LOCAL CONDITIONAL REGULARITY FOR THE LANDAU EQUATION WITH COULOMBO POTENTIAL

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Abstract

This paper studies the regularity of Villani solutions of the space homogeneous Landau equation with Coulomb interaction in dimension 3. Specifically, we prove that any such solution belonging to the Lebesgue space \( L^\infty_t \times L^q_v \) with \( q > 3 \) in an open cylinder \((0, S) \times B\), where \( B \) is an open ball of \( \mathbb{R}^3 \), must have Hölder continuous second order derivatives in the velocity variables, and first order derivative in the time variable locally in any compact subset of that cylinder.

1 Introduction

The objective of this work is to derive conditional regularity of certain weak solutions \( f = f(t, v) \geq 0 \) a.e. of the space homogeneous Landau equation with Coulomb potential. At the formal level, this equation reads (with the Einstein summation convention being used)

\[
\partial_t f - \overline{a_{ij}^f} \frac{\partial f}{\partial v_i \partial v_j} = 8\pi f^2,
\]

where

\[
\overline{a_{ij}^f}(t, v) = a_{ij} \ast f(t, \cdot),
\]

with

\[
a_{ij}(z) = \frac{1}{|z|} (\delta_{ij} - \frac{z_i z_j}{|z|^2}).
\]

A fundamental existence theorem due to Villani (theorem 3, (i) in [20]) provides the existence of a special class of weak solutions of the associated Cauchy problem for (1). For an earlier approach to the existence of weak solutions see [2]. These solutions are known as “H- solutions” or “Villani solutions” (see definition (2.1)). In this work we will be concerned only with this class of solutions.

We note the similarity between equation (1) and the semilinear heat equation

\[
u_t - \Delta v = v^2,
\]

for which finite time blow up may occur (see e.g. theorem 1 in [21]). The term \( 8\pi f^2 \) on the r.h.s. of (1) clearly promotes finite time blow-up. On the other hand, if \( f \) increases at some point, the diffusion matrix \( \overline{a_{ij}^f} \) in (1) increases as well. This is an important difference between the Landau equation (1) and the semilinear heat equation (2). Since any increase in the diffusion matrix offsets the effect of the quadratic source term \( 8\pi f^2 \), whether \( f \) blows up in finite time remains an major open problem at the time of this writing.

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One of the most major recent contributions in the study of equation (1) was produced in [4], where it is proved that for all \( T > 0 \) any H-solution \( f \) of (1) with finite mass, energy and entropy must satisfy the bound

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{f}|^2}{(1 + |v|^2)^{\frac{3}{2}}} dv dt < \infty.
\]

Observe that the bound above offers a striking similarity with Leray’s theory of weak solutions to the Navier-Stokes equation in space dimension 3. Indeed, \( \sqrt{f} \) belongs to \( L^\infty((0, +\infty); L^2(\mathbb{R}^3)) \) while \( \nabla \sqrt{f} \) belongs to \( L^3((0, +\infty); L^2_{locc}(\mathbb{R}^3)) \). Apart from the \((1 + |v|^2)^{-\frac{3}{2}}\) weight in the dissipation estimate, these bounds on \( \sqrt{f} \) are reminiscent of the bounds satisfied by Leray weak solutions of the Navier-Stokes equation in space dimension 3.

This loose analogy suggests the problem of proving conditional regularity results for H-solutions of the Landau equation (1). For instance, one could seek conditional, local regularity results analogous to those obtained in the work of Serrin [17] for the Navier-Stokes equation. Namely, Serrin proves that weak solutions of the Navier-Stokes equations lying in an appropriate mixed Lebesgue space are smooth with respect to the spatial variable and locally Hölder continuous with respect to time. In view of the analogy observed above, it is reasonable to expect that an analogous result holds for equation (1).

Silvestre’s result in [18] can be thought of as a global Serrin type theorem for the Landau equation (1). This result is complemented in some sense by the propagation estimates obtained in [4], which allow to get improved regularity on the solution by imposing integrability and suitable smallness assumptions on the initial data. Other conditional regularity results include e.g. [12], [13], [10] and [11]. In [12] the authors prove regularity of radially symmetric \( L^p \) solutions with \( p > \frac{3}{2} \). In [11], local Hölder continuity is proved for essentially bounded weak solutions of equation (1). In [13] regularity is proved provided the solutions satisfy a certain variant of Poincare inequality together with what is referred by the authors as the “local doubling property”, which is a reminiscent of the weak Harnack inequality for supersolutions to parabolic equations. Finally, in [10] regularity is proved provided the solutions satisfy a variant of the entropy inequality, which is a reminiscent of the Leray energy inequality, satisfied by some suitable weak solutions Navier Stokes equations. Regularity is of course related to uniqueness, and so we refer the reader to [8] for relevant uniqueness results.

Our new contribution here is obtaining a local Serrin type theorem for equation (1), that is, a localisation of Silvestre’s theorem. This is formulated precisely in theorem 2.6, which is the main result of the paper. Our approach here differs from the strategy introduced in [13], which is based on apriori estimates for the Landau equation. It is not clear whether it is possible to establish a local version of these apriori estimates, and we shall not pursue this direction here. Instead, our approach relies crucially on analyzing the regularity of the coefficients \( \overline{a}_{ij} f \), and can be roughly summarized as follows:

1. Obtaining regularity on the coefficients \( \overline{a}_{ij} f \) (this is the content of lemma 3.7).
2. Once regularity on the coefficients is achieved, we may apply classical existence and uniqueness results from the theory of linear parabolic PDE’s in order to show that the solution \( f \) enjoys 2 weak derivatives in space and one weak derivative in time (this is the content of theorem 3.1).
3. Once the solution is known to be (locally) in some appropriate space-time Sobolev space, we may improve the regularity of the solution via a bootstrap argument.
2 Preliminaries and Main Results

2.1 General Notations

We shall consider here space dimension 3. Let us fix some notation and recall some basic definitions. We shall denote by $\omega \subset (0, \infty) \times \mathbb{R}^3$ some cylinder, that is, a set of the form $J \times B$ where $J = (0, S) \subset (0, \infty)$ is some finite open interval and $B \subset \mathbb{R}^3$ is some open ball. Of course, upon shifting the spatial variable, it is sufficient to prove theorem 2.6 for $B$ centered at the origin. Let $1 \leq p \leq \infty, 1 \leq q \leq \infty, k \in \mathbb{R}$. We recall the following function spaces.

$L_p^q(\omega) := L_p^q(\omega)$ : The space of all measurable maps $f : \omega \to \mathbb{R}$ with $\|f\|_q < \infty$.

$L_p^q(\mathbb{R}^3)$ : the space of all measurable maps $f : \mathbb{R}^3 \to \mathbb{R}$ with $\|f\|_q < \infty$.

$L \log L(\mathbb{R}^3) := \{f \in L^1(\mathbb{R}^3) \mid \int |f(v)| \log |f(v)| dv < \infty\}$.

$W^{2,1}(\omega) :$ the Banach space consisting of all elements in $L^q(\omega)$ with generalized derivatives of the form $\partial_t^r \partial_x^s$ where $2r + s \leq 2$ and such that $\partial_t^r \partial_x^s \in L^q(\omega)$. The norm on $W^{2,1}(\omega)$ is defined by $\|u\|_q := \sum_{r=0}^{2} \sum_{s=0}^{2} \|\partial_t^r \partial_x^s u\|_q$.

