THE $p$-ADIC CM-METHOD FOR GENUS 2

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ABSTRACT. We present a nonarchimedian method to construct hyperelliptic CM-curves of genus 2 over finite prime fields.

Throughout the document we use the following conventions (this is only for the reference and use of the authors):

- $d$  degree of the base field of the curve, i.e. $C/F_{2d}$
- $s$  number of isomorphism classes, in elliptic curve case $s = h_K$
- $n$  degree of an irreducible component of class invariants
- $K$  a CM field
- $K_0$  the real subfield of $K$
- $K^*$  the reflex CM field of $K$
- $K^*_0$  the real subfield of $K^*$
- $j_1$  absolute Igusa invariant $J_2^5 J_{-10}^1$
- $j_2$  absolute Igusa invariant $J_2^3 J_4 J_{-10}^1$
- $j_3$  absolute Igusa invariant $J_2^2 J_6 J_{-10}^1$
- $N$  2-adic precision

1. Introduction

In 1991 Atkin proposed an algorithm for constructing elliptic curves over finite fields with a given endomorphism ring [Atk91, AM93]. This algorithm originally proposed to speed up the Goldwasser-Kilian primality test has several applications. Since the knowledge of the endomorphism ring, enables us to easily determine the number of points on the elliptic curve, it can for example be used to construct elliptic curves with a prime order which has applications to cryptography. The complex multiplication method has also become attractive to construct suitable curves for pairing based cryptography [DEM04, BLS02, BW03].

The usual CM-method works with floating point arithmetic. We first construct all $h = h(O)$ isomorphism classes of elliptic curves with complex multiplication by a given order $O$ of discriminant $D$ in an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. We then compute their $j$-invariants numerically and build the minimal polynomial

$$H_D(X) = \prod_{i=1}^{h}(X - j_i)$$

which by theory has integer coefficients, that can be recognized from their floating point value if the precision of the computation is high enough. The CM-method has been generalized to higher genus, i.e. to genus 2 curves and some special cases in genus 3 [Wen03, Wen01, KW04].
Recently, nonarchimedian approaches to the construction of class polynomials $H_D(X)$ and analogues have been developed (see [CH02, BS04]). In this setting, given an imaginary quadratic order $\mathcal{O}$ of discriminant $D$ we choose a prime $p$ of size roughly $D$ such that there exists an elliptic curve with complex multiplication by $\mathcal{O}$ over $\mathbb{F}_p$ — such a curve is found by exhaustive search. A canonical lift of the $j$-invariant of the initial curve is computed $p$-adically to sufficient precision to recover its minimal polynomial $H_D(X)$ over $\mathbb{Q}$.

In this paper we consider an analogue of the nonarchimedian approach to construct class polynomials of hyperelliptic curves of genus 2. We use a higher dimensional generalizations of the AGM over a 2-adic field.

Our paper is organized as follows. We first demonstrate the basic idea by using an example in genus 1 (see Section 2). We then recall some theoretical facts on complex multiplication of abelian varieties of dimension 2 (see Section 3). In Section 4 we give an overview of the complete algorithm.

For our algorithm we need to explain how to run over isomorphism classes of ordinary genus 2 curves in characteristic 2 (see Section 5). We need also to describe the AGM method for hyperelliptic curves of genus 2 (see Section 6). We revise the $p$-adic LLL-algorithm and describe some modifications which are specific to our situation (see Sections 7). We also discuss how to determine the endomorphism ring of a hyperelliptic curve over characteristic 2 in special cases (Section 8).

Finally, we give numerical examples which show that the $p$-adic method can be efficiently used to compute class polynomials of certain quartic CM fields (see Section 9).

2. Description of the basic AGM method for elliptic curves

We first recall the AGM method for elliptic curves and explain how it can be used to generate the class polynomial for imaginary quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with $D \equiv 1 \mod 8$. Let $k$ be a 2-adic local field with uniformizer $\pi$ and let $a, b \in k$ be two elements such that 

$$\frac{a}{b} \equiv 1 \mod (8\pi).$$

We can then take the square root $x = \sqrt{a/b}$ which is uniquely determined if we impose the condition $x \equiv 1 \mod 4\pi$. The sequence of pairs defined by

$$(a_{i+1}, b_{i+1}) = \left(\frac{a_i + b_i}{2}, b_i \sqrt{\frac{a_i}{b_i}}\right)$$

derive from 2-isogenies between elliptic curves. More precisely, if $E$ is an elliptic curve given by an equation of the form

$$E_i : y^2 = x(x - a_i^2)(x - b_i^2),$$

then the curve

$$E_{i+1} : y^2 = x(x - a_{i+1}^2)(x - b_{i+1}^2)$$

is 2-isogenous to $E_i$ (possibly over some extension). Moreover the value $t_i = a_i/b_i$ is an isomorphism invariant of the pair $(E_i, E_{i+1})$ with their full 2-torsion structures, and if $E_i$ is defined over the unramified extension of $\mathbb{Q}_2$, then $E_i \to E_{i+1}$ reduces to the Frobenius modulo 2.

Suppose that we are given an ordinary elliptic curve $E$ over $\mathbb{F}_q$ with $q = 2^d$ for some $d$. Let $\mathbb{Q}_q$ be the unique unramified extension of degree $d$ of $\mathbb{Q}_2$. Then by a Theorem of J. Lubin,
J.-P. Serre and J. Tate [LST64], there exists an elliptic curve $\tilde{E}/K_q$ such that
\[
\text{End}(E) \simeq \text{End}(\tilde{E}).
\]
This curve is called the canonical lift of $E$. Given $E$ with $\text{End}(E) \simeq O_K$ we want to construct the polynomial $H_D(X)$. Suppose that $h_K = t \cdot d$, i.e. the prime $p_2$ lying above 2 in $\mathbb{Q}(\sqrt{D})$ has order $d$ in the class group. We use the AGM method to construct a cycle of 2-isogenous elliptic curve
\[
E_0^1 = E \to E_1^1 \to \cdots \to E_d^1 = E_0^2
\]
where $\varphi_i : E_i \to E_{i+1}$ is a 2-isogeny. Repeating the cycle sufficiently many times, we get a sequence of elliptic curves such that $E_j^1$ is a good approximation for the canonical lift of $E_j$.

We can then recover the $j$-invariant of $\tilde{E}_j$ with high precision. If $d \neq h_K$, we have to repeat this process for other elliptic curves of $\mathbb{F}_q$ with the same endomorphism ring until we found all $j$-invariants in $\mathbb{Q}_q$. We then compute the class polynomial with coefficients in $\mathbb{Q}_q$ that we recognize as integers if the precision is high enough.

Example 2.1. Consider a simple example. Let $D = -15$. Then 2 splits into two nonprincipal prime ideals in $K = \mathbb{Q}(\sqrt{D})$ and we find the curve
\[
E : y^2 + xy = x^3 + \alpha^2
\]
over $\mathbb{F}_2(\alpha) = \mathbb{F}_2[x]/(x^2 + x + 1)$ with $\text{End}(E) = O_K = \mathbb{Z}[(1 + \sqrt{-15})/2]$. We lift $E$ to the curve $\tilde{E}/K_q$ where $Q_q$ is the unramified extension of $Q_2$ of degree 2 given by
\[
\tilde{E} : y^2 = x(x-a_0^2)(x-b_0^2)
\]
where $\beta$ is a lift of $\alpha$ to $K_q$ and $a_0 = 1 + 4\beta^2$ and $b_0 = 1 - 4\beta^2$. We now apply 13 rounds of the AGM and obtain
\[
\begin{align*}
j(E_1^{13}) &= 8026247402149799202321\beta - 6102896026815785332240, \\
j(E_2^{13}) &= 3730718496258231955951\beta + 2950325125578927178719.
\end{align*}
\]
The Hilbert class polynomial, determined modulo $2^{28}$, is given by
\[
H_{-15}(X) = X^2 + 191025X - 121287375.
\]

**N.B.** The size the coefficients of $H_D(X)$ can be explicitly bounded by
\[
\pi \sqrt{|D|} \sum_{a} \frac{1}{a} + 10,
\]
where the sum runs over all reduced quadratic forms $(a, b, c)$ of discriminant $D$ (see [Coh96, p. 416]), so the precision needed for this algorithm can be effectively determined.

There are two main obstructions to extending to an arbitrary discriminant $D$. First, the size of the coefficients of the output polynomial $H_D(X)$ makes the construction of the Hilbert class polynomial expensive even for $D$ of modest size, and second, the application of the AGM imposed a congruence condition $D \equiv 1 \mod 8$. In order to achieve a reduction in the coefficients size, one can use alternative modular functions, e.g. on some modular curve $X_0(N)$. In the AGM example, the modular invariant $t_i = a_i/b_i$ is a function on $X_0(8)$ of the form $(u_i + 4)/(u_i - 4)$, and the AGM recursion determines a lifted invariant which satisfies the smaller minimal polynomial
\[
X^4 - 9X^3 + 17X^2 + 24X + 16,
\]
any root $u$ of which determines a CM $j$-invariant by
\[ j = \frac{(u^4 + 224u^2 + 256)^3}{u^2(u + 4)^4(x - 4)^4}. \]

Existence of generalised AGM methods for elliptic curves in odd characteristic have been proved by Carls [Car02]. Explicit formulae for AGM recursions described for modular functions on various $X_0(pN)$ and small characteristics $p$ were determined by Kohel [Koh03] and by Bröker and Stevenhagen [BS04] using Weber functions (modular functions of level $N = 48$) and small characteristic $p$.

The method of Couveignes and Henocq [CH02] for level $N = 1$ imposes no congruence condition on the input discriminant, while variation of the level $N$ also varies of the congruence condition on $D$. In another direction, Lercier and Riboulet-Deyris [LRD04] use a $p$-adic lift of a CM order embedded in the endomorphism ring of a supersingular elliptic curve. For $p = 2$ this allows one to treat the complementary classes $D \equiv 5 \mod 8$ and (fundamental) $D \equiv 0 \mod 4$.

Remark 2.1. The case of genus 1 can be compared and contrasted with the problems which arise in the generalisation to higher dimension.

