Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order $\alpha_s^4$ in a General Gauge Theory

P. A. Baikov

Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia

K. G. Chetyrkin and J. H. Kühn

Institut für Theoretische Teilchenphysik, KIT, D-76128 Karlsruhe, Germany

We compute, for the first time, the order $\alpha_s^4$ contributions to the Bjorken sum rule for polarized electron-nucleon scattering and to the (non-singlet) Adler function for the case of a generic colour gauge group. We confirm at the same order a (generalized) Crewther relation which provides a strong test of the correctness of our previously obtained results: the QCD Adler function and the five-loop $\beta$-function in quenched QED. In particular, the appearance of an irrational contribution proportional to $\zeta_3$ in the latter quantity is confirmed. We obtain the commensurate scale equation relating the effective strong coupling constants as inferred from the Bjorken sum rule and from the Adler function at order $\alpha_s^4$.

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INTRODUCTION

The Crewther relation [1, 2] relates in a non-trivial way two seemingly disconnected quantities, namely, the (non-singlet) Adler function $\mathcal{D}$ and the coefficient function $C_{Bjp}$, describing the deviation of the Bjorken sum rule [3, 4] for polarized DIS from its naive-parton model value. The Adler function is defined through the correlator of the vector current $j_{\mu}$

$$3Q^2\Pi(Q^2) = i \int d^4xe^{iq \cdot x} \langle 0 | T j_{\mu}(x) j^{\mu}(0) | 0 \rangle, \quad (1)$$

as follows

$$D(Q^2) = -12 \pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2), \quad (2)$$

with $Q^2 = -q^2$. In fact, the Adler function is the main theoretical object required to study such important physical observables as the cross section for electron-positron annihilation into hadrons and the hadronic decay rates of both the $Z$-boson and the $\tau$-lepton (see, e.g. [3]). The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function $C_{Bjp}$:

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^{cp}(x, Q^2) - g_1^{cn}(x, Q^2)] dx$$

$$= \frac{g_A}{6} C_{Bjp}(a_s) + \sum_{i=2}^\infty \frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, \quad (3)$$

where $g_1^{cp}$ and $g_1^{cn}$ are the spin-dependent proton and neutron structure functions, $g_A$ is the nucleon axial charge as measured in neutron $\beta$-decay. The coefficient function $C_{Bjp}(a_s) = 1 + \mathcal{O}(a_s)$ is proportional to the flavour-nonsinglet axial vector current $\bar{\psi} \gamma_5 \gamma_{\mu} \gamma_5 \psi$ in the corresponding short distance Wilson expansion. The sum in the second line of (3) describes for the nonperturbative power corrections (higher twist) which are inaccessible for pQCD. Within perturbative QCD we define

$$D(Q^2) = d_R \left( 1 + \frac{\beta_0}{\alpha_s} C_F a_s + \sum_{i=2}^\infty d_i a_s^i(Q^2) \right),$$

$$C_{Bjp}(Q^2) = 1 - \frac{3}{4} C_F a_s + \sum_{i=2}^\infty c_i a_s^i(Q^2),$$

$$1/C_{Bjp}(Q^2) = 1 + \frac{3}{4} C_F a_s + \sum_{i=2}^\infty b_i a_s^i(Q^2),$$

where $d_R$ is the dimension of the quark colour representation (for QCD $d_R = 3$), $a_s \equiv \alpha_s / \pi$ and the normalization scale $\mu$ is set $\mu^2 = Q^2$. Note that we consider only the so-called “non-singlet” contribution to the Adler function and do not write explicitly a common factor $\sum_i Q_i^2$ (with $Q_i$ being the electric charge of the $i$-th quark flavour) for $R(s)$.

