Abstract

A subgroup $H$ of a finite group $G$ is wide if each prime divisor of the order of $G$ divides the order of $H$. We obtain the description of finite soluble groups with no wide subgroups. We also prove that a finite soluble group with nilpotent wide subgroups has the quotient group by its hypercenter with no wide subgroups.

Keywords: finite groups, soluble groups, nilpotent groups.

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1 Introduction

All groups in this paper are finite. All notations and terminology are standard. The reader is referred to [1, 2] if necessary.

Let $G$ be a group. We use $\pi(G)$ to denote the set of all prime divisors of $|G|$. By $|\pi(G)|$ we denote a number of different prime divisors of $|G|$. A subgroup $H$ of a group $G$ is said to be wide if $\pi(H) = \pi(G)$.

Let $k$ be a positive integer. A group $G$ is called $k$-primary if $|\pi(G)| = k$. If $|\pi(G)| = 1$ or $|\pi(G)| = 2$, then $G$ is said to be primary or biprimary, respectively. A group $G$ is quasi-$k$-primary if $|\pi(G)| > k$ and $|\pi(M)| \leq k$ for every maximal subgroup $M$ of $G$. Obviously, quasi-$k$-primary groups have no wide subgroups.

A quasi-1-primary group is also called quasiprimary, and a quasi-2-primary group is also called quasibiprimary [3]. It is clear that the order of a nilpotent quasiprimary group is equal to $pq$, where $p$ and $q$ are primes. A nonnilpotent quasiprimary group $G$ can be represented as the semidirect product $G = [E_{p^a}]Z_q$ of a normal elementary abelian group $E_{p^a}$, $|E_{p^a}| = p^a$, by a cyclic group $Z_q$, $|Z_q| = q$, where $a$ is the exponent of $p$. This follows from Schmidt theorem [4] on groups all of whose subgroups are nilpotent.
S. S. Levischenko investigated quasibiprimary groups [3]. A soluble quasibiprimary group $G$ can be represented as the semidirect product $[P]M$ of an elementary abelian Sylow $p$-subgroup $P$ and a quasiprimary subgroup $M$, which is a maximal subgroup of $G$ [3, Theorem 3.1]. In an insoluble quasibiprimary group $G$ the Frattini subgroup $\Phi(G)$ is primary [3, Theorem 2.2], the quotient group $G/\Phi(G)$ is a simple group, and all such simple groups are enumerated [3, Theorem 2.1].

It is natural to study the structure of quasi-$k$-primary groups for any positive integer $k$. Most of simple groups is quasi-$k$-primary. Simple groups that are not quasi-$k$-primary are enumerated in [5, 3.8]. It is satisfied the question of V. S. Monakhov in Kourovka notebook [6, 11.64].

We obtain the description of soluble quasi-$k$-primary groups and soluble groups with nilpotent wide subgroups.

2 Preliminaries

**Lemma 1.** A soluble quasi-$k$-primary group is $(k + 1)$-primary.

**Proof.** Let $G$ be a soluble quasi-$k$-primary group and $M$ be a maximal subgroup of $G$. Then $|\pi(G)| \geq k + 1$ and $|\pi(M)| \leq k$. In a soluble group maximal subgroups have prime indices. Therefore

$$|G : M| = p^\alpha, \ p \in \pi(G), \ \alpha \in \mathbb{N}.$$ 

Since $|G| = |M| \cdot |G : M|$, we have $|\pi(G)| \leq k + 1$, and $|\pi(G)| = k + 1$. Lemma 1 is proved.

**Lemma 2.** If $G$ is a soluble quasi-$k$-primary group and $N$ is its normal Hall subgroup, then $G/N$ is a quasi-$l$-primary group, where $l = k - |\pi(N)|$.

**Proof.** By Lemma 1, $G$ is $(k + 1)$-primary, hence

$$|\pi(G/N)| = |\pi(G)| - |\pi(N)| = k + 1 - |\pi(N)| = l + 1.$$ 

Let $M/N$ be a maximal subgroup of $G/N$. Then $M$ is a maximal subgroup of $G$. Therefore $|\pi(M)| < |\pi(G)|$. As $N$ is a Hall subgroup of $G$, we obtain

$$|\pi(M/N)| = |\pi(M)| - |\pi(N)| < |\pi(G)| - |\pi(N)| = |\pi(G/N)| = l + 1.$$ 

Thus $G/N$ is $(l + 1)$-primary, and any maximal subgroup of $G$ is not more than $l$-primary. Consequently, $G/N$ is quasi-$l$-primary. Lemma 2 is proved.