1. To determine the class polynomial of a maximal order $O$, we have to ensure that a selected curve $E/\mathbb{F}_q$ has complex multiplication by $O$ and not some suborder. Determining the correct order $\text{End}(E)$ requires a more detailed analysis (for example see [Koh96] and some extensions to genus 2 in [EL04]).

2. For an elliptic curve $E/\mathbb{F}_q$, such that its $j$-invariant generates $\mathbb{F}_q$ over $\mathbb{F}_p$, the class number must be divisible by the extension degree $d = [\mathbb{F}_q : \mathbb{F}_p]$. The order $\text{End}(E)$ of a randomly chosen $E$, however, has discriminant $D = t^2 - 4q$, whose class number tends to grow like $O(\sqrt{q})$. In the case of genus 2, the class number will tend to grow faster.

3. All elliptic curves over a finite field $\mathbb{F}_q$ which have complex multiplication by an imaginary quadratic order $O$ have the same field of definition. This follows from the Galois theory of class fields for imaginary quadratic fields; its generalization to higher dimension does not preserve this feature.

4. The $j$-invariant is an algebraic integer and we have explicit bounds on the size of $j$ in terms of the discriminant of the order. The lack of explicit bounds and the failure of the Igusa invariants to be algebraic integers provide both technical and theoretical obstacles. As a result, even proving the correctness of the result becomes more cumbersome (see Section 9).

3. The theoretic background

In this section we will summarize some basic facts on Jacobians of genus two curves and quartic CM fields needed to understand the algorithm represented in the next section.

3.1. The Frobenius endomorphism and its characteristic polynomial. Let $C$ be a hyperelliptic curve of genus 2 over a finite field $\mathbb{F}_q$ and let $J_C$ be the Jacobian of $C$. Note that $J_C$ is an abelian surface. Let $\pi_q$ be the Frobenius endomorphism on $J_C$. Let $T_{\ell}$ be the Tate module for $J_C$ for some prime $\ell$, $(\ell, q) = 1$. The Frobenius operates on the 4-dimensional vector space $T_{\ell} \otimes \mathbb{Q}_\ell$ and the characteristic polynomial $f_{\pi_q}(x) \in \mathbb{Z}[x]$ of this representation is independent of the prime $\ell$. It classifies the isogeny class of the Jacobian over $\mathbb{F}_q$. Any
root $w$ of the Frobenius polynomial has absolute value $\sqrt{q}$ and we have

$$f_{\pi}(1) = \# J_C(\mathbb{F}_q).$$

Given $f_{\pi}(x)$, we can determine $\text{End}(J_C) \otimes \mathbb{Q}$ [Wat69]. If $f_{\pi}(x)$ is irreducible, then the Frobenius endomorphism generates a CM field of degree 4, i.e. a totally imaginary quadratic extension of a real quadratic field.

In the opposite direction, in Section 4 we will try to construct a curve over a large finite field $\mathbb{F}_p$ whose Jacobian has complex multiplication by the maximal order in a given CM field $K$. Suppose we have given such a curve $C$. The Frobenius endomorphism on $J_C$ corresponds to an element $w \in \mathcal{O}_K$ with absolute value $\sqrt{q}$. If we know that $J_C$ is simple, then $\mathbb{Q}(w) = K$. This will always be the case if $K$ is non-normal or cyclic.

There are only finitely many elements in $\mathcal{O}_K$ such that $w\overline{w} = q$. For each $w$ we can compute the minimal polynomial $f_w(x) \in \mathbb{Z}[x]$. If the Jacobian is ordinary and $K$ does not contain any nontrivial roots of unity, we find precisely two different $w$ up to conjugation and two different group orders in the Galois case and two or four different $w$ up to conjugation and two, three or four different group orders in the non-normal case (cf. [Wen04]).

We can now find the right order $n$ by choosing random elements in the Jacobian and multiplying them with the possible values for $n$.

3.2. Quartic CM fields. Let $K$ be a quartic CM field and let $\Phi = \{\varphi_1, \varphi_2\}$ be a set of two different embeddings of $K$ into $\mathbb{C}$ such that $\varphi_1 \neq \varphi_2 \rho$ where $\rho$ is the complex conjugation. Then $(K, \Phi)$ is called a CM type; up to conjugation, there exist exactly two different CM types. To every abelian variety over $\mathbb{C}$ with complex multiplication by an order of $K$ we can assign a specific CM type. This CM type is called primitive if and only if the abelian variety is absolutely simple. A quartic CM field may be non-normal (whose normal closure is a $D_4$ extension of $\mathbb{Q}$), cyclic, or bicyclic Galois extensions of $\mathbb{Q}$. For the first two, every CM type is primitive, but every bicyclic CM type is nonprimitive, so we focus on the case that $K$ is non-normal or cyclic over $\mathbb{Q}$.

We can show that conjugate CM types will lead to the same set of isomorphism classes of abelian varieties. In the cyclic case, the set of isomorphism classes of one specific CM type coincides with the set of isomorphism classes of any other CM type. Hence it will be enough to consider only one fixed CM type (cf. [Spa94]). For a CM field $K$, we denote by $\mathcal{O}_K$ its maximal order and by $K_0$ its quadratic real subfield. In order to determine the number $s$ of isomorphism classes of principally polarized abelian varieties with CM by $\mathcal{O}_K$, we define an associated class group.

**Definition 3.1.** Let $\mathcal{I}(K)$ be the group of fractional ideals in $K$, and let $K^\times$ act on the group $\mathcal{I}(K) \times K_0^\times$ by $\mu(a, \alpha) = (\mu a, a \mu \overline{a})$. Then the subgroup of $\mathcal{I}(K) \times K_0^\times$ consisting of pairs $(a, \alpha)$ such that $a\overline{a} = (\alpha)$ for totally positive $\alpha \in K_0$ contains the image of $K^\times$, and we define $\mathcal{C}(\mathcal{O}_K)$ to be quotient of this subgroup by $K^\times$.

The following theorem summarises the results of §14.6 of Shimura [Shi98], and provides the explicit class number for the set of isomorphism classes of principally polarized CM abelian varieties.

**Theorem 3.1.** The set of isomorphism classes of principally polarized abelian varieties with CM by $\mathcal{O}_K$ is a principal homogeneous space over $\mathcal{C}(\mathcal{O}_K)$, in particular $s = |\mathcal{C}(\mathcal{O}_K)|$. 

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We note that $\mathfrak{C}(\mathcal{O}_K)$ is an extension by a group of order $1$ or $2$, of the kernel of the norm map $Cl(\mathcal{O}_K) \to Cl^+(\mathcal{O}_{K_0})$ \footnote{Here $Cl^+(\mathcal{O}_{K_0})$ is the group of ideals modulo totally positive principal ideals, and $Cl^+(\mathcal{O}_{K_0}) = Cl(\mathcal{O}_{K_0})$ if the fundamental unit of $\mathcal{O}_{K_0}$ has norm $-1$.} given by $a \mapsto a\overline{a}$. The class of $(\mathcal{O}_K, 1)$ is the identity element, and $\mathfrak{C}(\mathcal{O}_K)$ is a nontrivial extension of this kernel if and only if the fundamental unit $\epsilon_0$ of $\mathcal{O}_{K_0}$ has norm $1$ and is not in the image of a fundamental unit of $\mathcal{O}_K$. In this case $(\mathcal{O}_K, \epsilon_0)$ is a second element of $\mathfrak{C}(\mathcal{O}_K)$ which lies over the principal class of $Cl(\mathcal{O}_K)$.

**Corollary 3.1.** Let $s$ denote the number of isomorphism classes of principally polarised abelian varieties with CM by a maximal CM order $\mathcal{O}_K$, and let $h'$ be the order of the kernel of the norm map $Cl(\mathcal{O}_K) \to Cl^+(\mathcal{O}_{K_0})$. If $N(\epsilon_0)$ equals $-1$, then $s = h'$ if the field $K$ is normal and $s = 2h'$ if $K$ is non-normal. If $N(\epsilon_0)$ equals $1$, then $s = h'$ if $\epsilon_0$ is the norm of a unit in $\mathcal{O}_K$, and $s = 2h'$ otherwise.

The Cohen-Lenstra heuristics imply that the class number of the real quadratic field $K_0$ has class number $1$ with density greater than $3/4$. In this case we can express a more precise form of the theorem (see [Wen04]).

**Corollary 3.2.** Let $K$ be quartic CM field, with real quadratic subfield $K_0$ of class number $1$. If $K$ is cyclic over $\mathbb{Q}$, then there are $h_K$ isomorphism classes, and if $K$ is not normal over $\mathbb{Q}$ then there are $2h_K$ isomorphism classes, with $h_K$ classes associated to each CM type.

**N.B.** The enumeration of the isomorphism classes does not provide the Galois action on their moduli. The CM moduli determine an abelian extension of the Galois group $Cl(\mathcal{O}_K^*)$ of the reflex field $K^*$ via a map $Cl(\mathcal{O}_K^*) \to \mathfrak{C}(\mathcal{O}_K)$. In the cyclic case, $K^*$ and $K$ coincide, but in the non-normal case, $K$ and $K^*$ are nonisomorphic quartic CM fields embedded in the normal closure $L$ of $K$. In the latter case, the action on the CM isomorphism classes is given by $a \mapsto (g(a), N(a))$ where $g$ is the composition of ideal extension to $L$ with the norm of $L/K$, and $N = N_{K^*/\mathbb{Q}}$. Even if $Cl(\mathcal{O}_K^*) \cong Cl(\mathcal{O}_K)$, the map to $\mathfrak{C}(\mathcal{O}_K)$ may have a kernel which results in reducibility of the corresponding class equations (see Shimura [Shi98, Main Theorem 1, Note 3, pp. 112-113]).

### 3.3. The splitting of a prime in a given CM field

Let $K$ be a quartic CM field. Analogously to the elliptic curve case we can define invariants which classify the isomorphism class of the hyperelliptic curves of genus $2$ or equivalently the principally polarized abelian surfaces over $\mathbb{C}$ completely. In contrast to the elliptic curve case, the moduli space is $3$-dimensional and we find three $j$-invariants $j_1, j_2, j_3$. We define the class polynomial

$$H_k(X) = \prod_{\sigma \in \Sigma} (X - j_\sigma^k), \quad k = 1, 2, 3$$

where $\Sigma$ is the set of all isomorphism classes of principally polarized abelian surfaces with CM by the maximal order $\mathcal{O}_K$. Since we run over all isomorphism classes, the polynomials $H_k(X)$ are Galois invariants, i.e. $H_k(X) \in \mathbb{Q}[X]$.

