Fuzzy braid group: A concept

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Abstract. The braid group has role in invariant of structure. Based on definition of a braid group, a permutation group and the fuzzy set, there are relation between them such that fuzzy group is well-defined. However, fuzzy group is indirect related to a braid group. In this paper, we will disclose direct relation between the braid group and the fuzzy set so that we get the fuzzy braid group by using the measurement about composition of generators of the braids.

1. Introduction
One of the oldest works of human known is the woven [1], especially the so-called the braid [2]. The braid is widely used in life from daily activities to research activities [3]. The braids are not only a ready-made technology, but they are the basis of structural analysis in invariant form. Therefore, the braids have been a modeling object for a long time ago [4].

In the braid modeling, especially involving the group theory or an algebra [5]. There are several levels of group theory [6, 7]. The braid group has a close relationship with the permutation group from the point of the braid structure [8, 9], while the permutation group has implications for developing of the fuzzy group from the point of the implementation example [10]. If based on the structure of each braid has its own meaning, then of course can be derived a formulation to produce measurement based on the concept of the membership function of fuzzy set [11]. The paper addresses the possibility to construct a fuzzy braid group concept, by revealing some properties of the woven group, permutation group, and fuzzy group, then arranging an approach so that some of the relationships associated with the fuzzy braid group can be explained.

2. Review and Motivation
The braid group, is commonly recognized as Artin braid group, based on the form of woven like Fig. 1(a): $\sigma_i$, $\sigma_i^{-1}$ and $\sigma_i^0 = 1$, it can be defined as follows [8].

Definition 1. The braid group $A_n$ for a sequence $A_n \in A_n$, $n = 1, 2, \ldots$ on $n$ strings (we denoted a string as $s$), is presented on the generators $\sigma_1, \ldots, \sigma_{n-1}$, satisfy relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ 1 \leq i < j < n, \ j-i+1.$$  \hspace{1cm} (1)

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ 1 \leq i \leq n-1.$$  \hspace{1cm} (2)
Geometrically, each \( a_n \in A_n \), the braids \( A_n \) as elements of a braid group \( A_n \), with the woven patterns \( \sigma_i \) form a system [12]. The system consists of \( n \) string and the brushing. \( n \) strings connect to two sequences of points in parallel such that string at position \( i \) passing over/under the string at position \( i + 1 \) or other string direct connects the same points, i.e. \( i \) to \( i \). The brushing causes the woven patterns \( \sigma_i \) in parallel is not in same position [3]. Let us explain it by a braid \( A_{6,1} \in A_6 \) where \( A_{6,1} = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \), and so inverse of \( A_{6,1} \) is a reflection of the braid to a sequence of points or \( A_{6,1}^{-1} = (\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2)^{-1} = \sigma_2^{-1} \sigma_3^{-1} \sigma_3 \sigma_1 \sigma_4 \sigma_2^{-1} \sigma_2^{-1} \sigma_3 \) such that \( A_{6,1} A_{6,1}^{-1} = 1 \). Based on Eq. (1), \( A_{6,1} = \sigma_3^{-1} \sigma_4 \sigma_2 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \) also. While for \( A_{6,2} = \sigma_3^{-1} \sigma_4 \sigma_2 \sigma_1 \sigma_4 \sigma_3^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \), based on Eq. (2) \( A_{6,2} = \sigma_3^{-1} \sigma_4 \sigma_1 \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \).

Let \( \rho() \) is permutation of a given set \( I \), and there is \( \circ \) as a composition of permutations, we can define a permutation group as follows [9].

**Definition 2.** Let \( P_n \) is a group. If \( \rho() \in P_n \), then group \( P_n \) is a permutation group.

Geometrically, either \( \sigma_i \) or \( \sigma_i^{-1} \) change position of string by string based on a sequence of generators form a braid. For example \( A_{6,1} = \sigma_3^{-1} \sigma_4 \sigma_2 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \). String at position 3 passing under string at position 4 by \( \sigma_3^{-1} \), string at position 4 passing over string at position 5 by \( \sigma_4 \) and string at position 5 passing over string at position 6 by \( \sigma_5 \) such that point 3 connected to point 6. Similarly, by other generators in \( A_{6,1} \). Thus, we have a permutation

\[
\rho(A_{6,1}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 1 & 2 & 5 \end{pmatrix} \in P_6
\]

