LIBERATING EXOTIC SLAVES

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ABSTRACT

The introduction of confined, “slave” fields is frequently useful as a formal device in models of condensed matter physics; it becomes a conceptual necessity for describing possible phases of matter where the slaves are liberated. Here I discuss some aspects of the fractional quantum Hall effect from this point of view, emphasizing analogies with phenomena in other areas of physics, particularly to the Meissner and Higgs mechanisms, and to confinement-deconfinement transitions. In this application, and in some recent attempts to model the normal state of copper oxide superconductors, it is important to employ slave anyon fields.

I have long admired Yakir Aharonov’s style in physics: to continue to puzzle over that which is intrinsically strange, even in domains where more jaded spirits have lost, from mere familiarity, their sense of wonder. This child-like quality has led him to make fundamental discoveries where few would anticipate that fundamental discoveries could still be made, and—as we all must acknowledge on this occasion—it obviously has kept him young!

In that spirit, I hope, I would like to discuss with you today a personal perspective on the fascinating complex of new states of matter forming the “quantum Hall complex,” which I have developed in response to some simple puzzles that have bothered me for a long time. One of the puzzles, as I shall describe momentarily, has to do with gauge invariance. The other is broader: is the fractional quantized Hall effect as special and isolated as it seems at first sight, or can its occurrence be related to other deep ideas in theoretical physics? I have found my perspective quite comforting and informative, and I think it is different at least in emphasis and some significant details from what has appeared in the literature (including my own work.) However, I must quickly add that it in no way alters with Laughlin’s basic physical picture of an incompressible quantum liquid, nor will it be used here to derive new results that could not be found otherwise.

1. Critique of Laughlin’s Quantization Argument

1.1. The Argument

Shortly after the experimental discovery of the integer quantized Hall effect, Laughlin[4] proposed an argument, based on gauge invariance, that explains why the conductance is quantized. The argument proceeds from the physical hypothesis that in the conditions where the quantized Hall effect is observed the electrons form

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1Talk at the Celebration of the 60th Birthday of Yakir Aharonov, February 1993.
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an incompressible fluid in the bulk, to show that the conductivity of the fluid (to be defined, in a precise geometry, momentarily) must be an integer multiple of a certain combination of fundamental constants, viz. $c^2/h$. With some important refinements due to Halperin,[5] this argument remains the foundation of the theory of the effect. I would like briefly to recall its essence.

Imagine an annulus containing electrons held at low temperature and subject to a large perpendicular magnetic fields, and such that the inner and outer edges are connected by an ordinary wire and held at a voltage difference $V$. Suppose that we have the conditions of the quantized Hall effect, that is, by hypothesis, that within the bulk of the annulus there is an incompressible electron fluid. This means that there is, for each value of the current circulating around the annulus, a unique bulk state of minimum energy. It can be constructed, locally, from the unique, isolated ground state by a Galilean transformation.

Now let us suppose that there is a current $I$ circulating around the annulus, and consider the effect of switching on one quantum $h/e$ of flux in the void within the annulus. At the end of this operation we have produced a gauge field, that (for electrons within the annulus) is gauge equivalent to zero. Thus the bulk state, assumed unique, must return to its original form. The only change that can have occurred, is that some electrons from one edge might have been transferred to the other edge, through the wire.

We can calculate the work done during this operation in two different ways. On the one hand, we have transferred some charge $ne$ through a voltage $V$; thus the work is $neV$. On the other hand while the flux is being increased there is an azimuthal electric field, which does work on the circulating current. One easily computes in this way that the work done is $(h/e)I$. Upon equating these, one finds for the conductance:

$$V/I = ne^2/h .$$

Thus, this transverse conductance is quantized in terms of fundamental physical constants.

A slight variant of this argument corresponds less well to a practical experimental set-up, but is perhaps simpler conceptually and will be useful for my later purposes. Consider the same geometry and the same process of cranking on flux, but now with no transverse current and no voltage. As the flux is turned on, again some integer $k$ number of electrons is transported. There was an azimuthal electric field as the flux was turned on, and thus, for a determinate transverse conductivity, a radial current. The electric field is proportional to the time rate of change of the flux, so over the course of turning on one quantum of flux there is a definite integrated radial current, or in other words a definite charge transfer. Equating this charge transfer to $ke$, one finds the same quantization condition on the transverse conductivity as before.

