Geometrization of perfect fluid in 5-D Kaluza-Klein theory.

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Abstract

General formulation of geometrization matter problem by scalar field \( \varphi = \sqrt{-G_{55}} \) with the help of possibilities of classical 5-D Kaluza-Klein theory is given. Mathematical integrability conditions for such geometrization for the case of perfect fluid are derived.

1 Introduction

When GR had been built and gravitational interactions within its frame had been geometrized, Einstein himself considered this problem as half solved: energy-momentum tensor, that appeared in righthand part of field equations, in his mind, would had geometrical nature too [1].

Such a possibility naturally appears within the frame of multidimensional geometrical models type of Kaluza-Klein theory, of which the first (5-dimensional) version has been proposed by T.Kaluza in 1921 [2, 3]. Extracomponents, given by “scalar” sector of multidimensional metric have been used for a number of purposes. In present article the possibility of description of matter sources in terms of geometrical scalar field \( \varphi = \sqrt{-G_{55}} \) in effective 4-D space-time is analyzed. In present time such an approach is worked out by Wesson and others (see Ref. in [7, 8, 9, 10]), but some steps to the problem have been made by the number of authors earlier [4, 5].

2 Statement of the problem

The starting point of our investigation is vacuum 5-D Einstein equations, that we shall write in the form:

\[
5R_{AB} - \frac{1}{2} G_{AB} 5R = 0,
\]

where \( ^{5}R_{AB} \) - 5-D Ricci tensor, \( ^{5}R \) - 5-D curvature scalar, \( G_{AB} \) - 5-D metric, which in 4-D presentation has the following kind:

\[
G_{AB} = \frac{\tilde{g}_{\mu\nu} - (4k/c^4) \varphi^2 A_{\mu} A_{\nu} }{(2\sqrt{k/c^2}) \varphi^2 A_{\nu} } \left( \frac{(2\sqrt{k/c^2}) \varphi^2 A_{\mu} }{-\varphi^2} \right)
\]

Expression (3) for \( G_{AB} \) can be derived with the help of method of 1+4-splitting of 5-D manifold, which is outlined in monography [3]. “Vector” sector \( G_{5\mu} \) is identified

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with an electromagnetic potential: \( G_{5\mu} = (2\sqrt{\kappa/\epsilon^2})\phi^2 A_{\mu} \), that lets us consider 5-D Kaluza-Klein models as unified theory of gravitational and electromagnetic interactions.

Under the identification of metric of 4-D space-time section \( \tilde{g}_{\mu\nu} \) with observable 4-D metric \( g_{\mu\nu} \), there is a possibility of conformal transformation of the form:

\[
\tilde{g}_{\mu\nu} = F(\phi) g_{\mu\nu},
\]

where \( F(\phi) \) – arbitrary function of scalar field.

For identification of components of multidimensional metric with 4-D values it is necessary to impose the following conditions:

\[
\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^5} = 0, \quad \text{(cylindricity condition)}
\]

and restrictions on coordinate transformations:

\[
x'^{\mu} = x^{\mu}(x^0, x^1, x^2, x^3);
\]

\[
x'^5 = x^5 + f(x^0, x^1, x^2, x^3).
\]

Such transformations on the one hand conserve cylindricity conditions (4) and 4-D covariancy of \( \tilde{g}_{\mu\nu}, A_{\mu}, \phi \), and, on the other hand, are 4-D general coordinate transformations – (5) and gauge transformations of vector potentials \( A_{\mu} \) – (6).

The method of 1+4-splitting lets us split 5-D Einstein equations (1) in a ten 4-D Einstein equation:

\[
4 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} 4 R = \varphi^2 \kappa \kappa T^{(em)}_{\mu\nu} + \frac{1}{\varphi} (\varphi, \varphi, \tilde{\nabla}^2 \varphi);
\]

four equations, corresponding to Maxwell theory:

\[
\tilde{F}^{\alpha\beta}_{;\gamma} - \frac{2 \varphi_{,\alpha}}{\varphi} \tilde{F}^{\alpha\beta}_{;\gamma} = 0
\]

and the 15-th equation, that connects invariants of gravitational and electromagnetic fields:

\[
4 \tilde{R} + \frac{3}{8 \pi} \kappa \varphi^2 \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} = 0.
\]

In (7—9) values with “tilde” are related to starting metric \( \tilde{g}_{\mu\nu} \).

