THE BRAUER GROUP OF $\mathcal{M}_{1,1}$ OVER ALGEBRAICALLY CLOSED FIELDS OF CHARACTERISTIC 2

MINSEON SHIN

Abstract. We prove that the Brauer group of the moduli stack of elliptic curves $\mathcal{M}_{1,1,k}$ over an algebraically closed field $k$ of characteristic 2 is isomorphic to $\mathbb{Z}/(2)$. We also compute the Brauer group of $\mathcal{M}_{1,1,k}$ where $k$ is a finite field of characteristic 2.

1. Introduction

Let $\mathcal{M}_{1,1,Z}$ denote the moduli stack of elliptic curves over $\mathbb{Z}$. For any scheme $S$, we denote by $\mathcal{M}_{1,1,S} := S \times_{Z} \mathcal{M}_{1,1,Z}$ the restriction of $\mathcal{M}_{1,1,Z}$ to the category of schemes over $S$.

Antieau and Meier [AM16, 11.2] computed the Brauer group $\text{Br} \mathcal{M}_{1,1,S}$ for various base schemes $S$, and in particular proved that for any algebraically closed field $k$ of characteristic not 2 the Brauer group $\text{Br} \mathcal{M}_{1,1,k}$ is trivial. The purpose of this note is to compute $\text{Br} \mathcal{M}_{1,1,k}$ in the characteristic 2 case. This then completes the calculation of $\text{Br} \mathcal{M}_{1,1,k}$ over algebraically closed fields $k$. We summarize the result in the following theorem.

Theorem 1.1 ([AM16, 11.2] in char $k \neq 2$). Let $k$ be an algebraically closed field. Then $\text{Br} \mathcal{M}_{1,1,k}$ is 0 unless char $k = 2$, in which case $\text{Br} \mathcal{M}_{1,1,k} = \mathbb{Z}/(2)$.

To prove the theorem, we calculate the cohomology groups $H^2_{\text{ét}}(\mathcal{M}_{1,1,k}, \mu_n)$ for varying $n$. There are essentially two ways to approach this calculation: (1) using the coarse moduli space; (2) using a presentation of $\mathcal{M}_{1,1,k}$ as a quotient stack. In this paper we give a new proof of the Antieau-Meier result using approach (1), and calculate in characteristic 2 using approach (2).

We also compute the Brauer group of $\mathcal{M}_{1,1,k}$ where $k$ is a finite field of characteristic 2:

Theorem 1.2. Let $k$ be a finite field of characteristic 2. Then

$$\text{Br} \mathcal{M}_{1,1,k} = \begin{cases} \mathbb{Z}/(12) \oplus \mathbb{Z}/(2) & \text{if } x^2 + x + 1 \text{ has a root in } k \\ \mathbb{Z}/(24) & \text{otherwise.} \end{cases}$$

An outline of the paper is as follows.

In Section 2 we state definitions and recall general facts about the Brauer group of algebraic stacks.

In Section 3 we record some general remarks regarding $\text{Br} \mathcal{M}_{1,1,S}$. We show that if $S$ is a quasi-compact scheme admitting an ample line bundle and if at least one prime is invertible on $S$, then $\text{Br} \mathcal{M}_{1,1,S} \simeq \text{Br} \mathcal{M}_{1,1,S}$. The restriction of $\mathcal{M}_{1,1,Z}$ to the dense open substack of elliptic curves $E/S$ with $j$-invariant $j(E) \in \Gamma(S, \mathcal{O}_S)$ for which $j(E)$ and $j(E) - 1728$ are invertible is a trivial $\mathbb{Z}/(2)$-gerbe over the coarse space $\mathbb{A}^1_Z \setminus \{0, 1728\}$, and we use this fact.

Date: February 28, 2018.
to conclude that Br $\mathcal{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ for an algebraically closed field $k$ of arbitrary characteristic.

In Section 4 we give a second proof of Antieau and Meier’s result above (that Br $\mathcal{M}_{1,1,k} = 0$ if $k = \overline{k}$ and char $k \neq 2$). Using a dévissage argument, we study the relationship between the cohomology of $\mu_n$ on the stack $\mathcal{M}_{1,1,k}$ and on $\mathbb{A}^1_k$, in terms of the stabilizer groups of elliptic curves with $j$-invariant $0, 1728 \in \mathbb{A}^1_k$. This may be of independent interest for computing the Brauer groups of other separated Deligne-Mumford stacks whose coarse moduli space is a smooth curve over an algebraically closed field with vanishing Picard group.

In Section 5 we prove Theorem 1.1 and Theorem 1.2. Antieau and Meier suggest in [AM16, 11.3] that the characteristic 2 case can be settled using the GL smooth curve over an algebraically closed field with vanishing Picard group.

1.3 (Acknowledgements). I thank my advisor Martin Olsson for suggesting this research topic and for his generosity in sharing his ideas. I am also grateful to Benjamin Antieau, Siddharth Mathur, and Lennart Meier for helpful discussions. During this project, I received support from the Raymond H. Sciobereti Fellowship.

2. The Brauer group of algebraic stacks

Let $(X, \mathcal{O}_X)$ be a locally ringed site [Gir71, V, §4], [Sta18 04EU]. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we set $\text{GL}(\mathcal{E}) := \text{Aut}_{\mathcal{O}_X}^{\mathcal{mod}}(\mathcal{E})$ and let $\text{PGL}(\mathcal{E})$ be the sheaf quotient of $\text{GL}(\mathcal{E})$ by $\mathbb{G}_{m,X}$ via the diagonal embedding. We denote $\text{GL}_n(\mathcal{O}_X) := \text{GL}(\mathcal{O}_X^{\oplus n})$ and $\text{PGL}_n(\mathcal{O}_X) := \text{PGL}(\mathcal{O}_X^{\oplus n})$. A basic fact about these groups is the Skolem-Noether theorem, which states that the morphism

$$\text{PGL}_n(\mathcal{O}_X) \to \text{Aut}_{\mathcal{O}_X}^{\mathcal{alg}}(\text{Mat}_{n \times n}(\mathcal{O}_X))$$

is an isomorphism (see [Gir71 V.4.1]).

Definition 2.1 (Azumaya algebras). [Gro68a §2], [Gir71 V, §4] Let $(X, \mathcal{O}_X)$ be a locally ringed site. An Azumaya $\mathcal{O}_X$-algebra is a quasi-coherent (non-commutative, unital) $\mathcal{O}_X$-algebra $\mathcal{A}$ such that there exists a covering $\{X_i \to X\}_{i \in I}$, positive integers $n_i$, and $\mathcal{O}_X$-algebra isomorphisms $\mathcal{A}|_{X_i} \simeq \text{Mat}_{n_i \times n_i}(\mathcal{O}_{X_i})$.

Two Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent if there exist finite type locally free $\mathcal{O}_X$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, everywhere of positive rank, and an isomorphism

$$\mathcal{A}_1 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}^{\mathcal{mod}}(\mathcal{E}_1) \simeq \mathcal{A}_2 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}^{\mathcal{mod}}(\mathcal{E}_2)$$

of $\mathcal{O}_X$-algebras. Under tensor product of Azumaya algebras, Morita equivalence classes of Azumaya algebras form an abelian group $\text{Br}(X)$ called the (Azumaya) Brauer group of $X$ in which $[\mathcal{A}]^{-1} = [\mathcal{A}^{\text{op}}]$ and the identity element is the class of trivial Azumaya algebras $[\text{End}_{\mathcal{O}_X}^{\mathcal{mod}}(\mathcal{E})]$. 
Definition 2.2 (Gerbe of trivializations). [Gir71 IV, §4.2], [Ols16 12.3.5] There is a natural way to associate, to every Azumaya \( O_X \)-algebra \( \mathcal{A} \), a \( \mathbb{G}_m \times_X \)-gerbe \( \mathcal{G}_\mathcal{A} \) called the *gerbe of trivializations of \( \mathcal{A} \).* An object of \( \mathcal{G}_\mathcal{A} \) is a triple
\[
(U, \mathcal{E}, \sigma)
\]
consisting of an object \( U \in X \), a finite type locally free \( O_U \)-module \( \mathcal{E} \) (necessarily everywhere positive rank), and an isomorphism \( \sigma : \text{End}_{O_U \text{-mod}}(\mathcal{E}) \to \mathcal{A}|_U \) of \( O_U \)-algebras. A morphism \((f, f^\sharp) : (U_1, \mathcal{E}_1, \sigma_1) \to (U_2, \mathcal{E}_2, \sigma_2)\) consists of a morphism \( f \in \text{Mor}_X(U_1, U_2) \) and an isomorphism \( f^\sharp : f^* \mathcal{E}_2 \to \mathcal{E}_1 \) of \( O_{U_1} \)-modules such that \( \sigma_2 = \sigma_1 \circ \rho_f^\sharp \) where \( \rho_f^\sharp \) denotes conjugation by \( f^\sharp \). For any object \((U, \mathcal{E}, \sigma) \in \mathcal{G}_\mathcal{A}\), there is a canonical injection \( \iota(U, \mathcal{E}, \sigma) : \mathbb{G}_m, U \to \text{Aut}_{(U, \mathcal{E}, \sigma)} \) of sheaves on \( X/U \), sending \( u \mapsto (\text{id}_U, u) \); this is in fact an isomorphism, since if \((\text{id}_U, f^\sharp) \in \text{Aut}_{\mathcal{G}_\mathcal{A}(U)}((U, \mathcal{E}, \sigma))\) then \( f^\sharp \in Z(\text{End}_{O_U \text{-mod}}(\mathcal{E})) \), which coincides with \( O_U \) since \( Z(\text{Mat}_{n \times n}(A)) = A \) for any commutative, unital ring \( A \).

By the Skolem-Noether theorem, any two local trivializations of \( \mathcal{A} \) are locally related by an automorphism of the trivializing vector bundle \( \mathcal{E} \), i.e. any two objects of \( \mathcal{G}_\mathcal{A} \) are locally isomorphic. Furthermore, according to the definition, an Azumaya algebra is locally trivial, i.e. for any \( U \in X \) there exists a covering \( \{ U_i \to U \} \) such that the fiber category \( \mathcal{G}_\mathcal{A}(U_i) \) is nonempty. These considerations show that \( \mathcal{G}_\mathcal{A} \) is a \( \mathbb{G}_m \times_X \)-gerbe.

The assignment \( \mathcal{A} \mapsto \mathcal{G}_\mathcal{A} \) induces a group homomorphism
\[
\alpha'_X : \text{Br} X \to H^2(X, \mathbb{G}_m, X)
\]
which is injective since a \( \mathbb{G}_m \times_X \)-gerbe \( \mathcal{G} \) is trivial if and only if \( \mathcal{G}(X) \) is nonempty.

For a morphism \((f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) of locally ringed sites, the diagram
\[
\begin{array}{ccc}
\text{Br} X & \xrightarrow{\alpha'_X} & H^2(X, \mathbb{G}_m, X) \\
\uparrow f^* & & \uparrow f^* \\
\text{Br} Y & \xrightarrow{\alpha'_Y} & H^2(Y, \mathbb{G}_m, Y)
\end{array}
\]
is commutative.

Lemma 2.3. Let \( \mathcal{X} \) be a \( \mathbb{G}_m \times_X \)-gerbe over a locally ringed site \( X \). The class \([\mathcal{X}] \in H^2(X, \mathbb{G}_m, X)\) is in the image of \( \alpha'_X \) if and only if \( \mathcal{X} \) admits a 1-twisted finite locally free sheaf of everywhere positive rank.

The usual proof (c.f. [dJ03 2.14], [Lie08 3.1.2.1], [Ols16 12.3.11]) of Lemma 2.3 applies more generally to the case of \( \mathbb{G}_m \)-gerbes over an arbitrary locally ringed site.

We will only consider locally ringed sites \((X, \mathcal{O}_X)\) whose underlying site \( X \) is quasi-compact [Sta18 090G]. For such \( X \), the Brauer group \( \text{Br} X \) is a torsion group.
Definition 2.4. The torsion subgroup of $\text{H}^2(X, \mathbb{G}_m, X)$, denoted $\text{Br}' X$, is called the cohomological Brauer group and the restriction

\begin{equation}
\alpha_X : \text{Br} X \to \text{Br}' X
\end{equation}

of $\alpha'_X$ to $\text{Br}' X$ is called the Brauer map.

We will consider algebraic stacks using the étale topology except in Section 5 (the case of characteristic 2) in which we will require the flat topology.

Surjectivity of the Brauer map may be checked on a finite flat surjective covering (c.f. [Gab78 II, Lemma 4], [dJ03 2.15], [Lie08 3.1.3.5]):

Proposition 2.5. Let $f : X \to Y$ be a finitely presented, finite, flat, surjective morphism of algebraic stacks. A class $\beta \in \text{H}^2(Y, \mathbb{G}_m, Y)$ is in the image of $\alpha'_Y$ if and only if its pullback $f^* \beta \in \text{H}^2(X, \mathbb{G}_m, X)$ is in the image of $\alpha'_X$.

Proof. Let $\mathcal{Y}$ be the $\mathbb{G}_m,Y$-gerbe corresponding to $\beta$. Set $\mathcal{X} := X \times_Y \mathcal{Y}$ and let $F : \mathcal{X} \to \mathcal{Y}$ be the induced morphism of algebra stacks. If $\mathcal{X}$ is in the image of $\alpha'_X$, then there exists a 1-twisted finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of everywhere positive rank. The pushforward $F_* \mathcal{E}$ is a 1-twisted, finite locally free $\mathcal{O}_Y$-module of everywhere positive rank. Hence $\mathcal{Y}$ is in the image of $\alpha'_Y$.

The other direction follows from commutativity of the diagram (2.2.2) \qed

Corollary 2.6. Let $f : X \to Y$ be a finitely presented, finite, flat, surjective morphism of algebraic stacks. If $\alpha_X$ is an isomorphism, then $\alpha_Y$ is an isomorphism.

Corollary 2.7. Let $X$ be a smooth separated generically tame Deligne-Mumford stack over a field $k$ with quasi-projective coarse moduli space. Then the Brauer map $\alpha_X$ is surjective.

Proof. By Kresch-Vistoli [KV04 2.1.2.2], such $X$ has a finite flat surjection $Z \to X$ where $Z$ is a quasi-projective $k$-scheme. By Gabber’s theorem (see [dJ03 1.1]), the Brauer map is surjective for $Z$. Thus the Brauer map is surjective for $X$ by Proposition 2.5 \qed

Remark 2.8. If $\text{char} k \neq 2$, the stack $\mathcal{M}_{1,1}$ is generically tame and so [Corollary 2.7] implies surjectivity of the Brauer map $\alpha_{\mathcal{M}_{1,1}}$. For the case $\text{char} k = 2$, see [Lemma 3.1].

3. Preliminary observations

The purpose of this section is to prove [Lemma 3.1] below. Let us start, however, with a few preliminary observations about the stack $\mathcal{M}_{1,1}$ and its Brauer group.

The stack $\mathcal{M}_{1,1}$ is a Deligne-Mumford stack smooth and separated over $\mathbb{Z}$ [Ols16 13.1.2]; hence if $S$ is a regular Noetherian scheme then $\mathcal{M}_{1,1,S}$ is a regular Noetherian stack. For any locally Noetherian scheme $S$, the morphism

$$\pi : \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$$

sending an elliptic curve to its $j$-invariant identifies $\mathbb{A}^1_S$ with the coarse moduli space of $\mathcal{M}_{1,1,S}$ [FO10 4.4].

In general, if $\mathcal{X}$ is a separated Deligne-Mumford stack and $\pi : \mathcal{X} \to X$ is its coarse moduli space, then $\pi$ is initial among maps from $\mathcal{X}$ to an algebraic space, so the map $X(G) \to \mathcal{X}(G)$ is an isomorphism for any group scheme $G$; moreover if $U \to X$ is an étale morphism, then $\pi_U :$
$\mathcal{X} \times_X U \to U$ is a coarse moduli space. Applying these observations to $\mathcal{G} = \mathbb{G}_a, \mathbb{G}_m, \mu_n$ implies that the canonical maps $\mathcal{O}_X \to \pi_\ast \mathcal{O}_X, \mathcal{G}_{m,X} \to \pi_\ast \mathcal{G}_{m,X}, \mu_{n,X} \to \pi_\ast \mu_{n,X}$ are isomorphisms; thus we will omit subscripts and denote $\mu_n, \mathbb{G}_m$ for the corresponding sheaves on either $\mathcal{M}_{1,1,S}$ or $\mathbb{A}^1_S$.

