Class Numbers and Algebraic Tori

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Abstract. Let $p$ be an odd prime number, $D_p$ be the dihedral group of order $2p$, $h_p$ and $h_p^+$ be the class numbers of $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ respectively. Theorem. $h_p^+ = 1$ if and only if, for any field $k$ admitting a $D_p$-extension, all the algebraic $D_p$-tori over $k$ are stably rational. A similar result for $h_p = 1$ and $C_p$-tori is valid also.

\hspace{1cm}

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§1. Introduction

The initial goal of our investigation was to understand some rationality problem of algebraic tori defined over a non-closed field. Unexpectedly, we arrived at a result which related the rationality problem to a criterion of $h_p^+ = 1$ where $h_p^+$ is the second factor of the class number $h_p$. According to [Wa, page 420], it is notoriously difficult to determine $h_p^+$. Before stating the main results, let’s recall some terminology first.

Let $k \subset L$ be a field extension. The field $L$ is rational over $k$ (in short, $k$-rational) if, for some $n$, $L \simeq k(X_1, \ldots, X_n)$, the rational function field of $n$ variables over $k$. $L$ is called stably rational over $k$ (or, stably $k$-rational) if the field $L(Y_1, \ldots, Y_m)$ is $k$-rational where $Y_1, \ldots, Y_m$ are some elements algebraically independent over $L$. When $k$ is an infinite field, $L$ is called retract $k$-rational, if there exist an affine domain $A$ whose quotient field is $L$ and $k$-algebra morphisms $\phi: A \to k[X_1, \ldots, X_n][1/f]$, $\psi: k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \phi = 1_A$ the identity map on $A$ where $k[X_1, \ldots, X_n]$ is a polynomial ring over $k$, $f \in k[X_1, \ldots, X_n]\{0\}$ [Sa, Definition 3.1; Ka2, Definition 1.1].

It is known that “$k$-rational” $\Rightarrow$ “stably $k$-rational” $\Rightarrow$ “retract $k$-rational”. Moreover, if $k$ is an algebraic number field, retract $k$-rationality of $k(G)$ implies the inverse Galois problem for the field $k$ and the group $G$ [Sa, Ka2] (see Definition 6.7 for the definition of $k(G)$).

Let $k$ be a field, $G$ be a finite group. We will say that the field $k$ admits a $G$-extension if there is a Galois field extension $K/k$ such that $G \simeq Gal(K/k)$.

Let $k$ be a non-closed field. An algebraic torus $T$ defined over $k$ is an algebraic group such that $T \times_{\text{Spec}(k)} \text{Spec}(K) \simeq \mathbb{G}_{m,K}^d$ for some finite separable extension $K/k$, for some positive integer $d$ where $\mathbb{G}_{m,K}$ is the 1-dimensional multiplicative group defined over $K$; the field $K$ is called a splitting field of $T$.

**Definition 1.1** Let $G$ be a finite group, $k$ be a field admitting a $G$-extension. An algebraic torus $T$ over $k$ is called a $G$-torus if it has a splitting field $K$ which is Galois over $k$ with $Gal(K/k) \simeq G$.

An algebraic torus $T$ over $k$ is $k$-rational (resp. stably $k$-rational, retract $k$-rational) if so is its function field $k(T)$ over $k$.

The birational classification of algebraic tori was studied by Voskresenskii, Endo and Miyata, Kunyavskii, Colliot-Thélène and Sansuc, Klyachko, etc. For a survey, see [Vo, Ku4].

In particular, Voskresenskii proves that all the 2-dimensional algebraic tori are rational [Vo, page 57], and the birational classification of 3-dimensional algebraic tori was solved by Kunyavskii [Ku3].

**Definition 1.2** Let $n$ be a positive integer, $\zeta_n$ be a primitive $n$-th root of unity. Let $h_n$ and $h_n^+$ be the class numbers of $\mathbb{Q}(\zeta_n)$ and $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ respectively. It is known that $h_n^+$ divides $h_n$ (see [Wa, p.40, Theorem 4.14]). Thus we may write $h_n = h_n^+ \cdot h_n^-$.
where $h^-_n$ is a positive integer. The integers $h^-_n$ and $h^+_n$ are called the first factor and the second factor of $h_n$ respectively.

In the sequel we denote by $C_n$ and $D_n$ the cyclic group of order $n$ and the dihedral group of order $2n$ respectively.

**Theorem 1.3** ([Vo, page 64])

1. Let $k$ be a field admitting a $C_p$-extension where $p$ is a prime number with $h_p = 1$. Then all the $C_p$-tori defined over $k$ are $k$-rational.

2. Let $k$ be a field admitting a $C_4$-extension. Then all the $C_4$-tori over $k$ are $k$-rational.

**Theorem 1.4**

1. (Kunyavskii [Ku1]) Let $G = C_2 \times C_2$, the Klein four group, and $k$ be a field admitting a $G$-extension. Then a $G$-torus over $k$ is stably $k$-rational if and only if it is $k$-rational.

2. (Kunyavskii [Ku2]) Let $S_3$ be the symmetric group of degree 3, and $k$ be a field admitting an $S_3$-extension. Then all the $S_3$-tori over $k$ are $k$-rational.

The main results of this paper are about the stable rationality of $D_p$-tori.

**Theorem 1.5**

Let $p$ be an odd prime number, and $k$ be a field admitting a $D_p$-extension. Then $h^+_p = 1$ if and only if all the $D_p$-tori over $k$ are stably $k$-rational.

According to Washington [Wa, p.420], the calculation of $h^+_p$ is rather sophisticated. It is known that $h^+_p = 1$ if $p \leq 67$; if the generalized Riemann hypothesis is assumed, then $h^+_p = 1$ if $p \leq 157$, and $h^+_{163} = 4$ [Wa, page 421].

The proof of Theorem 1.5 will be given in Theorem 4.9 and Theorem 5.4. Using the same idea of the proof of Theorem 5.4, we may provide a supplement to Part (1) of Theorem 1.3 as follows (see Theorem 6.6).

**Theorem 1.6**

Let $p$ be a prime number, and $k$ be a field admitting a $C_p$-extension. Then $h_p = 1$ if and only if all the $C_p$-tori over $k$ are $k$-rational (resp. stably $k$-rational).

The main idea of our proof is to reduce the rationality problem of a $G$-torus over a field $k$ to that of its function field. If $M$ is the character module of $T$, i.e. $M = \text{Hom}(T \otimes_k K, \mathbb{G}_{m,K})$, then the function field of $T$ is $K(M)^G$ where $K/k$ is a Galois extension with $\text{Gal}(K/k) \cong G$ and $M$ is a $G$-lattice from its definition (see [Sw3, page 36]). On the other hand, by Theorem 2.4, the fixed field $K(M)^G$ is stably rational over $k$ if and only if the flabby class of $M$, $[M]^P$, is a permutation lattice. Thus it suffices to study which flabby lattices are stably permutation. The proof consists of two ingredients: the classification of integral representations of $D_p$ and the Steinitz class of an integral representation of $C_p$.

Theorem 3.4 and Theorem 3.5 show that the Krull-Schmidt-Azumaya Theorem fails in the case of integral representations [CR1, page 128]. The failure is not so desperate at first sight; on the contrary, it becomes a crucial step in proving some $D_p$-lattices are...
stably permutation. Such a phenomenon was observed for the case \( p = 3 \) and \( p = 5 \)
when we studied lower-rank lattices [HY]. By painstaking computer experiments, we
are led to the most general case, which is recorded in Theorem 3.4, Theorem 3.5 and
Theorem 3.7. The final result is summarized in Theorem 4.11.

On the other hand, we use the Steinitz class to detect whether a flabby lattice is
stably permutation or not, when we regard a \( D_p \)-lattice as a \( C_p \)-lattice by restriction
(for the Steinitz class, see Definition 5.2). Thus the proof of Theorem 1.5 is finished.

After a preprint of this article was posted in arXiv, Shizuo Endo kindly informed
us that he had another proof of Theoem 1.5 by applying Theorem 3.3 of his joint paper
with Miyata [EM2], which is included in the appendix of this paper.

This paper is organized as follows. Section 2 contains the preliminaries of \( G \)-lattices
and the flabby class monoid of \( G \)-lattices. In Section 3 we construct six \( D_n \)-lattices
where \( n \) is an odd integer. Then we show that the rank \( n + 1 \) lattices are stably
permutation. When \( n = p \) is an odd prime number, these lattices play an important role
in Section 4. Section 4 begins with the classification of indecomposable \( D_p \)-lattices due
to Myrna Pike Lee in [Le] [CR1 page 752]. Then we prove that, if \( h_p^+ = 1 \), all the flabby
\( D_p \)-lattices are stably permutation. This finishes the proof of one direction of Theorem
1.5 In Section 5 we recall Diederichsen-Reiner’s Theorem of integral representations
of \( C_p \) and their invariants [Di; Re; CR1 page 729]. Then we show that, the flabby
lattices constructed in Section 3 are the only indecomposable flabby \( D_p \)-lattices which
are stably permutation. Thus the proof of another direction of Theorem 1.5 is finished.
In the last section Theorem 1.6 is proved.

Terminology and notations. In this paper, \( C_n \) and \( D_n \) denote the cyclic group
of order \( n \) and the dihedral group of order \( 2n \) respectively. \( \zeta_n \) denotes a primitive \( n \)-th
root of unity. \( h_n \) and \( h_n^+ \) denote the class numbers of the \( n \)-th cyclotomic field \( \mathbb{Q}(\zeta_n) \)
and its real subfield \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) respectively.

If \( R \) is a Dedekind domain, recall the definition of the (ideal) class group \( C(R) \). Let
\( \text{Div}(R) \) denote the set of all non-zero fraction ideals of \( R \), which is a group under the
multiplication of ideals; let \( \text{Prin}(R) \) be the subgroup of \( \text{Div}(R) \) consisting of principal
ideals. The class group \( C(R) \) is the quotient group \( \text{Div}(R)/\text{Prin}(R) \); if \( I \) is a fractional
ideal, \([I]\) denotes the image of \( I \) in \( C(R) \).

§2. Preliminaries of \( G \)-lattices

Throughout this section, \( k \) is a field.

Let \( \Gamma_k = \text{Gal}(k_{\text{sep}}/k) \). A \( \Gamma_k \)-lattice \( M \) is a free abelian group of finite rank on which
\( \Gamma_k \) acts continuously. It is known that the category of algebraic tori defined over \( k \) is
anti-equivalent to the category of \( \Gamma_k \)-lattices [Vo] page 27; Sw2, page 36]. If \( T \) is a
torus, the \( \Gamma_k \)-lattice corresponding to \( T \) is its character module \( M := \text{Hom}(T, \mathbb{G}_{m,k_{\text{sep}}}) \).
Let $\Gamma_0$ be an open subgroup of $\Gamma_k$ such that $\Gamma_0$ acts trivially on $M$. Consider $M$ as a $G$-lattice where $G = \Gamma_k / \Gamma_0$ is a finite group. Thus we are led to the following formulation.

Let $G$ be a finite group. Recall that a finitely generated $\mathbb{Z}[G]$-module $M$ is called a $G$-lattice if it is torsion-free as an abelian group. We define rank$_\mathbb{Z} M = n$ if $M$ is a free abelian group of rank $n$.

A $G$-lattice $M$ is called a permutation lattice if $M$ has a $\mathbb{Z}$-basis permuted by $G$. A $G$-lattice $M$ is called stably permutation if $M \oplus P$ is a permutation lattice where $P$ is some permutation lattice. $M$ is called an invertible lattice if it is a direct summand of some permutation lattice. A $G$-lattice $M$ is called a flabby lattice if $H^1(S, M) = 0$ for any subgroup $S$ of $G$; it is called coflabby if $H^1(S, M) = 0$ for any subgroup $S$ of $G$. For details, see [CTS, Sw3, Lo].

Let $G$ be a finite group. Two $G$-lattices $M_1$ and $M_2$ are similar, denoted by $M_1 \sim M_2$, if $M_1 \oplus P_1 \cong M_2 \oplus P_2$ for some permutation $G$-lattices $P_1$ and $P_2$. The flabby class monoid $F_G$ is the class of all flabby $G$-lattices under the similarity relation. In particular, if $M$ is a flabby lattice, $[M] \in F_G$ denotes the equivalence class containing $M$; we define $[M_1] + [M_2] = [M_1 \oplus M_2]$ and thus $F_G$ becomes an abelian monoid [Sw3].

**Definition 2.1** Let $G$ be a finite group, $M$ be any $G$-lattice. Then $M$ has a flabby resolution, i.e. there is an exact sequence of $G$-lattices: $0 \to M \to P \to E \to 0$ where $P$ is a permutation lattice and $E$ is a flabby lattice. The class $[E] \in F_G$ is uniquely determined by the lattice $M$ [Sw3]. We define $[M]^f = [E] \in F_G$, following the nomenclature in [Lo, page 38]. Sometimes we will say that $[M]^f$ is permutation or invertible if the class $[E]$ contains a permutation or invertible lattice.

**Definition 2.2** Let $K/k$ be a finite Galois field extension with $G = \text{Gal}(K/k)$. Let $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot e_i$ be a $G$-lattice. We define an action of $G$ on $K(M) = K(x_1, \ldots, x_n)$, the rational function field of $n$ variables over $K$, by $\sigma \cdot x_j = \prod_{1 \leq i \leq n} a_{ij} e_i$ if $\sigma \cdot e_j = \sum_{1 \leq i \leq n} a_{ij} e_i \in M$, for any $\sigma \in G$ (note that $G$ acts on $K$ also). The fixed field is denoted by $K(M)^G$.

If $T$ is an algebraic torus over $k$ satisfying $T \times_{\text{Spec}(k)} \text{Spec}(K) \cong \mathbb{G}_{m,K}^n$ where $\mathbb{G}_{m,K}$ is the one-dimensional multiplicative group over $K$, then $M := \text{Hom}(T, \mathbb{G}_{m,K})$ is a $G$-lattice and the function field of $T$ over $k$ is isomorphic to $K(M)^G$ by Galois descent [Sw3, page 36]. Thus the stable rationality of $T$ over $k$ is equivalent to that of $K(M)^G$. Such a torus $T$ is called a $G$-torus over $k$ (see Definition [11]).

**Definition 2.3** We give a generalization of $K(M)^G$. Let $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot e_i$ be a $G$-lattice, $k'/k$ be a finite Galois extension field such that there is a surjection $G \to \text{Gal}(k'/k)$. Thus $G$ acts naturally on $k'$ by $k$-automorphisms. We define an action of $G$ on $k'(M) = k'(x_1, \ldots, x_n)$ in a similar way as $K(M)$. The fixed field is denoted by $k'(M)^G$. The action of $G$ on $k'(M)$ is called a purely quasi-monomial action in [HKK, Definition 1.1]; it is possible that $G$ acts faithfully on $k'$ (the case $k' = K$) or trivially on $k'$ (the case $k' = k$).
Theorem 2.4 Let $K/k$ be a finite Galois extension field, $G = \text{Gal}(K/k)$ and $M$ be a $G$-lattice.

