Demon driven by geometrical phase

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We theoretically study the entropy production and work extracted from a system connected to two reservoirs by periodic modulations of their electrochemical potentials of the reservoirs and one parameter in the system Hamiltonian under isothermal conditions. We find that the modulation of parameters can drive a geometrical state, which is away from a nonequilibrium steady state. With the aid of this property, we construct a demon in which the relative entropy increases with time such that we can extract the work if we begin with the nonequilibrium steady state without parameter modulations. We employ the Anderson model to demonstrate that the relative entropy can increase with time.

Introduction.- The second law of thermodynamics is one of the most fundamental laws in physics, which provides the upper bound of the available work that can be extracted from reservoirs. Maxwell proposed an idealistic setup to violate the second law, in which a demon quickly opens and closes the gate to allow only fast-moving molecules to pass through in one direction [1]. This leads to a decrease in entropy without applying any work, and thus violates the second law of thermodynamics. Because Maxwell’s original idea relies on the measurement of molecules, it is natural to combine the physical law and information science with information thermodynamics to realize Maxwell’s demon [2–7].

Nevertheless, the cost of implementation of informational Maxwell’s demon is expensive, although the theoretical formulation ignores this cost. Instead, we propose a geometrical demon with the aid of Berry’s phase [8] in a geometrical engine as an extension of the Thouless pumping [9–11]. We consider a small system sandwiched between two thermal reservoirs. If two parameters in the reservoirs and one parameter in the system Hamiltonian are controlled by an external agent, we can extract the work from the system. This is a natural application of the Thouless pumping [9–25] and geometrical thermodynamics [26–33]. It is known that the Kullback-Leibler (KL) divergence is positive semidefinite, where its zero is only achieved if the system is in a nonequilibrium steady state (NESS) [31, 32, 34–39]. We note that the relative entropy can differ from the KL divergence. Indeed, the KL divergence is always zero for a completely periodic modulation if we start from the NESS at which the KL divergence is zero, because the KL divergence cannot increase for any completely positive and trace-preserving (CPTP) processes [34–39]. However, the relative entropy of a system during a cyclic modulation can be positive and increase because of the existence of the geometrical phase. Thus, we can extract the work through this geometrical engine with the aid of an increment in the relative entropy.

Geometrical phase and entropy production.- In the present study, we focus on a system connected to two reservoirs. The left and right reservoirs are characterized by electrochemical potentials (μL and μR) and temperature T, respectively. We choose the control parameters to be the electrochemical potentials in the reservoirs and the confining potential of the system under isothermal conditions.

We modulate the electrochemical potentials through

μL = πθ, μR = π(1 + rR sin θ),

where

πθ := 1 2π ∫ π/2 θ dθμa(θ) is the one-cycle average of the electrochemical potential μa in a reservoir a (L or R). We assume that μa depends only on the modulation phase θ. We also assume that the system Hamiltonian (λ(θ)) is perfectly periodic, i.e., (λ(θ)) = (λ(θ + 2π)) through a parameter λ(θ), where λ(θ) := 1 + rH cos θ. To reduce the number of parameters, we consider only the case when r := rL = rR = rH. To maintain the positivity of the parameters, we assume |r| < 1. Thus, our system is characterized by a set of fixed parameters such as T, πθ, and two control parameters, r and δ. To express the control parameters, we introduce (λ(θ), μL(θ)/πθ, μR(θ, δ)/πθ) using λμ as one of its components.

We consider the master equation for the density matrix ̂ρ(θ, δ):

\[
\frac{d}{d\theta} ̂\rho(θ, δ) = e^{-1} K ̂\rho(θ, δ),
\]

where K is the evolution operator. We use the vector notation ̂|ρ(θ, δ)⟩ in Eq. (2), in which the components of the density matrix ̂ρ(θ, δ) align. Here, we use the scaled time θ := ωt, where ω is the modulation angular frequency. We also introduce the dimensionless parameter

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\(\epsilon\) in Eq. (2) as \(\epsilon := 2\pi\omega/\Gamma\) where \(\Gamma\) is the bandwidth used to characterize the hopping rate of the electrons from the reservoirs to the system. Introducing the eigenvalue \(\varepsilon_i\) with subscript \(i\) for \(K\), the corresponding left and right eigenstates \(|\ell_i\rangle\) and \(|r_i\rangle\) satisfy the orthonormal relation \(|\ell_i|\langle r_j|\rangle = \delta_{ij}\), if the eigenvalues are non-degenerate. We assume that there exists a non-degenerate largest eigenvalue \(\varepsilon_0 := 0\) corresponding to a steady-state. Probability conservation leads to the left-zero eigenvector \(|\ell_0\rangle\) defined as \(|\ell_0|K = 0\) whose diagonal components in the matrix form are 1 and 0 otherwise. The right zero eigenstate \(|r_0\rangle\) satisfying \(K|r_0\rangle = 0\) is also expressed as \(|\hat{\rho}^{SS}\rangle\) to specify the NESS.

Because all eigenvalues except \(\varepsilon_0 = 0\) are negative, the physical state is relaxed to \(|\hat{\rho}^{SS}\rangle\) in the absence of the modulations. This suggests that the natural choice for the initial state would be \(|\hat{\rho}^{SS}\rangle\). The stability of the steady-state has been discussed in Ref. [32].

The entropy production during one-cycle modulation starting from \(\theta\) is given by

\[
\Delta S(\theta, \delta) := S^{HS}(\hat{\rho}(\theta, \delta)||\hat{\rho}^{SS}(\theta, \delta)) - S^{HS}(\hat{\rho}(2\pi + \theta, \delta)||\hat{\rho}^{SS}(2\pi + \theta, \delta)),
\]

(3)

where we have introduced the Hatano-Sasa type relative entropy [40]:

\[
S^{HS}(\hat{\rho}||\hat{\sigma}) := \text{Tr}[\hat{\rho} \ln \hat{\rho} - \ln \hat{\sigma}],
\]

(4)

\(\Delta S(\theta, \delta)\) in Eq. (3) is expected to be \(\Delta S(\theta, \delta) \geq 0\) because the entropy increases with time as \(-S^{HS}(\theta, \delta) \geq 0\), where \(S^{HS}(\theta, \delta) := (\partial / \partial \theta) S^{HS}(\theta, \delta)\). Nevertheless, if we begin with the initial condition \(\hat{\rho}(0, \delta) = \hat{\rho}^{SS}(0, \delta)\), Eq. (3) leads to

\[
\Delta S(\delta) = -S^{HS}(\hat{\rho}(2\pi, \delta)||\hat{\rho}^{SS}(2\pi, \delta)) \leq 0,
\]

(5)

where hereafter we use \(\Delta S(\delta) := \Delta S(\theta = 0, \delta)\), because \(S^{HS}\) is positive semidefinite. Note that \(S^{HS}(\theta, \delta)\) decreases towards zero in the absence of the modulation [34–39]. The condition \(\Delta S(\delta) \geq 0\) is compatible with Eq. (5) only if \(\hat{\rho}(\theta, \delta)\) is always equal to \(\hat{\rho}^{SS}(\theta, \delta)\) for an arbitrary \(\theta\). In other words, if the density matrix can differ from \(\hat{\rho}^{SS}(\theta, \delta)\) at some \(\theta\), then \(\Delta S(\delta)\) is negative. Note that the decrease in entropy can be easily observed in physical situations if we begin with the equilibrium (maximum entropy) state [41].

