Effective Approaches to QCD

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Abstract In this lecture I will explain the established pictures of the QCD vacuum and, in particular, the underlying confinement mechanism. These are: the magnetic monopole condensation (dual Meißner effect), the center vortex picture and the Gribov–Zwanziger picture. I will start by giving a survey of the common order and disorder parameters of confinement: the temporal and spatial Wilson loop, the Polyakov loop and the ’t Hooft loop. Next the dual Meißner effect, which assumes a condensate of magnetic monopoles, will be explained as a picture of confinement. I will also show how magnetic monopoles arises in QCD after the so-called Abelian projection.

The second lecture is devoted to the center vortex picture of confinement. Center vortices will be defined both on the lattice and in the continuum. Within the center vortex picture the emergence of the area law for the Wilson loop as well as the deconfinement phase transition at finite temperature will be explained. Furthermore, lattice evidence for the center vortex picture will be provided. Finally, I will discuss the topological properties of center vortices and their relation to magnetic monopoles. I will provide evidence from lattice calculations that center vortices are not only responsible for confinement but also for the spontaneous breaking of chiral symmetry.

In the last lecture I will present the Hamiltonian approach to QCD in Coulomb gauge, which is then used to establish the Gribov–Zwanziger picture of confinement. Furthermore, I will also relate this scenario to the dual Meißner effect and the center vortex picture. Finally, I will study QCD at finite temperature within the Hamiltonian approach in a novel way by compactifying one spatial dimension. The Polyakov loop and the dual and chiral quark condensates will be evaluated as function of the temperature.

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1 QCD and phases of strongly interacting matter

By the current state of knowledge quarks and gluons are considered as elementary particles. Under normal conditions they are confined inside hadrons and they acquire mass through the mechanism of spontaneous breaking of chiral symmetry (SBCS). On the other hand, under extreme conditions – high temperature or high density – quarks and gluons loose their confinement and form the so called Quark-Gluon plasma. The deconfinement transition is accompanied by a restoration of chiral symmetry. The phase diagram of QCD is schematically pictured in Fig. 1. The detailed understanding of the phase diagram of QCD is one of the key challenges of particle and nuclear physics.

By means of ultra-relativistic heavy-ion collisions the properties of hadronic matter at high temperature and density are presently explored. From the theoretical point of view we have access to the finite temperature behavior of QCD by means of lattice calculations. Due to the notorious sign problem (the quark determinant becomes complex for gauge groups $SU(N \geq 3)$ and finite chemical potential), this method fails, however, to describe baryonic matter at high density or, more technically, at large chemical potential. Various methods have been invented to overcome this problem like e.g. Taylor expansion in the chemical potential or analytic continuation of the chemical potential to imaginary values. However, these methods work only up to chemical potential of about twice the value of the temperature. Therefore, alternative, non-perturbative continuum approaches to QCD, which do not suffer from the sign problem, are needed. Furthermore, we have not yet fully understood the confinement mechanism itself. A thorough understanding of this mechanism certainly will not come from numerical simulations alone. (A strict analytic proof of confinement was formulated as one of the millennium problems of the Clay Mathematics Institute.) However, during the last two decades a couple of consistent pictures of the QCD vacuum have emerged, which will be the subject of this lecture.
1.1 Confinement

All the presently known pictures of the QCD vacuum are formulated in or rely on a certain gauge. For example, the magnetic monopole condensation picture (dual Meißner effect) \[1, 2\] relies on the maximal Abelian gauge \[1\]. Similarly, the center vortex condensation picture \[3, 4, 5, 6, 7, 8\] was established in the maximal center gauge \[9\]. Finally, there is the Gribov–Zwanziger mechanism \[10, 11\], which relies on Coulomb gauge.

We may argue that confinement is a gauge invariant phenomenon, so we should also have a gauge invariant description of it. However, this may be too ambitious a goal. Let me remind you that even the parton picture makes sense only for certain gauges, like the light-cone gauge (see the lecture by J. P. Blaizot). So if the parton picture can be realized only for certain gauges we should not be surprised that in the non-perturbative regime the explanation of confinement requires fixing the gauge.

Before we develop the various pictures of confinement let us summarize some important phenomenological aspects of confinement and chiral symmetry breaking. In this context, I will also summarize the various order parameters frequently used in QCD to distinguish its phases. They will show up when we explain the pictures of confinement.

![Fig. 2 Illustration of the valence quark structure of hadrons: top left-baryon, top right-antibaryon, bottom-meson.](image)

The strongly interacting particles, i.e. the hadrons, are built up from so-called valence quarks, which interact via the exchange of gluons. Mesons are built up from a quark and an anti-quark while baryons are formed by three quarks. Besides these valence quarks there are also “sea” quarks which fill the negative energy Dirac sea. Although the quarks carry, besides electric charge, also a “color” charge all hadrons are colorless. Colored states do not exist in nature according to the confinement hypothesis. The simplest hadronic system is a heavy meson consisting of a heavy
quark and a heavy anti-quark. These particles can be described by potential models and the corresponding potential can be measured on the lattice, see Fig. 3.

The lattice QCD simulations show that at large distances the potential grows linearly with the distance between the two static color charges, with a coefficient referred to as string tension (taken in units of MeV$^2 \equiv$ energy per length)

$$V(r) = -\frac{\alpha_s}{r} + \sigma r, \quad \sigma = (440 \text{ MeV})^2. \quad (1)$$

Converting this quantity to more familiar units one obtains

$$\sigma = 157 \text{ kN}, \quad (2)$$

which corresponds to the weight of a mass of about 16 tons and which justifies calling this type of interactions ‘strong.’ At short distance the potential behaves like an ordinary Coulomb potential, with a coefficient given by the running coupling constant $\alpha_s$. This quantity can be evaluated in perturbation theory in the high-momentum regime (Fig. 4, left), while non-perturbative methods are required in the low-momentum regime, where it saturates (Fig. 4, right).

1.2 Chiral symmetry

The quarks receive a mass through the Higgs mechanism. This is the so-called “current mass”, which enters the QCD Lagrangian. This mass is small for the light quark flavors $u, d$ (and $s$), see Table I (top). There is, however, a large discrepancy between the sum of bare (current) masses of the valence quarks inside a hadron and the mass of the hadron that they form (see Table. I, bottom). Ordinary hadrons, like the proton or the neutron, receive most of their mass through the mechanism
of spontaneous breaking of chiral symmetry (SBCS), which converts the light bare quarks into massive constituent quarks (see the lecture by R. Pisarski).

\[ m_u \approx 2.3 \text{ MeV} \quad m_d \approx 4.8 \text{ MeV} \quad m_s \approx 95 \text{ MeV} \]

| Particle | Quark Content | Mass (in MeV) |
|----------|---------------|---------------|
| p        | uud           | 938           |
| n        | udd           | 939           |
| \( \rho^+ \) | ud       | 775           |
| \( \omega \) | \( \bar{u} \bar{d} + \bar{d} \bar{u} \) | 782           |
| \( \phi \) | \( \sqrt{2} \bar{s} \bar{s} \) | 1020          |

Table 1: Current quark masses (top) and masses of baryons (bottom). One can see that the small quark masses do not add up to the masses of the baryons that they form.

Let us focus now on the mass term of the QCD Lagrangian:

\[
\mathcal{L}_m = m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s = \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{3}
\]

Assuming the light quark masses to be equal \( m_u = m_d = m_s \), the QCD Lagrangian has the flavor symmetry, i.e., it is invariant under the transformation:

\[
\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow U \begin{pmatrix} u \\ d \\ s \end{pmatrix},
\tag{4}
\]

where \( U \) is an arbitrary unitary 3×3-matrix. The flavor transformation \( (4) \) mixes the light quark flavors and the entirety of such transformations forms the flavor group \( SU(N_F = 3) \). If we neglect the small current masses \( m_u = m_d = m_s = 0 \) the QCD Lagrangian is also invariant under the chiral transformation:
which differs from the flavor symmetry by the $\gamma_5$ in the exponent of the flavor matrix $U$. Expressing $\gamma_5$ in terms of the left and right projectors

$$P_{R/L} = \frac{1}{2} (1 \mp \gamma_5),$$

we find

$$U^{\gamma_5} = P_R U + P_L U^{-1}. \quad (7)$$

The left and right handed quarks transform oppositely in flavour space under a chiral transformation, while left and right quarks transform in the same way under a flavour transformation.

Obviously chiral symmetry makes no sense for the heavy quark flavors ($c, b, t$). It is, however, a good approximation for the up and down quarks and a good starting point for the strange quark sector.

In QCD chiral symmetry is broken in three different ways:

- Explicit breaking by the current quark masses, see eq. (3).
- Spontaneous symmetry breaking (most important in the context of this lecture).
- Anomalous breaking due to quantum effects (see R. Pisarski’s lecture).

![Fig. 5](image)

**Fig. 5** Consequences of the various forms of chiral symmetry breaking for the pseudoscalar meson masses.

The consequences of the various forms of breaking the chiral symmetry for the masses of the pseudo-scalar mesons are the following (cf. Fig. 5):

- If one ignores the current quark masses the pseudoscalar mesons (identified with Goldstone bosons of the SBCS) become exactly massless.
- Switching on the chiral (axial) anomaly (the violation of the chiral symmetry by quantum effects) induces a mass for the $\eta_1$ meson of about 1 GeV.
Including also the current quark masses, the pseudoscalar mesons become massive and furthermore $\eta_1$ and $\eta_8$ mix, resulting in $\eta$ and $\eta'$. 

### 1.3 Order parameters in QCD

As mentioned in the introduction QCD exists in several phases depending on the external conditions like temperature or matter density. The different phases are distinguished by order parameters, which I summarize below.

#### 1.3.1 Quark condensate

The order parameter of the chiral symmetry breaking is the **quark condensate** $\langle \bar{q}q \rangle$. It is non-zero in the hadronic phase and vanishes in the deconfined phase, see Fig. 6. From hadron phenomenology one extracts the value $\langle \bar{q}q \rangle = (-230 \text{ MeV})^3$. A non-zero quark condensate violates the chiral symmetry in analogy to what happens in superconductors — the electron pair condensation violates the particle number conservation. We will see later that in many respects the QCD vacuum indeed resembles a superconductor: the quark sector behaves like an ordinary superconductor (the quark vacuum wave functional looks, in a certain approximation, like a BCS-state, see Sec. 4.7), while the gluon sector can be interpreted as a **dual** superconductor (see Sec. 2).
1.3.2 The Wilson loop

At zero temperature the most familiar order parameter of confinement is the Wilson loop which is defined by\(^3\)

\[ W[A](C) = \frac{1}{d_r} \text{tr} P \exp \left( ig \oint_C dx^\mu A_\mu(x) \right), \tag{8} \]

where \( d_r \) is the dimension of the (group) representation of the gauge field \( A_\mu \). Furthermore \( C \) is an arbitrary closed loop in four-dimensional Euclidean space and \( P \) denotes the path ordering along this loop. For a non-Abelian gauge theory the gauge field \( A_\mu = A_\mu^a t_a \) is matrix-valued and the path ordering complicates the evaluation of \( W(C) \) considerably.

To exhibit the physical meaning of the Wilson loop let us rewrite it as

\[ W[A](C) = \text{tr} P \exp \left( i \int dx^4 \mathcal{J}^\mu(x; C) A_\mu(x) \right), \tag{9} \]

where

\[ \mathcal{J}^\mu(x; C) = g \oint_C dy^\mu \delta(x - y) \tag{10} \]

is the current generated by a point color charge on the trajectory \( C \). The integrand in Eq. (9) is the interaction Lagrangian of a unit point charge on the trajectory \( C \) with a gauge field \( A_\mu(x) \).

As we will see later, the expectation value of the Wilson loop obeys an area law in a confining theory and a perimeter law in a non-confining theory

\[ \langle W(C) \rangle = \begin{cases} e^{-\sigma A(C)}, & \text{confinement,} \\ e^{-\kappa P(C)}, & \text{deconfinement.} \end{cases} \tag{11} \]

Here \( A(C) \) is the area included by the loop \( C \) and \( P(C) \) is its perimeter. The coefficient \( \sigma \) is called string tension for reasons which will be explained below. Due to the property \((11)\) the Wilson loop is an order parameter of confinement.

At zero temperature the Euclidean space-time manifold is \( \mathbb{R}^4 \), and the theory exhibits an O(4)-symmetry, i.e. the position and orientation of the loop \( C \) in Euclidean space is irrelevant for the behavior of the Wilson loop \( \langle W(C) \rangle \). This changes at finite temperature \( T = L^{-1} \), where the fields in the functional integral of the grand canonical partition function \( \text{Tr} \exp(-L H) \) have to satisfy periodic (for Bose fields \( A \)) and anti-periodic (for Fermi fields \( \Psi \)) boundary conditions in the Euclidean time \( x^4 = i x^0, x^0 = ct \)

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\(^3\) This quantity was originally introduced as an order parameter by F. J. Wegner \cite{14}.

\(^4\) \( t_a \) are the (hermitian) generators of the gauge group satisfying \([t_a, t_b] = if_{abc} t_c\), where the \( f_{abc} \) are the structure constants.
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\[ A_\mu(x^4 = L/2) = A_\mu(x^4 = -L/2), \]
\[ \Psi(x^4 = L/2) = -\Psi(x^4 = -L/2). \]  

These boundary conditions compactify the Euclidean space-time manifold to \( S^1(L) \times \mathbb{R}^3 \) (see Fig. 7). In this case it matters whether \( C \) is a closed loop in \( \mathbb{R}^3 \) (existing only during a single time instant) or whether \( C \) evolves also along the (Euclidean) time axis. In the latter case \( W(C) \) is called temporal Wilson loop, in the former case spatial Wilson loop.

![Fig. 7 Illustration of the Euclidean space-time manifold \( S^1(L) \times \mathbb{R}^3 \) at finite temperature \( L^{-1} \).](image)

To exhibit the physical meaning of the temporal Wilson loop consider the case where \( C \) is a rectangular loop with its edges running parallel or perpendicular to the time-axis, see Fig. 8. This loop can be interpreted as the creation of a particle and an antiparticle at some initial time, which are separated then by a spatial distance \( R \), followed by the time-evolution of the particle and anti-particle over a time-period \( T \) and the subsequent annihilation of the particle and antiparticle pair. It can be shown that for \( T \gg R \) the temporal Wilson loop is related to the particle-antiparticle potential \( V(R) \) by

\[ \langle W(C) \rangle \sim e^{-V(R)/T}, \quad T \to \infty. \]  

In a confining theory with \( \langle W(C) \rangle \) obeying the area law (11) this potential obviously rises linearly at large distances

\[ V(R) \sim \sigma R, \quad R \to \infty. \]  

To exhibit the physical meaning of a spatial Wilson loop let us consider QED where the path ordering is irrelevant and the trace does not occur since \( d_r = 1 \):

\[ W(C) = \exp \left\{ ig \oint_C d\tau^\mu A_\mu(x) \right\}. \]  

For a spatial Wilson loop the path \( C \) is a closed loop in \( \mathbb{R}^3 \). Using Stoke’s theorem we obtain

\[ W(C) = \exp \left\{ ig \oint_{\Sigma(C)} df \cdot B \right\}, \]  

where \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic field and \( \Sigma(C) \) any area bounded by \( C \). The quantity in the exponent
\[ \Phi = \oint_{\Sigma(C)} df \cdot B \]  

(17)

is the magnetic flux through the loop \( C \). The spatial Wilson loop \( \langle W(C) \rangle \) measures the magnetic flux through \( C \) also in a non-Abelian gauge theory. However, in that case a non-Abelian version of Stokes’ theorem is required to establish this connection.