$W^{1,0}(\omega) :$ the Hilbert space consisting of all elements in $L^2(\omega)$ with generalized derivatives of the form $\partial_x$ such that $\partial_x \in L^2(\omega)$. The scalar product on $W^{1,0}(\omega)$ is defined by $(u, v) = \int_\omega u \partial_x v dx dt$.

Let $0 < l < 1$. We will use the following Hölder spaces.

$H^l(\omega)$: for each $P, Q \in \omega$ we introduce the metric $|P - Q| = \max\{|x^P - x^Q|, 1.5|t^P - t^Q|\}$. Write $P \sim Q$ to mean $P = Q + \eta e_k$ for some $\eta \in \mathbb{R}, 1 \leq k \leq 3$ (as customary $e_k$ stands for the unit vector in direction $k$ in $\mathbb{R}^3$). We consider the distances $d_P = \inf\{|P - Q|; Q \in T(P)\}$ where $T(P)$ is the set of points on the boundary of $\omega$ for which there exist a continuous arc connecting $P, Q$ along which the $t$ coordinate is nondecreasing from $Q$ to $P$. This gives rise to a distance $d_{PQ}$ defined by $d_{PQ} = \min\{d_P, d_Q\}$. For $m \in \mathbb{N}$ we then consider

$$\|d^m v\|^l_{P} = \sup_{P \in \omega} d^m_P |v(P)| + \sup_{P \sim Q} d^m_P |v(P) - v(Q)|.|P - Q|^l.$$ 

$(H^l(\omega), \|d \cdot \|_l^l)$ is the Banach space whose elements are all $v$ on $\omega$ admitting a (unique) $C^0$ extension to $\overline{\omega}$ and such that $\|dv\|_l^l < \infty$.

$H_{l+2}^l(\omega) :$ we consider the norm

$$\|v\|_{l+2} = \|v\|_l + \sum_{i=1}^{3} \|d\partial_{x_i}v\|_l + \sum_{1 \leq i,j \leq 3} \|d^2\partial_{x_i,x_j}v\|_l + \|d^2\partial_t v\|_l^l,$$

where

$$\|d^m v\|_l = \sup_{P \in \omega} d^m_P |v(P)| + \sup_{P \sim Q} d^m_P |v(P) - v(Q)|.|P - Q|^l, m \in \mathbb{N}.$$ 

$(H_{l+2}^l(\omega), \| \cdot \|_{l+2}^l)$ is the Banach space whose elements are all $v$ on $\omega$ admitting a (unique) $C^2$ extension to $\overline{\omega}$ and such that $\|v\|_{l+2} < \infty$.

$H^{1,\frac{1}{2}}(\mathbb{R}) :$ we consider the norm

$$\|v\|_{1,\frac{1}{2}} = \sup_{P \in \omega} |v(P)| + \sup_{(t,x), (t,y) \in \omega} \frac{|v(t,x) - v(t,y)|}{|x - y|^\frac{1}{2}} + \sup_{(t,x), (s,x) \in \omega} \frac{|v(t,x) - v(s,x)|}{|t - s|^\frac{1}{2}}.$$ 

$(H^{1,\frac{1}{2}}(\omega), \| \cdot \|_{1,\frac{1}{2}})$ is the Banach space whose elements are all $v$ on $\omega$ admitting a (unique) $C^0$ extension to $\overline{\omega}$ and such that $\|v\|_{1,\frac{1}{2}} < \infty$.
\[ H^{1+\frac{1}{2}}(\mathbb{R}) : \text{we consider the norm} \]
\[
||v||^{1+\frac{1}{2}} = \sup_{P \in \omega} |v(P)| + \sup_{(t,x),(t,y) \in \omega} \frac{|v(t, x) - v(t, y)|}{|x - y|^{\frac{1}{2}}} + \sup_{(t,x),(s,x) \in \omega} \frac{|v(t, x) - v(s, x)|}{|t - s|^{\frac{1}{2}}}
\]
\[
3 \sup_{P \in \omega} |\partial_x v(P)| + 3 \sup_{(t,x),(t,y) \in \omega} \frac{|\partial_{x,z} v(t, x) - \partial_{x,z} v(t, y)|}{|x - y|^{\frac{1}{2}}} + 3 \sup_{(t,x),(s,x) \in \omega} \frac{|\partial_{x,z} v(t, x) - \partial_{x,z} v(s, x)|}{|t - s|^{\frac{1}{2}}}.
\]

We remark that the notations used for the above Hölder type spaces has nothing to do with Sobolev spaces (which are frequently denoted by \( H^k \)).

### 2.2 The Landau Equation and H-Solutions

Following [20], let us briefly recall the Landau equation and the notion of H-solutions (also know as Villani solutions). We refer the reader to [20] for a elaborative and motivational discussion.

Define \( \Pi : \mathbb{R}^3 \setminus \{0\} \rightarrow M_3(\mathbb{R}) \) by \( \Pi(z) = I - (\frac{z}{|z|^2}) \otimes z \), so that \( \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2} \). Define \( a_{ij}(z) = \frac{1}{|z|^2} \Pi_{ij}(z) \) and \( b_i(z) = \frac{3}{|z|^2} \partial_i a_{ij}(z) \). For a function \( f \) on \( \mathbb{R}^3 \) consider the convolutions \( \Pi_{ij} := \int a_{ij}(v - z) f(z) dz \), \( b_i := \int b_i(v - z) f(z) dz \). As is customary, for a function \( f \) on \( \mathbb{R}^3 \) we shall write \( M(f) = \int f(v) dv, E(f) = \int f(v) |v|^2 dv, H(f) = \int f(v) \log(f(v)) dv \) whenever the quantity on the RHS is well defined. The quantities \( M(f), E(f), H(f) \) are called the mass, energy and entropy of \( f \) respectively.

**Definition 2.1.** Let \( f_0(v) = f_0 \) have finite mass, energy and entropy. A H-solution to equation \( (\Pi) \) with initial data \( f_0 \) on \( [0, T] \times \mathbb{R}^3 \) is an element \( f \in C([0, T], D'(\mathbb{R}^3)) \cap L^1((0, T), L^1_1(\mathbb{R}^3)) \) such that

1. \( f \geq 0 \) and \( \forall t \in [0, T] : f(t, \cdot) \in L^1_2(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3) \)
2. \( f(0, \cdot) = f_0(\cdot) \)
3. \( \forall t \in [0, T] : \int f(t, v) \log(f(t, v)) dv \leq \int f_0(v) \log(f_0(v)) dv \) and \( \int f(t, v) \psi(v) dv = \int f_0(v) \psi(v) dv \) where \( \psi = 1, v, |v|^2 \)
4. \( \forall \varphi \in C^1([0, T], C_0^\infty(\mathbb{R}^3)), \forall t \in [0, T] \)
   \[
   \int_{\mathbb{R}^3} f(t, v) \varphi(t, v) dv - \int_{\mathbb{R}^3} f_0(v) \varphi(0, v) dv - \int_0^t \int_{\mathbb{R}^3} f(\tau, \varphi(\tau, v) dv d\tau
   \]
   \[
   = - \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Pi(\nabla v, \nabla \varphi) \sqrt{\frac{f(\tau, v) f(\tau, w)}{|v - w|}} - \nabla \varphi \sqrt{\frac{f(\tau, v) f(\tau, w)}{|v - w|}} \sqrt{\frac{f(\tau, v) f(\tau, w)}{|v - w|}} (\nabla \varphi(v, t, v) - \nabla w \varphi(t, w)) dv dw d\tau.
   \]

The integral on the RHS in 4. is well defined, as explained in detail in [20]. Apriori, it is not clear what is the relation between the notion of a H-solution and a weak solution in the classical sense. As we will recall in the next
section, it can be shown that in fact H-solutions are weak solutions in the classical sense, but this is a consequence of a highly nontrivial theorem of Desvillettes [4]. In what follows, we shall refer to H-solutions of equation \( h \) simply as H-solutions.

