Spontaneous breaking of axial symmetry for Schrödinger’s equation in the presence of a magnetic field

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For appropriate parameters, the ground state for the Schrödinger and Ampère coupled equations in a cylindric domain does not have axial symmetry.

Symmetry breaking — a situation in which the solution of a problem has lower symmetry than the problem itself — is usually a room for interesting physics (e.g. [1]). It is therefore gratifying to note that symmetry breaking is found in one of the most common problems we encounter: that of a condensate of particles interacting with a magnetic field, in a time independent state.

Our model is as follows. We define a “thermodynamic potential” density

\[ g = |\nabla \times \mathbf{A}_i|^2 + |(i\nabla - \mathbf{A}_i - b \hat{\theta})\psi|^2 , \tag{1} \]

where \( \psi \) is a wave function, \( \mathbf{A}_i \) the magnetic potential induced by \( \psi \), \( b \) is the external magnetic field, which is taken as uniform, and \( r, \theta \) are cylindrical coordinates, with the \( z \)-axis parallel to the field. \( \mathbf{A}_i \) and \( \mathbf{A}_1 \) are normalized so that no coefficients are required in (1). The parameter: \( b \) a single particle we consider a "condensate"; by this we mean that instead of a single particle there are \( \rho \) particles per unit volume, but they don’t interact among themselves and they are all in the same state. The condensate obeys the same equation as a single particle, but its current and magnetic influence will be proportional to \( \rho \). In the limit \( \rho \to 0 \), the thermodynamic potential approaches \( \pi \mu \rho \) and Eq. (1) reduces to the Landau problem [2].

Variation of \( \mathbf{A}_i \) gives

\[ \nabla \times \nabla \times \mathbf{A}_i = \text{Re}[\bar{\psi}(i\nabla - \mathbf{A}_i - b \hat{\theta})\psi] , \tag{4} \]

where the bar denotes complex conjugation. This is just Ampère’s law. If \( \mathbf{A}_i \) (resp. \( \psi \)) is fixed, then Eq. (4) (resp. (3)) is linear, but the system of both equations is nonlinear due to their mutual interaction. We choose a gauge such that \( \mathbf{A}_i \) is parallel to \( \hat{\theta} \). The boundary condition for \( \mathbf{A}_i \) is continuity of \( \nabla \times \mathbf{A}_i \). For \( \psi \) we will take the natural condition that its normal derivative vanishes at the boundary.

There are solutions of the system (3)-4 with the axial symmetry of the problem. These have the form

\[ \psi = \mathcal{R}_m(r)e^{-mi\theta} , \]

\[ \mathbf{A}_i = \mathcal{A}_m(r)\hat{\theta} , \tag{5} \]

where \( m \) is an integer. For this form, (3)-4 reduces to a system of ordinary differential equations. The value of \( m \) has to be chosen such that the minimum value of the thermodynamic potential is obtained. This value of \( m \) is an increasing function of \( b \). Due to continuity, \( \psi \) has to vanish along the axis of the sample for \( m \neq 0 \).

We now ask whether there exist situations such that the minimizer of \( \int g dS \) is not in the family (3). In the following, we will no longer consider \( \rho \) and \( b \) as independent parameters, but will focus on the value of the magnetic field for which the lowest value of \( \int g dS \) among the solutions in the family (3) is shared by the winding numbers \( m = 0 \) and \( m = 1 \). For \( \rho \leq 10 \), it is found numerically that this approximately occurs at \( b = 1.924 + 0.171\rho + 0.00104\rho^2 - 0.000036\rho^3 \). We solve the problem in two stages: in the first stage we perform a variation in which \( \psi \) has the form \( p + q(r - 1/2)e^{-i\theta} \) and \( \mathbf{A}_1 = a_0(r - 2i\rho/3) + a_1(r^2 - 3r^2/4)e^{-i\theta} \). This variation can be performed analytically, enabling us to obtain a qualitative picture of the minima and saddle points of \( \int g dS \). After we know what to look for, the system (3)-4 is solved numerically.

Besides the symmetric solutions with \( m = 0 \) or \( m = 1 \), we always find a nonsymmetric solution, but this has...
a higher value of $\int gdS$. (By “a solution”, we mean a class of equivalent solutions.) However, for $\rho \gtrsim 2.7$, a bifurcation occurs from the symmetric solution with $m = 1$. This bifurcation is characterized by a migration of the nodal line away from the axis of the sample (Fig. 1). This behavior reminds of a transition numerically found for mesoscopic superconducting disks [3], except that in the present situation there is a single nodal line.

We find that this nonsymmetric solution has lower thermodynamical potential than any couple of fields in the family (5) with $m = 1$, the nodal line is at the axis of the sample. At $\rho \sim 2.7$, a new (asymmetric) solution bifurcates from it.

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One might suspect that the solutions of (3)-(4) we have found are not those with the lowest thermodynamical potential. This is unlikely. For $\rho \to 0$, our symmetric solutions approach well known analytic results; for $\rho = 10$ we have solved (3)-(4) for initial symmetric configurations in a broad range, and none of the trials lead to a lower potential. We might have missed a non-symmetric solution with lower potential, but this possibility would only strengthen our conclusion that axial symmetry is broken.

FIG. 1. Distance of the nodal line, $r_{\text{node}}$, from the axis of the sample (divided by the sample radius). For a solution in the family (5) with $m = 1$, the nodal line is at the axis of the sample. At $\rho \sim 2.7$, a new (asymmetric) solution bifurcates from it.

FIG. 2. Thermodynamic potential of asymmetric solutions, in comparison to those of symmetric solutions. $g_{\text{sym}}$ is the potential density that gives the lowest integral $\int g_{\text{sym}} dS$ in the symmetric family with $m = 0$ or $m = 1$. (For the magnetic field considered, $m = 0$ and $m = 1$ give the same integral.) For $\rho \gtrsim 2.7$, the lowest potential is obtained for an asymmetric solution.

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