The Crewther relation states that

$$D(a_s) C_{Bjp}(a_s) = d_R \left[ 1 + \frac{\beta_0}{\alpha_s^2} K(a_s) \right],$$

$$K(a_s) = K_0 + a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \ldots \quad (4)$$

Here $\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 2} \beta_i a_s^{i+2}$ is the QCD $\beta$-function describing the running of the coupling constant $a_s$ with respect to a change of the normalization scale $\mu$ and with its first term $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f$ being responsible for asymptotic freedom of QCD. The term proportional the $\beta$-function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order $\alpha_s^2$, and was suggested [2] on the basis of $O(\alpha_s^4)$ calculations of $D(a_s)$ [10, 11] and $C_{Bjp}(a_s)$ [9] with $O(\alpha_s^4)$ calculations of $D(a_s)$ [10, 11] and $C_{Bjp}(a_s)$ [9]. A formal proof was carried out in [10, 11]. The original relation without this term was first proposed in [1] (see, also, [12]).

At order $\alpha_s$ the Crewther relation is evidently fulfilled. The colour structures which appear in $d_n$ and $c_n$ (hence
also in $b_n$) for n=1,2,3, and 4 are:

$$
\begin{align*}
\alpha_1^3: & \quad C_F, \quad C_F^2, \quad C_F T_f, \quad C_F^2 C_A, \\
\alpha_2^3: & \quad C_F^2, \quad C_F^2 T_f, \quad C_F^2 T_f^2, \quad C_F T_f C_A, \quad C_F T_f C_A^2, \\
\alpha_3^4: & \quad \frac{dF_{abcd} dF_{ab'}}{dF}, \quad n_f dF_{abcd} dF_{ab'}, C_F^k, \\
\alpha_4^4: & \quad \frac{dF_{abcd} dF_{ab'}}{dF}, \quad n_f dF_{abcd} dF_{ab'}, C_F^k, \\
C^a T_f, \quad C^2 T_f^2, \quad C^2 T_f C_A, \quad C^2 T_f C_A^2, \quad C F T_f C_A, \\
\end{align*}
$$

(5)

Here $C_F$ and $C_A$ are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, $T$ is the trace normalization of the fundamental representation, $T_f = T_{n_f}$, with $n_f$ being the number of quark flavors. The exact definitions of $dF_{abcd} dF_{ab'}$, and $dF_{abcd} dF_{ab'}$ are given in [13]. For QCD (colour gauge group SU(3)):

$$
C_F = 4/3, \quad C_A = 3, \quad T = 1/2, \quad d_F = 3, \\
dF_{abcd} dF_{ab'} = \frac{15}{2}, \quad dF_{abcd} dF_{ab'} = \frac{5}{2}.
$$

(6)

Note, that all colour structures, apart of the $d^2$-terms which appear first at order $\alpha_2^4$, involve at least one factor $C_F$. As a consequence, $K_0$ must be set to zero. An inspection of eqs. (4) and (5) clearly shows that the colour structures which may appear in a coefficient $K_i$ are identical to those appearing in the coefficient $b_{i-1}$ and $c_{i-1}$, listed in eq. (5). Thus, at orders $\alpha_2^4$, $\alpha_3^4$, and $\alpha_4^4$ the Crewther relation puts as many as 2, 3 and, finally, 6 constraints on the differences $d_2 - b_2, d_3 - b_3$, and $d_4 - b_4$ respectively. The fulfillment of these constraints constitutes a powerful check of the correctness of the calculations of $D^{NS}(a_s)$ and $C^{BIP}(a_s)$.

Indeed, at orders $O(\alpha_2^4)$ and $O(\alpha_3^4)$ the results for $D^{NS}(a_s)$ and 1/$C^{BIP}(a_s)$:

$$
\begin{align*}
\alpha_2^4: & \quad C_F, \quad C_F^2, \quad C_F T_f, \quad C_F^2 C_A, \\
\alpha_3^4: & \quad C_F^2, \quad C_F^2 T_f, \quad C_F^2 T_f^2, \quad C_F T_f C_A, \quad C_F T_f C_A^2, \\
\alpha_4^4: & \quad \frac{dF_{abcd} dF_{ab'}}{dF}, \quad n_f dF_{abcd} dF_{ab'}, C_F^k, \\
C^a T_f, \quad C^2 T_f^2, \quad C^2 T_f C_A, \quad C^2 T_f C_A^2, \quad C F T_f C_A, \\
\end{align*}
$$

