On degenerate coupled transport processes in porous media with memory phenomena

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Abstract

In this paper we prove the existence of weak solutions to degenerate parabolic systems arising from the fully coupled moisture movement, solute transport of dissolved species and heat transfer through porous materials. Physically relevant mixed Dirichlet-Neumann boundary conditions and initial conditions are considered. Existence of a global weak solution of the problem is proved by means of semidiscretization in time, proving necessary uniform estimates and by passing to the limit from discrete approximations. Degeneration occurs in the nonlinear transport coefficients which are not assumed to be bounded below and above by positive constants. Degeneracies in transport coefficients are overcome by proving suitable a-priori $L^\infty$-estimates based on De Giorgi and Moser iteration technique.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let $\Gamma_D$ and $\Gamma_N$ be open disjoint subsets of $\partial \Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial \Omega\setminus (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0, \infty)$ be fixed throughout the paper, $I = (0, T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$. We shall study the following problem in $Q_T$:

\begin{align}
\partial_t [\phi(x, r) S(p)] &= \nabla \cdot [a(x, p, \vartheta, r) \nabla p] + \alpha_1 f(x, p, c, \vartheta, r), \quad (1.1) \\
\partial_t [\phi(x, r) S(p)c] &= \nabla \cdot [\phi(x, r) S(p) D_w(x, p) \nabla c] \\
&\quad + \nabla \cdot [ca(x, p, \vartheta, r) \nabla p], \quad (1.2) \\
\partial_t [\phi(x, r) S(p) \vartheta + \varrho(x, r) \vartheta] &= \nabla \cdot [\lambda(x, p, \vartheta, r) \nabla \vartheta] \\
&\quad + \nabla \cdot [\alpha a(x, p, \vartheta, r) \nabla p] + \alpha_2 f(x, p, c, \vartheta, r), \quad (1.3)
\end{align}

where

\begin{equation}
a(x, p, \vartheta, r) = \frac{k(x, r)}{k_H(S(p))}. \quad (1.4)
\end{equation}

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The system introduced above is coupled with an integral condition

$$r(x, t) = \int_0^t f(x, p(x, s), c(x, s), \vartheta(x, s), r(x, s)) \, ds$$  

(1.5)

and completed by the mixed-type boundary conditions

$$p = 0, \quad c = 0, \quad \vartheta = 0 \quad \text{on } \Gamma_{DT},$$

(1.6)

$$\nabla p \cdot n = 0, \quad \nabla c \cdot n = 0, \quad \nabla \vartheta \cdot n = 0 \quad \text{on } \Gamma_{NT}$$

(1.7)

and the initial conditions

$$p(\cdot, 0) = p_0, \quad c(\cdot, 0) = c_0, \quad \vartheta(\cdot, 0) = \vartheta_0 \quad \text{in } \Omega.$$  

(1.8)

The goal of this paper is to study the existence of the so-called weak solution to the degenerate fully coupled nonlinear system (1.1)–(1.8). The problem under consideration covers a large range of problems including memory phenomena. Namely, the system (1.1)–(1.8) arises from the coupled moisture movement, transport of dissolved species and heat transfer through the porous system [3, 40]. Equations (1.1) and (1.2) express the mass balance of water and dissolved species, respectively, in porous media and (1.3) represents the balance of heat energy in the porous system. For simplicity, the gravity terms are not included since they do not affect the analysis. For specific civil engineering applications, we refer the reader to e.g. [13, 39]. Our problem has been motivated by doubly nonlinear systems appearing in modelling of chemical reactions, heat transport and mass transfer in early age concrete [31, 32, 33, 46].

In (1.1)–(1.8), \(p : Q_T \to \mathbb{R}, \ c : Q_T \to \mathbb{R}, \ \vartheta : Q_T \to \mathbb{R} \) and \(r : Q_T \to \mathbb{R} \) are the unknown functions. In particular, \(p \) corresponds to the water pressure, \(c \) represents concentration of dissolved species and \(\vartheta \) represents the temperature of the complete porous system. Equations (1.1), (1.3) and (1.5) are encountered e.g. in the so-called problem of “hydratational heat” when inner moisture sinks and heat sources are of special types. In particular, the intensity of heat sources depends on the amount of heat already developed, \(f \) in (1.1) and (1.3) depends on the unknown function \(r \) (the so called “hydration degree”) given by (1.5). Further, \(\alpha : \Omega \times \mathbb{R}^3 \to \mathbb{R} \) represents the transport coefficient of the capillary water given by (1.4), where \(k \) is the intrinsic permeability, \(k_R \) denotes the relative permeability of the liquid water and \(\mu \) is the dynamic viscosity of the liquid water. \(D_w : \Omega \times \mathbb{R} \to \mathbb{R} \) is the capillary water diffusion coefficient, \(S : \mathbb{R} \to \mathbb{R} \) represents degree of saturation of the pores with liquid water, \(\phi : \Omega \times \mathbb{R} \to \mathbb{R} \) is porosity, \(\varrho : \Omega \times \mathbb{R} \to \mathbb{R} \) is the density of solid skeleton in the porous system. Further, \(\lambda : \Omega \times \mathbb{R}^3 \to \mathbb{R} \) is the thermal conductivity of porous material. \(\alpha_1 \) and \(\alpha_2 \) are material constants. Note that the density of water is assumed to be constant in the model and normalized to one. \(n \) is the outward unit normal vector with respect to the boundary of \(\Omega \). Finally, \(p_0 : \Omega \to \mathbb{R}, \ c_0 : \Omega \to \mathbb{R} \) and \(\vartheta_0 : \Omega \to \mathbb{R} \) are given functions describing initial state of the system.

Typical forms of \(S \) and \(k_R \) can be found e.g. in [16, 17, 18, 40] with applications to water movement in soils or structured rock masses or in [13, 14, 15] concerning transport processes in concrete. It follows that positive functions \(S(\cdot) \) and \(k_R(S(\cdot)) \) are typically increasing on \((-\infty, 0) \) and \(S'(\cdot) \) and \(k_R(S(\cdot)) \) tend to zero as \(p \to -\infty \).
Hence, (1.1) and (1.2) are degenerate parabolic equations where the degeneracy occurs in both elliptic as well as parabolic parts. The degeneracy in the elliptic part of (1.1) can be transformed only to the parabolic term using the so-called Kirchhoff transformation

\[ v := \kappa(p) = \int_0^p k_R(S(s)) \, ds. \]

The existence of the weak solution \( v \) for the resulting transformed problem follows from Alt and Luckhaus [2]. However, due to the degeneracy of the problem (\( k_R \) is not assumed to be bounded below by a positive constant) we are not able to ensure that \( p = \kappa^{-1}(v) \) solves the original problem. Therefore we treat directly the doubly degenerate problem (1.1) and omit degeneracies proving \( L^\infty \)-estimates for the solutions of the approximate problems.

**A brief bibliographical survey.** Nowadays, description of heat, moisture or soluble/non-soluble contaminant transport in concrete, soil or rock porous matrix is frequently based on time dependent models. Coupled transport processes (diffusion processes, heat conduction, moister flow, contaminant transport or coupled flows through porous media) are typically associated with systems of strongly nonlinear degenerate parabolic partial differential equations of type (written in terms of operators \( A, B, F \))

\[ \partial_t B(u) - \nabla \cdot A(u, \nabla u) = F(u), \]

(1.9)

where \( u \) stands for the unknown vector of state variables. There is no complete theory for such general problems. However, some particular results assuming special structure of operators \( A \) and \( B \) and growth conditions on \( F \) can be found in the literature. Most theoretical results on parabolic systems exclude the case of non-symmetrical parabolic parts \([2, 10, 23]\). Giaquinta and Modica in [19] proved the local-in-time solvability of quasilinear diagonal parabolic systems with nonlinear boundary conditions (without assuming any growth condition), see also [48]. The existence of weak solutions to more general non-diagonal systems like (1.9) subject to mixed boundary conditions has been proven in [2]. The authors proved an existence result assuming the operator \( B \) to be only (weak) monotone and subgradient. This result has been extended in [10], where the authors presented the local existence of the weak solutions for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in \( u \). These results, however, are not applicable if \( B \) does not take the subgradient structure, which is typical of coupled transport models in porous media. Thus, the analysis needs to exploit the specific structure of such problems. The existence of a local-in-time strong solution for moisture and heat transfer in multi-layer porous structures modelling by the doubly nonlinear parabolic system is provided in [7]. In [47], the author proved the existence of the solution to the purely diffusive hygro-thermal model allowing non-symmetrical operators \( B \), but requiring non-realistic symmetry in the elliptic part. In [8, 22], the authors studied the existence, uniqueness and regularity of coupled quasilinear equations modeling evolution of fluid species influenced by thermal, electrical and diffusive forces. In [28, 29, 30], the authors studied a model of specific structure of a heat and mass transfer arising
from textile industry and proved the global existence for one-dimensional problems in
[28, 29] and three-dimensional problems in [30]. In [21], the authors proved the exist-
tence of the weak solutions to systems modeling the consolidation of saturated porous
media.

In [42, 43], the author proved the local existence of weak solutions to degener-
ate quasilinear problems, where the coefficient function in front of the time derivative
may vanish at a set of zero measure. The main result is proved by means of semi-
discretization in time and proving $L^\infty$-estimates for approximates in order to omit a
growth limitations in nonlinearities and the right hand side.

In [38], the author studied an initial boundary value problem for the nonlinear de-
genrate parabolic equation of type (1.9) with evolutionary boundary conditions. Exis-
tence and uniqueness were established through some discrete schemes combined with
parabolic regularization and error estimates for these schemes were presented. In a
slightly different form, taking $\phi = 1$ and $f = 0$ and assuming different degeneration
features, problem (1.1)–(1.2) was studied in [35,36,37].

From the numerical point of view, scalar degenerate problems similar to (1.1) were
treated in [24,25]. The author proposed a nonstandard approximation scheme based
on the relaxation method in order to control the degeneracy in the problem.

