GLOBAL WELL-POSEDNESS AND LARGE TIME BEHAVIOR OF CLASSICAL SOLUTIONS TO THE DIFFUSION APPROXIMATION MODEL IN RADIATION HYDRODYNAMICS

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Abstract. We are concerned with the global well-posedness of the diffusion approximation model in radiation hydrodynamics, which describe the compressible fluid dynamics taking into account the radiation effect under the non-local thermal equilibrium case. The model consist of the compressible Navier-Stokes equations coupled with the radiative transport equation with non-local terms. Global well-posedness of the Cauchy problem is established in perturbation framework, and rates of convergence of solutions toward equilibrium, which are algebraic in the whole space and exponential on torus, are also obtained under some additional conditions on initial data. The existence of global solution is proved based on the classical energy estimates, which are considerably complicated and some new ideas and techniques are thus required. Moreover, it is shown that neither shock waves nor vacuum and concentration in the solution are developed in a finite time although there is a complex interaction between photons and matter.

1. Introduction. The aim of the radiation hydrodynamics is to include the radiation effects into hydrodynamics and the importance of thermal radiation in physical problems increases as the temperature is high. More precisely, the thermal effect usually varies as the fourth power of temperature. In this case, the radiation field significantly affects the dynamics of the field. This gives rise to the theory of radiation hydrodynamics, which is mainly concerned with the propagation of thermal radiation through a field or gas, and the effect of this radiation on the dynamics, see, for example, [1, 23, 29], and the references cited therein. The theory of radiation hydrodynamics finds a very broad range of applications, such as astrophysical, supernova explosions, laser fusion, and so on (cf. [17, 24, 29]). As in classical fluid mechanics, the equations of motion in radiation hydrodynamics are derived from the conservation laws for macroscopic quantities. However, due to the presence of
radiation, the classical “material” flow has to be coupled with radiation which is an assembly of photons (the photons are massless particles traveling at the speed of light) and need a priori a relativistic treatment. Hence, the whole problem to be considered is then a coupling between the standard hydrodynamics for the matter and a radiative transfer equation for the photon distribution. However, the equation of radiative transfer is very complicated, and hence, the physically valid approximate descriptions of radiative transfer have to be introduced (cf. [1]).

In this paper, we mainly consider the diffusion approximation (also are called the Eddington approximation), which is valid for optically thick regions where the photons emitted by the gas have a high probability of reabsorption within the region. The classical diffusion or Eddington approximation describes the energy flow due to radiative process in a semi-quantitative sense, and is particularly accurate if the specific intensity of radiation is almost isotropic (cf. [25]). Based on the standard hydrodynamics, the governing equations of the diffusion approximation in radiation hydrodynamics for 3-D flow of a viscous polytropic ideal heat-conducting radiative gas, can be written in terms of Euler coordinates as follows (see, Appendix and [1, 25]):

$$\rho_t + \text{div}(\rho u) = 0,$$

$$\rho(u_t + u \cdot \nabla u) + \nabla P = \mu \Delta u + (\lambda + \mu)\nabla \text{div} u,$$

$$c_v \rho(\theta_t + u \cdot \nabla \theta) = \kappa \Delta \theta - P \text{div} u + \lambda (\text{div} u)^2 + 2\mu D \cdot D - \theta^4 + n,$$

$$n_t - \Delta n = \theta^4 - n.$$

Here, the unknowns are \((\rho, u, \theta, n)\), where \(\rho = \rho(x, t) > 0\), \(\theta = \theta(x, t) > 0\), \(u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))\), for \(t \geq 0, x \in \Omega\) denote the mass density, temperature and velocity field of the fluid respectively, and \(n = n(x, t) \geq 0\) for \(t \geq 0, x \in \Omega\) denotes the radiation field. The spatial domain \(\Omega = \mathbb{R}^3\) or \(\mathbb{T}^3\). \(P = R\rho \theta\) is the material pressure, \(R, c_v, \kappa\) are positive constants; \(\lambda\) and \(\mu\) are the constant viscosity coefficients, \(\mu > 0\), \(3\lambda + 2\mu \geq 0\); \(D = D(u)\) is the deformation tensor

$$D_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad D \cdot D := \sum_{i,j=1}^{3} D_{ij}^2.$$

In this paper, we are interested in the global existence and asymptotics of smooth solutions of system (1)-(4) with the initial conditions:

\[ (\rho, u, \theta, n)|_{t=0} = (\rho_0, u_0, \theta_0, n_0)(x), \quad x \in \Omega. \]

The system (1)-(4), the conservation laws of mass, momentum, energy, and the radiative transfer equation, describe a non-equilibrium regime where the state of the radiation is determined by the transport equation (4). Since its physical importance, complexity, rich phenomena, and mathematical challenges, there is large of literature on the studies of radiation hydrodynamics from the mathematical/physical point of view, see, for example, [2, 4, 7, 11, 27] and [12, 13, 30]. Let us introduce some related mathematical result in radiation hydrodynamics. In [20], Lin, Coulombel and Goudon considered a situation where the gas is not in thermodynamical equilibrium with the radiation. They showed the existence of smooth traveling waves, called “shock profiles”, when the strength of shock is small. The governed system studied in [20] reads as follows:

$$\rho_t + (\rho u)_x = 0,$$
\[
\begin{align*}
(\rho u)_t + (\rho u^2 + P)_x &= 0, \\
(\rho E)_t + (\rho E u + Pu)_x &= n - \theta^4, \\
- n_{xx} &= \theta^4 - n.
\end{align*}
\]

which describes the interaction between an inviscid gas and photons. Here, \( P = R\rho\theta \) and \( E = e + \frac{u^2}{2} \). While, system (5) can be seen as an inviscid flow with no heat-conducting and stationary radiation field case of (1)-(4). In particular, (5) can be simplified to the radiating gases model [15, 16] as the following:

\[
\begin{align*}
u_t + \left(\frac{u^2}{2}\right)_x &= -q_x, \\
-q_{xx} + q &= -u_x,
\end{align*}
\]

which is indeed a hyperbolic-elliptic coupled system and can be recasted as:

\[
u_t + \left(\frac{u^2}{2}\right)_x = K * u - u
\]

by introducing the convolution operator

\[
K * u(x, t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} u(y, t) dy.
\]

The thorough study on (6) motivated a lot of works, see, for example, [19, 28] and the references cited therein. More models and results of radiation hydrodynamics can see [5, 6, 26].

Comparing the equations (1)-(4) with the systems (5)-(6), we find that the mathematical model problem (1)-(4) under the present consideration is more physically valid in radiation hydrodynamics, since the state of radiation is now described by the equation of radiative transfer and the effects of radiation on the full dynamics are completely taken into account.

Our main goal here is to establish the global existence of smooth solutions around a constant state \((1, 0, 1, 1)\), which is an equilibrium solution of system (1)-(4), and the decay rate of the global smooth solutions in time for the system (1)-(4). Therefore, it is natural to introduce the transforms

\[
\rho = 1 + \varrho, \quad \theta = 1 + \Theta, \quad n = 1 + \eta,
\]

and for simplicity of the presentation and without loss of generality, we assume the positive constants \( R = c_v = \kappa = \mu = \lambda \equiv 1 \). Then we can rewrite the system (1)-(4) as

\[
\begin{align*}
\varrho_t + (1 + \varrho)\text{div}\,u + \nabla \varrho \cdot u &= 0, \\
u_t + u \cdot \nabla u + \frac{1 + \Theta}{1 + \varrho} \nabla \varrho + \nabla \Theta &= \frac{\Delta u}{1 + \varrho} + \frac{2\text{div}\,u}{1 + \varrho}, \\
\Theta_t + u \cdot \nabla \Theta &= \frac{\Delta \Theta}{1 + \varrho} - (1 + \Theta)\text{div}\,u + \frac{(\text{div}\,u)^2}{1 + \varrho} + \frac{2D \cdot D}{1 + \varrho} - \frac{(1 + \Theta)^4}{1 + \varrho} + \frac{1 + \eta}{1 + \varrho}, \\
\eta_t - \Delta \eta &= (1 + \Theta)^4 - (1 + \eta).
\end{align*}
\]

with initial data

\[
(\varrho, u, \Theta, \eta)|_{t=0} = (\varrho_0(x), u_0(x), \Theta_0(x), \eta_0(x)) = (\rho_0(x) - 1, u_0(x), \theta_0(x) - 1, n_0(x) - 1)
\]