(1) (Voskresenskii, Endo and Miyata [EM1, Theorem 1.2; Len, Theorem 1.7]) $K(M)^G$ is stably $k$-rational if and only if $[M]^G$ is permutation, i.e. there exists a short exact sequence of $G$-lattices $0 \to M \to P_1 \to P_2 \to 0$ where $P_1$ and $P_2$ are permutation $G$-lattices.

(2) (Saltman [Sa, Theorem 3.14; Ka2, Theorem 2.8]) $K(M)^G$ is retract $k$-rational if and only if $[M]^G$ is invertible.

Theorem 2.5 (Endo and Miyata [EM2, Theorem 1.5; Sw3, Theorem 3.4; Lo, 2.10.1]) Let $G$ be a finite group. Then all the flabby $G$-lattices are invertible if and only if all the Sylow subgroups of $G$ are cyclic.

§3. Some $D_n$-lattices

Throughout this section, $G$ denotes the group $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle$ where $n \geq 3$ is an odd integer, i.e. $G$ is the dihedral group $D_n$. Define $H = \langle \tau \rangle$.

We will construct six $G$-lattices which will become indecomposable $G$-lattices if $n = p$ is an odd prime number (to be proved in Section 4).

Definition 3.1 Let $G = \langle \sigma, \tau \rangle$ be the dihedral group defined before. Define $G$-lattices $M_+$ and $M_-$ by $M_+ = \text{Ind}_H^G \mathbb{Z}$, $M_- = \text{Ind}_H^G \mathbb{Z}$ the induced lattices where $\mathbb{Z}$ and $\mathbb{Z}$ are $H$-lattices such that $\tau$ acts on $\mathbb{Z} = \mathbb{Z} \cdot u$, $\mathbb{Z} = \mathbb{Z} \cdot u'$ by $\tau \cdot u = u$, $\tau \cdot u' = -u'$ respectively (note that $u$ and $u'$ are the generators of $\mathbb{Z}$ and $\mathbb{Z}$ as abelian groups). By choosing a $\mathbb{Z}$-basis for $M_+$ corresponding to $\sigma^i u \in \text{Ind}_H^G \mathbb{Z}$ (where $0 \leq i \leq n - 1$), the actions of $\sigma$ and $\tau$ on $M_+$ are given by the $n \times n$ integral matrices

$$
\sigma \mapsto A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & & & \\
. & . & \ddots & . & \\
1 & 0 & 0 & & \\
1 & 0 & & & 
\end{pmatrix},
\tau \mapsto B = \begin{pmatrix}
1 & & & & \\
1 & \ddots & & & \\
& \ddots & 1 & & \\
& & \ddots & \ddots & \\
& & & \ddots & 1
\end{pmatrix}.
$$

Similarly, for a $\mathbb{Z}$-basis for $M_-$ corresponding to $\sigma^i u'$, the actions of $\sigma$ and $\tau$ are given by

$$
\sigma \mapsto A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & & & \\
. & . & \ddots & . & \\
1 & 0 & 0 & & \\
1 & 0 & & & 
\end{pmatrix},
\tau \mapsto -B = \begin{pmatrix}
1 & & & & \\
1 & \ddots & & & \\
& \ddots & 1 & & \\
& & \ddots & \ddots & \\
& & & \ddots & 1
\end{pmatrix}.
$$
Definition 3.2 As before, \( G = \langle \sigma, \tau \rangle \simeq D_n \). Let \( f(\sigma) = 1 + \sigma + \cdots + \sigma^{n-1} \in \mathbb{Z}[G] \). Since \( \tau \cdot f(\sigma) = f(\sigma) \cdot \tau \), the left \( \mathbb{Z}[G] \)-ideal \( \mathbb{Z}[G] \cdot f(\sigma) \) is a two-sided ideal; as an ideal in \( \mathbb{Z}[G] \), we denote it by \( \langle f(\sigma) \rangle \). The natural projection \( \mathbb{Z}[G] \to \mathbb{Z}[G]/\langle f(\sigma) \rangle \) induces an isomorphism of \( \mathbb{Z}[G]/\langle f(\sigma) \rangle \) and the twisted group ring \( \mathbb{Z}[\zeta_n] \circ H \) (see [CR1, p.589]). Explicitly, let \( \zeta_n \) be a primitive \( n \)-th root of unity. Then \( \mathbb{Z}[\zeta_n] \circ H = \mathbb{Z}[\zeta_n] \oplus \mathbb{Z}[\zeta_n] \cdot \tau \) and \( \tau \cdot \zeta_n = \zeta_n^{-1} \). If \( n = p \) is an odd prime number, \( \mathbb{Z}[\zeta_n] \circ H \) is a hereditary order [CR1 pages 593–595]. Note that we have the following fibre product diagram

\[
\begin{array}{ccc}
\mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[\zeta_n] \circ H \\
\downarrow & & \downarrow \\
\mathbb{Z}[H] & \longrightarrow & \mathbb{Z}[H]
\end{array}
\]

where \( \overline{\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} \) (compare with [CR1 page 748, (34.43)]).

Using the \( G \)-lattices \( M_+ \) and \( M_- \) in Definition 3.1 define \( N_+ = \mathbb{Z}[G]/\langle f(\sigma) \rangle \otimes_{\mathbb{Z}[G]} M_+ = M_+/f(\sigma)M_+ \), \( N_- = \mathbb{Z}[G]/\langle f(\sigma) \rangle \otimes_{\mathbb{Z}[G]} M_- = M_-/f(\sigma)M_- \).

The \( \mathbb{Z}[G]/\langle f(\sigma) \rangle \)-lattices \( N_+ \) and \( N_- \) may be regarded as \( G \)-lattices through the \( \mathbb{Z} \)-algebra morphism \( \mathbb{Z}[G] \to \mathbb{Z}[G]/\langle f(\sigma) \rangle \). By choosing a \( \mathbb{Z} \)-basis for \( N_+ \) corresponding to \( \sigma^iu \) where \( 1 \leq i \leq n-1 \), the actions of \( \sigma \) and \( \tau \) on \( N_+ \) are given by the \((n-1) \times (n-1)\) integral matrices

\[
\sigma \mapsto A' = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 1 \\
1 & -1 & 0 & \cdots & 1
\end{pmatrix}, \quad \tau \mapsto B' = \begin{pmatrix}
1 & & & & & \\
& 1 & & & & \\
& & \ddots & & & \\
& & & 1 & & \\
& & & & -1 & \\
& & & & & -1
\end{pmatrix}.
\]

Similarly, the actions of \( \sigma \) and \( \tau \) on \( N_- \) are given by

\[
\sigma \mapsto A' = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 1 \\
1 & -1 & 0 & \cdots & 1
\end{pmatrix}, \quad \tau \mapsto -B' = \begin{pmatrix}
1 & & & & & \\
& 1 & & & & \\
& & \ddots & & & \\
& & & 1 & & \\
& & & & -1 & \\
& & & & & -1
\end{pmatrix}.
\]

Definition 3.3 We will use the \( G \)-lattices \( M_+ \) and \( M_- \) in Definition 3.1 to construct \( G \)-lattices \( \tilde{M}_+ \) and \( \tilde{M}_- \) which are of rank \( n+1 \) satisfying the short exact sequences of \( G \)-lattices

\[
0 \to M_+ \to \tilde{M}_+ \to \mathbb{Z} \to 0 \\
0 \to M_- \to \tilde{M}_- \to \mathbb{Z}_- \to 0
\]
where the $\mathbb{Z}$-lattice structures of $\tilde{M}_+$ and $\tilde{M}_-$ will be described below and $\mathbb{Z} = \mathbb{Z} \cdot w$, $\mathbb{Z}_- = \mathbb{Z} \cdot w'$ are $G$-lattices defined by $\sigma \cdot w = w$, $\tau \cdot w = w$, $\sigma \cdot w' = w'$, $\tau \cdot w' = -w'$.

Let $\{w_i : 0 \leq i \leq n - 1\}$ be the $\mathbb{Z}$-basis of $M_+$ in Definition 3.1. As a free abelian group, $\tilde{M}_+ = (\bigoplus_{0 \leq i \leq n - 1} \mathbb{Z} \cdot w_i) \oplus \mathbb{Z} \cdot w$. Define the actions of $\sigma$ and $\tau$ on $\tilde{M}_+$ by the $(n + 1) \times (n + 1)$ integral matrices

$$\sigma \mapsto \tilde{A} = \begin{pmatrix} A & 0 \\ \vdots & \ddots \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \tilde{B} = \begin{pmatrix} B & 0 \\ \vdots & \ddots \\ 1 & 0\end{pmatrix}.$$ 

Similarly, let $\{w_i : 0 \leq i \leq n - 1\}$ be the $\mathbb{Z}$-basis of $M_-$ in Definition 3.1 and $\tilde{M}_- = (\bigoplus_{0 \leq i \leq n - 1} \mathbb{Z} w_i) \oplus \mathbb{Z} \cdot w'$. Define the actions of $\sigma$ and $\tau$ on $\tilde{M}_-$ by

$$\sigma \mapsto \tilde{A} = \begin{pmatrix} A & 0 \\ \vdots & \ddots \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \tilde{B} = \begin{pmatrix} -B & 0 \\ \vdots & \ddots \\ -1 & 0\end{pmatrix}.$$ 

In the remaining part of this section, we will show that $\tilde{M}_+$ and $\tilde{M}_-$ are stably permutation $G$-lattices, and $\tilde{M}_+ \oplus \tilde{M}_-$ is a permutation $G$-lattice.

**Theorem 3.4** Let $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \cong D_n$ where $n$ is an odd integer. Then $\tilde{M}_+ \oplus \mathbb{Z} \cong \mathbb{Z}[G/\langle \sigma \rangle] \oplus \mathbb{Z}[G/\langle \tau \rangle]$.

**Proof.** Let $u_0, u_1$ be the $\mathbb{Z}$-basis of $\mathbb{Z}[G/\langle \sigma \rangle]$ correspond to $1$, $\tau$. Then $\sigma : u_0 \mapsto u_0$, $u_1 \mapsto u_1$, $\tau : u_0 \leftrightarrow u_1$.

Let $\{v_i : 0 \leq i \leq n - 1\}$ be the $\mathbb{Z}$-basis of $\mathbb{Z}[G/\langle \tau \rangle]$ correspond to $\sigma^i$ where $0 \leq i \leq n - 1$. Then $\sigma : v_i \mapsto v_{i+1}$ (where the index is understood modulo $n$), $\tau : v_i \mapsto v_{n-i}$ for $0 \leq i \leq n - 1$.

It follows that $u_0, u_1, v_0, v_1, \ldots, v_{n-1}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[G/\langle \sigma \rangle] \oplus \mathbb{Z}[G/\langle \tau \rangle]$.

Define

$$t = u_0 + u_1 + \sum_{0 \leq i \leq n-1} v_i,$$

$$x = u_0 + u_1 + \sum_{1 \leq i \leq n-1} v_i, \quad y = \frac{n-1}{2} u_0 + \frac{n+1}{2} u_1 + \sum_{0 \leq i \leq n-1} v_i.$$ 

Since $\sum_{0 \leq i \leq n} \sigma^i \cdot (v_1 + \cdots + v_{n-1}) = (n-1) \sum_{0 \leq i \leq n-1} v_i$, it follows that $\tau \cdot y = -y + \sum_{0 \leq i \leq n-1} \sigma^i(x)$. Then it is routine to verify that

$$\left( \bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot \sigma^i(x) \right) \oplus \mathbb{Z} \cdot y \simeq \tilde{M}_+, \quad \mathbb{Z} \cdot t \simeq \mathbb{Z}$$
by checking the actions of $\sigma$ and $\tau$ on lattices in both sides.

Now we will show that $\sigma(x), \sigma^2(x), \ldots, \sigma^{n-1}(x), x, y, t$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[G/\langle \sigma \rangle] \oplus \mathbb{Z}[G/\langle \tau \rangle]$. Write the determinant of these $n + 2$ elements with respect to the $\mathbb{Z}$-basis $u_0, u_1, v_0, v_1, \ldots, v_{n-1}$. We get the coefficient matrix $T$ as

\[
T = \begin{pmatrix}
1 & 1 & \cdots & 1 & \frac{n-1}{2} & 1 \\
1 & 1 & \cdots & 1 & \frac{n+1}{2} & 1 \\
1 & 1 & 0 & \frac{n-1}{2} & 1 \\
0 & 1 & 1 & \frac{n-1}{2} & 1 \\
1 & 0 & 1 & \frac{n-1}{2} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \frac{n-1}{2} & 1 \\
\end{pmatrix}
\]

The determinant of $T$ may be calculated as follows: Subtract the last column from each of the first $n$ columns. Also subtract $\frac{n-1}{2}$ times of the last column from the $(n + 1)$-th column. Then it is easy to see $\det(T) = 1$.

**Theorem 3.5** Let $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n$ is an odd integer. Then $\widetilde{M} \oplus \mathbb{Z}[G/\langle \tau \rangle] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}$.

**Proof.** The idea of the proof is similar to that of Theorem 3.4.

Let $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}, t$ be a $\mathbb{Z}$-basis of $\mathbb{Z}[G] \oplus \mathbb{Z}$ where $u_i$, $v_j$ correspond to $\sigma^i, \sigma^j \tau$ in $\mathbb{Z}[G]$. The actions of $\sigma$ and $\tau$ are given by

$\sigma : u_i \mapsto u_{i+1}, \ v_j \mapsto v_{j+1}, \ t \mapsto t,$

$\tau : u_i \leftrightarrow v_{n-i}, \ t \mapsto t$

where the index of $u_i$ or $v_j$ is understood modulo $n$.

Define $x, y, z \in \mathbb{Z}[G] \oplus \mathbb{Z}$ by

$\begin{align*}
x &= u_0 - v_0, \quad y = \left( \sum_{0 \leq i \leq n-1} u_i \right) + t, \\
z &= \left( \sum_{1 \leq i \leq \frac{n-1}{2}} u_i \right) + \left( \sum_{\frac{n+1}{2} \leq j \leq n-1} v_j \right) + t.
\end{align*}$

We claim that

$\left( \bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot \sigma^i(x) \right) \oplus \mathbb{Z} \cdot y \simeq \widetilde{M}, \quad \bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot \sigma^i(z) \simeq \mathbb{Z}[G/\langle \tau \rangle]$. 