A profound investigation of problems with integral conditions connected with equa-
tions (1.1) and (1.5) can be found in [9] and [44] in case of nondegenerate linear elliptic
and parabolic parts and in [24] assuming strongly nonlinear and degenerate scalar
parabolic problems.

Outline of the paper. In the present paper we extend our previous existence result
for coupled heat and mass flows in porous media [4] to more general degenerate prob-
lem modeling coupled moisture, solute and heat transport in porous media including
memory phenomena. This leads to a fully nonlinear degenerate parabolic system cou-
pled with an integral condition and including natural (critical) growths and with de-
generacies in transport coefficients. The rest of this paper is organized as follows. In
Section 2 we briefly introduce basic notation and suitable function spaces and specify
our assumptions on data and coefficient functions in the problem. In Section 3 we for-
mulate the problem in the variational sense and state the main result, the global-in-time
existence of the weak solution to the problem (1.1)–(1.8). The main result is proved by
an approximation procedure in Section 4. First, we formulate the semi-discrete scheme
and prove the existence of the solution to the corresponding recurrence steady problem.
The crucial a-priori estimates and uniform boundness of time discrete approximations
are proved in Section 4.2. Finally, we conclude that solutions of the semi-discrete
scheme converge and that the limit corresponds to the solution of the original problem.

Remark 1.1 The present analysis can be straightforwardly extended to a setting with
nonhomogeneous boundary conditions (1.6) (see [4] for details or [2, Paragraph 1.10
on page 324]). Here we work with homogeneous boundary conditions to avoid unnec-
essary technicalities in the existence result.
2 Preliminaries

2.1 Notations and some function spaces

Vectors and vector functions are denoted by boldface letters. Throughout the paper, we will always use positive constants $C$, $C_1$, $C_2$, ..., which are not specified and which may differ from line to line. In what follows, we suppose $s, q, s' \in [1, \infty]$, $s'$ denotes the conjugate exponent to $s > 1$, $1/s + 1/s' = 1$. $L^s(\Omega)$ represents the usual Lebesgue space equipped with the norm $\| \cdot \|_{L^s(\Omega)}$ and $W^k,s(\Omega)$, $k \geq 0$ ($k$ need not to be an integer, see [20]), denotes the usual Sobolev-Slobodecki space with the norm $\| \cdot \|_{W^k,s(\Omega)}$. We define $W^{1,2}_r(\Omega) := \{ v \in W^{1,2}(\Omega): v|_{\Gamma_D} = 0 \}$. By $E^*$ we denote the space of all continuous, linear forms on Banach space $E$ and by $\langle \cdot, \cdot \rangle$ we denote the duality between $E$ and $E^*$. By $L^s(I; E)$ we denote the Bochner space (see [11]). Therefore, $L^s(I; E)^* = L^{s'}(I; E^*)$.

2.2 Structure and data properties

We start by introducing several structural assumptions on functions in (1.1)–(1.8):

(i) $S \in C^1(\mathbb{R})$ is a positive and strictly monotone function such that

$$
0 < S(\xi) \leq S_s < +\infty \quad \forall \xi \in \mathbb{R} \quad (S_s = \text{const}), \quad (2.1)
$$

$$
(S(\xi_1) - S(\xi_2))(\xi_1 - \xi_2) > 0 \quad \forall \xi_1, \xi_2 \in \mathbb{R}, \quad \xi_1 \neq \xi_2. \quad (2.2)
$$

(ii) $a, a_1, k, k_R, \mu, \varrho, D_w, \lambda$ are continuous functions, $a_1$ strictly increasing, satisfying

$$
0 < k_1 \leq k(x, \xi) \leq k_2 < +\infty \quad (k_1, k_2 = \text{const}) \quad \forall \xi \in \mathbb{R}, \quad x \in \Omega, \quad (2.3)
$$

$$
k_R \in C([0, S_s]), \quad (k_R(\xi_1) - k_R(\xi_2))(\xi_1 - \xi_2) > 0 \quad \forall \xi_1, \xi_2 \in [0, S_s], \quad (2.4)
$$

$$
0 < k_R(\xi) \quad \forall \xi \in [0, S_s], \quad (2.5)
$$

$$
0 < \mu_1 \leq \mu(\xi) \leq \mu_2 < +\infty \quad (\mu_1, \mu_2 = \text{const}) \quad \forall \xi \in \mathbb{R}, \quad (2.6)
$$

$$
0 < \varrho_1 \leq \varrho(x, \xi) \leq \varrho_2 < +\infty \quad (\varrho_1, \varrho_2 = \text{const}) \quad \forall \xi \in \mathbb{R}, \quad x \in \Omega, \quad (2.7)
$$

$$
0 < a_1(\xi_1) \leq a(x, \xi_1, \xi_2, \xi_3) \leq a_2 < +\infty \quad \forall \xi_1, \xi_2, \xi_3 \in \mathbb{R}, \quad x \in \Omega, \quad (a_2 = \text{const}), \quad (2.8)
$$

$$
0 < D_w(x, \xi) \quad \forall \xi \in \mathbb{R}, \quad x \in \Omega, \quad (2.9)
$$

$$
0 < \lambda(x, \xi_1, \xi_2, \xi_3) \quad \forall \xi_1, \xi_2, \xi_3 \in \mathbb{R}, \quad x \in \Omega. \quad (2.10)
$$

(iii) The function $\phi$ is Lipschitz continuous with respect to the second variable, i.e. there exists a constant $C_\phi > 0$ such that

$$
|\phi(x, \xi_1) - \phi(x, \xi_2)| \leq C_\phi |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in \mathbb{R}, \quad x \in \Omega \quad (2.11)
$$
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and

\[ 0 < \phi_1 \leq \phi(x, \xi) \leq \phi_2 < +\infty \quad \forall \xi \in \mathbb{R}, \ x \in \Omega \quad (\phi_1, \phi_2 = \text{const}). \tag{2.12} \]

(iv) \( f \) is Lipschitz continuous in all respective variables and there exists an increasing positive bounded function \( \tilde{f} \) such that \((C_f = \text{const})\)

\[ |f(x, \xi_1, \xi_2, \xi_3, \xi_4)| \leq \tilde{f}(\xi_1) \leq C_f \quad \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{R}, \ x \in \Omega. \tag{2.13} \]

(v) We assume that there exists a non-increasing positive function \( M \) such that

\[ M(\xi) \leq \frac{a_1(\xi)}{S'(\xi)} \quad \forall \xi \in \mathbb{R} \tag{2.14} \]

and

\[ \lim_{\delta \to 0^+} \frac{\tilde{f}(S^{-1}(\delta))}{M(S^{-1}(\delta))\delta} = 0. \tag{2.15} \]

(vi) (Initial data) Assume \( c_0, \vartheta_0 \in W^{1,2} \cap L^\infty(\Omega) \) and \( p_0 \in L^\infty(\Omega) \) such that

\[ -\infty < p_1 < p_0(\cdot) \leq 0 \quad \text{a.e. in } \Omega \quad (p_1 = \text{const}). \tag{2.16} \]

Throughout the paper the hypotheses (i)–(vi) will be assumed.

3 The main result

The aim of this paper is to prove the existence of a weak solution to the problem \((1.1)-(1.8)\). We first reformulate the problem in a variational sense.

**Definition 3.1** A weak solution of \((1.1)-(1.8)\) is a foursome \([p, c, \vartheta, r]\) such that

\[ p \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \]
\[ c \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \]
\[ \vartheta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \]
\[ r \in C([0,T]; L^\infty(\Omega)) \]

and

\[ - \int_{Q_T} \phi(x, r)S(p) \partial_t \zeta \, dx \, dt + \int_{Q_T} a(x, p, \vartheta, r) \nabla p \cdot \nabla \zeta \, dx \, dt \]
\[ = \int_{\Omega} \phi(x, r_0)S(p_0)\zeta(x, 0) \, dx + \int_{Q_T} \alpha_1 f(x, p, c, \vartheta, r) \zeta \, dx \, dt \tag{3.1} \]

for any \( \zeta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^1(\Omega)) \) with \( \zeta(\cdot, T) = 0; \)
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\[ - \int_{Q_T} \phi(x, r) S(p) c \partial_t \eta \, dx \, dt + \int_{Q_T} \phi(x, r) S(p) D_w(x, p) \nabla c \cdot \nabla \eta \, dx \, dt \]
\[ + \int_{Q_T} c a(x, p, \partial, r) \nabla p \cdot \nabla \eta \, dx \, dt = \int_{\Omega} \phi(x, r_0) S(p_0) c_0 \eta(x, 0) \, dx \]  
(3.2)

for any \( \eta \in L^2(I; W^{1,2}_D(\Omega)) \cap W^{1,1}(I; L^1(\Omega)) \) with \( \eta(\cdot, T) = 0 \);

\[ - \int_{Q_T} [\phi(x, r) S(p) + \varrho(x, r)] \vartheta \partial_t \psi \, dx \, dt \]
\[ + \int_{Q_T} \lambda(x, p, \vartheta, r) \nabla \vartheta \cdot \nabla \psi \, dx \, dt + \int_{Q_T} \partial_0 a(x, p, \vartheta, r) \nabla p \cdot \nabla \psi \, dx \, dt \]
\[ = \int_{\Omega} [\phi(x, r_0) S(p_0) + \varrho(x, r_0)] \vartheta_0 \varphi(x, 0) \, dx + \int_{Q_T} \alpha_2 f(x, p, c, \vartheta, r) \psi \, dx \, dt \]  
(3.3)

for any \( \psi \in L^2(I; W^{1,2}_D(\Omega)) \cap W^{1,1}(I; L^1(\Omega)) \) with \( \psi(\cdot, T) = 0 \), where

\[ r(t) = \int_0^t f(x, p(x, s), c(x, s), \partial(x, s), r(x, s)) \, ds \quad \text{in } L^\infty(\Omega) \text{ for all } t \in [0, T]. \]

(3.4)

The main result of this paper reads as follows:

**Theorem 3.2 (Main result)** Let the assumptions (i)–(vi) be satisfied. Then there exists at least one weak solution of the system (1.1)–(1.8).

