On constructing the Riemannian metrics in refraction tomography problems

E Yu Derevtsov*

The Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail: *dert@math.nsc.ru

Abstract. A problem of the refraction tomography consists in reconstructing a function or tensor field by their attenuated ray transforms. Usually an absorption coefficient and refraction in the medium are assumed to be known. A solution to the refraction tomography problems can be obtained by approximate methods within the corresponding mathematical models, and the modeling of refraction is one of important elements of the model. We adduce summary data on the Riemannian metrics, suitable for the implementation in numerical tests, methods of their construction and main characteristics.

1. Preliminaries

A phenomenon of the ray refraction, along which a signal propagates, arises in the process of probing an inhomogeneous medium by any physical field. In some settings, for example, in the seismic tomography, the refraction is so large that it is impossible not to take it into account. Thus, the influence of refraction on the accuracy of function reconstruction was investigated in [1]. A problem of reconstruction of 2D functions or tensor fields by their attenuated ray transforms with the known absorption coefficient and refraction was posed and numerically solved in [2]–[5]. A solution to the refraction tomography problems can be usually obtained by approximate methods within corresponding mathematical models, and the modeling of refraction is one of important elements of the model. We adduce summary data on the Riemannian metrics, suitable for the implementation in numerical tests, methods of their construction and main characteristics.

An element of the Riemannian metric length given in the bounded domain $D$ of $n$-dimensional Euclidean space $\mathbb{R}^n$ has the form

$$ds^2 = g_{ij}(x) dx^i dx^j,$$

where $x = (x^1, \ldots, x^n)$, and $g_{ij}(x)$, $i, j = 1, \ldots, n$ are sufficiently smooth in $D$ functions. Here and below we use the Einstein summation rule. An exception of this rule will be reserved.

The covariant components of the tensor $g$ in the matrix form are

$$g_{ij} =
\begin{pmatrix}
g_{11}(x) & g_{12}(x) & \cdots & g_{1n}(x) \\
g_{12}(x) & g_{22}(x) & \cdots & g_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
g_{1n}(x) & g_{2n}(x) & \cdots & g_{nn}(x)
\end{pmatrix},$$

(2)
the contravariant components are the elements of the inverse matrix, and \( g^{ik}g_{kj} = \delta^{i}_{j} \), where \( \delta^{i}_{j} \) are the components of Kronecker tensor. The designation \( g \) is used for the determinant of matrix (2), \( g \equiv \text{det}(g_{ij}) \). The Christoffel symbols of the second kind are defined by the formulas

\[
\Gamma^{k}_{ij} = \frac{1}{2}g^{kp}\left( \frac{\partial g_{jp}}{\partial x^{i}} + \frac{\partial g_{pi}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{p}} \right).
\]

Let \( x^{i}(t) \subset D \subset \mathbb{R}^{n} \) be an arbitrary parametrically defined curve, connecting any two points \( P_{1}(x^{i}(t_{1})), P_{2}(x^{i}(t_{2})) \in D \). The length of the curve is

\[
I = \int_{t_{1}}^{t_{2}} \sqrt{g_{ij}\frac{dx^{i}}{dt}\frac{dx^{j}}{dt}} \, dt.
\]

Here \( \sqrt{g_{ij}\frac{dx^{i}}{dt}\frac{dx^{j}}{dt}} \equiv F \) is the Lagrange function, \( \frac{dx^{i}}{dt} \equiv \dot{x}^{i} \). Let us consider the curve \( \bar{x}^{i}(t) = x^{i}(t) + \varepsilon u^{i}(t) \), also passing through the points \( P_{1} \) and \( P_{2} \). The first variation \( \delta I \) of the functional \( I \) of the curve length is

\[
\delta I = \varepsilon \int_{t_{1}}^{t_{2}} \left( \frac{\partial F}{\partial x^{i}} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}^{i}} \right) \right) u^{i} \, dt.
\]

The curve satisfying to equation \( \delta I = 0 \) for any \( u^{i} \), is \textit{geodesic line}. Along a geodesic line the Euler equations

\[
\lambda_{i} \equiv \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}^{i}} \right) - \frac{\partial F}{\partial x^{i}} = 0
\]

are valid. The values \( \lambda_{i} \) are covariant components of the vector. Passing to the contravariant components \( \lambda^{i} \) of the vector, defined by the the Euler equations (3), we obtain a system of ordinary differential geodesic equations

\[
\lambda^{i} \equiv \ddot{x}^{i} + \Gamma^{i}_{jk}\dot{x}^{j}\dot{x}^{k} - \dot{x}^{i}\dot{s}/\dot{s} = 0, \quad i = 1, \ldots, n.
\]

If we choose the length of the arc \( s \) as parameter of geodesic, which is definable up to a linear transform, then the system does not contain the last item to the left-hand side. Below we choose this type of parametrization.

