Hyperbolic inverse mean curvature flow

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Abstract

In this paper, we prove the short-time existence of hyperbolic inverse (mean) curvature flow (with or without the specified forcing term) under the assumption that the initial compact smooth hypersurface of $\mathbb{R}^{n+1} \ (n \geq 2)$ is mean convex and star-shaped. Several interesting examples and some hyperbolic evolution equations for geometric quantities of the evolving hypersurfaces have been shown. Besides, under different assumptions for the initial velocity, we can get the expansion and the convergence results of a hyperbolic inverse mean curvature flow in the plane $\mathbb{R}^2$, whose evolving curves move normally.

1 Introduction

Curvature flows is a hot topic in the research of Differential Geometry in the past several decades. It is well known that Perelman used the Ricci flow, an intrinsic curvature flow, to successfully solve the 3-dimensional Poincaré conjecture. Among extrinsic curvature flows, an important one is the mean curvature flow (MCF for short), which means a submanifold of a prescribed ambient space moves with a speed equal to its mean curvature vector. A classical result in the study of MCF due to Huisken [10] says that for a strictly convex, compact hypersurface immersed in $\mathbb{R}^{n+1} \ (n \geq 2)$, if it evolves along the MCF, then evolving hypersurfaces contract to a single point at some finite time, and moreover, after area-preserving rescaling, the rescaled evolving hypersurfaces converge to a round sphere in the $C^\infty$-topology as time tends to infinity. Many improvements have been obtained after this classical result. Besides, the theory of MCF also has some interesting applications. For instance, Topping [18] used curve shortening flow on surfaces, which is the lower dimensional version of MCF, to get isoperimetric inequalities on surfaces. The theory of curve shortening flow can also be used to do the image processing (see, e.g., [4]). The MCF is called inward flow, and conversely, the inverse mean curvature flow (IMCF for short), which means a submanifold of a prescribed ambient space moves in direction of the outward unit normal vector of the submanifold with a speed equal to $1/H$ ($H \neq 0$ denotes the mean curvature), is called outward flow.
flow. The IMCF is also a very important extrinsic flow, which has many interesting and important applications. For instance, the evolution of non-star-shaped initial surfaces under the IMCF may occur singularities in finite time, but, through defining a notion of weak solution to IMCF equation, Huisken-Ilmanen [11] proved the Riemannian Penrose inequality by using the method of IMCF (the Riemannian Penrose inequality can also be proved by applying the positive mass theorem, see [1] for details). Using the method of IMCF, Brendle, Hung and Wang [2] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurfaces in the $n$-dimensional $(n \geq 3)$ anti-de Sitter-Schwarzschild manifold, which generalized the related conclusions in the Euclidean space $\mathbb{R}^n$.

The corresponding author, Dr. Jing Mao, has been working on IMCF for several years and has also obtained some interesting results with his collaborators. For instance, Chen and Mao [5] considered the evolution of a smooth, star-shaped and $F$-admissible ($F$ is a 1-homogeneous function of principle curvatures satisfying some suitable conditions) embedded closed hypersurface in the $n$-dimensional $(n \geq 3)$ anti-de Sitter-Schwarzschild manifold along its outward normal direction has a speed equal to $1/F$ (clearly, this evolution process is a natural generalization of IMCF, and we call it inverse curvature flow). We write as ICF for short), and they proved that this ICF exists for all the time and, after rescaling, the evolving hypersurfaces converge to a sphere as time tends to infinity. This interesting conclusion has been improved by Chen, Mao and Zhou [6] to the situation that the ambient space is a warped product $I \times \lambda(r) N^n$ with $I$ an unbounded connected interval of $\mathbb{R}$ (i.e., the set of real numbers) and $N^n$ a Riemannian manifold of nonnegative Ricci curvature. Also for this kind of warped products $I \times \lambda(r) N^n$, under suitable growth assumptions on the warping function $\lambda(r)$, Chen, Mao, Xiang and Xu [7] successfully proved that if an $n$-dimensional $(n \geq 2)$ compact $C^{2,\alpha}$-hypersurface with boundary, which meets a given cone in $I \times \lambda(r) N^n$ perpendicularly and is star-shaped with respect to the center of the cone, evolves along the IMCF, then the flow exists for all the time and, after rescaling, the evolving hypersurfaces converge to a piece of the geodesic sphere as time tends to infinity, which generalized the main conclusion in [15].

We know that the MCF and the IMCF describe the motion of a prescribed submanifold, that is, the velocity $\frac{d}{dt}$ equals some scalar multiple of the unit normal vector of the submanifold. If the velocity $\frac{d}{dt}$ is replaced by the acceleration $\frac{d^2}{dt^2}$, what happens? Yau [19] suggested the following curvature flow

$$\frac{d^2X}{dt^2} = H\vec{n},$$

where, as before, $H$ denotes the mean curvature and $\vec{n}$ is the unit inner normal vector of the initial hypersurface $X(\cdot,0)$, and pointed out very little about the global time behavior of the evolving hypersurfaces. The curvature flow (1.1) can be seen as the hyperbolic version of MCF, and that is the reason why it is called hyperbolic mean curvature flow (HMCF for short). In fact, if $\mathcal{M}$ is an $n$-dimensional $(n \geq 2)$ smooth compact Riemannian manifold and $X(\cdot,t)$ is a one-parameter family of smooth hypersurface immersions in $\mathbb{R}^{n+1}$ satisfying (1.1), where $X(\cdot,0)$ is the hypersurface immersion of $\mathcal{M}$ into $\mathbb{R}^{n+1}$, then it is not hard to show that (1.1) is a second-order hyperbolic PDE, which is used to get the short time existence of the flow (see [9] Section 2) for details). Mao [13] considered a hyperbolic curvature flow whose form is given by (1.1) plus a forcing term in direction of the position vector, that is,$$

\frac{\partial^2 X}{\partial t^2} = H\vec{n} + c(t)X$$

where $c(t)$ is a smooth function of time.
with \(c(t)\) a bounded continuous function w.r.t. the time variable \(t\) only, and successfully improved most conclusions in [9] under suitable assumptions.

Based on our research experience on the ICF and the HMCF, it is natural to consider the hyperbolic version of the IMCF.

Let \(M_0\) be a compact, mean convex, star-shaped smooth hypersurface of the \((n+1)\)-dimensional Euclidean space \(\mathbb{R}^{n+1}\) \((n \geq 2)\), which is given as an embedding

\[ X_0 : S^n \rightarrow \mathbb{R}^{n+1}, \]

where \(S^n \subset \mathbb{R}^{n+1}\) denotes the unit sphere in \(\mathbb{R}^{n+1}\). Define a one-parameter family of smooth hypersurfaces embedding in \(\mathbb{R}^{n+1}\) given by

\[ X(\cdot, t) : S^n \rightarrow \mathbb{R}^{n+1} \]

with \(X(\cdot, 0) = X_0(\cdot)\), and we say that it is a solution of the hyperbolic inverse mean curvature flow (HIMCF for short) if \(X(\cdot, t)\) satisfies

\[ \frac{d^2}{dt^2}X(x, t) = H^{-1}(x, t)\vec{v}(x, t), \quad \forall x \in S^n, \ t > 0, \]

where \(H(x, t)\) is the mean curvature of \(X(x, t)\), \(\vec{v}(x, t)\) is the unit outward normal vector on \(X(x, t)\). If \(X(\cdot, 0) = X_0, \frac{dX}{dt}(\cdot, 0) = X_1(\cdot)\) with \(X_1(\cdot)\) a smooth vector-valued function on \(S^n\), then one can get the existence of the one-parameter family of smooth hypersurfaces \(X(\cdot, t)\) embedding in \(\mathbb{R}^{n+1}\) on the time interval \([0, T)\) with \(T < \infty\) (see Theorem 2.3 for the precise statement). Besides, under different assumptions for the initial velocity, we separately discuss the expansion and the convergence of a HIMCF in the plane \(\mathbb{R}^2\), whose evolving curves move normally, in the last section (see Theorem 5.11 for the precise statement).

**Remark 1.1.** As mentioned before, some interesting conclusions about IMCF or ICF can be generalized from the setting that the ambient space is the Euclidean space to the setting of warped products (see, e.g., [5, 6, 7]). Hence, one might ask the following question:

- If we consider the HIMCF or the HICF (see Remark 2.4 (2) below for this notion) in the warped product \(I \times \lambda(r)N^n\) with \(I\) an unbounded connected interval of \(\mathbb{R}\) and \(N^n\) a Riemannian manifold of nonnegative Ricci curvature, could we get similar results to this paper under some suitable assumptions on \(\lambda(r)\)?

This question has been solved in [14] and the answer is positive.

## 2 Local existence and uniqueness

In this section, we would like to use the star-shaped assumption to change the evolution equation in (2.6) into a single second-order hyperbolic PDE, which will lead to the short-time existence and the uniqueness of the flow (2.6).

Denote by \(M_t\) the evolving hypersurface under the flow (2.6). Since \(M_0\) is star-shaped, \(M_t\) also should be star-shaped on \([0, \varepsilon]\) for some small enough \(\varepsilon > 0\) by continuity. Let the surface \(M_t\) be represented as a graph over \(S^n\), i.e., the embedding vector \(x = (x^\alpha)\) now has the components

\[ x^{n+1} = u(x, t), \quad x^i = x^i(t), \]
with \((x^i)\) local coordinates of \(S^n\). Furthermore, let \(\xi = (\xi^i)\) be a local coordinate system of \(M_t\), which implies the graphic function \(u\) can be written as \(u = u(x(\xi), t)\). Clearly, the outward unit normal vector in \((x, u)\) has the form

\[
\bar{v} = v^{-1}(-D_iu, 1),
\]

where

\[
D_iu = \frac{\partial u}{\partial x^i},
\]

\[
v = (1 + u^{-2}|Du|^2)^{\frac{1}{2}} = (1 + u^{-2}\sigma^{ij}D_iuD_ju)^{\frac{1}{2}},
\]

\((\sigma_{ij})\) being the metric of \(S^n\) in the coordinates \((x^i)\) and naturally \((\sigma^{ij})\) being its inverse. Therefore, now, the Euclidean metric can be written as

\[
ds^2 = dr^2 + r^2\sigma_{ij}dx^idx^j.
\]

Then the evolution equation (2.6) now yields

\[
\frac{d^2u}{dt^2} = \frac{1}{Hu}, \quad \frac{d^2x^i}{dt^2} = -\frac{D_iu \cdot u^{-2}}{Hu}.
\]

On the other hand, by the chain rule, we have

\[
\frac{du}{dt} = \frac{\partial u}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial u}{\partial t},
\]

and

\[
\frac{d^2u}{dt^2} = \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} + \frac{\partial^2 u}{\partial x^i \partial t} \right) \frac{dx^j}{dt} + \frac{\partial u}{\partial x^i} \frac{d^2x^i}{dt^2} + \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} + \frac{\partial^2 u}{\partial t^2}.
\]

Substituting (2.1) into the above equation yields

\[
\frac{\partial^2 u}{\partial t^2} = \frac{d^2u}{dt^2} - \frac{\partial u}{\partial x^i} \frac{d^2x^i}{dt^2} - \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} \right) = \frac{1}{Hu} + D_iu \cdot \frac{D^i u \cdot u^{-2}}{Hu} - \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} \right) = \frac{v}{H} - \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} \right).
\]

Let \(\varphi = \log u\). For a graph \(M\) over \(S^n\), the metric has the components

\[
g_{ij} = u_iu_j + u^2\sigma_{ij} = u^2(\sigma_{ij} + \varphi_i\varphi_j),
\]

and their inverses are

\[
g^{ij} = u^{-2} \left( \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2} \right).
Besides, \( \upsilon \) can be expressed as

\[
\upsilon = \left( 1 + u^{-2} \sigma^{ij} D_i u D_j u \right)^{\frac{1}{2}} = \left( 1 + \sigma^{ij} D_i \varphi D_j \varphi \right)^{\frac{1}{2}} = \left( 1 + |D \varphi|^2 \right)^{\frac{1}{2}},
\]

and the second fundamental form can be given as the following

\[
h_{ij} = -\frac{1}{\upsilon} \left( u_{ij} - u \sigma_{ij} - \frac{2}{u} u_i u_j \right)
= \frac{u}{\upsilon} \left( \sigma_{ij} - u \frac{u_i u_j}{u^2} \right)
= \frac{u}{\upsilon} \left( \sigma_{ij} - \frac{uu_i u_j}{u^2} + \frac{u_i u_j}{u^2} \right)
= \frac{u}{\upsilon} \left( \sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j \right),
\]

Therefore, the mean curvature is

\[
H = g^{ij} h_{ij}
= u^{-2} \left( \sigma_{ij} - \frac{\varphi_i \varphi_j}{\upsilon^2} \right) \cdot \frac{u}{\upsilon} \left( \sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j \right)
= \frac{1}{u \upsilon} \left( n - \sigma^{ij} \varphi_{ij} + \sigma^{ij} \varphi_i \varphi_j - \frac{\sigma_{ij} \varphi_i \varphi_j}{\upsilon^2} + \frac{\varphi_i \varphi_j}{\upsilon^2} \varphi_{ij} - \frac{\varphi_i \varphi_j \varphi_{ij}}{\upsilon^2} \right)
= \frac{1}{u \upsilon} \left( n + \left( -\sigma^{ij} + \frac{\varphi_i \varphi_j}{\upsilon^2} \right) \varphi_{ij} \right).
\]

