Entanglement entropy of XX spin 1/2 chain with random partitioning at arbitrary temperature

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Abstract

We study the entanglement properties of random XX spin 1/2 chains at an arbitrary temperature \( T \) using random partitioning, where sites of a size-varying subsystem are chosen randomly with a uniform probability \( p \), and then an average over subsystem possibilities is taken. We show analytically and numerically, using the approximate method of real space renormalization group, that random partitioning entanglement entropy for the XX spin chain of size \( L \) behaves like \( \text{EE}(T, p) = a(T, p)L \) at an arbitrary temperature \( T \) with a uniform probability \( p \), i.e., it obeys volume law. We demonstrate that \( a(T, p) = \ln(2)(P_\uparrow + P_\uparrow\downarrow)p(1 - p) \), where \( P_\uparrow \) and \( P_\uparrow\downarrow \) are the average probabilities of having singlet and triplet, respectively, in the entire system, with \( \ln(2) \) being the logarithmic correction term. Also, people studied the entanglement properties of the system experimentally, which we would divide it in all possible ways. In this regard, we use the notion of random partitioning, in which we take an average over the EE’s corresponding to the randomly chosen subsystems. More accurately, to obtain random-partitioning EE, we do the following: first, we attribute a probability to each site based on a probability distribution, i.e., sites that belong to the subsystem are chosen randomly. Moreover, subsystem size varies between 1 to \( L \) (there are \( \binom{L}{n} \) different forms of having subsystems with \( n \) sites). We consider all of these ways of partitioning, and we take an average of the EE corresponding to them. This way of partitioning is entirely different from the usual method of bi-partitioning. In the bi-partitioning, the system’s middle is the boundary between the subsystem and its environment. In contrast, in the random bi-partitioning, there are many boundaries at different points. As a limiting case, if the subsystem sites are every other spin, we expect a volume law for the entanglement entropy.

Recently, this form of partitioning has been used for the ground state of a clean, free fermion system. They found a volume law dependence of the EE to the system size with a logarithmic correction term. Also, people studied the entanglement spectrum under random partitioning. Here, we advance these previous studies in two directions. First, we consider a disordered system with random impurities and second, we are concerned mainly with a typical excited state at an arbitrary temperature \( T \).

By analytical and numerical investigations, we find that the behavior of the EE with a random partitioning is volume law: \( \text{EE}(T, p) = a(T, p)L \), with a pre-factor \( a \) that depends on the probability and temperature. We obtain the analytical form for

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2. Analytical evaluation

We consider a one-dimensional XX spin 1/2 chain with \( L \) spins. They are coupled together locally and with random strength. Hamiltonian of the system is:

\[
H = \sum_{n=1}^{L-1} J_n (s_n^x s_{n+1}^x + s_n^y s_{n+1}^y),
\]

(1)

(with open boundary condition) where \( J_n \) are distributed randomly by a distribution function (to be determined later). The entanglement properties of this model have been studied before\(^{[35, 34, 33, 36]}\).

We want to obtain the entanglement properties of this model, and to do so, we use the notion of EE. In a bi-partitioned system, where the system is divided into two parts: subsystem \( A \) and its environment, the EE is the von Neumann entropy of the reduced density matrix of the subsystem:

\[
\text{EE} = - \text{Tr}[\rho_A \ln \rho_A],
\]

(2)

where \( \rho_A \) is the reduced density matrix for the subsystem \( A \) obtained by tracing out the environment degrees of freedom from the density matrix \( \rho \) of the entire system. For a pure state the density matrix is \( \rho = |\Psi\rangle \langle \Psi| \), where \( |\Psi\rangle \) is the state of the system. Thus, to calculate the EE in a bi-partitioned system in a brute-force method, first, we need to diagonalize the Hamiltonian to obtain its eigenvalues and eigenvectors \( |\Psi\rangle \). Then, by calculating the reduced density matrix of a chosen subsystem and using Eq. (2), we obtain the EE. To obtain the system’s state in the XX spin 1/2 chain with \( L \) spins, one deals with matrices with size \( 2^L \), which exponentially growing with the system size. By Jordan-Wigner transformation, the XX model is transformed into a free fermion model, where the size of the matrices to deal with in numerical calculations reduces to \( L \)\(^{[37, 38]}\). In addition to this direct and exact method, the XX spin 1/2 chain has been studied by approximate methods such as the real space renormalization group (RSRG) method\(^{[39, 40, 41]}\). In this method, the approximate ground state of a random XX spin 1/2 chain is obtained. An extension of this method is also developed to obtain an approximate typical excited state, namely the RSRG-X. We use these approximate methods to calculate the EE in random partitioning.

