TRANSVERSE SINGULARITIES OF MINIMAL TWO-VALUED GRAPHS IN
ARBITRARY CODIMENSION

SPENCER T. HUGHES

Abstract. We prove regularity results that give a detailed description of the structure of an n-
dimensional minimal two-valued Lipschitz graph and its singular set in a neighbourhood of any point
at which at least one tangent cone is equal to a union of four distinct multiplicity one n-dimensional
half-planes meeting only along an \((n-1)\)-dimensional axis and in a neighbourhood of any point
at which at least one tangent cone is equal to a transverse union of two distinct multiplicity one
n-dimensional planes. The key ingredient is a new Excess Improvement Lemma obtained via a
blow-up method (inspired by the work of L. Simon on the singularities of ‘multiplicity one’ classes
of minimal submanifolds) and which can be iterated unconditionally. We also show that any tangent
cone to a minimal two-valued Lipschitz graph whose spine is either \((n-1)\) or \((n-2)\)-dimensional
is indeed a cone of one of the two aforementioned forms, which yields a global decomposition result
about the singular set. The present work can be viewed as partial progress towards a more complete
understanding of the singular set of a minimal two-valued Lipschitz graph.

Contents

Introduction 1
1. Notation and Preliminaries 4
2. Proper Blow-Up Classes 8
3. Gaps In The Top Density Part 19
4. Partial Graphical Representation and \(L^2\) Estimates 22
5. Non-Concentration Estimates 24
6. Constructing Blow-Ups 26
7. Properties of Blow-Ups 28
8. Excess Improvement 31
9. Main \(\epsilon\)-Regularity Theorems 35
10. Proof of Theorem 4 37
Appendix A. 38
References 44

Introduction

In this paper, we study the local asymptotic structure of minimal two-valued Lipschitz graphs
(by which we mean any stationary integral \(n\)-varifold associated in the natural way to the graph
of a two-valued Lipschitz function). We completely describe the structure of such a varifold and
its singular set when it lies close to a pair of \(n\)-dimensional affine subspaces meeting along an axis
of dimension at most \((n-2)\) and when it lies close to a union of four \(n\)-dimensional half-spaces
meeting only along an \((n-1)\)-dimensional axis. In the first case, we obtain the following:

This work was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant
EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis.
When a minimal two-valued Lipschitz graph lies sufficiently close to a pair of planes meeting along an axis of dimension at most \( n - 2 \), it must be equal to the union of two smooth minimal submanifolds, each of which lies close to one of the two planes and which meet only along a subset of an \((n - 2)\)-dimensional smooth submanifold that is graphical over the axis of the pair of planes.

And in the second case:

When a minimal two-valued Lipschitz graph lies sufficiently close to a union of four \( n \)-dimensional half-planes that meet only along an \((n - 1)\)-dimensional axis, its singular set is contained in an \((n - 1)\)-dimensional \( C^{1,\alpha} \) submanifold that is graphical over the axis of the cone and at each singular point there is a unique tangent cone equal to either a transversely intersecting pair of planes or a union of four half-planes meeting only along an \((n - 1)\)-dimensional axis.

Note that in the first case the conclusion implies that the varifold is smooth as a two-valued graph. The conclusions of the second theorem however do not imply that it is \( C^{1,\alpha} \) as a two-valued graph. The following example makes this explicit:

**Example 1** Let \( f \) denote the two-\( \mathbb{R}^2 \)-valued function on \( \mathbb{R} \) given by \( f(t) = \{(t,0),(-t,0)\} \) for \( t \leq 0 \) and \( f(t) = \{(0,t),(0,-t)\} \) for \( t > 0 \). This is Lipschitz as a two-valued function and its graph is minimal (and indeed equal to a union of four smooth ‘sheets’ in \( \mathbb{R}^3 \)). However, it is clearly not \( C^1 \) as a two-valued function at the origin. The codimension of this example is irrelevant and so one can produce examples of any dimension and codimension by crossing this example with Euclidean space.

In general (i.e. not under the hypotheses of the main theorems herein), there are more ‘exotic’ singularities in Lipschitz minimal graphs, even in the single-valued setting, as the following example shows.

**Example 2** (Lawson-Osserman [LO77]) Consider \( S^3 \) to be the unit sphere in \( \mathbb{C}^2 \cong \mathbb{R}^4 \) and consider \( S^2 \) to be the unit sphere in \( \mathbb{R} \times \mathbb{C} \cong \mathbb{R}^3 \). Define \( \eta : S^3 \to S^2 \) by

\[
\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 \overline{z_2})
\]

(this is the Hopf map). It is shown in [LO77] that the homogeneous degree one function \( f : \mathbb{R}^4 \to \mathbb{R}^3 \) given by

\[
f(x) = \frac{\sqrt{5}}{2} |x| \eta \left( \frac{x}{|x|} \right) \text{ for } x \neq 0
\]

is a Lipschitz weak solution to the minimal surface system on \( \mathbb{R}^4 \). Note that the graph of the two-valued function \( g(x) = \{f(x),-f(x)\} \) is an example of a minimal two-valued Lipschitz graph which is a cone and which is not equal to a union of planes or half-planes.

