Anomalous low-energy $E2$-related behavior in triaxial nuclei

Yu Zhang,† Ying-Wen He,‡ D. Karlsson,§ Chong Qi,∥ Feng Pan,∥∥ and J. P. Draayer∥∥∥

1Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China
2Department of Physics, KTH Royal Institute of Technology, Stockholm 10691, Sweden
3Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA
(Dated: July 14, 2022)

The emergence of collective features is one of the most important and striking characteristics of complex nuclear many-body systems. How nuclear collectivity emerges from collective modes (shapes and deformations thereof) can be realized theoretically from both macroscopic and microscopic perspectives. Specifically, various collective modes can be explained within a Bohr-Mottelson picture of the dynamics using geometric language [1], which includes the notion of a spherical vibrator, that of either an axially-deformed or a triaxially-deformed rotor of the Davydov and Filippov [2] type, or even the γ-unstable rotational motion introduced by Wilets and Jean [3] which stands in sharp contrast with that of a rigid or γ-stable characterization of the dynamics.

On the other hand, in contrast with the above, the interacting boson model (IBM) [4], which is an algebraic theory, has also demonstrated excellent success in elucidating collective nuclear modes. A big advantage of the IBM is that not only are there three distinguishable collective limits that can be realized, but even more importantly, one can use the theory to study the results of the mixing of modes, and as well the fact that the spatial reach or size of the model space can be controlled by the number of bosons that are allowed to participate in the dynamics. In what follows below, we capitalize on this flexibility of the IBM, and in particular show that the boson-number dependence on the low-energy $E2$-related behavior of triaxially deformed nuclei can be used to explain the origin of the anomalous low-energy $E2$-related behavior that has recently been observed in some heavy neutron-deficient nuclei.

How to understand nuclear collectivity in terms of its microscopic roots is certainly a highly desirable proposition, but such approaches are usually plagued by various high levels of complexities that often include in addition major computational challenges. Nevertheless, it should be noted that much progress toward such a goal has been made based on so-called $ab$ initio shell-model theories. The seminal work of Elliott [5] opened the door to a fully microscopic pathway for achieving a truly microscopic description of rotational motion in light nuclei. Furthermore, some major steps forward in this direction have been achieved more recently using what has been dubbed by its founders as a no-core shell model theory, through which the emergent collectivity in light nuclei and the symmetries that underpin such modes have been addressed starting with $ab$ initio (from first principles) interactions that are parameter free [6–10]. Also, and much earlier than the latter, for heavy systems the so-called pseudo-SU(3) shell model was advanced and shown to be able to successfully describe many features of low-lying collective phenomena in strongly deformed heavy nuclei [11]. In addition, the proxy-SU(3) scheme has also been proffered as a similar methodology [12] along with some analytic results for considering associated collective features. All these developments, from Elliott and forward, including the SU(3) limit of the IBM - suggest that SU(3) - which is the symmetry group of the 3D-Harmonic Oscillator - plays an essential role in attempts to gain a deeper understanding of the origin of collectivity from a more microscopic perspective.

Overall, the nature of collective modes in atomic nuclei is best revealed through the low-lying spectroscopic features of nuclei and the associated $E2$-related electromagnetic transitions. However, recent experimental measurements [13–17] suggest a puzzling anomalous phenomenon among some low-lying yrast states in certain neutron-deficient nuclei including $^{166}$W, $^{168,170}$Os and $^{172}$Pt, within which the excitation energy ratio $R_{4/2} = E(4_1^+)/E(2_1^+) > 2.0$ shows the collective nature of these states while...
the results are accompanied by rare and anomalous $B(E2)$ ratio with $B_{4/2} \equiv B(E2; 4_1^+ \rightarrow 2_1^+)/B(E2; 2_1^+ \rightarrow 0_1^+) < 1.0$. This anomalous behavior seems to persist in the neighboring odd-$A$ nuclei, for example in $^{160}$Os, with an odd neutron outside of an even-even core serving as a spectator. What is clear is that this well-documented phenomenon does not seem to belong to any particular set of the more familiar conventional collective modes, nor has it been addressed in a convincing way in large-scale shell model approaches [15] or within self-consistent mean-field analyses [16].