### 2.3 Known and New Results

First and most foremost we recall that Villani proved the global existence of H-solutions for all initial data with finite mass, energy and entropy. In addition he proved that these solutions are weakly Hölder continuous in time, as described in the following

**Theorem 2.2.** ([20], Theorem 3, (i)) Let \( f_0 : \mathbb{R}^3 \to \mathbb{R} \) have finite mass, energy and entropy. Then there exist a H-solution \( f \) with initial datum \( f_0 \). Moreover, if \( f \) is a H-solution with initial datum \( f_0 \), then, for all \( \varphi \in W^{2,\infty}(\mathbb{R}^3) \), \( t \mapsto \int_{\mathbb{R}^3} f(t,v)\varphi(v)dv \) is Hölder continuous with exponent \( \frac{1}{2} \).

The next most important result for our purposes is the following weighted \( L^2 \) estimate on the distributional derivative of \( \sqrt{f} \)

**Theorem 2.3.** ([4], Theorem 1) Let \( f \) be a H-solution on \([0,T] \times \mathbb{R}^3\). Then
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla \sqrt{f(t,v)}|^2 \frac{dv}{1+|v|^2} < \infty.
\]

We remark that theorem 2.3 implies in particular that any \( L^2(\omega) \) H-solution is in \( W^{1,0}(\omega) \), as can be seen from the identity \( \partial_v f = 2\sqrt{f} \partial_v \sqrt{f} \). A list of important conclusions is derived from theorem 2.3 in [4]. The first one, is that H-solutions are in fact “usual” weak solutions in the sense of integration against test functions

**Corollary 2.4.** ([4], Corollary 1.1) Let \( f \) be a H-solution with initial datum \( f_0 \). Then \( f \in L^1((0,T),L^3_{-3}(\mathbb{R}^3)) \) and for all \( \varphi \in C^0_0((0,T) \times \mathbb{R}^3) \) it holds that
\[
-\int_{\mathbb{R}^3} f_0(v)\varphi(0,v)dv - \int_0^T \int_{\mathbb{R}^3} f(t,v)\partial_t \varphi(t,v)dvdt =
\]
\[
\frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t,v)f(t,w)a_{ij}(v-w)(\partial_{ij}\varphi(t,v) + \partial_{ij}\varphi(t,w))dvdwdt
\]
\[
+ \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t,v)f(t,w)b_i(v-w)(\partial_i\varphi(t,v) - \partial_i\varphi(t,w))dvdwdt.
\]
Remark 2.5. If \( f \) satisfies the condition \( \|f(t, \cdot)\|_{L^q(B)} \leq S_0 \) for some \( S_0 > 0, q > 3 \) then for any \( \varphi \in C_0^\infty((0, S) \times B) \) it holds that

\[
- \int_0^T \int_{\mathbb{R}^3} f \partial_t \varphi + \int_0^T \int_{\mathbb{R}^3} \Pi_{ij} \partial_i f \partial_j \varphi = \int_0^T \int_{\mathbb{R}^3} B_{ij} \partial_i \varphi(t, v). \tag{3}
\]

As we will see in lemma (3.7), subject to the above condition \( \Pi_{ij}, B_{ij} \in L^\infty_{\text{loc}}(\omega) \), which together with theorem (2.3), justifies that the above integrals are well defined.

Corollary 2.4 and the above remark will enable us to apply classical regularity, uniqueness and existence results from the theory of linear parabolic PDEs, which have the above weak formulation. We finish this section by stating the main result of this work

Theorem 2.6. Let \( f \) be a H-solution on \([0, T] \times \mathbb{R}^3\). Let \( \omega = (0, S) \times B \) be an open cylinder in \([0, T] \times \mathbb{R}^3\). Suppose there exist \( S_0 > 0, q > 3 \) such that for all \( t \in (0, S) \) one has \( \|f(t, \cdot)\|_{L^q(B)} \leq S_0 \). Then for all \( 0 < \alpha < 1 \) we have \( f \in H_{\alpha+2}^* (\Omega) \) for each \( \Omega \Subset \omega \).

Remark 2.7. We point out that theorem 2.6 implies that \( f \) is in fact a classical solution to equation (1) in \( \omega \). Indeed, from the last assertion of theorem 3.8 we know that \( f \) is a strong solution to equation (1) while theorem 2.6 in particular implies that the weak time derivative and the first and second order weak spatial derivatives are continuous.

3 Conditional Regularity

3.1 From H-Solutions To \( W_2^{2,1} \)

This subsection is the first step towards the conditional local regularity of H-solutions as stated in Theorem 2.6. We recall that we have adapted the notation \( \omega = J \times B = (0, S) \times B \). We prove

Theorem 3.1. Let \( f \) be a H-solution. Suppose there exist \( S_0 > 0 \) and \( q > 3 \) such that for all \( t \in J \) one has \( \|f(t, \cdot)\|_{L^q(B)} \leq S_0 \). Then \( f \in W_2^{2,1}(\Omega) \) for all \( \Omega \Subset \omega \). Moreover, \( f \) is a strong solution to equation (1) in \( \omega \), that is

\[
\partial_t f - \alpha_{ij} f \frac{\partial f}{\partial v_i \partial v_j} = 8\pi f^2
\]

for a.e. \((t, x) \in \omega\).

First, we recall that local coercivity for the coefficients \( \Pi_{ij}, B_{ij} \) has been established in proposition 2.3 of [1]. A straightforward conclusion of the latter is local ellipticity of the coefficients, as summarized in the following
Proposition 3.2. (Local Ellipticity) There are constants $0 < c = c(M_0, E_0, H_0, K), C = C(M_0, S_0, q)$ with the following property. Suppose $f \in L^2_1(R^3) \cap L \log L(R^3)$ satisfy $f \geq 0$ a.e. and $M(f) = M_0, E(f) = E_0, H(f) = H_0, \|f\|_{L^q(B)} \leq S_0$ for some $\frac{3}{2} < q \leq \infty$. Let $K \in R^3$. Then:

$$\forall \xi \in \mathbb{R}^n, v \in K : |\xi|^2 \leq |\pi_{ij}^f(v)\xi_i\xi_j| \leq C|\xi|^2.$$  

Proof. The coercivity estimate $c|\xi|^2 \leq |\pi_{ij}^f(v)\xi_i\xi_j|$ was established in proposition 2.3 of [1]. In addition for all $\xi \in S^1$ we have

$$|\pi_{ij}^f(v)\xi_i\xi_j| \leq \int_{R^3} \frac{2}{|z|}|f(v - z)|dz = \int_{B} \frac{2}{|z|}|f(v - z)|dz + \int_{R^3 - B} \frac{2}{|z|}|f(v - z)|dz \leq A\|f\|_{L^q(B)} + 2M_0$$

$$\leq AS_0 + 2M_0 := C(q, M_0, S_0),$$

where $A = A(q)$ is some constant. 

