DOMINATING CAT (−1) SURFACE GROUP REPRESENTATIONS BY FUCHSIAN ONES

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ABSTRACT. We show that for every representation \( \rho : \pi_1(S_g) \to \text{Isom}(X) \) of the fundamental group of a genus \( g \geq 2 \) surface to the isometry group of a complete CAT (−1) metric space \( X \) there exists a Fuchsian representation \( j \) and a \( (j, \rho) \)-equivariant map from \( \mathbb{H}^2 \) to \( X \) which is \( c \)-Lipschitz for some \( c < 1 \), or \( \rho \) restricts to a Fuchsian representation.

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1. Introduction

Let \( S \) be a closed surface of genus \( g \geq 2 \). We denote its fundamental group by \( \Gamma \). Let \( X_1 \) and \( X_2 \) be metric spaces. For two representations \( \rho_i : \Gamma \to \text{Isom}(X_i) \) \( (i = 1, 2) \), we say that \( \rho_1 \) dominates \( \rho_2 \) if there exists a 1-Lipschitz and \( (\rho_1, \rho_2) \)-equivariant map from \( X_1 \) to \( X_2 \). Such a map is called a domination. We say that \( \rho_1 \) dominates strictly \( \rho_2 \) if there exists a map which is \( (\rho_1, \rho_2) \)-equivariant and \( c \)-Lipschitz for a \( c < 1 \). We say that a representation \( j : \Gamma \to \text{Isom}(\mathbb{H}^2) \) is Fuchsian if it is the holonomy of a hyperbolic structure on \( S \).

**Theorem 1.** Let \( X \) be a CAT (−1) complete metric space and \( \rho : \Gamma \to \text{Isom}(X) \) a representation. Then there exists a Fuchsian representation \( j \) which dominates \( \rho \). Moreover, either \( j \) dominates strictly \( \rho \) or the domination is an isometric embedding.
Remark that in the second case, \( \rho \) stabilises a subset of \( X \) which is isometric to \( \mathbb{H}^2 \) and in restriction to which it is Fuchsian.

This theorem is known in several special cases. Guéritaud, Kassel and Wolff [10] proved it for \( X = \mathbb{H}^2 \), by showing that every such \( \rho \) is the holonomy of a folded hyperbolic structure. Deroin and Tholozan [7] proved it for \( X \) a smooth, complete, simply connected Riemannian manifold of sectional curvature bounded above by \(-1\). They construct an equivariant harmonic map (with respect to an arbitrary conformal structure) and show that one can chose a hyperbolic structure on \( S \) that make this map \( 1 \)-Lipschitz. Daskalopoulos, Mese, Sanders and Vdvina [6] proved it for \( X \) a complete CAT \((-1)\) metric space and \( \rho \) convex cocompact. Their proof relies on harmonic analysis in singular spaces, as developed by Koraveer-Schoen and Mese. They need the convex cocompact assumption to construct harmonic conformal equivariant maps.

Our approach to tackle this problem share an important feature with the works of Deroin-Tholozan and Daskalopoulos-Mese-Sanders-Vdvina: we use harmonic maps to construct domination. However, we use discrete harmonic maps. This makes the proof less technically involved and more combinatorial, as we don’t rely on the machinery of Sobolev maps with metric space target.

As observed by Deroin-Tholozan and Daskalopoulos-Mese-Sanders-Vdvina, this result gives another proof of a particular case of a result of Bonk-Kleiner [3] (conjectured by Bourdon [4]):

**Theorem 2 [3].** If \( \rho : \Gamma \to \text{Isom}(X) \) is a convex cocompact isometric acting on a CAT \((-1)\) space \( X \) then the Hausdorff dimension of the limit set of \( \rho \) is \( \geq 1 \), with equality if and only if \( \rho \) fixes a geodesically embedded copy of \( \mathbb{H}^2 \) in \( X \) whose induced action is Fuchsian.

The relation of domination has been studied in order to classify the compact anti-de Sitter spaces of dimension 3. These spaces are classified by pairs \((j, \rho)\) of representations \( \Gamma \to \text{Isom}(\mathbb{H}^2) \) which act faithfully and properly discontinuously on \( \text{PSL}(2, \mathbb{R}) \), by simultaneous multiplication on the left and on the right (and \( \Gamma \) is the fundamental group of a surface of genus \( g \geq 2 \)), by [8], [13], [16]. Such a pair is called admissible. By results of Salein [18] and Kassel [14], a pair is admissible if and only if \( j \) is Fuchsian and \( \rho \) is strictly dominated by \( j \). By the theorem [11] such pairs exist for any \( \rho : \Gamma \to \text{PSL}(2, \mathbb{R}) \) of non maximal Euler class, and it is proved in [10] that each Fuchsian representations \( j \) dominates strictly a \( \rho \) whose non maximal Euler class can be specified. Moreover, Tholozan [10] describes the space of Fuchsian representations which dominates a given \( \rho \); he produces an explicit diffeomorphism between this space and the Teichmüller space of the surface.
The notion of domination has been generalized to higher rank linear representations and has been proved to be a useful concept to construct Anosov representations, see [9].

The article is organized as follows. In section 2 we review the standard notions of metric geometry we need. In section 3 we study metric triangulation and show the existence of harmonic equivariant map. This allow us to construct, in section 4, a triangulated conical hyperbolic structure which dominates the representation $\rho$. In section 6 we explain how to handle the eventual singularities of this conical structure. Finally, in section 7 we show that the domination is an isometric embedding when the domination is not strict.

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2. Notions of metric geometry

We introduce the notions of metric geometry we use in this article. Our references are [11] and [2].

We denote the distance between two points $x$ and $y$ in a metric space $X$ by $|xy|$, $|x,y|$ or $|xy|_X$ depending on the context. A geodesic in a metric space is an isometric embedding of a closed interval of the real line. A metric space is geodesic if every two points are joined by a geodesic. We work principally with complete geodesic metric space. In a geodesic space, we denote by $[xy]$ any geodesic between $x$ and $y$.

A metric is $D$-geodesic if every two points at distance less that $D$ are joined by a geodesic.

Let $\kappa \in \mathbb{R}$ define $\mathcal{M}_\kappa$ to be the model plane of curvature $\kappa$, that is the unique simply connected manifold of dimension 2 with constant sectional curvature $\kappa$. In this article we work with $\kappa = -1$, $\kappa = 1$ and $\kappa = 0$, and in these cases we have $\mathcal{M}_{-1} = \mathbb{H}^2$ the hyperbolic plane, $\mathcal{M}_1 = S^2$ the unit sphere of dimension 2 and $\mathcal{M}_0 = \mathbb{R}^2$ the Euclidian plane. Denote by $D_\kappa$ the diameter of $\mathcal{M}_\kappa$ which is infinite for $\kappa \leq 0$ and $\pi \sqrt{|\kappa|}$ for $\kappa > 0$.

