Distributed static output feedback control for interconnected systems in finite-frequency domain

Xuefeng Chen | Huiling Xu | Hui Wang

School of Science, Nanjing University of Science and Technology, Nanjing, Jiangsu, China

Abstract

This paper aims at designing a distributed static output feedback (DSOF) controller for the interconnected system in finite-frequency domain. Motivated by the fact that the signal transmitting process is usually affected by surrounding environments, the interconnection communication between subsystems are assumed to be non-ideal. By introducing some dilated multipliers into the extended condition that guarantees the well-posedness, stability and finite-frequency $H_\infty$ performance of non-ideally interconnected systems, a new equivalent condition is derived, which is more effective for parameterizing the DSOF controller. A two-stage approach is then developed for constructing a DSOF controller: design an initial distributed full information (DFI) controller first and then derive a desired DSOF from the DFI controller. Moreover, two iterative linear matrix inequality based algorithms are proposed to improve the solvability of the DFI controller and DSOF controller design problems. Finally, an example is given to show the effectiveness of the proposed two-stage DSOF controller design method.

1 | INTRODUCTION

An interconnected system is a large-scale coupling system which consists of multiple subsystems with interconnected terms. In the last few decades, due to the broad range of practical applications in vehicle platoons, urban traffic systems, power networks and other areas, the interconnected systems have received significant attention and a series of control schemes have been developed. Traditionally, the centralized control scheme is adopted to achieve the closed-loop stability and performance, but the computational burden of this control scheme is extremely heavy as it needs to deal with all states information centrally [1]. To overcome this disadvantage, the decentralized control scheme is proposed, where the sub-controllers only utilize the local state information to control subsystems [2–4]. However, when the interconnection between subsystems is strong, the decentralized control scheme seems inappropriate either as the ignorance of communication will lead to poor performance. Trading off between calculation cost and control performance, researchers retain some level of communication between the sub-controllers and establish the distributed control scheme. In [5], authors proposed a state-space description for the interconnected system and analyzed the well-posedness, stability, contractiveness and distributed control of such interconnected system. On this basis, considerable works on distributed control of interconnected systems were carried out, such as $H_2$ distributed control [6], distributed finite-time control [7], distributed control of discrete-time interconnected systems [8–10] etc.

As a significant component of control theory, static output feedback (SOF) control problem has been extensively studied in recent years [11]. Compared with the dynamic output feedback controller, SOF controller has a simpler structure and is less expensive to be implemented. Moreover, it is shown in [12] that the dynamic output feedback control problem can be transformed into an SOF control problem. The SOF control problem is essentially a bilinear matrix inequality problem and generally considered to be NP-hard. In the last two decades, we have witnessed lots of research on SOF controller design for linear time-invariant systems through linear matrix inequality (LMI) based methods, to name a few, iterative LMI methods [13, 14], direct LMI methods with extra constraints [15, 16]. Recently, the two-stage approach has been shown to be a
promising method for dealing with the SOF controller design problem [17–20]. The idea of the two-stage approach is finding a state feedback (SF) controller at the first stage and then explore an SOF controller from the obtained SF controller at the second stage. For interconnected systems, most efforts have been devoted to decentralized SOF controller design till now but few research is focused on distributed SOF (DSOF) controller design. Thus, in this paper, one of our motivations is to extend the two-stage approach for designing a DSOF controller for the interconnected system.

Moreover, it is notable that most efforts so far have been devoted to controller design for the interconnected system in the entire-frequency (EF). Actually, in some real applications, there is no need to consider the system properties and design specifications in the EF range, as many systems may just belong to certain frequency range and the energies of many signals concentrate only in some finite-frequency (FF) ranges [21]. The generalized Kalman–Yakubovich–Popov (GKYP) lemma establishes the equivalence between an infinite number of frequency domain inequalities characterizing the properties of a transfer function in a finite-frequency range and a numerically tractable linear matrix inequality for its state space realization, which enables one to deal with the restricted frequency-domain specifications [22–28]. Based on the GKYP lemma developed for interconnected systems, a great quantity of results on control synthesis with restricted frequency-domain specifications have been obtained [7, 29–32]. However, no research on frequency-domain constrained feedback controller design for interconnected systems has been reported, which is another motivation of our work.

In this paper, we are going to study the DSOF controller design problem for interconnected systems subject to restricted frequency-domain specifications. Based on the GKYP lemma, an efficient two-stage approach is developed for constructing an FF DSOF controller. Moreover, for improving the solvability of the developed two-stage approach, we look into the conditions in depth and propose iterative LMI algorithms for getting a desired initial distributed full information controller and exploring the DSOF controller. Finally, an example is given to illustrate the validity and effectiveness of the proposed methods. The contributions of this paper are threefold:

1. Compared with the existing results based on a common communication channel [6, 10, 31], the communication channels between subsystems we considered in this paper are affected by uncertainty and attenuation, which is more general and can reflect the reality.
2. By introducing some dilated multipliers, we derive a new equivalent condition for characterizing the performance of interconnected systems with a finite-frequency constraint, which is more effective for parameterizing the DSOF controller.
3. A two-stage approach is proposed, as a first attempt, to design a DSOF for the interconnected system subject to restricted frequency-domain specifications, which fills the blank of distributed static output feedback controller design for interconnected systems.

The remainder of this paper is organized as follows. The system description and some primers are given in Section 2. Section 3 presents the two-stage approach of designing the DSOF controller for interconnected systems. Section 4 gives a way to construct a desired initial distributed full information controller. An example is included in Section 5 to demonstrate the validity and effectiveness of the derived results. Section 6 concludes this paper.

**Notations.** The sets of real numbers and nonnegative integers are denoted by \( \mathbb{R} \) and \( \mathbb{Z}^+ \), respectively. \( \mathbb{C}^n \) denotes \( n \)-dimensional complex space. \( \mathbb{C}^{m \times n} \) and \( \mathbb{H}_n \) refer to the \( m \times n \) complex matrix set and \( n \times n \) Hermitian matrix set, respectively. For a matrix \( A \in \mathbb{C}^{m \times n} \), \( A^* \) denotes its complex conjugate transpose; \( H(A) \) indicates \( A + A^*; A < 0 \) (\( > 0 \)) means that \( A \) is negative (positive) definite, where 0 denotes a zero matrix with appropriate dimension; \( \sigma_{\text{max}}(A) \) denotes the maximum singular value of matrix \( A \). For matrices \( A_1 \) and \( A_2, A_1 \otimes A_2 \) means the Kronecker product; \( [A_1; A_2] \) denotes the matrix \([A_1^*, A_2^*]^T\). The block-diagonal matrix is denoted by \( \text{diag} \{\cdots \} \), while \( \text{cof} \{\cdots \} \) denotes the block-column vector/matrix. The notation \( [A_{ij}]_{\text{soc}} \) is defined as:

\[
[A_{ij}]_{\text{soc}} := \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix},
\]

where \( A_{ij}, i = 1, \ldots, m, j = 1, \ldots, n \) are matrices with appropriate dimensions. \( T := [\mathcal{E}^I \mathcal{E}^I] \) represents a permutation matrix where the \( i \)-th block of row-partitioned matrix \( \mathcal{E}^I \) is an appropriately dimensioned identity matrix. The “\( \ast \) in a matrix block denotes the transposition of its conjugate symmetric term.

