The Joker: A Custom Monte Carlo Sampler for Binary-star and Exoplanet Radial Velocity Data

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Received 2016 October 24; revised 2017 January 31; accepted 2017 February 1; published 2017 February 28

Abstract

Given sparse or low-quality radial velocity measurements of a star, there are often many qualitatively different stellar or exoplanet companion orbit models that are consistent with the data. The consequent multimodality of the likelihood function leads to extremely challenging search, optimization, and Markov chain Monte Carlo (MCMC) posterior sampling over the orbital parameters. Here we create a custom Monte Carlo sampler for sparse or noisy radial velocity measurements of two-body systems that can produce posterior samples for orbital parameters even when the likelihood function is poorly behaved. The six standard orbital parameters for a binary system can be split into four nonlinear parameters (period, eccentricity, argument of pericenter, phase) and two linear parameters (velocity amplitude, barycenter velocity). We capitalize on this by building a sampling method in which we densely sample the prior probability density function (pdf) in the nonlinear parameters and perform rejection sampling using a likelihood function marginalized over the linear parameters. With sparse or uninformative data, the sampling obtained by this rejection sampling is generally multimodal and dense. With informative data, the sampling becomes effectively unimodal but too sparse; in these cases we follow the rejection sampling with standard MCMC. The method produces correct samplings in orbital parameters for data that include as few as three epochs. The Joker can therefore be used to produce proper samplings of multimodal pdfs, which are still informative and can be used in hierarchical (population) modeling. We give some examples that show how the posterior pdf depends sensitively on the number and time coverage of the observations and their uncertainties.

Key words: binaries: spectroscopic – methods: data analysis – methods: statistical – planets and satellites: fundamental parameters – surveys – techniques: radial velocities

1. Introduction

Precise radial velocity measurements of stars have transformed astrophysics in the past decades. They have permitted the discovery and confirmation of companions (planetary, stellar, and otherwise) orbiting thousands of stars. Radial velocity surveys hold the promise of delivering the full population statistics for binary and trinary star systems (e.g., Raghavan et al. 2010; Tokovinin 2014; Troup et al. 2016).

With large stellar spectroscopic surveys operating or under construction, we expect to have good-quality spectra for millions of stars in the next few years. Most of these surveys have at least some targets—and many have many targets—that are observed multiple times (e.g., Majewski et al. 2015). Whether as a primary or secondary goal of their observing strategies, these surveys can generate discoveries of planetary, substellar, and stellar companions. These discoveries, in turn, will feed population inferences, follow-up programs, and projects to refine precise stellar models.

However, when radial velocity observations are not designed with unambiguous detection and discovery in mind, there are usually multiple possible star-companion (orbit) models that are consistent with any modest number of radial velocity measurements that show stellar acceleration. That is, a small number of radial velocity measurements—even when the uncertainties are small—will lead to posterior beliefs about companion orbits and masses that put substantial plausibility onto multiple qualitatively different solutions. This is then reflected in a likelihood function that is highly multimodal in the relevant parameter spaces (e.g., Keplerian orbital parameters). While multimodal orbit likelihoods may be frustrating when studying individual systems of particular interest, extensive sets of such likelihoods can be powerful constraints for hierarchical modeling, inferring, e.g., the characteristics of the binary star, or the exoplanet population.

While the problem has of course been recognized for a long time, there are currently no methods that explore these highly multimodal distributions with good guarantees of converged samplings (but see Gregory 2005; Brewer & Donovan 2015). Converged, independent samplings are essential to delivering correct posterior samplings and reliable probabilistic statements about detection and characterization. With general Markov chain Monte Carlo (MCMC) methods, the returned samplings are inherently correlated and must therefore be run longer than the autocorrelation time. However, computing the autocorrelation time (a two-point statistic of a chain) is not simple to compute and can be misleading, especially when the target distribution is highly multimodal with widely separated modes.

Here we present a path to address this problem. Our approach is to build a custom posterior sampling method that capitalizes on the structure of the binary-star (or star–exoplanet) kinematics to generate robust samplings of multiple solutions at manageable computational cost. In what follows,
we use the term “binary” for any system with an observed source (the primary, e.g., a star) whose radial velocity variations are explained through gravitational two-body interactions with another object (the companion, e.g., star, exoplanet, stellar remnant), i.e., we restrict our analysis to single-line spectroscopic binary systems.

The structure of this article is as follows: We lay out our assumptions and implementation approach, and we demonstrate that we have a method that is reliable and essentially always correct under those assumptions. We perform experiments with the method to understand its properties and limitations. We then discuss the astrophysical applicability and potential of the method. We finish by describing the changes we would have to make if we weakened our assumptions, or if we do not weaken our assumptions but they indeed prove to be far from correct.