To prove the main result of the paper we use the method of semidiscretization in time by constructing temporal approximations and limiting procedure. The proof can be divided into three steps. In the first step, we approximate our problem by means of a semi-implicit time discretization scheme (which preserve the pseudo-monotone structure of the discrete problem) and prove the existence and \( W^{1,s}(\Omega) \)-regularity (with some \( s > 2 \)) of discrete approximations. In the second step we derive necessary a-priori estimates. The key point is to establish \( L^\infty \)-estimates to overcome degeneracies in transport coefficients. Finally, in the third step we construct temporal interpolants and pass to the limit from discrete approximations.

### 4 Proof of the main result

#### 4.1 Approximations

Applying the method of discretization in time, we divide the interval \([0, T]\) into \( n \) subintervals of lengths \( h := T/n \) (a time step), replace the time derivatives by the corresponding difference quotients and the integral in (3.4) by a sum. In this way, we approximate the problem (1.1)–(1.8) by a semi-implicit time discretization scheme and re-formulate the problem in a weak sense.
Let us consider $p_n^0 := p_0$, $c_n^0 := c_0$, $\vartheta_n^0 := \vartheta_0$ and $r_n^0 := 0$ a.e. on $\Omega$. We now define, in each time step $i = 1, \ldots, n$, a foursome $[p_n^i, c_n^i, \vartheta_n^i, r_n^i]$ as a solution of the following recurrence steady problem: for a given foursome $[p_n^{i-1}, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}]$, $i = 1, 2, \ldots, n$, $p_n^{i-1} \in L^\infty(\Omega)$, $c_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\vartheta_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $r_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, find $[p_n^i, c_n^i, \vartheta_n^i, r_n^i]$, such that $p_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $c_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\vartheta_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $r_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$$
\int_\Omega \frac{\phi(x, r_n^i) S(p_n^i) - \phi(x, r_n^{i-1}) S(p_n^{i-1})}{h} \zeta \, dx 
+ \int_\Omega a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla \zeta \, dx 
= \int_\Omega \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \zeta \, dx \quad (4.1)
$$

for any $\zeta \in W^{1,2}_D(\Omega)$;

$$
\int_\Omega \frac{\phi(x, r_n^i) S(p_n^i) c_n^i - \phi(x, r_n^{i-1}) S(p_n^{i-1}) c_n^{i-1}}{h} \eta \, dx 
+ \int_\Omega \phi(x, r_n^i) S(p_n^i) D_w(x, p_n^i) \nabla c_n^i \cdot \nabla \eta \, dx 
+ \int_\Omega c_n^i a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla \eta \, dx 
= 0 \quad (4.2)
$$

for any $\eta \in W^{1,2}_D(\Omega)$;

$$
\int_\Omega \frac{\phi(x, r_n^i) S(p_n^i) \vartheta_n^i - \phi(x, r_n^{i-1}) S(p_n^{i-1}) \vartheta_n^{i-1}}{h} \psi \, dx 
+ \int_\Omega \frac{\vartheta_n^i - \vartheta_n^{i-1}}{h} \vartheta_n^i \, dx 
+ \int_\Omega \lambda(x, p_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \nabla \vartheta_n^i \cdot \nabla \psi \, dx 
+ \int_\Omega \vartheta_n^i a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla \psi \, dx 
= \int_\Omega \alpha_2 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \psi \, dx \quad (4.3)
$$

for any $\psi \in W^{1,2}_D(\Omega)$ and

$$
r_n^i = h \sum_{j=1}^i f(x, p_n^j, c_n^{j-1}, \vartheta_n^{j-1}, r_n^{j-1}), \quad i = 1, \ldots, n \quad (4.4)
$$

$$
r_n^0(x) = 0 \quad (4.5)
$$
Theorem 4.1 (Existence of the solution to (4.1)–(4.4)) Let \( p_n^{i-1} \in L^\infty(\Omega), c_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \vartheta_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) and \( r_n^{i-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) be given and the Assumptions (i)–(vi) satisfies. Then there exists \([p_n^i, c_n^i, \vartheta_n^i, r_n^i]\), such that \( p_n^i \in W^{1,s}(\Omega), c_n^i \in W^{1,s}(\Omega), \vartheta_n^i \in W^{1,s}(\Omega) \) with some \( s > 2 \), and \( r_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \), satisfying (4.1)–(4.4).

Remark 4.2 By Theorem 4.1 and the embeddings \( W^{1,s}(\Omega) \hookrightarrow L^\infty(\Omega) \) (recall that \( s > 2 \) and \( \Omega \subset \mathbb{R}^2 \)) and \( W^{1,s}(\Omega) \hookrightarrow W^{1,2}(\Omega) \), we are able to solve (4.1)–(4.4) recursively for \([p_n^i, c_n^i, \vartheta_n^i, r_n^i]\) by the already known \([p_n^{i-1}, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}]\), such that we obtain

\[
\begin{align*}
p_n^i &\in W^{1,2}(\Omega) \cap L^\infty(\Omega), \\
c_n^i &\in W^{1,2}(\Omega) \cap L^\infty(\Omega), \\
\vartheta_n^i &\in W^{1,2}(\Omega) \cap L^\infty(\Omega), \\
r_n^i &\in W^{1,2}(\Omega) \cap L^\infty(\Omega)
\end{align*}
\]

for all \( i = 1, \ldots, n \).

Before proving Theorem 4.1, we present two auxiliary results, formulated in Theorem 4.3 and Lemma 4.4.

Theorem 4.3 (Weak maximum principle for pressure approximations) Let \( p_n^i \in W^{1,s}(\Omega), c_n^i \in W^{1,s}(\Omega), \vartheta_n^i \in W^{1,s}(\Omega) \) with some \( s > 2 \), and \( r_n^i \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) solve (4.1)–(4.5) successively for \( i = 1, \ldots, n \). Then there exists \( \ell \) (independent of \( n \)) such that

\[
p_n^i \geq \ell \text{ almost everywhere in } \Omega \text{ and for all } i = 1, 2, \ldots, n. \tag{4.6}
\]

To prove Theorem 4.3 we need the following lemma:

Lemma 4.4 (See e.g. Proposition 4.2 in [36]) If a nonnegative sequence \( \{Z_j\} \) satisfies

\[
Z_{j+1} \leq \gamma 4^j Z_j^\tau + 1 \quad (\tau > 0),
\]

then

\[
\lim_{j \to +\infty} Z_j = 0 \tag{4.7}
\]

provided that

\[
\gamma \leq Z_0^{-\tau} 4^{-1/\tau}.
\]

Proof. The proof follows from the proof of Lemma 4.1.1 in [49].

Proof of Theorem 4.3 The proof is based on the De Giorgi iteration technique, see e.g. [49 Chapter 4] or [27]. Let \( k \in \mathbb{R} \) and set

\[
(\phi - k)_- \equiv \begin{cases} 
\phi - k, & \phi < k, \\
0, & \phi \geq k.
\end{cases}
\]
For \( k < p_1 \) (here, \( p_1 \) is taken from (2.16)) we have \( \zeta = (S(p_n^i) - S(k))_+ \in W^{1,2}_{\text{loc}}(\Omega) \) and thus we may choose \( \zeta = (S(p_n^i) - S(k))_+ \) as a test function in (4.1). It is a matter of a simple technical computation to arrive at the estimate

\[
\frac{1}{2h} \int_\Omega \phi(x, r_n^i)\left[(S(p_n^i) - S(k))_+\right]^2 \, dx
\]

\[
- \frac{1}{2h} \int_\Omega \phi(x, r_n^i-1)\left[(S(p_n^{i-1}) - S(k))_+\right]^2 \, dx
\]

\[
+ \int_\Omega a(x, p_n^i, \varrho_n^{i-1}, r_n^{i-1}) \frac{1}{S'(p_n^i)} |\nabla(S(p_n^i) - S(k))_+|^2 \, dx
\]

\[
\leq \int_\Omega \alpha_1 f(x, p_n^i, \epsilon_n^{i-1}, \varrho_n^{i-1}, r_n^{i-1})(S(p_n^i) - S(k))_+ \, dx
\]

\[
- \frac{1}{2h} \int_\Omega \left[ \phi(x, r_n^i) - \phi(x, r_n^{i-1}) \right] (S(p_n^i) + S(k))(S(p_n^i) - S(k))_+ \, dx. \tag{4.8}
\]

Using the Lipschitz continuity of \( \phi \) with respect to \( r \), see (2.11), and using (4.4), we can write

\[
|\phi(x, r_n^i) - \phi(x, r_n^{i-1})| \leq C_\phi |r_n^i - r_n^{i-1}|
\]

\[
\leq hC_\phi f(x, p_n^i, \epsilon_n^{i-1}, \varrho_n^{i-1}, r_n^{i-1})). \tag{4.9}
\]

Let us denote

\[
I_k(i) := \int_\Omega \phi(x, r_n^i)\left[(S(p_n^i) - S(k))_+\right]^2 \, dx, \quad i = 1, \ldots, n \tag{4.10}
\]

and let \( I_k \) attains its maximum at \( i = m \), i.e.

\[
I_k(m) = \max_{i=1,\ldots,n} I_k(i) \tag{4.11}
\]

and, in other words,

\[
I_k(m) \geq I_k(i) \quad \text{for all } i = 1, 2, \ldots, n. \tag{4.12}
\]

From this and in view of (4.8) and (4.9) we have

\[
\int_\Omega a(x, p_n^m, \varrho_n^m, r_n^m, \varrho_n^m, r_n^m)(S(p_n^m) - S(k))_+ \, dx
\]

\[
\leq (C_\phi S_2 + |\alpha_1|) \int_\Omega f(x, p_n^m, \epsilon_n^m, \varrho_n^m, r_n^m, \varrho_n^m, r_n^m, 1)(S(p_n^m) - S(k))_+ \, dx. \tag{4.13}
\]

