We consider the problem of LOCC discrimination between two bipartite pure states of Fermionic systems. We show that, contrarily to the case of quantum systems, for Fermionic systems it is generally not possible to achieve the ideal state discrimination performances through LOCC measurements. On the other hand, we show that an ancillary system made of two Fermionic modes in a maximally entangled state is a sufficient additional resource to attain the ideal performances via LOCC measurements. The stability of the ideal results is studied when the probability of preparation of the two states is perturbed, and a tight bound on the discrimination error is derived.

The very concept of quantum information theory requires encoding distinguishable pieces of information on quantum states. In the simplest instance of encoding of classical information, the decoding procedure corresponds to the widely studied task of quantum state discrimination [1–8]. In turn, the state discrimination task has been extensively studied in the scenario where states are shared by distant agents that are only allowed to use Local Operations and Classical Communication (LOCC) [9–11]. These tasks are now exhaustively understood in the quantum realm.

On the other hand, real physical systems are Bosons or Fermions, and the latter are ruled by a theory that is a slight variation of the standard quantum one. The study of information processing in Fermionic theory has then various reasons, that are both practical and fundamental [12]. Of particular importance is establishing analogies and differences between quantum and Fermionic implementation of specific information processing tasks. For example, it is known that quantum and Fermionic computation are equivalent, meaning that every quantum algorithm can be efficiently mapped to a Fermionic one, and viceversa [12]. This implies e.g. that Fermionic processes are efficiently simulated by quantum computers [13]. In many other respects, however, the two theories present significant differences [14,15].

In the present Letter, we study the task of LOCC state discrimination in the Fermionic theory. We show that, unlike the quantum case, in the typical situation LOCC discrimination is strictly suboptimal. We also derive conditions where ideal discrimination performances can be achieved via a LOCC protocol. These conditions are very sensitive to prior information about the probability of occurrence of the two states. Therefore, we study the behavior of LOCC protocols in the presence of a small perturbation of the ideal conditions. Moreover, we show that a pair of Fermionic systems in a maximally entangled state is a sufficient resource in addition to LOCC to achieve discrimination performances equivalent to the optimal one.

We briefly introduce the Fermionic quantum theory as the theory dealing with systems made of local Fermionic modes [12,13,15,17,18]. A Fermionic mode represents the counterpart of a qubit in the quantum theory and can be either empty or occupied by a single “excitation.” The states of Fermionic systems satisfy the parity superselection rule [19,21], i.e. superpositions of vectors having even or odd excitation numbers are forbidden. The latter can be derived as a consequence of the assumption that the elements of the Fermionic algebra are Kraus operators of local Fermionic transformations [15]. The generators of the Fermionic algebra $\varphi_i$, $i$ running over arbitrary sets of $N$ modes, fulfill the canonical anticommutation relations $\{\varphi_i,\varphi_j^\dagger\} = \delta_{ij}$ and $\{\varphi_i,\varphi_j\} = \{\varphi_i^\dagger,\varphi_j^\dagger\} = 0 \forall i,j$. Once we define the vacuum state $|\Omega\rangle$ as the common eigenvector of operators $\varphi_i^\dagger \varphi_i$ with null eigenvalues, the Fermionic operators enable us to define the Fock states as $|n_1\ldots n_N\rangle := (\varphi_1^{\dagger})^{n_1}\cdots(\varphi_N^{\dagger})^{n_N} |\Omega\rangle$ and the antisymmetrized Fock space $\mathcal{F}$ through the linear combination of all Fock states. We may label with the lowercase letters $e$, $o$ those sectors of the Fock space featuring even and odd parity, respectively. The Jordan-Wigner isomorphism [22,24] is a crucial tool to handle the transformations and informational protocols in Fermionic theory. Indeed, it maps non-locally the Fermionic operator algebra to an algebra of transformations on qubits, thus allowing us to proceed with the usual quantum notation.

The orthogonal case.—In quantum theory, we may perfectly discriminate between any two orthogonal states $|\psi\rangle$, $|\phi\rangle$ of a bipartite system AB through LOCC measurements [9]. We remind that the most general case of a quantum measurement is represented by a positive-operator valued measure (POVM), i.e. a collection of effects (positive operators $0 \leq S \leq I$) that sum to the identity operator $I$. A necessary condition for a POVM to represent a LOCC measurement is to be separable (SEP). The effect $S$ is separable if there exists some operators $0 \leq A_i, B_i \leq I$ such that $S = \sum_i A_i \otimes B_i$, and a POVM represents a separable measurement if it is exclusively made of separable effects. Moreover, we recall that LOCC POVMs are a proper subset of SEP POVMs [25]. In the following we will use the acronyms LOCC and SEP to denote the corresponding subsets of POVMs.

