BANACH SPACES FOR THE SCHWARTZ DISTRIBUTIONS

TEPPER L. GILL

ABSTRACT. This paper is a survey of a new family of Banach spaces $B$ that provide the same structure for the Henstock-Kurzweil (HK) integrable functions as the $L^p$ spaces provide for the Lebesgue integrable functions. These spaces also contain the wide sense Denjoy integrable functions. They were first used to provide the foundations for the Feynman formulation of quantum mechanics. It has recently been observed that these spaces contain the test functions $D$ as a continuous dense embedding. Thus, by the Hahn-Banach theorem, $D' \subset B'$.

A new family that extends the space of functions of bounded mean oscillation $BMO[\mathbb{R}^n]$, to include the HK-integrable functions are also introduced.

1. Introduction

Since the work of Henstock [HS1] and Kurzweil [KW], the most important finitely additive measure is the one generated by the Henstock-Kurzweil integral (HK-integral). It generalizes the Lebesgue, Bochner and Pettis integrals. The HK-integral is equivalent to the Denjoy and Perron integrals. However, it is much easier to understand (and learn) compared to the these and the Lebesgue integral. It provides useful variants of the same theorems that have made the Lebesgue integral so important. We assume that the reader is acquainted with this integral,

1991 Mathematics Subject Classification. Primary (46) Secondary(47).

Key words and phrases. Henstock-Kurzweil integral, Schwartz distributions, path integral, Navier-Stokes, Markov Processes.

1
but more detail can be found in Gill and Zachary [GZ]. (For different perspectives, see Gordon [GO], Henstock [HS], Kurzweil [KW], or Pfeffer [PF].)

The most important factor preventing the widespread use of the HK-integral in mathematics, engineering and physics is the lack of a Banach space structure comparable to the $L^p$ spaces for the Lebesgue integral. The purpose of this paper is to provide a survey of some new classes of Banach spaces, which have this property and some with other interesting properties, but all contain the HK-integrable functions.

The first two classes are the $KS^p$ and the $SD^p$ spaces, $1 \leq p \leq \infty$. These are all separable spaces that contain the corresponding $L^p$ spaces as dense, continuous, compact embeddings. We have recently discovered that these two classes also contain the test functions $D$ as a continuous dense embedding. This implies the each dual space contains the Schwartz distributions. The family of $SD^p$ spaces also have the remarkable property that $\|D^\alpha f\|_{SD} = \|f\|_{SD}$.

The other main classes of spaces $Z^p$ and the $Z^{-p}$ spaces, $1 \leq p \leq \infty$, are related to the space of functions of bounded mean oscillation, $BMO$. We also introduce an extended version of this space, which we call the space of functions of weak bounded mean oscillation, $BMO^w$.

We provide a few applications of the first two families of spaces, which either provide simpler solutions to old problems or solve open problems.

The main tool for the work in this paper had its beginnings in 1965, when Gross [G] proved that every real separable Banach space contains a separable Hilbert space as a continuous dense embedding, which is the support of a Gaussian measure. This was a generalization of Wiener’s theory, which used the (densely embedded Hilbert)
Sobolev space $H^1_0[0, 1] \subset C_0[0, 1]$. In 1972, Kuelbs [KH] generalized Gross’ theorem to include the Hilbert space rigging $H^1_0[0, 1] \subset C_0[0, 1] \subset L^2[0, 1]$. For our purposes, a general version of this theorem can be stated as:

**Theorem 1.1.** (Gross-Kuelbs) Let $B$ be a separable Banach space. Then there exist separable Hilbert spaces $H_1, H_2$ and a positive trace class operator $T_{12}$ defined on $H_2$ such that $H_1 \subset B \subset H_2$ all as continuous dense embeddings, with

$$\left(T_{12}^{1/2} u, T_{12}^{1/2} v\right)_1 = (u, v)_2 \text{ and } \left(T_{12}^{-1/2} u, T_{12}^{-1/2} v\right)_2 = (u, v)_1. $$

A proof can be found in [GZ]. The space $H_1$ is part of the abstract Wiener space method for extending Wiener measure to separable Banach spaces. The space $H_2$ is a major tool in the construction Banach spaces for HK-integrable functions. (We call it the natural Hilbert space for $B$.)

1.1. **Summary.** In Section 2, we establish a number of background results to make the paper self contained. The major result is Theorem 2.5 (and Corollary 2.6). It allows us to construct the path integral in the manner originally suggested by Feynman. In Section 3, after a few examples, we construct the KS-spaces and derive some of their important properties. In Section 4, we construct the SD-spaces and discuss their properties. In Section 5 we discuss the family of spaces related to the functions of bounded mean oscillation.

In Section 6, we give a few applications. The first application uses $KS^2$ to provide a simple solution to the generator (with unbounded coefficients) problem for Markov processes. The second application uses $KS^2$ and Corollary 2.6 to construct the Feynman path integral. The third application uses $SD^2$ to provide the best possible a priori bound for the nonlinear term of the Navier-Stokes equation.
2. Background

In this section, we provide some background results, which are required in the paper. Let $\mathcal{B}$ be a Banach space, with dual $\mathcal{B}^*$.

2.1. Bounded Operator Extension. We are interested in the problem of operator extensions from $\mathcal{B}$ to $\mathcal{H}(= \mathcal{H}_2)$. It is not hard to see that, since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, every closed densely defined linear operator on $\mathcal{B}$ has a closed densely defined extension to $\mathcal{H}$, $\mathcal{C}[\mathcal{B}] \xrightarrow{\text{ext}} \mathcal{C}[\mathcal{H}]$ (see Theorem 2.5 for a proof). In this section, we show that this also holds for bounded linear operators, $L[\mathcal{B}] \xrightarrow{\text{ext}} L[\mathcal{H}]$. This important result depends on the following theorem by Lax [L]. It is not well known, so we include a proof.

**Theorem 2.1.** (Lax’s Theorem) Let $\mathcal{B}$ be a separable Banach space continuously and densely embedded in a Hilbert space $\mathcal{H}$ and let $T$ be a bounded linear operator on $\mathcal{B}$ which is symmetric with respect to the inner product of $\mathcal{H}$ (i.e., $(Tu, v)_\mathcal{H} = (u, Tv)_\mathcal{H}$ for all $u, v \in \mathcal{B}$). Then:

1. The operator $T$ is bounded with respect to the $\mathcal{H}$ norm and

\[ \|T^*T\|_{\mathcal{H}} = \|T\|^2_{\mathcal{H}} \leq k\|T\|^2_{\mathcal{B}}, \]

where $k$ is a positive constant.

2. The spectrum of $T$ relative to $\mathcal{H}$ is a subset of the spectrum of $T$ relative to $\mathcal{B}$.

3. The point spectrum of $T$ relative to $\mathcal{H}$ is a equal to the point spectrum of $T$ relative to $\mathcal{B}$.
Proof. To prove (1), let \( u \in \mathcal{B} \) and, without loss, we can assume that \( k = 1 \) and \( \|u\|_\mathcal{H} = 1 \). Since \( T \) is selfadjoint,

\[
\|Tu\|_\mathcal{H}^2 = (Tu, Tu) = (u, T^2u) \leq \|u\|_\mathcal{H} \|T^2u\|_\mathcal{H} = \|T^2u\|_\mathcal{H}.
\]

Thus, we have \( \|Tu\|_\mathcal{H}^4 \leq \|T^4u\|_\mathcal{H} \), so it is easy to see that \( \|Tu\|_{\mathcal{H}}^2 \leq \|T^{2n}u\|_\mathcal{H} \) for all \( n \). It follows that:

\[
\|Tu\|_{\mathcal{H}} \leq (\|T^{2n}u\|_{\mathcal{H}})^{1/2n} \leq (\|T^{2n}u\|_{\mathcal{B}})^{1/2n} \leq (\|T^{2n}\|_{\mathcal{B}})^{1/2n} (\|u\|_{\mathcal{B}})^{1/2n} \leq \|T\|_{\mathcal{B}} (\|u\|_{\mathcal{B}})^{1/2n}.
\]

Letting \( n \to \infty \), we get that \( \|Tu\|_{\mathcal{H}} \leq \|T\|_{\mathcal{B}} \) for \( u \) in a dense set of the unit ball of \( \mathcal{H} \). It follows that

\[
\|T\|_{\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}} \leq 1} \|Tu\|_{\mathcal{H}} \leq \|T\|_{\mathcal{B}}.
\]

To prove (2), suppose \( \lambda_0 \notin \sigma_T \), the spectrum of \( T \) over \( \mathcal{B} \) so that \( T - \lambda_0 I \) has a bounded inverse \( S \) on \( \mathcal{B} \). Since \( T \) is symmetric on \( \mathcal{H} \), so is \( S \). It follows from (1) that \( T \) and \( S \) extend to bounded linear operators \( \tilde{T} \) and \( \tilde{S} \) on \( \mathcal{H} \). We also see that \( \lambda_0 \notin \sigma_{\tilde{T}} \). It follows from this that \( T \) has inverse \( \tilde{S} \) and the spectrum of \( \tilde{T} \) on \( \mathcal{H} \) is a subset of the spectrum of \( T \) on \( \mathcal{B} \) (i.e., \( \sigma_{\tilde{T}} \subset \sigma_T \)).

To prove (3), suppose that \( \lambda \in \sigma_p \), the point spectrum of \( T \), so that \( T - \lambda I \) has a finite dimensional null space \( N \) and \( \dim N = \dim (\mathcal{B}/J) \), where \( J = (T - \lambda I)(\mathcal{B}) \).

Since \( T \) is symmetric, every vector in \( J \) is orthogonal to \( N \). Conversely, from \( \dim N = \dim (\mathcal{B}/J) \) we see that \( J \) contains all vectors that are orthogonal to \( N \). It follows that, \( (T - \lambda I) \) is a one-to-one, onto mapping of \( J \to J \), so that \( T - \lambda I = S \) has an inverse on \( J \), which is bounded (on \( J \)) by the Closed Graph Theorem. It follows that the extension \( \tilde{S} \) of \( S \) to the closure of \( J, \tilde{J} \) in \( \mathcal{H} \) is bounded on \( \tilde{J} \). This means that \( (\tilde{T} - \lambda I) \) is the orthogonal compliment of \( N \) over \( \mathcal{H} \), so that \( \lambda \) belongs
to the point spectrum of \( \tilde{T} \) on \( \mathcal{H} \) and the null space of \((\tilde{T} - \lambda I)\) over \( \mathcal{H} \) is \( N \). It follows that the point spectrum of \( T \) is unchanged on extension to \( \mathcal{H} \). \( \square \)

Let \( \mathcal{H} \) be the natural Hilbert space for \( \mathcal{B} \), let \( J \) be the standard linear mapping from \( \mathcal{H} \to \mathcal{H}^* \) and let \( J_B \) be its restriction to \( \mathcal{B} \). Since \( \mathcal{B} \) is a continuous dense embedding in \( \mathcal{H} \), \( J_B \) is a (conjugate) isometric isomorphism of \( \mathcal{B} \) onto \( J_B(\mathcal{B}) \subset \mathcal{B}^* \).

**Definition 2.2.** If \( u \in \mathcal{B} \), set \( u_h = J_B(u) \) and define

\[
\mathcal{B}^*_h = \{ u_h \in \mathcal{B}^* : u \in \mathcal{B} \},
\]

so that \( \langle u, u_h \rangle = (u, u)_\mathcal{H} = \|u\|^2_\mathcal{H} \). It is clear from our construction of \( \mathcal{B}^*_h \) that the mapping taking \( \mathcal{B} \to \mathcal{B}^*_h \) is a (conjugate) isometric isomorphism. We call \( \mathcal{B}^*_h \) the \( h \)-representation for \( \mathcal{B} \) in \( \mathcal{B}^* \).

It’s easy to prove that following result.

**Theorem 2.3.** If \( \mathcal{B} \) is a reflexive Banach space. Then \( \mathcal{B}^*_h \) is bijectively related to \( \mathcal{B}^* \).

**Remark 2.4.** In general, the embedding of \( \mathcal{B}^*_h \) is a proper subspace of \( \mathcal{B}^* \).

We can now state and prove the following fundamental theorem.