**Lemma 3.1.** Let $S$ be a quasi-compact scheme admitting an ample line bundle, and suppose that at least one prime $p$ is invertible in $S$. Then the Brauer map $\alpha_{\mathcal{M}_{1,1,S}} : \text{Br} \mathcal{M}_{1,1,S} \to \text{Br}' \mathcal{M}_{1,1,S}$ is an isomorphism.

*Proof.* By [KM85, 4.7.2], for $N \geq 3$ the moduli stack of full level $N$ structures is representable by an affine $\mathbb{Z}[\frac{1}{N}]$-scheme $Y(N)$. Set $Y(N)_S := Y(N) \times_{\mathbb{Z}[\frac{1}{N}]} S$; the projection $Y(N)_S \to S$ is an affine morphism, hence $Y(N)_S$ is quasi-compact and admits an ample line bundle, hence the Brauer map $\alpha_{Y(N)_S}$ is surjective by Gabber’s theorem (see [dJ03]), and, since the map $Y(N)_S \to \mathcal{M}_{1,1,S}$ is finite locally free, we have by Corollary 2.6 that $\alpha_{\mathcal{M}_{1,1,S}}$ is surjective. $\square$

**Lemma 3.2.** Let $U := \text{Spec} \mathbb{Z}[t, (t(t - 1728))^{-1}] \subset \mathbb{A}^1_\mathbb{Z}$ and let $\mathcal{M}_{0,1,Z} := U \times_{\mathbb{A}^1_\mathbb{Z}} \mathcal{M}_{1,1,Z}$. Then the restriction $\pi^0 : \mathcal{M}_{0,1,Z} \to U$ of $\pi$ to $U$ is a trivial $\mathbb{Z}/(2)$-gerbe, i.e. $\mathcal{M}_{0,1,Z} \simeq \mathcal{B}(\mathbb{Z}/(2))_U$.

*Proof.* Let $S$ be a scheme and let $E_1, E_2$ be two elliptic curves over $S$. If $j(E_1) = j(E_2) \in \Gamma(S, \mathcal{O}_S)$ and $j(E_1), j(E_2) \neq 1728$ are units of $\Gamma(S, \mathcal{O}_S)$, then by [Del75, 5.3] one can find a finite étale cover $S' \to S$ such that there is an isomorphism $S' \times_S E_1 \simeq S' \times_S E_2$ of elliptic curves over $S'$. For any connected scheme $S$ and an elliptic curve $E/S$ for which $j(E)$ and $j(E) - 1728$ are invertible, we have $\text{Aut}(E/S) \simeq \mathbb{Z}/(2)$ by [KM85, (8.4.2)]. It suffices now to show that there is an elliptic curve $E_U$ over $U$ with $j$-invariant $t$. For this we may take the elliptic curve $E_U$ defined by the Weierstrass equation

$$Y^2 Z + X Y Z = X^3 - \frac{36}{t - 1728} X Z^2 - \frac{1}{t - 1728} Z^3$$

which satisfies $\Delta(E_U) = \frac{t^2}{(t - 1728)^3}$ and $j(E_U) = t$ (see [Sil09, Proposition III.1.4(c)]). $\square$

**Lemma 3.3.** Let $k$ be an algebraically closed field and let $U$ be a smooth curve over $k$. If $\text{Pic}(U) = 0$, then $\text{Br}' \mathcal{B}(\mathbb{Z}/(2))_U \simeq (\mathbb{G}_m(U))/(2)$.

*Proof.* The cohomological descent spectral sequence associated to the cover $U \to \mathcal{B}(\mathbb{Z}/(2))_U$ is of the form

$$E_2^{p,q} = H^p(\mathbb{Z}/(2), H^q_{\text{ét}}(U, \mathbb{G}_m)) \implies H^{p+q}_{\text{ét}}(\mathcal{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m)$$

with differentials $E_2^{p,q} \to E_2^{p+2, q-1}$. We have by [Mil80, III.2.22 (d)] that $H^q_{\text{ét}}(U, \mathbb{G}_m) = 0$ for all $q \geq 2$. Moreover, we have $H^0_{\text{ét}}(U, \mathbb{G}_m) = \text{Pic}(U) = 0$ by assumption. Thus the only row of the $E_2$-page of (3.3.1) containing nonzero entries is $q = 0$, which gives an isomorphism

$$H^0_{\text{ét}}(\mathcal{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m) \simeq H^2(\mathbb{Z}/(2), H^0_{\text{ét}}(U, \mathbb{G}_m)) \simeq (\mathbb{G}_m(U))/(2)$$

of abelian groups. $\square$

**Lemma 3.4.** Let $k$ be an algebraically closed field. If $\text{char } k \neq 2, 3$, then $\text{Br}' \mathcal{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. If $\text{char } k$ is 2 or 3, then $\text{Br}' \mathcal{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2)$.

*Proof.* We have that $\mathcal{M}_{1,1,k}$ is regular Noetherian and that $\mathcal{M}_{1,1,k} := \mathcal{M}_{1,1,Z} \times_Z k$ is a dense open substack; thus by [AM16, 2.5(iv)] the map $\text{Br}' \mathcal{M}_{1,1,k} \to \text{Br}' \mathcal{M}_{1,1,k}$
induced by restriction is an injection. Here \([\text{Lemma 3.2}]\) implies \(\text{Br}'\mathcal{M}_{1,1,k} = \text{Br}'\mathbb{B}(\mathbb{Z}/(2))_U\) for \(U = \text{Spec} k[t, (t(t - 1728))^{-1}]\), and \([\text{Lemma 3.3}]\) implies \(\text{Br}'\mathbb{B}(\mathbb{Z}/(2))_U\) is \(\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)\) if \(\text{char } k \neq 2, 3\) and \(\mathbb{Z}/(2)\) otherwise (here we use that \(k^\times = (k^\times)^2\) since \(k\) is algebraically closed).

\[\square\]

### 4. The case \(\text{char } k\) is not 2

Antieau and Meier [AM16] compute the Brauer group \(\text{Br}\mathcal{M}_{1,1,S}\) for various base schemes \(S\), including algebraically closed fields \(k\) of odd characteristic [AM16, 11.2] (the case \(\text{char } k \neq 2\) in Theorem 1.1). In this section we give a proof via a dévissage argument, using the fact that the coarse moduli space morphism \(\pi : \mathcal{M} \to \mathbb{A}^1_k\) is a trivial \(\mathbb{Z}/(2)\)-gerbe away from \(0, 1728 \in \mathbb{A}^1_k\) (see \([\text{Lemma 3.2}]\)). Our proof is divided into two cases, depending on whether \(\text{char } k = 3\) or \(\text{char } k \neq 3\) (this will determine whether we puncture \(\mathbb{A}^1_k\) at one or two points, respectively). We first fix notation and record some observations that apply to both cases.

#### 4.1. We abbreviate \(\mathcal{M} := \mathcal{M}_{1,1,k}\). By \([\text{Lemma 3.1}]\) the Brauer map \(\alpha_{\mathcal{M}} : \text{Br}\mathcal{M} \to \text{Br}'\mathcal{M}\) is an isomorphism. By \([\text{Lemma 3.4}]\) the main task is to show that the 2-torsion in \(\text{Br}\mathcal{M}\) is 0.

For any integer \(n \geq 1\), the étale Kummer sequence

\[1 \to \mu_{2^n} \to \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \to 1\]

gives an exact sequence

\[0 \to (\text{Pic }\mathcal{M})/(2^n) \to H^2(\mathcal{M}, \mu_{2^n}) \to H^2(\mathcal{M}, \mathbb{G}_m)[2^n] \to 0\]

of abelian groups. Since we have \(\text{Pic }\mathcal{M} \simeq \mathbb{Z}/(12)\) by [FO10], we wish to compute \(H^2(\mathcal{M}, \mu_{2^n})\).

Set

\[U := \text{Spec } k[t, (t(t - 1728))^{-1}] = \mathbb{A}^1_k \setminus \{0, 1728\}\]

with inclusion \(j : U \to \mathbb{A}^1_k\) and let \(i : Z \to \mathbb{A}^1_k\) be the complement with reduced induced closed subscheme structure. (Thus, if \(\text{char } k = 2\) or \(3\) then \(Z \simeq \text{Spec } k\), otherwise \(Z \simeq \text{Spec } k \amalg \text{Spec } k\).) Set

\[\mathcal{M}^\circ := U \times_{\mathbb{A}^1_k} \mathcal{M}\]

\[\mathcal{M}_Z := Z \times_{\mathbb{A}^1_k} \mathcal{M}\]

with projections \(\pi^\circ : \mathcal{M}^\circ \to U\) and \(\pi_Z : \mathcal{M}_Z \to Z\). We have a commutative diagram

\[\begin{array}{ccc}
\mathcal{M}^\circ & \longrightarrow & \mathcal{M} \\
\pi^\circ \downarrow & & \pi \downarrow \\
U & \longrightarrow & \mathbb{A}^1_k \\
\end{array}\]

with cartesian squares.

We have a distinguished triangle

\[j_* j^* \mathbb{R}\pi_* \mu_{2^n} \to \mathbb{R}\pi_* \mu_{2^n} \to i_* i^* \mathbb{R}\pi_* \mu_{2^n} \underset{+1}{\to}\]

\[\end{array}\]
in the derived category of bounded-below complexes of abelian sheaves on the étale site of $\mathbb{A}_{k}^{1}$, whose associated long exact sequence has the form

$$\begin{align*}
H^0(\mathbb{A}^1_k, j_! R\pi_* \mu_{2n}) &\to H^0(\mathcal{M}, \mu_{2n}) \to H^0(Z, i^* R\pi_* \mu_{2n}) \\
H^1(\mathbb{A}^1_k, j_! R\pi_* \mu_{2n}) &\to H^1(\mathcal{M}, \mu_{2n}) \to H^1(Z, i^* R\pi_* \mu_{2n}) \\
H^2(\mathbb{A}^1_k, j_! R\pi_* \mu_{2n}) &\to H^2(\mathcal{M}, \mu_{2n}) \to H^2(Z, i^* R\pi_* \mu_{2n})
\end{align*}$$

(4.1.4)

since $j^* R\pi_* \mu_{2n} \cong R\pi_* \mu_{2n}$ and

$$\begin{align*}
H^s(\mathbb{A}^1_k, R\pi_* \mu_{2n}) &\cong H^s(\mathcal{M}, \mu_{2n}) \\
H^s(\mathbb{A}^1_k, i_* i^* R\pi_* \mu_{2n}) &\cong H^s(Z, i^* R\pi_* \mu_{2n})
\end{align*}$$

for all $s$. We will first compute the groups $H^s(\mathbb{A}^1_k, j_! j^* R\pi_* \mu_{2n})$ in the left column of (4.1.4).

**Lemma 4.2.** Let $k$ be an algebraically closed field, let $x_1, \ldots, x_r \in \mathbb{A}^1_k$ be $r$ distinct $k$-points, set

$$Z := \text{Spec } k(x_1) \amalg \cdots \amalg \text{Spec } k(x_r)$$

and let $U = \mathbb{A}^1_k \setminus Z$ be the complement with inclusion $j : U \to \mathbb{A}^1_k$. For any positive integer $\ell$ invertible in $k$, we have

$$H^s(\mathbb{A}^1_k, j_! \mu_{\ell}) = \begin{cases} 0 & s \neq 1 \\ \langle \mu(\ell(k)) \rangle & s = 1 \end{cases}.$$

**Proof.** Let $i : Z \to \mathbb{A}^1_k$ be the inclusion. We have a distinguished triangle

$$j_! \mu_{\ell}|_U \to \mu_{\ell} \to i_* i^* \mu_{\ell} \xrightarrow{+1}$$

in the derived category of bounded-below complexes of abelian sheaves on the big étale site of $\mathbb{A}^1_k$, which gives a long exact sequence

$$\begin{align*}
H^0(\mathbb{A}^1_k, j_! \mu_{\ell}|_U) &\to H^0(\mathbb{A}^1_k, \mu_{\ell}) \to H^0(Z, \mu_{\ell}) \\
H^1(\mathbb{A}^1_k, j_! \mu_{\ell}|_U) &\to H^1(\mathbb{A}^1_k, \mu_{\ell}) \to H^1(Z, \mu_{\ell}) \\
H^2(\mathbb{A}^1_k, j_! \mu_{\ell}|_U) &\to H^2(\mathbb{A}^1_k, \mu_{\ell}) \to H^2(Z, \mu_{\ell}) \\
H^3(\mathbb{A}^1_k, j_! \mu_{\ell}|_U) &\to \cdots
\end{align*}$$

in cohomology. The map $H^0(\mathbb{A}^1_k, \mu_{\ell}) \to H^0(Z, \mu_{\ell})$ is identified with the diagonal map $\mu_{\ell}(k) \to (\mu_{\ell}(k))^s$. Since $k$ is algebraically closed, the étale site of $Z$ is trivial, hence $H^s(Z, \mu_{\ell}) = 0$ for $s \geq 1$. By [De77, Exp. 1, III, (3.6)] we have $H^s(\mathbb{A}^1_k, \mu_{\ell}) = 0$ for $s \geq 2$. We have $\mathbb{G}_m(\mathbb{A}^1_k) \cong \mathbb{G}_m(k)$ and the multiplication-by-$\ell$ map $\times \ell : \mathbb{G}_m(k) \to \mathbb{G}_m(k)$ is surjective; thus $H^1(\mathbb{A}^1_k, \mu_{\ell}) = H^1(\mathbb{A}^1_k, \mathbb{G}_m) \langle \ell \rangle = (\text{Pic } \mathbb{A}^1_k) \langle \ell \rangle = 0$ by the Kummer sequence.
Lemma 4.3. In the setup of Lemma 4.2, let \( n \) be any positive integer and let \( \pi^\circ : B(\mathbb{Z}/(n))_U \to U \) be the trivial \( \mathbb{Z}/(n) \)-gerbe over \( U \). Then

\[
H^s(A^1_k, j_! R\pi^\circ_* \mu_\ell) = \begin{cases} 
0 & \text{if } s = 0, \\
(\mu_\ell(k))^{(r-1)} & \text{if } s = 1, \\
(\mu_{\gcd(n, \ell)}(k))^{(r-1)} & \text{if } s = 2.
\end{cases}
\]

Proof. We set

\[
C := j_! R\pi^\circ_* \mu_\ell
\]

for convenience. We will compute the groups \( H^s(A^1_k, C) \) using the fact that the canonical truncations \( \tau_{\leq s} C \) satisfy

\[(4.3.1) \quad H^s(A^1_k, \tau_{\leq t} C) \simeq H^s(A^1_k, C)\]

for \( s \leq t \). For any \( s \in \mathbb{Z} \), the distinguished triangle

\[(4.3.2) \quad \tau_{\leq s-1} C \to \tau_{\leq s} C \to (h^s C)[s] \xrightarrow{+1}\]

gives a long exact sequence

\[(4.3.3) \quad \begin{array}{ccc}
H^0(A^1_k, \tau_{\leq s-1} C) & \longrightarrow & H^0(A^1_k, \tau_{\leq s} C) \\
\longrightarrow & H^1(A^1_k, \tau_{\leq s-1} C) & \longrightarrow H^1(A^1_k, \tau_{\leq s} C) \\
\longrightarrow & H^2(A^1_k, \tau_{\leq s-1} C) & \longrightarrow H^2(A^1_k, \tau_{\leq s} C)
\end{array}\]

where

\[
h^s C \simeq j_! R^s \pi^\circ_* \mu_\ell
\]

since \( j_! \) is exact.

