Geometric phases for wave packets in a uniform magnetic field

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A wave packet of a charged particle always make cyclic circular motion in a uniform magnetic field, just like a classical particle. The nonadiabatic geometric phase for an arbitrary wave packet can be expressed in terms of the mean value of a number operator. For a large class of wave packets, the geometric phase is proportional to the magnetic flux encircled by the orbit of the wave packet. For more general wave packets, however, the geometric phase contains an extra term.

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I. INTRODUCTION

Since the discovery of the geometric phase [1, 2, 3], the subject has been studied for various systems [4, 5, 6, 7, 8], especially particles with spin in time-dependent magnetic fields [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. To our knowledge the geometric phases for wave packets that undergo cyclic motion in the configuration space were not considered in the literature. Of special interest is the wave packets of a charged particle moving in a uniform magnetic field, because this represents a rather simple and realizable physical situation. It should be noted that a geometric phase can occur even though the system is time independent, just like spin in a uniform magnetic field [2].

In this paper we consider a charged particle moving in a uniform magnetic field. In Sec. II we briefly review the energy levels and the eigenstates of the system. Though the solutions to the system is well known in the textbooks [20, 21], we present the results somewhat differently. In Sec. III we consider the time evolution of an arbitrary wave packet. The time evolution is always cyclic and the center of the wave packet makes circular motion like a classical particle regardless of the initial condition. The nonadiabatic geometric phase in a cycle can be expressed in terms of the mean value of a number operator. This is similar to the case for a particle with spin in a rotating magnetic field, where the nonadiabatic geometric phase can be expressed in terms of the mean value of the component of the angular momentum along the rotating axis [4, 7, 8]. We also establish a linear relation between the nonadiabatic geometric phase and the magnetic flux encircled by the circular orbit of the wave packet. There is an inhomogeneous...
term in this relation, which we call an extra term. A large class of wave packet is presented in Sec. IV for which the extra term vanishes and thus the nonadiabatic geometric phase is proportional to the magnetic flux. For wave packets that do not belong to this class, the extra term depends on the initial condition. Again this is similar to the case for particles with spin in a rotating magnetic field [19]. A brief summary is given in Sec. V.

II. REVIEW OF THE MODEL

Consider a charged particle with electric charge \( q \) and mass \( M \), moving on the \( xy \) plane under the influence of a uniform magnetic field \( \mathbf{B} = B \mathbf{e}_z \) where \( \mathbf{e}_z \) is the unit vector in the \( z \) direction and \( B \) is a constant which can be taken as positive without loss of generality. We take the symmetric gauge \( A_x = -B y / 2, A_y = B x / 2 \), then the Hamiltonian reads

\[
H = \frac{1}{2M} \left( p_x^2 + p_y^2 \right) + \frac{1}{8} M \omega_B^2 (x^2 + y^2) - \frac{1}{2} \epsilon \omega_B L_z, \tag{1}
\]

where \( \omega_B = |q|B/Mc > 0 \), \( L_z = xp_y - yp_x \), and \( \epsilon = 1 \ (-1) \) if \( q \) is positive (negative). We define

\[
a_1 = \frac{M \omega_B x + i2p_x}{2\sqrt{M\hbar\omega_B}}, \quad a_2 = \frac{M \omega_B y + i2p_y}{2\sqrt{M\hbar\omega_B}}. \tag{2}
\]

The nonvanishing commutators among these operators and their Hermitian conjugates are \([a_i, a_j^\dagger] = \delta_{ij}\). The Hamiltonian can be recast as

\[
H = \frac{1}{2} \hbar \omega_B \left[ (a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + i\epsilon (a_1^\dagger a_2 - a_1 a_2^\dagger) \right]. \tag{3}
\]

Next we define

\[
a = \frac{1}{\sqrt{2}} (a_1 + i\epsilon a_2), \quad b = \frac{1}{\sqrt{2}} (a_1 - i\epsilon a_2), \tag{4}
\]

and their Hermitian conjugates. They satisfy

\[
[a, a^\dagger] = [b, b^\dagger] = 1, \tag{5}
\]

and all other commutators vanish. In terms of these operators, we have

\[
H = \hbar \omega_B (a^\dagger a + \frac{1}{2}), \tag{6}
\]

and

\[
L_z = \epsilon \hbar (b^\dagger b - a^\dagger a). \tag{7}
\]

