Hyperbolicity of Divergence Cleaning and Vector Potential Formulations of GRMHD

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(Dated: December 11, 2018)

We examine hyperbolicity of general relativistic magnetohydrodynamics with divergence cleaning, a flux-balance law form of the model not covered by our earlier analysis. The calculations rely again on a dual-frame approach, which allows us to effectively exploit the structure present in the principal part. We find, in contrast to the standard flux-balance law form of the equations, that this formulation is strongly hyperbolic, and thus admits a well-posed initial value problem. Formulations involving the vector potential as an evolved quantity are then considered. Carefully reducing to first order, we find that such formulations can also be made strongly hyperbolic. Despite the unwieldy form of the characteristic variables we therefore conclude that of the free-evolution formulations of general relativistic magnetohydrodynamics presently used in numerical relativity, the divergence cleaning and vector potential formulations are preferred.

I. INTRODUCTION

It is well appreciated [1, 2] that the numerical modeling of binary neutron star spacetimes plays, and will continue to play, an important role in the new field of gravitational wave astronomy, particularly in the case of multi-messenger events. Such simulations are, however, hampered by relatively poor error behavior as compared with their vacuum, black hole counterparts. This is in part because the equations of motion of these models have a more complicated structure than those of pure general relativity, and are hence less well understood, but also because solutions naturally develop non-smooth features, not to mention the ever present complication of the stellar surface.

In a recent contribution [3] we employed a new tool, the dual-frame formalism [4–7], to analyze well-posedness of various fluid models. Well-posedness is the weakest necessary condition to require of a set of evolution partial differential equations (PDE) so that numerical approximation to their solutions may be meaningfully sought. The formalism can be used to exploit structure in the field equations and hence simplifies earlier treatments. This should allow more sophisticated results to be shown in the future.

One of the models treated in Ref. [3] was (ideal) general relativistic magnetohydrodynamics (GRMHD), taken in two different guises. In the Valencia flux-balance law form [8] we found that the field equations are only weakly hyperbolic, and therefore have an ill-posed initial value problem. Here we attend to two flavors of GRMHD un-touched by our earlier study, namely the hyperbolic divergence cleaning (HDC) and vector potential (VP) formulations. Our main result is that both are strongly hyperbolic, provided suitable choices are made in the first-order reduction of the latter.

We work in 3+1 dimensions in geometric units with $c = G = 1$. Our calculations were performed primarily with xTensor for Mathematica [9]; our notebooks are available online in Ref. [10].

II. MATHEMATICAL BACKGROUND

We start with a short overview of the relevant theory, definitions, and results to the PDE analysis and the dual-frame (DF) formalism. These are taken in a highly summarized form from Refs. [2, 3, 7].

Index notation. Latin letters $a–e$ are used as abstract indices. We also use $p$ as an abstract index, placing it always on the spatial derivative appearing on the right-hand side of our first-order PDE system. The four-metric $g_{ab}$ is the only object permitted to raise and lower indices. The symbol $\partial_a$ stands for a flat covariant derivative. Indices $u, S, s, \hat{s}$ and \hat{s} label contraction in that slot with $u^a$ or $\nu_a$ and so on, respectively. Capital Latin letters $A–C$ are taken as abstract indices and denote application of the projection operators $q_{\perp}$ and $q_{\perp}$, to be defined later. Similarly, we use indices $A–C$ and $\hat{A}–\hat{C}$ to denote the application of the projection operator $q_{\perp}$ over a vector or dual-vector, respectively.

DF formalism. We describe a region of spacetime in two different frames, namely the lowercase and the uppercase frame. We take the lowercase frame as an Eulerian frame, associated with a coordinate basis as is standard in numerical relativity. We denote the future pointing timelike unit vector to spatial slices of constant time, as usual, by $n^a$. Additionally, we take any three linearly independent vector fields orthogonal to $n^a$ to form a basis of the four-dimensional spacetime. Tensors orthogonal to $n^a$ are called lowercase spatial, or just lowercase. The uppercase frame consists of a future pointing timelike unit normal vector to spatial slices of constant time, by $n^a$. Additionally, we take any three linearly independent vector fields orthogonal to $n^a$ to form a basis of the four-dimensional spacetime. Tensors orthogonal to $n^a$ are likewise called uppercase spatial, or just uppercase. The future pointing unit vectors of the lower- and uppercase frames can be mutually $3+1$ decomposed as