We should note that, in this paper, we do not use the corresponding free fermion model of Eq. (1) to obtain the EE in random partitioning. Using the free fermion method in calculating EE for disjoint blocks is incorrect (since the Jordan-Wigner transformation of \( n \)th site depends on the previous sites). Instead, we use RSRG/RSRG-X approximate method to obtain EE, which is based on the number of singlets and triplets that cross the boundary of the subsystem. These methods do not fail for a disjoint subsystem. In what follows, we explain both methods and describe how to use them to calculate approximately the EE.

2.1. Real space renormalization group

To obtain the approximate ground state of a XX spin 1/2 chain by RSRG method, we go through the following steps\(^{[42]}\). Consider the Hamiltonian of Eq. (1) in which coupling constants \( |J| \) are distributed randomly. First, we pick up the largest coupling \( J_{\max} \). In the ground state of the system, we put the two spins that are coupled with \( J_{\max} \) in a singlet state \( |\text{singlet}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \). Next, we remove these two spins and couple the two closest spins with an effective coupling \( J = \frac{J_{\max}}{\alpha} \) (see Fig.1 for a schematic representation).

By repeating this process, the selected spin pairs are in the singlet state. Thus, the ground state of the system is the direct product of singlets with arbitrary bond length (the so-called random singlet phase). Since \( J \) is smaller than both of \( J_L, J_R \), by repeating the RSRG process, we will get smaller and smaller values for \( J \). So, the probability of \( |J| \) distributions will be a power-law: \( P(J) = \alpha J^{\nu-1} \) for \( 0 \leq J \leq 1 \). The RSRG fixed point is the infinite randomness fixed point and corresponds to \( \alpha \to 0 \). In this regard, small values of \( \alpha \) correspond to the strong disorder regime that RSRG yields to an asymptotically correct ground state.

For a highly excited state, a modified version of the RSRG is developed, namely the RSRG-X\(^{[43]}\). We should note that, at \( T = 0 \), the state of the system is pure and we obtained it approximately using the RSRG method, which in this approximation is the product state of the singlet state pairs. On the other hand, for \( T \neq 0 \), the state of the system is mixed which represents a combination of possible states weighted with a probability. In this paper, we do not consider a mixed state for a non-zero temperature, but we consider a typical state which is one of the possible states. This typical state (which in the RSRG-X approximation is a product state of singlet and each of the triplet

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{A part of the spin chain is presented in which we assume \( J_{\max} > J_L, J_R \). In real space renormalization group method, after picking up \( J_{\max} \), we remove spin 2 and 3 and couple spin number 1 and 4 with \( J = \frac{J_{\max}}{\alpha} \) (see Fig 1 for a schematic representation).}
\end{figure}
Like the RSRG method, the fixed point corresponds to a totally correct, corresponds to small values of $\alpha$ and the strong disorder regime, where the RSRG-X is asymptotically correct. Namely, the probability of having a singlet state or each of the triplet states based on the Boltzmann distribution function:

$$P_B = \frac{1}{Z} \exp(-E/T),$$

where $E$ is the energy of the singlet/triplet state of the two spins, and $Z = 2 + 2 \cosh T$ to have a normalized probability (see Table 1). The effective coupling, $J$ depends on which singlet or triplets are chosen by the Boltzmann distribution. In doing the RSRG-X method, the probability of getting smaller magnitude values for couplings increases, and thus we have a power-law distribution:

$$P(J) = \frac{\alpha}{2} |J|^\alpha - 1, \text{ for } |J| \leq 1$$

Like the RSRG method, the fixed point corresponds to $\alpha \rightarrow 0$ and the strong disorder regime, where the RSRG-X is asymptotically correct, corresponds to small values of $\alpha$. The outcome of the RSRG-X method, a typical excited state, is the direct product of singlets and triplets. We work in the sector of half-filling in the corresponding free fermion representation, which is equivalent to $S^\text{total}_z = 0$ in the spin representation of the Hamiltonian.

We note that the singlet and triplet$_{↑↓}$ states are entangled states with the value of the EE equal to $\ln(2)$; but triplet$_{↑↑}$ and triplet$_{↓↓}$ states are not entangled. Since the ground state (a typical excited state) of the system in the RSRG (RSRG-X) method is the product state of singlets (singlets and triplets), only those singlets (singlets and triplet$_{↑↓}$) that cross the boundary contribute to the EE. Thus, for a bi-partitioned system, to calculate the EE, we count the number of singlets (number of singlets and triplet$_{↑↓}$) crossing the boundary and multiply it by $\ln(2)$.