### 2 | SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the interconnected system \( G \) which is composed of \( L \) heterogeneous subsystems and the subsystems are interconnected over a communication network. The topology of such interconnected system can be modeled by an undirected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). The vertex set \( \mathcal{V} \) of the graph is defined as \( \mathcal{V} = \{G_i, i = 1, \ldots, L\} \), where \( G_i \) represents \( i \)-th subsystem and has the following dynamics:

\[
G_i: \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A_{TX}^i & A_{TS}^i & B_T^i & B_L^i \\ A_{TX}^i & A_{TS}^i & B_T^i & B_L^i \\ C_i^T & C_i^T & D_{yT}^i & D_{yL}^i \\ C_i^T & C_i^T & D_{yT}^i & D_{yL}^i \end{bmatrix} \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + \begin{bmatrix} d_i(t) \\ u_i(t) \end{bmatrix} \tag{1}
\]

where \( t \in \mathbb{R}^+; x_i(t) \in \mathbb{R}^{n_i}, \ z_i(t) \in \mathbb{R}^{n_i}, d_i(t) \in \mathbb{R}^{p_i}, y_i(t) \in \mathbb{R}^{q_i} \) and \( u_i(t) \in \mathbb{R}^{r_i} \) denote the state, performance output, external disturbance, measured output and control input of subsystem \( G_i \), respectively; \( v_i(t), u_i(t) \in \mathbb{R}^{r_i} \) are the input signal...
from the neighboring subsystems and output signal to neighbors, respectively.

The set of undirected edges is defined by $E := \{(G_i, G_j), i, j = 1, ..., L\}$ and represents the interconnection between subsystems. The weight of edge $(G_i, G_j)$ is an integer $n_{ij}$. The situation that the subsystem $G_i$ and $G_j$ are not interconnected can also be captured by allowing $n_{ij} = 0$. It is assumed that vector $v(t)$ and $w(t)$ can be subdivided into $v(t) = \text{col}[v_j(t)]_{j=1}^f$ and $w(t) = \text{col}[w_j(t)]_{j=1}^f$ with $v_j(t)$, $w_j(t) \in \mathbb{C}^{n_{ij}}$, where $v_j(t)$ and $w_j(t)$ are the internal input and output transferring between subsystem $G_i$ and $G_j$. Moreover, consider the interconnection relationships between subsystems to be

$$v_{ij}(t) = \Delta_{ji}w_{ji}(t), \quad \Delta_{ji} \in \Delta_{ij},$$

for all $i, j = 1, ..., L$, $t \in \mathbb{R}^+$, where

$$\Delta_{ij} = \left\{ \text{diag}(\delta_{ij}^1I_{n_{ij}}, ..., \delta_{ij}^fI_{n_{ij}}) : \sum_{j=1}^f \delta_{ij}^j = n_{ij}, \right\}$$

$$\delta_{ij}^l \in \mathbb{R}, |\delta_{ij}^l| \leq 1, l = 1, ..., f$$

denotes a class of systems which capture the nature of the interconnection between $G_i$ and $G_j$, such as being lossy, affected by possible uncertainties or attacked by adversaries in the transmission. Notably, the situation that the interconnection is ideal, that is, $\Delta_{ji} = I$, is included in this interconnection relationship. Figure 1 gives an intuitive view of the interconnected system for a clear illustration.

In this paper, we are expected to design a distributed static output feedback controller $K$ with sub-controller $K_j$ being

$$K_j : \begin{bmatrix} u_{ij}(t) \\ w_{ji}^k(t) \\ \end{bmatrix} = \begin{bmatrix} K_{ij}^c \\ K_{ij}^s \\ K_{ij}^x \\ \end{bmatrix} \begin{bmatrix} y_{ij}(t) \\ v_{ij}^k(t) \\ \end{bmatrix},$$

where $v_{ij}^k(t), w_{ji}^k(t) \in \mathbb{R}^{n_{ij}}$ are the input signal from the neighboring sub-controllers and output signal to neighbors, respectively. The interconnection relationship between sub-controllers follows with the one between subsystems, that is, $v_{ij}^k(t) = \Delta_{ji}w_{ji}^k(t)$. In Figure 2, we have depicted the distributed control structure of the interconnected system for a clear illustration.

Substituting the distributed controller into the interconnected system leads to the closed-loop system $\hat{G}$ with subsystem $\hat{G}_i$ being

$$\hat{G}_i : \begin{bmatrix} x_i(t) \\ \end{bmatrix} = \begin{bmatrix} (A_i^T)C_i & (A_i^T)C_i & (B_i^T)C_i \\ (A_i^T)C_i & (A_i^T)C_i & (B_i^T)C_i \\ (A_i^T)C_i & (A_i^T)C_i & (B_i^T)C_i \\ \end{bmatrix} \begin{bmatrix} x_i(t) \\ \end{bmatrix} + \begin{bmatrix} v_i(t) \\ \end{bmatrix},$$

and the system matrices are shown as follows:

$$\begin{bmatrix} A_i^T & B_i^T & K_i^c & C_i^s & C_i^x \\ A_i^T & B_i^T & K_i^c & C_i^s & C_i^x \\ A_i^T & B_i^T & K_i^c & C_i^s & C_i^x \\ \end{bmatrix} \begin{bmatrix} K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s \\ K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s \\ K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s & K_i^c C_i^s \\ \end{bmatrix} + \begin{bmatrix} D_i^c & D_i^c & D_i^c & D_i^c \\ D_i^c & D_i^c & D_i^c & D_i^c \\ D_i^c & D_i^c & D_i^c & D_i^c \\ \end{bmatrix}.$$  

Then the transfer function matrix of closed-loop system $\hat{G}$ can be compactly given as:

$$\hat{G}(j\omega) = \hat{C}(j\omega) - \hat{A}^{-1}\hat{B} + \hat{D},$$

where

$$\hat{A} := \text{diag}(\{A_i^T\}_{i=1}^L), \quad \hat{B} := \text{diag}(\{B_i^T\}_{i=1}^L), \quad \hat{D} := \text{diag}(\{D_i^c\}_{i=1}^L),$$

$$\hat{C} := \text{diag}(\{C_i^s\}_{i=1}^L), \quad \hat{D} := \text{diag}(\{D_i^c\}_{i=1}^L).$$
with $m = \sum_{i=1}^{n} m_i$, $n = \sum_{i=1}^{n} n_i$ and permutation matrices $P$; $T$, respectively, satisfying

$$P \times \text{col}(v_i(t)|_{[m_i]}) = \text{col}(\text{col}(v_j(t)|_{[m_j]})|_{[n_j]}),$$

$$T \times \text{col}(v_i(t), w_i(t)|_{[m_i]}) = \text{col}(\text{col}(v_j(t)|_{[m_j]}), \text{col}(w_j(t)|_{[n_j]})).$$

Finally, the finite-frequency distributed static output feedback controller design problem can be expressed as follows:

1. The closed-loop system $\mathcal{G}$ with subsystem $\mathcal{G}_i$ in (4) is well-posed and asymptotically stable;
2. The transfer function matrix $G(j\omega)$ of closed-loop system $\mathcal{G}$ satisfies

$$\|G(j\omega)\|_{\infty}^\Omega := \sup_{\omega \in \Omega} \sigma_{\max}[G(j\omega)] < \gamma, \ \forall \omega \in \Omega,$$  \tag{7}

for all $i = 1, \ldots, L$, where $\Phi := [0 \ 1 \ 0]$, $\Pi_i := \text{diag}(I_p, I_p^2, I_p)$,

$$F_i := \begin{bmatrix}
(A^{C}_{TT})_i & (A^{C}_{TD})_i & (B^{C}_{TD})_i \\
I & 0 & 0 \\
0 & I & 0 \\
(C^{C}_{iC})_i & (C^{C}_{iD})_i & (D^{C}_{iD})_i \\
0 & 0 & I
\end{bmatrix},$$

$$F_i^* := \begin{bmatrix}
(A^{C}_{TT})^*_i & (A^{C}_{TD})^*_i \\
I & 0 \\
0 & I \\
(C^{C}_{iC})^*_i & (C^{C}_{iD})^*_i & (D^{C}_{iD})^*_i \\
0 & 0 & I
\end{bmatrix},$$