$$n^a = W(N^a + V^a), \quad N^a = W(n^a + v^a), \quad (1)$$
with the Lorentz factor \( W = (1 - V^a v_a)^{-1/2} = (1 - v^a v_a)^{-1/2} \). The vectors \( v^a = \hat{v}^a / W \) and \( V^a \) are the boost vectors orthogonal to \( n^a \) and \( N^a \), respectively. We define projection operators by
\[
\gamma^b_a = \delta^b_a + n^b n_a, \quad (\gamma)_{ab} = \delta^b_a + N^b N_a, \quad (2)
\]
which are also denoted as the lowercase and uppercase spatial metrics, respectively. By definition, the relations \( \gamma^b_a n_a = 0, (\gamma)^{ab} N_b = 0 \) hold. We define furthermore the lowercase and uppercase boost metrics and their inverses, which are presented in Table I.

**PDE analysis.** We consider a quasilinear system of first order evolution PDEs, in this case GRMHD with HDC, written in the form
\[
\nabla a U = A^p \nabla_p U + S, \quad (3)
\]
with the covariant derivative along the streamlines of the fluid elements \( \nabla_a \equiv u^a \nabla_a \) of the vector of evolved variables, called the state vector \( U \), on the left-hand side. On the right-hand side, the covariant derivative of the state vector is contracted with the principal part \( A^p \). The symbol \( S \) stands for the source term which does not affect the level of hyperbolicity. We need only analyze the system of evolution equations for the matter variables, since they are minimally coupled to the Einstein equations for the components of the metric tensor.

**Strong hyperbolicity.** For the hyperbolicity analysis, we have to perform a 2 + 1 decomposition against lower- and uppercase spatial variables and their respective orthogonal spatial projectors. The relevant quantities are defined in Table I. Taking an arbitrary uppercase unit spatial 1-form \( S_a \), we define the uppercase principal symbol of the system \( (3) \) as
\[
P^a \equiv A^p S_p. \quad (4)
\]
We call the system \( (3) \) weakly hyperbolic, if for each \( S_a \) the eigenvalues of \( P^a \) are real. We call the system \( (3) \) strongly hyperbolic, if the system is weakly hyperbolic and for each \( S_a \) the principal symbol \( P^a \) has a complete set of right eigenvectors written as columns in a matrix \( T_S \) and there exists a constant \( K > 0 \), independent of \( S_a \), such that \( |T_S| + |T_S^T| \leq K \). Similar definitions are made if we \( 3 + 1 \) decompose the system against \( n^a \) rather than \( u^a \), and the initial value problem, where data is given at \( t = 0 \), can be well-posed only if it satisfies these lowercase strong hyperbolicity conditions \([11,13]\).
can be expressed in terms of $\rho_0$, $p$, and the specific internal energy $\varepsilon$ as

$$h = 1 + \varepsilon + \frac{p}{\rho_0}.$$  \hfill (7)

The local speed of sound $c_s$ is defined by the relation

$$c_s^2 = \frac{1}{\rho} \left( \chi + \frac{p}{\rho_0^2} \right), \quad \chi = \left( \frac{\partial p}{\partial \rho_0} \right), \quad \kappa = \left( \frac{\partial p}{\partial \varepsilon} \right)_{\rho_0}.$$  \hfill (8)

We assume an equation of state (EOS) of the form

$$p = p(\rho_0, \varepsilon),$$  \hfill (9)

with $p > 0$ is given satisfying furthermore that the local speed of sound lies in the range $0 < c_s \leq 1$.

