Integrated-Quantile-Based Estimation for First-Price Auction Models

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This article considers nonparametric estimation of first-price auction models under the monotonicity restriction on the bidding strategy. Based on an integrated-quantile representation of the first-order condition, we propose a tuning-parameter-free estimator for the valuation quantile function. We establish its cube-root-n consistency and asymptotic distribution under weaker smoothness assumptions than those typically assumed in the empirical literature. If the latter are true, we also provide a trimming-free smoothed estimator and show that it is asymptotically normal and achieves the optimal rate of Guerre, Perrigne, and Vuong (2000). We illustrate our method using Monte Carlo simulations and an empirical study of the California highway procurement auctions. Supplementary materials for this article are available online.

KEY WORDS: First price auctions; Monotone bidding strategy; Nonparametric estimation; Tuning-parameter-free.

1. INTRODUCTION

Since the seminal work of Guerre, Perrigne, and Vuong (2000, GPV hereafter), the nonparametric estimation of auction models has received enormous attention from both the perspectives of econometric analysis and empirical applications. In this article, we revisit the first-price auction models and propose a novel estimation procedure for the valuation quantile function. Our approach is appealing both computationally and theoretically. We first construct a quantile estimator that is tuning-parameter-free and robust in the sense it is consistent under weaker smoothness assumptions than typically imposed in the literature (details later). Whenever the typical smoothness assumptions are satisfied, we can construct a trimming-free and asymptotically normal second step estimator that achieves the optimal rate of GPV. Furthermore, our estimator explicitly incorporates the restriction of the monotone bidding strategy and is monotone in finite samples, which is important for empirical work but not ensured by most of the existing estimators.

To better illustrate the features of our estimator, we begin by reviewing existing approaches in the literature. We focus on the baseline case of homogeneous auctions and will show it can be extended to incorporate auction specific characteristics in Section 2.3. We consider the standard GPV setup of independent private value (IPV) first price auction. Their novel approach is to transform the first-order condition for optimal bids and express a bidder’s value as an explicit function of the submitted bid, the probability density function (PDF) and cumulative distribution function (CDF) of bids:

\[ v = s^{-1}(b) \equiv b + \frac{1}{I-1} \frac{G(b)}{g(b)}, \]

where \( b \) is the bid, \( I \) is the number of bidders, and \( G(\cdot) \) and \( g(\cdot) \) are the distribution and density of bids, respectively. A two-step estimation method follows from this observation: first, construct a pseudo value for each bid and then apply kernel density estimation to the sample of pseudo values. GPV establishes the consistency of their estimator and the optimal rate.

Based on the insight of Haile, Hong, and Shum (2003), who considered a quantile-based-test for the existence of common values, Marmer and Shneyerov (2012, MS hereafter) first proposed to estimate the valuation distribution based on the quantile representation of the first-order condition, that is, when the equilibrium bidding strategy is strictly monotone, valuation quantile function \( Q_v(\cdot) \) can be expressed as

\[ Q_v(\alpha) = Q_b(\alpha) + \frac{1}{I-1} \frac{\alpha}{g(Q_b(\alpha))}, \quad 0 \leq \alpha \leq 1, \tag{2} \]

where \( Q_b(\cdot) \) is the bid quantile function. Note that the right-hand side must be strictly increasing in \( \alpha \), too. MS proposed to first estimate \( Q_v(\cdot) \) using plug-in estimators for \( g(\cdot) \) and \( Q_b(\cdot) \) and subsequently estimate the valuation density using the relationship \( f(v) = 1/Q_v(Q_v^{-1}(v)) \). MS show that their estimator is asymptotically normal and achieves the optimal rate of GPV. Guerre and Sabbah (2012, GS) observed that the second term on the right-hand side of Equation (2) is a known linear function of \( \alpha \) multiplied by the quantile derivative and proposed an optimal local polynomial quantile estimator. Gimenes and Guerre (2013) further explored the observation made by GS and proposed an augmented-quantile regression to overcome the difficulty of incorporating many covariates.

In both estimators of GPV and MS, the bid density \( g(\cdot) \) appears in the denominator of the first step estimation; in MS, the derivative of the bid quantile also appears in the denominator of the second step. In practice, trimming near the boundaries is needed but can be troublesome as it is well known that there is no generic guidance. GS does not require trimming but a choice of a bandwidth is needed. In addition, all quantile estimators discussed above may not satisfy the monotonicity restriction imposed by the model.
In this article, we propose to use the integrated quantile representation of the first-order condition to construct our estimator. We use \( V(\beta) \) to denote the integrated quantile function of valuation. When the bidding strategy is strictly monotone, there is

\[
V(\beta) \equiv \int_{0}^{\beta} Q_\alpha(a)\,da = \frac{1 - 2}{\beta - 1} \int_{0}^{\beta} Q_\alpha(a)\,da + \frac{1}{\beta - 1} Q_\alpha(\beta), \quad 0 \leq \beta \leq 1. (3)
\]

Note that since \( Q_\alpha \) is strictly increasing, \( V \) is necessarily strictly convex and identical to its greatest convex minorant (g.c.m.).

