A FAMILY OF STABLE DIFFUSIONS

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Abstract. Consider a $C^\infty$ closed connected Riemannian manifold $(M, g)$ with negative curvature. The unit tangent bundle $SM$ is foliated by the (weak) stable foliation $W^s$ of the geodesic flow. Let $\Delta^s$ be the leafwise Laplacian for $W^s$ and let $X$ be the geodesic spray, i.e., the vector field that generates the geodesic flow. For each $\lambda$, the operator $L_\lambda^s = \Delta^s + \lambda X$ generates a diffusion for $W^s$. We show that, as $\lambda \to -\infty$, the unique stationary probability measure for the leafwise diffusion of $L_\lambda$ converges to the normalized Lebesgue measure on $SM$.

1. Statement of the result

Let $(M, g)$ be an $m$-dimensional closed connected negatively curved $C^\infty$ Riemannian manifold. We shall study a class of probability measures on the unit tangent bundle $SM$.

Endow the universal cover space $(\tilde{M}, \tilde{g})$ with the lifted Riemannian metric. The fundamental group $G = \pi_1(M)$ acts on $\tilde{M}$ as isometries such that $M = \tilde{M}/G$. Let $\partial \tilde{M}$ be the geometric boundary of $\tilde{M}$, i.e., the collection of equivalent classes of unit speed geodesic rays that remain a bounded distance apart. Since $\tilde{g}$ is negatively curved, there is a natural homeomorphism from $\partial \tilde{M}$ to the unit sphere $S_\tilde{x} \tilde{M}$ in the tangent space at $x \in \tilde{M}$, sending $\xi$ to the initial vector of the geodesic starting from $x$ in the equivalent class of $\xi$ (EO).

Hence we identify the unit tangent bundle $SM = \bigcup_{x \in \tilde{M}} S_{\tilde{x}} \tilde{M}$ with $\tilde{M} \times \partial \tilde{M}$.

For each $v = (x, \xi) \in S\tilde{M}$, its (weak) stable manifold for the geodesic flow $\Phi_t$ on $S\tilde{M}$, denoted $\tilde{W}^s(v)$, is the collection of initial vectors of geodesics in the equivalent class of $\xi$ and can be identified with $\tilde{M} \times \{\xi\}$. The collection of $\tilde{W}^s(v)$ form the stable foliation $\tilde{W}^s$ of $S\tilde{M}$. Extend the action of $G$ continuously to $\partial \tilde{M}$. Then $SM$ can be identified with the quotient of $\tilde{M} \times \partial \tilde{M}$ under the diagonal action of $G$. Since $\psi(\tilde{W}^s(v)) = \tilde{W}^s(D\psi(v))$ for $\psi \in G$, the collection of quotients of $\tilde{W}^s(v)$ defines a lamination $W^s$ on $SM$, the so-called (weak) stable foliation of $SM$. The leaves of $W^s$ are discrete quotients of $\tilde{M}$, which are naturally endowed with the Riemannian metric induced from $\tilde{g}$. For $v \in SM$, let $W^s(v)$ be

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the leaf of $\mathcal{W}$ containing $v$. Then $W^s(v)$ is a $C^\infty$ immersed submanifold of $SM$ depending Hölder continuously on $v$ in the $C^\infty$-topology (SFL).

Let $\mathcal{L}$ be a Markovian operator (i.e., $\mathcal{L}1 = 0$) on (the smooth functions on) $SM$ with continuous coefficients. It is subordinated to the stable foliation $W^s$, if for every smooth function $f$ on $SM$, the value of $\mathcal{L}(f)$ at $v \in SM$ only depends on the restriction of $f$ to $W^s(v)$. A Borel probability measure $m$ on $SM$ is called $\mathcal{L}$-harmonic if it satisfies

$$\int \mathcal{L}(f) \, dm = 0$$

for every smooth function $f$ on $SM$. Extend $\mathcal{L}$ to be a $G$-equivariant operator on $S\tilde{M} = \tilde{M} \times \partial \tilde{M}$, which we shall denote with the same symbol, and, for $v = (x, \xi) \in S\tilde{M}$, let $\mathcal{L}^v$ denote the laminated operator of $\mathcal{L}$ on $\tilde{W}^s(v) = \tilde{M} \times \{\xi\}$. Call $\mathcal{L}$ weakly coercive, if its lifted leafwise operators $\mathcal{L}^v$, $v \in S\tilde{M}$, are weakly coercive in the sense that there are a number $\varepsilon > 0$ (independent of $v$) and, for each $v$, a positive $(\mathcal{L}^v + \varepsilon)$-superharmonic function $F$ on $\tilde{M}$ (i.e., $(\mathcal{L}^v + \varepsilon)F \geq 0$). It is known that for a weakly coercive operator, there exists a unique harmonic measure ([Ga], [H]).

One classical example of weakly coercive operators is $\mathcal{L} = \Delta^s$, the laminated Laplacian for $W^s$, whose unique $\mathcal{L}$-harmonic measure is always referred to as the harmonic measure (Ga). Many interesting open problems in dynamics are concerned with the relationship of the harmonic measure with the normalized Lebesgue measure (Liouville measure) and the normalized maximal entropy measure (Bowen-Margulis measure) for the geodesic flow, and the applications of these relations to the characterizations of the locally symmetric property of the underlying space (see K2, Su and see also Kai, L2, Y for more descriptions).

In this paper, we are interested in the family $\mathcal{L}_\lambda = \Delta^s + \lambda X$, where $\lambda$ is a real number and $X$ is the geodesic spray. Since $X$ is tangent to the stable manifold, the operators $\mathcal{L}_\lambda$ are subordinated to the stable foliation.

Let $V$ denote the volume entropy of $(M, g)$:

$$V = \lim_{r \to +\infty} \frac{\log \text{Vol}(B(x, r))}{r},$$

where $B(x, r)$ is the ball of radius $r$ in $(\tilde{M}, \tilde{g})$ and Vol is the volume. The volume entropy coincides with the topological entropy of the geodesic flow on $SM$ since $g$ has negative sectional curvature ([M]). For $\lambda < V$, the operator $\mathcal{L}_\lambda$ is weakly coercive ([H]) and hence there is a unique $\mathcal{L}_\lambda$-harmonic measure, which we will denote by $m_\lambda$.

Clearly, $m_0$ is the classical harmonic measure. When $\lambda \to V$, $m_\lambda$ tends to the unique harmonic measure $m_V$ for $\mathcal{L}_V$ ([L1]), where $m_V$ is actually the harmonic measure for the Laplacian subordinated to the strong stable foliation and is such that its transversal distribution in the weak unstable coordinate is the same as for the maximal entropy measure of the geodesic flow. When $\lambda \to -\infty$, the main result of this note is:
Theorem 1.1. Let $(M, g)$ be an $m$-dimensional closed connected negatively curved $C^\infty$ Riemannian manifold. As $\lambda \to -\infty$, the $L_\lambda$-harmonic measure $m_\lambda$ converges to the normalized Lebesgue measure on $SM$.

Roughly speaking, since the measure $m_\lambda$ is $L_\lambda$-harmonic, it is also stationary for the operator $-\overline{\nabla} - (1/\lambda)\Delta^s$ (see Section 2 for a precise definition). In particular, any limit measure of the family $m_\lambda$ as $-1/\lambda \to 0$ is invariant under the (reversed) geodesic flow. For a limit of random perturbations of a conservative Anosov flow, the convergence of the stationary measures to a SRB measure has been shown by several authors, in particular Kifer ([K1]), under the condition that the operator $\Delta$ is hypoelliptic, so that the Markov kernels have a density with respect to Lebesgue on $SM$. We can not apply this to show Theorem 1.1 since in our case, the operators are subordinated to the stable foliation and the Markov kernels $p_\lambda(t, (x, \xi), d(y, \eta))$ are singular. Another approach by Cowieson-Young ([CY]) uses the variational principle from thermodynamical formalism and we show that such an approach can be used in our case in spite of the singularity of the Markov kernels. We shall show any limiting measure $m$ of $m_\lambda$ (as $\lambda \to -\infty$) would satisfy Pesin entropy formula for the geodesic flow. Theorem 1.1 follows since the normalized Lebesgue measure on $SM$ is indeed characterized by Pesin formula among invariant measures for the geodesic flow ([BR]). More precisely, we will define a stochastic flow on a bigger space and consider a special stationary measure $\overline{m}_\lambda$ for that stochastic flow that projects in $m_\lambda$ on $SM$. We then introduce a relative entropy like quantity $h^s_\lambda$ for $\overline{m}_\lambda$ and show $h_m$, the entropy of $m$ for the reversed geodesic flow, satisfies

$$h_m \geq \limsup_{\lambda \to -\infty} h^s_\lambda.$$  

(1.1)

This can be done (see Proposition 3.2) along the lines of Cowieson-Young ([CY]) and Kifer-Yomdin ([KY]) for the upper semi-continuity of the relative entropy. To conclude Theorem 1.1 we verify that $\limsup_{\lambda \to -\infty} h^s_\lambda$ has a lower bound given by Pesin entropy integral for $m$ using the SRB like properties of $\overline{m}_\lambda$ (see Proposition 3.1) and their nice convergence property inherited from our stochastic flow system (see Proposition 2.7).

We arrange the paper as follows. In Section 2, we will give preliminaries on the properties of the $L$-harmonic measures and the dynamics of the associated stochastic flows. In Section 3, we will introduce the random system to define $h^s_\lambda$ and reveal its relation with Pesin entropy formula. The upper semi-continuity equality (1.1) will be shown in the final step.

2. Harmonic measure and stochastic flow

We begin with some understanding of the $L_\lambda$-harmonic measure $m_\lambda$ ($\lambda < V$) by analyzing the dynamics of its $G$-invariant extension on $\tilde{M} \times \partial\tilde{M}$, which is denoted by $\tilde{m}_\lambda$.

Consider the $G$-equivariant extension of $L_\lambda$ to $S\tilde{M} = \tilde{M} \times \partial\tilde{M}$, which we shall denote with the same symbol. It defines a Markovian family of probabilities on $\tilde{\Omega}_+$, the space of paths of $\tilde{\omega} : [0, +\infty) \to S\tilde{M}$, equipped with the smallest $\sigma$-algebra $A$ for which the
projections $R_t: \tilde{\omega} \to \tilde{\omega}(t)$ are measurable. Indeed, for $v = (x, \xi) \in S\tilde{M}$, the laminated operator $L^\lambda_X$ on $\tilde{W}^s(v)$ can be regarded as an operator on $\tilde{M}$ with corresponding heat kernel functions $p^\lambda_X(t, y, z)$, $t \in \mathbb{R}_+$, $y, z \in \tilde{M}$. Define

$$p^\lambda_X(t, (x, \xi), d(y, \eta)) = p^\lambda_X(t, x, y) \det(y) \delta_\xi(\eta),$$

where $\delta_\xi(\cdot)$ is the Dirac function at $\xi$. Then the diffusion process on $\tilde{W}^s(v)$ with infinitesimal operator $L^\lambda_X$ is given by a Markovian family $\{P^\lambda_X\}_{x \in \tilde{M} \times \{\xi\}}^\dagger$, where for every $t > 0$ and every Borel set $A \subset \tilde{M} \times \partial \tilde{M}$ we have

$$P^\lambda_X(\{\tilde{\omega}: \tilde{\omega}(t) \in A\}) = \int_A p^\lambda_X(t, w, d(y, \eta)).$$

**Proposition 2.1.** (Ga, H) With the above notations, the following are true.

i) The measure $\tilde{m}_\lambda$ satisfies, for all $f \in C^2(\tilde{M} \times \partial \tilde{M})$ with compact support,

$$\int_{\tilde{M} \times \partial \tilde{M}} \left( \int_{\tilde{M} \times \partial \tilde{M}} f(y, \eta) p^\lambda_X(t, (x, \xi), d(y, \eta)) \right) d\tilde{m}_\lambda(x, \xi) = \int_{\tilde{M} \times \partial \tilde{M}} f(x, \xi) d\tilde{m}_\lambda(x, \xi).$$

ii) The measure $\tilde{P}_\lambda = \int P^\lambda_X \tilde{m}_\lambda(v)$ on $\tilde{\Omega}_+$ is invariant under the shift map $\{\sigma_t\}_{t \in \mathbb{R}_+}$ on $\tilde{\Omega}_+$, where $\sigma_t(\tilde{\omega}(s)) = \tilde{\omega}(s + t)$ for $s > 0$ and $\tilde{\omega} \in \tilde{\Omega}_+$.

iii) The measure $\tilde{m}_\lambda$ can be expressed locally at $v = (x, \xi) \in S\tilde{M}$ as $d\tilde{m}_\lambda = k_\lambda(y, \eta)(dy \times d\nu_\lambda(\eta))$, where $\nu_\lambda$ is a finite measure on $\partial \tilde{M}$ without atoms and, for $\nu_\lambda$-almost every $\eta$, $k_\lambda(y, \eta)$ is a positive function on $\tilde{M}$ satisfying the equation

$$\Delta(k_\lambda(y, \eta)) - \lambda \text{Div}(k_\lambda(y, \eta)\overline{X}(y, \eta)) = 0,$$

where we continue to use $X$ to denote the geodesic spray for $S\tilde{M}$.

