Asymptotics for the Eigenvalues of the Harmonic Oscillator with a Quasi-Periodic Perturbation

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Abstract

We consider operators of the form $H + V$ where $H$ is the one-dimensional harmonic oscillator and $V$ is a zero-order pseudo-differential operator which is quasi-periodic in an appropriate sense (one can take $V$ to be multiplication by a periodic function for example). It is shown that the eigenvalues of $H+V$ have asymptotics of the form $\lambda_n(H+V) = \lambda_n(H) + W(\sqrt{n}) n^{-1/4} + O(n^{-1/2} \ln(n))$ as $n \to +\infty$, where $W$ is a quasi-periodic function which can be defined explicitly in terms of $V$.

1 Introduction

The one-dimensional harmonic oscillator is the operator

$$H = -\frac{d^2}{dx^2} + (\alpha x)^2,$$

where $\alpha$ is a positive parameter. We can consider $H$ as an unbounded self-adjoint operator acting on $L^2(\mathbb{R})$. The determination of the spectrum of $H$ is a classical problem — virtually any introductory book on quantum mechanics has a section devoted to this topic. In particular $H$ has a compact resolvent and hence a discrete spectrum. Furthermore, the eigenvalues of $H$ are simple and can be enumerated as

$$\lambda_n(H) = \alpha(2n + 1), \quad n \in \mathbb{N}_0.$$

A normalised eigenfunction corresponding to $\lambda_n(H)$ can be chosen as

$$\phi_n(x) = \frac{\alpha^{1/4}}{\sqrt{n!2^n\sqrt{\pi}}} e^{-\alpha x^2/2} \mathcal{H}_n(\sqrt{\alpha}x), \quad (1)$$

where $\mathcal{H}_n$ is the $n$–th Hermite polynomial.
The purpose of this paper is to study the large $n$ asymptotics of the eigenvalues of the perturbed operator $H + V$ when $V$ is a self-adjoint quasi-periodic pseudodifferential operator of order 0. More precisely, we assume $V$ can be written in the form

$$V = \sum_{a \in \Lambda} V_a U_a$$

(2)

where $\Lambda \subset T^*\mathbb{R} \cong \mathbb{R}^2$ is a countable discrete index set and, for each $a = (a_x, a_\xi) \in T^*\mathbb{R}$, we define $U_a$ to be the unitary operator on $L^2(\mathbb{R})$ given by

$$U_a \phi(x) = e^{ia_x a_\xi / 2} e^{ia_x x} \phi(x + a_\xi).$$

(3)

The $V_a$’s are just complex coefficients.

Since $U_a^* = U_{-a}$ for any $a \in T^*\mathbb{R}$, the condition that $V$ is self-adjoint can be rewritten as the requirement

$$a \in \Lambda \implies -a \in \Lambda \quad \text{and} \quad V_{-a} = \overline{V_a}, \quad a \in \Lambda.$$

We will also assume the $V_a$’s satisfy the following condition (essentially a regularity assumption);

$$\sum_{a \in \Lambda} |a|^3 |V_a| < +\infty.$$  

(4)

In particular, this condition ensures that the right hand side of (2) is absolutely convergent in operator norm, making $V$ a well defined bounded operator. Since $H$ has a compact resolvent the same must then be true for $H + V$; it follows that the spectrum of $H + V$ also consists of discrete eigenvalues.

Remark: If we take $\Lambda = \{ (\omega m, 0) \mid m \in \mathbb{Z} \}$ then $V$ is the operator of multiplication by a function with period $\omega$ whose $m$-th Fourier coefficient is simply $\omega^{1/2} V_{\omega m, 0}$. Condition (4) becomes a standard regularity requirement (that the function $V$ should be a “bit more” than $C^3$).

In general we may consider $V$ to be a zero-order pseudo-differential operator with Weyl-symbol $\sum_{a \in \Lambda} V_a e^{i(a_x x + a_\xi \xi)}$ (n.b., $U_a$ is the operator with Weyl-symbol $e^{i(a_x x + a_\xi \xi)}$). If $\Lambda$ is a rational periodic lattice then $V$ will be a periodic operator (in the sense that it commutes with a specific translation operator). Taking $\Lambda$ to be an irrational periodic lattice, or an irregular discrete set, leads to a generalisation of such periodic operators; when we apply “quasi-periodic” to $V$ we mean this particular type of generalisation.

If $0 \in \Lambda$ then the corresponding term in $V$ is $V_0$ times the identity operator and will thus cause a simple shift in the spectrum of $H$ by $V_0$. This term is included in the statement of the main result (Theorem 1.1 below) but thereafter we shall assume $V_0 = 0$. We also set $\Lambda' = \Lambda \setminus \{0\}$; since $\Lambda$ is discrete, $T^*\mathbb{R} \setminus \Lambda'$ contains a neighbourhood of $0$.  

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Define a metric $|\cdot|_\alpha$ on $T^*\mathbb{R}$ by $|a|_\alpha = (\alpha^{-1}a^2 + \alpha a^2)^{1/2}$. This metric is equivalent to the usual metric $|\cdot|$ so condition (4) can be rewritten as

$$\sum_{a \in \Lambda'} |a|^p |V_a| < +\infty \quad \text{for all } p \leq 3. \quad (5)$$

The main result of the paper is the following.

**Theorem 1.1.** Suppose $V$ given by (2) satisfies (4) (or equivalently (5)). Then the eigenvalues of the operator $H + V$ satisfy

$$\lambda_n(H + V) = \alpha(2n+1) + V_0 + W(\sqrt{n}) n^{-1/4} + O(n^{-1/2} \ln(n))$$

as $n \to \infty$, where $W : \mathbb{R} \to \mathbb{R}$ is the quasi-periodic function defined by

$$W(\lambda) = \frac{2^{1/4}}{\sqrt{\pi}} \sum_{a \in \Lambda'} V_a |a|^{1/2} \cos \left( \sqrt{2} |a|_\alpha \lambda - \frac{\pi}{4} \right). \quad (6)$$

The presence of the quasi-periodic function $W$ means the first order asymptotics given by Theorem 1.1 contain considerably more information about the operator $V$ than one might expect (c.f. the simple power type asymptotics for the case when $V$ is given as multiplication by an element of $C_0^\infty$ ([PS]) or for the operator $-d^2/d\theta^2 + V(\theta)$ on $S^1$ (see Theorem 4.2 in [MO])). In particular we note that if $V$ is given as multiplication by a periodic function, knowledge of the first order asymptotics of $\lambda_n(H + V)$ allows the Fourier coefficients of $V$ to be “half” determined (the values of $V(-m\omega,0) + V(m\omega,0)$, $m \in \mathbb{N}$, can be determined from $W$).

