Radiative corrections and Lorentz violation

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Abstract

Radiative corrections in Lorentz violating (LV) models have already received a lot of attention in the literature in recent years, with many instances where a LV operator in one sector of the Standard Model Extension (SME) generates, via loop corrections, one of the LV coefficients in the photon sector, which is probably the most understood and well constrained part of the SME. In many of these works, however, the now standard notation of the SME is not used, which can obscure the comparison of different results, and their possible phenomenological relevance. In this work, we fill this gap, trying to build up a more general perspective on the topic, bringing many of the results to the SME conventional notation and commenting on their possible phenomenological relevance. We uncover one example where a result already presented in the literature can be used to place a stronger bound on the temporal component of the $b_{\mu}$ coefficient of the fermion sector of the SME.

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I. INTRODUCTION

The idea that Lorentz symmetry might be violated by new physics at the Planck scale is one of the motivations for the development of the Standard Model Extension (SME) [1, 2] as an effective field theory based on the internal symmetries and field content of the Standard Model, incorporating a very general set of Lorentz violating operators. In the minimal SME, power counting renormalizability is enforced, so that only operators with mass dimensions four or less are included, while the non-minimal extension of the SME includes all operators with higher mass dimensions [3–6], thus incorporating a huge set of new terms and presumably new physics. Lorentz symmetry being one of the cornerstones of QFT as we know it, Lorentz violation (LV) brings with it many interesting theoretical questions; at the same time, a fruitful experimental programme has used the SME framework to obtain new and improved tests of Lorentz invariance from many experiments and astrophysical observations [7, 8].

The source of LV in the SME is generally assumed to be new physics at some high energy scale, for example spontaneous symmetry breaking in a more fundamental theory such as string theory [9], in which case the constant tensors that couple to the LV operators arise as vacuum expectation values of tensor fields in this theory. One may also consider explicit breaking, which amounts to assume some unknown mechanism generating LV at the fundamental level, and therefore the LV background tensors are taken to assume, in principle, unspecified non-zero values. In the non-gravitational sector of the SME, the difference between explicit and spontaneous LV breaking is not essential in principle, but in the gravitational sector, one finds that explicit breaking is in general incompatible with the usual geometric picture of general relativity [10]. In this work, we will be mostly interested in the non-gravitational sector, so LV can be assumed to be explicit for simplicity.

As an example, Lorentz violation can be included in the standard Maxwell theory, leading to the photon sector of the SME, which is probably its most well studied part, being of utmost importance from the phenomenological viewpoint since the most stringent constraints on Lorentz violation are generally obtained by studying LV effects on photon propagation. It
is described by the Lagrangian density

\[ \mathcal{L}_{\text{photon}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} A_\lambda (\hat{k}_{AF})_\kappa F_{\mu\nu} \]

\[ -\frac{1}{4} F_{\kappa\lambda} (\hat{k}_F)^{\kappa\lambda\mu\nu} F_{\mu\nu}, \]

where

\[ (\hat{k}_{AF})_\kappa = \sum_{d \text{ odd}} (\hat{k}_{AF})^{(d)}_\kappa \alpha_1 \cdots \alpha_{(d-3)} \partial_{\alpha_1} \cdots \partial_{\alpha_{(d-3)}}, \]  

\[ (\hat{k}_F)^{\kappa\lambda\mu\nu} = \sum_{d \text{ even}} (\hat{k}_F)^{(d)}_{\kappa\lambda\mu\nu\alpha_1} \cdots \alpha_{(d-4)} \partial_{\alpha_1} \cdots \partial_{\alpha_{(d-4)}}, \]

\( d \geq 3 \) being the dimension of the corresponding operator (for more details see [3]). The minimal photon sector of the SME is obtained by the restriction to \( d = 3 \) and \( d = 4 \) for the CPT-odd and CPT-even terms, corresponding to the original \( k_{AF} \) and \( k_F \) defined in [2]. Some of the fundamental references regarding this sector of the SME are presented by [11–14], and the most recent experimental limits can be found in [7]. It is noteworthy to recall that the \( k_{AF} \) coefficient, and parts of the \( k_F \), induce birefringence in the vacuum, which leads to very strong experimental constraints, from astrophysical observations: components of \( k_{AF} \) are constrained to the order \( 10^{-43} \)GeV, while the birefringent components of \( k_F \) to the order of \( 10^{-37} \).

From the theoretical viewpoint, the quantum impacts of the LV operators in the SME are a matter that already received much attention. Lorentz symmetry being one of the starting points of the usual approach to quantum field theory, whether the full renormalization program can be carried out consistently or not for the minimal SME, which is power counting renormalizable, is an interesting question, which has been positively answered for the QED [15], electroweak [16], scalar and Yukawa [17] sectors. Interesting results have also been obtained regarding the Källén-Lehman representation [18] and the properties of asymptotic states [19], due to Lorentz violation, showing that much of the structure of QFT survives the introduction of LV as done in the SME, but space still remains for nontrivial modifications. Even from the phenomenological point of view, the study of quantum corrections might be of interest, since it can connect LV coefficients in different sectors of the framework, thus making it possible to transfer bounds found in one sector to the other. The general motivation can be explained by recalling the first example of this mechanism [2, 20–23], where integration over a fermion loop, including a LV operator involving an axial vector \( b^\mu \), was
shown to generate a finite correction to the photon effective action, proportional to the well
known Carroll-Field-Jackiw (CFJ) term [11] (in the SME nomenclature, that means the $b$
term in the QED fermion sector generating, by radiative correction, the $k_{AF}$ coefficient in the
photon sector). Strong experimental bounds on the CFJ coefficient (which are still among
the strongest found in the literature so far [7]) were already derived in [11], and theoretical
consistency of the mechanism presented in [23] would allow one to translate this bounds to
the original $b^\mu$ coefficient. However, from the start it was recognized that the generated
CFJ term was finite, yet ambiguous, its value depending on the regularization scheme, thus
being one more example of "finite but undetermined" quantum corrections [24]. Several
different approaches have been developed to remove this ambiguity (see for example [25, 26]
and references therein), and the conclusions seems to be that, in this particular case, gauge
invariance enforces the generated CFJ term to vanish. In any case, this very first example
in the study of quantum corrections induced by the LV coefficients of the SME already
shows the complexity and the potential for interesting theoretical and phenomenological
discussions related to this matter.

These calculations naturally motivated many other investigations, where specific LV oper-
ators where generated as a radiative corrections. Restriction to the minimal SME allows for
more systematical studies such as the ones presented in [15], since renormalizability greatly
reduces the number of allowed terms in the effective action. When the extension to the non-
minimal SME is considered, operators with mass dimension greater than four are included,
which are non renormalizable by power counting, so that the theory can only be understood
as an effective field theory. Still, one can look for specific cases where non minimal LV
operators can induce interesting results in the quantum theory. It is clear that the case
of finite and non ambiguous LV corrections is particularly interesting, since it could relate
LV couplings from different sectors of the framework, and present itself as a consistent way
to generate LV from some more "fundamental" setup (assuming for instance the integrated
field not to be one of the fields in the Standard Model). When divergent corrections to a
given LV operator are generated, this operator has to be introduced from the very begin-
ing to act as a counterterm to cancel this divergence, leaving behind an arbitrary finite
constant that has to be fixed by some physical condition. In this case, as in the "finite but
undetermined" scenario described previously, one cannot claim to have generated a given LV
operator without some degree of ambiguity, thus rendering less clear any phenomenological
application of this result. Even so, this approach have been used in literature to infer bounds on LV coefficients, either by implicitly assuming all finite constants to be of order one, or by fixing it using minimal subtraction [27, 28].

Interestingly enough, it has been shown that, in certain cases, even the non-renormalizable couplings can yield finite, well-defined results. The first known example of such a situation is the generation of a finite aether term from the magnetic Lorentz-breaking coupling [29]. This is a motivation for further examining quantum corrections in the non-minimal extensions of QED. Among the interesting results in this context one can mention, for example, the generation of the axion term from the Lorentz-breaking couplings [30] and the CPT-even terms proportional to the fourth-rank tensor [28]. It is noteworthy to mention that LV operators can also be generated by other mechanisms, such as spacetime varying couplings constants, either at the classical level [31], or in the computation of loop corrections [32].