The numerical verification of the RSRG and RSRG-X methods to calculate the EE have been studied before.

At $T = 0$, all spins that are decimated in the RSRG method are in the singlet state. Namely, the probability of having a singlet, $P_s$, is 1, and the probability of having each of the triplets is 0. Thus we expect $L/2$ singlets across the entire system. On the other hand, we expect that each of the singlet and the triplets are chosen with the same probability for a large $T$ in the RSRG-X method. I.e., $P_s = P_{\text{triplet}_1} = P_{\text{triplet}_2} = \frac{1}{2}$. See Table 1. As $T \rightarrow \infty$, the Boltzmann probability of the singlet and triplets are the same. Therefore, there are $L/4$ singlets and $L/4$ triplet$_{↑↓}$s. In addition, for an arbitrary $T \neq 0$, we need to calculate the average probabilities weighted with $P(J)$:

$$\langle P_s \rangle = \int_{-1}^{1} dJ P(J) e^{J/2T} T,$$

$$\langle P_{\text{triplet}_1} \rangle = \int_{-1}^{1} dJ P(J) e^{-J/2T} T,$$

and thus, the average probability of having a singlet and a triplet$_{↑↓}$ at an arbitrary temperature $T$ is:

$$\langle P_s + P_{\text{triplet}_1} \rangle = \alpha(2T)^\alpha \int_0^{\pi} dx \frac{x^{\alpha-1}}{1 + \text{sech}(x)}$$

There is no simple analytical solution for this integral, so we will calculate it numerically. Finally, we should note that the RSRG is not an exact method, and it is asymptotically correct.

2.2. EE in random partitioning

Now, we explain how to calculate the EE for a bi-partitioned system in which the sites that belong to the subsystem are chosen randomly. First, we specify a probability $p_i$ for each site $i$ to belong to the subsystem based on a probability distribution. The subsystem size $n$, can vary from 1 to $L$, and for each of them, there are $\binom{L}{n}$ different ways of choosing $n$ sites out of $L$ sites. In addition, for each of these choices, there is a corresponding probability that each of the $n$ sites belongs to the subsystem and other sites do not belong to the subsystem: $\prod_{i \in A} p_i \prod_{i \notin A} (1 - p_i)$. Thus the EE for a specific probability distribution in the random partitioning EE($T, \{p\}$) is the following:

$$\text{EE}(T, \{p\}) = \sum_{n=1}^{L-1} \sum_{j=1}^{\binom{L}{n}} \text{EE}_n(T) \left( \prod_{i \in A} p_i \right) \left( \prod_{i \notin A} (1 - p_i) \right),$$

where $\text{EE}_n$ is one of the $\binom{L}{n}$ calculated EE’s corresponding to the case of having $n$ sites in the subsystem.

In the particular case of uniform probability, where the probability for each site to belong to the subsystem is the same for all sites ($p_i = \text{constant} = p$) we can use the above equation to obtain the EE in random partitioning with constant probability distribution EE($T, p$):

$$\text{EE}(T, p) = \sum_{n=1}^{L-1} \sum_{j=1}^{\binom{L}{n}} \text{EE}_n(T) p^n(1 - p)^{L-n},$$

In practice, we can not go over all samples of $\binom{L}{n}$ choices for a large $L$; instead, we take the average over enough large number of samples to obtain $\text{EE}_n$, and thus we have:

$$\text{EE}(T, p) = \sum_{n=1}^{L-1} \overline{\text{EE}_n(T)} \binom{L}{n} p^n(1 - p)^{L-n},$$
Since \( \binom{L}{n} = \binom{L-n}{n} \), we can deduce from Eq. (10) that \( \text{EE}(T, p) = \text{EE}(T, 1-p) \), and thus we would expect a symmetric plot for \( \text{EE}(p) \) versus \( p \) about \( p = 1/2 \).