(simply \( \rho(A_{6,1}) = (136524) \), or generally

\[
\rho() = \begin{pmatrix} \cdots & i & i+1 & \cdots \\ s(\cdots) & s(i) & s(i+1) & s(\cdots) \end{pmatrix} \in P_n,
\]

where \( s \) is string. In common we know that \( i = 1, 2, \ldots, n \in I \) depend on modulo \( n \) with well-defined binary operation is a group. Inverse of permutation is

\[
\rho()^{-1} = \begin{pmatrix} \cdots & i & i+1 & \cdots \\ s(\cdots) & s(i) & s(i+1) & s(\cdots) \end{pmatrix} \in P_n.
\]
Figure 2. An equilateral triangle and permutation

Figure 3. Relation between the braid group, the permutation group and [0, 1]

In implementation

\[ \rho(A_{6,1}^{-1}) = \left( \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 1 & 2 & 5 \end{array} \right) \in \mathcal{P}_6, \]

or simply \( \rho(A_{6,1}^{-1}) = (136524)^{-1} = (142563). \) Therefore, this proposition is proven adaptively.

**Proposition 1.** If \( A_n \) is a braid group and \( \mathcal{P}_n \) is a permutation group, then there is a function \( h \) such that \( h : A_n \rightarrow \mathcal{P}_n. \)

On the opposite side, let \( \mathcal{P} \) is a set, the concept of the fuzzy set states that a mapping \( \mu : \mathcal{P} \rightarrow [0, 1] \) is called a fuzzy subset of \( \mathcal{P} \) [11], being an inspiration to a fuzzy group expressed as follows [13].

**Definition 3.** For \( \mathcal{P} \) as a group, \( \mu \subseteq \mathcal{P} \) is a fuzzy subgroup of \( \mathcal{P} \) if \( \mu \) meets two conditions: (i) \( \{ \min, \max \} \{ \mu(x), \mu(y) \} \{ \leq, \geq \} \mu(xy) \), and (ii) \( \mu(x) = \mu(x^{-1}) \) for every \( x, y \in \mathcal{P} \).

Suppose the permutation is based on an equilateral triangle, we have a group \( \mathcal{S}_3 \) as a symmetric group where \( \mathcal{S}_3 \) has three conjugacy classes [14]: \( \mathcal{S}_3 = \{ \epsilon, \xi_1, \xi_2, \xi_3, \zeta_1, \zeta_2 \} = \{(1), (12), (23), (13), (123), (132)\} \), see Fig. 2. Based Eq. (7) we define \( \mu(\epsilon) = 0.1, \mu(\xi_i) = 0.2, i = 1, 2, 3, \) and \( \mu(\zeta_j) = 0.3, j = 1, 2. \) We see that \( \max\{\mu(\xi_1), \mu(\xi_2)\} \geq \mu(\zeta_2) \) or \( 0.2 \geq 0.3 \), thus \( \min\{\mu(\xi_1), \mu(\xi_2)\} \leq \mu(\zeta_2) \). Similarly for another such that first condition is proven. Second condition, we can see that \( \mu(\xi_1) = \mu(\xi_1^{-1}) = 0.2. \) In this case the inverse of each \( \xi_i i = 1, 2, 3 \) is permutation itself. While \( \mu(\zeta_1) = \mu(\zeta_1^{-1}) = \mu(\zeta_2) = \mu(\zeta_2^{-1}) = \mu(\zeta_1) = 0.3 \) [10], see Fig. 2. Thus, \( \mu \) is a fuzzy group based on \( \mathcal{S}_3 \).
### Table 1. The braids have permutation equal to identity in the permutation group

| Sequence of generators | Permutation | Power |
|------------------------|-------------|-------|
| $\sigma_i$             | $\langle i \rangle$ | 2     |
| $\sigma_i\sigma_{i+1}$ | $(i)(i+2)(i+1)$  | 3     |
| $\sigma_i\sigma_{i+1}\sigma_{i+2}$ | $(i)(i+3)(i+1)(i+2)$ | 4     |
| $\vdots$               | $\vdots$    | $\vdots$ |
| $\sigma_{i+1}\cdots\sigma_{m-1}$ | $(i)(m)(i+1)\cdots(m-1)$ | $m$  |

3. An Approach

It has been disclosed that for generating any fuzzy group needs to show the relationship between the fuzzy set and the permutation group. Further, to declare a fuzzy braid group there is a need to disclose a relationship between the braid group and the fuzzy set, such that each braid has a value in $[0, 1]$.

The approach taken is as follows: Based on Proposition 1., it has been stated that there is a function $h$ as relation between the braid group and the permutation group, while between the permutation group and the fuzzy set there is a mapping $\mu$, see Fig. 3. Nevertheless, this approach also needs to reveal that $h$ builds a certain type of relationship between the braid group and the permutation group.