![Fig. 8 The temporal Wilson loop.](image)

At zero temperature the spatial and temporal Wilson loops behave in exactly the same way due to the exact O(4) symmetry of the Euclidean space-time manifold \( \mathbb{R}^4 \). This symmetry is spoiled at finite temperature \( L^{-1} \) due to the finite length \( L \) of the Euclidean time axis and its compactification caused by the (anti-)periodic boundary condition (12). For non-zero temperatures below the deconfinement transition the spatial and temporal string tension are approximately equal to their common zero temperature value. During the deconfinement transition the temporal string tension drops to zero, while the spatial string tension slightly increases with the temperature.

At finite temperatures \( L^{-1} \geq R \) the temporal Wilson loop can no longer serve as order parameter of confinement, since the static quark–anti-quark potential can no longer be extracted from it. In this case there exists a more efficient order parameter, which is the Polyakov loop.

### 1.3.3 The Polyakov loop

Consider a rectangular temporal Wilson loop in finite temperature QED which extends over the whole Euclidean time axis. Due to the compactification of the time axis the two spatial pieces (of length \( R \)) of the loop \( C \) fall on top of each other, see Fig. [9]. Since they are traveled in opposite direction their contribution to the Wilson loop cancel. What is left from the original Wilson loop are two loops (both of length \( L \)) a spatial distance \( R \) apart, running in opposite direction along the compactified time-axis. Wilson loops running along the whole compactified time axis
are referred to as Polyakov loops. What we have shown above is that (in QED) a temporal rectangular Wilson loop of spatial extension $R$ running along the whole compactified time-axis is equivalent to the product $P(x)P^\dagger(x + R)$ of two Polyakov loops. With Eq. (13) this implies that

$$\langle P(x)P^\dagger(x + R) \rangle \sim e^{-LV(R)}. \quad (19)$$

It can be shown that this equation remains valid in the non-Abelian case provided the r.h.s. is replaced by its color average \[15\]

$$e^{-LV(R)} \rightarrow e^{-LV_{\text{singlet}}(R)} + (N^2 - 1)e^{-LV_{\text{adjoint}}(R)}, \quad (20)$$

where “singlet” and “adjoint” refer to the color representations of the external color sources.

Instead of the Polyakov loop correlator \[19\] one can use the Polyakov loop itself as order parameter. This is because the Polyakov loop is related to the free energy of a static quark:

$$\langle P(x) \rangle = e^{-L(F_q - F_0)}, \quad (21)$$

Here $F_q$ and $F_0$ are the free energies in the presence and absence, respectively, of an isolated quark. For confined systems isolated quarks have infinite free energy and the Polyakov loop vanishes:

$$(F_q - F_0) \rightarrow \infty \quad \Rightarrow \quad \langle P(x) \rangle = 0. \quad (22)$$

In the deconfined phase, however, the free energy of a single quark is finite and the Polyakov loop is non-zero (see Fig. \[10\])

$$(F_q - F_0) = \text{finite} \quad \Rightarrow \quad \langle P(x) \rangle \neq 0. \quad (23)$$

Therefore, at finite temperature the Polyakov loop can serve as order parameter for confinement.
1.3.4 The spatial ’t Hooft loop

There is yet another order parameter, or precisely a disorder parameter, of confinement, which is the spatial ’t Hooft loop \[1\]. The proper definitions will be given later in Eq. (104), but it is worthwhile to introduce it already here in the context of the other order parameters.

While the spatial Wilson loop measures the magnetic flux, the spatial ’t Hooft loop \( V(C) \) measures the electric flux through the loop \( C \) (Fig. 11). One can show that its expectation value obeys a perimeter and an area law in the confined and deconfined, respectively, phase \[16\]

\[
\langle V(C) \rangle = \begin{cases} 
  e^{-\tilde{\kappa} P(C)}, & \text{confinement,} \\
  e^{-\tilde{\sigma} A(C)}, & \text{deconfinement.}
\end{cases}
\] (24)

This is opposite to the temporal Wilson loop, c.f. Eq. \[11\].

After having summarized some basic features of QCD we will now attempt to explain how these features emerge from the underlying theory. We will focus our attention on the confinement phenomenon and the deconfinement phase transition at finite temperature. Further issues will be the spontaneous breaking of chiral symmetry and the topological properties of gauge fields. We begin with the dual Meißner effect as a possible explanation of confinement.
2 The Magnetic monopole picture of confinement

When a normal material is brought into a magnetic field the field lines are somewhat disturbed but pass through the material, see Fig. 12. This is different for a superconductor. The magnetic field lines cannot penetrate into a type I-superconductor but are expelled from it (Meißner effect) due to presence of the so-called London currents which are induced at its surface by the magnetic field. The situation is different in a type II superconductor. If the magnetic field exceeds a critical strength it can pass through a type II superconductor in the form vortex lines, i.e. small magnetic flux tubes. Inside these flux tubes the material is in its normal conducting phase.

2.1 The dual Meißner effect

In the vacuum the magnetic field of a (hypothetical) magnetic monopole-antimonopole pair is given by Coulomb’s law like the electric field of two opposite (electric) point charges. When the monopole-antimonopole pair is brought into a type II supercon-
ductor the field lines are squeezed into magnetic flux tubes and as a consequence the static potential between the magnetic point charges is no longer Coulombic but linearly rising. This implies that magnetic monopoles are confined inside a type II superconductor.

Fig. 13 (left) Confinement of color charges in a condensate of magnetic monopoles. (right) (Electric) charge-anticharge potential in a normal material (or in the vacuum) and in a dual superconductor, respectively.

The QCD vacuum can be understood as a so-called dual superconductor. Dual means that the roles of electric and magnetic fields and charges are interchanged. While the ordinary superconductor consists of a condensation of electron pairs, the dual superconductor consists of a condensation of magnetic monopoles.

In a dual superconductor the electric field lines emitted from a pair of opposite electric charges are squeezed into flux tubes resulting in a linearly rising potential and thus confinement of electric charges (dual Meißner effect). The quarks carry color electric charges and are then confined in a dual (color) superconductor. The dual superconductor picture of the QCD vacuum was suggested independently by Nambu [17], ’t Hooft [1] and Mandelstam [2]. Dual superconductor models of the QCD vacuum analogous to the Ginzburg–Landau type theories of the ordinary superconductor were developed and are reviewed in Ref. [18].

2.2 Emergence of magnetic monopoles in QCD

The dual superconductor picture of the QCD vacuum, giving rise to a linear quark–anti-quark potential, has been confronted with lattice QCD calculations. Strong evidence for the realization of the dual Meißner effect in the QCD vacuum was found in the so-called maximal Abelian gauge (suggested by ’t Hooft in Ref. [1]). Let us consider as an example the color group SU(2). We choose the generator $T_3$ to generate the Abelian subgroup U(1) while the remaining generators $T_1$, $T_2$ belong to the coset SU(2)/U(1).
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\[ SU(2) = \frac{U(1)}{SU(2)/U(1)} \times SU(2)/U(1) \]  

(25)

The non-Abelian components of the gauge field \( A^1, A^2 \) are charged with respect to the Abelian U(1) subgroup while the Abelian field \( A^3(x) \) is, of course, (color) neutral, like the photon.

The maximal Abelian gauge is defined by minimizing the norm (module) of the non-Abelian components of the gauge field

\[ \int d^3x \left[ (A^1_\mu(x))^2 + (A^2_\mu(x))^2 \right] \longrightarrow \text{min} \]  

(26)

Having implemented such a gauge, one performs an Abelian projection, putting the non-Abelian components of the gauge fields to zero

\[ A^1_\mu(x) = A^2_\mu(x) = 0. \]  

(27)

On the lattice one finds that after Abelian projection the (temporal) string tension is found to be about 95% of the string tension of the full (i.e. not Abelian projected) theory

\[ \sigma_{\text{AP}} \sim 0.95 \sigma_{\text{exp}}. \]  

(28)

Contrary to instantons, magnetic monopoles do not arise as stable classical solutions of Yang–Mills theory but are artifacts of the Abelian projection, i.e. they show up only in the Abelian projected configuration but not in the full (unprojected) gauge field:

Let \( V \) be the gauge transformation required to bring a given gauge field configuration \( A_\mu = gA_\mu^aT_a \) into the (maximal) Abelian gauge

\[ A^V_\mu = V\partial_\mu V^\dagger + V A_\mu V^\dagger. \]  

(29)

The magnetic monopoles show up in the Abelian projection of the induced gauge field \( V\partial_\mu V^\dagger =: a_\mu \), i.e. the Abelian magnetic field

\[ \mathbf{b} = \nabla \times \mathbf{a}^3 \]  

(30)

contains an ordinary Dirac monopole with a Dirac string. If this is the case then the Abelian component \( B^3 \) of the non-Abelian magnetic field

\[ B_k = \frac{1}{2} \varepsilon_{kij} [a_i, a_j] \]  

(31)

\[ ^5 \text{We use here anti-hermitian generators } T_a \text{ satisfying } [T_a, T_b] = if_{abc}T_c. \text{ Furthermore, we have absorbed the coupling constant } g \text{ into the gauge field. Sometimes we will also use hermitian generators } t_a \text{ satisfying } [t_a, t_b] = if_{abc}t_c, \text{ which are related to the anti-hermitian ones by } T_a = -it_a. \]
contains an anti-monopole without a Dirac string so that in the Abelian component of the total field strength

\[ F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \] (32)

of the induced gauge field \( a_\mu \)

\[ B_k^a = \frac{1}{2} \varepsilon_{kij} F_{ij}^a[a] = b_k + \bar{B}_k^a \] (33)

only the unobservable Dirac string survives but not a magnetic monopole, see ref. [19] for more details.

An interesting remark is that the charge of the magnetic monopole arising after Abelian projection is topologically quantized. To be more precise the magnetic charge of monopoles arising after Abelian projection is given by the winding number of the mapping of a two-sphere \( S^2 \) around the monopole onto the coset \( SU(2)/U(1) \) of the gauge group. These mappings fall into the second homotopy group \( \Pi_2(SU(2)/U(1) = \mathbb{Z}) \).

Magnetic monopoles arise in the Abelian projected field \( a^3 = (V \nabla V^\dagger)^3 \) if the original gauge field configuration contains topological gauge fixing obstructions, which turn into localized singularities (magnetic monopoles) under Abelian projection[6]. Though the magnetic monopoles are “artifacts” of the Abelian projections the topological obstructions are gauge invariant features of the original gauge field. It is therefore not surprising that the topological charge of a gauge field (see also Sec. 3.4)

\[ \nu = \frac{1}{32} \int d^4x \varepsilon^{\mu\nu\kappa\lambda} F_{\mu\nu}[A] F_{\kappa\lambda}[A] \] (34)

can be entirely expressed by the magnetic monopoles contained in the Abelian projected gauge field configuration, for more details see ref. [19], [20].

The lattice allows one to calculate observables from only the magnetic monopole content of the gauge fields by taking into account only Abelian projected configurations \( A^3 \) which contain magnetic monopoles. One finds then still about 90% of the full string tension. This fact was interpreted as evidence that magnetic monopoles are the dominant IR degrees of freedom which are responsible for confinement.

Lattice calculation in the maximal Abelian gauge were initiated in Ref. [21] and later on performed on large scale in Refs. [22; 23; 24].

A final comment is in order: One may ask about the role of instantons in the confinement mechanism. After all instantons are classical solutions of Yang–Mills theory in Euclidean space. On the lattice the instanton content of a gauge field can be extracted by the so-called cooling method. One finds that the instantons account only for 10% of the string tension. Hence these objects seem to be of minor relevance for confinement.

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6 This has been studied in detail in the Polyakov gauge (131) (which is another Abelian gauge), see refs. [19], [20].
3 The Center vortex picture of confinement

3.1 Introduction

In everyday life one can find vortices in fluids, like water or air. Intuitively, the vortex is the part of a fluid which rotates around some line, either closed or open. Its topology depends on the dimensionality of the considered system. In two dimensions vortices are isolated points while in three dimensions they form closed loops. Finally, in four dimensions vortices become closed surfaces, which may be self-intersecting and non-oriented. In a fluid the characteristic quantity of a vortex is its vorticity $\omega$, which is defined as the curl of the velocity field $\mathbf{v}$, $\omega = \nabla \times \mathbf{v}$.

There is a formal analogy between hydrodynamics and a gauge field theory: The gauge potential $A$ plays the role of the velocity field $\mathbf{v}$, while the magnetic field $\mathbf{B} = \nabla \times A$ corresponds to the vorticity $\omega$. This analogy is presented in more details in the following table.

| Fluid dynamics | Gauge theory |
|----------------|--------------|
| Velocity field $\mathbf{v}$ | Gauge potential $A$ |
| Vorticity | Magnetic field |
| $\omega = \nabla \times \mathbf{v}, \nabla \cdot \omega = 0$ | $\mathbf{B} = \nabla \times A, \nabla \cdot \mathbf{B} = 0$ |
| Circulation | Magnetic flux |
| $\oint_C \mathbf{v} \cdot d\mathbf{x}$ | $\oint_C \mathbf{A} \cdot d\mathbf{S}$ |
| Magnus force | Lorentz force |
| $\omega \times \mathbf{x}$ | $q\mathbf{B} \times \mathbf{x}$ |

This lecture is devoted to special vortices occurring in non-Abelian gauge theories: the center vortices. Lattice calculations give strong evidence that these objects are the dominant IR degrees of freedom of QCD, which are responsible for confinement and spontaneous breaking of chiral symmetry, as I will show later. Before defining the center vortices it is worthwhile to recall some facts from group theory.

The center $Z$ of a group $G$ contains all elements $z \in G$ which commute with every other element $g$ of this group,

$$Z := \{ z \in G \mid zg = zg \quad \forall g \in G \}. \quad (35)$$

The center of a group forms itself a group. The center of an Abelian group is the whole group, while some groups, for example $G(2)$, have a trivial center containing only the neutral group element. Due to their physical significance we are particularly interested in the special unitary groups $SU(N)$. The center of the $SU(2)$ group is the following set

$$Z(2) = \{-\mathbf{i}, \mathbf{i}\} = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \quad (36)$$

In general, the center of the $SU(N)$ group contains $N$ elements,

$$Z(N) = \{ z_k \mid k = 0, 1 \ldots N - 1 \}, \quad (37)$$
which are the $N$th roots of unity,

$$z_k = e^{2\pi i k/N},$$  \hspace{1cm} (38)

where $\hat{1}$ is the $N \times N$ unit matrix. Particularly, for the SU(3) group, which is the gauge group of QCD, the center is

$$Z(3) = \{ z_0 = \hat{1}, z_1 = e^{2\pi i/3} \hat{1}, z_2 = e^{4\pi i/3} \hat{1} \}.$$  \hspace{1cm} (39)

The center vortex is a specific configuration of a gauge field, which is defined as follows:

Consider a gauge field $A(C_1)$, whose (magnetic) flux is localized on a closed curve $C_1$ in three dimensions. This field forms a center vortex if a Wilson loop $W$ calculated along a curve $C_2$ picks up a non-trivial center element of the gauge group if the curves $C_1$ and $C_2$ are linked in a non-trivial way. Mathematically speaking, the gauge field $A(C_1)$ is a center vortex if

$$W[A(C_1)](C_2) = z^{L(C_1,C_2)},$$  \hspace{1cm} (40)

where

$$L(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx_i \oint_{C_2} d\mathbf{x}' e_{ijk} \frac{x_k - x'_k}{|\mathbf{x} - \mathbf{x}'|^3}$$  \hspace{1cm} (41)

is the Gauss’ linking number of the closed curves $C_1$ and $C_2$ and $z$ is a non-trivial center element. Since the linking number is a topological invariant, it does not depend on details of the shape of $C_2$ or $C_1$. For example, the group SU(2) has only one non-trivial center element, $-\hat{1}$, and hence there is only one type of center vortex field, satisfying $W[A(C_1)](C_2) = (-\hat{1})^{L(C_1,C_2)}$.