To obtain an analytical expression for \( \overline{\text{EE}}_n(T) \), we do the following: at \( T = 0 \), since the state of the system is pure, we expect that \( \overline{\text{EE}}_n \approx \text{EE}_{L-n} \); thus, we guess that we can write \( \overline{\text{EE}}_n \approx n(L-n) \). To obtain the proportionality at zero temperature, we note that all bonds are singlet; therefore, having one site as the subsystem will yield to \( \overline{\text{EE}}_{n+1} = \ln(2) \times 1 \); thus the proportionality is \( \frac{\ln(2)}{L-1} \), and:

\[
\overline{\text{EE}}_n(T = 0) = \frac{\ln(2)}{L-1} n(L-n). \tag{11}
\]

Replacing this result in Eq. (10), we obtain the following expression for the \( \text{EE}(p) \) at \( T = 0 \):

\[
\text{EE}(T = 0, p) = \ln(2)p(1-p)L. \tag{12}
\]

In addition, for an arbitrary \( T \), as we argued above, on average, only \( (P_s + T_{\uparrow\downarrow}) \) fraction of the bonds contribute to the EE and thus:

\[
\text{EE}(T, p) = \ln(2)(P_s + T_{\uparrow\downarrow})p(1-p)L \tag{13}
\]

\[
= a(T, p) L \tag{14}
\]

where, \( a(T, p) = \ln(2)(P_s + T_{\uparrow\downarrow})p(1-p) \). In conclusion, we see a volume law expression for the random partitioning EE with \( a(T, p) \) as the pre-factor as a function of temperature and probability. We present numerical calculations in the next section that verify our analytical results.

3. Numerical verification

In our numerical calculations of the EE in a random partitioning with a uniform probability, we do the following. First, we apply the RSRG/RSRG-X to obtain the approximate ground state/typical excited state of the system corresponding to a specific system size. To work in the strong disorder regime where the RSRG method is asymptotically correct, we set \( \alpha = 0.2 \). Then we randomly choose sites that belong to the subsystem. By counting the number of singlets and triplets \( \uparrow\downarrow \) that cross the boundary of the chosen subsystem, we calculate EE for that chosen subsystem. We repeat this process large enough times and calculate its average \( \overline{\text{EE}}_n \). In the RSRG-X process, since each singlet or triplet is chosen based on the Boltzmann distribution, we need to take the ensemble average for each temperature \( T \). In addition, since the coupling constants are random, we also take the disorder average over random \( (J) \) realizations. After doing these averaging calculations, the \( \text{EE}(T, p) \) is obtained.

First, we check the symmetric property of \( \overline{\text{EE}}_n = \overline{\text{EE}}_{L-n} \), meaning that the \( \overline{\text{EE}}_n \) has to be symmetric about \( n = L/2 \). In addition, we compare the \( \overline{\text{EE}}_n \) calculated numerically with Eq. (11). As we can see in Fig. 2, the plot of \( \overline{\text{EE}}_n \) numerically obtained is symmetric about \( L/2 \), and also it fairly matches with Eq. (11).

Next, we check the \( L \) dependence of the \( \text{EE}(T, p) \). We plot the numerical data of EE versus system size \( L \) and the fitted straight line. This comparison is plotted in Fig. 3 for different temperatures and probabilities. The sum of squared residuals of the least squares fits are also denoted; since they have very small values, we can conclude that the straight-fitted lines are fitted the numerical data very well. As a double check, we also fit the logarithm of the EE versus the logarithm of the system size with a straight line, and we find that the slope is very close to 1 (see Fig. 4). Thus, we conclude that \( \text{EE} \propto L \), i.e., the EE has power-law behavior for system size \( L \).

Now that we know the system size dependence is a power-law with power 1, we write \( \text{EE} = a(T, p)L + c \) (we add the y-intercept \( c \) to be determined with numerical calculations) and study the behavior of coefficient \( a \) and \( c \) as a function of temperature \( T \) and probability \( p \). One way to do this, is to fit the data of EE versus \( L \) with a straight line and obtain the slope and the y-intercept of the fitted line numerically. The results of these calculations are plotted in Figs. 5 and 6. From the behavior of \( a \) versus probability, we can see that it is symmetric about \( p = 1/2 \), consistent with the analytical result of Eq. (13) (see left panel of Fig. 5).

In addition, from the behavior of \( a \) versus temperature, we can see that it approaches constant values at low and high temperatures. These constants are consistent with the analytical values of \( a \) in Eq. (14), which are \( \ln(2) \times p(1-p) \) in low temperatures, and \( \frac{1}{2} \ln(2) \times p(1-p) \) for high temperatures (Since \( \langle P_s + T_{\uparrow\downarrow} \rangle \) goes to 1 in the low \( T \) limit and it goes to \( \frac{1}{2} \) in the high \( T \) limit. See the middle panel of Fig. 5).

