Characterizations of BMO through commutators of bilinear singular integral operators

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Abstract. In this paper we characterize BMO in terms of the boundedness of commutators of various bilinear singular integral operators with pointwise multiplication. In particular, we study the commutators of the bilinear fractional integral operator and bilinear Calderón-Zygmund operators of convolution type with certain minor kernel conditions.

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1. Introduction and statements of main results

Recall that the space of functions with bounded mean oscillation, denoted BMO, consists of all locally integrable functions, $b$, such that

$$\|b\|_* := \sup_Q \int_Q |b(x) - b_Q| \, dx < \infty,$$

where $Q$ is a cube with sides parallel to the axes, and $b_Q$ is the average of $b$ over $Q$. Recall also that functions in BMO are identified up to a constant.

In the linear setting, we define the commutator of a function, $b$, with an operator, $T$, acting on a function $f$ as

$$[b, T](f)(x) := b(x)T(f)(x) - T(bf)(x).$$

In [3], Coifman, Rochberg, and Weis showed that the linear commutator was bounded for the Hilbert operator if and only if $b \in \text{BMO}$. Note that for $f \in L^p$ and $g \in L^{p'}$ we have

$$\langle [b, T](f), g \rangle = \langle T(f)g - fT^*(g), b \rangle,$$

where $T^*$ denotes the transpose of $T$. In this light, we see that the characterization of the boundedness of the commutator with BMO functions means $T(f)g - fT^*(g)$, which is clearly in $L^1$, is in fact in the Hardy space $H^1$, the pre-dual of BMO. This allowed Coifman et al. to achieve a factorization of $H^1$ in a higher dimensional setting than had been done previously. Janson and Uchiyama each extended this characterization of BMO, in [5] and [9] respectively, to commutators of Calderón-Zygmund operators of convolution type with smooth homogeneous kernels, and Chanillo did the same for commutators of the fractional integral operator in [1].

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The boundedness of commutators in the multilinear setting has been extensively studied already, as in Chen and Xue’s [2], P´erez and Torres’ [7], Lerner, Ombrosi, P´erez, Torres, and Trujillo-Gonz´alez’s [6], and Tang’s [8]. However, it has been an open question until now whether they can be used to characterize BMO. In this paper we will indeed show that the characterizations of BMO can be extended to a multilinear setting. For readability we will state and prove our results only for the bilinear cases.

The bilinear commutators we will be examining will be of the following forms

\[ [b, T]_1(f, g)(x) := bT(f, g)(x) − T(bf, g)(x), \]

and

\[ [b, T]_2(f, g)(x) := bT(f, g)(x) − T(f, bg)(x), \]

where \( b \) is a locally integral function and \( T \) is a bilinear singular integral operator. We define the bilinear fractional integral as follows,

\[ I_\alpha(f, g)(x) := \int \int f(y)g(z) \frac{dydz}{(|x − y| + |x, z|)^{2n − \alpha}}, \]

and we state our first result for \( T = I_\alpha \),

**Theorem 1.1.** For \( b \in L^1_{loc}, 0 < \alpha < 2n \) and \( 1 < p_1, p_2, q \) satisfying

\[ \frac{1}{p_1} + \frac{1}{p_2} − \frac{\alpha}{n} = \frac{1}{q} < 1, \]

we have

\[ \| [b, I_\alpha]_j \|_{L^{p_1} \times L^{p_2} \to L^q} \approx \| b \|_* \text{ for } j = 1 \text{ or } 2 \]

In particular, for \( j = 1 \) or \( 2 \) we have

\[ [b, I_\alpha]_j : L^{p_1} \times L^{p_2} \to L^q \iff b \in BMO. \]

To state our second result we must first define an \( m \)-linear Calderón-Zygmund operator, and in order to do this, we define the class of Calderón-Zygmund kernels. Let \( K(x, y_1, ..., y_m) \) be a locally integrable function defined away from the diagonal \( x = y_1 = ... = y_m \). If for some parameters \( A \) and \( \varepsilon \), both positive, we have

\[ |K(y_0, y_1, ..., y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn}} \]

and

\[ |K(y_0, ..., y_j, ..., y_m) − K(y_0, ..., y'_j, ..., y_m)| \leq \frac{A|y_j − y'_j|^\varepsilon}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn+\varepsilon}} \]

whenever \( 0 \leq j \leq m \) and \( |y_j − y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j − y_k| \), then we say \( K \) is an \( m \)-linear Calderón-Zygmund kernel. Suppose for some \( m \)-linear operator, \( T \), we have

\[ T(f_1, ..., f_m)(x) = \int K(x, y_1, ..., y_m) \prod_{j=1}^m f_j(y_j) d\vec{y} \]

for all \( x \notin \bigcap_{j=1}^m \text{supp}(f_j) \), where \( K \) is a Calderón-Zygmund kernel. Then if

\[ T : L^{p_1} \times ... \times L^{p_m} \to L^p, \]
for some $1 < p_1, ..., p_m$ satisfying $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$, we say $T$ is an $m$-linear Calderón-Zygmund operator. Many basic properties of these operators were thoroughly studied in by L. Grafakos and R. H. Torres in [4]. Lastly, we say that an operator is of ‘convolution type’ if the kernel $K(x, y, z)$ is actually of the form $K(x − y, x − z)$.