1.2. Too good to be true?

The Laughlin quantization argument is so simple and beautiful, and so directly addresses the central phenomenon, that one cannot seriously doubt its essen-
tial correctness. Unfortunately, it is too good. Shortly after it was proposed and digested, experimentalists discovered states where the conductance is quantized, but now as a definite fraction of $e^2/h$ rather than as an integer multiple. These states occur when the density is close to (the same) definite fraction of the density corresponding to a full Landau level. The jargon here is that there is a plateau in the resistivity around filling fraction $\nu = \rho/(eB/\hbar c)$; meaning that when the ratio of density to magnetic field is close to this value the conductivity remains at the quantized value $n e^2/h$. The first discovered and most robust such state (as reflected in the width of the associated plateau and the allowed range of impurities and temperatures) occurs at $\nu = 1/3$. For simplicity and concreteness I shall mainly focus the discussion on that state, although by now quantized Hall states at many other fractions have been observed and there is a beautiful, extensive theory of them—in fact several such theories[6].

1.3. The microscopic perspective

There is a successful microscopic theory of the fractional quantized Hall effect. So before I get carried away with grandiose rhetoric about breaking and amending gauge invariance, it behooves me to demonstrate how one understands at a “mechanical” level how the general gauge invariance argument, which seems so clear-cut in leading to integer quantized conductance, develops the necessary subtleties in the microscopic theory.

1.4. Lightning Review of Incompressible Hall States

As we have already seen in our discussion of the integer effect, the quantized conductance is a fairly direct manifestation of the existence of an incompressible quantum fluid. That is, the electron fluid has a preferred density pinned to the value of the external magnetic field. There must be an energy gap to deviations from this preferred density: such deviations must be accommodated by localized inhomogeneities, rather than in arbitrarily long wavelength “sound waves” which—if they existed—could have arbitrarily small energy. In the case of the integer quantized Hall effect the preferred density simply corresponds to filling an integer number of Landau levels, and the gap is quite easy to understand. Indeed, to raise the density here and lower it there we must excite a particle to the next Landau level here, which costs a finite minimum amount of energy equal to the splitting between Landau levels, that is not compensated by allowing a hole there[3].

Laughlin himself[8] was quick not only to recognize the physical meaning of the new observations, but also to propose a rationale for why specific special (non-

\[^3\]The lowest energy density fluctuations actually occur at a finite wavevector. These excitations, the so-called magnetorotons[7] can be regarded, intuitively, as bound states of quasiparticles and quasiholes. They therefore bear a family resemblance excitons in semiconductors; however unlike most excitons they do not easily cascade down and annihilate, because semiclassically the Coulomb attraction between them—in the presence of the strong ambient magnetic field—causes a drift in the perpendicular direction, and thus induces orbital motion. Of course the magnetorotons, unlike the quasiholes and quasiparticles discussed below, carry no net charge.
integer) filling fractions should be preferred. Let me very briefly recall the main points, since I shall want to build on them.

First I need to remind you of some basic results about electrons in a strong magnetic field (here, as throughout, I am assuming that the motion of the electrons is confined to a plane.) The energy levels are highly degenerate Landau levels, with a density of states $2\pi/\ell^2$ per unit area per Landau level, where the magnetic length $\ell$ is defined through $\ell^2 \equiv eB/\hbar c$. The splitting between levels is $\hbar$ times the cyclotron frequency, viz. $\Delta E = \hbar(\epsilon B/mc)$. At low temperatures and for densities small compared $2\pi\ell^2$ it ought to be a good approximation to restrict attention to states formed from single-particle states confined taken from the lowest Landau level, unless there is some very special energetic advantage to admixing higher levels (so as to minimize the interaction energy.) Within the lowest Landau level, the single particle wave functions take a particularly attractive form if one employs the so-called symmetric gauge, defined by the vector potentials $A_x = By/2$, $A_y = -Bx/2$. With this gauge choice, the wave functions in the lowest Landau level take the form

$$\psi = f(z)e^{-\frac{1}{4}|z|^2}$$

where $f(z)$ is an arbitrary analytic function of $z \equiv x + iy$, subject to a reasonable growth condition so that the wave function is normalizable, and distances are measured in units of the magnetic length. A basis of orthogonal vectors in this Hilbert space is provided by the functions with $f_l(z) = z^l$. $l$ is the canonical angular momentum around the origin, which here is intrinsically non-negative. For reasonably large $l$, the corresponding wave function is concentrated in a circular ring of radius $\sqrt{2l}$ and width $\sqrt{2\pi}$ around the origin. It follows, by comparing the size of the region where the wavefunction is large to the inverse density, or by direct calculation, that the supports of these wave functions are highly overlapping.