We now give a sketchy summary of the result of Ref. [9]. Let us introduce the orthonormal basis $\{|\psi_A\rangle\}$ for Alice party. Due to the Schmidt decomposition, every bipartite
pure state can be written as

\[ |\psi\rangle = \sum_{i=1}^{n} |i\rangle_A |\eta_i\rangle_B, \quad |\phi\rangle = \sum_{i=1}^{n} |i\rangle_A |\nu_i\rangle_B, \]

(1)

where \(\{|\eta_i\rangle_B\}\) and \(\{|\nu_i\rangle_B\}\) are suitable sets of orthogonal states in Bob’s Hilbert space. Alice has to measure her system in a properly selected basis and send the outcome to Bob, who in turn menages to locally discriminate between two orthogonal states and infer the correct result. Such a basis always exists, as shown in [9].

We follow here a strategy similar to the quantum one in order to distinguish between two pure orthogonal states \(|\psi\rangle, |\phi\rangle\) of a bipartite Fermionic system. First of all, we notice that whenever the two preparations have different parity, e.g. \(|\psi\rangle \in \mathcal{F}_e(AB)\) and \(|\phi\rangle \in \mathcal{F}_o(AB)\), it is always possible to perfectly discriminate between the two just through local measurements. Indeed, Alice and Bob have to locally measure the parity of their subsystems and if their outcomes match, then the provided state was even, otherwise it was the odd one. The nontrivial case then is that of two pure states with the same parity. Since the even and odd sector are equivalent under LOCC, it is not restrictive to focus on even vectors only. We introduce the following convenient notation for the even vectors \(|\psi\rangle, |\phi\rangle \in \mathcal{F}_e(AB)\)

\[ |\psi\rangle = |\psi_E\rangle + |\psi_O\rangle, \]

\[ |\phi\rangle = |\phi_E\rangle + |\phi_O\rangle, \]

(2)

and recalling the decomposition in Ref. [9], we decompose \(|\psi_E\rangle = \sum_{i=1}^{n} |e_i\rangle_A |\eta_i\rangle_B, |\phi_O\rangle = \sum_{i=1}^{n} |\nu_i\rangle_A |\phi_i\rangle_B, |\phi_E\rangle = \sum_{i=1}^{n} |\phi_i\rangle_A |\nu_i\rangle_B, |\phi_O\rangle = \sum_{i=1}^{n} |\nu_i\rangle_A |\phi_i\rangle_B,\) where \(\{|e_i\}\), \(\{|\nu_i\}\) are Alice orthonormal bases of even and odd sectors, respectively, while \(\{|\eta_i\rangle\}_B\) and \(\{|\phi_i\rangle\}_B\), for \(x = e, o\) are Bob vectors resulting from the Schmidt decomposition. In general, the latter are not normalized and \(\{|\eta_i\rangle\}_B \neq 0\). We may indicate with the capitalized letters \(E\) or \(O\) those entities pertaining to the \(E\) and \(O\)-spaces of \(\mathcal{F}_e(AB)\), i.e. those subspaces where the parities of Alice’s and Bob’s subsystems are both even or odd, respectively. E.g. the \(E\)-part of vector \(|\psi\rangle\) is defined as \(|\psi_E\rangle = \sum_{i=1}^{n} |e_i\rangle_A |\eta_i\rangle_B,\) whereas the \(O\)-part is \(|\psi_O\rangle = \sum_{i=1}^{n} |\nu_i\rangle_A |\phi_i\rangle_B.\) The orthogonality condition \(\langle \psi | \phi \rangle = 0\) generally reads

\[ \langle \psi_E | \phi_E \rangle + \langle \psi_O | \phi_O \rangle = 0. \]

(3)

Let us consider as the first case the scenario where the two preparations have only one component. Then they have components either in complementary subspaces, e.g. \(|\psi\rangle = |\psi_E\rangle\) and \(|\phi\rangle = |\phi_O\rangle\), and it is trivially possible to discriminate via LOCC by measuring the local parities, or in the same subspace, e.g. \(|\psi\rangle = |\psi_E\rangle\) and \(|\phi\rangle = |\phi_E\rangle.\) In the latter case the protocol reduces to the quantum one. Indeed, Alice selects the right basis \(\{|e_i\}\) and lets Bob perfectly discriminate between \(|\eta_i\rangle\) and \(|\nu_i\rangle\), which are now orthogonal thanks to the result of Ref. [9]. Moreover, as proved in Ref. [18], product POVMs in the Jordan-Wigner representation correspond to LOCC Fermionic POVMs.

As the second case, we consider the situation where only one component out of the four \(|\psi_E\rangle, |\psi_O\rangle, |\phi_E\rangle, |\phi_O\rangle\) is null. Perfect discrimination is implementable through LOCC in this case as well. Let us take for instance the vectors \(|\psi\rangle = |\psi_E\rangle + |\psi_O\rangle\) and \(|\phi\rangle = |\phi_O\rangle;\) Alice and Bob firstly measure the parity of their subsystem and if the outcome is even, they know for sure that the system has been prepared in the state \(|\psi\rangle\). Otherwise, the state after the measurement is either \(|\psi_O\rangle / ||\psi_O||\) or \(|\phi_O\rangle\), and the above strategy for the first case applies.