### 2. The scalar curvature of an isotropic metric

Let in \( \mathbb{R}^{n} \), a bounded domain \( D \) with a diagonal Riemannian metric

\[
ds^{2} = g_{ii}dx^{i}\,dx^{i} \equiv g_{11}(dx^{1})^{2} + g_{22}(dx^{2})^{2} + \ldots + g_{nn}(dx^{n})^{2}
\]

be given. In particular, if \( g_{11} = g_{22} = \ldots = g_{nn} = \lambda(x) \), then we obtain an isotropic metric,

\[
ds^{2} = \lambda(x)((dx^{1})^{2} + (dx^{2})^{2} + \ldots + (dx^{n})^{2}).
\]

As metric (4) is valid, the Christoffel symbols can be calculated as follows:

\[
\Gamma^{k}_{ij} = \frac{1}{2}g^{kk}\left( \frac{\partial g_{jk}}{\partial x^{i}} + \frac{\partial g_{ki}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{k}} \right).
\]

Let us divide them into four groups. Every group of the symbols has the same type depending on a value of the indexes.

\[
\Gamma^{i}_{ii} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^{i}}, \quad i = j = k, \quad \Gamma^{i}_{ii} = -\frac{1}{2g_{kk}} \frac{\partial g_{ii}}{\partial x^{k}}, \quad i = j, \ k \neq i.
\]
\[ \Gamma^k_{ki} = \frac{1}{2g_{kk}} \frac{\partial g_{kk}}{\partial x^i}, \quad k = i, \ j \neq i, \quad \Gamma^k_{ij} = 0, \quad i \neq j \neq k. \tag{8} \]

It is easy to see the validity of the following Lemma.

**Lemma 1.** Let \( \{ \Gamma^k_{ij} \mid i, j, k = 1, \ldots, n; i \leq j \} \) be a set of the Christoffel symbols (6) of diagonal metric (4). Then the aggregate of sets (7)--(8) of the Christoffel symbols is the partition.

The Einstein rule does not work from here up to the end of proof of Theorem 1.

**Theorem 1.** The scalar curvature of isotropic Riemannian metric (5), given in a bounded domain \( D \subset \mathbb{R}^n \), is calculated by the formula

\[ R = -\frac{n-1}{\lambda^2} \Delta \lambda - \frac{(n-1)(n-6)}{4\lambda^3} |\nabla \lambda|^2, \]

where \( \Delta \) is the Laplace operator and \( \nabla \) is the gradient.

**Proof.** The scalar curvature \( R = \sum_{i,j=1}^{n} g^{ij} R_{ij} \) is expressed through the Ricci tensor \( R_{ij} \), which is connected with the Riemann-Christoffel curvature tensor [6], \( R_{ij} = \sum_{p=1}^{n} R_{ijp}^p \). Taking into account (4), we get

\[ R = \sum_{i=1}^{n} g^{ii} R_{ii} = \sum_{i,j=1}^{n} g^{ii} R_{ijp}^p. \]

By means of the formula

\[ R_{ijkl} = \frac{\partial}{\partial x^l} \Gamma^i_{jk} - \frac{\partial}{\partial x^k} \Gamma^i_{jl} + \sum_{p=1}^{n} \Gamma^p_{jk} \Gamma^i_{pl} - \sum_{s=1}^{n} \Gamma^s_{jl} \Gamma^i_{sk}, \]

for the curvature tensor components we obtain

\[ R = \sum_{i=1}^{n} g^{ii}\left( \sum_{p=1}^{n} \frac{\partial}{\partial x^p} \Gamma^p_{ii} - \sum_{p=1}^{n} \frac{\partial}{\partial x^i} \Gamma^p_{ip} + \sum_{p,s=1}^{n} \Gamma^s_{ii} \Gamma^p_{ps} - \sum_{p,s=1}^{n} \Gamma^s_{ip} \Gamma^p_{is} \right). \tag{9} \]

Let us calculate the first two terms in the circular parenthesis, considering \( i \) to be fixed,

\[ \sum_{k=1}^{n} \frac{\partial}{\partial x^k} \Gamma^k_{ii} = \frac{\partial}{\partial x^i} \Gamma^i_{ii} + \sum_{k=1,k\neq i}^{n} \frac{\partial}{\partial x^k} \Gamma^k_{ii}. \]

