mapsto |E \cap K|$ is strictly quasiconcave with respect to this affine structure.

In fact we will prove a more general version of Theorem 1.4: see Theorem 3.7 below.

1.4 Maximum Intersection Position

Let us say that a centrally symmetric convex body $K \subset \mathbb{R}^d$ is in maximum intersection (MI) position if the euclidian unit ball in $\mathbb{R}^d$ is an MI ellipsoid for $K$. Every centrally symmetric convex body can be put in MI position by applying an appropriate invertible linear map, whose determinant can be any prescribed non-zero number.

In Sect. 5 we will give a simple characterization of MI position in the plane under a transversality hypothesis, which has the following interesting consequence:

Proposition 1.6 Let $K \subset \mathbb{R}^2$ be a compact convex centrally symmetric set whose boundary $\partial K$ is transverse to the unit circle $S^1$. Then:

(a) if the intersection consists of four points then $K$ cannot be in MI position;
(b) if the intersection consists of eight points then $K$ is in MI position if and only if $\partial K \cap S^1$ is invariant by a quarter turn (i.e., rotation of $\pi/2$).

Figure 1 shows an example of situation (b).

In Sect. 5 we also discuss the classical characterization of the John position (that is, the situation when the John ellipsoid is round) and how it relates to MI position.

1.5 Strategy of the Proofs and Organization of the Paper

The proof of Theorem 1.4 occupies Sects. 2 and 3. In order to make this proof more digestible, let us highlight the key ideas. We perform a local study of the function $I_K(t) = |K \cap E_t|$. It is essentially sufficient to consider a neighborhood of $t = 0$. We initially assume that the curve $\partial K$ is smooth and crosses the unit circle $S^1 = \partial E_0$ at finitely many points, making nonzero angles. This transversality condition implies that the function $I_K$ is of class $C^2$ on a neighborhood of $0$; furthermore, there are explicit formulas for the first two derivatives of $I_K$ at $0$, with $I'_K(0)$ depending on the locations of the crossings between $\partial K$ and $S^1$, and $I''_K(0)$ depending also on the crossing angles: see Proposition 2.1. Another observation (Proposition 2.2) is that we can allow certain types of “tame” tangencies between $\partial K$ and $S^1$ and the function $I_K(t)$ will still be $C^1$ on a neighborhood of $t = 0$, though $I''(0)$ may fail to exist. Then we reach the heart of the whole proof, Proposition 2.3, which essentially says that if $\partial K$ is transverse to $S^1$ and $I'_K(0) = 0$ then $I_K$ is strictly concave around $0$, that is, $I''_K(0) < 0$. The proof of this key proposition relies on a lower bound (2.15) for $-I''_K(0)$ which, like the formula for $I'_K(0)$, depends only on the locations and not on the angles of the crossings between $\partial K$ and $S^1$. A quick inspection of this bound reveals that it has a
Fig. 1  Example of a centrally symmetric body \( K \) in MI position; the unit circle and the John and Loewner ellipses of \( K \) are also pictured. The curve \( \partial K \) satisfies the equation \( 1.355x^2 - 0.58xy + 1.005y^2 - 0.1264x^4 + 0.58x^3y - 1.041x^2y^2 + 0.58xy^3 + 0.2236y^4 = 1 \). Apart from being centrally symmetric, \( K \) has no other linear symmetries

strong tendency to be positive: for example, if each pair of consecutive crossings is separated by a circle arc of length \(< \pi/2 \) then the bound is automatically positive. The actual proof of Proposition 2.3 is done by a case-by-case analysis, which occupies Sect. 2.5. All estimates are explicit and ultimately we obtain a positive lower bound for \( \max\{|I'_K(0)|, -I''_K(0)\} \) that does not depend on \( K \), but only on the areas of \( K \) and \( K \cap E_0 \). This uniformity with respect to \( K \) is crucial for the second part of the proof of Theorem 1.4, presented in Sect. 3. There, we argue that any convex body \( K \in \mathcal{C}_\pi \) admits small perturbations \( \tilde{K} \) with respect to the symmetric difference metric that have the same area as \( K \) and are “regular” in the following sense: the boundary \( \partial \tilde{K} \) is smooth and transverse to all ellipses \( \partial E_t \), except for a finite number of “tame” tangencies. Using the estimates obtained previously, we conclude that the resulting function \( I_{\tilde{K}} \) is strictly quasiconcave in a quantitative sense that is independent of the size of the perturbation. This uniformity allows us to take a limit and conclude that \( I_K \) is strictly quasiconcave as well, therefore proving Theorem 1.4.

The paper has three additional short sections. In Sect. 4 we prove Lemma 1.3 and therefore conclude the proof of Theorem 1.2. In Sect. 5 we study MI positions. These two sections may be read independently from the previous ones, except that we use Proposition 2.1 and Theorem 3.7. Finally, Sect. 6 discusses possible extensions of our results.
2 Derivatives of the Intersection Function Under Regularity Assumptions

2.1 Differentiability of the Intersection Function

Consider a pair of Jordan curves $\Gamma_1, \Gamma_2$ in the plane. A point $p$ of intersection between the curves is called:

- a **crossing** if each curve admits a $C^1$ parameterization at a neighborhood of $p$, and the pair of tangent vectors at $p$ is linearly independent;
- a **quadratic tangency** if each curve admits a $C^2$ parameterization at a neighborhood of $p$, and these parametrized curves have a first- but not a second-order contact at $p$.

We say that the curves $\Gamma_1, \Gamma_2$ are:

- **transverse** if every point of intersection is a crossing;
- **quasitransverse** if every point of intersection is either a crossing or a quadratic tangency.

In either case, the number of intersections is finite.

Now consider a centrally symmetric convex body $K \subset \mathbb{R}^2$ whose boundary $\partial K$ is transverse to the unit circle $\partial E_0 = S^1$. Then the two curves cross at $4n$ points. If $K \in C^2_\pi$ then necessarily $n \geq 1$. We list the crossing points in counterclockwise order as $\zeta_1, \ldots, \zeta_{4n}$. Since $K$ is centrally symmetric, we have $\zeta_{j+2n} = -\zeta_j$. Shifting indices by 1 (mod $4n$) if necessary, we assume that the following condition holds: if the curve $\partial K$ is traversed counterclockwise, then it exits the unit disk $E_0$ at the points $\zeta_j$ with $j$ even, and enters it at the points $\zeta_j$ with $j$ odd; see Fig. 2. Let $\alpha_j > 0$ denote the non-oriented angles of intersection; note that $\alpha_j < \pi/2$ since $K$ is centrally symmetric. Fix numbers $\xi_1 < \xi_2 < \cdots < \xi_{4n} < \xi_1 + 2\pi$ such that $\zeta_j = (\cos \xi_j, \sin \xi_j)$.

**Proposition 2.1** Let $K \subset \mathbb{R}^2$ be a centrally symmetric convex body whose boundary $\partial K$ is transverse to the unit circle $\partial E_0$ and intersects it at $4n > 0$ points. Let $(\xi_j)$
and \((\alpha_j)\) be the crossing positions and angles as defined above. Then the intersection function \(I = I_K\) is \(C^2\) at a neighborhood of 0 and

\[
I'(0) = \frac{1}{2} \sum_{j=1}^{2n} (-1)^j \sin 2\xi_j, \tag{2.1}
\]

\[
- I''(0) = \frac{1}{4} \sum_{j=1}^{2n} \left[ (-1)^j \sin 4\xi_j + \frac{1 + \cos 4\xi_j}{\tan \alpha_j} \right]. \tag{2.2}
\]

Proof In polar coordinates \((r, \theta)\), the ellipse \(\partial E_t\) has equation

\[
r^2 = \frac{1}{e^t \cos^2 \theta + e^{-t} \sin^2 \theta}.
\]

Similarly, the curve \(\partial K\) is represented by some equation \(r^2 = G(\theta)\), where \(G\) is a positive function on the circle \(\mathbb{R}/2\pi \mathbb{Z}\) which satisfies \(G(\theta + \pi) = G(\theta)\). Furthermore, \(G - 1\) vanishes exactly on the points \(\xi_1, \ldots, \xi_{4n}\), is \(C^1\) on a neighborhood of these points, and

\[
(-1)^j G'(\xi_j) > 0 \tag{2.3}
\]

for each \(j\).

