Exponential stabilization and continuous dependence of solutions on initial data in different norms for space-time-varying linear parabolic PDEs

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Abstract

For an arbitrary parameter $p \in [1, +\infty]$, we consider the problem of exponential stabilization in the spatial $L^p$-norm, and $W^{1,p}$-norm, respectively, for a class of anti-stable linear parabolic PDEs with space-time-varying coefficients in the absence of a Gevrey-like condition, which is often imposed on time-varying coefficients of PDEs and used to guarantee the existence of smooth (w.r.t. the time variable) kernel functions in the literature. Then, based on the obtained exponential stabilities, we show that the solution of the considered system depends continuously on the $L^p$-norm, and $W^{1,p}$-norm, respectively, of the initial data. In order to obtain time-independent (and thus sufficiently smooth) kernel functions without a Gevrey-like condition and deal with singularities arising in the case of $p \in [1, 2)$, we apply a combinatorial method, i.e., the combination of backstepping and approximation of Lyapunov functionals (ALFs), to stabilize the considered system and establish the continuous dependence of solutions on initial data in different norms.

Key words: Exponential stability; continuous dependence; backstepping; approximation of Lyapunov functional; parabolic PDEs; time varying system.

1 Introduction

In the last decades, the backstepping method has proved to be a powerful tool for the boundary control of distributed parameter systems governed by 1-D partial differential equations (PDEs); see, e.g., [1]. The main idea of the method is to transform the anti-stable PDEs into stable target systems by using a boundary feedback control and a change of variable of the PDEs, i.e., two integral transformations, so as to eliminate the anti-stable term of the considered PDEs. The process of conducting stability analysis via this method mainly consists of two steps: (i) proving the existence of the needed transformation and its inverse in a certain function space, which are often accordant with the existence and regularity of two kernel functions, and can be proceeded in the standard way; and (ii) establishing the stability estimates for the target systems, which may be based on different methods of PDEs or functional analysis.

In the context of parabolic systems, significant progress has been obtained in exponential stabilization in $L^2$-norm or $H^1$-norm for linear parabolic PDEs by using the method of backstepping; see, e.g., [1,2,3,4,5,6,7,8,9,10,11,12,13,14]. In general, when transformations are applied, the target systems are expected to be in a simple form, whose stability can be established easily. For example, the target systems usually have the following form:

$$u_t(x,t) = u_{xx}(x,t) - \lambda u(x,t),$$

where $\lambda$ is an arbitrary constant; see, e.g., [1,2,3,4,5,6,7,8,9,10,11]. In particular, for $\lambda > 0$, the stability in different norms of (1) has been well studied, and is used together with the inverse transformations to obtain the stability in $L^2$-norm or $H^1$-norm. 

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It should be mentioned that, for parabolic PDEs with time-varying coefficients, certain Gevrey-like condition, i.e., the time-varying coefficients of the reaction terms are supposed to be in Gevrey class, is often imposed to guarantee the existence of sufficiently smooth (w.r.t. the time variable) kernel functions when the method of backstepping is applied; see, e.g., [6,9,10,11]. It is reasonable to believe that the method of backstepping along with the Gevrey-like condition can be adopted to the exponential stabilization in $L^p$-norm or $W^{1,p}$-norm, and to the establishment of continuous dependence of solutions on the initial data in $L^p$-norm or $W^{1,p}$-norm, for linear parabolic PDEs with space-time varying coefficients whenever $p \in [1, +\infty]$. Nevertheless, it is challenging to obtain sufficiently smooth (w.r.t. the time variable) kernel functions via backstepping if no Gevrey-like condition is imposed on time-varying coefficients. Indeed, for a target system chosen as (1), the equations of kernel functions are relevant to the reaction coefficients of the original systems, and the proof of existence and higher regularity of kernel functions depends strongly on the application of the property of the reaction coefficients, namely, a Gevrey-like condition seems to be needed; see, e.g., [6,9,10,11].

An alternative is to expect that the equations of kernel functions are independent of the time variable, which may extensively reduce the complications of deriving kernel functions. In this case, a target system should have the following form:

$$u_t(x,t) = u_{xx}(x,t) - \lambda(x,t)u(x,t),$$

(2)

where $\lambda(x,t) > 0$ depends on the reaction coefficients of the original system, and hence is space-time varying. However, for space-time varying $\lambda(x,t)$, it is difficult to establish the stability and the continuous dependence of solutions on the initial data involved with the spatial $L^p$-norm or $W^{1,p}$-norm for system (2) whenever $p \in [1, +\infty]$.

Let us take an example by considering system (2) that is defined over $(0,1) \times (0, +\infty)$, and with homogeneous Dirichlet boundary conditions and $L^p$-initial data $u_0$. It is well-known that the Lyapunov method is a well-suited tool for stability analysis of parabolic PDEs. In particular, for the linear system (2), the functional

$$V(u) := \int_0^1 |u(x,t)|^p dx$$

is an suitable Lyapunov candidate that can be used to establish the exponential stability in $L^p$-norm having the form

$$\|u[t]\|_{L^p(0,1)} \leq C_1 e^{\sigma t}\|u_0\|_{L^p(0,1)}, \forall t \geq 0,$$

(3)

when $p \geq 2$, where $C_1 > 0$ and $\sigma > 0$ are some constants; see, e.g., [1,15].

In the case of $p \in [1,2)$, it may be expected that the functional $V(u)$ could be used to establish the stability in $L^p$-norm of system (2). However, as indicated in [16,17], the choice of $V(u)$ may lead to singularities when $p \in [1, 2)$. Indeed,

- for $p = 1$, due to the fact that the function $g(s) := |s|$ is not differentiable at $s = 0$, the functional $\frac{d}{dt}\int_0^1 |u|^p dx$ is singular;
- for $1 < p < 2$, by integrating by parts, it holds (formally) that

$$\frac{d}{dt}\int_0^1 |u|^p dx = p\int_0^1 |u|^{p-1} \text{sgn}(u)u_t dx = p\int_0^1 |u|^{p-1} \text{sgn}(u)(u_{xx} - \lambda u) dx = -p\int_0^1 \frac{d}{du}(|u|^{p-1} \text{sgn}(u))u_x^2 dx - p\int_0^1 |u|^p dx.$$

But $\frac{d}{du}(|u|^{p-1} \text{sgn}(u))$ is singular at $u = 0$, where $\text{sgn}(\cdot)$ denotes the standard sign function.

It may be also expected that the stability in $L^p$-norm could be established by using the Sobolev embedding $L^2(0,1) \hookrightarrow L^p(0,1)$. For instance, for $p \in [1,2)$, it follows from $L^2(0,1) \hookrightarrow L^p(0,1)$ and (3) that

$$\|u[t]\|_{L^p(0,1)} \leq C_2 \|u[t]\|_{L^2(0,1)} \leq C_1 C_2 e^{\sigma t}\|u_0\|_{L^2(0,1)}, \forall t \geq 0,$$
where $C_2 > 0$ is the Sobolev embedding constant\(^1\). However, the stability estimate obtained in this way involves the $L^2$-norm of the initial data, which does not imply the continuous dependence of solutions on the $L^p$-norm of the initial data.

It is worth mentioning that, compared with the upper case of $p \in [2, +\infty]$, the intermediate case of $p \in (1, 2]$ and the lower case of $p = 1$ make the problem more meaningful. For example, many physical models lead to PDEs with $L^1$-data and should be estimated in $L^1$-norm, or need to be shown that the solutions are continuously dependent on the $L^1$-data; see, e.g., [18,19,20,21]. It is also worth noting that the study on the continuous dependence of solutions on initial data for PDEs is important not only for the theory of regularity of evolution equations, but also for the analysis of stability and robustness of PDE systems, e.g., the input-to-state stability (ISS) and relative stability; see [22] for instance, where by using the De Giorgi iteration the continuous dependence of solutions on the $L^\infty$-norm of initial data was established for a class of nonlinear parabolic PDEs with Robin boundary conditions, and used to prove the ISS in $L^\infty$-norm for the considered systems and the stability in $L^\infty$-norm for a class of cascaded nonlinear parabolic systems, respectively.

Nevertheless, due to the aforementioned difficulties, few literature concerns the stabilization and continuous dependence of solutions on initial data in the spatial $L^p$-norm (or $W^{1,p}$-norm) for anti-stable parabolic PDEs with time-varying coefficients when $p$ is restricted to be in $[1, 2]$ and smooth kernel functions are adopted without a Gevrey-like condition in backstepping. This is the main motivation of the work.

In this paper, without any Gevrey-like condition and for arbitrary $p \in [1, +\infty]$, we consider the problem of exponential stabilization and continuous dependence of solutions on initial data for a class of space-time-varying linear parabolic PDEs, and establish for solutions the estimates containing the spatial $L^p$-norm, and $W^{1,p}$-norm, of the initial data, respectively. The method adopted in this paper is based on the combination of backstepping and approximations of Lyapunov functionals (ALFs), among which the latter were used in, e.g., [16,17], to analyze stability of nonlinear PDEs with integrable inputs. More specifically,

- in order to eliminate the anti-stable term of the considered parabolic PDEs, we design a boundary feedback control by using the method of backstepping;
- in the absence of a Gevrey-like condition, we consider time independent kernel functions, and transform the original system into certain target system that has space-time-varying coefficients;
- in order to deal with singularities that may arise in the case of $p \in [1, 2)$, we apply the technique of ALFs to analyze the stability and continuous dependence of solutions on initial data in different norms for the target system whenever $p \in [1, +\infty]$.

