Extended sigma-model in nontrivially deformed field-antifield formalism

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Abstract

We propose an action for the extended sigma-models in the most general setting of the kinetic term allowed in the nontrivially deformed field-antifield formalism. We show that the classical motion equations do naturally take their desired canonical form.

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1 Introduction and summary

The field-antifield formalism [1, 2] is known as the most powerful method for covariant quantization of general gauge theories. The method operates with two fundamental objects: the antibracket and the odd nilpotent second-order Delta operator. In the standard version of the method, it is formulated in the Darboux coordinates. Various local deformations of the Delta operator [3] and antibracket [4, 5, 6, 7] elucidate the geometry underlying the field-antifield formalism [8]. The deformed Delta operator and antibracket do play the central role in formulating the path integral version of the nontrivially deformed field-antifield formalism [9]. It has been shown recently [10] that the nontrivially deformed formalism does exist, as a consistent approach, at the classical level. In the present article, we confirm the latter result as applied to the important case of the topological sigma-models [8, 11, 12, 13, 14, 15, 16, 17]. We propose an action of the deformed sigma-models in the most general setting for the kinetic term allowed in the formalism, and show that the motion equations do naturally take their desired canonical form. We define the kinetic part of the action directly in terms of the zero modes for the τ-extended Euler operator $N_\tau$, that generalizes naturally the standard power counting operator. We show that these zero modes, although they are nontrivial functions of the original phase variables, do satisfy very simple antibracket algebra, so that they generate effectively their own antisymplectic structure, very similar to the original one. The latter circumstance makes it possible to deduce the desired canonical form of the motion equations in a very simple and natural way. Moreover, we have deduced the so-called canonical "Ward Identities" the complete action does satisfy to at the general (arbitrary) trajectory in the antisymplectic phase space. These identities, in their own turn, are necessary when deducing the functional master equation for the complete action. Thus, we have got to a considerable new progress in our study of the canonical structure in topological field theories.

2 Elements of the nontrivially deformed field-antifield formalism

Let \( \{Z^A, A = 1, ..., 2M\} \), \( \varepsilon(Z^A) = \varepsilon_A \), be original field-antifield phase variables with constant invertible antisymplectic metric \( E_{AB} \), \( \varepsilon(E_{AB}) = \varepsilon_A + \varepsilon_B + 1 \),

\[
E_{AB} = -E_{BA}(-1)^{\varepsilon_A\varepsilon_B} = \text{const}(Z). \tag{2.1}
\]

Thereby, we have defined the antibracket,

\[
(F, G) = F \delta_A E^{AB} \delta_B G, \quad E^{AB} E_{BC} = \delta^A_C, \tag{2.2}
\]

\[
\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad E^{AB} = -E^{BA}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \tag{2.3}
\]

with all standard properties. Then, the Fermionic Delta - operator, \( \Delta, \varepsilon(\Delta) = 1 \), is given by

\[
\Delta = \frac{1}{2}(-1)^{\varepsilon_A} \partial_A E^{AB} \partial_B, \quad \Delta^2 = 0. \tag{2.4}
\]
Vice versa, the antibracket (2.2) can be deduced from the Delta - operator (2.4),

\[(F, G) = (-1)^{ep} [[\Delta, F], G] \cdot 1. \tag{2.5}\]

Now, let \(\{t, \theta\}, \varepsilon(t) = 0, \varepsilon(\theta) = 1\), be a pair of new antisymplectic variables. Let \((F, G)_\tau\) and \(\Delta_\tau\) be the extended counterparts to \((F, G)\) and \(\Delta\), respectively,

\[(F, G)_\tau = t^2(F, G) + (N_\tau F)\partial_\theta G - F\partial_\theta N_\tau G, \tag{2.6}\]

\[\Delta_\tau = t^2 \Delta + N_\tau \partial_\theta, \quad N_\tau = N + t\partial_t, \quad N = N^A\partial_A, \tag{2.7}\]

\[\Delta_\tau^2 = 0, \quad [\Delta, N] = 2\Delta, \quad [\Delta_\tau, N_\tau] = 0. \tag{2.8}\]