2. Assumptions and Method

In order to set up a well-posed problem and build a path to a definite solution, we make a set of sensible but nontrivial assumptions about the stellar systems we will use and observations thereof. Ultimately, we assume that the radial velocity curve of a single-line spectroscopic binary system can be specified by six parameters (Kepler 1609). Here we adopt a parameterization for this problem that is similar to Murray & Correia (2010, pp. 15–23): $P$, $e$, $\omega$, $K$, and $v_0$, which are the period, eccentricity, pericenter phase and argument, velocity semi-amplitude, and the barycenter velocity, respectively. As is well known, this does not fully specify the binary system itself, because of the inclination ($\sin i$) and mass function degeneracies. To proceed, we assume the following:

1. We have measurements of the radial velocity of a star, and that the time dependence of the expectation of that radial velocity is well described by the gravitational orbit of a pair of point masses (the two-body problem). We assume that the uncertainties in the observation epochs are negligible and specified in an inertial frame (e.g., solar system barycentric Julian date). We only consider the case of a single-line spectroscopic binary system, where we do not have measurements of the (fainter) companion’s projected orbit.
2. Each star has exactly one companion, and that the radial velocity measurements are not contaminated by or affected by any other bodies. Our analysis allows the effective mass of the exactly one companion to go to zero with finite probability; this encompasses the case of no companion.
3. The noise contributions to individual radial velocity measurements are well described as samples from zero-mean normal (Gaussian) distributions with correctly known variances convolved with a zero-mean normal distribution with an additional “jitter” variance, $s^2$. We assume that there are no outliers beyond this flexible noise model.
4. We can put particular, fairly sensible but not highly restrictive, prior probability density functions (pdfs) on all the orbital parameters, as we describe below.

Each of these assumptions could be challenged: in particular, we expect some stars to have additional companions, and we expect there to be outliers and unaccounted sources of noise. We will return to these assumptions, and the consequences of relaxing them, in Section 4.

The radial velocity $v$ at time $t$ is then given by (see also Equation (63) in Murray & Correia 2010, pp. 15–23)

$$v(t; \theta) = v_0 + K [\cos(\omega + f) + e \cos \omega],$$  \hspace{1cm} (1)

where the $\theta$ represents the free parameters, $f$ is the true anomaly given by

$$\cos f = \frac{\cos E - e}{1 - e \cos E},$$  \hspace{1cm} (2)

and the eccentric anomaly, $E$, must be solved for with the mean anomaly, $M$,

$$M = \frac{2\pi t}{p} - \phi_0,$$  \hspace{1cm} (3)

$$M = E - e \sin E.$$  \hspace{1cm} (4)

Of these parameters, four ($P$, $e$, $\omega$, $\phi_0$) are nonlinearly related to the radial velocity expectation, and two ($K$, $v_0$) are linearly related. We additionally allow that the radial velocity curve of any star has an overall jitter, $s^2$, to vary to partially account for imperfect knowledge of the radial velocity uncertainties and any intrinsic radial velocity scatter; the jitter must also be treated as a nonlinear parameter.

With such a parameterization, the problem is then to construct the posterior pdf for these seven parameters, accounting for the fact that this pdf may have very complex, multimodal structure. Fundamentally, the method we describe and demonstrate here is to perform rejection sampling in the nonlinear parameters, but with analytic marginalization over the two linear parameters. The method capitalizes on the unique problem structure:

1. There are both linear and nonlinear parameters, and we can treat them differently; in particular, it is possible to analytically marginalize out the linear parameters (provided that the noise model is well behaved and the prior pdf is conjugate).
2. There is a finite, time-sampling-imposed minimum size or resolution—in the period—of any features in the likelihood function. That is, there cannot be arbitrarily narrow modes in the multimodal posterior pdf.

The method described above is a specific case of rejection sampling (von Neumann 1951) in which we densely sample (generate many samples with typical spacing smaller than the time-sampling-imposed resolution) from the prior pdf and use the likelihood evaluated at these samples as the rejection scalar. In detail, the rejection step works as follows:

1. For each sample $j$ in the prior pdf sampling of the four nonlinear parameters, there is a (linear, not logarithmic) marginal likelihood value $L_j$ (a probability for the data given the nonlinear parameters).
2. There is a maximum value $L_{\text{max}}$ that is the largest value of $L_j$ found across all of the samples in the prior sampling.
3. For each sample $j$, choose a random number $r_j$ between 0 and $L_{\text{max}}$.
4. Reject the sample $j$ if $L_j < r_j$.
5. The number of samples that survive the rejection is (hereafter) $M$. 

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Note that this algorithm is guaranteed to produce at least one surviving sample; of course, if only one sample survives (or any very small number), the sampling is not guaranteed to be fair. That said, if the original sampling of the prior pdf is dense enough that many survive the rejection step, the surviving samples do, by construction, constitute a fair, uncorrelated sampling from the posterior pdf.

