Bulk reconstruction in rotating BTZ black hole

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Abstract

The bulk reconstruction program aims to obtain representations of bulk fields as operators in the boundary CFT. In this paper we extend the program by obtaining the boundary representation for a scalar field in a rotating BTZ black hole. We find that the representation of the field near the inner horizon shows novel features. We also obtain a representation for fields inside the horizon as operators in a single boundary CFT using mirror operator construction.
I. INTRODUCTION

According to the AdS/CFT conjecture, quantum gravity in \( d + 1 \)-dimensional asymptotically AdS spacetime and a conformal field theory living on the \( d \)-dimensional boundary of the spacetime are equivalent. However, our knowledge of the AdS/CFT correspondence is still incomplete – we have only a partial dictionary between the objects on the two sides. Therefore, we cannot yet translate all questions about bulk physics to questions about the boundary theory.

The aim of the bulk reconstruction program is to complete this dictionary by obtaining representations of bulk fields as operators in the boundary CFT \([1–13]\). Concretely, for an asymptotically AdS geometry \( g \) dual to a CFT state \(|\psi_g\rangle\), the extrapolate dictionary tells us:

\[
\lim_{r \to \infty} r^n \Delta \langle \phi(r_1, x_1) \phi(r_2, x_2) \ldots \phi(r_n, x_n) \rangle_g = \langle \psi_g | \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) | \psi_g \rangle
\]  

(1)

Here \( x_i \) are the boundary coordinates and \( r, x_i \) are the bulk coordinates.

The extrapolate dictionary does not directly give us a way to recover the bulk correlation function for interior points in the bulk from the boundary CFT. In the bulk reconstruction program, the aim is to construct a boundary operator \( \phi_{CFT} \) which satisfies the following relation:

\[
\langle \phi(r_1, x_1) \ldots \phi(r_n, x_n) \rangle_g = \langle \psi_g | \phi_{CFT}(r_1, x_1) \ldots \phi_{CFT}(r_n, x_n) | \psi_g \rangle
\]  

(2)

This problem has been solved in pure AdS and certain other asymptotically AdS backgrounds. It turns out that \( \phi_{CFT} \) is a non local operator in the boundary theory. It can be written as:

\[
\phi_{CFT}(r, x) = \int d x' K(r, x; x') \mathcal{O}(x')
\]  

(3)

where \( K(r, x) \) is known as the smearing function. This is referred to as HKLL construction (after Hamilton, Kabat, Lifshytz and Lowe). From hereon we will drop the suffix ’CFT’ in referring to the boundary representation of the bulk field.

Bulk reconstruction has been studied in in black hole backgrounds \([14–19]\). Two-sided black holes are dual to an entangled state in a pair of CFTs, one on each boundary. As we will review shortly, a bulk field at a point inside the horizon is represented as a sum of operators on the left and right boundary theories.
A peculiarity of bulk reconstruction in black hole backgrounds is that the smearing function does not exist as a function\textsuperscript{5, 6, 16} but as a distribution\textsuperscript{20}. This is not a problem as such, as the correct correlators are still obtained from the boundary representation. However to obtain a convergent smearing function, one has to consider reconstruction of wave packets instead of fields.

In this paper, we extend the bulk reconstruction program to spinning BTZ black holes. Spinning BTZ black holes have both an outer and an inner horizon\textsuperscript{1}. We carry out the bulk reconstruction program for spinning BTZ black holes and, using appropriate wave packets, obtain smearing function\textsuperscript{2}. In the regime where the wave packet is high frequency, we obtain plots for the smearing function near the outer and inner horizons. We find that novel features emerge for the smearing function near the inner horizon.

For fields inside the horizon, the usual HKLL construction gives us a representation which is a sum of operators of the two CFTs. The mirror operator construction of \textsuperscript{27–29} gives a representation as an operator on a single CFT. We carry out the mirror operator construction for the spinning BTZ and obtain the corresponding smearing function.

In the next section, we briefly recall the basics of spinning BTZ black holes and those of bulk reconstruction in black holes. The third section presents our results. We conclude with a summary.

II. PRELIMINARIES

A. Spinning BTZ black hole

The rotating BTZ metric is given by:

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\phi - \Omega dt)^2 \]  (4)

where

\[ f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} \]

The spacetime has two horizons: \( r = r_- \) is the inner Cauchy horizon and \( r = r_+ \) is the outer event horizon. The causal structure is shown in the figure.

\textsuperscript{1} There has been much recent interest in the question of stability of inner horizons\textsuperscript{21–26}.

\textsuperscript{2} An alternative to using wave packets that works for non-rotating black holes is to Wick rotate to de-Sitter space. Here one obtains a representation in a complexified boundary\textsuperscript{6, 14}. This approach does not work in this case as Wick rotation yields a complex bulk metric.
FIG. 1: Causal structure of a spinning BTZ black hole. $I_R$ and $I_L$ are the two boundaries. $H_L^+$ and $H_L^-$ are the future and past parts of the left outer horizon, similarly $H_R^+$ and $H_R^-$ are the future and past right outer horizon. $CH^+$ and $CH^-$ denote the future and past inner horizons.