The main contribution of this paper is to apply such a combinatorial method to establish various estimates of solutions for a class of anti-stable parabolic PDEs, and is two-fold:

- exponential stability and continuous dependence of solutions on initial data are established in different norms;
- kernel functions are allowed to be independent of the time variable, and therefore the Gevrey-like condition can be avoided.

In the rest of the paper, we introduce first some basic notations. In Section 2, we present the problem formulation, main result and technical line of the proof. In Section 3, by using the technique of ALFs, we analyze the exponential stability and continuous dependence of solutions on initial data in different norms for the target systems. In Section 4, we present a result of well-posedness and regularity of kernel functions. In Section 5, we prove the main result stated in Section 2. Some conclusions are given in Section 6.

**Notation.** Let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ be the set of nonnegative integers, positive integers, real numbers, nonnegative real numbers, and positive real numbers, respectively.

For $q \in [1, +\infty]$, the notations of $L^q(0,1)$ and $W^{1,q}(0,1)$ denote the standard Lebesgue spaces and Sobolev spaces with elements defined over $(0,1)$ and equipped with the norm $\|v\|_q := \|v\|_{L^q(0,1)}$ and $\|v\|_{1,q} := \|v\|_{W^{1,q}(0,1)}$, respectively; see, e.g., [23].

Let $L^\infty((0,1) \times \mathbb{R}_{>0}) := \{v : (0,1) \times \mathbb{R}_{>0} \to \mathbb{R} | v \text{ is measurable and satisfies } \text{ess sup}_{(y,t) \in (0,1) \times \mathbb{R}_{>0}} |v(y,t)| < +\infty\}$.

\(^1\) It can be also determined by using the Hölder’s inequality.
For $v : (0, 1) \times \mathbb{R}_{>0} \to \mathbb{R}$, the notation $v[t]$ (or $v[y]$) denotes the profile at certain $t \in \mathbb{R}_{>0}$ (or $y \in (0, 1)$), i.e., $v[t](y) = v(y, t)$ (or $v[y](t) = v(y, t)$) for all $y \in (0, 1)$ (or $t \in \mathbb{R}_{>0}$). For $T \in \mathbb{R}_{>0}$ and a Banach space $X$ equipped with the norm $\| \cdot \|_X$, let $C([0, T]; X) := \{ v : (0, 1) \times [0, T] \to \mathbb{R} | v[t] \in X \text{ and satisfies } \sup_{t \in [0, T]} \| v[t] \|_X < +\infty \}$, which is a Banach space equipped with the norm $\| v \|_{C([0, T]; X)} := \max_{t \in [0, T]} \| v[t] \|_X$.

For $Q = \mathbb{R}_{>0}$ (or $[0, 1]$), let $C^1([0, 1] \times Q) := \{ v : [0, 1] \times Q \to \mathbb{R} | v \text{ has continuous derivatives up to order 1} \}$.

For $l \in \mathbb{R}_{>0}$ and $T \in \mathbb{R}_{>0}$, the notions of $\mathcal{H}^{2+l}([0, 1])$ and $\mathcal{H}^{2+l, 1+rac{1}{2}}([0, 1] \times [0, T])$ denotes Hölder spaces that are defined by [24, Chapter I].

2 Problem Setting, Main Result and Technical Line

2.1 Problem Setting and Main Result

Given certain initial data $w_0$, we study the problem of stabilization and continuous dependence of solutions on initial data in different norms for the linear parabolic PDEs with space and time dependent coefficients:

\begin{align*}
\frac{\partial w}{\partial t}(x, t) &= w_{xx}(x, t) + c(x, t) w(x, t) + \int_0^x w(y, t) f(x, y) \, dy, \quad (x, t) \in (0, 1) \times \mathbb{R}_{>0}, \tag{4a} \\
w_x(0, t) &= 0, \quad t \in \mathbb{R}_{>0}, \tag{4b} \\
w_x(1, t) &= U(t), \quad t \in \mathbb{R}_{>0}, \tag{4c} \\
w(x, 0) &= w_0(x), \quad x \in (0, 1), \tag{4d}
\end{align*}

where $c : [0, 1] \times \mathbb{R}_{>0} \to \mathbb{R}$ and $f : [0, 1] \times [0, 1] \to \mathbb{R}$ are given functions, and $U$ is the control input to be determined to stabilize the system.

For the system (4), the reactive coefficient $c$ often represents characteristic quantities for heat exchange and reaction, e.g., [25, 26]. Moreover, if $c$ is positive and sufficiently large, the open-loop system (i.e., $U(t) = 0$) is unstable; see, e.g., [1].

Throughout this paper, we assume that

\begin{align*}
p \in [1, +\infty) & \text{ is an arbitrary constant, or } p = +\infty, \\
f \in C^1([0, 1] \times [0, 1]), c \in C^1([0, 1] \times \mathbb{R}_{>0}) \cap L^\infty((0, 1) \times \mathbb{R}_{>0}).
\end{align*}

Moreover, assume that $c$ has a form

\begin{equation*}
c(x, t) := c_1(x) + c_2(t) \tag{5}
\end{equation*}

with some functions $c_1 : [0, 1] \to \mathbb{R}$ and $c_2 : \mathbb{R}_{>0} \to \mathbb{R}$.

It is worth noting that stabilization of anti-stable parabolic PDEs under such kind of structural condition (5) was firstly considered in [11], where boundary feedback controls were designed to exponentially stabilize equation (4a) (with $f(x, t) \equiv 0$) in $L^2$ and $W^{1, 2}$ norms for $c(x, t) \equiv c_1(x)$ (see also [12] with $f(x, t) \not\equiv 0$), and for $c(x, t) \equiv c_2(t)$, respectively.

Let $\lambda_0$ be an arbitrary constant satisfying

\begin{equation*}
\lambda_0 > \sup_{(x, t) \in (0, 1) \times \mathbb{R}_{>0}} c(x, t). \tag{6}
\end{equation*}

Let $D := \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq x \leq 1 \}$ and

\begin{equation*}
\mu(x, y) := \lambda_0 - c_1(x) + c_1(y), \quad \forall (x, y) \in D. \tag{7}
\end{equation*}
In order to design a boundary feedback law to stabilize system (4) in the sense of different norms, we consider the kernel function \( k : D \rightarrow \mathbb{R} \) that is independent of the time variable \( t \) and satisfies

\[
k_{xx}(x,y) - k_{yy}(x,y) = \mu(x,y)k(x,y) + f(x,y) + \int_{y}^{x} k(x,z) f(z,y) dz, \tag{8a}
\]

\[
2 \frac{d}{dx} (k(x,x)) = \lambda_0, \tag{8b}
\]

\[
k_y(x,0) = 0, \tag{8c}
\]

\[
k(0,0) = 0, \tag{8d}
\]

where \( \frac{d}{dx} (k(x,x)) := k_{x}(x,x) + k_y(x,x) \). The existence and regularity of the kernel function \( k \) that satisfies (8) will be specified in Section 4.

Define the boundary feedback control law as

\[
U(t) := -k(1,1)w(1,t) - \int_{0}^{1} k_x(1,y)w(y,t)dy. \tag{9}
\]

Let \( \beta \in (0,1) \) be a fixed constant. Define for the initial data the set

\[
W_0 := \left\{ w \mid w \in \mathcal{H}^{2+\beta}([0,1]), w_x(0) = 0, w_x(1) = -k(1,1)w(1) - \int_{0}^{1} k_x(1,y)w(y)dy \right\}.
\]

Concerning with the well-posedness and stability in different norms of system (4), we have the following theorem, which is the main result obtained in this paper.

