The Continuum is Countable: Infinity is Unique

Laurent Germain*

September 2008

*I thank Hervé Boco and also Sylvain Bourjade, Jean Claude Gabilllon, Anne Vanhems, Richard for their helpful comments which have improved the quality of this article. I thank also Sandrine, David, Lisa and Théo. Laurent Germain, ESC Toulouse, Université de Toulouse, ISAE. Correspondence to ESC Toulouse, 20 boulevard Lascrosses, BP 7010, 31068 Toulouse cedex 7, France. E-mail: l.germain@esc-toulouse.fr
Abstract

Since the theory developed by Georg Cantor, mathematicians have taken a sharp interest in the sizes of infinite sets. We know that the set of integers is infinitely countable and that its cardinality is \(\aleph_0\). Cantor proved in 1891 with the diagonal argument that the set of real numbers is uncountable and that there cannot be any bijection between integers and real numbers. Cantor states in particular the Continuum Hypothesis. In this paper, I show that the cardinality of the set of real numbers is the same as the set of integers. I show also that there is only one dimension for infinite sets, \(\mathbb{R}\).

Keywords: Infinity, Aleph, Set theory.
1 Introduction

Since the theory developed by Georg Cantor, mathematicians have taken a sharp interest in the sizes of infinite sets. We know that the set of integers is infinitely countable and that its cardinality is $\aleph_0$. Cantor proved in 1891 with the diagonal argument that the set of real numbers is uncountable and that there cannot be any bijection between integers and real numbers. The cardinality of the set of real numbers, the continuum, is $c = \aleph_1$. Cantor states, in particular, the Continuum Hypothesis (CH) by which there is no set the size of which is between the set of integers and the set of real numbers. The power of the continuum is equal to $2^{\aleph_0}$ which represents the cardinal of $P(\mathbb{N})$, the set of all the subsets of $\mathbb{N}$. Kurt Godel in 1939 and Paul Cohen in the 1960s have shown that the CH, which was mentioned by Hilbert as one of the more acute problems in mathematics in 1900 in Paris during the International Congress of Mathematics, was not provable. Paul Cohen showed that the Continuum Hypothesis is not provable under the Zermelo-Fraenkel set theory even if the axiom of choice is adopted (ZFC).

In this article, I show that the dimension of the set of integers is the same as the dimension of the real line. Cantor’s theory mentioned in fact that there were several dimensions for infinity. This, however, is questionable. Infinity can be thought as an absolute concept and there should not exist several dimensions for the infinite. As a matter of fact, if the set $\mathbb{N}$ is an infinite set it should be of same power as the set $\mathbb{R}$. This has not been proven so far. We knew by several arguments, and in particular by the argument of Cantor’s diagonal, that the set $\mathbb{R}$ and $\mathbb{N}$ do not have the same cardinality and that there is no bijection between these two sets. In this paper, I show that there is a bijection between the set of integers and the set of real numbers. I show that the set $\mathbb{R}$ is countable and that there is only one dimension for infinite sets, $\aleph$. The paper is organized as follows. In a first section, I show that the set $\mathbb{N}$ can be represented by an infinite tree and that the cardinal of the set of integers is the same as the cardinal of the powerset of integers. In a second section, I show that the set of real numbers can be represented by an infinite tree and that it is countable. In a third section, I show again that the cardinality of $P(\mathbb{N})$ is the same as the cardinality of $\mathbb{N}$. In a fourth section, I define infinity. Finally, I make some concluding remarks.

2 The Cardinal of the Set of Integers is the Same as the Cardinal of the Powerset of Integers

In this section, I define an infinite tree bijective with $\mathbb{N}$ and I show that the cardinal of $P(\mathbb{N})$ is the same as the cardinal of $\mathbb{N}$.

Proposition 2.1 There exists an infinite tree bijective with the set $\mathbb{N}$.
**Proof:** Let us consider an infinite tree starting with 10 nodes (0,1,2,3,4,5,6,7,8,9). Each of these nodes except the node 0 gives rise to 10 branches. Each of these 10 branches defines another 10 nodes. Let us define each of these nodes by the numbers characterizing the unique path to reach a particular node. The first node of the tree is the number 0, below 0 there are the numbers (1,2,\ldots,9). If now we consider the branches that start at node 1 they reach another 10 nodes. These nodes are defined by the numbers ((10),(11), (12), (13), (14), (15), (16), (17), (18), (19)). This tree is infinite and any node of the tree gives rise to another 10 nodes. All the numbers of these nodes describe, indeed, the set \( \mathbb{N} \). Hence, for any integer there exists a unique path in the tree and to any node of the tree there corresponds an integer (see Figure 1 and Figure 2). \( \mathbb{N} \) is therefore represented by this infinite tree. The set of the nodes of this infinite tree is bijective with the set \( \mathbb{N} \).