Our second theorem can now be stated as follows,

**Theorem 1.2.** For $b \in L^1_{loc}(\mathbb{R}^n)$, and $T$ a bilinear Calderón-Zygmund operator of convolution type with a homogeneous kernel of degree $-2n$, $K$, such that on some ball, $B$, in $\mathbb{R}^{2n}$ we have that the Fourier series of $\frac{1}{K}$ is absolutely convergent. We then have that for $1 > \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and for $j = 1$ or $2$,

$$[b, T]_j : L^{p_1} \times L^{p_2} \to L^p \iff BMO(\mathbb{R}^n).$$

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2. Proofs of the theorems

The sufficiency of $b \in BMO$ in the above theorems has been shown already in the bilinear setting. For Theorem 1.1, the sufficiency was shown in a weighted setting by X. Chen and Q. Xue in [2] in their Theorem 2.7 for a class of weights which includes the unweighted case. We state without proof a particular case of this which suits our needs,

**Proposition 2.1.** Let $0 < \alpha < 2n$, and $1 \leq p_1, p_2,$ and $q$ be such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q}$. Then we have that

$$\|[b, I_{\alpha}]_j(f, g)\|_{L^q} \lesssim \|b\|_* \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for $j = 1$ or $2$.

With this result in hand, we present the following theorem to demonstrate the necessity, which completes the desired characterization.

**Theorem 2.2.** Let $b \in L^1_{loc}$. If

$$\|[b, I_{\alpha}]_j(f, g)\|_q \leq C_0 \|f\|_{p_1} \|g\|_{p_2},$$

for $j = 1$ or $2$, where $1 > \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$, then $b \in BMO$ with $\|b\|_* \lesssim C_0$

To prove this, we adapt the techniques utilized by Chanillo in [1] to the multilinear setting, and we note that by the symmetry of the kernel of $I_{\alpha}$, it is enough to prove this theorem for $[b, I_{\alpha}]_1$. 
Proof of Theorem 2.2. Let $Q$ be a cube in $\mathbb{R}^n$, and let $\ell(Q)$ denote the side length of $Q$. We may assume that $b$ is real valued and let $\gamma(x) = \chi_Q(x)\text{sgn}(b(x) - b_Q)$. We then have the following,

$$|Q|^2 |b(x) - b_Q| \chi_Q(x)$$

$$= |Q| \gamma(x) \left( |Q| |b(x) - \int_{Q} b(y) \, dy \right)$$

$$= |Q| \gamma(x) \int (b(x) - b(y)) \chi_Q(y) \, dy$$

$$= \gamma(x) \int \int (b(x) - b(y)) \chi_Q(y) \chi_Q(z) \, dydz$$

$$= \gamma(x) \int \int \frac{(b(x) - b(y)) \chi_Q(y) \chi_Q(z)}{|x - y| + |x - z|} 2^{n-\alpha} \, dydz$$

$$\leq \gamma(x) \left( 2\sqrt{n} \ell(Q) \right)^{2n-\alpha} \int \int \frac{(b(x) - b(y)) \chi_Q(y) \chi_Q(z)}{|x - y| + |x - z|} 2^{n-\alpha} \, dydz$$

$$\simeq \gamma(x) |Q|^{2n-\alpha} [I, b]_1(\chi_Q, \chi_Q)(x),$$

$$= |Q|^{2n-\alpha} |\chi_Q(x)[I, b]_1(\chi_Q, \chi_Q)(x)|$$

We now divide through by $|Q|^3$, integrate both sides over $\mathbb{R}^n$, and continue with

$$\frac{1}{|Q|} \int_{Q} |b(x) - b_Q| \, dx \lesssim |Q|^{-1 - \frac{n}{q}} \int \int |\chi_Q(x)[I, b]_1(\chi_Q, \chi_Q)(x)| \, dx$$

$$\leq |Q|^{-1 - \frac{n}{q}} \left( \int_{Q} 1^q \right)^{\frac{1}{q}} \left( \int \int |I(\chi_Q, \chi_Q)(x)|^q \, dx \right)^{\frac{1}{q}}$$

$$\lesssim C_0 |Q|^{-1 - \frac{n}{q}} |Q|^{\frac{1}{p_1}} |Q|^{\frac{1}{p_2}}$$

$$= C_0.\]
some ball $B$ in $\mathbb{R}^{2n}$, the Fourier series of $\frac{1}{K}$ is absolutely convergent. If 
$$[b,T]_j : L^{p_1} \times L^{p_2} \to L^p,$$
for $j = 1$ or 2, where $1 > \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then $b \in \text{BMO}(\mathbb{R}^n)$.

