ON CLASSES OF C3 AND D3 MODULES

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Abstract. The aim of this paper is to study the notions of $A$-C3 and $A$-D3 modules for some class $A$ of right modules. Several characterizations of these modules are provided and used to describe some well-known classes of rings and modules. For example, a regular right $R$-module $F$ is a $V$-module if and only if every $F$-cyclic module $M$ is an $A$-C3 module where $A$ is the class of all simple submodules of $M$. Moreover, let $R$ be a right artinian ring and $A$, a class of right $R$-modules with local endomorphisms, containing all simple right $R$-modules and closed under isomorphisms. If all right $R$-modules are $A$-injective, then $R$ is a serial artinian ring with $J^2(R) = 0$ if and only if every $A$-C3 right $R$-module is quasi-injective, if and only if every $A$-C3 right $R$-module is C3.

1. Introduction and notation.

The study of modules with summand intersection property was motivated by the following result of Kaplansky: every free module over a commutative principal ideal ring has the summand intersection property (see [14, Exercise 51(b)]). A module $M$ is said to have the summand intersection property if the intersection of any two direct summands of $M$ is a direct summand of $M$. This definition is introduced by Wilson [18]. Dually, Garcia [10] consider the summand sum property. A module $M$ is said to have the summand sum property if the sum of any two direct summands is a direct summand of $M$. These properties have been studied by several authors (see [1, 3, 11, 12, 17,...]). Moreover, the classes of C3-modules and D3-modules have recently studied by Yousif et al. in [4, 20]. Some characterizations of semisimple rings and regular rings and other classes of rings are studied via C3-modules and D3-modules. On the other hand, several authors investigated some properties of generalizations of C3-modules and D3-modules in [6, 13]; namely, simple-direct-injective modules and simple-direct-projective modules.

A right $R$-module $M$ is called a C3-module if, whenever $A$ and $B$ are submodules of $M$ with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, then $A \oplus B \subset_d M$. $M$ is called simple-direct-injective in [6] if the submodules $A$ and $B$ in the above definition are simple. Dually, $M$ is called a D3-module if, whenever $M_1$ and $M_2$ are direct summands of $M$. 

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and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of $M$. $M$ is called \textit{simple-direct-projective} in \cite{12} if the submodules $M_1$ and $M_2$ in the above definition are maximal.

In Section 2, we introduce the notions of $A$-C3 modules and $A$-D3 modules, where $A$ is a class of right modules over the ring $R$ and closed under isomorphisms. It is shown that if each factor module of $M$ is $A$-injective, then $M$ is an $A$-D3 module if and only if $M$ satisfies D2 for the class $A$, if and only if $M$ have the summand intersection property for the class $A$ in Proposition \ref{prop2.7}. On the other hand, if every submodule of $M$ is $A$-projective, then $M$ is an $A$-C3 module if and only if $M$ satisfies C2 for the class $A$, if and only if $M$ have the summand sum property for the class $A$ in Proposition \ref{prop2.13}. Some well-known properties of other modules are obtained from these results.

In Section 3, we provide some characterizations of serial artinian rings and semisimple artinian rings. The Theorem \ref{thm3.2} and Theorem \ref{thm3.3} are indicated that let $R$ be a right artinian ring and $A$, a class of right $R$-modules with local endomorphisms, containing all simple right $R$-modules and closed under isomorphisms:

(1) If all right $R$-modules are $A$-injective, the following conditions are equivalent for a ring $R$:
   (i) $R$ is a serial artinian ring with $J^2(R) = 0$.
   (ii) Every $A$-C3 right $R$-module is quasi-injective.
   (iii) Every $A$-C3 right $R$-module is C3.

(2) If all right $R$-modules are $A$-projective, then the following conditions are equivalent for a ring $R$:
   (i) $R$ is a serial artinian ring with $J^2(R) = 0$.
   (ii) Every $A$-D3 right $R$-module is quasi-projective.
   (iii) Every $A$-D3 right $R$-module is D3.

Moreover, we give an equivalent condition for a regular $V$-module. It is shown that a regular right $R$-module $F$ is a $V$-module if and only if every $F$-cyclic module is simple-direct-injective in Theorem \ref{thm3.9}. It is an extension the result of rings to modules.

Throughout this paper $R$ denotes an associative ring with identity, and modules will be unitary right $R$-modules. The Jacobson radical ideal in $R$ is denoted by $J(R)$. The notations $N \leq M$, $N \leq_e M$, $N \leq M$, or $N \subset_d M$ mean that $N$ is a submodule, an essential submodule, a fully invariant submodule, and a direct summand of $M$, respectively. Let $M$ and $N$ be right $R$-modules. $M$ is called $N$-injective if for any right $R$-module $K$ and any monomorphism $f : K \to N$, the induced homomorphism $\text{Hom}(N, M) \to \text{Hom}(K, M)$ by $f$ is an epimorphism. $M$ is called $N$-projective if for any right $R$-module $K$ and any epimorphism $f : N \to K$, the induced homomorphism $\text{Hom}(M, N) \to \text{Hom}(M, K)$ by $f$ is an epimorphism. Let $A$ be a class of right modules over the ring $R$. $M$ is called $A$-injective ($A$-projective) if $M$ is $N$-injective (resp., $N$-projective) for all $N \in A$. We refer to \cite{3}, \cite{7}, \cite{10}, and \cite{11} for all the undefined notions in this paper.
2. On \( \mathcal{A} \)-C3 modules and \( \mathcal{A} \)-D3 modules

Let \( \mathcal{A} \) be a class of right modules over a ring \( R \) and closed under isomorphisms. We call that a right \( R \)-module \( M \) is an \( \mathcal{A} \)-C3 module if, whenever \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \) are submodules of \( M \) with \( A \subset_d M \), \( B \subset_d M \) and \( A \cap B = 0 \), then \( A \oplus B \subset_d M \). Dually, \( M \) is an \( \mathcal{A} \)-D3 module if, whenever \( M \) is a direct sum of \( M/M_1, M/M_2 \in \mathcal{A} \) with \( M = M_1 + M_2 \), then \( M_1 \cap M_2 \) is a direct summand of \( M \).