In present article all investigations are carried out, at first, without an electromagnetic field \( (F_{\mu\nu} = 0) \), and secondly, conformal factor in (3) is taken in the form:

\[
F(\phi) = \varphi^{2n},
\]

where \( n \) – arbitrary real constant. Then equations (7) and (9) with account of conformal transformation (10) can be put to the form:

\[
4 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} 4 R = (1 + 2n) \phi_{,\mu;\nu} - (2n^2 + 2n - 1) \phi_{,\mu} \phi_{,\nu} - \left[ g_{\mu\nu}(1 + 2n) \nabla^2 \phi + (n^2 + n + 1)(\nabla \phi)^2 \right];
\]

\[
n\nabla^2 \phi + n^2 (\nabla \phi)^2 - 1/6 4 R = 0,
\]

where \( \phi = \ln \varphi \). Equations (8) are satisfied identically.
So, when all outlined above conditions are satisfied, then vacuum Einstein equations (1) (after 1+4-splitting procedure) take the form of nonvacuum 4-D Einstein equations (11), (12).

Let us assume now, that we have some exact solutions of 4-D nonvacuum Einstein equations with energy-momentum tensor of perfect fluid in right-hand side ($\kappa = 1$):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (p + \varepsilon)u_\mu u_\nu - pg_{\mu\nu} \equiv T^{(hd)}_{\mu\nu} \quad (13)$$

If scalar field $\phi$ and constant $n$ are so, that

$$(p + \varepsilon)u_\mu u_\nu - pg_{\mu\nu} = (1 + 2n)\phi_{;\mu;\nu} - (2n^2 + 2n - 1)\phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}((1 + 2n)\nabla^2 \phi + (n^2 + n + 1)(\nabla \phi)^2).$$

and equation (12) is satisfied, then we can say, that such $T^{(hd)}_{\mu\nu}$ can be geometrized within the frame of 5-D Kaluza-Klein theory and 4-D metric, generated by this sources, has a purely 5-D origin. Below we shall derive restrictions on the geometry of 4-D space-time and movement of matter there, under which the equation (14) is satisfied.

## 3 Integrability conditions

Equations (14) can be considered as a system of differential equations in partial derivatives of the second order for components of the gradient $\phi_{;\mu}$, taking $u_\mu, \varepsilon, p, g_{\mu\nu}$ as prescribed functions. For integrability analysis of this system it is convenient to rewrite it in following more general form:

$$\phi_{;\mu;\nu} = k\phi_{;\mu}\phi_{;\nu} + au_\mu u_\nu + bg_{\mu\nu}, \quad (15)$$

where:

$$k = -\frac{k_2}{k_1}; \quad a = \frac{p + \varepsilon}{k_1};$$

$$b = -\frac{p - k_3(\nabla \phi)^2}{k_1} = c + k(\nabla \phi)^2; \quad (16)$$

$$c = -\frac{p}{k_1}; \quad k = -\frac{k_3}{k_1};$$

$$k_1 = 1 + 2n \neq 0; \quad k_2 = -2n^2 - 2n + 1; \quad k_3 = 3n^2 + 3n$$

The equation (15) can be derived from (14) with using the relation:

$$\nabla^2 \phi = -(1 + 2n)(\nabla \phi)^2, \quad (17)$$

which, in turn, follows from linear combination of equation (12) and contracted with $g_{\mu\nu}$ equation (11).

