A numerical method for the determination of a piecewise-constant conductivity

A El Badia¹, J Giroire¹, C Hollandts-Lechevalier¹

¹ Université de Technology de Compiègne, Laboratoire de Mathématiques Appliquées, B.P. 20529, 60205 Compiègne Cedex, France
E-mail: cholland@dma.utc.fr

Abstract. In the context of EEG, the head is usually modeled by disjoint coated domains (brain, skull, scalp...), in each of them the conductivity is constant. We present in this paper a numerical method to determine this conductivity, based on the Kohn-Vogelius cost function and using BIE as discretization method to solve direct problems.

1. Introduction
Let \( \Omega \subseteq \mathbb{R}^N \), \( N = 2 \) or 3, be a bounded domain with a sufficiently regular boundary \( \Gamma \), made out of \( m \) disjoint coated subdomains. Let \( \Omega_1 \) be the innermost subdomain and \( \Omega_i, i = 2, \ldots, m \) the successive layers so that their boundaries can be described as follows:

\[ \partial \Omega_1 = \Gamma_1, \partial \Omega_i = \Gamma_i \cup \Gamma_{i-1}, i = 2, \ldots, m, \text{ and } \partial \Omega = \Gamma_m = \Gamma \]

where \( \Gamma_i \) is the interface between domains \( \Omega_i \) and \( \Omega_{i+1} \) such that \( \Gamma_i \cap \Gamma_{i-1} = \emptyset \). Moreover, we suppose that these surfaces are known.

We are interested in determining the conductivity \( \sigma \) in the following elliptic equation

\[
(CP) : \begin{align*}
\nabla(\sigma \nabla u) &= 0 \quad \text{in } \Omega \\
u &= f \quad \text{on } \Gamma
\end{align*}
\]

from measurements of the normal trace \( g = \sigma_m \frac{\partial u}{\partial n} \), where the conductivity function \( \sigma \) is assumed to be constant in each subdomain \( \Omega_i \) and \( \sigma_i \) denotes the conductivity in \( \Omega_i \).

The map :

\[
\Lambda_\sigma : f \rightarrow \sigma_m \frac{\partial u}{\partial n}|_\Gamma
\]

is called the Dirichlet-to-Neumann map. The problem proposed in [2] by Calderon is to study whether it is possible to determine \( \sigma(x) \) from the knowledge of the Dirichlet-to-Neumann map \( \Lambda_\sigma \). This problem has received considerable contributions from the mathematical community. In particular, for regular \( \sigma \) function, Sylvester and Uhlmann [8] proved in 1987 the identifiability for the 3D case. When \( \sigma \) is piecewise constant with known subdomains as in our head model, one guess is that the whole operator (that is, an infinity of measurements i.e. of couples \( (f, g) \)) should not be required to determine this function. This paper is devoted to this problem from the...
numerical point of view. We first formulate it as an optimization problem in a finite dimension space. This problem is based on the minimization of a cost function. Two such cost functions are presented here. The first one is the usual Least Squares functional and the second one was proposed by Kohn and Vogelius in [4]. We use the classical BFGS iterative method to solve the corresponding optimization problems, and the boundary element method for the resolution of direct problems in iterative steps.

The efficiency of the algorithm will be illustrated by numerical results.

2. The optimization procedure

For the sake of clarity, we will expose the method for the case \( m = 2 \).

To solve our inverse conductivity problem numerically, Least Squares \( J_1 \) and Kohn-Vogelius \( J_2 \) cost functions

\[
J_1(\sigma) = \frac{1}{2} \int_{\Gamma} |\sigma_2 \frac{\partial u_D}{\partial n} - g|^2 ds
\]

\[
J_2(\sigma) = \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_i} \sigma_i |\nabla u_D - \nabla u_N|^2 dx + \frac{1}{2} \int_{\Gamma_2} |u_N - u_D|^2 ds
\]

are considered by using the classical BFGS iterative method (the Kohn-Vogelius cost function has been introduced and used in [4, 5, 6]). The functionals and their gradients are obtained by solving direct Dirichlet for the first method and Dirichlet and Neumann problems for the second one. Here \( u_D \) and \( u_N \) respectively denote the solutions of the Dirichlet and Neumann problems defined below in \( (P_D) \) and \( (P_N) \).