Remark 2.1. Let \( M \) be a right \( R \)-module and \( \mathcal{A} \), a class of right \( R \)-modules.

(1) If \( M \) is a C3 (D3) module, then \( M \) is an \( \mathcal{A} \)-C3 (resp., \( \mathcal{A} \)-D3) module.

(2) If \( \mathcal{A} = \text{Mod} - R \), then \( \mathcal{A} \)-C3 modules (\( \mathcal{A} \)-D3 modules) are precisely the C3 modules (resp., D3) modules.

(3) If \( \mathcal{A} \) is the class of all simple submodules of \( M \), then \( \mathcal{A} \)-C3 (\( \mathcal{A} \)-D3) modules are precisely the simple-direct-injective (resp., simple-direct-projective) modules and studied in [6, 13].

(4) If \( \mathcal{A} \) is a class of injective right \( R \)-modules, then \( M \) is always an \( \mathcal{A} \)-C3 module.

(5) If \( \mathcal{A} \) is a class of projective right \( R \)-modules, then \( M \) is always an \( \mathcal{A} \)-D3 module.

Lemma 2.2. Let \( \mathcal{A} \) be a class of right \( R \)-modules and closed under isomorphisms. Then every summand of an \( \mathcal{A} \)-C3 module (\( \mathcal{A} \)-D3 module) is also an \( \mathcal{A} \)-C3 module (resp., \( \mathcal{A} \)-D3 module).

Proof. The proof is straightforward. \( \square \)

Proposition 2.3. Let \( \mathcal{A} \) be a class of right \( R \)-modules and closed under direct summands. Then the following conditions are equivalent for a module \( M \):

(1) \( M \) is an \( \mathcal{A} \)-C3 module.

(2) If \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \) are submodules of \( M \) with \( A \subset_d M \), \( B \subset_d M \) and \( A \cap B = 0 \), there exist submodules \( A_1 \) and \( B_1 \) of \( M \) such that \( M = A \oplus B_1 = A_1 \oplus B \) with \( A \leq A_1 \) and \( B \leq B_1 \).

(3) If \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \) are submodules of \( M \) with \( A \subset_d M \), \( B \subset_d M \) and \( A \cap B \subset_d M \), then \( A + B \subset_d M \).

Proof. It is similar to the proof of Proposition 2.2 in [4]. \( \square \)

Dually Proposition 2.3, we have the following proposition.

Proposition 2.4. Let \( \mathcal{A} \) be a class of right \( R \)-modules and closed under isomorphisms. Then the following conditions are equivalent for a module \( M \):

(1) \( M \) is an \( \mathcal{A} \)-D3 module.

(2) If \( M/A, M/B \in \mathcal{A} \) with \( A \subset_d M \), \( B \subset_d M \) and \( M = M/A + M/B \), then \( M = M/A \oplus B_1 = A_1 \oplus B \) with \( A \leq A_1 \) and \( B \leq B_1 \).

(3) If \( M/A, M/B \in \mathcal{A} \) with \( A \subset_d M \), \( B \subset_d M \) and \( A + B \subset_d M \), then \( A \cap B \subset_d M \).
Let $f : A \to B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows:

$$\langle f \rangle = \{ a + f(a) \mid a \in A \}.$$ 

The following result is proved in Lemma 2.6 of [15].

**Lemma 2.5.** Let $M = X \oplus Y$ and $f : A \to Y$, a homomorphism with $A \leq X$. Then the following conditions hold

(1) $A \oplus Y = \langle f \rangle \oplus Y$.

(2) $\text{Ker}(f) = X \cap \langle f \rangle$.

**Proposition 2.6.** Let $M$ be an $A$-$D3$ module with $A$ a class of right $R$-modules and closed under isomorphisms and summands. If $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ is a homomorphism with $\text{Im}(f) \subseteq_d M_2$ and $\text{Im}(f) \in A$, then $\text{Ker}(f)$ is a direct summand of $M_1$.

**Proof.** Assume that $M = M_1 \oplus M_2$ and a homomorphism $f : M_1 \to M_2$ with $\text{Im}(f) \subseteq_d M_2$ and $\text{Im}(f) \in A$. Call $M' := M_1 \oplus \text{Im}(f)$. Then $M'$ is a direct summand of $M$ and so is an $A$-$D3$ module. It follows that $M' = M_1 \oplus \text{Im}(f) = \langle f \rangle \oplus \text{Im}(f)$ by Lemma 2.5. It is easy to check $M'/M_1, M'/\langle f \rangle \in A$ and $M' = M_1 + \langle f \rangle$. As $M'$ is an $A$-$D3$ module and by Lemma 2.5, $\langle f \rangle \cap M_1 = \text{Ker}(f)$ is a direct summand of $M'$. Thus $\text{Ker}(f)$ is a direct summand of $M_1$. \qed

**Proposition 2.7.** Let $M$ be a right $R$-module and $A$, a class of right $R$-modules and closed under isomorphisms and summands. If each factor module of $M$ is $A$-injective, then the following conditions are equivalent:

(1) For any two direct summands $M_1, M_2$ of $M$ such that $M/M_1, M/M_2 \in A$, $M_1 \cap M_2$ is a direct summand of $M$.

(2) $M$ is an $A$-$D3$ module.

(3) Any submodule $N$ of $M$ such that the factor module $M/N \in A$ is isomorphic to a direct summand of $M$, is a direct summand of $M$.

(4) For any decomposition $M = M_1 \oplus M_2$ with $M_2 \in A$, then every homomorphism $f : M_1 \to M_2$ has the kernel a direct summand of $M_1$.

