SL(2,R) YANG-MILLS THEORY ON A CIRCLE

Ingemar Bengtsson
Fysikum
University of Stockholm
Box 6730, S-113 85 Stockholm, Sweden

Joakim Hallin
Institute of Theoretical Physics
Chalmers University of Technology
and University of Göteborg
S-412 96 Göteborg, Sweden

Abstract

The kinematics of SL(2,R) Yang-Mills theory on a circle is considered, for reasons that are spelled out. The gauge transformations exhibit hyperbolic fixed points, and this results in a physical configuration space with a non-Hausdorff "network" topology. The ambiguity encountered in canonical quantization is then much more pronounced than in the compact case, and can not be resolved through the kind of appeal made to group theory in that case.

1Email address: ingemar@vana.physto.se
2Email address: tfejh@fy.chalmers.se
We have studied Yang-Mills theory on a cylindrical space-time, choosing the non-compact group $SL(2, \mathbb{R})$ for our structure group. Since this undertaking may appear peculiar, we will begin by spelling out our motivation. First of all the Yang-Mills Hamiltonian is not positive definite whenever the structure group is non-compact. However, this is of no concern to us, since this operator will have nothing to do with the time-development of our model. Actually, we will not be concerned with time-development at all, so that we are really interested only in setting up the model on a space which has the topology of the circle - “Yang-Mills theory”, here, refers only to the phase space of the model. Our interest has to do with gravity. We know that there are four choices of structure group for Yang-Mills theory that are of physical interest: $U(1)$, $SU(2)$ and $SU(3)$ - all of which are compact - and $SL(2, \mathbb{C})$, which is non-compact. The latter case can be used to formulate Einstein’s theory \[1\]. We believe that it is important to gain a broad experience of non-compact gauge theories, and this is one of two reasons for studying the toy model that we will describe. (It is true that the group theory of $SL(2, \mathbb{R})$ differs in important ways from the group theory of $SL(2, \mathbb{C})$, but the real case has certain simplifying features, and it seemed worthwhile to do a separate study of this case.)

This is our first motivation. The second motivation is more vague, but at least as important. So far, almost all our intuition about gauge theories comes from Yang-Mills theory with compact structure groups. If we step back a bit from the problem, and view the gauge transformations in the same way as we might view the Hamiltonian flow of a dynamical system, we observe that we are then dealing with gauge transformations of a very simple, “integrable” kind. Presumably, this is not a generic case, and presumably the “gauge flow” in any theory of gravitation - where time development itself may be viewed as a gauge transformation - is a very different kettle of fish. We believe that there is a risk that our experience from compact Yang-Mills theory may be qualitatively misleading when it comes to defining a quantum theory of gravity. Our toy model is of some interest here - although the gauge flow remains integrable, it exhibits hyperbolic fixed points, which is at least a step in the “chaotic” direction.

We hope that we have convinced the reader that we have chosen an interesting subject, and we will now briefly review the properties of Yang-Mills theory on a circle, and how one may set up the quantum theory in the compact case. To make a long story \[2\] short, the physical configuration space of the model is the space of conjugacy classes in the group. This is the basic fact on which our analysis is based. Hence we have a finite dimensional problem in front of us. Nevertheless, as readers familiar with the original references will recognize, there is enough structure left to make the toy model interestingly analogous to the 3+1 dimensional theory. Let us draw attention to some 1+1 dimensional peculiarities that are of particular interest to us. In the spirit of the “loop quantization program” \[3\], we wish to use traces of holonomies, with inserted momenta, as coordinates on the physical phase space. So we define

$$T^0(n) = \text{tr} \, h(x)^n \quad T^1(x, n) = \text{tr} \, E(x)h(x)^n,$$

where $h(x)$ is the holonomy. We also define an object which we will call the Hamiltonian, although we will not use it to generate time-evolution:
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It is then a peculiarity of 1+1 dimensions that

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In other words, \( H \) can be used to generate the higher \( T \)-variables \( [4] \). Therefore it suffices to consider \( T^0 \) and \( H \). A 1+1 dimensional peculiarity which is of some interest to us is that the generator of spatial diffeomorphisms is related to the generator of gauge transformations by

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where \( A \) is the connection, a spatial scalar density. This means that a gauge invariant object is also diffeomorphism invariant, and it follows that the Hamiltonian, as defined above, is weakly \( x \)-independent.

