Abstract

In a recent paper, the authors have shown that the secondary reduction of $\mathcal{W}$-algebras provides a natural framework for the linearization of $\mathcal{W}$-algebras. In particular, it allows in a very simple way the calculation of the linear algebra $\mathcal{W}(G,H)_{\geq 0}$ associated to a wide class of $\mathcal{W}(G,H)$ algebras, as well as the expression of the $W$ generators of $\mathcal{W}(G,H)$ in terms of the generators of $\mathcal{W}(G,H)_{\geq 0}$.

In this paper, we present the extension of the above technique to $\mathcal{W}$-superalgebras, i.e. $\mathcal{W}$-algebras containing fermions and bosons of arbitrary (positive) spins. To be self-contained the paper recall the linearization of $\mathcal{W}$-algebras. We include also examples such as the linearization of $\mathcal{W}_n$ algebras; $\mathcal{W}(s\ell(3|1), s\ell(3))$ and $\mathcal{W}(osp(1|4), sp(4)) \equiv WB_2$ superalgebras.
1 Introduction

Recently there has been much interest in the linearization of $\mathcal{W}$-algebras. This linearization consists in adding a number of fields to a $\mathcal{W}$-algebra, in such a way that the resulting algebra is equivalent, by a nonlinear, invertible basis-transformation, to a linear algebra. Such a linearization was first proposed for $\mathcal{W}_3$ and $\mathcal{W}_3^2$ by Krivonos and Sorin in [1], and for $\mathcal{W}_{(2,4)}$ and $\mathcal{W}B_2$ in [2]. In [3] the present authors proposed a general method for the linearization of a large class of quantum $\mathcal{W}$ algebras, using the method of secondary quantum hamiltonian reduction. The advantage of such a technique is to give a general framework for the linearization and to use only fields of positive spin. In [4], a different method of linearization for superconformal algebras (i.e. algebras with one spin 2 and spin $\frac{3}{2}$ and spin 1 fields) was given. These algebras are not linearizable by our techniques, since their linearization uses fields of spin $-\frac{1}{2}$. Note that a conjecture about the linearization of the other $\mathcal{W}$-algebras, using superconformal algebras, spin ($-\frac{1}{2}$) fields and secondary reductions was also given in the same paper.

The purpose of the present paper is to explain in more detail the linearization proposed in [3], to show how to apply the method to classical $\mathcal{W}$ algebras, and to extend the method to $\mathcal{W}$ algebras that includes fermionic fields, i.e. $\mathcal{W}$ algebras that are obtained by the hamiltonian reduction of Lie superalgebras. In particular, we want to stress that our procedure indeed provides explicit realizations for the linearization of $\mathcal{W}$-algebras and superalgebras.

In a first section, we recall the linearization of $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebras and treat explicitly the case of $\mathcal{W}_n \equiv \mathcal{W}(\mathfrak{sl}(n))$ algebras. Then, in the second section, we show that the same procedure can be applied to superalgebras. Section 3 is devoted to two examples, one based on $\mathfrak{sl}(3\vert 1)$, and the other on $\mathfrak{osp}(1\vert 4)$. Finally, the last section conclude on open questions. We have also added a technical appendix where the most important parts of the conjecture given in [3] is proven.

For explicit calculations, we have used the Mathematica package of K. Thielemans [5].

2 Linearization of $\mathcal{W}$-algebras

2.1 Secondary reductions

To be self-contained, we briefly recall the framework of secondary reductions. For details, see the original papers on secondary reductions[3,4], for a review on $\mathcal{W}$-algebras, see [7], and [8] for a review on $\mathcal{W}$-algebras in the context of Hamiltonian reduction. We start with a $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebra. A realization of this algebra can be obtained from the Hamiltonian reduction of the affine Lie algebra $\mathcal{G}$ w.r.t. the constraints associated to the $\mathcal{G}$-subalgebra $\mathcal{H}$. More precisely, starting with the principal $\mathfrak{sl}(2)$ in $\mathcal{H}$, \{\(E_-, H, E_+\}\), we first define the gradation of $\mathcal{G}$ w.r.t. the Cartan generator $H$:

\[
\mathcal{G} = \bigoplus_{i=-h}^{h} \mathcal{G}_i = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+ \quad \text{with} \quad [H, X_i] = i \ X_i \quad \forall X_i \in \mathcal{G}_i
\] (2.1)
Then, we impose the following constraints on a general Lie algebra element \( G \):

\[
J(z)|_{<0} = E_-
\]  

(2.2)
i.e. the negative grade part of \( J(z) \) is equal to \( E_- \). At the classical level, these constraints generate gauge transformations

\[
J \rightarrow J^g = gJg^{-1} + k(\partial g)g^{-1} \quad \text{with} \quad g \in G_+ \quad \text{and} \quad \text{Lie}(G_+) = \mathcal{G}_+
\]

and the \( W \) generators can be viewed as a basis for the gauge invariant polynomials. The Poisson brackets of the original affine Lie algebra \( \mathcal{G} \) induce a symplectic structure on the space of invariant polynomials, hence providing a realization of the \( \mathcal{W} \)-algebra using Poisson brackets.

At the quantum level, we have to use ghosts and a BRST operator to take into account the constraints (2.2). We find the \( W \) generators as generators of the zeroth cohomological space \( H_0(\Omega, s) \). \( \Omega \) is the enveloping algebra of \( \mathcal{A} = \mathcal{G} \otimes \Gamma \), where \( \Gamma \) is the algebra generated by the ghosts (and anti-ghosts), and \( s \) the BRST operator. \( H_0(\Omega, s) \) possesses an algebraic structure which is just the quantum version of the \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra [4]. In particular, the classical limit of this quantum algebra is the \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra as it has been defined in terms of Poisson brackets.

Note that \textit{a priori}, the fundamental object in the Hamiltonian reduction is not the \( sl(2) \) but the gradation \( H \) (since it defines the constraints). However, the \( \mathcal{W} \)-algebras we get are not classified by these different gradations, but are in one to one correspondence with the \( sl(2) \) embeddings in \( \mathcal{G} \) [10]. Thus, we get classes of gradations that lead to the same \( \mathcal{W} \)-algebra, each class being represented by the Cartan generator of a \( sl(2) \)-subalgebra. For a given class, two different gradations will differ by a \( U(1) \) factor that commute with the whole \( sl(2) \)-subalgebra, and which will satisfy a non-degeneration condition [11]. This \( U(1) \) factor (say \( Y \)) leads to non-trivial informations about the structure of the \( \mathcal{W} \)-algebra, but taking as grading operator \( H \) or \( H' = H + Y \) provides the same \( \mathcal{W} \)-algebra in two different bases. However, the gauge group (in the classical case) or the BRST operator (in the quantum case) will not be the same, although the final results are identical. This \( U(1) \) factor plays also an important role for the secondary reductions [3].

Now, instead of starting with an affine Lie algebra, and then impose constraints on its generators, one can think of starting directly with a \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra and impose constraints on some of the \( W \) generators themselves [3]. This can effectively be done in certain cases, and the resulting algebra is a \textit{(a priori)} new \( \mathcal{W} \)-algebra. In fact, starting with a \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra such that there exists another subalgebra \( \mathcal{H}' \) with \( \mathcal{H} \subset_{\text{reg}} \mathcal{H}' \subset_{\text{reg}} \mathcal{G} \) (where the subscript \( \text{reg} \) indicates that the embeddings are regular), one can find a set of constraints such that the Hamiltonian reduction of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) leads to the \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra [2]. Let us add that there is a natural gradation on \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) such that the constrained generators are just the \( W \) generators of negative grades.

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1Strictly speaking, this has being proved only for \( \mathcal{G} = sl(n) \) in the classical case [3], and most of the cases where \( \mathcal{G} = sl(n) \), and some other cases with \( \mathcal{G} = so(n) \) or \( sp(2n) \), in the quantum case [3]. There are however strong arguments (and examples) in favor of the generality of this statement.
The secondary reductions lead to chains of $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebras that reproduce the chains of embeddings of the $\mathcal{G}$-subalgebras. Note that in a secondary reduction, we express the resulting $\mathcal{W}(\mathcal{G},\mathcal{H}')$ algebra in terms of the polynomials in the generators of the starting $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebra: this remark is fundamental for the linearization of $\mathcal{W}$-algebras.

### 2.2 Linearization of $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebras

Roughly speaking, the linearizations we present are just a special case of secondary reductions. In fact, when doing a (primary or secondary) Hamiltonian reduction, we realize the generators of the resulting algebra only in terms of the positive grade generators of the starting algebra (because the negative grades are all constrained). Then, even if the $\mathcal{W}(\mathcal{G},\mathcal{H})$ algebra is not linear, it may happen that its positive grade subalgebra $\mathcal{W}(\mathcal{G},\mathcal{H})_{\geq 0}$ is linear: it depends both of $\mathcal{W}(\mathcal{G},\mathcal{H})$ and of the gradation we choose (i.e. of the $\mathcal{W}(\mathcal{G},\mathcal{H}')$ algebra we want to obtain). If $\mathcal{W}(\mathcal{G},\mathcal{H})_{\geq 0}$ is linear, the secondary reduction $\mathcal{W}(\mathcal{G},\mathcal{H}) \to \mathcal{W}(\mathcal{G},\mathcal{H}')$ will provide a linearization of $\mathcal{W}(\mathcal{G},\mathcal{H}')$. The cases where this happens have been studied in [3].