As an approach to determine type of relation between a braid group and a permutation group, we use the composition of generators that causes the element of permutation group do not change. Based on Eq. (5) and Table 1., we have the braids $\rho((\sigma_i\sigma_{i+1}\cdots\sigma_{m-1})^m) = (i)$. Therefore, next Lemma is proven.

#### Lemma 1.

If the braids have the composition of generators in $(\sigma_i\sigma_{i+1}\cdots\sigma_{m-1})^m$, then their permutations to be an identity element of the permutation group or $(i) \in P_n$.

Based on this lemma we obtain

$$h : A_n \xrightarrow{N!} P_n$$ (8)

Next, we define some of measurements about the braid group and the permutation group as follows.

**Definition 4.** Each permutation in a permutation group has scales of calculation:

(i) $n_n() = n$ is a number of numbers allowed in the permutation group $P_n$.

(ii) $n_p()$ is a number of numbers in same positions in a permutation.

(iii) $n_q()$ is a number of numbers in different positions in a permutation.

**Definition 5.** Each braid in a braid group has scales of calculation:

(i) $n_{n-1}() = n - 1$ is a number of generators allowed in the braid group $A_n$.

(ii) $n_c() = \| \cdot \|$ is a number of generators that appears in a braid.

(iii) $n_{\sigma}() = | \cdot |$ is a size of the generators composition in a braid.

4. Generating the fuzzy braid group

Based on Definition 4, we obtain a probability of the same position numbers in permutation

$$r_p() = \frac{n_p()}{n_n()} \in [0, 1],$$ (9)
or because \( n_p \leq n_n \) we have \( 0 \leq r_p \leq 1 \). If \( n \) modulo 2 = 0, then a number of numbers in same position is \( 2, 3, 4, \ldots, n \), else \( 1, 2, 3, \ldots, n \). While the probability of the different position numbers in permutation is

\[
r_q() = 1 - r_p = \frac{n_q}{n_n} \in [0, 1],
\]

or because \( n_q \leq n_n \) we have \( 0 \leq r_q \leq 1 \). Therefore, a transformation of \( r_q() \) to \( \mu \) forms function one by one. For case \( S_3 \), the values of \( r_q() \) are in \( \{0, \frac{1}{3}, \frac{2}{3}, 1\} \), and we can define a function \( \tau \) such that \( \tau(\frac{1}{3}) = 0.1, \tau(\frac{2}{3}) = 0.2, \tau(1) = 0.3 \). In other words, \( \tau : \{r_q()\} \rightarrow \mu \).

Based on Definition 5, we obtain a ratio between the generators that appear in a braid and the generators allowed in the braid group, i.e.

\[
r_c() = \frac{n_c}{n_{n-1}} \in [0, 1].
\]

In this case \( n_c() \leq n_{n-1} \), so \( 0 \leq r_c \leq 1 \). Whereas a ratio between the generators that appear in a braid and the size of the generators composition in a braid, i.e.

\[
r_\sigma() = \frac{n_c}{n_\sigma()},
\]

because \( n_c() \leq n_\sigma() \) and \( n_n() \leq n_\sigma() \), then \( 0 \leq r_\sigma \leq 1 \). However, minimum of \( n_\sigma() = n_c() \) produces \( r_\sigma = 1 \), then for calculating the strength of a braid we model a strength formulation as follows,

\[
r_s() = 1 - r_\sigma() = 1 - \frac{n_c}{n_\sigma} \in [0, 1].
\]

The membership function of the fuzzy set provides specific meaning for an object. Suppose every braid has power seen from its structure angle based on Eq. (13) or the strength probability based on the Eq. (11), with which the composition of a plaeting generator plays a role, thus giving meaning levels such as strongest, stronger, strong, commonly, weak, weaker, weakest. Thus it may be stated that this applies

\[
\mu : A_n \rightarrow [0, 1]
\]

Although the composition size of the generators of a braid is unpredictable, it must be in \([0,1]\) (Eqs. (11) and (13)), and the level of meaning based on the membership function can provide the appropriate values so as to fulfill the Definition 3., and we can use \( \mu \) to state the fuzzy braid group.

5. Conclusion

It has been disclosed that it is possible to construct a fuzzy group based on a braid group, it recognized as a fuzzy braid group, by providing a level of meaning to the braids structure by involving the composition of generators that builds a braid. The next study is to prove the existence of the fuzzy braid group.

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