Now what we want to do is to obtain an explicit description of the physical configuration space for specific choices of the structure groups, and then to set up a quantum version of the theory, in which the wave functions are functions of the physical configuration space and the operators \( T^0 \) and \( H \) are realized as self-adjoint operators. Let us first recall how this goes for compact structure groups \( [2] \), choosing \( SU(2) \) as our example. In this case the group manifold is \( S^3 \), the three-sphere. The unit and the anti-unit elements - which we can imagine as sitting at the South and North Pole of the sphere, respectively, form conjugacy classes by themselves.

![SU(2) and its conjugacy classes](image)

Figure 1: \( SU(2) \) and its conjugacy classes

A line of constant longitude between the poles is a Cartan subgroup, and the conjugacy class to which any point in the Cartan subgroup belongs is given by all the points on the three-sphere that have the same latitude as the given points. In this way, the group manifold is nicely foliated by the conjugacy classes, and the space of conjugacy classes is simply a closed interval. For some purposes (notably for the generalisation to arbitrary compact groups), it is more suitable to define this interval as \( S^1/\pi^2 \), where the permutation group \( \pi^2 \) is in fact the Weyl group, acting on the circle in a suitable way. Since the action of the discrete group has fixed points, this is an orbifold rather than a manifold. From
it is clear that the orbifold singularities appear because there are elliptic fixed points in the gauge flow. This picture generalizes in a straightforward manner to arbitrary compact structure groups. Moreover, the essential feature that the physical configuration space is an orbifold generalizes to the 3+1 dimensional case [5].

After canonical quantization [2], the operators to be made self-adjoint become

\[ T_0(n) = 2 \cos n\phi \quad H = -\partial^2_{\phi}, \]  

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Let us now see how things change when we move on to the non-compact structure group SL(2, R). The group manifold is now three dimensional anti-de Sitter space, which we can depict as the Penrose diagram drawn in fig. 2. We also use the G = KAN decomposition of the group, which we can write in terms of matrices as

\[
\begin{pmatrix}
Y + Z & X + T \\
X - T & -Y + Z
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
et \quad 0 \\
0 \quad e^{-t}
\end{pmatrix}
\begin{pmatrix}
1 & s \\
0 & 1
\end{pmatrix};
\]

\[-X^2 - Y^2 + Z^2 + T^2 = 1.\]  

(6)

In the Penrose diagram, the compact subgroup K corresponds to the line \( r = 0 \), the hyperbolic subgroup A corresponds to the line \( \theta = 0 \), and the parabolic subgroup N corresponds to the lightcone with vertex at the origin, i.e. at the unit element. We note that

\[\text{Tr}G = 2Z \quad \text{Tr}K = 2 \cos \theta \quad \text{Tr}A = 2 \cosh t \quad \text{Tr}N = 2.\]  

(7)

It is then easy to deduce how the group is foliated by the conjugacy classes. We need only one additional piece of information, which is that the parameter \( s \) in N may be scaled to plus or minus one, but it can not be set equal to zero by conjugation. Hence the unit and anti-unit element form conjugacy classes by themselves, and the backwards and

\(^3\)The figures are incomplete. A complete version will be supplied to the appropriate journal.
forwards light cones from these points also form conjugacy classes. The conjugacy class that contains a given element of $K$ (not equal to the unit or anti-unit elements) forms one sheet of a two-sheeted hyperboloid lying within these light cones, and the conjugacy class that contains a given element of $A$ forms a one-sheeted hyperboloid surrounding the same light cones. This is depicted in fig. 2. It is intuitively obvious what the topology of the space of conjugacy classes is, given the topology of the group, and this has also been drawn in the figure. The resulting topology exhibits three features that are new compared to the $SU(2)$ case: It is non-compact, it is non-Hausdorff, and it has the structure of a “network” rather than that of a manifold of a fixed dimension.

