Abstract

We compute the absolute Poisson’s ratio $\nu$ and the bending rigidity exponent $\eta$ of a free-standing two-dimensional crystalline membrane embedded into a space of large dimensionality $d = 2 + d_c$, $d_c \gg 1$. We demonstrate that, in the regime of anomalous Hooke’s law, the absolute Poisson’s ratio approaches material independent value determined solely by the spatial dimensionality $d_c$: $\nu = -1 + 2/d_c - a/d_c^2 + \ldots$ where $a \approx 1.76 \pm 0.02$. Also, we find the following expression for the exponent of the bending rigidity: $\eta = 2/d_c + (73 - 68\zeta(3))/(27d_c^2) + \ldots$. These results cannot be captured by self-consistent screening approximation.

Keywords: crystalline membrane, tethered membrane, graphene, Poisson’s ratio

1. Introduction

The Mermin—Wagner theorem states that in two–dimensional (2D) crystals long–range order is destroyed due to thermal fluctuations \cite{1, 2}. For $D = 2$–dimensional membrane embedded into $d = 3$–dimensional space, this means that transition from $O(d)$–symmetric crumpled phase to $O(D) \times O(d - D)$–symmetric flat phase governed by order parameter $m_{ab} = \partial_\beta r_a$ leads to existence of $d_c = d - D$ “massless” boson modes. Here $d$–dimensional vector $r$ parametrizes the position of a point at the membrane, latin indices $a$ and $b$ indicate the spatial components of $r$. Physically, this Goldstone boson corresponds to out-of-plane (or flexural) phonon mode $h(x)$, where 2D vector $x = (x, y)$ parametrizes the surface of 2D membrane. The flexural phonon produces divergent contribution into thermal fluctuations of order parameter in the thermodynamic limit:

$$\langle (\nabla h)^2 \rangle \propto T \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{\kappa q^4} \propto T \ln \frac{L}{l}. \quad (1)$$

Here $\kappa$ denotes the bending rigidity, $T$ is the temperature, $L$ stands for the size of the membrane, and $l$ is the ultra-violet cutoff of the order of the lattice spacing. In defiance to the statement of
Eq. (1), free–standing 2D membrane, e.g. graphene, do exist experimentally. Resolution of the seeming paradox lies in the fact that Mermin—Wagner theorem applies only to the systems with short–range interactions. Crystalline membranes posses long–range interaction between flexural phonons mediated by in–plane phonons. Effectively such interaction leads to the stiffening of the membrane at large scales. In particular, at small momenta, \( q \ll q_c \), the bending rigidity becomes renormalized [3]:

\[
\kappa_q = \kappa(q_c/q)^q. 
\]

Here \( q_c \sim \sqrt{YT_0/\kappa} \) is the so-called inverse Ginzburg length, where \( Y_0 = \frac{4\mu(\mu + \lambda)}{2\pi r^4} \) denotes the Young modulus of the 2D crystalline membrane with \( \mu \) and \( \lambda \) being the Lamé coefficients of a material. The stiffening of the membrane accounts for the existence of the flat phase:

\[
\langle (\nabla h)^2 \rangle \propto T \frac{\int d^2q q^2}{(2\pi)^2 \kappa q^q} \propto \frac{T}{\eta \kappa}. 
\]

Thus, interaction between phonons is crucial for stability of the 2D membrane.

Since harmonic approximation (\( q = 0 \)) does not suffice to be even a zeroth–order approximation and exact analytical solution of fully interacting problem of phonons modes is not feasible, one has to develop other methods. Up to date, the exponent \( \eta \) was determined within several approximate analytical schemes [3–8]. However, none of these approaches being controllable in the physical case \( D = 2 \) and \( d = 3 \). Numerical simulations for the latter case yielded \( \eta = 0.60 \pm 0.10 [9], \eta = 0.72 \pm 0.04 [10], \eta = 0.85 [11], \) and \( \eta = 0.795 \pm 0.01 [12]. \)

Non-trivial scaling of the bending rigidity, Eq. (2), results in failure of the linear Hooke’s law and in emergence of universal (i.e. material independent) Poisson’s ratio in the regime of small tensions \( \sigma \ll \sigma_r \), where \( \sigma_r = \kappa q_c^2 \sim Y_0 T/\kappa [6–7,13–17]. \) Recently, in the regime \( \sigma \ll \sigma_r \), the anomalous Hooke’s law, i.e. nonlinear dependence of the deformation on the stress, has been experimentally measured in graphene [18].

Most utilized analytical method to study the anomalous elastic properties of 2D crystalline membranes is the self–consistent screening approximation (SCSA) developed in seminal paper [7]. As any other self–consistent scheme, SCSA takes into account some subclass of diagrams in perturbation theory which is typically not preferable with respect to the others. This scheme becomes exact only in the limit \( d_c \to \infty \) where it corresponds to the summation of the leading order logarithmic corrections in perturbation theory. Within SCSA the following results for the bending rigidity exponent and the Poisson’s ratio at zero external stress have been obtained (see Ref. [19] for a review): \( \eta_{\text{SCSA}} = 4/[d_c + \sqrt{16 - 2d_c + d_c^2}] \) and \( \nu_{\text{SCSA}} = -1/3. \) Surprisingly, for \( d_c = 1 \) the value of \( \eta_{\text{SCSA}} = 0.82 \) is very close to the result for \( \eta \) reported from numerics. The value of the Poisson’s ratio \( \nu_{\text{SCSA}} = -1/3 \) is close to some numerical results for the Poisson’s ratio in the physical case \( d_c = 1 [10,20]. \) The “super-universal” (i.e. independent of \( d_c \)) SCSA result for the Poisson’s ratio has been checked to be stable against inclusion of more diagrams in the self–consistent scheme [21]. Also the “super-universal” SCSA result for the Poisson’s ratio has been supported by the non-perturbative renormalization group treatment of the problem [22,23].

The issue of the Poisson’s ratio of 2D crystalline membrane occurs to be more complicated than it has been thought originally. In fact, due to the anomalous Hooke’s law, the Poisson’s ratio can be defined in many ways. Recently, two Poisson’s ratios, differential and absolute, have been introduced and their behaviour has been studied [24]. The differential Poisson’s ratio is determined as the ratio of change in displacements after application of the infinitesimally small uniaxial stress in addition to a finite isotropic tension. The absolute Poisson’s ratio corresponds
to the conventional definition of the Poisson’s ratio, i.e. it is the ratio of displacements after application of a finite uniaxial stress. For $\sigma \ll \sigma_c$, both differential and absolute Poisson’s ratios have universal but different values. In the limit $\sigma \to 0$ they coincide as follows from their definitions. However, their limiting value depends on the boundary conditions since the limit $\sigma \to 0$ and the membrane size $L \to \infty$ do not commute. Also, the differential and absolute Poisson’s ratio coincide in the limit of linear Hooke’s law, $\sigma \gg \sigma_c$, where they both are equal to the value $\lambda/(2\mu + \lambda)$ given by the classical elasticity theory [25]. The universal regime for the Poisson’s ratio is realized at $\sigma_c \gg \sigma \gg \sigma_L$ where $\sigma_L = \sigma_c(q, L)^{n-2}$.

For some reasons, the corrections in $1/d_c$ to the results obtained within SCSA have not been analysed thoroughly. Recently, we have computed the differential Poisson’s ratio to the first order in $1/d_c$ within the universal range of tensions, $\sigma_L \ll \sigma \ll \sigma_c$. We derived the following result: $\nu_{\text{diff}} = -1/3 + c_{\text{diff}}/d_c + \ldots$ where the numerical constant $c_{\text{diff}} \approx 0.016$ [26]. This result indicates that the value of differential Poisson’s ratio in the regime of the anomalous Hooke’s law does depend on the number of flexural phonon modes, $d_c$.

In this paper we extend analysis of $1/d_c$ corrections and compute them for the absolute Poisson’s ratio $\nu$ and the bending rigidity exponent $\eta$. In particular, we find that

$$\nu = -1 + \frac{2}{d_c} - \frac{a}{d_c^2} + \ldots, \quad (4)$$

where $a \approx 1.76 \pm 0.02$ and

$$\eta = \frac{2}{d_c} \frac{68\xi(3) - 73}{27d_c^2} + \ldots \quad (5)$$

The paper is organized as follows. In the section 2, a reader will find the description of the model we use to study 2D crystalline membrane and formal definitions of the absolute Poisson’s ratio. In the Sec. 3 the perturbation theory in flexural phonon interaction is used to obtain expression for $\nu$ up to the second order in $1/d_c$. In the section 4 we calculate the critical exponent $\eta$ up to the second order in $1/d_c^2$. We end the paper with a summary of results, Sec. 5. Technical details are given in Appendices.

2. Formalism

As our starting point we choose the effective action of the Landau–Ginzburg type introduced in the seminal papers [3, 4] for a free-standing 2D membrane. Its imaginary-time Lagrangian is written in terms of the $d$-dimensional vector $r$:

$$\mathcal{L}[r] = \rho(\partial_t r)^2 + \frac{\kappa}{2}(\Delta r)^2 + \frac{\mu}{4}(\partial_x r \partial_y r - \delta_{xy})^2 + \frac{1}{8}(\partial_x r \partial_y r - 2)^2. \quad (6)$$

Here $\rho$ stands for the mass density of the membrane. The Greek indices correspond to the 2D coordinates $(x, y) \equiv x$ parameterizing the membrane. To take into account the effect of external stress we introduce the stretching factors $\xi_x$ and $\xi_y$ such that: $r = [\xi_x x + u_x, \xi_y y + u_y, h_1, \ldots, h_d]$. Here $u = [u_x, u_y]$ corresponds to the in-plane phonons whereas $h = [h_1, \ldots, h_d]$ describes the out-of-plane (or flexural) phonons. Substituting the reparametrization into Eq. (6) allows one to write the partition function for a 2D crystalline membrane in terms of the following (see Refs. [27] for details):

$$Z = \int \mathcal{D}[u, h] \exp(-S). \quad (7)$$

3
Here the action in the imaginary time is given by (\beta = 1/T)

\[
S = \int_0^\beta d\tau \int d^2x \left\{ \frac{1}{8} \left[ (2\mu_k \delta_{\alpha\beta} + \lambda) \left( \xi_\alpha^2 - 1 + K_\alpha \right) \left( \xi_\beta^2 - 1 + K_\beta \right) - K_\alpha K_\beta \right] + \frac{\beta}{2} \left( \partial_\tau u_\alpha^2 + (\partial_\tau h)^2 \right) \right. \\
+ \frac{\kappa}{2} \left( (\Delta h)^2 + (\Delta u)^2 \right) + \mu u_\alpha u_\beta + \frac{\lambda}{2} u_\alpha u_\beta \right\}.
\]

(8)

where

\[
u_{\alpha\beta} = \frac{1}{2} \left( \xi_\alpha \partial_\alpha u_\beta + \xi_\beta \partial_\beta u_\alpha + \partial_\alpha u_\beta \partial_\alpha h + \partial_\beta u_\alpha \partial_\beta h \right),
\]

(9)

and

\[
K_\alpha = \frac{1}{\beta L^2} \int_0^\beta d\tau \int d^2x \left( \partial_\alpha u_\alpha u + \partial_\alpha h \partial_\alpha h \right).
\]

(10)

In this paper we limit the analysis of the absolute Poisson’s ratio to the case of low enough temperature, \(T \ll \kappa\). This condition allows us to neglect the term \(\partial_\alpha u_\beta \partial_\beta u\) in comparison with \(\partial_\alpha h \partial_\beta h\) (see the expressions for \(u_{\alpha\beta}\) and \(K_\alpha\) in Eqs. (9) and (10), respectively) [28]. Then we can simplify the effective action (8) by integrating the in-plane phonons [29] such that the partition function becomes an integral over static flexural phonons only (see Ref. [27] for details):

\[
Z = \int \mathcal{D}[h] \exp(-E/T).
\]

(11)

Here the energy \(E\) of a given configuration of the flexural phonon field \(h(x)\) is as follows

\[
E = \frac{L^2}{8} \left( \xi_\alpha^2 - 1 + K_\alpha \right) M_{\alpha\beta} \left( \xi_\beta^2 - 1 + K_\beta \right) + \frac{\mu}{2L^2} \left( \int d^2x \partial_\alpha h_\beta h_\beta \right)^2 + \frac{\kappa}{2} \int d^2x (\Delta h)^2 \\
+ \frac{\kappa}{2} \int d^2x (\Delta u)^2 \left( \frac{1}{(2\pi)^6} \int d^2k d^2k' d^2q \int \frac{d^2q'}{q^2} \frac{d^2q''}{q'^2} \frac{d^2q'''}{q''^2} \langle h_{k+q} h_{-q} \rangle \langle h_{k'} q \rangle \langle h_{-k' q'} \rangle \right).
\]

(12)

Here we introduced the \(2 \times 2\) matrix \(M_{\alpha\beta} = 2\mu_k \delta_{\alpha\beta} + \lambda\). The quantity \(\tilde{K}_\alpha\) is obtained from \(K_\alpha\) by omitting the term \(\partial_\alpha u_\beta \partial_\beta u\):

\[
\tilde{K}_\alpha = \frac{1}{L^2} \int d^2x \partial_\alpha h \partial_\alpha h.
\]

(13)

The ‘prime’ sign in the last integral on the right hand side of Eq. (12) indicates that the interaction of the flexural phonons with \(q = 0\) is excluded.