Vice versa, the extended antibracket \((F, G)_\tau\) can be deduced from the extended Delta-operator, \(\Delta_\tau\),

\[(F, G)_\tau = (-1)^{ep} [[\Delta_\tau, F], G] \cdot 1. \tag{2.9}\]

Now, let the antisymplectic phase variables \(\{Z^A; t, \theta\}\) be functions of \(2n\) Bosonic variables \(u^a\) and \(2n\) Fermionic variables \(\xi^a\),

\[Z^A = Z^A(u, \xi), \quad t = t(u, \xi), \quad \theta = \theta(u, \xi). \tag{2.10}\]

Let \(D, \varepsilon(D) = 1\), be the De Rham differential in the variables \(u, \xi\),

\[D = \xi^a\partial_a, \quad \partial_a = \frac{\partial}{\partial u^a}, \quad D^2 = 0. \tag{2.11}\]

Further, let \(\kappa\) be a Bosonic deformation parameter, and \(\Delta_{\tau^*}\) be the extended trivially deformed Delta operator

\[\Delta_{\tau^*} = \Delta_\tau(1 - \kappa N_\tau)^{-1}, \quad \Delta_{\tau^*}^2 = 0, \tag{2.12}\]

and let \(T\) be the corresponding trivial deformation operator,

\[T = 1 + \kappa \theta \Delta_{\tau^*}, \quad T^{-1} = 1 - \kappa \theta \Delta_\tau, \tag{2.13}\]

so that the trivially deformed extended antibracket and Delta operator rewrite as

\[(F, G)_{\tau^*} = T^{-1}(TF, TG)_\tau = (F, G)_\tau + (\kappa N_\tau F)(\Delta_{\tau^*} G) + (\Delta_{\tau^*} F)(\kappa N_\tau G)(-1)^{ep}, \tag{2.14}\]

\[\Delta_{\tau^*} = T^{-1}\Delta_\tau T. \tag{2.15}\]


3 Trivially deformed sigma-model

The extended trivially deformed sigma-model is formulated via the action

$$\Sigma = \int [du][d\xi] \mathcal{L},$$

(3.1)

where the Lagrangian $\mathcal{L}$ is defined via $S$ satisfying the extended trivially deformed master equation,

$$(S, S)_{\tau^*} = 0,$$

(3.2)

or equivalently,

$$(TS, TS)_{\tau} = 0.$$  (3.3)

One should seek for a solution to the equation (3.2)/(3.3) in the form

$$S = \sum_{k=-2}^{\infty} S(\langle k \mid 0) t^k + \theta \sum_{k=1}^{\infty} S(\langle k \mid 1) t^k,$$

(3.4)

with the component $S(-2\mid 0) = S^*$ identified with the nontrivially deformed $S^*$,

$$(S^*, S^*)_{\tau} = 0,$$

(3.5)

where the nontrivially deformed antibracket is given by

$$(F, G)_{\tau} = (F, G) + (\kappa(N - 2)F)(\Delta, G)) + (\Delta, F)(\kappa(N - 2)G)(-1)^{\varepsilon_F}.$$  (3.6)

$$\Delta_{\tau} = \Delta(1 - \kappa(N - 2))^{-1}. $$  (3.7)

Let us consider our new Lagrangian for the deformed sigma-model in the most general setting,

$$\mathcal{L} = \frac{1}{2} \tilde{Z}^A E_{AB} D \tilde{Z}^B (-1)^{\varepsilon_B} + \frac{1}{2}(\theta D \ln t + \ln t D \theta) + T S,$$

(3.8)

where

$$\tilde{Z}^A = \exp\{- (\ln t)N\} Z^A, \quad N_{\tau} \tilde{Z}^A = 0,$$

(3.9)

is the zero mode for the $N_{\tau}$ operator. By making a variation $\delta \tilde{Z}^C$ in (3.8), we find

$$\partial_C \tilde{Z}^A E_{AB} D \tilde{Z}^B (-1)^{\varepsilon_B} + \partial_C(T S) = 0.$$  (3.10)