Then, the main results in this paper read as follows:
Theorem 1.1. Let $\Omega = \mathbb{R}^3$. Suppose that $\|q_0, u_0, \Theta_0, \eta_0\|_{H^3}$ is small enough. Then, the Cauchy problem \((7) - (11)\) admits a unique global solution $(q(x,t), u(x,t), \Theta(x,t), \eta(x,t))$ satisfying
\[
\begin{align*}
q, u, \Theta, \eta & \in C([0, \infty); H^3), \\
\sup_{t \geq 0} \|q, u, \Theta, \eta\|_{H^3} & \leq C\|q_0, u_0, \Theta_0, \eta_0\|_{H^3}.
\end{align*}
\]
Moreover, if $\|q_0, u_0, \Theta_0, \eta_0\|_{H^3} \cap L^1$ is sufficiently small then
\[
\|q, u, \Theta, \eta\|_{H^3} \leq C(1 + t)^{-\frac{2}{3}},
\]
for all $t \geq 0$.

Theorem 1.2. Let $\Omega = T^3$. Suppose that $\|q_0, u_0, \Theta_0, \eta_0\|_{H^3}$ is small enough, and
\[
\begin{align*}
\int_{T^3} q_0 dx = 0, \quad & \int_{T^3} (1 + q_0)u_0 dx = 0, \\
\int_{T^3} \left(\frac{1}{2}(1 + q_0)|u_0|^2 + q_0 + \Theta_0 + \rho_0 \Theta_0 + \eta_0\right) dx = 0
\end{align*}
\]
Then, the Cauchy problem \((7) - (11)\) admits a unique global solution $(q(x,t), u(x,t), \Theta(x,t), \eta(x,t))$ satisfying
\[
\begin{align*}
q, u, \Theta, \eta & \in C([0, \infty); H^3), \\
\sup_{t \geq 0} e^{\gamma t}\|q, u, \Theta, \eta\|_{H^3} & \leq C\|q_0, u_0, \Theta_0, \eta_0\|_{H^3}.
\end{align*}
\]
where $\gamma > 0$ is a constant.

The proof of Theorem 1.1 is based on the careful energy methods and the Fourier multiplier technique. The first key step is to establish the global a priori high order Sobolev’s energy estimates in time by using the careful energy methods. The second key step is to obtain the $L^p - L^q$ time decay rate of the linearized operator for the \((7) - (10)\) by using the Fourier technique. The third step of the proof is to obtain the time decay rate in Theorem 1.1 by combining the previous two steps and apply the energy estimate technique to the nonlinear problem, whose solutions can be represented by the solutions-semigroup operator for the linearized problem by using the Duhamel Principle. We notice that \((7) - (10)\) are Navier-Stokes coupled with parabolic type equations. When we establish both a priori energy estimates in step 1 and $L^p - L^q$ time decay rate in step 2, different from general compressible Navier-Stokes equations, in our system including some low-order terms like $\frac{1}{1+\rho}, \frac{\eta}{1+\eta}$ in \((9)\) and $4\Theta, \eta$ in \((10)\). There terms may lead to the solutions of which $H^3$ norm can not be control by the initial data. To overcome there difficulties, we shall modify some method motivated by \([21, 22]\) and \([8, 9]\) to deal with the compressible Navier-Stokes equations and other related models. Moreover, we need to construct some novel functionals which coorporate there low-order terms as a whole new one. We need to obtain more subtle energy estimates.

The remainder of this paper is organized as follows. In the next section, we derive the uniform-in-time a priori estimates and then establish the existence of global solution. In Section 3, we prove the rate of convergence of solutions. In Section 4, we adapt our proof to the periodic domain case. Through the paper, we use $\| \cdot \|$ to denote norm $L^2(\mathbb{R}^3)$, $C$ denotes a positive (generally large) constant and $\gamma$ a positive (generally small) constant, where both $C$ and $\gamma$ may take different values in different places. $A \sim B$ means $CA \leq B \leq \frac{1}{C}A$ for a generic constant $C > 0$. 
2. Global existence. In this section, we shall establish the global existence of classical solutions to Cauchy problem \([7]-[11]\) in the whole space \(\mathbb{R}^3\). At first, we give the uniform a priori estimates.

2.1. A priori estimates. We will show the uniform-in-time a priori estimates in the whole space \(\mathbb{R}^3\) under the assumption

\[
\sup_{t \geq 0} \|\varrho, u, \Theta, \eta\|_{H^3} \leq \delta,
\]

where \(0 < \delta < 1\) is a generic constant small enough and \((\varrho, u, \Theta, \eta)\) is the smooth solution to the Cauchy problem \([7]-[11]\) on \(0 \leq t < T\) for \(T > 0\). First, we introduce two useful lemma:

**Lemma 2.1.** (see [3]) There exist a positive constant \(C\), such that for any \(f, g \in H^3(\mathbb{R}^3)\) and any multi-index \(\alpha\) with \(1 \leq |\alpha| \leq 3\),

\[
\|f\|_{L^\infty(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{1/2},
\]

\[
\|fg\|_{H^2(\mathbb{R}^3)} \leq C \|f\|_{H^2(\mathbb{R}^3)} \|g\|_{H^2(\mathbb{R}^3)},
\]

\[
\|\partial^\alpha (fg)\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla f\|_{H^2(\mathbb{R}^3)} \|\nabla g\|_{H^2(\mathbb{R}^3)}.
\]

**Lemma 2.2.** (Moser-type calculus inequalities) (see [18]) Let \(s \geq 1\) be an integer. Suppose \(f \in H^s(\mathbb{R}^3), \nabla f \in L^\infty(\mathbb{R}^3)\) and \(g \in H^{s-1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\). Then for all multi-index \(\alpha\) with \(|\alpha| \leq s\), we have

\[
\|\partial^\alpha (fg) - f\partial^\alpha g\| \leq C_s \left( \|\nabla f\|_{L^\infty} \|D^{s-1} g\| + \|D^s f\| \|g\|_{L^\infty} \right),
\]

where

\[
\|D^s f\| = \sum_{|\alpha| = s} \|\partial^\alpha f\|.
\]

Then we begin to give the priori estimate of \(\varrho, u, \Theta, \eta\).

**Lemma 2.3.** For smooth solutions of the system \([7]-[11]\), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|\varrho\|^2 + 2\|u\|^2 + 2\|\Theta\|^2 + 2\|\eta\|^2 \right)
\]

\[
+ \gamma \left( \|\nabla u\|^2 + \|\text{div} u\|^2 + \|\nabla \Theta\|^2 + \|\nabla \eta\|^2 + 4\Theta - \eta \right)^2
\]

\[
\leq C \left( \|\varrho, u, \Theta, \eta\|_{H^2} + \|\varrho, u, \Theta, \eta\|_{H^3}^2 \right) \left( \|\nabla (\varrho, u, \Theta, \eta)\|^2 + \|\text{div} u\|^2 + 4\Theta - \eta \right)^2
\]

\[
(14)
\]

for any \(0 \leq t \leq T\) and any \(T > 0\) with \(C\) and \(\gamma\) not depending on \(T\).