The partition function \(Z\) depends on the stretching factors \(\xi_\alpha\) and \(\xi_\gamma\). The diagonal components of the tension tensor are determined as

\[
\sigma_\xi = \frac{1}{\xi_\xi} \frac{\partial f}{\partial \xi}, \quad \sigma_\gamma = \frac{1}{\xi_\gamma} \frac{\partial f}{\partial \xi},
\]

(14)

where \(f = -TL^{-2} \ln Z\) denotes the free energy per unit area. We note that Eq. (14) is the equation of state which determines the relation between the tension tensor \(\{\sigma_\xi, \sigma_\gamma\}\) and the stretching tensor \(\{\xi_\xi, \xi_\gamma\}\).
In what follows, it will be more convenient to choose the diagonal components of the tension tensor as independent variables rather than \( \xi_x \) and \( \xi_y \). As usual, the corresponding free energy \( g(\sigma_x, \sigma_y) \) can be constructed from \( f(\xi_x, \xi_y) \) via the Legendre transform:

\[
g(\sigma_x, \sigma_y) = f(\xi_x, \xi_y) - \sigma_x(\xi_x^2 - 1)/2 - \sigma_y(\xi_y^2 - 1)/2,
\]

where \( \xi_x \) is expressed in terms of \( \sigma_x \) with the help of the equation of state \[14\]. In the thermodynamic limit, \( L \to \infty \), one can explicitly write that

\[
g(\sigma_x, \sigma_y) = f(\xi_x, \xi_y) - \sigma_x(\xi_x^2 - 1)/2 - \sigma_y(\xi_y^2 - 1)/2,
\]

\[15\]

where

\[
E = -\frac{L^2}{2} \sigma_a M_{\alpha\beta} \sigma_{\beta} + \frac{\mu}{2 L^2} \left( \int d^2 x \partial_x h \partial_y h \right)^2 + \frac{\epsilon}{2} \int d^2 x (\Delta h)^2 + \frac{\sigma_a}{2} \int d^2 x (\nabla_x h)^2
\]

\[16\]

\[
+ \frac{Y_0}{8} \int \frac{d^2 k d^2 k' d^2 q [k \times q]^2}{q^2 q'^2} \left( h_{k+q}^x h_{-k}^y - h_{-k+q}^x h_{k}^y \right).
\]

\[17\]

In terms of \( g(\sigma_x, \sigma_y) \) the equation of states can be written as

\[
\epsilon_x \equiv \frac{\xi_x^2 - 1}{2} = -\frac{\partial g}{\partial \sigma_x}, \quad \epsilon_y \equiv \frac{\xi_y^2 - 1}{2} = -\frac{\partial g}{\partial \sigma_y},
\]

\[18\]

or more explicitly,

\[
\begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} = M \begin{pmatrix} \epsilon_x + \langle K_x \rangle/2 \\ \epsilon_y + \langle K_y \rangle/2 \end{pmatrix}.
\]

\[19\]

Here the average \( \langle \ldots \rangle \) is with respect to the energy \[17\].

Now the absolute Poisson’s ratio can be defined as follows. Let us apply the uniaxial stress, \( \sigma_x = \sigma \) and \( \sigma_y = 0 \), and consider the change of the stretching factors:

\[
\delta \epsilon_\beta(\sigma) = \epsilon_\beta(\sigma, 0) - \epsilon_\beta(0, 0).
\]

\[20\]

Then the absolute Poisson’s ratio is given as

\[
\nu = -\frac{\delta \epsilon_y}{\delta \epsilon_x} = \frac{\nu_0 - Y_0 \delta K_y/(2\sigma)}{1 + Y_0 \delta K_x/(2\sigma)},
\]

\[21\]

where \( \delta K_\beta(\sigma) = (\bar{K}_\beta(0, 0)) - (\bar{K}_\beta(\sigma, 0)) \) and \( \nu_0 = \lambda/(2\mu + \lambda) \) stands for the classical value of the Poisson’s ratio.

The overall behavior of the absolute Poisson ratio on \( \sigma \) has been discussed recently in Ref. \[24\]. Since \( Y_0 \delta K_\beta(\sigma)/\sigma \sim (\sigma_*/\sigma)^{\alpha} \) with the exponent \( \alpha = \eta/(2 - \eta) \) \[14\], the general expression \[21\] for the absolute Poisson’s ratio can be simplified in the universal regime \( \sigma \ll \sigma_* \):

\[
\nu = -\frac{\delta \bar{K}_y}{\delta \bar{K}_x}
\]

\[22\]
3. Perturbation theory in $1/d_c$ for the absolute Poisson’s ratio

In order to proceed with the computation of the absolute Poisson’s ratio, Eq. (22), we need to compute the $\delta \tilde{K}_\beta(\sigma)$. We can write the following formal expression in terms of the exact Green’s function of flexural phonons:

$$\langle \tilde{K}_\beta(\sigma, 0) \rangle = d_c \int \frac{d^2q}{(2\pi)^2} q_y^2 \tilde{G}_q, \quad \tilde{G}_q = \frac{T}{\pi q^4 + \sigma q_x^2 - \Sigma_{\sigma}(q^2)}. \quad (23)$$

As usual, the self-energy $\Sigma_{\sigma}(q)$ is due to the interaction between out-of-plane phonons. Although at $\sigma \ll \sigma_c$ this interaction is effectively controlled by $1/d_c$, we cannot develop the expansion in $1/d_c$ by expanding of the Green’s function in powers of $\Sigma_{\sigma}(q)$. The point is the infra-red divergence of the expression for $K_\beta(\sigma, 0)$ in the absence of $\Sigma_{\sigma}(q)$. In order to resolve this problem we construct an expansion in difference $\delta \Sigma_{\sigma}(q) = \Sigma_{\sigma}(q) - \Sigma_{\sigma}(q_0)$. Then, we find

$$\delta \tilde{K}_\beta(\sigma) = d_c T \int \frac{d^2q}{(2\pi)^2} \frac{q_x^2}{\pi q^4} \frac{\sigma q_y^2 - \delta \Sigma_{\sigma}(q)}{\pi q^4 + \sigma q_x^2 - \delta \Sigma_{\sigma}(q)}, \quad (24)$$

where $\pi q^4 = \pi q^4 - \Sigma_{\sigma}(q)$. Now we can expand the self-energy difference $\delta \Sigma_{\sigma}(q)$ in powers of $1/d_c$:

$$\delta \Sigma_{\sigma}(q) = \delta \Sigma_{\sigma}^{(1)}(q) + \delta \Sigma_{\sigma}^{(2)}(q) + \ldots \quad (25)$$

Such an expansion results in regular perturbation series for $\delta \tilde{K}_\beta(\sigma)$:

$$\delta \tilde{K}_\beta(\sigma) = \delta \tilde{K}_\beta^{(0)}(\sigma) + \delta \tilde{K}_\beta^{(1)}(\sigma) + \ldots \quad (26)$$

Then, we find

$$\nu = -\frac{\delta \tilde{K}_\beta^{(0)}(\sigma)}{\delta \tilde{K}_\beta^{(1)}(\sigma)} \left\{ 1 + \frac{\delta \tilde{K}_\beta^{(1)}(\sigma)}{\delta \tilde{K}_\beta^{(0)}(\sigma)} = \frac{\delta \tilde{K}_\beta^{(1)}(\sigma)}{\delta \tilde{K}_\beta^{(0)}(\sigma)} + \ldots \right\}. \quad (27)$$

Here the functions $\delta \tilde{K}_\beta^{(0,1)}(\sigma)$ are given explicitly as

$$\delta \tilde{K}_\beta^{(0)}(\sigma) = d_c T \int \frac{d^2q}{(2\pi)^2} \frac{\sigma q_y^2 q_x^2}{\pi q^4 \left[ \pi q^4 + \sigma q_x^2 \right]} \quad (28)$$

and

$$\delta \tilde{K}_\beta^{(1)}(\sigma) = -d_c T \int \frac{d^2q}{(2\pi)^2} \frac{\pi q^4 \delta \Sigma_{\sigma}^{(1)}(q)}{\left[ \pi q^4 + \sigma q_x^2 \right]^2}. \quad (29)$$

Both expressions are manifestly convergent in the ultra-violet. For the convergence of $\delta \tilde{K}_\beta^{(0)}(\sigma)$ in the infrared, the power-law renormalization of the bending rigidity (due to the interaction-induced self-energy $\Sigma_{\sigma}(q)$) is crucial. The expression $\delta \tilde{K}_\beta^{(1)}(\sigma)$ is convergent in the infrared even in the absence of the bending rigidity renormalization.

Performing evaluation of the integral over momentum we find

$$\delta \tilde{K}_\beta^{(0)}(\sigma) = \frac{d_c T}{2(2 - \eta) \pi \eta} \frac{\pi \eta}{\pi \eta} \left( \frac{\sin \frac{\pi \eta}{2 - \eta}}{n^2 \eta \beta} \right)^{-1} \langle n^2 \eta \beta \rangle. \quad (30)$$
We note that since $\eta = 2/d_c + O(1/d_c^2)$ we obtain the following result at $d_c \to \infty$ ($\eta \to 0$)

$$\delta \tilde{K}^{(0)}(\sigma) = \frac{d_c^2 T}{8\pi \varkappa}.$$  (31)

Now using the relation

$$\frac{\langle n_x^2 n_y^2 \rangle}{\langle n_x^2 n_x^2 \rangle} = \frac{1}{1 + 2\alpha},$$  (32)

we find

$$\nu = -\frac{1}{1 + 2\alpha} \left(1 - \frac{c}{d_c^2} + \ldots\right), \quad c = \frac{8\pi \varkappa}{T} \left[\delta \tilde{K}^{(1)}(\sigma) - \delta \tilde{K}^{(0)}(\sigma)\right].$$  (33)

Since the integrals in $\delta \tilde{K}^{(1)}(\sigma)$ are convergent even in the absence of the renormalization of the bending rigidity, we can set $\eta = 0$ in $\varkappa$ for the computation of the constant $c$. Taking into account the deviation of $\eta$ from 0 results in the $1/d_c$ correction to $c$ which is beyond our accuracy.

