Shells analysis in orthogonal curvilinear coordinate system with variation-difference method

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Abstract. The variation-difference method is a convenient numerical method for shells of complex forms. It is enough when only cinematic boundary conditions are satisfied because the method is based on the principle of Lagrange. Another advantage of the variation-difference method is the better opportunity to create computer programs based on it. For shell analysis in orthogonal coordinate system as well as for shell analysis in principal curvatures the system of equations describing stress-strain state can be simplified. In this paper the difference between analysis in orthogonal coordinate system and analysis in principal curvatures of the surface is considered. The main distinction of the analysis of shells in orthogonal curvilinear coordinate system is the necessity of determination of components which include curvature of torsion of coordinate lines. The addition of these components in the equations of the theory of shells for the coordinate system in principal curvatures gives possibility to analyze shells in common orthogonal coordinate system. In this article shell analysis in orthogonal coordinate system is applied to shells based on normal cyclic surfaces.

1. Introduction
The object of this work is the analysis of shells of complex forms with the variation-difference method. The most often used shells are shells of simple forms such as some surfaces of revolution, analytic formulas of which are less complex in comparison with other forms of shells [1-5]. Using different types of shells in construction requires studying of geometry of these surfaces that is considered in Encyclopaedia [6], the technology of construction as well as methods of analysis.

Analytic methods [7] in most cases cannot be applied for shells of non-canonic forms at all or there are difficulties in this application. That is the reason for which numerical methods are used: the difference method, the finite element method [8, 9] and the variation-difference method.

The disadvantage of the difference method is the necessity to satisfy both cinematic and static boundary conditions. This is not necessary if the finite element method or the variation-difference method, which are based on the principle of Lagrange, are used. The software such as ANSYS, NISA, NASTRAN etc. is based on the finite element method [10-11].

When the variation-difference method is used the construction is covered by the difference mesh and the derivatives in the functional of total energy of deformations are replaced by difference equations. Minimizing the functional of total energy of deformations (the function of nodal displacements) and obtaining a system of algebraic equations nodal displacements are got. Using difference derivatives internal forces and deformations in nods are obtained.

The analysis of the stress-strain state is more precise for shells of complex form if the variation-difference method is used. The finite element method takes insufficient into account the external and internal geometry. The variation-difference method is easier to realize in program. The usage of variation-difference method of calculation of plates and shells was considered in the work [7].
At the Department of Civil Engineering of People’s Friendship University of Russia (RUDN University) the software for analysis of shells in principal curvatures on the basis of variation-difference method was developed. The subject of this paper is the analysis of shells in orthogonal curvilinear coordinate system. The objective of this work is to find the way to simplify this analysis. In this paper the necessary changes in formulas of shells analysis in principal curvatures are given and program realization is discussed.

2. Materials and methods

In this paper the variation-difference method is used which is convenient to the analysis of shells based on normal cyclic shells in the orthogonal coordinate system. In this article the common orthogonal curvilinear coordinate system is considered, in which coefficient of the second fundamental form of surfaces $\mathcal{M}$ is not equal to zero.

Orthogonal curvilinear coordinate system is a coordinate system, where the curvilinear coordinate lines of the surface $u$ and $v$ form a right angle $\chi = 90^\circ$ and, therefore, the coefficient of the first fundamental form $F$, which is expressed as a dot product equal zero: $F = (\tilde{\rho}_u, \tilde{\rho}_v) = |\tilde{\rho}_u| \cdot |\tilde{\rho}_v| \cdot \cos \chi = AB \cdot \cos \sqrt{A^2B^2 - F^2}$. For a shell is analyzed in the orthogonal coordinate system the equations which characterize its stress-strain state are simplified.

For the coordinate system, in which the coefficient of the second fundamental form $M$ is different from zero:

$$\varphi_2 = (d^2 \tilde{\rho} \cdot \tilde{m}) = -(d\tilde{\rho} \cdot d\tilde{m}) = L d^2 u + 2M d u d v + N d v^2,$$

$$L = \frac{[\tilde{\rho}_u \tilde{\rho}_v \tilde{u} \tilde{u}]}{\Sigma}, M = \frac{[\tilde{\rho}_u \tilde{\rho}_v \tilde{u} \tilde{v}]}{\Sigma}, N = \frac{[\tilde{\rho}_u \tilde{\rho}_v \tilde{v} \tilde{v}]}{\Sigma},$$

where $\Sigma$ is a discriminant of a surface: $\Sigma = \sqrt{A^2B^2 - F^2}$. It is convenient to analyze normal cyclic surfaces in the orthogonal coordinate system. Coefficients of fundamental form of these surfaces are given in the paper [12]. Some examples of using cyclic shells are considered in the articles [13-15].