(5) Whenever $X_1, \ldots, X_n$ are direct summands of $M$ and $M/X_1, \ldots, M/X_n \in A$, then $\cap_{i=1}^n X_i$ is a direct summand of $M$.

**Proof.** (2) $\Rightarrow$ (1). Let $M_1, M_2$ be direct summands of $M$ such that $M/M_1, M/M_2 \in A$. Then $M = M_1 \oplus M'_1$. Without loss of generality we can assume that $M_2 \not\subseteq M_1, M_2 \not\subseteq M'_1$. From our assumption, $\pi(M_2)$ is a direct summand of $M'_1$. Then we can write $M'_1 = \pi(M_2) \oplus M''_1$ for some $M''_1 \leq M'_1$. Since the class $A$ is closed under direct summands, $M''_1 \in A$. It is easy to see that $M_1 + M''_1$ is a direct summand of $M$. We have $M/(M_1 + M''_1) \in A$ and $M_1 + M''_1 + M_2 = M$. It follows that $M_1 \cap M_2 = (M_1 + M''_1) \cap M_2$ is a direct summand of $M$.

(3) $\Rightarrow$ (2). It is obvious.
(1) $\Rightarrow$ (4). Assume that $M = M_1 \oplus M_2$ with $M_2 \in \mathcal{A}$ and a homomorphism $f : M_1 \to M_2$. It follows that $M = M_1 \oplus M_2 = \langle f \rangle \oplus M_2$ by Lemma 2.5. Note that $M/M_1, M/\langle f \rangle \in \mathcal{A}$. By (1) and Lemma 2.5, $\langle f \rangle \cap M_1 = \text{Ker}(f)$ is a direct summand of $M$. Thus $\text{Ker}(f)$ is a direct summand of $M_1$.

(4) $\Rightarrow$ (3). Let $M_1, M_2$ be submodules of $M$ such that $M = M_1 \oplus A, M/M_2 \cong A$ and $A \in \mathcal{A}$. Call $\pi_1 : M \to M_1$ and $\pi_2 : M \to A$ the projections. By the hypothesis, $\pi_2(M_2)$ is a direct summand of $A$ and hence $A = \pi_2(M_2) \oplus B$ for some submodule $B$ of $A$. Call $p : M \to M/M_2$ the canonical projection and isomorphism $\phi : M/M_2 \to A$. Take the homomorphism $f = \phi \circ (p|_{M_1}) : M_1 \to A$. It follows that $\text{Ker}(f) = M_1 \cap M_2$.

By (4), $\text{Ker}(f) = M_1 \cap M_2$ is a direct summand of $M_1$. Call $N_1$ a submodule of $M_1$ with $M_1 = N_1 \oplus (M_1 \cap M_2)$. Note that $M_1 + M_2 = M_1 \oplus \pi_2(M_2)$ and $N_1 \cap M_2 = 0$. This gives that

\[
\begin{align*}
M &= M_1 \oplus \pi_2(M_2) \oplus B \\
   &= (M_1 + M_2) \oplus B \\
   &= [N_1 \oplus (M_1 \cap M_2) + M_2] \oplus B = (N_1 + M_2) \oplus B \\
   &= (N_1 \oplus M_2) \oplus B.
\end{align*}
\]

(1) $\Rightarrow$ (5). We prove this by induction on $n$. When $n = 2$, the assertion is true from (1). Suppose that the assertion is true for $n = k$. Let $X_1, X_2, \ldots, X_{k+1}$ be summands of $M$ and $M/X_1, M/X_2, \ldots, M/X_{k+1} \in \mathcal{A}$. We can write $M = \cap_{i=1}^{k} X_i \oplus N$ for some submodule $N$ of $M$. Without loss of generality we can assume that $\cap_{i=1}^{k} X_i \not\subset X_{k+1}$. Let $f : M \to M/X_{k+1}$ be the natural projection. Then $(\cap_{i=1}^{k} X_i)/(\cap_{i=1}^{k} X_i \cap X_{k+1})$ is $\mathcal{A}$-injective, and therefore, it is isomorphic to a direct summand of $M/X_{k+1} \in \mathcal{A}$. This gives that $\cap_{i=1}^{k} X_i/\cap_{i=1}^{k} X_i$ is isomorphic to a direct summand of $M$ and

\[
M/(\cap_{i=1}^{k+1} X_i \oplus N) = (\cap_{i=1}^{k} X_i \oplus N)/(\cap_{i=1}^{k+1} X_i \oplus N) \in \mathcal{A}.
\]

Since the equivalence of (1) and (3), $(\cap_{i=1}^{k+1} X_i) \oplus N$ is a direct summand of $M$. Thus $\cap_{i=1}^{k+1} X_i$ is a direct summand of $M$. $\square$

**Corollary 2.8.** The following conditions are equivalent for a module $M$:

1. If $M/A$ is a semisimple module and $B$, a submodule of $M$ with $M/A \cong B \subset M$, then $A \subset M$.

2. For any two direct summands $A, B$ of $M$ with $M/A$ and $M/B$ are semisimple modules, then $A \cap B \subset M$.

3. For any two direct summands $A, B$ of $M$ such that $M/A, M/B$ are semisimple modules and $A + B = M$, then $A \cap B$ is a direct summand of $M$.

4. Whenever $X_1, X_2, \ldots, X_n$ are direct summands of $M$ and $M/X_1, M/X_2, \ldots, M/X_n$ are semisimple modules, then $\cap_{i=1}^{n} X_i$ is a direct summand of $M$.

**Corollary 2.9.** Let $P$ be a quasi-projective module. If $X_1, \ldots, X_n$ are summands of $P$ and $P/X_1, \ldots, P/X_n$ are semisimple modules, then $\cap_{i=1}^{n} X_i$ is a direct summand of $P$. 