Fix a $\kappa \in \mathbb{R}$. In a geodesic space $X$, consider three points $x, y, z$. A triangle $\Delta = [x, y, z]$ is a choice of geodesic between each pair of points. If $|xy| + |yz| + |zx| < 2D_\kappa$, there exists a unique (up to isometry) triangle in $\mathcal{M}_\kappa$ with the same side lengths. Denote by $\tilde{\Delta} = [\tilde{x}\tilde{y}\tilde{z}]$ this triangle. We call it the comparison triangle of $\Delta$. Define a map $\tilde{\Delta} \to \Delta$ by sending $\tilde{x} \mapsto x$, $\tilde{y} \mapsto y$, $\tilde{z} \mapsto z$, and each side isometrically to the corresponding side. This map is called the comparison map of $\Delta$. To a point $p$ of $\Delta$ corresponds a point $\tilde{p}$ of $\tilde{\Delta}$ that we call the comparison point of $p$. 
**Definition 1** ([11, Ch. II.1]). A $D_\kappa$-geodesic metric space $X$ is a CAT ($\kappa$) space if for every geodesic triangle $\Delta$ with perimeter $< 2D_\kappa$, the comparison map $\tilde{\Delta} \to \Delta$ is 1-Lipschitz. This means that for every $x, y \in \Delta$, the comparison points $\tilde{x}, \tilde{y} \in \tilde{\Delta}$ satisfy

$$|xy|_X \leq |\tilde{x}\tilde{y}|_{M_\kappa}.$$  

We say that the triangle $\Delta$ satisfies the CAT ($\kappa$) inequality (see figure 1). A space which is locally CAT ($\kappa$) is said to have curvature $\leq \kappa$.

![Figure 1. A comparison triangle for $\kappa = 0$](image)

We fix a $\kappa \in \mathbb{R}$ and $X$ a CAT ($\kappa$) space. A CAT ($\kappa$) space is $D_\kappa$-uniquely geodesic: two points at distance less than $D_\kappa$ are joined by a unique geodesic.

We define the angle between two geodesic in a CAT ($\kappa$) space. Let $c_1$ and $c_2$ be two geodesics starting at a point $x \in X$. For a small $t > 0$, consider the triangle $\Delta = [xc_1(t)c_2(t)]$. Denote by $\tilde{\angle}_x^\kappa(c_1(t), c_2(t))$ the angle at $\tilde{x}$ of the comparison triangle $\tilde{\Delta}$ in $M_\kappa$. Then the function $t \mapsto \tilde{\angle}_x^\kappa(c_1(t), c_2(t))$ is non-increasing and its limit when $t$ goes to 0 is denoted $\angle(c_1, c_2)$ and is called the angle between $c_1$ and $c_2$. We use the notation $\angle_x(y, z) = \angle([xy], [xz])$. The angle function satisfies a triangle inequality

$$\angle(c_1, c_3) \leq \angle(c_1, c_2) + \angle(c_2, c_3),$$

for every geodesics $c_1, c_2, c_3$ starting at the same point. This means that the angle function induces a pseudo-metric on the set of geodesic starting at a point $x$. Two geodesics with angle 0 define the same direction. The associated metric space is the space of direction of $X$ at $x$ and is denoted $\Sigma_xX$. Element of this space are directions and the distance between two such directions is given by the angle between two representatives. The completion of $\Sigma_xX$ is a CAT (1) space [11, Th. II.3.19] which we confound with $\Sigma_xX$.

A subset $C$ of $X$ is convex if for each $x, y \in C$, the geodesic $[xy]$ lies entirely in $C$. Suppose in this paragraph that $\kappa \leq 0$. A closed convex subset of a CAT ($\kappa$) space is itself a CAT ($\kappa$) space. If $C$ is a closed subset of $X$ there exists a retraction $\pi : X \to C$ called the projection...
on the convex $C$ which maps a point $x$ to the closed point $\pi(x) \in C$ [11, Prop. II.2.19]. This projection is a 1-Lipschitz map.

Let $X_1$ and $X_2$ be two CAT ($\kappa$) spaces with convex subspaces $C_1 \subset X_1$ and $C_2 \subset X_2$. If $i : C_1 \to C_2$ is an isometry, by Reshetnyak’s gluing theorem [11, Th. II.11.1], the gluing of $X_1$ and $X_2$ along $i$ is a CAT ($\kappa$) space.

Consider a triangle $\Delta = [xyz]$ with perimeter $< 2D_\kappa$ in $X$. By the CAT ($\kappa$) inequality we know that for each pair of points $p, q \in \Delta$ we have

$$|pq| \leq |\tilde{p}\tilde{q}|,$$

where $\tilde{p}$ and $\tilde{q}$ are the comparison points in the comparison triangle $\tilde{\Delta}$. If for $p, q$ on different sides of $\Delta$ this inequality is an equality: $|pq| = |\tilde{p}\tilde{q}|$, we say we are in the rigidity case of the CAT ($\kappa$) inequality. In this case, the comparison map $\tilde{\Delta} \to \Delta$ extends to an isometry between the convex hull of $\tilde{\Delta}$ in $\mathcal{M}_\kappa$ and the convex hull of $\Delta$ in $X$ [11, Prop. II.2.9]. If one of the angle of $\Delta$ is equal to the corresponding comparison angle in $\tilde{\Delta}$ we are also in the rigidity case of the CAT ($\kappa$) inequality and the same conclusion holds.

The CAT ($\kappa$) inequality for triangles can be extended to general curves:

**Theorem 3** (Reshetnyak’s majorization theorem [2, p. 8.12.4]). A closed curve $\sigma$ of length $< 2D_\kappa$ in a CAT ($\kappa$) space $X$ is majorized by a convex $C \subset \mathcal{M}_\kappa$. This means that there exists a convex $C \subset \mathcal{M}_\kappa$ and a 1-Lipschitz map $M : C \to X$ such that the restriction of $M$ to $\partial C$ is a length parametrization of $\sigma$.

An useful case of this theorem is when the curve $\sigma$ is a geodesic closed polygon $[x_1 \cdots x_n]$, that is the curve obtained by concatenating the geodesic segments $[x_i, x_{i+1}]$. In this case the convex $C$ can be choosen such that its boundary $\partial C$ is a geodesic polygon with same side lengths and the map $M$ sends vertices to corresponding vertices, see [2 p. 8.12.14] and also the discussion of section 7.

For a triangle $\Delta$ in $X$, the comparison map $\tilde{\Delta} \to \Delta$ is defined on the sides of the triangle $\tilde{\Delta}$. The majorization theorem 3 allows us to extend this map to the convex hull of $\tilde{\Delta}$.

### 3. Harmonic triangulation

The first part of the proof is to find a conical hyperbolic structure which dominates $\rho$. To construct this structure, we start with a topological triangulation of the surface, and then upgrade it to a hyperbolic simplicial complex. For that we construct a discrete harmonic map.

**3.1. Triangulation, metric triangulation.** We fix a triangulation $\mathcal{T}$ of the surface $S$, where $\mathcal{T} = (V, E, F)$ and $V$ is the set of vertices,
$E$ the set of edges and $F$ the set of faces. We can choose, for example, the Riemann triangulation: in the standard $4g$-gon folding model of $S$, connect one vertex to all of the other (see figure 2). We lift it to a triangulation of $\tilde{S}$ that we denote $\tilde{T} = (\tilde{V}, \tilde{E}, \tilde{F})$.

![Figure 2. The Riemann triangulation for a genus 2 surface.](image)

A length function on a triangulation $T$ is a function $\ell : E \to \mathbb{R}_{\geq 0}$ which satisfies the triangle inequality: for all faces $f$, if $e_1, e_2, e_3$ are the edges adjacent to $f$ then we have

$$\ell(e_1) \leq \ell(e_2) + \ell(e_3).$$

If for a face $f$, the inequality above is an equality, we say that $\ell$ flatten the face $f$. It happens in particular if one of the edges $e$ of $f$ is given length 0: in this case we say that $\ell$ flatten the edge $e$.

The data $(T, \ell)$ of a triangulation with a length function is called a metric triangulation. If $\ell$ is a length function on $\tilde{T}$ which is invariant by deck transformations, then it induces a length function on $T$ that we denote by the same symbol.

A map $F : \tilde{V} \to X$ which is $\rho$-equivariant. Let $T_0 = (V_0, E_0, F_0)$ be a subtriangulation of $\tilde{T}$ which is a fundamental domain for the action by deck transformations. The energy of $F$ is the following quantity:

$$E(F) = \sum_{e \in E_0} \ell_F(e)^2 = \sum_{e \in E_0} |F(e_-)F(e_+)|^2.$$  

We say that such an equivariant map $F : \tilde{E} \to X$ is harmonic if it minimizes the energy among all such maps.