In this subsection, we would like to discuss the properties of these class polynomials and their splitting modulo a prime $p$. This is used twice in our algorithm: first with $p = 2$, since we are going to start from a curve in characteristic $2$; and second with a large odd $p$, after the class polynomials have been computed, in order to build CM curves over large finite fields. By abuse of notation, we also use $p$ to denote the prime ideal generated by the
rational prime \( p \). For simplicity, we restrict to \( h_{K_0} = 1 \), in which case we know the number of isomorphism classes (see Subsection 3.2). Similar arguments will apply in the general case.

Let \( A \) be an abelian surface with principal polarization \( E \) of \( CM \) type \((K, \Phi)\) with \( j \)-invariants \( j_1, j_2, j_3 \) and let \( k_0 \) be the field of moduli which is the unique subfield of \( \mathbb{C} \) with the property: An automorphism \( \sigma \) of \( \mathbb{C} \) is the identity on \( k_0 \) if and only if there exists an isomorphism \( \lambda : (A, E) \to (A^\sigma, E^\sigma) = (A, E)^\sigma \). Obviously, we have \( k_0 := \mathbb{Q}(j_1, j_2, j_3) \). Let \((K^*, \Psi)\) be the reflex type of \((K, \Phi)\). We can characterize \( k_0^* := k_0K^* \) in terms of class field theory.

**Theorem 3.2** (Main Theorem of Complex Multiplication, [Shi98]). Given a \( CM \)-type \((K, \Phi)\) with reflex type \((K^*, \psi)\). Consider the ideal group \( H_0 \) of ideals \( a \) in \( K^* \) for which there exists an element \( \mu \) in \( K \) such that

\[
\prod_j \psi_j(a) = (\mu) \text{ and } N(a) = \mu \overline{\mu}.
\]

The group \( H_0 \) contains the principal ideals, and the corresponding unramified class field over \( K^* \) is the field \( k_0^* \).

For every \( CM \) type \((K, \Phi)\) we find \( h_K \) isomorphism classes of principally polarized abelian varieties (cf. [Wen04]) and the polynomial

\[
G_k^\Phi(X) = \prod_{\sigma \in \Sigma^\Phi} (X - j_k^\sigma)
\]

(where \( \Sigma^\Phi \) is the set of isomorphism classes of principally polarized abelian surfaces with \( CM \) type \((K, \Phi)\)) lies in \( K^*[X] \) by Theorem 3.2. Since it is invariant under complex conjugation, we even get \( G_k^\Phi(X) \in K_0^*[X] \) where \( K_0^* \) is the real subfield of \( K^* \). If \( K \) is Galois, \( H_k(X) = G_k^\Phi(X) \) and if \( K \) is non-normal, \( H_k(X) = G_k^\Phi(X)G_k^\Psi(X) \) where \( \Phi \) and \( \Psi \) are the two different \( CM \) types. The polynomial \( G_k^\Phi(X) \) does not need to be irreducible over \( K_0^* \), since \([k_0^*: K^*]\) can be smaller than \( h_K \). More precisely, we have

\[
[k_0^*: K^*] = |I_{K^*/K_0}| = |I_{K^*/K_0}/U_{K_0}| \times U_0/U_1
\]

where \( I_{K^*/K_0} \) is the ideal class group of \( K^* \), \( I_{K^*/K_0} \) is the group of ideals in \( I_K \) which are of the form \( \prod_j \psi_j(A^\sigma) \) for some \( A \in I_{K^*/K_0} \), \( U_{K_0} \) is the subgroup of principal ideals of \( I_{K^*/K_0} \), \( U_0 \) is the group of units in \( K_0 \) which are of the form \( N_{K/Q}(B)(\beta)^{-1} \) where \( \beta = N_{K/Q}(B) \) and \( U_1 \) the subgroup of units in \( K_0 \) which are a norm of a unit in \( K \) (see [Shi98], p. 112, Note 3 and p. 114, Example 15.4 (3)).

If \( h_K \) is odd, \( K \) is non-normal and \( N(\epsilon_0) = -1 \), we can deduce \([k_0^*: K^*] = h_K \) [Hec13, Shi98]. We expect \( G_k^\Phi(X) \) to be irreducible over \( K_0^*[X] \) (in general this might not be true, since \( K^*(j_1, j_2, j_3) = k_0^* \) does not imply \( K^*(j_k) = k_0^* \) for a single \( j_k \)).

We now consider the abelian variety obtained by reducing the invariants modulo a prime in \( k_0^* \). Let \( p \) be a rational prime and \( \mathfrak{P} | p \) be a prime ideal of degree \( f \) in \( k_0^* \) such that \( v_{\mathfrak{P}}(j_k) \geq 0 \) for all \( k \). By reducing \( j_k \) mod \( \mathfrak{P} \) we obtain a curve \( C \) over \( \mathbb{F}_p \). Let \( J_C \) be its Jacobian.

Note that \( v_{\mathfrak{P}}(j_k) < 0 \) only if the reduction of the corresponding principally polarized abelian variety \( A \) defined over a number field \( k \supseteq k_0^* \) modulo some prime \( \frak{q} \) lying above \( \mathfrak{P} \) is superspecial, i.e. \( A \mod \frak{q} \) is isomorphic to the product of two supersingular elliptic curves [dSG97].
We will determine the $p$-rank of $J_C$. For this we use to following theorem.

**Theorem 3.3.** ([Lan83, Chapter 4 Theorem 1.1], [Shi98, Section 19]) Let $\pi$ be the Frobenius endomorphism on the Jacobian $J_C$ obtained by reducing $j_1$, $j_2$ and $j_3$ modulo $\mathfrak{p}$. There exists an element $\pi_0$ in $K$ such $i(\pi_0) = \pi$ where $i$ denotes the embedding $\mathcal{O}_K \to \text{End}(A)$. Moreover, with this $\pi_0$ one has $g(N_{k_0^*/K}(\mathfrak{p})) = \pi_0 \mathcal{O}_K$ where $g(\mathfrak{p})$, $\mathfrak{p}$ ideal in $K^*$, is the ideal $a$ in $\mathcal{O}_K$ such that $\mathfrak{a} \mathcal{O}_L = \prod_{\psi_0 \in \psi} \mathfrak{p}^{\psi_0} \mathcal{O}_L$ where $L$ is the Galois closure of $K$.

The theorems so far allow us to determine the Frobenius endomorphism for every prime ideal in $k_0^*$. We now consider the case that the field $k_0 = \mathbb{Q}(j_1, j_2, j_3)$ does not contain $K^*$.

Let $G = \text{Gal}(k_0^*|k_0)$. There exists an injective homorphism $\pi : G \to \text{Aut}(K)$ defined by $i(\alpha) = i(\alpha^{\pi(\alpha)})$. The image $\pi(G)$ is a Galois group $\text{Gal}(K|M)$ for some subfield $M$.

To determine the characteristic polynomial of the Frobenius endomorphism over a smaller subfield we use the following theorem:

**Theorem 3.4** ([Lan83], Chapter 4, Theorem 6.2). Let $\mathfrak{p}_0$ be a prime in $k_0$ where $A$ has good reduction and let $\pi_0$ be the corresponding Frobenius. Let $T = \text{End}(K/\mathbb{C})/\pi(G)$ be a set of representatives of embeddings of $K$ into $\mathbb{C}$ modulo $\pi(G)$. Then for every $l \neq p$ the characteristic polynomial of the Frobenius is given by

$$\prod_{\tau \in T} \prod_{\mathfrak{p} \mid \mathfrak{p}_0} (X^{f(\mathfrak{P})} - \alpha(\mathfrak{P}^\tau))$$

where $f(\mathfrak{P})$ is the degree of the prime ideal $\mathfrak{P}$ in $k_0^*$ over $\mathfrak{p}_0$ and $\alpha(\mathfrak{P})$ is up to a root of unity equal to $g(\mathfrak{p}_1)$ where $\mathfrak{p}_1$ is the prime ideal in $K^*$ lying below $\mathfrak{P}$.

**Theorem 3.5.** Let $A$ be a principally polarized abelian surface with CM type $(K, \Phi)$ with complex multiplication by $\mathcal{O}_K$ with invariants $j_1$, $j_2$ and $j_3$ in $k_0$. Let $(K^*, \Psi)$ be the reflex CM type. Consider the abelian variety $\overline{A}$ obtained by reducing $j_k$ mod $\mathfrak{p}_0$ for some ideal $\mathfrak{p}_0$ above $p$ in $k_0$. Depending on the splitting of $p$ in $K$, we get

1. if $p$ splits completely, the abelian variety $\overline{A}$ is ordinary and has complex multiplication by $\mathcal{O}_K$;
2. if $p$ is unramified, inert or splits only in $K_0/\mathbb{Q}$ but not any further, the abelian variety $\overline{A}$ is supersingular; the same is true if $p$ ramifies completely, if $(p) = \mathfrak{p}^2$ and if $(p) = \mathfrak{p}_1 \mathfrak{p}_2$ but $p$ does not ramify in $K_0/\mathbb{Q}$;
3. if $p$ splits into three prime ideals, the abelian variety will have $p$-rank 1; the same is true if $(p) = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3^2$;
4. if $p$ is inert in $K_0/\mathbb{Q}$ but splits in $K/K_0$, the abelian variety will either be supersingular or ordinary with complex multiplication by $\mathcal{O}_K$ (depending on the CM type chosen); the same happens if $(p) = \mathfrak{p}_1 \mathfrak{p}_2^2$ where $p$ ramifies in the extension $K_0/\mathbb{Q}$.