Further, using (2.8) leads to

\[
\int_\Omega \alpha_1 \frac{(p_n^m)}{S'(p_n^m)} |\nabla(S(p_n^m) - S(k))_+|^2 \, dx
\]

\[
\leq (C_\phi S_2 + |\alpha_1|) \int_\Omega f(x, p_n^m, \epsilon_n^m, \varrho_n^m, r_n^m, \varrho_n^m, r_n^m, 1)(S(p_n^m) - S(k))_+ \, dx. \tag{4.14}
\]
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On the other hand, by Assumption (v), namely, the inequality (2.14), we see that

\[ M(k) \int_\Omega |\nabla (S(p^m_n) - S(k))_-|^2 \, dx \leq \int_\Omega \frac{\alpha(p^m_n)}{S'(p^m_n)} |\nabla (S(p^m_n) - S(k))_-|^2 \, dx. \quad (4.15) \]

Using the embedding theorem gives (recall that \( \Omega \) is a two-dimensional domain)

\[ M(k) \left( \int_\Omega |(S(p^m_n) - S(k))_-|^q \, dx \right)^{2/q} \leq C_E M(k) \int_\Omega |\nabla (S(p^m_n) - S(k))_-|^2 \, dx, \quad (4.16) \]

where \( 2 < q < +\infty \) and, here, the embedding constant \( C_E \) depends only on \( \Omega \). We may now combine (4.14)–(4.16) to obtain

\[ M(k) \left( \int_\Omega |(S(p^m_n) - S(k))_-|^q \, dx \right)^{2/q} \leq C_E(C_\phi S_s + |\alpha_1|) \int_\Omega |f(x, p^m_n, c^m_{n-1}, r^m_{n-1}, \vartheta^m_{n-1})(S(p^m_n) - S(k))_-| \, dx. \quad (4.17) \]

Taking into account (2.13) we have

\[ \int_\Omega |f(x, p^m_n, c^m_{n-1}, r^m_{n-1}, \vartheta^m_{n-1})(S(p^m_n) - S(k))_-| \, dx \leq \tilde{f}(k) \int_\Omega |(S(p^m_n) - S(k))_-| \, dx \quad (4.18) \]

and applying the Hölder’s inequality to the right hand side in (4.18) and combining (4.17) and (4.18) we arrive at the estimate

\[ \left( \int_{A_k(m)} |(S(p^m_n) - S(k))_-|^q \, dx \right)^{1/q} \leq \frac{C_E(C_\phi S_s + |\alpha_1|)\tilde{f}(k)}{M(k)} \left( \int_{A_k(m)} 1 \, dx \right)^{1/q'}, \quad (4.19) \]

where

\[ A_k(m) = \{ x \in \Omega ; \ p^m_n(x) < k \}. \]

On the other hand, applying the Hölder’s inequality we have

\[ \int_{A_k(m)} |(S(p^m_n) - S(k))_-|^2 \, dx \leq \left( \int_{A_k(m)} |(S(p^m_n) - S(k))_-|^q \, dx \right)^{2/q} \left( \int_{A_k(m)} 1 \, dx \right)^{(q-2)/q} \]
and using (4.19) yields
\[
\int_{A_{k(m)}} |(S(p_n^m) - S(k))^-|^2 \, dx \\
\leq \left( \frac{C_E(C_\phi S_s + |\alpha_1|)\bar{f}(k)}{M(k)} \right)^2 \left( \int_{A_{k(m)}} 1 \, dx \right)^{(3q-4)/q}.
\] (4.20)

In view of (4.11) and employing (2.12) we can write
\[
\phi_1 \int_{A_{k(i)}} |(S(p_n^i) - S(k))^-|^2 \, dx \leq I_k(i) \\
\leq I_k(m) \leq \phi_2 \int_{A_{k(m)}} |(S(p_n^m) - S(k))^-|^2 \, dx
\] (4.21)

for all \(i = 1, 2, \ldots, n\). Since \(\ell < k\) implies \(A_\ell(i) \subset A_k(i)\), and \((S(p_n^\ell) - S(k))^- \leq (S(\ell) - S(k)) < 0\) on \(A_\ell(i)\), we have
\[
|S(\ell) - S(k)|^2 \leq |(S(p_n^\ell) - S(k))^-|^2 \text{ on } A_\ell(i)
\]
and thus
\[
(S(\ell) - S(k))^2 |A_\ell(i)| \leq \int_{A_{\ell(i)}} |(S(p_n^\ell) - S(k))^-|^2 \, dx \\
\leq \int_{A_{k(i)}} |(S(p_n^m) - S(k))^-|^2 \, dx \\
\leq \frac{\phi_2}{\phi_1} \int_{A_{k(m)}} |(S(p_n^m) - S(k))^-|^2 \, dx.
\] (4.22)

Finally, from this and (4.20) we deduce
\[
|A_\ell(i)| \leq \frac{\phi_2}{\phi_1} \left( \frac{C_E(C_\phi S_s + |\alpha_1|)\bar{f}(k)}{M(k)(S(\ell) - S(k))} \right)^2 |A_k(m)|^{(3q-4)/q} \text{ for all } i = 1, \ldots, n.
\] (4.23)

To conclude the proof of Theorem 4.3 we define
\[
\mu_k = \max_{i=1,\ldots,n} |A_k(i)|.
\]

Now, (4.23) implies (recall \(\ell < k\))
\[
|\mu_\ell| \leq \frac{\phi_2}{\phi_1} \left( \frac{C_E(C_\phi S_s + |\alpha_1|)\bar{f}(k)}{M(k)(S(\ell) - S(k))} \right)^2 |\mu_k|^{(3q-4)/q}.
\] (4.24)

Next, we are going to apply Lemma 4.4. In particular, we define a decreasing sequence
\[
d_j = \frac{\delta}{2} \left( 1 + \frac{1}{2^j} \right), \quad j = 0, 1, 2, \ldots,
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where \( \delta \) is a small positive real number and let

\[
k_j = S^{-1}(d_j).
\]

Since \( S \) is strictly increasing function, it is clear that

\[
k_{j+1} < k_j, \quad j = 0, 1, 2, \ldots
\]

Then from (4.24) we have

\[
Z_{j+1} \leq \left( \frac{C_1(C_0 S_s + |\alpha_1|)}{M(S^{-1}(d_j)(d_{j+1} - d_j))} \right)^2 Z_j^{(3q-4)/q} Z_j^{1+2(q-2)/q},
\]

where

\[
Z_j = |\mu_k_j|.
\]

Recall that \( 2 < q < +\infty \). In view of Assumption (v) and taking \( \delta > 0 \) "small enough" we apply Lemma 4.4 to get (4.7). Explicitly, this means that there exists \( \ell \) (independent of \( n \)) such that

\[
|\mu_\ell| = 0,
\]

in other words,

\[
p_n^i \geq \ell \quad \text{almost everywhere in } \Omega \text{ and for all } i = 1, \ldots, n.
\]

The proof of Theorem 4.3 is complete.

Now we are ready to prove Theorem 4.1

**Proof of Theorem 4.1** The proof rests on the \( W^{1,s} \)-regularity of elliptic problems presented in [12, 20] and the embedding \( W^{1,s}_{\Gamma_D}(\Omega) \subset L^\infty(\Omega) \) if \( s > 2 \) (recall that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \)).

We begin by proving the existence of \( p_n^i \in W^{1,2}_{\Gamma_D}(\Omega) \), being the solution to problem (4.1). Due to Theorem 4.3 we may consider the truncated function \( \tilde{k}_r \) defined by

\[
\tilde{k}_r(\xi) = \begin{cases} k_r(S(\xi)), & \xi > \ell, \\ k_r(S(\ell)), & \xi \leq \ell, \end{cases}
\]

where \( \ell \) is taken from (4.6). Recall that \( k_R \) is positive and strictly increasing on \([0, S_s]\) and \( S \) is positive and strictly increasing on \( \mathbb{R} \). Hence \( \tilde{k}_r \) is the increasing function such that

\[
0 < K_0 \leq \tilde{k}_r(\xi) \leq K_1 \quad \forall \xi \in \mathbb{R}
\]
with appropriate chosen constants $K_0$ and $K_1$, say $K_0 = k_r(S_0)$ and $K_1 = k_r(S_1)$. Hence, problem (4.1) takes the form

$$\int_{\Omega} k(x, r_n^{-1}(x)) \frac{1}{\mu(\theta_n^{-1}(x))} \nabla p_n \cdot \nabla \xi \, dx$$

$$+ \frac{1}{h} \int_{\Omega} \phi(x, r_n^{-1}) S(p_n^i) \xi \, dx - \int_{\Omega} \alpha_1 f(x, p_n^i, c_n^{-1}, \theta_n^{-1}, r_n^{-1}) \xi \, dx$$

$$= \frac{1}{h} \int_{\Omega} \phi(x, r_n^{-1}) S(p_n^i) \xi \, dx$$

for any $\xi \in W^{1,2}_2(\Omega)$. Note that the unknown $r_n^i$ in the second line of (4.28) can be easily eliminated using the equation (4.4), which can be rewritten as

$$r_n^i = r_n^{-1} + hf(x, p_n^i, c_n^{-1}, \theta_n^{-1}, r_n^{-1}).$$

We now define the so called Kirchhoff transformation, which employs the primitive function $\beta : \mathbb{R} \to \mathbb{R}$, $\zeta = \beta(\xi)$, defined by

$$\beta(\xi) = \int_{0}^{\xi} k_r(s) \, ds.$$

It is worth noting that (4.27) implies $\beta$ to be continuous and increasing, and one-to-one with $\beta^{-1}$ Lipschitz continuous. Hence, with the notation $u(x) = \beta(p_n^i(x))$, problem (4.28) can be rewritten in terms of a new variable $u$ as

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \xi \, dx + \int_{\Omega} B(x, u) \xi \, dx = \int_{\Omega} g(x) \xi \, dx$$

for any $\xi \in W^{1,2}_{1,1}(\Omega)$, where we denote briefly

$$A(x) = \frac{k(x, r_n^{-1}(x))}{\mu(\theta_n^{-1}(x))},$$

$$B(x, u) = \frac{S(\beta^{-1}(u))}{h} \phi(x, r_n^{-1}(x)) + hf(x, \beta^{-1}(u), c_n^{-1}(x), \theta_n^{-1}(x), r_n^{-1}(x)))$$

$$- \alpha_1 f(x, \beta^{-1}(u), c_n^{-1}(x), \theta_n^{-1}(x), r_n^{-1}(x)), $$

$$g(x) = \frac{\phi(x, r_n^{-1}(x)) S(p_n^i)}{h}.$$

Note that $g \in L^{\infty}(\Omega)$ and

$$0 < A_1 < A(\cdot) < A_2 < +\infty \quad (A_1, A_2 = \text{const}) \quad \text{a.e. in } \Omega,$$

$$|B(\cdot, \xi)| \leq C \quad \forall \xi \in \mathbb{R} \text{ and a.e. in } \Omega.$$

The existence of $u \in W^{1,2}_{1,1}(\Omega)$, the solution of problem (4.30), follows from Chapter 2.4. With $u \in W^{1,2}_{1,1}(\Omega)$ in hand, the weak maximum principle for the problem

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \xi \, dx = \int_{\Omega} g(x) \xi \, dx - \int_{\Omega} B(x, u) \xi \, dx$$
for any $\zeta \in W^{1,2}_{\Gamma_D} (\Omega)$, gives the regularity $u \in L^\infty (\Omega)$, see e.g. [49] Chapter 4.1.2.

