Improved Brauer-Type Eigenvalue Localization Sets for Tensors with Their Applications

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Abstract. In this paper, by excluding some sets from the Brauer-type eigenvalue inclusion sets for tensors developed by Bu et al. (Linear Algebra Appl. 512 (2017) 234-248) and Li et al. (Linear and Multilinear Algebra 64 (2016) 727-736), some improved Brauer-type eigenvalue localization sets for tensors are given, which are proved to be much tighter than those put forward by Bu et al. and Li et al. As applications, some new criteria for identifying the nonsingularity of tensors are developed, which are better than some previous results. This fact is illustrated by some numerical examples.

1. Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, $n$ be a positive integer with $n \geq 2$, and $N = \{1, 2, \ldots, n\}$. The tensor $A = (a_{i_1 \ldots i_m})$ is called a complex (real) order $m$ dimension $n$ tensor, denoted by $A \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if $a_{i_1 \ldots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j \in N$ for $j = 1, 2, \ldots, m$ [17].

The tensor $A \in \mathbb{R}^{[m,n]}$ is called the unit tensor [14], denoted by $I$, if its entries $\delta_{i_1 \ldots i_m}(i_1, \ldots, i_m \in N)$ satisfy the following conditions:

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \ldots = i_m, \\ 0, & \text{otherwise,} \end{cases}$$

and for $x \in \mathbb{C}^n$. $A^x^{m-1}$ is a column vector of dimension $n$ and its $i$-th entry is

$$(A^x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \quad i \in N.$$ 

Some notations used in this paper are given. For $A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, $i, j \in N$, $j \neq i$, we denote

$$\Delta_i = \{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j = 2, 3, \ldots, m\},$$

$$\overline{\Delta}_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j = 2, 3, \ldots, m\}.$$

2010 Mathematics Subject Classification. Primary 15A18; Secondary 15A69

Keywords. Tensor eigenvalue; Localization set; Exclusion sets; Nonsingularity.

Received: 31 January 2020; Revised: 05 March 2020; Accepted: 15 April 2020

Communicated by Yimin Wei

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Research supported by the National Natural Science Foundation of China (No. 11901123), the Guangxi Natural Science Foundation (No. 2018BJ110062) and the Xiangshu Young Scholars Innovative Research Team of Guangxi University for Nationalities (No. 2019RSCXSHQN03).

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Lemma 1.1. A pair \((\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})\) is called an eigenpair of \(\mathcal{A}\) if

\[
\mathcal{A}x^{m-1} = \lambda x^{m-1},
\]

where \(x^{m-1} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T\). Here \(x^T\) denotes the transpose of \(x\). Furthermore, we call \((\lambda, x)\) an \(H\)-eigenpair, if \(\lambda\) is a real number and \(x\) is a real vector.

Recently, Che et al. \cite{3} consider the homogeneous dynamical system related to the tensor \(\mathcal{A}\) and derived the definition of \(\epsilon\)-pseudospectrum of \(\mathcal{A}\).

Definition 1.2. \cite{3} Let \(\epsilon \geq 0\). The \(\epsilon\)-pseudospectrum of \(A = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{m,n}\) is defined as

\[
\Lambda_{\epsilon}(\mathcal{A}) = \{\lambda \in \mathbb{C} : (\mathcal{A} + \epsilon)x^{m-1} = \lambda x^{m-1} \text{ for } \epsilon \in \mathbb{C}^{m,n} \text{ with } \|\epsilon\|_F \leq \epsilon \text{ and some } x \in \mathbb{C}^n \setminus \{0\}\},
\]

where \(\|\epsilon\|_F\) is the Frobenius norm of \(\epsilon = (\epsilon_{i_1i_2\ldots i_m}) \in \mathbb{C}^{m,n}\), i.e., \(\|\epsilon\|_F = \sqrt{\sum_{i_1=1}^n \sum_{i_2=1}^n \ldots \sum_{i_m=1}^n |\epsilon_{i_1i_2\ldots i_m}|^2}\).

Next we exhibit the definition of symmetry of tensor, which was put forward firstly by Qi \cite{21}.

Definition 1.3. \cite{11,12,15,16,21,24} A real tensor \(\mathcal{A} = (a_{i_1i_2\ldots i_m})\) is called symmetric if its entries satisfy

\[
a_{i_1\ldots i_m} = a_{\pi(i_1\ldots i_m)}, \forall \pi \in \Pi_m,
\]

where \(\Pi_m\) is the permutation group of \(m\) indices.

Eigenvalue problems of tensors play significant roles in many fields, and they have wide practical applications, such as magnetic resonance imaging \cite{22}, higher order Markov chains \cite{20}, spectral hypergraph theory \cite{4} and so forth. Due to this fact and the difficulty of computing eigenvalues of tensors directly, it is vital to study the eigenvalue inclusion sets for tensors. As observed in \cite{12,14,17}, we can utilize the smallest \(H\)-eigenvalue of an even-order real symmetric tensor to determine its positive (semi-)definiteness, but getting the smallest \(H\)-eigenvalue of tensors is a task work for us on many occasions. In addition, \cite{13} posed that the determinant of the tensor \(\mathcal{A}\), denoted by \(\text{det}(\mathcal{A})\), is the resultant of the ordered system of homogeneous equations \(\mathcal{A}x^{m-1} = 0\) and is closely related to the eigenvalues of \(\mathcal{A}\). If \(\text{det}(\mathcal{A}) \neq 0\), i.e., 0 is not an eigenvalue of \(\mathcal{A}\), then \(\mathcal{A}\) is nonsingular. While the nonsingularity of tensors is hard to be identified by computing their eigenvalues directly. Considering above situations, a set containing all eigenvalues of tensors should be derived. Much literature have been devoted to this topic recently, refer to \cite{1,2,6–17,21} for more details. A great eigenvalue localization set is conducive to judge the positive definiteness and the nonsingularity of tensors, so we establish the new eigenvalue localization sets called improved Brauer-type eigenvalue localization sets for tensors in this paper, which are proved to be tighter than those in \cite{1,13,18,23}.

Before establishing the new eigenvalue inclusion sets for tensors in this paper, we first review some related results. For the real supersymmetric tensors, Qi in \cite{21} gave the Geršgorin eigenvalue localization sets as follows.

Lemma 1.1. \cite{21} Let \(\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{m,n}\), \(n \geq 2\). Then

\[
\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in \mathbb{N}} \Gamma_i(\mathcal{A}),
\]

where \(\sigma(\mathcal{A})\) is the set of all the eigenvalues of \(\mathcal{A}\) and

\[
\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\ldots i}| \leq r_i(\mathcal{A})\}.
\]
This result is also valid for general tensors [15, 24]. To improve the accuracy of \( \Gamma(\mathcal{A}) \), Bu et al. [1] derived the following eigenvalue localization set recently for tensors.

**Lemma 1.2.** [1] Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]} \). Then

\[
\sigma(A) \subseteq \mathcal{B}(A) = \bigcup_{i,j \in \mathbb{N}, i \neq j} \mathcal{B}_{i,j}(A),
\]

where

\[
\mathcal{B}_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{i \ldots i}|^{m-1}|z - a_{j \ldots j}| \leq (r_i(A))^{-1}r_j(A) \}.
\]

\( \mathcal{B}(A) \) is called the Brauer-type eigenvalue localization set. Besides, another Brauer-type eigenvalue localization set is also proposed by the authors in [1] as follows.