For $\mathcal{G}=\mathfrak{sl}(n)$, they correspond to the starting algebras $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(2))$, called quasi-superconformal algebras. These algebras have generators only of spins 1, $\frac{3}{2}$, and one spin 2. The non-linear terms appears exclusively in the fundamental Poisson brackets (or OPEs) between two spin $\frac{3}{2}$ fields. More precisely, the set of spin $\frac{3}{2}$ fields can be divided into two subsets $S_\pm$ such that $S_+$ and $S_-$ are Abelian, while the quadratic terms appears in the Poisson brackets (or OPEs) of one $S_+$-field with one $S_-$-field. Then, if the (secondary) constraints on $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(2))$ are such that all the $S_-$-fields are constant, $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(2))_{\geq 0}$ will be obviously linear. This requirement is satisfied most of the times:

All the $\mathcal{W}(\mathfrak{sl}(n), \oplus_{i=1}^m \mathfrak{sl}(p_i))$ algebras are linearizable if we have $p_1 > p_i + 1$ ($i = 2, \ldots, m$).

We give hereafter concrete formulae for the linearization of (quantum and classical) $\mathcal{W}_n$ algebras.

For $\mathcal{G}=\mathfrak{sp}(2n)$, the calculation is the same: we start with the $\mathcal{W}(\mathfrak{sp}(2n), \mathfrak{sl}(2))$ algebra to linearize almost all the $\mathcal{W}(\mathfrak{sp}(2n), \mathcal{H})$ algebra. More precisely:

The linearization of the $\mathcal{W}(\mathfrak{sp}(2n), \mathcal{H})$ is possible when the subalgebra $\mathcal{H}$ takes the form $\mathcal{H} = \oplus_i \mathfrak{sl}(m_i) \oplus_{\mu} \mathfrak{sp}(2n_\mu)$ with

- $m_1 \in 2IN + 1$, $m_1 > m_i + 2$ ($\forall i \neq 1$), and $m_1 > 2n_\mu + 2$ ($\forall \mu$).
- $m_1 \in 2IN$, $m_1 \geq m_i + 2$ ($\forall i \neq 1$), and $m_1 \geq 2n_\mu + 2$ ($\forall \mu$).
- $n_1 > \frac{1}{2}(m_i + 2)$ ($\forall i$), and $n_1 > n_\mu + 1$ ($\forall \mu \neq 1$).

For $\mathcal{G}=\mathfrak{so}(n)$, the procedure is more restrictive. This is mainly due to the fact in the $\mathcal{W}(\mathfrak{so}(n), \mathfrak{sl}(2))$ algebra, there is no natural $U(1)$ factor that divides in two the spin $\frac{3}{2}$ generators, preventing us to find a "natural" linearized subalgebra.
We can linearize only the $\mathcal{W}(so(m), so(m'))$ algebras when $m = m' = 5$ or 6.

In all the linearizations we perform, the secondary reduction one has to perform is of the type $\mathcal{W}(G, s\ell(2)) \to \mathcal{W}(G, H)$, except for the $\mathcal{W}(so(5), so(5))$ algebra which is linearized through $\mathcal{W}(so(5), so(3)) \to \mathcal{W}(so(5), so(5))$, and the $s\ell(2)$ algebra is embedded into the "distinguished" subalgebra of $H$ ($s\ell(m_1)$ for $G = s\ell(n)$, and $sp(2n_1)$ or $s\ell(m_1)$ for $G = sp(2n)$).

2.2.1 Linearization of classical $\mathcal{W}_n$ algebras

We start with the $\mathcal{W}(s\ell(n), s\ell(2))$ algebra as it is obtained from the primary reduction of $s\ell(n)$. The fields are gathered in an $n \times n$ matrix:

\[
\begin{pmatrix}
U & T \\
1 & U \\
0 & G_1^+ \\
0 & G_2^- \\
\vdots & \vdots \\
0 & G_{n-2}^-
\end{pmatrix}
\begin{pmatrix}
G_1^+ \\
0 \\
0 \\
\vdots \\
0 \\
M
\end{pmatrix}
= \begin{pmatrix}
G_2^+ \\
\cdots \\
G_{n-2}^+ \\
0 \\
\cdots \\
0
\end{pmatrix}
\]

with
\[
\begin{aligned}
T & \text{ the spin 2 generator} \\
G_1^+ & \text{ the spin $\frac{3}{2}$ generators} \\
G_2^- & \text{ an } s\ell(n-2) \text{ matrix} \\
U & \text{ } U(1) \text{ generator} \\
M & \text{ a } s\ell(n-2) \text{ matrix}
\end{aligned}
\]

The subsets $S_\pm$ already mentioned are formed with the $G_i^\pm$ generators and correspond to the eigenvalues of these fields under the action of $U$. From the form of the above matrix, it is easy to see that the secondary constraints leading to the $\mathcal{W}_n$ algebra are

\[
G_i^- = \delta_{i,1} \quad \text{and} \quad M_\alpha = \chi_\alpha \quad (\alpha \text{ positive root})
\]

where $\chi_\alpha$ are the constraints of the Abelian Toda model built on $s\ell(n-2)$ (i.e. $\chi_\alpha = 1$ if $\alpha$ simple root of $s\ell(n-2)$ and 0 otherwise). These constraints generate gauge transformations on the $\mathcal{W}(s\ell(n), s\ell(2))$-fields. In [6], it has been shown that a correct gauge fixing for these secondary reductions is:

\[
(M_{\geq 0})^g = 0 \quad \text{and} \quad U^g = 0
\]

A priori, to get the realization of the resulting $\mathcal{W}$-algebra, one should first compute all the Poisson brackets of the $\mathcal{W}(s\ell(n), s\ell(2))$ algebra and then make the gauge transformations. Fortunately, for our purpose, there is a simpler way to get the realization.

The idea is to consider the general group transformations associated to $\mathcal{N}$, the algebra spanned by the positive root generators:

\[
J \to J^g = g J g^{-1} + k(\partial g) g^{-1} \quad \text{with} \quad g = \exp \begin{pmatrix}
0 & \cdots & \ast \\
\vdots & \ddots & \vdots \\
0 & \cdots & \ast
\end{pmatrix}
\]

(2.6)

For a general element $g$, (2.6) do not respect the form (2.3) with constraints (2.4) assumed. However, demanding the transformations (2.6) to send the matrix (2.3) with constraints (2.4)
to a matrix of the form:

\[
\begin{pmatrix}
0 & W_1 & \ldots & W_{n-2} & W_{n-1} \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

(2.7)

completely fixes the parameters of \( g \) and give the expression of the \( W_i \) in terms of the elements \( T(w), G_i^+(w), \) and \( M_{\geq 0}(w), U(w) \).

Thus it is a matter of straightforward calculations to get an expression for the parameters in \( g \) such that (2.4) is satisfied. Once this is done, the expressions of \( [T(w)]^g \) and \( [G_i^+(w)]^g \) will provide the linearization of the \( \mathcal{W}_n \) algebra in terms of the fields \( T(w), G_i^+(w), \) and \( M_{\geq 0}(w), U(w) \). Note that we only need to know the Poisson brackets (or OPEs) in the linearized algebra, not in the full \( \mathcal{W}(s\ell(n), s\ell(2)) \) algebra. The calculation of these OPEs has been replaced by extra equations coming from the general transformations (2.6).

To be complete, we give hereafter the OPEs of \( \mathcal{W}(s\ell(n), s\ell(2))_{\geq 0} \):

\[
T(z)T(w) = \frac{c/2}{(z-w)^2} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad T(z)U(w) = \frac{U(w)}{(z-w)^2} + \frac{\partial U(w)}{(z-w)}
\]

\[
T(z)M_a(w) = \frac{M_a(w)}{(z-w)^2} + \frac{\partial M_a(w)}{(z-w)} \quad T(z)G_i^+(w) = \frac{3}{2} \frac{G_i^+(w)}{(z-w)^2} + \frac{\partial G_i^+(w)}{(z-w)}
\]

\[
M_a(z)M_b(w) = \frac{k_{ab}}{(z-w)^2} + f_{ab} \frac{M_c(w)}{(z-w)} \quad U(z)U(w) = \frac{k'}{(z-w)^2}
\]

\[
M_a(z)G_i^+(w) = F_{ai}^j \frac{G_j^+(w)}{(z-w)} \quad U(z)G_i^+(w) = \frac{G_i^+(w)}{(z-w)}
\]

\[
G_i^+(z)G_j^+(w) = \text{regular} \quad U(z)M^\alpha(w) = \text{regular}
\]

(2.8)

**2.2.2 Linearization of quantum \( \mathcal{W}_n \) algebras**

For the quantum version of the above linearization, we need to calculate the cohomology space of a given BRST operator. The calculation is very similar to the primary reduction case [2]. We first have to introduce a pair of ghosts for each constraints (2.4): \((c_i, b^i) \ i = 1, \ldots, n-2\) and \((c^\alpha, b^\alpha)\), \(\alpha\) positive root. Then, the BRST operator acts as

\[
s(\varphi(w)) = \oint w \ dz \ j(z) \varphi(w) \quad \text{with} \quad j(z) = (M_a(z) - \chi_a)c^a(z) + (G_i^+(z) - \delta_{i,1})c^i(z) + \frac{1}{2} f_{\alpha\beta\mu}(b_\mu c^\beta c^\alpha)_0(z) + f_{\alpha j}^i(b_j c^i c^\alpha)_0(z)
\]

where \((\ )_0\) denotes the normal ordering \((AB)_0(w) = \oint w \frac{dw}{\bar{z}-w} A(z) B(w)\). This operator can be divided into two parts \( s = s_0 + s_1 \), and a tic-tac-toe procedure leads to the \( W \) generators. More precisely, we first divide the set \( \Omega \) in two subsets \( \bar{\Omega} \) and \( \mathcal{B} \) such that \( \Omega = \bar{\Omega} \otimes \mathcal{B} \), \( s(\mathcal{B}) \subset \mathcal{B} \), \( H(\mathcal{B}, s) = \mathcal{C} \), and \( \bar{\Omega} \) satisfying \( s(\bar{\Omega}) \subset \bar{\Omega} \), which leads to \( H_s(\bar{\Omega}, s) = H_s(\bar{\Omega}, s) \).