By multiplying the equation (3.10) by the coefficients $N^C$ from the left, we get

$$-(t \partial_t \tilde{Z}^A) E_{AB} D \tilde{Z}^B (-1)^{\varepsilon_B} + N(T S) = 0.$$  (3.11)
On the other hand, by making a variation $\delta \ln t$ in (3.8), we have
\[
(t\partial_t \bar{Z}^A) E_{AB} D\bar{Z}^B (-1)^{\varepsilon_B} + t\partial_t (TS) + D\theta = 0,
\] (3.12)
It follows from (3.11), (3.12) that
\[
D\theta + N_\tau (TS) = 0,
\] (3.13)
or, equivalently, in terms of the extended $\tau$-antibracket,
\[
D\theta + (TS, \theta)_\tau = 0.
\] (3.14)
That is exactly the desired canonical motion equation for the variable $\theta$. The two other canonical motion equations do follow from (3.8) in the usual way, as well,
\[
D \ln t + \partial_\theta (TS) = 0,
\] (3.15)
or in its canonical form,
\[
D \ln t + (TS, \ln t)_\tau = 0,
\] (3.16)

The complete set of the canonical equations (3.14), (3.16), (3.17) tells us that the action (3.8) yields the correct general canonical description to the deformed sigma-model at the classical level.

For the sake of completeness, an explicit derivation of the canonical form (3.17) directly from the original one (3.10) is given below. As the operator $(N_\tau - 2)$ does differentiate the antibracket (due to the second in (2.8)), we have
\[
(N_\tau + 2)(\bar{Z}^A, \bar{Z}^B) = 0, \quad (\bar{Z}^A, \bar{Z}^B)|_{t=1} = E^{AB}.
\] (3.18)
It follows from (3.18) that, within the class of regular functions of $\ln t$,
\[
(\bar{Z}^A, \bar{Z}^B) = t^{-2} \exp\{-(\ln t)N\} E^{AB} = t^{-2} E^{AB}.
\] (3.19)
The latter rewrites in the explicit form,
\[
\bar{Z}^A \partial_C E^{CD} \partial_D \bar{Z}^B = t^{-2} E^{AB},
\] (3.20)
which implies in turn,
\[
(\partial_A \bar{Z}^C) E_{CD}(\bar{Z}^D \partial_B) = t^{-2} E_{AB}.
\] (3.21)
Also, we have

$$D\ddot{Z}^A = (DZ^B - S\overset{\leftarrow}{\partial}_\theta NZ^B)\overset{\rightarrow}{\partial}_B Z^A,$$

(3.22)

where we have used (3.15). By inserting (3.22) into (3.10), and using (3.21), we get

$$DZ^A + t^2(TS, Z^A) - S\overset{\leftarrow}{\partial}_\theta NZ^A = 0.$$  

(3.23)

In terms of the extended antibracket (2.6), the equation (3.23) takes immediately its desired canonical form (3.17).

Although we have shown that all the motion equations generated by the action (3.1)/(3.8) have the canonical form, that is not the case for functional derivatives of the action taken at the general trajectory in the phase space. Instead, the functional derivatives do satisfy the so-called canonical "Ward Identities",

$$\frac{\delta}{\delta \theta} \Sigma = D \ln t + (TS, \ln t)_{\tau},$$

(3.24)

$$\left[t \frac{\delta}{\delta t} + (NZ^A) \frac{\delta}{\delta Z^A}\right] \Sigma = D\theta + (TS, \theta)_{\tau},$$

(3.25)

$$\left[t^2E^{AB} \frac{\delta}{\delta Z^B} + (NZ^A) \frac{\delta}{\delta \theta}\right] \Sigma = [DZ^A + (TS, Z^A)_{\tau}] (-1)^{\varepsilon_A}.$$  

(3.26)