With the usage of (7), (8), and \( g_{ii} = \lambda \) for \( i = 1, \ldots, n \), we obtain the first item

\[ \sum_{k=1}^{n} \frac{\partial}{\partial x^k} \Gamma^k_{ii} = \frac{1}{\lambda} \frac{\partial^2 \lambda}{(\partial x^i)^2} - \frac{1}{\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 - \left( \frac{1}{2\lambda} \Delta \lambda - \frac{1}{2\lambda^2} |\nabla \lambda|^2 \right). \tag{10} \]

For the second term, we get a similar expression,

\[ \sum_{k=1}^{n} \frac{\partial}{\partial x^i} \Gamma^k_{ki} = \frac{n}{2\lambda} \frac{\partial^2 \lambda}{(\partial x^i)^2} - \frac{n}{2\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2. \tag{11} \]

With the designation

\[ C_1 = \sum_{i=1}^{n} g^{ii}\left( \sum_{p=1}^{n} \frac{\partial}{\partial x^p} \Gamma^p_{ii} - \sum_{p=1}^{n} \frac{\partial}{\partial x^i} \Gamma^p_{ip} \right), \]
applying (10), (11), we obtain

\[
C_1 = \sum_{i=1}^{n} \left( \frac{1}{\lambda} \left( \frac{\partial^2 \lambda}{(\partial x^i)^2} - \frac{1}{\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 \right) - \frac{1}{2\lambda} \Delta \lambda + \frac{1}{2\lambda^2} |\nabla \lambda|^2 \right)
- \frac{n}{2\lambda} \frac{\partial^2 \lambda}{(\partial x^i)^2} + \frac{n}{2\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 \right) = -\frac{n-1}{\lambda^2} \Delta \lambda + \frac{n-1}{\lambda^3} |\nabla \lambda|^2 = -\frac{n-1}{\lambda} \Delta (\ln \lambda),
\]

(12)

Using the designation

\[
C_2 = \sum_{i=1}^{n} g^{ii} \left( \sum_{p,s=1}^{n} (\Gamma^p_{ip} \Gamma^{ps} - \Gamma^p_{ip} \Gamma^{ps}) \right)
\]

and the partition of the Christoffel symbols (7), (8), for the first item we derive

\[
\sum_{s=1}^{n} \sum_{p=1}^{n} \Gamma^s_{ip} \Gamma^{ps} = (\Gamma^i_{ii})^2 + \sum_{p=1}^{n} \Gamma^i_{ii} \Gamma^{ip} + \sum_{s=1}^{n} \sum_{s \neq i} \Gamma^s_{ii} \Gamma^{is} + \sum_{s=1}^{n} \sum_{s \neq i} \Gamma^s_{ii} \Gamma^{is} + \sum_{s=1}^{n} \sum_{s \neq i} \Gamma^s_{ip} \Gamma^{ps} = \frac{1}{4\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 + \frac{n-1}{4\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2
- \frac{1}{2\lambda^2} \sum_{p=1}^{n} \left( \frac{\partial \lambda}{\partial x^p} \right)^2 - \frac{1}{4\lambda^2} \sum_{s=1}^{n} \sum_{s \neq i} \left( \frac{\partial \lambda}{\partial x^s} \right)^2
= \frac{n}{2\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 - \frac{n}{4\lambda^2} |\nabla \lambda|^2.
\]

(13)

The second item can be obtained by analogy,

\[
\sum_{s=1}^{n} \sum_{p=1}^{n} \Gamma^s_{ip} \Gamma^{ps} = \frac{n+2}{4\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 - \frac{1}{2\lambda^2} |\nabla \lambda|^2.
\]

(14)

By means of (13), (14), we obtain for \( C_2 \) the following formula

\[
C_2 = \sum_{i=1}^{n} \left( \frac{n}{2\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 - \frac{n}{4\lambda^2} |\nabla \lambda|^2 - \frac{n+2}{4\lambda^2} \left( \frac{\partial \lambda}{\partial x^i} \right)^2 - \frac{1}{2\lambda^2} |\nabla \lambda|^2 \right),
\]

which is simplified as

\[
C_2 = -\frac{(n-1)(n-2)}{4\lambda^3} |\nabla \lambda|^2 = -\frac{(n-1)(n-2)}{4\lambda} |\nabla (\ln \lambda)|^2.
\]

(15)

With the usage of (12), (15) for the scalar curvature \( R \) we derive the formula

\[
R = C_1 + C_2 = -\frac{n-1}{\lambda^2} \Delta \lambda - \frac{(n-1)(n-6)}{4\lambda^3} |\nabla \lambda|^2 = -\frac{n-1}{4\lambda} \left( 4\Delta (\ln \lambda) + (n-2) |\nabla (\ln \lambda)|^2 \right).
\]

(16)

The theorem is proved.
3. The 3D-metrics
The following corollaries are simple. They are formulated for the fixed values of the scalar curvatures, which are important for certain refraction tomography models.