**Proof.** Multiplying \([7]-[10]\) by \(4\varrho, 4u, 4\Theta\) and \(\eta\) respectively and then taking integration and summation, one has

\[
\frac{1}{2} \frac{d}{dt} \left( \|\varrho\|^2 + 2\|u\|^2 + 2\|\Theta\|^2 + 2\|\eta\|^2 \right)
\]

\[
+ \int \frac{4\|\nabla u\|^2}{1 + \varrho} \, dx + \int \frac{8\|\text{div} u\|^2}{1 + \varrho} \, dx + \int \frac{4\|\nabla \Theta\|^2}{1 + \varrho} \, dx + \|\nabla \eta\|^2 + 4\Theta - \eta \right)^2
\]

\[
= - \int 2\varrho^2 \text{div} u \, dx - \int 4\left( \Theta - \frac{\Theta}{1 + \varrho} \right) \nabla \varrho \cdot u \, dx - \int 4(u \cdot \nabla u) \cdot u \, dx - \int 4\Theta \nabla \Theta \cdot u \, dx
\]

\[
- \int 4\left( \nabla \frac{1}{1 + \varrho} \cdot \nabla u \right) \cdot u \, dx - \int 8\nabla \frac{1}{1 + \varrho} \cdot u \text{div} u \, dx - \int 4\Theta^2 \text{div} u \, dx
\]

\[
+ \int \frac{4\Theta(\text{div} u)^2}{1 + \varrho} \, dx + \int \frac{8\Theta D \cdot D}{1 + \varrho} \, dx - \int 4\Theta \nabla \frac{1}{1 + \varrho} \cdot \nabla \Theta dx + \int \frac{4\varrho(4\Theta - \eta)}{1 + \varrho} \, dx
\]
Using Hölder’s, Sobolev’s inequalities, for $I_1$ to $I_{11}$, we have

- $I_1 \leq C\|\varrho\|_{L^2} \|\text{div}u\|_{L^2} \|u\|_{L^6}$
- $I_3 \leq C\|\varrho\|_{H^1} \|\text{div}u\|_{L^2} \|\nabla \varrho\|_{L^2}$,
- $I_5 \leq C\|\varrho\|_{L^3} \|\varrho\|_{L^6} \|\text{div}u\|_{L^2}$,
- $I_7 \leq C\|\varrho\|_{H^1} \|\varrho\|_{L^2} \|\varrho\|_{L^6}$.

For the last two terms, under the assumption $[13]$, we have

- $I_{12} \leq C\|\varrho\|_{L^2} \left(\|\varrho\|_{L^3} \|\varrho\|_{L^6} + \|\varrho\|_{L^6} \|\varrho\|_{L^6} + \|\varrho\|_{L^6} \|\varrho\|_{L^6} \|\nabla \varrho\|_{L^2}\right)$
- $I_{13} \leq C\|\varrho\|_{L^3} \|\nabla \varrho\|_{L^2} \|\nabla \varrho\|_{L^2} \|\varrho\|_{L^6}$.

Then, $15$ follows by plugging all estimates above into $15$, and hence Lemma $2.3$ is proved.
Lemma 2.4. For smooth solutions of the system \([7], [11]\), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|2\partial^\alpha \varrho\|^2 + \|2\partial^\alpha u\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2 \right) + \gamma \sum_{1 \leq |\alpha| \leq 3} \left( \|\nabla \partial^\alpha u\|^2 + \|\div \partial^\alpha u\|^2 + \|\nabla \partial^\alpha \Theta\|^2 + \|\nabla \partial^\alpha \eta\|^2 + \|\partial^\alpha (4\Theta - \eta)\|^2 \right)
\leq C \left( \|\varrho, u, \Theta, \eta\|_{H^3} + \|\varrho, u, \Theta, \eta\|_{L^2}^2 \right) \left( \|\nabla (\varrho, u, \Theta, \eta)\|_{L^2}^2 + \|\div u\|_{L^2}^2 \right)
\]

(16)

for any \(0 \leq t \leq T\) and any \(T > 0\) with \(C\) and \(\gamma\) not depending on \(T\).

Proof. Applying \(\partial^\alpha\) with \(1 \leq |\alpha| \leq 3\) to \([7] - [10]\) and multiplying by \(4\partial^\alpha \varrho\), \(4\partial^\alpha u\), \(4\partial^\alpha \Theta\) and \(\partial^\alpha \eta\) respectively and then taking integration and summation, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|2\partial^\alpha \varrho\|^2 + \|2\partial^\alpha u\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2 \right) + \gamma \sum_{1 \leq |\alpha| \leq 3} \left( \|\nabla \partial^\alpha u\|^2 + \|\div \partial^\alpha u\|^2 + \|\nabla \partial^\alpha \Theta\|^2 + \|\nabla \partial^\alpha \eta\|^2 + \|\partial^\alpha (4\Theta - \eta)\|^2 \right)
\leq C \left( \|\varrho, u, \Theta, \eta\|_{H^3} + \|\varrho, u, \Theta, \eta\|_{L^2}^2 \right) \left( \|\nabla (\varrho, u, \Theta, \eta)\|_{L^2}^2 + \|\div u\|_{L^2}^2 \right)
\]

(16)
\[- \int \partial^\alpha \left( \frac{6 \Theta^2 + 4 \Theta^3 + \Theta^4}{1 + \varrho} \right) \partial^\alpha \Theta dx + \int \partial^\alpha \left( 6 \Theta^2 + 4 \Theta^3 + \Theta^4 \right) \partial^\alpha \varrho dx \equiv \sum_{j=1}^{27} I_j,\]

where \([A, B] = AB - BA\) for two operators \(A\) and \(B\), \(C_{\alpha, \beta}\) is constant depending only on \(\alpha\) and \(\beta\). Each term can be estimated as follows. For \(I_1, I_2, I_7, I_8, I_{14}, I_{18}\), with help of Lemma \([14]\) we obtain
\[
I_1 \leq C \| \varrho \| \left( \| \nabla \varrho \|_{L^\infty} \| D^2 \text{div} \varrho \| + \| D^3 \varrho \| \| \text{div} \varrho \|_{L^\infty} \right) \\
\leq C \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \text{div} \varrho \|_{H^2}.
\]

Similarly,
\[
I_2 \leq C \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2},
\]
\[
I_7 \leq C \| \varrho \|_{H^3} \left( \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^2} \| \nabla \Theta \|_{H^2} \right) + C \| \varrho \|_{H^3} \| \varrho \|_{H^3} \| \nabla \Theta \|_{H^2} \| \nabla \Theta \|_{H^2},
\]
\[
I_8 \leq C \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2}^2,
\]
\[
I_{14} \leq C \| \Theta \|_{H^3} \| \varrho \|_{H^2} \| \nabla \Theta \|_{H^2},
\]
\[
I_{18} \leq C \| \Theta \|_{H^3} \| \text{div} \varrho \|_{H^2} \| \nabla \Theta \|_{H^2}.
\]

For \(I_{10}\), we have
\[
I_{10} \leq C \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2,
\]
with \(\epsilon > 0\) a small constant, where the first inequality follows that for \(\beta < \alpha\),
\[
\int \partial^{\alpha - \beta} \left( \frac{1}{1 + \varrho} \right) \partial^\beta \Delta \varrho dx \leq \begin{cases} \\
|\| \partial^\alpha \left( \frac{1}{1 + \varrho} \right) \| \| \Delta \varrho \|_{L^\infty} \| \partial^\alpha \varrho \|, \quad (\| \beta \| = 0); \\
|\| \partial^\alpha - \beta \left( \frac{1}{1 + \varrho} \right) \| \| \partial^\beta \Delta \varrho \|_{L^2} \| \partial^\alpha \varrho \|, \quad (\| \beta \| = 1); \\
|\| \partial^\alpha - \beta \left( \frac{1}{1 + \varrho} \right) \| \| \partial^\beta \Delta \varrho \|_{L^\infty} \| \partial^\alpha \varrho \|, \quad (\| \beta \| = 2), \\
\end{cases}
\]
and Sobolev’s and Young’s inequalities were further used. Similarly, we have
\[
I_{12} \leq \epsilon \| \nabla \varrho \|_{H^2}^2 + C_\epsilon \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2,
\]
\[
I_{16} \leq \epsilon \| \partial^\alpha \Theta \|_{H^2}^2 + C_\epsilon \| \Theta \|_{H^2} \| \nabla \varrho \|_{H^2}^2,
\]
\[
I_{20} \leq C \| \Theta \|_{H^3} \| \nabla \varrho \|_{H^2} \| \text{div} \varrho \|_{H^2}^2 + C \| \Theta \|_{H^3} \| \text{div} \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2
\]
\[
\leq C \| \Theta \|_{H^3} \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \text{div} \varrho \|_{H^2} + C_\epsilon \| \Theta \|_{H^3} \| \text{div} \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2
\]
\[
I_{21} \leq C \| \Theta \|_{H^3} \| \nabla \varrho \|_{H^2} \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2
\]
\[
\leq C \| \Theta \|_{H^3} \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} + C_\epsilon \| \nabla \varrho \|_{H^2}^2 + C_\epsilon \| \Theta \|_{H^3} \| \nabla \varrho \|_{H^2}^2.
\]

