Massless propagators: applications in QCD and QED

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+ 2 new results → will be for first time reported now

RAD COR 2007
In this talk I will mainly concentrate on the famous $R$-ratio:

$$R(s) = \frac{\sigma_{tot}(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

which is the main theoretical object appearing in precise extraction of $\alpha_s$ from inclusive hadronic $Z$ and $\tau$ decays.
From measurements at \(Z\)-peak LEPEWWG arrives at:

\[
\alpha_s(M_Z) = 0.1186(27)
\]

with predominantly theoretical error from uncalculated higher orders \(\leftrightarrow \alpha_s^4\) (massless diagrams!)

smaller than present experimental error (but not much!)

higher QCD corrections are even more important for

\[
R_\tau = \frac{\Gamma(\tau \rightarrow \nu\text{ had})}{\Gamma(\tau \rightarrow e\nu\nu)}
\]
Theoretical Framework

$R(s)$ is related (via unitarity) to the correlator of the EM quark currents:

$$R(s) = \Re \Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx} \langle 0 | T[ j^\nu_\mu(x) j^\nu_\mu(0) ] | 0 \rangle dx$$

To conveniently sum the RG-logs one uses the Adler function:

$$D(Q^2) = Q^2 \frac{d}{dQ^2} \Pi(q^2) = Q^2 \int \frac{R(s)}{(s + Q^2)^2} ds$$

or ($a_s \equiv \alpha_s / \pi$)

$$R(s) = \frac{1}{2\pi i} \int_{-s - i\delta}^{-s + i\delta} dQ^2 \frac{D(Q^2)}{Q^2} = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots$$
Current Status of $R(s)$:

$$R(s) = 1 + \frac{\alpha_s}{\pi} + 1.409 \frac{\alpha_s^2}{\pi^2} - 12.767 \frac{\alpha_s^3}{\pi^3} + ? \frac{\alpha_s^4}{\pi^4}$$

/Gorishnii, Kataev, Larin, (1991)/
$R(s)$ at five loops is contributed by $\approx 17 \cdot 10^3$ of nonabelian or/and non-quenched diagrams like

as well as 2671 purely abelian quenched diagrams like
masslessness $\leftrightarrow$ simplicity:

5-loop $R(s)$ is reducible$^\star$

to 4-loop massless propagators ($\equiv$ p-integrals)

main object to compute

$^\star$ (i) the same is true for massive corrections like $m_q^2/s$, etc.
/J. Kühn, K.Ch (91,94)/
Tool Box

- IRR / Vladimirov, (78)/ + IR $R^*$ -operation /K. Ch., Smirnov (1984)/
  + resolved combinatorics /K. Ch., (1997)/
- reduction to Masters: “direct and automatic” construction of CF’s through $1/D$ expansion—made with BAICER—within the Baikov’s representation for Feynman integrals$^1$
- all 4-loop master $p$-integrals are known analytically /P. Baikov and K.Ch. (2004)/
- computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

* NO IBP identities are ever used at any step!

$^1$Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003
\[ d_4(N_F = 3) = \]
\[ \frac{78631453}{20736} - \frac{1704247}{432} \zeta_3 + \frac{4185}{8} \zeta_3^2 + \frac{34165}{96} \zeta_4 - \frac{1995}{16} \zeta_7 \]
\[ \approx 49.0757 \]

and, finally, for the very \( R(s) \):

\[ 1 + a_s + 1.6398 a_s^2 + 6.3710 a_s^3 - 106.8798 a_s^4 \]
or with kinematical (trivial!) \( \pi^2 \) terms separated

\[ r_4 = 49.0757 - 155.956 \]
Let us compare our exact result with the (12 years old!) PMS/FAQ predictions by Kataev & Starshenko:

\[ d_4(FAC/PMS) = 27 \rightleftharpoons d_4(exact) = 49.1 \]

\[ r_4(FAC/PMS) = (27-156) = -129 \rightleftharpoons r_4(exact) = -107. \]