**Corollary 1.** The scalar curvature of the Riemannian domain $D \subset \mathbb{R}^3$ with given isotropic Riemannian metric is calculated according to the formulas

$$ R = -\frac{2}{\lambda^2} \Delta(\lambda) + \frac{3}{2\lambda^3} |\nabla(\lambda)|^2 = -\frac{2}{\lambda} \Delta(\ln \lambda) - \frac{1}{2\lambda} |\nabla(\ln \lambda)|^2 $$

**Corollary 2.** Let in the half-space $\mathbb{R}^3_{x^3+} = \{x \in \mathbb{R}^3 \mid x^3 \geq 0\} \subset \mathbb{R}^3$ the isotropic Riemannian metric (5), $n = 3$, be given for $\lambda > 0$, $\lambda(x^1, x^2, x^3) \equiv \lambda(x^3)$. Then the scalar curvature of the metric given in $\mathbb{R}^3_{x^3+}$, depends on $x^3$ only, and it is expressed by the formula

$$ R = -\frac{1}{\lambda^2(x^3)} \left(2\lambda''(x^3) - \frac{3}{2} \left|\frac{\lambda'(x^3)}{\lambda(x^3)}\right|^2\right). $$

**Corollary 3.** Let in the half-space $\mathbb{R}^3_{x^3+} = \{x \in \mathbb{R}^3 \mid x^3 \geq 0\} \subset \mathbb{R}^3$ the isotropic Riemannian metric (5), $n = 3$, be given for $\lambda(x^3) = (ax^3 + b)^a$, $a, b > 0$, $\alpha \in \mathbb{R}$. Then the scalar curvature of the metric given in $\mathbb{R}^3_{x^3+}$, depends on $x^3$ only, and it is expressed by the formula

$$ R = -\frac{a^2\alpha(a - 4)}{2} (ax^3 + b)^{-\alpha - 2} = -\frac{a^2\alpha(a - 4)}{2} \frac{1}{(ax^3 + b)^2 \lambda(x^3)}. $$

It can be easily seen that at $\alpha = 0$, $\lambda(x^3) \equiv 1$ the metric is Euclidean. At $\alpha = 4$ geodesics should be straight lines but the Christoffel symbols are not equal to zero. At $\alpha < 0$, we obtain the negative curvature metric, and at $0 < \alpha < 4$ we have the metrics of the positive curvature. At $\alpha > 4$, the curvature of the metric becomes negative again.

A value $\alpha = -2$ must be especially marked. The metric has a constant negative curvature $K = -6a^2$. The function $ax^3 + b$ describes the velocity in a half-space $\mathbb{R}^3_{x^3+}$ with a given isotropic metric, and $\lambda(x) = 1/(ax^3 + b)^2$. The geodesics of such a metric are the arc of circles that are intersections of spheres of any radius $\rho > b/a$ and a center at any point of the plane $x^3 = -b/a$, with any “vertical” plane parallel to the axis $x^3$ [8]. This metric was the main under an investigation of a certain seismology and seismic problems in the framework of the refraction tomography [9], [10].

**Remark 1.** It is interesting to consider a case $K = 0$ with $\alpha = 4$, $\lambda(x) = (ax^3 + b)^4$. If we interpret $\lambda$ as the square of velocity, then $v(x^3)$ in the medium has the form $1/(ax^3 + b)^2$ decreasing as the square-law depending of the depth. This metric is the only one among those under a consideration, that has two intervals $\alpha < 0$, and $\alpha > 4$ of the parameter $\alpha$, for which the metric has a negative curvature.

