On the arrowhead-Fibonacci numbers

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Abstract: In this paper, we define the arrowhead-Fibonacci numbers by using the arrowhead matrix of the characteristic polynomial of the $k$-step Fibonacci sequence and then we give some of their properties. Also, we study the arrowhead-Fibonacci sequence modulo $m$ and we obtain the cyclic groups from the generating matrix of the arrowhead-Fibonacci numbers when read modulo $m$. Then we derive the relationships between the orders of the cyclic groups obtained and the periods of the arrowhead-Fibonacci sequence modulo $m$.

Keywords: The arrowhead-Fibonacci Numbers, Sequence, Matrix, Period

MSC: 11K31, 11B50, 11C20, 15A15

1 Introduction

It is well-known that a square matrix is called an arrowhead matrix if it contain zeros all in entries except for the first row, first column, and main diagonal. In other words, an arrowhead matrix $M = [m_{i,j}]_{(n) \times (n)}$ is defined as follows:

$$
\begin{bmatrix}
  m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & \cdots & m_{1,n} \\
  m_{2,1} & m_{2,2} & 0 & 0 & \cdots & 0 \\
  m_{3,1} & 0 & m_{3,3} & 0 & \cdots & 0 \\
  \vdots & 0 & 0 & \ddots & \vdots & \vdots \\
  m_{n-1,1} & \vdots & 0 & m_{n-1,n-1} & 0 \\
  m_{n,1} & 0 & \cdots & 0 & 0 & m_{n,n}
\end{bmatrix}
$$

The $k$-step Fibonacci sequence $\{F_n^k\}$ is defined recursively by the equation

$$
F_{n+k}^k = F_{n+k-1}^k + F_{n+k-2}^k + \cdots + F_n^k
$$

for $n \geq 0$, where $F_0^k = F_1^k = \cdots = F_{k-2}^k = 0$ and $F_{k-1}^k = 1$.

For detailed information about the $k$-step Fibonacci sequence, see [1, 2]. It is clear that the characteristic polynomial of the $k$-step Fibonacci sequence is as follows:

$$
P_F^k(x) = x^k - x^{k-1} - \cdots - x - 1.
$$

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding $k$ terms:

$$
a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1}
$$

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where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [1], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix $A$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\
 c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix},$$

for $n \geq 0$.

Many of the obtained numbers by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by many authors; see, for example, [3–11]. Arrowhead-Fibonacci numbers for the 2-step Pell and Pell-Lucas sequences were illustrated in [12]. In Section 2, we define the arrowhead-Fibonacci numbers by using the arrowhead matrix $N$, which is defined by the aid of the characteristic polynomial of the $k$-step Fibonacci sequence. Then we derive their miscellaneous properties such as the generating matrix, the combinatorial representation, the Binet formula, the permanental representations, the exponential representation and the sums.

The study of recurrence sequences in groups began with the earlier work of Wall [13], where the ordinary Fibonacci sequence in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by some authors; see, for example, [3, 14–16]. In [3, 15, 17], the authors obtained the cyclic groups via some special matrices. In Section 3, we study the arrowhead-Fibonacci sequence modulo $m$. Also in this section, we obtain the cyclic groups from the multiplicative orders of the generating matrix of the arrowhead-Fibonacci sequence such that the elements of the generating matrix when read modulo $m$. Then we obtain the rules for the orders of the obtained cyclic groups and we give the relationships between the orders of those cyclic groups and the periods of the arrowhead-Fibonacci sequence modulo $m$.

### 2 The arrowhead-Fibonacci numbers

We next define the arrowhead matrix $N = [n_{i,j}]_{(k+1) \times (k+1)}$ by using the characteristic polynomial of the $k$-step Fibonacci sequence $P_k^F(x)$ as follows:

$$N = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & -1 & 0 \\ -1 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$ 

Now we consider a new $(k + 1)$-step sequence which is defined by using the matrix $N$ and is called the arrowhead-Fibonacci sequence. The sequence is defined by integer constants $a_{k+1}(1) = \cdots = a_{k+1}(k) = 0$ and $a_{k+1}(k+1) = 1$ and the recurrence relation

$$a_{k+1}(n + k + 1) = a_{k+1}(n + k) - a_{k+1}(n + k - 1) - \cdots - a_{k+1}(n)$$

for $n \geq 1$, where $k \geq 2$. 
Example 2.1. Let \( k = 3 \), then we have the sequence
\[
\{a_4 (n)\} = \{0, 0, 0, 1, 1, 0, -2, -4, -3, 3, 12, 16, 4, -27, -59, \ldots \}.
\]