3. Results and discussion

The coefficient of the first and the second fundamental forms of the normal cyclic surface in the orthogonal coordinate system are expressed as:

$$A = R^2 + [s' - Rk_s \cos \omega]^2; F = 0; B = R^2;$$

$$L = \frac{R'T_{33} - T_{13}T_{31}; M = \frac{1}{\sigma}}{\Sigma}, \frac{[s' - Rk_s (\bar{e}\bar{v})]}{\Sigma} = \frac{RT_{13}}{\sigma},$$

where $T_{13} = s' - Rk_s \cos \omega; T_{31} = s'k_s \cos \omega - Rk_s^2 \cos^2 \omega + R''$;

$$T_{33} = s'' - 2k_s R' \cos \omega - Rk_s' \cos \omega + Rk_s \chi_s (\bar{e}\bar{v}); \sigma = \frac{\Sigma}{R} = \sqrt{\frac{R^2}{\Sigma} + \frac{s' - Rk_s (\bar{e}\bar{v})}}.$$

The stress-strain state of elastic shells in the linear analysis is expressed as a system of equations, which consists of equilibrium equations, which are satisfied, when the functional of total energy of deformations is minimum; geometric equations; Hooke’s equations; expressions of potential energy of deformation and work of external forces. For calculation using programming methods it is more convenient to use a vector-matrix formulation.

Let coordinate lines of the surface be signified as $\alpha_1 = u, \alpha_2 = v$, coefficients of the first fundamental form $A = A_1, B = A_2$, components of the vector of displacements $u_1, u_2, u_3$, components of tangential deformations $\varepsilon_1, \varepsilon_2, \varepsilon_12$, components of bending deformations $\chi_1, \chi_2, \chi_3$, components of tangential forces $N_1, N_2, S$, bending forces $M_1, M_2, H$, shearing forces $Q_1, Q_2$.

Deformations and curvatures are expressed by geometric equations. The system of equations in the orthogonal curvilinear coordinate system is written in the linear theory of shells as:
\[
\varepsilon_1 = \frac{1}{A_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_2 + k_1 u_3; \quad \varepsilon_2 = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u_1 + \frac{1}{A_2} \frac{\partial u_2}{\partial \alpha_2} + k_2 u_3; \\
\varepsilon_{12} = -\frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_1 + \frac{1}{A_2} \frac{\partial u_1}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u_2 + \frac{1}{A_1} \frac{\partial u_2}{\partial \alpha_1} - 2k_{12} u_3; \\
\chi_1 = \frac{1}{A_1} \frac{\partial y_1}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} y_2 - k_{12} \omega_3; \quad \chi_2 = \frac{1}{A_2} \frac{\partial y_2}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} y_1 + k_{12} \omega_3; \\
\chi_3 = \tau_2 + k_2 \omega_1 + k_{12} \varepsilon_2 = \tau_1 + k_1 \omega_2 + k_{12} \varepsilon_1,
\]

where

\[
\gamma_1 = -\frac{1}{A_1} \frac{\partial u_3}{\partial \alpha_1} + k_1 u_1 - k_{12} u_2; \quad \gamma_2 = -\frac{1}{A_2} \frac{\partial u_3}{\partial \alpha_2} + k_2 u_2 - k_{12} u_1; \\
\omega_1 = \frac{1}{A_1} \frac{\partial u_2}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_1 - k_{12} u_3; \quad \omega_2 = \frac{1}{A_2} \frac{\partial u_1}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u_2 - k_{12} u_3; \\
\omega_3 = \frac{1}{2A_1 A_2} \left( \frac{\partial A_2 u_2}{\partial \alpha_1} - \frac{\partial A_1 u_1}{\partial \alpha_2} \right); \\
\tau_1 = \frac{1}{A_1} \frac{\partial y_2}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} y_1; \quad \tau_2 = \frac{1}{A_2} \frac{\partial y_1}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} y_2,
\]

where \(\gamma_1, \gamma_2\) are the angles made by unit vectors \(\hat{e}_i\) at the direction to the normal \(\vec{m}(\hat{e}_i)\) in the plane \((\hat{e}_i, \vec{m})\).

Unlike the analysis of surfaces in principal curvatures, it is required to calculate the coefficients, which contain \(k_{12}\).

