Abstract: In this paper, the boundedness properties for some Toeplitz type operators associated to the Riesz potential and general integral operators from Lebesgue spaces to Orlicz spaces are proved. The general integral operators include singular integral operator with general kernel, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Keywords: Toeplitz type operator, Singular integral operator, Orlicz space, BMO space, Lipschitz function.

MSC: 42B20, 42B25

1 Introduction and theorems

As the development of singular integral operators, their commutators have been well studied (see [3–7, 16–20]). Let $T$ be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman, Rochberg and Weiss (see [3]) stated that the commutator $[b, T](f) = T(bf) - bT(f)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [1], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by $BMO$ and Lipschitz functions are obtained (see [1, 8]). In [5], Janson proved the boundedness for the commutators generated by the singular integral operators and $BMO$ functions from Lebesgue spaces to Orlicz spaces. Motivated by these papers, in this paper, we will introduce some Toeplitz type operator associated to the Riesz potential and general integral operators (see [1, 8, 13]), and prove the boundedness properties of the Toeplitz type operators from Lebesgue spaces to Orlicz spaces. The general integral operators include singular integral operator with general kernel, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

In this paper, we are going to consider some integral operators as following (see [1]).

Definition 1.1. Let $T : S \to S'$ be a linear operator such that $T$ is bounded on $L^2(R^n)$ and has a kernel $K$, that is there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function $f$, where $K$ satisfies:

$$\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$
and there is a sequence of positive constant numbers \( \{C_k\} \) such that for any \( k \geq 1 \),

\[
\left( \int_{2^k |z-y| \leq |x-y| < 2^{k+1} |z-y|} \left( |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \right)^q dy \right)^{1/q} \leq C_k \left( 2^k |z-y| \right)^{-n/q'}.
\]

where \( 1 < q' < 2 \) and \( 1/q + 1/q' = 1 \). Let \( b \) be a locally integrable function on \( \mathbb{R}^n \). The Toeplitz type operator related to \( T \) is defined by

\[
T^b = \sum_{k=1}^{m} (T^{k, 1} M^b I_\alpha T^{k, 2} + T^{k, 3} I_\alpha M^b T^{k, 4}),
\]

where \( T^{k, 1} \) are \( T \) or \( \pm I \) (the identity operator), \( T^{k, 2} \) and \( T^{k, 4} \) are the bounded linear operators on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \), \( T^{k, 3} = \pm I \) for \( k = 1, \ldots, m \), \( M^b(f) = bf \) and \( I_\alpha \) is the Riesz potential operator \( (0 < \alpha < n) \) (see [2]).

**Definition 1.2.** Let \( F(x, y, t) \) be defined on \( \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty) \) and \( b \) be a locally integrable function on \( \mathbb{R}^n \), set

\[
F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x, y) f(y) dy
\]

and

\[
F_t^b(f) = \sum_{k=1}^{m} (F_t^{k, 1} M^b I_\alpha F_t^{k, 2} + F_t^{k, 3} I_\alpha M^b F_t^{k, 4}),
\]

for every bounded and compactly supported function \( f \), where \( F_t^{k, 1} \) are \( F_t(f) \) or \( \pm I \) (the identity operator), \( F_t^{k, 2} \) and \( F_t^{k, 4} \) are linear operators, \( F_t^{k, 3} = \pm I \) for \( k = 1, \ldots, m \).

Let \( H \) be the Banach space \( H = \{ h : \| h \| < \infty \} \). For each fixed \( x \in \mathbb{R}^n \), we view \( F_t(f)(x) \) and \( F_t^b(f)(x) \) as the mappings from \( [0, +\infty) \) to \( H \), and \( F \) satisfies:

\[
\int_{|y-z| < |x-y|} \left( \| F(x, y, t) - F(x, z, t) \| + \| F(y, x, t) - F(z, x, t) \| \right) dx \leq C,
\]

and there is a sequence of positive constant numbers \( \{C_k\} \) such that for any \( k \geq 1 \),

\[
\left( \int_{2^k |z-y| \leq |x-y| < 2^{k+1} |z-y|} \left( |F(x, y, t) - F(x, z, t)| + |F(y, x, t) - F(z, x, t)| \right)^q dy \right)^{1/q} \leq C_k \left( 2^k |z-y| \right)^{-n/q'},
\]

where \( 1 < q' < 2 \) and \( 1/q + 1/q' = 1 \). Set \( S(f)(x) = \| F_t(f)(x) \| \). The Toeplitz type operator related to \( F_t \) is defined by

\[
S^b(f) = \| F_t^b(f) \|.
\]

where \( S^{k, 2}(f) = \| F^{k, 2}(f) \| \) and \( S^{k, 4}(f) = \| F^{k, 4}(f) \| \) are the bounded operators on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) and \( k = 1, \ldots, m \).

