Puzzles at Large N

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We consider the AdS/CFT correspondence in the context of 2d CFT and find that essentially all “single particle” primary fields with higher spin, predicted from the CFT side are missing from the Kaluza-Klein excitations of the AdS supergravity. The high mass extension of these missing states gives rise to the macroscopic entropy of extremal 5d black holes.
1. Introduction

Recently there has been a renewed interest in studying the large $N$ limit of quantum field theories. In particular it was suggested in [1] following earlier work in connection with black holes [2] that the fundamental strings in appropriate backgrounds may provide the long sought after stringy interpretation of large $N$ gauge theories in the regime of strong coupling ($g_s N \gg 1$). This proposal has been sharpened in [3][4]. In particular, a precise relation between Kaluza-Klein states of the supergravity in these backgrounds and the spectrum of conformal operators at large $N$ was suggested. In the case of $\mathcal{N} = 4$ gauge theory in $d = 4$ it was shown that the KK modes give rise to all the operators of the form $\text{Tr}(X_{i_1}...X_{i_n})$ and its descendants [3][4][5][6] (see also the related work [7]). Of course, part of the conjecture is that the relation between large $N$ gauge theory and gravity modes is one to one. In other words, for every conformal operator one should find a state (in the fock space of) KK excitations in supergravity.

Recently large $N$ limit of certain $\mathcal{N} = (4,4)$ gauge theories in 2 dimensions was considered from this point of view in [8][9] and a large set of chiral primary operators, which can be viewed as elements of one of the $(c, c), (c, a), (a, c), (a, a)$ rings, whose dimensions are protected from receiving quantum corrections were identified with specific Kaluza-Klein excitations. However as we shall see a large number of states expected from the CFT side are actually missing from the Kaluza-Klein excitations. The aim of this note is to elaborate on these missing states.

2. The Basic Puzzle

In this section we state the basic puzzle in general terms. The manifestation of this puzzle for the 2d is discussed in the subsequent section.

Consider a conformal theory in $d$ dimensions. Let us consider the Euclidean version. The rotation group is $SO(d)$, and the conformal group is $SO(1, d+1)$. The set of operators in this theory will form representations of this group. Let us consider the primary operators of the conformal group and ask what spins they carry under $SO(d)$. A priori one would expect that if we consider sufficiently high dimension primary operators we should be able to get sufficiently high spin representation of $SO(d)$.

We generally have “single particle” operators which correspond to a single particle state in supergravity and multiparticle operators which correspond to many particle states. It is natural to assume that single particle operators are represented in the field theory by
a single trace while multiparticle operators correspond to operators with multiple traces (in the case of U(N) super-Yang-Mills). The AdS/CFT correspondence predicts that as \( N \to \infty \) (and \((g_s N) \to \infty\)) no matter which high dimensional primary operator we consider we will never find a “single particle” operator with spin greater than 2! This follows from the fact that when we expand the supergravity modes and find the Kaluza Klein states we are dealing with spins which go at most to spin 2. (Of course it is possible to have primary multiparticle operators with spins greater than one). This is a rather counter-intuitive statement, at least for 2d CFT’s. In this case, if the central charge \( c \) is bigger than one then one gets infinitely many primary states, and moreover, their spins \( L_0 - \bar{L}_0 \) are not bounded. Sometimes what one can do, even if \( c > 1 \) is to consider an enlarged symmetry algebra for which the primary fields of the enlarged algebra are finite in number and all have zero spin (for example the left-right symmetric WZW models). Note, however, that if we consider a very large value of \( c \), as is the analogue of the large \( N \) theories, for this to happen we need to have an infinitely big symmetry algebra. This is not the case for CFT’s in the context of AdS/CFT correspondence where the symmetry group is the superconformal group, and which is independent of \( N \). Thus since we have a finite symmetry algebra we expect to get many primary states with arbitrarily large spins. This is the basic puzzle.

3. The 2d Example

Here we consider type IIB strings and the system of large number of 1-branes and 5-branes wrapped around \( T^4 \) or \( K^3 \), recently analyzed from the viewpoint of AdS/CFT correspondence in [5][4]. We now discuss the missing states in this case. In the context of D1 and D5 branes this system was already analyzed in [1][4] and applied to the question of microstates of 5 dimensional black holes in [1]. In particular the left- and right-moving ground states of the left-over 1+1 CFT in the Ramond-Ramond sector was analyzed in [1][4] and was shown to correspond to BPS states anticipated from string/string duality. Moreover the left-moving excited states and right-moving ground states in the Ramond-Ramond sector, which have non-zero spin \( L_0 - \bar{L}_0 \) (and correspond to non-zero momentum in spacetime) were used in [1][1] to account for microstates of certain extremal black holes in 5 dimensions. For that particular application only the degeneracy of such states with large values of \( L_0 - \bar{L}_0 \) was relevant, but as was argued [1][1] there is a lower bound on such values (at least in the \( K^3 \) case) for all values of \( L_0 - \bar{L}_0 \) coming from the consideration of
elliptic genus of symmetric products. In particular even for small values of $L_0 - \mathcal{L}_0$ there is a non-zero lower bound for the number of such states (see the analysis of the elliptic genus of symmetric products of $K3$ in [12]). Moreover this lower bound is independent of any deformations of the theory, such as the moduli of $K3$ or the coupling constant. In the NS-NS sector these states get translated to states which are chiral (or anti-chiral) primary on the right-mover side but arbitrary state on the left-mover side. The corresponding operators form an infinite dimensional chiral algebra [13]. To get a feeling for such states in the present example, consider the primary states which are purely left-mover (i.e. tensored with the identity operator on the right-mover side). In particular consider $N$ symmetric products of $K3$. Let $T_i$ denote the left-moving energy momentum tensor for the $i$-th $K3$. For the sigma model on