So, together with (2.1), we can obtain the following equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{u \upsilon^2}{n + \left( -\sigma^{ij} + \frac{\varphi_i \varphi_j}{\upsilon^2} \right) \varphi_{ij}} \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} \right). \tag{2.2}
\]

Note that

\[
\frac{\partial \varphi}{\partial t} = \frac{1}{u} \frac{\partial u}{\partial t},
\]

then together with (2.2), we have

\[
\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{u} \frac{\partial^2 u}{\partial t^2} - \frac{1}{u^2} \left( \frac{\partial u}{\partial t} \right)^2
= \frac{\upsilon^2}{n + \left( -\sigma^{ij} + \frac{\varphi_i \varphi_j}{\upsilon^2} \right) \varphi_{ij}} \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \frac{\partial^2 u}{\partial x^i \partial t} \frac{dx^i}{dt} \right) - \left( \frac{\partial \varphi}{\partial t} \right)^2
= \left( \frac{\upsilon^2}{n + \left( -\sigma^{ij} + \frac{\varphi_i \varphi_j}{\upsilon^2} \right) \varphi_{ij}} \right) \left( \varphi_{ij} + \varphi_i \varphi_j \right) \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \left( \varphi_{ii} + \varphi_i \varphi_i \right) \frac{dx^i}{dt} \left( \frac{\partial \varphi}{\partial t} \right)^2. \tag{2.3}
\]
Let
\[ \phi(x, \varphi_{ij}, \varphi_{it}, \varphi_i, \varphi_t, \varphi) := \frac{v^2}{n + \left( -\sigma_{ij} + \frac{\varphi_i \varphi_j}{v^2} \right) \varphi_{ij}} - \left( \varphi_{ij} + \varphi_i \varphi_j \right) \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 \left( \varphi_{it} + \varphi_i \varphi_t \right) \frac{dx^i}{dt} - \left( \frac{\partial \varphi}{\partial t} \right)^2. \]

Consider the following equation

\[
\begin{cases}
\frac{\partial^2 \varphi}{\partial t^2} = \phi(x, \varphi_{ij}, \varphi_{it}, \varphi_i, \varphi_t, \varphi), & \forall x \in S^n, \ t > 0, \\
\frac{\partial \varphi}{\partial t} (\cdot, 0) = \varphi_1(x), \\
\varphi(\cdot, 0) = \varphi_2(x),
\end{cases}
\]

where \( \varphi_1(x), \varphi_2(x) \) are smooth functions on \( S^n \).

First, by the standard theory of second-order hyperbolic PDEs, we have the following conclusion.

**Lemma 2.1.** Assume that \( M_0 \) given as before (which, of course, is a graph over \( S^n \)) has strictly positive mean curvature \( H_0 \in C^\infty(S^n) \), and \( \varphi_1(x), \varphi_2(x) \) are given as in (2.4). Then the following wave equation

\[
\begin{cases}
\frac{\partial^2 \varphi}{\partial t^2} = \Delta \varphi + \frac{1}{H_0}, & \forall x \in S^n, \ t > 0, \\
\frac{\partial \varphi}{\partial t} (\cdot, 0) = \varphi_1(x) \\
\varphi(\cdot, 0) = \varphi_2(x)
\end{cases}
\]

has a unique solution \( \varphi_0 \in C^\infty(S^n \times [0, T_1]) \) with some \( T_1 > 0 \).

Next, we want to consider the linearization of (2.4) around \( \varphi_0 \).

**Lemma 2.2.** Let \( \varphi_0 \in C^\infty(S^n \times [0, T_1]) \) be the unique solution of the wave equation (2.5) and \( \xi \in C^\infty(S^n \times [0, T_1]) \). Then there exists some \( T > 0 \) such that the linearization of (2.4) around \( \varphi_0 \) given by

\[
\begin{cases}
L_{\varphi_0} \varphi := \frac{\partial^2 \varphi}{\partial t^2} - [a^{ij} \varphi_{ij} + b^{it} \varphi_{it} + c^i \varphi_i + d^t \varphi_t + e \varphi] = \xi, & \forall x \in S^n, \ t > 0, \\
\frac{\partial \varphi}{\partial t} (\cdot, 0) = \varphi_1 \\
\varphi(\cdot, 0) = \varphi_2
\end{cases}
\]

has a unique solution \( \varphi \in C^\infty(S^n \times [0, T]) \).
Proof. Let \( \varphi_\varepsilon := \varphi_0 + \varepsilon \varphi \). We obtain the linearized operator \( L_{\varphi_0} \) of \( \frac{\partial^2}{\partial t^2} - \varphi \) around \( \varphi_0 \) as

\[
L_{\varphi_0} \varphi := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left( \frac{\partial^2}{\partial t^2} \varphi - \varphi (x, (\varphi_\varepsilon)_{ij}, (\varphi_\varepsilon)_{it}, (\varphi_\varepsilon)_i, (\varphi_\varepsilon)_t, (\varphi_\varepsilon)) \right)
\]

\[
= \left. \frac{\partial^2}{\partial t^2} \varphi - \left( \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_{ij}} \frac{d (\varphi_\varepsilon)_{ij}}{d \varepsilon} + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_{it}} \frac{d (\varphi_\varepsilon)_{it}}{d \varepsilon} + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_i} \frac{d (\varphi_\varepsilon)_i}{d \varepsilon} + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_t} \frac{d (\varphi_\varepsilon)_t}{d \varepsilon} \right) \right|_{\varepsilon=0}
\]

\[
= \left. \frac{\partial^2}{\partial t^2} \varphi - \left( \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_{ij}} \varphi_{ij} + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_{it}} \varphi_{it} + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_i} \varphi_i + \frac{\partial \varphi}{\partial (\varphi_\varepsilon)_t} \varphi_t \right) \right|_{\varepsilon=0}
\]

So, we have

\[
a^{ij} := \frac{g^{ij}}{H^2} (\varphi_{ij}, (\varphi_0)_{ij}, (\varphi_0)_{it}, (\varphi_0)_i, (\varphi_0)_t, (\varphi_0)) - \frac{dx^i}{dt} \frac{dx^j}{dt},
\]

\[
b^{it} := -2 \frac{dx^i}{dt},
\]

and the first equation in (2.4) has the form

\[
\frac{\partial^2 \varphi}{\partial t^2} = a^{ij} \varphi_{ij} + b^{it} \varphi_{it} + I(x, \varphi_t, \varphi, \varphi),
\]

where the last term \( I(x, \varphi_t, \varphi, \varphi) \) depends on \( x, \varphi_t, \varphi, \varphi \). Consider the coefficient matrix of terms involving second-order derivatives of \( \varphi \), and then we have

\[
\begin{pmatrix}
-1 & \frac{dx^1}{dt} & \cdots & \frac{dx^n}{dt} \\
-\frac{dx^1}{dt} & \frac{1}{H^2} g^{11} - \frac{dx^1}{dt} & \cdots & \frac{dx^1}{dt} - \frac{dx^n}{dt} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{dx^n}{dt} & \frac{dx^n}{dt} & \cdots & \frac{1}{H^2} g^{nn} - \frac{dx^n}{dt}
\end{pmatrix}
\]

which, by a suitable linear transformation, becomes

\[
\begin{pmatrix}
-1 & 0 & \cdots & 0 \\
0 & \frac{1}{H^2} g^{11} & \cdots & \frac{1}{H^2} g^{1n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{H^2} g^{n1} & \cdots & \frac{1}{H^2} g^{nn}
\end{pmatrix}
\]

At \( t = 0 \), since \( H_0 \) is strictly positive, thus \( L_{\varphi_0} \) is uniformly hyperbolic in some small time interval \([0, \ell]\). Therefore, the theory of second-order linear hyperbolic PDEs yields the result.

Therefore, we have the following short-time existence.

**Theorem 2.3.** (Local existence and uniqueness) If the initial hypersurface \( M_0 \) is a compact, mean convex, star-shaped smooth hypersurface of \( \mathbb{R}^{n+1} \) \((n \geq 2)\), which is given as an embedding

\[
X_0 : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1},
\]
then there exists a constant $T_{\text{max}} > 0$ such that the initial value problem (IVP for short)
\[
\begin{aligned}
\frac{d^2}{dt^2} X(x,t) &= H^{-1}(x,t) \mathbf{\bar{v}}(x,t), \quad \forall x \in S^n, \ t > 0, \\
\frac{dX}{dt}(x,0) &= X_1(x), \\
X(x,0) &= X_0(x),
\end{aligned}
\]  
(2.6)

has a unique smooth solution $X(x,t)$ on $S^n \times [0,T_{\text{max}})$, where $X_1(x)$ is a smooth vector-valued function on $S^n$.

**Remark 2.4.** (1) If the IVP (2.6) is replaced by
\[
\begin{aligned}
\frac{d^2}{dt^2} X(x,t) &= H^{-1}(x,t) \mathbf{\bar{v}}(x,t) + c(t) X(x,t), \quad \forall x \in S^n, \ t > 0, \\
\frac{dX}{dt}(x,0) &= X_1(x), \\
X(x,0) &= X_0(x),
\end{aligned}
\]  
(2.7)

with $c(t)$ a bounded continuous function w.r.t. to $t$, and other assumptions are the same to those in Theorem 2.3, then one can also get the local existence and uniqueness of the forced HIMCF (2.7) since the first evolution equation in (2.7) is a second-order hyperbolic PDE by nearly the same argument in this section. Although we only add a forcing term $c(t)X(x,t)$ in direction of the position vector, the convergent situation of (2.7) will be much different from (2.6), which can be seen from examples shown in Section 3 and Remark 3.4.

(2) Let $F$ be a symmetric, positive, 1-homogeneous function defined on an open cone $\Gamma$ of $\mathbb{R}^n$ with vertex in the origin, which contains the positive diagonal, i.e., all $n$-tuples of the form $(\lambda, \cdots, \lambda)$, $\lambda > 0$. Assume that $F \in C^0(\Gamma) \cap C^2(\Gamma)$ is monotone, concave, i.e.,
\[
\frac{\partial F}{\partial \lambda^i} > 0, \quad i = 1, 2, \cdots, n, \text{ in } \Gamma,
\]
\[
\frac{\partial^2 F}{\partial \lambda^i \partial \lambda^j} \leq 0,
\]
and that
\[
F = 0 \text{ on } \partial \Gamma.
\]
We also use the normalization convention $F(1, \cdots, 1) = n + 1$. Based on Gerhardt [8] on the ICF in $\mathbb{R}^{n+1}$, we can consider the following IVP
\[
\begin{aligned}
\frac{d^2}{dt^2} X(x,t) &= F^{-1}(x,t) \mathbf{\bar{v}}(x,t), \quad \forall x \in S^n, \ t > 0, \\
\frac{dX}{dt}(x,0) &= X_1(x), \\
X(x,0) &= X_0(x),
\end{aligned}
\]  
(2.8)

where $F$ defined on $\Gamma$ is a function of principle curvatures described as above, and other assumptions are the same to those in Theorem 2.3. Clearly, the IVP (2.6) is a special case of the IVP (2.8), and the first evolution equation in (2.8) is a hyperbolic version of the ICF, which leads to the fact that we call it hyperbolic inverse curvature flow (HICF for short). We claim that the hyperbolic
flow (2.8) also has a unique smooth solution \( X(x,t) \) on \( S^n \times [0,T_2) \) with some \( T_2 > 0 \). As the argument in Section 2, together with the first evolution equation of (2.8), one can obtain the following evolution equation

\[
\frac{\partial^2 \phi}{\partial t^2} = \frac{\nu}{uF} - \left[ (\varphi_{ij} + \varphi_i \varphi_j) \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 (\varphi_{it} + \varphi_i \varphi_t) \frac{dx^i}{dt} \right] - \left( \frac{\partial \varphi}{\partial t} \right)^2. \tag{2.9}
\]

Denote by \( \mathcal{M}(\Gamma) \) the class of all real \((n \times n)\)-matrices whose eigenvalues belong to \( \Gamma \). Then one can define a function \( \mathcal{F} \) on \( \mathcal{M}(\Gamma) \) as

\[
\mathcal{F}(\tilde{a}^{ij}) = F(\lambda^i),
\]

where the \((\lambda^i)\) are eigenvalues of the matrix \((a^{ij})\). It has been proven in [3] the monotonicity and concavity of \( F \) now take the form

\[
\mathcal{F}_{ij} = \frac{\partial \mathcal{F}}{\partial a^{ij}} \quad \text{is positive definite}, \tag{2.10}
\]

and

\[
\mathcal{F}_{ij,rs} = \frac{\partial^2 \mathcal{F}}{\partial a^{ij} \partial a^{rs}} \quad \text{is negative semidefinite.} \tag{2.11}
\]

Consider the tensor

\[
h^{ij} = g^{ik} h_{kj} = \frac{1}{uv} \left[ \delta^i_j + \left( -\sigma^{ik} + \frac{\varphi^i \varphi^k}{\nu^2} \right) \varphi_{kj} \right].
\]

Define the symmetric tensor

\[
\tilde{h}_{ij} = \frac{1}{2} (\tilde{\sigma}_{ik} h^{k}_{ij} + \tilde{\sigma}_{jk} h^{k}_{ij}),
\]

where

\[
\tilde{\sigma}_{ij} = \sigma_{ij} + \varphi_i \varphi_j.
\]

Set

\[
\tilde{h}_{ij} := \frac{u}{v} \tilde{h}_{ij} = \nu^{-2} \left( \sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j \right),
\]

then, together with (2.9), we have

\[
\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\mathcal{F}(\tilde{h}_{ij})} - \left[ (\varphi_{ij} + \varphi_i \varphi_j) \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 (\varphi_{it} + \varphi_i \varphi_t) \frac{dx^i}{dt} \right] - \left( \frac{\partial \varphi}{\partial t} \right)^2,
\]

where the nonlinearity \( \mathcal{F} \) only depends on \( D\varphi \) and \( D^2\varphi \).