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1.1 with \( C_j = 2^{-j/\delta} \) (see [4, 19]). Note that the commutator \([b, T](f) = bT(f) - T(bf)\) is a particular operator of the Toeplitz type operators \( T^b \) and \( S^b \). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 5–14, 16–18, 21, 22]). The main purpose of this paper is to prove the boundedness properties for the Toeplitz type operators \( T^b \) and \( S^b \) from Lebesgue spaces to Orlicz spaces.
Let us introduce some notations. Throughout this paper, \( Q \) will denote a cube of \( \mathbb{R}^n \) with sides parallel to the axes. For any locally integrable function \( f \), the sharp function of \( f \) is defined by

\[
f^*(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy.
\]

where, and in what follows, \( f_Q = |Q|^{-1} \int_Q f(x) \, dx \). It is well-known that (see [4, 19])

\[
f^*(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.
\]

Let \( M \) be the Hardy-Littlewood maximal operator defined by

\[
M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]

We write that \( M_p f = (M(f^p))^1/p \) for \( 0 < p < \infty \). For \( 1 \leq r < \infty \) and \( 0 < \delta < n \), let

\[
M_{\delta, r}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-\delta/r}} \int_Q |f(y)|^r \, dy \right)^{1/r}.
\]

We say that \( f \) belongs to \( BMO(R^n) \) if \( f^* \) belongs to \( L^\infty(R^n) \) and \( \|f\|_{BMO} = \|f^*\|_{L^\infty} \). More generally, let \( \rho \) be a non-decreasing positive function on \([0, +\infty)\) and define \( BMO_\rho(R^n) \) as the space of all functions \( f \) such that

\[
\frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y) - f_Q| \, dy \leq C \rho(r).
\]

For \( \beta > 0 \), the Lipschitz space \( Lip_\beta(R^n) \) is the space of functions \( f \) such that

\[
\|f\|_{Lip_\beta} = \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\beta < \infty.
\]

For \( f, m_f \) denotes the distribution function of \( f \), that is \( m_f(t) = \|x \in R^n : |f(x)| > t\| \).

Let \( \rho \) be a non-decreasing convex function on \([0, +\infty)\) with \( \rho(0) = 0 \). \( \rho^{-1} \) denotes the inverse function of \( \rho \). The Orlicz space \( L_\rho(R^n) \) is defined by the set of functions \( f \) such that \( \int_{R^n} \rho(\lambda |f(x)|) \, dx < \infty \) for some \( \lambda > 0 \). The norm is given by

\[
\|f\|_{L_\rho} = \inf_{\lambda > 0} \lambda^{-1} (1 + \int_{R^n} \rho(\lambda |f(x)|) \, dx).
\]

We shall prove the following theorems in Section 2.

**Theorem 1.3.** Let \( 0 < \beta \leq 1, q' < p < n/(\alpha + \beta) \) and \( \psi, \varphi \) be the two non-decreasing positive functions on \([0, +\infty)\) with \( \psi^{-1}(t) = t^{1/p} \varphi(t^{-1}/n) \). Suppose that \( \psi \) is convex, \( \psi(0) = 0 \), \( \psi(2t) \leq C \psi(t) \). Let \( T \) be the same as in Definition 1.1 and the sequence \( \{JC_t\} \in \ell^1 \). If \( T^1(g) = 0 \) for any \( g \in L^\mu(R^n) \) \( (1 < u < \infty) \), then \( T^b \) is bounded from \( L^\mu(R^n) \) to \( L_\psi(R^n) \) if \( b \in BMO(R^n) \).

**Theorem 1.4.** Let \( 0 < \beta \leq 1, q' < p < n/(\alpha + \beta) \) and \( \psi, \varphi \) be the two non-decreasing positive functions on \([0, +\infty)\) with \( \psi^{-1}(t) = t^{1/p} \varphi(t^{-1}/n) \). Suppose that \( \psi \) is convex, \( \psi(0) = 0 \), \( \psi(2t) \leq C \psi(t) \). Let \( S \) be the same as in Definition 1.2 and the sequence \( \{JC_t\} \in \ell^1 \). If \( F^1(g) = 0 \) for any \( g \in L^\mu(R^n) \) \( (1 < u < \infty) \), then \( S^b \) is bounded from \( L^\mu(R^n) \) to \( L_\psi(R^n) \) if \( b \in BMO(R^n) \).

**Remark 1.5.**

(a) If \( \varphi(t) \equiv 1 \) and \( \psi(t) = t^\theta \) for \( 1 < p < \infty \), then \( T^b \) and \( S^b \) are all bounded on \( L^\mu(R^n) \) under the conditions of Theorems 1.3 and 1.4.

(b) If \( \psi(t) = t^s \) and \( \varphi(t) = t^{n/(1/p - 1/s)} \) for \( 1 < p < s < \infty \), then, by \( BMO_{||p}(R^n) = Lip_\beta(R^n) \) (see [5, Lemma 4]), \( T^b \) and \( S^b \) are all bounded from \( L^\mu(R^n) \) to \( L^s(R^n) \) under the conditions of Theorems 1.3 and 1.4.
2 Proofs of theorems

We begin with the following preliminary lemmas.