Assume that the initial state is the steady state \(|\hat{\rho}(0, \delta)\rangle = |r_0\rangle = |\hat{\rho}^{SS}\rangle\). As shown in Refs. [25, 42], we obtain:

\[
|\hat{\rho}(\theta, \delta)\rangle \simeq |\hat{\rho}^{SS}(\theta, \delta)\rangle + \sum_{i\neq 0} C_i(\theta, \delta)|r_i(\theta, \delta)\rangle,
\]

(6)

where

\[
C_i(\theta, \delta) = -\int_0^{\theta} d\phi e^{-i} \int_0^{\theta} d\epsilon (z, \delta) \langle \ell_i(\phi, \delta)|\partial z| r_0(\phi, \delta)\rangle.
\]

(7)

Note that the trace preserving is always satisfied for an arbitrary \(\theta\) from \(|\ell_0||r_i\rangle = \delta_{i0}\).

The expressions in Eqs. (6) and (7) are compatible with the slow-modulation approximation employed in Ref. [32]. The leading contribution of the modulation to the entropy production is indicated by the second term on the right-hand side (RHS) of Eq. (6).

Equation (6) is an important relation, because \(\hat{\rho}(\theta, \delta)\) deviates from \(\hat{\rho}^{SS}(\delta, \delta)\) if \(C_i \neq 0\). Therefore, if \(C_i\) for some \(i\) is non-zero, the relative entropy \(S^{HS}\) is positive. Namely, \(S^{HS}\) can increase if the geometrical phase exists. Thus, our system can automatically extract work from the reservoirs. This is the essence of the geometrical demon. Note that the density matrix is reduced to \(|\hat{\rho}^{SS}\rangle\) at \(\delta \gg 1\), regardless of the initial condition, as shown in Ref. [42], although the geometrical term becomes negligibly small for large \(\delta\).

The second term on the RHS of Eq. (6) is the Berry-Sinitsyn-Nemenman (BSN) connection. For a cyclic modulation satisfying \(|r_i(2\pi, \delta)\rangle = |r_i(0, \delta)\rangle\), the deviation from the initial state after one modulation cycle becomes

\[
\Delta |\hat{\rho}\rangle := |\hat{\rho}(2\pi, \delta)\rangle - |\hat{\rho}(0, \delta)\rangle = \sum_{i\neq 0} C_i(\delta)|r_i(0, \delta)\rangle,
\]

(8)

where

\[
C_i(\delta) := \int_0^{2\pi} d\phi e^{-i} \int_0^{\theta} d\epsilon (z, \delta) \Lambda^\mu_i(\lambda) \frac{\partial \Lambda^\mu_i}{\partial \lambda} r_0(\lambda, \phi, \delta)).
\]

(9)

Here, the BSN connection \(\Lambda^\mu_i\) is defined as

\[
\Lambda^\mu_i(\phi, \delta) := -\langle \ell_i(\Lambda(\phi, \delta))|\partial z| r_0(\Lambda(\phi, \delta))\rangle.
\]

(10)

According to \(\Lambda^\mu_i\), we define the BSN curvature as

\[
F_{i\mu}^\nu(\theta, \delta) := \left(\frac{\partial \Lambda^\nu_i}{\partial \lambda}\right) - \left(\frac{\partial \Lambda^\mu_i}{\partial \lambda}\right).
\]

(11)

Because of the damping factor in Eq. (9), the contribution of the BSN curvature is localized in time. If the BSN curvature is zero inside the trajectory, we find that \(\Delta S(\delta) = 0\), whereas it can be nonzero if any BSN curvature exists. These results can be used if we begin with the general initial condition given in Ref. [42].

Now, we discuss the thermodynamic relations used to construct the geometrical demon. Let us introduce the work \(W(\delta)\) as [43, 44]

\[
W(\delta) := \int_0^{2\pi} d\theta \mathcal{P}(\theta, \delta),
\]

(12)

where

\[
\mathcal{P}(\theta, \delta) := \text{Tr}[\hat{\rho}(\theta, \delta) \frac{\partial \hat{H}(\lambda(\theta))}{\partial \lambda(\theta)} \hat{\lambda}(\theta)].
\]

(13)

The work \(W(\delta)\) and power \(\mathcal{P}(\theta, \delta)\) can be positive or negative depending on the situation. A positive \(\mathcal{P}(\theta, \delta)\)
is interpreted as the power supply by the external agent, whereas a negative $\mathcal{P}(\theta, \delta)$ can be regarded as the power loss. One can introduce

$$\mathcal{P}_{A/R}(\theta, \delta) := \frac{\mathcal{P}(\theta, \delta) \pm |\mathcal{P}(\theta, \delta)|}{2},$$

which satisfies $\mathcal{P}_{A}(\theta, \delta) = \mathcal{P}(\theta, \delta)$ ($\mathcal{P}_{R}(\theta, \delta) = -\mathcal{P}(\theta, \delta)$) if $\mathcal{P}(\theta, \delta) > 0$ ($\mathcal{P}(\theta, \delta) < 0$), whereas $\mathcal{P}_{A}(\theta, \delta) = 0$ ($\mathcal{P}_{R}(\theta, \delta) = 0$) otherwise. We also introduce $Q_{A/R}$

$$Q_{A/R}(\delta) := \int_{0}^{2\pi} d\theta \mathcal{P}_{A/R}(\theta, \delta). \quad (15)$$

$Q_{A}(\delta)$ is interpreted as the absorbing heat of the system, whereas $Q_{R}(\delta)$ represents the heat released by the system. By definition, there exists a trivial relation $Q_{A}(\delta) \geq |W(\delta)| \geq W(\delta)$. If the work $W(\delta)$ is negative, the system can be regarded as an engine in which the work done by the system is greater than the work done by the reservoirs. In this situation ($W(\delta) < 0$), we can define the efficiency $\eta(\delta)$ as

$$\eta(\delta) := \frac{|W(\delta)|}{Q_{A}(\delta)}. \quad (16)$$

Thus, to construct a geometrical demon, we require that $W(\delta) < 0$ to utilize the negative entropy production $\Delta S(\delta) < 0$. To the best of our knowledge, we cannot determine the sign of $W(\delta)$ in general. Therefore, we demonstrate that $W(\delta)$ can be negative using the Anderson model.

