Approach to shape factor determination in estimating horizontal wells productivity for pressure transient model of Mutalik et al.

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Abstract
Horizontal well productivity \((J_h)\) equation at pseudo-steady state (PSS) of Mutalik et al. (in: Presented at SPE 63rd annual technical conference and exhibition, Houston, Texas, 1988) has long and widely been known to estimate well performance. One key parameter of that equation is the so-called shape factor \((C_A)\) that characterizes the effect of the pressure transient behavior at PSS depending on well and reservoir configuration. For limited well and reservoir configurations, \(C_A\) tables had been generated decades ago, but at unknown accuracy they are still used widely via tedious correlation or interpolation from those reported \(C_A\) tables to estimate \(J_h\) for different configurations even today. The innovation points in this study are as follows: (1) The pressure transient model of Mutalik et al. (in: Presented at SPE 63rd annual technical conference and exhibition, Houston, Texas, 1988) (PTM) is improved to generalized pressure transient model (GPTM) and verified by converting several dimensional parameters to dimensionless forms. (2) An efficient algorithm is developed, and accurate \(C_A\) and \(J_h\) are obtained for any well and reservoir configuration. (3) The accuracy of the reported \(C_A\) tables is quantified, and the consequences are determined when they are used to estimate \(J_h\). (4) New \(C_A\) tables and their correspondence dimensionless time based on drainage area at PSS \((T_{DA})_{PSS}\) tables are generated for a wider range of well and reservoir configurations.

Keywords Horizontal well productivity · Shape factor · Pressure transient model · Rectangular bounded reservoir · Instantaneous source functions

Introduction

\(J_h\) equation of Mutalik et al. (1988) is well known and presented by many authors (Duda et al. (1991), Economides et al. (1991b), Economides et al. (1991a), Frick and Economides (1993), Thomas et al. (1998), Penmatcha et al. (1999), Saavedra and Joshi (2002), Cho (2003), Jackson et al. (2011), Adesina et al. (2011), Ouyang (2015), Cetkovic et al. (2016), Wu et al. (2018), Luo et al. (2018), and Zhang et al. (2020)). In reservoir simulators, \(J_h\) equation of Mutalik et al. (1988) has been used to compute well productivity index (Krawchuk et al. 2006; Pinzon et al. 2007; Huebsch et al. 2008; Haidar et al. 2008; Brown and Tiwari 2010; Wood et al. 2010; and Leone et al. 2015). The \(J_h\) equation is defined as follows:

\[
J_h = \frac{0.007078kh}{\mu_oB_o} - \frac{2.1243 - \ln(L/4r_w) + \ln\sqrt{30.8828/C_A}}{\ln\left(\frac{\sqrt{A/x}}{r_w}\right)} + \ln\left[2\left(2\pi(T_{DA})_{PSS} - (P_{D}T)_{PSS}\right)\right]
\]

\(C_A = \frac{8.9832x_ey_eL^2}{2\pi r_w^2} - \exp\left[2\left(2\pi(T_{DA})_{PSS} - (P_{D}T)_{PSS}\right)\right]
\]

where \(J_h\) is the horizontal well productivity (STB/day.psi), \(k\) is the reservoir permeability (md), \(h\) is the reservoir thickness (ft), \(\mu_o\) is the oil viscosity (cp), \(B_o\) is the oil formation volume factor (bbl/STB), \(A\) is the drainage area (ft\(^2\)), \(r_w\) is the well radius (ft), \(L\) is the length of horizontal well (ft), \(x_e\) and \(y_e\) are the reservoir aspect ratio.
is the reservoir length in the X-direction (ft), the horizontal well is placed along the X-direction, \( y_e \) is the reservoir length in the Y-direction (ft), and \((P_D T)_{\text{PSS}}\) is the total dimensionless pressure at PSS.

The \( J_h \) equation [Eq. (1)] is for rectangular bounded reservoir at depletion stage (i.e., PSS). In \( C_A \) equation [i.e., Eq. (2)], \((P_D T)_{\text{PSS}}\) was obtained by the PTM:

\[
P_D T = 2\pi \int_0^{T_{\text{DA}}} \left\{ \left[ 1 + \frac{8x_e}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2\pi^2y_e T_{\text{DA}}}{x_e} \right) \right] \cos \left( \frac{n\pi L x_{\text{WD}}}{4x_e} \right) \right. \\
\cos \left( \frac{n\pi L (x_{\text{WD}} + x_D)}{4x_e} \right) \cos \left( \frac{n\pi L (y_{\text{WD}} + y_D)}{4y_e} \right) \left[ 1 + 2 \sum_{n=1}^{\infty} \exp \left( -\frac{n^2\pi^2y_e T_{\text{DA}}}{y_e} \right) \right] \\
\left. \sum_{n=1}^{\infty} \exp \left( -\frac{n^2\pi^2y_e T_{\text{DA}}}{h^2} \right) \cos \left( n\pi z_{\text{WD}} \right) \cos \left( n\pi z_D \right) \right\} dT_{\text{DA}}
\]

where \( P_D T \) is the total dimensionless pressure, \( T_{\text{DA}} \) is the dimensionless time based on drainage area, \( x_{\text{WD}}, y_{\text{WD}}, \) and \( z_{\text{WD}} \) are the dimensionless well location coordinates, \( x_D \) is the X-coordinate for a point in the porous media. \( (x_D = 0) \) at the horizontal well center, and \( x_D = 1 \) and \(-1\) at the well tips. \( y_D \) is the Y-coordinate for a point in the porous media. \( (y_D = y_{\text{WD}}) \) if the point is at the well wall, \( z_D \) is the Z-coordinate for a point in the porous media. \( (z_D = z_{\text{WD}}) \) if the point is at the well wall.

