Noncommutative Geometry as the Key to Unlock the Secrets of Space-Time

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Abstract
I give a summary of the progress made on using the elegant construction of Alain Connes noncommutative geometry to explore the nature of space-time at very high energies. In particular I show that by making very few natural and weak assumptions about the structure of the noncommutative space, one can deduce the structure of all fundamental interactions at low energies.
1. Introduction

This article is dedicated to Alain Connes on the occasion of his 60th birthday. I have come to know Alain well during my first visit to IHES in 1996. I was immediately overwhelmed with his brilliance and the overflow of his ideas, and within a short time started to collaborate with him on the interface of noncommutative geometry, his invention, and the ideas of unification in theoretical physics. This collaboration has been very fruitful, and we have come to appreciate the mysterious links between geometry and physics. Many problems remain, but I am optimistic that the challenge of finding a quantum theory of gravity using the geometric tools that Alain developed, is within reach. At the personal level, I discovered that Alain is a very warm person, full of life, and has fantastic sense of humor. I am proud of his friendship.

What I will present here, is a summary of a forthcoming long article in collaboration with Alain, which hopefully will appear in the near future, where a self contained exposition of the methods of noncommutative geometry applied to particle physics are explained in a language accessible to physicists [1]. A good part of this forthcoming article will elaborate and build on the results that were first obtained with the crucial input of the collaboration with Marcolli [2]. In addition, the introduction present in a recent paper [3] can be used to help introduce the reader to the general philosophy of our program. Our aim is to provide enough material to help students and young researchers who wish to learn about this promising direction of research.

The laws of physics at low energies are well encoded by the action functional which is the sum of the Einstein-Hilbert action and that of the standard model. These two parts have different properties, the first being dependent on the geometry of the underlying manifold \((M, g)\) where \(g\) is the metric, while the other is governed by internal symmetries of a gauge group \(G\) which can be well described using the language of vector bundles. The underlying symmetries are also different. General relativity is governed by diffeomorphism invariance (outer automorphisms of \((M, g)\)) while gauge symmetries are based on local gauge invariance (inner automorphisms).

Thus the natural group of invariance is the semi direct product

\[
G = U \rtimes \text{Diff} (M)
\]

where

\[
U = C^\infty (M, U(1) \times SU(2) \times SU(3)) .
\]

It is possible to trace back the failure of finding a unified theory of all interactions including quantum gravity to the difference between these two kinds of symmetries. In addition, there are many unanswered questions within the established formulation of the standard model. For example, the following questions have no compelling answer: Why the gauge group is specifically given by \(U(1) \times SU(2) \times SU(3)\)? Why the fermions occupy the particular representations that they do? Why there are three families and why there are 16 fundamental fermions per family? What is the theoretical origin of the Higgs mechanism and spontaneous breakdown of gauge symmetries? What is the Higgs mass and how to explain all the fermionic masses? These are only few of the questions that have to be answered by the ultimate unified theory of all interactions. We shall attempt to answer some of these questions taking as a starting point the following observations. At energies well below the
Planck scale $M_P = \sqrt{\frac{1}{8\pi G}} = \frac{1}{\kappa} = 2.43 \times 10^{18}$ Gev

gravity can be safely considered as a classical theory. But as energies approach the Planck scale one expects the quantum nature of space-time to reveal itself, and for the Einstein-Hilbert action to become an approximation of some deformed theory. In addition the other three forces must be unified with gravity in such a way that all interactions will correspond to one underlying symmetry. One thus would expect that the nature of space-time, and thus of geometry, would change at Planckian energies, in such a way that at lower energies, one recovers the above picture of diffeomorphism and internal gauge symmetries. It is not realistic to guess the exact properties of space-time at Planckian energies and to make directly an extrapolation of 17 orders of magnitude from our present energies. We are therefore led to take an indirect approach where we search for the hidden structure in the functional of gravity coupled to the standard model at present energies. To do this we shall make a basic conjecture which we will take as a starting point:

**Conjecture 1.** At some energy level, space-time is the product of a continuous four-dimensional manifold times a discrete space $F$.

The aim then is to find supporting evidence for this conjecture. Once this is done the next step would be to find the true geometry at Planckian energies, for which this product in turn is a limit.

This is the minimal extension where no new extra dimensions are assumed. The task now is to determine with minimal input the properties of the discrete space $F$, and construct the associated physical theory. Remarkably, we will show that this information will allow us to determine the hidden structure of space-time, and answer some, but not all (so far) of the questions posed above.

### 2. A Brief Summary of Alain Connes NCG

The basic idea is based on physics. The modern way of measuring distances is spectral. The unit of distance is taken as the wavelength of atomic spectra. To adapt this geometrically the notion of real variable which one takes as a function $f$ on a set $X$ where $f : X \rightarrow \mathbb{R}$ has to be replaced. This is now taken to be a self adjoint operator in a Hilbert space as in quantum mechanics. The space $X$ is described by the algebra $\mathcal{A}$ of coordinates which is represented as operators in a fixed Hilbert space $\mathcal{H}$. The geometry of the noncommutative space is determined in terms of the spectral data $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$. A real, even spectral triple is defined by [4], [5]

- $\mathcal{A}$ is an associative algebra with unit 1 and involution $\ast$.
- $\mathcal{H}$ is a complex Hilbert space carrying a faithful representation $\pi$ of the algebra.
- $D$ is a self-adjoint operator on $\mathcal{H}$ with the resolvent $(D - \lambda)^{-1}, \lambda \in \mathbb{R}$ of $D$ compact.
- $J$ is an anti–unitary operator on $\mathcal{H}$, which is a real structure (charge conjugation.)
- $\gamma$ is a unitary operator on $\mathcal{H}$, the chirality.

We require the following axioms to hold:
- $J^2 = \varepsilon$, ($\varepsilon = 1$ in zero dimensions and $\varepsilon = -1$ in 4 dimensions).
• \([a, b^o] = 0\) for all \(a, b \in A\), where \(b^o = Jb^*J^{-1}\). This is the zeroth order condition. This is needed to define the right action on elements of \(\mathcal{H}\): 
\[
\zeta b = b^o \zeta,
\]
and is a statement that left action and right action commute.
• \([DJ, a] = 0\) for all \(a, b \in A\). This is the first order condition.
• \(\gamma^2 = 1\) and \([\gamma, a] = 0\) for all \(a \in A\). These properties allow the decomposition \(H = H_L \oplus H_R\).
• \(H\) is endowed with \(A\) bimodule structure \(a \eta b = ab^o \eta\).

The reality conditions resemble the conditions governing the existence of Majorana (real) fermions.

The properties listed above of the anti-linear isometry \(J : \mathcal{H} \to \mathcal{H}\) are characteristic of a real structure of \(KO\)-dimension \(n \in \mathbb{Z}/8\) on the spectral triple \((A, \mathcal{H}, D)\).