**Theorem 2.5.** Let \( \mathcal{B} \) be a separable Banach space. If \( A \in \mathcal{C}[\mathcal{B}] \), then there is a unique operator \( A^* \in \mathcal{C}[\mathcal{B}] \) satisfying:

1. \( (aA)^* = \bar{a}A^* \),
2. \( A^{**} = A \),
3. \( (A^* + B^*) = A^* + B^* \),
(4) \((AB)^* = B^*A^*\) on \(D(A^*) \cap D(B^*)\),

(5) \((A^*A)^* = A^*A\) on \(D(A^*A)\) (self adjoint),

(6) if \(A \in L[\mathcal{B}]\), then \(\|A^*A\|_\mathcal{H} \leq k \|A^*A\|_\mathcal{B}\) and

(7) if \(A \in L[\mathcal{B}]\), then \(\|A^*A\|_\mathcal{B} \leq c \|A\|^2_\mathcal{B}\), for some constant \(c\).

Proof. Recall that \(J\) is the natural linear mapping from \(\mathcal{H}_2 = \mathcal{H} \to \mathcal{H}^*\) and \(J_\mathcal{B}\) is the restriction of \(J\) to \(\mathcal{B}\), so that \(J_\mathcal{B}(\mathcal{B}) = \mathcal{B}_h^*\). If \(A \in C[\mathcal{B}]\), then \(A' : \mathcal{B}^* \to \mathcal{B}^*\) and \(A'J_\mathcal{B} : \mathcal{B} \to \mathcal{H}^*\). Since \(\mathcal{B}\) is dense in \(\mathcal{H}\), \(\mathcal{B}_h^*\) is dense in \(\mathcal{H}^*\). It follows that \(A'J_\mathcal{B}\) is a closed densely define operator on \(\mathcal{H}^*\), \(J_\mathcal{B}^{-1}A'J_\mathcal{B} : \mathcal{B} \to \mathcal{B}\) is a closed and densely defined linear operator on \(\mathcal{B}\). We define \(A^* = [J_\mathcal{B}^{-1}A'J_\mathcal{B}] \in C[\mathcal{B}]\). If \(A \in L[\mathcal{B}]\), \(A^* = J_\mathcal{B}^{-1}A'J_\mathcal{B}\) is defined on all of \(\mathcal{B}\) so that, by the Closed Graph Theorem, \(A^* \in L[\mathcal{B}]\). The proofs of (1)-(3) are straightforward. To prove (4), let \(u \in D(A^*) \cap D(B^*)\), then

\[
(2.1) \quad (BA)^*u = [J_\mathcal{B}^{-1}(BA)'J_\mathcal{B}]u = [J_\mathcal{B}^{-1}A'B'J_\mathcal{B}]u = [J_\mathcal{B}^{-1}A'J_\mathcal{B}] [J_\mathcal{B}^{-1}B'J_\mathcal{B}] u = A'B^*u.
\]

If we replace \(B\) by \(A^*\) in equation (2.1), noting that \(A^{**} = A\), we also see that \((A^*A)^* = A^*A\), proving (5). To prove (6), we first see that:

\[
\langle A^*Av, J_\mathcal{B}(u) \rangle = \langle A^*Av, u_h \rangle = \langle A^*Av, u \rangle_\mathcal{H} = \langle v, A^*Au \rangle_\mathcal{H},
\]

so that \(A^*A\) is symmetric. Thus, by Theorem (Lax), \(A^*A\) has a bounded extension to \(\mathcal{H}\) and \(\|A^*A\|_\mathcal{H} = \|A\|^2_\mathcal{H} \leq k \|A^*A\|_\mathcal{B}\), where \(k\) is a positive constant.

We also have that

\[
\|A^*A\|_\mathcal{B} \leq \|A^*\|_\mathcal{B} \|A\|_\mathcal{B} \leq \|J_\mathcal{B}\|_\mathcal{B} \|J_\mathcal{B}^{-1}\|_\mathcal{B} \|A'\|_\mathcal{B} \|A\|_\mathcal{B} = c \|A\|^2_\mathcal{B},
\]

(2.2)
proving (7). It also follows that

\[ \|A\|_H \leq \sqrt{ck} \|A\|_B. \]  

(2.3)

\[ \square \]

If \( c = 1 \) and equality holds in (2.2) for all \( A \in L[\mathcal{B}] \), then \( L[\mathcal{B}] \) is a \( C^* \)-algebra. In this case, \( \mathcal{B} \) is a Hilbert space. Thus, in general the inequality in (2.2) is strict. From (2.3), we see the following.

**Corollary 2.6.** Let \( \mathcal{B} \) be a separable Banach space. If \( A \in L[\mathcal{B}] \), then there is a unique operator \( \bar{A} \in L[\mathcal{H}] \) (i.e., \( L[\mathcal{B}] \xrightarrow{ext} L[\mathcal{H}] \)).

**Theorem 2.7.** (Polar Representation) Let \( \mathcal{B} \) be a separable Banach space. If \( A \in \mathbb{C}[\mathcal{B}] \), then there exists a partial isometry \( U \) and a self-adjoint operator \( T, T = T^* \), with \( D(T) = D(A) \) and \( A = UT \).

**Proof.** Let \( \bar{A} \) be the closed densely defined extension of \( A \) to \( \mathcal{H} \). On \( \mathcal{H} \), \( T^2 = \bar{A}^* \bar{A} \) is self-adjoint and there exist a unique partial isometry \( \bar{U} \), with \( \bar{A} = \bar{U}T \). Thus, the restriction to \( \mathcal{B} \) gives us \( A = UT \) and \( U \) is a partial isometry on \( \mathcal{B} \). (It is easy to check that \( A^*A = T^2 \).) \[ \square \]

3. **The Kuelbs-Steadman \( KS^p \) Spaces**

3.1. **Special Constructions.** Our first construction is based on an extension of a norm due to Alexiewicz [AL]. We first recall that the HK-integral is equivalent to the strict Denjoy integral (see Henstock [HS] or Pfeffer [PF]). In the one-dimensional case, Alexiewicz [AL] has shown that the class \( D(\mathbb{R}) \), of Denjoy integrable functions,
can be normed in the following manner: for $f \in D(\mathbb{R})$, define $\|f\|_{A_1}$ by

$$\|f\|_{A_1} = \sup_s \left| \int_{-\infty}^{s} f(r) d\lambda(r) \right|.$$ 

It is clear that this is a norm, and it is known that $D(\mathbb{R})$ is not complete (see Alexiewicz [AL]). If we replace $\mathbb{R}$ by $\mathbb{R}^n$, let $[a_i, b_i] \subset \mathbb{R} = [-\infty, \infty]$, and define $[a, b] \in \mathbb{R}^n$ by $[a, b] = \prod_{k=1}^{n} [a_i, b_i]$. If $f \in D(\mathbb{R}^n)$ we define the norm of $f$ by (see [OS]):

$$(3.1) \quad \|f\|_{A_n} = \sup_{x_1, \ldots, x_n} \left| \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y) d\lambda_n(y) \right|.$$ 

**Lemma 3.1.** If $HK[\mathbb{R}^n]$ is the completion of $D(\mathbb{R}^n)$ in the above norm, then $L^1_{\text{Loc}}(\mathbb{R}^n) \subset HK[\mathbb{R}^n]$, as a continuous embedding.

The completion of $C_0[\mathbb{R}]$ in the $L^1$ norm leads to all absolutely integrable functions (i.e., the norm limit of $L^1$ functions is an $L^1$ function). The completion of $C_0[\mathbb{R}]$ in the Alexiewicz norm leads to $HK[\mathbb{R}^n]$, but the class of HK-integrable functions is a proper subset of $HK[\mathbb{R}^n]$. This is the case for all suggested norms and Hönig has conjectured that there is no “natural norm” for the HK-integrable functions (see [HO]). He suggests that the following is the best we can hope for:

**Lemma 3.2.** (Hönig) Let $HK[a, b]$ be the space of HK-integrable functions on $[a, b] \subset \mathbb{R}$ and let $C_a[a, b]$ be the continuous functions $f$ on $[a, b]$, with $f(a) = 0$. If $\|\cdot\|_{A_1}$ is the Alexiewicz norm:

$$\|f\|_{A_1} = \sup_s \left| \int_{a}^{s} f(r) d\lambda(r) \right|,$$

then the completion of $HK[a, b]$ is the set of all distributions that are weak derivatives of functions in $C_a[a, b]$. 

BANACH SPACES FOR THE SCHWARTZ DISTRIBUTIONS
Talvila [TA] independently obtained the same result for $\mathbb{R}$ and used it to motivate his definition of a distributional integral.

**Definition 3.3.** If there is a continuous function $F(x)$ with real limits at infinity such that $F'(x) = f(x)$ (weak derivative), then the distributional integral of $f(x)$ is defined to be $D \int_{-\infty}^{\infty} f(x) \, dx = F(\infty) - F(-\infty)$.

Talvila shows that the Alexiewicz norm leads to a Banach space of integrable distributions that is isometrically isomorphic to the space of continuous functions on the extended real line with uniform norm. He also shows that the dual space can be identified with the space of functions of bounded variation.

The following two spaces are also closely related to $HK[\mathbb{R}^n]$.

**Theorem 3.4.** Let $\{u_i\}_{i=1}^{\infty} \subset C_0^1[\mathbb{R}^n]$ be $S$-basis for $B = L^1[\mathbb{R}^n]$ or an orthonormal basis for $H = L^2[\mathbb{R}^n]$.

1. Then, for $L^1[\mathbb{R}^n]$,

$$\|f\|_{A_1^n} = \sup_i \left| \int_{\mathbb{R}^n} f(x) u_i(x) \, d\lambda_n(x) \right|,$$

defines a weaker norm on $L^1[\mathbb{R}^n]$ and $f \in L^1[\mathbb{R}^n] \Rightarrow \|f\|_{A_1^n} \leq \|f\|_1$, and

2. If $B_{A_1^n}$ is the completion of $B$ in this norm, then $L^1_{Loc}[\mathbb{R}^n] \subset B_{A_1^n}$, as a continuous embedding.

3. For $L^2[\mathbb{R}^n]$

$$\|f\|_{A_2^n} = \sup_i \left| \int_{\mathbb{R}^n} f(x) u_i(x) \, d\lambda_n(x) \right|,$$

defines a weaker norm on $L^2[\mathbb{R}^n]$ and $f \in L^2[\mathbb{R}^n] \Rightarrow \|f\|_{A_2^n} \leq \|f\|_2$.

4. If $H_{A_2^n}$ is the completion of $H$ in this norm, then $L^2_{Loc}[\mathbb{R}^n] \subset H_{A_2^n}$, as a continuous embedding.
Remark 3.5. The last two spaces are close but not the same. A proof is actually required to show that $B_{A_n^1}$ and $H_{A_n^2}$ contains the HK-integrable functions. The condition $\{u_i\}_{i=1}^\infty \subset C_0^1[\mathbb{R}^n]$ is sufficient and the proof is the same as for Lemma 4.10 (see Section 4.3).

3.2. General Construction. For our first general construction, fix $n$, and let $Q^n$ be the set $\{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\}$ such that $x_i$ is rational for each $i$. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $Q^n = \{x_1, x_2, x_3, \cdots\}$.

For each $l$ and $i$, let $B_l(x_i)$ be the closed cube centered at $x_i$, with sides parallel to the coordinate axes and edge $e_l = \frac{1}{2 \sqrt{n}} l \in \mathbb{N}$. Now choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$, and let $\{B_k, k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{B_l(x_i) \mid (l, i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $Q^n$. Let $\mathcal{E}_k(x)$ be the characteristic function of $B_k$, so that $\mathcal{E}_k(x)$ is in $L^p[\mathbb{R}^n] \cap L^\infty[\mathbb{R}^n]$ for $1 \leq p < \infty$.

