Iterative Rational Krylov Algorithms for model reduction of a class of constrained structural dynamic system with Engineering applications

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Abstract

This paper discusses model order reduction of large sparse second-order index-3 differential algebraic equations (DAEs) by applying Iterative Rational Krylov Algorithm (IRKA). In general, such DAEs arise in constraint mechanics, multibody dynamics, mechatronics and many other branches of sciences and technologies. By deflecting the algebraic equations the second-order index-3 system can be altered into an equivalent standard second-order system. This can be done by projecting the system onto the null space of the constraint matrix. However, creating the projector is computationally expensive and it yields huge bottleneck during the implementation. This paper shows how to find a reduced order model without projecting the system onto the null space of the constraint matrix explicitly. To show the efficiency of the theoretical works we apply them to several data of second-order index-3 models and experimental resultants are discussed in the paper.

keywords: Structured index-3 differential algebraic equations, sparsity, Model order reduction, Iterative Rational Krylov Algorithms.

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1 Introduction

In mechanics or multibody dynamics linearized equation of motion with holonomically constraint has the following form \[1, 2\]

\[ M \ddot{x}(t) + D \dot{x}(t) + Kx(t) + G^Tz(t) = Fu(t), \quad (1a) \]

\[ Gx(t) = 0, \quad (1b) \]

\[ y(t) = Lx(t), \quad (1c) \]

where \( M, K, D \in \mathbb{R}^{n_1 \times n_1} \) are sparse matrices known as mass, stiffness and damping matrices respectively. \( x(t) \in \mathbb{R}^{n_1}, u(t) \in \mathbb{R}^{n_2} \) and \( y(t) \in \mathbb{R}^{n_2} \) are respectively known as states, inputs and outputs vectors. The constraint matrix \( G \in \mathbb{R}^{n_2 \times n_1} \) (with \( n_1 < n_2 \)) is associated with the given algebraic constraints \( z(t) \in \mathbb{R}^{n_2} \). Furthermore, \( F \in \mathbb{R}^{n_1 \times n_2} \) is the input matrix corresponding to the input vector \( u(t) \) and \( L \in \mathbb{R}^{n_2 \times n_1} \) is the output matrix associated to the measurement output vector \( y(t) \). Such structured dynamical system also appear in mechatronics where electrical and mechanical parts are coupled or in the electric circuits \[3\].

If we convert the system into first-order form then it becomes first-order index-3 system \[2\]. Therefore the system in \[11\] is called second-order index-3 descriptor system. If the system becomes very large then it is very expensive to simulate, control and optimize. Therefore, we want to reduce the complexity of the model through model order reduction (MOR) \[4, 5, 6\]. Among different MOR methods \[4, 5, 6\] the two most frequently applied modern MOR methods are the balanced truncation (BT) \[7\] and the rational interpolation of the transfer function by the iterative rational Krylov algorithm (IRKA) \[8\]. Both approaches have been extended to first-order descriptor systems \[9, 10\]. The balancing based model order reduction of second-order index-3 system \[11\] has been investigated for second-order-to-first-order and second-order-to-second-order reductions in \[2\] and \[11\] respectively. On the other hand, the authors in \[12\] discussed IRKA for the model reduction of the underlying descriptor system. In order to follow the proposed algorithm one has to convert the system into a first-order form. Besides at each iteration one has to solve a linear system with dimension \( 2(n_1 + n_2) \) which is computationally expensive tasks.

In this paper we discuss second-order-to-second-order model reduction of second-order index-3 descriptor system via IRKA without converting the system into first-order form. In the literature second-order-to-second-order reduction is called structure preserving model order reduction (SPMOR). IRKA based SPMOR for the standard second-order system was developed by Wyatt in his P.hD., thesis \[13\]. This idea was generalized for the second-order index-1 system which is slightly different from \[11\] in \[14\]. Like index-1 system the second-order index-3 system can be converted into standard second-order system. In this case instead of using Schur complement techniques as used in \[14\] we apply projection onto hidden manifold. This idea was already found in \[10, 9, 15\] for the first-order index-2 systems. On the other hand, for the second-order index-3 system the technique was implemented using a balancing based model order reduction. However, there was no investigation of this idea for this system using the IRKA. This paper contributes to close this gap. That is we mainly devote to second-order-to-second-order model reduction of second-order index-3 system using IRKA. Following the procedure in \[10, 9\] first we show that the second-
order index-3 descriptor system can be projected onto the null space of the constraint matrix which we call hidden manifold to obtain a standard second-order system. Then we can apply the technique as in [13] to obtain a standard second-order reduced order system. It is shown in the paper that the explicit computation of hidden manifold projector is not required. This is important because creating a hidden manifold projector demands a lot of computational times. Moreover, the projected system is converted into a dense form which yields huge bottleneck in implementing the reduced order model. The proposed method is applied to several models coming from Engineering applications. The performances of the proposed algorithm seems to be promising and the results are better than balancing based techniques in both approximation accuracy and computational time which appears in numerical results.