Lemma 2.1 ([1]). Let $T$ and $S$ be the same as Definitions 1.1 and 1.2, the sequence $\{C_j\} \in l^1$. Then $T$ and $S$ are bounded on $L^p(R^n)$ for $1 < p < \infty$.

Lemma 2.2 ([4]). Let $0 < p < \infty$. Then, for any smooth function $f$ for which the left-hand side is finite,
\[
\int_{R^n} M(f)(x)^p \, dx \leq C \int_{R^n} f^p(x) \, dx.
\]

Lemma 2.3 ([2]). Suppose that $0 < \delta < n$, $1 \leq s < n/\delta$ and $1/r = 1/p - \delta/n$. Then
\[
\|I_\delta(f)\|_{L^r} \leq C \|f\|_{L^p}
\]
and
\[
\|M_{S, \delta}(f)\|_{L^r} \leq C \|f\|_{L^p}.
\]

Lemma 2.4 ([5]). Let $\rho$ be a non-decreasing positive function on $[0, +\infty)$ and $\eta$ be an infinitely differentiable function on $R^n$ with compact support such that $\int_{R^n} \eta(x) \, dx = 1$. Denote that $b_\gamma(x) = \int_{R^n} b(x - t) \eta(y) \, dy$. Then
\[
\|b - b_\gamma\|_{BMO} \leq C \rho(t) \|b\|_{BMO}.
\]

Lemma 2.5 ([5]). Let $0 < \beta < 1$ or $\beta = 1$ and $\rho$ be a non-decreasing positive function on $[0, +\infty)$. Then
\[
\|b_I\|_{Lip} \leq C \rho(t) \|b\|_{BMO}.
\]

Lemma 2.6 ([5]). Suppose $1 \leq p_1 < p < p_1 < \infty$, $\rho$ is a non-increasing function on $R^+$, $B$ is a linear or sublinear operator such that $m_{B(f)}(t^{1/p_1} \rho(t)) \leq C t^{-1}$ if $\|f\|_{L^{p_1}} \leq 1$ and $m_{B(f)}(t^{1/p_2} \rho(t)) \leq C t^{-1}$ if $\|f\|_{L^{p_2}} \leq 1$. Then
\[
\int_0^{\infty} m_{B(f)}(t^{1/p} \rho(t)) \, dt \leq C \|f\|_{L^p} \leq (p/p_1)^{1/p}.
\]

To prove the theorems of the paper, we need the following

Key Lemma. Let $T$ and $S$ be the same as in Definitions 1.1 and 1.2. Suppose that $Q = Q(x_0, d)$ is a cube with $\text{supp} \; f \subset (2Q)^c$ and $x, \hat{x} \in Q$. (I) If $b \in BMO(R^n)$ and the sequence $\{jC_j\} \in l^1$, then
\[
\|T^{(b-b_0)}_{\chi_{2Q^c}}(f)(x) - T^{(b-b_0)}_{\chi_{2Q^c}}(f)(x_0)\|
\leq C \|b\|_{BMO} \sum_{k=1}^m (M_{r}(|I_{a}T_{k}.2(f))\hat{x} + M_{a,r}(T_{k}.A(f))(\hat{x})) \text{ for } r > q';
\]

(II) If $0 < \beta \leq 1, b \in Lip(B(R^n))$ and the sequence $\{C_j\} \in l^1$, then
\[
\|T^{(b-b_0)}_{\chi_{2Q^c}}(f)(x) - T^{(b-b_0)}_{\chi_{2Q^c}}(f)(x_0)\|
\leq C \|b\|_{Lip} \sum_{k=1}^m (M_{\beta,r}(|I_{a}T_{k}.2(f))\hat{x} + M_{\beta+a,r}(T_{k}.A(f))(\hat{x})) \text{ for } r > q';
\]

(III) If $b \in BMO(R^n)$ and the sequence $\{jC_j\} \in l^1$, then
\[
\|F^{(b-b_0)}_{t} \chi_{2Q^c}(f)(x) - F^{(b-b_0)}_{t} \chi_{2Q^c}(f)(x_0)\|
\leq C \|b\|_{BMO} \sum_{k=1}^m (M_{r}(|I_{a}T_{k}.2(f))\hat{x} + M_{a,r}(T_{k}.A(f))(\hat{x})) \text{ for } r > q';
\]

(IV) If $0 < \beta \leq 1, b \in Lip(B(R^n))$ and the sequence $\{jC_j\} \in l^1$, then
\[
\|F^{(b-b_0)}_{t} \chi_{2Q^c}(f)(x) - F^{(b-b_0)}_{t} \chi_{2Q^c}(f)(x_0)\|
\leq C \|b\|_{BMO} \sum_{k=1}^m (M_{\beta,r}(|I_{a}T_{k}.2(f))\hat{x} + M_{\beta+a,r}(T_{k}.A(f))(\hat{x})) \text{ for } r > q';
\]


\[ \leq C \|b\|_{L^{p,\rho}} \sum_{k=1}^{m} (M_{\beta,r}(I_{\alpha} T^{k,2}(f))(\hat{x})) + M_{\beta+\alpha,r}(T^{k,4}(f))(\hat{x})) \text{ for any } r > q'. \]