Now, we do the linearization process. Set

\[
Q(\varphi, D\varphi, D^2\varphi) := \frac{1}{\mathcal{F}(\tilde{h}_{ij})} - \left[ (\varphi_{ij} + \varphi_i \varphi_j) \frac{dx^i}{dt} \frac{dx^j}{dt} + 2 (\varphi_{it} + \varphi_i \varphi_t) \frac{dx^i}{dt} \right] - \left( \frac{\partial \varphi}{\partial t} \right)^2,
\]
then one can obtain

\[ Q_{ij} = \frac{\partial Q}{\partial \phi_{ij}} = -\frac{1}{\mathcal{F}^2(h_{ij})} \frac{\partial \mathcal{F}}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial \phi_{ij}} - \frac{dx^i}{dt} \frac{dx^j}{dt}. \]

Therefore, we have

\[ \frac{\partial^2 \phi}{\partial t^2} = Q_{ij} \phi_{ij} - 2 \frac{dx^i}{dt} \frac{\phi_{it}}{dt} + I(x, \phi_t, \phi_t, \phi), \]

where the last term \( I(x, \phi_t, \phi_t, \phi) \) only depends on \( x, \phi_t, \phi_t, \phi \). The coefficient matrix of terms involving second-order derivatives of \( \phi \) in the above evolution equation is

\[
\begin{pmatrix}
-1 & \frac{dx^1}{dt} & \cdots & \frac{dx^n}{dt} \\
\frac{dx^1}{dt} & \frac{\partial \mathcal{F}}{\partial h_{11}} & \cdots & \frac{\partial \mathcal{F}}{\partial h_{1n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{dx^n}{dt} & \frac{\partial \mathcal{F}}{\partial h_{n1}} & \cdots & \frac{\partial \mathcal{F}}{\partial h_{nn}}
\end{pmatrix}
\]

which, by a suitable linear transformation, becomes

\[
\begin{pmatrix}
-1 & 0 & \cdots & 0 \\
0 & \frac{\partial \mathcal{F}}{\partial h_{11}} & \cdots & \frac{\partial \mathcal{F}}{\partial h_{1n}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{\partial \mathcal{F}}{\partial h_{n1}} & \cdots & \frac{\partial \mathcal{F}}{\partial h_{nn}}
\end{pmatrix}
\]

which, by (2.10) and (2.11), implies that the matrix (2.12) is negative definite. So, the equation is a second-order linear hyperbolic PDE. Our claim follows by the standard theory of second-order linear hyperbolic PDEs.

(3) Although we can also get the short-time existence of the IVP (2.8), in this paper we mainly discuss the IVP (2.6) since if the initial hypersurface \( M_0 \) is more special (e.g., sphere, cylinder), the evolution equation of the flow, which in general is a second-order hyperbolic PDE, degenerates into a second-order ordinary differential equation (ODE for short) and then the convergent situation of the evolving hypersurfaces can be easily known by directly checking the explicit solution to the ODE (for details, see examples shown in Section 3).

3 Examples

In order to possibly understand the convergence of HIMCF (2.6) well, we would like to consider the following two interesting examples in this section.
Example 3.1. Consider a family of spheres in $\mathbb{R}^3$

$$X(x,t) = r(t)(\cos \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha),$$

where $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\beta \in [0, 2\pi]$. By straightforward computation, the induced metric and the second fundamental form are given as follows

$$g_{11} = r^2, \quad g_{22} = r^2 \cos^2 \alpha, \quad g_{12} = g_{21} = 0,$$

and

$$h_{11} = r, \quad h_{22} = r \cos^2 \alpha, \quad h_{12} = h_{21} = 0,$$

which implies the mean curvature is

$$H = g^{ij}h_{ij} = \frac{2}{r}.$$ 

Besides, the outward unit normal vector of each $X(\cdot, t)$ is $\vec{v} = (\cos \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha)$. Therefore, in this setting, the HIMCF (2.6) becomes

$$\left\{ \begin{array}{l}
r_{tt} = \frac{\bar{r}}{2}, \\
r(0) = r_0 > 0, \quad r_t(0) = r_1,
\end{array} \right. \quad (3.1)$$

with $X_1(x) = r_1(\cos \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha)$ for some constant $r_1$. Solving (3.1) directly yields

$$r(t) = \frac{r_0 + \sqrt{2}r_1}{2}e^{\frac{\sqrt{2}t}{2}} + \frac{r_0 - \sqrt{2}r_1}{2}e^{-\frac{\sqrt{2}t}{2}}$$

on $[0, T_{\text{max}})$ for some $0 < T_{\text{max}} \leq \infty$. It is not difficult to know that

- if $r_0 + \sqrt{2}r_1 > 0$, then $T_{\text{max}} = \infty$ (i.e., the flow exists for all the time). Moreover, if furthermore, $r_0 - \sqrt{2}r_1 \leq 0$, the evolving spheres expand exponentially to the infinity, and if furthermore, $r_0 - \sqrt{2}r_1 > 0$, then the evolving spheres converge first for a while and then expand exponentially to the infinity;

- if $r_0 + \sqrt{2}r_1 = 0$, then $r(t) = \sqrt{2}r_0 e^{-\frac{\sqrt{2}t}{2}}$, which implies $T_{\text{max}} = \infty$ and the evolving spheres converge to a single point as time tends to infinity;

- if $r_0 + \sqrt{2}r_1 < 0$, then $T_{\text{max}} = \frac{\sqrt{2}}{2} \ln \left( \frac{\sqrt{2}r_1 - r_0}{\sqrt{2}r_1 + r_0} \right)$ and the evolving spheres converge to a single point as $t \to T_{\text{max}}$.

From the above argument, at least we can get an impression that the convergent situation of the HIMCF (2.6) is much complicated and has close relation with the initial data.

Based on Example 3.1, one can consider the following high-dimensional case.
Example 3.2. Consider a family of spheres in $\mathbb{R}^{n+1}$ $(n \geq 2)$

$$X(x,t) = r(t)(\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \cdots, \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{n-1} \cos \theta_n, \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{n-1} \sin \theta_n),$$

where $\theta_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\theta_\beta \in [0, 2\pi]$ for $\beta = 2, 3, \cdots, n$. By straightforward computation, the induced metric and the second fundamental form are given as follows

$$g_{11} = r^2 \cos^2 \alpha, \quad g_{22} = g_{33} = \cdots = g_{nn} = r^2, \quad g_{ij} = g_{ji} = 0 \text{ for } i \neq j,$$

and

$$h_{11} = r \cos^2 \alpha, \quad h_{22} = h_{33} = \cdots = h_{nn} = r, \quad h_{ij} = h_{ji} = 0 \text{ for } i \neq j,$$

which implies the mean curvature is

$$H = g^{ij} h_{ij} = \frac{n}{r}.$$

Similar to Example 3.1, in this setting, the HIMCF (2.6) becomes

$$\begin{cases} r_{tt} = \frac{r}{n}, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases} \tag{3.2}$$

for some constant $r_1$. Solving (3.2) directly yields

$$r(t) = \frac{r_0 + \sqrt{n}r_1}{2} e^{\frac{\sqrt{n}t}{2}} + \frac{r_0 - \sqrt{n}r_1}{2} e^{-\frac{\sqrt{n}t}{2}},$$

on $[0, T_{\text{max}})$ for some $0 < T_{\text{max}} \leq \infty$, and then we have

- if $r_0 + \sqrt{n}r_1 > 0$, then $T_{\text{max}} = \infty$ (i.e., the flow exists for all the time). Moreover, if furthermore, $r_0 - \sqrt{n}r_1 \leq 0$, the evolving spheres expand exponentially to the infinity, and if furthermore, $r_0 - \sqrt{n}r_1 > 0$, then the evolving spheres converge first for a while and then expand exponentially to the infinity;

- if $r_0 + \sqrt{n}r_1 = 0$, then $r(t) = \sqrt{n}r_0 e^{-\frac{\sqrt{n}t}{2}}$, which implies $T_{\text{max}} = \infty$ and the evolving spheres converge to a single point as time tends to infinity;

- if $r_0 + \sqrt{n}r_1 < 0$, then $T_{\text{max}} = \frac{\sqrt{n}}{n} \ln \left(\frac{\sqrt{n}r_1 - r_0}{\sqrt{n}r_1 + r_0}\right)$ and the evolving spheres converge to a single point as $t \to T_{\text{max}}$.

Example 3.3. Now, we would like to consider cylinder solution for the HIMCF (2.6) in $\mathbb{R}^3$ which has the following form

$$X(x,t) = (r(t) \cos \alpha, r(t) \sin \alpha, \rho),$$

where $\alpha \in [0, 2\pi], \rho \in [0, \rho_0]$ for some $\rho_0 > 0$. Clearly, the induced metric and the second fundamental form can be easily computed as follows

$$g_{11} = r^2, \quad g_{22} = r^2, \quad g_{12} = g_{21} = 0,$$
and

\[ h_{11} = r, \quad h_{22} = 0, \quad h_{12} = h_{21} = 0, \]

which implies the mean curvature is

\[ H = g^{ij} h_{ij} = \frac{1}{r}. \]

Besides, the outward unit normal vector of each \( X(\cdot, t) \) is \( \vec{v} = (\cos \alpha, \sin \alpha, 0) \). Therefore, in this setting, the HIMCF (2.6) becomes

\[
\begin{cases}
    r_{tt} = r, \\
    r(0) = r_0 > 0, \quad r_t(0) = r_1,
\end{cases}
\tag{3.3}
\]

with \( X_1(x) = (r_1 \cos \alpha, r_1 \sin \alpha, \rho) \) for some constant \( r_1 \). Solving (3.3) directly yields

\[ r(t) = \frac{r_0 + r_1}{2} e^t + \frac{r_0 - r_1}{2} e^{-t} \]

on \([0, T_{\text{max}}]\) for some \( 0 < T_{\text{max}} \leq \infty \). It is not difficult to know that

- if \( r_0 + r_1 > 0 \), then \( T_{\text{max}} = \infty \) (i.e., the flow exists for all the time). Moreover, if furthermore, \( r_0 - r_1 \leq 0 \), the evolving cylinders expand exponentially to the infinity, and if furthermore, \( r_0 - r_1 > 0 \), then the evolving cylinders converge first for a while and then expand exponentially to the infinity;

- if \( r_0 + r_1 = 0 \), then \( r(t) = r_0 e^{-t} \), which implies \( T_{\text{max}} = \infty \) and the evolving cylinders converge to a straight line as time tends to infinity;

- if \( r_0 + r_1 < 0 \), then \( T_{\text{max}} = \ln \left( \frac{r_1 - r_0}{r_0 + r_1} \right) \) and the evolving cylinders converge to a straight line as \( t \to T_{\text{max}} \).

Of course, as shown in Example 3.2, one can also consider the high-dimensional case of Example 3.3 i.e., the generalized cylinder solutions to the HIMCF (2.6). However, through a simple calculation, one can easily find that, similar to the sphere case, there is no obvious difference between Example 3.3 and its high-dimensional version.

**Remark 3.4.** If the HIMCF (2.6) is replaced by the forced HIMCF (2.7) in examples shown above, then the convergent situation will be more complicated. For instance, if the replacement has been made in Example 3.1 then (3.1) will become

\[
\begin{cases}
    r_{tt} = \frac{r}{2} + c(t) r, \\
    r(0) = r_0 > 0, \quad r_t(0) = r_1.
\end{cases}
\]

Denote the solution to the above IVP by \( r(t) \). Since \( c(t) \) is bounded continuous, there exist \( c_-, c^+ \) such that \( c^- \leq c(t) \leq c^+ \). Consider the following IVPs

\[
\begin{cases}
    r_{tt} = \frac{r}{2} + c^- r, \\
    r(0) = r_0 > 0, \quad r_t(0) = r_1,
\end{cases}
\]

and

\[
\begin{cases}
    r_{tt} = \frac{r}{2} + c^+ r, \\
    r(0) = r_0 > 0, \quad r_t(0) = r_1,
\end{cases}
\]

On the other hand, the conclusions are more complicated when the forced HIMCF (2.7) is considered. However, through a simple calculation, one can easily find that, since \( c(t) \) is bounded continuous, there exist \( c_-, c^+ \) such that \( c^- \leq c(t) \leq c^+ \). Consider the following IVPs
and
\[
\begin{aligned}
 r_{tt} &= \frac{\gamma}{r} + c^+ r, \\
 r(0) &= r_0 > 0, \quad r_t(0) = r_1,
\end{aligned}
\]
whose solutions are denoted by \( r^-(t) \) and \( r^+(t) \) respectively. Clearly, \( r^-(t) \leq r(t) \leq r^+(t) \). So, the convergent situation of \( r(t) \) deeply depends on that of \( r^-(t), r^+(t) \) which is not simple. This is because that one has to discuss the sign of \( (c^+ - \frac{1}{r}) \), \( (c^+ + \frac{1}{r}) \), which leads to the fact that the convergent situation of \( r(t) \) here will be more complicated that of the one described in Example 3.1.