The main purpose of this paper is to tackle this problem within the IBM framework, and in particular, within its representation in terms of the SU(3) symmetry limit of the theory. Specifically, in what follows we show how this novel collective feature emerges naturally within the SU(3) realization of a triaxially-deformed rotor, when a finiteness effect is added to the theory, which yields $R_{4/2} > 2.0$ ratios and $B_{4/2} < 1.0$ values simultaneously, and in so doing this picture provides a relatively simple explanation for the observed anomaly. We also proffer that it seems reasonable to suggest that this feature should as well as be found in other models that incorporate SU(3)-defined basis states [11, 12, 18, 21].

To describe the conventional collective modes within the IBM, we adopt the well-known consistent-$Q$ Hamiltonian [22]

$$\hat{H}_{CQ} = \epsilon \hat{n}_d + \kappa \frac{1}{N} \hat{Q}_d^2$$

(1)

with $\hat{n}_d = d^\dagger \cdot \hat{d}$ and $\hat{Q}_d^2 = (d^\dagger s + s^\dagger d)^2 + \chi (d^\dagger \times \hat{d})^2$, where $\epsilon$, $\kappa$ and $\chi$ are real parameters and $N$ is the total boson number. Different dynamical symmetries (DSs) in the IBM can be characterized as: the U(5) when $\epsilon > 0$ and $\kappa = 0$; the O(6) when $\epsilon = 0$, $\kappa < 0$ and $\chi = 0$; and the SU(3) when $\epsilon = 0$, $\kappa < 0$ and $\chi = \pm \sqrt{7}/2$. These DSs in turn describe the corresponding collective modes in an algebraic way, including the spherical vibrator (U(5)), $\gamma$-unstable rotor (O(6)), and axially-deformed rotor (SU(3)) [23]. However, the triaxially-deformed rotor, which is also a typical collective mode in the Bohr-Mottelson model, is out of reach of the consistent-$Q$ Hamiltonian.

The classical limit of the IBM Hamiltonian can be worked out by using coherent state of the system defined as [4]

$$|\beta, \gamma, N\rangle = N_A |s^+, \beta \cos \gamma d_0^+ + \frac{1}{\sqrt{2}} \beta \sin \gamma (d_2^0 + d_{-2}^0)|0\rangle$$

(2)

with $N_A = 1/\sqrt{N!(1+\beta^2)^N}$. The classical potential corresponding to $\hat{H}_{CQ}$ is then given as $V(\beta, \gamma) = 1/\beta |\beta, \gamma, N\rangle |\hat{H}_{CQ}| \beta, \gamma, N\rangle |N\rangle$. The potential configurations and the corresponding ratios of $\hat{H}_{CQ}$ are shown in FIG. 1. As expected, different modes indeed exhibit different types of potential minimum with different $R_{4/2}$ and $B_{4/2}$ ratios. Specifically, it is given by $R_{4/2} \approx 2.0$ in the U(5) mode, $R_{4/2} \approx 2.5$ in the O(6) mode, and $R_{4/2} \approx 3.3$ in the SU(3) modes for both prolate and oblate but with different quadrupole moments [23], of which the common feature is $R_{4/2} \geq 2.0$ and $B_{4/2} > 1.0$. It should also be mentioned that the pairing dominant situation in the shell model, which belongs to a non-collective mode, may result in $B_{4/2} \ll 1.0$ but accompanied with $R_{4/2} < 2.0$. Nevertheless, no triaxial minimum with $0^\circ < \gamma_{min} < 60^\circ$ appears in FIG. 1 described by the consistent-$Q$ Hamiltonian.

To obtain a potential configuration with triaxial minimum at a mean-field level, higher-order terms have to be introduced in the IBM [24]. For example, the stable triaxial minimum at $\gamma = 30^\circ$ can be induced by the cubic term $(d^\dagger \times d^\dagger \times d)^3 \cdot (d \times d \times d)^3$ [24, 25]. A region of triaxiality with $0^\circ < \gamma < 60^\circ$ may be allowed in an extension of the consistent-$Q$ Hamiltonian by adding the cubic term $(\hat{Q}_d^2 \times \hat{Q}_d^2 \times \hat{Q}_d^0)^0$ [26]. However, the existence of a triaxial minimum is insufficient in producing a triaxially-deformed rotor in the IBM. In turn, a group-guided approach first established in the shell model description.
of a quantum rotor \cite{27,28} was employed to realize the triaxial rotor mode in the IBM \cite{29}, wherein the Hamiltonian was constructed with the symmetry-conserving operators of the SU(3) ⊃ SO(3) integrity basis \cite{30}. From a group (algebra) theory point of view \cite{31}, the su(3) algebraic relations in the large-N limit contract to those of the semi-simple Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{so}(3)$ of a quantum rotor, within which an exact mapping \cite{32} between the triaxial rotor and its IBM image was established for any $\gamma$-deformation based on the formulism developed in \cite{27,28}.