**Application to the Anderson model.** Let us apply the general framework to the Anderson model for a quantum dot (QD) in which a single dot is coupled to two electron reservoirs [45]. Thus, the total Hamiltonian $\hat{H}^{\text{tot}}$ can be written as

$$\hat{H}^{\text{tot}} := \hat{H} + \hat{H}^{r} + \hat{H}^{\text{int}}, \quad (17)$$

where $\hat{H}$, reservoir Hamiltonian $\hat{H}^{r}$ and interaction Hamiltonian $\hat{H}^{\text{int}}$ are, respectively, given by

$$\hat{H} = \sum_{\sigma} \epsilon_{0} d_{\sigma}^\dagger d_{\sigma} + U(\theta) \hat{n}_{\uparrow} \hat{n}_{\downarrow}, \quad (18)$$

$$\hat{H}^{r} = \sum_{\alpha,k,\sigma} \epsilon_{k} a_{\alpha,k,\sigma}^{\dagger} a_{\alpha,k,\sigma}, \quad (19)$$

$$\hat{H}^{\text{int}} = \sum_{\alpha,k,\sigma} V_{\alpha} d_{\alpha}^\dagger a_{\alpha,k,\sigma} + \text{h.c.}, \quad (20)$$

where $\hat{a}_{\alpha,k,\sigma}^{\dagger}$ and $a_{\alpha,k,\sigma}$ are, respectively, the creation and annihilation operators for the electrons in the reservoirs $\alpha = \text{L or R}$ with the wave number $k$, energy $\epsilon_{k}$, and spin $\sigma = \uparrow$ or $\downarrow$. Moreover, $d_{\sigma}^\dagger$ and $d_{\sigma}$ are those in the QD, and $\hat{n}_{\sigma} = d_{\sigma}^\dagger d_{\sigma}$. $U(\theta) := U_{0}(\theta)$ and $V_{\alpha}$ are, respectively, the time-dependent electron-electron interaction in the QD and the transfer energy between the QD and the reservoir $\alpha$. We adopt a model in the wide-band limit for the reservoirs. We denote, in this paper, the line width $\Gamma$ by $\tau$.

The Anderson model for the QD has the four states: the double-occupied, singly occupied with an up-spin, singly occupied with a down-spin, and empty. Therefore, the density matrix is expressed as a $4 \times 4$ matrix. As shown in Ref. [32], however, $\rho(\theta, \delta)$ of the Anderson model under the wideband approximation is reduced to a diagonal matrix, where the diagonal elements correspond to the probability of finding the states in the empty state $\rho_{e}$, the down-spin state $\rho_{d}$, the up-spin $\rho_{u}$, and the double occupied state $\rho_{o}$, respectively. The trace-preserving condition $\text{Tr} \rho = \rho_{e} + \rho_{u} + \rho_{d} + 1$ reduces to the conservation of probability condition. This implies that the model is a quasi-classical model. The explicit forms of the evolution matrix $K$ and the corresponding eigenstates $|\ell_{i}\rangle$ and $|r_{i}\rangle$ are summarized in Ref. [42].

For explicit calculation, we use Eqs. (6) and (7), with $\theta = 2\pi$. Integrating by parts, one can show that $C_{2}(\theta, \delta) = 0$ because $|\ell_{2}\rangle$ is independent of $\theta$.

One can verify that $\Delta S(\delta) = 0$ for $\beta U_{0} \to \infty$ [42]. This implies that $\rho(\theta, \delta)$ has only a nonzero component $\rho_{e}$ throughout the process. As a result, the entropy production by the geometrical phase is absent in this limit.

Hereafter, we set $\epsilon = 0.1$ to obtain some explicit results for the Anderson model. Recasting $|\rho(\theta)\rangle$ in matrix form $|\rho(\theta, \delta)\rangle$ plugging it into Eq. (3), we obtain $S_{\text{HS}}(\rho(\theta, \delta))$ and $\Delta S$. The time evolution of $S_{\text{HS}}(\rho(\theta, \delta))$ is shown in Fig. 1 for $\delta = 0$, $r = 0.9$, and $\beta U_{0} = 0.1$. This figure clearly indicates the oscillation of $S_{\text{HS}}(\rho(\theta, \delta))$, which increases in some instances. It is easy to verify that all components of $\rho(\theta)$ maintain positivity during the dynamics [42], and thus, the dynamics preserve the CPTP. Thus, $S_{\text{HS}}(\rho(\theta, \delta))$ cannot be regarded as the KL-divergence of the initial state with the quasi-periodic state regardless of the ini-

![Figure 1. Time evolution of relative entropy $S_{\text{HS}}(\rho(\theta, \delta) = 0)|\rho^{\text{SS}}(\theta, 0))$ for $r = 0.9$ and $\beta U_{0} = 0.1$.](image)
The parameters are set to be $\beta U_0$, $\beta \varpi$, $\mu$, and $\delta$ for various $\theta$, $\vartheta$, $\delta$, and $\mu$. As done in Ref. [25], we ignored the energy cost of controlling parameters is the most important task to be clarified, whereas the contribution of the quantum coherence [30, 47, 48]. (iii) We ignored the energy cost of controlling $\lambda(\theta)$, $\mu_L(\theta)$, and $\mu_R(\theta)$; however, it is important to determine this cost when we consider the application of this geometrical demon. The estimation of the cost of controlling parameters is the most important task to be clarified, whereas the contribution of the quantum coherence [30, 47, 48].

Concluding Remarks. - We have implemented a geometrical demon through the modulations of the electrochemical potentials in the two reservoirs and the repulsion $U(\theta)$ in the system Hamiltonian under isothermal conditions. We can automatically extract the work from this engine with an increment in relative entropy if we begin with the nonequilibrium steady state. Our geometrical demon does not require any observation of states to decrease the entropy. In this sense, our geometrical demon can be easily implemented in realistic situations; thus, we expect wide applications of this demon, although it does not work after the second cycle. This means that we have to stop the modulation after one cycle, wait until the system reached the NESS again, and then restarted the modulation to extract additional work.

Our future tasks are as follows. (i) Because the present method for the argument is restricted to the case $\epsilon \ll 1$, we will need to extend the analysis to the regime of larger $\epsilon$ as done in Ref. [25]. (ii) Although we have analyzed a quantum system, our treatment remains quasi-classical. Thus, we were unable to clarify the role of the quantum coherence [30, 47, 48]. (iii) We ignored the energy cost of controlling $\lambda(\theta)$, $\mu_L(\theta)$, and $\mu_R(\theta, \delta)$; however, it is important to determine this cost when we consider the application of this geometrical demon. The estimation of the cost of controlling parameters is the most important task to be clarified, whereas the contribution of the quantum coherence [30, 47, 48].

Thus, our engine is suitable for calling the geometrical demon.

Figure 2. Plots of $\Delta S(\theta, \delta = 0)$ against $\theta$ for $\beta U_0 = 0.3$ and $r = 0.9$.

