Phase Synchronization and invariant measures in sinusoidally perturbed chaotic systems

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Abstract.
We show that, in periodically perturbed chaotic systems, Phase Synchronization appears, associated to a special type of stroboscopic map, in which not only averages quantities are equal to invariants of the perturbation, the angular frequency, but also it exists a very large number of non-transient transformations, possibly infinity. In cases where there is not phase synchronization there is either only transitive transformations on the attractor, or a finite number of non-transitive transformations. We base our statements in experimental and numerical results from the sinusoidally perturbed Chua’s circuit.

INTRODUCTION

In observing a dynamical system it is important to measure physical quantities that are time invariant. So, different experiments, considering different sampling time observations, still produce the same quantity. In chaotic isolated flows, the most basic invariant is the chaotic attractor. Representing $X \in \mathbb{R}^d$ to be the chaotic attractor, generated by the flow $\xi_t$, invariance implies that $\xi_t(X) = X$. For many experimental practical reasons, often one observes not the attractor, but measures of a subspace of it, for example the probability measure of a subspace $A \in X$, defined as $\rho(A) = \int_A d\mu$, with $\mu(C)$ being the distribution function of finding trajectories in the subspace $C$, and $\rho$ the probability. This invariant measure implies $\rho(A) = \rho(\xi_t(A))$. A consequence of this property is that as one observes the chaotic attractor for instant times $t = n\tau$, and therefore, for the discrete set represented by $D \in \mathbb{R}^d$, $\rho(D) = \rho(X)$, what means that the natural probability density of an attractor can be calculated by discrete observations of it, independently of the sampling time.

This invariant scenario, so important for experiments, radically changes when one periodically perturbs a chaotic system. Being $X'$ some subset of $X$ and $\omega$ the angular frequency of the perturbation, one can find two possible discrete dynamics $\mathcal{D}$ with respect to natural measure. For sufficiently small perturbation amplitudes, it is still true that $\rho(D) = \rho(X')$, however for sufficiently large perturbation amplitudes $\rho(D) \neq \rho(X)$, which means that the natural measure cannot be obtained for arbitrary sampling observations.

However, in a perturbed system, there are still measures that are invariant under
different time scales and different initial conditions on subspaces of the attractor. One example of such a measure is the ratio between the growing of the phase with respect to the time. As we will see, when there is Phase Synchronization (PS) between the chaotic system and the perturbation, this ratio is equal to the angular frequency of the perturbation. As a consequence, we can find a large number of discrete subsets of the chaotic attractor, \( \rho(D) \), such that
\[
\rho(D) \neq \rho(X').
\]
When there is not phase synchronization, either \( \rho(D) = \rho(X') \), or there is only a finite number of discrete subsets \( \rho(D) \).

In a sinusoidally perturbed chaotic system Phase Synchronization exists, between a subspace of \( X \) and the perturbation if the following two conditions Eqs. (1) and (2), apply:
\[
|\phi(t) - r\omega t| < c,
\]
where the phase \( \phi(t) = \phi[X(t)] \), and \( c \in \mathbb{R} \) is a constant chosen according to the particular system studied. We consider \( r \) to be a rational number, although it can also be irrational as shown in [2], and \( \omega \) is the phase of the chaotic attractor [4, 5, 6]. A necessary condition to PS is
\[
\langle \frac{d\phi(t)}{dt} \rangle - r\omega = 0
\]
where \( \langle \cdot \rangle \) is the average taken over an infinity time interval.

