Classical tensors from quantum states

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Abstract

The embedding of a manifold $M$ into a Hilbert-space $\mathbb{H}$ induces, via the pull-back, a tensor field on $M$ out of the Hermitian tensor on $\mathbb{H}$. We propose a general procedure to compute these tensors in particular for manifolds admitting a Lie-group structure.

1 Introduction

The geometrical identification of mathematical structures of quantum mechanics goes back to Dirac \cite{1} \cite{2}, with the introduction of quantum Poisson brackets, and to Weyl, Segal and Mackey \cite{3} \cite{4} \cite{5} \cite{6} \cite{7} \cite{8} \cite{9} who identified the role of the symplectic structure both in quantum mechanics and quantum field theory. In the same circle of ideas one may include the paper by Strocchi \cite{10}. A strict geometrical formulation, however, is more recent (Heslot, Rowe, Cantoni, Cirelli et al., Ashtekar, Gibbons, Brody, Hughston, de Gosson) \cite{11} \cite{12} \cite{13} \cite{15} \cite{16} \cite{17} \cite{18} \cite{19} \cite{20} \cite{21} \cite{22} \cite{23} \cite{24}, and it has also been used systematically to introduce and analyze in the quantum setting the role of bi-Hamiltonian description of evolution equations \cite{25} \cite{26}. The geometrical formulation of quantum mechanics in the Dirac approach goes along the following lines. As a first step, one replaces the Hilbert space
with the tangent bundle $T\mathbb{H}$ constructed on the real differential Hilbert manifold $\mathbb{H}^\mathbb{R} := \text{Re}(\mathbb{H}) \oplus \text{Im}(\mathbb{H})$, which replaces the usual complex separable Hilbert space. The Hermitian inner product $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$ on quantum states is then replaced by an Hermitian tensor on quantum-state-valued sections of the tangent bundle $T\mathbb{H}$ defining a Riemannian tensor for the real part, and a symplectic structure for the imaginary part. Roughly speaking, this amounts to identify the Hilbert space $\mathbb{H}$ with the tangent space $T_\phi\mathbb{H}$ at each point $\phi$ of the base manifold. By using the Hermitian structure one may start equally well with $\mathbb{H}^*$, the dual vector space of $\mathbb{H}$. By using sections of $T^*\mathbb{H}$ we define a Riemannian tensor in contravariant form and a Poisson tensor (i.e. a symplectic form written in contravariant form).

It has been remarked several times that in this formulation, quantum evolution described by the Schrödinger equation defines a Hamiltonian vector field which, in addition, preserves a complex structure and a related Riemannian metrics. With other words, vector fields representing quantum systems are not only Hamiltonian, they are also Killing vector fields. These remarks point out that several aspects of Hamiltonian dynamics may be also used with advantages in connection with quantum mechanics. On the other hand, it is well known that classical dynamics is fully described by using symplectic manifolds or, more generally, by means of Poisson manifolds when constraints are also taken into account. However on a selected family of quantum states parametrized by a real differential manifold $M$ some shadows of the additional structures existing in quantum mechanics appear also in this “classical framework” (This point of view has been emphasized many times by J. Klauder [27][28][29][30]).

In this paper, we would like to investigate how to define classical tensors from quantum states by considering in particular manifolds admitting a Lie-Group structure $G \cong M$. Thus our procedure may be considered as way to implement Klauder’s point of view. This approach is closely related to the mathematical setting appearing in the generalized coherent states [31][32][33]. The identified manifold will depend on the initial fiducial state we start with to define the orbit. In this way we will have the possibility to define embeddings $M \hookrightarrow \mathbb{H}$ and the associated pull-backs in a natural framework by using unitary (vector or ray) representations $G \to U(\mathbb{H})$. To clearly illustrate our procedure, we shall consider first the action of a group $G$ on a finite dimensional Hilbert-space. Then we shall move to the more realistic case of infinite dimensions by means of specific examples rather than dealing with general aspects. Depending on the particular nature of these Lie-group-manifolds we
may identify them with configuration spaces or phase spaces. Of course, these
constructions are not without relations with the quantum-classical transition,
nevertheless they should not be considered equivalent to the classical coun-
terpart. To be specific, finite-level quantum systems prevalently considered
for quantum computing and quantum information are systems defined on fi-
nite dimensional manifolds, or even stratified real differential manifolds [34],
and do not correspond to any classical limit - for this reason one should avoid
considering our procedure as a way to explicitly define classical dynamical
systems corresponding to quantum ones. Moreover the tensor fields defined
here will have only a kinematical interpretation. The family of quantum evo-
lutions or transformations admitting a counterpart on the finite dimensional
manifolds has to be analyzed separately.