**Proof.**

1. Let $p\mathcal{O}_K = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 \mathfrak{p}_4$ with all these prime ideals being distinct (since $p$ is unramified) and let $\mathfrak{P} \mid \mathfrak{p}_0$ be the prime ideal in $k_0^*$. Then $g(N_{k_0^*/K}(\mathfrak{P})) = (\mathfrak{p}_1 \mathfrak{p}_2)^f = \mu \mathcal{O}_K$ is principal where $f$ is the degree of $N_{k_0^*/K}(\mathfrak{P}) \subseteq K^*$ or equivalently the smallest integer such that $(\mathfrak{p}_1 \mathfrak{p}_2)$ is principal. We have $(\mathfrak{p}_1 \mathfrak{p}_2)^f$ is coprime to $(\mathfrak{p}_1 \mathfrak{p}_2)^f$. Hence, the abelian variety is ordinary and by [Shi98], p.100, its endomorphism ring is equal to $\mathcal{O}_K$. 8
(2) If \( p \) is inert, the abelian variety modulo \( p_0 \) is defined over \( \mathbb{F}_p \), \( \mathbb{F}_{p^2} \) or \( \mathbb{F}_{p^4} \) depending on whether \( k_0 \cap K^* = \mathbb{Q}, K^*_0 \) or \( K^* \). Using Theorem 3.3 and 3.4, we see that in this case the characteristic polynomial of the power of the Frobenius is equal to \( (X^2 \pm p^4)^2 \) and the abelian variety is supersingular.

If \( p \) splits in \( K_0/\mathbb{Q} \) but not any further, the abelian variety is defined over \( \mathbb{F}_p \) or \( \mathbb{F}_{p^2} \) and the characteristic polynomial of a power of the Frobenius is \( (X^2 \pm p^2)^2 \). Again, it is supersingular.

If \( (p) = p^4 \), we have \( g(N_{k_0/K^*}(\mathfrak{P})) \) is an even power of \( p \). Considering the characteristic polynomial of the Frobenius we see that its \( p \)-rank must be equal to 0. The same argument works for \( (p) = p^2 \) and \( (p) = p_1^2 p_2^2 \).

(3) If \( p \) splits in three prime ideals, then \( p \) is inert in \( K^*_0/\mathbb{Q} \). This can only occur if \( K \) is non-normal and in this case the field generated by the invariants will always contain \( K^*_0 \). Hence the field of definition will always contain \( \mathbb{F}_{p^2} \).

Let us consider the following diagrams of fields:

\[
\begin{array}{ccc}
\langle r,s \rangle & \langle r \rangle & \langle r \rangle \\
\langle s^2 \rangle & K^*_0 & K_0 \\
\langle r^2 \rangle & \langle r \rangle & \langle s^2 \rangle \\
K_0 & Q & \langle rs \rangle \\
\end{array}
\]

where \( \langle r, s; s^4 = r^2 = 1, rs^3 = sr \rangle \) is a representation of the Galois group of the Galois closure \( L \) of \( K \).

Let \( (p) = p_1^2 p_2^2 \) in \( K \) and \( q_1 q_2 \overline{q_1} \overline{q_2} \) in \( L \). Then \( (p) = \mathfrak{R} \mathfrak{R}^* \), where \( \mathfrak{R} = q_1 q_2 \), is the prime ideal decomposition of \( p \) in \( K^* \).

The automorphism \( r \) leaves \( q_1, \overline{q}_1 \) invariant and interchanges \( q_2 \) and \( \overline{q}_2 \). The automorphism \( s^2 \) maps \( q_1 \) to \( \overline{q}_1 \) and \( q_2 \) to \( \overline{q}_2 \).

The automorphism \( r \) is a continuation of the real conjugation of \( K^*_0/\mathbb{Q} \). We get \( \mathfrak{R} \mathfrak{R}^* = q_1 q_2 \overline{q}_1 \overline{q}_2 = q_1^2 q_2 \overline{q}_2 \). Hence, \( g(\mathfrak{R}) = p_1^2 p_2 \). The invariants are defined over the field \( \mathbb{F}_{p^2 f} \) where \( f \) is the smallest number such that \( (p_1^2 p_2)^f \) is principal.

In this case the Frobenius element is \( (w) = (p_1^2 p_2)^f \) and the Frobenius polynomial is the minimal polynomial of \( w \). By considering the Newton polygon of the characteristic polynomial of the Frobenius we see that its \( p \)-rank is equal to 1. The case \( (p) = p_1^2 p_2^2 \) can be treated similarly.

(4) Now consider the case where \( p \) is inert in \( K_0/\mathbb{Q} \) but splits in \( K/K_0 \), i.e. \( p = p \mathfrak{P} \). Here, \( p \) splits in three prime ideals \( (p) = \mathfrak{R}_1 \mathfrak{R}_1^* \mathfrak{R}_2 \mathfrak{R}_2^* \) in \( K^* \). We consider the same diagram as above.

We find \( \mathfrak{R}_1 \mathfrak{R}_1^* = p \mathcal{O}_L \), hence \( g(\mathfrak{R}_1) = p \mathcal{O}_K \), and \( \mathfrak{R}_2 \mathfrak{R}_2^* = p \mathcal{O}_L \), hence \( g(\mathfrak{R}_2) = p \mathcal{O}_K \).

In the first case, the invariants are defined over the field \( \mathbb{F}_{p^f} \) where \( f \) is the smallest integer such that \( p^f \) is principal in \( K \). The Frobenius element \( w \) is then given by \( (w) = p^f \).
In the second case, the invariants are always defined over $\mathbb{F}_{p^2}$. We get $\pi_0 = \pm p$. The abelian variety is $\mathbb{F}_{p^2}$-isogenous to the product of two supersingular elliptic curves. The case where $(p) = p_1^2p_2^2$ is similar. □

Our algorithm will start with an ordinary hyperelliptic curve of genus 2 defined over a finite field $\mathbb{F}_{2^d}$. Hence, we are only interested in CM field where 2 splits completely, 2 is inert in $K_0/\mathbb{Q}$ but splits in $K/K_0$ or 2 ramifies in $K_0$ and is of the form $p_1^2p_2^2$ in $K$.

We can then compute the extension degrees $d$ over which we expect to find a hyperelliptic curve with complex multiplication by $\mathcal{O}_K$ as follows:

1. If 2 splits completely, i.e. $(2) = p_1^2p_2^2$ where $\overline{p}_i$ is the complex conjugate of $p_i$.
   Let $f_1$ be the smallest integer such that $p_1^2p_2^2$ is principal and let $f_2$ be the smallest integers such that $p_1^2\overline{p}_2^2$ is principal. Then we will find $h_K$ isomorphism classes of hyperelliptic curves defined over $\mathbb{F}_{2^f_1}$ and $h_K$ isomorphism classes of hyperelliptic curves defined over $\mathbb{F}_{2^f_2}$.

2. If 2 is inert in $K_0/\mathbb{Q}$ but splits into two prime ideals $p\overline{p}$ in $K/K_0$, we find $h_K$ isomorphism classes of hyperelliptic curves with CM by $\mathcal{O}_K$ over $\mathbb{F}_{2^f}$ where $f$ is the smallest number such that $p^f$ is principal.

3. If 2 ramifies in $K_0/\mathbb{Q}$ and is of the form $p_1^2p_2^2$ in $K/\mathbb{Q}$, we find $h_K$ isomorphism classes of hyperelliptic curves with CM by $\mathcal{O}_K$ over $\mathbb{F}_{2^f}$ where $f$ is the smallest number such that $(p_1^2p_2^2)^2$ is principal.

4. **The Algorithm**

   We now describe an algorithm for constructing hyperelliptic curves over finite fields with complex multiplication by a given maximal order $\mathcal{O}_K$. We will restrict to specific CM-fields, e.g. there should exist an ordinary hyperelliptic curve with complex multiplication by $\mathcal{O}_K$ over a field of characteristic 2 (see Subsection 3.3). The algorithm differs from the analytic approach mainly in the computation of the class polynomials. Hence, we will first explain the construction of the class polynomial.

   **Input:** An ordinary hyperelliptic curve over $\mathbb{F}_{2^d}$ with complex multiplication by a maximal order $\mathcal{O}_K$ in a CM field $K$.

   **Output:** Irreducible factors $\tilde{H}_k(X)$ of the class polynomials $H_k(X) = \prod_{i=1}^{s}(X - j_k^{(i)})$ of degree $n \leq s$.

   1. Compute the number of isomorphism classes $s$ of principally polarized abelian varieties over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_K$ using Subsection 3.2. This gives an upper bound for the degree of $\tilde{H}_k(X)$.
   2. Lift the curve to a 2-adic field.
   3. Compute the Serre-Tate-Lubin lift using AGM.
   4. Recover the absolute Igusa $j$-invariants $j_1$, $j_2$, $j_3$ (see Section 5) with high $p$-adic precision.
   5. Apply LLL to find the minimal polynomial $\tilde{H}_1(x) \in \mathbb{Q}[x]$ of $j_1$ of degree $n \leq s$.
   6. Apply LLL to find the minimal polynomials $\tilde{H}_2(x)$ and $\tilde{H}_3(x) \in \mathbb{Q}[x]$ of $j_2$ and $j_3$ of degree $n$.
   7. Output $\tilde{H}_1$, $\tilde{H}_2$ and $\tilde{H}_3$. 
To get a curve over $\mathbb{F}_p$ for $p$ a large prime with complex multiplication by $\mathcal{O}_K$, we now choose a prime $p$ such that there exists an $w \in \mathcal{O}_K$ with

$$w\overline{w} = p$$

and such that $f_w(1)$ is prime where $f_w$ is the minimal polynomial of $w$. We determine $(\overline{j}_{1}, \overline{j}_{2}, \overline{j}_{3}) \in \mathbb{F}_p^3$ and compute the curve using Mestre’s algorithm (cf. [Wen03]).

**Remark 4.1.** The determination of the number $s = s(K)$ of isomorphism classes is useful for the application of the LLL algorithm. Note that $H_1(X)$ does not have to be irreducible (cf. Remark 2.1, (3)), but in many cases, we have $H_1(X) = \widetilde{H}_1(X)$ and using the algorithm above we can recover the complete polynomial $H_1(X)$. In general, we expect $n$ to be equal to $s$ or $s/2$. This is true, for example, if $K$ is Galois, the real quadratic subfield has class number one, the fundamental unit is negative, and the class number $h_K$ is odd [Hec13]. Hence, given $s$, we first try to take $n = s$ and $2n = s$ in step (5) of the algorithm.