In the following, we revisit and reformulate a generic SU(3)-based theory for realizing a triaxial rotor geometry, one that can be exercised within any application that uses SU(3) basis states, such as the shell model \cite{5,6,11} and the IBM \cite{29,32}. In the SU(3) algebraic realization of a triaxial rotor, the Hamiltonian is divided into its static and dynamic parts as

$$\hat{H}_{\text{Tri}} = \hat{H}_S + \hat{H}_D$$

(3)

where

$$\hat{H}_S = \frac{a_1}{N} \hat{C}_2[\text{SU}(3)] + \frac{a_2}{N} \hat{C}_2[\text{SU}(3)]^2 + \frac{a_3}{N} \hat{C}_3[\text{SU}(3)],$$

(4)

$$\hat{H}_D = t_1 \hat{L}^2 + t_2 (\hat{L} \times \hat{\phi})_0 + t_3 (\hat{L} \times \hat{\phi})_1$$

(5)

Here, $\hat{L}$ and $\hat{\phi}$ are the angular momentum and quadrupole momentum operators, respectively, while $a_i$ and $t_i$ with $(i = 1, 2, 3)$ are real parameters. The SU(3) Casimir operators are defined as

$$\hat{C}_2[\text{SU}(3)] = 2 \hat{\phi} \cdot \hat{\phi} + \frac{3}{7} \hat{L}^2,$$

(6)

$$\hat{C}_3[\text{SU}(3)] = -\frac{4}{\sqrt{3}} (\hat{\phi} \times \hat{\phi}_0) + \frac{2}{\sqrt{2}} (\hat{L} \times \hat{\phi}_0).$$

(7)

It is important to note that by including scalar polynomial forms in $\hat{\phi}$ up to $(\hat{\phi} \cdot \hat{\phi})^2$ the Hamiltonian \cite{41} is able to generate a collective potential of a stable axially-symmetric system. This feature is consistent with the analysis discussed in \cite{18}, except that in the present case the $\hat{\phi}$ that is used \cite{40} and \cite{74} are generators of SU(3). And further, note that scalar polynomial forms that include $\hat{L}$ contribute nothing to the ground state itself, with their action separately or in conjunction with $\hat{\phi}$ serving to define the effective moments of inertia of the system. And moreover, eigenvalues of the Casimir operators of SU(3) can be expressed in terms of the SU(3) irreducible representation (irreps) labels $(\lambda, \mu)$; that is,

$$\langle \hat{C}_2[\text{SU}(3)] \rangle = \lambda^2 + \mu^2 + 3\lambda + 3\mu + \lambda \mu,$$

(8)

$$\langle \hat{C}_3[\text{SU}(3)] \rangle = \frac{1}{6} (\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3).$$

(9)

The static triaxiality is determined by $\langle \hat{H}_S \rangle = f(\lambda, \mu)$ with the $\gamma$-deformation \cite{28}

$$\gamma_S = \tan^{-1} \left( \frac{\sqrt{3}(\mu + 1)}{2\lambda + \mu + 3} \right)$$