**Theorem 2.1** Given initial data in \( W_0 \) and considering system (4) under the feedback control law (9), the following statements hold true:

(i) system (4) admits a unique classical solution \( w \), which belongs to \( \mathcal{H}^{2+\beta,1+\frac{\beta}{2}}([0,1] \times [0,T]) \) and has the derivative \( w_{xt} \in L^2((0,1) \times (0,T)) \) for any \( T \in \mathbb{R}_{>0} \);

(ii) system (4) is exponentially stabilized in \( L^p \)-norm, and \( W^{1,p} \)-norm, having the estimates

\[
\|w[t]\|_p \leq C_1 e^{-\Delta t} \|u_0\|_p, \forall t \in \mathbb{R}_{>0}, \tag{10}
\]

and

\[
\|w[t]\|_{1,p} \leq C_2 e^{-\Delta t} \|w_0\|_{1,p}, \forall t \in \mathbb{R}_{>0}, \tag{11}
\]

respectively, where \( w_0 \in W_0 \) denotes the initial data, \( \Delta \) is a positive constant given by

\[
\Delta := \inf_{(x,t) \in (0,1) \times \mathbb{R}_{>0}} (\lambda_0 - c(x,t)), \tag{12}
\]

and \( C_1, C_2 \) are positive constants depending only on \( p, k \) and the solution \( l \) of (17) for \( p \in [1, +\infty) \), and on \( k \) and \( l \) for \( p = +\infty \), respectively;

(iii) the solution of system (4) is continuously dependent on the \( L^p \)-norm, and \( W^{1,p} \)-norm, of the initial data, having the estimates

\[
\|w_1 - w_2\|_{C((0,T];L^p(0,1))} \leq C_1 \|w_{01} - w_{02}\|_p, \forall T \in \mathbb{R}_{>0},
\]

and

\[
\|w_1 - w_2\|_{C((0,T];W^{1,p}(0,1))} \leq C_2 \|w_{01} - w_{02}\|_{1,p}, \forall T \in \mathbb{R}_{>0},
\]

respectively, where \( w_i \) denotes the solution of system (4) with the initial data \( w_{0i} \in W_0(i = 1, 2) \), and \( C_1, C_2 \) are the same as in (ii).
Remark 2.1 It is worth mentioning that under certain Gevrey-like conditions of the time-varying coefficient \( c \), e.g., with the assumption that \( c[x] \in C^\infty(\mathbb{R}_0) \) and satisfies the following inequality for some positive constant \( R \) and \( i \in \mathbb{N}_0 \):

\[
\sup_{t \in \mathbb{R}_0} |\partial_t^i c(x, t)| \leq R^{i+1} (i!)^\alpha, \forall x \in [0, 1],
\]

the exponential stabilization in \( L^2 \)-norm was studied in, e.g., [6,9,10,11], for a class of time-varying systems with different boundary feedback controls. The kernel functions obtained in the existing literature depend on the time variable and are differentiable (w.r.t. \( t \)) up to the order of \( i \). Since \( i \) can be given arbitrarily, the kernel functions can be sufficiently smooth.

In this paper, by using a combinatorial method, no Gevrey-like condition is imposed on the time-varying coefficient \( c \), and the stability is established in \( L^p \)-norm or \( W^{1,p} \)-norm for any \( p \in [1, +\infty] \). In addition, the kernel functions considered in this paper are independent of the time variable (and therefore naturally infinitely differentiable w.r.t. \( t \)), which reduce the associated computations extensively.

2.2 Technical Line

The proof of Theorem 2.1 mainly consists of 3 steps that are included in Section 4, Section 3 and Section 5, respectively.

Step 1: take into account an equivalent system by using transformations.

Indeed, in order to avoid using a Gevrey-like condition, we transform system (4) into a target system that has also a space-time-varying coefficient. More precisely, for \( \lambda_0 \) satisfying (6), let first

\[
\lambda(x, t) := \lambda_0 - c(x, t).
\]

Then using the integral transformation

\[
u(x, t) := w(x, t) + \int_0^x k(x, y) w(y, t) dy,
\]

we transform system (4) into the following target system:

\[
\begin{align}
\nu_t(x, t) &= \nu_{xx}(x, t) - \lambda(x, t) \nu(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}_0, \\
\nu_x(0, t) &= 0, \quad t \in \mathbb{R}_0, \\
\nu_x(1, t) &= 0, \quad t \in \mathbb{R}_0, \\
\nu(x, 0) &= \nu_0(x), \quad x \in (0, 1),
\end{align}
\]

where \( \nu_0(x) := w_0(x) + \int_0^x k(x, y) w_0(y) dy \).

Conversely, using the integral transformation

\[
w(x, t) := u(x, t) - \int_0^x l(x, y) u(y, t) dy,
\]

we transform system (15) into the original system (4), where \( l : D \to \mathbb{R} \) satisfies

\[
\begin{align}
l_{xx}(x, y) - l_{yy}(x, y) &= \phi(x, y) l(x, y) + f(x, y) - \int_y^x l(x, z) f(z, y) dz, \\
2 \frac{d}{dx} (l(x, x)) &= \lambda_0, \\
l_y(x, 0) &= 0, \\
l(0, 0) &= 0,
\end{align}
\]

with \( \phi(x, y) := -\lambda_0 - c(x) + c(y) \).
The equivalence of the original system (4) and system (15) can be proved in a direct way. For instance, we only show that if \( w \) is the solution of system (4), then \( u \) is the solution of system (15). Indeed, the partial derivatives of (14) w.r.t. \( x \) and \( t \) give
\[
\begin{align*}
  u_x(0, t) &= w_x(0, t) + k(0, 0)w(0, t), \\
  u_x(1, t) &= w_x(1, t) + k(1, 1)w(1, t) + \int_0^1 k_x(1, y)w(y, t)\,dy
\end{align*}
\]
and
\[
\begin{align*}
  u_t(x, t) - u_{xx}(x, t) + \lambda(x, t)u(x, t)
  &= w(x, t)
  \left(-k_y(x, x) - \frac{d}{dx}(k(x, x)) - k_x(x, x) + \lambda(x, t) + c(x, t)\right) + \int_0^x w(y, t)
  \left(k_{yy}(x, y) - k_{xx}(x, y) + \lambda(x, t)k(x, y) + c(y, t)k(x, y) + f(x, y) + \int_y^x k(z, y)\,dz\right)\,dy \\
  &\quad + k_y(x, 0)w(0, t) + \int_y^x k(x, z)\,dz\,dy + k_y(x, 0)w(0, t) \\
  &= w(x, t)
  \left(-k_y(x, x) - \frac{d}{dx}(k(x, x)) - k_x(x, x) + \lambda_0\right) + \int_0^x w(y, t)
  \left(k_{yy}(x, y) - k_{xx}(x, y) + \lambda_0 - c_1(x) - c_2(t)\right)k(x, y) + (c_1(y) + c_2(t))k(x, y) + f(x, y) + \int_y^x k(z, y)\,dz\,dy + k_y(x, 0)w(0, t) \\
  &\quad + f(x, y) + \int_y^x k(x, z)\,dz\,dy + k_y(x, 0)w(0, t) \\
  &= 0.
\end{align*}
\]
Since \( k \) satisfies (8) and the boundary control law \( U(t) \) is defined by (9), we deduce that \( u \) satisfies (15).

**Step 2:** establish stability in \( L^p \)-norm, and \( W^{1, p} \)-norm, for the target system (15) with initial data belonging to \( W_0 \) by using the technique of ALFs.

As mentioned in Introduction, singularities may appear in the Lyapunov arguments when \( p \in [1, 2] \). To overcome this obstacle, we apply the technique of ALFs that was introduced in [16, 17] for the analysis of integral input-to-state stability of nonlinear parabolic PDEs with external disturbances. Specifically, for any \( \rho \in \mathbb{R}_{>0} \), let
\[
\rho_\tau(s) := \begin{cases} 
  |s|, & |s| \geq \tau, \\
  -\frac{s^4}{8\tau^4} + \frac{3s^2}{4\tau} + \frac{3\tau}{8}, & |s| < \tau.
\end{cases}
\] (18)
that is \( C^2 \)-continuous in \( s \) and satisfies for any \( s \in \mathbb{R} \):
\[
\begin{align*}
  \rho'_\tau(0) &= 0, \quad 0 \leq |s| \leq \rho_\tau(s), \quad |\rho'_\tau(s)| \leq 1, \quad \text{(19a)} \\
  0 \leq \rho''_\tau(s) &= \begin{cases} 
  0, & |s| \geq \tau, \\
  \frac{3}{2\tau} \left(1 - \frac{s^2}{\tau^2}\right), & |s| < \tau,
\end{cases} \quad \text{(19b)} \\
  0 \leq \rho_\tau(s) - \frac{3\tau}{8} \leq \rho'_\tau(s)s \leq \rho_\tau(s) \leq |s| + \frac{3\tau}{8}. \quad \text{(19c)}
\end{align*}
\]
The second inequality of (19a) and the fourth inequality of (19c) guarantee that
\[
\int_0^1 \rho^p_\tau(v)\,dx \to \int_0^1 |v|^p\,dx \quad \text{as} \quad \tau \to 0^+,
\]
Proposition 3.1. Consider system (15) with initial data in $U_0$. The following statements hold true:

(i) system (15) admits a unique classical solution $u$, which belongs to $H^{2+\beta}([0,1]) \times H^{1+\frac{\beta}{2}}([0,1] \times [0,T])$ and has the derivative $u_t \in L^2((0,1) \times (0,T))$ for any $T > 0$;

(ii) the solution of system (15) depends continuously on the initial data $u_0 \in U_0$, $\lambda$ is a positive constant defined by (12);

(iii) the solution of system (15) is exponentially stable in $L^p$-norm, and $W^{1,p}$-norm, of the initial data, having the estimates

$$\|u(t)\|_p \leq e^{-\lambda t}\|u_0\|_p, \ \forall t \in \mathbb{R}_{>0},$$

(20)

and

$$\|u(t)\|_{1,p} \leq e^{-\lambda t}\|u_0\|_{1,p}, \ \forall t \in \mathbb{R}_{>0},$$

(21)

respectively, where $u$ denotes the solution of system (15) corresponding to the initial data $u_0 \in U_0$, $\lambda$ is a positive constant defined by (12);

Proof. The assertion (i) follows from [24, Chap. V, Theorem 7.4] and the proof of [24, Chap. V, Lemma 7.2].