Rest of the article is organized as follows. Section 2 briefly discuss IRKA based SPMOR of second-order system and reformulation of second-order index-3 system from previous literature which are the main ingredient to obtain the new results of this paper. The main contribution of this paper will be discussed in Section 3. In this section we developed IRKA based SPMOR of second-order index-3 system. The subsequent section illustrates numerical results. At the end, Section 5 presents the conclusive remarks.

2 Background

In the following texts at first we briefly discuss IRKA for standard second-order system to obtain second-order reduced order model. Then we will show how to convert the second-order index-3 system into second-order standard system by projecting onto the null space of the constraint matrix. In fact we establish some definitions and notations based on the previous literature that will be in the upcoming sections.

2.1 IRKA for second-order system

Structure preserving IRKA (SPIRKA) for a second-order standard system was proposed in [13]. The SPIRKA is mainly based on the IRKA of first-order system which was originally proposed in [8]. This prominent algorithm was developed by Gugercin et al., in [8] to achieve the $H_2$-optimal model reduction via interpolatory projection technique. To explain the SPIRKA let us consider a second-order linear time-invariant (LTI) continuous-time system

$$\begin{align*}
\ddot{\xi}(t) + D\dot{\xi}(t) + K\xi(t) &= F u(t), \\
y(t) &= L \xi(t),
\end{align*}$$

(2)

where $M$, $D$ and $K$ are non-singular, and $\xi(t)$ is the $n$ dimensional state vector. Consider that the system is MIMO and its transfer function is defined by

$$T(s) = L (s^2 M + sD + K)^{-1} F; \quad s \in \mathbb{C}. $$

(3)

Our goal is to obtain an $r$ dimensional ($r \ll n$) reduce order model

$$\begin{align*}
\ddot{\hat{\xi}}(t) + \hat{D}\dot{\hat{\xi}}(t) + \hat{K}\hat{\xi}(t) &= \hat{F} u(t), \\
\hat{y}(t) &= \hat{L} \hat{\xi}(t),
\end{align*}$$

(4)
where the reduced coefficient matrices are constructed as
\[
\hat{M} = W^T M V, \quad \hat{D} = W^T D V, \quad \hat{K} = W^T K V, \\
\hat{F} = W^T F, \quad \hat{L} = L V,
\]
and the transfer function of the reduced order model can be defined as
\[
\hat{T}(s) = \hat{L}(s^2 \hat{M} + s \hat{D} + \hat{K})^{-1} \hat{F}; \quad s \in \mathbb{C}.
\] (6)

According to [13] the procedure of IRKA for second-order system is same as the first-order system. We want to construct reduced order model (4) in such way that the reduced transfer function (6) interpolate to the original transfer function (3) at some interpolation points. Moreover the reduced order model satisfies the interpolation conditions mentioned below [13].

Given a set of interpolation points \(\{\alpha_1, \alpha_2, \cdots, \alpha_r\} \subset \mathbb{C}\), and sets of left and right tangential directions \(\{c_1, c_2, \cdots, c_r\} \subset \mathbb{C}^m\), \(\{b_1, b_2, \cdots, b_r\} \subset \mathbb{C}^p\) are respectively defined by
\[
V = \begin{bmatrix} v_1 F b_1 & v_2 F b_2 & \cdots & v_r F b_r \end{bmatrix},
\]
\[
W = \begin{bmatrix} w_1 L^T c_1 & w_2 L^T c_2 & \cdots & w_r L^T c_r \end{bmatrix},
\]
(7)

Where \(v_i = (\alpha_2^2 M + \alpha_i D + K)^{-1}\) and \(w_i = (\alpha_2^2 M^T + \alpha_i D^T + K^T)^{-1}; i = 1, 2, \cdots, r\).