The above given definition can be generalized to arbitrary vortex surfaces in four dimensions. Let $A(\partial \Sigma)$ be a gauge field whose (magnetic or electric) flux is localized on a closed surface $\partial \Sigma$ in $\mathbb{R}^4$. This field forms a center vortex if its Wilson loop is given by

$$W[A(\partial \Sigma)](C) = z^{L(\partial \Sigma, C)},$$  \hspace{1cm} (42)

where $L(\partial \Sigma, C)$ is the linking number between the closed surface $\partial \Sigma$ and the closed loop $C$. (Note in $\mathbb{R}^4$ loops can be non-trivially linked only to two-dimensional surfaces.)

Lattice data suggests that center vortices are the dominant field configurations in the infrared region and are responsible for both confinement and the spontaneous breaking of chiral symmetry. Therefore it is worthwhile to discuss the properties of center vortices. These can best be studied on discretized space-time, i.e. on the lattice.
3.2 Center vortices on the lattice

In the lattice approach to gauge field theory one approximates the continuous spacetime by a discrete lattice of spacing $a$. Gauge fields are represented by link variables, $U_\mu(x) = \exp[i a A_\mu(x)]$. The calculation of the Wilson loop on the lattice is greatly simplified in comparison to continuum calculations. On the lattice the Wilson loop $W(C)$ is given by the ordered product of link variables $U_\mu$ along the chosen curve $C$

$$W[U](C) = P \prod_{\mu \in C} U_\mu.$$

The partition function on the lattice is given by a sum over all link configurations

$$Z[U] = \int \prod_x \prod_{\mu=0}^3 dU_\mu(x) e^{-S[U]},$$

where $S[U]$ is a lattice action. Its simplest form, the Wilson action, is given by

$$S[U] = \beta \sum_{\mu<\nu, x} \left[ 1 - \frac{1}{2N} (\text{Tr} P_{\mu\nu}(x) + \text{c.c.}) \right],$$

where $P_{\mu\nu}(x) = U_\mu(x) U_\nu(x+x+\mu) U^\dagger_\mu(x+x+\nu) U^\dagger_\nu(x)$ is the product of link variables around an elementary lattice square called a plaquette. This action is invariant with respect to local gauge transformations:

$$U_\mu(x) \rightarrow e^{i \Theta(x)} U_\mu(x) e^{i \Theta(x+\mu)},$$

where $\Theta(x) = \Theta^\mu(x) t_\mu$ . Lattice gauge theory is a powerful tool to study non-perturbative effects in gauge field theories. For more details, see the lecture by O. Kaczmarek.

By Eq. (40), center vortices can in principle be defined in a gauge invariant way. For practical purposes, however, one has to fix the gauge to isolate the center vortex content of the gauge fields. Then it turns out that the properties of the identified center vortices depend on the chosen gauge. So far, one has found only one gauge in which the emerging center vortices are physical in the sense that their density scales properly in the continuum limit [25]. This is the so-called maximal center gauge [9].

Let the gauge group be SU(2), for simplicity. In this case, link variables can be expressed by the $S_4$ parametrization,

$$U_\mu = \alpha_0 + i \alpha \cdot \tau,$$

where $\tau$ are the Pauli matrices and the real parameters $\alpha_\mu, \mu = 0, 1, 2, 3$ satisfy the constraint

$$\alpha_0^2 + \alpha^2 = 1,$$
which defines the unit sphere $S^3$ in the four-dimensional space with coordinates $\alpha_\mu$. The maximal center gauge is defined by bringing each link as close as possible to its nearest center element, which in this case is either 1 or $-1$. In other words, one exploits the gauge freedom to maximize $(\alpha_0)^2$. By the constraint \[ \text{Eq. (48)} \] this is equivalent to minimize $\alpha^2$. On the lattice this is done by maximization of the following functional:

$$\sum_{x,\mu} |\text{Tr} \, U_\mu(x)|^2 \to \text{max} \quad (49)$$

with respect to gauge transformations $g$. Once a gauge field configuration is brought into the ‘maximal center gauge’, the second step is to perform the center projection — each link is replaced by its nearest center element \[ \text{Eq. (9)} \]. In case of the SU(2) group the non-Abelian part $\alpha$ is set to zero and the Abelian part $\alpha_0$ is set to either 1 or $-1$. As a result of the center projection one obtains a lattice in which all links belong to the center of the gauge group, i.e. all links are center elements. With all links being equal to (trivial or non-trivial) center elements, the only non-trivial field configurations are center vortices. To see this, consider a line of non-trivial center elements on a two-dimensional lattice (purple lines on the left panel of Fig. 14, trivial center elements are not shown). The Wilson loop calculated around one of the ends of this line (the black contour) yields $(-1)$, while Wilson loops which do not encircle precisely one of the ends of the string of non-trivial center elements yield the trivial center element 1. Therefore, \[ \text{cf. Eq. (40)} \] there is a center vortex at the end of such a one-dimensional domain of non-trivial center elements.

On a three-dimensional lattice one finds surfaces of non-trivial center elements (see the right panel of Fig. 14) and the center vortices in this case are located at the boundaries of such surfaces. In this case, they form closed loops on the dual lattice or closed strings of plaquettes being equal to a non-trivial center element on the original lattice. Similarly, on a four-dimensional lattice, center vortices form closed surfaces. In general, on a $D$-dimensional lattice, the center projected vortices are given by the $D-2$-dimensional boundaries of $D-1$-dimensional domains of links being equal to a non-trivial center element. Strictly speaking these boundaries live on the dual lattice and are linked by plaquettes with non-trivial center value on the original lattice. Because of the Bianchi identity center vortices have to be closed. Finally, one should also note that the Wilson loop can be used as a vortex counter.
In the center projected theory, one finds $W(C) = z^N$, where $N$ is the number of center vortices piercing any area encircled by the loop $C$.

It is also interesting to consider the physical implication of the center projection. Originally, center vortices are smooth configurations of the gauge field. These objects have a finite thickness of order $0.8$ fm as measured on the lattice [26]. In the procedure of maximal center gauge fixing and center projection, thick center vortices are replaced by the thin ones on the $Z(N)$ lattice.

On the lattice, one can remove center vortices by hand [28]. To remove vortices from the lattice each link $U_\mu(x)$ is multiplied by its center-projected image $Z_\mu(x)$: $U_\mu(x) \to U_\mu(x)Z_\mu(x)$. This adds an (oppositely oriented) center-projected vortex on top of the physical center vortex and their effects cancel.

Lattice calculations provide strong evidence that center vortices are the field configurations responsible for confinement. Figure 15 shows the static quark potential for the SU(2) group together with the potentials one obtains after center projection and after center vortex removal, respectively. The full potential grows linearly at long distances and behaves Coulomb-like at short distances. When center vortices are removed the heavy quark potential becomes flat at large distances and hence loses its confining property. In the center projected theory, i.e. when only the center vortices are kept from the ensemble of gauge fields (and converted into the center projected vortices), one loses the Coulombic part of the potential and finds just the confining (linearly rising) part. This shows that center vortices are indeed responsible for the confining part of the heavy quark potential and thus for confinement.

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7 The continuum version of the thin center projected vortices were referred to as ideal center vortices in ref. [27].
3.3 The random vortex model

The confining properties of center vortices can be exhibited by means of a simple random vortex model \[30\]. Consider an ensemble of randomly distributed center vortices, such that their intersection points with a two-dimensional plane in space-time can be found at random and uncorrelated locations. Assume that the space-time is a hypercube of length \(L\) and consider its two-dimensional slice of area \(L^2\), containing a Wilson loop which circumscribes an area \(A\), see fig. [16]. The probability that one vortex pierces this area is \(P_1 = A/L^2\), independently of its location. Accordingly, the probability that the vortex does not pierce that area is \(P_{10} = 1 - P_1 = 1 - A/L^2\). When there are \(N\) center vortices piercing a slice of the Universe, the probability that \(n\) of these pierce the Wilson loop is binomial,

\[
P^a_N = \binom{N}{n} \left( \frac{A}{L^2} \right)^n \left( 1 - \frac{A}{L^2} \right)^{N-n},
\]

because center vortices, as classical configurations of a gauge field in the Yang–Mills functional integral, are distinguishable. As shown above each vortex contributes a factor \(-1\) to the Wilson loop. Hence, its average can be evaluated as

\[
\langle W \rangle = \sum_{n=0}^{N} (-1)^n P^a_N = \sum_{n=0}^{N} \binom{N}{n} \left( \frac{A}{L^2} \right)^n \left( 1 - \frac{A}{L^2} \right)^{N-n}
= \left( 1 - 2 \frac{A}{L^2} \right)^N = \left( 1 - 2\rho \frac{A}{N} \right)^N \xrightarrow{N,L \to \infty} \exp(-2\rho A),
\]

where in the last step the size of the universe \(L\) and the number of vortices \(N\) have been sent to infinity with constant planar density \(\rho = N/L^2\). Hence, the random vortex picture provides an area-law for the Wilson loop with the string tension \(\sigma_{rvm} = 2\rho\). Lattice calculations of the vortex density yield \(\rho \approx 3.4 \text{ fm}^{-2}\), which leads to \(\sigma_{rvm} \approx (521 \text{ MeV})^2\) \[30\]. This result overestimates “experimental” string tension, which is \((440 \text{ MeV})^2\) and which sets the scale in the lattice calculation. This is due to the lack of correlations between vortices in the random vortex model. Correlations between the vortices make them less random and thus reduce the string tension.

Lattice calculations show that the density of the center vortices detected by center projection in the maximal center gauge shows the proper scaling \[25\] with the lattice spacing, which proves that these vortices are indeed physical objects, which survive in the continuum limit.

The center vortex picture gives not only a natural explanation of confinement, that is an area-law falloff of the Wilson loop, but also explains the deconfinement phase transition. In the finite-temperature quantum field theory the time dimension becomes compactified and its length gives the inverse of the temperature, \(L_t = 1/T\). Center vortices have finite thickness \(d \approx 0.8 \text{ fm}\), and at sufficiently high temperature space-like vortices do no longer fit in the lattice universe and they align along the
time axis, see Fig. 17. This happens when $L_t \approx d$ from which the deconfinement temperature is estimated as $T_c \approx 1.25 \text{ fm}^{-1} \approx 250 \text{ MeV}$ \[31\].

Due to the alignment of the vortices parallel to the time axis, in a three-dimensional time-space volume center vortices do no longer percolate. Therefore the deconfinement phase transition might be seen as a transition from a phase of large vortices percolating through space-time to a phase in which vortices are small and do not percolate.

This picture is supported by lattice results, shown in Figs. 18 and 19. As one can see, in the confined phase vortices form large clusters, which extend over the size of the temporal dimension, leading to an area law of both spatial and temporal Wilson loops. On the other hand, in the deconfined phase vortices form small clusters, aligned mostly along the time direction. Because of fluctuations in the cluster length, there is still a possibility that some vortices cross a \textit{temporal} Wilson loop, see Fig. 20. In this case the intersection points, however, are correlated—they occur pairwise, thus giving no contribution to the Wilson loop when they occur in the interior of the Wilson loop. The only non-trivial contribution comes from the edges of the loop, when one of the intersection points is located outside the loop—this results in a perimeter law of the \textit{temporal} Wilson loop. On the other hand intersec-
**Fig. 18** Typical vortex configurations in the confining (left panel) and the deconfined phase (right panel). From [31].

**Fig. 19** Vortex material distributions in the confined phase (left panel) and in the deconfined phase (right panel) obtained in SU(2) lattice gauge calculations. Cluster extensions are normalized to the extension of the lattice. From [31].

**Fig. 20** Center vortices in the deconfined phase. Left: alignment of the center vortices parallel to the time-axis, which runs upwards. Right: intersection points of center vortices with a time-space plane, which contains a (temporal) Wilson loop.

The intersection points of center vortices with a spatial plane are still uncorrelated, which leads to the area law for the spatial Wilson loop and thus to a non-vanishing spatial string tension.

The center vortex picture of the QCD vacuum can be extended to higher color groups SU($N > 2$). For SU($N$), the center is $Z(N)$ and there are $N - 1$ non-trivial
center elements $z_k = e^{i2\pi k/N}$, $k = 1, 2, \ldots, N-1$. Accordingly, there are $N-1$ different species of center vortices. Below, we will confine ourselves to the generic case of SU(3) \cite{32, 33}. The gauge group SU(3) has two non-trivial center elements, $z_1 = \exp(2\pi i/3)$ and $z_2 = \exp(4\pi i/3)$. This means that there are two species of center vortices in SU(3) gauge theory, which, however, are not independent. Since $z_2 = (z_1)^2$, a $z_2$ vortex can split into two $z_1$ vortices and conversely, two $z_1$ vortices can merge into one $z_2$ vortex. Since also $z_1 = z_2^2$, a $z_1$ vortex can split into two $z_2$ vortices. Therefore, in case of three colors center vortices can not only cross, but also branch and merge.

To investigate the role of vortex branching and merging in the deconfinement phase transition, in Refs. \cite{32, 33} a SU(3) random vortex model was constructed and studied numerically.

At a given lattice link a center vortex can be characterized by the number $\nu$ of vortex plaquettes meeting at this link. The following cases can be distinguished:

- $\nu = 0$ – there is no vortex at the given link
- $\nu = 1$ is impossible since it would lead to an open vortex
- $\nu = 2$ – usual center vortex
- $\nu = 3$ – vortex splitting
- $\nu = 4$ – vortex crossing
- $\nu = 5 = 2 + 3$ – vortex splitting
- $\nu = 6 = 2 + 4 = 3 + 3$ – vortex crossing and splitting

Vortex branching has not been studied in the SU(3) lattice theory yet, but was studied within the effective vortex model of Refs. \cite{32, 33}. Distributions of the branching number $\nu$ in three-dimensional slices of the four dimension lattice universe obtained in this model are shown in Fig. \ref{fig:branching}. The left panel shows this distribution in the confined phase – one can see that links are mainly attached to ordinary vortices ($\nu = 2$), but there is also a significant contribution from vortex branching ($\nu = 3$). Middle and right panels show the distributions in the deconfined phase for a temporal and a spatial, respectively, slice of the four dimension Euclidean universe. The temporal slice is the three-dimensional space $\mathbb{R}^3$ at a fixed time while a spatial slice arises when one space coordinate is kept fixed, i.e. a spatial slice is a three-dimensional space spanned by the time and two spatial axes. In the temporal slice (the middle panel), in which the time coordinate is fixed, vortex branching slightly increases compared to the confined phase (left panel). (Note that the spatial string tension also increases.) In the deconfined phase in a spatial slice (right panel), in which one space coordinate is fixed, vortex splitting and vortex crossing vanish entirely (and the temporal string tension also vanishes). Therefore the vortex branching behaves as an order parameter for the SU(3) deconfinement phase transition. The open question is whether the first order character of this transition is related to the vortex branching. In the case of the SU(2) group vortices can only cross and the transition is second order, while for $N \geq 3$ vortices can also split as well as merge and the transition is first order.
3.4 Topology of center vortices

We have shown that the center vortex picture provides a qualitative explanation of the confinement and deconfinement phase transition. In the remaining part of this lecture I will discuss the topological properties of center vortices and explore their role in the spontaneous breaking of chiral symmetry in QCD.