**Proof.** For \( \text{supp} f \subset (2Q)^c \) and \( x, \hat{x} \in Q \), we have

\[ |T^{(b-hQ)\times(2Q)^c}(f)(x) - T^{(b-hQ)\times(2Q)^c}(f)(x_0)| \]

\[ \leq \sum_{k=1}^{m} |I_{\alpha} T^{k,2}(f)(x) - I_{\alpha} T^{k,2}(f)(x_0)| \]

\[ + \sum_{k=1}^{m} |I_{\alpha} T^{k,3}(f)(x) - I_{\alpha} T^{k,3}(f)(x_0)|. \]

(I). Note that \( |x-y| \sim |x_0-y| \) for \( x \in Q \) and \( y \in \mathbb{R}^n \setminus 2Q \). Recalling \( r > q' \), taking \( 1 < u < \infty, 1 < v < r \) with \( 1/q + 1/u + 1/v = 1 \), we obtain, by the conditions on \( K \),

\[ |I_{\alpha} T^{k,2}(f)(y)|dy \]

\[ \leq \sum_{j=1}^{\infty} \int_{|x-y|<2^{-j+1}d} |K(x,y) - K(x_0,y)||b(y) - b_Q||I_{\alpha} T^{k,2}(f)(y)|dy \]

\[ \leq C \|b\|_{BMO} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'} j(2^j d)^n/u (2^j d)^n/v \left( \frac{1}{2^j+1 Q} \int_{2^j+1 Q} |I_{\alpha} T^{k,2}(f)(y)|^r dy \right)^{1/r} \]

\[ \leq C \|b\|_{BMO} M_r(I_{\alpha} T^{k,2}(f))(\hat{x}) \sum_{j=1}^{\infty} j C_j \]

\[ \leq C \|b\|_{BMO} M_r(I_{\alpha} T^{k,2}(f))(\hat{x}), \]

\[ |I_{\alpha} T^{k,3}(f)(y)|dy \]

\[ \leq \sum_{j=1}^{\infty} \int_{|x-y|<2^{-j+1}d} |b(y) - b_Q| \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,3}(f)(y)|dy \]

\[ \leq C \sum_{j=1}^{\infty} \int_{|x-y|<2^{-j+1}d} |b(y) - b_Q| \left| \frac{d}{|x_0-y|^{n-\alpha+1}} \right| |T^{k,3}(f)(y)|dy \]

\[ \leq C \sum_{j=1}^{\infty} d(2^j d)^{-n+\alpha-1} (2^j d)^n(1-r)(2^j d)^n/r-\alpha \]

\[ \times \left( \frac{1}{2^j+1 Q} \int_{2^j+1 Q} |b(y) - b_Q|^r dy \right)^{1/r} \left( \frac{1}{2^j+1 Q} \int_{2^j+1 Q} |T^{k,3}(f)(y)|^r dy \right)^{1/r} \leq C \|b\|_{BMO} \sum_{k=1}^{m} M_{\alpha,r}(T^{k,3}(f))(\hat{x}) \sum_{j=1}^{\infty} j 2^{-j} \]
\[ \leq C \|b\|_{BMO} \sum_{k=1}^{m} M_{\alpha, r}(T^{k,A}(f))(\tilde{x}). \]

thus
\[
|T^{(b-b_0)\chi_{\Omega^\infty}}(f)(x) - T^{(b-b_0)\chi_{\Omega^\infty}}(f)(x_0)| \\
\leq C \|b\|_{BMO} \sum_{k=1}^{m} (M_f(I_{\alpha} T^{k,2}(f))(\tilde{x}) + M_{\alpha, r}(T^{k,A}(f))(\tilde{x})).
\]

(II). Note that, for \( b \in Lip_\beta(\mathbb{R}^n) \),
\[
|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q |b(y) - b_Q| |x-y|^\beta dy \leq C \|b\|_{Lip_\beta} |x-x_0| + d|^\beta,
\]
similarly as in the proof of (I), we obtain
\[
|T^{k,1}M^{(b-b_0)\chi_{\Omega^\infty}} I_{\alpha} T^{k,2}(f)(x) - T^{k,1}M^{(b-b_0)\chi_{\Omega^\infty}} I_{\alpha} T^{k,2}(f)(x_0)| \\
\leq \int_{(2Q)^\infty} |K(x, y) - K(x_0, y)||b(y) - b_Q||I_{\alpha} T^{k,2}(f)(y)| dy \\
= \sum_{j=1}^{\infty} \int_{2^j Q} |b(y) - b_Q||K(x, y) - K(x_0, y)||I_{\alpha} T^{k,2}(f)(y)| dy \\
\leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1} Q|^\beta/n \left( \int_{2^j Q \setminus 2^{j+1} Q} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \\
\times \left( \int_{2^{j+1} Q} |I_{\alpha} T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} \\
\leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1} Q|^\beta/n C_j (2^{j}d)^{-n/q} |2^{j+1} Q|^{1/q' - \beta/n} \\
\times \left( \int_{2^{j+1} Q} |I_{\alpha} T^{k,2}(f)(y)|^{q'} dy \right)^{1/r} \\
\leq C \|b\|_{Lip_\beta} M_{\beta, r}(I_{\alpha} T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} C_j
\]