The condition on the Fourier coefficients will, for example, be satisfied if $K$ is smooth. The proof of this theorem uses techniques applied by Jans on in [5], modified to suit the multilinear setting, and as with the fractional integral commutator, we need only prove this for $[b,T]_1$.

**Proof of Theorem 2.4.** Let $B = B((y_0,z_0), \delta \sqrt{2n})$, be the ball for which we can express $\frac{1}{K(y,z)}$ as an absolutely convergent Fourier series of the form 
$$\frac{1}{K(y,z)} = \sum_j a_j e^{i\delta \nu_j \cdot (y,z)}.
$$

The specific vectors, $\nu_j$, will not play a role in this proof. Note that due to the homogeneity of $K$, we can take $(y_0,z_0)$ such that $|(y_0,z_0)| \geq 2\sqrt{n}$ and take $\delta < 1$ small such that $B \cap \{0\} = \emptyset$. We do not care about the specific vectors $\nu_j \in \mathbb{R}^{2n}$, but we will at times express them as $\nu_j = (\nu_j^1, \nu_j^2) \in \mathbb{R}^n \times \mathbb{R}^n$.

Set $y_1 = \delta^{-1}y_0$ and $z_1 = \delta^{-1}z_0$, and note that for all $(y,z)$ such that 
$$\left( |y - y_1|^2 + |z - z_0|^2 \right)^{1/2} < \sqrt{2n},$$
we have 
$$\frac{1}{K(y,z)} = \frac{\delta^{-2n}}{K(\delta y, \delta z)} = \delta^{-2n} \sum_j a_j e^{i\delta \nu_j \cdot (y,z)}.
$$

Let $Q = Q(x_0,r)$ be an arbitrary cube in $\mathbb{R}^n$. Set $\tilde{y} = x_0 + ry_1$, $\tilde{z} = x_0 + rz_1$, and take $Q' = Q(\tilde{y}, r)$ and $Q'' = Q(\tilde{z}, r)$. Then for any $x \in q$ and $y \in Q'$, we have 
$$\left| \frac{x - y}{r} - y_1 \right| \leq \left| \frac{x - x_0}{r} \right| + \left| \frac{y - \tilde{y}}{r} \right| \leq \sqrt{n}.
$$

The same estimate holds for $x \in Q$ and $z \in Q''$, and so we have 
$$\left( \left| \frac{x - y}{r} - y_1 \right|^2 + \left| \frac{x - z}{r} - z_1 \right|^2 \right)^{1/2} \leq \sqrt{2n}.
$$

Let $\sigma(x) = \text{sgn}(b(x) - b_{Q'})$, and let 
$$f_j(y) = e^{-i\phi \nu^1_j \cdot y} \chi_{Q'}(y)
$$
$$g_j(z) = e^{-i\phi \nu^2_j \cdot z} \chi_{Q''}(z)
$$
$$h_j(x) = e^{i\phi \nu^\cdot (x,x) \cdot \sigma(x)} \chi_{Q}(x)
$$

Note by the size condition on $(y_0,z_0)$ we have that $Q \cap Q' \cap Q'' = \emptyset$ since at least one of $Q'$ and $Q''$ must be disjoint from $Q$. We also have that each of the above functions has an $L^q$ norm of $|Q|^{1/q}$ for any $q \geq 1$, and that for all $x$, $y$, and $z$ in the supports of their respective characteristic functions, $(x - y, x - z)$ avoids the singularity of $K$. In particular, this means that when the time comes, the use of
the kernel representation of \([b, T](f_j, g_j)\) is valid for all \(x \in Q\). We now have the following,

