ALMOST SURE MIXING RATES FOR NON-UNIFORMLY EXPANDING MAPS

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Abstract. We consider random perturbations of non-uniformly expanding maps, possibly having a non-degenerate critical set. We prove that, if the Lebesgue measure of the set of points failing the non-uniform expansion or the slow recurrence to the critical set at a certain time, for almost all random orbits, decays in a (stretched) exponential fashion, then the decay of correlations along random orbits is stretched exponential, up to some waiting time. As applications, we obtain almost sure stretched exponential decay of random correlations for Viana maps, as for a class of non-uniformly expanding local diffeomorphisms and a quadratic family of interval maps.

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1. Introduction

One of the interests of smooth ergodic theory is to study iterations of smooth maps $f$ on a Riemann manifold $M$ through measures $\mu$ preserved by $f$, which describes the asymptotic behaviours of typical orbits $\{f^k(x)\}_{k \in \mathbb{N}}$, i.e.,

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \xrightarrow{n \to \infty} \mu$$

in the weak* topology. The absolutely continuous (with respect to Lebesgue measure) invariant probability measures are of greatest relevance. If such measures are also ergodic, the statistical prediction given by (1) holds for a positive Lebesgue measure of initial states $x \in M$. The measures possessing such a rich property are called $SRB$ (Sinai-Ruelle-Bowen) measures, and were introduced by Sinai, Ruelle and Bowen in [29, 28, 22, 20] for Anosov maps and Axiom A attractors.

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If μ is mixing, we have the decay of correlations
\[
\lim_{n \to \infty} \left( \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right) = 0
\]
for regular enough observables φ, ψ. We are then interested in describing how fast the convergence is. Among the available techniques we are particularly concerned to the use of induced schemes, popularized for this purpose by the works of L.-S. Young [31, 32].

We are going to address random perturbation of dynamical systems, by replacing the original dynamics \( f \) by a close map \( f_t, t \in T \), chosen independently according to some probability law \( \theta \). In a nutshell, the study of correlations for random perturbations can follow two approaches. We could focus on the stationary measure \( \mu_\epsilon \), which satisfies
\[
\int \varphi(f_t(x)) d\mu_\epsilon(x) d\theta_\epsilon(t) = \int \varphi d\mu_\epsilon \int \psi d\mu_\epsilon.
\]
for every continuous function \( \varphi \). In this average (“annealed”) setting, the correlation functions are expressed as
\[
\int (\varphi \circ (f_{t_{n-1}} \circ \cdots \circ f_{t_0})(x)) \psi(x) d\mu_\epsilon(x) d \prod_{i=0}^{n-1} d\theta_\epsilon(t_i) - \int \varphi d\mu_\epsilon \int \psi d\mu_\epsilon.
\]
Essentially, we are averaging over all possible realizations, which due to the i.i.d. setting can be done by averaging at each time-step. This is the natural setting to formalize the correlations in terms of a Markov chain. Alternatively, we could look for an almost sure approach, by considering the product space \( T^\mathbb{Z} \) and the usual skew product dynamics \( S(\omega, x) = (\sigma(\omega), f_{\omega}(x)) \) in \( T^\mathbb{Z} \times M \). We focus now on the invariant probability measures for \( S \) that disintegrates as \( d\mu_\omega(x) d\theta_\mathbb{Z}(\omega) \). The correlation is expressed in the following fiberwise (“quenched”) future and past random correlations:
\[
C^+_{\omega}(\varphi, \psi, \mu, n) = \left| \int (\varphi \circ f_{\sigma^n}^n) \psi d\mu_\omega - \int \varphi d\mu_{\sigma^n(\omega)} \int \psi d\mu_\omega \right|,
\]
\[
C^-_{\omega}(\varphi, \psi, \mu, n) = \left| \int (\varphi \circ f_{\sigma^{-n}}^n) \psi d\mu_{\sigma^{-n}(\omega)} - \int \varphi d\mu_\omega \int \psi d\mu_{\sigma^{-n}(\omega)} \right|.
\]
There are several works dealing either with the annealed approach (e.g., [15, 13, 12]) or the almost sure (e.g. [14, 13, 23, 16, 11]).

This paper concerns to almost sure decay of correlations for random perturbations of non-uniformly expanding (NUE for short) maps. The strategy is to build induced Gibbs-Markov-Young structures with (stretched) exponential decay of recurrence times for random orbits. We use random Young towers and a coupling argument to estimate stretched exponential decay of correlations over the abstract random induced dynamic. This estimates gives rise to the almost sure decay of random correlations.

We give applications to some known families of NUE systems. We present new results respecting to Viana maps, which are a higher-dimensional maps with critical set given in [30], and to an open class of local diffeomorphisms given in [4]. We also apply our results to the unimodal maps as considered in [16] to illustrate our strategy under the weaker hypotheses.

The strategy could be used for dynamical systems with other rates of decay (e.g. polynomial), but the hypotheses could be harder to achieve for known examples. Some other questions arise: Is the decay of correlations actually stretched exponential? Does a random
central limit theorem hold? Do we have the parallel result in partially hyperbolic attractors admitting a non-uniformly expansion direction? Can we replace lim sup in the definition of random non-uniformly expanding to the weaker assumption lim inf, as in the recent work [3]?

This paper is organized as follows. In §2 we give our basic definitions of random perturbations for non-uniformly expanding and in §3 we state the main results. We describe the main steps of our strategy in §4 where we have the principal intermediate results, which are proved in §6 (existence of random induced structures with stretched exponential decay of return times) and §7 (decay of correlations for abstract random induced dynamics). The applications to Viana maps, local diffeomorphisms and unimodal interval maps is given at §5.

2. RANDOM PERTURBATIONS FOR NON-UNIFORMLY EXPANDING MAPS

Let M be a compact Riemannian manifold endowed with a normalized volume measure m (Lebesgue measure), and f : M → M be a C^2 map. We assume that f is a local diffeomorphism in the whole manifold except, possibly, in a set C ⊂ M containing the critical points of f and ∂M. We say that the set C is non-degenerate if it has zero Lebesgue measure and there are constants B > 1 and β > 0 such that for every x ∈ M \ C

\[ \frac{1}{B} \text{dist}(x, C)^\beta \leq \frac{\|Df(x)v\|}{\|v\|} \leq B \text{dist}(x, C)^{-\beta} \forall v \in T_xM, \]

and, for every x, y ∈ M \ C with dist(x, y) < dist(x, C)/2, we have

\[ |\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\| | \leq \frac{B}{\text{dist}(x, C)^\beta} \text{dist}(x, y) \]

\[ |\log |\det Df(x)| - \log |\det Df(y)| | \leq \frac{B}{\text{dist}(x, C)^\beta} \text{dist}(x, y). \]

Roughly speaking, condition (2) says that f behaves like a power of the distance to C, meanwhile (3) and (4) say that the functions log |\det Df| and log \|Df^{-1}\| are locally Lipschitz in M \ C, with the Lipschitz constant depending on the distance to C. Given δ > 0 and x ∈ M \ C we define the δ-truncated distance from x to C as dist_δ(x, C) = dist(x, C) if dist(x, C) < δ and dist_δ(x, C) = 1 otherwise.

The idea we adopt for random perturbations is to replace the original deterministic orbits by random orbits generated by an (independent and identically distributed) random choice of map at each iteration. We are interested in systems whose random perturbation exhibit a non-uniform expansive behavior along the orbits generated by the successive composition of random elected maps. To be more precise we consider a metric space T and a continuous map

\[ \Phi : T \rightarrow C^2(M, M) \]

\[ t \mapsto \Phi(t) = f_t \]

such that f = f_{t^*} for some t^* ∈ T, and a family (\theta_\epsilon)_{\epsilon > 0} of Borel probability measures in T. We will refer to such a pair \(X = \{\Phi, (\theta_\epsilon)_{\epsilon > 0}\}\) as a random perturbation of f. We consider the product space \(\Omega = T^\mathbb{Z}\) endowed with the Borel product probability measure \(P = P_\epsilon = \theta_\epsilon^\mathbb{Z}\). For a realization \(\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Omega\) and \(n \geq 1\) we define

\[ f_\omega^n(x) = (f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_1} \circ f_{\omega_0})(x), \]
and set $f^0_\omega(x) = Id_M$. Given $x \in M$ and $\omega \in \Omega$ we call the sequence $(f^n_\omega(x))_{n \in \mathbb{N}}$ a random orbit of $x$. Note that $\omega^* = (\ldots, t^*, t^*, \ldots)$ gives rise to the unperturbed deterministic orbits given by the original dynamics $f$. We define the two-sided skew-product map

$$S : \Omega \times M \rightarrow \Omega \times M,$$

$$(\omega, z) \mapsto (\sigma(\omega), f_{\omega_0}(z)),$$

where $\sigma : \Omega \rightarrow \Omega$ is the left shift map. A Borel probability measure $\mu^*$ in $\Omega \times M$ invariant by $S$ (in the usual deterministic sense) is characterized by an essentiality unique disintegration $d\mu^*(\omega, x) = d\mu_\omega(x)dP(\omega)$ given by a family $\{\mu_\omega\}_\omega$ of sample measures on $M$ satisfying the quasi-invariance property $f_{\omega_*}\mu_\omega = \mu_{\sigma(\omega)}$, and such that for each Borel set $A \subset \Omega \times M$ we have $\mu^*(A) = \int \mu_\omega(A_\omega) dP(\omega)$, where $A_\omega = \{x \in M : (\omega, x) \in A\}$. For a complete introduction on random dynamical systems we refer for [10].

Consider a random perturbation $\chi_\epsilon$ of a map $f$ such that

$${\rm supp}(\theta_\epsilon) \rightarrow \{t^*\}, \quad \epsilon \rightarrow 0. $$

Due to the presence of the critical set, we assume that all the maps $f_t$ have the same critical set $C$:

$$Df_t(x) = Df(x), \quad \text{for every } x \in M \setminus C \text{ and } t \in T.$$  

(5)

We could implement this setting, for instance, in parallelizable manifolds (with an additive group structure, e.g. tori $\mathbb{T}^d$ (or cylinders $\mathbb{T}^{d-k} \times \mathbb{R}^k$), by considering $T = \{t \in \mathbb{R}^d : \|t\| \leq \epsilon_0\}$ for some $\epsilon_0 > 0$, and taking $f_t = f + t$, that is, adding at each step a random noise to the unperturbed dynamics.

**Definition 2.1.** We say that $f$ is non-uniformly expanding on random orbits if the following conditions hold, at least for small $\epsilon > 0$:

1. there is $\alpha > 0$ such that for $P \times m$ almost every $(\omega, x) \in \Omega \times M$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\sigma^j(\omega)}(f^j_\omega(x))^{-1}\| < -\alpha.$$  

(6)

2. given any small $\gamma > 0$ there is $\delta > 0$ such that for $P \times m$ almost every $(\omega, x) \in \Omega \times M$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_{\delta}(f^j_\omega(x), C) < \gamma.$$  

(7)

When $C = \emptyset$ we simply disregard condition (7) and assumption (5). We say that the original map $f$ is a non-uniformly expanding map if (5) and (7) holds for $\omega^*$ and $m$ almost every $x$. We will refer to the second condition by saying that the random orbits of points have slow recurrence to $C$. Condition (5) implies that for $P$ almost every (a.e.) $\omega \in \Omega$, the expansion time function

$$E_\omega(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\sigma^j(\omega)}(f^j_\omega(x))^{-1}\| \leq -\alpha, \text{ for all } n \geq N \right\}$$

is defined and finite Lebesgue almost everywhere in $M$. We notice that condition (7) is not needed in all its strength, and we just need to ensure that it holds for suitable $\delta, \gamma$ so that the proof of Proposition 6.3 works (see Remark 4.5 in [9]). Hence, we may consider
\(\gamma > 0\) and \(\delta > 0\) such that for \(P\) a.e. \(\omega \in \Omega\) we can define the recurrence time function Lebesgue almost everywhere in \(M\):

\[
\mathcal{R}_\omega(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} - \log \delta(f_j^\omega(x), C) \leq \gamma, \quad \text{for all } n \geq N \right\}.
\]

We introduce the tail set (at time \(n\))

\[
\Gamma_\omega^n = \{ x \in M : E_\omega(x) > n \text{ or } \mathcal{R}_\omega(x) > n \}.
\]

This is the set of points in \(M\) whose random orbit at time \(n\) has not yet achieved the uniform exponential growth of derivative or the slow recurrence given by conditions (6) and (7). If the critical set is empty, we simply ignore the recurrence time function in the definition of \(\Gamma_\omega^n\).

3. Main results

We assume that \(f\) is a topologically transitive non-uniformly expanding map and non-uniformly expanding on random orbits. We start with the case where we have a uniform (stretched) exponential decay of tail sets.

**Theorem A.** Assume there exist \(C, \gamma > 0\) and \(0 < \nu \leq 1\) such that \(m(\Gamma_\omega^n) < Ce^{-\gamma n^\nu}\) for \(P\) a.e. \(\omega \in \Omega\). Then, if \(\epsilon > 0\) is small, for some integer \(q \geq 1\) we have:

(i) for \(P\) a.e. \(\omega\) there is an absolutely continuous probability measure \(\mu_\omega = h_\omega dm\) satisfying \((f^n_\omega)^* \mu_\omega = \mu_{\phi^n(\omega)}\);

(ii) there exist \(C_i, \gamma_i > 0, i = 1, 2\), and for \(P\) a.e. \(\omega\) a positive integer \(n_0(\omega)\), such that for each Lipschitz function \(\psi : M \to \mathbb{R}\) and every bounded function \(\varphi : M \to \mathbb{R}\) we have

\[
C_\omega^\pm(\varphi, \psi, \mu, qn) \leq C_1 \sup |\varphi| \text{Lip}(\psi)e^{-\gamma n^{\nu/2}}, \ \forall n \geq n_0(\omega)
\]

and

\[
P(\{n_0(\omega) > n\}) \leq C_2 e^{-\gamma n^{\nu/2}}, \ \forall n \geq 1.
\]

We can interpret \(n_0(\omega)\) as the waiting time we have to consider before we see the stretched exponential behavior on the estimates of the decay of random correlations. In many cases, the estimates on the tail can be hard to achieve, in particular its uniformity over distinct realizations. However, we can state the following.