Corollary 2.10. The following conditions are equivalent for a module $M$:

1. For any maximal submodule $A$ of $M$ and any submodule $B$ of $M$ such that $M/A \cong B \subset_d M$, $A \subset_d M$.
2. For any two maximal summands $A, B$ of $M$, $A \cap B \subset_d M$.
3. If $M/A$ is a finitely generated semisimple module with $M/A \cong B \subset_d M$, then $A \subset_d M$.
4. Whenever $X_1, X_2, \ldots, X_n$ are maximal summands of $M$, then $\cap_{i=1}^n X_i$ is a direct summand of $M$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). Follow from Proposition 2.7.

(3) $\Rightarrow$ (1). Clearly.

(1) $\Rightarrow$ (3). Assume that $M/A$ is a finitely generated semisimple module and isomorphic to a direct summand of $M$. Write $M/A = M_1/A \oplus \cdots \oplus M_n/A$ with simple submodules $M_i/A$ of $M/A$. Then $M_i \cap (\sum_{j \neq i} M_j) = A$ for all $i = 1, 2, \ldots, n$. For any subset $\{i_1, i_2, \ldots, i_{n-1}\}$ of the set $I := \{1, 2, \ldots, n\}$, it is easily to see that

$$M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}) \cong M_k/A$$

for some $k \in I \setminus \{i_1, i_2, \ldots, i_{n-1}\}$. It follows that $M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}})$ is isomorphic to a simple summand of $M$. By (1), $M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}$ is a maximal summand of $M$. On the other hand, we can check that

$$A = \bigcap_{\{i_1, i_2, \ldots, i_{n-1}\} \subset I} (M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}).$$

So, by (4), $A$ is a direct summand of $M$. \qed

Proposition 2.11. Let $M$ be an $A$-$C3$ module with $A$ a class of right $R$-modules and closed under isomorphisms and summands. If $M = A_1 \oplus A_2$ and $f : A_1 \rightarrow A_2$ is a homomorphism with $\text{Ker}(f) \in A$ and $\text{Ker}(f) \subset_d A_1$, then $\text{Im}(f)$ a direct summand of $A_2$.

Proof. Let $f : A_1 \rightarrow A_2$ be an $R$-homomorphism with $\text{Ker}(f) \in A$. By the hypothesis, there exists a decomposition $A_1 = \text{Ker}(f) \oplus B$ for a submodule $B$ of $A_1$. Then $B \oplus A_2$ is a direct summand of $M$. Note that every direct summand of an $A$-$C3$ module is also an $A$-$C3$ module. Hence $B \oplus A_2$ is an $A$-$C3$ module. Let $g = f|_B : B \rightarrow A_2$. Then $g$ is a monomorphism and $\text{Im}(g) = \text{Im}(f)$. It is easy to see that $B \oplus A_2 = \langle g \rangle \oplus A_2$, $\langle g \rangle \cap B = 0$ and $\langle g \rangle \cong B$. Note that $B, \langle g \rangle \in A$. As $B \oplus A_2$ is an $A$-$C3$ module, $B \oplus \langle g \rangle$ is a direct summand of $B \oplus A_2$. Thus $B \oplus \langle g \rangle = B \oplus \text{Im}(g)$, which implies that $\text{Im}(g)$ or $\text{Im}(f)$ is a direct summand of $A_2$. \qed

Proposition 2.12. Let $M$ be a right $R$-module and $A$, a class of right $R$-modules and closed under isomorphisms and summands. If every submodule of $M$ is $A$-projective, the following conditions are equivalent:

1. For any two direct summands $M_1, M_2$ of $M$ such that $M_1, M_2 \in A$, $M_1 + M_2$ is a direct summand of $M$. 

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). Follow from Proposition 2.7.

(3) $\Rightarrow$ (1). Clearly.

(1) $\Rightarrow$ (3). Assume that $M/A$ is a finitely generated semisimple module and isomorphic to a direct summand of $M$. Write $M/A = M_1/A \oplus \cdots \oplus M_n/A$ with simple submodules $M_i/A$ of $M/A$. Then $M_i \cap (\sum_{j \neq i} M_j) = A$ for all $i = 1, 2, \ldots, n$. For any subset $\{i_1, i_2, \ldots, i_{n-1}\}$ of the set $I := \{1, 2, \ldots, n\}$, it is easily to see that

$$M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}) \cong M_k/A$$

for some $k \in I \setminus \{i_1, i_2, \ldots, i_{n-1}\}$. It follows that $M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}})$ is isomorphic to a simple summand of $M$. By (1), $M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}$ is a maximal summand of $M$. On the other hand, we can check that

$$A = \bigcap_{\{i_1, i_2, \ldots, i_{n-1}\} \subset I} (M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}).$$

So, by (4), $A$ is a direct summand of $M$. \qed
Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (3) Let \( f : A_1 \to A_2 \) be an \( R \)-homomorphism with \( A_1 \in \mathcal{A} \). By the hypothesis, \( \text{Ker}(f) \) is a direct summand of \( A_1 \). The rest of proof is followed from Proposition 2.11.

