ON KNOTTED SPHERES IN EUCLIDEAN 4-SPACE $\mathbb{E}^4$

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Abstract

In the present study we consider knotted spheres in Euclidean 4-space $\mathbb{E}^4$. Firstly, we give some basic curvature properties of knotted spheres in $\mathbb{E}^4$. Further, we obtained some results related with the conjugate nets and Laplace transforms of these kind of surfaces.

1 Introduction

Let us consider a unit speed regular curve $\gamma$ in $\mathbb{E}^4$ and a unit speed spherical curve $\rho$ in $\mathbb{E}^2$. Then, the rotation of $\gamma$ around $\rho$ give rise a surface $M$ in $\mathbb{E}^4$, which is called rotational surface. The rotational surfaces in $\mathbb{E}^4$ was first introduced by C. Moore in 1919 (see, [12]). Further, many researchers concentrated these studies on this subject, see for example [3], [4], [7] and [8]. The rotational surfaces in $\mathbb{E}^4$ with constant curvatures are studied in [13].

Let us denote the half-space $x_3 \geq 0, x_4 = 0$ by $\mathbb{E}_3^+(0)$ and take an arc $\alpha$ with the end point in the plane $x_3 = 0, x_4 = 0$ (denoted by $\Pi$). The rotation the half space plane $\mathbb{E}_3^+(0)$ by the angle $\nu$ around the plane $\Pi$ is denoted by $\mathbb{E}_3^+(\nu)$. Consequently, after the rotation the point with coordinates

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\(x_1, x_2, x_3, x_4\) passes into the point with the coordinates \(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\) by

\[
\begin{align*}
\tilde{x}_1 &= x_1 \\
\tilde{x}_2 &= x_2 \\
\tilde{x}_3 &= x_3 \cos v - x_4 \sin v \\
\tilde{x}_3 &= x_3 \sin v - x_4 \cos v.
\end{align*}
\]

In rotation by 360° the points of \(\alpha\) being in \(E_3^4(v)\), form the set \(M\) homeomorphic to \(S^2\) [2]. Let \(\alpha\) be a smooth curve with tangent vectors at \(p\) and \(q\) orthogonal to \(\Pi\). Then \(M\) is a smooth surface which is called knotted sphere in \(E^4\) [1].

A net of curves on a surface \(M\) is called conjugate, if at every point the tangent directions of the curves of the net separate harmonically the asymptotic directions [11]. Consequently, for a surface \(M\) with a patch \(X(u, v)\), a net of curves on \(M\) are conjugate if and only if the second partial derivative \(X_{uv}\) of \(X\) lies in the subspace spanned by \(X_u\) and \(X_v\) [10].

This paper is organized as follows: In section 2 we give some basic concepts of the second fundamental form and curvatures of the surfaces in \(E^4\). In Section 3 we consider knotted spheres in \(E^4\). Firstly, we give some basic curvature properties of knotted spheres in \(E^4\). Further, we introduce some kind of knotted spheres and obtained some results related with their curvatures. In section 4 we give some basic curvature properties of the conjugate nets on a surface in \(E^n\). Further, we calculated the Laplace invariants and the Laplace transforms of the knotted sphere in \(E^4\).

## 2 Preliminaries

Let \(M\) be a local surface in \(E^n\) given with position vector \(X(u, v)\). The tangent space \(T_pM\) is spanned by the vector fields \(X_u\) and \(X_v\). In the chart \((u, v)\) the coefficients of the first fundamental form of \(M\) are given by

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,
\]

where \(\langle, \rangle\) is the inner product in \(E^n\). We assume that \(X(u, v)\) is regular i.e., \(W^2 = EG - F^2 \neq 0\) [9].

Consequently, the Gaussian curvature of \(M\) is given by

\[
K = -\frac{1}{4W^2} \begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2W} \left( \frac{E_v - F_u}{W} v - \frac{F_v - G_u}{W} u \right).
\]

2
Let $\nabla$ be the Riemannian connection of $\mathbb{E}^4$, and $X_1 = X_u$, $X_2 = X_v$ tangent vector fields of $M$ then Gauss equation gives

$$\nabla_{X_i} X_j = \sum_{k=1}^{2} \Gamma^k_{ij} X_k + h(X_i, X_j); \quad 1 \leq i, j \leq 2,$$

(4)

where $h$ is the second fundamental form and $\Gamma^k_{ij}$ are the Christoffel symbols of $M$.