$$
\begin{align*}
b_3 = & - \frac{69}{128} C_F^3 + C_F^2 T_f \left[ \frac{299}{576} \frac{5}{123} \right] + C_F T_f \left[ \frac{115}{216} \right] + C_F^2 C_A \left[ \frac{1}{576} + \frac{11}{12} \right] \\
+ & C_F T_f C_A \left[ \frac{3535}{864} \frac{3}{4} + \frac{5}{6} \right] + C_F C_A^2 \frac{5437}{864} - \frac{55}{24} C_A^5.
\end{align*}
$$

are well consistent with all 5 constraints on the coefficients $d_2, d_3, b_2$ and $b_3$ and imply

$$
\begin{align*}
K_1 = C_F \left[ \frac{21}{8} + 3 \right], \quad K_2 = C_F T_f \left[ \frac{163}{24} - \frac{19}{3} \right] \\
+ C_F C_A \left[ \frac{27}{32} + \frac{221}{12} \right] + C_F C_A^2 \left[ \frac{397}{96} + \frac{17}{2} \frac{15}{15} \right] - 15 \frac{15}{15}.
\end{align*}
$$

The next, $O(\alpha_4^4)$, contribution to $D(a_s)$ has been recently computed [14] for QCD, i.e. setting the colour structures to their SU(3) numerical values (eq. [13]). The function $C^{BIP}(a_s)$ is known to order $\alpha_4^4$ only.

The importance of computation of the $O(\alpha_4^4)$ contribution to the both coefficients $d_4$ and $b_4$ for a generic colour gauge group comes from a few reasons.

First, the knowledge of $c_4$ in the Bjorken sum rule is vital for proper extraction of higher twist contributions. Indeed, in [15] the recent Jefferson Lab data on the spin-dependent proton and neutron structure functions [16 [20]] were used to extract the leading and subleading higher twist parameters $\mu_4$ and $\mu_5$. It has been demonstrated that, say, the twist four term $\mu_4$ approximately halves its value in transition from LO to NLO, and from NLO to NNLO. This duality between perturbative and non-perturbative contributions has been observed before for the structure function $F_3$ [21] (for a related recent discussion see also [22]).

Second, the Bjorken sum rule provides us with a very convenient definition of the effective strong coupling constant (ECC) [21, 23], namely:

$$
6 \Gamma_1 e^{-a_s(Q^2)} = g_A \left( 1 - a_{g_1}(Q^2) \right). \quad (7)
$$

This quantity is directly measurable down to vanishing values of $Q^2$ and, due to eq. (14), approaches to the standard $\alpha_s(Q)$ at large $Q^2$. It is by definition gauge and scheme invariant. Another convenient ECC, $a_D$, comes from the Adler function [24]:

$$
D(Q^2) = 1 + a_D(Q^2). \quad (8)
$$

As its perturbative expansion is available to $O(\alpha_4^4)$ [13] the knowledge of $c_4$ will allow for the first time to compare two ECC’s with the help of a commensurate scale relation [25] at an order unprecedented to date.

Third, the six constraints imposed by eq. (13) provide a highly nontrivial and welcome check of the calculation of $d_4$ in QCD [14]. In particular, in [26] we computed a part of the full result for $d_4$, namely, the term proportional to the colour structure $C_F^k$. As is well-known, an
interesting object – the $\beta$-function of quenched QED —
can be inferred from the part of the Adler function which
depends on $C_F$ only by setting $C_F = 1$ and adjusting a
global normalization factor. The result ($A \equiv \frac{\alpha}{\pi}$)

$$\beta^\text{QED} = \frac{4}{3} A + 4 A^2 - 2 A^3 - 46 A^4$$

$$+ \left( \frac{1157}{6} + 128 \zeta_3 \right) A^5$$

(9)

revealed an unexpected appearance of the irrational constant $\zeta_3$ at five loops and had cast doubt on the cor-
rectness of the full QCD result for $d_4$ \[27\].