By (1), we can write a generating matrix for the arrowhead-Fibonacci numbers as follows:
\[
G_{k+1} = \begin{bmatrix}
1 & -1 & -1 & \cdots & -1 & -1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}_{(k+1) \times (k+1)}.
\]

The matrix \( G_{k+1} \) is said to be an arrowhead-Fibonacci matrix. It is clear that
\[
(G_{k+1})^n = \begin{bmatrix}
\begin{bmatrix}
a_{k+1} (k + 1) \\
a_{k+1} (k) \\
\vdots \\
a_{k+1} (1)
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
a_{k+1} (n + k + 1) \\
a_{k+1} (n + k) \\
\vdots \\
a_{k+1} (n + 1)
\end{bmatrix}
\end{bmatrix}
\]

for \( n \geq 0 \). Again by an inductive argument, we may write
\[
(G_{k+1})^n = \begin{bmatrix}
\begin{bmatrix}
a_{k+1}^{n+k+1} \\
a_{k+1}^{n+k} \\
\vdots \\
a_{k+1}^{n+1}
\end{bmatrix} & -a_{k+1}^{n+k} \\
a_{k+1}^{n+k} & -a_{k+1}^{n+k-1} \\
\vdots & \vdots \\
a_{k+1}^{n+1} & -a_{k+1}^n
\end{bmatrix}
\]

where \( n \geq k \), \( a_{k+1}^n \) is denoted by \( a_{k+1}^n \) and \( G_{k+1} \) is a \((k+1) \times (k+1)\) matrix as follows:
\[
G_{k+1}' = \begin{bmatrix}
\begin{bmatrix}
-a_{k+1}^{n+k} & -a_{k+1}^{n+k-1} & \cdots & -a_{k+1}^{n+1}
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
-a_{k+1}^{n+k} & -a_{k+1}^{n+k-1} & \cdots & -a_{k+1}^{n+1}
\end{bmatrix}
\end{bmatrix}
\]

It is important to note that \( \det G_{k+1} = (-1)^{k+1} \) and the Simpson identity for a recursive sequence can be obtained from the determinant of its generating matrix. From this point of view, we can easily derive the Simpson formulas of the arrowhead-Fibonacci sequences for every \( k \geq 2 \).

Example 2.2. Since \( \det G_3 = -1 \) and
\[
(G_3)^n = \begin{bmatrix}
a_3^{n+3} - (a_3^{n+2} + a_3^{n+1}) & -a_3^{n+2} \\
a_3^{n+2} & -a_3^{n+1} & -a_3^{n} \\
a_3^{n+1} & -a_3^{n+1} & -a_3^{n-1} & -a_3^{n}
\end{bmatrix}
\]

for \( n \geq 2 \), the Simpson formula of sequence \( \{a_3 (n)\} \) is
\[
a_3^{n+3} (a_3^n)^2 - 2a_3^{n+2}a_3^{n+1}a_3^n + (a_3^{n+2})^2 a_3^{n-1} + (a_3^{n+1})^3 - a_3^{n+3}a_3^{n+1}a_3^{n-1} = 1.
\]

Let \( C (c_1, c_2, \ldots, c_v) \) be a \( v \times v \) companion matrix as follows:
\[
C (c_1, c_2, \ldots, c_v) = \begin{bmatrix}
c_1 & c_2 & \cdots & c_v \\
1 & 0 & 0 & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{bmatrix}
\]
See [18, 19] for more information about the companion matrix.

**Theorem 2.3** (Chen and Louck [20]). The \((i, j)\) entry \(c_{i,j}^{(u)}(c_1, c_2, \ldots, c_v)\) in the matrix \(C^u(c_1, c_2, \ldots, c_v)\) is given by the following formula:

\[
c_{i,j}^{(u)}(c_1, c_2, \ldots, c_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \left( \frac{t_1 + \cdots + t_v}{t_1, \ldots, t_v} \right) c_1^{t_1} \cdots c_v^{t_v}
\]

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + vt_v = u - i + j\), \(\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}\) is a multinomial coefficient, and the coefficients in (3) are defined to be 1 if \(u = i - j\).