$$\Psi_i := \begin{bmatrix}
-1 & j\omega \\
-j\omega & -\omega_j \omega_k
\end{bmatrix},$$

$$Z_i := \begin{bmatrix}
(Z^{11}_{ij})_C & (Z^{12}_{ij})_C \\
(Z^{12}_{ij})^*_C & (Z^{22}_{ij})_C
\end{bmatrix}, \quad Z_i := \begin{bmatrix}
(Z^{11}_{ij})_C & (Z^{12}_{ij})_C \\
(Z^{12}_{ij})^*_C & (Z^{22}_{ij})_C
\end{bmatrix},$$

where $(Z^{11}_{ij})_C$, $(Z^{12}_{ij})_C$ and $(Z^{22}_{ij})_C$ can be partitioned as

$$(Z^{11}_{ij})_C := \begin{bmatrix}
(Z^{11}_{ij})_C & (Z^{11}_{ij})^*_{JK} \\
(Z^{11}_{ij})^*_{JK} & (Z^{11}_{ij})_K
\end{bmatrix},$$

$$(Z^{12}_{ij})_C := \begin{bmatrix}
(Z^{12}_{ij})_C & (Z^{12}_{ij})^*_{JK} \\
(Z^{12}_{ij})^*_{JK} & (Z^{12}_{ij})_K
\end{bmatrix},$$

$$(Z^{22}_{ij})_C := \begin{bmatrix}
(Z^{22}_{ij})_C & (Z^{22}_{ij})^*_{JK} \\
(Z^{22}_{ij})^*_{JK} & (Z^{22}_{ij})_K
\end{bmatrix},$$

with

$$(Z^{11}_{ij})_C:= \text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}), \ (Z^{12}_{ij})_C:= -\text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}),$$

$$(Z^{11}_{ij})_K := \text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}), \ (Z^{12}_{ij})_K := -\text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}),$$

$$(Z^{11}_{ij})_{JK} := \text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}), \ (Z^{12}_{ij})_{JK} := -\text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}),$$

$$(Z^{11}_{ij})_{K_{i=1}^{L_i}} := \text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}), \ (Z^{12}_{ij})_{K_{i=1}^{L_i}} := -\text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}),$$

$$(Z^{11}_{ij})_{K_{i=1}^{L_i}} := \text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}), \ (Z^{12}_{ij})_{K_{i=1}^{L_i}} := -\text{diag}(\{x^{11}_{ij}\}_{i=1}^{L_i}),$$

and $(Z^{11}_{ij})_C$, $(Z^{12}_{ij})_C$ and $(Z^{22}_{ij})_C$ are defined analogously.

**Proof.** The proof of Lemma 1 is close to the one of Theorem 2 in [31] and thus is omitted here.

**Remark 3.** Lemma 1 is the continuous-time version of the result given in [31] for discrete-time interconnected systems.
In addition, the difference between the proof of Lemma 1 and the one of Theorem 2 in [51] lies in that \((Z_i^i)^{1071}\) and \((Z_i^i)^{1072}\) in Lemma 1 and be dilated into the following matrix

\[
\begin{bmatrix}
    K_i^i C_i^i & K_i^i D_i^j
    K_i^j C_i^j & K_i^j D_i^j
    K_i^j C_i^j & K_i^j D_i^j
    K_i^j C_i^j & K_i^j D_i^j
\end{bmatrix} - [K_{ij}^i]_{2 \times 2}.
\]

Next, based on the dilated conditions in (10) and (11), we now give conditions which are equivalent to (8) and (9) in Lemma 1.

**Theorem 1.** Consider the closed-loop system \(\bar{\mathcal{G}}\) with transfer function matrix \(G(j \omega)\) given in (6). Let scalars \(\omega_1, \omega_2 \in \mathbb{R}\) be given. There exist matrices such that the conditions (8) and (9) in Lemma 1 are satisfied if and only if there exist symmetric matrices \(X_i^i, X_i^j, Y_i^i, Z_i, Z_i, Z_i, 0\), and matrices \([K_{ij}^i]_{2 \times 2} \in \mathbb{C}^{i_1 + j_1} \times (i_1 + j_1), [K_{ij}^j]_{2 \times 2} \in \mathbb{C}^{i_2 + j_2} \times (i_2 + j_2), [X_i^i]_{3 \times 3} \in \mathbb{C}^{2i_1 \times 2j_1}, [X_i^j]_{3 \times 3} \in \mathbb{C}^{2i_2 \times 2j_2}\) such that \(X_i^i > 0, Y_i^i > 0, (Z_i^i)^{1071}, (Z_i^j)^{1072} > 0\) and

\[
\begin{align*}
    \text{diag}(\Phi \otimes X_i^i + \Psi \otimes X_i^j, Z_i, \Pi_i, 0) & > H(x, Y_i) = 0, \quad (12) \\
    \text{diag}(\Phi \otimes Y_i^i, Z_i, 0) & > H(x, Y_i) = 0, \quad (13)
\end{align*}
\]

for all \(i = 1, \ldots, L, \) where \(b_i = m_i + 2n_i, X_i := \text{diag}([X_i^i]_{3 \times 3}, [R_{ij}^i]_{2 \times 2}), Y_i := \text{diag}([X_i^i]_{3 \times 3}, [R_{ij}^j]_{2 \times 2})\).

**Proof.** (\(\iff\)): Pre-multiplying and post-multiplying the matrix inequality (12) by \(Y_i^{1071}\) and \(Y_i^{1072}\), and pre-multiplying and post-multiplying the matrix inequality (12) by \(Y_i^{1071}\) and \(Y_i^{1072}\), respectively, we can get

\[
\begin{align*}
    Y_i^{1071} & \geq \text{diag}(\Phi \otimes X_i^i + \Psi \otimes X_i^j, Z_i, \Pi_i, 0) Y_i^{1072} \geq 0, \quad (16)
\end{align*}
\]
Note that the null spaces of $Y_i$ and $Y_i^\perp$ can be, respectively, selected as

$$Y_{i}^\perp = \begin{bmatrix} \mathcal{K}_i \end{bmatrix}, \quad Y_{i}^{\perp} = \begin{bmatrix} \mathcal{K}_i^\perp \end{bmatrix}. $$

Then we can find that the condition (16) and (17) are exactly the conditions (8) and (9) in Lemma 1. Thus the sufficiency has been proved.

$(\Rightarrow)$: Define

$$\bar{P}_i := \begin{bmatrix} -I & 0 & 0 & 0 \\ (A_{1i}^T)^C & 0 & (A_{2i}^T)^C & 0 \\ 0 & (A_{3i}^T)^C & 0 & 0 \\ (C_{1i}^T)^C & 0 & (C_{2i}^T)^C & 0 \end{bmatrix},$$

$$\bar{P}_i^* := \begin{bmatrix} -I & 0 & 0 & 0 \\ (A_{1i}^T)^C & 0 & (A_{2i}^T)^C & 0 \\ 0 & (A_{3i}^T)^C & 0 & 0 \\ (C_{1i}^T)^C & 0 & (C_{2i}^T)^C & 0 \end{bmatrix}. $$

Thus it can be found that

$$\bar{P}_i^\perp = \bar{P}_i, \quad (\bar{P}_i^\perp)^\perp = \bar{P}_i^\,*.$$

Thus the inequalities (8) and (9) can be equivalently represented as

$$(\bar{P}_i^\perp)^\perp \cdot \text{diag}(\Phi \otimes Y_i^\perp, \Psi \otimes X_i^\perp, Z_i, \Pi_i) \cdot \bar{P}_i^\perp < 0, \quad (18)$$

$$(\bar{P}_i^\perp)^\perp \cdot \text{diag}(\Phi \otimes X_i^\perp, Z_i) \cdot (\bar{P}_i^\perp)^\perp < 0. \quad (19)$$