In this work we denote \( \phi(\tau) \) to phase at instants \( \tau_i \). The quantity \( \langle \Delta \phi(\tau) \rangle = \langle \frac{\phi(\tau_{i+1}) - \phi(\tau_i)}{\tau_i} \rangle \) is an invariant, which means that it has the same value independently of the initial conditions, and the time interval for its calculation \( \tau_i \). Whenever Eqs. (1) and (2) are satisfied, thus it is also true that
\[
\langle \Delta \phi(\tau) \rangle = \omega
\]

**The sinusoidally perturbed Chua’s circuit:** It is represented by:
\[
\begin{align*}
C_1 \frac{dx_1}{dt} & = g(x_2 - x_1) - i_{NL}(x_1) \\
C_2 \frac{dx_2}{dt} & = g(x_1 - x_2) + x_3 \\
L \frac{dx_3}{dt} & = -x_2 - V \sin(2\pi ft)
\end{align*}
\]
Where \( x_1 \), \( x_2 \), and \( x_3 \) represent, respectively, the tension across two capacitors and the current through the inductor (See [3] for more details), where the \( f \) is the frequency and \( V \) the amplitude of the perturbation and they are the control parameters. The circuit’s attractors are setup by the \( g \) parameter, and here we adjusted it to obtain a Rössler-like chaotic attractor. The perturbation has a well defined phase \( 2\pi ft = \omega t \).

Next we present some formalism using elements of the Ergodic Theory. Given a group of diffeomorhisms \( F_t \), that is the flow, we call \( \mathcal{X} \) its \( \omega \)-limit set. If \( \mathcal{X} \) is chaotic then \( F_t(\mathcal{X}) \) is mixing, transitive and ergodic. However, others transformations applied on \( \mathcal{X} \), as stroboscopic mapping, may not possesses all these properties.

The notion of stroboscopic mapping of a flow \( F_t(\mathcal{X}) \) can be formalized defining a transformation \( T^{\tau_i} : \mathcal{X} \rightarrow \mathcal{X} \), a discretization of the attractor.
FIGURE 1. (a) Experimental PS parameter space. b) Simulated PS parameter space, with parameters \( g = 0.574, C_1 = 10, C_2 = 6, \) and \( L = 6. \) Black points represent parameters for which Eq. (1) is satisfied. In both figures, the horizontal axis represents the perturbing frequency \( f \) and the vertical axis its amplitude \( V. \) Variables in (b) are dimensionless and \( f_0 \) is the frequency of the non perturbed circuit.

Given a point \( x_0 \in \mathcal{X} \) we have \( T^{\tau_i}(x_0) = F_{\tau_i}(x_0) \), so we constructed the orbit \( \{x_0, T^{\tau_1}(x_0), T^{\tau_2}(x_0), \ldots, T^{\tau_N}(x_0), \ldots\} \). We call \( D \) the \( \omega \)-limit set generated by \( T^{\tau_i}. \)

A subspace \( \mathcal{A} \) of \( \mathcal{X} \), is said invariant by transformation \( T^{\tau_i} \) if \( T^{\tau_i}(\mathcal{A}) = \mathcal{A}, \forall \tau_i \), and a physical measure \( \rho \) is said invariant (\( T^{\tau_i} \)-invariant) on \( \mathcal{A} \) if \( \rho(T^{-\tau_i}(\mathcal{A})) = \rho(\mathcal{A}). \) \( T^{\tau_i} \) is topologically transitive in \( \mathcal{A} \) [8] if for any two open sets \( B, C \subset \mathcal{A}, \exists \tau_i / T^{\tau_i}(B) \cap C \neq \emptyset \) (7)

**Definition 1:** Two sets \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent, \( \mathcal{A} \equiv \mathcal{B}, \) if \( \forall x_i \in \mathcal{A}, \) a set \( C \) can be constructed by the union of open sets \( B_i(x_i), \) open \( \mathcal{R}^d \) volumes centered at \( x_i \) with length \( \ell, \) such that \( \forall y_i \in B \implies y_i \in C, \) and \( \mathcal{A} \neq \mathcal{B} \) if \( y_i \notin C. \)