We see that the Hamiltonian becomes that for a simple harmonic oscillator. Thus the energy levels are

\[
E_n = \hbar \omega_B (n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots. \tag{8}
\]
One can take the eigenstates to be the common ones of $N = a^\dagger a$ and $N' = b^\dagger b$, denoted by $|nn'\rangle$, satisfying

$$N|nn'\rangle = n|nn'\rangle, \quad N'|nn'\rangle = n'|nn'\rangle, \quad n, n' = 0, 1, 2, \ldots . \quad (9)$$

They are also common eigenstates of $H$ and $L_z$, with eigenvalues $E_n$ and $\epsilon(n' - n)\hbar$, respectively.

These states are given by

$$|nn'\rangle = \frac{1}{\sqrt{n!n'!}}(a^\dagger)^n(b^\dagger)^{n'}|00\rangle, \quad (10)$$

where the ground state $|00\rangle$ satisfies

$$a|00\rangle = b|00\rangle = 0. \quad (11)$$

One can work out the wave functions in the configuration space for these eigenstates and show that they are essentially the same as those obtained by solving the Schrödinger equation in the cylindrical coordinates. However, we are not interested in the quantum number $n'$ in this paper, so we will only deal with the quantum number $n$. We consider the eigenstates $|n\rangle$ of $N$ or $H$, satisfying

$$N|n\rangle = n|n\rangle, \quad H|n\rangle = E_n|n\rangle, \quad n = 0, 1, 2, \ldots . \quad (12)$$

These are also called number states in the following. They are given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad (13)$$

where the ground state $|0\rangle$ satisfies

$$a|0\rangle = 0. \quad (14)$$

Obviously, the state $|n\rangle$ is a linear combination of $|nn'\rangle$, that is

$$|n\rangle = \sum_{n'=0}^{\infty} C_{n'}|nn'\rangle, \quad (15)$$

where the coefficients $C_{n'}$ are arbitrary except satisfying $\sum_{n'=0}^{\infty} |C_{n'}|^2 = 1$ such that $|n\rangle$ is normalized. Therefore there must exist some big freedom in the wave function for the state $|n\rangle$.

We define a complex number $z$ and its complex conjugate $z^*$ as

$$z = \sqrt{\frac{M\omega_B}{4\hbar}}(x + i\epsilon y), \quad z^* = \sqrt{\frac{M\omega_B}{4\hbar}}(x - i\epsilon y), \quad (16)$$

then in the configuration space we have

$$a = \frac{1}{\sqrt{2}}(z + \partial z^*), \quad a^\dagger = \frac{1}{\sqrt{2}}(z^* - \partial z). \quad (17)$$

The wave function for the ground state $|0\rangle$ is obviously

$$\psi_0(z, z^*) = \exp(-z^*z)f(z), \quad (18)$$
where the function $f(z)$ is such that $\psi_0(z,z^*)$ is well behaved everywhere and is normalizable. Thus $f(z)$ is rather arbitrary. In particular, any polynomial satisfies the requirement. We assume that $\psi_0(z,z^*)$ has been normalized, then the normalized wave function for the higher excited state $|n\rangle$ is

$$\psi_n(z,z^*) = \frac{(-)^n}{\sqrt{2^n n!}} \exp(z^*z) \partial_z^n \exp(-2z^*z) f(z).$$

(19)

Therefore the freedom in these eigenstates lies in the arbitrariness of $f(z)$. Of course this freedom corresponds to the degeneracy of the Landau energy levels \([8]\). An arbitrary state of the system can be expressed as a linear combination of the above number states.