Since the mass and energy of H-solutions are constant in time and the entropy is uniformly bounded in time we immediately get

Corollary 3.3. There are constants $0 < c = c(M_0, E_0, H_0, K), C = C(M_0, S_0, q)$ with the following property. Let $f$ be a H-solution with $M(f) = M_0, E(f) = E_0, H(f) = H_0, ||f(t, \cdot)||_{L^q(B)} \leq S_0$ for some $\frac{3}{2} < q \leq \infty$. Let $K \in R^3$. Then for all $\xi \in \mathbb{R}^n$ $|\xi|^2 \leq |\pi_{ij}^f(t,v)\xi_i\xi_j| \leq C|\xi|^2$ for all $(t, v) \in J \times K$.

The strategy of the proof of theorem 3.1 will be roughly as follows. We start by obtaining regularity of the coefficients $\pi_{ij}^f$. Then we localize the solution in order to obtain a linear parabolic PDE in divergence form. This will allow us to apply classical existence and uniqueness results from the theory of linear parabolic PDE’s. The local ellipticity of the coefficients (Corollary 3.3) will be freely and frequently used in the sequel. We start by recalling the following parabolic version of the celebrated De Giorgi-Nash-Moser method

Theorem 3.4. ([19], Theorem 18) Let $V : B \rightarrow R^3$ be a vector field such that $|V|^2 \in L^pL^q(\omega)$, where $1 < p < \infty, 1 < q < \infty$ satisfy $\frac{2}{p} + \frac{3}{q} < 2$. Suppose $u \in L^\infty L^2(\omega)$ is a weak subsolution to

$$\partial_t u - \partial_j(\pi_{ij}^f \partial_i u) + \nabla u \cdot V \geq 0$$

(4)

Then there is some $\alpha > 0$ such that $u \in H^{\alpha, \frac{3}{2}}(\Omega)$ for all $\Omega \in \omega$.

We will also heavily rely on the following Calderon Zygmund type theorem

Theorem 3.5. ([5], Theorem 4.12) Suppose $\nu \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ has the form $\nu(y) = \mu(\frac{y}{|y|})$ where $\mu \in L^q(S^2)$ for some $q > 1$ is an even function such that $\int_{S^2} \mu(z)dr(z) = 0$. Then for each $1 < p < \infty$ the operator

$T : L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)$ defined by $Tf = (\frac{\nu(y)}{|y|} * f(y))(x)$ is bounded.
We shall first verify that the conditions imposed in theorem 3.5 are indeed verified for the second derivatives of $a_{ij}$. This verification is based on elementary (yet somewhat tedious) calculations, and is the content of the following

**Lemma 3.6.** For each $1 \leq k, l \leq 3$ it holds that $\partial_{kl} a_{ij}(y) = \frac{\nu_{kl}(y)}{|y|^3}$ where $\nu_{kl}$ is as in theorem 3.5.

**Proof.** We differentiate

$$\partial_k a_{ij}(z) = \partial_k \left( \frac{\delta_{ij} z_j}{|z|^3} + \frac{z_i z_j}{|z|^5} \right) = -\frac{\delta_{ij} z_k}{|z|^3} + \frac{\delta_{ik} z_j + \delta_{jk} z_i}{|z|^3} - \frac{3z_i z_j z_k}{|z|^5} = \frac{\delta_{ik} z_j + \delta_{jk} z_i - \delta_{ij} z_k}{|z|^3} - \frac{3z_i z_j z_k}{|z|^5}.$$

$$\partial_l \left( \frac{\delta_{ij} z_j}{|z|^3} + \frac{z_i z_j}{|z|^5} \right) = \frac{\delta_{ik} \delta_{lj} + \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{kl}}{|z|^3} - \frac{3z_i (\delta_{ik} z_j + \delta_{jk} z_i - \delta_{ij} z_k)}{|z|^5} - \frac{3\delta_{il} z_j z_k + 3\delta_{ij} z_i z_k + 3\delta_{kl} z_j z_i}{|z|^7} = \frac{1}{|z|^3} \left( \delta_{ik} \delta_{lj} + \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{kl} - \frac{3z_i (\delta_{ik} z_j + \delta_{jk} z_i - \delta_{ij} z_k) + 3\delta_{il} z_j z_k + 3\delta_{ij} z_i z_k + 3\delta_{kl} z_j z_i}{|z|^2} + \frac{15z_i z_j z_k z_l}{|z|^4} \right).$$

Hence

$$\partial_{kl} a_{ij}(z) = \partial_{kl} \left( \frac{\delta_{ij} z_j}{|z|^3} + \frac{z_i z_j}{|z|^5} \right) = \frac{\delta_{ik} \delta_{lj} + \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{kl}}{|z|^3} - \frac{3z_i (\delta_{ik} z_j + \delta_{jk} z_i - \delta_{ij} z_k)}{|z|^5} - \frac{3\delta_{il} z_j z_k + 3\delta_{ij} z_i z_k + 3\delta_{kl} z_j z_i}{|z|^7} = \frac{1}{|z|^3} \left( \delta_{ik} \delta_{lj} + \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{kl} - \frac{3z_i (\delta_{ik} z_j + \delta_{jk} z_i - \delta_{ij} z_k) + 3\delta_{il} z_j z_k + 3\delta_{ij} z_i z_k + 3\delta_{kl} z_j z_i}{|z|^2} + \frac{15z_i z_j z_k z_l}{|z|^4} \right).$$

Denote by $P_i, 1 \leq i \leq 3$ the projection on the $i$-th coordinate. With this notation we recognize the following identity

$$\partial_{kl} (a_{ij}) = \frac{1}{|z|^3} \left( \delta_{ik} \delta_{lj} + \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{kl} - 3\delta_{ij} P_i \left( \frac{z}{|z|} \right) P_j \left( \frac{z}{|z|} \right) - 3\delta_{lk} P_i \left( \frac{z}{|z|} \right) P_j \left( \frac{z}{|z|} \right) + 3\delta_{ij} P_i \left( \frac{z}{|z|} \right) P_k \left( \frac{z}{|z|} \right) - 3\delta_{il} P_j \left( \frac{z}{|z|} \right) P_k \left( \frac{z}{|z|} \right) - 3\delta_{lj} P_i \left( \frac{z}{|z|} \right) P_k \left( \frac{z}{|z|} \right) + 3\delta_{kj} P_i \left( \frac{z}{|z|} \right) P_l \left( \frac{z}{|z|} \right) + 15 P_i \left( \frac{z}{|z|} \right) P_j \left( \frac{z}{|z|} \right) P_k \left( \frac{z}{|z|} \right) \right) := \frac{\mu_{kl}(\frac{z}{|z|})}{|z|^3}.$$