Since \( \pi^\circ : B(\mathbb{Z}/(n))_U \to U \) is a trivial \( \mathbb{Z}/(n) \)-gerbe, by Lemma B.1, we have

\[(4.3.4) \quad R^s \pi^\circ_* \mu_\ell \simeq \begin{cases} 
\mu_\ell & s = 0 \\
\mu_\ell[n] & s = 1, 3, 5, \ldots \\
\mu_\ell/(n) & s = 2, 4, 6, \ldots
\end{cases}\]

where \( \mu_\ell[n] \) and \( \mu_\ell/(n) \) are defined by the exact sequence

\[
1 \to \mu_\ell[n] \to \mu_\ell \times n \mu_\ell \to \mu_\ell/(n) \to 1
\]

of abelian sheaves. Since \( k \) is algebraically closed of characteristic prime to \( \ell \), the sheaves \( \mu_\ell[n] \) and \( \mu_\ell/(n) \) are both isomorphic to \( \mu_{\gcd(n, \ell)} \), but for us the difference is important for reasons of functoriality (as \( \ell \) is allowed to vary). More precisely, if \( \ell_1 \) divides \( \ell_2 \), then the inclusion \( \mu_{\ell_1} \to \mu_{\ell_2} \) induces an inclusion

\[
\mu_{\ell_1}[n] \to \mu_{\ell_2}[n]
\]

whereas

\[(4.3.5) \quad \mu_{\ell_1}/(n) \to \mu_{\ell_2}/(n)\]
is not necessarily injective since an element \( x \in \mu_{\ell_1} \) which is not an \( n \)th power of any \( y_1 \in \mu_{\ell_1} \) may be an \( n \)th power of some \( y_2 \in \mu_{\ell_2} \) (in particular, if \( \ell_2 = n\ell_1 \), then \([4.3.5]\) is the zero morphism).

We have

\[ \tau_{\leq 0} \mathcal{C} \simeq h^0 \mathcal{C} \simeq j_! \mathbf{R}^0 \tau_{\leq 0} \mathcal{C} \simeq j_! \mathbb{R}^0 \pi_{s*} \mathcal{C} \simeq j_! \mathbb{R}^0 \pi_{s*} \mathcal{M} \simeq j_! \mu_{\ell} \]

since \( \pi^{\circ} \) is a coarse moduli space morphism and \( \mathbf{R}^1 \pi_{s*} \mathcal{M} \simeq \mu_{\text{gcd}(n,\ell)} \) by \([4.3.4]\). Applying \( \text{Lemma 4.2} \) to the case \( s = 1 \) in \([4.3.3]\) implies \( H^0(\mathbb{A}^1_k, \tau_{\leq 1} \mathcal{C}) = 0 \) and gives isomorphisms \( H^1(\mathbb{A}^1_k, j_! \mu_{\ell}) \simeq H^1(\mathbb{A}^1_k, \tau_{\leq 1} \mathcal{C}) \) and \( H^2(\mathbb{A}^1_k, \tau_{\leq 1} \mathcal{C}) \simeq H^1(\mathbb{A}^1_k, j_! \mu_{\text{gcd}(n,\ell)}) \).

Since \( \mathbf{R}^2 \pi_{s*} \mathcal{M} \simeq \mu_{\text{gcd}(n,\ell)} \) by \([4.3.4]\) and \( H^s(\mathbb{A}^1_k, j_! \mu_{\text{gcd}(n,\ell)}) = 0 \) for \( s = -2, -1, 0 \), the case \( s = 2 \) in \([4.3.3]\) gives isomorphisms \( H^s(\mathbb{A}^1_k, \tau_{\leq 1} \mathcal{C}) \simeq H^s(\mathbb{A}^1_k, \tau_{\leq 2} \mathcal{C}) \) for \( s = 0, 1, 2 \), which implies the desired result. \( \square \)

**4.4 (Proof of Theorem 1.1 for char \( k = 3 \)).** If char \( k = 3 \), then \( Z \) consists of one point, so taking \( r = 1 \) in \( \text{Lemma 4.3} \) implies

\[ H^s(\mathbb{A}^1_k, j_! \mathbf{R}^s \pi_{s*} \mu_{2^n}) = 0 \]

for \( s = 0, 1, 2 \). Therefore, to compute \( H^2(\mathcal{M}, \mu_{2^n}) \), it now remains to compute \( H^2(\mathbb{Z}, i^* \mathbf{R} \pi_{s*} \mu_{2^n}) \) in \([4.4.1]\). The stabilizer of any object of \( \mathcal{M} \) of lying over \( i : Z \rightarrow \mathbb{A}^1_k \) is the automorphism group of an elliptic curve with \( j \)-invariant 0, which is the semidirect product \( \Gamma = \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4) \) since \( k \) has characteristic 3. The underlying reduced stack \( (\mathcal{M}/Z)_{\text{red}} \) is the residual gerbe associated to the unique point of \( |\mathcal{M}/Z| \) and is isomorphic to the classifying stack \( B\Gamma_k \). We have natural isomorphisms

\[ H^2(Z, i^* \mathbf{R} \pi_{s*} \mu_{2^n}) \simeq i^* \mathbf{R}^2 \pi_{s*} \mu_{2^n} \simeq H^2(\mathcal{M}/Z, \mu_{2^n}) \simeq H^2(B\Gamma_k, \mu_{2^n}) \simeq H^2(\Gamma, \mu_{2^n}(k)) \]

where isomorphism 1 follows from proper base change \([\text{Ols05, 1.3}]\), isomorphism 2 is by invariance of étale site for nilpotent thickenings and the fact that \( 2^n \) is invertible on \( \mathcal{M}/Z \), and isomorphism 3 is by the cohomological descent spectral sequence for the covering \( \text{Spec} k \rightarrow \text{Spec} \mathbb{Q} \) (and the fact that \( H^i(\text{Spec} k, \mu_{2^n}) = 0 \) for \( i > 0 \) since \( k \) is algebraically closed). The Hochschild-Serre spectral sequence for the exact sequence

\[ 1 \rightarrow \mathbb{Z}/(3) \rightarrow \Gamma \rightarrow \mathbb{Z}/(4) \rightarrow 1 \]

gives an isomorphism

\[ H^2(\Gamma, \mu_{2^n}(k)) \simeq H^2(\mathbb{Z}/(4), \mu_{2^n}(k)) \simeq \mu_{2^n}(k)/4 \]

where \( H^i(\mathbb{Z}/(3), \mu_{2^n}(k)) = 0 \) for \( i > 0 \) since 3 is coprime to the order of \( \mu_{2^n}(k) \). Since the first term in the last row of the diagram \([4.4.1]\) is zero by \([4.4.1]\), the above observations imply that we have natural inclusions

\[ H^2(\mathcal{M}, \mu_{2^n}) \rightarrow \mu_{2^n}(k)/4 \]

compatible with the inclusions \( \mu_{2^n} \subset \mu_{2^{n+1}} \) for all \( n \). The inclusion \( \mu_{2^n} \subset \mu_{2^{n+2}} \) induces the zero map \( \mu_{2^n}(k)/4 \rightarrow \mu_{2^{n+2}}(k)/4 \), so \( H^2(\mathcal{M}, \mu_{2^n}) \rightarrow H^2(\mathcal{M}, \mu_{2^{n+2}}) \) is the zero map as well, hence

\[ \lim_{\rightarrow n \in \mathbb{N}} H^2(\mathcal{M}, \mu_{2^n}) = 0 \]

which by \([4.4.1]\) gives \( H^2(\mathcal{M}, \mathbb{G}_m)[2^n] = 0 \) for all \( n \).
4.5 (Proof of Theorem 1.1 for char \( k \neq 2, 3 \)). We describe the terms in (4.1.4). For the right column, we have

\[
H^s(Z, i^* R\pi_* \mu_{2^n}) \simeq H^s(Z/(4), \mu_{2^n}(k)) \oplus H^s(Z/(6), \mu_{2^n}(k))
\]

by [ACV03, A.0.7]. For the middle column, we have

\[
H^0(\mathcal{M}, \mu_{2^n}) \simeq H^0(\mathbb{A}^1_k, \mu_{2^n}) \simeq \mu_{2^n}(k)
\]

since \( \mathbb{A}^1_k \) is the coarse moduli space of \( \mathcal{M} \), and we have

\[
H^1(\mathcal{M}, \mu_{2^n}) \overset{1}{\simeq} H^1(\mathcal{M}, \mathbb{G}_m)[2^n] \overset{2}{\simeq} (Z/(12))[2^n] \overset{3}{\simeq} Z/(4)
\]

where isomorphism 1 follows since \( k^\times = (k^\times)^{2^n} \), isomorphism 2 is by [Mum65], and isomorphism 3 holds for \( n \gg 0 \). For the left column, we have

\[
H^s(\tau \leq 1 j! R\pi_* \mu_{2^n}) = \begin{cases} 
0 & s = 0 \\
\mu_{2^n} & s = 1 \\
\mu_2 & s = 2
\end{cases}
\]

by Lemma 4.3.

To summarize, (4.1.4) simplifies to

\[
\begin{array}{c}
0 \\
\mu_{2^n} \\
\mu_2
\end{array} \longrightarrow
\begin{array}{c}
\mu_{2^n} \\
\mathbb{Z}/(4) \\
\mathbb{H}^2(\mathcal{M}, \mu_{2^n})
\end{array} \longrightarrow
\begin{array}{c}
\mu_{2^n} \oplus \mu_{2^n} \\
\mu_4 \oplus \mu_2 \\
\mu_{2^n}/(4) \oplus \mu_{2^n}/(6)
\end{array}
\]

(4.5.1)

for \( n \gg 0 \), and counting the number of elements in each group in (4.5.1) implies that the last morphism

\[
H^2(\mathcal{M}, \mu_{2^n}) \rightarrow \mu_{2^n}/(4) \oplus \mu_{2^n}/(6)
\]

is injective. Furthermore, the inclusion

\[
\mu_{2^n} \subset \mu_{2^{n+2}}
\]

induces the zero map

\[
\mu_{2^n}/(4) \oplus \mu_{2^n}/(6) \rightarrow \mu_{2^{n+2}}/(4) \oplus \mu_{2^{n+2}}/(6)
\]

so the map \( H^2(\mathcal{M}, \mu_{2^n}) \rightarrow H^2(\mathcal{M}, \mu_{2^{n+2}}) \) is the zero map as well, hence

\[
\lim_{\rightarrow n \in \mathbb{N}} H^2(\mathcal{M}, \mu_{2^n}) = 0
\]

which by (4.1.1) gives \( H^2(\mathcal{M}, \mathbb{G}_m)[2^n] = 0 \) for all \( n \).
5. THE CASE char $k = 2$

In this section we prove \[\text{Theorem 1.1}\] (in case char $k = 2$) and \[\text{Theorem 1.2}\]. For convenience, we denote $GL_{n,p} := GL_n(\mathbb{Z}/(p))$ and $SL_{n,p} := SL_n(\mathbb{Z}/(p))$. We denote by $e$ the identity element of $GL_{n,p}$.

5.1 (Hesse presentation of $\mathcal{M}_{1,1,k}$. By [FO10, 6.2] (and explained in more detail in A.6), there is a left action of $GL_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-algebra

$$A_H := \mathbb{Z}[\frac{1}{3}, \mu, \omega, \frac{1}{\mu-1}]/(\omega^2 + \omega + 1)$$

sending

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ast (\mu, \omega) = (\mu, \omega^2)$$
$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \ast (\mu, \omega) = (\omega \mu, \omega)$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ast (\mu, \omega) = (\frac{\mu + 2}{\mu - 1}, \omega)$$

for which the corresponding right action of $GL_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-scheme $S_H := \text{Spec } A_H$

gives a presentation

$$\mathcal{M}_{1,1,\mathbb{Z}[\frac{1}{3}]} \simeq [S_H/ GL_{2,3}]$$

of $\mathcal{M}_{1,1,\mathbb{Z}[\frac{1}{3}]}$ as a global quotient stack. The morphism

$$S_H \to \mathcal{M}_{1,1,\mathbb{Z}[\frac{1}{3}]}$$

is given by the elliptic curve

$$X^3 + Y^3 + Z^3 = 3\mu XYZ$$

over $S_H$.

5.2 (Cohomological descent). Let $k$ be an algebraically closed field of characteristic 2. The Brauer map $\alpha: \mathcal{M}_{1,1,k} \to \text{Br} \mathcal{M}_{1,1,k}$ is an isomorphism by \[\text{Lemma 3.1}\]. By \[\text{Lemma 3.4}\] there is only 2-torsion in $\text{Br} \mathcal{M}_{1,1,k}$. By Grothendieck’s fpf-etale comparison theorem for smooth commutative group schemes [Gro68b, (11.7)], it suffices to compute the 2-torsion in $H^2_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mathbb{G}_m)$. Since Spec $k$ is a reduced scheme, we have

$$H^1_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mathbb{G}_m) = \text{Pic}(\mathcal{M}_{1,1,k}) = \mathbb{Z}/(12)$$

by [FO10, 1.1]. Thus, for any integer $n$, the fpf Kummer sequence

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{\times^2} \mathbb{G}_m \to 1$$

gives an exact sequence

$$1 \to \mathbb{Z}/(2) \xrightarrow{\partial} H^2_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mu_2) \to H^2_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mathbb{G}_m)[2] \to 1$$

of abelian groups. It remains to compute the middle term $H^2_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mu_2)$. 

The cohomological descent spectral sequence associated to the cover \((5.1.3)\) is of the form
\[
E_2^{p,q} = H^p(\text{GL}_{2,3}; H_{\text{fppf}}^q(S_{H,k}, \mu_2)) \implies H_{\text{fppf}}^{p+q}(\mathcal{M}_{1,1,k}, \mu_2)
\]
with differentials \(E_2^{p,q} \to E_2^{p+2,q-1}\).

Let \(\xi \in k\) be a fixed primitive 3rd root of unity. By the Chinese Remainder Theorem, there is a \(k\)-algebra isomorphism
\[
A_{H,k} = k[\mu, \omega, \frac{1}{\mu^3-1}]/(\omega^2 + \omega + 1) \to k[\nu_1, \frac{1}{\nu_1-1}] \times k[\nu_2, \frac{1}{\nu_2-1}]
\]
sending \(\mu \mapsto (\nu_1, \nu_2)\) and \(\omega \mapsto (\xi, \xi^2)\). Since \(S_{H,k}\) is a smooth curve over an algebraically closed field, we have by \([\text{Mil80}, \text{III.2.22 (d)}]\) that \(H_{\text{et}}^q(S_{H,k}, \mathbb{G}_m) = 0\) for all \(q \geq 2\); since \(S_{H,k}\) is a disjoint union of two copies of a distinguished affine open subset of \(\mathbb{A}^1_k\), we have \(H^1_{\text{et}}(S_{H,k}, \mathbb{G}_m) = \text{Pic}(S_{H,k}) = 0\). By \([\text{Gro85}, (11.7)]\) we have \(H_{\text{fppf}}^q(S_{H,k}, \mathbb{G}_m) = H_{\text{et}}^q(S_{H,k}, \mathbb{G}_m)\) for all \(q \geq 0\); thus the fppf Kummer sequence implies \(H_{\text{fppf}}^q(S_{H,k}, \mu_2) = 0\) for all \(q \geq 2\). Furthermore, we have \(H_{\text{fppf}}^0(S_{H,k}, \mu_2) = 0\) since \(S_{H,k}\) is the product of two integral domains of characteristic 2. Thus the only nonzero terms on the \(E_2\)-page of \((5.2.3)\) occur on the \(q = 1\) row, so we have an isomorphism
\[
H_{\text{fppf}}^{p+1}(\mathcal{M}_{1,1,k}, \mu_2) \simeq H^p(\text{GL}_{2,3}; H_{\text{fppf}}^1(S_{H,k}, \mu_2))
\]
for all \(p \geq 0\). We are interested in the case \(p = 1\).