We first show that harmonic maps exist.
**Theorem 4.** If \( \rho : \Gamma \to \text{Isom}(X) \) is a representation which does not fix a point on the boundary at infinity of \( X \) then there exists a harmonic \( \rho \)-equivariant map.

To prove existence, we use results from [13] and [5] that we state now. Let \((N, d)\) be a complete CAT(0) metric space. Let \( \Phi : N \to \mathbb{R} \) be a convex, lower semicontinuous function which is bounded from below. For \( \lambda > 0 \) we define the Moreau-Yosida approximation of \( \Phi \) by:

\[
\Phi^\lambda = \inf_{y \in N} \left( \lambda \Phi(y) + |py|^2 \right),
\]

where \( p \in N \) is some fixed base point. For every \( \lambda > 0 \) there exists a unique \( y^\lambda \in N \) such that \( \Phi^\lambda = \Phi(y^\lambda) \) [13, Lemma 3.1.2].

**Theorem 5 ([13, Theorem 3.1.1]).** For such a function \( \Phi \), if for some sequence \((\lambda_n)\) of real numbers going to infinity the sequence \((y^\lambda_n)\) is bounded, then \( y^\lambda \) converges to a minimizer of \( \Phi \) when \( \lambda \) goes to infinity.

Following [5], we say that the action of a group \( G \) on a CAT(0) space is evanescent if there exists an unbounded set \( T \) such that for all compact \( K \subset G \), the set \( \{ |x, gx| ; x \in T, g \in K \} \) is bounded.

**Theorem 6 ([5, Proposition 1.8]).** Let \( G \) act on a CAT(−1) complete metric space \( X \). If the action is evanescent then \( G \) fixes a point in \( X \cup \partial X \).

We can now prove the existence of harmonic map.

**Proof of Theorem 4.** We will apply Theorem 5 to the function \( \mathcal{E} \). This function is defined on \( \text{Eq}(\rho) \), the space of \( \rho \)-equivariant maps from \( \tilde{E} \) to \( X \). This is a complete CAT(0) space with the product metric. The function \( \mathcal{E} \) satisfies the regularity assumptions of the theorem. If the sequence \((F_n)\) such that \( \mathcal{E}^n = \mathcal{E}(F_n) \) is bounded in \( \text{Eq}(\rho) \), then the theorem applies and \( \mathcal{E} \) admits a minimizer.

If on the contrary the sequence \((F_n)\) is unbounded in \( \text{Eq}(\rho) \), we show that a minimizer still exists.

For a function \( F \in \text{Eq}(\rho) \) denote by \( \text{Lip}(F) \) its Lipschitz constant with respect to the graph metric on \( \tilde{V} \) (that is, such that adjacent vertices are at distance 1). Observe that \( \text{Lip}(F) \leq \sqrt{\mathcal{E}(F)} \).

Now, if the sequence \((F_n)\) is unbounded in \( \text{Eq}(\rho) \), then there exists \( v \in \tilde{V} \) such that \( (F_n(v)) \) is unbounded in \( X \), and we have for all \( \gamma \in \Gamma \):

\[
|F_n(v), \rho(\gamma)F_n(v)| \leq \text{Lip}(F_n)|v, \gamma v| \leq \sqrt{\mathcal{E}(F_n)}|v, \gamma v|,
\]

and as \( \mathcal{E}(F_n) \) is bounded, the function \( x \mapsto |x, \rho(\gamma)x| \) is bounded on the unbounded set \( \{ F_n(v) ; n \in \mathbb{N} \} \). This implies that the action of \( \Gamma \) on \( X \) given by \( \rho \) is evanescent. By Theorem 6 \( \rho \) has a fixed point in \( X \cup \partial X \). By assumption, \( \rho \) does not fix a point in \( \partial X \) so it fixes a point \( p_0 \) in \( X \). In this case, the constant function \( v \mapsto p_0 \) is \( \rho \)-equivariant and is of energy 0, so is a minimizer. \( \square \)
The harmonic map is not unique in general. Understanding when it is is interesting question.

**Example 3.1.** Consider a real tree $T$ and two hyperbolic isometries $a$ and $b$. Recall that the axe of an hyperbolic isometry $\gamma$ of $T$ is the geodesic where $|x, \gamma x|$ is minimal. Suppose that the axes of $a$ and $b$ intersect along a non-trivial geodesic segment.

Remark that for the standard presentation $\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] = e \rangle$ of $\Gamma$, $a_1$ and $a_2$ generate a free group. Consider the representation $\rho: \Gamma \to \text{Isom}(T)$ obtained by sending $a_1$ to $a$, $a_2$ to $b$, and the other generators to the identity.

Let $\mathcal{T} = (V, E, F)$ be the Riemann triangulation on $S$ (see figure 2). The energy of a $\rho$-equivariant map $F: \tilde{V} \to T$ is

$$E(F) = |F(x_0), aF(x_0)|^2 + |F(x_0), bF(x_0)|^2,$$

where $x_0$ is any lift of the unique vertex of $\mathcal{T}$. The energy is minimized exactly when $F(x_0)$ belongs to the axes of $a$ and $b$. As these axes intersect along a segment, the harmonic map is not unique.

### 3.3. Properties of harmonic map

Given a $\rho$-equivariant harmonic map $F: \tilde{V} \to X$, we say that the metric triangulation $(\tilde{T}, \ell_F)$ is a harmonic triangulation. In this section we prove that the conical angles of an harmonic triangulation are all greater than $2\pi$ and we construct the singular hyperbolic surface associated to a harmonic triangulation.

First we prove the following result:

**Proposition 3.2.** Let $F: \tilde{V} \to X$ be an harmonic, $\rho$-equivariant map which does not flatten any edge, and $x \in \tilde{V}$. Denote by $(y_i)$ the cyclically ordered neighbours of $x$ in $\tilde{T}$, and $\alpha_i = \angle_{F(x)}(F(y_i), F(y_{i+1}))$. Then:

$$\sum_i \alpha_i \geq 2\pi.$$

The proof of proposition 3.2 use the following result of Izeki and Natayani: [12]

**Proposition 3.3 ([12, prop. 3.5]).** If $F: \tilde{V} \to X$ is an harmonic, $\rho$-equivariant map, then for every vertex $x \in \tilde{V}$ and every direction $v \in \Sigma_{F(x)}X$ at $x$:

$$\sum_{e \in \tilde{E} | v_+ = x} |xe_+| \cos \angle_x(v, e_+) \leq 0.$$

We call this inequality the *harmonic critical inequality*, because it expresses the fact that a harmonic map is a critical point of the energy. It means that every vertex is the barycenter of its neighbours, in a weak sense.
We also need a lemma about short geodesic polygons in CAT(1)-spaces. Recall that the radius of a set $A$ in a metric space is the infimum of the radius of balls that contain $A$.

**Lemma 7.** Let $\sigma$ be a geodesic polygon of length $< 2\pi$ in a CAT(1)-space $\Sigma$. Then the radius of $\sigma$ is $< \frac{\pi}{2}$.