Figure 3. (a) Schematics of a contour of the integral of $C_1$, where the black solid line is the trajectory of the parameters. The color scale at a value of $\varpi$ expresses $P^{\mu L, \mu R}$. (b) and (c) The BSN curvatures $F^{\mu L, \mu R}$ at $\theta = 0$ are plotted. The parameters are set to be $\beta \varpi = 0.1$, $\beta U_0 = 0.1$ and $\beta \vartheta = 0.1$ for all figures.

Figure 4. Plots of $\Delta S(\delta)$ versus $\delta$ for $\beta U_0 = 0.3$ (solid line), 0.5 (dotted line), 0.7 (dashed line) with fixing $r = 0.9$.

Figure 5. (a) Plots of the work done on the system in one-cycle modulation versus $\delta$ with $\beta U_0 = 0.1$ and $r = 0.9$ (solid line) and fitting by the sinusoidal function (dotted line). (b) Plots of the efficiency $\eta(\delta)$ for $\beta U_0 = 0.1$ and $r = 0.9$ (solid line) and fitting by the sinusoidal function (dotted line). The curves for (a) and (b) are fitted by $-0.3(\cos \delta + 1)$ and $0.0766(\cos \delta + 1)$, respectively.
entropic fluctuation in our system is positive semidefinite [42].

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[41] As can be seen in the relation $S^{SS}(\hat{\rho}^{SS}(\theta)|\hat{\rho}^{SS}(\theta)) = 0$, our formulation has already eliminated the energy supply (from the connection with higher electrochemical potential) and dissipation (from the connection with the lower electrochemical potential terms), which are perfectly balanced with each other in the nonequilibrium steady state. Thus, the Joule heat term does not appear explicitly in our formulation. See Ref. [42] for the role of the housekeeping entropy.

[42] See Supplemental Material.

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[45] Though there exist higher energy levels in realistic quantum dot systems, here we only consider the single energy level for simplicity.

[46] Needless to say, the system with $\Delta S < 0$ is not perfectly periodic even if $H(\lambda(\theta))$ and the control parameters are periodic. Since there exists a damping factor in the geometrical contribution, $|\Delta S|$ in the second cycle is much smaller than that in the first cycle as explained in the main text. In the long time limit, the system asymptotically reaches the NESS in which the geometrical contribution asymptotically is negligible.

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This Supplemental Material explains the details of calculations that are not included in the main text. In Sec.I we derive a general expression for the time evolution of the density matrix $\hat{\rho}(\theta, \delta)$ without a specific choice of model and initial condition. In Sec.II we explain the contribution of the housekeeping entropy, which has been ignored in the main text. In Sec.III we present the detailed properties of the Anderson model. In Sec.IV we briefly summarize the differences between our analysis and the analysis in Ref. [S1].

I. TIME EVOLUTION OF THE DENSITY MATRIX

This section consists of two subsections. In the first part, IA, we derive the time evolution of $\hat{\rho}(\theta, \delta)$ from the general initial condition to demonstrate the universality of Eqs. (6) and (7) for $\theta \gg 1$. In the second part, IB, we present the detailed derivations of the BSN connection and BSN curvature.

A. Derivation of the time-dependent expression of the density matrix starting from the general initial condition

The purpose of this section is to derive Eqs. (6) and (7). Although we assumed that the initial state is given by $\hat{\rho}^{SS}(\theta = 0, \delta)$ in the main text, we can derive these equations even when we begin with the generalized initial condition

$$|\hat{\rho}_{ini}\rangle = \sum_{i} a_i |r_i(0, \delta)\rangle,$$

where $a_i$ is given by

$$a_i := \langle \ell_i(0, \delta) | \hat{\rho}_{ini}\rangle.$$

We note that the normalization of the density matrix fixes the coefficient $a_0$ to 1, because

$$\text{Tr} \hat{\rho}_{ini} = \langle \ell_0 | \hat{\rho}_{ini}\rangle = \sum_{i} a_i \langle \ell_0 | r_i(0, \delta)\rangle = a_0 = 1.$$

One can derive $|\hat{\rho}(\theta, \delta)\rangle$ for $|\hat{\rho}(0, \delta)\rangle = |\hat{\rho}_{ini}\rangle$ as

$$|\hat{\rho}(\theta + \Delta \theta, \delta)\rangle \approx \sum_{i,j,k,\ldots,l,m} a_m |r_i(\theta, \delta)\rangle e^{\epsilon_{j}(\theta, \delta)} \langle \ell_j(\theta, \delta) | r_j(\theta - \Delta \theta, \delta)\rangle \times e^{\epsilon_{k}(\theta - \Delta \theta, \delta)} \langle \ell_k(\theta - \Delta \theta, \delta) | r_k(\theta - 2\Delta \theta, \delta)\rangle \times \cdots \times e^{\epsilon_{l}(\Delta \theta)} \langle \ell_l(0, \delta) | r_l(0, \delta)\rangle \times e^{\epsilon_{m}(\theta, \delta)} | r_m(\theta, \delta)\rangle + \sum_{m,n} C_{mn} a_n | r_m(\theta, \delta)\rangle,$$

(S4)
where $C_{mn}$ is given by
\[
C_{mn}(\theta, \delta) = - \int_0^\theta d\phi e^{\int_0^\phi \varepsilon_m(\zeta, \delta) \frac{d\zeta}{\epsilon}} e^{\int_0^\phi \varepsilon_n(\zeta, \delta) \frac{d\zeta}{\epsilon}} \times \langle \ell_m(\phi, \delta) | \frac{d}{d\phi} | r_n(\phi, \delta) \rangle.
\] (S5)

The phase factor in the first term on the RHS of Eq. (S4) is the dynamical phase, whereas the second term on the RHS is the term generated by the geometrical phase. Substituting $\int_0^\theta \varepsilon_m(\xi, \delta) \frac{d\xi}{\epsilon} = \int_0^\theta \varepsilon_m(\xi, \delta) \frac{d\xi}{\epsilon} - \int_0^\phi \varepsilon_m(\xi, \delta) \frac{d\xi}{\epsilon}$ in Eq. (S5) and substituting this result into Eq. (S4), we obtain
\[
|\dot{\rho}(\theta, \delta)| \simeq \sum_m a_m |r_m(\theta, \delta)| + \sum_{m,n} \tilde{C}_{mn} a_n |r_m(\theta, \delta)|,
\] (S6)
where we have introduced
\[
\tilde{C}_{mn}(\theta, \delta) := - \int_0^\theta d\phi \langle \ell_m(\phi, \delta) | \frac{d}{d\phi} e^{\int_0^\phi \varepsilon_m(\phi, \delta) \frac{d\phi}{\epsilon}} \rangle | r_n(\phi, \delta) \rangle,
\] (S7)
\[
|\tilde{r}_m(\theta, \delta)| := e^{\int_0^\phi \varepsilon_m(\phi, \delta) \frac{d\phi}{\epsilon}} | r_m(\theta, \delta) \rangle,
\] (S8)
and
\[
\langle \ell_m(\theta, \delta) | e^{\int_0^\phi \varepsilon_m(\phi, \delta) \frac{d\phi}{\epsilon}} \rangle = \delta_{mn}.
\] (S9)

