Log-concavity of characteristic polynomials and the Bergman fan of matroids

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Abstract In a recent paper, the first author proved the log-concavity of the coefficients of the characteristic polynomial of a matroid realizable over a field of characteristic 0, answering a long-standing conjecture of Read in graph theory. We extend the proof to all realizable matroids, making progress towards a more general conjecture of Rota–Heron–Welsh. Our proof follows from an identification of the coefficients of the reduced characteristic polynomial as answers to particular intersection problems on a toric variety. The log-concavity then follows from an inequality of Hodge type.

1 Introduction

In a recent paper [9], the first author proved that if $M$ is a rank $r + 1$ matroid realizable over a field of characteristic 0 with characteristic polynomial,

$$\chi_M(q) = \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1}$$

then the sequence $\mu_0, \ldots, \mu_{r+1}$ is log-concave, that is, for $1 \leq i \leq r$,

$$\mu_{i-1}\mu_{i+1} \leq \mu_i^2.$$
Because graphic matroids are realizable over any field, this result proved a conjecture due to Read [17] that chromatic polynomials of graphs are unimodal. There is a more general conjecture of Rota–Heron–Welsh [18] that the coefficients of the characteristic polynomial of any finite matroid form a log-concave sequence. The purpose of this paper is to extend the result of the first author to include all realizable matroids and to give some hints to an approach for proving the Rota–Heron–Welsh conjecture in general. A nice overview of the conjecture can be found in [1].

Let us explain the first author’s proof and our extension. His proof uses a Morse-theoretic argument to relate $\mu_i$ to Milnor numbers of the singularity at the origin of a hyperplane arrangement with matroid $\mathcal{M}$. These numbers are mixed multiplicites and are log-concave by the Khovanskii–Teissier inequality [13, Example 1.6.4]. Our method in this paper is to interpret the numbers $\mu_i$ as intersection numbers and apply the Khovanskii–Teissier inequality in a more classical framework. To identify the coefficients as intersection numbers, we use the combinatorial interpretation of the intersection theory on toric varieties developed by Fulton–Sturmfels [8] and studied in the context of tropical intersection theory by Mikhalkin [16], Allermann–Rau [3], and the second author [10,11]. We use the fact that there is an explicit Poincaré dual to a compactification of the complement of a hyperplane arrangement in a particular toric variety. The Poincaré dual arises from the description of the Bergman fan studied by Ardila–Klivans [2].

Let $\mathcal{A}$ be an arrangement of hyperplanes on an $r$-dimensional projective subspace $V \subset \mathbb{P}^n$ realizing $\mathcal{M}$. Let $\tilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$ be the closure of the graph of the Cremona transformation

$$\text{Crem} : \mathbb{P}^n \to \mathbb{P}^n, \quad (z_0 : \cdots : z_n) \mapsto (z_0^{-1} : \cdots : z_n^{-1})$$

restricted to $V \setminus \mathcal{A}$. $\tilde{V}$ is a compactification of $V \setminus \mathcal{A}$ whose boundary is a divisor with normal crossings. By virtue of the description of the class of $\tilde{V}$, we have the following result:

**Theorem 1.1** Write

$$\overline{\chi}_M(q) := \chi_M(q)/(q - 1) = \sum_{i=0}^r (-1)^i \mu_i q^{r-i}.$$ 

Then

$$[\tilde{V}] = \sum_{i=0}^r \mu_i [\mathbb{P}^{r-i} \times \mathbb{P}^i] \in A_r(\mathbb{P}^n \times \mathbb{P}^n)$$

in the Chow homology group of $\mathbb{P}^n \times \mathbb{P}^n$.

The log-concavity of $\mu_0, \ldots, \mu^r$, and hence that of $\mu_0, \ldots, \mu_{r+1}$, follows from applying the Khovanskii–Teissier inequality to the irreducibility variety $\tilde{V}$. Our proof is largely combinatorial except for establishing the Khovanskii–Teissier inequality in Lemma 3.3 which requires the work of Fulton–Sturmfels and a classical proof of...