### 4 Evolution equations of some geometric quantities

Form the evolution equation for the HIMCF (2.6), we can derive evolution equations for some geometric quantities of the hypersurface \( X(\cdot, t) \), and these equations will play an important role in the future study on the HIMCF.

**Lemma 4.1.** Under the HIMCF (2.6), the following identities hold
\[
\Delta h_{ij} = \nabla_i \nabla_j H + H h_{ig} h_{mj} - |A|^2 h_{ij},
\]
\[
\Delta |A|^2 = 2 g^{ik} g^{jl} h_{kl} \nabla_i \nabla_j H + 2 |\nabla A|^2 + 2 H r(A^3) - 2 |A|^4,
\]
where
\[
|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}, \quad tr(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}.
\]

The proof of Lemma 4.1 can be found in Zhu [20].

**Lemma 4.2.** Under the HIMCF (2.6), it holds that
\[
\frac{\partial^2 g_{ij}}{\partial t^2} = 2 H^{-1} h_{ij} + 2 e^2 \phi (\phi_{it} + \phi_i \phi_t)(\phi_{jt} + \phi_j \phi_t),
\]
where \( \phi \) is defined as in Section 2.

**Proof.** Denote by \( \langle , \rangle \) the standard Euclidean metric in \( \mathbb{R}^{n+1} \) in this section. By a direct computation, we have
\[
\frac{\partial^2 g_{ij}}{\partial t^2} = \frac{\partial^2}{\partial t^2} \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle = \frac{\partial}{\partial t} \left( \left\langle \frac{\partial^2 X}{\partial x^i \partial t}, \frac{\partial X}{\partial x^j} \right\rangle + \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial^2 X}{\partial x^j \partial t} \right\rangle \right)
= \left\langle \frac{\partial^3 X}{\partial t^2 \partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle + \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \right\rangle + \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle + \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle
= 2 \left\langle \frac{\partial^3 X}{\partial t^2 \partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle + 2 \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \right\rangle
= 2 H^{-1} \left\langle h_{ik} g^{kl} \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle + 2 \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle
= 2 H^{-1} h_{ij} + 2 \left\langle \frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \right\rangle
= 2 H^{-1} h_{ij} + 2 u_{it} u_{jt} = 2 H^{-1} h_{ij} + 2 e^2 \phi (\phi_{it} + \phi_i \phi_t)(\phi_{jt} + \phi_j \phi_t),
\]
which completes the proof of Lemma 4.2.

**Lemma 4.3.** Under the HIMCF (2.6), we have

\[
\frac{\partial^2 \vec{V}}{\partial t^2} = H^{-2} g^{ij} \frac{\partial H}{\partial x^i} \frac{\partial X}{\partial x^j} - \frac{1}{\nu} g^{ij} e^\Phi (\Phi_{lt} + \Phi_{lt}) \frac{\partial^2 X}{\partial t \partial x^j} + \frac{1}{\nu^2} g^{ij} g^{kl} e^{3\Phi} (\Phi_{lt} + \Phi_{lt}) (\Phi_{lt} \Phi_{lt} + 3 \Phi_{lt} \Phi_{lt} + 2 \Phi_{lt} \Phi_{lt}) \frac{\partial X}{\partial x^k},
\]

where \( \Phi \) and \( \nu \) are given as in Section 2.

**Proof.** First, we have

\[
\frac{\partial \vec{V}}{\partial t} = \left( \frac{\partial \vec{V}}{\partial t}, \frac{\partial X}{\partial x^i} \right) g^{ij} \frac{\partial X}{\partial x^j} = - \left( \vec{V}, \frac{\partial^2 X}{\partial t \partial x^j} \right) g^{ij} \frac{\partial X}{\partial x^j}.
\]

Then, by a direct computation, it follows that

\[
\frac{\partial^2 \vec{V}}{\partial t^2} = - \left( \frac{\partial \vec{V}}{\partial t}, \frac{\partial^2 X}{\partial t \partial x^j} \right) g^{ij} \frac{\partial X}{\partial x^j} - \left( \vec{V}, \frac{\partial^2 X}{\partial t \partial x^j} \right) g^{ij} \frac{\partial X}{\partial x^j} - \left( \vec{V}, \frac{\partial^2 X}{\partial t \partial x^j} \right) g^{ij} \frac{\partial X}{\partial x^j} - \left( \vec{V}, \frac{\partial^2 X}{\partial t \partial x^j} \right) g^{ij} \frac{\partial X}{\partial x^j}
\]

which completes the proof of Lemma 4.3.

**Lemma 4.4.** Under the HIMCF (2.6), we have

\[
\frac{\partial^2 h_{ij}}{\partial t^2} = H^{-2} \Delta h_{ij} - 2 H^{-3} \nabla_i H \nabla_j H + |A|^2 h_{ij} + \frac{1}{\nu} g^{ij} g^{kl} e^{2\Phi} (\Phi_{kl} + \Phi_{kl}) (\Phi_{lt} + \Phi_{lt}) + \frac{2}{\nu} \frac{\partial \Gamma^k_{ij}}{\partial t} e^\Phi (\Phi_{lt} + \Phi_{lt}),
\]

where \( \Phi \) and \( \nu \) are defined as in Section 2.
Proof. Since
\[
\frac{\partial h_{ij}}{\partial t} = - \frac{\partial}{\partial t} \left( \bar{v}, \frac{\partial^2 X}{\partial x^i \partial x^j} \right) = - \left\langle \frac{\partial \bar{v}}{\partial t}, \frac{\partial^2 X}{\partial x^i \partial x^j} \right\rangle - \left\langle \bar{v}, \frac{\partial^3 X}{\partial t \partial x^i \partial x^j} \right\rangle,
\]
we have
\[
\frac{\partial^2 h_{ij}}{\partial t^2} = - \left\langle \frac{\partial^2 \bar{v}}{\partial t^2}, \frac{\partial^2 X}{\partial x^i \partial x^j} \right\rangle - 2 \left\langle \frac{\partial \bar{v}}{\partial t}, \frac{\partial^3 X}{\partial t \partial x^i \partial x^j} \right\rangle - \left\langle \bar{v}, \frac{\partial^4 X}{\partial t^2 \partial x^i \partial x^j} \right\rangle.
\]
By Lemma 4.1
\[
\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij},
\]
we can obtain
\[
\frac{\partial^2 h_{ij}}{\partial t^2} = H^{-2} \Delta h_{ij} - 2 H^{-3} \nabla_i \nabla_j H + |A|^2 h_{ij} + \frac{1}{v^2} g^{kl} h_{ij}u_ku_l + \frac{2}{v} \frac{\partial \Gamma^k_{ij}}{\partial t} u_k + \frac{2}{v} \frac{\partial \Gamma^k_{ij}}{\partial t} e^\varphi (\varphi_{kl} + \varphi_k \varphi_l)
\]
which completes the proof of Lemma 4.4.
Lemma 4.5. Under the HIMCF \cite{2,6}, we have

\begin{align*}
\frac{\partial^2 H}{\partial t^2} &= H^{-2} \Delta H - 2H^{-3}|\nabla H|^2 - H^{-1}|A|^2 + 2g^i_k g^j_p g^l_q h_{ij} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - 2g^i_k g^j_p g^l_q \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} \\
&\quad - \left( 2g^i_k g^j_p h_{ij} - \frac{1}{v^2} H g^k_l \right) e^{2\varphi} (\varphi_k + \varphi_k \varphi_l) (\varphi_l + \varphi_l \varphi_k) + \frac{2}{v} g^{i_j} \frac{\partial \Gamma^i_k}{\partial t} e^{\varphi} (\varphi_k + \varphi_k \varphi_l),
\end{align*}

where \( \varphi \) and \( v \) are defined as in Section 2.

Proof. Noting that \( g^{lm} g_{ml} = \delta^i_j \), we can get \( \frac{\partial g^{ij}}{\partial t} = -g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \), which implies

\begin{align*}
\frac{\partial^2 g^{ij}}{\partial t^2} &= -\frac{\partial g^{ik}}{\partial t} g^{jl} \frac{\partial g_{kl}}{\partial t} - g^{ik} \frac{\partial g^{jl}}{\partial t} \frac{\partial g_{kl}}{\partial t} - g^{ik} g^{jl} \frac{\partial^2 g_{kl}}{\partial t^2} \\
&= 2g^{ik} g^{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - g^{ik} g^{jl} \frac{\partial^2 g_{kl}}{\partial t^2}.
\end{align*}

By a direct calculation, we have

\begin{align*}
\frac{\partial^2 H}{\partial t^2} &= \frac{\partial^2}{\partial t^2} (g^{ij} h_{ij}) \\
&= \frac{\partial^2 g^{ij}}{\partial t^2} h_{ij} + 2 \frac{\partial g^{ij}}{\partial t} \frac{\partial h_{ij}}{\partial t} + g^{ij} \frac{\partial^2 h_{ij}}{\partial t^2} \\
&= \left( 2g^{ik} g^{jl} g^{pq} \frac{\partial g_{kl}}{\partial t} - g^{ik} g^{jl} \frac{\partial^2 g_{kl}}{\partial t^2} \right) h_{ij} - 2g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} + g^{ij} \frac{\partial^2 h_{ij}}{\partial t^2} \\
&= 2g^{ik} g^{jl} g^{pq} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} - g^{ik} g^{jl} h_{ij} \left( 2H^{-1} h_{kl} + 2 \left( \frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l} \right) \right) \\
&\quad + g^{ij} \left( H^{-2} \Delta H - 2H^{-3} \nabla_i \nabla_j H - 2H^{-3} \nabla_i \nabla_j H + H^{-1} h_{kl} g^{kl} h_{ij} \\
&\quad + g^{kl} h_{ij} \left. \left( \nabla \cdot \nabla \right) \left( \frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l} \right) + 2 \frac{\partial \Gamma^i_k}{\partial t} \left( \nabla \cdot \nabla \right) \left( \frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l} \right) \right) \right) \\
&= H^{-2} \Delta H - 2H^{-3} |\nabla H|^2 - H^{-1} |A|^2 - 2g^{ik} g^{jl} h_{ij} \left( \frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l} \right) + \\
&\quad + 2g^{ik} g^{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - 2g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} \\
&\quad + 2g^{ik} g^{jl} g^{pq} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - 2g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} \\
&= H^{-2} \Delta H - 2H^{-3} |\nabla H|^2 - H^{-1} |A|^2 + 2g^{ik} g^{jl} g^{pq} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{kl}}{\partial t} - 2g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial h_{ij}}{\partial t} \\
&\quad - \left( 2g^{ik} g^{jl} h_{ij} - \frac{1}{v^2} H g^{kl} \right) e^{2\varphi} (\varphi_k + \varphi_k \varphi_l) (\varphi_l + \varphi_l \varphi_k) + \frac{2}{v} g^{i_j} \frac{\partial \Gamma^i_k}{\partial t} e^{\varphi} (\varphi_k + \varphi_k \varphi_l),
\end{align*}

which completes the proof of Lemma 4.5 \( \square \).
Lemma 4.6. Under the HIMCF (2.6), we have

\[
\frac{\partial^2}{\partial t^2} |A|^2 = H^{-2}\Delta (|A|^2) - 2H^{-2}\nabla |A|^2 + 2H^{-2}|A|^4 - 4H^{-2}\nabla H^2 + 4H^{-1}tr(A^3)
\]

\[
+ 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} \frac{\partial h_{jl}}{\partial t} - 8g^{im} g^{jn} g^{kl} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial h_{ik}}{\partial t}
\]

\[
+ 2g^{im} h_{ik} h_{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{mn}}{\partial t} (2g^{ip} g^{nq} g^{kl} + g^{jn} g^{kp} g^{lq})
\]

\[
+ \frac{2}{v^2} |A|^2 g^{pq} e^{2\Phi} (\Phi_{pt} + \Phi_p \Phi_t) (\Phi_{qt} + \Phi_q \Phi_t) + \frac{4}{v} g^{ij} g^{kl} h_{jl} \frac{\partial \Gamma^p_{ik}}{\partial t} e^{\Phi} (\Phi_{pt} + \Phi_p \Phi_t)
\]

\[- 4g^{im} g^{jn} g^{kl} h_{ik} h_{jl} e^{2\Phi} (\Phi_{mt} + \Phi_m \Phi_t) (\Phi_{nt} + \Phi_n \Phi_t),
\]

where \( \Phi \) is defined as in Section 2.