Note that for most application (e.g. constructing curves over large prime fields with given group order) it is sufficient to compute an irreducible factor over $\mathbb{Q}[x]$ of the class polynomial.

**Remark 4.2.** It is classical to consider that $\widetilde{H}_1(X)$, $\widetilde{H}_2(X)$ and $\widetilde{H}_3(X)$ are to be called the class polynomials. However, they do not describe fully the CM points in the moduli space, since the relations between the invariants are missing. We therefore modify the above algorithm as follows.

Instead of computing $\widetilde{H}_2(X)$ and $\widetilde{H}_3(X)$ in (6), we compute polynomials $G_2(X)$ and $G_3(X)$ of degree $n - 1$ such that

$$j_2 \widetilde{H}_1'(j_1) = G_2(j_1) \quad \text{and} \quad j_3 \widetilde{H}_1'(j_1) = G_3(j_1)$$

(see in Subsection 7.3 why this is better than classical interpolation).

This approach is only possible if the coordinate $j_1$ is a separating function for the CM points, or equivalently if $\widetilde{H}_1(X)$ is a squarefree polynomial of maximal degree. This is usually the case, and then there is a major advantage for the second part of the algorithm, the application of Mestre’s algorithm [Mes91].

For Mestre’s algorithm we reduce the polynomials modulo $p$ and we try to find a suitable triple $(\overline{j}_{1}, \overline{j}_{2}, \overline{j}_{3}) \in \mathbb{F}_p^3$. Given $\widetilde{H}_1(x)$, $\widetilde{H}_2(x)$ and $\widetilde{H}_3(x)$ we have to loop through all possible triples $(x_1, x_2, x_3)$ where $\widetilde{H}_k(x_i) = 0$. In our situation we can compute a root $\overline{j}_1$ of $\widetilde{H}_1(X)$ mod $p$ and then determine $\overline{j}_2$ and $\overline{j}_3$ directly from $G_2(x)$ and $G_3(x)$. We need to factor only one polynomial modulo $p$ and the set $(\overline{j}_{1}, \overline{j}_{2}, \overline{j}_{3}) \in \mathbb{F}_p^3$ can be deduced directly. This is much more efficient. Note that this trick can also be applied in the analytic approach.

5. **Isomorphism classes in characteristic 2**

In this section we discuss the choice of suitable invariants $j_1$, $j_2$, $j_3$ for the algorithm described in Section 4 and describe how to choose suitable curves in characteristic 2 which we can use as an input for the algorithm described in the previous section.

We have to be careful to choose the right invariants, since we are in characteristic 2. In the literature and the computer algebra package Magma we usually find three different sets of invariants, known as Igusa-Clebsch invariants, Clebsch invariants and Igusa invariants. These can easily be transformed into each other (see [Mes91]).
We are only interested in the so-called Igusa invariants \([J_2, J_4, J_6, J_8, J_{10}]\) since they also make sense in characteristic 2. Two curves are isomorphic over some extension if and only if their Igusa invariants agree as points in weighted projective space. The subspace of curves with ordinary Jacobian in \(\mathbb{F}_2\) is defined by the condition \(J_2 \neq 0\) (see [CNP04]). From the projective Igusa invariants we define absolute invariants

\[
(j_1, j_2, j_3, j_4, j_5) = \left( \frac{J_5^3}{J_{10}}, \frac{J_3^2}{J_{10}}, \frac{J_2^2 J_4}{J_{10}}, \frac{J_2 J_8}{J_{10}}, \frac{J_4 J_6}{J_{10}} \right).
\]

The absolute invariants are well-defined since \(J_{10} = 0\) if and only if a curve is singular. However, since \(4J_8 = J_2^2 - J_2 J_6\), in characteristic 2 we obtain the relation \(j_1 j_3 = j_2^2\) and also \(j_2 j_3 = j_1 j_5\). For an ordinary curve, the absolute invariant \(j_1\) is nonzero (since \(J_2 \neq 0\)) so we may eliminate the invariant \(j_3\) in determining a parametrization of such curves, but use the triple \((j_1, j_2, j_4)\) for the invariants of a lifted curve. In order to classify curves of nonordinary Jacobian, it is necessary to define additional absolute invariants (see [Igu60]).

We now would like to enumerate all isomorphism classes of hyperelliptic genus 2 curves over \(k\) which are defined over \(k = \mathbb{F}_{2^d}\) to find suitable CM fields as input to our algorithm. We note that over a finite field, a curve is defined over its field of moduli, hence the field of definition of the point \((j_1, j_2, j_3, j_4, j_5)\) is the field of definition for a curve. (For a classification of curves and their twists, we refer to a paper by Cardona, Nart and Pupolás [CNP04].)

Following Igusa [Igu60], every ordinary curve of genus 2 in characteristic 2 has a normal form

\[
y^2 - y = ax + bx^{-1} + c(x - 1)^{-1}, \quad abc \neq 0,
\]

isomorphic via \((x, y) \mapsto (x, y(x(x-1))^{-1})\) to the curve

\[
C : y^2 - x(x-1)y = x(x-1)(ax^3 + ax^2 + (b + c)x + b).
\]

We define \(s_1(C) = a + b + c\), \(s_2(C) = ab + bc + ac\), and \(s_3(C) = abc\). The absolute Igusa invariants can be expressed in terms of these invariants (cf. [Igu60, p. 623]); in particular

\[
\begin{align*}
    j_1^{-1} &= J_2^{-5} J_{10} = s_3(C)^2, \\
    j_2 j_1^{-1} &= J_2^2 J_4 = s_1(C)^2, \\
    j_4 j_1^{-1} &= J_2^4 J_8 = s_2(C)^2 + s_1(C)^3 + s_1(C)^4.
\end{align*}
\]

Thus the maps

\[
(s_1, s_2, s_3) \mapsto \left( \frac{1}{s_3^2}, \frac{s_1^2}{s_3}, \frac{s_2^2 + s_1^3 + s_3^3}{s_3^2} \right),
\]

and

\[
(j_1, j_2, j_4) \mapsto \left( \frac{\sqrt{j_2}}{\sqrt{j_1}}, \frac{\sqrt{j_4}}{\sqrt{j_1}} + \frac{j_2 \sqrt{j_2}}{j_1 \sqrt{j_1}}, \frac{1}{\sqrt{j_1}} \right),
\]

define mutual inverses between triples \((s_1, s_2, s_3)\) with \(s_3 \neq 0\) and \((j_1, j_2, j_4)\) with \(j_1 \neq 0\).

Conversely given a triple \((s_1, s_2, s_3) \in k^3\), with \(s_3 \neq 0\), there exists a curve in normal form

\[
C : y^2 - x(x-1)y = x(x-1)(ax^3 + ax^2 + (b + c)x + b),
\]

where \(x^3 + s_1 x^2 + s_2 x + s_3 = (x - a)(x - b)(x - c)\), over an extension of degree at most 3.

**Remark 5.1.** Cardona, Nart and Pupolás [CNP04] show for a finite field of characteristic 2, that one can in fact find a representative curve \(C/k\) given any triple \((s_1, s_2, s_3)\) in \(k^3\) with \(s_3 \neq 0\). This implies that triples \((s_1, s_2, s_3)\) are in bijection with \(\overline{k}\)-isomorphism classes of
6. Higher dimensional generalization of AGM

In order to apply the algorithm, we need a method to compute the Serre-Tate-Lubin lift of a genus 2 curve over a 2-adic field. We use an algorithm due to Mestre [Mesb] which uses the explicit formulae usually called “Richelot isogeny”. Later on, Mestre [Mesa] proposed another method, based on Borchardt’s mean. The latter has been implemented by Lercier and Lubicz [LL]. Borchardt’s mean involves simpler formulae and extends to higher genus. Since we are interested in the genus 2 case, we stick to the first “Richelot” algorithm. This variant is not well described in the literature, so we give now a few details about it.

6.1. AGM lifting via Richelot’s isogeny. The basic idea of the genus 2 AGM lifting algorithm is to have explicit formulae that describe fully a \((2,2)\)-isogeny between jacobians of curves. This can also be viewed as an explicit modular equation relating the invariants of the curves. The following can be found in [BM88]:

**Theorem 6.1.** If \(S\) and \(T\) are monic polynomials of degree 2, define

\[
[S,T](x) = S'(x)T(x) - S(x)T'(x).
\]

Let \(C\) be a genus 2 curve of equation \(y^2 = P(x)Q(x)R(x)\), where \(P\), \(Q\), \(R\) are monic of degree 2. Let \(C'\) be the curve given by the equation

\[
\Delta y^2 = [Q,R](x) [R,P](x) [P,Q](x),
\]

where \(\Delta\) is the determinant of \(P, Q, R\) in the basis \(1, x, x^2\).

Then \(\text{Jac}(C)\) and \(\text{Jac}(C')\) are \((2,2)\)-isogenous abelian varieties. Moreover the kernel and the expression of the isogeny can be made explicit.

This theorem is valid over any field of odd characteristic, including a 2-adic field. The next task is then to put the curve we have in a form suitable to apply the theorem, and then to make the right choice for \(P\), \(Q\) and \(R\), so that the \((2,2)\)-isogeny corresponds to the second power Frobenius isogeny, when we reduce everything modulo 2.

A convenient form to work with is a Rosenhain form: we find \(\lambda_0\), \(\lambda_1\) and \(\lambda_{\infty}\) such that the curve of equation \(y^2 = x(x-1)(x-\lambda_0)(x-\lambda_1)(x-\lambda_{\infty})\) is isomorphic to \(C\). By considering the reduction of the 2-torsion divisors, one can show that the \(\lambda_i\) can be chosen such that \(\lambda_1 \equiv 1 \mod 4\), \(\lambda_0 \equiv 0 \mod 4\) and \(\text{val}(\lambda_{\infty}) = -2\).