In the most general case all four components are non-null. If the two \(E\)- and \(O\)-parts are orthogonal—that is when \(\langle \psi_E | \phi_O \rangle = \langle \psi_O | \phi_E \rangle = 0—\) Alice and Bob can measure locally the parity of their systems, thus obtaining the post-measurement states \(|\psi\rangle = |\psi_E\rangle / ||\psi_E||\) and \(|\phi\rangle = |\phi_E\rangle / ||\phi_E||\) if the outcomes are both even, \(|\psi\rangle = |\psi_O\rangle / ||\psi_O||\) and \(|\phi\rangle = |\phi_O\rangle / ||\phi_O||\) if the outcomes are both odd. Consequently they reduced to the first case.

There is one situation left fulfilling condition [3], i.e. when \(|\psi_E|\phi_O\rangle \neq 0\) and \(|\psi_O|\phi_E\rangle \neq 0\). This case exhibits the main difference with respect to quantum theory. Consider for instance the states \(1/\sqrt{2}(|00\rangle_A |00\rangle_B \pm |01\rangle_A |01\rangle_B)\). In this case, the decompositions in Eq. (1) involves bases \(\{|\eta_i\rangle\}_B\) and \(\{|\nu_i\rangle\}_B\) where superpositions forbidden by the Fermionic superselection rule appear. Indeed, one has \(i = \pm\) and

\[ |\pm\rangle_A := \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle), \]

\[ |\eta_{\pm}\rangle_B = |\nu_{\mp}\rangle_B := \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle). \]

The last case can thus not be treated by straightforwardly applying the quantum strategy of Ref. [9]. The following theorem summarizes what we discussed so far, and shows that it is not possible to perfectly discriminate two states with \(\langle \psi_E | \phi_E \rangle \neq 0\) and \(\langle \psi_O | \phi_O \rangle \neq 0\) through POVMs in SEP, thus neither by means of LOCC.

**Theorem 1.** Let \(|\psi\rangle\) and \(|\phi\rangle\) be two pure, normalized and orthogonal states. Then the following statements are equivalent: (1) The even and odd parts are separately orthogonal, i.e.

\[ \langle \psi_E | \phi_E \rangle = \langle \psi_O | \phi_O \rangle = 0. \]

(4)

(2) The two states are perfectly discriminable through LOCC. (3) The two states are perfectly discriminable through SEP.

**Proof.** It is trivial to see that (2) \(\Rightarrow\) (3) whereas we have already shown above that (1) \(\Rightarrow\) (2) thanks to [9]. We
now focus on the implication \((3) \Rightarrow (1)\) and wonder under what conditions one has

\[
\max_{S \in \text{SEP}} \text{Tr}[(|\psi \rangle \langle \psi| - |\phi \rangle \langle \phi|) S] = 1,
\]

namely the condition for perfect discriminability via SEP. The expression in Eq. \((3)\) clearly involves only the component of \(S\) supported on the even subspace \(F_e(AB)\). Now, a necessary condition for a Fermionic effect \(S\) supported on \(F_e(\Lambda)\) to be SEP it that \(S = S_E + S_O\), where \(S_E\) and \(S_O\) have their support on the \(E\)-space and \(O\)-space, respectively (see the Supplemental Material). Consequently, the condition in Eq. \((3)\) is equivalent to

\[
\begin{align*}
\text{Tr} \left[ (|\psi_E \rangle \langle \psi_E| - |\phi_E \rangle \langle \phi_E|) S_E \right] &= 1, \\
\text{Tr} \left[ (|\psi_O \rangle \langle \psi_O| - |\phi_O \rangle \langle \phi_O|) S_O \right] &= 1,
\end{align*}
\]

for \(S = S_E + S_O\) representing an effect in SEP. Thus, it is possible to perfectly discriminate the two states through separable effects only if the \(E\)- and \(O\)-parts are perfectly discriminable separately, as required in Eq. \((4)\). \(\blacksquare\)

Ancilla assisted discrimination.—We now show that one can overcome the limits of Theorem \((1)\) by providing the two parties with an ancillary system prepared in a suitable pure entangled state \(|\omega\rangle\). Let us take

\[
|\omega\rangle_{AB} := a |00\rangle + b |11\rangle \quad \text{for} \quad a, b \neq 0,
\]

and consider the task of distinguishing the new vectors \(|\psi'\rangle := |\psi \rangle \otimes |\omega\rangle\) and \(|\phi'\rangle := |\phi \rangle \otimes |\omega\rangle\). In particular, we will see that only a maximally entangled ancillary state—i.e. with \(|a|^2 = |b|^2 = 1/2\)—enables perfect discrimination between every two pure Fermionic states, regardless of condition \((4)\).

