A FATOU-BIEBERBACH DOMAIN IN $\mathbb{C}^2$ WHICH IS NOT RUNGE

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Abstract. Since a paper by J.P. Rosay and W. Rudin from 1988 there has been an open question whether all Fatou-Bieberbach domains are Runge. We give an example of a Fatou-Bieberbach domain $\Omega$ in $\mathbb{C}^2$ which is not Runge. The domain $\Omega$ provides (yet) a negative answer to a problem of Bremermann.

1. Introduction

We give a negative answer to the problem, initially posed by J.P. Rosay and W. Rudin in [6] and later in [4], as to whether all Fatou-Bieberbach domains are Runge:

Theorem 1. There is a Fatou-Bieberbach domain $\Omega$ in $\mathbb{C}^* \times \mathbb{C}$ which is Runge in $\mathbb{C}^* \times \mathbb{C}$ but not in $\mathbb{C}^2$.

A Fatou-Bieberbach domain is a proper subdomain of $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$, and a domain $\Omega \subset \mathbb{C}^n$ is said to be Runge (in $\mathbb{C}^n$) if any holomorphic function $f \in O(\Omega)$ can be approximated uniformly on compacts in $\Omega$ by polynomials.

It should be noted that although the domain $\Omega$ is not Runge it still has the property that the intersection of $\Omega$ with any complex line $L$ is simply connected: Let $V$ be a connected component of $\Omega \cap L$, let $\Gamma \subset V$ be a simple closed curve, and let $D$ denote the disk in $L$ bounded by $\Gamma$. Since $\Gamma$ is null-homotopic in $\Omega$ we have that $D$ is contained in $\mathbb{C}^* \times \mathbb{C}$ and so the claim follows from the fact that $\Omega$ is Runge in $\mathbb{C}^* \times \mathbb{C}$. Intersecting $\Omega$ with a suitable bounded subset of $\mathbb{C}^2$ this gives a negative answer to the problem of Bremermann: "Suppose that $D$ is a Stein domain in $\mathbb{C}^n$ such that for every complex line $l$ in $\mathbb{C}^n$, $l \setminus D$ is connected. Is it true that $D$ is Runge in $\mathbb{C}^n$?". Negative answers to this problem have also recently been given in [1] and [5]. One can in fact show, using an argument as above together with the argument principle, that if $R$ is a smoothly bounded planar domain and if $\varphi(R)$ is a holomorphic embedding of $R$ into $\mathbb{C}^2$ with $\varphi(\partial R) \subset \Omega$, then $\varphi(R) \subset \Omega$.

The idea of the proof is the following: Observe first that if $\Omega$ is a Fatou-Bieberbach domain in $\mathbb{C}^2$ which is Runge, then $\Omega$ has the property that if $Y \subset \Omega$ is compact then
its polynomially convex hull
\[ \hat{Y} := \{(z, w) \in \mathbb{C}^2; |P(z, w)| \leq \|P\|_Y \forall P \in \mathcal{P}(\mathbb{C}^2)\} \]
is contained in \( \Omega \). To prove the theorem we will construct a domain \( \Omega \) such that \( \hat{Y} \setminus \Omega \neq \emptyset \) for a certain compact set \( Y \). For a compact subset \( Y \subset \mathbb{C}^* \times \mathbb{C} \) let \( \hat{Y}_* \) denote the set
\[ \hat{Y}_* := \{(z, w) \in \mathbb{C}^2; |P(z, w)| \leq \|P\|_Y \forall P \in \mathcal{O}(\mathbb{C}^* \times \mathbb{C})\}. \]

We say that the set \( Y \) is holomorphically convex if \( \hat{Y}_* = Y \). We will first construct (a construction by Stolzenberg) a holomorphically convex compact set \( Y \subset \mathbb{C}^* \times \mathbb{C} \) having the property that \( \hat{Y} \cap (\{0\} \times \mathbb{C}) \neq \emptyset \). \( Y \) is the disjoint union of two disks is \( \mathbb{C}^* \times \mathbb{C} \). We will then use the fact that \( \mathbb{C}^* \times \mathbb{C} \) has the density property to construct a Fatou-Bieberbach domain \( \Omega \subset \mathbb{C}^* \times \mathbb{C} \) such that \( Y \subset \Omega \). The domain \( \Omega \) cannot be Runge.