In this work, we revisit the question of the perturbative generation of quantum corrections in various sectors of SME, revisiting the relevant literature and presenting some new results. One objective is to put in a more general and systematic perspective the problem of radiative corrections in the SME. We will fill some gaps in the literature, in the sense of presenting the results in the standard SME notation, which greatly facilitates the comparison with experimental results. In doing so, we will show that new and interesting information can be obtained. For example, we will argue that higher order corrections induced from the $b^\mu$ coefficient in the birefringent part of the $k_F$ term of the photon sector may provide stronger constraints on the temporal component of $b^\mu$ than the ones currently known. We therefore provide a "road map" for obtaining more results of this kind while, on the same time, motivating the adherence to the SME notation in these studies.

The structure of the paper looks like follows. In section [II] we look at the minimal interactions between gauge and spinor fields, and the corresponding operators generated in the gauge sector, extending this study for the non minimal QED extension in section [III]. In section [IV] we carry the same study for scalar-spinor couplings, and in section [V] for the contributions with external spinors. Section [VI] is devoted to a discussion of Lorentz-breaking quantum corrections in a curved space-time. Finally, section [VII] contains our conclusions and final remarks.
II. THE MINIMAL QED EXTENSION

The most generic Lorentz-breaking extension of QED containing only terms of renormalizable dimensions, also called the minimal QED extension, is given by the following Lagrangian [2],

\[ \mathcal{L} = \bar{\psi} (i \Gamma^\nu D_\nu - M) \psi + \mathcal{L}^{(3,4)}_{\text{photon}}, \]  

where

\[ \Gamma^\nu = \gamma^\nu + c^{\mu \nu} \gamma_\mu + d^{\mu \nu} \gamma_\mu \gamma_5 + e^\nu + i f^{\nu \gamma_5} + \frac{1}{2} g^{\lambda \mu \nu \sigma} \sigma_{\lambda \mu}, \]  

\[ M = m + a_\mu \gamma^\mu + b_\mu \gamma^\mu \gamma_5 + \frac{1}{2} H^{\mu \nu} \sigma_{\mu \nu}, \]

\[ D_\mu = \partial_\mu - ie A_\mu \] is the usual \( U(1) \) covariant derivative, \( \mathcal{L}^{(3,4)}_{\text{photon}} \) is the restriction of the Lagrangian in Eq. (1) to minimal (dimension three and four) operators, and \( a^\mu, b^\mu, c^{\mu \nu}, d^{\mu \nu}, e^\mu, f^\mu, g^{\lambda \mu \nu}, H^{\mu \nu} \) are constant (pseudo)tensors, collectively known as the LV coefficients of the QED sector of the minimal SME, together with the \( k_\ell^{(4)} \) and \( k_{AF}^{(3)} \) defined by Eq. (2).

While generality motivates equation (3), incorporating into the QED Lagrangian all possible terms respecting gauge invariance, power counting renormalizability and observer Lorentz invariance, one should be aware that some of the LV couplings present in (3) are actually not relevant to physics in most cases. A first example is the vector \( a_\mu \), which can be eliminated in a single fermion theory by a field redefinition \( \psi \rightarrow e^{-ia \cdot x} \chi \). While in a theory with different fermion species and different \( a \)'s one might find some non-trivial effects arising from this coefficient, it is usually disregarded; the situation changes when gravity is taken into account, however [10]. The antisymmetric part of the \( c^{\mu \nu} \) coefficient can be removed by a redefinition of the gamma matrices, so \( c^{\mu \nu} \) is usually taken to be symmetric; also, the antisymmetric part of \( d^{\mu \nu} \), the trace and totally antisymmetric parts of the \( g^{\lambda \mu \nu \sigma} \) terms are not expected to generate independent physical effects [2]. The \( f^\mu \) coefficient can be shown to generate effects that can be exactly mimicked by the symmetric part of \( c^{\mu \nu} \) [33], so it is also usually disregarded. A detailed discussion of the removal of spurious LV operators via field redefinitions can be found in [34] (see also [35] for a related discussion on the \( b^\mu \) coefficient, and [36] for a discussion involving singular spinor fields and torsion).

The photon sector of the SME being so well understood theoretically, together with the strong experimental constraints that can be obtained from it, makes particularly interesting the study of the structure of quantum corrections that can be induced from different LV
couplings in this specific sector of the SME. For the remainder of this section, restrict ourselves to the original couplings being present in the minimal QED extension, as described in (3) (notice, however, that in several instances, the generated LV operators are themselves non minimal), and discuss the most interesting cases of radiative corrections. Whenever relevant, we will rewrite the original results obtained in the literature in the standard SME notation and comment on possible phenomenological bounds that might be inferred from these results, an analysis which is mostly missing on many of those works.

Maybe the most studied instance of the mechanism we are interested in, as we already mentioned in the introduction, is the one involving the axial vector $b_\mu$, mainly due to the fact the relevant calculation is puzzled by a technical ambiguity. Specifically, we are interested in the corrections induced in the photon sector from the LV minimal operator

$$V_b = b_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi,$$

after integration of the fermion loop. This LV insertion in a fermion loop contributing to the two-point photon vertex function generates the CFJ term [2, 23],

$$\mathcal{L}_{\text{eff}} \supset C_0 e^2 \epsilon^{\mu\nu\lambda\rho} b_\mu A_\nu \partial_\lambda A_\rho,$$

$e$ being the electric charge. This amounts, in the SME notation, to the generation of a minimal $k_{AF}$ term with $k_{AF} \sim b$. In this result, however, $C_0$ is a finite and ambiguous constant. The ambiguity is no surprise since it comes from a triangular fermion loop with one insertion of the LV two-fermion vertex $b_\mu \gamma^\mu \gamma_5$, whose corresponding amplitude is therefore quite similar to the well known anomalous triangle diagram in the Standard Model, where one of the external photon lines is taken at zero external momenta: $A(p)_\mu \gamma^\mu \gamma_5 \rightarrow A(0)_\mu \gamma^\mu \gamma_5$.

It has also been shown that in the non-Abelian case, for $N > 1$ spinor fields, the $N$-fields generalization of the $V_b$ operator generates the non-Abelian extension of the CFJ term [37],

$$\mathcal{L}_{\text{eff}} \supset C_0 e^2 \epsilon^{\mu\nu\lambda\rho} b_\mu \text{tr} \left( g^2 A_\nu \partial_\lambda A_\rho + \frac{2g^3}{3} A_\nu A_\lambda A_\rho \right),$$

$C_0$ being the same ambiguous constant. Several works in the literature claim that gauge invariance actually enforces $C_0 = 0$ [38, 39], so this mechanism can hardly be argued to generate in a consistent way the minimal $k_{AF}$ term in the photon sector of the SME.

One might also study higher order corrections derived from the $V_b$ insertion, both in the sense of an expansion in derivatives of the electromagnetic field, as well as higher orders in
the LV coefficient. In the first sense, keeping only one $V_b$ insertion, but calculating higher derivative contributions, the result yields the higher-derivative CFJ-like term

$$\mathcal{L}_{\text{eff}} \supset \frac{C_1}{m^2} e^2 e_{\beta \mu \rho \delta} b_\beta A_\mu \Box F_{\nu \rho} ,$$  \hspace{1cm} (8)

with $C_1 = 1/24\pi^2$ being a finite, well defined constant [40]. In the SME notation, this amounts to the generation of a dimension five coefficient

$$(k_{AF}^{(5)})_\kappa \alpha \beta = C_1 \frac{e^2}{2m^2} b_\kappa \eta^{\alpha \beta} .$$ \hspace{1cm} (9)

The interesting aspect of this calculation is that the result is ambiguity-free, so one could hope to infer experimental bounds on $b$ from the constraints on the photon sector coefficient $k_{AF}^{(5)}$. However, at the moment, experimental constraints on dimension five photon coefficients are obtained only by the study of free propagation of photons [4], which means that leading LV effects may be obtained by imposing the usual dispersion relation $\eta_{\mu\nu} p^\mu p^\nu = 0$ in the general expressions for $k_{AF}$ given in Eq. (2). This, together with Eq. (9), means that the LV operator in Eq. (8) does not contribute to wave propagation (the same result can be seen simply by the fact that $\Box F^{\mu \nu} = 0$ for free photon propagation, at the leading order). As a conclusion, in this example, even if the generated LV operator is finite and free of ambiguities, its particular form is such that no experimental constraints can be inferred from this result at the present.