### 3.1. Evaluation of $\delta \Sigma^{(1)}(\sigma)$

In order to find the value of the coefficient $c$, we need to compute $\delta \tilde{K}^{(1)}(\sigma)$. The self-energy $\Sigma^{(1)}(\sigma)$ is shown in Fig. 1. It involves the screened interaction [7] between flexural phonons (for details see Appendix A of Ref. [26]):

$$N_q^{(\sigma)} = \frac{Y_0/2}{1 + 3Y_0\Pi^{(\sigma)}/2}. $$  (34)

Here $\Pi^{(\sigma)}$ denotes the irreducible polarization operator in the presence of uniaxial stress. To the leading order in $1/d_c$, it can be written as

$$\Pi^{(\sigma)}(q) = \frac{d_c T}{3} \int \frac{d^2 k}{(2\pi)^2} \left[\frac{1}{\varkappa |k - q|^2 + \sigma |k - q|^4}\right].$$  (35)

The polarization operator has the following scaling form:

$$\Pi^{(\sigma)}(q) = \frac{d_c T}{3\varkappa \sigma} \mathcal{P}(q/q_\sigma).$$  (36)

In the universal regime $\sigma \ll \sigma_*$ one can neglect unity in denominator of Eq. (34) and approximate $N_q^{(\sigma)}$ as $1/\{3\Pi^{(\sigma)}\}$. Then the corresponding self-energy difference becomes

$$\delta \Sigma^{(1)}(k) = \frac{2T}{3} \int \frac{d^2 q}{(2\pi)^2} \left[\frac{k \times q}{q^2} \right]^4 \left(\frac{1}{\varkappa |k - q|^2 \Pi^{(\sigma)}} - \frac{1}{\varkappa |k - q|^4} + \frac{1}{\varkappa |k - q|^2 \Pi^{(\sigma)}}\right).$$  (37)
where \( \hat{\eta} \) to compute \( Q \) at large asymptotic behaviour: \( \text{Appendix A} \) for details. Introducing the function \( A(Q) = \arcsinh(Q)/[Q \sqrt{1 + Q^2}] \) and the vector \( P = (Q_x + i, Q_y) \), we result can be written as

\[
\mathcal{P}(Q) = \left\{ \begin{array}{ll}
(1 - 4Q_x^2)\hat{Q}_x^2 \left[ (A(Q) - 1) - \text{Re} \left[ Q^2 P^{-2}(A(P) - 1) \right] \right] + (1 + Q_x^2)A(Q) + \text{Re} \left[ (1 - P^2)A(P) \right] \\
4\hat{Q}_x \hat{Q}_y (\hat{Q}_x^2 - \hat{Q}_y^2) \text{Im} \left[ Q^2 P^{-2}(A(P) - 1) \right] + 4\hat{Q}_x^2 \hat{Q}_y \left[ A(Q) - \text{Re} A(P) \right] + 2(\hat{Q}_x^2 - \hat{Q}_y^2) \text{Im} \left[ P \text{Re} A(P) \right] \\
4\hat{Q}_x \hat{Q}_y (\hat{Q}_x^2 - \hat{Q}_y^2) \left[ A(Q) - \text{Re} A(P) \right] + 2(\hat{Q}_x^2 - \hat{Q}_y^2) \text{Im} \left[ P \text{Re} A(P) \right] \end{array} \right. / (8\pi),
\]

where \( \hat{Q} = Q/Q = [\cos \phi, \sin \phi] \) and \( P = \sqrt{(Q_x + i)^2 + Q_y^2} \). The function \( \mathcal{P}(Q) \) has the following asymptotic behaviour:

\[
\mathcal{P}(Q) = \begin{cases} 
\cos^{7/2} \phi - 2 \cos(2\phi) + \cos(4\phi) & , \quad Q \ll 1 \\
\frac{8\sqrt{Q}}{3} + \frac{8\pi}{16\pi Q^2} \left[ 9[1 + 4 \ln(2Q)] - 8 \cos(2\phi)[4 - 3 \ln(2Q)] + \cos(4\phi) \right] & , \quad Q \gg 1
\end{cases}
\]

We note the strong anisotropy in angle dependence of \( \mathcal{P}(Q) \) at small \( Q \). The asymptotics of \( \mathcal{P}(Q) \) at large \( Q \) implies that

\[
\Pi^{(0)}(q) = \frac{d, T}{16\pi q^2 q_x}. \tag{40}
\]

Now using Eqs. (37), (38), and (40), we can rewrite Eq. (33) in the following form

\[
c = 16\pi \int \frac{d^2 Q d^2 K [K \times Q]^2}{(2\pi)^4} \frac{1}{\mathcal{P}(Q)[K^4 + K_z^2]^2 |K - Q|^4 + (K_x - Q_x)^2} \left[ 1 - \frac{\mathcal{P}_0(Q)[K^4 + K_z^2]^2 |K - Q|^4}{\mathcal{P}_0(Q)[K^4 + K_z^2]^2 |K - Q|^4} \right] \right| \tag{41}
\]

where \( \mathcal{P}_0(Q) = 3/(16\pi Q^3) \). Numerical evaluation of this integral (see \text{Appendix B} \) yields

\[
c = 0.56 \pm 0.02. \tag{42}
\]

Now using Eq. (33), we can write the expansion of the absolute Poisson ratio to the second order in \( 1/d_c^2 \):

\[
\nu = -1 + \frac{2}{d_c} - \frac{2 - b - c}{d_c^2} + \ldots. \tag{43}
\]

Here the coefficient \( b \) determines expansion of the bending rigidity exponent to the second order in \( 1/d_c^2 \):

\[
\eta = \frac{2}{d_c} + \frac{b}{d_c^2} + \ldots. \tag{44}
\]

Therefore, in order to determine the absolute Poisson ratio to the second order in \( 1/d_c \) one needs to compute \( \eta \) to the same order.
4. Evaluation of $\eta$ to the second order in $1/d_c$

Perturbative calculation of critical exponent $\eta$ describing softening of the flexural mode due to interaction between phonons is quite straightforward. General statement (2) for $\varkappa_q$ implies that exact self-energy $\Sigma_0(k)$ has the following expansion at small values of momenta, $k \ll q_*$, and for $\eta \ll 1$:

$$\Sigma_0(k) = -\varkappa^2 \left( \eta \ln \frac{q_*}{k} + \frac{\eta^2}{2} \ln^2 \frac{q_*}{k} + \ldots \right).$$

(45)

4.1. SCSA type contributions to $\eta$

4.1.1. First order in $1/d_c$

In order to set notations, we start from the self-energy correction in the first order in the screened interaction (see Fig. 1):

$$\Sigma^{(1)}_0(k) = -2 \int_q \frac{[k \times q]^4}{q^4} N_q G^{(0)}_{k-q}, \quad G^{(0)}_k = \frac{T}{\varkappa} k^4.$$  

(46)

Here we introduced for a brevity the following shorthand notation: $\int_q \equiv \int d^2 q / (2\pi)^2$. Let us define $\tilde{q}_* = \sqrt{3d_c Y_0 T / (32\pi \varkappa^2)}$. Then, we obtain

$$\Sigma^{(1)}_0(k) = -2 d_c 16\pi \varkappa \frac{1}{3} \int_q \frac{[k \times q]^4}{q^2} \frac{\tilde{q}_*^2}{\tilde{q}_*^2 + q^2} \frac{1}{|k - q|^4}.$$  

(47)

Using the following integral

$$\int_0^{2\pi} d\theta \frac{\sin^4 \theta}{2\pi (1 + q^2 - 2q \cos \theta)^2} = \frac{3}{8} \begin{cases} 1, & q \leq 1, \\ q^{-4}, & q > 1. \end{cases}$$

(48)

we obtain

$$\Sigma^{(1)}_0(k) = -\frac{2}{\varkappa} k^4 L(k/\tilde{q}_*),$$

(49)

where

$$L(K) = \frac{1}{K^2} \int_0^K dq \frac{q^3}{1 + q^2} + \int_K^{\infty} dq \frac{1}{q(1 + q^2)} = -\ln K + \frac{1}{2} \ln(1 + K^2) + \frac{K^2 - \ln(1 + K^2)}{2K^4}.$$  

(50)

For $k/\tilde{q}_* \ll 1$ we find

$$\Sigma^{(1)}_0(k) = -\frac{2}{\varkappa} k^4 \ln \frac{q_*}{k},$$

(51)

where $q_* = e^{1/4} \tilde{q}_*$. Comparison of this result with the expansion (45) yields $\eta = 2/d_c$.  

9
4.1.2. Second order self-energy correction

We start from the diagram (a) in Fig. 2. The corresponding contribution to the self-energy can be written as

\[
\Sigma^{(2,0)}_0(k) = -\frac{2}{T} \int_q \frac{(k \times q)^4}{q^4} G^{(0)}_{k-q} \tilde{\Sigma}^{(1)}(k-q) = \frac{4}{d_v} \frac{16\pi \varkappa}{3} \int_q \frac{(k \times q)^4}{q^2} \frac{\tilde{q}_z^2}{q^2 + q^2} |k-q|^3 L(|k-q|/\tilde{q}_z). \tag{52}
\]

This diagram diverges in the infrared as \(\ln^2(k/\tilde{q}_z)\). However, we are interested also in the next, subleading, term which behaves as \(\ln(k/\tilde{q}_z)\). Therefore, we cannot approximate the function \(L(|k-q|/\tilde{q}_z)\) by the logarithm. Instead, we rewrite \(\Sigma^{(2,0)}_0(k)\) as follows

\[
\Sigma^{(2,0)}_0(k) = \frac{4}{d_v} \frac{16\pi \varkappa}{3} \int_q \frac{(k \times q)^4}{q^2} \frac{\tilde{q}_z^2}{q^2 + q^2} |k-q|^3 \frac{1}{q^2 + q^2} [L(|k-q|/\tilde{q}_z) - L(q/\tilde{q}_z)]. \tag{53}
\]

The last integral in the right hand side of the above expression is convergent in both ultraviolet and infrared. Thus we are not interested in it. Then, we find

\[
\Sigma^{(2,0)}_0(k) = \frac{4}{d_v} \varkappa k^6 \left[ \frac{1}{K^2} \int_0^K dq \frac{q^3 L(q)}{1 + q^2} + \int_K^{\infty} dq \frac{L(q)}{q(1 + q^2)} \right]. \tag{54}
\]

Evaluating the integrals for \(K \ll 1\), we obtain

\[
\Sigma^{(2,0)}_0(k) = \frac{2}{d_v} \varkappa k^6 \left[ \ln^2 K - \ln K \right] = \frac{2}{d_v} \varkappa k^6 \ln^2(q_*/k) + \frac{1}{2} \ln(q_*/k) \tag{55}
\]

Next, we compute the diagram (b) in Fig. 2. The diagram can be considered as the first order correction to the self-energy in which the interaction line is changed due to correction to the polarization operator:

\[
\Sigma^{(2,0)}_b(k) = \frac{6}{T} \int_q \frac{(k \times q)^4}{q^4} \left[ N^{(0)}_q \right]^2 G^{(0)}_{k-q} \delta \Pi^{(0)}_q, \quad \delta \Pi^{(0)}_q = \frac{2d_v}{3T} \int_k \frac{(k \times q)^4}{q^4} G^{(0)}_{k-q} \left[ G^{(0)}_k \right]^2 \tilde{\Sigma}^{(1)}(k). \tag{56}
\]

The correction to the polarization operator becomes

\[
\delta \Pi^{(0)}_q = -\frac{4T^2}{3\varkappa^2} \int_k \frac{(k \times q)^4}{q^4} L(|k-q|/\tilde{q}_z) = -\frac{T^2}{4\pi \varkappa^2 q^2} \left[ \frac{T}{k^3} \int_0^1 dk k L(qk/\tilde{q}_z) + \frac{1}{k} \int_1^{\infty} \frac{dk}{k^2} L(qk/\tilde{q}_z) \right] \tag{57}
\]

\[
= -\frac{T^2}{4\pi \varkappa^2 q^2} L(q/\tilde{q}_z). \tag{58}
\]
where
\[ \tilde{L}(K) = -\frac{3 + K^2}{3} \ln K + \frac{(1 + K^2)^2}{6K^2} \left[ \frac{(1 + K^2)^2}{K^2} \ln(1 + K^2) - 1 \right] \]  

(58)

We note that the function \( \tilde{L}(K) \) has the same asymptotic behaviour at \( K \ll 1 \) as the function \( L(K) \). At \( K \gg 1 \) the asymptotic of \( \tilde{L}(K) \) is given as \( \frac{1}{2K^2} \). Then, we obtain
\[ \Sigma^{(2,b)}_0(k) = -\frac{8}{d_c^2} \frac{16\pi}{3} \int \frac{d^3 q}{q^4} \left[ k \times q \right] \tilde{q}^2 \tilde{L}(q) = -\frac{8}{d_c^2} \frac{4\pi}{3} \int_0^\infty \frac{d q}{q^4} \left[ k \times q \right] \tilde{q}^2 \tilde{L}(q) \]

(59)

At \( K \ll 1 \) we find
\[ \Sigma^{(2,b)}_0(k) = -\frac{4}{d_c^2} \frac{4\pi}{3} \int \frac{d^3 q}{q^4} \left[ k \times q \right] \tilde{q}^2 \tilde{L}(q) = -\frac{4}{d_c^2} \frac{4\pi}{3} \int_0^\infty \frac{d q}{q^4} \left[ k \times q \right] \tilde{q}^2 \tilde{L}(q) \]

(60)

Summing up the SCSA-type corrections to the self-energy, \( \Sigma_0^{(1)} \), \( \Sigma_0^{(2,a)} \), and \( \Sigma_0^{(2,b)} \), we find that
\[ \varkappa k^4 - \Sigma_0^{(1)}(k) - \Sigma_0^{(2,a)}(k) - \Sigma_0^{(2,b)}(k) = \varkappa k^4 \left[ 1 + \frac{2}{d_c} \ln \frac{q_c}{k} + \frac{2}{d_c^2} \ln^2 \frac{q_c}{k} + \frac{1}{d_c^2} \ln \frac{q_c}{k} \right] \approx \varkappa k^4 \eta_{SCSA} \]

(61)

where \( \eta_{SCSA} = \frac{2}{d_c} + \frac{1}{d_c^2} + \ldots \) which is nothing but expansion of the general SCSA result in \( 1/d_c \).