The calculation of the Christoffel symbols of the isotropic 3D-metric (5), $n = 3$, is not a complicated task. The simplest form they have for $\lambda(x) \equiv \lambda(x^3)$,

$$\Gamma^1_{13} = \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x^3}, \quad \Gamma^2_{23} = \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x^3}, \quad \Gamma^3_{11} = -\frac{1}{2\lambda} \frac{\partial \lambda}{\partial x^3}, \quad \Gamma^3_{22} = -\frac{1}{2\lambda} \frac{\partial \lambda}{\partial x^3}, \quad \Gamma^3_{33} = \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x^3}. $$

The other symbols are equal to zero.
3.1. The isotropic metric with linear dependence on $x^3$ given in $\mathbb{R}^3_{x^3+}$
Let us consider the closed half-space $\tilde{D} = \mathbb{R}^3_{x^3+} \equiv \{ x \in \mathbb{R}^3 | x^3 \geq 0 \}$. An element of the length in $\mathbb{R}^3_{x^3+}$ is
\[ ds^2 = \lambda^2(x^3)((dx^1)^2 + (dx^2)^2 + (dx^3)^2), \]
where $\lambda(x) \equiv \lambda(x^3) = 1/(ax^3 + b)^2$. We remind [8] that the geodesics of metric (17) are the arcs of “vertical” circles. More precisely, the geodesics connecting the two points $x_1(x_1^1, x_1^2, 0)$, $x_2(x_2^1, x_2^2, 0) \in \mathbb{R}^3_{x^3+}$ is the arc of a circle of radius $r$ at the center of the point $p$, lying in the plane $P_{x_1x_2}$. Here
\[ r = \sqrt{\frac{b^2}{a^2} + \frac{|x_1 - x_2|^2}{4}}, \quad p = \left( \frac{x_1^1 + x_2^1}{2}, \frac{x_1^2 + x_2^2}{2}, -\frac{b}{a} \right), \]
\[ P_{x_1x_2} = \{ x \in \mathbb{R}^3 | (x_2^2 - x_1^2)x^1 - (x_2^1 - x_1^1)x^2 - (x_1^1x_2^2 - x_1^2x_2^1) = 0 \}. \]

Let’s fix the following designations:
$S^2_{x_0,R}$ for the sphere of radius $R > b/a$ with the center at the point $x_0(x_0^1, x_0^2, -b/a)$;
$S^1_{x_0,\rho}$ for the circle of radius $\rho$ with the center at the point $y_0(y_0^1, y_0^2, 0)$;
$B^2_{y_0,\rho}$ for the disk of radius $\rho$ with the center at the point $y_0(y_0^1, y_0^2, 0)$.

The following lemma is valid.

**Lemma 2.** Let $a, b > 0, R > b/a$ be real. Then the surface $S^2_{x_0,R} \cap \mathbb{R}^3_{x^3+}$ is a totally geodesic submanifold of the sub-space $\mathbb{R}^3_{x^3+}$ with the Riemannian metric (17) given in it.

**Proof.** We remind that bounded surface $E$ belonging to the Riemannian manifold $D$ is called a totally geodesic submanifold if from $x, y \in E$ follows $\gamma_{x,y} \subset E$. Here $\gamma_{x,y}$ is the geodesic connecting points $x, y$.

Let’s fix a point $x_0 = (0, 0, -b/a)$ as the center of the sphere $S^2_{x_0,R}$. The intersection $S^2_{x_0,R} \cap \mathbb{R}^3_{x^3+}$ of the sets $S^2_{x_0,R}$, $\mathbb{R}^3_{x^3+}$ is the circle $S^4_{x,\rho}$ of radius $\rho = \sqrt{R^2 - (b/a)^2}$ and the center at the point $q = (0, 0, 0)$. We consider two arbitrary points $p_1 = (x_1^1, x_1^2, 0)$, $p_2 = (x_2^1, x_2^2, 0)$ on the circle $S^1_{q,\rho}$. Coordinates of an arbitrary point $x$ of the geodesic of metric (17), passing through the points $p_1, p_2$, satisfy the relations (see (18), (19))
\[ x \in P_{x_1x_2} \Rightarrow (x_2^2 - x_1^2)x^1 - (x_2^1 - x_1^1)x^2 - (x_1^1x_2^2 - x_1^2x_2^1) = 0, \]
\[ \left( x^1 - \frac{x_1^1 + x_2^1}{2} \right)^2 + \left( x^2 - \frac{x_1^2 + x_2^2}{2} \right)^2 + \left( x^3 + \frac{b}{a} \right)^2 = \frac{b^2}{a^2} + \frac{|p_2 - p_1|^2}{4}. \]