Using the ideal MHD condition, where the electric conductivity tends to infinity while the electric four-current remains bounded, the field strength tensor and its dual become

$$F^{ab} = \epsilon^{abcd} u_c b_d,$$  \hfill (10)

$$*F^{ab} = u^a b^b - b^a b^b,$$  \hfill (11)

respectively, where we introduced the upper case magnetic field vector $b^a$, satisfying $u_a b^a = 0$; and the Levi-Civita tensor

$$\epsilon^{abcd} = -\frac{1}{\sqrt{-g}} [abcd],$$  \hfill (12)

where $g$ is the determinant of the spacetime metric $g_{ab}$, $[abcd]$ is the completely antisymmetric Levi-Civita symbol, and $2^*F^{ab} = -\epsilon^{abcd} F_{cd}$ holds. Note that we use the sign convention of Ref. 17.

Taking the sum of Eqs. (4) and (5), and substituting the field strength tensor (10), the total energy-momentum tensor of GRMHD may be written as

$$T^{ab} = \rho_0 h u^a u^b + p^* g^{ab} - b^a b^b,$$  \hfill (13)

with $h^* = h + b^2 / \rho_0$, $p^* = p + b^2 / 2$, and shorthand $b^2 = b^a b^a$.

The covariant system of evolution equations is given by the conservation of the number of particles and the conservation of energy-momentum,

$$\nabla_a (\rho_0 u^a) = 0,$$  \hfill (14)

$$\nabla_b T^{ab} = 0,$$  \hfill (15)

plus the relevant Maxwell equations

$$\nabla_b (F^{ab} - g^{ab} \phi) = -\frac{1}{\tau} n^b \phi,$$  \hfill (16)

which are already augmented by the terms with the scalar field $\phi$ to drive the Gauss constraint. Elsewhere the notation $\kappa = \tau^{-1}$ is employed. The constant $\tau$ is the timescale for the exponential driving toward the Gauss constraint for the magnetic field. Usually, $\phi$ is set to zero in the initial and boundary conditions.

### IV. HYPERBOLICITY ANALYSIS OF GRMHD WITH HDC

Projecting Eqs. (12)–(16) along the four velocity of the fluid $u^a$ and perpendicular to it by $(\omega^a)^{ab}_b$, the nine evolution equations which determine the time evolution of the GRMHD system with HDC are

$$\nabla_a (\rho_0 u^a) = 0, \quad (\omega^a)_{ab} \nabla_c T^{bc} = 0,$$  \hfill (17)

supplemented with an EOS [9]. In the limit of $\phi \to 0$ we find the upper case Gauss constraint: $(\omega^a)^{bc} \nabla_d b_a = u_a \nabla^b (F^{bc} - g^{bc} \phi) = 0$.

Taking Eq. (17) and performing algebraic manipulations similar to the investigation of other formulations of GRMHD in Ref. 8, we derive the evolution equations for the pressure,

$$\nabla_a p = -e^2 \rho_0 h (\omega^a)^{bc}(\mathbf{g}^{-1})^{ce} \nabla_p \hat{v}_e + \frac{k}{\rho_0} b^p \nabla_p \phi + S^{(p)},$$  \hfill (18)

the boost vector,

$$\nabla_a (\mathbf{g}^{-1})^{bc} \nabla_u \hat{v}_c = - \left( \frac{b^p b_a}{\rho_0^{1/2} h^{1/2}} + \frac{(\omega^a)^{bc}}{\rho_0^{1/2} h^{1/2}} \right) \nabla_p p + \left( \frac{2}{\rho_0 h} \right)^{(\omega^a)^{bc}} (\omega^a)^{bc} \nabla_p b_a + S^{(\hat{v})},$$  \hfill (19)

the magnetic field,

$$\nabla_a (\mathbf{g}^{-1})^{bc} \nabla_u b_c = (\omega^a)^{bc} (\omega^a)^{bc} \nabla_p \hat{v}_c - (\omega^a)^{bc} \nabla_p \phi + S^{(b)};$$  \hfill (20)

the specific internal energy,

$$\nabla_u \varepsilon = - \frac{p}{\rho_0} (\omega^a)^{bc} (\mathbf{g}^{-1})^{ce} \nabla_p \hat{v}_c + \frac{b^p}{\rho_0} \nabla_p \phi + S^{(\varepsilon)},$$  \hfill (21)

and finally the scalar field variable,

$$\nabla_u \phi = - (\omega^a)^{bc} (\mathbf{g}^{-1})^{ce} \nabla_p b_c + S^{(\phi)}.$$  \hfill (22)
The auxiliary magnetic vector \( \gamma_a \) is defined by the relation
\[
(\gamma_a)_a := \gamma_a \nabla_{b\perp} \phi, \quad (\gamma_a)_a = (\gamma_a)_{a}^{\perp}. \quad (23)
\]