In the right-hand sides in these relations, one can recognize the canonical form of the left-hand sides of the motion equations (3.14), (3.16), (3.17). Now, let $[F,G]_{\tau}$ be the $\tau$ - extended antibracket in the space of functionals, generated by the ultralocal $\tau$ - extended antibracket $(F,G)_{\tau}$,

$$[F,G]_{\tau} = \int [du][d\xi]F \left[\frac{\delta}{\delta Z^A} t^2E^{AB} \frac{\delta}{\delta Z^B} + \frac{\delta}{\delta Z^A} (NZ^A) \frac{\delta}{\delta \theta} - \frac{\delta}{\delta \theta} (NZ^A) \frac{\delta}{\delta Z^A} + \frac{\delta}{\delta \ln t} \frac{\delta}{\delta \theta} - \frac{\delta}{\delta \theta} \frac{\delta}{\delta \ln t} \right] G.$$  

(3.27)

Due to the canonical "Ward Identities" (3.24), (3.25), (3.26), the action $\Sigma$ does satisfy the functional master equation in terms of the functional $\tau$ - antibracket (3.27),

$$\frac{1}{2} [\Sigma, \Sigma]_{\tau} = \int [du][d\xi] \left[ DL + \frac{1}{2} (S,S)_{\tau} \right] = 0.$$  

(3.28)

In terms of the functional antibracket (3.27), the relations (3.24), (3.25), (3.26) rewrite as

$$[\Sigma, \Gamma]_{\tau} = \nabla \Gamma, \quad \nabla = D + \text{ad}_\tau(TS), \quad \Gamma = \{ \ln t, \theta, Z^A \}.$$  

(3.29)
Due to the Jacobi identity for the functional antibracket, together with the functional master equation (3.28), it follows from the equations (3.29) that the compatibility relations hold

\[ [\Sigma, \nabla \Gamma]_\tau = 0. \]  

(3.30)

The first equality in (3.28) holds due to the following integral identity,

\[ \int [du][d\xi] (\nabla \Gamma^\alpha)(\nabla \Gamma^\beta) \omega_{\beta\alpha}(-1)^{\varepsilon_\alpha} = 0, \]

(3.31)

where

\[ \omega^{\alpha\beta} = (\Gamma^\alpha, \Gamma^\beta)_\tau, \]  

(3.32)

while \( \omega_{\alpha\beta} \) is an inverse to (3.32).

Notice that all the above reasoning did use no further restrictions to the Euler operator \( N \). However, it follows from the second in (2.8), that

\[ N = N_0 + 2\text{ad}(F), \quad N_0 = Z^A \partial_A, \quad \varepsilon(F) = 1, \quad \partial_A \Delta F = 0. \]  

(3.33)

The most general form allowed for \( F \) is

\[ 2F = Z^A F_{AB} Z^B + \Delta Y, \quad \partial_C F_{AB} = 0, \]

(3.34)

\[ \varepsilon(F_{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad \varepsilon(Y) = 0, \]

(3.35)

\[ F_{AB} = F_{BA}(-1)^{\varepsilon_A \varepsilon_B}. \]  

(3.36)

As the operator \( (N_0 - 2) \) does differentiate the antibracket, we have

\[ [N_0, \text{ad}(F)] = \text{ad}((N_0 - 2)F). \]  

(3.37)

In the sense of (3.37), the simplest possibility in (3.34) is

\[ Y = 0, \quad 2F = Z^A F_{AB} Z^B, \]  

(3.38)

which we will assume from now on. It follows from (3.37), that, in the case (3.38),

\[ [N_0, \text{ad}(F)] = 0, \]  

(3.39)

which implies for the zero modes

\[ Z^A = \exp\{-(\ln t)2\text{ad}(F)\} \exp\{-(\ln t)N_0\} Z^A = S^A_B t^{-1} Z^B, \]  

(3.40)

where \( S^A_B \) is an antisymplectic matrix given by

\[ S^A_B = (\exp\{((\ln t)G)\})^A_B, \quad G^A_B = 2E^{AC} F_{CB}, \]  

(3.41)

so that we have,

\[ (-1)^{\varepsilon_A+1} S^A_C E_{AB} S^B_D = E_{CD}. \]  

(3.42)

Notice that, in the case (3.38), one can deduce (3.17) from (3.10) even more explicitly by making use of the representation (3.40) for the zero modes.
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