Integrability conditions for system (18) imply identical fulfillment of the relationship

$$2\phi_{;\mu;\nu;\lambda} = R^{\sigma}_{\mu\nu\lambda} \phi_{;\sigma}, \quad (18)$$

if into the left-hand side expressions from right-hand side of (15) are inserted , with replacement of second covariant derivatives $\phi_{;\mu;\nu}$ through (15) again. Identical fulfillment (18) is necessary for the existence of a scalar field $\phi$, satisfied the equation (14).
Omitting intermediate calculations, we perform (18) in the form:

\[
R^\sigma_{\mu\lambda\nu} \phi_\sigma = u_\mu(u_\nu a_\lambda - u_\lambda a_\nu) + kau_\mu(u_\lambda \phi_\nu - u_\nu \phi_\lambda) + (c_\lambda g_{\mu\nu} - c_\nu g_{\mu\lambda}) + \xi_1(\phi_\lambda g_{\mu\nu} - \phi_\nu g_{\mu\lambda}) + \xi_2(u_\lambda g_{\mu\nu} - u_\nu g_{\mu\lambda}) + a(u_{\mu;\lambda} u_\nu - u_{\mu;\nu} u_\lambda + u_\mu(u_{\nu;\lambda} - u_{\lambda;\nu}))
\]  

(19)

where \( \xi_1 = 2\vec{F}(k(\nabla \phi)^2 + b) - kb; \quad \xi_2 = 2\vec{k}(\nabla \phi \cdot \vec{u}) \)

(20)

4 Tetrads formalism

For obtaining of invariant form of integrability condition (15) it is convenient to go to tetrad component of all tensors. As a tetrad basis we take Lorentz tetrad \( \vec{t}, \vec{x}, \vec{y}, \vec{z}, \) The first of noted vectors can be identified with 4-velocity of matter \( \vec{u} \):

\[
g = \vec{u} \otimes \vec{u} - \vec{x} \otimes \vec{x} - \vec{y} \otimes \vec{y} - \vec{z} \otimes \vec{z}. \]

(21)

Curvature tensor \( R_{\mu\nu\lambda\sigma} \) is represented in such basis in the following kind:

\[
R_{\mu\nu\lambda\sigma} = a_{ij} X^{(i)}_{\mu\nu} X^{(j)}_{\lambda\sigma}, \quad i, j = 1, 6,
\]

(22)

where \( X^{(i)}_{\mu\nu} \) – simple bivectors, that forms the basis of 6-dimensional linear bivector space and are built from basic tetrad vectors:

\[
X^{(1)}_{\mu\nu} = 2u_{[\mu} x_{\nu]}, \quad X^{(2)}_{\mu\nu} = 2u_{[\mu} y_{\nu]}, \quad X^{(3)}_{\mu\nu} = 2u_{[\mu} z_{\nu]}, \quad X^{(4)}_{\mu\nu} = 2x_{[\mu} y_{\nu]}, \quad X^{(5)}_{\mu\nu} = 2x_{[\mu} z_{\nu]}, \quad X^{(6)}_{\mu\nu} = 2y_{[\mu} z_{\nu}].
\]

(23)

Matrix \( a_{ij} \) – is symmetrical and its elements are tetrad components of curvature tensor, for example:

\[
R_{\mu\nu\lambda\sigma} u^{\mu\nu} x^{\lambda\sigma} = R_{(0)(1)(0)(1)} = a_{11} \quad \ldots \quad a_{16}
\]

(24)

Einstein equations (13) in tetrad form take the following kind:

\[
a_{35} = a_{24}; \quad a_{14} = a_{36}; \quad a_{26} = a_{15}; \quad a_{12} = -a_{56}; \quad a_{13} = -a_{46}; \quad a_{23} = -a_{45};
\]

(25)

\[-a_{44} - a_{55} - a_{66} = \varepsilon; \quad a_{66} - a_{22} - a_{33} = \rho; \quad a_{35} - a_{11} - a_{33} = \rho; \quad a_{44} - a_{11} - a_{22} = \rho.
\]