A few words about the density property and approximation by automorphisms. As defined in [9], a complex manifold \( M \) is said to have the density property if every holomorphic vector field on \( M \) can be approximated locally uniformly by Lie combinations of complete vector fields on \( M \). It was proved in [9] that \( \mathbb{C}^* \times \mathbb{C} \) has the density property. In Andersén-Lempert theory the density property corresponds to the fact that in \( \mathbb{C}^n \) every entire vector field can be approximated by sums of complete vector fields. This has been studied also in [10].

Using the density property of \( \mathbb{C}^* \times \mathbb{C} \) one gets as in [4] (by copying their arguments): Let \( \Omega \) be an open set in \( \mathbb{C}^* \times \mathbb{C} \). For every \( t \in [0, 1] \), let \( \varphi_t \) be a biholomorphic map from \( \Omega \) into \( \mathbb{C}^* \times \mathbb{C} \), of class \( C^2 \) in \( (t, z) \in [0, 1] \times \Omega \). Assume that \( \varphi_0 = Id \), and assume that each domain \( \Omega_t = \varphi_t(\Omega) \) is Runge in \( \mathbb{C}^* \times \mathbb{C} \). Then for every \( t \in [0, 1] \) the map \( \varphi_t \) can be approximated on \( \Omega \) by holomorphic automorphisms of \( \mathbb{C}^* \times \mathbb{C} \). In the proof of Theorem 1 we will construct such an isotopy.

We will let \( \pi \) denote the projection onto the first coordinate in \( \mathbb{C}^* \times \mathbb{C} \) and in \( \mathbb{C}^2 \), and we will let \( B_\varepsilon(p) \) denote the open ball of radius \( \varepsilon \) centered at a point \( p \).

2. CONSTRUCTION OF THE SET \( Y \)

We start by defining a certain rationally convex subset \( Y \) of \( \mathbb{C}^2 \). The set will be a union of two disjoint polynomially convex disks in \( \mathbb{C}^* \times \mathbb{C} \), but the polynomial hull of the union will contain the origin. This construction is taken from [8], page 392-396, and is due to Stolzenberg [7].

Let \( \Omega_1 \) and \( \Omega_2 \) be simply connected domains in \( \mathbb{C} \), as in Fig.1. below, with smooth boundary, such that if \( I_+ = [1, \sqrt{3}], I_- = [-\sqrt{3}, -1], \) then \( I_+ \subset \partial \Omega_1, I_- \subset \partial \Omega_2 \). Require that \( \partial \Omega_1 \) and \( \partial \Omega_2 \) meet only twice, that \( I_- \subset \Omega_1, I_+ \subset \Omega_2 \), and, finally, that \( \partial \Omega_1 \cup \partial \Omega_2 \) be the union of the boundary of the unbounded component of \( \mathbb{C} \setminus (\partial \Omega_1 \cup \partial \Omega_2) \), together with the boundary of the component of this set that contains the origin. Let the intersections of the boundaries be the points \( i \) and \( -i \).
We define

\[ V_1 = \{(z, w) \in \mathbb{C}^2; z^2 - w \text{ is real and lies in } [0, 1]\}, \]

\[ V_2 = \{(z, w) \in \mathbb{C}^2; w \text{ is real and lies in } [1, 2]\}, \]

\[ X_1 = \{(z, w) \in V_1; z \in \partial \Omega_2\}, \]

\[ X_2 = \{(z, w) \in V_2; z \in \partial \Omega_1\}, \]

Note that \( X_1 \) and \( X_2 \) are totally real annuli, that they are disjoint, and that the origin is contained in the polynomial hull of \( X_1 \). Next we want to remove pieces from \( X_1 \) and \( X_2 \) to create two disks.

Define

\[ \tilde{V}_1 = V_1 \cap \pi^{-1}(I_+), \]

\[ \tilde{V}_2 = V_2 \cap \pi^{-1}(I_-), \]
\[ Y_1 = X_1 \setminus V_2, \]
\[ Y_2 = X_2 \setminus V_1. \]

The set \( Y \) will be defined as \( Y = Y_1 \cup Y_2 \). Note that
\[ (*) \hat{V}_1 \subset \hat{X}_1, \hat{V}_2 \subset \hat{X}_2. \]

Let us describe what \( Y_1 \) and \( Y_2 \) looks like over \( I_- \) and \( I_+ \) respectively. By the equations we see that these sets are contained in \( \mathbb{R}^2 \). Let \((x, y)\) denote the real parts of \((z, w)\).