A class of states (or wave packets in the configuration space) that are of special interest in the following are the so called displaced number states. Similar to those for a simple harmonic oscillator \([21,22,23]\), they are defined as

$$|n,\alpha\rangle = D(\alpha)|n\rangle,$$

(20)

where $D(\alpha)$ is the unitary displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a),$$

(21)

where $\alpha$ is a complex number. One can define a more general displaced state $|\varphi,\alpha\rangle$ by acting $D(\alpha)$ on an arbitrary state $|\varphi\rangle$. An important property is that if the wave function for $|\varphi\rangle$ is $\varphi(z,z^*)$ or $\varphi(x,y)$, then that for $|\varphi,\alpha\rangle$ is

$$\varphi_\alpha(z,z^*) = \exp\left(\frac{\alpha z^* - \alpha^* z}{\sqrt{2}}\right) \varphi\left(z - \frac{\alpha}{\sqrt{2}}, z^* - \frac{\alpha^*}{\sqrt{2}}\right),$$

(22a)

or

$$\varphi_\alpha(x,y) = \exp\left[i \sqrt{\frac{M_\omega B}{2\hbar}}(\alpha_y x - \epsilon \alpha_x y)\right] \varphi\left(x - \sqrt{\frac{2\hbar}{M_\omega B}} \alpha_x, y - \epsilon \sqrt{\frac{2\hbar}{M_\omega B}} \alpha_y\right),$$

(22b)

where $\alpha_x = \text{Re}\alpha$, $\alpha_y = \text{Im}\alpha$. Thus the wave packet of the displaced state is essentially an entire displacement of the original one, and the above definition for the displacement operator $D(\alpha)$ seems to be sound.

### III. THE GEOMETRIC PHASE FOR AN ARBITRARY WAVE PACKET

Consider the time evolution of an arbitrary wave packet. The initial state at $t = 0$ may be expressed as

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

(23)

where the coefficients $c_n$ are arbitrary except satisfying $\sum_{n=0}^{\infty} |c_n|^2 = 1$ such that $|\psi(0)\rangle$ is normalized. Since the Hamiltonian is given by Eq. \([6]\), the state at a later time $t$ is

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n \exp[-i\omega_B t(n + \frac{1}{2})] |n\rangle.$$

(24)
At the time $T = 2\pi/\omega_B$, we have
\[ |\psi(T)\rangle = e^{-i\pi}|\psi(0)\rangle. \tag{25} \]
Therefore any state is cyclic, and the total phase change in a cycle is
\[ \delta = -\pi, \mod 2\pi, \tag{26} \]
which is independent of the initial condition. The expectation value of $H$ in the state $|\psi(t)\rangle$ is
\[ \langle H \rangle_t \equiv \langle \psi(t)|H|\psi(t)\rangle = \langle \psi(0)|H|\psi(0)\rangle = \hbar \omega_B (\langle N \rangle + \frac{1}{2}), \tag{27} \]
where the mean value $\langle N \rangle$ is evaluated in the initial state. The dynamic phase is
\[ \beta = -\hbar^{-1} \int_0^T \langle H \rangle_t \, dt = -\pi - 2\pi \langle N \rangle. \tag{28} \]
This depends on the initial state. The nonadiabatic geometric phase is
\[ \gamma = \delta - \beta = 2\pi \langle N \rangle, \mod 2\pi. \tag{29} \]
It is proportional to the mean value of the number operator $N$. This is similar to the case for a particle with spin in a rotating magnetic field, where the nonadiabatic geometric phase can be expressed in terms of the mean value of the component of the angular momentum along the rotating axis \[1, 17, 18\]. A number state has obviously a vanishing geometric phase (modulo $2\pi$) since its time evolution is trivial.