thus
\[
|T^{k,3}I_{\alpha} M^{(b-b_0)\chi_{\Omega^\infty}} T^{k,A}(f)(x) - T^{k,3}I_{\alpha} M^{(b-b_0)\chi_{\Omega^\infty}} T^{k,A}(f)(x_0)| \\
\leq \int_{(2Q)^\infty} \left| b(y) - b_Q \right| \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,A}(f)(y)| dy \\
\leq C \sum_{j=1}^{\infty} \|b\|_{Lip_\beta} |2^{j+1} Q|^\beta/n \int_{2^j d \setminus 2^{j+1} d} \frac{d}{|x_0-y|^{n-\alpha+1}} |T^{k,A}(f)(y)| dy \\
\leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} (2^{j} d)^\beta d(2^{j} d)^{-n+\alpha-1}(2^{j} d)^{n(1-1/r)}(2^{j} d)^{n/r-\beta-\alpha} \\
\times \left( \int_{2^{j+1} Q} |T^{k,A}(f)(y)|^{q'} dy \right)^{1/r}
\]
The same argument as in the proof of (I) and (II) will give the proof of (III) and (VI), we omit the details.

Thus

\[ |T^{(b-b_Q)\chi_{\Omega'}}(f)(x) - T^{(b-b_Q)\chi_{\Omega'}}(f)(x_0)| \]

\[ \leq C||b||_{Lip} \sum_{k=1}^{m} (M_{\alpha}(I_{a}T^{k,2}(f)))(\tilde{x}) + M_{\alpha}(T^{k,4}(f))(\tilde{x}). \]

The same argument as in the proof of (I) and (II) will give the proof of (III) and (VI), we omit the details.

Now we are in position to prove our theorems.

**Proof of Theorem 1.3.** Without loss of generality, we may assume \( T^{k,1} \) are \( T(k = 1, \ldots, m) \). We prove the theorem in several steps.

First, we prove, if \( b \in BMO(R^n) \),

\[ (T^b(f))^q \leq C||b||_{BMO} \sum_{k=1}^{m} (M_{\alpha}(I_{a}T^{k,2}(f))) + M_{\alpha}(T^{k,4}(f)) \]

for any \( r \) with \( q' < r < n / \alpha \).

Fix a cube \( Q = Q(x_0, d) \) and \( \tilde{x} \in Q \). By \( T_1(g) = 0 \), we have \( T^b(f) = T^{b-b_Q}(f) \), thus

\[ T^b(f)(x) = \sum_{k=1}^{m} T^{k,1}M^b I_{a}T^{k,2}(f)(x) + \sum_{k=1}^{m} T^{k,3}I_{a}M^b T^{k,4}(f)(x) = A^{b-b_Q}(x) + B^{b-b_Q}(x), \]

where

\[ A^{b-b_Q}(x) = \sum_{k=1}^{m} T^{k,1}M^{(b-b_Q)\chi_{\Omega}} I_{a}T^{k,2}(f)(x) + \sum_{k=1}^{m} T^{k,3}M^{(b-b_Q)\chi_{\Omega'}} I_{a}T^{k,2}(f)(x) = A_1(x) + A_2(x) \]

and

\[ B^{b-b_Q}(x) = \sum_{k=1}^{m} T^{k,3}I_{a}M^{(b-b_Q)\chi_{\Omega}} T^{k,4}(f)(x) + \sum_{k=1}^{m} T^{k,3}I_{a}M^{(b-b_Q)\chi_{\Omega'}} T^{k,4}(f)(x) = B_1(x) + B_2(x). \]

Then

\[ \frac{1}{|Q|} \int_{Q} \left| T^b(f)(x) - A_2(x_0) - B_2(x_0) \right| \, dx \]

\[ \leq \frac{1}{|Q|} \int_{Q} |A_1(x)| \, dx + \frac{1}{|Q|} \int_{Q} |B_1(x)| \, dx + \frac{1}{|Q|} \int_{Q} |A_2(x) - A_2(x_0)| \, dx + \frac{1}{|Q|} \int_{Q} |B_2(x) - B_2(x_0)| \, dx \]

\[ = I_1 + I_2 + I_3 + I_4. \]