We have to prove that $x \in S^2_{x_0,R} \cap \mathbb{R}^3_{x^3+}$, and it is sufficient to check the validity of the formula
\[ (x^1)^2 + (x^2)^2 = \left( x^1 - \frac{x_1^1 + x_2^1}{2} \right)^2 + \left( x^2 - \frac{x_1^2 + x_2^2}{2} \right)^2 + \rho^2 - \frac{|p_2 - p_1|^2}{4}. \]

A simple remake of the latter expression leads to the formula $(x_1^1 + x_2^1)x^1 + (x_1^2 + x_2^2)x^2 - x_1^1x_2^2 - x_1^2x_2^1 = \rho^2$. With the usage of (20), (21) we obtain
\[ \rho^2 + \frac{x_2^2 - x_1^2}{x_3^2 - x_1^2}((x_2^1)^2 + (x_2^2)^2 - (x_1^1)^2 - (x_1^2)^2) = \rho^2, \]
which becomes the identity as $p_1, p_2 \in S^4_{q,\rho}$. Lemma is proved.

**Lemma 3.** Let $a, b > 0, R > b/a$ be fixed and real, $\rho = \sqrt{R^2 - (b/a)^2}$. Then the totally geodesic submanifold $S^2_{x_0,R} \cap \mathbb{R}^3_{x^3+}$ of the half-space $\mathbb{R}^3_{x^3+}$ with the Riemannian metric (17) given in it, is isometric to the disk $B_{y_0,\rho} \subset \mathbb{R}^3_{x^3+}$ with the Riemannian metric (1), $n = 2$, where
\[ g_{jj} = \frac{r^2 + (x^j - x_0^j)^2}{a^2r^4}, \quad j = 1, 2, \quad g_{12} = \frac{(x^1 - x_0^1)(x^2 - x_0^2)}{a^2r^4}, \]
\[ g_{13} = \frac{(x^1 - x_0^1)(x^3 - x_0^3)}{a^2r^4}, \quad g_{23} = \frac{(x^2 - x_0^2)(x^3 - x_0^3)}{a^2r^4}. \]
and \( r = \sqrt{R^2 - \left(x^1 - x^1_0\right)^2 - \left(x^2 - x^2_0\right)^2} \).

**Proof.** An equation of the surface \( S^2_{x_0, R} \cap \mathbb{R}^2_{x^3} \) can be written down in the form
\[
x^3 = -b/a + \sqrt{R^2 - \left(x^1 - x^1_0\right)^2 - \left(x^2 - x^2_0\right)^2}, \quad x^3 \geq 0.
\]
Expressing the differential \( dx^3 \) through \( dx^1, dx^2 \), substituting \( (dx^3)^2 \) into (17) and taking into account \( ax^3 + b = ar \), we obtain the expression equivalent to (17)
\[
ds^2 = \frac{1}{(ax^3 + b)^2} \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)
= \frac{r^2 + (x^1 - x^1_0)^2}{a^2 r^4}(dx^1)^2 + 2\frac{(x^1 - x^1_0)(x^2 - x^2_0)}{a^2 r^4} dx^1 dx^2 + \frac{r^2 + (x^2 - x^2_0)^2}{a^2 r^4}(dx^2)^2
\]
for the element of length. As the mapping
\[
F : S^2_{x_0, R} \cap \mathbb{R}^2_{x^3} \to B_{y_0, \rho} \cap \{x \in \mathbb{R}^3| x^3 = 0\}, \quad F(x^1, x^2, x^3) = (x^1, x^2, 0)
\]
is one-to-one and (23) is valid, we obtain the property of isometry.

Remark 2. The geodesics of metric (22), given in the disk \( B_{y_0, \rho} \cap \{x \in \mathbb{R}^3| x^3 = 0\} \) of radius \( \rho \), are segments of straight lines. The element of length of these segments is calculated according to metric (23).

Lemmas 2, 3 show that a 3D problem in the half-space is reduced to a series of 2D problems. Any such problem is posed in a disk belonging to the plane \( x^3 = 0 \). Thus, we can apply approaches, tools and algorithms developed in [1]–[5] in order to solve the problem of the 2D refraction tomography.