Define $F_k(\cdot)$ on $L^1[\mathbb{R}^n]$ by

$$ F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(x) f(x) d\lambda_n(x). \quad (3.2) $$

It is clear that $F_k(\cdot)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each $k$, $\|F_k\|_\infty \leq 1$ and, if $F_k(f) = 0$ for all $k$, $f = 0$ so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^\infty t_k = 1$ and define a measure $dP(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by:

$$ dP(x,y) = \left[ \sum_{k=1}^\infty t_k \mathcal{E}_k(x) \mathcal{E}_k(y) \right] d\lambda_n(x) d\lambda_n(y). $$

We first construct our Hilbert space. Define an inner product $(\cdot, \cdot)$ on $L^1[\mathbb{R}^n]$ by

$$ (f, g) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x)g(y)^* dP(x,y) $$

$$ = \sum_{k=1}^\infty t_k \left[ \int_{\mathbb{R}^n} \mathcal{E}_k(x) f(x) d\lambda_n(x) \right] \left[ \int_{\mathbb{R}^n} \mathcal{E}_k(y) g(y) d\lambda_n(y) \right]^*. \quad (3.3) $$
We use a particular choice of $t_k$ in Gill and Zachary [GZ], which is suggested by physical analysis in another context. We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $KS^2[\mathbb{R}^n]$. Following suggestions of Gill and Zachary, Steadman [ST] constructed this space by adapting an approach developed by Kuelbs [KB] for other purposes. Her interest was in showing that $L^1[\mathbb{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbb{R}^n]$, then:

$$
\|f\|^2_{KS^2} = \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} E_k(x) f(x) d\lambda_n(x) \right| \leq \sup_k \left| \int_{\mathbb{R}^n} E_k(x) f(x) d\lambda_n(x) \right| \leq \|f\|_{A_n}^2,
$$

so $f \in KS^2[\mathbb{R}^n]$.

**Theorem 3.6.** For each $p$, $1 \leq p \leq \infty$, $KS^2[\mathbb{R}^n] \supset L^p[\mathbb{R}^n]$ as a dense subspace.

**Proof.** By construction, $KS^2[\mathbb{R}^n]$ contains $L^1[\mathbb{R}^n]$ densely, so we need only show that $KS^2[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ for $q \neq 1$. If $f \in L^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$
\|f\|^2_{KS^2} = \left[ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} E_k(x) f(x) d\lambda_n(x) \right|^2 \right]^{1/2}
\leq \left[ \sum_{k=1}^{\infty} t_k \left( \int_{\mathbb{R}^n} E_k(x) |f(x)|^q d\lambda_n(x) \right)^{2/2} \right]^{1/2}
\leq \sup_k \left( \int_{\mathbb{R}^n} E_k(x) |f(x)|^q d\lambda_n(x) \right)^{1/2} \leq \|f\|_q.
$$

Hence, $f \in KS^2[\mathbb{R}^n]$. For $q = \infty$, first note that $vol(B_k)^2 \leq \left[ \frac{1}{2\sqrt{n}} \right]^{2n}$, so we have

$$
\|f\|^2_{KS^2} = \left[ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} E_k(x) f(x) d\lambda_n(x) \right|^2 \right]^{1/2}
\leq \left[ \sum_{k=1}^{\infty} t_k [vol(B_k)]^2 \left( ess \sup |f|^2 \right)^{1/2} \right] \leq \left[ \frac{1}{2\sqrt{n}} \right]^{n} \|f\|_{\infty}.
$$

Thus $f \in KS^2[\mathbb{R}^n]$, and $L^\infty[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. \qed
Before proceeding to additional study, we construct the $KS^p[\mathbb{R}^n]$ spaces, for $1 \leq p \leq \infty$.

To construct $KS^p[\mathbb{R}^n]$ for all $p$ and for $f \in L^p$, define:

$$
\|f\|_{KS^p} = \begin{cases}
\left\{ \sum_{k=1}^{\infty} t_k \int_{\mathbb{R}^n} E_k(x) |f(x)| d\lambda_n(x) \right\}^{1/p}, & 1 \leq p < \infty, \\
\sup_{k \geq 1} \int_{\mathbb{R}^n} E_k(x) f(x) d\lambda_n(x), & p = \infty.
\end{cases}
$$

It is easy to see that $\|\cdot\|_{KS^p}$ defines a norm on $L^p$. If $KS^p$ is the completion of $L^p$ with respect to this norm, we have:

**Theorem 3.7.** For each $q$, $1 \leq q \leq \infty$, $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a dense continuous embedding.

**Proof.** As in the previous theorem, by construction $KS^p[\mathbb{R}^n]$ contains $L^p[\mathbb{R}^n]$ densely, so we need only show that $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in L^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$
\|f\|_{KS^p} \leq \left[ \sum_{k=1}^{\infty} t_k \left( \int_{\mathbb{R}^n} E_k(x) \|f(x)\|^q d\lambda_n(x) \right)^{1/q} \right]^{1/p} \leq \sup_{k} \left( \int_{\mathbb{R}^n} E_k(x) \|f(x)\|^q d\lambda_n(x) \right)^{1/q} \leq \|f\|_q.
$$

Hence, $f \in KS^p[\mathbb{R}^n]$. For $q = \infty$, we have

$$
\|f\|_{KS^p} \leq \left[ \sum_{k=1}^{\infty} t_k \left( \int_{\mathbb{R}^n} E_k(x) \|f(x)\|^q d\lambda_n(x) \right)^{1/q} \right]^{1/p} \leq \left[ \left( \sum_{k=1}^{\infty} t_k \text{vol}(B_k)^{p} \right) \|f\|\|f\|_\infty \right]^{1/p} \leq M \|f\|_\infty.
$$

Thus $f \in KS^p[\mathbb{R}^n]$, and $L^\infty[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$. The case $p = \infty$ is obvious. □

**Theorem 3.8.** For $KS^p$, $1 \leq p \leq \infty$, we have:
(1) If \( f, g \in KS^p \), then \( \| f + g \|_{KS^p} \leq \| f \|_{KS^p} + \| g \|_{KS^p} \) (Minkowski inequality).

(2) If \( K \) is a weakly compact subset of \( L^p \), it is a compact subset of \( KS^p \).

(3) If \( 1 < p < \infty \), then \( KS^p \) is uniformly convex.

(4) If \( 1 < p < \infty \) and \( p^{-1} + q^{-1} = 1 \), then the dual space of \( KS^p \) is \( KS^q \).

(5) \( KS^\infty \subset KS^p \), for \( 1 \leq p < \infty \).

Proof. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that, if \( \{f_m\} \) is any weakly convergent sequence in \( K \) with limit \( f \), then

\[
\int_{\mathbb{R}^n} E_k(x) \left| f_m(x) - f(x) \right| d\lambda_n(x) \to 0
\]

for each \( k \). It follows that \( \{f_m\} \) converges strongly to \( f \) in \( KS^p \).

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for \( L^p \) norms.

In order to prove (4), observe that, for \( p \neq 2, 1 < p < \infty \), the linear functional

\[
L_g(f) = \|g\|_{KS^p}^{2-p} \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} E_k(x)g(x)d\lambda_n(x) \right|^{p-2} \int_{\mathbb{R}^n} E_k(y)f(y)^*d\lambda_n(y)
\]

is a unique duality map on \( KS^q \) for each \( g \in KS^p \) and that \( KS^p \) is reflexive from (3). To prove (5), note that \( f \in KS^\infty \) implies that \( \left| \int_{\mathbb{R}^n} E_k(x)f(x)d\lambda_n(x) \right| \) is uniformly bounded for all \( k \). It follows that \( \left| \int_{\mathbb{R}^n} E_k(x)f(x)d\lambda_n(x) \right|^p \) is uniformly bounded for each \( p, 1 \leq p < \infty \). It is now clear from the definition of \( KS^\infty \) that:

\[
\left[ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} E_k(x)f(x)d\lambda_n(x) \right|^p \right]^{1/p} \leq \| f \|_{KS^\infty} < \infty
\]
Note that, since \( L^1[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n] \) and \( KS^p[\mathbb{R}^n] \) is reflexive for \( 1 < p < \infty \), we see that the second dual \( \{ L^1[\mathbb{R}^n] \}^{**} = \mathcal{M}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n] \). Recall that \( \mathcal{M}[\mathbb{R}^n] \) is the space of bounded finitely additive set functions defined on the Borel sets \( \mathcal{B}[\mathbb{R}^n] \).

In many applications, it is convenient to formulate problems on one of the standard Sobolev spaces \( W^{m,p}(\mathbb{R}^n) \). We can easily see that \( L^p_{loc}(\mathbb{R}^n) \subset KS^q(\mathbb{R}^n) \), \( 1 \leq q \leq \infty \), for all \( p, 1 \leq p \leq \infty \). This means that \( KS^q(\mathbb{R}^n) \) contains a large class of distributions (see Adams \[ A \]). There is more:

**Theorem 3.9.** For each \( p, 1 \leq p \leq \infty \), the test functions \( \mathcal{D} \subset KS^p(\mathbb{R}^n) \) as a continuous embedding.

**Proof.** Since \( KS^\infty(\mathbb{R}^n) \) is continuously embedded in \( KS^p(\mathbb{R}^n) \), \( 1 \leq q < \infty \), it suffices to prove the result for \( KS^\infty(\mathbb{R}^n) \). Suppose that \( \phi_j \to \phi \) in \( \mathcal{D}[\mathbb{R}^n] \), so that there exist a compact set \( K \subset \mathbb{R}^n \), containing the support of \( \phi_j - \phi \) and \( D^\alpha \phi_j \) converges to \( D^\alpha \phi \) uniformly on \( K \) for every multi-index \( \alpha \). Let \( L = \{ l \in \mathbb{N} : \text{the support of } \mathcal{E}_l, \ stp\{\mathcal{E}_l\} \subset K \} \), then

\[
\lim_{j \to \infty} \| D^\alpha \phi - D^\alpha \phi_j \|_{KS} = \lim_{j \to \infty} \sup_{l \in L} \left| \int_{\mathbb{R}^n} [D^\alpha \phi(x) - D^\alpha \phi_j(x)] \mathcal{E}_l(x) d\lambda_n(x) \right| \\
\leq \left( \frac{1}{2\sqrt{n}} \right)^n \lim_{j \to \infty} \sup_{x \in K} |D^\alpha \phi(x) - D^\alpha \phi_j(x)| = 0.
\]

It follows that \( \mathcal{D}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n] \) as a continuous embedding, for \( 1 \leq p \leq \infty \). Thus, by the Hahn-Banach theorem, we see that the Schwartz distributions, \( \mathcal{D}'[\mathbb{R}^n] \subset [KS^p(\mathbb{R}^n)]' \), for \( 1 \leq p \leq \infty \). □

We close this section with the following result that will be important later.
Lemma 3.10. Let the Fourier transform, $\mathcal{F}$, and the convolution operator, $\mathcal{C}$, be defined on $L^1[\mathbb{R}^n]$. Then each has a bounded extension to the linear operators on $KS^2[\mathbb{R}^n]$.

**Proof.** From Theorem 2.6, every bounded linear operator on $L^1[\mathbb{R}^n]$ extends to a bounded linear operator on $KS^2[\mathbb{R}^n]$. The theorem applies to $\mathcal{F}$ and $\mathcal{C}$. \hfill \Box

4. The Jones $SD^p$ Spaces

For our second class of spaces, we begin with the construction of a special class of functions in $C^\infty_c[\mathbb{R}^n]$ (see Jones, [J] page 249).

4.1. The remarkable Jones functions.

**Definition 4.1.** For $x \in \mathbb{R}$, $0 \leq y < \infty$ and $1 < a < \infty$, define the Jones functions $g(x, y)$, $h(x)$ by:

$$g(x, y) = \exp \left\{ -y^a e^{iay} \right\},$$

$$h(x) = \begin{cases} 
\int_0^\infty g(x, y)dy, & x \in (-\frac{\pi}{2a}, \frac{\pi}{2a}) \\
0, & \text{otherwise.}
\end{cases}$$

The following properties of $g$ are easy to check:

(1) $$\frac{\partial g(x, y)}{\partial x} = -iy^a e^{iay} g(x, y),$$

(2) $$\frac{\partial g(x, y)}{\partial y} = -ay^{a-1} e^{iay} g(x, y),$$

so that
\[ iy \frac{\partial g(x, y)}{\partial y} = \frac{\partial g(x, y)}{\partial x}. \]

It is also easy to see that \( h(x) \in L^1[-\frac{\pi}{2a}, \frac{\pi}{2a}] \) and,

\[ \frac{dh(x)}{dx} = \int_0^\infty \frac{\partial g(x, y)}{\partial x} dy = \int_0^\infty iy \frac{\partial g(x, y)}{\partial y} dy. \]

(4.1)

Integration by parts in the last expression in (1.1) shows that \( h'(x) = -ih(x) \), so that \( h(x) = h(0) e^{-ix} \) for \( x \in (-\frac{\pi}{2a}, \frac{\pi}{2a}) \). Since \( h(0) = \int_0^\infty \exp\{-y^a\} dy \), an additional integration by parts shows that \( h(0) = \Gamma(\frac{1}{a} + 1) \).