As usual, square brackets around indices denote antisymmetrization, so that \( 2\epsilon^{[ab]} = \epsilon^{ab} - \epsilon^{ba} \). We have shown explicitly that the set of equations (18)-(22) is, up to non-principal terms, which we have not carefully checked, simply a linear combination of the formulation of GRMHD with HDG used in numerical applications, see, for example, Ref. [10]. This verification can be found in the notebook that accompanies the paper [10].

Writing Eqs. (18)-(22) in a vectorial form with state vector \( \mathbf{U} = (\rho, \mathbf{v}, \mathbf{B}, \mathbf{s})^T \), we obtain, in the notation of Ref. [8], the principal part in the form,
\[
\mathbf{B}^\top \nabla_a \mathbf{U} = \mathbf{B}^\top \nabla_a \mathbf{U} + \mathbf{S}. \quad (24)
\]

Let \( S_a \) be an arbitrary unit spatial uppercase L-form, \( S_a S^a = 1 \), and \( \mathbf{q}_+ := (\gamma_a)_a - S^a S_a \) be the associated orthogonal projector. Let \( \mathbf{s}_+ \) and \( \mathbf{q}_+ \) be their lowercase projected versions, \( \mathbf{s}_+ = \gamma^a S_a, \mathbf{q}_+ = \gamma^a - (\mathbf{e})^a \mathbf{s}_+ \). Decomposing \( (\gamma_a)^b \) and \( \gamma^b \) against \( S_a \) and \( \mathbf{s}_+ \), respectively, Eq. (24) can be written as
\[
(\nabla_a \mathbf{U})_{\mathbf{s}_+} \simeq \mathbf{P}^S (\nabla_a \mathbf{U})_{\mathbf{s}_+}, \quad (25)
\]
where \( \simeq \) denotes equality up to non-principal terms and uppercase spatial derivatives transverse to \( S^a \). The uppercarse principal symbol is \( \mathbf{P}^S = \mathbf{B}^S = \)
\[
\begin{pmatrix}
0 & -c_r S_{rb}^p h & 0 & 0 & 0 & \frac{b^b h^c}{\rho_0} \\
\frac{b^b h^c}{\rho_0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{b^b h^c}{\rho_0} h & -\frac{b^b h^c}{\rho_0} h & 0 & 0 \\
-\frac{b^b h^c}{\rho_0} h & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & -b_A^A & \mathbf{Q}_+& 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad (26)
\]
with the state vector ordered as,
\[
(\delta \mathbf{U})_{\mathbf{s}_+} = (\delta \rho, (\delta \mathbf{v})_{\mathbf{s}_+}, (\delta \mathbf{B}), (\delta \mathbf{s}_+)^T. \quad (27)
\]

The characteristic polynomial \( P_\lambda \) for the principal symbol (26) is calculated to
\[
P_\lambda = \lambda^4 - \lambda^2 - \lambda^2 + \lambda^2 - 1 \quad (28)
\]
with the quadratic polynomial for Alfvén waves
\[
P_{\text{Alfvén}} = (b^b)^2 + \lambda^2 - \lambda^2 - \rho_0 h^* \quad (29)
\]
and the quartic polynomial for the magnetosonic waves
\[
P_{\text{mgs}} = (\lambda^2 - 1) (\lambda^2 - (b^b)^2)^2 + \lambda^2 (\lambda^2 - c_s^2) \rho_0 h. \quad (30)
\]
Comparing Eq. (30) with our earlier results for the flux-balance law formulation of GRMHD in Ref. [8], we see that the linear polynomial associated with the Gauss constraint is replaced by the quadratic polynomial \( 1 - \lambda^2 \).