This proposition shows that there exists an infinite tree that represents \( \mathbb{N} \). In the next proposition, I compute the cardinality of the powerset of \( \mathbb{N} \).

**Proposition 2.2** \( \aleph_0 \) the cardinal of the set \( \mathbb{N} \) is equal to \( 10^{\aleph_0} \).

**Proof:** Each integer is a node of the tree. The set of the nodes in the tree is bijective with the set of integers. When \( \mathbb{N} \) is large, counting the number of nodes is the same as counting the number of paths (to infinity) in the tree. The cardinality of the set of the nodes of this infinite tree is \( 10^{\aleph_0} \) counted 9 times. Indeed, we see in Figures 1 and 2 that the set \( \mathbb{N} \) is of cardinality \((0, 1, 2, 3, 4, 5, 6, 7, 8, 9)^{\mathbb{N}} \) counted 9 times. This implies that the cardinality of the set \( \mathbb{N} \) is \( 10^{\aleph_0} \). As \( \aleph_0 \) represents the cardinal of \( \mathbb{N} \), which is by definition the cardinal of the set of the nodes in the tree (bijective with \( \mathbb{N} \)), we get \( 10^{\aleph_0} = \aleph_0 \). Note already that this implies that \( 2^{\aleph_0} = \aleph_0 \).

This proposition shows that the cardinality of \( \mathbb{N} \), \( \aleph_0 \), is equal to \( 10^{\aleph_0} \). In the next proposition, I compute the cardinality of the powerset of \( \mathbb{N} \).

**Proposition 2.3** The cardinal of \( P(\mathbb{N}) \), the powerset of \( \mathbb{N} \), is \( \aleph_0 \).

**Proof:** Note that the tree representing the set \( \mathbb{N} \), which at each node links up with 10 branches, includes the infinite subtree with 2 branches that represents \( P(\mathbb{N}) \). Indeed, the infinite subtree \((0, 1)^{\mathbb{N}} \) that represents \( P(\mathbb{N}) \) is included in the tree in bijection with \( \mathbb{N} \). This implies that the cardinality of \( P(\mathbb{N}) \) is equal to the cardinality of \( \mathbb{N} \), \( 2^{\aleph_0} = \aleph_0 \).

This proposition shows that the cardinality of the powerset of \( \mathbb{N} \) is the same as \( \mathbb{N} \). This implies already that \( \aleph_1 = \aleph_0 \).

In this section, I have defined an infinite tree which is bijective with \( \mathbb{N} \), I have defined the cardinality of \( \mathbb{N} \) and shown that the cardinal of the powerset of \( \mathbb{N} \) is \( \aleph_0 \). This section proves
already that the power of the continuum is $\aleph_0$. In the next section, I show the bijection between integers and real numbers.

3 Real Numbers

In this section, I define an infinite tree which represents real numbers, then I define a set of subtrees bijective with $\mathbb{N}$ in the interval $(0, 1)$. Finally, I compute again the power of the continuum and the cardinal of the set of real numbers.

3.1 The Infinite Tree of Real Numbers

In the next proposition, I define an infinite tree which represents the set $\mathbb{R}$.

**Proposition 3.1** There exists an infinite tree bijective with the set of all real numbers defined by their decimal representation.

**Proof:** Let us consider an infinite tree with $(N + 1)$ nodes which give rise to 10 branches, as in Figure 3. Each of the $(N+1)$ nodes is linked to 10 branches which are again linked to another 10 branches and so on to infinity. The first $(N + 1)$ nodes (in the first column of the tree) represent the set $\mathbb{N} = (0, 1, 2, \ldots, n, \ldots)$. Each node of this infinite tree can be defined by the numbers of the branches characterizing the unique path in the tree to reach a particular node. Thus, the first node of the tree is the number 0. The 10 other nodes characterizing the branches that start on the right of the 0 node are therefore defined by the couples $((0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (0,0))$. For example, the node $(0,1,1)$ is the node starting at 0 when you take branch 1 and again the following branch 1 (see Figure 3).