Using the same techniques as in calculations of \[14\] and \[16\] we have computed the the Adler function and the function $C^{BF}
for a general gauge group to order $\alpha_s^4$.
Our results read

$$d_4 = d_4^{abcd, b c d} \left[ \frac{3}{16} - \frac{1}{4} \zeta_3 - \frac{5}{4} \zeta_5 \right]$$

$$+ n_f d_4^{abcd, b c d} \left[ \frac{13}{16} - \zeta_3 + \frac{5}{2} \zeta_5 \right] + C_F \left[ \frac{4157}{2048} + \frac{3}{8} \zeta_3 \right]$$

$$+ C_F^3 T_f \left[ \frac{1001}{384} + \frac{99}{32} \zeta_3 - \frac{125}{4} \zeta_5 + \frac{105}{4} \zeta_7 \right] + C_F^2 T_f^2 \left[ \frac{5713}{1728} - \frac{581}{24} \zeta_3 + \frac{125}{6} \zeta_5 + 3 \zeta_3^2 \right] + C_F T_f^3 \left[ \frac{163197}{2048} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_5 \right]$$

$$+ C_F^3 C_A \left[ \frac{253}{32} - \frac{139}{28} \zeta_3 + \frac{2255}{32} \zeta_5 - \frac{1155}{16} \zeta_7 \right] + C_F^2 T_f C_A \left[ \frac{32357}{13824} + \frac{10661}{96} \zeta_3 - \frac{5155}{48} \zeta_5 - \frac{33}{4} \zeta_3^2 - \frac{105}{8} \zeta_7 \right]$$

$$+ C_F T_f C_A^2 \left[ \frac{4379861}{207436} + \frac{8609}{72} \zeta_3 + \frac{18805}{288} \zeta_5 - \frac{11}{2} \zeta_3^2 + \frac{35}{16} \zeta_7 \right] + C_F C_A^3 \left[ \frac{52207039}{248832} - \frac{456223}{3456} \zeta_3 - \frac{77995}{1152} \zeta_5 + \frac{605}{32} \zeta_3^2 - \frac{385}{64} \zeta_7 \right]$$

(10)

$$b_4 = b_4^{abcd, b c d} \left[ \frac{3}{16} - \frac{1}{4} \zeta_3 - \frac{5}{4} \zeta_5 \right]$$

$$+ n_f b_4^{abcd, b c d} \left[ \frac{13}{16} - \zeta_3 + \frac{5}{2} \zeta_5 \right] + C_F \left[ \frac{4157}{2048} + \frac{3}{8} \zeta_3 \right]$$

$$+ C_F^3 T_f \left[ \frac{473}{2304} - \frac{391}{96} \zeta_3 + \frac{145}{24} \zeta_5 \right] + C_F^2 T_f^2 \left[ \frac{869}{576} - \frac{29}{24} \zeta_3 \right] + C_F T_f^3 \left[ - \frac{605}{972} \right]$$

$$+ C_F^3 C_A \left[ \frac{8701}{4608} + \frac{1103}{576} \zeta_3 - \frac{1045}{48} \zeta_5 \right] + C_F^2 T_f C_A \left[ \frac{17309}{13824} + \frac{1127}{144} \zeta_3 - \frac{95}{144} \zeta_5 - \frac{35}{4} \zeta_7 \right]$$

$$+ C_F T_f C_A^2 \left[ \frac{165283}{20736} + \frac{43}{144} \zeta_3 - \frac{5}{12} \zeta_5 - \frac{1}{6} \zeta_3^2 \right] + C_F^2 C_A^2 \left[ \frac{435125}{55296} - \frac{1591}{144} \zeta_3 + \frac{55}{9} \zeta_5 + \frac{385}{16} \zeta_7 \right]$$

$$+ C_F T_f C_A^3 \left[ \frac{1238827}{41472} - \frac{59}{64} \zeta_3 + \frac{1855}{288} \zeta_5 - \frac{11}{12} \zeta_3^2 + \frac{35}{16} \zeta_7 \right] + C_F C_A^4 \left[ \frac{8004277}{248832} - \frac{1069}{576} \zeta_3 - \frac{12545}{1152} \zeta_5 + \frac{121}{96} \zeta_3^2 - \frac{385}{64} \zeta_7 \right]$$

(11)