(10)

and the corresponding $\beta$-deformation

$$k_0 \beta_S = \sqrt{\lambda^2 + \mu^2 + \lambda \mu + 3\lambda + 3\mu + 3},$$

(11)

where $k_0$ is a scale factor. The ground-state energy of the triaxial Hamiltonian \cite{3} is given as $E_g = f(\lambda, \mu)$ at the optimal values $(\lambda_0, \mu_0)$, with which the parameters $a_i$ are thus determined, while $\hat{H}_D$ contributes nothing to the ground-state energy. The total Hamiltonian $\hat{H}_{\text{Tri}}$ is then applied to generate the SU(3) image of a triaxial rotor with the Hamiltonian

$$\hat{H}_{\text{Rot}} = A_1 \hat{I}_1^2 + A_2 \hat{I}_2^2 + A_3 \hat{I}_3^2,$$

(12)

for which the parameters $t_i$ are fixed by the exact mapping \cite{27,28}: $t_i = t_i(\lambda_0, \mu_0, A_1, A_2, A_3)$. In \cite{12}, $\hat{I}_\mu$ are the angular momentum operators in the intrinsic frame and the inertia parameters $A_i$ can be either extracted from the momentum of inertia formulas \cite{52} or just taken as independent parameters \cite{53}. For convenience, throughout this work, $A_1 : A_2 : A_3 = 3 : 1 : 4$ is taken in the analysis, which corresponds to a very asymmetric situation. It is important to reiterate that this SU(3) realization of a triaxial rotor, as shown above, is not restricted to a specific implementation of SU(3), it is a generic property of the operators that generate SU(3) regardless of its specific implementation.

In this work, we focus on the IBM realization of a triaxial rotor. In the IBM, the SU(3) generators are defined as $\hat{L}_\mu = \sqrt{10} (d^\dagger \times d^\dagger)_\mu$ and $\hat{Q}_\mu = \hat{Q}_\mu^Z$ with $\chi = -\sqrt{7}/2$, which is also taken as the E2 transition operator in the calculation. We take $N = 9$ as an example to illustrate the finite-N triaxial rotor mode described by \cite{4}. For this example, $a_1 : a_2 : a_3 = -27 + 10N : 1 : 1$.
is taken, with which the SU(3) irrep of the ground state determined by $\hat{H}_S$ is $(\lambda_0, \mu_0) = (6, 6)$ resulting in $\gamma_S = 30^\circ$ according to \cite{10}. The other parameters are thus determined by the mapping $t_i \equiv t_i(\lambda_0, \mu_0, A_1, A_2, A_3)$ with $t_1 = 3.0$, $t_2 = 0.553$, $t_3 = -0.227$ for the $A_1 : A_2 : A_3$ ratio shown above. The resulting triaxial structure is shown in the left panel of FIG. 2 where the low-lying states may group into the standard rotational bands. The odd-even staggering appearing in the $\gamma$ band confirms the spectrum to be a rigid triaxial rotational one. Most interestingly, the unusual small $\beta(E2)$ ratio, $B_{4/2} < 1.0$, appears with the level energies still following the normal collective excitation value with $R_{4/2} \in [2.0, 3.33]$. Therefore, it can be concluded that the unusual small $B_{4/2}$ ratio with normal $R_{4/2}$ ratio occurs in the finite-$N$ triaxial rotor, which applies to other nuclear models \cite{11, 12, 18–21} with the same triaxial rotor description as well. Moreover, in a recent work \cite{34}, small $B_{4/2}$ has been considered to be produced from the $E2$ transition prohibition between two different irreps in the SU(3) limit of the IBM, in which $0^+_1$, $2^+_1$ belong to the SU(3) irrep $(2N, 0)$, while other yrast states with $L \geq 4$ belong to other SU(3) irreps. It is obvious that the mechanism proposed in this work is completely different from that of \cite{34}. To test the finite-$N$ effect in the present triaxial system, the evolution of $R_{4/2}$ and $B_{4/2}$ as functions of the boson number $N$ is also worked out and the results are shown in the right panel of FIG. 2. In the calculation, the ground-state irrep is chosen to be $(\lambda_0, \mu_0) = (2N/3, 2N/3)$ corresponding to $\gamma_S = 30^\circ$ consistent to the parameter ratio $a_1 : a_2 : a_3 = -\frac{27+10N}{3N} : 1 : 1$, as the leading SU(3) irrep with $\lambda + 2\mu = 2N$ is usually assumed to be dynamically favored. It can be observed that the $B_{4/2}$ ratio monotonically increases with the increasing of $N$ from $B_{4/2} < 1.0$ to $B_{4/2} > 1.0$ and finally reaches the triaxial rotor limit value at very large $N$, which indicates that the unusual small $B_{4/2}$ ratio in the triaxial system with $\gamma_S = 30^\circ$ occurs mainly due to the finite-$N$ effect. Meanwhile, the energy ratio $R_{4/2}$ may well coincide with the rotor limit, which confirms the robustness of the SU(3) mapping procedure. As a more general asymmetric situation, FIG. 3 shows the $B_{4/2}$ evolution on the $\beta_S - \gamma_S$ sector with $N \leq 15$, where $k_0$ is taken to be a constant for all SU(3) irreps, in which the irreps, such as $(2, 2)$, with the corresponding $R_{4/2} > 3.33$ in the mapping are excluded. As shown in FIG. 3, though $N$ may be larger than that used in the previous case, $B_{4/2} < 1.0$ survives in the triaxial region with $\gamma_S \in [25^\circ, 45^\circ]$, which turns to increase toward the rotor limit with further increasing of $N$ consistent with the $N$-dependent behavior shown in the right panel of FIG. 2.
To reveal the mean-field picture of this triaxial rotor mode, the potential function for $\hat{H}_S$ is also calculated by using the coherent state method [24]. Specifically,\[ V_S(\beta, \gamma) = \frac{1}{N} \langle \beta, \gamma| \hat{H}_S| \beta, \gamma, N \rangle |_{N \to \infty} \]