**Theorem B.** Assume that there exist \(C, \gamma > 0\), \(0 < \nu \leq 1\) and for \(P\) a.e. \(\omega\) a positive integer \(g_0(\omega)\) such that

\[
\begin{align*}
&\left\{ m(\Gamma_\omega^n) \leq Ce^{-\gamma n^\nu}, \quad \forall n \geq g_0(\omega) \\
&P(\{g_0(\omega) > n\}) \leq Ce^{-\gamma n^\nu}, \quad \forall n \geq 1.
\end{align*}
\]

Then, if \(\epsilon > 0\) is small, for some integer \(q \geq 1\) we have:

(i) for \(P\) a.e. \(\omega\) there is an absolutely continuous probability measure \(\mu_\omega = h_\omega dm\) satisfying \((f^n_\omega)^* \mu_\omega = \mu_{\phi^n(\omega)}\);

(ii) there exist \(C_i, \gamma_i > 0, i = 1, 2\), and for \(P\) a.e. \(\omega\) a positive integer \(n_0(\omega)\) such that for each Lipschitz function \(\psi : M \to \mathbb{R}\) and every bounded function \(\varphi : M \to \mathbb{R}\) we have

\[
C_\omega^\pm(\varphi, \psi, \mu, qn) \leq C_1 \sup |\varphi| \text{Lip}(\psi)e^{-\gamma n^{\nu/4}}, \ \forall n \geq n_0(\omega),
\]

and

\[
P(\{n_0(\omega) > n\}) \leq C_2 e^{-\gamma n^{\nu/4}}, \ \forall n \geq 1.
\]
Corollary C. Set $\Gamma^n = \{ (\omega, x) \in \Omega \times M : x \in \Gamma^n \}$. Assume that there exist $C, \gamma > 0$ and $0 < \nu \leq 1$ such that

$$(P \times m)(\Gamma^n) < Ce^{-\gamma n \nu}.$$  

Then the same conclusions of Theorem B hold.

4. The strategy: an overview

For both Theorems A and B the proof consists in two main steps. First, we prove that the hypothesis on the tail sets imply the existence of induced structures with nice decay for the return times $R$. Moreover, as in the deterministic case, we will need a condition of type $\gcd\{R\} = 1$ in order to ensure some mixing properties in the induced dynamics. In view of this, in this greater generality we need eventually to look for some power of the random system. If we are able to construct random induced structures with $\gcd\{R\} = 1$ then the main results hold with $q = 1$; see Remark 7.1.1. We also notice that the hypothesis of transitivity are used for the existence of this suitable induced structures.

In a second moment we give random versions of the already classic procedures introduced by Young [31, 32], in order to derive the decay of correlations in the induced dynamics from the decay of return times, that is later carried to the decay of correlations along random orbits. Corollary C is an immediate consequence of Theorem B and Lemma 5.7.

We notice that even in the case of uniform exponential decay of the tail set we do not achieve an exponential estimate for the decay of random correlations. The main reason for the damage in the estimates is related to the construction of induced measures, whose density is not bounded from below. Besides in some cases it is not known the optimal result on decay of correlations (even in the deterministic case), if we think for instance in the uniform expanding case it becomes clear that the strategy could compromise the search for optimal estimates.

4.1. Random induced schemes. We start by setting the induced Gibbs-Markov-Young (GMY for short) structures for the random orbits.

Definition 4.1. We say that $\omega \in \Omega$ induces a GMY map $F_\omega$ in a ball $\Delta \subset M$ if:

1. there is a countable partition $P_\omega$ of open sets of a full $m$ measure subset $D_\omega$ of $\Delta$;
2. there is a return time function $R_\omega : D_\omega \to \mathbb{N}$, constant in each $U_\omega \in P_\omega$;
3. the map $F_\omega(x) := f_{R_\omega(x)}(x) : \Delta \to \Delta$ verifies:
   a. $F_\omega|_{U_\omega}$ is a $C^2$ diffeomorphism onto $\Delta$;
   b. there exists $0 < \kappa_\omega < 1$ such that for $x$ in the interior of $U_\omega$
      $$\|DF_\omega(x)^{-1}\| < \kappa_\omega;$$
   c. there is some constant $K_\omega > 0$ such that for every $U_\omega$ and $x, y \in U_\omega$
      $$\log \left| \frac{\det DF_\omega(x)}{\det DF_\omega(y)} \right| \leq K_\omega \text{dist}(F_\omega(x), F_\omega(y)).$$

It is known that a deterministic transitive non-uniformly expanding map induces a GMY map $F$ (consider $\omega = \omega^*$ in the definition above) in some ball. The next theorem ensures that almost all realizations induce a GMY map with some uniformity on the constants, and relates the decay of the return times with the the decay of the tail set.
Theorem 4.2. Let \( f : M \to M \) be a transitive non-uniformly expanding map and non-uniformly expanding on random orbits. There is some ball \( \Delta \subset M \) such that if \( \epsilon > 0 \) is small enough then \( P \) a.e. \( \omega \) induces a GMY map \( F_\omega \) in \( \Delta \), and

(i) if there exist \( C, \gamma > 0 \), \( 0 < v \leq 1 \) such that \( m(\Gamma^\omega_n) < Ce^{-\gamma n^v} \) for \( P \) a.e. \( \omega \), then there exist \( C_1, \gamma_1 > 0 \) such that \( m(\{R_\omega > n\}) \leq C_1e^{-\gamma_1 n^v} \);

(ii) if there exist \( C_i, \gamma_i > 0 \), \( i = 1, 2, 0 < v \leq 1 \) and for \( P \) a.e. \( \omega \) a positive integer \( g_0(\omega) \) such that

\[
\begin{align*}
\{ m(\Gamma^\omega_n) \leq C_1e^{-\gamma_1 n^v}, \quad \forall n \geq g_0(\omega) \\
P\{\{g_0(\omega) > n\} \leq C_2e^{-\gamma_2 n^v}, \quad \forall n \geq 1,
\end{align*}
\]

then there exist \( C_3, \gamma_3 > 0 \), \( 0 < v \leq 1 \) such that

\[
\begin{align*}
\{ m(\{R_\omega > n\}) \leq C_3e^{-\gamma_3 n^v}, \quad \forall n \geq g_0(\omega) \\
P\{\{g_0(\omega) > n\} \leq C_2e^{-\gamma_2 n^v}, \quad \forall n \geq 1,
\end{align*}
\]

The proof is given in [6] where we can also deduce an induced GMY for \( \omega^* \) and the following uniformity conditions:

(U1) Given \( \xi > 0 \) and an integer \( \hat{N} > 1 \), if \( \epsilon > 0 \) is sufficiently small then for \( P \) a.e. \( \omega \) we have

\[
m(\{R_\omega = j\} \Delta \{R_\omega^* = j\}) \leq \xi
\]

for \( j = 1, 2, \ldots, \hat{N} \), where \( \Delta \) stands for the symmetric difference of two sets.

(U2) If \( \epsilon > 0 \) is sufficiently small, the constants \( K_\omega \) and \( \kappa_\omega \) for the induced GMY maps can be chosen uniformly over \( \omega \). We will refer to them as \( K > 0 \) and \( \kappa > 0 \), respectively.

4.2. Decay of correlations for random induced schemes. We start by following [16] and lift the GMY structures to random Young towers over copies of \( \Delta \). Letting \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \), set

\[
\Delta_\omega = \{ (x, \ell) \in \Delta \times \mathbb{Z}_+ : x \in D_{\tau^{-\ell}(\omega)}, 0 \leq \ell \leq R_{\tau^{-\ell}(\omega)}(x) - 1 \}.
\]

We set the \( n \)th level of the tower \( \Delta_\omega \) as \( \Delta_{\omega, n} = \Delta_\omega \cap \{ \ell = n \} \). The basis of all towers are copies of \( \Delta \), and \( \ell \)th level \( \Delta_{\omega, \ell} \) is a copy of \( \{ x \in \Delta : R_{\tau^{-\ell}(\omega)}(x) > \ell \} \). Moreover, for each basis \( \Delta_{\omega, 0} \) we consider the corresponding partition \( \mathcal{P}_\omega \) that can be extended to a partition of the full towers \( \Delta_\omega \). Moreover we denote by \( \mathcal{B}_\omega \) the Borel \( \sigma \)-algebra of \( \Delta_\omega \) and by \( \mathcal{B} \) the corresponding family of \( \sigma \)-algebras \( \mathcal{B}_\omega \). We define the dynamics \( F_\omega : \Delta_\omega \to \Delta_{\sigma(\omega)} \) by

\[
F_\omega(x, \ell) = \begin{cases} 
(x, \ell + 1), & \text{if } \ell + 1 < R_{\tau^{-\ell}(\omega)}(x) \\
(f_{R_{\tau^{-\ell}(\omega)}}(x), 0), & \text{if } \ell + 1 = R_{\tau^{-\ell}(\omega)}(x).
\end{cases}
\]

This dynamics carries \( (x, \ell) \) in the \( \ell \)th level of \( \Delta_\omega \) into \( (x, \ell + 1) \in \Delta_{\sigma(\omega)} \), unless \( R_{\tau^{-\ell}(\omega)}(x) = \ell + 1 \), in which case it falls down into the 0th level of \( \Delta_{\sigma(\omega)} \) by the return map. We denote by \( m_0 \) the normalized Lebesgue measure on \( \Delta \) and, without risk of confusion, we denote by \( m \) the lift of this measure on \( \Delta_\omega \), also dropping the reference to \( \omega \) in the notation. We set \( \Delta \) for the family \( \{\Delta_\omega\}_\omega \) and \( \mathcal{P} \) for the corresponding partition introduced before. We define, for almost every \( \omega \), the separation time \( s_\omega : \Delta_\omega \times \Delta_\omega \to \mathbb{Z}_+ \cup \{\infty\} \) given by

\[
s_\omega(x, y) = \min \{ n \geq 0 : F_\omega^n(x) \text{ and } F_\omega^n(y) \text{ lie in distinct elements of } \mathcal{P} \}.
\]
We introduce a Lipschitz-type space of observables \( \varphi = \{ \varphi_\omega \} \) on \( \hat{\Delta} \),
\[
\mathcal{F}_\beta = \left\{ \varphi : \hat{\Delta} \to \mathbb{R} : \exists C_\varphi > 0 \text{ s.t. } |\varphi(x) - \varphi(y)| \leq C_\varphi \beta_s(x,y), \ \forall x, y \in \Delta_\omega \right\},
\]
a space of densities \( \varphi = \{ \varphi_\omega \} \),
\[
\mathcal{F}^+_\beta = \{ \varphi \in \mathcal{F}_\beta : \exists \hat{\mathcal{C}}_\varphi > 0 \text{ s.t. on each } U_\omega \in \mathcal{P}_\omega, \text{ either } \varphi_\omega|_{U_\omega} \equiv 0, \text{ or } \varphi_\omega|_{U_\omega} > 0 \text{ and } \left| \log \frac{\varphi_\omega(x)}{\varphi_\omega(y)} \right| \leq \hat{\mathcal{C}}_\varphi \beta_s(x,y), \ \forall x, y \in U_\omega \},
\]
and a space of random bounded functions \( \varphi = \{ \varphi_\omega \} \),
\[
\mathcal{L}^\infty = \{ \varphi : \hat{\Delta} \to \mathbb{R} : \exists \hat{\mathcal{C}}_\varphi > 0 \text{ s.t. } \sup_{x \in \Delta_\omega} |\varphi_\omega| \leq \hat{\mathcal{C}}_\varphi \}.
\]
Let us assume the conclusions of Theorem 4.2 either in the case (i) of uniform estimates on return times or case (ii) of non-uniform rates.

**Theorem 4.3.** For \( P \text{ a.e. } \omega \) there is an absolutely continuous probability measure \( \nu_\omega \) on \( \Delta_\omega \) such that \( (F_\omega)_* \nu_\omega = \nu_{\sigma(\omega)} \). Moreover, \( \rho_\omega = d\nu_\omega/dm \in \mathcal{F}^+_\beta \) and there is a constant \( K_1 > 0 \) such that \( \rho_\omega \leq K_1 \) for \( P \text{ a.e. } \omega \).

For the proof see [16, 17]. Roughly speaking, for each \( \omega \) and \( n \geq 0 \), we consider the push-forward \( \nu^n_\omega \) of \( m_0|_{\Delta_{\sigma^{-n}(\omega)},0} \) by \( F^n_{\sigma^{-n}(\omega)} \). This push-forward is a probability measure on the tower \( \Delta_\omega \), absolutely continuous with respect to \( m \). One can see (following, for instance, estimates (3.9) in [16]) that the densities \( \varphi^n_\omega \) of the \( \nu^n_\omega \) belong to \( \mathcal{F}^+_\beta \), with constants \( C_{\varphi^n_\omega} \) depending only on \( K \) and \( \kappa \) as in our condition (U2). The hypotheses on the decay of return times imply that for \( P \text{ a.e. } \omega \) there is a subsequence \( n_\ell \to \infty \) so that \( \frac{1}{n_\ell} \sum_{k=0}^{n_\ell-1} \nu^k_\omega \) converges in the weak* topology to a probability measure on \( \Delta_\omega \), absolutely continuous with respect to \( m \). By a diagonalization argument, for \( P \text{ a.e. } \omega \) we can find a sequence \( n_j \) such that, for each integer \( N \), \( \frac{1}{n_j} \sum_{k=N}^{n_j-1} \nu^k_{\sigma^{-N}(\omega)} \) converges to a probability measure \( \nu_{\sigma^{-N}(\omega)} \) on the tower \( \Delta_{\sigma^{-N}(\omega)} \) with the desired properties.