5.3 (Description of the \(\text{GL}_{2,3}\)-action on \(H_{\text{fppf}}^1(S_{H,k}, \mu_2)\)). We describe the abelian group
\[
M := H_{\text{fppf}}^1(S_{H,k}, \mu_2)
\]
and the left \(\text{GL}_{2,3}\)-module structure it inherits from \((5.1.1)\). Since \(k[\mu, (\mu^3-1)^{-1}]\) is a principal localization of the polynomial ring \(k[\mu]\) by a polynomial \((\mu^3-1) = (\mu-1)(\mu-\xi)(\mu-\xi^2)\) splitting into three distinct irreducible factors, we have an isomorphism
\[
(k[\mu, \frac{1}{\mu^3-1}])^\times \simeq k^\times \cdot (\mu-1)^\mathbb{Z} \cdot (\mu-\xi)^\mathbb{Z} \cdot (\mu-\xi^2)^\mathbb{Z}
\]
of abelian groups. Thus \((5.2.4)\) and the Kummer sequence \((5.2.1)\) gives an isomorphism
\[
M \simeq (\mathbb{Z}/(2))^{\oplus 6}
\]
of abelian groups, with generators given by the classes of \(\nu_i - \xi^j\) for \(i = 1, 2\) and \(j = 0, 1, 2\).

The isomorphism \((5.2.4)\) is given by the map
\[
s_1(\mu)\omega + s_0(\mu) \mapsto (s_1(\nu_1)\xi + s_0(\nu_1), s_1(\nu_2)\xi^2 + s_0(\nu_2))
\]
for \(s_0, s_1 \in k[\mu, \frac{1}{\mu^3-1}]\). The inverse of \((5.2.4)\) is given by the map
\[
(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu) \left(\frac{\omega}{\xi - \xi^2} + \frac{\xi}{\xi - 1}\right) + f_2(\mu) \left(\frac{-\omega}{\xi - \xi^2} + \frac{-1}{\xi - 1}\right)
\]
where \(f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]\). (Note that, if we set \(A_1(t) := \frac{t}{\xi - \xi^2} + \frac{\xi}{\xi - 1}\) and \(A_2(t) := \frac{-t}{\xi - \xi^2} + \frac{-1}{\xi - 1}\), then \(A_i(\xi^j)\) is the Kronecker delta function.)
A computation with \((5.1.1)\) shows that the action of \(GL_{2,3}\) on the right hand side of \((5.2.4)\) is given by
\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) = (f_2(\nu_1), f_1(\nu_2))
\]
\[(5.3.5)\]
\[
\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) = (f_1(\xi \nu_1), f_2(\xi^2 \nu_2))
\]
\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) = (f_1(\frac{\nu_1 + 2}{\nu_1 - 1}), f_2(\frac{\nu_2 + 2}{\nu_2 - 1}))
\]
for \(f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i - 1}]\). A computation with \((5.3.5)\) (and using that char \(k = 2\)) shows that the action of \(GL_{2,3}\) on \((5.3.2)\) is given by \((5.3.6)\) where every element is considered up to multiplication by \(k^\times\).

\[
M_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
\[
M_2 := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]
\[
i := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]
\[(5.3.6)\]

\[
\begin{array}{cccccc}
\nu_1 - 1 & \nu_1 - \xi & \nu_1 - \xi^2 & \nu_2 - 1 & \nu_2 - \xi & \nu_2 - \xi^2 \\
\nu_2 - 1 & \nu_2 - \xi & \nu_2 - \xi^2 & \nu_1 - 1 & \nu_1 - \xi & \nu_1 - \xi^2 \\
\nu_1 - \xi^2 & \nu_1 & \nu_1 - \xi & \nu_2 - \xi & \nu_2 - \xi^2 & \nu_2 - 1 \\
\nu_1 - \xi & \nu_1 - 1 & \nu_1 - \xi & \nu_2 - \xi & \nu_2 - \xi^2 & \nu_2 - 1 \\
\nu_1 - \xi^2 & \nu_1 - 1 & \nu_1 - \xi & 1 & \nu_2 - \xi^2 & \nu_2 - \xi \\
\nu_1 - 1 & \nu_1 - 1 & \nu_1 - 1 & \nu_2 - 1 & \nu_2 - 1 & \nu_2 - 1 \\
\end{array}
\]

5.4. We compute \(H^1(GL_{2,3}, M)\). (In Appendix C we provide MAGMA code that can be used to verify this computation.) We have a filtration of groups
\[(5.4.1)\]
\[
Q_8 \triangleleft SL_{2,3} \triangleleft GL_{2,3}
\]
where each is a normal subgroup of the next. Here \(Q_8\) denotes the quaternion group
\[
Q_8 = \{ \pm e, \pm i, \pm j, \pm k : \ ijk = i^2 = j^2 = k^2 = -e \}
\]
and is identified with the subgroup of \(GL_{2,3}\) as follows:
\[
i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]
\[
j = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}
\]
\[
k = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}
\]
The quotient \(GL_{2,3} / SL_{2,3}\) is cyclic of order 2 and is generated by \(M_1\) in \((5.3.6)\). The quotient \(SL_{2,3} / Q_8\) is cyclic of order 3 and is generated by \(M_2\) in \((5.3.6)\). For \(i = 1, 2\), let \(\langle M_i \rangle\) denote the subgroup of \(GL_{2,3}\) generated by \(M_i\). We note that \(SL_{2,3}\) is generated by \(i\) and \(M_2\).

Let
\[
F : (\mathbb{Z}[GL_{2,3}] - \text{Mod}) \to (\mathbb{Z}[SL_{2,3}] - \text{Mod})
\]
be the forgetful functor. An inspection of \((5.3.6)\) implies that \(F(M)\) is the direct sum \(N_1 \oplus N_2\) where \(N_i\) is the \(SL_{2,3}\)-submodule of \(F(M)\) generated by the classes of \(\nu_i - 1, \nu_i - \xi, \nu_i - \xi^2\), and moreover \(M_1\) switches the summands \(N_1\) and \(N_2\). Under the adjunction
\[
\text{Hom}_{SL_{2,3}}(F(M), N_1) \simeq \text{Hom}_{GL_{2,3}}(M, \text{Ind}^{GL_{2,3}}_{SL_{2,3}}(N_1))
\]
the projection map \(F(M) \simeq N_1 \oplus N_2 \to N_1\) onto the first factor corresponds to a morphism
\[(5.4.2)\]
\[
M \to \text{Ind}^{GL_{2,3}}_{SL_{2,3}}(N_1)
\]
of \(GL_{2,3}\)-modules. Given \(m \in M\), write \(m = n_1 + n_2\) for \(n_i \in N_i\); then the image of \(m\) under \((5.4.2)\) is the function \(\varphi_m \in \text{Hom}_{\mathbb{Z}[\text{SL}_{2,3}]}(\mathbb{Z}[\text{GL}_{2,3}], N_1)\) such that \(\varphi_m([e]) = n_1\) and \(\varphi_m([M]) = M_1 \cdot n_2\); thus \((5.4.2)\) is an isomorphism.

A computation using \((5.3.6)\) and the identities

\[
\begin{align*}
  k &= M_2^{-1} \cdot i \cdot M_2 \\
  i &= M_2^{-1} \cdot j \cdot M_2 \\
  j &= M_2^{-1} \cdot k \cdot M_2
\end{align*}
\]

shows that the action of an element \(g \in \text{SL}_{2,3}\) on \(N_i\) is by left multiplication by the matrix \(T_g\) as in \((5.4.4)\) with elements of \(N_1\) being viewed as vertical vectors. We note \(T_{-e} = T^2_i = T^2_j = T^2_k = \text{id}_{N_1}\), i.e. \(-e\) acts trivially on \(N_1\).

\[
\begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0 \\
\end{bmatrix}
\]

Since \(M\) is an induced module, the restriction map

\[
(5.4.5) \quad H^1(\text{GL}_{2,3}, M) \to H^1(\text{SL}_{2,3}, N_1)
\]

is an isomorphism so we reduce to computing \(H^1(\text{SL}_{2,3}, N_1)\).

The Hochschild-Serre spectral sequence for the inclusion \(Q_8 \leq \text{SL}_{2,3}\) degenerates on the \(E_2\) page since the order of the quotient group \(\langle M_2 \rangle\) is coprime to the order of \(N_1\). In particular the restriction map

\[
(5.4.6) \quad H^1(\text{SL}_{2,3}, N_1) \to H^0(\langle M_2 \rangle, H^1(Q_8, N_1))
\]

is an isomorphism.

Let \(C^i(Q_8, N_1) := \text{Fun}((Q_8)^i, N_1)\) denote the group of inhomogeneous \(i\)-cochains. By Remark 5.5, the group \(\text{SL}_{2,3}\) has a natural left action on \(C^i(Q_8, N_1)\) (by entrywise conjugation on the source \((Q_8)^i\) and by its usual action on \(N_1\)) such that the differentials in the inhomogeneous cochain complex

\[
C^0(Q_8, N_1) \xrightarrow{d_0} C^1(Q_8, N_1) \xrightarrow{d_1} C^2(Q_8, N_1) \to \cdots
\]

are \(\text{SL}_{2,3}\)-linear. Since the order of the subgroup \(\langle M_2 \rangle\) is coprime to the orders of \(C^i(Q_8, N_1)\), we have that \(H^0(\langle M_2 \rangle, H^1(Q_8, N_1)) \simeq (H^1(Q_8, N_1))_{M_2}\) is isomorphic to the middle cohomology of the sequence

\[
(C^0(Q_8, N_1))_{M_2} \xrightarrow{(d_0)_{M_2}} (C^1(Q_8, N_1))_{M_2} \xrightarrow{(d_1)_{M_2}} (C^2(Q_8, N_1))_{M_2}
\]

i.e. cohomology commutes with taking \(M_2\)-invariants.

We now describe \(\ker(\langle d_1 \rangle_{M_2})\) and \(\text{im}(\langle d_0 \rangle_{M_2})\).

An element \(f \in (C^1(Q_8, N_1))_{M_2}\) is a function \(f : Q_8 \to N_1\) satisfying

\[
(5.4.7) \quad f(\mathbf{g}) = M_2 \cdot f(M_2^{-1} g M_2)
\]

for all \(\mathbf{g} \in Q_8\). We have that \(f \in \ker d_1\) if

\[
(5.4.8) \quad f(\mathbf{g}_1 \cdot \mathbf{g}_2) = \mathbf{g}_1 \cdot f(\mathbf{g}_2) + f(\mathbf{g}_1)
\]
for all \( g_1, g_2 \in Q_8 \).

Suppose \( f \in \ker ((d_1)^{M_2}) = (\ker d_1) \cap (C^1(Q_8, N_1))^M_2 \); taking \((g_1, g_2) = (e, e)\) in (5.4.8) implies \( f(e) = 0 \); taking \( g = -e \) in (5.4.7) implies that
\[ f(-e) = (s, s, s) \]
for some \( s \in \mathbb{Z}/(2) \); taking \((g_1, g_2) = (-e, -e)\) in (5.4.8) and using the fact that \(-e\) acts trivially on \( N_1 \) implies that \( 2f(-e) = 0 \), which imposes no condition on \( s \). We note that
\[ g \cdot f(-e) = f(-e) \]
for any \( g \in \text{SL}_{2,3} \).

Setting \( g = i, j, k \) in (5.4.7) and using (5.4.3) gives
\[ f(i) = M_2 \cdot f(k) \]
\[ f(j) = M_2 \cdot f(i) \]
\[ f(k) = M_2 \cdot f(j) \]
respectively; thus we have
\[ f(i) = (s_1, s_2, s_3) \]
\[ f(j) = (s_2, s_3, s_1) \]
\[ f(k) = (s_3, s_1, s_2) \]
for some \( s_1, s_2, s_3 \in \mathbb{Z}/(2) \).

Setting either \( g_1 = -e \) or \( g_2 = -e \) in (5.4.8) implies
\[ f(-g) = f(g) + f(-e) \]
for any \( g \in Q_8 \).

Setting \((g_1, g_2) = (\pm i, \pm j), (\pm j, \pm k), (\pm k, \pm i)\) in (5.4.8) (where the signs can vary independently of each other) all impose the condition
\[ s_2 = 0 \] (5.4.11)
for \( s, s_2 \) (check the case \((g_1, g_2) = (i, j)\), then use (5.4.10) to show that changing the signs don’t give new relations, then use (5.4.9) to show that one can permute using left multiplication by \( M_2 \)).

Setting \((g_1, g_2) = (\pm i, \pm j), (\pm j, \pm k), (\pm i, \pm k)\) in (5.4.8) (where the signs can vary independently of each other) all impose the condition
\[ s = s_3 \] (5.4.12)
for \( s, s_3 \) (check the case \((g_1, g_2) = (j, i)\), then use (5.4.10) to show that changing the signs don’t give new relations, then use (5.4.9) to show that one can permute using left multiplication by \( M_2 \)).

Setting \((g_1, g_2) = (\pm g, \pm g)\) for \( g = i, j, k \) (where the signs can vary independently of each other) all impose the condition
\[ s = s_2 + s_3 \] (5.4.13)
on \( s, s_2, s_3 \) (check the case \( g = i \), then use (5.4.10) to show that changing the signs don’t give new relations, then use (5.4.9) to show that one can permute using left multiplication by \( M_2 \)), but (5.4.13) is implied by (5.4.11) and (5.4.12).
These are the only relations satisfied by the \(s, s_1, s_2, s_3\). Thus we have
\[
\ker((d_1)^{M_2}) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)
\]
since there are no relations on \(s, s_1 \in \mathbb{Z}/(2)\).

An element of \((C^0(Q_8, N_1))^{M_2}\) corresponds to an element \((t, t, t) \in N_1\); since every element of \(\text{SL}_{2,3}\) fixes elements of this form (see (5.4.4)), the image of \((t, t, t)\) under \((d_0)^{M_2}\) corresponds to the function \(f : Q_8 \to N_1\) sending every element to \((0, 0, 0)\), in other words
\[
\text{im}((d_1)^{M_2}) = 0
\]
which implies
\[
(5.4.14) \quad H^0((M_2), H^1(Q_8, N_1)) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)
\]
and so
\[
(5.4.15) \quad \text{Br.} \mathcal{M}_{1,1,k} = H^2_{\text{fppf}}(\mathcal{M}_{1,1,k}, \mathbb{G}_m)[2] = \mathbb{Z}/(2)
\]
by combining (5.4.14) with (5.4.6), (5.4.5), (5.2.5), and (5.2.2). □

**Remark 5.5** (The inhomogeneous cochain complex admits a left \(G\)-action). Let \(G\) be a group, let \(H \leq G\) be a normal subgroup, and let \(M\) be a left \(G\)-module. Set \(P_i := \mathbb{Z}[H^{i+1}]\); we denote by \([h_0, \ldots, h_i]\) the canonical \(\mathbb{Z}\)-basis of \(P_i\). We view \(P_i\) as a left \(H\)-module via the diagonal action \(h \cdot [h_0, \ldots, h_i] = [hh_0, \ldots, hh_i]\); then \(P_i\) is a free left \(\mathbb{Z}[H]\)-module with basis consisting of elements of the form \([e, h_1, \ldots, h_i]\). Applying the functor \(\text{Hom}_H(-, M)\) to the bar resolution
\[
\cdots \to P_2 \to P_1 \to P_0 \to 0
\]
gives the usual homogeneous cochain complex
\[
\text{Hom}_{\mathbb{Z}[H]}(P_0, M) \xrightarrow{\delta_1} \text{Hom}_{\mathbb{Z}[H]}(P_1, M) \xrightarrow{\delta_2} \text{Hom}_{\mathbb{Z}[H]}(P_2, M) \to \cdots
\]
whose cohomology gives \(H^i(H, M)\).