Using Hölder’s, Soblev’s, Young’s inequalities and Lemma \([13]\) we easily get the following bounds
\[
I_1 + I_5 \leq C \| \text{div} \varrho \|_{L^\infty} \left( |\| \varrho \|_{H^2}^2 + |\| \varrho \|_{H^2}^2 + |\| \Theta \|_{H^2}^2 \right)
\]
\[
\leq C \| \varrho \|_{H^3} \left( \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^2}^2 \right),
\]
\[
I_4 \leq \epsilon \| \varrho \|_{L^\infty} \| \text{div} \varrho \|_{H^2} \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2 + C \| \varrho \|_{H^2} \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^2}^2,
\]
\[
I_5 \leq \epsilon \| \text{div} \varrho \|_{H^2}^2 + C_\epsilon \| \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2}^2 + C_\epsilon \| \nabla \varrho \|_{H^2} \| \Theta \|_{H^2}^2,
\]
\[
I_6 \leq C \| \varrho \|_{H^3} \| \varrho \|_{H^3} \| \nabla \varrho \|_{H^2} \| \nabla \varrho \|_{H^2} \| \Theta \|_{H^2}^2 + \epsilon \| \text{div} \varrho \|_{H^2}^2 + C_\epsilon \| \Theta \|_{H^2} \| \nabla \varrho \|_{H^2}^2,
\]
Using Hölder’s, Sobolev’s, Young’s inequalities, we obtain

\[ I_{11} \leq C \| \nabla \theta \|_{H^2} \| \nabla \partial^\alpha u \| \| \partial^\alpha u \| \leq \epsilon \| \nabla \partial^\alpha u \|^2 + C \| u \|_{H^3}^2 \| \nabla \theta \|_{H^2}^2, \]

\[ I_{13} \leq C \| \nabla \theta \|_{H^2} \| \text{div} \partial^\alpha u \| \| \partial^\alpha u \| \leq \epsilon \| \text{div} \partial^\alpha u \|^2 + C \| u \|_{H^3}^2 \| \nabla \theta \|_{H^2}^2, \]

\[ I_{17} \leq C \| \nabla \theta \|_{H^2} \| \nabla \partial^\alpha \theta \| \| \partial^\alpha \theta \| \leq \epsilon \| \nabla \partial^\alpha \theta \|^2 + C \| \theta \|_{H^3}^2 \| \nabla \theta \|_{H^2}^2, \]

\[ I_{19} \leq \epsilon \| \text{div} \partial^\alpha u \|^2 + C \| \theta \|_{H^3}^2 \| \nabla \theta \|_{H^2}^2, \]

\[ I_{24} + I_{25} \leq C \| \theta \|_{H^2} \| \nabla \theta \|_{H^2} \left( \| \nabla \theta \|_{H^2} + \| \nabla \eta \|_{H^2} \right). \]

For the last two terms, under the assumption (13), we get

\[ I_{26} \leq C \| \theta \|_{H^3} \| \nabla \theta \|_{H^2} \left( \| \nabla \theta \|_{H^2} + \| \nabla \theta \|_{H^2}^2 + \| \nabla \theta \|_{H^2}^3 \right) \]

\[ \leq C \| \theta \|_{H^3} \| \nabla \theta \|_{H^2} \| \nabla \theta \|_{H^2}, \]

\[ I_{27} \leq C \| \eta \|_{H^3} \left( \| \nabla \theta \|_{H^2}^2 + \| \nabla \theta \|_{H^2} + \| \nabla \theta \|_{H^2}^4 \right) \]

\[ \leq C \| \eta \|_{H^3} \| \nabla \theta \|_{H^2}^2. \]

Plugging these estimates into (17) and taking the sum over \( 1 \leq |\alpha| \leq 3 \), (16) follows and thus Lemma 2.4 is proved. \( \square \)

At last, we give the estimate of \( \nabla \theta \).

**Lemma 2.5.** For smooth solutions of the system (7), (11), we have

\[
\frac{d}{dt} \sum_{|\alpha| \leq 2} \int \nabla \partial^\alpha \theta \cdot \partial^\alpha u dx + \gamma \| \nabla \theta \|_{H^2}^2 \\
\leq C \left( \| \nabla u \|_{H^3}^2 + \| \nabla \theta \|_{H^2}^2 \right) \\
+ C \left( \| \theta, u, \theta, \eta \|_{H^3} + \| \theta, u, \theta, \eta \|_{H^2}^2 \right) \left( \| \nabla (\theta, u, \theta, \eta) \|_{H^2}^2 + \| \text{div} u \|_{H^2}^2 \right)
\]

for any \( 0 \leq t \leq T \) and any \( T > 0 \) with \( C \) and \( \gamma \) not depending on \( T \).

**Proof.** Taking differentiation \( \partial^\alpha (|\alpha| \leq 2) \) to (8), and carrying an direct calculation, we get

\[
\int |\nabla \partial^\alpha \theta|^2 dx = - \int \nabla \partial^\alpha \theta \cdot \partial^\alpha u dx - \int \nabla \partial^\alpha \theta \cdot \nabla u dx \\
- \int \nabla \partial^\alpha \theta \cdot \nabla \partial^\alpha \Theta dx + \int \nabla \partial^\alpha \theta \cdot \partial^\alpha \left( \frac{\Delta u}{1 + \theta} \right) dx \\
+ \int \nabla \partial^\alpha \theta \cdot \partial^\alpha \left( \nabla \text{div} u \right) 1 + \theta dx - \int \nabla \partial^\alpha \theta \cdot \partial^\alpha \left( \left( \frac{\Theta - \theta}{1 + \theta} \right) \nabla \theta \right) dx = \sum_{j=1}^{6} I_j.
\]

(19)

For \( I_1 \), applying (7), we have

\[
I_1 = - \frac{d}{dt} \int \nabla \partial^\alpha \theta \cdot \partial^\alpha u dx + \int \partial^\alpha \text{div} u \partial^\alpha u (1 + \theta) \text{div} u + \nabla \theta \cdot u dx \\
\leq - \frac{d}{dt} \int \nabla \partial^\alpha \theta \cdot \partial^\alpha u dx + C \| \text{div} \partial^\alpha u \|^2 + C \| \theta \|_{H^3} \| \nabla u \|_{H^2}^2.
\]

Using Hölder’s, Sobolev’s, Young’s inequalities, we obtain

\[ I_2 \leq C \| \theta \|_{H^3} \| \nabla \theta \|_{H^2}^2, \]

\[ I_3 \leq \frac{1}{4} \| \nabla \partial^\alpha \theta \|^2 + C \| \nabla \theta \|_{H^2}^2, \]

\[ I_4 + I_5 \leq C \| \theta \|_{H^3} \| \nabla \theta \|_{H^2} \| \nabla u \|_{H^2} + C \| \nabla \partial^\alpha \theta \| \| \nabla u \|_{H^2} \]

\[ \leq \frac{1}{4} \| \nabla \partial^\alpha \theta \|^2 + C \| \theta \|_{H^3} \| \nabla \theta \|_{H^2} \| \nabla u \|_{H^2} + C \| \nabla u \|_{H^2}^2, \]
estimates. In fact, define the temporal energy functional
\[
I_0 \leq C\|\nabla \varrho\|_{H^2}^2 \|\nabla \partial^\alpha \varrho\| + C\|\nabla \Theta\|_{H^2}^2 (\|\nabla \varrho\|_{H^2}^2 + \|\nabla \varrho\|_{H^2}^3 + \|\nabla \varrho\|_{H^2}^3) \|\nabla \partial^\alpha \varrho\|
\leq C\|\varrho\|_{H^3}^2 \|\nabla \varrho\|_{H^2}^2 + C\|\varrho\|_{H^3}^2 \|\nabla \Theta\|_{H^2} + C\|\varrho\|_{H^3}^2 \|\nabla \Theta\|_{H^2}^3.
\]
Putting these estimates into \([19]\) and taking the sum over \(|\alpha| \leq 2\) gives \([18]\), Lemma \([2.5]\) is proved.

