Genealogy of Nonperturbative Quantum-Invariants of 3-Manifolds: The Surgical Family

THOMAS KERLER

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Abstract: We study the relations between the invariants $\tau_{RT}$, $\tau_{HKR}$, and $\tau_L$ of Reshetikhin-Turaev, Hennings-Kauffman-Radford, and Lyubashenko, respectively. In particular, we discuss explicitly how $\tau_L$ specializes to $\tau_{RT}$ for semisimple categories and to $\tau_{HKR}$ for Tannakian categories. We give arguments for that $\tau_L$ is the most general invariant that stems from an extended TQFT. We introduce a canonical, central element, $Q$, for a quasi-triangular Hopf algebra, $A$, that allows us to apply the Hennings algorithm directly, in order to compute $\tau_{RT}$, which is originally obtained from the semisimple trace-subquotient of $A \mod$. Moreover, we generalize Hennings’ rules to the context of cobordisms, in order to obtain a TQFT for connected surfaces compatible with $\tau_{HKR}$. As an application we show that, for lens spaces and $A = U_q(sl_2)$, the ratio of $\tau_{HKR}$ and $\tau_{RT}$ is the order of the first homology group. In the course of this paper we also outline the topology and the algebra that enter invariance proofs, which contain no reference to 2-handle slides, but to other moves that are local. Finally, we give a list of open questions regarding cellular invariants, as defined by Turaev-Viro, Kuperberg, and others, their relations among each other, and their relations to the surgical invariants from above.

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1) Survey of Surgical Quantum-Invariants

At the end of the last decade the studies of quantum-groups, subfactors of von Neumann algebras, and the Chern-Simons quantum field theory brought forward a large and seemingly very powerful new class of so called “quantum-invariants” of closed, compact three-dimensional manifolds. In more recent times the focus of research has shifted towards finding relations and “universal” formulations for these invariants, which very well may turn out to be as instructive and fruitful as finding the invariants themselves. A large number of contributions to this conference dealt with the connection of the invariant of Reshetikhin-Turaev with the geometrical data, that we expect from the Chern-Simons functional integral, and with other classical invariants by using conventional perturbative methods, or number theoretical expansions.

Yet, there are also still plenty of questions that can be addressed regarding relations among the invariants that are related to quantum-group constructions. Here, we mainly use (non-perturbative) methods from categorical algebra, and geometric topology. In particular, the considered quantum-groups do not have to be deformations of classical Lie algebras, but they do have to be finite dimensional.

The main purpose of this contribution is to explain the relation between three “surgical” invariants. They are based on surgery presentations of three-manifolds, as in [Ki], and use additional algebraic data given by, e.g., quantum-groups, or tensor categories, in order to produce a number. We shall see that the general philosophy of the constructions is to, first, associate to the complement $M^{split} = S^3 - U(\mathcal{L})$ of a tubular neighborhood of the surgery link $\mathcal{L}$ a vector in a tensor power of a vector space, using only the braiding and rigidity information of the algebraic input data.

The invariant is then given by an evaluation against an “integral”, which corresponds in the surgery operation to the regluing of the opposite full tori. The specific assignments are, however, not precisely the same, and may lead to different invariants. Other types of quantum-invariants, that we shall call here “cellular invariants”, start from a presentation of the manifold in terms of decompositions into three-dimensional handles. We will come back to them with a few more details and questions in Chapter 4.
In this chapter we summarize the relations between the surgical invariants $\tau_{RT}$, $\tau_{HKR}$, and $\tau_L$, that will be proven in Chapter 3. We also give an outline of the general, categorical structure of the axiomatics of an extended TQFT, and we will argue that $\tau_L$ is the most general invariant that is a specialization of such a structure. In the discussion, and also in the rest of the article, we shall use the usual notations of tensor categories with conjugate objects. The conventions of the rigidity morphisms are as in [Ke1], which is the flipped version of [D]. For definitions of the notions of abelian categories, tensor categories, etc., see also [McL].

1.1) The Reshetikhin-Turaev Invariant
The construction given in [RT] starts from any semisimple, abelian, modular, braided tensor category, $\mathcal{C}$, with a finite set, $\mathcal{J}$, of equivalence classes of irreducible objects. Here, modularity refers to a non-degeneracy of the braiding that we will define and discuss for general categories in Paragraph 2.4.

The first step in the construction is to assign to a generic planar projection of a framed link, $L \subset S^3$, with a generic height coordinate, a composite of morphisms, where we associate braid-morphisms to crossings and rigidity morphisms to maxima and minima. The objects are specified by an orientation of the link, and a coloring, that assigns to the $\nu$-th component an irreducible object, $j_\nu \in \mathcal{J}$. For a closed link with $k$ components we arrive at a morphism $1 \to 1$, i.e., a number $I_{RT}(\mathcal{L}; j_1, \ldots, j_k)$, which is an invariant of the link for all colorings.

A closed compact three-manifold $M$ can now be presented by a framed link. An invariant of $M$ should be, in particular, an invariant of the link. As an Ansatz for $\tau_{RT}$ let us therefore use a general linear combination of the above link invariants

$$\tau_{RT}(M, \mathcal{C}) = \sum_{j_1, \ldots, j_k \in \mathcal{J}_0} w(j_1, \ldots, j_k) I_{RT}(\mathcal{L}; j_1, \ldots, j_k)$$

where $\mathcal{J}_0$ is a system of representatives of $\mathcal{J}$, and we have weights given by a function $w : \mathcal{J}_0^k \to \mathbb{C}$. It turns out that the additional constraints imposed on the invariant by Kirby-moves $\mathcal{O}_1$ and $\mathcal{O}_2$ (see [Ki]), are solved by a unique function $w$ (for modular $\mathcal{C}$ and up to overall scalings). This will be related in Paragraphs 2.5 and 3.2 to the existence and uniqueness of integrals of Hopf algebras.

A very common abuse of language in this context is to associate an invariant, $\tau_{RT}(M, \mathcal{A})$, to a non-semisimple quantum-group, $\mathcal{A}$, as for example $U_q(g)'$ with $q$ a root of unity.

The implied meaning of this is, that we should consider the category of representations $\mathcal{A} - mod$ and use as input for the construction the semisimple trace-quotient

$$\mathcal{C}_\mathcal{A} := (\mathcal{A} - mod)^{tr}$$

which has been defined in [Ke1].

1.2) The Hennings-Invariant
Very soon after that M. Hennings constructed an invariant, starting from any finite dimensional, modular quantum-group, $\mathcal{A}$, see [H]. In this approach the representations theory of $\mathcal{A}$ is not needed. Instead a projected link, that describes a manifold, $M$, by surgery, is mapped to an element in a category of singular links, which are decorated with elements in $\mathcal{A}$. To this an
algorithm is applied, that involves a set of quite intuitive combinatorial rules, which had already appeared in [Re]. The elements in $A$ that are eventually computed are evaluated against a right integral of $A$. This way we obtain a number, $\tau_{HKR}(M, A)$, which turns out to be an invariant of $M$. In recent papers, e.g., [KR], L. Kauffman and D. Radford reformulated the invariant avoiding the use of orientations and gave a clearer and more natural proof of invariance under the $O_2$-move, i.e., 2-handle-slides. In special examples, with $A = U_q(sl_2)'$, they and T. Ohtsuki [O] established that $\tau_{HKR}$ differs from $\tau_{RT}$. Generalization of these results will be given in Paragraph 3.5 using results from [Ke2]. The following is a special case of the main result in Theorem 1 stated in the next paragraph, and proven in Paragraphs 3.2, 3.3, and 3.4.

**Lemma 1**

Suppose $A$ is a finite-dimensional, modular quantum-group with modular trace-quotient, $C_A$, of its representation category. Then the identity

$$\tau_{RT}(M, A) = \tau_{HKR}(M, A)$$

(1.3)

holds for all closed, (oriented), compact three-manifolds $M$ if and only if $A$ is semisimple.

If $A$ is not semisimple they differ on most manifolds that are not rational homology-spheres, since in this case $\tau_{HKR}(M, A) = 0$, see [O] for $U_q(sl_2)'$ and [Ke3] for general $A$. A question that makes sense in the light of the known examples is, whether the invariant are equal for integral homology-spheres, even if $A$ is not semisimple.

A particularly interesting case of a semisimple, modular algebra is the Drinfel’d double $D(G)$ of a finite group. It has been shown in [AC] that $\tau_{RT}(M) = \frac{1}{|G|} |\text{Hom}(\pi_1(M), G)|$, which required a discussion of the representations theory of $D(G)$. With the above correspondence a more direct proof is given in [KR] using the rules for computing $\tau_{HKR}$.

In the study of Chern-Simons theory with a finite gauge group, see [F] and [AC], one is also interested in the computation of $\tau_{RT}$ for the quasi-quantum-group $D(G)^\alpha$, as defined in [DPR], for a non-trivial 3-cocycle of $G$. This corresponds to the “quantized” theory. Moreover, $\alpha$ corresponds to the coupling of the closely related quantum field theory from [DW], in a way that we shall explain a little more in Chapter 4.

As we shall see in Chapter 3, it is undoubtly possible to extend Hennings’ rules to quasi-Hopf algebras with a non-trivial associator $\phi \in A^{\otimes 3}$. Computations in the Hennings setting might make it easier to shed some light on open questions regarding the field theory associated to $D(G)^\alpha$.

### 1.3) The Lyubashenko Invariant

In several papers, see [L], V. Lyubashenko constructs in a more abstract fashion for any modular, abelian, braided tensor category, $C$, “with enough limits” a set of morphisms, which satisfy the relations of the usual generators of the mapping class groups. Moreover, he gives a construction of an invariant, $\tau_L(M, C)$, of closed three-manifolds.

The novelty of this approach is, that it is neither assumed that $C$ is semisimple nor that $C$ is the representation category of any Hopf-algebra. Instead, *categorical* Hopf algebras, obtained from so called *coends*, will enter the construction of the invariant. These notions and related techniques will be summarized in Chapter 2. The basic idea of defining an invariant will, however, be the same. Only now the algebraic input is a generalization of the data for both of the previous two invariants.
The following correspondence, to be proven in Chapter 3, asserts that also the invariant $\tau_L$ computed from this is a generalization of both $\tau_{RT}$ and $\tau_{HKR}$.

**Theorem 1**

1. Suppose $C$ is a semisimple, modular, braided, tensor category, with $|J| < \infty$.

Then

$$\tau_{RT}(M,C) = \tau_L(M,C).$$

(1.4)

2. Suppose $A$ is a modular quantum-group with $\dim(A) < \infty$.

Then

$$\tau_{HKR}(M,A) = \tau_L(M,A^{-mod}).$$

(1.5)

The missing implication of Lemma 1 follows immediately, once we know that $C_A = A^{-mod}$ for semisimple $A$. This assertion is not entirely obvious, since we also have to make sure that the quantum-dimensions of the irreducibles of $A^{-mod}$ are not zero. It follows from the fact that the $K_0$- (or fusion- or Grothendieck-) ring over $\mathbb{Z}^+$ for a rigid, fully reducible category has no proper ideals. However, irreducibles with zero quantum-dimension would form such an ideal.

**1.4) Quantum-Invariants from TQFT’s**

There is a justification in calling $\tau_{RT}$ a quantum-invariant, since (for certain $C$) it is thought to be the same as the invariant defined from the Chern-Simons functional integral, see [Wi]. But for $\tau_L$ and $\tau_{HKR}$, as well as $\tau_{RT}$ obtained from other categories, similar action-functionals are not known and, in fact, are not likely to exist. We recall, however, that the characteristics of what we consider a quantum field theory in 2+1 space-time dimensions may also be given in the canonical or algebraic formalism.

The basics include, that we associate to a surface, $\Sigma_t$, given by space points at a given time, a Hilbert space of states or, with more general axioms, a vector space, $V(\Sigma_t)$. In the Hamiltonian picture we also have a propagator, i.e., a unitary map, between the vector spaces at different times.

Atiyah [A] generalized this axiom for *topological* quantum field theories (TQFT’s) by associating such a map, $V(M) : V(\Sigma_1) \rightarrow V(\Sigma_2)$ to any three-fold that cobords the parametrized surfaces $\Sigma_i$ to each other. The time-coordinate is now a Morse function. Dropping the assumptions, regarding hermitian structures, a TQFT is thus nothing else but a fiber-functor on the category $\text{Cob}_3$ of 2+1-dim cobordisms. If we assume $V(\emptyset) = C$ then this assigns in fact a number, $V(M) = \tau(M)1$, to each closed manifold, $M$.

The precise definition also involves a central extension of the cobordism category (or, equivalently, the use of projective functors), which we shall suppress in the following discussion. In this sense we shall give the following, provisional definition:

**Definition 1**  We say that $\tau : M \mapsto \tau(M) \in C$ is a quantum-invariant if it specializes from an extended TQFT, $V : \text{Cob}_3(\ast) \rightarrow \text{AbCat}$.

The notion of an extended TQFT used here is motivated by the situation of Chern-Simons with field insertions, or, in the general language of algebraic field theory, the inclusion of other sectors, that may not have vacua. The mathematical axioms require that we associate to every
compact, one-fold, \( S \), an abelian category, \( \mathcal{C}(S) \). This assignment shall be compatible with the tensor product structures given for abelian categories by Deligne’s product \( \odot \) (see \([D]\)), and for the topological categories by disjoint unions. To a surface, \( \Sigma \), with \( \partial \Sigma \cong S \), we associate an object, \( \mathcal{V}(\Sigma) \), in \( \mathcal{C}(S) \), and to cobordisms between these we associated morphisms in \( \mathcal{C}(S) \).

Suppose that \( \text{Cob}_3(N) \) is the category of cobordisms between compact surfaces with \( N \) holes. Then the axioms require a functor \( \mathcal{V}_N : \text{Cob}_3(N) \to \mathcal{C}(\Pi^N S^1) = \mathcal{C}(S^1) \odot \ldots \odot \mathcal{C}(S^1) \). The ideas for formulating an extended TQFT in this way go back to D. Kazhdan and N. Reshetikhin, \([Ka]\), and have been put in the concrete setting of Chern-Simons theory with finite gauge group (as well as into writing) by D. Freed \([F]\).

They imply Atiyah’s definition if we set \( \mathcal{C}(\emptyset) = \text{Vect}(\mathcal{C}) \). There is indeed little choice in this assignment, since the category of vector spaces is the only canonical one that acts as a unit with respect to the \( \odot \)-multiplication.

If we distinguish on a surface between start and target holes, there is a canonical way to reinterpret the object \( \mathcal{V}(\Sigma) \in \mathcal{C}(S \sqcup S) \) as a functor, \( \mathcal{V}(\Sigma) : \mathcal{C}(-S_s) \to \mathcal{C}(S_t) \); see the introduction of \([Ke2]\) or also \([KL]\). Similarly, the morphisms may be rewritten as natural transformation. This explains the notation in Definition \([L]\) where we think of \( \mathcal{V} \) as a 2-functor from the 2-category of 1+1+1-dimensional cobordisms to the 2-category of abelian categories.

Although this language might be somewhat unfamiliar to the non-categorist, it is in many ways a more natural point of view, and we obtain a few more, very useful constraints on \( \mathcal{V} \). Aside from a few arguments that we will need in the next paragraph, extended TQFT’s will not be used in the remainder of this paper. We will, however, continue to discuss ordinary TQFT’s, e.g., in Paragraph 3.5.

### 1.5 \( \tau_L \) as a Quantum-Invariant

An invariant that is given by the (renormalized) partition function of a functional integral, such as Chern-Simons, is quite obviously expected to be a quantum-invariant. Recall also, that if \( \tau(-M) = \tau(M) \), we can often (re-)construct a TQFT from the invariant alone by an obvious generalization of the GNS-procedure. (The signed, inner product spaces, that are first obtained, can be quite huge and need not be self-dual so that we usually have to impose further regularity conditions on \( \tau \)). Moreover, constructions have been proposed by Turaev in \([T]\), in which a TQFT is associated to \( \tau_{RT} \), starting again from a semisimple category.

Despite the similarities discussed in the previous sections TQFT’s do not exist for \( \tau_L \) in this naïve way. In particular we will see at the end of Paragraph 3.3 that

\[
\tau_{HKR}(S^1 \times S^2, \mathcal{A}) = 0 \quad \text{iff} \quad \mathcal{A} \text{ not semisimple.} \tag{1.6}
\]

For a TQFT in the usual framework this also implies that \( \mathcal{V}(S^2) = 0 \), which is possible only if \( \mathcal{V} \equiv 0 \).

Still, it can be shown that the consistency problem arising here is solely one of treating the connectivity of cobordisms and surfaces in the right way. The easiest way to circumvent it is to consider the categories \( \text{Cob}_3^{\text{con}}(N) \), which consist only of connected cobordisms between connected surfaces. We may of course also consider disjoint unions of such cobordisms, but the important features of a tensor category, such as rigidity, leading to the contradiction in \([L]\), are no longer present. In this restricted setting the following result permits us to view \( \tau_L \) still as a quantum-invariant:
Theorem 2 ([KL]) For any abelian, rigid, modular, balanced, braided tensor category, $\mathcal{C}$, with certain limits, we have a series of functors

$$\mathcal{V}_N : \text{Cob}_3(N)^{\text{conn}} \rightarrow \mathcal{C}^{\otimes N},$$

which respect the tensor products and 2-categorical compositions.

In this construction the choice of the circle category $\mathcal{C} = \mathcal{C}(S^1)$ determines the construction completely. Also, $\mathcal{V}$ specializes to the $\tau_L$, and we reconstruct the same representations of the mapping class groups as in [L], only now directly derived from tangle-presentations of the cobordism categories as in [Ke5].

In Paragraph 3.5 we shall give a construction of the functor $\mathcal{V}_0$ in the case $\mathcal{C} = \mathcal{A} - \text{mod}$.

Instead of the categorical picture in [KL], we use here a translation into the combinatorial Hennings picture, extended to manifolds with boundary.