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Since $\tau(x) = -x$, $\tau(z) = z$, it follows that $\tau \cdot \sigma^i(x) = -\sigma^{n-i}(x)$, $\tau \cdot \sigma^i(z) = \sigma^{n-i}(z)$. The remaining proof is omitted.

Now we will show that $\sigma(x), \sigma^2(x), \ldots, \sigma^{n-1}(x), x, y, \sigma(z), \ldots, \sigma^{n-1}(z), z$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[G] \oplus \mathbb{Z}$. Write the coefficient matrix of these elements with respect to the $\mathbb{Z}$-basis $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}, t$. We get $\det(T_n)$ where $T_n$ is a $(2n+1) \times (2n+1)$ integral matrix. For example,

$$T_3 = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

For any $n$, we evaluate $\det(T_n)$ by adding the $i$-th row to $(i+n)$-th row of $T_n$ for $1 \leq i \leq n$. We find that $\det(T_n) = \pm \det(T')$ where $T'$ is an $(n+1) \times (n+1)$ integral matrix. Note that all the entries of the $i$-th row of $T'$ (where $1 \leq i \leq n$) are one except one position, because of the definition of $z$ (and those of $\sigma^i(z)$ for $1 \leq i \leq n-1$). Subtract the last row from the $i$-th row where $1 \leq i \leq n$. We find $\det(T') = \pm 1$.

Before proving Theorem 3.7, we define the following matrix first. Let

$$\text{Circ}(c_0, c_1, \ldots, c_{n-1}) = \begin{pmatrix}
c_0 & c_{n-1} & \cdots & c_2 & c_1 \\
c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
\vdots & c_1 & c_0 & \ddots & \vdots \\
c_{n-2} & \ddots & \ddots & \ddots & c_{n-1} \\
c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{pmatrix}$$

be the $n \times n$ circulant matrix whose determinant is

$$\det(\text{Circ}(c_0, c_1, \ldots, c_{n-1})) = \prod_{k=0}^{n-1} (c_0 + c_1 \zeta_n^k + \cdots + c_{n-1} \zeta_n^{(n-1)k}).$$

**Lemma 3.6** Let $n \geq 3$ be an odd integer.

1. $\det(\text{Circ}(1, \ldots, 1, 0, \ldots, 0)) = \frac{n-1}{2}$.
2. $\det(\text{Circ}(-1, \ldots, -1, 0, 1, \ldots, 1, 0)) = -1$. 

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Proof. (1) follows from
\[
\det(Circ(1, \ldots, 1, 0, \ldots, 0)) = \prod_{k=0}^{n-1} \left( 1 + \zeta_n^k + \cdots + \zeta_n^{\frac{n-3}{2}k} \right) \\
= \frac{n-1}{2} \prod_{k=1}^{n-1} \left( 1 + \zeta_n^k + \cdots + \zeta_n^{\frac{n-3}{2}k} \right) \\
= \frac{n-1}{2}
\]
because \(1 + \zeta_n + \cdots + \zeta_n^{\frac{n-3}{2}} = \frac{1-\zeta_n^{\frac{n-1}{2}}}{1-\zeta_n}\) is a cyclotomic unit with
\[
\left( \frac{1-\zeta_n^{\frac{n-1}{2}}}{1-\zeta_n} \right)^{-1} = \frac{1-\zeta_n}{1-\zeta_n^{\frac{n-3}{2}}} = \frac{1-\zeta_n}{1-\zeta_n^{n-1}}(1 + \zeta_n^{\frac{n-1}{2}}) = -\zeta_n(1 + \zeta_n^{\frac{n-1}{2}}).
\]
(2) follows from
\[
\det(Circ(-1, \ldots, -1, 0, 1, \ldots, 1, 0)) \\
= \prod_{k=0}^{n-1} \left( -1 - \zeta_n^k - \cdots - \zeta_n^{\frac{n-3}{2}k} + \zeta_n^{\frac{n+1}{2}k} + \cdots + \zeta_n^{(n-2)k} \right) \\
= (-1) \prod_{k=1}^{n-1} \left( -1 - \zeta_n^k - \cdots - \zeta_n^{\frac{n-3}{2}k} + \zeta_n^{\frac{n+1}{2}k} + \cdots + \zeta_n^{(n-2)k} \right) \\
= -1
\]
because \(-1 - \zeta_n - \cdots - \zeta_n^{\frac{n-3}{2}} + \zeta_n^{\frac{n+1}{2}} + \cdots + \zeta_n^{n-2}\) is a unit with
\[
\left( -1 - \zeta_n - \cdots - \zeta_n^{\frac{n-3}{2}} + \zeta_n^{\frac{n+1}{2}} + \cdots + \zeta_n^{n-2} \right)^{-1} = \begin{cases} 
\sum_{k=1}^{\frac{n+1}{2}} (-1)^k \zeta_n^k & n \equiv 1 \pmod{4} \\
- \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \zeta_n^{-k} & n \equiv 3 \pmod{4}.
\end{cases}
\]

Theorem 3.7 Let \(G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n\) where \(n\) is an odd integer. Then \(\widetilde{M}_+ \oplus \widetilde{M}_- \simeq \mathbb{Z}[G] \oplus \mathbb{Z}[G/\langle \sigma \rangle].\)

Proof. Let \(u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\) be a \(\mathbb{Z}\)-basis of \(\mathbb{Z}[G]\) where \(u_i, v_j\) correspond to \(\sigma^i, \sigma^j\tau\) in \(\mathbb{Z}[G]\). The actions of \(\sigma\) and \(\tau\) are given by
\[
\sigma : u_i \mapsto u_{i+1}, \quad v_j \mapsto v_{j+1},
\]
\[
\tau : u_i \leftrightarrow v_{n-i}
\]
where the index of $u_i$ or $v_j$ is understood modulo $n$. Let $t_0$, $t_1$ be the $\mathbb{Z}$-basis of $\mathbb{Z}[G]/(\sigma)$ correspond to 1, $\tau$. Then $\sigma : t_0 \mapsto t_0$, $t_1 \mapsto t_1$, $\tau : t_0 \mapsto t_1$.

Define $x, y_0, z, y_1 \in \mathbb{Z}[G] \oplus \mathbb{Z}[G/(\sigma)]$ by

$$x = u_0 + \left( \sum_{\frac{n+1}{2} \leq i \leq n-1} u_i \right) + \left( \sum_{2 \leq j \leq \frac{n+1}{2}} v_j \right) + t_0 + t_1,$$

$$y_0 = \frac{n-1}{2} \left( \sum_{0 \leq j \leq n-1} v_j \right) + t_0 + (n-1)t_1,$$

$$z = u_0 + u_1 + \left( \sum_{\frac{n+2}{2} \leq i \leq n-1} u_i \right) - \left( \sum_{1 \leq j \leq \frac{n+1}{2}} v_j \right) + t_0 - t_1,$$

$$y_1 = \left( \sum_{0 \leq i \leq n-1} u_i \right) - \frac{n-1}{2} \left( \sum_{0 \leq j \leq n-1} v_j \right) + t_0 - (n-1)t_1.$$

It is easy to verify that

$$\left( \bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot \sigma^i(x) \right) \oplus \mathbb{Z} \cdot y_0 \simeq \widetilde{M}_+, \quad \left( \bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot \sigma^i(z) \right) \oplus \mathbb{Z} \cdot y_1 \simeq \widetilde{M}_-.$$

It remains to show that $\sigma^{\frac{n+3}{2}}(x), \ldots, \sigma^{-n-1}(x), x, \sigma(x), \ldots, \sigma^{\frac{n-3}{2}}(x), y_0, \sigma^{\frac{n-3}{2}}(z), \ldots, \sigma^{n-1}(z), z, \sigma(z), \ldots, \sigma^{\frac{n-3}{2}}(z), y_1$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[G] \oplus \mathbb{Z}[G/(\sigma)]$. Let $Q$ be the coefficient matrix of $\sigma^{\frac{n-3}{2}}(x), \ldots, \sigma^{n-1}(x), x, \sigma(x), \ldots, \sigma^{\frac{n-3}{2}}(x), y_0, \sigma^{\frac{n-3}{2}}(z), \ldots, \sigma^{n-1}(z), z, \sigma(z), \ldots, \sigma^{\frac{n-3}{2}}(z), y_1$ with respect to the $\mathbb{Z}$-basis $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}, t_0, t_1$.

The matrix $Q$ is defined as

$$Q = \begin{pmatrix}
\begin{array}{c|c|c|c}
\frac{n+1}{2} & \frac{n+1}{2} & \frac{n+1}{2} & 0 \\
\text{Circ}(1, \ldots, 1, 0, \ldots, 0) & \cdots & \text{Circ}(1, \ldots, 1, 0, \ldots, 0) & 1 \\
0 & 0 & \frac{n-1}{2} & \frac{n-1}{2} \\
\frac{n+1}{2} & \frac{n+1}{2} & \frac{n+1}{2} & \frac{n-1}{2} \\
\text{Circ}(0, \ldots, 0, 1, \ldots, 1) & \cdots & \text{Circ}(0, \ldots, 0, -1, \ldots, -1) & 1 \\
\frac{n-1}{2} & \frac{n-1}{2} & \frac{n-1}{2} & \frac{n-1}{2} \\
1 & 1 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots \\
n & n & n-1 & n-1 \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
\frac{n-1}{2} \\
\frac{n-1}{2} \\
\frac{n-1}{2} \\
\frac{n-1}{2} \\
1 \\
1 \\
\cdots \\
1 \\
n \\
\end{pmatrix}.$$

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For examples, when \( n = 3, 5, \), \( Q \) is of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & -1 & -1 & 0 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 2 & 0 & -1 & -1 \\
0 & 0 & 1 & 1 & 2 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 & -1 \\
1 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & -2 \\
1 & 1 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

We will show that \( \det(Q) = -1 \). For a given matrix, we denote by \((C_i)\) its \( i\)-th column. When we say that, apply \((C_i) + (C_1)\) on the \( i\)-th column, we mean the column operation by adding the 1-st column to the \( i\)-th column.

On the \((2n+2)\)-th column, apply \(C(2n+2) + C(n+1)\). On the \((n+1)\)-th column, apply \(C(n+1) + \frac{n-1}{2}(C(2n+2))\). On the \((n+1)\)-th column, apply \(C(n+1) - (C_1) - \cdots - (C_n)\). Then all the entries of the \((n+1)\)-th column are zero except for the last \((2n+2)\)-th entry, which is \(-1\). Hence it is enough to show \(\det(Q_0) = 1\) where \(Q_0\) is a \((2n+1) \times (2n+1)\) matrix defined by

\[
Q_0 = \begin{pmatrix}
\left( \begin{array}{c}
\frac{n-1}{2} \\
\frac{n+1}{2} \\
\vdots \\
n \end{array} \right)
& \left( \begin{array}{c}
\frac{n+1}{2} \\
\frac{n-1}{2} \\
\vdots \\
n \end{array} \right)
& \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \end{array} \right)
& \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \end{array} \right)
\end{pmatrix}
\begin{pmatrix}
\left( \begin{array}{c}
\frac{n+1}{2} \\
\frac{n-1}{2} \\
\vdots \\
n \end{array} \right)
& \left( \begin{array}{c}
\frac{n-1}{2} \\
\frac{n+1}{2} \\
\vdots \\
n \end{array} \right)
& \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \end{array} \right)
& \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \end{array} \right)
\end{pmatrix}
\end{pmatrix}
\]
On the \((n+i)\)-th column, apply \(C(n+i) + C(f(n+i))\) for \(i = 1, \ldots, n\) where
\[
f(k) = \begin{cases} k & k \leq n \\ k-n & k > n. \end{cases}
\]

On the \((n+i)\)-th column, apply \(C(n+i) - C(2n+1)\) for \(i = 1, \ldots, n\). On the \((2n+1)\)-th column, apply \(C(2n+1) - \frac{2}{n-1}\{(C1) + \cdots + (Cn)\}\). Thus we get \(\det(Q_0) = \det(Q_1)\) where

\[
Q_1 = \begin{pmatrix}
\text{Circ}(1, \ldots, 1, 0, \ldots, 0) & \mathbf{0} & \vdots & 0 \\
\text{Circ}(0, \ldots, 0, 1, \ldots, 1) & \text{Circ}(0, 1, \ldots, 1, 0, -1, \ldots, -1) & \vdots & 0 \\
1 & \cdots & 1 & 0 \\
0 & \cdots & 0 & -\frac{2}{n-1}
\end{pmatrix}
\]

Because of Lemma 3.6 and
\[
\det(\text{Circ}(0, 1, \ldots, 1, 0, -1, \ldots, -1)) = \det(\text{Circ}(-1, \ldots, -1, 0, 1, \ldots, 1, 0)),
\]
we find \(\det(Q_1) = 1\).

**Proposition 3.8** Let \(G \simeq D_n\) where \(n\) is an odd integer. The all the flabby \(G\)-lattices are invertible. Consequently, if \(k\) is a field admitting a \(D_n\)-extension, then all the \(G\)-tori over \(k\) are retract \(k\)-rational.

**Proof.** Since all the Sylow subgroups of \(G\) are cyclic, the flabby \(G\)-lattices are invertible by Theorem 2.5. For a \(G\)-torus over \(k\), its function field is \(K(M)^G\) for some \(G\)-lattice \(M\), some \(G\)-extension \(K/k\). Since \([M]^G\) is invertible, we may apply Theorem 2.4.

§4. Integral representations of \(D_p\)

Let \(G = \langle \sigma, \tau : \sigma^p = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \simeq D_p\) where \(p\) is an odd prime number. Define \(H = \langle \tau \rangle\).

Denote \(\zeta_p\) a primitive \(p\)-th root of unity, \(R = \mathbb{Z}[\zeta_p]\), \(R_0 = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]\), \(h_p^+\) the class number of \(R_0\), \(P = \langle 1 - \zeta_p \rangle\) the unique maximal ideal of \(R\) lying over \(\langle p \rangle \subset \mathbb{Z}\). We
may regard $R$ as a $G$-lattice by defining, for any $\alpha \in R$, $\sigma \cdot \alpha = \zeta_p \alpha$, $\tau \cdot \alpha = \bar{\alpha}$ the complex conjugate of $\alpha$. Note that $R$ is a $G$-lattice of rank $p - 1$; it is even a lattice over $\mathbb{Z}[G]/\Phi_p(\sigma) \simeq \mathbb{Z}[\zeta] \circ H$, the twisted group ring defined in Definition \ref{3.2}.

If $I \subset R$ is an ideal with $\sigma(I) \subset I$, $\tau(I) \subset I$, then $I$ may be regarded as a $G$-lattice also. In particular, if $A \subset R_0$ is an ideal, then $RA$, $PA$ are $G$-lattices of rank $p - 1$.