It is apparent that $\mu_{kl}$ is even and $\mu_{kl} \in L^q(S^2)$ for $q > 1$. In addition, the following calculations are an elementary exercise in calculus (see e.g. [7])

$$\int_{S^2} \delta_{ik} \delta_{lj} d\sigma(z) = 4\pi \delta_{ik} \delta_{lj},$$

$$\int_{S^2} P_i(z) P_j(z) d\sigma(z) = \frac{4\pi}{3} \delta_{ij}.$$
\[
\int_{S^2} P_t(z)P_j(z)P_k(z)\,d\sigma(z) = -\frac{4\pi}{15} (\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{jk}\delta_{il} + \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} - \delta_{ik}\delta_{lj} - \delta_{kl}\delta_{ij}).
\]

With the aid of the above identities it is readily checked that \(\int_{S^2} \mu(z)\,d\sigma(z) = 0.\)

\[\square\]

**Lemma 3.7.** Let \(f\) be a \(H\)-solution with \(\|f(t,\cdot)\|_{L^q(\Omega)} \leq S_0\) where \(3 < q \leq \infty\). Then:

1. There is some \(\alpha > 0\) such that \(f \in H^\alpha,\dot{E}(\Omega)\) for all \(\Omega \Subset \omega\).

2. (i) \(a_{ij} \in C^0(\omega)\).

2. (ii) For each \(t \in J\) the function \(a_{ij}(t,\cdot)\) is differentiable on \(B\) and \(\partial_t a_{ij} \in C^0(\omega)\).

**Proof.** We assume with out loss of generality \(q < \infty\). We start by showing \(\overline{b_t^f} \in L^\infty(\omega)\).

Let \(B' \Subset B\) be a ball and \(J' \Subset J\). Pick \(B'' \Subset B\) to be a ball such that \(B' \Subset B''\) and any \(2\epsilon\)-neighborhood of \(B'\) is \(\Subset B''\), for some sufficiently small \(\epsilon > 0\). Pick \(\chi \in C^\infty_0(\mathbb{R}^3)\) with \(\chi(z) \equiv 1\) on \(|z| \leq \epsilon\), \(\chi \equiv 0\) on \(|z| > 2\epsilon\). Then for all \((t,v) \in J' \times B'\) we have

\[
\left| \overline{b_t^f}(t,v) \right| \lesssim \int_{\mathbb{R}^3} \frac{1}{|v-w|^2} f(t,w)\,dw = \int_{\mathbb{R}^3} \frac{\chi(v-w)}{|v-w|^2} f(t,w)\,dw + \int_{\mathbb{R}^3} \frac{1-\chi(v-w)}{|v-w|^2} f(t,w)\,dw = \int_{\mathbb{R}^3} \frac{\chi(v-w)}{|v-w|^2} f(t,w)\,dw + \int_{\mathbb{R}^3} \frac{1-\chi(w)}{|w|^2} \|f(t,\cdot)\|_{L^1(\mathbb{R}^3)}\]

where the last inequality follows from Hölder inequality and the assumption \(q > 3\). Taking the double supremum on both sides gives \(\overline{b_t^f} \in L^\infty(\omega)\).

1. Now we show that \(f\) is locally Hölder continuous in \(\omega\). To this aim we wish to show that \(f\) (which by assumption is \(L^\infty L^q(\omega), q > 3\)) is a subsolution to an inequality of the form \([43]\). Let \(\omega' = (0, S') \times B' \Subset \omega\). Corollary \([24]\) and Remark \([25]\) imply in particular the following equation for all \(0 \leq \varphi \in C^2_0(\omega')\)

\[
- \int_{\omega'} f \partial_t \varphi + \int_{\omega'} \left( \overline{\nabla^f_{ij}} \partial_i f \right) (\partial_j \varphi) = \int_{\omega'} \overline{b_t^f} (\partial_t \varphi)(t,v).
\]

That \(f \in W^{1,0}_2(\omega')\) is a particular byproduct of theorem \([2,3]\). Furthermore, in step 2.i we will prove (independently) that \(\partial_t a_{ij}^f(t,\cdot) \in L^\infty_{\text{loc}}(B)\) for each fixed \(t\), so that \(\overline{b_t^f}(t,\cdot) \in W^1_2(B')\) for each fixed \(t\).

Therefore we may integrate by parts the RHS of equation \([35]\) and arrive at the equation
\[-\int_{\omega'} f \partial_i \varphi + \int_{\omega'} (\pi^f_{ij} \partial_i f)(\partial_j \varphi) = -\int_{\omega'} (\overline{B}^f_{ij} \partial_i f + f \partial_i \overline{B}^f_{ij}) \varphi.\]

Hence

\[-\int_{\omega'} f \partial_i \varphi + \int_{\omega'} (\pi^f_{ij} \partial_i f)(\partial_j \varphi) + \int_{\omega'} \nabla f \cdot V \varphi = -\int_{\omega'} f \varphi \partial_i \overline{B}^f_{ij}. \quad (6)\]

where $V = (\overline{B}^f_1, \overline{B}^f_2, \overline{B}^f_3)$. By an elementary calculation $-\partial_i (b_i)$ is positively proportional to the Dirac distribution, which in turn implies that $-\partial_i (\overline{B}^f_{ij})$ is positively proportional to $f$. Therefore the RHS of (6) is nonnegative. Thus

\[-\int_{\omega'} f \partial_i \varphi + \int_{\omega'} (\pi^f_{ij} \partial_i f)(\partial_j \varphi) + \int_{\omega'} \nabla f \cdot V \varphi - \int_{\omega'} f \varphi(0,v)dv \geq 0.\]

Since $\overline{B}^f_i \in L^\infty_{\text{loc}}(\omega)$ we know in particular that $|V|^2 \in L^q(\omega')$ for some $q > \frac{2}{\omega}$ which obviously implies the integrability condition imposed for $V$ in theorem 3.4. Therefore $f$ is locally Hölder continuous in $\omega$.

2. i. We move to show that $\lim_{t \to 0} H^{ij}(t, \cdot) \in L^q(\omega')$.

Let $B' \subseteq B, J' \subseteq J$. Denote by $\rho > 0$ the radius of $B'$ and pick $\epsilon > 0$ so small so that $B'' := B_{\rho + 2\epsilon} \subseteq B$.

\[1_{B'}(v)(\partial_{kl} \overline{B}^f_i(t, v)) = 1_{B'}(v) \int_{\mathbb{R}^3} \partial_{kl} a_{ij}(v - w)f(t, w)dw =\]

\[= 1_{B'}(v) \int_{\mathbb{R}^3} \partial_{kl} a_{ij}(v - w)1_{B''}(w)f(t, w)dw + 1_{B'}(v) \int_{\mathbb{R}^3} \partial_{kl} a_{ij}(v - w)1_{\mathbb{R}^3 - B''}(w)f(t, w)dw =\]

\[= 1_{B'}(v) \int_{\mathbb{R}^3} \partial_{kl} a_{ij}(v - w)1_{B''}(w)f(t, w)dw + 1_{B'}(v) \int_{\mathbb{R}^3} \partial_{kl} a_{ij}(v - w)1_{|v-w| \geq 2\epsilon}1_{\mathbb{R}^3 - B''}(w)f(t, w)dw := g_t(\cdot) + h_t(\cdot).\]