The surface gravities of the two horizons are given by:

$$\kappa_\pm = \frac{r_\pm^2 - r^2}{r_\pm}. \quad (5)$$

while the corresponding angular velocities are given by:

$$\Omega_\pm = \frac{r_\pm}{r_\pm}. \quad (6)$$

We now want to solve the Klein-Gordon equation for a massless field in this background:

$$\Box \phi = 0 \quad (7)$$

The radial coordinate $z$ will be useful to describe the solutions.

$$z = \frac{r^2 - r_\pm^2}{r_\pm^2 - r^2}. \quad (8)$$

In these coordinates the inner horizon is at $z = 0$, the outer horizon is at $z = 1$ and the boundary is at $z = \infty$.

Using the symmetries of the metric we may write the solution as:

$$\phi = e^{i\omega t} e^{i\phi} G(z) \quad (9)$$

Substituting (9) in the Klein-Gordon equation we obtain the hypergeometric equation in the radial coordinate:
\[ z(1 - z)F''(z) + [c - z(a + b + 1)]F'(z) - abF(z) = 0 \quad (10) \]

where \( F(z) = z^{-\frac{i\omega - m\Omega_+}{2\kappa_+}}(1 - z)^{-\frac{i\omega - m\Omega_+}{2\kappa_+}} \) and

\[
\begin{align*}
a &= \frac{1}{2} \left( 2 - i \frac{\omega - m\Omega_-}{\kappa_-} - i \frac{\omega - m\Omega_+}{\kappa_+} \right) \\
b &= -\frac{1}{2} \left( i \frac{\omega - m\Omega_-}{\kappa_-} + i \frac{\omega - m\Omega_+}{\kappa_+} \right) \\
c &= 1 - \frac{\omega - m\Omega_-}{\kappa_-} 
\end{align*}
\]

(11-13)

The hypergeometric function has singular points at 0, 1 and \( \infty \) and near each singular point, there is a convenient choice of basis. Of course, one can transform between the bases.

Near the boundary, the solutions are:

\[
\begin{align*}
G_{\omega,m}^{\text{norm,}\infty} &= z^{-\frac{1}{2}(2a-c+1)}(z - 1)^{\frac{1}{2}(a+b-c)} \binom{a}{a - c + 1; a - b + 1; \frac{1}{z}}, \\
G_{\omega,m}^{\text{non-norm,}\infty} &= z^{-\frac{1}{2}(2b-c+1)}(z - 1)^{\frac{1}{2}(a+b-c)} \binom{a}{b - c + 1; -a + b + 1; \frac{1}{z}}
\end{align*}
\]

(14-15)

Out of these, \( G_{\omega,m}^{\text{norm}} \) is the normalizable mode. It falls off as \( z^{-1} \) near the boundary. This is the one which we need for bulk reconstruction.

Near the outer horizon, the convenient basis for the solutions is:

\[
\begin{align*}
G_{\omega,m}^{R,\infty} &= z^{-\frac{1}{2}(1-c)}|1 - z|^{-\frac{1}{2}(a+b-c)} \binom{c - b, c - a; -a - b + c + 1; 1 - z}, \\
G_{\omega,m}^{L,\infty} &= z^{-\frac{1}{2}(1-c)}|1 - z|^{\frac{1}{2}(a+b-c)} \binom{a, b : a + b - c + 1; 1 - z}
\end{align*}
\]

(16-17)

At \( r = r_+ \) this basis behaves as:

\[
\begin{align*}
G_{\omega,m}^{R,\infty}|_{z \sim 1} &\sim |1 - z|^{\frac{\omega - m\Omega_+}{2\kappa_+}} \\
G_{\omega,m}^{L,\infty}|_{z \sim 1} &\sim |1 - z|^{-\frac{\omega - m\Omega_+}{2\kappa_+}}
\end{align*}
\]

(18-19)

The “tortoise” radial coordinate \( r_* \) is useful to describe near horizon solutions and we will use it (rather than \( z \)) when we work out smearing functions near the horizon in the next section:

\[
r_* = \frac{1}{2\kappa_+} \log \left| \frac{r - r_+}{r_+ + r_+} \frac{r + r_-}{r - r_-} \right|^{r_-/r_+}
\]
In terms of $r_*$, the near horizon behavior of these modes are:

\[ G^{R,+}_{\omega,m} \sim \left( \frac{4r_+}{r_+^2 - r_-^2} \right) \left( \frac{r_+ - r_-}{r_+ + r_-} \right) \frac{e^{i(\omega - m\Omega_+)r_*}}{r_+} e^{i(\omega - m\Omega_+)r_*} \]  

(21)

\[ G^{L,+}_{\omega,m} \sim \left( \frac{4r_+}{r_+^2 - r_-^2} \right) \left( \frac{r_+ - r_-}{r_+ + r_-} \right) \frac{e^{-i(\omega - m\Omega_+)r_+}}{r_+} e^{-i(\omega - m\Omega_+)r_*} \]  

(22)

These solutions behave like right and left moving waves near the horizon (hence the R/L labels).

