Nielsen equivalence in mapping tori over the torus

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Abstract

We use the geometry of the Farey graph to give an alternative proof of the fact that if \( A \in GL_2 \mathbb{Z} \) and \( G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) is generated by two elements, there is a single Nielsen equivalence class of 2-element generating sets for \( G_A \) unless \( A \) is conjugate to \( \pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), in which case there are two.

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1 Introduction

Let \( G \) be a finitely generated group. Two ordered \( n \)-element generating sets \( S, T \) for \( G \) are Nielsen equivalent if the associated surjections \( F_n \rightarrow G \) differ by precomposition with a free group automorphism. This is equivalent to requiring that \( S, T \) are related by a sequence of Nielsen moves:

1. if \( a \neq b \) are generators, replace \( a \) with \( ab \),
2. if \( a \neq b \) are generators, switch their places in the ordering.
3. if \( a \) is a generator, replace it with \( a^{-1} \),
as the associated automorphisms generate \( \text{Aut}(F_n) \), see [5, Chap. I, Prop. 4.1].

In [4], Levitt–Metaftsis studied Nielsen equivalence within groups of the form \( G_A = \mathbb{Z}^d \rtimes_A \mathbb{Z} \), where \( A \in GL_d \mathbb{Z} \). Using the Cayley–Hamilton theorem, they show that \( G_A \) is 2-generated exactly when there is a vector \( v \in \mathbb{Z}^d \) such that \( \langle v, Av \rangle = \mathbb{Z}^d \). They also show that the number of Nielsen equivalence classes of 2-element generating sets is the index of \( \langle A, -Id \rangle \) in its \( GL_d \mathbb{Z} \)-centralizer.

When \( d = 2 \), one can combine this with an observation of Cooper–Scharlemann [2, Lemma 5.1] to prove the following theorem.

**Theorem 1.1.** If \( A \in GL_2 \mathbb{Z} \) and \( G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) is 2-generated, there is a single Nielsen equivalence class of 2-element generating sets for \( G_A \) unless \( A \) is conjugate to \( \pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), in which case there are two.

Note that when \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), \( G_A \) is 2-generated, since \( \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle = \mathbb{Z}^2 \).
Our goal here is not to prove anything new, but rather to understand how to prove Theorem 1.1 using the geometry of the Farey graph $\mathcal{F}$. Algebraically, vertices of $\mathcal{F}$ are primitive elements $v = (p, q) \in \mathbb{Z}^2$ up to negation, and vertices $v, w$ are connected by an edge if together they generate $\mathbb{Z}^2$. Any matrix $A \in GL_2 \mathbb{Z}$ acts on $\mathcal{F}$, and it turns out that Nielsen equivalence classes of 2-element generating sets of $G_A$ correspond to geodesics in $\mathcal{F}$ on which $A$ acts as a unit translation, see §2. Using this perspective, one can then prove Theorem 1.1 just using separation properties of geodesics in $\mathcal{F}$.

In the paper referenced above, Cooper–Scharlemann were interested in an analogue of Theorem 1.1 in the world of Heegaard splittings. Recall that a closed surface $S$ in a closed, orientable 3-manifold is a Heegaard splitting if $M \setminus H$ has two components, each of which are (open) handlebodies. They showed that there is a unique minimal genus Heegaard splitting of $M_A$ up to isotopy unless $A$ is conjugate to $\pm \left( \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right)$, in which case there are two.

Any Heegaard splitting gives a pair of generating sets for $\pi_1 M$, just by taking free bases for the fundamental groups of the two handlebodies. These generating sets are well-defined up to Nielsen equivalence, and their Nielsen types certainly do not change if the Heegaard splitting $S$ is isotoped in $M$. However, in general it is hard to say when a generating set for $\pi_1 M$ is ‘geometric’, i.e. when its Nielsen class comes from a Heegaard splitting, and when two (say, nonisotopic) Heegaard splittings give the same Nielsen class, see e.g. Johnson [3].