In practice, \( \mathcal{B} \) is built on the ghosts \( b^i, b^\alpha \) and \( s(b^i), s(b^\alpha) \), while \( \bar{\Omega} \) is generated by a suitable redefinition of the fields \( J^\alpha, G^i \) and \( T \) (see below).
Then, defining \( j_0(z) = -\chi_\alpha c^\alpha(z) - \delta_{i,1} c^i(z) \) and \( j_1(z) = j(z) - j_0(z) \), one starts by proving that as vector spaces there is an isomorphism between \( H_0(\hat{\Omega}, s) \) and \( H_0(\hat{\Omega}, s_0) \). But, \( H_0(\hat{\Omega}, s_0) = \ker s_0|_{\hat{\Omega}} \) which we denote by \( \hat{\Omega}_0 \). Note that the fields of \( \hat{\Omega}_0 \) play the role of the highest weights in the primary reductions. The generators of the \( W \)-algebra are obtained through the recursive relations \( s_1(W_i) = s_0(W_{i+1}) \), \( W_0 \in \hat{\Omega}_0 \), and \( W = \sum_i (-)^i W_i \). There is a bi-gradation (built on the initial gradation and the ghost number) that ensures that the sum is finite. \( W_0 \) has a bi-grade \((p, -p)\) and at each step \( W_i \) has a definite bi-grading \((p - i, i - p)\), with \( i = 0, \ldots, p \). Thus, the only technical difficulty is to find a set of generators \( W_0 \) for \( \hat{\Omega} \).

The following procedure provides a simple and explicit realization for \( \hat{\Omega} \) generators in a finite number of steps:

We start with the \( J^\alpha \), \( G^i \) and \( T \) generators. The generators \( J^\alpha \) are just residual Kac-Moody currents, so that the corresponding “hatted” generators are known:

\[
\hat{J}^\alpha = J^\alpha + f^{\alpha d} (b^c d)_0
\]  

(2.9)

Now, we focus on the \( G^i \)'s. \( s(G^i) \) can be decomposed as \( s(G^i) = R^i_{\beta \gamma}(\hat{J}, c) \bar{\alpha} J^\beta + P^i_{\alpha}(\hat{J}, c) \bar{\alpha} c_{\gamma} \). We first define \( G^i_1 = \frac{1}{2} R^i_{\beta \gamma}(\hat{J}, c) (J^\alpha b^\beta b^\alpha + P^i_{\alpha}(\hat{J}, c) b^\alpha + S^i_{\alpha}(\hat{J}, c) b^\alpha) \) and compute \( s(G^i - G^i_1) \): as a differential polynomial in the constrained currents (say \( W^\alpha \)) and \( b^\alpha \)'s, it contains terms of the form \( W^\alpha b^\beta, W^\alpha, \) and \( \partial W^\alpha \). Then, we introduce a \( G^i_2 \), which is just given by \( s(G^i - G^i_1) \) with the replacement rules \( W^\alpha b^\beta \to \frac{1}{2} b^\alpha b^\beta \), \( W^\alpha \to b^\alpha \), \( \partial W^\alpha \to \partial b^\alpha \) and any term which does not contain any \( W^\alpha \) is replaced by 0 (see definition of \( G^i_1 \) starting from \( s(G^i) \)). The final expression for \( \hat{G}^i \) is \( \hat{G}^i = G^i - G^i_1 - G^i_2 \).

Finally, \( T \) can be computed explicitly for any \( n \); it reads:

\[
\hat{T} = T - \frac{3}{2} (b_i \partial c^i)_0 + \frac{1}{2} (\partial b_i c^i + b_i \partial c^i)_0.
\]  

(2.10)

The above procedure can be generalized to any linearizable \( W \)-algebra: see the appendix for the scheme of linearization and the proofs.

We have checked the expressions for \( n = 2, 3, 4 \) and 5. Note that for \( n \) bigger or equal to 5, there is a technical fact to take into account: if one computes the OPEs of the above generators using Kac-Moody OPEs, one realizes that \( \hat{\Omega} \) is not an algebra. New generators, built only on ghosts, but with total ghost-number 0 appear at the right hand side of the OPEs. To make \( \hat{\Omega} \) an algebra, one has to define new OPEs through the formula:

\[
A \circ B = \pi(A \circ B) \quad \forall \ A, B \in \hat{\Omega} \text{ with } \pi \text{ the projection onto } \hat{\Omega}
\]  

(2.11)

where as a notation we have used \( (A \circ B) \) to denote the OPE of \( A \) with \( B \).

Note that we have not rigorously proven that “\( \circ \)” does in fact give an associative algebra in the general case, but we give a number of arguments for the validity of the procedure in the appendix.

Let us also remark that an alternative (and less systematic) approach for the linearization of \( W \)-algebras exists. In this approach, we start from the classical linearization (obtained as above). We adjust the various coefficients (in the expressions for the \( W \)-generators and in the expressions for the OPEs of the linearizing algebra) in such a way that the quantum OPEs of
the $\mathcal{W}$-generators gives a closed algebra. In that case clearly no extra ghost-terms appear, and it is clear that we do not need to modify the OPEs: everything will work as in the classical case. It seems evident that the linearizing algebra obtained in this way is identical to the algebra defined by the “$\diamond$” composition.

3 Linearization of $\mathcal{W}$-superalgebras

The case of super $\mathcal{W}$-algebras is very similar to what we have described so far for bosonic $\mathcal{W}$-algebras. One has first to study secondary reductions for super $\mathcal{W}$-algebras, and then to separate the reductions that provide linearizations. We first recall some general features on the Hamiltonian reductions of superalgebras.

3.1 Generalities on Hamiltonian reduction of superalgebras

As for algebras, to perform a Hamiltonian reduction on a superalgebra $SG$, we need to consider a gradation $H$ that will ensure the nil-potency of our set of (first class) constraints. The different $\mathcal{W}$-superalgebras one can get are not classified by the different gradations, but more precisely by the $sl(2)$ embeddings: there is still a freedom in a shift by a $U(1)$ factor. The classification of the $sl(2)$-embeddings in $SG$ is of same type as for Lie algebra: each $sl(2)$ algebra can be seen as the principal embedding in a regular subalgebra of $SG$, up to some exceptions for the $osp(m|n)$ superalgebras. Note that the subalgebra is embedded in the bosonic part of $SG$: although they have strong effects in the Hamiltonian reduction, the fermions do not play any role in this classification. However, a super-symmetric treatment of Hamiltonian reduction can also be done \[13, 14, 3\]: in that case, the super-symmetric $\mathcal{W}$-algebras are classified by $osp(1|2)$ embeddings, and the fermions do enter in the game.

Once the gradation is determined, the reduction follows the same lines as the bosonic case. The only difference relies on the grades $\pm \frac{1}{2}$ (when they exist). Indeed, the $G_{\frac{1}{2}}$ part leads to second class constraints: to cure it in Lie algebra, one has to use a "halving" procedure or to shift the grading operator with a $U(1)$ factor. For superalgebras, these techniques are not always possible, and in general one adds free (fermionic) spin $\frac{1}{2}$ fields (see examples).

At the classical level, one has just to constrain the $G_{<\frac{1}{2}}$ part of the current $J$ as usual, constrain the $G_{-\frac{1}{2}}$ part to be the free fields, and then fix the gauge symmetry in the highest weights gauge for instance.

At the quantum level, the free fields induce a new term in the decomposition of $s$: $s = s_{(0,1)} + s_{(\frac{1}{2},\frac{1}{2})} + s_{(1,0)}$ where the indices refer to the bi-grading ($s_{(0,1)}$ is the $s_0$ part of section 2.2.2, while $s_{(1,0)}$ is $s_1$). Although more complicated, the calculation can still be done $[15]$, and a tic-tac-toe construction used to get the corresponding generators.

For the secondary reductions of $\mathcal{W}$-superalgebras, we have to find a gauge fixing that ensures the embeddings of the sets of constraints. As far as $sl(m|n)$ superalgebras are concerned, the calculation is very similar to the case of $sl(m + n)$ algebras. Thus, we can define a generalized horizontal gauge for the classical case, or tune a $U(1)$ factor for the quantum case.
More precisely, at the classical level, if one wants to reduce $\mathfrak{sl}(m|n)$ with respect to, for instance, $[\mathfrak{sl}(m_1) \oplus \mathfrak{sl}(m_2)] \oplus \mathfrak{sl}(n_1)$ (with $\mathfrak{sl}(m_i)$ in $\mathfrak{sl}(m)$ and $\mathfrak{sl}(n_1)$ in $\mathfrak{sl}(n)$), the gauge fixing will be of the form

$$
\begin{pmatrix}
\ast & \ast & \ast & 1 (m_1) & \ast & \ast & \ast \\
1 & 1 & 1 & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{pmatrix}
$$

where the double lines delimit fermions and bosons, while the single lines indicate the positions of the $\mathfrak{sl}(m_i)$ and $\mathfrak{sl}(n_1)$ subalgebras. The entries where a $W$ generator appears are indicated by a star ($\ast$). For more details about the generalized horizontal gauge, see [6]. This proves that the secondary reductions for the $W$ superalgebras based on $\mathfrak{sl}(m|n)$ are always possible (at the classical level). For $\mathfrak{osp}(m|2n)$ superalgebras, such a horizontal gauge does not exist, but numerous examples (at the classical level) indicates that the secondary reductions are possible at least when the subalgebras that define the different $\mathfrak{sl}(2)$ embeddings are taken regular (which is not always possible, contrarily to the $\mathfrak{sl}(m|n)$ case). Thus, we can say:

**At the classical level, we can perform the secondary reductions:**

$$\mathcal{W}(\mathfrak{sl}(m|n), \mathcal{H}) \rightarrow \mathcal{W}(\mathfrak{sl}(m|n), \mathcal{H}') \text{ as soon as } \mathcal{H} \subset \mathcal{H}'.$$

**For $\mathfrak{osp}(m|2n)$ superalgebras, there is no doubt that**

$$\mathcal{W}(\mathfrak{osp}(m|2n), \mathcal{H}) \rightarrow \mathcal{W}(\mathfrak{osp}(m|2n), \mathcal{H}')$$