For each \( k \in \mathbb{N} \) let \( a = a_k = \pi 2^{k-1} \), \( h(x) = h_k(x), \ x \in (-\frac{1}{2^k}, \frac{1}{2^k}) \) and set \( \varepsilon_k = \frac{1}{2^{k+1}}. \)

Let \( \mathbb{Q} \) be the set of rational numbers in \( \mathbb{R} \) and for each \( x^i \in \mathbb{Q} \), define

\[ f_k^i(x) = f_k(x - x^i) = \begin{cases} c_k \exp \left\{ \frac{i \varepsilon_k^2}{|x - x^i|^2 - \varepsilon_k^2} \right\}, & \text{if } |x - x^i| < \varepsilon_k, \\ 0, & \text{if } |x - x^i| \geq \varepsilon_k, \end{cases} \]

where \( c_k \) is the standard normalizing constant. It is clear that the support of \( f_k^i \) is

\[ \text{spt}(f_k^i) \subset [-\varepsilon_k, \varepsilon_k] = [-\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}}] = I_k^i. \]

If we set \( \chi_k^i(x) = (f_k^i * h_k)(x) \), its support is \( \text{spt}(\chi_k^i) \subset [-\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}}] \). For \( x \in \text{spt}(\chi_k^i) \), we can also write \( \chi_k^i(x) = \chi_k(x - x^i) \) as:

\[ \chi_k^i(x) \]

\[ = \int_{I_k^i} f_k \left[(x - x^i) - z\right] h_k(z) dz 
= \int_{I_k^i} h_k \left[ (x - x^i) - z \right] f_k(z) dz 
= e^{-i(x - x^i)} \int_{I_k^i} e^{iz} f_k(z) dz. \]
Thus, if \( \alpha_{k,i} = \int_{I_k} e^{iz^i f_i(z)} dz \), we can now define:

\[
\xi^i_k(x) = \alpha_{k,i}^{-1} \chi^i_k(-x) = \begin{cases} 
\frac{1}{n} e^{ix^i - x^i}, & x \in I^i_k \\
0, & x \notin I^i_k,
\end{cases}
\]

so that \( |\xi^i_k(x)| < \frac{1}{n} \).

4.2. The Construction. To construct our space on \( \mathbb{R}^n \), let \( Q^n \) be the set of all vectors \( x \) in \( \mathbb{R}^n \), such that for each \( j, 1 \leq j \leq n \), the component \( x_j \) is rational. Since this is a countable dense set in \( \mathbb{R}^n \), we can arrange it as \( Q^n = \{ x^1, x^2, x^3, \ldots \} \). For each \( k \) and \( i \), let \( B_k(x^i) \) be the closed cube centered at \( x^i \) with edge \( e_k = \frac{1}{2^{k/\sqrt{n}}} \).

We choose the natural order which maps \( \mathbb{N} \times \mathbb{N} \) bijectively to \( \mathbb{N} \):

\[
\{(1,1), (2,1), (1,2), (1,3), (2,2), (3,1), (3,2), (2,3), \ldots\}
\]

and let \( \{B_m, m \in \mathbb{N}\} \) be the set of closed cubes \( B_k(x^i) \) with \( (k,i) \in \mathbb{N} \times \mathbb{N} \) and \( x^i \in Q^n \). For each \( x \in B_m, x = (x_1, x_2, \ldots, x_n) \), we define \( E_m(x) \) by:

\[
E_m(x) = (\xi^i_k(x_1), \xi^i_k(x_2), \ldots, \xi^i_k(x_n)).
\]

It is easy to show that, for \( m = (k,i) \),

\[
|E_m(x)| < 1, \quad x \in \prod_{j=1}^n I^i_k,
\]

\[
E_m(x) = 0, \quad x \notin \prod_{j=1}^n I^i_k.
\]

It is also easy to see that \( E_m(x) \) is in \( L^p[\mathbb{R}^n] = L^p[\mathbb{R}^n] \) for \( 1 \leq p \leq \infty \). Define \( F_m(\cdot) \) on \( L^p[\mathbb{R}^n] \) by

\[
F_m(f) = \int_{\mathbb{R}^n} E_m(x) \cdot f(x) d\lambda_n(x).
\]
It is clear that $F_m(\cdot)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each $m$ with $\|F_m\| \leq 1$. Furthermore, if $F_m(f) = 0$ for all $m$, $f = 0$ so that $\{F_m\}$ is a fundamental sequence of functionals on $L^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$.

Set $t_m = \frac{1}{2^m}$ so that $\sum_{m=1}^{\infty} t_m = 1$ and define an inner product $(\cdot, \cdot)$ on $L^1[\mathbb{R}^n]$ by

$$(f, g) = \sum_{m=1}^{\infty} t_m \left[ \int_{\mathbb{R}^n} \mathcal{E}_m(x) \cdot f(x) d\lambda_n(x) \right] \left[ \int_{\mathbb{R}^n} \mathcal{E}_m(y) \cdot g(y) d\lambda_n(y) \right].$$

The completion of $L^1[\mathbb{R}^n]$ with the above inner product is a Hilbert space, which we denote as $SD^2[\mathbb{R}^n]$. For our next theorem, we recall that $\mathcal{M}[\mathbb{R}^n]$ is the space of all (finite) complex measures on $\mathcal{B}[\mathbb{R}^n]$ that are absolutely continuous with respect to Lebesgue measure $\lambda_n$ and that, a sequence of measures $(\mu_j) \subset \mathcal{M}[\mathbb{R}^n]$, converges weakly to a measure $\mu \in \mathcal{M}[\mathbb{R}^n]$ if and only if, for every bounded continuous function $h$ on $\mathbb{R}^n$, $h \in C(\mathbb{R}^n)$, $\lim_{j \to \infty} \left| \int_{\mathbb{R}^n} h(x) d\mu_j(x) - \int_{\mathbb{R}^n} h(x) d\mu(x) \right| = 0$. The proofs of the following theorem is the same as for the $K^p[\mathbb{R}^n]$ spaces, so we omit them.

**Theorem 4.2.** For each $p$, $1 \leq p \leq \infty$, we have:

1. The space $SD^2[\mathbb{R}^n] \supset L^p[\mathbb{R}^n]$ as a continuous, dense and compact embedding.

2. The space $SD^2[\mathbb{R}^n] \supset \mathcal{M}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, as a continuous dense and compact embedding.

**Definition 4.3.** We call $SD^2[\mathbb{R}^n]$ the Jones strong distribution Hilbert space on $\mathbb{R}^n$.

In order to justify our definition, let $\alpha$ be a multi-index of nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, with $|\alpha| = \sum_{j=1}^{k} \alpha_j$. If $D$ denotes the standard partial differential operator, let $D^\alpha = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_k}$. 
Theorem 4.4. Let $\mathcal{D}[\mathbb{R}^n]$ be $C^\infty_c[\mathbb{R}^n]$ equipped with the standard locally convex topology (test functions).

(1) If $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, then $\phi_j \to \phi$ in the norm topology of $SD^2[\mathbb{R}^n]$, so that $\mathcal{D}[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]$ as a continuous dense embedding.

(2) If $T \in \mathcal{D}'[\mathbb{R}^n]$, then $T \in SD^2[\mathbb{R}^n]'$, so that $\mathcal{D}'[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]'$ as a continuous dense embedding.

(3) For any $f, g \in SD^2[\mathbb{R}^n]$ and any multi-index $\alpha$, $(D^\alpha f, g)_{SD} = (-i)^{\alpha} (f, g)_{SD}$.

Proof. The proofs of (1) and (2) are easy. To prove (3), we use the fact that each $E_m \in C^\infty_c[\mathbb{R}^n]$. Thus, for any $f \in SD^2[\mathbb{R}^n]$ we have:

$$\int_{\mathbb{R}^n} E_m(x) \cdot D^\alpha f(x) d\lambda_n(x) = (-1)^{\vert \alpha \vert} \int_{\mathbb{R}^n} D^\alpha E_m(x) \cdot f(x) d\lambda_n(x).$$

An easy calculation shows that:

$$(-1)^{\vert \alpha \vert} \int_{\mathbb{R}^n} D^\alpha E_m(x) \cdot f(x) d\lambda_n(x) = (-i)^{\vert \alpha \vert} \int_{\mathbb{R}^n} E_m(x) \cdot f(x) d\lambda_n(x).$$

It now follows that, for any $g \in SD^2[\mathbb{R}^n]$, $(D^\alpha f, g)_{SD} = (-i)^{\alpha} (f, g)_{SD^2}$. □

4.3. Functions of Bounded Variation. The objective of this section is to show that every HK-integrable function is in $SD^2[\mathbb{R}^n]$. To do this, we need to discuss a certain class of functions of bounded variation. For functions defined on $\mathbb{R}$, the definition of bounded variation is unique. However, for functions on $\mathbb{R}^n$, $n \geq 2$, there are a number of distinct definitions.

The functions of bounded variation in the sense of Cesari are well known to analysts working in partial differential equations and geometric measure theory (see Leoni [GL]).
Definition 4.5. A function $f \in L^1[\mathbb{R}^n]$ is said to be of bounded variation in the sense of Cesari or $f \in BV_c[\mathbb{R}^n]$, if $f \in L^1[\mathbb{R}^n]$ and each $i$, $1 \leq i \leq n$, there exists a signed Radon measure $\mu_i$, such that
\[
\int_{\mathbb{R}^n} f(x) \frac{\partial \phi(x)}{\partial x_i} d\lambda_n(x) = -\int_{\mathbb{R}^n} \phi(x) d\mu_i(x),
\]
for all $\phi \in C^\infty_0[\mathbb{R}^n]$.

The functions of bounded variation in the sense of Vitali [TY1], are well known to applied mathematicians and engineers with interest in error estimates associated with research in control theory, financial derivatives, high speed networks, robotics and in the calculation of certain integrals. (See, for example [KAA], [NI], [PT] or [PTR] and references therein.) For the general definition, see Yeong ([TY1], p. 175). We present a definition that is sufficient for continuously differentiable functions.

Definition 4.6. A function $f$ with continuous partials is said to be of bounded variation in the sense of Vitali or $f \in BV_v[\mathbb{R}^n]$ if for all intervals $(a_i, b_i)$, $1 \leq i \leq n$,
\[
V(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left| \frac{\partial^n f(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} \right| d\lambda_n(x) < \infty.
\]

Definition 4.7. We define $BV_{v,0}[\mathbb{R}^n]$ by:
\[
BV_{v,0}[\mathbb{R}^n] = \{ f(x) \in BV_v[\mathbb{R}^n] : f(x) \to 0, \text{ as } x_i \to -\infty \},
\]
where $x_i$ is any component of $x$.

The following two theorems may be found in [TY1]. (See p. 184 and 187, where the first is used to prove the second.) Recall that, if $[a_i, b_i] \subset \bar{\mathbb{R}} = [-\infty, \infty]$, we
define $[a, b] \in \mathbb{R}^n$ by $[a, b] = \prod_{k=1}^n [a_i, b_i]$. (The notation $(RS)$ means Riemann-Stieltjes.)