For \( I_1 \), choose \( 1 < s < r \), by Hölder’s inequality and the boundedness of \( T \) (see Lemma 2.1), we obtain

\[ \frac{1}{|Q|} \int_{Q} |T^{k,1}M^{(b-b_Q)\chi_{\Omega}} I_{a}T^{k,2}(f)(x)| \, dx \]

\[ \leq \left( \frac{1}{|Q|} \int_{R^n} |T^{k,1}M^{(b-b_Q)\chi_{\Omega}} I_{a}T^{k,2}(f)(x)|^s \, dx \right)^{1/s} \]
\[
\leq C |Q|^{-1/s} \left( \int_{Q^n} |M^{(b-hO)\chi_{\Omega}} I_{\alpha} T^{k,2}(f)(x)|^2 \, dx \right)^{1/s}
\]
\[
\leq C |Q|^{-1/s} \left( \int_{\Omega} |I_{\alpha} T^{k,2}(f)(x)|^r \, dx \right)^{1/r} \left( \int_{\Omega} |b(x) - b_Q|^{r s/(r-s)} \, dx \right)^{(r-s)/rs}
\]
\[
\leq C \|b\|_{BMO} \left( \frac{1}{|Q|} \int_{\Omega} |I_{\alpha} T^{k,2}(f)(x)|^r \, dx \right)
\]
\[
\leq C \|b\|_{BMO} M_r(I_{\alpha} T^{k,2}(f))(\tilde{x}).
\]
thus
\[
I_1 \leq \sum_{k=1}^{m} \frac{1}{|Q|} \int_{Q} |T^{k,1} M^{(b-hO)\chi_{\Omega}} I_{\alpha} T^{k,2}(f)(x)| \, dx
\]
\[
\leq C \|b\|_{BMO} \sum_{k=1}^{m} M_r(I_{\alpha} T^{k,2}(f))(\tilde{x}).
\]
For \(I_2\), choose \(u, v\) with \(1 < v < r, 1/v = 1/u - \alpha/n\), by the \((L^u, L^v)\)-boundedness of \(I_{\alpha}\) (see Lemma 2.2), we get
\[
I_2 \leq \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{Q^n} |I_{\alpha} M^{(b-hO)\chi_{\Omega}} T^{k,4}(f)(x)|^v \, dx \right)^{1/v}
\]
\[
\leq C \sum_{k=1}^{m} \left( \int_{Q} |I_{\alpha} M^{(b-hO)\chi_{\Omega}} T^{k,4}(f)(x)|^u \, dx \right)^{1/u}
\]
\[
\leq C \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{Q^n} |b(x) - b_Q|^{r u/(r-u)} \, dx \right)^{(r-u)/ru} \left( \frac{1}{|Q|^{1-ra/n}} \int_{Q} |T^{k,4}(f)(x)|^r \, dx \right)^{1/r}
\]
\[
\leq C \|b\|_{BMO} \sum_{k=1}^{m} M_{\alpha, r}(T^{k,4}(f))(\tilde{x}).
\]
For \(I_3\) and \(I_4\), by using Key Lemma,
\[
I_3 + I_4 \leq C \|b\|_{BMO} \sum_{k=1}^{m} (M_r(I_{\alpha} T^{k,2}(f))(\tilde{x}) + M_{\alpha, r}(T^{k,4}(f))(\tilde{x})).
\]
We now put these estimates together and take the supremum over all \(Q\) such that \(\tilde{x} \in Q\), we obtain
\[
(T^b(f))^q(\tilde{x}) \leq C \|b\|_{BMO} \sum_{k=1}^{m} (M_r(I_{\alpha} T^{k,2}(f))(\tilde{x}) + M_{\alpha, r}(T^{k,4}(f))(\tilde{x})).
\]
Thus, taking \(r, u\) such that \(q' < r < p\) with \(1/u = 1/p - \alpha/n\), we obtain, by Lemmas 2.2 and 2.3,
\[
\|T^b(f)\|_{L^u} \leq \|M(T^b(f))\|_{L^u} \leq C \|(T^b(f))^q\|_{L^u}
\]
\[
\leq C \|b\|_{BMO} \sum_{k=1}^{m} \|M_r(I_{\alpha} T^{k,2}(f))\|_{L^u} + \|M_{\alpha, r}(T^{k,4}(f))\|_{L^u}
\]
\[
\leq C \|b\|_{BMO} \sum_{k=1}^{m} (\|I_{\alpha} T^{k,2}(f)\|_{L^u} + \|T^{k,4}(f)\|_{L^p})
\]
Thus, (3) holds. We take $r; w$ for any $b$

Secondly, we prove that, if $b \in Lip_\beta(R^n)$,

$$(T^b(f))^\# \leq C\|b\|_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,r}(I\alpha T^{k,2}(f)) + M_{\beta+\alpha,r}(T^{k,A}(f)))$$

for any $r$ with $q' < r < n/(\alpha + \beta)$. In fact, similarly as in the proof of (1) and by Key Lemma, we obtain, for $1/v = 1/r - \alpha/n$,