Topological properties of gauge fields are characterized by the so-called topological charge (Pontryagin index or second Chern number)

\[ \nu = \frac{1}{16\pi^2} \int d^4x \text{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}), \quad (52) \]

where \( \tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda} \) is the tensor dual to the field strength tensor \( F_{\mu\nu} \). The topological charge can be also expressed in terms of the electric, \( E^a_i = F^a_{0i} \), and the magnetic, \( B^a_i = \tilde{F}^a_{0i} \), fields,

\[ \nu = \frac{1}{4\pi^2} \int d^4x \mathbf{E} \cdot \mathbf{B}. \quad (53) \]

Two components of center-projected vortices can be distinguished — magnetic and electric ones. Magnetic vortices form closed lines of magnetic flux in the three-dimensional space. These vortex loops evolve in time direction for a finite time interval and form closed two-dimensional surfaces in four-dimensional space-time. On the other hand, electric vortices form closed surfaces in three-dimensional space which exist only at a single time instant and with the electric field directed along the surface normal. The presence of both types of vortices is obviously needed to have a non-vanishing Pontryagin index (53).

The study of the topological properties of center vortices is most conveniently done in the continuum where the gauge potential of a (closed) center vortex surface \( \partial \Sigma \) can be represented in \( D \) space-time dimensions as [27, 34].
Effective Approaches to QCD\textsuperscript{†}

\[ A_\mu(x, \Sigma) = 2\pi \tilde{\omega} \int_\Sigma \delta^{(D)}(x - \tilde{x}(\sigma)) . \]  

(54)

Here \( \tilde{\omega} = \tilde{\omega}^a h_a \) (with \( h_a \) the (hermitian) generators of the Cartan algebra) is a coweight satisfying

\[ e^{i2\pi \omega} = z, \]  

(55)

\( \delta^{(D)}(x) \) is the D-dimensional \( \delta \)-function and \( d^{D-1}\tilde{\sigma}_\mu \) is the \( D-1 \) dimensional (dual) surface element in \( D \)-dimension. Furthermore \( \tilde{x}'(\sigma) \) is a parametrization of the \( D-1 \)-dimensional volume \( \Sigma \) enclosed by the vortex surface \( \partial \Sigma \). An alternative continuum representation of the gauge potential of a center vortex surface \( \partial \Sigma \) is given by

\[ A_\mu(x, \partial \Sigma) = 2\pi \tilde{\omega} \int_{\partial \Sigma} d^{D-2}\tilde{\sigma}_{\mu\nu} \partial^\nu D(x - \tilde{x}(\sigma)) , \]  

(56)

where \( d^{D-2}\tilde{\sigma}_{\mu\nu} \) is the \( D-2 \) dimensional (dual) surface element in \( D \)-dimension and \( D(x) \) denotes the Green function of the \( D \)-dimensional Laplacian

\[ -\partial^\mu \partial^\nu D(x) = \delta^{(D)}(x) . \]  

(57)

Since the coweights \( \tilde{\omega} = \tilde{\omega}^a h_a \) live in the Cartan algebra the non-Abelian part of the field strength of a center vortex vanishes

\[ F_{\mu\nu}(x, \partial \Sigma) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) . \]  

(58)

Both representations (54) and (56) yield, of course, the same field strength

\[ F_{\mu\nu}(x, \partial \Sigma) = 2\pi \tilde{\omega} \int_{\partial \Sigma} d^{D-2}\tilde{\sigma}_{\mu\nu} \delta^{(D)}(x - \tilde{x}(\sigma)) . \]  

(59)

According to Eq. (53), each intersection point\textsuperscript{8} between an electric and a magnetic part of a vortex contributes to \( v \). Using eq. (59) one shows that the topological charge of a center vortex can be expressed by the self-intersection number \( I(\partial \Sigma, \partial \Sigma) \) of the vortex surface \( \partial \Sigma \) \cite{27}, \cite{34}

\[ v = \frac{1}{4} I(\partial \Sigma, \partial \Sigma) , \]  

(60)

where the intersection number between two surfaces \( S_1 \) and \( S_2 \) in \( D = 4 \) is defined by

\[ I(S_1, S_2) = \frac{1}{2} \int_{S_1} d\sigma_{\mu\nu} \int_{S_2} d\tilde{\sigma}'_{\mu\nu} \delta(\tilde{x}(\sigma) - \tilde{x}(\sigma')) . \]  

(61)

Here \( \tilde{x}'(\sigma) \) denotes a parametrization of the vortex surface \( S_1 \), \( d\sigma_{\mu\nu} \) is an infinitesimal area element in four dimensions and \( d\tilde{\sigma}'_{\mu\nu} = \frac{1}{2} \epsilon_{\rho\nu\kappa\lambda} d\sigma_{\rho\kappa} \) is its dual.

\textsuperscript{8} In \( D = 4 \) surfaces intersect generically in points. This is analogous to the intersection of lines in \( D = 2 \), see Fig. 22.
Fig. 22 Two-dimensional analog of two intersecting closed surfaces in $D = 4$. Closed loops intersect always in an even number of points.

Fig. 23 Intersecting electric and magnetic vortices. Left panel: both vortices are oriented; middle panel: oriented electric and non-oriented magnetic vortices; right panel oriented magnetic and non-oriented electric vortices.

It is not difficult to see that Eq. (60) yields integer topological charge. To see this let us assume that the vortex sheet $\partial \Sigma$ consists of an electric and a magnetic piece

$$\partial \Sigma = \partial \Sigma_e + \partial \Sigma_m.$$  \hfill (62)

From the definition (61) of the intersection number follows

$$I(\partial \Sigma, \partial \Sigma) = I(\partial \Sigma_e, \partial \Sigma_m) + I(\partial \Sigma_m, \partial \Sigma_e) = 2I(\partial \Sigma_e, \partial \Sigma_m)$$ \hfill (63)

since $I(\partial \Sigma_e, \partial \Sigma_e) = 0 = I(\partial \Sigma_m, \partial \Sigma_m)$. As explained above, electric or magnetic vortex sheets cannot intersect with themselves in points in $D = 4$. With (63) we obtain from (60)

$$\nu = \frac{1}{2}I(\partial \Sigma_e, \partial \Sigma_m).$$ \hfill (64)

Since closed surfaces intersect in an even number of intersection points (see Fig. 22) $\nu$ is indeed integer valued.

Moreover, the intersection number and thus the topological charge depends also on the orientation of a vortex surface. Assume that electric and magnetic vortices are both oriented, see Fig. 23 then for any intersection point between the electric vortex surface and the magnetic vortex line which gives a positive contribution to the topological charge there is one intersection point which gives the opposite contribution. In general, one can show that the self-intersection number of oriented surfaces in four-dimensional space is always zero and hence the topological charge of an ori-
Fig. 24 Splitting a Dirac string (left panel) into two center vortices (right panel)

entire center vortex vanishes. To be topologically non-trivial, a center vortex surface has to be non-oriented.

The middle and right panel of Fig. 23 show the intersection of an oriented and a non-oriented center vortex. In the latter case the two intersection points add coherently to the intersection number, which is two in this example.

The orientation of center vortices is irrelevant for their confining properties. It is, however, crucial for their topological properties and thus for spontaneous breaking of chiral symmetry. What makes a center vortex surface non-oriented? The answer is magnetic monopoles. Consider a monopole–anti-monopole pair, connected with the Dirac string, see Fig. 24. The Dirac string itself is not observable. However, it represents a magnetic flux which is twice the flux of a center vortex. Therefore the Dirac string can be split into two center vortices resulting in a center vortex loop with two magnetic monopoles on it. As can be seen from Fig. 24, the magnetic monopoles change the orientation of the vortex flux.

In four dimensions, center vortices are closed surfaces while the trajectories of magnetic monopoles form closed loops, placed on center vortex surfaces after center projection. It can be shown rigorously that the topological charge of a center vortex is given by the linking number between the monopole loop and the vortex surface [34]:

$$\nu = \frac{1}{4} L(C_{\text{monopole}}, \partial \Sigma_{\text{vortex}}).$$

Equation (65)

It should come as no surprise that the magnetic monopoles are here required for a non-zero topological charge \(\nu\). This is because in the Abelian gauges \(\nu\) can be entirely expressed in terms of the magnetic monopole content of the gauge fields. In the Polyakov gauge (see section 4.8) the topological charge can be expressed as

$$\nu = \sum_{i} n_{i} m_{i},$$

Equation (66)

where \(m_{i}\) is the integer valued magnetic charge of a monopole and \(n_{i}\) is an integer related to the Dirac string, and the summation runs over all magnetic monopoles contained after Abelian projection in the gauge field configuration considered.

There are two types of intersection points [35]: the transverse intersection point and the writhing point, also called twist. Transverse or generic intersection points are those were in Eq. (61) \(x^{\mu}(\sigma) = x^{\mu}(\sigma')\) for \(\sigma \neq \sigma'\). They contribute \(\pm \frac{1}{2}\) to the topological charge \(\nu\) (60). Writhing points contribute to the intersection number \(I\) (61) for \(\sigma = \sigma'\). This means that at a writhing point the vortex surface twist in such a way that the tangent vectors to the vortex surface elements emanating from the writhing point span the full four dimensional space. They contribute less than \(1/2\) to the topological charge \(|\nu|\). Lattice realizations of these points are shown in Fig. 25.
where the left panel corresponds to a transverse intersection point and the right panel to a writhing point. Dashed lines show the time direction. The topological charge on the lattice can be expressed as

\[ \nu \equiv \frac{1}{4} I(\partial \Sigma, \partial \Sigma) = \sum_{\text{site}} \nu_{\text{site}} \]  \hspace{1cm} (67) \]

where \[ \nu_{\text{site}} = \frac{1}{32} \#_{\text{site}}, \] \hspace{1cm} (68) \]

and \#_{\text{site}} is the number of pairs of plaquettes which share this site and which are completely orthogonal to one another (i.e. their tangent vectors span the four-dimensional space). In case of transverse intersection points there are four plaquettes which extend in one space dimension and the time dimension and four plaquettes which extend in spatial direction only. Using the prescription (68) one finds that \( \nu = \frac{1}{32} \times 4 \times 4 = \frac{1}{2} \). For the writhing point shown in Fig. 25b one finds two plaquettes extending in one space dimension and in time and two plaquettes extending in spatial directions only. This yields \( \nu = \frac{1}{32} \times 2 \times 2 = \frac{1}{8} \).

Figure 26 shows the time evolution (along the \( n_0 \) axis) of a generic center vortex on the lattice, which has both intersection and writhing points, while Figure 27 shows a sequence of snapshots of this vortex in \( \mathbb{R}^3 \) where this vortex shows up as a (time-dependent) closed loop: At some time the vortex loop emerges from the vacuum, and during its evolution it can intersect with itself. Finally, the loop disappears. On our spatial manifold \( \mathbb{R}^3 \), the four-dimensional transverse intersection points show up as (time-dependent) true crossings of vortex loop segments, while the writhing points become “fractional crossing” or turning points of vortex line segments.

The time-evolution of a center vortex in \( \mathbb{R}^3 \) is sufficient to calculate its topological charge, which can be expressed as

\[ \nu = \frac{1}{4} \int dt \partial_t W_r(C(t)), \] \hspace{1cm} (69) \]

where \( W_r(C(t)) \) is the writhing number, defined as the self-linking number of the closed loop \( C \)

\[ W_r(C(t)) = L(C(t), C(t)) \] \hspace{1cm} (70)
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Fig. 26 Lattice realization of a generic center vortex surface. $n_0$ denotes the lattice time. At each lattice time $n_0$, shaded plaquettes are part of the vortex surface. These plaquettes are connected to plaquettes running in the time direction. The two non-shaded plaquettes at $n_0 = 2$ are not part of the vortex but the two sets of links bounding them are. From [36].

with $L(C_1, C_2)$ being Gauss’ linking number [41]. For more details see Ref. [34].

3.5 Chiral symmetry breaking

Below we show that center vortices also induce spontaneous breaking of chiral symmetry.

Consider the eigenvalue equation for the Dirac operator with massless quarks,

$$D \Psi = \gamma^\mu D_\mu \Psi = \lambda \Psi,$$

(71)

where $D_\mu = \partial_\mu - A_\mu, A_\mu = g A_\mu^a T_a$, is the covariant derivative (see footnote 5 on page 15). When the eigenvector $\Psi$ with eigenvalue $\lambda$ is multiplied by $\gamma_5$ one obtains a new eigenvector with the eigenvalue $-\lambda$. This follows from the fact that $\gamma_5$ anticommutes with all Dirac matrices $\gamma^{\mu=1,2,3,4}$. Therefore the states $\Psi$ and $\gamma_5 \Psi$ are linearly independent and, hence, non-zero eigenvalues of the Dirac operator come in pairs $\pm \lambda$. For the zero modes (eigenvectors with $\lambda = 0$) it follows from $[\gamma^\mu, \gamma_5] = 0$ that $D \Psi = 0$ implies also $D \gamma_5 \Psi = 0$. Therefore the zero modes can be chosen as left- or right-handed modes, such that

$$D \Psi_{L/R} = 0, \quad \gamma_5 \Psi_{L/R} = \mp \Psi_{L/R}, \quad \Psi_{L/R} = P_{L/R} \Psi_{L/R},$$

(72)

Fig. 27 The center vortex surface of Fig. 26 shown in $\mathbb{R}^3$ at different time instants. From [34].
Fig. 28 The probability density of the zero mode of quarks plotted against the two-dimensional cut of the four-dimensional universe. (a) Results obtained in the continuum limit with center vortices as a background field, taken from Ref. [37]. (b) Corresponding lattice result, taken from Ref. [38].

where $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ are the left/right projectors. From the Atiyah–Singer index theorem

$$\nu[A] = n_L - n_R$$

(73)

follows that the topological charge $\nu$ of a field configuration $A$ is given by the difference between the numbers $n_{L/R}$ of left- and right-handed zero modes. This means that the zero modes of the Dirac operator are related to the topology of gauge fields. In topologically non-trivial field configurations, $\nu \neq 0$, quarks must have zero modes.

It is interesting to study the quark zero modes emerging in the presence of center vortices. As shown above, in order to be topologically non-trivial the vortices must intersect. Figure 28a shows the probability density of Dirac zero modes in the continuum limit, calculated in the background field of two pairs of intersecting vortex sheets, plotted for a two-dimensional slice of the four-dimensional universe [37]. In this cut center vortices appear as intersecting lines. One can see that zero modes are concentrated along these vortex sheets and the concentration is largest at the intersection points. Therefore center vortices act as quark guides in the QCD vacuum. Similar calculations have been done on the lattice (see Ref. [38]) and the result is shown in Fig. 28b. As one can see, it is qualitatively similar to the continuum limit but the probability density is smeared out on the lattice, as it should, due to the finite lattice spacing.

Spontaneous breaking of chiral symmetry requires a non-vanishing density of quark (near) zero modes. This follows from the Banks–Casher relation

$$\langle \bar{q}q \rangle = -\pi \rho(0),$$

(74)

which relates the quark condensate $\langle \bar{q}q \rangle$, the order parameter of chiral symmetry breaking, to the spectral density $\rho(\lambda) = dN_{\lambda}/d\lambda$, where $dN_{\lambda}$ is the number of states with eigenvalues in an interval $[\lambda, \lambda + d\lambda]$. To have a non-vanishing quark conden-
sate, one needs a non-zero density of near zero Dirac modes. The eigenmodes of the Dirac operator have been calculated for the gauge field configurations generated on the lattice. Figure 29a shows the 50 smallest eigenvalues of the Dirac operator for 10 different configurations for the gauge group SU(2). As we can see, the density of near-zero eigenmodes is non-zero and from the Banks–Casher relation follows that the quark condensate is also non-vanishing. When one removes center vortices from the gauge field configurations considered in Fig. 29a, a gap opens up in the Dirac spectrum around zero virtuality \( \lambda = 0 \) (Fig. 29b), and hence the quark condensate vanishes. This suggests that center vortices are responsible for the spontaneous breaking of the chiral symmetry. If this is indeed the case one would expect that the density of quark models near \( \lambda = 0 \) increases after center projection. Surprisingly, the gap is even larger than in the case of vortex removal, see Fig. 29c. The reason is that the Dirac operator used in these calculations is not sensitive enough to see the (rather singular) center projected vortices. Lattice calculations with a more sophisticated Dirac operator yield the expected result — after center projection only near-zero modes are left, see Fig. 30. This shows that the center vortices are indeed the dominant IR degrees of freedom which are responsible not only for quark confinement but also for the spontaneous chiral symmetry breaking.