Now, we will calculate the gradient of the functional.

2.1. The Least Squares functional \( J_1 \)

\( u_D \) is solution of the direct Dirichlet problem:

\[
(P_D) : \begin{aligned}
\nabla (\sigma \nabla u_D) &= 0 \text{ in } \Omega \\
u_D &= f \text{ on } \Gamma_2
\end{aligned}
\]

and \( g \) is \( \Lambda_\sigma f = \sigma_2 \frac{\partial u_D}{\partial n} \) the Neumann measured combined data.

We will look for an optimum \( \sigma_{\text{opt}} = (\sigma_{1\text{opt}}, \sigma_{2\text{opt}}) \) such that:

\[
\sigma_{2\text{opt}} \frac{\partial u_D}{\partial n} |_{\Gamma_2} \approx g.
\]

Hence, the cost function \( J_1 \) is defined by the following relation:

\[
J_1(\sigma) = \frac{1}{2} \int_{\Gamma_2} |\sigma_2 \frac{\partial u_D}{\partial n} - g|^2 ds.
\]

so, that :

\[
\nabla_\sigma J_1(\sigma) = \int_{\Gamma_2} (\sigma_2 \frac{\partial u_D}{\partial n} - g) \nabla_\sigma (\sigma_2 \frac{\partial u_D}{\partial n}) ds
\]
where $\nabla_\sigma(\alpha)$ denotes the vector:

$$
\begin{pmatrix}
\frac{\partial \alpha}{\partial \sigma_1} \\
\frac{\partial \alpha}{\partial \sigma_2}
\end{pmatrix}
$$

In order to calculate the value of $\nabla_\sigma J_1(\sigma)$ and $J_1(\sigma)$, $g$, $\sigma_2 \frac{\partial u_D}{\partial n}$ and $\nabla_\sigma (\sigma_2 \frac{\partial u_D}{\partial n})$ are required. The value of $g$ is known while the value of $\sigma_2 \frac{\partial u_D}{\partial n}$ and $\nabla_\sigma (\sigma_2 \frac{\partial u_D}{\partial n})$ in $\Gamma_2$ will be obtained by using the Dirichlet problem. We will use the formulation of the forward Dirichlet problem in the form of integral equations. In order to obtain this formulation, we use the representation formula of the harmonic function in $\Omega$ [7] whose trace on the boundary $\Gamma$ is:

$$
C(y)u(y) = \int_{\Gamma} \left( \frac{\partial u(x)}{\partial n_x} G(x,y) - u(x) \frac{\partial G}{\partial n_x} \right) d\gamma_x
$$

(2)

with

$$
C(y) = \begin{cases} 
1 & \text{if } y \in \Omega, \\
\frac{1}{2} & \text{if } y \in \Gamma, \\
0 & \text{else}.
\end{cases}
$$

Hence, we write equation (2) on both boundary $\Gamma_1$ and boundary $\Gamma_2$ and we obtain the system:

$$
A(\sigma) \tau(u_D, \sigma) = c(\sigma)
$$

(3)

$$
A(\sigma) = \begin{pmatrix}
\frac{(\sigma_1+\sigma_2)}{2} I_1 + (\sigma_1 - \sigma_2) D_{11} & S_{21} \\
(\sigma_1 - \sigma_2) D_{12} & S_{22}
\end{pmatrix}
$$

(4)

$$
c(\sigma) = \begin{pmatrix} 
-\sigma_2 D_{21}(f) \\
-\sigma_2 (D_{22} + \frac{I_2}{2})(f)
\end{pmatrix}
$$

(5)

$$
\tau(\sigma, u_D) = \begin{pmatrix}
\frac{u_D|\Gamma_1}{\partial u_D/\partial n|_{\Gamma_2}} \\
-\sigma_2 \partial u_D/\partial n|_{\Gamma_2}
\end{pmatrix}
$$

(6)

where the single and double potentials for $h$ regular are defined by the following relation:

$$
S_d: \forall y \in \Gamma_i, \quad h(y) \rightarrow \int_{\Gamma_i} (h G(x,y)) d\gamma(x), \quad x \in \Gamma_i
$$

$$
D_d: \forall y \in \Gamma_i, \quad h(y) \rightarrow \int_{\Gamma_i} (h \frac{\partial G(x,y)}{\partial n}) d\gamma(x), \quad x \in \Gamma_i
$$

Solving the system (3), we obtain $-\sigma_2 \frac{\partial u_D}{\partial n}|_{\Gamma_2}$, and consequently $J_1(\sigma)$. 