We adopt the technique of ALFs to prove the assertions (ii) and (iii).

Indeed, let $u$ be the solution of system (15) with the initial data $u_0 \in U_0$. For any $\tau \in \mathbb{R}_{>0}$, let $\rho_{\tau} : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ be defined by (18).

For $p \in [1, +\infty)$, by integrating by parts, we get

$$\int_0^1 \rho_{\tau}^p(x)dx \rightarrow \int_0^1 |v_x|^pdx \text{ as } \tau \rightarrow 0^+,$$

for any $v \in W^{1,p}(0,1)$. Therefore, instead of choosing $\int_0^1 |v|^pdx$ or $\int_0^1 |v|^pdx + \int_0^1 |v_x|^pdx$ as a Lyapunov functional, we take into account the functional $\int_0^1 \rho_{\tau}^p(x)dx$ or $\int_0^1 \rho_{\tau}^p(x)dx + \int_0^1 \rho_{\tau}^p(x)dx$, to establish the stability in $L^p$-norm or $W^{1,p}$-norm for the target system (15) in the Lyapunov arguments.

Step 3: establish the exponentially stability in $L^p$-norm, and $W^{1,p}$-norm, for the original system (4) with initial data in $W_0$ by using the inverse transformation (16), the estimates of $u$, and the properties of the kernel functions $k$ and $l$.

3 Well-posedness and Stability of the Target System

In accordance with $W_0$, we define

$$U_0 := \{u | u \in H^{2+\beta}([0,1]), u_x(0) = u_x(1) = 0\}$$

for the initial data of the target system (15).

Proposition 3.1. Consider system (15) with initial data in $U_0$. The following statements hold true:

(i) system (15) admits a unique classical solution $u$, which belongs to $H^{2+\beta}([0,1]) \times H^{1+\frac{\beta}{2}}([0,1] \times [0,T])$ and has the derivative $u_t \in L^2((0,1) \times (0,T))$ for any $T > 0$;

(ii) system (15) is exponentially stable in $L^p$-norm, and $W^{1,p}$-norm, having the estimates

$$\|u(t)\|_p \leq e^{-\lambda t}\|u_0\|_p, \ \forall t \in \mathbb{R}_{>0},$$

(20)

and

$$\|u(t)\|_{1,p} \leq e^{-\lambda t}\|u_0\|_{1,p}, \ \forall t \in \mathbb{R}_{>0},$$

(21)

respectively, where $u$ denotes the solution of system (15) corresponding to the initial data $u_0 \in U_0$, $\lambda$ is a positive constant defined by (12);

(iii) the solution of system (15) depends continuously on the initial data $u_0 \in U_0$, $\lambda$ is a positive constant defined by (12);

Proof. The assertion (i) follows from [24, Chap. V, Theorem 7.4] and the proof of [24, Chap. V, Lemma 7.2].

We adopt the technique of ALFs to prove the assertions (ii) and (iii).

Indeed, let $u$ be the solution of system (15) with the initial data $u_0 \in U_0$. For any $\tau \in \mathbb{R}_{>0}$, let $\rho_{\tau} : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ be defined by (18).

For $p \in [1, +\infty)$, by integrating by parts, we get

$$\int_0^1 \frac{d}{dt} \rho_{\tau}^p(x)dx = \int_0^1 \rho_{\tau}^{p-1}(u)\rho_{\tau}'(u)u_tdx.$$
By Lemma A.1 in Appendix, for any 

Note that

Letting 

which along with (6), (12) and (13) implies

By (19c), (24) and (25), we deduce that

which is (20) in the assertion (ii) for 

Analogously, for 

we calculate

\[
\frac{1}{p} \frac{d}{dt} \int_0^1 \rho_p^p(u) \rho'(u) \, dx = \int_0^1 \rho_p^p(u) |u_x| \, dx
\]

\[
= (\rho_p^p(u) \rho'(u)) \bigg|_{x=0}^{x=1} - \int_0^1 (\rho_p^p(u) \rho'(u)) \, dx
\]

\[
= -\int_0^1 (\rho_p^p(u) \rho'(u)) \, dx
\]

\[
= -\int_0^1 \phi(u) u_x^2 \, dx + \int_0^1 (\rho_p^p(u) \rho'(u)) \, dx
\]

\[
= -\int_0^1 \phi(u) u_x^2 \, dx - \int_0^1 \rho_p^p(u) \rho'(u) \, dx.
\]
By (19c), (25) and (29), we get
\[
\frac{d}{dt} \int_0^1 \rho_t^p(u_x)dx \leq -\lambda p \int_0^1 \rho_t^p(u_x)dx + \frac{3}{8} \tau p \int_0^1 \lambda \rho_t^{p-1}(u_x)dx.
\] (30)

By (26) and (30), we have
\[
\frac{d}{dt} \left( \int_0^1 \rho_t^p(u)dx + \int_0^1 \rho_t^p(u_x)dx \right) \leq -\lambda p \int_0^1 (\rho_t^p(u) + \rho_t^p(u_x)) dx + \frac{3}{8} \tau p \int_0^1 \lambda (\rho_t^{p-1}(u) + \rho_t^{p-1}(u_x)) dx.
\] (31)

By Lemma A.1 in Appendix, for any \( t \in [0, T] \), it holds that
\[
\int_0^1 (\rho_t^p(u(x, t)) + \rho_t^p(u_x(x, t))) dx \leq e^{-\Delta t} \int_0^1 (\rho_t^p(u_0(x)) + \rho_t^p(u_x(x, 0))) dx + \tau \int_0^t e^{-\Delta p(t-s)} \psi_2(s)ds,
\] (32)

where \( \psi_2(s) := \frac{3}{8} p \int_0^1 \lambda (\rho_t^{p-1}(u(x, s)) + \rho_t^{p-1}(u_x(x, s))) dx \).

Letting \( \tau \to 0^+ \), we infer from (32) that
\[
\|u(t)\|_{1,p} \leq e^{-\Delta t} \|u_0\|_{1,p}, \forall t \in [0, T],
\] (33)

which is (21) in the assertion (ii) for \( p \in [1, +\infty) \).

Now letting \( p \to +\infty \), by the classical result of functional analysis, e.g., [27, pp.74], we deduce that (20) and (21) also hold true for \( p = +\infty \).

In order to prove the assertion (iii), letting \( v := u_1 - u_2 \) and \( v_0 := u_{01} - u_{02} \), we infer from the linearity of system (15) and the assertion (ii) that
\[
\|v(t)\|_p \leq e^{-\Delta t} \|v_0\|_p, \forall t \in \mathbb{R}_{>0},
\]

and
\[
\|v(t)\|_{1,p} \leq e^{-\Delta t} \|v_0\|_{1,p}, \forall t \in \mathbb{R}_{>0},
\]

hold true for all \( p \in [1, +\infty] \). It follows that
\[
\max_{t \in [0,T]} \|v[t]\|_p \leq \|v_0\|_p, \forall T \in \mathbb{R}_{>0},
\]

and
\[
\max_{t \in [0,T]} \|v[t]\|_{1,p} \leq \|v_0\|_{1,p}, \forall T \in \mathbb{R}_{>0}.
\]

Therefore, the assertion (iii) is true. The proof of Proposition 3.1 is complete. \( \blacksquare \)

4 Well-posedness and Regularity of Kernel Functions

In this section, we prove the existence, uniqueness and regularity of kernel functions \( k \) and \( l \) by using the method of successive approximation as in, e.g., [1,11,10]. Although the proof is standard, we would like to provide details for the completeness. In particular, as indicated later in Remark 4.1, we fix a gap that exists in the proof of uniqueness of kernel functions in the existing literature.

**Proposition 4.1** The problem (8), and (17), has a unique solution \( k \) and \( l \), respectively, both of which belong to \( \mathcal{H}^{2+\beta}(D) \).
\textbf{Proof.} Let }\xi := x + y \text{ and } \eta := x - y. \text{ It follows that } \eta \in [0, 1] \text{ and } \xi \in [\eta, 2 - \eta]. \text{ Let }

\begin{align*}
G(\xi, \eta) := k(x, y) &= k\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right).
\end{align*}

Then (8) is changed into

\begin{align*}
4G_{\xi\eta}(\xi, \eta) &= \tilde{\mu}(\xi, \eta)G(\xi, \eta) + f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right) + \int_{\xi - 2\eta}^{\xi + \eta} \tilde{G}(z, \xi, \eta)dz, \\
G(\xi, 0) &= \frac{\lambda_0}{4}\xi, \\
G_\xi(\xi, \xi) &= G_\eta(\xi, \xi), \\
G(0, 0) &= 0,
\end{align*}

where

\begin{align*}
\tilde{\mu}(\xi, \eta) := \mu(x, y) &= \lambda_0 - c_1\left(\frac{\xi + \eta}{2}\right) + c_1\left(\frac{\xi - \eta}{2}\right), \\
\tilde{G}(z, \xi, \eta) := f\left(z, \frac{\xi - \eta}{2}\right)G\left(\frac{\xi + \eta}{2} + z, \frac{\xi + \eta}{2} - z\right).
\end{align*}

Integrating (34a) w.r.t. \eta from 0 to \eta and differentiating (34b) w.r.t. \xi give

\begin{align*}
G_\xi(\xi, \eta) &= \frac{\lambda_0}{4} + \frac{1}{4} \int_0^\eta \tilde{\mu}(\xi, s)G(\xi, s)ds + \frac{1}{4} \int_0^\eta f\left(\frac{\xi + s}{2}, \frac{\xi - s}{2}\right)ds + \frac{1}{4} \int_0^\eta \int_{\xi}^{\xi + \eta - s} \tilde{G}(\tau, s, \xi)d\tau ds, \\
\end{align*}

where \(\tilde{G}(\tau, s, \xi) := f\left(\frac{\tau - s}{2}, \xi - \frac{\tau - s}{2}\right)G(\tau, s)\).

