Analysis of the loop length distribution for the negative weight percolation problem in dimensions \( d = 2 \) through \( 6 \)

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We consider the negative weight percolation (NWP) problem on hypercubic lattice graphs with fully periodic boundary conditions in all relevant dimensions from \( d = 2 \) to the upper critical dimension \( d = 6 \). The problem exhibits edge weights drawn from disorder distributions that allow for weights of either sign. We are interested in the full ensemble of loops with negative weight, i.e. non-trivial (system spanning) loops as well as topologically trivial (“small”) loops. The NWP phenomenon refers to the disorder driven proliferation of system spanning loops of total negative weight. While previous studies where focused on the latter loops, we here put under scrutiny the ensemble of small loops. Our aim is to characterize -using this extensive and exhaustive numerical study- the loop length distribution of the small loops right at and below the critical point of the hypercubic setups by means of two independent critical exponents. These can further be related to the results of previous finite-size scaling analyses carried out for the system spanning loops. For the numerical simulations we employed a mapping of the NWP model to a combinatorial optimization problem that can be solved exactly by using sophisticated matching algorithms. This allowed us to study here numerically exact very large systems with high statistics.

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I. INTRODUCTION

The statistical properties of lattice-path models on graphs, equipped with quenched disorder, have experienced much attention during the last decades. They have proven to be useful in order to characterize, e.g., linear polymers in disordered/random media \([1,2]\), vortex loops in high-\(T_c\) superconductivity at zero field \([3,4]\) and the \( d = 3 \) XY model \([10,11]\), networks of vortex strings found after a symmetry-breaking phase transition in field theories \([12,13]\), as well as domain wall excitations in disordered media such as spin glasses \([13,16]\) and the solid-on-solid model \([17]\). The precise computation of these paths can often be formulated in terms of a combinatorial optimization problem and hence might allow for the application of exact optimization algorithms \([18]\) developed in computer science.

For an analysis of the statistical properties of these lattice path models, geometric observables and scaling concepts similar to those developed in percolation theory \([19,21]\) have been used conveniently. In the past decades, a large number of percolation problems in various contexts have been investigated through numerical simulations. Among those are problems, where the fundamental entities are string-like, similar to the lattice-path models mentioned in the beginning, rather than clusters consisting of occupied nearest neighbor sites as in the case of usual random bond percolation.

In a sequence of recent articles we have introduced \([22]\) and investigated (see below) the negative-weight percolation (NWP), a problem with subtle differences as compared to other string-like percolation problems. In the most basic NWP setup, one considers a regular lattice graph with periodic boundary conditions (BCs), where adjacent sites are joined by undirected edges. Weights are assigned to the edges, representing quenched random variables drawn from a distribution that allows for edge weights of either sign. The properties of the weight distribution are further controlled by a tunable disorder parameter, signified by \( \rho \). For a given realization of the disorder, one then computes a configuration of loops, i.e. closed paths on the lattice graph, such that the sum of the edge weights that build up the loops is minimal and negative. As an additional optimization constraint we impose the condition that the loops are not allowed to intersect; consequently there is no definition of clusters in the NWP model. Regardless of the spacial dimension of the underlying (hypercubic) lattice graph, the observables are always line-like, i.e. have an intrinsic dimension of \( d = 1 \). Nevertheless, the loops may be fractal with fractal dimensions \( d_f > 1 \), see Ref. \([23]\).

The problem of finding these loops numerically can be cast into a minimum-weight path (MWP) problem, outlined in sect. \([11]\) in more detail. A pivotal observation is that, as a function of the disorder parameter \( \rho \), the NWP model features a disorder driven, geometric phase transition, \([22,24]\) triggered by a vital change of the typical loop size (as discussed below in more detail). In this regard, depending on the precise lattice setup and on the value of \( \rho \), one can identify two different phases: (i) a phase where the loops are “small”, meaning that the lin-
FIG. 1: Samples of minimum weight configurations of loops for a 2D square lattice with side length $L = 64$ and fully periodic boundary conditions. The snapshots relate to different values of the disorder parameter $\rho$, where (a) $\rho \approx \rho_c$, (b) $\rho' < \rho_c$, and, (c) $\rho'' < \rho'$. In the limit of large system sizes and above the critical point $\rho_c$, loops might span the lattice along at least one direction as, e.g., the gray loop in (a). For small values of $\rho$, loops with a comparatively large length appear to be suppressed exponentially.

ear extensions of the loops are small in comparison to the system size, see Figs. 1(b-c) (therein, the linear extension of a loop refers to its projection onto the independent lattice axes), and, (ii) a phase where “large” loops exist that span the entire lattice, see Fig. 1(a). Regarding these two phases and in the limit of large system sizes, there is a particular value of the disorder parameter, signified as $\rho_c$, at which system spanning (or “percolating”) loops appear for the first time.