It is likely that there exists a full asymptotic expansion for $\lambda_n(H + V)$, involving further terms with quasi-periodic functions multiplying increasingly negative powers of $n$. Judging by numerical evidence (for example with the potential $V(x) = \cos(x)$) the second term in the asymptotics is $O(n^{-3/4})$. This order (even as an improvement of the remainder estimate in Theorem 1.1) appears to involve reasonable subtle cancellation effects within the series giving the second term of the asymptotics; no attempt to deal with this analysis is made here.

**Remark.** With an obvious modification to the definition of $W$ and a remainder estimate of $O(n^{-1/3} \ln(n))$, Theorem 1.1 also holds for operators $V$ of the form

$$V = \int_{T^*\mathbb{R}} V_a U_a d^2a \quad \text{where } V_a \text{ satisfies } \int_{T^*\mathbb{R}} (|a|^{-3/2} + |a|^3 |V_a| d^2a < +\infty. \quad (7)$$

In this case $V$ is a pseudo-differential operator of order zero whose Weyl-symbol has Fourier transform $2\pi V_a$. The $|a|^3$ term in the condition on $V_a$ is then a regularity condition, while the $|a|^{-3/2}$ term is a generalisation of quasi-periodicity.
The proof of Theorem 1.1 is given in Section 4 using standard ideas to express the eigenvalues of $H + V$ in terms of a series involving the resolvent of $H$ and the operator $V$. The non-triviality of Theorem 1.1 is contained in technical results used to establish the convergence of these series. These results are obtained in Sections 2 and 3; estimates for the elements $\langle V \phi_k, \phi_{k'} \rangle$ of the matrix of $V$ with respect to the eigenbasis $\{ \phi_k | k \in \mathbb{N}_0 \}$ are obtained in the former and are then combined to give resolvent estimates in the latter.

**Notation.** We use $C$ to denote any positive real constant whose exact value is not important but which may depend only on the things it is allowed to in a given problem. Appropriate function type notation is used in places to make this clearer whilst subscripts are added if we need to keep track of the value of a particular constant (e.g. $C_1(V)$ etc.).

We use $\| T \|$, $\| T \|_1$ and $\| T \|_2$ to denote the operator, trace class and Hilbert-Schmidt norms of the operator $T$ respectively.

## 2 Estimates for Matrix Elements

The aim of this section is to obtain the necessary estimates for the matrix elements $\langle V \phi_k, \phi_{k'} \rangle$ for all $k, k' \in \mathbb{N}_0$. In turn these will be estimated via

$$U_a^{k,k'} := \langle U_a \phi_k, \phi_{k'} \rangle$$

(7)

defined for all $a \in T^* \mathbb{R}$ and $k, k' \in \mathbb{N}_0$. Since the operator $U_a$ is unitary we immediately get

$$|U_a^{k,k'}| \leq 1.$$  \hfill (8)

To obtain more precise estimates we can use the following special function identity (see 7.377 on page 844 of [GRJ]) to find an explicit formula for $U_a^{k,k'}$; for any $0 \leq k \leq k'$ and $y, z \in \mathbb{C}$ we have

$$\int_{\mathbb{R}} e^{-x^2} \mathcal{H}_k(x + y) \mathcal{H}_{k'}(x + z) \, dx = 2^{k'} \sqrt{\pi} k! z^{k' - k} L_k^{(k' - k)}(-2yz),$$

(9)

where $L_k^{(k' - k)}$ is the generalised Laguerre polynomial.

**Lemma 2.1.** For any $0 \leq k \leq k'$ and $a \in T^* \mathbb{R} \setminus \{0\}$ we have

$$U_a^{k,k'} = \sqrt{\frac{k!}{k'!}} (\sqrt{2} \rho e^{i\theta})^{k' - k} e^{-\rho^2} L_k^{(k' - k)}(2\rho^2)$$

for some $\theta \in \mathbb{R}$, where

$$\rho = \frac{1}{2} \left( \frac{a^2}{\alpha} + \alpha a^2 \right)^{1/2} = \frac{1}{2} |a|_\alpha.$$  \hfill (10)
Proof. Introduce the complex number
\[ \omega = \frac{\sqrt{\alpha}a_\xi}{2} - i\frac{a_\xi}{2\sqrt{\alpha}}. \]

From (7), (3) and (1) we get
\[ U_a^{k,k'} = \langle U_a\phi_k, \phi_{k'} \rangle = \sqrt{\alpha}^2 \frac{e^{i a_\xi a_\xi/2}}{k! k'! \pi} \int_\mathbb{R} e^{ia_\xi x} e^{-\alpha(x + a_\xi)^2/2} e^{-ax^2/2} \mathcal{H}_k(\sqrt{\alpha}(x + a_\xi)) \mathcal{H}_k'(\sqrt{\alpha}x) dx \]
\[ = \frac{2^{-(k+k')/2}}{k! k'! \pi} e^{\omega^2 - a_\xi^2/2 + ia_\xi a_\xi/2} \int_\mathbb{R} e^{-x^2} \mathcal{H}_k(x - \omega + \sqrt{\alpha}a_\xi) \mathcal{H}_k'(x - \omega) dx \]
\[ = \sqrt{\frac{k!}{k'!}} 2^{(k' - k)/2} (-\omega)^{k' - k} e^{\omega^2 - a_\xi^2/2 + ia_\xi a_\xi/2} L_k^{(k' - k)}(-2\omega(\omega - \sqrt{\alpha}a_\xi)) \]
where the last line follows from (9). Now \( |\omega| = \rho \) while
\[ \omega^2 - \frac{a_\xi^2}{2} + \frac{ia_\xi a_\xi}{2} = -\frac{\alpha\xi^2}{\rho^2} - \frac{a_\xi^2}{4\alpha} - \frac{a_\xi^2}{2} + \frac{ia_\xi a_\xi}{2} = -|\omega|^2 \]
and
\[ -2\omega(\omega - \sqrt{\alpha}a_\xi) = -2\omega(-\omega) = 2|\omega|^2. \]

The result follows. \( \square \)

Throughout the remainder of this section we will assume \( a \in T^*\mathbb{R}\{0\} \) is fixed and \( \rho > 0 \) is given by (10).