(3) ⇒ (1) Let \( N \) and \( K \) be direct summands of \( M \) such that \( N, K \in \mathcal{A} \). Write \( M = N \oplus N' \) and \( M = K \oplus K' \) for some submodules \( N', K' \) of \( M \). Consider the canonical projections \( \pi_K : M \to K \) and \( \pi_{N'} : M \to N' \). Let \( A = \pi_{N'}(\pi_K(N)) \). Then \( A = (N + K) \cap (N + K') \cap N' \) is a direct summand of \( M \) by (3). Write \( M = A \oplus L \) for some submodule \( L \) of \( M \). Clearly,

\[(N + K) \cap [(N + K') \cap (N' \cap L)] = 0.\]

Hence, \( N' = A \oplus (N' \cap L) \) and \( M = (N \oplus A) \oplus (N' \cap L) \). Since \( A \leq N + K \) and \( A \leq N + K' \), we get

\[N + K = (N \oplus A) \cap [(N + K) \cap (N' \cap L)]\]

and

\[N + K' = (N \oplus A) \cap [(N + K') \cap (N' \cap L)].\]

They imply

\[M = N + K' + K = (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \leq (N + K) + [(N + K') \cap (N' \cap L)].\]

Thus \( M = (N + K) \oplus [(N + K') \cap (N' \cap L)]. \)

\[\square\]

**Proposition 2.13.** Let \( M \) be a right \( R \)-module and \( \mathcal{A} \), a class of artinian right \( R \)-modules and closed under isomorphisms and summands. If every submodule of \( M \) is \( \mathcal{A} \)-projective, then the following conditions are equivalent:

1. \( M \) is an \( \mathcal{A} \)-C3 module.
2. Every submodule \( N \in \mathcal{A} \) of \( M \) that is isomorphic to a direct summand of \( M \) is itself a direct summand.
3. Whenever \( X_1, X_2, \ldots, X_n \) are direct summands of \( M \) and \( X_1, X_2, \ldots, X_n \in \mathcal{A} \), then \( \sum_{i=1}^{n} X_i \) is a direct summand of \( M \).

**Proof.** (1) ⇒ (2). Let \( M_1 \) be submodule of \( M \) and isomorphic to a direct summand \( M_2 \) of \( M \) and \( M_1 \in \mathcal{A} \). Then \( M = M_2 \oplus M_2' \). If \( M_1 \subset M_2 \), then by \( M_2 \) is artinian and \( M_1 \cong M_2 \), implies that \( M_1 = M_2 \). Let \( M_1 \nsubseteq M_2 \) and \( \pi : M_2 \oplus M_2' \to M_2' \) be projection. According to the hypothesis, \( \text{Ker}(\pi|_{M_1}) \) is a direct summand of \( M_1 \). It follows that \( M_1 = M_1 \cap M_2 \oplus N_1 \). Since \( N_1 \cong \pi(M_1), M_1 \cong M_2 \), then there is an isomorphism \( \phi : N' \to \pi(M_1) \), where \( N' \) is a direct summand of \( M_1 \). Since \( \langle \phi \rangle \in \mathcal{A} \) and \( \langle \phi \rangle \cap M_2 = 0, M_2 + \langle \phi \rangle = M_2 \oplus N_1 \) is a direct summand of \( M \). Therefore, \( N_1 \) is a non-zero direct
summand of $M$. It is clear that $M_1 \cap M_2 \in \mathcal{A}$ and $M_1 \cap M_2$ is isomorphic to a direct summand of $M$. If $M_1 \cap M_2$ is not a direct summand of $M$, by using an argument that are similar to the argument presented above, we can show that $M_1 \cap M_2 = N_2 \oplus N'_2$, where $N_2 \in \mathcal{A}$ is a non-zero direct summand of $M$ and $N'_2 \in \mathcal{A}$ is a submodule of $M$ isomorphic to a direct summand of $M$. Since each module of the class $\mathcal{A}$ is artinian, by conducting similar constructions continue for some $k$, we obtain a decomposition $M_1 = N_1 \oplus \ldots \oplus N_k$, where $N_i \in \mathcal{A}$ for each $i$. Since $M$ is an $\mathcal{A}$-C3 module, $N_1 \oplus N_2 \oplus \ldots \oplus N_k$ is a direct summand of $M$.

$(2) \Rightarrow (1)$. It is obvious.

$(1) \Rightarrow (3)$. We prove this by induction on $n$. When $n = 2$, the assertion follows from Proposition 2.12. Suppose that the assertion is true for $n = k$. Let $X_1, X_2, \ldots, X_{k+1}$ be summands of $M$ and $X_1, X_2, \ldots, X_{k+1} \in \mathcal{A}$. Then there exists a submodule $N$ of $M$ such that $M = (\sum_{i=1}^{k} X_i) \oplus N$. Let $\pi : (\sum_{i=1}^{k} X_i) \oplus N \rightarrow N$ be the natural projection. As $\pi(X_{k+1})$ is $\mathcal{A}$-projective, then $X_{k+1} = ((\sum_{i=1}^{k} X_i) \cap X_{k+1}) \oplus S$ for some submodule $S$ of $M$. Since the equivalence of $(1)$ and $(2)$, $\pi(X_{k+1})$ is a direct summand of $M$ and, therefore, $N = \pi(X_{k+1}) \oplus T$ with $T$ a submodule $M$. It follows that $\sum_{i=1}^{k+1} X_i = (\sum_{i=1}^{k} X_i) \oplus \pi(X_{k+1})$ and $M = (\sum_{i=1}^{k} X_i) \oplus \pi(X_{k+1}) \oplus T$. Thus, $\sum_{i=1}^{k+1} X_i$ is a direct summand of $M$. \hfill $\Box$

**Remark 2.14.** Let $F$ be any nonzero free module over $\mathbb{Z}$ and $\mathcal{A}$, a class of all free $\mathbb{Z}$-modules. It is well known that $F$ is a quasi-continuous module and $F$ is not a continuous module. Thus, $F$ is an $\mathcal{A}$-C3 module and satisfies the property: there exists a submodule $N \in \mathcal{A}$ of $F$ that is isomorphic to a direct summand of $F$ is not a direct summand.

**Proposition 2.15.** Let $M$ be a right $R$-module and $\mathcal{A}$, a class of right $R$-modules and closed under isomorphisms and summands. If every factor module of $M$ is $\mathcal{A}$-projective, then the following conditions are equivalent:

$(1)$ For any two direct summands $M_1, M_2$ of $M$ such that $M_1, M_2 \in \mathcal{A}$, $M_1 + M_2$ is a direct summand of $M$.

$(2)$ $M$ is an $\mathcal{A}$-C3 module.