The relations between vectors of internal forces and vectors of tangential and bending strains of a mid-surface are determined according to the Hook’s law:

\[
\vec{T} = c[N] \vec{\varepsilon}; \quad \vec{M} = d[N] \vec{\chi}; \quad [N] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{bmatrix}; \quad c = \frac{Eh}{1 - \nu^2}; \quad d = \frac{Eh^3}{12(1 - \nu^2)},
\]

where \(\vec{T}\) is a vector of tangential internal forces; \(\vec{M}\) is a vector of bending internal forces; \([N]\) is a elasticity matrix; \(c\) is a tangential rigidity of the shell; \(d\) is a flexural rigidity of the shell; \(E\) is a modulus of elasticity of a material of the shell; \(\nu\) is a Poisson’s ratio; \(h\) is a thickness of the shell.

Tangential and bending potential energies can be expressed in terms of an integral taken over a mid-surface of the shell \(\Omega\):

\[
U_T = \frac{1}{2} \iint_{\Omega} \vec{T}^* \vec{\varepsilon} \, d\Omega = \iint_{\Omega} \left( N_1 \varepsilon_1 + N_2 \varepsilon_2 + 2S \varepsilon_3 \right) \, d\Omega; \\
U_u = \frac{1}{2} \iint_{\Omega} \vec{M}^* \vec{\chi} \, d\Omega = \iint_{\Omega} \left( M_1 \chi_1 + M_2 \chi_2 + 2H \chi_3 \right) \, d\Omega; \quad d\Omega = A_1 A_2 d\alpha_1 d\alpha_2.
\]

When the Hook’s law is taken into account, the formulas of potential energy are written as:

\[
U_T = \frac{1}{2} \iint_{\Omega} \vec{T}^* \vec{\varepsilon} \, d\Omega = \frac{c}{2} \iint_{\Omega} \vec{\varepsilon}^* [N] \vec{\varepsilon} \, d\Omega; \quad U_u = \frac{1}{2} \iint_{\Omega} \vec{M}^* \vec{\chi} \, d\Omega = \frac{d}{2} \iint_{\Omega} \vec{\chi}^* [N] \vec{\chi} \, d\Omega.
\]

The index * means that it is a transpose of the matrix.
where $T_1$, $T_2$ are normal tangential forces, $T_3 = S$ is a shearing tangential force; $M_1$ and $M_2$ are bending moments, $M_3 = M_{12}$ is a twisting moment; $\varepsilon_1$ and $\varepsilon_2$ are tangential deformations; $\varepsilon_3 = \gamma_{12}$ is a bending deformation; $\chi_1$ and $\chi_2$ are changes of principle curvature of a shell, $\chi_3 = 2\chi_{12}$ is a twist of a shell.

The work of external forces can be expressed as:

$$A = \iint_D \overline{\mathbf{q}} \cdot \overline{\mathbf{u}} d\Omega + \sum p_i \overline{\mathbf{p}}_i$$

where $\overline{\mathbf{q}}$ is a vector of loadings and $\sum \overline{\mathbf{p}}_i$ is a vector of projections of single loads onto the coordinate system of the surface and onto the normal to the surface, $\overline{\mathbf{p}}$ is a vector of projections of single loads; $\overline{\mathbf{u}}$ is a vector of tangential and normal displacements of the mid-surface of the shell; $\overline{u}_i$ is a vector of these displacements in the point where the single load $\overline{p}_i$ is applied.

The expression of tangential and bending deformation in matrix form is:

$$[\varepsilon] = \begin{bmatrix}
\varepsilon_1^1(u_1) & \varepsilon_1^2(u_2) & \varepsilon_1^3(u_3) \\
\varepsilon_2^1(u_1) & \varepsilon_2^2(u_2) & \varepsilon_2^3(u_3) \\
\varepsilon_3^1(u_1) & \varepsilon_3^2(u_2) & \varepsilon_3^3(u_3)
\end{bmatrix};
[\chi] = \begin{bmatrix}
\chi_1^1(u_1) & \chi_1^2(u_2) & \chi_1^3(u_3) \\
\chi_2^1(u_1) & \chi_2^2(u_2) & \chi_2^3(u_3) \\
\chi_3^1(u_1) & \chi_3^2(u_2) & \chi_3^3(u_3)
\end{bmatrix}.
$$

In $\varepsilon^k$ and $\chi^k$ the index $k$ means the number of the column and the index $\ell$ the number of a distortion, the index $\bar{I}$ means the number of the row and a number of a deformation $\varepsilon_{\bar{I}}$ and $\chi_{\bar{I}}$.