Despite $V_b$ involving an assumedly very small LV coefficient, it might be still interesting to look for corrections of higher order in $b$, since these may provide unique effects, which might not be obscured from lower order corrections. In particular, at second order, $V_b$ can contribute to the CPT-even, minimal, $k_F$ coefficient. Considering two $V_b$ insertions, one indeed obtains the aether term

$$\mathcal{L}_{\text{eff}} \supset - C_2 \frac{e^2}{m^2} b_\mu b_\lambda F_{\mu \nu} F^{\lambda \nu} ,$$ \hspace{1cm} (10)

with $C_2 = 1/6\pi^2$. This result is finite by power counting and therefore ambiguity-free [41, 42], and corresponds to the SME minimal coefficient

$$(k_F^{(4)})_{\mu \nu \alpha \beta} = - C_2 \frac{e^2}{m^2} \left( b_\mu b_\alpha \eta^{\nu \beta} - b_\nu b_\alpha \eta^{\mu \beta} - b_\mu b_\beta \eta^{\nu \alpha} + b_\nu b_\beta \eta^{\mu \alpha} \right) .$$ \hspace{1cm} (11)

Since this is a well defined quantum correction, we may question whether this result can induce competitive constraints on the $b_\mu$ coefficients, given that very strong constraints exist
on $k_F$ due to birefringence effects. One simple parametrization for the components of $k_F$ relevant for birefringence is in terms of the ten $k^a$ coefficients defined in [12], which are constrained at the order $10^{-37}$ from the study of birefringence in gamma ray bursts [43]. Therefore, from Eq. (11) we could expect components of $b$ to be limited by $b^2 < 6\pi^2 m^2 / e^2 \times 10^{-37}$, amounting to $|b_p| < 3 \times 10^{-15}$ GeV for protons and $|b_e| < 1.5 \times 10^{-20}$ GeV for electrons, for example: these are not better than the limits already known for the spatial components of $b_\mu$, but are potentially better than the ones for the temporal components of $b_\mu$, which are $|(b_T)_p| < 7 \times 10^{-8}$ GeV for protons and $|(b_T)_e| < 10^{-15}$ GeV for electrons [7], where the $T$ subscript is the standard notation for the temporal component of a vector in the Sun centered frame. To verify this, we assume for the moment that $b^\mu = \left( b, 0 \right)$ in the Sun centered frame, which is adopted as the standard frame of reference for quoting experimental constraints on the SME coefficients, and we take into account that quantum corrections will induce a corresponding $(k_F)^{\mu\alpha\beta}$ of the form given in Eq. (11). It is easy to verify that we generate this way non-vanishing birefringent coefficients $k^a = C_2 e^2 b^2 / m^2$ for $a = 3, 4$, which are subjected to the aforementioned $10^{-37}$ constraints. Alternatively, one may obtain the matrices $\kappa_{DE}$, $\kappa_{DB}$ and $\kappa_{HB}$ as defined in [13] as $\kappa_{DE} = -(2 C_2 e^2 b^2 / m^2) \mathbf{1}$ and $\kappa_{DB} = \kappa_{HB} = \mathbf{0}$, and from this, $\tilde{\kappa}_{e\pm} = \mp (C_2 e^2 b^2 / m^2) \mathbf{1}$, $\tilde{\kappa}_{o\pm} = 0$ and $\tilde{\kappa}_{tr} = -2 C_2 e^2 b^2 / m^2$, and the birefringent matrix $\tilde{\kappa}_{e^+}$ is to be subjected to the $10^{-37}$ limit. From this argument, we conclude that the consistency of the quantum corrections given by Eq. (10) imply in the following constraints for the temporal component of the $b_\mu$ pseudo-vector, written in general as

$$|b_T| < \pi m / e \sqrt{6} \times 10^{-37}.$$  (12)

This implies in new stronger experimental constraints on $b_T$ coefficients, for example

$$|(b_T)_p| < 3 \times 10^{-15} \text{GeV}$$  (13)

for protons and

$$|(b_T)_e| < 1.5 \times 10^{-20} \text{GeV}$$  (14)

for electrons. It is interesting to note that we are able to obtain these new constraints, despite the induced operator being of the second order in $b_\mu$, and this happens because one of the components of $b_\mu$ (the temporal one) have not been so tightly constrained so far.

Higher orders corrections in $V_b$ are not expected to lead to competitive bounds, yet they have been studied for theoretical reasons. Considering three $V_b$ insertions, one obtains a
linear combination of the higher-derivative CFJ-like term \( \text{[8]} \) (with a coefficient proportional to \( b^2 \)) and the Myers-Pospelov term \( \text{[44]} \):

\[
\mathcal{L}_{\text{eff}} \supset \frac{e^3}{m_4} C_3 b^\alpha F_{\alpha \mu} (b \cdot \partial) e^{\beta \mu \rho} b_\beta F_{\nu \rho},
\]

(15)

where \( C_3 \) is also an ambiguity-free constant \( \text{[45]} \). This model can be obtained from the general SME formalism by a specific choice of \( \hat{k}_F^{(5)} \), corresponding to a particular isotropic limit of Lorentz violation, leading to modified dispersion relations for photons (which were the original motivation for the introduction of the model in \( \text{[44]} \)), see section IV-F in \( \text{[3]} \) for more details. It is interesting to note that the higher-derivative CFJ-like term generated in this case does not appear for a light-like vector \( b_\mu \). For more insertions, it is natural to expect the appearance of terms including fourth and higher orders in derivatives, meaning contributions to \( k_F^{(d)} \) for \( d \geq 6 \). Up to now, apart from the general discussion in \( \text{[3]} \), further consequences of these dimension six terms have only been studied at the tree level \( \text{[46]} \).

After this discussion of the quantum consequences of the \( V_b \) operator, we will consider the other relevant minimal couplings. We start with the operator proportional to the tensor \( c_{\mu \nu} \),

\[
V_c = i e^{\mu \nu} \bar{\psi} \gamma_\mu (\partial_\nu - i e A_\nu) \psi.
\]

(16)

Without loss of generality, \( c^{\mu \nu} \) is taken to be symmetric, as discussed before. A minimal scenario including only the \( c^{\mu \nu} \) coefficient was discussed in \( \text{[47]} \), where its presence was shown to be equivalent to the redefinition of the Dirac matrices through

\[
\gamma^\mu \rightarrow \gamma^\mu + c^{\mu \nu} \gamma_\nu,
\]

(17)

with the subsequent arising of the deformed metric

\[
M^{\mu \nu} = (\delta^\mu_\alpha + c^\mu_\alpha)(\delta^\nu_\beta + c^\nu_\beta) \eta^{\alpha \beta}.
\]

(18)

As a result, one can arrive at quantum corrections involving contractions of the gauge field \( A_\mu \) and the stress tensor \( F_{\mu \nu} \) with the constant tensors \( \delta^\mu_\alpha + c^\mu_\alpha \). The Adler-Bell-Jackiw anomaly was the main interest of the reference \( \text{[47]} \), and it was shown that the presence of the \( c^{\mu \nu} \) coefficient as the only LV in the model does not modify the usual picture of the anomaly and index theorem. As for the radiative generation of corrections in the photon sector, calculations have been performed only considering a particular form of the \( c_{\mu \nu} \) coefficient,
parametrized by a constant vector $u$, 
\begin{equation}
    c_{\mu\nu} = u_\mu u_\nu - \frac{\zeta}{4} \eta_{\mu\nu} u^2,
\end{equation}
so that, at $\zeta = 0$ we have the simplest form $c_{\mu\nu} = u_\mu u_\nu$, and at $\zeta = 1$ the $c_{\mu\nu}$ is traceless. Respecting CPT invariance, quantum corrections involving the $V_c$ insertion will contribute to the CPT-even photon coefficient $k_F$, starting at the first order, and the explicit form of this contribution involving up to three $c_{\mu\nu}$ insertions has been calculated in [48]. The most interesting case to quote is the first order in the traceless $c_{\mu\nu}$, where the leading (divergent) corrections generate the simple term
\begin{equation}
    L_{\text{eff}} \supset \frac{e^2}{2\pi^2 \epsilon^2} c_{\mu\nu} \eta_{\mu \sigma} F^{\mu\nu} F^{\sigma\rho},
\end{equation}
corresponding to
\begin{equation}
    (k_F)_{\mu\rho\sigma} \sim c_{\mu\sigma} \eta_{\rho\nu} - c_{\nu\sigma} \eta_{\rho\mu} - c_{\mu\rho} \eta_{\sigma\nu} + c_{\nu\rho} \eta_{\sigma\mu}.
\end{equation}
For $\zeta = 0$, one obtains in particular the aether-like photon coefficient in Eq. (11), together with a rescaled Maxwell term proportional to $u^2 F^{\mu\nu} F_{\mu\nu}$. However, unlike in the results generated from the CPT-odd couplings [41, 42], in the case of the $c_{\mu\nu}$ insertions the aether term logarithmically diverges.