According to Finlser's lemma [33], we have that (18) and (19) are, respectively, equivalent to

$$\exists \tilde{X}_i, \text{diag}(\Phi \otimes Y_i^\perp, \Psi \otimes X_i^\perp, Z_i, \Pi_i) + \text{He}(\tilde{X}_i \bar{P}_i) < 0,$$

$$\exists \tilde{X}_i^\perp, \text{diag}(\Phi \otimes X_i^\perp, Z_i) + \text{He}(\tilde{X}_i^\perp \bar{P}_i) < 0.$$

Then it is easy to see that there must exist a sufficiently large scalar $\varepsilon$ such that the following inequalities hold:

$$\begin{bmatrix} \text{diag}(\Phi \otimes Y_i^\perp, \Psi \otimes X_i^\perp, Z_i, \Pi_i) + \text{He}(\tilde{X}_i \bar{P}_i) \end{bmatrix} * \begin{bmatrix} (B_{1i}^*)^* & (B_{2i}^*)^* & 0 & (D_{1i}^*)^* \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_i^* \\ -2\varepsilon I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \text{diag}(\Phi \otimes X_i^\perp, Z_i) + \text{He}(\tilde{X}_i^\perp \bar{P}_i) \end{bmatrix} * \begin{bmatrix} (B_{1i}^*)^* & (B_{2i}^*)^* & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{X}_i^\perp^* \\ -2\varepsilon I \end{bmatrix} < 0.$$
for all \( i = 1, \ldots, I \), where

\[
X_i := \text{diag} \{ A_{\text{mi}}^i \}_{n \times 4}, \quad I_{n \times n}
\]

\[
X_i' := \text{diag} \{ P_{\text{mi}}^i \}_{n \times 3}, \quad I_{n \times n}
\]

\[
\Xi_i := \begin{bmatrix}
-I & \Xi_1^i & \Xi_2^i & 0 \\
0 & \Xi_3^i & -[R_{ik}^l]_{2 \times 2}
\end{bmatrix}
\]

\[
\Xi_i' := \begin{bmatrix}
-I & (\Xi_j)_1 & (\Xi_j)_2 \\
0 & (\Xi_j)_3 & -[R_{ik}^l]_{2 \times 2}
\end{bmatrix}
\]

\[
T_i := [\xi_1^i \xi_2^i \xi_3^i \xi_4^i \xi_5^i \xi_6^i \xi_7^i \xi_8^i]
\]

\[
T_i' := [\xi_1^i \xi_4^i \xi_2^i \xi_5^i \xi_3^i \xi_6^i \xi_7^i \xi_8^i]
\]

with

\[
\Xi_1 := \begin{bmatrix}
A_{ij} + B_{ja}K_{j1}^i & A_{ij} + B_{ja}K_{j2}^i & A_{ij} + B_{ja}K_{j3}^i & A_{ij} + B_{ja}K_{j4}^i \\
A_{ij} + B_{ja}K_{j1}^i & A_{ij} + B_{ja}K_{j2}^i & A_{ij} + B_{ja}K_{j3}^i & A_{ij} + B_{ja}K_{j4}^i \\
0 & 0 & 0 & 0 \\
K_{i1} & K_{i2} & K_{i3} & K_{i4}
\end{bmatrix}
\]

\[
(\Xi)_1 := \begin{bmatrix}
A_{ij} + B_{ja}K_{j1}^i & A_{ij} + B_{ja}K_{j2}^i & A_{ij} + B_{ja}K_{j3}^i & A_{ij} + B_{ja}K_{j4}^i \\
A_{ij} + B_{ja}K_{j1}^i & A_{ij} + B_{ja}K_{j2}^i & A_{ij} + B_{ja}K_{j3}^i & A_{ij} + B_{ja}K_{j4}^i \\
0 & 0 & 0 & 0 \\
K_{i1} & K_{i2} & K_{i3} & K_{i4}
\end{bmatrix}
\]

\[
\Xi_2 := \begin{bmatrix}
B_{ja} & 0 \\
B_{ja} & 0 \\
0 & I \\
0 & 0
\end{bmatrix}, \quad (\Xi)_2 := \begin{bmatrix}
B_{ja} & 0 \\
B_{ja} & 0 \\
0 & I \\
0 & 0
\end{bmatrix}
\]

\[
\Xi_3 := [I_{\text{ny}}]_{2 \times 2} \begin{bmatrix}
C_{ji} & C_{ji} & 0 & D_{ji} \\
0 & 0 & I & 0
\end{bmatrix} - [R_{ij}^l]_{2 \times 2} [K_{jy}^l]_{2 \times 4}
\]

\[
(\Xi)_3 := [I_{\text{ny}}]_{2 \times 2} \begin{bmatrix}
C_{ji} & C_{ji} & 0 \\
0 & 0 & I
\end{bmatrix} - [R_{ij}^l]_{2 \times 2} [K_{jy}^l]_{2 \times 3}
\]

Moreover, if the previous conditions are satisfied, a realization of the DSOF controller can be obtained by

\[
\begin{bmatrix}
K_{jy}^l & K_{jy}^l \\
K_{jy}^l & K_{jy}^l
\end{bmatrix} = \left[ R_{ij}^l \right]_{2 \times 2}^{-1} \cdot [I_{\text{ny}}]_{2 \times 2}. \tag{25}
\]

Proof. Firstly, since \([R_{ij}^l]_{2 \times 2} > 0\) can be derived from (22) and (23), \([R_{ij}^l]_{2 \times 2}\) is invertible. Then if there exist matrices such that the conditions in (22) and (23) are satisfied, the conditions in (12) and (13) are also satisfied by setting the gain of the DSOF controller as (25). Thus according to Theorem 1, the DSOF controller with its gain given in (25) can guarantee the well-posedness, stability and the finite-frequency specification in (7). The proof is completed.

Note that the conditions in (22) and (23) are not LMIs, as there still exists the multiplication between unknown matrices \(R_{ik}^l\) and \(K_{jy}^l\). Thus if the matrices \(K_{jy}^l\) can be specified a priori, the conditions in (22) and (23) will turn into LMIs with respect to other variables. Fortunately, it is not difficult to find that the matrices \(K_{jy}^l\) can be interpreted as the system matrices of a distributed full information controller

\[
\begin{bmatrix}
x_i(t) \\
y_i(t)
\end{bmatrix} = \begin{bmatrix}
K_{i1}^l & K_{i2}^l & K_{i3}^l & K_{i4}^l \\
K_{i1}^l & K_{i2}^l & K_{i3}^l & K_{i4}^l
\end{bmatrix} \begin{bmatrix}
x_i(t) \\
y_i(t)
\end{bmatrix}, \tag{26}
\]

which can guarantee the well-posedness, stability and the finite-frequency specification in (7) of the interconnected system \(G\).

In the following, we will assume that matrices \(K_{jy}^l\) are known and give an algorithm (Algorithm 1) for designing a DSOF controller.

Remark 5. It can be observed from Algorithm 1 that \(\xi_1^l\) and \(\xi_2^l\) satisfy \(\xi_1^l \geq \xi_2^l \geq \xi_1^{l+1}\), that is, both \(\xi_1^l\) and \(\xi_2^l\) are non-increasing. Thus, when we find that \(\xi_1^l\) or \(\xi_2^l\) is less than 0, we can compute the DSOF controller according to (25).

Algorithm 1 gives a procedure of computing a desired DSOF controller when there exists a priori known distributed full information controller. Thus, it is the main part of the second stage. In next section, we will focus on constructing and optimizing a distributed full information controller given in (26) for Algorithm 1, in other words, the first stage.