We observe that the picture can be generalized to $SL(N, \mathbb{R})$. Of more importance is the fact that the topology of the physical configuration space of non-compact Yang-Mills theory is known to be non-Hausdorff also in $3+1$ dimensions [3]. Hence our toy model captures some aspects of the physically interesting case.

Let us comment on the non-Hausdorff property. The enlarged dots in the figure each denote three separate points, corresponding to the backwards and forwards light cone and its vertex, respectively. These points can not be separated by any continuous function. It is geometrically clear how this complication arises when we take the quotient of the group manifold by conjugations (indeed non-Hausdorff spaces typically arise in some such way, when they arise at all). It is also clear that this is a harmless complication for many purposes, especially when we go on to consider quantum mechanics on the space of conjugacy classes. The wave function at a point does not matter. Hence the non-Hausdorff property can be safely ignored in the sequel.

The network structure is important - it will be necessary to supply appropriate boundary conditions at the vertices in order to ensure that the Hamiltonian be self-adjoint. Quantum theory on networks has been considered in quantum chemistry [7] and more recently in connection with mesoscopic networks [8] - in which case the networks are typically made out of gold films, say ten nanometers thick. Let us review the simplest case of three half-lines meeting at a point, as in fig. 3. The Hilbert space is $L^2$ of the union of the three halflines $[0, \infty]$, and we choose the Hamiltonian to be the flat Laplacian on each half-line. This is a symmetric operator, and it follows from general theory [9] that it admits a nine parameter family of self-adjoint extensions (while the translation operator

\[ \theta = 2\pi \]

\[ \theta = \pi \]

\[ r = 0 \]

\[ \theta = 0 \]

Figure 2: $SL(2,\mathbb{R})$ and its conjugacy classes
admits none). The domain of definition of a self-adjoint extension is given explicitly by the three conditions

$$\Psi'_i(V) = K_{ij} \Psi_j(V),$$  \hspace{1cm} (8)

where $K_{ij}$ is a hermitian matrix. In applications to solid state physics, the network is only an approximation to a network of finite thickness. Physical intuition dictates that the wave function should be continuous at the vertex (possibly up to a phase). Then there is only a one-parameter family of self-adjoint extensions left, defined by

$$\Psi'_1(V) + \Psi'_2(V) + \Psi'_3(V) = \lambda \Psi_1(V) = \lambda \Psi_2(V) = \lambda \Psi_3(V); \quad \lambda \in \mathbb{R}.  \hspace{1cm} (9)$$

The condition on the derivatives enforces conservation of the probability current, and a non-zero value of the parameter $\lambda$ can be thought of as a delta function potential at the vertex. When setting boundary conditions for an entire network, the vertices are treated separately.

Let us now turn to our network, which is the space of conjugacy classes of $\text{SL}(2,\mathbb{R})$. We begin by defining the coordinates that we will use, as well as the form of the Hamiltonian operator on the various segments. This is done in fig. 4. The $T^0$-variable is self-adjoint as it stands. The fact that the Hamiltonian is no longer positive definite causes no particular problems - for a single vertex, all that happens is that the matrix $K_{ij}$ in eq. (8) becomes pseudo-hermitian. The deficiency indices of the Hamiltonian on the full network are $(6,6)$, so there is a 36-parameter family of self-adjoint extensions of this operator. Unless we add further requirements, all of them are in a sense “correct”. It does seem reasonable to require that the conditions should be set at each vertex separately, but a large ambiguity remains. Clearly, physical intuition is not necessarily a good guide for setting boundary conditions here. In the compact case, we saw that an appeal to group theory was enough to single out a preferred answer. Now the qualitative properties of the spectrum of $H$ will be the same for most choices. There will be a discrete set of levels bounded from below, and a doubly degenerate continuous set bounded from above. This is reasonable from the point of view of group theory, and corresponds very roughly speaking to the discrete and principal series of representations, respectively (the supplementary series plays no rôle in harmonic analysis). Important qualitative issues are nevertheless at stake, in particular whether superselection rules will occur. We could impose (say) Dirichlet conditions on

Figure 3: Three half-lines meeting at a point.
all ends, in which case they certainly do. On the other hand, with the “solid state” conditions in eq. (8), there are no superselection rules: The wave function will leak through the vertex when \( H \) is applied to it.