Let \(f(\theta, t) := G(\theta) - 1/(e^t \cos^2 \theta + e^{-t} \sin^2 \theta)\). By the Implicit Function Theorem, for \(t\) sufficiently close to 0, the function \(f(\cdot, t)\) vanishes on \(4n\) points \(\xi_1(t), \ldots, \xi_{4n}(t)\); moreover each function \(\xi_j(\cdot)\) is \(C^1\) and satisfies \(\xi_j(0) = \xi_j\) and

\[
\xi_j'(t) = - \frac{f_t(\xi_j(t), t)}{f_0(\xi_j(t), t)}.
\]

Consider the function

\[
A(t) := d_{sym}(E_t, K) = \frac{1}{2} \int_0^{2\pi} |f(\theta, t)| \, d\theta = \int_0^\pi |f(\theta, t)| \, d\theta.
\]

Equivalently,

\[
A(t) = \sum_{j=1}^{2n} (-1)^j \int_{\xi_j(t)}^{\xi_{j+1}(t)} f(\theta, t) \, d\theta. \tag{2.4}
\]

By the Leibniz integral rule,

\[
A'(t) = \sum_{j=1}^{2n} (-1)^j \left[ \int_{\xi_j(t)}^{\xi_{j+1}(t)} f_t(\theta, t) \, d\theta + \frac{f(\xi_{j+1}(t), t)\xi_{j+1}'(t) - f(\xi_j(t), t)\xi_j'(t)}{0} \right]. \tag{2.5}
\]
\[ A''(t) = \sum_{j=1}^{2n} (-1)^j \left[ \int_{\xi_j(t)}^{\xi_{j+1}(t)} f_{tt}(\theta, t) \, d\theta + f_{t}(\xi_{j+1}(t), t)\xi_{j+1}'(t) - f_{t}(\xi_{j}(t), t)\xi_{j}'(t) \right] \]

\[ = \sum_{j=1}^{2n} (-1)^j \left[ \int_{\xi_j(t)}^{\xi_{j+1}(t)} f_{tt}(\theta, t) \, d\theta - 2f_{t}(\xi_{j}(t), t)\xi_{j}'(t) \right] \]

\[ = \sum_{j=1}^{2n} (-1)^j \left[ \int_{\xi_j(t)}^{\xi_{j+1}(t)} f_{tt}(\theta, t) \, d\theta + \frac{2f_{t}(\xi_{j}(t), t)}{f_{0}(\xi_{j}(t), t)} \right]^2. \quad \text{(2.6)} \]

In particular, \( A \) is a \( C^2 \) function on a neighborhood of 0. Since the functions \( A \) and \( I \) are related by formula (1.2), \( I \) is also \( C^2 \) on a neighborhood of 0.

Now we consider \( t = 0 \). A computation gives

\[ f_{t}(\theta, 0) = \cos 2\theta \quad \text{and} \quad f_{tt}(\theta, 0) = -\cos 4\theta. \]

Plugging into (2.5),

\[ I'(0) = -\frac{1}{2} A'(0) = -\frac{1}{4} \sum_{j=1}^{2n} (-1)^j \left[ \sin 2\xi_{j+1} - \sin 2\xi_j \right] = \frac{1}{2} \sum_{j=1}^{2n} (-1)^j \sin 2\xi_j, \]

proving (2.1). Analogously, from (2.5) we obtain

\[ -I''(0) = \frac{1}{2} A''(0) = \frac{1}{2} \sum_{j=1}^{2n} (-1)^j \left[ \frac{-\sin 4\xi_{j+1} + \sin 4\xi_j}{4} + \frac{2\cos^2 2\xi_j}{G'(\xi_j)} \right] \]

\[ = \frac{1}{2} \sum_{j=1}^{2n} \left[ (-1)^j \sin 4\xi_j \frac{1 + \cos 4\xi_j}{2} + \frac{1 + \cos 4\xi_j}{|G'(\xi_j)|} \right], \]

where in the last step we have used (2.3). Since \(|G'(\xi_j)| = 2\tan \alpha_j\), we obtain formula (2.2).

**Proposition 2.2** Let \( K \subset \mathbb{R}^2 \) be a centrally symmetric convex body whose boundary \( \partial K \) is quasitransverse to the unit circle \( S^1 = \partial E_0 \). Suppose the points of tangency are not \((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\). Then the intersection function \( I_K \) is \( C^1 \) at a neighborhood of 0.

**Proof** Assume there is at least one tangency between \( \partial K \) and the unit circle \( \partial E_0 \), otherwise the proposition follows from Proposition 2.1. Fix numbers \( \tau_1 < \tau_2 < \cdots < \tau_{2\ell} \) with \( \tau_{i+\ell} = \tau_i + \pi \) such that the tangencies between occur at the points \((\cos \tau_i, \sin \tau_i)\). By assumption, these tangencies are quadratic and do not occur at the points \((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\). Also fix small neighborhoods \( V_i = (a_i, b_i) \ni \tau_i \).

Define functions \( G(\theta) \) and \( f(\theta, t) \) as in the proof of Proposition 2.1, and note that \( G(\theta) = 1 + f(\theta, 0) \). These functions are continuous everywhere and are \( C^2 \) if \( \theta \) is restricted to the set \( \bigcup_i V_i \) and \( t \) is close to zero. Furthermore, for each \( i \) we have
\(G(\tau_i) = 1, G'(\tau_i) = 0, \) and \(G''(\tau_i) \neq 0; \) the latter inequality expresses the fact that each tangency is quadratic. Note also that \(f_i(\tau_i, 0) \neq 0; \) indeed, along the proof of Proposition 2.1 we computed \(f_i(\theta, 0) = \cos 2\theta, \) and since the tangency points are not \((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\), we have \(\cos 2\tau_i \neq 0.\)

We will show that the function \(A(t) := d_{\text{sym}}(E_t, K)\) is \(C^1\) on a neighborhood of \(t = 0; \) then it will follow from the relation (1.2) that \(I_K(t)\) is also \(C^1\) on a neighborhood of \(t = 0.\)

If there are no tangencies then \(A(t)\) is given by formula (2.4). In order to take the tangencies into account, for each \(i\) we need to add a certain correction term \(C_i(t)\) to the formula. More precisely, let \(\varsigma_i \in \{+1, -1\}\) be the sign of \(G\) on the neighborhood \(V_i \ni \tau_i,\) in the sense that \(\varsigma_i G \geq 0\) there; then the correction term \(C_i(t)\) satisfies

\[
\int_{a_i}^{b_i} |f(\theta, t)| \, d\theta = \varsigma_i \int_{a_i}^{b_i} f(\theta, t) \, d\theta + C_i(t).
\]

Once we prove that each function \(C_i\) is \(C^1\) at a neighborhood of 0, we will conclude that so are the functions \(A\) and \(I_K.\)

For definiteness, consider the case where \(G''(\tau_i) < 0\) (i.e. \(\varsigma_i = -1\)) and \(f_i(\tau_i, 0) > 0; \) the other three cases are analogous. For each \(t\) sufficiently close to zero, consider the equation \(f(\theta, t) = 0\) for \(\theta \in V_i;\) it has no solution for \(t > 0,\) exactly one solution \(\tau_i\) for \(t = 0,\) and exactly two solutions \(\tau_i^- < \tau_i^+\) for \(t < 0.\) Then the correction term is

\[
C_i(t) = \int_{a_i}^{b_i} \left[ |f(\theta, t)| + f(\theta, t) \right] \, d\theta = \begin{cases} 2 \int_{\tau_i^-}^{\tau_i^+} f(\theta, t) \, d\theta & \text{if } t < 0, \\ 0 & \text{if } t \geq 0. \end{cases}
\]

For \(t < 0\) close to 0, the width \(\tau_i^+(t) - \tau_i^-(t)\) is \(O(|t|^{1/2}),\) and so \(C_i(t) = O(|t|^{3/2}).\) In particular, \(C_i'(0) = 0.\) Still assuming \(t < 0\) close to 0, by Leibniz integral rule we have

\[
\frac{1}{2} C_i'(t) = \int_{\tau_i^-}^{\tau_i^+} f_i(\theta, t) \, d\theta + f(\tau_i^+(t), t) \cdot (\tau_i^+)'(t) - f(\tau_i^-(t), t) \cdot (\tau_i^-)'(t),
\]

which tends to 0 as \(t \searrow 0.\) Hence \(C_i\) is a function of class \(C^1,\) as we wanted to show. \(\square\)

The proof also shows that formula (2.1) still holds in the situation of Proposition 2.2, but we will not use this fact.