Our estimator imposes the monotonicity by taking the g.c.m. of the integrated valuation quantile function. The g.c.m. of \( V_\alpha(\cdot) \) is easy to compute since it is piecewise linear. Indeed, satisfying monotonicity in finite samples is a desirable feature of a quantile function estimator in general; see discussions in Chernozhukov, Fernandez-Val, and Galichon (2010). Chernozhukov, Fernandez-Val, and Galichon (2010) proposed a “rearrangement” approach to achieve the monotonicity. We take the g.c.m. approach on the integrate-quantile function in our context because it not only delivers the monotonicity, but also circumvents the necessity of estimating the bid density function in the denominator.

Our estimator is constructed using order statistics of the bids. Indeed, using order statistics is not uncommon in the literature of nonparametric estimation of auction models (see, e.g., Athey and Haile 2007). Recently, Menzel and Morganti (2013) discussed estimation of value distribution based on the distributions of order statistics. They show that the mapping between distribution of order statistics and valuation distribution is in general non-Lipschitz continuous and established optimal rate for varies of parameters of interest. Our main motivation is to provide computationally easy estimators for the classical IPV setup of GPV with all bids being observed, which is, as mentioned by Menzel and Morganti (2013), a scenario for which the irregularity of inverting order statistics distribution does not rise. The reason that our first estimator converges at an irregular cube-root-\( n \) rate is that we impose weaker smoothness assumption on the valuation distribution, rather than the non-Lipschitz continuous feature of the mapping.

We illustrate our method using the California Highway Procurement auction dataset. In practice, it is common that researchers observe auction-specific characteristics. It is worth noting that our method applies naturally if the observed auction-specific characteristics are discrete-valued (or discretization of continuous variables) by conditioning on realizations. The estimator will then be interpreted as conditional valuation quantiles on observed auction characteristics. When the observed auction-specific characteristics are continuous, GPV and MS propose to estimate the conditional valuation density by the kernel method, which suffers the “curse of dimensionality” when the covariates are high dimensional. In Section 2.3, we show our estimation procedure can also take GS’s estimator for bids quantile function as an input and deliver a consistent and monotone conditional valuation quantile estimator for each realization of the continuous covariates. Finally, we can also use the homogenization method proposed by Haile, Hong, and Shum (2003) and apply our estimation methods to the homogenized bids. The homogenization approach requires additional additive separability structure on how valuation depends on observed characteristics. As a result, it is easier to compute and has a faster convergence rate.

The rest of the article is organized as follows. We lay out the model and propose our estimator in Section 2. We examine the performance of our estimator in Section 3. Section 4 is the empirical illustration. We conclude the article in Section 5.

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1See Hickman and Hubbard (2014) for a modified version of the GPV estimator which replaces trimming with boundary correction.

2In general, the first price auction model is not identified if there is unobserved heterogeneity across auctions, see Armstrong (2013b).
2. MODEL AND MAIN RESULTS

We consider the first-price sealed-bid auction model with independent private values. A single and indivisible object is auctioned. We make the following assumptions.

**Assumption 1.** There are \( L \rightarrow \infty \) identical auctions, and for each auction, there are \( I \) symmetric and risk neutral bidders. Their private values are iid draws from a common distribution \( F(\cdot) \).

Let the total number of bids be \( n = LI \). Asymptotics are on the number of auctions, that is, \( L \rightarrow \infty \). The assumption that number of bidders \( I \) is constant across auctions is just for simplifying notation; our analysis can be easily extended to conditional on \( I \).

**Assumption 2.** \( F(\cdot) \) is continuously differentiable over its compact support \([v, \bar{v}]\). There exists \( \lambda > 0 \) such that \( \inf_{v \in [v, \bar{v}]} F(v) \geq \lambda > 0 \).

Assumption 2 only requires \( F(\cdot) \) is continuously differentiable, which is weaker than the twice continuously differentiability, as assumed in the literature, for example, GPV and MS. It is well known that the equilibrium strategy is

\[
b = s(v|F, I) = v - \frac{1}{F(v)^{n-1}} \int_0^v F(x)^{1-1} dx.
\]

GPV show that the first-order condition can be written as Equation (3). We do not prove it here due to space constraints.