The term proportional to the coefficient $d_{\mu\nu}$ is
\begin{equation}
    V_d = i d^{\mu\nu} \bar{\psi} \gamma_\mu \gamma_5 (\partial_\nu - ieA_\nu) \psi.
\end{equation}
Early works concerning this term include [15], where the study of one-loop renormalizability of the extended QED involved the calculation of divergent corrections induced by all the minimal coefficients in (3), and also [47], where it was shown that compatibility with chiral symmetry allows for a $d_{\mu\nu}$ coefficient which is not independent, but actually defined in terms of the $c_{\mu\nu}$ coefficient by $d^{\mu}_\nu = Q(\delta^{\mu}_\nu + c^{\mu}_\nu)$, leading to the particular QED extension involving only the $c^{\mu\nu}$ independent coefficient that was already mentioned in the previous paragraph.

From a more general perspective, the $d_{\mu\nu}$ is CPT even and therefore can only contribute to the CPT even photon coefficient $k_F$, however, since $d_{\mu\nu}$ is a pseudotensor, the possible LV contributions generated in the photon sector will involve even orders in $d_{\mu\nu}$, starting from a minimal term of the general form $d^{\mu\alpha} d^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}$, corresponding to the generation of a $k_F$ term with
\begin{equation}
    (k_F)^{\mu\alpha\beta} \sim d^{\mu\alpha} d^{\nu\beta} - d^{\nu\alpha} d^{\mu\beta}.
\end{equation}
A first-order term, whose only possible structure respecting observer Lorentz invariance would be like \( \varepsilon_{\mu\nu\alpha\beta} F^\mu_{\nu} d^\alpha_{\lambda} F^\lambda_{\rho} \), would correspond to \( (k_F)^{\mu\nu\alpha\beta} \sim \varepsilon_{\mu\nu\alpha\beta} d^\lambda_{\alpha} \), which does not possess the necessary symmetry properties except if \( d^\lambda_{\nu} \sim \delta^\lambda_{\nu} \), which is evidently trivial since there is no breaking of Lorentz symmetry in this case; moreover, the absence of first order in \( d_{\mu\nu} \) corrections has been verified through direct calculations [15]. The second order contribution is divergent: actually, its pole part has been shown in [49] to possess the same structure as the second order in \( c_{\mu\nu} \) corrections found in [48]. Notice however that explicit results have only been obtained so far for \( c_{\mu\nu} \) only for the particular form in Eq. (19), while \( d_{\mu\nu} \) being a pseudotensor, it cannot be cast in the same form, and therefore, no explicit results for the calculation of the \( d_{\mu\nu} \) corrections are available.

Despite \( V_c \) contributing to \( k_F \) already at the first order, and \( V_d \) at the second order, a phenomenological analysis of these results is obscured by the fact that these corrections are divergent, and that a particular choice of these tensors have been used in the literature to obtain explicit results.

The term proportional to \( e^\mu \) can contribute in the quantum corrections starting in the second order: being a vector instead of a pseudo-vector, it cannot be used at first order to construct the \( k_A F \) term, and being a CPT-odd term, it can only contribute to \( k_F \) at second order. The same applies to \( f^\mu \), as can be checked through straightforward calculations [49]. It turns out that both corrections have exactly the same form, amounting to divergent aether-like corrections like the ones in [10]. Since the calculations in [49] where performed with an implicit regularization method, we can quote the explicit result for the generated corrections to \( k_F \) as

\[
(k_F)^{\mu\nu\rho\sigma} = \frac{e^2}{12} I_{\log}(m^2) \left( \eta^{\rho\mu} e^\nu e^\sigma - \eta^{\rho\nu} e^\mu e^\sigma - \eta^{\rho\sigma} e^\mu e^\nu + \eta^{\rho\nu} e^\mu e^\sigma \right),
\]

with the corresponding expression for \( f^\mu \) being obtained by simply substituting \( e^\mu \) by \( f^\mu \), and where \( I_{\log}(m^2) \) is a logarithmically divergent expression that may be calculated in different regularization schemes. In dimensional regularization, for example, one has

\[
I_{\log}(m^2) = \frac{i}{16\pi^2} \Gamma \left( \frac{\epsilon}{2} \right) \left( \frac{4\pi m^2}{\mu^2} \right)^{\epsilon/2}.
\]

The term proportional to \( g^{\mu\nu\lambda} \) will yield the finite and well defined higher-derivative contribution [50]

\[
\mathcal{L}_{eff} \supset \frac{e^2}{24m^2 \pi^2} G^{\mu\nu\rho\beta} A_\alpha \partial_\rho \partial_\beta A_\nu,
\]

with the corresponding expression for \( f^\mu \) being obtained by simply substituting \( e^\mu \) by \( f^\mu \), and where \( I_{\log}(m^2) \) is a logarithmically divergent expression that may be calculated in different regularization schemes. In dimensional regularization, for example, one has

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\]

The term proportional to \( g^{\mu\nu\lambda} \) will yield the finite and well defined higher-derivative contribution [50]
\(e\) being the charge and \(m\) the mass of the integrated fermion, and
\[
G^{\mu\nu\rho\alpha\beta} = g^{\mu\rho\alpha} \eta^{\nu\beta} + g^{\mu\nu\beta} \eta^{\rho\alpha} - g^{\mu\nu\alpha} \eta^{\rho\beta} - g^{\mu\rho\beta} \eta^{\nu\alpha} - g^{\rho\nu\alpha} \eta^{\mu\beta} - g^{\rho\nu\beta} \eta^{\mu\alpha}. 
\] (27)

In the SME notation, this amounts to a contribution to \(\hat{k}_{AF}^{(5)}\) of the form
\[
(\hat{k}_{AF}^{(5)})_{\kappa}^{\alpha\beta} = \frac{e^2}{24 \times 3! m \pi^2} \epsilon_{\kappa\lambda\mu\nu} G^{\lambda\mu\nu\alpha\beta}. 
\] (28)

Despite being finite and well defined, this correction actually does not contribute to photon propagation in leading order. This can be seen by noticing that either the leading order dispersion relation \[89\]
\[
(p^2)^2 - 4 \left( p^\mu (k_{AF})_\mu \right)^2 \approx 0 
\] (29)
or the relevant Stokes parameter \[90\]
\[
\varsigma^2 = -p^\mu (k_{AF})_\mu / \omega^2 
\] (30)
are modified by the combination \(p^\mu (k_{AF})_\mu\), which can be shown to vanish. Indeed, from the antisymmetry of \(g^{\mu\nu\lambda}\) in the first two indices, it can be shown that
\[
(k_{AF})_\mu = (\hat{k}_{AF}^{(5)})_{\mu}^{\alpha\beta} p_\alpha p_\beta = 6 \epsilon_{\mu\nu\sigma\rho} g^{\nu\rho\alpha} p_\alpha p_\sigma, 
\] (31)
and therefore \(p^\mu (k_{AF})_\mu = 0\) as a result of the contraction of the epsilon with two momenta. Current limits on dimension five photon coefficients all derive from astrophysical observations of photon propagation and, therefore, cannot be used to impose limits on \(g^{\mu\nu\lambda}\) based on the induced term \[26\].

Finally, we note that for the particular case of completely antisymmetric \(g_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\rho} \hat{h}^\rho\), Eq. \[26\] yields the finite higher-derivative CFJ-like result \[8\], with an appropriate multiplying factor. The finite temperature behavior of this term is discussed in \[51\]. In the second order in \(g_{\mu\nu\lambda}\), for the same case of a completely antisymmetric \(g_{\mu\nu\lambda}\), one arrives at the logarithmically divergent aether-like result \[10\], with \(b_\mu\) replaced by \(h_\mu\).

To close the discussion of the minimal part, it remains to discuss the impacts of the \(H_{\mu\nu}\) term. One can naturally make the conclusion that the lower possible contribution involving this insertion should be at least of the second order (the first-order contribution evidently vanishes by symmetry reasons), and it must be superficially finite by dimensional arguments. It is natural to expect expression of the form \(H^{\mu\nu} H^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\). However, explicit calculation shows that this term identically vanishes at the one-loop order \[49\].
III. NON-MINIMAL EXTENSIONS OF QED

It is very natural to study quantum corrections in the minimal SME, which is proven to be a renormalizable model. In a more general perspective, however, the SME is an effective field theory which naturally includes non-minimal operators of mass dimension greater than four, naturally appearing in various phenomenological models (see f.e. [52]), and in this section we want to discuss the effects of these in the quantum corrections.