**Proof.** The lemma is true if $\Sigma = S^2$, see for example [2, Lem. 1.2.1]. For general $\Sigma$, as the length of $\sigma$ is $< 2\pi$, we apply Reshetnyak’s majorization theorem 3 to $\sigma$. It gives a convex set $C \subset S^2$ whose boundary $\tilde{\sigma}$ have the same length as $\sigma$, and a 1-Lipschitz map $M : C \to \Sigma$ which sends $\tilde{\sigma}$ to $\sigma$. We can apply the lemma to $\tilde{\sigma} \subset S^2$, and we have that the radius of $\tilde{\sigma}$ is $< \frac{\pi}{2}$. Because $M$ is 1-Lipschitz and maps $\tilde{\sigma}$ to $\sigma$, we have that the radius of $\sigma$ is $< \frac{\pi}{2}$. □

We can now prove proposition 3.2.

**Proof of proposition 3.2.** We work in the space $\Sigma = \Sigma_{F(x)}X$. Recall that it is a CAT(1)-space and that the distance in $\Sigma$ is given by $|uv| = \angle(u, v)$ for two geodesics $u, v$ starting at $F(x)$. Consider the geodesic polygon $P$ whose vertices are the $[F(x)F(y_i)]$. The length of $P$ is $\sum_i \alpha_i$. Suppose, for the sake of contradiction, that $\sum_i \alpha_i < 2\pi$. Then applying lemma 7 to $P$ gives that the radius of $P$ is $< \frac{\pi}{2}$. It means that there exists a direction $v \in \Sigma$ such that for all $u \in P$, $|uv| < \frac{\pi}{2}$. Now, by proposition 3.3 we have

$$\sum_i |F(x)F(y_i)| \cos \angle_{F(x)}(v, F(y_i)) \leq 0,$$

but $\cos \angle_{F(x)}(v, F(y_i))$ is positive because $\angle_{F(x)}(v, F(y_i)) < \frac{\pi}{2}$. This is a contradiction. We infer that $\sum_i \alpha_i \geq 2\pi$. □

4. **Triangulated conical hyperbolic surfaces**

In this section we explain the concept of triangulated conical hyperbolic surface and how a metric triangulation allows us to construct a domination of $\rho$ by a triangulated conical hyperbolic surface.

We use a definition suited for our purposes:

**Definition 2.** A triangulated conical hyperbolic surface is a connected, triangulated metric space such that each face of the triangulation is isometric to a triangle in $\mathbb{H}^2$.

A triangulated conical hyperbolic surface is a $\mathbb{H}^2$-simplicial complex, in the sense of [11, Chapter I.7]. We will only work with compact, or covering of compact triangulated conical hyperbolic surfaces: this implies that such a surface has finitely many isometry type of shapes, and by [11, Th. I.7.19], it is a complete geodesic space.

For a triangulated conical hyperbolic surface $C$, we say that $C$ is non-degenerate if each face of the triangulation is a non-flat triangle. In this case, $C$ is a topological surface. If no edge of the triangulation
is flattened, we say that the conical angle $\theta_x$ of $S$ at a vertex $x$ of the triangulation is the sum of the angles at $x$ of the triangles adjacent to $x$. We have the following condition for a triangulated conical hyperbolic surface to be of curvature $\leq -1$:

**Theorem 8.** [[14, th. I.7.5.2]] A triangulated conical hyperbolic surface with no edge flattened is of curvature $\leq -1$ if and only if for each vertex $x$ of the triangulation the conical angle $\theta_x$ is $\geq 2\pi$.

We explain how to construct triangulated conical hyperbolic surfaces. Given a topological surface $S$ and a metric triangulation $(T, \ell)$ on $S$ we construct a conical hyperbolic surface: for each face $f$ of $T$ with sides $e_1, e_2$ and $e_3$, consider the triangle $\Delta_f$ in $H^2$ with side lengths given by $\ell(e_1), \ell(e_2)$ and $\ell(e_3)$. Then glue the triangles $\Delta_f$ along their common side following the combinatoric of $T$. This gives a triangulated conical hyperbolic surface $C_T$ and a continuous surjection $p_T : S \to C_T$. If $\ell$ does not flatten any face, this map is a homeomorphism. We call this triangulated conical hyperbolic surface $C_T$ a conical structure on $S$ (even when $p_T$ is not a homeomorphism).

Now suppose that we are given a topological surface $S$ and a metric triangulation $(T, \ell)$ on $S$. The fundamental group $\pi_1 C_T$ acts by deck transformation on its fundamental group $\tilde{C}_T$ by isometries. This gives a representation $\pi_1 C_T \to \text{Isom}(\tilde{C}_T)$. Composing with the morphism induced on $\pi_1$ by $p_T$, this gives a representation $\rho_T : \pi_1 S \to \text{Isom}(\tilde{C}_T)$.

We call it the representation associated to the conical structure $C_T$ on $S$. When the metric triangulation is induced by a $\rho$-equivariant map $F : \tilde{V} \to X$, we denote by $C_F$ the associated conical structure and by $\rho_F$ the associated representation.

If the associated conical structure $C_T$ satisfies the condition of theorem [remark that this condition depends only of $(T, \ell)$], by theorem [the space $C_T$ is of curvature $\leq -1$]. The universal covering $\tilde{C}_T$ is then a CAT ($-1$) space.

**Proposition 4.1.** Let $T = (V, E, F)$ be a triangulation on $S$ and $F : \tilde{V} \to X$ a $\rho$-equivariant map. Then the representation $\rho_F$ associated to the conical structure $C_F$ on $S$ given by the metric triangulation $(T, \ell_F)$ dominates the representation $\rho$.

**Proof.** Consider the universal covering $Y = \tilde{C}_F$. Remark that the triangulation $\tilde{T}$ induces a triangulation $\tilde{T}' = (V', E', F')$ on $Y$ and that $F$ factors through $p_T$ and defines a map $V' \to X$, equivariant with respect to the deck transformation and $\rho$. Each triangle $\Delta$ of the triangulation is, by construction, the comparison triangle of the triangle $F(\Delta) \subset X$, that is the triangle in $H^2$ with the same side lengths. Because $X$ is a CAT ($-1$) space, the comparison map $\Delta \to F(\Delta)$ (which maps the corresponding sides isometrically) is 1-Lipschitz. If we extend $F$ on each triangle by the comparison map, equivariantly,
then $F$ defines a 1-Lipschitz map $Y \to X$, equivariant with respect to the deck transformations and $\rho$. Pulling back $F$ to $\tilde{S}$ by $p_T$ gives a 1-Lipschitz map $\tilde{S} \to X$ which is $(\rho_F, \rho)$-equivariant. It follows that $\rho_F$ dominates $\rho$. □

When we are in the hypothesis of proposition 4.1 we say that the conical structure $C_F$ dominates $\rho$.

We study the conformal structure on a non-degenerate triangulated conical hyperbolic surface. If $C = C_T$ is a non-degenerate conical structure on $S$ with triangulation $\mathcal{T} = (V, E, F)$, then $C \setminus V$ is a surface of constant curvature $-1$. It implies that it exists a unique conformal structure on $S \setminus V$ compatible with this metric. We equip $C \setminus V$ with this structure of Riemann surface. It is well known that we can extend this structure on all of $C$:

**Proposition 4.2.** Using the same notations, there exists a structure of Riemann surface on $C$ compatible with the one on $C \setminus V$. Denote by $g$ the metric tensor defining the metric on $C \setminus V$. On each point $z_0 \in V$, there exists a holomorphic chart centered at $z_0$ such that in the coordinate $z$ given by this chart, $g$ has the expression:

$$|z|^2(\frac{\theta}{2\pi} - 1)\phi(z)|dz|^2,$$

where $\theta$ is the conical angle at $z_0$ and $\phi$ is a positive function.

Remark that when the conical structure has curvature $\leq -1$, the metric tensor does not blow up at the vertices of the triangulation (because $\theta \geq 2\pi$).