### 4.2. Non-SCSA-type corrections to \( \eta \)

In addition to the SCSA type diagrams there exist four more diagrams for the self-energy in the second order in \( 1/d_c \), shown in Fig. 3. Contrary to the SCSA type diagrams the diagrams in Fig. 3 have only logarithmic divergence at the infrared. This allows us, on the one hand, to send \( \tilde{q} \) to \( \infty \) in the expressions for \( N^{(0)}_q \), and, on the other, send the external momentum \( k \to 0 \) whenever it is possible. After such the procedure, we shall compute the integral by restoring the ultraviolet (\( \tilde{q} \)) and infrared cutoffs (\( k \)) in the most convenient way. However, contrary to the SCSA type diagrams, the evaluation of each of the diagrams in Fig. 3 is still involved. Details of the analytical evaluation of the integrals determining the non-SCSA corrections to \( \eta \) can be found in Appendices C and D.
4.2.1. Evaluation of $\Sigma^{(2,c)}_0(k)$

The correction to the self-energy shown in Fig. 3a) can be written as

$$
\Sigma^{(2,c)}_0(k) = \frac{4}{T} \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} G_{k-q}^{(0)} G_{k-q}^{(0)} G_{k-q}^{(0)} N_{Q}^{(0)} N_{Q}^{(0)}.
$$

(62)

Taking the limit $q_s$ to $\infty$ and neglecting the external momentum $k$ in comparison with $q$ and $Q$, we find

$$
\Sigma^{(2,c)}_0(k) = \left( \frac{32\pi}{3d_c} \right)^2 \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4
\begin{align}
\Sigma^{(2,c)}_0(k) &= \left( \frac{32\pi}{3d_c} \right)^2 \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4
\end{align}
\text{(63)}

Now since $\Sigma^{(2,c)}_0(k)$ depends only on the absolute value of $k$, we can perform averaging of $\Sigma^{(2,c)}_0(k)$ over directions of $k$. Then we find

$$
\Sigma^{(2,c)}_0(k) = \left[ \frac{1}{2} \right] \left( \frac{64\pi}{3d_c} \right)^2 \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4.
\begin{align}
\Sigma^{(2,c)}_0(k) &= \left[ \frac{1}{2} \right] \left( \frac{64\pi}{3d_c} \right)^2 \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4.
\end{align}
\text{(64)}

Now using the following integral

$$
\int_0^{2\pi} d\theta (1 + 2 \cos^2 \theta) \sin^4 \theta = \frac{1}{16} \left( \frac{8q^2 + 5Q^2}{q^6} \right) Q \leq q,
\begin{align}
\int_0^{2\pi} d\theta (1 + 2 \cos^2 \theta) \sin^4 \theta = \frac{1}{16} \left( \frac{8q^2 + 5Q^2}{q^6} \right) Q \leq q,
\end{align}
\text{(65)}

we find

$$
\Sigma^{(2,c)}_0(k) = \frac{4}{9d_c^2} \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4.
\begin{align}
\Sigma^{(2,c)}_0(k) &= \frac{4}{9d_c^2} \int_{q_0} \frac{|k \times q|^2}{q^4} \frac{|k \times Q|^2}{q^4} \frac{|(k - q) \times Q|^2}{q^4} \frac{|(k - Q) \times q|^2}{q^4} q^6 Q^6 |q - Q|^4.
\end{align}
\text{(66)}

4.2.2. Evaluation of $\Sigma^{(2,d)}_0(k)$

The correction to the self-energy shown in Fig. 3b) has the following form

$$
\Sigma^{(2,d)}_0(k) = \frac{4d_c}{T} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2} \times G_{p-p'}^{(0)} G_{p-p}^{(0)} G_{p-p}^{(0)} G_{p-p}^{(0)} N_{Q}^{(0)} N_{Q}^{(0)}.
\begin{align}
\Sigma^{(2,d)}_0(k) &= \frac{4d_c}{T} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2} \times G_{p-p'}^{(0)} G_{p-p}^{(0)} G_{p-p}^{(0)} G_{p-p}^{(0)} N_{Q}^{(0)} N_{Q}^{(0)}.
\end{align}
\text{(67)}

Again taking the limit $q_s$ to $\infty$ and neglecting the external momentum $k$ in the argument of the Green’s function, we find after averaging over directions of $k$:

$$
\Sigma^{(2,d)}_0(k) = -\frac{16\pi}{3} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2}.
\begin{align}
\Sigma^{(2,d)}_0(k) &= -\frac{16\pi}{3} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2}.
\end{align}
\text{(68)}

Let us introduce the angles $\varphi = \angle(Q, p)$ and $\varphi' = \angle(Q, p')$. Then, we obtain

$$
\Sigma^{(2,d)}_0(k) = -\frac{16\pi}{3} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2}.
\begin{align}
\Sigma^{(2,d)}_0(k) &= -\frac{16\pi}{3} \int_{p,p'} \frac{|p \times (p - p')|^4}{p^4 (p - p')^4} \frac{|p \times Q|^2}{p^2 Q^2} \frac{|p \times (p - Q)|^2}{p^2 Q^2} \frac{|p' \times Q|^2}{p'^2 Q^2} \frac{|p' \times (p' - Q)|^2}{p'^2 Q^2}.
\end{align}
\text{(69)}

Let us make a change of variables \( p \rightarrow pQ \) and \( p' \rightarrow p'Q \), then we find
\[
\Sigma_0^{(2,0)}(k) = \frac{b^{(2,0)}_{\beta}}{d^2_{\beta}} \propto k^4 \ln \frac{q_\beta}{k},
\] (70)
where
\[
b^{(2,0)}_{\beta} = -\left( \frac{16}{3} \right)^2 \int_0^\infty dp dp' \int_{2\pi}^\infty d\varphi d\varphi' \frac{p' \sin^2 \varphi \sin^2 \varphi' \sin^2(\varphi - \varphi')}{(p^2 + 1 - 2p \sin \varphi)^2}
\times \frac{[p' \sin \varphi' - p \sin \varphi - pp' \sin(\varphi' - \varphi)]^2}{(p^2 + 1 - 2p \sin \varphi)^2(2p^2 + 2p^2 - 2pp' \cos(\varphi - \varphi))^2}.
\] (71)

We note that the expression under the integral signs is symmetric under the interchange of \( p \) and \( p' \). The explicit calculation of the integral in Eq. (71) presented in Appendix C-1 yields
\[
b^{(2,0)}_{\beta} = -2.
\]

Thus, we find
\[
\Sigma_0^{(2,0)}(k) = -\frac{2}{d^2_{\beta}} \propto k^4 \ln \frac{q_\beta}{k}.
\] (72)

4.2.3. Evaluation of \( \Sigma_0^{(2,0)}(k) \)

The correction to the self-energy shown in Fig. 3e) has the following form
\[
\Sigma_0^{(2,0)}(k) = -\frac{8d_\beta}{T^2} \int_{Q,p,Q} \frac{[k \times q]^2 [k \times Q]^2 [(k - q) \times (Q - q)]^2 [p \times Q]^2 [p \times Q]^2}{Q^2 q^2 |Q - Q|^2}
\times \frac{1}{|Q - Q|^2} G^{(0)}_{p} G^{(0)}_{p} G^{(0)}_{p} G^{(0)}_{p} G^{(0)}_{p} G^{(0)}_{p} N^{(0)}_{Q} N^{(0)}_{Q} N^{(0)}_{Q} N^{(0)}_{Q}.
\] (73)

Again we take the limit \( q_\beta \rightarrow \infty \). Next neglecting the external momentum \( k \) in the argument of the Green’s functions, we find after averaging over directions of \( k \):
\[
\Sigma_0^{(2,0)}(k) = -\left( \frac{16\pi}{3} \right)^2 \propto k^4 \int_{Q,p,q} \frac{q^2 Q^2 + 2(q \cdot Q)^2 |q \times Q|^2 |p \times Q|^2 |p \times Q|^2 |(p - Q) \times (q - Q)|^2}{Q^2 q^2 p^4 |p - Q|^4 |q - Q|^4}.
\] (74)

Next we introduce angles \( \varphi = \angle(p, q) \) and \( \varphi' = \angle(p, Q) \). Then, we find
\[
\Sigma_0^{(2,0)}(k) = -\left( \frac{8}{3d_\beta} \right) \propto k^4 \int_{Q,p,q} \frac{d^2 p d^2 q d^2 \varphi d^2 \varphi'}{2(2\pi)^2}
\times \frac{pqQ^2 \sin^2 \varphi \sin^2 \varphi' (1 + 2 \cos^2(\varphi - \varphi') - 2pq \cos \varphi)^2}{(p^2 + q^2 - 2pq \cos \varphi)^2}
\times \frac{\sin^2(\varphi' - \varphi) |pqQ \sin \varphi - pq \sin \varphi - qQ \sin(\varphi' - \varphi)|^2}{(p^2 + Q^2 - 2pqQ \cos \varphi')^2 |q^2 + Q^2 - 2pqQ \cos(\varphi - \varphi)|^2}.
\] (75)

Let us change variables \( Q \rightarrow pQ \) and \( q \rightarrow pq \) then we find
\[
\Sigma_0^{(2,0)}(k) = \frac{b^{(2,0)}_{\beta}}{d^2_{\beta}} \propto k^4 \int_0^\infty \frac{dp}{p} \frac{b^{(2,0)}_{\beta}}{d^2_{\beta}} \propto k^4 \ln \frac{q_\beta}{k}.
\] (76)
where

\[
b^{(2,e)} = -\frac{64}{27} \int_0^\infty dQ dq \int_0^{2\pi} \frac{d\varphi d\varphi'}{(2\pi)^2} \frac{qQ \sin^2 \varphi \sin^2 \varphi' (1 + 2 \cos^2 (\varphi - \varphi'))}{(q^2 + 1 - 2q \cos \varphi)^2} \\
\times \frac{\sin^2 (\varphi - \varphi') |Q \sin \varphi' - q \sin \varphi - qQ \sin (\varphi' - \varphi)|^2}{(1 + Q^2 - 2Q \cos \varphi')^2 (q^2 + Q^2 - 2qQ \cos (\varphi - \varphi'))}.
\]

(77)

We note that the expression under the integral sign is symmetric under the interchange of \(q\) and \(Q\). The calculation presented in Appendix C yields

\[
b^{(2,e)} = -\frac{58}{27}.
\]

(78)

Hence, we obtain

\[
\Sigma_0^{(2,f)}(k) = -\frac{58}{27} \frac{k^4}{k} \ln \frac{q_s}{k}.
\]

(79)

4.2.4. Evaluation of \(\Sigma_0^{(2,f)}(k)\)

The correction to the self-energy shown in Fig. 4 can be written as

\[
\Sigma_0^{(2,f)}(k) = \frac{8d^2}{F^2} \int_{p,q,q',\theta} \left[ k \times q \right]^2 \left[ p \times q \right]^2 \left[ (p - Q) \times (q - Q) \right]^2 G^{(0)}_{k-q} G^{(0)}_{p} G^{(0)}_{q} G^{(0)}_{k}
\]

\[
\times \left[ p' \times Q \right]^2 \left[ p' \times Q \right]^2 \left[ (p' - Q) \times (q - Q) \right]^2 G^{(0)}_{p'} G^{(0)}_{p} G^{(0)}_{q} G^{(0)}_{q'} \Lambda_Q \nu_Q.
\]

(80)

Again we take the limit \(q_s \to \infty\). Next neglecting the external momentum \(k\) in the argument of the Green’s functions, we find after averaging over directions of \(k\):

\[
\Sigma_0^{(2,f)}(k) = \left( \frac{16\pi}{3} \right)^4 \frac{3 \chi k^4}{d_e^2} \int_{q,q,q',\theta} \frac{|p \times Q|^2 |p \times q|^2 |(p - Q) \times (q - Q)|^2}{q^2 Q^2 |p - Q|^2 |p' - Q|^2 |p - q|^2}
\]

\[
\times \left[ p' \times Q \right]^2 \left[ p' \times Q \right]^2 \left[ (p' - Q) \times (q - Q) \right]^2
\]

\[
\frac{p}{p^2 + Q^2} |p' - Q|^2 |p' - q|^2.
\]