A complete list of non-isomorphic indecomposable $G$-lattices was proved by Myrna Pike Lee [Le]. In the following we adopt the reformulation of Lee’s Theorem in [CR1, page 752, Theorem (34.51)]. In the following theorem $\mathbb{Z}_-$ is the $G$-lattice on which $\sigma$ acts trivially, and $\tau$ acts as multiplication by $-1$.

**Theorem 4.1** (M. P. Lee [Le], [CR1, page 752]) Let $G \simeq D_p$ where $p$ is an odd prime number. Let $\mathcal{A}$ range over a full set of representatives of the $h_p^+$ ideal classes of $R_0$ where $R_0 = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$, $R = \mathbb{Z}[\zeta_p]$, $P = \langle 1 - \zeta_p \rangle$. Then there are precisely $7h_p^+ + 3$ isomorphism classes of indecomposable $G$-lattices, and there are represented by

\[
\mathbb{Z}, \mathbb{Z}, \mathbb{Z}[H] \simeq \mathbb{Z}[G/\langle \sigma \rangle]; \ RA, PA;
\]

and the non-split extensions

- $0 \to PA \to V_A \to \mathbb{Z} \to 0$,
- $0 \to RA \to (Y_0)_A \to \mathbb{Z}[H] \to 0$,
- $0 \to RA \oplus P \to (Y_2)_A \to \mathbb{Z}[H] \to 0$.

**Remark.** In the above theorem, the words “the non-split extensions $0 \to PA \to V_A \to \mathbb{Z} \to 0$,...” means that, if $M$ is an indecomposable $G$-lattice satisfying that $0 \to PA \to M \to \mathbb{Z} \to 0$, then $M \simeq V_A$ as $G$-lattices, i.e. there is essentially a unique indecomposable lattice arising from an extension of $\mathbb{Z}$ by $PA$. See [CR1] pages 711-730 and the proof in Lee’s paper [Le].

**Definition 4.2** In Theorem 4.1, when $\mathcal{A}$ is a principal ideal in $R_0$, we will write the corresponding $G$-lattices by $R$, $P$, $0 \to P \to V \to \mathbb{Z} \to 0$, $0 \to R \to X \to \mathbb{Z} \to 0$, $0 \to R \to Y_0 \to \mathbb{Z}[H] \to 0$, $0 \to P \to Y_1 \to \mathbb{Z}[H] \to 0$, $0 \to R \oplus P \to Y_2 \to \mathbb{Z}[H] \to 0$.

If $l$ is a prime number of $\mathbb{Z}$, denote by $\mathbb{Z}_l = \{m/n : m, n \in \mathbb{Z}, l \nmid n\}$ the localization of $\mathbb{Z}$ at the prime ideal $\langle l \rangle$. Since $(R_0)_l = \mathbb{Z}_l[\zeta_p + \zeta_p^{-1}]$ is a semi-local principal ideal domain, we find that $\mathcal{A}_l$ is a principal ideal in $(R_0)_l$ for any prime number $l$.

It follows that, if $\mathcal{A}$ is any ideal in $R_0$, then $R$ and $RA$, $P$ and $PA$, $V$ and $(V)_{\mathcal{A}}$, $X$ and $X_{\mathcal{A}}$, ... belong to the same genus, i.e. they become isomorphic after localization at any prime ideal $\langle l \rangle$ of $\mathbb{Z}$ (see [CR1] page 642)).

We will show that $M_+, M_-$, $N_+, N_-$, $\tilde{M}_+, \tilde{M}_-$ defined in Section 3 are isomorphic to $V$, $X$, $R$, $P$, $Y_0$, $Y_1$ when $n = p$ is an odd prime number.

**Lemma 4.3** Let $N_+$ and $N_-$ be the $G$-lattices with $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n$ is an odd integer. If $n = p$ is an odd prime number, then $N_+ \simeq R$ and $N_- \simeq P$. 

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Proof. By definition, when \( n = p \), \( N_+ \) has a \( \mathbb{Z} \)-basis \( \{ u_i : 1 \leq i \leq p - 1 \} \) with \( \sigma : u_1 \mapsto u_2, u_2 \mapsto \cdots, u_{p-1} \mapsto -(u_1 + u_2 + \cdots + u_p) \), \( \tau : u_i \mapsto u_{p-i} \). On the other hand, \( R = \sum_{0 \leq i < p} \mathbb{Z} \cdot \zeta^i_p \) has a \( \mathbb{Z} \)-basis \( \{ \zeta_p^i : 1 \leq i \leq p - 1 \} \) with \( \sigma : \zeta_p^i \mapsto \zeta_p^{2i} \mapsto \cdots \mapsto \zeta_p^{p-1} \mapsto -(\zeta_p^1 + \cdots + \zeta_p^i) \), \( \tau : \zeta_p^i \mapsto \zeta_p^{p-i} \). Hence the result.

For the proof of \( N_- \simeq P \), note that \( P = R(1 - \zeta_p) = \sum_{0 \leq i < p} \mathbb{Z}(\zeta_p^i - \zeta_p^{i+1}) \). Define \( v_0 = \zeta_p^{i+1} - \zeta_p^{-1} \), \( v_i = \sigma^i(v_0) \) for \( 0 \leq i \leq p - 1 \). Then \( \{ v_1, v_2, \ldots, v_{p-1} \} \) is a \( \mathbb{Z} \)-basis of \( P \) with \( \sigma : v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p-1} \mapsto -(v_1 + v_2 + \cdots + v_{p-1}) \) and \( \tau : v_i \mapsto -v_{p-i} \) because \( \tau(\zeta_p) = \zeta_p^{-1} \). Thus \( P \simeq N_- \). \( \blacksquare \)

**Lemma 4.4** Let \( M_+, M_-, N_+, N_- \) be \( G \)-lattices with \( G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n \) where \( n \) is an odd integer. Then there are non-split exact sequences of \( G \)-lattices \( 0 \to N_- \to M_+ \to \mathbb{Z} \to 0, 0 \to N_+ \to M_- \to \mathbb{Z} \to 0 \). When \( n = p \) is an odd prime number, then \( M_+ \simeq V, M_- \simeq X \).

**Proof.** Case 1. \( M_+ \).

By definition, choose a \( \mathbb{Z} \)-basis \( \{ x_i : 0 \leq i \leq n - 1 \} \) of \( M_+ \) such that \( \sigma : x_i \mapsto x_{i+1}, \tau : x_i \mapsto x_{n-i} \) where the index is understood modulo \( n \).

Define \( u_0 = x_{n-1}^{-1} - x_{n+1}^{-1}, t = x_{n-1}, u_i = \sigma^i(u_0) \) for \( 0 \leq i \leq n - 1 \).

It follows that \( \sum_{0 \leq i < n} u_i = 0 \) and \( \{ u_1, u_2, \ldots, u_{n-1}, t \} \) is a \( \mathbb{Z} \)-basis of \( M_+ \) with \( \sigma \) and \( \tau \) acting by

\[
\sigma : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{n-1} \mapsto u_0 = -(u_1 + u_2 + \cdots + u_{n-1}),
\]

\[
t \mapsto t + u_1 + u_2 + \cdots + u_{n-1},
\]

\[
\tau : u_i \mapsto -u_{n-i}, t \mapsto t + u_1 + u_2 + \cdots + u_{n-1}.
\]

Note that \( \sum_{1 \leq i \leq n-1} \mathbb{Z} \cdot u_i \simeq N_- \) and \( M_+/(\sum_{1 \leq i \leq n-1} \mathbb{Z} \cdot u_i) \simeq \mathbb{Z} \). Hence we get the sequence \( 0 \to N_- \to M_+ \to \mathbb{Z} \to 0 \).

This sequence doesn’t split. Otherwise, there is some element \( s \in M_+ \) such that \( \sigma(s) = \tau(s) = s \), and \( \{ u_1, u_2, \ldots, u_{n-1}, s \} \) is a \( \mathbb{Z} \)-basis of \( M_+ \).

Write \( s = \sum_{1 \leq i \leq n-1} a_i \cdot u_i + b \cdot t \) where \( a_i, b \in \mathbb{Z} \). Because \( \{ u_1, \ldots, u_{n-1}, s \} \) is a \( \mathbb{Z} \)-basis of \( M_+ \), we find that \( b = \pm 1 \).

Consider the case \( b = -1 \) (the situation \( b = 1 \) can be discussed similarly). Since \( \tau(\sum_{1 \leq i \leq n-1} a_i u_i - t) = \sum_{1 \leq i \leq n-1} a_i u_i - t \), we find that \( a_i - 1 = a_{n-i} \) and \( a_{n-i} - 1 = a_i \) for all \( 1 \leq i \leq n - 1 \). This is impossible.

Now assume that \( n = p \) is an odd prime number. We will show that \( M_+ \simeq V \).

By Lemma 4.3 \( N_- \simeq P \). Thus we have a non-split extension \( 0 \to P \to M_+ \to \mathbb{Z} \to 0 \). Then apply the remark after Theorem 4.1 More precisely, it is proved in [Le] page 221] that, up to \( G \)-lattice isomorphisms, there is precisely one indecomposable \( G \)-lattice arising from extensions of \( \mathbb{Z} \) by \( P \), although \( \operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, P) = \mathbb{Z}/p\mathbb{Z} \) by [Le] Lemma 2.1. Since \( 0 \to P \to V \to \mathbb{Z} \to 0 \) is a non-split extension by Theorem 4.1 we conclude that \( M_+ \simeq V \).

Case 2. \( M_- \).
The proof is similar to Case 1. Choose a $\mathbb{Z}$-basis $\{x_i : 0 \leq i \leq n - 1\}$ of $M_-$ with

$\sigma : x_i \mapsto x_{i+1}$, $\tau : x_i \mapsto -x_{n-i}$.

Define $u_0 = x_{n+1} - x_{n+1}$, $t = x_{n+1}$, $u_i = \sigma^i(u_0)$ for $0 \leq i \leq n - 1$. We find that

$\sigma : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{n-1} \mapsto -(u_1 + u_2 + \cdots + u_{n-1})$,

$t \mapsto t + u_1 + u_2 + \cdots + u_{n-1}$,

$\tau : u_i \leftrightarrow u_{n-i}$, $t \mapsto -t - u_1 - u_2 - \cdots - u_{n-1}$.

Thus $\sum_{0 \leq i \leq n-1} \mathbb{Z} \cdot u_i \simeq N_+$ and $M_-/(\sum_{1 \leq i \leq n-1} \mathbb{Z} u_i) \simeq \mathbb{Z}_-$.

Similarly, the sequence $0 \to N_+ \to \tilde{M}_- \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0$, $0 \to N_- \to \tilde{M}_+ \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0$. When

$n = p$ is an odd prime number, then $\tilde{M}_- \simeq Y_0$, $\tilde{M}_+ \simeq Y_1$.

**Lemma 4.5** Let $N_+$, $N_-$, $\tilde{M}_+$, $\tilde{M}_-$ be $G$-lattices with $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n$ is an odd integer. Then there are non-split exact sequences of $G$-lattices $0 \to N_+ \to \tilde{M}_- \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0$, $0 \to N_- \to \tilde{M}_+ \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0$. When

$n = p$ is an odd prime number, then $\tilde{M}_- \simeq Y_0$, $\tilde{M}_+ \simeq Y_1$.

Proof. **Case 1.** $\tilde{M}_+$. We adopt the same notations $x_0, x_1, \ldots, x_{n-1}, u_1, \ldots, u_{n-1}$ in the proof of Lemma 4.4 Write $\tilde{M}_+ = (\bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot x_i) \oplus \mathbb{Z} \cdot w$ with

$\sigma : x_i \mapsto x_{i+1}$, $w \mapsto w$,

$\tau : x_i \mapsto x_{n-i}$, $w \mapsto -w + x_0 + x_1 + \cdots + x_{n-1}$

where the index is understood modulo $n$.

Define $u_0 = x_{n+1} - x_{n+1}$, $t = x_{n+1}$, $u_i = \sigma^i(u_0)$ for $0 \leq i \leq n - 1$.

We claim that $\sum_{0 \leq i \leq n-1} x_i = nt - (n-1)u_0 - (n-2)u_1 - \cdots - u_{n-2}$.

Since $x_{n+1} = t$, $x_{n+1} = t - u_0$, we find that $x_{n+1} = x_{n+1} - u_1 = t - u_0 - u_1$. By induction, we may find similar formulae for $x_{n+1}, \ldots, x_{n-1}, x_0, \ldots, x_{n-2}$. In particular, $x_{n+1} = t - u_0 - u_1 - \cdots - u_{n-2}$. Thus the formula of $\sum_{0 \leq i \leq n-1} x_i$ is found.

Note that $\{u_1, \ldots, u_{n-1}, t, w\}$ is a $\mathbb{Z}$-basis of $\tilde{M}_+$ and

$\sigma : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{n-1} \mapsto -(u_1 + u_2 + \cdots + u_{n-1})$, $t \mapsto t + \sum_{1 \leq i \leq n-1} u_i$, $w \mapsto w$,

$\tau : u_i \leftrightarrow u_{n-i}$, $t \mapsto t + \sum_{1 \leq i \leq n-1} u_i$, $w \mapsto -w + u_1 + 2u_2 + \cdots + (n-1)u_{n-1} + nt$.

Define $w_0 = -\frac{n-1}{2}t + w$, $w_1 = \frac{n+1}{2}t - w$. Then $\{u_1, \ldots, u_{n-1}, w_0, w_1\}$ is also a $\mathbb{Z}$-basis of $\tilde{M}_+$ with

$\sigma : w_0 \mapsto w_0 - \frac{n-1}{2} \sum_{1 \leq i \leq n-1} u_i$, $w_1 \mapsto w_1 + \frac{n+1}{2} \sum_{1 \leq i \leq n-1} u_i$,

$\tau : w_0 \mapsto w_1 - \frac{n-3}{2}u_1 - \frac{n-5}{2}u_2 - \cdots - u_{n-3} + u_{n+1} + 2u_{n+2} + \cdots + \frac{n+1}{2}u_{n-1}$,

$w_1 \mapsto w_0 + \frac{n+1}{2}u_1 + \frac{n-3}{2}u_2 + \cdots + u_{n+1} - u_{n+3} - 2u_{n+5} - \cdots - \frac{n-3}{2}u_{n-1}$.
Note that \( \sum_{1 \leq i \leq n-1} \mathbb{Z} \cdot u_i \cong \mathbb{Z} \) and \( \tilde{M}_+/(\sum_{1 \leq i \leq n-1} \mathbb{Z} \cdot u_i) \cong \mathbb{Z}[G/\langle \sigma \rangle] \). It follows that we get an exact sequence \( 0 \to \mathbb{Z} \to \tilde{M}_+ \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0 \).