PTM [i.e., Eq. (3)] is based on 3D instantaneous source functions with Newman product method under homogenous and isotropic rectangular bounded reservoir at constant well flow rate and uniform flux wellbore condition of Gringarten and Ramey (1973). It is assumed that horizontal well is analogous to a vertical well of a vertical fracture. The equivalent infinite conductivity wellbore condition was obtained by solving \( P_D T \) at \( x_D \) equal to \( 0.732 \) (Mutalik et al. (1988) and Gringarten et al. (1974)). “Appendix 1” presents a summary of PTM.

When \( P_D T \) slope with time becomes constant at PSS and is equal to \( 2\pi \), both the \( T_{\text{DA}} \) and \( (P_D T)_{\text{PSS}} \) have been obtained to use for \( C_A \) calculation. \( C_A \) at PSS also becomes constant for a given well and reservoir configuration.

PTM and their reported \( C_A \) tables have identified three issues. Firstly, PTM as defined in Eq. (3) contains several dimensional parameters (i.e., \( x_e, y_e, h, \) and \( L \) ) and this makes the solution not only awkward, but also difficult to use PTM to expand the reported \( C_A \) tables to any other well and reservoir configurations. Secondly, PTM has not been verified in terms of \( P_D T \) which may lead to errors in the reported \( C_A \) tables. The third issue is the discrepancy in the assumed input data of the reported \( C_A \) tables. The first assumption of the input data which simply interpolation or extrapolation of those tables is required as the correlative models previously presented by Bahadori (2012), Bahadori et al. (2013), and Ahmadi et al. (2015).

The main purposes of this study are in four parts. (1) To overcome the first weakness in PTM (i.e., the dimensional parameters) by recasting PTM into a fully dimensionless form (i.e., GPTM) and to verify GPTM in terms of \( P_D T \). (2) To develop an efficient algorithm to compute \( C_A \) and \( J_h \) accurately for any well and reservoir configuration. Because \( C_A \) depends on the all-configuration parameters, unlimited tables of \( C_A \) can be generated; therefore, the MATLAB code is provided. (3) To quantify the accuracy of the reported \( C_A \) tables in their both assumed input data (i.e., constant and variable \( r_{\text{WD}} \)), and to determine the consequences of any identified errors in the reported \( C_A \) tables when they are used to estimate \( J_h \). (4) To generate new \( C_A \) tables for a wider spectrum of well and reservoir configurations.

The generalized pressure transient model (GPTM)

The PTM is improved to fully dimensionless form (i.e., to GPTM) by applying the following dimensionless parameters into Eq. (3):

\[
L_D = \frac{L}{2h}
\]

\[
x_{\text{WD}} = \frac{2x_e}{L}
\]
where \( x_{eD} \) is the dimensionless reservoir length in the \( X \)-direction and \( y_{eD} \) is the dimensionless reservoir length in the \( Y \)-direction.

For anisotropic reservoir solution, the dimensionless parameters are presented in “Appendix 2.”

The following meaningful mathematical relationship should be noted in the well and reservoir configuration:

\[
\begin{align*}
\frac{y_{wD}}{y_{eD}} & = \frac{y_{w}}{y_{e}} \\
\text{Used for centrally and off-centrally well location (OCWL) in the} \ Y \text{-direction.} \\
\frac{x_{wD}}{x_{eD}} & = \frac{x_{w}}{x_{e}} \\
\text{Used for centrally and OCWL in the} \ X \text{-direction.} \\
\frac{z_{wD}}{z_{eD}} & = \frac{z_{w}}{z_{e}} \\
\text{Used for centrally and OCWL in the} \ Z \text{-direction.} \\
\text{PR} & = \frac{L_D}{2x_{eD}} \\
\text{PR is also equal to (1/}x_{eD}) \ [i.e., \text{the inverse of Eq. (5)].}
\end{align*}
\]

Note that OCWL is used to indicate the offset of the well relative to the central well location (i.e., OCWL = 1) at a respective coordinate direction for the purpose of classifying the configurations only.