The formulas for the potential energy will be:

$$U_T = \frac{c}{2} \iint_D \varepsilon^1 [N][\varepsilon] d\Omega = \frac{c}{2} \iint_D \bar{\varepsilon}^1 [N][\varepsilon] d\Omega; U_u = \frac{d}{2} \iint_D \varepsilon^2 [N][\varepsilon] d\Omega = \iint_D \bar{\varepsilon}^2 [N][\varepsilon] d\Omega.$$

The components of the deformation matrix are expressed as:

$$\varepsilon_1^1 = \frac{1}{A_1} \frac{\partial u_1}{\partial \alpha_1}; \quad \varepsilon_1^2 = \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_2; \quad \varepsilon_1^3 = k_1 u_3; \quad \varepsilon_2^1 = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u_1; \quad \varepsilon_2^2 = \frac{1}{A_2} \frac{\partial u_2}{\partial \alpha_2}; \quad \varepsilon_2^3 = k_2 u_3;$$

$$\varepsilon_3^1 = -\frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_1 + \frac{1}{A_2} \frac{\partial u_2}{\partial \alpha_2}; \quad \varepsilon_3^2 = -\frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u_2 + \frac{1}{A_1} \frac{\partial u_2}{\partial \alpha_1}; \quad \varepsilon_3^3 = -2k_{12} u_3;$$

$$\chi_1^1 = \frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{k_{12}}{2 A_2} \frac{\partial A_1}{\partial \alpha_2} \right) u_1 + \frac{k_1}{A_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_2} \frac{k_{12}}{2 A_1} \frac{\partial u_1}{\partial \alpha_1};$$

$$\chi_1^2 = \left( -\frac{1}{A_1} \frac{k_{12}}{A_2} + \frac{k_2}{A_1} \frac{\partial A_1}{\partial \alpha_2} - \frac{k_{12}}{2 A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \right) u_2 - \frac{3}{2} A_1 \frac{k_{12}}{A_1} \frac{\partial u_2}{\partial \alpha_1};$$

$$\chi_1^3 = \frac{1}{A_1} \frac{\partial A_1}{\partial \alpha_3} u_3 - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u_2 - \frac{1}{A_1} \frac{\partial^2 u_3}{\alpha_1^2};$$

$$\chi_2^1 = \left( -\frac{1}{A_2} \frac{k_{12}}{A_2} + \frac{k_1}{A_2} \frac{\partial A_2}{\partial \alpha_1} - \frac{k_{12}}{2 A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \right) u_1 + \frac{3}{2} \frac{k_{12}}{A_2} \frac{\partial u_1}{\partial \alpha_1};$$

$$\chi_2^2 = \frac{1}{A_2} \frac{\partial^2 u_2}{\partial \alpha_2^2} + \frac{k_1}{A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{k_{12}}{2 A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \frac{\partial u_2}{\partial \alpha_1}$$

$$\chi_2^3 = \frac{1}{A_2} \frac{\partial A_2}{\partial \alpha_3} u_3 + \frac{1}{A_2} \frac{\partial A_2}{\partial \alpha_2} u_2 + \frac{1}{A_2} \frac{\partial^2 u_3}{\partial \alpha_2^2};$$
\[
\chi_2^3 = -\frac{1}{A_2^2} \frac{\partial A_2}{\partial \alpha_1} \frac{\partial u_3}{\partial \alpha_1} + \frac{1}{A_2^2} \frac{\partial A_2}{\partial \alpha_2} \frac{\partial u_3}{\partial \alpha_2} - \frac{1}{A_2^2} \frac{\partial^2 u_3}{\partial \alpha_1^2}; \\
\chi_3^3 = \left( \frac{1}{A_2} \frac{\partial k_1}{\partial \alpha_2} + \frac{2k_{12}}{A_1A_2} \frac{\partial A_2}{\partial \alpha_1} \right) u_1 + \frac{k_1}{A_2} \frac{\partial u_1}{\partial \alpha_2}; \\
\chi_2^3 = \left( -\frac{1}{A_2} \frac{\partial k_{12}}{\partial \alpha_2} - \frac{k_2}{A_1A_2} \frac{\partial A_2}{\partial \alpha_1} \right) u_2 + \frac{k_2}{A_1} \frac{\partial u_2}{\partial \alpha_1}; \\
\chi_3^3 = \frac{1}{A_2^2} \frac{\partial A_3}{\partial \alpha_1} + \frac{1}{A_2^2} \frac{\partial A_3}{\partial \alpha_2} - \frac{1}{A_2^2} \frac{\partial^2 u_3}{\partial \alpha_1^2}. 
\]

The obtained components of deformations can be written as:

\[
\varepsilon_k^e (u_k) = \sum_{k=0}^n a^e_{k,m} m^m u_k; \quad \text{and} \quad \chi_k^e (u_k) = \sum_{k=0}^n b^e_{k,m} m^m u_k,
\]

where

\[
\partial^0 u_k = u_k; \quad \partial^1 u_k = \frac{\partial u_k}{\partial \alpha_1}; \quad \partial^2 u_k = \frac{\partial^2 u_k}{\partial \alpha_2}; \quad \partial^3 u_k = \frac{\partial^3 u_k}{\partial \alpha_1^2}; \quad \partial^4 u_k = \frac{\partial^4 u_k}{\partial \alpha_1 \partial \alpha_2}; \quad \partial^5 u_k = \frac{\partial^5 u_k}{\partial \alpha_1^2},
\]

where \( a^e_{k,m}, b^e_{k,m} \) are coefficients of derivatives of the components of deformations.