2.2. **Proof of global existence.** It is now immediate to obtain the global a priori estimates. In fact, define the temporal energy functional
\[
\mathcal{E}(t) = \|2\varrho\|^2 + \|2u\|^2 + \|2\Theta\|^2 + \|\eta\|^2
+ \sum_{1 \leq |\alpha| \leq 3} (\|2\partial^\alpha \varrho\|^2 + \|2\partial^\alpha u\|^2 + \|2\partial^\alpha \Theta\|^2 + \|\partial^\alpha \eta\|^2) + \tau_1 \sum_{|\alpha| \leq 2} \int \nabla \partial^\alpha \varrho \cdot \partial^\alpha u dx,
\]
and the corresponding dissipation rate functional
\[
\mathcal{D}(t) = \|\nabla \varrho\|_{H^2}^2 + \|\nabla (u, \Theta, \eta)\|_{H^3}^2 + \|\text{div} u\|_{H^2}^2 + \|4\Theta - \eta\|_{H^3}^2,
\]
where \(0 < \tau_1 < 1\) are constant. Notice that since \(\tau_1 > 0\) are sufficiently small, under the assumption \([13]\), it holds that
\[
\mathcal{E}(t) \sim \|\varrho, u, \Theta, \eta(t)\|_{H^3}^2,
\]
uniformly for all \(0 < t < T\). Moreover, by suitably choosing constant \(\tau_1\), the sum of equations \([14], [16], \tau_1 \times [18]\) gives
\[
\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}(t) \leq C \left[ \mathcal{E}^{1/2}(t) + \mathcal{E}(t) \right] \mathcal{D}(t),
\]
for all \(0 \leq t < T\). By \([13]\), one has \(\mathcal{E}^{1/2}(t) + \mathcal{E}(t) \leq C (\delta + \delta^2)\). Thus, as long as \(0 < \delta < 1\) is small enough, the time integration of \([22]\) yields
\[
\mathcal{E}(t) + \gamma \int_0^t \mathcal{D}(s) ds \leq \mathcal{E}(0),
\]
for all \(0 \leq t < T\). Besides, \([13]\) can be justified by choosing
\[
\mathcal{E}(0) \sim \|\varrho_0, u_0, \Theta_0, \eta_0\|_{H^3},
\]
and \([23]\) holds true for all \(t \geq 0\).

3. **Time-decay of solutions.** In this section, so as to obtain the time-decay rates of solutions to the nonlinear system \([7]-[10]\), firstly, we consider the following initial problem on the linearized homogeneous equations corresponding to system \([7]-[10]\):
\[
\varrho_t + \text{div} u = 0,
\]
\[
u_t + \nabla \varrho + \nabla \Theta - \Delta u - 2\nabla \text{div} u = 0,
\]
\[
\Theta_t - \Delta \Theta + \text{div} u + 4\Theta - \eta = 0,
\]
\[
\eta_t - \Delta \eta + \eta - 4\Theta = 0,
\]
with initial data
\[
(\varrho, u, \Theta, \eta)|_{t=0} = (\varrho_0, u_0, \Theta_0, \eta_0)(x), \quad x \in \mathbb{R}^3.
\]
In this section, we use $U(t) = (\rho(t), u(t), \Theta(t), \eta(t))$ to denote the solution of system (24)-(28), and denote $U_0 = (\rho_0, u_0, \Theta_0, \eta_0)$. Define $\mathcal{A}(t)$ to be the solution operator of (24)-(28), then, $U(t)$ can be presented as

$$U(t) = \mathcal{A}(t)U_0.$$ 

We can now show the following uniform estimates on $U(t)$.

**Theorem 3.1.** Let $1 \leq q \leq 2$. For any $\alpha, \alpha'$ with $\alpha' \leq \alpha$ and $m = |\alpha - \alpha'|$, 

$$\|\partial^\alpha \mathcal{A}(t)U_0\|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(1-\frac{1}{q})-\frac{m}{2}} \left(\|\partial^\alpha' U_0\|_{L^q} + \|\partial^\alpha U_0\|_{L^2}\right),$$ 

(29)

hold for all $t \geq 0$.

**Proof.** By Fourier transforming (24)-(27) in $x$, one has

$$\hat{u}_t + ik \cdot \hat{u} = 0,$$  

(30)

$$\hat{u}_t + ik\hat{\varrho} + i\hat{k}\Theta + |k|^2 \hat{u} + 2k(k \cdot \hat{u}) = 0,$$  

(31)

$$\hat{\Theta}_t + |k|^2 \hat{\Theta} + ik \cdot \hat{u} + 4\hat{\Theta} - \hat{\eta} = 0,$$  

(32)

$$\hat{\eta}_t + |k|^2 \hat{\eta} + \hat{\eta} - 4\hat{\Theta} = 0,$$  

(33)

where $k \in \mathbb{R}^3$, $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit.

Multiplying (30)-(33) by $4\hat{\varrho}, 4\hat{u}, 4\hat{\Theta}$ and $\hat{\eta}$ respectively, its real part gives

$$\partial_t \left(\sqrt{2}\hat{\varrho}, \sqrt{2}\hat{u}, \sqrt{2}\hat{\Theta}, \hat{\eta}\right)^2 + 4|k|^2 \hat{u}^2 + 8|k \cdot \hat{u}|^2 + 4|k|^2 |\hat{\Theta}|^2 + |k|^2 |\hat{\eta}|^2 + 4|\hat{\Theta} - \hat{\eta}|^2 = 0.$$  

(34)

Multiplying (31) by $ik\hat{\varrho}$, utilizing integration by parts in $t$ and replacing $\partial_t \hat{\varrho}$ by (30), we have

$$\partial_t (\hat{u} |ik\hat{\varrho} + |k|^2 \hat{\varrho}|^2 = |k \cdot \hat{u}|^2 + 3|k|^2 ik \cdot \hat{u} \hat{\varrho} - |k|^2 \hat{k} \hat{\Theta} \hat{\varrho},$$  

(35)

here $(\cdot, \cdot)$ means the complex inner product. Then, taking the real part of (35) and utilizing the Cauchy-Schwarz inequality, one has

$$\partial_t \text{Re}(\hat{u} |ik\hat{\varrho}) + |k|^2 \hat{\varrho}^2 \leq |k \cdot \hat{u}|^2 + \epsilon |k|^2 |\hat{\varrho}|^2 + C\epsilon |k|^2 |\hat{\varrho}|^2 + 4k^2 |\hat{\Theta}|^2$$  

with $\epsilon > 0$ a small constant. Multiplying it by $\frac{1}{1 + |k|^2}$, we conclude that there exist $\gamma > 0$ such that

$$\partial_t \text{Re}(\hat{u} |ik\hat{\varrho}) + \gamma |k|^2 |\hat{\varrho}|^2 \leq C |k \cdot \hat{u}|^2 + \frac{C|k|^2 |\hat{\Theta}|^2}{1 + |k|^2}.$$  

(36)

Now, as in (29), we define

$$\mathcal{E}(\hat{U}(t,k)) = \left|\sqrt{2}\hat{\varrho}, \sqrt{2}\hat{u}, \sqrt{2}\hat{\Theta}, \hat{\eta}\right|^2 + \tau_2 \frac{\text{Re}(\hat{u} |ik\hat{\varrho})}{1 + |k|^2},$$

where $0 < \tau_2 \ll 1$ are constant. It is also immediate to verify that $\mathcal{E}(\hat{U}) \sim |\hat{U}|^2$. Moreover, by suitably choosing constant $\tau_2$, the sum of equations (34), $\tau_2 \times (36)$ gives

$$\partial_t \mathcal{E}(\hat{U}(t,k)) + \frac{\gamma |k|^2}{1 + |k|^2} \mathcal{E}(\hat{U}(t,k)) \leq 0.$$  

As in [10,13], the desired time decay estimates (29) directly follows from the above estimate, and the detailed proof is omitted for brevity. 

