SCHRÖDINGER OPERATORS ON STAR GRAPHS WITH SINGULARLY SCALED POTENTIALS SUPPORTED NEAR THE VERTICES

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Abstract. We study Schrödinger operators on star metric graphs with potentials of the form \( \alpha \varepsilon^{-2} Q(\varepsilon^{-1} x) \). In dimension 1 such potentials, with additional assumptions on \( Q \), approximate in the sense of distributions as \( \varepsilon \to 0 \) the first derivative of the Dirac delta-function. We establish the convergence of the Schrödinger operators in the uniform resolvent topology and show that the limit operator depends on \( \alpha \) and \( Q \) in a very nontrivial way.

1. Introduction

Currently, there is considerable interest in theory of differential operators on graphs. The reason for this lies in a great deal of progress in fabricating graph-shaped structures of submicron sizes, for which differential operators on graphs represent a natural model. A case of special interest arises when operators appearing in such models have coefficients supported on graph vertices. Such singular operators are widely used in various application to atomic, nuclear, and solid state physics (see the survey [25] for details). By a quantum graph we mean a metric graph together with a second order self-adjoint differential operator, which is determined by differential operators on the edges and certain interface conditions at the vertices. A quantum graph is a natural generalization of the one-dimensional Schrödinger operator; it can be applied to describe the transition of a quantum particle along the graph and serves as a model of wave propagation in “thin” media. The differential operators determine the motion of the quantum particle along the edges, while the interface conditions describe the movement across the vertices. The interface conditions have to be chosen to make the Hamiltonian self-adjoint or, in physical terms, to ensure the conservation of the probability current at the vertex. Basic notions and many references on this subject can be found in [20, 21, 31, 36, 37].

The idea to investigate quantum mechanics of particles confined to a graph originated with the study of free-electron models of organic molecules [32, 33, 34]. Quantum graphs have recently found numerous applications in physics, chemistry, engineering and quantum computing. Among the systems that were successfully modeled by quantum graphs, we mention, e.g. single-mode acoustic and electromagnetic waveguide networks [14], Anderson transition [4] and quantum Hall systems with long-range potential [7], fracton excitations in fractal structures [9], and mesoscopic quantum systems [24]. Applications also arise in quantum wires, quantum chaos and photonic crystals (see [22, 23, 27]). For surveys of physical systems giving rise to boundary value problems on graphs see [25, 26] and the references given there.
Many important one-dimensional models have their analogues on graphs. One of the best studied graph Laplacians is given by the Schrödinger differential expressions on the graph edges and the $\delta$ coupling at $N$ edge vertices

$$\psi_1(a) = \ldots = \psi_N(a), \quad \sum_{n=1}^{N} \psi_n'(a) = \alpha \psi(a)$$

(see, e.g., [8, 9, 11, 12]). Such a model is a generalization of the Schrödinger operator with the potential $\alpha \delta(x)$. The waveguide approximation to the $\delta$ coupling was proposed in [13].

Our aim in this paper is to study a family of Schrödinger operators of the form

$$-\frac{d^2}{dx^2} + \alpha \varepsilon^{-2} Q(\varepsilon^{-1}x)$$

(1)
on a star metric graph. Here $\varepsilon$ is a small positive parameter, $\alpha \in \mathbb{R}$, and $Q$ is a real-valued integrable function. We shall establish the convergence of this family as $\varepsilon \to 0$ in the norm resolvent topology. In this paper we generalize the results of [17], where such a problem was considered on the line (the graph with two edges). In dimension one such potentials are often referred to as $\delta'$-like potentials, since if $Q$ has zero mean and its first moment is equal to $-1$, then the sequence $\varepsilon^{-2} Q(\varepsilon^{-1}x)$ converges in the sense of distributions as $\varepsilon \to 0$ to $\delta'(x)$. Here $\delta'$ is the first derivative of the Dirac delta-function. The Schrödinger operators with the Dirac delta-function and its derivatives in potentials have widely been used in quantum mechanics and mathematical physics; see [2, 3] and the references given there in various physical models leading to such potentials.

In 1986 Šeba [35] considered a formal Schrödinger operator with the potential $\delta'(x)$; he approximated $\delta'$ by regular potentials $\varepsilon^{-2} Q(\varepsilon^{-1}.)$ and then investigated the convergence of the corresponding family of regular Schrödinger operators of the form [1]. Šeba claimed that the regularized operators converge in the uniform resolvent sense to the direct sum of the unperturbed half-line Schrödinger operators subject to the Dirichlet boundary conditions at the origin. From the viewpoint of the scattering theory it means that the $\delta'$-barrier is completely opaque, i.e., in the limit $\varepsilon \to 0$ the potential $\varepsilon^{-2} Q(\varepsilon^{-1}.)$ becomes a totally reflecting wall at the origin splitting the system into two independent subsystems lying on the half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$. Until recently, it was thought that the $\delta'$-potential is physically trivial in the above sense. However, in 2003 Zolotaryuk et al. [10] observed resonances in the transmission probability for piecewise constant $\delta'$-like potentials, which conflicted with conclusions reached by Šeba. Although in [10] the total reflection for the limiting $\delta'$-potential was demonstrated to take place for almost all $\alpha \in \mathbb{R}$, the authors found a discrete set of $\alpha$ (the roots of a transcendental equation depending on $Q$), for which partial transmission through the limiting $\delta'$-potential occurs. Exactly solvable models with other piecewise constant potentials as well as nonrectangular regularizations of $\delta'$ have later been studied in [38]–[42] and the same conclusion has been drawn. All these findings were generalized in [28], where the scattering on an arbitrary potential of the form $\alpha \varepsilon^{-2} Q(\varepsilon^{-1}x)$ with $Q$ supported by $[-1,1]$ has been considered. It was proved that such a potential is asymptotically transparent only if the constant $\alpha$ is such that the problem

$$-f'' + \alpha Q f = 0 \quad \text{on} \quad (-1,1), \quad f'(-1) = f'(1) = 0$$

(2)
admits a nontrivial solution.