Note that, in view of (4.27), the Kirchhoff transformation preserves $L^\infty$ space for the problem. We now set $p_n^i (x) := \beta^{-1} (u(x))$ a.e. in $\Omega$ to get the representation

$$\nabla p_n^i = \frac{1}{k_r (\beta^{-1} (u))} \nabla u, \quad \text{i.e.} \quad k_r (p_n^i) \nabla p_n^i = \nabla u$$

and hence

$$p_n^i \in W^{1,2}_{\Gamma_D} (\Omega) \cap L^\infty (\Omega) \quad \text{iff} \quad u \in W^{1,2}_{\Gamma_D} (\Omega) \cap L^\infty (\Omega).$$

We now conclude that $p_n^i$ solves (4.1).

With $p_n^i \in W^{1,2}_{\Gamma_D} (\Omega) \cap L^\infty (\Omega)$ in hand, we rewrite the equation (4.1) in the form (transferring the lower-order terms to the right hand side)

$$\int _\Omega \alpha (x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla \zeta \, dx = \int _\Omega \alpha _1 f (x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \zeta \, dx - \int _\Omega \frac{\phi (x, r_n^i, S (p_n^i) - \phi (x, r_n^{i-1}, S (p_n^{i-1})}{h} \zeta \, dx$$

for any $\zeta \in W^{1,2}_{\Gamma_D} (\Omega)$. In view of Assumptions (i), (iii) and (iv), both integrals on the right hand side make sense for any $\zeta \in W^{1,2}_{\Gamma_D} (\Omega)$, $r' = r/(r - 1)$ with some $r > 2$.

Now we are able to apply [12] Theorem 4 to obtain $p_n^i \in W^{1,s}_{\Gamma_D} (\Omega)$ with some $s > 2$.

Now with $c_n^{i-1} \in W^{1,2} (\Omega)$, $\vartheta_n^{i-1} \in W^{1,2} (\Omega)$, $r_n^{i-1} \in W^{1,2} (\Omega)$ and $p_n^i \in W^{1,s}_{\Gamma_D} (\Omega)$ (with some $s > 2$) in hand, one obtains $r_n^i$ directly from (4.28). Since $f$ is supposed to be Lipschitz continuous, we easily deduce $r_n^i \in W^{1,2} (\Omega)$ (c.f. [45] Proposition 1.28). Moreover, by (2.13) we have $r_n^i \in L^\infty (\Omega)$.

The existence of $c_n^i \in W^{1,2}_{\Gamma_D} (\Omega)$ and $\vartheta_n^i \in W^{1,2}_{\Gamma_D} (\Omega)$, being the solutions to problems (4.2) and (4.3), respectively, can be proven in the same way as [4] Theorem 6.5. In particular, with $p_n^i \in W^{1,s}_{\Gamma_D} (\Omega)$, $s > 2$, and $r_n^i \in L^\infty (\Omega)$, given by (4.4), in hand, (4.2) and (4.3) represent semilinear equations which can be solved by the approach in [45] Chapter 2.4. Analysis similar to the above yields $c_n^i, \vartheta_n^i \in W^{1,s}_{\Gamma_D} (\Omega)$ with some $s > 2$. By embedding theorem we have $c_n^i, \vartheta_n^i \in L^\infty (\Omega)$. The proof of Theorem 4.1 is complete.

### 4.2 A-priori estimates for discrete approximations

In this part of the paper, which is rather technical, we prove some uniform estimates (with respect to $n$) for the discrete approximations of the solution. In the following estimates, many different constants will appear. For simplicity of notation, as above, $C$, $C_1$, $C_2$, $\ldots$, represent generic constants which may change their numerical values from one formula to another but do not depend on $n$ and the functions under consideration.
4.2.1 Uniform bounds in $L^\infty$

We first prove the apriori $L^\infty$-estimate for $c_n^i$, $i = 1, \ldots, n$, being the solution to (4.2). By Theorem 4.1 we have $c_n^i \in W^{1,2}_D(\Omega) \cap L^\infty(\Omega)$. Hence, $(c_n^i)^\ell \in W^{1,2}_D(\Omega) \cap L^\infty(\Omega)$ for all $\ell = 1, 2, \ldots$. The following procedure is similar to that used e.g. in [11, 41] for scalar problems. Let $\ell$ be an odd integer. Using $\zeta = \lfloor \ell/(\ell + 1) \rfloor (c_n^i)^{\ell + 1}$ as a test function in (4.1) and $\eta = (c_n^i)^\ell$ in (4.2) and combining both equations we obtain

\[
\frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
- \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
+ \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
+ \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
- \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
+ \int_\Omega \ell[c_n^i]^{\ell - 1} \phi(x, r_n^{i-1}) S(p_n^{i-1}) D_w(p_n^{i-1}) \nabla c_n^i \cdot \nabla c_n^i \, dx \\
= 0. 
\]

(4.31)

Applying the Young’s inequality, for the term in the fifth line in (4.31) we can write

\[
\frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
\leq \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
+ \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx. 
\]

(4.32)

Taking (4.31) and (4.32) together we deduce

\[
\frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
- \frac{1}{h} \int_\Omega \phi(x, r_n^{i-1}) S(p_n^{i-1})[c_n^i]^{\ell + 1} \, dx \\
+ \int_\Omega \ell[c_n^i]^{\ell - 1} \phi(x, r_n^{i-1}) S(p_n^{i-1}) D_w(p_n^{i-1}) |\nabla c_n^i|^2 \, dx \\
\leq 0. 
\]

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Now, we sum (4.33) for \( i = 1, \ldots, j \) to get

\[
\int_{\Omega} \phi(x, r_n^i) S(p_n^i)[c_n^i]^{\ell+1} \, dx \\
+ h \sum_{i=1}^{j} \int_{\Omega} \ell(\ell + 1)[c_n^i]^{\ell-1} \phi(x, r_n^{i-1}) S(p_n^{i-1}) D_w(p_n^{i-1}) |\nabla c_n^i|^2 \, dx \\
\leq \int_{\Omega} \phi(x, r_n^0) S(p_n^0)[c_n^0]^{\ell+1} \, dx. \tag{4.34}
\]

Note that the second integral in (4.34) is nonnegative (\( \ell \) is supposed to be the odd integer). Moreover, in view of (4.34), (2.12) and (4.6) we have

\[
\|c_n^j\|_{L^{\ell+1}(\Omega)} \leq C\|c_0\|_{L^{\ell+1}(\Omega)}, \tag{4.35}
\]

where the constant \( C \) is independent of \( \ell \) and \( n \). Now, letting \( \ell \to +\infty \) in (4.35), we obtain

\[
\|c_n^j\|_{L^\infty(\Omega)} \leq C, \quad j = 1, \ldots, n. \tag{4.36}
\]

The same \( L^\infty \)-estimate can be drawn for temperature approximations, i.e.

\[
\|\vartheta_n^j\|_{L^\infty(\Omega)} \leq C, \quad j = 1, \ldots, n. \tag{4.37}
\]

Because many steps of the proof of (4.37) are similar to those from the preceding estimate (4.36), we shall proceed more rapidly here, without explaining particular steps once more.

At the same time, from (4.4) and using (2.13) we have

\[
\|r_n^i\|_{L^\infty(\Omega)} = \|h \sum_{j=1}^{i} f(x, p_n^j, c_n^{j-1}, \vartheta_n^{j-1}, r_n^{j-1})\|_{L^\infty(\Omega)} \\
\leq ihC_f \leq TC_f, \quad i = 1, \ldots, n. \tag{4.38}
\]

4.2.2 Energy estimates for discrete approximations of primary unknowns

We start with the uniform estimate for pressure approximations \( p_n^i \). We test (4.1) with \( \zeta = p_n^i \) to get

\[
\int_{\Omega} \left[ \phi(x, r_n^i) - \phi(x, r_n^{i-1}) \right] S(p_n^i)p_n^i \, dx \\
+ \int_{\Omega} \phi(x, r_n^{i-1}) \left[ S(p_n^i) - S(p_n^{i-1}) \right] p_n^i \, dx \\
+ h \int_{\Omega} a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla p_n^i \, dx \\
= h \int_{\Omega} a_3 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) p_n^i \, dx. \tag{4.39}
\]
Define the function $\Theta_S : \mathbb{R} \to \mathbb{R}$ given by the equation

$$\Theta_S(\xi) = \int_0^\xi S'(z)dz, \quad \xi \in \mathbb{R}. \quad (4.40)$$

It is easy to check that

$$\Theta_S(\xi_1) - \Theta_S(\xi_2) \leq [S(\xi_1) - S(\xi_2)]\xi_1 \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \quad (4.41)$$