If one does have \( C^{1,\alpha} \) regularity for a minimal two-valued graph, then one may apply the results of L. Simon and N. Wickramasekera ([SW10]) to deduce that in fact the two-valued function in question is \( C^{1,1/2} \). And if one starts from \( C^{1,\alpha} \), then this is the best possible general result for the regularity of such objects, as the example of the irreducible holomorphic variety \( I := \{(z,w) \in \mathbb{C} \times \mathbb{C} : z^2 = w^3\} \subset \mathbb{R}^4 \) shows: It is well known that such a variety is area-minimizing and therefore minimal and yet it is easy to see that if viewed as the two-valued graph of \( w \mapsto w^{3/2} \), then the regularity at the origin is no better than \( C^{1,1/2} \). In codimension one, a minimal two-valued Lipschitz graph is necessarily \( C^{1,\alpha} \) (this is shown by the author in [Hug14], relying on work due to...
B. Krummel and Wickramasekera on the structure of the branch set, [KW13], and recent regularity theory due to Wickramasekera [Wic].

It is a well-known fact in the study of stationary integral varifolds that the singular set of such an object can be ‘stratified’ in a particularly useful way: For a stationary cone C, we write $S(C) := \{ Z \in \mathbb{R}^{n+k} : \Theta_C(Z) = \Theta_C(0) \}$. We call this set the spine of C and it is not difficult to show that it is a linear subspace of $\mathbb{R}^{n+k}$. Given a stationary varifold V we write

$$(*) \quad S_j = \{ X \in \text{sing} V : \dim S(C) \leq j \forall C \in \text{Var Tan}(V, X) \}$$

(where Var Tan(V, X) is the set of all varifolds obtained as tangent cones to V at X). Then $\dim \mathcal{H} S_j \leq j$. This was first shown for stationary integral varifolds by F. Almgren [Alm00], but is true in other settings in the study of solutions to geometric variational problems (see e.g. [Sim83], [Sim96] and [Whi97]). With this in mind, we show the following:

Any tangent cone to a minimal two-valued graph that has a spine of dimension $(n-2)$ must be equal to a union of two distinct multiplicity one $n$-dimensional planes intersecting only along an $(n-2)$-dimensional subspace. Any tangent cone with a spine of dimension $(n-1)$ must be equal to a union of four distinct multiplicity one $n$-dimensional half-planes meeting only along their common boundary, an $(n-1)$-dimensional subspace.

So, in light of our two main $\epsilon$-regularity theorems, we have a complete description of a minimal two-valued Lipschitz graph near points in $S_{n-1} \setminus S_{n-2}$ and points in $S_{n-2} \setminus S_{n-3}$.

Except for one-dimensional varifolds (a detailed description of which was given by Allard and Almgren in [AA76]), very little is known about the structure of general stationary integral varifolds and their singular sets (in arbitrary codimension). Allard’s seminal work [All72] shows that the regular part is non-empty, open and dense, but stationary varifolds with Cantor set-like singular sets of large measure have not at present been ruled out. In fact, there are currently no general results on the size of the singular set of a stationary integral n-varifold (in light of the simple example of a transverse union of hyperplanes, the best possible general result one can hope for here is $\dim_\mathcal{H}(\text{sing} V) \leq n - 1$).

Remarks On The Proof. Our method is inspired by the blow-up method used by L. Simon in [Sim93]. By deriving estimates that hold for any ‘multiplicity one’ minimal submanifold lying close to a cylindrical tangent cone with arbitrary vertex density, Simon was able to control blow-ups off these non-planar cones. Such decay estimates have already been used in contexts in which the multiplicity one hypothesis fails (e.g. in [Wic04] and [Wic14]) and indeed we use them here too. They hold at all scales in balls centred at ‘good density points’, by which we mean points on the varifold with density at least that of the vertex of the cone, which in our case is two. However, in all previous manifestations of this blow-up method, it was either assumed (Remark 1.14 of [Sim93]) or could be checked (e.g. in [Wic04] and [Wic14]) that there were ‘lots’ of good density points in the sense that a $\delta$-neighbourhood of $\{ X : \Theta_V(X) \geq 2 \}$ contained the axis of the cone. In these cases, it was possible to use Simon’s $L^2$ estimates to show that the excess does not concentrate near the axis and that the blow-ups are ‘regular up to the axis’ in the appropriate sense. The key difference therefore in the present work at the technical level is that we have no control over the set of good density points and we must deal with ‘gaps’ in the good density part near the axis. In these gaps - regions in which every point has density less than two and where we do not have good decay estimates - we show that the two-valued graph decomposes as two minimal single-valued Lipschitz graphs. If such gaps persist in the limit as the varifold approaches a union of four half-planes, then the cone must necessarily be a pair of planes. Thus, locally, we can separately blow-up each of the two single-valued graphs in the decomposition off each of the two planes that constitute the cone. Novel problems are encountered and overcome in proving regularity of the blow-ups: apriori the...
blow-up is regular only away from the set $D$ of points on the axis of the cone at which the good density points congregate - a set about which we assume nothing. Technical modifications of the work of [Sim33] and [Wic13], together with some intricate arguments utilising various removability theorems for harmonic functions must be employed. By being able to establish the appropriate regularity for blow-ups without making any assumptions on $D$, we get an excess improvement lemma which can be iterated unconditionally (i.e. there is no chance that after finitely many iterations we encounter some scale at which the hypotheses suddenly fail), and this allows us to conclude (via what are by now quite standard arguments) that in some neighbourhood the entire singular set is contained in a single $C^{1,o}$ submanifold.