Our prior pdf in the nonlinear parameters is very straightforward:

\[
p(\ln P) = \mathcal{U}(\ln P_{\text{min}}, \ln P_{\text{max}}),
\]

\[
p(e) = \text{Beta}(a, b) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} e^{a-1} [1 - e]^{b-1},
\]

\[
p(\omega) = \mathcal{U}(0, 2\pi) \text{ (rad)},
\]

\[
p(\phi_0) = \mathcal{U}(0, 2\pi) \text{ (rad)},
\]

\[
p(\ln s) = \mathcal{N}(\mu_s, \sigma_s^2),
\]

where \( \mathcal{U}(x_1, x_2) \) is the uniform distribution over the domain \((x_1, x_2)\), \( \mathcal{N}(\mu, \sigma^2) \) is the normal distribution with mean \( \mu \) and variance \( \sigma^2 \), and the prior pdf over eccentricity is the Beta distribution with \( a = 0.867, b = 3.03 \) (Kipping 2013). To simplify the marginalization integrals below, we assume that the priors over the linear parameters \((K, v_0)\) are very broad and Gaussian—or at least very flat over the range of relevance—and that they do not depend on the nonlinear parameters in any way (which is a substantial restriction; see Section 4). Of course, we may actually have stronger prior beliefs about the systemic velocity of the binary, \( v_0 \) (i.e., we may want to impose that it belong to the Galactic disk, or use a mixture model for the different kinematic components of the Galaxy). Using a more informative prior would require only a minimal change to the method.

In Section 3 or in the subsections specific to the different experiments we specify the values for hyperparameters \( P_{\text{min}}, P_{\text{max}}, \mu_s, \) and \( \sigma_s^2 \). In practice, the choice of \( \mu_s \) and \( \sigma_s^2 \) can be tuned appropriately depending on knowledge about intrinsic variability of the source or suspicions about the reported uncertainties. We sample the prior pdf directly and explicitly with standard \texttt{numpy.random} calls (Van der Walt et al. 2011). In practice, we are usually required to take around \( J = 2^{28} \) samples (indeed, a quarter billion samples) from the prior pdf to produce sufficient final samplings; in each experiment below, we state the total number of prior samples generated and the number of surviving samples.

The un marginalized likelihood function \( L \) is

\[
\ln L = \frac{1}{2} \sum_{n=1}^{N} \left[ \frac{v_n - v(t_n; \theta)}{\sigma_n^2} + \ln(2\pi [\sigma_n^2 + s^2]) \right],
\]

where \( n \) indexes the individual data points \( v_n \), \( v(t) \) is the radial velocity prediction at time \( t \) given the orbital parameters \( \theta \), the data-point times are the \( t_n \), and \( \sigma_n^2 \) is the Gaussian noise variance for data point \( n \). Note that the form of this likelihood function is fully specified by the assumptions, given above.

We rejection-sample, however, using a marginalized likelihood, where we analytically marginalize out the linear parameters \((K, v_0)\). We construct an \( N \times 2 \) design matrix consisting of a column of unit-[\( K \)] predictions (given the nonlinear parameters) and a column of ones. We perform standard linear least-squares fitting with this design matrix to obtain the best-fit values for the two linear parameters, and the standard \( 2 \times 2 \) linear-fitting covariance matrix \( C \) for their uncertainties. With these, we can construct—for each prior sample—the marginalized likelihood \( Q_j \)

\[
\ln Q_j = -\frac{1}{2} \sum_{n=1}^{N} \left[ \frac{[v_n - v(t_n; \theta_j)]^2}{\sigma_n^2 + s^2} \right] + \ln(2\pi [\sigma_n^2 + s^2]) - \frac{1}{2} \ln|2\pi C|,
\]

where the prediction \( v(t_n; \theta_j) \) is taken at the best-fit values of the linear parameters given sample \( j \) of the nonlinear parameters, and the log-determinant term \( \ln|C| \) accounts for the volume in the marginalization integral. These \( Q_j \) are used in the rejection sampling algorithm described above.

There are three possible outcomes of this rejection sampling, based on two thresholds: we set a minimum number of samples \( M_{\text{min}} = 128 \). We also set a period resolution \( \Delta = [4 P^2]/[2\pi T] \), with \( P \) set to the median period across the surviving samples and \( T \) set to the epoch span of the data. This \( \Delta \) is an expansion of the period resolution expected from an information-theory (sampling theorems) perspective: for a periodic signal with frequency \( \omega \) observed over a window with size \( T \), the smallest resolvable frequency differences will be \( \Delta \omega \approx T^{-1} \), corresponding to period differences of \( \Delta P \approx \frac{P^2}{2\pi T} \). The three possible outcomes are as follows:

1. \( M > M_{\text{min}} \) samples survive the rejection. In this case, we are done.
2. \( M < M_{\text{min}} \) samples survive the rejection, and these samples have a root variance \( \text{rms} \) in the period parameter \( P \) that is smaller than \( \Delta \) (i.e., they give no indication of period ambiguity). In this case we assume that the posterior pdf is effectively unimodal, and we use the surviving samples (or sample) to initialize an MCMC sampling using the emcee package (Foreman-Mackey et al. 2013).

3. \( M < M_{\text{min}} \) samples survive the rejection, and these samples span a period range larger than \( \Delta \). In this case, we iterate the rejection-sampling procedure: we generate new prior pdf samplings and rejection-sample until the number of surviving samples is larger than \( M_{\text{min}} \). This is expensive.