We can now define these nodes by decimal numbers, where the first figure corresponds to the integer in the first column of the tree (see Figure 4). For example, the nodes $(0,1), (0,2)$ and $(0,1,1)$ are defined by the decimal numbers $0, 1, 0, 2$ and $0, 1, 1$. In fact, we can define any node in this infinite tree by the decimal number characterizing the path of the tree to reach a node. For example, the node $(3,1,4,1,5,9,2,6)$ can be defined by the decimal number $(3,1415926)$ which is the number $\pi$. To any integer, or algebraic or transcendental number, there corresponds a unique path characterized by a decimal number (see Figure 4). This is true also for negative numbers that would be drawn on the left of the Figure. We can notice that, as in Hilbert’s Hotel, one can add as many branches as necessary in the tree to correspond with any sequence of figures.

This proposition shows that we can establish a bijection between the infinite tree of Figure 4 and the real numbers. We now give a lemma that characterize decimal numbers in the tree.
Lemma 3.1 If we consider the first \((N + 1)\) nodes \((0, 1, 2, \ldots, n, \ldots)\), the other \(10(N + 1)\) branches starting from these nodes define all the 1 decimal real numbers and the other \((N + 1)10^i\) branches define all the \(i\) decimal real numbers, \(i\) describing all the set \(\mathbb{N}\).

**Proof:** For any real number represented by a decimal there exists a unique path in the tree and vice versa. We can notice, of course, that in the tree certain real numbers are counted several times. For example, 0,1 is the same real number as 0,01. The tree represents all the possibilities to write a real number with decimals.

Therefore, in this section, I have defined an infinite tree which represents all real numbers with decimals.

### 3.2 Numbers between 0 and 1

In this section, I define a set of subtrees that are bijective with \(\mathbb{N}\) in the interval \((0, 1)\).

Let us consider all the nodes that depart from the node 0 and that are between the node 0 and the node 1. Let us define \(t_0\) the particular tree which represents all the paths starting at the node 0.

**Proposition 3.2** Any real number in the interval \((0, 1)\) is defined by a node of the infinite tree \(t_0\).

**Proof:** Any decimal number between 0 and 1 is characterized by a unique path in the infinite tree (see Figure 6). For example, let us consider the real number \(\pi - 3\). There exists a path in the tree characterizing this particular number.

We can now establish that to any node of the tree one can associate an integer.

**Proposition 3.3** To any node of the tree there corresponds an integer and there exists a set of subtrees the dimension of which is the same as \(\mathbb{N}\).

**Proof:** Let us consider all the nodes that start at the node 0 and that exclude all the nodes that are linked to the node \((00)\). One can associate an integer with any node of this truncated infinite tree. This integer is the number of the node i.e. the numbers which characterize the unique path to reach this node. Therefore, all the nodes that start with the number 0 when we exclude all the branches that depart from the node \((0,0)\) describe the set \(\mathbb{N}\). The application which associates to the nodes of the tree, starting with the node 0 and excluding the paths starting at the node \((0,0)\), an integer is bijective (see Figures 5, 6, 7 and 8).
Definition 3.1 Let $t(0)$ be the set of all the subtrees $t_{0^i} = t_{(0...0)}$, 0 is counted $i$ times for $i = 1$ to $i = N$, where $t_{0^i}$ is the subtree that represents all the paths that start from the node $(0^i)$ and exclude all the nodes that are linked to the node $(0^{i+1})$.

One can associate an integer with any node of any truncated infinite tree of $t(0)$. All the nodes that start with the number $(0^i)$ when we exclude all the branches that depart from the node $(0^{i+1})$ describe the set $\mathbb{N}$. The application which associates the nodes of the tree starting with the node $(0^i)$ and excluding the paths starting at the node $(0^{i+1})$ is bijective with the set of integers (see Figures 7 and 8). This is true for any subtree $t_{0^i}$.

This section defines a set of subtrees bijective with $\mathbb{N}$ in the interval $(0,1)$.

In the next section, I compute again the power of the continuum.

3.3 The Power of the Continuum

In this section, I compute again the power of the continuum showing a bijection between integers and real numbers.