In [15, 16] the authors studied Schrödinger operators of the form

$$-\frac{d^2}{dx^2} + \alpha \varepsilon^{-2} Q(\varepsilon^{-1} x) + q(x)$$

with $q$ being a real-valued potential tending to $+\infty$ as $|x| \to \infty$; such a behavior ensures that the spectrum of the considered operators is discrete. For each pair $(\alpha, Q)$ a limit self-adjoint operator was constructed there. The choice of the limit operator was determined by proximity of its eigenvalues and eigenfunctions to those for the Schrödinger operators with regularized potentials for small $\varepsilon$. It was established that for a fixed $Q$ and almost all constants $\alpha \in \mathbb{R}$ the limit operator is just the direct sum of the Schrödinger operators with potential $q$ on half-axes subject to the Dirichlet boundary conditions at the origin. But in the exceptional case, when problem (2) admits a nontrivial solution $f_\alpha$, functions $\psi$ in the domain of the limit operators satisfy the interface conditions

$$f_\alpha(-1)\psi(+0) = f_\alpha(1)\psi(-0), \quad f_\alpha(1)\psi'(+0) = f_\alpha(-1)\psi'(-0).$$

Studies of [15, 16] were continued in [17], where the question on correct definition of the formal Schrödinger operator with the potential $\delta'$ was finally answered. The authors not only pointed out a mistake in Šeba’s proof, but also showed that the operators considered in [15, 16] converge in the uniform resolvent sense as $\varepsilon \to 0$ to the limit one obtained there. The results of [17] were derived for the special case $q = 0$; but the same can be obtained without difficulty for $q \neq 0$ as well. In [11, 16] the problem of approximating a smooth quantum waveguide with a quantum graph was analyzed; the authors encountered the question on the convergence of a family of one-dimensional Schrödinger operators of the form (1) and obtained similar results under the assumption that the mean value of $Q$ is different from zero. For a treatment of a more general case of the $\alpha \delta' + \beta \delta$-like potential we refer the reader to the recent papers [18, 19].

In the following section we briefly sketch the findings of [29, 30], where an analogue for the one-dimensional Schrödinger operator with the $\delta'$-like potential was considered on the metric graph. In Section 3 we prove our main theorem, which generalizes the results of [17] to the metric graph.

## 2. Preliminaries and main results

Let us recall the basic notions of the theory of differential equations on graphs. By a metric graph $G = (V, E)$ we mean a finite set $V$ of points in $\mathbb{R}^3$ (vertices) together with a set $E$ of smooth regular curves connecting the vertices (edges). A map $f : G \to \mathbb{R}$ is said to be a function on the graph, and the restriction of $f$ onto the edge $g \in E(G)$ will be denoted by $f_g$. Each edge is equipped with a natural parametrization; the differentiation is performed with respect to the natural parameter. We denote by $\frac{df}{dg}(a)$ the limit value of the derivative at the point $a \in V(G)$ taken in the direction away from the vertex. The integral of $f$ over $G$ is the sum of integrals over all edges. Let $\mathcal{L}_2(G)$ be a Hilbert space with the inner
product \((f, h) = \int_G f \overline{h} \, dG\). We also introduce the spaces
\[
\mathcal{C}^\infty(G) = \{ f \mid f_g \in C^\infty(\bar{g}) \text{ for all } g \in E(G) \},
\]
\[
\mathcal{A}\mathcal{C}^\infty(G) = \{ f \mid f_g \in AC(g) \text{ for all } g \in E(G) \},
\]
\[
\mathcal{W}_2^2(G) = \{ f \in \mathcal{L}_2(G) \mid f_g \in W_2^2(g) \text{ for all } g \in E(G) \}.
\]
We say that a function \(f\) satisfies the Kirchhoff interface conditions at the vertex \(a \in V(G)\) if \(f\) is continuous at this vertex and \(\sum \frac{df}{dg}(a) = 0\), where the sum is taken over all \(g \in E(G)\) such that \(a \in \bar{g}\); in the latter case we shall write \(f \in \mathcal{K}(G; a)\).

Let us consider a noncompact star graph \(\Gamma\) consisting of three edges \(\gamma_1, \gamma_2, \gamma_3\). All edges are connected at the origin of \(\mathbb{R}^3\), denoted by \(a\). Then \(E(\Gamma) = \{ \gamma_1, \gamma_2, \gamma_3 \}\) and suppose that all edges are half-lines. We write \(a_n^\gamma\) for the point of intersection of the \(\varepsilon\)-sphere, centered at \(a\), with the edge \(\gamma_n\). Let \(\Gamma^\gamma\) be a star subgraph of \(\Gamma\) such that \(V(\Omega^\gamma) = \{ a, a_1^\gamma, a_2^\gamma, a_3^\gamma \}\) and \(E(\Omega^\gamma) = \{ \omega_1^\gamma, \omega_2^\gamma, \omega_3^\gamma \}\).

Set \(\Omega := \Omega_1, \omega_n := \omega_n^1\) and \(a_n := a_n^1\).

In [29], the family of Schrödinger operators on the star graph
\[
H_\varepsilon(\alpha, Q) = -\frac{d^2}{dx^2} + q(x) + \alpha \varepsilon^{-2} Q(\varepsilon^{-1}x),
\]
dom \(H_\varepsilon(\alpha, Q) = \{ f \in \mathcal{L}_2(\Gamma) \mid f, f' \in \mathcal{A}\mathcal{C}^\infty(\Gamma), \ f'' + qf \in \mathcal{L}_2(\Gamma), \ f \in \mathcal{K}(\Gamma; a) \}\)
was studied. Here \(Q \in \mathcal{Q} := \{ Q \in \mathcal{C}^\infty(\Gamma) \mid \text{supp } Q = \Omega, \ \int_{\Omega} Q \, d\Omega = 0 \}\) and the potential \(\varepsilon^{-2} Q(\varepsilon^{-1}x)\) is referred to as the \(\delta\)-like potential; \(q\) is a smooth real-valued function such that \(q_n(x) \to +\infty\) as \(|x| \to +\infty\) for all \(\gamma \in E(\Gamma)\) (this ensures the spectrum discreteness for \(H_\varepsilon(\alpha, Q)\)). An operator \(H(\alpha, Q)\) was assigned to each pair \((\alpha, Q) \in \mathbb{R} \times \mathcal{Q}\. The choice of the operator was suggested by the proximity of the eigenvalues and eigenfunctions for \(H_\varepsilon(\alpha, Q)\) and \(H(\alpha, Q)\) respectively. The following problem is introduced in [29]
\[
-f'' + \alpha Qf = 0 \quad \text{on } \Omega, \quad f \in \mathcal{K}(\Omega; a),
\]
\[
\frac{df}{d\omega_1}(a_1) = \frac{df}{d\omega_2}(a_2) = \frac{df}{d\omega_3}(a_3) = 0. \tag{3}
\]
The choice of \(H(\alpha, Q)\) depends on whether the above problem has a nontrivial solution which can be regarded as an eigenfunction corresponding to the eigenvalue \(\alpha\). Therefore three different cases are distinguished. In the simplest non-resonant case, when \(\alpha\) is not in the spectrum of problem (3), \(H(\alpha, Q)\) is the direct sum of the Schrödinger operators with the potential \(q\) on edges, subject to the Dirichlet boundary conditions at the vertex \(a\). In the (simple or double) resonant case the coupling vector \((\theta_1, \theta_2, \theta_3)\) is introduced.