The orthonormal relation $\langle \ell_m | r_n \rangle = \delta_{mn}$ leads to
\[
\langle \ell_m(\theta, \delta) | \tilde{r}_n(\theta, \delta) \rangle = e^{-\int_0^\phi \varepsilon_m(\phi, \delta) \frac{d\phi}{\epsilon}} e^{\int_0^\phi \varepsilon_n(\phi, \delta) \frac{d\phi}{\epsilon}} \delta_{mn} = \delta_{mn}.
\] (S10)

Substituting this relation into Eq. (S7), we obtain
\[
\tilde{C}_{mn}(\theta, \delta) = \frac{\delta_{mn}}{\epsilon} \int_0^\theta d\phi e^{\int_0^\phi \varepsilon_n(\phi, \delta) \frac{d\phi}{\epsilon}} \bigg( \frac{d}{d\phi} e^{\int_0^\phi \varepsilon_m(\phi, \delta) \frac{d\phi}{\epsilon}} \bigg).
\] (S11)

Equations (S6) and (S11) are expressions of the time evolution starting from the general initial condition in Eq. (S1).

Note that the trace of $\dot{\rho}(\theta, \delta)$ is always equal to unity. This can be proven as follows. The coefficient $C_{0j}$ identically vanishes because
\[
\langle \ell_0(\phi, \delta) | \frac{d}{d\Lambda_\mu} | \tilde{r}_j(\phi, \delta) \rangle = \frac{\partial \langle \ell_0(\phi, \delta) | \tilde{r}_j(\phi, \delta) \rangle}{\partial \Lambda_\mu} = 0.
\] (S12)

Thus, $\text{Tr} \dot{\rho}(\theta, \delta)$ satisfies
\[
\text{Tr} \dot{\rho}(\theta, \delta) = \langle \ell_0 | \dot{\rho}(\theta, \delta) \rangle = a_0 = 1.
\] (S13)

where we have used $\varepsilon_0(\phi, \delta) = 0$ and Eq. (S10) to obtain this result.

Now, let us consider the behavior for $\theta/\epsilon \gg 1$. It is evident that the first term on the RHS of Eq. (S6) is reduced to $|\dot{\rho}^{\text{SS}}(\theta)|$ because the exponential damping factor for $m \neq 0$ is negligible. Thus, Eq. (S6) is reduced to
\[
|\dot{\rho}(\theta)| \simeq |\dot{\rho}^{\text{SS}}(\theta)| + \sum_{m,n} C_{mn}(\theta, \delta) a_n |r_m(\theta, \delta) \rangle
\] (S14)
for $\theta/\epsilon \gg 1$. It is straightforward to evaluate $C_{mn}(\theta, \delta)$ for $n \neq 0$ as
\[
|C_{mn}(\theta)| = \left| \int_0^\theta d\phi e^{\int_0^\phi \varepsilon_m(\zeta, \delta) \frac{d\zeta}{\epsilon}} e^{\int_0^\phi \varepsilon_n(\zeta, \delta) \frac{d\zeta}{\epsilon}} \langle \ell_m(\phi) | \frac{d}{d\phi} | r_n(\phi) \rangle \right|
\] (S15)
where we have omitted $\delta$ dependence in the expressions in Eq. (S15). Thus, $C_{mn}$ with $n \neq 0$ is much smaller than $C_{m0}$ because of the existence of the exponential factor in Eq. (S15).

Equation (S14) implies that the relative entropy $\Delta S(\theta, \delta)$ approaches 0 in the absence of the geometrical phase, but not in the presence of the geometrical phase. Moreover, the relative entropy becomes non-zero, as indicated by the results in the main text. Thus, the geometrical phase prevents the system from the relaxation towards the NESS (at which the relative entropy $S^{\text{HS}}$ is zero). Once we start to modulate the parameters, the system is driven from the NESS state. However, because of the exponential factor in Eq. (S5), which truncates the contribution in the integration except for $\phi \approx \theta$, the system again exhibits periodic behavior in $\theta$ after the characteristic time given by the eigenvalues of $\tilde{K}$. As a result, the relative entropy $S^{\text{HS}}$ also becomes periodic in $\theta$ after some characteristic time $\theta_c$. Thus, the relative entropy $\Delta S(\theta, \delta)$ is expected to be zero for $\theta > \theta_c$.

In Fig. 2, we plot the $\theta$ dependence of $\Delta S(\theta, \delta)$, which clearly shows that $\Delta S(\theta, \delta)$ exponentially approaches zero.

B. Derivation of BSN connection and BSN curvature

For the cyclic modulation $|r_i(2\pi, \delta) \rangle = |r_i(0, \delta) \rangle$, the time evolution from the general initial state becomes
\[
\Delta |\rho\rangle = \sum_{i \neq 0} a_i \left( e^{\int_0^{2\pi} \varepsilon_i(\phi, \delta) \frac{d\phi}{\epsilon}} - 1 \right) |r_i(0, \delta) \rangle
\] (S16)
where

\[ C_{ij} := \delta_{ij} \int_{\partial \Omega} d\phi \frac{\xi_i(\phi, \delta)}{\epsilon} - \int_{\partial \Omega} d\Lambda_\mu A^\mu_{ij} \]  

(S17)

with the introduction of the BSN connection \( A^\mu_{ij} \):

\[ A^\mu_{ij} := \langle \hat{\xi}_i(\phi, \delta) \rangle \frac{\partial}{\partial \Lambda_\mu} |\hat{\xi}_j(\phi, \delta)\rangle. \]  

(S18)

Here, the summation of the first term on the RHS of Eq. (S16) is considered except for \( i = 0 \) because \( \epsilon_0(\phi, \delta) = 0 \) always yields \( e^{i\frac{\epsilon_0}{\epsilon}(\phi, \delta)} \delta = 1 \). Using the Stokes theorem, we can rewrite Eq. (S17) as

\[ C_{ij} = \delta_{ij} \int_{\partial \Omega} d\phi \frac{\xi_i(\phi, \delta)}{\epsilon} - \int_{\Omega} dS_{\mu\nu} F_{ij}^{\mu\nu}, \]  

(S19)

where \( \Omega \) is the area enclosed by the closed trajectory \( \partial \Omega \), \( S_{\mu\nu} = \frac{i}{2} d\Lambda_\mu \wedge d\Lambda_\nu \), and \( F_{ij}^{\mu\nu} \) represents the BSN curvatures defined as

\[ F_{ij}^{\mu\nu} := \frac{\partial}{\partial \Lambda_\nu} \langle \hat{\xi}_i(\phi, \delta) \rangle - \frac{\partial}{\partial \Lambda_\mu} \langle \hat{\xi}_j(\phi, \delta) \rangle. \]  

(S20)

It is also possible to rewrite Eq. (S19) as

\[ C_{ij} = \delta_{ij} \int_{\partial \Omega} d\phi \frac{\xi_i(\phi, \delta)}{\epsilon} + \frac{1}{2} \int_{\Omega} d\hat{\xi}_i \wedge d\hat{\xi}_j. \]  

(S21)

II. CONTRIBUTION OF THE HOUSEKEEPING ENTROPY

The nonequilibrium system we consider is sustained by an external agent, which requires housekeeping heat as well as excess heat, although the main text only contains the description for the excess heat [S2, S3]. In this section, we evaluate the housekeeping entropy production in our system.