2 Hermitian tensor fields on the Hilbert man-
ifold

We consider a separable complex Hilbert space $\mathbb{H}$. On its realification $\mathbb{H}^\mathbb{R}$ we
construct then tangent and cotangent bundles $T\mathbb{H}^\mathbb{R}$ and $T^*\mathbb{H}^\mathbb{R}$. To introduce
coordinate functions we use an orthonormal basis $\{|e_j\rangle\}_{j \in \mathcal{J}}$ where the index
set may be finite or infinite dimensional ($\mathcal{J} := \mathbb{N}$). For any vector $|\psi\rangle$ we set:

$$z^j(\psi) = \langle e^j | \psi \rangle = q^j(\psi) + ip^j(\psi).$$

(2.1)

Usually we shall simply write $z^j$ or $(q^j, p^j)$, respectively, for complex or real
coordinates, and drop the argument. When we need to use a continuous
basis, say for the coordinate or momentum representation, we write

$$|\psi\rangle = \int dx \ |x\rangle \langle x |\psi\rangle = \int dx \ |x\rangle \ \psi(x)$$

(2.2)

with $dx$ representing the Lebesque-measure. In what follows our statements
should be considered to be always mathematically well defined whenever
the Hilbert space we are considering is finite dimensional. In the case of
an infinite-dimensional Hilbert space, additional qualifications are needed
whenever we have to deal with unbounded operators; these cases will be
handled separately when it is the case instead of making general claims. To
make computations easy to follow we shall use symbols like $d|\psi\rangle := |d\psi\rangle$. 
They should be understood as defining vector-valued differential forms, i.e.

\[ |d\psi\rangle = d(z^j|e_j) = dz^j|e_j \] . \hspace{1cm} (2.3)

Specifically, we assume that an orthonormal basis has been selected once and it does not depend on the base point. To deal with "moving frames" one should introduce a connection. Of course this can be done and in some specific situations it is useful and convenient. This is the case when we deal with Berry phases or the non-abelian generalization of Wilczek and Zee \[35\][36].

In this respect, \(|d\psi\rangle\) should be thought of as a section of the cotangent bundle \(H \to T^*H \cong H \times H^*\) tensored with the Hilbert space \(H\). With this notation, the usual Hermitian inner product

\[ \langle \psi | \psi \rangle = \langle z^j e_j | z^k e_k \rangle = \langle e_j | e_k \rangle \bar{z}^j z^k = \delta_{jk} \bar{z}^j z^k \] \hspace{1cm} (2.4)

is easily promoted to a tensor field (Hermitian or Kählerian tensor field) by setting

\[ \langle d\psi \otimes d\psi \rangle := \langle e_j | e_k \rangle d\bar{z}^j \otimes dz^k = \delta_{jk} d\bar{z}^j \otimes dz^k . \] \hspace{1cm} (2.5)

By factoring out real and imaginary part we find:

\[ \langle d\psi \otimes d\psi \rangle = \delta_{jk}(dq^j \otimes dq^k + dp^j \otimes dp^k) + i\delta_{jk}(dq^j \otimes dp^k - dp^j \otimes dq^k) . \] \hspace{1cm} (2.6)

Thus the Hermitian tensor decomposes into an Euclidean metric and a symplectic form. Clearly, infinitesimal generators of one-parameter groups of unitary transformations will be at the same time Hamiltonian vector fields. In this sense for quantum evolution we may be able to use most of the mathematical tools which have been elaborated for Hamiltonian dynamics (Arnold, Abraham-Mardsen, Marmo et al., Lieberman-Marle) \[37\][38][39][40].

If with any vector \(|\psi\rangle\) we associate a vector field

\[ X_\psi : H \to T^*H; \phi \mapsto (\phi, \psi), \] \hspace{1cm} (2.7)

then it is possible to write a contravariant Hermitian tensor which we may write in the form

\[ \left\langle \frac{\delta}{\delta \psi} \otimes \frac{\delta}{\delta \psi} \right\rangle := \langle e_j | e_k \rangle \frac{\partial}{\partial \bar{z}^j} \otimes \frac{\partial}{\partial z^k} \] \hspace{1cm} (2.8)

