The Evolution of Harmonic Progression Analysis: 
Ultimate CAT

In this article we introduce an improved version of CAT (Cataldo Advanced Transformations). The purpose of CAT fundamentally lies in providing musicians with a reliable instrument that may effectively assist them in carrying out chord progressions analysis. The correct application of CAT allows to convert any harmonic progression, net of some very rare exceptions, into a mere sequence of plagal and perfect cadences. Amongst other features, the improved version of CAT, unlike the previous one, takes also into account Modal Interchange and Tonicization.
The Evolution of Harmonic Progression Analysis: Ultimate CAT

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Abstract

In this article we introduce an improved version of CAT (Cataldo Advanced Transformations). The purpose of CAT fundamentally lies in providing musicians with a reliable instrument that may effectively assist them in carrying out chord progressions analysis. The correct application of CAT allows to convert any harmonic progression, net of some very rare exceptions, into a mere sequence of plagal and perfect cadences. Amongst other features, the improved version of CAT, unlike the previous one, takes also into account Modal Interchange and Tonicization.

Keywords

CAT, Cataldo Advanced Transformations, Chord Progression Analysis, Harmonic Substitutions, Jazz.

Introduction

Net of a single exception (a routine named “Structure Reduction”), the whole method is based on the mere application of a series of (inverse) harmonic transformations [1] [2]. The above-mentioned transformations, herein introduced in a considerably improved version, turn out to be nothing but inverse chord substitutions characterized by specific conditions and restrictions. As elsewhere underlined [7] [8], the method arises from the analysis of a considerable number of chord progressions, devoting particular (although not exclusive) attention to traditional jazz compositions. A significant improvement of CAT has been achieved by conducting an extremely thorough analysis of a huge amount of LEGO Bricks (public domain harmonic patterns) [5] [6]. Unlike the previous version [3] [4], the one herein introduced has no limitation concerning the key (any song written in both major and minor key can be analysed), exploits a more rigorous definition of “Similitude”, and takes into consideration “Modal Interchange” (comparing 35 scales/modes) and “Tonicization” (actually not to be regarded as a real substitution, but rather as a “Harmonic Enrichment”) [7] [8]. The time signature must always be imagined as being equal to 4/4. For example, even if we deal with a 3/4, we have to consider four pulses per measure (four beats per bar): each beat, in this case, will be characterized by a duration equivalent to a dotted quaver.

1. KEY Selection and Writing of the SCALE and HARMONIZATION VECTORS

If we denote with X a generic note belonging to the Chromatic scale, and with t a whole tone interval, we can represent the Ionian and the Aeolian scales as vectors [9] [10], denoted by $s_{\text{ion}}$ and $s_{\text{aeo}}$ respectively [11] [12]. Similarly, we can consider two vectors, denoted by $h_{\text{ion}}$ and $h_{\text{aeo}}$, whose components are the seventh chords that arise from the harmonization of the above-mentioned scales.
2. STRUCTURE REDUCTION

Structure Reduction – Without “Structure Reduction” a correct application of CAT (Cataldo Advanced Transformations) is de facto impossible. Very simply, the number of bars and the duration of the chords are iteratively halved. The procedure is stopped the moment in which even a single chord characterized by a duration equal to a beat appears. “Structure Reduction” is applied every time it is possible, so as to obtain the highest simplification level. [3] [4]

Very concisely, we have:

\[ n_{\text{max}} = \text{current number of chords} \]
\[ k_{\text{max}} = \text{current number of bars} \]
\[ a_n = n - \text{th chord} \]
\[ T(a_n) = \text{current duration of the n – th chord} \]

\[ \forall n \in (1, n_{\text{max}}) \quad T(a_n) > 1\text{ beat} \quad \text{and} \quad k_{\text{max}} \quad \frac{1}{2} \in \mathbb{N} \quad T(a_n) \quad \text{red.T(a_n)} \quad \frac{1}{2} \quad \text{and} \quad k_{\text{max}} \quad \text{red.k}_{\text{max}} \quad \frac{1}{2} \]