**is possible as soon as**

$$\mathcal{H} \subset_{reg} \mathcal{H}'$$

At the quantum level, we have to look at the sets of first class constraints of the different reductions, and see whether they can be embedded one into each other. The calculation is rather cumbersome: using $U(1)$ factors if necessary, we have to check case by case if the sub-superalgebra $SG_+$ is embedded in $SG'_+$. At the end, we are led to the result:
At the quantum level, we can perform the secondary reductions of the type:

\[ \mathcal{W}(s\ell(m|n), \mathcal{H}) \rightarrow \mathcal{W}(s\ell(m|n), \mathcal{H}') \quad \text{with} \quad \mathcal{H} = \oplus_i s\ell(m_i) \quad \text{and} \quad \mathcal{H}' = \oplus_{\mu} s\ell(m'_{\mu}) \quad \text{as soon as} \quad |m_i - m_j| > |m'_{\mu} - m'_{\nu}| \quad \forall i,j,\mu,\nu \quad \text{with} \quad i \neq j \quad \text{and} \quad \mu \neq \nu. \]

\[ \mathcal{W}(osp(m|2n), \mathcal{H}) \rightarrow \mathcal{W}(osp(m|2n), \mathcal{H}') \quad \text{is possible when} \]

- \( \mathcal{H} = \mathfrak{s}\ell(2) \subset \mathfrak{sp}(2N) \) and \( \mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2' \) where \( \mathcal{H}_1' = [ \oplus_i \mathfrak{so}(m_i) \oplus_{\mu} \mathfrak{s}\ell(m_{\mu}) ] \subset \mathfrak{so}(M) \), \( \mathcal{H}_2' = [ \oplus_j \mathfrak{sp}(2n_j) \oplus_{\nu} \mathfrak{s}\ell(p_{\nu}) ] \subset \mathfrak{sp}(2N) \) obey to one of the three conditions
  1. \( n_1 > n_j + 2 \quad (\forall j \neq 1), \quad n_1 > \frac{1}{2} (m_\mu + 2) \quad (\forall \mu), \quad n_1 \geq \left[ \frac{m_\mu - 1}{2} \right] + 2 \quad (\forall i) \) and
     \( n_1 > \frac{1}{2} (p_\nu + 2) \quad (\forall \nu) \).
  2. \( p_1 \in 2IN, \quad p_1 \geq p_\nu + 2 \quad (\forall \nu \neq 1), \quad p_1 \geq m_\mu + 2 \quad (\forall \mu), \quad p_1 \geq 2 \left[ \frac{m_\mu - 1}{2} \right] + 3 \quad (\forall i) \) and
     \( p_1 \geq 2n_j + 2 \quad (\forall j) \).
  3. \( p_1 \in 2IN + 1, \quad p_1 > p_\nu + 2 \quad (\forall \nu \neq 1), \quad p_1 > m_\mu + 2 \quad (\forall \mu), \quad p_1 > 2 \left[ \frac{m_\mu - 1}{2} \right] + 3 \quad (\forall i) \) and \( p_1 > 2n_j + 2 \quad (\forall j) \).
- \( \mathcal{H} = \mathfrak{s}\ell(2) \subset \mathfrak{sp}(2N) \) and \( \mathcal{H}' = \mathcal{H}_1' \oplus \mathfrak{sp}(4) \) where \( \mathcal{H}_1' = \left[ m_1 \mathfrak{s}\ell(3) \oplus m_2 \mathfrak{so}(3) \right] \subset \mathfrak{so}(M) \)
- \( \mathcal{H} = \mathfrak{s}\ell(2) \subset \mathfrak{sp}(2N) \) and \( \mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2' \) where \( \mathcal{H}_1' = m \mathfrak{s}\ell(2) \subset \mathfrak{so}(M) \), and \( \mathcal{H}_2' = (\mathfrak{sp}(4) \oplus_{\nu} \mathfrak{s}\ell(p_{\nu})) \subset \mathfrak{sp}(2N) \)

We conjecture that the general secondary reduction \( \mathcal{W}(SG, \mathcal{H}) \rightarrow \mathcal{W}(SG, \mathcal{H}') \) will be possible as soon as \( \mathcal{H} \subset_{\text{reg}} \mathcal{H}' \).

Note that what we have presented is a classification of algebras in which the secondary reductions can be carried out using our procedure of modifying the set of first class constraints by adding \( U(1) \) currents to the grading. It is not intended as an exhaustive classification of algebras where a secondary hamiltonian reduction is possible. On the contrary, as mentioned above, we conjecture that the secondary reductions are possible as soon as the embeddings are regular. However, to show this general property, other methods will have to be used, while the "\( U(1) \) factor technique" not only ensures the secondary reduction but also provide an explicit realization for it.

As we are interested here only in the linearization of \( \mathcal{W} \)-superalgebras, we will not go further into details. The proofs of the above properties are very similar to the bosonic case, and we refer to [3] for the interested reader. We rather focus on the linearization.

### 3.2 Linearizations

The linearization of \( \mathcal{W} \)-superalgebras is very similar to the linearization of \( \mathcal{W} \)-algebras. For a given \( \mathcal{W}(SG, \mathcal{H}) \) superalgebra, and among all the possible secondary reductions, we look for \( \mathcal{W}(SG, \mathcal{H}_0) \) superalgebra such that the reduction \( \mathcal{W}(SG, \mathcal{H}_0) \rightarrow \mathcal{W}(SG, \mathcal{H}) \) is possible and \( \mathcal{W}(SG, \mathcal{H}_{\geq 0}) \) is a linear superalgebra.
3.2.1 \( \mathfrak{sl}(m|n) \) superalgebras

As already mentioned, the reduction of \( \mathfrak{sl}(m|n) \) superalgebras is very similar to the case of \( \mathfrak{sl}(m+n) \) algebras. Indeed, for the linearization, the results are still very similar:

The linearization of \( \mathcal{W}[\mathfrak{sl}(n|m), \mathcal{H}] \) superalgebras can be done by the secondary reductions \( \mathcal{W}(\mathfrak{sl}(n|m), \mathfrak{sl}(2)) \rightarrow \mathcal{W}(\mathfrak{sl}(n|m), \mathcal{H}) \) as soon as \( \mathcal{H} \) satisfies: \( \mathcal{H} = \oplus_i \mathfrak{sl}(p_i) \) with \( p_1 > p_i + 1 \) (\( i \neq 1 \)). In that case, the \( \mathcal{H}_0 = \mathfrak{sl}(2) \) algebra will be embed in \( \mathfrak{sl}(p_1) \) for the secondary reduction.

In particular, the superalgebras \( \mathcal{W}[\mathfrak{sl}(n|m), \mathfrak{sl}(n) \oplus \mathfrak{sl}(m)] \) are linearizable as soon as \( m \neq n, n \pm 1 \).

Let us first describe the \( \mathcal{W}(\mathfrak{sl}(n|m), \mathfrak{sl}(2)) \) superalgebra; it contains one spin 2 generator \( T \) (stress energy tensor), \( 2m \) fermionic spin \( \frac{3}{2} \) fields \( G_i^\pm \) (super-symmetry charges), a \( \mathfrak{sl}(n-2) \oplus \mathfrak{sl}(m) \oplus g(1) \) Kac-Moody subalgebra (spin 1 generators \( M_a \)), as well as \( 2n(m-2) \) spin 1 fermionic fields \( Q_i^\pm \) and \( 2m \) spin \( \frac{3}{2} \) bosonic fields \( D_i^\pm \).

The fields \( G_i^\pm \) and the fields \( D_i^\pm \) form a \( m \oplus m \) representation of the \( \mathfrak{sl}(m) \) subalgebra, they are trivial under \( \mathfrak{sl}(n-2) \) transformations, and the index \( \pm \) denotes the \( g(1) \) charge. The fields \( Q_i^\pm \) form two \( (n-2,m) \) representations of \( \mathfrak{sl}(n-2) \oplus \mathfrak{sl}(m) \).

At the classical level, for the linearization, we start with the \( \mathcal{W}(\mathfrak{sl}(n|m), \mathfrak{sl}(2)) \) superalgebra in the generalized horizontal gauge (as defined above, see example (3.1)), and we perform a gauge transformation that will lead to the \( \mathcal{W}[\mathfrak{sl}(n|m), \mathcal{H}] \) superalgebra in the generalized horizontal gauge. The expression of the transformed fields as functions of the fields of \( \mathcal{W}[\mathfrak{sl}(n|m), \mathcal{H}]_{\geq 0} \) provides the linearization. Note that the techniques described in section 2.2.1 still work.

At the quantum level, we introduce the index \( \alpha \) and \( \bar{\alpha} \) for the respectively negatively and non-negatively graded Kac-Moody generators. Then, the BRST current is

\[
j(z) = (M_a(z) - \chi_\alpha)c^a(z) + (G_{i}^-(z) - \chi_i)\gamma^i(z) + (Q_{ij}^-(z) - \chi_{ij})\gamma^{ij}(z) + (D_{i}^-(z) - \chi^i)c^i(z) + \frac{1}{2} \epsilon_{abc} c^b (B^c C_b C_a)_{0}(z)
\]

(3.2)

where we have introduced fermionic ghosts \( (b^a, c_\alpha) \) and \( (\bar{b}^i, c_i) \), and bosonic ghosts \( (\beta^{ij}, \gamma_{ij}) \). We have denoted generically by \( B \) and \( C \) the ghosts and by \( a, b, c, \ldots \) the indices; \( f_{abc} \) are the structure constants of the (linear) algebra \( \mathcal{W}_{\geq 0} \), and \( \epsilon_{a} = 1 \) (resp. \( \epsilon_{a} = -1 \)) if \( C_b \) is a bosonic (resp. fermionic) ghost. The cohomology of \( s \) will provide the linearization. We present hereafter an example for the calculation of this cohomology (hence for the linearization).