We note that there is a natural left \(G\)-action on \(\text{Hom}_{\mathbb{Z}[H]}(P_i, M)\) for which the differential \(\delta_i : \text{Hom}_{\mathbb{Z}[H]}(P_i, M) \to \text{Hom}_{\mathbb{Z}[H]}(P_{i+1}, M)\) is \(G\)-linear. Namely, the action of \(g \in G\) on \(\varphi_i \in \text{Hom}_{\mathbb{Z}[H]}(P_i, M)\) is described by
\[
(g \varphi_i)([h_0, \ldots, h_i]) := g \cdot (\varphi_i([g^{-1}h_0g, \ldots, g^{-1}h_ig]))
\]
for all \(h_0, \ldots, h_i \in H\). Let
\[
C^i(H, M) := \text{Fun}(H^i, M)
\]
denote the abelian group of functions \(H^i \to M\). Via the usual abelian group isomorphism
\[
\text{Hom}_{\mathbb{Z}[H]}(P_i, M) \simeq C^i(H, M)
\]
sending \(\varphi_i \mapsto \{(h_1, \ldots, h_i) \mapsto \varphi_i(e, h_1, h_1h_2, \ldots, h_1 \cdots h_i)\}\), the abelian group \(C^i(H, M)\) inherits a left action of \(G\) described by
\[
(5.5.1) \quad (gf_i)(h_1, \ldots, h_i) = g \cdot (f_i(g^{-1}h_1g, \ldots, g^{-1}h_ig))
\]
for \(g \in G\) and \(f_i \in C^i(H, M)\). The inhomogeneous cochain complex
\[
C^0(H, M) \xrightarrow{d_0} C^1(H, M) \xrightarrow{d_1} C^2(H, M) \to \cdots
\]
is \(G\)-linear as well.

For \(f_0 \in C^0(H, M)\), we have \((d_0f_0)(h_1) = h_1 \cdot f_0(e) - f_0(e)\).
For \( f_1 \in C^1(H, M) \), we have \((d_1 f_1)(h_1, h_2) = h_1 \cdot f_1(h_2) - f_1(h_1 h_2) + f_1(h_1)\).

Let \( \Sigma := G/H \) be the quotient; then there is an induced left action of \( \Sigma \) on the cohomology \( H^1(C^*(H, M)) \). In case \( G \to \Sigma \) has a section, in which case \( G \) is the semi-direct product \( G \simeq H \rtimes \Sigma \), then this \( \Sigma \)-action coincides with the one obtained by restricting the \( G \)-action on \( C^*(H, M) \) to \( \Sigma \).

**Remark 5.6.** The arguments used in 5.3 and 5.4 are similar to those of Mathew and Stojanoska [MS16, Appendix B], who show \( H^1(GL_{2,3}, (TMF(3)_0)^\times) = \mathbb{Z}/(12) \) where \( GL_{2,3} \) acts on

\[
TMF(3)_0 = \mathbb{Z}[\frac{1}{3}, \zeta, \frac{1}{t}, \frac{1}{1-t}, \frac{1}{1+\zeta t}] / (\zeta^2 + \zeta + 1)
\]

as in [Sto14] §4.3.

**Note 5.7** (Explicit description of inhomogeneous 1-cocycles). We describe the 1-cocycles \( GL_{2,3} \to M \) obtained via the compositions \((5.4.6)\) and \((5.4.5)\). By our computation in 5.4, the 1-cocycles

\[
f_{Q_8} : Q_8 \to N_1
\]

are of the form

\[
e \mapsto (0, 0, 0) \quad -e \mapsto (s, s, s) \\
i \mapsto (s_1, 0, s) \quad -i \mapsto (s_1 + s, s, 0) \\
j \mapsto (0, s, s_1) \quad -j \mapsto (s, 0, s_1 + s) \\
k \mapsto (s, s_1, 0) \quad -k \mapsto (0, s_1 + s, s)
\]

for some \( s, s_1 \in \mathbb{Z}/(2) \). Suppose

\[
f_{SL_{2,3}} : SL_{2,3} \to N_1
\]

is a 1-cocycle such that \( f_{SL_{2,3}} \) is fixed by the action of \( M_2 \) (see \((5.5.1)\)) and which satisfies \( f_{SL_{2,3}}(g) = f_{Q_8}(g) \) for \( g \in Q_8 \). We have

\[
M_2 \cdot f_{SL_{2,3}}(M_2^{-1} \cdot g \cdot M_2) = f_{SL_{2,3}}(g)
\]

for all \( g \in SL_{2,3} \); taking \( g = M_2 \) gives \( M_2 \cdot f_{SL_{2,3}}(M_2) = f_{SL_{2,3}}(M_2) \). Taking \( g_1 = g_2 = M_2 \) in the 1-cocycle condition \((5.4.8)\) then gives \( f_{SL_{2,3}}(M_2) = 0 \). Thus we have

\[
f_{SL_{2,3}}(g \cdot M_2) = f_{SL_{2,3}}(g)
\]

for any \( g \in SL_{2,3} \), again by \((5.4.8)\).

By Shapiro’s lemma \((5.4.5)\), there is a 1-cocycle

\[
f_{GL_{2,3}} : GL_{2,3} \to Ind_{SL_{2,3}}^{GL_{2,3}}(N_1)
\]

such that precomposing with the inclusion \( SL_{2,3} \subset GL_{2,3} \) and postcomposing with the projection \( Ind_{SL_{2,3}}^{GL_{2,3}}(N_1) \to N_1 \) gives \( f_{SL_{2,3}} \). After altering \( f_{GL_{2,3}} \) by a 1-coboundary, we may assume by \Note 5.8 that \( f_{GL_{2,3}} \) is given by the formula \((5.8.1)\), namely

\[
f_{GL_{2,3}}(g \cdot M_i^j) := f_{SL_{2,3}}(M_i^j \cdot g \cdot M_i^{-j})
\]

for any \( i, j \in \{0, 1\} \) and \( g \in SL_{2,3} \). Any element \( g \in GL_{2,3} \) may be expressed in the form

\[
h \cdot M_2^j \cdot M_1^i
\]
where \( i_1 \in \{0, 1\} \) and \( i_2 \in \{0, 1, 2\} \) and \( h \in Q_8 \). We have formulas
\[
M_1 \cdot M_2^{-1} \cdot M_1 = M_2^{-1} \\
M_1 \cdot i \cdot M_1^{-1} = -i \\
M_1 \cdot j \cdot M_1^{-1} = -k \\
M_1 \cdot k \cdot M_1^{-1} = -j
\]
and so
\[
f_{GL_{2,3}}(h \cdot M_2^{i_2} \cdot M_1^{i_1})([M_2^j]) \overset{1}{=} f_{SL_{2,3}}(M_1^j \cdot h \cdot M_2^{i_2} \cdot M_1^{-j}) \\
= f_{SL_{2,3}}((M_1^j \cdot h \cdot M_1^{-j}) \cdot (M_1^i \cdot M_2^{i_2} \cdot M_1^{-j})) \\
\overset{2}{=} f_{SL_{2,3}}(M_1^j \cdot h \cdot M_1^{-j}) \\
\overset{3}{=} f_{Q_8}(M_1^j \cdot h \cdot M_1^{-j})
\]
where equality 1 is by \((5.7.2)\) and equality 2 is by \((5.7.1)\) and \((5.7.3)\) and equality 3 is since \(M_1^j \cdot h \cdot M_1^{-j} \in Q_8\) (see \((5.7.3)\)). This is summarized in \((5.7.4)\) below.
\[
f_{GL_{2,3}}(e) = (f_{Q_8}(e), f_{Q_8}(e)) = ((0, 0, 0), (0, 0, 0)) \\
f_{GL_{2,3}}(i) = (f_{Q_8}(i), f_{Q_8}(-i)) = ((s_1, 0, s), (s_1 + s, s, 0)) \\
f_{GL_{2,3}}(j) = (f_{Q_8}(j), f_{Q_8}(-k)) = ((0, s, s_1), (0, s_1 + s, s)) \\
f_{GL_{2,3}}(k) = (f_{Q_8}(k), f_{Q_8}(-j)) = ((s, s_1, 0), (s, 0, s_1 + s))
\]

Note 5.8 (The Shapiro isomorphism and inhomogeneous 1-cocycles). \[\] Let \( G \) be a group, let \( H \subseteq G \) be a normal subgroup of finite index such that the projection \( G \to G/H \) has a section \( G/H \to G \) whose image corresponds to a subgroup \( \Sigma \) of \( G \). Let \( N \) be a left \( H \)-module and let \( \text{Ind}^G_H N := \text{Hom}_{Z[H]}(Z[G], N) \) denote the associated induced left \( G \)-module. We recall that the left \( G \)-action on \( \text{Ind}^G_H N \) sends \( \varphi \mapsto g \varphi \) where \((g \varphi)(x) = \varphi(xg)\).

We describe the inverse of the Shapiro isomorphism \( H^1(G, \text{Ind}_H^G N) \to H^1(H, N) \) in terms of inhomogeneous cochains. Suppose given a function
\[
f : H \to N
\]
which satisfies
\[
f(h_1 h_2) = h_1 \cdot f(h_2) + f(h_1)
\]
for all \( h_1, h_2 \in H \). We construct a 1-cocycle
\[
s : G \to \text{Ind}^G_H(N)
\]
which restricts to \( f \), i.e. satisfies \( s(h)(1 \cdot [e]) = f(h) \) for all \( h \in H \). Note that every element of \( g \in G \) may be written uniquely in the form
\[
g = h \sigma
\]
for \( h \in H \) and \( \sigma \in \Sigma \), hence the collection \( \{[\sigma]\}_{\sigma \in \Sigma} \) forms a basis for \( Z[G] \) as a left \( Z[H] \)-module. We set
\[
s(h \sigma)([\xi]) := f(\xi h \xi^{-1})
\]
\[\text{Ehud Meir’s MathOverflow post [Mei16] was helpful in working out the details of this section.}\]
for \( h \in H \) and \( \sigma, \xi \in \Sigma \) and extend \( \mathbb{Z}[H] \)-linearly. Given \( g_1, g_2 \in G \) where \( g_i = h_i \sigma_i \) with \( h_i \in H \) and \( \sigma_i \in \Sigma \), for any \( \xi \in \Sigma \) we have

\[
s(g_1g_2)([\xi]) = s(h_1 \sigma_1 h_2 \sigma_2)([\xi])
\]

\[
= s(h_1(\sigma_1 h_2 \sigma_1^{-1}) \sigma_2)([\xi])
\]

\[
= f(\xi h_1(\sigma_1 h_2 \sigma_1^{-1}) \xi^{-1})
\]

and

\[
(g_1 \cdot s(g_2))([\xi]) = s(h_2 \sigma_2)([\xi h_1 \sigma_1])
\]

\[
= s(h_2 \sigma_2)([\xi h_1 \xi^{-1} \sigma_1])
\]

\[
= (\xi h_1 \xi^{-1}) \cdot s(h_2 \sigma_2)([\xi \sigma_1])
\]

\[
= (\xi h_1 \xi^{-1}) \cdot f((\xi \sigma_1) h_2 (\xi \sigma_1)^{-1})
\]

and

\[
s(g_1)([\xi]) = s(h_1 \sigma_1)([\xi]) = f(\xi h_1 \xi^{-1})
\]

which implies

\[
s(g_1g_2) = g_1 \cdot s(g_2) + s(g_1)
\]

by \( \mathbb{Z}[H] \)-linearity and since \( f \) is a 1-cocycle; hence \( s \) is a 1-cocycle. \( \square \)

5.9 (Proof of Theorem 1.2). Let \( k^{\text{sep}} \) be a fixed separable closure of \( k \) and let \( G_k := \text{Gal}(k^{\text{sep}}/k) \simeq \hat{\mathbb{Z}} \) be the absolute Galois group. Set \( \mathcal{M} := \mathcal{M}_{1,1,k} \) and \( \mathcal{M}^{\text{sep}} := \mathcal{M}_{1,1,k^{\text{sep}}} \). We have \( \text{Br.}\mathcal{M} = \text{Br.}\mathcal{M}^{\text{sep}} \) by Lemma 3.1. The Leray spectral sequence for the map \( \mathcal{M} \to \text{Spec} k \) is of the form

\[
E_2^{p,q} = H^p(G_k, H^q_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m)) \implies H^{p+q}_{\text{et}}(\mathcal{M}, \mathbb{G}_m)
\]

with differentials \( E_2^{p,q} \to E_2^{p+2,q-1} \). Here we have \( \Gamma(\mathcal{M}^{\text{sep}}, \mathbb{G}_m) = \Gamma(\mathcal{A}^{\text{sep}}_{k^{\text{sep}}}, \mathbb{G}_m) = (k^{\text{sep}})^{\times} \) since \( \mathcal{M}^{\text{sep}} \to \mathcal{A}^{\text{sep}}_{k^{\text{sep}}} \) is the coarse moduli space map. Since \( k \) is a finite field, we have that \( H^0_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m) \) is a torsion group. Moreover \( H^1_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m) \simeq \text{Pic}(\mathcal{M}^{\text{sep}}) \simeq \mathbb{Z}/(12) \) is a torsion group by [FO10]. Thus by e.g. [Fu11 4.3.7] or [GS06 6.1.3] we have \( E_2^{0,q} = 0 \) for \( (p,q) \in \mathbb{Z}_{\geq 2} \times \{0,1\} \). This means there is an exact sequence

\[
0 \to E_2^{1,1} \to H^1_{\text{et}}(\mathcal{M}, \mathbb{G}_m) \to E_2^{0,2} \to 0
\]

of abelian groups.

By [FO10], we have that \( \text{Pic}(\mathcal{M}^{\text{sep}}) \simeq \mathbb{Z}/(12) \) is generated by the class of the Hodge bundle; since \( G_k \) acts trivially on invariant differentials of elliptic curves \( E \to S \) where \( S \) is a \( k \)-scheme, the action of \( G_k \) on \( \text{Pic}(\mathcal{M}^{\text{sep}}) \) is trivial. Hence we have

\[
E_2^{1,1} = H^1(G_k, H^1_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m)) \overset{1}{=} \text{Hom}_{\text{cont}}(G_k, \text{Pic}(\mathcal{M}^{\text{sep}})) \overset{2}{=} \mathbb{Z}/(12)
\]

where equality 1 is by [Fu11 4.3.7] and equality 2 is since \( G_k \simeq \hat{\mathbb{Z}} \). We have

\[
E_2^{0,2} = H^0(G_k, H^2_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m)) \overset{1}{=} (\mathbb{Z}/(2))^{G_k} \overset{2}{=} \mathbb{Z}/(2)
\]

where equality 1 is by the computation for an algebraically closed field ([Theorem 1.1] and also the fact that \( H^2_{\text{et}}(\mathcal{M}^{\text{sep}}, \mathbb{G}_m) \) is a torsion group (see [AM16 Proposition 2.5 (iii)]) and

\[\text{Br.}\mathcal{M}_{1,1,k} = \mathbb{Z}/(2) \text{ FOR } k = \mathbb{F} \text{ AND char } k = 2\]
equality 2 is because any group action on the group of order 2 is necessarily trivial. Thus (5.9.1) reduces to a natural extension
\[(5.9.2) \quad 0 \to \mathbb{Z}/(12) \to \text{Br} \mathcal{M} \to \mathbb{Z}/(2) \to 0\]
and it remains to see whether (5.9.2) is split. It suffices to compute the size of \((\text{Br} \mathcal{M})[2]\), since \((\text{Br} \mathcal{M})[2]\) has 4 or 2 elements depending on whether (5.9.2) is split or not, respectively.

As in (5.2), the fppf Kummer sequence
\[(5.9.3) \quad 1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{\times 2} \mathbb{G}_m \to 1\]
gives an exact sequence of abelian groups. We compute \(H^2_{\text{fppf}}(\mathcal{M}, \mu_2)\) using the Leray spectral sequence which is of the form
\[E_2^{p,q} = H^p(G_k, H^q_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)) \Rightarrow H^{p+q}_{\text{fppf}}(\mathcal{M}, \mu_2)\]
with differentials \(E_2^{p,q} \to E_2^{p+2,q-1}\). We have
\[H^p_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2) = \begin{cases} 0 & \text{if } p = 0 \\ \mathbb{Z}/(2) & \text{if } p = 1 \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) & \text{if } p = 2 \end{cases}\]
from the fppf Kummer sequence on \(\mathcal{M}_{\text{sep}}\), where the \(p = 0\) case follows since we are in characteristic 2 and \(\Gamma(\mathcal{M}_{\text{sep}}, \mathbb{G}_m) = \Gamma(k_{\text{sep}}^1, \mathbb{G}_m) = (k_{\text{sep}}^x)\), the \(p = 1\) case is since the multiplication-by-2 map on \(\Gamma(\mathcal{M}_{\text{sep}}, \mathbb{G}_m) = (k_{\text{sep}}^x)\) is an isomorphism, and the \(p = 2\) case is by the computation in the algebraically closed case (combine (5.2.5), (5.4.5), (5.4.6), (5.4.14)).