\[ \quad = a_1 \frac{\beta^2}{(1 + \beta^2)^2} \left[ 8 + 4\sqrt{2} \beta \cos(3\gamma) + \beta^2 \right] \\
\quad + a_2 \frac{\beta^4}{(1 + \beta^2)^4} \left[ 64 + 32 \beta^2 + \beta^4 + 16 \beta^2 \cos(6\gamma) \right] \\
\quad + 8\sqrt{2} (8\beta + \beta^3) \cos(3\gamma) \\
\quad + a_3 \frac{2\beta^3}{9(1 + \beta^2)^3} \left[ 24\beta + 16\sqrt{2} \cos(3\gamma) \right] \\
\quad + 6\sqrt{2} \beta^2 \cos(3\gamma) + \beta^3 \cos(6\gamma) \right]. \] (13)

The potential $V_S(\beta, \gamma)$ simultaneously describes the classical limit of the triaxial Hamiltonian $H_{\text{Tri}}$ since its dynamical part $\langle \hat{H}_D \rangle/N$ may disappear in the large-$N$ limit through setting an $N$-dependent form of the parameter $t_i$ in (5). One can check that the minimum of $V_S(\beta, \gamma)$ coincides exactly with the ground-state energy of $\hat{H}_S/N$ in the large-$N$ limit and even the $\gamma$-deformation determined by $V_S(\beta, \gamma)$ agrees well with that obtained from (10) in the axially-deformed situation. FIG.4 shows four examples with different $a_1 : a_2 : a_3$ ratios all generating the triaxial irrep $(\lambda_0, \mu_0) = (6, 6)$ in the $N = 9$ case. As shown in FIG.4, the potential minimum in panel (A) and (B) with nonzero $a_i$ ($i = 1, 2, 3$) locates at $\gamma \simeq 43^\circ$ and $\gamma \simeq 30^\circ$, respectively, while the minima in (C) with $a_1 = 0$ and (D) with $a_2 = 0$ are within $\gamma \in [0^\circ, 45^\circ]$ and $\gamma \in [0^\circ, 60^\circ]$ region, respectively, showing unfixed asymmetric $\gamma$-deformation in the latter two cases. It means that an asymmetric deformation can indeed be generated by the static part $\langle \hat{H}_S \rangle$, but a stable triaxial minimum can be achieved only when both $a_2$ and $a_3$ are nonzero. The case (D) actually represents the critical point situation in the prolate-oblake shape phase transition generated by a combination of $\hat{C}_2[\text{SU}(3)]$ and $\hat{C}_3[\text{SU}(3)]$ [35]. Clearly, stable triaxial deformation does not occur even at the critical point of the prolate-oblake shape phase transition. Meanwhile, the $\hat{C}_2[\text{SU}(3)]^2$ term is indispensable in generating a stable triaxial deformation at either finite-$N$ case or large-$N$ limit. Moreover, (10) gives $\gamma_S \simeq 30^\circ$ and $\gamma_S \simeq 14^\circ$ in the large-$N$ limit for case (A) and (B), respectively. Though the $\gamma_S$ value obtained from (10) is smaller than that obtained from the coherent state method, the parameter conditions for the triaxial deformation are consistent with each other. For finite $N$, the parameter relations shown in (A) and (C) may always guarantee $(\lambda_0, \mu_0) = (2N/3, 2N/3)$ corresponding to $\gamma_S = 30^\circ$, while those of (D) provides degenerate ground-state irrep of $\lambda_0 + 2\mu_0 = 2N$ for any $N$ resulting in the prolate-oblake shape (phase) transition due to level crossing [35]. Furthermore, the finite-$N$ correction to $V_S(\beta, \gamma)$ may result in a very complicated potential function but only with $1/N$ order contribution to (13). Therefore, (10) seems more convenient in estimating the $\gamma$-deformation when $N$ is finite.