When we trigger the initialization and operation of emcee, we do the following:

1. Randomly generate \( M_{\text{min}} \) sets of parameters \( \theta_m \) (linear and nonlinear parameters) in a small, Gaussian ball around the highest-\( Q_j \) sample from the rejection sampling.
2. Run emcee on this ensemble (with \( M_{\text{min}} \) walkers) for \( 2^{16} \) steps (this number is arbitrary and can be tuned if the walkers converge much faster or slower).
3. Take the final state of the \( M_{\text{min}} \)-element ensemble as an independent sampling of the posterior pdf.

This procedure ensures that no matter what path we take, we end up with at least \( M_{\text{min}} \) samples from the posterior for any input data. Within the initial assumptions, this procedure almost inevitably results in a correct sampling of the parameter pdf. This is ensured by the density of the prior sampling in the nonlinear parameters and also borne out in the numerical experiments in the following section.

Figure 2. Projections of the 678 surviving posterior samples when considering the five velocity measurements from Figure 1 with fixed jitter (gray points). The values used to generate the input orbit are shown as green cross-hairs.
3. Experiments and Results

In what follows, we use (1) simulated data with known properties and (2) actual spectroscopic data from Data Release 13 (DR13; SDSS Collaboration et al. 2016) of the Apache Point Observatory Galactic Evolution Experiment (APOGEE; Majewski et al. 2015) in a series of experiments that demonstrate the reliability and utility of The Joker. APOGEE is one of the four subsurveys of the Sloan Digital Sky Survey-III (SDSS-III; Eisenstein et al. 2011) and utilized a new infrared spectrograph to obtain moderate-resolution, $H$-band spectra for over 160,000 stars throughout the Galaxy. From these spectra, high-precision radial velocities, chemical abundances, and stellar parameters have been derived and released as a part of DR13 (Holtzman et al. 2015; Nidever et al. 2015).

As a part of the observing strategy of APOGEE, most stars are observed multiple times and binned by day into “visit” spectra. Though a typical star is only observed a few times, (1) at least one pair of visits are separated by 1 month or longer, and (2) thousands of stars have been observed more than 10 times (for a more detailed look at the cadence and number of visits for APOGEE stars, see Figure 1 in Troup et al. 2016). Radial velocities (and stellar parameters) are derived for each of the visit spectra, affording a sparse and sporadic time-domain sampling of the radial velocity variations of most stars in the survey. This time-domain information was recently used to identify a sample of candidate stellar and substellar companions to APOGEE stars (Troup et al. 2016).

This search was conducted after data quality and data quantity cuts that were designed to keep the number of data points larger than the number of parameters in the model. In their case, the model parameters are six Keplerian orbital parameters plus a long-term (linear) velocity trend (seven in total). Under this criterion, stars with fewer than eight visits were eliminated from further consideration, leaving $\approx 15,000$ stars. For each of the remaining stars, the radial velocity curves were searched for significant periods, which were then used to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Same as Figure 2, but for the 580 surviving posterior samples when also sampling over an unknown jitter parameter, $s^2$.}
\end{figure}
Figure 4. Same as Figure 1, but for the three cases in this experiment: (a) known uncertainties and jitter fixed to \(s^2 = 0\), (b) underestimated uncertainties and jitter fixed to \(s^2 = 0\), and (c) underestimated uncertainties and jitter allowed to vary.

initialize \(\chi^2\) fits of a Keplerian orbit using a custom least-squares jitter (MPRVFIT; De Lee et al. 2013). In cases where multiple significant periods were found, the parameters obtained from the fit with the best modified \(\chi^2\) value were retained (modulo a number of other considerations described in Section 3.3.4 of Troup et al. 2016).

The complexity of this pipeline and logic needed to identify a single optimal set of orbital parameters from a set of solutions highlights the fact that the likelihood function for a Keplerian orbit model is generically multimodal. When there are few data points or poor phase coverage, this is especially true. While useful for searching for new candidate binaries, such a pipeline does not easily fit within hierarchical probabilistic modeling of the population of companions.

3.1. Experiment 1: Validation with Simulated Data

As a first demonstration, we generate fake radial velocity observations (with uncertainties) that are consistent with our assumptions (Section 2) and then sample from the posterior pdf over orbital parameters with \textit{The Joker}. The eccentricity, period, and velocity semi-amplitude are chosen to be broadly consistent with the typical substellar companion found in recent analysis of the APOGEE data (Troup et al. 2016), the angle parameters \((\omega, \phi_0)\) are sampled from a uniform distribution, and the barycenter velocity is drawn from a zero-mean Gaussian with variance \(\sigma_v^2 = (30 \text{ km s}^{-1})^2\). The values of the parameters used for the simulated data (i.e., the truth) are \(P = 103.71\text{ day}\), \(e = 0.313\), \(\omega = 68^\circ 95\), \(\phi_0 = 223^\circ 96\), \(K = 8.134 \text{ km s}^{-1}\), \(v_0 = 42.98 \text{ km s}^{-1}\). We randomly sample five observation epochs uniformly over the interval \((0, 1095)\) days imagining a 3 yr survey with no observing strategy and arbitrarily set the survey start date (in barycentric MJD) to be 55,555. The radial velocity measurement uncertainties are drawn from a uniform distribution over the interval \((0.1, 0.2)\ \text{km s}^{-1}\), motivated by current APOGEE radial velocity uncertainties.