Now let us consider an assembly of (non-interacting) electrons. Let us suppose that they subject to a very small potential that draws them toward the origin, but does not appreciably change the form of the wave functions (that is a second order effect). Then the ground state will be composed out of the wave functions with the smallest values of $l$, consistent with Fermi statistics. It will be the Slater determinant

$$\psi_1 = \det\{z_c^r\}e^{-\frac{1}{4}\sum|z_k|^2}.$$ 

Now Laughlin’s inspiration was to notice that the cube of this wave function has remarkable qualities, that make it a particularly attractive trial wave function
for an assembly of interacting electrons. The Gaussian factor is then not appropriate
for the lowest Landau level, but this can be compensated by a trivial redefinition
of the length unit, which we suppose done. Then clearly one has a wavefunction
again describing a uniform droplet centered at the origin, now with radius \( \sqrt{2N/3} \),
density \( 2\pi/3 \) (that is, filling factor \( 1/3 \)) and fuzziness in an annulus of width \( 1/\sqrt{3} \)
after the rescaling. The Laughlin wave function is particularly advantageous if the
electrons have repulsive short-range interactions, because it enforces a triple zero as
one electron approaches another. A large number of numerical studies have shown
that it is a very good representation of the ground state wave function, for a variety
of repulsive interactions.

From a physical point of view, the most remarkable thing about the Laughlin
wave function (and its various generalizations—see below) is its rigidity. It picks
out a particular filling factor in the bulk. Deviations from this average density will
have to be accommodated by localized disturbances. As we shall make much more
precise below, the situation is analogous to what one has for type II superconductors,
where magnetic fields are not allowed in the bulk, but can penetrate only in localized
vortices. Laughlin proposed a form for these disturbances, that compares very well
with numerical and experimental data. It is that a minimal quasihole localized
around \( z_0 \) is represented by multiplying the wave function with a factor that pushes
electrons away from \( z_0 \) by adding one unit of angular momentum around that point:

\[
\text{quasihole factor} = \prod_{k=1}^{N} (z_k - z_0). \tag{5}
\]

This gives a density deficit; there is an analogous but slightly more complicated
construction for an enhancement, the quasiparticle. There is an important \textit{gedanken}
production process for the quasihole: it is what you get by adiabatically switching
on one unit of magnetic flux at \( z_0 \). The quasiholes are rather exotic: they carry
fractional charge and fractional statistics. These properties can be shown directly
from the microscopic theory\[9\]. I will forego that pleasure here, however the result
will be central to our later considerations.

1.5. The Gauge Argument, Reconsidered

With this background, let us return to the gauge invariance argument. The
second form of the argument is a little easier to discuss, so let’s consider it.

There appears to be a technical awkwardness at the outset, in that we would
like to work in an annular geometry for the fluid and to include some mechanism for
taking electrons in one side and out the other, whereas the simple wave functions
are for a droplet geometry. Fortunately there is a way around this that is quite
simple and instructive for our purposes. We have already mentioned that wave
functions with a high power \( z^l \) times the usual exponential \( e^{-\frac{1}{4}|z|^2} \) are concentrated
in a small ring of radius \( \sqrt{2l} \) and width \( \sqrt{2\pi} \) around the origin Thus to put a hole in
the droplet of radius \( R \), and produce an annulus of quantized Hall fluid, we should
multiply the wave function by a factor

\[ \text{Annulizing factor} = \prod_k z_k^{R^2/2}. \]  

(6)

Now you will not fail to notice that the annulizing factor is nothing but \( R^2/2 \) quasiholes at the origin. A large number of quasiholes do literally make a (classical, spatial) hole in the fluid! Also, since the quasiholes are the end result of adiabatic insertion of a unit of magnetic flux—that’s how we (following, of course, Laughlin) constructed them—we conclude that adiabatic insertion of flux drills a hole in the droplet.