Since \(k\) has characteristic 2, the 2-cohomological dimension of \(k\) satisfies \(\text{cd}_2(k) \leq 1\) by e.g. [GS06, 6.1.9]; hence \(E_2^{p,q} = 0\) for \(p \geq 2\) and any \(q\). Hence there is an exact sequence
\[(5.9.5) \quad 0 \to H^1(G_k, H^1_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)) \to H^2_{\text{fppf}}(\mathcal{M}, \mu_2) \to H^0(G_k, H^2_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)) \to 0\]
of abelian groups. As above, the \(G_k\)-action on \(H^1_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)\) is necessarily trivial so we have an isomorphism \(H^1(G_k, H^1_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)) \simeq \text{Hom}_{\text{cont}}(G_k, \mathbb{Z}/(2)) \simeq \mathbb{Z}/(2)\).

To describe \(H^0(G_k, H^2_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2))\), we describe the \(G_k\)-action on \(H^2_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)\). Let \(\xi \in k_{\text{sep}}\) be a fixed root of \(x^2 + x + 1\) (i.e. a primitive 3rd root of unity).

If \(\xi \in k\), then \(G_k\) acts trivially on \(H^2_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2)\); hence \(H^0(G_k, H^2_{\text{fppf}}(\mathcal{M}_{\text{sep}}, \mu_2))\) has 4 elements, hence \(H^2_{\text{fppf}}(\mathcal{M}, \mu_2)\) has 8 elements by (5.9.5), hence \((\text{Br} \mathcal{M})[2]\) has 4 elements by (5.9.4), hence \(\text{Br} \mathcal{M} \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(12)\).

Suppose \(\xi \notin k\). The \(k\)-algebra map
\[k[\mu, \omega, \frac{1}{\mu^3-1}]/(\omega^2 + \omega + 1) \to k_{\text{sep}}[\nu_1, \frac{1}{\nu_1^2-1}] \times k_{\text{sep}}[\nu_2, \frac{1}{\nu_2^2-1}]\]
sending \(\mu \mapsto (\nu_1, \nu_2)\) and \(\omega \mapsto (\xi, \xi^2)\) induces an isomorphism
\[(5.9.6) \quad k[\mu, \omega, \frac{1}{\mu^3-1}]/(\omega^2 + \omega + 1) \otimes_k k_{\text{sep}} \to k_{\text{sep}}[\nu_1, \frac{1}{\nu_1^2-1}] \times k_{\text{sep}}[\nu_2, \frac{1}{\nu_2^2-1}]\]
of \( k^{\text{sep}} \)-algebras. The inverse to (5.9.6) sends
\[
(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu) \left( \omega \otimes \frac{1}{\xi - \xi^2} + 1 \otimes \frac{\xi}{\xi - 1} \right) + f_2(\mu) \left( (-\omega) \otimes \frac{1}{\xi - \xi^2} + (-1) \otimes \frac{1}{\xi - 1} \right)
\]
for \( f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i - 1}] \).

Let
\[
\lambda \in G_k
\]
be an automorphism of \( k^{\text{sep}} \) such that \( \lambda(\xi) = \xi^2 \). Then the \( k \)-algebra automorphism of \( k^{\text{sep}}[\nu_1, \frac{1}{\nu_1 - 1}] \times k^{\text{sep}}[\nu_2, \frac{1}{\nu_2 - 1}] \) induced by (5.9.6) sends \((\nu_1, 0) \mapsto (0, \nu_2)\) and \((\nu_1, 0) \mapsto (\nu_1, 0)\) and \((\xi, 0) \mapsto (0, \xi^2)\) and \((\xi, 0) \mapsto (\xi^2, 0)\). We see that the action of \( \lambda \) on \( M \) (see (5.3.2)) is given by (5.9.7)
\[
(5.9.7) \quad \begin{pmatrix} \nu_1 - 1 & \nu_1 - \xi \\ \nu_2 - \xi & \nu_2 - 1 \end{pmatrix} \lambda \begin{pmatrix} \nu_1 - 1 & \nu_1 - \xi \\ \nu_2 - \xi & \nu_2 - 1 \end{pmatrix} = \begin{pmatrix} \nu_2 - 1 & \nu_2 - \xi \\ \nu_1 - \xi & \nu_1 - 1 \end{pmatrix}
\]
A computation with (5.9.7) and (5.3.6) shows that
\[
\lambda g \lambda^{-1} \cdot m = g \cdot m
\]
for any \( m \in M \) and \( g \in \text{GL}_{2,3} \).

Let \( f_{\text{GL}_{2,3}} : \text{GL}_{2,3} \to M \) be an inhomogeneous 1-cocycle as in Note 5.7. Multiplying the 1-cocycle condition (5.4.8) on the left by \( \lambda \) gives
\[
\lambda \cdot f_{\text{GL}_{2,3}}(g_1 \cdot g_2) = \lambda g_1 \cdot f_{\text{GL}_{2,3}}(g_2) + \lambda \cdot f_{\text{GL}_{2,3}}(g_1)
\]
where equality 1 follows from (5.9.8) Hence the function \( \lambda \cdot f_{\text{GL}_{2,3}} : \text{GL}_{2,3} \to M \) sending \( g \mapsto \lambda \cdot f_{\text{GL}_{2,3}}(g) \) is a 1-cocycle as well. Using (5.9.7) and (5.7.4) we have that
\[
(5.9.9) \quad \begin{align*}
(\lambda \cdot f_{\text{GL}_{2,3}})(e) &= ((0, 0, 0), (0, 0, 0)) \\
(\lambda \cdot f_{\text{GL}_{2,3}})(i) &= ((s_1 + s, 0, s), (s_1, s, 0)) \\
(\lambda \cdot f_{\text{GL}_{2,3}})(j) &= ((0, s, s_1 + s), (0, s_1, s)) \\
(\lambda \cdot f_{\text{GL}_{2,3}})(k) &= ((s, s_1 + s, 0), (s, 0, s_1))
\end{align*}
\]
and so
\[
(5.9.10) \quad \begin{align*}
f_{\text{GL}_{2,3}}(e) - (\lambda \cdot f_{\text{GL}_{2,3}})(e) &= ((0, 0, 0), (0, 0, 0)) \\
f_{\text{GL}_{2,3}}(i) - (\lambda \cdot f_{\text{GL}_{2,3}})(i) &= ((s, 0, 0), (s, 0, 0)) \\
f_{\text{GL}_{2,3}}(j) - (\lambda \cdot f_{\text{GL}_{2,3}})(j) &= ((0, 0, s), (0, s, 0)) \\
f_{\text{GL}_{2,3}}(k) - (\lambda \cdot f_{\text{GL}_{2,3}})(k) &= ((0, s, 0), (0, 0, s))
\end{align*}
\]
for the same \( s, s_1 \in \mathbb{Z}/(2) \) as in (5.7.4)

Suppose \( f_{\text{GL}_{2,3}} \) and \( \lambda \cdot f_{\text{GL}_{2,3}} \) differ by a 1-coboundary, in other words there exists an element
\[
m := ((m_1^1, m_2^1, m_3^1), (m_1^2, m_2^2, m_3^2)) \in M
\]
such that
\[
(5.9.11) \quad f_{\text{GL}_{2,3}}(g) - (\lambda \cdot f_{\text{GL}_{2,3}})(g) = g \cdot m - m
\]
for all \( g \in \text{GL}_2 \). By (5.9.10) taking \( g = M_2 \) in (5.9.11) gives \( m^i = m_1^i = m_2^i = m_3^i \) for \( i = 1, 2 \); then taking \( g = M_1 \) gives \( m^1 = m_2^1 \); then taking \( g = i \) gives \( m = 0 \). We see that \( f_{\text{GL}_2} \) and \( \lambda \cdot f_{\text{GL}_2} \) differ by a 1-coboundary if and only if \( s = 0 \).

Hence we have that \( H^0(\text{Gr}, H^2_{\text{fppf}}(\mathcal{M}^\text{sep}, \mu_2)) \simeq \mathbb{Z}/(2) \), hence \( H^2_{\text{fppf}}(\mathcal{M}, \mu_2) \) has 4 elements by (5.9.5) hence (Br \( \mathcal{M} \))[2] has 2 elements by (5.9.4) hence \( \text{Br} \mathcal{M} \simeq \mathbb{Z}/(24) \). □

**Appendix A. The Weierstrass and Hesse presentations of \([\Gamma(3)]\)**

The purpose of this section is to prove [Proposition A.4] below, which we could not find proved in the literature. For completeness of exposition, we first recall the definition of a full level \( N \) structure on an elliptic curve \( E/S \).

**A.1 (Full level \( N \) structure).** [KM85, Ch. 3] Let \( N \) be a positive integer. We define \([\Gamma(N)]\) to be the category of pairs

\[
(\mathcal{E}/\mathcal{S}, \xi)
\]

where

\[
\mathcal{E}/\mathcal{S} = (f: \mathcal{E} \to \mathcal{S}, e: \mathcal{S} \to \mathcal{E})
\]

is an elliptic curve and

\[
\xi: (\mathbb{Z}/(N))_S^2 \to \mathcal{E}
\]

is a morphism of \( S \)-group schemes inducing an isomorphism \((\mathbb{Z}/(N))_S^2 \simeq \mathcal{E}[N]\). A morphism

\[
(\mathcal{E}_1/\mathcal{S}_1, \xi_1) \to (\mathcal{E}_2/\mathcal{S}_2, \xi_2)
\]

is a pair

\[
(\alpha: \mathcal{E}_1 \to \mathcal{E}_2, \beta: \mathcal{S}_1 \to \mathcal{S}_2)
\]

of morphisms of schemes such that the diagram

\[
\begin{array}{ccc}
(\mathbb{Z}/(N))_S^2 & \xrightarrow{(\mathbb{Z}/(3))_S^2} & (\mathbb{Z}/(N))_{S_1}^2 \\
\downarrow \alpha \downarrow & \xrightarrow{\text{id} \times \beta} & \downarrow f_1 \downarrow \\
S_1 & \xrightarrow{f_2} & S_2
\end{array}
\]

(A.1.1)

commutes, where the morphism \( \text{id} \times \beta \) is the one induced by the identity on \((\mathbb{Z}/(3))_{S_1}^2\) and \( \beta \), and such that \( \alpha \) induces an isomorphism of \( S_1 \)-group schemes \( \mathcal{E}_1 \simeq S_1 \times_{\beta, S_2} \mathcal{E}_2 \).

There is a functor

\[
[\Gamma(N)] \to \mathcal{M}_{1,1,\mathbb{Z}}
\]

sending \((\mathcal{E}/\mathcal{S}, \xi) \mapsto E/S\) on objects and \((\alpha, \beta) \mapsto (\alpha, \beta)\) on morphisms. If \( E/S \) admits a full level \( N \) structure, then \( N \) is invertible on \( S \) by [KM85 2.3.2], hence the above functor factors through \( \mathcal{M}_{1,1,\mathbb{Z}}[1] \). If \( N \geq 3 \), then for any scheme \( \mathcal{S} \) the fiber category \([\Gamma(N)](\mathcal{S})\) is equivalent to a set by [KM85 2.7.2], so \([\Gamma(N)]\) is fibered in sets over the category of schemes.
A.2 (The $\text{GL}_2(\mathbb{Z}/(N))$-action on $[\Gamma(N)]$). Fix a scheme $S$. For any element
\[
\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}
\]
in $\text{GL}_2(\mathbb{Z}/(N))$, let
\[
\varphi_\sigma : (\mathbb{Z}/(N))^2_S \to (\mathbb{Z}/(N))^2_S
\]
be the $S$-group scheme automorphism of $(\mathbb{Z}/(N))^2_S$ corresponding to the abelian group homomorphism $(\mathbb{Z}/(N))^2 \to (\mathbb{Z}/(N))^2$ defined by
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}x_1 + \sigma_{12}x_2 \\ \sigma_{21}x_1 + \sigma_{22}x_2 \end{bmatrix}
\]
for $x_1, x_2 \in \mathbb{Z}/(N)$, i.e. acting by multiplication on the left on $(\mathbb{Z}/(N))^2$ viewed as vertical vectors. We have
\[
\varphi_{\sigma_1}\varphi_{\sigma_2} = \varphi_{\sigma_1\sigma_2}
\]
for $\sigma_1, \sigma_2 \in \text{GL}_2(\mathbb{Z}/(N))$.

Fix an object $(E/S, \xi) \in [\Gamma(N)](E/S)$; then $(E/S, \xi \circ \varphi_\sigma)$ is another object of $[\Gamma(N)](E/S)$, i.e. corresponds to another full level $N$ structure on $E/S$. This implies that there is a natural action of $\text{GL}_2(\mathbb{Z}/(N))$ on each fiber category $[\Gamma(N)](E/S)$; the action is a right action since it is defined by precomposition.

**Theorem A.3.** [KM85, 4.7.2] If $N \geq 3$, the category $[\Gamma(N)]$ is representable by a smooth affine curve $Y(N)$ over $\mathbb{Z}[\frac{1}{N}]$.

We are primarily interested in the case $N = 3$. The 3-torsion points of an elliptic curve correspond to its inflection points (also “flex points”). In [KM85 (2.2.11)] it is shown that $Y(3) \simeq \text{Spec } A_W$ where
\[
A_W := \mathbb{Z}[\frac{1}{3}, B, C, \frac{1}{C}, \frac{1}{a_3}, \frac{1}{a_3^2 - 27a_3}]/(B^3 - (B + C)^3)
\]
and the universal elliptic curve over $A_W$ with full level 3 structure is the pair
\[
\begin{aligned}
E_W := \text{Proj } A_W[X, Y, Z]/(Y^2Z + a_1XYZ + a_3YZ^2 = X^3) \\
[0 : 0 : 1], [C : B + C : 1]
\end{aligned}
\]
where
\[
\begin{aligned}
a_1 &= 3C - 1 \\
a_3 &= -3C^2 - B - 3BC
\end{aligned}
\]
The formulas \((A.3.2)\) and \((A.3.3)\) are obtained by imposing the condition that the line $Y = X + BZ$ is a flex tangent to $E_W$ at $[C : B + C : 1]$. The ring $A_W$ is isomorphic to $\text{TMF}(3)_0$ \((5.6.1)\) with mutually inverse ring isomorphisms $\text{TMF}(3)_0 \to A_W$ and $A_W \to \text{TMF}(3)_0$ given by $(\zeta, t) \mapsto (\frac{B}{3C}, \frac{1}{3C})$ and $(B, C) \mapsto (\frac{1}{3(\zeta - 1)^2}, \frac{1}{3})$ respectively.

In this paper, however, we use the “Hesse presentation” of $Y(3)$ as in [FO10, 5.1]. The following is claimed without proof in the Introduction to [DR73] and [Har11 5.2.30].
Proposition A.4. There is an isomorphism $Y(3) \cong \text{Spec } A_H$ where
$$A_H := \mathbb{Z}[\frac{1}{3}, \mu, \omega, \frac{1}{\mu-1}]/(\omega^2 + \omega + 1)$$
and the universal elliptic curve over $A_H$ with full level 3 structure is the pair
$$\begin{align*}
E_H := & \text{Proj } A_H[X, Y, Z]/(X^3 + Y^3 + Z^3 = 3\mu XYZ) \\
& [-1 : 0 : 1], [1 : -\omega : 0]
\end{align*}$$
with identity section $[1 : -1 : 0]$.

The explicit $\mathbb{Z}[\frac{1}{3}]$-algebra isomorphisms $A_H \to A_W$ and $A_W \to A_H$ are given in (A.8.7) and (A.8.8) respectively.