Let us now define the correlations on \( \hat{\Delta} \). Given a family of measures \( \nu = \{ \nu_\omega \} \) in \( \hat{\Delta} \) and \( \varphi, \psi : \hat{\Delta} \to \mathbb{R} \), we set the **future correlation** as
\[
\hat{C}_\varphi^+(\varphi, \psi, \nu, n) = \int_{\Delta_\omega} (\varphi_{\sigma^n(\omega)} \circ F^n_\omega) \psi_\omega d\nu_\omega - \int_{\Delta_\omega} \varphi_{\sigma^n(\omega)} d\nu_{\sigma^n(\omega)} \int_{\Delta_\omega} \psi_\omega d\nu_\omega
\]
and, similarly, the **past correlation**
\[
\hat{C}_\varphi^-(\varphi, \psi, \nu, n) = \int_{\Delta_{\sigma^{-n}(\omega)}} (\varphi_\omega \circ F^n_{\sigma^{-n}(\omega)}) \psi_{\sigma^{-n}(\omega)} d\nu_{\sigma^{-n}(\omega)} - \int_{\Delta_\omega} \varphi_\omega d\nu_\omega \int_{\Delta_{\sigma^{-n}(\omega)}} \psi_{\sigma^{-n}(\omega)} d\nu_{\sigma^{-n}(\omega)}.
\]
We relate now the estimates on return times with the induced decay of correlations.

**Theorem 4.4.** Let \( \varphi = \{ \varphi_\omega \} \in \mathcal{L}^\infty \) and \( \psi = \{ \psi_\omega \} \in \mathcal{F}_\beta \).

(i) If there exist \( C_1, \gamma_1 > 0 \) and \( 0 < \nu \leq 1 \) such that \( m(R_\omega > n) \leq C_1 e^{-\gamma_1 n^\nu} \) then there exist \( C_i, \gamma_i > 0 \), \( i = 1, 2 \), and for \( P \text{ a.e. } \omega \) a positive integer \( n_0(\omega) \), such that
\[
\begin{align*}
\left\{ \begin{array}{ll}
\hat{C}_\varphi^+(\varphi, \psi, \nu, n) &\leq C_2 e^{-\gamma_2 n^\nu / 2}, \\
P(\{n_0(\omega) > n\}) &\leq C_3 e^{-\gamma_3 n^\nu / 2},
\end{array} \right. \quad \forall n \geq n_0(\omega)
\end{align*}
\]
for random perturbations of uniformly expanding maps see [14].

the decay of correlations along the random orbits. For estimates on decay of correlations perturbations for this maps were considered in [3, 9]. We obtain exponential estimates for isotopy inside some small region. In general, these maps are not uniformly expanding: maps and can be obtained, e.g. through deformation of a uniformly expanding map by unicity of SRB probability measures for this maps was proved in [4, 2]. This classes of expanding local diffeomorphisms (with no critical set) introduced in [4]. The existence and volume form. Let

\[ f \]

5.1. Local diffeomorphisms. We recall a robust \((C^1\) open) classes of non-uniformly expanding local diffeomorphisms (with no critical set) introduced in [4]. The existence and unicity of SRB probability measures for this maps was proved in [3, 2]. This classes of maps and can be obtained, e.g. through deformation of a uniformly expanding map by isotopy inside some small region. In general, these maps are not uniformly expanding: deformation can be made in such way that the new map has periodic saddles. Random perturbations for this maps were considered in [3, 9]. We obtain exponential estimates for the decay of correlations along the random orbits. For estimates on decay of correlations for random perturbations of uniformly expanding maps see [13].

Let \( M \) be the \( d \)-dimensional torus \( \mathbb{T}^d \), for some \( d \geq 2 \), and \( m \) the normalized Riemannian volume form. Let \( f_0: M \to M \) be a uniformly expanding map and \( V \subset M \) be a small neighborhood of a fixed point \( p \) of \( f_0 \) so that the restriction of \( f_0 \) to \( V \) is injective. Consider a \( C^1 \)-neighborhood \( \mathcal{U} \) of \( f_0 \) sufficiently small so that any map \( f \in \mathcal{U} \) satisfies:
(i) \( f \) is expanding outside \( V \): there exists \( \lambda_0 < 1 \) such that
\[
\|Df(x)^{-1}\| < \lambda_0 \quad \text{for every } x \in M \setminus V;
\]
(ii) \( f \) is volume expanding everywhere: there exists \( \lambda_1 > 1 \) such that
\[
|\det Df(x)| > \lambda_1 \quad \text{for every } x \in M;
\]
(iii) \( f \) is not too contracting on \( V \): there is some small \( \gamma > 0 \) such that
\[
\|Df(x)^{-1}\| < 1 + \gamma \quad \text{for every } x \in V;
\]
and, moreover, the constants \( \lambda_0, \lambda_1 \) and \( \gamma \) are the same for all \( f \in U \).

It was shown in \([9]\) how to perform the construction a bit more carefully in order to have topologically mixing maps, and thus transitive, by considering a map \( \bar{f} : M \to M \) in the boundary of the set of uniformly expanding maps which satisfies (i), (ii) and (iii) as the cartesian product of one-dimensional maps \( \varphi_1 \times \cdots \times \varphi_d \), with \( \varphi_1, \ldots, \varphi_{d-1} \) uniformly expanding in \( S^1 \), and \( \varphi_d \) the intermittent map in \( S^1 \): it can be written as
\[
\varphi_d(x) = x + x^{1+\alpha}, \quad \text{for some } 0 < \alpha < 1,
\]
in a neighborhood of 0 and \( \varphi'_d(x) > 1 \) for every \( x \in S^1 \setminus \{0\} \). If \( f \) is in a sufficiently small \( C^1 \) neighborhood \( \bar{U} \) of \( \bar{f} \), it satisfies (i), (ii) and (iii) for convenient choice of constants \( \lambda_0, \lambda_1 \), and a neighborhood \( V \) of the fixed point \( p = 0 \in \mathbb{T}^d \), and is topologically mixing.

For \( f \in U \) we introduce random perturbations \( \{\Phi, (\theta_\epsilon)_{\epsilon > 0}\} \). In particular, we consider a continuous map
\[
\Phi : T \to U, \quad t \mapsto f_t,
\]
where \( T \) is a metric space and \( f = f_t \), for some \( t^* \in T \). Consider a family \( (\theta_\epsilon)_{\epsilon > 0} \) of probability measures on \( T \) such that their supports are non-empty and satisfies \( \text{supp}(\theta_\epsilon) \to \{t^*\} \), when \( \epsilon \to 0 \). According to \([3]\), we can choose appropriately the constants \( \lambda_0, \lambda_1 \) and \( \gamma \) so that every map \( f \in U \) is non-uniformly expanding on all random orbits with uniform exponential decay of the Lebesgue measure of the tail sets \( \Gamma^n \) given by \([8]\), ignoring naturally the recurrence time function:

**Proposition 5.1.** Consider \( f \in U \) and \( \{\Phi, (\theta_\epsilon)_{\epsilon > 0}\} \) as before. There exists \( \alpha > 0 \) such that for every \( \omega \in \text{supp}(\theta^N) \) and \( m \) a.e. \( x \in M \)
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\sigma^j(\omega)}(f^j_\epsilon(x))^{-1}\| \leq -\alpha.
\]
Moreover, there is \( 0 < \tau < 1 \) such that \( m(\Gamma^n \omega) \leq \tau^n \), for \( n \geq 1 \) and \( \omega \in \text{supp}(\theta^N) \).

As application of Theorem \([A]\) we have the following.

**Theorem 5.2.** Let \( f \in \bar{U} \). Then, for some integer \( q \geq 1 \) and all sufficiently small \( \epsilon > 0 \):

(i) for \( P \) a.e. \( \omega \) there is an absolutely continuous probability measure \( \mu_\omega = h_\omega dm \) satisfying \( \langle f^*_\epsilon \rangle_{\mu_\omega} = \mu_{\sigma^q(\omega)} \);

(ii) there exist \( C_i, \gamma_i > 0 \), \( i = 1, 2 \), and for \( P \) a.e. \( \omega \) a positive integer \( n_0(\omega) \), such that for each Lipschitz function \( \psi : M \to \mathbb{R} \) and every bounded function \( \varphi : M \to \mathbb{R} \) we have
\[
C^{\pm}_\omega(\varphi, \psi, \mu, qn) \leq C_1 \sup |\varphi| \text{Lip}(\psi)e^{-\gamma_1 \sqrt{n}}, \forall n \geq n_0(\omega)
\]
and
\[
P(\{n_0(\omega) > n\}) \leq C_2 e^{-\gamma_2 \sqrt{n}}, \forall n \geq 1.
\]
5.2. Viana maps. We consider now an important open class of non-uniformly expanding maps with critical sets in higher dimensions introduced in [30]. Without loss of generality we discuss the two-dimensional case and we refer [30] for details.

Let \( p_0 \in (1, 2) \) be such that the critical point \( x = 0 \) is pre-periodic for the quadratic map \( Q(x) = p_0 - x^2 \). Let \( S^1 = \mathbb{R}/\mathbb{Z} \) and \( b : S^1 \to \mathbb{R} \) be a Morse function, for instance, \( b(s) = \sin(2\pi s) \). For fixed small \( \alpha > 0 \), consider the map

\[
\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad (s, x) \mapsto (\hat{g}(s), \hat{q}(s, x))
\]

where \( \hat{g} \) is the uniformly expanding map of the circle defined by \( \hat{g}(s) = ds \) (mod \( \mathbb{Z} \)) for some \( d \geq 16 \), and \( \hat{q}(s, x) = a(s) - x^2 \) with \( a(s) = p_0 + \alpha b(s) \). As it is shown in [3], it is no restriction to assume that \( C = \{(s, x) \in S^1 \times I : x = 0\} \) is the critical set of \( \hat{f} \) and we do so. If \( \alpha > 0 \) is small enough there is an interval \( I \subset (-2, 2) \) for which \( \hat{f}(S^1 \times I) \) is contained in the interior of \( S^1 \times I \). Any map \( f \) sufficiently close to \( \hat{f} \) in the \( C^3 \) topology has \( S^1 \times I \) as a forward invariant region (in fact, here it suffices to be \( C^1 \) close). We consider a small \( C^3 \) neighborhood \( \mathcal{V} \) of \( \hat{f} \) as before and will refer to maps in \( \mathcal{V} \) as Viana maps. Thus, any Viana map \( f \in \mathcal{V} \) has \( S^1 \times I \) as a forward invariant region, and so an attractor inside it, which is precisely

\[
\Lambda = \bigcap_{n \geq 0} f^n(S^1 \times I).
\]

In [8, Theorem C] it was proved a topological mixing property.

**Proposition 5.3.** For every \( f \in \mathcal{V} \) and every open set \( A \subset S^1 \times I \) there is some \( n_A \in \mathbb{N} \) for which \( f^{n_A}(A) = \Lambda \).

We introduce the random perturbations \( \{\Phi, (\theta_i)\}_i \) for this maps. We set \( T \subset \mathcal{V} \) to be a \( C^3 \) neighborhood of \( \hat{f} \) consisting in maps \( f \) restricted to the forward invariant region \( S^1 \times I \) for which \( Df(x) = D\hat{f}(x) \) if \( x \notin C \), the map \( \Phi \) to be the identity map at \( T \) and \( (\theta_i)_i \) a family of Borel measures on \( T \) such that their supports are non-empty and satisfy \( \text{supp}(\theta_i) \to \{f\} \), when \( \epsilon \to 0 \), for \( f \in T \). In [8] the authors realized that Viana maps are non-uniformly expanding and non-uniformly expanding on random orbits, and that there exist \( C, \gamma > 0 \) such that \( m(\Gamma^\omega_n) < Ce^{-\gamma n} \), for almost every \( \omega \in \text{supp}(\theta_i^n) \). We may conclude the following from Theorem A.

**Theorem 5.4.** Let \( f \in \mathcal{V} \) be a Viana map. Then, for some integer \( q \geq 1 \) and all sufficiently small \( \epsilon > 0 \):

(i) for \( P \) a.e. \( \omega \) there is an absolutely continuous probability measure \( \mu_\omega = h_\omega dm \) satisfying \( (f^n_\omega)_* \mu_\omega = \mu_\omega \);  

(ii) there exist \( C_i, \gamma_i > 0 \), \( i = 1, 2 \), and for \( P \) a.e. \( \omega \) a positive integer \( n_0(\omega) \), such that for each Lipschitz function \( \psi : M \to \mathbb{R} \) and every bounded function \( \varphi : M \to \mathbb{R} \) we have

\[
C_1^\pm(\varphi, \psi, \mu, qn) \leq C_2 \sup |\varphi| \text{Lip}(\psi)e^{-\gamma_1 n^{1/4}}, \quad \forall n \geq n_0(\omega)
\]

and

\[
P(\{n_0(\omega) > n\}) \leq C_2 e^{-\gamma_2 n^{1/4}}, \quad \forall n \geq 1.
\]
5.3. Unimodal maps. We consider now random perturbations for a class of unimodal maps as in [10]. In that paper the authors construct directly the induced structures with non-uniform decay of return times, meanwhile we are going to check the hypothesis of Theorem [3]. The improvements in the stretched exponential rates for the decay of random correlations as compared with [10] are not relevant, and the main motivation is to illustrate our techniques in a case where we are not in conditions to obtain uniform estimates for the decay of the tail sets along random orbits. In [15] the authors consider a similar family of unimodal maps, for which obtain uniform exponential decay of correlations with respect to the stationary measure, in contrast to our almost sure results.