Although it is somewhat off the point for this talk, it is quite interesting and appropriate to the occasion to note that by redistributing flux that lies entirely in the empty void within the fluid annulus, one changes the shape of the annulus. Thus some of the factors of \( \prod \hat{z} \) in the annulizing factor could be changed to \( \prod(z-\alpha) \). This is a truly remarkable example of an Aharonov-Bohm type effect, in my opinion. That is, although one has “pure gauge” outside the flux tube, by moving the tube around one produces definite physical effects. (There is a pedestrian explanation for this—the moving flux tube produces an electric field at distant points.) The dynamics of motion within this manifold of quasi-degenerate states, produced by moving flux in the void, is governed by the theory of edge excitations. Perhaps it is even a practical proposition to produce these excitations by manipulating flux in this way. (End of digression.)

So now we should be able to see, in the microscopic theory, how it can be that the gauge invariance argument becomes subtle, in such a way that inserting a single unit \( h/e \) of flux does not transport an integral number of electrons—while inserting three units does.

It is really quite simple and beautiful. The point is that when the power in the annulizing factor is a multiple of three, we can again write the wavefunction in Vandermonde-Laughlin form. That is (stripping away the Gaussian factors):

\[
\prod_{k=1}^{N} z_k^{3L} \prod_{(k<l):k,l=1}^{N} (z_k - z_l)^3 = \\
\prod_{k=1}^{N} z_k^{3L} (\det\{z_{r}^{c-1}\})^3 = \\
(\det\{z_{r}^{c+L-1}\})^3,
\]

(7)

where one has \( N \times N \) determinants with row index \( r \) and column index \( c \). Thus to change \( L \) by one unit, to \( L + 1 \), we need only to change the wavefunction of one electron, changing a \( z^L \) to a \( z^{L+N} \). In physical terms, this means removing an electron from the inner edge and transporting it to the outer edge. (Note that the minimum occupied level has been emptied, and the minimum available unoccupied level has been filled.) That is the sort of operation an ordinary wire is happy to do. The remaining electrons in the annular drop can be entirely passive, and need not re-arrange their correlated wavefunctions.
It is quite a different story if you change the flux by one unit. That does not correspond to transport of an electron from the inner edge to the outer edge, leaving the bulk intact. Indeed, as we have just seen, the latter operation in its minimal form unambiguously corresponds to changing the flux by three units. The physical operation that corresponds to one flux unit, is creation of a quasihole-quasiparticle pair at the inner edge, followed by transport of the quasiparticle to the outer edge. This is not an operation an ordinary wire will do for you. There is an amplitude for it to occur by the quasiparticle tunneling across the sample, but since it requires a simultaneous rearrangement of all the electrons this amplitude will be exponentially small. In the thermodynamic limit of an infinite number of electrons, at zero temperature, it will not occur at all. Then we are justified in saying that gauge invariance has been spontaneously violated, in the only sense it ever is: while the gauge transformation with three flux units connects one accessible state to another, and represents a legitimate symmetry; but the transformation with a single flux unit, although formally valid, is useless because it relates amplitudes for processes in our world only to amplitudes for processes in another, inaccessible one.

2. Introducing, and Liberating, Confined Slaves

2.1. Analogies of iQHE and Superconductivity

One cannot long reflect on the properties of the incompressible Hall states without noticing many analogies between their properties and those of ordinary superconductors. Let me mention a few of the most striking ones:

- In the quantum Hall system, there is a vanishing longitudinal resistivity. Thus the current flow is non-dissipative, as in a superconductor. Strictly speaking, this is true only at zero temperature. However, this fact does not spoil the analogy: we are dealing with a two-dimensional system, and in two dimensions the superconducting transition is also at zero temperature. Indeed, the reason is the same in both cases: there is a finite energy gap to vortex production, which leads to finite though exponentially small dissipation at any non-zero temperature.

- In both cases, one has an energy gap to charged excitations.

- In both cases, one has rigidity against an applied magnetic field. In the case of superconductors this is of course the famous Meissner effect, but it may seem to be a rather peculiar thing to say about iQHE states, since they occur immersed in a magnetic field from the start. Nevertheless they exhibit a form of rigidity, in that changes of the field away from a preferred value, pinned to the effective density, are disfavored. Here by effective density I mean the nominal density as given by the Hall coefficient, which is constant over a given plateau – in the analogy, we could call this the superfluid density.

- In both cases, one has vortex-like objects. We have of course just seen this in our discussion of the iQHE, where the quasiparticles are in some sense vortices,

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4I shall use this notation for the incompressible quantum Hall effect, which is a mouthful. The lower case i is used here, because IQHE is already used to indicate the integer quantized Hall effect.
and it is a famous fact for type II superconductors.