**Theorem 4.8.** Let $f$ be HK-integrable on $[a, b]$ and let $g \in BV_{v,0}[\mathbb{R}^n]$, then $fg$ is HK-integrable and

$$\left(\text{HK}\right) \int_{[a,b]} f(x)g(x)d\lambda_n(x) = (RS) \int_{[a,b]} \left\{ \left(\text{HK}\right) \int_{[x,x']} f(y)d\lambda_n(y) \right\} dg(x)$$

**Theorem 4.9.** Let $f$ be HK-integrable on $[a, b]$ and let $g \in BV_{v,0}[\mathbb{R}^n]$, then $fg$ is HK-integrable and

$$\left| \left(\text{HK}\right) \int_{[a,b]} f(x)g(x)d\lambda_n(x) \right| \leq \|f\|_{A_n} V_{[a,b]}(g)$$

**Lemma 4.10.** The space $HK[\mathbb{R}^n]$, of all HK-integrable functions is contained in $SD^2[\mathbb{R}^n]$. 

**Proof.** Since each $\mathcal{E}_m(x)$ is continuous and differentiable, $\mathcal{E}_m(x) \in BV_{v,0}[\mathbb{R}^n]$, so that for $f \in HK[\mathbb{R}^n]$,

$$\|f\|^2_{SD^2} = \sum_{m=1}^{\infty} t_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(x) \cdot f(x)dx \right|^2 \leq \sup_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(x) \cdot f(x)dx \right|^2 \leq \|f\|_{A_n}^2 \left( \sup V(\mathcal{E}_m) \right)^2 < \infty.$$

It follows that $f \in SD^2[\mathbb{R}^n]$. \hfill $\square$

### 4.4. The General Case, $SD^p$, $1 \leq p \leq \infty$

To construct $SD^p[\mathbb{R}^n]$ for all $p$ and for $f \in L^p$, define:

$$\|f\|_{SD^p} = \begin{cases} \left\{ \sum_{m=1}^{\infty} t_m \left| \int_{\mathbb{R}^n} \mathcal{E}_m(x) \cdot f(x)d\lambda_n(x) \right|^p \right\}^{1/p}, & 1 \leq p < \infty, \\ \sup_{m \geq 1} \left| \int_{\mathbb{R}^n} \mathcal{E}_m(x) \cdot f(x)d\lambda_n(x) \right|, & p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{SD^p}$ defines a norm on $L^p$. If $SD^p$ is the completion of $L^p$ with respect to this norm, we have:
**Theorem 4.11.** For each $q$, $1 \leq q \leq \infty$, $SD^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a continuous dense embedding.

**Theorem 4.12.** For $SD^p$, $1 \leq p \leq \infty$, we have:

1. If $p^{-1} + q^{-1} = 1$, then the dual space of $SD^p[\mathbb{R}^n]$ is $SD^q[\mathbb{R}^n]$.

2. For all $f \in SD^p[\mathbb{R}^n], g \in SD^q[\mathbb{R}^n]$ and all multi-index $\alpha$, $\langle D^{\alpha}f, g \rangle = (-i)^{|\alpha|} \langle f, g \rangle$.

3. The test function space $\mathcal{D}[\mathbb{R}^n]$ is contained in $SD^p[\mathbb{R}^n]$ as a continuous dense embedding.

4. If $K$ is a weakly compact subset of $L^p[\mathbb{R}^n]$, it is a strongly compact subset of $SD^p[\mathbb{R}^n]$.

5. The space $SD^\infty[\mathbb{R}^n] \subset SD^p[\mathbb{R}^n]$.

**Remark 4.13.** The fact that the families $KS^p[\mathbb{R}^n]$ and $SD^p[\mathbb{R}^n]$ are separable and contain $L^\infty$ as a continuous dense compact embedding, makes it clear that the relationship between analysis and topology is not as straightforward as one would expect from past history. Thus, from an analysis point of view they are big, but from a topological point of view they are relatively small (separable).

5. **Zachary Spaces**

In this section, we discuss one new space and two other families of spaces that naturally flow from the existence of a Banach space structure for functions with a bounded integral.

5.1. **Functions of Bounded and Weak Bounded Mean Oscillation.** In this section, we first define the functions of bounded mean oscillation (BMO) based
on the sharp maximal function ($M^\#$). In the following section, we define a weak maximal function ($M^w$) and use it construct the space of functions $BMO^w$, which extends $BMO$ to include the functions with a bounded integral.

5.1.1. Sharp maximal function and $BMO$.

**Definition 5.1.** Let $f \in L^1_{\text{loc}}[\mathbb{R}^n]$ and let $Q$ be a cube in $\mathbb{R}^n$.

1. We define the average of $f$ over $Q$ by

$$\text{Avg}_Q f = \frac{1}{\lambda_n(Q)} \int_Q f(y) d\lambda_n(y).$$

2. We defined the sharp maximal function $M^\#(f)(x)$, by

$$M^\#(f)(x) = \sup_Q \frac{1}{\lambda_n(Q)} \int_Q \left| f(y) - \text{Avg}_Q f \right| d\lambda_n(y),$$

where the supremum is over all cubes containing $x$.

3. If $M^\#(f)(x) \in L^\infty[\mathbb{R}^n]$, we say that $f$ is of bounded mean oscillation. More precisely, the space of functions of bounded mean oscillation are defined by:

$$BMO[\mathbb{R}^n] = \{ f \in L^1_{\text{loc}}[\mathbb{R}^n] : M^\#(f) \in L^\infty[\mathbb{R}^n] \}$$

and

$$\|f\|_{BMO} = \|M^\#(f)\|_{L^\infty}.$$
5.1.2. Weak maximal function and $\text{BMO}^w$.

**Definition 5.2.** Let $f \in L^1_{\text{loc}}[\mathbb{R}^n]$ and let $Q$ be a cube in $\mathbb{R}^n$.

1. We define $f_{aQ}$ over $Q$ by
   
   $f_{aQ} = \left| \frac{1}{\lambda_n(Q)} \int_Q f(y) d\lambda_n(y) \right| = \left| \text{Avg}_Q f \right|.$

2. We defined the weak maximal function $M^w(f)(x)$, by
   
   $M^w(f)(x) = \sup_Q \left| \frac{1}{\lambda_n(Q)} \int_Q [f(y) - f_{aQ}] d\lambda_n(y) \right|,$

   where the supremum is over all cubes containing $x$.

3. If $M^w(f)(x) \in L^\infty[\mathbb{R}^n]$, we say that $f$ is of weak bounded mean oscillation.

We define $BM$ by

$BM = \{ f(x) \in L^1_{\text{loc}}[\mathbb{R}^n] : M^w(f)(x) \in L^\infty[\mathbb{R}^n] \}$

and define a seminorm on $BM$ by

$\| f \|_{\text{BMO}^w} = \| M^w(f) \|_\infty.$

**Definition 5.3.** We define $BMO^w[\mathbb{R}^n]$ to be the completion of $BM$ in the seminorm $\| \cdot \|_{\text{BMO}^w}$.

**Remark 5.4.** If $\| f \|_{\text{BMO}^w} = 0$, then for every cube $Q$ containing $x$,

$\frac{1}{\lambda_n(Q)} \int_Q f(y) d\lambda_n(y) - f_{aQ} = 0 \Rightarrow \frac{1}{\lambda_n(Q)} \int_Q f(y) d\lambda_n(y) = \left| \frac{1}{\lambda_n(Q)} \int_Q f(y) d\lambda_n(y) \right|.$

Since $a = |a|$ if and only if $a \geq 0$, we see that $f(x)$ is a nonnegative constant (a.e).

It follows that $BMO^w[\mathbb{R}^n]$ is not a Banach space, but becomes one if we identify terms that differ by a nonnegative constant.
Theorem 5.5. The space $BMO[\mathbb{R}^n] \subset BMO^w[\mathbb{R}^n]$ as a continuous dense embedding.

Proof. It is easy to see that $BMO[\mathbb{R}^n]$ is a dense subset. To prove that its a continuous embedding, we note that

$$M^w(f)(x) = \sup_Q \frac{1}{\lambda_n(Q)} \left| \int_Q f(y) - \frac{1}{\lambda_n(Q)} \int_Q f(y) \, d\lambda_n(y) \right| \leq \sup_Q \frac{1}{\lambda_n(Q)} \int_Q \left| f(y) - \frac{1}{\lambda_n(Q)} \int_Q f(y) \, d\lambda_n(y) \right| = M^w(f)(x).$$

□

Corollary 5.6. The space $BMO^w[\mathbb{R}^n] \subset KS^\infty[\mathbb{R}^n]$ as a continuous dense embedding.

Proof. The proof is easy since $L^1_{loc}[\mathbb{R}^n] \cup L^\infty[\mathbb{R}^n] \subset KS^\infty[\mathbb{R}^n]$, with $L^\infty[\mathbb{R}^n]$ dense and

$$\|M^w(f)\|_{KS^\infty} = \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(x) M^w(f)(x) \, d\lambda_n(x) \right| \leq \left[ \frac{1}{2\sqrt{n}} \right] \left[ \left. \|M^w(f)\|_{L^\infty} = \left[ \frac{1}{2\sqrt{n}} \right] \|f\|_{M^w} \right. \right].$$

□

5.2. Zachary Functions of Bounded Mean Oscillation. We now construct another class of functions. Let the family of cubes $\{Q_k\}$ centered at each rational point in $\mathbb{R}^n$ be the ones generated by the indicator functions $\{\mathcal{E}_k(x)\}$, for $KS^2[\mathbb{R}^n]$.

Let $f \in L^1_{loc}[\mathbb{R}^n]$ and as before, we define $f_{ak}$ by

$$f_{ak} = \frac{1}{\lambda_n(Q_k)} \int_{Q_k} f(y) \, d\lambda_n(y) = \frac{1}{\lambda_n(Q_k)} \int_{\mathbb{R}^n} \mathcal{E}_k(y) f(y) \, d\lambda_n(y).$$
Definition 5.7. If \( p, 1 \leq p < \infty \) and \( t_k = 2^{-k} \), we define \( \|f\|_{Z^p} \) by
\[
\|f\|_{Z^p} = \left\{ \sum_{k=1}^{\infty} t_k \left| \frac{1}{\lambda_n(Q_k)} \int_{Q_k} [f(y) - f_{ak}] d\lambda_n(y) \right|^p \right\}^{1/p}.
\]
The set of functions for which \( \|f\|_{Z^p} < \infty \) is called the Zachary functions of bounded mean oscillation and order \( p, 1 \leq p < \infty \). If \( p = \infty \), we say that \( f \in Z^\infty[\mathbb{R}^n] \) if
\[
\|f\|_{Z^\infty} = \sup_k \left| \frac{1}{\lambda_n(Q_k)} \int_{Q_k} [f(y) - f_{ak}] d\lambda_n(y) \right| < \infty.
\]

The following theorem shows how the Zachary spaces are related to the space of functions of Bounded mean oscillation \( BMO[\mathbb{R}^n] \). (We omit proofs.)

Theorem 5.8. If \( Z^p[\mathbb{R}^n] \) is the class of Zachary functions of bounded mean oscillation and order \( p, 1 \leq p \leq \infty \), then \( Z^p[\mathbb{R}^n] \) is a linear space and

1. \( \|\lambda f\|_{Z^p} = |\lambda| \|f\|_{Z^p} \).
2. \( \|f + g\|_{Z^p} \leq \|f\|_{Z^p} + \|g\|_{Z^p} \).
3. \( \|f\|_{Z^p} = 0 \Rightarrow f \geq 0 \) is a nonnegative constant \((a.e.)\).
4. The space \( Z^\infty[\mathbb{R}^n] \subset Z^p[\mathbb{R}^n] \), \( 1 \leq p < \infty \), as a dense continuous embedding.
5. \( BMO^w[\mathbb{R}^n] \subset Z^\infty[\mathbb{R}^n] \), as a dense continuous embedding.