**Theorem 2.** It is always possible to perfectly discriminate between every two pure, normalized and orthogonal preparations \(|\psi\rangle\) and \(|\phi\rangle\) with LOCC and an ancillary system in a pure maximally entangled state

\[
|\omega\rangle_{AB} = \frac{1}{\sqrt{2}} \left( |00\rangle + e^{i\varphi} |11\rangle \right), \quad \varphi \in [0, 2\pi).
\]

Moreover, the same does not hold if the ancillary state is not maximally entangled.

**Proof.** We show here a sketch of the proof, the full rigorous derivation being given in the Supplemental Material. Let us consider the states

\[
|\psi\rangle' := |\psi \rangle \otimes |\omega\rangle = |\psi'\rangle_{O} + |\psi'\rangle_{E},
\]

\[
|\phi\rangle' := |\phi \rangle \otimes |\omega\rangle = |\phi'\rangle_{O} + |\phi'\rangle_{E},
\]

with \(|\psi'\rangle_{E} = a |\phi_E 00\rangle + b |\phi_O 11\rangle\), \(|\psi'\rangle_{O} = b |\phi_E 11\rangle + a |\phi_O 00\rangle\), \(|\phi'\rangle_{E} = a |\phi_E 00\rangle + b |\phi_O 11\rangle\), and \(|\phi'\rangle_{O} = b |\phi_E 11\rangle + a |\phi_O 00\rangle\), and evaluate for \(|a|^2 = |b|^2 = 1/2\) the scalar products

\[
\langle \psi'_{E}\rangle |\phi'_{E}\rangle = \langle \psi'_{O}\rangle |\phi'_{O}\rangle = \frac{1}{2} \langle \psi|\phi\rangle = 0.
\]

The vectors \(|\psi'\rangle\) and \(|\phi'\rangle\) do satisfy Eq. \((4)\), even if \(|\psi\rangle\) and \(|\phi\rangle\) may not. Thus, we are now able to apply the protocol of Theorem \((1)\) to the new states as shown above. Condition \((8)\) is also necessary for perfect discrimination, as shown in the Supplemental Material. \(\blacksquare\)

**Optimal discrimination.**—If the orthogonality condition \(\langle \psi|\phi\rangle = 0\) is relaxed, the two states are clearly not perfectly discriminable. Hence, one looks for the protocol which minimizes the error probability—i.e. the probability of wrong detection. For this purpose, it is necessary to introduce our prior probabilities for the two states, given by the distribution \(\{p, q\}\). In this case, the error probability reads

\[
P_{\text{err}} := \text{Tr}[p |\psi\rangle \langle 
\]

\[
p |\phi\rangle + q |\phi\rangle \langle \phi| \Pi_{\psi}],
\]

where \(\{\Pi_{\psi}, \Pi_{\phi}\}\) is the binary POVM describing the discrimination protocol. We remind that by definition the POVM satisfies \(\Pi_{\psi}, \Pi_{\phi} \geq 0\) and \(\Pi_{\psi} + \Pi_{\phi} = I\). In the quantum case, the optimal discrimination strategy corresponds to the POVM \(\{|+, -\rangle \langle +|, \langle -| \rangle \langle -|\}\) diagonalizing the operator

\[
\Delta := p |\psi\rangle \langle |\psi| - q |\phi\rangle \langle \phi| = \lambda_+ |+\rangle \langle +| + \lambda_- |\rangle \langle -|,
\]

where \(\lambda_+ > 0\), \(\lambda_- < 0\) are the eigenvalues of \(\Delta\), and \(\langle ++|\rangle = 0\) (see Ref. \([2]\)) . The corresponding error probability is \(\[2\]

\[
P_{\text{err}} = \frac{1}{2} \left( 1 - ||\Delta||_1 \right).
\]

In Ref. \([10]\), the authors observe that optimal discrimination through LOCC of \(|\psi\rangle\) and \(|\phi\rangle\) with prior probabilities \(p\) and \(q\), respectively, is equivalent to perfect LOCC discrimination between \(|+\rangle\) and \(|\rangle\rangle\) (see also Ref. \([2]\)), thus reducing the optimal case to an instance of perfect discrimination. While the latter is always possible in quantum theory, we know from Theorem \((1)\) that in Fermionic theory this is true only if the eigenvectors satisfy

\[
\langle +E|\rangle = \langle +O|\rangle = 0.
\]