4. The 2D metrics
One of methods for constructing the Riemannian metrics is as follows. Let in \( \mathbb{R}^3 \) a surface \( z = z(x, y) \) be given (here and below we use the denotation \((x, y, z)\) for a point in \( \mathbb{R}^3 \)). An element of length in \( \mathbb{R}^3 \) has the form \( ds^2 = dx^2 + dy^2 + dz^2 \). The main step is to find projections of geodesic lines, lying at the surface, on the unit disk \( B = \{(x, y) | x^2 + y^2 \leq 1, z = 0, B \subset \text{OXY}\} \). We offer that mapping from the surface onto the disk \( B \) is one-to-one. By means of the formula
\[
dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy
\]
the differential \( dz \) is substituted into the length element \( ds^2 \) in \( \mathbb{R}^3 \) and we have an element of length of the Riemannian metric given in the unit disk,
\[
ds^2 = g_{11} dx^2 + 2g_{12} dx dy + g_{22} dy^2.
\]
The Riemann-Christoffel curvature tensor \( R_{jkl} \) is expressed through the Christoffel symbols and their derivatives. The covariant components \( R_{jkl} \) can be calculated as follows:
\[
R_{jkl} = \frac{1}{2} \left( \frac{\partial^2 g_{kl}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{jk}}{\partial x^l \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^l} \right) - g_{rs} \left( \Gamma_{jkl}^r \Gamma_{il}^s - \Gamma_{jil}^r \Gamma_{kl}^s \right).
\]
In the 2D Riemannian manifold, only one component (in particular \( R_{1212} \)) is linearly independent,
\[
R_{1212} = \frac{1}{2} \left( \frac{\partial^2 g_{11}}{\partial y^2} + \frac{\partial^2 g_{22}}{\partial x^2} - \frac{\partial^2 g_{21}}{\partial x \partial y} \right) - g_{11} \left( \Gamma_{211}^1 \Gamma_{12}^1 - \Gamma_{212}^1 \Gamma_{11}^1 \right) - g_{22} \left( \Gamma_{211}^2 \Gamma_{12}^2 - \Gamma_{212}^2 \Gamma_{11}^2 \right)
\]
An operation of tensors contraction leads to the Ricci tensor \( R_{jk} = R_{jkl}^l \), and then to the scalar curvature \( R = g^{jk} R_{jk} \). Closely connected with it, the Gaussian curvature is calculated by the formula \( K = -\frac{R_{1212}}{g} \), where \( g = g_{11} g_{22} - g_{12}^2 \).
Below in detail we describe few metrics that are the most useful in mathematical models of the refraction tomography. The scalar curvature (16) for \( n = 2 \) has a simple form, 
\[
R = -\Delta(\ln \lambda)/\lambda.
\]
The Gaussian curvature [7] of a surface in \( \mathbb{R}^3 \) with isotropic metrics (5) has the form 
\[
K = -\Delta(\ln \lambda)/(2\lambda).
\]

1. Generalization of the constant positive curvature metric (GCP-metric).

\[
g_{11}(x, y) = g_{22}(x, y) \equiv \lambda(x, y) = \left( \frac{2R}{R^2 + a^2 r^2} \right)^\alpha, \quad a^2 < R^2, \quad \alpha \in \mathbb{R}.
\]

The Christoffel symbols of the GCP-metric are as follows:

\[
\Gamma^1_{11} = -\frac{\alpha a^2 x}{R^2 + a^2 r^2}, \quad \Gamma^1_{12} = -\frac{\alpha a^2 y}{R^2 + a^2 r^2}, \quad \Gamma^1_{22} = \frac{\alpha a^2 x}{R^2 + a^2 r^2},
\]

\[
\Gamma^2_{11} = \frac{\alpha a^2 y}{R^2 + a^2 r^2}, \quad \Gamma^2_{12} = -\frac{\alpha a^2 x}{R^2 + a^2 r^2}, \quad \Gamma^2_{22} = -\frac{\alpha a^2 y}{R^2 + a^2 r^2}.
\]

As 
\[
\Delta(\ln \lambda) = -\alpha a^2 \left( \frac{2R}{R^2 + a^2 r^2} \right)^2,
\]
the Gaussian curvature is
\[
K = -\frac{1}{2n} \Delta(\ln \lambda) = -\frac{\alpha a^2}{2} \left( \frac{R^2 + a^2 r^2}{2R} \right)^{\alpha - 2}.
\]

At \( \alpha = 2 \), we get the metric of constant positive curvature \( a^2 \). At \( \alpha < 0 \), the Gaussian curvature of the GCP-metric is negative. At \( \alpha > 0 \) it is positive.