In the right-hand side of (13) all tensors can be decomposed by a tetrad basis as follows:

\[
\vec{\nabla} \phi = \phi_0 \vec{u} + \phi_1 \vec{x} + \phi_2 \vec{y} + \phi_3 \vec{z};
\]

(26)

\[
\vec{\nabla} \varepsilon = \varepsilon_0 \vec{u} + \varepsilon_1 \vec{x} + \varepsilon_2 \vec{y} + \varepsilon_3 \vec{z};
\]

(27)

\[
\vec{\nabla} a = \frac{1 + p \varepsilon}{k_1} \vec{\varepsilon}; \quad \vec{\nabla} c = -\frac{p \varepsilon}{k_1} \vec{\varepsilon}.
\]

(28)

Here \( p \varepsilon = dp/d\varepsilon \).

For decomposition of the covariant derivative of vector \( u_\mu \) we use the well known expression (1):

\[
u_{;\mu} = F_\mu u_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu},
\]

(29)
where
\[ F_\mu = u_{\mu} u^\nu \]

- the acceleration vector of a comoving reference frame, \( F_\nu u^\nu = 0 \);

\[ \omega_{\mu\nu} = u_{[\mu} F_{\nu]} \]

- the antisymmetric tensor of angular velocity of the reference frame, \( \omega_{\mu\nu} u^\nu = 0 \);

\[ \sigma_{\mu\nu} = u_{(\mu} F_{\nu)} + \frac{\theta}{3} h_{\mu\nu} \]

- the symmetric shear tensor (traceless part of strain tensor) of the reference frame, \( \sigma_{\mu\nu} u^\nu = 0 \);

\[ \theta = u^\nu ; \]

- the stretch scalar of the reference frame (trace of strain tensor);

\[ h_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu} \]

- the metric of the local 3-D space section, orthogonal to \( u_\mu \).

Tensors of the reference frame \( F, \omega, \sigma \) in a chosen tetrad basis are given by the following expressions:

\[ \vec{F} = F_1 \vec{x} + F_2 \vec{y} + F_3 \vec{z}; \]

\[ \omega_{\mu\nu} = \omega_4 X_{\mu\nu}^{(4)} + \omega_5 X_{\mu\nu}^{(5)} + \omega_6 X_{\mu\nu}^{(6)}; \]

\[ \sigma_{\mu\nu} = \sigma_2 Y_{\mu\nu}^{(2)} + \sigma_3 Y_{\mu\nu}^{(3)} - (\sigma_2 + \sigma_3) Y_{\mu\nu}^{(4)} + \sigma_8 Y_{\mu\nu}^{(8)} + \sigma_9 Y_{\mu\nu}^{(9)} + \sigma_{10} Y_{\mu\nu}^{(10)}; \]

In the last expression \( Y_{\mu\nu}^{(i)} \) - diadic basis of 10-dimensional linear space of symmetric tensors, formed by basic vectors:

\[ Y_{\mu\nu}^{(1)} = u_\mu u_\nu; \]

\[ Y_{\mu\nu}^{(2)} = x_\mu x_\nu; \]

\[ Y_{\mu\nu}^{(3)} = y_\mu y_\nu; \]

\[ Y_{\mu\nu}^{(4)} = z_\mu z_\nu; \]

\[ Y_{\mu\nu}^{(5)} = 2 u_{(\mu} x_{\nu)}; \]

\[ Y_{\mu\nu}^{(6)} = 2 u_{(\mu} y_{\nu)}; \]

\[ Y_{\mu\nu}^{(7)} = 2 u_{(\mu} x_{\nu)}; \]

\[ Y_{\mu\nu}^{(8)} = 2 x_{(\mu} z_{\nu)}; \]

\[ Y_{\mu\nu}^{(9)} = 2 x_{(\mu} z_{\nu)}; \]

\[ Y_{\mu\nu}^{(10)} = 2 y_{(\mu} z_{\nu)}; \]

The tensor \( h_{\mu\nu} \) has the following form:

\[ h = \vec{x} \otimes \vec{x} + \vec{y} \otimes \vec{y} + \vec{z} \otimes \vec{z}. \]

Integrability conditions (15), being rewritten in tetrad form are invariant under 4-D general coordinates transformations. Freedom in the choosing of tetrad basic vectors is reduced to local space rotations of triad \( \vec{x}, \vec{y}, \vec{z} \), and can be used for the simplification of the obtained in the next section integrability equations. For the most generality we suppose that triad \( \vec{x}, \vec{y}, \vec{z} \) is arbitrary oriented.