Now, derivating (3), one gets:
\[
\frac{\partial A}{\partial \sigma_i} \tau + A \frac{\partial \tau}{\partial \sigma_i} = \frac{\partial c}{\partial \sigma_i} \text{ for } i=1,2,
\]
(7)

In order to determine the value of \(\frac{\partial \tau}{\partial \sigma_i}\), for \(i=1,2\), we have to solve the systems:
\[
A\mathbf{x}_i = \frac{\partial c}{\partial \sigma_i} - \frac{\partial A}{\partial \sigma_i} \tau, \quad i = 1, 2
\]
(8)
where \(\mathbf{x}_i\) represents \(\frac{\partial \tau}{\partial \sigma_i}\) for \(i=1,2\).

We can remark from (4) and (5) that \(\frac{\partial A}{\partial \sigma_i}\) and \(\frac{\partial c}{\partial \sigma_i}\) are independent of \(\sigma\) for \(i=1,2\).

Hence, the vector \(\mathbf{x}_i\) is obtained and we get:
\[
\nabla_\sigma J_1(\sigma) = -\left( \begin{array}{c}
\int_{\Gamma_2} (\sigma_2 u_D \frac{\partial u_D}{\partial n} - g) (\mathbf{x}_1)_2 ds \\
\int_{\Gamma_2} (\sigma_2 u_D \frac{\partial u_D}{\partial n} - g) (\mathbf{x}_2)_2 ds
\end{array} \right)
\]
where \((\mathbf{x}_i)_2\) denotes the second component of the vector \(\mathbf{x}_i\).

2.2. The Kohn-Vogelius functional \(J_2\)

In the case of the Kohn-Vogelius functional, we look for the value of \(\sigma\) which minimizes the difference of energy between \(u_D\) and \(u_N\) such that \(u_D\) and \(u_N\) respectively are the solutions of the direct Dirichlet and the direct Neumann problem:
\[
J_2(\sigma) = \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_i} \sigma_i |\nabla u_D - \nabla u_N|^2 dx + \frac{1}{2} \int_{\Gamma_2} |u_N - u_D|^2 ds
\]
and using the Green’s formula:
\[
J_2(\sigma) = \frac{1}{2} \int_{\Gamma_2} \left( g u_N + f \sigma_2 \frac{\partial u_D}{\partial n} \right) ds - \int_{\Gamma_2} f g ds + \frac{1}{2} \int_{\Gamma_2} |u_N - u_D|^2 ds
\]
\[
\nabla_\sigma J_2(\sigma) = \frac{1}{2} \int_{\Gamma_2} \nabla \sigma \left( g u_N + f \sigma_2 \frac{\partial u_D}{\partial n} \right) ds - \int_{\Gamma_2} \nabla (f g) ds + \frac{1}{2} \int_{\Gamma_2} \nabla (|u_N - u_D|^2) ds
\]
\[
= \frac{1}{2} \int_{\Gamma_2} \left( g \nabla \sigma (u_N) + f \nabla \sigma (\sigma_2 \frac{\partial u_D}{\partial n}) \right) ds + \int_{\Gamma_2} (u_N - u_D) (\nabla \sigma (u_N)) ds
\]

Indeed, \(f\) et \(g\) are known so that they are independent from the value of variable \(\sigma\) in the optimization procedure.

In order to calculate the value of the functional \(J_2(\sigma)\) and the gradient \(\nabla_\sigma J_2(\sigma)\), it is necessary to known the value of \(f, g, u_N, u_D, \nabla \sigma (\sigma_2 \frac{\partial u_D}{\partial n})\) and \(\nabla \sigma (u_N)\) on \(\Gamma_2\).