We take the algebra \(A\) to be given by a tensor product which geometrically corresponds to a product space. The spectral geometry of \(A\) is given by the product rule \(A = C^\infty (M) \otimes A_F\) where the algebra \(A_F\) is finite dimensional, and
\[
\mathcal{H} = L^2 (M, S) \otimes \mathcal{H}_F, \quad D = D_M \otimes 1 + \gamma_5 \otimes D_F,
\]
where \(L^2 (M, S)\) is the Hilbert space of \(L^2\) spinors, and \(D_M\) is the Dirac operator of the Levi-Civita spin connection on \(M\). The Hilbert space \(\mathcal{H}_F\) is taken to include the physical fermions. The chirality operator is \(\gamma = \gamma_5 \otimes \gamma_F\) and the reality operator is \(J = C \otimes J_F\), where \(C\) is the charge conjugation matrix.

In order to avoid the fermion doubling problem where the fermions \(\zeta, \zeta^c, \zeta^*, \zeta^{c*}\), \(\zeta \in \mathcal{H}\), should not be all independent, it was shown that the finite dimensional
space must be taken to be of K-theoretic dimension 6 [6], [7], where in this case 
\((\varepsilon, \varepsilon', \varepsilon'') = (1, 1, -1)\) (so as to impose the condition \(J\xi = \xi\)). This makes the total
K-theoretic dimension of the noncommutative space to be 10 and would allow to
impose the reality (Majorana) condition and the Weyl condition simultaneously
in the Minkowskian continued form, a situation very familiar in ten-dimensional
supersymmetry. In the Euclidean version, the use of the \(J\) in the fermionic action,
would give for the chiral fermions in the path integral, a Pfaffian instead of deter-
minant [6], and will thus cut the fermionic degrees of freedom by a factor of 2. In
other words, in order to have the fermionic sector free of the fermionic doubling
problem we must make the choice
\[ J_F^2 = 1, \quad J_F D_F = D_F J_F, \quad J_F \gamma_F = -\gamma_F J_F. \]
In what follows we will restrict our attention to determination of the finite algebra,
and will omit the subscript \(F\).

3. Classification of Finite Noncommutative Spaces

There are two main constraints on the algebra from the axioms of noncom-
mutative geometry. We first look for involutive algebras \(A\) of operators in \(H\) such that,
\[ [a, b^0] = 0, \quad \forall a, b \in A, \]
where for any operator \(a\) in \(H\), \(a^0 = Ja^*J^{-1}\). This is called the order zero
condition. We shall assume that the representations of \(A\) and \(J\) in \(H\) are
irreducible.

The classification of the irreducible triplets \((A, H, J)\) leads to the following
theorem [8], [9]:

**Theorem 2.** The center \(Z(AC)\) is \(\mathbb{C}\) or \(\mathbb{C} \oplus \mathbb{C}\).

If the center \(Z(AC)\) is \(\mathbb{C}\) then we can state the following theorem:

**Theorem 3.** Let \(H\) be a Hilbert space of dimension \(n\). Then an irreducible
solution with \(Z(AC) = \mathbb{C}\) exists iff \(n = k^2\) is a square. It is given by \(AC = M_k(\mathbb{C})\)
acting by left multiplication on itself and anti-linear involution
\[ J(x) = x^*, \quad \forall x \in M_k(\mathbb{C}). \]

For \(AC = M_k(\mathbb{C})\) we have \(A = M_k(\mathbb{C}),\ M_k(\mathbb{R})\) or \(M_a(\mathbb{H})\) for even \(k = 2a\),
where \(\mathbb{H}\) is the field of quaternions [10]. These correspond respectively to the
unitary, orthogonal and symplectic case.

If the center \(Z(AC)\) is \(\mathbb{C} \oplus \mathbb{C}\) then we can state the theorem:

**Theorem 4.** Let \(H\) be a Hilbert space of dimension \(n\). Then an irreducible
solution with \(Z(AC) = \mathbb{C} \oplus \mathbb{C}\) exists iff \(n = 2k^2\) is twice a square. It is given by \(AC = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})\)
acting by left multiplication on itself and anti-linear involution
\[ J(x, y) = (y^*, x^*), \quad \forall x, y \in M_k(\mathbb{C}). \]

With each of the \(M_k(\mathbb{C})\) in \(AC\) we can have the three possibilities \(M_k(\mathbb{C}),\ M_k(\mathbb{R}),\) or \(M_a(\mathbb{H})\), where \(k = 2a\). At this point we make the hypothesis that we are in the “symplectic–unitary” case, thus restricting the algebra \(A\) to the form
\[ A = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \quad k = 2a. \]
The dimension of the Hilbert space is \( n = 2k^2 \), however, because of the reality condition, these correspond to \( k^2 \) fundamental fermions, where \( k = 2a \) is an even integer. The first possible value for \( k \) is 2 corresponding to a Hilbert space of four fermions and an algebra \( A = \mathbb{H} \oplus M_2(\mathbb{C}) \). This is ruled out because it does not allow to impose grading on the algebra. It is also ruled out by the existence of quarks. The next possible value for \( k \) is 4 thus predicting the number of fermions to be 16.

In the \( Z(A_{\mathbb{C}}) = \mathbb{C} \) case, one can show that it is not possible to have the finite space to be of K-theoretic dimension 6 consistent with the relation \( J\gamma = -\gamma J \). We therefore can proceed directly to the second case.

One then takes the grading \( \gamma \) of \( \mathcal{H} \) so that the K-theoretic dimension of the finite space is 6 and this is consistent with the condition \( J\gamma = -\gamma J \). It is given by

\[
\gamma (\zeta, \eta) = (\gamma \zeta, -\gamma \eta).
\]

This grading breaks the algebra \( A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C}) \), which is non-trivially graded only for the \( M_2(\mathbb{H}) \) component, to its even part:

\[
A^{ev} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})
\]

The Dirac operator must connect the two pieces non-trivially, and therefore must satisfy

\[
[D, Z(A)] \neq \{0\}.
\]

The physical meaning of this constraint, is to allow some of the fermions to acquire Majorana masses, realizing the see-saw mechanism, and thus connecting the fermions to their conjugates.

We have to look for subalgebras \( A_F \subset A^{ev} \), the even part of the algebra \( A \), for which \( [[D, a], b^n] = 0, \forall a, b \in A_F \). We can state the theorem:

**Theorem 5.** Up to automorphisms of \( A^{ev} \), there exists a unique involutive subalgebra \( A_F \subset A^{ev} \) of maximal dimension admitting off-diagonal Dirac operators

\[
A_F = \{ (\lambda \oplus \bar{\lambda}) \oplus q, \lambda \oplus m | \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C}) \}
\]

\[
\subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})
\]

It is isomorphic to \( \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \).