From the formal viewpoint, the presence of such couplings is a natural consequence of the fact that the SME is an effective field theory arising in the low-energy limit of some fundamental theory at a very high energy scale $\Lambda$, therefore it depends on this characteristic energy scale, with non-minimal vertices being proportional to negative powers of $\Lambda$ [53]. While the restriction to dimension three and four operators, corresponding to the minimal SME, leads to a consistent quantum field theory by itself (which, being renormalizable, is actually independent of the scale $\Lambda$), the general picture is certainly less clear for the non-minimal SME, since higher-dimension kinetic operators, due to the presence of higher derivatives, typically yield ghost excitations, while higher-derivative interactions are essentially non-renormalizable. So, it is not expected that consistent quantum corrections can be calculated in general, however specific terms can be shown to provide interesting results, and indeed several examples have been reported in the literature so far. Up to now, most of these studies focused on the leading, dimension-five operators, with the dimension six case being discussed recently in [46] for the gauge sector, and in [54] for the spinor sector.

The first non-minimal coupling whose quantum impacts were studied at the perturbative level is the dimension five magnetic one [29], involving a single LV vector $u_\beta$,

$$\mathcal{V}_1 = g u_\beta \bar{\psi} \gamma_\alpha \psi \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta},$$

or, in the SME notation [6],

$$\mathcal{V}_1 = -\frac{1}{2} (a^{(5)})^{\alpha\beta\gamma} \bar{\psi} \gamma_\alpha \psi F_{\beta\gamma},$$

where $(a^{(5)})^{\alpha\beta\gamma} = 2g \epsilon^{\alpha\beta\gamma\rho} u_\rho$. There are at the moment no experimental constraints reported on these non-minimal coefficients [7]. It is interesting to notice that in the works mentioned by us, $u_\beta$ is assumed to be a mass dimension one LV coefficient, just as the minimal extended QED coefficient $b^{3}$ (actually, in many instances, the coupling $V_5$ is also considered in the
calculation, and it is assumed that \( u^\beta = b^\beta \), so that \( g \) has mass dimension \(-2\). One remarkable fact related to this vertex is that the contribution to the two-point function of the gauge field generated by two such vertices, although quadratically divergent by power counting, unexpectedly yields a finite aether-like result

\[
\mathcal{L}_{\text{eff}} \supset C_4 m^2 g^2 u^\mu u^\lambda F_{\mu\nu} F^{\lambda\nu},
\]

(34)

\( C_4 \) being an ambiguous dimensionless finite constant studied in detail in [42]. When this mechanism is considered at finite temperature, the situation becomes more involved, for example, an aether-like term involving only spatial components like \( u_i u_k F_{ij} F_{kj} \) becomes possible [55]. The non-Abelian generalization of this calculation is possible as well. The ambiguity in the calculation of the constant \( C_4 \) in principle precludes a confident phenomenological analysis, with the objective of transferring the bounds on \( (k_F)^{\mu\nu\alpha\beta} = -4C_4 m^2 g^2 \left( u^\mu u^\lambda \eta^{\rho\sigma} - u^\nu u^\lambda \eta^{\mu\rho} - u^\mu u^\rho \eta^{\lambda\sigma} + u^\nu u^\rho \eta^{\mu\lambda} \right) \), which corresponds in the SME notation to Eq. (34), to a bound in \( a^{(5)}_F \), which would be a very interesting result.

Other corrections arising from the presence of the \( \mathcal{V}_1 \) vertex are possible. If one consider the contribution involving one \( \mathcal{V}_1 \) and one usual vertex \(-e\bar{\psi} A\psi\), for example, the CFJ term proportional to the same ambiguous constant \( C_4 \) would be obtained. Calculating the higher-derivative contributions to the two-point function generated by Feynman diagrams involving either two non-minimal \( \mathcal{V}_1 \) vertices or one \( \mathcal{V}_1 \) and one usual QED vertex, with one or two minimal \( V_b \) insertions, the result will be superficially finite, being a linear combination of the Myers-Pospelov term [15] and the higher-derivative CFJ term [5], just as it occurs for the case when both vertices are minimal [45]. One of the contributions to each of these terms will be ambiguous. The complete result for the linear combination of these terms, generated by the presence of both interactions, contains [13] as one of the contributions, and looks like

\[
\mathcal{L}_{\text{eff}} \supset \left( 2g^2 C_1 + \frac{e g}{6\pi^2 m^2} + \frac{4e^2}{45\pi^2 m^4} \right) u^\alpha F_{\alpha\mu}(b \cdot \partial) u_\beta \epsilon^{\beta\mu\nu\lambda} F_{\nu\lambda}
+ \left( 2g^2 C_1 + \frac{e g}{6\pi^2 m^2} + \frac{e^2}{9\pi^2 m^4} \right) u^2 u_\beta \epsilon^{\beta\mu\nu\lambda} A_\mu \Box F_{\nu\lambda},
\]

(35)

where \( C_1 \) is the same finite and ambiguous constant involved in the generation of the CFJ term [5], as described in the previous section. Again, the remarkable property is the finiteness of this result, despite the initial power counting of the Feynman diagrams involved.

Another non-minimal, CPT odd vertex have been discussed in the literature in connection
with axion physics, to wit,

$$\mathcal{V}_2 = v^\beta \bar{\psi} \gamma^\alpha \psi F_{\alpha \beta}, \quad (36)$$

which is also of the same form as Eq. (33), but with

$$(a_f^{(5)})_{\alpha \beta \gamma} = - (v^\beta \eta^{\alpha \gamma} - v^\alpha \eta^{\beta \gamma}) \quad (37)$$

Notice that, here, $v^\beta$ has dimension of inverse of mass. It has been shown in [30] that a triangle graph similar to that one studied in [23], but with one external field $eA_\alpha$ replaced by the $\mathcal{V}_2$ vertex and the insertion $\bar{\psi} \gamma_5$ replaced by $\partial \phi \gamma_5$, with $\partial = \partial(x)$ being the axion field, will generate in the effective action the usual coupling between the photon and an axion-like-particle, i.e.,

$$\mathcal{L}_{eff} \supset C_1 e^{\mu \alpha \beta} b_\alpha u^\rho \partial F_{\mu \nu} F_{\rho \beta} = 2C_1 e g (u \cdot b) \bar{\psi} \left( \vec{E} \cdot \vec{B} \right), \quad (38)$$

where $C_1$ is the same ambiguous constant defined in (6). In obtaining this result, it was assumed that the integrated fermion $\psi$ is very massive, so that it makes sense to extract from the relevant integrals only the dominant results when its mass is very large compared to any other scale (so it is sufficient to keep the first term in a derivative expansion for $\partial^\alpha(x)$). Interestingly enough, this LV mechanism yields an isotropic correction that exactly mimics the standard axion-photon coupling, which is relevant for many experimental searches for axion-like-particles [56]. Despite being finite, this calculation suffers from the same sort of ambiguities present in the generation of the CFJ term. Notice, however, that the arguments used to argue that $C_1$ should vanish, involving the gauge symmetry of a correction to the photon propagator, do not necessarily apply in this case, which concerns an interaction term involving photons and a light pseudoscalar. The possibility of a phenomenological relevance of the generated term (38) was hinted in [30], however, a proper examination of the ambiguity in this correction is still missing. Finally, as a comment, we note that if we replace the magnetic coupling in (32) by the one in (36) within the study of the aether term carried out in the paper [29], in the four-dimensional case, we also will obtain the aether term with the same ambiguous multiplier $C_4$ defined in (34).

The interaction vertex in (36) was further studied in a series of articles devoted to its quantum effects in the photon sector. The groundwork was developed in [57], considering the functional determinant

$$S_{eff} \supset i \text{Tr} \ln \left( i\partial - eA - m - \gamma^\alpha F_{\alpha \beta} v^\beta \right), \quad (39)$$
which was calculated using the zeta function method, and the result was expressed in a power series of the electromagnetic field strength. Wave propagation was studied with the dominant LV corrections that are generated in the photon sectors, as well as the leading non-linear corrections, i.e.,

\[
S_{\text{eff}} \supset \int d^4x \left( \mathcal{L}_{F^4} + \frac{g}{12\pi^2} \ln \left( \frac{M^2}{\mu^2} \right) v_\alpha F_{\mu\nu} \partial^\mu F^{\nu\alpha} \right),
\]

where \( \mathcal{L}_{F^4} \) stands for the usual Lorentz invariant Euler-Heisenberg Lagrangian, with the surprising result that the LV background decouples from wave propagation in vacuum, in this approximation. It is interesting to notice that this calculation does not suffer from any ambiguities of the sort involved in the generation of the CFJ term, yet the leading LV correction present in the last equation is divergent, thus needing a renormalization, as indicated by the presence of the renormalization scale \( \mu \). Also, from these results one could extract additional LV non-linear terms for the field strength, which is a topic still quite unexplored in the literature.