We perform two samplings as a validation of \textit{The Joker}: (a) we fix the jitter parameter \(s^2 = 0\), i.e., we assume that the uncertainties are known perfectly and there is no intrinsic scatter, and (b) we sample over the jitter parameter as well, setting \((\mu_s, \sigma_s^2) = (0, 8)\) (note that these are dimensionless because they set the scale of a Gaussian in \(\text{ln s}^2\)). We start by generating \(J = 2^{28}\) prior samples over the nonlinear parameters with a period domain of \((P_{\text{min}}, P_{\text{max}}) = (16, 8192)\) days (see Section 2) and use these same prior samples for both runs. For sampling (a), 678 samples pass our rejection sampling step, and for sampling (b), 580 samples survive.

Figure 1 shows the simulated data (black points) along with the true orbit (dashed green line) and orbits computed from samples from the posterior pdf (gray lines) for the sampling with fixed jitter (panel (a)) and the sampling including jitter as a nonlinear parameter (panel (b)). Because we use the same prior samples in each, these look almost identical. Figures 2–3 show all projections of the posterior samples (gray points) and the true, input parameters (green lines and markers) for the case with fixed jitter and the case where jitter is treated as a free parameter. The surviving samples in each case look very similar, as expected. The typical uncertainties for these simulated data are \(\sigma_j \approx 0.15 \text{ km s}^{-1}\); for values of \(\text{ln s} \lesssim 3\) (where most are concentrated), this extra jitter is negligible compared to the uncertainties. Note that samples above \(\text{ln s} \approx 5\) are mostly rejected, indicating that, as constructed, the uncertainties are purely Gaussian and known.

In both cases, the modes in period space are narrow with a variety of separations, as can be seen in the radial velocity curves plotted in the top panel. For small numbers of precise observations with poor phase coverage, the posterior pdf over orbital parameters is extremely complex and structured, but we are still able to generate samples using \textit{The Joker} that capture the complexity. It is obvious that these multimodal samples are essentially useless when trying to understand one particular system at hand. Yet, they rule out almost all of the parameter space encompassed by the prior distribution: even a few radial velocity measurements are manifestly very informative.

3.2. Experiment 2: Underestimated Uncertainties

Besides the number and spacing of the observation epochs, the precision of the individual radial velocity measurements matters. Less precise data, compared to the radial velocity amplitude, will admit more qualitatively different orbital solutions. The structure and complexity of the posterior pdf (at fixed jitter) will therefore depend strongly on knowing the measurement uncertainties and the intrinsic velocity variability of the system. We illustrate this using another simulated radial velocity curve with lower signal-to-noise ratio. We show that for underestimated uncertainties, the posterior pdf over orbital parameters can look well constrained, but may be discrepant with the true orbital parameters. Finally, we show that by simultaneously sampling over an unknown extra variance in
the data (the jitter), we can account for additional, unaccounted sources of noise.

For this experiment, we use the following parameter values to generate the simulated data: $P = 103.71$ days, $e = 0.313$, $\omega = 250^\circ73$, $\phi_0 = 103^\circ01$, $K = 1.134$ km s$^{-1}$, $v_0 = 8.489$ km s$^{-1}$. We again uniformly sample five observation epochs over the interval $(0, 1095)$ days and use radial velocity measurement uncertainties drawn from a uniform distribution over the interval $(-0.1, 0.2)$ km s$^{-1}$ (the median true uncertainty is $\ln(\sigma_\rho/\text{m s}^{-1}) \approx 5$).

When running *The Joker* for this experiment, we consider three cases: (a) the uncertainties are (correctly) known and jitter is fixed and ignored ($s^2 = 0$), (b) the uncertainties are underestimated by a factor of 8 and jitter is fixed and ignored ($s^2 = 0$), and (c) the uncertainties are underestimated by a factor of 8 and we treat the jitter as a free parameter. For each case, we again generate $2^{28}$ prior samples over the nonlinear parameters with a period domain of $(P_{\text{min}}, P_{\text{max}}) = (16, 8192)$ days and reuse these prior samples for each case. In case (c), we generate samples in the jitter by setting $(\mu_\sigma, \sigma_s^2) = (10, 1)$—here we are assuming that we have some suspicion about the true magnitude of the uncertainties.