### 2.2 The Key Proposition

**Proposition 2.3** For every \(\varepsilon > 0\) there exists \(\delta > 0\) with the following properties. Suppose that \(K \subset \mathbb{R}^2\) is a centrally symmetric convex body whose boundary \(\partial K\) is transverse to the unit circle \(\partial E_0,\) and
\[ \varepsilon \leq |K \cap E_0| \leq \min\{\pi, |K|\} - \varepsilon. \]  
(2.7)

Then

\[ \max \{|I'_{K}(0)|, -I''_{K}(0)|\} > \delta, \]  
(2.8)

The proof of Proposition 2.3 occupies the rest of this section. Fix the convex body \( K \) as above, and write \( I = I_{K} \).

2.3 Geometric Inequalities

Let us establish some preliminary inequalities.

It is convenient to reparameterize the sequence \( (\xi_j) \) differently. For each \( i \in \{1, \ldots, n\} \), let

\[ \sigma_i := \xi_{2i} + \xi_{2i-1}, \quad \omega_i := \xi_{2i} - \xi_{2i-1}. \]  
(2.9)

So \( \sigma_{i+n} = \sigma_i + 2\pi, \omega_{i+n} = \omega_i \), and \( 0 < \omega_i < \pi \).

**Lemma 2.4** For each \( i \) we have \( \max\{\alpha_{2i-1}, \alpha_{2i}\} \leq \omega_i / 2. \)

**Proof** Figure 3 shows how to bound \( \alpha_{2i} \). The bound for \( \alpha_{2i-1} \) is analogous. \( \square \)

Next, we want some bounds on the parameters \( \omega_i \). Shifting indices if necessary, we assume that

\[ \omega_1 = \max\{\omega_1, \omega_2, \ldots, \omega_n\}. \]
If \( n \geq 2 \), we fix \( s \in \{2, \ldots, n\} \) such that \( \omega_s = \max\{\omega_2, \ldots, \omega_n\} \). Note that \( \sum_{i=1}^{n} \omega_i < \pi \) and in particular

\[
\omega_s < \pi - \omega_1 \leq \frac{\pi}{2} .
\] (2.10)

The following lemma uses hypothesis (2.7) from Proposition 2.3, namely that \( I(0) = |K \cap E_0| \) is not too close to \( 0 \) nor to \( \min\{\pi, |K|\} \).

**Lemma 2.5** *For every \( \varepsilon > 0 \) there exists \( \kappa > 0 \), not depending on \( K \), such that if \( \varepsilon \leq I(0) \leq \min\{\pi, |K|\} - \varepsilon \) then

\[
\kappa < \omega_1 < \pi - \kappa
\] (2.11)

and, if \( n \geq 2 \),

\[
\omega_s < \frac{\pi}{2} - \kappa .
\] (2.12)

*Proof* Note that \( K \) contains the disk of radius \( \cos \omega_1/2 \) centered at the origin (see Fig. 4), and in particular \( I(0) \geq \pi \cos^2 \omega_1/2 \). By assumption, \( I(0) \leq \pi - \varepsilon \), and so \( \omega_1 \) cannot be too small, proving the first inequality in (2.11).

Now consider (2.12): if this inequality does not hold then, by (2.10), both \( \omega_1 \) and \( \omega_s \) are approximately \( \pi/2 \). Then \( K \setminus E_0 \) is contained in the union of the four small regions represented in Fig. 5. This contradicts the fact that \( |K \setminus E_0| = |K| - I(0) \geq \varepsilon \) is not too small.

The second inequality in (2.11) is the trickiest one. Let \( L_1 \) (resp. \( L_2 \)) be the tangent line to \( \partial K \) at the point \( \zeta_1 \) (resp. \( \zeta_{2n+2} \)), oriented so that \( K \) sits to the left of this line. The lines \( L_1 \) and \( L_2 \) cross the circle \( \partial E_0 \) forming angles \( \alpha_1 \) and \( \alpha_{2n+2} = \alpha_2 \), respectively. Let \( R_i \) be the part of the disk \( E_0 \) to the right of the line \( L_i \); see Fig. 6.
The regions $R_1$ and $R_2$ are disjoint and their interiors are contained in $E_0 \setminus K$. In particular,
\[ |R_1| + |R_2| \leq |E_0 \setminus K| = \pi - I(0) \leq \pi - \varepsilon. \]
So the areas $|R_1|$ and $|R_2|$ cannot be both too close to $\pi/2$. On the other hand, these areas are related to the crossing angles $\alpha_i$ as follows:
\[ |R_i| = \alpha_i - \frac{1}{2} \sin 2\alpha_i. \]
Therefore the angles $\alpha_1$ and $\alpha_2$ cannot be both too close to $\pi/2$. By Lemma 2.4, we have $\max\{\alpha_1, \alpha_2\} \leq \omega_1/2$, and in particular the quantity $\gamma := \omega_1 - \alpha_1 - \alpha_2$ is nonnegative. If $\gamma$ is zero or small then $\omega_1$ is not too close to $\pi$, as desired. So assume from now on that $\gamma$ is not too close to 0. Then the lines $L_1$ and $L_2$ cannot be parallel; indeed they cross forming angle $\gamma$ at some point $z$. Recall that the centrally symmetric
convex body $K$ sits to the left of each oriented line $L_1$ and $L_2$; furthermore, the arc of the counterclockwise-oriented Jordan curve $\partial K$ from $\zeta_1$ to $\zeta_2$ is contained in the disk $E_0$. It follows from these observations that $K \subseteq E_0 \cup G \cup (-G)$, where $G$ is the (filled) triangle with vertices $\zeta_1, \zeta_{2n+2}, z$. In particular,

$$2|G| = |G \cup (-G)| \geq |K \setminus E_0| = \pi - I(0) \geq \varepsilon,$$

Note that the triangle $G$ has a side $[\zeta_1, \zeta_{2n+2}]$ of length $\ell := 2 \cos \omega_1/2$, and therefore its area cannot exceed the area of an isosceles triangle with angle $\gamma$ and opposite side $\ell$, that is,

$$|G| \leq \frac{1}{4} \ell^2 \cot \frac{\gamma}{2}.$$ 

Since $\gamma$ and $|G| \geq \varepsilon/2$ are bounded away from 0, so is $\ell$. It follows that $\omega_1$ cannot be too close to $\pi$. This completes the proof of the second inequality in (2.11) and of the lemma. 

2.4 More Manipulation of the Derivatives

We will now come back to the formulas obtained in Proposition 2.1 and rewrite them in terms of the new parameters (2.9); we will also use Lemma 2.4 to obtain a convenient lower bound for minus the second derivative.

Lemma 2.6

$$I'(0) = \sum_{i=1}^{n} \sin \omega_i \cos \sigma_i,$$  \hfill (2.13)

$$-I''(0) \geq \frac{1}{4} \sum_{i=1}^{n} \left[ \sin \omega_i \sin^2 \sigma_i + \left( \cot \frac{1}{2} \omega_i - \sin \omega_i \right) \cos^2 \sigma_i \right].$$ \hfill (2.14)

Proof We can rewrite (2.1) as

$$I'(0) = \frac{1}{2} \sum_{i=1}^{n} \left( \sin 2\xi_{2i} - \sin 2\xi_{2i-1} \right),$$

so (2.13) follows from the identity

$$\sin x - \sin y = 2 \sin \frac{x - y}{2} \cos \frac{x + y}{2}$$

[together with the definitions (2.9)]. Analogously, rewriting (2.2) as

$$-I''(0) = \frac{1}{4} \sum_{i=1}^{n} \left[ \sin 4\xi_{2i} - \sin 4\xi_{2i-1} + \frac{1 + \cos 4\xi_{2i}}{\tan \alpha_{2i}} + \frac{1 + \cos 4\xi_{2i-1}}{\tan \alpha_{2i-1}} \right].$$
By Lemma 2.4, \( \max(\alpha_{2i}, \alpha_{2i-1}) \leq \omega_i/2 \). So, using the identity
\[
\cos x + \cos y = 2 \cos \frac{x - y}{2} \cos \frac{x + y}{2}
\]
we obtain
\[
- I''(0) \geq \frac{1}{2} \sum_{i=1}^{n} \left[ \sin 2\omega_i \cos 2\sigma_i + \cot \frac{1}{2} \omega_i \left( 1 + \cos 2\omega_i \cos 2\sigma_i \right) \right].
\]