Laguerre polynomials can be expressed in terms of the confluent hypergeometric function; using 22.5.54 in [AS] we get
\[ L_k^{(k' - k)}(2\rho^2) = \binom{k'}{k} M(-k, k' - k + 1, 2\rho^2). \]

The confluent hypergeometric function can, in turn, be written as a pointwise absolutely convergent series of Bessel functions; from 13.3.7 in [AS] we get
\[ M(-k, k' - k + 1, 2\rho^2) = (k' - k)! e^{\rho^2/2} (k' + k + 1)^{-(k' - k)/2} \sum_{j=0}^{\infty} A_j \frac{\rho}{(k' + k + 1)^{1/2}} J_{k' - k + j}(2\rho\sqrt{k' + k + 1}), \]
where
\[ A_0 = 1, \quad A_1 = 0, \quad A_2 = \frac{1}{2}(k' - k + 1) \quad (11) \]
and, for $j \geq 2$,

$$(j + 1)A_{j+1} = (j + k' - k)A_{j-1} - (k' + k + 1)A_{j-2}. \quad (12)$$

It follows from Lemma 2.1 that

$$U^k_{a} = e^{i(k' - k)\theta} \sqrt{F_{k',k}} \sum_{j=0}^{\infty} A_j \left( \frac{\rho}{(k' + k + 1)^{1/2}} \right)^j J_{k'-k+j}(2\rho\sqrt{k' + k + 1}), \quad (13)$$

where

$$F_{k',k} := \frac{k'!}{k!} \left( \frac{2}{k' + k + 1} \right)^{k'-k}.$$

The next two results give estimates for the constants appearing in (13).

**Lemma 2.2.** Suppose $k' \geq 2$ and $0 \leq k' - k \leq k'^{2/3}$. Then

$$|A_j| \leq (k' + k + 1)^{j/3}.$$  

**Proof.** Set $m = k' - k$ and $n = k' + k + 1$ so

$$0 \leq m \leq k'^{2/3} \leq (k' + k + 1)^{2/3} = n^{2/3}$$

while $k' \geq 2$ and $k \geq 0$ so $n \geq 3$.

We have $A_0 = 1 = n^0$, $A_1 = 0 \leq n^{1/3}$ and $m, 1 \leq n^{2/3}$ so $A_2 = \frac{1}{2}(m + 1) \leq n^{2/3}$.

Now let $J \geq 2$ and suppose the result hold for $j \leq J$. Since

$$A_{J+1} = \frac{J + m}{J + 1} A_{J-1} - \frac{n}{J + 1} A_{J-2}$$

we then get

$$|A_{J+1}| \leq \frac{J + m}{J + 1} n^{(J-1)/3} + \frac{n}{J + 1} n^{(J-2)/3} = \frac{n^{(J+1)/3} (J + m)n^{-2/3} + 1}{J + 1}.$$

Now $mn^{-2/3} \leq 1$ while

$$n \geq 3 \implies n^{-2/3} \leq 3^{-2/3} \leq \frac{1}{2}$$

$$\implies J(1 - n^{-2/3}) \geq 1 \quad \text{(as } J \geq 2)$$

$$\implies 1 + Jn^{-2/3} \leq J.$$

Thus $(J + m)n^{-2/3} + 1 \leq J$. Therefore $|A_{J+1}| \leq n^{(J+1)/3}$ and the result follows by induction.

**Lemma 2.3.** If $0 \leq k \leq k'$ then $F_{k',k} \leq 1$.  

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Proof. We have
\[ F_{k',k} = \frac{k'(k' - 1) \ldots (k + 1)}{\frac{1}{2}(k' + k + 1) \ldots \frac{1}{2}(k' + k + 1)}, \]
where the numerator and denominator both contain \( k' - k \) terms. Now set \( m = \frac{1}{2}(k' - k - 1) \) and \( n = \frac{1}{2}(k' + k + 1) \) so \( m \leq n \) while
\[ F_{k',k} = \frac{(n + m)(n - m)}{n^2} \frac{(n + m - 1)(n - m - 1)}{n^2} \ldots \frac{(n - m - 1)(n - m)}{n^2}. \]
If \( k' - k \) is odd this can be rearranged as
\[ F_{k',k} = \frac{(n + m)(n - m)}{n^2} \frac{(n + m - 1)(n - m - 1)}{n^2} \ldots \frac{(n + \frac{1}{2})(n - \frac{1}{2})}{n^2}. \]
The result now follows from the fact that
\[ \frac{(n + m')(n - m')}{n^2} = \frac{n^2 - m'^2}{n^2} \leq 1 \]
for any \( 0 \leq m' \leq n \).

Next we obtain some estimates for the Bessel functions appearing in (13).

Lemma 2.4. For any \( x, \varepsilon > 0 \) and \( n \in [0, x/2] \)
\[ \left| \{ \theta \in [0, \pi] \mid |x \cos(\theta) - n| < \varepsilon \} \right| \leq \frac{4\pi \varepsilon}{3x}. \]

Proof. Set \( \delta = \varepsilon/x \), \( y = n/x \) and \( \Omega_{y,\delta} = \text{Cos}^{-1}([y - \delta, y + \delta]) \); we need to show that \( |\Omega_{y,\delta}| \leq 4\pi \delta/3 \).

Now set \( \theta_0 = \text{Cos}^{-1}(y) \) and let \( \ell(\theta) \) denote the affine function with \( \ell(0) = 1 \) and \( \ell(\theta_0) = y \). It is easy to see that \( |\cos(\theta) - y| \geq |\ell(\theta) - y| \) which implies \( |\Omega_{y,\delta}| \leq 2\delta/|L| \) where \( L \) is the gradient of \( \ell(\theta) \). On the other hand, \( y \in [0, \frac{1}{2}] \) so the minimum value for \( |L| \) occurs when \( y = 1/2 \); hence \( 1/|L| \leq 2\text{Cos}^{-1}(1/2) = 2\pi/3 \) and the result follows.

Lemma 2.5. For any \( n \in \mathbb{N}_0 \) and \( x \geq 2n \) we have \( |J_n(x)| \leq 4x^{-1/2} \).