Figure 4 shows the simulated data and presumed uncertainties (black points), the true orbit (dashed green line), and orbits computed from samples from the posterior pdf (gray lines) for the three different samplings. In all cases, a wide variety of orbit solutions is permitted by the data. Note that far fewer modes are present in panel (b) (case (b)), when the uncertainties are underestimated. Figures 5–7 show the corresponding corner plots of the parameter pdfs for these three cases. For this data set, when the uncertainties are correctly known, the posterior pdf is highly multimodal (case (a)). When the uncertainties are underestimated, the posterior pdf has fewer modes, but the true orbital parameters do not appear consistent with any of the strongest modes (case (b)). When the uncertainties are (severely) underestimated but the jitter is allowed to vary (case (c)), the model prefers solutions with a finite jitter comparable to the input true uncertainties ($\ln(\sigma_\rho/\text{m s}^{-1}) \approx 5$). We have therefore shown that *The Joker* is useful even when uncertainties are underestimated or the intrinsic velocity variability of a system is unknown.
3.3. Experiment 3: Varying the Number of Data Points

When the phase coverage of the radial velocity observations is good and the number of observation epochs is large, the posterior pdf over orbital parameters effectively becomes unimodal. Under these conditions, The Joker is of course a very inefficient sampler for this problem and will return very few samples (as few as one). As we have seen in the previous experiments, when the number of data points is small or the uncertainties are large, the posterior pdf is generally multimodal. In this experiment we explore the dependence of the posterior pdf’s complexity on the number of observation epochs by generating radial velocity curves with initially 11 epochs. We use the following parameter values to generate the simulated data: \( P = 103.71 \text{ days} \), \( e = 0.313 \), \( \omega = 134.83 \), \( \phi_0 = 342.26 \), \( K = 4.227 \text{ km s}^{-1} \), \( v_0 = 19.431 \text{ km s}^{-1} \). After running The Joker with the full 11 observations, we successively remove two data points and rerun the sampling until we are left with three observation epochs as input data (a total of five consecutive runs).

Specifically, we again generate \( 2^{28} \) prior samples over the nonlinear parameters with a period domain of \( (P_{\text{min}}, P_{\text{max}}) = (16, 8192) \) days and reuse these prior samples for each subsampling of the data. We fix the jitter to \( s^2 = 0 \) and assume that the uncertainties are known. Figure 8 shows the simulated data and orbits computed from posterior samplings. Starting from the top of Figure 8 with the full set of 11 data points, each panel below has two epochs fewer than the previous. The data used for the pdf sampling shown in a given panel are plotted as black circles, and the number of data points \( N \) used in each panel is indicated. As described in Section 2, when the number of surviving samples \( M < M_{\text{min}} = 128 \) after rejection sampling, we either (1) initialize emcee using the remaining sample(s) if the periods of the surviving sample(s) are sufficiently close, or (2) rerun The Joker with a new set of prior samples until we have at least \( M_{\text{min}} \) samples from the posterior pdf. In all panels, 128 orbits computed from the posterior samples are shown.

The structure in the posterior samples in one projection of the posterior pdf (period and eccentricity) is shown in the right...
Experiment 2(c): sample jitter, underestimated errors

3.4. Experiment 4: Real Data for a Known Binary

For a more realistic application of The Joker, we choose an APOGEE target with a previously identified companion (2M00110648+6669349) but with few radial velocity measurements (Troup et al. 2016). Figure 9 shows radial velocity data for the star (black points). Similar to Experiment 1, these data are sparse and have poor phase coverage. However, this epoch sampling is quite different and is representative of realistic survey design choices.

We again generate $2^{28}$ prior samples over the nonlinear parameters with a period domain of $(P_{\text{min}}, P_{\text{max}}) = (16, 8192)$ days and with $(\mu_1, \sigma_1^2) = (10.5, 1)$, of which 22,313 samples pass our rejection sampling step. Overplotted as gray lines on Figure 9 are 256 orbits computed from these samples. Already from visualizing these orbits, it becomes clear that there are at least a few distinct period modes, and a wide variety of eccentricities, represented in the posterior sampling.

Figure 10 shows projections of all posterior samples in different parameter combinations. Here it is clear that there are at least three period modes: the dominant mode at $P \approx 300$ days is broadly consistent (but not coincident) with the previously measured period (Troup et al. 2016), but other modes are clearly present at shorter periods with low eccentricity and longer periods with higher eccentricity. Interestingly, we also find that the model prefers having a finite jitter around $s \approx 116$ m s$^{-1}$, which would imply that the panels of Figure 8. For the cases with nine and 11 data points, the posterior pdf appears to be unimodal. Multiple modes first appear when $N = 7$, and the posterior pdf becomes more structured in further subsamplings of the data until ultimately forming a harmonic series of modes in the final case of $N = 3$. It is worth emphasizing that even for the case with three data points, <1% of the prior samples pass the rejection step: even three radial velocity observations are informative!
effective radial velocity uncertainties might be underestimated by a factor of $\approx 2$.

3.5. **Experiment 5: Prospects for Observation Planning**

A noticeable difference between the $N = 5$ and $N = 7$ panels in Figure 8 is that the posterior pdf collapses significantly between these cases (from $\gtrsim 20$ period modes to 3); this implies that the two added observations are extremely informative. Inverting this idea, it also suggests that (1) we can use The Joker to predict the observation time that maximally collapses the posterior pdf for a previously measured source, and (2) for an expected population of sources, we can identify the optimal sampling pattern to maximize discovery or characterization of the sources. We will explore these ideas...
in detail in future work, but here we simply show that the timing of subsequent observations can lead to very different structure in the posterior samples.