**Lemma 3.** ([2, IV.2.11]) If all Sylow subgroups of a group $G$ are cyclic, then the derived subgroup $G'$ is a cyclic Hall subgroup and the quotient group $G/G'$ is cyclic.
3 The structure of soluble quasi-\( k \)-primary groups

**Theorem 1.** Let \( G \) be a soluble group. Then the following statements are equivalent.

1. \( G \) is a quasi-\( k \)-primary group.
2. Every normal subgroup of \( G \) is a Hall subgroup.
3. Every maximal subgroup of \( G \) is a Hall subgroup.
4. \( G = [N]M \), where \( N \) is a minimal normal and Sylow subgroup of \( G \), \( M \) is a quasi-(\( k-1 \))-primary and maximal subgroup.

**Proof.** Check that (1) implies (2). Let \( G \) be a soluble quasi-\( k \)-primary group and \( N \) be a normal subgroup of \( G \), \( \tau = \pi(G) \setminus \pi(N) \). Suppose that \( N \) is not a Hall subgroup of \( G \). Since \( G \) is a quasi-\( k \)-primary group and \( N \) is its proper subgroup, we have \( |\pi(N)| \leq k \) and \( \tau \neq \emptyset \).

As \( G \) is a soluble group, in \( G \) there is \( \tau \)-Hall subgroup \( M \). Now, \( (|M|, |N|) = 1 \), so \( M \cap N = 1 \) and \( NM = [N]M < G \). But

\[ \pi([N]M) = \pi(N) \cup \pi(M) = \pi(N) \cup \tau = \pi(G), \quad |\pi(G)| = k + 1, \]

that is, in a quasi-\( k \)-primary group \( G \) there is a proper \((k + 1)\)-primary subgroup \([N]M\), a contradiction. Thus \( N \) is a Hall subgroup of \( G \).

Now we prove that (3) follows from (2). Assume that in a soluble group \( G \) every normal subgroup is a Hall subgroup. Let \( M \) be a maximal subgroup and \( N \) a minimal normal subgroup of \( G \). Then \( N \) is a Sylow subgroup of \( G \). If \( N \) does not belong to \( M \), then \( G = NM \). And since \( N \) is abelian, \( N \cap M = 1 \) and subgroup \( M \) is a Hall subgroup. Let \( N \subseteq M \). Then \( M/N \) is a maximal subgroup of \( G/N \). It is clear that every normal subgroup of \( G/N \) is a Hall subgroup. Then by induction \( M/N \) is a Hall subgroup of \( G/N \). Therefore \( M \) is also a Hall subgroup of \( G \).

Check that (3) implies (1). Assume that every maximal subgroup of a soluble group \( G \) is a Hall subgroup. Let \( M \) be a maximal subgroup of \( G \) and \( |\pi(G)| = k + 1 \). Since \( M \) is a Hall subgroup, we have \( |\pi(M)| = |\pi(G)| - 1 = k \), and \( G \) is a quasi-\( k \)-primary group.

Thus, (1), (2) and (3) are equivalent.

Check that (1) implies (4). Let \( G \) be a soluble quasi-\( k \)-primary group and \( N \) its minimal normal subgroup. In view of (2), \( N \) is a Sylow \( p \)-subgroup of \( G \). By SchurZassenhaus theorem \([\text{III, Theorem 4.32}]\), in \( G \) there is a subgroup \( M \) such that \( G = [N]M \). Applying Lemma 1, \( G \) is \((k + 1)\)-primary. Hence \( M \) is \( k \)-primary. Suppose that in \( M \) there is a proper \( k \)-primary subgroup \( M_1 \). Then in \( G \) there is a proper \((k + 1)\)-primary subgroup \([N]M_1\), a contradiction. Consequently, \( M \) is quasi-(\( k-1 \))-primary.

Finely we prove that (1) follows from (4). Let \( G = [N]M \), where \( N \) is a minimal normal and Sylow subgroup of \( G \), \( M \) is its quasi-(\( k-1 \))-primary and maximal subgroup. Then by
Lemma 1,
\[ |\pi(M)| = k, \quad |\pi(G)| = |\pi([N]M)| = |\pi(N)| + |\pi(M)| = k + 1. \]

Show that every maximal subgroup \( K \) of \( G \) is not more than \( k \)-primary. If \( NK = G \), then \( N \cap K = 1 \) and \( K \simeq M \). Since \( M \) is a quasi-\((k - 1)\)-primary soluble subgroup of \( G \), using Lemma 1, we obtain that \( M \) is \( k \)-primary. And so \( K \) is also a \( k \)-primary group. Assume that \( NK \) is a proper subgroup of \( G \). Then \( N \subseteq K \) and \( K = N(K \cap M) \) by Dedekind identity. As \( M \) is quasi-\((k - 1)\)-primary, we obtain that \( L \cap M \) is not more than \((k - 1)\)-primary. Hence \( K = [N](K \cap M) \) is not more than \( k \)-primary. Thus (4) and (1) are equivalent. Theorem 1 is proved.