3.2.2 \( \mathfrak{osp}(M|2N) \) superalgebras

The calculation for \( \mathfrak{osp}(M|2N) \) superalgebras resembles the one for \( \mathfrak{sl}(m|n) \). Using the results for \( \mathfrak{so}(M) \) and \( \mathfrak{sp}(2N) \) algebras, one sees that we will mostly use the \( \mathfrak{sp}(2N) \) part to linearize, except for few particular values of \( M \) in \( \mathfrak{so}(M) \). More precisely:
The linearization of $\mathcal{W}[osp(2M+1|2N),\mathcal{H}]$ superalgebras is possible through the secondary reduction $\mathcal{W}[osp(2M+1|2N),\mathcal{H}_0] \to \mathcal{W}[osp(2M+1|2N),\mathcal{H}]$ when $\mathcal{H}_0 = \mathfrak{sl}(2) \subset \mathfrak{sp}(2N)$ and $\mathcal{H} = \oplus_p \mathfrak{so}(m_p) \oplus_i \mathfrak{sl}(m_i) \oplus_\mu \mathfrak{sp}(2n_\mu)$ with

- Either $m_1 \in 2N$, $m_1 \geq m_i + 1 \ (\forall i \neq 1)$, $m_1 \geq 2m_p + 1 \ (\forall p)$, and $m_1 \geq 2n_\mu + 1 \ (\forall \mu)$.
- Or $n_1 \geq \frac{1}{2}(m_i + 1) \ (\forall i)$, $n_1 \geq m_p + 1 \ (\forall p)$, and $n_1 \geq 2n_\mu + 1 \ (\forall \mu \neq 1)$.

In particular, the linearization of $\mathcal{W}[osp(2m + 1|2n), \mathfrak{so}(2m + 1) \oplus \mathfrak{sp}(2n)]$ superalgebras is possible when $m < n - 1$.

As for $\mathfrak{sl}(m|n)$ superalgebras, the linearization is done using a quadratic superconformal algebra. The calculations are of same type: for the classical level, we have to compute the gauge transformations that leads to the (secondary) $\mathcal{W}$-superalgebra, and at the quantum level, it is once more the cohomology of the BRST operator that will give rise to the linearization. The new feature is the emergence of auxiliary fields: see example below.

## 4 Examples

### 4.1 Case of $\mathcal{W}(\mathfrak{sl}(3|1), \mathfrak{sl}(3))$

#### 4.1.1 Classical Linearization

In order to demonstrate the linearization procedure, we will consider the linearization of the algebra $\mathcal{W}(\mathfrak{sl}(3|1), \mathfrak{sl}(3))$ in some detail. This superalgebra is comparatively simple, but it still shows most of the characteristics of our linearization procedure. Note that this example has already been done in [16]; we repeat it here only to demonstrate our method.

We start with the superalgebra $\mathfrak{sl}(3|1)$, parameterized by

$$J(z) = \begin{pmatrix}
H_1(z) + Y(z) & J_1(z) & J_2(z) & j_1(z) \\
J_4(z) & H_2(z) - H_1(z) + Y(z) & J_3(z) & j_2(z) \\
J_5(z) & J_6(z) & -H_2(z) + Y(z) & j_3(z) \\
j_1(z) & j_2(z) & j_3(z) & 3Y(z)
\end{pmatrix}$$

The constraints and highest weight gauge resulting in the algebra $\mathcal{W}(\mathfrak{sl}(3|1), \mathfrak{sl}(2))$ are

$$J_c(z) = \begin{pmatrix}
H_1(z) + Y(z) & J_1(z) & J_2(z) & j_1(z) \\
1 & H_2(z) - H_1(z) + Y(z) & J_3(z) & j_2(z) \\
0 & J_6(z) & -H_2(z) + Y(z) & j_3(z) \\
0 & 0 & j_2(z) & j_3(z) & 3Y(z)
\end{pmatrix}$$

$$J_{hw}(z) = \begin{pmatrix}
U(z) + Y(z) & \tilde{T}(z) & W^+(z) & G^+(z) \\
1 & U(z) + Y(z) & 0 & 0 \\
0 & W^-(z) & -2U(z) + Y(z) & B^-(z) \\
0 & G^-(z) & B^+(z) & 3Y(z)
\end{pmatrix}.$$  \hspace{1cm} (4.1)
The secondary first class constraints that lead to the algebra $\mathcal{W}(s\ell(3|1), s\ell(3))$ are $W(z) = 1$ and $G(z) = 0$, and these first class constraints induce a gauge-invariance which can be used to choose the gauge $U(z) = B(z) = 0$, so the result is

$$J_w(z) = \begin{pmatrix}
Y(z) & \tilde{T}(z) & W(z) & G(z) \\
1 & Y(z) & 0 & 0 \\
0 & 1 & Y(z) & 0 \\
0 & 0 & B(z) & 3Y(z)
\end{pmatrix}. \quad (4.2)$$

$U$ and $Y$ are two $U(1)$ Kac-Moody currents (spin 1 primary fields), $W^\pm$ are primary spin $\frac{3}{2}$ bosonic fields with $(U, Y)$ charges $(\pm \frac{1}{2}, 0)$, $G^\pm$ are primary spin $\frac{3}{2}$ fermionic fields with charges $(\pm \frac{1}{6}, \pm \frac{1}{3})$, and $B^\pm$ are fermionic Kac-Moody currents with charges $(\pm \frac{1}{3}, \mp \frac{1}{3})$.

Again, doing the finite gauge transformation $J \to J^g = g J g^{-1} + k (\partial g) g^{-1}$ with $g = \exp(\Lambda)$ and

$$\Lambda = \begin{pmatrix}
0 & \epsilon_1 & \epsilon_2 & \gamma_1 \\
0 & 0 & 0 & 0 \\
0 & \epsilon_3 & 0 & 0 \\
0 & \gamma_2 & 0 & 0
\end{pmatrix}$$

and requiring the result to respect the secondary constraints and take the form (4.2), we find conditions for all the gauge parameters, and the result $J^g$ gives the classical linearization of $\mathcal{W}(s\ell(3|1), s\ell(3))$:

$$\mathcal{T} = \frac{1}{k} \left( T + 3Y^2 - (3k - 1) \partial U + \partial Y \right)$$

$$W = W^+ - k B^- \partial B^+ - 2k \partial B^- B^+ + 2kTU + (6k + 2)U \partial U + 2U \partial Y + 4B^- B^+U - 2B^- B^+Y - 8U^3 + 6UY^2 - 2k \partial^2 U$$

$$G = G^+ - k B^- T - B^- \partial U - k B^- \partial U - B^- \partial Y + 2k B^- \partial Y - 2k \partial B^- + 4k Y \partial B^- + 4B^- U^2 - 4B^- UY + B^- Y^2 + k^2 \partial^2 B^-$$

$$B = B^+$$

$$Y = Y \quad (4.3)$$

Note that $T$ and $\mathcal{T}$ are the normalized energy-momentum tensors, corrected with quadratic terms in the Kac-Moody currents, such that all fields are primary.

### 4.1.2 Quantum Linearization

In order to perform the quantum linearization, we need to know the operator product expansions of the algebra $\mathcal{W}(s\ell(3|1), s\ell(2))$. As noted above, this algebra consists of the energy-momentum tensor $T$ with central charge $-\frac{(2k+1)(3k+4)}{k+2}$, two $U(1)$ Kac-Moody currents $U$ and $Y$, two spin $\frac{3}{2}$ bosonic fields $W^\pm$ with $(U, Y)$ charges $(\pm \frac{1}{2}, 0)$, two fermionic spin $\frac{3}{2}$ fields $G^\pm$ with charges $(\pm \frac{1}{6}, \pm \frac{1}{3})$, and fermionic Kac-Moody currents $B^\pm$ with charges $(\pm \frac{1}{3}, \mp \frac{1}{3})$. The non-trivial operator product expansions not already implicitly given are:
\begin{align}
U(z)U(w) &= \frac{(3k + 4)/18}{(z - w)^2} \\
U(z)Y(w) &= \frac{-1/18}{(z - w)^2} \\
Y(z)Y(w) &= \frac{-(3k + 2)/18}{(z - w)^2} \\
B^+(z)B^-(w) &= \frac{-(k + 1)}{z - w} + \frac{-2Y - 2U}{z - w} \\
B^+(z)G^+(w) &= \frac{W^\pm}{z - w} \\
B^+(z)W^+(w) &= \frac{\pm G^\mp}{z - w} \\
W^+(z)G^+(w) &= \frac{2(1 + k)^2}{(z - w)^3} + \frac{2Y - (6k + 4)U}{(z - w)^2} + \frac{(k + 2)T + 2(B^+B^0 - 12(UU)_0 + 3(YY)_0 - 3k\partial U + 3\partial Y)}{z - w} \\
G^+(z)G^-(w) &= \frac{-2(1 + k)^2}{(z - w)^3} + \frac{-2kU + (4k + 6)Y}{(z - w)^2} + \frac{(k + 2)T - 4(UU)_0 + 4(UY)_0 - (YY)_0 - k\partial U + (2k + 3)\partial Y}{z - w}
\end{align}

We now introduce a fermionic ghost pair \((b, c)\) corresponding to the secondary first class constraint \(W^- = 1\), and a bosonic ghost pair \((\beta, \gamma)\) corresponding to the constraint \(G^- = 0\), and we define the BRST current

\[ j = (W^- - 1)c + G^-\gamma \]

