Corrections to the running of gauge couplings due to quantum gravity

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Concerning the gravitational corrections to the running of gauge couplings two different results were reported. Some authors claim that gravitational correction at the one-loop level indicates an interesting effect of universal gravitational decreasing of gauge couplings, that is, gravitational correction works universally in the direction of asymptotic freedom no matter how the gauge coupling behaves without gravity, while others reject the presence of gravitational correction at the one-loop level at all. Being these calculations done in the framework of an effective field theory approach to general relativity, we wanted to draw attention to a recently discovered profound quantum-gravitational effect of space-time dimension running that inevitably affects the running of gauge couplings. The running of space-time dimension indicating gradual reduction of dimension as one gets into smaller scales acts on the coupling constants in the direction of asymptotic freedom and therefore in any case manifests the plausibility of this quantum-gravitational effect. Curiously enough, the results are also in perfect quantitative agreement with those of Robinson and Wilczek.

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Introduction

Treating the general relativity as an effective field theory, that is, to consider it as an effective low energy approximation to some as yet unknown fundamental theory of quantum gravity, offers a way to get round the familiar renormalization difficulties of general relativity in the low energy regime [1]. Namely, based on the linearized theory of general relativity one can make reliable predictions about the quantum-gravitational corrections in the low energy limit \( E \ll E_P \). Following this way of reasoning, Robinson and Wilczek considered the correction to the running of gauge coupling due to graviton exchange in the vertex diagram [2]. Using the cut-off regularization, they found gravitationally corrected \( \beta \) function of the form

\[
\beta(g, E) = -\frac{b_0 g^3}{(4\pi)^2} \frac{3g}{\pi} \left( \frac{E}{E_P} \right)^2. \tag{1}
\]

This result indicates that gravitational correction always works in the direction of asymptotic freedom, that is, it always diminishes the coupling. However, subsequent analysis of this problem has cast serious doubt on the results of paper [2]. It was found that this result was a consequence of an incorrect gauge fixing and that at the one-loop level there is in fact no gravitational correction to the Callan-Symanzik \( \beta \) function [3]. Further gauge invariant study of the one-loop effective action for the gravity-Maxwell theory confirmed the absence of gravitational correction [4]. This problem was studied further in a diagrammatic approach as well, analyzing carefully all the relevant one-loop diagrams no gravitational correction was found [4]. Recently, claiming that an appropriate approach to this problem ought both to preserve gauge invariance and keep quadratical divergences from the gravitational contributions, the old results were revisited by using the loop regularization method [5]. This approach appears to corroborate the conclusions made in [2]. While the discussion of this problem in the framework of an effective field theory approach to general relativity proceeds, we wanted to draw attention on a recently discovered profound quantum-gravitational effect of space-time dimension running [7] that certainly contributes to the running of gauge couplings. In view of this profound discovery, we will try to understand the effect it exerts on the running of gauge couplings. In implementing of this program we will use a simple analytic expression of running dimension that comes from a fairly generic consideration without resorting to any particular approach to quantum gravity.

QG running/reduction of space-time dimension

Because of quantum gravity the dimension of space-time appears to depend on the size of region, it is somewhat smaller than 4 and monotonically increases with increasing of size of the region [7]. We can account for this effect in a simple and physically clear way that allows us to write simple analytic expression for space-time dimension running. Let us consider a set \( F \) that is understood to be a subset of four dimensional Euclidean space \( \mathbb{R}^4 \), and let \( l^4 \) be a smallest box containing this set, \( F \subseteq l^4 \). The mathematical concept of dimension tells us that for estimating the dimension of \( F \) we have to cover it by \( \epsilon^4 \) cells and counting the minimal number of such cells, \( N(\epsilon) \), we can determine the dimension, \( d = \dim(F) \) as a limit \( d = d(\epsilon \to 0) \), where \( \epsilon \to 0 \). For more details see [8]. This definition can be written in a more familiar form as

\[
d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}}.
\]
Certainly, in the case when $F = l^4$, by taking the limit $d(\epsilon \to 0)$ we get the dimension to be 4. From the fact that we are talking about the dimension of a set embedded into the four dimensional space, $F \subset \mathbb{R}^4$, it automatically follows that its dimension can not be greater than 4, $d \leq 4$. We see that the volume of a fractal $F$ uniformly filling the box $l^4$ is reduced

$$V(F) = \lim_{\epsilon \to 0} N(\epsilon) \epsilon^4 = \lim_{\epsilon \to 0} n(\epsilon)^d(\epsilon) \epsilon^4,$$

in comparison with the four dimensional value $l^4$. Introducing $\delta N = n(\epsilon)^4 - N(\epsilon)$, the reduction of dimension $\varepsilon = 4 - d$ can be written as

$$\varepsilon(\epsilon) = -\frac{\ln \left(1 - \frac{\delta N(\epsilon)}{n(\epsilon)^2}\right)}{\ln n(\epsilon)} \approx \frac{1}{n(\epsilon)^2} \frac{\delta N(\epsilon)}{n(\epsilon)^4}. \quad (2)$$