Over \( I_- \) we have that \( Y_1 \) is the union of the two sets defined by

\[(a)\]
\[ 2 \leq y \leq x^2 \text{ if } -\sqrt{3} \leq x \leq -\sqrt{2}, \]
\[ (b) \]
\[ x^2 - 1 \leq y \leq 1 \text{ if } -\sqrt{2} \leq x \leq -1. \]

Over \( I_+ \) we have that \( Y_2 \) is the union of the sets defined by

\[(c)\]
\[ x^2 \leq y \leq 2 \text{ if } 1 \leq x \leq \sqrt{2}, \]
\[ (d) \]
\[ 1 \leq y \leq x^2 - 1 \text{ if } \sqrt{2} \leq x \leq \sqrt{3}. \]

From these equations we see that \( Y_1 \) and \( Y_2 \) are disks.

We have that
\[ (**) \hat{Y} \text{ contains the origin} \]

because of the following: We already noted that the origin is contained in \( \hat{X}_1 \), so the claim follows from \((*)\) and the following simpler version of Lemma 29.31, [8], page 392: Let \( X_1 \) and \( X_2 \) be disjoint compact sets in \( \mathbb{C}^N \), and let \( S_1 \) and \( S_2 \) be relatively open subsets of \( X_1 \) and \( X_2 \) respectively such that \( S_1 \subset \hat{X}_2, S_2 \subset \hat{X}_1 \). Then \( \hat{X}_1 \cup \hat{X}_2 = (X_1 \setminus S_1) \cup (X_2 \setminus S_2) \). The reason for this, which was pointed out by the referee, is simply that neither \( S_1 \) nor \( S_2 \) can contain peak points for the algebra generated by the polynomials on \( X_1 \cup X_2 \).

3. Proof of Theorem [11]

It is proved in [8] that the set \( Y \) is rationally convex, and that the sets \( Y_j \) are polynomially convex separately. For our construction we need to know that \( Y \) is holomorphically convex, so we prove the following:

3.1. Lemma. We have that \( Y \) is holomorphically convex in \( \mathbb{C}^* \times \mathbb{C} \).

Proof. For \( j = 1, 2 \), let \( Y^+_j \) and \( Y^-_j \) denote the sets \( Y_j \cap \{ \text{Re}(z) \geq 0 \} \) and \( Y_j \cap \{ \text{Re}(z) \leq 0 \} \) respectively. Let \( Y^+ = Y^+_1 \cup Y^+_2 \) and \( Y^- = Y^-_1 \cup Y^-_2 \).

Observe first that \( Y^+ \) and \( Y^- \) are polynomially convex separately: Assume to get a contradiction that \( \hat{Y}^- \) contains nontrivial points. In that case there exists a graph \( G(f) \) of a bounded holomorphic function defined on the topological disk \( U \) bounded by
π(\(Y^−\)), such that \(G(f) \subset \hat{Y}^−\), and such that \((z, f(z)) \in \hat{Y}^−\) for a.a. (in terms of radial limits if we regard \(U\) as a proper disk) \(z \in π(Y^−)\) (Theorem 20.2. in [2], page 172, holds by the discussion on page 171 even though the fibers over \(±i\) are not convex). Then for continuity reasons \(G(f)\) would have to contain nontrivial points of \(\hat{Y}^−\) in the fibers \(\{±i\} \times \mathbb{C}\) - but as this clearly cannot be the case, we have our contradiction. The case of \(Y^+\) is similar.

Next assume to get a contradiction that there is a point \((z_0, w_0) \in \hat{Y}^* \setminus Y\) with \(\text{Re}(z_0) < 0\). The function \(f(z)\) defined to be \((z + i)(z - i)\) on \(π(Y^−) \cup \{z_0\}\) and zero on \(π(Y^+)\) can be uniformly approximated on \(π(Y) \cup \{z_0\}\) by polynomials in \(z\) and \(\frac{1}{z}\), and so any representing Jensen measure (see [8] Chapter 2) for the functional \(g \mapsto g(z_0, w_0)\) would have to be supported on \(Y^−\). But then the point \((z_0, w_0)\) would have to be in the hull of \(Y^−\) which is a contradiction. The corresponding conclusion holds for \(\text{Re}(z_0) > 0\).

