MATCHED PAIRS OF DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT. Matched pairs of Lie groupoids and Lie algebroids are studied. Discrete Euler-Lagrange equations are written for the matched pairs of Lie groupoids. As such, a geometric framework to couple two mutually interacting discrete systems in the Lie groupoid setting is obtained. We particularly exhibit matched pair dynamics on Lie groups as well. Two examples are provided. One is the discrete dynamics on the matched pair decomposition of the trivial groupoid. The other is the discrete dynamics on matched pair decomposition of the special linear group.

Key words: Discrete dynamics, Lie groupoids, matched pairs.

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1. Introduction

Consider two dynamical/mechanical systems, and their equations of motion presented either in Lagrangian or in Hamiltonian framework. Let also that the systems are mutually interacting so that they cannot preserve their individual motions. As a result, the equation of motion governing the coupled system is much more complicated than merely putting together the equations of motions of the individual systems. Such systems have been first studied within the semidirect product theory [6, 33, 36], where only one of the systems is allowed to act on the other. Many physical systems fit into this geometry; such as the heavy top [42], Maxwell-Vlasov equations [15].

The present paper is a part of the project which concerns to find purely geometric frameworks in order to exhibit the (discrete) dynamical equations for a coupled system under the mutual (nontrivial) interactions of the individual systems. More precisely, we derive dynamical equations, that we call the matched pair dynamics, involving terms that indicate the representations of both systems on each other [12, 13]. We further observe that if one of the actions is assumed to be trivial, then the matched pair dynamics reduce to the semidirect product dynamics.

The fundamental geometric object in the matched pair dynamics is a matched pair of Lie groups [46, 27]. That is, a pair of Lie groups that act on each other subject to compatibility conditions ensuring a group structure on their cartesian product. We call the total space the matched pair Lie group in order to emphasize the “matching”, yet it is called a bicrossedproduct group in [29, 28], the twilled extension in [20], the double Lie group in [23], or the Zappa-Szép product in [11], see also [43, 44, 45, 49]. Conversely, if a Lie group is isomorphic (as topological sets) to the cartesian product of two of its subgroups (with trivial intersection), then it is a matched pair Lie group. In this case, the mutual actions of the subgroups are derived from the group multiplication.

A few comments on previous results are in order. In [13], given a pair of two mutually interacting systems, with configuration spaces as Lie groups, the equations of motion of the matched system in the Lagrangian framework have been presented on the matched pair tangent
group of the individual spaces. The matched Euler-Lagrange equations, and the matched Euler-Poincaré equations have been obtained. The Hamiltonian counterpart, on the other hand, was studied in [12]. The matched Hamilton’s equations, and the matched Lie-Poisson equations were obtained. For an application of the matched pair Hamiltonian dynamics in the field of fluid dynamics, we refer to [11].

The present paper, now, concerns the discrete dynamics [37] on the Lie groupoid setting [32]. In this geometry, a discrete system is generated by a Lagrangian function defined on a Lie groupoid. The dynamical equations are obtained by the directional derivatives of the Lagrangian with left and right invariant vector fields at a finite sequence of composable elements. The theory of discrete dynamics on the Lie groupoid framework is studied extensively in the literature. We refer the reader to [16, 31] for the discrete dynamics involving constraints, and to [47] for the field theoretic approach. In particular, the theory of discrete dynamics on Lie groups, as expected, has a richer literature. We cite [5] for the discrete time Lagrangian mechanics on Lie groups, [34] for the discrete Lie-Poisson and the discrete Euler-Poincaré equations, and [35, 18] for the reduction of discrete systems under symmetry. For the discrete Hamiltonian dynamics in the realm of variational integrators we refer to [21], and [8, 7] for the higher order discrete dynamics. The local description of discrete mechanics can be found in [30], whereas the inverse problem of the calculus of variations in [1]. An implicit formulation of the discrete dynamics has been introduced in [17].

Much in the same way as the matched pair of Lie groups, the Lie groupoids (as well as their Lie algebroids) can be matched. Below, we shall study the discrete dynamics of a system consisting of a matched pair of Lie groupoids. We derive the equations of motion of such a system involving the mutual actions of the individual groupoids (and their Lie algebroids). Accordingly, the paper is organized into four sections. In the following section, the basics of the Lie groupoids and Lie algebroids will be recalled. In Section 3, definitions of the matched pair of Lie groupoids and the matched pair of Lie algebroids will be presented. Novelty of the paper is mostly in Section 4, where we shall calculate the discrete dynamics on the matched pair of Lie groupoids. This will present a geometric pathway to match (couple) two discrete systems under mutual interaction. In particular, we shall obtain the (matched) discrete dynamics in the Lie group setting. Section 5 will be reserved for concrete examples. In the first example we discuss the discrete dynamics on the trivial Lie groupoid from the the matched pair point of view, whereas in the second example we consider the discrete dynamics on the Lie group $SL(2, \mathbb{C})$ via its Iwasawa decomposition.

2. Lie groupoids and Lie algebroids

In order to fix the notation, as well as the convenience of the reader, we devote the present section to a brief summary of the basics of Lie groupoids and Lie algebroids. For further details, we refer the reader to the seminal book [24].

2.1. Lie groupoids and their actions.

2.1.1. Definition of a Lie groupoid.
We shall begin with the notion of the Lie groupoid. Let \( \mathcal{G} \) and \( B \) be two manifolds, and let there be two surjective submersions 
\[
\begin{array}{c}
  \mathcal{G} \\
  \alpha \downarrow \quad \beta \\
  B
\end{array}
\]
called the source map and the target map, respectively. We assume also that, there exists a smooth map 
\[
\varepsilon : B \longrightarrow \mathcal{G}, \quad b \mapsto \tilde{b},
\]
called the object inclusion. The product space 
\[
\mathcal{G} \ast \mathcal{G} := \{ (g, g') \in \mathcal{G} \times \mathcal{G} \mid \beta(g) = \alpha(g') \}
\]
the space of composable elements, and is equipped with the partial multiplication 
\[
\mathcal{G} \ast \mathcal{G} \longrightarrow \mathcal{G}, \quad (g, g') \mapsto gg'.
\]
The five-tuple \((\mathcal{G}, B, s, t, \varepsilon)\) with a partial multiplication is called a Lie groupoid if

\begin{enumerate}
\item \( \alpha(gg') = \alpha(g) \), and \( \beta(gg') = \beta(g') \),
\item \( g(g'g'') = (gg')g'' \),
\item \( \alpha(b) = \beta(b) = b \),
\item \( g\beta(g) = g = \alpha(g)g \),
\item there is \( g^{-1} \in \mathcal{G} \) such that \( \alpha(g^{-1}) = \beta(g) \) and \( \beta(g^{-1}) = \alpha(g) \), and that 
\[
 g^{-1}g = \beta(g), \quad gg^{-1} = \alpha(g),
\]
\end{enumerate}
for any \((g, g'), (g', g'') \in \mathcal{G} \ast \mathcal{G} \), any \( b \in B \), and any \( g \in \mathcal{G} \).

The elements of \( B \) are called “objects”, whereas the elements of \( \mathcal{G} \) are referred as “arrows” or “morphisms”. A groupoid may also be considered as a category such that all arrows are invertible. We shall denote a Lie groupoid by \( \mathcal{G} \Rightarrow B \), or simply by \( \mathcal{G} \) when there is no confusion on the base. Let us present some examples of Lie groupoids.

**Example 2.1.** Any Lie group \( G \) gives rise to a Lie groupoid over the identity element \( \{ e \} \): the source map and the target map being the constant maps \( \alpha = \beta : G \rightarrow \{ e \} \), and the object inclusion map being the obvious inclusion \( \tilde{e} = e \). Moreover, the partial multiplication of this groupoid is the group multiplication. We denote this Lie groupoid by \( G \Rightarrow \{ e \} \).

**Example 2.2.** Let \( G \) be a Lie group, and \( M \) a manifold with a smooth \( G \)-action \( M \times G \rightarrow M \) from the right. Then \( M \times G \Rightarrow M \) has the structure of a Lie groupoid over \( M \) equipped with the source map, the target map, and the object inclusion given by 
\[
\begin{align*}
  \alpha & : M \times G \longrightarrow M, \quad s(m, g) := m, \\
  \beta & : M \times G \longrightarrow M, \quad t(m, g) := mg, \\
  \varepsilon & : M \longrightarrow M \times G, \quad \varepsilon(m) := (m, e).
\end{align*}
\]
The partial multiplication is given by 
\[
(2.1) \quad (m, g) \cdot (m', g') := (m, gg'), \quad \text{if} \quad mg = m'.
\]
The groupoid \( M \times G \) is called as the “action groupoid”.

Example 2.3. Let \( M \) be a manifold. Then the cartesian product \( M \times M \) of \( M \) with itself is a groupoid over \( M \) via the source, target, and the object inclusion maps given by
\[
\begin{align*}
\alpha &: M \times M \rightarrow M, \quad s(m, m') := m, \\
\beta &: M \times M \rightarrow M, \quad t(m, m') := m', \\
\varepsilon &: M \rightarrow M \times M, \quad \varepsilon(m) := (m, m).
\end{align*}
\]
The partial multiplication is
\[(m, m') \cdot (n, n') := (m, n'), \quad \text{if} \quad m' = n.\]
This groupoid \( M \times M \rightrightarrows M \) is called the “coarse groupoid”, the “pair groupoid”, or the “banal groupoid”.

Example 2.4. Given a manifold \( M \), and a Lie group \( G \), the triple product \( M \times G \times M \) is a Lie groupoid over \( M \) by the source, target, and the object inclusion maps
\[
\begin{align*}
\alpha &: M \times G \times M \rightarrow M, \quad s(m, g, m') := m, \\
\beta &: M \times G \times M \rightarrow M, \quad t(m, g, m') := m', \\
\varepsilon &: M \rightarrow M \times G \times M, \quad \varepsilon(m) = \tilde{m} := (m, e, m),
\end{align*}
\]
and the partial multiplication
\[(m, g, m') \cdot (n, g', n') := (m, gg', n'), \quad \text{if} \quad m' = n.\]
The groupoid \( M \times G \times M \rightrightarrows M \) is called the “trivial groupoid”.

2.1.2. Left and right invariant vector fields.

Let \( \mathcal{G} \rightrightarrows B \) be a Lie groupoid over the base \( B \). A vector field \( Z \) on \( \mathcal{G} \) is called left invariant if
\[
Z(gg') = T_g' \ell_g Z(g'),
\]
for any \((g, g') \in \mathcal{G} \ast \mathcal{G}\), where \( \ell_g \) is the left translation on the total space \( \mathcal{G} \) of the Lie groupoid induced from the partial multiplication, and \( T_g' \ell_g \) is the tangent lift of this mapping at \( g' \). Similarly, a right invariant vector field \( Z \) on \( \mathcal{G} \) is the one satisfying
\[
Z(gg') = T_g r_{g'} Z(g)
\]
for any \((g, g') \in \mathcal{G} \ast \mathcal{G}\), where \( r_{g'} \) is the right translation mapping on \( \mathcal{G} \), and \( T_g r \) is the tangent lift of \( r_{g'} \) at \( g \) in \( \mathcal{G} \).

2.1.3. Morphism of Lie groupoids.

Let \( \mathcal{G} \) be a Lie groupoid over the base manifold \( B \), \( \mathcal{H} \) be another Lie groupoid over the base \( C \). A morphism of Lie groupoids is a pair of smooth maps \( \Phi : \mathcal{G} \rightarrow \mathcal{H} \) and \( \Phi_0 : B \rightarrow C \) respecting the groupoid multiplications, source, target and inclusion maps. More precisely, a Lie groupoid morphism is a pair \((\Phi, \Phi_0)\) satisfying
\[
\begin{align*}
(i) \quad (\Phi(g), \Phi(g')) & \in \mathcal{H} \ast \mathcal{H}, \\
(ii) \quad \Phi(gg') & = \Phi(g)\Phi(g'), \\
(iii) \quad a(\Phi(g)) & = \Phi_0(a(g)), \\
(iv) \quad \beta(\Phi(g)) & = \Phi_0(\beta(g)), \\
(v) \quad \Phi(b) & = \Phi_0(b),
\end{align*}
\]
for any \((g, g') \in \mathcal{G} \ast \mathcal{G}\), and any \(b \in B\). The requirements may be summarized by the commutativity of the diagram

\[
\begin{array}{c}
\mathcal{G} \quad \Phi \\
\downarrow \quad \downarrow \\
\mathcal{H} \\
\downarrow \Phi_0 \\
B \quad C
\end{array}
\]

for each source, target and inclusion map. In order to avoid the notation inflation, we shall not distinguish the source maps of the Lie groupoids \(\mathcal{G}\) and \(\mathcal{H}\) with different notations. Instead, we shall make it clear from the context.

2.1.4. Action of a Lie groupoid on a function.

We now recall the action of a Lie groupoid on a smooth map from [24]. Let \(\mathcal{G}\) be a Lie groupoid over the base \(B\), and consider a smooth map \(f : P \to B\) from a manifold \(P\) to the base manifold \(B\). Given the product space

\[P \ast \mathcal{G} := \{(p, g) \in P \times \mathcal{G} \mid f(p) = \alpha(g)\},\]

a smooth map

\[\triangleright : P \ast \mathcal{G} \to P, \quad (p, g) \mapsto p \triangleright g\]

is called the (right) action of \(\mathcal{G}\) on \(f\) if

(i) \(f(p \triangleright g) = \beta(g)\),

(ii) \((p \triangleright g) \triangleright g' = p \triangleright (gg')\),

(iii) \(p \triangleright f(p) = p\),

for any \((p, g) \in P \ast \mathcal{G}\), any \((g, g') \in \mathcal{G} \ast \mathcal{G}\), and any \(p \in P\). The definition may be summarized by the commutativity of the diagram

\[
\begin{array}{c}
P : \hspace{1cm} p \quad \triangleright \quad p \triangleright g \quad \triangleright \quad (p \triangleright g) \triangleright g' \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B : \hspace{1cm} \alpha(g) \quad \rightarrow \quad \beta(g) = \alpha(g') \quad \rightarrow \quad \beta(g')
\end{array}
\]

Here, \(P\) on the beginning of the first row means that all the elements in this row are the elements of the manifold \(P\), whereas \(B\) on the beginning of the second row indicates that all the elements in this line are the elements of the manifold \(B\).