Proof. The first four are clear. To prove (5), suppose that \( f \in BMO^w[\mathbb{R}^n] \), then by definition of \( \|\cdot\|_{BMO^w} \), the supremum is over the set of all cubes in \( \mathbb{R}^n \). Since this set is much larger than the countable number used to define \( \|\cdot\|_{Z^\infty} \). It follows that \( BMO^w[\mathbb{R}^n] \subset Z^\infty[\mathbb{R}^n] \) as a dense continuous embedding. \( \square \)

We now consider the Carleson measure characterization of \( BMO[\mathbb{R}^n] \) which will prove useful in constructing another class of Zachary spaces that are Banach spaces.
(see Koch and Tataru [KT]). If \( u(x, t) \) is a solution of the heat equation:

\[
    u_t - \Delta u = 0, \quad u(x, 0) = f(x),
\]

where \( f \in L^1_{loc}([\mathbb{R}^n]) \), it can be shown that

\[
    \| f \|_{BMO} = \sup_{x, r} \left\{ \frac{1}{\lambda_n [Q(x, r)]} \int_{Q(x, r)} \int_0^{r^2} |\nabla u(y, s)|^2 ds \lambda_n(y) \right\}^{1/2},
\]

where the gradient is in the weak sense. Since

\[
    \sup_{k, r} \left\{ \frac{1}{\lambda_n [Q_k]} \left| \int_{Q_k} \int_0^{r^2} \nabla u(y, s) ds \lambda_n(y) \right|^2 \right\}^{1/2} \leq \sup_{x, r} \left\{ \frac{1}{\lambda_n [Q(x, r)]} \int_{Q(x, r)} \int_0^{r^2} |\nabla u(y, s)|^2 ds \lambda_n(y) \right\}^{1/2},
\]

we see that we can also define the seminorms on \( BMO^w([\mathbb{R}^n]) \) and \( Z^p([\mathbb{R}^n]) \) by:

\[
    \| f \|_{BMO^w} = \sup_{x, r} \left\{ \frac{1}{\lambda_n [Q(x, r)]} \left| \int_{Q(x, r)} \int_0^{r^2} \nabla u(y, s) ds \lambda_n(y) \right|^2 \right\}^{1/2},
\]

\[
    \| f \|_{Z^p} = \sup_{r} \left\{ \sum_{k=1}^\infty t_k \frac{1}{\lambda_n [Q_k]} \left| \int_{Q_k} \int_0^{r^2} u(y, s) ds \lambda_n(y) \right|^p \right\}^{1/p}.
\]

### 5.3. The Space of Functions \( BMO^{-1}([\mathbb{R}^n]) \)

We define the class of functions \( BMO^{-1}([\mathbb{R}^n]) \), as those for which:

\[
    \| f \|_{BMO^{-1}} = \sup_{x, r} \left\{ \frac{1}{\lambda_n [Q(x, r)]} \int_{Q(x, r)} \int_0^{r^2} |u(y, s)|^2 ds \lambda_n(y) \right\}^{1/2} < \infty.
\]

It is known that \( BMO^{-1}([\mathbb{R}^n]) \) is a Banach space in the above norm.

**Definition 5.9.** We say \( f \in Z^{-p}([\mathbb{R}^n]) \), \( 1 \leq p < \infty \) if

\[
    \| f \|_{Z^{-p}} = \sup_{r} \left\{ \sum_{k=1}^\infty t_k \frac{1}{\lambda_n [Q_k]} \left| \int_{Q_k} \int_0^{r^2} u(y, s) ds \lambda_n(y) \right|^p \right\}^{1/p} < \infty.
\]
If \( p = \infty \), we say that \( f \in \mathcal{Z}^{-\infty}[\mathbb{R}^n] \) if

\[
\|f\|_{\mathcal{Z}^{-\infty}} = \sup_{k, r} \frac{1}{\lambda_n(Q_k)} \left| \int_{Q_k} \int_0^{r^2} u(y, s) dsd\lambda_n(y) \right| < \infty.
\]

**Theorem 5.10.** For the class of spaces \( \mathcal{Z}^{-p}[\mathbb{R}^n] \), we have:

1. For each \( 1 \leq p \leq \infty \), \( \mathcal{Z}^{-p}[\mathbb{R}^n] \) is a Banach space.
2. The space \( \mathcal{Z}^{-\infty}[\mathbb{R}^n] \subset \mathcal{Z}^{-p}[\mathbb{R}^n] \), \( 1 \leq p < \infty \), as a dense continuous embedding.
3. The space \( BMO^{-1}[\mathbb{R}^n] \subset \mathcal{Z}^{-\infty}[\mathbb{R}^n] \) as a dense continuous embedding.

**Proof.** The first two are obvious. To prove (3), if \( f \in BMO^{-1}[\mathbb{R}^n] \), then

\[
\|f\|_{\mathcal{Z}^{-\infty}} = \sup_{k, r} \frac{1}{\lambda_n(Q_k)} \left| \int_{Q_k} \int_0^{r^2} u(y, s) dsd\lambda_n(y) \right| \\
= \sup_{k, r} \frac{1}{\lambda_n(Q_k)} \left( \int_{Q_k} \int_0^{r^2} u(y, s) dsd\lambda_n(y) \right)^{1/2} \\
\leq \sup_{x, r} \frac{1}{\lambda_n(Q(x,r))} \left\{ \int_{Q(x,r)} \int_0^{r^2} |u(y, s)|^2 dsd\lambda_n(y) \right\}^{1/2} = \|f\|_{BMO^{-1}}.
\]

\[ \square \]

**Remark 5.11.** We could also define \( BMO^{-w}[\mathbb{R}^n] \) by:

\[
\|f\|_{BMO^{-w}} = \sup_{x, r} \left\{ \frac{1}{\lambda_n(Q(x,r))} \left| \int_{Q(x,r)} \int_0^{r^2} u(y, s) dsd\lambda_n(y) \right|^2 \right\}^{1/2}.
\]

It is easy to see that \( BMO^{-w}[\mathbb{R}^n] \) is a Banach space and that \( BMO^{-1}[\mathbb{R}^n] \subset BMO^{-w}[\mathbb{R}^n] \subset \mathcal{Z}^{-\infty}[\mathbb{R}^n] \) as a continuous embeddings. We conjecture that \( BMO^{-w}[\mathbb{R}^n] \) and \( \mathcal{Z}^{-\infty}[\mathbb{R}^n] \) allow us to replace solutions of the heat equation by those of the Schrödinger equation: \( iu_t - \Delta u = 0, \ u(x,0) = f(x) \).
6. Applications

In this section, we consider a few applications associated with the families $KS^p[\mathbb{R}^n]$ and $SD^p[\mathbb{R}^n]$, $1 \leq p \leq \infty$. In each case, we either solve an open problem or provide a substantial improvement in methods used in a given area.

6.1. Markov Processes. In the study of Markov processes, it is well-known that semigroups associated with processes whose generators have unbounded coefficients, are not necessarily strongly continuous when defined on $C_b[\mathbb{R}^n]$, the space of bounded continuous functions, or $UBC[\mathbb{R}^n]$, the bounded uniformly continuous functions. These are the natural spaces on which to formulate the theory. The problem is that the generator of such a semigroup does not exist in the standard sense (is not $C_0$). As a consequence, a number of equivalent weaker definitions have been developed in the literature. A good discussion of this and related problems see Lorenzi and Bertoldi [LB]. The following is one version of convergence used to define semigroups using the generator in these cases.

**Definition 6.1.** A sequence of functions $\{f_k\}$ in $C_b[\mathbb{R}^n]$ is said to converge to $f$ in the mixed topology, written $\tau^M\text{-lim} f_k = f$, if and only if $\sup_{k \in \mathbb{N}} \|f_k\|_\infty \leq M$ and $\|f_k - f\|_\infty \to 0$ uniformly on every compact subset of $\mathbb{R}^n$.

It is clear that the family of bounded continuous functions, $C_b[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ as a continuous dense embedding.

**Theorem 6.2.** If $\{f_k\}$ converges to $f$ in the mixed topology on $C_b[\mathbb{R}^n]$, then $\{f_n\}$ converges to $f$ in the norm topology of $KS^p[\mathbb{R}^n]$ for each $1 \leq p \leq \infty$. 
Proof. It suffices to prove the result for $K S^\infty$. Since $\sup_{k \in \mathbb{N}} \|f_k\|_{K S^\infty} \leq \sup_{k \in \mathbb{N}} \|f_k\|_{C_b}$, we must prove that $\tau^M$-$\lim f_k = f \Rightarrow \lim_{k \to \infty} \|f_k - f\|_{K S^\infty} = 0$. This follows from the fact that each cube used in the definition of the $K S^\infty[\mathbb{R}^n]$ norm, is a compact subset of $\mathbb{R}^n$. □

**Theorem 6.3.** Suppose that $\hat{T}(t)$ is a transition semigroup defined on $C_b[\mathbb{R}^n]$, with weak generator $\hat{A}$. Let $T(t)$ be the extension of $\hat{T}(t)$ to $K S^p[\mathbb{R}^n]$. Then $T(t)$ is strongly continuous, and the extension $A$ of $\hat{A}$ to $K S^p[\mathbb{R}^n]$ is the strong generator of $T(t)$.

Proof. Since $C_b[\mathbb{R}^n] \subset K S^p[\mathbb{R}^n]$ as a continuous embedding, for $1 \leq p \leq \infty$, we can apply Corollary 2.6 to show that $\hat{T}(t)$ has a bounded extension to $K S^2[\mathbb{R}^n]$. It is easy to see that the extended operator $T(t)$ is a semigroup. Since the $\tau^M$ topology on $C_b[\mathbb{R}^n]$ is stronger than the norm topology on $K S^p[\mathbb{R}^n]$, we see that the generator $A$, of $T(t)$ is strong. □

### 6.2. Feynman Path Integral

Feynman’s introduction of his path integral into quantum theory created a major mathematical problem, that centered around the nonexistence of a measure for his integral. A number of analysts, beginning with Henstock [HS] have advocated the HK-integral as a perfect substitute for this problem (see also Muldowney [MD]). Since then, a number of researchers have addressed the problem. A fairly complete list of papers and books can be found in Gill and Zachary [GZ]. The book by Johnson and Lapidus [JL] also contains additional sources.

The space $L^2[\mathbb{R}^n]$ is perfect for the Heisenberg and Schrödinger formulations of quantum mechanics, but fails for the Feynman formulation. In addition, neither the
physically intuitive nor computationally efficient methods of Feynman are revealed on $L^2[\mathbb{R}^n]$. In this section we briefly show that $KS^2[\mathbb{R}^n]$ is the natural Hilbert space for the Feynman formulation of quantum mechanics. This space makes it possible to preserve all the physically intuitive and computational advantages discovered by Feynman and to represent the Heisenberg and Schrödinger formulations.

We assume that the reader is familiar with the HK-integral, but give a brief discussion in one dimension to establish notation. (A full discussion with proofs and some interesting examples can be found in Gill and Zachary [GZ].)

**Definition 6.4.** Let $[a, b] \subset \mathbb{R}$, let $\delta(t)$ map $[a, b] \to (0, \infty)$, and let $P = \{t_0, \tau_1, t_1, \tau_2, \ldots, \tau_n, t_n\}$, where $a = t_0 \leq \tau_1 \leq t_1 \leq \cdots \leq \tau_n \leq t_n = b$. We call $P$ an HK-partition for $\delta$, if for $1 \leq i \leq n$, $t_{i-1}, t_i \in (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$.

**Definition 6.5.** The function $f(t), t \in [a, b]$, is said to have a HK-integral if there is a number $F[a, b]$ such that, for each $\varepsilon > 0$, there exists a function $\delta$ from $[a, b] \to (0, \infty)$ such that, whenever $P$ is a HK-partition for $\delta$, then (with $\Delta t_i = t_i - t_{i-1}$)

$$\left| \sum_{i=1}^{n} \Delta t_i f(\tau_i) - F[a, b] \right| < \varepsilon.$$

To understand Feynman’s path integral in a natural setting, we consider the free particle in non-relativistic quantum theory in $\mathbb{R}^3$:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} - \frac{\hbar^2}{2m} \Delta \psi(x, t) = 0, \quad \psi(x, s) = \delta(x - y).$$

The solution can be computed directly:

$$\psi(x, t) = K[x, t; y, s] = \left( \frac{2\pi i \hbar (t - s)}{m} \right)^{-3/2} \exp \left( \frac{im |x - y|^2}{2\hbar (t - s)} \right).$$
Feynman wrote the above solution to equation (6.1) as

\[ K[x(t), t; y(s), s] = \int_{x(s)=y} \mathcal{D}x(\tau) \exp \left\{ \frac{im}{2\hbar} \int_{t}^{s} \left| \frac{dx}{d\tau} \right|^2 d\tau \right\}, \]

where

\[ \lim_{N \to \infty} \left[ \frac{m}{2\pi i \varepsilon(N)} \right]^{3N/2} \int_{R^3} \prod_{j=1}^{N} d\lambda_{3}(y) \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2\pi i \varepsilon(N)} (x_j \cdot x_{j-1})^2 \right] \right\}, \]

with \( \varepsilon(N) = (t - s)/N \). Equation (6.3) is an attempt to define an integral over the space of all continuous paths of the exponential of an integral of the classical Lagrangian on configuration space. This approach has led to a new approach for quantizing physical systems, called the path integral method.