As shown in Refs. [S2, S3], we introduce a set of counting fields \( \chi \) to calculate physical observables. As a result, Eq. (2) is formally modified as

\[ \frac{d}{dt} \hat{\rho}(\theta, \delta, \chi) = e^{-i\hat{K}^\chi} \hat{\rho}(\theta, \delta, \chi), \]  

(S22)

where the set of counting fields contains two components \( \chi = (\chi_L, \chi_R) \), which are inserted to monitor the time evolution of the housekeeping entropy production in the left and right lead, respectively, and \( i\hat{K}^\chi \) is the generalized density matrix and the evolution operator, respectively. Since \( \hat{\rho}(\theta, \delta, \chi) \) behaves as \( \hat{\rho}(\theta, \delta, \chi) \sim \exp[\lambda_0(\Lambda, \chi) / \theta / \epsilon] \) for large \( \theta / \epsilon \) with the smallest eigenvalue \( \lambda_0(\Lambda, \chi) \) of \( \hat{K}^\chi \) (which is reduced to zero in the limit \( \chi \to 0 \)) under a fixed \( \Lambda \), the housekeeping entropy flux [S2, S3] is given by

\[ J_{hk}(\phi, \delta) := \left. \frac{\partial \lambda_0(\Lambda(\phi, \delta), \chi)}{\partial (i\chi_L)} \right|_{\chi = 0} + \left. \frac{\partial \lambda_0(\Lambda(\phi, \delta), \chi)}{\partial (i\chi_R)} \right|_{\chi = 0}. \]  

(S23)

Note that \( \chi_L \) and \( \chi_R \) couple to the housekeeping entropy production in the left reservoir (\( \hat{S}_L = \beta(\hat{H}_L - \mu_L) \)) and that in the right reservoir (\( \hat{S}_R = \beta(\hat{H}_R - \mu_R) \)), respectively, where \( \hat{H}_a \) (\( a = L, R \)) stands for the Hamiltonian of the reservoir \( a \). This housekeeping entropy flux is dominant to maintain the steady-state.

More explicitly, \( \hat{K}^\chi \) in Eq. (S22) can be written as

\[ \hat{K}^\chi = \hat{K} + i \sum_{a=L,R} \chi_a \hat{K}_a + O(\chi^2). \]  

(S24)

For the explicit calculation of Eq. (S23), we employ the Anderson model, as in the main text. In the following, we consider the generating function defined by

\[ \text{Tr}_\rho(\theta, \delta, \chi) = \ln \left\langle e^{i(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta))} \right\rangle, \]  

(S25)

where the bracket stands for the thermal and quantum averages. In the present case, only \( \hat{H}^\text{int} \) in the total Hamiltonian does not commute with \( e^{i(\chi_L \hat{S}_L + \chi_R \hat{S}_R)}/\theta \). In this case, one can use the technique with that used in Ref. [S4], namely, the counting fields can be absorbed as the phases of the interaction part,

\[ e^{-i\frac{\epsilon}{\epsilon}(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta)) \delta} \]  

(S26)

\[ e^{-i\frac{\epsilon}{\epsilon}(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta)) \delta} = e^{i\frac{\epsilon}{\epsilon}(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta)) \delta}, \]  

(S27)

where \( \hat{S}_{a,k} = \beta(\epsilon_k - \mu_a) \) and \( \delta_{a\beta} \) is Kronecker's delta. Then, we just need to proceed the following transposition to calculate the generating function.

\[ a_{a,k}^\dagger \rightarrow e^{i\frac{\epsilon}{\epsilon}(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta))} a_{a,k}, \]  

(S28)

\[ a_{a,k} \rightarrow e^{-i\frac{\epsilon}{\epsilon}(\chi_L \hat{S}_L(\theta) + \chi_R \hat{S}_R(\theta, \delta))} a_{a,k}. \]  

(S29)

As a consequence, the similar calculation in Ref. [S4] yields \( \hat{K}^\chi \) as

\[ \hat{K}^\chi = \Gamma \begin{pmatrix} -2f^{(1)}_+ & f^{(1)}_+ & 0 & 0 \\ f^{(1)}_- & f^{(1)}_+ & 0 & 0 \\ 0 & f^{(0)}_- & 0 & 0 \\ 0 & 0 & f^{(0)}_- & 0 \end{pmatrix}, \]  

(S30)

where

\[ f^{(j)}_+ = e^{i\epsilon_L \hat{S}_L(\theta) \hat{f}^{(j)}(\epsilon_0) + e^{i\epsilon_R \hat{S}_R(\theta) \hat{f}^{(j)}(\epsilon_0)} \]  

(S31)

\[ f^{(j)}_+ = e^{-i\epsilon_L \hat{S}_L(\theta) \hat{f}^{(j)}(\epsilon_0) + e^{i\epsilon_R \hat{S}_R(\theta) \hat{f}^{(j)}(\epsilon_0)}, \]  

\( \hat{S}_L = \beta(\epsilon_0 + jU - \mu_a) \). Here, \( f^{(j)}(\epsilon_0) = [1 + e^{i\epsilon_0(\epsilon_0 + jU - \mu_a)}]^{-1} \) corresponds to the Fermi distribution function. Substituting Eq. (S30) into Eq. (S24), we can write \( \hat{K}_a \) as

\[ \hat{K}_a = \begin{pmatrix} S_{a,k}^0(g_0^a - 1) & 0 & 0 & 0 \\ 0 & S_{a,k}^0(g_0^a - 1) & 0 & 0 \\ 0 & 0 & S_{a,k}^0(g_0^a - 1) & 0 \\ 0 & 0 & 0 & S_{a,k}^0(g_0^a - 1) \end{pmatrix}, \]  

(S31)
with \( g^0_\alpha := (1 + e^{\beta(\epsilon_\alpha + jU - \mu_\alpha)})^{-1} \), \( S^j_\alpha = \beta(\mu_\alpha - \epsilon_0 - jU) \) \((j = 1, 2, \alpha = L, R)\). Using the relation
\[
\lambda^{(1)}_{0,\alpha}(\Lambda(\phi, \delta)) = \langle \ell_0| \hat{K}_\alpha(\Lambda(\phi, \delta))|r_0 \rangle, \tag{S32}
\]
and the expansion
\[
\lambda_0(\Lambda(\phi, \delta), \chi) = i \sum_\alpha \chi_\alpha \lambda^{(1)}_{0,\alpha}(\Lambda(\phi, \delta)) + O(\chi^2), \tag{S33}
\]
we obtain \( \lambda_0(\Lambda, \chi) \) and \( J_{hk} \).