Integrating (35) w.r.t. \xi from \eta to \xi gives

\begin{align*}
G(\xi, \eta) &= G(\eta, \eta) + \frac{\lambda_0}{4}(\xi - \eta) + \frac{1}{4} \int_\eta^\xi \int_0^\eta \tilde{\mu}(\tau, s)G(\tau, s)d\tau ds + \frac{1}{4} \int_\eta^\xi \int_0^\eta f\left(\frac{\tau + s}{2}, \frac{\tau - s}{2}\right)ds d\tau \\
&\quad + \frac{1}{4} \int_\eta^\xi \int_0^\eta \int_{\xi}^{\xi + \eta - s} \tilde{G}(\tau, s, \xi)d\tau ds dz.
\end{align*}

By (34c), we have

\begin{equation}
\frac{d}{d\xi} (G(\xi, \xi)) = 2G_\xi(\xi, \xi).
\end{equation}

Substituting \(\eta = \xi\) into (35), then integrating it over \([0, \xi]\) and applying (34d) and (37), we obtain

\begin{align*}
G(\xi, \xi) &= \frac{\lambda_0}{2}\xi + \frac{1}{2} \int_0^\xi \int_0^\tau \tilde{\mu}(\tau, s)G(\tau, s)d\tau ds + \frac{1}{2} \int_0^\xi \int_0^\tau f\left(\frac{\tau + s}{2}, \frac{\tau - s}{2}\right)ds d\tau \\
&\quad + \frac{1}{2} \int_0^\xi \int_0^\tau \int_{\xi}^{\xi + \xi - s} \tilde{G}(\tau, s, \xi)d\tau ds dz,
\end{align*}

which gives

\begin{align*}
G(\eta, \eta) &= \frac{\lambda_0}{2}\eta + \frac{1}{2} \int_0^\eta \int_0^\tau \tilde{\mu}(\tau, s)G(\tau, s)d\tau ds + \frac{1}{2} \int_0^\eta \int_0^\tau f\left(\frac{\tau + s}{2}, \frac{\tau - s}{2}\right)ds d\tau \\
&\quad + \frac{1}{2} \int_0^\eta \int_0^\tau \int_{\xi}^{\xi + \xi - s} \tilde{G}(\tau, s, \xi)d\tau ds dz.
\end{align*}
We deduce by (36) and (38) that
\[
G(\xi, \eta) = \frac{\lambda_0}{4} (\xi + \eta) + \frac{1}{4} \int_0^\eta \int_0^\xi f\left(\frac{\tau + s, \tau - s}{2}\right) ds \, d\tau + \frac{1}{4} \int_0^\eta \int_0^\eta f\left(\frac{\tau + s, \tau - s}{2}\right) ds \, d\tau + \frac{1}{2} \int_0^\xi \int_0^\xi \mu(\tau, s)G(\tau, s) ds \, d\tau + \frac{1}{2} \int_0^\eta \int_0^\eta \mu(\tau, s)G(\tau, s) ds \, d\tau + \frac{1}{4} \int_0^\xi \int_0^\xi \int_0^{z+\eta-s} \hat{G}(\tau, s, z) d\tau \, dz \\
+ \frac{1}{2} \int_0^\eta \int_0^\eta \int_0^{z^{2s-s}} \tilde{G}(\tau, s, z) d\tau \, dz.
\]  
(39)

We rewrite the integral equation (39) as
\[
G(\xi, \eta) = G_0(\xi, \eta) + \Phi_G(\xi, \eta),
\]  
(40)

where we defined
\[
G_0(\xi, \eta) := \frac{\lambda_0}{4} (\xi + \eta) + \frac{1}{4} \int_0^\xi \int_0^\eta f\left(\frac{\tau + s, \tau - s}{2}\right) ds \, d\tau + \frac{1}{4} \int_0^\eta \int_0^\eta f\left(\frac{\tau + s, \tau - s}{2}\right) ds \, d\tau,
\]
and \(\Phi_G(\xi, \eta)\) via
\[
\Phi_H(\xi, \eta) := \frac{1}{4} \int_0^\xi \int_0^\eta \mu(\tau, s)G(\tau, s) ds \, d\tau + \frac{1}{2} \int_0^\xi \int_0^\xi \mu(\tau, s)G(\tau, s) ds \, d\tau + \frac{1}{4} \int_0^\xi \int_0^\eta \int_0^{z+\eta-s} \hat{H}(\tau, s, z) d\tau \, dz \\
+ \frac{1}{4} \int_0^\eta \int_0^\eta \int_0^{z^{2s-s}} \tilde{H}(\tau, s, z) d\tau \, dz,
\]
\[
\hat{H}(\xi, \eta, z) := f\left(\frac{\xi - \eta}{2}, z - \frac{\xi + \eta}{2}\right) H(\xi, \eta),
\]  
(41)

for any \(\eta \in [0, 1], \xi \in [\eta, 2 - \eta], z \in [\eta, 2 - \eta]\), and any function \(H\).

Define the sequence \(\{G_n(\xi, \eta)\}\) with \(n \in \mathbb{N}_0\) via
\[
G_{n+1}(\xi, \eta) := G_0(\xi, \eta) + \Phi_{G_n}(\xi, \eta),
\]  
(42)

and let \(\Delta G_n(\xi, \eta) := G_{n+1}(\xi, \eta) - G_n(\xi, \eta)\). It holds that
\[
\Delta G_{n+1}(\xi, \eta) = \Phi_{\Delta G_n}(\xi, \eta),
\]

and
\[
G_{n+1}(\xi, \eta) = G_0(\xi, \eta) + \sum_{j=0}^n \Delta G_j(\xi, \eta).
\]  
(43)

From (42), we see that if \(\{G_n(\xi, \eta)\}\) converges uniformly w.r.t. \((\xi, \eta)\) when \(n \to \infty\), then \(G(\xi, \eta) := \lim_{n \to \infty} G_n(\xi, \eta)\) is the solution of the integral equation (40), and therefore the existence of a kernel function \(k\) of (8) is guaranteed. In addition, in view of (43), the convergence of \(\{G_n(\xi, \eta)\}\) is equivalent to the convergence of the series \(\sum_{n=0}^\infty \Delta G_n(\xi, \eta)\). So it suffices to show that the series \(\sum_{n=0}^\infty \Delta G_n(\xi, \eta)\) is uniformly convergent w.r.t. \((\xi, \eta)\). Furthermore, recalling the Weierstrass M-test, it suffices to show that the following inequality
\[
|\Delta G_n(\xi, \eta)| \leq \frac{M^{n+2}}{(n+1)!} (\xi + \eta)^{n+1},
\]  
(44)

holds true for all \(n \in \mathbb{N}_0\), where
\[
M := \frac{\lambda_1 + \tilde{f}}{2}, \lambda_1 := \max \left\{ |\lambda_0|, \max_{(x,y) \in [0,1] \times [0,1]} |\lambda_0 - c_1(x) + c_1(y)| \right\}, \tilde{f} := \max_{(x,y) \in [0,1] \times [0,1]} |f(x,y)|.
\]
Now we prove (44) by induction. First of all, since

\[ |G_0(\xi, \eta)| \leq \left| \frac{\lambda_0}{4} (\xi + \eta) + \frac{7}{4} (\xi \eta - \eta^2) + \frac{1}{4} \eta^2 \right| \]
\[ \leq \left| \frac{\lambda_0}{4} (\xi + \eta) + \frac{7}{4} \xi \eta \right| \]
\[ \leq \frac{\lambda_0}{4} \times 2 + \frac{7}{4} \times 2 \times 1 \]
\[ \leq M, \]

it follows that

\[ |\Delta G_0(\xi, \eta)| = |\Phi G_0(\xi, \eta)| \]
\[ \leq \frac{\lambda_1}{4} \int_\eta^\xi \int_0^\eta M ds d\tau + \frac{\lambda_1}{4} \int_0^\eta \int_0^\tau M ds d\tau + \frac{7}{4} \int_\eta^\xi \int_0^\eta \int_z^z + \eta^s - \mu d\tau d\sigma d\tau \]
\[ = M \frac{\lambda_1}{4} (\xi \eta - \eta^2) + M \frac{\lambda_1}{4} \eta^2 + M \frac{7}{8} (\xi \eta^2 - \eta^3) + M \frac{7}{12} \eta^3 \]
\[ \leq M \left( \frac{\lambda_1}{4} \xi \times 1 + \frac{7}{8} \times 2 \times 1 \times \eta - 0 \right) \]
\[ \leq M \left( \frac{\lambda_1}{4} + \frac{7}{4} \right) (\xi + \eta) \]
\[ \leq M^2 (\xi + \eta), \quad (45) \]

which shows that (44) holds for \( n = 0 \).