$(3)$ For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \rightarrow A_2$ has the image a direct summand of $A_2$.

$(4)$ Every submodule $N \in \mathcal{A}$ of $M$ that is isomorphic to a direct summand of $M$ is itself a direct summand.

$(5)$ Whenever $X_1, X_2, \ldots, X_n$ are direct summands of $M$ and $X_1, X_2, \ldots, X_n \in \mathcal{A}$, then $\sum_{i=1}^{n} X_i$ is a direct summand of $M$.

**Proof.** $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3) \Rightarrow (1)$ are proved similarly to the argument proof of Proposition 2.12.

$(4) \Rightarrow (2)$ is obvious.

$(3) \Rightarrow (4)$. Let $\sigma : A \rightarrow B$ be an isomorphism with $A \in \mathcal{A}$ a summand of $M$ and $B \leq M$. We need to show that $B$ is a direct summand of $M$. Write $M = A \oplus T$ for
some submodule $T$ of $M$. We have $A/A \cap B$ is an image of $M$ and obtain that $A \cap B$ is a direct summand of $A$. Take $A = (A \cap B) \oplus C$ for some submodule $C$ of $A$. Now $M = (A \cap B) \oplus (C \oplus T)$. Clearly, $A \cap [(C \oplus T) \cap B] = 0$ and $B = (A \cap B) \oplus [(C \oplus T) \cap B]$. Let $H := \sigma^{-1}((C \oplus T) \cap B)$. Then $H$ is a submodule of $A$, $H \cap [(C \oplus T) \cap B] = 0$ and $A = H \oplus (H' \cap T)$. Consider the projection $\pi : M \rightarrow H' \cap T$. Then

$$H \oplus [(C \oplus T) \cap B] = H \oplus \pi((C \oplus T) \cap B).$$

By (3), the image of the homomorphism $\pi |_{(C \oplus T) \cap B} \circ \sigma |_H : H \rightarrow H' \oplus T$ is a direct summand of $H' \oplus T$ since $H$ is contained in $A$. Write $H' \oplus T = \pi |_{(C \oplus T) \cap B} \circ \sigma (H) \oplus K$ for some submodule $K$ of $H' \oplus T$. Then $H' \oplus T = \pi((C \oplus T) \cap B) \oplus K$. It follows that

$$M = H \oplus \pi((C \oplus T) \cap B) \oplus K = H \oplus [(C \oplus T) \cap B] \oplus K.$$

By the modular law, $C \oplus T = [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)]$. Thus

$$M = (A \cap B) \oplus [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)]$$

$$= B \oplus [(H \oplus K) \cap (C \oplus T)].$$

The implication $(1) \Rightarrow (5)$ is proved similarly to the argument proof of Proposition 2.13

\[\square\]

**Corollary 2.16.** The following conditions are equivalent for a module $M$:

1. For any semisimple submodules $A, B$ of $M$ with $A \cong B \subset M$, $A \subset M$.
2. For any semisimple summands $A, B$ of $M$, $A + B \subset M$.
3. For any semisimple summands $A, B$ of $M$ with $A \cap B = 0$, $A + B \subset M$.
4. Whenever $X_1, \ldots, X_n$ are semisimple summands of $M$ and $X_1, \ldots, X_n \in A$, then $\sum_{i=1}^n X_i$ is a direct summand of $M$.

**Corollary 2.17.** Let $Q$ be a quasi-injective module. If $X_1, \ldots, X_n$ are semisimple summands of $Q$, then $\sum_{i=1}^n X_i$ is a direct summand of $Q$.

**Corollary 2.18 ([6 Proposition 2.1]).** The following conditions are equivalent for a module $M$:

1. For any simple submodules $A, B$ of $M$ with $A \cong B \subset M$, $A \subset M$.
2. For any simple summands $A, B$ of $M$ with $A \cap B = 0$, $A \oplus B \subset M$.
3. For any finitely generated semisimple submodules $A, B$ of $M$ with $A \cong B \subset M$, $A \subset M$.
4. For any finitely generated semisimple summands $A, B$ of $M$ with $A \cap B = 0$, $A \oplus B \subset M$.

3. Characterizations of rings

**Lemma 3.1.** Let $A$ be a class of right $R$-modules with local endomorphisms and closed under isomorphisms. Assume that $K$ and $M$ are indecomposable right $R$-modules and not contained in $A$. Then
(1) $N = M \oplus P$ is an $A$-D3 module for all projective modules $P$.
(2) $N = M \oplus E$ is an $A$-C3 module for all injective modules $E$.
(3) $N = M \oplus K$ is an $A$-D3 module and an $A$-C3 module.

Proof. (1) Let $N/A \cong S \subset_d N$ with $S \in \mathcal{A}$. By [5] Lemma 26.4, there exist a direct summand $M_1$ of $M$ and a direct summand $P_1$ of $P$ such that $N = S \oplus M_1 \oplus P_1$. Write $P = P_1 \oplus P_2$ for some submodule $P_2$ of $P$. Since $M$ is an indecomposable module, we have either $M_1 = 0$ or $M = M_1$. If $M_1 = 0$, then $N = S \oplus P_1 = (M \oplus P_2) \oplus P_1$ and it follows that $M \oplus P_2 \cong S$, and hence $M \in \mathcal{A}$ contradicting. So $M_1 = M$. Then $N = S \oplus (M \oplus P_1) = (M \oplus P_1) \oplus P_2$. This gives $S \cong P_2$, and consequently $N/A \cong S$ is projective. Hence, $A$ is a direct summand of $N$ and (1) holds.

(2) Suppose that $A$ is a submodule of $N$ such that $A \cong S$ with $S$ a submodule of $N$ and $S \in \mathcal{A}$. As in (1), we see that $N = S \oplus M_1 \oplus E_1$ with $M = M_1 \oplus M_2$ and $E = E_1 \oplus E_2$. Also, as in (1), $M_1 = M$. Therefore, $N = S \oplus M \oplus E_1 = M \oplus E = (M \oplus E_1) \oplus E_2$. It follows that $S \cong E_2$ is an injective module. Thus $A$ is a direct summand of $N$.