In the right panel of Fig. 5, we do the following. First, for the numerically obtained values of \( a \), we plot \( \frac{a}{p(1-p)} \) for some selected values of \( p \). As we can see, they all coincide with each other. I.e., the only probability dependence is in the form of \( p(1-p) \). In addition, we can see that the numerically obtained value of \( a \) goes to \( \ln(2) \) in the low \( T \) limit, and it goes to \( \frac{1}{4} \ln(2) \) in the high \( T \) limit, which are consistent with the analytical result of the \( \frac{a}{p(1-p)} = \ln(2)(P_s + T_{\uparrow\downarrow}) \). For an arbitrary temperature \( T \), numerical data and the analytical predictions of \( a \) are in a fair agreement. The difference between the numerically and analytically obtained values of \( a \) stems from the fact that we use an approximate RSRG method to calculate the EE. We also note that, in a numerical calculation, it is always possible to benefit from larger system sizes to avoid finite-size scaling. Finally, we can see that the numerically obtained values of \( c \) plotted in Fig. 6 are very small compared to the EE values; they are thus negligible.

4. Conclusion and outlook

The usual way of measuring the entanglement properties of a system is to bipartite it into two subsystems, and then obtain the non-local entanglement properties by calculating the EE. What we did in this paper is distinct: each site has a chance to be part of the subsystem, subsystem size is also varying, and in addition, for each subsystem size, we do all different ways of partitioning, and then we take the average over the calculated EE’s.
In such wise, we measure long-range as well as short-range correlations in the system. In other words, when we calculate the EE for a bi-partitioned system, we measure how much subsystem is entangled with the environment, in which long-range correlations and also near-to-the-boundary short-range correlations take part. On the other hand, we are measuring both the short-range and long-range correlations in the entire system when we randomly partition the system in all possible ways.

Considering the XX spin chain, if we cut the system in the middle and then calculate the EE, we are counting the number of entangled bonds (singlet and triplet) that cross the middle of the system, and thus we are counting the long bonds or those short bonds that are close to the boundary. But, since in random partitioning the partitions are random and they can be disconnected as well, we are counting the entangled bonds, both with short and long lengths. In averaging over all such partitioning, we thus count the number of singlets and triplet all over the entire system. This is, of course, in agreement with the analytical point of view that \( N_s + N_{t_1} = \langle P_s + P_{t_1}^s \rangle \times \frac{L}{2} \), and thus we can rewrite the EE(\( T, p \)) of Eq. (13) as the following:

\[
EE(T, p) = 2 \ln(2)p(1 - p)(N_s + N_{t_1})
\]  

(15)

In Fig. 4, we plot the number of singlets and triplet all forming in the entire system as a function of the temperature. In the low \( T \) limit, all bonds are only singlet and triplet, and the sum goes to \( L/2 \). On the other hand, in the high-temperature limit, the singlet and the three triplets have the same probability, so the sum goes to \( L/8 \). As a numerical check, we also plot and compare Eq. (15) with the numerically obtained data for the EE. We see full agreement.

The XX spin chain, which we employed in this paper, gives us a schematic representation of the bonds forming in the system. However, this picture is not always available. So in general, to see both short- and long-range correlations, we can use the random partitioning method. In particular, the behavior of the EE in a Hamiltonian with local and non-local interactions would be interesting. The random partitioning can also be used in characterizations of the phase transition. For example, in the Anderson delocalized-localized phase transition.
Figure 5: The EE versus system size $L$ is fitted with the straight line $aL + c$, and the results of the behavior of $a$ as a function of probability (left panel) and temperature (middle panel) are plotted. In the middle panel, the horizontal lines are $\ln(2)p(1-p)$ and $\frac{1}{2}\ln(2)p(1-p)$ for each $p$. In the right panel, the numerically obtained $\frac{a(T,p)}{p(1-p)}$ are plotted for different probabilities, $p$. We can see that they coincide. In addition, The result of the analytically obtained $\frac{a(T,p)}{p(1-p)}$ based on Eq. (15), is plotted with a dashed line. These two numerical and analytical results nearly match. Two horizontal lines of $\ln(2)$ and $\ln\frac{1}{2}$ are also plotted. For each data point, we take disorder average over $\sim 10^5$ samples, subsystem average over $\sim 10^5$ samples, ensemble average over $\sim 10$ samples and thus we take the average over $\sim 10^5$ samples in total.

Besides, we only considered the uniform probability distribution, i.e., each site has the same chance to belong to the subsystem. Considering non-uniform probabilities could also be helpful and give us more physical insights.

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