Again, the decomposition into real and imaginary part would give

\[ \delta^{jk}(\frac{\partial}{\partial q^j} \otimes \frac{\partial}{\partial q^k} + \frac{\partial}{\partial p^j} \otimes \frac{\partial}{\partial p^k}) \] \hspace{1cm} (2.9)
for the real part, and
\[
\delta_{jk} \left( \frac{\partial}{\partial q^j} \otimes \frac{\partial}{\partial p^k} - \frac{\partial}{\partial p^j} \otimes \frac{\partial}{\partial q^k} \right),
\]
(2.10)
for the imaginary part. As the probabilistic interpretation of quantum mechanics requires that the identification of quantum states is made of with rays of \( \mathbb{H} \) (one dimensional complex vector spaces) rather than with vectors, our tensors should be defined on the ray space \( \mathcal{R}(\mathbb{H}) \); i.e. the complex projective space \( \mathbb{CP}(\mathbb{H}) \), instead of \( \mathbb{H} \). Equivalence classes of vectors are defined by \( \psi \sim \varphi \) iff \( \psi = \lambda \varphi \) for \( \lambda \in \mathbb{C}_0 := \mathbb{C} - \{0\} \). At the infinitesimal level, the action of the group \( \mathbb{C}_0 \) is generated by the vector fields
\[
\Delta := q^j \frac{\partial}{\partial q^j} + p^j \frac{\partial}{\partial p^j} : \mathbb{H} \to T\mathbb{H}; \psi \mapsto (\psi, \psi) \quad (2.11)
\]
and
\[
\Gamma := p^j \frac{\partial}{\partial q^j} + q^j \frac{\partial}{\partial p^j} : \mathbb{H} \to T\mathbb{H}; \psi \mapsto (\psi, J\psi). \quad (2.12)
\]
Here \( J \) is the one-one tensor field representing the complex structure on the realified version of the complex Hilbert space. For contravariant tensor fields \( \tau \) on \( \mathbb{H} \) to be projectable onto \( \mathcal{R}(\mathbb{H}) \) one has to require that \( L_\Delta \tau = 0 \) and \( L_\Gamma \tau = 0 \). On the other hand, for covariant tensor fields, to be the pull-back of tensor fields on \( \mathcal{R}(\mathbb{H}) \) it is necessary that \( L_\Delta \alpha = 0 \), \( L_\Gamma \alpha = 0 \) and moreover \( i_\Delta \alpha = 0 \), \( i_\Gamma \alpha = 0 \).

These remarks allow to conclude that the Hermitian tensor on \( \mathbb{H}_0 \) (the Hilbert space \( \mathbb{H} \) without the zero vector), which is the pull-back of the Kählerian tensor on \( \mathcal{R}(\mathbb{H}) \), has the form
\[
\frac{\langle d\psi \otimes d\psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle} \otimes \frac{\langle d\psi | \psi \rangle}{\langle \psi | \psi \rangle}. \quad (2.13)
\]
For further details on these tensors we refer the reader to [18] [19] [20] [21] [25] [41].

### 3 Tensors on Lie groups from finite dimensional representations

Let us consider a Lie group \( G \) acting on a vector space \( V \). This means that there exists a Lie-Group homomorphism
\[
\pi : G \to \text{Aut}(V) \quad (3.14)
\]
or an action

\[ \phi : G \times V \rightarrow V. \tag{3.15} \]

For each fiducial vector \( v_0 \in V \) we define a submanifold in \( V \) given by

\[ \phi(G \times \{v_0\}) = \{\pi(g) \cdot v_0\} \subset V. \tag{3.16} \]

By considering the tangent bundle construction, we find an action of \( TG \) on \( TV = V \times V \)

\[ T\phi : TG \times TV \rightarrow TV. \tag{3.17} \]

Because \( TG \cong G \times g \), \( g \) being the Lie-Algebra of \( G \), the tangent map \( T\phi \) requires the existence of a representation on \( V \) of the Lie algebra \( g \). In general, in the finite dimensional case, the representation of \( g \) extends naturally to a representation of the enveloping algebra \( U(g) \) on \( V \). In infinite dimensions when we start with unitary representations of \( G \) on \( \mathbb{H} \), the fiducial vector should be chosen to be smooth or analytic, so that again we have a natural extension of the representation of the Lie-algebra to the enveloping algebra [42].

The main idea we use to construct covariant tensors on \( G \) out of covariant tensors on \( V \) is to consider the map

\[ \phi_{v_0} : G \rightarrow V \tag{3.18} \]

as an embedding so that we may pull-back to \( G \) the algebra of functions \( \phi_{v_0}^*(\mathcal{F}(V)) \subset \mathcal{F}(G) \), and, along with the relation connecting with the exterior differential on the two spaces

\[ d\phi_{v_0}^* = \phi_{v_0}^* d, \tag{3.19} \]

we are able to pull-back all the algebra of exterior forms, but also the tensor algebra generated by one-forms with real or complex valued functions as coefficients.

When the group acts directly on the space of rays, \( \mathcal{R}(V) \), for a complex vector space \( V \), by means of

\[ \phi(g) \cdot [v] = [\pi(g) \cdot v], \tag{3.20} \]

the corresponding action on \( V \), by means of \( \pi(g) \), need not be a true representation but it is enough that is defined up to a multiplier, i.e.

\[ \pi(g) \cdot \pi(h) = m(g, h)\pi(g, h) \tag{3.21} \]
with \( m(g, h) \) a non-zero complex number. Thus the quantum mechanical
probabilistic interpretation does not require that \( \pi \) is a vector representation
but only that it is a ray-representation. In many cases we have to deal with
this additional freedom.