The left action of a Lie groupoid on a smooth map is defined similarly, [24, 25]. Let \(\mathcal{H}\) be a Lie groupoid over the base manifold \(B\), and let \(f : P \to B\) be a smooth function from a manifold \(P\) to \(B\). Then for the product space

\[\mathcal{H} \ast P := \{(h, p) \in \mathcal{H} \times P \mid \beta(h) = f(p)\},\]
the smooth mapping

\[ \triangleright : \mathcal{H} \ast P \longrightarrow P, \quad (h, p) \mapsto h \triangleright p \]

is called a left action of \( \mathcal{H} \) on \( f \) if

(i) \( f(h \triangleright p) = \alpha(h) \)

(ii) \( h' \triangleright (h \triangleright p) = (h' h) \triangleright p \),

(iii) \( f(p) \triangleright p = p \)

for any \((h, p) \in \mathcal{H} \ast P\), any \((h', h) \in \mathcal{H} \ast \mathcal{H}\), and any \(p \in P\). In other words, the following diagram is commutative:

\[ \text{(2.4)} \]

\[ P : \quad h' \triangleright (h \triangleright p) \quad \quad h \triangleright p \quad \quad p \]

\[ B : \quad \alpha(h') \quad \quad \beta(h') = \alpha(h) \quad \quad \beta(h) \]

We conclude this subsection by defining left and right action of a Lie groupoid on another Lie groupoid. Let \( \mathcal{G} \) be a Lie groupoid over the base manifold \( B \), and \( \mathcal{H} \) be another Lie groupoid over the base \( C \). The left action of \( \mathcal{H} \) on \( \mathcal{G} \) is defined to be the (left) action of \( \mathcal{H} \) on the source map \( \alpha : \mathcal{G} \rightarrow B \). The right action of \( \mathcal{G} \) on \( \mathcal{H} \) is similarly defined to be the right action of \( \mathcal{G} \) on the target map \( \beta : \mathcal{H} \rightarrow C \). For further details on the representations of Lie groupoids we refer the reader to [5, 25].

2.2. Lie algebroids and their actions.

2.2.1. Definition of a Lie algebroid.

A Lie algebroid over a manifold \( M \) may be thought of a generalization of the tangent bundle \( TM \) of \( M \), [24, 39, 41]. More technically, given a manifold \( M \), a Lie algebroid \( A \) over the base \( M \) is a (real) vector bundle \( \tau : A \rightarrow M \), together with a map \( a : A \rightarrow TM \) of vector bundles, called the anchor map, and a Lie bracket \( [\cdot, \cdot] \) (bilinear, anti-symmetric, satisfying the Jacobi identity) on the sections of the vector bundle \( \tau : A \rightarrow M \) so that

(i) \( a([X, Y]) = [a(X), a(Y)] \),

(ii) \( [X, fY] = f[X, Y] + \mathcal{L}_{a(X)}(f)Y \),

for any \( X, Y \in \Gamma(A) \), and any \( f \in C^\infty(M) \), where, \( \mathcal{L}_{a(X)}(f) \) stands for the directional derivative of \( f \in C^\infty(M) \) in the direction of \( a(X) \in TM \). Accordingly, a Lie algebroid is denoted by a quintuple \( (A, \tau, M, a, [\cdot, \cdot]) \), or occasionally by a triple \( (A, \tau, M) \) when there is no confusion on the bracket, and the anchor map. Let us now take a quick tour on a bunch of critical examples.

Example 2.5. Any Lie algebra \( g \) is a Lie algebroid \( \tau : g \rightarrow \{*\} \); taking the base manifold \( B = \{*\} \) to be a one-point set, and the anchor map \( a : g \rightarrow TB \) to be the zero map.
Example 2.6. Given any manifold $M$, the tangent bundle $TM$ is a Lie algebroid over the base $M$. Indeed, setting $\tau$ to be the tangent bundle projection $\tau_M : TM \to M$, and the anchor $a : TM \to TM$ to be the identity mapping, one can easily verify the conditions (i) and (ii).

Example 2.7. Let $M$ be manifold admitting an infinitesimal left action of a Lie algebra $\mathfrak{g}$. As such, there exists a linear map from the Lie algebra $\mathfrak{g}$ to the sections of the tangent bundle $TM$ of $M$ preserving the Lie brackets. This linear map takes a Lie algebra element $\xi \in \mathfrak{g}$ and maps it to a vector field $X_\xi \in \mathfrak{X}(M)$. Consider the cartesian product $M \times \mathfrak{g}$ as the total space of the trivial bundle $M \times \mathfrak{g} \to M$ via the projection onto the first component, it is possible to construct a Lie algebroid structure on this trivial bundle; \cite{14, 19, 22, 39}. The (fiber preserving) anchor map of this Lie algebroid is given by

$$a : M \times \mathfrak{g} \to TM, \quad a(m, \xi) := X_\xi(m).$$

Furthermore, viewing the sections of this trivial bundle as maps $u, v : M \to \mathfrak{g}$, the Lie bracket is the one given by

$$[u, v](m) = [u(m), v(m)]_\mathfrak{g} + (\mathcal{L}_{X_{u(m)}} v)(m) - (\mathcal{L}_{X_{v(m)}} u)(m).$$

Here, $u(m)$ and $v(m)$ are the elements of the Lie algebra $\mathfrak{g}$, and the bracket $[\cdot, \cdot]_\mathfrak{g}$ is the Lie bracket on $\mathfrak{g}$. Note that, $X_{u(m)}$ is a vector field on $M$ generated by the Lie algebra element $u(m)$, and $\mathcal{L}_{X_{u(m)}} v$ is the derivative of the Lie algebra valued function $v$ by the vector field $X_{u(m)}$. In the literature, this Lie algebroid is called the action Lie algebroid (or the transformation Lie algebroid).

2.2.2. Lie algebroid of a Lie groupoid.

We now recall from \cite{10, 26, 41} that how a Lie algebroid associates to a given Lie groupoid in a canonical way. To this end, we start with a Lie groupoid $\mathcal{G}$ over the base manifold $B$, and consider the source map $\alpha$ projecting a mapping $g$ in the Lie groupoid $\mathcal{G}$ to its initial point. The tangent lift of the source map is a linear operator $T_{\tilde{\alpha}}s$ from the tangent space $T_g\mathcal{G}$ to the tangent space $T_{\alpha(b)}B$ for each $g$. Along the lines of \cite{32}, we identify the fiber of the algebroid $(\mathcal{A}\mathcal{G}, \tau, B)$ at the point $b$ in $B$ with the kernel of $T_{\tilde{\beta}}\alpha$, that is,

$$\mathcal{A}_b\mathcal{G} := \ker T_{\tilde{\beta}}\alpha,$$

where $\bar{b}$ is the image of $b$ in $B$ under the inclusion mapping $\epsilon$. Hence, $\mathcal{A}\mathcal{G}$ corresponds to the vertical bundle on $\mathcal{G}$ with respect to the fibration $\alpha : \mathcal{G} \to B$. We denote a section of the fibration $\mathcal{A}\mathcal{G} \to B$ by $X$. The anchor map $a$ is from $\mathcal{A}\mathcal{G}$ to the tangent bundle $TB$ of the base manifold $B$ and it is defined as

$$a(X(b)) = T_{\tilde{\beta}}\beta \circ X(b)$$

where $T_{\tilde{\beta}}\beta$ is the tangent lift of the target map $\beta$. Finally, the definition of the Lie bracket $[\mathfrak{A}\mathcal{G}]$ on the sections of $\mathcal{A}\mathcal{G} \to B$ follows from the direct analogy with that of the Lie groups. More precisely, the bracket on the (associated) Lie algebroid is defined by means of the Jacobi-Lie bracket of the left (or the right invariant) vector fields on the groupoid; \cite{9, 24, 32}.

For a Lie groupoid $\mathcal{G} \to B$, there exists an isomorphism of $C^\infty(\mathcal{G})$-modules between the sections $\Gamma(\mathcal{A}\mathcal{G})$ of the Lie algebroid of the Lie groupoid and the left-invariant (resp. the right invariant) vector fields on $\mathcal{G}$; \cite{26}. For a section $X$, of the Lie algebroid bundle $\mathcal{A}\mathcal{G} \to B$, a left invariant vector field $\overline{X}$ on $\mathcal{G}$ is defined to be

$$\overline{X}(g) := T_{\beta(g)}\ell_g X(\beta(g)),$$
where $\beta(g)$ is the projection of $g$ to $B$ under the target map on $G$, and $\tilde{\beta}(g)$ is the image of the projection $\beta(g)$ under the object inclusion map $\epsilon$.

Note that, then, a left invariant vector field satisfies

$$\tilde{X}(g_1g_2) = T_{g_2}\ell_{g_1}\tilde{X}(g_2),$$

for any $g_1, g_2 \in G$. Conversely, if $\tilde{X}$ is a left invariant vector field then the equation

$$X(b) := \tilde{X}(\tilde{b})$$

defines a section of the Lie algebroid. After identifying the sections of the Lie algebroid with the left invariant vector fields, let us now define the Lie bracket on the sections. If $X$ and $Y$ are two sections of the Lie algebroid $\mathcal{A}G$ then we define the Lie bracket of $X$ and $Y$ by

$$(2.7) \quad [X, Y]_{\mathcal{A}G}(b) = [\tilde{X}, \tilde{Y}](\tilde{b})$$

where the bracket on the right hand side is the Jacobi-Lie bracket of left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ on $G$. Here, $b$ is an element of $B$, whereas $\tilde{b}$ is the image of $b$ under the object inclusion hence it is an element of $G$.

For the right invariant vector fields, the picture is as follows. Assume that $X$ be a section of $\mathcal{A}G \rightarrow B$, we define a right invariant vector field $\overrightarrow{X}$ on $G$ as

$$(2.8) \quad \overrightarrow{X}(g) := -T_{\alpha(g)}r_g \circ T_{\alpha(g)}\text{inv}(X(\alpha(g))),$$

where “inv” stands for the inversion operator on $G$ which maps an element $g$ in $G$ to its inverse $g^{-1}$ in $G$. Here, $\alpha(g)$ in $B$ is the projection of $g$ by the source map, and $\alpha(g)$ in $G$ is its image under the object inclusion mapping $\epsilon$. Let $x(t)$ be the curve through $G$ which is tangent to $X$, that is $x(0) = X \in \mathcal{A}_{\alpha(g)}G$. We note also that, since $\alpha(x(t))$ is constant and is equal to $\alpha(g)$, we can define the curve $x^{-1}(t)g$ in $G$. As a result, the right invariant vector field corresponding to $X$ can be given as

$$(2.9) \quad \overrightarrow{X}(g) = -\frac{d}{dt}{\bigg|}_{t=0} x^{-1}(t)g.$$ 

It follows at once from the definition that

$$\overrightarrow{X}(g_1g_2) = Tr_{g_2}\overrightarrow{X}(g_1)$$

for any $g_1, g_2 \in G$. In this case, the relation between the Lie brackets are given by

$$[X, Y]_{\mathcal{A}G} = -[\overrightarrow{X}, \overrightarrow{Y}],$$

where the bracket on the left side is the one on the Lie algebroid level, whereas the bracket on the right hand side is the Jacobi-Lie bracket of vector fields on the manifold $G$.

Let us next present examples on the construction of the left and the right invariant vector fields on the Lie groupoids in Example 2.1 - 2.4 which will be needed in the sequel.

**Example 2.8.** Recall Example 2.1 where we consider a Lie group $G$ as a Lie groupoid $G \rightarrow \{e\}$ over the identity element $e$. Under the inclusion, we have that $\tilde{\epsilon} = e$, and the tangent space at $\tilde{\epsilon}$ is $T_\epsilon G = g$. So that, kernel of the tangent lift of the source map results with that the total space of
the associated Lie algebroid $\mathcal{A}G$ can be identified with the Lie algebra $\mathfrak{g}$ of the group $G$. The right and the left invariant vector fields associated to $\xi \in \mathfrak{g}$ are

\begin{equation}
\vec{\xi} : G \longrightarrow TG, \quad g \mapsto T_{e}r_{g}(\xi),
\end{equation}

\begin{equation}
\check{\xi} : G \longrightarrow TG, \quad g \mapsto T_{e}l_{g}(\xi).
\end{equation}

**Example 2.9.** The Lie algebroid of the coarse groupoid $M \times M \rightrightarrows M$ of Example 2.3 is the Lie algebroid $(TM, \tau_{M}, M)$ of Example 2.6. Indeed, consider a curve $(m, n_{t}) \in M \times M$ with constant source, so that $n_{0} = n \in M$ and $n_{0} = X \in T_{n}M$. Then its derivative (at $t = 0$) yields a vector $(\theta_{m}, X) \in T_{n}M \times T_{n}M$. Note that, in this case, the anchor map is the inclusion mapping. Given any section $(\theta_{m}, X) \in \mathcal{A}_{m}(M \times M)$, the corresponding right and the left invariant vector fields on $M \times M$ are computed to be

\begin{equation}
(\theta_{m}, X) : M \times M \longrightarrow T(M \times M) = TM \times TM, \quad (m, n) \mapsto (-X, \theta_{m}),
\end{equation}

\begin{equation}
(\theta_{m}, X) : M \times M \longrightarrow T(M \times M) = TM \times TM, \quad (m, n) \mapsto (\theta_{m}, X).
\end{equation}

**Example 2.10.** The Lie algebroid of the action groupoid $M \times G \rightrightarrows M$ of Example 2.2 is the transformation Lie algebroid $(M \times \mathfrak{g}, pr_{1}, M)$ of Example 2.7. Indeed, the derivative of a curve $(m, g_{t}) \in M \times G$ of constant source, such that $g_{0} = e$ and that $g_{0} = \xi \in \mathfrak{g}$, yields a vector $(\theta_{m}, \xi) \in T_{n}M \times T_{G}G$. As such, the total space $\mathcal{A}(M \times G)$ may be identified with the cartesian product $M \times \mathfrak{g}$. For any $(\theta_{m}, \xi) \in \mathcal{A}_{m}(M \times G)$, the left invariant and the right invariant vector fields are given by

$$
(\check{\theta}_{m}, \check{\xi})(m, g) = \left(-\xi^{\sharp}(m), \vec{\xi}(g)\right), \quad (\check{\theta}_{m}, \check{\xi})(m, g) = \left(\theta_{m}, \vec{\xi}(g)\right),
$$

where the infinitesimal generator of the right action is computed to be

$$
\xi^{\sharp}(m) := \left. \frac{d}{dt} \right|_{t=0} me^{t}
$$

for a curve $e_{t} \in G$ with $e_{0} = e \in G$, and $\dot{e}_{0} = \xi \in \mathfrak{g}$.