Now let us first propose a tuning-parameter-free estimator for the valuation quantile function. Let \( b(\cdot) \) be the \( \ell \)th order statistic of a sample of bids \( \{b_i\}_{i=1}^n \). Employing Equation (3), we construct a raw estimator \( V_n(\cdot) \) for \( V(\cdot) \) as follows. Let \( V_n(0) = 0 \). For \( \alpha \in \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\} \),

\[
V_n(\alpha) = \frac{I - 2}{n(I - 1)} \sum_{i=1}^n b(i) + \frac{1}{I - 1} \alpha b(n).
\]

For \( \alpha \in \left(\frac{j}{n}, \frac{j+1}{n}\right) \), \( j = 1, \ldots, n \), define

\[
V_n(\alpha) = (j - \alpha n)V_n \left(\frac{j - 1}{n}\right) + (\alpha n - j + 1)V_n \left(\frac{j}{n}\right).
\]

Note \( V_n(\cdot) \) may not be convex in finite samples. Let \( \hat{V}(\cdot) \) be the g.c.m. of \( V_n(\cdot) \). To obtain a quantile estimator which respects the monotonicity property, we consider to use the left-derivative of \( \hat{V}(\cdot) \). Define \( \hat{Q}_n(0) = b(1) \) and for \( \alpha \in \left(\frac{j}{n}, \frac{j+1}{n}\right) \), \( j = 1, \ldots, n \),

\[
\hat{Q}_n(\alpha) = n \left\{ \hat{V} \left(\frac{j}{n}\right) - \hat{V} \left(\frac{j - 1}{n}\right) \right\}.
\]

By definition, \( \hat{Q}_n(\cdot) \) is a left-continuous and weakly increasing step function.

Constructing the g.c.m. \( \hat{V}(\cdot) \) for a piecewise linear function \( V_n(\cdot) \) is computationally easy. While many algorithms are proposed, the most widely used one is the pooled adjacent violators algorithm (PAVA, see, e.g., Robertson et al. 1988; Groeneboom, Jongbloed, and Wellner 2014). We can envision \( \hat{V}(\cdot) \) as a taut string tied to the leftmost point \((0, 0)\) and pulled up and under the graph of \( V_n(\cdot) \), ending at the last point \((1, V_n(1))\). See Section 6 for details.

**Theorem 1.** Suppose Assumptions 1 and 2 are satisfied at a given \( \alpha_0 \in (0, 1) \), then

\[
n^\gamma \left( \hat{Q}_n(\alpha_0) - Q_v(\alpha_0) \right) \xrightarrow{d} C(\alpha_0) \argmax \{B(t) - t^2\},
\]

where \( C(\alpha_0) \) is a constant that depends on \( \alpha_0 \) and \( B \) is a two-sided Brownian motion process.

**Proof.** See Section 6.

We have a few comments on Theorem 1. First, \( C(\alpha_0) \) depends on \( \alpha_0, g, \) and \( Q_v(\cdot) \) and is estimable (detailed expression in Section 6). To conduct inference on \( Q_v(\alpha_0) \), one can obtain the critical values by estimating \( C(\alpha_0) \) and simulating the one-dimensional Brownian motion \( B \), which is straightforward to implement, and an alternative way is subsampling whose validity follows straightforwardly from Theorem 1. Second, the \( n^{1/3} \)-consistency of our quantile estimator is obtained under weak assumptions on value distribution \( F(\cdot) \) and without choosing any tuning parameters. It is slower than the optimal rate of \( n^{2/5} \) when \( F(\cdot) \) is twice continuously differentiable, as established in GPV. This is similar to the well-known results in the literature on isotonic estimation: without imposing additional smoothness assumptions on the model primitives and without introducing smoothing, one can at most get cube-root-\( n \) rate. Third, Theorem 1 shows that our estimator is consistent at quantile level \( \alpha \in [\epsilon, 1 - \epsilon] \) for any fixed \( \epsilon > 0 \). In addition, our estimator is super-consistent at \( \alpha_0 = 0 \) by construction. To see this, note that in equilibrium, the bidder with the lowest valuation bids his/her own valuation, which implies that \( \hat{Q}_n(0) = b(1) = \hat{Q}_n(0) = Q_v(0) \) at a super-consistent rate under current assumptions. However, our estimator need not be consistent at quantile levels at the right boundary 1. This is because the construction of the g.c.m. starts from the left boundary 0 and tends to use one-sided information as it approaches to the right boundary 1. In the context of isotonic regression, a common way to correct the bias is to use the “penalized cusum diagram,” as suggested by Groeneboom et al. (2013). Since our main target is imposing the monotonicity constraint and using fewer tuning parameters, we leave this issue for future research.