In his seminal paper [43], Bargmann associated a vector-representation of a
central extension of \( G \) by means of the multiplier \( m \) (the so called Bargmann
group of \( G \)) with a ray representation of \( G \). The most important example
is provided by the Abelian vector group which may be centrally extended to
the Heisenberg-Weyl group. Another important example is provided by the
Galilei group.

Let us start with a vector representation of \( G \) on a vector space
\( V \). The orbit
of the action of \( G \) on \( V \), starting with the fiducial vector \( v_0 \) will be denoted
by \( M = \phi(G \times \{v_0\}) \subset V \). We shall use for convenience the bra-ket notations
of Dirac. We have

\[
U(g) |0\rangle = |g\rangle; \quad \{|g\rangle\}_{g \in G} = M.
\] (3.22)

It should be noticed that \( M \) will not be a vector space and may be given
a manifold structure by using the differential structure on \( G \). If \( G_0 \) is the
isotropy group of \( |0\rangle \), we find \( M := G/G_0 \). The vectors parametrized by
\( M \) may generate the full vector space by means of linear combinations.
We may use an orthonormal basis for \( V \) and define coordinate functions
\( z^j(g) = \langle e_j | g \rangle \), The vector-valued one-forms we obtain by taking the exterior derivative

\[
d|g\rangle = dU(g) |0\rangle = dU(g)U^{-1}(g) |g\rangle
\] (3.23)

and the Hermitian tensor \( \langle d\psi \otimes dg \rangle \), when calculated on the manifold \( M \)
(the pulled-back tensor) will be

\[
\langle dg \otimes dg \rangle := \langle g| (dU(g)U^{-1}(g))^\dagger \otimes dU(g)U^{-1}(g) |g\rangle.
\] (3.24)

If we denote by \( X^1, X^2, ..., X^n \) the generators of the left action of \( G \) on
itself, i.e. the right invariant infinitesimal generators and by \( \theta_1, \theta_2, ... \theta_n \) the
corresponding dual basis of one forms, i.e. \( \theta_j(X^k) = \delta^k_j \), we consider \( U(t) = e^{itR(X)} \) and we find

\[
dU(g)U^{-1}(g) = iR(X^j)\theta_j
\] (3.25)

along with

\[
(dU(g)U^{-1}(g))^\dagger = -iR(X^j)\theta_j
\] (3.26)

7
because the infinitesimal generators are skew-Hermitian. In conclusion:

\[ \langle dg \otimes dg \rangle = \langle g | R(X^j)R(X^k) | g \rangle \theta_j \otimes \theta_k . \]  \hspace{1cm} (3.27)

By decomposing the basis elements \( \theta_j \otimes \theta_k \) into

\[ \frac{1}{2}(\theta_j \otimes \theta_k + \theta_j \otimes \theta_k) + \frac{1}{2}(\theta_j \otimes \theta_k - \theta_j \otimes \theta_k) := \frac{1}{2}\theta_j \otimes \theta_k + \frac{1}{2}\theta_j \land \theta_k , \]  \hspace{1cm} (3.28)

it is possible to extract the real part

\[ \frac{1}{2}\langle g | R(X^j)R(X^k) + R(X^k)R(X^j) | g \rangle \theta_j \otimes \theta_k \]  \hspace{1cm} (3.29)

and the imaginary part

\[ \frac{1}{2}\langle g | R(X^j)R(X^k) - R(X^k)R(X^j) | g \rangle \theta_j \land \theta_k \]  \hspace{1cm} (3.30)

in the usual way. Because the commutator of Hermitian operators is skew hermitian, the second term is imaginary and we have derived a Riemannian tensor along with a (pre-)symplectic structure. It should be remarked that the Riemannian tensor is the expectation value of an element of order two in the homomorphic image, provided by the representation of the enveloping algebra of the Lie algebra \( G \). Thus in the infinite dimensional situation we have to consider whether \( |g\rangle \) is in the domain of the operator of order two which appears in the definition of the Riemannian tensor. Some theorems are available \[42\] \[44\] \[45\], but we shall not be concerned with these problems here. By using the fact that \( R \) is associated with the tangent map of a vector or ray unitary representation, we find

\[ R(X^j)R(X^k) - R(X^k)R(X^j) = iR([X^j, X^k]) + i\omega(X^j, X^k) , \]  \hspace{1cm} (3.31)

where \( \omega \) is a closed 2-form on the group associated with the multiplier \( m \) when we deal with a ray representation instead of a vector representation of \( G \).