Note that \( \pi \)-soluble groups with certain Hall maximal subgroups are investigated in \([7]\).

**Corollary 1.1.** In a soluble quasi-\( k \)-primary group the Frattini subgroup is trivial.

**Proof.** The Frattini subgroup of any group can never be a Hall subgroup \([1, \text{Theorem 4.33}]\). It remains only to use Statement (2) of Theorem 1.

**Corollary 1.2.** In a soluble quasi-\( k \)-primary group the Fitting subgroup is a Hall subgroup and every its Sylow subgroup is a minimal normal subgroup.

**Proof.** By Theorem 1 (2) the Fitting subgroup of a soluble quasi-\( k \)-primary group \( G \) is a Hall subgroup. And so every its Sylow subgroup \( P \) is a Sylow subgroup of \( G \). Moreover, since the Fitting subgroup is nilpotent, \( P \) is a characteristic subgroup. Consequently, \( P \) is normal in \( G \). At the same time by Theorem 1 (2), a minimal normal subgroup of a soluble quasi-\( k \)-primary group is a Sylow subgroup. Corollary 1.2 is proved.

**Corollary 1.3.** Let \( N \) be a normal subgroup of a group \( G \). If \( G \) is a soluble quasi-\( k \)-primary group, then the quotient group \( G/N \) is a soluble quasi-\( l \)-primary group, where \( l = k - |\pi(N)| \).

**Proof.** It follows from Theorem 1 (2) and Lemma 2.

If a group \( G \) has a normal series whose factors are isomorphic to Sylow subgroups, then we say \( G \) has a Sylow tower.

**Corollary 1.4.** A soluble quasi-\( k \)-primary group has a Sylow tower.

**Proof.** We use induction on \( k \). Let \( G \) be a soluble quasi-\( k \)-primary group. Then by Statement (2) and (4) of Theorem 1, \( G \) can be represented as \( G = [P_1]M_1 \), where \( P_1 \) is a minimal normal and Sylow subgroup of \( G \), \( M_1 \) is a quasi-\((k - 1)\)-primary and maximal subgroup. By the induction hypothesis, \( M_1 \) has a Sylow tower. Hence \( G \) also has a Sylow tower. Corollary 1.4 is proved.
A positive integer $n$ is said to be squarefree if $p^2$ does not divide $n$ for all primes $p$. A group is supersolvable if all its chief factors are of prime orders.

**Corollary 1.5.** The order of a group $G$ is squarefree if and only if $G$ is a supersolvable quasi-$k$-primary group, where $k = |\pi(G)| - 1$. In particular, a supersolvable quasi-$k$-primary group is metacyclic.

**Proof.** Let the order of a group $G$ be squarefree and $k = |\pi(G)| - 1$. Then all Sylow subgroups of $G$ are cyclic and $G$ is supersolvable and metacyclic by Lemma 3. It is clear that $|\pi(X)| < |\pi(G)|$ for every proper subgroup $X$ of $G$, that is, $G$ is quasi-$k$-primary.

Converse, let $G$ be a supersolvable quasi-$k$-primary group. Apply induction on $k$. By Theorem 1 (4), $G = [N]M$, where $N$ is a minimal normal and Sylow subgroup of $G$, $M$ is a quasi-$(k-1)$-primary and maximal subgroup. In view of [1, Theorem 4.48], a minimal normal subgroup of a supersolvable group is of prime order, hence $|N| = p, p \in \pi(G)$. A subgroup $M$ is supersolvable and quasi-$(k-1)$-primary. By the induction hypotheses, the order of $M$ is squarefree. Hence the order of $G$ is also squarefree. And so $G$ is metacyclic by Lemma 3. Corollary 1.5 is proved.

**Corollary 1.6.** The derived length of a soluble quasi-$k$-primary group $G$ does not exceed

$$\min\{|\pi(G)|, \max\{1 + a_i \mid i = 1, 2, \ldots, t\}\},$$

where $|F(G)| = p_1^{a_1}p_2^{a_2}\ldots p_t^{a_t}$.