Since \( L^2[\mathbb{R}^3] \) is the standard state space for quantum physics, from a strictly mathematical point of view equation (6.3) has two major problems:

1. The kernel \( K[x, t; y, s] \) and \( \delta(x) \) are not in \( L^2[\mathbb{R}^3] \).
2. The kernel \( K[x, t; y, s] \) cannot be used to define a measure.

Since \( KS^2[\mathbb{R}^3] \) contains the space of measures \( \mathcal{M}[\mathbb{R}^3] \), it follows that all the approximating sequences for the Dirac measure converge strongly to it in the \( KS^2[\mathbb{R}^n] \) topology. (For example, \( [\sin(\lambda \cdot x)/(\lambda \cdot x)] \in KS^2[\mathbb{R}^n] \) and converges strongly to \( \delta(x) \).) Thus, the finitely additive set function defined on the Borel sets (Feynman kernel \( \mathcal{K}_{t} \)): (with \( m = 1 \) and \( \hbar = 1 \))

\[ \mathcal{K}_{t}[t, x; s, B] = \int_{B} (2\pi i(t - s))^{-n/2} \exp\{i|x - y|^2/2(t - s)\} d\lambda_3(y) \]
is in $KS^2[\mathbb{R}^n]$ and $\|\mathbb{K}_f[t, x; s, B]\|_{K_S} \leq 1$, while $\|\mathbb{K}_f[t, x; s, B]\|_{\mathbb{M}} = \infty$ (the total variation norm) and

$$\mathbb{K}_f[t, x; s, B] = \int_{\mathbb{R}^3} \mathbb{K}_f[t, x; \tau, d\lambda_3(z)]\mathbb{K}_f[\tau, z; s, B], \text{ (HK-integral)}. $$

**Definition 6.6.** Let $P_n = \{t_0, \tau_1, t_1, \tau_2, \ldots, \tau_n, t_n\}$ be a HK-partition for a function $\delta_n(s), s \in [0, t]$ for each $n$, with $\lim_{n \to \infty} \Delta \mu_n = 0$ (mesh). Set $\Delta t_j = t_j - t_{j-1}, \tau_0 = 0$ and, for $\psi \in KS^2[\mathbb{R}^n]$, define

$$\int_{\mathbb{R}^n[0, t]} \mathbb{K}_f[\mathcal{D}_x(\tau); x(0)] = e^{-\lambda t} \sum_{k=0}^{[\lambda t]} \frac{\lambda^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbb{R}^n} \mathbb{K}_f[t_j, x(\tau_j); t_{j-1}, d\lambda_3(\tau_{j-1})] \right\},$$

and

$$(6.4) \int_{\mathbb{R}^n[0, t]} \mathbb{K}_f[\mathcal{D}_x(\tau); x(0)]\psi[x(0)] = \lim_{\lambda \to \infty} \int_{\mathbb{R}^n[0, t]} \mathbb{K}_f[\mathcal{D}_x(\tau); x(0)]\psi[x(0)]$$

does not exist.

**Remark 6.7.** In the above definition we have used the Poisson process. This is not accidental but appears naturally from a physical analysis of the information that is knowable in the micro-world (see [GZ]). This approach also provides the mathematical foundations for Feynman’s sum over all paths theory (see Section 7.7 in [GZ]). It has been suggested by Kolokoltsov [KO] that such jump processes provide another way to give meaning to Feynman diagrams.

By Corollary 2.8 (with $\mathcal{B} = L^2[\mathbb{R}^n]$), $KS^2[\mathbb{R}^n]$ is closed under convolution, so that the following is elementary.

**Theorem 6.8.** The function $\psi(x) = 1 \in KS^2[\mathbb{R}^n]$ and

$$\int_{\mathbb{R}^n[0, t]} \mathbb{K}_f[\mathcal{D}_x(\tau); x(s)] = \mathbb{K}_f[t, x; s, y] = \frac{1}{\sqrt{[2\pi(t-s)]^n}} \exp\{i|x - y|^2 / 2(t - s)\}.$$
Remark 6.9. The above result is what Feynman was trying to obtain without the appropriate space. When a potential is present, one uses the standard perturbation methods (i.e., Trotter-Kato theorems). A more general (sum over paths) result, that covers all application areas can be found in [GZ].

It is clear that the position operator $x$, and the momentum operator $p$ have closed densely defined extensions to $KS^2$. Treating the Fourier transform as an unitary operator, it has a bounded (unitary) extension to $KS^2$ so that $x$ and $p$ are still canonically conjugate pairs. Thus, both the Heisenberg and Schrödinger theories also have natural formulations on $KS^2$. It is in this sense that we say that $KS^2$ is the most natural Hilbert space for quantum mechanics.

If we replace the Feynman kernel by the heat kernel, we have:

$$K_h[t, x; s, B] = \int_B (2\pi(t-s))^{-n/2} \exp\left\{-|x-y|^2/2(t-s)\right\} d\lambda_n(y)$$

is in $KS^2[\mathbb{R}^n]$ and $\|K_h[t, x; s, \mathbb{R}^n]\|_{KS} = 1$ and

$$K_h[t, x; s, B] = \int_{\mathbb{R}^n} K_h[t, x; \tau, d\lambda_n(z)] K_h[\tau, z; s, B], \text{ (HK-integral).}$$

Theorem 6.10. For the function $\psi(x) \equiv 1 \in KS^2[\mathbb{R}^n]$, we have

$$\int_{\mathbb{R}^n[s,t]} K_h[Dx(\tau); x(s)] = K_h[t, x; s, y] = \frac{1}{\sqrt{2\pi(t-s)^n}} \exp\left\{-|x-y|^2/2(t-s)\right\}.$$  

This result implies that all known results for the Weiner path integral also have extensions to $KS^2[\mathbb{R}^n]$, with initial data in $HK[\mathbb{R}^n]$. Furthermore, the strong continuity of the semigroup generating the heat equation means that the integral can still be concentrated on the space of continuous paths.
6.3. **Examples.** If we treat $K[x, t; y, s]$ as the kernel for an operator acting on good initial data, then a partial solution has been obtained by a number of workers. (See [GZ] for references to all the important contributions in this direction.) The standard method is to compute the Wiener path integral for the problem under consideration and then use analytic continuation in the mass to provide a rigorous meaning for the Feynman path integral. The standard reference is Johnson and Lapidus [JL].

The following example provides a path integral representation for a problem that cannot be solved using analytic continuation via a Gaussian kernel. (For the general non-Gaussian case, see [GZ].) It is shown that, if the vector $A$ is constant, $\mu = mc/\hbar$, and $\beta$ is the standard beta matrix of relativistic quantum theory, then the solution to the square-root equation for a spin 1/2 particle:

$$i\hbar \partial \psi(x, t)/\partial t = \left\{ \beta \sqrt{c^2 (p - eA)^2 + m^2 c^4} \right\} \psi(x, t), \psi(x, 0) = \psi_0(x),$$

is given by:

$$\psi(x, t) = U[t, 0] \psi_0(x) = \int_{\mathbb{R}^3} \exp \left\{ \frac{ie}{2\hbar c} (x - y) \cdot A \right\} K[x, t; y, 0] \psi_0(y) dy,$$

where

$$K[x, t; y, 0] = \frac{ict^2 \beta}{4\pi} \left\{ \begin{array}{ll}
-H_2^{(1)}(\mu(c^2t^2 - ||x - y||^2)^{1/2}) & , \text{ct} < -||x - y||, \\
-2iK_2(\mu(||x - y||^2 - c^2t^2)^{1/2}) & , \frac{ct}{||x - y||^2 - c^2t^2} \\
-H_2^{(2)}(\mu(c^2t^2 - ||x - y||^2)^{1/2}) & , \text{ct} > ||x - y||.
\end{array} \right.$$
free-particle Feynman kernel and, if we set $\mu = 0$, we get the kernel for a (spin 1/2) massless particle.

6.4. The Navier-Stokes Problem. In this section, we use $SD^2[\mathbb{R}^3]$ to provide the strongest possible a priori estimate for the nonlinear term of the classical Navier-Stokes equation.

6.4.1. Introduction. Let $[L^2(\mathbb{R}^3)]^3$ be the Hilbert space of square integrable functions on $\mathbb{R}^3$, let $\mathbb{H}[\mathbb{R}^3]$ be the completion of the set of functions in $\{u \in C^\infty_0[\mathbb{R}^3]^3 \mid \nabla \cdot u = 0\}$ which vanish at infinity with respect to the inner product of $[L^2(\mathbb{R}^3)]^3$. The classical Navier-Stokes initial-value problem (on $\mathbb{R}^3$ and all $T > 0$) is to find a function $u : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t) \text{ in } (0, T) \times \mathbb{R}^3,$$

$$\nabla \cdot u = 0 \text{ in } (0, T) \times \mathbb{R}^3 \text{ (in the weak sense),}$$

$$u(0, x) = u_0(x) \text{ in } \mathbb{R}^3.$$  

(6.5)

The equations describe the time evolution of the fluid velocity $u(x, t)$ and the pressure $p$ of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient $\nu$ in terms of a given initial velocity $u_0(x)$ and given external body forces $f(x, t)$.

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $L^2[\mathbb{R}^3]^3$ onto $\mathbb{H}[\mathbb{R}^3]$ and define the Stokes operator by: $Au \equiv -\mathbb{P} \Delta u$, for $u \in D(A) \subset \mathbb{H}^2[\mathbb{R}^3]$, the domain of $A$.

If we apply $\mathbb{P}$ to equation (6.5), with $B(u, u) = \mathbb{P}(u \cdot \nabla)u$, we can recast it into the
standard form:

$$\partial_t u = -\nu Au - B(u, u) + Pf(t) \text{ in } (0, T) \times \mathbb{R}^3,$$

(6.6)

$$u(0, x) = u_0(x) \text{ in } \mathbb{R}^3,$$

where the orthogonal complement of $\mathbb{H}$ relative to $\{L^2(\mathbb{R}^3)\}^3$, \{v : v = \nabla q, q \in H^1[\mathbb{R}^3]\}, is used to eliminate the pressure term (see [GA] or [SY], [T1], [T2]).

**Definition 6.11.** We say that a velocity vector field in $\mathbb{R}^3$ is **reasonable** if for $0 \leq t < \infty$, there is a continuous function $m(t) > 0$, depending only on $t$ and a constant $M_0$, which may depend on $u_0$ and $f$, such that

$$0 < m(t) \leq \|u(t)\|_{\mathbb{H}} \leq M_0.$$

The above definition formalizes the requirement that the fluid has nonzero but bounded positive definite energy. However, this condition still allows the velocity to approach zero at infinity in a weaker norm.