Let us begin the discussion by imposing the solid state conditions at each vertex. This will provide us with an explicit example of a quantum version of our model. More precisely, taking the various signs into account and using the coordinates given in fig. 4, we impose

\[
\begin{align*}
\Psi'_1(0) - \Psi'_2(2\pi) - \Psi'_3(0) &= \lambda \Psi_1(0) = \lambda \Psi_2(2\pi) = \lambda \Psi_3(0) \\
-\Psi'_1(\pi) + \Psi'_2(\pi) - \Psi'_4(0) &= \lambda \Psi_1(\pi) = \lambda \Psi_2(\pi) = \lambda \Psi_4(0)
\end{align*}
\]

where \( \lambda \) is a real parameter. The Hamiltonian \( H \) is given by \( -\partial^2_\theta \) on the circle and \( \partial^2_t \) on the half-lines. Note also that \( T^0(n) \) is given by \( 2 \cos(n\theta) \) on the circle, \( 2 \cosh(nt) \) on half-line \( I \) and \( -2 \cosh(nt) \) on half-line \( II \). The eigenvalues \( E \) of \( H \) are

\[
E = k^2
\]

for suitable values of \( k \). The possible values of \( k \) are as follows. There is a discrete set of states with support on the entire network and oscillatory behaviour on the circle, for which

\[
e^{2\pi ik} = \left( \frac{1 - \lambda/k + 2i}{1 - \lambda/k - 2i} \right)^2, \quad k > 0.
\]

For all values of \( \lambda \) there will be an additional set of discrete states with support confined to the circle, and

\[
k = n \in \mathbb{Z}_+.
\]

Precisely when \( \lambda \) is a positive integer, there are two additional states, smooth on the circle and with support on the entire network, for which

\[
k = \lambda = n.
\]

A state with zero eigenvalue of \( H \) exists only when
\[ \lambda = 0 \quad \text{or} \quad \lambda = -\frac{4}{\pi}. \]  

(15)

For all values of \( \lambda \) there is a doubly degenerate set of states with oscillatory behaviour on the half-lines and with continuous negative eigenvalues of \( H \).

Finally, we discuss whether an appeal to group theory will help to cut down the ambiguity that we have encountered. (A good reference for harmonic analysis on \( \text{SL}(2, \mathbb{R}) \) is the book by Varadarajan.) The first observation is that the characters of \( \text{SL}(2, \mathbb{R}) \) do not obey boundary conditions that satisfy (8). The characters of the discrete series are

\[ \Theta_n(\theta) = -\text{sgn}(n) \frac{e^{in\theta}}{e^{i\theta} - e^{-i\theta}} \]  

(16)

on the circle,

\[ \Theta_n(t) = \frac{e^{-|n|t}}{e^t - e^{-t}}, \]  

(17)

on half line I and

\[ \Theta_n(t) = (-1)^n e^{-|n|t} \frac{e^{t}}{e^t - e^{-t}}, \]  

(18)

on half line II. The next, trivial but important, observation is that a class function can not be square integrable on the group, when the latter is non-compact. What we can do is to associate a class function with any smooth function \( f(g) \) of compact support on the group, by means of an orbital integral. Let \( B \) denote the elliptic Cartan subgroup (which corresponds to the circle), and \( L \) denote the hyperbolic Cartan subgroup (the half-lines). Then the orbital integral associated to \( B \) is

\[ F_{f,B}(\theta) = (e^{i\theta} - e^{-i\theta}) \int_{G/B} f(gu_\theta g^{-1}) d\hat{g}, \]  