**Proof.** Let $d(G)$ and $r(G)$ be the derived length and the rank of a soluble quasi-$k$-primary group $G$, respectively. By Theorem 1 (4), all Sylow subgroups of $G$ are elementary abelian and the length of chief series of $G$ equals $|\pi(G)|$. Hence $d(G) \leq |\pi(G)|$. Soluble groups with abelian Sylow subgroups are considered in paragraph VI.14 [2]. Therefore all statements applying to $A$-groups (soluble groups with abelian Sylow subgroups) of this paragraph is correct for quasi-$k$-primary groups. In particular, $d(G) \leq 1 + r(G)$ by [2] VI.14.18, and $r(G) \leq u$ by [2] VI.14.31, where $u$ is the maximal number of generating Sylow subgroups in the Fitting subgroup $F(G)$. Since Sylow subgroups of $G$ are elementary abelian, we have $u = \max\{a_i \mid i = 1, \ldots, t\}$. It follows that

$$d(G) \leq \min\{|\pi(G)|, \max\{1 + a_i \mid i = 1, \ldots, t\}\}.$$  

Corollary 1.6 is proved.

Substituting $k = 2$ in Theorem 1, we obtain the result of S.S. Levischenko.

**Corollary 1.7.** ([3, Theorem 3.1]) A soluble group $G$ is quasibiprimary if and only if $G$ is equal to the semidirect product $[P]M$ of its elementary abelian Sylow $p$-subgroup $P$ and quasiprimary subgroup $M$, which is also a maximal subgroup of $G$.  

5
4 Soluble groups with nilpotent wide subgroups

Let $G$ be a nontrivial group, $Z_0(G) = 1$, $Z_1(G) = Z(G)$, $Z_2(G)/Z_1(G) = Z(G/Z_1(G))$, ..., $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$, ... . Then the subgroup $Z_\infty(G) = \bigcup_{i=0}^{\infty} Z_i(G)$ is called the hypercenter of $G$.

Obviously, $Z(G/Z_\infty(G)) = 1$.

**Theorem 2.** If in a soluble group $G$ every maximal subgroup $M$ such that $\pi(M) = \pi(G)$ is nilpotent, then the quotient group $G/Z_\infty(G)$ is quasi-$k$-primary, where $k = |\pi(G/Z_\infty(G))| - 1$.

**Proof.** Suppose that the quotient group $\overline{G} = G/Z_\infty(G)$ is not quasi-$k$-primary, where $k = |\pi(G/Z_\infty(G))| - 1$. Then in $\overline{G}$ there is a maximal subgroup $\overline{M} = M/Z_\infty(G)$ such that $\pi(\overline{M}) = \pi(G)$. By the hypotheses, $M$ is nilpotent. Therefore $\overline{M}$ is also nilpotent. Since $\overline{M}$ is a maximal subgroup of a soluble group $\overline{G}$, we obtain that $|\overline{G} : \overline{M}| = p^\alpha$ for some $p \in \pi(G)$ and positive integer $\alpha$. Let $\overline{M}_p$ be a Sylow $p$-subgroup of $\overline{M}$, $G_p$ be a Sylow $p$-subgroup of $G$ containing $\overline{M}_p$. Then $\overline{M}_p$ is a proper subgroup of $\overline{G}_p$. Therefore $\overline{M}_p$ is normal in $\overline{G}$. Now there is a nontrivial element $x$ of $\overline{M}_p \cap Z(\overline{G}_p)$ such that it belongs to the center of $\overline{G}$. This contradicts the fact that $Z(\overline{G}/Z_\infty(G)) = 1$. Theorem 2 is proved.

**Corollary 2.1.** If all wide subgroups of a group $G$ are nilpotent, then $G/Z_\infty(G)$ is quasi-$k$-primary, where $k = |\pi(G/Z_\infty(G))| - 1$.

The converse of Theorem 2 is not true. For example, let $G = S_3 \times Z_6$. Here $S_3$ is the symmetric group of order 3, $Z_6$ is the cyclic group of order 6. Then $Z_\infty(G) = Z_6$, and so $G/Z_\infty(G) \simeq S_3$ is quasiprimary, but the wide maximal subgroup $M = S_3 \times Z_2$ is nonnilpotent.

The following question becomes natural.

*What is the structure of a finite soluble group all of whose wide subgroups are supersoluble?*

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