### 4. Tensor Notation

It is helpful to write the results obtained about the standard model using tensor notation. The Dirac action must take the form

\[
\Psi^*_M D_M^N \Psi_N
\]

where \( \Psi_M = \begin{pmatrix} \psi_A^* \\ \psi_{A'}^* \end{pmatrix} \) and we have denoted \( \psi_{A'} = \psi_A^* \), the conjugate spinor. We start with the algebra

\[
A = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})
\]

and denote the spinors by \( \psi_A = \psi_{\alpha I}, A = \alpha I, \alpha = 1, \cdots, 4, I = 1, \cdots, 4 \), and thus \( D^B_A = D^B_{\alpha I} \). The Dirac operator takes the form

\[
D = \begin{pmatrix}
D^R_A & D^B_{A'} \\
D^B_A & D^B_{A'}
\end{pmatrix},
\]
where $A' = \alpha' I'$, $\alpha' = 1', \cdots, A'$, $I' = 1', \cdots, A'$, and $D_{A'}^{B'} = \overline{D_{A}^{B}}$, $D_{A}^{B} = \overline{D_{A'}^{B'}}$ and overbar denotes complex conjugation.

Elements of the algebra $\mathcal{A}$ are matrices $a_{M}^{N}$ of the special form:

$$a = \left( \begin{array}{cc} X_{\alpha}^{\beta} \delta_{I}^{J} & 0 \\ 0 & \delta_{\alpha}^{\beta'} Y_{I'}^{J'} \end{array} \right),$$

where $X_{\alpha}^{\beta}$ is an element of the first $M_{4} (\mathbb{C})$ and $Y_{I'}^{J'}$ is an element of the second $M_{4} (\mathbb{C})$. The reality operator $J$ is defined by

$$J = \left( \begin{array}{cc} 0 & \delta_{\alpha}^{\beta'} \\ \delta_{\alpha}^{\beta} & 0 \end{array} \right) \times \text{complex conjugation}.$$ 

In this representation we deduce that $a^{o}$ takes the form

$$a^{o} = J a^{*} J^{-1} = \left( \begin{array}{cc} \delta_{\alpha}^{\beta} \overline{Y_{I}^{J}} & 0 \\ 0 & \overline{X_{\alpha}^{\beta'} \delta_{I'}^{J'}} \end{array} \right),$$

where $\overline{\cdot}$ denotes transposition. It is trivial to verify that $[a, b^{c}] = 0$.

The order one condition is

$$[[D, a], b^{o}] = 0$$

If we write

$$b^{o} = \left( \begin{array}{cc} \delta_{\alpha}^{\beta} W_{I}^{J} & 0 \\ 0 & Z_{\alpha}^{\beta'} \delta_{I'}^{J'} \end{array} \right),$$

then

$$[[D, a], b^{o}] = \begin{pmatrix} [[D, X], W_{I}^{J}]_{A}^{B} & ((DY - XD)Z - W (DY - XD))_{A}^{B'} \\ ((DX - YD)W - Z (DX - YD))_{A}^{B} & [[D, Y], Z_{I}^{J}]_{A}^{B'} \end{pmatrix} = 0.$$ 

Explicitly, the first two equations read:

\[
\left( D_{\alpha I}^{\gamma K} X_{\gamma}^{\beta} - X_{\alpha}^{\gamma} D_{\gamma I}^{\beta K} \right) W_{K}^{J} - W_{I}^{K} \left( D_{\alpha K}^{\gamma J} X_{\gamma}^{\beta} - X_{\alpha}^{\gamma} D_{\gamma K}^{\beta J} \right) = 0,
\]

\[
\left( D_{\alpha I}^{\gamma K} Y_{K'}^{J'} - X_{\alpha}^{\gamma} D_{\gamma I}^{\beta K} \right) Z_{\gamma'}^{\beta'} - W_{I}^{K} \left( D_{\alpha K}^{\gamma J} Y_{K'}^{J'} - X_{\alpha}^{\gamma} D_{\gamma K}^{\beta J'} \right) = 0.
\]

We have shown [8], [2], that the only non-zero solution of the second equation is

$$D_{\alpha I}^{\beta K'} = \delta_{\alpha}^{\gamma} \delta_{\gamma}^{\beta'} \delta_{I}^{J} \delta_{I'}^{J'} k^{\nu K},$$

which means that there can be only one non-zero single entry in the off-diagonal $16 \times 16$ matrix $D_{A}^{B}$, and this implies that

$$D_{\alpha I}^{\beta J} = D_{\alpha (I)}^{\beta} \delta_{I}^{J} + D_{\alpha (\alpha)}^{\beta} \delta_{I}^{J} \delta_{i}^{J},$$

$$Y_{I'}^{J'} = \delta_{I}^{J} \delta_{J'}^{I'} Y_{I}^{J} + \delta_{I}^{J} \delta_{J'}^{I'} Y_{I}^{J'},$$

$$X_{i}^{J} = Y_{I}^{J'}, \; X_{1}^{\alpha} = 0, \; \alpha \neq 1.$$
where we have split the index $I = 1, i$, and $I' = 1', i'$. From the property of commutation of the grading operator
$$
g^a_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$
the algebra $M_4(\mathbb{C})$ reduces to $M_2(\mathbb{C})_R \oplus M_2(\mathbb{C})_L$. We further impose the condition of symplectic isometry on $M_2(\mathbb{C})_R \oplus M_2(\mathbb{C})_L$
$$
s_2 \otimes 1_2(\mathbb{T})s_2 \otimes 1_2 = a,
$$
which reduces it to $\mathbb{H}_R \oplus \mathbb{H}_L$. We will be using the notation
$$
\alpha = \hat{1}, \hat{2}, a \quad \text{where} \quad \xi_{1,2} \in \mathbb{H}_R, \xi_a \in \mathbb{H}_L.
$$
Together with the above condition this implies that
$$
X_\alpha = \delta^1_\alpha X^1 + \delta^2_\alpha X^2 + \delta^3_\alpha X^3
$$
and the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ reduces to
$$
\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
$$
because $X^1 = Y^1$. Expanding the Dirac action we get
$$
\psi^*_A D^B_A \psi_B + \psi^*_{1'1'} D^B_{1'1'} \psi_B + \psi^*_A D^B_{1'1'} \psi_{1'1'} + \psi^*_A D^B_{1'1'} \psi_{1'1'}
$$
The spinors can thus be denoted by
$$
\psi_A = (\psi_{\alpha, 1}, \psi_{\alpha, i}) = (\psi_{1, 1}, \psi_{1, 2}, \psi_{1', 1', 1}, \psi_{1', 1', 2}, \psi_{1', \alpha, i}) = (\nu_R, e_R, l_a, u_R, d_R, q_{\alpha, i}),
$$
where $l_a = (\nu_L, e_L)$ and $q_{\alpha, i} = (u_{L, i}, d_{L, i})$. The component $\psi_{1', 1'} = \psi^c_{11} = \nu_R^c$ which implies that the Dirac action
$$
\psi^*_A D^B_A \psi_B + \nu^c_R k^{\mu \nu} \nu_R + c.c
$$
has only a mixing term for the right-handed neutrinos.