Then we can give a combinatorial representation for the arrowhead-Fibonacci numbers by the following Corollary.

**Corollary 2.4.** Let \(a_{k+1+1}(n)\) be the \(n\)th the arrowhead-Fibonacci number for \(k \geq 2\). Then

\[
a_{m,n}(n) = \sum_{(t_1, t_2, \ldots, t_{k+1})} \frac{t_{k+1}}{t_1 + t_2 + \cdots + t_{k+1}} \times \left( \frac{t_1 + \cdots + t_{k+1}}{t_1, \ldots, t_{k+1}} \right) (-1)^{k+1}
\]

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + (k + 1)t_{k+1} = n\).

**Proof.** In Theorem 2.3, if we choose \(v = k + 1, u = n, i = j = k + 1, c_1 = 1\) and \(c_2 = \cdots c_{k+1} = -1\), then the proof is immediately seen from \((G_{k+1})^0\). \(\square\)

Now we consider the Binet formulas for the arrowhead-Fibonacci numbers by using the determinantal representation.

**Lemma 2.5.** The characteristic equation of the arrowhead-Fibonacci sequence \(x^{k+1} - x^k + x^{k-1} + x^{k-2} + \cdots + 1 = 0\) does not have multiple roots.

**Proof.** Let \(f(x) = x^{k+1} - x^k + x^{k-1} + x^{k-2} + \cdots + 1\), then \(f(x) = x^{k+1} - x^k + x^{k-1} + x^{k-2} + \cdots + 1\). It is clear that \(f(0) = 1\) and \(f(1) = k\) for all \(k \geq 2\). Let \(u\) be a multiple root of \(f(x)\), then \(u \notin \{0, 1\}\). If possible, \(u\) is a multiple root of \(f(x)\) in which case \(f(u) = 0\) and \(f'(u) = 0\). Now \(f'(u) = 0\) and \(u \neq 0\) give \(u_1 = 1 + i \frac{k+1}{k+2}\) and \(u_1 = 1 - i \frac{k+1}{k+2}\) while \(f(u) = 0\) shows \(u^k \left(u-1\right)^2 + 1\) so that \((u_1)^k = \frac{(k+2)^2}{2^k+3}\) and \((u_2)^k = \frac{(k+2)^2}{(k+2)^2 + (k+2)}\) which are contradictions since \(k \geq 2\). \(\square\)

If \(x_1, x_2, \ldots, x_{k+1}\) are roots of the equation \(x^{k+1} - x^k + x^{k-1} + x^{k-2} + \cdots + 1 = 0\), then by Lemma 2.5, it is known that \(x_1, x_2, \ldots, x_{k+1}\) are distinct. Let \(V^{k+1}\) be \((k + 1) \times (k + 1)\) Vandermonde matrix as follows:

\[
V^{k+1} = \begin{bmatrix}
(x_1)^k & (x_2)^k & \cdots & (x_{k+1})^k \\
(x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_{k+1})^{k-1} \\
\vdots & \vdots & & \vdots \\
x_1 & x_2 & \cdots & x_{k+1} \\
1 & 1 & \cdots & 1
\end{bmatrix}
\]

Let

\[
U^{k+1}_i = \begin{bmatrix}
(x_1)^{n+k+1-i} \\
(x_2)^{n+k+1-i} \\
\vdots \\
(x_{k+1})^{n+k+1-i}
\end{bmatrix}
\]

and suppose that \(V^{k+1}_{i,j}\) is a \((k + 1) \times (k + 1)\) matrix obtained from \(V^{k+1}\) by replacing the \(j\)th column of \(V^{k+1}\) by \(U^{k+1}_i\).
Theorem 2.6. For $n \geq k \geq 2$,
\[ g_{i,j}^{(n)} \frac{\det V_{i,j}^{k+1}}{\det V^{k+1}}, \]
where $(G_{k+1})^n = \left[ g_{i,j}^{(n)} \right]$.

Proof. Since the eigenvalues of the matrix $G_{k+1}$, $x_1, x_2, \ldots, x_{k+1}$ are distinct, the matrix $G_{k+1}$ is diagonalizable.