Surely this estimate (or an improvement) lies in a book somewhere!
Proof. Define a function by \( f(\theta) = x\sin(\theta) - n\theta \) so we have the following integral representation for the Bessel function \( J_n \) (see 9.1.21 in [AS]):

\[
J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(f(\theta)) \, d\theta. \tag{14}
\]

Now set
\[
\Omega_0 = \{ \theta \in [0, \pi] \mid |f'(\theta)| < x^{1/2} \}
\]
and \( \Omega_1 = [0, \pi] \setminus \Omega_0 \)
so \( J_n(x) = (I_0 + I_1)/\pi \) where \( I_k = \int_{\Omega_k} \cos(f(\theta)) \, d\theta \) for \( k = 0, 1 \). Lemma 2.4 gives
\[
|I_0| \leq |\Omega_0| \leq \frac{4\pi}{3} x^{-1/2}. \tag{15}
\]

On the other hand
\[
I_1 = \left[ \frac{\sin(f(\theta))}{f'(\theta)} \right]_{\partial \Omega_1} + \int_{\Omega_1} \frac{f''(\theta)}{(f'(\theta))^2} \sin(f(\theta)) \, d\theta.
\]

Now \( f''(\theta) = -x\sin(\theta) \leq 0 \) on \([0, \pi]\) while \((f'(\theta))^2 > 0\) on \( \Omega_1 \). Thus
\[
\left| \int_{\Omega_1} \frac{f''(\theta)}{(f'(\theta))^2} \sin(f(\theta)) \, d\theta \right| \leq - \int_{\Omega_1} \frac{f''(\theta)}{(f'(\theta))^2} \, d\theta = \left[ \frac{1}{f'(\theta)} \right]_{\partial \Omega_1}.
\]

Furthermore \( f'(\theta) \) is decreasing on \([0, \pi]\) so \( \Omega_0 \) consists of a single interval. Hence \( \partial \Omega_1 \setminus \{0, \pi\} \) contains at most 2 points. Since \( f(0) = 0 \) and \( f(\pi) = -n\pi \) we then get
\[
|I_1| \leq \left| \left[ \frac{\sin(f(\theta))}{f'(\theta)} \right]_{\partial \Omega_1} \right| + \left[ \frac{1}{f'(\theta)} \right]_{\partial \Omega_1} \leq 6 \max_{\theta \in \Omega_1} \frac{1}{|f'(\theta)|} \leq 6x^{-1/2}. \tag{16}
\]

Combining (15), (16) we now get
\[
|J_n(x)| \leq \frac{1}{\pi} (|I_0| + |I_1|) \leq \frac{1}{\pi} \left( \frac{4\pi}{3} + 6 \right) x^{-1/2} \leq 4x^{-1/2},
\]
completing the result.

Lemma 2.6. Suppose \( k' \geq 2 \), \( 0 \leq k' - k \leq \rho(k' + k + 1)^{1/2} \) and \( 2\rho \leq (k' + k + 1)^{1/6} \).
Then
\[
|U_{a}^{k,k'}| \leq \left( 4(2\rho)^{-1/2} + \frac{1}{2}(2\rho)^2 \right) (k' + k + 1)^{-1/4}.
\]

Before starting, note that as a clear consequence of (14) we have
\[
|J_n(x)| \leq 1. \tag{17}
\]
Proof. Since \(2, k \leq k'\)
\[
k' - k \leq \frac{1}{2}(k' + k + 1)^{2/3} \leq \frac{1}{2}\left(\frac{5}{2}\right)^{2/3} k'^{2/3} \leq k'^{2/3}.
\]

Now combining (13) with (11), (17) and Lemmas 2.2 and 2.3 we get
\[
|U_a^{k,k'}| \leq \sqrt{F_{k',k}} \sum_{j=0}^{\infty} |A_j| \frac{\rho^j}{(k' + k + 1)^{j/2}} |J_{k-k'}(2\rho\sqrt{k' + k + 1})| \\
\leq |J_{k-k'}(2\rho\sqrt{k' + k + 1})| + \sum_{j \geq 2} \rho^j (k' + k + 1)^{-j/6} \\
\leq |J_{k-k'}(2\rho\sqrt{k' + k + 1})| + \frac{1}{2}(2\rho)^2(k' + k + 1)^{-1/3},
\]
where the last line follows from the hypothesis that \(\rho(k' + k + 1)^{-1/6} \leq 1/2\). Lemma 2.5 can now be used to estimate the remaining Bessel function term.

Main estimate

The next result is the main estimate we will need for the matrix elements \(|\langle V\phi_k, \phi_{k'} \rangle|\).
This estimate is valid in a parabolic region around the diagonal \(k = k'\); the width of this region is governed by the quantity
\[
\gamma := \min_{a \in A'} |a|_\alpha,
\]
which is positive since \(A'\) is discrete and doesn’t contain 0. Although not required in this paper, we remark that for a general parabolic region around the diagonal one is restricted to estimates of the form \(|\langle V\phi_k, \phi_{k'} \rangle| \leq C(V)(k' + k + 1)^{-1/6}\).

**Proposition 2.7.** Suppose \(V\) satisfies condition (5) and set
\[
\kappa = \min\{1/3, \gamma/(2\sqrt{3})\}. \tag{18}
\]
If \(n \in \mathbb{N}\) and \(k, k' \in \mathbb{N}_0\) satisfy \(|k-n|, |k'-n| \leq \kappa n^{1/2}\) then
\[
|\langle V\phi_k, \phi_{k'} \rangle| \leq C(V)n^{-1/4}. \tag{19}
\]

**Proof.** We have \(|\langle V\phi_k, \phi_{k'} \rangle| \leq \|V\|\) for any \(k, k' \in \mathbb{N}_0\) so we can increase \(C(V)\) if necessary to ensure that (19) is satisfied for \(n = 1, 2\). Furthermore \(V\) is self-adjoint so \(|\langle V\phi_{k'}, \phi_k \rangle| = |\langle V\phi_k, \phi_{k'} \rangle|\). It thus suffices to prove the result assuming \(n \geq 3\) and \(k', k \in \mathbb{N}_0\) satisfy \(k' \geq k\) and \(|k-n|, |k'-n| \leq \kappa n^{1/2}\). Then \(k', k \geq n - \frac{1}{3}n^{1/2} \geq \frac{4}{3}n\) so \(k' \geq 2\),
\[
k' + k + 1 \geq \frac{4}{3}n. \tag{20}
\]
and
\[0 \leq k' - k \leq 2\kappa n^{1/2} \leq \frac{\gamma}{2}(k' + k + 1)^{1/2}.
\]