If the reduced-order model (4) is constructed by \(V\) and \(W\), the reduced transfer-function (6) tangentially interpolates (3), satisfies the interpolation conditions
\[
T(\alpha_i) b_i = \hat{T}(\alpha_i) b_i, \\
c_i^T T(\alpha_i) b_i = c_i^T \hat{T}(\alpha_i) b_i, \\
c_i^T T'(\alpha_i) b_i = c_i^T \hat{T}'(\alpha_i) b_i,
\]
for \(i = 1, 2, \cdots, r\), which is known as Hermite bi-tangential interpolation conditions. One of the challenging parts of SPIRKA is to find a set of optimal interpolation points as well as tangential directions since they are not predefined. In [13] author shows several remedies. Among them this paper consider the following strategy. Construct
\[
(E, A, B, C) := \left( \begin{bmatrix} I & 0 \\ 0 & \hat{M} \end{bmatrix}, \begin{bmatrix} 0 & I \\ -\hat{K} & -\hat{D} \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{F} \end{bmatrix}, \begin{bmatrix} \hat{L} & 0 \end{bmatrix} \right).
\]
(9)

Then find \(r\) dimensional reduced order model \((\hat{E}, \hat{A}, \hat{B}, \hat{C})\) from \((E, A, B, C)\). The interpolation points and tangential directions for the next iteration step are constructed from the mirror image of the eigenvalues and the eigenvectors of \((A, E)\). The reduced order model \((\hat{E}, \hat{A}, \hat{B}, \hat{C})\) can be constructed again by the IRKA. Note that IRKA of first-order system is presented in [6, Algorithm 1]. The whole procedure for SPIRKA is summarized in Algorithm [1].

2.2 Reformulation of second-order index-3 descriptor system

We already have mentioned in earlier section that projecting the index-3 system (4) onto the hidden manifold we can convert the system into an index-0 i.e., second-order standard system like (2). However, such conversion for a large
position of $\Pi$ into (12) and considering $\tilde{l}$ of $\Pi$. Those equations can be avoided by splitting $\Pi = \Psi$

The dynamical system (12) still has unnecessary equations due to the singularity where $I$

Now applying these identities into (2) we obtain

Readers are referred to e.g., [11] to see the details of these properties with proofs.

scale dynamical system is practically impossible due to additional complexities. The idea of conversion is already developed in [11]. For our convenience, we briefly introduce this in the following.

Let us consider the projector onto the null-space of $G$

$$\Pi := I - G^T(GM^{-1}G^T)^{-1}GM^{-1},$$

which satisfies $\Pi M = M\Pi^T$, $\text{Null}(\Pi) = \text{Rang}(G^T)$, $\text{Rang}(\Pi) = \text{Null}(GM^{-1})$
and the most importantly

$$Gx = 0 \iff \Pi^T x = x.$$ (11)

The dynamical system (12) still has unnecessary equations due to the singularity of $\Pi$. Those equations can be avoided by splitting $\Pi = \Psi_l\Psi_r$, where $\Psi_l, \Psi_r \in \mathbb{R}^{n_1 \times (n_1 - n_2)}$ and they satisfy

$$\Psi_l^T \Psi_r = I_{n_1 - n_2},$$ (13)

where $I_{n_1 - n_2}$ is an $(n_1 - n_2) \times (n_1 - n_2)$ identity matrix. Inserting the decomposition of $\Pi$ into (12) and considering $\dot{x}(t) = \Psi_l^T x(t)$, the resulting dynamical

