Optical tomography of residual stresses in GRIN rod lenses with transverse and longitudinal translucence

D Karov\textsuperscript{1}, A Puro\textsuperscript{2}, A Fadeev\textsuperscript{1}, A Kuzmina\textsuperscript{1}

\textsuperscript{1}Higher School of Applied Physics and Space Technologies, Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russia
\textsuperscript{2}Department of Mathematics and Statistics, Euroacademy, Tallinn, Estonia

E-mail: dmkarov@yandex.ru

Abstract. The residual stresses (RS) in gradient-index (GRIN) rod lens are formed due to the material characteristics gradient. RS can lead to the birefringence effect. If a GRIN lens is birefringent, its applications may be limited, because the anisotropy can cause the image quality degradation. According to this, the urgent task is to develop a non-destructive diagnostic method of RS in the gradient optical structures. This paper presents the relations of the generalized method of integral photoelasticity for reconstructions of RS (optical tomography of the stress field) for the case of the bending probing rays in GRIN optical rods with a radial distribution of the refractive index. The problem is solved with plane strain approximations. RS Distributions are given reconstructed from tomographic data obtained with transverse and longitudinal (coaxial) translucence of GRIN rod lens.

1. Introduction

A number of optical elements (light guides, image translators — GRIN lenses) have cylindrical shape with chemical composition radial ($r$) gradients, which are specified by ion-exchange diffusion, paraphase deposition, etc. These phenomena induce a number of gradients of physical-mechanical properties such as refractive index $n(r)$, thermal expansion coefficient, molar volume, etc. When the structure is cooled, residual stresses (RS) $\sigma(r)$ are formed. In the elements of communication optics, RS can lead to the transfer characteristics deterioration due to aberrations induced by optical anisotropy (OA). According to this, the urgent task is the diagnosis of RS in the gradient optical structures. For non-destructive stress diagnostics in transparent objects, the integral photoelasticity method (optical tomography of the stress field) is used based on the presence of a spatially inhomogeneous OA induced by stresses in a sample [1].

The polarized beam, entering stressed media, splits into two beams which have mutually perpendicular polarizations along the directions ($n$, $l$) of the ellipse axes which are the cross section of the stress ellipsoid (or dielectric constant) by the plane of the wave front (quasi-principal directions). For these rays, due to OA, the distributions of refractive index (RI) are different, which leads to the fact that the components of the oscillations that have passed through the body acquire an integral phase difference $\Delta$. The relation between $\Delta$ and the differences in the quasi-principal values of the RI $(n_n$, $n_l$) or the differences in the quasi-principal components of the stresses $(\sigma_n$, $\sigma_l$) has the form of Wertheim's integral law:
\[ \Delta = \frac{2\pi}{\lambda} \int (n_i - n_l) dl = C_0 \int (\sigma_n - \sigma_l) dl. \]  

where \( C_0 \) is the stress-optical coefficient. Integrals are taken over the entire path of the ray \( l; \) \( dl \) is an element of the length trajectory, \( \lambda \) – wavelength of beam.

The reconstruction of the local distribution of the principal stresses components according to the measurements of \( \Delta(x) \) for many transmission parameters \( x \) constitutes an incorrect inverse problem of integral photoelasticity. The method of integral photoelasticity is developed for optically homogeneous structures for the approximation of small OA. Thus, it is assumed that the probing rays in the object propagate in a straight line and the trajectories of both polarized rays coincide.

The RS reconstruction tasks in elements with a gradient of RI lead to the necessity of generalizing the classical method under the assumption of the rays curvature and taking into account the bending of probing rays.

For the general case of an axisymmetric structure with a radial gradient RI, the tomographic problem with transverse translucence is considered in [10 – 13].

In these works as well as in the framework of the analysis proposed below, it is also assumed that in the gradient structures considered the quasi-isotropic approximation of geometrical optics for weakly anisotropic inhomogeneous media is valid [2].

In this paper the problem is solved with plane strain approximations – we review long gradient index cylindrical rod and use an assumption that the parameters of the rod do not depend on the axial coordinate \( Z \), and there is no gradient along this axis.