Having determined the structure of the Dirac operator of the discrete space, we can form the Dirac operator of the product space of this discrete space times a four-dimensional Riemannian manifold:
$$
D = D_M \otimes 1 + \gamma_5 \otimes D_F.
$$
Since $D_F$ is a $32 \times 32$ matrix tensored with the $3 \times 3$ matrices of generation space and with the Clifford algebra, $D$ is $384 \times 384$ matrix.

To take inner automorphisms into account, we have to evaluate the Dirac operator
$$
D_A = D + A + JAJ^{-1},
$$
where
$$
A = \sum a [D, b].
$$
In particular
$$
A^B_A = \sum a^C_A (D^D_C b^B_D - b^D_C D^B_D).
$$
Note there are no mixing terms like $D^D_C b^B_D$, because $b$ is block diagonal.
Evaluating all components of the full Dirac operator $D_M^N$, quoting only the result, the full derivation will be given in a forthcoming paper [1], we obtain:

$$(D)_{11}^{11} = \gamma^\mu \otimes D_\mu \otimes 1_3, \quad D_\mu = \partial_\mu + \frac{1}{2} \omega^c_{\mu} (e) \gamma^cd, \quad 1_3 = \text{generations}$$

$$(D)_{11}^{a1} = \gamma_5 \otimes k^{\sigma} \otimes e^{ab} H_b \quad k^{\sigma} = 3 \times 3 \text{ neutrino mixing matrix}$$

$$(D)_{21}^{21} = \gamma^\mu \otimes (D_\mu + ig_1 B_\mu) \otimes 1_3$$

$$(D)_{a1}^{a1} = \gamma_5 \otimes k^{ce} \otimes \overline{T}^c$$

$$(D)_{a1}^{11} = \gamma_5 \otimes k^{\nu} \otimes e_{ab} \overline{T}^b$$

$$(D)_{a1}^{21} = \gamma_5 \otimes k^{\sigma} \otimes H_a$$

$$(D)_{a1}^{b1} = \gamma^\mu \otimes \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^b_a - \frac{i}{2} g_2 W_\mu^a (\sigma^b)^a \right) \otimes 1_3, \quad \sigma^a = \text{Pauli}$$

$$(D)_{i1}^{ij} = \gamma^\mu \otimes \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^j_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)^j_i \otimes 1_3, \quad \lambda^i = \text{Gell-Mann}$$

$$(D)_{i1}^{ai} = \gamma_5 \otimes k^{e} \otimes H_i$$

$$(D)_{i1}^{aj} = \gamma_5 \otimes k^{\mu} \otimes e_{ab} \overline{H}^a \delta^j_i$$

$$(D)_{i1}^{bj} = \gamma^\mu \otimes \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta^b_i - \frac{i}{2} g_2 W_\mu^a (\sigma^b)^a \delta^b_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)^b_i \otimes 1_3$$

$$(D)_{i1}^{a1} = \gamma_5 \otimes k^{\nu} \otimes \overline{T}^a \delta^j_i$$

$$(D)_{i1}^{a1} = \gamma_5 \otimes k^{e} \otimes H_a \delta^j_i$$

$$(D)_{i1}^{b1} = \gamma_5 \otimes k^{\sigma} \otimes e_{ab} \overline{H}^a \delta^j_i$$

$$(D)_{i1}^{a1} = \gamma_5 \otimes k^{\nu} \otimes \overline{T}^a \delta^j_i$$

$$(D)_{i1}^{11} = \gamma_5 \otimes k^{\sigma} \otimes \overline{T}^a \delta^j_i$$

where $B_\mu, W_\mu^a$ and $V_\mu^m$ are the $U(1)$, $SU(2)$ and $SU(3)$ gauge fields, and $H$ is a complex doublet scalar field and $\sigma$ is a singlet real scalar field. We have assumed that the unitary algebra $\mathcal{U}(A)$ is restricted to $SU(A)$ to eliminate a superfluous
U(1) gauge field. Pictorially, the matrix $D_N^M$ has the structure:

$$
\left(
\begin{array}{cccccc}
11 & 21 & a1 & i_l & 2i & ai \\
v_R & e_R & l_a & u_{iR} & d_{iR} & q_{iL} \\
\end{array}
\right)
$$

$$(D)^{11}_{11} 0 (D)^{a1}_{11} 0 0 0
\begin{array}{cc}
0 & (D)^{21}_{21} (D)^{a3}_{21} 0 0 0 \\
(D)^{b1}_{11} (D)^{21}_{b1} (D)^{a1}_{b1} 0 0 0 \\
(D)^{1j}_{1j} 0 0 0 0 (D)^{2i}_{1j} (D)^{ai}_{1j} \\
0 0 0 0 (D)^{2i}_{2j} (D)^{ai}_{2j} \\
0 0 0 0 (D)^{b1}_{b1} (D)^{2i}_{bj} (D)^{ai}_{bj}
\end{array}
$$

Needless to say the term $\psi^*_M D_M^N \psi_N$ contains all the fermionic terms and their interactions in the standard model.

5. The Spectral Action Principle

There is a shift of point of view in NCG similar to Fourier transform, where the usual emphasis on the points $x \in M$ of a geometric space is now replaced by the spectrum $\Sigma$ of the operator $D$. The existence of Riemannian manifolds which are isospectral but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity.

The physical action depends only on the $\Sigma$

This is the spectral action principle [11]. The spectrum is a geometric invariant and replaces diffeomorphism invariance. We now apply this basic principle to the noncommutative geometry defined by the spectrum of the standard model to show that the dynamics of all interactions, including gravity is given by the spectral action

$$\text{Trace } f \left( \frac{D_A}{\Lambda} \right) + \frac{1}{2} \langle J\Psi, D_A\Psi \rangle,$$

where $f$ is a positive function, $\Lambda$ a cutoff scale needed to make $D_A/\Lambda$ dimensionless, and $\Psi$ is a Grassmann variable which represents fermions.

In the case of the cut-off function, $f$ only plays a role through its momenta $f_0, f_2, f_4$ where

$$f_k = \int_0^\infty f(v)v^{k-1}dv, \quad \text{for } k > 0, \quad f_0 = f(0).$$

These will serve as three free parameters in the model. In this case the action $S_A[D_A]$ is the number of eigenvalues $\lambda$ of $D_A$ counted with their multiplicities such that $|\lambda| \leq \Lambda$. 
To illustrate how this comes about, expand the function $f$ in terms of its Laplace transform

$$\text{Trace} f (P) = \sum_s f_s \text{Trace} (P^{-s})$$

$$\text{Trace} (P^{-s}) = \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} \text{Trace} (e^{-tP}) \, dt \quad \text{Re} (s) \geq 0$$

$$\text{Trace} (e^{-tP}) \simeq \sum_{n \geq 0} t^{-\frac{m}{d}} \int M a_n (x, P) \, dv (x),$$

where $m = 4$ is the dimension of the manifold $M$ and $d = 2$ is the order of the elliptic operator $D^2$. Gilkey gives generic formulas for the Seeley-deWitt coefficients $a_n (x, P)$ for a large class of differential operators $P$. The details are explained in preceding papers [11], [2] or using the tensorial notation, in a forthcoming paper [1].