We now need two technical lemmas for the later proof.
Lemma 3.2. (see [3]) Given any $0 < \beta_1 \neq 1$ and $\beta_2 > 1$,
\[
\int_0^t (1 + t - s)^{-\beta_1} (1 + s)^{-\beta_2} \, ds \leq C (1 + t)^{-\min\{\beta_1, \beta_2\}}
\]
for all $t \geq 0$.

Lemma 3.3. Let $\lambda > 1$ and $g_1, g_2, g_3 \in C^0(\mathbb{R}_+, \mathbb{R}_+, \mathbb{R}_+)$ with $g_1(0) = 0$. For $A \in \mathbb{R}_+$, define $B_A := \{ y \in C^0(\mathbb{R}_+, \mathbb{R}_+, \mathbb{R}_+) \mid y \leq A + g_1(A)y + g_2(A)y^2 + g_3(A)y^\lambda, y(0) \leq A \}$. Then, there exists a constant $A_0 \in (0, \min\{A_1, A_2, A_3\})$ such that for any $0 < A \leq A_0$,
\[
y \in B_A \Rightarrow \sup_{t \geq 0} y(t) \leq 2A,
\]
where $A_i = \sup g_i > 0$, $i = 1, 2, 3$ are small enough.

Proof. One can fix $A_0 \in (0, \min\{A_1, A_2, A_3\})$ such that
\[
\sup_{0 \leq A \leq A_0} \left[ g_1(A) + g_2(A)(2A) + g_3(A)(2A)^{\lambda-1} \right] \leq \frac{1}{3}
\]
due to $\lambda > 1$ and assumptions on functions $g_i(\cdot)$, $i = 1, 2, 3$. Take $0 < A \leq A_0$. Define
\[
t_* = \sup \left\{ t \geq 0 \text{ such that } \sup_{0 \leq s \leq t} y(s) \leq 2A \right\}.
\]
Notice $t_* > 0$ since $y(0) \leq A$ and $y(t)$ is continuous. We claim $t_* = \infty$. Otherwise, $t_* > 0$ is finite. Thus, from the definition of $t_*$, $y(t_*) = 2A$ and $y(t) \leq 2A$ for any $0 \leq t \leq t_*$. The latter implies that for $0 \leq t \leq t_*$,
\[
y(t) \leq A + g_1(A)y(t) + g_2y^2(t) + g_3(A)y^\lambda(t) \\
\leq A + \sup_{0 \leq A \leq A_0} \left[ g_1(A) + g_2(A)(2A) + g_3(A)(2A)^{\lambda-1} \right] y(t),
\]
which by the choice of $A_0$, further gives
\[
\sup_{0 \leq t \leq t_*} y(t) \leq \frac{A}{1 - \sup_{0 \leq A \leq A_0} \left[ g_1(A) + g_2(A)(2A) + g_3(A)(2A)^{\lambda-1} \right]} \leq \frac{3}{2} A < 2A.
\]
This is a contradiction to $y(t_*) = 2A$. Therefore, $t_* = \infty$ follows. 

Now, we continue to proof the rate of convergence. We can rewrite the nonlinear Cauchy problem \([7], [11]\) as
\[
U(t) = \mathcal{A}(t)U_0 + \int_0^t \mathcal{A}(t-s)(G_1, G_2, G_3, G_4) \, ds 
\]
with
\[
G_1 = -\varrho \text{div} u - \nabla \varrho \cdot u, \\
G_2 = -u \cdot \nabla u - \frac{\Theta - \varrho}{1 + \varrho} \nabla \varrho - \frac{\varrho}{1 + \varrho} \Delta u - \frac{2\varrho}{1 + \varrho} \nabla \text{div} u, \\
G_3 = -\frac{\varrho}{1 + \varrho} \Delta \Theta - \Theta \text{div} u + \left( \frac{\text{div} u}{1 + \varrho} \right)^2 + \frac{2D \cdot D}{1 + \varrho} - \frac{\varrho \Theta}{1 + \varrho} + \frac{4\varrho}{1 + \varrho} - \frac{6\Theta^2 + 4\Theta^3 + \Theta^4}{1 + \varrho}, \\
G_4 = 6\Theta^2 + 4\Theta^3 + \Theta^4.
\]
To estimate $\|U(t)\|_{L^2}$, we further rewrite \([37]\) as
\[
U(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t),
\]
Therefore, it follows that
\[ J_1(t) = \mathcal{A}(t)U_0, \]
\[ J_2(t) = \int_0^t \mathcal{A}(t-s)(G_1, 0, 0, 0)ds, \]
\[ J_3(t) = \int_0^t \mathcal{A}(t-s)(0, G_2, 0, 0)ds, \]
\[ J_4(t) = \int_0^t \mathcal{A}(t-s)(0, 0, G_3, 0)ds, \]
\[ J_5(t) = \int_0^t \mathcal{A}(t-s)(0, 0, 0, G_4)ds. \]

Define
\[ \mathcal{E}_\infty(t) = \sup_{0 \leq s \leq t} (1 + s)^{\frac{3}{2}} \mathcal{E}(s). \]

One has that from (29) and Lemma 3.2
\[ \| J_1(t) \|_{L^2} \leq C(1 + t)^{-\frac{3}{2}} \| U_0 \|_{L^2} \cap L^1, \]
\[ \| J_2(t) \|_{L^2} + \| J_3(t) \|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \left( \| G_1 \|_{L^2} \cap L^1 + \| G_2 \|_{L^2} \cap L^1 \right) ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} ds \mathcal{E}_\infty(t) \]
\[ \leq C(1 + t)^{-\frac{3}{2}} \mathcal{E}_\infty(t), \]
\[ \| J_4(t) \|_{L^2} + \| J_5(t) \|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \left( \| G_3 \|_{L^2} \cap L^1 + \| G_4 \|_{L^2} \cap L^1 \right) ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} [ \mathcal{E}(s) + \mathcal{E}^2(s) ] ds \]
\[ \leq C(1 + t)^{-\frac{3}{2}} [ \mathcal{E}_\infty(t) + \mathcal{E}^2_\infty(t) ]. \]

Therefore, it follows that
\[ \| U \|_{L^2}^2 \leq C(1 + t)^{-\frac{3}{2}} \left\{ \| U_0 \|_{L^2}^2 \cap L^1 + \mathcal{E}_\infty^2(t) + \mathcal{E}^4_\infty(t) \right\}. \]  

(38)

On the other hand, by the definitions of \( \mathcal{E}(t) \) and \( D(t) \) in (20) and (21)
\[ \mathcal{E}(t) \leq C \left( D(t) + \| U \|_{L^2}^2 \right). \]

From (22), we have
\[ \frac{d}{dt} \mathcal{E}(t) + \frac{\gamma}{C} \left( \mathcal{E}(t) - \| U \|_{L^2}^2 \right) \leq \frac{d}{dt} \mathcal{E}(t) + \gamma D(t) \leq 0, \]
which implies
\[ \frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{E}(t) \leq C \| U \|_{L^2}^2. \]  

(39)

Then, by Gronwall’s inequality, Lemma 3.2 (38) and (39), we get
\[ \mathcal{E}(t) \leq e^{-\gamma t} \mathcal{E}(0) + C(1 + t)^{-\frac{3}{2}} \left\{ \| U_0 \|_{L^2}^2 \cap L^1 + \mathcal{E}_\infty^2(t) + \mathcal{E}^4_\infty(t) \right\}. \]
and hence
\[ \mathcal{E}_\infty(t) \leq C \left\{ \|U_0\|_{H^3}^2 \{L^1 + \mathcal{E}_\infty^2(t) + \mathcal{E}_\infty^4(t) \} \right\} \]
Thus, since \( \|U_0\|_{H^3}^2 \{L^1 \) can be small enough, by Lemma 3.3 one has
\[ \mathcal{E}_\infty(t) \leq C\|U_0\|_{H^3}^2 \{L^1 \]
for all \( t \geq 0 \), that is,
\[ \mathcal{E}(t) \leq C(1 + t)^{-\frac{3}{2}} \|U_0\|_{H^3}^2 \{L^1 \]
which means
\[ \|\varrho, u, \Theta, \eta\|_{H^3} \leq C(1 + t)^{-\frac{3}{2}}, \]
for all \( t \geq 0 \). This completes the proof of Theorem 1.1.