Proof. By a direct calculation, we have

\[
\frac{\partial^2}{\partial t^2} |A|^2 = 2g^{kl} h_{ik} h_{jl} \frac{\partial^2 g^{ij}}{\partial t^2} + 2\frac{\partial g^{ij}}{\partial t} \frac{\partial g^{kl}}{\partial t} h_{ik} h_{jl} + 8g^{kl} h_{jl} \frac{\partial g^{ij}}{\partial t} \frac{\partial h_{ik}}{\partial t}
\]

\[
+ 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} \frac{\partial h_{jl}}{\partial t} + 2g^{ij} g^{kl} \frac{\partial^2 h_{ik}}{\partial t^2}
\]

\[
= 2g^{kl} h_{ik} h_{jl} \left( 2g^{im} g^{jp} g^{nq} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{mn}}{\partial t} - g^{im} g^{jn} \frac{\partial^2 g_{mn}}{\partial t^2} \right) +
\]

\[
2g^{im} g^{jn} g^{kp} g^{lq} h_{ik} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial g_{pq}}{\partial t} - 8g^{im} g^{jn} g^{kl} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial h_{ik}}{\partial t} + 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} \frac{\partial h_{jl}}{\partial t}
\]

\[
+ 2g^{ij} g^{kl} h_{jl} \left( H^{-2} \nabla_i \nabla_k H - 2H^{-3} \nabla_i H \nabla_j H + H^{-1} h_{ip} g^{pq} h_{qk} \right)
\]

\[
+ g^{pq} h_{ij} \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x_p} \right) \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x^q} \right) + 2 \frac{\partial \Gamma^p_{ij}}{\partial t} \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x^q} \right)
\]

\[
= 4g^{im} g^{jp} g^{nq} g^{kl} h_{ik} h_{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{mn}}{\partial t} - 2g^{im} g^{jn} g^{kl} h_{ik} h_{jl} \times
\]

\[
\left( 2H^{-1} h_{mn} + 2 \left( \frac{\partial^2 X}{\partial t \partial x^m} \frac{\partial^2 X}{\partial t \partial x^n} \right) \right)
\]

\[
+ 2g^{im} g^{jn} g^{kp} g^{lq} h_{ik} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial g_{pq}}{\partial t} - 8g^{im} g^{jn} g^{kl} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial h_{ik}}{\partial t} + 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} \frac{\partial h_{jl}}{\partial t}
\]

\[
+ 2H^{-2} g^{ij} g^{kl} h_{jl} \nabla_i \nabla_k H + 2H^{-1} tr(A^3) - 4H^{-3} g^{kl} h_{kl} |\nabla H|^2
\]

\[
+ 2g^{pq} |A|^2 \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x_p} \right) \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x^q} \right) + 4g^{ij} g^{kl} h_{jl} \frac{\partial \Gamma^p_{ik}}{\partial t} \left( \vec{\nabla}, \frac{\partial^2 X}{\partial t \partial x^q} \right).
\]

Noting, by Lemma 4.1

\[
\Delta |A|^2 = 2g^{jk} g^{ij} h_{kl} \nabla_i \nabla_j H + 2|\nabla A|^2 + 2H tr(A^3) - 2|A|^4,
\]
then

\[
\frac{\partial^2}{\partial t^2}|A|^2 = \nabla^2 A - 2H^{-2}|\nabla A|^2 + 2H^{-2}|A|^4 - 4H^{-2}|\nabla H|^2 - 4H^{-1} tr(A^3)
\]

\[
+ \frac{2}{v^2}|A|^2 g^{pq} u_p u_q + 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} \frac{\partial h_{jl}}{\partial t} - 8g^{im} g^{jn} g^{kl} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial h_{ik}}{\partial t}
\]

\[
- 4g^{im} g^{jn} s^{kl} h_{ik} h_{jl} u_m u_n + \frac{4}{v} g^{ij} g^{kl} h_{jl} \frac{\partial \Gamma^p_{ik}}{\partial t} u_p
\]

\[
+ 2g^{im} h_{ik} h_{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{mn}}{\partial t} (2g^{ip} g^{nq} g^{kl} + g^{jn} g^{kp} g^{lq})
\]

\[
= \nabla^2 (|A|^2) - 2H^{-2}|\nabla A|^2 + 2H^{-2}|A|^4 - 4H^{-2}|\nabla H|^2 - 4H^{-1} tr(A^3)
\]

\[
+ 2g^{ij} g^{kl} \frac{\partial h_{ik}}{\partial t} h_{jl} - 8g^{im} g^{jn} g^{kl} h_{jl} \frac{\partial g_{mn}}{\partial t} \frac{\partial h_{ik}}{\partial t}
\]

\[
+ 2g^{im} h_{ik} h_{jl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{mn}}{\partial t} (2g^{ip} g^{nq} g^{kl} + g^{jn} g^{kp} g^{lq})
\]

\[
+ \frac{2}{v^2}|A|^2 g^{pq} e^{2\phi} (\phi_p + \phi_q \phi_t) (\phi_q + \phi_p \phi_t) + \frac{4}{v} g^{ij} g^{kl} h_{jl} \frac{\partial \Gamma^p_{ik}}{\partial t} e^\phi (\phi_p + \phi_q \phi_t)
\]

\[
- 4g^{im} g^{jn} s^{kl} h_{ik} h_{jl} e^{2\phi} (\phi_m + \phi_q \phi_t) (\phi_m + \phi_n \phi_t),
\]

which completes the proof of Lemma 4.6.

As we can see from complicated evolution equations in this section, it is difficult to get gradient estimates and higher-order estimates for the mean curvature and the second fundamental forms, which leads to the result that so far we cannot say something about the convergence of the HIMCF (2.6) and also the hyperbolic flows (2.7), (2.8). However, for the lower dimensional case (i.e., the HIMCF in the plane \( \mathbb{R}^2 \)), we can get the expanding and convergent conclusions, which will be shown clearly in the following section.

5 HIMCF in the plane \( \mathbb{R}^2 \)

5.1 The short-time existence

Consider a family of closed plane curves \( F : S^1 \times [0,T) \rightarrow \mathbb{R}^2 \) which satisfy the following IVP

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} F(u,t) &= k^{-1}(u,t) \tilde{\nu}(u,t) - \nabla \rho(u,t), \quad \forall u \in S^1, \ t \in [0,T) \\
\frac{\partial F}{\partial t} (\cdot,0) &= f(u) \tilde{\nu}_0,
\end{align*}
\]

\[
F(\cdot,0) = F_0,
\]

where \( k(u,t) \) and \( \tilde{\nu} \) are the curvature and the unit outward normal vector of the plane curve \( F(u,t) \) respectively, \( f(u) \in C^\infty(S^1) \) is the initial normal velocity, and \( \tilde{\nu}_0 \) is the unit outward normal vector of the smooth strictly convex plane curve \( F_0(u) \). Besides, \( \nabla \rho \) is defined by

\[
\nabla \rho := \left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle \tilde{\nu}(u,t),
\]
where, by abuse of a notation, $\langle , \rangle$ denotes the standard Euclidean metric in $\mathbb{R}^2$ also, and $\vec{T}$, $s$ are the unit tangent vector of $F(u,t)$ and the arc-length parameter respectively.

Now, we would like to show that the HIMCF (5.1) is a normal flow. However, before that, we need the following definition which has been mentioned in [12, 13].

**Definition 5.1.** A curve $F : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ evolves normally if and only if its tangential velocity vanishes.

**Lemma 5.2.** The hyperbolic curvature flow (5.1) is a normal flow.

**Proof.** By a direct computation, we have

$$\frac{d}{dt} \left< \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u} \right> = \left< \frac{\partial^2 F}{\partial t^2}, \frac{\partial F}{\partial u} \right> + \left< \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial u} \right> = \left< -\nabla \rho, \frac{\partial F}{\partial u} \right> + \left< \frac{\partial F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right> \cdot \left< \frac{\partial F}{\partial s}, \frac{\partial F}{\partial u} \right> + \left< \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial u} \right> = 0,$$

which, together with the fact that the initial velocity of the IVP (5.1) is normal, implies the conclusion of Lemma 5.2. \qed

By the IVP (5.1) and Lemma 5.2, it is easy to get the following

$$\begin{cases}
\frac{\partial F}{\partial t}(u,t) = \sigma(u,t)\vec{N}
F(u,0) = F_0(u),
\end{cases}$$

(5.2)

where $\sigma(u,t) = f(u) + \int_0^t k^{-1}(u, \xi) d\xi$. So, we have

$$\frac{\partial \sigma}{\partial t} = k^{-1}(u,t), \quad \frac{\partial \sigma}{\partial s} = \left< \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right>,$$

where $s = s(\cdot, t)$ is the arc-length parameter of curve $F(\cdot, t) : \mathbb{S}^1 \rightarrow \mathbb{R}^2$. By arc-length formula, we have

$$\frac{\partial}{\partial s} = \frac{1}{\left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial y}{\partial u} \right)^2 \frac{\partial}{\partial u}} = \frac{1}{\left| \frac{\partial F}{\partial u} \right|} = \frac{1}{v} \frac{\partial}{\partial u},$$

where $(x,y)$ is the cartesian coordinates, and $v = \sqrt{\frac{\partial x}{\partial u}^2 + \frac{\partial y}{\partial u}^2} = \left| \frac{\partial F}{\partial u} \right|$. For the orthogonal field $\{\vec{N}, \vec{T}\}$ of $\mathbb{R}^2$, by Frenet formula, we have

$$\frac{\partial \vec{T}}{\partial s} = -k\vec{N}, \quad \frac{\partial \vec{N}}{\partial s} = k\vec{T}.$$  

(5.3)
Denote by $\theta$ the unit inner normal angle for a convex closed curve $F : \mathbb{S}^1 \to \mathbb{R}^2$. Then, we have
\[ \vec{v} = (\cos \theta, \sin \theta), \quad \vec{T} = (-\sin \theta, \cos \theta). \]
Together with (5.3), we have
\[ \frac{\partial \vec{T}}{\partial s} = \frac{\partial \vec{v} \partial \theta}{\partial \theta} = -\vec{v} \frac{\partial \theta}{\partial s} = -k\vec{v}, \]
which implies $\frac{\partial \theta}{\partial s} = k$. Moreover,
\[ \frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{\partial \vec{T}}{\partial t} = \frac{\partial \vec{T}}{\partial \theta} \frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial t} \vec{v}. \quad (5.4) \]

**Lemma 5.3.** The derivative of $v$ with respect to $t$ is $\frac{\partial v}{\partial t} = k\sigma v$.

**Proof.** By a direct computation, we have
\[
\frac{\partial}{\partial t} (v^2) = \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial u} \right) = 2 \left( \frac{\partial F}{\partial u} \cdot \frac{\partial^2 F}{\partial t \partial u} \right) = 2 \left( \frac{\partial F}{\partial u} \right) \cdot \left( \frac{\partial}{\partial u} \left( \sigma \vec{v} \right) \right)
\]
\[ = 2 \left( v \vec{T}, \sigma \frac{\partial v}{\partial u} \right) = 2 \left( v \vec{T}, \sigma \frac{\partial v}{\partial s} \frac{\partial s}{\partial u} \right)
\]
\[ = 2 \left( v \vec{T}, \sigma k \vec{T} \right)
\]
\[ = 2v^2k\sigma,
\]
which implies the conclusion of Lemma 5.3. \qed

By Lemma 5.3, we can obtain
\[
\frac{\partial^2}{\partial t \partial s} = \frac{\partial}{\partial t} \left( \frac{1}{v} \frac{\partial v}{\partial u} \right) = -\frac{1}{v^2} \frac{\partial^2}{\partial t \partial u} + \frac{1}{v} \frac{\partial v}{\partial u} \frac{\partial}{\partial t} = -k\sigma \frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t}.
\]

Therefore, together with (5.2), we have
\[
\frac{\partial \vec{T}}{\partial t} = \frac{\partial^2 F}{\partial t \partial s}
\]
\[ = -k\sigma \frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s \partial t}
\]
\[ = -k\sigma \vec{T} + \frac{\partial}{\partial s} \left( \sigma \vec{v} \right)
\]
\[ = \frac{\partial \sigma}{\partial s} \vec{v},
\]
which, combining with (5.4), yields
\[
\frac{\partial \sigma}{\partial s} = -\frac{\partial \theta}{\partial t}, \quad \frac{\partial \vec{v}}{\partial t} = -\frac{\partial \sigma}{\partial s} \vec{T}.
\]
Assume $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ is a family of convex curves satisfying the flow (5.1), and we can use the normal angle to parameterize the evolving curve $F(\cdot, t)$, which will give the notion of support function used to get the short-time existence of the flow. Set

$$
\bar{F}(\theta, \tau) = F(u(\theta, \tau), t(\theta, \tau)),
$$

where $t(\theta, \tau) = \tau$. By the chain rule, we have

$$
0 = \frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t},
$$

and then

$$
\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial u} \frac{\partial u}{\partial \tau} = -k \nu \frac{\partial u}{\partial \tau}.
$$

Therefore, a direct calculation yields

$$
\frac{\partial \bar{T}}{\partial \tau} = \frac{\partial \bar{T}}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \bar{T}}{\partial t} = \frac{\partial \bar{T}}{\partial s} \frac{\partial u}{\partial \tau} - \frac{\partial \theta}{\partial \nu} \nu = -\left(k \nu \frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t}\right) \bar{\nu} = 0,
$$

and

$$
\frac{\partial \bar{\nu}}{\partial \tau} = \frac{\partial \bar{\nu}}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \bar{\nu}}{\partial t} = \frac{\partial \bar{\nu}}{\partial s} \frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial \bar{T}} \bar{T} = \left(k \nu \frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t}\right) \bar{T} = 0,
$$

which implies $\bar{\nu}$ and $\bar{T}$ are independent of the parameter $\tau$.