Then the corresponding Rosenhain form for the curve \(C'\), so that the isogeny reduces to the second power Frobenius modulo 2, is given by invariants \(\lambda'_i\) satisfying

\[
\lambda'_\infty = \frac{(u_1 - v_\infty)(u_\infty - v_0)}{(u_1 - v_0)(u_\infty - v_\infty)}, \quad \lambda'_1 = \frac{(u_1 - v_\infty)(w_1 - v_0)}{(u_1 - v_0)(w_1 - v_\infty)}, \quad \lambda'_0 = \frac{(u_1 - v_\infty)(w_0 - v_0)}{(u_1 - v_0)(w_0 - v_\infty)},
\]

where \(u_1\) and \(u_\infty\) are the solutions of the equation

\[
U^2 - 2\lambda_{\infty}U + \lambda_{\infty}(1 + \lambda_1) - \lambda_1 = 0,
\]

\(v_0\) and \(v_{\infty}\) are the solutions of the equation

\[
V^2 - 2\lambda_{\infty}V + \lambda_0\lambda_{\infty} = 0,
\]
and \( w_0 \) and \( w_1 \) are the solutions of the equation
\[
(\lambda_0 - 1 - \lambda_1)W^2 + 2\lambda_1 W - \lambda_0 \lambda_1 = 0.
\]

In all these formulae, the subscript indicates the value of the variable modulo 2 (and an infinity subscript means that the valuation is negative). Hence, the distinction between the roots of the equations of degree 2 is easy.

As a consequence, we can derive a genus 2 AGM lifting procedure just like in genus 1 as recalled in Section 2. At each step, we have to compute three square roots (for solving the three equations of degree 2) and a few products, additions and inversions. If the curve \( C \) we started with is ordinary, then the sequence converges (in the same sense as in Section 2) to the canonical lift of \( C \). The theoretical explanation for that is given in [Car04].

To complete the algorithm, we still need to explain how to initialize the AGM iteration. Since the formulae involve the 2-torsion points, we need to have them defined in the base field that we consider. In other words, when looking at the starting curve defined over the finite field \( y^2 + h(x)y = f(x) \), it is necessary that \( h(x) \) splits completely. We restrict to the case where \( \deg h = 2 \) and \( \deg f = 5 \). Also, since the curve is supposed to be ordinary, the polynomial \( h \) is squarefree. Let us write \( h(x) = x^2 + h_1 x + h_0 = (x - \rho_0)(x - \rho_1) \). Then, by doing the transformation to the Rosenhain form, and keeping everything formal, we can derive the following values for the initialization of the AGM iteration:
\[
\lambda_\infty = 4/h_1^2, \quad \lambda_0 = 4f(\rho_0)h_1^2 + f'(\rho_0^2) / h_1^6, \quad \lambda_1 = 1 + 4f(\rho_1)h_1^2 + f'(\rho_1^2) / h_1^6.
\]

6.2. Asymptotically fast lifting algorithm. In the \( p \)-adic CM method, we might need to lift the curve to a very high precision. The plain AGM method that we have just sketched has a complexity which is at best quadratic in the precision. This quickly becomes a problem. A first subquadratic algorithm was designed by Satoh, Skjernaa and Taguchi [SST03], then an almost-linear lifting method was designed by Kim et al. [KPC+02] in the case where the base field admits a Gaussian normal basis, and finally Harley obtained an almost-linear lifting method that works for any base field. A precise description and comparison of these methods in the elliptic case can be found in [Ver03].

We have used the asymptotically fast variant of Harley, that we now explain briefly.

Instead of going around the cycle of isogenous curves, getting closer and closer to the canonical lift, we consider only two curves \( C \) and \( C' \) and their canonical lifts. Once lifted, the Rosenhain invariants of \( C \) should annihilate the Frobenius-twisted modular equations corresponding to the equations above: we should have
\[
\Phi(\Lambda, \Lambda') = 0,
\]
where \( \Lambda \) is the vector \((\lambda_0, \lambda_1, \lambda_\infty)\) of Rosenhain invariants, \( \sigma \) is the Frobenius substitution in a 2-adic field \( \mathbb{Q}_q \), and \( \Phi \) is the function from \( \mathbb{Q}_q^6 \) to \( \mathbb{Q}_q^3 \) that corresponds to the Richelot equations above, where the intermediate variables \( u_i, v_i \) and \( w_i \) have been eliminated.

Then an adaptation of the Newton lifting method can be used to compute a solution \( \Lambda \) to that equation, thus yielding the invariants of the canonical lift. A key ingredient of that method is that we have to be able to compute the action \( \sigma \) quickly. To this effect, the 2-adic field is represented in a polynomial basis, with a generator that is a root of unity (a Teichmüller lift of a generator of the underlying finite field). Then the computation of the Frobenius image of an element has a cost bounded by the cost of a few multiplications.
in the field. We skip the details and refer to [Ver03] for a precise description and analysis. Adapting the algorithm given there to the genus 2 case is essentially a multivariate rewriting of the algorithm for elliptic curves. Not surprisingly the jacobian matrix of \( \Phi \) is involved in place of just the two partial derivatives.

7. The \( p \)-adic LLL Algorithm and Lagrange Interpolation

As pointed out in Remark 2.1, (3), the hyperelliptic curves with complex multiplication by \( \mathcal{O}_K \) in characteristic 2 do not all have the same field of definition. Moreover, given a class polynomial \( H_K(X) \in \mathbb{Q}[X] \), not all roots in a field of characteristic 2 need to lead to hyperelliptic curve with complex multiplication by the field \( K \) (see Subsection 3.3 and the discussion following Theorem 3.5). This happens for example for a non-normal \( K \) whose real subfield has class number one if 2 is inert in the real subfield \( K_0 \) but splits in \( K/K_0 \). In this case we find only \( h_K \) hyperelliptic curves over a field of characteristic 2 with complex multiplication by \( \mathcal{O}_K \) although there exist \( 2h_K \) isomorphism classes over \( \mathbb{C} \).

Hence, it is more convenient to compute only one root up to a high precision and then apply the LLL algorithm to recover the minimal polynomial. Note that using this approach we will only find a irreducible factor of the class polynomials and there are in general not irreducible.

7.1. The \( p \)-adic LLL-algorithm. Given a lattice \( \Lambda = \langle b_1, \ldots, b_m \rangle \) the LLL algorithm produces a short lattice basis. This can be used to determine the minimal polynomial of an algebraic element given by a floating point representation. Let \( \det(\Lambda) \) be the determinant of \( \Lambda \). Using Minkowski’s inequality we can approximate the shortest lattice vector by

\[
\sqrt{\frac{m}{2\pi e}} \det(L)^{1/m}.
\]

If \( v \in \Lambda \) has length much smaller than this bound, it will be the shortest vector with high probability.

Let \( \mathbb{Z}_q \) be an extension of \( \mathbb{Z}_2 \) of degree \( d \) with \( \mathbb{Z}_2 \) basis 1, \( w_1, \ldots, w_{d-1} \). Let \( \alpha \in \mathbb{Z}_q \) generating \( \mathbb{Z}_q \), and \( \tilde{\alpha} \) be an approximation of \( \alpha \) modulo a high power of 2, say \( \alpha \equiv \tilde{\alpha} \mod 2^N \). We assume that we know the degree \( n \) of its minimal polynomial \( f(x) \in \mathbb{Z}[x] \), i.e.

\[
f(x) = a_n x^n + \ldots + a_0
\]

where \( a_i \in \mathbb{Z} \) are unknown. In order to determine \( a_i \), we determine a basis of the left kernel in \( \mathbb{Z}^{n+d+1} \) of the matrix

\[
\begin{pmatrix}
A \\
2^N I_d
\end{pmatrix}
\]

where \( A \) is the \((n+1) \times d \) matrix

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
\alpha_{10} & \alpha_{11} & \ldots & \alpha_{1,(d-1)} \\
\vdots & \vdots & & \vdots \\
\alpha_{n0} & \alpha_{n1} & \ldots & \alpha_{n,(d-1)}
\end{pmatrix}
\]

with \( \alpha_{jk} \) defined by

\[
\alpha^j = \alpha_{j0} + \alpha_{j1} w_1 + \ldots + \alpha_{j,(d-1)} w_{d-1}.
\]
This kernel is a lattice $\Lambda$, in which the coefficients of the minimal polynomial of $\alpha$ are part of a short vector. Indeed, if $a_0, \ldots, a_n$ are integers such that
\[ a_n\alpha^n + \ldots + a_0 \equiv 0 \mod 2^N \]
then $(a_0, \ldots, a_n, *, \ldots, *)$ will be a short vector in $\Lambda$ that we expect to find in a $LLL$-reduced basis.

7.2. Lagrange interpolation. In Section 4, Remark 4.2, we mention that we do not compute $\tilde{H}_1(X), \tilde{H}_2(X)$ and $\tilde{H}_3(X)$ but $\tilde{H}_1(X)$ and two polynomials $G_2(X), G_3(X)$ with the property that
\[ j_2 \cdot \tilde{H}_1'(j_1) = G_2(j_1) \quad \text{and} \quad j_3 \cdot \tilde{H}_1'(j_1) = G_3(j_1). \]
Let us first consider the usual Lagrange interpolation, i.e. suppose we compute $F_k(X) \in \mathbb{C}[X]$ with
\[ j_k = F_k(j_1) \quad \text{for} \quad k = 2, 3. \]
Let us assume that the conjugates $j^{(i)}_1$ for $i = 1, \ldots, n$ are all distinct (see Remark 7.1). Then $F_k(X)$ is given by
\[ \sum_{i=1}^{n} j^{(i)}_k \prod_{\ell \neq i} \frac{X - j^{(i)}_1}{j^{(i)}_1 - j^{(\ell)}_1}. \]
Since $F_k(X)$ is easily seen to be Galois invariant, we have $F_k(X) \in \mathbb{Q}[X]$. Unfortunately, due to the factor
\[ \prod_{\ell \neq i} \frac{1}{j^{(i)}_1 - j^{(\ell)}_1} \]
the coefficients of $F_k(X)$ have usually a much larger height than those of $\tilde{H}_k(X)$. Hence, we prefer to compute $G_k(X)$ with the property
\[ (7.1) \quad j_k \tilde{H}_1'(j_1) = G_k(j_1). \]
A formula for $G_k$ is then given by
\[ G_k(X) = \sum_{i=1}^{n} j^{(i)}_k \tilde{H}'(j^{(i)}_1) \prod_{\ell \neq i} \frac{X - j^{(i)}_1}{j^{(i)}_1 - j^{(\ell)}_1}. \]
Since
\[ H'(j^{(i)}_1) = \text{leadcoeff}(\tilde{H}_1(X)) \cdot \prod(j^{(i)}_1 - j^{(\ell)}_1) \]
where $\text{leadcoeff}(\tilde{H}_1(X))$ denote the leading coefficient of $\tilde{H}_1(X)$, we expect $G_k(X)$ to have approximately the same height as $\tilde{H}_1(X)$.