Now set \(K = (k + k + 1)^{1/6}\). Using (2), (7) and (8) we have
\[
|\langle V\phi_k, \phi_{k'} \rangle| \leq \sum_{a \in \Lambda'}|U_{a}^{k,k'}| |V_a| \leq \sum_{a \in \Lambda', |a|_\alpha \leq K} |U_{a}^{k,k'}| |V_a| + \sum_{a \in \Lambda', |a|_\alpha > K} |V_a|. \tag{22}
\]

Since \(1 < K^{-3/2}|a|^{3/2}_\alpha \) whenever \(|a|_\alpha > K\), (5) and (20) give us
\[
\sum_{a \in \Lambda', |a|_\alpha > K} |V_a| \leq K^{-3/2} \sum_{a \in \Lambda'} |a|^{3/2}_\alpha |V_a| \leq C(V) n^{-1/4}.
\]

Now let \(a \in \Lambda'\). Since \(|a|_\alpha = 2\rho\) (see (10)) the definition of \(\gamma\) implies \(\gamma/2 \leq \rho\) and thus \(k' - k \leq \rho K^3\) by (21). Lemma 2.6, (5) and (20) then give
\[
\sum_{a \in \Lambda', |a|_\alpha \leq K} |U_{a}^{k,k'}| |V_a| \leq K^{-3/2} \sum_{a \in \Lambda'} (4|a|^{-1/2}_\alpha + \frac{1}{2}|a|^{2}_\alpha) |V_a| \leq C(V) n^{-1/4}.
\]

The result follows.

**First order term**

The next result is used to obtain the explicit form for the first order correction term in the asymptotics for \(\lambda_n(H + V)\).

**Proposition 2.8.** Suppose \(V\) satisfies condition (5). Then
\[
\langle V\phi_n, \phi_n \rangle = W(\sqrt{n}) n^{-1/4} + O(n^{-1/2})
\]
as \(n \to +\infty\), where \(W\) is defined by (6).

**Proof.** Let \(a \in T^*\mathbb{R} \setminus \{0\}\) and set \(\rho = |a|_\alpha/2\). Using (13) and the fact that \(F_{n,n} = 1\) we get
\[
U_{a}^{n,n} = \sum_{j=0}^{\infty} A_j \frac{\rho^j}{(2n + 1)^{j/2}} J_j(2\rho \sqrt{2n + 1}).
\]

Now suppose \(2\rho \leq N\) where \(N := (2n + 1)^{1/6}\). Using (11), (17) and Lemma 2.2 we have
\[
|U_{a}^{n,n} - J_0(2\rho \sqrt{2n + 1})| \leq \frac{1}{2} \rho^2 (2n + 1)^{-1} + \sum_{j=3}^{\infty} \rho^j (2n + 1)^{-j/6}
\leq \frac{1}{8} |a|^{2}_\alpha (2n + 1)^{-1} + \frac{1}{4} |a|^{3}_\alpha (2n + 1)^{-1/2}.
\]
Standard asymptotic forms for Bessel functions (see 9.2.1 in [AS]) give us

\[ J_0(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) + O(z^{-3/2}) \]

while

\[ \left| \frac{d}{dz} \left( \frac{1}{\sqrt{z}} \cos\left(z - \frac{\pi}{4}\right) \right) \right| \leq z^{-1/2} + \frac{1}{2}z^{-3/2} \]

and \(2\rho\sqrt{2n+1} - 2\rho\sqrt{2n} \leq 2^{-1/2}\rho n^{-1/2}\). It follows that

\[
\begin{align*}
&\left| J_0(2\rho\sqrt{2n+1}) - \sqrt{\frac{2}{\pi}}(2\rho)^{-1/2}(2n)^{-1/4}\cos\left(2\rho\sqrt{2n} - \frac{\pi}{4}\right) \right| \\
&\quad \leq C(2\rho)^{-3/2}(2n+1)^{-3/4} \\
&\quad \quad + \sqrt{\frac{2}{\pi}}\left((2\rho)^{-1/2}(2n)^{-1/4} + \frac{1}{2}(2\rho)^{-3/2}(2n)^{-3/4}\right)2^{-1/2}\rho n^{-1/2} \\
&\quad \leq C((2\rho)^{-3/2} + (2\rho)^{1/2})n^{-3/4}.
\end{align*}
\]

Combining the above estimates we thus obtain

\[
\begin{align*}
U^{n,n}_a - \frac{2^{1/4}}{\sqrt{\pi}}|a|^{-1/2}n^{-1/4}\cos\left(|a|\sqrt{2n} - \frac{\pi}{4}\right) &\leq C(|a|^{-3/2} + |a|^3)\alpha^{-1/2} \\
\end{align*}
\]

whenever \(|a| \leq N\). Using (2), (6), (7) and (8) we thus have

\[
\left| \langle V\phi_n, \phi_n \rangle - W(\sqrt{n})n^{-1/4} \right| \\
\leq Cn^{-1/2} \sum_{a\in A'} (|a|^{-3/2} + |a|^3)|V_a| + \sum_{a\in A', |a|>N} (1 + |a|^{-1/2})|V_a|. 
\]

Since \(1 < N^{-3}|a|^3 < n^{-1/2}|a|^5\) whenever \(|a| > N\) the term inside the last sum can be replaced with \(n^{-1/2}(|a|^3 + |a|^5)|V_a|\). Using (5) we then get

\[
\left| \langle V\phi_n, \phi_n \rangle - W(\sqrt{n})n^{-1/4} \right| \leq Cn^{-1/2} \sum_{a\in A'} (|a|^{-3/2} + |a|^3)|V_a| \leq C(V)n^{-1/2},
\]

completing the result.

\[ \square \]

### 3 Resolvent Estimates

For any \(\lambda \in \mathbb{C}\setminus\sigma(H)\) let \(R(\lambda) = (H - \lambda)^{-1}\) denote the resolvent of the operator \(H\); we will also write \(R\) for \(R(\lambda)\) where this should not cause confusion.
Let $\kappa$ denote the constant defined in (18). For a given $n \in \mathbb{N}$ we will make repeated use of the partition of $\mathbb{N}_0$ defined by

$$I = \{ k \in \mathbb{N}_0 \mid |k - n| \leq \kappa n^{1/2} \} \quad \text{and} \quad J = \mathbb{N}_0 \setminus I.$$  \hspace{1cm} (23)