We again simulate a data set of four noisy radial velocity measurements, shown as black points in the top left panel of Figure 11. We use the following parameter values to generate the simulated data: $P = 127.31$ days, $e = 0.213$, $\omega = 137^\circ 23$, $\phi_0 = 36^\circ 23$, $K = 8.996$ km s$^{-1}$, $v_0 = 17.643$ km s$^{-1}$. Uncertainties were chosen to match the APOGEE data ($\sigma_x \approx 0.2$ km s$^{-1}$) and are shown as error bars, but they are often comparable to or smaller than the marker size. The top right panel shows posterior samples produced with The Joker again in the space of log-period, $\ln P$, and eccentricity, $e$. The three lower rows all have six observation epochs: the same four from the top row, but now with two additional observations spaced, in phase, by $\Delta \phi = 0.04$ but with a different starting phase for the new observations. As is shown by the right panels, the observation times of the new observations can greatly affect the compactness of the posterior pdf. In particular, the placement of the observations in the bottom row of panels rules out most of the long-period modes from the top panels, as well as many of the short-period modes, whereas for the middle two cases the new data are not as informative.

4. Discussion

We have built a Monte Carlo sampler—The Joker—to draw samples from the full posterior pdf over orbital parameters for single-companion systems, given a set of multiepoch, single-line radial velocity measurements. The Joker has important properties that differ from other sampling methods: (1) it produces independent samplings even when the likelihood (and hence posterior pdf) is highly multimodal; (2) the method is based on Simple Monte Carlo, in some sense a pure brute-force method, which parallelizes trivially; and (3) the samplings are guaranteed to be correct under the sensible assumptions presented here, without the need for convergence or other diagnostic checks. If the pdf is effectively unimodal, The Joker tells us that the solution is unique. If the pdf is multimodal, The Joker captures all relevant different solutions.

Our experiments show that The Joker can be used for discovery and characterization of stellar binaries or exoplanets, even with the presence of unrecognized or unaccounted noise contributions. However, for exoplanets, we emphasize that while The Joker could in principle be used for any exoplanet system, the simplicity of our noise model and single-companion assumption strongly suggest that it will be most useful in the study of massive exoplanets.

Perhaps the primary innovation The Joker brings is a separation of the parameters into linear and nonlinear subsets. The brute-force sampling is only required in the nonlinear subspace, aiding computational feasibility. We further capitalize on the problem structure by identifying effectively unimodal and effectively multimodal posterior pdfs using the minimum possible width of a likelihood peak in the period direction, given the time sampling.

Interestingly, as we show in Section 3.3, even very sparse samplings of the radial velocity history of a star provide highly informative posterior pdfs. The bottom right panel of Figure 8 shows a highly multimodal posterior pdf. Nonetheless, the vast majority of prior pdf samples have been eliminated, and only a tiny subset of periods, eccentricities, and amplitudes are consistent with the data. These posterior pdfs may look bewilderingly complex, but they can contribute extremely valuable information to any hierarchical inference, or provide a very informative prior pdf for further observing campaigns.

Indeed, The Joker can be used to generate inputs for a hierarchical inference. In previous work (Hogg et al. 2010b; Foreman-Mackey et al. 2014) we have shown that posterior samplings under an interim prior can be importance-sampled with a hierarchical inference to generate posterior beliefs about the full population. These hierarchical inferences are the only population inferences that properly propagate nontrivial uncertainties at the individual-system level to the conclusions at the population level (see Strader et al. 2004; Mandel et al. 2011; Brewer et al. 2013, 2014, for other examples of hierarchical inference in astronomy).

In the experiments above we have used massive prior samplings, starting with $2^{28}$ samples before rejection sampling. When the data are sparse or have a low signal-to-noise ratio (e.g., bottom panels of Figure 8), many prior samples pass the rejection-sampling step; if the goal is to learn about an individual system and the data are poor, many fewer prior samples can be used to initialize The Joker. In this limit, generating a set number of posterior samples is very fast because of the easy parallelization of the likelihood calls. The same is true when the data are of high quality, and the samples that pass the rejection step will be used to initialize an MCMC sampling. A large prior sampling is needed when (a) many posterior pdf samples are needed for hierarchical inference, or (b) the data are of intermediate quality (the exact definition of which is problem specific). The Joker is most valuable in the low- to intermediate-quality range, especially when the samples will be used for hierarchical inference, when a small but converged sampling is needed for many (thousands to millions of) sources.

The Joker should also be valuable in observation planning, or cadence evaluation, or survey strategy: as Section 3.5 shows, The Joker could be used to plan the times of next observations to maximize their expected information content (see also Ford 2004; Loredo 2004). That is beyond the scope of this article and will be explored in future work.
The Joker is based on a set of assumptions, itemized in Section 2. The method delivers correct samplings when these specific assumptions hold. Of course, these specific assumptions do not hold sufficiently well!