Previously, we have investigated the NWP phenomenon for 2D lattice graphs [22] using finite-size scaling (FSS) analyses, where we characterized the underlying transition by means of a set of critical exponents. Considering different disorder distributions and lattice geometries, the exponents where found to be universal in 2D and clearly distinct from those describing other percolation phenomena. In a subsequent study we investigated the effect of dilution on the critical properties of the 2D NWP phenomenon [24]. Therefore we performed FSS analyses to probe critical points along the critical line in the disorder-dilution plane that separates domains that exhibit or do not exhibit system spanning loops. One conclusion of that study was that bond dilution changes the universality class of the NWP problem. Further we found that, for bond-diluted lattices prepared at the percolation threshold of 2D random percolation and at full disorder, the geometric properties of the system spanning loops compare well to those of ordinary self-avoiding walks. We performed further simulations for the NWP model on hypercubic lattice graphs in dimensions $d = 2$ through 7 [23], where we found evidence for an upper critical dimension $d_u = 6$ of the NWP phenomenon. This result was based on monitoring the critical exponents related to the NWP transition (one expects them to stay fixed for $d \geq d_u$). We also studied numerically as well as analytically a variant of the NWP transition on 3-regular random graphs (RRGs), i.e. graphs where each node has exactly 3 neighbors and where there is no regular lattice structure. Hence, we obtained direct access to the mean-field exponents that govern the model for $d \geq d_u$. We obtained excellent agreement between numerical and analytic results and could provide further support for the claim $d_u = 6$.

All of the studies mentioned above where focused on the statistical properties of the largest loop (or more precisely, the longest loop) for a given realization of the disorder and the critical properties of the NWP model that derive from an analysis of these loops in the vicinity of the critical point $\rho_c$. Up to now, limited attention was payed to the ensemble of “small” loops that actually comprise the major part of loop segments in the vicinity of $\rho_c$. As we found earlier, at this critical point the loops are rather isolated and well separated from each other, resembling a dilute gas of loops (cf. Fig. 1). Further, the normalized and ensemble-averaged probability mass function (pmf) $n_{\ell}$ of loops having length $\ell$ right at $\rho_c$ was studied in Refs. [22] [23]. It exhibits an algebraic decay similar to the distribution of cluster sizes at the critical point in ordinary random percolation [13] [20], i.e.

$$n_{\ell}(\rho_c) \propto \ell^{-\tau} \quad \text{at} \quad \rho = \rho_c. \tag{1}$$

The numerical values of the decay exponent $\tau$ (also termed “Fisher exponent”) found for the NWP model in $d = 2$ through 7 are listed in Tab. I. Note that the Fisher exponent is only one out of two independent exponents that characterize the whole ensemble of loops.

In the present article, the second critical exponent that characterizes the ensemble of small loops is addressed. In this regard, the present article discusses the pmf $n_{\ell}(\rho)$ as function of the disorder parameter $\rho$. Consequently, the numerical effort to obtain these distributions, in several dimensions $d = 2, \ldots, 6$ was much larger, compared to the previous studies where the distribution was obtained just for $\rho \approx \rho_c$. One of our main results is that for values $\rho < \rho_c$, the pmf appears to scale similar to the distribution of cluster sizes in usual percolation [13] [20], i.e.