### Algorithm 1: SPIRKA for standard Second-Order Systems.

| Step | Description |
|------|-------------|
| 1    | Consider: interpolation points $\{\alpha_i\}_{i=1}^r \subset \mathbb{C}$, left and right tangential directions $\{b_i\}_{i=1}^r \subset \mathbb{C}^p$ and $\{c_i\}_{i=1}^r \subset \mathbb{C}^m$. |
| 2    | Form $V = [v_1 \mathcal{F}b_1, v_2 \mathcal{F}b_2, \cdots, v_r \mathcal{F}b_r]$, $W = [w_1 \mathcal{L}^T c_1, w_2 \mathcal{L}^T c_2, \cdots, w_r \mathcal{L}^T c_r]$, where $v_i = (\alpha_i^2 M + \alpha_i D + K)^{-1}$ and $w_i = (\alpha_i^2 M^T + \alpha_i D^T + K^T)^{-1}$; $i = 1, 2, \cdots, r$. |
| 3    | while (not converged) do |
| 4    | $\mathcal{M} = W^T MV$, $\mathcal{D} = W^T DV$, $\mathcal{K} = W^T K^T$, $\mathcal{F} = W^T F$, $\mathcal{L} = LV$. |
| 5    | Use the first-order representation $(E, A, B, C)$ as in [9] find the reduced-order matrices $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$. |
| 6    | Compute $\hat{A}z_i = \lambda_i E z_i$ and $y_i^T \hat{A} = \lambda_i y_i^T \hat{E}$ for $\alpha_i \leftarrow -\lambda_i$, $b_i^* \leftarrow -y_i^T \hat{B}$ and $c_i^* \leftarrow \hat{C} z_i^*$ for all $i = 1, \cdots, r$. |
| 7    | Repeat Step 2. |
| 8    | $i = i + 1$; |
| 9    | end while |
| 10   | Construct the reduced matrices by repeating Step 4. |
This system is now a standard second-order system as described by (2). In fact system in (14) can be seen as the system (12) with the redundant equations being removed through the $\Psi_r$ projection. Note that the coefficient matrices of (14) are dense if compared to (1). Therefore, for the large-scale index-3 system explicit computation of (14) is forbidden. In fact the dynamical systems (1), (12) and (14) are equivalent in the sense that they are the different realizations of the same transfer function. Moreover, their finite spectra are the same which has been proven in the sequel. Once the index-3 system (1) is converted into the index-0 system (14) then the SPIRKA i.e., Algorithm 1 can be applied to the converted system. However, as the converted system is dense, the computational costs of Algorithm 1 becomes high. For a very high dimensional index-3 descriptor system computing (14) is not possible due to the restriction of computer memory. Therefore we are motivated to construct reduced order model without forming system (14) explicitly.

3 IRKA for second-order index-3 descriptor systems

The SPIRKA introduced in Section 2 can be applied to the projected system (14). As already mentioned, this is infeasible for a large-scale system. Therefore, the technique can be applied to the equivalent system (12) instead. For this purpose, following the discussion in Section 2, we can create the right and left projectors as

\begin{align*}
V &= [v_1, v_2, \cdots, v_r], \\
W &= [w_1, w_2, \cdots, w_r],
\end{align*}

where

\begin{align*}
v_i &= (\alpha_i^2 \bar{M} + \alpha_i \bar{D} + \bar{K})^{-1} \bar{F} b_i, \\
w_i &= (\alpha_i^2 \bar{M}^T + \alpha_i \bar{D}^T + \bar{K}^T)^{-1} \bar{L}^T c_i,
\end{align*}

for $i = 1, 2, \cdots, r$ and in which $\bar{M} = \Pi M \Pi^T$, $\bar{D} = \Pi D \Pi^T$, $\bar{K} = \Pi K \Pi^T$, $\bar{F} = \Pi F \Pi^T$ and $\bar{L} = \Pi L \Pi^T$. The main expensive task here is to compute each vector inside the projectors by solving a linear system. For example to construct $V$, at $i$-th iteration we find $(\alpha_i^2 \bar{M} + \alpha_i \bar{L} + \bar{K})^{-1} \bar{F} b_i$ to solve the linear system

\[(\alpha_i^2 \bar{M} + \alpha_i \bar{D} + \bar{K}) v_i = \bar{F} b_i,
\]

which implies

\[\Pi(\alpha_i^2 \bar{M} + \alpha_i \bar{D} + \bar{K}) \Pi^T v_i = \bar{F} b_i.\]  