However, inspired by the fact that the Cooper–Scharlemann result also applies when the minimal genus of a Heegaard splitting is 3, we ask:

**Question 1.** Is it true that if rank($G_A$) = 3, there is a single Nielsen equivalence class of 3-element generating sets?

Here, rank is the minimal size of a generating set. In [1], the author and Souto studied rank and Nielsen equivalence for mapping tori $M_\phi$, where $\phi : S \rightarrow S$ is a pseudo-Anosov homeomorphism of a closed orientable surface of genus $g \geq 2$. We showed that as long as $\phi$ has large translation distance in the curve complex $C(S)$, the group $\pi_1 M_\phi$ has rank $2g + 1$ and all minimal size generating sets are Nielsen equivalent.

From above, when $A \in GL_2 \mathbb{Z}$ the group $G_A$ has rank 2 exactly when there was some $v \in \mathbb{Z}^d$ such that $\langle v, Av \rangle = \mathbb{Z}^d$. The Farey graph is the curve graph of $T^2$, and $\langle v, Av \rangle = \mathbb{Z}^d$ exactly when $v, Av \in \mathcal{F}$ are adjacent, so in the Euclidean setting the analogue of the rank part of our theorem in [1] still holds, and says that rank($G_A$) = 3 if the translation distance of $A$ on $\mathcal{F}$ is at least two. The analogue of the Nielsen equivalence part is (a weaker version of) Question 1.

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2 The proof

We will first show that for a general \( A \in GL_2 \mathbb{Z} \), there can be at most two Nielsen equivalence classes of 2-element generating sets for \( G_A \). We’ll then show that the conjugates of \( (\frac{3}{1}) \) are the only \( A \) that realize this bound.

The beginning of this argument overlaps with that of Levitt–Metaftsis [4], so we will just outline it and give citations when necessary. Suppose that \( G_A \) is 2-generated. By [4, Proposition 4.1], every minimal size generating set for \( G_A \) is Nielsen equivalent to a generating set of the form

\[
\begin{align*}
x &= (v, 0), \\
y &= (0, 1),
\end{align*}
\]

where \( v \in \mathbb{Z}^2 \).

Set \( S_A = \{ v \in \mathbb{Z}^2 \mid \langle v, Av \rangle = \mathbb{Z}^2 \} \). Again by [4, Proposition 4.1], if \( v, v' \in S_A \), then \( \{(v, 0), (0, 1)\} \) and \( \{(v', 0), (0, 1)\} \) are Nielsen equivalent if and only if \( v, v' \) lie in the same \( \langle A \rangle \times \mathbb{Z}/2\mathbb{Z} \)-orbit on \( S_A \), where \( \mathbb{Z}/2\mathbb{Z} \) acts by \( v \mapsto -v \).

We now reinterpret this in terms of the Farey graph \( F \). Recall from the introduction that the vertex set of \( F \) consists of primitive elements of \( \mathbb{Z}^2 \) up to negation, so can be identified with \( \mathbb{Q} \cup \{\infty\} \) through the map

\[
\mathbb{Z}/\mathbb{Z}^2 \rightarrow \mathbb{Q}/\mathbb{Z}^2, \quad \pm \frac{a}{b} \mapsto \frac{a}{b}.
\]

Below, we will regard \( \mathbb{Q} \cup \{\infty\} \) as a subset of \( \mathbb{R} \cong \partial_{\infty} \mathbb{H}^2 \), where \( \mathbb{H}^2 \) is considered in the upper half plane model, and we will identify edges of \( F \) with the corresponding geodesics in \( \mathbb{H}^2 \). (See all figures below.) This embedding of \( F \) has some convenient properties. All edges of \( F \) separate \( \mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2 \), and also \( F \), into two connected components. Every component of \( \mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2 \setminus F \) is an ideal hyperbolic triangle, which we will call a complementary triangle below. Finally, the action of \( A \in GL_2 \mathbb{Z} \) on \( F \) is the restriction of its action on \( \mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2 \) as a fractional linear transformation.