**Lemma 1.3.** [1] Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]} \) and \( r_i(A) \neq 0 \) (\( i \in \mathbb{N} \)). Then

\[
\sigma(A) \subseteq \mathcal{Z}(A) = \bigcup_{a_{ij_1 \ldots j_m} \neq 0} \left\{ z \in \mathbb{C} : \sum_{j=1}^{m} |z - a_{i \ldots i}| \leq \sum_{j=1}^{m} r_j(A) \right\}.
\]

The set in Lemma 1.3 was confirmed to be better than that in Lemma 1.1. In addition, by dividing the set \( N \) into two disjoint parts, Li et al. in [17] constructed the new Brauer-type eigenvalue localization set for tensors in the following lemma.

**Lemma 1.4.** [17] Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]} , n \geq 2 \). Then

\[
\sigma(A) \subseteq \Omega(A) = \left( \bigcup_{i \in \mathbb{N}} \Omega_i(A) \right) \bigcup \left( \bigcup_{i \in \mathbb{N}, j \neq i} (\Omega_{i,j}(A) \cap \Gamma_i(A)) \right),
\]

where

\[
\Omega_i(A) = \{ z \in \mathbb{C} : |z - a_{i \ldots i}| \leq r_i^\lambda(A) \},
\]

\[
\Omega_{i,j}(A) = \{ z \in \mathbb{C} : (|z - a_{i \ldots i}| - r_i^\lambda(A))(|z - a_{j \ldots j}| - r_j^\lambda(A)) \leq r_i^\lambda(A)r_j^\lambda(A) \}.
\]

In [13], the authors excluded some proper subsets, which do not include any eigenvalues of tensors, from eigenvalue localization set in Lemma 1.1. And they skillfully constructed a tighter eigenvalue localization set as follows.

**Lemma 1.5.** [13] Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]} , n \geq 2 \). Then

\[
\sigma(A) \subseteq \Upsilon(A) = \bigcup_{i \in \mathbb{N}} \Upsilon_i(A),
\]

where

\[
\Upsilon_i(A) = \Gamma_i(A) \setminus \Delta_i(A),
\]

\[
\Delta_i(A) = \bigcup_{j \neq i} \Delta_{ij}(A),
\]

and

\[
\Delta_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{j \ldots j}| \leq 2|a_{i \ldots i}| - r_j(A) \}.
\]

Note that in recent published literature [5], He et al. made use of the idea of excluding the subsets, and constructed the exclusion set for the pseudospectrum of tensors, which is significant in practical applications. Moreover, when the tensor \( \epsilon = 0 \), the exclusion set for the pseudospectrum of tensors in [5] reduces to an eigenvalue inclusion set for tensors, whose form is similar to that in Lemma 1.5.

In this work, motivated by the idea of [13, 18], several improved Brauer-type eigenvalue localization sets are established, which are sharper than those in Lemmas 1.2-1.4. And their forms are different from those of the exclusion sets in Lemma 1.5 and [5] as \( \epsilon = 0 \). As applications of the new sets, some new criteria for identifying the nonsingularity of tensors are given, which have advantages over some existing ones.
2. Improved Brauer-type eigenvalue localization sets for tensors

In this section, we construct the improved Brauer-type eigenvalue localization sets, and the comparisons between the new sets and those in Lemmas 1.2-1.4 are given.

Theorem 2.1. Let \( \mathcal{A} = (a_{i_1 \cdots i_n}) \in \mathbb{C}^{m,n} \). Then

\[
\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} (\mathcal{B}_{ij}(\mathcal{A}) \setminus \Omega_{ij}(\mathcal{A}))
\]  

(3)

where

\[
\mathcal{B}_{ij}(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{i_1 \cdots i_n}|^m |z - a_{j_1 \cdots j_n}| \leq (r_p(\mathcal{A}))^{m-1} r_f(\mathcal{A}) \},
\]

\[
\Omega_{ij}(\mathcal{A}) = \{ z \in \mathbb{C} : (|z - a_{i_1 \cdots i_n}| + r_f(\mathcal{A}))^{m-1} |z - a_{j_1 \cdots j_n}| < |a_{i_1 \cdots i_n}|^{m-1} (2|a_{j_1 \cdots j_n}| - r_f(\mathcal{A})) \}.
\]

Proof. For any \( \lambda \in \sigma(\mathcal{A}) \), let \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^m \setminus \{0\} \) be an associated eigenvector, i.e.,

\[
\mathcal{A}x^m = \lambda x^m,
\]

(4)

In view of the proof of Theorem 3.1 in [1], let \( |x_p| \geq |x_q| \geq \max \{|x_i|, i \in N, i \neq p, i \neq q\} \). Then, \( |x_p| > 0 \). It follows from the \( p \)-th equation of (4) that

\[
(\lambda - a_{p \cdots p})x_p^m = \sum_{\delta_{p_2, i_m = 0}} a_{pi_2 \cdots i_n} x_{i_2} \cdots x_{i_n}.
\]

(5)

Taking absolute values in Equation (5) and applying the triangle inequality yield

\[
|\lambda - a_{p \cdots p}| |x_p|^{m-1} \leq \sum_{\delta_{p_2, i_m = 0}} |a_{pi_2 \cdots i_n}| |x_{i_2}| \cdots |x_{i_n}|
\]

\[
\leq \sum_{\delta_{p_2, i_m = 0}} |a_{pi_2 \cdots i_n}| |x_p|^{m-1} |x_q|
\]

\[
= r_p(\mathcal{A}) |x_p|^m |x_q|,
\]

which leads to

\[
|\lambda - a_{p \cdots p}| |x_p|^{m-1} \leq r_p(\mathcal{A}) |x_p|^m |x_q|.
\]

(6)

If \( |x_p| = 0 \), then it follows from (6) that \( |\lambda - a_{p \cdots p}| \leq 0 \) by \( |x_p| > 0 \), which implies that \( \lambda = a_{p \cdots p} \). Evidently, \( \lambda \in \mathcal{B}_{p \cdots p}(\mathcal{A}) \). Otherwise, \( |x_q| > 0 \). Then \( q \)-th equation of (4) gives

\[
(\lambda - a_{q \cdots q})x_q^m = \sum_{\delta_{p_2, i_m = 0}} a_{qi_2 \cdots i_n} x_{i_2} \cdots x_{i_n},
\]

(7)

and it follows that

\[
|\lambda - a_{q \cdots q}| |x_q|^{m-1} \leq \sum_{\delta_{p_2, i_m = 0}} |a_{qi_2 \cdots i_n}| |x_{i_2}| \cdots |x_{i_n}|
\]

\[
\leq \sum_{\delta_{p_2, i_m = 0}} |a_{qi_2 \cdots i_n}| |x_q|^{m-1} |x_q|
\]

\[
= r_q(\mathcal{A}) |x_q|^m |x_q|.
\]

(8)

Combining (6) with (8) results in

\[
|z - a_{p \cdots p}|^{m-1} |z - a_{q \cdots q}| \leq (r_p(\mathcal{A}))^{m-1} r_q(\mathcal{A})
\]

between the new sets and those in Lemmas 1.2-1.4 are given.
by \(|x_p| \geq |x_q| > 0\), which means that \(\lambda \in B_{p,q}(A)\) holds true. It follows from (5) that

\[
a_{p\ldots q}x_{m-2}^p x_q = (\lambda - a_{p\ldots q})x_{m-1}^p - \left( \sum_{\delta_{p2} = 0} \sum_{\delta_{q2} = \lambda - a_{p\ldots q}} |a_{p2\ldots q}||x_{q2}| \cdots |x_{qn}| - |a_{p\ldots q}||x_p||x_{m-2}^p x_q| \right). \tag{9}
\]