The bosonic part of the spectral action, gives an action that unifies gravity with $SU(2) \times U(1) \times SU(3)$ Yang-Mills gauge theory, with a Higgs doublet $H$ and spontaneous symmetry breaking and a real scalar field $\sigma$. It is given by [11], [2]

$$S = \frac{48}{\pi^2} f_4 \Lambda^4 \int d^4 x \sqrt{g}$$

$$- \frac{4}{\pi^2} f_2 \Lambda^2 \int d^4 x \sqrt{g} \left( R + \frac{1}{2} a \overline{H} H + \frac{1}{4} c \right)$$

$$+ \frac{1}{2 \pi^2} f_0 \int d^4 x \sqrt{g} \left[ \frac{1}{30} \left( -18C_{\mu \nu \rho \sigma}^2 + 11R^* R^* \right) + \frac{5}{3} g_1^2 B_{\mu \nu}^2 + g_2^2 (W_{\mu \nu})^2 + g_3^2 (V_{\mu \nu}^m)^2 \right.$$

$$+ \frac{1}{6} a R H_a H^a + b \left( \overline{H} H \right)^2 + a | \nabla H_a |^2 + 2e \overline{H} H \sigma^2 + \frac{1}{2} \sigma^4 + \frac{1}{2} a_R \sigma^2 + \frac{1}{2} c (\partial_\mu \sigma)^2 \right]$$

$$+ f \lambda^2 a_0 + \cdots$$

This can be rearranged, after normalizing the kinetic energies and ignoring the $\sigma$ field which only plays a role in cosmology, to the form:

$$S = \int \left( \frac{1}{2 \kappa_0} R + \alpha_0 C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma} + \gamma_0 + \tau_0 R^* R^* \right.$$

$$+ \frac{1}{4} G_{\mu \nu}^i G^{\mu \nu i} + \frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} + \frac{1}{4} B_{\mu \nu} B^{\mu \nu}$$

$$+ \frac{1}{2} | D_\mu H |^2 - \mu_0^2 | H |^2 - \xi_0 R | H |^2 + \lambda_0 | H |^4 \right) \sqrt{g} d^4 x,$$

where

$$\frac{1}{\kappa_0^2} = \Lambda^2 \left( \frac{96}{12 \pi^2} f_2 - f_0 c \right)$$

$$\mu_0^2 = \Lambda^2 \left( 2 \frac{f_2}{f_0} - \frac{e}{a} \right)$$

$$\alpha_0 = - \frac{3 f_0}{10 \pi^2}, \quad \tau_0 = \frac{11 f_0}{60 \pi^2}, \quad \lambda_0 = \frac{\pi^2}{2} \frac{b}{f_0 a^2}$$

$$\gamma_0 = \Lambda^4 \frac{1}{\pi^2} (48 f_4 - f_2 c + \frac{1}{4} f_0 d), \quad \xi_0 = \frac{1}{12}.$$
The parameters $a, b, c, d, e$ are all dimensionless and related to the Yukawa couplings that give the fermionic masses after the spontaneous breaking of symmetry:

\[
    a = \text{Tr} \left( k^e c^e + k^{u*} k^u + 3 k^{d*} k^d \right)
\]

\[
    b = \text{Tr} \left( \left( k^e c^e \right)^2 + \left( k^{u*} k^u \right)^2 + 3 \left( k^{d*} k^d \right)^2 \right)
\]

\[
    c = \text{Tr} \left( k_R^e k_R^c \right), \quad d = \text{Tr} \left( \left( k_R^e \right)^2 \right), \quad e = \text{Tr} \left( k_R^e k_R^{u*} k^u \right)
\]

### 6. Predictions of Spectral Action for Standard Model

We shall first perform our analysis by assuming that the function $f$ is well approximated by the cut-off function, thus allowing us to ignore higher order terms. We will determine, to what extent such an approximation could be made, and its effects on the predictions. The normalization of the kinetic terms imposes a relation between the coupling constants $g_1, g_2, g_3$ and the coefficient $f_0$, of the form

\[
    g_3^2 f_0 = \frac{1}{4}, \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.
\]

This gives the relation $\sin^2 \theta_W = \frac{3}{8}$ a value also obtained in $SU(5)$ and $SO(10)$ grand unified theories. The three momenta of the function $f_0$, $f_2$ and $f_4$ can be used to specify the initial conditions on the gauge couplings, the Newton constant and the cosmological constant. We deduce that the geometrical picture is valid at high energies, and the spectral action must be considered in the Wilsonian approach, where all coupling constants are energy dependent and follow the renormalization group equations. For example, The fine structure constant $\alpha_{em}$ is given by

\[
    \alpha_{em} = \sin^2(\theta_w) \alpha_2, \quad \alpha_i = g_i^2 / 4\pi.
\]

Its infrared value is $\sim 1/137.036$ but it is running as a function of the energy and increases to the value $\alpha_{em}(M_Z) = 1/128.09$ already, at the energy $M_Z \sim 91.188$ Gev.

Assuming the “big desert” hypothesis, the running of the three couplings $\alpha_i$ is known. With 1-loop corrections only, it is given by [13]

\[
    \beta_{g_i} = \frac{(4\pi)^{-2}}{b_i} g_i^3, \quad \text{with} \quad b = \left( \frac{41}{6}, -\frac{19}{6}, -7 \right),
\]

so that

\[
    \alpha_1^{-1}(\Lambda) = \alpha_1^{-1}(M_Z) - \frac{41}{12\pi} \log \frac{\Lambda}{M_Z},
\]

\[
    \alpha_2^{-1}(\Lambda) = \alpha_2^{-1}(M_Z) + \frac{19}{12\pi} \log \frac{\Lambda}{M_Z},
\]

\[
    \alpha_3^{-1}(\Lambda) = \alpha_3^{-1}(M_Z) + \frac{42}{12\pi} \log \frac{\Lambda}{M_Z},
\]

where $M_Z$ is the mass of the $Z^0$ vector boson.

In fact, if one considers the actual experimental values

\[
    g_1(M_Z) = 0.3575, \quad g_2(M_Z) = 0.6514, \quad g_3(M_Z) = 1.221,
\]

one obtains the values

\[
    \alpha_1(M_Z) = 0.0101, \quad \alpha_2(M_Z) = 0.0337, \quad \alpha_3(M_Z) = 0.1186.
\]
The graphs of the running of the three constants $\alpha_i$ do not meet exactly, hence do not specify a unique unification energy.