**Remark 2.2.** Let $\tilde{m}$ be any weak* limit of the probability measures $m_\lambda$ on $SM$ as $\lambda \to -\infty$ and let $\tilde{m}$ be the $G$-invariant extension of $\tilde{m}$ to $\tilde{M} \times \partial \tilde{M}$. Clearly, Theorem 1.1 follows if we can show $\tilde{m}$ has absolutely continuous conditional measures on leaves $\tilde{M} \times \{\xi\}$. But this does not follow directly from equation (2.1) since the Harnack inequality used for each $\lambda$ finite is worse and worse when $\lambda$ goes to $-\infty$ and hence we have less and less control of the density functions $k_\lambda$.

For Theorem 1.1 we will further explore the invariant dynamics of $m_\lambda$ from the stochastic flow point of view and use it to establish the entropy formula for the limit measures.

We first recall some classical theories from the Stochastic Differential Equations (SDE). Let $\{B_t = (B^1_t, \cdots, B^d_t)\}_{t \in \mathbb{R}_+}$ be a $d$-dimensional Euclidean Brownian motion starting from the origin with the Euclidean Laplacian generator (so the covariance matrix is $2\text{Id}$) and let $(\Omega, \mathbb{P})$ denote the corresponding Wiener space. Let $\Gamma = (X_0, X_1, \cdots, X_d)$, where $\{X_i\}_{i \leq d+1}$ are bounded vector fields on a smooth finite dimensional Riemannian manifold $(N, \langle \cdot, \cdot \rangle)$. The pair $(\Gamma, \{B_t\}_{t \in \mathbb{R}_+})$ consists of a stochastic dynamical system (SDS) on $N$ and it is $C^j$
(j ≥ 1 or j = ∞) if all X_i are C^j bounded ([EI]). An N-valued semimartingale \((x_t)_{t \in \mathbb{R}_+}\) defined up to a stopping time τ is said to be a solution of the following Stratonovich SDE

\[(2.2) \quad dx_t(\omega) = X_0(x_t(\omega)) \, dt + \sum_{i=1}^d X_i(x_t(\omega)) \circ dB_t^i(\omega),\]

if for all \(f \in C^\infty(N),\)

\[f(x_t(\omega)) = f(x_0(\omega)) + \int_0^t X_0f(x_s(\omega)) \, ds + \int_0^t \sum_{i=1}^d X_if(x_s(\omega)) \circ dB_s^i(\omega), \quad 0 \leq t < \tau.\]

The solution to (2.2) always exists and is essentially unique when all \(X_i\)'s are \(C^1\) bounded ([EI]). Moreover, for \(\mathbb{P}\) almost all \(\omega,\) the mapping

\[F_t(\cdot, \omega) : x_0(\omega) \mapsto x_t(\omega)\]

has the following property.

**Proposition 2.3.** ([EI Chapter VIII]) Let \((X, (B_t)_{t \in \mathbb{R}_+})\) be a \(C^j\) SDS on \(N,\) where \(j \geq 1\) or \(j = \infty.\) There is a version of the explosion time map \(x \mapsto \tau^x,\) defined for \(x \in N,\) and a version of \(F_t(x, \omega),\) defined when \(t \in [0, \tau^x(\omega)),\) such that if \(N(t, \omega) = \{x \in N : t < \tau^x\},\) then the following are true for each \((t, \omega) \in \mathbb{R}_+ \times \Omega.\)

- i) The set \(N(t, \omega)\) is open in \(N.\)
- ii) For almost all \(\omega,\) \(x_0 \in N\) and \(0 < t < t' < \tau^{x_0},\) we have the cocycle equality

\[F_{t'}(x_0, \omega) = F_{t-t}(x_t, \sigma_t(\omega)) \circ F_t(x_0, \omega),\]

where \(\sigma_t\) is the shift transformation on \(\Omega:\)

\[\sigma_t((B^1_{\tau}, \ldots, B^m_{\tau})_{\tau \geq 0}) = ((B^1_{\tau-t}, \ldots, B^m_{\tau-t})_{\tau \geq 0}) - (B^1_t, \ldots, B^m_t).\]

- iii) The map \(F_t(x, \omega) : N(t, \omega) \rightarrow N\) is \(C^{j-1}\) (or \(C^\infty\) when \(j = \infty)\) and is a diffeomorphism onto an open subset of \(N.\) Moreover, the map \(\tau \mapsto F_\tau(\cdot, \omega)\) of \([0, t]\) into \(C^{j-1}\) (or \(C^\infty\) when \(j = \infty)\) mappings of \(N(t, \omega)\) is continuous.

- iv) For \(1 \leq l \leq j - 1,\) denote by \(D^{(l)}F_t(\cdot, \omega)\) the \(l\)-th tangent map of \(F_t.\) Then, for any \(q \in [1, \infty),\) there is a bounded function \(c_l(t, q),\) which depends on \(t, m, q,\) and the bounds of \(\{\nabla^l X_0\} \subset L^1, \{\nabla^l X_i\} \subset L^1 \cap L^{1+t} \subset L^{1+t} \subset L^{1+t+1}\) such that

\[\|D^{(l)}F_t(\cdot, \omega)\|_{L^q} < c_l(t, q),\]

where \(\cdot \in L^q\) is the \(L^q\)-norm and \(\nabla^l\) denotes the \(l\)-th covariant derivative and \(R\) is the curvature tensor.

When \(N(t, \omega) \equiv N,\) the maps \(\{F_t(\cdot, \omega)\}_{t \in \mathbb{R}_+}\) induce a flow on \(N,\) which we shall call the stochastic flow associated to the SDS \((X, (B_t)_{t \in \mathbb{R}_+})\) or (2.2). A direct consequence of Proposition 2.3 is the following regularity of a one-parameter family of stochastic flows.

**Corollary 2.4.** Let \((X^a, (B_t)_{t \in \mathbb{R}_+})\) be a one-parameter family of SDS on \(N \) with \(N^a(t, \omega) \equiv N.\) Assume \(X^a_t\)'s are all \(C^k\) \((k \geq 1\) or \(= \infty)\) on \(N \times A\) in the product differentiable structure. Then for any \(t > 0,\) and \(j \leq k - 1,\) \(a \mapsto F^a_t(\cdot, \omega)\) is \(C^j\) in the space of \(C^{k-1-j}\) maps of \(N.\)
Proof. Let \( x_t^n \) be the solution for the SDS \((X^a, (B_t)_{t \in \mathbb{R}_+})\). Then \((x_t^n, a)\) solves the new SDS \(((X^a, 0), (B_t)_{t \in \mathbb{R}_+})\) on \( N \times A \). The regularity in \( a \) is a straightforward application of Proposition 2.3 by treating \( a \) as a part of the initial value. \( \square \)

Corollary 2.4 does not apply when we only have Hölder continuity of \( X^a \) in \( a \). However, it is still possible to discuss the regularities of \( a \mapsto F_t^a(\cdot, \omega) \) by using Kolmogorov’s criterion:

**Proposition 2.5.** (cf. [Ku] Theorem 1.4.1) Let \( T > 0 \) and let \( \{Y_t^a(\omega)\}_{t \in [0, T], a \in A} \) be a one parameter family of random processes on a complete manifold, where \( A \) is some bounded \( n \)-dimensional Euclidean domain. Suppose there are positive constants \( b, b_0, b_1, \ldots, b_n \), with \( \sum_{i=0}^{n} (b_i)^{-1} < 1 \), and \( C_0(b) \) such that for all \( t, t' \in [0, T] \) and \( a = (a_1, \ldots, a_n), a' = (a'_1, \ldots, a'_n) \in A \),

\[
\mathbb{E} \left[ |Y_t^a - Y_t^{a'}|^b \right] \leq C_0(b) \left( |t - t'|^{b_0} + \sum_{i=1}^{n} |a_i - a'_i|^{b_i} \right),
\]

then \( Y_t^a \) has a continuous modification with respect to the parameter \((t, a)\).

Let \( \beta_i, i = 0, \ldots, n \), be arbitrary positive numbers less than \( b_1(1 - \sum_{i=0}^{n} (b_i)^{-1})/b \). Then for any hypercube \( D \) in \( A \), there exists a positive random variable \( k(\omega) \) with \( \mathbb{E}[k(\omega)^b] < \infty \) such that for any \( t, t' \in [0, T] \) and \( a, a' \in D \),

\[
|Y_t^a - Y_t^{a'}| \leq k(\omega) \left( |t - t'|^{\beta_0} + \sum_{i=1}^{n} |a_i - a'_i|^{\beta_i} \right).
\]

Next, we consider \( \lambda < 0, \varepsilon := 1/\sqrt{-\lambda} \) and \( \mathcal{L}'_{\varepsilon} := -X + \varepsilon^2 \Delta^s \). Extend \( \mathcal{L}'_{\varepsilon} \) to be a \( G \)-equivariant operator on \( \widetilde{S}\widetilde{M} \), which we shall denote with the same symbol. Its associated leafwise diffusions can be visualized using the classical Eells-Elworthy-Malliavin construction.

Recall that, for \( \nu = (x, \xi) \in \widetilde{S}\widetilde{M} \), we have identified the stable manifold \( \widetilde{W}^s(\nu) \) with \( \widetilde{M} \times \{\xi\} \) and endowed it with the Riemannian metric on \( \widetilde{M} \). In the same way, we can identify an orthogonal frame in the tangent space \( T_x \widetilde{W}^s \) with \( O_x \times \{\xi\} \), where \( O_x = (e_1, \ldots, e_m) \) is an element in \( O_x(\widetilde{M}) \), the collection of the orthogonal frames in \( T_x \widetilde{M} \). Set \( \mathcal{O}^s(\widetilde{S}\widetilde{M}) \) for the bundle of such stable orthogonal frames:

\[
\mathcal{O}^s(\widetilde{S}\widetilde{M}) := \left\{(x, \xi) \mapsto O_x \times \{\xi\} : O_x = (e_1, \ldots, e_m) \in O_x(\widetilde{M}), x \in \widetilde{M} \right\}.
\]

We carry to \( \mathcal{O}^s(\widetilde{S}\widetilde{M}) \) all the Riemannian geometry from \( \mathcal{O}(\widetilde{M}) = \bigcup_{x \in \widetilde{M}} O_x(\widetilde{M}) \). In particular, if \( H_x \) denotes the horizontal lift from \( T_x \widetilde{M} \) to \( TO_{\xi}O(\widetilde{M}) \), we can define the horizontal lift \( \tilde{H}_x \) from \( T_x \mathcal{O}^s(\widetilde{S}\widetilde{M}) \) by \( \tilde{H}_x(w, \xi) = (H_x(w), \xi) \) for \( w \in T_x \widetilde{M} \).