For any $\varepsilon \in (0, \alpha)$ and $n \in \mathbb{N}_0$, let $\Gamma_{\varepsilon,n}$ be the anti-clockwise circular contour in $\mathbb{C}$ centred at $\lambda_n = \lambda_n(H) = \alpha(2n + 1)$. If $\lambda \in \Gamma_{\varepsilon,n}$ then $\lambda = \alpha(2n + 1) + \varepsilon e^{i\theta}$ for some $\theta \in [0, 2\pi)$. It follows that $|\lambda - \lambda_k| = |2\alpha(n-k) + \varepsilon e^{i\theta}|$ for any $k \in \mathbb{N}_0$. Straightforward arguments then lead to the following estimates;

$$\sum_{k \in I} |\lambda - \lambda_k|^{-1} \leq C(\varepsilon) \ln(n),$$ \hspace{1cm} (24)

$$\sum_{k \in \mathbb{N}_0} |\lambda - \lambda_k|^{-2} \leq C(\varepsilon),$$ \hspace{1cm} (25)

$$\sum_{k \in J} |\lambda - \lambda_k|^{-2} \leq C n^{-1/2}$$ \hspace{1cm} (26)

and

$$|\lambda - \lambda_k| \geq C n^{1/2} \quad \text{for any } k \in J.$$ \hspace{1cm} (27)

The first two results in this section relate to the operator $R(\lambda)VR(\lambda)$, which is clearly bounded whenever $\lambda$ is in the resolvent set of $H$. We show that it is in fact trace class while its operator norm decreases as $n^{-1/4}$ for $\lambda \in \Gamma_{\varepsilon,n}$.

**Lemma 3.1.** For any $n \in \mathbb{N}$ and $\lambda \in \Gamma_{\varepsilon,n}$ we have

$$\|R(\lambda)VR(\lambda)\| \leq \|R(\lambda)VR(\lambda)\|_2 \leq C(V, \varepsilon) n^{-1/4}.$$  

We remark that since $\{\phi_k \mid k \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(\mathbb{R})$

$$\sum_{k' \in \mathbb{N}_0} |\langle V\phi_k, \phi_{k'} \rangle|^2 = \|V\phi_k\|^2 \leq \|V\|^2.$$ \hspace{1cm} (28)

**Proof.** Using the orthonormal basis $\{\phi_k \mid k \in \mathbb{N}_0\}$ we have

$$\|RVR\|_2^2 = \sum_{k,k' \in \mathbb{N}_0} |\langle RVR\phi_k, \phi_{k'} \rangle|^2 = \sum_{k,k' \in \mathbb{N}_0} \frac{|\langle V\phi_k, \phi_{k'} \rangle|^2}{|\lambda_k - \lambda|^2|\lambda_{k'} - \lambda|^2}.$$ \hspace{1cm} (29)

We will split this sum using the partition (23). Firstly Proposition 2.7 and (25) imply

$$\sum_{k,k' \in I} \frac{|\langle V\phi_k, \phi_{k'} \rangle|^2}{|\lambda_k - \lambda|^2|\lambda_{k'} - \lambda|^2} \leq C(V)n^{-1/2} \left( \sum_{k \in I} \frac{1}{|\lambda_k - \lambda|^2} \right)^2 \leq C(V, \varepsilon)n^{-1/2}.$$
Now using (27), (28) and (25) we get
\[
\sum_{k \in \mathbb{N}_0} \frac{|\langle V \phi_k, \phi_k' \rangle|^2}{|\lambda_k - \lambda|^2 |\lambda_k' - \lambda|^2} \leq C n^{-1} \sum_{k \in \mathbb{N}_0} \frac{1}{|\lambda_k - \lambda|^2} \sum_{k' \in J} |\langle V \phi_k, \phi_k' \rangle|^2 \leq C (V, \varepsilon) n^{-1}.
\]

The remaining part of the sum on the right hand side of (29) involves \(k \in J\) and \(k' \in I \subset \mathbb{N}_0\); thus we can estimate this part using an argument similar to the last one with \(k\) and \(k'\) swapped.

**Lemma 3.2.** For any \(n \in \mathbb{N}_0\) and \(\lambda \in \Gamma_{\varepsilon,n}\) the operator \(R(\lambda)VR(\lambda)\) is trace class. Furthermore \(\|R(\lambda)VR(\lambda)\|_1\) is uniformly bounded (in \(n\) and \(\lambda \in \Gamma_{\varepsilon,n}\)).

**Proof.** The set \(\{\phi_k \mid k \in \mathbb{N}_0\}\) is an orthonormal eigenbasis for \(R\) with corresponding eigenvalues \((\lambda_k - \lambda)^{-1}, k \in \mathbb{N}_0\) so (25) implies
\[
\|R\|^2_2 = \sum_{k \in \mathbb{N}_0} |\lambda_k - \lambda|^{-2} \leq C(\varepsilon).
\]
Thus \(\|RV\|_1 = \|VR^2\|_1 \leq \|V\| \|R^2\|_1 \leq \|V\| \|R\|^2_2 \leq C(\varepsilon) \|V\|\).

Suppose \(n \in \mathbb{N}_0\) and \(j \in \mathbb{N}\). From the previous result we know that \(R(\lambda)VR(\lambda)\) is trace class for any \(\lambda \in \Gamma_{\varepsilon,n}\). On the other hand \(R(\lambda)V\) is bounded (in fact \(\|R(\lambda)V\| \leq \varepsilon^{-1} \|V\|\)). It follows that
\[
(R(\lambda)V)^j R(\lambda) = (R(\lambda)V)^{j-1} R(\lambda) VR(\lambda)
\]
is also trace class with trace norm uniformly bounded for \(\lambda \in \Gamma_{\varepsilon,n}\). The work in the remainder of this section leads to Proposition 3.5 where we obtain an estimate for the trace of an integral of such operators.