On the other hand, we consider the motion of the center of the wave packet. The position of it is characterized by the mean value of the coordinate variables $x$ and $y$. We denote
\[ \beta_a = \langle \psi(0)|a|\psi(0)\rangle, \quad \beta_b = \langle \psi(0)|b|\psi(0)\rangle. \tag{30} \]
Using the relations
\[ e^{iHt/\hbar}ae^{-iHt/\hbar} = a \exp(-i\omega_B t), \quad e^{iHt/\hbar}be^{-iHt/\hbar} = b, \tag{31} \]
which can be easily verified, we have
\[ \langle \psi(t)|a|\psi(t)\rangle = \beta_a \exp(-i\omega_B t), \quad \langle \psi(t)|b|\psi(t)\rangle = \beta_b. \tag{32} \]
This leads to
\[ \bar{x}_t - \sqrt{\frac{2\hbar}{M\omega_B}} \beta_{bx} = \sqrt{\frac{2\hbar}{M\omega_B}} |\beta_a| \cos(\omega_B t - \arg \beta_a), \tag{33a} \]
\[ \bar{y}_t + \epsilon \sqrt{\frac{2\hbar}{M\omega_B}} \beta_{by} = -\epsilon \sqrt{\frac{2\hbar}{M\omega_B}} |\beta_a| \sin(\omega_B t - \arg \beta_a), \tag{33b} \]
where $\bar{x}_t = \langle \psi(t)|x|\psi(t)\rangle$, $\bar{y}_t = \langle \psi(t)|y|\psi(t)\rangle$, and $\beta_{bx} = \text{Re} \beta_b$, $\beta_{by} = \text{Im} \beta_b$. Therefore the center of the wave packet always makes a circular motion. The angular frequency is $\omega_B$. The motion is clockwise when $q > 0$ and anticlockwise when $q < 0$. These are all the same as for
a classical particle. The radius of the circle depends only on $\beta_a$, while $\beta_b$ only determines the position of the center of the circle. The magnetic flux go through the area encircled by the circular orbit is defined as positive (negative) if the motion is anticlockwise (clockwise). This turns out to be

$$\Phi = -\frac{hc}{q} 2\pi |\beta_a|^2 = -\frac{hc}{q} 2\pi \langle a^\dagger \rangle \langle a \rangle = -\frac{hc}{q} 2\pi |\langle a \rangle|^2,$$

where the expectation values are evaluated in the initial state. Compared with Eq. (29) we obtain

$$\gamma = -\frac{q\Phi}{hc} + 2\pi (\Delta a)^2, \mod 2\pi.$$  

where

$$(\Delta a)^2 \equiv \langle a^\dagger a \rangle - \langle a \rangle^2 = \langle (a - \langle a \rangle)^\dagger (a - \langle a \rangle) \rangle.$$  

Thus the nonadiabatic geometric phase contains two terms, the first is proportional to the magnetic flux encircled by the orbit of the center of the wave packet, the second is an “extra term”. In Sec. [V] we will show that the extra term vanishes (modulo $2\pi$) if the wave packet is initially a displaced number state. For a more general wave packet, however, the extra term depends on the initial condition. Again this is similar to the case for particles with spin in a rotating magnetic field [19].

**IV. THE GEOMETRIC PHASE FOR A DISPLACED NUMBER STATE**

Consider the special case where the initial state is a displaced number state

$$|\psi(0)\rangle = |n, \alpha\rangle = D(\alpha)|n\rangle.$$  

Using the relations

$$D^\dagger(\alpha)aD(\alpha) = a + \alpha, \quad D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*,$$

which can be easily verified, we have

$$\langle a \rangle = \langle \psi(0)|a|\psi(0)\rangle = \alpha, \quad \langle a^\dagger \rangle = \langle \psi(0)|a^\dagger|\psi(0)\rangle = \alpha^*,$$

and

$$\langle a^\dagger a \rangle = \langle \psi(0)|a^\dagger a|\psi(0)\rangle = n + |\alpha|^2.$$  

Therefore in these states

$$(\Delta a)^2 = n,$$

and

$$\gamma = -\frac{q\Phi}{hc}, \mod 2\pi.$$
This holds for all numbers \( n = 0, 1, 2, \ldots \), and for any complex number \( \alpha \). Once again this result is similar to the case for spin in a time-dependent magnetic field where for special initial conditions the nonadiabatic geometric phase is always proportional to the solid angle subtended by the trace of the spin vector. (For a rotating magnetic field this is well known. For an arbitrarily varying magnetic field, see [13, 14, 19].)