Let \( (B_t^1, \ldots, B_t^n)_{t \geq 0} \) be an \( m \)-dimensional Euclidean Brownian motion starting from the origin with the Euclidean Laplacian generator (and covariance matrix \( 2\text{Id} \)) and let \( (\Omega, \mathbb{P}) \) be the Wiener space. Set \( \tilde{X} \) as the horizontal lift of \( X \) to \( TO^s(\widetilde{S}\widetilde{M}) \). We can realize the
diffusion for \( \mathcal{L}_\varepsilon' \) as the projection to \( \tilde{S}\tilde{M} \) of the solution of the Stratonovich Stochastic Differential Equation on \( \mathcal{O}^s(\tilde{S}\tilde{M}) \):

\[
du_t = -\hat{X}(u_t) \, dt + \varepsilon \sum_{i=1}^{m} \hat{H}(u_t(e_i)) \circ dB^i_t.
\]

Let \( \hat{\pi} : \mathcal{O}^s(\tilde{S}\tilde{M}) \to \tilde{S}\tilde{M} \) be the natural projection and denote \( \hat{W}^s \) for the foliation of \( \mathcal{O}^s(\tilde{S}\tilde{M}) \) that projects on \( \hat{W}^s \). Let \( D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \) be the space of homeomorphisms of \( \mathcal{O}^s(\tilde{S}\tilde{M}) \) that preserve the leaves of \( \hat{W}^s \) and are \( C^\infty \)-diffeomorphisms along the leaves. We endow \( D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \) with the \( C^0,\infty \) topology: \( \varphi, \varphi' \in D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \) are close if, for all \( r > 0 \), the \( r \)-germs of \( \varphi \) and \( \varphi' \) are uniformly close on compact sets and the \( r \)-germs of \( \varphi^{-1} \) and \( (\varphi')^{-1} \) are uniformly close on compact sets.

**Proposition 2.6.** With the above notations, for a.e. \( \omega \in \Omega \), for all \( \varepsilon > 0, t > 0 \), there exists \( \varphi_{\varepsilon,t}(\omega) \in D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \) such that the following hold true.

i) For all \( u \in \mathcal{O}(\tilde{S}\tilde{M}), (\omega,t) \to \varphi_{\varepsilon,t}(\omega)(u) \) is a solution of the equation \( \text{(2.3)} \); in particular, for all \( T \geq 0, \omega \to \varphi_{\varepsilon,t}(\omega) \) is measurable with respect to the \( \sigma \)-algebra generated by \( (B^1_t, \ldots, B^m_t), 0 \leq t \leq T \).

ii) For almost all \( \omega \), all \( t, s > 0 \), \( \varphi_{\varepsilon,t+s}(\omega) = \varphi_{\varepsilon,t}(\sigma_s(\omega)) \circ \varphi_{\varepsilon,s}(\omega) \), where \( \sigma_s \) is the shift map on \( \Omega \).

iii) For all \( \psi \in G \), \( D\psi \circ \varphi_{\varepsilon,t}(\omega) = \varphi_{\varepsilon,t}(\omega) \circ D\psi \).

iv) The map \( \varepsilon \mapsto \varphi_{\varepsilon,t}(\omega) \) is continuous in \( D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \).

v) For fixed \( r \in \mathbb{N}, t > 0 \),

\[
\mathbb{E} \left[ \max_u \|\varphi_{\varepsilon,t}(\omega)(u)|_{\hat{W}^s(\omega)}\|_{C^r} \right] < +\infty.
\]

**Proof.** Since both \( \hat{X} \) and \( \hat{H} \) are tangent to \( \hat{W}^s \), the solution to \( \text{(2.3)} \) is constrained in \( \hat{W}^s \). For fixed \( \xi \) and \( \varepsilon \), equation \( \text{(2.3)} \) can be seen as a SDE on \( \mathcal{O}(\tilde{M}) \times \{\xi\} \) and is solvable with infinite explosion time. Hence properties i) and ii) are given by Proposition 2.3. Property iii) follows from the uniqueness of the solution to \( \text{(2.3)} \). Considering \( \varepsilon \) as a parameter, we get the continuity of the solution in \( \varepsilon \) by Corollary 2.4. Considering \( \xi \) as a parameter, the leaves of \( \hat{W}^s \) and \( \hat{X}, \hat{H} \) vary Hölder continuously with respect to \( \xi \). Hence, by a standard estimation using Burkholder’s inequality and Gronwall’s Lemma and applying Proposition 2.5, we can obtain the continuity of the solution to \( \text{(2.3)} \) in \( \xi \), so that we can consider it as an element of \( D^\infty(\mathcal{O}^s\tilde{S}\tilde{M}) \). This shows iv). Finally we show v). Using a fundamental domain for the action of \( G \) on \( \tilde{M} \), we may regard \( \mathcal{O}^s(M) \) as a subset of \( \mathcal{O}^s(\tilde{S}\tilde{M}) \). By the \( G \)-equivariance property of the diffusion, we can restrict \( u \) in the left hand side of \( \text{(2.4)} \) to \( \mathcal{O}^s(M) \). By continuity of \( \varphi_{\varepsilon,t}(\omega)(u)|_{\hat{W}^s(\omega)} \) in \( u \), the compactness of \( \mathcal{O}^s(M) \) and Proposition 2.5 for \( \text{(2.4)} \), it suffices to show for each \( u \in \mathcal{O}^s(M), r \in \mathbb{N} \) and \( t > 0 \),

\[
\mathbb{E} \left[ \|\varphi_{\varepsilon,t}(\omega)(u)|_{\hat{W}^s(\omega)}\|_{C^r} \right] < +\infty.
\]
This is an application of Proposition 2.6 iv) by using the SDE (2.3).

Equation (2.3) for \( \varepsilon = 0 \) is the ordinary differential equation \( d\mathbf{u} = -\mathbf{X}(\mathbf{u}) \, dt \). Its solution is the extension \( \{ \hat{\Phi}_{-t} \}_{t \geq 0} \) of the reversed geodesic flow to \( \mathcal{O}(\mathcal{S}\tilde{M}) \) by parallel transport along the geodesics. Write \( \nu_\lambda \) for the probability measure on \( D^\infty(\mathcal{O}(\mathcal{S}\tilde{M})) \) that is the distribution of \( \varphi_{\varepsilon,1} = \varphi_{1/\sqrt{-\lambda},1} \) in Proposition 2.6. Every element \( \varphi \in D^\infty(\mathcal{O}(\mathcal{S}\tilde{M})) \) preserves each leaf \( \tilde{W}^s(u) \) and is a \( C^\infty \) diffeomorphism along it. We write \( J(\varphi, u) \) for the Jacobian determinant of the tangent map of \( \varphi_{\tilde{W}^s(u)} \) at \( u \). For later use, we state a proposition concerning the limit behavior of \( \varphi_{1/\sqrt{-\lambda},t} \) when \( \lambda \to -\infty \).

**Proposition 2.7.** With the above notations, the following are true.

i) For a.e. \( \omega \in \Omega \), all \( t > 0 \), as \( \lambda \to -\infty \), \( \varphi_{1/\sqrt{-\lambda},1} \) converge to \( \hat{\Phi}_{-t} \) in \( D^\infty(\mathcal{O}(\mathcal{S}\tilde{M})) \), in particular, \( \varphi_{1/\sqrt{-\lambda},1} \) converge to the time 1 map of the reverse \( \hat{\Phi}_{-1} \).

ii) For any \( -\infty < \lambda < 0 \) and \( r, M \) positive integers,

\[
B_{r,M}(\lambda) := \mathbb{E} \left[ \max_u \| \varphi_{1/\sqrt{-\lambda},M}(\omega)(u) \|_{C^r} \right] < +\infty \text{ and } B_{r,M} := \limsup_{\lambda \to -\infty} B_{r,M}(\lambda) < +\infty.
\]

iii) For any \( r \in \mathbb{N} \),

\[
\lim_{\lambda \to -\infty} \int \max_u \| \varphi_{\tilde{W}^s(u)} \|_{C^r} - \| \hat{\Phi}_{-1} \tilde{W}^s(u) \|_{C^r} \, d\nu_\lambda(\varphi) = 0.
\]

iv) We have

\[
\lim_{\lambda \to -\infty} \int \log J(\varphi, u) \, d\nu_\lambda(\varphi) = \log J(\hat{\Phi}_{-1}, u)
\]

and the convergence is locally uniform in \( u \).

**Proof.** The proof of continuity of the solution to (2.3) in Proposition 2.6 extends to \( \varepsilon = 0 \). This shows i). When \( \lambda \to -\infty \), \( \varepsilon = 1/\sqrt{-\lambda} \to 0 \). For ii), note that \( B_{r,M}(\lambda) \) is finite by Proposition 2.6 v) (applied for \( t = M \)). Then \( B_{r,M} \) is also finite by using the continuity in \((\varepsilon, u)\) in the estimation of the expectation in (2.5) in the proof of Proposition 2.6 v). Similarly, we have the continuity in \( \varepsilon \) of the tangent maps (and its derivatives in \( \varepsilon \)) of the solution to (2.3). Following Proposition 2.6 iv), it is easy to deduce from (2.3) the continuity in \( \varepsilon \) of the norm of the tangent maps and of the Jacobian of the first order tangent map. This shows iii) and iv).

It follows from Proposition 2.6 i) and ii) that we can consider \( \varphi_{\varepsilon,n} \), \( n \in \mathbb{N} \), as an independent product of the homeomorphisms \( \varphi_{\varepsilon,1} \) and that we can apply the theory of independent random mappings. For any \( C^2 \) compactly supported function \( f \) on \( \mathcal{S}\tilde{M} \), \((x, \xi) \in \mathcal{S}\tilde{M} \) and any frame \( u \in \mathcal{O}(\mathcal{S}\tilde{M}) \) in the fiber \( \hat{\pi}^{-1}(x, \xi) \), we have

\[
\int_{\mathcal{S}\tilde{M}} f(y, \eta) \, d\mathbf{p}_\lambda(\varepsilon, (x, \xi), d(y, \eta)) = \int_{D^\infty(\mathcal{O}(\mathcal{S}\tilde{M}))} f(\hat{\pi}(\varphi)(u)) \, d\nu_\lambda(\varphi).
\]
Let $\tilde{m}_\lambda$ be the measure on $\mathcal{O}^s(\tilde{M})$ that projects on $\tilde{m}_\lambda$ on $\tilde{M}$ and such that the conditional measures on fibers of the projection on $\tilde{\pi}$ are proportional to Lebesgue measure on $m$-dimensional frames. The following is true.

**Proposition 2.8.** The measure $\tilde{m}_\lambda$ is stationary under $\nu_\lambda$, i.e., it satisfies, for any $C^2$ compactly supported function $f$ on $\mathcal{O}^s(\tilde{M})$, 
\[
\int f(\varphi(u)) \, d\nu_\lambda(\varphi) \, d\tilde{m}_\lambda(u) = \int f(u) \, d\tilde{m}_\lambda(u).
\]
Moreover, the conditional measures $\tilde{m}_{s,u}^\lambda$ of $\tilde{m}_\lambda$ with respect to the leaves of the $\tilde{W}^s$ foliation are absolutely continuous with respect to Lebesgue.