This linear system can be solved efficiently by applying the following Lemma.
Lemma 3.1. The matrix $\kappa$ satisfies $\kappa = \Pi^T \kappa$ and $\Pi(\alpha^2 M + \alpha D + K)\Pi^T \kappa = \Pi F b$ if and only if
\[
\begin{bmatrix}
\alpha^2 M + \alpha D + K & G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
\kappa \\
\Lambda
\end{bmatrix} =
\begin{bmatrix}
F b \\
0
\end{bmatrix}.
\]
(17)

Proof. If $\kappa = \Pi^T \kappa$, then by using (11) we have
\[
G \kappa = 0
\]
which is the second block of equation (20). Furthermore $\Pi(\alpha^2 M + \alpha D + K)\Pi^T \kappa = \Pi F b$ implies
\[
\Pi(\alpha^2 M + \alpha D + K) \kappa = \Pi F b,
\]
\[
\Pi((\alpha^2 M + \alpha D + K) \kappa - F b) = 0.
\]
This means $(\alpha^2 M + \alpha D + K) \kappa - F b$ is in the null space of $\Pi$. We know that $\text{Null}(\Pi) = \text{Rang}(G^T)$. Therefore, there exists $\Lambda$ such that
\[
(\alpha^2 M + \alpha D + K) \kappa - F b = -G^T \Lambda
\]
which implies
\[
(\alpha^2 M + \alpha D + K) \kappa + G^T \Lambda = F b.
\]
Equations (18) and (19) yield (20). Conversely, we assume (20) holds. From the second line of (20) we obtain $G \kappa = 0$ and thus $\kappa = \Pi^T \kappa$. Now from first equation we obtain
\[
(\alpha^2 M + \alpha D + K) \kappa + G^T \Lambda = F b.
\]
Applying (11) this equation gives
\[
(\alpha^2 M + \alpha D + K) \Pi^T \kappa + G^T \Lambda = F b.
\]
Multiplying both sides by $\Pi$ and since $\Pi G^T = 0$ we have
\[
\Pi(\alpha^2 M + \alpha D + K)\Pi^T \kappa = \Pi F b.
\]
This completes the proof.

Following Lemma 3.1 instead of solving (16) we can solve
\[
\begin{bmatrix}
\alpha_i^2 M + \alpha_i D + K & G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
\xi_i \\
\Lambda
\end{bmatrix} =
\begin{bmatrix}
F b_i \\
0
\end{bmatrix},
\]
(20)
for $\xi$. Although this system is larger than its projected system. Therefore we solve this linear system to avoid constructing the projector. Similarly to construct $W$ at each iteration we compute $w_i = (\alpha_i^2 M^T + \alpha_i D^T + K^T)^{-1} L^T c_i$ by solving the following linear system
\[
\begin{bmatrix}
\alpha_i^2 M^T + \alpha_i D^T + K^T & G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
w_i \\
\Gamma
\end{bmatrix} =
\begin{bmatrix}
L^T c_i \\
0
\end{bmatrix}.
\]
Once we have $V$ and $W$, apply them to (12) to find the reduce order model
\[
\tilde{M} \ddot{x}(t) + \tilde{D} \dot{x}(t) + \tilde{K} x(t) = \tilde{F} u(t),
\]
(21a)
\[
y(t) = \tilde{L} \dot{x}(t),
\]
(21b)
in which the reduced matrices are constructed as follows
\[
\hat{\mathbf{M}} = \mathbf{W}^T \mathbf{Π} \mathbf{Π}^T \mathbf{V}, \quad \hat{\mathbf{D}} = \mathbf{W}^T \mathbf{Π} \mathbf{Π}^T \mathbf{V}, \\
\hat{\mathbf{K}} = \mathbf{W}^T \mathbf{Π} \mathbf{K} \mathbf{Π}^T \mathbf{V}, \quad \hat{\mathbf{F}} = \mathbf{W}^T \mathbf{Π} \mathbf{F}, \quad \hat{\mathbf{L}} = \mathbf{L} \mathbf{Π}^T \mathbf{V}.
\]