The efficient algorithm to solve Eq. (7) is developed and implemented in MATLAB code and is given in “Appendix 3.”

\[ P_{DT} = 2\pi \int_{0}^{\pi} \left\{ 1 + 4x_{eD} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n^2\pi^2 y_{eD}^2 T_{DA}}{x_{eD}} \right) \cos \left( n\pi x_{wD} \right) \right. \]
\[ \times \cos \left( \frac{n\pi (x_{wD} + x_{D})}{2x_{eD}} \right) \sin \left( \frac{n\pi x_{wD}}{2x_{eD}} \right) \left[ 1 + 2 \sum_{n=1}^{\infty} \exp \left( -\frac{n^2\pi^2 y_{eD}^2 T_{DA}}{y_{eD}} \right) \right] \]
\[ \left. \cos \left( \frac{n\pi (y_{wD} + y_{D})}{2y_{eD}} \right) \cos \left( \frac{n\pi y_{wD}}{2y_{eD}} \right) \right] \right\} dT_{DA} \]

Fig. 1 Comparison of \( P_{DT} \) by GPTM and by the pressure transient model of Ozkan and Raghavan (1988) for a centrally well located in a square reservoir.
(x_D = 0), and (r_wD = 10^{-4}). Figure 2 shows the identical results of P_D T by both the models.

In the both previous verifications, the time intervals of T_D A in Eq. (7) are to be taken: (10^{-9}–10^{-8}), (10^{-8}–10^{-7}), (10^{-7}–10^{-6}), (10^{-6}–10^{-5}), (10^{-5}–10^{-4}), (10^{-4}–10^{-3}), (10^{-3}–10^{-2}), (10^{-2}–10^{-1}), (1–10), and (10–100), and within each of these intervals its T_D A increment is taken to be (10^{-9}), (10^{-9}), (10^{-9}), (10^{-9}), (10^{-9}), (10^{-8}), (10^{-7}), (10^{-6}), (10^{-5}), (10^{-5}), and (10^{-2}), respectively. These T_D A intervals and their increments are sufficient to yield accurate P_D T. It is found that the first-time logarithmic cycle (FTLC) [i.e., (0–10^{-9})] plays a crucially important role in determining the accuracy of P_D T at any period even at PSS and then in C_A calculation as to be shown in the next sections.

The C_A equation [i.e., Eq. (2)] can be written in general form by substituting Eqs. (5) and (6) into Eq. (2):

\[
C_A = 2.2458 x_{vD} v_{eD} \exp \left[ 2 \left( 2 \pi (T_D A)_{PSS} - (P_D T)_{PSS} \right) \right]
\]  

(8)

For vertical fracture well (i.e., when (L_D > 10^3)), a comparison between the C_A(s) by this study and their counterparts by Earlougher (1977) and Hagoort (2009) is presented in Table 1. For FTLC equal to (0–10^{-9}) or (0–10^{-5}), excellent matching is found with C_A(s) by Hagoort (2009) at any PR and with C_A(s) by Earlougher (1977) at just high PR.

In what follows, the reported C_A tables by PTM for selected well and reservoir configurations are examined, and their well and reservoir configurations can be classified into four cases in terms of OCWL. Table 2 lists the four cases of the reported C_A tables. The parameters in Cases (1) to (3) correspond to tables (4), (7), and (8) in Mutalik et al. (1988) paper, respectively. Note that parameters in Case (4) are deduced from Fig. 12 in Mutalik et al. (1988) paper.

The accuracy of the reported C_A tables by PTM (case 1)

Although the accuracy of the reported C_A tables by PTM for other cases has been examined, here only the results for the Case 1 are presented for both assumed input data (i.e., the discrepancy issue). For each C_A by PTM, the counterpart C_A

| Table 1 Comparison of C_A by this study, Earlougher (1977) and Hagoort (2009) |
|---|---|---|---|
| PR | C_A (X_D=0.732) | C_A (X_D=0.740098) |
|---|---|---|
| Earlougher (1977) | FTLC of this study | FTLC of this study |
|---|---|---|
| Hagoort (2009) | 0.7881 | 0.788619 | 0.788444 |
| 1 | 0.7881 | 0.788619 | 0.788444 |
| 0.7 | 1.3127 | 1.309808 | 1.309011 |
| 0.5 | 1.6620 | 1.589124 | 1.587596 |
| 0.3 | 1.9986 | 1.789103 | 1.785893 |
| 0.2 | 2.0348 | 1.855014 | 1.849753 |
| 0.1 | 2.6541 | 1.899797 | 1.888444 |
| 0.7881 | 0.788619 | 0.788444 |
| 1.3127 | 1.309808 | 1.309011 |
| 1.6620 | 1.589124 | 1.587596 |
| 1.9986 | 1.789103 | 1.785893 |
| 2.0348 | 1.855014 | 1.849753 |
| 2.6541 | 1.899797 | 1.888444 |
is calculated by GPTM. The relative errors of the reported $C_A$ tables are determined [i.e., $C_A \% \text{Err}= \left( \left( C_A \text{PTM} - C_A \text{GPTM} \right) / C_A \text{GPTM} \right) \times 100$] for the both assumed input data.