Near the inner horizon, the solutions are:

\[ G^{R,-}_{\omega,m} = z^{-\frac{1}{2}(1-c)} (1 - z)^{\frac{1}{2}(a+b-c)} \, _2F_1(a, b; c; z) \]  

(23)

\[ G^{L,-}_{\omega,m} = z^{\frac{1}{2}(1-c)} (1 - z)^{\frac{1}{2}(a+b-c)} \, _2F_1(a - c + 1, b - c + 1; 2 - c; z) \]  

(24)

The near-horizon behavior of the two solutions is given by:

\[ G^{R,-}_{\omega,m} \mid_{z \to 0} \sim \left| 1 - z \right|^{-\frac{\omega - m\Omega}{2\kappa_-}} \]  

(25)

\[ G^{L,-}_{\omega,m} \mid_{z \to 0} \sim \left| 1 - z \right|^{-\frac{\omega - m\Omega}{2\kappa_-}} \]  

(26)

Again, in terms of the tortoise coordinate they are given by:

\[ G^{R,-}_{\omega,m} \mid_{r_+ \to 0} \sim \left( \frac{4r_+}{r_+^2 - r_-^2} \right) \left( \frac{r_+ - r_-}{r_+ + r_-} \right) \frac{e^{i(\omega - m\Omega_+)r_*}}{r_+} e^{i(\omega - m\Omega_+)r_*} \]  

(27)

\[ G^{L,-}_{\omega,m} \mid_{r_+ \to 0} \sim \left( \frac{4r_+}{r_+^2 - r_-^2} \right) \left( \frac{r_+ - r_-}{r_+ + r_-} \right) \frac{e^{-i(\omega - m\Omega_+)r_+}}{r_+} e^{-i(\omega - m\Omega_+)r_*} \]  

(28)

One can go from one basis to the other. Later we will need to know how the normalizable mode at the boundary can be written in terms of the basis we used near the outer horizon. This is given by:

\[ G_{\omega,m}^{\text{norm}} = A(\omega, m)G_{\omega,m}^{R,+} + B(\omega, m)G_{\omega,m}^{L,+} \]  

(29)

where

\[ A(\omega, m) = \frac{\Gamma(a - b + 1)\Gamma(a + b - c)}{\Gamma(a)\Gamma(a - c + 1)} \]  

(30)

\[ B(\omega, m) = \frac{\Gamma(-a + b + 1)\Gamma(a - b + 1)}{\Gamma(1 - b)\Gamma(c - b)} \]  

(31)
We will also need to know how to write the basis near the outer horizon in terms of the one near the inner horizon:

\[
G_{\omega,m}^{R,+} = C(\omega, m)G_{\omega,m}^{R,-} + D(\omega, m)G_{\omega,m}^{L,-} \\
G_{\omega,m}^{L,+} = \tilde{C}(\omega, m)G_{\omega,m}^{L,-} + \tilde{D}(\omega, m)G_{\omega,m}^{R,-}
\]

where

\[
C(\omega, m) = \frac{\Gamma(1 - c)\Gamma(1 - a - b + c)}{\Gamma(1 - a)\Gamma(1 - b)} \\
D(\omega, m) = \frac{\Gamma(c - 1)\Gamma(1 - a - b + c)}{\Gamma(c - a)\Gamma(c - b)} \\
\tilde{C}(\omega, m) = \frac{\Gamma(c - 1)\Gamma(a + b - c + 1)}{\Gamma(a)\Gamma(b)} \\
\tilde{D}(\omega, m) = \frac{\Gamma(1 - c)\Gamma(a + b - c + 1)}{\Gamma(a - c + 1)\Gamma(b - c + 1)}
\]

**B. Bulk reconstruction in two-sided black holes**

In this section, we outline the general strategy of obtaining a CFT representation of a bulk field in a black hole background. We will consider the bulk to be 2+1 dimensional, but the method holds for all dimensions in principle.

There are two key inputs needed for the construction. First, we define the following operators on the boundary:

\[
\mathcal{O}_{\omega,m} = \int dt \, d\theta \, e^{-i\omega t + im\theta} \mathcal{O}(t, \theta) \\
\mathcal{O}_{-\omega,-m} = \int dt \, d\theta \, e^{i\omega t - im\theta} \mathcal{O}(t, \theta).
\]

These operators are boundary analogues of the bulk creation and annihilation operators \(a_{\omega,m}, a_{\omega,m}^\dagger\) respectively. From hereon we refer to \(\mathcal{O}_{-\omega,-m}\) as \(\dagger\mathcal{O}_{\omega,m}\).

With the annihilation and creation operators in hand, the mode solutions of the Klein-Gordon equation constitute the second input needed for the construction.
First, let us consider the exterior of the black hole, i.e., regions I and III. To obtain the boundary representation, we need the normalizable modes.