Otherwise, by Theorem \((2)\) perfect LOCC discrimination requires a maximally entangled ancilla. As for the perfect discrimination case, the conditions for optimal LOCC discrimination in Fermionic theory differ from the quantum ones only when the \(E\)- and \(O\)-components of \(|+\rangle\) and \(|\rangle\rangle\) are all non-zero, and \(\langle +E|\rangle, \langle +O|\rangle \neq 0\) . For the latter case, we now prove a necessary and sufficient condition for achievability of optimal discrimination with LOCC that does not require diagonalization of \(\Delta\).
Theorem 3. Let $\rho = p \ket{\psi} \bra{\psi}$ and $\sigma = q \ket{\phi} \bra{\phi}$ be two pure and sub-normalized states for $p, q > 0$ and $p + q = 1$. They are optimally discriminable through LOCC if and only if they satisfy
\[ [\Delta, P_E] = 0, \tag{12} \]
where $\Delta$ is defined in Eq. (9) and $P_E$ is the projector onto the $E$-subspace.

Proof. Since optimal discrimination between $\ket{\psi}$ and $\ket{\phi}$ is equivalent to perfect discrimination between $\ket{+}$ and $\ket{-}$, by Eq. (11) optimal discriminability of the states $\ket{\psi}$ and $\ket{\phi}$ by LOCC is equivalent to the condition
\[ \langle + | (P_E - P_O) | - \rangle = 0, \tag{13} \]
where $P_O$ is the projector onto the $O$-subspace. Now, taking the difference of the first two members of Eq. (13), we can then express the LOCC-discriminability condition through the single expression
\[ \langle + | (P_E - P_O) | - \rangle = 0. \tag{14} \]
Indeed, since $P_O = P_e - P_E$, where $P_e$ is the projection on the even subspace $F_e(AB)$ of system $AB$, Eq. (14) is equivalent to the requirement that the restriction of the projector $P_e$ onto the space $\text{Span}\{\ket{\psi}, \ket{\phi}\}$ is diagonal in the basis $\{|+, -\rangle\}$. The operators $\Delta$ and $P_E$ are simultaneously diagonalizable if and only $[\Delta, P_E] = 0$. Eq. (12) is then equivalent to attainability of optimal discrimination between the two states $\rho$ and $\sigma$ via LOCC. \hfill\Box

We may wonder what happens when condition (12) is not satisfied. As we show in the next theorem, the best discrimination strategy through SEP corresponds to measuring in the basis of eigenvectors of $\Delta_E$ and $\Delta_O$, defined as the restriction of the operator $\Delta$ onto the $E$- and $O$-subspaces, respectively. Such a strategy is LOCC.

Theorem 4. Let $\rho = p \ket{\psi} \bra{\psi}$ and $\sigma = q \ket{\phi} \bra{\phi}$ be two pure sub-normalized states for $p, q > 0$ and $p + q = 1$. The optimal SEP discrimination protocol is locally implementable through LOCC and its error probability reads
\[ P_{err}^{\text{SEP}} = P_{err}^{\text{LOCC}} = \frac{1}{2} (1 - \| \Delta_E + \Delta_O \|_1), \tag{15} \]
where $\Delta_E = P_E \Delta P_E$ and $\Delta_O = P_O \Delta P_O$.

Proof. This result can be obtained considering that
\[ P_{err}^{\text{SEP}} = p - \max_{\Pi_{\psi} \in \Pi_{\psi}(AB)} \text{Tr}[\Pi_{\psi} \Delta], \]
where $\Pi_{\psi}$ must be of the form $\Pi_{\psi} = \Pi_{\psi}^E + \Pi_{\psi}^O$ in order to comply with the separability condition, as observed in the proof of Theorem 4. The result then follows. \hfill\Box

The above result allows us to treat the case where we are restricted only to local measurements and Eq. (12) does not hold for the preparations $\rho$, $\sigma$. Once we are given the pure states $\ket{\psi}$ and $\ket{\phi}$, the condition for optimal LOCC-discrimination of Eq. (12) is fulfilled either for the vectors laying in the $E$- or $O$-space, i.e. $[\ket{\psi} \bra{\psi}, P_E] = [\ket{\phi} \bra{\phi}, P_E] = 0$, or if the probability $p$ satisfies
\[ [\ket{\psi} \bra{\psi}, P_E] = \frac{1 - p}{p} [\ket{\phi} \bra{\phi}, P_E]. \tag{16} \]
Condition (16) can be satisfied by a unique value of the prior probability $p$, unless $[\ket{\psi} \bra{\psi}, P_E] = [\ket{\phi} \bra{\phi}, P_E] = 0$. However, we now show that optimal LOCC discrimination can achieve the performances of unconstrained protocols, provided that two ancillary Fermionic systems are used in a maximally entangled state. As discussed above, indeed, the problem of optimal discrimination between two pure states reduces to that of the orthogonal vectors $\{|+, -\rangle\}$ in Eq. (9). Considering Theorem 2, we know that orthogonal states can be perfectly discriminated via LOCC provided a maximally entangled ancillary system is available. These two observations immediately lead to our last result.