Coefficients of tangential deformations for a displacement \( u_1 \):

for \( \varepsilon_1^1 \): \( a_{1,0}^1 = 0; a_{1,1}^1 = \frac{1}{A_1}; a_{1,m}^1 = 0 \) if \( m > 1 \); for \( \varepsilon_1^2 \): \( a_{2,0}^1 = \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1}; a_{2,m}^1 = 0 \) if \( m > 0 \);

for \( \varepsilon_2^1 \): \( a_{3,0}^1 = 0; a_{3,1}^1 = \frac{1}{A_2}; a_{3,m}^1 = 0 \) if \( m > 1 \).

Coefficients of tangential deformations for a displacement \( u_2 \):

for \( \varepsilon_1^2 \): \( a_{1,0}^2 = \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2}; a_{1,m}^2 = 0 \) if \( m > 0 \);

for \( \varepsilon_2^2 \): \( a_{2,0}^2 = 0; a_{2,1}^2 = \frac{1}{A_2}; a_{2,m}^2 = 0 \) if \( m > 2 \);

for \( \varepsilon_3^2 \): \( a_{3,0}^2 = 0; a_{3,1}^2 = \frac{1}{A_1}; a_{3,m}^2 = 0 \) if \( m > 1 \).

Coefficients of tangential deformations for a displacement \( u_3 \):

\( a_{1,0}^3 = k_1; a_{2,0}^3 = k_2; a_{3,0}^3 = -2k_{12}; a_{1,m}^3 = a_{2,m}^3 = a_{3,m}^3 = 0 \) if \( m > 0 \).

Coefficients of bending deformations for a displacement \( u_1 \):

for \( \chi_1^1 \): \( b_{1,0}^1 = \frac{1}{A_1} \left( \frac{\partial k_1}{\partial \alpha_1} - k_{12} \frac{\partial A_1}{\partial \alpha_1} \right) \); \( b_{1,1}^1 = k_1; b_{1,2}^1 = \frac{1}{2} k_{12} \frac{\partial A_1}{\partial \alpha_1} \); \( b_{1,m}^1 = 0 \) if \( m > 2 \);

for \( \chi_1^2 \): \( b_{2,0}^1 = -\frac{1}{A_2} \frac{\partial k_{12}}{\partial \alpha_2} + \left( \frac{k_1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{k_{12}}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_2} \right) b_{2,1}^1 = 0; b_{2,2}^1 = \frac{-3}{2} k_{12} \frac{\partial A_1}{\partial \alpha_1} \); \( b_{2,m}^1 = 0 \) if \( m > 2 \);

for \( \chi_2^3 \): \( b_{3,0}^1 = \frac{-1}{2} \frac{\partial k_{12}}{\partial \alpha_1} + \frac{k_{12}}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} b_{3,1}^1 = 0; b_{3,2}^1 = \frac{k_1}{A_2} \); \( b_{3,m}^1 = 0 \) if \( m > 2 \).

Coefficients of bending deformations for a displacement \( u_2 \):

for \( \chi_2^1 \): \( b_{1,0}^2 = -\frac{1}{A_1} \frac{\partial k_{12}}{\partial \alpha_1} + \left( \frac{k_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} - \frac{k_{12}}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_1} \right) \); \( b_{2,1}^2 = \frac{k_{12}}{2 A_1} \); \( b_{2,2}^2 = \frac{k_2}{A_2} \); \( b_{2,m}^2 = 0 \) if \( m > 2 \);

for \( \chi_2^2 \): \( b_{3,0}^2 = -\frac{1}{A_2} \frac{\partial k_{12}}{\partial \alpha_2} - \frac{k_{12}}{2 A_1 A_2} \); \( b_{3,1}^2 = \frac{k_{12}}{A_1} \); \( b_{3,2}^2 = \frac{k_2}{A_2} \); \( b_{3,m}^2 = 0 \) if \( m > 0 \).