(19)

where \( u_\theta \) is an element of \( B \) and \( d\hat{g} \) is an invariant measure on \( G/B \). There is a similar formula for \( L \). The normalizing factor in front of the integral has been chosen so that the Laplacian \( \Omega \) on the group gets pushed down precisely to our network Hamiltonian:

\[ F_{\Omega f,B} = -\frac{d^2}{d\theta^2} F_{f,B} = HF_{f,B} \]  

(20)

\[ F_{\Omega f,L} = \frac{d^2}{dt^2} F_{f,L} = HF_{f,L}. \]  

(21)

It is now crucial to determine the behaviour of the orbital integrals at the network vertices. One finds e.g.

\[ \frac{1}{t} (F_{f,B}(0^+) - F_{f,B}(0^-)) = F_{f,L}(0). \]  

(22)
Also the first derivative of $F_{f,B}$ is continuous on $B$, and $F_{f,L}$ obeys the Neumann condition. The discontinuity of $F_{f,B}$ is called the Harish-Chandra jump relation, and - apart from factors which depend on the specific normalization of the orbital integrals used - it is intuitively clear why it occurs, since the integral over the one-sheeted hyperboloids in anti-de Sitter space will tend to the sum of the integrals over the two sheets of the two-sheeted hyperboloid as both surfaces approach the light cone. Unfortunately, for our purposes, this is not an acceptable behaviour for a wave function, since these boundary condition do not give the domain of definition of a self-adjoint Hamiltonian.

The objects that are naturally integrated against the orbital integrals are characters (times a normalizing factor, so that the denominators in eqs. (16 - 18) is removed), which obey e.g.

$$-i\Phi'_B(0) = \Phi'_{LI}(0).$$

(23)

Also $\Phi_B$ is a smooth function on $B$. These are the matching conditions of Harish-Chandra. Then integration by parts can be done in

$$\int \Theta f dG = -\int_0^{2\pi} \Phi_B F_{f,B} d\theta + \int_0^\infty \Phi_{LI} F_{f,LI} dt + \int_0^\infty \Phi_{LII} F_{f,LII} dt,$$

(24)

with no boundary terms. Unfortunately, this is not useful for our purposes.

In conclusion, we have investigated the physical configuration space of SL(2, $\mathbb{R}$) Yang-Mills theory in 1+1 dimensions. The “gauge flow” exhibits hyperbolic fixed points, which leads to topological complications that are not present for compact structure groups, and results in a considerable ambiguity in the quantum theory. Unlike the ambiguity that arises in the compact case, this ambiguity affects broad qualitative issues such as the appearance of superselection rules, and we were not able to resolve it through any straightforward appeal to group theory. A further study seems called for - as a natural second step, one could consider BRST quantization.

Acknowledgements:

We thank those of our friends who listened. Special thanks to Ralph Howard.

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$T^0$ causes no problems, but $H$ does not have a unique self-adjoint extension. To make it self-adjoint, we may specify Dirichlet or Neumann conditions, or any combination of those, at the ends of the interval. The choice will affect the spectrum of $H$. If one wants a unique, or at least a preferred, answer one has to add further rules to the game. We might insist that the result should be the same if we choose to quantize before constraining the theory, or that the result should be in some sense stable against addition of matter degrees of freedom to the model. In the present case, a preferred choice also emerges after an appeal to group theory. After a canonical transformation of the operators given in eq. (5), one can choose the boundary conditions such that the Hilbert space measure becomes the Haar measure restricted to conjugacy classes, the Hamiltonian becomes the restriction of the Laplacian defined on the group manifold, and its eigenstates become the characters of SU(2). (The interplay between the canonical transformation and the Weyl group gives rise to a subtlety here, which does not arise in the SL(2,R) case. We refer to the literature for this.) This choice is clearly in a sense to be preferred.