Define the support function of the evolving curve $\bar{F}(\theta, \tau) = (x(\theta, \tau), y(\theta, \tau))$ as follows

$$
S(\theta, \tau) = \langle \bar{F}(\theta, \tau), \bar{\nu} \rangle = x(\theta, \tau) \cos \theta + y(\theta, \tau) \sin \theta.
$$

Then we have

$$
S_\theta(\theta, \tau) = \langle \bar{F}(\theta, \tau), \bar{T} \rangle = -x(\theta, \tau) \sin \theta + y(\theta, \tau) \cos \theta,
$$

and

$$
\left\{ \begin{array}{l}
x(\theta, \tau) = S \cos \theta - S_\theta \sin \theta, \\
y(\theta, \tau) = S \sin \theta + S_\theta \cos \theta.
\end{array} \right.
$$
By a direct computation, we have
\[
S_{\theta \theta} + S = \langle \tilde{F}_\theta(\theta, \tau), \vec{T} \rangle + \langle \tilde{F}(\theta, \tau), -\vec{v} \rangle + \langle \tilde{F}(\theta, \tau), \vec{v} \rangle \\
= \langle \tilde{F}_\theta(\theta, \tau), \vec{T} \rangle \\
= \left\langle \frac{\partial F}{\partial u} \frac{\partial s}{\partial \theta} \frac{\partial \theta}{\partial \tau}, \vec{T} \right\rangle \\
= \frac{1}{k}.
\]
The above expression makes sense, since the evolving curve is strictly convex.

On the other hand, since \(\vec{v}\) and \(\vec{T}\) are independent of the parameter \(\tau\), together with (5.2) and the definition of the support function \(S\), we can get
\[
S_\tau = \left\langle \frac{\partial \tilde{F}}{\partial \tau}, \vec{v} \right\rangle = \left\langle \frac{\partial F}{\partial u} \frac{\partial s}{\partial \tau} + \frac{\partial F}{\partial \tau}, \vec{v} \right\rangle = \left\langle \frac{\partial F}{\partial t}, \vec{v} \right\rangle = \tilde{\sigma}(\theta, \tau),
\]
and
\[
S_{\tau \tau} = \left\langle \frac{\partial^2 \tilde{F}}{\partial \tau^2}, \vec{v} \right\rangle \\
= \left\langle \frac{\partial F}{\partial u} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 F}{\partial u^2} \left( \frac{\partial u}{\partial \tau} \right)^2 + 2 \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} \frac{\partial F}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, \vec{v} \right\rangle \\
= \left\langle \frac{\partial^2 F}{\partial u^2} \frac{\partial u}{\partial \tau} \frac{\partial \tau}{\partial \tau} + \frac{\partial^2 F}{\partial u \partial t} \frac{\partial \tau}{\partial \tau} \frac{\partial \tau}{\partial \tau}, \vec{v} \right\rangle + \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, \vec{v} \right\rangle \\
= \frac{\partial u}{\partial \tau} \left\langle \frac{\partial F}{\partial u}, \vec{v} \right\rangle + \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, \vec{v} \right\rangle \\
= \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, \vec{v} \right\rangle + k^{-1}.
\]
Since \(F : S^1 \times [0, T) \rightarrow \mathbb{R}^2\) is a normal flow (see Lemma 5.2), which implies
\[
\left\langle \frac{\partial F}{\partial t}, \vec{T} \right\rangle (u, t) = 0,
\]
for all \(t \in [0, T)\), we have
\[
S_{\tau \theta} = \frac{\partial}{\partial \tau} \left\langle \tilde{F}, \vec{T} \right\rangle = \left\langle \frac{\partial F}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial \tau}, \vec{T} \right\rangle = \nu \frac{\partial u}{\partial \tau}
\]
and
\[
S_{\theta \tau} = \frac{\partial}{\partial \theta} \left\langle \frac{\partial F}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u \partial \theta} \frac{\partial u}{\partial \tau}, \vec{v} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \theta} + \frac{\partial^2 F}{\partial u \partial \tau} \frac{\partial u}{\partial \theta}, \vec{v} \right\rangle = \frac{1}{k \nu} \left\langle \frac{\partial^2 F}{\partial u \partial t}, \vec{v} \right\rangle
\]
by straightforward computation. Hence, \( S(\theta, \tau) \) satisfies

\[
S_{\tau \tau} = \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau}, \vec{v} \right\rangle + k^{-1} = k v S_{\theta \tau} \frac{\partial u}{\partial \tau} + k^{-1} = k S_{\theta \tau}^2 + k^{-1},
\]

which is equivalent to

\[
S_{\tau \tau} = \frac{S_{\theta \tau}^2}{S_{\theta \theta} + S} + (S_{\theta \theta} + S), \quad \forall (\theta, \tau) \in S^1 \times [0, T).
\]

Together with (5.1), we know that

\[
\begin{aligned}
SS_{\tau \tau} + S_{\tau \tau} S_{\theta \theta} - S_{\theta \tau}^2 - (S_{\theta \theta} + S)^2 &= 0, \\
S(\theta,0) &= h(\theta), \\
S_{\tau}(\theta,0) &= \bar{f}(\theta),
\end{aligned}
\]

(5.5)

where \( h(\theta) \) and \( \bar{f}(\theta) \) are the support function of the initial curve \( F_0(u(\theta)) \) and the initial velocity of this initial curve respectively.

Similar to the high-dimensional case mentioned in Section 2, here we would like to get the short-time existence of the IVP (5.5) by the linearization method. Let

\[
Q(S_{\theta \theta}, S_{\theta \tau}, S) := \frac{S_{\theta \tau}^2}{S_{\theta \theta} + S} + (S_{\theta \theta} + S),
\]

then we have

\[
S_{\tau \tau} = \frac{\partial Q}{\partial S_{\theta \theta}} S_{\theta \theta} + \frac{\partial Q}{\partial S_{\theta \tau}} S_{\theta \tau} + \frac{\partial Q}{\partial S},
\]

(5.6)

where

\[
\frac{\partial Q}{\partial S_{\theta \theta}} = 1 - \frac{S_{\theta \tau}^2}{(S_{\theta \theta} + S)^2}, \quad \frac{\partial Q}{\partial S_{\theta \tau}} = \frac{2S_{\theta \tau}}{S_{\theta \theta} + S}.
\]

Consider the coefficient matrix of terms in (5.6) involving second-order derivatives of \( S \), and then we have

\[
\begin{pmatrix}
-1 & \frac{S_{\theta \tau}}{S_{\theta \theta} + S} \\
\frac{S_{\theta \tau}}{S_{\theta \theta} + S} & 1 - \frac{S_{\theta \tau}^2}{(S_{\theta \theta} + S)^2}
\end{pmatrix}
\]

which, by a suitable linear transformation, we have

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

So (5.6) is a second-order hyperbolic PDE. By the standard theory of second-order hyperbolic PDEs, we have the following.

**Theorem 5.4.** (Local existences and uniqueness) Assume that \( F_0 \) is a smooth strictly convex closed plane curve. Then there exist a positive \( T_{\text{max}} > 0 \) and a family of strictly convex closed curves \( F(u,t) \) satisfying the IVP (5.1) on \( S^1 \times [0, T_{\text{max}}) \), provided \( f(u) \) is a smooth function on \( S^1 \).
5.2 Expansion and Convergence

As in Section 3, we would like to understand further and then try to get more evolution information about the hyperbolic flow (5.1) through the following interesting example. For simplicity, we replace $\tau$ by $t$.

**Example 5.5.** Let $F(u,t)$ be a family of round circles in $\mathbb{R}^2$ with the radius $r(t)$ centered at the origin, i.e.,

$$F(u,t) = r(t)(\cos \theta, \sin \theta).$$

Then the support function $S$ and the curvature $k$ are given by

$$S(\theta,t) = r(t), \quad k(\theta,t) = \frac{1}{r(t)},$$

which implies IVP (5.1) becomes

$$
\begin{align*}
  r_{tt} &= r(t), \\
  r(0) &= r_0 > 0, \\
  r_t(0) &= r_1.
\end{align*}
$$

Solving (5.7) directly yields

$$r(t) = \frac{r_0 + r_1}{2} e^t + \frac{r_0 - r_1}{2} e^{-t}$$
on $[0, T_{\text{max}})$ for some $0 < T_{\text{max}} \leq \infty$. As Example 3.1 we know that

- if $r_0 + r_1 > 0$, then $T_{\text{max}} = \infty$ (i.e., the flow exists for all the time). Moreover, if furthermore, $r_0 - r_1 \leq 0$, the evolving curves expand exponentially to the infinity, and if furthermore, $r_0 - r_1 > 0$, then the evolving curves converge first for a while and then expand exponentially to the infinity;

- if $r_0 + r_1 = 0$, then $r(t) = r_0 e^{-t}$, which implies $T_{\text{max}} = \infty$ and the evolving curves converge to a single point as time tends to infinity;

- if $r_0 + r_1 < 0$, then $T_{\text{max}} = \frac{1}{2} \ln \left( \frac{r_1 - r_0}{r_1 + r_0} \right)$ and the evolving curves converge to a single point as $t \to T_{\text{max}}$.

**Remark 5.6.** From the above example, we know that although the initial curve is so special (i.e., circles), the evolution of the flow (5.1) is complicated which deeply depends on the initial values of the flow. It seems like it is very difficult to accurately describe the evolution of the HIMCF (5.1) as time tends to the maximal existence time (i.e., as $t \to T_{\text{max}}$). Fortunately, using the containment principle we have derived (see Proposition 5.8 below), we can overcome this difficulty.

In order to get the containment principle, we need to use the maximum principle for a strip (see Lemma 5.7 below) which has been shown in [16]. However, in order to state the conclusion of Lemma 5.7 clearly, we need to introduce some preliminaries first, which has been mentioned in [12]. Consider the general second-order operator $L$

$$L[\omega] := a\omega_{\theta\theta} + b\omega_{\theta t} + c\omega_t + d\omega_{\theta} + e\omega$$

(5.8)
where \( a, b, c \) are twice continuously differentiable and \( d, e \) are continuously differentiable of \( \theta \) and \( t \). If \( b^2 - ac > 0 \) at a point \((\theta, t)\), then the operator \( L \) is said to be hyperbolic at this point. It is hyperbolic in a domain \( D \) if it is hyperbolic at each point of \( D \), and uniformly hyperbolic in a domain \( D \) if there exists a constant \( \mu \) such that \( b^2 - ac \geq \mu > 0 \) in \( D \).

Assume that \( \omega \) and the conormal derivative

\[
\frac{\partial \omega}{\partial \nu} = -b \frac{\partial \omega}{\partial \theta} - c \frac{\partial \omega}{\partial t}
\]

are given at \( t = 0 \). The adjoint operator \( L^* \) associated with \( L \) can be defined by

\[
L^*[\omega] \triangleq (a\omega)_{\theta\theta} + 2(b\omega)_{\theta t} + (c\omega)_{tt} - (d\omega)_\theta - (e\omega)_t
\]

\[
= a\omega_{\theta\theta} + 2b\omega_{\theta t} + c\omega_{tt} + (2a\theta + 2bt - d)\omega_\theta + (2b\theta + 2ct - e)\omega_t
\]

\[
+ (a\theta\theta + 2b\theta_t + ctt - d\theta - e_t)\omega.
\]

Set

\[
K_+(\theta, t) := \left( \sqrt{b^2 - ac} \right)_\theta + \frac{b}{c} \left( \sqrt{b^2 - ac} \right)_\theta + \frac{1}{c} (b\theta + ct - e) \sqrt{b^2 - ac}
\]

\[
- \left[ -\frac{1}{2c} (b^2 - ac)_\theta + a\theta + bt - d - \frac{b}{c} (b\theta + ct - e) \right],
\]

and

\[
K_-(\theta, t) := \left( \sqrt{b^2 - ac} \right)_\theta + \frac{b}{c} \left( \sqrt{b^2 - ac} \right)_\theta + \frac{1}{c} (b\theta + ct - e) \sqrt{b^2 - ac}
\]

\[
- \left[ -\frac{1}{2c} (b^2 - ac)_\theta + a\theta + bt - d - \frac{b}{c} (b\theta + ct - e) \right].
\]

As shown in [12, pp. 502-503], we know that for

\[
l(\theta, t) := 1 + \alpha t - \beta t^2
\]

with \( \alpha, \beta \) sufficiently large such that

\[
\begin{cases}
2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_+ \geq 0 \\
2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_- \geq 0 \\
-2c\beta + (2\beta t + 2ct - e)(\alpha - 2\beta t) + (a\theta\theta + 2b\theta_t + ctt - d\theta - e_t + g)(1 + \alpha t + \beta t^2) \geq 0
\end{cases}
\]

(5.10)

and \( l(\theta, t) > 0 \) on a sufficiently small strip \( 0 \leq t \leq t_0 \), the hyperbolic operator defined by (5.8) satisfies

\[
\begin{cases}
2\sqrt{b^2 - ac} \left[ l_t - \frac{1}{c} \left( \sqrt{b^2 - ac} - b \right) l_\theta \right] + lK_+ \geq 0 \\
2\sqrt{b^2 - ac} \left[ l_t + \frac{1}{c} \left( \sqrt{b^2 - ac} - b \right) l_\theta \right] + lK_- \geq 0 \\
(L^* + g)[w] \geq 0
\end{cases}
\]
on the same strip $0 \leq t \leq t_0$. It is easy to check that with $l$ defined as (5.2), the condition on the conormal derivative

$$\frac{\partial \omega}{\partial \nu} + (b_\theta + c_t - e + c\alpha) \omega \leq 0,$$

becomes at $t = 0$. Besides, if we select a constant $M$ so large that

$$M \geq -(b_\theta + c_t - e + c\alpha), \quad \text{on } \Gamma_0,$$

then the following maximum principle for the strip adjacent to the $\theta$-axis can be obtained.