In a cylindrical coordinate system the principal axes of the stress ellipsoid coincide with the axial, radial, and tangential directions (\( \sigma_z, \sigma_r, \sigma_\theta \), are the principal components of stresses).

For further analysis, we have to write expression (1), which links the principal and quasi-principal stress components for a specific geometry of the translucence (transverse or longitudinal – figure 1 and 2). Beam trajectory: \__________________\ in a gradient environment; \__________________\ in the absence of a gradient.

**Figure 1.** Transverse translucence of a GRIN rod. **Figure 2.** Longitudinal translucence of a GRIN rod.

2. Transverse translucence

In case of transverse translucence of the axisymmetric bodies, ray trajectories of the ordinary and extraordinary beams can be considered as geometrically identical [15, 16] and corresponding to the original axisymmetric distribution of RI for gradient index cylindrical rod. The beam trajectory is a flat curve lying in the cross-sectional plane (figure 1), which form is given by the Bouguer formula \( n_r \sin \varphi = a \), where \( \varphi \) is the angle between the radius vector and the tangent to the beam, \( a = xn_r \) is the
constant for the beam, which transmission parameter is \( x \); \( n_R \) is the value of the refractive index \( n(r) \) on the surface of the cylinder and an element of the trajectory \( dl \) has the form [3]:

\[
dl = \frac{nrdr}{\sqrt{n^2 r^2 - a^2}}. \tag{2}\]

As it is shown in [7, 8] the expressions connecting the principal and quasi-principal stress components have the form:

\[
\sigma_l = \sigma_r \sin^2 \varphi + \sigma_p \cos^2 \varphi, \\
\sigma_n = \sigma_z. \tag{3}\]

Taking into account (2) and (3), after some transformations, we obtain for the integral phase difference \( \Delta \) from Wertheim’s integral law (1):

\[
\Delta = \Delta_0 + \Delta_1; \Delta_0 = 2C_0 \int_{r_1}^{R} \frac{nRdr}{\sqrt{n^2 r^2 - a^2}}, \Delta_1 = 2C_0 \int_{r_1}^{R} \frac{a^2 n'}{m^3} \sigma_r - \frac{nrdr}{\sqrt{n^2 r^2 - a^2}}, \tag{4}\]

\( n' = \frac{dn}{dr} \); \( r \) is the radial coordinate; \( r_1 \) is determined from the condition \( r_1 n(r_1) = a \). For the gradient-free case \( (n' = 0) \Delta = \Delta_0 \) and the expression for \( \Delta_0 \) is reduced to as it is obtained in [4, 5]. Estimation of the factor before \( \sigma_r \) shows that for GRIN rod lenses with the distribution of RI of the following form:

\[
n(r) \approx n_0 (1 - \frac{g^2 r^2}{2}), g = \frac{1}{R} \sqrt{\frac{2 \Delta n}{n_0}}, \tag{5}\]

it has the value of:

\[
\left| \frac{a^2 n'}{m^3} \right| \leq g^2 R^2, \tag{6}\]

where \( g \) is power constant. For many types of glass GRIN rod lenses usually \( gR << 0.1 \div 0.2 \). Thus, it follows from this estimation that in this case the factor before \( \sigma_r \) does not exceed 0.04, in that way the main contribution to the phase difference is made by the first and the main term \( \Delta_0 \). If the second term \( \Delta_1 \) is neglected, the expression for the phase difference can be reduced by a number of transformations to the Abel integral equation:

\[
\Delta^T(a) = 2 \int_{a}^{\infty} C_0 \sigma_z \frac{du}{\sqrt{u^2 - a^2}} \tag{7}\]

and after its inversion we have [7, 8]:

\[
\sigma_z(r) = -(\pi C_0)^{-1} \frac{du}{dr} \int_{u(r)}^{\infty} \frac{d\Delta^T}{da} \frac{da}{\sqrt{a^2 - u^2}} \tag{8}\]
where $\Delta'(x)$ is the integral phase difference of 2 plane-polarized oscillations in transverse translucence of the GRIN rod lens, $u = m(r)$, $n_R$ is the value of RI $n(r)$ on the surface of the rod. Here it is assumed that $C_0(r) \approx C_0(0)$.