**Proof.** The stationarity follows from relation (2.7), the stationarity of $\tilde{m}_\lambda$ and the fact that the flow preserves the orthogonal group on the fibers. The leaves of $\tilde{W}^s$ are made of whole fibers and project on the leaves of $\tilde{W}^s$. The conditional measures on the leaves of $\tilde{W}^s$ are given by the extension by Lebesgue on the fibers of the conditional measures on the leaves of $\tilde{W}^s$. By Proposition 2.1 iii), they are therefore absolutely continuous. \qed

Let $\pi : \mathcal{O}^s(SM) \to SM$ be the quotient of the map $\tilde{\pi}$ by the action of $G$ and let $W^s = \{ W^s(u) \}_{u \in \mathcal{O}^s(SM)}$ denote the corresponding quotient foliation of $\tilde{W}^s$. Let $D^\infty(\mathcal{O}^sSM)$ be the space of homeomorphisms of $\mathcal{O}^s(SM)$ that preserve the leaves of $W^s$ and are $C^\infty$-diffeomorphisms along the leaves. We endow $D^\infty(\mathcal{O}^sSM)$ with the $C^{0,\infty}$ topology: $\varphi, \varphi' \in D^\infty(\mathcal{O}^sSM)$ are close if, for all $r > 0$, the $r$-germs of $\varphi$ and $\varphi'$ are uniformly close and the $r$-germs of $\varphi^{-1}$ and $(\varphi')^{-1}$ are uniformly close. By Proposition 2.6 iii), we can consider $\nu_\lambda$ as a probability measure on $D^\infty(\mathcal{O}^sSM)$.

We define the measure $\overline{m}_\lambda$ on $\mathcal{O}^s(SM)$ such that its $G$-invariant extension to $\mathcal{O}^s(\tilde{M})$ is $\tilde{m}_\lambda$. We see that $\overline{m}_\lambda$ is a probability measure that projects to $m_\lambda$ on $SM$ and is such that the conditional measures on fibers of $\pi$ are proportional to Lebesgue on $m$-dimensional frames. As a consequence of Proposition 2.8 we have

**Corollary 2.9.** The measure $\overline{m}_\lambda$ is stationary under $\nu_\lambda$, i.e., it satisfies, for any continuous function $f$ on $\mathcal{O}^s(SM)$,
\[
\int f(\varphi(u)) \, d\nu_\lambda(\varphi) \, d\overline{m}_\lambda(u) = \int f(u) \, d\overline{m}_\lambda(u).
\]
Moreover, the conditional measures $\overline{m}_{s,u}^\lambda$ of $\overline{m}_\lambda$ with respect to the leaves of the $\overline{W}^s$ foliation are absolutely continuous with respect to Lebesgue.

We are interested in the limit measures of $\overline{m}_\lambda$’s when $\lambda$ goes to $-\infty$. Let $m$ be such a limit point and let $\overline{m}$ be the probability measure on $\mathcal{O}^s(SM)$ that projects to $m$ on $SM$ and is such that the conditional measures on fibers of $\pi$ are proportional to Lebesgue on $m$-dimensional frames. Then $\overline{m}$ is the limit of $\overline{m}_s$ along the same subsequence. Let $\{ \overline{\mu}_t \}_{t \geq 0}$ be the extension of the reversed geodesic flow to $\mathcal{O}^s(SM)$ by parallel transportation along
the geodesics. Then \( \overline{m} \) is invariant under \( \Phi_{-t} \). To show \( m \) is Lebesgue, it suffices to show the conditional measures of \( \overline{m} \) on the leaves of \( \overline{W}^s \) are absolutely continuous. But this does not follow from Corollary 2.9 by the same reason that we mentioned in Remark 2.2. What we are going to do in the next section is to analyze the entropy related to the natural random dynamics for \( \overline{m} \) that arises in the stationarity relation (2.8).

3. Entropy of random mappings

We consider the action on \( O^s(SM) \) of the random elements of \( D^\infty(O^sSM) \) with distribution \( \nu_\lambda, -\infty \leq \lambda < 0 \). Namely, let \( S := (D^\infty(O^sSM))^{[0,1]} \), endowed with the product measures \( \nu_\lambda^{\otimes [0,1]} \) (with the convention that \( \nu_{-\infty} \) is the Dirac measure at \( \Phi_{-1} \)) and the shift transformation \( \sigma \). On the space \( \mathcal{T} := S \times O^s(SM) \), define the transformation \( \tau \) by:

\[
\tau(\varphi, u) := (\sigma\varphi, \varphi_0 u).
\]

For \(-\infty < \lambda < 0 \), let \( \overline{m}_\lambda \) be the stationary measure from Corollary 2.9 and for \( \lambda = -\infty \), let \( \overline{m}_{-\infty} = \overline{m} \) be some weak* limit of \( \overline{m}_\lambda \) as \( \lambda \to -\infty \). For \(-\infty \leq \lambda < 0 \), the measure \( \mu_\lambda := \nu_\lambda^{\otimes [0,1]} \otimes \overline{m}_\lambda \) is invariant under the transformation \( \tau \).

Let \( \mathcal{P} \) be a measurable partition of \( \mathcal{T} \) with finite or countably many elements. We assume \(-\int \log(\overline{m}_\lambda(\mathcal{P}))\) \( d\mu_\lambda < \infty \). For \( n \in \mathbb{N} \), set \( \mathcal{P}_{-1} = \mathcal{P} \) and \( \mathcal{P}_{n} := \mathcal{P} \bigvee \tau^{-1} \mathcal{P} \bigvee \cdots \bigvee \tau^{-(n-1)} \mathcal{P} \) for \( n > 1 \), where \( \bigvee \) denotes the join of partitions, i.e., the refinement of partitions by taking intersections. For \( (\varphi, u) \in \mathcal{T} \), let \( \mathcal{P}_{-n}(\varphi, u) \) denote the element of \( \mathcal{P}_{-n} \) that contains \( (\varphi, u) \). We define the entropy \( h^s_\lambda \) for \( \overline{m}_\lambda \) as

\[
h^s_\lambda := \sup_{\mathcal{P}} \underline{h}^s_{\lambda, \mathcal{P}},
\]

where

\[
\underline{h}^s_{\lambda, \mathcal{P}} := \liminf_{n \to \infty} -\frac{1}{n} \int \log \overline{m}_{\lambda,u} (\mathcal{P}_{-n}(\varphi, u)) \, d\mu_\lambda(\varphi, u).
\]

For a formal definition of \( \overline{m}_{\lambda,u} \), we should use a measurable partition subordinated to \( \overline{W}^s \) (see Section 4 for details). But the value of \( \underline{h}^s_{\lambda, \mathcal{P}} \) does not depend on the choice of the subordinated partition and is thus well-defined. Observe that

\[
-\int \log \overline{m}_{\lambda,u} (\mathcal{P}_{-n}(\varphi, u)) \, d\mu_\lambda(\varphi, u) \leq -\int \log \overline{m}_{\lambda} (\mathcal{P}_{-n}(\varphi, u)) \, d\mu_\lambda(\varphi, u).
\]

Using the random Ruelle inequality (cf. [BB] [K12]), we obtain that \( \underline{h}^s_{\lambda, \mathcal{P}} \) is bounded independent of \( \mathcal{P} \). Hence \( h^s_\lambda \) is finite. Note also that \( \overline{m}_{\lambda,u} \) is absolutely continuous with respect to Lebesgue with a smooth density.

For the computation of \( \underline{h}^s_{\lambda, \mathcal{P}} \), we can restrict the conditional measure \( \overline{m}_{\lambda,u} \) to the local stable leaf \( \overline{W}^s_{loc,\epsilon}(u) := \{ w \in \overline{W}^s(u) : d\overline{W}^s(w, u) < \epsilon \} \) for \( \epsilon \) small enough. Recall that \( \varphi \in D^\infty(O^sSM) \) preserves each leaf \( \overline{W}^s(u) \) and is a \( C^\infty \) diffeomorphism along it. Write \( J(\varphi, u) \) for the Jacobian determinant of the tangent map of \( \varphi|_{\overline{W}^s(u)} \) at \( u \). We will conclude Theorem 1.1 from the following two propositions.
Proposition 3.1. For $-\infty < \lambda < 0$,

$$h^s_\lambda \geq \int \log J(\varphi, u) \, d\nu(\varphi) \, d\mathbb{M}(u).$$

Proposition 3.2. Let $\lambda_p, p \in \mathbb{N}$, be a sequence such that $\mathbb{M}_{\lambda_p}$ converges to the probability measure $\mathbb{m}$ as $p \to +\infty$ and let $\mathbb{M}$ be as above. Then

$$h^s_{\mathbb{M}} := h^s_{\mathbb{M}} \geq \limsup_{p \to +\infty} h^s_{\lambda_p}.$$

The proofs of Proposition 3.1 and Proposition 3.2 use completely different techniques and will be presented in this section and the following section, respectively.

In the following, we shall use $H_{\vartheta}(A)$ to denote the entropy of a measurable partition $A$ with respect to a measure $\vartheta$ of some space and use $H_{\vartheta}(A|B)$ to denote the entropy of $A$ conditioned on some measurable partition $B$, whenever these entropies are well-defined.

We shall denote $m$ for the dimension of $W^s$; for $\varphi, u \in T$, we shall write $\varphi|_0 = \text{Id}$ and $\varphi|_n = \varphi_{n-1} \circ \cdots \circ \varphi_0$ for $n \geq 1$, and $J(\varphi|_n, u)$ for the Jacobian determinant of the tangent map of $\varphi|_n$ at $u$. Clearly, we have $J(\varphi|_1, u) = J(\varphi_0, u)$ for $\varphi = (\varphi_0, \varphi_1, \cdots) \in S$.

Proof of Theorem 1.1. Let $\lambda_p, p \in \mathbb{N}$, be a sequence such that $\mathbb{M}_{\lambda_p}$ converges to the probability measure $\mathbb{m}$ and let $\mathbb{M}$ be as above. Recall that $\Phi_{-1}$ is the time one map of the reversed frame flow on $O^s(SM)$ which is a compact isometric extension of the time one map of the reversed geodesic flow $\Phi_{-1}$ on $SM$. Hence,

$$h_m = h_\mathbb{M}.$$ 

On the other hand, we have:

$$h_\mathbb{M} = \sup_p \lim_{n \to +\infty} \frac{1}{n} H_{\mathbb{M}}(P_n) \geq \sup_p \liminf_{n \to +\infty} \frac{1}{n} H_{\mathbb{M}}(P_n) = h^s_{\mathbb{M}}.$$ 

Assume Proposition 3.1 and Proposition 3.2 hold true. Then,

$$h^s_{\mathbb{M}} \geq \limsup_{p \to +\infty} \int \log J(\varphi, u) \, d\nu(\varphi) \, d\mathbb{M}(\lambda_p)(u) = \int \log J(\Phi_{-1}, u) \, d\mathbb{M}(u),$$

where the last equality holds by Proposition 2.7(iii). Altogether, we obtain

$$h_m \geq \int \log J(\Phi_{-1}, u) \, d\mathbb{M}(u).$$

Note that $\mathbb{W}^s$ is the central unstable foliation for $\Phi_{-1}$, so that $\int \log J(\Phi_{-1}, u) \, d\mathbb{M}(u)$ is the integral of the sum of the nonnegative exponents of $\Phi_{-1}$ for $\mathbb{M}$; neither the direction of the flow nor the vertical directions tangent to the fibers provide positive exponents, so that $\int \log J(\Phi_{-1}, u) \, d\mathbb{M}(u)$ is the integral of the sum of the positive exponents of $\Phi_{-1}$ for $\mathbb{M}$. By [BR], $\mathbb{M}$ is the normalized Lebesgue measure. \qed
Clearly, Proposition 3.1 follows if we can show the sample measures are SRB. This approach might work since in a recent preprint, Blumenthal-Young ([BY]) announced that, in a similar context, the sample measures are SRB. We didn’t try that way since we don’t need that strong conclusion and the intuition for Proposition 3.1 is relatively simpler.