Supposing that the inequality (44) holds true for a general \( n \in \mathbb{N} \), we need to show that (44) also holds true for \( n + 1 \). Indeed, we get

\[ |\Delta G_{n+1}(\xi, \eta)| = |\Phi \Delta G_n(\xi, \eta)| = |I_1 + I_2 + I_3 + I_4|, \quad (46) \]

where

\[ I_1 := \frac{1}{4} \int_\eta^\xi \int_0^\eta \tilde{\mu}(\tau, s) \Delta G_n(\tau, s) ds d\tau, \]
\[ I_2 := \frac{1}{2} \int_0^\eta \int_0^\tau \tilde{\mu}(\tau, s) \Delta G_n(\tau, s) ds d\tau, \]
\[ I_3 := \frac{1}{4} \int_\eta^\xi \int_0^\eta \int_z^z + \eta^s - \mu \tilde{\Delta G}_n(\tau, s, z) ds d\tau dz, \]
\[ I_4 := \frac{1}{2} \int_0^\eta \int_0^\tau \int_z^z + \eta^s - \mu \tilde{\Delta G}_n(\tau, s, z) ds d\tau dz, \]

with \( \tilde{\Delta G}_n \) defined via (41).

Since (44) holds true for \( n \), it holds that

\[ |I_1| \leq \frac{\lambda_1 M^{n+2}}{4(n+1)!} \int_\eta^\xi \int_0^\eta (\tau + s)^{n+1} ds d\tau \]
\[ = \frac{\lambda_1 M^{n+2}}{4(n+1)!} \frac{1}{n+2} \int_\eta^\xi ((\tau + \eta)^{n+2} - \tau^{n+2}) d\tau \]
suffices to show that any twice continuously differentiable solution of (39) can be approximated by the sequence \( k \) admit a solution with arbitrary exponent \( 0 < \beta < 1 \), we conclude that the series \( \sum_{n=0}^{\infty} G_n(\xi, \eta) \) converges absolutely and uniformly in \( \eta \in [0, 1] \) and \( \xi \in [\eta, 2 - \eta] \), and \( G(\xi, \eta) := \lim_{n \to \infty} G_n(\xi, \eta) \) is a solution of (40). Moreover, \( G(\xi, \eta) \) is continuous in \( \xi \) and \( \eta \). Furthermore, in view of (39) with \( f \in C^1([0, 1] \times [0, 1]) \) and \( c_1 \in C^1([0, 1]) \), we deduce that \( G_{\xi\xi}, G_{\eta\eta}, G_{\xi\eta} \) exist and are Hölder continuous with an arbitrary exponent \( \beta' \in (0, 1) \). In particular, \( G_{\xi\xi}, G_{\eta\eta}, G_{\xi\eta} \) are Hölder continuous with the exponent \( \beta \). Therefore, the problem (8) admits a solution \( k \in H^{2+\beta}(D) \).

Then we get

\[
|\Delta G_{n+1}(\xi, \eta)| \leq (|I_1| + |I_2|) + (|I_3| + |I_4|) \\
\leq \frac{\lambda_1 M^{n+2}}{4(n+1)!(n+2)} \int_0^\infty (\xi + \eta)^{n+3} d\tau + \frac{\lambda_1 M^{n+2}}{2(n+1)!(n+2)(n+3)} (\xi + \eta)^{n+3} \\
\leq \left( \frac{\lambda_1}{2(n+3)} + \frac{T}{(n+2)!} \right) M^{n+2} (\xi + \eta)^{n+2} \\
\leq \frac{M^{n+3}}{(n+2)!} (\xi + \eta)^{n+2},
\]

which implies that (44) holds for \( n + 1 \). Therefore, (44) holds true for \( n \in \mathbb{N}_0 \).

We conclude that the series \( \sum_{n=0}^{\infty} \Delta G_n(\xi, \eta) \) converges absolutely and uniformly in \( \eta \in [0, 1] \) and \( \xi \in [\eta, 2 - \eta] \), and \( G(\xi, \eta) := \lim_{n \to \infty} G_n(\xi, \eta) \) is a solution of (40). Moreover, \( G(\xi, \eta) \) is continuous in \( \xi \) and \( \eta \). Furthermore, in view of (39) with \( f \in C^1([0, 1] \times [0, 1]) \) and \( c_1 \in C^1([0, 1]) \), we deduce that \( G_{\xi\xi}, G_{\eta\eta}, G_{\xi\eta} \) exist and are Hölder continuous with an arbitrary exponent \( \beta' \in (0, 1) \). In particular, \( G_{\xi\xi}, G_{\eta\eta}, G_{\xi\eta} \) are Hölder continuous with the exponent \( \beta \). Therefore, the problem (8) admits a solution \( k \in H^{2+\beta}(D) \).

Regarding the uniqueness of solution of (8), which is equivalent to the uniqueness of solution of the integral equation (39), it suffices to show that any twice continuously differentiable solution of (39) can be approximated by the sequence \( \{G_n(\xi, \eta)\} \).
constructed via (42). More precisely, for any twice continuously differentiable solution \( \overline{G}(\xi, \eta) \) of (39), we intend to show that the estimate

\[
|G_n(\xi, \eta) - \overline{G}(\xi, \eta)| \leq \frac{LM^n}{n!} (\xi + \eta)^n,
\]

holds true for all \( n \in \mathbb{N}_0 \), where

\[
L := \max_{\eta \in [0,1], \xi \in [\eta, 2-\eta]} |\Phi_{\overline{G}}(\xi, \eta)|.
\]

This is because (47) implies the uniform convergence of \( \{G_n(\xi, \eta)\} \) to \( \overline{G}(\xi, \eta) \), while the uniqueness of \( G(\xi, \eta) := \lim_{n \to \infty} G_n(\xi, \eta) \) leads to \( \overline{G}(\xi, \eta) \equiv G(\xi, \eta) \). So the solution of (39) is unique, and therefore the solution of (39) is also unique.

Now we prove (47) by induction. First, since \( \overline{G}(\xi, \eta) \) is a solution (39), using the definition of \( \Phi_{\overline{G}} \), we obtain

\[
|G_0(\xi, \eta) - \overline{G}(\xi, \eta)| = |G_0(\xi, \eta) - (G_0(\xi, \eta) + \Phi_{\overline{G}}(\xi, \eta))| = |\Phi_{\overline{G}}(\xi, \eta)| \leq L,
\]

which shows that (47) holds true for \( n = 0 \).

Suppose that (47) holds true for a general \( n \in \mathbb{N} \). Then by this assumption and the linearity of \( \Phi_H(\xi, \eta) \) w.r.t. \( H \), we have

\[
|G_{n+1}(\xi, \eta) - \overline{G}(\xi, \eta)| = |\Phi_{G_n}(\xi, \eta) - \Phi_{\overline{G}}(\xi, \eta)| = |\Phi_{\overline{G}}(\xi, \eta)| \leq I_5 + I_6 + I_7 + I_8,
\]

where

\[
I_5 := L \frac{\lambda_1 M^n}{4n!} \int_\eta^\xi \int_0^n (\tau + s)^n ds d\tau,
I_6 := L \frac{\lambda_1 M^n}{2n!} \int_\eta^n \int_0^\tau (\tau + s)^n ds d\tau,
I_7 := L \frac{\int M^n}{4n!} \int_\eta^\xi \int_0^n \int_z^{z+s-n} (\tau + s)^n d\tau ds dz,
I_8 := L \frac{\int M^n}{2n!} \int_0^n \int_0^z \int_z^{2z-s} (\tau + s)^n d\tau ds dz.
\]

Analogous to the estimates of \( I_1, I_2, I_3 \) and \( I_4 \), it holds that

\[
I_5 \leq L \frac{\lambda_1 M^n}{4n!} \frac{1}{(n+1)(n+2)} ((\xi + \eta)^{n+2} - (2\eta)^{n+2}),
I_6 \leq L \frac{\lambda_1 M^n}{4n!} \frac{1}{(n+1)(n+2)} (2\eta)^{n+2},
I_7 \leq L \frac{\int M^n}{4n!} \frac{1}{(n+1)(n+2)} ((\xi + \eta)^{n+2} - (2\eta)^{n+2}),
I_8 \leq L \frac{\int M^n}{2n!} \frac{1}{(n+1)(n+2)} (2\eta)^{n+2}.
\]

Then we obtain

\[
|G_{n+1}(\xi, \eta) - \overline{G}(\xi, \eta)| \leq L \left( \frac{\lambda_1}{2(n+2)} + \frac{\int}{n+2} \right) \frac{M^n}{(n+1)!} (\xi + \eta)^{n+1} \leq \frac{LM^{n+1}}{(n+1)!} (\xi + \eta)^{n+1},
\]

which implies that (47) holds true for \( n + 1 \). Thus, (47) holds true for all \( n \in \mathbb{N}_0 \). We conclude that the solution of (8) is unique.