**Lemma 3.3.** Let \(n \geq 2\), \(\lambda \in \Gamma_{\varepsilon,n}\) and suppose \(f : \mathbb{N}_0 \to \mathbb{C}\) satisfies
\[
\sum_{k \in \mathbb{N}_0} |f(k)|^2 \leq C_1^2 \quad \text{and} \quad |f(k)| \leq C_1 n^{-1/4} \text{ when } k \in I
\]
for some constant \(C_1\). For each \(k \in \mathbb{N}_0\) set
\[
g(k) = \sum_{k' \in \mathbb{N}_0} \frac{f(k') \langle V \phi_k', \phi_k \rangle}{\lambda - \lambda_{k'}}.
\]
Then there exists a constant \(K = K(V, \varepsilon)\) such that
\[
\sum_{k \in \mathbb{N}_0} |g(k)|^2 \leq C_1^2 K^2 n^{-1/2} \ln^2(n) \quad \text{and} \quad |g(k)| \leq C_1 K n^{-1/2} \ln(n) \text{ when } k \in I.
\]
Proof. Since
\[ \sum_{k \in \mathbb{N}_0} |g(k)|^2 = \sum_{k, k', k'' \in \mathbb{N}_0} \frac{f(k')f(k'')\langle V\phi_{k'}, \phi_k \rangle \langle \phi_k, V\phi_{k''} \rangle}{(\lambda - \lambda_{k'}) (\lambda - \lambda_{k''})} \]
and
\[ \left| \sum_{k \in \mathbb{N}_0} \langle V\phi_{k'}, \phi_k \rangle \langle \phi_k, V\phi_{k''} \rangle \right| = \left| \langle V\phi_{k'}, V\phi_{k''} \rangle \right| \leq \|V\|^2 \]
it follows that
\[ \sum_{k \in \mathbb{N}_0} |g(k)|^2 \leq \|V\|^2 \left( \sum_{k \in I} \frac{|f(k)|}{|\lambda - \lambda_k|} + \sum_{k \in J} \frac{|f(k)|}{|\lambda - \lambda_k|} \right)^2. \]
Using the second part of (30) and (24) we get
\[ \sum_{k \in I} \frac{|f(k)|}{|\lambda - \lambda_k|} \leq C_1 n^{-1/4} \sum_{k \in I} |\lambda - \lambda_k|^{-1} \leq C_1 C_2 (\varepsilon) n^{-1/4} \ln(n). \]
On the other hand the first part of (30) and (26) give
\[ \sum_{k \in J} \frac{|f(k)|}{|\lambda - \lambda_k|} \leq \left( \sum_{k \in J} |f(k)|^2 \right)^{1/2} \left( \sum_{k \in J} |\lambda - \lambda_k|^2 \right)^{1/2} \leq C_1 C_3 n^{-1/4} \leq 2C_1 C_3 n^{-1/4} \ln(n). \]
Putting these estimates together now leads to
\[ \sum_{k \in \mathbb{N}_0} |g(k)|^2 \leq C_1^2 K_1^2 n^{-1/2} \ln^2(n), \]
with \( K_1 = \|V\|(C_2(\varepsilon) + 2C_3). \) Now suppose \( k \in I \) and write \( g(k) = g_I(k) + g_J(k) \) where
\[ g_I(k) = \sum_{k' \in I} \frac{f(k') \langle V\phi_{k'}, \phi_k \rangle}{\lambda - \lambda_{k'}} \quad \text{and} \quad g_J(k) = \sum_{k' \in J} \frac{f(k') \langle V\phi_{k'}, \phi_k \rangle}{\lambda - \lambda_{k'}}. \]
From Proposition 2.7 the second part of (30) and (24) we get
\[ |g_I(k)| \leq C_1 C(V) n^{-1/2} \sum_{k' \in I} |\lambda - \lambda_{k'}|^{-1} \leq C_1 C_4(V, \varepsilon) n^{-1/2} \ln(n). \]
On the other hand (27), the first part of (30) and (28) give us
\[ |g_J(k)| \leq C(\varepsilon) n^{-1/2} \left( \sum_{k' \in J} |f(k')|^2 \right)^{1/2} \left( \sum_{k' \in J} |\langle V\phi_{k'}, \phi_k \rangle|^2 \right)^{1/2} \leq C_1 C_5(\varepsilon) \|V\| n^{-1/2} \leq 2C_1 C_5(\varepsilon) \|V\| n^{-1/2} \ln(n). \]
Putting these estimates together now leads to \( |g(k)| \leq C_1 K_2 n^{-1/2} \ln(n) \) with \( K_2 = C_4(V, \varepsilon) + 2C_5(\varepsilon) \|V\|. \) Taking \( K = \max\{K_1, K_2\} \), completes the result. \( \blacksquare \)
Proposition 3.5. Suppose

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Lemma 3.4. from Proposition 2.7 and Lemma 3.3 respectively.

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we have

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Since
\[ \frac{d}{d\lambda} \lambda \left( \prod_{i=0}^{j-1} \frac{1}{\lambda k_i - \lambda} \right) = \prod_{i=0}^{j-1} \frac{1}{\lambda k_i - \lambda} + \lambda \sum_{i=0}^{j-1} \frac{1}{\lambda k_i - \lambda} \left( \prod_{i=0}^{j-1} \frac{1}{\lambda k_i - \lambda} \right) \]
for any \( k_0, \ldots, k_{j-1} \in \mathbb{N}_0 \), we can rewrite the above equation as
\[ \text{Tr} \, \lambda R(V R)^j = \frac{d}{d\lambda} (\lambda A(\lambda)) - A(\lambda). \]

Integrating around the contour \( \Gamma_{\varepsilon,n} \) it follows that
\[ \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(V R)^j d\lambda = -\frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} A(\lambda) d\lambda. \tag{32} \]

The poles of the meromorphic function \( A(\lambda) \) occur at the points \( \lambda = \lambda_k \) for \( k \in \mathbb{N}_0 \). Since the only such point enclosed by the contour \( \Gamma_{\varepsilon,n} \) is \( \lambda = \lambda_n \), it follows that the only terms in the series \( \text{Tr} \lambda R(V R)^j \) which contribute to the right hand side of \( \text{Tr} \lambda R(V R)^j \) are those with at least one of \( k_0, \ldots, k_{j-1} \) equal to \( n \). With the help of symmetry we then obtain the identity
\[ \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(V R)^j d\lambda = -\frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} A(\lambda) d\lambda. \tag{33} \]