We will show that this exact sequence doesn’t split.

Suppose not. Then there exists \( s \in \tilde{M}_+ \) such that \( \sigma(s) = s \) and \( \{u_1, \ldots, u_{n-1}, s, \tau(s)\} \) is a \( \mathbb{Z} \)-basis of \( \tilde{M}_+ \).

Write \( s = \sum_{1 \leq i \leq n-1} a_i u_i + b_0 w_0 + b_1 w_1 \) where \( a_i, b_j \in \mathbb{Z} \). Since \( \tau(s) = \sum_{1 \leq i \leq n-1} a'_i u_i + b_1 w_0 + b_0 w_1 \) for some integers \( a'_i \in \mathbb{Z} \), it follows that \( b_1^2 - b_0^2 = \pm 1 \) (remember that \( \{u_1, \ldots, u_{n-1}, s, \tau(s)\} \) is a \( \mathbb{Z} \)-basis of \( \tilde{M}_+ \)). It follows that the only solutions for the pair \( (b_0, b_1) = (-1,0), (0,\pm 1) \).

We consider the situation \( (b_0, b_1) = (1,0) \) (the other situations may be discussed similarly). Write \( s = \sum_{1 \leq i \leq n-1} a_i u_i + w_0 \) as before. Since \( \sigma(s) = s \), we find an identity of the ordered \((n - 1)\)-tuples: \( (a_1, a_2, \ldots, a_{n-1}) = (0, a_1, a_2, \ldots, a_{n-2}) - a_{n-1}(1, 1, \ldots, 1) - n \frac{1}{2}(1, 1, \ldots, 1) \). Solve \( a_1, a_2, \ldots, a_{n-1} \) inductively in terms of \( a_{n-1} \).

We find \( a_1 = -(a_{n-1} + \frac{n}{2}) \), \( a_2 = -2(a_{n-1} + \frac{n-1}{2}) \), \ldots, \( a_{n-1} = -(n-1)(a_{n-1} + \frac{n-1}{2}) \). Hence \( na_{n-1} = -(n-1)^2/2 \). But gcd\{\( n, n-1 \}\} = 1. Thus we find a contradiction.

In conclusion, \( 0 \to \mathbb{Z} \to \tilde{M}_+ \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0 \) doesn’t split.

When \( n = p \) is an odd prime number, we get a non-split extension \( 0 \to \mathbb{Z} \to \tilde{M}_+ \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0 \). It is proved by Lee (see the last paragraph of [Le, page 221]) that the non-split extensions of \( \mathbb{Z}[G/\langle \sigma \rangle] \) by \( \mathbb{P} \) give rise to precisely one indecomposable \( G \)-lattice, although \( \text{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}[G/\langle \sigma \rangle], P) \cong \mathbb{Z}/p\mathbb{Z} \) by [Le, Lemma 2.1]. Since \( 0 \to \mathbb{Z} \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0 \) is also a non-split extension, we conclude that \( \tilde{M}_+ \cong Y_1 \).

**Case 2.** \( \tilde{M}_- \)

The proof is similar. We adopt the notations \( x_0, x_1, \ldots, x_{n-1}, u_1, \ldots, u_{n-1} \) in the proof of Case 1. Write \( \tilde{M}_- = (\bigoplus_{0 \leq i \leq n-1} \mathbb{Z} \cdot x_i) \oplus \mathbb{Z} \cdot w \) such that

\[
\sigma : x_i \mapsto x_{i+1}, \quad w \mapsto w,
\]

\[
\tau : x_i \mapsto -x_{n-i}, \quad w \mapsto w - \sum_{0 \leq i \leq n-1} x_i
\]

where \( 0 \leq i \leq n-1 \) and the index is understood modulo \( n \).

Define \( u_0 = x_{\frac{n-1}{2}} - x_{\frac{n+1}{2}}, \quad t = x_{\frac{n+1}{2}} \), \( u_i = \sigma^i(u_0) \) for \( 0 \leq i \leq n-1 \). Then \( \{u_1, \ldots, u_{n-1}, t, w\} \) is a \( \mathbb{Z} \)-basis of \( \tilde{M}_- \) and

\[
\sigma : u_i \mapsto u_{i+1} \mapsto \cdots \mapsto u_{n-1} \mapsto -(u_1 + \cdots + u_{n-1}),
\]

\[
t \mapsto t + u_1 + u_2 + \cdots + u_{n-1}, \quad w \mapsto w,
\]

\[
\tau : u_i \mapsto u_{n-i}, \quad t \mapsto -t - \left( \sum_{1 \leq i \leq n-1} u_i \right),
\]

\[
w \mapsto w - u_1 - 2u_2 - \cdots - (n-1)u_{n-1} - nt.
\]
Define $w_0 = \frac{n-1}{2}t - w$, $w_1 = \frac{n+1}{2} - w$. We find that

\[
\sigma : w_0 \mapsto w_0 + \frac{n-1}{2} \sum_{1 \leq i \leq n-1} u_i, \quad w_1 \mapsto w_1 + \frac{n+1}{2} \sum_{1 \leq i \leq n-1} u_i,
\]

\[
\tau : w_0 \mapsto w_0 - \frac{n-3}{2} u_1 - \frac{n-5}{2} u_2 - \cdots - u_{n-3} + u_{n+1} + 2u_{n+3} + \cdots + \frac{n-1}{2} u_{n-1},
\]

\[
w_1 \mapsto w_0 - \frac{n-1}{2} u_1 - \frac{n-3}{2} u_2 - \cdots - u_{n-1} + u_{n+1} + 2u_{n+3} + \cdots + \frac{n-3}{2} u_{n-1}.
\]

The remaining proof is similar and is omitted. ■

**Lemma 4.6** Let $N_+, N_-$ be $G$-lattices with $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n$ is an odd integer. Then there are non-split exact sequences of $G$-lattices $0 \to N_+ \oplus N_- \to \mathbb{Z}[G] \to \mathbb{Z}[G/\langle \tau \rangle] \to 0$. When $n = p$ is an odd prime number, then $Y_2 \simeq \mathbb{Z}[G]$.

**Proof.** This lemma was proved by Lee for the case when $n = p$ is an odd prime number in (i) of Case 1 of [Le, pages 222–224]. There was also a remark in the first paragraph of [Le, page 229, Section 4].

Here is a proof when $n$ is an odd integer. Once the first part is proved, we may deduce the second part when $n = p$ is an odd prime number because $N_+ \simeq R$, $N_- \simeq P$ (by Lemma 4.3) and there is a unique indecomposable $G$-lattice arising from non-split extensions of $\mathbb{Z}[G/\langle \tau \rangle]$ by $R \oplus P$ (see [Le, page 222]). Hence $\mathbb{Z}[G] \simeq Y_2$.

Now we start to prove the first part with $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n$ is an odd integer.

From now on till the end of the proof, denote $\zeta = \zeta_n$ a primitive $n$-th root of unity, $R = \mathbb{Z}[\zeta]$, $R_0 = \mathbb{Z}[\zeta + \zeta^{-1}]$, $H = \langle \tau \rangle$.

Step 1. Let $\{\sigma^i, \sigma^i \tau : 0 \leq i \leq n-1\}$ be a $\mathbb{Z}$-basis of $\mathbb{Z}[G]$.

Let $\{t_0, t_1\}$ be a $\mathbb{Z}$-basis of $\mathbb{Z}[H]$ with $\sigma(t_i) = t_i$, $\tau : t_0 \leftrightarrow t_1$.

Define a $G$-lattice surjection $\varphi : \mathbb{Z}[G] \to \mathbb{Z}[H]$ by $\varphi(\sigma^i) = t_0$, $\varphi(\sigma^i \tau) = t_1$. Define a $G$-lattice $\mathcal{M}$ by $\mathcal{M} = \text{Ker}(\varphi)$. We will prove that $\mathcal{M} \simeq N_+ \oplus N_-$ (note that $\mathbb{Z}[G]$ is indecomposable [Sw1]).

Define $u_i, v_i \in \mathcal{M}$ as follows. Define $u_0 = \sigma^{(n-1)/2} - \sigma^{(n+1)/2}$, $v_0 = \sigma^{(n+1)/2} - \sigma^{(n-1)/2} \tau$, and $u_i = \sigma^i(u_0)$, $v_i = \sigma^i(v_0)$ for $0 \leq i \leq n-1$.

It follows that $\sum_{0 \leq i \leq n-1} u_i = \sum_{0 \leq i \leq n-1} v_i = 0$, and $\{u_i, v_i : 1 \leq i \leq n-1\}$ is a $\mathbb{Z}$-basis of $\mathcal{M}$. Moreover, it is easy to see that $\sigma : u_i \mapsto u_{i+1}, v_i \mapsto v_{i+1}, \tau : u_i \mapsto v_{n-i}, v_i \mapsto u_{n-i}$ where the index is understood modulo $n$.

Step 2

Define $x_i = u_i + v_i$, $y_i = u_{i-1} - v_{i+1}$ where $0 \leq i \leq n-1$. Clearly $\sum_{0 \leq i \leq n-1} x_i = \sum_{0 \leq i \leq n-1} y_i = 0$. We claim that $\{x_i, y_i : 1 \leq i \leq n-1\}$ is a $\mathbb{Z}$-basis of $\mathcal{M}$.

Assume the above claim. Define $M_1 = \bigoplus_{1 \leq i \leq n-1} \mathbb{Z} \cdot x_i$, $M_2 = \bigoplus_{1 \leq i \leq n-1} \mathbb{Z} \cdot y_i$. It is easy to verify that $M_1 \simeq N_+$ and $M_2 \simeq N_-$. Hence the proof that $\mathcal{M} \simeq N_+ \oplus N_-$ is finished.
Step 3

We will prove that \( \{x_i, y_i : 1 \leq i \leq n - 1\} \) is a \( \mathbb{Z} \)-basis of \( M \).

Let \( Q \) be the coefficient matrix of \( x_1, x_2, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \) with respect to the \( \mathbb{Z} \)-basis \( u_1, u_2, \ldots, u_{n-1}, v_1, \ldots, v_{n-1} \). For the sake of visual convenience, we will consider the matrix \( P \) which is the transpose of \( Q \). We will show that \( \det(P) = 1 \).

The matrix \( P \) is defined as

\[
P = \begin{pmatrix}
1 & 1 \\
& \ddots & \ddots \\
& 1 & -1 & -1 & 0 & -1 \\
& 1 & 0 & \ddots & \ddots \\
& 0 & 0 & \ddots & 0 & -1 \\
1 & 0 & 1 & \cdots & 1
\end{pmatrix}.
\]

For examples, when \( n = 3, 5 \), it is of the form

\[
P = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & -1 & 0 & -1 \\
1 & 0 & 1 & 1
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

In the case \( n = 3, 5 \), it is routine to show that \( \det(P) = 1 \). When \( n \geq 7 \), we will apply column operations on the matrix \( P \) and then expand the determinant along a row. Thus we are reduced to matrices of smaller size.

For a given matrix, we denote by \((Ci)\) its \( i \)-th column. When we say that, apply \((Ci) + (C1)\) on the \( i \)-th column, we mean the column operation by adding the 1-st column to the \( i \)-th column.

Step 4

We will prove \( \det(P) = 1 \) where \( P \) is the \((2n-2) \times (2n-2)\) integral matrix defined in Step 3. Suppose \( n \geq 7 \).
Apply column operations on the matrix $P$. On the $(n + i)$-th column where $0 \leq i \leq n - 2$, apply $C(n + i) - C(i + 1)$.

Thus all the entries of the right upper part of the resulting matrix vanish. We get $\det(P) = \det(P_0)$ where $P_0$ is an $(n - 1) \times (n - 1)$ integral matrix defined as

$$P_0 = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & -1 \\
-1 & 0 & -1 \\
-1 & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
-1 & 0 & -1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1
\end{pmatrix}.$$  

Step 5
Apply column operations on $P_0$. On the 3rd column, apply $(C3) - (C1)$. Then, on the 4th column, apply $(C4) - (C2)$.

In the resulting matrix, each of the 2nd row and the 3rd row have only one non-zero entry.

Thus $\det(P_0) = \det(P_1)$ where $P_1$ is an $(n - 3) \times (n - 3)$ integral matrix defined as

$$P_1 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & -1 \\
-1 & 0 & -1 \\
-1 & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
-1 & 0 & -1 \\
0 & 0 & 1 & \cdots & 1 & 0 & 1
\end{pmatrix}.$$  

Step 6
Apply column operations on $P_1$. On the 3rd column, apply $(C3) - (C1)$. On the 4th column, apply $(C4) - (C2)$.

Then expand the determinant along the 2nd row and the 3rd row. We get $\det(P_1) = \det(P_2)$ where $P_2$ is an $(n - 5) \times (n - 5)$ integral matrix defined as

$$P_2 = \begin{pmatrix}
1 & 0 & 1 & \cdots & 1 \\
-1 & 0 & -1 \\
\ddots & \ddots & \ddots \\
-1 & 0 & -1 \\
1 & \cdots & 1 & 0 & 1
\end{pmatrix}.$$  

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Lemma 4.7  Let $M_1$ and $M_2$ be $G$-lattices belonging to the same genus. Then $M_1$ is flabby if and only if so is $M_2$.

Proof. For any subgroup $S$ of $G$, $H^{-1}(S, M_1)$ is a finite abelian group. If $H^{-1}(S, M_1) \neq 0$ for some subgroup $S$, then there is some prime number $l$ such that

\[ H^{-1}(S, \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M_1) \neq 0 \]

because localization commutes with taking Tate cohomology. Thus $H^{-1}(S, M_2) \neq 0$ because $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M_1 \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M_2$.

We thank Kunyavskii for providing the following alternative proof. By Roiter’s Theorem [CR1, page 660], there is a rank-one projective module $L$ over $\mathbb{Z}[G]$ such that $M_1 + L \simeq M_2 + \mathbb{Z}[G]$. Hence the result.

Proposition 4.8  Let the notations be the same as in Theorem 4.1. Then the indecomposable $G$-lattices which are flabby are

\[ \mathbb{Z}, \mathbb{Z}[H], V_A, (Y_0)_A, (Y_1)_A, (Y_2)_A, \]

while the remaining ones are not flabby.

Proof. Remember that $V_A$ and $V$ belonging to the same genus (see Definition 4.2). Apply Lemma 4.7. It suffices to check whether the following ten lattices

\[ \mathbb{Z}, \mathbb{Z}[H], V, Y_0, Y_1, Y_2, \mathbb{Z}_-, R, P, X \]

are flabby lattices.