So for instance, in region I one solves the Klein Gordon equation and chooses the modes which are normalizable:

$$\lim_{r \to \infty} f_{\omega,m}^{\text{norm}} \propto r^{-\Delta}$$  \hspace{1cm} (41)

Near the horizon the normalizable modes will behave as linear combinations of left-moving and right-moving modes

$$f_{\omega,m}^{L} \approx e^{i\omega t + im\phi + \omega r_*}$$ and $$f_{\omega,m}^{R} \approx e^{i\omega t + im\phi - \omega r_*}

f_{\omega,m}^{\text{norm}} \propto f_{\omega,m}^{(L)} + e^{-2i\delta_{\omega,m}} f_{\omega,m}^{(R)}$$  \hspace{1cm} (42)

The CFT representation in region I is then obtained as:

$$\phi^I(x, r) = \sum_{m} \int d\omega \left[ O_{\omega,m} f_{\omega,m}^{\text{norm}}(r, t, \theta) + O_{\omega,m}^\dagger (f_{\omega,m}^{\text{norm}}(r, t, \theta))^* \right]$$  \hspace{1cm} (43)

One can formally write down the smearing function as:

$$K(r, t, \theta; t', \theta') = \sum_{m} \int d\omega e^{i(\omega(t-t')-m(\theta-\theta'))} f_{\omega,m}^{\text{norm}}(r, t, \theta)$$  \hspace{1cm} (44)

A similar procedure can be followed for region III.

Now we consider the interior or region II. Again, we solve the Klein Gordon equation and obtain the mode solutions. However, now there are no boundary conditions, there are only matching conditions at the horizon. One obtains the mode solutions $$\chi_{\omega,m}^{L}(r, t, \theta)$$ and $$\chi_{\omega,m}^{R}(r, t, \theta)$$ which behave near the horizon as $$e^{i\omega t + im\phi - \omega r_*}$$ and $$e^{i\omega t + im\phi + \omega r_*}$$ respectively. Now we can use matching at the horizon. The solution inside the horizon must match with $$f_{\omega,m}^{\text{norm}}$$ at the horizon between regions I and III, for instance. This gives $$\chi_{\omega,m}^{L}(r, t, \theta) \sim e^{i\omega t + im\phi} f_{\omega,m}^{L}$$ (up to a scaling factor). Similarly $$\chi_{\omega,m}^{R}(r, t, \theta)$$ is obtained by matching at the horizon between regions II and III.

The absence of a boundary condition means that the number of independent modes inside the horizon is double that of outside the horizon. This is expected as independent modes move into region II from both regions I and III.

The CFT representation is given by:

$$\phi^{II}(r, t, \theta) = \sum_{m} \int d\omega \left[ O_{\omega,m} \chi_{\omega,m}^{(L)}(r, t, \theta) + \tilde{O}_{\omega,m}^\dagger \chi_{\omega,m}^{(R)}(r, t, \theta) + \text{h.c.} \right].$$  \hspace{1cm} (45)

Here, as later, we refer to the CFT operators on the right as $$\tilde{O}.$$
This can be re-written as:

\[
\phi^{I}(r, t, \theta) = \int dt' d\theta' K_{L}(r, t, \theta; t', \theta') \mathcal{O}(t', \theta') + \int dt'' d\theta'' K_{R}(r, t, \theta; t'', \theta'') \tilde{\mathcal{O}}(t'', \theta'') \tag{46}
\]

where

\[
K_{L}(r, t, \theta; t', \theta') = \sum_{m} \int d\omega e^{-i(\omega t' - m\theta')} \chi_{\omega, m}^{(L)}(r, t, \theta) + h.c \tag{47}
\]

It has support on both the left and right boundaries. The expressions for smearing functions in (44), (46) diverge. However one can construct a wave packet by smearing the field over a region in spacetime and obtain a convergent expression for the CFT representation of the wave packet.

III. SMEARING FUNCTION FOR SPINNING BTZ BLACK HOLE

In this section, we present our results. First, we present the usual HKLL representation where the fields in the interior are represented as a sum of the operators on the left and right boundaries. Then we present a ‘mirror operator’-like construction where the representation is on a single boundary.

A. HKLL representation of the smearing function

Using the mode solutions in section II A and following the general strategy outlined in section II B, we can now obtain the boundary representation for bulk fields in a spinning BTZ. However, as we observed before, the smearing function obtained from (44) diverges. One needs to introduce wave packets to obtain a convergent answer.