The housekeeping entropy production during one cycle is given by
\[
S_{hk}(\theta, \delta) := \int_{\theta}^{\theta + 2\pi} d\phi J_{hk}(\phi, \delta). \tag{S34}
\]
Substituting Eqs. (S23), (S32) and (S33) into Eq. (S34), we obtain
\[
S_{hk}(\theta, \delta) = \sum_{\alpha=L,R} \int_{\theta}^{\theta + 2\pi} d\phi \langle \ell_0| \hat{K}_\alpha(\Lambda(\phi, \delta))|r_0 \rangle. \tag{S35}
\]
We note that \( S_{hk}(\theta, \delta) \) is independent of \( \theta \), because \( \lambda_0(\Lambda(\phi), \chi) \) depends on \( \phi \) only through \( \Lambda(\phi, \delta) \) which is a periodic function of \( \phi \). In Fig. S6, we plot the time dependence of the housekeeping entropy production given by Eq. (S23) in the case of the Anderson model, where we set \( \delta = \pi \). The housekeeping entropy production is always non-negative as it is expected. We also plot the housekeeping entropy production during one cycle as a function of \( \delta \) in Fig. S7. As shown in Fig. S7, the housekeeping entropy production during one cycle is positive irrespective of \( \delta \), except for \( \delta = 0 \). In the case of \( \delta = 0 \), the average bias voltage is absent and thus the housekeeping entropy production, which maintain the steady-state, is zero.

### III. PROPERTIES OF THE ANDERSON MODEL

This section summarizes the properties of the Anderson model in greater detail, which consists of four subsections. In the first subsection, we summarize the evolution matrix and eigenstates in the Anderson model. In the second subsection, we present the explicit form of the density matrix. In the third subsection, we provide the explicit forms of the BSN connection and BSN curvature as well as the expansion coefficients. In the last subsection, we present some detailed calculations for the Anderson model. For simplicity, we do not write \( \delta \) dependence of variables explicitly in this section.

#### A. Evolution matrix and eigenstates in the Anderson model

In this subsection, we summarize the evolution matrix and eigenstates in the Anderson model. A similar discussion can be found in Ref. [S1].

Because \( \hat{\rho} \) is a diagonal matrix, \( |\hat{\rho}\rangle \) also has only four components, and the transition matrix \( \hat{K} \) in Eq. (2) in the wideband approximation is given by the \( 4 \times 4 \) matrix:
\[
\hat{K} = -\begin{pmatrix}
2f^{(1)}_+ & -f^{(1)}_- & -f^{(1)}_+ & 0 \\
-f^{(1)}_- & f^{(0)}_+ + f^{(1)}_+ & 0 & -f^{(0)}_+ \\
f^{(1)}_+ & 0 & f^{(1)}_- + f^{(1)}_+ & -f^{(0)}_+ \\
0 & -f^{(0)}_+ & -f^{(0)}_- & 2f^{(0)}_+
\end{pmatrix}, \tag{S36}
\]
where we have introduced
\[
f^{(j)}_+ := f^{(j)}_L + f^{(j)}_R, f^{(j)}_- := 2 - f^{(j)}_+, \tag{S37}
\]
with the Fermi distribution
\[
f^{(j)}_{\alpha}(\mu_\alpha(\theta), U(\theta)) := \frac{1}{1 + e^{\beta(\epsilon_\alpha + jU(\theta) - \mu_\alpha(\theta))}} \tag{S38}
\]
in the lead $\alpha (= L$ or $R$) for the single occupancy $j = 0$
and double occupancy $j = 1$.

It is straightforward to obtain the eigenvalues of $K(\Lambda(\theta))$ in Eq. (S36) as $\varepsilon_0 = 0$, $\varepsilon_1 = -f_+^{(0)} - f_-^{(1)}$, $\varepsilon_2 = -4 - \varepsilon_1$, $\varepsilon_3 = -4$. The left and right eigenfunctions corresponding to $\varepsilon_0 = 0$ for $K$ are given by

$$\langle \ell_0 \rangle = (1, 1, 1, 1),$$
$$|r_0 \rangle = \alpha_0 (f_+^{(0)} f_+^{(1)}, f_+^{(0)} f_-^{(1)}, f_-^{(0)} f_-^{(1)} f_-^{(1)})^T,$$

respectively, where $\alpha_0 = [2(f_+^{(0)} + f_-^{(1)})]^{-1}$ is the normalization factor. As discussed previously, the eigen state corresponding to the zero eigenvalue denotes the nonequilibrium steady state, namely $|r_0 \rangle = \rho^{\infty}$. Note that $|r_0 \rangle$ satisfies $\langle \ell_0 |r_0 \rangle = \operatorname{Tr} \rho^{\infty} = 1$. The left and right eigenfunctions corresponding to $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are respectively given by

$$\langle \ell_1 \rangle = \left( f_-^{(1)}, \gamma, \gamma, -f_+^{(0)} \right),$$
$$|r_1 \rangle = \alpha_1 \left( f_+^{(1)}, \gamma, \gamma, -f_-^{(0)} \right)^T,$$

and

$$\langle \ell_2 \rangle = (0, 1, -1, 0), |r_2 \rangle = \frac{1}{2}(0, 1, -1, 0),$$

and

$$\langle \ell_3 \rangle = \left( f_-^{(0)} f_+^{(1)}, -f_-^{(0)} f_-^{(1)}, -f_-^{(0)} f_-^{(1)} + f_-^{(1)} \right),$$
$$|r_3 \rangle = \alpha_3 (1, -1, -1, 1)^T,$$

where $\alpha_1 = 2[(f_+^{(0)} + f_-^{(1)})(f_-^{(1)} + f_-^{(0)})]^{-1}$, $\alpha_3 = [2(f_+^{(0)} + f_-^{(1)})]^{-1}$, and $\gamma = (-f_+^{(0)} + f_-^{(1)})/2$.

**B. Time evolution of the density matrix for the Anderson model**

We are interested in the entropy production in the first cycle $\Delta S$ given by Eq. (5). For this purpose, we must know $\dot{\rho}(2\pi)$ and $\rho(0)$. The former is given by

$$|\dot{\rho}(2\pi)\rangle = |\dot{\rho}^{\infty}(2\pi)\rangle + \sum_{i=1}^{3} C_i |r_i(2\pi)\rangle$$

$$= \left( \alpha_0 f_+^{(0)} f_+^{(1)} + C_1 \alpha_1 f_+^{(1)} + C_3 \alpha_3 f_-^{(0)} f_-^{(1)} \right)$$
$$\left( \alpha_0 f_-^{(0)} f_-^{(1)} + C_1 \alpha_1 f_-^{(1)} - C_3 \alpha_3 f_-^{(0)} f_-^{(1)} \right)$$
$$\left( \alpha_0 f_-^{(0)} f_-^{(1)} - C_1 \alpha_1 f_-^{(1)} + C_3 \alpha_3 f_-^{(0)} f_-^{(1)} \right)$$

where $\alpha_0 = [2(f_+^{(0)} + f_-^{(1)})]^{-1}$, $\alpha_1 = 2[(f_+^{(0)} + f_-^{(1)})(f_-^{(1)} + f_-^{(0)})]^{-1}$, $\alpha_3 = [2(f_+^{(0)} + f_-^{(1)})]^{-1}$, and $\gamma = (-f_+^{(0)} + f_-^{(1)})/2$.