Due to the properties of the projector, as mentioned in [11] we have \( \mathbf{Π}^T \mathbf{V} = \mathbf{V} \) and \( \mathbf{Π}^T \mathbf{W} = \mathbf{W} \) or \( (\mathbf{Π}^T \mathbf{W})^T = \mathbf{W}^T \). Therefore the reduced matrices can be constructed without using \( \mathbf{Π} \) as follows
\[
\tilde{\mathbf{M}} = \mathbf{W}^T \mathbf{M} \mathbf{V}, \quad \tilde{\mathbf{D}} = \mathbf{W}^T \mathbf{D} \mathbf{V}, \quad \tilde{\mathbf{K}} = \mathbf{W}^T \mathbf{K} \mathbf{V}, \\
\tilde{\mathbf{F}} = \mathbf{W}^T \mathbf{F}, \quad \tilde{\mathbf{L}} = \mathbf{L} \mathbf{V}.
\]
The whole procedure to construct reduced order model from second-order index-3 system [1] is summarize in Algorithm 2.

**Update interpolation points and tangential directions.** In IRKA we need to update the interpolation points and tangential direction at each iteration steps which is often a challenging task. Usually, the interpolation points and tangential direction are updated by using mirror image of the eigenvalues and eigenvector of the reduced order model. In SPIRKA this task is complicated because we need to solve the quadratic eigenvalue problems. If we solve a quadratic eigenvalue problem [10] using \( (\hat{\mathbf{M}}, \hat{\mathbf{D}}, \hat{\mathbf{K}}) \) we obtain \( 2r \) number of eigenvalues and eigenvectors. Selecting \( r \) number of optimal interpolation points and corresponding tangential direction is challenging task. To resolve this complexity [13] propose a techniques which is discussed in Algorithm 2. To follow this idea, we need to apply standard IRKA onto the converted first order system from the reduced second order system. However, this is again an iterative method which is computationally expensive. In this paper to construct the interpolation points and tangential directions we construct
\[
\begin{align*}
\hat{\mathbf{E}} & := \begin{bmatrix} 0 & \mathbf{I} \\ \hat{\mathbf{M}} \end{bmatrix}, & \hat{\mathbf{A}} & := \begin{bmatrix} 0 & \mathbf{I} \\ -\hat{\mathbf{K}} & -\hat{\mathbf{D}} \end{bmatrix}, \\
\hat{\mathbf{B}} & := \begin{bmatrix} 0 \\ \hat{\mathbf{F}} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{C}} & := \begin{bmatrix} \hat{\mathbf{L}} & 0 \end{bmatrix}.,
\end{align*}
\]

Then we apply MATLAB function `balred` to compute \( r \) dimensional reduced-order model from \( (\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}) \). The interpolation points are updated by choosing the mirror images of the eigenvalues of the pair \( (\hat{\mathbf{A}}, \hat{\mathbf{E}}) \) as the next interpolations points. This seems to be more efficient than the existing one.

**4 Numerical results**

To assess the efficiency of the proposed algorithm, i.e., Algorithm 2 we have applied this to two sets of data. First one is coming from a damped spring-mass system (DSMS) with holonomic constraint which is taken from [17]. The second set of data set is a constrain triple chain oscillator model (TCOM). This data is originated in [15] but with the index-3 setup described in [2]. The details of the models is also available in [11]. We intentionally have considered the same data for the both model examples as in [11] since we want to compare the results of
Algorithm 2: IRKA for Second-Order Index-3 Descriptor Systems.

Input: $M, D, K, F, L$

Output: $\hat{M}, \hat{D}, \hat{K}, \hat{F}, \hat{L}$

1. Select randomly a set of the interpolation points $\{\alpha_i\}_{i=1}^r$ and the tangential directions $\{b_i\}_{i=1}^r$ and $\{c_i\}_{i=1}^r$.

2. Construct the projection matrices
   
   $V_s = [v_1, v_2, \ldots, v_r]$ and $W_s = [w_1, w_2, \ldots, w_r],$

   where $v_i \& w_i; i = 1, \ldots, r$ are the solutions of the linear systems
   
   $[\alpha_i^2M + \alpha_iD + K G^T G_0] \begin{bmatrix} v_i \end{bmatrix} = \begin{bmatrix} F b_i \end{bmatrix}$

   and
   
   $[\alpha_i^2M^T + \alpha_iD^T + K^T G^T G_0] \begin{bmatrix} w_i \end{bmatrix} = \begin{bmatrix} L^T c_i \end{bmatrix},$

   respectively.