Using the inequality (4.41) in the equation (4.39) we arrive at

$$\int_{\Omega} \left[ \phi(x, r_n^i - \phi(x, r_n^{i-1})) \left[ S(p_n^i)p_n^i - \Theta_S(p_n^i) \right] \right] dx$$

$$+ \int_{\Omega} \left[ \phi(x, r_n^i)\Theta_S(p_n^i) - \phi(x, r_n^{i-1})\Theta_S(p_n^{i-1}) \right] dx$$

$$+ h \int_{\Omega} a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot \nabla p_n^i dx$$

$$\leq h \int_{\Omega} \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) p_n^i dx.$$

From this we have

$$\int_{\Omega} \left[ \phi(x, r_n^i)\Theta_S(p_n^i) - \phi(x, r_n^{i-1})\Theta_S(p_n^{i-1}) \right] dx$$

$$+ h \int_{\Omega} a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) |\nabla p_n^i|^2 dx$$

$$\leq h \int_{\Omega} \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) p_n^i dx$$

$$- \int_{\Omega} \left( \phi(x, r_n^i) - \phi(x, r_n^{i-1}) \right) \left[ S(p_n^i)p_n^i - \Theta_S(p_n^i) \right] dx. \quad (4.42)$$

Taking into account (2.11) and (2.13) and estimating the right-hand side in (4.42) we deduce

$$h \int_{\Omega} \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) p_n^i dx$$

$$- \int_{\Omega} \left( \phi(x, r_n^i) - \phi(x, r_n^{i-1}) \right) \left[ S(p_n^i)p_n^i - \Theta_S(p_n^i) \right] dx$$

$$\leq h \int_{\Omega} \left| \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \right| p_n^i dx$$

$$+ \int_{\Omega} C_\phi |r_n^i - r_n^{i-1}| S(p_n^i)p_n^i - \Theta_S(p_n^i) dx$$

$$\leq |\alpha_1| h \int_{\Omega} \left| f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \right| p_n^i dx$$

$$+ h \int_{\Omega} C_\phi |f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1})| S(p_n^i)p_n^i - \Theta_S(p_n^i) dx$$

$$\leq C_1 h \int_{\Omega} |p_n^i| dx + C_2 h \int_{\Omega} \Theta_S(p_n^i) dx. \quad (4.43)$$
Combining (4.42) and (4.43), using (4.6) and applying the Young's inequality to the first term on the right-hand side in (4.43) we get

\[
\int_{\Omega} \left[ \phi(x, r_{n}^{i}) S(p_{n}^{i}) - \phi(x, r_{n}^{i-1}) S(p_{n}^{i-1}) \right] \, dx + C_{1} h \int_{\Omega} |\nabla p_{n}^{i}|^2 \, dx \\
\leq C_{2} h + C_{3} h \int_{\Omega} S(p_{n}^{i}) \, dx. \tag{4.44}
\]

Sum (4.44) for \( i = 1, 2, \ldots, k \). We have

\[
\int_{\Omega} \phi(x, r_{n}^{i}) S(p_{n}^{i}) \, dx + C_{1} h \sum_{i=1}^{k} \int_{\Omega} |\nabla p_{n}^{i}|^2 \, dx \\
\leq \int_{\Omega} \phi(x, r_{n}^{0}) S(p_{n}^{0}) \, dx + C_{2} k h + C_{3} h \sum_{i=1}^{k} \int_{\Omega} S(p_{n}^{i}) \, dx, \quad k = 1, 2, \ldots, n. \tag{4.45}
\]

We now apply the discrete version of the Gronwall’s inequality (see e.g. [45] Chapter 1, inequality (1.67)) to get

\[
\int_{\Omega} S(p_{n}^{i}) \, dx + h \sum_{i=1}^{k} \int_{\Omega} |\nabla p_{n}^{i}|^2 \, dx \leq C, \quad k = 1, 2, \ldots, n. \tag{4.46}
\]

In what follows, we proceed by proving a similar uniform estimate for approximations of \( c \). Using \( \eta = 2c_{n}^{i} \) as a test function in (4.2) we have

\[
\int_{\Omega} \phi(x, r_{n}^{i}) S(p_{n}^{i}) (c_{n}^{i})^2 - \phi(x, r_{n}^{i-1}) S(p_{n}^{i-1}) (c_{n}^{i-1})^2 \, dx \\
+ \int_{\Omega} \left[ \phi(x, r_{n}^{i}) S(p_{n}^{i}) - \phi(x, r_{n}^{i-1}) S(p_{n}^{i-1}) \right] (c_{n}^{i})^2 \, dx \\
+ \int_{\Omega} \phi(x, r_{n}^{i-1}) S(p_{n}^{i-1}) (c_{n}^{i} - c_{n}^{i-1})^2 \, dx \\
+ 2h \int_{\Omega} \phi(x, r_{n}^{i}) S(p_{n}^{i}) D_{a}(x, p_{n}^{i}) \nabla c_{n}^{i} \cdot \nabla c_{n}^{i} \, dx \\
+ h \int_{\Omega} a(x, p_{n}^{i}, \vartheta_{n}^{i-1}, r_{n}^{i-1}) \nabla p_{n}^{i} \cdot 2c_{n}^{i} \nabla c_{n}^{i} \, dx \\
= 0. \tag{4.47}
\]

One is allowed to use \( \zeta = (c_{n}^{i})^2 \) as a test function in (4.1) to obtain

\[
\int_{\Omega} \left[ \phi(x, r_{n}^{i}) S(p_{n}^{i}) - \phi(x, r_{n}^{i-1}) S(p_{n}^{i-1}) \right] (c_{n}^{i})^2 \, dx \\
+ h \int_{\Omega} a(x, p_{n}^{i}, \vartheta_{n}^{i-1}, r_{n}^{i-1}) \nabla p_{n}^{i} \cdot (c_{n}^{i})^2 \, dx \\
= h \int_{\Omega} \alpha_{1} f(x, p_{n}^{i}, c_{n}^{i-1}, \vartheta_{n}^{i-1}, r_{n}^{i-1}) (c_{n}^{i})^2 \, dx. \tag{4.48}
\]
Subtracting (4.48) from (4.47) gives
\[
\int_{\Omega} \phi(x, r_n^i) S(p_n^i)(c_n^i)^2 - \phi(x, r_n^{i-1}) S(p_n^{i-1})(c_n^{i-1})^2 \, dx \\
+ \int_{\Omega} \phi(x, r_n^{i-1}) S(p_n^{i-1}) (c_n^i - c_n^{i-1})^2 \, dx \\
+ 2h \int_{\Omega} \phi(x, r_n^i) S(p_n^i) D_w(x, p_n^i) \nabla c_n^i \cdot \nabla e_n^i \, dx \\
+ h \int_{\Omega} \alpha_1 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1})(c_n^i)^2 \, dx \\
= 0.
\]  
(4.49)

Upon addition (4.49) for \( i = 1, 2, \ldots, j \) and taking into account (2.13), we can write
\[
\int_{\Omega} \phi(x, r_n^i) S(p_n^i)(c_n^i)^2 \, dx \\
\leq \phi(x, r_n^0) S(p_n^0)(c_n^0)^2 + h|\alpha_1|C \sum_{i=1}^j \int_{\Omega} (c_n^i)^2 \, dx, \quad j = 1, \ldots, n.
\]  
(4.50)

Noting that, in view of (4.49), (2.1) and (2.12), there exists a positive constant \( C \) (independent of \( n \)) such that
\[
\phi(x, r_n^i), S(p_n^i), D_w(x, p_n^i) > C \quad \text{in} \, \Omega, \quad i = 1, \ldots, n,
\]  
(4.51)
the inequality (4.50) can be simplified to
\[
\int_{\Omega} (c_n^i)^2 \, dx \leq C_1 + C_2 h \sum_{i=1}^j \int_{\Omega} (c_n^i)^2 \, dx, \quad j = 1, \ldots, n.
\]  
(4.52)

Now, similarly as in (4.45), we can use the Gronwall’s inequality. By doing that, in view of (4.49), (4.51) and (4.52), we obtain the estimate
\[
\max_{i=1, \ldots, n} \int_{\Omega} |c_n^i|^2 \, dx + h \sum_{i=1}^n \int_{\Omega} |\nabla c_n^i|^2 \, dx \leq C.
\]  
(4.53)

The same uniform estimate can be drawn for the temperature approximations \( \vartheta_n^i \). We use \( \psi = 2\vartheta_n^i \) as a test function in (4.3) to obtain
\[
\int_{\Omega} \phi(x, r_n^i) S(p_n^i) \vartheta_n^i - \phi(x, r_n^{i-1}) S(p_n^{i-1}) \vartheta_n^{i-1} \, dx \\
+ \int_{\Omega} \phi(x, r_n^{i-1}) S(p_n^{i-1}) (\vartheta_n^i - \vartheta_n^{i-1}) \, dx \\
+ 2 \int_{\Omega} \lambda(x, p_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) |\nabla \vartheta_n^i|^2 \, dx \\
+ \int_{\Omega} a(x, p_n^i, \vartheta_n^{i-1}, r_n^{i-1}) \nabla p_n^i \cdot 2\vartheta_n^i \nabla \vartheta_n^i \, dx \\
= 2 \int_{\Omega} \alpha_2 f(x, p_n^i, c_n^{i-1}, \vartheta_n^{i-1}, r_n^{i-1}) \vartheta_n^i \, dx.
\]
The above equation may be written as

\[
\int_{\Omega} \frac{\phi(x, r_n^i) S(p_n^i)[{\varphi}_n^i]^2 - \phi(x, \{r_n^{i-1}\}) S(p_n^{i-1})[{\varphi}_n^{i-1}]^2}{h} \, dx \\
+ \int_{\Omega} \frac{\phi(x, r_n^i) S(p_n^i) - \phi(x, \{r_n^{i-1}\}) S(p_n^{i-1})}{h} [\varphi_n^i]^2 \, dx \\
+ \int_{\Omega} \frac{\varphi(x, r_n^i) \varphi_n^i - \varphi(x, \{r_n^{i-1}\}) \varphi_n^{i-1}}{h} \, dx \\
+ \int_{\Omega} \frac{\varphi(x, r_n^i) \varphi_n^i - \varphi(x, \{r_n^{i-1}\}) \varphi_n^{i-1}}{h} \, dx \\
+ \frac{1}{\Omega} \int_{\Omega} \lambda(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) |\nabla \varphi_n^i|^2 \, dx \\
+ \int_{\Omega} \alpha(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) \nabla p_n^i \cdot \nabla |\varphi_n^i|^2 \, dx \\
= 2 \int_{\Omega} \alpha_2 f(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) |\varphi_n^i| \, dx.
\] (4.54)