In practice, it is often useful to conduct joint inferences on a set of quantile levels. For example, the test for common values in Haile, Hong, and Shum (2003) and test for different models of entry in Marmer, Shneyerov, and Xu (2013) are characterized by stochastic dominance relations between distributions. The following Corollary shows the quantile estimator is independent across a fixed vector of quantile levels asymptotically.

**Corollary 1.** Let \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_J < 1 \) be a vector of fixed quantile levels. Then

\[
\mathbb{P} \left( \cap_{j=1,2,\ldots,J} \left\{ \sqrt{n} \left( \hat{Q}_n(\alpha_j) - Q_v(\alpha_j) \right) \leq z_j \right\} \right)
\]

\[
= \Pi_{j=1,2,\ldots,J} \mathbb{P} \left( C(\alpha_j) \argmax \{B(t) - t^2\} \leq z_j \right).
\]
where $B_j$, $j = 1, 2, \ldots, J$, are independent two-sided Brownian motions.\footnote{Please see the online supplementary materials (Luo and Wan 2016) for the proof of all corollaries in this article.}

The result in Corollary 1 does not hold in general for quantile estimations. It is useful when researchers would like to compare multiple quantile levels simultaneously, which in practice, is a useful approximation for comparing the whole distribution. One possible choice is the multiple testing procedure of Holm (1979), which controls the familywise error rate of one false rejection. For example, one can calculate the $p$-values $p_j$, $j = 1, 2, \ldots, J$ for each of the $J$ hypotheses. Rank all the $p$-values such that $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(J)}$. Let $\alpha$ be the significance level. If $p_{(j)} \geq \alpha / J$, then no hypothesis is rejected; otherwise the procedure rejects hypotheses $H_{(1)}, \ldots, H_{(k)}$, where $k$ is the largest integer such that $p_{(j)} \leq \alpha / (J - j + 1)$. One important source of conservativeness of Holm’s procedure—dependence among $p$-values—does not rise here.

Theorem 1 also provides a basis for constructing a simple trimming-free smoothed quantile estimator that converges at the optimal rate of GPV under appropriate smoothness conditions as listed in Assumption 3 below. Numerous smooth quantile function estimators have been studied, see, for example, Nadaraya (1964) for inverting a kernel distribution function estimator, Harrell and Davis (1982) for using generalized order statistics and Cheng (1995) for a Bernstein polynomial estimator. We adopt the kernel estimator used in Yang (1985), which dates back to Parzen (1979). Specifically, for any $0 < \alpha < 1$, let

$$
\hat{Q}_v(\alpha) = \frac{1}{h} \int_0^1 K \left( \frac{\alpha - u}{h} \right) \hat{Q}_v(u) du,
$$

where $h$ is a bandwidth and $K(\cdot)$ is a kernel with a compact support. Note that by construction, $\hat{Q}_v(\cdot)$ is necessarily increasing since $\hat{Q}_v(\cdot)$ is increasing.

**Assumption 3.** The valuation density $f$ is continuously differentiable.

**Assumption 4.** Let $K'$ be the first-order derivative of $K$. Then, $K$ satisfies (1) $K$ has compact support and take value zero on the boundary, (2) $\int K'(u) du = 0$, (3) $\int u^2 K'(u) du = 0$, (4) $\int u^3 K'(u) du \neq 0$.

Assumption 3 requires same smoothness as in GPV and MS. Assumption 4 is satisfied by commonly used kernel functions such as second order Epanechnikov or triweight kernels.

**Theorem 2.** Suppose Assumptions 1 to 4 are satisfied, and let $\alpha \in (0, 1)$,

(i) if $nh^5 \to c \in (0, \infty)$, then $\sqrt{n}h \hat{q}_v(\alpha) - Q_v(\alpha) \to N(0, \gamma')$.