Tables 3 and 4 show the relative errors in the reported $C_A$ tables by PTM for the variable $r_{WD}$ and the constant $r_{WD}'$, respectively. For the first assumed input data (variable $r_{WD}$), Table 3 shows that in $x_e/y_e = (1)$ and (2), the $C_A$ errors of any PR are in reverse order with $L_D$, while in $x_e/y_e = (5)$, they are in non-order with $L_D$. The maximum $C_A$ error is (50%) at (PR = 1), ($L_D = 1$), and ($x_e/y_e = 2$). For the second assumed input data (constant $r_{WD}$), Table 4 shows that in $x_e/y_e = (1)$ and (2), the $C_A$ errors of any PR are in reverse order with $L_D$, while in $x_e/y_e = (5)$, they are in non-order with $L_D$. The maximum $C_A$ error is (1215%) at (PR = 0.2), ($L_D = 1$), and ($x_e/y_e = 1$). For any assumed input data, $T_{\text{DA/PPS}}$ by GPTM is noticeably close to its counterpart by PTM and both are independent of $L_D$.

The errors in the reported $C_A$ tables are also found in the cases of off-center well location in $x$- and $y$-directions for both assumed input data but are not reported here.

To determine the consequences of the identified errors in the reported $C_A$ tables of Mutalik et al. (1988) when they are used to estimate $J_h$, an example problem is applied. The reservoir of ($y_e = 933.33$ ft) and ($r_w = 0.225$ ft) is for the first assumed input data, and the reservoir of ($y_e = 2640$ ft) and ($r_w = 10^{-4}$) is for the second assumed input data, and all are at ($k = 1$ md), ($\mu_o = 0.5$ cp), and ($B_o = 1.2$ rbbl/STB). The $J_h$ is calculated by using Eq. (1), and the $C_A(s)$ by PTM and GPTM is applied by using Tables 3 and 4. For the first assumed input data, PR = (0.4), (1), and (0.8) for $x_e/y_e = (1)$, (2), and (5), respectively. For the second assumed input data, PR = (0.2) and (0.8) for $x_e/y_e = (1)$, and for $x_e/y_e = (2)$ and (5) they are not reported here. ($OCWL = 1$) and $L_D$ is from (1) to (100). Figures 3 and 4 show a comparison of those $J_h(s)$ and $C_A(s)$ for the both assumed input data, respectively.

Figure 3 shows that the reported $C_A$ tables of PTM lead to underestimation in the $J_h$ for $x_e/y_e = (1)$ and (2) and lead to overestimation in the $J_h$ for $x_e/y_e = (5)$. The relative error in the $J_h$ for each selected $x_e/y_e = (1)$, (2), and (5) follows the same trend as those in $C_A$ table. The maximum underestimation in $J_h$ is (8.29%) corresponding to the error in $C_A$ at ($x_e/y_e = 1$) and ($L_D = 1$), while the maximum overestimation in $J_h$ is (1037%) corresponding to the error in $C_A$ at ($x_e/y_e = 5$) and ($L_D = 100$). For both $x_e/y_e = (1)$ and (2), although the maximum error in $J_h$ (i.e., underestimation) at ($L_D = 1$), a rather unrealistic case in most real fields, from Fig. 3 the error in $J_h$ can still reach (6.38%) underestimation for even a realistic configuration of $L_D$ equal to (10). For ($x_e/y_e = 5$), the error in $J_h$ (i.e., overestimation) is much more significant and starts from (46.43%) to (1037%) overestimation for $L_D = (10)$ to (100), respectively.

Figure 4 shows that the reported $C_A$ tables of PTM lead to overestimation in the $J_h$ for both PR = (0.2) and (0.8). The relative error in the $J_h$ for each PR follows the same trend as those in $C_A$ table. The maximum overestimation in $J_h$ is (32%) corresponding to the error in $C_A$ at ($L_D = 1$) and (PR = 0.2). Although the maximum error in $J_h$ (i.e., overestimation) is at ($L_D = 1$), a rather unrealistic case in most real fields, from Fig. 4 the error in $J_h$ can still reach (7%) overestimation for even a realistic configuration of $L_D$ equal to (10).

### The source of errors in the reported $C_A$ tables

Given that both PTM and GPTM solutions are derived identically, except re-casting some variables into dimensionless forms in the latter model, the source of the errors in the reported $C_A$ tables may be originated from the evaluation of the $P_{DT}$ which was not verified by Mutalik et al. (1988).

Figure 5 shows the effect of using FTLC = (0–10–9) and (0–10–5) on the $C_A$ accuracy. GPTM is used to calculate $C_A$ for square reservoir, ($r_{WD} = 10^{-4}$), ($OCWL = 1$), PR = (0.1), (0.5), and (1), and $L_D$ from (1) to (10^3). The maximum error in $C_A$ is (3630%) which appears at the lowest PR (i.e., PR = 0.1) and $L_D$ (i.e., $L_D = 1$). Notice that the error in $C_A$ becomes small and not significant at ($L_D > 100$) and suggests that the FTLC of (0–10–5) would be sufficient for $C_A$ accuracy when ($L_D > 100$) for any well and reservoir configuration. This has also been shown previously in Table 1 for vertical fracture well case when ($L_D > 10^3$).