Let us manipulate the quantity between square brackets. For simplicity of writing, we omit the \( i \) indices
\[
[\ldots] = \frac{\sin 2\omega}{2 \sin \omega \cos \omega} \cos 2\sigma + \cot \frac{1}{2} \omega \left[ \frac{(-1 + \cos 2\omega) \cos 2\sigma + (1 + \cos 2\sigma)}{-2 \sin^2 \omega} \right]
\]
\[
= 2 \sin \omega \left( \cos \omega - \cot \frac{1}{2} \omega \sin \omega \right) \frac{\cos 2\sigma + 2 \cot \frac{1}{2} \omega \cos^2 \sigma}{\cos^2 \sigma - \sin^2 \sigma}
\]
\[
= 2 \sin \omega \sin^2 \sigma + 2 \left( \cot \frac{1}{2} \omega - \sin \omega \right) \cos^2 \sigma,
\]
yielding (2.14). \( \square \)

2.5 Proof of the Key Proposition 2.3

The proof is a case-by-case analysis; in most of the cases we will show that \( I''(0) \) is negative and away from zero, but in a few cases the conclusion is that \( I'(0) \) is away from zero. All the estimates on those derivatives will be obtained from Lemma 2.6, which will not be explicitly mentioned each time. All estimates are explicit and ultimately we will obtain a lower bound for \( \max(|I'(0)|, -I''(0)|) \) that depends only on \( \kappa \) from Lemma 2.5, and therefore is a function of \( \varepsilon \) which is independent of \( K \).

Let us introduce some notation
\[
f(\omega) := \cot \frac{1}{2} \omega - \sin \omega = \cot \omega - \csc \omega - \sin \omega,
\]
\[
g(\omega, \sigma) := \sin \omega \sin^2 \sigma + f(\omega) \cos^2 \sigma.
\]

So the fundamental inequality (2.14) can be rewritten as
\[
- I''(0) \geq \sum_{i=1}^{n} g(\omega_i, \sigma_i).
\] (2.15)

Note that \( g(\omega, \sigma) \) is a convex combination of the two functions \( \sin \omega \) and \( f(\omega) \), which are plotted in Fig. 7. We will use this fact repeatedly to obtain bounds. Note that the
Fig. 7  Plot of the functions \( \sin \) and \( f \)

Abscissa of the crossing between the two graphs is \( \pi/3 \), that \( \min g = \min f > -0.31 \), and that

\[
\frac{1}{2} \sin \omega \quad \text{for all } \quad \omega \in (0, \pi).
\]

(2.16)

Recall that \( \omega_1 \) is the biggest of all angles \( \omega_i \)'s and so it is the only angle that can be bigger than \( \pi/2 \); therefore the sum in (2.15) contains at most one negative term.

**Case 1** \( |\cos \sigma_1| \leq 0.8 \). Then

\[
-I''(0) \geq g(\omega_1, \sigma_1)
= \sin \omega_1 \sin^2 \sigma_1 + f(\omega_1) \cos^2 \sigma_1
\geq \sin \omega_1 \left( \sin^2 \sigma_1 - \frac{1}{2} \cos^2 \sigma_1 \right) \quad \text{[by (2.16)]}
\geq \sin \omega_1 \left( 0.36 - \frac{1}{2} \cdot 0.64 \right)
\geq 0.02 \sin \kappa \quad \text{[by the first inequality in (2.11)]}
> 0.
\]

In the remaining cases, we assume \( |\cos \sigma_1| > 0.8 \).

**Case 2** \( n = 1 \). Then, by the first inequality in (2.11),

\[
|I'(0)| = \sin \omega_1 |\cos \sigma_1| > 0.8 \sin \kappa > 0.
\]

In the remaining cases, we assume \( n \geq 2 \).

**Case 3** \( \omega_1 \leq 3\pi/8 \). Then, by the first inequality in (2.11),

\[
-I''(0) \geq g(\omega_1, \sigma_1) \geq \min \{\sin \omega_1, f(\omega_1)\} \geq \min \{\sin \kappa, f(3\pi/8)\} > 0.
\]

In the remaining cases, we assume \( \omega_1 > 3\pi/8 \). Recall from Sect. 2.3 that \( \omega_s \) is the second biggest of the \( \omega_i \)'s.
**Case 4** \( \omega_s \geq \pi / 3 \). So
\[
- I''(0) \geq g(\omega_1, \sigma_1) + g(\omega_s, \sigma_s) \\
\geq f(\omega_1) + f(\omega_s) \\ (\text{since } \omega_1 \geq \omega_s \geq \pi / 3).
\]
Now, \( \omega_1 < \pi - \omega_s \leq 2\pi / 3 \) and the function \( f \) is decreasing on the interval \((0, 2\pi / 3)\) (actually, it is decreasing on a slightly bigger interval), so
\[
- I''(0) \geq f(\pi - \omega_s) + f(\omega_s) = 2(\csc - \sin)(\omega_s).
\]
Now, using (2.12) we obtain
\[
- I''(0) \geq 2(\csc - \sin)(\frac{\pi}{2} - \kappa) > 0
\]
and we are done in this case.

In the remaining cases, we assume \( \omega_s < \pi / 3 \). Define the following numbers:
\[
\kappa' := 0.1 \sin \kappa, \\
\Lambda := \max \{ - g(\omega_1, \sigma_1), 0 \} + \kappa', \\
\Sigma := \sum_{i=2}^{n} g(\omega_i, \sigma_i).
\]

**Case 5** \( \Sigma \geq \Lambda \). Then
\[
- I''(0) = \Sigma + g(\omega_1, \sigma_1) \\
\geq \sum_{i=2} g(\omega_i, \sigma_i) + \kappa' - \Lambda \\
\geq \kappa'
\]
and we are done.

In the final and most interesting case, we assume \( \Sigma < \Lambda \).

**Case 6** Let us establish two upper estimates for \( \Lambda \); the first one is
\[
\Lambda \leq \max(-g) + \kappa' < 0.31 + \kappa' < 0.41, \\
\tag{2.17}
\]
and the second one is
\[
\Lambda \leq - g(\omega_1, \sigma_1) + \kappa' \\
\leq - f(\omega_1) + \kappa' \\
\leq \frac{1}{2} \sin \omega_1 + \kappa' \\
\tag{2.18}
\]
[by (2.16)].
Define
\[ \Delta := \arcsin \Lambda, \]
which by (2.17), satisfies \( \Delta < 0.43 \).

Note that
\[ \sin \Delta = \Lambda > \Sigma \geq g(\omega_s, \sigma_s) \geq \sin \omega_s \quad (\text{since } \omega_s < \pi/3), \]
that is, \( \omega_s \leq \Delta \).

Let \( \varphi(\omega) := f(\omega) - \sin(\omega) \). We can rewrite \( \Sigma \) as
\[ \Sigma = \sum_{i=2}^{n} \left[ \sin \omega_i + \varphi(\omega_i) \cos^2 \sigma_i \right]. \]

Since \( \omega_s \leq \Delta < \pi/3 \) and the function \( \varphi \) is positive and decreasing on the interval \( (0, \pi/3) \), the assumption \( \Sigma < \Lambda \) yields two other inequalities
\[ \sum_{i=2}^{n} \sin \omega_i < \Lambda, \]
\[ \sum_{i=2}^{n} \cos^2 \sigma_i < \frac{\Lambda}{\varphi(\Delta)}. \]

So, on the one hand,
\[ \sum_{i=2}^{n} \sin^2 \omega_i \leq \sin \omega_s \sum_{i=2}^{n} \sin \omega_i \leq \Lambda^2. \]

On the other hand, recalling that \( \Lambda = \sin \Delta \) and \( \varphi(\Delta) = \cot \Delta + \csc \Delta - 2 \sin \Delta \),
\[ \sum_{i=2}^{n} \cos^2 \sigma_i < \frac{\Lambda}{\varphi(\Delta)} = \frac{\Lambda^2}{\cos \Delta + 1 - 2 \sin^2 \Delta} \leq \frac{\Lambda^2}{\cos 0.43 + 1 - 2 \sin^2 0.43} < 0.65\Lambda^2. \]

By the Cauchy–Schwarz inequality,
\[ \left| \sum_{i=2}^{n} \sin \omega_i \cos \sigma_i \right| \leq \sqrt{\sum_{i=2}^{n} \sin^2 \omega_i \times \sum_{i=2}^{n} \cos^2 \sigma_i} < \sqrt{\Lambda^2 \times 0.65\Lambda^2} < 0.81\Lambda^2 < \Lambda. \]
This inequality together with (2.18) allows us to show that $A'(0)$ is not too close to zero:

$$|I'(0)| \geq \left| \sin \omega_1 \cos \sigma_1 - \sum_{i=2}^{n} \sin \omega_i \cos \sigma_i \right|$$

$$\geq 0.8 \sin \omega_1 - \Lambda$$

$$\geq 0.3 \sin \omega_1 - \kappa'$$

$$\geq 0.2 \sin \kappa$$

$$> 0.$$ 

This concludes the proof of Proposition 2.3.