Let $D_{k+1} = (x_1, x_2, \ldots, x_{k+1})$, we easily see that $G_{k+1}V^{k+1} = V^{k+1}D_{k+1}$. Since $\det V^{k+1} \neq 0$, the matrix $V^{k+1}$ is invertible. Then it is clear that $(V^{k+1})^{-1} G_{k+1} V^{k+1} = D_{k+1}$. Thus, the matrix $G_{k+1}$ is similar to $D_{k+1}$.

So we get $(G_{k+1})^n V^{k+1} = V^{k+1} (D_{k+1})^n$ for $n \geq k \geq 2$. Then we can write the following linear system of equations:
\[
\begin{align*}
&g_{1,1}^{(n)} (x_1)^k + g_{1,2}^{(n)} (x_1)^{k-1} + \cdots + g_{1,k+1}^{(n)} = (x_1)^{n+k+1-1} \\
&g_{1,1}^{(n)} (x_2)^k + g_{1,2}^{(n)} (x_2)^{k-1} + \cdots + g_{1,k+1}^{(n)} = (x_2)^{n+k+1-1} \\
&\vdots \\
&g_{1,1}^{(n)} (x_{k+1})^k + g_{1,2}^{(n)} (x_{k+1})^{k-1} + \cdots + g_{1,k+1}^{(n)} = (x_{k+1})^{n+k+1-1}
\end{align*}
\]

for $n \geq k \geq 2$. So, for each $i, j = 1, 2, \ldots, k+1$, we obtain $g_{i,j}^{(n)}$ as follows
\[ g_{i,j}^{(n)} = \frac{\det V_{i,j}^{k+1}}{\det V^{k+1}}. \]

Then we can give the Binet formulas for the arrowhead-Fibonacci numbers by the following corollary.

Corollary 2.7. Let $a_{k+1} (n)$ be the $n$th the arrowhead-Fibonacci number for $k \geq 2$. Then
\[ a_{k+1} (n) = \frac{\det V_{k+1,1}^{n+1} - \det V_{k+1,1}^{n+1}}{\det V^{k+1}}. \]

Now we consider the permanent representations of the arrowhead-Fibonacci numbers.

Definition 2.8. A $u \times v$ real matrix $M = \left[ m_{i,j} \right]$ is called a contractible matrix in the $k^{th}$ column (resp. row) if the $k^{th}$ column (resp. row) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix $M$. If $M$ is contractible in the $k^{th}$ column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{i,j}^{\alpha:k}$ obtained from $M$ by replacing the $i^{th}$ row with $m_{i,j}x_j + m_{j,k}x_i$ and deleting the $j^{th}$ row. The $k^{th}$ column is called the contraction in the $k^{th}$ column relative to the $i^{th}$ row and the $j^{th}$ row.

In [21], Brualdi and Gibson obtained that $\text{per} (M) = \text{per} (N)$ if $M$ is a real matrix of order $\alpha > 1$ and $N$ is a contraction of $M$.

Let $M_{k+1}^{\alpha} (n) = \left[ m_{i,j}^{(n)} \right]$ be the $n \times n$ super-diagonal matrix, defined by
\[
m_{i,j}^{(n)} = \begin{cases} 1 & \text{if } i = j = \alpha \text{ for } 1 \leq \alpha \leq n \\ 0 & \text{otherwise} \end{cases}
\]

such that
\[
\begin{align*}
&i = \alpha + 1 \text{ and } j = \alpha \text{ for } 1 \leq \alpha \leq n - 1, \\
&i = \alpha - u \text{ and } j = \alpha \text{ for } u + 1 \leq \alpha \leq n \\
&1 \leq u \leq k,
\end{align*}
\]
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that is,

\[
M_{k+1}(n) = \begin{bmatrix}
1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 \\
1 & 1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 1 & 1 & -1 & \cdots & -1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & 1 & -1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & 1 & 1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & 0 & 1 & 1 & -1 \\
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

Then we have the following theorem.

**Theorem 2.9.** For \( n \geq 1 \) and \( k \geq 2 \),

\[
\text{per}M_{k+1}(n) = a_{k+1}(n+k+1),
\]

where \( \text{per}M_{k+1}(1) = 1 \).