5 Integrability conditions in tetrads representation.

Equating scalar coefficient in (15) under the similar combinations of basic vectors in right-hand and in the left-hand sides we get the following system of scalar equations, that is equivalent to the original system (15):

\[ a_{11} \phi_1 + a_{12} \phi_2 + a_{13} \phi_3 = e_1 + \lambda \phi_1 - f_1; \]
\[a_{12}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 = e_2 + \lambda\phi_2 - f_2;\]
\[a_{13}\phi_1 + a_{23}\phi_2 + a_{33}\phi_3 = e_3 + \lambda\phi_3 - f_3;\]
\[a_{14}\phi_1 + a_{24}\phi_2 + a_{34}\phi_3 = 2W_4;\]
\[a_{15}\phi_1 + a_{25}\phi_2 + a_{35}\phi_3 = 2W_5;\]
\[a_{16}\phi_1 + a_{26}\phi_2 + a_{36}\phi_3 = 2W_6;\]
\[a_{11}\phi_0 + a_{14}\phi_1 - a_{15}\phi_3 = \Delta_2 + S;\]
\[a_{12}\phi_0 + a_{24}\phi_2 - a_{25}\phi_3 = \Delta_8 + W_4;\]
\[a_{13}\phi_0 + a_{34}\phi_2 - a_{35}\phi_3 = \Delta_9 - W_5;\]
\[a_{14}\phi_0 + a_{44}\phi_2 - a_{45}\phi_3 = p_\varepsilon e_2 - \xi_1\phi_2;\]
\[a_{15}\phi_0 + a_{45}\phi_2 - a_{55}\phi_3 = -p_\varepsilon e_3 + \xi_1\phi_3;\]
\[a_{16}\phi_0 + a_{46}\phi_2 - a_{56}\phi_3 = 0;\]
\[a_{12}\phi_0 - a_{14}\phi_1 + a_{16}\phi_3 = \Delta_8 - W_4;\]
\[a_{22}\phi_0 - a_{24}\phi_1 + a_{26}\phi_3 = \Delta_3 + S;\]
\[a_{23}\phi_0 - a_{34}\phi_1 + a_{36}\phi_3 = \Delta_{10} + W_6;\]
\[a_{24}\phi_0 - a_{44}\phi_1 + a_{46}\phi_3 = -p_\varepsilon e_1 + \xi_1\phi_1;\]
\[a_{25}\phi_0 - a_{45}\phi_1 + a_{55}\phi_3 = 0;\]
\[a_{26}\phi_0 - a_{46}\phi_1 + a_{66}\phi_3 = p_\varepsilon e_3 - \xi_1\phi_3;\]
\[a_{13}\phi_0 + a_{15}\phi_1 - a_{16}\phi_2 = \Delta_9 + W_5;\]
\[a_{23}\phi_0 + a_{25}\phi_1 - a_{26}\phi_2 = \Delta_{10} - W_6;\]
\[a_{33}\phi_0 + a_{35}\phi_1 - a_{36}\phi_2 = \Delta_4 + S;\]
\[a_{34}\phi_0 + a_{44}\phi_1 - a_{46}\phi_2 = 0;\]
\[a_{35}\phi_0 + a_{55}\phi_1 - a_{56}\phi_2 = p_\varepsilon e_1 - \xi_1\phi_1;\]
\[a_{36}\phi_0 + a_{56}\phi_1 - a_{66}\phi_2 = -p_\varepsilon e_2 + \xi_1\phi_2.\]