Let us conclude this paragraph with a few remarks on the construction of so called half-projective TQFT’s in the disconnected, non-semisimple case, given a TQFT for connected surfaces as in [KL]:

As in the semisimple case, we first write a connected cobordism, $M : \Sigma_s \rightarrow \Sigma_t$, in the form $M = \Pi_{\Sigma} \circ \tilde{M} \circ \Pi_{\Sigma}$, where $\tilde{M}$ is a cobordism between connected surfaces. Here we used a choice of cobordisms, $\Pi_{\Sigma} : \Sigma_1 \Pi \ldots \Pi \Sigma_K \rightarrow \Sigma_1 \# \ldots \# \Sigma_K$, for every surface, $\Sigma$, with $K$ connected components $\Sigma_j$. The cobordism $\Pi_{\Sigma}^\dagger$ is an arrow in the opposite direction with the property that $\Pi_{\Sigma} \circ \Pi_{\Sigma}^\dagger = \mathbb{1}_{\Sigma_1 \# \ldots \# \Sigma_K}$. From this we easily see that $\Lambda_{\Sigma} := \Pi_{\Sigma} \circ \Pi_{\Sigma}^\dagger$ obeys

$$\Lambda_{\Sigma} \circ \Lambda_{\Sigma} = \Lambda_{\Sigma} \# \left( S^1 \times S^2 \right) \# \ldots \# \left( S^1 \times S^2 \right)$$

which $K-1$ times and

$$\Lambda_{\Sigma} \in \text{Cob}_3^{\text{conn}}.$$ (1.8)

In the semisimple case $\tau(S^1 \times S^2)$ can be normalized to 1 so that a TQFT, $\mathcal{V}$, associates to $\Lambda_{\Sigma}$ a projector. The map $\mathcal{V}(M)$ is the reduction of $\mathcal{V}(\tilde{M})$ to the respective subspaces $\mathcal{V}_{\Sigma_{s/t}} := \text{im}(\mathcal{V}(\Lambda_{\Sigma_{s/t}}))$.

In the non-semisimple case it follows for $K \geq 2$ that $\mathcal{V}(\Lambda_{\Sigma})$ is merely nilpotent of order two, and in fact will be zero for a slight variation of the connected TQFT from [KL]. In [Ke3] we use this observation to construct a map $\mathcal{V} : \text{Cob}_3 \rightarrow \text{Vect}(\mathcal{C})$, which extends the functor on $\text{Cob}_3^{\text{conn}}$ and has all the properties of a functor, except that it is has half-projective compositions. This means

$$\mathcal{V}(M_2 \circ M_1) = x^{\mu(M_2,M_1)} \mathcal{V}(M_2) \mathcal{V}(M_1),$$

where $\mu$ is a cocycle on $\text{Cob}_3$ with values in $\mathbb{Z}^{0,+}$, and $x$ is not necessarily invertible. In our case we have $x = 0$ (recall that $0^0 = 1$), and $\mu$ is $K-1$ if two connected cobordisms are glued over $K$ boundary components.

These properties also imply that a cobordisms, whose “interior” homology or “interior” fundamental group (see [Ke3]) is non-trivial, is mapped to zero. This generalizes a vanishing result for closed manifolds and $U_q(s\ell_2)$, that was obtained in [O] by a more or less direct computation.

In the following, if we speak of extended TQFT’s or quantum-invariants, we shall assume that we have taken care of connectivity questions in some way. In particular, we count $\tau_L$ as a quantum-invariant.
1.6) Some Arguments for Exhaustiveness of $\tau_L$

Theorem 1 does in fact suggest that $\tau_L$ is a most general invariant in some class of invariants. Looking at Definition 1 it is a-priori not clear why this should be the class of quantum-invariants. For example the definition does not refer to any braided tensor structure or quantum-groups. One might thus hope to find TQFT’s coming from more general and more exotic categories.

This is, however, not possible. The mentioned quantum-algebraic structures are already contained in the topology of three-dimensional manifolds with corners. In particular, the generating category $C = C(S^1)$ must reflect the properties of $\text{Cob}_3(1)$. In [Ke4] we will give an abstract algebraic characterization of this topological category as follows:

**Theorem 3** Suppose $G$ is the free strict, balanced, braided tensor category, freely generated by a self-dual, braided Hopf algebra-object, $F$, with integrals and a non-degenerate Hopf pairing, $F \otimes F \to 1$.

Then there is a surjective functor:

$$T : G \longrightarrow \text{Cob}_3(1)^{\text{conn}}.$$  

(Here the topological category shall have only one object in each isomorphism class.)

Since elementary Hopf algebra relations are quite naturally identified by $T$ with elementary Cerf theoretical relations, we are led to conjecture here that $T$ is in fact an isomorphism of categories.

Hopf algebra structures - in a topological sense - for the punctured torus were independently also formulated by Crane and Yetter in [CY]. The generators described there, however, are only those appearing in the axioms for Hopf algebras. Consequently, the corresponding cobordisms are only those that are embeddable into $\mathbb{R}^3$. Yet, in order to guarantee the asserted surjectivity, we also need to include generators, which account for surgery on cobordisms. Those are precisely the integrals, entering the definition of $G$. Moreover, the Hopf pairing is used to describe a Dehn twists between neighboring one-handles. It is closely related to the modularity requirement, to be discussed in Section 2.4. For these reasons the proof of Theorem 3 cannot be done using simply diagrams on a surface, as in [CY], but higher dimensional presentations, as in [Ke5], have to be employed.

It is relatively easy to see that gluing surfaces to a three holed sphere, $\Sigma_{0,3}$, allows us to define a tensor product in $\text{Cob}_3(1)^{\text{conn}}$. There are also easy choices of cobordisms that make it into a strict and braided category, see, e.g., [KL].

The generating object is obviously the punctured torus, $T(F) = \Sigma_{1,1}$. Since the classical notion of a Hopf algebra makes sense only in a symmetric category, we have to use the natural, braided modification introduced by Majid [M1]. The generating morphisms in $G$, like (co-)products and (co-)integrals, can be naturally represented by elementary cobordisms, i.e., ones that correspond to simple handle-attachments of dimensions three or four.

Now, the composite $V_1 \circ T : G \to C$ shows that $C$ must have all the properties required in Theorem 2. For example we have a tensor structure given by

$$\otimes := V(\Sigma_{0,3}) : C \otimes C \longrightarrow C.$$  

(1.10)

The only thing that still needs explanation, is the relation between the existence of a Hopf algebra object and the existence of certain limits. Specifically, the limit we need is going to be the following coend:

$$F := \int X^\vee \otimes X \in C.$$  

(1.11)
We shall give a meaning of to this formula in Chapter 2. Let us outline here, how this formula follows from the axioms of an extended TQFT and basic topology. We shall freely use the coend notation; the details of its definition are not relevant to the general line of arguments presented next:

The disc, \( D^2 \), seen as a 1+1-cobordism from \( S^1 \) to \( \emptyset \) induces by the axioms a basic fiber-functor \( \mathcal{I} := \mathcal{V}_1(\Sigma_{0,1}) : \mathcal{C} \to \text{Vect}(\mathcal{C}) \). The choice that appears to be the most natural one coming from the physical examples is the invariance-functor:

\[
\mathcal{I} = \text{Inv} : X \mapsto \text{Hom}_\mathcal{C}(1, X) \quad (1.12)
\]

Although there are some constraints on \( \mathcal{I} \), this choice is by no means mathematically stringent. A slightly different fiber-functor is for example chosen in \([\text{Ke}3]\). Still, let us assume it here as an additional axiom of extended TQFT’s.

Now, the two-holed sphere can be thought of as a cobordism \( \Sigma_{0,2} : S^1 \coprod S^1 \to \emptyset \), which in turn is the composite of the pair of pants \( \Sigma_{0,3} \) and a disc. Using the representations of the latter in (1.10) and (1.12), we find that \( \Sigma_{0,2} \) is represented by the following functor:

\[
\Omega : \mathcal{C} \otimes \mathcal{C} \longrightarrow \text{Vect}(\mathcal{C}) : X \otimes Y \mapsto \text{Inv}(X \otimes Y) \quad . (1.13)
\]

But \( \Sigma_{0,2} \) has an obvious, two-sided inverse given by the same cylinder in opposite direction, \(-\Sigma_{0,2} : \emptyset \to S^1 \coprod S^1 \). The existence of an extended TQFT thus implies that \( \Omega \) has a two-sided inverse, \( \Omega^{-1} : \text{Vect}(\mathcal{C}) \to \mathcal{C} \otimes \mathcal{C} \). A functor of this form is up to isomorphisms given by an object \( \mathcal{F} \in \mathcal{C} \otimes \mathcal{C} \) (e.g., the image of \( \mathcal{C} \)). It turns out that \( \mathcal{F} \) inverts \( \Omega \) iff it is the coend:

\[
\mathcal{F} = \bigint X^\vee \otimes X \quad . (1.14)
\]

This does not always exist (like \( \Omega \) may not be invertible in some categories). In order to say it is an inverse we actually also have to specify an isomorphism of the composite to the identity, which gives rise to the transformations that are also part of the definition of a coend. The precise meaning of the additional requirement about “enough limits” is thus that \( \mathcal{C} \) contains the element \( \mathcal{F} \). We shall see in the examples of Paragraph 2.3 that it has to be understood as a finiteness condition for \( \mathcal{C} \).

In light of Theorem 3 we should be really interested in the functor associated to \( \Sigma_{1,1} : \emptyset \to S^1 \), which is given by an object \( F \) in \( \mathcal{C} \). We notice that \( \Sigma_{1,1} \) is the composite of \(-\Sigma_{0,2} \) with \( \Sigma_{0,3} \) over \( S^1 \coprod S^1 \). This results in the following identity, from which we also obtain formula (1.11):

\[
F = \otimes(\mathcal{F}) \quad . (1.15)
\]

At this point one should compare the statements of Theorem 2 and Theorem 3, keeping in mind that, so far, we have made no assumptions on the structure of \( F \) in \( \mathcal{C} \). The observant reader will notice that this must imply a non-trivial theorem on the structure of coends in abelian, braided tensor categories in general. The corresponding, more precise statement is given in the next theorem, and enters crucially in the construction of \( \tau_L \):

**Theorem 4 ([L])** Suppose that \( \mathcal{C} \) is an abelian, rigid, balanced, modular, braided tensor category, for which the coend \( F \), as in (1.11), exists.

Then \( F \) has the structure of a braided Hopf-algebra in \( \mathcal{C} \). It has unique, categorical (co-)integrals, and its non-degenerate pairing is a pairing of Hopf-algebras. In short, there is a functor

\[
\mathcal{R} : \mathcal{G} \to \mathcal{C} \quad ,
\]

which maps \( \mathcal{F} \) to \( F \).
It is intriguing to see that, although we were starting from quite different and very general assumptions, we can extract, both in the topological as well as in the abelian setting, distinguished objects, $\Sigma_1$ and $F$, respectively, which automatically have the properties of $F$. This may be seen as the abstract reason, why it is possible to construct the functor from Theorem 3.

It is not hard to see from the constructions, that the functors $T$ and $R$ are also unique up to obvious isomorphisms and a few choices of orientations. It thus follows that the extended TQFT $\mathcal{V}$ is basically unique, too.

The line of arguments given in this paragraph leaves thus no alternatives to the construction of quantum-invariant as in Definition 4. It should now be a matter of filling in the details and making the proper formalization that would complete them into a proof of the uniqueness of $\tau_L$.

2) Coends, Universal Liftings, Modularity, Integrals, and All That

In this chapter we will discuss some of the properties of a coend as in (1.11), especially the consequences of its universality, which are crucial in the construction of $\tau_L$. We will compute $F$ explicitly for semisimple categories and for representations categories. Moreover, we shall determine the integrals and cointegrals of $F$ in these cases. Let us start with some general, preparatory remarks on the the two types of categories in question:

2.1) Remarks on Semisimple and Tannakian Categories

An important fact about a rigid, balanced, braided tensor category, $\mathcal{C}$, is that it always admits a canonical system of traces,

$$tr_X : \text{End}_\mathcal{C}(X) \rightarrow \mathcal{C},$$

which is generally cyclic and respects the $\otimes$-product. Important ingredients in the construction are the usual rigidity morphisms, $ev_X : X^\vee \otimes X \rightarrow 1$ and $coev_X : 1 \rightarrow X \otimes X^\vee$, but also a canonical set of flipped rigidities,

$$\tilde{ev}_X : X \otimes X^\vee \rightarrow 1 \quad \text{and} \quad \tilde{coev}_X : 1 \rightarrow X^\vee \otimes X,$$

that are constructed from the ordinary ones using the balancing and braiding in $\mathcal{C}$, see, e.g., [Ke1]. The trace allows us to define the so-called quantum-dimensions

$$d(X) = tr_X(1 \mathbb{I}_X) \quad \text{for all objects } X,$$

and with the properties $d(X \oplus Y) = d(X) + d(Y)$ and $d(X \otimes Y) = d(X) \cdot d(Y)$. Also, we obtain a symmetric pairing

$$\text{Hom}_\mathcal{C}(Y, X) \otimes \text{Hom}_\mathcal{C}(X, Y) \xrightarrow{\circ} \text{End}_\mathcal{C}(X) \xrightarrow{tr_X} \mathcal{C}. \quad (2.19)$$

As in [Ke1] we say that a category is \textit{semisimple} iff this pairing is non-degenerate for all $X$ and $Y$. It follows that a morphism, $I : X \rightarrow Y$, is completely determined by the matrix elements $gIf \in \mathcal{C}$, where $f \in \text{Hom}_\mathcal{C}(j, X)$, $g \in \text{Hom}_\mathcal{C}(X, j)$, and $j$ is irreducible. More precisely, we have the following isomorphisms.

$$\text{Hom}_\mathcal{C}(X, Y) \cong \bigoplus_{j \in J_0} \text{Hom}_\mathcal{C}(\text{Hom}_\mathcal{C}(j, X), \text{Hom}_\mathcal{C}(j, Y)). \quad (2.20)$$
Here $\mathcal{J}_0$ is a set of representatives of the set of classes of irreducibles $\mathcal{J}$ as in Paragraph 1.1.

Another class of categories are the so called Tannakian ones, i.e., those which admit an exact $\otimes$-functor into the category of vector spaces:

$$V : \mathcal{C} \rightarrow \text{Vect}(\mathcal{C}) \quad (2.21)$$

The cardinal example for such a category is given by the representation category $\mathcal{A} - \text{mod}$ of a Hopf algebra. $V$ is then simply the “forgetful functor”.

An obvious, necessary condition for a category, $\mathcal{C}$, to be Tannakian, is that we have dimensions, $X \mapsto d_X$, which assign to each object a value in $\mathbb{Z}^{+,0}$, and which respect the sums and products of objects in the way the canonical dimensions from above do. It follows from Perron-Frobenius theory that for categories, with finitely many isomorphism classes of irreducibles, there exists exactly one such dimension with values in $\mathbb{R}^{+,0}$. As a matter of fact, most of the categories $\mathcal{C}_A$, with $\mathcal{A} = U_q(\mathfrak{g})$, admit a dimension that is positive but not integral, and hence these $\mathcal{C}_A$ can be excluded from the list of Tannakian categories.

In the case of a symmetric category, $\mathcal{C}$, and if the canonical dimensions take values in $\mathbb{Z}^{+,0}$, it is a result of [D] that $\mathcal{C}$ is Tannakian.

A very important question is, whether a Tannakian category is always of the form $\mathcal{A} - \text{mod}$ for some Hopf-algebra $\mathcal{A}$. Given the functor $V$, a natural candidate for the algebra is $\mathcal{A} = \text{End}_{\text{Cat}}(V)$. Under a few technical assumptions (which, e.g., allow us identify $\mathcal{A} - \text{mod}$ with $\mathcal{A}^* - \text{comod}$) reconstruction results are obtained - in several stages of generality - in [Ta]. In this sense we shall often use the notions of a representation category and a Tannakian category synonymously.

It is quite useful to try to represent fiber-functors as in (2.21). This means we want to find an object $Q$ in $\mathcal{C}$ itself such that $\mathcal{F}$ can also be expressed by the functor $\text{Hom}_{\mathcal{C}}(Q, -)$, which associates to each $X$ the vector space $\text{Hom}_{\mathcal{C}}(Q,X)$.

If we have already made the identification $\mathcal{C} = \mathcal{A} - \text{mod}$, with $\text{dim}(\mathcal{A}) < \infty$, it is easy to see that $Q$ is given, as a vector space, by $\mathcal{A}$ itself, and as module it is equipped with the left regular action $a.q = aq$.

The dual $Q^\vee$ is thus, as a vector space, given by the space of functions $A^*$. The action is given by $a.\rho = \rho \leftarrow S(a)$ for all $a \in \mathcal{A}$ and $\rho \in A^*$. Here, $S$ is the antipode, and $\leftarrow$ is defined by $(\rho \leftarrow b)(y) = \rho(by)$.

It is not true that $Q$ is an algebra in $\mathcal{C}$ since the multiplication does not intertwine the action of $A$. However, since the comultiplication intertwines, $Q^\vee$ is an algebra, which is commutative in symmetric categories. This property is crucial in the construction of a fiber functor as in [D].

Another algebra, in fact even a Hopf algebra, in $\mathcal{C}$ was proposed in Theorem 4. We shall see that $\mathcal{F}$ is also given by $\mathcal{A}^*$, but carries the coadjoint action instead.

### 2.2) General Characterizations of Coends

Quite generally, coends are associated to a pair of categories, $\mathcal{C}$ and $\mathcal{B}$, and a functor

$$S : \mathcal{C}^{\text{opp}} \times \mathcal{C} \rightarrow \mathcal{B}, \quad (2.22)$$

and are denoted by $\int S(X,X)$. The aim of this paragraph is to make the elegant but also rather abstract definition in, e.g., [McL] more concrete in the example, where $\mathcal{B} = \mathcal{C}$ and $S(Y,X) = Y^\vee \otimes X$, and thus make sense of the notation in (1.11).
We start by introducing the category $dn(S)$ of “dinatural transformations of $S$ to a constant”. Specifically, this means that the objects of $dn(S)$ are given by pairs $\langle Z, \xi \rangle$. The first argument $Z$ is an object in $C$, and the second is a map that assigns to every object $X$ in $C$ a morphism $\xi_X : X^\vee \otimes X = S(X, X) \to Z$. They shall be such that for any pair of objects, $X$ and $Y$, and any morphism $f : X \to Y$ the following diagram commutes:

\[
\begin{array}{ccc}
Y^\vee \otimes X & \xrightarrow{I_Y \otimes f} & Y^\vee \otimes Y \\
\downarrow f^\vee \otimes 1_X & & \downarrow \xi_Y \\
X^\vee \otimes X & \xrightarrow{\xi_X} & Z
\end{array}
\]  

(2.23)

A morphism, $h : \langle Z, \xi \rangle \to \langle Z', \xi' \rangle$, in $dn(S)$ is first of all given by a morphism $h \in \text{Hom}_C(Z, Z')$. It shall also map the second arguments into each other, i.e., we have the following, additional requirement:

\[h \circ \xi_X = \xi_X' \quad \text{for all objects } X \quad .\]  

(2.24)

It is often instructive to consider a slightly different but equivalent definition of $dn(S)$. The objects here are again pairs $\langle Z, \delta \rangle$, but now the second is a natural transformation, $\delta : id \to - \otimes Z$, i.e., a family of morphisms $\delta_X : X \to X \otimes Z$ with $\delta_Y f = f \otimes 1_Z \delta_X$ for $f : X \to Y$. Because of the intertwining relation we may think of such a transformation as a coaction of $Z$ on the objects of $C$, if it were also compatible with a coproduct of $Z$.