**Example 2.11.** The Lie algebroid of the trivial groupoid of Example 2.4 is the Lie algebroid $(M \times \mathfrak{g}) \oplus TM$. Indeed, given a curve $(m, e_{t}, n_{t}) \in M \times G \times M$ with $e_{0} = e \in G, m_{0} = n_{0} = m \in M, \dot{e}_{0} = \xi \in \mathfrak{g}, m_{0} = X \in T_{m}M$, and $n_{0} = Y \in T_{n}M$,

$$
T_{(m,e_{m},m)}(\alpha(X, \xi), Y) = X.
$$

As such,

$$
\mathcal{A}_{m}(M \times G \times M) = \{(\theta_{m}, \xi, Y) \mid \xi \in \mathfrak{g}, Y \in T_{m}M\}
$$

which yields $\mathcal{A}(M \times G \times M) \cong (M \times \mathfrak{g}) \oplus TM$. In order to compute the left (resp. right) invariant vector fields, let $(m, g, n) \in M \times G \times M$, and let $(\theta_{m}, \xi, X) \in \mathcal{A}_{n}(M \times G \times M)$. Accordingly, let $(n, e_{t}, n_{t}) \in M \times G \times M$ be a curve such that $e_{0} = n, n_{0} = n, \dot{e}_{0} = \xi$, and $n_{0} = Y \in T_{n}M$. We then see that

\begin{equation}
(\check{\theta}_{m}, \check{\xi})(m, g, n) = \left. \frac{d}{dt} \right|_{t=0} (m, g, n)(e_{t}, n_{t}) = \left. \frac{d}{dt} \right|_{t=0} (m, ge_{t}, n_{t}) = (\theta_{m}, \check{\xi}(g), Y).
\end{equation}

Similarly, given $(\theta_{m}, \xi, X)$, if $(m, e_{t}, m_{t}) \in M \times G \times M$ be a curve such that $e_{0} = e, m_{0} = m, \dot{e}_{0} = \xi$, and $m_{0} = X \in T_{m}M$, then

\begin{equation}
(\check{\theta}_{m}, \check{\xi}, X)(m, g, n) = -\left. \frac{d}{dt} \right|_{t=0} (m, e_{t}, m_{t})^{-1} (m, g, n) = -\left. \frac{d}{dt} \right|_{t=0} (m, e_{t}^{-1}, m)(m, g, n) = -\left. \frac{d}{dt} \right|_{t=0} (m_{t}, e_{t}^{-1}g, n) = (-X, \vec{\xi}(g), \theta_{n}).
\end{equation}
2.2.3. Lie algebroid morphisms.

Given two Lie algebroids \((\mathcal{A}, \tau, M)\) and \((\mathcal{A}', \tau', M')\), a morphism from \((\mathcal{A}, \tau, M)\) to \((\mathcal{A}', \tau', M')\) is a vector bundle morphism preserving the anchors as well as the brackets. That is, a pair \((\phi : \mathcal{A} \to \mathcal{A}', \phi_0 : M \to M')\) such that

\[
\begin{array}{ccc}
\mathcal{A} & \phi & \mathcal{A}' \\
\tau \\
M & \phi_0 & M',
\end{array}
\]

that

\[a' \circ \phi = T\phi_0 \circ a,\]

where \(a : \mathcal{A} \to TM\) and \(a' : \mathcal{A}' \to TM'\) are the respective anchor maps, and finally that

\[
\phi([X,Y]) = [\phi(X), \phi(Y)].
\]

It is possible to derive a Lie algebroid morphism starting from a Lie groupoid morphism as follows. Given two Lie groupoids \(\mathcal{G} \rightrightarrows B\) and \(\mathcal{H} \rightrightarrows C\). Let

\[
\Phi : \mathcal{G} \to \mathcal{H}; \quad \Phi_0 : M \to N
\]

be a morphism of Lie groupoids. Then, for \(\mathcal{A}_m\mathcal{G} \ni X = \frac{d}{dt}_{t=0} x_t\), where \(a(x_t) = m\),

\[
\mathcal{A}_m\phi : \mathcal{A}_m\mathcal{G} \to \mathcal{A}_{\Phi_0(m)}\mathcal{H}; \quad X \mapsto \frac{d}{dt}_{t=0} \Phi(x_t)
\]

defines a morphism \(\mathcal{A}\mathcal{G} \to \mathcal{A}\mathcal{H}\) of Lie algebroids associated with the Lie groupoids \(\mathcal{G}\) and \(\mathcal{H}\), respectively. We refer [24, Sect. 3.5] for the details.

3. Matched pairs of Lie groupoids and matched pairs of Lie algebroids

3.1. Matched pairs of Lie groupoids.

3.1.1. Definition of a matched Lie groupoid.

In this subsection we recall, mainly from [39, 25], the groupoid level of the matched pair theory of [28, 29]. Let \(\mathcal{G} \rightrightarrows B\) and \(\mathcal{H} \rightrightarrows B\) be two Lie groupoids over the same base \(B\), and let \(\mathcal{H}\) act on \(\mathcal{G}\) from the left by

\[
\triangleright : \mathcal{H} \ast \mathcal{G} \longrightarrow \mathcal{G}, \quad (h, g') \mapsto h \triangleright g'.
\]

Here, the set \(\mathcal{H} \ast \mathcal{G}\) of composable elements consists of the pairs \((h, g)\) in the cartesian product \(\mathcal{H} \times \mathcal{G}\) such that \(\beta(h) = \alpha(g)\). Being a left action, \(\triangleright\) satisfies the identities

(i) \(\alpha(h) = \alpha(h \triangleright g')\),
(ii) \((h'h) \triangleright g' = h' \triangleright (h \triangleright g')\)
(iii) \(\alpha(h) \triangleright g' = g'\) for any \(h \in \mathcal{H}\)
for any \((h, g') \in \mathcal{H} \ast \mathcal{G}\), any \((h', h) \in \mathcal{H} \ast \mathcal{H}\), and any \(h \in \mathcal{H}\). Let also \(\mathcal{G}\) act on \(\mathcal{H}\) from right by

\[ (h, g') \mapsto h \ast g' \quad \text{for any } (h, g') \in \mathcal{H} \ast \mathcal{G}. \]

Then, being a right action, \(\ast\) satisfies the identities

\[
\begin{align*}
(iv) & \quad \beta(g') = \beta(h \ast g'), \\
(v) & \quad h \ast g' g = (h \ast g') \ast g, \\
(vi) & \quad h \ast \beta(g') = h,
\end{align*}
\]

for any \((h, g') \in \mathcal{H} \ast \mathcal{G}\), any \((g', g) \in \mathcal{G} \ast \mathcal{G}\), and any \(g' \in \mathcal{G}\). Now, the pair \((\mathcal{G}, \mathcal{H})\) is called a “matched pair of Lie groupoids” if, in addition, the compatibilities

\[
\begin{align*}
(vii) & \quad \beta(h \triangleright g') = \alpha(h \ast g'), \\
(viii) & \quad h \triangleright (g' g) = (h \triangleright g')(h \ast g' \triangleright g), \\
(ix) & \quad (h' h) \ast g' = (h' \ast (h \triangleright g'))(h \ast g'),
\end{align*}
\]

are also satisfied for any \((h, g') \in \mathcal{H} \ast \mathcal{G}\), any \((g', g) \in \mathcal{G} \ast \mathcal{G}\), and any \((h', h) \in \mathcal{H} \ast \mathcal{H}\).

Then, the product space

\[
\mathcal{G} \bowtie \mathcal{H} := \mathcal{G} \ast \mathcal{H} = \{(g, h) \in \mathcal{G} \times \mathcal{H} \mid \beta(g) = \alpha(h)\}
\]

becomes a Lie groupoid by the partial multiplication, on

\[
(\mathcal{G} \bowtie \mathcal{H}) \ast (\mathcal{G} \bowtie \mathcal{H}) := \{((g, h), (g', h')) \in (\mathcal{G} \bowtie \mathcal{H}) \times (\mathcal{G} \bowtie \mathcal{H}) \mid \beta(h) = \alpha(g')\},
\]

given by

\[ ((g, h), (g', h')) \mapsto (g(h \triangleright g'), (h \ast g')h'), \]

where \(\triangleright\) stands for the left action of \(\mathcal{H}\) on \(\mathcal{G}\) in \((3.1)\) whereas \(\ast\) stands for the right action of \(\mathcal{G}\) on \(\mathcal{H}\) in \((3.2)\).

The source and target maps of the “matched pair Lie groupoid” \(\mathcal{G} \bowtie \mathcal{H}\) are given by

\[
\begin{align*}
\alpha : \mathcal{G} \bowtie \mathcal{H} & \longrightarrow B, \quad (g, h) \mapsto \alpha(g), \\
\beta : \mathcal{G} \bowtie \mathcal{H} & \longrightarrow B, \quad (g, h) \mapsto \beta(h),
\end{align*}
\]

respectively. The object inclusion map of the matched pair Lie groupoid is defined in terms of those of \(\mathcal{G}\) and \(\mathcal{H}\) as

\[
\epsilon : B \longrightarrow \mathcal{G} \bowtie \mathcal{H}, \quad b \mapsto (\tilde{b}, b).
\]

The relation between the matched pair Lie groupoid \(\mathcal{G} \bowtie \mathcal{H}\) and the individual Lie groupoids \(\mathcal{G}\) and \(\mathcal{H}\) is given in \([25, \text{Thm. 2.10}]\) which we record below.

**Proposition 3.1.** A pair \((\mathcal{G}, \mathcal{H})\) of groupoids is a matched pair of groupoids if and only if the manifold \(\mathcal{G} \ast \mathcal{H}\) has the structure of a Lie groupoid, such that

\[
\begin{align*}
(i) & \quad \text{the maps } \mathcal{G} \rightarrow \mathcal{G} \ast \mathcal{H} \text{ given by } g \mapsto (g, \tilde{\beta}(g)) \text{ and } \mathcal{H} \rightarrow \mathcal{G} \ast \mathcal{H} \text{ given by } h \mapsto (\tilde{\alpha}(h), h) \\
(ii) & \quad \text{the multiplication } ((g, \tilde{\beta}(g)), (\tilde{\alpha}(h), h)) \mapsto (g, h) \in \mathcal{G} \ast \mathcal{H} \text{ is a diffeomorphism.}
\end{align*}
\]
Picturing an element \((g, h) \in \mathcal{G} \rightharpoonup \mathcal{H}\) by a corner

\[
\begin{array}{ccc}
\alpha(g) & \xrightarrow{g} & \beta(g) = \alpha(h), \\
\beta(h) & \xleftarrow{h} & \\
\end{array}
\]

the matched pair partial multiplication \((3.3)\) may be summarized by

\[
\begin{array}{ccc}
\alpha(g) & \xrightarrow{g} & \beta(g) = \alpha(h) = \alpha(g \triangleright g') \\
\beta(h) = \alpha(g') & \xleftarrow{g'} & \beta(h \triangleright g') = \alpha(h \triangleright g'). \\
\end{array}
\]

Let us conclude the present subsection with the following remark about the actions on the identity elements.

**Remark 3.2.** Let \((\mathcal{G}, \mathcal{H})\) be a matched pair of Lie groupoids. Given \(g \in \mathcal{G}\) with \(\alpha(g) = b_1\) and \(\beta(g) = b_2\), and \(h \in \mathcal{H}\) with \(\alpha(h) = b_3\) and \(\beta(h) = b_4\), we see at once that

\[
h \triangleright \widetilde{b}_4 = \widetilde{b}_3 \in \mathcal{G}, \quad \widetilde{b}_1 \triangleleft g = \widetilde{b}_2 \in \mathcal{H}.
\]

and that,

\[
\widetilde{b}_1 \triangleright g = g, \quad h \triangleright \widetilde{b}_4 = h.
\]

3.1.2. **Matched pair decomposition of the trivial groupoid.**

Given the action groupoid \(\mathcal{G} = M \times G\) of Example 2.2 and the coarse groupoid \(\mathcal{H} = M \times M\) of Example 2.3 let us define the set

\[
\mathcal{H} \ast \mathcal{G} = (M \times M) \ast (M \times G) = \{(m', m; m, g) \in (M \times M) \times (M \times G)\}
\]

of composable elements. The left action

\[
\triangleright : (M \times M) \ast (M \times G) \longrightarrow (M \times G), \quad (m', m) \triangleright (m, g) := (m', g)
\]

of the action groupoid \(\mathcal{G} = M \times G\) on the coarse groupoid \(\mathcal{H} = M \times M\), and the right action

\[
\triangleleft : (M \times M) \ast (M \times G) \longrightarrow (M \times M), \quad (m', m) \triangleleft (m, g) := (m'g, mg)
\]

of the action groupoid \(\mathcal{G} = M \times G\) on the coarse groupoid \(\mathcal{H} = M \times M\) satisfies the conditions (i)-(ix) of the previous subsection. Thus, the set

\[
\mathcal{G} \ast \mathcal{H} = (M \times G) \ast (M \times M) = \{(m, g; mg, m') \in (M \times G) \times (M \times M)\}
\]
such that

\[ \Phi(\gamma, \alpha) = (M \times G) \rightarrow (M \times M), \quad (m, g; mg, m') \mapsto m, \]

and

\[ \Phi(\gamma, \beta) = (M \times G) \rightarrow (M \times M), \quad (m, g; mg, m') \mapsto m', \]

and

\[ \Phi(\gamma, \epsilon) = M \rightarrow (M \times G) \rightarrow (M \times M), \quad m \mapsto (m, e; m, m). \]

In order to proceed to the partial multiplication, we consider the product space

\[ ((M \times G) \rightarrow (M \times M)) \ast ((M \times G) \rightarrow (M \times M)) \]

\[ := \{(m, g; mg, m'), (m', h; m'h, n) : m', m, n \in M \text{ and } g, h \in G\}. \]

The partial multiplication, given by (3.3), then appears as

\[ (m, g; mg, m') \ast (m', h; m'h, n) = ((m, g)((mg, m') \ast (m', h)); ((mg, m') \ast (m', h))(m'h, n)) \]

\[ = (m, g)(mg, h); (mg, m'h)(m'h, n)) = (m, gh; mgh, n). \]

Accordingly, the inversion is computed to be

\[ (m, g; mg, n)^{-1} = (n, g^{-1}; ng^{-1}, m). \]

The matched pair Lie groupoid \((M \times G) \rightarrow (M \times M)\) is identified with the trivial Lie groupoid \(M \times G \times M\) via

\[ \Phi: M \times G \times M \rightarrow (M \times G) \rightarrow (M \times M), \quad (m, g, n) \mapsto (m, g; mg, n), \]

see for instance, [39].