By taking modulus in both sides of (9) and utilizing the triangle inequality, it has

\[
|a_{p\ldots q}||x_p||x_{m-2}^p x_q| \leq |\lambda - a_{p\ldots q}||x_{m-1}^p| + \left( \sum_{\delta_{p2} = 0} \sum_{\delta_{q2} = \lambda - a_{p\ldots q}} |a_{p2\ldots q}||x_{q2}| \cdots |x_{qn}| - |a_{p\ldots q}||x_p||x_{m-2}^p x_q| \right)
\]

which results in

\[
|a_{p\ldots q}||x_p||x_{m-2}^p x_q| \leq (|\lambda - a_{p\ldots q}| + \gamma_p(\mathcal{A}))(|x_{m-1}^p|).
\tag{10}
\]

Furthermore, from (7), it has

\[
a_{q\ldots p}x_{m-1}^q = (\lambda - a_{q\ldots p})x_{m-1}^q - \sum_{\delta_{q2} = 0, \delta_{p2} = \lambda - a_{q\ldots p}} a_{q2\ldots p}x_{q2} \cdots x_{qn}.
\tag{11}
\]

Applying the same operations utilized in (10) to (11) results in

\[
|a_{q\ldots p}||x_p||x_{m-1}^q| \leq |\lambda - a_{q\ldots p}||x_{m-1}^q| + \sum_{\delta_{q2} = 0, \delta_{p2} = \lambda - a_{q\ldots p}} |a_{q2\ldots p}||x_{q2}| \cdots |x_{qn}|
\]

which yields that

\[
(2|a_{q\ldots p} - r_q(\mathcal{A})||x_{m-1}^p \leq |\lambda - a_{q\ldots p}||x_{m-1}^q|.
\tag{12}
\]

If \(|x_p| > 0\), then combining (10) with (12) leads to

\[
|a_{p\ldots q}||x_p||x_{m-2}^p x_q| \geq (|\lambda - a_{p\ldots q}| + \gamma_p(\mathcal{A}))(|\lambda - a_{q\ldots p}|)(|x_{m-1}^p|),
\]

and hence

\[
|a_{p\ldots q}||x_p||x_{m-2}^p x_q| \leq (|\lambda - a_{p\ldots q}| + \gamma_p(\mathcal{A}))(|\lambda - a_{q\ldots p}|).
\tag{13}
\]

as \(|x_p| \geq |x_q| > 0\). If \(|x_q| = 0\), then (12) implies that \(2|a_{p\ldots q} - r_q(\mathcal{A})| \leq 0\), and (13) is also valid. (13) means that \(\lambda \notin \Omega_{p,q}(\mathcal{A})\). Therefore, \(\lambda \in (B_{p,q}(\mathcal{A}) \setminus \Omega_{p,q}(\mathcal{A}))\).

It is uncertain which \(p\) and \(q\) are appropriate to each eigenvalue \(\lambda\), we conclude that

\[
\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,j \neq \lambda, p} (B_{i,j}(\mathcal{A}) \setminus \Omega_{i,j}(\mathcal{A})),
\]

which completes the proof of Theorem 2.1. \(\square\)

Next, we prove that \(\Theta(\mathcal{A})\) is better than \(B(\mathcal{A})\) in Lemma 1.2.

**Theorem 2.2.** Let \(\mathcal{A} = (a_{i\ldots n-i}) \in C^{[m,n]}\), then

\[
\Theta(\mathcal{A}) \subseteq B(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).
\]
Thus \( \Omega \) since \( and \left| i \right| \leq \left| \bar{\Omega} \right| \leq \left| \bar{\bar{\Omega}} \right| \leq \left| \bar{\bar{\bar{\Omega}}} \right| \leq \left| \bar{\bar{\bar{\bar{\Omega}}}} \right| \), we see that \( \bar{\bar{\Omega}} \) is valid for this case. Thus we conclude that \( \bar{\bar{\bar{\Omega}}} \) is tighter than \( \bar{\bar{\bar{\bar{\Omega}}}} \) as shown in Theorem 2.2.

The following example is given to compare the sets in Theorem 2.1 and Theorem 3.1 of [1], and we depict them in Figure 1.

**Example 2.4.** Consider the tensor \( A = (a_{ij}) \in \mathbb{C}^{[m,n]} \) with elements defined as follows:

\[
\begin{align*}
 a_{11} &= 60, \quad a_{22} = 5, \quad a_{33} = 90 + 30i, \quad a_{44} = 15, \quad a_{114} = 1, \quad a_{122} = 30 + i, \quad a_{331} = 1 - i, \quad a_{444} = 1 + i, \\
 a_{211} &= 2, \quad a_{221} = 120, \quad a_{223} = 1, \quad a_{33} = 1, \quad a_{331} = 1, \quad a_{332} = 1, \quad a_{334} = 2, \quad a_{441} = 2, \quad a_{442} = 1
\end{align*}
\]

and other elements of \( A \) are zeros.

The localization sets \( \Theta(A) \) and \( \Theta(A) \) are plotted in Figure 1. Besides, all eigenvalues of the tensor \( A \) computed by the Matlab code `eig`, are depicted in Figure 1 with the black plus. It is clear that \( \Theta(A) \subseteq \Theta(A) \) and all eigenvalues of the tensor \( A \) are included in \( \Theta(A) \), which are in accordance with the results of Theorem 2.1 and Theorem 2.2 (see Figure 1).

**Theorem 2.5.** Let \( A = (a_{ij}) \in \mathbb{C}^{[m,n]} \) and \( r_i(A) \neq 0 \) \( (i \in N) \). Then

\[
\sigma(A) \subseteq \Psi(A) = (\Psi_1(A)) \cup (\Psi_2(A)),
\]

where

\[
\begin{align*}
\Psi_1(A) &= \bigcup_{i \in N} \{a_{i,\ldots}\}, \\
\Psi_2(A) &= \bigcup_{a_{i,j}, a_{i,j} \neq 0} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i,j}| \leq \prod_{j=1}^{m} r_j(A) \right\} \\
&\quad \setminus \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i,j}| < \prod_{j=1}^{m} (2|a_{ij}| - r_j(A)); 2|a_{ij}| - r_j(A) \geq 0, j = 1, \ldots, m \right\},
\end{align*}
\]

and \( |a_{ij}| = \max_{\pi \in \Pi_{m-1}} |a_{i,\pi|_{\{i,\ldots,i_j,\ldots,i_m\}}}| \) with \( \Pi_{m-1} \) being the permutation group of \( m - 1 \) indices.
Proof. For any $\lambda \in \sigma(A)$, let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be an associated eigenvector, i.e.,

$$A^m x = \lambda^m x.$$  \hfill (17)

By making use of the technique of Theorem 3.3 in [1], let $|x| = \max\{|x_{i_1}| |x_{i_2}| \cdots |x_{i_m}| : a_{i_1 i_2 \cdots i_m} \neq 0, \delta_{i_1 i_2 \cdots i_m} = 0, i_1, \ldots, i_m \in N\}$. Then for all $i \in N$, it has

$$(\lambda - a_{i_1 \cdots i_m})^m x = \sum_{\delta_{i_1 i_2 \cdots i_m} = 0} a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$  \hfill (18)

Taking absolute values in Equation (18) and applying the triangle inequality give

$$|\lambda - a_{i_1 \cdots i_m}| |x|^m \leq \sum_{\delta_{i_1 i_2 \cdots i_m} = 0} |a_{i_1 i_2 \cdots i_m}| |x_{i_2}| \cdots |x_{i_m}| \leq r_1(A) |x|.$$  \hfill (19)