The entropy, Alfvén, and slow and fast magnetosonic uppercarse eigenvalues remain the same, as before, and are given by
\[
\lambda_{(e)} = 0, \quad \lambda_{(a)} = \pm \frac{b^b}{\sqrt{\rho_0 h^*}}, \quad (31)
\]
respectively, with shorthands
\[
\lambda_{(e)} = \pm \sqrt{\lambda_{(e)} \lambda_{(e)} - \lambda_{(e)}}, \quad \lambda_{(a)} = \pm \sqrt{\lambda_{(a)} \lambda_{(a)} - \lambda_{(a)}}, \quad (31)
\]

The remaining two speeds can be associated with the scalar field and the longitudinal magnetic field [16], and are given by
\[
\lambda_{\pm} = \pm 1. \quad (33)
\]
Since all upuppercarse eigenvalues have absolute value smaller than or equal to one, the relation \( |\lambda_{\pm}| V < 1 \) is satisfied, so we may analyze hyperbolicity independently of the frame [8]. Therefore, we analyze the characteristic structure of the principal symbol in the upuppercarse frame and the result of the analysis applies directly to the numerically used system (in the lowercase).

Continuing the characteristic analysis, we find the left entropy, scalar field and longitudinal magnetic field, Alfvén, and magnetosonic eigenvectors being
\[
(\rho_0 - \frac{\rho p}{c_s^2 \rho_0 h}) \frac{\rho^2 \rho_0}{c_s^2} 0 0 0^A (0 0 0^A 0 0 1 0),
(0 0 0^A \pm 1 0 0 0 0),
(0 0 \mp \sqrt{c_s^2 AC} b_C \sqrt{\rho_0 h} 0 0),
\]
respectively, where we defined the antisymmetric uppercase two- and three-Levi-Civita tensors as \( \varepsilon^{AB} = S_d^{\nu} \delta^{AB} = \pm \delta^{AB} \). We employ furthermore the shorthands
\[
K = \left( \rho h^* (\lambda_{m(\pm)}^*)^2 - b^2 \right),
L = \frac{\rho h^* (\lambda_{m(\pm)}^*)^2 - (b^2)(\lambda_{m(\pm)}^*)^2}{(1 - c_s^2 \rho_0 h^* \lambda_{m(\pm)}^*)^2}.
\]
The right eigenvectors can be computed and are presented in the same order,
\[
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\mp \rho h (\lambda_{m(\pm)}^*)^2 \rho_0 h^* (\lambda_{m(\pm)}^*)^2 - 2 \zeta S) \left( b^2 + \rho h^* (\lambda_{m(\pm)}^*)^2 - 2 \zeta S \right) b_B \\
\rho_0 \lambda_{m(\pm)}^* \rho h^* \rho_0 \lambda_{m(\pm)}^* \rho h^* b_B \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]
with the abbreviations in Eq. (37) given by
\[
H = \frac{|b_\perp|}{c_s^2 - \lambda^2}^2,
F^A = \frac{b^A}{(\rho h^* \lambda^2 - b^2)^2},
\]
where for type II and even for type II' degeneracy we take \( Q_1^a \) and \( Q_2^a \) such that in the degenerate limit we have
\[
\frac{b_\perp^2}{|b_\perp|^2} = \frac{1}{\sqrt{2}} (Q_1 C + Q_2 C),
H = 0,
F^A = 0^A.
\]
For further explanations concerning degeneracies and rescaling, see also Ref. [18].
Using the recovery procedure given in Ref. [3], the lowercase characteristic quantities such as eigenvalues and eigenvectors can be derived. The calculation can be found in the notebook [10], but results in rather long expressions which we suppress here. Both the lowercase left magnetosonic eigenvectors and the lowercase right eigenvectors associated with the scalar field and longitudinal magnetic field eigenvalues have a particularly complicated structure, for which a useful simplification seems difficult. In applications it may therefore be appropriate to compute the characteristics numerically.