For an endomorphism of a compact manifold preserving an absolutely continuous measure, the corresponding measure theoretical entropy is at least the integral of the logarithm of the Jacobian, which coincides with the so-called folding entropy (cf. [R], [LS]). Proposition 3.1 is intuitively a random conditional version of this phenomenon. But it might be subtle since we are considering the conditional measures and are in the random case. So we will give some details for the key steps.

We first recall some notations and results concerning Pesin local Lyapunov charts theory for random diffeomorphisms. In many places, we have to take invariant variables instead of constants since our system \((T, \tau, \mu_\lambda)\) is invariant, but not necessarily ergodic in general.

**Lemma 3.3.** ([Os]) For each \(\lambda < 0\), there is a measurable \(\Omega \subset T\) with \(\mu_\lambda(\Omega) = 1\) such that for \((\varphi, u) \in \Omega\), there exist \(r(\varphi, u) \in \mathbb{N}\) and, for \(i, 1 \leq i \leq r(\varphi, u)\), \(\chi_i(\varphi, u), d_i(\varphi, u)\) and a filtration

\[
\{0\} = V_{r(\varphi, u)+1} \subset V_{r(\varphi, u)} \subset \cdots \subset V_1 = T_u \overline{W^s}(u)
\]

with the following properties:

1. all of \(r, \chi_i, d_i, V_i\)’s depend measurably on \((\varphi, u)\);
2. \(\lim_{n \to \infty} \frac{1}{n} \log \|D_u(\varphi_n)(e)\| = \chi_i(\varphi, u)\) for \(e \in V_i(\varphi, u) \setminus V_{i+1}(\varphi, u)\);
3. \(d_i(\varphi, u) = \dim V_i(\varphi, u) - \dim V_{i+1}(\varphi, u)\) and \(\sum_{i=1}^{r(\varphi, u)} d_i(\varphi, u) = m\);
4. \(\int \sum_{i=1}^{r(\varphi, u)} \chi_i(\varphi, u)d_i(\varphi, u)\ d\mu_\lambda(\varphi, u) = \int \log J(\varphi, u)\ d\nu_\lambda(\varphi)\ d\mu_\lambda(\varphi).\)

**Lemma 3.4.** (cf. [LQ], Chapter III, Section 1) For each \(\lambda < 0\), given a small enough positive \(\tau\)-invariant function \(\epsilon\) on \(T\), there is a positive function \(\kappa\) on \(\Omega \times \{\mathbb{N} \cup \{0\}\}\) such that for \(n \in \mathbb{N} \cup \{0\}\),

\[\kappa((\varphi, u), n + 1) \leq \epsilon' \cdot \kappa((\varphi, u), n),\]

a positive constant \(\kappa_0\) and a sequence Euclidean metrics \(\| \cdot \|_{\varphi_n(u)}\) on \(T_{\varphi_n(u)}\overline{W^s}(\varphi_n(u))\) such that for all \(n \in \mathbb{N} \cup \{0\}\),

1. \(\kappa_0 \| \cdot \|_{\varphi_n(u)} \leq \| \cdot \|_{\varphi_n(u)}' \leq \kappa((\varphi, u), n) \| \cdot \|_{\varphi_n(u)}\), where \(\| \cdot \|_{\varphi_n(u)}\) is the Riemannian norm on \(T_{\varphi_n(u)}\overline{W^s}(\varphi_n(u))\);
2. \(F_{(\varphi, u), n}(e) := \exp^{-1}_{\varphi_{n+1}(u)} \circ \varphi_n \circ \exp_{\varphi_n(u)}(e)\) is defined for \(e\) with \(\| e \|_{\varphi_n(u)}' \leq \epsilon((\varphi, u), n)\);
3. \(F_{(\varphi, u), n}\) is \(C^2\) and the norm \(\| D^{(2)} F_{(\varphi, u), n} \|_{\varphi_n(u)}'_{\varphi_n(u)} \) is \(\| \cdot \|_{\varphi_n(u)}'_{\varphi_n(u)} \) for \(n \leq \kappa((\varphi, u), n)\), where by \(\| \cdot \|_{\varphi_n(u)}'_{\varphi_n(u)} \) we mean the norm of the tangent map calculated using \(\| \cdot \|_{\varphi_n(u)}\) and \(\| \cdot \|_{\varphi_n(u)}'\).
iv) the map $DF([\varphi, u], n)$ satisfies
\[ \|D_0F([\varphi, u], n) - D_0F([\varphi, u], n, n+1)\| \leq \epsilon([\varphi, u])\|e\|([\varphi, u], n); \]

v) the map $D_0F([\varphi, u], n)$ satisfies
\[ e^{\chi(\varphi, u) - \epsilon([\varphi, u])}\|e\|([\varphi, u], n) \leq \|D_0F([\varphi, u], n)(e)\|([\varphi, u], n, n+1) \leq e^{\chi(\varphi, u) + \epsilon([\varphi, u])}\|e\|([\varphi, u], n); \]

for all $e \in T_1\tau^n(\varphi, u))$. Moreover, for $i, 1 \leq i \leq r(\varphi, u)$, the spaces $E_i(\varphi, u)$, $j \geq i$, generate $V_i(\varphi, u)$.

(Since elements of $D^\infty(O^\ast SM)$ preserve the leaves of $\overline{W}^s$, Lemma 3.4 can be obtained as in [LQ] Chapter III, Section 1) using the natural auxiliary charts $E_j(\varphi, u)$'s and Lemma 3.3]

Let $\epsilon$ be as in Lemma 3.4. For $\eta > 0, u \in O^\ast (SM), \varphi \in \mathcal{S}$ such that $\eta < \epsilon(\varphi, u)$, and $n \in \mathbb{N}$, let us define the modified random $W$-Bowen ball by
\[ B^s(\varphi, u, \eta, n) := \left\{ e \in T_u\overline{W}^s(u) : \|e\|([\varphi, u], n) < \eta\kappa^{-1}((\varphi, u), 0), \text{ and for } k, 1 \leq k \leq n, \right. \]
\[ \left. \|F([\varphi, u], k)(e)\|([\varphi, u], k) < \eta\kappa^{-1}((\varphi, u), k) \right\}, \]

where $F([\varphi, u], k) := F([\varphi, u], k-1 \circ \cdots \circ F([\varphi, u], 0)$.

The following can be considered as a coarse local version of Proposition 3.1 whose deterministic diffeomorphism version estimation is standard using Pesin theory (cf. [Ma]).

**Lemma 3.5.** Let $\epsilon_0 > 0$, $-\infty < \lambda < 0$ and $\eta > 0$ be fixed. Choose $\epsilon < \epsilon_0$ as in Lemma 3.4. Then, there is a constant $C_1$ such that for $\mu_\lambda$, almost all $(\varphi, u) \in T$,
\[ -\frac{1}{n}\log m_{\lambda, u}(\exp_u(B^s(\varphi, u, \eta, n))) \geq -\frac{1}{n}\log J(\varphi|_n, u) - 3m\epsilon_0 - \frac{1}{n}C_1. \]

**Proof.** The set $B^s(\varphi, u, \eta, n)$ is empty if $\eta \geq \epsilon(\varphi, u)$. Otherwise, by definition and Lemma 3.4 iv), $B^s(\varphi, u, \eta, n)$ is contained in the set of vectors $e \in T_u\overline{W}^s(u)$ such that
\[ \|e\|([\varphi, u], 0) < \eta\kappa^{-1}((\varphi, u), 0), \|F([\varphi, u], n)(e)\|([\varphi, u], n) < \eta\kappa^{-1}((\varphi, u), n) \text{ and } \]
\[ |\det D_0F([\varphi, u]|_n)| \geq |\det D_0F([\varphi, u]|_n)(1 - \epsilon([\varphi, u]))|^\overline{m}, \]

where $\det'$ is the determinant of a linear mapping in the metrics $\|\cdot\|([\varphi, u], n)$ and $\|\cdot\|([\varphi, u], n+1)$. Therefore, there exists a geometric constant $C$ such that $\exp_uB^s(\varphi, u, \eta, n)$ is contained in the set $B^s(\varphi, u, \eta, n)$ of points $w \in \overline{W}^s(u)$ such that
\[ d_{\overline{W}^s(u)}(w, u) < C\kappa^{-1}\eta\kappa^{-1}((\varphi, u), \overline{W}^s(\varphi|_n, u) < C\kappa^{-1}\eta\kappa^{-1}((\varphi, u), n) \text{ and } \]
\[ J([\varphi, u]|_n, w) \geq (C^{-1}\kappa_0)^n(1 - \epsilon([\varphi, u]))^{n\overline{m}} J([\varphi, u]|_n, u). \]
It follows that
\[
\overline{m}_{\lambda,u}(\exp_u(B^*(\varphi, u, \eta, n))) \\
\leq \int_{\mathcal{F}_{\eta}(\mathbb{B}(\varphi, u, \eta, n))} J^{-1}(\varphi_n, w) \, d\overline{m}_{\lambda,u}(w) \\
\leq J^{-1}(\varphi_n, u)(C\kappa_0^{-1})^{\overline{m}}(1 - \epsilon(\varphi, u))^{-n\overline{m}}\overline{m}_{\lambda,u}^{\lambda,\varphi_n}(B_{\varphi_n}(\varphi_n, u, C\kappa_0^{-1}\eta^{-1}((\varphi, u), n))).
\]

Let $C_1$ be a constant such that
\[
(C\kappa_0^{-1})^{\overline{m}}\overline{m}_{\lambda,u}^{\lambda,\varphi_n}(B_{\varphi_n}(\varphi_n, u, C\kappa_0^{-1}\eta^{-1}((\varphi, u), n))) \leq e^{C_1+n\overline{m}\epsilon(\varphi,u)}.
\]

We obtain
\[
-\frac{1}{n} \log \overline{m}_{\lambda,u}^{\lambda,\varphi_n}(\exp_u(B^*(\varphi, u, \eta, n))) \geq \frac{1}{n} \log J(\varphi_n, u) + \overline{m}\log(1 - \epsilon(\varphi, u)) - \overline{m}\epsilon(\varphi, u) - \frac{1}{n}C_1.
\]

The estimation in (3.1) follows for $\epsilon_0$ small enough. \hfill $\square$

**Lemma 3.6.** Let $\epsilon_0 > 0$, $-\infty < \lambda < 0$ be fixed. There exists a finite measurable partition $\mathcal{P}$ of $\mathcal{T}$ satisfying
\[
(3.2) \quad h_{\lambda,\mathcal{P}} \geq \int \log J(\varphi, u) \, d\nu(\varphi) \, d\overline{m}(u) - 4\overline{m}\epsilon_0.
\]

**Proof.** Let $a > 0$ be small. By Lemma 3.5 for $\eta > 0$ small enough (for instance, smaller than $\eta_0$ so that $\mu(\{(\varphi, u) : \epsilon(\varphi, u) > \eta_0\}) \geq 1 - a/4$), there exists some set $A \subset \mathcal{T}$ with $\mu(A) > 1 - a/2$, such that for $(\varphi, u) \in A$, there are $\eta, \epsilon(\varphi, u)$ with $0 < \eta < \epsilon(\varphi, u) < \epsilon_0$ satisfying
\[
(3.3) \quad \liminf_{n \to +\infty} -\frac{1}{n} \log \overline{m}_{\lambda,u}^{\lambda,\varphi_n}(\exp_u(B^*(\varphi, u, \eta, n))) \geq \liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log J(\tau^j(\varphi, u)) - 3\overline{m}\epsilon_0.
\]

For $(\varphi, u) \in A$, $\kappa_1 > 0$ and $n \in \mathbb{N}$, set
\[
B^{\kappa_1}(\varphi, u, \eta, n) := \left\{ w \in \mathbb{W}^u(\varphi) : d(\varphi_k(w), \varphi_k(u)) < \eta\kappa_1(\kappa((\varphi, u), k))^{-2} \text{ for } 0 \leq k \leq n \right\}.
\]