We start by recalling the setting and refer for [10] for details. Let $I = [L, R]$ be a compact interval containing 0 in its interior and $f : I \rightarrow I$ be a $C^2$ unimodal map ($f$ is increasing on $[L, 0]$ and decreasing on $[0, R]$) satisfying: $f^n(0) \neq 0$, sup$_f |f'| < 8$, and

(H1) There are $0 < \alpha < 1$ and $1 < \lambda \leq 4$ with $200\alpha < (\log \lambda)^2$ for which

(i) $|(f^n)'(f(0))| \geq \lambda^n$, for all $n \geq 0$.

(ii) $|f^n(0)| \geq e^{-\alpha n}$, for all $n \geq 0$.

(H2) For each small enough $\delta > 0$, there is $N = N(\delta) \geq 0$ for which

(i) If $x, \ldots, f^{N-1}(x) \notin (-\delta, \delta)$ then $|(f^N)'(x)| \geq \lambda^N$.

(ii) For each $n$, if $x, \ldots, f^{n-1}(x) \notin (-\delta, \delta)$ and $f^n(x) \in (-\delta, \delta)$, then $|(f^n)'(x)| \geq \lambda^n$.

(H3) $f(I)$ is a subset of the interior of $I$.

(H4) $f$ is topologically mixing on $[f^2(0), f(0)]$.

Examples of unimodal maps satisfying this hypothesis are the quadratic maps $f_a(x) = a - x^2$ for a positive Lebesgue measure set of (Benedicks-Carleson. [18]) parameters $a$.

Fixing $\epsilon_0 > 0$ small enough to guarantee $f(x) \pm \epsilon_0 \in I$ for all $x \in I$, we assume that we are given a constant $D > 0$ and for each $0 < \epsilon < \epsilon_0$ a probability measure $\theta_\epsilon$ on $T = T_\epsilon = [-\epsilon, \epsilon]$, such that for any subinterval $J \subset T$,

$$\theta_\epsilon(J) \leq \frac{D|J|}{\epsilon}$$  \hspace{1cm} (9)

Assumption [3] may be relaxed, but it cannot be completely suppressed since there are open intervals of parameters corresponding to periodic attractors arbitrarily close to $a$. Assumption [3] holds for instance if $\theta_\epsilon$ has a density with respect to Lebesgue which is bounded above by $D/\epsilon$. We stress that this does not imply that 0 belongs to the support of $\theta_\epsilon$. We consider $\Omega = \Omega_\epsilon = T^2$, $P = P_\epsilon = \theta_\epsilon^2$ and for $t \in T$ we set $\Phi(t) = f_t(x) = f(x) + t$.

We assume that $f$ is a transitive non-uniformly expanding and non-uniformly expanding (with slow recurrence to the critical set $C = \{0\}$) and notice that it holds for $f_a$ with Benedicks-Carleson parameter $a$; see [25]. We will see that we are in conditions to apply Theorem [13] to get the following result.

**Theorem 5.5.** For some integer $q \geq 1$, if $\epsilon > 0$ is small:

(i) For $P$ a.e. $\omega$ there is an absolutely continuous probability measure $\mu_\omega = h_\omega dm$ satisfying $(f_q^*)_* \mu_\omega = \mu_{\sigma^q(\omega)}$;

(ii) there exist $C_1, \gamma_i > 0$, $i = 1, 2$, and for $P$ a.e. $\omega$ a positive integer $n_0(\omega)$ such that for each Lipschitz function $\psi : I \rightarrow \mathbb{R}$ and every bounded function $\varphi : I \rightarrow \mathbb{R}$ we have

$$C_1 \sup |\varphi| \text{ Lip}(|\psi|) e^{-\gamma_1 n^{1/4}}, \forall n \geq n_0(\omega),$$

where $C_1$ is a constant.
and
\[ P\{\{n_0(\omega) > n\} \leq C_2 e^{-\gamma_0 n^{1/4}}, \quad \forall n \geq 1. \]

### 5.3.1. Non-uniform expansion and slow recurrence.
In \[7.1\] and \[7.2\] the authors followed some ideas from \[30\] and also \[1\] in order to have some estimates on the recurrence near the critical set for random orbits of points in the interval. We are going to discuss how to translate those estimates to our framework. Consider \( \eta > 0 \) such that
\[
\frac{2\alpha}{\log \lambda} < \eta < \frac{1}{10}.
\]

For \( r \in \mathbb{Z}_+ \) let \( I_r = (\sqrt{e}^{-r}, \sqrt{e}^{-(r-1)}) \), and \( I_r = -I_{|r|} \) for \( r \leq -1 \). For \( k \geq 1 \) we introduce the functions \( r_k: \Omega \times I \to \mathbb{Z}_+ \), by setting \( r_k(\omega, x) = |r| \) if \( f^k_\omega(x) \in I_r \) and \( r_k(\omega, x) = 0 \) otherwise, and sets
\[
G_k(\omega, x) = G_k(\omega, x) = \left\{ 0 \leq j \leq k : r_j(\omega, x) \geq \max \left\{ 1, \left( \frac{1}{2} - 2\eta \right) \log(1/\epsilon) \right\} \right\}.
\]

There are suitably small \( c > 0 \) and large \( C > 1 \) such that for sufficiently small \( \epsilon > 0 \), large enough \( n \gg C \log(1/\epsilon) \) and all \((\omega, x)\) for which
\[
\sum_{j \in G_n(\omega, x)} r_j(\omega, x) \leq cn, \tag{10}
\]
we have \( |(f^n_\omega)'(x)| > e^{n/C} \). From \[16\] Corollary 7.5 we have the following.

**Lemma 5.6.** There are \( C(\epsilon) > 1, \gamma(\epsilon) > 1/(C \log(1/\epsilon)) \), and for each \( n \geq 1 \) there are sets \( E_n \subset \Omega \times I \) with \( (P \times m)(E_n) \leq C(\epsilon) e^{-\gamma(\epsilon)n} \), such that if \( (\omega, x) \notin E_n \) then condition \( \lambda \) holds.

From Lemma 5.6 we have \( \sum_{n \geq 1}(P \times m)(E_n) < \infty \), and by Borel-Cantelli’s lemma
\[
(P \times m) \left( \bigcap_{n \geq 1} \bigcup_{k \geq n} E_k \right) = 0.
\]

This means that for \( (P \times m) \) a.e. \((\omega, x)\) condition \( \lambda \) holds for every \( n \) large enough, implying that \( |(f^n_\omega)'(x)| > e^{n/C} \), and thus that \( f \) is non-uniformly expanding on random orbits. Moreover, given \( \zeta > 0 \) if we set
\[
E^n_\zeta = \left\{ (\omega, x) : \sum_{0 \not\in G_n(\omega, x)} r_j(\omega, x) \geq \zeta n, \right\},
\]
then, for small \( \zeta > 0 \), \( (P \times m)(E^n_\zeta) \leq C(\epsilon) e^{-\gamma(\epsilon)n} \). If we take \( \delta = (1/2 - 2\eta) \log(1/\epsilon) \), then, for \((\omega, x) \notin E^n_\zeta \) we have
\[
\frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta(f^j_\omega(x), C) \leq \zeta.
\]

Proceeding as before, we may conclude that \( f \) has slow recurrence to the critical set on random orbits. The tail sets can be considered as \( \Gamma^n = \{ x : (\omega, x) \in E_n \cup E^n_\zeta \} \subset I \). The rate of decay of the Lebesgue measure of this sets is estimated to be exponential, but not uniform on \( \omega \): by Fubini’s theorem there exists \( C = C(\epsilon), \gamma = \gamma(\epsilon) > 0 \) such that
\[
(P \times m)(\Gamma^n) \leq Ce^{-\gamma n},
\]
where $\Gamma^n = \{ (\omega, x) : x \in \Gamma^n_\omega \}$. With the following lemma we are in conditions to apply Theorem \[3\] and conclude Theorem \[5.5\]

**Lemma 5.7.** If there exist $C, \gamma > 0$ and $0 < \nu \leq 1$ such that for all $n \geq 1$

$$ (P \times m)(\Gamma^n) < Ce^{-\gamma n^\nu} $$

then there exist $C_i, \gamma_i > 0$, $i = 1, 2$, and for $P$ a.e. $\omega$ a positive integer $g_0(\omega)$ such that

$$
\begin{cases}
    m(\Gamma^n_\omega) \leq C_1 e^{-\gamma n^\nu}, & \forall n \geq g_0(\omega) \\
    P(\{g_0(\omega) > n\}) \leq C_2 e^{-\gamma n^\nu}, & \forall n \geq 1.
\end{cases}
$$

**Proof.** By Fubini’s theorem,

$$ P \left( \left\{ \omega : m(\Gamma^n_\omega) > \sqrt{Ce^{-\gamma n^\nu}} \right\} \right) \leq \sqrt{Ce^{-\gamma n^\nu}}, \quad (11) $$

otherwise we are lead into a contradiction: $(P \times m)(\Gamma^n) = \int_\Omega m(\Gamma^n_\omega) dP > Ce^{-\gamma n^\nu}$. Set

$$ B_n = \left\{ \omega : \exists m \geq n \text{ s.t. } m(\Gamma^n_\omega) > \sqrt{Ce^{-\gamma n^\nu}} \right\}. $$

By (11), we have

$$ P(B_n) \leq \sum_{k=n}^{\infty} P \left( \left\{ \omega : m(\Gamma^n_\omega) > \sqrt{Ce^{-\gamma k^\nu}} \right\} \right) < \infty, $$

so that $\lim_{n \to \infty} P(B_n) = 0$. For $\omega$ in the $P$ full measure subset $\cup_n (\Omega \setminus B_n)$ of $\Omega$ we define

$$ g_0(\omega) = \min \{ n \geq 0 : \omega \notin B_n \}. $$

\[ \square \]

6. Random Gibbs-Markov-Young structures

In this part we get the random GMY structure for the random NUE system. We prove Theorem \[4.2\] in \[6.3\]. We firstly derive uniform expansion and bounded distortion, then we simulate the GMY structure given in \[6\] for partially hyperbolic attractors with non-uniformly expanding direction. Fix $B > 1$ and $\beta > 0$ as in the definition of the critical set $C$, and take a constant $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$.

**Definition 6.1.** For $0 < \lambda < 1, \delta > 0$, we call $n$ a $(\lambda, \delta)$-hyperbolic time for $(\omega, x) \in \Omega \times M$ if for all $1 \leq k \leq n$

$$ \prod_{j=n-k+1}^{n} \| Df_{\sigma^j(\omega)}(f^j_\omega(x))^{-1} \| \leq \lambda^k, \quad \text{and} \quad \text{dist}_\delta(f^{n-k}_\omega(x), C) \geq \lambda^k. $$

In the case of $C = \emptyset$ the definition of $(\lambda, \delta)$-hyperbolic time reduces to the first condition. Hyperbolic times were introduced in \[1\] for deterministic systems and extended in \[3\] to a random context. We recall the following results from \[9\].

**Proposition 6.2.** Given $\lambda < 1$ and $\delta > 0$, there exist $\delta_1, C_0 > 0$ only depending on $\lambda, \delta$ and $f$, such that, if $n$ is $(\lambda, \delta)$-hyperbolic time for $(\omega, x) \in \Omega \times M$, then there is a neighbourhood $V^n_\omega(x)$ of $x$ in $M$ s.t.:  

1. $f^n_\omega$ maps $V^n_\omega(x)$ diffeomorphically onto $B(f^n_\omega(x), \delta_1)$;
2. for every $y \in V^n_\omega(x)$ and $1 \leq k \leq n$ we have $\| Df^{n-k}_\omega(y)^{-1} \| \leq \lambda^{k/2}$. 

for every $1 \leq k \leq n$ and $y, z \in V^+_n$,
$$\text{dist}(f^{-k}_\omega(y), f^{-k}_\omega(z)) \leq \lambda^{k/2} \text{dist}(f^n_\omega(y), f^n_\omega(z));$$
(4) (Bounded Distortion) for any $y, z \in V^+_n(x)$,
$$\log \frac{\left| \det Df^n_\omega(y) \right|}{\left| \det Df^n_\omega(z) \right|} \leq C_0 \text{dist}(f^n_\omega(y), f^n_\omega(z)).$$

We call the sets $V^+_\omega$ as hyperbolic pre-balls and $B(f^n_\omega(x), \delta_1) = f^n_\omega(V^+_\omega)$ hyperbolic balls.

Proposition 6.3. There exist $0 < \lambda < 1$, $\delta > 0$ and $0 < \zeta \leq 1$ such that for every $\omega \in \Omega$ and every $x \in M$ with $E_\omega(x) \leq n$ and $R_\omega(x) \leq n$, there exist $(\lambda, \delta)$-hyperbolic times $1 \leq n_1 < \cdots < n_t \leq n$ for $(\omega, x)$ with $l \geq \zeta n$.

For technical reasons, more precisely in Item (1) of Lemma 6.9, we shall take $\delta'_1 = \frac{\delta}{\lambda^2} > 0$ and consider $V^+_\omega(x)$ the part of $V^+_\omega(x)$ which is sent by $f^n_\omega$ onto $B(f^n_\omega(x), \delta'_1)$. The sets $V^+_\omega(x)$ will also be called hyperbolic pre-balls. The next lemma is an immediate consequence from [7, Lemma 2.4, 2.5]; see also [9, Lemma 4.14].