4 | CONSTRUCTION OF INITIAL DISTRIBUTED FULL INFORMATION CONTROLLERS

Since not all choices of \([K_{jy}^l]_{2 \times 4}\) can lead to a desired DSOF controller through Algorithm 1, it is significant to find a suitable \([K_{jy}^l]_{2 \times 4}\) which has more possibility to produce a desired DSOF controller. Thus in this section, we will focus on constructing and optimizing an initial distributed full information controller (26) for Algorithm 1.

4.1 | Computation of a initial controller

First, we have the following result on the existence of a distributed full information controller which can guarantee the well-posedness, stability and the finite-frequency specification in (7) of the closed-loop system.
Theorem 3. Consider the interconnected system $G$ and let scalars $\omega_a, \omega_b \in \mathbb{R}$ be given. There exist a distributed full information controller such that the resulting closed-loop system is well-posed, stable and satisfies the finite-frequency specification (7) if there exist symmetric matrices $\tilde{Y}_i, \tilde{Y}_i^T, Z_i, \tilde{Z}_i$ and matrices $[\tilde{Y}_i]_{2\times 4}$ with $Y_i^1 \in \mathbb{C}^{n_i \times n_i}, Y_i^2 \in \mathbb{C}^{n_i \times n_i}, Y_i^3 \in \mathbb{C}^{n_i \times n_i}, Y_i^4 \in \mathbb{C}^{n_i \times n_i}$ with $Y_i^2 = Y_i^3$ and $Y_i^3 = Y_i^4$ such that

$$\text{diag}(\Phi \otimes \tilde{Y}_i, \Psi \otimes \tilde{Y}_i^T, Z_i, \Pi_i) + He(H_i \left[ -\tilde{Z}_i, \tilde{M}_i \right] T_i) < 0,$$

for all $i = 1, \ldots, I$, where

$$\tilde{Y}_i := \text{diag}(Y_i^1, Y_i^2, Y_i^3, I_{y_i}), \quad \tilde{Y}_i^T := \text{diag}(Y_i^2, Y_i^3, Y_i^4),$$

$$Y_i^3 := \text{diag}(Y_i^{3, l_{i-1}}), \quad Y_i^4 := \text{diag}(Y_i^{3, h_{i-1}}),$$

$$H_i := \text{diag}\left(\begin{bmatrix} a_i^I \ a_i^I \\ \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

and its transpose, respectively, and we can get

$$\text{diag}(\Phi \otimes \tilde{Y}_i, \Psi \otimes \tilde{Y}_i^T, Z_i, \Pi_i) + He(\tilde{Y}_i \left[ -I, \Xi_i \right] T_i) < 0,$$

where

$$\tilde{Y}_i := (Y_i^1)^{-*}\tilde{Y}_i^T(Y_i^1)^{-1},$$

$$Y_i^T := (Y_i^1)^{-*}\tilde{Y}_i^T(Y_i^1)^{-1},$$

$$Z_i := (\star)^* \cdot Z_i \cdot \text{diag}(Y_i^2, Y_i^3, Y_i^2, Y_i^3)^{-1},$$

$$\tilde{Y}_i := \begin{bmatrix} a_i^1(Y_i^1)^{-*} \\ a_i^2(Y_i^1)^{-*} \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_i := \begin{bmatrix} b_i^1(Y_i^2)^{-*} \\ b_i^2(Y_i^2)^{-*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_y := \begin{bmatrix} \Psi_i^T \\ 0 & \Psi_i^T \\ 0 & \Psi_i^T \\ 0 & \Psi_i^T \end{bmatrix}.$$
Then, pre-multiply and post-multiply inequality (29) by $[(\Xi^i)^* I] T_i^*$ and its transpose, respectively, we have
\[
[(\Xi^i)^* I] T_i^* \cdot \text{diag} \{\Phi \otimes Y_i^i + \Psi \otimes X_i^i, Z_i, \Pi_i\} \cdot (T_i^*)^* [\Xi^i] I < 0.
\]

Analogously, pre-multiply and post-multiply inequality (28) by
\[
[(\Xi^i)_1 I]^* T_i^j \cdot \text{diag} \{Y_j^i, Y_j^1, Y_j^2, Y_j^3\}^{-*},
\]
and its transpose, respectively, we have
\[
[(\Xi^i)_1 I]^* T_i^j \cdot \text{diag} \{\Phi \otimes X_j^i, Z_j\} \cdot (T_i^j)^* [\Xi^i]_1 I < 0. \quad (30)
\]

Note that, as the controller is replaced by the distributed full information controller in (26), the terms $K_{ij}^0 C_{ij}^0, K_{ij}^0 C_{ij}^0, K_{ij}^0 D_{ij}^0, K_{ij}^0 C_{ij}^0, K_{ij}^0 C_{ij}^0, K_{ij}^0 D_{ij}^0$ and $K_{ij}^0 D_{ij}^0$ in (4) are replaced by $K_{ij}^0, K_{ij}^0, K_{ij}^0, K_{ij}^0, K_{ij}^0, K_{ij}^0$ respectively. Thus, according to inequalities (29), (30), and Lemma 1, the closed-loop system resulting from the distributed full information controller in (26) is well-posed, stable and satisfies the finite-frequency specification (7). The proof is completed.

\[\square\]

4.2 Optimization of the initial controller

In Theorem 3, we have constructed a distributed full information controller as the initial controller for Algorithm 1. However, here comes the question: Is the constructed distributed full information controller a suitable choice for generating a desired DSOF controller? Taking a deep sight into conditions (22) and (23) in Theorem 2, we can find that a distributed full information controller with the following form:
\[
\begin{bmatrix}
K_{i1}^i & K_{i2}^i & K_{i3}^i & K_{i4}^i \\
K_{i1}^i & K_{i2}^i & K_{i3}^i & K_{i4}^i \\
K_{i1}^i & K_{i2}^i & K_{i3}^i & K_{i4}^i \\
K_{i1}^i & K_{i2}^i & K_{i3}^i & K_{i4}^i
\end{bmatrix} = \begin{bmatrix}
K_{ij}^0 & K_{ij}^0 & C_{ij}^0 & 0 & D_{ij}^0 \\
K_{ij}^0 & K_{ij}^0 & C_{ij}^0 & 0 & I & 0
\end{bmatrix}, \quad (31)
\]
could be an ideal initial controller. In the following, we will present an algorithm to generate a new initial controller based on the constructed distributed full information controller in Theorem 3. First, we give the following theorem which will be used to update the initial controller.

**Theorem 4.** Given a distributed full information controller produced by Theorem 3, there exist symmetric matrices $X_i^r, Y_i^r, Z_i, Z_i$ and matrices $[X_{ij}^r]_{2n \times 2n}$, $[X_{ij}^r]_{2n \times 3}, [K_{ij}^0]_{2n \times 2n}$ such that $X_i^r > 0, Y_i^r > 0, (Z_i^r)^T C_{ij}^0 > 0, (Z_i^r)^T D_{ij}^0 > 0$, and $I_{ij}^r I_{ij}^r$ are replaced by $L_{ij}^r L_{ij}^r, L_{ij}^r L_{ij}^r, L_{ij}^r C_{ij}^0, L_{ij}^r C_{ij}^0, L_{ij}^r C_{ij}^0, L_{ij}^r D_{ij}^0$ respectively.