The mean curvature vector $\vec{H}$ of $M$ is given by

$$\vec{H} = \frac{1}{2W^2} (Eh(X_v, X_v) - 2Fh(X_u, X_v) + Gh(X_u, X_u)).$$

(5)

The norm of the mean curvature vector $\vec{H}$ is known as mean curvature of $M$. Recall that, a surface $M$ is said to be minimal (resp. flat) if its mean curvature (resp. Gaussian curvature) vanishes identically [5].

### 3 Knotted Spheres in $\mathbb{E}^4$

Let $M$ be a knotted sphere given with (1), then the position vector of $M$ can be represented as:

$$X(u, v) = (x_1(u), x_2(u), x_3(u) \cos v - x_4(u) \sin v, x_3(u) \sin v + x_4(u) \cos v).$$

(6)

where

$$\gamma(u) = (x_1(u), x_2(u), x_3(u), x_4(u)),$$

is the profile curve of the surface [1]. Then, the tangent space $T_p M$ of $M$ is spanned by

$$X_u = (x'_1(u), x'_2(u), x'_3(u) \cos v - x'_4(u) \sin v, x'_3(u) \sin v + x'_4(u) \cos v),$$

$$X_v = (0, 0, -x_3(u) \sin v - x_4(u) \cos v, x_3(u) \cos v - x_4(u) \sin v).$$

(7)

Consequently, the coefficients of first fundamental form become

$$E = \langle X_u, X_u \rangle = 1,$$

$$F = \langle X_u, X_v \rangle = x_3(u)x'_4(u) - x'_3(u)x_4(u),$$

$$G = \langle X_v, X_v \rangle = x_3^2(u) + x_4^2(u).$$

(8)
The Christoffel symbols $\Gamma^k_{ij}$ of the canal surface $M$ are given by

$$
\Gamma^1_{11} = -\frac{FF_u}{W^2}, \quad \Gamma^1_{12} = -\frac{FG_u}{2W^2}, \quad \Gamma^1_{22} = -\frac{GG_u}{2W^2}, \\
\Gamma^2_{11} = \frac{F}{W^2}, \quad \Gamma^2_{12} = \frac{G}{2W^2}, \quad \Gamma^2_{22} = \frac{F^2}{2W^2},
$$  

(9)

which are symmetric with respect to the covariant indices ([9], p.398).

The second partial derivatives of $r$ are expressed as follows:

$$
X_{uu} = (x_1''(u), x_2''(u), x_3''(u) \cos v - x_4''(u) \sin v, x_4''(u) \sin v + x_3''(u) \cos v), \\
X_{uv} = (0, 0, -x_3'(u) \sin v - x_4'(u) \cos v, x_3'(u) \cos v - x_4'(u) \sin v), \\
X_{vv} = (0, 0, -x_3'(u) \cos v + x_4'(u) \sin v, -x_3'(u) \sin v - x_4'(u) \cos v).
$$

(10)

Using (7) with (10) we get

$$
\langle X_{uu}, X_{uv} \rangle = -(x_3(u)x_3''(u) + x_4(u)x_4''(u)), \\
\langle X_{uv}, X_u \rangle = -(x_3(u)x_3'(u) + x_4(u)x_4'(u)), \\
\langle X_{uu}, X_v \rangle = x_3(u)x_4'(u) - x_3'(u)x_4(u), \\
\langle r_{u\theta}, X_{uv} \rangle = (x_3'(u))^2 + (x_4'(u))^2, \\
\langle X_{uu}, X_u \rangle = 0, \\
\langle X_{uv}, X_v \rangle = 0.
$$

(11)

Hence, taking into account (11), the Gauss equation implies the following equations:

$$
\bar{\nabla}_u X_u = X_{uu} = \Gamma^1_{11} X_u + \Gamma^1_{12} X_v + h(X_u, X_u), \\
\bar{\nabla}_u X_v = X_{uv} = \Gamma^1_{12} X_u + \Gamma^1_{22} X_v + h(X_u, X_v), \\
\bar{\nabla}_v X_v = X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + h(X_v, X_v).
$$