The pressure drop that occurs at the very early time in the $P_{DT}$ solution, especially when $L_D$ is small (i.e., when the reservoir thickness is large), has significant effects on the $C_A$ accuracy. The evaluation of $P_{DT}$ in that period requires...
| $L_D$ | Mutalik et al. (1988) | This study | $C_A$ % Err |
|------|----------------------|-----------|------------|
| 1    | 0.0163 | 0.4 | 0.0147 | 0.4075 | 10.8994 |
| 5    | 0.9813 | 0.4 | 0.9608 | 0.4075 | 2.1390 |
| 10   | 1.4438 | 0.4 | 1.4286 | 0.4075 | 1.0640 |
| 20   | 1.6917 | 0.4 | 1.6827 | 0.4075 | 0.5349 |
| 50   | 1.8178 | 0.4 | 1.8138 | 0.4075 | 0.2205 |
| 100  | 1.834  | 0.4 | 1.8445 | 0.4075 | 0.5693 |

Table 3 Comparison between $C_A$ and $(T_{DA})_{PSS}$ of this study and Mutalik et al. (1988) (when $r_{wD} = 0.225$ ft and $y_e = 933.33$ ft) (variable $r_{wD}$)

| $L_D$ | Mutalik et al. (1988) | This study | $C_A$ % Err |
|------|----------------------|-----------|------------|
| 1    | 0.0034 | 0.4 | 0.0039 | 0.3944 | 11.92 |
| 5    | 0.6268 | 0.4 | 0.6075 | 0.3944 | 3.17 |
| 10   | 1.0241 | 0.4 | 1.0082 | 0.3944 | 1.58 |
| 20   | 1.2644 | 0.4 | 1.2546 | 0.3944 | 0.78 |
| 50   | 1.402  | 0.4 | 1.3923 | 0.3944 | 0.32 |
| 100  | 1.439  | 0.4 | 1.4367 | 0.3944 | 0.16 |

| $L_D$ | Mutalik et al. (1988) | This study | $C_A$ % Err |
|------|----------------------|-----------|------------|
| 1    | 0.00085 | 0.4 | 0.0012 | 0.3944 | 28.79 |
| 5    | 0.2773 | 0.4 | 0.2968 | 0.3944 | 6.56 |
| 10   | 0.501  | 0.4 | 0.5183 | 0.3944 | 3.34 |
| 20   | 0.6506 | 0.4 | 0.6617 | 0.3944 | 1.67 |
| 50   | 0.7435 | 0.4 | 0.7485 | 0.3944 | 0.66 |
| 100  | 0.771  | 0.4 | 0.7734 | 0.3944 | 0.31 |

Table 3 Comparison between $C_A$ and $(T_{DA})_{PSS}$ of this study and Mutalik et al. (1988) (when $r_{wD} = 0.225$ ft and $y_e = 933.33$ ft) (variable $r_{wD}$)
to use a much shorter FTLC [i.e., FTLC = (0–10–9)] to gain sufficient accuracy.

Due to the absence of the information about the selected FTLC in the calculation of the reported \( C_A \) tables by PTM, \( C_A \) tables have errors in reverse order with \( L_D \) as shown previously for \( x_e/y_e = (1) \) and (2) in Tables 3 and 4. Therefore, the source of errors in the reported \( C_A \) tables by PTM could be originated and justified from the using of long FTLC in the \( P_D T \) evaluation. About (\( x_e/y_e = 5 \)), no analysis can be achieved.

**Additional characteristics of PTM and GPTM**

Figure 6 shows the effect of OCWL in the Z-direction on \( C_A \) for square reservoir, \( r_{WD} = 10^{-3} \) and PR is equal to (0.1) and (0.9). The effect of \( z_{WD} \) (i.e., \( z_D = z_{WD} \)) on the \( C_A \) that is calculated by GPTM becomes insignificant when \( L_D > 35 \) (\( C_A \) error < 5%) which is inconsistent with their \( C_A \) counterparts by PTM [i.e., Fig. 12 in Mutalik et al. (1988) paper] because \( z_{WD} \) will be insignificant when \( L_D > 5 \). Therefore, \( z_{WD} \) should be considered as a major parameter for any correlation.