**Remark:** By using the adjoint action of \( G \) on its Lie algebra, it is possible to go from right invariant vector fields to left-invariant ones. In this way the expectation values of operators generated by right-invariant infinitesimal generators on the states \( |g\rangle \) may be replaced by the expectation values of
the corresponding operators written in terms of left-invariant infinitesimal
generators evaluated on the initial fiducial state $|0\rangle$.

If we introduce $Y_1, Y_2, \ldots, Y_n$ generators of the right action along with
$
\alpha_1, \alpha_2, \ldots, \alpha_n$ dual one-forms, $\alpha_j(Y^k) = \delta^k_j$, we have also

$$
\langle dg \otimes dg \rangle = \langle 0 \mid R(Y^j)R(Y^k) \mid 0 \rangle \alpha_j \otimes \alpha_k. 
$$

(3.32)

In this way the role of the fiducial vector and the requirement that it should
be in the domain of the operators of order two in the enveloping algebra of
the left invariant generators becomes more clear. It may be convenient to
derive in general form the pull-back Kählerian tensor when we start with an
action on the ray space, the complex projective space, instead of the Hilbert
space.

Here we have to start not with $\langle d\psi \otimes d\psi \rangle$ in (2.5) but with

$$
\frac{\langle d\psi \otimes d\psi \rangle}{\langle \psi \mid \psi \rangle} = \frac{\langle \psi \mid d\psi \rangle \otimes \langle d\psi \mid \psi \rangle}{\langle \psi \mid \psi \rangle} 
$$

(3.33)
in (2.13). Therefore the pulled-back tensor becomes

$$
\frac{\langle dg \otimes dg \rangle}{\langle g \mid g \rangle} = \frac{\langle g \mid dg \rangle \otimes \langle dg \mid g \rangle}{\langle g \mid g \rangle} 
$$

(3.34)

After simple computations we find

$$
\left( \frac{\langle g \mid R(Y^j)R(Y^k) \mid g \rangle}{\langle g \mid g \rangle} - \frac{\langle g \mid R(Y^j) \mid g \rangle \langle g \mid R(Y^k) \mid g \rangle}{\langle g \mid g \rangle} \right) \theta_j \otimes \theta_k 
$$

(3.35)

The net result is that the closed 2-form will not be effected, except for the
normalization, while the metric tensor will be modified by the addition of an
extra term

$$
\langle g \mid R(Y^j) \mid g \rangle \langle g \mid R(Y^k) \mid g \rangle \theta_j \otimes \theta_k 
$$

(3.36)

Few comments are in order. From the expression of the $jk$—th coefficient of
the pulled back tensor

$$
\langle 0 \mid R(Y^j)R(Y^k) \mid 0 \rangle - \langle 0 \mid R(Y^j) \mid 0 \rangle \langle 0 \mid R(Y^k) \mid 0 \rangle 
$$

(3.37)

we notice that when

$$
R(Y^k) \mid 0 \rangle = \lambda^k \mid 0 \rangle 
$$

(3.38)
we find
\[ \lambda^k \langle 0 | R(Y^j) | 0 \rangle - \langle 0 | R(Y^j) | 0 \rangle \lambda^k = 0. \]  
(3.39)

It means that the subalgebra of $g$ of the subgroup of $G$ which acts on $|0\rangle$ simply by multiplication by a phase will give rise to ”degeneracy directions” for the Hermitian tensor. In more specific terms the tensor we are pulling back provides a tensor on $G/G_0$, i.e. on the homogeneous space defined by the isotropy subgroup (up to a phase) of the fiducial vector.

In the coming sections we are going to consider some specific examples which have been selected because of their relevance for physical problems.

## 4 Pulled-back tensors on a compact space: $SU(2)$

The simplest non-trivial compact Lie-group is given by $SU(2) \cong S^3$. An embedding of this group into the Hilbert space

\[ \mathbb{H} = L^2(SU(2)) := \bigoplus_s \mathbb{C}^{2s+1}, s \text{ integer or half integer} \]  
(4.40)

can be realized in different ways, since it will depend on the choice of the spin-$s$-representations

\[ U^s : SU(2) \rightarrow \text{Aut}(\mathbb{C}^{2s+1}), \ g \mapsto U^s(g). \]  
(4.41)

By using

\[ dU^s(g)^\dagger = -U^s(g)^\dagger dU^s(g) U^s(g)^\dagger \]  
(4.42)

the pulled back tensor reads

\[ \langle 0 | (dU^s(g))^\dagger \otimes dU^s(g) | 0 \rangle = \langle 0 | R^s(Y^j) R^s(Y^k) | 0 \rangle \theta_j \otimes \theta_k. \]  
(4.43)

The pulled back tensor associated to the pulled back tensor on $SU(2)/U(1) \cong S^2$ provides on the other hand the structure

\[ \left( \langle 0 | R^s(Y^j) R^s(Y^k) | 0 \rangle - \langle 0 | R^s(Y^j) | 0 \rangle \langle 0 | R^s(Y^k) | 0 \rangle \right) \theta_j \otimes \theta_k. \]  
(4.44)

If we choose $|0\rangle$ to be an eigenvector of $R^s(Y^3)$ it follows that the symmetric tensor and skew-symmetric form are both degenerate.