A.5. By [Sma01, §4], the group law of an elliptic curve $E = \text{Proj } A[X, Y, Z]/(X^3 + Y^3 + Z^3 = 3\mu XYZ)$ in Hessian form over a ring $A$ is as follows. If $P = [x : y : z]$, then $2P = [x' : y' : z']$ where
$$\begin{align*}
x' &= y(z^3 - x^3) \\
y' &= x(y^3 - z^3) \\
z' &= z(x^3 - y^3)
\end{align*}$$
and if $P_i = [x_i : y_i : z_i]$ are points of $E_H$ for $i = 1, 2, 3$ satisfying $P_1 + P_2 = P_3$, then
$$\begin{align*}
x_3 &= x_2y_1^2z_2 - x_1y_2^2z_1 \\
y_3 &= x_1y_2^2z_2 - x_2y_1^2z_1 \\
z_3 &= x_2y_2z_1^2 - x_1y_1z_2^2
\end{align*}$$
which only makes sense if $P_1 \neq P_2$.

Using the above formulas, we may check that the full level 3 structure $\xi_H : (\mathbb{Z}/(3))^2_{A_H} \to E_H$ is given by the table (A.5.1)
$$\begin{align*}
\xi_H & \left( \begin{array}{ccc}(0, 0) & (1, 0) & (2, 0) \\
(0, 1) & (1, 1) & (2, 1) \\
(0, 2) & (1, 2) & (2, 2) \end{array} \right) = \left( \begin{array}{ccc}[1 : -1 : 0] & [1 : 0 : 1] & [0 : 1 : -1] \\
[1 : -\omega : 0] & [-\omega : 0 : 1] & [0 : 1 : -\omega] \\
[1 : -\omega^2 : 0] & [-\omega^2 : 0 : 1] & [0 : 1 : -\omega^2] \end{array} \right)
\end{align*}$$

The Hesse presentation (A.4.1) is sometimes easier to work with than the Weierstrass presentation (A.3.1) since the equation of the universal elliptic curve is symmetric in $X, Y, Z$, which means that there is also considerable symmetry in the 3-torsion points (A.5.1).

A.6. We describe the $\text{GL}_2(\mathbb{Z}/(3))$-action on $E_H/A_H$. Set $S_H := \text{Spec } A_H$. The functor $[\Gamma(3)]$ being representable by $S_H$ means explicitly that for any $\mathbb{Z}[\frac{1}{3}]$-scheme $T$ and object $(E/T, \xi) \in ([\Gamma(3)])(T)$, there exists a unique pair $(\alpha, \beta)$ of morphisms of schemes $\alpha : E \to E_H$ and $\beta : T \to S_H$ such that the diagram
commutes and induces an isomorphism of $T$-group schemes $E \simeq T \times_{\beta} S_H E_H$ as in \cite{A.1.1}.

As in \cite{A.2}, for every $\sigma \in \text{GL}_2(\mathbb{Z}/(3))$, let $\varphi_\sigma$ be the $S_H$-automorphism of $(\mathbb{Z}/(3))^2_{S_H}$ induced by $\sigma$; then precomposition $\xi_H \varphi_\sigma$ defines another full level 3 structure on $E_H/S_H$. Taking $T = S_H$ and $\xi = \xi_H \varphi_\sigma$ above, there is a unique pair $(\alpha_\sigma, \beta_\sigma)$ of morphisms of schemes $\alpha_\sigma : E_H \to E_H$ and $\beta_\sigma : S_H \to S_H$ such that the diagram

\[
\begin{array}{c}
E & \xrightarrow{\alpha} & E_H \\
\downarrow{\xi} & & \downarrow{\xi_H} \\
(\mathbb{Z}/(3))^2_T & \xrightarrow{id \times \beta} & (\mathbb{Z}/(3))^2_{S_H} \\
\downarrow{f_T} & & \downarrow{f_{S_H}} \\
T & \xrightarrow{\beta} & S_H \\
\end{array}
\]

commutes and induces an isomorphism of $S_H$-group schemes $E_H \simeq S_H \times_{\beta_\sigma} S_H E_H$. Given two elements $\sigma_1, \sigma_2 \in \text{GL}_2(\mathbb{Z}/(3))$, we have a commutative diagram

\[
\begin{array}{c}
E_H & \xrightarrow{\alpha_{\sigma_1}} & E_H & \xrightarrow{\alpha_{\sigma_2}} & E_H \\
\downarrow{\xi_H \varphi_{\sigma_1} \varphi_{\sigma_2}} & & \downarrow{\xi_H} & & \downarrow{f_{S_H}} \\
(\mathbb{Z}/(3))^2_{S_H} & \xrightarrow{id \times \beta_{\sigma_1}} & (\mathbb{Z}/(3))^2_{S_H} & \xrightarrow{id \times \beta_{\sigma_2}} & (\mathbb{Z}/(3))^2_{S_H} \\
\downarrow{f_{S_H}} & & \downarrow{f_{S_H}} & & \downarrow{f_{S_H}} \\
S_H & \xrightarrow{\beta_{\sigma_1}} & S_H & \xrightarrow{\beta_{\sigma_2}} & S_H \\
\end{array}
\]

which implies

$$\beta_{\sigma_2} \beta_{\sigma_1} = \beta_{\sigma_1} \sigma_2$$

since $\varphi_{\sigma_1} \sigma_2 = \varphi_{\sigma_1} \varphi_{\sigma_2}$ (see \cite{A.2}). Thus the assignment

(A.6.1) \hspace{1cm} \sigma \mapsto \beta_\sigma

defines a right action of $\text{GL}_2(\mathbb{Z}/(3))$ on the scheme $S_H$. 
In terms of the generators

\[ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

of \( \text{GL}_2(\mathbb{Z}/(3)) \), the action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on \( E_H/A_H \) is as follows. (We refer to (A.5.1) for the additive structure on \( E_H[3] \).)

1. For \( \sigma = M_1 \), the new level 3 structure \( \xi_{H\varphi_{M_1}} \) is

\[ \left[ [-1 : 0 : 1] \ [1 : -\omega : 0] \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left[ [-1 : 0 : 1] \ [1 : -\omega^2 : 0] \right] \]

and the scheme morphisms \( \alpha_{M_1} : E_H \to E_H \) and \( \beta_{M_1} : S_H \to S_H \) correspond to the ring homomorphisms sending

\[
\begin{cases} 
(X, Y, Z) & \mapsto (X, Y, Z) \\
(\mu, \omega^2) & \mapsto (\mu, \omega)
\end{cases}
\]

respectively.

2. For \( \sigma = M_2 \), the new level 3 structure \( \xi_{H\varphi_{M_2}} \) is

\[ \left[ [-1 : 0 : 1] \ [1 : -\omega : 0] \right] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \left[ [-\omega^2 : 0 : 1] \ [1 : -\omega : 0] \right] \]

and the scheme morphisms \( \alpha_{M_2} : E_H \to E_H \) and \( \beta_{M_2} : S_H \to S_H \) correspond to the ring homomorphisms sending

\[
\begin{cases} 
(X, Y, \omega^2 Z) & \mapsto (X, Y, Z) \\
(\omega \mu, \omega) & \mapsto (\mu, \omega)
\end{cases}
\]

respectively.

3. For \( \sigma = i \), the new level 3 structure \( \xi_{H\varphi_i} \) is

\[ \left[ [-1 : 0 : 1] \ [1 : -\omega : 0] \right] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \left[ [1 : -\omega : 0] \ [0 : 1 : -1] \right] \]

and the scheme morphisms \( \alpha_i : E_H \to E_H \) and \( \beta_i : S_H \to S_H \) correspond to the ring homomorphisms sending

\[
\begin{cases} 
(\omega X + \omega^2 Y + Z, \omega^2 X + \omega Y + Z, X + Y + Z) & \mapsto (X, Y, Z) \\
(\mu + 2 \mu^2, \omega) & \mapsto (\mu, \omega)
\end{cases}
\]

respectively.

Remark A.7. According to our convention, the action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on the fiber category \( [\Gamma(3)](E_H/\text{Spec } A_H) \) is by precomposition, hence the action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on pairs of points on the right hand side of (A.5.1) is a right action; thus the induced action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on the scheme \( \text{Spec } A_H \) is a right action (as described in (A.6.1)) and the corresponding action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on the coordinate ring \( A_H \) is a left action.
A.8 (Proof of Proposition A.4). In fact, it turns out that the identities
\[(A.8.1) \quad a_1^3 - 27a_3 = (3C + 9B - 1)^3\]
\[(A.8.2) \quad a_3 = B(6C + 9B - 1)\]
hold in \(A_W\) which yields a simpler description
\[A_W \simeq \mathbb{Z}[\frac{1}{3}, B, C, \frac{1}{\mathfrak{c}C + 9B - 1}, \frac{1}{6C + 9B - 1}]/(C^2 + 3CB + 3B^2)\]
of \(A_W\). (For (A.8.1) write out \(a_3^3 - 27a_3\) in terms of \(B, C\) and notice that it is of the form \(9C + 27B - 1\) plus higher order terms; then check that the naive guess works. To see (A.8.2) substitute \(C^2 = -3CB - 3B^2\) into (A.8.3).)

We follow the argument of [AD09, 2.1]; see also [Con96, §1.4.1, §1.4.2]. Working “generically”, we will assume that \(a_1\) is a unit to obtain the coordinate change formula (A.8.9) then observe that it applies also to the case when \(a_1\) is not a unit. Starting with
\[(A.8.3) \quad Y_1Z_1(Y_1 + a_1X_1 + a_3Z_1) = X_1^3\]
we define \(X_2, Y_2, Z_2\) by the system
\[
\begin{bmatrix}
X_1 \\
Y_1 \\
Z_1
\end{bmatrix} =
\begin{bmatrix}
1 & u^2 & u^3 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
X_2 \\
Y_2 \\
Z_2
\end{bmatrix}
\]
where \(u = a_1/3\) and substitute into (A.8.3) to get
\[(A.8.4) \quad Y_2Z_2(Y_2 + 3X_2 + \frac{27a_3}{a_1}Z_2) = X_2^3.\]
We define \(X_3, Y_3, Z_3\) by the system
\[
\begin{bmatrix}
1 & 1 & \frac{27a_3}{a_1} \\
1 & 1 & \frac{27a_3}{a_1}
\end{bmatrix}
\begin{bmatrix}
X_2 \\
Y_2 \\
Z_2
\end{bmatrix} =
\begin{bmatrix}
\omega & \omega^2 & \omega^3 \\
\omega^2 & \omega & 1
\end{bmatrix}
\begin{bmatrix}
X_3 \\
Y_3 \\
Z_3
\end{bmatrix}
\]
where \(\omega = \frac{C + B}{B}\) and substitute into (A.8.4) to get
\[
(\omega X_3 + \omega^2 Y_3 - Z_3)(\omega^2 X_3 + \omega Y_3 - Z_3)(-X_3 - Y_3 + Z_3) = \frac{27a_3}{a_1}Z_3^3
\]
or equivalently
\[(A.8.5) \quad X_3^3 + Y_3^3 + \frac{27a_3}{a_1}Z_3^3 = -3X_3Y_3Z_3.\]
We know that the coefficient of \(Z_3^3\) in (A.8.5) is a cube (A.8.1) so we normalize by defining \(X_4, Y_4, Z_4\) by the system
\[
\begin{bmatrix}
X_3 \\
Y_3 \\
Z_3
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & -a_1 \\
& & \frac{-a_1}{3C + 9B - 1}
\end{bmatrix}
\begin{bmatrix}
X_4 \\
Y_4 \\
Z_4
\end{bmatrix}
\]
and substitute into (A.8.5) to get
\[(A.8.6) \quad X_4^3 + Y_4^3 + Z_4^3 = 3\frac{a_1}{3C + 9B - 1}X_4Y_4Z_4.\]

\(^2\)Since 3 is invertible, if \(x\) is a root of the polynomial \(T^2 + 3T + 3\) then \(x + 1\) is a root of the polynomial \(T^2 + T + 1\), thus it is natural to take \(\frac{C + B}{B}\) as our \(\omega\).
To summarize the above, there is a ring homomorphism \( \varphi_{21} : A_H \to A_W \) sending
\[
\begin{align*}
\mu &\mapsto \frac{3C - 1}{3C + 9B - 1} \\
\omega &\mapsto \frac{C + B}{B}
\end{align*}
\] (A.8.7)
and solving for \( B, C \) in terms of \( \mu, \omega \) implies that the inverse \( \varphi_{12} : A_W \to A_H \) sends
\[
\begin{align*}
B &\mapsto \frac{\mu - 1}{3(\omega + 2)(\mu - \omega)} \\
C &\mapsto \frac{1}{3}(\omega - 1)(\mu - 1)
\end{align*}
\] (A.8.8)
where \( \omega + 2 \) is a unit of \( A_H \) since \( (\omega + 2)(\omega - 1) = -3 \) and \( \mu - \omega \) is a unit of \( A_H \) since \( \mu^3 - 1 = (\mu - 1)(\mu - \omega)(\mu - \omega^2) \). We may check that the product
\[
\begin{bmatrix}
u^2 \\ u^3 \\ 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \frac{27a_3}{a_1^2} \\
1 & \frac{\omega^2}{a_3} & \frac{\omega}{a_3} \\
1 & \frac{\omega^2}{a_3} & \frac{\omega}{a_3}
\end{bmatrix}^{-1}
\begin{bmatrix}
\omega & \omega^2 \\ \omega & 1 \\ \frac{1}{3} - \frac{a_1}{3C + 9B - 1}
\end{bmatrix}
\] is “projectively equivalent” to the matrix
\[
X := 
\begin{bmatrix}
0 & 0 & -\frac{3}{3C + 9B - 1} \\
\omega & \omega^2 & \frac{3a}{3C + 9B - 1} \\
\frac{\omega^2}{a_3} & \frac{\omega}{a_3} & \frac{3a}{a_3(3C + 9B - 1)}
\end{bmatrix}
\] (A.8.9)
whose determinant is a unit of \( A_W \). Given a section \([s_X : s_Y : s_Z]\) of (A.8.3), the corresponding section of (A.8.6) is \( X^{-1} \cdot [s_X : s_Y : s_Z]^T \) where
\[
X^{-1} = 
\begin{bmatrix}
-\frac{a_1}{3} & B & -\frac{9CB - 18B^2 - C}{3} \\
-\frac{a_1}{3} & \frac{-B}{C + 3B} & -\frac{9CB - 9B^2 + C + 3B}{3} \\
-\frac{3C - 9B + 1}{3} & 0 & \frac{3}{3}
\end{bmatrix}
\]
The above implies that the sections
\[
[0 : 1 : 0], [0 : 0 : 1], [C : B + C : 1]
\] of (A.8.3) (i.e. the identity section and ordered basis for the 3-torsion) correspond to the sections
(A.8.10) \([1 : -\omega : 0], [1 : -\omega^2 : 0], [-1 : 0 : 1]\)
of (A.8.6). We may apply an automorphism of the pair \((A_H, E_H/A_H) \in \mathcal{M}_{1,1,Z}\) of the form (A.6(2)) (for \( Y \) instead of \( Z \)) to (A.8.10) to get
(A.8.11) \([1 : -1 : 0], [1 : -\omega : 0], [-1 : 0 : 1]\)
and using the fact that there is a simply transitive action of \( \text{GL}_2(\mathbb{Z}/(3)) \) on the set of ordered bases of the 3-torsion in \( E_H/A_H \), we may switch the second and third sections of (A.8.11) to obtain
(A.8.12) \([1 : -1 : 0], [-1 : 0 : 1], [1 : -\omega : 0]\)
as desired. \(\square\)
Remark A.9. For (A.8.1) see also Stojanoska’s derivation [Sto14, §4.1].