Since $\mathbb{Z}, \mathbb{Z}[H]$ are permutation lattices, they are flabby.

By Lemma 4.4, Lemma 4.5 and Lemma 4.6, we find that $V \simeq M_+, Y_0 \simeq \tilde{M}_-, Y_1 \simeq \tilde{M}_+, Y_2 \simeq \mathbb{Z}[G]$. $M_+$ and $\mathbb{Z}[G]$ are permutation lattices. Hence they are flabby. As to $\tilde{M}_-$ and $\tilde{M}_+$. Applying Theorem 3.3 and Theorem 3.5, we find that they are stably permutation. Thus they are flabby also.

Now we turn to the non-flabby cases. Since $G = D_p$, a flabby lattice is necessarily an invertible lattice by Theorem 2.5; thus it is also coflabby. In summary, if we want to show that a $G$-lattice $M$ is not flabby, we may show that it is not coflabby or it is not invertible.

For $\mathbb{Z}_-$, $H^1(G, \mathbb{Z}_-) = \mathbb{Z}/2\mathbb{Z}$. For, write $\mathbb{Z}_- = \mathbb{Z} \cdot w$ with $\sigma(w) = w, \tau(w) = -w$. Apply the Hochschild-Serre spectral sequence $0 \to H^1(\langle \sigma, \mathbb{Z}_-^{(\sigma)} \rangle) \to H^1(G, \mathbb{Z}_-) \to H^1(\langle \sigma \rangle, \mathbb{Z}_-^{(\sigma)})$. Because $H^1(\langle \sigma \rangle, \mathbb{Z}_-) = 0$ and $H^1(\langle \sigma \rangle, \mathbb{Z}_-^{(\sigma)}) = \mathbb{Z}/2\mathbb{Z}$, we find that $H^1(G, \mathbb{Z}_-) = \mathbb{Z}/2\mathbb{Z} \neq 0$.

For $R$, if $R$ is a flabby $G$-lattice, then it is invertible. Restricted to the subgroup $S = \langle \sigma \rangle$, $R$ become an invertible $S$-lattice.

From the short exact sequence of $S$-lattices $0 \to R \to \mathbb{Z}[S] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where $\varepsilon$ is the augmentation map, since $R$ is $S$-invertible and $\mathbb{Z}$ is $S$-permutation, it follows that
the sequence splits [Len. Proposition 1.2]. Thus $\mathbb{Z}[S] \simeq R \oplus \mathbb{Z}$ is not indecomposable as an $S$-lattice. This leads to a contradiction.

For $P$, if $P$ is a flabby $G$-lattice, then it is invertible. From the short exact sequence of $G$-lattices $0 \to P \to \mathbb{Z}[G/\langle \tau \rangle] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where $\varepsilon$ is the augmentation map, since $P$ is $G$-invertible and $\mathbb{Z}$ is $G$-permutation, the sequence splits by [Len. Proposition 1.2]. Thus $\mathbb{Z}[G/\langle \tau \rangle]$ is a decomposable $G$-lattice. But $\mathbb{Z}[G/\langle \tau \rangle] = \text{Ind}_{\langle \varepsilon \rangle} G \mathbb{Z} = M_+$ by Definition 3.1. We find a contradiction again.

For $X$, we know that $X \simeq M_-$ by Lemma 4.3. Let $S = \langle \tau \rangle$. Regard $M$ as an $S$-lattice. By Definition 3.1 as an $S$-lattice, $M_- \simeq N \oplus \mathbb{Z}$ where $N$ is isomorphic to $\frac{r-1}{2}$ copies of $\mathbb{Z}[S]$. Thus $H^1(S, M_-) = H^1(S, N \oplus \mathbb{Z}_-) \simeq H^1(S, \mathbb{Z}_-) = \mathbb{Z}/2\mathbb{Z} \neq 0$. Hence $X \simeq M_-$ is not coflabby.

**Theorem 4.9** Let $G = \langle \sigma, \tau : \sigma^p = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_p$ where $p$ is an odd prime number. Assume that $h_p^+ = 1$.

(1) Then there are precisely ten non-isomorphic indecomposable $G$-lattices

$$\mathbb{Z}, \mathbb{Z}_-, \mathbb{Z}[H], R, P, V, X, Y_0, Y_1, Y_2.$$

Among these lattices, $\mathbb{Z}, \mathbb{Z}[H], V, Y_2$ are permutation $G$-lattices, $Y_0$ and $Y_1$ are stably permutation $G$-lattices, $\mathbb{Z}_-, R, P, X$ are not flabby $G$-lattices.

(2) If $k$ is a field admitting a $G$-extension, then any $G$-torus defined over $k$ is stably $k$-rational. In other words, if $K/k$ is a Galois extension field with $\text{Gal}(K/k) \simeq G$ and $M$ is any $G$-lattices, then $K(M)^G$ is stably $k$-rational.

**Proof.** (1) follows from Theorem 4.1 and the proof of Proposition 4.8.

For the proof of (2), apply Theorem 2.4. It suffices to show that $[M]^G$ is permutation for any $G$-lattice $M$.

Choose any exact sequence of $G$-lattices $0 \to M \to Q \to E \to 0$ where $Q$ is a permutation $G$-lattice and $E$ is a flabby $G$-lattice. Write $E$ as a direct sum of indecomposable $G$-lattices $E = \bigoplus_{1 \leq i \leq m} E_i$ where each $E_i$ is indecomposable. It is necessary that each $E_i$ is flabby. Apply the result in (1), we find that $E$ is stably permutation, i.e. there is an permutation $G$-lattice $Q'$ such that $E \oplus Q'$ is a permutation $G$-lattice. Thus we get an exact sequence $0 \to M \to Q \oplus Q' \to E \oplus Q' \to 0$ where $Q \oplus Q'$ and $E \oplus Q'$ are permutation $G$-lattices, i.e. $[M]^G$ is permutation.

**Theorem 4.10** Let $G = \langle \sigma, \tau : \sigma^p = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_p$ where $p$ is an odd prime number. Let $M$ be a $G$-lattice such that $[M]^G$ is a direct sum of $\mathbb{Z}, \mathbb{Z}[H], V, Y_0, Y_1, Y_2$ (up to a direct summand of some permutation $G$-lattice). If $K/k$ is a Galois field extension with $\text{Gal}(K/k) \simeq G$, then $K(M)^G$ is stably $k$-rational.

**Proof.** Follow the proof of (2) of Theorem 4.9.
**Theorem 4.11** Let $G \simeq D_p$ where $p$ is a prime number. If $h_p^+ = 1$ and $M$ is a $G$-lattice which is flabby and coflabby, then $M$ is stably permutation.

In fact, when $h_p^+ = 1$ and $p$ is an odd prime number, then a $G$-lattice is flabby (resp. flabby and coflabby) if and only if it is stably permutation.

**Proof.** If $p$ is an odd prime number, then a $G$-lattice is flabby if and only if it is both flabby and coflabby by Theorem 2.5. Then apply Theorem 4.9.

If $p = 2$, i.e., $G = C_2 \times C_2$ is the Klein-four group, by Colliot-Thélène and Sansuc’s result [CTS, Proposition 4], a $G$-lattice which is flabby and coflabby is necessarily stably permutation. □

**Remark.** A similar result also due to Colliot-Thélène and Sansuc is that, if $G$ is the quaternion group of order 8, an invertible $G$-lattice is stably permutation [CTS, R5, page 187]. See [EM2] for related results.

§5. **Steinitz classes**

Let $p$ be an odd prime number. Suppose that $h_p^+ \geq 2$ and $A$ is a non-principal ideal in $R_0 = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$. In this section we will show that $V_A, (Y_0)_A, (Y_1)_A, (Y_2)_A$ are not stably permutation $G$-lattices by applying the Steinitz classes of these $G$-lattices.

Recall the integral representations of cyclic groups of prime order.

**Theorem 5.1** (Diederichsen and Reiner [Di; Re; CR1, page 729, Theorem 34.31; Sw1, page 74, Theorem 4.19]) Let $S = \langle \sigma \rangle \simeq C_p$ the cyclic group of prime order $p$, $h_p$ be the class number of $\mathbb{Q}(\zeta_p)$. Let $B$ range over a full set of representatives of the $h_p$ ideal classes of $\mathbb{Z}[\zeta_p]$. Then there are precisely $2h_p + 1$ isomorphism classes of indecomposable $S$-lattices, and there are represented by

$$\mathbb{Z}, B$$

and the non-split extensions

$$0 \rightarrow B \rightarrow W_B \rightarrow \mathbb{Z} \rightarrow 0.$$ 

The $S$-lattices $W_B$ are rank-one projective modules over $\mathbb{Z}[S]$.

**Definition 5.2** Let $R = \mathbb{Z}[\zeta_p]$ and $C(R)$ be the ideal class group of $R$ (written multiplicatively). For any $S$-lattice $N$, we define the Steinitz class of $N$, denoted by $cl(N)$, by $cl(\mathbb{Z}) = [R]$, the equivalence class containing the principal ideal $R$, $cl(B) = [B] \in C(R)$, $cl(W_B) = [B] \in C(R)$. Furthermore, it satisfies the condition: If $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is an exact sequence of $S$-lattices, then $cl(N) = cl(N') \cdot cl(N'')$ in $C(R)$ (see [Sw2, page 73; CR, page 729]). For any $S$-lattice $N$, its Steinitz class $cl(N)$ is uniquely determined by $N$ [Sw2, page 75].
Lemma 5.3 ([Sw2, pages 78–80]) Let $S = \langle \sigma \rangle \simeq C_p$ where $p$ is a prime number, $\Phi_p(X) = 1 + X + X^2 + \cdots + X^{p-1} \in \mathbb{Z}[X]$ be the $p$-th cyclotomic polynomial. For any $S$-lattice $N$, define $N_0 = \{ x \in N : \sigma(x) = x \}$, $N_1 = \{ x \in N : \Phi_p(\sigma) \cdot x = 0 \}$. Then $N/N_0$ may be regarded as a module over $\mathbb{Z}[S]/\Phi_p(\sigma) \simeq \mathbb{Z}[\zeta_p]$. Thus, as a module over $\mathbb{Z}[\zeta_p]$, $N/N_0 \simeq \bigoplus_{1 \leq i \leq t} I_i$ where each $I_i$ is an ideal in $\mathbb{Z}[\zeta_p]$. The Steinitz class $cl(N)$ is equal to $[I_1 \cdot I_2 \cdot \cdots \cdot I_t] \in C(\mathbb{Z}[\zeta_p])$; it is also equal to $cl(N_1)$.

Proof. Following the presentation of [Sw2, page 78], we get the fibre product diagram

\[
\begin{array}{c}
\mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]/\langle \Phi_p(\sigma) \rangle \simeq \mathbb{Z}[\zeta_p] \\
\downarrow \quad \downarrow \\
\mathbb{Z} \simeq \mathbb{Z}[S]/\langle \sigma - 1 \rangle \longrightarrow \mathbb{Z}/p\mathbb{Z}
\end{array}
\]

For any $S$-lattice $N$, $N/N_0$ is a lattice over $\mathbb{Z}[S]/\Phi_p(\sigma)$, $N/N_1$ is a lattice over $\mathbb{Z}[S]/\langle \sigma - 1 \rangle$. Moreover, we get the following diagram

\[
\begin{array}{c}
N \longrightarrow N/N_0 \\
\downarrow \quad \downarrow \\
N/N_1 \longrightarrow N/N_0 + N_1
\end{array}
\]

It follows that $N$ is isomorphic to the pull-back of $N/N_0$ and $N/N_1$ along $N/\langle N_0 + N_1 \rangle$. The Steinitz class $cl(N)$ is uniquely determined by $N/N_0$ (see [Sw2, page 79]). In [Sw2, page 79], $N/N_0$ is written as a normalized form $\bigoplus_{1 \leq i \leq t} I_i$ where $I_2 \simeq I_3 \simeq \cdots \simeq I_t \simeq \mathbb{Z}[\zeta_p]$ and $I_1 \simeq \mathcal{B}$ is a non-zero ideal of $\mathbb{Z}[\zeta_p]$.

The formula $cl(N) = cl(N_1)$ follows from the exact sequence $0 \to N_1 \to N \to N/N_1 \to 0$ and $cl(N) = cl(N_1) \cdot cl(N/N_1)$ (see Definition 5.2), because $N/N_1$ is a lattice over $\mathbb{Z}[S]/\langle \sigma - 1 \rangle \simeq \mathbb{Z}$ and thus $N/N_1 \simeq \mathbb{Z}^m$ for some integer $m$. $\blacksquare$

Theorem 5.4 Let $G = \langle \sigma, \tau : \sigma^p = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_p$ where $p$ is an odd prime number. Assume that $h_\mathcal{A}^+ \geq 2$. For any non-principal ideal $\mathcal{A}$ of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$, let $M$ be one of the $G$-lattices $V_\mathcal{A}$, $(Y_0)_\mathcal{A}$, $(Y_1)_\mathcal{A}$ or $(Y_2)_\mathcal{A}$ in Theorem 4.4. Then $[M]^\mathcal{A}$ is not permutation.

Proof. Step 1. By Proposition 4.8 $M$ is flabby. Hence it is invertible by Theorem 2.5. Choose a $G$-lattice $M$ such that $M \oplus N$ is a permutation $G$-lattice. Write $Q := M \oplus N$. It follows that $0 \to M \to Q \to N \to 0$ is a flabby resolution of $M$ and $[M]^\mathcal{A} = [N]$. We will show that $N$ is not stably permutation.

Suppose not. There is a permutation $G$-lattice $Q_1$ such that $N \oplus Q_1$ is a permutation $G$-lattice. Write $Q_2 := N \oplus Q_1$. Then $0 \to N \to Q_2 \to Q_1 \to 0$ is exact.

Let $S = \langle \sigma \rangle$ be the subgroup of $G$. By restricting to the subgroup $S$, we may regard the exact sequences of $G$-lattices $0 \to M \to Q \to N \to 0$, $0 \to N \to Q_2 \to Q_1 \to 0$ as exact sequences of $S$-lattices. We will find a contradiction by evaluating the Steinitz classes of these $S$-lattices.
Step 2. Recall $R = \mathbb{Z}[\zeta_p]$. If $Q_0$ is a permutation $S$-lattices, we will show that $cl(Q_0) = [R]$.

Since $|S| = p$ is a prime number, any permutation $S$-lattice is a direct sum of $\mathbb{Z}$ and $\mathbb{Z}[S]$. In particular, $Q_0 = \mathbb{Z}^{(s)} \oplus (\mathbb{Z}[S])^{(t)}$ for some non-negative integers $s$ and $t$. Note that $cl(\mathbb{Z}) = [R]$. We will show that $cl(\mathbb{Z}[S]) = [R]$ also.