Instead of considering the field at a point, we smear them using a wave packet:

\[
\Phi(r, t_{0}, \theta_{0}) = \int dt d\theta \xi_{\omega_{0}, t_{0}}^{*} \eta_{\theta_{0}, m_{0}}^{*} \phi^{+}(r, t, \theta) + h.c \tag{48}
\]

where \(\phi^{+}\) denotes the positive frequency part of the field. We follow the wavepacket construction of [17]:

\[
\xi_{\omega_{0}, t_{0}} = e^{-i\omega_{0}(t-t_{0})} \frac{\sin(\epsilon (t - t_{0}))}{\sqrt{\epsilon(t-t_{0})}}, \eta_{m_{0}, \theta_{0}} = e^{im_{0}(\theta - \theta_{0})} \frac{\sin(\epsilon (\theta - \theta_{0}))}{\sqrt{\epsilon(\theta - \theta_{0})}} \tag{49}
\]

This gives a wave packet centered around \(t_{0}, \theta_{0}\).
Now using (43) and (44) we can obtain the CFT representation of the wave-packet $\Phi$ in region I:

$$\Phi^I(r, t_0, \theta_0) = \int dt' K_{\omega, m_0}^I(r, t_0, \theta_0; t', \theta') \mathcal{O}(t', \theta)$$ (50)

where:

$$K_{\omega, m_0}^I(t_0 - t, r, \phi_0 - \phi) = \sum_m \frac{1}{(2\pi)^2} \int d\omega \left[ e^{i\omega t - i m \phi} \tilde{\xi}_{\omega, t_0}^*(\omega) \tilde{\eta}_{m_0, \phi_0}^*(m) + \text{h.c.} \right] G_{\omega, m}^{\text{norm,} \infty}(r)$$

$$= \sum_{m=m_0-\frac{1}{2} \epsilon}^{m_0+\frac{1}{2} \epsilon} \frac{1}{(2\pi)^2} \int_{\omega_0-\frac{1}{2} \epsilon}^{\omega_0+\frac{1}{2} \epsilon} d\omega \cos [\omega(t_0 - t) - m(\phi_0 - \phi)] G_{\omega, m}^{\text{norm,} \infty}(r).$$ (51)

Here $G_{\omega, m}^{\text{norm,} \infty}$ is the normalizable mode at the boundary given by (14).

Similarly from (45) and (46) we obtain the CFT representation of the wave packet inside the horizon in region II:

$$\Phi^{II}(r, t_0, \theta_0) = \int dt' K_{\omega, m_0}^{II}(r, t_0, \theta_0; t', \theta) \mathcal{O}_L(t', \theta) + \int dt'' K_{\omega, m_0}^{IR}(r, t_0, \theta_0; t'', \theta'') \mathcal{O}_R(t'', \theta'').$$ (52)

where:

$$K_{0}^{II}(t_0 - t, r, \phi_0 - \phi) = \sum_m \frac{1}{(2\pi)^2} \int d\omega \left[ e^{i\omega t - i m \phi} \tilde{\xi}_{\omega, t_0}^*(\omega) \tilde{\eta}_{m_0, \phi_0}^*(m) A(\omega, m) G_{L, m_0, \phi_0}^{L,+}(r) + \text{h.c.} \right]$$

$$= \sum_{m=m_0-\frac{1}{2} \epsilon}^{m_0+\frac{1}{2} \epsilon} \frac{1}{(2\pi)^2} \int_{\omega_0-\frac{1}{2} \epsilon}^{\omega_0+\frac{1}{2} \epsilon} d\omega \left[ e^{-i\omega(t_0 - t) - i m(\phi_0 - \phi)} A(\omega, m) G_{L, m_0, \phi_0}^{L,+}(r) + \text{h.c.} \right].$$ (53)

where $G_{L, m_0, \phi_0}^{L,+}$ is given by the equation (17). A similar equation holds for the right smearing function.

To obtain a representation of the smearing function near the inner horizon we can use the basis of hypergeoemtric functions that is convenient in that region. To do this we use (32) and obtain:

$$K_{0}^{L}(t_0 - t, r, \phi_0 - \phi) = \sum_{m=m_0-\frac{1}{2} \epsilon}^{m_0+\frac{1}{2} \epsilon} \frac{1}{(2\pi)^2} \int_{\omega_0-\frac{1}{2} \epsilon}^{\omega_0+\frac{1}{2} \epsilon} d\omega \left[ e^{-i\omega(t_0 - t) - i m(\phi_0 - \phi)} A(\omega, m) \left( \tilde{C}(\omega, m) G_{L, m_0, \phi_0}^{L,-}(r) + \tilde{D}(\omega, m) G_{L, m_0, \phi_0}^{R,-}(r) \right) + \text{h.c.} \right].$$ (54)

where $G_{L, m_0, \phi_0}^{L,-}$ and $G_{L, m_0, \phi_0}^{R,-}$ are given by (23) and (24) respectively and $\tilde{C}(\omega, m)$ and $\tilde{D}(\omega, m)$ are given by (34).
While these expressions of the smearing function are convergent, they cannot be written as closed-form expressions and offer little insight. One may obtain useful approximations near the outer and inner horizons.