As we already pointed out, a very important property of a coend for the construction of $\tau_L$ is the universality property. More precisely, we say that $\langle F, i \rangle$ is a coend iff it is an initial object in $dn(S)$. This means that for any other object $\langle Z, \xi \rangle$ there exists exactly one morphism $r^Z : \langle F, i \rangle \to \langle Z, \xi \rangle$.

In other words, whenever we have a system of morphisms that fulfill the dinaturality condition from (2.23) there exists a map $r^Z : F \to Z$, such that each of them factors through $r^Z$ as below.

\[\xi_X : X^\vee \otimes X \xrightarrow{i_X} F \xrightarrow{r^Z} Z \quad .\]  

(2.25)

It follows immediately from general nonsense that if $F$ exists it is also unique up to isomorphisms.

The transformations $i_X$ allow us define for every object, $X$, in a balanced, braided category a canonical invariance, $tr^q_X \in \text{Hom}_C(1, F)$, which is given by

\[tr^q_X : 1 \xrightarrow{\text{coev}_X} X^\vee \otimes X \xrightarrow{i_X} F \quad .\]  

(2.26)

It has properties similar to those of the trace in (2.16), e.g., we have $tr^q_{X \oplus Y} = tr^q_X + tr^q_Y$ and $tr^q_{X \otimes Y} = tr^q_X \cdot tr^q_Y$, where the multiplication is in the sense of Theorem 4.

### 2.3) Coends for Special Categories

In this paragraph we wish to give some more reality to the formalism described above, by computing the coend explicitly for semisimple categories and representation categories. For $\mathcal{C} = G-mod$, 

12
where $G$ is a finite group, $F$ will be given by the space of functions on $G$ with the coadjoint action on it. It is well known that the matrix elements of irreducibles are a basis of this space so that by

$$C(G) = \bigoplus_{j \in J_0} \text{End}_C(V_j)^* \cong \bigoplus_{j \in J_0} V_j^* \otimes V_j,$$

we can present $F$ also as a sum (or tensors) of irreducibles. The other cases are given by the obvious generalizations of these two pictures.

The answer for semisimple situation is given by the next easy result:

**Lemma 2** Suppose $C$ is an abelian, semisimple, rigid tensor category, for which the set $J$ of isomorphism classes of irreducible objects is finite. Further, let $J_0$ be any set of representatives of $J$.

Then the coend from (1.11) exists and is given by

$$F = \bigoplus_{j \in J_0} j^\vee \otimes j.$$

The summand $i_X^j : X^\vee \otimes X \to j^\vee \otimes j$ of the transformation is the canonical (identity) element in

$$i_X^j \in \text{Hom}_C(X^\vee, j^\vee) \otimes \text{Hom}_C(X, j) \cong \text{End}_C(\text{Hom}_C(j, X)).$$

**Proof:** In the last identity we used the isomorphism $\text{Hom}_C(X^\vee, j^\vee) \cong \text{Hom}_C(j, X)$ obtained from rigidity. Moreover, we used the duality $\text{Hom}_C(X, j) \cong \text{Hom}_C(j, X)^*$ with respect to the pairing $f, g \mapsto f \circ g \in \text{End}_c(j) = C$, which is non-degenerate by our definition of semisimplicity.

Let us introduce dual bases, $\{\epsilon^\alpha_j\}$ of $\text{Hom}_C(j, X)$, and $\{f^\alpha_j\}$ of $\text{Hom}_C(X, j)$. It follows from the presentation in (2.20) that a natural transformation, $\zeta : id \to \mathcal{T}$, from the identity functor to another functor, $\mathcal{T} : C \to C$, is determined entirely by the special morphisms $\zeta_j : j \to \mathcal{T}(j)$ on a representing set of irreducibles. In fact, any such set of morphisms yields by

$$\zeta_X = \sum_{j \in J_0, \alpha} \mathcal{T}(\epsilon^\alpha_{jX}) \circ \zeta_j \circ f^\alpha_{jX} \quad (2.27)$$

a natural transformation. Thus in the semisimple setting a morphism in $dn(S)$, in the picture of coactions, is given by a set of morphisms $\delta_j : j \to j \otimes Z$ for $j \in J_0$.

In the case of $F$ as above we can rewrite the transformations $i_X$ into coactions $\theta^k_X : X \to X \otimes (k^\vee \otimes k)$ for the individual summands. They are determined by $\theta^k_j = \text{coev}_j \otimes \mathbb{I}_j : j \to j \otimes j^\vee \otimes j$ if $j = k$, and $\theta^k_j = 0$ if $j \neq k$.

By rigidity we can rewrite the basic morphisms of a general coaction always in the form $\delta_j = (\mathbb{1}_j \otimes \hat{\delta}_j) \circ \theta^j_j$, where $\delta_j : j^\vee \otimes j \to Z$ is unique. Now, it follows directly from (2.27) that $\delta_X = (\mathbb{1}_X \otimes \hat{\delta}) \circ \theta_X$, where $\hat{\delta} := \oplus_j \delta_j : F \to Z$. In particular, we can think of $r^Z = \hat{\delta}$ as a morphism in $dn(S)$.

Notice also that $i^j_j = \mathbb{1}_{j^\vee \otimes j}$ and $i^k_j = 0$ if $k \neq j$ so that $r^Z \circ i_j = 0$ for all $j \in J_0$ implies that $r^Z = 0$. Hence a morphism starting at $(F, i)$ is also unique in $dn(S)$, which completes the proof. □

The condition that $|J| < \infty$ cannot be dropped here. In fact, we see from the proof that a coend does not exist in a category, which is both Nötherian, i.e., the objects have only finite decompositions, and which has infinite set $J$. An example are the finite dimensional representations of a compact, continuous Lie-group, $G$. Here, the algebra of functions on $G$ does not
belong to $G - mod$. Summarily, we can say that in the semisimple setting the condition $|\mathcal{J}| < \infty$ that entered the construction of $\tau_{RT}$ in Paragraph 1.1 is the exact equivalent of the condition on "enough limits" alluded to in the construction of $\tau_{L}$ in Paragraph 1.3.

Next, let us discuss the coend for the case of a representation category. Here, finite dimensionality of $A$ substitutes the condition $|\mathcal{J}| < \infty$ as a prerequisite for existence.

**Lemma 3** Suppose that $\mathcal{C} = A - mod$, where $A$ is a Hopf algebra with $\dim(A) < \infty$. Then the coend from (1.11) exists and can be chosen as the dual linear space $A^*$ endowed with the coadjoint action as follows

$$ a, \rho = a'' \rightarrow \rho \leftarrow S(a') $$

Here $a \in A$, $\rho \in A^*$, the arrows $\rightarrow$ and $\leftarrow$ indicate the canonical left and right regular actions, and we use the shorthand $\Delta(a) = a' \otimes a''$ for the coproduct.

The transformations $i_X : X^\vee \otimes X \to F$ are defined from the canonical maps on the corresponding vector spaces. I.e., the element $l \otimes v$, where $v$ is a vector and $l$ a linear form, is mapped to the matrix element $\langle l, \ldots, v \rangle \in A^*$.

**Proof:** Let $Q$ and $Q^\vee$ be the projective representations as in the end of Paragraph 2.1. For an object, $(Z, \xi)$, we consider the commutative diagram (2.23) in the special case, where $X = Y = Q$, and $f = \phi_a : Q \to Q$ is given for arbitrary $a \in A$ by right multiplication $\phi_a(y) = ya$, which obviously intertwines the action of $A$. We obtain the relation

$$ \xi_Q(\rho \otimes a) = \xi_Q(a \rightarrow \rho \otimes 1) = r^Z(i_Q(\rho \otimes a)) $$

Here we have set $r^Z : F \to Z : \psi \mapsto \xi_Q(\psi \otimes 1)$, which also intertwines the action of $A$. Since $i_Q$ is onto it follows immediately that if $r^Z$ is actually a morphism in $dn(S)$ it is unique.

It therefore remains to show that $r^Z \circ i_Y = \xi_Y$ also holds for a general $A$-module, $Y$. We look again at (2.23), now with $X = Q$ and $f_v : Q \to Y : a \to a.v$. If we check commutativity for the special vector $l \otimes 1 \in Y^\vee \otimes Q$ the assertion follows. $\square$

Another choice for the functor in (2.22) is given by $S(Y, X) = Y^\vee \otimes X$, if we redefine $B = \mathcal{C} \circ \mathcal{C}$. Let us specify here also the coend, which is formally given by $\mathcal{I}F$ from (1.14), in both considered cases:

The irreducible objects of $\mathcal{C} \circ \mathcal{C}$ for a semisimple category are given by $i \otimes j$, where $i$ and $j$ run over the irreducibles of $\mathcal{C}$. The coend is thus given by the obvious sum

$$ \mathcal{I}F = \sum_{j \in \mathcal{J}} j^\vee \circ j $$

and the dinatural transformations are completely analogous to those of Lemma 2. In fact, the corresponding proof is much easier for $\mathcal{I}F$, since $\text{Hom}(a \otimes b, c \circ d) = \text{Hom}(a, c) \otimes \text{Hom}(b, d)$, and implies by (1.13) the one for $F$.

In the Tannakian case we have $\mathcal{C} \circ \mathcal{C} = (\mathcal{A} \otimes \mathcal{A}) - mod$, see [D]. As a vector space $\mathcal{I}F$ is again $A^*$ but now with $A \otimes A$-action given by

$$ (a \otimes b), \rho := b \rightarrow \rho \leftarrow S(a) $$

As for the semisimple case we can use also here (1.13) in order to find an easier, though more abstract, proof of Lemma 3.
The invariances $tr^q_X$ from (2.26) are also easily identified. For the semisimple case it is enough to consider irreducible $X$ because $tr^q$ respects direct sums. Since the $i_j$ are simply injections of summands the $tr^q_j$ are thus directly identified with the $\tilde{\text{coev}}_j$’s.

For the Tannakian case $tr^q_X : 1 \to F$ can be thought of as a state on $\mathcal{A}$. It is given by the following “quantum-trace”:

$$tr^q_X = tr^\text{can}_{\mathcal{V}_X} - G \quad .$$  \hspace{1cm} (2.28)

Here $tr^\text{can}$ is the canonical trace over the vector space $\mathcal{V}_X$ of the module $X$. Moreover, $G$ is the special, group like element that defines a balancing of $\mathcal{A}$. Equation (2.28) follows directly from the relation

$$\tilde{\text{coev}}_{\mathcal{V}_X} := (G \otimes 1) \circ T \circ \text{coev}_{\mathcal{V}_X}$$  \hspace{1cm} (2.29)

where $T$ is the transposition and $\text{coev}$ is the canonical coevaluation. This relation and its analogue for $\text{ev}_{\mathcal{V}_X}$ will be important in the combinatorial description of rigidity in Paragraph 3.4.

2.4) Special Liftings, Pairings, and Modularity

It will be useful and instructive to consider a few special situations, to which we can apply the lifting of a dinatural transformation to a coend. For the case with $Z = 1$, we find the following:

**Lemma 4** For a category $\mathcal{C}$ as above there is a canonical isomorphism,

$$\text{Hom}_{\mathcal{C}}(F, 1) \cong \text{Nat}(\text{id}_\mathcal{C}, \text{id}_\mathcal{C})$$  \hspace{1cm},

which respects the algebra structure on each space.

Moreover, for semisimple categories this space is one-to-one with the space of functions on $\mathcal{J}_0$; and for $\mathcal{C} = \mathcal{A} - \text{mod}$ it is canonically isomorphic to the center $Z_{\mathcal{A}} = \mathcal{A}' \cap \mathcal{A}$. 

**Proof:** An element in $\text{Nat}(\text{id}_\mathcal{C}, \text{id}_\mathcal{C})$, i.e., a natural transformation of the identity functor, is given by system of morphisms $\delta_X : X \to X$ that commute with morphisms in $\mathcal{C}$. Hence they form an element in $\text{dn}(S)$ in the coaction picture. The morphisms of the corresponding element $(1, \xi)$ in the picture of dinatural transformations are $\xi_X = (\text{ev}_X \otimes \delta_X) \circ \text{ev}_X$. As remarked in the proof of Lemma 4 a natural transformation is determined by the special morphisms $\delta_j \in \text{End}_\mathcal{C}(j) = \mathcal{C}$, for $j \in \mathcal{J}_0$, which proves the second assertion in the Lemma 4. Finally, for $\mathcal{C} = \mathcal{A} - \text{mod}$ a linear map $l : F = \mathcal{A}^* \to \mathcal{C}$ is clearly given by the evaluation on a unique element, $a_l \in \mathcal{A}$. In order for $l$ to be a morphism, $a_l$ must be invariant under the adjoint action of $\mathcal{A}$. It is an elementary fact for Hopf algebras that this is equivalent to saying that $a_l$ lies in the center.

This allows us also to relate the two notions of a trace from (2.16) and (2.26) for a natural transformation, $\delta$, by the identity:

$$\delta \circ tr^q_X = tr_X(\delta_X) \quad .$$  \hspace{1cm} (2.30)

Let us denote by $1^* : F \to 1$ the morphisms associated to the identity transformation of $id_\mathcal{C}$ or the dinatural transformation given by the evaluations $ev_X$. It corresponds to the unit element in $\mathcal{A}$, and to the constant function with value $1$ for semisimple categories.

Next, suppose that for $\mathcal{C}$ as above we have an element in $\text{dn}(S)$ given by transformations $\xi_X : X^\vee \otimes X \to A \otimes B \otimes C^\vee$. Using rigidity we can rewrite them as morphisms of the form

$$\hat{\xi}_X : A^\vee \otimes X^\vee \otimes X \otimes C \longrightarrow B \quad .$$  \hspace{1cm} (2.31)
The dinaturality condition from (2.23) can thus be reformulated more generally for the transformations $\xi$. Conversely, any set of morphisms of this form that satisfy this general dinaturality can be lifted to the coend. I.e., we find a morphism $r^{ABC} : A^\vee \otimes F \otimes C \to B$, such that we have the factorization $\xi_X = r^{ABC} \circ (1_{A^\vee} \otimes i_X \otimes 1_C)$. For balanced or braided categories we also do not have to assume that the first tensor factor in (2.31) is in fact a conjugate.

The universality of the coend is also used in [L] to define the morphisms of a categorical Hopf algebra. For example the morphisms

$$
\Delta_X : X^\vee \otimes X \xrightarrow{\text{coev}_X \otimes \text{ev}_X} X^\vee \otimes X \otimes X^\vee \otimes X \xrightarrow{i_X \otimes i_X} F \otimes F
$$

(2.32)

are dinatural and thus lift to a coproduct $\Delta : F \to F \otimes F$.

As another application let us explain the condition of modularity (in its strongest version) as the non-degeneracy of a pairing, $\omega$, of the coend. It is constructed from the monodromy transformation $\gamma(X,Y) := \epsilon(Y,X)\epsilon(X,Y) \in \text{End}_C(X \otimes Y)$, where $\epsilon(X,Y) : X \otimes Y \to Y \otimes X$ is the braid constraint of $C$. Specifically, we consider the morphisms

$$
\omega_{X,Y} : X^\vee \otimes X \otimes Y^\vee \otimes Y \xrightarrow{1_{X^\vee} \otimes \gamma(X,Y^\vee) \otimes 1_Y} X^\vee \otimes X \otimes Y^\vee \otimes Y \xrightarrow{\text{ev}_X \otimes \text{ev}_Y} 1.
$$

(2.33)

For fixed $Y$ this is a system of morphisms as in (2.31) with $A, B = 1$, and $C = Y^\vee \otimes Y$. The corresponding dinaturality condition follows from the naturality of $\gamma$ so that we can lift it to a map $\omega_Y : F \otimes Y^\vee \otimes Y \to 1$. Now, for every $X$ the $\omega_{X,Y}$ are dinatural also with respect to $Y$. From the uniqueness of liftings it thus follows that $\omega_Y$ shares this property, and therefore lifts itself to a morphism

$$
\omega : F \otimes F \to 1.
$$

(2.34)

Summarily, we find that $\omega_{X,Y} = \omega \circ (i_X \otimes i_Y)$. An important property that was already implied in Theorem [L] is given in the next lemma.

**Lemma 5** The pairing $\omega$ from (2.34) is a Hopf pairing, i.e., $m : F \otimes F \to F$ and $\Delta : F \to F \otimes F$ are adjoints of each other.

We now define a category with a coend to be modular if the pairing $\omega$ is non-degenerate. To illustrate the meaning of this condition let us remark that a symmetric category is never modular unless it is $\cong \text{Vect}(C)$, for which $F = C$. This follows since $\gamma(X,Y) = 1$ and hence $\omega = 1^* \otimes 1^*$, which is of rank one. In this sense modularity expresses the fact that all objects have non-trivial braiding.

For $C = A - \text{mod}$ it can be easily worked out that the corresponding linear map $A^* \otimes A^* \to C$ is given by evaluation on the element

$$
\omega = 1 \otimes S(\mathcal{R}'\mathcal{R}) = \sum_{ij} f_i e_j \otimes S(e_i f_j).
$$

(2.35)

Here $\mathcal{R} = \sum_j e_j \otimes f_j \in A \otimes A$ is the so called $R$-matrix of the quantum-group $A$. An interesting special case is the double $A = D(\mathcal{B})$ of a Hopf algebra $\mathcal{B}$. It is characterized by the properties that $A$ contains both $\mathcal{B}$ and $\mathcal{B}^\circ$ (where the latter is $\mathcal{B}^*$ with opposite comultiplication) as sub Hopf algebras, and that both maps $\mathcal{B} \otimes \mathcal{B}^\circ \to A$ and $\mathcal{B}^\circ \otimes \mathcal{B} \to A$, given by multiplication in $A$, are isomorphisms of vector spaces. For the $R$-matrix the $e_j$ form a basis of $\mathcal{B}$ and the $f_j \in \mathcal{B}^*$ are given by the dual basis. It is now clear from (2.33) that doubles always give rise to modular categories.
In the semisimple case the pairing is given by its summands $\omega_{i,j}$, i.e., the morphisms $\gamma(i, j^\vee)$. A popular modularity condition that was used, e.g., in the works of [RT], is to require non-degeneracy for the restriction of the pairing to invariance, i.e., for

$$Inv(\omega) : Inv(F) \otimes Inv(F) \to C : f \otimes g \mapsto \omega(f \otimes g) \quad (2.36)$$

In the basis $\{\tilde{coev}_j\}$ of $Inv(F)$ the matrix of this pairing is what is commonly called the $S$-matrix:

$$S_{i,j} = tr_{i\otimes j^\vee}(\gamma(i, j^\vee)) \quad (2.37)$$

The non-degeneracy of $Inv(\omega)$ is then reexpressed as the invertibility of the $S$-matrix. This does in fact also imply non-degeneracy of $\omega$, but for non-semisimple categories $Inv(\omega)$ may be degenerate although the category is modular, see the example at the end of Paragraph 3.5. In the next paragraph we shall find an even weaker and more natural condition that is equivalent to modularity for general categories.