3. Inverse Substitutions (Modal Interchange and Similitude) involving AUGMENTED MAJOR SEVENTH CHORDS

Modal Interchange – Two chords that arise from the harmonization of two different scales characterized by the same tonic (generic parallel keys) are interchangeable if they are placed in the same position (if they represent the same harmonic degree). [7] [8]

| Ionian | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
|-------|--------|-----|-----|-----|----|-----|-------|
| Dorian | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
| Phrygian | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
| Lydian | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
| Mixolydian | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
| Aolion | C-maj7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |
| Locrian | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bm7b5 |

| Harmonic Minor | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Locior #6 | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Lydian Augmented | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Phrygian Dominant | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Lydian Dominant | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| UltraLocrian | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |

| Harmonic Major | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Locior #6 | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Lydian Augmented | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Lydian Dominant | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| UltraLocrian | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |

| Double Harmonic | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Locior #6 | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| UltraLocrian | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Hungarian Major | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Oriental | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Ionian Augmented #2 | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
| Locrian #6 | Cm7 | Dm7 | Em7 | Fm7 | G7 | An7 | Bdim7 |
**Similitude Substitution** – Any Augmented Major Seventh Chord can be approximately identified with a Dominant Seventh Chord (provided with the flat thirteenth) distant an ascending major third. Any Minor Major Seventh Chord can be approximately identified with a Dominant Seventh Chord (provided with the sharp eleventh) distant an ascending perfect fourth [3] [4].

From now onwards, the examined chord (the one which must undergo substitution) will be denoted by $a_n$. The analysis must be carried out by starting from the final chord of the progression.

| MAJOR Key | MINOR Key |
|-----------|-----------|
| $a_n = Ym^7$ | $a_n = Ym^7$ |
| $s_1^m$ $\in$ MAJ 7#5 $\equiv$ $h_1^{ Lydia Augmented}$ | $s_1^a$ $\in$ MAJ 7#5 $\equiv$ $h_1^{ Lydia Augmented}$ |
| $(s_1^m + \frac{t}{2}) m^a$ $\equiv$ $h_2^{Super Locrian}$ | $(s_1^a + \frac{t}{2}) m^a$ $\equiv$ $h_2^{Super Locrian}$ |
| $(s_2^m + \frac{t}{2}) m^a$ $\equiv$ $h_3^{ Mixolydian Aug}$ | $(s_2^a + \frac{t}{2}) m^a$ $\equiv$ $h_3^{ Mixolydian Aug}$ |
| $(s_3^m + \frac{t}{2}) m^a$ $\equiv$ $h_4^{ Lyd Dominant}$ | $(s_3^a + \frac{t}{2}) m^a$ $\equiv$ $h_4^{ Lyd Dominant}$ |
| $(s_4^m + \frac{t}{2}) m^a$ $\equiv$ $(s_4 + \frac{t}{2}) 7^b_{13}$ | $(s_4^a + \frac{t}{2}) m^a$ $\equiv$ $(s_4 + \frac{t}{2}) 7$ |
| $(s_5^m + \frac{t}{2}) m^a$ $\equiv$ $(s_5 + \frac{t}{2}) 7_{13}$ | $(s_5^a + \frac{t}{2}) m^a$ $\equiv$ $(s_5 + \frac{t}{2}) 7$ |

4. Inverse Substitutions (Modal Interchange and Similitude)* involving MINOR MAJOR SEVENTH CHORDS

*See point 3 for the definitions of Modal Interchange and Similitude Substitution.
5. Inverse Substitutions (Modal Interchange* and Inverse Diminished Substitution) involving DIMINISHED CHORDS followed by Half-Diminished or Diminished Chords

Diminished Substitution – Any Dominant Seventh Chord, especially if provided with the flat ninth, can be replaced, even if it were to arise from a previous harmonic substitution, by a Diminished Chord distant a major third, a perfect fifth, a minor seventh or a flat ninth from the initial chord. All the above-mentioned intervals must be regarded as ascending [1] [2] [3] [4] [7] [8].

In other terms, if we denote with \( P \) a generic note belonging to the Chromatic scale, we can write, with obvious meaning of the notation, as follows:

\[
p^{7(b)}_{\text{dim}} \frac{Y}{Y \dim 7} = P + 2t + \frac{3n}{2} t \quad n = 0,1,2,3
\]

More explicitly, we have:

\[
p^{7(b)}_{\text{dim}} \frac{(P + 2t) \dim 7}{\text{n = 0, major third}}
\]

\[
p^{7(b)}_{\text{dim}} \frac{(P + \frac{7}{2} t) \dim 7}{\text{n = 1, perfect fifth}}
\]

\[
p^{7(b)}_{\text{dim}} \frac{(P + 5t) \dim 7}{\text{n = 2, minor seventh}}
\]

\[
p^{7(b)}_{\text{dim}} \frac{(P + \frac{13}{2} t) \dim 7}{\text{n = 3, flat ninth}}
\]