To describe realistic nuclear systems, a more general IBM Hamiltonian with

$\hat{H} = \hat{H}_{\text{CQ}} + \hat{H}_{\text{Tri}}$ (14)

may be adopted, which covers all typical collective modes including spherical vibrator, $\gamma$-unstable, axially-deformed, and triaxially-deformed rotor. Since $\hat{H}_{\text{CQ}}$ includes SU(3)-symmetry-breaking terms, (14) can also be applied to describe a shape (phase) transitional situation. As a preliminary application, $^{172}\text{Pt}$ [15] and $^{166}\text{Os}$ [13], of which unusual small $B_{4/2}$ ratio was observed, are fitted by the Hamiltonian (14) all with $N = 8$. Due to the scarcity of experimental data, the model parameters have been fully
TABLE I: The model fits for $^{172}$Pt [13] and $^{168}$Os [13] with $B(E2; L_i^0 \rightarrow L_j^0)$ values normalized to $B(E2; 2_1^+ \rightarrow 0_1^+)=1.0$, where “-” indicates the corresponding value is unknown in the experiments.

| Transition | $^{172}$Pt | $^{168}$Os |
|------------|------------|------------|
| $E(2_1^+)$ | 0.458      | 0.341      |
| $E(4_1^+)$ | 1.070      | 0.857      |
| $E(6_1^+)$ | 1.753      | 1.499      |
| $E(8_1^+)$ | 2.405      | 2.222      |
| $E(0_2^+)$ | -          | -          |
| $E(0_3^+)$ | -          | -          |

| Transition | IBM$_a$ | IBM$_b$ | IBM$_a$ | IBM$_b$ |
|------------|---------|---------|---------|---------|
| $2_1^+ \rightarrow 0_1^+$ | 1.0      | 1.0      | 1.0      |
| $4_1^+ \rightarrow 2_1^+$ | 0.55(19) | 0.677    | 0.34(18) | 0.656    |
| $6_1^+ \rightarrow 4_1^+$ | -        | 0.174    | -        | 0.111    |
| $2_2^+ \rightarrow 0_1^+$ | -        | 0.005    | -        | 0.001    |
| $0_2^+ \rightarrow 2_1^+$ | -        | 0.331    | -        | 0.237    |

The finite-N effect suppresses the $B_{4/2}$ ratio but keeps the $R_{4/2}$ ratio nearly unchanged, so that the mode with $B_{4/2} < 1.0$ appears when the low-lying yrast states are dominated by SU(3) irreps with small ($\lambda, \mu$). As a preliminary application, the low-lying energy levels and related $B(E2)$ values of $^{172}$Pt and $^{168}$Os are calculated from the IBM Hamiltonian involving the triaxial mode. It is shown that the yrast band and the depressed $B_{4/2}$ ratio can be excellently reproduced from the model calculation. The finite-N triaxial rotor mode proposed provides a simple yet promising mechanism of the anomalous $B(E2)$ values in the yrast band of neutron-deficient nuclei. Since the triaxial mode can also be realized in other models similarly under the SU(3) basis [11, 12, 18–21], the mechanism proposed in this work is expected to be solid.

Support from the National Natural Science Foundation of China (11875158, 12175097) and the US National Science Foundation (PHY-1913728) is acknowledged.
[1] A. Bohr and B. R. Mottelson, *Nuclear Structure II* (Benjamin, New York, 1975).