The components $\sigma_\theta(r)$, $\sigma_r(r)$ can be found, if equilibrium condition of stresses in the cylinder is used and the assumption that in our case so called "law of the sum" $\sigma_\theta(r) = \sigma_\theta(r) + \sigma_r(r)$ [6] is true.

Expression (8) includes a singular convergent integral; $d\Delta'/da = 0$ for $a = 0$ due to symmetry; $\Delta'/da = \infty$ for $a = u_R$ (as a result, $\Delta'(x)$ cannot be measured precisely when $x \to R$). Taking these circumstances into account, a regularized algorithm was developed in [7, 8, 14] based on the spline approximation of $\Delta'(x)$ and optimization of boundary conditions for ensuring the stability of the solution near the boundaries of the interval $(0, R)$.

3. Longitudinal (coaxial) translucence

In case of longitudinal translucence of a GRIN rod lens (figure 2), the general equation of the trajectory of the meridional beam is [8]:

$$\frac{dz}{dl} = b = \text{const},$$

where $b = n(x)\cos\phi_0$ is the optical directing cosine, $x$ is the height of the beam entrance to the GRIN rod lens, $\phi_0$ – the initial angle of inclination of the beam relative to the optical axis of the GRIN rod lens. Here we consider the case when the beam enters the media parallel to the axis, i.e. $\cos\phi_0 = 1$, since this corresponds to the most often realized conditions of translucence. Consequently, the cosine of the angle of inclination of the trajectory to the axis is $\cos\phi = b/n$.

The elementary segment of the trajectory $dl$ is equal to:

$$dl = \frac{dz}{\cos\phi} = -\frac{dz}{b}.$$  \hfill (10)

In this case, one of the quasi-principal directions is tangential, and the expressions connecting the principal and quasi-principal stress components have the form:

$$\sigma_i = \sigma_r \cos^2 \phi + \sigma_\theta \sin^2 \phi,$$

$$\sigma_n = \sigma_\theta.$$  \hfill (11)

Using these relations and the Wertheim law (1), we can write the integral phase difference for a GRIN rod lens with length $L$ [7–9]:

$$\Delta' = \int_0^L C_0 [\sigma_\theta - \sigma_i + (1 - \frac{b^2}{n^2})(\sigma_r - \sigma_i)] n dz,$$

$$\Delta' = \int_0^L C_0 [\sigma_\theta - \sigma_i + (1 - \frac{b^2}{n^2})(\sigma_r - \sigma_i)] n dz.$$  \hfill (12)

The magnitude of the multiplier in front of the difference $\sigma_i - \sigma_r$ for $gR < 0.2$, which corresponds to the majority of modern GRIN rod lenses, is estimated as follows [7, 9]:

$$1 - \frac{b^2}{n^2} \leq 1 - \frac{n_S^2}{n_0^2} = 1 - \left(1 - \frac{g^2 R^2}{2}\right)^2 \approx g^2 R^2 \leq 0.04$$  \hfill (13)
with \( gR \leq 0.2 \). So in expression (12), we can neglect the term containing the difference \( \sigma_r - \sigma_z \). Because

\[
\mathrm{d}z = \mathrm{d}r \cdot \cot \varphi = -\frac{\cos \varphi}{\sqrt{1 - \cos^2 \varphi}} \mathrm{d}r = \frac{b}{\sqrt{n^2 - b^2}} \mathrm{d}r,
\]

an integral equation (6) can be reduced to:

\[
\Delta(x) = \int_x^b C_0(\sigma_0 - \sigma_r) \frac{ndr}{\sqrt{n^2 - b^2}},
\]

where \( r_L \) is the height of the exit point of the beam from the GRIN rod lens. It is known that for ideal (hyper-secant) distribution of the RI:

\[
n(r) = n_0 \text{sech } (gr) = n_0 \left(1 - \frac{1}{2} (gr)^2\right)
\]