Let us now see how things change when we move on to the non-compact structure group SL(2,R). The group manifold is now three dimensional anti-de Sitter space, which we can depict as the Penrose diagram drawn in fig. 2. We also use the $G = KAN$ decomposition of the group, which we can write in terms of matrices as

$$\begin{pmatrix} Y + Z & X + T \\ X - T & -Y + Z \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix};$$

$$-X^2 - Y^2 + Z^2 + T^2 = 1. \quad (6)$$

In the Penrose diagram, the compact subgroup $K$ corresponds to the line $r = 0$, the hyperbolic subgroup $A$ corresponds to the line $\theta = 0$, and the parabolic subgroup $N$ corresponds to the lightcone with vertex at the origin, i.e. at the unit element. We note that

$$TrG = 2Z \quad TrK = 2\cos \theta \quad TrA = 2\cosh t \quad TrN = 2. \quad (7)$$

It is then easy to deduce how the group is foliated by the conjugacy classes. We need only one additional piece of information, which is that the parameter $s$ in $N$ may be scaled to plus or minus one, but it cannot be set equal to zero by conjugation. Hence the unit and anti-unit element form conjugacy classes by themselves, and the backwards and
forwards light cones from these points also form conjugacy classes. The conjugacy class that contains a given element of $K$ (not equal to the unit or anti-unit elements) forms one sheet of a two-sheeted hyperboloid lying within these light cones, and the conjugacy class that contains a given element of $A$ forms a one-sheeted hyperboloid surrounding the same light cones. This is depicted in fig. 2. It is intuitively obvious what the topology of the space of conjugacy classes is, given the topology of the group, and this has also been drawn in the figure. The resulting topology exhibits three features that are new compared to the SU(2) case: It is non-compact, it is non-Hausdorff, and it has the structure of a “network” rather than that of a manifold of a fixed dimension.

We observe that the picture can be generalized to SL(N, $\mathbb{R}$). Of more importance is the fact that the topology of the physical configuration space of non-compact Yang-Mills theory is known to be non-Hausdorff also in 3+1 dimensions [6]. Hence our toy model captures some aspects of the physically interesting case.

Let us comment on the non-Hausdorff property. The enlarged dots in the figure each denote three separate points, corresponding to the backwards and forwards light cone and its vertex, respectively. These points can not be separated by any continuous function. It is geometrically clear how this complication arises when we take the quotient of the group manifold by conjugations (indeed non-Hausdorff spaces typically arise in some such way, when they arise at all). It is also clear that this is a harmless complication for many purposes, especially when we go on to consider quantum mechanics on the space of conjugacy classes. The wave function at a point does not matter. Hence the non-Hausdorff property can be safely ignored in the sequel.

The network structure is important - it will be necessary to supply appropriate boundary conditions at the vertices in order to ensure that the Hamiltonian be self-adjoint. Quantum theory on networks has been considered in quantum chemistry [7] and more recently in connection with mesoscopic networks [8] - in which case the networks are typically made out of gold films, say ten nanometers thick. Let us review the simplest case of three half-lines meeting at a point, as in fig. 3. The Hilbert space is $L^2$ of the union of the three halflines $[0, \infty]$, and we choose the Hamiltonian to be the flat Laplacian on

![Figure 2: SL(2, $\mathbb{R}$) and its conjugacy classes](image_url)
each half-line. This is a symmetric operator, and it follows from general theory that it admits a nine parameter family of self-adjoint extensions (while the translation operator admits none). The domain of definition of a self-adjoint extension is given explicitly by the three conditions

$$\Psi'_i(V) = K_{ij} \Psi_j(V),$$

where $K_{ij}$ is a hermitian matrix. In applications to solid state physics, the network is only an approximation to a network of finite thickness. Physical intuition dictates that the wave function should be continuous at the vertex (possibly up to a phase). Then there is only a one-parameter family of self-adjoint extensions left, defined by

$$\Psi'_1(V) + \Psi'_2(V) + \Psi'_3(V) = \lambda \Psi_1(V) = \lambda \Psi_2(V) = \lambda \Psi_3(V); \quad \lambda \in \mathbb{R}. (9)$$

The condition on the derivatives enforces conservation of the probability current, and a non-zero value of the parameter $\lambda$ can be thought of as a delta function potential at the vertex. When setting boundary conditions for an entire network, the vertices are treated separately.