Let \(\alpha\) be a simple eigenvalue of the problem (3) (the simple resonant case) with an eigenfunction \(\phi_\alpha\); then introduce
\[
\theta_1 := \phi_\alpha(a_1), \quad \theta_2 := \phi_\alpha(a_2), \quad \theta_3 := \phi_\alpha(a_3).
\]
In the double resonant case, when \(\alpha\) is a double eigenvalue, the coupling vector is defined as the vector product of \((\varphi_\alpha(a_1), \varphi_\alpha(a_2), \varphi_\alpha(a_3))\) and \((\psi_\alpha(a_1), \psi_\alpha(a_2), \psi_\alpha(a_3))\), where \(\varphi_\alpha\) and \(\psi_\alpha\) form a basis in the corresponding eigenspace, i.e.,
\[
\begin{bmatrix}
\varphi_\alpha(a_2) & \varphi_\alpha(a_3) \\
\psi_\alpha(a_2) & \psi_\alpha(a_3)
\end{bmatrix}, \quad
\begin{bmatrix}
\varphi_\alpha(a_3) & \varphi_\alpha(a_1) \\
\psi_\alpha(a_3) & \psi_\alpha(a_1)
\end{bmatrix}, \quad
\begin{bmatrix}
\varphi_\alpha(a_1) & \varphi_\alpha(a_2) \\
\psi_\alpha(a_1) & \psi_\alpha(a_2)
\end{bmatrix}.
\]
In the simple resonant case the operator $H(\alpha, Q)$ acts via $H(\alpha, Q)f = -f'' + qf$ on an appropriate set of functions obeying the interface conditions

$$\frac{f_{\gamma_1}(a)}{\theta_1} = \frac{f_{\gamma_2}(a)}{\theta_2} = \frac{f_{\gamma_3}(a)}{\theta_3}, \quad \sum_{n=1}^{3} \theta_n \frac{df}{d\gamma_n}(a) = 0.$$  

In the double resonant case the interface conditions may be written as

$$\frac{1}{\theta_1} \frac{df}{d\gamma_1}(a) = \frac{1}{\theta_2} \frac{df}{d\gamma_2}(a) = \frac{1}{\theta_3} \frac{df}{d\gamma_3}(a), \quad \sum_{n=1}^{3} \theta_n f_{\gamma_n}(a) = 0.$$  

If $\lambda$ is an eigenvalue of the operator $H(\alpha, Q)$, we shall denote by $P_\lambda$ the orthogonal projector onto the corresponding eigenspace. Let $P(\lambda, \varepsilon)$ stand for the orthogonal projector onto the finite dimensional space spanned by all eigenfunctions corresponding to those eigenvalues $\lambda_\varepsilon$ of $H(\alpha, Q)$ that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$. Here $\lambda_\varepsilon = \lambda_{j,\varepsilon}$ continuously depends on $\varepsilon$. The results of [29] may be summarized in the following theorem.

**Theorem A.** All eigenvalues of $H(\alpha, Q)$ (except at most a finite number) are bounded as $\varepsilon \to 0$. Let $\lambda_\varepsilon$ be an eigenvalue of $H(\alpha, Q)$ bounded as $\varepsilon \to 0$, then $\lambda_\varepsilon$ has a finite limit $\lambda$ that is a point of the spectrum of $H(\alpha, Q)$. Moreover, $|P(\lambda, \varepsilon) - P(\lambda)| \to 0$ as $\varepsilon \to 0$. Conversely, if $\lambda$ is an eigenvalue of $H(\alpha, Q)$, then there exists an eigenvalue $\lambda_\varepsilon$ of $H(\alpha, Q)$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$.

From this point forward we assume that $q \equiv 0$, i.e., the operator $H(\alpha, Q)$ involves no potential and $H(\alpha, Q)$ is

$$H(\alpha, Q) = -\frac{d^2}{dx^2} + \alpha \varepsilon^{-2}Q(\varepsilon^{-1}x), \quad \text{dom} H(\alpha, Q) = \mathcal{H}_2^2(\Gamma) \cap \mathcal{K}(\Gamma; a).$$

The scattering properties of the Hamiltonians $H(\alpha, Q)$ on the graph $\Gamma$ with the finite-range potentials $\alpha \varepsilon^{-2}Q(\varepsilon^{-1}x)$ in the limit $\varepsilon \to 0$ were studied in [30]. It was proved that the scattering coefficients depend on $\alpha$ and $Q$. In the generic (nonresonant) case the barrier $\alpha \varepsilon^{-2}Q(\varepsilon^{-1}x)$ is asymptotically opaque as $\varepsilon \to 0$. An exception to this occurs when problem [3] admits a nontrivial solution. In [30] it was also shown that the scattering matrix for the operators $H(\alpha, Q)$ and $H_0$ converges as $\varepsilon \to 0$ to that for the limiting operator $H(\alpha, Q)$ and $H_0$. Here the operator $H_0$ (the Hamiltonian of a free particle on $\Gamma$) acts via $H_0f = -f''$ on its domain $\mathcal{H}_2^2(\Gamma) \cap \mathcal{K}(\Gamma; a)$. The following theorem summarizes the main advances in [30].

**Theorem B.** The scattering matrix for the operators $H(\alpha, Q)$ and $H_0$ converges as $\varepsilon \to 0$ to the scattering matrix for $H(\alpha, Q)$ and $H_0$.

We observe that the above results on proximity for small $\varepsilon$ of the eigenvalues, eigenfunctions, and scattering quantities for the limiting and regularized operators do not shed any light on convergence as $\varepsilon \to 0$ of the operators $H(\alpha, Q)$ in whatever topology. Such a convergence, however, would imply convergence of many other characteristics of interest. Our objective in this paper is to study this question and the main result is contained in the following theorem.

**Theorem 1.** As $\varepsilon \to 0$, the operator family $H(\alpha, Q)$ converges in the norm resolvent sense to $H(\alpha, Q)$. Moreover, for a fixed $\zeta \in \mathbb{C} \setminus \mathbb{R}$ there exists a constant $C$ such that

$$\|H(\alpha, Q) - \zeta\|^{-1} - \|H(\alpha, Q) - \zeta\|^{-1}\|B(\mathcal{H}^2(\Gamma)) \leq C\varepsilon^2.$$


for every $\varepsilon \in (0, 1)$.

Throughout the paper, $C_n$ and $c_n$ will denote different constants independent of $f$ and $\varepsilon$, and $\| \cdot \|$ will denote the $L_2(\Gamma)$-norm.