Lemma 6.4. There are $\delta_0 > 0$, a point $p \in M$ and $N_0 \in \mathbb{N}$ s.t., if $\varepsilon$ is sufficiently small, for any hyperbolic pre-ball $V^+_\omega(x)$ and every $\omega \in \text{supp}(P)$ there exists $0 \leq m \leq N_0$ for which $\Delta = B(p, \delta_0) \subset f^{n+m}_\omega(V^+_\omega(x))$ and $f^{n+m}_\omega(V^+_\omega(x))$ is disjointed from the critical set $\mathcal{C}$. Moreover, there are $D_0, K_0$ such that, for every $\omega \in \text{supp}(P)$ we have

(1) for each $x, y \in V$
$$\log \frac{\left| \det Df^n_\omega(x) \right|}{\left| \det Df^n_\omega(y) \right|} \leq D_0 \text{dist}(f^n_\omega(x), f^n_\omega(y));$$
(2) for each $0 \leq j \leq m$ and for all $x \in f^j_\omega(V)$ we have
$$K_0^{-1} \leq \| Df^j_\omega(x) \|, \| (Df^j_\omega(x))^{-1} \|, \left| \det Df^j_\omega(x) \right| \leq K_0;$$

in particular $f^j_\omega(V) \cap \mathcal{C} = \emptyset$.

In the following, we fix the two disks centered at $p$
$$\Delta^0 = \Delta = B(p, \delta_0) \quad \text{and} \quad \Delta^1 = B(p, 2\delta_0).$$
We actually need $\Delta^1 = B(p, 2\delta_0) \subset f^{n+m}_\omega(V^+_\omega(x))$.

6.1. The auxiliary partition. We construct the random Markov partition $\mathcal{P}_\omega$ on the reference disk $\Delta$ found in the previous section, for each $\omega \in \Omega$. Basically, it is a random version of the construction in [6]. For $\omega \in \Omega$, $n \geq 1$, we define
$$H^0_\omega = \{ x \in M : n \text{ is a } (\lambda, \delta)-\text{hyperbolic time for } (\omega, x) \}.$$ 

We set $\Delta^c = M \setminus \Delta$. Given a point $x \in H^0_\omega \cap \Delta$, there is a hyperbolic pre-ball $V^+_\omega(x)$, and for $0 \leq m \leq N_0$ as in Lemma 6.4 we define
$$U_{n,m}^i(x) = (f^{n+m}|_{V^+_\omega(x)})^{-1}(\Delta^i), \quad i = 0, 1.$$ 

These $U_{n,m}^i$ are the candidates for $\mathcal{P}_\omega$ with uniform expansion and bounded distortion of $f^{n+m}_\omega$. Notice that the recurrence time is given by
$$R_\omega(x) = n + m \quad \text{for} \quad x \in U_{n,m}^i.$$
We introduce sets $\Delta^n$ and $S^n_\omega$: $\Delta^n$ is the part of $\Delta$ that has not been chosen until time $n$; $S^n_\omega$, the satellite set, is to make sure we may collect the remaining part of the hyperbolic pre-ball when the $n$-step’s elements of partition have been taken, more precisely, to get 

$$H_\omega^n \cap \Delta \subset S^n_\omega \bigcup \{R_\omega < n + N_0\}.$$ 

At each step of the algorithm there is a unique hyperbolic time and possibly several return times. For $k \geq n$, we construct the annulus around the element $U^n_\omega = U^n_{n,m}$.

$$A^k_\omega(U^n_\omega) = \{y \in V^n_\omega(x): 0 \leq \text{dist}(f^k_{\omega}(U^n_\omega)(y), \Delta) \leq \delta_0 \lambda^{\frac{k-n}{2}}\}. \quad (13)$$

Obviously

$$A^n_\omega(U^n_\omega) \cup U^n_\omega = U^{1,x}_{n,m}.$$ 

First step of induction. Given the initial time $R_0 \in \mathbb{N}$ and consider the dynamics after time $R_0$ (can be taken independent of $\omega$); to be determined in Section 6.2 this paragraph we omit $\epsilon$ in $R_0$. There are finitely many points $I_{R_0} = \{ z_1, \ldots, z_{N_{R_0}} \} \in H_{R_0} \cap \Delta$ such that

$$H_{R_0} \cap \Delta \subset V_{R_0}^R(z_1) \cup \ldots \cup V_{R_0}^R(z_{N_{R_0}}).$$

We take a maximal family of pairwise disjoint sets of type (12) contained in $\Delta$,

$$\{U^{1,x_0}_{R_0,m_0}, U^{1,x_1}_{R_0,m_1}, \ldots, U^{1,x_k\prime}_{R_0,k_{R_0}}\},$$

where $\{x_0, \ldots x_{k_{R_0}}\} \subset I_{R_0}$.

And set

$$U_{R_0} = \{U^{0,x_0}_{R_0,m_0}, U^{0,x_1}_{R_0,m_1}, \ldots, U^{0,x_{k\prime}_{R_0}}_{R_0,k_{R_0}}\}$$

Now we get the elements of the partition $P_\omega$ at $R_0$-step.

The recurrence time is $R_\omega(U_{R_0,m_i}) = R_0 + m_i$ with $0 \leq i \leq k_{R_0}$. Recalling (13), we define

$$A_{R_0}^0(U_{R_0}) = \bigcup_{U \in U_{R_0}} A_{R_0}^0(U_{\omega}).$$

We pay attention to the sets $\{U^{1,x_m}_{R_0,m}: z \in I_{R_0}, 0 \leq m \leq N_0\}$ which intersect $U_{R_0} \cup A^0_{R_0}(U_{R_0})$ or $\Delta^c$. Given $U_{\omega} \in U_{R_0}^\prime$, for each $0 \leq m \leq N_0$, we define

$$I^m_{R_0}(U_\omega) = \{x \in I_{R_0} : U^{1,x}_{R_0,m} \cap (U_\omega \cup A_{R_0}^0(U_\omega)) \neq \emptyset\};$$

the $R_0$-satellite around $U_\omega$ is

$$S_{R_0}^0(U_\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I^m_{R_0}(U_\omega)} V^0_{R_0}(x) \cap (\Delta \setminus U_\omega),$$

The union

$$S_{R_0}^0(\Delta) = \bigcup_{U_\omega \in U_{R_0}^\prime} S_{R_0}^0(U_\omega).$$

Similarly, the $R_0$-satellite for $\Delta^c$ is

$$S_{R_0}^0(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in U^{1,x}_{R_0,m} \cap \Delta^c \neq \emptyset} V^0_{R_0}(x) \cap \Delta.$$
We will show in the general step, the volume of $S^{R_0}_ω(\Delta^c)$ is exponentially small. The ‘global’ $R_0$-satellite is

$$S^{R_0}_ω = \bigcup_{U_ω \in U^{R_0}_ω} S^{R_0}_ω(U_ω) \cup S^{R_0}_ω(\Delta^c).$$

The remaining portion at step $R_0$ is

$$\Delta^{R_0}_ω = \Delta \setminus U^{R_0}_ω.$$

Clearly,

$$H^{R_0}_ω \cap \Delta \subset S^{R_0}_ω \cup U^{R_0}_ω.$$

**General step of induction.** The general step of the construction follows the ideas in the first step with minor modifications. We assume $U^1_ω, S^1_ω, A^1_ω, \Delta^1_ω$, $\{R_ω = j + m\}$ are defined for all $0 \leq j \leq n - 1$. As before, there is a finite set of points $I_n = \{z_1, \ldots, z_{N_n}\} \in H^n_ω \cap \Delta$ such that

$$H^n_ω \cap \Delta \subset V^m_ω(z_1) \cup \cdots \cup V^m_ω(z_{N_n}).$$

We get $U^i_ω, A^i_ω$ and $S^i_ω$ for $i \leq n - 1$. Assuming

$$U^i_ω = \{U^{i,x_0,0}_ω, U^{i,x_1,0}_ω, \ldots, U^{i,x_{k_ν}}_ω\}$$

for $R_0 \leq \ell \leq n - 1$, let

$$A^n_ω(U^\ell_ω) = \bigcup_{U_ω \in U^\ell_ω} A^n_ω(U_ω).$$

We get a maximal family of pairwise disjoint sets of type \([12]\) contained in $\Delta^{n-1}_ω$,

$$\{U^{1,x_0,0}_ω, U^{1,x_1,0}_ω, \ldots, U^{1,x_{k_n}}_ω\}$$

where $\{x_0, \ldots, x_{k_n}\} \subset I_n$, satisfying

$$U^{1,x_i}_ω \cap (U^{n-1}_ω(A^n_ω(U^\ell_ω) \cup U^\ell_ω)) = \emptyset, \quad i = 0, \ldots, k_n,$$

and define

$$U^n_ω = \{U^{i,x_0,0}_ω, U^{i,x_1,0}_ω, \ldots, U^{i,x_{k_n}}_ω\}.$$

They are the $n$-step’s elements of the partition $P_ω$.

For $0 \leq i \leq \ell_n$, $x \in U^{i,x_i}_ω$, $R_ω(x) = n + m_i$. Given $U_ω \in U^{R_0}_ω \cup \cdots \cup U^{n}_ω$, $0 \leq m \leq N_0$,

$$I^m_n(U_ω) = \{x \in I_n : U^{i,x}_ω \cap (U_ω \cup A^n_ω(U_ω)) \neq \emptyset\},$$

define

$$S^m_n(U_ω) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I^m_n(U_ω)} V^m_ω(x) \cap (\Delta \setminus U_ω)$$

and

$$S^m_n(\Delta) = \bigcup_{U_ω \in U^{R_0}_ω \cup \cdots \cup U^\ell_ω} S^m_n(U_ω).$$

Similarly, the $n$-satellite associated to $\Delta^c$ is

$$S^m_n(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I^m_n(U_ω) \cap \Delta^c \neq \emptyset} V^m_ω(x) \cap \Delta.$$
Remark 6.5. By Proposition 6.2,
\[ S_n^\omega(\Delta^c) \subset \{ x \in \Delta : \text{dist}(x, \partial \Delta) \leq 2\delta_0 \lambda^{n/2} \}. \]
So there exists \( \rho > 0 \) such that \( m(S_n^\omega(\Delta^c)) \leq \rho \lambda^{n/2} \).

Finally we define the \( n \)-satellite for \( U^{\omega_R}_0 \cup \cdots \cup U^{\omega}_n \)
\[ S_n^\omega = S_n^\omega(\Delta) \cup S_n^\omega(\Delta^c) \]
and
\[ \Delta_n^\omega = \Delta \setminus \bigcup_{i=R_0}^n U_i^\omega. \]
Obviously
\[ H_n^\omega \cap \Delta \subset S_n^\omega \cup \bigcup_{i=R_0}^n U_i^\omega. \]

6.2. Expansion, bounded distortion and uniformity. The return time \( R_\omega \) for an element \( U_\omega \) of the partition \( P_\omega \) of \( \Delta \) is made by a hyperbolic time \( n \) plus \( m \leq N_0 \). We know \( f_\omega^{n+m}(V^{\omega_0}_n) \) covers \( \Delta \) completely. Then by Proposition 6.2 and Lemma 6.4,
\[ \| Df_\omega^{n+m}(x)^{-1} \| \leq \| Df_\omega^{n}(f_\omega^{m}(x))^{-1} \| \cdot \| Df_\omega^{n}(x)^{-1} \| \leq K_0 \lambda^{n/2} \leq K_0 \lambda^{(R_0 - N_0)/2}. \]
If we take \( R_0 \) sufficiently large, this is smaller than some \( \kappa < 1 \). We also need to show that there exists a constant \( K > 0 \) such that for any \( x, y \in U_\omega \) with return time \( R_\omega \), we have
\[ \log \left| \frac{\det Df_\omega^{R_\omega}(x)}{\det Df_\omega^{R_\omega}(y)} \right| \leq K \text{dist}(f_\omega^{R_\omega}(x), f_\omega^{R_\omega}(y)). \]
By Proposition 6.2 and Lemma 6.4 we choose \( K = D_0 + C_0 K_0 \).

Since \( K_0, C_0, D_0 \) and \( N_0 \) are independent of \( \omega \) then \( \kappa \) and \( K \) could be taken the same for all \( \omega \), leading us to condition (U2). Moreover, by the continuity of \( \Phi \) in the random perturbation \( \{ \Phi, \{ \theta_\epsilon \}_{\epsilon > 0} \} \), the algorithm provides partitions such that for any two realizations \( \omega, \omega' \) in \( \Omega \) and any natural number \( N \), the Lebesgue measure of the symmetric difference of sets \( \{ R_\omega = j \} \) and \( \{ R_{\omega'} = j \} \), for \( j = 1, \ldots, N \), is smaller than any given \( \xi > 0 \), as long as we take \( \epsilon \) sufficiently small. Since we do not assume a particular behavior for the decay of the tail set for the deterministic dynamics given by \( f \), we are not able to conclude the (stretched) exponential decay of the corresponding return times. However we can construct a partition for for the original dynamics \( f \) (given by the realization \( \omega^\ast \)) and consider it as reference to construct the elements of each partition \( P_\omega \) with return time lower than \( N \), obtaining condition (U1). This is of great utility to ensure condition (\( \ast \)); see Remark 7.1.1.
6.3. Tail set estimates. In this section we will show that if the tail set decays (stretched) exponentially fast, then the tail of the recurrence times decays (stretched) exponentially fast too. More precisely, given a local unstable disk $\Delta \subset M$ and constants $\gamma > 0$ and $0 < \nu \leq 1$, there is $\gamma_1 > 0$ such that

$$m\{\Gamma_n^\nu > n\} = O(e^{-\gamma_1 n^\nu}) \Rightarrow m\{R_\omega > n\} \leq O(e^{-\gamma_1 n^\nu}).$$

(15)

This is case (i) of Theorem 4.2. Case (ii) follows in the same way.