Next we study the running of the Higgs quartic coupling $\lambda$ [14]:

$$\frac{d\lambda}{dt} = \lambda \gamma + \frac{1}{8\pi^2} (12\lambda^2 + B),$$

where

$$\gamma = \frac{1}{16\pi^2} (12k_t^2 - 9g_2^2 - 3g_1^2)$$

$$B = \frac{3}{16} (3g_2^4 + 2g_1^2 g_2^2 + g_1^4) - 3k_t^4.$$

The Higgs mass is then given by

$$m_H^2 = 8\lambda \frac{M^2}{g^2}, \quad m_H = \sqrt{2\lambda} \frac{2M}{g}.$$

The numerical solution to these equations with the boundary value $\lambda_0 = 0.356$ at $\Lambda = 10^{17}$ Gev gives $\lambda(M_Z) \sim 0.241$ and a Higgs mass of the order of 170 Gev. This specific value has been recently ruled out experimentally. However, this is to be expected, because of the non unification of the three gauge couplings.

The mass of the top quark is governed by the top quark Yukawa coupling $k_t$ and is given by the equation

$$m_{top}(t) = \frac{1}{\sqrt{2}} \frac{2M}{g} k_t = \frac{1}{\sqrt{2}} v k_t,$$

where $v = \frac{2M}{g}$ is the vacuum expectation value of the Higgs field. There is a relation between the Yukawa and the gauge couplings which emerges as a consequence of normalizing the Higgs interactions. This relation is a consequence of the fact that all fermions get their masses by coupling to the same Higgs through interactions of the form

$$kH\bar{\psi}\psi.$$
After normalizing the kinetic energies of the Higgs field through the redefinition \( H \rightarrow \frac{k}{\sqrt{\alpha f_0}} H \), the mass terms take the form
\[
\frac{\pi}{\sqrt{\alpha f_0}} H \bar{\psi} \psi.
\]
Using the identity \( \sum_i \left( \frac{k_i}{\sqrt{\alpha}} \right)^2 = 1 \) gives a relation among the fermions masses and W mass [2]
\[
\sum_{\text{generations}} m_e^2 + m_{\mu}^2 + 3m_d^2 + 3m_u^2 = 8M_W^2.
\]
The value of \( g \) at a unification scale of \( 10^{17} \) Gev is \( \sim 0.517 \). Thus, neglecting the \( \tau \) neutrino Yukawa coupling, we get the simplified relation
\[
k_t = \frac{2}{\sqrt{3}} g \sim 0.597.
\]
The numerical integration of the differential equation with the boundary condition gives the value \( k_0 \sim 1.102 \) and a top quark mass of the order of \( \frac{1}{\sqrt{2}} k_0 v \sim 173.683 k_0 \) Gev. The value of \( k_0 \) improves to \( k_0 \sim 1.04 \) when the \( \tau \)-neutrino Yukawa coupling is taken into account, which yields an acceptable value for the top quark mass of 179 Gev [2]. One reason why the resulting top quark mass is acceptable while the Higgs mass is not, is because the later is dependent on the cut-off function.

The fact that the coupling constants do not meet is giving us information about the nature of the function \( f \) used in the spectral action. Our results were obtained under the assumption that the function \( f \) is the cut-off function for which all coefficients of the higher order terms in the asymptotic expansion vanish. These coefficients are given by derivatives of the function evaluated at zero. We can infer from these results, especially from the near meeting of the coupling constants, the good approximate values for \( \sin^2 \theta \) and the top quark mass, that the function \( f \) is well approximated by the cut-off function, but deviates slightly from it. What is needed then is for the Taylor coefficients of the function to be very small but not zero.

To prove that this is indeed the case we compute the gauge and Higgs contributions to the next order i.e. \( a_6 \), in the asymptotic expansion. It is enough to look only at the non gravitational terms [1]:
\[
\begin{align*}
- \frac{f''(0)}{12\pi^2 \Lambda^2} & \left[ c_1 \mathcal{H} H \left( \frac{1}{4} g_2^2 (W_{\mu \nu}^a)^2 \right) + c_2 \mathcal{H} H \left( g_2^3 (V_{\mu \nu}^m)^2 \right) + c_3 \mathcal{H} \sigma^a H \left( \frac{1}{2} g_1 g_2 B_{\mu \nu} W_{\mu \nu}^a \right) \\
& + c_4 (\mathcal{H} H)^3 + c_5 (\mathcal{H} H)^2 \sigma^2 + c_6 \left( (\mathcal{H} \nabla_{\mu} H)^2 + (\nabla_{\mu} \mathcal{H})^2 \right) \\
& + c_7 \left( \nabla_{\mu} \nabla_{\nu} \mathcal{H} \right) \left( \nabla_{\mu} \nabla_{\nu} H \right) + c_8 \left( \mathcal{H} H \left| \nabla_{\mu} H \right|^2 + \mathcal{H} \nabla_{\mu} H \nabla_{\nu} H \right) + c_9 \left| \nabla_{\mu} (H \sigma) \right|^2 \\
& + c_{10} \left| e^{\alpha \beta} H_{\alpha} \nabla_{\mu} H_{\beta} \right|^2 + c_{11} \nabla_{\mu} \mathcal{H} \nabla_{\nu} H \left( \frac{3}{2} i g_1 B_{\mu \nu} \right) + c_{12} \nabla_{\mu} \mathcal{H} \sigma^a \nabla_{\nu} H \left( \frac{3}{2} i g_2 W_{\mu \nu}^a \right) \right]
\end{align*}
\]
where the coefficients \( c_1, \cdots, c_{12} \) depend only on the Yukawa couplings. The exact expression will be given in reference [1]. This clearly shows that the kinetic terms of the gauge fields get modified, and are all multiplied with the coefficients
\[ f_{-2} = f'(0). \] The remarkable thing is that if we rescale the Higgs field by
\[ H = \frac{\Lambda}{|k^t|}, \]
assuming the top quark mass dominate the other fermion masses, then the potential will depend on \( \Lambda \) through an overall scale and the \( |k^t| \) dependence drops out
\[ V = \frac{3\Lambda^4}{\pi^2} \left( -2f_2\phi\phi + \frac{1}{2}f_0 (\phi\phi)^2 + \frac{1}{3}f_{-2} (\phi\phi)^3 + \cdots \right). \]
Now since \( \phi \) is a dimensionless doublet field, the vev
\[ \langle \phi \rangle = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
will have a numerical value that depends only on the coefficients \( f_2, f_0 \)
\[ v_0^2 = \frac{f_0}{2f_{-2}}, \]
and will be perturbed very slightly by the higher coefficients \( f_{-2}, f_{-4}, \cdots \), provided they decrease very rapidly. Looking at the minimum of the potential with the three terms above we have
\[ v^2 = \frac{f_0}{2f_{-2}} \left( -1 + \sqrt{1 + \frac{f_2f_{-2}}{f_0^2}} \right). \]
Thus the condition that the higher order term in the potential perturb the minimum \( v_0 \) slightly requires the condition
\[ f_{-2} \ll \frac{f_0}{8f_2}, \]
so that
\[ v^2 \simeq v_0^2 \left( 1 - 4\frac{f_2f_{-2}}{f_0^2} \right). \]
We can get a rough estimate of the coefficients \( f_0 \) and \( f_2 \) at unification scale by setting
\[ \frac{4f_2\Lambda^2}{\pi^2} = \frac{1}{2\kappa^2}, \quad \kappa = 4.2 \times 10^{-19}\text{Gev}^{-1} \]
which implies that
\[ f_2 \simeq \left( \frac{\pi^2}{8} \right) \left( \frac{1}{\kappa\Lambda} \right)^2. \]
Thus if \( \Lambda \) is of the order of \( M_{\text{Planck}} \) then \( f_2 \sim 1 \) while if \( \Lambda \sim 10^{17} \) then \( f_2 \sim 10^2 \). We also have
\[ \frac{f_0g_3^2}{2\pi^2} = \frac{1}{4}, \]
thus
\[ f_0 = \frac{\pi}{8\alpha_s} \sim 20, \quad \alpha_s = \frac{g_3^2}{4\pi} \]
at unification scale. Therefore we must have
\[ f_{-2} \ll \frac{10^2}{f_2} \]
and this can be anywhere between \( 10^2 \) and \( 10^{-2} \) depending whether \( \Lambda \) is at the Planck mass or two orders less.
We can now speculate on the form of the function $F(D^2) = f(D)$. This function must have rapidly decreasing Taylor coefficients (these are $F_0 = F(0)$, $F_{-2} = -F'(0)$, $F_{-4} = F''(0) \cdots$) while the Mellin coefficients $F_2, F_4$ should behave independently. The cut-off function can be approximated by the sequence $F_{\{N\}}(x)$