**Theorem 3.1** The dimension of the interval $(0,1)$ of real numbers is $\aleph_0$ the dimension of $\mathbb{N}$.

We know that all the subtrees $t_{0^i}$ for all $i$ and the set $\mathbb{N}$ have the same cardinality. There are $\aleph_0$ subtrees in the set $t(0)$ as there are $\aleph_0$ nodes starting with $(0^i)$, $i$ describing the set $\mathbb{N}$ (see Figures 7 and 8). As the countable union of countable sets is countable, we get that the cardinality of $t(0)$ is $\aleph_0$. Hence, there is a bijection between the set of integers and the interval $(0,1)$. Therefore, the cardinality of the set of real numbers is indeed $\aleph_0$. We can therefore state the following theorem.

**Theorem 3.2** The power of the continuum is $\aleph = \aleph_0$.

**Proof:** By the same reasoning as before, we find that there are $\aleph_0$ real numbers in the infinite tree when considering the union of all the intervals $(0,1), (1,2), \ldots (n-1,n)$. The power of the continuum is indeed $\aleph = \aleph_0$.

This section proves again that the continuum is countable. In the next section, I compute the cardinality of $\mathbb{R}$.

3.4 Cardinal of Real Numbers

In this section, I compute again the cardinal of the set of real numbers.
Proposition 3.4 The cardinality of $\mathbb{R}$ is $\aleph_10^\aleph = \aleph$.

Proof: The cardinality of the set of the nodes of the infinite tree which represents real numbers is $\aleph_10^\aleph$, that is to say $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)^\mathbb{N}$ counted $\aleph$ times (see Figure 4). As $10^\aleph = \aleph$ (Proposition 2.2), the cardinality of $\mathbb{R}$ is equal to $\aleph \times \aleph$ which is equal to $\aleph$.

This section defines another way to compute the cardinality of the set of real numbers. In the next section, I compute again the cardinality of $P(\mathbb{N})$.

4 Cardinality of the set $P(\mathbb{N})$

In this section, I compute again the cardinality of the powerset of $\mathbb{N}$.

Theorem 4.3 $\aleph_1 = \aleph_0 = \aleph$.

Proof: $P(\mathbb{N})$ is the powerset of $\mathbb{N}$, the set of all subsets of $\mathbb{N}$. The cardinality of this set of all subsets is $2^{\aleph_0}$ equal to $\aleph_0 = \aleph$, the power of the continuum. Hence the cardinality of $P(\mathbb{N})$ is $\aleph$.

This section shows again that continuum is countable. In the next section, I define infinity.

5 Unique Infinity

This section defines the dimension of infinity.

Theorem 5.4 There is a unique dimension for the infinity which is $\aleph$.

Proof We know that the dimension of $P(\mathbb{N})$ is $2^{\aleph_0}$. Therefore, since we know that $2^{\aleph_0} = \aleph_0$, by induction showing that $\aleph_{\alpha+1} = \aleph_\alpha = \aleph_0 = \aleph$ is straightforward.

As a consequence, there is only one dimension for infinity.

This section shows that there is one unique dimension for infinity.

6 Conclusion

In this article, I prove that the cardinality of infinite sets is always $\aleph$. There is a unique dimension for infinity. I also prove that infinity is always countable. The consequence of this result is that the continuum is a countable set. This result has several consequences in Mathematics, Probability and Statistics. It modifies not only our vision of the world, but also that of modeling.
in Physics, Economics, Biology and Computer Science, among other fields. Moreover, it opens
the door to new concepts in Philosophy.
References

1. Cantor, G.: Über eine elementare Frage zur Mannigfaltigkeitslehre, Jahresbericht der Deutschen Mathematiker-Vereinigung (1891).
2. Cohen, Paul J.: The Independence of the Continuum Hypothesis. Proceedings of the National Academy of Sciences of the United States of America (1963).
3. Gödel, K.: The Consistency of the Continuum-Hypothesis. Princeton University Press (1940).
4. Gödel, K.: What is Cantor’s Continuum Problem?. Amer. Math. Monthly 54, 515–525 (1947).
Figure 1: This figure shows the infinite tree where each node gives rise to 10 branches.
Figure 2: This figure shows the infinite tree representing the set $\mathbb{N}$. 
Figure 3: This figure shows the infinite tree where each node is defined by the unique path to reach a node.
Figure 4: This figure shows the infinite tree where all real numbers are represented.
Figure 5: This figure shows the infinite tree representing all the real numbers in the interval $(0,1)$. 
Figure 6: This figure shows the infinite tree of the interval (0,1) rearranged with nodes starting with zeros in the downside.
Figure 7: This figure shows the truncated infinite trees of dimension $\kappa_0$. 
Figure 8: This figure shows the bijection between the truncated trees and $\mathbb{N}$. 