Table 5 which is presented at the end of the paper shows the effect of the wellbore radius on \( C_A \) for square reservoir and PR = (0.1) and (1). When \( r_{WD} \) increases two times (i.e., from \( 5 \times 10^{-5} \) to \( 10^{-3} \)), a significant error occurs when \( L_D < 15 \), and when \( r_{WD} \) increases (10) to (20) times [i.e., from \( (5 \times 10^{-4}) \) to \( (10^{-3}) \)], a significant error occurs when \( L_D < 50 \). This is also inconsistent with their \( C_A \) counterparts by PTM (i.e., table (9) in Mutalik et al. (1988) paper), the significant error will occur when \( L_D < 3 \) if \( r_{WD} \) increases 10 times. Therefore, \( r_{WD} \) should be considered as a major parameter for any correlation.
Table 4 Comparison between $C_A$ and $(T_{DA})_{PSS}$ of this study and Mutalik et al. (1988) (when constant $r_wD = 10^{-4}$)

| $L_D$ | Mutalik et al. (1988) | This study | $C_A$ % Err |
|-------|-----------------------|------------|-------------|
| PR = 0.2, Central well, $x_eD/y_eD = 1$, $r_wD = 10^{-4}$ | | | |
| $C_A$ | $(T_{DA})_{PSS}$ | $C_A$ | $(T_{DA})_{PSS}$ |
| 1 | 0.0163 | 0.4 | 0.0012 | 0.4075 | 1215.58 |
| 5 | 0.9813 | 0.4 | 0.5858 | 0.4075 | 67.50 |
| 10 | 1.4438 | 0.4 | 1.1155 | 0.4075 | 29.43 |
| 20 | 1.6917 | 0.4 | 1.4869 | 0.4075 | 13.77 |
| 50 | 1.8178 | 0.4 | 1.7261 | 0.4075 | 5.31 |
| 100 | 1.834 | 0.4 | 1.7990 | 0.4075 | 1.95 |
| PR = 0.4, Central well, $x_eD/y_eD = 1$, $r_wD = 10^{-4}$ | | | |
| $C_A$ | $(T_{DA})_{PSS}$ | $C_A$ | $(T_{DA})_{PSS}$ |
| 1 | 0.0034 | 0.4 | 0.0010 | 0.3944 | 253.80 |
| 5 | 0.6268 | 0.4 | 0.4600 | 0.3944 | 36.26 |
| 10 | 1.0241 | 0.4 | 0.8773 | 0.3944 | 16.73 |
| 20 | 1.2644 | 0.4 | 1.1703 | 0.3944 | 8.04 |
| 50 | 1.402 | 0.4 | 1.3592 | 0.3944 | 3.15 |
| 100 | 1.439 | 0.4 | 1.4168 | 0.3944 | 1.57 |
| PR = 0.6, Central well, $x_eD/y_eD = 1$, $r_wD = 10^{-4}$ | | | |
| $C_A$ | $(T_{DA})_{PSS}$ | $C_A$ | $(T_{DA})_{PSS}$ |
| 1 | 0.00085 | 0.4 | 0.0005 | 0.3924 | 278.01 |
| 5 | 0.2773 | 0.4 | 0.2489 | 0.3924 | 11.41 |
| 10 | 0.501 | 0.4 | 0.4747 | 0.3924 | 5.55 |
| 20 | 0.6506 | 0.4 | 0.6332 | 0.3924 | 2.75 |
| 50 | 0.7435 | 0.4 | 0.7354 | 0.3924 | 1.10 |
| 100 | 0.771 | 0.4 | 0.7666 | 0.3924 | 0.58 |
| PR = 0.8, Central well, $x_eD/y_eD = 2$, $r_wD = 10^{-4}$ | | | |
| $C_A$ | $(T_{DA})_{PSS}$ | $C_A$ | $(T_{DA})_{PSS}$ |
| 1 | 0.00061 | 0.6 | 0.0009 | 0.708 | 278.01 |
| 5 | 0.544 | 0.6 | 0.4180 | 0.708 | 30.14 |
| 10 | 0.9095 | 0.6 | 0.7972 | 0.708 | 14.08 |
| 20 | 1.1359 | 0.6 | 1.0635 | 0.708 | 6.81 |
| 50 | 1.2868 | 0.6 | 1.2351 | 0.708 | 2.71 |
| 100 | 1.3046 | 0.6 | 1.2875 | 0.708 | 1.33 |
| PR = 1, Central well, $x_eD/y_eD = 2$, $r_wD = 10^{-4}$ | | | |
| $C_A$ | $(T_{DA})_{PSS}$ | $C_A$ | $(T_{DA})_{PSS}$ |
| 1 | 0.00061 | 0.6 | 0.0006 | 0.6077 | 237.91 |
| 5 | 0.3051 | 0.6 | 0.2087 | 0.6077 | 30.14 |
| 10 | 0.5825 | 0.6 | 0.5878 | 0.6077 | 14.08 |
| 20 | 0.7825 | 0.6 | 0.7852 | 0.6077 | 6.81 |
| 50 | 0.9109 | 0.6 | 0.9120 | 0.6077 | 2.71 |
| 100 | 0.95 | 0.6 | 0.9507 | 0.6077 | 1.33 |
Table 4 (continued)