3 Proof of the Quasiconcavity Theorem 1.4

3.1 Setting Up the Proof

Let us say that a centrally symmetric body $K \in C^2$ is regular if it satisfies the following conditions:

(a) the boundary $\partial K$ is a $C^2$ curve;
(b) there is a finite set $T \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus T$, the curves $\partial K$ and $\partial E_t$ are transverse;
(c) for every $t \in \mathbb{R}$, the curves $\partial K$ and $\partial E_t$ are quasitransverse, and the points of (necessarily quadratic) tangency do not belong to the envelope hyperbolas $xy = \pm1/2$ of the family of curves $(\partial E_t)_{t \in \mathbb{R}}$.

We will prove that regularity is dense in $C^2$; actually we will show more:

**Proposition 3.1** (Regularization) For every $K \in C^2$ and every $\varepsilon > 0$ there exists a regular $\tilde{K} \in C^2$ such that $|\tilde{K}| = |K|$ and $d_{\text{sym}}(\tilde{K}, K) < \varepsilon$.

On the other hand, using Propositions 2.1–2.3 one can check that Theorem 1.4 holds for regular convex bodies in $C^2_\pi$, that is, the associated intersection functions are strictly quasiconcave. Actually, the uniformity provided by Proposition 2.3 will allow us to prove a more precise property:

**Proposition 3.2** (Quantitative quasiconcavity) Given $\varepsilon > 0$ and $r > 0$, there exists $\eta > 0$ with the following properties. For every regular $K \in C^2_\pi$, if $t_0, t_1, t_2 \in \mathbb{R}$ are such that

$$t_0 + r \leq t_1 \leq t_2 - r \quad \text{and} \quad 2\varepsilon \leq I_K(t_1) \leq \max \{\pi, |K|\} - 2\varepsilon,$$

then

$$I_K(t_1) > \min \{I_K(t_0), I_K(t_2)\} + \eta.$$
Let us postpone the proofs of Propositions 3.1 and 3.2, and use them to deduce the theorem:

**Proof of Theorem 1.4** Fix $K \in C^2_\pi$ and arbitrary numbers $t_0 < t_1 < t_2$. Let

$$r := \min\{t_1 - t_0, t_2 - t_1\} \quad \text{and} \quad \varepsilon := \frac{1}{3} \min\{I_K(t_1), \max\{\pi, |K|\} - I_K(t_1)\}.$$ 

Let $\eta = \eta(\varepsilon, r)$ be given by Proposition 3.2. Reducing $\eta$ if necessary, we assume $0 < \eta \leq 4\varepsilon$. By Proposition 3.1, there exists a regular body $\tilde{K} \in C^2_\pi$ with the same area as $K$ such that $d_{\text{sym}}(\tilde{K}, K) < \eta / 2$. Recalling that $C^2_\pi$ is open in $C^2_\pi$, we can assume that $\tilde{K} \in C^2_\pi$. As a consequence of relation (1.2), for every $t \in \mathbb{R}$ we have $|I_{\tilde{K}}(t) - I_K(t)| \leq \eta / 4 \leq \varepsilon$. In particular,

$$I_{\tilde{K}}(t_1) \geq I_K(t_1) - \varepsilon \geq 3\varepsilon - 2\varepsilon,$$

$$I_{\tilde{K}}(t_1) \leq I_K(t_1) + \varepsilon \leq \max\{\pi, |\tilde{K}|\} - 3\varepsilon + \varepsilon.$$

This allows us to apply Proposition 3.2 to the convex body $\tilde{K}$ and obtain $I_{\tilde{K}}(t_1) \geq \min\{I_{\tilde{K}}(t_0), I_{\tilde{K}}(t_1)\} + \eta$. It immediately follows that $I_K(t_1) \geq \min\{I_K(t_0), I_K(t_1)\} + \eta / 2$. This proves that the function $I_K$ is quasiconcave. \hfill \Box

**3.2 Proof of the Regularization Proposition 3.1**

Let $\mathbb{P}^1$ denote the projective space of $\mathbb{R}^2$, i.e. the set of all lines through the origin. Let $[v] \in \mathbb{P}^1$ denote the line determined by a nonzero vector $v \in \mathbb{R}^2$.

If $\Gamma \subset \mathbb{R}^2$ is any smooth 1-dimensional submanifold, denote by $\hat{\Gamma} \subset \mathbb{R}^2 \times \mathbb{P}^1$ the set of pairs $(u, [v])$ such that $u \in \Gamma$ and $v$ is tangent to $\Gamma$ at $u$. Define the following sets:

$$Z_1 := \hat{H}$$

where $H$ is the pair of hyperbolas $xy = \pm 1/2$;

$$Z_2 := \bigcup_{t \in \mathbb{R}} \partial E_t$$

where $E_t$ are the ellipses (1.3).

The latter union is disjoint, because any two distinct ellipses in our family have transverse boundaries.

**Lemma 3.3** $Z_1$ and $Z_2$ are closed smooth submanifolds of $\mathbb{R}^2 \times \mathbb{P}^1$ of respective dimensions 1 and 2, and $Z_1 \subset Z_2$.

The lemma is intuitively clear, but for completeness we provide a proof at the end of this subsection.

Let $\mathbb{T} := \mathbb{R} / 2\pi \mathbb{Z}$, the additive group of real numbers mod $2\pi$. A regular parametrization of a smooth Jordan curve $\Gamma \subset \mathbb{R}^2$ is a map $g : \mathbb{T} \to \mathbb{R}^2$ that is a smooth diffeomorphism onto $\Gamma$. In that case, let $\hat{g} : \mathbb{T} \to \mathbb{R}^2 \times \mathbb{P}^1$ denote the map $\hat{g}(\theta) := (g(\theta), [g'(\theta)])$, which is a smooth diffeomorphism onto $\hat{\Gamma}$. \hfill \copyright Springer
Lemma 3.4 Suppose \( K \in \mathcal{C}^2 \) has smooth boundary, and \( g : \mathbb{T} \to \mathbb{R}^2 \) is a regular parametrization of it. If \( \hat{g} \) is transverse to both submanifolds \( Z_1 \) and \( Z_2 \) then the body \( K \) is regular.

Proof Let \( K \in \mathcal{C}^2 \) and suppose that \( \partial K \) has a regular parametrization \( g \) such that \( \hat{g} \) is transverse to both \( Z_1 \) and \( Z_2 \). The first regularity condition (a) is automatic: the boundary \( \partial K \) is actually smooth.

Since the ambient space \( \mathbb{R}^2 \times \mathbb{P}^1 \) is 3-dimensional, transversality implies that there are finitely many (if any) parameters \( \theta_i \) such that the point \( \hat{g}(\theta_i) \) belongs to the surface \( Z_2 \). Each of these points belongs to a unique curve \( \overline{\partial E_{t_i}} \). Let \( T \subset \mathbb{R} \) be the set of the \( t_i \)'s. If \( t \notin T \) then the image of \( \hat{g} \) does not intersect the curve \( \overline{\partial E_{t_i}} \), which means that the plane curves \( \partial K \) are transverse. This shows that \( K \) meets regularity condition (b).

On the other hand, for each \( i \), the plane curves \( \partial K \) and \( \partial E_i \) are tangent at the point \( g(\theta_i) \). Suppose for a contradiction that this tangency is not quadratic, i.e., the curves have a second-order contact. Choose a regular parametrization \( h_i : \mathbb{T} \to \mathbb{R}^2 \) of \( \partial E_{t_i} \) such that \( h_i(\theta_i) = \hat{g}(\theta_i) \). Then parameterized curves \( \hat{g} \) and \( \hat{h}_i \) have a first-order contact (i.e., are tangent) at parameter \( \theta_0 \). Since \( \hat{h}_i \) is an immersion whose image is contained in the surface \( Z_2 \), we conclude that \( \hat{g} \) is not transverse to \( Z_2 \), which is a contradiction.