### 2.5) Integrals and Cointegrals

Integrals play an important rôle in the constructions of the invariants $\tau_{HKR}$ and $\tau_L$, but they enter also cellular invariants, as for example $\tau_{Ku}$ of G. Kuperberg, see [Ku]. Moreover, they are at the center of study for Hopf algebras themselves.

A left integral of a Hopf algebra $\mathcal{A}$, in the classical sense, is defined as a linear form, $\mu \in \mathcal{A}^*$, with the property

$$1\mu(y) = 1_A \otimes \mu(\Delta(y)) \equiv y'\mu(y'') \quad \text{for all } y \in \mathcal{A} \quad (2.38)$$

It is an easy exercise to show that $\mu$ for the functions on a finite group $G$ is exactly the integration against the Haar-measure on $G$. The notion of a right integral is analogous. We also have cointegrals, which are integrals for the dual algebra $\mathcal{A}^*$. If we assume further that they are given by an evaluation on an element $\lambda \in Inv\mathcal{A}(Q)$, with $Q$ as in Paragraph 2.2.

It has been shown by Sweedler, [Sw], they exist iff $\mathcal{A}$ is finite dimensional, in which case they are unique up to scalars, i.e., $dim(Inv\mathcal{A}(Q)) = 1$.

The notion of integrals and cointegrals can also be put in the more general categorical language if we think of the coend as the Hopf algebra dual to $\mathcal{A}$. Since $F$ is now an abstract object it does not make sense to say that $\mu$ is an element of $F$, but it is still an element in the invariance $Inv(F)$. The integral and the cointegral of $F$ are thus defined as morphisms,

$$\mu : 1 \to F \quad \text{and} \quad \lambda : F \to 1 \quad (2.40)$$

The characterizing relations for left integrals from (2.38) and (2.39) can now be rewritten as the following pair of commutative diagrams, that make sense also in non-Tannakian categories:
The existence and uniqueness of integrals, proven in the classical case in [Sw], has been generalized to the category situation in [L]. For general, braided Hopf algebras $H$ in $C$ we actually only have $\mu : \alpha \to H$, where $\alpha$ is an invertible object, i.e., $\alpha^\vee \otimes \alpha = 1$. However, for the algebra obtained from the coend $F$ of a modular category we obtain $\alpha = 1$, and, further, that the integrals are two-sided (in the braided setting).

In [LS] it is shown that integrals and cointegrals of a Hopf algebra give rise to non-degenerate pairings. They are given by the morphisms

$$\beta : F \otimes F \xrightarrow{m} F \xrightarrow{\lambda} 1,$$

and

$$\beta^\dagger : 1 \xrightarrow{\mu} F \xrightarrow{\Delta} F \otimes F .$$

(2.42)

In fact, $\beta$ and $\beta^\dagger$ are almost inverses of each other. The following is a categorization of results in [LS] and [Rd].

**Lemma 6** Suppose $\lambda$ and $\mu$ are a cointegral and an integral of a braided Hopf algebra in $C$. Then they can be normalized such that $\lambda \circ \mu = 1$.

Suppose $\lambda$ and $\mu$ are a right integral and a left cointegral in this normalization. Then we have

$$\Gamma^{-1} = (1_F \otimes \beta) \circ (\beta^\dagger \otimes 1_F) ,$$

where $\Gamma$ is the braided antipode, and the pairings are as in (2.43).

**Proof:** The fact that the composition is non-zero and thus can be normalized to 1 follows from the Fundamental Theorem for Hopf modules, which is also used to prove the existence of integrals, and is proven to hold for braided categories in [L].

The axioms of a braided Hopf algebra with braided antipode $\Gamma$ imply that the following diagram commutes:

$$\begin{array}{ccccccc}
F \otimes F & \xrightarrow{\Delta \otimes 1_F} & F \otimes F \otimes F & \xrightarrow{1_F \otimes \epsilon(F,F)} & F \otimes F \otimes F & \xrightarrow{m \otimes 1_F} \\
1_F \otimes \Delta & | & F \otimes F \otimes F & \xrightarrow{(\Delta \circ m) \otimes \Gamma} & F \otimes F \otimes F & \xrightarrow{1_F \otimes m} & F \otimes F
\end{array}$$

(2.43)

If we multiply the morphisms in $End_C(F \otimes F)$ identified in (2.43) with $1_F \otimes \mu$ from the left and with $\lambda \otimes 1_F$ from the right we obtain the identity from Lemma 6. $\square$

Since we defined another canonical pairing in the previous paragraph we obtain a canonical endomorphism:

$$S_F : F \xrightarrow{\beta^\dagger \otimes 1_F} F \otimes F \otimes F \xrightarrow{1_F \otimes \omega} F .$$

(2.44)

The pairing from (2.34) can be written as $\omega = \beta \circ (1_F \otimes (\Gamma \circ S_F))$, and modularity is the same as invertibility of $S_F$. Also, the matrix in (2.37) is equivalent to $Inv(S_F)$ in the semisimple case.

The following reformulation of modularity is motivated by topological considerations that we shall discuss in Paragraph 3.1.

**Theorem 5** Suppose $\omega$ is a Hopf pairing of a braided, categorical Hopf algebra, $F$, with cointegral, $\lambda : F \to 1$. Then $\omega$ is non-degenerate if and only $\lambda$ lies in the image of $\omega$. I.e., there is $\mu \in Inv(F)$, such that

$$\lambda : F \xrightarrow{\mu \otimes 1_F} F \otimes F \xrightarrow{\omega} 1 .$$

(2.45)

If it exists, $\mu$ has to be an integral of $F$. 18
Proof: Since \( \omega \) is a Hopf pairing it is clear that, if it is also non-degenerate, the unique solution to (2.45) is given by an integral. Conversely, assume we have a solution. The equation that characterizes \( \omega \) as a Hopf pairing is

\[
\begin{array}{ccc}
F \otimes F \otimes F & \xrightarrow{\Delta \otimes 1 I_{F} \otimes F} & F \otimes F \otimes F \\
\otimes m & \downarrow & \otimes \omega \\
F \otimes F & \xrightarrow{1 I_{F} \otimes \omega \otimes 1 I_{F}} & F \otimes F
\end{array}
\]

If we multiply to this \( \mu \otimes 1 I_{F} \otimes F \) from the left, where \( \mu \) is a solution to (2.45), and use the definition in (2.44) we arrive at

\[
\beta = \omega \circ (S_{F} \otimes 1 I_{F})
\]

This shows by Lemma 6 that \( \omega \) is invetible.

We conclude this chapter with a characterization of the integrals in the two prominent, special cases:

By Lemma 4 there is a natural transformation of the identity associated to the integral, which is given by morphisms \( \lambda_{X} : X \to X \), for every \( X \). In the semisimple setting it is thus given by a function on \( J_{0} \). From the definition of the coproduct in (2.32) and axiom (2.41) we find that the numbers have to solve the equation \( \lambda_{j}i_{j} = \lambda_{j}(i_{1} \circ \text{ev}_{j}) \). Thus we have \( \lambda_{j} = 0 \) if \( j \neq 1 \) so that \( \lambda_{X} \) is by formula (2.27) a multiple of the full projection onto the invariance of \( X \).

The substitute for the modularity condition from Theorem 3 was already proposed in [Ke5] in the semisimple case. It was expressed by the requirement that \( 1 \in \text{im}(S) \), where \( S \) is the matrix from (2.37), and \( 1 \) is the vector with \( 1_{j} = \delta_{1,j} \), i.e., it is proportional to \( \lambda \). In the modular case the preimage is necessarily given by the quantum-dimensions, i.e.,

\[
\sum_{j \in J_{0}} S_{i,j}^{-1} d(j) = D^{2} \delta_{1,1} \quad \text{with} \quad D^{2} = \sum_{j \in J_{0}} d(j)^{2}
\]

This is shown in [T], but see also [Ke5] for a short computation. The integrals, related as in Theorem 3, and with normalization \( \lambda \circ \mu = 1 \), are thus given by

\[
\mu = D^{-1} \sum_{j \in J_{0}} d(j) \tilde{c} \circ \text{ev}_{j} \equiv D^{-1} \sum_{j \in J_{0}} d(j) tr_{j}^{q} \quad \text{and} \quad \lambda = D \text{ev}_{1} \equiv D 1,
\]

for a semisimple category with \( |J_{0}| < \infty \). The alternate notation for the basis vectors in \( \text{Inv}(F) \) is obtained from (2.26).

Finally, for \( C = A - \text{mod} \), we know that for a modular Hopf algebra the (classical) right integral \( \mu^{R} \in A^{*} \) is invariant under the adjoint action - i.e., \( \mu^{R} \) is an element in \( \text{Hom}_{A}(F,1) \) - which, by relations from [Rd], is equivalent to saying that the comodulus of \( A \) is trivial. In fact, it is noticed in [FLM] and also [Ke2] that \( \mu^{R} \) does coincide with the categorical integral of \( C = A - \text{mod} \) we have discussed above if \( A \) is a double.

Also, triviality of the comodulus means exactly that the left cointegral \( \lambda \in A \) is two-sided, which in turn is the same as saying that \( \lambda \) lies in the center of \( A \). By Lemma 4 the evaluation on \( \lambda \) gives us therefore also the categorical cointegral, and the associated natural transformations \( \lambda_{X} \) are given by acting with \( \lambda \in A \) on the respective module. For details, see, e.g., [Ke2].
3) Lyubashenko’s Invariant, and Derivations of $\tau_{HKR}$ and $\tau_{RT}$:

In this chapter we will discuss the general construction of three-manifold invariants starting from a modular, abelian, braided tensor category, $\mathcal{C}$. We shall apply the special results on coends for semisimple and Tannakian categories from the previous chapter, in order to show that $\tau_L$ specializes to the invariants of Reshetikhin-Turaev and Hennings, respectively, thus proving Theorem 1.

3.1) Construction of $\tau_L$

As for $\tau_{RT}$ the construction of $\tau_L$ starts by presenting a three-manifold $M$ by surgery along a framed link $L \subset S^3$, or, equivalently, a link of ribbons.

The link can be thought of as a special element in the category of isotopy classes of oriented, framed tangles, $\mathcal{T}_{gl}$. An object is an ordered set of labels $\{a_1, \ldots, a_N\}$, and a morphism $\{a_1, \ldots, a_N\} \to \{b_1, \ldots, b_M\}$ is the generic projection of a framed tangle in the strip $\mathbb{R} \times [0, 1]$. It consists of ribbons that end in $N$ intervals along the top-line $\mathbb{R} \times \{1\}$ and in $M$ intervals along the bottom-line. We require that the labels of the intervals, that are connected by a ribbon, are the same. We also admit closed ribbons so that $L \in \text{End}_{\mathcal{T}_{gl}}(\emptyset)$ for the projection of a closed link.

For a given abelian category $\mathcal{C}$ we call a coloring, $\text{Col}$, an assignment, $a \mapsto X_a$, of labels to general objects of $\mathcal{C}$, together with a sense of direction for each component of a given tangle. To every coloring we then associate a functor:

$$I_{\text{Col}} : \mathcal{T}_{gl} \longrightarrow \mathcal{C}$$

(3.49)

The construction of $I_{\text{Col}}$ proceeds as in [RT]. To a label set, $\{a_1, \ldots, a_N\}$, we associate the object $X_{a_1}^# \otimes \ldots \otimes X_{a_N}^#$, where $X_{a}^\# = X_a$ if the attached strand is directed downward and $X_{a}^\# = X_a^\vee$ if the strand is directed upward. A tangle projection is always isotopic to one with a generic height coordinate, in which the tangle is presented as the composite of elementary tangles. The latter are allowed to contain only one crossing, one maximum, or one minimum. They are mapped by $I_{\text{Col}}$ to the braid constraint $\epsilon(X, Y)$ and the rigidity morphisms, respectively. In our convention the latter assignment is as follows:

$$\begin{align*}
\begin{array}{c}
\xymatrix{ \circlearrowleft & \mapsfrom \text{coev}_X \\
\end{array} & \begin{array}{c}
\xymatrix{ \circlearrowleft & \mapsfrom \text{ev}_X \\
\end{array} & \begin{array}{c}
\xymatrix{ \circlearrowright & \mapsfrom \text{coev}_X \\
\end{array} & \begin{array}{c}
\xymatrix{ \circlearrowright & \mapsfrom \text{ev}_X \\
\end{array}
\end{align*}$$

(3.50)

Moreover, we assign a balancing element, $v_{X^\#}$, with $v \in \text{Nat}_{\mathcal{C}}(id, id)$ to every local $2\pi$-twist of a ribbon with coloring $X$. By the framed version of the first Reidemeister move, it relates the rigidity morphisms to their flipped counterparts.

With $I_{\text{Col}}(\emptyset) = 1$, this procedure assigns to every link, $\mathcal{L}$, with $N$ components and every coloring with objects $(X_1, \ldots, X_N)$ a number $I^{(X_1, \ldots, X_N)}(\mathcal{L}, \mathcal{C})$. We could now, as for $\tau_{RT}$, try to find a combination of such invariants that is invariant under 2-handle slides. The following lemma shows that it is not possible to find anything new for non-semisimple categories in this way:
Lemma 7 To an object, $X$, of a Noetherian, abelian, braided, tensor category, $C$, we associate the set $J(X)$, which consists of all irreducible factors of a Jordan-Hölder series of $X$ that have non-vanishing quantum-dimensions (with repeated isomorphic objects).

For any coloring we then have

$$I^{(X_1,\ldots,X_N)}(L,C) = \sum_{j_1 \in J(X_1),\ldots,j_N \in J(X_N)} I^{(j_1,\ldots,j_N)}(L,\overline{C}^f) .$$

This follows from the fact that for each $X_\nu$ the invariant can be expressed as the canonical trace $tr_{X_\nu}$ over a morphism that depends on $X_\nu$ like a natural transformation. The value thus only depends on its image in the semisimple trace-quotient of $C$. The object $X_\nu$ itself factors into the object $\overline{X_\nu} = \oplus_{j_\nu \in J(X_\nu)} j_\nu$ in $\overline{C}^f$.

In Lemma 3 we learned that natural transformations of the identity, as in the computation alluded to above, are naturally identified with a coinvariance-morphism or state on the coend. In order to obtain such a state from a given link, $L$, we have to open the components of $L$ in a controlled way. The procedure can be described as follows:

We first introduce a horizontal line $R_1$ above a region in the plane that contains a projection of $L$. For each component, $C \cong S^1 \times [0,1]$, of the link we introduce a ribbon, $R_C$, that starts at an interval in $R_1$ and ends in an interval on $\partial C$. Then we split $R_C$ down the middle so that we obtain two parallel ribbons, $R'_C$ and $R''_C$. We also cut the component $C$, where $R_C$ has been attached, as indicated in the diagram \((3.51)\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_C \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Thus we obtain a tangle, $L^{split}$, which contains no closed components anymore, but only ribbons that start and end in $R_1$. Moreover, the start and end points are always neighbors. Hence, as a morphism $\{a_1,\ldots,a_{2N}\} \rightarrow \emptyset$ in $T gl$, where $N$ is the number of components of $L$, we have for the labels $a_{2i-1} = a_{2i}$.

As described in [KL] this tangle represents the manifold $M^{split}$, defined in the introduction of Chapter 1, (see also [Ke5] of Paragraph 3.5).

Let us choose directions such that the strand at $a_{2i}$ is going downwards. A coloring, $Col$, is now uniquely given by the $N$ objects from $a_{2i} \mapsto X_i$. Applying the functor $I^{Col}$ from \((3.49)\), we obtain morphisms

$$I^{(X_1,\ldots,X_N)}(L^{split},C) : X_1^\vee \otimes X_1 \otimes \ldots \otimes X_N^\vee \otimes X_N \rightarrow 1 .$$

\((3.52)\)

These maps are dinatural in any of the arguments. This means for example that for fixed $X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_N$ they have the property of the morphisms described in \((2.31)\) with respect to $X = X_j$, setting $A^\vee = X_j^\vee \otimes X_1 \otimes \ldots \otimes X_{j-1}$, $C = X_{j+1}^\vee \otimes \ldots \otimes X_N$, and $B = 1$. This follows from the fact that all elementary morphisms are natural so that a morphism $f : X_j \rightarrow Y_j$ can be pushed through an entire component of a diagram. The transformation can thus be lifted to the coend. Repeated application yields a state on the $N$-fold tensor product of the coend, i.e., a morphism $\hat{I}(L^{split},C) : F^{\otimes N} \rightarrow 1$, from which we recover the ones in \((3.52)\) by multiplying the transformations $i_{X_j}$. The question how the morphism $\hat{I}$ relates to invariants is answered next:
Lemma 8 For a sequence of invariances, \( \mu_j \in \text{Hom}_C(1, F) \), define a number by
\[
\tau_L(\mathcal{L}^{\text{split}}, C, \vec{\mu}) := \tilde{I}(\mathcal{L}^{\text{split}}, C) \circ (\mu_1 \otimes \ldots \otimes \mu_N) .
\] (3.53)

1. If all of the \( \mu_j \) are invariant under the braided antipode \( \Gamma \) then \( \tau_L \) only depends on \( \mathcal{L} \), i.e., not on the choice of splitting ribbons \( R_C \).

2. Suppose \( \mu_j = tr^{q_j}_X \), with \( tr^{q_j}_X \) as in (2.26). Then \( \tau_L \) is identical to the invariant from Lemma 7.

3. If \( \mu_j = \mu \) are integrals of \( F \), then \( \tau_L \) is invariant under 2-handle slides.

Proof: In order to verify independence of the choice of the splitting ribbons we have to be able to switch an under-crossing with \( R_C \) to an over-crossings, which is possible since all \( \mu_j \) are in the invariance. It remains to ensure that we can switch the attachment from one side of \( C \) to the other. This is taken care of by the antipode. In a precise version of this fact we also have to take care of orientations of \( R_C \) as a surface with boundary, see [L] or [KL] for details.

It is easy to see that we can recover \( \mathcal{L} \) up to isotopy from \( \mathcal{L}^{\text{split}} \) by placing \( N \) small, flat arcs on top of \( R_t \), joining an interval with label \( a_{2i-1} \) to an interval with label \( a_{2i} \). With given colorings and the conventions from (3.50) this corresponds to the insertion of flipped coevaluations for the \( \tilde{I} \) factored into the respective transformation.