The values of \(f\) and \(g\) are known on \(\Gamma_2\) while the determination of the other values requires to solve the problem with Dirichlet’s condition and the problem with Neumann’s condition.
Solving the direct Dirichlet and Neumann problem at the same time

- In the previous method: the calculation of the Least Squares cost function requires, among others, the value of $\nabla_\sigma (\sigma^2 \frac{\partial u}{\partial n})$. This value is obtained by solving the system (8). It is the derivative of the formulation in the form of integral equations of the direct Dirichlet problem.

- Now, we only have to determine the value of $u_N$ and $\nabla_\sigma (u_N)$ on $\Gamma_2$ where $u_N$ is solution of:

$$ (P_N) \begin{cases} -\text{div} (\sigma \text{ grad } u_N) = 0 & \text{in } \Omega \\ \sigma^2 \frac{\partial u_N}{\partial n} = g & \text{on } \Gamma_2 \end{cases} $$

In order to obtain their values, we will use the formulation of the direct Neumann problem in the form of boundary integral equations:

$$ B(\sigma) \phi(\sigma, u_N) = d(\sigma) $$

$$ B(\sigma) = \begin{pmatrix} \frac{(\sigma_1 + \sigma_2)}{2} I_1 - (\sigma_2 - \sigma_1) D_{11} & \sigma_2 D_{21} \\ -(\sigma_2 - \sigma_1) D_{12} & \sigma_2 D_{22} \end{pmatrix} $$

$$ d(\sigma) = \begin{pmatrix} S_{21}(g) \\ S_{22}(g) \end{pmatrix} $$

$$ \phi(\sigma, u_N) = \begin{pmatrix} u_N|_{\Gamma_1} \\ u_N|_{\Gamma_2} \end{pmatrix} $$

In the aim of normalizing Neumann’s problem, we impose the value of potential at one point on $\Gamma_2$ denoted $p_0$ on $\Gamma_2$:

$$ u_N(p_0) = f(p_0). $$

By using this normalization condition, we have the following system:

$$ B_{(l-1),(l-1)} \phi_{(l-1)}(\sigma, u_N) = d_{(l-1)} - u_N(p_0) - b_{(l-1)} $$

where $b_{(l-1)}$ is the $p_0$ column of $B(\sigma)$ without the line $p_0$.

$B_{(l-1),(l-1)}$, $\phi_{(l-1)}$ and $d_{(l-1)}$ respectively represent the matrix $B$ without the line and the column $p_0$, the vector $\phi$ without the line $p_0$, and $d$ without the line $p_0$.

Since $b_{(l-1)}$ is a column of $B(\sigma)$ which corresponds to the second layer, it takes the shape of:

$$ \begin{pmatrix} \sigma_2 D_{21} \\ \frac{\sigma_2}{2} I_2 + \sigma_2 D_{22} \end{pmatrix} $$
Derivating the expression (12), we have that:

\[
\frac{\partial B_{(l-1),(l-1)}}{\partial \sigma_i} \phi_{(l-1)} + B_{(l-1),(l-1)} \frac{\partial \phi_{(l-1)}}{\partial \sigma_i} = \frac{\partial d_{(l-1)}(\sigma)}{\partial \sigma_i} - u_N(p_0). \frac{\partial b_{(l-1)}}{\partial \sigma_i} \text{ for } i = 1, 2
\]

In order to determine the value of \( \frac{\partial \phi_{(l-1)}}{\partial \sigma_i} \) for \( i = 1, 2 \), we have to solve for \( i = 1, 2 \) the following system:

\[
B_{(l-1),(l-1)} y_1 = \frac{\partial d_{(l-1)}}{\partial \sigma_i} - u_N(p_0). \frac{\partial b_{(l-1)}}{\partial \sigma_i} - \frac{\partial B_{(l-1),(l-1)}}{\partial \sigma_i} \phi_{(l-1)}.
\]  
\tag{13}

where:

\[
y_1 = \left( \frac{\partial \phi_1}{\partial \sigma_i}, \frac{\partial \phi_2}{\partial \sigma_i} \right)_{(l-1)} \text{ for } i = 1, 2.
\]

As already done for the first method, we can remark from (9) and (10) that \( \frac{\partial B_{(l-1),(l-1)}}{\partial \sigma_i} \) and \( \frac{\partial d_{(l-1)}}{\partial \sigma_i} \) and \( \frac{\partial b_{(l-1)}}{\partial \sigma_i} \) for \( i = 1, 2 \) are also independent to \( \sigma \).