Let us finally note that the map (3.14) that gives the matched pair decomposition of the trivial groupoid is differentiated to the Lie algebroid morphism

\[ \mathcal{A}_m \Phi: \mathcal{A}_m(M \times G \times M) \rightarrow \mathcal{A}_m((M \times G) \rightarrow (M \times M)), \]

\[ (\theta, \xi, Y) \mapsto (\theta, \xi; \xi'(m), Y) \]

Indeed, for a curve \((m, e_t, m_t) \in M \times G \times M\) with \(e_0 = e, m_0 = m, e_0 = \xi\), and \(m_0 = Y\), we compute

\[ (\mathcal{A}_m \Phi)(\theta, \xi, Y) = \left. \frac{d}{dt} \Phi(m, e_t, m_t) \right|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} (m, e_t; me_t, m_t) = (\theta, \xi; \xi'(m), Y). \]

3.2. Matched pairs of Lie algebroids.

Let us begin with a brief discussion on the representation of a Lie algebroid on a vector bundle from [14, 39, 22].

Let \((\mathcal{A}, \tau, M)\) be a Lie algebroid with an anchor map \(a\), and let \((E, \pi, M)\) be a vector bundle over the same base manifold \(M\). A left representation of \((\mathcal{A}, \tau, M)\) to \((E, \pi, M)\) is a bilinear map

\[ \rho: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E), \quad (X, s) \mapsto \rho_X(s) = \rho(X, s) \]

such that

(i) \(\rho_f Y = f \rho_X(Y)\),

(ii) \(\rho_X(f Y) = f \rho_X(Y) + (a(X)f)Y\),

(iii) \(\rho_{X,Y} = \rho_X(\rho_Y(Y)) - \rho_Y(\rho_X(Y))\).
for any \( X, \tilde{X} \in \Gamma(A) \), any \( Y \in \Gamma(E) \), and any \( f \in C^\infty(M) \). A right representation of a Lie algebroid on a vector bundle is defined similarly.

Two Lie algebroids with the same base manifold form a matched pair of Lie algebroids if the direct sum of the total spaces of the Lie algebroids has a Lie algebroid structure on the same base such that individual Lie algebroids are Lie subalgebroids of the direct sum, \([39]\). More precisely, let \((\mathcal{A}, \tau, M)\) and \((\mathcal{B}, \kappa, M)\) be two Lie algebroids over the same base \( M \) with the anchor maps \( a : \mathcal{A} \to TM \) and \( b : \mathcal{B} \to TM \), and mutual representations

\[
\rho : \Gamma(\mathcal{B}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A}), \quad \rho' : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{B}) \to \Gamma(\mathcal{B}),
\]

satisfying

(i) \( \rho_Y[X, \tilde{X}] = [\rho_Y(X), \tilde{X}] + [X, \rho_Y(\tilde{X})] - \rho_{\rho'_X(Y)}(\tilde{X}) + \rho_{\rho'_Y(X)}(\tilde{Y}) \)

(ii) \( \rho'_X[Y, \tilde{Y}] = [\rho_X(Y), \tilde{Y}] + [Y, \rho_X(\tilde{Y})] - \rho'_{\rho_Y(X)}(\tilde{X}) + \rho'_{\rho_Y(\tilde{X})}(X) \)

(iii) \( [b(Y), a(X)] = a(\rho_Y(X)) - b(\rho'_Y(Y)) \)

for any \( X, \tilde{X} \in \Gamma(\mathcal{A}) \), and any \( Y, \tilde{Y} \in \Gamma(\mathcal{B}) \). Then, the direct sum vector bundle \( \mathcal{A} \bowtie \mathcal{B} := \mathcal{A} \oplus \mathcal{B} \) has the structure of a Lie algebroid by the bracket given by

\[
[Y, X] = \rho(Y, X) - \rho'(X, Y)
\]

for any \( X \in \Gamma(\mathcal{A}) \), and any \( Y \in \Gamma(\mathcal{B}) \). The pair \((\mathcal{A}, \mathcal{B})\) of Lie algebroids is called a “matched pair of Lie algebroids”, whereas the vector bundle \( \mathcal{A} \bowtie \mathcal{B} \) is called a “matched pair Lie algebroid”.

### 3.3. Lie algebroid actions induced from Lie groupoid actions.

We devote the present subsection on the infinitesimal versions of the mutual actions of a matched pair \((\mathcal{G}, \mathcal{H})\) of Lie groupoids, over a base manifold \( B \).

To this end, we first consider an element \( h \in \mathcal{H} \), and a curve \( x_t \) so that \( \beta_H(h) = \alpha_G(x_t) = b \). Since the curve \( x_t \) has a constant source, its derivative at \( t = 0 \) qualifies as a Lie algebroid element, which we denote by \( X \in \mathcal{A}_b \mathcal{G} \). Now, using the left action, we define a curve \( \tilde{x}_t := h \triangleright x_t \) in \( \mathcal{G} \), whose source \( \alpha(h \triangleright x(t)) \) is a single point \( \alpha(h) = c \) in \( B \). As such, the time derivative of the curve \( h \triangleright x_t \) lies in the kernel of the tangent mapping of the source map. At \( t = 0 \), we arrive at an element

\[
h \triangleright X := \left. \frac{d}{dt} \right|_{t=0} (h \triangleright x_t) \in \mathcal{A}_c \mathcal{G}.
\]

in the Lie algebroid \( \mathcal{A}_c \mathcal{G} \). On the other hand, we can also define a curve \( h \bullet x_t \) in \( \mathcal{H} \) which passes through \( h \) at \( t = 0 \) by \((3.7)\). We note that the source map is not constant now, and that the time derivative

\[
X^\triangleright(h) = h \bullet X := \left. \frac{d}{dt} \right|_{t=0} (h \bullet x_t) \in T_h \mathcal{H}
\]

of the curve is not necessarily in the Lie algebroid \( \mathcal{A}_c \mathcal{H} \).

Let \( y_t \) be a curve in \( \mathcal{H} \) passing through \( \epsilon_H(c) \) at \( t = 0 \), and generating a Lie algebroid element \( Y \) in \( \mathcal{A}_c \mathcal{H} \). So that we are assuming that the source map \( \alpha_H \) takes \( y_t \) to a single point \( \alpha_H(g) \) for
a $g$ in $\mathcal{G}$. We define the left infinitesimal action of $\mathcal{A}\mathcal{H}$ on $\mathcal{G}$ as the time derivative of the curve $y_t^{-1} \triangleright g$ that is
\begin{equation}
Y^\triangleright (g) := \left. \frac{d}{dt} \right|_{t=0} y_t^{-1} \triangleright g \in T_g \mathcal{G}.
\end{equation}
Similarly, we can lift the right action of $\mathcal{G}$ on $\mathcal{H}$ to the right action of $\mathcal{G}$ on $\mathcal{A}\mathcal{H}$ as follows. Starting with the curve $y_t$, we define the curve $(y_t^{-1} \triangleleft g)^{-1}$ in $\mathcal{H}$. The source of the latter curve is constant, hence, at $t = 0$ the curve is tangent to an element in $\mathcal{A}_c \mathcal{H}$, which is given by
\begin{equation}
Y \triangleleft g := \left. \frac{d}{dt} \right|_{t=0} (y_t^{-1} \triangleleft g)^{-1} \in \mathcal{A}_c \mathcal{H}.
\end{equation}

### 3.4. Lie algebroid of a matched pair Lie groupoid.

In Subsection 3.2.3, we have exhibited how a Lie algebroid is derived from a Lie groupoid. In this subsection, we will repeat this construction for a matched Lie groupoid, say $\mathcal{G} \bowtie \mathcal{H}$. At the end, we shall arrive at the Lie algebroid $\mathcal{A}(\mathcal{G} \bowtie \mathcal{H})$ of the matched pair groupoid, and present an isomorphism between $\mathcal{A}(\mathcal{G} \bowtie \mathcal{H})$ and the matched pair of Lie algebroids $\mathcal{A}\mathcal{G} \bowtie \mathcal{A}\mathcal{H}$.

To begin with, let $\mathcal{G} \bowtie \mathcal{H}$ be a matched pair of Lie groupoids. Let also $X \in \mathcal{A}_b \mathcal{G}$, and $Y \in \mathcal{A}_b \mathcal{H}$; that is, there are curves $x_t \in \mathcal{G}$ and $y_t \in \mathcal{H}$ so that $s(x_t) = b = s(y_t)$, and that $x_0 = \epsilon_G(b) \in \mathcal{G}$ with $y_0 = \epsilon_H(b) \in \mathcal{H}$. Let us next consider the groupoid embeddings
\begin{align}
\mathcal{G} & \rightarrow \mathcal{G} \bowtie \mathcal{H}, \quad x_t \mapsto (x_t, (\epsilon_H \circ \beta)(x_t)) \\
\mathcal{H} & \rightarrow \mathcal{G} \bowtie \mathcal{H}, \quad y_t \mapsto (\epsilon_G(b), y_t).
\end{align}
These maps are morphisms of Lie groupoids, see Proposition 3.1. As such, they induce morphisms
\begin{align}
\mathcal{A}_b \mathcal{G} & \rightarrow \mathcal{A}_b (\mathcal{G} \bowtie \mathcal{H}), \quad X \mapsto (X, T(\epsilon_H \circ \beta)(X)) \\
\mathcal{A}_b \mathcal{H} & \rightarrow \mathcal{A}_b (\mathcal{G} \bowtie \mathcal{H}), \quad Y \mapsto (\theta_{\epsilon_G(b)}Y)
\end{align}
of Lie algebroids. Furthermore, we have the following proposition. See [39, Prop. 5.1].

**Proposition 3.3.** Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie groupoids, then the map
\begin{equation}
\mathcal{A}_b \mathcal{G} \oplus \mathcal{A}_b \mathcal{H} \rightarrow \mathcal{A}_b (\mathcal{G} \bowtie \mathcal{H}), \quad (X, Y) \mapsto (X, T(\epsilon_H \circ \beta)(X) + Y).
\end{equation}
is an isomorphism.

**Proof.** Let $(x_t, z_t) \in \mathcal{G} \bowtie \mathcal{H}$ be a curve such that $\alpha(x_t) = b$, and that $\beta(x_t) = \alpha(z_t)$. Let also $x_0 = X \in \mathcal{A}_b \mathcal{G}$, and $z_0 = Z \in T_{\epsilon_H(b)} \mathcal{H}$. Consider the following multiplication in $\mathcal{G} \bowtie \mathcal{H}$:
\begin{equation}
((\epsilon_G \circ \beta)(z_t), z_t^{-1})(x_t^{-1}, \epsilon_H(b))(x_t, z_t) = ((\epsilon_G \circ \beta)(z_t), (\epsilon_H \circ \beta)(z_t)).
\end{equation}
Multiplying both sides by $((\epsilon_G \circ \alpha)(z_t), z_t) \in \mathcal{G} \bowtie \mathcal{H}$, from the left, we obtain
\begin{equation}
(x_t^{-1}, \epsilon_H(b))(x_t, z_t) = ((\epsilon_G \circ \alpha)(z_t), z_t).
\end{equation}
Differentiating this equality, noticing that we can indeed apply the product rule (in the multiplication of $\mathcal{G} \bowtie \mathcal{H}$ here), we obtain
\begin{equation}
\left( \frac{d}{dt} \right|_{t=0} (x_t^{-1}, \epsilon_H(b)) + \left( \frac{d}{dt} \right|_{t=0} (x_t, z_t) \right) = \left( \frac{d}{dt} \right|_{t=0} ((\epsilon_G \circ \alpha)(z_t), z_t),
\end{equation}
that is,

\[ (3.26) \quad \left( \frac{d}{dt} \right)_{t=0} (x_t, z_t) = \left( \frac{d}{dt} \right)_{t=0} ((\varepsilon_G \circ \alpha)(z_t), z_t) - \left( \frac{d}{dt} \right)_{t=0} (x_t^{-1}, \varepsilon_H(b)). \]

On the other hand, differentiating

\[ (x_t^{-1}, \varepsilon_H(b))(x_t, (\varepsilon_H \circ \beta)(x_t)) = ((\varepsilon_G \circ \beta)(x_t), (\varepsilon_H \circ \beta)(x_t)), \]

we arrive at

\[ \left( \frac{d}{dt} \right)_{t=0} (x_t^{-1}, \varepsilon_H(b)) = \left( \frac{d}{dt} \right)_{t=0} ((\varepsilon_G \circ \beta)(x_t), (\varepsilon_H \circ \beta)(x_t)) - \left( \frac{d}{dt} \right)_{t=0} (x_t, (\varepsilon_H \circ \beta)(x_t)). \]

that is,

\[ (3.27) \quad \left( \frac{d}{dt} \right)_{t=0} (x_t^{-1}, \varepsilon_H(b)) = \left( \frac{d}{dt} \right)_{t=0} ((\varepsilon_G \circ \beta)(x_t), (\varepsilon_H \circ \beta)(x_t)) - \left( \frac{d}{dt} \right)_{t=0} (x_t, (\varepsilon_H \circ \beta)(x_t)). \]