*See point 3 for the definition of Modal Interchange

6. Inverse Diminished Substitutions involving DIMINISHED CHORDS not followed by Half-Diminished or Diminished Chords

*See point 5 for the definition of Diminished Substitution

Unlike the previous version of CAT [3] [4], we herein consider two cases altogether:
6.1. Inverse Diminished Substitutions involving Diminished chords followed by Dominant Seventh Chords

\[ a_n = Y_{\text{dim}} \]

\[ \begin{cases} 
   a_{n+1} = Z7 \\
   a_n \in \text{sub}^{\text{dim}}(Z7) \Rightarrow a_n \leftrightarrow Z7 
\end{cases} \]

\[ \begin{cases} 
   a_{n+1} = Z7 \\
   a_n \in \text{sub}^{\text{dim}}[(Z + \frac{5}{2} t)7] \Rightarrow a_n \leftrightarrow (Z + \frac{5}{2} t)7 
\end{cases} \]

\[ \begin{cases} 
   a_{n+1} = Z7 \\
   a_n \in \text{sub}^{\text{dim}}[(Z + \frac{9}{2} t)7] \Rightarrow a_n \leftrightarrow (Z + \frac{9}{2} t)7 
\end{cases} \]

6.2. Inverse Diminished Substitutions involving Diminished chords followed by Major Seventh or Minor Seventh Chords

\[ a_n = Y_{\text{dim}} \]

\[ \begin{cases} 
   a_{n+1} = Z_{\text{maj}}7, Z_{\text{m7}} \\
   a_n \in \text{sub}^{\text{dim}}[(Z + \frac{5}{2} t)7] \Rightarrow a_n \leftrightarrow (Z + \frac{5}{2} t)7 
\end{cases} \]

\[ \begin{cases} 
   a_{n+1} = Z_{\text{maj}}7, Z_{\text{m7}} \\
   a_n \in \text{sub}^{\text{dim}}[(Z + \frac{7}{2} t)7] \Rightarrow a_n \leftrightarrow (Z + \frac{7}{2} t)7 
\end{cases} \]

\[ \begin{cases} 
   a_{n+1} = Z_{\text{maj}}7, Z_{\text{m7}} \\
   a_n \in \text{sub}^{\text{dim}}[(Z + \frac{9}{2} t)7] \Rightarrow a_n \leftrightarrow (Z + \frac{9}{2} t)7 
\end{cases} \]

7. CONTRACTION (Inverse EXPANSION)

Expansion – Any Dominant Seventh Chord, by forgoing half of its duration, can be imagined as being preceded by a Minor Seventh or a Half-Diminished Chord distant a descending perfect fourth and characterized by a duration equal to half of the one of the initial chord. In other terms, any Dominant Seventh Chord can be converted into a half-cadence [3] [4].

\[ a_n = Y_{\text{m7}}, Y_{\text{m7b5}} \]

\[ \begin{cases} 
   a_{n+1} = (Y + \frac{5}{2} t)7 \\
   a_n, a_{n+1} \in \text{bar} \Rightarrow a_n | a_{n+1} \leftrightarrow a_{n+1} | a_n \\
   T(a_n) = T(a_{n+1}) \\
   \text{heat}(a_n) = \text{on} 
\end{cases} \]

8. STRUCTURE REDUCTION*

Note – Every time a Reduction has been carried out, it is necessary to evaluate the possibility of carry out Contractions the application of which, in turn, could make feasible another Structure Reduction, and so on. In other terms, we have a LOOP that can be stopped only if there are no conditions that make possible further Contractions and Structure Reductions [3] [4].

*See point 2 for the definition of Structure Reduction

9. Inverse Substitutions (Modal Interchange*) involving MAJOR SEVENTH CHORDS different from \( h_1 \) and \( h_4 \), if the Key is Major, or \( h_2 \) and \( h_3 \), if the Key is Minor (Major Seventh Chords that does not belong to the Harmonization Vector).