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - A_2(x_0) - B_2(x_0)| \, dx$$

$$\leq \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{R^n} |T^{k,1} M^{(b-b_Q)\chi_{2Q}} I\alpha T^{k,2}(f)(x)|^r \, dx \right)^{1/r}$$

$$+ \frac{1}{|Q|} \int_Q |A_2(x) - A_2(x_0)| \, dx + \frac{1}{|Q|} \int_Q |B_2(x) - B_2(x_0)| \, dx$$

$$\leq C \sum_{k=1}^{m} |Q|^{-1/r} \left( \int_{2Q} (|b(x) - b_Q||I\alpha T^{k,2}(f)(x)|)^r \, dx \right)^{1/r}$$

$$+ C \sum_{k=1}^{m} |Q|^{-1/v} \left( \int_{2Q} (|b(x) - b_Q||T^{k,A}(f)(x)|)^v \, dx \right)^{1/r}$$

$$+ C\|b\|_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,r}(I\alpha T^{k,2}(f))(\bar{\xi}) + M_{\beta+\alpha,r}(T^{k,A}(f))(\bar{\xi}))$$

$$\leq C \sum_{k=1}^{m} |Q|^{-1/r} |b|_{Lip_\beta} |2Q|^{\beta/n}|Q|^{1/r-\beta/n} \left( \frac{1}{|Q|^{1-\beta/n}} \int_{2Q} |I\alpha T^{k,2}(f)(x)|^r \, dx \right)^{1/r}$$

$$+ C\|b\|_{Lip_\beta} \sum_{k=1}^{m} |Q|^{-1/v} |2Q|^{\beta/n}|Q|^{1/r-(\beta+\alpha)/n} \left( \frac{1}{|Q|^{1-r(\beta+\alpha)/n}} \int_{2Q} |T^{k,A}(f)(x)|^v \, dx \right)^{1/r}$$

$$+ C\|b\|_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,r}(I\alpha T^{k,2}(f))(\bar{\xi}) + M_{\beta+\alpha,r}(T^{k,A}(f))(\bar{\xi}))$$

$$\leq C\|b\|_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,r}(I\alpha T^{k,2}(f))(\bar{\xi}) + M_{\beta+\alpha,r}(T^{k,A}(f))(\bar{\xi})).$$

Thus, (3) holds. We take $r, w$ with $q' < r < p < n/(\alpha + \beta), 1/w = 1/p - (\alpha + \beta)/n$ and obtain, by Lemmas 2.2 and 2.3,

$$(T^b(f))_w \leq M(T^b(f))_w \leq C\|b\|_{Lip_\beta} \left( \sum_{k=1}^{m} (||M_{\beta,r}(I\alpha T^{k,2}(f))||_{L^w} + ||M_{\beta+\alpha,r}(T^{k,A}(f))||_{L^w}) \right)$$

$$\leq C\|b\|_{Lip_\beta} \sum_{k=1}^{m} (||T^{k,2}(f)||_{L^p} + ||T^{k,A}(f)||_{L^p}) \leq C\|b\|_{Lip_\beta} \|f\|_{L^p}.$$  (4)
Now we verify that \( T^b \) satisfies the conditions of Lemma 2.6. In fact, for any \( 1 < p_i < n/(\alpha + \beta) \) with \( 1/u_i = 1/p_i - \alpha/n, 1/v_i = 1/p_i - (\alpha + \beta)/n \) and \( \| f \|_{L^{p_i}} \leq 1 \) (\( i = 1, 2 \)), note that \( T^b(f)(x) = T^{b-b_\xi}(f)(x) + T^{b_\xi}(f)(x), b - b_\xi \in BMO(R^n) \) and \( b_\xi \in LIP(R^n) \), by (2) and Lemma 2.4, we obtain

\[
\|T^{b-b_\xi}(f)\|_{L^{u_i}} \leq C\|b - b_\xi\|_{BMO}\|f\|_{L^{p_i}} \\
\leq C\|b - b_\xi\|_{BMO} \leq C\|b\|_{BMO}\varphi(s),
\]

and by (4) and Lemma 2.5, we obtain

\[
\|T^{b_\xi}(f)\|_{L^{v_i}} \leq C\|b\|_{LIP}\|f\|_{L^{p_i}} \leq Cs^{-\beta}\varphi(s)\|b\|_{BMO}.\]

Thus, for \( s = t^{-1/n} \) and \( i = 1, 2, \)

\[
m_{T^b(f)}(\psi^{-1}(t)) \leq m_{T^{b-b_\xi}(f)}(t^{1/u_i}\varphi(t^{-1/n})) \\
\leq m_{T^{b-b_\xi}(f)}(t^{1/u_i}\varphi(t^{-1/n})/2) + m_{T^{b_\xi}(f)}(t^{1/u_i}\varphi(t^{-1/n})/2) \\
\leq C \left[ \left( \frac{\varphi(s)}{t^{1/u_i}\varphi(s)} \right)^{u_i} + \left( \frac{s^{-\beta}\varphi(s)}{t^{1/u_i}\varphi(s)} \right)^{v_i} \right] = Ct^{-1}.
\]

Taking \( 1 < p_2 < p < p_1 < n/(\alpha + \beta) \) and by Lemma 2.6, we obtain, for \( \| f \|_{L^p} \leq (p/p_1)^{1/p}, \)

\[
\int_{R^n} \psi(|T^b(f)(x)|)dx = \int_0^\infty m_{T^b(f)}(\psi^{-1}(t))dt \leq C,
\]

then, \( \|T^b(f)\|_{L^p} \leq C. \)

This completes the proof of Theorem 1.3.

By using the same arguments as in the proof of Theorem 1.3 will give the proof of Theorem 1.4, we omit the details.