Hence, we have the vector \( y_1 \). Since the value of the potential is imposed at a point \( p_0 \), the derivative \( \frac{\partial \phi}{\partial \sigma_i} \) at a point \( p_0 \) is null.

For \( i = 1, 2 \), we define the vector \( z_1 \) by the following relation:

\[
z_1 = \begin{cases} y_1(j) & \text{if } j < p_0 \\ y_1(j-1) & \text{if } j > p_0 \\ 0 & \text{else.} \end{cases}
\]  
\tag{14}

Hence, we have the value of \( \frac{\partial \phi}{\partial \sigma_i} \) at all points on \( \Gamma_2 \) and we obtain:

\[
\nabla_\sigma J_2(\sigma) = \left( \frac{1}{2} \int_{\Gamma_2} (g (z_1)_2 - f (x_1)_2) \, ds + \int_{\Gamma_2} (u_N - u_D) (z_1)_2 \, ds \right) \left( \frac{1}{2} \int_{\Gamma_2} (g (z_2)_2 - f (x_2)_2) \, ds + \int_{\Gamma_2} (u_N - u_D) (z_2)_2 \, ds \right)
\]

\]
3. Numerical Results and Conclusion

Figure 1. $\sigma_1$ with respect to $\%$ of points whose $f$ is null on $\Gamma_2$ by the Least Squares Method

Figure 2. $\sigma_1$ with respect to $\%$ of points whose $f$ is null on $\Gamma_2$ by the Kohn-Vogelius Method
In the two previous graphs, we have plotted the value of $\sigma_1$ (for $\sigma_2$ we have the same results) with respect to the percentage of points where $f$ is null on $\Gamma_2$ by using the two methods (Least Squares and Kohn-Vogelius). We used the spherical mesh (42 nodes and 80 triangles by layer). Numerical results are presented in the case $m = 2$, where the Dirichlet data $f$ is chosen equal to zero in a part $\Gamma^* \subset \Gamma_2$ and non zero on $\Gamma_2 \setminus \Gamma^*$. The quality of the obtained numerical results depends on the support of the test function $f$. In fact, when it tends to $\Gamma_2$ or to $\emptyset$ the numerical results are bad. The best results are those obtained with a test function $f$ for which the identifiability has been proved [3].

Right now we are trying to improve the results of the Kohn-Vogelius method by adding a regularization term.

We consider the case where 50% of the points whose Dirichlet datum is null on $\Gamma_2$ and $(\sigma_{1\text{sol}}, \sigma_{2\text{sol}}) = (1, 1.25 \times 10^{-2})$ and we use the same initial point $(2, 5 \times 10^{-2})$ for the optimization method (in the case, the Least Squares functional is used).

We perturb the Dirichlet data, by adding a Gaussian noise of standard deviation : 1%, 3% and 6%. We use the realistic mesh constructed by segmentation of MRI images obtained using the Asa Software [1] (510 nodes and 1016 triangles by layer). These images come from the hospital of Amiens.

### Table 1. Error on $\sigma_1$

| perturbation | $\sigma_{1\text{comp}}$ | error on $\sigma_1$ |
|--------------|--------------------------|---------------------|
| 1%           | 0.953                    | 7.57%               |
| 3%           | 1.086                    | 20.47%              |
| 6%           | 1.433                    | 43.38%              |

### Table 2. Error on $\sigma_2$

| perturbation | $\sigma_{2\text{comp}}$ | error on $\sigma_2$ |
|--------------|--------------------------|---------------------|
| 1%           | $1.26 \times 10^{-2}$    | 1.35%               |
| 3%           | $1.23 \times 10^{-2}$    | 2.74%               |
| 6%           | $1.19 \times 10^{-2}$    | 4.67%               |

It can be seen that the calculation of $\sigma_2$ is robust towards an important perturbation (up to 6%) whereas the calculation of $\sigma_1$ is not. In effect, the problem is ill-conditioned, because it is impossible to make any measurement in the innermost layer of the head. Thus, the conductivity $\sigma_1$ can only be approximated. On the contrary, the conductivity $\sigma_2$ corresponds to the outermost layer, on which boundary $\Gamma_2$ measurements can be made.

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