Finally, Rossi’s local maximum principle excludes the possibility of there being nontrivial points in the hull contained in \(\{±i\} \times \mathbb{C}\). □

3.2. Lemma. Let \(p = (z_0, w_0) \in \mathbb{C}^* \times \mathbb{C}\) and let \(ε > 0\). Then there exists an automorphism \(ψ\) of \(\mathbb{C}^* \times \mathbb{C}\) such that \(ψ(Y) \subset B_ε(p)\).

Proof. We need to argue that there exists an isotopy as described in the introduction, and we content ourselves by demonstrating that there exist isotopies mapping \(Y_1\) and \(Y_2\) into separate arbitrarily small balls - the rest is trivial. Let \(q_j \in Y_j\) be a point for \(j = 1, 2\), and let \(δ > 0\). Since \(Y_j\) is a smooth disk there clearly exists a smooth map \(f^j : [0, 1] \times Y_j \to Y_j\) such that for each fixed \(t\) the map \(f^j_t : Y_j \to Y_j\) is a smooth diffeomorphism, such that \(f^j_0\) is the identity, and such that \(f^j_1(Y_j) \subset B_δ(q_j)\). Since \(Y_j\) is totally real there exists, by [3] Corollary 3.2, for each \(ε > 0\) a real analytic map \(Φ^j : [0, 1] \times \mathbb{C}^2 \to \mathbb{C}^2\) such that \(Φ^j_t \in \text{Aut}_{\text{hol}}(\mathbb{C}^2)\) for each \(t\), \(Φ^j_0\) is the identity, and \(\|f^j - Φ^j_t\|_{[0,1] \times Y_j} < ε\). For small enough \(ε\) we restrict \(Φ^j\) to a sufficiently small Runge neighborhood of \(Y_j\).

□

Proof of Theorem 1: Let \(G\) be an automorphism of \(\mathbb{C}^* \times \mathbb{C}\) with an attracting fixed point \(p \in \mathbb{C}^* \times \mathbb{C}\). It is well known that the basin of attraction of the point \(p\) is a Fatou-Bieberbach domain. This domain is clearly contained in \(\mathbb{C}^* \times \mathbb{C}\). Denote this domain by \(Ω(G)\). Let \(ε\) be a positive real number such that \(B_ε(p) \subset Ω(G)\). By Lemma 3.2 there is an automorphism \(ψ\) of \(\mathbb{C}^* \times \mathbb{C}\) such that \(ψ(\hat{Y}) \subset B_ε(p)\). Then \(Y \subset ψ^{-1}(Ω(G))\). The set \(ψ^{-1}(Ω(G))\) is biholomorphic to \(\mathbb{C}^2\), and from (**) in Section 2 we have that \(\hat{Y}\) contains the origin. On the other hand it is clear that \(Ω(G)\) is Runge in \(\mathbb{C}^* \times \mathbb{C}\), and so \(ψ^{-1}(Ω(G))\) is Runge in \(\mathbb{C}^* \times \mathbb{C}\).

□

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REFERENCES

1. M. Abe: Polynomial Convexity and Strong Disk Property. *J. Math. Anal. Appl.*, **321** (2006), 32–36.
2. H. Alexander, J. Wermer: Several Complex Variables and Banach Algebras, 3rd ed., *Springer-Verlag New York, Inc.* (1998).
3. F. Forstnerič: Approximation by automorphisms on smooth sumbanifolds of $\mathbb{C}^n$. *Math. Ann.* **300** (1994), 719–738.
4. F. Forstnerič, J.P. Rosay: Approximation of Biholomorphic Mappings by Automorphisms of $\mathbb{C}^n$. *Invent. Math.* **112** (1993), 112–123.
5. C. Joiţa: On a Problem of Bremermann Concerning Runge Domains. *Math. Ann.* **337**, (2007), 395–400.
6. J.P. Rosay, W. Rudin: Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. *Trans. Amer. Math. Soc.* **310** (1988), 47–86.
7. G. Stolzenberg: On the analytic part of a Runge hull. *Math. Ann.* **164** (1966), 286–290.
8. E.L. Stout: The Theory of Uniform Algebras. *Bogden and Quigly, Inc.* (1971).
9. D. Varolin: The Density Property for Complex Manifolds and Geometric Structures. *J. Geom. Anal.* **11** (2001), 135–160.
10. D. Varolin: The Density Property for Complex Manifolds and Geometric Structures II. *Internat. J. Math.* **11** (2000), 837–847.

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