Since $x \neq 0$, there exists one index $k$ such that $x_k \neq 0$. Taking $i = k$ in (19) leads to

$$|\lambda - a_{k \cdots k}| |x|^m \leq r_k(A) |x|.$$  \hfill (20)

If $|x| = 0$, then it follows from (20) that $\lambda = a_{k \cdots k}$ and therefore $\lambda \in \Psi_1(A)$.
For the case that $|x_0| \neq 0$, without loss of generality, we assume that $|x_0| = |x_{j_1}| |x_{j_2}| \cdots |x_{j_m}|$. Then from (20), it holds that
\[
|\lambda - a_{j_1}||x_{j_1}|^m \leq r_{j_1}(\mathcal{A})|x_0|,
\]
\[
|\lambda - a_{j_2}||x_{j_2}|^m \leq r_{j_2}(\mathcal{A})|x_0|,
\]
\[
\vdots
\]
\[
|\lambda - a_{j_m}||x_{j_m}|^m \leq r_{j_m}(\mathcal{A})|x_0|,
\]
which yields that
\[
\prod_{i=1}^{m} |\lambda - a_{j_i}||x_{j_i}|^m \leq |x_0|^m \prod_{i=1}^{m} r_{j_i}(\mathcal{A}),
\]
and hence
\[
\prod_{i=1}^{m} |\lambda - a_{j_i}| \leq \prod_{i=1}^{m} r_{j_i}(\mathcal{A}),
\]
which also implies that
\[
\lambda \in \bigcup_{a_{j_1}, \ldots, a_{j_m} \neq 0} \left\{ z \in \mathbb{C} : \prod_{i=1}^{m} |z - a_{j_i}| \leq \prod_{i=1}^{m} r_{j_i}(\mathcal{A}) \right\}.
\]
Considering $i = j_1$ in (18):
\[
(\lambda - a_{j_1})x_{j_1}^{m-1} = \sum_{\delta_{j_1 \rightarrow r_{j_1} \rightarrow 0}} a_{j_2 \ldots j_m} x_{j_2} \cdots x_{j_m}.
\]
Without loss of generality, we assume $|a_{j_1 j_2 \ldots j_m}| = \max_{\pi \in \Pi_{m-1}} |a_{j_1 \pi(j_2 \ldots j_m)}|$ with $\Pi_{m-1}$ being the permutation group of $m-1$ indices. Then it follows from (23) that
\[
a_{j_1 j_2 \ldots j_m} x_{j_1} \cdots x_{j_m} = (\lambda - a_{j_1 \ldots j_m})x_{j_1}^{m-1} - \left( \sum_{\delta_{j_1 j_2 \ldots j_m} = 0} a_{j_2 \ldots j_m} x_{j_2} \cdots x_{j_m} - a_{j_1 j_2 \ldots j_m} x_{j_2} \cdots x_{j_m} \right).
\]
Taking modulus in the above equation and applying the triangle inequality lead to
\[
|a_{j_1 j_2 \ldots j_m}| |x_0| \leq |\lambda - a_{j_1 \ldots j_m}| |x_{j_1}|^m + (r_{j_1}(\mathcal{A}) - |a_{j_1 j_2 \ldots j_m}|) |x_0|,
\]
which is equivalent to
\[
(2|a_{j_1} - r_{j_1}(\mathcal{A})||x_0|) |x_0| \leq |\lambda - a_{j_1}||x_{j_1}|^m.
\]
Similarly, for $i = j_2$, $i = j_3$, $\ldots$, $i = j_m$ in (18), we have
\[
(2|a_{j_2} - r_{j_2}(\mathcal{A})||x_0|) |x_0| \leq |\lambda - a_{j_2}||x_{j_2}|^m,
\]
\[
(2|a_{j_3} - r_{j_3}(\mathcal{A})||x_0|) |x_0| \leq |\lambda - a_{j_3}||x_{j_3}|^m,
\]
\[
\vdots
\]
\[
(2|a_{j_m} - r_{j_m}(\mathcal{A})||x_0|) |x_0| \leq |\lambda - a_{j_m}||x_{j_m}|^m,
\]
which together with (24) gives
\[
\prod_{i=1}^{m} |\lambda - a_{j_i}||x_{j_i}|^m = |x_0|^m \prod_{i=1}^{m} |\lambda - a_{j_i}||x_0| \geq |x_0|^m \prod_{i=1}^{m} (2|a_{j_i} - r_{j_i}(\mathcal{A}))
\]
under the condition $2|a_j| - r_j(\mathcal{A}) \geq 0$ ($j = 1, \ldots, m$). Then it follows that
\[
\prod_{j=1}^{m} |\lambda - a_{\beta_{j}}| \geq \prod_{j=1}^{m} (2|a_j| - r_j(\mathcal{A}))
\]
in terms of $|x_{\beta}| > 0$. This implies that
\[
\lambda \not\in \left\{ z \in \mathbb{C} : \prod_{i=1}^{m} |z - a_{\beta_{j}}| < \prod_{j=1}^{m} (2|a_j| - r_j(\mathcal{A})); 2|a_j| - r_j(\mathcal{A}) \geq 0, j = 1, \ldots, m \right\}.
\] (27)
By combining (22) with (27), we have $\lambda \in \Psi_2(\mathcal{A})$. This proof is completed. \hfill \square

**Remark 2.6.** In the proof of Theorem 2.5, Inequality (26) is valid under the assumptions $2|a_j| - r_j(\mathcal{A}) \geq 0$ ($j = 1, \ldots, m$). Actually, (26) also holds true in other cases. For example, existing even number of $2|a_j| - r_j(\mathcal{A}) \leq 0$ in Inequalities (24)-(25) or satisfying other proper restrictions, which is not easy to be described, may result in (26). Hence it is convenience for us to prove our theorem under the condition $2|a_j| - r_j(\mathcal{A}) \geq 0$ ($j = 1, \ldots, m$).

The following theorem illustrates that $\Psi(\mathcal{A})$ in Theorem 2.2 is sharper than $\mathcal{Z}(\mathcal{A})$ in Lemma 1.3.

**Theorem 2.7.** Let $\mathcal{A} = (a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}) \in \mathbb{C}^{[m,n]}$ and $r_j(\mathcal{A}) \neq 0$ ($i \in \mathbb{N}$). Then
\[
\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).
\]

**Proof.** Theorem 3.3 of [1] has proven $\mathcal{Z}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. Thus, we only need to prove $\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$. First, we show that
\[
\mathcal{Z}(\mathcal{A}) = \mathcal{Z}_1(\mathcal{A}) := \left( \bigcup_{i \in \mathbb{N}} [a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}] \right) \bigcup \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}| \leq \prod_{j=1}^{m} r_j(\mathcal{A}) \right\}.
\]

Obviously, $\mathcal{Z}(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A})$. So it is remain to prove $\mathcal{Z}_1(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$.