By Lemma 3.3 i), we see that there exists some constant $\kappa_1$ depending on the geometry of $(M, g)$ such that, for all $(\varphi, u) \in A$ and $n \in \mathbb{N}$, $B^{\kappa_1}(\varphi, u, \eta, n) \subset \exp_u(B^*(\varphi, u, \eta, n))$. Following Mañé [Ma] (see also [Th]), we can obtain a countable partition $\mathcal{Q}$ of $\mathcal{T}$ with $-\int \log(\overline{m}(\mathcal{Q})) \, d\mu < \infty$ and a set $A' \subset A$ with $\mu_A(A') > 1 - a$ such that for all $n$ such that $\tau^n(\varphi, u) \in A'$,
\[
\mathcal{Q}_n(\varphi, u) \subset B^{\kappa_1}(\varphi, u, \eta, n), \quad \forall (\varphi, u) \in A'.
\]

Hence, for $\mu_A$ almost all $(\varphi, u) \in A'$,
\[
\liminf_{n \to +\infty} -\frac{1}{n} \log \overline{m}_{\lambda,u}^{\lambda,\varphi_n}(\mathcal{Q}_n(\varphi, u)) \geq \liminf_{n \to +\infty} \frac{1}{n} \log \overline{m}_{\lambda,u}^{\lambda,\varphi_n}(\exp_u(B^*(\varphi, u, \eta, n))).
\]
By Fatou Lemma,
\[ \mathcal{H}_{\lambda, Q}^* \geq \int_{\mathcal{A}_1} \liminf_{n \to +\infty} \frac{1}{n} \log \mathbb{M}_{\lambda, u} (Q_{-n}(\varphi, u)) \, d\mu_{\lambda}(\varphi, u). \]
Since the function $\log J(\varphi, u)$ is integrable and $a > 0$ can be arbitrary small, by (3.3), we can find a partition $Q$ of $\mathcal{T}$ as above satisfying
\[ \mathcal{H}_{\lambda, Q}^* \geq \int \log J(\varphi, u) \, d\nu_{\lambda}(\varphi) \, d\mathbb{M}_{\lambda}(u) - 4m\epsilon_0. \]
Note that $Q$ is such that $-\int \log(\mathbb{M}_{\lambda}(Q)) \, d\mu_{\lambda} < \infty$ and for any finite partition $\mathcal{P}$ such that $Q$ is finer than it,
\[ \mathcal{H}_{\lambda, Q} - \mathcal{H}_{\lambda, \mathcal{P}} \leq \limsup_{n \to +\infty} \frac{1}{n} \int \mathcal{H}_{\mathbb{M}_{\lambda}(Q_{-n}|\mathcal{P}_{-n})} \, d\mu_{\lambda} \leq \limsup_{n \to +\infty} \frac{1}{n} \int \mathcal{H}_{\mathbb{M}_{\lambda}(Q_{-n}|\mathcal{P}_{-n})} \, d\mu_{\lambda} \leq \int \mathcal{H}_{\mathbb{M}_{\lambda}(Q|\mathcal{P})} \, d\mu_{\lambda}. \]
We can group the tail elements in $Q$ (see its construction below) together with some care to obtain a finite partition $\mathcal{P}$ satisfying the requirement in (3.2).

Some details of the construction of $Q$ in our random setting are as follows. By the ergodicity of $\nu^{\otimes \mathbb{N} \cup \{0\}}$ with respect to $\sigma$ and the integrability property (2.6), for a.e. $\varphi$,
\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \left( \log \max_{u} \|\varphi_i|_{\mathcal{W}^s(u)}\|_{C^2} \right)^+ = \int \left( \log \max_{u} \|\varphi|_{\mathcal{W}^s(u)}\|_{C^2} \right)^+ \, d\nu_{\lambda}(\varphi) =: L < +\infty. \]
For any $b > 0$, let
\[ A(b) := \left\{ \varphi \in \mathcal{S} : \prod_{i=0}^{n-1} \max_{u} \|\varphi_i|_{\mathcal{W}^s(u)}\|_{C^2} \leq be^{2Ln} \text{ for all } n \in \mathbb{N} \cup \{0\} \right\}. \]
For any $a > 0$, there exists $b > 0$ large such that
\[ \nu^{\otimes \mathbb{N} \cup \{0\}}(A(b)) > 1 - \frac{1}{2}a. \]
Let $b$ be as in (3.4). For $l > 0$, set
\[ A(b, l) := \left\{ (\varphi, u) \in A : \varphi \in A(b), \eta_{\epsilon_0}(\kappa((\varphi, u), 0))^{-2} > l \right\}. \]
Choosing $\eta, \epsilon$ and then $l$ to be small enough, we can obtain a set $A' := A(b, l)$ with $\mu_{\lambda}$ measure greater than $1 - a$. For each $n \in \mathbb{N} \cup \{0\}$, let $A'_n \subset A'$ be the collection of points with $n$ as the first return time to $A'$ with respect to the map $\tau$. Recall that the local stable leaf $\mathcal{W}_{\lambda, \epsilon_0}(u) = \{ w \in \mathcal{W}^s(u) : d_{\mathcal{W}^s}(w, u) < \epsilon_0 \}$ depends continuously on $u$ and for each $n$, we can choose in a continuous way a maximal $(4l^2b)^{-1}e^{-2(L+\epsilon_0)n}$ separated set in $\mathcal{W}_{\lambda, \epsilon_0}(u)$. The cardinality $N_n$ of such sets satisfies $N_n \leq K^n$ for some $K$. Using
these points, we can further slice $A_n'$ into $\{A_{n,\ell}'\}_{\ell \leq N_n}$ such that each $A_{n,\ell}' \cap W^{\star}_{loc,\ell} (u)$ has diameter less than $(2l^2b)^{-1} e^{-2(L+\epsilon_0)n}$. The partition $Q$ can be chosen to be

$\{A_{n,\ell}', n \in \mathbb{N} \cup \{0\}, \ell \leq N_n, T \setminus A'\}$.

□

Proof of Proposition 3.1. Let $\epsilon_0 > 0$ and let $P$ be as in Lemma 3.6. Then, by definition of $h^s_\lambda$ and (3.2),

$$h^s_\lambda \geq h^s_{\lambda,P} \geq \int \log J(\varphi,u) \, d\nu_\lambda(\varphi) \, d\underline{m}_\lambda(u) - 4m \epsilon_0.$$ 

This concludes the proof of Proposition 3.1 since $\epsilon_0$ is arbitrary. □

4. The proof of Proposition 3.2

Let $\underline{m}$ be as in Proposition 3.2. To compare $h^s_\lambda$ with $h^s_{\lambda,P}$, we first formulate the entropy $h^s_{\lambda,P}$ in terms of some conditional entropy for the unconditional measure $\mu_\lambda$.

Let $W$ be a lamination of a compact metric space. A measurable partition is said to be subordinated to $W$ if its elements are bounded subsets of the leaves of $W$ with non-empty interiors in the topology of the leaf. We can construct a partition $R$ subordinated to $W$ by choosing a finite partition $X$ of $O_{SM}$ into sufficiently small sets with non-empty interiors and subdivide each element of $X$ into the connected components of its intersection with the leaves. The partition $R$ is measurable if it is constructed as an intersection of an increasing family $R_j, j \in \mathbb{N}$, of finite partitions into measurable sets.

Let $P$ be a finite partition of $O_{SM}$ and we assume that we have chosen $X, R$ as above and that $P$ refines $X$. We may assume that the boundaries of the elements of $P, X$ and $R_j$ are all $\underline{m}$-negligible. The conditional measures $\underline{m}_{\lambda,u}$ in the definition of $h^s_{\lambda,P}$ can be taken on any measurable partition subordinated to $W$, so that

$$h^s_{\lambda,P} = \liminf_{n \to +\infty} - \frac{1}{n} \int \log \underline{m}_{\lambda,u} (P_{-n}(\varphi,u)) \, d\mu_\lambda(\varphi,u) = \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_\lambda} (P_{-n}|R).$$

Proving Proposition 3.2 amounts to proving that, if $\underline{m}_{\lambda,p} \to \underline{m}$ as $p \to +\infty$, then

$$h^s_{\underline{m}} \geq \limsup_{p \to +\infty} \sup_{P} \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda,p}} (P_{-n}|R).$$

This is true, if we can show, for any $\alpha > 0$, there are partitions $P, R$ and $n$ large, such that for all $p$ large enough,

$$h^s_{\underline{m}} \geq \frac{1}{n} H_{\mu_{\lambda,p}} (P_{-n}|R) - 2\alpha \geq h^s_{\lambda,P} - 3\alpha \geq h^s_{\lambda,p} - 5\alpha.$$ 

The first inequality in (4.1) can be achieved if we can find good $P, R$ for $\underline{m}$ with $h^s_{-\infty,p}$ being close to $h^s_{\lambda,p}$. So we will show the other two inequalities in (4.1) first.
We begin with the second inequality in (4.1), which is not trivial in our setting since the conditional entropy sequence \( H_{\mu_{\lambda}}(\mathcal{P}_{-n}|\mathcal{R}) \) is not necessarily a subadditive sequence in \( n \).

**Lemma 4.1.** Given \( \mathcal{X}, \mathcal{R} \) and \( \mathcal{P} \) as above, there exists a countable partition \( \mathcal{Q} \) of \( \mathcal{T} \) such that the partition \( \mathcal{R} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \) is finer than \( \tau^{-1}\mathcal{R} \). Moreover, given \( \alpha > 0 \), there is \( \delta \) and \( \Lambda \) such that if the diameters of the elements of \( \mathcal{X} \) are smaller than \( \delta \) and if \( \lambda < \Lambda \), one can choose \( \mathcal{Q} \) with \( H_{\mu_{\lambda}}(\mathcal{Q}) < \alpha \).

**Proof.** For \( u, w \in \mathcal{O}^s(SM) \) in the same \( \mathcal{W} \) leaf, write \( d^s(u, w) \) for the distance between \( u \) and \( w \) along their common leaf. For any \( \delta > 0 \), there are two constants \( c(\delta) \) and \( C(\delta) \) such that if \( u \) and \( w \) are on the same leaf and \( d(u, w) < \delta \), then either \( d^s(u, w) < c(\delta) \) or \( d^s(u, w) \geq C(\delta) \). We can ensure that \( c(\delta) \to 0 \) as \( \delta \to 0 \) and that \( C(\delta) \to \infty \) as \( \delta \to 0 \). Suppose \( u \) and \( w \) are in the same element of the partition \( \mathcal{R} \) and that \( \varphi_0u \) and \( \varphi_0w \) are in the same element of \( \mathcal{X} \). If \( d^s(\varphi_0u, \varphi_0w) < C(\delta) \), in particular, as soon as \( d^s(u, w) < C(\delta)/\|\varphi_0\|_{C^1} \), then \( \varphi_0u \) and \( \varphi_0w \) are in the same connected component of \( \mathcal{W} \) and thus in the same element of \( \mathcal{R} \).

To obtain Lemma 4.1 it is therefore enough to take the partition \( \mathcal{Q} \) of \( \mathcal{T} \) as follows: the projection on \( \mathcal{S} \) depends only on the first coordinate \( \varphi_0 \) and is the partition \( \mathcal{A}_n, n \geq 0 \), where \( \mathcal{A}_n := \{ \varphi : nC(\delta) \leq \|\varphi\|_{C^1} \leq (n + 1)C(\delta) \}; \mathcal{A}_0 \times \mathcal{O}^s(SM) \) is one element of \( \mathcal{Q} \); on each \( \mathcal{A}_n, n > 0 \), we further cut \( \mathcal{O}^s SM \) into \( N_n \) pieces of diameter smaller than \( 1/(n + 1) \).