### 3 Applications

In this section we shall apply the Theorems 1.3 and 1.4 to some particular operators such as the Calderón-Zygmund singular integral operator and Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

**Application 1. Calderón-Zygmund singular integral operator**

Let \( T \) be the Calderón-Zygmund operator (see [3, 4, 19, 20]), the Toeplitz type operator related to \( T \) is defined by

\[
T^b = \sum_{k=1}^m (T^{k.1}M^b I_\alpha T^{k.2} + T^{k.3}I_\alpha M^b T^{k.4}).
\]

Then it is easy to verify that Key Lemma holds for \( T^b \) (see [3, 4, 19, 20]), thus \( T \) satisfies the conditions in Theorem 1.3 and Theorem 1.3 holds for \( T^b \).

**Application 2. Littlewood-Paley operator**

Let \( \varepsilon > 0 \) and \( \psi \) be a fixed function which satisfies the following properties:

1. \( |\psi(x)| \leq C(1 + |x|)^{-(\alpha + 1)} \),
2. \( |\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)} \) when \( 2|y| < |x| \).
The Toeplitz type operator related to the Littlewood-Paley operator is defined by

\[
g^b_\psi(f)(x) = \left( \int_0^\infty |F^b_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\]

where

\[
F^b_t(f) = \sum_{k=1}^m (F^k_{t,1} M^b I^d_{t} F^k_{t,2}(f) + F^k_{t,3} I^d_{t} M^b F^k_{t,4}(f))
\]

and \(\psi_t(x) = t^{-\eta} \psi(x/t)\) for \(t > 0\). We write that \(F_t(f) = \psi_t * f\). We also define that

\[
g_\varphi(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\]

which is the Littlewood-Paley operator (see [20]).

Let \(H\) be the space \(H = \{ h : \|h\| = \left( \int_0^\infty |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \}\), then, for each fixed \(x \in \mathbb{R}^n\), \(F^b_t(f)(x)\) may be viewed as a mapping from \([0, +\infty)\) to \(H\), and it is clear that

\[
g_\varphi(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad g^b_\psi(f)(x) = \|F^b_t(f)(x)\|.
\]

It is easy to see that \(g^b_\psi\) satisfies the conditions of Theorem 1.4 (see [9–13]), thus Theorem 1.4 holds for \(g^b_\psi\).

**Application 3. Marcinkiewicz operator**

Let \(\Omega\) be homogeneous of degree zero on \(\mathbb{R}^n\) and \(\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0\). Assume that \(\Omega \in Lip_\gamma(S^{n-1})\) for \(0 < \gamma \leq 1\), that is there exists a constant \(M > 0\) such that for any \(x, y \in S^{n-1}\), \(|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma\). The Toeplitz type operator related to the Marcinkiewicz operator is defined by

\[
\mu^b_\Omega(f)(x) = \left( \int_0^\infty |F^b_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F^b_t(f) = \sum_{k=1}^m (F^k_{t,1} M^b I^d_{t} F^k_{t,2}(f) + F^k_{t,3} I^d_{t} M^b F^k_{t,4}(f)).
\]

We write that

\[
F_t(f)(x) = \int_{|x - y| \leq t} \frac{\Omega(x - y)}{|x - y|^{n-1}} f(y)dy.
\]

We also define that

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

which is the Marcinkiewicz operator (see [21]).

Let \(H\) be the space \(H = \{ h : \|h\| = \left( \int_0^\infty |h(t)|^2 \frac{dt}{t^3} \right)^{1/2} < \infty \}\). Then, it is clear that

\[
\mu_\Omega(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad \mu^b_\Omega(f)(x) = \|F^b_t(f)(x)\|.
\]

It is easy to see that \(\mu^b_\Omega\) satisfies the conditions of Theorem 1.4 (see [10–14, 21]), thus Theorem 1.4 holds for \(\mu^b_\Omega\).
Application 4. Bochner-Riesz operator

Let $\delta > (n - 1)/2$, $F^\delta_t(f)(\xi) = (1 - t^2|\xi|^2)^\delta t^\delta \hat{f}(\xi)$ and $B^\delta_t(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [15])

$$B^\delta_{\ast\ast}(f)(x) = \sup_{t > 0} |F^\delta_t(f)(x)|.$$  

Let $H$ be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$. The Toeplitz type operator related to the maximal Bochner-Riesz operator is defined by

$$B^b_{\ast\ast}(f)(x) = \sup_{t > 0} |F^b_{\ast\ast,t}(f)(x)|,$$

where

$$F^b_{\ast\ast,t}(f) = \sum_{k=1}^m (F^1_k M^b I_{\alpha} F^{2}_k (f) + F^{3}_k M^b F^{4}_k (f)).$$

We know

$$B^b_{\ast\ast}(f)(x) = ||B^b_{\ast\ast,t}(f)(x)||.$$  

It is easy to see that $B^b_{\ast\ast}$ satisfies the conditions of Theorem 1.4 (see [10–14, 22]), thus Theorem 1.4 holds for $B^b_{\ast\ast}$.

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