3. while (not converged) do

4. Construct

5. $M = W^T M V, \quad D = W^T D V, \quad K = W^T K V, \quad F = W^T F, \quad L = L V.$

   Compute $(E, A, B, C)$ as in (22), then using MATLAB function balred compute $r$ dimensions model $(\hat{E}, \hat{A}, \hat{B}, \hat{C}).$

6. Compute $\hat{A}z_i = \lambda_i \hat{E}z_i$ and $y_i^* A = \lambda_i y_i^* \hat{E}$ for $\alpha_i \leftarrow -\lambda_i, b_i^* \leftarrow -y_i^* \hat{B}$

   and $c_i^* \leftarrow \hat{C}z_i^*.$

7. Repeat Step 2.

8. $i = i + 1.$

9. end while

10. Finally construct the reduced-order matrices

   $\hat{M} = W^T M V, \quad \hat{D} = W^T D V, \quad \hat{K} = W^T K V, \quad \hat{F} = W^T F, \quad \hat{L} = L V.$

This experiment was carried out with MATLAB® R2015a (8.5.0.197613) on a board with 4×INTELCore™i5-4460s CPU with a 2.90 GHz clock speed and 16 GB RAM.

We apply Algorithm 2 to both models and find 30 dimensional reduced-order models. Figures[1] and[2] show the frequency domain analysis of the original and reduced models for the DSMS and TCOM, respectively. The frequency responses of the full and the reduced-order models and their absolute and relative errors for the DSMS are shown in Figure[3] over the frequency interval $[10^{-2}, 10^0].$

Sub-figure[4] shows that the frequency responses of the reduced-order models are matching with the original model with good accuracy. The absolute and rel-
models & dimension & $n_1$ and $n_2$ & inputs/outputs \\
DSMS & 2200 & 2000 and 200 & 1/3 \\
TCOM & 11001 & 6001 and 5000 & 1/1 \\

Table 1: The dimension of the tested models including number of differential and algebraic variables and, inputs and outputs.

Comparisons of Balanced truncation and IRKA. To compare the performance of IRKA and balanced truncation we compute 30 dimensional reduced-order model applying [11, Algorithm 2] to the TCOM. This algorithm can compute several reduced-order models based on different balancing criterion. Here we consider the velocity-velocity balancing label which gives the best approximation. Figure 3 shows the approximation errors of 30 dimensional reduced-order models computed by the BT and IRKA. From Figure 3 it seems that the performance of IRKA is better than the BT. Both the absolute error and relative errors as shown in Figures 3a and 3b respectively, IRKA depicts better accuracy than the balanced truncation. On the other hand, when we consider the computation time, again the performance of IRKA is far better than the BT which is reflected in Figure 4. We know that balanced truncation is expensive method since it requires to solve two continuous-time algebraic Lyapunov equations. The solution of the Lyapunov equations involved the computation of shift parameters which is a computational. We have solved the Lyapunov equations by [11, Algorithm 3] using adaptive shift parameters. See, e.g., [11] for details. Note that the computational time of IRKA is increasing if the dimension of reduced order model and the number of iterations are increased gradually.

5 Conclusions

In this paper we have discussed a IRKA based technique to find a reduced second-order system from a large-scale sparse second-order index-3 system. In particular, we have linearized equation of motion with holonomic constrains which arise in constrained mechanics or multibody dynamics. It has been shown that the index-3 system can be converted into index-0 by projecting onto the hidden manifold to apply the standard second-order IRKA. But creating projector is often computationally expensive task and it yields system matrices dense. Therefore we have modified the standard IRKA for the underlying index-3 descriptor system. We also have shown a clever techniques to compute the interpolation points and tangential directions. The proposed algorithm was applied
Figure 1: Comparison of original and the 30 dimensional reduced models for the DSMS.

to several data of second-order index-3 models. Numerical results showed that the proposed algorithm can generated lower dimensional model with higher accuracy. The IRKA based method is better than Balanced truncation in terms of accuracy and computational complexity as well.

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Figure 2: Comparison of the original and 30 dimensional reduced models for the TCOM.
Figure 3: Comparison of the original and 30 dimensional reduced models computed by IRKA and balanced truncation for the TCOM.

Figure 4: Time comparisons of both balanced truncation and IRKA for the TCOM.
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