$$F_{\{N\}}(x) = A \left(1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{N!}x^N\right)e^{-x}$$

where

$$A \sim 20.$$ 

This function has the property that the first $N$ coefficients in the Taylor expansion vanish, and is thus a very good approximation to a cut-off function. A slightly perturbed form of this function is given by

$$F_{\{N\}}(x, \epsilon) = e^{-\epsilon x}F_{\{N\}}(x)$$

where $\epsilon \leq \pm 10^{-2}$. In this case, we have $f_{-2} = A\epsilon$, $f_{-4} = A\epsilon^2$. To have a feeling about this function we can plot $F_{\{10\}}(x, \epsilon)$

This shows that $\epsilon$ should be at least of order $10^{-2}$ to $10^{-3}$ in order not to disturb the cut-off function much, in the region where the scale is comparable to $\Lambda$. As seen from the plot, the function $F_N(x, \epsilon)$ is indistinguishable from $F_N(x)$ for $\epsilon \sim 10^{-3}$. From this we deduce that higher order terms in the heat kernel expansion will be suppressed by the Taylor coefficients of the function, and the perturbation can be trusted to within one order from the Planck scale. This property will insure that the initial conditions on the RG equations for the gauge coupling constant get modified. To see this, we have, to lowest order, the modification to the gauge
It remains to show that this form, for some value of $f$, can provide a mechanism for the unification of the three gauge couplings at some energy not far from the Planck scale. Similarly, the contributions to the Higgs potential are expected to modify the prediction of the Higgs mass [15]. The analysis of the running of the gauge coupling constants and the Higgs mass, taking these higher order terms into account is presently under study. We hope to report on this in the near future.

7. Spectral Action for Noncommutative Spaces with Boundary

In the Hamiltonian quantization of gravity it is essential to include boundary terms in the action as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action is well defined. The reason for this is that the curvature scalar $R$ contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. To see this note that the curvature $R \sim \partial \Gamma + \Gamma \partial$ where $\Gamma \sim g^{-1} \partial g$ has two parts, one part is of second order in derivatives of the form $g^{-1} \partial^2 g$ and the second part is the square of derivative terms of the form $\partial g \partial g$. To define the conjugate momenta in the Hamiltonian formalism, it is necessary to integrate by parts the term $g^{-1} \partial^2 g$ and change it to the form $\partial g \partial g$. These surface terms, which turned out to be very important, are canceled by modifying the Euclidean action to

$$I = \frac{1}{16\pi} \int_M d^4 x \sqrt{g} R - \frac{1}{8\pi} \int_{\partial M} d^3 x \sqrt{h} K,$$

where $\partial M$ is the boundary of $M$, $h_{ab}$ is the induced metric on $\partial M$ and $K$ is the trace of the second fundamental form on $\partial M$. Notice that there is a relative factor of 2 between the two terms, and that the boundary term has to be completely fixed.

This is a delicate fine tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand [16]. In contrast we can compute the spectral action for manifolds with boundary. The hermiticity of the Dirac operator

$$(\psi \mid D \psi) = (D \psi \mid \psi),$$

is satisfied provided that $\pi_- \psi_{|\partial M} = 0$ where $\pi_- = \frac{1}{2} (1 - \chi)$ is a projection operator on $\partial M$ with $\chi^2 = 1$. To compute the spectral action for manifolds with boundary we have to specify the condition $\pi_- D \psi_{|\partial M} = 0$. The result of the computation gives the remarkable result that the Gibbons-Hawking boundary term is
generated without any fine tuning [17]. Adding matter interactions, does not alter the relative sign and coefficients of these two terms, even when higher orders are included. The Dirac operator for a product space such as that of the standard model, must now be taken to be of the form

\[ D = D_1 \otimes \gamma_F + 1 \otimes D_F, \]

instead of

\[ D = D_1 \otimes 1 + \gamma_5 \otimes D_F, \]

because \( \gamma_5 \) does not anticommute with \( D_1 \) on \( \partial M \).

\section{8. Dilaton and the dynamical generation of scale}

Replacing the cutoff scale \( \Lambda \) in the spectral action, replacing \( f(D^2) \) by \( f(P) \) where \( P = e^{-\phi}D^2 e^{-\phi} \) modifies the spectral action with dilaton dependence to the form [18]

\[ \text{Tr } F(P) \approx \sum_{n=0}^{6} f_{4-n} \int dx \sqrt{g} e^{(4-n)\phi} a_n (x, D^2). \]

One can then show that the dilaton dependence almost disappears from the action if one rescales the fields according to

\[ G_{\mu\nu} = e^{2\phi} g_{\mu\nu}, \]
\[ H' = e^{-\phi} H, \]
\[ \psi' = e^{-\frac{3}{2}\phi} \psi. \]

With this rescaling one finds the result that the spectral action

\[ I (g_{\mu\nu}, H, \psi, \phi) = I (G_{\mu\nu}, H', \psi', \phi = 0) + \frac{24f_2}{\pi^2} \int dx \sqrt{GG} \partial_{\mu} \phi \partial_{\nu} \phi \]

is scale invariant (independent of the dilaton field) except for the kinetic energy of the dilaton field \( \phi \). The dilaton field has no potential at the classical level. It acquires a Coleman-Weinberg potential [19] through quantum corrections, and thus a vev and a very small mass. [20]. The Higgs sector in this case becomes identical with the Randall-Sundrum model [21]. In that model there are two branes in a five dimensional space, one located at \( x_5 = 0 \) representing the invisible sector, and another located at \( x_5 = \pi r_c \), the visible sector. The physical masses are set by the symmetry breaking scale \( v = v_0 e^{-kr_c \pi} \) so that \( m = m_0 e^{-kr_c \pi} \). If the bare symmetry breaking scale is taken at \( m_0 \sim 10^{19} \) Gev, then by taking \( kr_c \pi = 10 \) one gets the low-energy mass scale \( m \sim 10^2 \) Gev. It is not surprising that the Randall-Sundrum scenario is naturally incorporated in the noncommutative geometric model [22], [23], because intuitively one can think of the discrete space as providing the different right-handed and left-handed brane sectors.