First we consider the field at a point in the exterior of the outer horizon which is close to the horizon. In the exterior region, the smearing function is given by (51). We can rewrite $G_{\omega,m,\infty}^{\text{norm}}$ in terms of $G_{\omega,m}^{R,+}$ and $G_{\omega,m}^{L,+}$ using (30). Close to the outer horizon we can use (21) and (22). Converting to the tortoise coordinate we finally get:

$$G_{\omega,m,\infty}^{\text{norm}} \approx |a_{\omega,m}| \cos(\omega r_* + \delta_{\omega,m}) \quad (55)$$

where

$$|a_{\omega,m}| = |A(\omega, m)\alpha(\omega, m)|^{1/2} \quad (56)$$
$$e^{2i\delta_{\omega,m}} = \frac{A(\omega, m)\alpha(\omega, m)}{B(\omega, m)\alpha^*(\omega, m)} \quad (57)$$

where

$$\alpha(\omega, m) = \left( \frac{4r_+}{r_+^2 - r_-^2} \left| \frac{r_+ - r_-}{r_+ + r_-} \right| \right)^{i(\omega - m\Omega_+)} \quad (58)$$

By choosing a wave packet in the high frequency regime $\omega_0 \gg |m_0| \gg 1$ we can obtain an expression for the smearing function. We use the following identities to simplify the formulae:

$$\Gamma(ix) = \frac{\pi}{x \sinh x} \quad (59)$$
$$\Gamma(1 + ix) = \frac{\pi x}{\sinh x} \quad (60)$$

Further using the approximation which holds for $x \gg 1$:

$$i \log \left( \frac{\Gamma(ix)}{\Gamma(-ix)} \right) = 2x(\log x - 1) - \frac{\pi}{2} + \mathcal{O}(x^{-1}) \quad (61)$$

Using the above identities we get:

$$|a_{\omega,m}| \approx \frac{2}{\sqrt{2\pi r_+}} \frac{r_+ - r_-}{\omega^{3/2}} \quad (62)$$
$$\delta_{\omega,m} \approx \frac{\pi}{4} \quad (63)$$
The smearing function is peaked around two points $t_0 + r_*$ and $t_0 - r_*$ in one of the boundaries. (b) Plot of the smearing function for a point in the exterior near the outer horizon with boundary time. $\phi_0 - \phi$ is taken to be zero. $\omega_0$ is chosen to be 40 and $\epsilon = 1$.

Then the smearing function for a high-frequency wave packet is given by:

$$K(r, t - t_0, \phi - \phi_0) \approx \frac{2(r_+ - r_-)}{\sqrt{2\pi r_+}} \int \frac{d\omega}{\omega^{3/2}} \left( \cos[\omega(t_0 - t + r_*) - \pi/4]
+ m_0(\phi_0 - \phi)] + \cos[\omega(t_0 - t - r_*) + \pi/4 + m_0(\phi_0 - \phi)] \right)$$

(64)

Here we have taken $\epsilon$ to be 1, thereby reducing the sum over $m$ to just a single $m_0$. Except for the prefactor, this agrees with the result derived for the non-rotating BTZ in [17]. This can be integrated using Mathematica. The results are summarized in figure 2.

The second case we may consider is that of a wave packet close to the outer horizon, but in the interior of the black hole.

In this case, the smearing function is given by (53). Once again we use (21) and the approximations above. The resulting expression for smearing function is given by:

$$K^L(r, t - t_0, \phi - \phi_0) \approx \frac{2(r_+ - r_-)}{\sqrt{2\pi r_+}} \int \frac{d\omega}{\omega^{3/2}} \cos[\omega(t_0 - t - r_*) + m_0(\phi_0 - \phi)]$$

(65)
FIG. 3: (a) Bulk Reconstruction for wave packet in the interior close to the outer horizon.

The smearing function is peaked around two points $t_0 + r_*$ and $t_0 - r_*$, one in each boundary. (b) Plot of the smearing function for a point in the interior near the outer horizon with boundary time. $\phi_0 - \phi$ is taken to be zero. $\omega_0$ is chosen to be 40 and $\epsilon = 1$.

A similar expression holds for the $K^R$.

Finally we consider a wave packet close to the inner horizon. In this case the formula (54) applies for the smearing function. In this case we have:

$$G_{\text{out},-} = |C_n| \cos(\omega r_* + \delta_{\omega,m}^{(1)}) + |D_n| \cos(\omega r_* + \delta_{\omega,m}^{(2)})$$  \hfill (66)$$

where

$$|c_{\omega,m}| = |A(\omega, m)\tilde{C}(\omega, m)\beta(\omega, m)|^{1/2}$$ \hfill (67)

$$|d_{\omega,m}| = |A(\omega, m)\tilde{D}(\omega, m)\beta(\omega, m)|^{1/2}$$ \hfill (68)

$$\delta_{\omega,m}^{(1)} = \frac{\tilde{C}^*(\omega, m)\beta^*(\omega, m)}{\tilde{C}(\omega, m)\beta(\omega, m)}$$ \hfill (69)

$$\delta_{\omega,m}^{(2)} = \frac{\tilde{D}^*(\omega, m)\beta^*(\omega, m)}{\tilde{D}(\omega, m)\beta(\omega, m)}$$ \hfill (70)
where

$$\beta(\omega, m) = \left( 4r_- \frac{r_+ - r_-}{r_+ + r_-} \right)^{\frac{1}{2}} \left( \frac{1}{r_+ - r_-} - \frac{1}{r_+ + r_-} \right)^{\frac{1}{2}} (\omega - m \Omega)$$