**Proof.** Suppose that a distributed full information controller with system matrices $[K_{ij}^0]_{2n \times 2n}$ is derived from Theorem 3. According to the proof of Theorem 3, there exist $X_i^r, Y_i^r, Z_i$ and $Z_i$ such that (29) is satisfied for all $i = 1, \ldots, L$. Then there exist a sufficiently large scalar $\epsilon$ such that inequality (32) is satisfied.

\[
\text{diag} \{\Phi \otimes Y_i^r + \Psi \otimes X_i^r, Z_i, \Pi_i\} + H(\bar{Y}_i) \begin{bmatrix} -I & -M \end{bmatrix} T_i^* < 0.
\]

(32)

Note that the terms in (22) are replaced as stated above. Hence, the condition in (32) is that in (22) by assigning:
\[
[X_{ij}^r]_{2n \times 2n} = Y_i^r, [K_{ij}^0]_{2n \times 2n} = \epsilon I, [L_{ij}^r]_{2n \times 2n} = [K_{ij}^0]_{2n \times 2n}.
\]

Analogously, it can be proved that there exist matrices such that (23) is also satisfied. Therefore, the distributed full information controller produced by Theorem 3 guarantees that there exist appropriate matrices such that (22) and (23) are satisfied. The proof of this theorem is completed.

\[\square\]

The basic thought of Algorithm 1 is to produce a DSOF controller with a known distributed full information controller. With some modifications, Algorithm 1 can also be used to update the initial distributed full information controller. Theorem 4 shows that, if the distributed full information controller generated from Theorem 3 is used as the initial controller, the optimization problem in Step 2-1 of Algorithm 1 with $L_{ij}^r C_{ij}^0, L_{ij}^r C_{ij}^0, L_{ij}^r D_{ij}^0, L_{ij}^r C_{ij}^0, L_{ij}^r C_{ij}^0, L_{ij}^r D_{ij}^0$ replaced by $L_{ij}^r, L_{ij}^r, L_{ij}^r, L_{ij}^r, L_{ij}^r, L_{ij}^r$ must have a feasible solution when $\epsilon > 0$ is sufficiently large. Then, by solving this optimization problem, we can get a new distributed full information controller with $[K_{ij}^0]_{2n \times 2n}$ as $([K_{ij}^0]_{2n \times 2n})^{-1} \cdot [L_{ij}^r]_{2n \times 2n}$.

Moreover, noting that we are expected to produce a controller with its form given in (31), which is equivalent to
\[
\epsilon_i := [K_{ij}^0]_{2n \times 2n} \begin{bmatrix} C_{ij}^0 & 0 & 0 \\
0 & C_{ij}^0 & 0 \\
0 & 0 & I
\end{bmatrix} = 0, \quad (33)
\]
A PRACTICAL EXAMPLE

Algorithm 2 Optimization of the initial controller

1. Construct an initial distributed full information controller \([K_{pq}^{(0)}]_{2 \times 4}\) according to Theorem 3. Let \(\delta\) be a specified tolerance and set \(\eta = 1\).

2. Solve the following optimization problem:

\[
\begin{align*}
\min & \quad \varepsilon \\
\text{s.t.} & \quad \begin{cases}
X_i^0 > 0, X_i^0 > 0, X_i^0 > 0, \forall i = 1, \ldots, L \\
K_{pq}^0 = [K_{pq}^{(0)}], I_{pq}^0 = [I_{pq}^{(0)}] \\
K_{pq}^0 \tau_{pq} = [K_{pq}^{(0)}]_{2 \times 2}, \varepsilon \end{cases}
\end{align*}
\]

3. Let \(\varepsilon = 0\) and set \(\eta = 1\) and go back to Step 1.

5 | A PRACTICAL EXAMPLE

In this section, we will apply the proposed method to the control of a vehicle platoon which simulates a spring-mass-damper system. Consider the situation that the transmissions of the position and velocity information between vehicles are not ideal. The system dynamics can be given as follows:

\[
\begin{align*}
m_i \ddot{y}_i &= - (k_i + k_{i+1}) \dot{y}_i - (c_i + c_{i+1}) \dot{y}_i + \delta_{1,i} \dot{y}_{i-1} + \delta_{2,i} \dot{y}_{i+1} \\
&\quad + \delta_{3,i} \dot{y}_{i-1} \dot{y}_{i+1} + m_i y_{i-1} \\
\end{align*}
\]

where \(y_i\) is the error between equilibrium position and real-time position of \(i\)-th vehicle; \(k_i, c_i, m_i\) are the stiffness coefficient, damping constant, mass of vehicle, respectively; \(u_i\) is the power provided by vehicle engine and is the control signal; \(y_i\) is the measured output; \(\delta_{1,i}, \delta_{2,i} \in (0, 1)\) is used to describe the non-ideal transmission of the position and velocity information between vehicles and \(\delta_{3,i}\). Moreover, as in horizontal direction, the body is sensitive to acceleration in the frequency range 1–2 Hz, we define the controlled output as \(\ddot{y}_i = \ddot{y}_i\). Our goal is to improve the driving comfort by reducing \(\ddot{y}_i\) in the frequency range 1–2 Hz. Denoting \(x_i(t) = e_i, x_i(t) := e_i\) and taking the noises during the movement of vehicle into consideration, the above vehicle platoon system can be converted into the an interconnected system with subsystems given in (36) at the top of next page, and the interconnection relationship follows:

\[
\begin{align*}
\nu_{ij} &= \delta_{ij} \dot{w}_{ij} \\
\end{align*}
\]

In this example, we consider there are \(L = 5\) vehicles and \(\delta_{ij} = 0.9\) for all \(i, j = 1, \ldots, 5\). The masses, stiffness coefficients and damping constants are randomly chosen as follows:

\[
\begin{align*}
m_1 &= 7.08, \quad m_2 = 7.46, \quad m_3 = 7.57, \quad m_4 = 5.33, \quad m_5 = 7.12; \\
k_1 &= 6.66, \quad k_2 = 6.56, \quad k_3 = 5.94, \quad k_4 = 5.48, \quad k_5 = 5.96; \\
c_1 &= 4.34, \quad c_2 = 4.05, \quad c_3 = 4.22, \quad c_4 = 4.52, \quad c_5 = 3.33.
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1^1(t) \\
\dot{x}_1^2(t) \\
\dot{w}_{i2}(t) \\
\dot{z}_1(t) \\
\dot{y}_1^1(t) \\
\dot{y}_1^2(t)
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0.1 & 0 \\
-k_1 + k_2 & -c_1 + c_2 & k_2 & c_2 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-k_1 + k_2 & -c_1 + c_2 & k_2 & c_2 & 1 & 1 \\
1 & 0 & 0 & 0 & 0.1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1^1(t) \\
x_1^2(t) \\
w_{i2}(t) \\
z_1(t) \\
y_1^1(t) \\
y_1^2(t)
\end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1^1(t) \\
\dot{x}_1^2(t) \\
\dot{w}_{i,j-1}(t) \\
\dot{w}_{i,j+1}(t) \\
\dot{y}_1^1(t) \\
\dot{y}_1^2(t)
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0.1 & 0 \\
-k_i + k_{i+1} & -c_i + c_{i+1} & k_i & c_i & k_{i+1} & c_{i+1} & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_i + k_{i+1} & -c_i + c_{i+1} & k_i & c_i & k_{i+1} & c_{i+1} & 1 & 1 \\
1 & 0 & 0 & 0 & 0.1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1^1(t) \\
x_1^2(t) \\
w_{i,j-1}(t) \\
w_{i,j+1}(t) \\
y_1^1(t) \\
y_1^2(t)
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1^1(t) \\
\dot{x}_1^2(t) \\
\dot{w}_{i,j-1}(t) \\
\dot{z}_1(t) \\
\dot{y}_1^1(t) \\
\dot{y}_1^2(t)
\end{bmatrix} &=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0.1 & 0 \\
-k_L + c_L & k_L & c_L & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-k_L + c_L & k_L & c_L & 1 & 1 \\
1 & 0 & 0 & 0 & 0.1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1^1(t) \\
x_1^2(t) \\
w_{i,j-1}(t) \\
z_1(t) \\
y_1^1(t) \\
y_1^2(t)
\end{bmatrix}.
\end{align*}
\]
In addition to the finite-frequency specification in the frequency range $\Omega = [1, 2] \times \pi$, that is, $\|G(\omega)\|_\Omega < \gamma$, we also include another constraint $\|G(\omega)\|_\infty < \gamma_\infty$ to avoid a bad disturbance attenuation performance in the entire-frequency range. We let $\gamma = 0.07$ and $\gamma_\infty = 0.11$.