(ii) if $nh' \to c$ for some $2 < r \leq 5$, then

$$\sqrt{nh} (\hat{q}_v(\alpha) - Q_v(\alpha)) \to N(0, \gamma').$$

**Proof.** See Section 6. \hfill $\square$

We have some remarks on Theorem 2. First, the variance and bias depend on $c$ analytically. One can estimate the optimal choice of $c$ that minimizes the asymptotic mean squared error, provided that the model has enough smoothness for consistent estimation of $Q_v(\cdot)$. Part (ii) of the theorem suggests that we can use undersmoothing to eliminate the asymptotic bias. Second, note from the quantile representation Equation (2) that the valuation quantile function is the sum of the bid quantile and a term containing quantile derivative (also known as quantile density function). It is the latter which determines the convergence rate of our estimator since the bid quantile can be estimated at root-$n$ rate. When there is sufficient amount of smoothness, the bid quantile derivative can be estimated at a faster rate, which results in a faster convergence rate than what we obtained in Theorem 1. Third, another way of estimating the quantile derivative is to use the relationship $Q_v(\alpha) = 1/g(Q_v(\alpha))$ and plug in a kernel density estimator for $g$, which is the method adopted in Marmer, Shneyerov, and Xu (2013). It is quite interesting to note that by choosing the two bandwidths proportionally, the asymptotic variances of our estimator and Marmer, Shneyerov, and Xu (2013) are identical in the current context.\footnote{Marmer, Shneyerov, and Xu (2013) considered entry. In our model, all the potential bidders enter with probability one. Let $L_{MAX}$ be the bandwidth of Marmer, Shneyerov, and Xu (2013) and $h_{LY}$ be ours. Then, the equality of variances holds by letting $h_{LY}f = h_{LY}f(Q(\alpha))$ and observing that $Lh^r \to c \not\equiv nh' \to I$ since $n = LI$ and $I$ is fixed. As a matter of fact, taking derivative of a smoothed quantile estimator and taking reciprocal of a kernel density estimator are two general approaches to estimate quantile derivatives. For general comparison of these two approaches, see Jones (1992).}

In the rest of the section, we discuss several interesting extensions of our method.

### 2.1 Generalization to Procurement Auctions

Our method can be easily adapted to first price procurement auction settings. Suppose that there are $I$ bidders competing for a contract in a first-price sealed bid auction. For each auction, every bidder $i$ simultaneously draws an iid cost $c_i$ from a common distribution $F(\cdot)$ and submits a bid to maximize his/her expected profit $E[\min_{j \neq i} c_j]$. The lowest bid wins the contract, and the bidder is paid the amount he/she bid.

Differentiating the expected profit with respect to $b_i$ gives the following system of first-order differential equations that define the equilibrium strategy $s(\cdot)$:

$$
(b_i - c_i)(1 - n^{-1}) f[s^{-1}(b_i)] \frac{1}{[1 - F(s^{-1}(b_i))] s'[s^{-1}(b_i)]} = 1,
$$

which can be rewritten as

$$
c_i = b_i \frac{1}{1 - n^{-1} \frac{1 - G(b_i)}{g(b_i)}}.
$$

Therefore, the quantile relationship becomes

$$
Q_v(\alpha) = Q_v(b_i - (1 - \alpha)/[(1 - n^{-1}) g(Q_v(\alpha))]).
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where \( Q_c(\cdot) \) represents the cost quantile function. The integrated quantile function becomes

\[
C(\beta) = \int_0^\beta Q_c(\alpha)d\alpha = \frac{I - 2}{I - 1} \int_0^\beta Q_s(\alpha)d\alpha - \frac{1}{1 - I}Q_s(\beta)(1 - \beta) + \frac{1}{I - 1}Q_s(0). \tag{3}'
\]

To impose the monotonicity constraint, we consider the g.c.m. of the empirical counterpart of the following function:

\[
\tilde{C}(\beta) = C(1 - \beta),
\]

which is the reflection of the integrated quantile function over the line \( \beta = 1/2 \). The idea is to use the prior information that the maximum possible bid equals the maximum cost in procurement auctions, that is, \( Q_b(1) = Q_c(1) \).

As the pseudo values are constructed sequentially, consider the g.c.m. of \( \tilde{C}(\cdot) \) preferable to \( C(\cdot) \). To see this, note that \( [\tilde{C}(1) - \tilde{C}(\frac{n-1}{n})]/(j/n) = \frac{I - 2}{n-1} \sum_{k=0}^{n-j-1} b_{k+1}/j + \frac{1}{n-1}b_{N-j} \) and \( [\tilde{C}(1/n) - \tilde{C}(0)]/(1/n) = b_1 \). By definition, the preferred method starts with the largest pseudo valuation \( \tilde{C}(n) = \frac{I - 2}{n}b_n + \frac{1}{n}b_{n-1} \). Note the right-hand side converges to \( Q_b(1) = Q_c(1) \) at a fast rate. On the other hand, considering the g.c.m. of \( C(\cdot) \), we would start with an estimate of the smallest pseudo valuation \( \tilde{C}(1) \leq b_1 \). Although \( b_1 \) converges to \( Q_s(0) \) at a fast rate, it does not guarantee that \( \tilde{C}(1) \) converges to \( Q_s(0) \) at a fast rate.