Let us compute this pulled back tensors in the defining representation $s = 1/2$
with \( R(Y^j) := \sigma^j \), the Pauli-matrices explicitly. Here we get based on the fiducial state
\[
|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2
\]
(4.45)
a symmetric tensor
\[
\frac{1}{2} \langle 0 | \sigma^j \sigma^k + \sigma^k \sigma^j | 0 \rangle \theta_j \otimes \theta_k = \frac{1}{2} \delta^{jk} \theta_j \otimes \theta_k = \frac{1}{2} \theta_j \otimes \theta_j
\]
(4.46)
and an antisymmetric tensor
\[
\frac{1}{2i} \langle 0 | \sigma^j \sigma^k - \sigma^k \sigma^j | 0 \rangle \theta_j \wedge \theta_k = -d\theta_3,
\]
(4.47)
where we have used the decomposition
\[
\sigma^j \sigma^k = \delta^{jk} \sigma^0 + i \epsilon^{jk}_r \sigma^r,
\]
(4.48)
and the Maurer-Cartan relation
\[
d\theta_r + \frac{1}{2} c_{rjk} \theta_j \wedge \theta_k = 0
\]
(4.49)
with \( c_{rjk} = \epsilon_{rjk} \) for \( G = SU(2) \). By using furthermore the right invariant one-forms on \( SU(2) \) given by
\[
\begin{align*}
\theta_1 &= \sin(\alpha) d\beta - \sin(\beta) \cos(\alpha) d\gamma, \\
\theta_2 &= \cos(\alpha) d\beta + \sin(\beta) \sin(\alpha) d\gamma, \\
\theta_3 &= d\alpha + \cos(\beta) d\gamma,
\end{align*}
\]
(4.50 - 4.52)
we see that the symmetric tensor (4.46) coincides with the Riemannian tensor
\[
\frac{1}{2} (d\alpha \otimes d\alpha + d\beta \otimes d\beta + 2 \cos(\beta) d\alpha \otimes d\gamma),
\]
(4.53)
which is induced on a three-sphere by an embedding in a four dimensional Euclidean space, where else its “projective” counterpart
\[
\left( \langle 0 | \sigma^j \sigma^k | 0 \rangle - \langle 0 | \sigma^j | 0 \rangle \langle 0 | \sigma^k | 0 \rangle \right) \theta_j \otimes \theta_k,
\]
(4.54)
coincides after symmetrization with the induced metric
\[
\frac{1}{2} (d\beta \otimes d\beta + \sin^2(\beta) d\gamma \otimes d\gamma),
\]
(4.55)
on a two-sphere being embedded in three dimensional Euclidean space. The antisymmetric part (4.47) turns out to be equal to
\[
\sin(\beta) d\beta \wedge d\gamma.
\]
(4.56)
5 Weyl systems and pulled-back tensors on a symplectic vector space

We consider now a symplectic vector space \((V, \omega)\). A Weyl system is defined by a map from \(V\) to the set of unitary operators on a Hilbert space \(\mathbb{H}\). This map is required to be strongly continuous and satisfying the following properties:

1. \(W(v) \in U(\mathbb{H})\), for all \(v \in V\);
2. \(W(v_1)W(v_2)W^\dagger(v_1)W^\dagger(v_2) = e^{i\omega(v_1,v_2)}I\).

Here \(\omega\) is the symplectic structure on \(V\) \([47]\). The symplectic structure is the "infinitesimal form" of the multiplier \(m(v_1,v_2)\) appearing in the definition of "ray-representations" for the abelian vector group \(V\) \([46]\). It should be remarked that for different orderings, the symplectic structure is actually replaced by an Hermitian product on \(V\). Let us now carry on the general procedure on this specific example - For simplicity we introduce a basis in \(V\), say \(\{e_1, e_2, ..., e_{2n}\}\), so that \(v = v^i e_i\).