(81)

Now we introduce three angles: \(\varphi = \angle(k, Q)\), \(\varphi' = \angle(k', Q)\), and \(\theta = \angle(q, Q)\). Then, we find

\[
\Sigma_0^{(2,f)}(k) = \left( \frac{8}{3} \right)^4 \frac{3 \chi k^4}{d_e^2} \int_{p,p',p,q} \frac{d^2 \varphi d\varphi' d\theta}{(2\pi)^3} \frac{pp'}{q^2 Q(q^2 + Q^2 - 2qQ \cos \theta)}
\]

\[
\times \left[ p'Q^2 \sin^2 \varphi \sin^2 (\varphi - \theta) |pQ \sin \varphi - qQ \sin \theta - pq \sin (\varphi - \theta)|^2
\]

\[
(\varphi^2 + Q^2 - 2pQ \cos \varphi)^2 (\varphi^2 + Q^2 - 2pq \cos (\varphi - \theta))^2
\]

\[
\times \left[ q^2 Q^2 \sin^2 \varphi \sin^2 (\varphi' - \theta) |p'Q \sin \varphi' - qQ \sin \theta - pq \sin (\varphi' - \theta)|^2
\]

\[
(\varphi'^2 + Q^2 - 2p'Q \cos \varphi)^2 (\varphi'^2 + Q^2 - 2pq' \cos (\varphi' - \theta))^2.
\]

(82)

Next we make a change of variables: \(p = Qe^x\), \(p' = Qe^{x'}\) and \(q = Qe^z\). Then we find

\[
\Sigma_0^{(2,f)}(k) = \frac{b^{(2,f)}}{d_e^2} \chi k^4 \int_0^\infty \frac{dQ}{Q} \frac{b^{(2,f)}}{d_e^2} \chi k^4 \ln \frac{q_s}{k}.
\]

(83)
where
\[ b^{(2,f)} = \frac{3}{2} \left( \frac{8}{3} \right)^2 \int_{-\infty}^{\infty} dz \int_{0}^{2\pi} d\theta \frac{e^{-z} \Phi^2(z, \theta)}{2\pi \cosh z - \cos \theta}. \] (84)

Here the function \( \Phi(z, \theta) \) is defined as follows:
\[ \Phi(z, \theta) = \frac{1}{16} e^{-z} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\varphi \frac{\sin^2 \varphi}{2\pi (\cosh x - \cos \varphi)^2 (\cosh(x - z) - \cos(\varphi - \theta))^2} \times \left[ e^{-z} \sin \varphi - e^{-x} \sin \theta - \sin(\varphi - \theta) \right]^3. \] (85)

We note that \( \Phi(z, \theta) = \Phi(-z, \theta) = \Phi(z, -\theta) \).

The integral in Eq. (84) is evaluated in Appendix C.3, yielding
\[ b^{(2,f)} = \frac{1}{9} + \frac{68\zeta(3)}{27} \] (86)

Hence, we find
\[ \Sigma^{(2,f)}_0(k) = \frac{3 + 68\zeta(3)}{27d_c^2} \kappa k^4 \ln \frac{q_+}{k}. \] (87)

In total, using Eqs. (66), (72), (79), and (72), we obtain the contribution of the non-SCSA-type diagrams of Fig. 3 to the self-energy
\[ \Sigma^{(2,e)}_0(k) + \Sigma^{(2,d)}_0(k) + \Sigma^{(2,e)}_0(k) + \Sigma^{(2,f)}_0(k) = \frac{68\zeta(3)}{27d_c^2} \kappa k^4 \ln \frac{q_+}{k}. \] (88)

4.3. Final result for \( \eta \)

Now we can add up the contributions of the SCSA and non-SCSA types to the self energy up to the second order in \( 1/d_c^2 \). Using Eqs. (61) and (85), we obtain
\[ \kappa k^4 - \Sigma^{(1)}_0(k) - \Sigma^{(2)}_0(k) = \kappa k^4 \left[ 1 + \frac{2}{d_c} \ln \frac{q_+}{k} + \frac{2}{d_c^2} \ln^2 \frac{q_+}{k} + \frac{73 - 68\zeta(3)}{27d_c^2} \ln \frac{q_+}{k} \right]. \] (89)

As one can see, the non-SCSA-type diagrams have no smallness in comparison with the diagrams which are taken into account within SCSA. The result (89) translates into the result (5) for the bending rigidity exponent. The obtained result for \( \eta \) implies that the value of the coefficient \( b \) in Eq. (43) is equal to
\[ b = (73 - 68\zeta(3))/27 \approx -0.32. \] (90)

Using this value we obtain the result (4) for the absolute Poisson’s ratio.

5. Conclusions

To summarize, in this paper we studied a suspended 2D crystalline membrane embedded into a space of large dimensionality \( d = 2 + d_c \gg 1 \). We computed the absolute Poisson’s ratio \( \nu \) and the bending rigidity exponent \( \eta \) to the second order in \( 1/d_c \).
Our result (4) demonstrates that, for $\sigma_L \ll \sigma \ll \sigma_*$, the absolute Poisson’s ratio of a 2D crystalline membrane is a universal but non-trivial function of $d_c$. Interestingly, the simple relation between the absolute Poisson’s ratio and the exponent $\alpha$, see Eq. (33), proposed in Ref. [24] breaks down at the order $1/d_c^2$ only.

Our result (5) for the bending rigidity exponent $\eta$ indicates that, in agreement with general expectations, at each order of expansion in $1/d_c$ the non-SCSA-type diagrams provide the contribution of the same order as diagrams which are included into SCSA scheme. Therefore, the coincidence of $\eta^{\text{SCSA}}$ at $d_c = 1$ with numerical result for the bending rigidity exponents is a surprising occasion.

Finally, we note that our results have been restricted to clean 2D membranes. It would be interesting to extend our analytical results for the $1/d_c$-expansion to the the case of a 2D disordered membrane.

6. Acknowledgements

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Appendix A. The polarization operator in the presence of uniaxial stress

In this Appendix we present details of analytical calculation of the polarization operator in the presence of the uniaxial stress, see Eq. (35). First step is to rewrite integral in real space representation

$$
\mathcal{P}(Q) = \int \frac{d^2 k}{(2\pi)^2} \frac{[k \times \mathbf{Q}]^2}{k^4 + k_s^4} \frac{[(Q - k) \times \mathbf{Q}]^2}{(Q - k)^4 + (Q_s - k)^4} = \int d^2 x \ e^{-Q x} \left[ f(x, \mathbf{Q}) \right]^2,
$$

(A.1)

where \( \mathbf{Q} = Q / Q \) and

$$
f(x, \mathbf{Q}) = \int \frac{d^2 k}{(2\pi)^2} e^{i k x} [k \times \mathbf{Q}]^2 / k^4 + k_s^4.
$$

(A.2)

Let us introduce the following notations

$$
f^{(a,b)}(x) = \int \frac{d^2 k}{(2\pi)^2} e^{i k x} k_a k_b / k^4 + k_s^4,
$$

(A.3)

where \( a \) and \( b \) are non-negative integers. Then we can write

$$
f(x, \mathbf{Q}) = \hat{Q}^2 f^{(0,2)}(x) - 2 \hat{Q} \hat{Q}_s f^{(1,1)}(x) + \hat{Q}_s^2 f^{(2,0)}(x).
$$

(A.4)

In order to find the three functions \( f^{(0,2)}, f^{(1,1)}, \) and \( f^{(2,0)}, \) we first compute analytically the functions \( f^{(1,0)} \) and \( f^{(2,0)} + f^{(0,2)}: \)

$$
f^{(1,0)}(x) = \sum_{\sigma = \pm} \frac{e_x x}{2} \int \frac{d^2 k}{(2\pi)^2} e^{i k x} \frac{1}{k^4 + i \sigma k_s} = \sum_{\sigma = \pm} \frac{e_x x}{2} \int_0^\infty dk \frac{J_0(k \sqrt{x^2 + y^2})}{k^2 + 1/4}
$$

(A.5)

$$
= \frac{i}{2 \pi} \sinh \left( \frac{x}{2} \right) K_0 \left( \sqrt{x^2 + y^2} / 2 \right).
$$

(A.6)

$$
f^{(2,0)}(x) + f^{(0,2)}(x) = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \frac{e^{i k x}}{k^2 + i \sigma k_s} = \frac{1}{2} \left[ \int_0^\infty dk \frac{J_0(k \sqrt{x^2 + y^2})}{k^2 + 1/4} \right]
$$

(A.7)

$$
= \frac{1}{2 \pi} \cosh \left( \frac{x}{2} \right) K_0 \left( \sqrt{x^2 + y^2} / 2 \right).
$$

(A.8)

Here \( J_0(z) \) and \( K_0(z) \) stands for the Bessel and modified Bessel functions of the zeroth order. Next we use the relation \( f^{(a+m,b+m)}(x) = (-i \partial_x)^m (-i \partial_y)^m f^{(a,b)}(x) \) in order to find the three required functions:

$$
f^{(2,0)}(x) = \frac{1}{4 \pi} \cosh \left( \frac{x}{2} \right) K_0 \left( \frac{\sqrt{x^2 + y^2}}{2} \right) - \frac{x}{\sqrt{x^2 + y^2}} \sinh \left( \frac{x}{2} \right) K_1 \left( \frac{\sqrt{x^2 + y^2}}{2} \right),
$$

$$
f^{(1,1)}(x) = -\frac{1}{4 \pi} \frac{y}{\sqrt{x^2 + y^2}} \sinh \left( \frac{x}{2} \right) K_1 \left( \frac{\sqrt{x^2 + y^2}}{2} \right),
$$

$$
f^{(0,2)}(x) = \frac{1}{4 \pi} \cosh \left( \frac{x}{2} \right) K_0 \left( \frac{\sqrt{x^2 + y^2}}{2} \right) + \frac{x}{\sqrt{x^2 + y^2}} \sinh \left( \frac{x}{2} \right) K_1 \left( \frac{\sqrt{x^2 + y^2}}{2} \right).
$$

(A.9)
Next step is to perform inverse Fourier transform. We introduce the following notations $F^{(a,b,c,d)}(x) = f^{(a,b)}(x) f^{(c,d)}(x)$ and express the polarization operator as

$$
\mathcal{P}(Q) = \hat{Q}^1 F^{(0,2,0,2)}(Q) + 4\hat{Q}^2 \hat{Q}^3 F^{(1,1,1,1)}(Q) + \hat{Q}^4 F^{(0,2,0,2)}(Q) + 2\hat{Q}^5 \hat{Q}^6 F^{(0,2,2,0)}(Q) - 4\hat{Q}^7 \hat{Q}^8 F^{(2,0,1,1)}(Q).
$$

where $F^{(a,b,c,d)}(Q) = \int d^2 x \exp(-iQx) F^{(a,b,c,d)}(x)$. In order to evaluate the functions $F^{(a,b,c,d)}(Q)$, we shall use the following identities:

\begin{align}
\int_0^\infty dx x K^2_0 \left( \frac{x}{2} \right) J_0(Qx) &= \frac{2 \text{arcsinh} Q}{Q \sqrt{1 + Q^2}} \equiv 2A(Q), \\
\int_0^\infty dx x K_1 \left( \frac{x}{2} \right) J_1(Qx) &= \frac{2 \text{arcsinh} Q}{\sqrt{1 + Q^2}} \equiv 2\Omega A(Q), \\
\int_0^\infty dx K^2_1 \left( \frac{x}{2} \right) \left[ J_1(Qx) - \frac{Qx}{2} \right] &= 2 \left[ Q - \sqrt{1 + Q^2} \right] \text{arcsinh} Q.
\end{align}