We have shown that the tangencies between the plane curves \( \partial K \) and \( \partial E_i \) are all quadratic, i.e., the curves are quasitransverse. Furthermore, the fact that the mapping \( \hat{g} \) is transverse to the 1-dimensional submanifold \( Z_1 \subset Z_2 \) means that its image does not intersect \( Z_1 \). That is, all tangency points \( g(\theta_i) \) are outside the forbidden hyperbolas \( xy = \pm 1/2 \). This concludes the proof that the body \( K \) is regular.

Proposition 3.5 If \( K \in \mathcal{C}^2 \) has smooth boundary then there is an open dense subset \( U \) of \( \text{SL}(2, \mathbb{R}) \) such that if \( L \in U \) then the body \( LK \) is regular.

Proof Let \( Y \subset \mathbb{R}^2 \times \mathbb{P}^1 \) be the set of pairs \((u, [v])\) such that \( u \) and \( v \) are linearly independent. Note that \( Z_1, Z_2 \) are subsets of \( Y \). The group \( \text{SL}(2, \mathbb{R}) \) acts on \( Y \) in the obvious way: \( L(u, [v]) = (Lu, [Lv]) \). This action is smooth, transitive, and faithful; in particular \( \text{SL}(2, \mathbb{R}) \) and \( Y \) are diffeomorphic.

Let \( g : \mathbb{T} \to \mathbb{R}^2 \) be a regular parametrization of \( \partial K \). Since \( K \) is centrally symmetric, \( \hat{g} \) takes values in \( Y \). The map \( f : \mathbb{T} \times \text{SL}(2, \mathbb{R}) \to Y \) defined by \( f(\theta, L) := L\hat{g}(\theta) \) is a submersion. Therefore, by the transversality theorem [12, p. 68] (or see [14, Thm. 2.7] for a more precise version), the set \( U \subset \text{SL}(2, \mathbb{R}) \) formed by those \( L \) such that \( f(\cdot, L) : \mathbb{T} \times \text{SL}(2, \mathbb{R}) \to Y \) is transverse to \( Z_1 \) and to \( Z_2 \) is open and dense in \( \text{SL}(2, \mathbb{R}) \). Take \( L \in U \). Noting that \( f(\cdot, L) = L \circ \hat{g} \), it follows from Lemma 3.4 that the body \( LK \) is regular.

The previous proposition implies the result we are looking for:

Proof of Proposition 3.1 Given \( K \in \mathcal{C}^2 \), we initially perturb it so that the area is unchanged and the boundary becomes smooth: for example, we can take an inscribed polygon, smoothen the corners, and inflate it to recover the area. \( \mathcal{C}^2_\pi \) is open in \( \mathcal{C}^2 \). Then by Proposition 3.5 we can apply a element of \( \text{SL}(2, \mathbb{R}) \) close to the identity and so obtain the desired regular body approximating \( K \).\( \square \)
Finally, we check that $Z_1$ and $Z_2$ are indeed submanifolds.

**Proof of Lemma 3.3** The QR decomposition comes in handy: there is a diffeomorphism $\Phi: \mathbb{R} \times \mathbb{R} \times \mathbb{T} \to \text{SL}(2, \mathbb{R})$ given by

$$
\Phi(\tau, \rho, \xi) := \begin{pmatrix}
e^{-\tau/2} & \rho \\ 0 & e^{\tau/2}\end{pmatrix} \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}.
$$

Changing coordinates under $\Phi$, the left action of the diagonal subgroup corresponds to translation of the first coordinate.

Let $Y$ be as in the proof of Proposition 3.5. Define a diffeomorphism $\Psi: \text{SL}(2, \mathbb{R}) \to Y$ by $\Psi(L) := (L e_1, [L e_2])$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. So the map $\Psi \circ \Phi$ allows us to put global coordinates $(t, \rho, \xi)$ on $Y$.

Note that, in the coordinates just described, $\hat{\partial} E_0$ is given by equations $t = 0, \rho = 0$. So, applying the diagonal subgroup, we conclude that $Z_2$ is the surface $\rho = 0$. Analogously, $Z_1$ corresponds to $\rho = 0$ and $\xi \in \{\pm \pi/4, \pm 3\pi/4\}$. This proves Lemma 3.3. \hfill \Box

### 3.3 Proof of the Quantitative Quasiconcavity Proposition 3.2

Let us begin by collecting the more direct consequences of Propositions 2.1–2.3 in the following:

**Lemma 3.6** Let $K \in C^2_\pi$ be regular, and let $T \subset \mathbb{R}$ be the corresponding (finite) set of tangency parameters. Then the intersection function $I = I_K$ has the following properties:

(a) $I$ is of class $C^1$.
(b) The restriction of $I$ to the set $\mathbb{R} \setminus T$ is of class $C^2$.
(c) For every $t \in \mathbb{R} \setminus T$ and $\varepsilon > 0$ we have

$$
\varepsilon \leq I(t) \leq \min\{\pi, |K|\} - \varepsilon \quad \Rightarrow \quad \max\{|I'(t)|, -I''(t)|\} > \delta(\varepsilon),
$$

where $\delta(\cdot)$ is the function from Proposition 2.3.
(d) $I$ has a unique critical point $t_*$.
(e) $I$ is increasing on $(-\infty, t_*]$ and decreasing on $[t_*, +\infty)$.

**Proof** Fix a regular $K \in C^2_\pi$. Consider the one-parameter subgroup $L_t := \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2}\end{pmatrix}$ of $\text{SL}(2, \mathbb{R})$. Then $L_t(E_s) = E_{t+s}$ and therefore intersection functions of the images of $K$ under the subgroup are identical up to translations

$$
I_K(t + s) = I_{L_{t}^{-1}K}(s).
$$

Also note that regularity is invariant under the action of the subgroup. The regularity property (b) guarantees that $K$ fulfills the hypothesis of Proposition 2.2 and therefore the function $I_K$ is $C^1$ on a neighborhood of 0. By invariance, $I_K$ is $C^1$ on the whole line,
which is statement (a) of the lemma. Similarly, bearing in mind regularity property (b), we see that Proposition 2.1 implies that $I_K$ is $C^2$ on the set $\mathbb{R}\setminus T$, which is statement (b), and that Proposition 2.3 implies that the derivatives of $I_K$ satisfy the bounds stated in (c).

The function $I_K$ obeys the inequalities $0 < I_K < \min\{\pi, |K|\}$ on the whole line; the second inequality is a consequence of the assumption that $K \in C^2_\pi$. Since the function $I_K$ vanishes at $\pm \infty$, it has critical points. Let $t_\circ$ be one of these. On the one hand, if $t_\circ \notin T$ then $I_K$ is actually $C^2$ on a neighborhood $V$ of $t_\circ$ and $I''_K(t_\circ) < 0$ there. So, reducing the neighborhood $V$ if necessary, the function $I''_K$ becomes decreasing on $V$. On the other hand, if $t_\circ \in T$ then we can find a neighborhood $V$ of $t_\circ$ such that $I_K$ is $C^2$ on $V \setminus \{t_\circ\}$ and $I''_K < 0$ there. So the function $I''_K$ is decreasing on $V \setminus \{t_\circ\}$, and since it is continuous, it is actually decreasing on $V$.

We have shown that every critical point of $I_K$ is isolated and is a local maximum. Therefore the critical point, which exists, is unique, proving statement (d). We have also seen that the critical point is a local maximum, and so statement (e) follows.

**Proof of Proposition 3.2** Let $\varepsilon > 0$ and $r > 0$ be given. Without loss of generality, we assume $r \leq 1$. Let $\delta = \delta(\varepsilon) > 0$ be given by Proposition 2.3 and let $\eta := \min\{\varepsilon, \delta r^2/2\}$. Fix a regular $K \in C^2_\pi$ and for simplicity write $I := I_K$ and $b := \min\{\pi, |K|\}$. Fix the three numbers $t_0 < t_1 < t_2$ satisfying the assumptions, namely $t_1 \in [t_0 + r, t_2 - r]$ and $I(t_1) \in [2\varepsilon, b - 2\varepsilon]$. We suppose that $t_1 \geq t_\circ$, where $t_\circ$ is the critical point of $I$, the other case being analogous. Since $I(t_2) \leq I(t_1 + r)$, it is sufficient to prove that

$$I(t_1 + r) \leq I(t_1) - \eta.$$ 

This clearly holds if $I(t_1 + r) < \varepsilon$, so assume that $I(t_1 + r) \geq \varepsilon$.