Now, (3.26) and (3.27) together imply

\[ \left( \frac{d}{dt} \right)_{t=0} (x_t, z_t) = \left( \frac{d}{dt} \right)_{t=0} (x_t, (\varepsilon_H \circ \beta)(x_t)) - \left( \frac{d}{dt} \right)_{t=0} (x_t, (\varepsilon_H \circ \beta)(x_t)) = \left( \frac{d}{dt} \right)_{t=0} ((\varepsilon_G \circ \beta)(x_t), (\varepsilon_H \circ \beta)(x_t)) = \left( \frac{d}{dt} \right)_{t=0} (x_t^{-1}, \varepsilon_H(b)). \]

As such, \( Z - T(\varepsilon_H \circ \beta)(X) \in A_b(G \bowtie H). \)

Example 3.4. Let us see the isomorphism (3.25) for the matched pair \((M \times G, M \times M)\) of the action Lie groupoid of Example 2.2 and the pair groupoid \(M \times M\) of Example 2.3. Differentiating a curve \((m, g_t; m g_{\xi_t} n_t) \in (M \times G) \bowtie (M \times M)\) of constant source at \(t = 0\), with \(g_0 = e \in G, n_0 = m, g_0 = \xi, n_0 = Y \in T_m M\), we obtain a generic element \((\theta_m, \xi, \xi^\top(m), Y) \in \mathcal{A}_m((M \times G) \bowtie (M \times M))\). On the other hand, differentiating the inclusion

\[ M \times G \ni (m, g_t) \mapsto (m, g_t; (\varepsilon_{M \times M} \circ \beta)(m, g_t)) = (m, g_t; m g_{\xi_t} m g_{\xi_t}) \in (M \times G) \bowtie (M \times M) \]

we arrive at

\[ \mathcal{A}_m(M \times G) \ni (\theta_m, \xi) \mapsto (\theta_m, \xi, \xi^\top(m), \xi^\top(m)) \in \mathcal{A}_m((M \times G) \bowtie (M \times M)), \]
while differentiating
\[ M \times M \ni (m, n_t) \mapsto (m, e; m, n_t) \in (M \times G) \bowtie (M \times M) \]
we obtain
\[ A_m(M \times M) \ni (\theta_m, X) \mapsto (\theta_m, \theta; \theta_m, X) \in A_m((M \times G) \bowtie (M \times M)). \]
As a result, we see the isomorphism
\[ (3.29) \]
\[ \mathcal{A}_m(M \times G) \oplus \mathcal{A}_m(M \times M) \ni (\theta_m, \xi) \mapsto ((\theta_m, \xi; \xi^\dagger(m), X + \xi^\dagger(m)) \in \mathcal{A}_m((M \times G) \bowtie (M \times M)). \]

3.5. Left and right invariant vector fields on matched pairs of Lie groupoids.

In this subsection we shall attempt to determine the nature of the left invariant (resp. right invariant) vector fields on a matched pair Lie groupoid. Let \( \mathcal{G} \bowtie \mathcal{H} \) be a matched pair of Lie groupoids over the base \( B \), and \( (g, h) \) be a fixed element of this groupoid.

**Proposition 3.5.** Let \( \mathcal{G} \bowtie \mathcal{H} \) be a matched pair of Lie groupoids over a base manifold \( B \). Then, the left invariant vector field corresponding to \( U \in A_b(\mathcal{G} \bowtie \mathcal{H}) \) is given by
\[ (3.30) \]
\[ \hat{U}(g, h) = (X, T(\xi \circ \beta)(X))(g, h) \]
where \( g \in \mathcal{G}, h \in \mathcal{H} \), so that \( \beta(h) = b, X \in A_b \mathcal{G}, \text{ and } Y \in A_b \mathcal{H} \).

**Proof.** We have seen in Subsection 3.4 that any \( U \in A_b(\mathcal{G} \bowtie \mathcal{H}) \) is a sum
\[ U = (X, T(\xi \circ \beta)(X)) + (\theta_{\xi \circ \beta}(b), Y) \]
for some \( X \in A_b \mathcal{G} \), and \( Y \in A_b \mathcal{H} \). As such,
\[ \hat{U}(g, h) = (X, T(\xi \circ \beta)(X))(g, h) + (\theta_{\xi \circ \beta}(b), Y)(g, h). \]
More precisely, assuming \( \alpha(x_t) = \beta(h) \), we have
\[ (X, T(\xi \circ \beta)(X))(g, h)(x_t, (\xi \circ \beta)(x_t)) = \frac{d}{dt} \bigg|_{t=0} (g, h)(x_t, (\xi \circ \beta)(x_t)) = \]
\[ \frac{d}{dt} \bigg|_{t=0} (g(h \triangleright x_t), (h \triangleright x_t)((\xi \circ \beta)(x_t))) = \]
\[ \frac{d}{dt} \bigg|_{t=0} (g(h \triangleright x_t), (h \triangleright x_t)) = (h \triangleright X(g), h \triangleright X) = (h \triangleright X(g), X^\dagger(h)). \]
Similarly, assuming \( \beta(y_t) = \alpha(g) \) and \( \beta(h) = b = \alpha(y_t) \), we have
\[ (\theta_{\xi \circ \beta}(b), Y)(g, h) = \frac{d}{dt} \bigg|_{t=0} (g, h)((\xi \circ \beta)(b), y_t) = \frac{d}{dt} \bigg|_{t=0} (g(h \triangleright \xi \circ \beta)(b), (h \triangleright \xi \circ \beta)(b))y_t) = \]
\[ \frac{d}{dt} \bigg|_{t=0} (g(\xi \circ \beta)(\alpha(h)), hy_t) = (\theta_g, Y(h)). \]
The result follows. \[ \square \]

**Proposition 3.6.** Let \( \mathcal{G} \bowtie \mathcal{H} \) be a matched pair of Lie groupoids over a base manifold \( B \). Then, the right invariant vector field corresponding to \( U \in A_b(\mathcal{G} \bowtie \mathcal{H}) \) is given by
\[ (3.30) \]
\[ \hat{U}(g, h) = (X, T^\dagger(Y))(g, h), \]
where \( g \in \mathcal{G} \), so that \( \alpha(g) = b, h \in \mathcal{H}, X \in A_b \mathcal{G}, \text{ and } Y \in A_b \mathcal{H} \).
Proof. This time, we note for any $U \in A_b(G \rightrightarrows H)$ that

$$\overrightarrow{U}(g, h) = (X, T(\varepsilon \circ \beta)(X))(g, h) + (\theta_{\varepsilon G(b)}, Y)(g, h).$$

Accordingly, assuming $\alpha(x_t) = b = \alpha(g)$,

$$\overrightarrow{(X, T(\varepsilon \circ \beta)(X))}(g, h) = - \frac{d}{dt} \bigg|_{t=0} (x_t, (\varepsilon_H \circ \beta)(x_t))^{-1}(g, h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} ((\varepsilon_H \circ \beta)(x_t)^{-1} \triangleright x_t^{-1}, (\varepsilon_H \circ \beta)(x_t)^{-1} \triangleleft x_t^{-1})(g, h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} (x_t^{-1}, \varepsilon_H(b))(g, h) = - \frac{d}{dt} \bigg|_{t=0} (x_t^{-1}(\varepsilon_H(b) \triangleright g), (\varepsilon_H(b) \triangleleft g)h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} (x_t^{-1}g, h) = (\overrightarrow{X}(g), \theta_h),$$

and

$$\overrightarrow{(\theta_{\varepsilon G(b)}, Y)}(g, h) = - \frac{d}{dt} \bigg|_{t=0} (\varepsilon_G(b), y_t)^{-1}(g, h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} (y_t^{-1} \triangleright \varepsilon_G(b)^{-1}, y_t^{-1} \triangleleft \varepsilon_G(b)^{-1})(g, h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} ((\varepsilon_G \circ \beta)(y_t), y_t^{-1})(g, h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} ((\varepsilon_G \circ \beta)(y_t^{-1} \triangleright g), (y_t^{-1} \triangleleft g)h) =$$

$$- \frac{d}{dt} \bigg|_{t=0} (y_t^{-1} \triangleright g, (y_t^{-1} \triangleleft g)h) = (-Y^\dagger(g), \overrightarrow{Y}(g)).$$

The result follows. \qed

**Example 3.7.** Let us now derive the left (resp. right) invariant vector fields on $(M \times G) \rightrightarrows (M \times M)$ in view of the isomorphism (3.28).

Given $(m, g; mg, n) \in (M \times G) \rightrightarrows (M \times M)$ and $(\theta_n, \xi; \xi^\dagger(n), Y) \in A_n((M \times G) \rightrightarrows (M \times M))$, we have

$$\overrightarrow{(\theta_n, \xi; \xi^\dagger(n), Y)}(m, g; mg, n) =$$

$$\frac{d}{dt} \bigg|_{t=0} (m, g; mg, n)(n, e_t; ne_t, n_t) =$$

$$= \frac{d}{dt} \bigg|_{t=0} ((m, g)((mg, n) \triangleright (n, e_t)); ((mg, n) \triangleleft (n, e_t))(ne_t, n_t)) =$$

$$= \frac{d}{dt} \bigg|_{t=0} ((m, g)(mg, e_t); (mge_t, ne_t)(ne_t, n_t)) = \frac{d}{dt} \bigg|_{t=0} (m, g; mg, n) =$$

$$(\theta_m, \overrightarrow{\xi}(g); \xi^\dagger(mg), Y).$$
Similarly, for \((m, g; mg, n) \in (M \times G) \rtimes (M \times M)\) and \((\theta_m, \xi; \xi^\dagger(m), Y) \in \mathcal{A}_m((M \times G) \rtimes (M \times M))\),

\[
(\theta_m, \xi; \xi^\dagger(m), Y)(m, g; mg, n) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} (m, e_t; me_t, n_t)^{-1}(m, g; mg, n) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( (me_t, n_t)^{-1} \circ (me_t, e_t)^{-1} \circ (me_t, n_t)^{-1} \circ (m, e_t)^{-1} \right)(m, g; mg, n) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( (n_t, me_t) \circ (me_t, e_t)^{-1}; (n_t, me_t) \circ (me_t, e_t)^{-1} \right)(m, g; mg, n) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( n_t, e_t^{-1}; n_t e_t^{-1}, m \right)(m, g; mg, n) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( (n_t, e_t^{-1})(n_t e_t^{-1}, m) \circ (m, g) \circ (n_t, e_t^{-1}, m) \circ (m, g) \right) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( (n_t, e_t^{-1})(n_t e_t^{-1}, g); (n_t e_t^{-1}, g, mg)(mg, n) \right) =
\]

\[
= \quad \frac{d}{dt} \bigg|_{t=0} \left( n_t, e_t^{-1}; g; n_t e_t^{-1}, g, n \right) = (\xi \circ g; (Y - \xi^\dagger(m)) \circ g, \theta_n).
\]

On the other hand, in view of (3.28), for any \((m, g; mg, n) \in (M \times G) \rtimes (M \times M)\), and

\[
(\theta_n, \xi; \xi^\dagger(n), Y) = (\theta_n, \xi; \xi^\dagger(n), \xi^\dagger(n)) + (\theta_n, \theta; \theta_n, X) \in \mathcal{A}_n((M \times G) \rtimes (M \times M)),
\]

we have

\[
\begin{aligned}
(\theta_n, \xi; \xi^\dagger(n), Y)(m, g; mg, n) &= \\
(\theta_n, \xi; \xi^\dagger(n), \xi^\dagger(n))(m, g; mg, n) + (\theta_n, \theta; \theta_n, X)(m, g; mg, n) &= \\
(\theta_m, \xi; \xi^\dagger(g); \xi^\dagger(m), \xi^\dagger(n)) + (\theta_m, \theta; \theta_{mg}, X) &= \\
(\theta_m, \xi; \xi^\dagger(g); \xi^\dagger(m), \xi^\dagger(n) + X),
\end{aligned}
\]

and for any

\[
(\theta_m, \xi; \xi^\dagger(m), Y) = (\theta_m, \xi; \xi^\dagger(m), \xi^\dagger(m)) + (\theta_m, \theta; \theta_m, Z) \in \mathcal{A}_m((M \times G) \rtimes (M \times M)),
\]

we obtain

\[
\begin{aligned}
(\theta_m, \xi; \xi^\dagger(m), Y)(m, g; mg, n) &= \\
(\theta_m, \xi; \xi^\dagger(m), \xi^\dagger(m))(m, g; mg, n) + (\theta_m, \theta; \theta_m, Z)(m, g; mg, n) &= \\
(\xi^\dagger(m), \xi^\dagger(g); \theta_{mg}, \theta_n) + (-Z, \theta_g; -Z \circ g, \theta_n) &= \\
(-\xi^\dagger(m) - Z, \xi^\dagger(g); -Z \circ g, \theta_n).
\end{aligned}
\]

**Remark 3.8.** We can relate the above calculations to the left (resp. right) invariant vector fields on the trivial groupoids as follows. The (groupoid) isomorphism (3.14) induces the isomorphism (3.15) on the level of Lie algebroids, it also induces an isomorphism on the level of left (resp. right)
invariant vector fields. Indeed,
\[
(\theta, \xi; \xi^\dagger(n), Y) (m, g; mg, n) = \mathcal{A}_n \Phi(\theta, \xi, Y)(m, g; mg, n)
\]
\[
\frac{d}{dt} \bigg|_{t=0} (m, g; mg, n)(n, e; ne, n_t) = 
\frac{d}{dt} \bigg|_{t=0} \Phi((m, g, n)(n, e, n_t)) = 
T_{(m,g,n)} \Phi(\theta, \xi, Y)(m, g, n).
\]
Similarly,
\[
\frac{d}{dt} \bigg|_{t=0} (m, g; mg, n) = \mathcal{A}_m \Phi(\theta, \xi, Y)(m, g; mg, n)
\]
\[
\frac{d}{dt} \bigg|_{t=0} (n, e; ne, n_t)^{-1}(m, g; mg, n) = 
\frac{d}{dt} \bigg|_{t=0} \Phi((n, e, n_t)^{-1}(m, g, n)) = 
T_{(m,g,n)} \Phi(\theta, \xi, Y)(m, g, n).
\]

4. DISCRETE DYNAMICS ON MATCHED PAIRS

4.1. DISCRETE EULER-LAGRANGE EQUATIONS. We shall recall briefly the discrete Euler-Lagrange equations from [32] Subsect. 4.1. Let \( \mathcal{G} \) be a Lie groupoid, and \( \mathcal{A}\mathcal{G} \) be its associated Lie algebroid. For any fixed \( g \in \mathcal{G} \), and \( N \geq 1 \),
\[
C^N_g = \{(g_1, \ldots, g_N) \in \mathcal{G}^N \mid (g_k, g_{k+1}) \in \mathcal{G} \ast \mathcal{G}, \ 1 \leq k \leq N - 1, \ g_1 \cdots g_N = g\}
\]
where \( \mathcal{G}^N \) is the \( N \) times cartesian product of \( \mathcal{G} \), is called the set of admissible sequences with values in \( \mathcal{G} \).