| $L_D$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ |
|-------|----------------------------------------|----------------------------------------|
|       | Mutalik et al. (1988) | This study | C_A % Err | Mutalik et al. (1988) | This study | C_A % Err |
| 1     | 0.00019 | 2 | 0.0004 | 1.75 | −47.51 | 1 | 0.0003 | 2 | 0.0003 | 1.703 | −13.29 |
| 5     | 0.1869 | 2 | 0.1777 | 1.75 | 5.19 | 5 | 0.2111 | 2 | 0.1789 | 1.703 | 17.97 |
| 10    | 0.3876 | 2 | 0.3389 | 1.75 | 14.39 | 10 | 0.4144 | 2 | 0.3413 | 1.703 | 21.43 |
| 20    | 0.5412 | 2 | 0.4520 | 1.75 | 19.73 | 20 | 0.5609 | 2 | 0.4552 | 1.703 | 23.21 |
| 50    | 0.645  | 2 | 0.5250 | 1.75 | 22.86 | 50 | 0.6571 | 2 | 0.5287 | 1.703 | 24.28 |
| 100   | 0.6782 | 2 | 0.5472 | 1.75 | 23.93 | 100 | 0.687 | 2 | 0.5511 | 1.703 | 24.65 |

PR = 0.6, Central well, $x_{eD/y_{eD}} = 5, r_{wD} = 10^{-4}$

| $L_D$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ |
|-------|----------------------------------------|----------------------------------------|
|       | Mutalik et al. (1988) | This study | C_A % Err | Mutalik et al. (1988) | This study | C_A % Err |
| 1     | 0.00011 | 0.4 | 0.0002 | 0.4832 | −42.41 | 1 | 0.0003 | 2 | 0.0003 | 1.703 | −13.29 |
| 5     | 0.1241 | 0.6 | 0.1150 | 0.0747 | 7.87 |
| 10    | 0.2548 | 0.6 | 0.2194 | 0.0747 | 16.13 |
| 20    | 0.3527 | 0.6 | 0.2927 | 0.0747 | 20.50 |
| 50    | 0.4187 | 0.6 | 0.3399 | 0.0747 | 23.17 |
| 100   | 0.4397 | 0.6 | 0.3543 | 0.0747 | 24.09 |

PR = 0.8, Central well, $x_{eD/y_{eD}} = 5, r_{wD} = 10^{-4}$

| $L_D$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ | $x_{eD/y_{eD}}$ = 5, $r_{wD} = 10^{-4}$ |
|-------|----------------------------------------|----------------------------------------|
|       | Mutalik et al. (1988) | This study | C_A % Err | Mutalik et al. (1988) | This study | C_A % Err |
| 1     | 0.00011 | 0.4 | 0.0002 | 0.4832 | −42.41 | 1 | 0.0003 | 2 | 0.0003 | 1.703 | −13.29 |
| 5     | 0.1241 | 0.6 | 0.1150 | 0.0747 | 7.87 |
| 10    | 0.2548 | 0.6 | 0.2194 | 0.0747 | 16.13 |
| 20    | 0.3527 | 0.6 | 0.2927 | 0.0747 | 20.50 |
| 50    | 0.4187 | 0.6 | 0.3399 | 0.0747 | 23.17 |
| 100   | 0.4397 | 0.6 | 0.3543 | 0.0747 | 24.09 |

PR = 1, Central well, $x_{eD/y_{eD}} = 5, r_{wD} = 10^{-4}$

**Fig. 3** The $J_0$ comparison by using the reported $C_A$ tables and this study $C_A$ tables (variable $r_{wD}$)
Conclusions

1. The PTM was improved to GPTM and verified by converting several dimensional parameters to dimensionless forms.
2. An efficient algorithm was developed, and accurate $C_A$ and $J_h$ were obtained for any well and reservoir configuration.
3. The accuracy of the reported $C_A$ tables was quantified, and the consequences were determined when they were used to estimate $J_h$.
4. New $C_A$ tables and their $(T_{Da})_{PSS}$ tables were generated for a wider range of well and reservoir configurations. The tables are provided in the Supplementary Material.
5. Because $C_A$ depends on the all-configuration parameters, unlimited tables of $C_A$ can be generated; therefore, the MATLAB code was provided.
6. Although horizontal wells usually have been drilled with high $L_D$ industrially (i.e., $L_D > 10$), it is still a significant error in $J_h$ by ($< 7\%$) that has been determined when using the reported $C_A$ of PTM for ($x_e/y_e = 1$) and (2) in the both assumed input data, while for ($x_e/y_e = 5$), a significant error in $J_h$ starts from (46.43\%) to (1037\%) for $L_D = (10)$ to (100), respectively, in the first assumed
input data (i.e., variable $r_{WD}$) and expected the same in the second assumed input data (i.e., constant $r_{WD}$).

\[ P_D T = \frac{0.007078kh\Delta P}{q_uB_o\mu_o} \]  

\[ I_D = \frac{0.000264kr}{\phi\mu_o c_i(L/2)^2} \]  

\[ T_{DA} = \frac{0.000264kt}{\phi\mu_o c_i4x_e y_e} \]  

\[ x_D = \frac{2x}{L} \]