$$n_{\ell}(\rho) \propto \ell^{-(3 - \gamma_{\ell})/\sigma} \quad \text{for} \quad \rho < \rho_c. \tag{2}$$

| $d$ | $L$ | $\rho_c$ | $\nu_p d_{f,p}$ | $\gamma_p$ | $\tau$ | $\sigma$ | $(3 - \gamma)/\sigma$ |
|-----|-----|--------|----------------|--------|------|-------|------------------|
| 2   | 512 | 0.340(1) | 0.53(3) | 0.77(7) | 2.59(3) | 0.53(3) | 0.77(10) |
| 3   | 64  | 0.1273(3) | 0.69(2) | -0.09(3) | 3.07(1) | 0.71(1) | -0.10(1) |
| 4   | 21  | 0.0640(2) | 0.78(3) | -0.66(5) | 3.55(2) | 0.78(2) | -0.71(1) |
| 5   | 12  | 0.0385(2) | 0.86(4) | -1.06(7) | 3.86(3) | 0.88(2) | -0.98(1) |
| 6   | 6   | 0.0265(2) | 1.00(3) | -0.99(3) | 4.00(2) | 0.97(4) | -1.03(2) |
Therein, the exponential factor accounts for the observation that below the critical point \( \rho_c \), the proliferation of “long” (still non-spanning) loops is suppressed due to some finite “loop size cut-off parameter”. The latter might be captured by means of a scaling parameter \( T_L(\rho) \) which depends on the subtleties of the disorder. Its inverse \( \ell_0(\rho) = 1/T_L(\rho) \) relates to a typical length scale to which the perimeter of the loops is limited at a given value of \( \rho \) and it should not depend on the side length \( L \) of the system (at least in the limit of large system sizes where a loop of, say, length \( \ell_0 \) fits well into the simulation box). Therefore, loop configurations a\(^{13,20}\) small values of \( \rho \) are consistent with a spanning probability \( P_L(\rho < \rho_c) \to 0 \) in the limit of large system sizes \( (L \to \infty) \). As the critical point is approached from below the loop size cut-off parameter vanishes, giving rise to the purely algebraic decay of \( n_L \) observed at \( \rho_c \), as in Eq. \( (1) \), featuring loops with length \( \ell \) on virtually all length scales. (A qualitatively similar observation in the context of high-\( T_c \) superconductors is referred to as “Onsager vortex-loop unbinding” that signals the superconductor to normal metal transition \( \[6,7\] \). Further, in string theory, the analog observation is referred to as “Hagedorn transition” \( \[12,13\] \).)

The decrease of the parameter \( T_L(\rho) \) can be related to a second, independent exponent that, in addition to \( \tau \), serves to characterize the ensemble of small loops. The respective critical exponent \( \sigma \) is defined via

\[
T_L(\rho) \propto |\rho - \rho_c|^{1/\sigma},
\]

where \( \sigma \) might be referred to as “loop-size cut-off” exponent (i.e. the critical exponent related to the loop-size cut-off parameter \( T_L \)) and where \( \rho \) approaches \( \rho_c \) from below. Similarly, the corresponding lengthscale \( \ell_0 \), to which the loops are confined, diverges. This implies that loops might get arbitrarily long, limited only by the finite size of the underlying lattice. One might expect a maximal loop length of \( \ell_{\text{max}} \sim L^d \), where \( d_f \) denotes the fractal scaling dimension of the loops. Thus, at \( \rho_c \) and in the limit \( L \to \infty \), the distribution of the loop perimeter exhibits an algebraic decay, solely governed by the Fisher exponent \( \tau \). Finally, according to scaling theory \( \[12,20\] \), the scaling relations

\[
\nu_p d_f, p = 1/\sigma, \tag{4a}
\]
\[
\gamma_p = (3 - \tau)/\sigma \tag{4b}
\]

should hold, relating \( \sigma \) (as measured for the small loops) to \( \nu_p, d_f, p \) and \( \gamma_p \) (all measured from the system spanning loops; indicated by the subscript \( p \)). These three exponents signify the critical exponents that describe the divergence of the correlation length, the scaling dimension of the loops, and the fluctuations of the loop order-parameter, respectively.

The remainder of the present article is organized as follows. In section \( \[14\] \) we introduce the model in more detail and we outline the algorithm used compute the loop configurations. In section \( \[14\] \) we list the results of our numerical simulations and in section \( \[14\] \) we conclude with a summary. Note that an extensive summary of this paper is available at the papercore database \( \[25\] \).