We will use the following strong version of Ahlfors' lemma [1] due to Minda to compare two conformal metrics:

**Theorem 9** [17]. Let $\Sigma$ be a Riemann surface and $g_0$ the conformal metric with constant curvature $-1$. If $g$ is another conformal metric, allowed to vanish, with curvature $\leq -1$ at the points where it does not vanish then $g < g_0$ everywhere or else $g = g_0$ everywhere.

5. PROOF OF THE THEOREM

In this section we prove the main theorem [1]. Recall that $S$ is a topological surface, $X$ is a CAT ($-1$) space, $\rho : \Gamma \to \text{Isom}(X)$ a representation from the fundamental group of $S$ to the isometry group of $X$.

Start with a triangulation $\mathcal{T} = (V, E, F)$ of $S$ with only one vertex. If the representation $\rho$ fixes a point on the boundary at infinity $\partial_{\infty}X$, the argument of [7] sec. 2.1 allows us to conclude that $\rho$ satisfies the conclusion of the theorem. If $\rho$ does not fix a point of $\partial_{\infty}X$, we apply theorem [4] which gives the existence of an harmonic, $\rho$-equivariant $F : V \to X$. 
Consider the metric triangulation \((T, \ell_F)\) and conical structure \(C_F\) induced on \(S\) by \(F\). We first suppose that \(C_F\) is non-degenerate. Applying proposition \(3.2\), we know that all the conical angles of \(C_F\) are \(\geq 2\pi\). By theorem \(8\) this means that \(C_F\) is of curvature \(\leq -1\). According to proposition \(4.1\), the representation \(\rho_F\) dominates \(\rho\). By proposition \(4.2\), \(C_F\) can be equipped with a structure of Riemann surface \(\Sigma\) such that the tensor \(g\) defining the conical metric is conformal (\(g\) is allowed to vanish at the vertices of the triangulation) and of curvature \(-1\) where it does not vanish. Let \(g_0\) be the conformal metric on \(S\) with constant curvature \(-1\). The metrics \(g\) and \(g_0\) satisfies the hypothesis of theorem \(9\), so we either have \(g < g_0\) or \(g = g_0\) everywhere.

If \(g = g_0\) it means that \(C_F\) is a (non-conical) hyperbolic surface and that \(\rho_F\) is a Fuchsian representation. As we have seen that \(\rho_F\) dominates \(\rho\), we have proved that \(\rho\) is dominated by a Fuchsian representation. By proposition \(4.2\) because \(g\) does not vanish, the conical angles of \(C_F\) are all equals to \(2\pi\) and we prove in section \(7\) that it implies that the domination extends to an isometric embedding.

If \(g < g_0\), it means that the identity map \((S, g_0) \to (S, g)\) is \(c\)-Lipschitz for some \(c < 1\). Denoting by \(j\) the Fuchsian representation which is the holonomy of the hyperbolic surface \(\Sigma\), it means that \(j\) strictly dominates \(\rho_F\). Composing the domination between \(j\) and \(\rho_F\) and between \(\rho_F\) and \(\rho\) gives a domination between \(j\) and \(\rho\). This domination is strict because the one between \(j\) and \(\rho_F\) is strict.

We have proved the theorem, assuming that \(C_F\) is non-degenerate. We treat the case where \(C_F\) is degenerate in section \(6\).

### 6. Desingularization

In this section we show that given a degenerate conical structure \(C_T\) on \(S\), with either no edges flattened and conical angle \(\geq 2\pi\) or some edge flattened, we have two cases: either we can find a non-degenerate conical structure \(C'\) on \(S\) which dominates \(C_T\), whose conical angle is \(> 2\pi\) (and the domination preserves the triangulation) or no edge is flattened, the conical angle is \(2\pi\) and exactly one face is flattened. In the first case, we can apply the argument of section \(5\) to \(C'\) and in the second case, we can apply directly apply the rigidity argument of section \(7\) to \(C_T\). In both cases, this proves theorem \(1\).

Denote by \((T, \ell)\) the metric triangulation defining the conical structure. The idea is to perturb \(\ell\) so that it becomes non-degenerate. First we state a result which allows us to perturb individual triangles.

**Proposition 6.1.** Let \(\Delta\) be an hyperbolic triangle with sides \((l_1, l_2, l_3)\) and \(\varepsilon > 0\). Define \(\Delta_\varepsilon\) to be the hyperbolic triangle with sides \((l_1 + \varepsilon, l_2 + \varepsilon, l_3 + \varepsilon)\) Then there exists a \(1\)-Lipschitz map \(\Delta_\varepsilon \to \Delta\) which maps vertex to corresponding vertex and side to corresponding side.
The restriction of this map to each side depends only of the length of this side and \( \varepsilon \).

We say that \( \Delta_\varepsilon \) dominates \( \Delta \).

**Proof.** Let \( x, y, z \) be the vertices of \( \Delta \). Consider the space \( \Delta_\varepsilon^* \) obtained by gluing a segment of length \( \varepsilon/2 \) to each vertex of \( \Delta \) (see figure 3). Call \( x', y', z' \) the extremities of these segments. By Reshetnyak’s gluing theorem this is a CAT \((-1)\) space. The map that crushes these segments to the corresponding vertex is a 1-Lipschitz \( \Delta_\varepsilon^* \rightarrow \Delta \), because it is the projection on the convex \( \Delta \subset \Delta_\varepsilon^* \).

![Figure 3. The triangle \( \Delta_\varepsilon^* \)](image)

Now, consider the triangle with vertices \( x', y', z' \) in \( \Delta_\varepsilon^* \). This is a triangle with sides \((l_1+\varepsilon, l_2+\varepsilon, l_3+\varepsilon)\). Applying the CAT \((-1)\) inequality we have that the comparison map between the comparison hyperbolic triangle \( \Delta_\varepsilon \) and the triangle \( \Delta_\varepsilon^* \) is 1-Lipschitz. Composing the comparison map \( \Delta_\varepsilon \rightarrow \Delta_\varepsilon^* \) and the crushing map \( \Delta_\varepsilon^* \rightarrow \Delta \) we get the desired 1-Lipschitz map \( \Delta_\varepsilon \rightarrow \Delta \). Its restriction to each side is the projection on the convex subset \([\varepsilon/2, l_i+\varepsilon/2]\) of \([0, l_i+\varepsilon]\).

\( \Box \)

Observe that if we apply this proposition to a flat triangle \( \Delta \), then \( \Delta_\varepsilon \) is not flat for any \( \varepsilon > 0 \). Also, the angles of \( \Delta_\varepsilon \) are as close as desired to the angles of \( \Delta \) when \( \varepsilon \) is small enough. If we apply this proposition to every triangle of the triangulation we get:

**Proposition 6.2.** Let \((\mathcal{T}, \ell)\) be a metric triangulation, with associated conical structure \( C, \varepsilon > 0 \) and let \((\mathcal{T}, \ell_\varepsilon)\) be the metric triangulation where \( \ell_\varepsilon = \ell + \varepsilon \), with associated conical structure \( C' \). Then \((\mathcal{T}, \ell_\varepsilon)\) does not flatten any triangle and we have a 1-Lipschitz map \( C' \rightarrow C \).

6.0.1. No edge flattened. We first suppose that \( C = C_F = C_T \) is degenerate but does not flatten edges. Remember that the triangulation has only one vertex. Because \( C_T \) is degenerate, some triangle is flattened. We separate two cases: the conical angle is \( > 2\pi \) or it is \( 2\pi \).
Consider the first case. We apply proposition 6.2 to \((T, \ell)\) with some \(\varepsilon > 0\). The conical structure \(C'\) associated to the metric triangulation \((T, \ell_\varepsilon)\) is non-degenerate and we have a 1-Lipschitz map \(C' \to C\). This means that \(C'\) dominates \(C\). If \(\varepsilon\) is small enough, the conical angle of \(C'\) is close enough to the conical angle of \(C\), so it is \(> 2\pi\). This proves the result in this case.