Firstly, according to Theorem 3, we find a distributed full information controller as

$$
K_{11}^1 = [4.89, 0.02], K_{12}^1 = [-6.65, -4.34],
K_{14}^1 = [-0.98], K_{23}^1 = \text{diag}[1.64, 1.64],
K_{11}^2 = [10.14, 2.42], K_{12}^2 = [-6.65, 4.34, 6.56, 4.05],
K_{14}^2 = [-0.99], K_{23}^2 = \text{diag}[0.13, 0.13, 0.64, 0.64],
K_{11}^3 = [2.74, -14.90], K_{12}^3 = [-6.56, 4.05, 5.94, 4.22],
K_{14}^3 = [-1.06], K_{23}^3 = \text{diag}[0.55, 0.55, 0.04, 0.04],
K_{11}^4 = [4.47, -6.98], K_{12}^4 = [-5.94, 4.22, 5.48, 4.52],
K_{14}^4 = [-1.04], K_{23}^4 = \text{diag}[1.15, 1.15, 1.31, 1.31],
K_{11}^5 = [8.53, 0.87], K_{12}^5 = [-5.48, -4.52],
K_{14}^5 = [-1.00], K_{23}^5 = \text{diag}[0.27, 0.27],
$$

and other gain matrices are approximately zero matrices. Computing the $\varepsilon$ in (33) with $[K_{pj}^0]_{2 \times 4}$ shown in (37), we have $\|\varepsilon_1\| = 8.08$, $\|\varepsilon_2\| = 11.29$, $\|\varepsilon_3\| = 10.61$, $\|\varepsilon_4\| = 10.21$ and $\|\varepsilon_5\| = 7.36$, which indicate that the distributed full information controller in (37) may not be a good initial controller for Algorithm 1. Now, based on the above controller, we run Algorithm 2 with $\delta = 0.01$ and get a new distributed full information controller as

$$
K_{11}^1 = [-2.02, -8.69], K_{14}^1 = [-1.07],
K_{23}^1 = \text{diag}[1.76, 1.76],
K_{11}^2 = [-0.43, -9.59], K_{14}^2 = [-1.00],
K_{23}^2 = \text{diag}[-0.20, -0.20, 0.45, 0.45],
K_{11}^3 = [0.63, -11.56], K_{14}^3 = [-1.09],
K_{23}^3 = \text{diag}[-0.15, -0.15, -0.26, -0.26],
K_{11}^4 = [1.49, -11.85], K_{14}^4 = [-1.04],
K_{23}^4 = \text{diag}[1.39, 1.39, 1.36, 1.36],
K_{11}^5 = [-0.68, -9.66], K_{14}^5 = [-1.03],
K_{23}^5 = \text{diag}[-0.28, -0.28],
$$

and other gain matrices are approximately zero matrices. Computing the $\varepsilon$ in (33) with $[K_{pj}^0]_{2 \times 4}$ in (38), we have $\|\varepsilon_1\| = 1.18 \times 10^{-6}$, $\|\varepsilon_2\| = 1.06 \times 10^{-6}$, $\|\varepsilon_3\| = 2.18 \times 10^{-6}$, $\|\varepsilon_4\| = 2.12 \times 10^{-6}$ and $\|\varepsilon_5\| = 1.77 \times 10^{-6}$, which are much less than those obtained with $[K_{pj}^0]_{2 \times 4}$ shown in (37), that is, the distributed full information controller in (38) produced by Algorithm 2 has the excepted form presented in (31) and is more suitable than the controller in (37) to be the initial controller for Algorithm 1. Then, using this distributed full information controller in (38) as the initial controller for Algorithm 1, a desired DSOF controller is produced as

$$
K_{pj}^1 = [-1.85, 8.85], K_{pj}^1 = \text{diag}[1.98, 1.98],
K_{pj}^2 = [-0.29, 9.73], K_{pj}^2 = \text{diag}[-1.84, -1.84, 1.42, 1.42],
K_{pj}^3 = [0.75, -11.67], K_{pj}^3 = -\text{diag}[1.49, 1.49, 1.56, 1.56],
K_{pj}^4 = [1.65, -12.01], K_{pj}^4 = \text{diag}[1.67, 1.67, 1.64, 1.64],
K_{pj}^5 = [-0.53, 9.81], K_{pj}^5 = \text{diag}[-1.53, -1.53],
$$

and other gain matrices are approximately zero matrices. In order to show the advantage of controller design in finite-frequency range, we also design a conventional $H_\infty$ DSOF which satisfies $\|G(\omega)\|_\infty < 0.11$. The obtained $H_\infty$ DSOF is produced by Algorithm 1 and Algorithm 2 as

$$
K_{pj}^1 = [-4.18, 11.38], K_{pj}^1 = \text{diag}[2.90, 2.90],
K_{pj}^2 = [-2.44, 12.74], K_{pj}^2 = \text{diag}[-2.91, -2.91, 3.04, 3.04],
K_{pj}^3 = [-2.90, 12.34], K_{pj}^3 = -\text{diag}[3.05, 3.05, 2.78, 2.78],
K_{pj}^4 = [-2.19, 9.37], K_{pj}^4 = \text{diag}[2.78, 2.78, 2.71, 2.71],
K_{pj}^5 = [-3.03, 11.97], K_{pj}^5 = \text{diag}[-2.71, -2.71],
$$

and other gain matrices are approximately zero matrices. Computing the $\varepsilon$ in (33) with $[K_{pj}^0]_{2 \times 4}$ in (38), we have $\|\varepsilon_1\| = 1.18 \times 10^{-6}$, $\|\varepsilon_2\| = 1.06 \times 10^{-6}$, $\|\varepsilon_3\| = 2.18 \times 10^{-6}$, $\|\varepsilon_4\| = 2.12 \times 10^{-6}$ and $\|\varepsilon_5\| = 1.77 \times 10^{-6}$, which are much less than those obtained with $[K_{pj}^0]_{2 \times 4}$ shown in (37), that is, the distributed full information controller in (38) produced by Algorithm 2 has the excepted form presented in (31) and is more suitable than the controller in (37) to be the initial controller for Algorithm 1. Then, using this distributed full information controller in (38) as the initial controller for Algorithm 1, a desired DSOF controller is produced as

$$
K_{pj}^1 = [-4.18, 11.38], K_{pj}^1 = \text{diag}[2.90, 2.90],
K_{pj}^2 = [-2.44, 12.74], K_{pj}^2 = \text{diag}[-2.91, -2.91, 3.04, 3.04],
K_{pj}^3 = [-2.90, 12.34], K_{pj}^3 = -\text{diag}[3.05, 3.05, 2.78, 2.78],
K_{pj}^4 = [-2.19, 9.37], K_{pj}^4 = \text{diag}[2.78, 2.78, 2.71, 2.71],
K_{pj}^5 = [-3.03, 11.97], K_{pj}^5 = \text{diag}[-2.71, -2.71],
$$

and other gain matrices are approximately zero matrices. In Figure 3, we have shown the maximum singular values of open-loop system, the closed-loop system with finite-frequency controller in (39) and the closed-loop system with entire-frequency controller in (40). From Figure 3, we can see that the closed-loop system with the finite-frequency DSOF controller (39) has a least gain over frequency range 1–2 Hz, which implies that the finite-frequency DSOF controller (39) can provide a better drive comfort. Moreover, in order to further illustrate the effectiveness of the finite-frequency DSOF controller (39), we assume that the first and third vehicles suffer the following disturbance signals:

$$
d_i(t) = \begin{cases} 
A_1 \sin(2\pi \cdot 1.25 \cdot t), & 0 \leq t \leq 0.8 \\
0, & t > 0.8 
\end{cases}
$$

with $A_1 = 0.3$ and $A_3 = 0.5$. The frequency of the above disturbance is 1.25 Hz, which is covered by 1–2 Hz. We can see from Figure 4 that the horizontal acceleration of closed-loop system with finite-frequency DSOF controller is the smallest.
among those three systems, which shows that a better disturbance attenuation performance can be achieved by the finite-frequency DSOF controller in (39).