Let us now turn to our network, which is the space of conjugacy classes of $\text{SL}(2, \mathbb{R})$. We begin by defining the coordinates that we will use, as well as the form of the Hamiltonian operator on the various segments. This is done in fig. The $T^0$-variable is self-adjoint as it stands. The fact that the Hamiltonian is no longer positive definite causes no particular problems - for a single vertex, all that happens is that the matrix $K_{ij}$ in eq. (8) becomes pseudo-hermitian. The deficiency indices of the Hamiltonian on the full network are $(6, 6)$, so there is a 36-parameter family of self-adjoint extensions of this operator. Unless we add further requirements, all of them are in a sense “correct”. It does seem reasonable to require that the conditions should be set at each vertex separately, but a large ambiguity remains. Clearly, physical intuition is not necessarily a good guide for setting boundary conditions here. In the compact case, we saw that an appeal to group theory was enough to single out a preferred answer. Now the qualitative properties of the spectrum of $H$ will be the same for most choices. There will be a discrete set of levels bounded from below, and a doubly degenerate continuous set bounded from above. This is reasonable from the point of view of group theory, and corresponds very roughly speaking to the discrete and principal series of representations, respectively (the supplementary series plays no rôle in

![Figure 3: Three half-lines meeting at a point.](image-url)
harmonic analysis). Important qualitative issues are nevertheless at stake, in particular whether superselection rules will occur. We could impose (say) Dirichlet conditions on all ends, in which case they certainly do. On the other hand, with the “solid state” conditions in eq. (9), there are no superselection rules: The wave function will leak through the vertex when $H$ is applied to it.

Let us begin the discussion by imposing the solid state conditions at each vertex. This will provide us with an explicit example of a quantum version of our model. More precisely, taking the various signs into account and using the coordinates given in fig. 4, we impose

$$
\Psi_1'(0) - \Psi_2'(2\pi) - \Psi_3'(0) = \lambda \Psi_1(0) = \lambda \Psi_2(2\pi) = \lambda \Psi_3(0)
$$

(10)

$$
- \Psi_1'(\pi) + \Psi_2'(\pi) - \Psi_4'(0) = \lambda \Psi_1(\pi) = \lambda \Psi_2(\pi) = \lambda \Psi_4(0)
$$

(11)

where $\lambda$ is a real parameter. The Hamiltonian $H$ is given by $-\partial^2_\theta$ on the circle and $\partial^2_t$ on the half-lines. Note also that $T^0(n)$ is given by $2 \cos(n\theta)$ on the circle, $2 \cosh(nt)$ on half-line $I$ and $-2 \cosh(nt)$ on half-line $II$. The eigenvalues $E$ of $H$ are

$$
E = k^2
$$

for suitable values of $k$. The possible values of $k$ are as follows. There is a discrete set of states with support on the entire network and oscillatory behaviour on the circle, for which

$$
e^{2\pi ik} = \left( \frac{1 - \lambda/k + 2i}{1 - \lambda/k - 2i} \right)^2, \quad k > 0.
$$

(12)

For all values of $\lambda$ there will be an additional set of discrete states with support confined to the circle, and

$$
k = n \in \mathbb{Z}_+.
$$

(13)

Precisely when $\lambda$ is a positive integer, there are two additional states, smooth on the circle and with support on the entire network, for which

$$
k = \lambda = n.
$$

(14)
A state with zero eigenvalue of \( H \) exists only when
\[
\lambda = 0 \quad \text{or} \quad \lambda = -\frac{4}{\pi}.
\] (15)

For all values of \( \lambda \) there is a doubly degenerate set of states with oscillatory behaviour on the half-lines and with continuous negative eigenvalues of \( H \).