### 3.6 Center vortex dominance

One may ask: Why are center vortices the dominant IR degrees of freedom? What distinguishes center vortices from other gauge field configurations, for instance from
vortices with a flux different from that of center vortices? The latter question was investigated in Ref. [41], where the energy density of a straight magnetic vortex was calculated at one-loop level as function of its flux $\phi$ for pure Yang–Mills theory and for QCD. The result is shown in Fig. 31. The flux is normalized such that $\phi = 1$ corresponds to the flux of the center vortex, while $\phi = 0$ refers to the perturbative vacuum. In the pure Yang–Mills case one finds that the energy density of the center vortex is the same as the one of the vacuum, while all other fluxes have higher energy. In the presence of quarks the energy density corresponding to the center vortex flux exceeds the energy density of the perturbative vacuum, but the center vortex flux still represents a local minimum. This provides a qualitative explanation of the center vortex dominance in the IR sector of QCD as compare to other flux tube configurations.
3.7 Conclusions

In this lecture the center vortex picture of the QCD vacuum has been presented. Center vortices seem to be dominant infra-red configurations of the gauge field and provide appealing pictures of confinement (center vortices percolating through space-time lead to an area law for the Wilson loop) and the deconfinement phase transition (as the temperature increases, center vortices align in the temporal direction which leads to a vanishing temporal string tension).

Topological properties of center vortices and their relevance for the spontaneous breaking of chiral symmetry have also been discussed. The topological charge of an oriented vortex sheet vanishes and to be topologically non-trivial, a vortex sheet has to be non-oriented. Non-orientability of vortex sheets is caused by magnetic monopoles, which change the direction of the vortex flux. The topological charge is concentrated on vortex intersection points, which give fractional contributions to the topological charge. Nevertheless the total topological charge is integer valued. Finally, the quark zero modes are concentrated on center vortices and, in particular, at their intersection points. Furthermore, the low-lying Dirac modes disappear when center vortices are removed from the Yang–Mills ensemble while only near zero modes survive after center projection. This shows that center vortices are responsible for chiral symmetry breaking.

4 Hamiltonian approach to QCD in Coulomb gauge

4.1 Introduction

In the previous lectures we have discussed two scenarios of quark confinement: magnetic monopole condensation (dual Meißner effect), which utilizes the maximal Abelian gauge, and center vortex condensation, which was established by lattice calculations in the maximal center gauge. In this lecture the Gribov–Zwanziger mechanism of confinement is presented. This picture of confinement is formulated in Coulomb gauge and naturally emerges within a Hamiltonian approach as I will show in this lecture. I will also exhibit various connections between the different pictures of confinement. Before I expand the Hamiltonian approach to QCD in Coulomb gauge let me give some arguments why this approach is advantageous in non-perturbative studies.

Nowadays the most popular approach to quantum field theory is the path integral formulation, which is definitely advantageous in perturbation theory, which can be formulated in terms of Feynman diagrams. Furthermore, the functional integral formulation is also the basis for the numerical lattice calculation, which are fully non-perturbative. On the other hand in ordinary (non-relativistic) quantum mechanics it is usually much simpler to solve the Schrödinger equation than calculating the corresponding functional integral. The hydrogen atom is a good example. One can
easily find the exact solution of the Schrödinger equation for this problem, while it is an extremely difficult task to find the exact solution within the path integral formalism. Therefore, for non-perturbative studies in continuum quantum field theory we expect the Hamiltonian approach to be more efficient than approaches based on the functional integral formulation like for e.g. Dyson–Schwinger equations or functional renormalization group flow equations. The Hamiltonian approach to QCD has been worked out mainly in collaboration with C. Feuchter, W. Schleifenbaum, D. Campagnari and P. Vastag [42,43,13,44,45]. For didactic reasons I will develop the Hamiltonian approach first for pure Yang–Mills theory.

4.2 Canonical quantization of Yang–Mills theory

The starting point for the canonical quantization of Yang–Mills theory is the classical action,

\[ S = \frac{1}{4g^2} \text{Tr} \int d^4x F_{\mu\nu} F^{\mu\nu}, \]  

(75)

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]  

(76)

is the field strength tensor. In the canonical quantization the components of the gauge field \( A_\mu^a(x) \) itself serve as coordinates. The canonical momenta conjugate to the spatial components of the gauge field are given by the color electric field

\[ \Pi^a_i(x) = \frac{\delta S}{\delta \dot{A}^a_i(x)} = E^a_i(x), \]  

(77)

while the momenta conjugate to the temporal components of the gauge field vanish, \( \Pi^a_0 = 0 \). This causes a problem in the canonical quantization. To avoid this problem one can choose Weyl gauge, \( A^a_0 = 0 \), in which the classical Yang–Mills Hamiltonian becomes

\[ H = \frac{1}{2} \int d^3x [\Pi^a(x) \cdot \Pi^a(x) + B^a(x) \cdot B^a(x)]. \]  

(78)

In order to quantize the system, one has to replace the canonical conjugate coordinates and momenta by operators, satisfying the following commutation relations:

\[ [A^a_\xi(x), \Pi^b_\eta(y)] = i\delta^{ab}\delta_\xi\delta(x-y) \]  

(79)

\[ [A^a_\xi(x), A^b_\eta(y)] = [\Pi^a_\xi(x), \Pi^b_\eta(y)] = 0. \]  

(80)

In the “coordinate representation” the gauge fields are classical functions while the canonical momentum operator is given by

\[ \Pi^a_\xi(x) = \frac{\delta}{i\delta A^a_\xi(x)}. \]  

(81)
In Weyl gauge Gauss' law is lost as an equation of motion and has to be imposed as a constraint on the wave functional $\Psi[A]$}

$$\hat{D}^{ab}_{k}(x)\Pi^{b}_{k}(x)\psi[A] = \rho_{m}(x)\psi[A], \quad (82)$$

where

$$\hat{D}^{ab}_{k}(x) = \delta^{ab}\partial^{k} + g\epsilon^{abc}A^{c}_{k}(x) \quad (83)$$

is the covariant derivative in the adjoint representation of the gauge group and $\rho_{m}$ is the color charge density of the matter fields. Moreover, the operator on the left-hand side, $\hat{D} \cdot \Pi$, is the generator of space-dependent but time-independent gauge transformations. When matter fields are not present, $\rho_{m} = 0$, the right-hand side of Eq. (82) vanishes and the wave functional must be invariant under time-independent gauge transformations $U(x), \psi[A] = \psi[U^{A}]$.

After quantization the Hamiltonian (78) becomes an operator, acting in the Hilbert space of gauge invariant wave functionals with the scalar product

$$\langle \Phi | \ldots | \Psi \rangle = \int D\Phi \Phi^{\ast}[A] \ldots \Psi[A]. \quad (84)$$

Here $\int D\Phi$ is the functional integral over the time-independent spatial components of the gauge field. The main objective in the Hamiltonian approach is to solve the Schrödinger equation

$$H\Psi[A] = E\Psi[A] \quad (85)$$

for the wave functional $\Psi[A]$. Usually one is interested in the wave functional of the vacuum. In 1+1 dimensions the Yang-Mills Schrödinger equation (85) can be solved exactly [46]. Approximate solutions for the gauge invariant vacuum state have been found in 2+1 dimensions [47] [48] and for a limiting case in 3+1 dimensions [49]. In general, the construction of the gauge invariant wave functional is extremely difficult and it is much more efficient to choose a specific gauge. A convenient choice for this problem is the Coulomb gauge, $\partial \cdot A = 0$. Implementing the Coulomb gauge by means of the Faddeev–Popov method the scalar product (84) becomes

$$\langle \Phi | \ldots | \Psi \rangle = \int D\Phi \Phi^{\ast}[\perp A] \ldots \Psi[\perp A], \quad (86)$$

where the functional integration extends over the transverse part, $\perp A$, of the gauge field only and

$$J(\perp A) = \text{Det}(\perp \hat{D} \cdot \partial) \quad (87)$$

is the Faddeev–Popov determinant in Coulomb gauge. The Coulomb gauge fixing may be seen as a transition from the Cartesian $A$ to "curvilinear" $\perp A$ coordinates with the Faddeev–Popov determinant corresponding to the Jacobian of this transformation.

Coulomb gauge fixing eliminates the longitudinal components of the (spatial) gauge field. The momentum operator, however, still contains transverse and longitudinal components, $\Pi = \Pi^{\perp} + \Pi^{\parallel}$, where the transverse part is still given by eq. (81).
The longitudinal components of the momentum operator can be determined by resolving Gauss’ law, which leads to

$$\Pi^\parallel \psi[A] = -\partial(-\hat{D}\partial)^{-1}\rho \psi[A], \quad (88)$$

where

$$\rho^a(x) = -f^{abc}A^b_a(x)\Pi^c_k(x) + \rho^a_m(x) \quad (89)$$

is the total color charge density, composed of the color charge density of matter, $\rho_m$, and the color charge density of the gauge field, $\rho^a_g(x) = -f^{abc}A^b_a(x)\Pi^c_k(x)$. The latter exists only in non-Abelian gauge theories. It should be emphasized that Eq. (88) is not a true operator identity, i.e. the operator expression implied by Eq. (88) for $\Pi^\parallel \psi[A]$ acts on the wave functional. Using $\Pi = \Pi^\perp + \Pi^\parallel$ and the relations (88) and (89) one derives from Eq. (78) the Yang–Mills Hamiltonian in the Coulomb gauge

$$H = \frac{1}{2} \int d^3x \left( J^{-1} J^\parallel \psi^\parallel \psi + B^a(x) \cdot B^a(x) \right) + H_C \equiv H_{YM} + H_C \quad (90)$$

where

$$H_C = \frac{1}{2} \int d^3x \int d^3y J^{-1} \rho^a(x)J\left[(-\hat{D}\cdot\partial)^{-1}(-\partial^2)(-\hat{D}\cdot\partial)^{-1}\right]^{ab}(x,y)\rho^b(y) \quad (91)$$

is the Coulomb term which arises from the kinetic energy of the longitudinal components of the momentum operator. In the case of QED this term reduces to the ordinary Coulomb interaction between the electric charge distribution $\rho^m$.

The Yang–Mills Hamiltonian in the Coulomb gauge, Eq. (90), is more complicated than the original gauge invariant one, Eq. (78). The kinetic energy of the transverse degrees of freedom (the first term of Eq. (90)) contains the Faddeev–Popov determinant. Moreover, the Coulomb term is a highly non-local object. The Faddeev–Popov determinant is also present in the scalar product (86). It is, however, still more convenient to work with the complicated gauge fixed Hamiltonian (90) than with gauge invariant wave functionals. It should also be stressed that by implementing Gauss’ law in the gauge fixed Hamiltonian gauge invariance has been fully accounted for.

### 4.3 Variational solution for the Yang–Mills vacuum wave functional

To solve the Yang–Mills Schrödinger equation for the gauge-fixed Hamiltonian (90) one can exploit the variational principle. This was first done by D. Schutte who assumed a Gaussian ansatz for the vacuum wave functional.
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Fig. 32 The gluon energy $\omega(p)$ (left panel) and the ghost form factor $d$ (right panel) obtained in the variational approach for SU(2) Yang–Mills theory. From [53].

and derived a set of coupled integral equations for the gluon propagator, the ghost propagator and the Coulomb potential. This set of equations was rederived and solved numerically in ref. [52]. An improved variational approach to the Yang–Mills Schrödinger equation has been developed in refs. [42; 43]. This approach differs from previous works in: i) the form of the vacuum wave functional, ii) the treatment of the Faddeev–Popov determinant (which turns out to be crucial for the confining properties of the theory) and iii) the renormalization. The trial ansatz used in ref. [42] for the vacuum wave functional is

$$\Psi[A] = \exp \left[ -\frac{1}{2} \int d^3x \int d^3y A^\gamma_a(x) \omega(x, y) A^\gamma_b(y) \right],$$

(92)

and

$$\Psi[A] = \frac{1}{\sqrt{J(A)}} \exp \left[ -\frac{1}{2} \int d^3x \int d^3y A^\gamma_a(x) \omega(x, y) A^\gamma_b(y) \right],$$

(93)

where $\omega(x, y)$ is the variational kernel which is determined by minimizing the energy $\langle \Psi[H]\Psi \rangle$. The advantage of this ansatz is that the static gluon propagator is essentially given by the inverse of the variational kernel:

$$\langle A^\gamma_a(x) A^\gamma_b(y) \rangle = \delta^{ab} t_{kl}(x) \frac{1}{2} \omega^{-1}(x, y),$$

(94)

where $t_{kl}(x) = \delta_{kl} - \partial^a_i \partial^a_k / \partial^a_i$ is the transverse projector. From the form of this propagator follows that its Fourier transform $\omega(p)$ represents the single-particle gluon energy. Minimization of the energy density with respect to $\omega(x, y)$ leads to the result shown in the left panel of Fig. 32. For large momenta the gluon energy behaves like the photon energy, $\omega(p) \sim |p|$, while at small momenta it diverges, $\omega(p) \sim 1/|p|$. This means that there are no free gluons in the infrared, which is a manifestation of gluon confinement.

The general form of the gluon gap equation obtained by minimizing the energy density is

$$\omega^2(p) = p^2 + \chi^2(p) + I_{\text{tad}} + I_C(p),$$

(95)
Fig. 33 Diagrammatic representation of the ghost loop (left panel) and the tadpole $I_{\text{tad}}$ (right panel). A wavy and a dashed, respectively, line represents the gluon (94) and ghost (97), respectively, propagator.

Fig. 34 Dyson–Schwinger equation for the ghost propagator.

where $I_{\text{tad}}$ is the tadpole, see Fig. 33 and

$$\chi_{kl}(x, y) = -\frac{1}{2} \left\langle \Psi \left( -\hat{D} \partial \right)^{-1} \partial A_{k}^{\mu} (x) \partial A_{l}^{\mu} (y) \right\rangle \Psi = \delta_{ij}t_{ij}(x - y)\chi(x - y) \quad (96)$$

is the ghost loop, $(t_{ij}(x) = \delta_{ij} - \partial_i \partial_j / \partial^2$ is the transversal projector.) Furthermore, $I_{C}(p)$ follows from the Coulomb term. Both $I_{\text{tad}}$ and $I_{C}$ contain UV-divergencies, which are removed by adding two counter terms to the gauge fixed Yang–Mills Hamiltonian (90), see refs. [54], [55] for details. The gluon gap equation (95) has the form of a relativistic dispersion relation. To calculate the ghost loop $\chi$ one needs the ghost propagator $G = \langle \psi \left( -\hat{D} \bar{\partial} \right)^{-1} | \psi \rangle$. Once the vacuum wave functional $| \Psi \rangle$ is known the ghost propagator can, in principle, be evaluated. However, this cannot be done in closed form even for the variational ansatz (93). To handle this problem one can expand the inverse of the Faddeev–Popov operator $-\hat{D} \cdot \partial$ in powers of the gauge field $A_{\perp}$ and then resum it after certain approximations. This, however, does not lead to a closed form of the ghost propagator, but to the Dyson–Schwinger equation for this propagator (shown in Fig. 34). In principle, it is neither clear nor trivial that the gap equation (95) coupled to the ghost Dyson–Schwinger equations does have a solution.\footnote{In principle, there is only one variational equation since we have only one variational kernel, $\omega$. The ghost Dyson–Schwinger equation only comes into the game since we are unable to calculate the ghost propagator with the trial wave functional (93) in closed form.}

It is convenient to represent the ghost propagator as

$$G \equiv \langle \psi (-\hat{D} \bar{\partial})^{-1} \psi \rangle = \frac{d(\Lambda)}{-\Delta}, \quad (97)$$

where $d(\Lambda)$ is called the ghost form factor, containing all the deviations of the gluon propagator from the photon propagator. For the latter the form factor is just unity.