(3) We show that $N$ has no a nonzero direct summand $S$ with $S \in \mathcal{A}$. Assume on the contrary that there exists a non-zero summand $S \subset_d N$ with $S \in \mathcal{A}$. As, in (1), $N = S \oplus M_1 \oplus K_1$ with $M = M_1 \oplus M_2$ and $K = K_1 \oplus K_2$. Also, as in (1), $M_1 = M$. Therefore, $N = S \oplus M \oplus K_1 = M \oplus K$.

Since $K$ is indecomposable, $K = K_1$ or $K = K_2$. If $K = K_1$, then $S \oplus M \oplus K = M \oplus K$ and consequently $S = 0$, a contradiction. If $K = K_2$, then $K_1 = 0$ and so $S \oplus M = M \oplus K$. Therefore, $K \cong S$ and hence $K \in \mathcal{A}$, a contradiction. □

Recall that a module is uniserial if the lattice of its submodules is totally ordered under inclusion. A ring $R$ is called right uniserial if $R_R$ is a uniserial module. A ring $R$ is called serial if both modules $R_R$ and $R_R$ are direct sums of uniserial modules.

Theorem 3.2. Let $R$ be a right artinian ring and $A$, a class of right $R$-modules with local endomorphisms, containing all right simple right $R$-modules and closed under isomorphisms. If all right $R$-modules are $A$-injective, then the following conditions are equivalent for a ring $R$:

(1) $R$ is a serial artinian ring with $J^2(R) = 0$.
(2) Every $A$-C3 module is quasi-injective.
(3) Every $A$-C3 module is C3.

Proof. (1) ⇒ (2) Assume that $R$ is an artinian serial ring with $J^2(R) = 0$. Then every right $R$-module is a direct sum of a semisimple module and an injective module. Furthermore, every injective module is a direct sum of cyclic uniserial modules. Let $M$ be an $A$-C3 module. We can write $M = (\bigoplus_{i \in I} S_i) \oplus (\bigoplus_{j \in J} E_j)$ where each $S_i$ is simple if $i \in I$ and $\bigoplus_{j \in J} E_j$ is injective where each $E_j$ is cyclic uniserial non-simple if $j \in J$. Note
that any \( E_j \) has length at 2 by [4, 13.3]. We show that \( M \) is a quasi-injective module. To show that \( M \) is quasi-injective, by [16, Proposition 1.17] it suffices to show that \( \bigoplus I S_i \) is \( \bigoplus J E_j \)-injective. By [16, Theorem 1.7], \( \bigoplus I S_i \) is \( \bigoplus J E_j \)-injective if and only if \( S_i \) is \( \bigoplus J E_j \)-injective for all \( i \in I \). Furthermore, for any \( i \in I \), if \( S_i \) is \( E_j \)-injective for all \( j \in J \), then \( S_i \) is \( \bigoplus J E_j \)-injective by [16, Proposition 1.5]. So, it suffices to show that \( S_i \) is \( E_j \)-injective for each \( i \in I \) and \( j \in J \). Suppose that \( E_j \) has a series \( 0 \subset X \subset E_j \). Let \( f : A \rightarrow S_i \) be a homomorphism with \( A \leq E_j \). If \( A = 0 \) or \( A = E_j \) then it is obvious that \( f \) is extended to a homomorphism from \( E_j \) to \( S_i \). Assume that \( A = X \). If \( f \) is non-zero, then \( X \cong S_i \). As \( M \) is an \( A \)-C3 module, \( X \) is a direct summand of \( M \). It follows that \( X = E_j \), a contradiction. Hence \( S_i \) is \( E_j \)-injective and so \( M \) is quasi-injective.

(2) \( \Rightarrow \) (3) This is clear.

(3) \( \Rightarrow \) (1) Let \( M \) be an indecomposable module. If \( M \in A \), then it is quasi-injective. Now, suppose that \( M \not\in A \) and let \( \iota : M \rightarrow E(M) \) be the inclusion. Then, by Lemma 3.1, \( M \oplus E(M) \) is \( A \)-C3 and by assumption, \( M \oplus E(M) \) is a C3-module. It follows that \( \text{Im}(\iota) \) is a direct summand of \( E(M) \) by [4, Proposition 2.3]. Hence \( M \) is injective. Inasmuch as every indecomposable right \( R \)-module is quasi-injective, we infer from [9, Theorem 5.3] that \( R \) is an artinian serial ring. By [5, Theorem 25.4.2], every right \( R \)-module is a direct sum of uniserial modules. Now, by [4, 13.3], we only need to show that each uniserial module, say \( M \), has length at most 2. Suppose that \( M \) has a series \( 0 \subset X \subset Y \subset M \) of length 3. Assume that \( Y \in A \). Then \( X \) is \( Y \)-injective and hence \( X \) is a direct summand of \( Y \), a contradiction. It follows that \( Y \not\in A \). By Lemma 3.1, \( M \oplus Y \) is an \( A \)-C3 module and then, by hypothesis, is a C3-module. Consequently, the natural inclusion, \( \eta : Y \rightarrow M \) splits; i.e. \( Y \subset_d M \) and so \( Y = M \), a contradiction. Hence, \( R \) is an artinian ring with \( J^2(R) = 0 \).

\[ \square \]

**Theorem 3.3.** Let \( R \) be a right artinian ring and \( A \), a class of right \( R \)-modules with local endomorphisms, containing all right simple right \( R \)-modules and closed under isomorphisms. If all right \( R \)-modules are \( A \)-projective, then the following conditions are equivalent for a ring \( R \):

1. \( R \) is a serial artinian ring with \( J^2(R) = 0 \).
2. Every \( A \)-D3 module is quasi-projective.
3. Every \( A \)-D3 module is D3.