Remark 7.1. In order to be able to apply the Lagrange interpolation formula we need the roots of the polynomial $\tilde{H}_1(X)$, to be distinct. In practice we do not expect it to have any multiple roots. If this happens to be the case, we solve the problem by choosing some linear combinations of $j_1, j_2, j_3$ such that all roots are distinct.
7.3. Lagrange interpolation and LLL. We now modify the lattice given in Subsection 7.1 to work for determining \( G_2(X) \) and \( G_3(X) \). Let \( d_k \) be the denominator of \( G_k(X) \). Then equation (7.1) becomes

\[
\tilde{G}_k(X) = d_k G_k(X) \in \mathbb{Z}[X].
\]

where \( \tilde{G}_k(X) \) is the vector \((a, a, \ldots, a)\)

We consider the lattice \( \Lambda \) which is the kernel of the matrix

\[
\begin{pmatrix} A \\ 2^{N} I_d \end{pmatrix}
\]

where the rows of \( A \) contain the coefficients of \( 1, j_1, j_2, \ldots, j_{n-1} \), \( \tilde{H}_i(j_1)j_2 \), expressed on the \( \mathbb{Z}_2 \)-basis. If we have

\[
\tilde{G}_k(X) = a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \ldots + a_0,
\]

then the vector \((a_0, \ldots, a_{n-1}, d_k, *, \ldots, *)\) will be a short vector in \( \Lambda \) that we expect to find in a LLL-reduced basis.

7.4. Starting from several triples. It is often possible to compute \( p \)-adic approximations of several triples \((j_1, j_2, j_3)\) of invariants of curves having CM by \( \mathcal{O}_K \). Furthermore, it can be the case that those triples form an orbit under the action of a subgroup of the Galois group of the field generated by the invariants.

Before showing how this can be used to speed up the computations, let us give two examples of situations where we get such information.

- Once we have lifted one triple \((j_1, j_2, j_3)\) of elements in \( \mathbb{Z}_q \) where \( q = 2^4 \), we can easily compute the \( d \) conjugate triple by applying the Frobenius automorphism of \( \mathbb{Z}_q \).
- It is possible that by enumerating all isomorphism classes over the finite fields we have found several nonconjugate curves having CM by \( \mathcal{O}_K \). For instance, if \( K \) is non-normal, \( h_{K_0} = 1, N(\epsilon_0) = 1 \) and the class number \( h_K \) is odd, we expect to find at least \( h_K \) isomorphism classes over the finite field.

Let \( (j_1^{(i)}, j_2^{(i)}, j_3^{(i)})_{1 \leq i \leq k} \) be such a set of conjugate triples, with \( k \) divides \( n \). Then the symmetric functions of these triples are in an extension of degree \( n/k \) of \( \mathbb{Q} \). It is then possible to build appropriate symmetric functions, so that applying the LLL algorithm to recognize algebraic numbers of degree \( n/k \) will allow to reconstruct the polynomials \( H_1, G_2, G_3 \). We expect this approach to be faster than applying the LLL algorithm to reconstruct elements of degree \( n \) directly, since the complexity of lattice reduction depends badly on the dimension of the lattice (hence of the degree of the elements to recognize).

On the other hand, having the possibility to recognize elements of smaller degree implies more involved computations to deduce the polynomials \( H_1, G_2, G_3 \). This is based essentially on resultant computations. We give now more details about this approach.

We start by building the polynomial \( M_1(X) \) whose roots are the \( j_1^{(i)} \):

\[
M_1(X) = (X - j_1^{(1)}) (X - j_1^{(2)}) \ldots (X - j_1^{(k)}) = X^k + m_{k-1}X^{k-1} + \ldots + m_1X + m_0.
\]

By the discussion above, the coefficients \( m_i \) of \( M(X) \) are algebraic elements of degree \( n/k \). We use the LLL algorithm to compute the minimal polynomial \( P(X) \in \mathbb{Q}[X] \) of \( m_0 \). Let us call \( K_P \) the number field \( \mathbb{Q}[X]/(P(X)) \), which is a degree \( n/k \) subfield of the field \( k_0 \) of degree \( n \) containing the CM invariants. Then we recognize the other \( m_i \) as elements of \( K_P \),
expressed in terms of $m_0$. For that we use again the LLL algorithm, but with the modified lattice as in Section 7.3. Hence $M_1$ has been rewritten as a bivariate polynomial

$$X^k + m_{k-1}(Y)X^{k-1} + \cdots + m_1(Y)X + Y,$$

with rational coefficients, where $Y$ is a root of the polynomial $P(Y)$. The resultant in $Y$ of $M_1(X,Y)$ and $P(Y)$ is the polynomial $H_1(X)$ we are looking for, perhaps up to a multiplicative factor.

We can perform the same kind of computation for $j_2$ and $j_3$, so as to obtain $H_2(X)$ and $H_3(X)$. However, we would prefer to obtain $G_2(X)$ and $G_3(X)$ that give more information. Let us explain how to get them using the LLL algorithm, with the modified lattice, and we get a bivariate polynomial

$$G_2(X,Y) = n_{k-1}(Y)X^{k-1} + \cdots + n_1(Y)X + n_0(Y),$$

defined over $\mathbb{Q}$, where $Y$ is again a root of $P(Y)$. To convert back into a univariate representation, we need an explicit expression for the embedding of the subfield $K_P$ into $\mathbb{Q}[X]/(H_1(X))$.

The computation of this embedding can be handled by various algorithms. We suggest the following: the polynomial $H_1(X)$ is obtained as the resultant of $M_1(X,Y)$ and $P(Y)$. If this resultant is computed by the subresultant algorithm, on the way to the solution we compute a polynomial of degree 1 in $Y$ that belongs to the ideal generated by $M_1(X,Y)$ and $P(Y)$. Let us denote this polynomial by $S(X,Y) = S_1(X)Y + S_0(X)$. Then as an element of $\mathbb{Q}[X]/(H_1(X))$, a root of $P$ is given by $-S_0(X)/S_1(X)$, thus yielding the required embedding.

Once $M_2$ has been recognized as an element of $\mathbb{Q}[X]/(H_1(X))$, we just have to renormalize it with $H_1'(X)$, to obtained $G_2(X)$.

**Remark 7.2.** In the description of our method, we have overlooked two problems that we encounter when actually implementing these algorithms:

- The elements are not algebraic integers, so we have to take care of denominators everywhere. This is not a big difficulty but can induce many programming mistakes.
- If we implement line by line the method, there is a huge explosion of the sizes of the coefficients in the middle of the algorithm. Once $p$-adic elements are recognized as algebraic elements, we therefore have to switch to modular computation: resultants, subresultants, and computations in $\mathbb{Q}[X]/(H_1(X))$ must be handled by computing modulo sufficiently enough primes, and we switch back to integers only for the final reconstruction of $H_1$, $G_2$ and $G_3$, when we know that the integers have a reasonable size.

**Remark 7.3.** As before, in this algorithm we made some genericity assumptions. Indeed, it could well be that the coefficient $m_0$ that we used to defined the subfield $K_P$ is in a fact in a subfield of degree less than $n/k$. In that case, we just have to choose another element to define the field $K_P$ we work with.
8. Determining the endomorphism ring

A critical issue is the identification of a representative curve whose Jacobian has maximal endomorphism ring. It is necessary to have a mechanism to discard curves associated to the nonmaximal orders. The following proposition gives a partial answer.

**Proposition 8.1.** Let \( f \) be the minimal polynomial of the Frobenius endomorphism on the Jacobian \( J_C \) of a genus 2 curve \( C \) defined over \( \mathbb{F}_q \) of characteristic \( p \). Let \( \pi \) be any root of this polynomial and set \( K = \mathbb{Q}(\pi) \). Let the set
\[
\left\{ \frac{g_1(\pi)}{p^{e_1}m_1}, \ldots, \frac{g_t(\pi)}{p^{e_t}m_t} \right\}
\]
generate the maximal order \( \mathcal{O}_K \) over \( \mathbb{Z}[\pi, \overline{\pi}] \) with \((m_i, p) = 1\). Then \( g_i(\pi)/m_i \) is in \( \text{End}(J_C) \) if and only if \( g_i(\pi) \) is the zero map on \( J_C[m_i](\mathbb{F}_q) \).

**Remark 8.1.** If all the \( e_i = 0 \), then we can really test the maximality, as mentioned in [EL04]. However, and unlike the genus 1 case, it is possible that \( \mathbb{Z}[\pi, \overline{\pi}] \) is not \( p \)-maximal in \( \mathcal{O}_K \) and then we cannot answer the problem.

Besides this algorithm, there are some other strategies which can be applied:

1. Suppose we have given a curve \( C \) of genus two with field of definition \( \mathbb{F}_q = \mathbb{F}_{2^d} \) and Frobenius polynomial \( f_C(x) \). Let \( K \) be the quartic CM field generated by \( f_C(x) \). Using the discussion following Theorem 3.5, we can compute the degree \( f_1 \) (resp. \( f_2 \)) of the field of definitions of the curves in characteristic 2 with complex multiplication by \( \mathcal{O}_K \). If \( d \neq f_1 \) and \( d \neq f_2 \), the endomorphism ring of \( C \) cannot be maximal.

2. Furthermore we can use the fact that the endomorphism ring of the maximal order is in general as unicyclic as possible (a similar idea has been mentioned in [EL04]). By this we mean the following: Suppose we find two hyperelliptic curves \( C_1 \) and \( C_2 \) with the same characteristic polynomial i.e. \( f_{C_1}(x) = f_{C_2}(x) \). Then over every field extension of \( \mathbb{F}_{2^d} \) the group of rational points on the Jacobian will have the same order but not necessary the same group structure. Suppose we have a prime \( \ell \) such that \( J_{C_1} \) has all \( \ell \) torsion points rational (\( \ell \neq p \)) and not \( J_{C_2} \), then the conductor of the order of the endomorphism ring of \( J_{C_2} \) will contain the prime \( \ell \). Indeed, \((\pi - 1)/\ell \in J_{C_1}\) but is not in \( J_{C_2} \).