Numerical calculations of the coupled gluon gap equation and ghost DSE show that in the pure glue sector the tadpole contribution $I_{\text{tad}}$ and the Coulomb term $I_{C}(p)$ can be neglected. The latter, however, has to be included when quarks are present, for it is responsible for chiral symmetry breaking. Hence, in the Yang–Mills sector
the gap equation (95) can be reduced to
\[ \omega^2(p) = p^2 + \chi^2(p), \] (98)
with the ghost loop \( \chi \) given in the left panel of Fig. 33 in terms of the ghost propagator, which in turn has to be found by solving the ghost Dyson–Schwinger equation, shown in Fig. 34.

The infrared analysis of these two equations has been carried out in Refs. [13], where power laws for the gluon energy, \( \omega = A/p^\alpha \), and the ghost form factor, \( d(p) = B/p^\beta \), have been assumed. Furthermore, the ghost form factor is assumed to fulfill the so-called horizon condition, \( d^{-1}(0) = 0 \), which is the crucial part of the Gribov–Zwanziger confinement scenario (see the discussion following eq. (103). Assuming also that the ghost-gluon vertex is bare, one finds the following sum rule for the IR exponents:
\[ \alpha = 2\beta + 2 - d, \] (99)
where \( d \) is the number of spatial dimensions. For \( d = 3 \) one finds from the variational equations two solutions for the IR exponents: either \( \alpha = \beta = 1 \) or \( \alpha = 0.6 \) and \( \beta = 0.8 \) of which only the first one is physical. To find the whole momentum dependence of the gluon energy and the ghost form factor one has to solve the variational equations numerically. The corresponding results are shown in Fig. 32.

In the Coulomb gauge the ghost form factor has a physical meaning [56]. Its inverse gives the dielectric function of the Yang–Mills vacuum, \( \epsilon(p) = d^{-1}(p) \). This function calculated in the variational approach of Ref. [42] is shown in Fig. 35. The horizon condition \( d^{-1}(0) = 0 \) implies that the dielectric function vanishes in the infrared regime. This means that there are no free color charges since for \( \epsilon = 0 \) the electrical displacement \( \mathbf{D} = \epsilon \mathbf{E} \) vanishes and from Gauss’ law, \( \nabla \cdot \mathbf{D} = \rho_{\text{free}} \), follows that the density of free (color) charges has to vanish. A medium with vanishing dielectric constant is a perfect dielectric or a dual superconductor. In this way the Hamiltonian approach to Yang–Mills theory in the Coulomb gauge establishes the connection to the dual superconductor picture of confinement, discussed in the first lecture.
4.4 Comparison with the lattice

To check the quality of the variational approach let us confront the obtained propagators with the lattice data. We will confine ourselves to the gauge group SU(2). Figure 36 shows a comparison between lattice data and the variational solution for the gluon energy (a) and the ghost form factor (b) in 2+1 dimensions. The variational solution correctly reproduces both infrared and ultraviolet behaviors, but somewhat deviates from the lattice data in the intermediate momentum regime. The left panel of Fig. 37 shows the static gluon propagator \( D(p) = (2\omega(p))^{-1} \) in 3+1 dimensions, obtained on the lattice (points) and using the variational approach (dot-dashed line). The solid line shows a fit to Gribov’s formula [10] for the gluon energy,

\[
\omega(p) = \sqrt{p^2 + \frac{M^2}{p^2}},
\]

with \( M = 0.88 \) GeV. Similarly to the 2+1 dimensional case, the variational approach reproduces correctly high and low momentum regimes and deviates from the lattice results at intermediate momenta. The deviations from the lattice data in the mid-momentum regime can be largely removed by using a non-Gaussian ansatz for the vacuum wave functional [44] (see also Ref. [60]), which reproduces the lattice data much better, see fig. 37.

In the original lattice calculations of the ghost form factor in \( D = 3 + 1 \) [61] an infrared exponent of \( \beta \approx 0.5 \) was found while the IR exponent of the gluon energy was determined as \( \alpha \approx 1 \) [59]. This result is surprising since it violates the sum rule (99). It turns out that the lattice results for the ghost form factor depend on the way the Coulomb gauge is implemented. An alternative method of the gauge fixing leads

![Graphs showing comparison between lattice data and variational solution](image)

Fig. 36 Comparison between the lattice data [57] and the variational solution (black line) in 2 + 1 dimensions for the gluon energy (a) and the ghost form factor (b). From [58].
to results compatible with $\beta \approx 1$ [62]. The gluon propagator, on the other hand, seems to be insensitive to the choice of the gauge fixing method.

### 4.5 The non-Abelian Coulomb potential

So far we have not discussed the Coulomb term, Eq. (91), of the gauge fixed Yang–Mills Hamiltonian, Eq. (90). In the presence of matter fields, this term contains a part which is quadratic in the color charge $\rho_m$ – it represents a two body interaction induced by the Yang–Mills vacuum. The vacuum expectation value of this part,

$$ V_C(|x - y|) = g^2 \langle x | (-\hat{D} \cdot \partial)^{-1} (-\Delta) (-\hat{D} \cdot \partial)^{-1} | y \rangle, \quad (101) $$

is the static color charge potential. To simplify its evaluation we use the following factorization
\[ V_C(|x - y|) \approx \int d^3 w \int d^3 z G(x, w) \langle w | - \Delta | z \rangle G(z, y), \tag{102} \]

where \( G(x, y) \) is the ghost propagator, given by Eq. (97). Using for \( G \) the result of the variational approach one finds the potential is shown in Fig. 38. For small distances it behaves like the ordinary (QED) Coulomb potential, \( V_C(r) \sim 1/r \), while for large distances it rises linearly, \( V_C(r) \sim \sigma_C r \), where \( \sigma_C \) is the so-called Coulomb string tension, which can be shown rigorously to represent an upper bound for the Wilson string tension, \( \sigma_W < \sigma_C \) [63]. On the lattice one finds that \( \sigma_C \) is about 2...3 times the Wilson string tension \( \sigma_W \). Fourier transforming the potential (102)

\[ V_C(|x - y|) = \int \frac{d^3 p}{(2\pi)^3} e^{i p \cdot (x - y)} V_C(p), \tag{103} \]

one finds \( V_C(p) = (d(p))^2/p^2 \). A linearly rising potential \( V_C(p) \sim 1/p^4 \) requires \( d(p) \sim 1/p \), and hence an infrared exponent \( \beta = 1 \), which is obtained for one of our two variational solutions. Such a ghost form factor obviously satisfies the horizon condition \( d^{-1}(0) = 0 \).

As we have seen, the horizon condition is crucial for confinement. An interesting question is therefore: what are the field configurations which trigger the horizon condition? An infrared singular ghost form factor arises from field configurations which are on, or near, the Gribov horizon. Center vortices and magnetic monopoles can be shown to lie exactly on the Gribov horizon [66]. Using the lattice methods presented in the second lecture one can calculate the contributions of center vortices to the ghost form factor [64]. Figure 39 shows the ghost form factor obtained in the full lattice gauge theory together with the one obtained when the center vortices are removed from the gauge field ensemble. After removal of center vortices the ghost form factor is no longer divergent at \( p \to 0 \) and the confining properties of the theory are lost. A similar result holds for the Coulomb potential (the right panel in Fig. 39)

Fig. 39  Left panel: the renormalized ghost form factor obtained on the lattice. Green points – the full result. Red and black points – the form factor after removal of center vortices (prefix s stands for spatial). Right panel: lattice calculations of the Coulomb potential. Green points – the full result. Red points – the potential after removal of center vortices. Purple points – the potential obtained after the center projection. Both figures from [64].
Fig. 40 (left) The Coulomb potential obtained from the Wilson loop using the Dyson–Schwinger equation. (right) The potential after subtraction of the perturbative part. From [65].

of Fig. 39 – after removal of center vortices the Coulomb string tension vanishes and the potential is no longer confining. It was also shown in ref. [64] that the Coulomb string tension $\sigma_C$ is not related to the temporal $\sigma_W^t$ but to the spatial Wilsonian string tension $\sigma_W^s$. This explains also the finite temperature behaviour of the Coulomb string tension, which does not disappear but slightly increases above the deconfinement phase transition, see ref. [64].

4.6 Spatial Wilson and ’t Hooft loops

Since $\sigma_C \geq \sigma_W^t$ the confining Coulomb potential is a necessary but not sufficient condition for confinement. To really show that the variational approach yields confinement, one has to calculate the vacuum expectation value of the Wilson loop, the true order parameter of confinement. As discussed in the first lecture in the confined phase this quantity exhibits an area-law falloff, while in the deconfined phase it falls off with the perimeter. The calculation of the Wilson loop is, unfortunately, difficult in a continuum theory due to the path ordering in this operator. In Ref. [67] a Dyson–Schwinger equation for the Wilson loop in supersymmetric Yang–Mills theory has been derived. Although this equation is strictly valid only in supersymmetric Yang–Mills theory this equation can be used for an approximate evaluation of the Wilson loop in the non-supersymmetric theory as well [65]. The static potential extracted from the obtained Wilson loop is shown in Fig. 40. Unfortunately the method used to solve the DSE for the Wilson loop works only up to intermediate distances, where the potential is still strongly affected by the Coulomb-like behavior. After subtraction of the Coulombic part one obtains the linearly rising potential, as shown in the right panel of Fig. 40.

An alternative order parameter (or rather a disorder parameter) of confinement, which is easier to calculate in a continuum theory is the (spatial) ’t Hooft loop [3]. This object is the expectation value of the operator $\hat{V}(C)$ defined by the following
commutation relation
\[ \hat{V}(C_1)W(C_2) = z^{L(C_1, C_2)}W(C_2)\hat{V}(C_1), \]  
(104)
where \( C_1 \) and \( C_2 \) are closed curves in \( \mathbb{R}^3 \), \( W(C_2) \) is the Wilson loop, \( z \) is a non-trivial center element of the gauge group and \( L(C_1, C_2) \) is the Gauss linking number [41]. As discussed in the first lecture the ’t Hooft loop \( -\ln\langle \hat{V}(C) \rangle \) measures the electric flux through a surface enclosed by \( C \). It exhibits an area law falloff in the deconfined phase and a perimeter law in the confined phase, see Eq. (24). In Ref. [68] the following continuum representation of the ’t Hooft loop operator was derived
\[ \hat{V}(C) = \exp\left[ i \int_{\mathbb{R}^3} A(C)\Pi \right], \]  
(105)
where \( \Pi = \delta/(i\delta A) \) is the momentum operator of the gauge field and \( A(C) \) is the gauge potential of a thin center vortex located at the loop \( C \). A realization of such a gauge potential is given by
\[ A(C)(x) = 2\pi\mu \int_{\Sigma(C)} d\Sigma(x') \delta(x - x'), \]  
(106)
where \( \Sigma(C) \) is a surface with boundary \( C \) and \( \mu \) is a co-weight, which satisfies
\[ e^{i2\pi\mu} = z. \]  
(107)
With the gauge potential (106) the ’t Hooft loop operator (105) becomes
\[ \hat{V}(C) = \exp\left[ i2\pi\mu \int_{\Sigma(C)} d\Sigma(x') \cdot \Pi(x') \right]. \]  
(108)
Since \( \Pi(x) \) is the operator of the electric field, see Eq. (75), this representation (108) shows that \( \hat{V}(C) \) measures indeed the electric flux through \( C \).

When the ’t Hooft loop operator acts on the wave functional, it displaces its argument by the center vortex field \( A(C) \):
\[ \hat{V}(C)\Psi[A] = \Psi[A + A(C)]. \]  
(109)
This is obvious if one notices that \( \Pi = \delta/(i\delta A) \) is the momentum operator and recalls a similar relation from the usual quantum mechanics, \( \exp(i\alpha\hat{p})\psi(x) = \psi(x + \alpha) \). The ’t Hooft loop has been calculated within the variational approach in Ref. [55] at zero temperature and a perimeter law was found.
4.7 Hamiltonian approach to QCD in Coulomb gauge

The Hamiltonian approach to pure Yang–Mills theory presented above can be extended to full QCD. The QCD Hamiltonian in Coulomb gauge is

\[ H_{\text{QCD}} = H_{\text{YM}} + H_{C} + H_{q}, \]  

(110)

where \( H_{\text{YM}} \) and \( H_{C} \) are the Yang–Mills Hamiltonian, Eq. (90), and the Coulomb term, Eq. (91), respectively. The latter contains now also the color charge density of the quarks

\[ \rho_{m}^{a}(x) = \psi^{\dagger}(x) t^{a} \psi(x), \]  

(111)

where \( \psi(x) \) is the quark field and \( t^{a} \) is the \( a \)th generator of the gauge group in the fundamental representation. The last term of Eq. (110) is the Dirac Hamiltonian of quarks coupled to the spatial gauge field

\[ H_{q} = \int d^{3}x \psi^{\dagger}(x) \left[ \alpha(p + gA) + \beta m_{0} \right] \psi(x), \]  

(112)

where \( \alpha \) and \( \beta \) are Dirac matrices.

For the vacuum wave functional of full QCD the following variational ansatz was used [45]

\[ |\Phi(A)_{\text{QCD}} \rangle \sim |\Psi[A]|_{q}, \]  

(113)

where \( \Psi[A] \) is the wave functional (91) of the Yang–Mills sector and

\[ |\phi[A]_{q} \rangle = \exp \left[ \int \psi^{\dagger}_{+}(s\beta + vA + wA) \right] |0_{q} \rangle \]  

(114)

is the wave functional of the quarks coupled to the gauge field. Here \( \psi^{+}_{+} \) and \( \psi^{+}_{-} \) are the positive and negative energy components of the quark field, respectively, and \( s, v, w \) are variational kernels. When \( v \) and \( w \) are set to zero, the wave functional (114) takes a form reminiscent of the Bardeen–Cooper–Schrieffer (BCS) wave function of superconductivity. A wave functional of this type was used in Refs. [69; 70; 71].

When one varies the vacuum expectation value \( \langle \Phi|H_{\text{QCD}}|\Phi \rangle \) with respect to the variational kernels, one finds four coupled equations for \( s, v, w \), and the gluon energy \( \omega \). The equations for \( v \) and \( w \) can be explicitly solved in terms of the scalar kernel \( s \) and the gluon energy \( \omega \),

\[ w(p, q) = f_{w}[s, \omega] \quad v(p, q) = f_{v}[s, \omega], \]  

(115)

while for the scalar kernel \( s \) and the gluon energy \( \omega \) one finds non-linear integral equations referred to as gap equations. The gluon gap equation (95) of the pure Yang-Mills sector is then modified by additional quark loop terms, see ref. [72], while the equation for the scalar kernel \( s \) has the form

\[ s(p) = f_{s}[s, v, w; p]. \]  

(116)
An important advantage of the quark wave functional (114) is that, as opposed to previous ansätze, all the UV divergences in the gap equation (116) strictly cancel. The gap equation (116) was numerically solved in Ref. [72] using the Gribov formula (98) as input for the gluon energy. The obtained scalar form factor $s(p)$ and the effective quark mass

$$M(p) = \frac{2ps(p)}{1 - s(p)},$$

are shown in Fig. 41. The red curve corresponds to a calculation without the quark-gluon coupling, i.e. with a BCS type wave functional, and the blue line shows the result obtained with this coupling included. The inclusion of the coupling of the quarks to the transverse gauge field influences the UV behavior but does not significantly change the IR behavior. This is not surprising, because the IR behavior is dominated by the Coulomb term. There is, however, an increase of $S(p)$ and $M(p)$ in the UV which increases the quark condensate from $(-185 \text{ MeV})^3$ towards the phenomenological value of $(-235 \text{ MeV})^3$. Let us also stress that without the Coulomb term there is no spontaneous breaking of chiral symmetry as the numerical calculations show [72].