**Proof.** First we start with considering the case \( n < 4 \). The matrices \( M_{k+1}(2) \) and \( M_{k+1}(3) \) are reduced to the following forms:

\[
\text{per}M_{k+1}(2) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

and

\[
\text{per}M_{k+1}(3) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

It is easy to see that \( \text{per}M_{k+1}(2) = 0 \) and \( \text{per}M_{k+1}(3) = -2 \). From definition of arrowhead-Fibonacci sequence it is clear that \( a_{k+1}(k+2) = 1, a_{k+1}(k+3) = 1 \) and \( a_{k+1}(k+4) = -2 \). So we have the conclusion for \( n < 4 \).

Let the equation hold for \( n \geq 4 \), then we show that the equation holds for \( n \geq 1 \). If we expand the \( \text{per}M_{k+1}(n) \) by the Laplace expansion of permanent with respect to the first row, then we obtain

\[
\text{per}M_{k+1}(n+1) = \text{per}M_{k+1}(n) - \text{per}M_{k+1}(n-1) - \cdots - \text{per}M_{k+1}(n-k).
\]

Since \( \text{per}M_{k+1}(n) = a_{k+1}(n+k+1), \text{per}M_{k+1}(n-1) = a_{k+1}(n+k), \ldots, \text{per}M_{k+1}(n-k) = a_{k+1}(n+1) \), we easily obtain that \( \text{per}M_{k+1}(n+1) = a_{k+1}(n+k+2) \). So the proof is complete.

Let \( n > k + 1 \) such that \( k \geq 2 \) and let \( R_{k+1}(n) = \begin{bmatrix} r_{i,j}^{(n)} \end{bmatrix} \) be the \( n \times n \) matrix, defined by

\[
r_{i,j}^{(n)} = \begin{cases} 
1 \quad \text{if } i = j = \alpha \text{ for } 1 \leq \alpha \leq n, \\
1 = \alpha + 1 \text{ and } j = \alpha \text{ for } 1 \leq \alpha \leq n-k-1 \\
-1 \quad \text{if } i = n-k+\alpha \text{ and } n-k+\alpha \leq j \leq n \text{ for } 1 \leq \alpha \leq k-1, \\
\text{such that} \\
1 \leq u \leq k, \\
0 \quad \text{otherwise}.
\end{cases}
\]

that is,
Assume that the $n 	imes n$ matrix $T_{k+1} (n) = \left[ t_{i,j}^{(n)} \right]$ is defined by

$$R_{k+1} (n) = \begin{bmatrix}
1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 \\
1 & 1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & 1 & 1 & -1 & \cdots & -1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 1 & -1 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \rightarrow (n - k) \text{ th}
$$

where $n > k + 2$ such that $k \geq 2$.

Then we can give more general results by using other permanent representations than the above.

**Theorem 2.10.** Let $a_{k+1} (n)$ be the $n$th the arrowhead-Fibonacci number for $k \geq 2$. Then

(i) For $n > k + 1$,

$$\text{per} R_{k+1} (n) = a_{k+1} (n + 1).$$

(ii) For $n > k + 2$,

$$\text{per} T_{k+1} (n) = \sum_{i=1}^{n} a_{k+1} (i).$$

**Proof.** (i). Let the equation hold for $n > k + 1$, then we show that the equation holds for $n + 1$. If we expand the $\text{per} R_{k+1} (n)$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per} R_{k+1} (n + 1) = \text{per} R_{k+1} (n) - \text{per} R_{k+1} (n - 1) - \cdots - \text{per} R_{k+1} (n - k).$$

Also, since

$$\text{per} R_{k+1} (n) = a_{k+1} (n + 1), \text{per} R_{k+1} (n - 1) = a_{k+1} (n), \ldots, \text{per} R_{k+1} (n - k) = a_{k+1} (n - k + 1),$$

it is clear that

$$\text{per} R_{k+1} (n + 1) = a_{k+1} (n + 2).$$

(ii) It is clear that expanding the $\text{per} T_{k+1} (n)$ by the Laplace expansion of permanent with respect to the first row, gives us

$$\text{per} T_{k+1} (n) = \text{per} T_{k+1} (n - 1) + \text{per} R_{k+1} (n).$$

Then, by the result of Theorem 2.10. (i) and an induction on $n$, the conclusion is easily seen.

It is easy to show that the generating function of the arrowhead-Fibonacci sequence $\{a_{k+1} (n)\}$ is as follows:

$$g (x) = \frac{x^k}{1 - x + x^2 + \cdots + x^{k+1}}.$$
where \( k \geq 2 \).

Then we can give an exponential representation for the arrowhead-Fibonacci numbers by the aid of the generating function with the following theorem.