Next, suppose $I' < -\delta$ over the interval $J := [t_1, t_1 + r]$. Then, by the Mean Value Theorem, $I(t_1 + r) - I(t_1) \leq -\delta r < -\eta$, completing the proof in this case. So assume that $I' > -\delta$ somewhere on $J$.

Note that $I' \leq 0$ and $\varepsilon \leq I \leq b - 2\varepsilon$ on the interval $J$. It follows from part c of Lemma 3.6 that $\max(-I'(t), -I''(t)) > \delta$ for every $t \in J$ except a finite number of points where the second derivative may not be defined. So $I'$ is decreasing on the set $S := \{t \in J : I'(t) < -\delta\}$, which is nonempty by assumption. It follows that $S$ must be an interval with left endpoint $t_1$. Let $s$ be the right endpoint. Then

$$t \in [t_1, s] \Rightarrow I'(t) = I'(t_1) + \int_{t_1}^{t} I''(u) \, du \leq -\delta(t - t_1),$$

while

$$t \in [s, t_1 + r] \Rightarrow I'(t) \leq -\delta \leq -\delta(t - t_1)$$ as well.
(using that \( r \leq 1 \)). So
\[
I(t_1 + r) - I(t_1) \leq \int_{t_1}^{t_1 + r} -\delta(t - t_1) \, dt = -\frac{1}{2} \delta r^2 \leq -\eta,
\]
as we wanted to show. \( \square \)

As explained in Sect. 3.1, Theorem 1.4 follows.

### 3.4 An Extension of Theorem 1.4

The intersection function \( I_K \) of any \( K \in C_2 \) is always bounded by the value \( \min\{\pi, |K|\} \). If \( K \notin C_2^\pi \) then \( I_K \) may have a plateau at this value and therefore may fail to be strictly quasiconcave. Therefore the assumption \( K \in C_2^\pi \) cannot be removed altogether from Theorem 1.4. On the other hand, this assumption is only used to guarantee that \( I_K < \min\{\pi, |K|\} \) everywhere. In fact, it is straightforward to modify the proof of Theorem 1.4 and obtain the following result:

**Theorem 3.7** Let \( K \in C_2 \). Let \( J \subseteq \mathbb{R} \) be an interval such that \( I_K(t) < \max\{\pi, |K|\} \) for every \( t \in J \). Then the restriction of \( I_K \) to \( J \) is a strictly quasiconcave function.

### 4 Discarding Ellipses with Displaced Centers: Proof of Lemma 1.3

Let us finally prove Lemma 1.3, which, as seen in the introduction, allows us to deduce the uniqueness Theorem 1.2 from Theorem 1.4. We rely on the following result, which is essentially a corollary of the Brunn–Minkowski inequality and holds in arbitrary dimension:

**Proposition 4.1** (Zalgaller [20]) Let \( K_1, K_2 \subseteq \mathbb{R}^d \) be convex bodies. Let \( M \) be the set of \( v \in \mathbb{R}^d \) that maximize the volume of \( K_1 \cap (K_2 + v) \). Then \( M \) is a nonempty compact convex set, and the sets \( K_1 \cap (K_2 + v) \) with \( v \in M \) are identical up to translation.

**Proof of Lemma 1.3** Let \( K \subseteq \mathbb{R}^2 \) be a centrally symmetric convex body. For a contradiction, suppose that \( K \) admits an MI ellipse with area \( \lambda \) in the range \( |\mathcal{J}_K| < \lambda < |\mathcal{L}_K| \) which is not centrally symmetric, and write it as \( E + v_0 \), where \( E \) is centrally symmetric and \( v_0 \neq 0 \). Applying an appropriate linear map if necessary, we can assume that \( E \) is the unit disk \( E_0 \) and that \( v_0 \) is horizontal, i.e. \( v_0 = (\varepsilon_0, 0) \).

Let \( M \ni v_0 \) be the set of \( v \in \mathbb{R}^2 \) such that that maximize \( |K \cap (E_0 + v)| = |(K - v) \cap E_0| \), which by Proposition 4.1 is compact and convex. Since \( K \) and \( E_0 \) are centrally symmetric, so is \( M \). In particular, \( 0 \in M \) and \( E_0 \) is also an MI ellipse for \( K \).

The proposition also says that for each \( v \in M \) the set \( (K + v) \cap E_0 \) is a translate of \( K \cap E_0 \), say \( (K \cap E_0) + u \). Consider some \( z \in K \cap \partial E_0 \) (which exists since \( K \notin E_0 \)). Then both points \( z + u \) and \( -z + u \) belong to \( E_0 \), which forces \( u = 0 \). We have shown that the sets \( (K + v) \cap E_0 \) with \( v \in M \) are actually identical: no translation is needed.
Since $M$ contains the segment $[-v_0, v_0]$, for every $z \in K \cap E_0$, the intersection of the segment $z + [-v_0, v_0]$ with $E_0$ is contained in $K$. By overlapping such segments, we conclude that the intersection of the line $z + \mathbb{R}v_0$ with $E_0$ is contained in $K$. This property implies that $K \cap E_0$ equals $S \cap E_0$, where $S = \mathbb{R} \times [-b, b]$ is a strip in the plane. Since $E_0 \not\subseteq K$, we must have $0 < b < 1$. Using that $K \cap (E_0 \pm v_0) = (K \cap E_0) \pm v_0$, we conclude that there is a neighborhood $V$ of the unit disk $E_0$ such that $V \cap K = V \cap S$. In particular, $\partial K$ is transverse to the unit circle $\partial E_0$ and there are 4 crossings, namely $(\pm \sqrt{1 - b^2}, \pm b)$. Therefore we may apply Proposition 2.1, and conclude that if $\xi_1 := \arcsin b$ then $I'_K(0) = \sin 2\xi_1 > 0$. So for sufficiently small $t > 0$, the set $E_0 \cap K$ has a bigger area than $E_0 \cap K$, which contradicts the fact that $E_0$ is an MI ellipse for $K$. $\square$

5 Analysis of Maximum Intersection Positions

5.1 Characterization of MI Positions for the Transverse Case

Recall from Sect. 2 that two Jordan curves in the plane are called transverse if each of them is of class $C^1$ at a neighborhood of each point of intersection, and that these intersections are transverse in the usual sense.

**Theorem 5.1** Let $K \subset \mathbb{R}^2$ is a compact convex centrally symmetric set whose boundary $\partial K$ is transverse to the unit circle $S^1$. Let $\zeta_1, \ldots, \zeta_n$ be the points of intersection, cyclically ordered. Then $K$ is in MI position if and only if

$$\sum_{j \text{ odd}} \zeta_j^2 = \sum_{j \text{ even}} \zeta_j^2, \quad (5.1)$$

where we identify $\mathbb{R}^2$ and $\mathbb{C}$ in the usual way.

**Proof** Write $\zeta_j = e^{i\xi_j}$. If $K$ is in MI position then the derivative given by formula (2.1) vanishes; moreover, the same is true if we apply a rotation to $K$, i.e., replace each $\xi_j$ by $\xi_j + \phi$. Therefore

$$\sum_{j=1}^{2n} (-1)^j \sin 2(\xi_j + \phi) = 0 \quad \text{for all } \phi \in \mathbb{R}.$$ 

Using a trigonometric identity, we see that the latter condition is equivalent to

$$\sum_{j=1}^{2n} (-1)^j \cos 2\xi_j = \sum_{j=1}^{2n} (-1)^j \sin 2\xi_j = 0,$$

which is condition (5.1).

Conversely, suppose that condition (5.1) holds. Then, reversing the arguments above, we obtain that for every $\phi \in \mathbb{R}$, the intersection function of the rotated convex
body \( e^{i\varphi}K \) is \( C^1 \) and its derivative at \( t = 0 \) vanishes. Furthermore, by Proposition 2.3, the second derivative is also defined and is negative. Hence, among centrally symmetric ellipses of area \( \pi \), the unit disk \( E_0 \) attains a local maximum for the area of intersection with \( K \). If we knew that \( K \in C^2_\pi \) (and therefore \( e^{i\varphi}K \in C^2_\pi \) for every \( \varphi \)) then we could apply the strict quasiconcavity Theorem 1.4 and conclude that this local maximum is the global maximum, that is, \( K \) is in MI position. In order to conclude the proof we will show that \( K \in C^2_\pi \), that is, \( |J_K| < \pi < |L_K| \).