6 CONCLUSION

In this paper, we have investigated the problem of DSOF controller design under the restricted frequency-domain specifications for continuous-time non-ideally interconnected systems. Based on the extended condition for the well-posedness, stability and finite-frequency $H_\infty$ performance of continuous-time non-ideally interconnected systems, a new equivalent condition has been obtained by the introduction of matrix multipliers. A two-stage approach is then proposed to construct a desired DSOF controller: find a suitable distributed full information controller at the first stage and then compute a DSOF controller with the distributed full information controller. Moreover, two iterative LMI-based algorithms have been proposed to improve the solvability of the two-stage approach. The illustrated example has well shown the effectiveness of the DSOF controller design approach in this paper.

ORCID
Huiling Xu https://orcid.org/0000-0003-4545-3428

REFERENCES
1. Lavaei, J.: Decentralized implementation of centralized controllers for interconnected systems. IEEE Trans. Autom. Control 57(7), 1860–1865 (2012)
2. Bakule, L.: Decentralized control: An overview. Annual Reviews in Control 32(1), 87–98 (2008)
3. Panagi, P., Polycarpou, M.M.: Decentralized fault tolerant control of a class of interconnected nonlinear systems. IEEE Trans. Autom. Control 56(1), 178–184 (2011)
4. Mahmoud, M.S., et al.: Quantised feedback stabilisation of interconnected discrete-delay systems. IET Control Theory Appl. 5(6), 795–802 (2011)
5. Langbort, C., et al.: Distributed control design for systems interconnected over an arbitrary graph. IEEE Trans. Autom. Control 49(9), 1502–1519 (2004)
6. Chen, X., et al.: $H_2$ performance analysis and $H_\infty$ distributed control design for systems interconnected over an arbitrary graph. Syst. Control Lett. 124, 1–11 (2019)
7. Xue, X., et al.: Distributed finite-time control for markovian jump systems interconnected over undirected graphs with time-varying delay. IET Control Theory Appl. 13(18), 2969–2982 (2019)
8. Mazen, F., et al.: Distributed control of linear time-varying systems interconnected over arbitrary graphs. Int. J. Robust Nonlinear Control 25(2), 179–206 (2014)
9. Vamshi, A.S.M., Elia, N.: Optimal distributed controllers realizable over arbitrary networks. IEEE Trans. Autom. Control 61(1), 129–144 (2016)
10. van Horssen, E.P., Weiland, S.: Distributed control design for systems interconnected over an arbitrary graph. IEEE Trans. Autom. Control 57(6), 1532–1537 (2012)
11. Sadabadi, M.S., Peaucelle, D.: From static output feedback to structured robust static output feedback: A survey. Ann. Rev. Control 42, 11–26 (2016)
12. Syrmos, V.L., et al.: Static output feedback—A survey. Automatica 33(2), 125–137 (1997)
13. CAO, Y.Y., et al.: Static output feedback stabilization: An ILMI approach. Automatica 34(12), 1641–1645 (1998)
14. Bhattacharyya, S., Patra, S.: Static output-feedback stabilization for MIMO LTI positive systems using LMI-based iterative algorithms. IEEE Control Syst. Lett. 2(2), 242–247 (2018)
15. Dong, J., Yang, G.H.: Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties. Automatica 49(6), 1821–1829 (2013)
16. Qiu, J., et al.: Static-output-feedback $H_\infty$ control of continuous-time t - s fuzzy affine systems via piecewise lyapunov functions. IEEE Trans. Fuzzy Syst. 21(2), 245–261 (2013)
17. Shu, Z., Lam, J.: An augmented system approach to static output-feedback stabilization with $H_\infty$ performance for continuous-time plants. Int. J. Robust Nonlinear Control 19(7), 768–785 (2009)
18. Shu, Z., et al.: Static output-feedback stabilization of discrete-time markovian jump linear systems: A system augmentation approach. Automatica 46(4), 687–694 (2010)
19. Agulhari, C.M., et al.: LMI relaxations for reduced-order robust $H_\infty$ control of continuous-time uncertain linear systems. IEEE Trans. Autom. Control 57(6), 1532–1537 (2012)
20. Sereni, B., et al.: New gain-scheduled static output feedback controller design strategy for stability and transient performance of LPV systems. IET Control Theory Appl. 14(5), 717–725 (2020)
21. Wang, M., et al.: Finite frequency filtering design for uncertain discrete-time systems using past output measurements. IEEE Trans. Circuits Syst. I Regul. Pap. 65(9), 3005–3013 (2018)
22. Iwasaki, T., Hara, S.: Generalized KYP lemma: Unified frequency domain inequalities with design applications. IEEE Trans. Autom. Control 50(1), 41–59 (2005)
23. Li, X., Gao, H.: A heuristic approach to static output-feedback controller synthesis with restricted frequency-domain specifications. IEEE Trans. Autom. Control 59(4), 1008–1014 (2014)
24. Li, X., Gao, H.: Robust frequency-domain constrained feedback design via a two-stage heuristic approach. IEEE Trans. Cybern. 45(10), 2065–2075 (2015)
25. Hao, Y., Duan, Z.: Static output-feedback controller synthesis with restricted frequency domain specifications for time-delay systems. IET Control Theory Applications 9(10), 1608–1614 (2015)
26. Kazemy, A., et al.: Dynamic output feedback $H_{\infty}$ design in finite-frequency domain for constrained linear systems. ISA Trans. 96, 185–194 (2020)
27. Wang, M. et al.: Membership-function-dependent fault detection filtering design for interval type-2 T-S fuzzy systems in finite frequency domain. IEEE Trans. Fuzzy Syst. (2020)
28. Ren, Y. et al.: Finite-frequency memory filter design for uncertain linear discrete-time systems: A polynomially parameter-dependent approach. ISA Trans. (2020)
29. Wang, G., et al.: N-D representation and generalised Kalman-Yakubovich-Popov lemma of spatially interconnected systems with interconnected chains. Int. J. Syst. Sci. 48(15), 3160–3171 (2017)
30. Chen, X. et al.: Robust finite frequency $H_{\infty}$ distributed memory filter design for networked control systems with polytopic uncertainty. Asian J. Control (2020)
31. Xu, H., et al.: Performance analysis and distributed filter design for networked dynamic systems over finite-frequency ranges. Neurocomputing 334, 143–155 (2019)
32. Chen, X. et al.: Generalised KYP lemma with its application in finite frequency $H_{\infty}$ distributed filter design for nonideally interconnected networked control systems. Int. J. Syst. Sci. 1–22 (2020)
33. de Oliveira, M.C., Skelton, R.E.: Stability tests for constrained linear systems. In: Moheimani, S.O.R., ed. Perspectives in Robust Control. pp. 241–257. Springer, London (2001)

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