Let $z \in \mathcal{Z}_1(\mathcal{A})$. For the case that $z \in \bigcup_{i \in \mathbb{N}} [a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}]$, then there exists $p \in \mathbb{N}$ such that $z = a_{p_{-} \ldots p_{+}}$. Since $r_p(\mathcal{A}) > 0$, there exists $a_{p_{-} \ldots p_{+}} \neq 0$ and
\[
|z - a_{p_{-} \ldots p_{+}}| = 0 \leq r_p(\mathcal{A}) \prod_{i=2}^{m} r_i(\mathcal{A}),
\]
which means that
\[
z \in \bigcup_{a_{p_{-} \ldots p_{+}} \neq 0} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}| \leq \prod_{j=1}^{m} r_j(\mathcal{A}) \right\} = \mathcal{Z}(\mathcal{A}).
\]

Moreover, if
\[
z \in \bigcup_{a_{p_{-} \ldots p_{+}} \neq 0} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_{-} \ldots i_{-} i_{+} \ldots i_{+}}| \leq \prod_{j=1}^{m} r_j(\mathcal{A}) \right\},
\]
then it is easy to see that $z \in \mathcal{Z}(\mathcal{A})$. Therefore, $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}_1(\mathcal{A})$.

In the sequel, we prove that
\[
\Psi(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A}) = \mathcal{Z}(\mathcal{A}).
\]
For any \( a_{i_1i_2...i_m} \neq 0 \) and \( \delta_{i_1i_2...i_m} = 0 \), if \( \prod_{j=1}^{m} (2|a_{ij}| - r_j(\mathcal{A})) \leq 0 \), then
\[
\left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{ij}| < \prod_{j=1}^{m} (2|a_{ij}| - r_j(\mathcal{A})) \right\} = \emptyset,
\]
and then
\[
\left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{ij}| < \prod_{j=1}^{m} (2|a_{ij}| - r_j(\mathcal{A})) \right\} \subseteq \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{ij}| \leq \prod_{j=1}^{m} r_j(\mathcal{A}) \right\}.
\]
(28)

Now we consider the case that \( \prod_{j=1}^{m} (2|a_{ij}| - r_j(\mathcal{A})) > 0 \). By the definition of \( |a_{ij}| \) in Theorem 2.5, it can be seen that \( 0 \leq |a_{ij}| \leq r_j(\mathcal{A}) \) and therefore
\[
2|a_{ij}| - r_j(\mathcal{A}) \leq 2r_j(\mathcal{A}) - r_j(\mathcal{A}) = r_j(\mathcal{A}).
\]
In addition, we see that \(-r_j(\mathcal{A}) \leq 2|a_{ij}| - r_j(\mathcal{A}) \) and hence
\[
|2|a_{ij}| - r_j(\mathcal{A})| \leq r_j(\mathcal{A}),
\]
which leads to
\[
\prod_{j=1}^{m} |z - a_{ij}| < \prod_{j=1}^{m} (2|a_{ij}| - r_j(\mathcal{A})) \leq \prod_{j=1}^{m} r_j(\mathcal{A}).
\]
Thus (28) also holds true and \( \Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) \). \( \Box \)

Example 2.8. Consider the tensor \( \mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]} \) with elements defined as follows:
\[
a_{111} = 2, \ a_{222} = 2, \ a_{333} = 50, \ a_{444} = 50, \ a_{122} = 30 + i, \ a_{133} = 3 - i, \n\]
\[
a_{221} = 30, \ a_{233} = 1, \ a_{311} = 1, \ a_{334} = 20, \ a_{443} = 50
\]
and other elements of \( \mathcal{A} \) are zeros.

The localization sets \( \Psi(\mathcal{A}) \) and \( \mathcal{Z}(\mathcal{A}) \) are plotted in Figure 2 where all eigenvalues of \( \mathcal{A} \) are indicated by the plus. It can be seen that \( \Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) \) and the new set \( \Psi(\mathcal{A}) \) contains all eigenvalues of the tensor \( \mathcal{A} \) (see Figure 2), which confirms the correctness of Theorem 2.7 and the feasibility of the new set \( \Psi(\mathcal{A}) \).

In the sequel, we establish another new Brauer-type eigenvalue localization set for tensors in the following theorem, which is better than that in Lemma 1.4.

Theorem 2.9. Let \( \mathcal{A} = (a_{i_1...i_m}) \in \mathbb{C}^{[m,n]} \), \( n \geq 2 \). Then
\[
\sigma(\mathcal{A}) \subseteq \hat{\Omega}(\mathcal{A}) = \left( \bigcup_{i \in N} \hat{\Omega}_i(\mathcal{A}) \right) \bigcup \left( \bigcup_{i \in N, j \neq j'} \left( \hat{\Omega}_{ij}(\mathcal{A}) \setminus \hat{\Omega}_{ij'}(\mathcal{A}) \right) \right) \cap \Gamma_i(\mathcal{A}),
\]
(29)
where
\[
\hat{\Omega}_i(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{i...i}| \leq r_i^\alpha(\mathcal{A}) \},
\]
\[
\hat{\Omega}_{ij}(\mathcal{A}) = \{ z \in \mathbb{C} : (|z - a_{i...i}| - r_j^\alpha(\mathcal{A}))(|z - a_{j...j}| - r_j^\alpha(\mathcal{A})) \leq r_j^\alpha(\mathcal{A})r_j^\alpha(\mathcal{A}) \},
\]
\[
\hat{\Omega}_{ij'}(\mathcal{A}) = \{ z \in \mathbb{C} : (|z - a_{i...i}| + r_j^\alpha(\mathcal{A}))(|z - a_{j...j}| - r_j^\alpha(\mathcal{A})) < |a_{ij...i}|(2|a_{ij...i}| - r_j^\alpha(\mathcal{A})) \}.
\]
Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{m-1}. \quad (30)$$

Let $|x_p| \geq |x_q| \geq \max\{|x_i| : i \in N, i \neq p, i \neq q\}$. Then $|x_p| > 0$. According to the proof of Theorem 2.1 in [17], $p$th and $q$th equations of (30) give

$$(|\lambda - a_{p-q}| - r_p^\lambda(\mathcal{A}))|x_p|^{m-1} \leq r_p^\lambda(\mathcal{A})|x_q|^{m-1} \quad (31)$$

and

$$(|\lambda - a_{q-p}| - r_q^\lambda(\mathcal{A}))|x_q|^{m-1} \leq r_q^\lambda(\mathcal{A})|x_p|^{m-1}. \quad (32)$$

If $|x_q| = 0$, then Equation (31) is equivalent to $|\lambda - a_{p-q}| \leq r_p^\lambda(\mathcal{A})$ and hence $\lambda \in \mathring{\mathcal{G}}_p(\mathcal{A}) \subseteq \bigcup_{i \in N} \mathring{\mathcal{G}}_i(\mathcal{A}) \subseteq \mathring{\mathcal{G}}(\mathcal{A})$. If $|x_q| > 0$, then it follows from (31) and (32) that

$$(|\lambda - a_{p-q}| - r_p^\lambda(\mathcal{A}))(|\lambda - a_{q-p}| - r_q^\lambda(\mathcal{A})) \leq r_p^\lambda(\mathcal{A})r_q^\lambda(\mathcal{A}) \quad (33)$$

in view of $|x_p| \geq |x_q| > 0$, which means that $\lambda \in \mathring{\mathcal{G}}_{p,q}(\mathcal{A})$. Moreover, it follows from (31) that $|\lambda - a_{p-q}| \leq r_p(\mathcal{A})$, which together with $\lambda \in \mathring{\mathcal{G}}_{p,q}(\mathcal{A})$ results in $\lambda \in (\mathring{\mathcal{G}}_{p,q}(\mathcal{A}) \cap \mathcal{G}_p(\mathcal{A}))$.