3. Proof of Theorem 1

The underlying idea is to construct a function $\tilde{y}_\varepsilon$ that is a good approximation to the functions $y_\varepsilon := (H_\varepsilon(\alpha, Q) - \zeta)^{-1}f$ and $y := (H(\alpha, Q) - \zeta)^{-1}f$, uniformly for $f$ in bounded subsets of $L_2(\Gamma)$. Here $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a fixed number. More precisely, to establish Theorem 1, we shall rely on the fact that for every $f \in L_2(\Gamma)$ and $\varepsilon > 0$ there exists a function $\tilde{y}_\varepsilon$ with the property that

$$\| \tilde{y}_\varepsilon - y_\varepsilon \| \leq C_1 \varepsilon^{\frac{1}{2}} \| f \|,$$  
$$\| \tilde{y}_\varepsilon - y \| \leq C_2 \varepsilon^{\frac{1}{2}} \| f \|. \quad (4)$$

From this we find that

$$\| (H_\varepsilon(\alpha, Q) - \zeta)^{-1}f - (H(\alpha, Q) - \zeta)^{-1}f \| \leq \| \tilde{y}_\varepsilon - y_\varepsilon \| + \| y_\varepsilon - \tilde{y}_\varepsilon \| \leq C_1 \varepsilon^{\frac{1}{2}} \| f \|.$$

Our next aim is to construct such an approximation. The proof proceeds differently in the three cases: non-resonant, simple and double resonant.

3.1. Proof of Theorem 1 in the non-resonant case. In this subsection we shall prove Theorem 1 in the non-resonant case, when $\alpha$ does not belong to the resonant set. We shall construct an approximation $\tilde{y}_\varepsilon$.

Denote by $u_\varepsilon$ a solution of the problem

$$\begin{cases}
-u'' + \alpha Qu = \varepsilon f(\varepsilon \cdot) & \text{on } \Omega, \\
\frac{du}{dn}(a_n) = \frac{du}{dn}(a^\varepsilon_n), & n = 1, 2, 3.
\end{cases}$$

It is clear that $u_\varepsilon$ belongs to the Sobolev space $W_2^2(\Omega)$ for every positive $\varepsilon$.

Lemma 1. Suppose that $f \in L_2(\Gamma)$ and $\varepsilon \in (0, 1)$; then

$$\| u_\varepsilon \|_{W_2^2(\Omega)} \leq C \| f \|.$$

Proof. Observe that $(H(\alpha, Q) - \zeta)^{-1}$ is a bounded operator from $L_2(\Gamma)$ to the domain of $H(\alpha, Q)$ equipped with the graph norm. The latter space is a subspace of $W_2^2(\Gamma)$, hence,

$$\| y \|_{W_2^2(\Gamma)} \leq c_1 \| f \|.$$  

Since by the Sobolev embedding theorem $W_2^2(\Gamma) \subset C^1(\Gamma)$, we have

$$\| y \|_{C^1(\Gamma)} \leq c_2 \| f \|. \quad (5)$$

The solution $u_\varepsilon$ obeys the a priori estimate

$$\| u_\varepsilon \|_{W_2^2(\Omega)} \leq c_3 \left( \sum_{n=1}^{3} \left| \frac{dy}{dy_n}(a_n) \right| + \varepsilon \| f(\varepsilon \cdot) \|_{L_2(\Omega)} \right).$$

This estimate can be obtained, roughly speaking, by removing the nonhomogeneity from the boundary conditions of the problem for $u_\varepsilon$ to the right hand side of the equation and using the fact that the resolvent of the corresponding differential operator is a bounded operator from $L_2(\Omega)$ to the domain of the operator equipped with the graph norm. Applying (5), the a priori estimate and taking into account that

$$\| f(\varepsilon \cdot) \|_{L_2(\Omega)} \leq \varepsilon^{-\frac{1}{2}} \| f \|,$$  
$$\| u_\varepsilon \|_{W_2^2(\Omega)} \leq c_4 \| f \|. \quad (6)$$
we, therefore, easily derive the inequality \( \|u_\varepsilon\|_{\mathcal{W}^2_2(\Omega)} \leq C\|f\| \), which completes the proof. 

We introduce the function \( v_\varepsilon := (1 - \chi_\varepsilon)y + \varepsilon\chi_\varepsilon u_\varepsilon \), where \( \chi_\varepsilon \) is the characteristic function of \( \Omega_\varepsilon \), i.e., \( v_\varepsilon = y \) on \( \Gamma \setminus \Omega_\varepsilon \) and \( v_\varepsilon = \varepsilon u_\varepsilon(\varepsilon^{-1}.) \) on \( \Omega_\varepsilon \). The function \( v_\varepsilon \) is almost the desired approximation; the only problem is that it is discontinuous at the points \( a_\varepsilon^n \). However the jumps of \( v_\varepsilon \) at these points are small. Indeed,

\[
[v_\varepsilon]_{a_\varepsilon^n} = y(a_\varepsilon^n) - \varepsilon u_\varepsilon(a_\varepsilon^n), \quad [v_\varepsilon']_{a_\varepsilon^n} = 0,
\]

where \([h]_{a_\varepsilon^n} = h_{\gamma_\varepsilon^n}(a_\varepsilon^n) - h_{\omega_\varepsilon^n}(a_\varepsilon^n) \) and \([h']_{a_\varepsilon^n} = \frac{dh}{d\gamma_\varepsilon^n}(a_\varepsilon^n) + \frac{dh}{d\omega_\varepsilon^n}(a_\varepsilon^n) \) are jumps of a function \( h \) and its first derivative at \( x = a_\varepsilon^n \). In view of Lemma 1 and the relations

\[
|y(a_\varepsilon^n)| \leq \int_a^{a_\varepsilon^n} |y'| \, d\gamma_n \leq c_1 \varepsilon \|y\|_{\mathcal{W}^2_2(\Gamma)} \leq c_2 \varepsilon^{1/2} \|f\|
\]

we get

\[
|\{v_\varepsilon\}_{a_\varepsilon^n}| \leq |y(a_\varepsilon^n)| + \varepsilon \|u_\varepsilon\|_{\mathcal{W}^2_2(\Omega)} \leq c_3 \varepsilon^{1/2} \|f\|. \tag{7}
\]