Theorem 5. Let $\rho = p \ket{\psi} \bra{\psi}$ and $\sigma = q \ket{\phi} \bra{\phi}$ be two pure sub-normalized states for $p, q > 0$ and $p + q = 1$. It is always possible to optimally discriminate between the two preparations via LOCC if we use an ancillary system in a pure maximally entangled state.

Equation (16) introduces a strict condition on the prior probability of the preparations, which are always subject to noise. We show hereafter that if we introduce a small perturbation $\epsilon$ on the preparation probabilities of pair of states satisfying Eq. (12), the discrimination error probability increases at most linearly in $\epsilon$ with respect to the appropriate optimal LOCC protocol. Thus, we map $p \mapsto p + \epsilon$ and attain
\[ \Delta' := (p + \epsilon) \ket{\psi} \bra{\psi} - (q - \epsilon) \ket{\phi} \bra{\phi} = \Delta^0 + \epsilon (\ket{\psi} \bra{\psi} + |\phi \rangle \langle \phi|), \]
where $[\Delta^0, P_E] = 0$. At this stage, we estimate the error difference between the optimal POVM $\mathcal{P}_0 := \{\Pi_{\psi}, \Pi_{\phi}\}$ for $\epsilon = 0$, which is LOCC thanks to Theorem 3, and the LOCC-optimal POVM for the perturbed case $\Delta'$. The error increases as $\delta P_{err} := P_{err}(\mathcal{P}_0) - P_{err}(\mathcal{P}_0) \geq 0$ where $P_{err}(\mathcal{P}_0) = \text{Tr}[\mathcal{P}_0 (\Delta')] \geq 0$ and $P_{err}(\Delta') = \frac{1}{2} (1 - \| \Delta_E + \Delta_O \|_1)$ as in Eq. (15). Suitably manipulating the expression for $\delta P_{err}$ one obtains
\[ \delta P_{err} \leq k |\epsilon| + g \epsilon, \tag{17} \]
where $k$, $g$ are suitable constants depending only on $|\psi \rangle, |\phi \rangle$. The former inequality is as tight as possible:
let us take indeed the states $|\psi\rangle = 1/\sqrt{2} |00\rangle + 1/\sqrt{2} |11\rangle$ and $|\phi\rangle = \alpha |00\rangle + \sqrt{1-\alpha^2} |11\rangle$, where $\alpha := (1/\sqrt{2} + \xi)$, and $\xi$ belongs to a neighborhood of zero. In such a case, we have numerically assessed that the error difference $\delta P_{\text{err}}$ exhibits a corner in $\epsilon = 0$ as $\xi \to 0$ (more details can be found in the Supplemental Material).

We also investigate the performance of the optimal LOCC protocol for $\epsilon \neq 0$ in the neighborhood of a prior probability $p$ satisfying condition (16), by comparing its efficiency to that of the optimal unconstrained (i.e. entanglement-assisted LOCC) POVM. Thus we estimate $\delta P'_{\text{err}} := P_{\text{err}}^{\text{LOC}}(\Delta') - P_{\text{err}}(\Delta')$ by means of Eqs. (10) and (15), obtaining

$$\delta P'_{\text{err}} \leq \kappa |\epsilon|,$$

for a suitable $\kappa$, thanks to the triangle inequality.

We remark that, in the case of a mismatch in the assessment of the prior probability $p$, also for unconstrained optimal discrimination—coinciding with ancilla-assisted LOCC—one has the same bound as in Eq. (17), with possibly different constants $k$ and $g$. This feature, however, must not be considered as an artefact of Fermionic theory. Indeed, the technique used to derive the bound in Eq. (17) is very general and leads to the same behavior in the quantum case as well.

Discussion.—As in the quantum case, discrimination with separable and LOCC POVMs in the Fermionic case achieve the same performances. Unlike in quantum theory, on the other hand, in Fermionic theory ideal state discrimination through LOCC is subject to non-trivial conditions. In this Letter, we derived the conditions under which LOCC discrimination achieves the ideal performances of unconstrained discrimination protocols. However, in the Fermionic case, ancilla-assisted LOCC protocols achieve ideal discrimination. One has to remark, though, that this is the case only for maximally entangled ancillary states. We finally studied the behavior of optimal protocols—which depend on prior probabilities of the states to be discriminated—if the prior conditions are subject to perturbation. A remarkable instability is observed, corresponding to a corner point in the curve representing the error probability excess due to non-optimized POVMs. We stress that the latter phenomenon is not exclusive of Fermionic theory, as it occurs also in the quantum case.