Consider the second case. A flat triangle has angles \((\pi, 0, 0)\). As all of these angles meet at the only vertex, they contribute to the conical angle with \(\pi\). If we had at least 2 flat triangles, they would contribute to the conical angle with \(2\pi\). As there is at least one other triangle which contribute with a positive angle, the conical angle would be \(> 2\pi\). As we assumed that the conical angle is \(2\pi\), it means that there is only one flat triangle. We can apply the rigidity argument of section 7 to \(C_T\).

6.0.2. Some edge flattened. Suppose now that \(C\) is degenerate, and flatten some edges. We can suppose that at least one edge is not flattened. Otherwise, it means that \(F\) is constant, and the unique point in its image is fixed by the representation \(\rho\). In this case, \(\rho\) is trivially dominated by any Fuchsian representation.

Because at least one edge is not flattened, there exists a triangle \(\Delta\) with side lengths \((a, a, 0)\) for some \(a > 0\). We apply proposition 6.2 to \((T, \ell)\) with some \(\varepsilon > 0\). The conical structure \(C'\) associated to the metric triangulation \((T, \ell_\varepsilon)\) is non-degenerate and we have a 1-Lipschitz map \(C' \to C\). This means that \(C'\) dominates \(C\). It remains to show that the conical angle of \(C'\) is \(> 2\pi\).

The triangle \(\Delta\) with sides \((a, a, 0)\) in \((T, \ell)\) gives the triangle \(\Delta_\varepsilon\) with sides \((a + \varepsilon, a + \varepsilon, \varepsilon)\) in \((T, \ell_\varepsilon)\). An application of the hyperbolic law of cosines show that when \(\varepsilon\) goes to 0, the angles of \(\Delta_\varepsilon\) go to \((\frac{\pi}{2}, \frac{\pi}{2}, 0)\). So the triangle \(\Delta_\varepsilon\) contributes with \(\pi + o(1)\) to the conical angle of \(C'\).

Consider the triangle \(\Delta'\) which share with \(\Delta\) the edge of length 0. We have two case: either \(\Delta\) has sides \((0, 0, 0)\) or it has sides \((b, b, 0)\) with \(b > 0\). If it has sides \((0, 0, 0)\), then \(\Delta_\varepsilon\) has sides \((\varepsilon, \varepsilon, \varepsilon)\), so its angles go to \((\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\) when \(\varepsilon\) goes to 0. The triangle \(\Delta'_\varepsilon\) contributes \(\pi - o(1)\) to the conical angles of \(C'\). If it has sides \((b, b, 0)\), we argue as in the previous paragraph,
Consider the triangle $\Delta''$ which share with $\Delta$ one of the two edge of length $a$. The sum of the angles of this triangle is some $\alpha > 0$. The triangle $\Delta''_\varepsilon$ has the sum of its angles which goes to $\alpha$ when $\varepsilon$ goes to 0. So the triangle $\Delta''_\varepsilon$ contributes with $\alpha + o(1)$ to the conical angle of $C'$.

Summing all of these contributions we have that the conical angle of $C'$ is at least $2\pi + \alpha + o(1)$ when $\varepsilon$ goes to 0. For $\varepsilon$ small enough, this quantity is $> 2\pi$.

7. Rigidity case

We first assume that $X$ is proper, that is the closed ball are compact. We suppose that we are in the case where $C_F$ is either non-degenerate, or flatten at most one face and no edge, and that the conical angle is $2\pi$.

Let $\xi$ be one vertex in the triangulation lifted to the universal cover and $(\eta_i)$ its neighbours, ordered such that the faces of the triangulations adjacent to $\xi$ are $[\eta_i\xi\eta_{i+1}]$ (we consider the indices $i$ cyclically). We denote $x = F(\xi)$ and $y_i = F(\eta_i)$.

The conical angle of the conical hyperbolic surface equals $2\pi$, that is the sum of the comparison angles of the triangles at $x$ equals $2\pi$. We show that in this case, $F$ is in fact an isometry.

Let $\Delta_i = [x, y_i, y_{i+1}]$ be the triangles adjacent to $x$. We denote $\alpha_i = \angle_x(y_i, y_{i+1})$ and $\tilde{\alpha}_i = \tilde{\angle}_x^{-1}(y_i, y_{i+1})$ the corresponding comparison angle. Our hypothesis is

$$\sum_i \tilde{\alpha}_i = 2\pi.$$ 

Because $F$ is harmonic we have $\sum_i \alpha_i \geq 2\pi$ and we always have $\tilde{\alpha}_i \geq \alpha_i$. Then:

$$2\pi = \sum_i \tilde{\alpha}_i \geq \sum_i \alpha_i \geq 2\pi,$$

and all inequalities become equalities, so $\alpha_i = \tilde{\alpha}_i$. We are in the rigidity case of the CAT ($-1$) inequality for $\Delta_i$ so the comparison map from $\Delta_i$ to $\Delta_i$ is an isometry. By construction, $F$ is an isometry in restriction to each $\Delta_i$.

Now, it is enough to show that the union of triangles $\Delta_i \cup \Delta_{i+1}$ in $X$ is isometric via $F$ to the union of the comparison triangles $\tilde{\Delta}_i \cup \tilde{\Delta}_{i+1}$ in $\mathbb{H}^2$, as it would imply that $F$ is a global isometry.

We show this result, assuming first that $\alpha_i + \alpha_{i+1} = \angle_x(y_i, y_{i+2})$.

As $F$ is an isometry from $\tilde{\Delta}_j$, to $\Delta_j$ it is enough to show that for each $z = F(\tilde{z}) \in \Delta_j$ et $z' = F(\tilde{z'}) \in \Delta_{j+1}$ the distance $|z'z|$ in $X$ equals the distance $|\tilde{z}\tilde{z'}|$ in $\tilde{\Delta}_j \cup \tilde{\Delta}_{j+1}$.
First we compute $\angle_x(z, z')$. We have:

$$\angle_x(y_i, z) + \angle_x(z, z') + \angle_x(z', y_{i+2})$$

$$\leq \angle_x(y_i, z) + \angle_x(z, y_{i+1}) + \angle_x(y_{i+1}, z') + \angle_x(z', y_{i+2})$$

$$= \angle_x(y_i, y_{i+1}) + \angle_x(y_{i+1}, y_{i+2})$$

$$= \angle_x(y_i, y_{i+2}),$$

where we used the fact that the triangles $\Delta_j$ are isometric to hyperbolic triangles and so that

$$\angle_x(y_i, z) + \angle_x(z, y_{i+1}) = \angle_x(y_i, y_{i+1})$$

and

$$\angle_x(y_{i+1}, z') + \angle_x(z', y_{i+2}) = \angle_x(y_{i+1}, y_{i+2}).$$

Every intermediate inequality is then an equality and so $\angle_x(z, z') = \angle_x(y_i, y_{i+1}) + \angle_x(y_{i+1}, z')$.

**Figure 5.** The triangles $\Delta_1$ and $\Delta_2$

Now consider the triangle $[\tilde{z}, \tilde{x}, \tilde{z'}]$ in $\mathbb{H}^2$. Its angle at $\tilde{x}$ is $\angle_x(z, y_{i+1}) + \angle_x(y_{i+1}, z') = \angle_x(z, z')$. The comparison triangle $\tilde{\Delta}$ of $(z, x, z')$ is an hyperbolic triangle whose adjacent sides at $x$ are of the same sizes as the one of $[\tilde{z}, \tilde{x}, \tilde{z'}]$ at $\tilde{x}$ and whose angle at $\tilde{x}$ is greater or equal than $\angle_x(z, z')$. As the opposite side of an hyperbolic triangle is an increasing function of the angle we have $|\tilde{z}\tilde{z'}| \leq |zz'|$ and so $|\tilde{z}\tilde{z'}| = |zz'|$. We conclude that $F$ is an isometry from $\tilde{\Delta}_i \cup \tilde{\Delta}_{i+1}$ to $\Delta_i \cup \Delta_{i+1}$.