Now we focus our attention to non-minimal CPT-even couplings. The first calculation of quantum corrections involving one of these was presented in [28], involving the dimension five vertex

\[
V_3 = \frac{1}{2} \kappa^{\mu\nu\lambda\rho} \bar{\psi} \sigma_{\mu\nu} \psi F_{\lambda\rho} = -\frac{1}{4} H_F^{(5)\mu\nu\alpha\beta} \bar{\psi} \sigma_{\mu\nu} \psi F_{\alpha\beta},
\]

where \( H_F^{(5)\mu\nu\alpha\beta} = -\frac{1}{2} \kappa^{\mu\nu\lambda\rho} \) in this case, \( \kappa^{\mu\nu\lambda\rho} \) having dimension of inverse of mass. In the QED extension with this additional vertex, radiative corrections at first and second order of the LV coefficients were presented in the literature. The dominant contributions are given by [28]

\[
\mathcal{L}_{\text{eff}} \supset \frac{m e}{8\pi^2\epsilon} \kappa^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \text{finite},
\]

which matches the minimal CPT-even \( k_F \) term in the SME, contributing to its renormalization. At the second order, besides a minimal \( k_F \) term of the form \( (k_F)_{\mu\nu\lambda\rho} \propto \kappa^{\mu\nu\alpha\beta} (k)_{\alpha\beta}^{\lambda\rho} \), one also will obtain the higher-derivative terms

\[
\mathcal{L}_{\text{eff}} \supset \frac{1}{\pi^2\epsilon} \left( C_5 \kappa^{\mu\nu\alpha\beta} \kappa_{\alpha\beta}^{\lambda\rho} F_{\mu\nu} \square F_{\lambda\rho} + C_6 \kappa^{\mu\nu\alpha\beta} \kappa_{\beta}^{\gamma\lambda\rho} F_{\mu\nu} \partial_\alpha \partial_\gamma F_{\lambda\rho} \right) + \text{finite},
\]

where \( C_5 \) and \( C_6 \) are dimensionless numerical constants. In the two last expressions, finite parts, in the UV leading order, reproduce the same tensorial structures as the pole parts.
A natural modification of the previous example consists in introducing a pseudotensor coupling [58],

\[ V_4 = -ig\kappa^{\mu\nu\lambda\rho} \bar{\psi}\sigma_{\mu\nu}\gamma_5\psi F_{\lambda\rho} = -\frac{1}{4} H_F^{(5)\mu\nu\alpha\beta} \bar{\psi}\sigma_{\mu\nu}\psi F_{\alpha\beta}, \]

(44)

where now \( H_F^{(5)\mu\nu\alpha\beta} = -2g\kappa^{\mu\nu\lambda\rho}\epsilon_{\lambda\rho}^{\alpha\beta} \). In this case, the resulting quantum corrections to the photon sector we will involve a "twisted" tensor \( \tilde{F}_{\mu
u} = \kappa^{\mu\nu\lambda\rho} F_{\lambda\rho} \), together with the dual \( \tilde{F}_{\mu
u} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \). Again, as in the previous case, we will have contributions involving both second and higher derivatives, and they are divergent [59]. The explicit result reads

\[ \mathcal{L}_{\text{eff}} \supset \frac{1}{16\pi^2\epsilon} \left( 8\epsilon F^{\mu\nu} \tilde{F}_{\mu\nu} + 8m^2 F^{\mu\nu} \tilde{F}_{\mu\nu} + 8m^2 F^{\alpha\gamma} \partial_\alpha \partial_\beta \tilde{F}_{\gamma} + 2m^2 \lambda_2^2 F^{\alpha\gamma} \partial^\beta \partial^\lambda \tilde{F}_\rho \epsilon_{\alpha\beta\mu} \epsilon_{\rho\lambda\mu} \right) + \text{finite}. \]

(45)

This result includes terms both of first and in second orders in the LV coefficient \( \kappa^{\mu\nu\lambda\rho} \). We note that some of these contributions are CPT-odd despite the even order in derivatives, because of the Levi-Civita symbol. The tensorial structure of the finite parts is again analogous to that one of pole parts. Some consequences of these generated terms have been studied in [59].

Another interaction considered in the literature, in order to generate higher-derivatives contributions in the gauge sector, is based on the Myers-Pospelov approach [44]. The idea is that additional derivatives appear in the action being contracted to some constant vector, which as a result prevents the arising of ghosts, and in the Lorentz-invariant limit the higher derivatives disappear completely. One may start by adding to the QED Lagrangian the following term,

\[ V_5 = \frac{1}{M^{n-1}} \bar{\psi}\gamma_5\psi (v \cdot D)^n \psi, \]

(46)

with \( n \geq 2 \). Here \( M \) is the energy scale supposed of the order of the Planck mass. This operator has mass dimensions equal to \( n + 3 \), and have been studied in the case \( n = 2 \) [60], where the linear combination of the higher-derivative CFJ-like term [8] and the Myers-Pospelov term [15] was generated, both being divergent. In principle, many other terms can be generated from the couplings [16], for example, it is natural to expect that the aether term can arise at least for some values of \( n \).
IV. SPINOR-SCALAR LV COUPLINGS

The spinor-scalar LV couplings are studied in a smaller number of papers compared with the ones discussed so far, yet several interesting results have been presented in the literature. For example, the Yukawa potential was calculated in [61] considering LV just in the scalar sector, and in [62] for a LV modification of the spinor-scalar coupling of the form $\bar{\psi} G \psi \phi$, with $G$ being a matrix of the form

$$G = g + ig' \gamma_5 + a^\mu \gamma_\mu + b^\mu \gamma_5 \gamma_\mu + \frac{1}{2} L^\mu_\nu \sigma_{\mu\nu}.$$

Also, the one loop renormalization of a general model including fermions and scalars interacting including the aforementioned general LV Yukawa coupling was worked out in [17].

Lorentz violating Yukawa couplings have also appeared in other studies that looked into radiative corrections. The particular LV coupling

$$Y_1 = a^\mu \bar{\psi} \gamma^\mu \psi \phi,$$

has been used in [29] in order to generate the CPT-even aether-like term for the scalar field,

$$\mathcal{L}_{\text{eff}} \supset C_7 \phi (a \cdot \partial)^2 \phi,$$

where $C_7$ is a constant which diverges in four-dimensional space-time. This contributions amounts to $(k_{c(4)}^{(4)})^{\mu\nu} = C_7 a^{\mu} a^{\nu}$ in the SME conventions put forth in [70]. Actually, the aether term for the scalar field is the simplest Lorentz-breaking contribution for the scalar sector in the four-dimensional space-time, and in [63], it has been shown to arise also for the Lorentz-breaking spinor-scalar theory with the usual Yukawa coupling, but with Myers-Pospelov-like higher-derivative modified kinetic term for the spinor, being finite in this case.

Another interesting coupling in this sector is the pseudoscalar one,

$$Y_2 = b^\mu \bar{\psi} \gamma^\mu \gamma_5 \psi \vartheta,$$

where $\vartheta(x)$ is a pseudoscalar field. This vertex has been used in [30] as a part of a mechanism to generate the photon-axion term (38), as discussed in the previous section.

It is clear that, in principle, terms with more derivatives can be generated in the scalar sector as well, by the same couplings above, considering the derivative expansion of the two-point vertex function of the scalar, where the finiteness of these terms will be guaranteed.
by the renormalizability of the couplings \(48, 50\). In principle, these couplings can be used to generate the interaction terms for the Lorentz-breaking Higgs sector, although these calculations were not carried out up to now, at least in the approach we are considering. It is easy to see that since these couplings are dimensionless, the corresponding contributions to the vertices in the Higgs sector will be logarithmically divergent. Finally, it is worth mentioning the study of the Lorentz-breaking Higgs sector carried out for the scalar QED in [64], where the spontaneous symmetry breaking is considered in detail, including one-loop quantum corrections.

V. LV CONTRIBUTIONS IN THE SPINOR SECTOR

The number of studies concerning the generation of LV corrections in the fermion sector of the SME is much smaller than for the scalar and especially the gauge sectors. In the context of LV theories arising as effective field theories, it seems natural to start with some basic Lorentz-breaking theory where the scalar and gauge fields are originally coupled to spinor fields, which are then integrated out. This was the basic mechanism considered in most of the works mentioned in the previous sections. Nevertheless, the study of the spinor-dependent LV contributions is an interesting task, as we discuss in this section.