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Separable effects

In order to implement the parity superselection rule, the operator $0 \leq S \leq I$ representing a separable effect supported on $F_\epsilon(AB)$ must be of the form

$$S = S_E + S_O,$$

where $S_E = \sum_i e_i \otimes e'_i$, $S_O = \sum_j o_j \otimes o'_j$, and $e_i, e'_i, o_j, o'_j \geq 0$, with

$${\text{Supp}}(e_i) \subseteq F_\epsilon(A) \quad \text{Supp}(e'_i) \subseteq F_\epsilon(B)$$

$${\text{Supp}}(o_j) \subseteq F_o(A) \quad \text{Supp}(o'_j) \subseteq F_o(B).$$

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1. C. W. Helstrom, Journal of Statistical Physics 1, 231 (1969)
Once the effect is applied to a Fermionic state $\tau \in \mathcal{S}(AB)$, the Born rule returns

$$\text{Tr}[\tau S] = \text{Tr}[P_E \tau P_E S_E + P_O \tau P_O S_O].$$  \hfill (20)

The above expression shows that any separable POVM operates on the $E$- and $O$-parts of $\tau$ independently. In particular, in the proof of Theorem 1 we seek the maximum of $r := \text{Tr}[(\rho - \sigma)S]$, with $\rho = |\psi \rangle \langle \psi|$ and $\sigma = |\phi \rangle \langle \phi|$, i.e.

$$r = \text{Tr}\left[\left(|\psi_E \rangle \langle \psi_E| - |\phi_E \rangle \langle \phi_E|\right) S_E\right]$$

$$+ \left(|\psi_O \rangle \langle \psi_O| - |\phi_O \rangle \langle \phi_O|\right) S_O].$$

The latter achieves unit value iff one can find $S_E$ and $S_O$ such that

$$\text{Tr}[\rho S] = ||\psi_E||^2 \langle \tilde{\psi}_E | S_E | \tilde{\psi}_E \rangle + ||\psi_O||^2 \langle \tilde{\psi}_O | S_O | \tilde{\psi}_O \rangle = 1,$$

$$\text{Tr}[\sigma S] = ||\phi_E||^2 \langle \tilde{\phi}_E | S_E | \tilde{\phi}_E \rangle + ||\phi_O||^2 \langle \tilde{\phi}_O | S_O | \tilde{\phi}_O \rangle = 0,$$

where $\tilde{\eta} := |\eta\rangle / ||\eta||$. However, due to the hypotheses assumed so far, we achieve the above conditions if and only if Eq. (6) is satisfied.

**Proof of Theorem 2**

Alice and Bob are provided with an entangled ancilla in the state $|\omega\rangle$, as in Eq. (7). They now share two bipartite systems in the possible states $|\psi\rangle = |\psi\rangle \otimes |\omega\rangle$ or $|\phi\rangle = |\phi\rangle \otimes |\omega\rangle$, whose full expression can be obtained from

$$|\psi_E\rangle = a \sum_{i=0}^n |e_i1\rangle_A |\eta_1^E\rangle_B + b \sum_{j=0}^n |o_j1\rangle_A |\eta_j^E\rangle_B,$$

$$|\phi_E\rangle = a \sum_{i=0}^n |e_i1\rangle_A |\eta_1^E\rangle_B + b \sum_{j=0}^n |o_j1\rangle_A |\eta_j^E\rangle_B,$$

$$|\psi_O\rangle = b \sum_{i=0}^n |e_i1\rangle_A |\eta_1^O\rangle_B + a \sum_{j=0}^n |o_j0\rangle_A |\eta_j^O\rangle_B,$$

$$|\phi_O\rangle = b \sum_{i=0}^n |e_i1\rangle_A |\eta_1^O\rangle_B + a \sum_{j=0}^n |o_j0\rangle_A |\eta_j^O\rangle_B.$$

Let $\Sigma_E := \langle \psi_E | \phi_E \rangle$ and $\Sigma_O := \langle \psi_O | \phi_O \rangle = -\Sigma_E$, where the last equality follows from the fact that $\Sigma_E + \Sigma_O = \langle \psi | \phi \rangle = 0$. As shown in the body, there are cases where the ancilla is not needed, and clearly its absence cannot reduce the performances of LOCC discrimination. The remaining case is that where $\Sigma_E \neq 0$. The necessary and sufficient condition for perfect LOCC discrimination between $|\psi\rangle$ and $|\phi\rangle$ of Eq. (4) can then be written using Eq. (21) as

$$\langle \psi_E | \phi_E \rangle = (|a|^2 - |b|^2) \Sigma_E = 0.$$

For $|a|^2 = |b|^2 = \frac{1}{2}$ the above condition is clearly satisfied. On the other hand, if $\Sigma_E \neq 0$, discrimination by LOCC is not possible for $|a| \neq |b|$.