*See point 3 for the definition of Modal Interchange

| MAJOR Key |
|---------------------------------|
| \( a_n = Y_{\text{maj}}7 \equiv Y^\Delta \equiv h_1^{10n}, h_2^{60n} \) |
| \( \langle s_1^{10n} + \frac{1}{2} \rangle \text{maj}7 = h_2^{2n} \text{prygian} \leftrightarrow \text{int} \) |
| \( s_2^{10n} \text{maj}7 = h_3^{Loci} \text{bb}3 \text{bb}7 \text{int} \leftrightarrow h_3^{10n} \) |

| MINOR Key |
|---------------------------------|
| \( a_n = Y_{\text{maj}}7 \equiv Y^\Delta \equiv h_2^{20e}, h_3^{60e} \) |
| \( s_1^{20e} \text{maj}7 = h_1^{10n} \text{ion} \leftrightarrow \text{int} \) |
| \( s_1^{20e} + \frac{1}{2} \text{maj}7 = h_2^{2n} \text{prygian} \leftrightarrow \text{int} \) |
| \( \langle s_1^{20e} + \frac{1}{2} \rangle \text{maj}7 = h_3^{Loci} \text{bb}3 \text{bb}7 \text{int} \leftrightarrow h_3^{10n} \) |
10. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]

11. Inverse Substitutions (Modal Interchange and Similitude) involving HALF – DIMINISHED CHORDS different from $h_7$, if the Key is Major, or $h_2$; if the Key is Minor (Half – Diminished Chords that does not belong to the Harmonization Vector).

*See point 3 for the definitions of Modal Interchange and Similitude*
12. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]

13. Inverse Substitutions (Modal Interchange*) involving MINOR SEVENTH CHORDS different from \( h_2, h_3, \) and \( h_6, \) if the Key is Major, or \( h_1, h_4, \) and \( h_5, \) if the Key is Minor (Minor Seventh Chords that does not belong to the Harmonization Vector).

*See point 3 for the definition of Modal Interchange

### MAJOR Key

\[
a_n = Ym7 \neq h_2^m, h_3^m, h_6^m
\]

\[
\begin{align*}
a_n &= \left( s_i^m \right)^{1/2} + t \times m7 \\
a_n + 1 &= s_i^m \\
\text{bar}_k \n, a_n, a_n + 1 \in \text{bar}_k \\
T(a_n) &= T(a_n + 1) \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

\[
\begin{align*}
a_n &= s_i^m \times m7 \\
a_n + 1 &= \left( s_i^m + \frac{11}{2} \times t \right)^7 \\
\text{bar}_k \n, a_n, a_n + 1 \in \text{bar}_k \\
T(a_n) &= T(a_n + 1) \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

### MINOR Key

\[
a_n = Ym7 \neq h_2^m, h_3^m, h_6^m
\]

\[
\begin{align*}
a_n &= \left( s_i^m \right)^{1/2} + t \times m7 \\
a_n + 1 &= s_i^m \\
\text{bar}_k \n, a_n, a_n + 1 \in \text{bar}_k \\
T(a_n) &= T(a_n + 1) \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

\[
\begin{align*}
a_n &= s_i^m \times m7 \\
a_n + 1 &= \left( s_i^m + \frac{11}{2} \times t \right)^7 \\
\text{bar}_k \n, a_n, a_n + 1 \in \text{bar}_k \\
T(a_n) &= T(a_n + 1) \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

\[
\begin{align*}
s_1^m \times m7 &= h_2^m \text{ mod. int. } h_2^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_2^m \text{ mod. int. } h_2^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_3^m \text{ mod. int. } h_3^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_4^m \text{ mod. int. } h_4^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_5^m \text{ mod. int. } h_5^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_6^m \text{ mod. int. } h_6^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_7^m \text{ mod. int. } h_7^m \\
\left( s_i^m + \frac{1}{2} \times t \right)^{m7} &= h_8^m \text{ mod. int. } h_8^m
\end{align*}
\]

14. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]

15. (Inverse) TRITONE Substitutions

**Tritone Substitution** – Any Dominant Seventh Chord, especially if altered, can be replaced, even if it were to arise from a previous harmonic substitution, by a chord of the same kind (a Dominant Seventh Chord) distant three whole tones from the initial chord.