It follows from (30) that

$$a_{p-q}x_q^{m-1} = (\lambda - a_{p-q})x_p^{m-1} - \sum_{(i_2, \ldots, i_m) \in \Delta r, \delta_{i_2-ma} = 0} a_{p_2-i_m}x_{i_2} \cdots x_{i_m} - \sum_{(i_2, \ldots, i_m) \in \Delta r, \delta_{i_2-ma} = 0} a_{p_2-i_m}x_{i_2} \cdots x_{i_m}. \quad (34)$$
By taking modulus in both sides of (34) and utilizing the triangle inequality, it has
\[ |a_{pq}| |x|^m - 1 \leq |\lambda - a_{pq}| |x|^m + \sum_{(i_2, \ldots, i_n) \in \partial \Omega_p, \delta_{i_2} = 0} |a_{pip_1}||x_{i_1}| \cdots |x_{i_n}| + \sum_{(i_2, \ldots, i_n) \in \partial \Omega_p, \delta_{i_2} = 0} |a_{pip_1}||x_{i_1}| \cdots |x_{i_n}| \]
\[ \leq |\lambda - a_{pq}| |x|^m + \sum_{(i_2, \ldots, i_n) \in \partial \Omega_p, \delta_{i_2} = 0} |a_{pip_1}||x_{i_1}| \cdots |x_{i_n}| \]
\[ = |\lambda - a_{pq}| |x|^m + r^{\lambda}_{\gamma}(\mathcal{A})|x|^m - 1, \]
which yields that
\[ (2 |a_{pq}| - r^{\lambda}_{\gamma}(\mathcal{A}))|x|^m - 1 \leq (|\lambda - a_{pq}| + r^{\lambda}_{\gamma}(\mathcal{A}))|x|^m - 1. \] (37)

If \(|x| > 0\), then multiplying (35) with (37) leads to
\[ |a_{pq}|(2 |a_{pq}| - r^{\lambda}_{\gamma}(\mathcal{A}))|x|^m - 1 \leq (|\lambda - a_{pq}| + r^{\lambda}_{\gamma}(\mathcal{A}))(|\lambda - a_{pq}| + r^{\lambda}_{\gamma}(\mathcal{A}))|x|^m - 1, \]
and therefore
\[ |a_{pq}|(2 |a_{pq}| - r^{\lambda}_{\gamma}(\mathcal{A})) \leq (|\lambda - a_{pq}| + r^{\lambda}_{\gamma}(\mathcal{A}))(|\lambda - a_{pq}| + r^{\lambda}_{\gamma}(\mathcal{A})) \] (38)
as \(|x| \geq |x| > 0\). If \(|x| = 0\), then (37) implies that \(2 |a_{pq}| - r^{\lambda}_{\gamma}(\mathcal{A}) \leq 0\), and (38) is also valid. (38) means that \(\lambda \notin \Omega_{\mathcal{A}}(\mathcal{A})\). Therefore, \(\lambda \in \left( (\Omega_{\mathcal{A}}(\mathcal{A})) \Omega_{\mathcal{A}}(\mathcal{A}) \right) \cap \Gamma_{\mathcal{A}}(\mathcal{A}) \subseteq \Omega(\mathcal{A}). \)

Next theorem shows that \(\Omega(\mathcal{A})\) is sharper than \(\Omega(\mathcal{A})\) in Lemma 1.4.

**Theorem 2.10.** Let \(\mathcal{A} = (a_{i_1}) \in C^{[i_n]}\), then
\[ \Omega(\mathcal{A}) \subseteq \Omega(\mathcal{A}). \]

**Proof.** For any \(i, j \in N\) and \(j \neq i\), if \(|a_{i-j}|(2 |a_{i-j}| - \lambda_{ji})(\mathcal{A}) \leq 0\), then \(\hat{\Omega}_{i,j}(\mathcal{A}) = \emptyset\), and therefore \(\hat{\Omega}_{i,j}(\mathcal{A}) \subseteq \hat{\Omega}_{i,j}(\mathcal{A})\).

Now we consider the case that \(|a_{i-j}|(2 |a_{i-j}| - \lambda_{ji})(\mathcal{A}) > 0\). Since \(r^*_{\gamma}(\mathcal{A}) \geq r^*_{\gamma}(\mathcal{A})\), it has
\[ (|z - a_{i-j}|(2 |a_{i-j}| - \lambda_{ji})(\mathcal{A})) = (|z - a_{i-j}| + \lambda_{ji} A) - (|z - a_{i-j}| - \lambda_{ji} A) \]
\[ = 2 |z - a_{i-j}|r^*_{\gamma} A + |z - a_{i-j}|(r^*_{\gamma} A + r^*_{\gamma} A) \geq 0. \] (39)
Moreover, it is not difficult to verify that $|a_{i...j}| \leq r_i^A$ and $2|a_{i...j}| \leq 2r_j^A$, that is $0 < 2|a_{i...j}| - r_j^A \leq r_j^A$, which implies that

$$|a_{i...j} - r_j^A| \leq r_j^A,$$

which together with (39) shows that $\bar{\Omega}_{i,j}(A) \subseteq \tilde{\Omega}_{i,j}(A)$ and $\tilde{\Omega}_{i,j}(A) \setminus \bar{\Omega}_{i,j}(A) \subseteq \tilde{\Omega}_{i,j}(A)$. Thus

$$\left( \left( \tilde{\Omega}_{i,j}(A) \setminus \bar{\Omega}_{i,j}(A) \right) \cap \Gamma_j(A) \right) \subseteq \left( \tilde{\Omega}_{i,j}(A) \cap \Gamma_j(A) \right),$$

which leads to $\bar{\Omega}(A) \subseteq \Omega(A)$. This proof is completed. \hfill \Box

Remark 2.11. For a tensor $A = (a_{i...j}) \in \mathbb{C}^{[m,n]}$, the set $\bar{\Omega}(A)$ in Theorem 2.9 contains $n$ sets $\bar{\Omega}_i(A)$, $\frac{n(n+1)}{2}$ sets $\bar{\Omega}_{ij}(A)$, $\frac{n(n+1)}{2}$ sets $\bar{\Omega}_{ij}(A)$ and $n$ sets $\Gamma_i(A)$. Hence there are $n^2 + 3n$ sets in $\bar{\Omega}(A)$. By Lemma 1.4, it can be seen that $\Omega(A)$ contains $\frac{n(n+1)}{2}$ sets. Thus computing $\Omega(A)$ requires more computations than $\Omega(A)$. However, Theorem 2.10 reveals $\bar{\Omega}(A)$ is sharper than $\Omega(A)$.

Example 2.12. Consider the tensor $A = (a_{i...j}) \in \mathbb{C}^{[3,2]}$ with elements defined as follows:

$$a_{111} = 10, \  a_{222} = 3, \ a_{112} = 1, \ a_{121} = 1,$$
$$a_{122} = 8, \ a_{211} = 20, \ a_{212} = 2, \ a_{221} = 0.1$$

and other elements of $A$ are zeros.
3. Some new sufficient criteria for nonsingularity of tensors

As applications of the sets proposed in Section 2, we develop new sufficient criteria for the nonsingularity of tensors in this section. Additionally, we use several examples to show the advantages of the proposed criteria over the existing ones in [1, 13, 18, 23].

Theorem 3.1. Let \( \mathcal{A} = (a_{ih..}) \in \mathbb{C}^{[m,n]} \). If for all \( i, j \in N \) and \( i \neq j \), one of the following two conditions holds:

(i) \(|a_{i..}|^{m-1}a_{j..} > (r_j(\mathcal{A}))^{m-1}r_i(\mathcal{A})|\)

(ii) \(|a_{i..}| + r_j^h(\mathcal{A})|^{m-1}|a_{j..}| < |a_{i..}|^{m-1}(2|a_{j..}| - r_j(\mathcal{A}))|\)

then \( \mathcal{A} \) is nonsingular.