It should be noticed that the time evolution of a displaced number state is rather simple. Using relations similar to Eq. (31), it is easy to show that

\[
e^{-i\frac{HT}{\hbar}}D(\alpha)e^{i\frac{HT}{\hbar}} = D(\alpha_t),
\]

(43)

where

\[
\alpha_t = \alpha \exp(-i\omega_B t).
\]

(44)

Thus for the initial condition (37), the time evolution is given by

\[
|\psi(t)\rangle = \exp[-i\omega_B t(n + \frac{1}{2})]D(\alpha_t)|n\rangle = \exp[-i\omega_B t(n + \frac{1}{2})]|n, \alpha_t\rangle.
\]

(45)

We see that it remains to be a displaced number state at all latter times, except that the displacement parameter varies with time, and an overall phase factor is gained. Since the displacement only changes the position of the wave packet but not the shape of it, we conclude that a displaced number state keeps its shape unchanged while it is making circular motion.

V. SUMMARY

In this paper we have studied the time evolution and the associated nonadiabatic geometric phase for a wave packet of a charged particle moving in a uniform magnetic field. Any state of this system is cyclic, and the center of it always makes circular motion like a classical particle. The nonadiabatic geometric phase in a cycle can be expressed in terms of the mean value of a number operator. A linear relation between the nonadiabatic geometric phase and the magnetic flux encircled by the circular orbit of the wave packet is established. For wave packets that are initially displaced number states the nonadiabatic geometric phase is proportional to the magnetic flux. For more general wave packets it contains an extra term which depends on the initial condition. This shows that the nonadiabatic geometric phase is not always proportional to the geometric object (the magnetic flux in the present problem is a geometric object since it is proportional to the area through with it penetrates) involved in the cyclic motion, but may be a more general function of the latter. Similar results were encountered in the problem of particles with spin moving in a rotating magnetic field [19]. We also showed that a displaced number state keeps its shape unchanged while it is making circular motion.

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[1] M. V. Berry, Proc. R. Soc. Lond. A 392, 45 (1984).
[2] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
[3] J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
[4] H.-Z. Li, *Global Properties of Simple Physical Systems–Berry’s Phase and Others* (Shanghai Scientific & Technical, Shanghai, 1998) (in Chinese).
[5] Y.-S. Wu and H.-Z. Li, Phys. Rev. B 38, 11907 (1988).
[6] T. F. Jordan, Phys. Rev. A 38, 1590 (1988).
[7] S.-Q. Chen, and G.-J. Ni, Phys. Lett. A 178, 339 (1993).
[8] M. V. Berry, J. Mod. Opt. 34, 1401 (1987).
[9] S.-J. Wang, Phys. Rev. A 42, 5107 (1990).
[10] A. G. Wagh and V. C. Rakhecha, Phys. Lett. A 170, 71 (1992).
[11] A. G. Wagh and V. C. Rakhecha, Phys. Rev. A 48, R1729 (1993).
[12] D. J. Fernández C, L. M. Nieto, M. A. del Olmo and M. Santander, J. Phys. A 25, 5151 (1992).
[13] E. Layton, Y. Huang and S-I Chu, Phys. Rev. A 41, 42 (1990).
[14] X.-C. Gao, J.-B. Xu and T.-Z. Qian, Phys. Lett. A 152, 449 (1991).
[15] G.-J. Ni, S.-Q. Chen, and Y.-L. Shen, Phys. Lett. A 197, 100 (1995).
[16] S.-L. Zhu, Z. D. Wang and Y.-D. Zhang, Phys. Rev. B 61, 1142 (2000).
[17] Q.-G. Lin, Phys. Rev. A 63, 012108 (2001).
[18] Q.-G. Lin, J. Phys. A 34, 1903 (2001).
[19] Q.-G. Lin, J. Phys. A 35, 377 (2002).
[20] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1977).
[21] G.-J. Ni and S.-Q. Chen, *Advanced Quantum Mechanics* (Fudan Univ. Press, Shanghai, 2000) (in Chinese).
[22] X.-L. Ka, *Advanced Quantum Mechanics*, 1st ed. (Higher Education Press, Beijing, 1999); 2nd ed. (Higher Education Press, Beijing, 2001). (in Chinese).
[23] M. M. Nieto, Phys. Lett. A 229, 135 (1997).