Suppose for a contradiction that the John ellipse \( J_K \) has area \( \geq \pi \). Since \( J_K \) is centrally symmetric, it follows that \( K \) contains a centrally symmetric ellipse \( E \) of area \( \pi \). This ellipse \( E \) cannot be the disk \( E_0 \) since we are assuming that \( K \) and \( E_0 \) have transverse boundaries. Applying a rotation if necessary, we can assume that \( E = E_t \) for some \( t > 0 \). Then the intersection function \( I_K \) satisfies \( I_K(t) = \pi \). Reducing \( t \) if necessary, we can assume that \( I_K(t) < \pi \) for all \( t \) in the interval \( J = [0, t] \). As seen before, the function \( I_K \) attains a local maximum at \( 0 \). It follows the function \( I_K \) attains a local minimum somewhere in the interior of \( J \). This contradicts Theorem 3.7. Therefore \( |J_K| \leq \pi \). A similar reasoning proves that \( |L_K| \geq \pi \). So \( K \in C^2_\pi \), as claimed, and the theorem follows.

Proposition 1.6 is actually a corollary:

**Proof of Proposition 1.6** Suppose that \( K \) is MI position with boundary transverse to the unit circle and intersecting it at the points \( \zeta_1, \ldots, \zeta_{4n} \), listed in counterclockwise order. Note that \( \zeta_{j+2n} = -\zeta_j \).

If \( n = 1 \) then by (5.1) we would have \( \zeta_1^2 = \zeta_2^2 \), i.e., \( \zeta_1 \) and \( \zeta_2 \) are antipodal; absurd. This proves part (a).

Now suppose \( n = 2 \). We want to prove that \( \zeta_{j+2} = i\zeta_j \). Condition (5.1) becomes

\[
\zeta_1^2 + \zeta_3^2 = \zeta_2^2 + \zeta_4^2.
\]

So we must prove that both sides of this equation vanish. Suppose that is not the case. Observe that a pair of non-antipodal points in the unit circle is uniquely determined (modulo permutation) by their midpoint. Therefore \( \{\zeta_1^2, \zeta_2^2\} = \{\zeta_3^2, \zeta_4^2\} \). But \( \zeta_1^2, \zeta_2^2, \zeta_3^2, \zeta_4^2 \) are distinct (and cyclically ordered). We have reached a contradiction. This proves part (b).

**5.2 Comparison with the Classical Characterization of John Position**

Let \( \mu \) be a positive (and nonzero) Borel measure on the unit sphere \( S^{d-1} \). We say that \( \mu \) is *balanced* if

\[
\int_{S^{d-1}} u \, d\mu(u) = 0,
\]

that is, the center of mass of \( \mu \) is the origin. We say that \( \mu \) is *isotropic* if, for some \( c > 0 \),
for all \( v \in \mathbb{R}^d \),
\[
\int_{S^{d-1}} \langle u, v \rangle^2 \, d\mu(u) = c \|v\|^2
\]
(where \( \|\cdot\| \) denotes euclidian norm), that is, the inertia ellipsoid of \( \mu \) with respect to
the origin is round. One necessarily has \( c = \mu(S^{d-1})/d \). See e.g. [18, § 10.13] for
several uses of isotropic measures in convex geometry.

We say that a convex body in \( K \subset \mathbb{R}^d \) is in John position if its John ellipsoid is the
euclidian unit ball. The following theorem is well-known:

**Theorem 5.2** (John) *If a convex body \( K \subset \mathbb{R}^d \) is in John position then there exists a*
balanced isotropic measure supported on \( S^{d-1} \cap \partial K \).

Here we prove a similar result for the planar MI position:

**Proposition 5.3** *If a centrally symmetric body \( K \subset \mathbb{R}^2 \) is in MI position then there*
exists a balanced isotropic measure supported on \( S^1 \cap \partial K \).

See the paper [1] for another result on the existence of balanced isotropic measures
for bodies in MI position, under certain generic assumptions, and without restriction
on dimension.

**Proof of Proposition 5.3** It suffices to consider the case of \( \partial K \) transverse to \( S^1 \); the
general case then follows by perturbation and using the fact that balanced isotropic
measures form a weakly-*closed set.

Let \( \zeta_1, \ldots, \zeta_{4n} \) be the points in \( S^1 \cap \partial K \), cyclically ordered. Note that \( \zeta_{j+2n} = -\zeta_j \).

We claim that the convex hull of the points \( \zeta^2_j \) contains the origin. If not, there exists
a line \( L \subset \mathbb{R}^2 \) through the origin such that all points \( \zeta^2_j \) belong to the same connected
component of \( \mathbb{R}^2 \subset L \). Let \( P : \mathbb{R}^2 \to L \) be the orthogonal projection onto \( L \). Since the
points \( \zeta^2_1, \ldots, \zeta^2_{2n} \in S^1 \) are distinct, their projections \( z_1 := P(\zeta^2_1), \ldots, z_{2n} := P(\zeta^2_{2n}) \)
are distinct. Color each \( z_j \) red or blue depending on whether \( j \) is odd or even. Since the
points \( \zeta^2_1, \ldots, \zeta^2_{2n} \in S^1 \) are cyclically ordered, the colors of the points \( z_1, \ldots, z_{2n} \in L \)
alternate. In particular, (5.1) cannot hold, since the two sums have different projections.
By Theorem 5.1, the set \( K \) is not in MI position. This contradicts the assumption, and
therefore we proved that the convex hull of the points \( \zeta^2_j \) contains the origin.

Hence there exist weights \( p_1, \ldots, p_{4n} \geq 0 \) such that \( \sum_j p_j = 1 \) and \( \sum_j p_j \zeta^2_j = 0 \).
Since \( \zeta^2_{j+2n} = \zeta^2_j \), we can assume that the weights satisfy \( p_{j+2n} = p_j \) (indices taken
mod \( 4n \)). Then the measure on \( S^1 \cap \partial K \) defined by \( \mu := \sum_j p_j \delta_{\zeta_j} \) is balanced. Let
us check that \( \mu \) is isotropic. Note that the (real) euclidian inner product in \( \mathbb{C} \) is given
by the formula \( \langle u, v \rangle = \Re(u \bar{v}) \) and so satisfies the identity
\[
\langle u, v \rangle^2 = \frac{|u^2 v^2| + \langle u^2, v^2 \rangle}{2}.
\]
Using this, we calculate, for arbitrary \( v \in \mathbb{C} \),
\[
\int_{S^1} \langle u, v \rangle^2 \, d\mu(u) = \sum_j p_j \langle \zeta_j, v \rangle^2 = \frac{\sum_j p_j |v|^2}{2} + \frac{1}{2} \sum_j p_j \zeta_2, v^2 \rangle = \frac{|v|^2}{2},
\]
proving that \( \mu \) is isotropic. \( \square \)
Remark 5.4 The converse of Theorem 5.2 also holds, as shown by Ball [2]. However, the converse of Proposition 5.3 is false. Indeed, fix any $\varepsilon \neq \pi/6$ in the range $0 < \varepsilon < \pi/4$, and let $Z$ be the set consisting of 12 points in the unit circle that is invariant by a quarter turn and contains \{e$^{-i\varepsilon}$, 1, e$^{i\varepsilon}$\}. Then the equidistributed probability measure on $Z$ is balanced and isotropic. Take a centrally symmetric convex body $K \subset \mathbb{R}^2$ whose boundary $\partial K$ is transverse to the unit circle $S^1$ and intersects it exactly on $Z$. Then condition (5.1) does not hold; indeed the two sums are $2 - 4 \cos 2\varepsilon \neq -2 + 4 \cos 2\varepsilon$.

By Theorem 5.1, the set $K$ is not in MI position.

6 Directions for Future Research

We pose a few questions:

6.1 Can the assumption of central symmetry be removed from the main Theorem 1.2?

6.2 Given a (say, centrally symmetric) convex body $K \subset \mathbb{R}^2$, is there a unique ellipse that best approximates it with respect to symmetric difference metric (without constraining its area)?

6.3 The previous question for the normalized symmetric difference metric $d_{\text{nsym}}$, defined by (1.1).

6.4 Given an arbitrary $K \in C^2_\pi$, is the intersection function $I_K \log$-concave? (See [1, § 4] for a stronger conjecture, motivation, and relations with known results.)

Some of these questions are possibly accessible with the methods of this paper. In any case, the investigation of the higher-dimensional versions of Theorems 1.2 and 1.4 and of the questions above will require new methods.

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