Denote by \( \eta_{0,n} \) functions on the graph \( \Gamma \) that are smooth outside the point \( x = a_\varepsilon^n \), have supports \( [a_\varepsilon^n, a_\varepsilon^{n+1}] \), and \( \eta_{0,1} = 1 \) on \( [a_\varepsilon^n, a_\varepsilon^{n+1}] \) for \( n = 1, 2, 3 \). The function \( \hat{\eta}_{0,1} \) can be constructed in the following way. On the edge \( \gamma_1 \) we consider a smooth function \( \eta_{0,1} \) with the properties that \( \text{supp } \eta_{0,1} = [a, a_{1\varepsilon}] \) and \( \eta_{0,1} = 1 \) on \( [a, a_{1\varepsilon}] \); then \( \eta_{0,1} \) is just a translation of \( \eta_{0,1} \) extended by zero to the whole graph \( \Gamma \). The functions \( \eta_{0,2} \) and \( \eta_{0,3} \) can be constructed in a similar manner. We set

\[
w_\varepsilon(x) := \sum_{n=1}^{3} [v_\varepsilon]_{a_\varepsilon^n} \eta_{0,n}(x).
\]

Taking into account (7), we obtain

\[
\max_{x \in \Omega_{x+1} \setminus \Omega_\varepsilon} |w_\varepsilon^{(j)}(x)| \leq c_4 \varepsilon^{1/2} \|f\| \tag{8}
\]

for \( j = 0, 1, 2 \). By construction,

\[
y_\varepsilon := v_\varepsilon - w_\varepsilon = \begin{cases} y - w_\varepsilon & \text{on } \Gamma \setminus \Omega_\varepsilon, \\ \varepsilon u_\varepsilon(\varepsilon^{-1}..) & \text{on } \Omega_\varepsilon \end{cases}
\]

is a \( C^1(\Gamma) \)-function and belongs to the domain of \( H_\varepsilon(\alpha, Q) \). Now we show that \( y_\varepsilon \) is a desired approximation for \( y_\varepsilon = (H_\varepsilon(\alpha, Q) - \zeta)^{-1}f \) and \( y = (H(\alpha, Q) - \zeta)^{-1}f \).

**Proof of Theorem 1 in the non-resonant case.** Rewrite \( y_\varepsilon \) in the form

\[
y_\varepsilon(x) = \begin{cases} (1 - \chi_\varepsilon(x))y(x) + \varepsilon u_\varepsilon(\varepsilon^{-1}x) - w_\varepsilon(x) & \text{on } \Gamma \setminus \Omega_\varepsilon, \\ \varepsilon u_\varepsilon(\varepsilon^{-1}x) & \text{on } \Omega_\varepsilon \end{cases}
\]

where \( \chi_\varepsilon \) is the characteristic function of \( \Omega_\varepsilon \), \( u_\varepsilon \) and \( w_\varepsilon \) are extended by zero to the whole graph \( \Gamma \). Recalling the definition of \( y, u_\varepsilon, \) and \( w_\varepsilon \), we find that

\[
(H_\varepsilon(\alpha, Q) - \zeta)y_\varepsilon(x) = (-\frac{d^2}{dx^2} - \zeta)(y(x) - w_\varepsilon(x)) = f(x) + w_\varepsilon''(x) + \zeta w_\varepsilon(x)
\]

for \( x \in \Gamma \setminus \Omega_\varepsilon \), and that

\[
(H_\varepsilon(\alpha, Q) - \zeta)y_\varepsilon(x) = \varepsilon \left\{-\frac{d^2}{dx^2} + \alpha \varepsilon^{-2}Q(\varepsilon^{-1}x) - \zeta \right\} u_\varepsilon(\varepsilon^{-1}x)
\]

\[
= \varepsilon \left\{-u_\varepsilon'' + \alpha Q u_\varepsilon \right\} - \zeta y_\varepsilon(x) = f(x) - \varepsilon \zeta u_\varepsilon(\varepsilon^{-1}x).
\]
for \( x \in \Omega_\varepsilon \). Therefore \((H_\varepsilon(\alpha, Q) - \zeta)\hat{y}_\varepsilon(x) = f + r_\varepsilon\), where
\[
    r_\varepsilon(x) = \begin{cases} 
        w_\varepsilon''(x) + \zeta w_\varepsilon(x) & \text{if } x \in \Gamma \setminus \Omega_\varepsilon, \\
        -\varepsilon \zeta u_\varepsilon(x) & \text{if } x \in \Omega_\varepsilon.
    \end{cases}
\]
From this we conclude that \(\hat{y}_\varepsilon - y_\varepsilon = (H_\varepsilon(\alpha, Q) - \zeta)^{-1}r_\varepsilon\), and thus
\[
    \|\hat{y}_\varepsilon - y_\varepsilon\| \leq \|(H_\varepsilon(\alpha, Q) - \zeta)^{-1}\| r_\varepsilon \leq |3\zeta|^{-1} \|r_\varepsilon\|.
\]
We can now use Lemma 1 and estimate (3) to arrive at the relation
\[
    \|r_\varepsilon\| \leq c_1 \|w_\varepsilon'' + \zeta w_\varepsilon\|_{L^2(\Omega_{\varepsilon+1})} + c_2 \|u_\varepsilon(\varepsilon^{-1}.))\|_{L^2(\Omega_\varepsilon)}
\]
\[
    \leq c_3 \max_{x \in \Omega_{\varepsilon+1}} |w_\varepsilon| + c_4 \varepsilon^{\frac{3}{2}} \|u_\varepsilon\|_{L^2(\Omega)} \leq c_5 \varepsilon^{\frac{3}{2}} \|f\|.
\]
This proves the first inequality in (1). Similarly,
\[
    \|\hat{y}_\varepsilon - y\| = \|\varepsilon u_\varepsilon(\varepsilon^{-1}.)) - w_\varepsilon - \chi_\varepsilon y\| \leq c_6 \varepsilon^{\frac{3}{2}} \|u_\varepsilon\|_{L^2(\Omega)}
\]
\[
    + c_7 \max_{x \in \Omega_{\varepsilon+1}} |w_\varepsilon| + c_8 \|y\|_{H^1(\Gamma)} \|\chi_\varepsilon\| \leq c_9 \varepsilon^{\frac{3}{2}} \|f\|
\]
as required. \(\Box\)

3.2. Proof of Theorem 1 in the simple resonant case. Our aim in this subsection is to prove Theorem 1 in the simple resonant case, when \(\alpha\) is a simple eigenvalue of problem (3).