(71)

Once again we use a high frequency wave packet. In this case we find that while the pre-factors are similar, the $\delta$ s turn out to be proportional to $\omega$. We write $\delta^{(1)}_{\omega,m} = \omega \Delta^{(1)}$ and $\delta^{(2)}_{\omega,m} = \omega \Delta^{(2)}$ where:

$$\Delta^{(1)} = \frac{1}{2} \left( \frac{1}{r_+ + r_-} \log \frac{r_+ - r_-}{2r_+} + \frac{1}{r_+ - r_-} \log \frac{2r_+}{r_+ + r_-} \right)$$

(72)

$$\Delta^{(2)} = \frac{1}{2} \left( \frac{1}{r_+ + r_-} \log \frac{2r_+}{r_+ + r_-} + \frac{1}{r_+ - r_-} \log \frac{r_+ + r_-}{2r_+} \right)$$

(73)

Then the smearing function for a wave packet close to the inner horizon is given by:

$$K^L(r, t - t_0, \phi - \phi_0) \approx \frac{2(r_+ - r_-)}{\sqrt{2\pi r_+}} \int \frac{d\omega}{\omega^{3/2}} \left( \frac{\kappa_-}{\kappa_+} \cos[\omega(t_0 - t + r_+ + \Delta^{(1)}) + m_0(\phi_0 - \phi)] + \frac{\kappa_+}{\kappa_-} \cos[\omega(t_0 - t - r_+ - \Delta^{(2)}) + m_0(\phi_0 - \phi)] \right)$$

(74)

We note two striking features in this last case. First, unlike previous cases, the smearing function is not exactly peaked at where the light ray reaches, but at a distance $\Delta^{(1)}$ or $\Delta^{(2)}$ from them. If the inner and outer horizons are close, this deviation can be significant. This is a rather surprising feature of the smearing function for points near the inner horizon, which demands further study.

Second, the smearing function is peaked on four points, two each on each boundary. Two of the points are similar to the ones for the outer horizon – the peaks occur at points close to where the past light-ray from the bulk point reaches the boundary. But in this case, we get that the smearing function is peaked at two more points. These two points can be seen to be close to the ones where the future light ray from the bulk point would reach if the past inner horizon and the future inner horizon were to be identified. This novel feature also demands further study.

**B. Mirror operator representation of smearing function**

In the previous section we obtained the HKLL representation of the smearing function which has support on both the boundaries. We can also use the mirror operator construction
FIG. 4: (a) Bulk reconstruction construction for wave packet centered around the point \( x \) in the interior close to the inner horizon. The smearing function is peaked around the four points a,b,c,d two at each boundary. (b) Plot of the smearing function for a point in the interior near the inner horizon with boundary time. \( \phi_0 - \phi \) is taken to be zero. \( \omega_0 \) is chosen to be 40 and \( \epsilon = 1 \).

to obtain a representation which only has support on one boundary. For the CFT state \( |\psi\rangle \) dual to the spinning BTZ black hole, we can carry out the mirror operator construction starting from the following observation from [22]:

\[
\tilde{\mathcal{O}}_{\omega,m} |\psi\rangle = e^{-\pi \omega} \mathcal{O}^\dagger_{\omega,m} |\psi\rangle \tag{75}
\]
\[
\tilde{\mathcal{O}}_{\omega,m}^\dagger |\psi\rangle = e^{\pi \omega} \mathcal{O}_{\omega,m} |\psi\rangle \tag{76}
\]

We note that this is a state-dependent equation. For this particular state, one may represent the creation and annihilation operators on the right CFT by the operators on the right-hand side of the equation.

The expression of a field in region II as given by [45] now becomes:
\[ \phi^{II}(r, t, \theta) = \sum_m \int d\omega \left[ O_{\omega,m} \chi^{(L)}_{\omega,m}(r, t, \theta) + \text{h.c.} + e^{i\omega}\chi^{(R)}_{\omega,m}(r, t, \theta) + e^{-i\omega}\chi^{(R)*}_{\omega,m}(r, t, \theta) \right]. \]  

(77)

Which translates to:

\[ \phi^{II}(x, r) = \int dt' d\phi' (K_L(r, t, \theta; t', \phi') + K_{\text{mirror}}(r, t, \theta; t', \phi')) \mathcal{O}_R(t'', \theta'') \]  

(78)

where

\[ K_{\text{mirror}}(r, t, \theta; t', \phi') = \sum_m \int d\omega e^{i\omega t'} \chi^{(R)}_{\omega,m}(r, t, \theta) + e^{-i\omega t'} \chi^{(R)*}_{\omega,m}(r, t, \theta) \]  

(79)