Here

\[e_1 = \varepsilon_1/k_1; \quad f_i = aF_i; \quad W_i = aw_i; \quad \Delta_i = a\sigma_i; \quad \lambda = \xi_1 - ka. \quad (39)\]

\[S = \frac{-p_\varepsilon}{k_1} \varepsilon_0 + \xi_1\phi_0 + \xi_2 - a\theta/3.\]

The obtained equations connect components of a curvature tensor of 4-D space with a characteristics of motion of a matter there. Besides, this connections are additional to Einstein equations \([3]\). If space-time allows the identical fulfillment of the obtained integrability equations and scalar field satisfy the equation \([12]\), then this space-time has 5-D nature and 5-dimensionally geometrized matter.
6 Example: potential motion in the Friedman flat cosmological model.

Let us suppose, that expansion $\phi,\mu$ over basic tetrad vectors has the following form:

$$\phi,\mu = \phi_0 u,\mu. \tag{40}$$

This means, that a vector field $u,\mu$ is orthogonal to hypersurface $\phi = \text{const}$. For simplicity we restrict our attention to the case of motion in the Friedman flat cosmological model:

$$ds^2 = dt^2 - e^{2\lambda}(dx^2 + dy^2 + dz^2). \tag{41}$$

Nonzero components of curvature tensor are:

$$a_{11} = a_{22} = a_{33} = \ddot{\lambda} + \lambda^2, \quad -a_{44} = -a_{55} = -a_{66} = \dot{\lambda}^2. \tag{42}$$

Tetrad Einstein equations (25) take the following form:

$$3\ddot{\lambda}^2 = \varepsilon; \quad -2\dot{\lambda} - 3\lambda^2 = p. \tag{43}$$

4-velocity $u,\mu$ has the following components:

$$u_0 = 1, \quad u_1 = u_2 = u_3 = 0. \tag{44}$$

From the tensors characteristics of the reference frame only stretch coefficient is nonzero:

$$\theta = 3\dot{\lambda}. \tag{45}$$

Integrability conditions of the previous section are reduced to only:

$$a_{11}\phi_0 = S. \tag{46}$$

Taking into account that $\phi_0 = \dot{\phi}$, writing out abbreviated designations by formulae (20) and (39) and expressing $\dot{\varepsilon}$ from Einstein equations (43), condition (46) can be put to the form:

$$\mu_1\dot{\phi}^3 + \phi(\lambda^2(\mu_2 - 1) - \lambda) - \mu_3\dot{\lambda}\ddot{\lambda} = 0, \tag{47}$$

where

$$\mu_1 = \frac{k(k + 2k)}{2}; \quad \mu_2 = 3(2k/k + p_c k/k_1); \quad \mu_3 = 2(3p_c - 1)/k_1$$

and state equation is taken in the following form:

$$p = p_c\varepsilon, \quad p_c = \text{const}. \tag{48}$$

Equation (12), that in metric (41) has the form:

$$\dddot{\phi} + 3\lambda\dot{\phi} + (1 + 2n)\phi^2 = 0 \tag{49}$$

has the following first integral:

$$\phi = \frac{1}{C + (1 + 2n) \int e^{-3\lambda} dt}, \tag{50}$$

where $C$ – is a constant integration, of which general solution can be easily obtained:

$$\phi = \frac{1}{1 + 2n} \ln(C + (1 + 2n) \int e^{-3\lambda} dt) + C_1 \tag{51}$$
With the substitution of $\dot{\phi}$ from (50) to (49) the expression in the left-hand must be equal zero identically. Einstein equations for a chosen class state equation (48) can be easily integrated:

$$\lambda = \begin{cases} 
(2/3(1 + p_\varepsilon))\ln[(3/2)(1 + p_\varepsilon)t + C_0], & p_\varepsilon > -1; \\
C_0t, & p_\varepsilon = -1.
\end{cases}$$