With the help of the Stone-von Neumann theorem it is possible to write

\[
W(v) = e^{iR(v)}.
\] (5.57)

In particular this relation implies

\[
[R(v_1), R(v_2)] = i\omega(v_1, v_2)
\] (5.58)

Now, after the selection of a fiducial vector \(|0\rangle\), we have to compute

\[
\langle 0 | (dW)^\dagger \otimes dW | 0 \rangle
\] (5.59)

First we notice that unitarity of \(W(v)\) implies that \(d(W^\dagger) = (dW)^\dagger\). Than, by using the decomposition \(v = v^i e_i\), we find for the pulled back tensor:

\[
\langle 0 | R(e_j)R(e_k) | 0 \rangle dv^j \otimes dv^k.
\] (5.60)

By considering the real part and the imaginary part respectively, we find

\[
\frac{1}{2} \langle 0 | R(e_j)R(e_k) + R(e_k)R(e_j) | 0 \rangle dv^j \otimes dv^k
\] (5.61)
and
\[ \frac{1}{2i} \langle 0 | R(e_j)R(e_k) - R(e_k)R(e_j) | 0 \rangle \, dv^j \wedge dv^k = \omega_{jk} dv^j \wedge dv^k . \tag{5.62} \]

If, as we should for physical interpretation, we consider the pull-back of the Kählerian tensor from the complex projective space associated with \( \mathbb{H} \), we should find the same imaginary term, but the symmetric part should be evaluated as
\[ \frac{1}{2} \langle 0 | R(e_j)R(e_k) + R(e_k)R(e_j) | 0 \rangle - \langle 0 | R(e_j) | 0 \rangle \langle 0 | R(e_k) | 0 \rangle . \tag{5.63} \]

From this expression it follows clearly that the fiducial vector \( |0\rangle \) should be selected such that it belongs to the domain of \( R(e_j) \) for all \( j \in \{1, 2, \ldots, 2n\} \) and to the domain of the elements of order two in the enveloping algebra.

Remark: Note that, when we consider only an Abelian vector subgroup of \( V \) which is a Lagrangian subspace, the pull-back tensor only contains the Euclidean part because \( \omega_{jk} \) restricted to the subgroup will vanish identically.

To evaluate (5.61) and (5.63) we have to give a realization of \( \mathbb{H} \). This is done by considering the decomposition of the symplectic vector space into \( V = \mathbb{R}^n \oplus (\mathbb{R}^n)^* \). In the realization \( \mathbb{H} = L^2(\mathbb{R}^n) \) we may compute the expectation values of \( R(e_j), R(e_k) \) and combination of these based on a Gaussian function
\[ |0\rangle := Ne^{-\frac{1}{2}q^2} \in L^2 \cap C^\infty(\mathbb{R}^n) . \tag{5.64} \]

Here we get due to the realizations
\[ R(e_j) |0\rangle := Q^j |0\rangle = Q^j (Ne^{-\frac{1}{2}q^2}) = q_j |0\rangle \tag{5.65} \]

for \( j \in \{1, 2, \ldots, n\} \) and
\[ R(e_j) |0\rangle := P^j |0\rangle = i \frac{\partial}{\partial q_j} (Ne^{-\frac{1}{2}q^2}) = -iq_j |0\rangle \tag{5.66} \]

for \( j \in \{n + 1, n + 2, \ldots, 2n\} \) the \( L^2(\mathbb{R}^n) \) inner products
\[ \langle 0 | Q^j P^k |0\rangle = -i \langle 0 | q_j q_k |0\rangle \tag{5.67} \]
\[ \langle 0 | P^j Q^k |0\rangle = i \langle 0 | q_j q_k |0\rangle \tag{5.68} \]
\begin{align}
\langle 0 | Q^j Q^k | 0 \rangle &= \langle 0 | q_j q_k | 0 \rangle \quad (5.69) \\
\langle 0 | P^j P^k | 0 \rangle &= \langle 0 | q_j q_k | 0 \rangle , \quad (5.70)
\end{align}

which can be made explicit by the the integrals

\begin{align}
I_{jk} &:= \langle 0 | q_j q_k | 0 \rangle = N^2 \int_{-\infty}^{\infty} d^n q e^{-q^2} q_j q_k . \quad (5.71)
\end{align}

They get zero for \( j \neq k \) due to

\begin{align}
I_{jk} &= N^2 \left( \int_{-\infty}^{\infty} dq_i e^{-q^2} \right)^{n-2} \left( \int_{-\infty}^{\infty} dq_j e^{-q^2} q_j \right)^2 = 0 \quad (5.72)
\end{align}

and non-zero for \( j = k \) due to

\begin{align}
I_{jj} &= N^2 \left( \int_{-\infty}^{\infty} dq_i e^{-q^2} \right)^{n-1} \int_{-\infty}^{\infty} dq_j e^{-q^2} q_j^2 = \frac{1}{2} N^2 \pi^{n/2} . \quad (5.73)
\end{align}

By setting \( N^2 \pi^{n/2} \equiv 1 \) we can summarise this into

\begin{align}
I_{jk} &= \frac{1}{2} \delta_{jk} \quad (5.74)
\end{align}

and since we have furthermore \( \langle P^j \rangle = \langle Q^j \rangle = 0 \) we can conclude that both relations in (5.61) and (5.63) define each of them a metric tensor field

\begin{align}
g_{jk} dv^j \otimes dv^k &= \frac{1}{2} \delta_{jk} dv^j \otimes dv^k , \quad (5.75)
\end{align}

giving rise to an Euclidean metric on \( \mathbb{R}^{2n} \).