for \( \chi_3^2 \): \( b_{3,0}^3 = -\frac{1}{A_2} \frac{\partial k_{12}}{\partial \alpha_2} - \frac{k_{12}}{2 A_1 A_2} \); \( b_{3,1}^3 = \frac{k_{12}}{A_1} \); \( b_{3,2}^3 = \frac{k_2}{A_2} \); \( b_{3,m}^3 = 0 \) if \( m > 1 \).
Coefficients of bending deformations for a displacement $u_3$:

for $\chi^3$: $b_{3,0}^3 = 0; b_{3,1}^3 = \frac{1}{A_1^2} \frac{\partial A_1}{\partial a_1}; b_{3,2}^3 = -\frac{1}{A_1 A_2^2} \frac{\partial A_2}{\partial a_2}; b_{3,3}^3 = -\frac{1}{A_1^3}; b_{3,m}^3 = 0$ if $m > 3$;

for $\chi^2$: $b_{2,0}^3 = 0; b_{2,1}^3 = \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_1} + \frac{1}{A_2} \frac{\partial A_2}{\partial a_2}; b_{2,2}^3 = \frac{1}{A_2^2} \frac{\partial A_2}{\partial a_2}; b_{2,3}^3 = b_{2,4}^3 = 0; b_{2,5}^3 = -\frac{1}{A_2}$;

for $\chi^1$: $b_{1,0}^3 = 0; b_{1,1}^3 = \frac{1}{A_1 A_2^2} \frac{\partial A_1}{\partial a_1}; b_{1,2}^3 = \frac{1}{A_1 A_2^2} \frac{\partial A_2}{\partial a_2}; b_{1,3}^3 = 0; b_{1,4}^3 = -\frac{1}{A_1 A_2}; b_{1,5}^3 = 0$.

The matrices of these coefficients will be:

$$\bar{a}^1 = \begin{bmatrix} a_{1,0}^k & a_{1,1}^k & a_{1,2}^k & a_{1,3}^k & a_{1,4}^k & a_{1,5}^k \\ a_{2,0}^k & a_{2,1}^k & a_{2,2}^k & a_{2,3}^k & a_{2,4}^k & a_{2,5}^k \\ a_{3,0}^k & a_{3,1}^k & a_{3,2}^k & a_{3,3}^k & a_{3,4}^k & a_{3,5}^k \end{bmatrix}$$

$$\bar{a}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{a}^3 = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 & 0 & 0 \\ -2k_{12} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{b}^1 = \begin{bmatrix} \frac{1}{A_1} \frac{\partial A_1}{\partial a_1} + \frac{k_{12}^2}{2A_2^2} \frac{\partial A_2}{\partial a_2} \\ -\frac{3}{2} k_{12} \frac{\partial A_2}{\partial a_2} - \frac{k_{12}^2}{2A_2^2} \frac{\partial A_1}{\partial a_1} \\ \frac{1}{2} k_2 \frac{\partial A_2}{\partial a_2} + \frac{k_{12}^2}{2A_2^2} \frac{\partial A_2}{\partial a_2} + \frac{k_{12}^2}{2A_2^2} \frac{\partial A_1}{\partial a_1} \end{bmatrix}$$

$$\bar{b}^2 = \begin{bmatrix} 0 & \frac{1}{A_1} \frac{\partial A_1}{\partial a_1} - \frac{k_{12}^2}{2A_2^2} \frac{\partial A_2}{\partial a_2} \\ 0 & \frac{1}{A_1} \frac{\partial A_1}{\partial a_1} - \frac{k_{12}^2}{2A_2^2} \frac{\partial A_2}{\partial a_2} \\ -2k_{12} \frac{\partial A_2}{\partial a_2} - \frac{k_{12}^2}{2A_2^2} \frac{\partial A_2}{\partial a_2} \end{bmatrix}$$

$$\bar{b}^3 = \begin{bmatrix} 0 & -\frac{1}{A_1} \frac{\partial A_1}{\partial a_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{A_1} \frac{\partial A_1}{\partial a_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{A_1} \frac{\partial A_1}{\partial a_1} & 0 & 0 & 0 \end{bmatrix}$$

Let us use the vector of derivatives:

$$\tilde{\partial}^* = \{\partial^0, \partial^1, \partial^2, \partial^3, \partial^4, \partial^5\} = \{1, \frac{\partial}{\partial a_1}, \frac{\partial^2}{\partial a_1}, \frac{\partial^3}{\partial a_1}, \frac{\partial^4}{\partial a_1}, \frac{\partial^5}{\partial a_1}\}.$$

Then the matrix formulation of the vector of components of deformations can be written as:

$$\tilde{\varepsilon}(u_k) = \tilde{a}^k \tilde{\partial} u_k; \tilde{\chi}(u_k) = \tilde{b}^k \tilde{\partial} u_k; \tilde{\varepsilon} = \sum_{i=1}^3 \tilde{a}^k \tilde{\partial} u_k; \tilde{\chi} = \sum_{i=1}^3 \tilde{b}^k \tilde{\partial} u_k.$$