For $p_\varepsilon = -1$ ($p + \varepsilon = 0$), omitting all intermediate calculations, we obtain that identically satisfied under $n = 1, -2$. In this case $\phi = C_0t$. Case $p_\varepsilon = 1$ ($p = \varepsilon$) will be considered in subsection (3) of Conclusion. For $p_\varepsilon \neq \pm 1$ we obtain the following square equation for the index of conformal transformation $n$:

$$\alpha_1 n^2 + \alpha_1 n + \alpha_2 = 0,$$

where

$$\alpha_1 = 27p_\varepsilon^3 + 63p_\varepsilon^2 + 33p_\varepsilon + 5; \quad (53)$$

$$\alpha_2 = 18p_\varepsilon^2 + 24p_\varepsilon - 10.$$  

Its solution is:

$$n = \frac{3p_\varepsilon(\sigma - 1) - (3\sigma + 1)}{2(3p_\varepsilon + 1)}, \quad \sigma = \pm 1 \quad (54)$$

For dust ($p_\varepsilon = 0$) we have $n = 1, -2$, for radiation ($p_\varepsilon = 1/3$) $n = 0, -1$. Corresponding 5-D vacuum metrics that in all cases can be obtained by the inverse conformal transformation are only of the following two types:

$$dI_1^2 = dt^2 - dx^2 - dy^2 - dz^2 - t^2(dx^5)^2; \quad (55)$$

$$dI_2^2 = dt^2 - t(dx^2 + dy^2 + dz^2) - \frac{1}{t}(dx^5)^2 \quad (56)$$

Its relations with considered 4-D metrics are shown in the table:

| $p_\varepsilon$ | $n$ | $p_\varepsilon = 0$ | $p_\varepsilon = 1/3$ |
|----------------|-----|---------------------|----------------------|
| $p_\varepsilon = -1$ | $n = 1$ | $n = -2$ | $n = -1$ |
| $p_\varepsilon = 0$ | $n = -2$ | $n = 1$ | $n = 0$ |

Cases with $n = 1$ have been considered in |11, 12], the case with $n = 0$ have been analyzed in Wesson’s work |7].

7. Conclusion

Completing our account we make some remarks:

1. The approach, proposed here, can be called “4-dimensional”, because 4-D metric and an energy-momentum tensor are given. From integrability conditions scalar field $\phi$ and index $n$ can be found, or, that is the same, can be found 5-D vacuum space-time which geometrizes beforehand given matter. As an example of another approach, “5-dimensional”, we can consider the approach of Wesson, where known 5-D vacuum solution is used. 5-D Einstein equations are split in to parts: one – 4-D Einstein tensor, other – a combination
of derivatives of the scalar field, which is declared as an effective energy-momentum tensor of induced matter. Type of this tensor in such approach is, in general, arbitrary. In both [8, 9] obtained energy-momentum tensor was anisotropic. All investigations in Wesson’s group are carried out under \( n = 0 \). Note, that using of the conformal transformation alleviates statement made in [10]: there is no rigid necessity to introduce the 5-th coordinate into metric to obtain state equation for effective matter other than radiationlike.

2. The problem of the geometrization of matter in Kaluza-Klein theory has many formal analogies with the known problem of isometric embedding of 4-D Riemannian space into flat space of more dimensions [6].

3. The restriction on index \( n \neq -1/2 \) involves the fact that in this case the second derivatives of a scalar field in equations (14) vanish. Equations (14) became algebraic with respect to the gradient \( \phi_\mu \) and integrability conditions will be

\[
\phi_{\mu,\nu} - \phi_{\nu,\mu} = 0,
\]

where \( \phi_\mu \) is algebraically expressed through \( u_\mu \), \( p \) and \( \varepsilon \). In the considered example a peculiar case \( p = 1 \) \( (p = \varepsilon) \) is realized namely under \( n = -1/2 \) from both 5-D metrics (55,56).

4. Cosmological models of open and closed types have been considered under \( n = 1 \) in [5] (p.230-234) from the viewpoint of “5-dimensional” approach.

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