2. Generalization of the constant negative curvature metric (GCN-metric).

\[
g_{11}(x, y) = g_{22}(x, y) \equiv n(x, y) = \left( \frac{2R}{R^2 - a^2 r^2} \right)^\alpha, \quad a^2 < R^2, \quad \alpha \in \mathbb{R}.
\]

The Christoffel symbols are

\[
\Gamma^1_{11} = \frac{\alpha a^2 x}{R^2 - a^2 r^2}, \quad \Gamma^1_{12} = \frac{\alpha a^2 y}{R^2 - a^2 r^2}, \quad \Gamma^1_{22} = -\frac{\alpha a^2 x}{R^2 - a^2 r^2},
\]

\[
\Gamma^2_{11} = -\frac{\alpha a^2 y}{R^2 - a^2 r^2}, \quad \Gamma^2_{12} = \frac{\alpha a^2 x}{R^2 - a^2 r^2}, \quad \Gamma^2_{22} = \frac{\alpha a^2 y}{R^2 - a^2 r^2}.
\]

As 
\[
\Delta(\ln n) = \alpha a^2 \left( \frac{2R}{R^2 - a^2 r^2} \right)^2,
\]
the Gaussian curvature of the GCN-metric is
\[
K = -\frac{1}{2n} \Delta(\ln n) = -\frac{\alpha a^2}{2} \left( \frac{R^2 - a^2 r^2}{2R} \right)^{\alpha - 2}.
\]

At \( \alpha = 2 \), we have the metric of the constant negative curvature \( -a^2 \). At \( \alpha > 0 \), the Gaussian curvature is negative, at \( \alpha < 0 \) it is positive.

3. Generalization of the elliptic paraboloid metric

\[
z = \left( \frac{h}{R^2} \left( R^2 - \frac{ax^2}{2} - \frac{by^2}{2} \right) \right)^\alpha, \quad \alpha \in \mathbb{R}.
\]
With the designations
\[ P = \frac{h}{R^2} \left( R^2 - \frac{ax^2}{2} - \frac{by^2}{2} \right), \quad S := R^4 + P^{2\alpha - 2} \alpha^2 h^2 (a^2 x^2 + b^2 y^2) \]
we obtain the components
\[ g_{11} = \frac{R^4 + P^{2\alpha - 2} \alpha^2 h^2 a^2 x^2}{R^4}, \quad g_{12} = \frac{P^{2\alpha - 2} \alpha^2 h^2 abxy}{R^4}, \quad g_{22} = \frac{R^4 + P^{2\alpha - 2} \alpha^2 h^2 b^2 y^2}{R^4} \]
of the metric tensor and the Christoffel symbols
\[ \Gamma^1_{11} = \frac{x a^2 h^2 \alpha^2 P^{2\alpha - 3} (a h x^2 (1 - \alpha) + PR^2)}{R^2 S}, \quad \Gamma^2_{11} = \frac{y a^2 h^2 \alpha^2 P^{2\alpha - 3} (a h x^2 (1 - \alpha) + PR^2)}{R^2 S}, \]
\[ \Gamma^1_{12} = \frac{x^2 y a^2 b h^3 \alpha^2 (1 - \alpha) P^{2\alpha - 3}}{R^2 S}, \quad \Gamma^2_{12} = \frac{x y^2 a^2 h^3 \alpha^2 (1 - \alpha) P^{2\alpha - 3}}{R^2 S}, \]
\[ \Gamma^1_{22} = \frac{x a^2 h^2 \alpha^2 P^{2\alpha - 3} (b h y^2 (1 - \alpha) + PR^2)}{R^2 S}, \quad \Gamma^2_{22} = \frac{y a^2 h^2 \alpha^2 P^{2\alpha - 3} (b h y^2 (1 - \alpha) + PR^2)}{R^2 S}. \]
The determinant \( g = g_{11} g_{22} - g_{12}^2 \) of the matrix \((g_{ij})\) is equal to \( g = S/R^4 \). The elements of inverse matrix \((g^{ij})\) are
\[ g^{11} = \frac{R^4 + P^{2\alpha - 2} \alpha^2 h^2 b^2 y^2}{S}, \quad g^{12} = -\frac{P^{2\alpha - 2} \alpha^2 h^2 abxy}{S}, \quad g^{22} = \frac{R^4 + P^{2\alpha - 2} \alpha^2 h^2 a^2 x^2}{S}. \]
The Gaussian curvature is
\[ K = \frac{ab h^2 \alpha^2 P^{2\alpha - 3} R^2 (h(1 - \alpha)(a x^2 + b y^2) + PR^2)}{S^2}. \]

**Acknowledgements**

The work was partially supported by the Program for Fundamental Researches of SB RAS (project No. 0314-2016-0013) and by RFBR according to the RFBR-DFG project No. 19-51-12008.

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