Using these formulas we will have the expressions of the potential energy of deformation as:

$$U_T = \frac{c}{2} \int_\Omega \sum_{k=1}^3 \sum_{i=1}^3 (\tilde{a}^k \tilde{\partial} u_k)^* \{N\}(\tilde{\varepsilon} \tilde{\partial} u_k) \, d\Omega;$$

$$U_u = \frac{d}{2} \int_\Omega \sum_{k=1}^3 \sum_{i=1}^3 (\tilde{b}^k \tilde{\partial} u_k)^* \{N\}(\tilde{\chi} \tilde{\partial} u_k) \, d\Omega.$$
Calculating with the variation-difference method includes replacing derivatives in the functional of potential energy with difference equations, and the shell is covered by the mesh of constant or variable step (Figure 1). Variable step is ordinary used, if there are domains where the function is changing fast.

Figure 1. Coordinate mesh at region of nod $ij$.

Let us consider a coordinate mesh with the axes $\alpha_1, \alpha_2$. Let the domain of the mesh, which surround the node $ij$ located at the intersection of axes $i$ and $j$ be divided in four quadrants (Fig. 2). There will be four types of difference equations (left and right) for these quadrants. In this case the solution will be more precise. For the first quadrant a step forward in both coordinates is taken $(++)$, for the second quadrant it is a step back in $\alpha_1$ and a step forward in $\alpha_2$ ($-+$), for the third it is a step back in both $\alpha_1$ and $\alpha_2$ ($- -$) and for the fourth it is a step forward in $\alpha_1$ and a step back in $\alpha_2$ ($+ -$).

Figure 2. Nod points at the region of nod $ij$ and subregions of integration

$$d\Omega_{ij}^1 = \frac{h_{i+1}h_{j+1}}{4};$$
$$d\tilde{\Omega}_{ij}^2 = \frac{h_{i}h_{j+1}}{4}, d\tilde{\Omega}_{ij}^3 = \frac{h_{i+1}h_{j}}{4}$$

Using this division first derivatives are written as:

$$\frac{\partial u_k^{ij}}{\partial \alpha_1} = \frac{u_k^{i+1,j} - u_k^{i,j}}{h_{i+1}}; \frac{\partial u_k^{ij}}{\partial \alpha_2} = \frac{u_k^{i,j+1} - u_k^{i,j}}{h_{j+1}}; \frac{\partial u_k^{i,j}}{\partial \alpha_1} = \frac{u_k^{i,j+1} - u_k^{i,j}}{h_{j+1}}; \frac{\partial u_k^{i,j}}{\partial \alpha_2} = \frac{u_k^{i+1,j} - u_k^{i,j}}{h_{i+1}}.$$
Second-order mixed partial derivatives are written as:

\[
\frac{\partial^2 u_k}{\partial \alpha_1 \partial \alpha_2} \bigg|_{ij}^{+} = \frac{u_k^{i+1,j+1} - u_k^{i-1,j+1} - u_k^{i+1,j-1} + u_k^{i-1,j-1}}{h_{1i+1} h_{2j+1}},
\]

\[
\frac{\partial^2 u_k}{\partial \alpha_1 \partial \alpha_2} \bigg|_{ij}^{-} = \frac{u_k^{i+1,j-1} - u_k^{i-1,j-1} - u_k^{i+1,j+1} + u_k^{i-1,j+1}}{h_{1i+1} h_{2j+1}}.
\]

Second-order non-mixed partial derivatives based on central difference equations are written as:

\[
\frac{\partial^2 u_k}{\partial \alpha_i^2} \bigg|_{ij}^{\pm} = \frac{2(\beta_1 \delta_k^{-} - (1+\beta_1)^{-1}\delta_k^{i+j} + \delta_k^{i-j})}{\beta_1^2 (1+\beta_1)^2 h_i^2}.
\]

where \( \beta_1 = \frac{h_{1i+1}}{h_i}; \beta_2 = \frac{h_{2j+1}}{h_j}; u_k^{ij} \) is a displacement of a nod \( ij \) with a number \( k \) of the displacement; \( h_{1i}, h_{2j}, h_{1i+1}, h_{2j+1} \) are steps of the mesh at the left side and at the right side of the nod in \( \alpha_1, \alpha_2 \), and analogous steps in \( \alpha_2 \).

According to the formulas of derivatives, nine displacements of the nod determine difference derivatives near to this nod. The vector of the displacement in nod is:

\[
\{\delta_k^{ij}\} = \{u_k^{i-1,j-1}, u_k^{i-1,j+1}, u_k^{i-1,j}, u_k^{i-1,j-1}, u_k^{i,j}, u_k^{i,j+1}, u_k^{i,j-1}, u_k^{i+1,j}, u_k^{i+1,j+1}\},
\]

where \( k = 1, 2, 3 \).