Generally, if we denote with \( P \) a generic note belonging to the **Chromatic** scale, and with \( t \) a whole tone interval, we can write:

\[
X^\text{dim.} \rightarrow Y \\
Y = X + 3t
\]

### MAJOR Key

\[
\begin{align*}
a_n &= Y7 \\
Y = \neq s_i^m \implies a_n & \leftarrow (Y + 3t)^7 \\
i &= 1, \ldots, 7
\end{align*}
\]

### MINOR Key

\[
\begin{align*}
a_n &= Y7 \\
Y = \neq s_i^m \implies a_n & \leftarrow (Y + 3t)^7 \\
i &= 1, \ldots, 7
\end{align*}
\]
16. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]

17. SECONDARY DOMINANTS Inverse Substitutions

Secondary Dominant Substitution – Any chord, even if it were to arise from a previous harmonic substitution, can be converted into a Dominant Seventh Chord [1] [2] [3] [4] [7] [8].

**MAJOR Key**

\[
\begin{align*}
a_n &= Y_7 \\
a_n &= s^i_{1\text{ion}} \iff a_n &\leftarrow h^I_{1\text{ion}} \quad i \neq 5
\end{align*}
\]

**MINOR Key**

\[
\begin{align*}
a_n &= Y_7 \\
a_n &= s^i_{1\text{Aeo}} \iff a_n &\leftarrow h^I_{1\text{Aeo}} \quad i \neq 5, 7
\end{align*}
\]

The Secondary Dominants Inverse Substitutions involving \( s^5 \) are procrastinated in order to facilitate possible Contractions.

18. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]

19. (Inverse) DIATONIC Substitutions

Diatonic Substitution – Two chords that arise from the harmonization of the same scale are interchangeable if the distance between them (between the roots) is equal to a diatonic third (both ascending and descending) [1] [2] [3] [4] [7] [8].

**Substitutions Chart – MAJOR Key**

\[
\begin{align*}
&\text{Imaj7} \\
&\text{VIIm7b5} \\
&\text{VIIIm7} \\
&\text{VIIm7} \\
&\text{V7} \\
&\text{IVmaj7}
\end{align*}
\]

**Substitutions Chart – MINOR Key**

\[
\begin{align*}
&\text{Im7} \\
&\text{VIIm7b5} \\
&\text{VIImaj7} \\
&\text{Vim7} \\
&\text{IVm7}
\end{align*}
\]

19.1. (Inverse) DIATONIC Substitutions involving \( h_6 \)

**MAJOR Key**

\[
\begin{align*}
a_n &= h^I_{6\text{ion}} \iff a_n &\leftarrow h^I_{1\text{ion}}
\end{align*}
\]

**MINOR Key**

\[
\begin{align*}
a_n &= h^I_{6\text{Aeo}} \iff a_n &\leftarrow h^I_{1\text{Aeo}}
\end{align*}
\]
19.2. (Inverse) DIATONIC Substitutions involving $h_1$

**MAJOR Key**

\[
\begin{align*}
a_n &= h_1^{\text{Ion}} \\
(a_{n+1} &= h_1^{\text{Ion}}) \implies a_n \leftrightarrow h_1^{\text{Ion}} \\
a_n &= h_1^{\text{Ion}} \\
(a_{n+1} \neq h_2^{\text{Ion}}, h_4^{\text{Ion}}) \implies a_n \leftrightarrow h_1^{\text{Ion}} \\
\text{beat}(a_n) &= \text{off}
\end{align*}
\]

**MINOR Key**

\[
\begin{align*}
a_n &= h_1^{\text{Aeo}} \\
(a_{n+1} &= h_1^{\text{Aeo}}) \implies a_n \leftrightarrow h_1^{\text{Aeo}} \\
a_n &= h_1^{\text{Aeo}} \\
(a_{n+1} \neq h_2^{\text{Aeo}}, h_4^{\text{Aeo}}) \implies a_n \leftrightarrow h_1^{\text{Aeo}} \\
\text{beat}(a_n) &= \text{off}
\end{align*}
\]

*otherwise: $a_n = h_1^{\text{Ion}} \leftrightarrow h_1^{\text{Aeo}}$*

19.3. (Inverse) DIATONIC Substitutions involving $h_7$

**MAJOR Key**

\[
\begin{align*}
a_n &= h_7^{\text{Ion}} \\
(a_{n+1} &= h_7^{\text{Ion}}) \implies a_n \leftrightarrow h_7^{\text{Ion}} \\
a_n &= h_7^{\text{Ion}} \\
(a_{n+2} = h_5^{\text{Ion}}) \implies a_n \leftrightarrow h_5^{\text{Ion}} \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