Let \(\phi_\alpha\) be an eigenfunction of problem (3) corresponding to the eigenvalue \(\alpha\). Since all \(\phi_\alpha(a_n)\) cannot be zero, we assume without loss of generality that \(\phi_\alpha(a_1) = \pm 1\). Denote by \(u_\varepsilon\) a solution of the problem
\[
    \begin{cases} 
        -u'' + \alpha Q u = \varepsilon f(\varepsilon \cdot) & \text{on } \Omega, \\
        \frac{du}{dx_1}(a_1) = \kappa_\varepsilon, \\
        \frac{du}{dx_n}(a_n) = \frac{du}{d\gamma_n}(a_n) & n = 2, 3,
    \end{cases}
\]
(9)
obeying the condition \(u_\varepsilon(a_1) = 0\). Such a solution exists and is unique if
\[
    \kappa_\varepsilon = -\left(\theta_2 \frac{dy}{d\gamma_2}(a_2) + \theta_3 \frac{dy}{d\gamma_3}(a_3^\varepsilon) - \varepsilon \int_\Omega \phi_\alpha(t) f(t \varepsilon) d\Omega\right);
\]
(10)
this follows from the Fredholm alternative. In order to obtain \(\kappa_\varepsilon\), we multiplied Eq. (9) by the eigenfunction \(\phi_\alpha\) and integrated by parts, bearing in mind that \(\theta_1 := \phi_\alpha(a_1) = 1\).

Lemma 2. For any \(f \in L^2(\Gamma)\) and \(\varepsilon \in (0, 1)\) the following inequalities hold:
\[
    \left| \kappa_\varepsilon - \frac{d y}{d\gamma_1}(a) \right| \leq C_1 \varepsilon^{\frac{3}{2}} \|f\|, \quad \|u_\varepsilon\|_{W^2(\Omega)} \leq C_2 \|f\|.
\]

Proof. As in the proof of Lemma 1 we find that
\[
    \|y\|_{W^2(\Gamma)} \leq c_1 \|f\|, \quad \|y\|_{H^1(\Gamma)} \leq c_2 \|f\|.
\]
Subtracting the relation \(\frac{dy}{d\gamma_1}(a) = -\theta_2 \frac{dy}{d\gamma_2}(a) - \theta_3 \frac{dy}{d\gamma_3}(a)\) from (10) gives
\[
    \left| \kappa_\varepsilon - \frac{dy}{d\gamma_1}(a) \right| \leq |\theta_2| \left| \frac{dy}{d\gamma_2}(a_2) - \frac{dy}{d\gamma_2}(a) \right| + |\theta_3| \left| \frac{dy}{d\gamma_3}(a_3^\varepsilon) - \frac{dy}{d\gamma_3}(a) \right|
\]
\[
    + \varepsilon \|f(\varepsilon \cdot)\|_{L^2(\Omega)} \|\phi_\alpha\|_{L^2(\Omega)} \leq C_1 \varepsilon^{\frac{3}{2}} \|f\|.
\]
Here we used \((6)\) and the following estimates
\[
\left| \frac{dy}{d\gamma_n}(a^*_n) - \frac{dy}{d\gamma_n}(a) \right| \leq \int_a^{a^*_n} |\gamma''| \, d\gamma_n \leq c_3 \varepsilon^{1/2} \|y\|_{W^2_2(\Gamma)} \leq c_4 \varepsilon^{1/2} \|f\| \tag{11}
\]
for \(n = 1, 2, 3\).

Observe that the restriction \(u_{\varepsilon, \omega_1}\) of \(u_{\varepsilon}\) onto \(\omega_1\) is a solution of the Cauchy problem
\[
\begin{align*}
-u'' + \alpha Q u &= \varepsilon f(\varepsilon \cdot) \quad \text{on} \quad \omega_1, \\
u(a) &= 0, \quad \frac{du}{d\omega_1}(a_1) = \kappa_\varepsilon;
\end{align*}
\]
thus, using \((6)\) and properties of solutions of this problem \((19)\), we get the estimate
\[
\|u_{\varepsilon}\|_{W^2_2(\omega_1)} \leq c_5 (|\kappa_\varepsilon| + \varepsilon \|f(\varepsilon \cdot)\|_{L^2(\Omega)}) \leq c_6 \|f\| \tag{12}
\]
Next, we claim that \(\alpha\) does not belong to the intersection of spectra of the following problems
\[
\begin{align*}
-f'' + \alpha Q f &= 0 \quad \text{on} \quad \omega_2, \\
 f(a) &= 0, \quad \frac{df}{d\omega_2}(a_2) = 0, \\
 f(a) &= 0, \quad \frac{df}{d\omega_3}(a_3) = 0.
\end{align*}
\]
Assume the contrary, i.e., let there exist nonzero solutions of the above problems. From these eigenfunctions one can construct in a straightforward manner an eigenfunction of the problem \((3)\) vanishing on \(\omega_1\), which is impossible in view of the equality \(\phi_\alpha(a_1) = 1\). Without loss of generality we suppose that \(\alpha\) does not belong to the spectrum of the problem on \(\omega_2\). Therefore the nonhomogenous problem
\[
\begin{align*}
-u'' + \alpha Q u &= \varepsilon f(\varepsilon \cdot) \quad \text{on} \quad \omega_2, \\
u(a) &= u_{\varepsilon, \omega_1}(a), \quad \frac{du}{d\omega_2}(a_2) = \frac{dy}{d\gamma_2}(a^*_2)
\end{align*}
\]
admits a unique solution which coincides with \(u_{\varepsilon, \omega_2}\). Moreover,
\[
\|u_{\varepsilon}\|_{W^2_2(\omega_2)} \leq c_5 \left( |u_{\varepsilon, \omega_1}(a)| + \left| \frac{du}{d\gamma_2}(a^*_2) \right| + \varepsilon \|f(\varepsilon \cdot)\|_{L^2(\Omega)} \right) \leq c_8 \|f\| \tag{13}
\]
by the a priori estimate of \(u_{\varepsilon, \omega_2}, \tag{6}\) and \((12)\).

Next, the Cauchy problem
\[
\begin{align*}
-u'' + \alpha Q u &= \varepsilon f(\varepsilon \cdot) \quad \text{on} \quad \omega_3, \\
u(a) &= u_{\varepsilon, \omega_1}(a), \quad \frac{du}{d\omega_3}(a) = -\frac{du}{d\omega_2}(a) - \frac{du}{d\omega_3}(a),
\end{align*}
\]
gives us \(u_{\varepsilon, \omega_3}\). From this, using \((12)\) and \((13)\), we find that
\[
\|u_{\varepsilon}\|_{W^2_2(\omega_3)} \leq c_9 \left( |u_{\varepsilon, \omega_1}(a)| + \left| \frac{du}{d\omega_1}(a) \right| + \left| \frac{du}{d\omega_2}(a) \right| + \varepsilon \|f(\varepsilon \cdot)\|_{L^2(\Omega)} \right) \leq c_{10} \|f\|.
\]
Combining the above estimate with \((12)\) and \((13)\), we arrive at the desired inequality.