All six constraints from the generalized Cawthrew relation are indeed met with

$$K_3 = C_F^3 \left( \frac{2471}{768} + \frac{61}{8} \zeta_3 - \frac{715}{8} \zeta_5 + \frac{315}{4} \zeta_7 \right) + C_F^2 T_f \left( \frac{7729}{1152} - \frac{917}{16} \zeta_3 + \frac{125}{2} \zeta_5 + \frac{9}{4} \zeta_3^2 \right)$$

$$+ C_F T_f^2 \left( \frac{-307}{18} + \frac{203}{18} \zeta_3 + 5 \zeta_5 \right) + C_F^2 C_A \left( \frac{199757}{2304} + \frac{8285}{96} \zeta_3 - \frac{1555}{12} \zeta_5 - \frac{105}{8} \zeta_7 \right)$$

$$+ C_F T_f C_A \left( \frac{1055}{9} - \frac{2521}{36} \zeta_3 - \frac{125}{3} \zeta_5 - 2 \zeta_3^2 \right) + C_F C_A^2 \left( \frac{-406043}{2304} + \frac{18007}{144} \zeta_3 + \frac{2975}{48} \zeta_5 - \frac{77}{4} \zeta_3^2 \right).$$

Note, that coefficients in front of first three colour structures in eqs. \[10-11\] ($C_F, n_f d_4^{abcd, b c d}$ and $d_4^{abcd, b c d}$) are
equal, as they should. The $C_F^3$-term, in particular, provides us with a beautiful confirmation of the correctness of the result \[9\] for the $Q$-function (the test was originally suggested in \[27\]).

It is interesting to note that the results do not depend on $\zeta_n$ with $n = 2, 4, 6$. Also, unexpected feature of our results is the separate proportionality all terms of highest and sub-highest transcendentality in a given loop order (that is $\zeta_3^2$ and $\zeta_7$ at $\alpha_s^4$, $\zeta_5$ at $\alpha_s^5$ and, at last, $\zeta_3$ at $\alpha_s^6$) to $\beta_0$. This feature is not required by \[9\], the latter
essentially constraints only the difference $d_i - b_i$.

In numerical form $C^{Bjp}$ reads (with all colour factors set to their QCD values)

$$C^{Bjp} = 1 - a_s + \left( -4.583 + 0.3333 n_f \right) a_s^2$$
$$+ a_s^3 \left( -41.44 + 7.607 n_f - 0.1775 n_f^2 \right) a_s^3$$
$$+ \left( -479.4 + 123.4 n_f - 7.697 n_f^2 + 0.1037 n_f^3 \right) a_s^4.$$

It is of interest to compare the newly found coefficient in front of the $a_s^4$ term with well-known predictions [28]

$$c_4^{\text{pred}}(n_f = 3, 4, 5, 6) = -130, -58, -18, 22$$

and

$$c_4^{\text{exact}}(n_f = 3, 4, 5, 6) = -175.7, -102.4, -41.96, 6.2.$$  

At last, we derive the commensurate relation connecting two the ECC’s $a_{g_i}$ and $a_D$ as defined in eqs. (78). Following ref. [29] we get for QCD

$$(1 + a_D(Q^{2*}))(1 - a_{g_i}(Q^2)) = 1,$$  

with

$$(a_D^* = a_D(Q^{2*})).$$

$$\ln \left( \frac{Q^{2*}}{Q^2} \right) = -K_1 + a_D^* \left[ \beta_0 K_1^2 + 2d_2 K_1 - K_1 - K_2 \right]$$
$$+ (a_D^*)^2 \left[ \beta_0 ( -6d_2 K_1^2 + 2 K_1^2 + 3 K_2 K_1) - 2 \beta_0^2 K_1^3 \right]$$
$$+ K_1 \left( \frac{3}{2} \beta_1 K_1 - 6d_2 + 2d_2 + 3d_3 \right) + K_2 (3d_2 - 1) - K_3$$
$$= -1.30823 + a_D^* [0.80241 - 0.03933 n_f]$$
$$+ (a_D^*)^2 [-16.9020 + 2.62311 n_f - 0.10202 n_f^2].$$

In conclusion we want to mention that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel [30] as well as thread-based [31] versions of FORM [32]. For evaluation of color factors we have used the FORM program COLOR [33].

The diagrams have been generated with QGRAF [34].

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* Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia

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[37] Unexpected, because there existed a wide-spread belief that the rationality property is not accidental but holds also in higher orders [33, 34].