Write $L := \mathbb{Z}[S]$. Define $L_0 = \{x \in \mathbb{Z}[S] : (\sigma - 1) \cdot x = 0\}$. It is easy to see that $L_0 = (\Phi_p(\sigma))$ the ideal generated by $\Phi_p(\sigma)$. Thus $L/L_0 = \mathbb{Z}[S]/\Phi_p(\sigma) \simeq R$. By Lemma 5.3 $cl(L) = [R]$.

Step 3. From the exact sequence $0 \to N \to Q_2 \to Q_1 \to 0$, we find that $cl(N) = [R]$ because $[R] = cl(Q_2) = cl(N) \cdot cl(Q_1) = cl(N) \cdot [R]$.

On the other hand, from the exact sequence $0 \to M \to Q \to N \to 0$, we have $[R] = cl(Q) = cl(M) \cdot cl(N) = cl(M) \cdot [R]$. Thus $cl(M) = [R]$. We will show that $cl(M) = [R]$ is impossible. Thus a contradiction is obtained.

Step 4. Recall that $M = V_A, (Y_0)_A, (Y_1)_A, (Y_2)_A$. We will consider the case $M = (Y_1)_A$; the other cases may be proved similarly.

By Theorem 4.11 we have an exact sequence of $G$-lattices $0 \to PA \to (Y_1)_A \to \mathbb{Z}_\tau \to 0$. Regard it as an exact sequence of $S$-lattices by restriction. When the action of $\tau$ is forgotten, then $P \simeq R, \mathbb{Z}_\tau \simeq \mathbb{Z}$ as $S$-lattices. Hence, as $S$-lattices, we have $0 \to RA \to (Y_1)_A \to \mathbb{Z} \to 0$ and $cl((Y_1)_A) = cl(RA) \cdot cl(\mathbb{Z}) = [RA]$.

By [Wa, page 40, Theorem 4.14], the natural map of the class group of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ to that of $R = \mathbb{Z}[\zeta_p]$ is injective. Since we choose $A$ to be a non-principal ideal of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$, it follows that $RA$ is a non-principal ideal of $R$. Thus $cl(M) = cl((Y_1)_A) = [RA] \neq [R]$ as we claimed before.

Proof of Theorem 1.5

If $h^+_p = 1$, then all the $D_p$-tori are stably rational by Theorem 4.9.

If $h^+_p \geq 2$ let $G = D_p$, choose a $G$-lattice $M$ such that $[M]^R$ is not permutation by Theorem 5.4. Let $T$ be the $G$-torus defined over $k$ with character module $M$. By Theorem 2.4, $T$ is not stably $k$-rational (but $T$ is retract $k$-rational by Proposition 3.8).

§6. Some related rationality problems

By Theorem 4.9 if $h^+_p = 1$, $M$ is any $D_p$-lattice and $K/k$ is a Galois extension with $Gal(K/k) \simeq D_p$, then $\hat{K}(M)^D_p$ is stably $k$-rational. In this section we will estimate the number of variables $m$ (which depends on $M$ and its decomposition; see Lemma 6.4) such that $K(M)^D_p(x_1, \ldots, x_m)$ is $k$-rational. The key idea of our method is the notion of anisotropic lattices exploited by Voskresenskii and his school (see [Ku1, Ku2]). Before the proof, we recall two known rationality criteria.
Theorem 6.1 ([Ka1, Theorem 2.1]) Let $L$ be a field and $G$ be a finite group acting on $L(x_1, \ldots, x_m)$, the rational function field of $m$ variables over $L$. Suppose that

(i) for any $\sigma \in G$, $\sigma(L) \subset L$;
(ii) the restriction of the action of $G$ to $L$ is faithful;
(iii) for any $\sigma \in G$,

$$
\begin{pmatrix}
\sigma(x_1) \\ \vdots \\ \sigma(x_m)
\end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)
$$

where $A(\sigma) \in GL_m(L)$ and $B(\sigma)$ is an $m \times 1$ matrix over $L$. Then $L(x_1, \ldots, x_m) = L(z_1, \ldots, z_m)$ where $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq m$. In particular, $L(x_1, \ldots, x_m)^G = L^G(z_1, \ldots, z_m)$.

Proposition 6.2 Let $G$ be a finite group, $M$ be a $G$-lattice. Let $k'/k$ be a finite Galois extension such that there is a surjection $G \to \text{Gal}(k'/k)$. Suppose that there is an exact sequence of $G$-lattices $0 \to M_0 \to M \to Q \to 0$ where $Q$ is a permutation $G$-lattice. If $G$ is faithful on the field $k'(M_0)$, then $k'(M) = k'(M_0)(x_1, \ldots, x_m)$ for some elements $x_1, x_2, \ldots, x_m$ satisfying $m = \text{rank}_\mathbb{Z} Q$, $\sigma(x_j) = x_j$ for any $\sigma \in G$, any $1 \leq j \leq m$.

Proof. Note that the action of $G$ on $k'(M)$ is the purely quasi-monomial action in Definition 2.1.

Write $M_0 = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot u_i$, $Q = \bigoplus_{1 \leq i \leq m} \mathbb{Z} \cdot v_j$. Choose elements $u_1, \ldots, u_n, v_1, \ldots, v_m \in M$ such that $w_j$ is a preimage of $v_j$ for $1 \leq j \leq m$. It follows that $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ is a $\mathbb{Z}$-basis of $M$.

For each $\sigma \in G$, since $Q$ is permutation, $\sigma(w_j) - w_l \in M_0$ for some $w_l$ (depending on $j$). In the field $k'(M)$, if we write $k'(M) = k(u_1, \ldots, u_n, v_1, \ldots, v_m)$ as the rational function field in $m + n$ variables over $k'$, then $\sigma(w_j) = \alpha_j(\sigma) w_l$ for some $\alpha_j(\sigma) \in k'(M_0)$.

Since $G$ is faithful on $k'(M_0)$, apply Theorem 6.1.

Definition 6.3 (Kunyavskii [Ku1]) Let $G$ be a finite group, $M$ be a $G$-lattice. $M$ is called an anisotropic lattice if $M^G = 0$ where $M^G := \{x \in M : \sigma \cdot x = x \forall \sigma \in G\}$. For a $G$-lattice $M$, define $M_0 := \{x \in M : (\sum_{\sigma \in G} \sigma) \cdot x = 0\}$. Then $M_0$ is an anisotropic sublattice of $M$. Moreover, $(M/M_0)^G = M/M_0$ (for, if $\tilde{x} \in M/M_0$ and $\sigma \in G$, then $(\sigma - 1) \cdot \tilde{x} = 0$, because $(\sigma - 1) \cdot x \in M_0$).

Lemma 6.4 Let $G = \langle \sigma, \tau : \sigma^p = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_p$ where $p$ is an odd prime number. Let $M$ be any $G$-lattice and $M_0 = \{x \in M : (\sum_{\sigma \in G} \sigma) \cdot x = 0\}$. If $h_p^+ = 1$, then $M_0 \simeq X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)} \oplus \mathbb{Z}_+^{(t)}$ for some non-negative integers $s_0, s_1, s_2, t$, which may not be uniquely determined by the lattice $M_0$.

Proof. Note that $(\sum_{\sigma \in G} \sigma) \cdot M_0 = 0$, by definition. From Theorem 3.9 the only indecomposable $G$-lattices annihilated by $\sum_{\sigma \in G} \sigma$ are $\mathbb{Z}_-$, $R$, $P$ and $X$.

Note that the Krull-Schmidt-Azumaya Theorem (see [CRI, page 128]) is not valid in the category of $G$-lattices; for examples, Theorem 3.4, Theorem 3.5 and Theorem
Theorem 6.5 Let the notations and assumptions be the same as in Lemma 6.4. Let \( K/k \) be a Galois extension with \( G \simeq \operatorname{Gal}(K/k) \). Define \( m = \operatorname{rank}_\mathbb{Z} M - \operatorname{rank}_\mathbb{Z} M_0 \), \( n = s_0(p+1) + s_1(p+2) + s_2 - t - m \), which may depends on the decomposition of \( M_0 \) in Lemma 6.4. If \( n < 0 \), then \( K(M)^G \) is k-rational. If \( n \geq 0 \), then \( K(M)^G(z_1, z_2, \ldots, z_n) \) is k-rational where \( z_1, z_2, \ldots, z_n \) are elements algebraically independent over \( K(M)^G \).

Proof. Step 1. Note that \( G \) acts trivially on \( M/M_0 \). By Proposition 6.2 \( K(M)^G = K(M_0)^G(x_1, \ldots, x_m) \) where \( m = \operatorname{rank}_\mathbb{Z}(M/M_0) \).

Note that \( K(M_0)^G = K(X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)})(y_1, y_2, \ldots, y_t) \) where \( \sigma \cdot y_i = y_i \), \( \tau \cdot y_i = 1/y_i \) for \( 1 \leq i \leq t \). Define \( v_i = (1+y_i)/(1-y_i) \) if \( \text{char } k \neq 2 \), define \( v_i = 1/(1+y_i) \) if \( \text{char } k = 2 \). Then \( \sigma \cdot v_i = v_i \), \( \tau \cdot v_i = -v_i \) or \( v_i+1 \) depending on \( \text{char } k \neq 2 \) or \( \text{char } k = 2 \). Apply Theorem 6.1 we find that \( K(M_0)^G = K(X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)})(w_1, \ldots, w_l) \) where \( \sigma(w_i) = \tau(w_i) = w_i \) for \( 1 \leq i \leq t \).

It remains to add variables \( z_1, \ldots, z_l \) such that \( K(X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)})(z_1, \ldots, z_l) \) is G-isomorphic to \( K(N_1 \oplus N_2 \oplus \cdots \oplus N_t) \) where each \( N_i \) is a G-lattice satisfying the condition that \( K(N_1 \oplus N_2 \oplus \cdots \oplus N_d)^G \) is rational over \( K(N_1 \oplus \cdots \oplus N_{d-1})^G \) for all \( 1 \leq d \leq r \).

This condition may be fulfilled if (i) \( N_d \) is a permutation lattice by applying Theorem 6.1 or (ii) \( \operatorname{rank}_\mathbb{Z} N_d = 2 \) by applying Voskresenskii’s Theorem for 2-dimensional tori [Vo] page 57, or (iii) \( \operatorname{rank}_\mathbb{Z} N_d = 3 \) and \( N_d \) gives rise to a rational torus in Kunyavskii list [Ku] Theorem 1]. Once the lattices \( N_1, \ldots, N_t \) are found, we may show that \( K(N_1 \oplus \cdots \oplus N_r)^G \) is k-rational inductively. Hence \( K(X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)})^G(z_1, \ldots, z_l) \) is k-rational. But we have \( t+m \) variables arising from \( \mathbb{Z}^{(t)} \) and \( M/M_0 \). Thus \( l-(t+m) \) extra variables is required. This explains the definition of \( n \) in the statement of the theorem.

Step 2. For simplicity we consider how manay variables we should add to \( K(X), K(R), K(P) \) to achieve the goal in Step 1.

Consider \( K(P) \) first. By Theorem 4.1 \( 0 \to P \to V \to \mathbb{Z} \to 0 \). Thus \( K(V) = K(P)(x) \) by Proposition 6.2. By Lemma 4.3 \( V \simeq M_{+} \) is a permutation G-lattice. Hence one more variable is enough for \( K(P) \).

Consider \( K(X) \). By Lemma 4.4 \( X \simeq M_{-} \). By Definition 8.3 we have an exact sequence \( 0 \to M_{-} \to \widetilde{M}_{-} \to \mathbb{Z} \to 0 \). Thus \( K(\widetilde{M}_{-}) \) is G-isomorphic to \( K(X)(y) \) by Proposition 6.2. By Theorem 3.3 \( \widetilde{M}_{-} \oplus \mathbb{Z}[G/\langle \tau \rangle] \) is a permutation lattice. But \( K(\widetilde{M}_{-} \oplus \mathbb{Z}[G/\langle \tau \rangle]) = K(\widetilde{M}_{-})(u_1, u_2, \ldots, u_p) \) by Proposition 6.2. Thus \( p+1 \) variables is required for \( K(X) \).

Consider \( K(R) \). By Theorem 4.1 we have \( 0 \to R \to Y_0 \to \mathbb{Z}[H] \to 0 \). From Lemma 4.5 \( Y_0 \simeq \widetilde{M}_{-} \).

Use the fact \( \widetilde{M}_{-} \oplus \mathbb{Z}[G/\langle \tau \rangle] \) is permutation again. Thus we need \( p+2 \) extra variable this time.
In summary, for $K(X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)})$, we need $s_0(p + 1) + s_1(p + 2) + s_2$ extra variables. Subtract the $t + m$ variables which were obtained previously.

The same method may be used to prove Theorem 1.3 and Theorem 1.4. For the convenience of the reader we indicate some crucial steps of the proof because [Ku2] has only the Russian version. We emphasize that our proof is almost the same as those given by Voskresenskii and Kunyavskii in [Vo] [Ku1] [Ku2].

Proof of Theorem 1.3 and Theorem 1.4.

(A) Let $G = C_p$ (where $h_p = 1$), $C_4$ or $S_3 \simeq D_3$, $M$ be any $G$-lattice. Suppose $K/k$ is a Galois extension with $G \simeq \text{Gal}(K/k)$. We will show that $K(M)^G$ is $k$-rational.

Define $M_0 = \{x \in M : (\sum_{g \in G} g) \cdot x = 0\}$. As in the proof of Theorem 6.5, it remains to show that $K(M_0)^G$ is $k$-rational.

Case 1. $G = \langle \sigma \rangle \simeq C_p$ with $h_p = 1$.

By Theorem 5.1, $M_0 \simeq R^{(m)}$ where $R = \mathbb{Z}[\zeta_p]$. Note that $K(R) = K(x_1, x_2, \ldots, x_{p-1})$ with $\sigma : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto 1/(x_1 x_2 \cdots x_{p-1})$. Define

$$y_0 = 1 + x_1 + x_1 x_2 + \cdots + x_1 x_2 \cdots x_{p-1},$$
$$y_1 = 1/y_0, \ y_2 = x_1/y_0, \ \ldots, \ y_{p-1} = x_1 x_2 \cdots x_{p-2}/y_0.$$

Then $K(x_1, x_2, \ldots, x_{p-1}) = K(y_1, y_2, \ldots, y_{p-1})$ with $\sigma : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{p-1} \mapsto 1 - y_1 - y_2 - \cdots - y_{p-1}$. Hence $K(y_1, y_2, \ldots, y_{p-1})^{(\sigma)} = K(z_1, z_2, \ldots, z_{p-1})^{(\sigma)}$ where $\sigma(z_i) = z_i$ for $1 \leq i \leq p - 1$ by Theorem 6.1. The case $m \geq 2$ can be proved similarly.