(12)

Taking in mind (11), (9) and (12) we get

$$
h(X_u, X_u) = X_{uu} + \frac{FF_u}{W^2} X_u - \frac{F_u}{W^2} X_v, \\
h(X_u, X_v) = X_{uv} + \frac{FG_u}{2W^2} X_u - \frac{G_u}{2W^2} X_v, \\
h(X_v, X_v) = X_{vv} + \frac{GG_u}{2W^2} X_u - \frac{F^2}{2W^2} X_v.
$$

(13)

Consequently, by the use of (3), (5), (8) and (11) with (13) the Gaussian curvature and mean curvature vector of $M$ become

$$
K = -\frac{1}{2W} \left( \frac{G_u}{W} \right)_u,
$$

(14)
and
\[ \vec{H} = \frac{1}{2W^2} (Eh(X_v, X_v) - 2Fh(X_u, X_v) + Gh(X_u, X_u)), \] (15)
respectively.

In the sequel, we consider some special cases;

**Case I.** Suppose
\[ x_3 = \cos \varphi(u), \quad x_4 = \sin \varphi(u), \] (16)
then the position vector of the knotted sphere \( M \) can be represented as
\[ X(u, v) = (x_1, x_2, \cos \varphi(u) \cos v - \sin \varphi(u) \sin v, \cos \varphi(u) \sin v + \sin \varphi(u) \cos v). \] (17)

Hence, the coefficients of the first fundamental form become
\[ E = \langle X_u, X_u \rangle = 1, \]
\[ F = \langle X_u, X_v \rangle = \varphi'(u), \]
\[ G = \langle X_v, X_v \rangle = 1, \] (18)
where \( \varphi \) is a differentiable (angle) function.

Summing up (9)-(18) the following results are proved;

**Proposition 3.1** The surface \( M \) given with the position vector (17) is a flat surface.

**Proposition 3.2** Let \( M \) be a surface given with the position vector (17). Then, the mean curvature \( \vec{H} \) of \( M \) at point \( p \) is given by
\[ \| \vec{H} \| = \frac{1}{4 (1 - (\varphi'(u))^2)} \left( \kappa_\gamma^2 + 1 - 2 (\varphi'(u))^2 - \frac{(\varphi''(u))^2}{1 - (\varphi'(u))^2} \right), \] (19)
where \( \kappa_\gamma \) is the curvature of the profile curve \( \gamma \).

**Proof 3.3** With the help of (13), (15) and (18) the Gaussian curvature vector of \( M \) becomes
\[ 2 \vec{H} = \frac{1}{1 - (\varphi')^2} (\vec{x}_1, \vec{x}_2, \vec{x}_3 \cos v + \vec{x}_4 \sin v, \vec{x}_3 \sin v - \vec{x}_4 \cos v), \] (20)
where,

\[
\begin{align*}
\mathbf{\tau}_1 &= x''_1 + \frac{\varphi'\varphi''}{1 - (\varphi')^2} x'_1, \\
\mathbf{\tau}_2 &= x''_1 + \frac{\varphi'\varphi''}{1 - (\varphi')^2} x'_2, \\
\mathbf{\tau}_3 &= \frac{\varphi'\varphi'' x'_3 + \varphi'' x'_4}{1 - (\varphi')^2} + x''_3 - x'_3 + 2\varphi' x'_4 = -\left(1 - (\varphi')^2\right) \cos \varphi, \\
\mathbf{\tau}_4 &= \frac{\varphi'' x'_3 - \varphi'\varphi'' x'_4}{1 - (\varphi')^2} - x''_4 + x'_4 + 2\varphi' x'_3 = \left(1 - (\varphi')^2\right) \sin \varphi
\end{align*}
\]

are differentiable functions. Taking the norm of the vector (20) and using (16) with (21) we obtain (19). This completes the proof of the proposition.

As a consequence of Proposition 2 we obtain the following result.