$$(K3)^N/S_N$$

we can construct $N$ new holomorphic currents out of $T$, namely we consider the permutation invariant operators

$$T^l = \sum_{i=1}^{N} T_i^l$$

for $l = 1, ..., N$. It is easy to see for each $l$ (except $l = 1$) we can form one primary field (using products of $T^{k_i}$ with $\sum k_i = l$). In fact these currents lead to a kind of $W$ algebra [14] (for a review of $W$ algebra see [15] and references therein). We can repeat this for all the $\mathcal{N} = 4$ superconformal generators and we obtain some kind of $\mathcal{N} = 4$ $W$-algebra.

We thus conclude that there are infinitely many primaries of the $\mathcal{N} = 4$ superconformal algebras with arbitrary integer $L_0 - \mathcal{L}_0$ spin, in contradiction with anticipations based on gravity which predicts only a small bounded set. In fact the states that are missing from the AdS/CFT correspondence in this case are huge and in the asymptotic regime form the bulk of the microstates of the black hole. In fact note that if the problem was only occuring for extremely high values of spin, then one could have perhaps matched it to some very high spin states that are also expected in the AdS side, such as massive string states, etc. However we have shown here that for any finite value of spin there are such missing states.

Even though this example demonstrates that something is missing in the AdS side to account for all the expected CFT states, it is not clear from this example alone whether or not this is connected with special features of conformal theories in 2d.

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1 Here we mean the modified elliptic genus which on the non-supersymmetric side measures the fermion number by the insertion of $\exp(i\theta F_L)$ where $F_L$ is the left-moving fermion number.
4. Discussion

What we have shown here is that the proposed closed string theories on AdS do not give a full account of all the states that we expect from the 2d CFT side. That there should be some difficulty in getting some states is perhaps to be expected. After all, in the 2d example, had we been able to identify all these higher spin states in a purely gravitational setup we would have most probably been able to account for the black hole entropy directly in a semiclassical gravitational setup.

Our general discussion, as well as the concrete example in the 2d case suggests that the organizing principle for conformal fields at large $N$ should be a much bigger algebra than simply the superconformal algebra. We should perhaps expect some kind of infinite dimensional $W$ algebras to be present, such as volume preserving diffeomorphisms in the corresponding dimension. However if one is to get such a symmetry from an underlying closed string theory, the corresponding gravity theory would have to involve something analogous to $W_{\infty}$ gravity studied in the 2d case. In particular the higher spin gravity states would couple to higher spin primary fields, and the KK excitations of such a gravity theory may have a chance of reproducing such states. It is an extremely interesting question to find the right ‘gravity’ theory which could account for the spectrum of all operators at large $N$. It should be possible to make this more concrete in the context of the large $N$ limit of 2d conformal field theories. In fact in such a case some connections between large $N$ and $W_{\infty}$ algebra and self-dual gravity in four dimensions has already been noted \[16\][17]. One would then expect that the large $N$ limit of 2d conformal theories should be related to the $N = 2$ string \[18\] which quantizes self-dual gravity in 2+2 dimensions. It would also be interesting to find the analogous proposal in higher dimensions and also find how it fits with the AdS/CFT correspondence. Note that this is giving a hint that a $d$ dimensional conformal theory may be related to a theory in $d + 2$ dimensions. This is in accord with the fact that $SO(d - 1, 1)$ Lorentz group is to be extended to $SO(d, 2)$ conformal group (i.e. it would be related to including the extra dimension in which the AdS is embedded in). This is somewhat reminiscent of F-theory.

One could also look for less radical solutions. An interesting feature of the missing states is that the mismatch is not so bad as it could be. Consider all primary states with levels smaller than a given level $\Delta$. Then the number of states is independent of $N$ (or $c$) for small values of $\Delta$. This is not what we would expect for a generic conformal field theory (for example consider $c$ free bosons in 1+1 dimensions). This is suggesting that probably
the solution might be related to understanding more properly the dynamics of the fields that might live at the boundary of AdS. In fact higher spin fields were analyzed in [19] but these do not seem to be the states we need because they can be viewed as two particle states. Another possibility, in the 1+1 dimensional case, is that we have a Chern-Simons theory which might give some anyon statistics to the fields living in the bulk of AdS. This might lead to new states coming from bound states of the particle states that we already had.

Note that the question of extending the missing states to the higher dimensional case, for example $N = 4$ in four dimensions is a rather interesting one (in fact the simplest guess for potentially missing states in the form of $\text{Tr} W_+^k$ where $W_+$ is the gaugino chiral field suffers from the fact that they are not $N = 1$ primary states). One way one can imagine getting such states is by considering $N = 4$ Yang-Mills on $T^3$, and viewing the 4 dimensional theory from the 2d perspective in which case we would expect to get similar operators related to the elliptic genus. It is conceivable that they are related to ‘t Hooft flux operators in this context.

Despite the fact that we have found many states missing on the AdS side in the 2 dimensional example, given the nice results that have already been obtained from the conjectured AdS/CFT correspondence, and the fact that there are no obvious states missing in the higher dimensional cases, one has the feeling that even if this proposal is not complete, it is not too far from being complete.

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