Case 2. $G = \langle \sigma \rangle \simeq C_4$.

The indecomposable $G$-lattices are listed in [Vo, page 64]. We choose only these lattices which are annihilated by $1 + \sigma + \sigma^2 + \sigma^3$. They are the lattices listed below

$$(-1), \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The first one gives rise to $K(x)$ with $\sigma(x) = 1/x$. As before, the action can be linearized by setting $y = (1 + x)/(1 - x)$ or $1/(1 + x)$ depending on $\text{char} \ k \neq 2$ or $\text{char} \ k = 2$.

The second is a rank-two lattice. Thus it is rational by Voskresenskii’s Theorem of 2-dimensional tori [Vo, page 57].

The third one is the kernel of the augmentation map $\mathbb{Z}[G] \to \mathbb{Z}$ by [Vo, page 65, line 7]. It gives rise to $K(x_1, x_2, x_3)$ with $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto 1/(x_1 x_2 x_3)$. This action can be linearized by the same method of Case 1.

Case 3. $G = \langle \sigma, \tau : \sigma^3 = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq S_3$.

As in the proof of Theorem 6.5 with $p = 3$, $M_0 = X^{(s_0)} \oplus R^{(s_1)} \oplus P^{(s_2)} \oplus \mathbb{Z}^{(t)}$. The action for $\mathbb{Z}^{(t)}$ can be linearized as before. Since $\text{rank}_\mathbb{Z} R = \text{rank}_\mathbb{Z} P = 2$, Voskresenskii’s Theorem takes care of these situations; thus it is unnecessary to add new variables to
ensure \( k \)-rational. Since \( X \) is a rank-3 \( D_3 \)-lattice, we apply Kunyavskii’s Theorem \[Ku3\]. Thus no new variables are needed and we find that \( K(X) \) is \( k \)-rational. Done.

(B) Let \( G = \langle \sigma, \tau : \sigma^2 = \tau^2 = 1, \sigma \tau = \tau \sigma \rangle \simeq C_2 \times C_2 \), \( M \) be any \( G \)-lattice. Suppose that \( K/k \) is a Galois extension with \( G \simeq \text{Gal}(K/k) \). If \( K(M)^G \) is stably \( k \)-rational (resp. retract \( k \)-rational), it is \( k \)-rational. Consequently, if \([M]^R\) is flabby and coflabby, then \([K(M)^G] \) is \( k \)-rational.

Define \( M_0 = \{ x \in M : (\sum_{g \in G} g) \cdot x = 0 \} \). It follows that \( K(M_0)^G \) is also stably \( k \)-rational (resp. retract \( k \)-rational by \[Ka2\] Lemma 3.4]). Note that \( M_0 \) is a direct sum of indecomposable \( G \)-lattices annihilated by \( \sum_{g \in G} b \). These “special” indecomposable lattices were enumerated by Kunyavskii in \[Ku\] page 537–538. Except for two of them, the ranks of these lattices are \( \leq 2 \). Hence we may apply Voskresenskii’s Theorem again. The remaining two lattices are of rank 3: One is \( N_1 \) which is the kernel of the augmentation map \( \mathbb{Z}[G] \to \mathbb{Z} \), the other is \( N_2 = \text{Hom}(N_1, \mathbb{Z}) \) (see \[Ku\] page 540). By \[Ku3\], \( K(N_1)^G \) is \( k \)-rational and \( K(N_2)^G \) is not retract \( k \)-rational.

Since \( K(M_0)^G \) is retract \( k \)-rational, we find that \( N_2 \) will not appear as a direct summand of \( M_0 \). Hence \( M_0 = M_1 \oplus M_2 \oplus \cdots \oplus M_\ell \) where \( M_i \) is either \( N_1 \) or is of rank \( \leq 2 \). Thus \( K(M_0)^G \) is \( k \)-rational.

Finally, when \([M]^R\) is flabby and coflabby, apply Theorem 4.1.1

\[■\]

Note that Theorem 1.6 is a consequence of the following theorem.

**Theorem 6.6** Let \( p \) be a prime number, \( G = \langle \sigma \rangle \simeq C_\ell \), and \( k \) be a field admitting a \( G \)-extension.

1. If \( T \) is a \( G \)-torus over \( k \) which is stably rational, then \( T \) is rational.

2. \( h_p = 1 \) if and only if all the \( G \)-tori over \( k \) are stably rational.

**Proof.** (1) Working on the character module \( M \) of \( T \), it suffices to show that, if \( K/k \) is a Galois extension with \( \text{Gal}(K/k) \simeq G \) and \( K(M)^G \) is stably rational, then it is \( k \)-rational.

Define \( M_0 = \{ x \in M : (\sum_{g \in G} g) \cdot x = 0 \} \). By assumption, \( K(M_0)^G \) is stably \( k \)-rational. Since \( M_0 \) is annihilated by \( \sum_{g \in G} g \), \( M_0 \) is a direct sum of the ideals \( B \)'s by Theorem 5.1 where \( B \)'s are ideals of \( \mathbb{Z}[\zeta_\ell] \). Without loss of generality, we may write \( M_0 = B \oplus (\mathbb{Z}[\zeta_\ell])^{(m)} \) for some ideal \( B \) and some non-negative integer \( m \).

Since \( K(M_0)^G \) is stably rational, the class \([M_0]^R\) is stably permutation. Because \([\mathbb{Z}[\zeta_\ell]]^R\) is permutation, we find that \([B]^R\) is stably permutation.

As before, checking the Steinitz class, we find that \( B \) is a principal ideal. In other words, \( M_0 = \mathbb{Z}[\zeta_\ell]^{(m')} \) for some integer \( m' \). But then the action of \( \sigma \) on \( K(M_0) \) may be linearized as in the proof of Case 1 of (A) for the proof of Theorem 1.3 and Theorem 1.4. Thus \( K(M_0)^G \) is \( k \)-rational.

(2) It remains to show that, if \( h_p \geq 2 \), then there is a \( G \)-torus which is not stably rational.
Use the same method as in the proof of Theorem 5.4. It suffices to find a $G$-lattice $M$ such that $[M]^{fl}$ is not stably permutation.

Since $h_{p} \geq 2$, there is a non-principal ideal $B$ in $\mathbb{Z}[\zeta_{p}]$. Define $M = B$. If $[M]^{fl}$ is stably permutation, then there exist permutation $G$-lattices $Q_1$ and $Q_2$ such that $0 \to M \to Q_1 \to Q_2 \to 0$ is exact. Hence the Steinitz class $cl(M) = [\mathbb{Z}[\zeta_{p}]]$. However we know that $cl(M) = [B]$ and $B$ is not a principal ideal. A contradiction.

Remark. Part (1) of the above theorem is just a special case of a more general result. Let $k$ be a field admitting a $C_n$-extension and $T$ be a $C_n$-torus over $k$. Thanks to the works of Endo and Miyata, Voskresenskii, Chistov, Bashmakov and Klyachko (see [Vo, pages 62-63, 69-71]), if $n = p^a q^b$ where $p, q$ are prime numbers and $a, b$ are non-negative integers, then a $C_n$-torus $T$ is stably $k$-rational if and if it is $k$-rational.

The proof of Theorem 6.6 may be adapted to solve another rationality problem.

Definition 6.7 Let $G$ be any finite group, $k$ be any field. Let $k(x_g : g \in G)$ be the rational function field in $|G|$ variables over $k$ with a $G$-action via $k$-automorphism defined by $h \cdot x_g = x_{h g}$ for any $h, g \in G$. Define $k(G) := k(x_g : g \in G)^G$ the fixed field. Noether’s problem asks whether $k(G)$ is $k$-rational.

Theorem 6.8 Let $k$ be any field, $G = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau \sigma \tau^{-1} = \sigma^{-1} \rangle \simeq D_n$ where $n \geq 3$ is an odd integer. Define an action of $G$ on the rational function field $k(x_1, x_2, \ldots, x_{n-1})$ through $k$-automorphisms defined by

$$\sigma : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{n-1} \mapsto \frac{1}{x_1 x_2 \cdots x_{n-1}},$$

$$\tau : x_i \mapsto x_{n-i}.$$

Then $k(x_1, \ldots, x_{n-1})^G$ is stably $k$-rational if and only if $k(G)$ is stably $k$-rational.

Proof. We may write $k(x_1, \ldots, x_{n-1})^G = k(M)^G$ where $M$ is the $G$-lattice $\mathbb{Z}[G/\langle \tau \rangle]$ (for the definition of the field $k(M)$ with $G$ actions, see Definition 2.3). Note that $M$ is nothing but $N_+$ in Definition 3.2.

By Lemma 4.5 we find that $0 \to N_+ \to \tilde{M}_- \to \mathbb{Z}[G/\langle \sigma \rangle] \to 0$ is an exact sequence of $G$-lattices. By Proposition 6.2 $k(\tilde{M}_-)$ is $G$-isomorphic to $k(M)(y_1, y_2)$ where $\sigma(y_i) = \tau(y_i) = y_i$ for $1 \leq i < 2$.

By Theorem 3.5 $\tilde{M}_- \oplus \mathbb{Z}[G/\langle \tau \rangle] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}$. Thus, by Proposition 6.2 again, $k(\tilde{M}_-)(z_1, \ldots, z_n)$ is $G$-isomorphic to $k(\mathbb{Z}[G])(z_0)$ where $\sigma(z_i) = \tau(z_i) = z_i$ for $0 \leq i \leq n$.

Hence $k(M)^G$ is stably isomorphic to $k(M)^G$ with $M = \mathbb{Z}[G]$, which is nothing but $k(G)$. ■
Appendix

The ideas in this appendix were communicated by Shizuo Endo to the second-named author.

First we recall some terminology in [EM2].

Let $G$ be a finite group. Define an equivalence relation in the set of $G$-lattices: Two $G$-lattices $M$ and $N$ are equivalent, denoted by $M \sim N$, if, for any field $k$ admitting a $G$-extension, for any Galois extension $K/k$ with $\text{Gal}(K/k) \cong G$, the fields $K(M)^G$ and $K(N)^G$ are stably isomorphic over $k$, i.e. $K(M)^G(X_1, \ldots, X_m) \cong K(N)^G(Y_1, \ldots, Y_n)$ for some algebraically independent elements $X_i, Y_j$.

**Lemma A1** Let $G$ be a finite group, $M$ and $N$ be $G$-lattices. Then $M \sim N$ if and only if $[M]^{fl} = [N]^{fl}$.

**Proof.** Using the same idea in the proof of [Len, Theorem 1.7], it is not difficult to show that $K(M)^G$ and $K(N)^G$ are stably isomorphic over $k$ if and only if there exist exact sequences $0 \to M \to E \to P \to 0$ and $0 \to N \to E \to Q \to 0$ where $P$ and $Q$ are permutation $G$-lattices and $E$ is some $G$-lattice. The latter condition is equivalent to $[M]^{fl} = [N]^{fl}$ by [Sw3, Lemma 8.8].

**Definition A2** Let $G$ be a finite group. The commutative monoid $T(M)$ is defined as follows. As a set, $T(M)$ is the set of all equivalence classes $[M]$ under the equivalence relation "~" defined above where $M$ is any $G$-lattice and $[M]$ is the equivalence class containing $M$. The addition in $T(M)$ is defined by $[M] + [N] = [M \oplus N]$.

Recall the flabby class monoid $F_G$ defined in Section 2. By Lemma A1 it is easy to see that $T(M) \to F_G$ is an isomorphism by sending $[M]$ to $[M]^{fl}$.

**Definition A3** For a finite group $G$, let $\Lambda$ be a $\mathbb{Z}$-order satisfying $\mathbb{Z}[G] \subset \Lambda \subset \mathbb{Q}[G]$. We will define the locally free class group of $\Lambda$ following [EM1]. Let $K_0(\Lambda)$ be the Grothendieck group of the category of locally free $\Lambda$-modules of finite constant ranks. Define a subgroup $C(\Lambda)$ of $K_0(\Lambda)$ by $C(\Lambda) = \{ [M] - n[\Lambda] : M \text{ is locally free of rank } n, \text{ where } n \text{ runs over all positive integers} \}$. The group $C(\Lambda)$ is called the locally free class group of $\Lambda$.

For an idele definition of $C(\Lambda)$, see [CR2] page 219.

**Theorem A4** (Endo and Miyata [EM2]) Let $G = C_n$ or $D_p$ where $n$ is any positive integer and $p$ is an odd prime number. Then $T(G) \cong C(\Omega_{\mathbb{Z}[G]})$ where $\Omega_{\mathbb{Z}[G]}$ is a maximal order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$.

**Remark.** The above theorem is just a special case of Theorem 3.3 in [EM2] (see [EM3] also). When $G$ is a cyclic group, besides the proof given in [EM2], Swan has another proof in [Sw4]. When $G$ is a non-cyclic group (see [EM3] page 189, line -15), similar arguments and similar exact sequences as in [EM2] page 96 may be applied to obtain a proof.
The following theorem gives a generalization of Theorem 1.6.

**Theorem A5** Let $G = C_n$ where $n$ is a positive integer, and $k$ be a field admitting a $G$-extension. Then $h_n = 1$ if and only if all the $G$-tori over $k$ are stably $k$-rational.

*Proof.* Apply Theorem A4. Since $T(G) \simeq F_G$, it remains to show that $C(\Omega_{\mathbb{Z}[G]}) = 0$. When $G = C_n$, it is not difficult to verify that the maximal order $\Omega_{\mathbb{Z}[G]}$ is isomorphic to $\prod_{d|n} \mathbb{Z}[\zeta_d]$ [CR2, page 243]. Hence the result.

**Theorem A6** Let $p$ be an odd prime number, $G = D_p$, and $k$ be a field admitting a $G$-extension. Then $h_p^+ = 1$ if and only if all the $G$-tori over $k$ are stably $k$-rational.

*Proof.* Apply Theorem A4 again. We will show that $C(\Omega_{\mathbb{Z}[G]}) = 0$.

Note that $C(\mathbb{Z}[G]) \to C(\Omega_{\mathbb{Z}[G]})$ is surjective [CR2, page 230, Theorem 49.25]. Define $D(\mathbb{Z}[G])$ by the exact sequence $0 \to D(\mathbb{Z}[G]) \to C(\mathbb{Z}[G]) \to C(\Omega_{\mathbb{Z}[G]}) \to 0$ (see [CR2, page 234]).

By [CR2, page 259, Theorem 50.25], we find that $C(\Omega_{\mathbb{Z}[G]}) \simeq C(\mathbb{Z}[G]) \simeq C[\zeta_p + \zeta_p^{-1}]$. Hence the result.
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