**Corollary 3.4** Let \( M \) be a surface given with the position vector (17). Then \( M \) is a minimal surface if and only if the curvature \( \kappa_\gamma \) of the profile curve \( \gamma \) satisfies the equality

\[
\kappa^2_\gamma = \frac{(\varphi''(u))^2}{1 - (\varphi'(u))^2} + 2 (\varphi'(u))^2 - 1, \tag{22}
\]

in such a way that the (angle) function \( \varphi \) is non-constant.

**Case II.** Suppose \( x_4 = \lambda x_3, \lambda \in \mathbb{R} \), then the position vector of the knotted sphere \( M \) can be represented as

\[
r(s, \theta) = (x_1(u), x_2(u), x_3(u) (\cos \nu - \lambda \sin \nu), x_3(u) (\sin \nu + \lambda \cos \nu)). \tag{23}
\]

Hence, the coefficients of the first fundamental form of \( M \) become

\[
\begin{align*}
E &= \langle X_u, X_u \rangle = 1, \\
F &= \langle X_u, X_v \rangle = 0, \\
G &= \langle X_v, X_v \rangle = (1 + \lambda^2) x^2_3(u).
\end{align*}
\]

By the use of (24) with (3) we obtain the following result.
Proposition 3.5 Let $M$ be a surface given with the position vector $\mathbf{X}$. Then, the Gaussian curvature of $M$ is given by

$$K = -\frac{x_3''(u)}{x_3(u)}.$$ 

As a consequence of Proposition 4 we obtain the following result.

Corollary 3.6 Let $M$ be a surface given with the position vector $\mathbf{X}$. Then we have the following statements

i) If $x_3(u) = ae^{cu} + be^{-cu}$ then the corresponding surface is pseudospherical, i.e., it has negative Gaussian curvature $K = -\frac{1}{c^2}$.

ii) If $x_3(u) = a\cos cu + b\sin cu$ then the corresponding surface is spherical, i.e., it has negative Gaussian curvature $K = \frac{1}{c^2}$, where $a$, $b$ and $c$ are real constants.

iii) If $x_3(u) = au + b$ then the corresponding surface is flat.

4 The Conjugate Nets and Laplace Transforms of Knotted Sphere

In the present section, we will give some basic relations of the conjugate net of curves on a surface in $\mathbb{R}^n$. A net of curves on a surface $M$ is called conjugate, if at every point the tangent directions of the curves of the net separate harmonically the asymptotic directions. Taking the net to be parametric net with parameters $u$ and $v$, the classical notion of the conjugate net usually can be stated in [11] as follows:

Definition 4.1 Let $M$ be a smooth surface given with the position vector $X : U \subset \mathbb{R}^2 \to \mathbb{R}^n$, and $N_1, ..., N_{n-2}$ normal vector fields of $M$ in $\mathbb{R}^n$. If $X_{uv} = \frac{\partial^2 X}{\partial u \partial v}$ satisfies

$$\langle X_{uv}, N_\alpha \rangle = 0, 1 \leq \alpha \leq n - 2,$$

then $(u,v)$ is called conjugate coordinates of $X$ and the net woven by coordinate curves is called the conjugate net. For convenience, we denote the conjugate net by $(u,v)$. Here, $\langle , \rangle$ denotes the inner product on $\mathbb{R}^n$ [11].
Equation (25) is equivalent to the condition that $X_{uv}$ lies in the subspace spanned by $X_u$ and $X_v$; i.e.,

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v.$$ (26)

Now, for the surface with normal conjugate net, we have two transforms

$$X_1 = X - \frac{X_v}{\Gamma^1_{12}}, \quad X_{-1} = X - \frac{X_u}{\Gamma^2_{12}},$$ (27)

which are called the Laplace transforms of surface $M$ [10].

Furthermore, the functions

$$h = \frac{\partial_u \Gamma^1_{12} - \Gamma^1_{12} \Gamma^2_{12}}{\Gamma^1_{12}}, \quad k = \frac{\partial_v \Gamma^2_{12} - \Gamma^1_{12} \Gamma^2_{12}}{\Gamma^2_{12}}$$ (28)

are called the Laplace invariants.