[2] A. S. Davydov and G. F. Filippov, Nucl. Phys. **8**, 237 (1958).

[3] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).

[4] F. Iachello and A. Arima, *The Interacting Boson Model* (England: Cambridge University, 1987).

[5] J. P. Elliott, Proc. R. Soc. A **245**, 128 (1958); **245**, 562 (1958); J. P. Elliott and M. Harvey, Proc. R. Soc. A **272**, 557 (1963); J. P. Elliott and C. E. Wilsdon, Proc. R. Soc. A **302**, 509 (1968).

[6] J. P. Draayer, T. Dytrych, K. D. Launey, and D. Langr, Prog. Part. Nucl. Phys. **67**, 516 (2012).

[7] B. R. Barrett, P. Navrátil, and J. P. Vary, Prog. Part. Nucl. Phys. **69**, 131 (2013).

[8] T. Dytrych, K. D. Launey, J. P. Draayer, P. Maris, J. P. Vary, E. Saule, U. Catalyurek, M. Sosonkina, D. Langr, and M. A. Caprio, Phys. Rev. Lett. **111**, 252501 (2013).

[9] T. Dytrych, K. D. Launey, J. P. Draayer, D. J. Rowe, J. L. Wood, G. Rosensteel, C. Bahri, D. Langr, and R. B. Baker, Phys. Rev. Lett. **124**, 042501 (2020).

[10] A. E. McCoy, M. A. Caprio, T. Dytrych, and P. J. Fasano, Phys. Rev. Lett. **125**, 102505 (2020).

[11] J. P. Draayer and K. J. Weeks, Phys. Rev. Lett. **51**, 1422 (1983); J. P. Draayer, S. C. Park, and O. Castaños, Phys. Rev. Lett. **62**, 20 (1989).

[12] D. Bonatsos, I. E. Assimakis, N. Minkov, A. Martinou, R. B. Cakirli, R. F. Casten, and K. Blaum, Phys. Rev. C **95**, 064325 (2017); D. Bonatsos, I. E. Assimakis, N. Minkov, A. Martinou, S. Sarantopoulou, R. B. Cakirli, R. F. Casten, and K. Blaum, Phys. Rev. C **95**, 064326 (2017).

[13] T. Grahn *et al.*, Phys. Rev. C **94**, 044327 (2016).

[14] B. Saygı *et al.*, Phys. Rev. C **96**, 021301(R) (2017).

[15] D. Bonatsos, I. E. Assimakis, N. Minkov, A. Martinou, S. Sarantopoulou, R. B. Cakirli, R. F. Casten, and K. Blaum, Phys. Rev. C **95**, 064326 (2017).

[16] A. Goasduff *et al.*, Phys. Rev. C **100**, 034302 (2019).

[17] T. Grahn *et al.*, Phys. Rev. C **94**, 044327 (2016).

[18] B. Saygı *et al.*, Phys. Rev. C **96**, 021301(R) (2017).

[19] B. Cederwall *et al.*, Phys. Rev. Lett. **121**, 022502 (2018).

[20] A. M. Oros-Peusquens, R. Zaballa, J. M. Allmond, and W. D. Kulp, Phys. Rev. C **70**, 024308 (2004).

[21] Y. Zeng, F. Pan, L. R. Dai, and J. P. Draayer, Phys. Rev. C **90**, 044310 (2014).

[22] J. L. Wood, A. M. Oros-Peusquens, R. Zaballa, J. M. Allmond, and W. D. Kulp, Phys. Rev. C **70**, 024308 (2004).

[23] T. Wang, EPL **129**, 52001 (2020).

[24] Y. Zhang, F. Pan, L. R. Dai, and J. P. Draayer, Phys. Rev. C **90**, 044310 (2014).

[25] J. L. Wood, A. M. Oros-Peusquens, R. Zaballa, J. M. Allmond, and W. D. Kulp, Phys. Rev. C **70**, 024308 (2004).

[26] T. Wang, EPL **129**, 52001 (2020).

[27] Y. Zhang, F. Pan, L. R. Dai, and J. P. Draayer, Phys. Rev. C **90**, 044310 (2014).