## II. Model and Algorithm

In the present article we consider hypercubic lattice graphs \( G = (V, E) \) with side length \( L \) and fully periodic boundary conditions (BCs) for all relevant dimensions \( d = 2 \) through 6. The considered graphs have \( N = |V| = L^d \) sites \( i \in V \) and a number of \( |E| = zN/2 \) undirected edges \( \{i, j\} \in E \) that join adjacent sites \( i, j \in V \). Above, \( z \) signifies the coordination number of the lattice geometry, where \( z = 2d \). We further assign a weight \( \omega_{ij} \) to each \( \{i, j\} \in E \). These weights represent quenched random variables that introduce disorder to the lattice. Here we consider independent identically distributed weights which either have just weight one (probability \( 1-\rho \)) or are drawn (probability \( \rho \)) from a Gaussian distribution with zero mean and variance one. Hence, the disorder distribution is given by

\[
P(\omega) = \rho \exp(-\omega^2/2)/\sqrt{2\pi} + (1 - \rho)\delta(\omega - 1), \tag{5}\]

that explicitly allows for loops \( L \) with a negative total weight \( \omega_L = \sum_{\{i,j\} \in L} \omega_{ij} \). To support intuition: For any nonzero value of the disorder parameter \( \rho \), a sufficiently large lattice will exhibit at least “small” loops that exhibit a negative weight, see Fig. \( \[14\] (c). If the disorder parameter is large enough, system spanning loops with negative weight will exist, see Figs. \( \[14\] (a).

The NWP problem then reads as follows: Given a realization of the disorder for a hypercubic lattice graph \( G \), determine a set \( C \) of loops such that the configuration energy, defined as the sum of all the loop-weights \( E = \sum_{L \in C} \omega_L \), is minimized. As further optimization constraint, the loops are not allowed to intersect. Note that due to the “energy minimization principle” of the optimization procedure, the weight of an individual loop is necessarily smaller than zero. The configuration energy \( E \) is the quantity subject to optimization and the result of the optimization procedure is a set of loops \( C \), obtained using an appropriate transformation of the original graph \( \[20\] \). For the transformed graphs, minimum-weight perfect matchings (MWPMs) \( \[27,29\] \) are calculated, that serve to identify the loops for a given realization of the disorder. Since exact MWPMs can be obtained in polynomial time, this procedure allows for an efficient implementation \( \[30\] \) of the simulation algorithms. Here, we give a brief description of the algorithmic procedure that yields a minimum-weight set of loops for a given realization of the disorder. Fig. \( \[2\] \) illustrates the three basic steps, detailed below:

1. (1) each edge, joining adjacent sites on the original graph \( G \), is replaced by a path of 3 edges. Therefore, 2 “additional” sites have to be introduced for each edge in \( E \). Therein, one of the two edges connecting an additional site to an original site gets the same weight as
The resulting auxiliary graph consists of an even number of sites and thus guarantees that a perfect matching exists. Note that a MWPM can be computed in polynomial time as a function of the number of sites, hence large systems with hundreds of thousands of sites are feasible.

(3) Finally, it is possible to find a relation between the matched edges \( M \) on \( G_A \) and a configuration of negative-weighted loops \( C \) on \( G \) by tracing back the steps of the transformation (1). As regards this, note that each edge contained in \( M \) that connects an additional site (square) to a duplicated site (circle) corresponds to an edge on \( G \) that is part of a loop, see Fig. 2(d). Note that, by construction of the auxiliary graph, for each site \( i_1 \) or \( i_2 \) matched in this way, the corresponding “twin” site \( i_2/i_1 \) must be matched to an additional site as well. This guarantees that wherever a path “enters” a site of the original graph, the paths also “leaves” the site, corresponding to the defining condition of loops. All the edges in \( M \) that connect like sites (i.e. duplicated-duplicated, or additional-additional) carry zero weight and do not contribute to a loop on \( G \). Once all loop segments are found, a depth-first search \([26, 28]\) can be used to identify the loop set \( C \) and to determine the geometric properties of the individual loops. Here, the exemplary weight assignment illustrated in Fig. 2(a) yields 2 loops, i.e. \( C = \{L_1, L_2\} \), with weights \( \omega_{C_1} = \omega_{C_2} = -4 \) and lengths \( \ell_1 = \sum_{\{i,j\} \in C_1} 1 = 8, \ell_2 = 4 \). Hence, the configurational energy reads \( \mathcal{E} = -8 \).