Returning to the proof, vertices \( v, w \in F \) are adjacent if \( \langle v, w \rangle = \mathbb{Z}^2 \), so \( S_A \) is exactly the set of vertices in \( F \) that \( A \) translates a distance of 1. Also, in the Farey graph we have identified primitive pairs up to negation, so the action of \( \langle A \rangle \times \mathbb{Z}/2\mathbb{Z} \) on \( S_A \) is just the \( A \)-action on the corresponding set of vertices of \( F \). Define a 1-orbit of \( A \cap F \) to be an orbit all of whose points are translated a distance of 1 by \( A \). Theorem [1.1] then becomes the following lemma.

Lemma 2.1. The action of \( A \cap F \) has a single 1-orbit unless \( A \) is conjugate to \( \pm (\frac{3}{1}) \), in which case it has two.

Fix a matrix \( A \in GL_2 \mathbb{Z} \) and let \( \ell \) be a 1-orbit of \( A \). Adding in edges connecting each \( v \in \ell \) to \( Av \), we will regard \( \ell \) as an oriented path in \( C(T^2) \). At each of its vertices \( v \), a path \( \ell \) has a turning number, whose absolute value is one more than the number of Farey graph edges that separate the two edges of \( \ell \) incident to \( v \). The turning number at \( v \) is positive if the turn is counterclockwise when \( \ell \) is traversed positively, and negative when the turn is clockwise. (Remember that we are viewing \( F \) as a subset of the upper half plane in \( \mathbb{R}^2 \).) When \( v = \infty \),
the turning number is just $A(v) - A^{-1}(v)$. For instance, in Figure 2 all turning numbers on the red 1-orbit are 3, on the blue 1-orbit they are $-3$.

When $A$ is orientation preserving, all the turning numbers on a given 1-orbit coincide. On the other hand, if $A$ is orientation reversing then the turning numbers on a 1-orbit all have the same absolute value and alternate sign. As $GL_2 \mathbb{Z}$ acts edge transitively on $\mathcal{F}$, any 1-orbit of $A$ may be translated to pass through $\infty, 0$, which conjugates $A$ so that it has the form

$$A = \begin{pmatrix} 0 & \epsilon \\ 1 & x \end{pmatrix}, \ x \in \mathbb{Z}, \ \epsilon = \pm 1. \quad (1)$$

When $A$ is as above, the turning number at 0 is $-\epsilon x$. Checking eigenvalues, two matrices $(\begin{smallmatrix} 0 & \epsilon \\ 1 & x \end{smallmatrix})$, where $i = 1, 2$, are conjugate in $PGL_2 \mathbb{Z}$ if and only if $\epsilon_1 = \epsilon_2$ and $|x_1| = |x_2|$. This implies that the turning numbers of all the 1-orbits of a matrix $A$ have the same absolute value.

It suffices to prove the lemma when $A = (\begin{smallmatrix} 0 & \epsilon \\ 1 & x \end{smallmatrix})$ as above. Here, the conjugacy classes of $\pm (\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix})$ correspond to the cases $\epsilon = -1, \ x = \pm 3$, so the goal is to prove that there are two 1-orbits in those cases, and one otherwise.

- If $x = 0$, then $A^2 = \pm 1$ and one can check directly that the only 1-orbit of $A$ is the edge connecting $\infty, 0$.

- If $\epsilon = -1$ and $x = \pm 1$, then $A$ is orientation preserving and $A^3 = \pm 1$. Each of its 1-orbits has turning number either 1 or $-1$, so bounds a complementary triangle in $\mathcal{F}$. But then $A$ is a rotation around the barycenter of this triangle in $\mathbb{H}^2$, so this 1-orbit is the only one.

- If $\epsilon = -1$ and $x = \pm 2$, then $A$ is parabolic. Its 1-orbit has turning number $\pm 2$, so consists of all vertices in the $\mathcal{F}$-link of the fixed point of $A$.

When $A$ is hyperbolic, its 1-orbits are simple, biinfinite paths in $\mathcal{F}$ that accumulate onto the attracting and repelling fixed points $\lambda_{+}(A), \lambda_{-}(A)$.