In quantum gravity the space-time resolution is set by the Planck length $\epsilon = l_P$. The local fluctuations $\sim l_P$ add up over the length scale $l$ to $\delta l = (l l_P)^{1/2}$ [3]. Respectively, for the region $l^4$ we have the fluctuation (deviation) of volume of the order $\delta V = \delta l(l)^4$. Thus in quantum gravity we expect the Poisson fluctuation of volume $l^4$ of the order $\delta V = (l^2/l_P^2) l_P$. One naturally finds that this fluctuation of volume has to account for the reduction of dimension. Respectively, from Eq. (2) one gets, $n = l/l_P$, $\delta N = l^2/l_P^2$,

$$\varepsilon = \frac{1}{\ln \left(\frac{l}{l_P}\right)} \left(\frac{l_P}{l}\right)^2.$$

This equation gives the running of dimension with respect to the size of region $l$. We can write this expression in terms of energy, $E_P = l_P^3$, $E = l^{-1}$, as

$$\varepsilon(E) = \frac{1}{\ln \left(\frac{E}{E_P}\right)} \left(\frac{E}{E_P}\right)^2. \quad (3)$$

From the derivation of this equation it is clear that its validity condition is set by $\delta V = (l^2/l_P^2) \ll V = l^4$; $\Rightarrow (E/E_P)^2 \ll 1$.

**Gravitational correction to the $\beta$ function**

Usually, mass independent renormalization procedure allows one to write a renormalization group equation in a very simple way [13]. Bare coupling in the dimensional regularization approach has the following general form

$$g_B = E^{\beta} \left[ g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\epsilon^n}\right].$$

Differentiating this equation with respect to $E$ and recalling that $\lambda_B$ does not depend on $E$ one finds

$$\varepsilon \left[ g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\epsilon^n}\right] + E \frac{dg}{dE} \left[ 1 + \sum_{n=1}^{\infty} \frac{da_n(g)}{dg} \epsilon^{-n}\right] = 0.$$

Since $\beta$ function, $\beta = E (dg/dE)$, is analytic for $\varepsilon \to 0$, for small values of $\varepsilon$ one can write

$$\beta = A + \varepsilon B.$$

Identifying the coefficients for various powers of $\varepsilon$ we get

$$B + g = 0, \quad a_1 + A + B \frac{da_1}{dg} = 0.$$

That is, we have

$$A = -\left(1 - g \frac{d}{dg}\right) a_1, \quad B = -g,$$

and correspondingly

$$\beta = \left(1 - g \frac{d}{dg}\right) a_1 - \varepsilon g. \quad (4)$$

One might wonder about the $d\varepsilon/dE$ terms, but by using the Eq. (3) one easily verifies that such terms can be safely ignored as far as $E \ll E_P$. So we have a general expression enabling one to estimate the correction to the $\beta$ function coming from the dimension running. Using the Eq. (4)

$$\beta = \left[1 - g \frac{d}{dg}\right] a_1 - \frac{g}{\ln \left(\frac{E}{E_P}\right)} \left(\frac{E}{E_P}\right)^2,$$

and then an explicit form of $a_1$ function, from Eq. (4) we find RG equation for the gauge coupling

$$E \frac{dg}{dE} = -\frac{b_0 g^3}{(4\pi)^2} - \frac{g}{\ln \left(\frac{E}{E_P}\right)} \left(\frac{E}{E_P}\right)^2.$$

Integrating this equation one finds

$$\frac{f(E/E_P)}{g^2(E)} - \frac{f(E_0/E_P)}{g^2(E_0)} = \frac{2b_0}{(4\pi)^2} \int_{E_0}^{E} \frac{f(\xi/E_P)}{\xi} d\xi, \quad (5)$$

where

$$f(x) = \exp \left(2 \int_{\xi}^{x} \frac{d\xi}{\ln(\xi)}\right) = e^{2\ln(x^2)}.$$
The logarithmic integral \( \text{li}(x) \) decays monotonically from \( \text{li}(0) = 0 \) to \( \text{li}(1) = -\infty \) in the interval \( 0 \leq x < 1 \). Thus, in the limit \( E_P \to \infty \) one recovers familiar logarithmic running of the inverse coupling.