- In this vein, there is also the analogy that the non-dissipative state requires that the vortices be pinned. The pinning is much easier in the iQHE case, because the vortices are electrically charged and subject to a large magnetic field, so they will be happy to make closed orbits on electric field equipotentials. (Nevertheless some impurities must be present to make these equipotentials form closed lines, or else there will be no plateau. Indeed for a translationally invariant system the Hall constant must be equal to the carrier density, by Galilean invariance, and it cannot “stick” at a preferred value as the density varies.) At finite density the quasiparticles would presumably, given their large effective band mass and repulsive interactions, form a Wigner crystal, analogous to the Abrikosov flux lattice.

On the other hand one has the apparent contrast, that the iQHE states but not ordinary superconductors support exotic charge and statistics for the quasiparticles. Also, as I discussed in the first part of this talk, the breaking of gauge invariance is rather different in the two cases. For an ordinary superconductor, the periodicity in the Aharonov-Bohm type gedanken experiments we considered there would be $\hbar/2e$ instead of the $\hbar/(e/3)$ we encountered for the $\nu = 1/3$ state. The difference is profound: whereas in the first case one has a higher degree of flux-periodicity (that is, a smaller flux quantum) than might of been anticipated, reflecting a pairing order parameter, in the later case one has a subharmonic periodicity.

2.2. Introducing Exotic Slaves

The subharmonic periodicity in flux coexists, in the iQHE, with the existence of fractional charge, and one would like to think that there is an organic connection between them. Such a connection will arise, similarly to what one has in superconductivity, if one requires that the integral

$$\text{charge transport phase} = e^{iq \oint A d\phi} = e^{iq\Phi},$$

(8)

describing the phase acquired by a particle of charge $q$ transported around a closed loop enclosing flux $\Phi$ to be unity, for a fractional charge $q = e/3$. This single-valuedness, in turn, will have to be imposed if there is condensation of a field with charge $e/3$. The case for an organic connection thus becomes compelling. For the existence of fractionally charged quasiparticles supplies, on the face of it, a natural candidate for the desired condensate field: namely, of course, the field $\psi$ that creates the fractionally charged quasiparticles.

There is a difficulty, however. If $\psi$ is to condense one would like it to be bosonic. But that desire appears to conflict with another: one would also like to be able to have possibility for an electron decay into three identical quasiparticles. For the quasiparticles are supposed to be the important charged low-energy excitations, and this is the minimal decay channel that allows an electron to communicate with them, while conserving charge. Clearly, if the quasiparticles are bosons this decay is not going to be possible. One needs particles with exotic anyon quantum statistics, in order that a state of three identical particles can have the quantum
numbers of a fermion. Furthermore the microscopic theory teaches us that the quasiparticles are in fact anyons, and an electron can in fact decay into three of them. (Another possibility would have been to have more than one kind of quasiparticle: for example, one could reproduce the electron quantum numbers if there were in addition a light neutral fermion excitation, so that an electron could decay into three identical bosons and the neutral fermion. There may be iQHE states with this kind of non-minimal structure—a candidate $\nu = 1/2$ state of this kind has been described\cite{10}. However for the more conventional iQHE states, a minimalist procedure works out quite elegantly, as we shall see.)

So we seem to have arrived at a dilemma: on the one hand we want to have a bosonic field to create the quasiparticles, so that the field can condense; but on the other hand we want the quasiparticles to be anyons, so that they can reproduce the electron’s fermion statistics. Fortunately, these requirements only appear to be contradictory. Theoretical work on quantum statistics in 2+1 dimensions has shown that a bosonic field, properly coupled to a gauge field, can create anyons of any type\cite{3}. The way of this is done is called the Chern-Simons construction. It works as follows. One couples the field $\psi$ using the minimal coupling procedure to a gauge field $a$ that does not have an ordinary Maxwell kinetic energy term, but instead only a “Chern-Simons” term

$$\Delta L_{\text{CS}} = \frac{n}{4\pi} \int \epsilon^{\alpha\beta\gamma} a_\alpha f_{\beta\gamma}.$$  \hspace{1cm} (9)

Now one can demonstrate, without much difficulty, that the quanta produced by $\psi$ will have their quantum statistics altered, by the presence of the so-called Chern-Simons gauge field $a$ of which they are a source. And—at least in the point particle limit, for which the concepts are clearly defined—this change in the statistics of the quanta is the only effect of coupling in $a$. This construction is therefore a valid, and the minimal, way of implementing statistical transmutation—that is, the creation of quanta of one statistics by fields with another.