Lemma 3.6, theorem 3.5 and the assumption on $f$ imply that $g_t \in L^q(B')$. Furthermore, by Young’s convolution inequality:

\[||h_t(\cdot)||_{L^q(B')} = ||h_t(\cdot)||_{L^q(\mathbb{R}^3)} = ||(1_{|\cdot| \geq 2\epsilon} \partial_{kl} a_{ij})(1_{\mathbb{R}^3 - B''}f)||_{L^q(\mathbb{R}^3)} \leq ||1_{|\cdot| \geq 2\epsilon} \partial_{kl} a_{ij}||_{L^q(\mathbb{R}^3)} ||1_{\mathbb{R}^3 - B''}f(t, \cdot)||_{L^q(\mathbb{R}^3)} < \infty.\]

We move to show that $\overline{\pi}^{ij}(t, \cdot) \in L^q(\omega)$. The proof is similar to the argument presented at the begining. Denote by $R$ the radius of the ball $B$. We have

\[\int_{\mathbb{R}^3} |a_{ij}(v - z)|f(t, z)dz \leq 2 \int_{\mathbb{R}^3} \frac{1}{|v - z|} f(t, z)dz = 2 \int_{\mathbb{R}^3} \frac{1}{|z|} f(t, v - z)dz =\]

\[2 \int_{|z| \leq R} \frac{1}{|z|} f(t, v - z)dz + 2 \int_{|z| > R} \frac{1}{|z|} f(t, v - z)dz \leq C(||f(t, \cdot)||_{L^q(B)} + ||f(t, \cdot)||_1).\]

for some constant $C = C(q)$ (The finiteness of $\int_{|z| \leq R} \frac{1}{|z|} dz$ is guaranteed because of the assumption $q > 3$). So

\[\sup_{\omega} \int_{\mathbb{R}^3} |a_{ij}(v - z)|f(t, z)dz \leq C(q)(||f||_{L^q(\omega)} + ||f||_{L^q(\mathbb{R}^3)})\]

and in particular $\overline{\pi}^{ij}(t, \cdot) \in L^q(\omega)$. Thus we have
proved $\overline{a_{ij}}(t, \cdot) \in W^{2,q}(B')$ for $q > 3$ which by Sobolev embedding implies that $\overline{a_{ij}}(t, \cdot)$ is $C^1(B')$.

We prove continuity with respect to $t$. Pick $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\chi \equiv 1$ on $B_r(0)$ and supp$(\chi) \Subset B_{2\varepsilon}(0)$.

\[
1_{B'}(v)\overline{a_{ij}}(t, v) = 1_{B'} \int_{\mathbb{R}^3} a_{ij}(v-w)f(t, w)dw = 1_{B'}(\int_{\mathbb{R}^3} \chi(v-w)a_{ij}(v-w)f(t, w)dw + \int_{\mathbb{R}^3} (1-\chi(v-w))a_{ij}(v-w)f(t, w)dw =
\]

\[
= 1_{B'}(\int_{|v-w| \leq 2\varepsilon} \chi(v-w)a_{ij}(v-w)f(t, w)dw + \int_{|v-w| \leq 2\varepsilon} (1-\chi(v-w))a_{ij}(v-w)f(t, w)dw =
\]

\[
= 1_{B'} \int_{|v-w| \leq 2\varepsilon} \chi(v-w)a_{ij}(v-w)1_{B'}(w)f(t, w)dw + 1_{B'} \int_{|v-w| \leq 2\varepsilon} \chi(v-w)a_{ij}(v-w)1_{\mathbb{R}^3 \setminus B'}(w)f(t, w)dw
\]

\[
+ 1_{B'} \int_{\mathbb{R}^3} (1-\chi(v-w))a_{ij}(v-w)f(t, w)dw =
\]

\[
1_{B'} \int_{|v-w| \leq 2\varepsilon} \chi(v-w)a_{ij}(v-w)1_{B'}(w)f(t, w)dw + 1_{B'} \int_{\mathbb{R}^3} (1-\chi(v-w))a_{ij}(v-w)f(t, w)dw := \tilde{g}_t(v) + \tilde{h}_t(v).
\]

Keep $v \in B'$. Fixed. That $t \mapsto \tilde{g}_t(v)$ is continuous is an immediate consequence of 1 and the CS inequality. Furthermore it is clear that $w \mapsto (1-\chi(v-w))a_{ij}(v-w) \in W^{2,\infty}(\mathbb{R}^3)$, which by theorem 2.2 implies that $t \mapsto \tilde{h}_t(v)$ is continuous. So $\overline{a_{ij}}(\cdot, v)$ is continuous on $J$ as a sum of such functions.

ii. We already know that $\partial_k a_{ij}(t, \cdot)$ is continuous by 2.i. Continuity with respect to $t$ is achieved as in 2.i (here we use Hölder’s inequality instead of CS).

Before giving the proof of theorem 3.1 we will need the following existence and uniqueness results, which will also prove themselves useful in section 4.

**Theorem 3.8.** (Theorem 6.1,III in [15]) Suppose $A_{ij}(t, v)$ satisfies the following conditions:

1. $A_{ij}$ are locally uniformly elliptic on $\omega$.
2. For all $t \in J$, $A_{ij}(t, \cdot)$ are differentiable with respect to $v$ and $\text{ess sup}_{v \in \omega} |\partial_{v_k} A_{ij}| < \infty$ for all $1 \leq k \leq 3$.

Suppose $F \in L^2(\omega)$. Then the problem

\[
\begin{cases}
\partial_t u - \partial_j(A_{ij}\partial_i u) = F & \omega \\
u = 0 & [0, S] \times \partial B \\
u = 0 & \{0\} \times B
\end{cases}
\]

(7)

has a unique solution from $W^{2,1}\omega$. Moreover, this solution is a strong solution.
**Theorem 3.9.** (Theorem 3.3, III [15]) Suppose $A_{ij}(t, v)$ are locally uniformly elliptic. Let $F \in L^2(\omega)$. Then problem (7) cannot have more than one weak solution in $W^{2,0}_2(\omega)$.

**Proof of theorem 3.4.1.** Suppose $\Omega = (T_1, T_2) \times B' \subset \omega$. Let $\omega' := (T_1', T_2') \times B''$ such that $\Omega \subset \omega' \subset \omega$. Let $\zeta \in C_{0}^{\infty} (\omega')$ such that $\zeta \equiv 1$ on $\Omega$ and put $u = \zeta f$. Fix some test function $\chi \in C_{0}^{\infty} ([0, T) \times \mathbb{R}^3)$. We compute

$$
\int_{\omega'} - u \partial_t \chi + (\overline{\pi}_{ij}) \partial_i u \partial_j \chi = \int_{0}^{T} \int_{\mathbb{R}^3} - u \partial_t \chi + (\overline{\pi}_{ij}) \partial_i u \partial_j \chi = \int_{0}^{T} \int_{\mathbb{R}^3} - \zeta f \partial_t \chi + \int_{0}^{T} (\overline{\pi}_{ij}) (\partial_i \zeta f + \partial_i f \zeta) \partial_j \chi
$$

$$
= \int_{0}^{T} \int_{\mathbb{R}^3} - \zeta f \partial_t \chi + \int_{0}^{T} (\overline{\pi}_{ij}) \partial_i f (\partial_j (\zeta \chi) - \chi \partial_j \zeta) + \int_{0}^{T} (\overline{\pi}_{ij}) f \partial_j \chi \partial_i \zeta = 0.
$$