Putting \( \zeta = [\varphi_n^i]^2 \) into (4.1), we get

\[
\int_{\Omega} \frac{\phi(x, r_n^i) S(p_n^i) - \phi(x, \{r_n^{i-1}\}) S(p_n^{i-1})}{h} [\varphi_n^i]^2 \, dx \\
+ \int_{\Omega} \alpha(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) \nabla p_n^i \cdot \nabla |\varphi_n^i|^2 \, dx \\
= \int_{\Omega} \alpha_1 f(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) |\varphi_n^i|^2 \, dx.
\] (4.55)

Substituting (4.55) into (4.54), multiplying by \( h \) and taking into account (4.4) we deduce

\[
\int_{\Omega} \left( \frac{\phi(x, r_n^i) S(p_n^i)[{\varphi}_n^i]^2 - \phi(x, \{r_n^{i-1}\}) S(p_n^{i-1})[{\varphi}_n^{i-1}]^2}{h} \right) \, dx \\
+ \int_{\Omega} \left( \frac{\varphi(x, r_n^i)[{\varphi}_n^i]^2 - \varphi(x, \{r_n^{i-1}\})[{\varphi}_n^{i-1}]^2}{h} \right) \, dx \\
+ \frac{2}{\Omega} \int_{\Omega} \lambda(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) |\nabla \varphi_n^i|^2 \, dx \\
\leq h \int_{\Omega} \left( 2|\alpha_2| |\varphi_n^i| + |\alpha_1| \varphi_n^i + [{\varphi}_n^{i-1}]^2 \right) \left| f(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) \right| \, dx.
\] (4.56)

We now apply the estimate

\[
h \int_{\Omega} \left( 2|\alpha_2| |\varphi_n^i| + |\alpha_1| \varphi_n^i + [{\varphi}_n^{i-1}]^2 \right) \left| f(x, p_n^i, \varphi_n^i, \{r_n^{i-1}\}) \right| \, dx \\
\leq h \int_{\Omega} \left( C_1 + C_2 [{\varphi}_n^{i-1}]^2 \right) C_f \, dx,
\]
which holds for “sufficiently large” $C_1$, and sum (4.56) for $i = 1, \ldots, j$ to obtain

$$
\int_\Omega (\phi(x, r^j_n) S(p^j_n) + \varrho(x, r^j_n)) [\vartheta^j_n]^2 \, dx
+ 2h \sum_{i=1}^j \int_\Omega \lambda(x, p^i_{n-1}, \vartheta^{i-1}_n, r^{i-1}_n) |\nabla \vartheta^i_n|^2 \, dx
\leq \int_\Omega (\phi(x, r^0_n) S(p^0_n) + \varrho(x, r^0_n)) [\vartheta^0_n]^2 \, dx
+ h \sum_{i=1}^j \int_\Omega (C_1 + C_2 [\vartheta^i_n]^2) C_f \, dx.
$$

By (2.7), (2.10), (2.12) and (4.6), the above estimate may be simplified as

$$
\int_\Omega |\vartheta^j_n|^2 \, dx + h \sum_{i=1}^j \int_\Omega |\nabla \vartheta^i_n|^2 \, dx \leq C_1 + C_2 T + C_3 h \sum_{i=1}^j \int_\Omega |\vartheta^i_n|^2 \, dx.
$$

As before, we can now use the Gronwall’s inequality. As a consequence, we obtain

$$
\int_\Omega |\vartheta^j_n|^2 \, dx + h \sum_{i=1}^j \int_\Omega |\nabla \vartheta^i_n|^2 \, dx \leq C, \quad j = 1, 2, \ldots, n.
$$

(4.57)

Finally, using Assumptions (iv) and (vi), (4.46), (4.53) and (4.57) and applying [45, Proposition 1.28], the uniform estimate

$$
h \sum_{i=1}^j \int_\Omega |\nabla r^i_{n-1}|^2 \, dx \leq C, \quad j = 1, 2, \ldots, n.
$$

(4.58)

can be obtained directly from (4.4). Note that (4.58) together with (4.38) yields

$$
h \sum_{i=1}^j \int_\Omega \|r^i_n\|_{W^{1,2}(\Omega)}^2 \, dx \leq C, \quad j = 1, 2, \ldots, n.
$$

(4.59)

Moreover, from (4.4) (see also (4.29)) one observes immediately that

$$
\frac{r^i_n - r^{i-1}_n}{h} = f(x, p^i_n, c^i_{n-1}, \varrho^{i-1}_n, r^{i-1}_n)
$$

and therefore, in view of (2.13), we have

$$
\left\| \frac{r^i_n - r^{i-1}_n}{h} \right\|_{L^\infty(\Omega)} \leq C_f, \quad i = 1, 2, \ldots, n.
$$

(4.60)

### 4.2.3 Further estimates

We now derive additional auxiliary estimates which will be used in the following section. Such estimates play the crucial role in compactness arguments (see [2, Lemma 1.9]) and taking the limit $n \to +\infty$. 
Let us sum up (4.1) for $i = j + 1, \ldots, j + k$ and then put $\zeta = p_{n}^{i+k} - p_{j}^{i}$. This leads to

$$
\int_{\Omega} \left[ \phi(x, r_{n}^{i+k}) S(p_{n}^{i+k}) - \phi(x, r_{n}^{i}) S(p_{j}^{i}) \right] (p_{n}^{i+k} - p_{j}^{i}) \, dx
$$

$$
+ h \sum_{i=j+1}^{j+k} \int_{\Omega} a(x, p_{j}^{i}, \vartheta_{j}^{i-1}, r_{n}^{i-1}) \nabla p_{j}^{i} \cdot \nabla (p_{n}^{i+k} - p_{j}^{i}) \, dx
$$

$$
= h \sum_{i=j+1}^{j+k} \int_{\Omega} \alpha_{1} f(x, p_{n}^{i}, c_{n}^{i-1}, \vartheta_{j}^{i-1}, r_{n}^{i-1}) (p_{n}^{i+k} - p_{j}^{i}) \, dx. \tag{4.61}
$$

From this and in view of (2.8) and (2.13) we have

$$
\int_{\Omega} \left[ \phi(x, r_{n}^{i+k}) S(p_{n}^{i+k}) - \phi(x, r_{n}^{i}) S(p_{j}^{i}) \right] (p_{n}^{i+k} - p_{j}^{i}) \, dx
$$

$$
\leq C_{h} \sum_{i=j+1}^{j+k} \int_{\Omega} \left| \nabla p_{j}^{i} \right| \left| \nabla (p_{n}^{i+k} - p_{j}^{i}) \right| \, dx
$$

$$
+ kh|\alpha_{1}|C_{f} \int_{\Omega} \left| p_{n}^{i+k} - p_{j}^{i} \right| \, dx. \tag{4.62}
$$

Using (2.11) and (4.4), the above estimate can be further rewritten as

$$
\int_{\Omega} \phi(x, r_{n}^{i+k}) \left[ S(p_{n}^{i+k}) - S(p_{j}^{i}) \right] (p_{n}^{i+k} - p_{j}^{i}) \, dx
$$

$$
\leq C_{h} \sum_{i=j+1}^{j+k} \int_{\Omega} \left| \nabla p_{j}^{i} \right| \left| \nabla (p_{n}^{i+k} - p_{j}^{i}) \right| \, dx
$$

$$
+ (kh|\alpha_{1}|C_{f} + khS_{s}C_{e}C_{f}) \int_{\Omega} \left| p_{n}^{i+k} - p_{j}^{i} \right| \, dx. \tag{4.63}
$$

Again, we sum (4.63) for $j = 1, \ldots, n - k$, multiply it by $h$ and use (2.12) and (4.46) to arrive at

$$
h \sum_{j=1}^{n-k} \int_{\Omega} \left[ S(p_{n}^{j+k}) - S(p_{j}^{j}) \right] (p_{n}^{j+k} - p_{j}^{j}) \, dx \leq C_{kh}, \quad 0 \leq k < n. \tag{4.64}
$$

Further, using the same arguments as in (4.61)–(4.64), we arrive at

$$
h \sum_{j=1}^{n-k} \int_{\Omega} \left| c_{n}^{j+k} - c_{n}^{j} \right|^{2} \, dx \leq C_{kh}. \tag{4.65}
$$

Finally, derivation similar to that presented above leads to

$$
h \sum_{j=1}^{n-k} \int_{\Omega} \left| \vartheta_{n}^{j+k} - \vartheta_{n}^{j} \right|^{2} \, dx \leq C_{kh}. \tag{4.66}
$$
At the same time, it is easily deduced from (4.29) that
\[ h \sum_{j=1}^{n-k} \int_{\Omega} |r_n^{j+k} - r_n^j|^2 \, dx \leq C h. \quad (4.67) \]

### 4.3 Temporal interpolants and uniform estimates

By means of the sequences \( p^i_n, c^i_n, \vartheta^i_n, r^i_n \) constructed in Section 4.1, we define the piecewise constant interpolants \( \bar{\varphi}_n(t) = \varphi^i_n \) for \( t \in ((i-1)h, ih] \) and, in addition, we extend \( \bar{\varphi}_n \) for \( t \leq 0 \) by \( \bar{\varphi}_n(t) = \varphi_0 \) for \( t \in (-h, 0] \). Here, \( \varphi^i_n \) stands for \( p^i_n, c^i_n, \vartheta^i_n \) or \( r^i_n \).