I originally called fields such as $a$ “fictitious” gauge fields. The newer terminology is in many ways preferable, but the old terminology did have the advantage of emphasizing that the $a$ do not introduce new local degrees of freedom. One can in principle fix a gauge and solve for the $a$s in terms of $\psi$. (The price for this is that the resulting action is complicated and no longer manifestly local.)

Although I do not intend to pause for a full demonstration here, it is especially appropriate on this occasion to note that the Aharonov-Bohm effect lies close to the heart of statistical transmutation. For the essence of the matter is that one finds, on solving the equations of motion for the gauge fields $a$, that the effect of the Chern-Simons coupling is simply to turn each quantum created by $\psi$ into a source of flux, as well as charge. Indeed, on varying with respect to $a_0$ one finds the equation

$$\rho = -\frac{n}{2\pi} f_{12}$$  \hspace{1cm} (10)

relating the particle number density to the Chern-Simons magnetic field. Note that in two space dimensions one has flux points, as opposed to the familiar flux lines,
and one can properly speak of flux associated to a point particle. When one such particle circles around another the wave function acquires, as Aharonov and Bohm taught us, a phase proportional to the product of charge and flux. But such a phase is operationally indistinguishable from the effect of quantum statistics! And that’s why one can freely change the statistics of the quanta created by a given field $\psi$ by coupling $\psi$ to a Chern-Simons gauge field.

We can summarize these considerations succinctly as follows. As far as the quantum numbers of charge and statistics are concerned, we can represent a field capable of creating an electron as

$$ e \sim \psi \psi \psi, $$

where $\psi$ is a *bosonic* field with electric charge $e/3$, properly coupled as well to a Chern-Simons gauge field. With our conventions, the correct choice is simply $n = 3$ in Eq. (9).

It has frequently been useful in condensed matter problems to introduce, as a mathematical device, representations of electron fields as products of other “slave” fields. One might, for example, represent the electron as a product of a neutral fermion “spinon” field and a charged boson “holon” field. As long as there is a constraint in place, forbidding the separate propagation of quanta of these fields, this is just a mathematical device. One is then in a confined phase, analogous to the confined phase for quarks in QCD. What we have done here is introduce a particular exotic kind of slave field, with fractional charge and statistics. As long as its quanta are kept confined—as might be implemented by a $Z_3$ gauge field coupling—doing this is just a mathematical device. As long as we consider only scales much larger than the confinement scale, we will not have changed the physical content of the theory. The procedure will be useful if added flexibility introduced by the slave variables allows us to represent excitations or correlations that are awkward to describe (i.e. non-local) in terms of the original variables.

2.3. iQHE as a Modified Meissner Effect: Liberating the Slaves

We introduced the slave field $\psi$ with two purposes in mind: the straightforward one, that after all there are quasiparticle states with exotic quantum numbers in the iQHE, so we should have fields to create them; and the deeper one, that we would like to have a condensation, or vacuum expectation value, of charge $e/3$ fields, so as to understand the subharmonic flux periodicity in the Laughlin argument.

Can $\psi$ condense? At first hearing the idea might sound mad. After all $\psi$ is a charged field, and the essence of the Meissner effect is that charged fields cannot condense in the presence of a background magnetic field. They are, in the jargon, frustrated. Since the iQHE necessarily takes place in a large background magnetic field, the proposed condensation sounds to be grossly anti-Meissner.

On deeper consideration, however, one discovers within this seeming difficulty the central point of this circle of ideas. Let us recall how one understands the Meissner effect, in the language of condensation. In the free energy associated with
a charged condensing field $\eta$ one has a gradient term

$$|\nabla \eta|^2 = |\partial_\mu \eta - iqA_\mu \eta|^2$$

(12)

involving the gauge covariant derivative. Now a constant magnetic field introduces a vector potential $A$ which grows with the distance, and whose effect, since it is solenoidal, cannot be cancelled by the ordinary derivative term, which is longitudinal. Thus to maintain a non-zero expectation value for the magnitude of $\eta$ costs a free energy density which grows with the distance, and this can never be favorable.