\[
\int_Q |b(x) - b_{Q'}| \, dx \\
= \int_Q (b(x) - b_{Q'}) \sigma(x) \, dx \\
= \frac{1}{|Q''|} \frac{1}{|Q'|} \int_Q \int_Q \int_{Q''} (b(x) - b(y)) \sigma(x) \, dzdydx \\
= r^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^{2n} K(x - y, x - z)}{K(\frac{x - y}{r}, \frac{x - z}{r})} \\
\cdot \sigma(x) \chi_Q(x) \chi_Q'(y) \chi_{Q''}(z) \, dzdydx \\
= \int \int \int (b(x) - b(y)) K(x - y, x - z) \sum_j a_j e^{i \frac{2\pi}{2^n} x \cdot (y - z)} \\
\cdot \sigma(x) \chi_Q(x) \chi_Q'(y) \chi_{Q''}(z) \, dzdydx \\
= \sum_j a_j \int h_j(x) \int (b(x) - b(y)) \\
\cdot K(x - y, x - z) f_j(y) g_j(z) \, dzdydx \\
= \sum_j a_j \int h_j(x) [b, T](f_j, g_j)(x) \, dx \\
\leq \sum_j |a_j| \int |h_j(x)||[b, T](f_j, g_j)(x)| \, dx \\
\leq \sum_j |a_j| \left( \int |h_j(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int ||[b, T](f_j, g_j)(x)||^{p'} \, dx \right)^{\frac{1}{p'}} \\
\leq \sum_j |a_j||h_j||_{L^{p'}} ||[b, T]||_{L^{p_1} \times L^{p_2} \rightarrow L^{p_1}} ||f_j||_{L^{p_1}} ||g_j||_{L^{p_2}} \\
= ||[b, T]||_{L^{p_1} \times L^{p_2} \rightarrow L^{p_1}} \sum_j |a_j||Q||_{\frac{p'}{p}}|Q||_{\frac{p'}{p}} \\
= |Q||||[b, T]||_{L^{p_1} \times L^{p_2} \rightarrow L^{p_1}} \sum_j |a_j|
\]

Recall that \(\frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx \leq \frac{2}{|Q|} \int_Q |f(x) - C|\) for any \(C\), and so this gives us that for any arbitrary \(Q \subset \mathbb{R}^n\) we have

\[
\frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx \leq \frac{2}{|Q|} \int_Q |b(x) - b_{Q'}| \, dx \leq 2||[b, T]||_{L^{p_1} \times L^{p_2} \rightarrow L^{p_1}} \sum_j |a_j|.
\]

Therefore \(b \in \text{BMO}(\mathbb{R}^n)\) \(\square\)
3. Closing remarks

Note that in the proof of Theorem 2.4, the regularity of Calderón-Zygmund operators was never used, and indeed, this theorem holds for any bilinear operator which can be realized as an integral operator away from the support of the functions on which it is operating, which satisfies the aforementioned kernel conditions. Furthermore, if $K$ is real valued, we can replace the absolute convergence of the Fourier series with a simple bound away from zero, in particular, we get the following result with an almost identical proof.

**Theorem 3.1.** For $b \in L^1_{loc}(\mathbb{R}^n)$, and $T$ a bilinear Calderón-Zygmund operator of convolution type with a real-valued, homogeneous kernel of degree $-2n$, $K$, such that on some ball, $B$, in $\mathbb{R}^{2n}$ we have either

$$\text{esssup}\{K(x,y,z) : (x,y,z) \in B\} < 0$$

or

$$\text{essinf}\{K(x,y,z) : (x,y,z) \in B\} > 0$$

We then have that for $1 > \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and for $j = 1$ or $2$,

$$[b,T]_j : L^{p_1} \times L^{p_2} \to L^p \iff \text{BMO} (\mathbb{R}^n)$$

Note that this bound away from zero is trivially satisfied if there exists a point at which $K$ is nonzero and continuous.

With regards to our main results, it should be noted that the proof of Theorem 2.2 easily generalizes to commutators with the $m$-linear fractional integral, and the original statement of Proposition 2.1 in [2] is for $m$-linear commutators, so it can be shown that Theorem 1.1 holds for the commutator with the $m$-linear fractional integral as well. Similarly, the proof of Theorem 2.4 can easily generalize to the $m$-linear setting to match the $m$-linear setting in [7], and so $m$-linear results mirroring Theorems 1.2 and 3.1 are quickly forthcoming as well.

While these results give a characterization of BMO in terms of the boundedness of certain bilinear commutators, present in all is the requirement that the exponent in our target space must be larger than 1. We do not know if it is possible to characterize BMO in this manner when the commutators are bounded from $L^{p_1} \times L^{p_2} \to L^p$ for $\frac{1}{2} < p < 1$. This is of interest because bounds of this form have indeed been shown; in [6], Lerner et al. showed that commutators with $m$-linear Calderón-Zygmund operators are bounded from $\prod_{j=1}^m L^{p_j}$ to $L^p$, for any $1 < p_1, ..., p_m$ such that $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, provided that $b \in \text{BMO}$. In [8], Tang obtained this result for commutators of vector valued multilinear Calderón-Zygmund operators, again without the restriction that $p$ be greater than 1.

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