Now we once again consider a wave packet instead of a field point. The CFT representation of the wave packet inside the horizon then becomes:

\[ \Phi^{II}(r, t_0, \theta_0) = \int dt' \left( K_{\omega_0,m_0}^L(r, t_0, \theta_0; t', \theta') + K_{\omega_0,m_0}^{(\text{mirror})}(r, t_0, \theta_0; t', \theta') \right) \mathcal{O}_L(t', \theta) \]  

(80)

where

\[ K^{(\text{mirror})}(t_0 + t, r, \phi_0 + \phi) = \sum_{m=m_0}^{m_0+1/2} \frac{1}{(2\pi)^2 \epsilon} \int_{\omega_0-1/2}^{\omega_0+1/2} d\omega \left[ e^{-i\omega t_0} e^{i\omega(t+0+t)+im(\phi_0+\phi)} G^{(L)*}_{\omega,m}(r) + e^{-i\omega t_0} e^{i\omega(t+0+t)-im(\phi_0+\phi)} \left( G^{(L)}_{\omega,m}(r) \right)^* \right]. \]

Here we see that the peak of the mirror smearing function has a simple interpretation, it is obtained by reflecting the light ray connecting the bulk point to the left (or right) boundary (denoted by the dashed red line in the figure) back from the center to the right boundary.

Using the approximations for a high-frequency wave packet close to the outer horizon we get the expression:

\[ K^{(\text{mirror})}(t_0 + t, r, \phi_0 + \phi) \approx 2(r_+ - r_-) \int \frac{d\omega}{\omega^{3/2}} \left( \cosh \pi \omega \cos[\omega(t + t_0 + r_*) + m_0(\phi + \phi_0)] + i \sinh \pi \omega \sin[\omega(t + t_0 + r_*) + m_0(\phi + \phi_0)] \right) \]  

(81)
FIG. 5: (a) Mirror operator construction for wave packet in the interior close to the outer horizon. The smearing function is peaked around two points $t_0 + r_*$ and $r_* - t_0$, both at the same boundary. The mirror point is obtained by reflecting back the red light ray at the center. (b) Plot of the real part of the mirror smearing function for a point in the interior near the outer horizon. $\omega_0$ is chosen to be 40 and $\epsilon = 1$. The imaginary part is identical.

For a wave packet close to the inner horizon, using the same approximations one obtains:

$$K^{(\text{mirror})}(t_0 + t, r, \phi_0 + \phi) \approx \frac{2(r_+ - r_-)}{\sqrt{2\pi r_+}} \int \frac{d\omega}{\omega^{3/2}} \left( \frac{\kappa_-}{\kappa_+} e^{-\pi \omega} + \frac{\kappa_+}{\kappa_-} e^{\pi \omega} \right) \left( \cos[\omega(t_0 + t + r_+ + \Delta^{(1)}) + m_0(\phi_0 + \phi)] + \cos[\omega(t_0 + t - r_* - \Delta^{(1)}) + m_0(\phi_0 + \phi)] \right)$$

$$+ \left( \sin[\omega(t_0 + t + r_* + \Delta^{(2)}) + m_0(\phi_0 + \phi)] + \sin[\omega(t_0 + t - r_* - \Delta^{(2)}) + m_0(\phi_0 + \phi)] \right)$$

(82)

Again the mirror operator smearing function is obtained by reflecting the left-moving rays back from the center to the right boundary. Another interesting feature is that the mirror smearing function is complex. This bears further study in the future.
FIG. 6: (a) Mirror construction for wave packet centered around the point $x$ in the interior close to the inner horizon. The smearing function is peaked around the four points $a', b', c, d$ on one boundary. $a', b'$ are the mirrors of $a$ and $b$ in the previous diagram. (b) Plot of the real part of the mirror smearing function for a point in the interior near the inner horizon. $\omega_0$ is chosen to be 40 and $\epsilon = 1$.

IV. SUMMARY

In this paper, we carried out bulk reconstruction for a spinning BTZ black hole. We obtained boundary representations for a scalar field in both the exterior and interior of the horizon. Using high-frequency wave packets we obtained smearing functions near the inner and outer horizons. While the smearing function near the outer horizon had expected features, the wave packet near the inner horizon showed some novel and striking features. For one it was peaked around two points in each boundary, as opposed to one. The position of the second peak could be interpreted by identifying the past and future inner horizons and considering a light ray that passed through the future inner horizon and emerged from the past one. This is a surprising feature that calls for further investigation. The second
novel feature is that the peaks are not exactly at the boundary points hit by light rays from the bulk, but are displaced from them in time. This is also something to be understood better.

We also carried out a mirror operator construction for fields inside the horizon. This gives us a boundary representation of fields inside the horizon as operators on a single CFT. We saw that for high-frequency wave packets the peaks of the mirror operator smearing function on the, say, left boundary could be read off by reflecting the right moving rays from the center back to the left boundary. We obtained mirror operators smearing functions for points close to the inner as well as outer horizons. One interesting feature was that the mirror operator smearing functions are complex. The implication of this is unclear and bears further study.

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