One observes that the quality of the prediction is not especially good but in the R(s) it is getting better due to dominance of (exactly known!) \( \pi^2 \) terms.
It is instructive to compare the vector case to the scalar one*:

\[
\tilde{R} = 1 + 5.667 a_s + a_s^2 [51.57 - 15.63 - n_f (1.907 - 0.548)]
\]
\[
\quad + a_s^3 [648.7 - 484.6 - n_f (63.74 - 37.97) + n_f^2 (0.929 - 0.67)]
\]
\[
\quad + a_s^4 [9471. - 9431. - n_f (1454.3 - 1233.4) + n_f^2 (54.78 - 45.10)]
\]
\[
\quad - n_f^3 (0.454 - 0.433)
\]

remarkable mutual cancellations in all \(n_f\) powers!!!

for \(n_f = 3 \longrightarrow a_s^4(5589 - 6126) = -536.8\)
similar cancellations happen for \(\alpha_s^4 m_a^2/s\) and \(\alpha_s^4 n_f^2\) terms in R(s)

As a result the PMS/FAC predictions are very-very good for the dynamical (euclidean) terms but fail miserably after adding \(\pi^2\) terms!

* /P. Baikov, K. Ch, J.Kühn, PLR 96, 012003 (2006)/
Phenomenological Applications

B. Contour Improvement\(^1\) PT (CIPT) versus Fixed Order PT (FOPT)

in extracting \(\alpha_s\) from \(R_\tau\)

\[
R_\tau = \frac{\Gamma(\tau \to \nu_\tau \text{hadrons})}{\Gamma(\tau \to \nu_\tau e\bar{\nu}_e)} \sim \int_0^{M_\tau^2} ds \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) R(s)
\]

typical results look (using only \(O(\alpha_s^3)\) approx. for \(R(s)\)!)\(^1\)

\[
\alpha_s^{\text{FOPT}}(M_Z) = 0.1204 \pm 0.0036
\]

\[
\alpha_s^{\text{CIPT}}(M_Z) = 0.1223 \pm 0.002
\]

\(^1\) A.A. Pivovarov (1991,1992); F. Le Diberder and A. Pich (1992)/
$R_\tau$: the dependence on $d_4$ was thoroughly investigated in (P. Baikov and K.Ch., Phys. Rev. D67 (2003) 074026)

with $d_4 = 0$

$\alpha_s(M_\tau) = 0.34 \pm 0.035$ (FOPT) and $= 0.358 \pm 0.021$ (CIPT)

with old (guessed!) value of $d_4 = 26$

$\alpha_s(M_\tau) = 0.327 \pm 0.02$ (FOPT) and $= 0.351 \pm 0.01$ (CIPT)

with new (exact) value of $d_4 = 49.1$

$\alpha_s(M_\tau) = 0.326 \pm 0.02$ (FOPT) and $= 0.347 \pm 0.01$ (CIPT)

the difference (FOPT) - (CIPT) survives: $\pi^2$ terms seems to be overestimated in CIPT approach?
Finally, using only FOPT we get (preliminary; theoretical uncertainty from the scale variation):

$$\alpha_s^{\text{FOPT}}(M_Z) = 0.1183 \pm 0.0007_{\text{ex}} \pm 0.001_{\text{th}}$$
Consider more attentively the old result for the 4-loop $R(s)$ and note that the “pure” $C_F$ terms look astonishingly simple:

$$R(s) = r_0 + A_s(\mu) r_1 + A_s^2(\mu) \left[ r_2 + \ln \frac{\mu^2}{s} (r_1 \beta_0) \right]$$

$$+ A_s^3(\mu) \left[ r_3 + \ln \frac{\mu^2}{s} (2r_2 \beta_0 + r_1 \beta_1) + \ln \frac{2\mu^2}{s} (r_1 \beta_0^2) \right] + a_s^4(\mu) […]$$