We remark that the dichotomy between points of density $\geq 2$ and points with density $< 2$ to which we have just alluded does not appear to be exploitable in the same way in the 3-valued case: One cannot expect a minimal three-valued Lipschitz graph for instance to decompose into separate single-valued graphs in a region in which every point has density less than 3. Such a region can obviously still contain density two singularities on an $(n - 1)$-dimensional set.

Acknowledgements. This work is part of the author’s PhD thesis and was carried out at the University of Cambridge under the supervision of Dr. Neshan Wickramasekera. The author would like to thank him for many hours of helpful and stimulating discussions relating to the work and for a reading a draft of the manuscript in detail.

1. Notation and Preliminaries

We will use uppercase letters such as $X$ to denote points in $\mathbb{R}^{n+k}$.

When we write $X = (x, y)$ without comment, we mean $X = (x, y) \in \mathbb{R}^{k+l} \times \mathbb{R}^m$ for some fixed $l, m \geq 1$, where $l + m = n$.

When $X = (x, y)$, we will write $R = R(X) = |X|$ and $r = r(X) = |x|$.

For $X_0 \in \mathbb{R}^{n+k}$ and $\rho > 0$, $B_\rho(X_0) = \{X \in \mathbb{R}^{n+k} : |X_0 - X| < \rho\}$.

For $X_0 \in \mathbb{R}^n \times \{0\}^k$ and $\rho > 0$, $B_\rho^n(X_0) = \{X \in \mathbb{R}^n \times \{0\}^k : |X_0 - X| < \rho\}$.

For $X_0 \in \mathbb{R}^{n+k}$ and $\rho > 0$, we define the transformation $\eta_{X_0,\rho} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ by $\eta_{X_0,\rho}(X) = \rho^{-1}(X - X_0)$.

For $s \geq 0$, $\mathcal{H}^s$ denotes the $s$-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$ for any $s \geq 0$, and $\omega_n = \mathcal{H}^n(B^n(0))$.

For $A, B \subset \mathbb{R}^{n+k}$, $\text{dist}_H(A, B)$ denotes the Hausdorff distance between $A$ and $B$.

For $X \in \mathbb{R}^{n+k}$ and $A \subset \mathbb{R}^{n+k}$, $\text{dist}(X, A) = \inf_{Y \in A} |X - Y|$.

For $A \subset \mathbb{R}^{n+k}$ and $\rho > 0$, we write $(A)_\rho = \{X \in \mathbb{R}^{n+k} : \text{dist}(X, A) < \rho\}$.

By a plane we mean an affine $n$-dimensional subspace of $\mathbb{R}^{n+k}$ and for any plane $T$, we use $p_T$ to denote the orthogonal projection onto $T$. More commonly, we will use the shorthands $Y^T = p_T Y$ and $Y^{\perp_T} = p_{T^\perp} Y$.

By a half-plane, we mean any set which is the closure of one of the connected components of $T \setminus L$, where $T$ is any plane and $L$ is any $(n - 1)$-dimensional subspace of $T$. For any half-plane $H$, we write $p_H$ for the orthogonal projection onto the unique plane containing $H$.

$G_n$ denotes the space of $n$-dimensional subspaces of $\mathbb{R}^{n+k}$.

For an integral $n$-varifold $V$ (see [All72] or [Sim83] Chapter 4, 8)) in the open set $U$, we use the following notation:

The weight measure $\|V\|$ of $V$ is the Radon measure on $U$ given by $\|V\|(A) = V(\{(x, S) \in G_n(U) : x \in A\})$ and $\text{spt} |V|$ is called the support of the varifold $V$.

Given an $n$-rectifiable set $M$, $|M|$ denotes the multiplicity one varifold associated with $M$. 


where $f \in C^1$. Here, the Hölder coefficient is interpreted in the obvious way, i.e.:

$$[Df]_{\alpha;\Omega} = \sup_{x,y,\in \Omega, x \neq y} |x - y|^{-\alpha}\mathcal{G}(Df(x), Df(y)).$$

Note that $C^1(\Omega; A_2(\mathbb{R}^k))$ and $C^{1,\alpha}(\Omega; A_2(\mathbb{R}^k))$ are not linear spaces as there is in general no well-defined pointwise addition on two-valued functions. We define the graph of a two-valued function $f$ by

$$\text{graph } f := \{(x, y) \in \Omega \times \mathbb{R}^k : y \in \{f_1(x), f_2(x)\}\}.$$
1.3. Classes of Varifolds. We now define the main classes of varifolds with which we will work.

Write $C_{n-2}$ for the set of all integral $n$-varifolds in $\mathbb{R}^{n+k}$ which are of the form $C = |P_1| + |P_2|$, where $P_1, P_2$ are distinct planes meeting only along an affine subspace $A(C) := P_1 \cap P_2$ of dimension at most $n - 2$ (which we call the axis of $C$). We do not assume that $A(C) \neq \emptyset$, so varifolds in $C_{n-2}$ are not necessarily cones.

Write $C_{n-1}$ for the set of all integral $n$-varifolds in $\mathbb{R}^{n+k}$ which are of the form $C = \sum_{i=1}^4 |H_i|$, where for $i = 1, \ldots, 4$, the $H_i$ are distinct (closed) half-planes meeting only along their common boundary $A(C) = \cap_{i=1}^4 H_i$, the axis of $C$, which is an affine $(n - 1)$-dimensional subspace.