6.4.2. The Nonlinear Term: A Priori Estimates. The difficulty in proving the existence and uniqueness of global-in-time strong solutions for equation (6.6) is directly linked to the problem of getting good a priori estimates for the nonlinear term $B(u, u)$. For example, using standard methods on $\mathbb{H}$, the following estimates are known. If $u, v \in D(A)$, a typical bound in the $\mathbb{H}$ norm for the nonlinear term in equation (6.6) can be found in Sell and You [SY] (see page 366):

$$\max \{\|B(u, v)\|_{\mathbb{H}}, \|B(v, u)\|_{\mathbb{H}}\} \leq C_0 \left\| A^{5/8} u \right\|_{\mathbb{H}} \left\| A^{5/8} v \right\|_{\mathbb{H}}.$$

In this section, we show how $SD^2[\mathbb{R}^3]$ allows us to obtain the best possible a priori estimates. Let $\mathbb{H}_{sd}$ be the closure of $\mathbb{H} \cap SD^2[\mathbb{R}^3]$ in the $SD^2$ norm.
Theorem 6.12. Suppose that the reasonable vector field \( \mathbf{u}(x, t) \in SD^2[\mathbb{R}^3] \cap D(\mathbf{A}) \) satisfies (6.6) and \( \mathbf{A} \) is the Stokes operator. Then

\begin{enumerate}
    \item \begin{equation}
        \langle \nu \mathbf{A} \mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}_{sd}} = 3 \| \mathbf{u} \|_{\mathbb{H}_{sd}}^2.
    \end{equation}
    \end{enumerate}

\begin{enumerate}
    \item There exists a constants \( M_1, M_2; M_i = M_i(\mathbf{u}_0, \mathbf{f}) > 0 \), such that
    \begin{equation}
        |\langle \mathbf{B}(\mathbf{u}, \mathbf{u}), \mathbf{u} \rangle_{\mathbb{H}_{sd}}| \leq M_1 \| \mathbf{u} \|_{\mathbb{H}_{sd}}^3
    \end{equation}
    \end{enumerate}

\begin{enumerate}
    \item and
    \begin{equation}
        \max \{ \| \mathbf{B}(\mathbf{u}, \mathbf{v}) \|_{\mathbb{H}_{sd}}, \| \mathbf{B}(\mathbf{v}, \mathbf{u}) \|_{\mathbb{H}_{sd}} \} \leq M_2 \| \mathbf{u} \|_{\mathbb{H}_{sd}} \| \mathbf{v} \|_{\mathbb{H}_{sd}}.
    \end{equation}
\end{enumerate}

Proof. From the definition of the inner product, for \( \mathbb{H}_{sd}[\mathbb{R}^3] \) we have

\begin{equation}
    \langle \nu \mathbf{A} \mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}_{sd}} = \nu \sum_{m=1}^{\infty} t_m \left[ \int_{\mathbb{R}^3} \mathcal{E}_m(x) \cdot \mathbf{A} \mathbf{u}(x) d\lambda_3(x) \right] \left[ \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \mathbf{u}(y) d\lambda_3(y) \right].
\end{equation}

Using the fact that \( \mathbf{u} \in D(\mathbf{A}) \), it follows that

\begin{equation}
    \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \partial_y^2 \mathbf{u}(y) d\lambda_3(y) = \int_{\mathbb{R}^3} \partial_y^2 \mathcal{E}_m(y) \cdot \mathbf{u}(y) d\lambda_3(y) = (i)^2 \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \mathbf{u}(y) d\lambda_3(y).
\end{equation}

Using this in the above equation and summing on \( j \), we have (\( \mathbf{A} = -\mathcal{P} \Delta \))

\begin{equation}
    \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \mathbf{A} \mathbf{u}(y) d\lambda_3(y) = 3 \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \mathbf{u}(y) d\lambda_3(y).
\end{equation}

It follows that

\begin{equation}
    \langle \mathbf{A} \mathbf{u}, \mathbf{u} \rangle_{\mathbb{H}_{sd}}
    = 3 \sum_{m=1}^{\infty} t_m \left[ \int_{\mathbb{R}^3} \mathcal{E}_m(x) \cdot \mathbf{u}(x) d\lambda_3(x) \right] \left[ \int_{\mathbb{R}^3} \mathcal{E}_m(y) \cdot \mathbf{u}(y) d\lambda_3(y) \right]
    = 3 \| \mathbf{u} \|_{\mathbb{H}_{sd}}^2.
\end{equation}
This proves (6.7). To prove (6.8), let
\[
b(u, v, E_m) = \int_{\mathbb{R}^3} (u(x) \cdot \nabla v(x)) \cdot E_m(x) d\lambda_3(x)
\]
and define the vector \( I \) by \( I = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). We start with integration by parts and \( \nabla \cdot u = 0 \), to get
\[
b(u,v,E_m) = -b(u,E_m,v) = -i \int_{\mathbb{R}^3} (u(x) \cdot I) (E_m(x) \cdot v(x)) d\lambda_3(x).
\]
From the above equation, we have \((m \leftrightarrow (k,i))\)
\[
|b(u,v,E_m)| \leq \sqrt{3} \int_{\mathbb{R}^3} |u(x)||v(x)|d\lambda_3(x) \sup_k \|E_m\|_{\infty} \\
\leq C_1 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.
\]
We also have:
\[
\left| \int_{\mathbb{R}^3} w(x) \cdot E_m(x) d\lambda_3(x) \right| \leq C_2 \|w\|_{\mathcal{H}}.
\]
If we combine the last two results, we get that:
\[
|\langle B(u,v), w \rangle_{\mathcal{H}_{sd}}| \\
\leq \sum_{m=1}^{\infty} t_m |b(u,v,E_m)| \left| \int_{\mathbb{R}^3} w(y) \cdot E_m(y) d\lambda_3(y) \right| \\
\leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \|w\|_{\mathcal{H}}.
\]
Since \( u, v, w \) are reasonable velocity vector fields, there is a constant \( M_1 \) depending on \( u_0, v_0, w_0 \) and \( f \), such that
\[
(6.10) \quad C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \|w\|_{\mathcal{H}} \leq M_1 \|u\|_{\mathcal{H}_{sd}} \|v\|_{\mathcal{H}_{sd}} \|w\|_{\mathcal{H}_{sd}}.
\]
If \( w = v = u \), we have that:
\[
|\langle B(u,u), u \rangle_{\mathcal{H}_{sd}}| \leq M_1 \|u\|_{\mathcal{H}_{sd}}^3.
\]
This proves (6.8). The proof of (6.9) is a straightforward application of (6.10) and (6.11).

Conclusion

In this survey we have constructed a number of new classes of Banach spaces $KS^p$, $1 \leq p \leq \infty$, $SD^p$, $1 \leq p \leq \infty$, $Z^p$, $1 \leq p \leq \infty$, $Z^{-p}$, $1 \leq p \leq \infty$, $BMO^w$ and $BMO^{-1}$. These spaces are of particular interest because they contain the HK-integrable functions. (They also contain all functions that are integrable via any of the classical integrals.) The $KS^p$ and $SD^p$ class contain the standard $L^p$ spaces as dense compact embeddings. They also contain the test functions $D[\mathbb{R}^n]$, so that $D'[\mathbb{R}^n]$ is a continuous embedding into the corresponding dual space. The $SD^p$ class has the remarkable property that $\|D^\alpha f\|_{SD} = \|f\|_{SD}$, for every index $\alpha$.

The space $BMO^w$ and the families $Z^p$, $1 \leq p \leq \infty$ extend the space $BMO$ (i.e., functions of bounded mean oscillation). The space $BMO^{-w}$ and the families $Z^{-p}$, $1 \leq p \leq \infty$ extend the related space $BMO^{-1}$.

In the analytical theory of Markov processes, it is well-known that, in general, the semigroup $T(t)$ associated with the process is not strongly continuous on $C_b[\mathbb{R}^n]$, the space of bounded continuous functions or $UBC[\mathbb{R}^n]$, the bounded uniformly continuous functions. We have shown that the weak generator defined by the mixed locally convex topology on $C_b[\mathbb{R}^n]$ is a strong generator on $KS^p[\mathbb{R}^n]$ (e.g., $T(t)$ is strongly continuous on $KS^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$).

We also have used $KS^2$ to construct the free-particle path integral in the manner originally intended by Feynman. It is shown in [GZ] that $KS^2$ has a claim as the natural representation space for the Feynman formulation of quantum theory in that
it allows representations for both the Heisenberg and Schrödinger representations for quantum mechanics, a property not shared by $L^2$.

We also have used $SD^2$ to provide the strongest possible a priori bounds for the nonlinear term of the classical Navier-Stokes equation.

References

[A] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, (1975).

[AL] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, Colloq. Math., 1 (1948), 289-293.

[FH] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, (1965).

[GA] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, 2nd Edition, Vol. II, Springer Tracts in Natural Philosophy, Vol. 39 Springer, New York, 1997.

[G] L. Gross, Abstract Wiener spaces, Proc. Fifth Berkeley Symposium on Mathematics Statistics and Probability, 1965, pp. 31?42. MR 35:3027

[GL] G. Leoni, *A First Course in Sobolev Spaces*, AMS Graduate Studies in Math. 105, Providence, R.I, 2009.

[GO] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics, Vol. 4, Amer. Math. Soc., (1994).

[GRA] L. Grafakos, *Classical and Modern Fourier Analysis*, Person Prentice-Hall, New Jersey, 2004.

[GRRZ] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, (1965).

[GZ] T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman operator Calculus*, Springer New York, (2016).
[HO] C. S. Höning, *There is no natural Banach space norm on the space of Kurzweil-Henstock-Denjoy-Perron integrable functions*, 30º Seminário Brasileiro de Análise, (1989), 387-397.

[HS] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, (1991).

[HS1] R. Henstock, *A Riemann-type integral of Lebesgue power*, Canadian Journal of Mathematics **20** (1968), 79-87.

[J] F. Jones, *Lebesgue Integration on Euclidean Space*, Revised Edition, Jones and Bartlett Publishers, Boston, (2001).

[HL] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman’s Operational Calculus*, Oxford U. Press, New York, (2000).

[KAA] V. Kolokoltsov, C.T. Abdallah, M. Ariola, P. Dorato and D. Panchenko, *Improved Sample Complexity Estimates for Statistical Learning Control of Uncertain Systems*, IEEE Trans. Automatic Control **45** (2000), 2383-2388.

[KB] J. Kuelbs, *Gaussian measures on a Banach space*, Journal of Functional Analysis **5** (1970), 354–367.

[KO] V. Kolokoltschov, *A new path integral representation for the solutions of the Schrödinger equation*, Math. Proc. Cam. Phil. Soc. **32** (2002), 353-375.

[KT] H. Koch and D. Tataru. *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157**(1) (2001), 22-35.

[KW] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zur Mathematik, Band **26**, Teubner Verlagsgesellschaft, Leipzig, (1980).

[LB] L. Lorenzi and M. Bertoldi, *Analytical Methods for Markov Semigroups*, Monographs and Textbooks in Pure and Applied Mathematics, Chapman & Hall/CRC, New York, (2007).

[L] P. D. Lax, *Symmetrizable linear transformations*, Comm. Pure Appl. Math. **7** (1954), 633-647.

[MD] P. Muldowney, *A Modern Theory of Random Variation*, Wiley, (2012).
[NI] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, (1992).

[OS] K. M. Ostaszewski, *The space of Henstock integrable functions of two variables*, Internat. J. Math. and Math. Sci. **11** (1988), 15?22.

[PF] W. F. Pfeffer, *The Riemann Approach to Integration: Local Geometric Theory*, Cambridge Tracts in Mathematics **109**, Cambridge University Press, (1993).

[PT] A. Papageorgiou and J.G. Traub, *Faster Evaluation of Multidimensional Integrals*, Computers in Physics, Nov., (1997), 574-578.

[PTR] S. Paskov and J.G. Traub, *Faster Valuation of Financial Derivatives*, Journal of Portfolio Management, **22** (1995), 113-120.

[ST] V. Steadman, *Theory of operators on Banach spaces*, Ph.D thesis, Howard University, 1988.

[SY] G. R. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences, Vol. **143**, Springer, New York, 2002.

[TA] E. Talvila, *The distributional Denjoy integral*, Real Analysis Exchange, **33** (2008), 51-82.

[T1] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, AMS Chelsea Pub., Providence, RI, 2001.

[T2] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, Vol. **68**, Springer, New York, 1988.

[TY1] L. Tuo-Yeong, *Henstock-Kurweil Integration on Euclidean Spaces*, Series in Real Analysis-Vol 12 World Scientific, New Jersey, (2011).

(Tepper L. Gill) Department of Electrical & Computer Engineering, and Mathematics, Howard University, Washington DC 20059, USA, E-mail: tgill@howard.edu