\section{9. Speculations on the Structure of the Noncommutative Space and Quantum Gravity}

The small deviation from experimental results of the predictions of the standard model derived from the spectral action can have the following interpretation. This is an indication that the basic assumption we made about space-time as a product of a continuous four dimensional manifold times a discrete space breaks down at
energies just below the unification (Planck) scale. This will lead us to postulate that at Planckian energies, the structure of space time becomes noncommutative in a nontrivial way, which will change in an intrinsic way the particle spectrum. On the other hand, the encouraging results we obtained about the unique prediction of the spectrum of the standard model, the determination of the gauge group and for particle representations, can be taken as a guide that the true geometry should reproduce at lower energies, the product structure we assumed. The starting point is to look for a noncommutative space whose KO-dimension is ten (mod 8) and whose metric dimension is dictated by the growth of eigenvalues of the Dirac operator to be four. A good starting point would be to mesh in a smooth manner the four-dimensional manifold with the discrete space $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. The appearance of $4 \times 4$ matrices and their relation to a four-dimensional space-time may not be accidental. In particular, we can define the four-dimensional manifold through the following data. The $C^*$ algebra is generated by $M_2(\mathbb{H})$ and a projection $e = e^2 = e^*$ such that \[ \left\langle e - \frac{1}{2} \right\rangle = 0 \]
\[ \left\langle \left( e - \frac{1}{2} \right)[D, e]^{2n} \right\rangle = \begin{cases} 0, & n = 0, 1 \\ \gamma, & n = 2 \end{cases}, \]
where $\gamma$ is the chirality operator satisfying $\gamma^2 = \gamma, \quad \gamma = \gamma^*, \quad \gamma e = e \gamma, \quad D\gamma = -\gamma D$

The constraint on $e$ forces it to be of the form
\[
e = \begin{pmatrix}
\frac{1}{2} + t & 0 & \alpha & \beta \\
0 & \frac{1}{2} + t & -\beta^* & \alpha^* \\
\alpha^* & -\beta & \frac{1}{2} - t & 0 \\
\beta^* & \alpha & 0 & \frac{1}{2} - t
\end{pmatrix}
\]
where $t, \alpha, \alpha^*, \beta$ and $\beta^*$ all commute and satisfy the relation $t^2 + |\alpha|^2 + |\beta|^2 = \frac{1}{4}$.

One can then check that $A = C(S^4)$. The differential constraints are then satisfied by any Riemannian structure with a given volume form on $S^4$. This space can be deformed by considering the algebra to be generated by $M_4(\mathbb{C})$ and $e$ where \[ Q_{11} \quad Q_{12} \\
Q_{21} \quad Q_{22} \]
and each $Q$ is a $2 \times 2$ matrix of the form
\[
Q = \begin{pmatrix}
\alpha & \beta \\
-\lambda \beta & \alpha^*
\end{pmatrix}
\]
In this case the projection constraints imply
\[
e = \begin{pmatrix}
\frac{1}{2} + t & 0 & \alpha & \beta \\
0 & \frac{1}{2} + t & -\beta^* & \alpha^* \\
\alpha^* & -\lambda \beta & \frac{1}{2} - t & 0 \\
\beta^* & \alpha & 0 & \frac{1}{2} - t
\end{pmatrix}
\]
satisfying
\[
\alpha \alpha^* = \alpha^* \alpha, \quad \beta \beta^* = \beta^* \beta, \quad \alpha \beta = \lambda \beta \alpha, \quad \alpha^* \beta = \overline{\lambda} \beta \alpha
\]
giving rise to deformed $S^4$.

The idea now is to define the noncommutative space by marrying the concept of generating a manifold as instantonic solution of a set of equations, and to blend these with the finite space. We will report on this in the future.

10. Conclusions

We summarize the main assumptions made:

- Space-time is a product of a continuous four-dimensional manifold times a finite space.
- One of the algebras $M_4(\mathbb{C})$ is subject to symplectic symmetry reducing it to $M_2(\mathbb{H})$.
- The commutator of the Dirac operator with the center of the algebra is non trivial $[D, Z(A)] \neq 0$.
- The unitary algebra $U(A)$ is restricted to $SU(A)$.

These give rise to the following results:

- The number of fundamental fermions is 16.
- The algebra of the finite space is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
- The correct representations of the fermions with respect to $SU(3) \times SU(2) \times U(1)$ are derived.
- The Higgs doublet appears as part of the inner fluctuations of the metric, and spontaneous symmetry breaking mechanism appears naturally with the negative mass term without any tuning.
- Mass of the top quark of around 179 Gev.
- See-saw mechanism to give very light left-handed neutrinos.

The following problems are encountered:

- The unification of the gauge couplings with each other and with Newton constant do not meet at one point which is expected to be one order below the Planck scale.
- Mass of the Higgs field of around 170 Gev. This however, depends on the value of the gauge couplings at the unification scale, which is very uncertain.
- No new particles besides those of the Standard Model. This will be problematic if new physics is observed at LHC.
- No Explanation of the number of generations.
- No constraints on the values of the Yukawa couplings which are the non-zero entries in the Dirac operator of the finite space.

From these results we can deduce the following:

- It is necessary to include the higher order corrections to the spectral action using a convergent series for the heat kernel expansion. This step is now done, and shows clearly that the corrections cannot be ignored if the spectral function deviates even slightly from the cut-off function. What remains to be done is to input these corrections into the RG equations and prove that this mechanism does produce gauge couplings unification, and thus will enable us to get an accurate prediction for the Higgs mass.
- To get an insight on the problem of quantum gravity, it is essential to find the noncommutative space whose limit is the product $M_4 \times F$. We
speculated that this could be done by adopting the strategy of generating a continuous manifold through instantonic solutions of algebraic and differential constraints. This step has to be elaborated on and we must construct in detail the structure of such a space, to study its properties at the Planck scale and to show that the usual space-time can be recovered from the geometry of a non-trivial noncommutative space.

- The results obtained so far are very encouraging and we hope to report on future positive developments.

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