Now in the analogous considerations for our exotic slave field $\eta$, we must include not only the electromagnetic gauge field but also the Chern-Simons field $a$. And then we realize, that there is a possibility for $A$ and $a$ to cancel, thus allowing for the possibility of a uniform condensate. This will occur when the part of $\xi A + a$ that grows with the distance cancels. That, in turn, requires that the average flux density associated with this combination of fields vanishes. In view of Eq. (10), this occurs when one has the relation

$$\frac{e}{3}B = b = \frac{2\pi}{n} \rho = \pi \rho_e,$$

(13)

where in the third equality we have taken into account the $n = 3$ demanded by quantum statistics, and that the quasiparticle density is three times the electron density. Thus the cancellation takes place precisely at filling fraction $\nu = 1/3$. Whereas the ordinary Meissner effect for a superconductor tends to exclude magnetic field, the modified Meissner effect taking into account the statistical transmutation, excludes deviations of the magnetic field from a fixed multiple of the density (and, of course, vice versa). Deviations from zero field in the superconductor, or from the desirable density in the iQHE, are accommodated most cheaply by allowing inhomogeneities—vortices in the first case, quasiparticles in the second. In fact the quasiparticles are vortices too—but in the Chern-Simons field, not the electromagnetic field. Only by allowing such inhomogeneities can one preserve condensation in bulk, which requires the integrated form of Eq. (13). That is the essence of the modified Meissner effect.

Another feature of the situation is that the condensation of $\psi$ into a Higgs phase entails, as a consistency requirement, deconfinement of its quanta. One cannot, after all, confine vacuum quantum numbers! Thus the two purposes which motivated us to introduce the confined slaves, namely on the one hand to have fields which described the exotic quasiparticles once they are liberated, and on the other hand to have fields capable of condensation, are intimately related in their realization.

2.4. Past and Future

Well that concludes the main story I wanted to tell you today, and I think it is a very nice story as far as it goes. I hope I have conveyed how the concepts of fractional charge and statistics, the Chern-Simons construction of the latter, and the modified Meissner effect ineluctably come together in a coherent account encompassing both the iQHE and ordinary superconductivity. It does justice, I
believe, to the ‘paradoxical’ nature of gauge symmetry in the fractional quantum Hall states that one encounters upon taking the Laughlin quantization argument seriously, as we discussed above.

This story has both a history and, I hope, a future. I’d like briefly to comment very briefly on these, although you should be warned that in neither case do I speak with authority.

Girvin\[11\] stressed the analogies between superconductivity and the iQHE very early on, made pioneering attempts to construct a consistent, unfrustrated order parameter, and recognized the importance of the statistical gauge field in this regard. Girvin and MacDonald\[12\] made an important connection to the microscopic theory. The early ideas were refined and extended in important ways by Zhang, Kivelson, and Hansson\[13\], and by Read\[14\]. There is an interesting discussion of this body of work in Stone’s book\[2\].

In previous work, as far as I know, integrally charged condensates have been emphasized. For example in the approach of \[\text{[13]}\] one couples the statistical gauge field to the electron field to make it a “super-fermion”—though created by a bosonic field. This can be done with a Chern-Simons coupling $n = \frac{1}{3}$. With this value the modified Meissner argument gives the same relation between real magnetic field and electron density as was discussed above.

In this talk I have discussed how one is naturally led to the fractional charge condensate. Of course the existence of such a condensate does not contradict the existence of an electron condensate, but postulates additional structure. I think there are significant advantages to this point of view. For example the quantization of $n$ in integers is required, for consistency, when one considers carefully the quantization of the Chern-Simons theory on topologically non-trivial surfaces. The appearance of integers multiplying the Chern-Simons term, and more generally (for iQHE states at higher levels in the hierarchy) matrices of integers describing several coupled Chern-Simons theories, plays a crucial role in Wen’s theory of edge states\[15\]. Thus both for understanding the accuracy of the quantization in the FQHE in a fundamental way, and for connecting ideas about the bulk state to the successful theory of edge states, it is important to have integers.