We now show what was assumed until then, that

$$\alpha_i + \alpha_{i+1} = \angle_x(y_i, y_{i+2}).$$

The strategy is to interpret this equality in the space of directions at $x$. Let $p_i = [xy_i] \in \Sigma_x X$ the image of $y_i$ in the space of directions at $x$. 
By definition of the distance in the space of directions, the inequalities we want to prove are

\[ |p_i p_{i+1}| + |p_{i+1} p_{i+2}| = |p_i p_{i+2}|, \]

that is the points \( p_i, p_{i+1} \) and \( p_{i+2} \) are aligned on a geodesic. Another way to say that is to consider the geodesic polygon \( P = [p_1, \ldots, p_N] \). The equalities we want to prove are equivalent to the fact that \( P \) is a local geodesic. That’s what we are going to prove.

We argue by contradiction and suppose that \( P \) is not a local geodesic, that is we have \( |p_1 p_2| + |p_2 p_3| < |p_1 p_3| \) (up to renaming). We say that \( P \) has an angle at \( p_2 \). To obtain a contradiction, we will use a comparison polygon \( \tilde{P} \) in \( S^2 \). The harmonicity of the triangulation will force this polygon to be a great circle, but the fact that \( P \) has an angle will force \( \tilde{P} \) to also have an angle, which is absurd.

7.0.1. **Construction of the comparison polygon.** To construct the comparison polygon, we want to use Reshetnyak’s Majorization theorem. By hypothesis, \( P \) is a geodesic polygon with perimeter \( 2\pi \) so we can’t use the theorem directly. Using the fact that it has an angle, we will deform it slightly to obtain a polygon of perimeter less than \( 2\pi \).

Let \( T \) be the triangle \([p_1 p_2 p_3]\). We fix a small \( \varepsilon > 0 \). Let \( q_{1,\varepsilon} \) be the point on \([p_1 p_2]\) at distance \( \varepsilon \) from \( p_2 \), \( q_{3,\varepsilon} \) the point on \([p_3 p_2]\) at distance \( \varepsilon \) from \( p_2 \), and \( p_{2,\varepsilon} \) the middle point of \([q_{1,\varepsilon} q_{2,\varepsilon}]\). We denote \( T_\varepsilon \) the triangle \([p_1 p_{2,\varepsilon} p_3]\).

![Figure 6. The triangle \( T_\varepsilon \)](image)

By the first variation’s formula we have

\[ |q_{1,\varepsilon} q_{3,\varepsilon}| \sim \varepsilon \sin(\alpha), \]

when \( \varepsilon \) goes to 0, where \( \alpha \) is the angle \( \angle_{p_2}(p_1, p_3) \) (in \( \Sigma_x X \)), which is less than \( \pi \) because we assumed that \( P \) has an angle at \( p_2 \). We have,
for $\varepsilon$ small enough,
\[ |p_1 p_2, \varepsilon| \leq |p_1 q_1, \varepsilon| + |q_1, \varepsilon p_2, \varepsilon| \]
\[ \leq |p_1 p_2| - \varepsilon + \frac{2}{3} \sin(\alpha) \varepsilon \]
\[ = |p_1 p_2| - (1 - \frac{2}{3} \sin(\alpha)) \varepsilon \]
\[ < |p_1 p_2|, \]

and,
\[ |p_3 p_2, \varepsilon| \leq |p_3 p_2| - (1 - \frac{2}{3} \sin(\alpha)) \varepsilon < |p_3 p_2|. \]

Also, $|p_2 p_2, \varepsilon| \leq 2\varepsilon$ so $p_2, \varepsilon \to p_2$ as $\varepsilon$ goes to 0. So the perimeter of $T, \varepsilon$ goes to the one of $T$ as $\varepsilon$ goes to 0, while being smaller.

Let $P, \varepsilon$ be the polygon $P$ with $T$ replaced by $T, \varepsilon$, that is the polygon $[p_1, p_2, \varepsilon, p_3, \ldots, p_N]$. It is an approximation of $P$ of perimeter less than $2\pi$. We apply the Reshetnyak’s Majorization theorem to $P, \varepsilon$. We get
- a polygon $\tilde{P}, \varepsilon = [\tilde{p}_1, \varepsilon, \tilde{p}_2, \varepsilon, \ldots, \tilde{p}_N, \varepsilon]$ in $S^2$, whose sides’ length are the same as $P, \varepsilon$’s,
- a convex $D, \varepsilon$ in $S^2$ which is bounded by $\tilde{P}, \varepsilon$,
- the majorizer $M, \varepsilon : D, \varepsilon \to \Sigma_x X$, a 1-Lipschitz map which is length-preserving on $\tilde{P}, \varepsilon$ and which maps $\tilde{p}_i, \varepsilon$ to $p_i$, except for $\tilde{p}_2, \varepsilon$ which is mapped to $p_2, \varepsilon$.

Because $\tilde{P}, \varepsilon$ is of perimeter less than $2\pi$, its radius is less than $\frac{\pi}{2}$. We normalize the $D, \varepsilon$ using an isometry so that that their centers coincide as $\varepsilon$ vary. By compactness of $S^2$ and properness of $\Sigma_x X$, we can extract limits as $\varepsilon$ goes to 0 and obtain:
- a polygon $\tilde{P} = [\tilde{p}_1, \varepsilon, \tilde{p}_2, \varepsilon, \ldots, \tilde{p}_N, \varepsilon]$ in $S^2$, whose sides’ length are the same as $P$’s,
- a convex $D$ in $S^2$ which is bounded by $\tilde{P}$,
- the majorizer $M : D \to \Sigma_x X$, a 1-Lipschitz map which is length-preserving on $\tilde{P}$ and which maps $\tilde{p}_i$ to $p_i$.

7.0.2. The comparison polygon lies on a great circle. The perimeter of $\tilde{P}$ is $2\pi$ so it is contained in an half-hemisphere centered at a point $\tilde{q}$, that is for all $\tilde{p} \in \tilde{P}$ we have $|\tilde{p} \tilde{q}| \leq \frac{\pi}{2}$. Applying the map $M$ we have $|p_i q| \leq \frac{\pi}{2}$ (where $q = M(\tilde{q})$), that is $\angle_x (y_i, \gamma) \leq \frac{\pi}{2}$, where $\gamma$ is a geodesic representative of $q$. Now if we apply the harmonic critical inequality 3.3 to $\gamma$ we obtain:
\[ \sum_i |x y_i| \cos \angle_x (y_i, \gamma) \leq 0, \]
but the sum is also positive because $\angle_x (y_i, \gamma) \leq \frac{\pi}{2}$, so each cosine is null and $\angle_x (y_i, \gamma) = \frac{\pi}{2}$. This implies that $|\tilde{p} \tilde{q}| = \frac{\pi}{2}$, and so $\tilde{P}$ lies on the great circle centered at $\tilde{q}$. 
The comparison polygon has an angle. We present a slightly modified proof of the Reshetnyak’s Majorization theorem, which gives the fact that if the polygon has an angle, the comparison polygon can be constructed with an angle. We follow closely [2, Th. 8.12.14]. Let’s recall the statement of the theorem:

**Theorem 10.** Let \( P = [p_1 \ldots p_N] \) be a closed polygon \((p_1 = p_N)\) with geodesic side in a CAT \((\kappa)\) space \(Y\) and perimeter \(< 2D_\kappa\). Then there exists

- a polygon \( \tilde{P} = [\tilde{p}_1 \ldots \tilde{p}_N] \) in \( \mathcal{M}_\kappa \), whose sides’ length are the same as \( P \)’s,
- a convex \( D \) in \( \mathcal{M}_\kappa \) which is bounded by \( \tilde{P} \),
- a majorizer \( M : D \to Y \), a 1-Lipschitz map which is length-preserving on \( \partial \tilde{P} \) and which maps \( \tilde{p}_i \) to \( p_i \).