**Theorem 2.11.** The arrowhead-Fibonacci numbers have the following exponential representation:

\[
g(x) = x^k \exp \left( \sum_{i=1}^{\infty} \frac{x^i}{i} \left( 1 - x - \cdots - x^k \right)^i \right),
\]

where \( k \geq 2 \).

**Proof.** Since

\[
\ln g(x) = \ln x^k - \ln \left( 1 - x + x^2 + \cdots + x^{k+1} \right)
\]

and

\[
- \ln \left( 1 - x + x^2 + \cdots + x^{k+1} \right) = - \ln \left( 1 - \left( x - x^2 - \cdots - x^{k+1} \right) \right) = \sum_{i=1}^{\infty} \frac{x^i}{i} \left( 1 - x - \cdots - x^k \right)^i,
\]

it is clear that

\[
\ln g(x) - \ln x^k = \ln \frac{g(x)}{x^k} = \sum_{i=1}^{\infty} \frac{x^i}{i} \left( 1 - x - \cdots - x^k \right)^i.
\]

Thus we have the conclusion. \( \square \)

Now we consider the sums of arrowhead-Fibonacci numbers.

Let

\[
S_n = \sum_{i=1}^{n} a_{k+1}(i)
\]

for \( n \geq 1 \) and \( k \geq 2 \), and suppose that \( A_{k+2} \) is the \( (k + 2) \times (k + 2) \) matrix such that

\[
A_{k+2} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
& & \ddots & \ddots \\
& & & 1 & 0 & \cdots & 0
\end{bmatrix}.
\]

Then it can be shown by induction that

\[
(A_{k+2})^n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
S_{n+k} & \cdots & \cdots & \cdots \\
& & \ddots & \ddots \\
& & & S_n + (G_{k+1})^n
\end{bmatrix}.
\]

### 3 The arrowhead-Fibonacci sequence modulo \( n \)

If we reduce the arrowhead-Fibonacci sequence \( \{a_{k+1}(n)\} \) by a modulus \( m \), then we get the repeating sequence, denoted by

\[
\{a_{k+1}^m(n)\} = \{a_{k+1}^m(1), a_{k+1}^m(2), a_{k+1}^m(3), \ldots, a_{k+1}^m(i), \ldots\}
\]

where we denote \( a_{k+1}(i) \) \( \mod m \) by \( a_{k+1}^m(i) \). It has the same recurrence relation as in (1).
Theorem 3.1. The sequence $\{a_{k+1}^m(n)\}$ is simply periodic for $k \geq 2$. That is, the sequence is periodic and repeats by returning to its starting values.

Proof. Let $X = \{x_1, x_2, \ldots, x_{k+1}\}$ where $x_i$’s are integers such that $0 \leq x_i \leq m - 1$. Since there are $m^{k+1}$ distinct $(k + 1)$-tuples of elements of $Z_m$, at least one of the $(k + 1)$-tuples appears twice in the sequence $\{a_{k+1}^m(n)\}$. Thus, the subsequence following this $(k + 1)$-tuple repeats; hence, the sequence is periodic. Assume that $u > v$ and $a_{k+1}^m(u + 1) \equiv a_{k+1}^m(v + 1) \mod p$, $a_{k+1}^m(u + 2) \equiv a_{k+1}^m(v + 2) \mod p$, \ldots, $a_{k+1}^m(u + k + 1) \equiv a_{k+1}^m(v + k + 1) \mod p$, then $u \equiv v \mod (k + 1)$. From the definition of the arrowhead-Fibonacci sequence, we can easily derive

$$a_{k+1}^m(n) = -a_{k+1}^m(n + k + 1) + a_{k+1}^m(n + k) - a_{k+1}^m(n + k - 1) - \cdots - a_{k+1}^m(n + 1)$$

So using this equation, we easily obtain that

$$a_{k+1}^m(u) \equiv a_{k+1}^m(v) \mod p, a_{k+1}^m(u - 1) \equiv a_{k+1}^m(v - 1) \mod p, \ldots, a_{k+1}^m(u - v) \equiv a_{k+1}^m(1) \mod p,$$

which implies that $\{a_{k+1}^m(n)\}$ is a simply periodic sequence. □

We next denote the period of the sequence $\{a_{k+1}^m(n)\}$ by $P_{k+1}(m)$.