For the fact that the axiom on the left of (2.41) directly implies the 2-handle slide, see [L], but also [KR] in the Hennings-framework.

The invariant from Part 3 of the lemma is basically Lyubashenko’s invariant, \( \tau_L \). The only thing we have omitted from the discussion is the invariance under the signature-move (or \( \mathcal{O}_1 \)-move in [Ki]). As we remarked in Paragraph 1.4, our notion of TQFT’s is really defined for a central extension of the cobordisms categories by \( \Omega_4 \). The \( \mathcal{O}_1 \)-move is exactly the generator of this extension. We shall therefore substitute it by a move under which two link diagrams are equivalent if they not only describe the same three-fold but also have linking-matrices with the same signature, \( \sigma(L) \). It is given by the addition or removal of an isolated Hopf-link, \( 0 \otimes \otimes \), for which one component is 0-framed, see also the \( \eta \)-move in [Ke5]. Due to multiplicativity of \( \tau_L \) and 2-handle slides over the 0-frame, it is thus enough to check that
\[
\tau_L(0 \otimes 0) = \tau_L(0 \otimes 1) = 1
\] (3.54)

In a variation of Kirby’s calculus described next, it turns out that this imposes only a normalization condition on \( \mu \), which determines the integral up to a sign. We will not consider the usual renormalizations of the entire invariant, e.g., by \( \tau_L(\otimes \pm 1)^{\pm \sigma(L)} \), required by the original \( \mathcal{O}_1 \)-moves, or by \( \tau_L(S^1 \times S^2)^{\pm 1} \), which is not defined in the non-semisimple case.

Let us outline here the mentioned modification of Kirby’s calculus, namely the so-called “bridged link calculus” from [Ke5]. It has the advantage that it involves only local moves, at the price of introducing additional data, that indicates 1-handle surgeries. In one version of the calculus this data is represented by horizontal coupons, which may have strands \( R_1, \ldots, R_n \) entering at the top side and leaving at the bottom side, and is sometimes also expressed by a “Kirby’s dotted circle” around \( R_1, \ldots, R_n \). The relevant moves of this calculus are depicted in the following diagram. We assume as usual blackboard-framing, i.e., flat ribbon-projections:
In the diagrams we already indicated an extension of the construction of $\tau_L$ to links with coupons: If the strands going through a coupon from top to bottom are colored with objects $X_1, \ldots, X_n$, we insert here a morphism $\lambda_X : X \to X$ with $X = X_1 \otimes \ldots \otimes X_n$. In particular, the strands for the modification and the coupon-slide may be replaced by several parallel strands. The modification-move also imposes that the $\lambda_X$ do belong to a natural transformation of the identity. In addition to the moves in (3.55) we also have to require invariance of the balancing, $v_X \lambda_X = \lambda_X$, in order to guarantee invariance with respect to collective $2\pi$-twists of the strands entering a coupon. It is clear that these moves also imply the normalization conditions in (3.54).

In view of our discussion in Paragraph 2.5 it is not difficult to see that a natural candidate for $\lambda$ is the cointegral of $F$. It is has become evident in Hopf algebra theory that the interplay of integrals and cointegrals can often be very useful in the organization and clarification of involved structures and computations. It is in this sense, the formalism of bridged links enables us to apply the same principles in a purely topological setting to the relations between index 1 and index 2 surgeries on a three-fold.

If we are interested only in closed manifolds modularity, and the fact that $\mu$ and $\lambda$ are integrals are not absolutely necessary. We could try to define an invariant from general elements

$$\mu \in \text{Hom}_C(1, F) \quad \lambda \in \text{Hom}_C(F, 1).$$

Invariance under the special 1-2-cancellation from the above diagram imposes the constraint $\lambda \circ \mu = 1$. Moreover, the equation related to the modification-move holds for every $X$ iff the pairing $\omega$ maps $\mu$ to $\lambda$ as in Theorem 3. Let us call here a pair as in (3.56) with these two properties dual states.

To an invariance, $\mu$, we can compute the difference $D_\mu$ of the two morphisms from $F$ to itself in the left diagram of (2.41). In the same way we can define $D^\lambda$ from the diagram for cointegrals. It is clear that $\mu$ is an integral iff $D_\mu = 0$, and $\lambda$ analogously. In absence of strict modularity we can also introduce the weaker notion of $\omega$-integrals and $\omega$-cointegrals defined by the conditions

$$\omega \circ (\mathbb{1}_F \otimes D_\mu) = 0 \quad \text{and} \quad \omega \circ (\mathbb{1}_F \otimes D^\lambda) = 0,$$

respectively. The first two assertions of the following lemma are immediately obtained from liftings to coends and and the definitions of $F$ as an algebra. In the last one we use that $\omega$ is a Hopf pairing in a similar way as in the proof of Theorem 4.

**Lemma 9** Suppose $\mu$ and $\lambda$ are as in (3.56).

1. The 2-handle slide of an open strand with color $X$ over a closed one holds for all $X$, if and only if $\mu$ is an $\omega$-integral.
2. The coupon-slide as in diagram (3.55) holds for all colorings, if and only if \( \lambda \) is an \( \omega \)-cointegral.

3. Suppose that \( \lambda \) and \( \mu \) are dual states. Then \( \mu \) is an \( \omega \)-integral if and only if \( \lambda \) is an \( \omega \)-cointegral.

Let us remark here that we should have anticipated the last (purely algebraic) statement already from the first two, using the close correspondence between algebraic and topological structures. Specifically, this was to be expected since introducing the 2-handle slide in addition to the first two moves leads us to an equivalent calculus of surgery diagrams as introducing the coupon-slide. Finally, it is not hard to see that, with \( \lambda \) and \( \mu \) as in Lemma 9, the balancing \( v : F \to 1 \) is in the image of \( \omega \) so that \( v \cdot \lambda = (v \circ i_1)\lambda = v_1\lambda \). We thus have invariance under \( 2\pi \)-twists at coupons if \( v_1 = 1 \).

In the treatment of [L] another algebra, \( f \), is introduced by dividing the coend \( F \) by the null-space of \( \omega \). This allows us to construct also ordinary TQFT’s for categories with much weaker modularity conditions. We shall, however, focus in the following on the strictly modular case, where we can hope to represent extended structures, as discussed in Chapter 1.

3.2) Specialization to \( \tau_{RT} \)

The proof of the first part of Theorem 1 is now a straightforward application of the theory developed so far:

We insert the formula (2.48) for the integral of the semisimple category into (3.53) from Lemma 8), and reorganize the summation in the following form.

\[
\tau_L(\mathcal{L}, \mathcal{C}) = D^{-N} \sum_{j_1, \ldots, j_N \in J_0} \prod_{\nu=1}^N d(j_\nu) \tilde{I}(\mathcal{L}^{\text{split}}, \mathcal{C}) \circ (\text{tr}^q_{j_1} \otimes \ldots \text{tr}^q_{j_N}) ,
\]

where \( D \) is as in (2.47). Next, it follows from Part 3 of Lemma 8) that the composite of morphisms in this sum is identical to \( I^{(j_1, \ldots, j_N)}(\mathcal{L}, \mathcal{C}) \). If we set

\[
w(j_1, \ldots, j_N) = D^{-N} \prod_{\nu=1}^N d(j_\nu) ,
\]

we finally obtain the \( \tau_{RT} \)-invariant in the form given in (1.1). In this normalization \( \tau_{RT} \) is strictly speaking an invariant of pairs, given by a three-fold together with a signature. It was already observed in [RT] that there is basically no freedom in the choice of the weights, except for overall scalings. From the point of view we have taken here, this is simply a reflection of the uniqueness of the integral of a categorical Hopf algebra.

In the introduction we already pointed out that the \( \tau_{RT} \)-invariant is often also constructed for a non-semisimple category, \( \mathcal{C} \), like that of representations of a quantum group at a root of unity, by considering the semisimple trace-subquotient \( \mathcal{C}^{\text{tr}} \) instead. Yet, by virtue of Lemma 7 there is also a way to compute it from \( \tau_L \) for the original \( \mathcal{C} \) with some modifications.

To be more precise let us pick a set of objects, \( J_0^* \), of \( \mathcal{C} \), which factors into a set of representatives of irreducibles, \( J_0 \), for \( \mathcal{C}^{\text{tr}} \). It is not hard to see from the definition of the functor \( \mathcal{C} \to \mathcal{C}^{\text{tr}} \) that these objects can always be chosen as irreducibles themselves, and that any two choices of irreducibles are isomorphic in \( \mathcal{C} \). From Lemma 7 we have in particular for \( j_\nu \in J_0^* \) that

\[
I^{(j_1, \ldots, j_N)}(\mathcal{L}, \mathcal{C}) = I^{(\tilde{j}_1, \ldots, \tilde{j}_N)}(\mathcal{L}, \mathcal{C}^{\text{tr}}) .
\]
Let us also introduce the following invariance of the coend \( F \) in \( \mathcal{C} \):

\[
\mu^{\text{semis}} := D^{-1} \sum_{j \in J_0^*} d(j) \, tr^q_j,
\]

with \( tr^q \) as in (2.26). Since \( tr^q_X \) only depends on the equivalence class of \( X \) in \( \mathcal{C} \), \( \mu^{\text{semis}} \) is in fact a canonical invariance, independent of the choice of \( J_0^* \).

Using Part 2 of Lemma 8, it is now clear that we obtain the \( \tau_{RT} \)-invariant if we substitute \( \mu^{\text{semis}} \) for the integral \( \mu \) of \( \mathcal{C} \). More precisely:

\[
\tau_{RT}(\mathcal{L}, \mathcal{C}^{\text{tr}}) = \tau_L(\mathcal{L}, \mathcal{C}, \mu^{\text{semis}}).
\]  

(3.58)

It is often convenient to express this modification of \( \tau_L \) in a slightly different way. To this end we remark that the non-degenerate invariant form defined in (2.42) associates to the canonical invariance \( \mu^{\text{semis}} \) a unique, canonical state, \( Q : F \to 1 \), by the condition

\[
\mu^{\text{semis}} : 1 \xrightarrow{\beta^\dagger} F \otimes F \xrightarrow{\mathbb{I}_F \otimes Q} 1.
\]  

(3.59)

Quite obviously we have that \( \mathcal{C} \) is semisimple iff (up to rescalings) \( Q = 1^* \). Also, we find that \( Q \) is invariant under the braided antipode \( \Gamma \).

Now, in view of the correspondence of Lemma 4 we can think of \( Q \) also as a natural transformation of the identity, i.e., a set of morphisms \( Q_X : X \to X \). Semisimplicity of \( \mathcal{C} \) is then equivalent to the condition \( Q_X = \mathbb{I}_X \), \( \forall X \).

In order to relate the substitution of (3.58) to “\( Q \)-insertions”, let us describe next a generalization of the procedure from Paragraph 3.1, that associates an invariant \( \tau_L(\mathcal{L}^Q, \mathcal{C}) \), to a link, \( \mathcal{L} \), with inserted natural transformations. More precisely, we consider decorated links for which a strand may contain a chip labeled by a natural transformation of the identity, as depicted in the diagram (3.60) below.

\[
X \\
\downarrow Q_X \\
(3.60)
\]

The rule to compute \( I^\text{col} \) for a given coloring is now to insert for a chip at a downward strand with coloring \( X \) the morphism \( Q_X \). To be precise we also have to require

\[
Q_X^\vee = Q_X^\dagger,
\]  

(3.61)

in order for this to be well defined with respect to sliding a chip through extrema. Note, that coupons are also decorations in this sense, since the cointegral in our situation is two-sided.

Let us denote by \( \mathcal{L}^Q \) the link, where we have inserted exactly one chip into every component of \( \mathcal{L} \), and labeled them all with the same \( Q \). In a splitting the chips can all be moved to the top-line \( R_t \) at the intervals with labels \( a_{2i} \).

We now have the following modification of (3.58):
Theorem 6 Suppose $\mathcal{C}$ is a modular category with modular quotient $\mathcal{C}^{tr}$ and let $Q$ be as defined in (3.59).

Then

$$\tau_{RT}(\mathcal{L}, \mathcal{C}^{tr}) = \tau_L(\mathcal{L}^Q, \mathcal{C})$$

(3.62)

where the chosen integrals are those of the coends of the respective categories.

Proof: Suppose $q, q': F \rightarrow Z$ are related by $q \circ i_X(\mathbb{I}_{X^*} \otimes Q_X) = q' \circ i_X$. Then we find from the definition of the coproduct in (2.32) that $q' = (q \otimes Q) \circ \Delta$. Hence, with (3.59) we find $q' \circ \mu = q \circ \mu^{\text{semis}}$. If we substitute for $q$ the lifted transformation of $\int I(\mathcal{L}^{\text{split}})$ we see from the assumptions that the lift, $\mathbb{I}((\mathcal{L}^Q)^{\text{split}})$, where we inserted into the transformations from (3.52) a $Q_X$ at each second interval at the top-line, corresponds to $q'$. A comparison with (3.58) then completes the proof. □

An interesting situation would arise if we found for some category a (split) knot, $K$, such that $Q = \int I(K^{\text{split}}, \mathcal{C})$. In this case we can switch between the two invariants by connected summing of $K$ to every component of $\mathcal{L}$, i.e., $\tau_{RT}(\mathcal{L}, \mathcal{C}^{tr}) = \tau_L(\mathcal{L} \# (K^N), \mathcal{C})$. A more realistic question is, whether we can find at least a formal linear combination of knots that yields the canonical transformation $Q$.

In Paragraph 3.4 we will discuss the above correspondence between invariants via $Q$ in the more concrete context of Hennings-invariants.

3.3) Categories of Singular Tangles and Hennings’ Rules

The basic braid and rigidity morphisms of a strict, Tannakian, braided tensor category, $\mathcal{C} = \mathcal{A}^{\text{mod}}$, can be identified with the action of certain elements of the quantum-group $\mathcal{A}$ on given modules. Specifically, these are an $R$-matrix and a balancing element,

$$R = \sum_j e_j \otimes f_j \in \mathcal{A} \otimes \mathcal{A}, \quad \text{and} \quad G \in \mathcal{A}$$

(3.63)

It is a quite old idea to get rid of the modules in the computation of tangle invariants and try to find a combinatorics of these elements alone. The first such attempt was made by Reshetikhin in [Re], where he associated to open (one-component) tangles an element in the center $Z_\mathcal{A}$ of $\mathcal{A}$. At about the same time Hennings formulated the same combinatorial rules, but also included a (classical) right integral, $\mu^R$, of $\mathcal{A}$ in the picture. This allowed him to treat closed components of a tangle by evaluating the elements on this component against $\mu^R$. The number that is eventually computed for a closed link by this procedure is the three-manifold invariant, $\tau_{HKR}$.

In this paragraph we shall briefly review these rules, first without integrals, in the improved version of Kauffman and Radford, using a slightly more categorical language. The central gadget to be defined and studied is a category, $\mathcal{sT}(\mathcal{A})$, of “singular $\mathcal{A}$-tangles”, associated to a balanced Hopf-algebra $\mathcal{A}$.

$s\mathcal{T}(\mathcal{A})$ is defined quite analogously to $Tgl$. Its objects are again strings of labels and a morphism is again given by generic strands in a strip, that are either closed paths, or that start and end at boundary points of the strip with the same labels. They differ from the tangles in $Tgl$ in that we do not distinguish between under- and over-crossings, and that we can have blobs sitting on a strand (similar to the chips), which are labeled by elements in $\mathcal{A}$. Thus the category is generated by the following, elementary morphisms:
for any $x \in A$. (3.64)

Any other morphism can be obtained by composition, i.e., putting tangles on top of each other, by tensor products, i.e., juxtapositions of morphisms, and, finally, by (distributive) linear combinations of diagrams for a fixed topological tangle but different decorations by elements in $A$. The latter shall be indicated in a diagram by writing summation indices to the elements.

Given these generators we can formally characterize the category $\mathcal{ST}(A)$ by a list of relations among the generating morphisms. In $\mathcal{T}_{gl}$ these had been the three Reidemeister moves (the first with framing), and, further, the two moves due to a preferred vertical direction, namely the extrema-cancellations and the “crossing-symmetry”. Their singular versions shall also apply to parts of tangles that carry no Hopf algebra elements. Only the first Reidemeister move shall be replaced by the move indicated on the left of diagram (3.65), where $G \in A$ is a-priori any invertible element. An analogous relation at a maximum can be derived from this using the extrema-cancellation moves.

We also depicted here the form of the second Reidemeister move. Since we have singular crossings it implies that we have a presentation of the symmetric groups instead of the braid groups as in $\mathcal{T}_{gl}$.

Next, we give relations that allow us to move and combine elements along a component of a singular tangle:

Again, using extrema cancellations and crossing-symmetry, we can derive analogous relations for moving an element through a maximum or along the opposite strand of the crossing. If we have that $G$ implements the antipode $S$, i.e.,

$$S^2(x) = GxG^{-1} \quad \forall x \in A \quad ,$$

then an element $x$ can be moved through a piece of a tangle and emerge as $S^p(x)$, where the order $p$, by which the antipode acts, is invariant under the isotopy moves. Moreover, $p$ is even if
the element leaves the piece in the same direction as it entered, and odd if it reverses the direction. Also, two applications of a crossing in $Rm1$ of (3.63), and the $Me2$-move yield $S(G) = G^{-1}$, which we thus have to impose in order for $sT(A)$ to be non-trivial.

It is sometimes useful to have a “co-tensorproduct” on $sT(A)$, as in [KR] that acts on a morphism by pushing a strand with labeling $a$ off itself in the plane so that we get two parallel strands with labels $a'$ and $a''$. Further, a blob with an element $x \in A$ is duplicated, and the elements assigned to each of the new blobss are $x'_j$ and $x''_j$, where $\Delta(x) = \sum_j x'_j \otimes x''_j$, see [KR]. In order for this to be compatible with $Rm1$, we have to impose the somewhat stronger condition that $G$ is group-like, i.e., $\Delta(G) = G \otimes G$. A group-like element with (3.67) is now exactly what is meant by a balancing of a general Hopf algebra.