where $A_s \equiv \frac{\alpha_s}{4\pi}$ and

$$r_0 = 1, \quad r_1 = 3 C_F, \quad r_2 = -\frac{3}{2} C_F^2 + \ldots, \quad r_3 = -\frac{69}{2} C_F^3 + \ldots$$
\[
\begin{align*}
    r_0 &= 1, \quad r_1 = 3 \, C_F, \quad r_2 = -\frac{3}{2} \, C_F^2, \quad r_3 = -\frac{69}{2} \, C_F^3
\end{align*}
\]

cmp. \( r_i \) to the \( \beta \)-function of quenched QED:

\[
\beta^{\text{qQED}} = \frac{4}{3} \, A \left\{ 1 + 3 \, A - \frac{3}{2} \, A^2 - \frac{69}{2} \, A^3 \right\} \quad \text{with} \quad A = \frac{\alpha}{4\pi}
\]

It is not by chance, but well-known fact:

the \( C_F \)-only part of \( R(s) \) is given essentially by

the quenched QED \( \beta \)-function after replacement \( C_F \, \alpha_s \rightarrow \alpha \)
PUZZLE of $\beta^q_{\text{QED}}$

- it is scheme independent in all orders

- the coefficients are simple rational numbers at 1, 2, 3 and four loops: $(4/3, 4, -2, -46)$

- if

$$\beta^q_{\text{QED}}(\alpha_0) \equiv 0$$

then $\alpha = \alpha_0$ leads to self-consistent finite solution of (massless) QED

/K. Johnson and M. Baker, (1973)/
some people understand the observed rationality of $\beta^q_{\text{QED}}$ at 1, 2, 3 and 4 loops and hope that it is not a pure coincidence.

For instance: David Broadhurst: (in hep-th/9909185)

“Noting the profound work of Alain Connes and Dirk Kreimer [1], one arrives at the nub of the rationality of quenched QED: dimensional regularization of the derivative of the scheme-independent single-fermion-loop Gell-Mann-Low function, via Fock-Feynman-Schwinger formalism”
We have computed the $\beta_{QED}^q$ at 5 loops.

Our results reads:

$$\frac{4}{3}, ~ 4, ~ -2, ~ -46, \quad \frac{4157}{6} \quad + \quad 128 \zeta_3$$

Two comments:

1. no chance for finite QED (at this order)

2. It gives (numerically small) $O(C_F^4a_s^4)$ contribution to $R(s)$
2005 - 2007: Three $\mathcal{O}(\alpha_s^4)$ Calculations

used CPU time (measured in terms of a one 3 GH PC)

- **QCD:** scalar $R^{SS}(s)$: ($\approx 5$ years)
- **Quenched QED:** $\beta$-function ($\approx 25$ years)
- **QCD:** $R^{VV}(s)$ at $N_F = 3$ ($\approx 40$ years)
Reality Check

calculations of $R^{VV}(s)$ with $N_F = 3$ were made mainly on HP XC4000 supercomputer of the Karlsruhe University (claster of Dual Core AMD Opteron 2.6 GH). It took about 40 CPU years, but only 3 calendar months (up to 160 processors were simultaneously used).

To compare: scalar $R^{SS}(s)$ took about 5 CPU years and QQED about 25 CPU years, with calendar time about 1 year for each (but with older processors and less developed software).
Summary: Results

- complete results on $R^{SS}(s)$ (with full $N_F$ dependence) and $R^{VV}(s)$ (for $N_F = 3$ at the moment) at $\mathcal{O}(\alpha_s^4)$ order are available

- full $N_F$ dependence for $R^{VV}(s)$ is under way and expected soon

- the $C_F^4$ term in $R^{VV}(s)$ ($\equiv$ the five-loop qQED $\beta$-function) is finished and ceases to be purely rational number any more!

- higher order terms in these (and others quantities too !!) massless correlators display interesting cancellations between kinematical ($\sim \pi^2$) and dynamical contributions
Summary: Puzzles (irrelevant for physics?)

- Could one ever understand the mysterious absence of $\zeta_4$ in massless physical (that is scale-invariant quantities) like $D^{VV}$?

- Could one ever understand the mysterious structure of irrationalities in the quenched QED $\beta$-function?