Astrophysically, we know that a star’s radial velocity history need not be set entirely by a single companion, with no other perturbers or sources of radial velocity signal. Although the single-companion assumption is a severe assumption, it is pretty much required for the method to be tractable. Of course, in reality, it is likely that many stars reside in higher-order multiplets. Sampling over orbital parameters even for two companions, however, is already intractable as the nonlinear parameter space jumps to eight-dimensional, and (at least) 10-dimensional if there are companion–companion interactions. This would be absolutely intractable to sample by brute-force Simple Monte Carlo; our advice would be to switch to some kind of Markov chain method that deals as well as possible with multimodal posteriors, such as nested sampling (Skilling et al. 2004; Brewer et al. 2009). This change would be associated with the loss of the simple convergence criterion that the rejection sampling provides: If lots of samples survive the rejection step, the posterior has been sampled independently! There is no comparably simple way of determining that any nested sampling is converged.

That said, there is a simple N-body problem that can be solved tractably: for systems with one short-period companion and \( \geq 1 \) very long period companion(s), The Joker can be easily extended to include additional linear parameters that allow long-period velocity trends that are, e.g., polynomial in time; these additional parameters do not worsen the prior pdf sampling (which happens over the nonlinear parameters only). These extra linear parameters could alternatively include extra \( v_0 \) terms that come in when, say, the data come from a set of different radial velocity programs with different calibrations (as

![Figure 10. All projections of the 22,313 surviving posterior samples (gray points) for 2M00110648+6609349, with previously found orbital parameter values shown as the blue cross-hairs (Troup et al. 2016).](image)
is the case for the recent, impressive Proxima b discovery; Anglada-Escudé et al. (2016).

At a crucial practical level, the assumption of Gaussian noise and perfectly known noise variances is often violated. Here, introducing the jitter as an explicit fitting parameter should help to mitigate The Joker’s sensitivity to unknown noise sources. Nonetheless, presuming we understand the noise properties may still be the most problematic assumption made by The Joker: essentially all data sets show occasional outliers (catastrophic errors). There is nothing we can do simply here, if we want to capitalize on treating the linear and nonlinear parameters separately. To deal with very rare outliers (such that no star would be likely to suffer from more than one), one possible modification would be to do all leave-one-out samplings, take the union, and then importance-sample the results using some ratio of the mixture of leave-one-out Gaussian likelihoods to some more realistic likelihood that involves an outlier model (as, for example, we suggest in Hogg et al. 2010a). Such a modification to the method is beyond the scope of this article but not beyond the scope of our ambitions.

This project was started at AstroHackWeek 2016, organized by Kyle Barbary (UCB) and Phil Marshall (SLAC) at the Berkeley Institute for Data Science. It is a pleasure to thank Megan Bedell (Chicago), Will Farr (Birmingham), Ben Weaver (NOAO), Josh Winn (Princeton), and the participants at AstroHackWeek 2016 for valuable discussions. We thank the referee, Brendon Brewer, for extremely valuable feedback.

This research was partially supported by the NSF (grants IIS-1124794, AST-1312863, AST-1517237), NASA (grant NNX12AI50G), and the Moore-Sloan Data Science Environment at NYU. The data analysis presented in this article was partially performed on computational resources supported by the Princeton Institute for Computational Science and
Engineering (PICSciE) and the Office of Information Technology’s High Performance Computing Center and Visualization Laboratory at Princeton University. This work additionally used the Extreme Science and Engineering Discovery Environment (XSEDE; Towns et al. 2014), which is supported by National Science Foundation grant no. ACI-1053575.

This project made use of SDSS-III data. Funding for SDSS-III has been provided by the Alfred P. Sloan Foundation, the Participating Institutions, the National Science Foundation, and the U.S. Department of Energy Office of Science. The SDSS-III web site is http://www.sdss3.org/.

SDSS-III is managed by the Astrophysical Research Consortium for the Participating Institutions of the SDSS-III Collaboration, including the University of Arizona, the Brazilian Participation Group, Brookhaven National Laboratory, Carnegie Mellon University, University of Florida, the French Participation Group, the German Participation Group, Harvard University, the Instituto de Astrofísica de Canarias, the Michigan State/Notre Dame/JINA Participation Group, Johns Hopkins University, Lawrence Berkeley National Laboratory, Max Planck Institute for Astrophysics, Max Planck Institute for Extraterrestrial Physics, New Mexico State University, New York University, Ohio State University, Pennsylvania State University, University of Portsmouth, Princeton University, the Spanish Participation Group, University of Tokyo, University of Utah, Vanderbilt University, University of Virginia, University of Washington, and Yale University.

Software: The code used in this project is available from https://github.com/adm/thejoker (Price-Whelan & Hogg 2017). The code used to run the experiments and generate the figures for this article is available from https://github.com/adm/thejoker-paper. Both are released under the MIT open-source software license. This version was generated at git commit 3189dc9 (2017-01-30). This research additionally utilized Astropy (Astropy Collaboration et al. 2013), emcee (Foreman-Mackey et al. 2013), IPython (Pérez & Granger 2007), matplotlib (Hunter 2007), and numpy (Van der Walt et al. 2011).

Facilities: SDSS (APOGEE).

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