For estimation, we construct a raw estimator \( \tilde{C}_n(\cdot) \) for \( \tilde{C}(\cdot) \) by plugging in the bid quantile estimator. We then take the g.c.m. of \( \tilde{C}_n(\cdot) \). The pseudo cost of the bidder whose bid is the \( j \)th highest is constructed as the negative of the right-derivative of the g.c.m. at \( \beta = (j - 1)/n \), where \( j = 1, \ldots, n \). A smooth estimator for the cost quantile function follows naturally:

\[
\hat{q}_n(\alpha) = \int_0^{\frac{1}{n}} K \left( \frac{x - u}{h} \right) \tilde{Q}_n(u)du.
\]

Moreover, we can also apply a kernel density estimator on the sample of pseudo costs:

\[
f(\nu) = \frac{1}{nh} \sum_{j=1}^n K \left( \frac{\hat{v}_j - \nu}{h} \right).
\]

Since our first step estimator \( \tilde{Q}_n(\cdot) \) is tuning-parameter-free, our estimator of the valuation density function requires no trimming and only one tuning parameter \( h \).

2.3 Incorporating Auction-Level Heterogeneity

In practice, it is of empirical interest to incorporate auction-level observed heterogeneity. In this subsection, we discuss several ways of incorporating such heterogeneity.

In many applications, researchers are interested in the valuation quantile/distribution conditional on discrete or discretization of continuous variables. For example, in a procurement auction, researchers may be interested in the valuation (or cost) quantiles of “large” projects, which is defined by whether engineers’ estimates (a continuous variable) exceed certain cutoff values. In these cases, our estimation procedure can be directly applied to corresponding subsamples, and the estimates can be interpreted as the conditional valuation quantile.

Our method can also be applied together with the homogenization method (see Haile, Hong, and Shum 2003)—one common method of controlling both continuous and discrete observed heterogeneity in the empirical auction literature. The homogenization approach assumes valuations depend on auction-level characteristics in an additively (or multiplicatively) separable form, which implies the bids depend on characteristics in the same separable way. Under such an assumption, the effect of auction level characteristics can be controlled by focusing on the regression residuals (called homogenized bids) of the original bids (or log of bids in the multiplicative case) on those covariates. Since our estimators converge at rates which are slower than root-\( n \), their asymptotic properties are not affected by the homogenizing step.

If researchers would like to be agnostic about how valuations depend on the continuous covariates \( X \), and if they are interested in the valuations quantiles conditional on a particular realization \( x \), our method can also be applied with modification. In particular, since the quantile representation holds conditional on \( X \), we can define the (sample) conditional integrated-quantile functions as

\[
V_n(\beta|X = x) = \begin{cases} \sum_{i=1}^{n} Q_n(i/n|X = x) \\ + \frac{1}{n} |\beta| Q_n(n|X = x), & \beta \in \left\{ \frac{i}{n}, \ldots, 1 \right\} \\ (j - \beta n) V_n \left( \frac{i-1}{n} | X = x \right) + (\beta n - j + 1) V_n \left( \frac{j}{n} | X = x \right), & \beta \in \left( \frac{i-1}{n}, \frac{j}{n} \right) \end{cases}
\]
where \( Q_{n,b}(\alpha | X = x) \) is a suitable estimator for the conditional quantile function of bids given \( X = x \). Note that by definition, \( V_n(\cdot|X=x) \) is still a piecewise linear function and its g.c.m., denoted by \( \hat{V}(\cdot|X=x) \), is as easy to compute as in the unconditional case. Our estimator for conditional valuation quantile function, denoted by \( \hat{Q}_v(X=x) \), is then defined as the piecewise derivative of \( \hat{V}(\cdot|X=x) \).

For \( Q_{n,b}(\alpha | X = x) \), we adopt the local polynomial estimator proposed by GS and establish the convergence rate of our estimator based on GS’s uniform Bahadur representation for the conditional quantile function and its derivatives.

**Corollary 3.** Suppose that (i) \( X \) has a continuously differentiable density function which is bounded away from zero over its compact support \( \mathcal{X} \subseteq \mathbb{R}^d \); (ii) for every \( x \), the conditional valuation distribution \( F(\cdot|x) \) is continuously differentiable over its compact support \( [\underline{x}, \overline{x}] \). There exists \( \lambda(x) > 0 \) such that \( \inf_{v \in [\underline{x}, \overline{x}]} f(v|x) \geq \lambda(x) > 0 \); (iii) the bandwidth for the local polynomial regression is chosen such that \( h = cn^{-\gamma} \) for some \( r \) satisfying \( \frac{1}{n^{1/3-r}} < r < \frac{1}{2} \). The kernel function is chosen to satisfy Assumption 4. Then for a given \( x \) in the interior of \( \mathcal{X} \) and \( \alpha_0 \in (0, 1) \),

\[
\sqrt{n}h^{d/2}(\hat{Q}_v(\alpha_0 | X = x) - Q_v(\alpha_0 | X = x)) \xrightarrow{d} \tilde{C}(\alpha_0, x) \text{argmax}_t \left\{ \mathbb{E}(t) - t^2 \right\},
\]

where \( \tilde{C}(\alpha_0, x) \) is a constant that depends on \( \alpha_0, x \) and the kernel function.