V. DISCUSSION OF FORMULATIONS OF GRMHD WITH VP

The formulations of GRMHD we have thus far considered use the magnetic field as an evolved variable. Another possibility is to introduce the four-vector potential instead [19–21]. In practice, the potential is then $3 + 1$ decomposed. Such formulations have the advantage that the Gauss constraint is satisfied by construction, and is not minimally coupled rather than free-evolution. On the other hand one obtains a system of equations which is a priori not, from the PDE point of view, minimally coupled to the gravitational field equations. The resulting evolution equations for the GRMHD variables are moreover themselves not in first-order form, but rather first order in time and second order in space, and there is an additional gauge degree of freedom. Different choices in this freedom may have different PDE properties as the principal part of the evolution system is altered. We follow Ref. [20] and focus on the Lorenz gauge, but similar comments will hold elsewhere. Strong hyperbolicity of first-order in time, second order in space systems can be defined [22, 23] by the requirement that there exists a first order reduction which satisfies the definition given for first order PDEs in Sec. [11]. Therefore, we must reduce the governing system of equations as in Eq. [3], by introducing reduction variables. There are two natural ways to go about this.

The first, naive possibility is to introduce reduction constraints $c_{ab} = d_{ab} - \gamma^c a \gamma^d b \partial_c A_d$, which should vanish, for the lowercase spatial derivatives of the lowercase spatial part of the vector potential $A_a$, and likewise for the electric potential. The reduction variables $d_{ab}$ should satisfy also the ordering constraint,

$$\gamma^d c_{[ab]} f = 0 ,$$

and similarly for the electric potential reduction variables. The reduction constraints must then be added to the equations of motion to remove all second spatial derivatives. Besides that, both the reduction and ordering constraints can be added freely to try and find a hyperbolic reduction. Such a reduction does not use the special structure of the Maxwell equations, does not utilize the fact that the original system satisfies the Gauss constraint by construction, and is not minimally coupled to the evolution equations for the geometric variables. Worse, the resulting principal symbol does not have a clear structure, which makes the analysis very difficult.

The less obvious option is to bring back the magnetic field as a reduction variable for the curl of the spatial vector potential by defining a reduction constraint,

$$C_n = \epsilon_{abc} D_b A_c - B_a .$$

(44)

In this reduction we need not introduce a reduction variable to the electric potential as it appears with at most one spatial derivative. Part of the analog of the ordering constraint in such a reduction turns out to be simply the Gauss constraint,

$$C = -D_a C^a = D_a B^a .$$

(45)

A generic PDE system does not allow a reduction of this type, in which new variables that only capture part of the spatial derivatives are introduced. Due to the gauge freedom of the Maxwell equations however the ‘longitudinal’ part of the vector potential does not appear elsewhere in the remaining equations of motion, and so we can close the evolution system using only $B_a$. Note that such a restricted reduction does have consequences on the norms in which rigorous estimates would be demonstrated, and also that as usual first derivatives of the metric here are non-principal.

Ultimately we end up with evolution equations for the matter variables which are minimally coupled to the Einstein equations. Naively writing out the lowercase principal symbol of the matter variables we can obtain moreover a block-diagonal structure,

$$P^a = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ,$$

(46)

where block $A$ denotes the principal symbol of the system of evolution equations of the spatial part of the vector potential and the electric potential, whereas $B$ can be rendered identical to the principal symbol of the prototype algebraic constraint free formulation of GRMHD investigated in Ref. [3]. Here, crucially, we rely on the fact that, as it is not to be used in applications, this formal first order reduction need not be of a flux-balance form, and therefore we can add the ordering constraint $C$ as desired. The upper right block vanishes trivially and the lower left block vanishes by appropriate choice of reduction. We showed already that prototype algebraic constraint free formulation of GRMHD is strongly hyperbolic in the lowercase frame, with an EOS of the form [10] and $0 < c_s \leq 1$, so all that remains is to show that the block $A$ satisfies the conditions for strong hyperbolicity. This was done already in Ref. [20], but with the use of the reduction variable $B_a$ we can give a slightly cleaner treatment. The lowercase principal symbol can be read off from,

$$\nabla_n \Phi \simeq -\gamma^p \nabla_p A_e ,$$

(47)

$$\gamma^b A_b \simeq -\gamma^p \nabla_p \Phi .$$

(48)
Note that in Eq. (48) the term $D_n A_b - D_b A_n$ is written in terms of the reduction variable $B_n$ and does not contribute to the principal part. Let $s^2, s^a s^a = 1$, be an unit spatial lowercase vector and be $\Psi \perp s^a$, the orthogonal projector. The characteristic variables associated with this block are hence,

$$\delta \Phi \mp \delta (A)_s,$$

with speeds $\pm 1$, respectively, and,

$$\delta (A)_A,$$

with speed 0 for the two orthogonal directions to $s^a$. The calculation is provided in a notebook that accompanies the paper [10].