Finally, by changing the variables and using the method of successive approximation as above, we may prove the well-posedness and regularity of the kernel function \( l \) for (17).
Remark 4.1 For the stabilization problem of linear parabolic PDEs with time-independent coefficients, the “uniqueness of kernel functions” was claimed in [12] and proved by estimating

$$|\Delta G(\xi, \eta)| := |G'(\xi, \eta) - G''(\xi, \eta)| \leq \frac{C}{n!}(\xi + \eta)^n$$  \hspace{1cm} (48)

with some constant $C > 0$, where $G', G''$ are two kernel functions; see (39) and (40) of [12].

It is worth noting that a gap exists in the proof of [12]. On one hand, if $G'$ and $G''$ are given by (see (39) of [12])

$$G'(\xi, \eta) = \sum_{n=0}^{\infty} G_n(\xi, \eta), \quad G''(\xi, \eta) = \sum_{n=0}^{\infty} G_n(\xi, \eta),$$  \hspace{1cm} (49)

as indicated in [12], the estimate (48) can be obtained in the same way as in (38) of [12]. Therefore the equality of $G' = G''$ can be concluded by letting $n \to \infty$. However, since the limit of $\sum_{n=0}^{\infty} G_n(\xi, \eta)$ is unique, which is known by the limit theory, due to (49), the equality of $G' = G''$ holds true trivially. So the proof presented in [12] is essentially for the uniqueness of the limit of $\{G_n\}$ rather than the uniqueness of solution.

On the other hand, if $G'$ and $G''$ are not given by (49), without imposing more conditions, the estimate (48) can not be obtained in the same way as in (38) of [12].

In order to address the above-mentioned issue, as presented in this paper, a feasible idea of proving the uniqueness of kernel functions is to prove that all solutions of integral equations (e.g., (39) or (40)) can be approximated by the sequence $\{G_n\}$ constructed. That is to say, $G'$ and $G''$ in [12] have to be with a form of (49). Then the uniqueness of solution is determined by the uniqueness of the limit of $\{G_n\}$. Note that, by using this method, the estimate considered in this paper (see (47)) is different from (48). In particular, (47) implies (48).

Remark 4.2 It is worth noting that for general functions $f$ and $c$ it is impossible to obtain an analytic solution $G$ of the integral equation (39) by using the iterating formula (43) or (42). In some special cases, the solution $G$ of (39) can be given by a refined iteration formula. For example, consider $f \equiv 0$ and $c_1(x) = rx^2$ with a nonnegative constant $r$. The integral equation (39) becomes

$$G(\xi, \eta) = \frac{\lambda_0}{4} (\xi + \eta) + \frac{1}{4} \int_{\eta}^{\xi} \int_{0}^{\eta} (\lambda_0 - r\tau s)G(\tau, s)d\tau ds + \frac{1}{2} \int_{0}^{\eta} \int_{0}^{\tau} (\lambda_0 - r\tau s)G(\tau, s)d\tau ds,$$  \hspace{1cm} (50)

whose solution is uniquely determined by

$$G(\xi, \eta) = \frac{\lambda_0}{4} (\xi + \eta) + \frac{\lambda_0}{4} \sum_{n=1}^{\infty} \sum_{i=0}^{n} \left( \frac{1}{4} \right)^n \lambda_0^{-A_{n+1-i}} T_{2n-1}(\xi, \eta),$$  \hspace{1cm} (51)

where

$$A_{n+1-i} := \begin{cases} (-r)^n \Pi_{m \geq 1}^n C_{2m}, & i = 0, \\ (A_{n-i}^{n-1} + A_{n+1-i}^{n-1}) C_{2n-i}, & 0 < i < n, \\ \Pi_{m \geq 1}^n C_{m}, & i = n, \end{cases}$$

$$C_n := \frac{1}{n(n+1)},$$

$$T_{2n-1}(\xi, \eta) := \xi^{2n+1-i} \eta^{2n-1} + \xi^{2n-i} \eta^{2n+1-i}.$$  

The proof is provided in Appendix.

5 Well-posedness and Stability of the Original System

In this section, we prove the well-posedness and exponential stability in different norms for the original system, i.e., system (4).
Proof of Theorem 2.1. The assertion of Theorem 2.1 (i) follows immediately from the inverse transformation (16), Proposition 4.1 and Proposition 3.1 (i).

For the proof of Theorem 2.1 (ii), we define first the following qualities:

\[
\begin{align*}
\alpha_1 &:= \max_{(x,y) \in D} |k(x,y)|, \\ \beta_1 &:= \max_{(x,y) \in D} |l(x,y)|, \\ \alpha_2 &:= \max_{x \in [0,1]} |k(x,x)|, \\ \beta_2 &:= \max_{x \in [0,1]} |l(x,x)|, \\ \alpha_3 &:= \max_{(x,y) \in D} |k_x(x,y)|, \\ \beta_3 &:= \max_{(x,y) \in D} |l_x(x,y)|.
\end{align*}
\]

For \( p \in [1, +\infty) \), we infer from (16) and the Cauchy-Schwartz’ inequality that

\[
\|w[t]\|_p^p \leq 2^{p-1} (1 + \beta_1^p) \|u[t]\|_p^p.
\]  
(52)

Note that

\[
u_0(x) = w_0(x) + \int_0^x k(x,y) w_0(y) dy,
\]

which implies

\[
\begin{align*}
\|u_0\|_p^p &\leq 2^{p-1} (1 + \alpha_1^p) \|w_0\|_p^p, \\
\|u_{0x}\|_p^p &\leq 3^{p-1} (\alpha_2^p + \alpha_3^p) \|w_0\|_p^p + 3^{p-1} \|w_{0x}\|_p^p.
\end{align*}
\]
(53) (54)

By (20), (52) and (53), we obtain the exponential stability of system (4) in \( L^p \)-norm for \( p \in [1, +\infty) \):

\[
\|w[t]\|_p \leq C_1 e^{-\Delta t} \|w_0\|_p, \quad \forall t \in \mathbb{R}_{>0},
\]  
(55)

where \( C_1 := (4^{p-1} (1 + \alpha_1^p)(1 + \beta_1^p))^{\frac{1}{p}} \).

Now differentiating two-hand sides of (16) w.r.t. \( x \), taking \( L^p \)-norm, and using the Cauchy-Schwartz’ inequality, we have

\[
\|u_x[t]\|_p^p \leq 3^{p-1} \|u_x[t]\|_p^p + 3^{p-1} (\beta_2^p + \beta_3^p) \|u[t]\|_p^p,
\]

which along with (21), (54) and (55) implies the exponential stability of system (4) in \( W^{1,p} \)-norm for \( p \in [1, +\infty) \):

\[
\|w[t]\|_{1,p} \leq C_2 e^{-\Delta t} \|w_0\|_{1,p},
\]  
(56)

where \( C_2 := \max \left\{ 9^{p-1} \gamma_1^p : \gamma_2^p \right\} \) with

\[
\begin{align*}
\gamma_1 &:= \max \{1, \beta_2^p + \beta_3^p\}, \\
\gamma_2 &:= C_1^p + 9^{p-1} \gamma_1 (1 + \alpha_1^p + \alpha_2^p + \alpha_3^p).
\end{align*}
\]

Furthermore, letting \( p \to +\infty \) in (55) and (56), we obtain the exponential stability of system (4) in \( L^\infty \)-norm and \( W^{1,\infty} \)-norm, respectively:

\[
\begin{align*}
\|w[t]\|_{\infty} &\leq C_3 e^{-\Delta t} \|w_0\|_{\infty}, \quad \forall t \in \mathbb{R}_{>0}, \\
\|w[t]\|_{1,\infty} &\leq C_4 e^{-\Delta t} \|w_0\|_{1,\infty}, \quad \forall t \in \mathbb{R}_{>0},
\end{align*}
\]

where \( C_3 := 4 \gamma_3 \) and \( C_4 := \max \{9 \gamma_4, C_1 + 9 \gamma_4 (1 + \alpha_1 + \alpha_2 + \alpha_3)\} \) with

\[
\gamma_3 := \begin{cases} 
1, & 0 < \alpha_1 \leq 1, 0 < \beta_1 \leq 1, \\
\beta_1, & 0 < \alpha_1 \leq 1, \beta_1 > 1, \\
\alpha_1, & \alpha_1 > 1, 0 < \beta_1 \leq 1, \\
\alpha_1 \beta_1, & \alpha_1 > 1, \beta_1 > 1,
\end{cases}
\]
\[\gamma_4 := \max\{1, \beta_2 + \beta_3\}.\]

Therefore, the assertion of Theorem 2.1 (ii) holds true.