In the presence of covariates, the rate of convergence of our first stage estimator is \( \sqrt{n}h^{d/2} \), which is slower than the optimal \( n^{2/(5+d)} \) rate obtained by GPV and MS under stronger smoothness assumption on \( F(\cdot|x) \); when there is no covariates (i.e., \( d = 0 \)), this reduces to the \( \sqrt{n} \) versus \( n^{2/5} \) comparison in the homogeneous auction case.

### 3. SIMULATION

To study the finite sample performance of our estimation method, we conduct Monte Carlo experiments. We adopt the setup of the Monte Carlo simulations from MS. The true valuation distribution is

\[
F(v) = \begin{cases} 
0 & \text{if } v < 0, \\
v^\gamma & \text{if } 0 \leq v \leq 1, \\
1 & \text{if } v > 1,
\end{cases}
\]

where \( \gamma > 0 \). Such a choice of private value distributions is convenient since the distributions correspond to linear bidding strategies as:

\[
s(v) = \left( 1 - \frac{1}{\gamma(I-1)+1} \right) v.
\]

We consider \( I = 7 \) bidders, \( n = 4200 \), and \( \gamma \in \{0.5, 1, 2\} \). The number of Monte Carlo replications is 1000. For each replication, we first generate randomly \( n \) private values from \( F(\cdot) \). Second, we obtain the corresponding bids \( b_i \) employing the linear bidding strategy (5). Third, we construct a raw estimator \( V_n(\cdot) \) for \( \hat{V}(\cdot) \). Let \( \hat{V}(\cdot) \) be the g.c.m. of \( V_n(\cdot) \). Fourth, we obtain

| \( \gamma = 0.5 \) | MSE | Ours | Bias | MAD |
|----------------|------|------|------|------|
| \( v = 0.2 \) | 0.0068 | 0.0044 | -0.0041 | 0.0672 |
| \( v = 0.3 \) | 0.0073 | 0.0057 | -0.0019 | 0.0689 |
| \( v = 0.4 \) | 0.0103 | 0.0080 | -0.0006 | 0.0806 |
| \( v = 0.5 \) | 0.0131 | 0.0100 | 0.0029 | 0.0907 |
| \( v = 0.6 \) | 0.0132 | 0.0109 | -0.0159 | 0.0908 |
| \( v = 0.7 \) | 0.0188 | 0.0140 | -0.0156 | 0.1027 |
| \( v = 0.8 \) | 0.0163 | 0.0102 | -0.0185 | 0.1094 |

### Table 1. Simulation results for density estimation
Table 2. Regression results

|       | 2  | 3  | 4  | 5  | 6  | 7  |
|-------|----|----|----|----|----|----|
| logX  | 0.978*** | 0.966*** | 1.015*** | 0.957*** | 0.932*** | 0.938*** |
|       | (34.11) | (56.68) | (50.59) | (51.81) | (49.91) | (56.58) |
| logDays| 0.00650 | 0.00473 | −0.00271 | 0.0901*** | 0.138*** | 0.00430 |
|       | (0.15) | (0.25) | (−0.13) | (4.76) | (6.31) | (0.18) |
| type  | Yes | Yes | Yes | Yes | Yes | Yes |
| n     | 206 | 474 | 564 | 470 | 402 | 252 |
| adj. $R^2$ | 0.871 | 0.906 | 0.857 | 0.929 | 0.930 | 0.947 |

NOTE: $t$ statistics in parentheses. * $p<0.05$, ** $p<0.01$, *** $p<0.001$.

We compare our method with MS and GPV. For the MS and GPV methods, we use the same setups as in MS: the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidths in estimation of densities. For our method, we also use the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidth in estimation of $f$: $h = 1.06\hat{\sigma}_v N^{-1/5}$, where $\hat{\sigma}_v$ is the estimated standard deviation of the constructed pseudo valuations $\{\hat{v}_j\}_{j=1}^N$.

Table 1 shows the simulation results for density estimation. When the distribution is skewed to the left ($\gamma = 0.5$), our method improves MSE and MAD but seems to produce larger biases near the boundaries. While the MS and GPV methods behave similarly in terms of MSE and MAD, the former seems to produce larger biases. When the distribution is uniform or skewed to the right ($\gamma = 1$ or 2), our method performs similarly to the GPV method, both of which seem to perform slightly better than the MS method.

4. EMPIRICAL ILLUSTRATION

In this section, we implement our method using the California highway procurement data. In particular, we analyze the data used by Krasnokutskaya and Seim (2011). It covers highway and street maintenance projects auctioned by the California Department of Transportation (Caltrans) between January 2002 and December 2005. We focus on the procurement auctions with 2–7 bidders. For each auction, the data contain the engineer’s estimate of the project’s total cost, the type of work involved, the number of days allocated to complete the project, and the identity of the bidders and their bids.