The entropy of \( \mathcal{Q} \) satisfies

\[
H_{\mu_{\lambda}}(\mathcal{Q}) \leq H_{\mu_{\lambda}}(\{ A_n, n \geq 0 \}) + c \sum_{n=1}^{\infty} \nu_{\lambda}(A_n) \log n,
\]

where \( c \) is some constant depending on the geometry of \( \mathcal{W} \). Given \( \alpha > 0 \), we will have \( H_{\mu_{\lambda}}(\mathcal{Q}) < \alpha \) as soon as \( \nu_{\lambda}(\{ \varphi : \|\varphi\|_{C^1} > C(\delta) \}) \) and the integral \( \int_{\{ \varphi : \|\varphi\|_{C^1} > C(\delta) \}} \log \|\varphi\|_{C^1} d\nu_{\lambda} \) are sufficiently small. These two conditions can be realized by choosing \( \delta \) small and \( \lambda \) close enough to \( -\infty \). \( \square \)

**Proposition 4.2.** Given \( \alpha > 0 \), there is \( \delta > 0 \) and \( \Lambda \) such that, for all \( n > 0 \), if the diameter of the elements of \( \mathcal{X} \) are smaller than \( \delta \) and \( \lambda < \Lambda \),

\[
\frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}_{-n}|\mathcal{R}) \geq \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}_{-n}|\mathcal{R}) - \alpha.
\]

**Proof.** Let \( \mathcal{Q} \) be as in Lemma 4.1. Then we have that the mapping \( n \to H_{\mu_{\lambda}}(\mathcal{P}_{-n} \vee \mathcal{Q}_{-n}|\mathcal{R}) \) is subadditive. Indeed, for \( n, n' \in \mathbb{N}, \)

\[
H_{\mu_{\lambda}}(\mathcal{P}_{-(n+n')} \vee \mathcal{Q}_{-(n+n')}|\mathcal{R}) = H_{\mu_{\lambda}}(\mathcal{P}_{-n} \vee \mathcal{Q}_{-n}|\mathcal{R}) + H_{\mu_{\lambda}}(\mathcal{P}_{-(n+n')} \vee \mathcal{Q}_{-(n+n')}|\mathcal{R} \vee \mathcal{P}_{-n} \vee \mathcal{Q}_{-n}),
\]

where \( \mathcal{P}_{-(n+n')} := \tau^{-n}\mathcal{P} \vee \cdots \vee \tau^{-(n+n'-1)}\mathcal{P} \) and \( \mathcal{Q}_{-(n+n')} \) is defined in the same way. Moreover, by Lemma 4.1 the partition \( \mathcal{R} \vee \mathcal{P}_{-n} \vee \mathcal{Q}_{-n} \) is finer than \( \tau^{-n}\mathcal{R} \) and the last
term is smaller than $H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{(n+n')} \vee \mathcal{Q}^{-n}_{(n+n')})^{-\tau^n_R}$. The desired subadditivity follows by invariance of $\mu_{\lambda}$ under $\tau^n$. Hence (4.2) follows since

$$\liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{n}) \geq \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{n} \vee \mathcal{Q}^{-n}_{n})$$

$$= \inf_{n} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{n} \vee \mathcal{Q}^{-n}_{n})$$

$$\leq \inf_{n} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{n}) + H_{\mu_{\lambda}}(\mathcal{Q})$$

$$\leq \inf_{n} \frac{1}{n} H_{\mu_{\lambda}}(\mathcal{P}^{-n}_{n}) + \alpha.$$  

\[\square\]

Next we show the last inequality in (4.1). For this, we first state the results extending to our context the classical results of [B], [Y] and [Bu] (compare with [CY]).

For $u \in \mathcal{O}^s(SM), \varphi \in \mathcal{S}, \eta > 0$ and $n \in \mathbb{N}$, define the random $\overline{W}^s$-Bowen ball by

$$B^s(\varphi, u, \eta, n) := \{w \in \overline{W}^s(u) : d(\varphi|_k(w), \varphi|_k(u)) < \eta \text{ for } 0 \leq k \leq n\}.$$  

The following notion was introduced by Bowen ([B]) for a single map and by Cowieson-Young ([CY]) in the random case. Since our mappings are smooth only along the foliation $\overline{W}^s$, we introduce a variant by restricting to the leaves $\overline{W}^s$. Fix $\rho > 0$ and a sequence $\varphi \in \mathcal{S}$. We denote for $u \in \mathcal{O}^s(SM), \eta > 0$ and $n \in \mathbb{N}$, $r(\rho, \varphi, u, \eta, n)$ the smallest number of random $\overline{W}^s$-Bowen balls $B^s(\varphi, w, \eta, n)$ needed to cover the random $\overline{W}^s$-Bowen ball $B^s(\varphi, u, \rho, n)$. We then set

$$h^s_{loc}(\rho, \varphi) := \sup_{u \in \mathcal{O}^s(SM)} \limsup_{\eta \to 0} \inf_{n \to +\infty} \frac{1}{n} \log r(\rho, \varphi, u, \eta, n).$$

The function $\varphi \mapsto h^s_{loc}(\rho, \varphi)$ is $\sigma$-invariant; we denote $h^s_{loc, \lambda}(\rho)$ its $\nu_{\lambda}^{\boxtimes_{\lambda-1}}$-a.e. value.

The following three propositions (Proposition 4.3, Proposition 4.4 and Proposition 4.5) are proven in [CY] for the global entropy with the additional hypotheses that $\nu_{\lambda}$ are supported in a fixed neighborhood $\mathcal{N}$ of $\Phi_{-1}$ in $D^\infty(\mathcal{O}^sSM)$ and that $\nu_{\lambda}$ converge to $\nu_{-\infty}$ as $\lambda \to -\infty$, in the sense that any $D^\infty(\mathcal{O}^sSM)$ neighborhood of $\Phi_{-1}$ has eventually full measure for $\nu_{\lambda}$. In our case, we have two extensions of the argument in [CY]: one is that the distributions $\nu_{\lambda}$ are not supported on a neighbourhood of $\Phi_{-1}$, but there is a tail; the other extension is that our mappings are not smooth everywhere, but only along the leaves of the foliation $\overline{W}^s$.

**Proposition 4.3.** Given $\alpha > 0$, let $\mathcal{X}$ be as in Proposition 4.2. Assume that the diameters of the elements of $\mathcal{P} \cap \mathcal{R}$ are all smaller than $\rho$. Then, for all $\lambda$ close enough to $-\infty$, (4.3)

$$h^s_{\lambda} - h^s_{\lambda, \mathcal{P}} \leq h^s_{loc, \lambda}(\rho) + \alpha.$$
Proof. Let $M$ be a fixed positive integer and set $\mathcal{P}_{-n}^M := \mathcal{P} \sqrt{(\tau^M)^{-1}} \mathcal{P} \sqrt{\cdots} \sqrt{(\tau^M)^{-(n-1)}} \mathcal{P}$ for $n \in \mathbb{N}$. We define

$$h_{\lambda, \mathcal{P}}^s := \liminf_{n \to +\infty} \frac{1}{n} \int \log |M_{\lambda, u}(\mathcal{P}_{-n}^M, u)| \, d\mu_{\lambda, u} = \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda, u}}(\mathcal{P}_{-n}^M, u)$$

and

$$h_{\lambda, \mathcal{P}}^s, M := \sup_{\mathcal{P}} h_{\lambda, \mathcal{P}}^s.$$ 

Since

$$h_{\lambda, \mathcal{P}}^s, M = \liminf_{n \to +\infty} \frac{1}{n} H_{\mu_{\lambda, u}}((\mathcal{P}_{-n}^M, u) \mathcal{P} \mathcal{P} \sqrt{\cdots} \sqrt{(\tau^M)^{-(n-1)}} \mathcal{P}) = M \cdot \liminf_{n \to +\infty} \frac{1}{nM} H_{\mu_{\lambda, u}}((\mathcal{P}_{-n}^M, u) \mathcal{P} \mathcal{P} \sqrt{\cdots} \sqrt{(\tau^M)^{-(n-1)}} \mathcal{P}) \geq M \cdot h_{\lambda, \mathcal{P}}^s,$$

we have

(4.4) 

$$h_{\lambda, \mathcal{P}}^s, M \geq \sup_{\mathcal{P}} h_{\lambda, \mathcal{P}}^s, M \geq M \cdot \sup_{\mathcal{P}} h_{\lambda, \mathcal{P}}^s = M \cdot h_{\lambda}^s.$$ 

Let $\mathcal{P}, \mathcal{Q}$ be as in Lemma 4.1, we also have

(4.5) 

$$h_{\lambda, \mathcal{P}}^s \leq h_{\lambda, \mathcal{P}, \mathcal{Q}}^s \leq M \cdot \liminf_{n \to +\infty} \frac{1}{nM} H_{\mu_{\lambda, u}}((\mathcal{P}_{-n}^M, u) \mathcal{P} \mathcal{P} \sqrt{\cdots} \sqrt{(\tau^M)^{-(n-1)}} \mathcal{P}) \leq M \cdot \liminf_{n \to +\infty} \frac{1}{nM} H_{\mu_{\lambda, u}}((\mathcal{P}_{-n}^M, u) \mathcal{P} \mathcal{P} \sqrt{\cdots} \sqrt{(\tau^M)^{-(n-1)}} \mathcal{P}) \leq M \cdot (h_{\lambda, \mathcal{P}}^s + H_{\mu_{\lambda, u}}(\mathcal{Q})) \leq M \cdot (h_{\lambda, \mathcal{P}}^s + \alpha) .$$

Following [B] Section 3], we obtain in our random setting that there is some $c$ which depends on the geometry of $\mathcal{W}$ such that for any $\beta > 0$,

$$h_{\lambda, \mathcal{P}}^s, M \leq h_{\lambda, \mathcal{P}}^s, M \leq h_{\lambda, \mathcal{P}}^s, M + M(h_{\lambda, \mathcal{P}}^s, \rho) + \beta + c.$$ 

Using (4.4) and (4.5), we deduce that

$$h_{\lambda}^s \leq h_{\lambda, \mathcal{P}}^s + h_{\lambda, \mathcal{P}}^s, \rho + \alpha + \beta + \frac{1}{M} c .$$

Letting $\beta \to 0$ and then $M \to \infty$, we obtain the inequality (4.3). \qed 

Let $M$ be a fixed positive integer. We define for $u \in \mathcal{O}(\Sigma(SM), \varphi \in S, \eta > 0$ and $n \in \mathbb{N}$,

$$B^{s, M}(\varphi, u, \rho, n) := \{ w \in \mathcal{W}^n(u) : d(\varphi|_{\kappa}(w), \varphi|_{\kappa}(u)) < \eta \text{ for } 0 \leq k \leq n \} ,$$

$r^{M}(\rho, \varphi, u, \eta, n)$ the smallest number of $B^{s, M}(\varphi, u, \eta, n)$ balls needed to cover the $B^{s, M}(\varphi, u, \rho, n)$ ball,

$$h_{\rho, \varphi}^{s, M}(\rho, \varphi) := \sup_{u \in \mathcal{O}(\Sigma(SM), \eta > 0} \lim \limsup_{n \to +\infty} \frac{1}{n} \log r^{M}(\rho, \varphi, u, \eta, n)$$

and $h_{\rho, \varphi}^{s, M}(\rho, \varphi)$ the $\nu_{\alpha}^{\otimes N + \beta}$-a.e. value of $h_{\rho, \varphi}^{s, M}(\rho, \varphi)$. 