### 4.8 Alternative Hamiltonian approach to finite temperature QFT

In Refs. [73, 74] the variational approach to Yang–Mills theory in Coulomb gauge was extended to finite temperatures by making a quasi-particle ansatz for the density operator $\exp(-LH)$ of the grand canonical ensemble where the quasi-particle energy was determined by minimizing the free energy. The resulting variational equations could be solved analogously to the ones at zero temperature. There is, however, a more efficient way to treat Yang–Mills theory at finite temperature within the Hamiltonian approach. The motivation comes from the Polyakov loop (18).
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\[ P[A_4(x)] = \frac{1}{d_r} \text{tr} \ P \exp \left[ ig \int_0^L dx^4 A_4(x^4, x) \right], \quad (118) \]

where \( A_4 \) is the temporal gauge field in the fundamental representation, \( P \) is the path ordering prescription and \( L = 1/T \) is the length of the compactified Euclidean time axis, which represents the inverse temperature. Furthermore \( d_r \) is the dimension of the representation, which serves as normalization factor. The Polyakov loop cannot be calculated straightforwardly in the Hamiltonian approach due to the unrestricted time interval and the use of the Weyl gauge \( A_0 = -iA_4 = 0 \). Both problems are overcome in the more efficient Hamiltonian approach to finite-temperature quantum field theory developed in Ref. [75]. This novel approach does not require an \textit{ansatz} for the density operator of the grand canonical ensemble and allows the evaluation of the Polyakov loop. In this approach, one exploits the \( O(4) \) invariance to interchange the Euclidean time axis with one spatial axis. The temporal (anti-)periodic boundary conditions to the fields become then spatial boundary conditions, while the new (Euclidean) time axis has infinite extent as is required within the Hamiltonian approach (see below). The upshot is that the partition function at finite temperature \( L^{-1} \) is entirely given by the ground state calculated on the spatial manifold \( \mathbb{R}^2 \times S^1(L) \), where \( S^1(L) \) is a circle with length \( L \). The whole thermodynamics of the theory is then encoded in the vacuum calculated on the partially compactified spatial manifold \( \mathbb{R}^2 \times S^1(L) \). This approach was used to study Yang–Mills theory at finite temperature [76], to calculate the Polyakov loop within the Hamiltonian approach [77] and to evaluate the so-called dual quark condensate [78]. Let us briefly sketch its main features.

4.8.1 Finite temperature from the compactification of a spatial dimension

Consider finite-temperature quantum field theory in the standard functional integral approach. Here the finite temperature is introduced by going to Euclidean space and compactifying the Euclidean time dimension by imposing periodic and antiperiodic boundary conditions for Bose \( A \) and Fermi \( \psi \) fields, respectively,

\[
A(x^4 = L/2) = A(x^4 = -L/2), \quad (119a)
\]
\[
\psi(x^4 = L/2) = -\psi(x^4 = -L/2). \quad (119b)
\]

The length of the compactified dimension \( L \) represents then the inverse temperature \( T^{-1} = L \). One can now exploit the \( O(4) \) invariance of the Euclidean Lagrangian to rotate the Euclidean time axis \( x^4 \) into a spatial axis and, correspondingly, one spatial axis into the Euclidean time axis. Of course, thereby all vector quantities transform in the same way. For example, we can choose the transformation:
\[ x^4 \to x^3, \quad A^4 \to A^3, \quad \gamma^4 \to \gamma^3, \]
\[ x^3 \to x^2, \quad A^3 \to A^2, \quad \gamma^3 \to \gamma^2, \]
\[ x^2 \to x^1, \quad A^2 \to A^1, \quad \gamma^2 \to \gamma^1, \]
\[ x^1 \to x^4, \quad A^1 \to A^4, \quad \gamma^1 \to \gamma^4. \quad (120) \]

After this rotation we are left with the spatial periodic and antiperiodic boundary conditions
\[ A(x^3 = L/2) = A(x^3 = -L/2), \]
\[ \psi(x^3 = L/2) = -\psi(x^3 = -L/2). \quad (121a) \]

As a consequence of the \( O(4) \) rotation our spatial manifold is now \( \mathbb{R}^2 \times S^1(L) \) instead of \( \mathbb{R}^3 \) while the temporal manifold is \( \mathbb{R} \) independent of the temperature, i.e. the temperature is now encoded in one spatial dimension while time has infinite extension. We can now apply the usual canonical Hamiltonian approach to this rotated space-time manifold. As the new time axis has infinite extension \( \ell \to \infty \), the partition function is now given by
\[ Z(L) = \lim_{\ell \to \infty} \text{tr} \exp(-\ell H(L)), \quad (122) \]

where \( H(L) \) is the usual Hamiltonian obtained after canonical quantization, however, now defined on the spatial manifold \( \mathbb{R}^2 \times S^1(L) \). Taking the trace in the basis of the exact eigenstates of the Hamiltonian \( H(L) \), we obtain for the partition function [122]
\[ Z(L) = \lim_{\ell \to \infty} \sum_n \exp(-\ell E_n(L)) = \lim_{\ell \to \infty} \exp(-\ell E_0(L)). \quad (123) \]

The full partition function is now obtained from the ground state energy \( E_0(L) \) calculated on the spatial manifold \( \mathbb{R}^2 \times S^1(L) \). Introducing the energy density \( e(L) \) on \( \mathbb{R}^2 \times S^1(L) \) by separating the volume \( L\ell^2 \) of the spatial manifold from the energy we have
\[ E_0(L) = L\ell^2 e(L). \quad (124) \]

For the physical pressure
\[ P = \frac{1}{L} \frac{\partial \ln Z}{\partial V}, \quad V = \ell^3 \quad (125) \]

one finds from [123]
\[ P = -\frac{\partial (Ve(L))}{\partial V} = -e(L) - V \frac{\partial e(L)}{\partial V}, \quad (126) \]

while the physical energy density
\[ e = \frac{\langle H \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Z}{\partial L} + \frac{\mu}{V L \partial \mu} \quad (127) \]
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is obtained as
\[ e = \frac{\partial (L e(L))}{\partial L} - \mu \frac{\partial e(L)}{\partial \mu}. \] (128)

To distinguish this quantity from the (negative) Casimir pressure \( e(L) \) [Eq. (126)], which also appears as an energy density in our formalism after the transformation [Eq. (120)], we will denote \( e(L) \) as pseudo-energy density. Note also that the last term in eq. (126) vanishes for generic non-interacting systems such that \( P = -e(L) \).

Finally, after the \( O(4) \) rotation, Eq. (120), the finite chemical potential \( \mu \) enters the single-particle Dirac Hamiltonian \( h \) in the form
\[ h(\mu) = h(\mu = 0) + i \mu \alpha^3, \] (129)
where \( \alpha^3 \) is the third Dirac matrix and \( h(\mu = 0) \) is the usual single particle Dirac Hamiltonian.

4.8.2 Free Bose and Fermi gases

To illustrate the above approach let us first consider a relativistic Bose gas with dispersion relation \( \omega(p) = \sqrt{p^2 + m^2} \), where we assume for simplicity a vanishing chemical potential. The thermodynamical pressure obtained from the grand canonical ensemble for such a system is given by \( P = T^{-1} \)
\[ P = \frac{2}{3} \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\omega(p)} n(p), \quad n(p) = \frac{1}{\exp(\omega(p)/T) - 1}, \] (130)
where \( n(p) \) are the finite temperature Bose occupation numbers. On the other hand, for the ideal Bose gas with dispersion relation \( \omega(p) = \sqrt{p^2 + m^2} \) one finds the pseudo-energy density on the spatial manifold \( R^2 \times S^1(L) \) [75]
\[ e(L) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 L} \sum_{n=-\infty}^{\infty} \sqrt{p^2 + p_n^2 + m^2}, \quad p_n = \frac{2n\pi}{L}, \] (131)
where \( p_n \) are the bosonic Matsubara frequencies. This quantity does not look at all like the negative of the pressure (130), as it should by Eq. (126). In fact, as it stands \( e(L) \) is ill defined: the integral and the sum are both divergent. To make it mathematically well defined, we first use the proper-time regularization of the square root,
\[ \sqrt{A} = \frac{1}{\Gamma(-\frac{1}{2})} \lim_{\Lambda \to \infty} \left[ \int_{\Lambda/A^2}^{\infty} d\tau \tau^{-\frac{1}{2}} \exp(-\tau A) - 2\Lambda + O(\Lambda^{-1}) \right]. \] (132)
The divergent constant appears because the limit \( \Lambda \to \infty \) of the incomplete \( \Gamma \)-function \( \Gamma(-\frac{1}{2}, \Lambda) \) is not smooth; it drops out when taking the difference to the zero-temperature case after Eq. (137) below. To extract the zero temperature part of
eq. (131) for the Matsubara sum we use the Poisson resummation formula,

\[ \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(ikx) = \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n), \]

(133)

by means of which one derives the relation

\[ \frac{1}{L} \sum_{n=-\infty}^{\infty} f(p_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z) \sum_{k=-\infty}^{\infty} e^{ikzL}. \]

(134)

Inserting this relation into eq. (131) we obtain

\[ e(L) = \frac{1}{2} \int d^3p \sqrt{p^2 + m^2} \sum_{k=-\infty}^{\infty} e^{ikLp_3}, \]

(135)

where the integration variable \( z \) in (134) was renamed \( p_3 \) and interpreted as third component of the 3-momentum \( p = p_\perp + p_3 e_3 \). In eq. (135) \( \int d^3p = \int d^3p/(2\pi)^3 \) is the usual integration measure in flat momentum space \( \mathbb{R}^3 \). Obviously, the \( k = 0 \) term is just the zero temperature part of the vacuum energy density, which is an (infinite) temperature independent (and thus irrelevant) constant, which has to be omitted from the thermodynamical quantities. Then with the replacement (132) (taking the limit \( \Lambda \to \infty \) thereby skipping the divergent piece \( \Lambda \)) we find from (135)

\[ e(L) = \frac{1}{\Gamma(-\frac{1}{2})} \int_{0}^{\infty} d\tau \tau^{-1} \int d^3p e^{-\pi(p^2 + m^2)} \sum_{k=1}^{\infty} \cos(kLp_3). \]

(136)

After performing the momentum integrals the proper-time integral can also be carried out, yielding for the pseudo energy density (136)

\[ e(L) = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{m}{nL} \right)^2 K_2(nLm), \]

(137)

where \( K_\nu(z) \) is the modified Bessel function and we have used \( \Gamma(-\frac{1}{2}) = -2 \sqrt{\pi} \). The individual terms \( (n \neq 0) \) are all finite and also their sum converges. This sum, however, cannot be carried out analytically for massive bosons (the same applies to the integral in the grand canonical expression (130) for the pressure). In the zero-mass limit the expression Eq. (137) can be worked out analytically. Using the asymptotic form of the Bessel function

\[ K_\nu(z) = \frac{1}{2} \Gamma(\nu) \left( \frac{1}{2} z \right)^{-\nu}, \quad z \to 0, \quad \nu > 0 \]

(138)

we find from (137)

\[ e(L) = -\frac{1}{\pi^2} T^4 \zeta(4), \quad T = L^{-1}, \]

(139)
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where

\[ \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \]  

is the Riemann \( \zeta \)-function. With \( \zeta(4) = \frac{\pi^4}{90} \) we find for the pressure \( P = -e(L) \) of massless bosons

\[ P = \frac{\pi^2}{90} T^4, \]  

which is Stefan–Boltzmann law, the correct result also obtained from the grand canonical ensemble. For massive bosons the evaluation of the sum in Eq. (137) as well as the evaluation of the integral in Eq. (130) have to be done numerically. The result is shown in Fig. 42. As expected the pressure calculated from the compactified spatial dimension reproduces the result of the usual grand canonical ensemble. Figure 42 shows the various contributions to the pressure. It is seen that only a few terms in the sum of Eq. (137) are necessary to reproduce the result of the grand canonical ensemble to good accuracy.

In the case of the relativistic Fermi gas with dispersion relation \( \omega(p) = \sqrt{p^2 + m^2} \) the energy density on \( \mathbb{R}^2 \times S^1(L) \) is given by

\[ e(L) = -2 \int \frac{d^2 p}{(2\pi)^2} \frac{1}{L} \sum_{n=-\infty}^{\infty} \sqrt{p^2 + (p_n + i \mu)^2 + m^2}, \quad p_n = \frac{2n + 1}{L\pi}, \]

where we have now included a non-vanishing chemical potential \( \mu \). To make this expression mathematically well-defined one has to resort again to the proper-time regularization and Poisson resummation technique sketched above. The result is

\[ e(L) = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \cos \left( nL \frac{\pi}{L} - i \mu \right) \left( \frac{m}{nL} \right)^2 K_{-2}(nLm). \]

Again, the term with \( n = 0 \) represents the zero temperature vacuum energy density, which is divergent and has to be removed. As before, this expression can only be calculated in closed form for massless particles. Furthermore for the remaining sum to converge, an analytic continuation \( i \mu L \rightarrow \tilde{\mu} \in \mathbb{R} \) is required to carry out the sum

\[ \sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\tilde{\mu}L)}{n^4} = \frac{1}{48} \left[ -\frac{7}{15} \pi^2 + 2\pi^2 \tilde{\mu}^2 - \tilde{\mu}^4 \right]. \]

Continuing back to real chemical potentials one finds for the pressure \( P = -e(L) \)

\[ P = \frac{1}{12\pi^2} \left[ \frac{7}{15} \pi^4 T^4 + 2\pi^2 T^2 \mu^2 + \mu^4 \right], \]

which is the correct result obtained also from the usual grand canonical ensemble.

In Ref. [76], the above approach was used to study Yang–Mills theory at finite temperature. For this purpose, it was only necessary to repeat the variational Hamil-
4.8.3 The Polyakov loop

Consider SU($N$) gauge theory at finite temperature, where the temperature is introduced by the usual periodic boundary condition in the temporal direction (119). Gauge transformations preserving this boundary conditions need to be periodic only up to an element $z$ of the center $Z(N)$ of the gauge group,

$$U(x^4 = L) = zU(x^4 = 0), \quad z \in Z(N).$$

(146)

Since there are $N$ center elements, this theory has a residual global $Z(N)$ symmetry, which remains after local gauge fixing. However, there are quantities which are sensitive to such a $Z(N)$ symmetry transformation. The most prominent example is the Polyakov loop (118). A gauge transformation of the form (146) multiplies the Polyakov loop by the center element $z$, i.e.

$$P[A_4^U] = zP[A_4].$$

(147)

As discussed in sect.1.3.3 the expectation value of the Polyakov loop

$$\langle P[A_4]\rangle \sim \exp\left(-L(F_0(x) - F_0)\right)$$

(148)

is related to the free energy $F(x) - F_0$ of a static color point charge located at $x$ where $F_0$ is the (divergent) free energy of the vacuum [15]. In a confining theory this
quantity has to be infinite since there are no free color charges, while in a deconfined phase it is finite. Accordingly we find for the expectation value of the Polyakov loop

$$\langle P[A_4](x) \rangle = \begin{cases} 0 & \text{confined phase,} \\ \neq 0 & \text{deconfined phase.} \end{cases}$$ (149)

From Eq. (147) follows that a state which is invariant with respect to the global center transformation, has a vanishing expectation value of the Polyakov loop. Hence, in the deconfined phase the $Z(N)$ center symmetry is obviously broken.