**Lemma 5.7.** Suppose that the coefficients of the operator $L$ given by (5.8) are bounded and have bounded first and second derivatives. Let $D$ be an admissible domain. If $t_0$ and $M$ are selected in accordance with (5.10) and (5.11), then any function $\omega$ which satisfies

$$\begin{cases}
(L + g)[\omega] \geq 0 & \text{in } D, \\
\frac{\partial \omega}{\partial \nu} - M \omega \leq 0 & \text{on } \Gamma_0, \\
\omega \leq 0 & \text{on } \Gamma_0,
\end{cases}$$

also satisfies $\omega \leq 0$ in the part of $D$ which lies in the strip $0 \leq t \leq t_0$. The constants $t_0$ and $M$ depend only on lower bounds for $-c$ and $\sqrt{b^2 - ac}$ and on bounds for the coefficients of $L$ and their derivatives.

**Proposition 5.8.** (Containment principle) Let $F_1$ and $F_2 : S^1 \times [0, T) \to \mathbb{R}^2$ be two convex solutions of (5.7). Suppose that $F_2(u, 0)$ lies in the domain enclosed by $F_1(u, 0)$, and $f_2(u) \leq f_1(u)$. Then $F_2(u, t)$ is contained in the domain enclosed by $F_1(u, t)$ for all $t \in [0, T)$.

**Proof.** Let $S_1(\theta, t)$ and $S_2(\theta, t)$ be the support functions of $F_1(u, t)$ and $F_2(u, t)$, respectively. Then $S_1(\theta, t)$ and $S_2(\theta, t)$ satisfy the same equation (5.5) with $S_2(\theta, 0) \leq S_1(\theta, 0)$ and $S_2(\theta, 0) \leq S_1(\theta, 0)$.

Let

$$\omega(\theta, t) := S_2(\theta, t) - S_1(\theta, t).$$

Then we have

$$\begin{align*}
\omega_{tt} &= S_{2tt} - S_{1tt} = \frac{S_{2\theta t} + k_2^2}{S_2 + S_{2\theta \theta}} - \frac{S_{1\theta t} + k_1^2}{S_1 + S_{1\theta \theta}} \\
&= k_1k_2 \left( \frac{1}{k_1k_2} - S_{1\theta t} S_{2\theta t} \right) \omega_{\theta \theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t}) \omega_{\theta t} + k_1k_2 \left( \frac{1}{k_1k_2} - S_{1\theta t} S_{2\theta t} \right) \omega,
\end{align*}$$

which implies that $\omega$ satisfies the following system

$$\begin{cases}
\omega_{tt} = k_1k_2 \left( \frac{1}{k_1k_2} - S_{1\theta t} S_{2\theta t} \right) \omega_{\theta \theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t}) \omega_{\theta t} + k_1k_2 \left( \frac{1}{k_1k_2} - S_{1\theta t} S_{2\theta t} \right) \omega, \\
\omega(\theta, 0) = f_2(\theta) - f_1(\theta) = \omega_1(\theta), \\
\omega(\theta, 0) = h_2(\theta) - h_1(\theta) = \omega_0(\theta).
\end{cases}$$

(5.12)
Define the operator $L$ by
\[L[\omega] := k_1 k_2 \left( \frac{1}{k_1 k_2} - S_{1\theta t} S_{2\theta t} \right) \omega_{\theta \theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t}) \omega_{\theta t} - \omega_t,\]
and then we know that
\[a = k_1 k_2 \left( \frac{1}{k_1 k_2} - S_{1\theta t} S_{2\theta t} \right), \quad b = \frac{1}{2} (k_1 S_{1\theta t} + k_2 S_{2\theta t}), \quad c = -1\]
are twice continuously differentiable functions of $\theta$ and $t$. By a direct computation, we have
\[b^2 - ac = \frac{1}{4} (k_1 S_{1\theta t} + k_2 S_{2\theta t})^2 - k_1 k_2 \left( \frac{1}{k_1 k_2} - S_{1\theta t} S_{2\theta t} \right) \cdot (-1)\]
\[= \frac{1}{4} (k_1 S_{1\theta t} - k_2 S_{2\theta t})^2 + 1 > 0.\]
Hence, the operator $L$ is uniformly hyperbolic in $S^1 \times [0,T)$. By Lemma 5.7, we deduce that $S_2(\theta, t) \leq S_1(\theta, t)$ for all $t \in [0,T)$, which completes the proof.

**Proposition 5.9.** (Preserving convexity) Let $k_0(\theta)$ be the curvature function of $F_0$ and
\[\delta = \min_{\theta \in [0,2\pi]} \{k_0(\theta)\} > 0.\]
Then for a $C^4$-solution $S$ of (5.5), we have
\[k(\theta, t) \geq \delta,\]
for all $t \in [0,T_{\text{max}})$, where $[0,T_{\text{max}})$ is the maximal time interval for solution $F(\cdot, t)$ of (5.1).

**Proof.** Since the initial curve is strictly convex, by Theorem 5.4 we know that the solution of (5.5) remains strictly convex on some short time interval $[0, T)$ with some $T \leq T_{\text{max}}$ and its support function satisfies
\[S_{\theta t} = k S_{\theta \theta t}^2 + k^{-1}\]
for all $(\theta, t) \in S^1 \times [0,T)$. Taking derivative with respect to $t$, we have
\[k_t = \frac{1}{S + S_{\theta \theta t}} (S_t + S_{\theta \theta t}) = -k^2 (S_t + S_{\theta \theta t}).\]
Together with the fact $S_t = \bar{\sigma}$, it is easy to know $k_t = -k^2 (\bar{\sigma} + \bar{\sigma}_{\theta \theta})$. Therefore, we can obtain the followings
\[S_t + S_{\theta \theta t} = -(S + S_{\theta \theta})^2 k_t = -\frac{1}{k^2} k_t,\]
\[S_{\theta t} + S_{\theta \theta \theta t} = \left( \frac{1}{k^2} k_t \right) \hat{\theta} = \frac{2}{k^3} k_t k_\theta - \frac{1}{k^2} k_{\theta t},\]
and

\[
k_{tt} = \left(-\frac{1}{(S + S_t \theta)^2} (S_t + S_{t \theta \theta})\right)_t
\]

\[
= \frac{2}{(S + S_t \theta)^3} (S_t + S_{t \theta \theta})^2 - \frac{1}{(S + S_t \theta)^2} (S_{tt} + S_{t \theta \theta})
\]

\[
= 2k^3 \left(-\frac{1}{k^2} k_t\right)^2 - k^2 [(S_{tt})_{\theta \theta} + S_{tt}]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(kS_{t \theta}^2 - k + k + 1)_{\theta \theta} + (kS_{t \theta}^2 - k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + (S_{t \theta}^2 - 1)_{\theta \theta} + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2(S_{t \theta}^2 - 1)_{\theta \theta} + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 4S_{t \theta}S_{t \theta \theta}k_\theta + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(1 - \frac{1}{k^2})k_\theta + \frac{2}{k^3} k_\theta^2 + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2(S_{t \theta}^2 - 1)_{\theta \theta} + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 4k_\theta S_{t \theta}(S_{t \theta} + S - S_t)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(1 - \frac{1}{k^2})k_\theta + \frac{2}{k^3} k_\theta^2 + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2k^3 [(1 - \frac{1}{k^2})k_t^2 - 2(-\frac{1}{k^2} k_t)S_t + S_t^2 - S_{t \theta}^2]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(1 - \frac{1}{k^2})k_\theta + \frac{2}{k^3} k_\theta^2 + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2k^3 (k_\theta S_{t \theta}^2 - \frac{1}{k^2} k_t) + k_\theta S_{t \theta} S_t]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(1 - \frac{1}{k^2})k_\theta + \frac{2}{k^3} k_\theta^2 + (k + k + 1)]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2k^3 k_\theta S_{t \theta} + 4k^2 S_{t \theta}k_\theta + 4k^2 S_{t \theta}S_t]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(1 - \frac{1}{k^2})k_\theta + \frac{2}{k^3} k_\theta^2 + (k + k + 1)]
\]

\[
= k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2k^3 S_{t \theta}k_\theta + 4k^2 S_{t \theta}S_t]
\]

\[
= \frac{2}{k} k_i^2 - k^2 [(S_{t \theta}^2 - 1)_{\theta \theta} + 2k^3 S_{t \theta}k_\theta + 4k^2 S_{t \theta}S_t - k^2]
\].
So, the curvature $k$ satisfies the equation
\[ k_{tt} = k^2 \left( \frac{1}{k^2} - S_{\theta t}^2 \right) k_{\theta \theta} + 2kS_{\theta t}k_{\theta t} + 4k^2S_{\theta t}k_{\theta} - \frac{2}{k_3}k_{\theta}^2 - 4kS_{\theta}k_{t} + k^3(S_{\theta t}^2 - 2S_{t}^2 - k^{-2}). \]

Define the operator $L$ as
\[ L[k] := k^2 \left( \frac{1}{k^2} - S_{\theta t}^2 \right) k_{\theta \theta} + 2kS_{\theta t}k_{\theta t} - k_{tt} + 4k^2S_{\theta t}k_{\theta} - \frac{2}{k_3}k_{\theta}^2 - 4kS_{\theta}k_{t}. \]

We know that
\[ a = k^2 \left( \frac{1}{k^2} - S_{\theta t}^2 \right), \quad b = kS_{\theta t}, \quad c = -1 \]
are twice continuously differentiable functions of $\theta$ and $t$. So we have
\[ b^2 - ac = (kS_{\theta t})^2 - k^2 \left( \frac{1}{k^2} - S_{\theta t}^2 \right) \cdot (-1) = 1 > 0, \]
which implies that the operator $L$ is hyperbolic in $\mathbb{S}^1 \times [0, T)$.

Determining a function $k(\theta, t)$ which satisfies the following system
\[
\begin{cases}
(L + \tilde{h})[k] = 0 & \text{in } \mathbb{S}^1 \times [0, T), \\
k(\theta, 0) = k_0(\theta) & \text{on } \Gamma_0, \\
0 \leq \frac{\partial k}{\partial \nu} : = -bk_{\theta} - ck_{t} := \beta(\theta) & \text{on } \Gamma_0,
\end{cases}
\]
where the operator $\tilde{h}$ is defined as $\tilde{h}[k] := k^3(S_{\theta t}^2 - 2S_{t}^2 - k^{-2})$. It is easy to check that the function $\tilde{k}(\theta, t) = \min_{\theta \in [0, 2\pi]} \{k_0(\theta)\} = \delta$ satisfies
\[
\begin{cases}
(L + \tilde{h})[\tilde{k}] = 0 & \text{in } \mathbb{S}^1 \times [0, T), \\
\tilde{k}(\theta, 0) \leq k_0(\theta) & \text{on } \Gamma_0, \\
\frac{\partial \tilde{k}}{\partial \nu} - M\tilde{k} \leq \beta(\theta) - Mk_0(\theta) & \text{on } \Gamma_0,
\end{cases}
\]
where $\Gamma_0$ is the initial domain, and $M$ is the constant determined by (5.11). Applying Lemma 5.7 to $\tilde{k} - k$ yields
\[ \tilde{k} \leq k(\theta, t) \quad \text{in } \mathbb{S}^1 \times [0, t_0). \]
with $t_0 \leq T$. Hence, we know that the solution $F(\cdot, t)$ remains convex on $[0, T_{\max})$ and its curvature function $k(\theta, t)$ has a uniformly positive lower bound $\delta = \min_{\mathbb{S}^1} \{k_0(\theta)\}$ on $\mathbb{S}^1 \times [0, T_{\max})$, which completes the proof.

We need the following evolution equations of the arc-length of evolving curves.

**Lemma 5.10.** The arc-length $\mathcal{L}(t)$ of the closed curve $F(u, t)$ satisfies
\[ \frac{d\mathcal{L}(t)}{dt} = \int_0^{2\pi} \tilde{\sigma}(\theta, t)d\theta, \]
and
\[ \frac{d^2\mathcal{L}(t)}{dt^2} = \int_0^{2\pi} \left[ k \left( \frac{\partial \tilde{\sigma}}{\partial \theta} \right)^2 + k^{-1} \right] d\theta. \]
Proof. Since
\[ L(t) = \int_0^{2\pi} v(\theta, t) d\theta, \]
and \( \frac{\partial v}{\partial t} = kv \tilde{\sigma} \), by a direct calculation, we have
\[ \frac{dL}{dt} = \int_0^{2\pi} \frac{\partial v}{\partial t} d\theta = \int_0^{2\pi} k v \tilde{\sigma} d\theta = \int_0^{2\pi} \tilde{\sigma}(\theta, t) d\theta, \]
and
\[ \frac{d^2 L}{dt^2} = \int_0^{2\pi} \frac{\partial}{\partial t} \tilde{\sigma}(\theta, t) d\theta = \int_0^{2\pi} S_{tt} d\theta \]
\[ = \int_0^{2\pi} \left( k S_{\theta \theta}^2 + k^{-1} \right) d\theta = \int_0^{2\pi} \left[ k \left( \frac{\partial}{\partial \theta} S_{t} \right)^2 + k^{-1} \right] d\theta \]
\[ = \int_0^{2\pi} \left[ k \left( \frac{\partial \tilde{\sigma}}{\partial \theta} \right)^2 + k^{-1} \right] d\theta, \]
which completes the proof of Lemma 5.10.