VI. CONCLUSION

In previous work [3] we examined two formulations of ideal GRMHD, and showed that a formulation similar to that studied in Refs. [14, 23], which we call the prototype algebraic constraint free formulation is strongly hyperbolic. Unfortunately, this formulation is not in the flux-balance law form desirable for the application of standard numerical methods. Turning to GRMHD in flux-conservative form, we found the system to be only weakly hyperbolic. This formulation of GRMHD hence has an ill-posed initial value problem. Fortunately, two popular, applicable, alternative formulations of GRMHD were left untreated by that analysis. Presently, we have addressed this shortcoming with the outcome first, that formulations of GRMHD with HDC [16, 25, 26] are indeed strongly hyperbolic as long as the sound speed is suitably bounded $0 < c_s < 1$. In fact, it is straightforward to achieve hyperbolicity also in the case $c_s = 1$ by changing the speed of the cleaning in the formulation. Second, we have shown that by a careful reduction to first order, formulations of GRMHD with VP [20] can also be rendered strongly hyperbolic whenever $0 < c_s \leq 1$. The latter result is a corollary of strong hyperbolicity of the prototype algebraic constraint free formulation. Here we have discussed only the Lorenz gauge choice, but our results carry over trivially to generalized Lorenz-gauge, in which there is a modification by source terms, and a natural treatment will be very similar in other cases, too.

Both HDC and the VP formulations were introduced as strategies to control Gauss-constraint violation in applications. Another popular approach, called constrained transport (CT) [27–29], uses a carefully constructed discretization so that in a particular approximation the constraint is identically satisfied. There is some subtlety in precisely what continuum PDE should be analyzed given such a constrained evolution, but supposing that the constraints are identically satisfied, they may again be added arbitrarily to the evolution equations, and strong-hyperbolicity can again be achieved, in the restricted, constraint-satisfying phase space, as a corollary of hyperbolicity of the prototype algebraic constraint free formulation.

In Ref. [3] we discussed two minimally coupled formulations of resistive GRMHD with HDC, one with and one without the evolution of the charge density $q$. Both were found to be only weakly hyperbolic. A natural question is therefore whether the use of the VP approach could cure this problem. Replacing the divergence cleaning variables by $A_a$ and $\Phi$, and making a minimally coupled first order reduction as we did for GRMHD, one arrives with a lower block triangular structure in the principal symbol, with the lower-right block $C$ being precisely a subblock of the principal symbol of the original formulation of RGRMHD. Neither of the original two formulations were strongly hyperbolic because $C$ was not diagonalizable. Consequently, the vector potential formulations are also not strongly hyperbolic. Thus, at least if we insist on taking only minimally coupled first order reductions, use of a VP reformulation of RGRMHD does nothing to circumvent weak hyperbolicity of RGRMHD.

For numerical applications we therefore have the clear conclusion that, by the fundamental requirement of well-posedness, HDC and VP formulations (and likely also CT schemes) are preferred over their older variant which should henceforth be avoided. From the PDEs point of view it is, at this stage, difficult to choose between the favored formulations. One might be tempted to argue in favor of the vector potential formulation, as indeed it is true that there the characteristic structure, inherited from the prototype algebraic constraint free formulation, is simpler, but this is not a principle advantage. In the future it is hoped that the characteristic structure uncovered by our analysis can be put to good use in numerical work in both systems.

Acknowledgments

We are grateful to Sebastiano Bernuzzi and Bruno Giacomazzo for useful discussions and comments. This work was partially supported by the FCT (Portugal) IF Program No. IF/00577/2015 and the GWverse COST action Grant No. CA16104.

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