We write $C = C_{n-2} \cup C_{n-1}$.

For $C \in C$, when the coordinates of $\mathbb{R}^{n+k}$ are labelled in such a way that for some $1 \leq m \leq n - 1$ we have $A(C) = \{0\}^{1+k} \times \mathbb{R}^m \subset \mathbb{R}^{l+k} \times \mathbb{R}^m$, we will say that $C$ is properly aligned. In this case, $C = C_0 \times \mathbb{R}^m$, where $\text{sing} C_0 = \{0\}$ ($C_0$ is either the sum of two distinct $l$-dimensional subspaces of $\mathbb{R}^{l+k}$ meeting only at the origin or the sum of four distinct rays in $\mathbb{R}^{1+k}$ meeting only at the origin, depending on whether $C \in C_{n-2}$ or $C \in C_{n-1}$, respectively).

Write $V$ for the set of all minimal two-valued graphs in $B_2(0) \times \mathbb{R}^k$ that are associated to some Lipschitz function $f : B_2(0) \to \mathcal{A}_2(\mathbb{R}^k)$. We write $p$ for the orthogonal projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^n \times \{0\}^k$.

1.4. Main Results. The main lemma (Lemma 8.3) is a typical excess improvement lemma which says that if a minimal two-valued graph is sufficiently close to a cone in $\mathbb{C}$ at scale 1, then at a smaller scale $\theta$, the excess has decayed by a factor of $\theta^\mu$ for some $\mu \in (0, 1)$. By careful iteration of this Lemma and arguments of a well-known nature, we get the main results of the paper, which are the following two $\varepsilon$-regularity theorems:

**Theorem 1.** Let $C^{(0)} \in C_{n-2}$ be properly aligned. There exists $\varepsilon = \varepsilon(n, k, C^{(0)}) > 0$ such that the following is true. If $V \in V$ is such that $0 \in \text{spt} V$ and

$$Q_V(C^{(0)}) := \left(\int_{B_2^+} \text{dist}^2(X, \text{spt} \|C^{(0)}\|) \, d\|V\|(X)ight)^{1/2} < \varepsilon,$$

then we have the following conclusions:

1. $V[B_{1/2}(0)] = |M_1| + |M_2|$, where, for $i = 1, 2$, $M_i$ is a smooth, embedded $n$-dimensional minimal submanifold of $B_{1/2}(0)$.
2. $\text{sing} V \cap B_{1/2}(0) = M_1 \cap M_2$ is contained in the graph of a $C^{1,\alpha}$ function $\varphi : A(C^{(0)}) \cap B_{1/2}(0) \to A(C^{(0)})^\perp$ satisfying $\|\varphi\|_{C^{1,\alpha}}(A(C^{(0)}) \cap B_{1/2}(0)) \leq cQ_V(C^{(0)})$, for some $c = c(n, k, C^{(0)})$ and some $\alpha = \alpha(n, k, C^{(0)}) \in (0, 1)$.
3. At each $Z \in \text{sing} V \cap B_{1/2}(0)$, we have that $\text{Var} \text{Tan}(V, Z) = \{C_Z\}$ for some $C_Z \in C_{n-2}$ and we have the decay estimate

$$\rho^{-n-2} \int_{\mathbb{R}^k} \text{dist}^2(X, \text{spt} C_Z) \, d\|V\|(X) \leq cp^\alpha Q^2_V(C^{(0)}),$$

which holds for all $\rho \in (0, 1/8)$ and for some $c = c(n, k, C^{(0)}) > 0$. 
Theorem 2. Let $C^{(0)} \in C_{n-1}$ be properly aligned. There exists $\epsilon = \epsilon(n, k, C^{(0)}) > 0$ such that the following is true. If $V \in C$ is such that $0 \in \text{spt} V$ and

$$Q_V(C^{(0)}) := \left( \int_{B_2^\perp(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} ||C^{(0)}||) d||V||(X) \right)^{1/2} + \int_{(B_2^\perp(0) \times \mathbb{R}^k) \setminus \{r < 1/8\}} \text{dist}^2(X, \text{spt} ||V||) d||C^{(0)}||(X) < \epsilon,$$

then we have the following conclusions:

1. $\text{sing} V \cap B_{1/2}(0)$ is contained in the graph of a $C^{1,\alpha}$ function $\varphi : A(C^{(0)}) \cap B_{1/2}(0) \to A(C^{(0)})$ satisfying $\|\varphi\|_{C^{1,\alpha}(A(C^{(0)}) \cap B_{1/2}(0))} \leq cQ_V(C^{(0)})$ for some $c = c(n, k, C^{(0)})$ and some $\alpha = \alpha(n, k, C^{(0)}) \in (0, 1)$.

2. At each $Z \in \text{sing} V \cap B_{1/2}(0)$, we have that $\text{Var} \text{Tan}(V, Z) = \{C_Z\}$ for some $C_Z \in C$ and we have the decay estimate

$$\rho^{-n-2} \int_{B_{\rho}(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} C_Z) d||V||(X) \leq c\rho^\alpha Q_V^2(C^{(0)}),$$

which holds for all $\rho \in (0, 1/8)$ and for some $c = c(n, k, C^{(0)}) > 0$.