the meridional rays propagate along periodic trajectories close to sinusoidal, with a length of periodicity \( L_0 = 2\pi/g \), regardless of the translucence parameter (input parameters) \( x_{in} \). For GRIN rod lenses with a length of \( L_0/4 \), the output parameter \( r_L \) is 0 for any \( x_{in} \). For measurements, the GRIN rod lenses with a length of \( L_0/2 \) are more convenient, since the beam exits from it parallel to the optical axis. The trajectory of the beam in this case consists of two symmetrical parts and the integral phase difference must be doubled:

\[
\Delta^L(x) = 2 \int_x^b C_0(\sigma_0 - \sigma_r) \frac{ndr}{\sqrt{n^2 - b^2}}
\]

or

\[
\Delta^L(b) = 2 \int_b^\infty C_0(\sigma_0 - \sigma_r) \frac{ndn}{n^2 - b^2}.
\]

The resulting formula is the Abel integral equation for the function \( C_0(\sigma_0 - \sigma_r)/n' \). Using the inverse Abel transform, we obtain the solution of this equation:

\[
\sigma_0 - \sigma_r = -\frac{n'}{\pi C_0} \int_{x_{in}}^x \frac{d\Delta^L / db}{\sqrt{b^2 - n^2}} \mathrm{d}b,
\]

where \( b = n(x) \); \( x \) is the height of the beam entry point; \( \Delta^L(x) \) is the integral phase difference for the half-wave segment of the gradient with longitudinal translucence [8, 9].

Separation of stress components is not difficult here, since using the equilibrium condition, this expression can be integrated:

\[
\sigma_r = \int_{R}^{r} \frac{\sigma_0(r') - \sigma_z(r')} {r'} \mathrm{d}r' + C'.
\]

Using the boundary condition \( \sigma_0(R) = 0 \), we obtain the integration constant \( C' = 0 \). Axial stresses can be obtained from the use of “the rule of sum”. 

4. Comparison of reconstructed radial distributions of stresses in GRIN lens with longitudinal and transverse translucence

It is interesting to compare the distributions of RS reconstructed from tomographic data obtained for both considered variants of translucence. It is expedient to do this for the distribution of axial RS \( \sigma_z(r) \), which in the case of transverse translucence can be directly reconstructed from the \( \Delta T(x) \) data on the basis of relations (8). For the case of longitudinal translucence, \( \sigma_z(r) \) is found using the “law of sum” from the distributions \( \sigma_\theta(r) \) and \( \sigma_r(r) \), obtained by separating the difference between stresses \( \Delta\sigma_{\theta r}(r) = \sigma_\theta(r) - \sigma_r(r) \), reconstructed from \( \Delta L(r_L) \) data based on relations (19) using the equilibrium equation.

This comparison has been made on the basis of tomographic studies of a quarter-wave segment of a GRIN rod lens based on the glass TCM-412 with a distribution RI close to the ideal-focusing \((R = 4.95 \text{ mm}, n_0 = 1.523, gR = 0.105, L/4 = 75 \text{ mm}, C_0 = 2.7 \text{ Br})\). The GRIN lenses have been fabricated by a high-temperature ion exchange lithium (glass) \( \leftrightarrow \) sodium (a salt melt). The specimen is immersed in a fluid of matching refraction index to eliminate the refraction of light at the specimen's curved surface. Retardation is measured using a polarization-optical setup with a He-Ne laser and a Senarmon compensator. Two types of examination have been used, i.e. by a broad beam with scanning by a pinhole in the image plane of the object \( \Delta L(r_L) \) and scanning of the object itself by a focused beam \( \sim 40 \text{ mkm} \) in diameter \( \Delta T(x) \). RS distributions \( \sigma(r) \) were reconstructed by stabilizing algorithms using a spline approximation \( \Delta(x) \).

The results of the comparison of reconstructed axial stresses are presented in figure 3 [8].

\[ \text{Figure 3. Reconstructed axial stresses with transverse (— — —) and longitudinal (- - - -) translucence:} \]

- 1 – axial stresses;
- 2 – radial stresses;
- 3 – difference of tangential and radial stresses.