Recall that \(\chi_{\varepsilon}\) is the characteristic function of \(\Omega_{\varepsilon}\) and put
\[
v_{\varepsilon} := (1 - \chi_{\varepsilon})y + \chi_{\varepsilon} \left[ y(a^*_1)\phi_\alpha(\varepsilon^{-1} \cdot) + \varepsilon u_{\varepsilon}(\varepsilon^{-1} \cdot) \right],
\]
i.e., \(v_{\varepsilon} = y\) on \(\Gamma \setminus \Omega_{\varepsilon}\) and \(v_{\varepsilon} = y(a^*_1)\phi_\alpha(\varepsilon^{-1} \cdot) + \varepsilon u_{\varepsilon}(\varepsilon^{-1} \cdot)\) on \(\Omega_{\varepsilon}\). We now estimate the jumps of \(v_{\varepsilon}\) and its first derivative at the points \(a^*_n\). Direct calculations show that
\[
|v_{\varepsilon}|_{a^*_n} = y(a^*_n) - \theta_n y(a^*_1) - \varepsilon u_{\varepsilon}(a_n), \quad |v'_{\varepsilon}|_{a^*_n} = \frac{dy}{d\gamma_n}(a^*_n) - \frac{du_{\varepsilon}}{d\omega_n}(a_n).
\]
Using Lemma 2, the relations $y_{\gamma_n}(a) = \theta_n y_{\gamma_1}(a)$, and the estimates
\[
|y(a_\varepsilon^n) - y_{\gamma_n}(a)| \leq \int_{a}^{a_\varepsilon^n} |y'| \, d\gamma_n \leq c_1 \varepsilon^{\frac{1}{2}} \|y\|_{L^2(\Gamma)} \leq c_2 \varepsilon^{\frac{1}{2}} \|f\| \tag{14}
\]
holding for $n = 1, 2, 3$, we arrive at the bounds for the jumps
\[
|[v_\varepsilon]|_{\gamma_n} \leq |y(a_\varepsilon^n) - y_{\gamma_n}(a)| + |\theta_n||y(a_\varepsilon^n) - y_{\gamma_1}(a)| + \varepsilon \|u_\varepsilon\|_{L^2(\Gamma)} \leq c_3 \varepsilon^{\frac{1}{2}} \|f\|, \\
|[v'_\varepsilon]|_{\gamma_n} \leq \left| \frac{dy}{d\gamma_n}(a_\varepsilon^n) - \frac{dy}{d\gamma_n}(a) \right| + \left| \frac{dy}{d\gamma_n}(a) - \frac{du_\varepsilon}{d\omega_n}(a) \right| \leq c_4 \varepsilon^{\frac{1}{2}} \|f\|. \tag{15}
\]

Introduce now the function $\eta^\varepsilon_{\gamma_1,n}$ supported by $[a^n_\varepsilon, a^{n+\frac{1}{2}}_\varepsilon]$, that is smooth outside the point $a^n_\varepsilon$, linear on $[a^n_\varepsilon, a^{n+\frac{1}{2}}_\varepsilon]$ with $\eta^\varepsilon_{\gamma_1,n}(a^n_\varepsilon) = 0$ and $\eta^\varepsilon_{\gamma_1,n}(a^{n+\frac{1}{2}}_\varepsilon) = \frac{1}{2}$. Put
\[w_\varepsilon(x) := \sum_{n=1}^{3} ([v_\varepsilon]_{\gamma_n} \eta^\varepsilon_{\gamma_0,n}(x) + [v'_\varepsilon]_{\gamma_n} \eta^\varepsilon_{\gamma_1,n}(x)). \]

Inequalities (15) imply that
\[
\max_{x \in \Omega_{n+1} \setminus \Omega_{n}} |w^{(n)}_\varepsilon(x)| \leq c_5 \sqrt{\varepsilon} \|f\|
\]
for $n = 0, 1, 2$. By construction,
\[
\tilde{y}_\varepsilon := v_\varepsilon - w_\varepsilon = \begin{cases} y - w_\varepsilon & \text{on } \Gamma \setminus \Omega_\varepsilon, \\ y(a_\varepsilon^n) \phi_\alpha(\varepsilon^{-1}) + \varepsilon u_\varepsilon(\varepsilon^{-1}) & \text{on } \Omega_\varepsilon \end{cases}
\]
is a $C^1(\Gamma)$-function and belongs to the domain of $H(\alpha, Q)$.

The rest of the proof of Theorem 1 (i.e., proof of (4)) in the simple resonant cases is similar to that in the non-resonant case and is therefore omitted.

3.3. Proof of Theorem 1 in the double resonant case. Finally, we establish Theorem 1 in the case when $\alpha$ is a double eigenvalue of problem (3). Let $\psi_\alpha$ and $\phi_\alpha$ be a pair of linearly independent eigenfunctions of problem (3) corresponding to the eigenvalue $\alpha$. Let $u_\varepsilon$ be any solution of the following problem
\[
\begin{aligned}
- \Delta u + \alpha Qu &= \varepsilon f(\varepsilon \cdot) & \text{on } \Omega, & u \in K(\Omega; a), \\
\frac{du}{d\alpha}(a_1) &= \mu_\varepsilon, & \frac{du}{d\alpha}(a_2) &= \nu_\varepsilon, & \frac{du}{d\alpha}(a_3) &= \frac{du}{d\alpha}(a_3),
\end{aligned}
\]
where
\[
\begin{align*}
\mu_\varepsilon &= \frac{\theta_1}{\theta_3} \frac{dy}{d\gamma_3}(a_3^n) + \varepsilon \frac{\theta_2}{\theta_3} \int_{\Omega} (\psi_\alpha(a_2) \varphi_\alpha(t) - \varphi_\alpha(a_2) \psi_\alpha(t)) f(\varepsilon t) \, d\Omega, \\
\nu_\varepsilon &= \frac{\theta_2}{\theta_3} \frac{dy}{d\gamma_3}(a_3^n) + \varepsilon \frac{\theta_1}{\theta_3} \int_{\Omega} (\varphi_\alpha(a_1) \psi_\alpha(t) - \psi_\alpha(a_1) \varphi_\alpha(t)) f(\varepsilon t) \, d\Omega. \tag{16}
\end{align*}
\]
This problem admits solutions in view of the Fredholm alternative. We fix $u_\varepsilon$ by the conditions $u_\varepsilon(a_1) = u_\varepsilon(a_2) = 0$. The following statement is similar to Lemma 2.

Lemma 3. Suppose that $f \in L^2(\Gamma)$ and $\varepsilon \in (0, 1)$. Then
\[
|\mu_\varepsilon - \frac{dy}{d\gamma_3}(a_3^n)| \leq C_1 \varepsilon^{\frac{1}{2}} \|f\|, \quad |\nu_\varepsilon - \frac{dy}{d\gamma_3}(a_3^n)| \leq C_2 \varepsilon^{\frac{1}{2}} \|f\|, \quad \|u_\varepsilon\|_{L^2(\Omega)} \leq C_3 \|f\|.
\]
Similarly, with \( d \) in view of (6) and (17). The proof is complete.