4. The periodic case. In this section, we consider the spatial domain \( \Omega = \mathbb{T}^3 \).

It is easy to check that for smooth solution of the system \([1]-[4]\), the following quantities are conserved:
\[ \frac{d}{dt} \int_{\mathbb{T}^3} \rho dx = 0, \quad \frac{d}{dt} \int_{\mathbb{T}^3} \rho u dx = 0, \quad \frac{d}{dt} \int_{\mathbb{T}^3} (\frac{1}{2} \rho |u|^2 + \rho \Theta + n) dx = 0, \]
and by the assumption \([12]\), it follows that
\[ \int_{\mathbb{T}^3} \rho dx = 0, \quad \int_{\mathbb{T}^3} (1 + \varrho) u dx = 0, \quad \int_{\mathbb{T}^3} (\frac{1}{2} (1 + \varrho) |u|^2 + \varrho + \Theta + \varrho \Theta + \eta) dx = 0, \]
for all \( t \geq 0 \).

Now, we begin to prove Theorem 1.2. Here, we only give the proof of the global a priori estimates. First, we need to find out the zero-order dissipation of \((\varrho, u, \Theta, \eta)\), by using the conservation law \([40]\) and with the help of the Poincaré inequality, we have
\[ \|\varrho\|_{L^2} \leq C\|\nabla \varrho\|_{L^2}, \quad (41) \]
\[ \|u\|_{L^2} \leq \|u + \varrho u\|_{L^2} + \|\varrho u\|_{L^2} \leq C\|\nabla u\|_{L^2} + C\|\nabla (\varrho u)\|_{L^2} + \|u\|_{L^\infty}\|\varrho\|_{L^2} \]
\[ \leq C\|\nabla u\|_{L^2} + C\|u\|_{H^2}\|\nabla \varrho\|_{L^2} + C\|\varrho\|_{H^2}\|\nabla u\|_{L^2}, \quad (42) \]
and
\[ \|\Theta + \eta\|_{L^2} \leq \|\frac{1}{2} (1 + \varrho) |u|^2 + \varrho + \Theta + \varrho \Theta + \eta\|_{L^2} + \|\frac{1}{2} (1 + \varrho) |u|^2 + \varrho + \Theta\|_{L^2} \]
\[ \leq C\|u\|_{H^2}\|\nabla u\|_{L^2} + C\|\varrho\|_{L^2}\|u\|_{H^2}\|\nabla u\|_{L^2} + C\|u\|_{H^2}\|\nabla \varrho\|_{L^2} + C\|u\|_{H^2}\|\nabla \Theta\|_{L^2} + C\|\Theta\|_{H^2}\|\nabla \Theta\|_{L^2} + C\|\eta\|_{H^2}\|\nabla \Theta\|_{L^2}, \]
\[ \quad (43) \]
Let the temporal energy functional \( \mathcal{E}(t) \) and the corresponding dissipation rate functional \( \mathcal{D}(t) \) be defined in the same way as in \([20]\), \([21]\) for the case of the whole space \( \Omega = \mathbb{R}^3 \). Then, the similar process by making the energy estimates lead to
\[ \frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{D}(t) \leq C (\|\varrho, u, \Theta, \eta\|_{H^3} + \|\varrho, u, \Theta, \eta\|_{H^3}^2) \left( \|\nabla (\varrho, u, \Theta, \eta)\|_{L^2}^2 + \|\text{div}u\|_{L^2}^2 \right). \]
\[ \quad (44) \]
Define
\[ \mathcal{D}_\infty(t) = \mathcal{D}(t) + \tau_3 (\|\varrho\|_{L^2}^2 + \|u\|_{L^2}^2) + \tau_4 (\|\Theta + \eta\|_{L^2}^2), \]
\[ \quad (45) \]
where \(0 < \tau_3, \tau_4 \ll 1\) are constants. Notice
\[
D_T(t) \sim \|\theta\|^2_{H^3} + \|u, \Theta, \eta\|^2_{H^4},
\]
uniformly for all \(t \geq 0\). Moreover, by choosing \(0 < \tau_3, \tau_4 \ll 1\) suitably small, it follows from (41)-(44), we conclude that
\[
\frac{d}{dt} \mathcal{E}(t) + \gamma D_T(t) \leq C \left[ \mathcal{E}^{1/2}(t) + \mathcal{E}(t) + \mathcal{E}^2(t) \right] D_T(t).
\]
In fact, \(\mathcal{E}(t)\) is small enough uniformly in time, which implies
\[
\frac{d}{dt} \mathcal{E}(t) + \gamma D_T(t) \leq 0.
\]
Since \(\mathcal{E}(t) \leq CD_T(t)\), we get
\[
\frac{d}{dt} \mathcal{E}(t) + \gamma \mathcal{E}(t) \leq 0 \quad (45)
\]
for all \(t \geq 0\). Applying Gronwall’s inequality to (45), one has
\[
\mathcal{E}(t) \leq e^{-\gamma t} \mathcal{E}(0),
\]
which gives the desired exponential decay of \(\mathcal{E}(t) \sim \|g, u, \Theta, \eta\|^2_{H^3}\). The proof of Theorem 1.2 is complete.

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**Appendix.** We use \(I(x, t, \nu, \Omega)\) to denote the specific intensity of radiation (at time \(t\)) at spatial point \(x \in \mathbb{R}^3\), with frequency \(\nu > 0\) in a direction \(\Omega \in S^2\), the system of radiation hydrodynamics with viscous fluid consists of the following radiative transport equation:
\[
\frac{1}{c} \frac{\partial I(\nu, \Omega)}{\partial t} + \Omega \cdot \nabla I(\nu, \Omega) = S(\nu) - \sigma_s(\nu) I(\nu, \Omega)
\]
\[
+ \int_0^\nu \int_{S^2} \left( \frac{\nu}{\nu'} \sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega) I(\nu', \Omega') - \sigma_s(\nu \rightarrow \nu', \Omega \cdot \Omega') I(\nu, \Omega) \right) d\Omega',
\]
and the compressible Navier-Stokes equations:
\[
\rho_t + \text{div} (\rho u) = 0,
\]
\[
(\rho u + F_r)_t + \nabla P_m + \text{div} (\rho u \otimes u + F_r) = \text{div} \mathcal{S},
\]
\[
\left( \frac{1}{2} \rho u^2 + E_m + E_r \right)_t + \text{div} \left( \left( \frac{1}{2} \rho u^2 + E_m + P_m \right) u + F_r \right) = \text{div} (\mathcal{S} u + \kappa \nabla \theta),
\]
where \(c\) is the light speed, \(S(\nu) = S(x, t, \nu, \Omega)\) is the rate of energy emission due to spontaneous processes, and \(\sigma_s(\nu)\) is the absorption coefficient. Similar to absorption, a photon can undergo scattering interactions with matter, and the scattering interactions change the photon’s characteristics \(\nu'\) and \(\Omega'\) to a new set of characteristics \(\nu\) and \(\Omega\), which leads to the definition of the “differential scattering coefficient” \(\sigma_s(x, t, \nu' \rightarrow \nu, \Omega' \cdot \Omega, \rho, \theta)\). In the Navier-Stokes equations (47), \(\rho = \rho(x, t)\) is the density, \(u = (u_1, u_2, u_3)\) is the velocity, \(\theta = \theta(x, t)\) is the temperature, \(E_m(\rho, \theta)\) is the energy, \(P_m = P_m(\rho, \theta)\) is the pressure, \(\kappa\) is the heat-conducting coefficient of the fluid, and the symbol \(\mathcal{S}\) stands for the viscous stress tensor
\[
\mathcal{S} = \lambda (\text{div} u) I d + \mu (\nabla u + (\nabla u)^t),
\]
\( \lambda \) and \( \mu \) are the viscosity coefficients of the fluid satisfying \( 2\lambda + \mu > 0 \). \( E_r, F_r \) and \( P_r \) represent the radiative energy density, radiative flux and the radiative pressure respectively defined by

\[
E_r = \frac{1}{c} \int_0^\infty d\nu \int_{S^2} I(\nu, \Omega) d\Omega,
\]

\[
F_r = \int_0^\infty d\nu \int_{S^2} \Omega I(\nu, \Omega) d\Omega,
\]

\[
P_r = \frac{1}{c} \int_0^\infty d\nu \int_{S^2} \Omega \otimes \Omega I(\nu, \Omega) d\Omega.
\]