At tree level, the LV extension of the spinor sector of the Standard Model was first described in the seminal papers [1, 2], with the restriction of minimal (renormalizable) operators. More recently, the non-minimal fermionic sector with LV was described in [5], presenting a general parametrization for the LV coefficients, as well as discussing several aspects of these models such as dispersion relations, exact Hamiltonian and eigenstates, together with some first numerical estimations for these Lorentz-breaking parameters.

Regarding the quantum corrections, for the minimal sector, an exhaustive study of the one-loop divergent contributions to the spinor sector for the QED sector of the SME has been presented in [15], where the full one-loop renormalization of this sector was studied. In the non-minimal sector, one first result was that the CPT-odd term

\[
S_1 = c^{\mu \nu} \bar{\psi} \gamma_\mu \partial_\nu \psi,
\]  

was shown to arise in the non-minimal extension of the QED developed in [20] based on the non-minimal magnetic coupling \(32\), where \(c^{\mu \nu} \sim b^\mu b^\nu\), the proportionality involving a
divergent constant that needs to be renormalized. The contribution of the same structure was shown in \[65\] to arise also from the CPT-even coupling \(41\), where, however, a particular form of the \(\kappa_{\mu\nu\lambda\rho}\) tensor completely described by one vector \(u^\mu\) has been used. Again, this contribution diverges. Also, in \[63\], the Lorentz-breaking extension of the Yukawa model with the extra term

\[
S_2 = \bar{\psi}(\alpha m + g(a \cdot \partial)^2)\phi \psi
\]  

(52)

\(\alpha\) being a constant, has been considered and divergent quantum corrections where shown to arise for the first term in \(S_1\).

Taking into account all this, one can expect that the number of open problems in the spinor sector is still very large, and the most important among them are studying of quantum contributions to the kinetic term of the spinor field and to spinor-scalar (and similarly, spinor-vector) couplings, for various Lorentz-breaking extensions of QED and Yukawa model, both minimal and non-minimal ones.

VI. LORENTZ VIOLATION IN GRAVITY

The problem of quantum corrections with Lorentz-breaking extensions in gravity is much more complicated than with extension of other field theory models. The first reason for this is that general coordinate invariance, being the essential symmetry of the Einstein gravity, is known to play a double role, being at the same time the analogue not only of the Lorentz symmetry, but also of a gauge symmetry. Therefore, the breaking of the general coordinate invariance will be associated with the breaking of gauge symmetry (for an extensive discussion of possible implications of breaking the gauge symmetry in the linearized gravity, see \[66\]). The second reason is that in the case of the curved space-time, the constant vectors or tensors used in the non gravitational SME to introduce preferred space-time directions are almost useless since their covariant derivatives are not necessary equal to zero, implying in a great increasing of the number of new structures which involve covariant derivatives of these "constant" tensors, see f.e. \[67\]. Third, it has been noted that backgrounds with explicit Lorentz violation are not in general compatible with Riemannian geometry \[10\], and there are hints that an alternate geometrical description of gravity, based on Riemann-Finsler geometries, may be necessary to describe explicit LV backgrounds \[68–70\] in a consistent way. Besides these questions, one has the well-known difficulty of quantum
calculations in gravity arising from the fact that the Einstein gravity is a highly nonlinear, and, moreover, non-renormalizable theory. Apart from these theoretical questions, a strong phenomenological program have been developed, connecting the gravitational SME with the Post-Newtonian formalism [71], and setting the grounds for looking for Lorentz violation in short-range gravity experiments [72–74], gravitational Cerenkov radiation [75] and gravitational waves [76, 77], among others.

One of the early models including Lorentz violation in gravity that where studied was devoted to the four-dimensional Chern-Simons (CS) modified gravity which, besides the usual Einstein-Hilbert Lagrangian, includes the CS term [78]

\[
S_{CS} = \frac{1}{64\pi G} \int d^4x \vartheta^* RR, \tag{53}
\]

where

\[
*RR = *R^\tau_{\sigma\mu\nu} R^\tau_{\sigma\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R^\sigma_{\alpha\beta} R^\tau_{\sigma\mu\nu}. \tag{54}
\]

This term is a total derivative, just as the \( \tilde{F}^\mu\nu F_{\mu\nu} \) in the electromagnetism, therefore, there is a natural analogy between the CFJ term and the 4D gravitational CS term. In general, \( S_{CS} \) breaks only the CPT symmetry, becoming Lorentz-violating for a special choice of the \( \vartheta \) in the form \( \vartheta = k_\mu x^\mu \), with \( k_\mu \) being a constant axial vector. In [78], the consistency of such a choice for \( \vartheta \) has been proved for the case of the Schwarzschild metric; in general, it is achieved if the condition \( *RR = 0 \) is satisfied, and this indeed occurs for a wide class of metrics with spherical or axial symmetry [79]. In terms of the spin connection \( \omega_{\nu ab} \), this LV form of the 4D gravitational CS model can be written as

\[
S_{CS} = \frac{1}{64\pi G} \int d^4x \epsilon^{\mu\nu\lambda\rho} k_\mu \left( \partial_\lambda \omega_{\alpha\beta} \omega_{\beta\rho} - \frac{2}{3} \omega_{\rho\lambda} \omega_{\lambda} \omega_{\alpha} \right). \tag{55}
\]

While the study of classical aspects of the 4D CS modified gravity is clearly an interesting problem (for a review, see [80]), we are mainly interested in the perturbative generation of LV operators. It is clear that the consideration of the weak gravity approximation greatly simplifies the calculations. In the linear approximation, equation (55) can be written as

\[
S_{CS}^{lin} = \frac{1}{256\pi G} \int d^4x k^\lambda h^{\mu\nu} \epsilon_{\mu\nu\lambda\rho} \partial^\rho (\Box h^\mu_\alpha - \partial_\alpha \partial_\lambda h^{\gamma\mu}). \tag{56}
\]

The general formalism for LV in linearized gravity was developed in [66], and in term of these conventions, the linearized form of the gravitational CS term in the last equation
corresponds to the dimension five, gauge-invariant and CPT-odd coefficient
\[ q^{(5)\mu\rho_1\nu_2\sigma_3\epsilon_3} = \frac{3}{4} k_{\lambda} e^{\mu\rho_1\lambda} (\eta^{\epsilon_3\epsilon_4} \eta^{\nu\sigma} - \eta^{\epsilon_3\epsilon_4} \eta^{\nu\rho}) . \tag{57} \]

The linearized 4D gravitational CS term has been generated at the one-loop order for the first time in \[81\], starting with the following action of the spinor field in the curved space-time,
\[ S = \int d^4x \left( \frac{i}{2} e e_a^\mu \bar{\psi} \gamma^\alpha D_\mu \psi - e e_a^\mu \bar{\psi} b_\mu \gamma^\mu \gamma_5 \psi + e m \bar{\psi} \psi \right) , \tag{58} \]
where
\[ D_\mu \psi = \left( \partial_\mu + \frac{1}{2} \omega_{\mu cd} \sigma^{cd} \right) \psi , \]
\( e_a^\mu \) is the vielbein, \( e \) its determinant, and \( b_\mu \) is a constant axial vector. As a result, despite the corresponding Feynman diagrams being superficially divergent, the result of the calculations turns out to be finite due to the gauge symmetry, reproducing Eq. \(56\) with \( k_{\lambda} = \frac{1}{48 \pi^2} b_\lambda \). Similarly to the case of the quantum generation of the CFJ term \[24\], the corresponding mechanism for the generation of the 4D gravitational CS term was also shown to be ambiguous \[82, 83\].

The action \(58\) was also considered in \[84\], in order to generate the LV term
\[ \mathcal{L}_{\text{eff}} \supset C_9 m^2 \epsilon^{\mu\nu\lambda\rho} b_\rho \partial_\nu h_\lambda \partial_\mu h_\sigma , \tag{59} \]
corresponding, in the SME notation of \[66\], to the LV coefficient
\[ q^{(3,3)\mu\rho_1\nu_2\sigma} = 4 C_9 m^2 \epsilon^{\mu\rho_1\nu_2\sigma} \eta^{\nu\sigma} , \tag{60} \]
which was argued to be related to a non commutative geometric setup. However, the constant \( C_9 \) is divergent, hence, for a consistent renormalization it must be introduced from the very beginning. Besides of this, this term breaks gauge invariance in general. A particular setup of wave propagation was discussed in \[84\], exhibiting vacuum birefringence, but the question of the breaking of gauge invariance was not solved in general.