Before the key proof (proof for Proposition 6.10). We state some lemmas and notations for preparing. To simplify the notation, we avoid the superscript 0 in $U_{n,m}^{0,x}$. The next lemma and proposition are the random versions of [6, Lemma 3.5, Proposition 3.6].

**Lemma 6.6.** (1) There is $C_5 > 0$, for any $n \geq R_0$, $0 \leq m \leq N_0$ and finitely many $\{x_1, \ldots, x_N\} \in I_n$ satisfying $U_{n,m}^{x_i} = U_{n,m}^{x_1}$ ($1 \leq i \leq N$), we get

$$m\left(\bigcup_{i=1}^{N} V_{n,m}^{x_i}(x_i)\right) \leq C_5 m(U_{n,m}^{x_1}).$$

(2) There is $C_6 > 0$, for $k \geq R_0$, $U_\omega \in \mathcal{U}_k^\omega$ and $0 \leq m \leq N_0$, any $n \geq k$, we have

$$m\left(\bigcup_{x \in I_n(U_\omega)} U_{n,m}^{x}\right) \leq C_6 \lambda^\frac{n-k}{k} m(U_\omega).$$

**Proposition 6.7.** There is $C_7 > 0$ such that $\forall U_\omega \in \mathcal{U}_k^\omega$, and $n \geq k$, we get

$$m(S_{\omega}^m(U_\omega)) < C_7 \lambda^\frac{n-k}{k} m(U_\omega).$$

**Definition 6.8.** Given $k \geq R_0$ and $U_{k,m}^\omega \in \mathcal{U}_k^\omega$, $x \in \Delta$ and $0 \leq m \leq N_0$, for $n \geq k$ we define

$$B_n^k(x) = S_{\omega}^m(U_{k,m}^\omega) \cup U_{k,m}^x$$

and $t(B_n^k(x)) = k$.

Here $k$ and $n$ are hyperbolic times for points in $\Delta$. We call $U_{k,m}^x$ the core of $B_n^k(x)$ and sign it $C(B_n^k(x))$.

From Lemma 6.7 we easily get: $\forall n \geq k$ and $x$, we have

$$m(B_n^k(x)) \leq (C_7 + 1) m(C(B_n^k(x))).$$

The dependence of $\delta_1'$ on $\delta_1$ is clarified in the next lemma. The proof is similar with [6, Lemma 3.9, 3.10].

**Lemma 6.9.** (1) If $\delta_1' > 0$ is sufficiently small (only depending on $\delta_1$), for all $k' \geq k \geq R_0$, $n \geq k$, $n' \geq k'$ and $B_n^k(x) \cap B_n^{k'}(y) \neq \emptyset$, we have

$$C(B_n^k(x)) \cup C(B_n^{k'}(y)) \subset V_n^x(k);$$

(2) there exists $P \geq N_0$ such that for all $R_0 \leq t_1 \leq t_2$, $B_{t_2}^{t_2} \cap B_{t_1}^{t_1} = \emptyset$.

Now we come to the core of this section: to show (13), i.e. Theorem 4.2. Recalling Remark 6.5 similarly, there exists a constant $\rho > 0$ such that for all $n \in \mathbb{N}$

$$m\{x : \text{dist}(x, \partial \Delta) \leq 2\delta_0 \lambda^\frac{x}{x}\} \leq \rho \lambda^\frac{x}{x}.$$  

(16)

Recalling $\Delta_n^\omega$ is the complement part at step $n$, $\theta$ is defined in Proposition 6.3. We will show $m(\Delta_n^\omega)$ decays (stretched) exponentially. That is enough to conclude the proof since $m(\Gamma_n^\omega)$...
is (stretched) exponentially small and \( m\{x : \text{dist}(x, \partial \Delta) \leq 2\delta_0 \lambda^{\frac{m}{2}}\} \) decays exponentially as in (16).

Take \( x \in \Delta^n \), suppose \( x \notin \Gamma^n \cup \{x : \text{dist}(x, \partial \Delta) \leq 2\delta_0 \lambda^{\frac{m}{2}}\} \). By Proposition 6.3 for \( n \) large, \( x \) has at least \( \theta n \) hyperbolic times between 1 and \( n \), such that we have \( \frac{\theta n}{2} \leq t_1 < \cdots < t_k \leq n, k \geq \frac{\theta n}{2} \). We get \( x \in H_{\omega}^{\Delta_i} \cap \Delta \) for \( 1 \leq i \leq k \). Recalling (14),

\[
H_{\omega}^{\Delta_i} \cap \Delta \subset S_{\omega}^{t_i} \cup \bigcup_{j=R_0}^{t_i} U_j^{t_i}, \quad \text{for } 1 \leq i \leq k.
\]

If \( x \notin S_{\omega}^{t_i} \), \( x \in \bigcup_{j=R_0}^{t_i} U_j^{t_i} \) such that \( x \notin \Delta_n \). That is a contradiction. So \( x \in S_{\omega}^{t_i} \). Since \( x \in \{x \in \Delta : \text{dist}(x, \partial \Delta) > 2\delta_0 \lambda^{\frac{m}{2}}\} \), \( x \in H_{\omega}^{\Delta_i} \cap \{x \in \Delta : \text{dist}(x, \partial \Delta) > 2\delta_0 \lambda^{\frac{m}{2}}/2\} \), for \( 1 \leq i \leq k \). With Remark 6.5 we obtain \( x \notin S_{\omega}^{t_i} (\Delta^c) \). Consequently, \( x \in S_{\omega}^{t_i} (\Delta) \), for \( i = 1, \ldots, k \). We simply take \( k = \frac{\theta n}{2} \). Thus, \( x \) is contained in

\[
Z_\omega \left( \frac{\theta n}{2}, n \right) := \left\{ x : \exists t_1 < \cdots < t_\frac{\theta n}{2} \leq n, x \in \bigcap_{i=1}^{\frac{\theta n}{2}} S_{\omega}^{t_i} (\Delta) \right\} \cap \Delta_n.
\]

So we have

\[
\Delta_n \subset \Gamma^n \cup \{x \in \Delta : \text{dist}(x, \partial \Delta) \leq 2\delta_0 \lambda^{\frac{m}{2}}\} \cup Z_\omega (\theta n/2, n).
\]

See the first set in the union above decays exponentially fast from the assumption of Theorem 4.2; the second set in the union above is exponentially small by (16). In the following, we only need to show the measure of \( Z_\omega (\theta n/2, n) \) is exponentially small. That is Proposition 6.10.

Observe that if we have shown there exist \( C_1, \gamma_1 > 0 \) such that

\[
m(\Delta_n) \leq C_1 e^{-\gamma_1 n^\nu},
\]

then, for any large integer \( n \), we have \( R_n = \{R_\omega > n\} \subset \Delta_n \cap N_0 \), and so

\[
m(R_\omega > n) \leq m(\Delta_n - N_0) = C_1 e^{-\gamma_1 (n-N_0)^\nu} = C_1 e^{-\gamma_1 n^\nu}. \tag{17}
\]

We show the set of points which are contained in finitely many satellite sets and have not been chosen yet has a measure exponentially small.

**Proposition 6.10.** For \( k, N \in \mathbb{Z}_+ \),

\[
Z_\omega(k, N) = \left\{ x : \exists t_1 < \cdots < t_k \leq N, x \in \bigcap_{i=1}^{k} S_{\omega}^{t_i} (\Delta) \cap \Delta^N \right\}.
\]

There are \( D_3 > 0 \) and \( \lambda_3 < 1 \), for all \( N \) and \( 1 \leq k \leq N \),

\[
m(Z_\omega(k, N)) \leq D_3 \lambda_3^k m(\Delta).
\]

In order to prove this result we need several pre-lemmas in the sequel. We fix some integer \( P' \geq P \) (see \( P \) in Lemma 6.9 see the proof of Proposition 6.10) In the following, for some \( t_i, x, m_i \leq P' \) we denote \( B_i = B_{t_i+m_i}(x) \). The proof of the next two lemmas may be found in [6, Lemma 3.14, 3.15].
Lemma 6.11. Set

\[ Z^1_\omega(k, N) = \left\{ x : \exists t_1 < \ldots < t_k \leq N, m_1, \ldots, m_q < P', \right. \]
\[ \left. x \in S^t_{\omega_1 + m_1}(U^t_{\omega_1}) \cap \ldots \cap S^t_{\omega_k + m_k}(U^t_{\omega_k}) \cap \Delta^N_\omega \right\}. \]

There are constants \( D_1 > 0 \) and \( \lambda_2 < 1 \) (both independent of \( P' \)) such that, for all \( N \) and \( 1 \leq k \leq N \),

\[ m(Z^1_\omega(k, N)) \leq D_1 \lambda_2^k m(\Delta). \]

Lemma 6.12. Given \( B_1 = B^1_{\omega_1}(e_1) \), let

\[ Z^2_\omega(n_1, \ldots, n_k, B_1) = \left\{ x : \exists t_2, \ldots, t_k \text{ with } t_1 < \ldots < t_k; n_1, \ldots, n_k > P; \text{and } x_2, \ldots, x_k, \right. \]
\[ \left. \text{s.t. } x \in \bigcup_{i=1}^{k} B^i_{t_i + n_i}(x_i) \cap \Delta^N_\omega \right\}. \]

Then, there is \( D_2 > 0 \) (independent of \( B_1, n_1, \ldots, n_k \)) such that for \( n_1, \ldots, n_k > P \),

\[ m(Z^2_\omega(n_1, \ldots, n_k, B_1)) \leq D_2 (D_2 \lambda^{n_1/2} \ldots (D_2 \lambda^{n_k/2}) m(C(B_1)). \]

Then we complete the proof of the metric estimates.

Proof of Proposition 6.10 Take \( P' \geq P \) (recall \( P \) in Lemma 6.9) such that \( \lambda^{1/2} + D_1 \lambda^{P'/2} < 1 \).

Let \( x \in Z_\omega(k, N) \), we have all the instants \( u_i \) for which \( x \in S^t_{\omega_i + m_i}(U^t_{\omega_i,m_i}) \) with \( n_i \geq P' \), ordered as \( u_1 < \ldots < u_p \). Then \( x \in Z^2_\omega(n_1, \ldots, n_p, B_1) \) for some \( B_1 \). If \( \sum_{i=1}^{p} n_i \geq k/2 \), we are done. Otherwise, we have \( \sum_{i=1}^{p} n_i < k/2 \) and \( p < k/2P' \). Then \( v_1 < \ldots < v_q \) be the other instants for which \( x \in S^t_{\omega_i + m_i}(U^t_{\omega_i,m_i}) \), where \( m_1, \ldots, m_q < P' \). Obviously \( p + q \geq k \), so that \( q \geq \frac{(2P'-1)k}{2^{P'}} \geq \frac{k}{2^{P'}} \), where \( P' > 1 \). So \( P'q \geq \frac{k}{2} \). Thus we obtain

\[ Z_\omega(k, N) \subseteq \bigcup_{B_1} \bigcup_{\sum n_i \geq \frac{k}{2P'}} Z^2_\omega(n_1, \ldots, n_p, B_1) \cup Z^1_\omega \left( \frac{k}{2P'}, N \right). \]

By Lemma 6.11 and 6.12 we obtain

\[ m(Z_\omega(k, N)) \leq \sum_{B_1} \sum_{\sum n_i \geq \frac{k}{2P'}} D_2 (D_2 \lambda^{n_1/2} \ldots (D_2 \lambda^{n_p/2}) m(C(B_1)) + D_1 \lambda_2^{P'/2} m(\Delta). \]

We know \( \sum_{B_1} m(C(B_1)) \leq m(\Delta) < \infty \) as the cores \( C(B_1) \) are pairwise disjoint. There are constants \( D_4 > 0 \) and \( \lambda_4 < 1 \) such that

\[ \sum_{\sum n_i \geq \frac{k}{2P'}} (D_2 \lambda^{n_1/2} \ldots (D_2 \lambda^{n_p/2}) \leq D_4 \lambda_4^k. \]

Sum over \( n \geq k/2 \) and \( B_1 \), we obtain constants \( D_3 > 0 \), \( \lambda_3 < 1 \) such that

\[ m(Z_\omega(k, N)) \leq D_3 \lambda_3^k m(\Delta). \]

□
7. Decay of Correlations on Random Young Towers

In this section we prove Theorem 4.4. We start by compiling in §7.1 and §7.2 some definitions and key results from [16], which are randomised versions of that in [32]. Then, in §7.3 we transpose the hypotheses on the return times to the estimates on the (joint) return times. Finally, in §7.4 we give the estimates on the induced decay of correlations. We will focus on the future time results, being that the results for the past correlations are the recycling of the arguments for the future ones, as noticed in [16, §6].

7.1. Mixing. Recall the abstract setting $\hat{\Delta} = \{\Delta_\omega\}_\omega$ with the dynamics of the fibered map $F = \{F_\omega\}_\omega$ that we call the induced skew product. We concern now to the mixing properties of the induced skew product with respect to a measure $\nu$ whose disintegration $d\nu(\omega, x) = \nu_\omega(x) dP(\omega)$ is given by the family $\{\nu_\omega\}_\omega$ of sample measures constructed at Theorem 4.3 (for $A \in \mathcal{B}$, we have $\nu(A) = \int \nu_\omega(A_\omega) dP(\omega)$. For $n \geq 1$ we set $F^n_\omega$ for the compositions $F_{\sigma^{n-1}(\omega)} \circ \cdots \circ F_\omega$ and also $F^{-n}(\mathcal{B})$ for the family $\{(F^n_\omega)^{-1}(\mathcal{B}_{\sigma^n(\omega)})\}_\omega$. Let $L^2(\nu)$ denote the space of functions $\phi = \{\phi_\omega\}_\omega : \hat{\Delta} \to \mathbb{R}$ such that $\phi_\omega \in L^2(\mathcal{B}_{\omega}, \nu_\omega)$ for $P$ a.e. $\omega$, and $\int_\Omega \int_{\Delta_\omega} |\phi_\omega|^2 d\nu_\omega dP < \infty$.