Proof. Assume that \( \lambda \) is the eigenvalue of \( \mathcal{A} \). From Theorem 2.1, it has \( \lambda \in \Theta(\mathcal{A}) \), which implies that there are \( k, h \in N \) such that

\[
|\lambda - a_{k..}|^{m-1}|\lambda - a_{h..}| \leq (r_k(\mathcal{A}))^{m-1}r_h(\mathcal{A}),
\]

\[
(|\lambda - a_{k..}| + r_h^k(\mathcal{A}))^{m-1}|\lambda - a_{h..}| \geq |a_{k..}|^{m-1}(2|a_{h..}| - r_h(\mathcal{A})).
\]

If \( \lambda = 0 \), then it follows that

\[
|\lambda - a_{k..}|^{m-1}|\lambda - a_{h..}| = |a_{k..}|^{m-1}|a_{h..}| \leq (r_k(\mathcal{A}))^{m-1}r_h(\mathcal{A})
\]

and

\[
(|\lambda - a_{k..}| + r_h^k(\mathcal{A}))^{m-1}|\lambda - a_{h..}| = (|a_{k..}| + r_h^k(\mathcal{A}))^{m-1}|a_{h..}| \geq |a_{k..}|^{m-1}(2|a_{h..}| - r_h(\mathcal{A})),
\]

which contradict with the conditions of this theorem. Hence, \( \lambda \neq 0 \) and \( \mathcal{A} \) is nonsingular. \( \square \)

Example 3.2. Consider the tensor \( \mathcal{A} = (a_{ih..}) \in \mathbb{C}^{[5,3]} \) with elements defined as follows:

\[
a_{111} = 80, \quad a_{222} = 30, \quad a_{333} = 90, \quad a_{122} = 30, \quad a_{133} = 1, \\
a_{221} = 120, \quad a_{211} = 2, \quad a_{223} = 1, \quad a_{233} = 1, \quad a_{331} = 1, \quad a_{322} = 1
\]

and other elements of \( \mathcal{A} \) are zeros.

By some calculations, we have

\[
|a_{222}|^2|a_{111}| = 72000 < 476656 = (r_2(\mathcal{A}))^2r_1(\mathcal{A}),
\]

which implies that Corollary 3.2 in [1] can not be applied to identify the nonsingularity of \( \mathcal{A} \) in this example. And we can obtain

\[
(|a_{111}| - r_1^h(\mathcal{A}))|a_{222}| = 2400 < 3844 = (r_1^h(\mathcal{A}))^2r_2(\mathcal{A}),
\]

\[
(|a_{111}| + r_2^h(\mathcal{A}))(|a_{222}| + r_2^h(\mathcal{A})) = 2592 > -3540 = |a_{122}|(2|a_{211}| - r_2(\mathcal{A})).
\]

Thus Corollary 1 of [18] can not be used to determine the nonsingularity of \( \mathcal{A} \). Besides,

\[
|a_{222}| = 30 < 124 = r_2(\mathcal{A}), \quad |a_{222}| = 30 > -120 = 2|a_{211}| - r_2(\mathcal{A}), \quad |a_{222}| = 30 > -122 = 2|a_{233}| - r_2(\mathcal{A}).
\]
which leads to λ then |

Proof. Assume that (ii) for all a

Let Θ

depict the eigenvalue localization sets A which means that the tensor A is nonsingular. We depict the eigenvalue localization sets Θ(A) in Theorem 2.1, B(A) in Lemma 1.2 and the eigenvalues of A in Figure 4, where the eigenvalues of A are represented by the black plus. It can be observed that the new set Θ(A) can work, and (0, 0) ∈ Θ(A) while (0, 0) ∈ B(A), which is in accordance with the results in Theorems 2.1, 2.2 and 3.1.

Theorem 3.3. Let A = (a_{i_1...i_m}) ∈ C[m,n]. If the following conditions hold:
(i) a_{i,...} ≠ 0 for any i ∈ N;
(ii) for all a_{i_1i_2...i_m} ≠ 0 and δ_{i_1i_2...i_m} = 0, \( \prod_{j=1}^{m} |a_{i_j...i_m}| > \prod_{j=1}^{m} r_j(A) \) or \( \prod_{j=1}^{m} |a_{i_j...i_m}| < \prod_{j=1}^{m} (2|a_{i_j}| - r_j(A)) \) and \( 2|a_{i_j}| - r_i(A) ≥ 0 \), where \( |a_i| = \max_{n \in \Pi_{m-1}} (|a_{i_{n(i_{n-1})...i_1}}|) \) with \( \Pi_{m-1} \) being the permutation group of m - 1 indices, then A is nonsingular.

Proof. Assume that λ is the eigenvalue of A. From Theorem 2.5, it has λ ∈ Ψ(A) = (Ψ_1(A)) \cup (Ψ_2(A)), which leads to λ = a_{p,...} for some p ∈ N, or there exist k_1, k_2, ..., k_n satisfying a_{k_1k_2...k_n} ≠ 0 and δ_{k_1k_2...k_n} = 0 such

Then we also can not use Corollary 1 of [13] to determine the nonsingularity of A. However, we can derive the following results by Theorem 3.1.

\[
\begin{align*}
|\tilde{a}_{111}|^2|\bar{a}_{222}| &= 192000 > 119164 = (r_1(A))^2r_2(A), \\
|\tilde{a}_{111}|^2|\bar{a}_{333}| &= 576000 > 2883 = (r_1(A))^2r_3(A), \\
(\tilde{a}_{222} + \bar{a}_2^2(A))|\bar{a}_{111}| &= 92480 < 417600 = |\tilde{a}_{222}|^2(2|\bar{a}_{122}| - r_1(A)), \\
|\bar{a}_{222}|^2|\bar{a}_{333}| &= 81000 > 46128 = (r_2(A))^2r_3(A), \\
|\bar{a}_{333}|^2|\bar{a}_{111}| &= 648000 > 279 = (r_3(A))^2r_1(A), \\
|\bar{a}_{333}|^2|\bar{a}_{222}| &= 243000 > 1116 = (r_3(A))^2r_2(A),
\end{align*}
\]

which means that the tensor A satisfies the conditions of Theorem 3.1, and hence A is nonsingular.
that
\[ \prod_{j=1}^{m} |\lambda - a_{k_{j-1},k_{j}}| \leq \prod_{j=1}^{m} r_{k_{j}}(\mathcal{A}), \quad \text{and} \quad \prod_{j=1}^{m} |\lambda - a_{k_{j-1},k_{j}}| \geq \prod_{j=1}^{m} (2|z_{k_{j}}| - r_{k_{j}}(\mathcal{A})), \quad 2|\bar{a}_{k_{j}}| - r_{k_{j}}(\mathcal{A}) \geq 0. \]

If \( \lambda = 0 \), then we deduce that \( a_{p...p} = 0 \) for some \( p \in \mathbb{N} \) or
\[ \prod_{j=1}^{m} |a_{k_{j-1},k_{j}}| \leq \prod_{j=1}^{m} r_{k_{j}}(\mathcal{A}) \quad \text{and} \quad \prod_{j=1}^{m} |a_{k_{j-1},k_{j}}| \geq \prod_{j=1}^{m} (2|z_{k_{j}}| - r_{k_{j}}(\mathcal{A})), \quad 2|\bar{a}_{k_{j}}| - r_{k_{j}}(\mathcal{A}) \geq 0, \]
which contradicts with the conditions of this theorem. Hence, \( \lambda \neq 0 \) and \( \mathcal{A} \) is nonsingular. \( \Box \)

**Example 3.4.** Consider the tensor \( \mathcal{A} = (a_{ij}) \in \mathbb{C}^{[3,4]} \) with elements defined as follows:
\[ \begin{align*}
a_{111} &= 6, \quad a_{222} = 2, \quad a_{333} = 33, \quad a_{444} = 20, \quad a_{122} = 10, \quad a_{133} = 3, \\
a_{222} &= 30, \quad a_{233} = 1, \quad a_{311} = 1, \quad a_{444} = 5, \quad a_{443} = 40
\end{align*} \]
and other elements of \( \mathcal{A} \) are zeros.