Let \(w_1\) and \(w_2\) be the solution of system (4) with initial data \(w_{01} \in W_0\) and \(w_{02} \in W_0\), respectively. Using the linearity of system (4) and Theorem 2.1 (ii), the estimates (10) and (11) are satisfied with \(w := w_1 - w_2\), which imply the assertion of Theorem 2.1 (iii). The proof is complete. ■

6 Conclusion

In this paper, a combinatorial method, i.e., the method of backstepping and the technique of ALFs, was applied to address the exponential stabilization and continuous dependence of solutions on initial data for a class of 1-D space-time-varying linear parabolic PDEs without imposing a Gevrey-like condition on the time-varying coefficient. Compared to the results of the existing literature, the kernel functions obtained via the proposed method are independent of the time variable, therefore the associated calculations have been extensively simplified. In addition, exponential stability and continuous dependence of solutions on initial data could be established in different norms (\(L^p\) and \(W^{1,p}\) norms) by using the technique of ALFs, which was used to deal with singularities in the case of \(p \in [1, 2)\).

It is worth noting that, with a Gevrey-like condition, the proposed method is still suitable to address the exponential stabilization and continuous dependence of solutions on initial data in different norms for 1-D linear parabolic PDEs having a general form, e.g., \(c(x, t)\) need not to be divided into two terms as (5). However, without involving a Gevrey-like condition, it is still challenging to apply the method to obtain time-independent kernel functions and establish the stability estimates in different norms for parabolic PDEs having a general form, which will be studied in our future work.

A Appendix

A.1. The Bellman-Gronwall-Peano’s inequality

The following lemma used in this paper is concerned with the Bellman-Gronwall-Peano’s inequality, which can be found in, e.g., [28, pp. 94].

Lemma A.1 Let \(T \in \mathbb{R}_{>0}\), if the function \(z\) is a nonnegative, absolutely continuous function on \([0, T]\), and satisfies the inequality

\[\frac{dz}{dt} \leq q(t)z(t) + h(t) \text{ a.e. in } [0, T],\]

where \(q, h \in L^1(0, T)\), then

\[z(t) \leq e^{\int_0^t q(s)ds} z(0) + \int_0^t e^{\int_s^t q(\tau)d\tau} h(s)ds, \forall t \in [0, T].\]

A.2. Proof of (52) in Remark 4.2

We apply a new approximation to obtain (52), which is the solution of the integral equation (50).

Let \(G_0(\xi, \eta) := \frac{\lambda_0}{4} (\xi + \eta),\) and

\[G_{n+1}(\xi, \eta) := \frac{1}{4} \int_0^\xi \int_0^\eta (\lambda_0 - \tau s)G_n(\tau, s)d\tau d\sigma + \frac{1}{2} \int_0^\eta \int_0^\tau (\lambda_0 - \tau s)G_n(\tau, s)d\tau d\sigma.\]

According to the proof of Proposition 4.1, the solution of (50) exists and is unique. Therefore, as long as the series \(\sum_{n=0}^{\infty} G_n(\xi, \eta)\) converges, \(G(\xi, \eta) := \sum_{n=0}^{\infty} G_n(\xi, \eta)\) is the unique solution of (50).
Now we apply the induction to prove the convergence of \( \sum_{n=0}^{\infty} G_n(\xi, \eta) \) and the fact that

\[
G_n(\xi, \eta) = \frac{\lambda_0}{4} \left( \frac{1}{4} \right)^n \sum_{i=0}^{n} \lambda_0^i A_{n+1-i} T_{2n-i}(\xi, \eta).
\]  
(A.1)

Regarding the convergence of \( \sum_{n=0}^{\infty} G_n(\xi, \eta) \), it suffices to show that

\[
|G_n(\xi, \eta)| \leq \frac{M_1^{n+1}}{(n+1)!} \left( \xi^{n+1+1} + \xi^2 + \eta^{n+2} \right), \quad \forall n \in \mathbb{N}_0,
\]  
(A.2)

where \( M_1 := r + |\lambda_0| \). Firstly, the following result can be established:

\[
|G_0(\xi, \eta)| \leq M_1 (\xi + \eta),
\]

which shows that (A.2) holds true for \( n = 0 \). The next step is to prove that if (A.2) holds true for a general \( n \in \mathbb{N} \), then (A.2) also holds true for \( n + 1 \). Indeed, by direct computations, we have

\[
\begin{align*}
|G_{n+1}(\xi, \eta)| &\leq \frac{1}{4} \int_0^\xi \int_0^\eta \left( (\lambda_0 + r \tau s) G_n(\tau, s) \right) ds d\tau + \frac{1}{2} \int_0^\tau \left( (\lambda_0 + r \tau s) G_n(\tau, s) \right) ds d\tau \\
&\leq \frac{1}{4} \frac{M_1^{n+1}}{(n+1)!} \frac{1}{n+2} \left( \frac{1}{n+1} + \frac{r \xi \eta}{n+3} \right) \left( \xi^{n+2} + \eta^{n+2} \right) \\
&\leq \frac{M_1^{n+2}}{(n+2)!} \left( \xi^{n+2} + \eta^{n+2} \right),
\end{align*}
\]

which implies that (A.2) holds for \( n + 1 \). Thus, (A.2) holds true for all \( n \in \mathbb{N}_0 \). Therefore, \( \sum_{n=0}^{\infty} G_n(\xi, \eta) \) converges, and \( G := \sum_{n=0}^{\infty} G_n(\xi, \eta) \) is the solution of (50).

We need to prove (A.1). Note that

\[
\begin{align*}
\int_0^\xi \int_0^\eta \left( \tau^{n+1} + \tau^n s^{n+1} \right) ds d\tau &+ 2 \int_0^\eta \int_0^\tau \left( \tau^{n+1} s^n + \tau^n s^{n+1} \right) ds d\tau = \frac{1}{(n+1)(n+2)} \xi^{n+1} \eta^{n+1} (\xi + \eta).
\end{align*}
\]  
(A.3)

For any function \( \nu \), define \( K \) via

\[
K(\nu(\tau, s)) := \int_0^\xi \int_0^\eta \nu(\tau, s) ds d\tau + 2 \int_0^\eta \int_0^\tau \nu(\tau, s) ds d\tau.
\]

By (A.3), we have

\[
K(T_n(\tau, s)) = C_{n+1} T_{n+1}(\xi, \eta),
\]  
(A.4)

where

\[
T_n(\tau, s) := \tau^{n+1} s^n + \tau^n s^{n+1},
\]

\[
C_{n+1} := \frac{1}{(n+1)(n+2)}.
\]

Then, it holds that

\[
\begin{align*}
K(\lambda_0 T_n(\tau, s)) &= \lambda_0 C_{n+1} T_{n+1}(\xi, \eta), \\
K(\tau s T_n(\tau, s)) &= C_{n+2} T_{n+2}(\xi, \eta).
\end{align*}
\]  
(A.5)

(A.6)
By (A.4), (A.5) and (A.6), we obtain
\[
G_1(\xi, \eta) = \frac{\lambda_0}{4} \frac{1}{4} K((\lambda_0 - r\tau s)T_0(\tau, s)) = \frac{\lambda_0}{4} \frac{1}{4} (\lambda_0 C_1 T_1(\xi, \eta) - rC_2 T_2(\xi, \eta))
\]
which implies that (A.1) holds true for \( n = 1 \).

Supposing that (A.1) holds true for 1, 2, \ldots, \( n \), it follows that
\[
G_{n+1}(\xi, \eta) = \frac{1}{4} K(\lambda_0 - r\tau s) \left( \sum_{i=0}^{n} \frac{\lambda_0}{4} \lambda_0^{n+1} A_{n+1-i}^{n+1} T_{2n-i}(\xi, \eta) \right)
\]
\[
= \frac{\lambda_0}{4} \frac{1}{4} \left( \sum_{i=0}^{n} \frac{\lambda_0}{4} A_{n+1-i}^{n+1} C_{2n+1-i} T_{2n+1-i}(\xi, \eta) - \sum_{i=0}^{n} r \lambda_0^{n+1} A_{n+1-i}^{n+1} C_{2n+2-i} T_{2n+2-i}(\xi, \eta) \right)
\]
\[
= \frac{\lambda_0}{4} \frac{1}{4} \left( \sum_{i=0}^{n} \lambda_0^{n+1} A_{n+1-i}^{n+1} C_{2n+1-i} T_{2n+1-i}(\xi, \eta) \right)
\]
\[
+ \lambda_0^{n+1} A_{n+1}^{n+1} C_{n+1} T_{n+1}(\xi, \eta) - r A_{n+1}^{n+1} C_{2n+2} T_{2n+2}(\xi, \eta) + \sum_{i=1}^{n} \lambda_0^{n+1} A_{n+1-i}^{n+1} C_{2n+2-i} T_{2n+2-i}(\xi, \eta)
\]
\[
= \frac{\lambda_0}{4} \frac{1}{4} \left( \sum_{i=0}^{n} \lambda_0^{n+1} A_{n+1-i}^{n+1} C_{2n+1-i} T_{2n+1-i}(\xi, \eta) \right)
\]
which implies that (A.1) holds true for \( n + 1 \). Therefore, by induction, (A.1) holds true for all \( n \geq 1 \). The proof is complete.

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