In [KR] the connection between $T_{gl}$ and $sT(A)$ is being made. We shall review it here as a functor

$$\text{Dec}_A : T_{gl} \longrightarrow sT(A) \quad . \tag{3.68}$$

It is clear that for this to exist we must be able to represent braid group relations in $sT(A)$, too. Thus we have to assume that $A$ is in addition quasi-triangular. This means that $A$ shall have an element $R$ as in (3.63), which satisfies the Yang-Baxter equation. Hence, if we define $\text{Dec}_A$ on a crossing in $T_{gl}$, as in the right of diagram (3.69), tangles in $T_{gl}$, that are equivalent under the third Reidemeister move, are mapped to equivalent tangles in $sT(A)$.

$$\begin{array}{c}
\begin{array}{c}
\times \quad \text{Dec}_A \quad \times \\
\bullet e_j \quad \bullet f_j \quad \bullet S(e_j) \quad \bullet f_j
\end{array}
\end{array} \tag{3.69}$$

Now, we also can consider co-products of strands in $T_{gl}$. The condition that $\text{Dec}_A$ maps this operation to the co-tensor product in $sT(A)$ translates with the assignment in (3.69) into the familiar triangle equations for $R$ of a quasi-triangular Hopf algebra.

Maxima and minima in $T_{gl}$ shall be mapped by $\text{Dec}_A$ to their counterparts in $sT(A)$. Then the second assignment in (3.63) results from checking the crossing-symmetry. Since $S \otimes 1(R) = R^{-1}$ we find that $\text{Dec}_A$ is also compatible with the second Reidemeister move in $T_{gl}$.

It remains to verify the framed version of the first Reidemeister move in $T_{gl}$. To this end let us recall here that a quasi-triangular Hopf algebra has a number of canonical elements, which can be constructed from the $R$-matrix, see [Dr]. The first of these is $u = \sum_j S(f_j) e_j$. It does in fact implement the square of the antipode but it is not group like. Another element with these properties is $\hat{u} := S(u)^{-1}$. It turns out that $g := u \hat{u}$ is group like, with $Ad(g) = S^4$. In other words, although also quasi-triangular Hopf algebras do not have a canonical balancing, at least the square of the balancing is canonical. Indeed, it turns out that our functor is compatible with the first Reidemeister move iff we have

$$g = G^2 \quad . \tag{3.70}$$

A balancing with this property is what we mean by a balancing of a quasi-triangular Hopf algebra. Sometimes this is equivalently described by the central element $v = uG^{-1}$, which has to satisfy the equation

$$\mathcal{M} := R' R = v \otimes v \Delta(v^{-1}) \quad . \tag{3.71}$$

It is easily worked out that it is assigned to a local $2\pi$-twist of a ribbon in $T_{gl}$. Since $v$ is central this extension of $\text{Dec}_A$ is compatible with moving a $2\pi$-twist through a crossing. Imposing also $S(v) = v$ allows us to move such twists through extrema.
Let us summarize these findings in a more formal statement:

**Theorem 7 ([H],[KR])** For any balanced, quasi-triangular Hopf algebra, the functor $\text{Dec}_A$ is well defined.  
(I.e., it factors into the equivalence classes.)

The functor $\text{Dec}_A$ can also be extended quite naturally to tangles with coupons, given a two-sided cointegral, $\lambda \in A$. For a coupon, that has $m$ strands passing through we associate $m$ vertical strands, each with one blob as in diagram (3.72). Here the elements assigned to the blobs are from the $m$-fold coproduct $\Delta^{(m-1)}(\lambda) = \sum_j \lambda_j^{(1)} \otimes \ldots \otimes \lambda_j^{(m)}$. This is again compatible with the co-tensor product.

$$\text{Dec}_A \rightarrow \lambda_j^{(1)} \otimes \lambda_j^{(2)} \ldots \otimes \lambda_j^{(m)}$$

In general, we can extend $\text{Dec}_A$ to tangles with decorations by an assignment as in the right of diagram (3.72) if the chips are labeled by central, $S$-invariant elements $Q \in A$. The arguments are exactly the same as for $2\pi$-twists, which can be considered special examples of chips. Thus blobs with this labeling are of the same kind as the chips in the previous paragraph.

Now, the coupons were in fact substitutes for 1-handle attachments. Let us therefore impose in $T_{gl}$ also as additional equivalences the $2\pi$-twist at a coupon and the coupon-slide from (3.55). In order for $\text{Dec}_A$ to respect the $2\pi$-twist, we have to require again $v\lambda = \lambda$. The coupon-slide imposes, e.g., $M(\lambda \otimes 1) = (\lambda \otimes 1)$, where $M$ is as in (3.71). Given the $\varepsilon \otimes \text{id}(R) = 1$ and $\varepsilon(v) = 1$ this follows immediately from (2.39). Thus we can add the following:

**Remark 8** $\text{Dec}_A$ is also well defined for tangles with the coupons of 1-handle attachments and other decorations that are labeled by $S$-invariant elements in $Z_A$.

The algorithm to compute $\tau_{HKR}$ for a link, $L$, now proceeds by unknotting its image in $sT(A)$ into a bunch of disjoint circles using the Reidemeister moves, with the singular version $Rm2$ from (3.65). By pushing the elements along each component together and multiplying them, we arrive at only one blob on each component of the link. $\tau_{HKR}$ is finally obtained by evaluation of the corresponding elements against $\mu^R$.

In a more precise language, that is also closer to the strategy by which $\tau_{RT}$ and $\tau_L$ are constructed, the result of the first part of the algorithm, given by the rules in $sT(A)$, can be stated as follows:

**Lemma 10** Suppose $L$ is a closed, projected link with $N$ components, and $L^\text{split}$ is a splitting as described in (3.54). Moreover, let $A$ be a balanced, quasi-triangular Hopf algebra.

Then there exists a unique element

$$\mathcal{I}(L^\text{split}, A) = \sum_j A_j^{(1)} \otimes \ldots \otimes A_j^{(N)} \in A^\otimes N,$$

(3.73)
such that we have the following equivalence in $\mathcal{ST}(\mathcal{A})$:

\[
\text{Dec}_\mathcal{A}(\mathcal{L}^{\text{split}}) \simeq \begin{array}{c}
\mathcal{R}_t \\
\bullet \quad A_j^{(1)} \quad \bullet \quad A_j^{(2)} \quad \ldots \quad \bullet \quad A_j^{(N)}
\end{array}
\]

(3.74)

with corresponding labels at the top-line $\mathcal{R}_t$.

The existence is quite obvious from our description of the algorithm. A proof of uniqueness, i.e., the fact that the elements in (3.74) actually form a basis, can be drawn from the antipode rule for pushing elements through tangle pieces, together with the combinatorics given by isotopies and the new $Rm1$ rule. We shall not go into details here since Lemma 10 follows at once from the relations among categories in the next paragraph.

In this formalism the invariant is now obtained as follows:

**Theorem 9 ([H],[KR])** Suppose $\mathcal{L}$ is an $N$-component link with splitting $\mathcal{L}^{\text{split}}$. Let $\mathcal{A}$ be a balanced, quasi-triangular Hopf algebra with $\check{H}(\mathcal{L}^{\text{split}}, \mathcal{A})$ as in Lemma 10. For a tuple of states, $\mu_j \in \mathcal{A}^*$, with $j = 1, \ldots, N$ define the number

\[
\tau_{\text{HKR}}(\mathcal{L}^{\text{split}}, \mathcal{A}, \bar{\mu}) := \left\langle \mu_1 \otimes \ldots \otimes \mu_N, \check{H}(\mathcal{L}^{\text{split}}, \mathcal{A}) \right\rangle = \sum_j \mu_1(A_j^{(1)}) \cdot \ldots \cdot \mu_N(A_j^{(N)}).
\]

(3.75)

1. Suppose further that the states fulfill the following properties:

   (a) For all $j$, $\mu_j(xy) = \mu_j(S^2(y)x)$, or, equivalently, all $\mu_j$ are invariant under the $\text{ad}^*$-action as in Lemma 3.

   (b) For all $j$, $S^*(\mu_j) = \mu_j$, i.e., $\mu_j(S(x)) = \mu_j(x)$.

Then $\tau_{\text{HKR}}$ is independent of the splitting.

2. Suppose $\mu_j = \mu^R$, i.e., the right integral of $\mathcal{A}$. Then $\tau_{\text{HKR}}$ is invariant under 2-handle slides, and hence can be normalized to a three-manifold invariant.

The claim made in (1.6) of the introduction is now easily verified by direct evaluation of the invariant on $S^1 \times S^2$. A surgery diagram is given by an unframed unknot, $\mathcal{L} = \bigcirc^0$, so that we obtain $\check{H}(\mathcal{L}^{\text{split}}) = 1$, and hence

\[
\tau_{\text{HKR}}(S^1 \times S^2) = \mu^R(1).
\]

It follows from results in [LS] that $\mu^R(1) \neq 0$ is equivalent to $\mathcal{A}$ being cosemisimple. For modular Hopf algebras we have that the invariant is also equal to $\varepsilon(\lambda)$, i.e., what we obtain if we present $S^1 \times S^2$ by a single coupon without strands instead of $\bigcirc^0$. In particular, modular Hopf algebras are semisimple if and only if they are cosemisimple.

We shall see in the next paragraph that the constructions of $\tau_L$ and $\tau_{\text{HKR}}$ are equivalent so that Theorem 3 can be inferred from Lemma 8. Nevertheless, let us sketch a few elements of the combinatorial proof in the category $\mathcal{ST}(\mathcal{A})$:
The \( q \)-\textit{cyclicity} (or \( ad^* \)-invariance) in (a) ensures that for two consecutive elements on a component \( C \), we get the same result, whether we attach the splitting ribbon \( R_C \) before the pair or in between them. A change of sides of the attachment can be thought of an application of the braided antipode \( \Gamma \), which, by using \( q \)-cyclicity, can be reduced to invariance under the ordinary antipode \( S \). Also, changing an over-crossing of a splitting ribbon into an under-crossing can be found from the equation \( S \otimes 1(R) = R^{-1} \), making also use of \( M_{e2} \).

The argument that translates the 2-handle slide into the defining equation of \( \mu^R \) can be found in Kauffman’s work, or it may be derived from the analogous argument for categories used by Lyubashenko in \([L]\), combined with the equivalences shown here. Alternatively, we may also try to check the 1-2-cancellation and modification from (3.55). They translate into the conditions \( \mu(\lambda) = 1 \) and \( (\mathbb{1} \otimes \mu)(\omega) = \lambda \) with \( \omega \in A \otimes^2 \) as in (2.34). The latter was identified in Theorem 5 with the modularity property of \( A \).

3.4) Computing \( \tau_L \) and \( \tau_{RT} \) from \( \tau_{HKR} \): Two Fiber Functors and a Central Element

The strategy to prove the second part of Theorem 1 is to realize that the elements \( \hat{I}(\mathcal{L}) \), defined in Lemma 8 for the construction of \( \tau_L \) and in Lemma 10 for the computation of \( \tau_{HKR} \), are identified with the same vectors by a suitable pair of fiber-functors. For large enough vector spaces this identification becomes one-to-one and we can also relate their evaluations against integrals:

For \( C = \mathcal{A} \mod a \) fiber-functor is of course given by the forgetful functor \( V \) from (2.21). For \( s\mathcal{T}(\mathcal{A}) \) with the same algebra \( \mathcal{A} \), we first have to specify a coloring, \( \mathcal{C}o1 \). This consists again of a choice of directions for the strands of a tangle, and an assignment, \( a \mapsto V_a \). Here we think of \( V_a \) as a vector space, on which we can also declare an \( \mathcal{A} \)-action. In a more formal language, there shall be an object \( X_a \in \mathcal{C} \), such that

\[
V_a = V(X_a) \quad .
\]

(3.76)

The morals behind these formalities are that the objects in \( \mathcal{C} \) shall be thought of as abstract objects as opposed to the vector spaces, \( V_a \), and, furthermore, we wish to permit operations on the \( V_a \), like transpositions of tensor factors, that make no sense for the objects in \( \mathcal{C} \).

We then define a \( \otimes \)-functor

\[
W^{\mathcal{C}o1} : s\mathcal{T}(\mathcal{A}) \longrightarrow Vect(\mathcal{C})
\]

(3.77)
on the generating morphisms of \( s\mathcal{T}(\mathcal{A}) \). An object in \( s\mathcal{T}(\mathcal{A}) \), i.e., a string of labels \( \{a_1, \ldots, a_k\} \), is mapped to the tensor product \( V^\#_{a_1} \otimes \ldots \otimes V^\#_{a_k} \) in analogy to the definition of \( I^{\mathcal{C}o1} \) in (3.49), where \( V^\# \) is now simply the dual space \( V^* \).

Since \( W^{\mathcal{C}o1} \) is supposed to respect \( \otimes \), we map the juxtapositions of two elementary diagrams to the respective tensor products of linear maps. A functor is thus uniquely specified by the image of the elementary diagrams in (3.64).

The assignment for a single strand, carrying an element \( x \in \mathcal{A} \), is simply the application of \( x \) to \( V \) if the direction is downward, and of \( S(x)^* \) to \( V^* \) if it is upward.

To a crossing, \( X : \{a, b\} \rightarrow \{b, a\} \), we simply assign the transposition \( T_{a,b} : V^\#_a \otimes V^\#_b \rightarrow V^\#_b \otimes V^\#_a \) of vector spaces.

The assignments to maxima and minima are as in (3.50). Here \( ev_V \) and \( coev_V \) are the canonical pairing and its inverse, and the flipped morphisms are defined by formula (2.29).

It remains to show that \( W \) assigns the same linear maps to equivalent composites:
Lemma 11 The functor $W^{\text{Col}}$ from (3.77) is well defined for every coloring $\text{Col}$.

The relations that contain no blobs with elements from $A$ are immediate from the fact that $\text{Vect}(C)$ is a symmetric, rigid tensor category with respect to both choices of (co-)evaluations. Also $Me1$ and $Me2$ are obvious from the definition of the action of an element $x \in A$. The moves $Rm1$ and $Me2$ can be checked directly for both choices of directions through the minimum, from (2.29) and the fact that the condition $(a \otimes 1)\theta = (1 \otimes S(a))\theta$ for all $a \in A$ is equivalent to $\theta \in Inv_A(V_1 \otimes V_2)$.

We thus have fiber-functors on both $C$ and $sT(A)$. Recall, that we have also defined functors in (3.49) and (3.68) that start at the category $T$ and end in the latter two categories. The intuitive correspondence, that guided Hennings and Reshetikhin in formulating their rules, can thus be expressed in a more precise language as a commutative diagram of functors that describe different ways of assigning linear maps to purely topological tangles:

Theorem 10 Suppose we have chosen a colorings for $T^{\text{Col}}$ from (3.43) and the one for $W^{\text{Col}}$ obtained from (3.76). Then the following diagram of functors commutes.

\[
\begin{array}{ccc}
\tau_{gl} & \xrightarrow{T^{\text{Col}}} & A-\text{mod} \\
\text{Dec}_A & \downarrow & \downarrow V \\
sT(A) & \xrightarrow{W^{\text{Col}}} & \text{Vector}(C)
\end{array}
\]

(3.78)

Proof: The proof is easy since we only have to check the images of the generators. It is clear that a label $a$ is mapped in both cases to $V_a$ or $V_a^\ast$ depending on direction. The action of an element that lives on a strand in $sT(A)$ on the vector-spaces coincides in both cases with the module action of $A$ on $X_a$ and $X_a^\ast$.

Hence, the linear map $T_{a,b}R_{V_a^\ast,V_b^\ast}$, which we obtain by applying $W^{\text{Col}} \circ \text{Dec}_A$ to a braid on labels $a,b$, coincides with the definition of the commutativity $V(\epsilon(X_a^\ast,X_b^\ast))$ on specific $A$-modules. Consistency for coupons, $2\pi$-twists, and other central elements that may occur as decoration in $T_{gl}$, follows by exactly the same arguments.

Finally, we remark that the (co-)evaluations produced by $W^{\text{Col}}$ are in fact invariant under the corresponding $A$-actions. Thus they can be used to define a rigidity structure on $A-\text{mod}$ (and all such structures are isomorphic).

This result allows us to make the first identification between the computations of $\tau_{HKR}$ and $\tau_L$. Consider now a split link, $L^{\text{split}}$, whose closure represents a manifold. The first step to find $\tau_L$ was to compute the lift of the transformations in (3.52) to the coend. In the particular situation of $C=\text{mod}$ we have $V(F) = A^\ast$ so that the lift $\hat{\mu}(L^{\text{split}}) : F^{\otimes N} \to 1$ is realized by $V$ in $\text{Vector}(C)$ by a linear map $\hat{\mu}(L^{\text{split}},A-\text{mod})' : (A^\ast)^{\otimes N} \to 1$, or, equivalently, by an element

\[
\hat{\mu}(L^{\text{split}},A-\text{mod})' \in A^{\otimes N}.
\]

(3.79)

Recall from Lemma 11 that in the first step of computing $\tau_{HKR}$ we also constructed an element $\hat{\mu}(L^{\text{split}},A) \in A^{\otimes N}$ as in (3.73). Using Theorem 11 we can now conclude equality:

Corollary 11 Let $L^{\text{split}}$ be a split link, and $A$ a balanced, modular, quasi-triangular Hopf algebra. Then

\[
\hat{\mu}(L^{\text{split}},A-\text{mod})' = \hat{\mu}(L^{\text{split}},A).
\]

(3.80)
Proof: To begin with, let us choose a coloring, Co1, of $L^{\text{split}}$ given by a string of objects $X_1, \ldots, X_N$, and the corresponding vector spaces $V_j := V(X_j)$. We now determine the image of $L^{\text{split}}$ in Vect$(C)$ in the both ways suggested by Theorem 10, using the special elements in $A^{\otimes N}$.

Going through $C$, $L^{\text{split}}$ is mapped to the transformations $I^{(X_1, \ldots, X_N)}$ from (3.52), which can be written as a product of the lift $\mathcal{I}$ to the coend and the transformations, $i_{X_j}$. Thus the image in Vect$(C)$ is also the product of $\mathcal{I}'_L = V(\mathcal{I})$ and the maps $V(i_{X_j}) : V_j^* \otimes V_j \rightarrow A^*$, which are by Lemma 10 given as the evaluation of matrix elements. In summary, we have

$$V(I^{(X_1, \ldots, X_N)}(L^{\text{split}})) : V_1^* \otimes V_1 \otimes \ldots \otimes V_N^* \otimes V_N \rightarrow C$$

for any choice of $l_j \in V_j^*$ and $v_j \in V_j$.

In the composite over $\mathcal{W}(A)$ the link $L^{\text{split}}$ is first mapped to a series of $N$ arcs as in (3.74), each with a blob on it, labeled by an element $A_j^{(k)}$ of $A$. It follows easily from the rules that, with directions going from right to left, every one of these arcs is functioned by $W^{\text{co1}}$ into the linear map $ev_V(1 \otimes A) : V^* \otimes V \rightarrow C : l \otimes v \mapsto \langle l, Av \rangle$. We thus obtain the same formula for $W^{\text{co1}}(\text{Dec}_A(L^{\text{split}}))$ as in (3.81), only with $\mathcal{I}'_L$ replaced by $\mathcal{I}'_H$ from (3.73). Theorem 10 therefore implies that the equality (3.80) holds for all matrix elements of any choice of representations. Since, e.g., the projective representation $Q$ from Paragraph 2.1 is faithful, i.e., $A \rightarrow \text{End}_C(Q)$ is injective, this implies the equality of the elements in $A^{\otimes N}$ themselves.