Example 3.2. Since $\{a_2^3(n)\} = \{0, 0, 0, 1, 1, 0, 0, 0, 1, \ldots\}$, $P_4(2) = 5$.

Given an integer matrix $X = [x_{i,j}]$, $X \mod m$ means that all entries of $X$ are modulo $m$, that is, $X \mod m = (x_{i,j} \mod m)$. Let us consider the set $(X)_m = \{X^t \mod m \mid t \geq 0\}$. If $\gcd(m, \det X) = 1$, then $(X)_m$ is a cyclic group. Let $|\{(X)_m\}|$ denote the cardinality of the set $(X)_m$. Since $\det G_{k+1} = (-1)^{k+1}$, $(G_{k+1})_m$ is a cyclic group for every positive integer $m$. By (2), it is easy to show that $P_{k+1}(m) = |(G_{k+1})_m|$ for $k \geq 2$.

Now we give some useful properties for the period $P_{k+1}(m)$ by the following theorem.

Theorem 3.3.

(i) Let $p$ be a prime and suppose that $u$ is the smallest positive integer with $P_{k+1}(p^{u+1}) \neq P_{k+1}(p^u)$. Then $P_{k+1}(p^v) = P_{k+1}(p^u)$ for every $v > u$ and $k \geq 2$.

(ii) If $m$ has the prime factorization $m = \prod_{i=1}^{u} (p_i)^{\alpha_i}$, $(u \geq 1)$, then $P_{k+1}(m)$ equals the least common multiple of the $P_{k+1}(\{p_i\}^{\alpha_i})$’s for $k \geq 2$.

(iii) If $k$ is an even integer such that $k \geq 2$, then $P_{k+1}(m)$ is even for every positive integer $m$.

Proof. (i) If $I$ is the $(k + 1) \times (k + 1)$ identity matrix and $t$ is a positive integer such that $(G_{k+1})_{p^{t+1}} \equiv I \mod p^{t+1}$, then $(G_{k+1})_{p^{t+1}} \equiv I \mod p^t$. Then, it is clear that $P_{k+1}(p^t)$ divides $P_{k+1}(p^{t+1})$. On the other hand, if we denote $(G_{k+1})_{p^{t+1}}(p^t) = I + (a_{i,j}^{(t)} \cdot p^t)$, then by the binomial expansion, we may write

$$(G_{k+1})_{p^{t+1}}(p^t) = (I + \sum_{i=0}^{p} \binom{p}{i} a_{i,j}^{(t)} \cdot p^t)^{t+1} \equiv I \mod p^{t+1}.$$ 

This yields that $P_{k+1}(p^t) \cdot p$ is divisible by $P_{k+1}(p^t)$. Then, $P_{k+1}(p^{t+1}) = P_{k+1}(p^t) \cdot p$ or $P_{k+1}(p^{t+1}) = P_{k+1}(p^t) \cdot p$, and the latter holds if and only if there is a $a_{i,j}^{(t)}$ which is not divisible by $p$. Due to fact that we assume $u$ is the smallest positive integer such that $P_{k+1}(p^{u+1}) \neq P_{k+1}(p^u)$, there is an $a_{i,j}^{(u)}$ which is not divisible by $p$. Since there is a $a_{i,j}^{(u)}$ such that $p$ does not divide $a_{i,j}^{(u)}$, it is easy to see that there is an $a_{i,j}^{(u+1)}$ which is not divisible by $p$. This shows that $P_{k+1}(p^{u+2}) \neq P_{k+1}(p^{u+1})$. Then, we get that $P_{k+1}(p^{u+2}) = p \cdot P_{k+1}(p^{u+1}) = p \cdot (P_{k+1}(p^u)) = p \cdot (P_{k+1}(p^{u+1})) = p \cdot (P_{k+1}(p^{u+1})) = p \cdot p \cdot (P_{k+1}(p^u)) = p^2 \cdot P_{k+1}(p^u)$. So by induction on $u$ we obtain $P_{k+1}(p^v) = p^{u-v} \cdot P_{k+1}(p^u)$ for every $v > u$. In particular, if $P_{k+1}(p^2) \neq P_{k+1}(p)$, then $P_{k+1}(p^v) = p^{u-v} \cdot P_{k+1}(p)$.