As it can be seen from figure 3, the stress distributions reconstructed from the data of 2 different tomographic experiments are quite close to each other. Possible reasons for some differences in the resulting stress distributions may be associated to the difficulty of obtaining information about \( \Delta L(r_L) \) in areas near the cylinder contour in case of transverse translucence; the deviation of the distributions RI from the ideally focusing can be significant during reconstruction with the longitudinal translucence data \( \Delta L(r_L) \).

Finally, the application of the "law of the sum", which is strictly true only for thermoelastic stresses, can lead to additional errors.

5. Conclusions

The expressions have been obtained for the retardation at transverse and longitudinal translucence in the approximation of the flat deformed state of the rods with a radial refractive index gradient concerning the curvature of the probe beams. It is shown that the polarization state of the probe beam is affected by for axial stresses for transverse geometry and by tangential and radial stresses for...
longitudinal one. This approximation provides the reconstruction accuracy for low- and medium-gradient axisymmetric objects. An acceptable accordance has been demonstrated for the stress distributions in the gradient rod TSM-412 reconstructed from tomographic data and obtained using polarization-optical studies for both translucency cases.

A separate issue is the validity of the assumption that the trajectories of two orthogonally polarized components of the probe rays are identical for the longitudinal translucence of long GRIN rod lenses. It is also relevant to obtain equations for the reconstructed RS for longitudinal translucence of the rods with arbitrary distributions RI.

References

[1] Aben H K 1979 Integrated Photoelasticity (New York: McGraw-Hill)
[2] Fuki A A Kravtsov Y A and Naida O N 1998 Geometrical Optics of Weakly Anisotropic Media (Amsterdam: Gordon and Breach Science Publishers)
[3] Born M and Wolf E 1968 Principles of Optics (Oxford: Pergamon)
[4] Saenz A W 1950 Determination of residual stress of quenching origin in solid and concentric hollows cylinders from interferometric observations J. of Appl Phys. 21(10) 962-5
[5] O’Rourke R C and Saenz A W 1950 Quenching stresses in transparent isotropic media and photoelastic method Quart. Appl. Math. 8(3) 303-11
[6] O’Rourke R C 1951 Three-Dimensional Photoelasticity Appl. Phys. 22 872–8
[7] Fadeev A B 1988 The study of optical anisotropy and residual stresses in cylindrical rods with a gradient of the refractive index. Degree work (Leningrad: LPI)
[8] Karov D D 2012 Extended Abstract of Cand. Sci. Dissertation (St. Petersburg: SPb State Polytech. Univ.)
[9] Karov D D Fadeev A B and Ushakov S N 1991 Axial translucence of GRIN rod lenses: inverse and direct problems of integral photoelasticity (Zvenigorod: V All-Union Symp. on computed tomography, abstracts) 140-1
[10] Karov D D and Puro A E 2018 Polarization Tomography of Residual Stresses in Cylindrical Gradient-Index Lenses Optics and Spectroscopy 124 735–40
[12] Puro A and Karov D 2017 Optical polarimetric tomography of residual stresses with beam deflection (GRIN cylinder) In: Intern. Conf.: Quasilinear Equations, Inverse Problems and Their Applications" Moscow 5-7 dec. 2017
[13] Karov D D and Puro A E 2006 Optical tomography of the internal stresses in Cylindrical Phase Objects Works SPbGPU 500 237–46
[14] Melnikov N Y 1997 Development of effective algorithms for tomographic reconstruction and stress modeling in gradient-optical rods. Master. diss. (St. Peterburgh: SPbGTU)
[15] Aben H K and Kell K Y E 1983 Integral photoelasticity with measurement of light ray deflection In: Experimental Methods of Studying Stresses and Strains (Kiev: Inst. Elektrosvariki Ak Ld. Nauk) 3–11
[16] Aben H K, Krasnowski B R and Pindera J T 1984 Nonrectilinear Light Propagation in Integrated Photoelasticity of Axisymmetric Bodies Trans. CSME 8 (4) 195–200