Remarks Note that the conclusions of Theorem 2 imply in particular that at each singular point, there cannot be any tangent cones of the type alluded to in Example 2, or indeed anything more exotic.

By classifying tangent cones to minimal, two-valued Lipschitz graphs with spine dimension $(n-1)$ or $(n-2)$ and combining Theorems 1 and 2 we have the following description of the singular set: Suppose that $V$ is a minimal two-valued graph. Let $\mathfrak{B}$ denote the set of points $X \in \text{sing} V$ at which there exists $C \in \text{Var} \text{Tan}(V, X)$ equal to a multiplicity two hyperplane. Write $\bar{S}_{n-1}$ for the set of points $X \in \text{sing} V$ for which there exits $C \in \text{Var} \text{Tan}(V, X) \cap C_{n-1}$ and similarly $\bar{S}_{n-2}$ for the set of points $X \in \text{sing} V$ for which there exits $C \in \text{Var} \text{Tan}(V, X) \cap C_{n-2}$. Finally define $\bar{S}_{n-3} := \text{sing} V \setminus (\mathfrak{B} \cup \bar{S}_{n-1} \cup \bar{S}_{n-2})$.

Theorem 3. For any minimal two-valued graph $V$, $\text{sing} V$ is the disjoint union $\mathfrak{B} \cup \bar{S}_{n-1} \cup \bar{S}_{n-2} \cup \bar{S}_{n-3}$, where

1. By definition, for every $X \in \mathfrak{B}$, there is $C \in \text{Var} \text{Tan}(V, X)$ equal to a multiplicity two hyperplane.

2. $\dim_H \bar{S}_{n-1} \leq n - 1$, $\bar{S}_{n-1} \cup \bar{S}_{n-2}$ is relatively open in $\text{sing} V$ and for every $X \in \bar{S}_{n-1}$, we have that the conclusions of Theorem 2 hold in a neighbourhood of $X$.

3. $\dim_H \bar{S}_{n-2} \leq n - 2$, $\bar{S}_{n-2}$ is relatively open in $\text{sing} V$ and for every $X \in \bar{S}_{n-2}$, we have that the conclusions of Theorem 1 hold in a neighbourhood of $X$.

4. $\dim_H \bar{S}_{n-3} \leq n - 3$ and the closure of $\bar{S}_{n-3}$ does not intersect $\bar{S}_{n-1} \cup \bar{S}_{n-2}$.

One of our main results is to two-valued graphs that are locally area minimizing: It is not difficult to see that an $n$-dimensional area-minimizing current without boundary cannot have a tangent cone with spine dimension $(n-1)$. The work of Almgren ([Alm00]) implies that for such a current, the set of points where there is a multiplicity 2 tangent plane has Hausdorff dimension at most $(n-2)$. Thus we get the following corollary of our main theorems:

Corollary 1.1. If $V$ is a locally area-minimizing current in $B_1(0)$ with $\partial V = 0$ in $B_1(0)$ and which corresponds to a two-valued Lipschitz graph, then $V$ is smoothly immersed away from a closed set $S$ with $\dim_H(S) \leq n - 2$ and which can be written as the disjoint union $S = S_1 \cup S_2$, where
2. Proper Blow-Up Classes

Let $C^{(0)} \in C$ be properly aligned and write $A = A(C^{(0)})$. For $Y \in \mathbb{R}^{n+k}$ and $X \in C^{(0)} \cap \{r > 0\}$, we use the shorthand $Y^\perp C^{(0)} := Y^\perp T_X C^{(0)}$ (note that this notation suppresses a dependence on $X$). Also, when $C^{(0)} \in \mathcal{C}_{n-1}$ is properly aligned, we write $\{\omega_1, \ldots, \omega_4\} = \{r = 1\} \cap (\mathbb{R}^{1+k} \times \{0\}^{n-1}) \cap \text{spt} \|C^{(0)}\|$. We will define two special classes of functions, Firstly, we define the class $\mathcal{H} = \mathcal{H}(C^{(0)})$ of certain homogeneous degree one functions on $C^{(0)}$ for $C^{(0)} \in \mathcal{C}_{n-1}$:

**Definition.** For properly aligned $C^{(0)} \in \mathcal{C}_{n-1}$, each $\psi \in \mathcal{H}(C^{(0)})$ is of the following form: There are vectors $c_1, \ldots, c_{n-1} \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ and a function $\varphi : \{\omega_1, \ldots, \omega_4\} \to C^{(0)} \perp$ with $\varphi(\omega_j) \in T^{\perp}_{(\omega_j, 0)} C^{(0)}$ for $j = 1, \ldots, 4$ such that for $X = (x, y) \in C^{(0)} \cap \{r > 0\}$,

$$\psi(X) = \psi(x, y) = \sum_{p=1}^{n-1} y^p c_p \circ C^{(0)} + |x|\varphi(x/|x|).$$

Note that for $\psi \in \mathcal{H}(C^{(0)})$, there is a unique $C \in C$ for which graph $\psi \subset C$. Also note that if $C^{(0)} \in \mathcal{P}$, then the class $\mathcal{H}(C^{(0)})$ includes all functions $\psi$ such that $\psi|_{B^{(0)}_i}$ is linear.