Remark A.10. There are coordinate change formulas in [Sma01, §3] transforming a Weierstrass equation into Hesse normal form, but there it is assumed that the base ring is a finite field $\mathbb{F}_q$ where $q \equiv 2 \pmod{3}$, in order to take cube roots of $a_1^3 - 27a_3$, but from this description it is not clear that the cube root is an algebraic function. As shown in (A.8.1) it turns out that in fact $a_1^3 - 27a_3$ is a cube in the ring $A_W$. One suspects that this is the case after tracing through the proof of [AD09, 2.1] and arriving at the equation $x^3 + y^3 + 27a_3 - a_1^3 z^3 = 3xyz$, in which case we know that $\frac{27a_3 - a_1^3}{a_1^3}$ is a cube by Lemma A.11.

Lemma A.11. Let $k$ be a field of characteristic not 3, and let

\begin{align*}
(A.11.1) \quad & x^3 + y^3 + \beta = 3xy \\
(A.11.2) \quad & ax + by + c = 0
\end{align*}

be a curve in $\mathbb{A}_k^2$. Suppose that

\begin{align*}
(A.11.2) & ax + by + c = 0
\end{align*}

is the tangent line to a flex point of $E$ and suppose that $a^3 \neq b^3$. Then $\beta$ is a cube in $k$.

Proof. If $a = 0$, then $b \neq 0$ and substituting $y = -\frac{c}{b}$ into (A.11.1) and rearranging gives $x^3 + \frac{3}{b}x - \left(\frac{c}{b}\right)^3 + \beta = 0$ which by assumption is of the form $(x + \ell)^3$ for some $\ell \in k$. Comparing coefficients, we have $\ell = 0$ and so $\beta = \left(\frac{c}{b}\right)^3$.

By symmetry we may assume that $a, b \neq 0$. By scaling (A.11.2) we may assume that $b = -1$. Substituting $y = ax + c$ into $E$ gives

\begin{align*}
(a^3 + 1)x^3 + 3(a)(ac - 1)x^2 + 3(c)(ac - 1)x + (c^3 + \beta)
\end{align*}

and dividing by the leading coefficient gives

\begin{align*}
x^3 + 3\left(\frac{a(ac - 1)}{a^3 + 1}\right)x^2 + 3\left(\frac{c(ac - 1)}{a^3 + 1}\right)x + \left(\frac{c^3 + \beta}{a^3 + 1}\right)
\end{align*}

and comparing this to

\begin{align*}
x^3 + 3\ell x^2 + 3\ell^2 x + \ell^3
\end{align*}

gives either $ac - 1 = 0$ in which case $c^3 + \beta = 0$ as well (so that $\beta = (-1/a)^3 = (-c)^3$), otherwise if $ac - 1 \neq 0$ then

\begin{align*}
\frac{c}{a} &= a \left(\frac{ac - 1}{a^3 + 1}\right)
\end{align*}

which implies $c = -a^2$ so that the original equation of the tangent line is $y = ax - a^2$. Substituting this back into $E$ gives $\beta = (-a)^3$. \hfill $\square$

APPENDIX B. HIGHER DIRECT IMAGES OF SHEAVES ON CLASSIFYING STACKS OF DISCRETE GROUPS

The material in this section is standard and we claim no originality.

For a category $C$, we denote by $\text{PSh}(C)$ (resp. $\text{PAb}(C)$) the category of presheaves (resp. abelian presheaves) on $C$. If $C$ is a site, we denote by $\text{Sh}(C)$ (resp. $\text{Ab}(C)$) the category of sheaves (resp. abelian sheaves) on $C$. 
Let \( C \) be a site, let \( G \) be a finite (discrete) group, let \( B G_C \) be the classifying stack associated to \( G \) over \( C \). Let 
\[
\pi : B G_C \to C
\]
be the projection and let 
\[
\varphi : C \to B G_C
\]
be the canonical section of \( \pi \). We view any fibered category \( p : \mathcal{F} \to C \) as a site via the Grothendieck topology inherited from \( C \) via \( p \).

**Lemma B.1.** In the setup above, for any abelian sheaf \( \mathcal{F} \in \text{Ab}(B G_C) \) the higher pushforward \( R^i \pi_* \mathcal{F} \) is naturally isomorphic to the sheaf associated to the presheaf whose value on an object \( U \in C \) is \( H^i(G, \Gamma(U, \varphi^* \mathcal{F})) \).

**Proof.** Let \( PG_C \) denote the category whose objects are the objects of \( C \) and where a morphism \( X_1 \to X_2 \) in \( PG_C \) is a pair \((\varphi, g)\) where \( \varphi \in \text{Mor}_C(X_1, X_2) \) and \( g \in G \). (In other words, there is an equivalence of categories \( PG_C \simeq C \times [*/G] \) where \([*/G]\) is the category with one object \(*\) and where \( \text{Hom}_{[*/G]}(*, *) \) is isomorphic to \( G \).) The fibered category \( PG_C \) is a (separated) prestack whose associated stack is \( B G_C \), and the inclusion \( PG_C \to B G_C \) induces an equivalence of topoi \( \text{Sh}(PG_C) \simeq \text{Sh}(B G_C) \). Hence in the statement of the lemma we may replace \( B G_C \) by \( PG_C \) where by abuse of notation we also denote 
\[
\pi : PG_C \to C
\]
the projection morphism. Since sheafification is an exact functor, the diagram 
\[
\begin{array}{ccc}
\text{PAb}(PG_C) & \xrightarrow{\pi^\text{pre}} & \text{PAb}(C) \\
\text{sh} \downarrow & & \downarrow \text{sh} \\
\text{Ab}(PG_C) & \xrightarrow{\pi_*} & \text{Ab}(C)
\end{array}
\]
is (2-)commutative. For the same reason, we have a natural isomorphism 
\[
(R \pi^\text{pre}_*(\mathcal{F}))^\text{sh} \simeq R \pi_*(\mathcal{F}^\text{sh})
\]
in \( \text{D}^+(\text{Ab}(C)) \) for any abelian presheaf \( \mathcal{F} \in \text{PAb}(PG_C) \). Presheaves on \( PG_C \) correspond to presheaves \( \mathcal{F} \) on \( C \) equipped with a \( G \)-action, and under this identification \( \pi^\text{pre}_*(\mathcal{F}) = \mathcal{F}^G \) where \( \Gamma(U, \mathcal{F}^G) := (\Gamma(U, \mathcal{F}))^G \) for all \( U \in C \). Let \( \mathcal{F} \in \text{Ab}(PG_C) \) be an abelian sheaf, and let 
\[
\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots
\]
be a resolution of \( \mathcal{F} \) by injective abelian presheaves \( \mathcal{I}^i \in \text{PAb}(PG_C) \). Then \( R \pi^\text{pre}_*(\mathcal{F}) \) is isomorphic to 
\[
(\mathcal{I}^\bullet)^G = \{ (\mathcal{I}^0)^G \to (\mathcal{I}^1)^G \to (\mathcal{I}^2)^G \to \cdots \}
\]
in \( \text{D}^+(\text{PAb}(C)) \), and \( \Gamma(U, R \pi^\text{pre}_*(\mathcal{F})) \) is isomorphic to 
\[
\Gamma(U, (\mathcal{I}^\bullet)^G) = \{ (\Gamma(U, \mathcal{I}^0))^G \to (\Gamma(U, \mathcal{I}^1))^G \to (\Gamma(U, \mathcal{I}^2))^G \to \cdots \}
\]
in \( \text{D}^+(\text{PAb}(C)) \). Furthermore \( \Gamma(U, \mathcal{I}^i) \simeq (i_U)^* \mathcal{I}^i \) is an injective \( G \)-module for all \( i \) by Lemma B.2, thus we have an isomorphism 
\[
h^i(\Gamma(U, (\mathcal{I}^\bullet)^G)) \simeq H^i(G, \Gamma(U, \mathcal{F}))
\]
Lemma B.2. Let $C$ be a category, let $U \in C$ be an object, let $A_{C,U}$ denote the full subcategory of $C$ containing exactly $U$, and let $i_U : A_{C,U} \to C$ denote the inclusion. The inverse image functor $(i_U)^* : \text{PAb}(C) \to \text{PAb}(A_{C,U})$ preserves injectives.

Proof. The functor $(i_U)^* : \text{PAb}(PG_C) \to \text{PAb}(A_{C,U})$ has an exact left adjoint, namely the “extension by zero” functor $i_U \cdot : \text{PAb}(A_{C,U}) \to \text{PAb}(PG_C)$ which sends $M \in \text{PAb}(A_{C,U})$ to the abelian presheaf $i_U \cdot (M)$ where $\Gamma(V, i_U \cdot (M)) = M$ if $V = U$ and 0 otherwise (with the only nontrivial restriction morphisms being those corresponding to the endomorphisms of $U$).

Appendix C. Computation using Magma

We compute $H^1(\text{GL}_2(\mathbb{Z}/(3)), M)$ in 5.4 using MAGMA [BCP97]. Here $G$ is defined as the subgroup of $\text{GL}_2(\mathbb{Z}/(3))$ generated by the matrices in (5.3.6), but the specified matrices constitute a generating set so in fact $G = \text{GL}_2(\mathbb{Z}/(3))$. The group $G$ acts on the abelian group $M = (\mathbb{Z}/2) \oplus 6$ by the three specified elements of $\text{Mat}_{6 \times 6}(\mathbb{Z})$, where each $x \in M$ is viewed as a horizontal vector and each $6 \times 6$ matrix $A$ acts on $M$ by right multiplication $x \mapsto x \cdot A$. The last line computes $H^1(G, (\mathbb{Z}/(2)) \oplus 6)$.  

G := MatrixGroup< 2, FiniteField(3) | [ 1,0 , -1,1 ] , [ 0,-1 , 1,0] , [ 1,0 , 0,-1 ] >;

mats := [
  Matrix(Integers(), 6 , 6 , [ 
    0, 0, 1, 0, 0, 0, 
    1, 0, 0, 0, 0, 0, 
    0, 1, 0, 0, 0, 0, 
    0, 0, 0, 0, 1, 0, 
    0, 0, 0, 0, 0, 1, 
    0, 0, 0, 1, 0, 0 ]),
  Matrix(Integers(), 6 , 6 , [ 
    1, 0, 0, 0, 0, 0, 
    1, 0, 1, 0, 0, 0, 
    0, 0, 0, 0, 0, 0, 
    0, 0, 1, 0, 0, 0, 
    0, 0, 0, 1, 0, 1, 
    0, 0, 0, 1, 1, 0 ]),
  Matrix(Integers(), 6 , 6 , [ 
    0, 0, 0, 1, 0, 0, 
    0, 0, 0, 0, 1, 0, 
    0, 0, 0, 0, 0, 1, 
    1, 0, 0, 0, 0, 0, 
    0, 1, 0, 0, 0, 0, 
    0, 0, 1, 0, 0, 0, 
    0, 0, 1, 0, 0, 0 ])];

CM := CohomologyModule(G,[2,2,2,2,2],mats);
CohomologyGroup(CM,1);
References

[ACV03] D. Abramovich, A. Corti, and A. Vistoli. Twisted bundles and admissible covers. Communications in Algebra, 31(8):3547–3618, 2003.

[AD09] M. Artebani and I. V. Dolgachev. The Hesse pencil of plane cubic curves. LEnseignement Mathématique, 55(3):235–273, 2009. https://arxiv.org/abs/math/0611590

[AM16] Benjamin Antieau and Lennart Meier. The Brauer group of the moduli stack of elliptic curves. arXiv, 2016. preprint, http://arxiv.org/abs/1608.00851

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma Algebra System I: The User Language. Journal of Symbolic Computation, 24(34):235 – 265, 1997.

[Con96] I. Connell. Elliptic Curve Handbook. on-line notes, McGill University, 2 edition, 1996. available at http://www.math.mcgill.ca/connell/public/ECH1/

[Del75] P. Deligne. Courbes elliptiques: formulaire d’apres J. Tate. In Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), volume 476 of Springer Lecture Notes in Mathematics, pages 53–73, 1975.

[Del77] P. Deligne. Cohomologie étale, Seminaire de Géométrie Algébrique du Bois-Marie (SGA 4 1/2), avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. Lecture Notes in Mathematics, 569, 1977.

[dJ03] A. J. de Jong. A result of Gabber. preprint, http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf, 2003.

[DR73] Pierre Deligne and Michael Rapoport. Les schémas de modules de courbes elliptiques. In Modular functions of one variable, II, pages 143–316, Springer, 1973.

[FO10] William Fulton and Martin Olsson. The Picard group of $\mathcal{M}_{1,1}$. Algebra & Number Theory, 4(1):87–104, 2010.

[Fu11] Lei Fu. Etale Cohomology Theory, volume 13 of Nankai Tracts in Mathematics. World Scientific Publishing, 2011.

[Gab78] Ofer Gabber. Some theorems on Azumaya algebras. PhD thesis, Harvard, 1978. published in Groupes de Brauer, Lecture Notes in Math., vol. 844, Springer-Verlag, Berlin and New York, 1981, pp. 129-209.

[Gir71] Jean Giraud. Cohomologie non abélienne, volume 179 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, 1971.

[Gro68a] A. Grothendieck. Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses. In A. Grothendieck and N. H. Kuiper, editors, Dix Exposés sur la Cohomologie des Schémas, volume 3 of Advanced Studies in Pure Mathematics, chapter IV, pages 46–66. North-Holland Publishing, Amsterdam, 1968.

[Gro68b] A. Grothendieck. Le groupe de Brauer. III. Exemples et compléments. In A. Grothendieck and N. H. Kuiper, editors, Dix Exposés sur la Cohomologie des Schémas, volume 3 of Advanced Studies in Pure Mathematics, chapter VI, pages 88–188. North-Holland Publishing, Amsterdam, 1968.

[GS06] Phillipe Gille and Tamás Szamuely. Central Simple Algebras and Galois Cohomology, volume 101 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.

[Har11] G. Harder. Lectures on Algebraic Geometry I. Vieweg+ Teubner. Heidelberg, Germany, 2011.

[KM85] N. M. Katz and B. Mazur. Arithmetic Moduli of Elliptic Curves. Princeton University Press, Princeton, NJ, 1985.

[KV04] A. Kresch and A. Vistoli. On coverings of Deligne–Mumford stacks and surjectivity of the Brauer map. Bulletin of the London Mathematical Society, 36(02):188–192, 2004.

[Lie08] Max Lieblich. Twisted sheaves and the period-index problem. Compositio Mathematica, 144(01):1–31, 2008.

[Mei16] Ehud Meir. Shapiro’s lemma in the language of group extensions. MathOverflow, 2016. URL http://mathoverflow.net/q/256208 (version: 2016-12-02).

[Mil80] J. S. Milne. Etale Cohomology. Princeton University Press, Princeton, New Jersey, 1980.

[MS16] A. Mathew and V. Stojanoska. The Picard group of topological modular forms via descent theory. Geometry & Topology, 20(6):3133–3217, 2016.

[Mum65] D. Mumford. Picard groups of moduli problems. In Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), pages 33–81, 1965.
Br. \( \mathcal{M}_{1,k} = \mathbb{Z}/(2) \) FOR \( k = \overline{\mathbb{F}} \) AND char \( k = 2 \)

[Ols05] M. Olsson. On proper coverings of Artin stacks. *Advances in Mathematics*, 198(1):93–106, 2005.

[Ols16] M. Olsson. *Algebraic Spaces and Stacks*, volume 62 of *Colloquium Publications*. American Mathematical Society, 2016. [http://bookstore.ams.org/coll-62/](http://bookstore.ams.org/coll-62/)

[Sil09] J. H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2009.

[Sma01] N. P. Smart. The Hessian Form of an Elliptic Curve. In *International Workshop on Cryptographic Hardware and Embedded Systems*, pages 118–125. Springer, 2001.

[Sta18] The Stacks Project Authors. *Stacks Project*. [http://stacks.math.columbia.edu](http://stacks.math.columbia.edu), 2018.

[Sto14] V. Stojanoska. Calculating descent for 2-primary topological modular forms. *An Alpine Expedition through Algebraic Topology*, 617:241, 2014.

E-mail address: shinms@math.berkeley.edu

URL: [http://math.berkeley.edu/~shinms](http://math.berkeley.edu/~shinms)