In the continuum theory the Polyakov loop can be most easily calculated in the Polyakov gauge

$$\partial_4 A_4 = 0, \quad A_4 \text{ color diagonal.}$$ (150)

In this gauge one finds, for example, for the SU(2) gauge group that the Polyakov loop

$$P[A_4](x) = \cos \left( \frac{1}{2} g A_4(x) L \right)$$ (151)

is a unique function of the gauge field, at least in the fundamental modular region of this gauge. It can be shown, see Refs. [79; 80], that instead of the expectation value of the Polyakov loop $\langle P[A_4] \rangle$ one may alternatively use the Polyakov loop of the expectation value, $P[\langle A_4 \rangle]$, or the expectation value of the temporal gauge field itself, $\langle A_4 \rangle$, as order parameter of confinement in the gauge (150). This analysis also shows that the most efficient way to obtain the Polyakov loop is to carry out a so-called background field calculation with a temporal background field $a_4(x) = \langle A_4(x) \rangle$ chosen in the Polyakov gauge, and then calculate the effective potential $e[a_4]$ of that background field. From the minimum $\tilde{a}_4$ of this potential one evaluates the Polyakov loop $P[\langle A_4 \rangle] = P[\tilde{a}_4]$, which can then serve as an order parameter for confinement.

Such a calculation was done a long time ago in Ref. [81; 82], where the effective potential $e[a_4]$ was calculated in one-loop perturbation theory. The result is shown in Fig. 43a. The potential is periodic due to center symmetry. The minimum of the potential occurs at vanishing background field, which gives $P[a_4 = 0] = 1$ corresponding to the deconfined phase. This is, of course, expected due to the use of perturbation theory. Below, I present the results of a non-perturbative evaluation of $e[a_4]$ in the Hamiltonian approach in Coulomb gauge.

At first sight it seems that the Polyakov loop cannot be calculated in the Hamiltonian approach due to the use of the Weyl gauge $A_4 = 0$. However, we can now use the alternative Hamiltonian approach to finite temperature introduced in sect. 4.8.2 where the temperature is introduced by compactifying a spatial dimension. Here, we compactify the $x_3$-axis and consequently put also the background field along this axis$^{10}$ $a = a e_3$. The Polyakov loop is then given by$^{11}$ (c.f. eq. (118))

---

10 A constant background field directed along an uncompactified axis is irrelevant since it can be eliminated by a simple change of the corresponding momentum variable.

11 Since we are considering here only stationary vacuum wave functionals the time-argument can be skipped.
\[ P[A_3](x_{\perp}) = \frac{1}{d_r} \text{tr}\Phi \exp \left[ i g \int_0^L dx^3 A_3(x_{\perp}, x^3) \right]. \tag{152} \]

where \( x_{\perp} \) denotes the coordinates of the two uncompactified spatial dimensions. We are interested here in the quantity \( P[\langle A_3 \rangle](x_{\perp}) \). For this purpose we calculate the effective potential of \( \langle A_3 \rangle \) using the background field method.

In the Hamiltonian approach the effective potential of a spatial background field \( a \) can be easily calculated by minimizing the expectation value of the Hamiltonian under the constraint \( \langle A \rangle = a \) [83]. The resulting energy \( \langle H \rangle_a = L^2 e(a) \) is then (up to the spatial volume factor) the effective potential. So the effective potential \( e(a) \) is nothing but the pseudo energy density considered earlier, but now calculated with the constraint \( \langle A \rangle = a \), see ref. [77] for more details.

![Fig. 43](image)

Fig. 43 The effective potential of the Polyakov loop \( e(a, L) \) [153] as function of the background field \( x = a_3 L / 2\pi \). The curvature is neglected (\( \chi = 0 \)) and the gluon energy assumed to be (a) \( \omega(p) = p \) (UV-form) and (b) \( \omega(p) = M^2 / p \) (IR-form), respectively, for various temperatures. Note that the UV form of the potential is (up to a global factor of \( T^4 \)) independent on the temperature, see fig. (a). This is different for IR form of the potential, see fig. (b), which contains a mass scale. Here the temperature increases from bottom to top.

**4.8.4 The effective potential of the Polyakov loop**

After lengthy calculations, exploiting the gluon gap equation [75] and neglecting the Coulomb term, one finds for the effective potential of the Polyakov loop (or more precisely of \( \langle A_3 \rangle = a \)) the following expression [77]

\[ e(a, L) = \sum_{\sigma} \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \left( \omega(p^\perp) - \chi(p^\perp) \right), \tag{153} \]

where \( \omega(p) \) is the gluon energy and \( \chi(p) \) is the ghost loop. These quantities have to be taken with the momentum variable.
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\[ p^\sigma = p_\perp + (p_n - \sigma \cdot a) e_3, \]  

(154)

where \( p_\perp \) is the momentum corresponding to the two non-compactified space dimensions while \( p_n = 2\pi n/L \) is the Matsubara frequency resulting from the compactification of the third dimension. Furthermore, \( \sigma \cdot a \equiv \sigma^b a^b \) denotes the product of the color background field with the root vectors \( \sigma^b \) of the color group. Equation (153) includes also the summation over the roots \( \sigma \) of the gauge group. In Refs. [84; 77], the effective potential (153) was explicitly calculated using for \( \omega(p) \) and \( \chi(p) \) the results from the variational calculation in Coulomb gauge at zero temperature [53]. This represents certainly an approximation since, in principle, one should use the finite-temperature solutions obtained in Ref. [76].

Before I present the full results let me ignore the ghost loop \( \chi(p) \) in Eq. (153) and consider the ultraviolet and infrared limit of the gluon energy. If we choose the ultraviolet limit \( \omega(p) = p \), we obtain from Eq. (153) with \( \chi(p) = 0 \) precisely the Weiss potential, shown in Fig. 43a, which corresponds to the deconfined phase. Choosing for the gluon energy its infrared limit \( \omega(p) = M^2/p \), one finds from Eq. (153) with \( \chi(p) = 0 \) the (center symmetric) potential shown in Fig. 43b. From its center symmetric minimum \( \tilde{a} = \pi/L \) one finds a vanishing Polyakov loop \( P[\tilde{a}] = 0 \) corresponding to the confined phase. Obviously, the deconfining phase transition results from the interplay between the confining infrared and the deconfining ultraviolet dispersions. Choosing for the gluon energy the sum of its UV- and IR-parts \( \omega(p) = p + M^2/p \), which can be considered as an approximation to the Gribov formula (100), one has to add the UV and IR potentials and finds a phase transition at a critical temperature \( T_c = \sqrt{3} M/\pi \). With the Gribov mass \( M \approx 880 \) MeV this gives a critical value of \( T_c \approx 485 \) MeV for the color group SU(2), which is much too high as compared to the lattice value of 312 MeV [85]. One can show analytically [84; 77] that the neglect of the ghost loop \( \chi(p) = 0 \) shifts the critical temperature to higher values. If one uses the Gribov formula (100) for the gluon energy \( \omega(p) \) and includes the ghost loop \( \chi(p) \), one finds the effective potential shown in Fig. 44, which shows a second order phase transition and gives a transition temperature of \( T_c \approx 269 \) MeV for the gauge group SU(2), which is in the right ballpark. The Polyakov loop \( P[\tilde{a}] \) calculated from the minimum \( \tilde{a} \) of the effective potential \( e(a, L) \) (153) is plotted in Fig. 45 as function of the temperature.

The effective potential for the gauge group SU(3) can be reduced to that of the SU(2) group by noticing that the SU(3) algebra consists of three SU(2) subalgebras characterized by the three positive roots \( \sigma = (1, 0), (1/2, \sqrt{3}/2), (1/2, -\sqrt{3}/2) \). One finds

\[ e_{SU(3)}(a, L) = \sum_{\sigma > 0} e_{SU(2)}[\sigma](a, L). \]  

(155)

The resulting effective potential for SU(3) is shown in Fig. 46 as function of the components of the background field in the Cartan algebra, \( a_3 \) and \( a_8 \). Above and below \( T_c \) the absolute minima of the potential occur in both cases for \( a_8 = 0 \). Cutting the two-dimensional potential surface at \( a_8 = 0 \), one finds the effective potential shown in Fig. 44b, which shows a first order phase transition with a critical temperature of \( T_c \approx 283 \) MeV. The first order nature of the SU(3) phase transition is
Fig. 44 Effective potential of the Polyakov loop as function of the background field \( x = a_3 L/2\pi \) at various temperatures, for the gauge group (a) SU(2) and (b) SU(3). The temperature increases from bottom to top.

Fig. 45 The Polyakov loop as function of the temperature (a) for SU(2) and (b) for SU(3).

Fig. 46 The effective potential of the Polyakov loop for the gauge group SU(3) as function of the two Cartan components of the background field \( x = a_3 L/2\pi \) and \( y = a_8 L/2\pi \) for (a) \( T < T_c \) and (b) \( T > T_c \).
also seen in Fig. [45], where the Polyakov loop $P[\bar{a}]$ is shown as function of the temperature.

### 4.8.5 The dual quark condensate

The dual quark condensate was originally introduced in Ref. [86] and was discussed in a more general context in Ref. [87]. This quantity has been calculated on the lattice [88, 89], in the functional renormalization group approach [90] and in the Dyson–Schwinger approach [91]. The dual condensate is defined by

$$
\Sigma_n = \int_0^{2\pi} \frac{d\phi}{2\pi} \exp(-in\phi) \langle \bar{\psi} \psi \rangle \ ,
$$

(156)

where $\langle \bar{\psi} \psi \rangle$ is the quark condensate calculated with the $U(1)$-valued boundary condition

$$
\psi(x^\ell + L/2, x) = e^{i\phi} \psi(x^\ell - L/2, x) \ .
$$

(157)

For $\phi = \pi$ these boundary conditions reduce to the usual finite-temperature boundary conditions of the quark field in the functional integral representation of the partition function, see Eq. (119). On the lattice it is not difficult to show that the quantity $\Sigma_n$ (156) represents the vacuum expectation value of the sum of all closed Wilson loops winding precisely $n$-times around the compactified time axis. In particular, the quantity $\Sigma_1$ represents the expectation value of all closed loops winding precisely once around the compactified time axis and is therefore called the dressed Polyakov loop. The phase in the boundary condition (157) can be absorbed into an imaginary chemical potential

$$
\mu = \frac{\pi - \phi}{L} \ ,
$$

(158)

for fermion fields satisfying the usual antisymmetric boundary condition $\psi(x^\ell + L/2, x) = -\psi(x^\ell - L/2, x)$. In the Hamiltonian approach to finite temperatures of Ref. [75], where the compactified time axis has become the third spatial axis, the phase dependent boundary condition (157) or equivalently the imaginary chemical potential (158) manifests itself in the momentum variable along the (compactified) three-axis, which reads

$$
p_3 = p_n + i\mu = \frac{2\pi n + \phi}{L} \ , \hspace{1cm} p_n = \frac{2n + 1}{L}\pi \ ,
$$

(159)

where $p_n$ is the usual fermionic Matsubara frequency [Eq. (142)]. Using the zero-temperature quark mass function $M(p)$ calculated in Ref. [72], one finds in the Hamiltonian approach to QCD of Ref. [45] for the dual quark condensate after Poisson resummation (134) the leading expression (78).
\[
\Sigma_n = -\frac{N}{\pi^2} \int_0^\infty dp \frac{p^2 M(p)}{\sqrt{p^2 + M^2(p)}} \left[ \delta_{n0} + \frac{\sin(nLp)}{nLp} \right],
\]

(160)

where \( N \) denotes the number of colors. In the same way, one can compute the quark condensate \( \langle \bar{\psi} \psi \rangle \phi \) shown in Fig. 47a. For the dressed Polyakov loop \( \Sigma_1 \) one finds the temperature behavior shown in Fig. 47b, where we also compare with the result obtained when the coupling to the transverse gauge field degrees of freedom is neglected \((g = 0)\). As one observes there is no difference at small temperatures in accord with the fact that the mass function \( M(p) \) has the same infrared behavior, whether the coupling to the transverse gluons is included or not. The slower UV decrease of the full mass function causes the dual condensate to reach its high-temperature limit

\[
\lim_{L \to 0} \Sigma_1 = -\frac{N}{\pi^2} \int_0^\infty dp \frac{p^2 M(p)}{\sqrt{p^2 + M^2(p)}} = \lim_{L \to \infty} \langle \bar{\psi} \psi \rangle_{\phi=\pi}
\]

(161)

only very slowly. However, we expect that this limit is reached faster when the finite-temperature solutions are used. This will presumably also convert the crossover obtained for the chiral condensate, see Fig. 47b, into a true phase transition as expected for chiral quarks. From the inflexion points of the chiral and dual condensates one extracts the values of \( T_{pc}^{\chi} \approx 170 \text{ MeV} \) and \( T_{pc}^{c} \approx 198 \text{ MeV} \) for the pseudo-critical temperatures of the chiral and deconfinement transition, respectively. For comparison, on the lattice one finds for realistic quark masses \( T_{pc}^{\chi} \approx 155 \text{ MeV} \) and \( T_{pc}^{c} \approx 165 \text{ MeV} \) [92, 93].

Fig. 47 (a) Chiral quark condensate \( \langle \bar{\psi} \psi \rangle_{\phi} \) as function of the temperature \( T \) and the phase \( \phi \) of the boundary condition [157]. (b) Chiral and dual quark condensate as function of the temperature. Results are presented for both a coupling of \( g \approx 2.1 \) and \( g = 0 \).
4.9 Conclusions

In this talk I have presented a survey of results obtained within the Hamiltonian approach to QCD in Coulomb gauge. I have first considered the pure Yang-Mills sector and shown that the obtained gluon and ghost propagators reflect the confining properties of the theory and are in satisfactory agreement with the lattice data. The gluon energy and the ghost form-factor are both IR divergent in accordance with the Gribov picture of confinement. I have also established the connection of this picture with the magnetic monopole (dual Meißner effect) and the center vortex picture.

I have then studied the quark sector of QCD within the Hamiltonian approach in Coulomb gauge using a Slater determinant ansatz for the quark wave functional, which includes in particular the quark-gluon coupling with two different Dirac structures. Our calculations show that there is no spontaneous breaking of chiral symmetry when the (linearly rising) infrared part of the Coulomb potential is excluded. Furthermore, choosing the Coulomb string tension from the lattice data we can reproduce the phenomenological value of the quark condensate when the coupling of the quarks to the transverse gluons is included.

I have then extended the Hamiltonian approach to QCD in Coulomb gauge to finite temperatures by compactifying a spatial dimension [75]. Within this approach, I have calculated the effective potential of the Polyakov loop for pure Yang-Mills theory as well as the chiral and dual quark condensates for QCD as function of the temperature. Using our zero-temperature variational solution as input, from the Polyakov loop we predict a critical temperature for the deconfinement phase transition in pure Yang-Mills theory of about $T_c \sim 275$ MeV for SU(2), and $T_c \sim 280$ MeV for SU(3). Furthermore, the correct order of the phase transition was found for both SU(2) and SU(3). For full QCD our calculations of the dual and chiral quark condensate predict pseudo-critical temperatures of $T_{pc}^\chi \approx 170$ MeV for the chiral and $T_{pc}^c \approx 198$ MeV for the deconfinement transition. In all these finite-temperature calculations the zero-temperature variational solutions were used as input, which is likely the reason that the critical temperatures currently obtained are too high as compared to the lattice data. The solution of the variational principle at finite temperature will be the next step in our investigation of the QCD phase diagram.

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