(ii) It is clear that the sequence $\{a_{k+1}^m(p^u)\}$ repeats only after blocks of length $\lambda \cdot P_{k+1}(p^u)$ where $\lambda$ is a natural number. Since $P_{k+1}(m)$ is period of the sequence $\{a_{k+1}^m(n)\}$, the sequence $\{a_{k+1}^m(p^u)\}$ repeats after
\( P_{k+1}(m) \) terms for all values \( i \). Thus, we easily see that \( P_{k+1}(m) \) is of the form \( \lambda \cdot P_{k+1}\left((p_1)^{\alpha_1}\right) \) for all values of \( i \), and since any such number gives a period of \( P_{k+1}(m) \). Therefore, we conclude that

\[
P_{k+1}(m) = \text{lcm}\left[ P_{k+1}\left((p_1)^{\alpha_1}\right), \ldots, P_{k+1}\left((p_u)^{\alpha_u}\right) \right].
\]

(iii) Since \( \det G_{k+1} = -1 \) when \( k \) is an even integer and \( P_{k+1}(m) = \left| (G_{k+1})_m \right| \), we have the conclusion.

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References

[1] Kalman D., Generalized Fibonacci numbers by matrix methods, Fibonacci Quart., 1982, 20, 73–76
[2] Sloane N.J.A., Sequences A000045/M0692, A000073/M1074, A000078/M1108, A001591, A01622, A046698, A058265, A086088, and A118745 in The On-Line Encyclopedia of Integer Sequences
[3] Deveci O., Akuzum Y., Karaduman E., The Pell-Padovan p-sequences and its applications, Util. Math., 2015, 98, 327-347
[4] Falcon S., Plaza A., k-Fibonacci sequences modulo m, Chaos Solitons Fractals, 2009, 41, 497-504
[5] Gogin N.D., Myllari A.A., The Fibonacci-Padovan sequence and MacWilliams transform matrices, Programing and Computer Software, published in Programmirovanie, 2007, 33, 74-79
[6] Kilic E., The generalized Pell (p,i)-numbers and their Binet formulas, combinatorial representations, sums, Chaos, Solitons Fractals, 2009, 40, 2047-2063
[7] Kilic E., Stakhov A.P., On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices, Chaos Solitons Fractals, 2009, 40, 2210–2221
[8] Shannon A.G., Bernstein L., The Jacobi-Perron algorithm and the algebra of recursive sequences, Bull. Australian Math. Soc., 1973, 8, 261-277
[9] Stakhov A., Rozin B., Theory of Binet formulas for Fibonacci and Lucas p-numbers, Chaos Solitons Fractals, 2006, 27, 1162–1177
[10] Tasci D., Firengiz M.C., Incomplete Fibonacci and Lucas p-numbers, Math. and Compt. Modell., 2010, 52, 1763-1770
[11] Tuglu N., Kocer E.G., Stakhov A., Bivariate Fibonacci like p-polynomials, Appl. Math. and Compt., 2011, 217, 10239-10246
[12] Shannon A.G., Horadam A.F., Arrowhead curves in a tree of Pythagorean triples, Int. J. Math. Educ. Technol., 1994, 25, 255-261
[13] Wall D.D., Fibonacci series modulo m, Amer. Math. Monthly, 1960, 67, 525-532
[14] Deveci O., Karaduman E., Campbell C.M., The Chebyshev sequences and their applications, Iranian J. Sci. Technol. Trans. A, Sci., (in press)
[15] Lu K., Wang J., k-step Fibonacci sequence modulo m, Util. Math., 2006, 71, 169-177
[16] Ozkan E., On truncated Fibonacci sequences, Indian J. Pure Appl. Math., 2007, 38, 241-251
[17] Deveci O., Karaduman E., The cyclic groups via the Pascal matrices and the generalized Pascal matrices, Linear Algebra Appl., 2012, 437, 2538-2545
[18] Lancaster P., Tismenetsky M., The theory of matrices, Academic, 1985
[19] Lidi R., Niederreiter H., Introduction to finite fields and their applications, Cambridge U.P., 1994
[20] Chen W.Y.C., Louck J.D., The combinatorial power of the companion matrix, Linear Algebra Appl., 1996, 232, 261–278
[21] Brualdi R.A., Gibson P.M., Convex polyhedra of doubly stochastic matrices I: applications of permanent function, J. Combin. Theory, 1977, 22, 194–230