Here, it should be mentioned that various Lorentz-breaking terms in the gravitational sector like \( \phi^{\mu\nu} R_{\mu\nu} \), \( \phi^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \), with \( \phi^{\mu\nu} \) and \( \phi^{\mu\nu\alpha\beta} \) being constructed from covariant derivatives of constant tensors present in Eq. \(3\) have been generated in \[67\] for a theory represented itself as a general model like \(3\) embedded into a curved space-time. However, all these terms, and even terms of higher orders in the curvature tensor, are essentially divergent.
As noted before, the generic incompatibility of explicit Lorentz violation with the geometric interpretation of General Relativity means that scenarios of LV being spontaneously broken are particularly interesting. This idea was proposed already in [85] and can be naturally promoted to a curved space-time. One way to do it is based on the Einstein-aether model (see e.g. [86]) whose action looks like

$$S = -\frac{1}{16\pi G} \int d^{4}x \sqrt{|g|} \left( R + K^{\alpha\beta}_{\mu\nu} \nabla_{\alpha} u^\mu \nabla_{\beta} u^\nu + \lambda (u^\mu u_\mu - 1) \right),$$  

(61)

where

$$K^{\alpha\beta}_{\mu\nu} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + c_3 \delta^{\alpha}_{\nu} \delta^{\beta}_{\mu} + c_4 u^{\alpha} u^{\beta} g_{\mu\nu},$$  

(62)

with $c_1, c_2, c_3, c_4$ are some dimensionless constants, the $\lambda$ is a Lagrange multiplier used to implement the constraint $u^\mu u_\mu - 1 = 0$. Further development of this concept was presented by the bumblebee model [87], where, instead of the constraint on the vector field $B_\mu$, called now the bumblebee field, a potential for $B_\mu$ is introduced, whose various vacua generate different preferred space-time directions (nevertheless, it should be noted that the vacua in general are no more represented by constant vectors). Moreover, the vector field becomes a dynamical field in its own nature. The corresponding action is

$$S = -\frac{1}{16\pi G} \int d^{4}x \sqrt{|g|} \left( R + \xi B^\mu B^\nu R_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - V (B^\mu B_\mu - b^2) \right),$$  

(63)

with $\xi$ and $b^2$ being constants, and the $B_{\mu\nu}$ the stress tensor for the bumblebee field. In this case, the quantum dynamics of the bumblebee field could be considered, and in principle one could expect some generalization of the result from [88] for a curved space-time. However, up to now, there are no examples of loop calculations for a bumblebee theory on a curved background.

Without trying to exhaust the topic, we conclude that the problem of Lorentz violation in gravity is still a very open one, where one expects to see new developments, both on the many conceptual issues, as well as from the phenomenological standpoint. In particular, the matter of radiative corrections is still quite unexplored.

**VII. CONCLUSIONS**

There have been extensive activity in the search for possible Lorentz violation in the last decades, which have resulted in a solid experimental programme [7], as well as in a deep
understanding of the theoretical questions involved in incorporating CPT and/or Lorentz breaking in the context of effective field theory. From the theoretical viewpoint, the question of quantum corrections and its effects when LV operators are considered is one of the most studied, since the seminal papers [2, 23]. In this work, we revisit this question, filling many gaps present in the literature, when results were not written in the standard SME notation, or their phenomenological implications not fully addressed. In many instances, the generated operators are "finite but undetermined", or divergent, however there are examples where finite, well defined corrections can be shown to exist, and these may even lead to improved experimental bounds on some LV coefficients. The most natural scenario for these studies involves the calculation of radiative corrections generated by fermion loops involving the minimal LV coefficients of the QED sector of the SME. Since this represents a small set of the possible coefficients for LV, most of these cases have been already considered in the literature, at least in the leading order. In this work, however, we show that higher order corrections may still provide interesting results, such as the example contained in Eq. (10), and these have not been so systematically addressed. Also, when this idea is extended to the non minimal SME, despite a few specific setups that where devised to generate specific "notorious" LV operators, systematic studies are still missing, in particular in sectors others than the extended QED. We conclude that still there is space for studying quantum corrections in LV theories, and that this program may even help the experimental task of constraining Lorentz violation via different experiments.

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Appendix A: Summary Table

In this Appendix, we present a table summarizing the results concerning the generation of effective operators in the photon sector of the SME, originating from the LV operators in the spinor sector, as a summary of the results discussed in Sections II and III. In formulating these tables, it is assumed that the usual vertex $\sim e \bar{\psi} \gamma^{\mu} \psi A_{\mu}$ can be combined with each
of the Lorentz-violating vertices involving the gauge field. For the $d^\mu\nu$ coefficient, the form presented in the table is derived from covariance and symmetry arguments, since no explicit results are available. For simplicity, non-Abelian generalizations are not included, but they are cited in the main text. Also, all Lorentz invariant contributions that are generated are omitted in the table, exception made for the case involving the axion field $\vartheta$, which is also the only one where the generation mechanism involves two different LV couplings.

Table I: Summary table of the generation of Lorentz-breaking operators in the photon sector of the SME. Notice that CPT-odd coefficients can generate CPT-even corrections at second order. Also, the axion case (pseudo-scalar field $\vartheta$) is not written in SME notation since the induced correction is actually Lorentz invariant, so it is not included in the traceless coefficient $k_F$. The columns D and A indicate whether the diagrams generating the given operator are divergent or ambiguous, respectively.

| LV operator | CPT | Generated term | D | A | References |
|-------------|-----|----------------|---|---|------------|
| $a_\mu$     | O   | 0              |   |   | [49]       |
| $b_\mu$     | O   | $(k_{AF}^{(4)})_\mu \sim b_\mu$ | X |   | [2, 20–23] |
|             |     | $(k_{AF}^{(5)})_\alpha\beta \sim b_\kappa\eta^{\alpha\beta}$ |   |   | [40]       |
|             |     | $(k_F^{(4)})^{\mu\nu\beta} \sim b_\mu b^{\alpha\eta^{\nu\beta} + \ldots}$ |   |   | [41, 42]   |
| $e^{\mu\nu}$| E   | $(k_F^{(4)})_{\mu\nu\rho} \sim c_{\mu\rho}\eta_{\nu\sigma} + \ldots$ | X |   | [48]       |
| $d^{\mu\nu}$| E   | $(k_F^{(4)})_{\mu\nu\rho\sigma} \sim d^{\mu\rho}d^{\nu\sigma} - d^{\mu\sigma}d^{\nu\rho}$ | X |   | [49]       |
| $e^{\mu}$   | O   | $(k_F^{(4)})_{\mu\nu\rho\sigma} \sim \eta^{\rho\mu}e^\nu e^\sigma + \ldots$ | X |   | [49]       |
| $f^{\mu}$   | O   | $(k_F^{(4)})_{\mu\nu\rho\sigma} \sim \eta^{\rho\mu}f^{\nu}f^{\sigma} + \ldots$ | X |   | [49]       |
| $g^{\mu\nu\lambda}$ | O   | $(k_F^{(5)})_{\kappa} \sim \epsilon_{\kappa\mu\nu\lambda} (g^{\mu\nu\rho\sigma} \eta^{\rho\sigma} + \ldots)$ |   |   | [50]       |
| $(a_F^{(5)})^{\alpha\beta\gamma} = 2g e^{\rho\alpha\beta\gamma} u_\rho$ | O   | $(k_F^{(4)})_{\mu\nu\alpha\beta} \sim b_\mu b^{\alpha\beta} \eta^{\nu\gamma} + \ldots$ | X |   | [29]       |
| $(a_F^{(5)})^{\alpha\beta\gamma} = -(u_\beta \eta^{\gamma\alpha} - u^{\alpha\gamma}\eta^{\beta})$ | O   | $L_{eff} \supset \epsilon^{\mu\nu\alpha\beta}b_\alpha u^\nu \partial F_{\mu\nu} F_{\rho\beta}$ | X |   | [30]       |
| $\bar{\psi}\gamma_5 \gamma^\mu \gamma^\nu b_\mu \vartheta$ | O   | $(k_F^{(5)})_{\nu\nu\lambda\rho} \sim k_{\mu\lambda\rho}$ |   |   | [28]       |
| $H_F^{(5)}_{\mu\nu\alpha\beta}$ | E   | $(k_F^{(4)})_{\mu\nu\lambda\rho} \sim k_{\mu\nu\lambda\rho}$ |   |   | [28]       |
| $H_F^{(5)}_{\mu\nu\alpha\beta} = -2g_\kappa^{\mu\nu\lambda\rho} \epsilon_{\lambda\rho}^{\alpha\beta}$ | E   | $(k_F^{(4)})_{\nu\nu\lambda\rho} \sim k_{\mu\nu\lambda\rho} \epsilon^{\alpha\beta}_{\lambda\rho}$ | X |   | [58]       |
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[90] See section IIIC of [3].