As we remarked in the end of Paragraph 2.5, the categorical integral, $\mu$, of $F$ is identical with the classical, right integral, $\mu^R$, of $A$. We thus have the equality,

$$\tau_L = \langle \mu^{\otimes N}, \mathcal{I}'_L \rangle = \langle (\mu^R)^{\otimes N}, \mathcal{I}'_H \rangle = \tau_H \, ,$$

which completes the proof of Theorem 10. □

Recall, that the element from (3.74) actually defined an intertwiner, $\mathcal{I}'_L : (A^*)^{\otimes N} \rightarrow 1$, with the coadjoint action defined on $A^*$. We thus have as another, immediate application of the correspondence in Corollary 12 the following:

**Corollary 12** The element $\mathcal{I}'_H(L^{\text{split}}, A) \in A^{\otimes N}$, computed from the rules in $\mathcal{W}(A)$, is invariant under the $A$-action, defined by the $N$-fold-tensor product of the adjoint representations.

In particular, for a split knot $K^{\text{split}} (N = 1)$ we obtain a central elements, $\mathcal{I}'_H(K^{\text{split}}, A) \in Z_A$, from Hennings’ rules.

Since a split knot is the same as a one-component tangle the last assertion is exactly about the centrality of the elements computed in [Re] and the problem that L. Kauffman addressed in his lecture.

In Theorem 10 we learned about a modification of the construction of $\tau_L$ that yields $\tau_R$, even if $C$ is not semisimple. Let us use the equivalence between $\tau_L$ and $\tau_{HKR}$ in order to derive an analogous rule in the Hennings picture.

In the modification for general categories, we considered links, $L^Q$, where we have decorated each component with a chip labeled by a natural transformation of the identity, $Q$. By Lemma 10 this corresponds for $C = A - mod$ to the application of a central element

$$Q \in Z_A \, .$$
Moreover, condition \((3.61)\) translates into \(S\)-invariance of \(Q\). By Remark \(8\) and the rule indicated in \((3.60)\) we can extend the functors in Theorem \(1\) to tangles with decorations. If the corresponding natural transformation for \(f^{\text{cot}}\) is defined by application of the central element used for insertions in \(\text{Dec}_A\) it is clear that \((3.78)\) still commutes.

This implies that the invariant \(\tau_L(\mathcal{L}^Q)\) of a decorated link can be computed equivalently from \(\tau_H(\mathcal{L}^Q)\), generalizing Theorem \(1\). Before we use this to reformulate the result in Theorem \(3\), let us give a definition of the central element \(Q\) in the setting of an ordinary Hopf algebra:

The definition from \((3.59)\) translates to \(\mu^{\text{semis}} = \mu^R \hookrightarrow Q\), where we used the sum of quantum-traces from \((3.57)\). By virtue of \((2.28)\), we can characterize \(Q\) uniquely by the following concrete formula

\[
\mu^R(Q \cdot y) = \mathcal{D}^{-1} \sum_{j \in J_0} d(j) \text{tr}^\text{can}_{V_j}(G \cdot y) \quad \forall y \in A. \quad (3.82)
\]

The sum runs over a complete set of inequivalent, irreducible representations, \(V_j\), of \(A\), \(G\) is the balancing, \(\text{tr}^\text{can}\) is the canonical trace, and \(d(j) = \text{tr}^\text{can}_{V_j}(G)\). Since the \(\text{tr}^\text{can}\) \(\hookrightarrow G\) and \(\mu^R\) are all \(q\)-cyclic, and since \(\mu^R\) is non-degenerate, it follows that any solution, \(Q\), of \((3.82)\) is central. Because of \(d(j) = d(j')\) and the properties of the quantum-traces, we have that \(\mu^{\text{semis}}\) fulfills \(S^*(\mu^{\text{semis}}) = G^{-2} \hookrightarrow \mu^{\text{semis}}\). It follows from results in [Rd], see also [Ke2], that the same equation holds for \(\mu^R\). This can be used to show that \(Q\) is invariant under the antipode \(S\).

**Corollary 13** Suppose \(A\) is a balanced, quasi-triangular Hopf algebra and \(Q \in Z_A\) is as in \((3.82)\). Then

\[
\tau_{\text{RT}}(\mathcal{L}, A) = \tau_{\text{HKR}}(\mathcal{L}^Q, A),
\]

where the additional rule for evaluating the decorated tangle, is to insert the element \(Q\) in each component of the link in the \(\mathcal{ST}(A)\)-picture.

As opposed to expressing the relation between \(\tau_{\text{RT}}\) and \(\tau_{\text{HKR}}\) in a form analogous to \((3.58)\), the picture given above lends itself much better to explain relations of the corresponding TQFT’s, as we shall indicate in the next paragraph.

Quite obviously we have \(Q=1\) if \(A\) is semisimple, since the formula \(\mu^{\text{semis}}\) was computed for a general semisimple category. The converse is also true. If \(A\) is not semisimple it follows that \(\lambda^2 = 0\) so that we must have \(\text{tr}^\text{can}_{V_j}(G \cdot \lambda) = 0\). Thus if \(Q = 1\), i.e., if \(\mu^R\) is given by a sum of traces, this implies \(\mu^R(\lambda) = 0\). But that is not possible by the Lemma \(3\)

In fact a stronger statement holds, namely that \(Q\) is always nilpotent:

**Lemma 12** Suppose \(A\) is a non-semisimple, balanced, finite dimensional, quasi-triangular Hopf algebra, and \(Q\) is as above. Then

\[
Q \cdot Q = 0. \quad (3.83)
\]

**Proof:** Using the \(S\)-invariance of \(Q\) and a relation from [Rd], similar to the one from Lemma \(3\), we find that \(Q = \mu^{\text{semis}} \otimes 1(\Delta(\lambda))\), which can be rewritten as a weighted sum over the elements \(Q_j = \sum_{\nu} \text{tr}_j^\nu(\lambda''_{\nu})\lambda''_{\nu}\) for \(j \in J_0^*\). Using the identity \((1 \otimes S(a))\Delta(\lambda) = (a \otimes 1)\Delta(\lambda)\), and the multiplicativeness of the quantum-traces, we find for these elements

\[
S(Q_j)Q_k = \sum_{j, j' \otimes k} \text{tr}_{j \otimes k}^\nu(\lambda''_{\nu} \otimes \lambda''_{\nu}) S(\lambda''_{\nu})\lambda''_{\nu} = \sum_{\nu} \text{tr}_{j \otimes k}^\nu(\Delta(\lambda)(1 \otimes \lambda''_{\nu}))\lambda''_{\nu}. \]

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Now, the image of $\Delta(\lambda)$ on $V_j \otimes V_k$ is in the invariance, which is zero, unless $j \cong k^\vee$. Moreover, if $d(k) \neq 0$ the invariance is a direct summand of $V_k^\vee \otimes V_k$ so that $\Delta(\lambda)$ acts as $\varepsilon(\lambda) = 0$. It follows that $S(Q)Q = Q^2 = 0$.

We shall given a more explicit form of $Q$ in the case of $U_q(sl_2)$ in the next paragraph.

### 3.5) Generalization of Hennings’ Rules to TQFT’s

The extend, to which it is possible to construct TQFT’s from non-semisimple categories, has been explained in Paragraph 1.5. In particular, Theorem 2 together with the relation between $\tau_L$ and $\tau_{HKR}$ shows that there exists a TQFT for connected surfaces that specializes to $\tau_{HKR}$. The aim of this paragraph is to outline a set of rules, which generalize those of $\mathfrak{sl}_2(A)$, and which allow us to compute the linear maps associated to a cobordism.

From these rules we derive representations of the mapping class groups of the torus, that had been discovered before in [FLM]. In the work of Lyubashenko and Majid the formulas were obtained using rather similar ideas as the ones presented here. Quite interestingly, they also appeared in works of Felder et al. on local systems that arise in the study of integral representations of conformal blocks.

We shall review the detailed structure of these representation, that was obtained in [Ke2] for the case of $U_q(sl_2)$, and derive from this a relation between $\tau_{HKR}$ and $\tau_{RT}$ for lens spaces. We also show how the special element $Q$ can be used to directly relate the (inequivalent) representations arising in both pictures.

Let us sketch first a few ingredients of the combinatorial representation of the category $\text{Cob}_3^{conn}$ in terms of tangles, as proposed in [Ke5]. Here $\text{Cob}_3^{conn}$ is the category of connected three-manifolds, $M$, (or more precisely a central extension by $\Omega_4$ thereof) that cobord one connected surface to another connected surface, i.e., $\partial M \cong -\Sigma_{g_1} \amalg \Sigma_{g_2}$ so that $M : [g_1] \rightarrow [g_2]$. (compact and orientable always assumed).

A tangle $\mathcal{T}_M$ in $\mathcal{T}_{gl}$ that represents $M$ will have $2g_1$ endpoints at the top-line $R_t$ and $2g_2$ endpoints at the bottom-line $R_b$. In one version of the presentation-calculus every component of an admissible tangle must be one of the following three types:

1. A top-ribbon, that starts at the $2j$-th position at $R_t$ and returns to the $2j - 1$-st position at $R_t$.

2. a bottom-ribbon, that starts at the $2j$-th position at $R_b$ and returns to the $2j - 1$-st position at $R_b$, or

3. a closed ribbon, that is disjoint from the boundaries of the diagram.

Hence there are exactly $g_1$ top-ribbons, $g_2$ bottom-ribbons, and any number, $N$, of closed ribbons. Moreover, the label structure of $\mathcal{T}_M$ as a morphism in $\mathcal{T}_{gl}$ is as follows:

$$\mathcal{T}_M : \{a_1, a_1, \ldots, a_{g_1}, a_{g_1}\} \rightarrow \{b_1, b_1, \ldots, b_{g_2}, b_{g_2}\}$$

It will be convenient to use a slightly larger space, where we admit in addition pairs of through-ribbons that can be either regular or crossed. A pair of such ribbons is associated to a pair of labels, $(a_j, b_k)$, one from the top, one from the bottom. In a regular pair of through-strands one strand connects the first $a_j$-label at $R_t$ to the first $b_k$-label at $R_b$ and the second starts and ends at the other two labels. For a crossed through-pair we have one ribbon starting at the first $a_j$-label and ending at the second $b_k$-label and the other ribbon connects again the remaining two labels.
The moves for surgery presentations of closed manifolds shall also apply to regions of tangles in $\mathcal{T}gl$ that stay away from the boundaries of the diagram. In addition we have two moves in the vicinity of the boundary. The first is what we call the $\sigma$-move, which is depicted below in the version with coupons:

![Image of $\sigma$-move diagram]

Invariance of the general construction in [KL] under the $\sigma$-move translates directly to inversion-formulae as the one in Lemma 3. Under strong enough modularity conditions, or the canonical choices for $\mu$ and $\lambda$, the invariance with respect to $\sigma$-moves follows therefore immediately.

We also have the $\tau$-move, which is basically a special isotopy of tangles over the sphere through the point at infinity. It allows us to push a strand, that is parallel and close to $R_t$, and that crosses all strands emerging from the top-line, to the back of everything, as it is indicated in the next diagram:

![Image of $\tau$-move diagram]

As a matter of fact, the omission of this move would give us a tangle category that is equivalent to $\text{Cob}_3^{conn}(1)$, i.e., cobordisms between surfaces with one hole.

In [KL] we now construct a linear map for every cobordism $M$, represented by a tangle $\mathcal{T}_M$. The first step is, again, to consider a split tangle $\mathcal{T}^{\text{split}}$. In this generalization we attach one splitting-ribbon to every closed and every bottom-ribbon. The top- and through-ribbons are not split.

Hence, if $\mathcal{T}$ has $N$ closed ribbons, $M$ pairs of through-ribbons, $g_1-M$ top-ribbons, and $g_2-M$ bottom-ribbons, then the split tangle $\mathcal{T}^{\text{split}}$ will have $g_2$ pairs of through-ribbons, $(g_1-M)+N$ top-ribbons, no bottom-ribbons, and no closed ribbons. If we assume for simplicity that the labels of the through-ribbons in $\mathcal{T}$ are all to the left of the labels of the top-ribbons, we obtain a schematic label structure:

$$\mathcal{T}^{\text{split}} : \{L^{M}_\text{thr.}, L^{g_1-M}_{\text{top}}, L^{g_2-M}_{\text{bot}}, L^N_{\text{cl.}}\} \rightarrow \{L^{M}_{\text{thr.}}, L^{g_2-M}_{\text{bot.}}\},$$

where $L^n_\#$ has $2n$-labels associated to components of $\mathcal{T}$ with characteristic $\#$. 

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In the next step we construct again morphisms in \( \mathcal{C} \) by choosing colorings, \( a \to X_a \), and check their (di-)naturality properties. They allow us to lift everything to a single morphism

\[
\hat{\mathbb{A}}(\mathcal{T}^{\text{split}}) : F^{\otimes M} \otimes F^{\otimes (g_1-M)} \otimes F^{\otimes (g_2-M)} \otimes F^{\otimes N} \to F^{\otimes g_2}.
\]

¿From this we obtain a morphism \( \hat{\mathbb{I}}_L(\mathcal{T}) : F^{g_1} \to F^{g_2} \) by multiplying with \( \mathbb{I}_{F^{g_1}} \otimes \mu^{\otimes (g_2-M+N)} \) from the right. If the tangle represented a cobordism in \( \text{Cob}_3^1(1)^{\text{conn}} \), this would already be the construction of the morphism in \( \mathcal{C} \), that is associated by the TQFT-functor \( \mathcal{V}_1 \).

In order to get a TQFT for closed surfaces, we also have to take care of the \( \tau \)-move. As explained in Paragraph 1.6 from general arguments, this can be done by composing \( \mathcal{V}_1 \) with the invariance. The vector space associated to a surface \( \Sigma_g \) is then \( \text{Inv}_{\mathcal{C}}(F^{\otimes g}) \).

Our aim here is to find a functor that is explicitly described by Hopf algebra elements. We shall therefore construct instead the contravariant functor given by application of the coinvariance:

\[ \mathcal{V}_0^* = \text{Coinv} \circ \mathcal{V}_1 \]

The functor, \( \mathcal{V}_0^* \), respects now the \( \tau \)-move in the same way as \( \mathcal{V}_0 \) does. The vector space associated by \( \mathcal{V}_0^* \) to \( \Sigma_g \) is the \( \text{ad} \)-invariant subspace \( (A^{\otimes g})_{\text{ad}} \) of \( A^{\otimes g} \). The image of a tangle under \( \mathcal{V}_0^* \) is then obtained by restricting the \( \hat{\mathbb{A}}(\mathcal{T})^* : A^{\otimes g_2} \to A^{\otimes g_1} \) to the \( \text{ad} \)-invariant subspaces. The rules to compute the \( \hat{\mathbb{A}}(\mathcal{T})^* \) are the following:

We start from a split, \( \mathcal{T}^{\text{split}} \), of the representing tangle as above and map it to an element in \( s\mathcal{T}(A) \) by applying \( \text{Dec}_{\mathcal{A}} \) from (3.68). It is not hard to see that by invoking the relations of \( s\mathcal{T}(A) \) we can bring the singular tangle in the form described in the diagram below:

\[
\begin{array}{ccc}
M & & g_1-M & & g_2-M & & N \\
\end{array}
\]  

(3.86)

Here the strands of each of the \( g_2 \) pairs of through-strands are straight and parallel lines. Hence the diagram is uniquely characterized by a permutation \( \pi \in S_M \times S_{g_2-M} \) of label-pairs.

In the small strip at the top line we are allowed to have insertions of \( \mathcal{A} \)-elements and crossings of a pair of crossed through-strands. They are depicted below:

\[
P \qquad q \quad \sim x \mapsto S(p)xq , \qquad \sim x \mapsto S(x)G .
\]  

(3.87)

We consider possible insertions at the crossing as a composite of the two diagrams in (3.87). For the minima we can use relations in \( s\mathcal{T}(A) \) to assume a form as in diagram (3.74). Note that strands with different colorings, but also the strands of a pair of regular through-strands, do not cross in the upper part in this normal form.
A map $\mathcal{A}^{g_2} \to \mathcal{A}^{(M+(g_1-M)+(g_2-M)+N)}$ may now be obtained by thinking of a pair of through-strands as a single strand, and a diagrams as in (3.87) as one insertion. The translation into maps on tensor products of $\mathcal{A}$ is then as for the functor $W^{\text{co}}$, but now with morphisms going from bottom to top.

Specifically, we first apply to an element in $\mathcal{A}^{g_2}$ the permutation $\pi$ indicated in (3.86). The morphism for a fixed summation index of the insertions is further described by applying the maps associated in (3.87) to each tensor factor (Note, that $G$ on the right side is the balancing so that the operation associated to a crossing is in fact an involution). Finally, we also insert the elements in $\mathcal{A}^{(g_1-M)}$ and $\mathcal{A}^{\otimes N}$, defined as in Lemma 10, into the respective places in the tensor product, and take the sum of the morphisms over all labels.

This yields in fact a contravariant representation of $\text{Cob}_3^{\text{conn}}$. Generalizing the arguments of the previous chapter and using the results from [KL] we thus find the following:

**Theorem 14** The algorithm described above yields a well defined, contravariant functor

$$\mathcal{V}^*_0 : \text{Cob}_3^{\text{conn}} \to \text{Vect}(C)$$

with $\mathcal{V}_0^*(\Sigma g) \cong (\mathcal{A}^{g})_{ad}$.

It is dual to the functor constructed in [KL], and specializes to $\tau_{\text{HKR}}$ for closed manifolds.

In the remainder of this chapter let us consider as an application the case of the mapping class group $\pi_0(\text{Diff}(\Sigma_1)^+)$ of the torus. The standard generators, denoted by $T$ and $S$, are realized as invertible cobordisms $\Sigma_1 \to \Sigma_1$ by the following tangles:

$$T = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{tangleT.png}
\end{array} \quad S = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{tangleS.png}
\end{array}. \quad (3.88)$$

By a straightforward application of the above rules we find that the maps $T^*, S^* : \mathcal{A} \to \mathcal{A}$, that are associated by our algorithm to these tangles, have the form:

$$T^*(x) = xv = vx, \quad \text{and} \quad S^*(x) = \sum_j \omega_j(1) \mu(x \omega_j(2)), \quad (3.89)$$

where $v$ is the central balancing element from (3.71) and $\omega = \sum_j \omega_j(1) \otimes \omega_j(2)$ is as in (2.33). This representation of $\pi_0(\text{Diff}(\Sigma_1)^+)$ is equivalent to the ones found in [FLM].