Below we demonstrate as an example the details of calculation of integrals for the function $F^{(1,1,1,1)}(Q)$:

\begin{align}
F^{(1,1,1,1)}(Q) &= \int \frac{d^2 x}{(4\pi)^2} e^{-iQx} \frac{-y}{\sqrt{x^2 + y^2}} \sinh \left( \frac{x}{2} \right) K_1 \left( \frac{\sqrt{x^2 + y^2}}{2} \right)^2 = \sum_{\sigma = \pm} \int \frac{d^2 x}{(8\pi)^2} e^{-iQx} \\
\times \frac{y^2}{x^2 + y^2} (e^{\sigma x} - 1) K_1 \left( \frac{\sqrt{x^2 + y^2}}{2} \right)^2 &= \frac{1}{8\pi} \frac{\partial}{\partial Q_y} \left[ 2Q_y (1 + Q^2) A(Q) - P_y (1 + P^2) A(P) \right],
\end{align}

where we introduced the vector $P = (Q_1 + i, Q_2)$. The other required functions are computed in a similar way. The results are as follows:

\begin{align}
8\pi F^{(0,2,0,2)}(Q) &= (\hat{Q}_1^2 + \hat{Q}_2^2 + 1) A(Q) + \hat{Q}_1^2 - \text{Re} \left[ (\hat{P}_1^2 + P_1^2 - 1 + 2iP_y) A(P) + \hat{P}_1^2 \right], \\
8\pi F^{(2,0,2,0)}(Q) &= (\hat{Q}_1^2 + \hat{Q}_2^2 + 1) A(Q) + \hat{Q}_2^2 - \text{Re} \left[ (\hat{P}_2^2 + P_2^2 - 1 - 2iP_x) A(P) + \hat{P}_2^2 \right], \\
8\pi F^{(1,1,1,1)}(Q) &= (\hat{Q}_1^2 + \hat{Q}_2^2) A(Q) + \hat{Q}_3^2 - \text{Re} \left[ (\hat{P}_3^2 + P_3^2) A(P) + \hat{P}_3^2 \right], \\
-8\pi F^{(0,2,0,2)}(Q) &= (\hat{Q}_1^2 + \hat{Q}_2^2 - 1) A(Q) + \hat{Q}_2^2 - \text{Re} \left[ (\hat{P}_2^2 + P_2^2 + 1) A(P) + \hat{P}_2^2 \right], \\
-8\pi F^{(2,0,2,0)}(Q) &= (\hat{Q}_1^2 + \hat{Q}_2^2 - 1) A(Q) + \hat{Q}_1^2 - \text{Re} \left[ (\hat{P}_1^2 + P_1^2 - 1) A(P) + \hat{P}_1^2 \right], \\
-8\pi F^{(1,1,1,1)}(Q) &= \hat{Q}_1 \hat{Q}_2 [1 - A(Q)] - \text{Re} \left[ (-\hat{P}_1 \hat{P}_3 + iP_y) A(P) + \hat{P}_3 \hat{P}_1 \right], \\
-8\pi F^{(2,0,1,1)}(Q) &= \hat{Q}_1 \hat{Q}_3 (A(Q) - 1) - \text{Re} \left[ (\hat{P}_3 \hat{P}_1 + iP_y) A(P) - \hat{P}_1 \hat{P}_3 \right].
\end{align}

That together with (A.10) sums up to the answer for the polarization operator, Eq. (38).

**Appendix B. Some details of numerical computation of the coefficient $c$**

In this Appendix we present some details of numerical computation of the coefficient $c$ given by Eq. (41). It is convenient to write it as follows

\begin{align}
c &= c_1 - c_2, \\
c_\beta &= 16\pi \int \frac{d^2 Q}{(2\pi)^2} \left( \frac{L_{\beta}^{(1)}(Q)}{\mathcal{P}(Q)} - \frac{L_{\beta}^{(0)}(Q)}{\mathcal{P}_0(Q)} \right).
\end{align}
where
\[
L_{\beta}^{(j)}(Q) = \int \frac{d^2K}{(2\pi)^2} \frac{(K \times Q)^4}{Q^4} \left[ \frac{K_\beta^2}{[K^4 + K_\beta^4][K - Q]^4 + j(K_\beta - Q)^2}] \right].
\] (B.2)

We note that
\[
L_1^{(1)}(Q) = \frac{1}{2} \mathcal{P}(Q) + \frac{1}{2} Q_0 \frac{\partial \mathcal{P}(Q)}{\partial Q_0} + \frac{1}{2} Q_0 \frac{\partial \mathcal{P}(Q)}{\partial Q_0} \left. \right|_{Q=0}.
\] (B.3)

Also, we mention that similar to \( \mathcal{P}(Q) \) the functions \( L_1^{(1)}(Q) \) and \( L_1^{(1)}(Q) \) has very anisotropic behaviour close to \( Q = 0 \). This complicates numerical evaluation of the integrals. In order to circumvent this problem we make the following splitting \( c = c_1 + c_2 + c_3 \) where
\[
c_{\beta 1} = 16\pi \int \frac{d^2Q}{(2\pi)^2} \frac{L_{\beta}^{(j)}(Q)[\mathcal{P}_0(Q) - \mathcal{P}(Q)]}{\mathcal{P}(Q)\mathcal{P}_0(Q)}.
\] (B.4)

and
\[
c_{\beta 2} = 16\pi \int \frac{d^2Q}{(2\pi)^2} \frac{L_1^{(1)}(Q) - L_1^{(0)}(Q) - L_1^{(1)}(Q) + L_1^{(0)}(Q)}{\mathcal{P}_0(Q)}. \] (B.5)

Numerical evaluation yields
\[
c_{\beta 1} = 2.819 \pm 0.001, \quad c_{\beta 2} = 7.19 \pm 0.02. \] (B.6)

In order to evaluate \( c_{\beta 2} \) it is convenient to use the following transformations
\[
\int \frac{d^2Q}{(2\pi)^2} \frac{L_{\beta}^{(1)}(Q) - L_{\beta}^{(0)}(Q)}{\mathcal{P}_0(Q)} = -\frac{16\pi}{3} \int \frac{d^2P}{(2\pi)^2} \frac{d^2K}{(2\pi)^2} \frac{[K \times P]^4}{P_\beta^4} \frac{[K - Q]^4 + (K_\beta - Q_\beta)^2]}{[Q_0^4 + Q_0^4][Q - Q]^4 + j(Q_\beta - Q_\beta)^2]} \frac{K_\beta^2}{K^4 + K_\beta^4}. \] (B.7)

Now introducing \( K = P't \) and \( P' = \sqrt{y} \), we find that
\[
c_{\beta 2} = -\frac{8}{3\pi^2} \int_0^{2\pi} \! d\theta \int_0^{2\pi} \! d\varphi \int_0^{\infty} \! dy \int_0^{\infty} \! dt y \sin^4 \theta \cos^2 \varphi \cos[2(\theta + \varphi)] \left[ y \cos^2(\theta + \varphi) \right]^2 \left[ y + \cos^2 \varphi \right]^2 (1 + 2\cos \theta). \] (B.8)

Now using the following result
\[
\int_0^{\infty} \! dy \frac{y \sin^4 \theta \cos^2 \varphi \cos[2(\theta + \varphi)]}{[y \cos^2(\theta + \varphi) \varphi]^2} = \frac{\cos^2(\theta + \varphi) - \varphi^2 \cos^2 \varphi + \varphi^2 \cos^2 \varphi \ln \frac{\varphi \cos \theta \varphi}{\cos(2\theta + \varphi)}}{\varphi^2 [\cos^2(\theta + \varphi) - \varphi^2 \cos^2 \varphi]^2}, \] (B.9)

we obtain
\[
c_{\beta 2} = -\frac{8}{3\pi^2} \int_0^{2\pi} \! d\theta \int_0^{2\pi} \! d\varphi \int_0^{\infty} \! dt t \sin^4 \theta \cos^2 \varphi \cos[2(\theta + \varphi)] \left[ \cos^2(\theta + \varphi) - \varphi^2 \cos^2 \varphi + \varphi^2 \cos^2 \varphi \ln \frac{\varphi \cos \theta \varphi}{\cos(2\theta + \varphi)} \right] \times \left[ \cos^2(\theta + \varphi) - \varphi^2 \cos^2 \varphi + \varphi^2 \cos^2 \varphi \ln \frac{\varphi \cos \theta \varphi}{\cos(2\theta + \varphi)} \right], \] (B.10)
We note that the integrand is symmetric under simultaneous transformation \( \theta \to 2\pi - \theta \) and \( \varphi \to 2\pi - \varphi \). Therefore, we can rewrite the above integral as follows

\[
c_{y2} = -\frac{16}{3\pi^2} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \int_0^\infty dt \frac{t \sin^4 \theta \cos^2 \varphi \cos[2(\theta + \varphi)]}{[t^2 + 1 - 2\text{ct} \cos \theta]} \left[ \cos^2(\theta + \varphi) - t^2 \cos^2 \varphi + t^2 \cos^2 \varphi \ln \frac{\cos^2(\theta + \varphi)}{\cos^2(\theta + \varphi)} \right].
\]

(Numerical evaluation of this integral results in \( c_{y2} = 4.933 \pm 0.001 \). In total, we find \( c = 0.56 \pm 0.02 \).

**Appendix C. Evaluation of non-SCSA contributions to the self-energy**

In this Appendix, we present the detailed analytical calculation of the integrals involved in the non-SCSA contributions \( \Sigma_{0}^{(2, f)} \), \( \Sigma_{0}^{(2, f)} \), and \( \Sigma_{0}^{(2, f)} \) to the self-energy.

**Appendix C.1. Evaluation of \( b^{(2, f)} \)**

In this subsection, we calculate analytically the integral determining the coefficient \( b^{(2, f)}(k) \) in Eq. (71). Let us introduce the variables \( x \) and \( y \) such that \( p = \exp(x) \) and \( p' = \exp(y) \). Also we introduce variables \( u = \exp(i\phi) \) and \( v = \exp(i\varphi') \). Then, we find

\[
b^{(2, f)} = -\frac{2}{9} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{|u|=1} dv \int_{|v|=1} du \frac{(u^2 - 1)^2}{2\pi i u (u - e^x)^2 (u - e^{-x})^2 (v - e^y)^2 (v - e^{-y})^2} \times \left( \frac{v^2 - u^2}{u - e^{-x})v^2(u - e^{-x})v^2} \right).
\]

The positions of the poles in the expression under the integral signs in \( u \) and \( v \) complex planes depend on values of \( x \) and \( y \). Therefore it is convenient to consider the three domains of integration over \( x \) and \( y \):

(I) \( : 0 < y < x \), \quad (II) \( : x > 0 \) \& \( y < 0 \), \quad (III) \( : y < x < 0 \).

For each domain we can determine the poles of the expression inside the unit circle, \(|u| < 1\):

(I) \( : u = e^{-x}, u = ve^{-x} \), \quad (II) \( : u = e^{-x}, u = ve^{-x} \), \quad (III) \( : u = e^x, u = ve^{x} \). (C.3)

Also for all three domains there is a pole at \( u = 0 \). Performing integration over \( u \), we find that except the pole at \( v = 0 \) the obtained expression has the following poles inside the unit circle, \(|v| < 1\):

(I) \( : v = e^{-x}, v = e^{x-2x} \), \quad (II) \( : v = e^{x}, v = e^{x+2x} \), \quad (III) \( : v = e^{x} \). (C.4)
After integration over $v$, we find

$$b^{(2,d)} = -\frac{4}{9} \int_0^\infty dx \int_0^x dy \frac{2e^{2(1+y)} + 5e^{2(1+y)} - 6e^{4(1+y) - 6e^{2(1+y) + 6e^{2(1+y) + 2} + 8e^{2(1+y) + 4y} + 2e^{4y} - 4e^{6y} + 9e^{8y}}}{e^{2(1+y)} (e^{2x} - 1)^2}
- \frac{4}{9} \int_0^\infty dx \int_0^x dy \frac{2e^{4(1+y)} - 16e^{2(1+y) - 6e^{4(1+y) + 2} + 2e^{2(1+y) + 4y} + 5e^{4y} + 8e^{6y} + 9e^{8y}}}{e^{2x-2y} (e^{2x} - e^{2y})^2}
- \frac{4}{9} \int_{-\infty}^0 dx \int_{-\infty}^x dy \frac{9e^{4(1+y)} - 6e^{4(1+y) + 2} + 2e^{2(1+y) + 4} + 8e^{2x} + 5e^{4y} - 4e^{2y} + 2e^{4y}}{e^{2x-2y} (e^{2x} - 1)^2}
= -\frac{2}{9} \left[ \left( 3 - \frac{\pi^2}{6} \right) + \frac{11}{4} + \frac{13}{4} + \frac{\pi^2}{6} \right] = -2 \tag{C.5}$$

This value yields Eq. (72) of the main text.