Following Haile, Hong, and Shum (2003), we homogenize the bids before implementing our method to control for observable heterogeneity for each sample (with the same number of bidders). In particular, we regress the logarithm of the bid ($\log b$) on the logarithm of the engineer’s estimate ($\log X$), the logarithm of the number of days ($\log_{\text{Days}}$), and the project type dummies. Table 2 displays the results. The homogenized bids ($\text{bid}_\text{new}$) are calculated as the exponential of the differences between the logarithm of the original bids and the constructed pseudo valuations $\{\hat{v}_j\}_{j=1}^N$.

Table 3. Summary statistics

|       | 2  | 3  | 4  | 5  | 6  | 7  | Total |
|-------|----|----|----|----|----|----|-------|
| bid   | 993.8 | 967.6 | 757.7 | 1136.9 | 990.9 | 1769.7 | 1042.8 |
|       | (1644.5) | (1935.9) | (843.7) | (4584.7) | (3350.3) | (7288.0) | (3595.9) |
| bid_new| 652.5 | 587.7 | 566.3 | 508.9 | 464.4 | 478.5 | 540.0 |
|       | (208.4) | (190.6) | (178.6) | (129.0) | (135.0) | (137.4) | (174.0) |
| cost  | 402.1 | 468.0 | 477.8 | 453.8 | 423.6 | 441.7 | 451.5 |
|       | (259.6) | (223.6) | (218.9) | (164.4) | (156.3) | (159.5) | (200.0) |
| profit| 250.4 | 119.7 | 88.46 | 55.09 | 40.79 | 36.81 | 88.59 |
|       | (75.81) | (77.65) | (60.62) | (49.56) | (42.83) | (46.68) | (83.51) |
| profit rate | 0.439 | 0.244 | 0.197 | 0.136 | 0.109 | 0.0978 | 0.190 |
|       | (0.213) | (0.208) | (0.194) | (0.167) | (0.153) | (0.158) | (0.206) |

NOTE: Std. Dev. in parentheses. profit = bid_new − cost. Profit rate = profit/bid.

Figure 1. Estimation results.
measured fitted values of the regression. Table 3 displays the mean and standard deviation of the original and homogenized bids.

We estimate a first price auction model with each sample. Figure 1 displays the estimated inverse bidding strategies and the estimated valuation quantile functions without and with smoothing, respectively. The curves represented are: from the sample with 2 bidders (yellow solid line); 3 bidders (magenta dash-dotted line); 4 bidders (cyan solid line); 5 bidders (red dash-dotted line); 6 bidders (green solid line); 7 bidders (blue dash-dotted line), and the 45-degree line (black dashed line).

All inverse bidding strategies are increasing. The valuation quantile functions seem to be close except for \( I = 2 \). Table 3 displays some summary statistics of the estimated pseudo costs. The auctions with two bidders tend to be less costly to finish in percentage terms. In fact, the generated profit rate is almost twice that of the sample with three bidders. As the auction becomes more competitive when the number of bidders increases from two to seven, the profit rate decreases from 44% to about 10%.

5. CONCLUSION

This article considers nonparametric estimation of first-price auction models based on an integrated-quantile representation of the first-order condition. The monotonicity of bidding strategy is imposed in a natural way. We propose two estimators for the valuation quantile function and derive their asymptotics: a non-smoothed estimator that is tuning-parameter-free and a smoothed one that is trimming-free. We show the former is cube-root consistent under weaker smoothness assumptions, and the latter achieves the optimal rate of GPV under standard ones. Monte Carlo simulations show our method works well in finite samples. We apply our method to data from the California highway procurement auctions.

SUPPLEMENTARY MATERIALS

Description of IQest RR2_supp: This supplement provides the details of the proofs which were omitted from the main text.

ACKNOWLEDGMENTS

We thank the editor, the associate editor, and two anonymous referees for their useful suggestions that improved this article. We thank the valuable comments from Victor Aguirregabiria, Tim Armstrong, Yanqin Fan, Christian Gourieroux, Emmanuel Guerre, Sung Jae Jun, Shaehee Khan, Nianqing Liu, Ruixuan Liu, Vadim Marmer, Ismael Mourifie, Joris Pinkse, and Quang Vuong. We benefited from discussions with participants at CMES 2014, the 2nd CEC (Nanjing), the 5th Shanghai Econometrics Workshop at SUFE, ESWC 2015, and the CEME 2015 Conference at Cornell. Luo gratefully acknowledges the financial support from the National Natural Science Foundation of China (NSFC-71373283 and NSFC-71463019). All errors are ours.

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