Note that \( a_{222} = 10 \neq 0 \) and
\[ |a_{111}| \cdot |a_{222}|^2 = 24 < 12493 = r_1(\mathcal{A})(r_2(\mathcal{A}))^2. \]

It follows from Corollaries 3.2 and 3.4 in [1] that they cannot be applied to identify the nonsingularity of the tensor \( \mathcal{A} \). Besides, it can be seen that
\[ \begin{align*}
|a_{111} - r_1^2(\mathcal{A})| |a_{222}| &= 6 < 310 = |a_{122}| r_2(\mathcal{A}), \\
|a_{111} - r_1^2(\mathcal{A})| |a_{333}| &= -132 < 18 = |a_{133}| r_3(\mathcal{A}), \\
|a_{111} - r_1^2(\mathcal{A})| |a_{444}| &= -140 < 0 = |a_{144}| r_4(\mathcal{A}), \\
|a_{111} + r_1^2(\mathcal{A})| |a_{222}| &= 18 > -310 = |a_{122}| (2|a_{211}| - r_2(\mathcal{A})), \\
|a_{111} + r_1^2(\mathcal{A})| |a_{333}| &= 528 > -12 = |a_{133}| (2|a_{311}| - r_3(\mathcal{A})), \\
|a_{111} + r_1^2(\mathcal{A})| |a_{444}| &= 380 > 0 = |a_{144}| (2|a_{411}| - r_4(\mathcal{A})),
\end{align*} \]
hence Corollary 3 of [13] and Corollary 2.4 of [23] are invalid. While by Theorem 2.9, we obtain
\[ \begin{align*}
a_{111} &= 6 \neq 0, \quad a_{222} = 2 \neq 0, \quad a_{333} = 33 \neq 0, \quad a_{444} = 20 \neq 0, \\
|a_{111}| |a_{222}|^2 &= 24 < 5887 = (2|a_{11}| - r_1(\mathcal{A}))(2|a_{22}| - r_2(\mathcal{A}))^2, \\
|a_{111}| |a_{333}|^2 &= 6534 > 468 = r_1(\mathcal{A})(r_3(\mathcal{A}))^2, \\
|a_{222}||a_{111}| &= 24 < 5887 = (2|a_{22}| - r_2(\mathcal{A}))^2(2|a_{11}| - r_1(\mathcal{A})), \\
|a_{222}| |a_{333}|^2 &= 2178 > 1116 = r_2(\mathcal{A})(r_3(\mathcal{A}))^2, \\
|a_{333}| |a_{111}|^2 &= 1188 > 1014 = r_3(\mathcal{A})(r_1(\mathcal{A}))^2, \\
|a_{333}| |a_{444}|^2 &= 13200 > 9600 = r_3(\mathcal{A})(r_4(\mathcal{A}))^2, \\
|a_{444}| |a_{333}|^2 &= 21780 > 1440 = r_4(\mathcal{A})(r_3(\mathcal{A}))^2.
\end{align*} \]
Therefore we conclude that the tensor \( \mathcal{A} \) is nonsingular.

To illustrate the correctness of Theorem 3.3, the eigenvalue localization sets \( \Psi(\mathcal{A}) \) are drawn in Figure 5, where \( \Psi(\mathcal{A}) \), the exact eigenvalues of \( \mathcal{A} \) and the point \( (0, 0) \) are represented by the blue zones, the black plus and the red asterisk, respectively. From Figure 5, it is easy to see that \((0, 0) \notin \Psi(\mathcal{A})\).
Proof. By Theorem 2.9 and using the method applied in Theorems 3.1–3.3, we can prove the conclusion of this theorem. \(\Box\)

We will verify the advantages of Theorem 3.5 by Example 3.6.

**Example 3.6.** Consider the tensor \(\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,2]}\) with elements defined as follows:

\[
\begin{align*}
a_{111} &= 12, \ a_{222} = 3.5, \ a_{112} = 1, \ a_{121} = 1, \\
a_{122} &= 9, \ a_{211} = 10, \ a_{212} = 2, \ a_{221} = 0.1
\end{align*}
\]

and other elements of \(\mathcal{A}\) are zeros.

Since \(a_{221} = 0.1 \neq 0\), by direct computations, it follows that

\[
\begin{align*}
|a_{222}|^2|a_{111}| &= 147 < 1610.5 = (r_2(\mathcal{A}))^2 r_1(\mathcal{A}), \\
(|a_{222}| + r_2(\mathcal{A})^2|a_{111}|) &= 2883 > 0.7 = |a_{222}|^2(2|a_{122}| - r_1(\mathcal{A})),
\end{align*}
\]

which illustrates that the conditions of Corollaries 3.2 and 3.4 in [1] and Theorem 3.1 of this paper are not satisfied. According to Theorem 3.5 in this paper, we get

\[
\begin{align*}
|a_{111}| &= 12 > 11 = r_1^1(\mathcal{A}), \ |a_{222}| = 3.5 > 2.1 = r_2^1(\mathcal{A}), \\
(|a_{111}| + r_1(\mathcal{A}))(|a_{222}| + r_2^1(\mathcal{A})) &= 49 < 71.1 = |a_{122}|(2|a_{211}| - r_2^2(\mathcal{A})), \\
(|a_{222}| + r_1(\mathcal{A}))(|a_{111}| + r_2^1(\mathcal{A})) &= 67.2 < 70 = |a_{221}|(2|a_{122}| - r_1^2(\mathcal{A})),
\end{align*}
\]

which confirms that the tensor \(\mathcal{A}\) is nonsingular. To further verify this fact, the eigenvalue localization set \(\Omega(\mathcal{A})\), the exact eigenvalues of \(\mathcal{A}\) and the point \((0,0)\) are drawn in Figure 6, where \(\Omega(\mathcal{A})\), the exact eigenvalues of \(\mathcal{A}\) and the point \((0,0)\) are represented by the green zones, the black plus and the red asterisk, respectively. As observed in Figure 6, \((0,0) \notin \Omega(\mathcal{A})\) and the tensor \(\mathcal{A}\) is nonsingular.
4. Conclusions

In this paper, some improved Brauer-type eigenvalue localization sets for tensors are established, which are sharper than those in [1, 17]. Based on these sets, some new sufficient criterias are given, which have wider scope of applications compared with those of [1, 13, 18, 23] for the nonsingularity of tensors. In addition, we should investigate more tighter eigenvalue localization sets for tensors. Finally, based on the exclusion set for the pseudospectrum of tensors put forward recently [5], we should try to extend the proposed exclusion sets in this paper for the pseudospectrum of tensors, and investigate the more accurate exclusion sets for the pseudospectrum of tensors in our future work.

Competing interests

The authors declare that they have no competing interests.

Acknowledgments

We would like to express our sincere thanks to the editor and the anonymous reviewer for their valuable suggestions and constructive comments which greatly improved the presentation of this paper.

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