**Appendix C.2. Evaluation of $b^{(2,e)}$.**

Here, we calculate the integral in Eq. (77) for coefficient $b^{(2,e)}$. We introduce the following variables $x = \ln q$, $y = \ln Q$, $u = \exp(i\rho)$, and $v = \exp(i\rho')$. Then, we find

$$b^{(2,e)} = \frac{4}{27} \int_{-\infty}^\infty dx \int_{-\infty}^x dy \oint_{|\rho|=1} \oint_{|\rho'|=1} \frac{du}{2\pi i u^2} \oint_{|\rho'|=1} \frac{dv}{2\pi i v^2} \frac{(u^2 - 1)^2}{(u - e^x)^2 (u - e^{-x})^2} \frac{(v^2 - 1)^2}{(v - e^{-y})^2 (v - e^{-y'})^2} \times \frac{(x^2 - u^2)^2 [e^{-y}(v - 1/v) - e^{-y}(u - 1/u) - (v/u - u/v)]^2}{(u - e^{-y})^2 (u - e^{-y'})^2}
\times \left[ 1 + \frac{1}{2} \left( \frac{u}{v} + \frac{v}{u} \right) \right] \tag{C.6}$$

Again the pole structure of the expression under the integral signs in $u$ and $v$ complex planes depend on values of $x$ and $y$. Therefore it is convenient to consider the three domains of integration over $x$ and $y$ defined in Eq. (C.2). The poles inside the unit circle, $|\rho| < 1$, are the same as for the diagram on Fig. (3), see Eq. (C.3). Performing integration over $u$, one finds that the obtained expression has poles in $v$ inside the unit circle, $|v| < 1$. Their positions are exactly the same as for the coefficient $b^{(2,d)}$, see Eq. (C.4). After integration over $v$, we find

$$b^{(2,e)} = \frac{4}{27} \int_0^\infty dx \int_0^x dy \frac{e^{-4(1+y)} - 36e^{6(1+y)} + 7e^{4(1+y) + 2y} - 19e^{6(1+y) + 2y} + 14e^{8(1+y) + 5e^{6(1+y) + 2y} - 23e^{8(1+y) + 2y} + 19e^{10(1+y) - 7e^{2(1+y) + 4y} + 2e^{2(1+y) + 6y} - 14e^{4y} + 21e^{6y} - 5e^{8y} - 3e^{10y}} + 4}{27} \int_{-\infty}^0 dy \int_{-\infty}^x dy \frac{e^{2y} - 7e^{1(1+y)} - 4e^{3(1+y) + 2y} + 14e^{5(1+y) + 2y} + 2e^{2(1+y) + 4y} - 10e^{4y} - 28e^{6y} + 3e^{8y}}{-73e^{6(1+y)} - 36e^{5(1+y)} + 62e^{4(1+y) + 2y} - 4e^{4y} + 10e^{2(1+y) + 4y} + 59e^{6(1+y) + 2y} + 2e^{6y} + 2e^{6y} + 2e^{2(1+y) + 6y} + 23e^{4(1+y) + 6y} - 28e^{2y} - 10e^{4y} + 14e^{2y} + 2e^{6y}}
= \frac{4}{27} \left[ \frac{2}{9} (8 - 3\pi^2) - \frac{889}{144} \left( \frac{2\pi^2}{3} - \frac{485}{48} \right) \right] = -\frac{58}{27} \tag{C.7}$$

This results in Eq. (79) of the main text.
Appendix C.3. Evaluation of $b^{(2,f)}$

The evaluation of the integral in Eq. (84) for the coefficient $b^{(2,f)}$ turns out to be most involved among the integrals determining the non-SCBA contributions to the self-energy (Fig. 3). By introducing new variables $w = \exp(i\theta)$ and $u = \exp(i\varphi)$, we can write

$$b^{(2,f)} = -2 \left( \frac{16}{3} \right)^3 \int_0^\infty dz \int_{|w|=1} dw \frac{\cosh z}{2\pi i (w - e^z)(w - e^{-z})} \Phi(z, w), \quad \Phi(z, w) = \int_{-\infty}^0 dx F(x, z, w),$$

$$F(x, z, w) = -\frac{1}{64} e^z \int_{|w|=1} du \frac{(u^2 - 1)^2}{2\pi i u (u - e^z)^2(u - e^{-z})^2 (u - we^{x-x})^2 (u - we^{x-x})^2} \times \left[ e^{-z} (u - u^{-1}) - e^{-1} (w - w^{-1}) - (uw^{-1} - wu^{-1}) \right]^2. \quad (C.8)$$

Let us first, integrate over $u$ in the function $F(x, z, w)$ under assumption that $z > 0$. The result of integration depends on the intervals in which $x$ is situated. We split it into three domains:

(I) : $x < 0$, \quad (II) : $0 \leq x < z$, \quad (III) : $z \leq x$. \quad (C.9)

Then pole structure inside the unit circle, $|w| < 1$, can be summarized as follows

(I) : $u = e^z$, $u = we^{x-z}$, \quad (II) : $u = e^{-z}$, $u = we^{x-z}$, \quad (III) : $u = e^z$, $u = we^{-x-z}$. \quad (C.10)

Also there is always the pole at $u = 0$. Then, we can write that $\Phi(z, w) = \Phi_1(z, w) + \Phi_2(z, w)$, where

$$\Phi_1(z, w) = \int_0^\infty dx F_{II}(x, z, w), \quad \Phi_2(z, w) = \int_0^\infty dx \left[ F_{III}(x + z, z, w) + F_{II}(x, z, w) \right]. \quad (C.11)$$

The function $\Phi_1(z, w)$ can be found explicitly

$$\Phi_1(z, w) = \frac{e^{-4z}}{128w^2 (e^z - w)(w - e^{-z})} \left[ 3w (w^2 + 1)^2 - (w^2 + 1) e^z (3w^4 + 2w^2 (4z + 5) + 3) - (w^2 + 1) e^{-z} (w^4 (4z - 3) + 2w^2 (16z - 5) + 4z - 3) + 4we^{2z} (w^4 + 3w^2 + 8w^2 + 3) z + 1 \right]$$

$$+ we^{2z} \left( w^4 (8z - 7) + 6w^2 (4z - 1) + 8z - 7 \right). \quad (C.12)$$

whereas integration over $x$ in $\Phi_2(z, w)$ is not easy to perform analytically. Therefore, we rewrite the expression for $b^{(2,f)}$ in the following way

$$b^{(2,f)} = -2 \left( \frac{16}{3} \right)^3 \int_0^\infty dz \int_{|w|=1} dw \frac{\cosh z}{2\pi i (w - e^z)(w - e^{-z})} [\Phi_1^2(z, w) + 2\Phi_1(z, w)\Phi_2(z, w) + \Phi_2^2(z, w)]$$

$$\equiv b^{(2,f)}_{11} + 2b^{(2,f)}_{12} + b^{(2,f)}_{22}. \quad (C.13)$$

In order to compute $b^{(2,f)}_{11}$ we integrate over $w$. There are poles at $w = 0$ and $w = e^z$ inside the unit circle, $|w| < 1$, in the $w$ complex plane. Then, we obtain the following result:

$$b^{(2,f)}_{11} = \frac{1}{27} \int_0^\infty dz \frac{e^{9z} \cosh z}{(e^{2z} - 1)^3} \left[ 9 - 24e^{2z} (z + 1) + e^{2z} (16z^2 + 128z - 23) - 8e^{6z} (16z^2 + 34z - 13) \right.$$

$$+ e^{8z} \left( 368z^2 + 192z - 21 \right) - 8e^{10z} (52z^2 - 29z + 14) + e^{12z} \left( 352z^2 - 256z + 67 \right) \right]. \quad (C.14)$$
Finally, integrating over $z$, we find

$$b^{(2,f)}_{11} = \frac{20}{27}\zeta(3) + \frac{13253}{58320}.$$ (C.15)

Next we consider the contribution $b^{(2,f)}_{12}$. In order to proceed, we first integrate over $w$. Using the explicit expression \[C.12\] for the function $\Phi_1(z,w)$, we analyse the pole structure in the complex plane of $w$. Inside the unit circle $|w| < 1$, there are poles at $w = 0$, $w = e^{-z}$, and $w = e^{-2z}$, after integration over $w$, we obtain the following result:

$$b^{(2,f)}_{12} = \frac{2}{27} \int_0^\infty dz \int_0^\infty dx \frac{e^{-8x-9z} \cosh z}{(e^{2x+z} - 1)^3} \left[(134 - 436z)e^{6x+8z} - 21e^{2(x+z)} - 6e^{4x+2z} - 12e^{8x+2z}ight.$$

$$+(98 - 80z)e^{8(x+z)} + (20 - 40z)e^{8x+6z} + (6 - 12z)e^{6x+2z} - 4(z - 2)e^{10x+6z} + 8(z + 13)e^{6(x+z)}

$$+(8z - 169)e^{6(x+z)} + 2(8z - 7)e^{8x+4z} + (8z - 7)e^{10x+8z} - 5(8z - 7)e^{6x+10z} + 3(8z - 5)e^{6x+4z}

$$-4(25z + 33)e^{2x+4z} + 4(44z - 37)e^{4x+8z} + 4(52z - 23)e^{6x+10z} + (56z - 4)e^{10x+10z} + 3e^{10x+4z}

$$-3(64z - 51)e^{2z+6z} + (336z + 50)e^{4z+6z} + 9e^{6z} + 48e^{2z} + 16e^{4z} \right].$$ (C.16)

Next, integrating over $x$, we obtain

$$b^{(2,f)}_{12} = \frac{1}{81} \int_0^\infty dz e^{-9z} \cosh z \left[27 + 48z^2(e^z - e^{6z}) + 240e^{8z} - 32e^{z} + 44e^{5z}

$$+404e^{6z} - 96e^{8z} - 228e^{10z} + 12e^{12z} + 24e^{8z} - 60e^{4z} - 96e^{6z} + 111e^{8z} - 6e^{10z}

$$-6e^{2z}(e^{z}(14 - 4z) + e^{10z} + e^{z}(4z + 1) + e^{6z}(20z + 6) + e^{8z}(28z - 19) - 3) \ln (e^z - 1)

$$-6e^{2z}(e^{z}(14 - 4z) + e^{10z} + e^{z}(4z + 1) + e^{6z}(20z + 6) + e^{8z}(28z - 19) - 3) \ln (e^z + 1) \right].$$ (C.17)

Finally, integrating over $z$ we find

$$b^{(2,f)}_{12} = \frac{4}{9}\zeta(3) + \frac{5789}{19440} - \frac{23}{972}.$$. (C.18)

Now we turn our attention to the coefficient $b^{(2,f)}_{22}$. Again, at first, we integrate over $w$. Inside the unit circle, $|w| < 1$, there are four poles: at $w = 0$, $w = e^{-z}$, $w = e^{-2z}$, and $w = e^{-3z}$ (we took into account that $x, y, z > 0$). Integrating over $w$, we obtain the following result:

$$b^{(2,f)}_{22} = -2 \left(\frac{16}{3}\right)^3 \int_0^\infty dz \int_0^\infty dx \int_0^\infty dy X_{22}(x,y,z)$$ (C.19)

The resulting expression for $X_{22}(x,y,z)$ is rather cumbersome and we present it separately in
Appendix D. Next, after integration over $x$ and $y$, we find

$$b_{22}^{(2, f)} = -\frac{1}{243} \int_0^{\infty} \frac{e^{-9z} \cosh z}{\sqrt[3]{73728}} \left[ e^{3z} \left( -36e^{2z} (e^{2z} + 1)^2 (e^{2z} - 1)^3 \left( -\text{Li}_2(1 - e^{-2z}) \right) + \frac{1}{2} \left( \pi^2 \ln^2 (1 - e^{-2z}) \right) - \frac{\pi^2}{6} + 216e^{2z} - 216e^{2z} - 312e^{2z} - 312e^{2z} - 216e^{2z} - 216e^{2z} + 13e^{2z} + 179e^{4z} + 427e^{6z} + 351e^{8z} + 90e^{10z} + 36e^{12z} - 120e^{2z} + 804e^{4z} + 672e^{6z} - 1140e^{8z} - 264e^{10z} + 156e^{12z} - 108z + 144e^{2z} (e^{2z} + 1)^2 z \left( e^{2z} - 1 \right)^3 \ln \left( e^{2z} - 1 \right) \right] + 36e^{4z} \left( -9e^{2z} - 12e^{4z} - 12e^{6z} - 9e^{8z} + e^{10z} + 1 \right) + 117e^{2z} - 74e^{4z} - 406e^{6z} - 1225e^{8z} + 195e^{10z} - 246e^{12z} - 36e^{14z} - 108e^{2z} + 120e^{2z} + 804e^{6z} + 672e^{8z} - 1140e^{10z} - 264e^{12z} + 156e^{14z} - 6\pi^2 e^{6z} + 6\pi^2 e^{8z} + 12\pi^2 e^{10z} - 6\pi^2 e^{12z} + 6\pi^2 e^{14z} - 90e^{4z} \left( e^{2z} - 1 \right)^2 \left( e^{2z} - 1 \right)^3 \ln^2 \left( e^{2z} - 1 \right) + 12e^{2z} \left( 2e^{2z} (52 - 12z) + 48e^{2z} \right) + 9e^{6z} + e^{2z} (12z - 19) + e^{10z} (12z - 13) - 9 \left( e^{2z} - 1 \right) \ln \left( e^{2z} - 1 \right) + 81 \right]. \quad (C.20)$$

Here $\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ stands for the polylogarithm. Finally, integrating over $z$, we find

$$b_{22}^{(2, f)} = \frac{8}{9} \zeta(3) - \frac{41507}{58320} + \frac{23}{486} \pi^2. \quad (C.21)$$

Combing all contributions, $b_{11}^{(2, f)}$, $b_{12}^{(2, f)}$, $b_{22}^{(2, f)}$, together, we obtain

$$b^{(2, f)} = b_{11}^{(2, f)} + b_{12}^{(2, f)} + b_{22}^{(2, f)} = \frac{3 + 68\zeta(3)}{27}, \quad (C.22)$$

leading to Eq. (87) of the main text.