The next class of functions is the main focus of the section. Consider a family $\mathfrak{B} = \mathfrak{B}(C^{(0)})$ of functions $v : C^{(0)} \cap \{r > 0\} \cap B_1(0) \to C^{(0)} \perp$. We say that $\mathfrak{B}$ is a proper blow-up class if it satisfies the following properties.

(\mathfrak{B}1) $v \in L^2(C^{(0)} \cap B_1(0); C^{(0)} \perp) \cap C^\infty(C^{(0)} \cap \{r > 0\} \cap B_1(0); C^{(0)} \perp)$

(\mathfrak{B}2) $\Delta v = 0$ on $C^{(0)} \cap \{r > 0\} \cap B_1(0)$.

(\mathfrak{B}3) For each $v \in \mathfrak{B}$ there is a closed set $D_v \subset A \cap B_1(0)$ such that if $D_v \neq A \cap B_1(0)$, then $C^{(0)} \in \mathcal{P}$. In this case (i.e. when $D_v \neq A \cap B_1(0)$) we write $C^{(0)} = |P^{(0)}_1| + |P^{(0)}_2|$ and $v_i := v|_{P^{(0)}_i \cap \{r > 0\} \cap B_1(0)}$ extends to a smooth, harmonic function on $(P^{(0)}_i \setminus D_v) \cap B_1(0)$ for $i = 1, 2$.

(\mathfrak{B}4) When $C^{(0)} = \sum_{j=1}^4 |H^{(0)}_j| \in \mathcal{C}_{n-1}$, we have, for $p = 1, \ldots, n-1$, that

$$\sup_{|y| \leq 1/2} \left| \frac{\partial^2}{\partial r \partial y^p} \sum_{j=1}^4 v(r \omega_j, y) \right| \to 0 \text{ as } r \downarrow 0^+.$$  

(\mathfrak{B}5) For any $v \in \mathfrak{B}$, we have the following closure and compactness properties:

(\mathfrak{B}5I) For any $Y \in A \cap B_1(0)$ and $\rho \in (0, 1/2(1 - |Y|)]$, we have that

$$\tilde{v}_{Y, \rho}(X) := \frac{v(Y + \rho X)}{\|v(Y + \rho(\cdot))\|_{L^2(C^{(0)} \cap B_1(0))}} \in \mathfrak{B}.$$

(\mathfrak{B}5II) If $C^{(0)} \in \mathcal{C}_{n-2}$, then for any $\psi : C^{(0)} \cap \{r > 0\} \to C^{(0)} \perp$ such that $\psi|_{P^{(0)}_i}$ is affine for $i = 1, 2$ we have that

$$\frac{v(X) - \psi(X)}{\|\psi(\cdot) - \psi(\cdot)\|_{L^2(C^{(0)} \cap B_1(0))}} \in \mathfrak{B}.$$
And if \(C(0) \in C_{n-1}\), then for any \((\xi, 0) \in \mathbb{R}^{l+k} \times \{0\}^m\) and any \(\psi \in \mathcal{H}(C(0))\), we have that
\[
\frac{v(X) - (\xi, 0)^\perp C(0) - \psi(X)}{\|v(\cdot) - (\xi, 0)^\perp C(0) - \psi(\cdot)\|_{L^2(C(0) \cap B_1(0))}} \in \mathcal{B}.
\]

(2.3) If \(c(0) \in C_{n-1}\), then for any \((\xi, 0) \in \mathbb{R}^{l+k} \times \{0\}^m\) and any \(\psi \in \mathcal{H}(C(0))\), we have that
\[
\frac{v(X) - (\xi, 0)^\perp C(0) - \psi(X)}{\|v(\cdot) - (\xi, 0)^\perp C(0) - \psi(\cdot)\|_{L^2(C(0) \cap B_1(0))}} \in \mathcal{B}.
\]

(2.4) For any \(Y \in D_v \cap B_{1/2}(0)\), there is \(\kappa_Y \in \mathbb{R}^{l+k} \times \{0\}^m\) such that \(|\kappa_Y|^2 \leq c \int_{C(0) \cap B_1(0)} |v|^2\) for some \(c = c(n, k, C(0)) > 0\) and such that for all \(\rho \in (0, 1/2]\) we have the estimates
\[
\frac{v(X) - (\xi, 0)^\perp C(0) - \psi(X)}{\|v(\cdot) - (\xi, 0)^\perp C(0) - \psi(\cdot)\|_{L^2(C(0) \cap B_1(0))}} \in \mathcal{B}.
\]

The content of this section is essentially that the blow-up class consists of functions which are \(C^{1,\alpha}\), in a sense to be made precise (the main results are Theorems 2.2 and 2.6). The analysis does not quite neatly divide into cases, but different methods are applicable depending on the size and nature of the set \(D_v\):

I. \(\mathcal{H}^{n-2}(D_v) < \infty\). Note that this case includes all blow-ups off cones \(C(0) \in C_{n-2}\).

II. \(\mathcal{H}^{n-2}(D_v) = \infty\) and \(D_v \neq A \cap B_1(0)\).

III. \(\mathcal{H}^{n-2}(D_v) = \infty\) and \(D_v = A \cap B_1(0)\).

Of course in cases II and III it is necessarily true that \(C(0) \in C_{n-1}\) and (by (2.3)), it is only in case III that it may be the case that \(C(0) \notin \mathcal{P}\) (i.e. \(C(0)\) is a union of four half-planes not equal to a pair of planes). First we prove continuity of blow-ups up to the axis by verifying the hypotheses of Lemma A.2