On the other hand, a discrete Lagrangian is defined as a function \( L : \mathcal{G} \rightarrow \mathbb{R} \), and the discrete action sum associated to it is given by
\[
SL : C^N_g \rightarrow \mathbb{R}, \quad (g_1, \ldots, g_N) \mapsto \sum_{k=1}^{N} L(g_k).
\]
Now the discrete Hamilton’s principle is recalled, from [32], as; given \( g \in \mathcal{G} \) and \( N \geq 1 \), an admissible sequence \((g_1, \ldots, g_N)\) is a solution of the Lagrangian system if and only if \((g_1, \ldots, g_N)\) is a critical point of \( SL \). So, along the lines of [32] one arrives at the discrete Euler-Lagrange equations
\[
\sum_{k=1}^{N-1} [\dot{X}_k (g_k) (L) - \dot{X}_k (g_{k+1}) (L)] = 0,
\]
for any \( X_k \in \Gamma(\mathcal{A}\mathcal{G}) \). In particular, for \( N = 2 \), the discrete Euler-Lagrange equations are given by
\[
\dot{X}_1 (g_1) (L) - \dot{X}_2 (g_2) (L) = 0
\]
for every section \( X \in \Gamma(\mathcal{A}\mathcal{G}) \).
We review below the examples discussed in [32].

**Example 4.1.** Let $M \times M$ be the coarse (pair) groupoid of Example 2.3. Let us recall also that the left invariant vector fields, as well as the right invariant vector fields, corresponding to the Lie algebroid of $M \times M$ were obtained in Example 2.9.

Now, given a discrete Lagrangian density $L : M \times M \to \mathbb{R}$, the discrete Euler-Lagrange equations (4.1) takes the particular form

$$(4.2) \quad \overrightarrow{\xi}(g_k)(L) - \overrightarrow{\xi}(g_{k+1})(L) = 0.$$ 

In terms of the total derivatives on the product manifold $M \times M$, we may rewrite the discrete Euler-Lagrange equations as

$$(4.3) \quad D_2 L(x, y) + D_1 L(y, z) = 0,$$

see also [37].

**Example 4.2.** Let $G$ be a Lie group (with the Lie algebra $\mathfrak{g}$), considered as a groupoid over the identity $\{e\}$. We recall that the left invariant and the right invariant vector fields corresponding to the Lie algebroid $\mathfrak{g}$ of $G$ coincides with the left invariant and the right invariant vector fields on $G$.

Now, given a discrete Lagrangian density $L : G \to \mathbb{R}$ the discrete Euler-Lagrange equations are given by

$$(4.4) \quad \overrightarrow{\xi}(g_k)(L) - \overrightarrow{\xi}(g_{k+1})(L) = 0,$$

or equivalently

$$\langle dL(g_k), \overrightarrow{\xi}(g_k) \rangle - \langle dL(g_{k+1}), \overrightarrow{\xi}(g_{k+1}) \rangle = \langle dL(g_k), T_e^* \ell_{g_k}(\xi) \rangle - \langle dL(g_{k+1}), T_e^* r_{g_{k+1}}(\xi) \rangle = \langle T_e^* \ell_{g_k}(dL(g_k)) - T_e^* r_{g_{k+1}}(dL(g_{k+1})), \xi \rangle = 0,$$

for any $\xi \in \mathfrak{g}$, and any $g_k, g_{k+1} \in G$. As such, the discrete Euler-Lagrange equations may be written by

$$(4.5) \quad T_e^* \ell_{g_k}(dL(g_k)) - T_e^* r_{g_{k+1}}(dL(g_{k+1})) = 0.$$

Following [32], we set

$$\mu_k := \left( r_{g_k}^* dL(e) \right).$$

Then (4.1) takes the form

$$T_e^* \ell_{g_k}(dL(g_k)) - T_e^* r_{g_{k+1}}(dL(g_{k+1})) = T_e^* \ell_{g_k}(dL(g_k)) - \mu_{k+1} =$$

$$T_e^* \ell_{g_k} T_e^* r_{g_k} (dL(g_k)) - \mu_{k+1} = T_e^* \ell_{g_k} T_e^* r_{g_k} T_e^* r_{g_k} (dL(g_k)) - \mu_{k+1} =$$

$$T_e^* \ell_{g_k} T_e^* r_{g_k} (\mu_k) - \mu_{k+1} = \text{Ad}_{g_k}^*(\mu_k) - \mu_{k+1} = 0.$$ 

In other words,

$$\mu_{k+1} = \text{Ad}_{g_k}^*(\mu_k),$$

called the discrete Euler-Lagrange equations, see also [2, 34, 35].

**Remark 4.3.** We note for the adjoint action $\text{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, $(g, \xi) \mapsto \text{Ad}_g(\xi) =: g \triangleright \xi$ that

$$\langle \mu, \text{Ad}_g(\xi) \rangle = \langle \mu, g \triangleright \xi \rangle = \langle \mu \triangleright g, \xi \rangle = \langle g^{-1} \triangleright \mu, \xi \rangle = \langle \text{Ad}_{g^{-1}}^*(\mu), \xi \rangle.$$
Example 4.4. Let $M \times G$ be the action Lie groupoid of Example 2.2 with the left invariant and the right invariant vector fields from Example 2.10.

Given a Lagrangian $L : M \times G \rightarrow \mathbb{R}$, the sequence $((m, g_k), (mg_k, g_{k+1})) \in (M \times G) \ast (M \times G)$ is a solution of the discrete Euler-Lagrange equations for the Lagrangian, if

$$(\theta_m, \xi)(m, g_k)(L) - (\theta_m, \xi)(m \cdot g_k, g_{k+1})(L) = 0,$$

for any $\xi \in \mathfrak{g}$. Equivalently,

$$(T_e \ell_{g_k})(\xi)(L_m) - (T_e \ell_{g_{k+1}})(\xi)(L_{mg_k} + \xi^\ell (mg_k)(L_{g_{k+1}})) = 0,$$

where $L_m : G \rightarrow \mathbb{R}$ is the map given by $L_m(g) := L(m, g)$, and similarly $L_g : M \rightarrow \mathbb{R}$ is the one given by $L_g(m) := L(m, g)$.

Setting $\mu_k(m, g_k) = d(L_m \circ r_{g_k})(e)$ as above, the discrete Euler-Lagrange equations appear as

$$\mu_{k+1}(m \cdot g_k, g_{k+1}) = \text{Ad}_{g_k}^* \mu_k(m, g_k) + d(L_{g_{k+1}} \circ ((m \cdot g_k) \cdot))(e)$$

where $(m \cdot g_k) : G \rightarrow M$ is given by $(m \cdot g_k) \cdot (g) := m \cdot (g_k g)$.

If, in particular, $M$ is the orbit space in a representation $V$ of $G$, the corresponding equations were first obtained in [2,3], and they are called the discrete Euler-Poincaré equations.

Example 4.5. Let $M \times G \times M$ be the trivial groupoid of Example 2.4, whose left invariant and right invariant vector fields are obtained in Example 2.11. Accordingly, given a Lagrangian $L : M \times G \times M \rightarrow \mathbb{R}$, together with $(m_k, g_k, n_k), (m_{k+1}, g_{k+1}, n_{k+1}) \in M \times G \times M$ with

$$\beta(m_k, g_k, n_k) = n_k = m_{k+1} = \alpha(m_{k+1}, g_{k+1}, n_{k+1}),$$

the discrete Euler-Lagrange equations are given by

$$(\theta_{m_k}, \xi(g_k), X)(L) - (-X, \xi, (g_{k+1}), \theta_{n_{k+1}})(L) = 0.$$

Setting $\mu_k = T^*r_{g_k}d_2L(m_k, g_k, n_k)$, the discrete Euler-Lagrange equations appear as

$$(4.6) \quad \langle X(n_k), d_1L(n_k, g_{k+1}, n_{k+1}) + d_2L(m_k, g_k, n_k) \rangle + \langle \xi, \text{Ad}_{g_k}^*(\mu_k) - \mu_{k+1} \rangle = 0,$$

where, for $1 \leq i \leq 3$, $d_i$ stands for the derivative with respect to the $i$th variable.

4.2. Discrete dynamics on matched pairs of Lie groupoids.

In this section, we shall rewrite the discrete dynamical equations (4.1) in the case of a matched pair of Lie groupoids as

$$(4.7) \quad \overrightarrow{U}(g_k, h_k)(L) - \overrightarrow{U}(g_{k+1}, h_{k+1})(L) = 0,$$

for a Lagrangian function $L$ defined on the matched pair $\mathcal{G} \bowtie \mathcal{H}$. In view of (3.29) and (3.30), the equation (4.7) takes the form

$$(4.8) \quad \left( \overrightarrow{h_k \circ X}(g_k), X^\dagger(h_k) + \overrightarrow{Y}(h_k) \right)(L) - \left( \overrightarrow{X}(g_{k+1}) - Y^\dagger(g_{k+1}), \overrightarrow{Y}(g_{k+1}) \right)(L) = 0.$$

As such, we arrive at the following proposition.
Proposition 4.6. Let $L$ be Lagrangian function defined on the matched pair of Lie groupoids $G \rightrightarrows H$. The discrete Euler-Lagrange equations on $G \rightrightarrows H$ generated by $L$ is

\begin{equation}
\tag{4.9}
\overline{h_k} \triangleright X(g_k)(L) - \overline{X}(g_{k+1})(L) + Y(g_{k+1})(L) + X(h_k)(L) + Y(h_k)(L) - Y \triangleleft g_{k+1}(h_{k+1})(L) = 0.
\end{equation}

In particular, considering the left action of $H$ on $G$ to be trivial, the equation (4.9) reduces to

\begin{equation}
\tag{4.10}
\overline{X}(g_k)(L) - \overline{X}(g_{k+1})(L) + X(h_k)(L) + \overline{Y}(h_k)(L) - \overline{Y} \triangleleft g_{k+1}(h_{k+1})(L) = 0.
\end{equation}

Similarly, considering this time the right action of $G$ on $H$ to be trivial, the equation (4.9) takes the form of

\begin{equation}
\tag{4.11}
\overline{h_k} \triangleright X(g_k)(L) - \overline{X}(g_{k+1})(L) + Y(g_{k+1})(L) + \overline{Y}(h_k)(L) - \overline{Y} = 0.
\end{equation}

If both actions are trivial, the equation (4.9) simplifies to

\begin{equation}
\tag{4.12}
\overline{X}(g_k)(L) - \overline{X}(g_{k+1})(L) + \overline{Y}(h_k)(L) - \overline{Y}(h_{k+1})(L) = 0.
\end{equation}

We conclude the present section with yet another particular case of (4.1), or of (4.7), for a matched pair of Lie groups (regarded as Lie groupoids) to analyse the discrete dynamics on Lie groups from the matched pair point of view.

4.3. Discrete dynamics on matched pairs of Lie groups.

Let $G$ and $H$ be two Lie groups with mutual actions

\begin{align*}
\rho : H \times G &\to G, \quad (h, g) \mapsto h \triangleright g, \\
\sigma : H \times G &\to H, \quad (h, g) \mapsto h \triangleleft g,
\end{align*}

where $h, h_1, h_2 \in H$, $g, g_1, g_2 \in G$, $e_H$ is the identity element in $H$, and $e_G$ is the identity element in $G$. If the actions (4.12)-(4.13) satisfy

\begin{align*}
\tag{4.14}
h \triangleright (g_1 g_2) &= (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2), \\
\tag{4.15}(h_1 h_2) \triangleleft g &= (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g),
\end{align*}

then the pair $(G, H)$ is called a matched pair of Lie groups, \cite{27, 28}. In this case, the cartesian product $G \times H$ may be equipped with the group structure given by the multiplication

\begin{equation}
\tag{4.16}
(g_1, h_1)(g_2, h_2) = (g_1 h_1 \triangleright g_2, h_1 \triangleleft g_2) h_2 = (g_1 \rho (h_1, g_2), \sigma (h_1, g_2) h_2),
\end{equation}

and the unit element $(e_G, e_H)$. This matched pair group is denoted by $G \rightrightarrows H$. Conversely, if a Lie group $M$ is a cartesian product of two subgroups $G \hookrightarrow M \hookleftarrow H$, and if the multiplication on $M$ defines a bijection $G \times H \to M$, then $M$ is a matched pair, that is, $M \cong G \rightrightarrows H$. In this case, the mutual actions are given by

\begin{equation}
\tag{4.17}
h \cdot g = (h \triangleright g)(h \triangleleft g),
\end{equation}

for any $g \in G$, and any $h \in H$. Let us also record here the inversion in $G \rightrightarrows H$ as

\begin{equation}
\tag{4.18}(g, h)^{-1} = \left( h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1} \right).
\end{equation}

for later use.
We next consider the lifting of the group actions to the Lie algebra level. As for the left action, we have

\[ H \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (h, \xi) \mapsto h \triangleright \xi := \frac{d}{dt} \bigg|_{t=0} h \triangleright x_t, \]

\[ \mathfrak{h} \times G \longrightarrow TG, \quad (\eta, g) \mapsto \eta^\triangleright(g) := \eta \triangleright g := \frac{d}{dt} \bigg|_{t=0} y_t \triangleright g, \]

where \( \mathfrak{g} \) denotes the Lie algebra of the Lie group \( G \), \( \mathfrak{h} \) stands for the Lie algebra of \( H \), \( x_t \) is a curve in \( G \) passing through the identity at \( t = 0 \) in the direction of \( \xi \in \mathfrak{g} \), and finally \( y_t \) is a curve in \( H \) passing through the identity at \( t = 0 \) in the direction of \( \eta \in \mathfrak{h} \).