By equation (3) the last sum is

$$
= \int_{0}^{T} \int_{\mathbb{R}^3} - \zeta f \partial_t \chi + \int_{0}^{T} (\overline{\pi}_{ij}) \partial_i f \partial_j \chi + \int_{0}^{T} f \chi \partial_t \zeta + \int_{0}^{T} (\overline{\pi}_{ij}) \chi \partial_i f \partial_j \zeta + \int_{0}^{T} (\overline{\pi}_{ij}) f \partial_j \chi \partial_i \zeta.
$$

Since $f \in W^{2,0}_2(\omega')$ and $\overline{\pi}_{ij}(t, \cdot), \overline{\tau}_{ij}(t, \cdot) \in W^{4,3}_2(B')$ for each fixed $t$, the last expression may be integrated by parts and recasted as

$$
= -\int_{\omega'} (\zeta f \partial_t (\overline{b}_{ij}) + \zeta \overline{b}_{ij}(\partial_i f) \chi) + \int_{\omega'} \chi (f \partial_t \zeta - f \partial_j (\overline{\pi}_{ij}) \partial_i \zeta - \overline{\pi}_{ij} (f \partial_i \partial_j \zeta + \partial_j f \partial_i \zeta) - \overline{\pi}_{ij} \partial_i f \partial_j \zeta) = 0.
$$

Thus, we find that $u$ is a $W^{2,0}_2(\omega')$ solution to the linear equation

$$
\begin{cases}
\partial_t u - \partial_j (\overline{\pi}_{ij} \partial_i u) = F & \omega' \\
u = 0 & [T_1', T_2'] \times \partial B'' \\
u = 0 & \{T_1'\} \times B''
\end{cases}
$$

Keeping in mind Step 1 of lemma 3.4 (in particular $f \in L^{\infty,0}_2(\omega)$) and $f \in L^\infty L^s(\omega)$ where $s' > 3$, we make the following key observations

i. $f \in L^2(\omega')$

ii. By theorem 3.8 ($\sqrt{\mathcal{T}})_i \in L^2(\omega')$ and thus $f_i = 2\sqrt{\mathcal{T}}(\sqrt{\mathcal{T}})_i \in L^2(\omega')$

iii. $\overline{\pi}_{ij}, \overline{\tau}_{ij} \in L^\infty(\omega')$

iv. $(\overline{b}_{ij}), f$ is proportional to $f^2$.

From i-iv we immediately see that $F \in L^2(\omega')$. Moreover, lemma 3.7 shows that the condition $\text{ess sup}_{(T_1', T_2')}|(\overline{\pi}_{ij})_k| < \infty$ required in theorem 3.8 is verified, and so we know that equation (9) has a unique $W^{2,1}_2(\omega')$ solution, call it $\overline{u}$. In
particular $\tilde{u}$ is a $W^{1,0}_{2,0}(\omega')$ solution. Now, viewing equation (9) as an equation for the space $W^{1,0}_{2,0}(\omega')$ and owing to the uniqueness provided by theorem 3.9 it follows that $u = \tilde{u} \in W^{2,1}_{2,0}(\omega')$, which implies $f \in W^{2,1}_{2,0}(\Omega)$.

$\square$

With the aid of the following estimate we can improve the Lebesgue exponent of the first order spatial derivatives of $f$

**Lemma 3.10.** (Lemma 3.3, II in [15]) Suppose $u \in W^{2,1}_{q,0}(\omega)$ and $1 < q < \infty$. Then

$$||\partial_v u||_{r,\omega} \leq C_1(||u||_{q,\omega} + ||\partial_v u||_{q,\omega} + ||\partial_v \partial_j u||_{q,\omega}) + C_2 ||u||_{q,\omega}. \quad (10)$$

**Corollary 3.11.** Let $f$ be a $H$-solution. Suppose there exist $S_0 > 0, q > 3$ such that for all $t \in T$ one has $||f(t, \cdot)||_{L^q(\Omega)} \leq S_0$. Then $\partial_v f \in L^{\frac{10}{3}}(\Omega)$ for all $\Omega \in \omega$.

**Proof.** By Theorem 3.1 $f \in W^{2,1}_{2,0}(\Omega)$ and so taking $q = 2$ in Lemma 3.10 we find that $\partial_v f \in L^r(\Omega)$ as long as $1 \leq r \leq \frac{10}{3}$.

$\square$

In the next section we will iterate lemma 3.10 in order to show that $f$ lies locally in $W^{r,1}_{q,0}(\Omega)$ for arbitrary $1 \leq r < \infty$.

**4 From $W^{2,1}_{2,0}$ to $H^*_{\alpha+2}$**

We now wish to push further the main result obtained in the previous section, by proving that $f$ is locally $H^*_{\alpha+2}$, which in particular implies that $f$ is a classical solution to equation (1). To finish the proof of theorem 2.6 we will need

**Theorem 4.1.** (15, IV, Theorem 9.1) Suppose that $A_{ij}$ are locally uniformly elliptic and bounded continuous on $\omega$. Suppose $F \in L^q(\omega), 1 < q < \infty$. Then the problem

$$\begin{cases} 
\partial_t U - A_{ij}\partial_i \partial_j U = F, & \omega \\
U = 0, & [T_1, T_2] \times \partial B \\
U = 0, & \{T_1\} \times B
\end{cases} \quad (11)$$

has a unique solution $U \in W^{2,1}_{q,0}(\omega)$. 

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Lemma 4.2. ([15], Page 343) Let $5 < q < \infty$, $0 < \alpha < 1 - \frac{2}{q}$ and let the conditions of theorem 4.1 hold. The solution $u \in W^{2,1}_q(\omega)$ of equation (11) has $u \in H^{1+\alpha, 1+\alpha/2}(\omega)$.

In addition we need to slightly refine the conclusion of 2.i of lemma 3.7 by showing that $a_{ijf}$ are Hölder continuous in space uniformly with respect to time.

Lemma 4.3. Suppose $f \in H^{\alpha, \alpha/2}(\omega)$. Let $\Omega \subset \subset \omega$. Then there is some $C > 0$ such that for all $(t,v_1), (t,v_2) \in \Omega$ it holds that $|a_{ijf}(t,v_1) - a_{ijf}(t,v_2)| \leq C|v_1 - v_2|^\alpha$. In particular $a_{ijf} \in H^{\star, \alpha}(\Omega)$.

Proof. With the same notation of lemma 3.7 we have

$$1_B'(v)a_{ijf}(t,v) = \int_{B''} \chi(v-w)a_{ij}(v-w)f(t,w)dw + 1_{B'} \int_{\mathbb{R}^3} (1-\chi(v-w))a_{ij}(v-w)f(t,w)dw := g_t(\cdot) + h_t(\cdot).$$

Let $B''_v$ be the ball centered at $v$ with the same radius as $B''$.

$$|g_t(v_1) - g_t(v_2)| \leq \int_{B''_v} |\chi(w)a_{ij}(w)||f(t, v_1 - w) - f(t, v_2 - w)|dw \leq C_1|v_1 - v_2|^\alpha.$$  

In addition by the mean value theorem

$$|h_t(v_1) - h_t(v_2)| \leq ||\nabla h_t(\xi)|| \times |v_1 - v_2| \leq C_2|v_1 - v_2|,$$

where $\xi$ is an intermediate point and the second inequality is because $||\nabla h_t||_\infty$ is easily seen to be bounded independently of $t$.