Lemma 2.1. Let \(C(0) \in C\) be properly aligned and \(v \in \mathcal{B}(C(0))\). Then

1. There is a constant \(c = c(n, k, C(0)) > 0\) for which \(\sup_{C(0) \cap B_{1/2}(0)} |v| \leq c\).

2. If \(C(0) \in C_{n-1}\), then for \(j = 1, \ldots, 4\), we have that \(v|_{H^1_j(r > 0) \cap B_{1/2}(0)}\) extends continuously to \(\{r = 0\} \cap B_{1/2}(0)\). If \(C(0) \in C_{n-2}\), then for \(i = 1, 2\) \(v|_{P_i(0) \cap B_{1/2}(0)}\) extends continuously to \(D_v\).

Proof. For \(Y \in D_v \cap B_{3/4}(0)\), (2.3) gives that
\[
\sigma^{-n} \int_{C(0) \cap B_{\sigma}(Y)} |v - \kappa_y C(0)|^2 d\mathcal{H}^n \leq c \sigma^{7/4} \int_{C(0) \cap B_{\sigma}(Y)} |v - \kappa_y C(0)|^2 d\mathcal{H}^{n+7/4}.
\]

If \(C(0) \notin \mathcal{P}\), this shows immediately that \(v\) satisfies that hypotheses of Lemma A.2 with \(D = D_v \cap B_{3/4}(0) = A \cap B_{3/4}(0)\) and the conclusions follow from that Lemma. So we suppose on the other hand that \(C(0) \in \mathcal{P}\). In this case, since \(v_i\) is harmonic on \((P_i^{0}) \setminus D_v) \cap B_1(0)\), for any
\( X \in (P_i^{(0)} \setminus D_v) \cap B_{3/4}(0) \) and any constant vector \( \kappa \in P_i^{(0)\perp} \) we have (using the mean value property) that

\[
\sigma^{-n} \int_{P_i^{(0)} \cap B_\sigma(X)} |v_i - v_i(X)|^2 d\mathcal{H}^n \leq c \left( \frac{\sigma}{\rho} \right)^2 \rho^{-n} \int_{P_i^{(0)} \cap B_\rho(X)} |v_i - \kappa|^2 d\mathcal{H}^n,
\]

for \( 0 < \sigma \leq \rho/2 \leq 1/2 \min\left\{ \frac{1}{4}, \text{dist}(X, D_v) \right\} \) and where \( c = c(n, k) > 0 \). This verifies (A.13) and thus in this case we also satisfy the hypotheses of Lemma A.2 from which the conclusions follow. \( \square \)

2.1. Case I.

**Theorem 2.2.** Fix a properly aligned cone \( C^{(0)} \in C \). If \( v \in \mathcal{B}(C^{(0)}) \) is such that \( \mathcal{H}^{n-2}(D_v) < \infty \), then

1. \( C^{(0)} \in \mathcal{P} \).
2. For \( i = 1, 2 \), \( v_i \) extends over \( D_v \) to a smooth harmonic function defined on \( P_i^{(0)} \cap B_{1/2}(0) \).
3. There exists \( \psi : C^{(0)} \cap \{ r > 0 \} \to C^{(0)\perp} \) for which \( \psi|_{P_i^{(0)}} \) is affine for \( i = 1, 2 \) such that for any \( \theta = \theta(n, k, C^{(0)}) \in (0, 1/8) \) we have

\[
\theta^{-n-2} \int_{C^{(0)} \cap B_\theta(0)} |v - \psi|^2 d\mathcal{H}^n \leq c \theta^2 \int_{C^{(0)} \cap B_1(0)} |v|^2 d\mathcal{H}^n,
\]

for some \( c = c(n, k, C^{(0)}) > 0 \).

**Proof.** First we observe that if \( C^{(0)} \in C_{n-1} \), then the fact that \( \dim_{\mathcal{H}}(D_v) \leq n - 2 \) means that \( (A \setminus D_v) \cap B_1(0) \neq \emptyset \) and so (B.3) implies that \( C^{(0)} \in \mathcal{P} \). By Lemma 2.1, we have that \( v_i \) is bounded on \( P_i^{(0)} \cap B_{1/2}(0) \) and since \( (P_i^{(0)} \setminus D_v) \cap B_{1/2}(0) \) is a connected domain, we can then apply (1) of Lemma A.3 directly to \( v_i \) to deduce that \( v_i \) is smooth and harmonic on \( P_i^{(0)} \cap B_{1/2}(0) \). The existence of \( \psi \) and the estimate follows by setting \( \psi|_{P_i^{(0)}} = v_i^\alpha(0) + Dv_i^\alpha(0) \cdot X \) for \( \alpha = 1, \ldots, k \) and using standard derivative estimate for harmonic functions. \( \square \)

**Remarks.** Note that \( \mathcal{H}^{n-2}(D_v) < \infty \) for any \( v \in \mathcal{B}(C^{(0)}) \) such that \( C^{(0)} \in C_{n-2} \) and thus the hypotheses of the Lemma are satisfied whenever \( C^{(0)} \in C_{n-2} \).