As described before, we need to define modified “hatted” generators in such a way that the modified, unconstrained generators together with the anti-ghosts gives a sub-complex \(\hat{\Omega}\), i.e. such that the BRST operator acting on the unconstrained generators does not involve constrained generators. We find:

\begin{align}
\hat{B}^- &= B^- - b\gamma \\
\hat{B}^+ &= B^+ + \beta c \\
\hat{U} &= U - \frac{1}{2}(bc)_0 + \frac{1}{6}(\beta\gamma)_0 \\
\hat{Y} &= Y + \frac{3}{2}(\beta\gamma)_0 \\
\hat{T} &= T - \frac{1}{2}(\partial bc)_0 - \frac{3}{2}(b\partial c)_0 + \frac{1}{2}(\partial \beta\gamma)_0 + \frac{3}{2}(\beta\partial\gamma)_0
\end{align}
while $\hat{G}^+ = G^+$ and $\hat{W}^+ = W^+$.

the central charge of $\hat{T}$ is $\hat{c} = -\frac{(2k+1)(3k+4)}{k+2}$. The operator product expansions of the “hatted” generators are unchanged, except for:

\[
\hat{T}(z)\hat{U}(w) = -\frac{2}{3}(k+2) + \frac{\hat{U}}{(z-w)^2} + \frac{\partial \hat{U}}{z-w},
\]

\[
\hat{T}(z)\hat{Y}(w) = \frac{2}{3}(k+2) + \frac{\hat{U}}{(z-w)^2} + \frac{\partial \hat{U}}{z-w},
\]

\[
\hat{U}(z)\hat{U}(w) = \frac{(3k+8)/18}{(z-w)^2},
\]

\[
\hat{U}(z)\hat{Y}(w) = -\frac{1}{9},
\]

\[
\hat{Y}(z)\hat{Y}(w) = -\frac{(3k+4)/18}{(z-w)^2},
\]

\[
\hat{B}^+(z)\hat{B}^-(w) = -\frac{(k+2)}{(z-w)^2} + \frac{-2\hat{U} - 2\hat{Y}}{z-w}.
\]

(4.6)

We find the generators of the BRST-cohomology, and thereby the generators of $\mathcal{W}(sl(3|1), sl(3))$ to be:

\[
\mathcal{T} = \hat{T} - 3\partial \hat{U},
\]

\[
\mathcal{G} = \hat{G}^+ - (k+2)(\hat{B}\hat{Y})_0 - k(\hat{B}\partial \hat{U})_0 + (2k+3)(\hat{B}\partial \hat{Y})_0 - 2(k+2) \left[ (\hat{U}\partial \hat{B})_0 - 2(\hat{Y}\partial \hat{B})_0 \right] + 4(\hat{B}\hat{U}\hat{U})_0 - 4(\hat{B}\hat{U}\hat{Y})_0 + (\hat{B}\hat{Y}\hat{Y})_0 + \frac{1}{2}(k+2)(2k+1)\partial^2 \hat{B}^{-},
\]

\[
\mathcal{W} = \hat{W}^+ - k(\hat{B}\partial \hat{B}^+)_0 - 2(k+2) \left[ (\partial \hat{B}^+\hat{B})_0 - (\hat{T}\hat{U})_0 \right] + 2(4+3k)(\hat{U}\partial \hat{U})_0 + 4(\hat{B}^+\hat{U})_0 + 2(\hat{B}^+\hat{Y})_0 - 8(\hat{U}\hat{U}\hat{U})_0 + 6(\hat{Y}\hat{Y}\hat{U})_0 - 2(k+2)^2\partial^2 \hat{U} + 2(\hat{U}\partial \hat{Y})_0 - 4(\hat{Y}\partial \hat{U})_0
\]

\[
\mathcal{B} = \hat{B}^+,
\]

\[
\mathcal{Y} = \hat{Y}.
\]

(4.7)

Note that except for normal-ordering contributions, we recover the classical linearization (4.3).

### 4.2 Case of $\mathcal{WB}_2 \sim \mathcal{W}(osp(1|4), sp(4))$

#### 4.2.1 Classical Linearization

As a second example, we have chosen the linearization of the algebra $\mathcal{W}(osp(1|4), sp(4))$. 


The affine \( osp(1|4) \) can be parameterized by:

\[
J(z) = \begin{pmatrix}
0 & j_1(z) & j_2(z) & j_3(z) & j_4(z) \\
-j_4(z) & H_1(z) & j_1(z) & j_3(z) & j_4(z) \\
-j_3(z) & j_5(z) & H_2(z) & j_2(z) & j_3(z) \\
-j_2(z) & j_7(z) & j_6(z) & -H_2(z) & -j_1(z) \\
-j_1(z) & j_8(z) & j_7(z) & -j_5(z) & -H_1(z)
\end{pmatrix}
\]

\( j_i \) denotes the fermionic currents, while \( H_i \) and \( J_i \) denotes the bosonic currents. The \( \mathcal{W} \) algebra \( \mathcal{W}(osp(1|4), sp(1|4)) = \mathcal{W}B_2 \) is obtained by imposing the constraints

\[
J_c(z) = \begin{pmatrix}
0 & 0 & \psi(z) & j_3(z) & j_4(z) \\
-j_4(z) & H_1(z) & J_1(z) & J_3(z) & J_4(z) \\
-j_3(z) & 1 & H_2(z) & J_2(z) & J_3(z) \\
\psi(z) & 0 & 1 & -H_2(z) & -J_1(z) \\
0 & 0 & 0 & -1 & -H_1(z)
\end{pmatrix}
\]

where \( \psi(z) \) is an auxiliary free fermion, normalized such that

\[
\psi(z)\psi(w) = \frac{1}{2} \frac{1}{z - w}
\]

We can use the gauge-invariance induced by these first class constraints to choose the highest weight gauge, which takes the form:

\[
J_{hw}(z) = \begin{pmatrix}
0 & 0 & 0 & 0 & G(z) \\
-G(z) & 0 & 3T(z) & 0 & W(z) \\
0 & 1 & 0 & 4T(z) & 0 \\
0 & 0 & 1 & 0 & -3T(z) \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]

where \( T \) is the energy-momentum tensor, \( W \) is a spin 4 primary field, and \( G \) is a spin \( \frac{5}{2} \) primary fermionic field.

In order to linearize this algebra, we should consider the algebra \( \mathcal{W}(osp(1|4), s\ell(2)) \), obtained by imposing the constraints and the highest weight gauge

\[
J_c(z) = \begin{pmatrix}
0 & 0 & j_2(z) & j_3(z) & j_4(z) \\
-j_4(z) & H_1(z) & J_1(z) & J_3(z) & J_4(z) \\
-j_3(z) & 1 & H_2(z) & J_2(z) & J_3(z) \\
-j_2(z) & 0 & J_6(z) & -H_2(z) & -J_1(z) \\
0 & 0 & 0 & -1 & -H_1(z)
\end{pmatrix}
\]

(4.8)

\[
J_{hw}(z) = \begin{pmatrix}
0 & 0 & G^-(z) & 0 & G^+(z) \\
-G^+(z) & U(z) & T(z) & 0 & W_+(z) \\
0 & 1 & U(z) & 0 & 0 \\
G^- & 0 & W^-(z) & -U(z) & -T(z) \\
0 & 0 & 0 & -1 & -U(z)
\end{pmatrix}
\]

(4.9)
Using the soldering procedure, we can find the operator product expansions of this algebra. The normalized energy-momentum tensor is $T = \frac{2}{k} \tilde{T}$, with a central charge of $c = -12k$ ($k$ is the level of the affine $OSp(1|4)$). $U$ is a primary $U(1)$ current, $G^\pm$ are primary fermionic spin $\frac{3}{2}$ currents with $U(1)$ charge $\pm \frac{1}{2}$, and $W^\pm$ are primary bosonic spin 2 currents with $U(1)$ charge $\pm 1$. The rest of the non-trivial operator product are given in the quantum form in equation (4.14): to get the classical operator product expansions, one has simply, in each term, to discard all but the leading order in $k$.

In order to find the classical linearization of the $\mathcal{WB}_2$ algebra, we impose the secondary constraints $W(osp(1|4), s\ell(2))$: $G^-(z) = \psi(z)$ and $W^-(z) = 1$. We then make a finite gauge-transformation $J \to J^g = gJg^{-1} + k(\partial g)g^{-1}$ where $g = \exp(\Lambda), \Lambda \in \mathcal{N}$, such that the gauge-transformed current $J^g$ is of the form

$$J(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & G(z) \\ -G(z) & 0 & T(z) & 0 & W(z) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\tilde{T}(z) \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$ (4.10)

Note that while this is indeed the currents of the algebra $\mathcal{WB}_2$, it is not in the highest weight gauge, and therefore $G$ and $W$ may not be primary fields in this basis.

We find that the finite, field dependent gauge-transformation that takes the constrained current into the gauge (4.10) is

$$\Lambda = \begin{pmatrix} 0 & 0 & -\psi & \frac{3}{2}\psi U - k\partial \psi \\ -\frac{3}{2} + k\psi & 0 & -U & U^2 - k\partial U & \epsilon \\ \psi & 0 & 0 & -2U & U^2 - k\partial U \\ 0 & 0 & 0 & 0 & U \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ (4.11)

$$\epsilon = \psi G^+ - \frac{k^2}{2} \psi \partial^2 \psi + 2\tilde{T}U + kU\partial U + \frac{k}{6}U\psi\partial \psi - \frac{1}{3}U^3 - k^2\partial^2 U.$$. (4.12)

Performing the gauge-transformation, we find the linearization of the algebra:

$$G = G^+ + \frac{k}{2} \psi T + \frac{k}{2} \psi \partial U + kU\partial \psi - \frac{1}{2} \psi U^2 - k^2\partial^2 \psi$$

$$T = T - \psi \partial \psi + U^2 - 2\partial U$$

$$W = W^+ - 3kG^+ \partial \psi + k\psi \partial G^+ + 2G^+ \psi U + T\partial U + \frac{k^2}{2}U\partial T - \frac{k}{2}T U^2 - \frac{k^2}{2}T \psi \partial \psi + -\frac{k^3}{2}(\psi \partial^2 \psi + 3\partial \psi \partial^2 \psi) + \frac{k^2}{2}U \partial^2 U - \frac{k^2}{4}\partial U \partial U - \frac{k^3}{2}\partial^3 U + k^2(\psi U \partial^2 \psi - \psi \partial \psi \partial U)$$