Finally, we discuss whether an appeal to group theory will help to cut down the ambiguity that we have encountered. (A good reference for harmonic analysis on \( \text{SL}(2,\mathbb{R}) \) is the book by Varadarajan [11].) The first observation is that the characters of \( \text{SL}(2,\mathbb{R}) \) do not obey boundary conditions that satisfy (8). The characters of the discrete series are
\[
\Theta_n(\theta) = -\text{sgn}(n) e^{i n \theta} \frac{e^{in\theta}}{e^{i\theta} - e^{-i\theta}}
\] on the circle,
\[
\Theta_n(t) = \frac{e^{-|n|t}}{e^t - e^{-t}},
\] (17)
on half line I and
\[
\Theta_n(t) = (-1)^{n-1} \frac{e^{-|n|t}}{e^t - e^{-t}},
\] (18)on half line II. The next, trivial but important, observation is that a class function can not be square integrable on the group, when the latter is non-compact. What we can do is to associate a class function with any smooth function \( f(g) \) of compact support on the group, by means of an orbital integral. Let \( B \) denote the elliptic Cartan subgroup (which corresponds to the circle), and \( L \) denote the hyperbolic Cartan subgroup (the half-lines). Then the orbital integral associated to \( B \) is
\[
F_{\lambda, B}(\theta) = (e^{i\theta} - e^{-i\theta}) \int_{G/B} f(gu_\theta g^{-1}) dg,
\] (19)
where \( u_\theta \) is an element of \( B \) and \( dg \) is an invariant measure on \( G/B \). There is a similar formula for \( L \). The normalizing factor in front of the integral has been chosen so that the Laplacian \( \Omega \) on the group gets pushed down precisely to our network Hamiltonian:
\[
F_{\Omega f,B} = -\frac{d^2}{d\theta^2} F_{f,B} = HF_{f,B}
\] (20)
\[
F_{\Omega f,L} = \frac{d^2}{dt^2} F_{f,L} = HF_{f,L}.
\] (21)

It is now crucial to determine the behaviour of the orbital integrals at the network vertices. One finds e.g.
\[
\frac{1}{t}(F_{f,B}(0^+) - F_{f,B}(0^-)) = F_{f,L}(0).
\] (22)
Also the first derivative of $F_{f,B}$ is continuous on $B$, and $F_{f,L}$ obeys the Neumann condition. The discontinuity of $F_{f,B}$ is called the Harish-Chandra jump relation, and - apart from factors which depend on the specific normalization of the orbital integrals used - it is intuitively clear why it occurs, since the integral over the one-sheeted hyperboloids in anti-de Sitter space will tend to the sum of the integrals over the two sheets of the two-sheeted hyperboloid as both surfaces approach the light cone. Unfortunately, for our purposes, this is not an acceptable behaviour for a wave function, since these boundary condition do not give the domain of definition of a self-adjoint Hamiltonian.

The objects that are naturally integrated against the orbital integrals are characters (times a normalizing factor, so that the denominators in eqs. (16 - 18) is removed), which obey e.g.

$$- i\Phi_B' (0) = \Phi_{LI}' (0).$$

(23)

Also $\Phi_B$ is a smooth function on $B$. These are the matching conditions of Harish-Chandra. Then integration by parts can be done in

$$\int \Theta f dG = - \int_0^{2\pi} \Phi_B F_{f,B} d\theta + \int_0^\infty \Phi_{LI} F_{f,LI} dt + \int_0^\infty \Phi_{LII} F_{f,LII} dt,$$

(24)

with no boundary terms. Unfortunately, this is not useful for our purposes.

In conclusion, we have investigated the physical configuration space of SL(2, $\mathbb{R}$) Yang-Mills theory in 1+1 dimensions. The “gauge flow” exhibits hyperbolic fixed points, which leads to topological complications that are not present for compact structure groups, and results in a considerable ambiguity in the quantum theory. Unlike the ambiguity that arises in the compact case, this ambiguity affects broad qualitative issues such as the appearance of superselection rules, and we were not able to resolve it through any straightforward appeal to group theory. A further study seems called for - as a natural second step, one could consider BRST quantization.

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