In [1] it was shown that PS is always associated with special non-transitive transformations that decomposes the attractor in discrete chaotic subsets. To see this we introduce some notations and definitions from [1]. If there is a non-transitive transformation \( T^i, \) thus, the discrete set \( \mathcal{D} \) is a basic set if \( \mathcal{D} \neq \mathcal{X}, \) and it is formed by a finite union of open invariant and minimal sets \( \mathcal{D}_1, \ldots, \mathcal{D}_N, \) and for each \( \mathcal{D}_i \) there is a \( x \in \mathcal{D}_i \) which, under \( T^{\infty}(\mathcal{X}) \) has a dense orbit on \( \mathcal{D}_i. \) And, a minimal set \( D^i \) cannot be further decomposed. Also, if \( \mathcal{X} \) can be decomposed into a collection \( \mathcal{D} = \mathcal{D}^0, \mathcal{D}^1, \ldots, \mathcal{D}^{P-1} \) of subsets of \( \mathcal{X}, \) with \( P \geq 1, \) such that \( T(\mathcal{D}^i) \subseteq \mathcal{D}^{i+1(modP)}, \) and \( T^P(\mathcal{D}^i) \subseteq \mathcal{D}^i, \) we refer to each minimal set \( \mathcal{D}^i \) as a recurrent decomposition. The number of sets \( P \) is called length of the decomposition [7].

In Fig. (A), PS of the circuit with the perturbation is experimentally detected whenever there is a subset \( D, \) done by an observation time \( \tau_i = i\frac{1}{f}, \) such that \( \mu(\mathcal{X}) \) is very different from \( \mu(\mathcal{D}). \) In practice, we detect a subset \( \mathcal{D}, \) that contains points concentrated in an angular section smaller than \( 2\pi. \) Simulation is shown in Fig. (B), where black points represent perturbing parameters for which condition for PS is satisfied, where we
FIGURE 2. Filled balls shows projections of the discrete basic sets, and filled lines, a projection of the set $\mathcal{D}$, i.e., the sets $\mathcal{D}'$. In (A-B), we show two minimal sets of a length-2 basic set, constructed for a time series given by $i\tau, (2i + 1)/2\tau, (i + 1)\tau, ldots$. In (A) we plot the set $\mathcal{D}_0$, for points observed in the time series $i\tau, (i + 1)\tau, (i + 2)\tau$, and in (B) we plot the set $\mathcal{D}_1$ for points observed for the time series $(2i + 1)/2\tau, (2i + 3)/2\tau, (2i + 5)/2\tau, ldots$. In (C), we show a length-2 basic set, for parameters within the phase synchronization region ($V=2 \text{ mV}$ and $f=5294 \text{ Hz}$), for time series constructed as in Figs. (A)-(B). In (D), we show a length-3 basic set, constructed for a time series $i\tau, (2i + 1)/3\tau, 2(2i + 1)/3\tau, (i + 1)\tau, ldots$.

assumed that in Eq. (1) $r = 1, c_s = 2\pi$. Also, both sides of the $PS$ region are approximately symmetric. In this figure, the triangular shaped region denoted by the light gray dashed line represents the region where long $PS$ transient happens. This transient region is connected with a period-3 region. The dashed line in this figure indicate the parameter regions for which we can find $\mathcal{D} \neq \mathcal{D}'$, with equivalence as defined in Definition 1.

In Fig. 2 we show in (A) one minimal set $D_0$ and (B) another minimal set $D_1$ of a length-2 basic set $D$, for a situation for parameters for which there is not phase synchronization, indicated by the cross in Fig. 1(A). In (B) we show a length-2 basic...
set for parameters for which there is phase synchronization. Note that while \( D \) is approximately equivalent to \( X \) in (A-B), \( D \) in (C) as well \( D \) in (D) is absolutely not equivalent to \( X \).

In conclusion, we show that phase synchronization implies the existence of many invariant measure associated to sampling of perturbed chaotic attractor. In PS states these invariant measure are associated to basic sets. Experimental and Numerical examples of these statements are presented of Chua’s Circuit.

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