**C. BSN connection for Anderson model**

In this subsection, we present an explicit form of the BSN connection for the Anderson model. For the explicit calculation of the BSN connection, we use Eqs. (6) and (7) with $\theta = 2\pi$. Integrating by parts, one obtains

$$\int_0^{2\pi} \langle \ell_2(\phi) | \frac{d}{d\phi} | r_0(\phi) \rangle d\phi$$

$$= \int_0^{2\pi} e^{\int_0^\phi \varepsilon_2(\xi) d\xi} \langle \ell_2(\phi) | \frac{d}{d\phi} | r_0(\phi) \rangle$$

$$= \int_0^{2\pi} e^{\int_0^\phi \varepsilon_2(\xi) d\xi} \left[ \frac{d}{d\phi} \langle \ell_2(\phi) | r_0(\phi) \rangle - \frac{d}{d\phi} \langle \ell_2(\phi) | r_0(\phi) \rangle \right]$$

$$= 0,$$

where $\langle \ell_2(\phi) \rangle$ is independent of $\phi$. Thus, the summation of $i$ in Eq. (6) is reduced to the summation with $i = 1$ and 3. The differentiation of $|r_0(\phi)\rangle$ with respect to $\phi$
Substituting Eq. (S41) into Eq. (S51), we obtain
\[
C_1 = - \int_0^{2\pi} d\phi \epsilon f^2_\phi \epsilon_2(\xi) \frac{2 f^{(1)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} \bigg|_{\phi=0}^{2\pi} \int_0^{2\pi} d\phi \epsilon f^2_\phi \epsilon_2(\xi) \frac{2 f^{(1)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} \bigg|_{\phi=0}^{2\pi}.
\]
(S52)

Similarly, we obtain
\[
C_3 \approx \int_0^{2\pi} d\phi \epsilon f^2_\phi \epsilon_3(\xi) \frac{f^{(1)}_+ f^{(1)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} - \int_0^{2\pi} d\phi \epsilon f^2_\phi \epsilon_3(\xi) \frac{f^{(0)}_+ f^{(0)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} \bigg|_{\phi=0}^{2\pi}.
\]
(S53)

As shown by Eq. (S53), the factor \(e^{4\epsilon^{-1}(\phi - 2\pi)}\) in the integrand plays an important role. Owing to this factor, it is not necessary to consider the long-term memory in the dynamics. In the case of \(C_3\), the scaling factor does not depend on the choice of the trajectory, and only depends on \(\theta\). Thus, one can estimate the BSN curvature at \(\phi\).

For \(\epsilon \ll 1\), the exponential factor \(e^{4\epsilon^{-1}(\phi - 2\pi)}\) behaves as the cut-off function and thus
\[
C_3 \approx \int_0^{2\pi} d\phi \epsilon f^2_\phi \epsilon \bigg[ \frac{f^{(1)}_+ f^{(1)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} - \frac{f^{(0)}_+ f^{(0)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} \bigg] \bigg|_{\phi=0}^{2\pi}.
\]
(S54)

Similarly, we can write
\[
C_1 \approx \epsilon \bigg[ \frac{2 e^{\epsilon_1(2\pi)} f^{(1)}_+ f^{(0)}_+ d\phi}{f^{(0)}_+ + f^{(1)}_+} \bigg] \bigg|_{\phi=2\pi}^{\phi=0}.
\]
(S55)

From the above estimations, the leading order contribution of the geometrical phase is the order \(O(\epsilon)\).

Let us show the \(C_3\) reduces to zero in the noninteracting case \(\beta U_0 = 0\). In this case, \(f^{(0)} = f^{(1)} = g\pm\) and \(g_+ + g_- = 2\); thus \(C_3\) becomes
\[
\lim_{\beta U_0 \to 0} C_3 = 0.
\]
(S56)

In the opposite limit \(\beta U_0 \to \infty\) with \(f^{(1)} = 0\), \(f^{(1)} = 2\), \(C_3\) also becomes zero, because
\[
\lim_{\beta U_0 \to \infty} C_3 = 0.
\]
(S57)
IV. DIFFERENCE BETWEEN REF. [S1] AND THE PRESENT STUDY

Most important difference between this paper and Ref. [S1] exists in the setup. Namely, we mainly discuss the relaxation process from NESS in this paper, but Ref. [S1] discussed the behavior after the system reaches a quasi-periodic state, in which the geometrical term can be ignored.

If we assume that \(|\hat{\rho}(\theta, \delta)\rangle\) depends on \(\theta\) only through modulation parameters \(\Lambda\), as assumed in Ref. [S1], then we can express \(|\hat{\rho}^{(1)}(\Lambda(\theta, \delta))\rangle\) as

\[
|\hat{\rho}^{(1)}(\Lambda(\theta, \delta))\rangle = \hat{K}^+ (\Lambda(\theta, \delta)) \frac{d}{d\theta} |\hat{\rho}^{\Lambda}(\Lambda(\theta, \delta))\rangle
\]

which is the equation used in Ref. [S1].

However, if \(|\hat{\rho}(\theta, \delta)\rangle\) explicitly depends on \(\theta\) and the geometrical term still survives, it is dangerous to use Eq. (S58). To clarify this difference, consider the relaxation process in the absence of the parameter modulation, in which the time evolution of the density matrix is given by

\[
|\hat{\rho}(\theta, \delta)\rangle = |\hat{\rho}^{SS}\rangle + \sum_{i \neq 0} e^{\frac{i}{\epsilon} \theta} |\hat{r}_i\rangle
\]

\[
= |\hat{\rho}^{SS}\rangle + \sum_{i \neq 0} e^{\frac{i}{\epsilon} \theta} |\hat{r}_i\rangle
\]

\[
= |\hat{\rho}^{SS}\rangle + \sum_{i \neq 0} e^{-\frac{i}{\epsilon} \theta} |\hat{r}_i\rangle.
\]

In this case, the density matrix relaxes to the steady state \(|\hat{\rho}^{SS}\rangle\) in the long time limit \(\theta/\epsilon \gg 0\). We note that this...
limit is usually achieved not by $\theta \gg 1$ but by $\epsilon \to 0$. It is obvious that $\epsilon \to 0$ is the singular limit and the expansion via $\epsilon$ is not available because the convergence radius of $e^{-1/x}$ is zero. Thus, if the density matrix explicitly depends on $\theta$, it is dangerous to use the naive perturbation via $\epsilon$ used in Ref. [S1].

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