Freezing the group element in (4.18), we arrive at a linear mapping \( h \triangleright : \mathfrak{g} \rightarrow \mathfrak{g} \) for any \( h \in H \). We shall denote the transpose of this mapping by \( h^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \), which is given by

\[ \langle h^* \xi, \mu \rangle = \langle \xi, \mu \triangleright h \rangle, \]

for any \( \mu \in \mathfrak{g}^* \), where the pairing is the one between \( \mathfrak{g}^* \) and \( \mathfrak{g} \).

Similarly, freezing the group element (4.19), we arrive at a linear operator \( b_g : \mathfrak{h} \mapsto T_gG \), given by \( b_g(\eta) := \eta \triangleright g \). The transpose of this mapping shall be denoted by \( b_g^* : T_g^*G \mapsto \mathfrak{h}^* \), and it is given by

\[ \langle \eta \triangleright g, \mu_g \rangle = \langle b_g(\eta), \mu_g \rangle = \langle \eta, b_g^*(\mu_g) \rangle, \]

for any \( \mu_g \in T_g^*G \).

Now on the other hand, for the right \( G \) action on \( H \), we have the maps

\[ \mathfrak{h} \times G \longrightarrow \mathfrak{h}, \quad (\eta, g) \mapsto \eta \triangleleft g := \frac{d}{dt} \bigg|_{t=0} y_t \triangleleft g, \]

\[ H \times \mathfrak{g} \longrightarrow TH, \quad (h, \xi) \mapsto \xi^\triangleleft(h) := h \triangleleft \xi := \frac{d}{dt} \bigg|_{t=0} h \triangleleft x_t. \]

Similar to above, freezing the group element in (4.22) we arrive at a linear mapping \( \triangleleft : \mathfrak{h} \rightarrow \mathfrak{h} \) for any \( g \in G \). The transpose of this map will be denoted by \( g^\triangleleft : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \), and it is defined by

\[ \langle \eta \triangleleft g, \nu \rangle = \langle \eta, g^\triangleleft \nu \rangle, \]

for any \( \nu \in \mathfrak{h}^* \). The pairing here is the one between \( \mathfrak{h}^* \) and \( \mathfrak{h} \).

Finally, freezing the group element in (4.23) we obtain a mapping \( a_h : \mathfrak{g} \mapsto T_hH \), \( a_h(\xi) = h \triangleleft \xi \) for any \( \xi \in \mathfrak{g} \). The transpose \( a_h^* : T_h^*H \mapsto \mathfrak{g}^* \) of this linear mapping will be given by

\[ \langle h \triangleleft \xi, \nu_h \rangle = \langle a_h(\xi), \nu_h \rangle = \langle \xi, a_h^*(\nu_h) \rangle, \]

for any \( \nu_h \in T_h^*H \).

We note also that if \( G \rhd \bowtie H \) is a matched pair Lie group, then its Lie algebra is a matched pair Lie algebra \( \mathfrak{g} \bowtie \mathfrak{h} \). That is, the induced actions

\[ \triangleright : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{and} \quad \triangleleft : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{h} \]

of the Lie algebras satisfy

\[ \eta \triangleright [\xi_1, \xi_2] = [\eta \triangleright \xi_1, \xi_2] + [\xi_1, \eta \triangleright \xi_2] + (\eta \triangleleft \xi_1) \triangleright \xi_2 - (\eta \triangleleft \xi_2) \triangleright \xi_1 \]
and
\begin{equation}
[\eta_1, \eta_2] \triangleleft \xi = [\eta_1, \eta_2 \triangleleft \xi] + [\eta_1 \triangleleft \xi, \eta_2] + \eta_1 \triangleleft (\eta_2 \triangleright \xi) - \eta_2 \triangleleft (\eta_1 \triangleright \xi),
\end{equation}
for any $\eta, \eta_1, \eta_2 \in \mathfrak{g}$, and any $\xi, \xi_1, \xi_2 \in \mathfrak{g}$. Such a pair $(\mathfrak{g}, \mathfrak{h})$ is called a matched pair of Lie algebras, and the Lie algebra structure on $\mathfrak{g} \bowtie \mathfrak{h} := \mathfrak{g} \oplus \mathfrak{h}$ is given by
\begin{equation}
[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1, [\eta_1, \eta_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1).
\end{equation}
It is immediate that both $\mathfrak{g}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g} \bowtie \mathfrak{h}$ via the obvious inclusions. Conversely, given a Lie algebra $\mathfrak{m}$ with two subalgebras $\mathfrak{g} \hookrightarrow \mathfrak{m} \hookrightarrow \mathfrak{h}$, if $\mathfrak{m} \cong \mathfrak{g} \oplus \mathfrak{h}$ via $(\xi, \eta) \mapsto \xi + \eta$, then $\mathfrak{m} = \mathfrak{g} \bowtie \mathfrak{h}$ as Lie algebras. In this case, the mutual actions of the Lie algebras are uniquely determined by
\begin{equation}
[\eta, \xi] = (\eta \triangleright \xi, \eta \triangleleft \xi).
\end{equation}
On the other hand, there are integrability conditions under which a matched pair of Lie algebras can be integrated into a matched pair of Lie groups. For a discussion of this direction we refer the reader to [27], Sect. 4.

We shall also need the adjoint action of the matched pair Lie group $G \bowtie H$ on its Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$. For any $(g, h) \in G \bowtie H$, and any $(\xi, \eta) \in \mathfrak{g} \bowtie \mathfrak{h}$, we have
\begin{equation}
\text{Ad}_{(g, h)^{-1}}(\xi, \eta) = (h^{-1} \triangleright \xi, T_{h^{-1}} R_h (h^{-1} \triangleleft \xi) + \text{Ad}_{h^{-1}}(\eta \triangleleft g))
\end{equation}
where $\zeta := \text{Ad}_{h^{-1}}(\xi) + T_k L_{g^{-1}} (\eta \triangleright g) \in \mathfrak{g}$.

Furthermore, the tangent lifts of the left and right regular actions of $G \bowtie H$ are given by
\begin{equation}
T_{(g_2, h_2)} L_{(g_1, h_1)} (U_{g_2}, V_{h_2}) = \left( T_{h_2 \triangleright g_2} L_{g_1} (h_1 \triangleright U_{g_2}), T_{h_2 \triangleright g_2} R_{h_2} (h_1 \triangleleft U_{g_2}) + T_{h_2} L_{(h_1 \triangleright g_2)} V_{h_2} \right),
\end{equation}
\begin{equation}
T_{(g_1, h_1)} R_{(g_2, h_2)} (U_{g_1}, V_{h_1}) = \left( T_{g_1} R_{(h_1 \triangleright g_2)} U_{g_1} + T_{h_1 \triangleright g_2} L_{g_1} (V_{h_1} \triangleright g_2), T_{h_1 \triangleright g_2} R_{h_2} (V_{h_1} \triangleleft g_2) \right).
\end{equation}
We can thus compute the left and right invariant vector fields generated by a Lie algebra element $(\xi, \eta) \in \mathfrak{g} \bowtie \mathfrak{h}$ as
\begin{equation}
\xi(g, h) = T_{(e_G, e_H)} L_{(g, h)} (\xi, \eta) = (h \triangleright \xi(g), h \triangleleft \xi + \xi(h)),
\end{equation}
\begin{equation}
\eta(g, h) = T_{(e_G, e_H)} R_{(g, h)} (\xi, \eta) = (\xi(g) + \eta \triangleright g, \eta \triangleleft g(h)).
\end{equation}
Recalling the discrete Euler-Lagrange equations (4.4), discrete dynamics on $G \bowtie H$ generated by a Lagrangian function $L : G \bowtie H \to \mathbb{R}$ is given by
\begin{equation}
(\xi, \eta)(g_k, h_k) = (\xi(g_{k+1}), h_{k+1}) = 0.
\end{equation}
Let now the exterior derivative of the Lagrangian $L : G \bowtie H \to \mathbb{R}$ be a two-tuple $(d_1 L, d_2 L)$, where $d_1 L$ denotes the derivative with respect to group variable $g \in G$ whereas $d_2 L$ denotes the derivative with respect to group variable $h \in H$. Then, in view of the left and right invariant vector fields (4.30)-(4.31), we arrive at
\begin{equation}
\begin{aligned}
\left( h \triangleright \xi(g_k, h_k), d_1 L(g_k, h_k) \right) + \left( h \triangleleft \xi, d_2 L(g_k, h_k) \right) + \left( \xi(h_k), d_2 L(g_k, h_k) \right) \\
- \left( \xi(g_{k+1}), d_1 L(g_{k+1}) \right) - \left( \eta \triangleright g_{k+1}, d_1 L(g_{k+1}) \right) - \left( \eta \triangleleft g_{k+1}(h_{k+1}), d_2 L(g_{k+1}, h_{k+1}) \right) = 0.
\end{aligned}
\end{equation}
It is possible to single out $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$ from these equations, so that
\begin{equation}
\begin{aligned}
\langle \xi, (T^* L_{g_{k+1}} \cdot d_1 L(g_{k+1}, h_{k+1})) \rangle &+ a_{g_{k+1}}^{h_k} d_2 L(g_k, h_k) - T^* R_{g_{k+1}} \cdot d_1 L(g_{k+1}, h_{k+1}) \\
+ \langle \eta, T^* L_{h_k} \cdot d_2 L(g_k, h_k) - b_{g_{k+1}}^{h_{k+1}} d_1 L(g_{k+1}, h_{k+1}) - g_{k+1} \triangleright \eta \rangle = 0.
\end{aligned}
\end{equation}
Proposition 4.4. By setting the covectors
\[ T^*R_{g_k} \cdot d_1L(g_k, h_k) = \mu_k \in \mathfrak{g}^*, \quad T^*R_{h_k} \cdot d_2L(g_k, h_k) = \nu_k \in \mathfrak{h}^*, \]
we can write the discrete Euler-Lagrange equations on the matched pair Lie group \( G \bowtie H \) by
\[ (4.33) \quad \text{Ad}_{\mathfrak{g}_k}^*(\mu_k) \ast h_k + a_{h_k}^* d_2L(g_k, h_k) - \mu_{k+1} + \text{Ad}_{\mathfrak{h}_k}^*(\nu_k) - b_{\mathfrak{h}_k}^* d_1L(g_{k+1}, h_{k+1}) - g_{k+1} \triangleright \nu_{k+1} = 0. \]

In particular, when the (right) action of \( G \) on \( H \) is trivial, we have the discrete Euler-Lagrange equation
\[ (4.34) \quad \text{Ad}_{\mathfrak{g}_k}^*(\mu_k) \ast h_k - \mu_{k+1} + \text{Ad}_{\mathfrak{h}_k}^*(\nu_k) - b_{\mathfrak{h}_k}^* d_1L(g_{k+1}, h_{k+1}) - \nu_{k+1} = 0 \]
on the semidirect product Lie group \( G \bowtie H \).

On the other extreme, assuming the (left) action of \( H \) on \( G \) to be trivial, we arrive at the equation
\[ (4.35) \quad \text{Ad}_{\mathfrak{g}_k}^*(\mu_k) + a_{h_k}^* d_2L(g_k, h_k) - \mu_{k+1} + \text{Ad}_{\mathfrak{h}_k}^*(\nu_k) - g_{k+1} \triangleright \nu_{k+1} = 0 \]
on the semidirect product Lie group \( G \bowtie H \).

If both actions are trivial, then the equations reduce all the way down to
\[ (4.36) \quad \text{Ad}_{\mathfrak{g}_k}^*(\mu_k) - \mu_{k+1} + \text{Ad}_{\mathfrak{h}_k}^*(\nu_k) - \nu_{k+1} = 0. \]

5. Examples

5.1. Discrete dynamics on trivial groupoid.

In this subsection we shall drive the discrete Euler-Lagrange equation on the trivial groupoid of Example 2.4, regarded as the matched pair groupoid of the coarse (banal) groupoid of Example 2.3 and the action groupoid of Example 2.2. To this end, we first recall from Example 3.7 that given \((m, g; mg, n) \in (M \times G) \bowtie (M \times M)\) and \((\theta_n, \xi; \xi^\dagger, Y) \in \mathcal{A}_n((M \times G) \bowtie (M \times M))\), we have
\[ (\theta_n, \xi; \xi^\dagger, Y)(m, g; mg, n) = (\theta_n, \xi(g); \xi^\dagger(mg), \xi^\dagger(n) + X), \]
and similarly, for any \((\theta_n, \xi; \xi^\dagger, Y) \in \mathcal{A}_m((M \times G) \bowtie (M \times M))\),
\[ (\theta_n, \xi; \xi^\dagger, Y)(m, g; mg, n) = (-\xi^\dagger(m) - Z, \xi(g); -Z \ast g, \theta_n). \]

Now, given \((m_k, g_k; m_kg_k, n_k), (m_{k+1}, g_{k+1}; m_{k+1}g_{k+1}, n_{k+1}) \in (M \times G) \bowtie (M \times M)\), so that
\[ \beta((m_k, g_k; m_kg_k, n_k)) = n_k = m_{k+1} = \alpha(m_{k+1}, g_{k+1}; m_{k+1}g_{k+1}, n_{k+1}), \]
and a Lagrangian \( L : (M \times G) \bowtie (M \times M) \to \mathbb{R} \), the equation (4.9) yields
\[ \langle X(n_k), d_1L(n_k, g_{k+1}; m_{k+1}g_{k+1}, n_{k+1}) + d_4L(m_k, g_k; m_kg_k, n_k) \rangle + \langle \xi, \text{Ad}_{\mathfrak{g}_k}^*(\mu_k) - \mu_{k+1} \rangle + \langle \xi^\dagger(m_kg_k), d_3L(m_k, g_k; m_kg_k, n_k) \rangle \]
\[ = 0. \]
where \( \mu_k = T^* r_{g_k} d_2 L(m_k, g_k; m_k g_k, n_k) \), and for \( 1 \leq j \leq 4 \), the operator \( d_j \) denotes the derivative of the \( j \)th variable.