Definition 7.1. We say that the random skew product $(F_\nu)$ is

1. exact if each $B \in \mathcal{B}$ belonging to $F^{-n}(\mathcal{B})$ for all $n \geq 0$ is trivial (i.e., for almost every $\omega$, either $\nu_\omega(B) = 0$ or $\nu_\omega(B) = 1$);
2. mixing if for all $\varphi, \psi \in L^2(\nu)$,

$$\lim_{n \to +\infty} \left| \int_\Omega \int_{\Delta_\omega} (\varphi_{\sigma^n(\omega)} \circ F^n_\omega) \psi d\nu_\omega dP - \int_\Omega \int_{\Delta_\omega} \varphi_\omega \psi d\nu_\omega dP \int_\Omega \int_{\Delta_\omega} \psi d\nu_\omega dP \right| = 0.$$

Proposition 7.2. If $(F, \nu)$ is exact then it is mixing.

However, for the exactness (and mixing) of $(F, \nu)$ we need to assume the following:

(*) there are $L_0(\epsilon)$ and $t_i \in \mathbb{Z}_+, 1 \leq i \leq L_0$, with $g.c.d. \{t_i\} = 1$ such that for $P$ a.e. $\omega$, $m(\{R_\omega = t_i\}) > 0$.

Proposition 7.3. $(F, \nu)$ is exact (and thus mixing).

7.1.1. A remark on the return times. One knows that there exists a GMY structure for $f$. If $g.c.d. \{R_f\} = 1$ then there are $L_0$ and $t_i \in \mathbb{Z}_+, 1 \leq i \leq L_0$, with $g.c.d. \{t_i\} = 1$ such that $m(\{R_f = t_i\}) > 0$. Hence, condition (U1) ensures that (*) hold, just considering $\hat{N} = tL_0$ and $\xi$ sufficiently small (only depending on $\{R_f\}$). On the other hand, if $g.c.d. \{R_f\} = q > 1$ then, the previous partition related to $f$ also provides a partition for $f^q$, with $R_{f^q} = R_f/q$ and, in this case, $g.c.d. \{R_{f^q}\} = 1$. Thus, we look then for the $q$th iterate $f^q_{\omega}$ for the random systems. We consider the partitions $\{R_\omega = k\} = \{R_\omega = q \cdot k\}$ and the fibered dynamics $\hat{F}$ ("$F^q$") in the new towers $\hat{\Delta} = \{\Delta_\omega\}_\omega = \{\cup_{k=0}^{\infty} \Delta_{\omega^k}\}_\omega$. Similarly to Theorem 4.3 we may obtain a family of probability measures $\{\hat{\mu}_\omega\}$ that constitutes a disintegration of an $\hat{F}$-invariant probability measure $\hat{\nu}$, satisfying $(\hat{F}_\omega)_* \hat{\nu}_\omega = \hat{\nu}_{\sigma(\omega)}$. Their projections $\hat{\mu}_\omega = \pi_* \hat{\mu}_\omega$ are absolutely continuous probability measures for which $(f^q_{\omega})_* \hat{\mu}_\omega = \hat{\mu}_{\sigma^q(\omega)}$, and the measure $\hat{\mu} = \{\hat{\mu}_\omega\}$ is invariant for the power $S^q$ of the skew product. Once again, by (U1) the condition (*) hold provided $\epsilon$ is sufficiently small. In this case, our proofs yield the (stretched) exponential decay of correlations for the $q$th iterate of the perturbed system. If we are able to guarantee a GMY induced map for $f$ with $g.c.d. \{R_f\} = 1$, then the main
results hold with \( q = 1 \). For simplicity in the exposition, henceforth we will always assume that (\( \ast \)) hold.

7.2. Converging to equilibrium.

7.2.1. Stopping times and joint returns. Let us define the random variable

\[
V^\ell_\omega = m(\Delta_{\omega,0} \cap (F^\ell_\omega)^{-1}(\Delta_{\sigma^\ell(\omega),0})).
\]

From condition (\( \ast \)) we may consider \( \ell_0 \in \mathbb{N} \) such that, for \( P \) a.e. \( \omega \) and \( \ell \geq \ell_0 \), we have \( V^\ell_\omega > 0 \). For \( \omega \in \Omega \) and \( (x,x') \in \Delta_{\omega} \times \Delta_{\omega} \) we introduce the stopping times \( \tau^\ell_\omega(x,x') \) as follows:

\[
\begin{align*}
\tau^1_\omega(x,x') &= \inf\{n \geq \ell_0 : F^n_\omega(x) \in \Delta_{\sigma^n(\omega),0}\}, \\
\tau^2_\omega(x,x') &= \inf\{n \geq \ell_0 + \tau^1_\omega(x,x') : F^n_\omega(x') \in \Delta_{\sigma^n(\omega),0}\}, \\
\tau^3_\omega(x,x') &= \inf\{n \geq \ell_0 + \tau^2_\omega(x,x') : F^n_\omega(x) \in \Delta_{\sigma^n(\omega),0}\}, \\
&\vdots
\end{align*}
\]

with the action alternating between \( x \) and \( x' \). We define then the joint return time \( T^\ell_\omega(x,x') \) to be the smallest integer \( \tau^\ell_\omega = \tau^\ell_\omega(x,x') \geq \ell_0 \) such that \( (F^\ell_\omega(x), F^\ell_\omega(x')) \) belongs to \( \Delta_{\sigma^\ell(\omega),0} \times \Delta_{\sigma^\ell(\omega),0} \), with \( i \geq 2 \). Note that \( \tau^\ell_\omega - \tau^{i-1}_\omega \geq \ell_0 \) and \( T^\ell_\omega(x,x') \geq 2\ell_0 \). Given \( \omega \) and \( j \geq 1 \), we consider also the partition \( \xi^j_\omega \) of \( \Delta_{\omega} \times \Delta_{\omega} \) into maximal subsets on which \( \tau^i_\omega \) is constant for all \( 1 \leq i \leq j \).

We should notice that even under hypothesis of uniform decay of the tail sets we are not able to guarantee an uniform control of random variables \( V^\ell_\omega \). Indeed, the induced sample measures \( \{\nu_\omega\}_\omega \) have densities uniformly bounded from above by \( K_1 \) but not from below (see [16] and, in particular, its corrigendum). In view of this we cannot exploit the mixing properties of the induced skew product, and we are endorsed to a large deviation arguments. As we will see later, this is the principal cause for successive damages on the (stretched) exponential estimates.

For \( q \in \mathbb{N} \) and each fixed sequence of integers \( 0 = \tau_0 < \tau_1 < \ldots < \tau_q \), with \( \tau_i - \tau_{i-1} \geq \ell_0 \), we set

\[
Q^{(\tau_i)}_q(\omega) = \sum_{i=1}^q V^{\tau_i - \tau_{i-1}}_{\sigma^{\tau_{i-1}}(\omega)}.
\]

**Lemma 7.4.** There exists \( \rho > 0 \) and \( 0 < \varrho < 1 \) such that for each \( q \in \mathbb{N} \) and every fixed sequence of integers \( 0 = \tau_0 < \tau_1 < \ldots < \tau_q \), with \( \tau_i - \tau_{i-1} \geq \ell_0 \), there is a set \( M^{(\tau_i)}_q \subset \Omega \), with \( P(M_q^{(\tau_i)}) \leq \varrho^q \) and such that if \( \omega \notin M_q^{(\tau_i)} \) then \( Q^{(\tau_i)}_q(\omega) \geq \rho q \).

From now on, let \( \lambda, \lambda' \) be absolutely continuous probability measures on \( \Delta \) with densities \( \varphi, \varphi' \in F^+_{\beta} \), and set \( \Lambda = \lambda \times \lambda' \). In particular, we could take \( \lambda \) or \( \lambda' \) as \( \nu \). Let us state a lower bound for \( \Lambda_\omega(\{T^\ell_\omega = \tau^\ell_\omega\}) \), by transposing the previous large deviation arguments for estimates on the sets on towers.

**Corollary 7.5.** Assume additionally that \( \varphi, \varphi' \in L^\infty \). There exist \( C > 0, 0 < \varrho < 1 \) and a random variable \( n_4(\omega) \) defined on a full \( P \)-measure subset of \( \Omega \) such that

\[
\begin{align*}
&\left\{ \begin{array}{l}
K_1\Lambda_\omega(\{(x,x') \in \Delta_{\omega} \times \Delta_{\omega} : Q^{(\tau^\ell_\omega(x,x'))}_n(\omega) \leq \rho n\}) \leq \varrho^{n/2}, \quad \forall n \geq n_4(\omega) \\
P(\{n_4(\omega) > n\}) \leq C\varrho^{n/2}, \quad \forall n \geq 1.
\end{array} \right.
\]


We give now some estimates on stopping times and joint return times.

**Lemma 7.6.** If $\Gamma \in \xi_\omega$ is such that $T_\omega|\Gamma > \tau_\omega^{-1}$, then letting $V_{\sigma_\omega^{-1}(\omega)}$ be associated to $\tau_\omega^{+}(\Gamma)$,

$$
\Lambda_\omega(\{T_\omega > \tau_\omega^{+}\}|\Gamma) \leq 1 - V_{\sigma_\omega^{-1}(\omega)}/C
$$

where $C = C(\lambda, \lambda')$ depends on the Lipschitz constants of $\varphi$ and $\varphi'$. This dependence can be removed if we consider $i \geq i_0(\varphi, \varphi')$.

We relate now the stopping times with the return times.

**Lemma 7.7.** For all $i, n \geq 0$ and $\Gamma \in \xi_\omega$ we have

$$
\Lambda_\omega(\{\tau_\omega^{i} - \tau_\omega^{i-1} > \ell_0 + n\}|\Gamma) \leq K_2 m(\{R_{\sigma_\omega^{i}+t_0(\omega)} > n\}) \cdot m(\Delta_{\sigma_\omega^{i}+t_0(\omega)}),
$$

where $K_2 = K_2(\varphi, \varphi')$ depends on the Lipschitz constants of $\varphi$ and $\varphi'$. This dependence can be removed if we consider $i \geq i_0(\varphi, \varphi')$.

From now on we will assume that $\varphi, \varphi' \in F_\beta^+ \cap L^\infty$.

### 7.3. From return times to joint stopping times

In this section we relate the return times to the joint stopping times. We deal separately with the uniform and non-uniform decay of the return times, as in the hypotheses of Theorem 4.3. We adapt the strategy in [26] to optimise the stretched estimates when comparing to the random version of Young’s work [32] used in [10]. However, as we will see in §7.3.2 in the non-uniform case there are some damages on the estimates during the transposing from return times to joint stopping times, mainly due to Lemma 7.6 and Corollary 7.5.

#### 7.3.1. Uniform decay of return times

We consider the uniform decay of return times for the induced structure.

**Lemma 7.8.** If there exist $C_1, \gamma_1 > 0$ and $0 < \nu \leq 1$ such that for $P$ a.e. $\omega$ we have

$$
m(\{R_\omega > n\}) \leq C_1 e^{-\gamma_1 n^\nu}, \quad \forall n \geq 1,
$$

then there exist $C_i, \gamma_i, i = 1, 2$, and for $P$ a.e. $\omega$ a positive integer $n_2(\omega)$, such that

$$
\begin{align*}
\Lambda_\omega(\{T_\omega > n\}) & \leq C_1 e^{-\gamma_1 n^\nu}, \quad \forall n \geq n_2(\omega) \\
\nu(\{n_2(\omega) > n\}) & \leq C_2 e^{-\gamma_2 n^\nu}, \quad \forall n \geq 1,
\end{align*}
$$

**Proof.** Part of the proof is a randomised version of [26, lemma 4.2], where we must take (carefully) $L = 1$, $t = \tau$, $\tau = T$, and $\mu = \Lambda$. From Lemma 7.7 we may consider $a_n = C_1 e^{-\gamma_1 n^\nu}$ to be so that

$$
\Lambda_\omega(\{\tau_\omega^i - \tau_\omega^{i-1} = n|\tau_\omega^{i-1}, \ldots, \tau_\omega^{j-1}\}) \leq a_n.
$$

We recall that the convolution $b^1 \ast b^2$ of two real sequences $b^1$ and $b^2$ is given by

$$
(b^1 \ast b^2)_n = \sum_{i+j=n} b^1_i b^2_j.
$$

When $b$ is a sequence, we also write $b^{\ell}$ for the sequence obtained by convolving $\ell$ times the sequence $b$ with itself. As shown in [26], for large enough $K$ the sequence $b_n = 1_{n \geq K} a_n$ satisfies

$$
(b \ast b)_p \leq b_p, \quad \forall p \in \mathbb{N}.
$$
We define the measurable function \( k_\omega : \Delta \to \mathbb{N} \) as follows: if \((x, x') \in \Delta_\omega \times \Delta_\omega \) is such that 
\( T_\omega(x, x') = \tau^j_\omega(x, x') \), then we set \( k_\omega(x, x') = i \). Let \( k \geq 0 \) and \( A \subset \{1, \ldots, k\} \). For \( j \in A \), take \( n_j \geq 1 \). Set 
\( Y_\omega(A, n_j) = \{(x, x') \in \Delta_\omega \times \Delta_\omega : k_\omega(x, x') \geq \text{sup}(A) \) and \( \tau^j_\omega(x, x') - \tau^{j-1}_\omega(x, x') = n_j, \forall j \in A \} \).
Conditioning successively with respect to the different times, we get 
\[
\Lambda_\omega(Y_\omega(A, n_j)) \leq \prod_{j \in A} \Lambda_\omega(\{\tau^j_\omega - \tau^{j-1}_\omega = n_j | \tau^1_\omega, \ldots, \tau^{j-1}_\omega\}) \leq \prod_{j \in A} a_{n_j}.
\]