### 4.2.2 Quantum Linearization

To perform the quantum linearization, we perform first the quantum hamiltonian reduction leading to $\mathcal{W}(osp(1|4), s\ell(2))$. As a result of this procedure, we get the generators of the
algebra in terms of the generators of $OSp(1|4)$ (which we do not need for the linearization), and the operator product expansions. We find the central charge to be $c = \frac{2(4k+7)(6k+5)}{2k+5}$. $U$ is a primary $U(1)$ current, $G^\pm$ are primary fermionic spin $\frac{3}{2}$ currents with $U(1)$ charge $\pm\frac{1}{2}$, and $W^\pm$ are primary bosonic spin 2 currents with $U(1)$ charge $\pm 1$. The rest of the non-trivial operator product expansions are:

$$
U(z)U(w) = \frac{k + \frac{7}{4}}{(z-w)^2}
$$

$$
G^\pm(z)G^\pm(w) = \frac{\pm\frac{1}{2}W^\pm}{z-w}
$$

$$
G^+(z)G^-(w) = \frac{-(1+k)(7+4k)/4}{(z-w)^3} + \frac{-\frac{1}{2}(k+1)U}{(z-w)^2} + \frac{-\frac{1}{8}(2k+5)T - \frac{1}{4}(UU)_0 - \frac{1}{4}(k+1)\partial U}{z-w}
$$

$$
G^\pm(z)W^\mp(w) = \frac{\mp\frac{1}{2}(3k+5)G^\mp}{(z-w)^2} + \frac{\mp\frac{1}{2}(k+2)\partial G^\mp - (UG^\mp)_0}{z-w}
$$

$$
W^+(z)W^-(w) = \frac{(k+1)(3k+5)(4k+7)}{z-w^4} + \frac{(k+1)(3k+5)U}{(z-w)^3} + \frac{\frac{1}{8}(2k+3)(2k+5)T + \frac{1}{2}(5+4k)(UU)_0 + \frac{1}{2}(k+1)(3k+5)\partial U}{(z-w)^2} + \frac{\frac{1}{8}(2k+5)(TU)_0 + (UUU)_0 - 2(G^-G^+)_0 - \frac{1}{4}(k+1)(2k+5)\partial T}{z-w}
$$

$$
+ \frac{2(k+1)(U\partial U)_0 + \frac{1}{4}(2k^2 + 8k + 9)\partial^2 U}{z-w}
$$

(4.14)

In order to perform the secondary quantum hamiltonian reduction, we now need to impose the secondary constraints $G^-(z) = \psi(z)$ and $W^-(z) = 1$. In order to do this we introduce a ghost pair corresponding to each constraint, a fermionic pair $(b, c)$ for the constraint $W^-(z) = 1$, and a bosonic pair $(\beta, \gamma)$ (with OPE $\gamma(z)\beta(w) = \frac{1}{z-w}$) for the constraint $G^-(z) = \psi(z)$. The BRST current then takes the form

$$
j = (W^- - 1)c + (G^- - \psi)\gamma + \frac{1}{4}b\gamma\gamma
$$

Now, we have to introduce the “hatted” generators that are the starting point of the tic-tac-toe construction (see section 2.2.2). They are defined by

$$
\hat{G}^- = s(\beta) + \psi = G^- + \frac{1}{2}\gamma b
$$

$$
\hat{W}^- = s(b) + 1 = W^-
$$

$$
\hat{U} = U - \frac{1}{4}(\beta\gamma)_0 + \frac{1}{2}(bc)_0
$$

$$
\hat{T} = T - 2(b\partial c)_0 - (\partial bc)_0 - \frac{1}{2}(\gamma\partial \beta)_0 - \frac{3}{2}(\partial \gamma \beta)_0
$$

$$
\hat{G}^+ = G^+ + \frac{1}{2}(3k+5)\beta \partial c + (k+2)\partial \beta c - U\beta c - \frac{1}{4}(\beta \gamma bc)_0
$$

$$
\hat{W}^+ = W^+ + 2G^+ \beta c + \frac{3k+5}{2}\beta \partial cc
$$

(4.15)
The generators that will be used for the linearization are $\hat{U}$, $\hat{G}^+$ and $\hat{T}$, together with the free fermion $\psi$. The modified central charge is $\hat{c} = -\frac{2(12k^2 + 46k + 55)}{2k+5}$, and the operator product expansions of the “hatted” generators are:

$$
\hat{T}(z)\hat{U}(w) = -\frac{2}{(z-w)^3} + \frac{\hat{U}}{(z-w)^2} + \frac{\partial\hat{U}}{z-w}
$$

$$
\hat{U}(z)\hat{U}(w) = \frac{k + \frac{7}{2}}{(z-w)^2}
$$

$$
\psi(z)\psi(w) = \frac{1/2}{z-w}, \quad (4.16)
$$

while the rest of the operator product expansions are unchanged. We now find that the generators of the zeroth cohomology of $s$, i.e. the generators of $\mathcal{WB}_2$, are:

$$
T = \hat{T} - 2\partial\hat{U} - (\psi\partial\psi)_0
$$

$$
G = \hat{G}^+ + \frac{1}{4}(2k + 5)\psi\hat{T} + \frac{1}{2}(k+1)\psi\partial\hat{U} + (k+2)\partial\psi\hat{U} - \frac{1}{2}(\psi\hat{U})_0 - (2 + k)^2\partial^2\psi
$$

$$
W = \hat{W}^+ - (7 + 3k)\hat{G}^+\partial\psi - (1 + k)\partial\hat{G}^+\psi - \frac{(5 + 2k)(11 + 6k)(17 + 8k)}{192}(\psi\partial^3\psi)_0 + \frac{1}{4}(2 + k)(2k + 5)(\hat{T}\partial\hat{U})_0 + \frac{1}{4}(9 + 8k + 2k^2)(\hat{U}\partial^2\hat{U})_0 + \frac{3(5 + 2k)(69 + 66k + 16k^2)}{64}(\partial\psi\partial^2\psi)_0 - \frac{1}{4}(2 + k)(6 + k)(\partial\hat{U}\partial\hat{U})_0 + 2(\hat{G}^+\psi\hat{U})_0 - \frac{(5 + 2k)^2}{8}(\hat{T}\psi\partial\psi)_0 + \frac{1}{2}(k + 2)(2k + 5)(\hat{U}\psi\partial^2\psi)_0 + \frac{1}{4}(k + 2)(2k + 5)(\partial\hat{T}\hat{U})_0 + \frac{1}{4}(3 + 2k)(5 + 2k)(\partial\hat{U}\psi\partial\psi)_0 - \frac{1}{4}(2k + 5)(\hat{T}\hat{U}\hat{U})_0 - \frac{1}{2}(2k + 3)(\hat{U}\hat{U}\partial\hat{U})_0 + \frac{1}{4}(\hat{U}\hat{U}\hat{U}\hat{U})_0 - \frac{(2 + k)(73 + 58k + 12k^2)}{24}\partial^3\hat{U} \quad (4.17)
$$

Thus, we find in a very simple and natural way the quantum linearization of $\mathcal{WB}_2$, as it was given by brute force in [2]. We notice that if in each term we keep only the highest order in $k$, this expression becomes the classical linearization (4.13).

### 5 Conclusion

In this paper, we have presented a general framework and explicit realization for the linearization of $\mathcal{W}$-superalgebras. This linearization relies on the concept of secondary reductions, that is the Hamiltonian reduction of $\mathcal{W}$-algebras themselves. The techniques we use ensures that the linear algebras we obtain have only fields of positive spin. The price to pay is that some $\mathcal{W}$-(super)algebras are not linearizable through our procedure. For some of them (as superconformal algebras) we already know that they are linearizable with fields of negative spins: it should be interesting to see whether it is a general feature, or, on the contrary, if there are other schemes of linearization that use only positive spins.
When considering $\mathcal{W}$-superalgebras that contain a true $N = 1$ supersymmetric subalgebra (Ramon-Neveu-Schwarz superconformal algebra), one can directly perform the Hamiltonian reduction is $N = 1$ superfield formalism. In that case, one considers $osp(1|2)$ subalgebras instead of $s\ell(2)$ embeddings. This technique applies also to the secondary reductions, and therefore to the linearizations. We have not studied exhaustively this approach, but as the gradations one uses in $N = 1$ formalism are the same as ours, one can already conclude that this formalism does not provide new schemes of linearizations. In particular, the $\mathcal{W}[s\ell(m|m \pm 1), s\ell(m|m \pm 1)]$ are still not linearizable in super-fields formalism, although they are supersymmetric. The same thing appears for $osp(2m \pm 1|2m)$, $osp(2m|2m)$, and $osp(2m + 2|2m)$ algebras.

Finally, let us mention that the linearized $\mathcal{W}_3$ algebra has been used to build non-critical $\mathcal{W}_3$ BRST operators as well as new realization of the $\mathcal{W}_3$ algebra [17]: such an approach using the general framework of secondary reductions could indeed lead to a wide class of new realizations of $\mathcal{W}$-(super)algebras and also to their non-critical BRST operators.

Acknowledgments

We would like to thank Jan de Boer and Tjark Tjin for stimulating discussion. One of the authors (JOM) wishes to thank the Danish Natural Science Research Council for financial support.

A Construction of the algebra sub-complex $\hat{\Omega}$

A.1 $\hat{\Omega}$ is a sub-complex.

In this section, we show that we can define modified ("hatted") generators corresponding to the unconstrained currents, in such a way that the space $\hat{\Omega}$ generated by these hatted generators and the c’s is a subcomplex. We will do this by a double induction, using the conformal dimension and the $(H - H')$-grade of the generators as induction parameters.