(48)

By using (46) and (48), (47) can rewrite as

\[
\rho_t + \text{div}(\rho u) = 0,
\]

\[
(\rho u)_t + \nabla P_m + \text{div} (\rho u \otimes u) = \text{div} \mathcal{S} - \frac{1}{c} \int_0^\infty \int_{S^2} F(x, t, \nu, \Omega) d\Omega,
\]

\[
\left( \frac{1}{2} \rho u^2 + E_m \right)_t + \text{div} \left( \frac{1}{2} \rho u^2 + E_m + P_m \right) u = \text{div}(\mathcal{S} u + \kappa \nabla \theta) - \int_0^\infty \int_{S^2} F(x, t, \nu, \Omega) d\Omega,
\]

(49)

where

\[
F(x, t, \nu, \Omega) = S(\nu) - \sigma'_s(\nu) I(\nu, \Omega)
\]

\[
+ \int_0^\nu d\nu' \int_{S^2} \left( \frac{\nu'}{\nu} \sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega) I(\nu', \Omega') - \sigma_s(\nu \rightarrow \nu', \Omega' \cdot \Omega) I(\nu, \Omega) \right) d\Omega'.
\]

We notice that (46)-(47) are integro-differential equations, and have complex structure. It is so difficult to solve both numerically and analytically. Practical simplified models are introduced in some physical regions. From the physical and numerical points of view, these models can approximate the general equations of radiation hydrodynamics (46)-(47) very well in some particular physical situations. In particular, non-local thermodynamics equilibrium (non-LTE) assumption and the Eddington approximation or diffusion approximation model are mainly studied in this paper.

First we introduce the non-LTE assumption, we consider the manifestation in the equation of transfer of the quantum statistics (i.e. (46)) obeyed by photons. Since photons are bosons, both the processes of emissions and scattering are enhanced by the number of photons already in the final state following the interaction. This enhancement is generally referred to as resulting from “induced processes”. The quantitative statement of this enhancement is simply stated as: If \( Z \) represents the basic probability of a photon event (emission or scattering, i.e \( S(\nu) \) or \( \sigma_s(\nu) \)) then, due to induced effects, the actual probability \( Z' \) is given by

\[
Z' = Z(1 + \psi),
\]

where \( \psi \) is the number of photons in the final state of the transition. In “induced processes” case,

\[
\psi = \frac{c^2}{2h\nu^3} I(\nu, \Omega),
\]

and thus

\[
Z' = Z \left( 1 + \frac{c^2 I}{2h\nu^3} \right),
\]
where \( h \) is the Planck constant. To see the effect of the non-LTE assumption on the equation (46), it is convenient to eliminate \( S, \sigma_a' \) in (46) in favor of \( B \) and \( \sigma_a \) defined by the relationships

\[
S(\nu) = \sigma_a B(\nu), \quad \sigma_a'(\nu) = \sigma_a \left( 1 + \frac{c^2 B(\nu)}{2h\nu^3} \right),
\]

and assume that \( \sigma_s = 0 \), where \( B \) is Planck function, which is describing the (isotropic) specific intensity of radiation in the case of thermal equilibrium (cf. [11, 25]), defined by

\[
B(\theta) := B(\theta, \nu) = \frac{\alpha \nu^3}{\beta \nu / \theta - 1},
\]

where \( \alpha \) and \( \beta \) are some positive physical constants. Thus, from the “induced processes” and the non-LTE assumption together, \( S(\nu), \sigma_a'(\nu) \) in (46) can be written as

\[
S(\nu) = \sigma_a B(\nu) \left( 1 + \frac{c^2 I(\nu, \Omega)}{2h\nu^3} \right),
\]

\[
\sigma_a'(\nu) = \sigma_a \left( 1 + \frac{c^2 B(\nu)}{2h\nu^3} \right).
\]

Then, we can rewrite (46) and (49) as

\[
\rho_t + \text{div}(\rho u) = 0,
\]

\[
(\rho u)_t + \nabla P_m + \text{div}(\rho u \otimes u) = \text{div}S - \frac{1}{c} \int S \sigma_a (B(\nu) - I) d\Omega,
\]

\[
\left( \frac{1}{2} \rho u^2 + E_m \right)_t + \text{div} \left( \frac{1}{2} \rho u^2 + E_m + P_m \right) u = \text{div}(Su + \kappa \nabla \theta) - \int \sigma_a (B(\nu) - I) d\Omega,
\]

\[
\frac{1}{c} I_t + \Omega \cdot \nabla I = \sigma_a (B(\nu) - I).
\] (50)

The basic assumption underlying the classical diffusion or Eddington description of transfer is that the angular dependence of the specific intensity can be represented by the first two terms in aspherical harmonic expansion. That is, it is assumed that

\[
I(x, t, \nu, \Omega) = \frac{1}{4\pi} I_0(x, t, \nu) + \frac{3}{4\pi} \Omega \cdot I_1(x, t, \nu).
\] (51)

Integration of (51) over all solid angle gives

\[
I_0(x, t, \nu) = \int_{4\pi} I(x, t, \nu, \Omega) d\Omega,
\]

and multiplication of (51) by \( \Omega \) to a similar integration yields

\[
I_1(x, t, \nu) = \int_{4\pi} \Omega I(x, t, \nu, \Omega) d\Omega.
\]

The truncated spherical harmonic representation, (51), is only strictly valid if \( |I_1| \ll I_0 \), and \( I_0, I_1 \) satisfied the following Fick’s Law:

\[
I_1(x, t, \nu) = -D \nabla I_0(x, t, \nu),
\]

where \( D \) is the diffusion coefficient. In this paper, we consider a physical case that the energy transfer of photons to plasmas (with subcritical density) completely dominates a process. Therefore, the effect of radiation on the momentum could
be neglected in the process. Let \( n := n(x, t) = \int_0^\infty I_0(x, t, \nu) d\nu \), assume absorption coefficient \( \sigma_n \), diffusion coefficient \( D \) are positive constant, and notice that we integrate the fourth equation of (50) over all solid angle. Then, (50) can rewrite as
\[
\rho_t + \text{div}(\rho u) = 0,
\]
\[
(\rho u)_t + \nabla P_m + \text{div}(\rho u \otimes u) = \text{div} S,
\]
\[
\left(\frac{1}{2} \rho u^2 + E_m\right)_t + \text{div}\left(\frac{1}{2} \rho u^2 + E_m + P_m\right) u = \text{div}(S u + n \nabla \theta) - \tilde{\sigma} \theta^4 + \sigma_a n,
\]
\[
\frac{1}{c} \nabla n - D \Delta n = \tilde{\sigma} \theta^4 - \sigma_a n,
\]
where \( \tilde{\sigma} \) is a positive constant defined by
\[
0 < \tilde{\sigma} = 4\pi \sigma_a \alpha_\beta^{-4} \int_0^\infty \frac{s^3}{e^s - 1} ds < \infty.
\]
In this paper, we consider only polytropic ideal gases, namely,
\[
E_m = c_v \rho \theta, \quad P_m = R \rho \theta.
\]
For simplicity of the presentation and without loss of generality, we assume the positive constants in (52) that \( c = D = \tilde{\sigma} = \sigma_a = 1 \), then, we obtain the diffusion approximation model in radiation hydrodynamics of the form (1)-(4).

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