Proposition 4.4. With the above notations, we have, for all \( \lambda < 0, \rho > 0 \),

\[
(4.6) \quad h_{t_{oc,\lambda}}^s(\rho) \leq \frac{1}{M} h_{t_{oc,\lambda}}^{sM}(\rho).
\]

Proof. Observe that \( B^s(\varphi, u, \rho, nM) \) is a subset of \( B^{sM}(\varphi, u, \rho, n) \), so we are going to cover \( B^{sM}(\varphi, u, \rho, n) \) with \( B^s(\varphi, w, \eta, nM) \) balls, \( \eta \) arbitrarily small. Start with a cover of \( B^{sM}(\varphi, u, \rho, n) \) with \( B^s(\varphi, w, \eta, n) \) balls with \( 1 \leq \ell \leq r(M, \varphi, u, \rho, n) \) and fix \( K > 0 \) big. Let \( \varepsilon(\varphi) := \max(\|\varphi_k\|_{C^1} : 0 \leq k < M) \). If \( \varepsilon(\sigma^jM\varphi) \leq K \) for all \( j, 0 \leq j < n \), then each \( B^s(\varphi, w, \eta, n) \) ball is contained in \( B^s(\varphi, w, 2K\eta, nM) \) and we take these \( B^s(\varphi, w, 2K\eta, nM) \) balls to be our cover of \( B^{sM}(\varphi, u, \rho, n) \). Otherwise, assume, for instance, that \( \varepsilon(\varphi) > K \), we find, for each \( w_\ell \), at most \( c\varepsilon(\varphi)/K \) points \( w'_\ell \), such that the union of the \( B^s(\varphi, w'_\ell, 2K\eta, M) \) balls cover \( B^{sM}(\varphi, w, 1) \), where \( c \) is some constant depending on the geometry of \( \overline{W}^n \) and \([a]\) denotes the smallest integer greater than \( a \). Working inductively, we see that

\[
r(\varphi, u, 2K\eta, nM) \leq r(M, \varphi, u, \eta, n) \prod_{j=0}^{n-1} \frac{\varepsilon(\sigma^jM\varphi)/K}{\varepsilon(\varphi)/K} c \prod_{j=0}^{n-1} \lambda_{\{\varepsilon(\sigma^jM\varphi) \leq K\}}.
\]

It follows that for all \( K > 0, \varphi \in \mathcal{S} \),

\[
Mh_{t_{oc,\lambda}}^s(\rho, \varphi) \leq h_{t_{oc,\lambda}}^{sM}(\rho, \varphi) + \limsup_{n \to +\infty} \frac{M}{n} \sum_{j=0}^{n-1} \log\left(\frac{\varepsilon(\sigma^jM\varphi)/K}{\varepsilon(\varphi)/K}\right) + \limsup_{n \to +\infty} \frac{\log c}{n} \sum_{j=0}^{n-1} \lambda_{\{\varepsilon(\sigma^jM\varphi) \leq K\}}.
\]

Finally, we get, for all \( \lambda < 0, \rho > 0, K > 0 \),

\[
Mh_{t_{oc,\lambda}}^s(\rho, \varphi) \leq h_{t_{oc,\lambda}}^{sM}(\rho, \varphi) + \mathbb{E}\left[\log(\varepsilon(\varphi)/K)\right] + \mathbb{P}\left[\varepsilon(\varphi) > K\right] \log c.
\]

Since \( \mathbb{E}[\log\varepsilon] < +\infty \), Proposition 4.3 follows by letting \( K \) go to infinity. \( \square \)

Proposition 4.5. Fix \( \rho > 0 \) small and \( \lambda < 0 \). For all \( r \in \mathbb{N} \), there is a positive constant \( C(r) \) such that, for all \( M \in \mathbb{N} \),

\[
h_{t_{oc,\lambda}}^{sM}(\rho) \leq \frac{1}{r} \int \log \left( \max \left\{ \rho^{s-1}\|\varphi|_{\overline{W}^n(\mathcal{U})}\|_{C^s} : 1 \leq s \leq r, u \in \mathcal{O}^s(SM) \right\} \right) d\nu_{\lambda}(\varphi|M) + \log C(r).
\]

Proof. Fix \( r > 0, M \in \mathbb{N} \), a sequence \( \varphi \in \mathcal{S} \) and \( \rho > 0 \). Two points \( w, w' \in \overline{W}^n(\mathcal{U}) \) are said to be \((M, n, \eta)\)-separated if

\[
\max \left\{ d(\varphi|_{kM}(w), \varphi|_{kM}(w')) : 0 \leq k \leq n \right\} > \eta.
\]

It is clear that \( r(M, \varphi, u, \rho, n) \) is bounded from above by \( s^M(\varphi, u, \eta, n) \), the maximal cardinality of a set of \((M, n, \eta)\)-separated points in \( B^{sM}(\varphi, u, \rho, n) \). Consider the mappings

\[
\varphi_k' := \varphi|_M \circ \sigma^{(k-1)M} = \varphi_{kM-1} \circ \cdots \circ \varphi_{(k-1)M}
\]
and their standard magnifications \( \hat{\varphi}'_k : B(0,2) \to \mathbb{R}^m \) as explained in [CY], page 1129. In particular, we have \( \| \hat{\varphi}'_k \|_{C^s} \leq \rho^{s-1} \| \varphi' \|_{C^s} \). Using this, we can estimate \( s^M(\rho, \varphi, u, \eta, n) \) by following almost verbatim the argument for the proof of Proposition 3 in [CY] (which is based on the ‘Renormalization’ Theorem in [Yo] and a telescoping construction in [Bu]) and obtain some constant \( C_1(r, \overline{m}, \overline{m}) =: C(r) \) as in [CY] Theorem 4 such that

\[
\begin{aligned}
s^M(\rho, \varphi, u, \eta, n) \\
&\leq C(r)^n \left( \frac{4}{\eta} \right)^{\overline{m}} \prod_{k=1}^n \left( \max \left\{ \rho^{s-1} \| (\varphi'_k)_{\mathcal{T}^s(u)} \|_{C^s} : 1 \leq s \leq r, u \in \mathcal{O}^s(SM) \right\} \right)^{\overline{m}/r}.
\end{aligned}
\]

Since \( \varphi'_k \) are independent, the ergodic theorem gives Proposition 4.5.

\[ \square \]

**Corollary 4.6.** For any \( \alpha > 0 \), there exists \( \rho_0 \) such that if \( \rho \leq \rho_0 \), then

\[ \limsup_{\lambda \to -\infty} h^{\delta}_{\lambda}(\rho) < \alpha. \]

**Proof.** Fix \( r \geq 2 \). We choose \( M \in \mathbb{N} \) large such that \( \frac{1}{M} \log C(r) \leq \frac{\overline{m}}{r} B_{1,1} \), where \( B_{1,1} \) is defined in (2.6). Fix \( \rho \leq 1, \lambda < 0 \). By (1.6), \( h^\delta_{\lambda}(\rho) \leq \frac{1}{M} h^\delta_{\lambda}(\rho) \). Therefore,

\[ h^\delta_{\lambda}(\rho) \leq \frac{\overline{m}}{r} B_{1,1} + \frac{\overline{m}}{r M} \int \log \left( \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^1} + \rho \sum_{2 \leq s \leq r} \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^s} \right) \right) d\nu^M(\varphi_{M}). \]

Write, for \( \alpha > 0 \), \( \log^+ \alpha := \max \{ \log \alpha, 0 \} \). We have, using \( \log(\alpha_1 + \alpha_2) \leq \log^+ \alpha_1 + \alpha_2 \),

\[ \log \left( \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^1} + \rho \sum_{2 \leq s \leq r} \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^s} \right) \right) \]

\[ \leq \log^+ \left( \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^1} \right) \right) + \rho \sum_{2 \leq s \leq r} \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^s} \right) \]

\[ \leq \sum_{k=0}^{M-1} \log^+ \left( \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^1} \circ \sigma^k \right) \right) + \rho \sum_{2 \leq s \leq r} \max_u \left( \| \varphi_{M|\mathcal{T}^s(u)} \|_{C^s} \right). \]

We get by integrating with respect to \( \nu^M_{\lambda} \),

\[ h^\delta_{\lambda}(\rho) \leq \frac{\overline{m}}{r} B_{1,1} + \frac{\overline{m}}{r M} \sum_{2 \leq s \leq r} B_{s,M}(\lambda), \]

where \( B_{1,1}(\lambda), B_{2,M}(\lambda), \cdots, B_{r,M}(\lambda) \) are defined in (2.6). Note that

\[ B_{s,M} = \limsup_{\lambda \to -\infty} B_{s,M}(\lambda) < +\infty, \ \forall 1 \leq s \leq r, \]

by Proposition 2.7 ii), hence

\[ \inf \limsup_{\rho \to 0} h^\delta_{\lambda}(\rho) \leq \frac{2\overline{m}}{r} \inf_{\rho \to 0} \left( B_{1,1} + \rho \frac{1}{2M} \sum_{s=2}^r B_{s,M} \right) = \frac{2\overline{m}}{r} B_{1,1}. \]
Since \( r \) is arbitrary, the corollary follows.

**Proof of Proposition 3.2.** Fix \( \alpha > 0 \). We can choose the diameters of the elements of \( \mathcal{X} \) smaller than \( \rho_0 \), where \( c \) is a constant depending on the local geometry of the leaves so that the diameter of the elements of \( \mathcal{P} \cap \mathcal{R} \) are smaller than \( \rho_0 \) and Corollary 4.6 applies. We can also ensure that these diameters are smaller than \( \delta \) given by Proposition 4.2. We may assume that the boundaries of the elements of \( \mathcal{P}, \mathcal{X} \) and \( \mathcal{R}^j \) are all \( m \)-negligible.

By definition, \( h_{\mathcal{M}}^s \geq \lim \inf_{n \to +\infty} \inf_{j} \frac{1}{n} H_{\mathcal{M}}(\mathcal{P}-n|R^j) \). We can choose \( n \) and \( j \) so that

\[
(4.7) h_{\mathcal{M}}^s \geq \frac{1}{n} H_{\mathcal{M}}(\mathcal{P}-n|R^j) - \alpha.
\]

Consider now \( \lambda_p, p \in \mathbb{N} \), such that \( \mathcal{M}_{\lambda_p} \to \mathcal{M} \) as \( p \to +\infty \). For a.e. \( \omega \in \Omega \), each element of the partition \( \bigcap_{k=0}^{n} \bigcup_{\omega \in \mathcal{M}_p} (\omega)^{-1} \mathcal{P} \) converges in the Hausdorff metric towards the corresponding element \( \bigcap_{k=0}^{n} \bigcup_{\omega \in \mathcal{M}_p} (\omega)^{-1} \mathcal{P} \). Note that all these elements of \( \mathcal{P}_n \), and the elements of \( \mathcal{R}^j \) have \( \mathcal{M} \)-negligible boundaries. It follows that there exists \( P \in \mathbb{N} \) such that for \( p \geq P \),

\[
(4.8) \frac{1}{n} H_{\mathcal{M}}(\mathcal{P}_n|R^j) \geq \frac{1}{n} H_{\mu_{\lambda_p}}(\mathcal{P}_n|R^j) - \alpha \geq \frac{1}{n} H_{\mu_{\lambda_p}}(\mathcal{P}_n|R) - \alpha.
\]

The second inequality holds because the partition \( \mathcal{R} \) is finer than \( \mathcal{R}^j \). By Proposition 4.2 we have, by our choice of \( \delta \) and as soon as \( \lambda_p < \Lambda \),

\[
(4.9) \frac{1}{n} H_{\mu_{\lambda_p}}(\mathcal{P}_n|R) \geq h_{\lambda_p}^s(\mathcal{P}) - \alpha \geq h_{\lambda_p}^s - 2\alpha - h_{\text{loc},\lambda_p}(\rho),
\]

where the second equality follows from Proposition 4.3. Finally, using all the above inequalities (i.e., (4.7), (4.8) and (4.9)) and Corollary 4.6 we find that

\[
h_{\mathcal{M}}^s \geq \limsup_{p \to +\infty} h_{\lambda_p}^s - 5\alpha.
\]

Proposition 3.2 follows from the arbitrariness of \( \alpha \). \( \square \)

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