If we have several gauge couplings, \( g_i \), with corresponding values of \( b_{0i} \), the condition that they unify at a common value \( g(E_U) \) takes the form

\[
\frac{1}{g_i^2(E_0)} - \frac{1}{g_i^2(E_0)} = \frac{2}{(4\pi)^2} \frac{b_{0i} - b_{0j}}{f(E_0/E_P)} \int_{E_0}^{E_U} \frac{f(x/E_P)}{x} \, dx.
\]

Comparing this expression with the unification condition without the gravitational correction, that is \( f = 1 \), one finds the following relation between the uncorrected, \( E \), without the gravitational correction, that is \( f \equiv 1 \), and corrected unification scales

\[
\frac{1}{f(E_0/E_P)} \int_{E_0}^{E_U} \frac{f(x/E_P)}{x} \, dx = \ln \frac{E_*}{E_0}.
\]

Here we assumed that the couplings at \( E_0 \) are not affected by the gravity, which is certainly good approximation. As \( E_0/E_P \to 0 \) one can take \( f(E_0/E_P) \equiv 1 \) with a good accuracy. Because \( f(x) \) is a monotonically decreasing function one infers that the gravitational correction increases the unification scale, \( E_U > E_* \). In most applications \( E_0 = 10^{-17}E_P \), \( E_0 = 10^{-3}E_P \). To get an idea how large is the gravitational correction to the \( E_* \), one has to study the solutions of equation

\[
\int_{10^{-17}}^{E_U/E_P} \frac{f(x)}{x} \, dx = 32.2362.
\]

We will take less strict but easier way to get an idea about the gravitational increment of \( E_* \). Estimating the variation of \( E_* \) in terms of the length fluctuation \( \delta l_* = (l P l_*)^{1/2} \) one finds

\[
\delta E_* = \delta l_* = \left( (l P l_*)^{1/2} \right)^{1/2} = E_* \left( \frac{E_*}{E_P} \right)^{1/2} = 10^{-3/2} E_*.
\]

The value of the gravitationally corrected coupling at the unification is related with the uncorrected one, \( g_* \), through the relation

\[
\frac{f(E_0/E_P)}{g^2(E_0)} = \frac{1}{g^2(E_*)}.
\]

Finally, let us notice that by omitting a less important \( \ln(E_P/E) \) term in Eq. (3) we get \( f(x) = \exp(-2x^2) \), so in this case we arrive at the Eq. (1) and one can immediately go along the discussion of [2].

**Concluding remarks**

The results derived here are in perfect agreement with those of [2]. However, as it was indicated in the introduction, the paper [2] is understood to be incorrect for being treated properly the effective field theory approach to general relativity does not yield any gravitational correction to the gauge coupling at the one-loop level [3, 4, 5]. Nevertheless, we see that quantum gravity through the running of space-time dimension [7] inevitably affects the running of gauge couplings. Interestingly enough, the results appear to be in perfect qualitative and quantitative agreement with those of [2]. Namely, by taking into account that for monotonically increasing but otherwise arbitrary expression of \( \varepsilon(E) \) the function \( f \) is monotonically decreasing, one finds that for each value of \( E \) there is an intermediate scale \( E_0 < E < E \) such that

\[
\frac{1}{f(E/E_P)} \int_{E_0}^{E} \frac{f(\xi/E_P)}{\xi} \, d\xi = \frac{f(\tilde{E}/E_P)}{f(E/E_P)} \ln \frac{E}{E_0}.
\]

Denoting \( \delta f \equiv \left[ f(E_0/E_P) - f(\tilde{E}/E_P) \right]/f(E/E_P) \) one finds

\[
\frac{1}{g^2(E)} = \frac{\delta f}{g^2(E_0)} + \frac{f(\tilde{E}/E_P)}{f(E/E_P)} \frac{1}{g^2(E)},
\]

where we have again assumed \( g(E_0) = g_0(E_0) \). Thus on the quite general grounds, from Eq. (4) one infers that the gravitational correction works in the direction of asymptotic freedom irrespective to the sign of \( b_0 \). Quantitatively, the gravitational correction to the Callan-Symanzik \( \beta \) function we have derived above differs from the corresponding result of [2], Eq. (1), by the logarithmic term \( \ln(E_P/E) \) that is not essential in any way. Thus, irrespective to whether the effective quantum field theory approach to general relativity indicates or not the gravitational decay of gauge couplings [2, 3, 4, 5, 6], the profound quantum-gravitational effect of space-time dimension running [7] inevitably manifests the presence of this effect. Let us notice that this sort of investigation inspires interest to estimate the behavior of gauge couplings in models characterized with low quantum gravity scale [15] as this can be measured in near future high energy experiments [16].

Recently a new interesting paper about the one-loop effective action of Einstein-Maxwell theory with a cosmological constant appeared in arXive [17], demonstrating the gravitational decay of gauge coupling for a positive cosmological constant. Physically speaking, this result implicitly corroborates our discussion for the quantum-gravitational fluctuations of the background space that results in an effective reduction of space-time dimension contributes simultaneously to the dark energy, that is, to the dynamical cosmological constant [10, 18].
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