For a function \( \varphi \) we often use the simplified notation \( \varphi := \varphi(t), \varphi_h(t) := \varphi(t-h) \), \( \partial_t^{-h} \varphi(t) := \frac{\varphi(t) - \varphi(t-h)}{h} \), \( \partial_t^h \varphi(t) := \frac{\varphi(t+h) - \varphi(t)}{h} \). Then, following (4.1)–(4.3), the piecewise constant time interpolants \( \bar{p}_n \in L^\infty(I; W^{1,s}_D(\Omega)), \bar{c}_n \in L^\infty(I; W^{1,s}_D(\Omega)) \) and \( \bar{\vartheta}_n \in L^\infty(I; W^{1,s}_D(\Omega)) \) (with some \( s > 2 \)) satisfy the equations

\[
\int_{\Omega} \partial_t^{-h} [\phi(x, \bar{r}_n(t))] S(\bar{p}_n(t))] \zeta \, dx \\
+ \int_{\Omega} a(x, \bar{p}_n(t), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h)) \nabla \bar{p}_n(t) \cdot \nabla \zeta \, dx \\
= \int_{\Omega} \alpha_1 f(x, \bar{p}_n(t), \bar{c}_n(t-h), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h))) \zeta \, dx \quad (4.68)
\]

for any \( \zeta \in W^{1,2}_D(\Omega) \),

\[
\int_{\Omega} \partial_t^h [\phi(x, \bar{r}_n(t))] S(\bar{p}_n(t))] \bar{\vartheta}_n(t)] \eta \, dx \\
+ \int_{\Omega} \phi(x, \bar{r}_n(t)) S(\bar{p}_n(t)) D_u(\bar{p}_n(t-h)) \nabla \bar{c}_n(t) \cdot \nabla \eta \, dx \\
+ \int_{\Omega} \bar{c}_n(t)a(x, \bar{p}_n(t), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h)) \nabla \bar{p}_n(t) \cdot \nabla \eta \, dx \\
= 0 \quad (4.69)
\]

for any \( \eta \in W^{1,2}_D(\Omega) \) and

\[
\int_{\Omega} \partial_t^{-h} [\phi(x, \bar{r}_n(t))] S(\bar{p}_n(t)) \bar{\vartheta}_n(t)] + g(x, \bar{r}_n(t)) \bar{\vartheta}_n(t)] \psi \, dx \\
+ \int_{\Omega} \lambda(x, \bar{p}_n(t-h), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h)) \nabla \bar{\vartheta}_n(t) \cdot \nabla \psi \, dx \\
+ \int_{\Omega} \bar{\vartheta}_n(t)a(x, \bar{p}_n(t), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h)) \nabla \bar{p}_n(t) \cdot \nabla \psi \, dx \\
= \int_{\Omega} \alpha_2 f(x, \bar{p}_n(t), \bar{c}_n(t-h), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h))) \zeta \, dx \quad (4.70)
\]
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for any \( \psi \in W^{1,2}_{1,D}(\Omega) \). Finally, from (4.4) and (4.5) we have

\[
\bar{R}_n(t) = f(x, \bar{p}_n(t), \bar{c}_n(t-h), \bar{\vartheta}_n(t-h), \bar{r}_n(t-h))
\]  

(4.71)

for all \( t \in [0, T] \), where

\[
\bar{R}_n(t) = r_n^i - r_n^{i-1} \quad \text{for} \ t \in ((i-1)h, ih], \ i = 1, 2, \ldots, n
\]

and \( \bar{R}_n(0) = r_n^1/h, r_n^0 = 0 \). To be able to say something about the behaviour of the sequences \( \{\bar{p}_n\}, \{\bar{c}_n\}, \{\bar{\vartheta}_n\}, \{\bar{R}_n\}, \) and \( \{\bar{r}_n\} \), we now present some apriori estimates for solutions of the problem (4.68)–(4.71).

To this aim, from (4.36), (4.37), (4.38), (4.46), (4.53), (4.57) and (4.58) we see immediately that

\[
\sup_{0 \leq t \leq T} \int_\Omega \theta_S(\bar{p}_n(t))dx + \int_0^T \|\bar{p}_n(t)\|_{W^{1,2}_{1,D}(\Omega)}^2 dt \leq C, \tag{4.72}
\]

\[
\int_0^T \|\bar{c}_n(t)\|_{W^{1,2}_{1,D}(\Omega)}^2 dt \leq C, \tag{4.73}
\]

\[
\int_0^T \|\bar{\vartheta}_n(t)\|_{W^{1,2}_{1,D}(\Omega)}^2 dt \leq C, \tag{4.74}
\]

\[
\int_0^T \|\bar{r}_n(t)\|_{W^{1,2}_{1,D}(\Omega)}^2 dt \leq C, \tag{4.75}
\]

\[
\|\bar{c}_n\|_{L^\infty(Q_T)} \leq C, \tag{4.76}
\]

\[
\|\bar{\vartheta}_n\|_{L^\infty(Q_T)} \leq C; \tag{4.77}
\]

\[
\|\bar{r}_n\|_{L^\infty(Q_T)} \leq C. \tag{4.78}
\]

Moreover, the estimates (4.64)–(4.67) can be rewritten in the form

\[
\int_0^{T-kh} [S(\bar{p}_n(t + kh)) - S(\bar{p}_n(t))] \left( \bar{p}_n(t + kh) - \bar{p}_n(t) \right) dt \leq Ckh, \tag{4.79}
\]

\[
\int_0^{T-kh} \|\bar{c}_n(t + kh) - \bar{c}_n(t)\|^2 dt \leq Ckh, \tag{4.80}
\]

\[
\int_0^{T-kh} \|\bar{\vartheta}_n(t + kh) - \bar{\vartheta}_n(t)\|^2 dt \leq Ckh, \tag{4.81}
\]

\[
\int_0^{T-kh} \|\bar{r}_n(t + kh) - \bar{r}_n(t)\|^2 dt \leq Ckh. \tag{4.82}
\]

Now we are ready to complete the proof of the main result of this paper which is the conclusion of the following section.

4.4 Passage to the limit

The a-priori estimates (4.72)–(4.78) allow us to conclude that there exist \( p \in L^2(I; W^{1,2}_{1,D}(\Omega)), \)

\( c \in L^2(I; W^{1,2}_{1,D}(\Omega)) \cap L^\infty(Q_T), \)

\( \vartheta \in L^2(I; W^{1,2}_{1,D}(\Omega)) \cap L^\infty(Q_T) \) and \( r \in L^\infty(Q_T), \)
such that, letting \( n \to +\infty \) (along a selected subsequence),

\[
\begin{align*}
\bar{p}_n & \rightharpoonup p \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \\
\bar{c}_n & \rightharpoonup c \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \\
\bar{\vartheta}_n & \rightharpoonup \vartheta \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \\
\bar{\vartheta}_n & \rightharpoonup \vartheta \quad \text{weakly star in } L^\infty(Q_T), \\
\bar{r}_n & \rightharpoonup r \quad \text{weakly in } L^2(I; W^{1,2}(\Omega)), \\
\bar{r}_n & \rightharpoonup r \quad \text{weakly star in } L^\infty(Q_T).
\end{align*}
\]

Thus, we derived fundamental properties of the functions \( p, c, \vartheta \text{ and } r \). The crucial step to ensure that \( p, c, \vartheta \text{ and } r \) solve the problem (3.1)–(3.4) consists in showing that the sequences \( \{\bar{p}_n\}, \{\bar{c}_n\}, \{\bar{\vartheta}_n\} \text{ and } \{\bar{r}_n\} \), converge not only weakly in appropriate Bochner spaces, but even almost everywhere on \( Q_T \).

To this aim, in view of (4.72) and (4.79), using the compactness argument one can show in the same way as in [2, Lemma 1.9] and [10, Eqs. (2.10)–(2.12)] that

\[
S(\bar{p}_n) \to S(p) \quad \text{in } L^1(Q_T) \tag{4.83}
\]

and almost everywhere on \( Q_T \). Since \( S \) is strictly monotone, it follows from (4.83) that [23, Proposition 3.35]

\[
\bar{p}_n \rightharpoonup p \quad \text{almost everywhere on } Q_T. \tag{4.84}
\]

By similar arguments, using estimates (4.73)–(4.78) and (4.80)–(4.82), we have

\[
\begin{align*}
\bar{c}_n & \rightharpoonup c \quad \text{almost everywhere on } Q_T, \tag{4.85} \\
\bar{\vartheta}_n & \rightharpoonup \vartheta \quad \text{almost everywhere on } Q_T, \tag{4.86} \\
\bar{r}_n & \rightharpoonup r \quad \text{almost everywhere on } Q_T. \tag{4.87}
\end{align*}
\]

Finally, in consequence of (4.60), the norms \( \|\bar{R}_n(t)\|_{L^\infty(\Omega)} \) are uniformly bounded with respect to \( n \) and \( t \). Hence, there exists \( R \in L^2(I; L^\infty(\Omega)) \), such that

\[
\bar{R}_n \rightharpoonup R \quad \text{weakly star in } L^2(I; L^\infty(\Omega)) \tag{4.88}
\]

and \( R \) can be shown to satisfy (see [44, Chapter 11])

\[
\int_0^t R(s)ds = r(t) \tag{4.89}
\]

and

\[
R(t) = r'(t) \quad \text{in } L^2(I; L^\infty(\Omega)). \tag{4.90}
\]

It follows that

\[
r \in C([0, T]; L^\infty(\Omega)) \quad \text{ (and even } AC([0, T]; L^\infty(\Omega))\text{)}
\]
and \( r(0) = 0 \). In view of (4.84)–(4.87) and the assumption (iv) we have
\[
f(x, \bar{p}_n(t), \bar{c}_n(t - h), \bar{\vartheta}_n(t - h), \bar{r}_n(t - h)) \to f(x, p(t), c(t), \vartheta(t), r(t))
\]
almost everywhere on \( Q_T \) and on account of (4.71) and (4.88) we can write
\[
R = f \text{ in } L^2(I; L^\infty(\Omega)).
\]
This leads to (3.4). Moreover, the above established convergences are sufficient for taking the limit \( n \to \infty \) in (4.68)–(4.70) (along a selected subsequence) to get the weak solution of the system (1.1)–(1.3) in the sense of Definition 3.1. This completes the proof of the main result stated by Theorem 3.2.

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