We consider the “twisted” algebra, i.e. the algebra where the conformal dimensions are given by the $H'$-grade + 1. In this case the conformal dimensions of all the constrained generators is 1 and the $(H - H')$-grade of the constrained generators is less than zero.

We will need a lemma:

Lemma 1 Take an unconstrained generator $W^\alpha$ with conformal dimension $h$ and grade $n$, and consider $s(W^\alpha)$. All generators in this expression are the result of OPEs between a constrained generator in $j_{brs}$, and $W^\alpha$. Thus all monomials of generators occurring in $s(W^\alpha)$ must have conformal dimension $h$ and grade less than $n$. Write:

\[ s(W^\alpha) = P_\beta(c)W^\beta + Q_\sigma(c)W^\alpha W^\sigma + \cdots, \]

\[ ^2\text{We use the word "monomial", even though what we have is actually a normal-ordered product.} \]
then we see that the conformal dimension of $W_{\bar{\beta}}$ is $h$ and the grade is less than $n$. The conformal dimension of $W_{\bar{\gamma}}$ is $h-1$ etc., i.e. all unconstrained generators occurring in $s(W_{\bar{\alpha}})$ has either conformal dimension less than $h$ or conformal dimension $h$ and grade less than $n$.

Assume that we have already found hatted generators for all generators with conformal dimension less than $h$, and define $\hat{\Omega}^{h-1}$ to be the space generated by these hatted generators and the $c$'s. Assume that $W_{\bar{\alpha}}$ is any generator with conformal dimension $h$ and grade 0, we will show that we can define $\hat{W}_{\bar{\alpha}}$ such that $s(\hat{W}_{\bar{\alpha}}) \in \hat{\Omega}^{h-1}$. Consider $s(W_{\bar{\alpha}})$. According to the lemma, all unconstrained generators occurring in $s(W_{\bar{\alpha}})$ must have conformal dimension less than $h$. We can therefore write

$$s(W_{\bar{\alpha}}) = \sum_{i,j} A_{ij} B_j, \quad A_{ij} \in B, B_j \in \hat{\Omega}^{h-1}$$

where the $B_j$'s are chosen to be linearly independent. Since $j_{brs}$ is linear in the constrained currents, each of the terms $A_{ij}$ are monomials in the constrained currents, the $W_{\alpha}$'s. Let us consider only those terms that have the highest grade, considered as monomials in $W_{\alpha}$.

$$s(W_{\bar{\alpha}}) = \sum_{i,j} A_{ij}^m B_j + \text{lower orders terms}, \quad A_{ij}^m \text{ is order } m \text{ in } W_{\alpha}$$

Now apply $s$ once again. We get:

$$0 = \sum_{i,j} \left( s(A_{ij}^m) B_j \pm A_{ij}^m s(B_j) \right)$$

We know that $s(B_j) \in \hat{\Omega}^{h-1}$, and $s(A_{ij}^m) \in B$. We also know that $s(A_{ij}^m)$ is of order $m+1$ in the $W_{\alpha}$'s, and these are the only possible terms of order $m+1$; and since the expression must vanish order by order in the $W_{\alpha}$'s, we find

$$0 = \sum_{i,j} s(A_{ij}^m) B_j$$

Since the $B_j$'s are linearly independent we find that

$$0 = \sum_i s(A_{ij}^m)$$

Now we use the fact that $B$ has trivial cohomology. Since $\sum_i A_{ij}^m$ is in the kernel of $s$ it must be in the image of $s$, so we can find $X_j$ (of grade $m-1$ in the $W_{\alpha}$'s) such that $s(X_j) = \sum_i A_{ij}^m$. Define

$$W_{(1)} = W - \sum_j X_j B_j.$$

We find that:

$$s(W_{(1)}) = \sum_{i,j} A_{ij}^m B_j + \text{lower orders terms} - \sum_j (s(X_j) B_j \pm X_j s(B_j))$$

$$= \sum_{i,j} A_{ij}^m B_j + \text{lower order terms} - \sum_{i,j} A_{ij}^m B_j \mp \sum_j X_j s(B_j)$$

$$= \text{lower order terms} \mp \sum_j X_j s(B_j)$$
(the \(\pm\) depends on the Grassman parity of \(X_j\)). All these terms are of order at most \(m - 1\) in the \(W^{\alpha}\)'s. By induction we see that we can define \(\hat{W}^{\alpha}\) such that \(s(\hat{W}^{\alpha})\) is a polynomial of degree 0 in the constrained currents.

We want to show that in fact no \(b\)'s appear in \(s(\hat{W}^{\alpha})\) either. Actually this is quite simple: write
\[
s(\hat{W}^{\alpha}) = B + \sum_{\alpha} B_\alpha b^\alpha + \sum_{\alpha,\beta} B_{\alpha\beta} b^\alpha b^\beta + \cdots.
\]
Apply \(s\) again to get
\[
0 = s(B) + \sum_{\alpha} s(B_\alpha) b^\alpha \pm B_\alpha (\hat{J}^\alpha - \chi^\alpha) + \cdots
\]
Since \(s(B_\alpha)\) does not contain any constrained currents, we must have \(0 = \sum_{\alpha} B_\alpha \hat{J}^\alpha\), but this can only be true if \(B_\alpha = 0\) for all \(\alpha\). We see that indeed \(s(\hat{W}^{\alpha}) \in \hat{\Omega}\).

Now assume that we have found hatted generators for all generators with conformal dimension less than \(h\), and with conformal dimension \(h\) and grade less than \(n\), and define \(\hat{\Omega}^h_{n-1}\) to be the space generated by these hatted generators and the \(c\)'s. Assume that \(W^{\alpha}\) is any generator with conformal dimension \(h\) and grade \(n\), we want to show that we can define \(\hat{W}^{\alpha}\) such that \(s(\hat{W}^{\alpha}) \in \hat{\Omega}\).

We have shown that to any generator \(W^{\alpha}\) we can construct \(\hat{W}^{\alpha}\) such that \(s(\hat{W}^{\alpha}) \in \hat{\Omega}\). We have therefore shown that \(\hat{\Omega}\) is a sub-complex.

A.2 Construction of the algebra law in \(\hat{\Omega}\)

In general, the subcomplex \(\hat{\Omega}\) generated by \(\{\hat{W}^{\alpha}, c_\alpha\}\) is not \textit{a priori} a subalgebra. Actually it turns out that \(\hat{\Omega}\) is “often” a subalgebra in explicit examples, but we can find cases where this is not the case. It turns out that if the OPEs between \textit{constrained} and \textit{un-constrained} operators contains terms that are multi-linear in the \textit{constrained} generators, then extra operators (in the simplest cases of the form \((bc)_0\)) will appear in the OPEs of the generators of \(\hat{\Omega}\). We will argue that even in this case, it is consistent to project the OPEs on \(\hat{\Omega}\), thereby making \(\hat{\Omega}\) an algebra.

Define \(A\) to be the ghost-number zero subspace of \(\hat{\Omega}\), i.e. the space generated by the \(\{\hat{W}^{\alpha}\}\), and let \(a, b \in A\). Then, we have \(a \circ b\) in \(\hat{\Omega}\), where \(\circ\) is the algebra composition (the OPEs) in \(\hat{\Omega}\). We define a new composition on \(\hat{\Omega}\) through
\[
a \circ b = \pi(a \circ b), \text{ where } \pi \text{ is the projection on } A
\] (A.1)

The generators of the \(\mathcal{W}\)-algebra is constructed by the tic-tac-toe construction as polynomials in the generators of \(\hat{\Omega}\) with quantum number zero, i.e. the generators of the \(\mathcal{W}\)-algebra.
are polynomials of the generators in $\mathcal{A}$. Since the algebra is closed, we know that for any $\mathcal{W}$-generators $V$ and $W$, in $V \circ W$ appears only polynomial in the generators of $\mathcal{A}$. Thus from the point of view of the $\mathcal{W}$-algebra, one can consistently do the projection $\pi$. This does not change the OPEs of the $\mathcal{W}$-algebra.

It does not immediately follow, however, that the “$\diamond$” OPEs gives an associative algebra. We have investigated two explicit examples where the $\circ$ does not give an algebra on $\hat{\Omega}$ (the linearization of $\mathcal{W}_5$ and $\mathcal{W}_6$; for higher $n$ the calculations becomes extremely time consuming even on the computer), and in these cases the algebra is indeed associative. We expect this to be the case in general.

Note that the problem is inherent in the method that we use for the hamiltonian reduction, the BRST method; it is not connected directly to the quantization. Indeed, if we perform the classical hamiltonian reduction using the classical BRST method (see e.g. [9]), we find that also in that case the classical OPEs of the generators of $\hat{\Omega}$ contains extra operators, in the simplest examples of the form $bc$.

On the other hand, we can use the alternative quantization approach that has already been mentioned earlier. In this approach, we start from the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathcal{G}, \mathcal{H}^\prime)$ and find the classical expressions for the generators in $\mathcal{W}(\mathcal{G}, \mathcal{H})$ in terms of the unconstrained generators of $\mathcal{W}(\mathcal{G}, \mathcal{H}^\prime)$. We adjust the various coefficients (in the expressions for the generators of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ and in the expressions for the OPEs of the unconstrained generators of $\mathcal{W}(\mathcal{G}, \mathcal{H}^\prime)$), in such a way that the quantum OPEs of the generators of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ gives a closed algebra. In that case clearly no extra ghost-terms appear, and it is clear that we do not need to modify the OPEs; everything will work as in the classical case. It seems evident that the quantum OPEs of the unconstrained generators of $\mathcal{W}(\mathcal{G}, \mathcal{H}^\prime)$ obtained in this way are identical to the OPEs defined by the $\circ$.

In particular, if we focus on the classical hamiltonian reduction, it is clear that the gauge approach described in section 2.2.1 will provide a “good” linearization, while the classical BRST approach already leads to the emergence of $bc$-type terms. In that case it is obvious that the “$\circ$” composition law will just reproduce the classical Poisson brackets obtained by the gauge-method.
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