**Proof.** By Lemma 3.1 and [13, Theorem 4.4]. \[ \square \]

**Proposition 3.4.** Let \( A \) be a class of right \( R \)-modules and closed under isomorphisms and summands. Then the following conditions are equivalent:

1. All modules \( A \in A \) are injective.
2. Every right \( R \)-module is \( A \)-C3.

**Proof.** (1) \( \Rightarrow \) (2) is obvious.
(2) ⇒ (1). Suppose that \( A \in \mathcal{A} \). Then by (2), \( A \oplus E(A) \) is an \( \mathcal{A} \)-C3 module. Call \( \iota : A \to E(A) \) the inclusion map. By Proposition 2.11, \( \text{Im}(\iota) = A \) is a direct summand of \( E(A) \). Thus \( A = E(A) \) is an injective module.

\[ \square \]

**Corollary 3.5 (\[6\]).** The following conditions are equivalent for a ring \( R \):

1. \( R \) is a right \( \mathcal{V} \)-ring.
2. Every right \( R \)-module is simple-direct-projective.

**Proposition 3.6.** Let \( \mathcal{A} \) be a class of right \( R \)-modules and closed under isomorphisms and summands. Then the following conditions are equivalent:

1. All modules \( A \in \mathcal{A} \) are projective.
2. Every right \( R \)-module is \( \mathcal{A} \)-D3.

**Proof.** (1) ⇒ (2). Assume that \( M \) is a right \( R \)-module. Let \( M_1, M_2 \) be submodules of \( M \) with \( M/M_1, M/M_2 \in \mathcal{A} \) and \( M = M_1 + M_2 \). It follows that \( M/M_1, M/M_2 \) are projective modules and the following isomorphism

\[ M/(M_1 \cap M_2) = (M_1 + M_2)/(M_1 \cap M_2) \simeq M/M_1 \times M/M_2. \]

Then \( M/(M_1 \cap M_2) \) is a projective module. We deduce that \( M_1 \cap M_2 \) is a direct summand of \( M \). It shown that \( M \) is an \( \mathcal{A} \)-D3 module.

(2) ⇒ (1). Suppose that \( A \in \mathcal{A} \). Call \( \phi : R^{(I)} \to A \) an epimorphism. Then \( R^{(I)} \oplus A \) is an \( \mathcal{A} \)-D3 module. By Proposition 2.6, \( A \) is isomorphic to a direct summand of \( R^{(I)} \). Thus \( A \) is a projective module.

\[ \square \]

**Corollary 3.7 (\[6\]).** The following conditions are equivalent for a ring \( R \):

1. \( R \) is a semisimple artinian ring.
2. Every right \( R \)-module is simple-direct-projective.

Let \( M \) be a right \( R \)-module. \( M \) is called regular if every cyclic submodule of \( M \) is a direct summand. A right \( R \)-module is called \( M \)-cyclic if it is isomorphic to a factor module of \( M \).

**Lemma 3.8.** Let \( F \) be a regular module. Assume that \( A \neq 0 \) is a small finitely generated submodule of the factor module \( F/F_0 \) for some submodule \( F_0 \) of \( F \) and \( \mathcal{A} \) the class of all modules isomorphism to \( A \). Then there exists a \( F \)-cyclic module \( M \) and satisfies the property: there is a submodule \( N \in \mathcal{A} \) of \( M \) that is isomorphic to a direct summand of \( M \) and not a direct summand.

**Proof.** By the hypothesis we have \( ((x_1R + x_2R + \cdots + x_mR) + F_0)/F_0 = A \) for some \( x_1, x_2, \ldots, x_m \) of \( F \). Since \( F \) is a regular module, \( x_1R + x_2R + \cdots + x_mR = \pi(F) \), where \( \pi \in \text{End}(F) \) and \( \pi^2 = \pi \). Since \( A \) is a small submodule of \( F/F_0 \), we have \( F/F_0 = ((1-\pi)F + F_0)/F_0 \). It follows that there exist epimorphisms \( f_1 : \pi(F) \to A \), \( f_2 : (1-\pi)(F) \to F/F_0 \). It is easy to check \( A \oplus (F/F_0) \) is an \( F \)-cyclic module. Call \( M = A \oplus (F/F_0) \). Thus, the module \( N := 0 \oplus A \simeq A \) is not a direct summand of \( M \) and isomorphic to a direct summand of \( M \).

\[ \square \]
A module $M$ is called a $V$-module if every simple module in $\sigma[M]$ is $M$-injective (see [19]). $R$ is called a right $V$-ring if the right module $R_R$ is a $V$-module.

**Theorem 3.9.** The following conditions are equivalent for a regular module $F$:

(1) $F$ is a $V$-module.

(2) Every $F$-cyclic module $M$ is an $A$-$C_3$ module where $A$ is the class of all simple submodules of $M$.

**Proof.** The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $S \in \sigma[F]$ is a simple module and $E_F(S)$ is the injective hull of $S$ in the category $\sigma[F]$. Assume that $E_F(S) \neq S$. As $E_F(S)$ is generated by $F$, there exists a homomorphism $f : F \to E_F(S)$ such that $f(F) \neq S$. Then $S$ is a small submodule of $f(F) \cong F/\text{Ker}(f)$. Call $A$ the class of all modules isomorphism to $S$. By Lemma 3.8, there exists a $F$-cyclic module $M$ and satisfies the property: there is a submodule $N \in A$ of $M$ that is isomorphic to a direct summand of $M$ and not a direct summand. We infer from Proposition 2.15 that $M$ is not an $A$-$C_3$ module. This contradicts the condition of (2). $\square$

**Corollary 3.10** ([6, Theorem 4.4.]). A regular ring $R$ is a right $V$-ring if and only if every cyclic right $R$-module is simple-direct-injective.

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