If $\Gamma^1_{12} \neq 0$ (resp. $\Gamma^2_{12} \neq 0$), the conjugate net is called $v$–direction normal (resp. $u$–direction normal). To establish geometrically the notion of conjugate net in ambient space, the following result explain the real geometric meaning of the conjugate net defined in (25).

**Proposition 4.2** [11] $(u, v)$ is a $v$–direction normal conjugate net of $M$ if and only if there exists another surface $\widetilde{M}$ given with the position vector $X_1(u, v)$ such that, for any $(u, v) \in D \subset E^2$, the straight line $XX_1$ joining the points $X(u, v)$ and $X_1(u, v)$ is parallel to the vectors $X_v(u, v)$ and $X_u(u, v)$.

We obtain the following results,

**Theorem 4.3** Let $M$ be a surface given with the position vector (6). If $(u, v)$ are conjugate coordinates, then $M$ is a flat surface.

**Proof 4.4** Let $(u, v)$ be conjugate coordinates of the knotted sphere $M$ given with the parametrization (6). Then, by definition $h(X_u, X_v) = 0$. So, by the use of (13) we have

$$X_{uv} = \frac{G_u}{2W^2} X_v - \frac{FG_u}{2W^2} X_u.$$ (29)

Consequently, substituting (7) with (11) into (29) we get

$$x'_1(u) = 0,$$

$$x'_2(u) = 0,$$

$$x'_3(u) = x_3(u) \frac{G_u}{2W^2} - x'_4(u) \frac{FG_u}{2W^2},$$ (30)

$$x'_4(u) = x_4(u) \frac{G_u}{2W^2} + x'_3(u) \frac{FG_u}{2W^2}. $$
Summing up the last two equations of (30) we obtain
\[(x'_3(u))^2 + (x'_4(u))^2 = \frac{G_u}{2W^2} (x_3(u)x'_3(u) + x_4(u)x'_4(u)). \quad (31)\]

Moreover, the profile curve $\gamma$ has arclength parameter and the first two equations of (30) imply that
\[(x'_3(u))^2 + (x'_4(u))^2 = 1. \quad (32)\]

Hence, by the use of (3) with (32) the equation (31) reduces to
\[1 = \frac{G_u^2}{4W^2}, \quad W > 0. \quad (33)\]

Thus, substituting (33) into (14) we get $K = -\frac{1}{2W} \left( \frac{G_u}{W} \right)_u = 0$. This completes the proof of the proposition.

By the virtue of (14) the following results hold.

**Corollary 4.5** The coordinates $(u, v)$ of the surface $M$ given with the position vector (17) can not be conjugate.

**Proof 4.6** Suppose that $(u, v)$ are the conjugate coordinates of the surface given with the parametrization (17). Then, from (13) and (33) we get $4W^2 = G_u^2 = 0$. But, this contradicts the fact that $W > 0$. So, the coordinates $(u, v)$ can not be conjugate.

**Remark 4.7** Corollary 7 shows that the inverse statement of Theorem 7 may not be true.

**Proposition 4.8** Let $M$ be a knotted sphere given with the position vector (20). Then, the conjugate surface $\tilde{M}$ is a part of the rotation plane $\Pi$.

**Proof 4.9** Let $M$ be a knotted sphere given with the parametrization (6), then by the use of (9) we get
\[\Gamma_1 = -\frac{FG_u}{2W^2}, \quad \Gamma_2 = \frac{G_u}{2W^2}.\]


Now, assume that the surface $M$ with normal conjugate net, then equation (24) yields $\Gamma_{12}^1 = 0$ and $\Gamma_{12}^2 = \frac{x_3'(s)}{x_3(s)}$. Consequently, the Laplace transform $X$ becomes
\[ X = X - \frac{x_3(u)}{x_3'(u)} X_u. \]

Hence, using (23) with its partial derivative $X_u$ we obtain
\[ X = \left( x_1(u) - \frac{x_3(u)}{x_3'(u)} x_1'(u), x_2(u) - \frac{x_3(u)}{x_3'(u)} x_2'(u), 0, 0 \right). \quad (34) \]

This completes the proof of the proposition.

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