The result of the calculation is a collection \( C \) of loops such that the total loop weight, and consequently the configuration energy \( \mathcal{E} \), is minimized. Hence, one obtains a global collective optimum of the system. Obviously, all loops that contribute to \( C \) possess a negative weight. Also note that the choice of the weight assignment in step (1) is not unique, i.e. there are different possibilities to choose a weight assignment that all result in equivalent sets of matched edges on the transformed lattice, corresponding to the minimum-weight collection of loops on the original lattice. Some of these weight assignments result in a more symmetric transformed graph, see e.g. \([26]\). However, this is only a technical issue that does not affect the resulting loop configuration. Finally, for the purpose of illustration, a small 2D lattice graph with free BCs was chosen intentionally. The algorithmic procedure extends to higher dimensions and fully periodic BCs in a straightforward manner.

In the remainder of the article, we will use the procedure outlined above in order to study the pmf of loop lengths in the NWP model on hypercubic lattice graphs in \( d = 2 \) through 6.

**III. RESULTS**

Within our extensive numerical studies, we performed exact NWP loop calculations for dimensions \( d = 2 \) through 6 for various values \( \rho \leq \rho_c \) while averaging over many realizations of the disorder. Details are given in Tab. III.

In order to get a grip on the loop-size cut-off parameter
σ for a particular hypercubic lattice setup of dimension d, the pmf \( n_{L}(\rho) \) of the loop length needs to be obtained for different values of the disorder parameter \( \rho \leq \rho_{c} \). Then, a best fit to the form \( n_{L}(\rho) = n_{0} e^{-\tau} \exp(-T_{L}(\rho)/\ell) \) might be used to obtain the three fit parameters \( n_{0}, \tau, \) and \( T_{L}(\rho) \) for different values of \( \rho \). Finally, the sequence of fit parameters \( T_{L}(\rho) \) might be analyzed to yield the exponent \( \sigma \) according to Eq. (2).

However, a different procedure appears to be more appealing: from previous simulations in dimensions \( d = 2 \) through 7, reported in Refs. 22, 23, we found that the pmf \( n_{L}(\rho_{c}) \) exhibits an algebraic decay governed by the exponent \( \tau \). For the largest lattice graphs simulated for the various dimensions \( d \), we obtain the numerical estimates listed in Tab. II. For the corresponding data analyses, very small loops have to be neglected since they are affected by the granularity of the lattice and very large loops have to be withdrawn since they are affected by the lattice boundaries. Once the exponent \( \tau \) for a given dimension \( d \) is obtained, it can be utilized in the analysis of the loop length pmf at \( \rho < \rho_{c} \) to limit the number of fit parameters to only two (i.e. \( n_{0} \) and \( T_{L} \)), allowing for a more precise estimate of the individual values of \( T_{L}(\rho) \).

So as to get a grip on the loop size cut-off parameter \( \sigma \), the loop perimeter distribution was obtained for different \( \ell \) (at \( \rho = 0.02 \)), see Tab. II and Fig. 3, wherein the fit to functions of the form of Eq. (3), i.e. \( \ell(p) \) exhibits an algebraic decay governed by the critical exponent that describes the divergence of the effective critical points \( \rho_{c}^{e} \) can be expected to differ slightly from the asymptotic critical points listed in Tab. II (see discussion below). The resulting loop-length cut-off parameters \( \sigma \) are also listed in Tab. II.

For \( d = 2 \) systems of side length \( L = 512 \) the analysis yields \( \sigma = 0.53(3) \) and \( \rho_{c}^{e} = 0.344(2) \), see Fig. 4 (at \( L = 256 \) we find \( \rho_{c}^{e} = 0.346(2) \) and \( \sigma = 0.53(2) \)). For all values of \( \rho \) considered, the data curves of \( T_{L} \) at \( \rho = 0.24 \) and \( d_{f} \), and for a number \( n_{R} \) of realizations, for the different dimensions \( d \) and system sizes \( L \).

| \( d \) | \( L \) | \( [\rho_{L}, \rho_{R}] \) | \( n_{p} \) | \( n_{R} \) |
|---|---|---|---|---|
| 2 | 256 | [0.24, 0.34] | 11 | \( \sim 2 \times 10^{4} \) |
| 2 | 512 | [0.24, 0.34] | 16 | 6400 |
| 3 | 64 | [0.075, 0.1245] | 100 | 4800 |
| 4 | 21 | [0.022, 0.058] | 19 | 8000 |
| 5 | 12 | [0.02, 0.038] | 19 | 6400 |
| 6 | 6 | [0.015, 0.025] | 21 | \( \sim 5 \times 10^{4} \) |