- If $\epsilon = 1$ and $|x| \geq 1$, then $A$ is hyperbolic and orientation-reversing. The turning numbers on a 1-orbit $\ell$ alternate sign, so there is an edge of $\ell$ that separates $\lambda_{+}(A)$ from $\lambda_{-}(A)$ in the upper half plane. Any other 1-orbit would then have to intersect $\ell$, which is impossible, so $A$ has a single 1-orbit. See Figure 1 for an illustration of the case $\epsilon = 1, \ x = 1$.

Figure 1: There is a single 1-orbit for the action $(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}) \circ \mathcal{F}$, on which the turning numbers alternate between $\pm 1$. 

\[\text{Figure 1: There is a single 1-orbit for the action } (\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}) \circ \mathcal{F}, \text{ on which the turning numbers alternate between } \pm 1.\]
If $\epsilon = -1$ and $x = \pm 3$, then $A$ is orientation preserving, hyperbolic and conjugate to $\pm \left( \begin{smallmatrix} 0 & -1 \\ 1 & 3 \end{smallmatrix} \right)$. When $x = 3$, the orbits of $-1$ and $0$ are distinct, since they have opposite turning numbers (see Figure 2). Since the edge from $-1$ to $0$ in $\mathcal{F}$ separates the attracting and repelling fixed points of $A$, any 1-orbit of $A$ must pass through either $-1$ or $0$. So, the orbits of $\infty$ and $-1$ are the only 1-orbits. The argument when $x = -3$ is similar.

It remains to deal with the case $\epsilon = -1$, $|x| \geq 4$, in which case $A$ is again orientation preserving and hyperbolic. We claim that any biinfinite path $\ell$ whose turning numbers are all at least 3 in absolute value is a geodesic in $\mathcal{F}$, and that if the turning numbers are all at least 4 in absolute value then $\ell$ is the unique geodesic in $C(T^2)$ connecting its endpoints. This will imply that when $|x| \geq 4$, the matrix $A$ has only a single 1-orbit.

So, suppose that $\ell = (v_i)$ is a biinfinite path in $\mathcal{F}$ whose turning numbers are all at least 3 in absolute value. For each $i$, let $m_i$ be the edge of $\mathcal{F}$ incident to $v_i$ that lies between the edges $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$, and shares a complementary triangle of $\mathcal{F}$ with $[v_i, v_{i+1}]$, as in Figure 3. Each $m_i$ separates $m_{i-1}$ from $m_{i+1}$, so by planarity all the $m_i$ are disjoint. Two vertices $v_i$ and $v_j$, with $i < j$, are
disjoint from and separated by all the edges

\[ m_{i+1}, \ldots, m_{j-1}. \]

Any path from \( v_i \) to \( v_j \) must go through all of these edges, so must have length at least \(|i - j|\). Therefore, \( \ell \) is a geodesic in \( \mathcal{F} \).

Suppose now that all the turning numbers of \( \ell = (v_i) \) are at least 4 in absolute value. Choose for each \( i \) two more edges \( n_i, o_i \) incident to \( v_i \) that lie between \([v_{i-1}, v_i]\) and \([v_i, v_{i+1}]\), as in Figure 3. All the edges \( m_i, n_i, o_i \) separate the forward and backward limits of \( \ell \), so any geodesic \( \gamma \) in \( \mathcal{F} \) connecting these limits must pass through a vertex of each \( m_i, n_i, o_i \). As \( \gamma \) cannot pass through all three of the non-\( v_i \) vertices of \( m_i, n_i, o_i \), it must pass through \( v_i \), so \( \gamma = \ell \). Thus, \( \ell \) is the unique geodesic in \( \mathcal{F} \) connecting its endpoints. This concludes the proof of Lemma 2.1 and thus the proof of Theorem 1.1.

**Remark 2.2.** The educated reader will note that some of the simple properties of \( \mathcal{F} \) used above reflect (and probably inspired) deeper results about the curve complexes of higher genus surfaces. For instance, the argument used to prove that a path whose turning numbers are all at least 3 in absolute value is a geodesic is a simple version of Masur-Minsky’s bounded geodesic image theorem [?].

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