**Extremal case for $\delta P_{\text{err}}$ bound**

In the Letter we investigated the behavior of the error in the case where the prior probabilities slightly differ from the ideal ones. We are given two pure states $|\psi\rangle$, $|\phi\rangle$ and if there exists a probability distribution $\{p, q\}$ such that condition (16) is satisfied, we proved that such a solution is unique and the optimal discrimination strategy is LOCC-implementable, unless $[P_E, |\psi \rangle \langle \psi|] = [P_E, |\phi \rangle \langle \phi|] = 0$. Therefore, a small perturbation $\epsilon$ in the prior probability $p$ produces an increase of error probability of the LOCC protocol—which is optimized for the unperturbed case—with respect to the optimal LOCC one. For this purpose, we introduce the quantity

$$\delta P_{\text{err}} := P_{\text{err}}(\rho^0) - P_{\text{err}}^{\text{LOCC}}(\Delta^e).$$

(22)

Thanks to the triangle inequality we have that

$$\delta P_{\text{err}} \leq \frac{1}{2} \left( \|\Delta^e + \Delta^O - \Delta^O\| \right),$$

$$+ \epsilon \text{Tr}[\langle |\psi \rangle \langle \psi| + |\phi \rangle \langle \phi|) (\Pi_\phi - \Pi_\psi)],$$

for

$$k := \frac{1}{2} \|\Delta^e + \Delta^O - \Delta^O\|,$$

$$g := \frac{1}{2} \text{Tr}[\langle |\psi \rangle \langle \psi| + |\phi \rangle \langle \phi|) (\Pi_\phi - \Pi_\psi)].$$

Hence, the error difference $\delta P_{\text{err}}$ is sublinear.

We numerically assessed that the bound above is indeed achieved by the states

$$|\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle,$$

$$|\phi\rangle = \left( \frac{1}{\sqrt{2}} + \xi \right) |00\rangle + \frac{\gamma}{\sqrt{2}} |11\rangle,$$

where $\gamma := \sqrt{1 - 2\sqrt{2} - 2\sqrt[4]{2}}$ and $\xi$ belongs to a neighborhood of zero. The condition for optimality of Eq. (16) is fulfilled by

$$p(\xi) = \frac{\gamma + \sqrt{2}\gamma \xi}{1 + \gamma + \sqrt{2}\gamma \xi} \text{ for } \xi \in [0, 1 - \sqrt{2}/2)$$

and the terms of Eq. (22) read

$$P_{\text{err}}(\rho^0) = \text{Tr}[\langle |\psi \rangle \langle \psi| + |\phi \rangle \langle \phi|) (\Pi_\phi - \Pi_\psi)],$$

$$P_{\text{err}}^{\text{LOCC}}(\Delta^e) = \frac{1}{2} (1 - \|\Delta^e + \Delta^O\|).$$

In Fig. 1 we show a plot of the quantity $\delta P_{\text{err}}$ versus $\epsilon$ and $\xi$. We observe that, letting $\xi$ vary in a neighborhood of 0 one gets arbitrarily close to the bound in Eq. (17). On the other hand, the same analysis shows that one cannot find any lower bound for $\delta P_{\text{err}}$ better than $\delta P_{\text{err}} \geq 0$. Following exactly the same line as in the above derivation of the bound in Eq. (17), one can derive the bound in Eq. (18).
Figure 1. The plot shows the difference $\delta P_{err}$ between the error probability $P_{err}(\rho^0|\Delta)$ in discrimination between $\rho = (p + \epsilon)|\psi\rangle\langle\psi|$ and $\sigma = (1 - p - \epsilon)|\phi\rangle\langle\phi|$ with the POVM that is optimal for discrimination between $p|\psi\rangle\langle\psi|$ and $(1 - p)|\phi\rangle\langle\phi|$ and the error probability $P_{err}^{\text{LOCC}}(\Delta)$ in discrimination between the same states $\rho$ and $\sigma$ with the correct LOCC-optimal POVM, as a function of $\epsilon$ and $\xi$, where $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ and $|\phi\rangle = \alpha|00\rangle + \sqrt{1 - \alpha^2}|11\rangle$, and $\alpha = 1/\sqrt{2} + \xi$. The special value $p$ of the prior probability, corresponding to $\epsilon = 0$, is such that $p|\psi\rangle\langle\psi|$ and $(1 - p)|\phi\rangle\langle\phi|$ are ideally discriminable via LOCC. All the other values of $\epsilon$, on the other hand, lead to pairs of states $\rho$ and $\sigma$ that cannot be ideally discriminated via LOCC. For values of $\xi$ in a neighborhood of 0 the function $\delta P_{err}$ gets arbitrarily close to the bound $\delta P_{err} \leq k|\epsilon| + g\epsilon$ for suitable constants $k$ and $g$. 

Note: The figure and text are not visually aligned, and the text does not provide a full explanation of the figure. The figure shows a three-dimensional plot with $\epsilon$ and $\xi$ on the axes and $\delta P_{err}$ as the third dimension. The plot uses color to represent the values of $\delta P_{err}$. The text provides a mathematical description of the quantities and their relationship, including the conditions under which ideal discrimination is possible.