FIG. 3: Results for \( d = 2 \) square lattice graphs. The main plot shows the probability mass functions \( n_{L}(\rho) \) of the loop perimeter \( \ell \) for different values of the disorder parameter \( \rho \) for square systems of side length \( L = 512 \). The data curves illustrate the suppression of loops with a perimeter larger than some cut-off length scale \( \ell_{0} \), related to a finite “loop size cut-off parameter” \( T_{L}(\rho) \) for \( \rho < \rho_{c} \) (see text). The data curves fit well to functions of the form of Eq. (2). Right at \( \rho_{c} \), the data curve exhibits a pure algebraic decay according to Eq. (1). The inset compares the resulting values of the cut-off parameter \( T_{L}(\rho) \) for systems of side length \( L = 512 \) and \( L = 256 \), where the solid line indicates a best fit to the \( L = 512 \) data using a function with four free parameters as explained in the text.

\[ L = 256 \text{ and } L = 512 \text{ compare well as shown in the inset of Fig. 3.} \] This finding is further consistent with the usual scaling relation \( 1/(\nu_{p} d_{f,L}) = \sigma \) that relates \( \sigma \) to \( \nu_{p} \) and \( d_{f,L} \), where the latter two exponents are not of interest and the effective critical points \( \rho_{c}^{e} \) can be expected to differ slightly from the asymptotic critical points listed in Tab. II (see discussion below). The resulting loop-length cut-off parameters \( \sigma \) are also listed in Tab. II.

\[ \text{TABLE II: Simulation parameters: we performed our study for } n_{R} \text{ values of the disorder parameter } \rho \text{ in intervals } [\rho_{L}, \rho_{R}] \text{ and for a number } n_{R} \text{ of realizations, for the different dimensions } d \text{ and system sizes } L. \]
The resulting loop-length cut-off parameters $\sigma$ are also listed in Tab. I. The analysis of the cut-off parameter $T_l(\rho)$ in dimensions $d = 2$ through 6. The aim of the study was to characterize the ensemble of small loops in the NWP model using two independent critical exponents: the Fisher exponent $\tau$ (which was already known from previous studies, see Ref. [23]), and the loop-length cut-off exponent $\sigma$. Both exponents can be determined by means of an analysis of the probability mass function $n_L(\rho)$ measuring the distribution of loop lengths $L$, considering a sequence of different values $\rho$ close to but below the critical point. This implies a huge numerical effort, since we had to study in different dimensions large systems, for several values of the disorder parameter, while averaging over many realizations of the disorder. For the numerical simulations we used a mapping of the NWP model to a combinatorial optimization problem that allows to obtain configurations of minimum weight loops via exact algorithms. Note that due to the small side lengths of the lattice graphs that are accessible in high dimensions, the data analysis is notoriously difficult at large values of $d$. However, we find the results regarding the exponent $\sigma$ consistent with the scaling relations Eqs. (4a) and (4b) for any dimension considered (see Tab. I). Thus, via this extensive numerical study we have completed a comprehensive description of the static behavior of the NWP model in all relevant dimensions $d = 2, \ldots, 6$.

In particular, many of the $d = 3$ loop models studied in the literature report on values for the critical exponents $\tau$ and $\sigma$ that are close to $\tau = 3.07(1)$ and $\sigma = 0.71(1) \ (1/\sigma = 1.41(2))$ found here. E.g., Ref. [7] obtains $\tau = 2.4(1)$ (however, in a previous study they report $\tau = 3$, see Ref. [8]) and $1/\sigma = 1.45(5)$ (in that study the latter quantity was called $\gamma$) for the $d = 3$ uniformly frustrated XY model as well as for the lattice Ginzburg-Landau model in a frozen gauge approximation, and Ref. [5] yields $\tau = 2.8(1)$ and $\sigma = 0.6(1)$ for the strongly screened vortex glass model. Note that in $d = 3$ all the critical exponents of the NWP problem appear to be quite close to those that describe the strongly screened vortex glass model analyzed in Ref. [5]. As regards this, it appears to be tempting to conclude that in $d = 3$ both models are in the same universality class.

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