From Example 5.5 we know that the behavior of evolving plane curves of HIMCF (5.1) is complicated. However, using Propositions 5.8 and 5.9, Lemma 5.10, we can get the following conclusion about the asymptotic behavior of the hyperbolic flow (5.1).

**Theorem 5.11.** Suppose that \( F_0 \) is a smooth strictly convex closed plane curve with the curvature function \( k_0(\theta) \) whose minimum and maximum are given by \( \delta = \min_{\theta \in S^1} k_0(\theta) \) > 0 and \( \zeta := \max_{\theta \in S^1} \{ k_0(\theta) \} \) respectively. Then there exists a family of strictly convex closed plane curves \( F(\cdot, t) \) satisfying the IVP (5.1) on the time interval \([0, T_{\max})\) with \( 0 < T_{\max} \leq \infty \). Moreover, we have

(I) if \( \zeta^{-1} + \min_{u \in S^1} f(u) > 0 \), then \( T_{\max} = \infty \), i.e., the flow exists for all the time;

(II) if \( \delta^{-1} + \max_{u \in S^1} f(u) < 0 \), then \( T_{\max} < \infty \). Moreover, if furthermore \( \delta^{-1} T_{\max} + \max_{u \in S^1} f(u) < 0 \), then as \( t \to T_{\max} \), one of the following must be true:

- the solution \( F(\cdot, t) \) converges to a point as \( t \to T_{\max} \), i.e., the curvature of the limit curve becomes unbounded as \( t \to T_{\max} \);

- the curvature \( k \) of the evolving curve is discontinuous as \( t \to T_{\max} \), so the solution \( F(\cdot, t) \) converges to a piecewise smooth curve.

**Remark 5.12.** In Case (II) of Theorem 5.11 above, the condition \( \delta^{-1} T_{\max} + \max_{u \in S^1} f(u) < 0 \) is not easy to check, since for a general strictly convex closed plane curve evolving under the hyperbolic flow (5.1), it is difficult to get the accurate value of the maximal time \( T_{\max} \). However, as shown in the proof below, by Example 5.5 and Proposition 5.8 (Containment principle), we have \( T_{\max} \leq T^* = \frac{1}{2} \ln \left( \frac{-1 + \delta_{\max} f(u)}{1 + \delta_{\max} f(u)} \right) \). So, for the purpose of easily checking, one can use a weaker condition \( \delta^{-1} T^* + \max_{u \in S^1} f(u) < 0 \) to replace the assumption \( \delta^{-1} T_{\max} + \max_{u \in S^1} f(u) < 0 \). However, here we prefer to use the latter one, since it is sharper than the previous one.
Proof. Let $[0, T_{\text{max}})$ be the maximal time interval of the IVP (5.1) with $F_0$ and $f$ as the initial curve and initial velocity of the initial curve, respectively.

By Proposition 5.9, we know that the solution $F(\cdot,t)$ remains strictly convex on $[0, T_{\text{max}})$ and the curvature of $F(\cdot,t)$ has a uniformly positive lower bound $\delta > 0$ on $\mathbb{S}^1 \times [0, T_{\text{max}})$.

Case (I): When $\xi^{-1} + \min_{u \in \mathbb{S}^1} f(u) > 0$.

Since $\xi = \max_{\mathbb{S}^1} \{ k_0(\theta) \} \geq \delta > 0$, the initial curve $F_0$ can enclose a circle $\mathcal{C}_0$ with radius $\xi^{-1}$. Let the normal initial velocity of $\mathcal{C}_0$ be equal to $\min_{u \in \mathbb{S}^1} f(u)$. Evolving $\mathcal{C}_0$ by the hyperbolic flow (5.1) to get a solution $\mathcal{C}(\cdot, t)$. By Example 5.5, we know that if $\xi^{-1} + \min_{u \in \mathbb{S}^1} f(u) > 0$, the evolving circle $\mathcal{C}(\cdot, t)$ exists for all the time, and its radius tends to infinity as $t \to \infty$. By Proposition 5.8, we can get that $\mathcal{C}(\cdot, t)$ always lies in the domain $\mathcal{D}$ enclosed by the closed curve $F(\cdot, t)$ for all $t \geq 0$, and moreover, $\mathcal{D}$ tends to be the whole plane as $t \to \infty$. So, in this case, the IVP (5.1) has the long-time existence, i.e., $T_{\text{max}} = \infty$.

Case (II): When $\delta^{-1} + \max_{u \in \mathbb{S}^1} f(u) < 0$.

Since $\delta = \min_{\mathbb{S}^1} \{ k_0(\theta) \} > 0$, the initial curve $F_0$ can be enclosed by a circle $\mathcal{C}_1$ with radius $\delta^{-1}$. Let the normal initial velocity of $\mathcal{C}_1$ be equal to $\max_{u \in \mathbb{S}^1} f(u)$. Evolving $\mathcal{C}_1$ by the hyperbolic flow (5.1) to get a solution $\mathcal{C}(\cdot, t)$. By Example 5.5, we know that if $\delta^{-1} + \max_{u \in \mathbb{S}^1} f(u) < 0$, the solution exist at a finite time interval $[0, T^*)$ and the evolving circle $\mathcal{C}(\cdot, t)$ converges to a single point as $t \to T^*$. By Proposition 5.8, we know that the evolving curve $F(\cdot, t)$ always lies in the domain $\mathcal{D}$ (i.e., a disk) enclosed by $\mathcal{C}(\cdot, t)$ for all $t \in [0, T^*)$. Hence, we can get that $F(\cdot, t)$ must become singular at some time $T_{\text{max}} \leq T^* < \infty$.

Now, we need the following conclusion in convex geometry (see, e.g., [17]).

Blaschke Selection Theorem Let $K_j$ be a sequence of convex sets which are contained in a bounded set. Then there exists a subsequence $K_{jk}$ and a convex set $K$ such that $K_{jk}$ converges to $K$ in the Hausdorff metric.

In Case (II), since $\mathcal{C}(\cdot, t)$ shrinks as $t$ increases and the evolving curve $F(\cdot, t)$ is contained by the circle $\mathcal{C}(\cdot, t)$ for each $t \in [0, T_{\text{max}})$, this strictly convex closed plane curve $F(\cdot, t)$ is contained in the circle $\mathcal{C}_1$ for all $t \in [0, T_{\text{max}})$. By Blaschke Selection Theorem, we know that in the sense of the Hausdorff metric, $F(\cdot, t)$ converges to a weakly convex curve $F(\cdot, T_{\text{max}})$ which might be degenerated and non-smooth.

We claim that $F(\cdot, t)$ converges to either a single point or a limit curve which has the discontinuous curvature under the further assumption $\delta^{-1} T_{\text{max}} + \max_{u \in \mathbb{S}^1} f(u) < 0$.

By Proposition 5.9 and Lemma 5.10, we have
\[
\frac{d^2 \mathcal{L}(t)}{dt^2} = \int_0^{2\pi} \left[ k \left( \frac{\partial \sigma}{\partial \theta} \right)^2 + k^{-1} \right] d\theta > 0 \quad \text{for all } t \in [0, T_{\text{max}}).
\]

Besides, by Proposition 5.9, we have
\[
\bar{\sigma}(\theta, t) = \sigma(u, t) = f(u) + \int_0^t k^{-1}(u, \xi) d\xi \leq \delta^{-1} t + \max_{u \in \mathbb{S}^1} f(u) \leq \delta^{-1} T_{\text{max}} + \max_{u \in \mathbb{S}^1} f(u) < 0.
\]
for all \( t \in [0, T_{\max}) \), which implies
\[
\frac{d \mathcal{L}(t)}{dt} = \int_0^{2\pi} \tilde{\sigma}(\theta, t) d\theta < 0 \quad \text{for all } t \in [0, T_{\max}).
\]
So, for all \( t \in [0, T_{\max}) \), we have
\[
\frac{d \mathcal{L}(t)}{dt} < 0, \quad \frac{d^2 \mathcal{L}(t)}{dt^2} > 0,
\]
which implies that there exists a finite time \( T_0 \) such that \( \mathcal{L}(T_0) = 0 \). There will be the following two situations:

- \( T_0 \leq T_{\max} \). On one hand, by Theorem 5.4, there exists a unique classical solution \( F(\cdot, t) \) to the IVP (5.1) on \([0, T_0)\). On the other hand, since \( \mathcal{L}(t) \) is decreasing on \([0, T_0)\) and \( \mathcal{L}(T_0) = 0 \), we have \( \mathcal{L}(T_0) \to 0 \) as \( t \to T_0 \). This implies the curvature \( k \) tends to infinity as \( t \to T_0 \), and the solution will blow up at \( T_0 \). Therefore, by the definition of \( T_{\max} \), we have \( T_0 = T_{\max} \). So, \( F(\cdot, t) \) converges to a point as \( t \to T_{\max} \).

- \( T_0 > T_{\max} \). In this situation, \( \mathcal{L}(T_{\max}) > 0 \), which implies that \( F(\cdot, T_{\max}) \) must be non-smooth. Then there will be three possibilities:

  1. \( \|F(u, T_{\max})\| = \sup \|F(u, T_{\max})\| = \infty \). However, \( F(\cdot, t) \) is always contained in the circle \( \mathcal{C}_1 \), which implies that \( \|F(u, T_{\max})\| \) must be bounded. This is a contradiction. So, (1) is impossible.

  2. \( \|F_u(u, T_{\max})\| = \infty \). However, the length of the limit curve \( \mathcal{L}(T_{\max}) \) satisfies

\[
\mathcal{L}(T_{\max}) = \lim_{t \to T_{\max}} \int_{F(u,t)} ds = \lim_{t \to T_{\max}} \int_{F(u,t)} |F_u(u,t)| du = \int_{F(u,t)} \lim_{t \to T_{\max}} |F_u(u,t)| du = \infty
\]

which contradicts with \( \mathcal{L}(T_{\max}) < \mathcal{L}_0 \) with \( \mathcal{L}_0 \) the length of the initial curve \( F_0 \). So, (2) is also impossible.

  3. The curvature function \( k \) is discontinuous. We cannot exclude this possibility. This phenomena will be occurred if the above shocks are not possible.

Our claim before is true. The proof of Theorem 5.11 is finished.

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References

[1] H. Bray, Proof of the Riemannian Penrose inequality using the positive mass theorem, J. Differential Geom. 59 (2001) 177–267.

[2] S. Brendle, P.-K. Hung and M.-T. Wang, A Minkowski inequality for hypersurfaces in the anti-de Sitter-Schwarzschild manifold, Commun. Pure Appl. Math. 69 (2016) 124–144.

[3] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, III; Functions of the eigenvalue of the Hessian, Acta Math. 155 (1985) 261–301.

[4] F. Cao, Geometric Curve Evolution and Image Processing, Lecture Notes in Mathematics, Vol. 1805, Springer, 2003.

[5] L. Chen and J. Mao, Non-parametric inverse curvature flows in the AdS-Schwarzschild manifold, The Journal of Geometric Analysis, DOI:10.1007/s12220-017-9848-6.

[6] L. Chen, J. Mao and H.-Y. Zhou, Inverse curvature flows in warped product manifolds, preprint.

[7] L. Chen, J. Mao, N. Xiang and C. Xu, Inverse mean curvature flow inside a cone in warped products, submitted and available online at arXiv:1705.04865v3.

[8] C. Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differential Geom. 32 (1990) 299–314.

[9] C.-L. He, D.-X. Kong and K.-F. Liu, Hyperbolic mean curvature flow, J. Differential Equat. 246 (2009) 373–390.

[10] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. 20 (1984) 237–266.

[11] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differential Geom. 59 (2001) 353–437.

[12] D.-X. Kong, K.-F. Liu and Z.-G. Wang, Hyperbolic mean curvature flow: evolution of plane curves, Acta Math. Scientia 29(B)(3) (2009) 493–514.

[13] J. Mao, Forced hyperbolic mean curvature flow, Kodai Math. J. 35 (2012) 500–522.

[14] J. Mao, Q. Ming, C.-X. Wu and Z. Zhou, Hyperbolic inverse curvature flows in warped products, preprint.
[15] T. Marquardt, *Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone*, J. Geom. Anal. 23 (2013) 1303–1313.

[16] M.-H. Protter, H.-F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.

[17] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

[18] P. Topping, *Mean curvature flow and geometric inequalities*, J. reine angew. Math. 503 (1998) 47–61.

[19] S.-T. Yau, *Review of geometry and analysis*, Asian J. Math. 4 (2000) 235–278.

[20] X.-P. Zhu, *Lectures on Mean Curvature Flows*, Stud. Adv. Math., Vol. 32, AMS/IP, 2002.