As we remarked following its definition, the form $\omega$ has particularly nice properties if $\mathcal{A} = D(B)$ is a Drinfel’d double. In [Ke2] we find in this case a factorization, given by the following commutative diagram. In the last chapter we will attempt to relate this to a conjecture on relations between three-manifold invariants.

$$\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{\beta_1 \circ S^{-1} \otimes S \circ \beta_1} & \mathcal{B} \otimes \mathcal{B} \\
D(B) & \xrightarrow{S^*} & D(B)
\end{array} \quad (3.90)$$
Here, the vertical arrows are the isomorphisms given by multiplication in $D(\mathcal{B})$. The canonical maps in the upper horizontal arrow are defined by $\beta_l(x)(y) = \mu^L(yx)$, and $\beta_r(\rho) = \rho \mapsto \lambda^L$, where $\mu^L$ and $\lambda^L$ are the left integral and cointegral of $\mathcal{B}$.

In order to obtain maps for the closed torus we have to restrict $T^*$ and $S^*$ to the center $Z_A$ of $A$. The projective representations of $SL(2, \mathbb{Z})$ we get in this way from $U_q(sl_N)$ at a $p$-the root of unity are closely related to the representations we obtain from the Fourier transformations on the group $\mathbb{Z}/p$ and its powers. If $p$ is an odd prime this representation has two irreducible factors, given by the eigenspaces of the inversion-map:

$$C(\mathbb{Z}/p) = Y_{\frac{p+1}{2}}^+ \oplus Y_{\frac{p-1}{2}}^-,$$

where the subscripts indicate the dimensions.

Theorem 15 Suppose $\mathcal{A} = U_q(sl_2)$ with $q$ a primitive $p$-th root of unity and $p$ an odd prime. Then the (projective) representation of $SL(2, \mathbb{Z})$ on the center of $\mathcal{A}$, which is obtained from the TQFT described above, has the following structure:

$$Z_A = Y_{\frac{p+1}{2}}^+ \oplus \Pi \otimes Y_{\frac{p-1}{2}}^-.$$

In this decomposition the $Y$’s are the finite representations from (3.91), and $\Pi$ is the restriction of the two-dimensional fundamental representation of $SL(2, \mathbb{R})$.

Although this result was obtained mainly by involved computations, it is possible to give some algebraic meanings to the constituents of $Z_A$. To begin with, we know that in the pairing of invariance and coinvariance of an object in a non-semisimple category is usually degenerate. The null-space of this pairing is certainly an invariant subspace of the representation.

In our example this subspace turns out to be precisely the summand $Y_{\frac{p+1}{2}}^+$. A more concrete characterization results from the fact that if we consider $\mathcal{A}$ as an $\mathcal{A}$-module with adjoint action the invariance is $Z_A$ and the coinvariance consists of the $q$-cyclic states. $Y_{\frac{p+1}{2}}^+$ is thus given by the central elements on which every $q$-cyclic state vanishes.

It is also known that the representations we obtain from the semisimple picture are equivalent to the $Y$’s.

By its very property the null-space does not contribute to the invariants of closed manifolds so that it suffices to consider the respective factor representation. In the situation of Theorem 15 this is also a direct summand, and the generators are explicitly represented by the following matrices:

$$T_H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes T_{RT}, \quad S_H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes S_{RT}$$

(3.92)

Here, the subscript $RT$ indicates, as usual, that the matrices are those that we obtain from the TQFT associated to the semisimple quotient of $\mathcal{A} – mod$.

Recall from Corollary 13 that the computation of $\tau_{RT}$ can be done in the Hennings picture if we insert at each component of the link the special element $Q$, that was defined in (3.82).

Now, if a link is given by the composition of many admissible tangles, with ribbon types defined as in the beginning of this paragraph, then there is a one-to-one correspondence between

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the closed and bottom ribbons of the composites and the components of the link. Thus, if we
define a modified Hennings procedure for TQFT’s by inserting one $Q$ at every bottom or closed
ribbon in a given tangle, then the product of the assigned linear maps gives us $\tau_{RT}$.

It is, however, not true that this assignment defines a TQFT, since there are degeneracies so
that we usually do not preserve invariance under the $\sigma$-move. Often, this can be salvaged by
taking proper quotients of the vector spaces of the original TQFT:

We know from Lemma 12 that $Q$ is nilpotent. In fact, multiplication by $Q$ it is represented for
our special example (up to a scalar) on the second summand in Theorem 15 by

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes I$$

If we wish to apply our insertion procedure to the $T$-generator, we immediately see that nothing
has to be done, since we have only through-strands, i.e., $T^Q_H = T_H$. However, the $S$ generator
contains a bottom ribbon:

$$S^Q_H = S_H \cdot Q = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \otimes S_{RT}$$

It is quite obvious that the matrices $T^Q_H$ and $S^Q_H$ do not define a representation of $SL(2, \mathbb{Z})$
anymore. Also, while $S_H$ and $T_H$ define an irreducible representations, the matrices $T^Q_H$ and $S^Q_H$
have $\ker(S^Q_H) = \ker(Q)$ as a common subspace. If we divide by this subspace we get again a
representation of $SL(2, \mathbb{Z})$, namely exactly the one defined by $S_{RT}$ and $T_{RT}$.

Now, in a TQFT as in Theorem 14 we associate to a full torus the map $C \to \mathbb{Z}$ whose image
is the unit $1$, and to its complement in $S^3$ the counit $\varepsilon : \mathbb{Z} \to C$. Hence if an element of the
mapping class group is represented by $\psi : \mathbb{Z} \to \mathbb{Z}$, then the $\tau_{HKR}$-invariant of the lens-space $L(\psi)$
is $\varepsilon(\psi(1))$.

In the relevant part of the presentation of Theorem 15 we find that the units are given by

$$1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \mu \quad \varepsilon = [1, 0] \otimes \lambda,$$

where $\mu$ is the column vector whose components are the $d(j)$ and $\lambda$ is the projection on the trivial
object. (The switch from units to integrals in the semisimple part occurs since we also switched
from a covariant to a contravariant picture). From the factorized form of the matrices and vectors
we find:

**Corollary 16** Suppose $\mathcal{A} = U_q(sl_2)$ with $q$ a primitive, (prime) root of unity, and let $L(\psi)$ denote
the lens-space associated to an element $\psi \in \pi_0(Diff(S^3))$.

Then

$$\tau_{HKR}(L(\psi)) = \rho(L(\psi)) \cdot \tau_{RT}(L(\psi))$$

Here, $\rho(M)$ is the order of $H_1(M, \mathbb{Z})$ if that is finite, and zero else wise.

The basic form of this result is consistent with computations in [KR] for $L(k, 1)$ and $q$ an
8-th root of unity, especially the linear dependence on $k = \rho(M)$. In particular, we see here
that, even for fixed finite $\mathcal{A}$, the invariant $\tau_{HKR}$ is unbounded. This is very different from the
known semisimple invariants that are all bounded. The result also suggests that $\tau_{HKR}$ might be
computable from homological data and its semisimple counterpart also in more general situations.
We shall leave this as open problem.
4) Open Questions, and Relations with Cellular Invariants

In the last chapters we tried to give a survey over those quantum-invariants that are obtained from surgery presentations of 3-manifolds, but use different algebraic tools for the construction. A similar pattern of relations can be detected in the zoo of cellular invariants, i.e., invariants that use presentations of the manifold in terms of a simplicial subdivision, or the cell decomposition given by a Heegaard diagram. Since the theory is not as developed here as in the surgical case (and also since the author is not a true expert on the subject) we shall only pose a number of questions that arise naturally when we compare the two families. There are also relations of two different types between cellular and surgical invariants that we shall briefly talk about.

The constructions are all in some way connected to an abelian, balanced tensor category. Notions of braiding or modularity are not needed. In [TV] 6-j-symbols, which are given as the structure constants of $\mathcal{A}^{-mod}$, where $\mathcal{A} = U_q(\mathfrak{sl}_2)$, are used to define state sums over colorings of the edges of a simplicial complex, representing $M$, which turn out to be invariants.

The basic reason for invariance is the correspondence between subdivision-moves and the Biedenharn-Elliott (or pentagonal) relations. This observation has already been made earlier in a non-rigorous setting by Ponzano and Regge [PR]. Let us denote in general an invariant that is obtained in this way from semisimple, balanced tensor category, $\mathcal{C}$, by $\tau_{TV}(M, \mathcal{C})$.

In [DW] Dijkgraaf and Witten formulated a lattice version of a topological field theory associated to a finite gauge group, $G$, and a 3-cocycle, $\alpha$, of $G$, such that the partition function is the order of $\text{Hom}(\pi_1(M), G)$, when $\alpha = 1$. This is in fact another version of $\tau_{TV}$, if we set $\mathcal{C} = C(G) - mod$, with a possibly non-trivial associativity constraint given by $\alpha$, (which is in fact a special 6-j-symbol).

Generalizations of this construction to a more categorical - but still semisimple - context have been undertaken by many people since. See, e.g., [D...].

Another approach is due to G. Kuperberg, [Ku], who associates to the handle attachments of a Heegaard diagram the multiplication and comultiplication morphisms of a Hopf algebra, $\mathcal{A}$, which again does not have to be quasi-triangular. The invariant $\tau_{Ku}(M, \mathcal{A})$ can also be defined on a “combed” manifold, $M$, if $\mathcal{A}$ is not semisimple (and even without balancing if we consider framed manifolds, where the notion of balancing in [Ku] differs from the one in the braided case).

A relation of the form

$$\tau_{TV}(M, \mathcal{A} - mod) = \tau_{Ku}(M, \mathcal{A})$$

has been proven by [BW]. This and also the general ingredients in the constructions suggest that $\tau_{TV}$ and $\tau_{Ku}$ relate to each other in quite the same way as $\tau_{RT}$ and $\tau_{HKR}$ do. Our first question is thus, whether there is a counterpart to $\tau_L$:

**Question 1** *Is it possible to define for an abelian, balanced tensor category, $\mathcal{C}$, (that is not necessarily semisimple or braided) an invariant $\tau_X(M, \mathcal{C})$, which is constructed from a cell-decomposition of $M$, and for which we have the specializations $\tau_{TV} = \tau_X$ if $\mathcal{C}$ is semisimple, and $\tau_{Ku} = \tau_X$ if $\mathcal{C} = \mathcal{A} - mod$?*

Topological quantum field theories for $\tau_{TV}$ exists by quite general arguments. They are constructed by first associating vector spaces of “admissible” colorings of a triangulation of a surfaces $\Sigma$. We then use the state sums to define a “pre-TQFT”, which respects compositions but not
necessarily the identities. Reduction of the vector spaces by the projections associated to the cylinders $\Sigma \times [0, 1]$ then yields an actual TQFT. Extending the construction to more general simplicial subdivisions, it is shown in [KS] that a basis of the vector space associated to a surface is given by admissible colorings of the pair of spines that belong to the two handle bodies bounding $\Sigma \subset S^3$. Using the coend-object $F$ in the semisimple setting as in Lemma 2 the result may be expressed in the following form:

$$\mathcal{V}_{TV}(\Sigma_g) \cong Inv_C(F^{\otimes g}) \otimes Inv_C(F^{\otimes g})^* \cong End_C(Inv_C(F^{\otimes g})) \quad (4.93)$$

This formula makes sense in any tensor category with limits, even if it is not semisimple and has no braiding. Although their possibility has been mentioned, a construction of TQFT’s in the formalism of [Ku] is still missing. Moreover, we have $\tau_{\text{Ku}}(S^2 \times S^1) = 0$ if $\mathcal{A}$ is not semisimple, for the same reasons why $\tau_{\text{HKR}}$ vanishes. The next question is basically about filling this gap for the cellular situation as it had been done for the surgical in [KL]:

**Question 2** Is there a (extended) TQFT, $\mathcal{V} : \text{Cob}^{conn}(\ast)_3 \to \text{Vect}(\mathcal{C})$ (or $\to \text{AbCat}$), that specializes to $\tau_X$ for closed manifolds (or at least to $\tau_X = \tau_{\text{Ku}}$)?

It has been brought forward in [T] and [Wa] that the $\tau_{TV}$ invariant is the norm-square of $\tau_{RT}$ in the case of semisimple categories with hermitian structures. In fact, an argument of J. Roberts in [R] is easily generalized to semisimple categories (without *-structures) to give the identity:

$$\tau_{TV}(M, \mathcal{C}) = \tau_{RT}(M \#(-M), \mathcal{C}) = \tau_{RT}(M, \mathcal{C})\tau_{RT}(-M, \mathcal{C}) \quad (4.94)$$

The basic strategy is to replace the 3-dim handle attachments, given by a Heegaard diagram, by 4-dim handle attachments of same index, and identify the connected sum with the boundary of the resulting four-fold. The application of $\tau_{TV}$ to the Heegaard diagram is via a “chain-mail invariant” shown to be the same as $\tau_{RT}$ for the substituted link.

Notice that we need a braided category in order to determine $\tau_{RT}$, which is superfluous data for the computation of the left hand side of the equation. In any case we need that $\mathcal{C}$ is semisimple. Yet, it seems that this property is needed in the proofs only for technical reasons.

**Question 3** Is is possible to prove a relation between $\tau_X$ and $\tau_L$ (or at least between $\tau_{\text{Ku}}$ and $\tau_{\text{HKR}}$) that generalizes the one from (4.94)?

Notice that the $Inv_C(F^g)$-spaces that appear in (4.93) are those associated by the surgery TQFT to a surface. In particular, we can rewrite it as $\mathcal{V}_{TV}(\Sigma) = \mathcal{V}_{RT}(\Sigma \amalg (-\Sigma))$. There is also a natural identification of the two bases since the $\mathcal{V}_{RT}$-vector spaces can also be constructed from colorings of spines.

This situation for vector spaces has obvious similarities with (4.94), (we may replace $\#$ by $\amalg$), and the squaring of dimensions is consistent with the relation of the invariants for $S^1 \times \Sigma$. For these reasons it seems to be likely that there is a TQFT-generalization of (4.94).

**Question 4** Is there a topological “doubling”-functor $D : \text{Cob}_3 \to \text{Cob}_3$, which maps a surface $\Sigma$ to $\Sigma \amalg -\Sigma$ and a closed manifold $M$ to $M \amalg -M$ such that:

$$\mathcal{V}_{RT} \circ D = \mathcal{V}_{TV}$$

for a fixed braided category $\mathcal{C}$? What about non-semisimple generalizations?
Constructive answers to the above questions would give us a way to compute all known quantum-invariants and the corresponding TQFT’s from $\tau_L$, if we consider only cellular invariants that start from braided categories. Finding proofs should only be a matter of gaining a thorough and sufficiently deep understanding of the existing constructions.

There is, however, also an anticipated relation between cellular and surgical invariants that would allow us also to compute the cellular ones for non-braided categories from $\tau_L$. Unlike the previous relation there are still a few conceptual ideas missing to make it work in full generality.

The relation is with respect to a fixed manifold $M$. Instead we change the algebraic data in a non-trivial way. Specifically, if $\tau_{Ku}$ is defined for a Hopf algebra $B$ we can define $\tau_{HKR}$ for the corresponding Drinfel’d double $D(B)$, since this is quasi-triangular and, moreover, always modular. The notion of doubles has been extended by Majid [M2] to balanced, monoidal categories $\mathcal{C}$, such that $D(\mathcal{C})$ is a balanced, modular, braided tensor category. Following a conjecture of Gelfand and Kazhdan, we thus ask the next question.

**Question 5** Is there a relation of the form

$$\tau_L(M, D(\mathcal{C})) = \tau_X(M, \mathcal{C})$$

for general balanced, abelian tensor categories? What about TQFT’s?

There is in fact more than just aesthetical evidence for this fact:

First of all we already know that this relation is true for finite groups, comparing the results from [DW] for $\tau_{TV}(G)$ and from [AC] for $\tau_{RT}(D(G))$, where the answer is in both cases the order of $Hom(\pi_1(M), G)$. The vector spaces of $\mathcal{V}_{TV}(G)$ and $\mathcal{V}_{RT}(D(G))$ are identified by the easy but remarkable correspondence $C\left(\frac{Hom(\pi_1(\Sigma_g), G)}{G}\right) \cong \left(D(G)^\otimes\right)_{ad}$, which is given on representatives by

$$\phi \mapsto \sum_{\gamma \in Hom(\pi_1(\Sigma_g), G)} \phi(\gamma) \gamma(a_1)\delta_{\gamma(b_1)} \otimes \ldots \otimes \gamma(a_g)\delta_{\gamma(b_g)}$$

Here, the $a_j$ and $b_j$ are the usual generators of $\pi_1(\Sigma)$, and the basis vector $\delta_a \in C(G)$ is the characteristic function of $\{a\}$.

In the case of a general algebra, $\mathcal{A}$, we expect for $\mathcal{V}_{Ku}$ to associate an algebra, $\mathcal{A}$, to a generating $a$-cycle of $\Sigma$ and the dual $\mathcal{A}^*$ if the cycle lives in an opposite Lagrangian subspace. (Let us omit the passage to invariances, i.e., imagine we have introduced a puncture.) This would then reproduce the $\mathcal{V}_{HKR}$ vector spaces.

This also fits in nicely with the actions of the action of the $S^*$ and $T^*$ matrices for doubles, from (3.89). As the $S$-transformation flips $a$-cycles into $b$-cycles we expect it to be represented by a permutation and other canonical isomorphisms. From the $\mathcal{V}_L$-picture this is confirmed by the result in (3.90).

The relation can obviously be combined with the previous one, giving rise to a formula for $\tau_L$ only, with $\mathcal{C}$ a braided category:

$$\tau_L(M, D(\mathcal{C})) = \tau_L(M \amalg -M, \mathcal{C})$$

(4.95)

This would imply interesting formulae for quasi-triangular Hopf algebras. For example the dimension of vector spaces would obey $dim\left(Z_{D(\mathcal{A})}\right) = dim\left(Z_{\mathcal{A}}\right)^2$, which is wrong for plenty of examples of Hopf algebras, $\mathcal{A}$, that are not quasi-triangular.
More abstractly, if we think of a TQFT as a functor, $\text{Cob}_3 \times \text{AbCat} \rightarrow \text{AbCat}$, the relation in (4.95) suggests that there are topological operations, like the “doubling”-functor from Question 4, which are dual to the doubling of abelian categories. Let us thus conclude with the following esoterical question:

**Question 6** Is it possible to describe a topological doubling in a way that it is dual to the doubling of abelian categories, and consistent with an extended $V_L$?

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