9. Numerical examples

9.1. **Implementation.** We have implemented our algorithm using various computer algebra packages. The first implementation has been written at a high level, using the Magma system [BC97]. Then, to be able to deal with high precisions, the asymptotically fast lifting algorithm using Richelot isogeny has been implemented in C, based upon the Mploc package written by Emmanuel Thomé [Tho]. Finally, for the LLL computations, we have interfaced our programs with Victor Shoup’s NTL library [Sho]. Those three packages use the GMP library [Gra02] for their time-critical integer operations.

After these optimizations, the cost of computing the canonical lift of a curve is not so high, even if precision is huge. Therefore it appears that the bottleneck of our method is the LLL computation and the method of section 7.4 should be used for large examples.
9.2. A non-Galois example with \( n = 2h \). We start with the curve \( C \) of equation \( y^2 + h(x)y + f(x) = 0 \) over \( \mathbb{F}_8 = \mathbb{F}_2[t]/(t^3 + t + 1) \), with
\begin{align*}
  f(x) &= x^5 + t^6x^3 + t^5x^2 + t^3x, \\
  h(x) &= x^2 + t^9x.
\end{align*}

The curve is ordinary and has CM by the maximal order of \( K = \mathbb{Q}(i\sqrt{23 + 4\sqrt{5}}) \). The field \( K \) is non-normal and its class number is 3; so we have 6 isomorphism classes of principally polarized abelian varieties.

We apply our algorithm and compute the canonical lift of \( C \) to high precision (in fact, a posteriori, we see that 1200 bits are enough) and get its invariants. From this we reconstruct the minimal polynomial \( H_1 \) and the corresponding \( G_2 \) and \( G_3 \). As expected, the degree of \( H_1 \) is 6.

\[
H_1 = 2^{18}5^{36}7^{24}t^6 - 111877303992736897740097447014016967290295436515808105468750000 T^5 + 501512527690519679504420832767471412512684541043834547644662988263671875000 T^4 - 10112409242787391786678284637305750476145431357202566748622143704263857808262923 T^3 + 118287000250586667564544773490615439811539784477927719285455514209797386992091828213521875 T^2 - 2183651011313531701163169693873494948953569198870004032131912686578084990317 T + 356515234909179364113 T^6 \]

\[
G_2 = 2^{-3}(2734249284974589542086559792016565391133303228092193603515625000000 T^5 + 5755460727714979756884938796725835456245600247914401041193774531250000000 T^4 + 2420137816065408829663614041292340977297204376705010145338233818943868187923 T^3 - 75691166830755778249624043381642889715482810993181013834694650235981947587900092046875 T^2 + 21836510113582581967081231211744309693912640348471666514876459782054400437 T - 35841249700000000 T^1 + 133333032280921936035156250000000 T^0)
\]

\[
G_3 = 2^{-4}(20062022977265019387539624994338812342692117691040039906250000 T^5 - 23006467431764975697282555828188900514908689925547595360431355781250000000 T^4 + 615017294619678086113194171814416654558821826012411563850151290136646894987547 T^3 - 1314360974215340178789612716292992493175900244503557114370659520249839645781463319312875 T^2 - 2183651011361183739513266892571328858726120821201079572468058656150973410907 T + 357513323490923501244013117964641133451984602352017881724712641689 T)
\]

By looking at the Newton polygon of \( H_1 \) for the 2-adic valuation, we see that there are three roots that have valuation 0, and the others have negative valuation. Hence only three of the curves have good reduction modulo 2. However, since \( H_1 \) is irreducible over \( \mathbb{Q} \), starting with one curve (or from the 3 conjugate curves) yields the whole \( H_1 \).

This is consistent with Theorem 3.5. Indeed, 2 is inert in \( K_0 = \mathbb{Q}(\sqrt{5}) \) and splits in two prime ideals of degree 2 in \( K \). Hence we are in subcase (4). Furthermore, one can check that each of the prime ideals above 2 have order 3 in the class group of \( K \).

9.3. A large example. We start with the curve \( C \) of equation \( y^2 + h(x)y + f(x) = 0 \) over \( \mathbb{F}_{32} = \mathbb{F}_2[t]/(t^5 + t^2 + 1) \), with
\begin{align*}
  f(x) &= x^5 + t^{20}x^3 + t^{17}x^2 + t^{19}x, \\
  h(x) &= x^2 + t^9x.
\end{align*}

The curve is ordinary and has CM by the maximal order of \( K = \mathbb{Q}(i\sqrt{75 + 12\sqrt{17}}) \). The field \( K \) is non-normal and its class number is 50; so we have \( s = 100 \) isomorphism classes of principally polarized abelian varieties. The ideal (2) splits completely in \( K \), and the primes above 2 have order 5 and 25 in the class group.

However, when considering a minimal polynomial of the lifted value of \( j_1 \), the LLL algorithm produced a plausible answer of degree 50. In fact, it seems that the class polynomial
of degree $s$ is not irreducible over the rationals, but splits in two factors of degree $n = 50$. Using our method, we can only produce one of these factors $H_1(X)$, and the corresponding polynomials $G_2(X)$ and $G_3(X)$.

For this large example, this would have been much faster to use the 5 conjugate curves instead of only one. Indeed, with our implementation, using only one curve (and therefore, doing lattice reduction to recognize elements of degree 50) requires about one day for the whole computation on an Athlon64 processor, most of the time being spent in $LLL$.

For that case, we use a $p$-adic precision of 65000 bits. The running time to lift the curve and compute the invariants is 20 seconds.

The leading coefficient of $H_1$ is $3^{50}11^{156}17^{60}23^{72}41^{48}691^{12}$.

9.4. Checking the result. Since we cannot give a bound on the coefficients of the class polynomials, there is no way to prove the result of the computation. However there are some hints that indicate that the result is correct.

- The leading coefficient can be a large integer. However, we expect this integer to be very smooth. In particular, it should be easy to factor this number by trial division, even though the integer has several hundreds of decimal digits. This could not occur for a random integer. Therefore, if the answer of the $LLL$ algorithm has this property, then we probably had enough precision.
- When reducing the class polynomials modulo a suitable prime $p$, one should be able to recover curves with the prescribed complex multiplication. Hence, we can choose a prime $p$ small enough so that all the computations are easy, and check that everything is consistent. For instance, the large example of the previous section was checked with the prime $p = 47653$ which splits completely in $K$ into 4 prime ideals that are principal. Then we check that $H_1$ splits completely over $\mathbb{F}_p$, and from its roots we deduce invariants and then equation for curves (using Mestre’s algorithm) that have indeed CM by $O_K$.

10. COMPLEXITY

In this Section, we estimate the cost of our algorithm. The usual way of computing class polynomials was described in [Wen03]. One starts with a CM field, computes the period matrices $(\Omega_i)_i$, recovers the $j$-invariants by computing theta constants and computes the class polynomials by gathering all the $j$-invariants. Weng’s algorithm is dominated by the computation of theta constants. This computation depends on the value of the first minima of the period matrix, which makes the analysis of this part difficult. However a naive evaluation of the theta constants is quadratic in the precision. Our algorithm is linear in the precision. Let us give some details. We can distinguish two steps : the canonical lift of the curve and the $LLL$ part. Recursive programming based on the formulae of Richelot leads to a linear algorithm in the precision. More precisely the complexity is $O((nk)^{1+\epsilon})$ where $n$ is the degree of the extension, $k$ the final precision of the $p$-adic $j$-invariants and $\epsilon$ represents the logarithmic factors in $n$ and $k$. Then we use $LLL$ to recover the class polynomials. Given $\langle b_1, b_2, \ldots, b_m \rangle$ a basis of a lattice $\Lambda$ such that for all $i$ in $\{1, \ldots, m\}$, $\|b_i\|^2 \leq B$, LLL returns a $LLL$-reduced basis in a time $O(m^6 \log^3(B))$. In our case $\log(B)$ is the precision needed in order to make $LLL$ work, so this step is in $O(m^6k^3)$. The dimension of the lattice $m$ is here the degree $h$ of our class polynomials $\tilde{H}_k(X)$. Note that the floating-point version of
LLL had been improved by Nguyêñ and Stehlé in [NS05]. Their version has a complexity of \( O(m^5(m + \log(B)) \log(B)) \). When we look at the LLL complexity, we can see that the dimension of the lattice has a very bad influence on efficiency. To reduce the dimension, one can proceed as suggested in 7.4. For instance, if one looks at the example 9.2, we can see that \( H_1(X) \) has bad reduction modulo 2. It gives a degree 3 polynomial. Thus, if one seeks CM curves over \( \mathbb{F}_{2^n} \) with maximal order in \( K = \mathbb{Q}(i\sqrt{23 + 4\sqrt{5}}) \), one finds three such curves. Hence, LLL has to deal with a lattice of dimension only 2. However such an enumeration is quite expensive. It takes \( 2^{3n} \) operations to enumerate all the curves and therefore one can afford it only over small extensions of \( \mathbb{F}_2 \). Note that, in practice, this idea is still valuable because extensions of \( \mathbb{F}_2 \) of degree less than 10 provide already huge class number (for instance with \( n = 7 \), one can find a quartic CM field whose class number is 6496).

11. Conclusion

We have presented in this article a 2-adic construction of CM genus 2 curves based on the AGM. This construction seems more efficient than the existing complex method. However, as for genus 1, it does not allow to obtain all CM fields. To tackle this problem, one should first find analogues of the AGM method in characteristics greater than 2. Another possible generalization is to higher genus. Note that for generic genus 3 curves, any explicit method is known to construct a curve over \( \overline{\mathbb{Q}} \) whose Jacobian has complex multiplication. Such a construction can be done over the 2-adics with the AGM. However unlike the hyperelliptic case, one does not know a complete set of invariants for non hyperelliptic genus 3 curves which, for the moment, prevent to make the link with number fields.

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