**Appendix 1**

**Summary of pressure transient model of Mutalik et al. (1988)**

The mathematical expression of the dimensionless parameters of PTM (Eq. (3)) and the geometrical figure of PTM is given as follows:

\[ (9) \]

\[ (10) \]

\[ (11) \]

\[ (12) \]
Appendix 2

The dimensionless parameters in anisotropic reservoir

The mathematical expression of the dimensionless parameters for anisotropic reservoir by Ozkan et al. (1987), Ozkan and Raghavan (1988) and Besson (1990) in (') symbol is given as follows:

\[
y_D = \frac{2y}{L} = r_w D = \frac{2r_w}{L} \quad (13)
\]

\[
z_D = \frac{z}{h} \quad (14)
\]

\[
x_{wD}' = \frac{2x_w}{L} \quad (15)
\]

\[
y_{wD}' = \frac{2y_w}{L} \quad (16)
\]

\[
z_{wD}' = \frac{z_w}{h} \quad (17)
\]

\[
x_D' = \frac{2x}{L} \sqrt{\frac{k_{eq}}{k_x}} \quad (23)
\]

\[
y_D' = \frac{2y}{L} \sqrt{\frac{k_{eq}}{k_y}} = r_w D' = \frac{2r_w}{L} \sqrt{\frac{k_{eq}}{k_y}} \quad (24)
\]

\[
z_D' = \frac{z}{h} \sqrt{\frac{k_z}{k_{eq}}} \quad (25)
\]

\[
x_{wD}' = \frac{2x_w}{L} \sqrt{\frac{k_{eq}}{k_x}} \quad (27)
\]

\[
y_{wD}' = \frac{2y_w}{L} \sqrt{\frac{k_{eq}}{k_y}} \quad (28)
\]

\[
z_{wD}' = \frac{z_w}{h} \quad (29)
\]

Appendix 3

The MATLAB code
% SHAPE FACTORS (CA) COMPUTATIONS
% PD and TDA Calculation for Rectangular Bounded Reservoir
% THE GENERALIZED SOLUTION TO MUTALIX, Z.N., GOOBLE, AND JOSHI, S.D. "S. (1988) MODEL

clc

k=1; xy=1; rs=1; kh=(kx*ky)^0.5; keq=(kx*ky*ka)*(1/3);

PH=0.2; k0=1/(PH)^2*(keq/(kx*ky)^0.5); YD=2/(X0*1)*((keg/ky)^0.5); XWD=1*X0*1 *((keq/ky)^0.5);

YWD=1-YD; ID=1*(kx/ky*0.5); S0=0; A=0; sshift=100; slopem=100;

TDA= 0; PD=0; XD=0.732; YD=0.0001 *((keq/ky)^0.5); XWD=0.5; XD=5*2; INC=1*10^-6;

TDF=0; dify=0; TDF=10; PD=0;

% CALCULATION OF THE FIRST SUM. TERM, X-direction
RA1=0; RA2=0; RA3=0; PD=0; V1=0; V2=0; V3=0; SUM1=0; NN=0; P1=100;

while P1 > 0

NN = NN+1;
RA1=(NN/NN+1)*(XYD); RA2=(NN/NN+1)*(XWD); RA3=(NN/NN+1)*(XYD); PD=1*(XYD); V1=2/(X0*1)*((keg/ky)^0.5); V2=2/(X0*1)*((keg/ky)^0.5); V3=2/(X0*1)*((keg/ky)^0.5);

SUM1=SUM1+V1*V2*V3*NN;

end

% CALCULATION OF THE SECOND SUM. TERM, Y-direction
RA1=0; RA2=0; RA3=0; PD=0; V1=0; V2=0; V3=0; SUM2=0; MM = 0; P2=100;

while P2 > 0

MM = MM+1;
RA1=(MM/NN+1)*(XYD); RA2=(NN/NN+1)*(XWD); RA3=(NN/NN+1)*(XYD); PD=1*(XYD); V1=2/(X0*1)*((keg/ky)^0.5); V2=2/(X0*1)*((keg/ky)^0.5); V3=2/(X0*1)*((keg/ky)^0.5);

SUM2=SUM2+V1*V2*V3*MM;

end

% REM+CALCULATION OF THE THIRD SUM. TERM, Z-direction
RA1=0; RA2=0; RA3=0; PD=0; V1=0; V2=0; V3=0; SUM3=0; GG=0;

while P3 > 0

GG=GG+1;
RA1=(GG/NN+1); RA2=(GG/NN+1); RA3=(GG/NN+1); PD=1*(XYD); V1=2/(X0*1)*((keg/ky)^0.5); V2=2/(X0*1)*((keg/ky)^0.5); V3=2/(X0*1)*((keg/ky)^0.5);

SUM3=SUM3+(P3)*V111;

end

% TDA IN CTION
TDA= TDA+SUM1+SUM2+SUM3; PD= PD+SUM3/GG;

end

if TDA>10^-5

INC=10^-8;
end

if TDA>10^-4

INC=10^-7;
end

if TDA>10^-3

INC=10^-6;
end

if TDA>10^-2

INC=10^-5;
end

if TDA>10^-1

INC=10^-4;
end

if TDA> 1

INC=10^-3;
end
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Declarations

Conflict of interest  The authors declare no conflict of interest.

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