For any \( \lambda \in \Gamma_{\varepsilon,n} \) we have \( |\lambda_n - \lambda| = \varepsilon \) while
\[ \left| \sum_{k_{1}, \ldots, k_{j-1} \in \mathbb{N}_0} \frac{\langle V \phi_n, \phi_{k_1} \rangle \langle V \phi_{k_1}, \phi_{k_2} \rangle \cdots \langle V \phi_{k_{j-1}}, \phi_n \rangle}{(\lambda_{k_1} - \lambda) \cdots (\lambda_{k_{j-1}} - \lambda)} \right| \leq K^j n^{-j/4} \ln^{j-1}(n) \]
by Lemma 3.4. Since the length of \( \Gamma_{\varepsilon,n} \) is \( 2\pi \varepsilon \) we finally get
\[ \left| \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(V R)^j d\lambda \right| \leq \frac{1}{2\pi} \oint_{\Gamma_{\varepsilon,n}} \frac{1}{\lambda_n - \lambda} K^j n^{-j/4} \ln^{j-1}(n) d\lambda = K^j n^{-j/4} \ln^{j-1}(n), \]
completing the result. \( \blacksquare \)

Taking \( j = 1 \) in \( \text{Tr} \lambda R(V R)^j \) leads to the formula
\[ \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(\lambda) V R(\lambda) d\lambda = -\frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \frac{1}{\lambda_n - \lambda} \langle V \phi_n, \phi_n \rangle d\lambda = \langle V \phi_n, \phi_n \rangle. \tag{34} \]

This is needed to obtain the first order correction term in Theorem 1.1.
4 Proof of Theorem 1.1

Lemmas 3.1 and 3.2 give us \( \|R(\lambda)V R(\lambda)\| \leq C_1 n^{-1/4} \) and \( \|R(\lambda)V R(\lambda)\|_1 \leq C_2 \) for all \( n \in \mathbb{N} \) and \( \lambda \in \Gamma_{\varepsilon,n} \). In particular \( \|(V R(\lambda))^2\| \leq \|V\| C_1 n^{-1/4} \). We also note that \( \|R(\lambda)\| = \varepsilon^{-1} \). It follows that for any \( j \in \mathbb{N}_0 \) we get

\[
\|(V R(\lambda))^{2j}\| \leq \|(V R(\lambda))^2\|^j \leq (\|V\| C_1 n^{-1/4})^j
\]

and

\[
\|(V R(\lambda))^{2j+1}\| \leq \|V\|\|R(\lambda)\|\|(V R(\lambda))^{2j}\| \leq \|V\|\varepsilon^{-1}(\|V\| C_1 n^{-1/4})^j.
\]

Choose \( N' \in \mathbb{N} \) so that \( \|V\| C_1 N'^{-1/4} \leq 1/2 \). It follows that for any \( n \geq N' \) and \( \lambda \in \Gamma_{\varepsilon,n} \) the series

\[
(I + V R(\lambda))^{-1} = \sum_{j=0}^{\infty} (-V R(\lambda))^j
\]

is absolutely convergent and has norm bounded by \( 2(1 + \|V\|\varepsilon^{-1}) \). In particular, \( I + V R(\lambda) \) is invertible with a uniformly bounded inverse for all \( n \geq N' \) and \( \lambda \in \Gamma_{\varepsilon,n} \).

On the other hand, the series

\[
T(\lambda) := \sum_{j=1}^{\infty} R(\lambda)(-V R(\lambda))^j = -R(\lambda)V R(\lambda) \sum_{j=0}^{\infty} (-V R(\lambda))^j
\]

is convergent in trace class with

\[
\|T(\lambda)\|_1 \leq \|R(\lambda)V R(\lambda)\|_1 \|(I + V R(\lambda))^{-1}\| \leq 2C_2(1 + \|V\|\varepsilon^{-1})
\]

for \( n \geq N' \) and \( \lambda \in \Gamma_{\varepsilon,n} \). Setting

\[
T_n = -\frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda T(\lambda) \, d\lambda
\]

it follows that we have an absolutely convergent expansion

\[
\text{Tr} T_n = -\sum_{j=1}^{\infty} \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(\lambda)(-V R(\lambda))^j \, d\lambda
\]

whenever \( n \geq N' \).

Now choose \( N \geq N' \) so that \( KN^{-1/4} \ln(N) \leq 1/2 \) where \( K \) is the constant given by Proposition 3.5. Using this Proposition and the above results it follows that

\[
\left| \sum_{j=2}^{\infty} \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon,n}} \lambda R(\lambda)(-V R(\lambda))^j \, d\lambda \right| \leq 2K^2 n^{-1/2} \ln(n)
\]
for all $n \geq N$. Therefore

$$\text{Tr} T_n = \text{Tr} \frac{1}{2\pi i} \oint_{\Gamma_{\epsilon,n}} \lambda R(\lambda) V R(\lambda) d\lambda + O(n^{-1/2}\ln(n))$$

$$= \langle V \phi_n, \phi_n \rangle + O(n^{-1/2}\ln(n))$$

for all $n \geq N$, where we have used (34).

The argument can be tied together using a standard resolvent expansion. Set $R_V(\lambda) = (H + V - \lambda)^{-1}$ and let $n \geq N$. Then

$$R_V(\lambda) = R(\lambda)(1 + VR(\lambda))^{-1} = R(\lambda) \sum_{j=0}^{\infty} (-VR(\lambda))^j.$$

The right hand side of (35) will still converge if $V$ is replaced with $gV$ for some $g \in [0,1]$. Hence $\sigma(H + gV) \cap \Gamma_{\epsilon,n} = \emptyset$. Since the eigenvalues of $H + gV$ depend continuously on $g$, it follows that $\Gamma_{\epsilon,n}$ must enclose $\lambda_n(H + V)$ but no other points of $\sigma(H + V)$. Thus we can write

$$\lambda_n(H + V) - \lambda_n(H) = -\frac{1}{2\pi i} \text{Tr} \oint_{\Gamma_{\epsilon,n}} \lambda(R_V(\lambda) - R(\lambda)) d\lambda$$

$$= -\frac{1}{2\pi i} \text{Tr} \oint_{\Gamma_{\epsilon,n}} \lambda \sum_{j=1}^{\infty} R(\lambda)(-VR(\lambda))^j d\lambda$$

$$= \text{Tr} T_n = \langle V \phi_n, \phi_n \rangle + O(n^{-1/2}\ln(n)).$$

Theorem 1.1 now follows from Proposition 2.8.

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