First we outline the un-modified proof of the theorem. The proof is induction of \( N \), the number of vertices of the polygon \( P \). For \( N = 3 \), \( P \) is a triangle and there is a construction, the details of which are not important to us, which gives a majorizer with \( D \) which is taken to be the convex hull of the comparison triangle in \( \mathcal{M}_\kappa \) and which extends the natural comparison map.

Let’s suppose that the property is true for all number less than \( N \). We consider the polygon \( P \) as the union of \( T = [p_1p_2p_3] \) and \( Q = [p_1p_3 \ldots p_N] \) glued along the side \([p_1p_3]\). By the induction hypothesis, we can majorize \( T \) and \( Q \) by polygons \( \tilde{T} \) and \( \tilde{Q} \) in \( \mathcal{M}_\kappa \). If the gluing of \( \tilde{T} \) and \( \tilde{Q} \) along \([\tilde{p}_1\tilde{p}_3]\) is a convex subset of \( \mathcal{M}_\kappa \), we’re done. Otherwise, consider this gluing as new as metric space, with the induced length-metric. By Reshetnyak’s gluing theorem, this space is CAT \((\kappa)\). Because the gluing of \( \tilde{T} \) and \( \tilde{Q} \) along \([\tilde{p}_1\tilde{p}_3]\) is not convex, the gluing of \( \tilde{T} \) and \( \tau \) along \([\tilde{p}_1\tilde{p}_3]\) is not convex with respect to the ambient metric of \( \mathcal{M}_\kappa \), where \( \tau \) is either the triangle \([\tilde{p}_1\tilde{p}_3\tilde{p}_4]\) or the triangle \([\tilde{p}_N\tilde{p}_1\tilde{p}_3]\). So in the the length metric of the gluing \( \tilde{T} \cup \tilde{Q} \), the polygon \( \tilde{T} \cup \tau \) is a triangle. Consequently, the polygon \( \tilde{T} \cup \tilde{Q} \), is a \((N-1)\)-gon, and by induction hypothesis we can majorize \( P \).

Now, let’s suppose that \( P \) has an angle at \( p_2 \), that is the triangle \( T = [p_1p_2p_3] \) is not flat. We re-follow the proof to see what is the corresponding triangle in \( \tilde{P} \). As before, we majorize \( T \) and \( Q \) by \( \tilde{T} \) and \( \tilde{Q} \). If the gluing \( \tilde{T} \cup \tilde{Q} \) is convex, the construction is finished and the triangle corresponding to \( T \) is \( \tilde{T} \), its comparison triangle. As \( T \) is not flat, its comparison triangle is also not flat, and \( \tilde{P} \) has an angle at \( \tilde{p}_2 \). If the gluing is not convex, then \( \tilde{T} \cup \tau_1 \) is a triangle in the length-metric of \( \tilde{T} \cup \tilde{Q} \) (where we denote by \( \tau_1 \) what we denoted by \( \tau \) earlier) and we use the induction hypothesis. Let’s call depth of the induction, and denote it by \( d \), the number of times we have to use the induction hypothesis until we obtain a convex gluing. Then it is clear...
that the triangle corresponding to $T$ in the final polygon we obtain is the comparison triangle of $\tilde{T} \cup \tau_1 \cup \cdots \cup \tau_d$ (considering it a triangle for the length-metric of the gluing), where the $\tau_i$ are obtained as $\tau$ in the previous paragraph, and so are triangles whose vertices are vertices of $Q$. As this triangle is not flat, its comparison triangle is also not flat and $\tilde{P}$ has an angle. We also note that without knowing the depth and which comparison triangle correspond to $T$, there is a finite number of possibilities of such triangles.

Now we show that in our case, $\tilde{P}$ has an angle. The theorem does not apply directly because its perimeter is $2\pi$. We use the construction $P_\varepsilon$ of the previous paragraph. It is a polygon of length less than $2\pi$ which has an angle, so applying the modified version of the theorem, we can construct a majorizer whose boundary $\tilde{P}_\varepsilon$ has an angle. It remains to show that the limit $\tilde{P}$ when $\varepsilon$ goes to 0 still has an angle. According to the discussion at the end of the previous paragraph, the triangle corresponding to the vertex where $\tilde{P}_\varepsilon$ has an angle is the comparison triangle of a triangle of the form $\tilde{T}_\varepsilon \cup \tau_1 \cup \cdots \cup \tau_d$, where $d_\varepsilon$ is the depth for $P_\varepsilon$. The $\tau_i$ that appears are sub-triangle of the polygon $Q = [p_1 p_3 \ldots p_N]$, which does not depend of $\varepsilon$. Because $T_\varepsilon$ converges to $T$, all of these triangles converges to a non-flat triangle and the angle of $\tilde{P}_\varepsilon$ is bounded away from 0. We can conclude that $\tilde{P}$ has an angle.

**Conclusion.** Assuming that $P$ is not a local geodesic we have proved that $\tilde{P}$ is a great circle and that $\tilde{P}$ has an angle. This is absurd, and we conclude that $P$ is a local geodesic, and that the equalities hold.

Finally, $F$ induces an isometry on the reunion of two adjacent triangles $\Delta_i \cup \Delta_{i+1}$ and then $F$ is a (global) isometric embedding.

### 7.1. Non-proper $X$

We explain how to adjust the argument when $X$ is not proper. The only place where we used the properness of $X$ is to construct a majorizer of $P$ by taking a limit of majorizer of $P_\varepsilon$. If $X$ is not proper, a limit map may not exist in $X$, but it exists in the ultrapower $X^\omega$.

We define very briefly what is the ultrapower $X^\omega$, see [2, Ch. 3]. A selective ultrafilter $\omega$ on $\mathbb{N}$ allows to define the $\omega$-limit (in $\mathbb{R} \cup \{-\infty, +\infty\}$) of any sequence of real number $(x_n) \in \mathbb{R}^\mathbb{N}$. We denote this $\omega$-limit by $\lim_{n \to \omega} x_n$; intuitively, it is the choice of a convergent subsequence. We equip the product $X^\mathbb{N}$ with the pseudo-metric

$$| (x_n)(y_n) | = \lim_{n \to \omega} |x_n y_n|,$$

and the associated metric space is denoted by $X^\omega$. The point in $X^\omega$ associated to a sequence $(x_n)$ is denoted by $\lim_{n \to \omega} x_n$. There is an isometric embedding $X \to X^\omega$ which maps $x$ to $\lim_{n \to \omega} x$. We consider $X$ as an isometrically embedded subspace of $X^\omega$ and any sequence $(x_n)$ of $X$ has an $\omega$-limit $\lim_{n \to \omega} x_n$ in $X^\omega$, which is a limit of a convergent subsequence. The ultraprodut of a CAT $(\kappa)$ space is a CAT $(\kappa)$ space.
Going back to the argument; the sequence of geodesic polygons $P_\varepsilon$ has a $\omega$-limit $P$ in $X^\omega$ when $\varepsilon$ goes to $0$. We can proceed with the same argument, but in $X^\omega$ instead of $X$, and still reach a contradiction.

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