We see that the function \( \mu \) then, since (6) holds, we arrive at

\[
\frac{d\mu}{d\omega}(a) = \frac{\theta_2}{\theta_3} \frac{d\mu}{d\gamma_3}(a)
\]

\[
\frac{d\mu}{d\gamma_2}(a) - \frac{d\mu}{d\gamma_3}(a) \leq \left| \frac{\theta_2}{\theta_3} \frac{d\mu}{d\gamma_3}(a) \right| + c_3 \varepsilon \| f(\varepsilon \cdot) \|_{L^2(\Omega)} \leq C_1 \varepsilon^2 \| f \|
\]

Similarly, with \( \frac{d\mu}{d\gamma_2}(a) = \frac{\theta_2}{\theta_3} \frac{d\mu}{d\gamma_3}(a) \),

\[
\frac{d\mu}{d\omega_2}(a_2) - \frac{d\mu}{d\gamma_3}(a_2) \leq \left| \frac{\theta_2}{\theta_3} \frac{d\mu}{d\gamma_3}(a_2) \right| + c_4 \varepsilon \| f(\varepsilon \cdot) \|_{L^2(\Omega)} \leq C_2 \varepsilon^2 \| f \|
\]

By construction, the restrictions of \( u \) on \( \omega_1 \) and \( \omega_2 \) obey the Cauchy problems

\[
\begin{align*}
-u'' + \alpha Q u &= \varepsilon f(\varepsilon \cdot) \quad \text{on } \omega_2, \\
u(a_1) &= 0, \quad \frac{d\mu}{d\omega_1}(a_1) = \mu_\varepsilon,
\end{align*}
\]

then, since (6) holds, we arrive at

\[
\begin{align*}
\| u \|_{H^2(\omega_1)} &\leq c_5 (\mu_\varepsilon + \varepsilon \| f(\varepsilon \cdot) \|_{L^2(\Omega)}) \leq c_6 \| f \|, \\
\| u \|_{H^2(\omega_2)} &\leq c_7 (\mu_\varepsilon + \varepsilon \| f(\varepsilon \cdot) \|_{L^2(\Omega)}) \leq c_8 \| f \|.
\end{align*}
\]

We see that the function \( u_{\varepsilon,\omega_3} \) is a solution of the initial problem

\[
\begin{align*}
-u'' + \alpha Q u &= \varepsilon f(\varepsilon \cdot) \quad \text{on } \omega_3, \\
\mu(a_1) &= u_{\varepsilon,\omega_1}(a), \quad \frac{d\mu}{d\omega_1}(a) = -\frac{d\mu}{d\omega_2}(a)
\end{align*}
\]

and that, moreover,

\[
\| u \|_{H^2(\omega_3)} \leq c_9 \left( |u_{\varepsilon,\omega_1}(a_1)| + \frac{d\mu}{d\omega_1}(a_1) \right) + \frac{d\mu}{d\omega_2}(a_2) \leq c_{10} \| f \|
\]

in view of (6) and (17). The proof is complete. \( \Box \)

We set

\[
\begin{align*}
\Phi := (1 - \chi_3) y + \frac{\chi_3}{\theta_3} \left( \{ \psi_\alpha(a_1) y(a_2) - \psi_\alpha(a_2) y(a_1) \} \right) + \{ \varepsilon f(\varepsilon \cdot) \} \psi_\alpha(\varepsilon^{-1} \cdot) + \theta_3 \varepsilon u_{\varepsilon}(\varepsilon^{-1} \cdot).
\end{align*}
\]

Straightforward calculations show that

\[
\begin{align*}
[v_\varepsilon]_{a_\varepsilon} &= y(a_\varepsilon) - \frac{y(a_\varepsilon)}{\theta_3} \left( \psi_\alpha(a_\varepsilon) \phi_\alpha(a_2) - \psi_\alpha(a_2) \phi_\alpha(a_1) \right) \\
&\quad - \frac{y(a_\varepsilon)}{\theta_3} \left( \psi_\alpha(a_2) \phi_\alpha(a_n) - \psi_\alpha(a_1) \phi_\alpha(a_1) \right) \varepsilon u_{\varepsilon}(a_n),
\end{align*}
\]

\[
\begin{align*}
[v_\varepsilon]_{a_\varepsilon} &= \frac{d\mu}{d\omega_1}(a_\varepsilon) - \frac{d\mu}{d\omega_2}(a_\varepsilon).
\end{align*}
\]
In view of Lemma 3, 11, 13, and the relation $y_{\gamma_1}(a) = -\frac{\theta_2}{\theta_3} y_{\gamma_2}(a) - \frac{\theta_1}{\theta_3} y_{\gamma_3}(a)$, we get

$$|v_\varepsilon|_{a_2^\varepsilon} \leq |y(a_3^\varepsilon) - y_{\gamma_1}(a)| + \frac{\theta_1}{\theta_3} |y(a_2^\varepsilon) - y_{\gamma_1}(a)|$$

$$+ \frac{\theta_2}{\theta_3} |y(a_3^\varepsilon) - y_{\gamma_2}(a)| + \varepsilon \| u_\varepsilon \|_{\mathcal{W}_2(\Omega)} \leq c_3 \varepsilon^{\frac{2}{3}} \| f \|,$$

$$|v_\varepsilon|_{a_2^\varepsilon} \leq \left| \frac{dy}{d\gamma_n}(a_3^\varepsilon) - \frac{dy}{d\gamma_n}(a) \right| + \left| \frac{dy}{d\gamma_n}(a_n) - \frac{du}{d\omega_n}(a_n) \right| \leq c_4 \varepsilon^{\frac{2}{3}} \| f \|.$$  

Now we set

$$\tilde{y}_\varepsilon := \begin{cases} y - w_\varepsilon & \text{on } \Gamma \setminus \Omega_\varepsilon, \\ \tilde{\psi}_\varepsilon(a_3^\varepsilon) - \tilde{\psi}_\varepsilon(a_2^\varepsilon) + \varepsilon u_\varepsilon(\varepsilon^{-1} \cdot) & \text{on } \Omega_\varepsilon \\
\end{cases}$$

with

$$w_\varepsilon(x) := \sum_{n=1}^{3} \left( [\tilde{v}_\varepsilon|_{a_3^\varepsilon}] n^0 \tilde{\eta}_{0,n}(x) + [\tilde{v}_\varepsilon|_{a_2^\varepsilon}] n^1 \tilde{\eta}_{1,n}(x) \right).$$

The rest of the proof runs as before.

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