At the last step of the proof we will apply the following parabolic version of interior Schauder estimates.

Theorem 4.4. ([14], Theorem 1) Suppose:

1. There are constants $\nu, \mu > 0$ such that for all $\xi \in \mathbb{R}^3$ it holds that $\nu|\xi|^2 \leq A_{ij}(t,v)\xi_i\xi_j \leq \mu|\xi|^2$ for all $(t,v) \in \omega$.

2. There is a constant $\kappa > 0$ such that $||A_{ij}||_{H^{\alpha, \alpha}_2(\omega)} \leq \kappa$ and $||d^2F||^\star_{\alpha} < \infty$.

If $u \in H^{\alpha+2}_{\alpha+2}(\omega)$ is a solution to the equation

$$\partial_t u - A_{ij}\partial_i\partial_j u = F,$$

then $||u||_{H^{\alpha+2}_{\alpha+2}(\omega)} \leq c(||d^2F||_{H^2(\omega)} + ||u||_0)$ where $c = c(\Omega, \kappa, \nu, \alpha).$
Using the above Schauder estimates we can obtain the following uniqueness and existence result. The proof is based on a standard method of continuity argument. We include it here only for the sake of completeness, since it does not appear explicitly in the literature.

**Corollary 4.5.** Let the conditions 1+2 of theorem [4.4] hold. Write $\omega = (T_1, T_2) \times B$. The equation

$$
\begin{align*}
\partial_t u - A_{ij} \partial_i \partial_j u &= \mathcal{F} \quad (T_1, T_2) \times B \\
u &= 0 \quad [T_1, T_2] \times \partial B \\
u &= 0 \quad \{T_1\} \times B
\end{align*}
$$

has a unique $H^\alpha_{\alpha+2}(\omega)$ solution.

**Proof.** Denote $L = \partial_t u - \pi_{ij} \partial_i \partial_j u$. Consider the Banach spaces $\mathcal{B}_1 = H^\alpha_{\alpha+2}(\omega) \cap \{u = 0 \text{ on } [T_1, T_2] \times \partial B \cup \{T_1\} \times B\}$ and $\mathcal{B}_2 = H^\alpha_{\alpha}(\omega)$. For each $0 \leq s \leq 1$ consider the operator $L_s : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ defined by $L_s u = sLu + (1 - s)(\partial_t u - \Delta u)$. The solvability of equation (12) for arbitrary $\mathcal{F} \in H^\alpha_{\alpha}(\omega)$ is equivalent to the fact that $L_1$ is onto. Let $u \in \mathcal{B}_1$ and write $L_s u = \mathcal{F} \in H^\alpha_{\alpha}(\omega)$. By theorem [4.4] we have

$$
||u||_{\mathcal{B}_1} = ||u||_{H^\alpha_{\alpha+2}(\omega)} \leq c(||d^2 \mathcal{F}||^\alpha + ||u||_0) \leq c(||\mathcal{F}||_{H^\alpha_{\alpha}(\omega)} + \sup_{\omega} |\mathcal{F}|) \leq c ||\mathcal{F}||_{H^\alpha_{\alpha}(\omega)} = c ||Lu||_{H^\alpha_{\alpha}(\omega)},
$$

where the second inequality is by the maximum principle and $c$ is independent of $s$. Indeed, note that if $\nu$ stands for the lower ellipticity constant of $L$, then for each $s$ the lower ellipticity constant of $L_s$ can be taken to be $\nu_s = \min(1, \nu)$, which is independent of $s$. It is also apparent that the $H^\alpha_{\alpha}(\omega)$ norm of the coefficients of the second order derivatives of $L_s$ is bounded independently of $s$. Therefore the constant $c$ from theorem [4.4] can be taken to be the same for all the $L_s$. Furthermore, it is classical that the heat equation

$$
\begin{align*}
\partial_t u - \Delta u &= \mathcal{F} \quad (T_1, T_2) \times B \\
u &= 0 \quad [T_1, T_2] \times \partial B \\
u &= 0 \quad \{T_1\} \times B
\end{align*}
$$

has a $H^\alpha_{\alpha+2}(\omega)$ solution provided $\mathcal{F} \in H^\alpha_{\alpha}(\omega)$ (in fact less regularity on $\mathcal{F}$ is required here). Otherwise put, $L_0$ is onto $\mathcal{B}_2$. By the method of continuity (see e.g. theorem 5.2 in [9]) it follows that $L_1$ is onto, as desired. Uniqueness of the solution is a consequence of the maximum principle. 

\[\square\]

**Proof of theorem [2.0] 1.** Given a cylinder $(T_1, T_2) \times B' \Subset \omega$ pick a cylinder $(T_1, T_2) \times B' \Subset (T'_1, T'_2) \times B'' \Subset \omega$. Set $\omega' = (T'_1, T'_2) \times B''$. We localize the solution $f$ by defining $u = \zeta f$ for some $\zeta \in C^\infty_0((T'_1, T'_2) \times B'')$ satisfying $\zeta \equiv 1$ on $(T_1, T_2) \times B'$. First we wish to show that $u$ satisfies an equation of the form (11). As in theorem [3.1] we get that $u$ satisfies the equation

$$
\begin{align*}
\partial_t u - \pi_{ij} \partial_i \partial_j u &= \mathcal{F} \quad \omega' \\
u &= 0 \quad [T'_1, T'_2] \times \partial B'' \\
u &= 0 \quad \{T'_1\} \times B''
\end{align*}
$$

(13)
where $F = \zeta f - \pi^{ij}(\zeta f f_j \zeta - \zeta (\overline{b}^j_i) f)$. By corollary 3.11 we see that $F \in L^{\frac{10}{3}}(\omega')$ (recall that $(\overline{b}^j_i)_i$ is propotional to $f$) and so by theorem 4.1 $u \in W_{10}^{2,1}(\omega')$. We proceed by iterating lemma 3.10 utilizing lemma 3.10 with $q = 10, r = 10$ we find that $u \in W_{10}^{2,1}(\omega')$. Utilizing lemma 3.10 once again with $q = 10, 10 \leq r < \infty$ we get $u \in W_{10}^{2,1}(\omega')$. Thus, we have shown $u \in W_{r}^{2,1}(\omega')$ for all $1 \leq r < \infty$. Lemma 4.2 entails that $u \in H^{1+\alpha, 1+\alpha, r}(\omega')$ for all $0 < \alpha < 1$. We have thus shown that $f \in H^{1+\alpha, 1+\alpha, r}(\Omega)$ for any $\Omega \subset \omega$ and $0 < \alpha < 1$, which in turn implies that $F$ verifies the Hölder condition 2 in theorem 4.1 for all $0 < \alpha < 1$. In addition lemma 4.3 guarantees that $\pi^{ij}$ verifies the Hölder condition 2 imposed in Theorem 4.4 for all $0 < \alpha < 1$. Corollary 4.5 ensures that equation (13) has a unique $H_{\alpha+2}^{*}(\omega')$ solution (for arbitrary fixed $0 < \alpha < 1$), and a fortiori this solution must identify with $f$.

\[\Box\]

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