Appendix D. Expression for the function $X_{22}(x, y, z)$

In this Appendix we present the expression for the function $X_{22}(x, y, z)$ which is determined the right hand side of Eq. (C.19). This function can be written as

$$X_{22} = \frac{e^{-8x - 8y - 9z} \cosh z}{2048 (1 - e^{2ix+y}) (1 - e^{2iy+z}) (1 - e^{2ix+y+z})^2} \left[ X_{22}^{(1)}(x, y, z) + X_{22}^{(2)}(x, y, z) \right], \quad (D.1)$$
where

\[ y_{22}^{(1)}(x, y, z) = 9e^{6(x+y)} + 48e^{6(x+y)} - 48e^{8(x+y)} - 48e^{6(x+y)} - 32e^{4(x+y)} + 53e^{10(x+y)} + 290e^{6(x+y)} - 1081e^{10(x+y)} + 1510e^{15(x+y)} + 313e^{14(x+y)} + 3e^{16(x+y)} + 1074e^{24(x+y)} + 906e^{22(x+y)} - 185e^{27(x+y)} + 21e^{23(x+y)} + 377e^{8(3x+y+2)} - 567e^{4(x+3y+2)} - 6e^{6x+3y+2} + 48e^{6y+2} - 6e^{4x+6y+2} + 9e^{6y+2} - 3e^{6x+8y+2} - 3e^{6y+8y+2} - 45e^{8x+8y+2} - 15e^{6x+8y+3} + 316e^{4x+y+3} + 357e^{2(5x+y+3)} - 15e^{4(x+2y+3)} + 38e^{62(2x+2y+3)} - 83e^{6(3x+2y+3)} + 16e^{4(x+4y+3)} - 4e^{4(4+5y+3)} + 357e^{2(6x+y+5)} + 16e^{6y+4} - 80e^{8x+4y+4} + 167e^{2(5x+y+4)} - 123e^{6(3x+y+4)} + 36e^{4x+2y+4} - 112e^{6x+2y+4} - 169e^{8x+2y+4} + 978e^{6(3x+3y+4)} + 1260e^{4(4x+3y+4)} + 167e^{6(5x+y+4)} + 160e^{6y+4} - 123e^{6(x+y+6)} - 112e^{2x+6y+4} + 63e^{4x+6y+4} - 9e^{6x+6y+4} + 21e^{8x+6y+4} - 80e^{8y+4} - 169e^{2x+8y+4} + 21e^{6x+8y+4} - 44e^{8x+8y+4} + 15e^{10x+8y+4} + 15e^{8x+10y+4} + 90e^{10x+10y+4} + 69e^{3(4x+5y+5)} + 158e^{2(5x+y+5)} - 183e^{2(6x+y+5)} - 44e^{2(7x+y+5)} + 69e^{2(4x+5y+2)} - 183e^{2(6x+y+5)} - 48e^{2(5x+y+6)} + 370e^{2(6x+y+5)} + 60e^{2(5x+y+5)} - 50e^{2(5x+y+5)} + 35e^{2(7x+y+5)} - 15e^{2(8x+y+5)} - 15e^{2(6x+y+5)} - 116e^{6y+2y+6} + 128e^{8x+2y+6} - 328e^{6y+8y+6} + 39e^{2(7x+y+6)} + 82e^{8x+8y+6} + 3e^{2(9x+8y+6)} - 15e^{10x+y+6} + 3e^{2(8x+9y+6)} + 16e^{10y+6} + 143e^{4x+10y+6} - 211e^{6x+10y+6} - 15x^{8+10y+6} + 85e^{10x+y+6} + 30e^{12x+10y+6} - 30e^{10x+12y+6} - 116e^{2(5x+7y+7)} - 1012e^{2(6x+5y+7)} + 882e^{2(7x+y+7)} - 712e^{2(8x+y+7)} - 2e^{2(9x+y+7)} - 1021e^{2(5x+7y+7)} + 318e^{2(6x+7y+7)} - 2275e^{2(7x+y+7)} + 89e^{2(8x+7)} - 2e^{2(9x+y+7)} - 712e^{2(5x+7y+7)} + 89e^{2(6x+8y+7)} + 69e^{2(7x+y+7)} + 52e^{2(8x+y+7)} + 9e^{2(9x+y+7)} - 2e^{2(5x+y+7)} - 2e^{2(6x+7y+7)} - 2e^{2(7x+y+7)} + 9e^{2(8x+y+7)} - e^{2(9x+9y+7)} + 32e^{10x+8} + 16e^{12x+8} - 69e^{6x+2y+8} - 78e^{8x+2y+8} + 174e^{6x+4y+8} + 408e^{8x+4y+8} - 789e^{10x+4y+8} - 58e^{2(5x+5y+8)} + 317e^{2(6x+5y+8)} + 535e^{2(7x+5y+8)} - 1062e^{2(8x+5y+8)} + 87e^{29x+9y+8} - 69e^{21x+6y+8} + 174e^{4x+6y+8} + 317e^{2(5x+y+8)} + 560e^{6x+6y+8} - 731e^{2(7x+y+8)} + 53e^{3x+6y+8} - 17e^{2(9y+6y+8)} - 272e^{10x+y+8} + 149e^{12x+6y+8} + 535e^{2(5x+7y+8)} - 731e^{2(6x+7y+8)} + 2390e^{2(7x+7y+8)} - 166e^{2(8x+9y+8)} - 2e^{2(9x+7y+8)} + 87e^{2(5x+9y+8)} - 17e^{2(6x+9y+8)} - 2e^{2(7x+9y+8)} - 15e^{2(8x+9y+8)} - 5e^{2(9x+9y+8)} + 32e^{10x+8y} - 789e^{3x+10y+8} - 272e^{6x+10y+8} + 644e^{8x+10y+8} - 257e^{10x+10y+8} - 30e^{12x+10y+8} + 16e^{2x+2y+8} + 149e^{6x+12y+8} + 185e^{8x+12y+8} - 30e^{10x+12y+8} - 80e^{12x+12y+8} + 30e^{14x+12y+8} + 30e^{12x+14y+8} + 45e^{14x+14y+8} + 105e^{7x+5y+9} - 126e^{2(8x+5y+9)} + 21e^{2(9x+5y+9)},

\text{(D.2)}
\[ X_{22}^{(2)} (x, y, z) = -478e^{2(7x+6y+9z)} + 571e^{2(8x+6y+9z)} + 105e^{2(5x+7y+9z)} - 478e^{2(6x+7y+9z)} + 616e^{2(7x+y+9z)} - 953e^{2(8x+7y+9z)} + 19e^{2(9x+y+9z)} - 126e^{2(5x+8y+9z)} + 571e^{2(6x+8y+9z)} - 953e^{2(7x+8y+9z)} + 37e^{2(8x+8y+9z)} - 19e^{2(9x+8y+9z)} + 15e^{6x+4y+10z} + 185e^{8x+4y+10z} - 199e^{10x+4y+10z} - 28e^{12x+4y+10z} + 307e^{14x+4y+10z} + 15e^{4x+6y+10z} - 102e^{6x+6y+10z} - 846e^{8x+6y+10z} - 321e^{10x+6y+10z} + 1440e^{12x+6y+10z} + 273e^{14x+6y+10z} + 185e^{4x+8y+10z} - 846e^{6x+8y+10z} - 11358e^{8x+8y+10z} - 106e^{10x+8y+10z} - 588e^{12x+8y+10z} - 281e^{14x+8y+10z} + 28e^{16x+12y+10z} + 1440e^{16x+12y+10z} - 588e^{8x+12y+10z} + 307e^{10x+14y+10z} + 273e^{6x+14y+10z} - 281e^{8x+14y+10z} - 297e^{10x+4y+12z} + 367e^{14x+4y+12z} - 51e^{8x+6y+12z} + 47e^{10x+6y+12z} - 598e^{6x+8y+12z} - 273e^{16x+6y+12z} - 51e^{8x+8y+12z} + 554e^{8x+8y+12z} + 1539e^{10x+8y+12z} - 476e^{12x+8y+12z} - 834e^{14x+8y+12z} + 163e^{6x+8y+12z} - 291e^{10x+10y+12z} + 47e^{6x+10y+12z} + 1539e^{8x+10y+12z} + 367e^{4x+14y+12z} - 598e^{6x+14y+12z} - 273e^{16x+16y+12z} + 163e^{8x+16y+12z} + 75e^{10x+9y+14z} + 429e^{12x+6y+14z} - 287e^{14x+6y+14z} - 253e^{16x+6y+14z} + 36e^{18x+6y+14z} + 18e^{18x+8y+14z} - 21e^{10x+8y+14z} - 612e^{12x+8y+14z} + 146e^{14x+8y+14z} + 852e^{16x+8y+14z} - 23e^{18x+8y+14z} + 75e^{6x+10y+14z} - 21e^{8x+10y+14z} + 429e^{6x+12y+14z} - 612e^{8x+12y+14z} - 253e^{16x+16y+14z} + 852e^{8x+16y+14z} + 36e^{18x+16y+14z} - 23e^{18x+18y+14z} - 141e^{12x+8y+16z} - 177e^{8x+18y+16z} - 59e^{8x+18y+16z} - 50e^{14x+14y+20z} - 70e^{16x+14y+20z} + 15e^{18x+14y+20z} - 14e^{18x+16y+20z} + 15e^{14x+18y+20z} + 14e^{16x+18y+20z} + 23e^{8x+18y+20z} + 36e^{2x+2(8y+z)} + 408e^{4x+2(8y+z)} + 460e^{6x+(2y+3z)} + 377e^{8x+2(8y+z)} - 116e^{2x+3(8y+z)} - 77e^{4x+3(8y+z)} - 476e^{4x+2(3y+z)} - 83e^{6x+3(8y+z)} + 16e^{4x+3(8y+z)} - 78e^{6x+3(8y+z)} + 260e^{6x+3(8y+z)} + 63e^{6x+4y+z} + 158e^{3x+5y+z} - 199e^{2x+5y+z} - 321e^{2x+5y+z} - 106e^{2x+5y+z} - 481e^{2x+5y+z} - 50e^{2x+5y+z} - 100e^{2x+6y+z} + 753e^{3x+6y+z} - 240e^{3x+6y+z} - 328e^{3x+6y+z} - 211e^{10x+6y+z} - 287e^{2x+7y+z} + 146e^{2x+7y+z} + 882e^{2x+7y+z} + 2275e^{2x+7y+z} + 69e^{2x+7y+z} - 2e^{2x+7y+z} + 106e^{2x+8y+z} + 53e^{6x+8y+z} - 166e^{2x+8y+z} - 15e^{2x+9y+z} + 644e^{10x+8y+z} + 185e^{12x+8y+z} + 21e^{2x+9y+z} + 19e^{2x+9y+z} = 19e^{2x+9y+z} - 19e^{2x+9y+z} \] \hspace{1cm} (D.3)

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[28] For details see discussion after Eq. (15) in Ref. [27]. The role of the term $\delta_\alpha u_{\beta} \delta_\beta u_{\alpha}$ for the crumpling transition at $d_c \gg 1$ will be discussed elsewhere (D.R. Saykin, I.V. Gornyi, V.Yu. Kachorovskii, I.S. Burmistrov, in preparation).
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