Having identified something like an order parameter, one might like to continue the analogy with superconductivity by considering inhomogeneous situations, response to external fields, and so forth, by solving classical equations using an effective Lagrangian, in the style of Landau and Ginzburg. In attempting this, however, one must recognize that the fields involved in such an effective Lagrangian cannot be regarded as normal local 2+1 dimensional fields, because they should only create

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5The notion of “super-fermions,” that is of particles for which the wave function not only changes sign—that is, accumulates phase $\pi$—but accumulates phase $3\pi$, say, may appear incoherent at first sight. After all, there is no denying that $e^{i\pi} = e^{3i\pi}$. However, it does have a concrete meaning, operating among states within the lowest Landau level. For in that context the relative angular momentum must be positive, and the effect of boosting the angular momentum by two units is to change the spectrum of allowed values, so that the angular momentum has to be at least three. Without the positivity restriction on angular momenta that operates in the lowest Landau level the allowed spectrum would not be altered, and the notion of “super-fermion” would be quite dubious.
and destroy quanta in the lowest Landau level (which makes them effectively 1+1 dimensional).

As a concrete example, one would like to use an effective Lagrangian to describe the motion of quasiparticles in response to slowly varying external electric and magnetic fields, or their scattering at small momenta. Indeed these most basic processes involving quasiparticles are perhaps the most fundamental observable processes governed by their exotic charge and statistics, so one would like to have an explicit description of them. Even in the simplest case of the integer quantized Hall effect, where the quasiparticles are the electrons themselves, it would seem that a more direct approach to calculating charged particle drifts in the lowest Landau level is appropriate, and this has quite a different flavor from solving simple classical field equations. This subject needs more work [16].

2.5. Coda: Question of Statistics in Spin-Charge Separation

There are several indications that the normal state of the CuO high temperature superconductors, for the dopings at which they exhibit superconductivity, is an anomalous metal. Perhaps the most striking anomaly is the linear dependence of resistivity on temperature, down to quite low temperatures. This is different from what is expected for a Fermi liquid, even after allowing for various possible complications [17, 18]. On the other hand there definitely are indications that a Fermi surface exists, at least in the sense that there is a significant singularity in the density of states (imaginary part of the electron Green function) at a surface in momentum space. However, the size of the Fermi surface appears in some classes of experiments, particularly photoemission, to be roughly normal; whereas Hall effect measurements, if interpreted as reflecting Fermi surface parameters, give a very different picture. Although these experiments are not entirely straightforward to interpret (because the Fermi liquid theory fails to describe their temperature dependence correctly, the foundations of the analysis are insecure), on the face of it they seem to indicate a small Fermi surface for small doping, with positive (hole-like) carriers. Thus they seem to reflect not the entire electron density, but rather its deviation from half filling.

Motivated by these and other experimental results, which appear to require a 2-component model, and by experience with 1+1 dimensional models, Anderson and others have proposed that the anomalous state is characterized by spin-charge separation, that is the existence of separate spin and charge degrees of freedom—spinons and holons. Electrons are supposed to decompose into these more basic objects. This is known to happen in 1+1 dimensions, even for very weak coupling [19]. In 2+1 dimensions the situation is much less clear. The infrared singularities that drive 1+1 dimensional metals, even for small coupling, to qualitatively different behaviors are substantially weaker in 2+1 dimensions.

Nevertheless one is motivated by the phenomenology, by the 1+1 dimensional models, and by the “existence proof” provided by the foregoing analysis of the iQHE, to consider the possibility that in the CuO materials the transition to the normal state involves a liberation of exotic slaves. If there are states of matter in
2+1 dimensions wherein electrons do separate into spinons and holons, the question arises what is the statistics of these particles. The most obvious assignment is boson statistics for one, fermion statistics for the other [20]. On closer examination however this assignment appears to lead to severe difficulties [20]. The Bose condensation temperature tends to be very high, and if it occurred it would lead to striking effects, none of which are observed. My colleagues and I suggest instead [21] to consider the possibility that both species are half-fermions. This avoids the Bose condensation problem. Recent work on gauge theories [22] inspired by the Halperin-Lee-Read [23] theory of the compressible Hall states near $\nu = 1/2$ suggests another advantage of assigning fractional statistics to the spinons and holons, namely that they lead to a pattern of anomalous behaviors at least qualitatively suggestive of CuO phenomenology. There is a nominal Fermi surface, but as one approaches the Fermi momentum there is a severe renormalization of the effective mass, so that the singularities and temperature dependences are not of the form predicted by Fermi liquid theory.

A detailed account of this work will be appearing shortly. I wanted to mention it here as it is so closely allied to the ideas discussed in the body of the talk, and perhaps gains some credibility from the association.

References

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