After that four types of difference matrix \( \delta_k^{ij} \) and the matrix form of the vector of derivatives \( \delta u_k = d_t^i \delta_k^{ij} \) can be written, where \( t \) means the number of quadrant near to the nod \( i, j \), \( t = 1, 2, 3, 4 \).

The functional of potential energy of deformation can be written in the difference form as:

\[
U_{ij} = \frac{c}{2} \int_{\Omega} \left( \sum_{k=1}^{3} \sum_{t=1}^{3} \left( \tilde{a}_{ij}^{k} \delta_k^{ij} \right)^* [N] \left( \tilde{a}_{ij}^{k} \delta_k^{ij} \right) \right) d\Omega; \quad U_{ij} = \frac{d}{2} \int_{\Omega} \left( \sum_{k=1}^{4} \sum_{t=1}^{4} \left( \tilde{b}_{ij}^{k} \delta_k^{ij} \right)^* [N] \left( \tilde{b}_{ij}^{k} \delta_k^{ij} \right) \right) d\Omega;
\]

where \( \tilde{r}_k^{ij} \) is a matrix of tangential rigidity, \( \tilde{t}_k^{ij} \) is a matrix of flexural rigidity:

\[
\tilde{r}_k^{ij} = \frac{c}{2} \int_{\Omega} \left( \sum_{k=1}^{3} \left( \tilde{a}_{ij}^{k} d_k^{ij} \right)^* [N] \left( \tilde{a}_{ij}^{k} d_k^{ij} \right) \right) d\Omega; \quad \tilde{t}_k^{ij} = \frac{d}{2} \int_{\Omega} \left( \sum_{k=1}^{4} \left( \tilde{b}_{ij}^{k} d_k^{ij} \right)^* [N] \left( \tilde{b}_{ij}^{k} d_k^{ij} \right) \right) d\Omega;
\]

\( \tilde{a}_{ij}^{k}, \tilde{b}_{ij}^{k} \) are matrices for coefficients near this nod \( i, j \); \( \Omega_{ij}^{k} \) are subdomains near it.

The functional of potential energy is defined by summing nodal integrals. The condition of the minimum of the total potential energy of deformation is \( \frac{\partial^3}{\partial z_k^3} = \frac{\partial}{\partial z_k} - \frac{\partial}{\partial z_k} = 0 \), that is \( \frac{\partial^3}{\partial z_k^3} = \frac{\partial}{\partial z_k} \).

Tangential and bending deformations are calculated using difference derivatives, and internal forces are calculated with the formulas of Hook’s law. To obtain shear forces the formulas below are used:

\[
Q_1 = \frac{1}{A_1 A_2} \left[ \frac{\partial}{\partial \alpha_2} (A_2 M_2) - \frac{\partial A_1}{\partial \alpha_2} M_1 + \frac{1}{A_2} \frac{\partial}{\partial \alpha_1} (A_2^2 H) \right],
\]

\[
Q_2 = \frac{1}{A_1 A_2} \left[ \frac{\partial}{\partial \alpha_1} (A_2 M_2) - \frac{\partial A_2}{\partial \alpha_1} M_2 + \frac{1}{A_1} \frac{\partial}{\partial \alpha_2} (A_1^2 H) \right].
\]

Shell analysis in the orthogonal coordinate system is convenient because of the more simple form of the equations than in general form and it gives the opportunity to calculate shells which is impossible to analyze in the coordinate system in principal curvatures. For example, in this coordinate system it is possible to analyze normal cyclic shells, for which in general case generatrix circles are not the lines of principal curvatures. It is true only for tubular surfaces and surfaces of rotation.
The difference between the analysis of surfaces in principal curvatures and their analysis in orthogonal coordinate system is in the geometric equations of deformations and curvatures. In the case of the orthogonal coordinate system the deformation matrix include the components containing \( k_{12} \). With the addition of these components in the analysis it is possible to enlarge the field of application of the software which is developed for analysis of shells in principal curvatures. For the software it is convenient to use the variation-difference method.

Conclusions

Formulas for analysis in the orthogonal curvilinear coordinate system may be obtained through adding to formulas of strains and curvatures in the coordinate system in principal curvatures components which include \( k_{12} \). This allows analysis of structures for which it is inconvenient to use the coordinate system in principal curvatures. Changing of formulas of strains in the program for analysis of shells in the coordinate system in principal curvatures and consideration of the geometry of shells allow analyzing normal cyclic shells in the orthogonal coordinate system.

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