For each \( n \in \mathbb{N} \) we define \( q(n) = \lfloor \alpha n^\nu \rfloor \), where \( \alpha \) is to be determined later. Let \((x, x') \) be such that \( T_\omega(x, x') > n \). If \( k_\omega(x, x') = \ell \leq q(n) \), let \( n_j = \tau^j_\omega(x, x') - \tau^{j-1}_\omega(x, x') \) for \( j \leq \ell \), and \( A = \{j : n_j \geq K\} \). Thus, \((x, x') \in Y_\omega(A, n_j) \) and \( \sum_{j \in A} n_j \geq n/2 \) if \( n \) is large enough. Consequently,
\[
\{(x, x'): T_\omega(x, x') > n\} \subset \{k_\omega(x, x') > q(n)\} \cup \bigcup_{A \subset \{1, \ldots, q(n)\}} \bigcup_{n_j \geq K, \sum A n_j \geq n/2} Y_\omega(A, n_j).
\]

Following the estimates for part (II) in the proof of [16, Proposition 5.6], there are \( \hat{C}, \hat{\tilde{C}} \) (depending on the Lipschitz constants of \( \varphi \) and \( \varphi' \)), \( \hat{\gamma}, \tilde{\gamma} > 0 \) and a random variable \( n_4(\omega) \) on a full measure subset of \( \Omega \) (the same as in Corollary 7.5) such that
\[
\begin{cases}
\Lambda_\omega(\{k_\omega > q(n)\}) \leq \hat{C} e^{-\hat{\gamma} q(n)}, & \forall n \text{ such that } q(n) \geq n_4(\omega) \\
P(\{n_4(\omega) > n\}) \leq \hat{\tilde{C}} e^{-\tilde{\gamma} n}, & \forall n \geq 1.
\end{cases}
\]

For the measure of the second part in (18) we have
\[
\Lambda_\omega \left( \bigcup_{A \subset \{1, \ldots, q(n)\}} \bigcup_{n_j \geq K, \sum A n_j \geq n/2} Y_\omega(A, n_j) \right) \leq \sum_{A \subset \{1, \ldots, q(n)\}} \sum_{n_j \geq K, \sum A n_j \geq n/2} \prod_{j \in A} a_{n_j} \\
\leq \sum_{0 \leq \ell \leq q(n)} \binom{q(n)}{\ell} \sum_{p=n/2}^{\infty} b^p \cdot \sum_{p=n/2}^{\infty} b^p.
\]

Since \( b_p = \mathcal{O}(e^{-\tilde{\gamma} n^\nu}) \), we have
\[
\sum_{p=n/2}^{\infty} b_p = \mathcal{O}(n^{1-\nu} e^{-\tilde{\gamma} n^\nu}).
\]

We notice that the previous estimates are uniform over \( \omega \). The proof follows then by [18, 19] and [20] just by taking \( n_2 = n_4 \) and considering \( \alpha \) small enough.

### 7.3.2. Non-uniform decay of return times
We treat now the non-uniform decay of return times.
Lemma 7.9. If there exist $C_i, \gamma_i, i = 1, 2,$ and $0 < v \leq 1, and for $P$ a.e. $\omega$ a positive integer $g_0(\omega),$ such that
\[
\begin{align*}
\begin{cases}
m(\{R_\omega > n\}) \leq C_1 e^{-\gamma_1 n v}, & \forall n \geq g_0(\omega) \\
P(\{g_0(\omega) > n\}) \leq C_2 e^{-\gamma_2 n v}, & \forall n \geq 1,
\end{cases}
\end{align*}
\tag{22}
\]
then there exist $C_j, \gamma_j, j = 3, 4,$ and for $P$ a.e. $\omega$ a positive integer $n_2(\omega),$ such that
\[
\begin{align*}
\begin{cases}
\Lambda_\omega(\{T_\omega > n\}) \leq C_3 e^{-\gamma_3 n^{v/2}}, & \forall n \geq n_2(\omega) \\
P(\{n_2(\omega) > n\}) \leq C_4 e^{-\gamma_4 n^{v/2}}, & \forall n \geq 1,
\end{cases}
\end{align*}
\tag{23}
\]
Proof. Let $0 < \hat{v} < v$ to be fixed later. From [16, Remark 3.1] one can see that estimates [22] imply that there exist $C_3, C_4, \tilde{C}_3, \gamma_3, \gamma_4 > 0$ and a random variable $n_3(\omega)$ defined on a full $P$-measure subset of $\Omega$ so that
\[
\begin{align*}
\begin{cases}
m(\{R_\omega > n\}) \leq C_3 e^{-\gamma_3 n v}, & \forall n \geq n_3(\omega) \\
m(\Delta_\omega) \leq n_3(\omega) + C_3 \\
P(\{n_3(\omega) > n\}) \leq C_4 e^{-\gamma_4 n v}, & \forall n \geq 1,
\end{cases}
\end{align*}
\tag{23}
\]
Then, from Lemma 7.4 we may consider $a_n = \tilde{C}_1 e^{-\gamma_1 n^{\hat{v}}}$ to be so that
\[
\Lambda_\omega(\{r_j^1(x, x') - r_j^{j-1}(x, x') > n|\tau_j^1, \ldots, \tau_j^{j-1}\}) \leq a_n n_3(\sigma^{\tau_j^{j-1}+\rho}(\omega)).
\]
Let $k_\omega, Y_\omega(A, n_j), b_n$ and large $K$ be defined accordingly as in the proof of Lemma 7.8 and set $\tilde{q}(n) = [an^{\hat{v}}].$ Conditioning successively with respect to the different times, we get
\[
\Lambda_\omega(Y_\omega(A, n_j)) \leq \prod_{j \in A} \Lambda_\omega(\{r_j^1 - r_j^{j-1} = n_j|\tau_j^1, \ldots, \tau_j^{j-1}\}) \leq \prod_{j \in A} a_{n_j} n_3(\sigma^{\tau_j^{j-1}+\rho}(\omega)).
\]
We recall that
\[
\{(x, x') : T_\omega(x, x') > n\} \subset \{k_\omega(x, x') > \tilde{q}(n)\} \cup \bigcup_{A \in \{1, \ldots, \tilde{q}(n)\}} \bigcup_{n_j \geq K} \sum_{A \leq n_j \geq n/2} Y_\omega(A, n_j), \tag{24}
\]
and from [19] there are $\hat{C}, \tilde{C}, \tilde{\gamma}, \tilde{\gamma} > 0, 0 < \rho < 1$ and a random variable $n_4(\omega)$ such that
\[
\begin{align*}
\begin{cases}
\Lambda_\omega(\{k_\omega > \tilde{q}(n)\}) \leq \hat{C} e^{-\tilde{\gamma} \tilde{q}(n)}, & \forall n \text{ such that } \tilde{q}(n) \geq n_4(\omega) \\
P(\{n_4(\omega) > n\}) \leq \hat{C} e^{-\gamma n}, & \forall n \geq 1.
\end{cases}
\end{align*}
\tag{25}
\]
Let us now concentrate in the measure of the second part in (24). We have
\[
\Lambda_\omega\left(\bigcup_{A \in \{1, \ldots, \tilde{q}(n)\}} \bigcup_{n_j \geq K} \sum_{A \leq n_j \geq n/2} Y_\omega(A, n_j)\right) \leq \sum_{A \in \{1, \ldots, \tilde{q}(n)\}} \sum_{n_j \geq K} \sum_{A \leq n_j \geq n/2} \prod_{j \in A} a_{n_j} n_3(\sigma^{\tau_j^{j-1}+\rho}(\omega)) \\
\leq \prod_{j = 1}^{\tilde{q}(n)} \sum_{j = 1}^{\tilde{q}(n)} b_j n_3(\sigma^{\tau_j^{j-1}(\omega)}) \leq \rho n^{\rho} \tag{26}
\]
Fix $0 < \rho < 1.$ We say that $\Gamma \in \xi^{\tilde{q}(n)}$ is $n$-good if for all $(x, x') \in \Gamma$ and $\ell \leq \tilde{q}(n),$
\[
\sum_{j = 1}^{\ell} \sum_{j = 1}^{\tilde{q}(n)} n_3(\sigma^{\tau_j^{j-1}(\omega)})^v \leq \rho n^{\rho}.
\tag{27}
\]
The remaining cylinders are called $n$-bad. We claim that there is a random variable $\hat{n}_3(\omega) \geq n_3(\omega)$ on a full measure subset of $\Omega$ such that, for all $\dot{v}' = v - \dot{v}$ and $n \geq \hat{n}_3(\omega)$, the $\Lambda_\omega$-measure of the $n$-bad cylinders is less than $\hat{C}_3 e^{-q n \dot{v}'}$ and $P(\{\hat{n}_3(\omega) > n\}) \leq \hat{C}_3 e^{-q n \dot{v}'}$.

Indeed, note first that from (23), for each fixed $1 \leq \ell \leq \hat{q}(n)$ and $0 = \tau^0, \tau^1, \ldots, \tau^{\ell-1}$,

$$P \left( \left\{ \sum_{j=1}^{\ell} n_3(\sigma^{\tau_j-1}(\omega))^{\dot{v}} > \frac{\rho}{\ell} \right\} \right) \leq \sum_{j=1}^{\ell} P \left( \left\{ n_3(\sigma^{\tau_j-1}(\omega))^{\dot{v}} > \frac{\rho}{\ell} \right\} \right) \leq \hat{C}_3 e^{-q n \dot{v}' / \hat{q}(n)} \quad (28)$$

Let $M_n \subset \Omega \times \Delta$ be the set of points $(\omega, x, x')$ such that $(x, x')$ belongs to an $n$-bad $\Gamma_\omega \in e^{\hat{q}(n)}$. Thus (28) implies $(P \times \Lambda)(M_n) \leq \hat{C}_3 e^{-q n \dot{v}' / \hat{q}(n)}$. Fix any $0 < \eta < 1$. Setting

$$M'_n = \left\{ \omega : \int_{\Delta_n \times \Delta_\omega} \chi_{M_n} d\Lambda_\omega(x, x') > C'_3 e^{-q n \dot{v}' / \hat{q}(n)} \right\},$$

we must have

$$P (M'_n) < e^{-(1-\eta)q n \dot{v}' / \hat{q}(n)}, \quad (29)$$

otherwise, using Fubini’s theorem we are led into a contradiction. Define, for each $n$,

$$B_n = \{ \omega : \exists k \geq n \text{ s.t. } \Lambda_\omega(\{ (x, x') : (\omega, x, x') \in M_k \}) > C'_3 e^{-q n \dot{v}' / \hat{q}(n) / \hat{q}(k)} \}.$$ 

Then (29) implies that $P(B_n) \leq \sum_{k \geq n} e^{-(1-\eta)q n \dot{v}' / \hat{q}(k)}$, and therefore $\lim_{n \to \infty} P(B_n) = 0$. For $P$ a.e. $\omega \in \bigcup_n (\Omega \setminus B_n)$, we set $\hat{n}_3(\omega) = \sup\{n_3(\omega), \inf\{n \geq 1 : \omega \notin B_n\}\}$. We have then

$$P(\{\hat{n}_3(\omega) > n\}) \leq P(\{\omega : \exists m > n \text{ s.t. } \omega \in M'_m\}) + P(\{n_3(\omega) > n\}) \leq \sum_{k \geq n} e^{-(1-\eta)q n \dot{v}' / \hat{q}(k)} + C'_3 e^{-q n \dot{v}'} \leq \hat{C}_3 e^{-q n \dot{v}'}.$$

Moreover, the $\Lambda_\omega$-measure of the $n$-bad cylinders is less than $C'_3 e^{-q n \dot{v}' / \hat{q}(n)}$, for $n \geq \hat{n}_3(\omega)$, proving the claim. On the other hand, condition (27) leads to

$$\prod_{j=1}^{\hat{q}(n)} n_3(\sigma^{\tau_j-1+\ell_0}(\omega)) \leq e^{\hat{q}(n) \log(\rho \dot{v}')},$$

Taking into account the claim and (21), for $n \geq \hat{n}_3(\omega)$ we have that (26) is less than $\hat{C} e^{-\gamma n \dot{v}'}$, if $\alpha$ is small enough, which together with (21) and (25) implies that if $n \geq n_2(\omega) = \max\{\hat{n}_3(\omega), n_3(\omega)\}$, we have

$$\Lambda_\omega(\{T_\omega(x) > n\}) \leq C_3 e^{-\gamma n \dot{v}} \quad (27),$$

with $v^* = \min\{\dot{v}, \dot{v}'\}$. The optimal result occurs considering $v^* = v/2$. \hfill \Box

7.4. From stopping times to the decay of correlations.
Then there exist such that, for each pair of absolutely continuous probability measures \( \varphi, \varphi' \) such that for almost every \( \omega \),
\[
\nu \text{ gives Theorems A and B. Let }
\]
\[
\text{the convergence to equilibrium gives rise to similar estimates for the rates of decay of }
\]
\[
\text{Finally, Lemmas 7.8, 7.9 and Lemma 7.10 lead to the desired estimates just by taking }
\]
\[
\lambda' = \nu, \text{ for some } C' = C'(\varphi, \psi). \text{ For } \psi \in \mathcal{F}_\beta \text{ we obtain the same estimates for } \hat{\phi}_\omega = \psi_\omega + C_\psi + 1 \in F_\beta^+ \cap L^\infty.
\]

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