**Remark 5.1.** Let us note that the equations (5.1) above correspond to the discrete Euler-Lagrange equations (4.6) on the trivial groupoid \( M \times G \times M \), under the isomorphism \( 3.14 \). More precisely, on one hand we have

\[
(\theta, \xi, \xi^\dagger(n), Y)(m, g, mg, n) = (\theta, \xi^\dagger(g), \xi^\dagger(mg), \xi^\dagger(n) + X) = \\
\frac{d}{dt}_{|t=0} (m, ge_t, mg e_t, n, e_t) = \frac{d}{dt}_{|t=0} \Phi(m, ge_t, n, e_t) = T \Phi \left( \theta, \xi^\dagger(g), \xi^\dagger(n) + X \right) = \\
T \Phi \left( (\theta, \xi, \xi^\dagger(n) + X)(m, g, n) \right),
\]

and on the other hand,

\[
(\theta, \xi, \xi^\dagger(m), Y)(m, g, mg, n) = (-\xi^\dagger(m) - Z, \xi^\dagger(g); -Z \circ g, \theta_n) = \\
- \frac{d}{dt}_{|t=0} (m, e_t, e_t^{-1}g, m, g, n) = - \frac{d}{dt}_{|t=0} \Phi(m, e_t, e_t^{-1}g, n) = T \Phi \left( -\xi^\dagger(m) - Z, \xi^\dagger(g), \theta_n \right) = \\
T \Phi \left( (\theta, \xi, \xi^\dagger(m) + Z)(m, g, n) \right).
\]

The correspondence, then, follows at once.

### 5.2. Discrete Dynamics on \( SL(2, \mathbb{C}) = SU(2) \rtimes K \)

In this subsection, we shall study the discrete dynamical equations on the Lie group \( SL(2, \mathbb{C}) \) from the matched pair point of view. To this end, we recall its decomposition

\[
SL(2, \mathbb{C}) = SU(2) \rtimes K.
\]

from [27], see also [12, 13], the group structures, the mutual actions of the groups \( SU(2) \) and \( K \), together with their lifts.

The group

\[
SU(2) = \left\{ \begin{pmatrix} \omega & \bar{\theta} \\ -\bar{\theta} & \omega \end{pmatrix} \in SL(2, \mathbb{C}) : |\omega|^2 + |\bar{\theta}|^2 = 1 \right\}
\]

in the matched pair decomposition (5.2) is a universal double cover of the group \( SO(3) \). As such, for each element \( A \in SU(2) \) there exists a unique matrix \( \text{Rot}_A \in SO(3) \). The Lie algebra \( \mathfrak{su}(2) \) of the group \( SU(2) \) is the matrix Lie algebra

\[
\mathfrak{su}(2) = \left\{ \begin{pmatrix} -t & t \\ r + ts & -r - ts \end{pmatrix} : r, s, t \in \mathbb{R} \right\}
\]

of traceless skew-hermitian matrices. Following [27] we fix three matrices

\[
e_1 = \begin{pmatrix} 0 & -i/2 \\ -i/2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}
\]

as a basis of the Lie algebra \( \mathfrak{su}(2) \). We further make use of this to identify the matrix Lie algebra \( \mathfrak{su}(2) \) with the Lie algebra \( \mathbb{R}^3 \) by the cross product;

\[
re_1 + se_2 + te_3 \in \mathfrak{su}(2) \longleftrightarrow X = (r, s, t) \in \mathbb{R}^3.
\]
We also identify the dual space $su(2)^* \approx \mathbb{R}^3$ with $\mathbb{R}^3$ using the Euclidean dot product. Using this dualization, we can express the coadjoint action of the Lie algebra $su(2) \approx \mathbb{R}^3$ on $su^*(2) \approx \mathbb{R}^3$ as

$$\text{ad}^*: su(2) \times su^*(2) \to su^*(2), \quad (X, \Phi) \mapsto \text{ad}_X^* \Phi := X \times \Phi,$$

for any $X \in su(2) \approx \mathbb{R}^3$, and any $\Phi \in su^*(2) \approx \mathbb{R}^3$.

The simply-connected group $K$, on the other hand, may be represented by

$$K = \left\{ \frac{1}{\sqrt{1 + c}} \begin{pmatrix} 1 + c & 0 \\ a + ib & 1 \end{pmatrix} \in SL(2, \mathbb{C}) \mid a, b \in \mathbb{R} \text{ and } c > -1 \right\},$$

where the group operation is the matrix multiplication. The Lie algebra $\mathfrak{k}$ of the group $K$ is then given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} \frac{1}{2}c & 0 \\ a + ib & -\frac{1}{2}c \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \mid a, b, c \in \mathbb{R} \right\},$$

with matrix commutator being the Lie bracket. The group $K$ can also be presented as a subgroup of $GL(3, \mathbb{R})$ as

$$K = \left\{ \begin{pmatrix} 1 + c & 0 & 0 \\ 0 & 1 + c & 0 \\ -a & -b & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \mid a, b \in \mathbb{R} \text{ and } c > -1 \right\},$$

where the group operation is the matrix multiplication. In this case, its Lie algebra $\mathfrak{k}$ is given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ -a & -b & 0 \end{pmatrix} \in \mathfrak{gl}(3, \mathbb{R}) \mid a, b, c \in \mathbb{R} \right\},$$

where the Lie bracket is the matrix commutator. The group $K$ can, alternatively, be identified with the subspace

$$K = \{(a, b, c) \in \mathbb{R}^3 \mid a, b \in \mathbb{R} \text{ and } c > -1 \}$$

of $\mathbb{R}^3$ with a non-standard multiplication

$$(a_1, b_1, c_1) \ast (a_2, b_2, c_2) = (a_1, b_1, c_1)(1 + c_2) + (a_2, b_2, c_2),$$

in which case the Lie algebra $\mathfrak{k}$ is $\mathbb{R}^3$ via the Lie bracket

$$[Y_1, Y_2] = k \times (Y_1 \times Y_2),$$

where $k$ is the unit vector $(0, 0, 1) \in \mathbb{R}^3$. In this case, using the dot product, we may identify the dual space $\mathfrak{k}^*$ with $\mathbb{R}^3$ as well. Then, the coadjoint action of the Lie algebra $\mathfrak{k} \approx \mathbb{R}^3$ on its dual space $\mathfrak{k}^* \approx \mathbb{R}^3$ can be computed as

$$\text{ad}^*: \mathfrak{k} \times \mathfrak{k}^* \to \mathfrak{k}^*, \quad (Y, \Phi) \mapsto \text{ad}_Y^* \Phi := (k \cdot Y) \Phi - (\Phi \cdot Y) k,$$

for any $Y \in \mathfrak{k} \approx \mathbb{R}^3$, and any $\Phi \in \mathfrak{k}^* \approx \mathbb{R}^3$. The group isomorphisms relating (5.7), (5.9) and (5.11) are given by

$$\frac{1}{\sqrt{1 + c}} \begin{pmatrix} 1 + c \\ a + ib \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 + c \\ a + ib \end{pmatrix} \leftrightarrow (a, b, c).$$

The Lie algebra isomorphisms

$$\begin{pmatrix} \frac{1}{2}c \\ a + ib \end{pmatrix} \leftrightarrow \begin{pmatrix} c \\ a + ib \end{pmatrix} \leftrightarrow (a, b, c).$$
between (5.8), (5.10) and (5.12) are then obtained by differentiating (5.14).

We now move on to the mutual actions of the groups $SU(2)$ and $K$ on each other. Given any $A \in SU(2)$, and any $B \in K \subset SL(2, \mathbb{C})$, the left action of $K$ on $SU(2)$ is given by

\begin{equation}
    B \triangleright A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^{-1} BA \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + B^{-\dagger} A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\end{equation}

where $B^{-\dagger}$ stands for the inverse of the conjugate transpose of $B \in K$, and $\|B\|_M^2 = tr(B^\dagger B)$ refers to the matrix norm on $SL(2, \mathbb{C})$. The right action of $SU(2)$ on $K \subset \mathbb{R}^3$, on the other hand, is

\begin{equation}
    B \triangleright A = \frac{\|B\|^2_E}{2(c + 1)} e_3 + A \left( B - \frac{\|B\|^2_E}{2(c + 1)} e_3 \right) A^{-1}.
\end{equation}

Here $\|B\|^2_E$ denotes the Euclidean norm in $\mathbb{R}^3$ in view of the identification (5.14) of $B \in K \subset SL(2, \mathbb{C})$ with $B \in K \subset \mathbb{R}^3$.

Differentiating (5.16) with respect to $A \in SU(2)$, and regarding $B \in K \subset GL(3, \mathbb{R})$ via (5.14), we obtain

\begin{equation}
    \triangleright : K \times \mathfrak{su}(2) \to \mathfrak{su}(2), \quad (B, X) \mapsto B \triangleright X := BX,
\end{equation}

for any $X \in \mathfrak{su}(2) \cong \mathbb{R}^3$. Freezing the group element here, we get a linear operator $\triangleright : \mathfrak{su}(2) \to \mathfrak{su}(2)$. The transpose of this operator $\triangleright^T : \mathfrak{su}^* (2) \to \mathfrak{su}^* (2)$ is given by

\begin{equation}
    \Phi \triangleright^T B = B^T \Phi.
\end{equation}

Similarly, the derivative of (5.17) with respect to $A \in SU(2)$ renders the infinitesimal right action of the Lie algebra $\mathfrak{su}(2)$ on $K$ as

\begin{equation}
    \triangleleft : K \times \mathfrak{su}(2) \to TK, \quad (B, X) \mapsto B \triangleleft X = T_{e_k} r_B (X \times \tilde{B}),
\end{equation}

where $X \in \mathfrak{su}(2) \cong \mathbb{R}^3$, and

\begin{equation}
    \tilde{B} := \frac{1}{c + 1} B - \frac{\|B\|^2_E}{2(c + 1)^2} k
\end{equation}

identifying once again $B \in K \subset SL(2, \mathbb{C})$ with $B \in K \subset \mathbb{R}^3$ via (5.14). Here, $T_{e_k} r_B$ is the tangent lift of the right translation $r_B : K \to K$ by $B \in K$, and it acts simply by the matrix multiplication regarding $X \times \tilde{B} \in \mathfrak{R} \cong \mathbb{R}^3 \cong \mathfrak{gl}(3, \mathbb{R})$ via (5.15). Freezing the group element in (5.20) we arrive at a linear operator $\triangleleft : \mathfrak{su}(2) \to T_BK$, the transpose of which is the operator $\triangleleft^T : T^*_B K \to \mathfrak{su}^* (2)$ given by

\begin{equation}
    \triangleleft^T (\Psi_B) = T^*_B r_B (\Psi_B) \times \tilde{B}
\end{equation}

for any $\Psi_B \in T^*_B K$, where $\tilde{B}$ is the one in (5.21).

Next, the derivative of (5.16) with respect to $B \in K$ at the identity, in the direction of $Y \in \mathfrak{R} \subset \mathbb{R}^3$, yields

\begin{equation}
    \triangleright : \mathfrak{R} \times SU(2) \to TSU(2), \quad Y \triangleright A = Tr_A \left( Y \times (Ad_A(e_3) - e_3) \right),
\end{equation}

where we consider $Ad_A(e_3) - e_3 \in \mathbb{R}^3$ to perform the vector product, then we view the resulting element in $SU(2)$, i.e. as a $2 \times 2$ complex matrix. Freezing the group element in (5.23), we obtain
a mapping $b_A : \mathbb{R} \to T_A SU(2)$. The transpose of this operator $b_A^* : T_A^* SU(2) \to \mathbb{R}^*$ is given explicitly by

$$b_A^*(\Phi_A) = (\text{Ad}_A(e_3) - e_3) \times Tr^*_A \Phi_A.$$ 

(5.24)

Similarly, the derivative of (5.17) with respect to $B \in K$ in the direction of $Y \in \mathbb{R} \cong \mathbb{R}^3$ produces

$$\alpha : \mathbb{R} \times SU(2) \to \mathbb{R}, \quad Y \cdot A = \text{Rot}_A(Y),$$

and hence defines a linear mapping $\alpha A : \mathbb{R} \to \mathbb{R}$, whose transpose $A^* : \mathbb{R}^* \to \mathbb{R}^*$ may be given by

$$A^* \Psi := \text{Rot}^*_A \Psi,$$

for any $\Psi \in \mathbb{R}^*$.

Now we are ready to write the discrete dynamics on the matched pair Lie group given in (4.33) for the case of $SL(2, \mathbb{C}) = SU(2) \bowtie K$. Finally, substituting (5.19), (5.22), (5.24), and (5.25) into (4.33), we conclude that

$$B_{k+1}^T (\text{Ad}^*_A \Phi_k) + T^* r_{B_k} (d_2 L(A_k, B_k)) \times \mathbf{B} - \Phi_{k+1} + \text{Ad}_{B_k} \Psi_k - (\text{Ad}_{A_{k+1}}(e_3) - e_3) \times T^* r_{A_{k+1}} (d_1 L(A_{k+1}, B_{k+1})) - \text{Rot}^*_{A_{k+1}} \Psi_{k+1} = 0.$$ 

(6.1)

6. Conclusion and Discussions

In the present paper, we have studied the discrete dynamics on the matched pairs of Lie groupoids. This enables us to derive the equations of motions governing two mutually interacting discrete systems. More precisely, we have presented the (matched) discrete Euler-Lagrange equations on Lie groupoids in (4.6). Since Lie groups are the most immediate examples of Lie groupoids, we have introduced in particular the (matched) discrete Euler-Lagrange equations on Lie groups in (4.7). In Section 5 we have studied two concrete examples. One is the matched pair decomposition of the trivial Lie groupoid which led to the discrete Euler-Lagrange equations of the trivial Lie groupoid in terms of the Euler-Lagrange equations of the action groupoid, the Euler-Lagrange equations of the coarse groupoid, and the additional terms associated to the mutual actions of the action Lie groupoid and the coarse groupoid. The other tangible example we studied is the Iwasawa decomposition of the group $SL(2, \mathbb{C})$, as well as the matched pair decomposition of its discrete Euler-Lagrange equations.

We finally note that, the present paper concerns only the discrete dynamics generated by Lagrangian functions on Lie groupoids. It is very well known that there exists Lagrangian dynamics on the Lie algebroid level as well; [48, 38]. We, thus, plan to apply the matched pair strategy to the Lagrangian dynamics, from the point of view of Lie algebroids, on an upcoming paper.

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