\section{The pull-back on a manifold without group structure}

We consider a family of Hamiltonian operators \( H(\lambda) \) with \( \lambda \in M \), a smooth manifold and the eigenvalue problem

\begin{align}
H(\lambda) | \psi_0(\lambda) \rangle &= E_0(\lambda) | \psi_0(\lambda) \rangle , \quad (6.76)
\end{align}
where \( E_0 \) defines the lowest nonzero eigenvalue which is supposed to be non-degenerate. This association \( \lambda \mapsto |\psi_0(\lambda)\rangle \) defines an embedding of \( M \) into \( \mathcal{R}(\mathbb{H}) \), the ray space of \( \mathbb{H} \). Using the Hermitian tensor (2.13), we find

\[
\frac{\langle d\psi_0(\lambda) \otimes d\psi_0(\lambda) \rangle}{\langle \psi_0(\lambda) | \psi_0(\lambda) \rangle} - \frac{\langle \psi_0(\lambda) | d\psi_0(\lambda) \rangle \otimes \langle d\psi_0(\lambda) | \psi_0(\lambda) \rangle}{\langle \psi_0(\lambda) | \psi_0(\lambda) \rangle}.
\]

The external derivative \( d \) is meant to act on functions on \( M \). By using \( d = d\lambda^\mu \otimes \frac{\partial}{\partial \lambda^\mu} \), we find

\[
h_{\mu\nu} = \langle \partial^\mu \psi_0 | \partial^\nu \psi_0 \rangle - \langle \psi_0 | \partial^\nu \psi_0 \rangle \langle \partial^\mu \psi_0 | \psi_0 \rangle \quad (6.78)
\]

with the requirement \( \langle \psi_0 | \psi_0 \rangle = 1 \). Using more generally the spectrum of \( H(\lambda) \), say

\[
H(\lambda) |a; \lambda\rangle = E_a(\lambda) |a; \lambda\rangle, \quad (6.79)
\]

we have

\[
dH(\lambda) |a; \lambda\rangle = dE_a(\lambda) |a; \lambda\rangle + E_a(\lambda)d |a; \lambda\rangle - H d |a; \lambda\rangle. \quad (6.80)
\]

Taking the scalar product with \( \langle b; \lambda | \) we obtain

\[
\langle b; \lambda | dH(\lambda) |a; \lambda\rangle = (E_a - E_b) \langle b; \lambda | d |a; \lambda\rangle, \quad (6.81)
\]

i.e.

\[
d |a; \lambda\rangle = \sum_{b \neq a} \frac{|b; \lambda\rangle \langle b; \lambda | dH |a, \lambda\rangle}{E_a - E_b} \quad (6.82)
\]

Using this expression for \( a = 0 \), we get

\[
d |\psi(\lambda)_0\rangle = \sum_{b \neq a} \frac{|b; \lambda\rangle \langle b; \lambda | dH |\psi(\lambda)_0\rangle}{E_0 - E_b}, \quad (6.83)
\]

which allows to write the pull-back of the Hermitian tensor given by (6.77). It should be mentioned that a particular interesting application to physical systems has been provided by Zanardi et al.[48].

7 Conclusions and outlook

We have seen that, out of any unitary representation of a group on some Hilbert space, it is possible to identify a manifold by acting with the group
on some fiducial state. It is also possible to identify submanifolds by other procedures. On each submanifold, it is possible to consider the pullback of the Hermitian tensor and therefore obtain classical tensors out of the quantum states. In some sense, this procedure may give rise to a kind of "dequantization". As a matter of fact, this is not connected to any dynamics resp. quantum classical transition and we have to accept that there are quantum and classical-like structures in every quantum system, which should be considered as coexistent. In particular, when we consider the immersion of a symplectic vector space by means of a Weyl system, we obtain not only the original symplectic structure but also an Euclidean tensor and therefore a complex structure. With the help of this structure it is possible to define complex coordinates and a correspondence between them and creation/annihilation operators. We should stress that while the symplectic structure turns out to be independent of the fiducial vector we start with, the Riemannian tensor does depend on it. More likely it is this particular aspect that makes the symplectic structure more fundamental than the metric structure in classical mechanics. On the other hand, when we consider the imbedding of a Lagrangian subspace in the Hilbert space identified by a Weyl system, we find no symplectic structure but we find a metric tensor, this available metric tensor, intrinsically built, permits to define the velocity field associated with a wave function in Bohmian mechanics. Thus this procedure will allow us to define a Bohmian vector field on any manifold we may immerse in the Hilbert space. In a future paper we shall consider more closely this problem and provide a general setting for Bohmian vector fields.

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