**MINOR Key**

\[
\begin{align*}
a_n &= h_7^{\text{Aeo}} \\
(a_{n+1} &= h_7^{\text{Aeo}}) \implies a_n \leftrightarrow h_7^{\text{Aeo}} \\
a_n &= h_7^{\text{Aeo}} \\
(a_{n+2} = h_5^{\text{Aeo}}, s_7^{\text{Aeo}}) \implies a_n \leftrightarrow h_5^{\text{Aeo}} \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

*otherwise: $a_n = h_7^{\text{Ion}} \leftrightarrow h_7^{\text{Aeo}}$*

19.4. (Inverse) DIATONIC Substitutions involving $h_2$ and $h_4$

**MAJOR Key**

\[
\begin{align*}
a_n &= h_2^{\text{Ion}} \\
(a_{n+1} &= h_1^{\text{Ion}}, h_4^{\text{Ion}}) \implies a_n \leftrightarrow h_4^{\text{Ion}} \\
a_n &= h_2^{\text{Ion}} \\
(a_{n+1} = h_2^{\text{Ion}}, h_4^{\text{Ion}}) \implies a_n \leftrightarrow h_2^{\text{Ion}} \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

**MINOR Key**

\[
\begin{align*}
a_n &= h_2^{\text{Aeo}} \\
(a_{n+1} &= h_1^{\text{Aeo}}, h_4^{\text{Aeo}}) \implies a_n \leftrightarrow h_4^{\text{Aeo}} \\
a_n &= h_2^{\text{Aeo}} \\
(a_{n+1} = h_2^{\text{Aeo}}, h_4^{\text{Aeo}}) \implies a_n \leftrightarrow h_2^{\text{Aeo}} \\
\text{beat}(a_n) &= \text{on}
\end{align*}
\]

*otherwise: $a_n = h_2^{\text{Ion}} \leftrightarrow h_2^{\text{Aeo}}$*

20. CONTRACTIONS (Inverse EXPANSION) + STRUCTURE REDUCTION [LOOP]
In detail, a chord covering the beats from the first to the third must be rewritten and treated as two identical chords: the first chord covers the first two beats, the second exclusively covers the third. A chord covering the second and third beats must be rewritten and treated as two identical chords: the first chord covers the second beat, the second covers the third. A chord covering the beats from the second to the fourth must be rewritten and treated as two identical chords: the first chord exclusively covers the second beat, the second covers the last two.

21. Further Inverse SECONDARY DOMINANTS Substitutions

| MAJOR Key | MINOR Key |
|-----------|-----------|
| \[ a_n = Y7 \] | \[ a_n = Y7 \] |
| \[ a_{n+1} = a_{n-1} = (Y + \frac{5}{2}t) \text{ maj7} \] & \[ a_{n+1} = a_{n-1} = a_n \] |
| \( T(a_n) \leq \text{Min}(T(a_{n+1}), T(a_{n-1})) \) & \( T(a_n) \leq \text{Min}(T(a_{n+1}), T(a_{n-1})) \) |
| \[ h^t \text{ ton.} \] | \[ h^t \text{ ton.} \] |

EXTRA. Inverse TONICIZATION (to be applied every time it is possible)

**TONICIZATION** – Any (major seventh or minor seventh) chord, especially if it covers a certain number of consecutive bars, can be tonicized. The Tonicization, or Micro-Modulation, can be obtained by replacing a portion of the initial chord with a dominant seventh chord distant an ascending perfect fifth, so as to locally create a harmonically Perfect Cadence [1] [2] [3] [4] [7] [8].

\[
\begin{align*}
a_n &= Y7 \\
a_{n+1} &= a_{n-1} = (Y + \frac{5}{2}t) \text{ maj7} \iff a_n \leftarrow a_{n+1} = a_{n-1} \\
T(a_n) &\leq \text{Min}(T(a_{n+1}), T(a_{n-1})) \\
\end{align*}
\]

**Remarks**

It's worth underlining how chord progression analysis has almost nothing to do with the improvisation built on the original harmonic structure of a song. More precisely, net of a certain "Horizontalization" the musician can exploit by facing specific harmonic aggregates, such as "Tonicizations" and "Turnarounds", the improvisation should be carried out "vertically", abiding to the local harmony of the song.

The Local Tonal Centre can be characterized by significant fluctuations: sometimes, it is very difficult to identify it, in particular when a progression cannot be regarded as being manifestly "tonal" (built on cadences).

And especially in this case, paradoxically, the musician should improvise by abiding to the harmonic progression, chord by chord [13 – 30].

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