2.2. Cases II and III. The real hard work of this section goes into understanding the structure of a homogeneous degree one blow-up for which \( \mathcal{H}^{n-2}(D_v) = \infty \). This first lemma is a kind of non-concentration estimate for blow-ups.

**Lemma 2.3.** Fix a properly aligned cone \( C^{(0)} \in C_{n-1} \). For any \( v \in \mathcal{B}(C^{(0)}) \), there exists a function \( \kappa = \kappa_v : C^{(0)} \cap \{ r > 0 \} \cap B_1(0) \to \mathbb{R}^{n+k} \) of the form \( \kappa(X) = \kappa(r, y) \) that satisfies \( \sup |\kappa| \leq c \int_{C^{(0)} \cap B_1(0)} |v|^2 \) for some \( c = c(n, k, C^{(0)}) > 0 \) and is such that:

\[
\int_{C^{(0)} \cap B_{1/2}(0)} \frac{|v(X) - \kappa(X)|^2 + c^{(0)}|^2}{\text{dist}(X, D_v)^{5/2}} d\mathcal{H}^n(X) \leq c \int_{C^{(0)} \cap B_1(0)} |v(X)|^2 d\mathcal{H}^n(X)
\]

for every \( \rho \in (0, 1/4) \), where \( c = c(n, k, C^{(0)}) > 0 \).

**Proof.** For each \( (r, y) \) with \( r > 0 \) define \( \kappa(r, y) \in \mathbb{R}^{1+k} \times \{ y \} \) by

\[
\sum_{j=1}^4 |v(r\omega_j, y) - \kappa(r, y)^\perp c^{(0)}|^2 = \inf \sum_{j=1}^4 |v(r\omega_j, y) - \lambda^\perp c^{(0)}|^2,
\]

where \( \lambda^\perp c^{(0)} \) is the orthogonal projection of \( c^{(0)} \) onto the space spanned by \( \omega_1, \omega_2, \omega_3, \omega_4 \).
where the infimum is taken over $\lambda \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ with $|\lambda| \leq c \int_{C^{(0)} \cap B_1(0)} |v|^2$. By using (2.3) with $\rho = 1/2$, the definition of $\kappa$ and the coarea formula, it follows directly that for $\sigma \in (0, 1/4]$ we have

$$\int_{C^{(0)} \cap B_1(0)} |v - \kappa \frac{1}{\sigma} c(0)|^2 d\mathcal{H}^n \leq \int_{C^{(0)} \cap B_1(0)} |v|^2 d\mathcal{H}^n. \quad (2.9)$$

Then we cover $(D_v)_{\sigma/4} \cap B_1(0)$ with a collection of at most $c(n, k)\sigma^{-(n-1)}$ balls $\{B_\sigma(Y_j)\}$, where $Y_j \in D_v$ for each $j$ and sum up the integrals to get that

$$\int_{C^{(0)} \cap (D_v)_{\sigma/4} \cap B_{1/4}(0)} |v - \kappa \frac{1}{\sigma} c(0)|^2 d\mathcal{H}^n \leq \int_{C^{(0)} \cap B_1(0)} |v|^2 d\mathcal{H}^n. \quad (2.10)$$

When we multiply by $\sigma^{-1/4}$, integrate in $\sigma$ from 0 to $\rho$ and use Fubini’s theorem to carry out the $\sigma$ integral, a short computation gives

$$\int_{C^{(0)} \cap (D_v)_{\rho/4} \cap B_{1/4}(0)} |v - \kappa \frac{1}{\sigma} c(0)|^2 d\mathcal{H}^n \leq \int_{C^{(0)} \cap B_1(0)} |v|^2 d\mathcal{H}^n,$$

which establishes (2.7).

From this we deduce more information about exactly how a homogeneous degree one blow-up decays to its values on $D_v$.

**Lemma 2.4.** Fix a properly aligned cone $C^{(0)} \subset D_{n-1}$. Let $v \in \mathcal{B}(C^{(0)})$ be homogeneous degree one and suppose that $\mathcal{H}^{n-2}(D_v) = \infty$. There are $c_p \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ for $p = 1, \ldots, n-1$ such that for $j = 1, \ldots, 4$ we have

$$\lim_{\rho \to 0^+} \rho^{-5/4} \int_{H_j^{(0)} \cap (D_v)_{\rho} \cap B_{1/4}(0)} |v(r\omega_j, y) - \sum_{i=1}^{n-1} y^p b_p H_j^{(0)}| d\mathcal{H}^n = 0. \quad (2.12)$$

**Proof.** Using (2.8) and the reflection principle for harmonic functions, we deduce that the function

$$\Psi_{y_p}(r, y) := \frac{\partial}{\partial y^p} \sum_{j=1}^4 v(r\omega_j, y), \quad (2.13)$$

defined initially on the domain $(0, \infty) \times \mathbb{R}^{n-1}$, extends to a homogeneous degree zero harmonic function on the whole of $\mathbb{R}^n$. Such functions are necessarily constant and since this holds for each $p \in \{1, \ldots, n-1\}$, we deduce that

$$\Psi(r, y) := \sum_{j=1}^4 v(r\omega_j, y) = ra + \sum_{p=1}^{n-1} y^p b_p, \quad (2.14)$$

for some $a, b_p \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ (where we have also used the fact that $v$ is homogeneous degree one to deduce the form of the dependence on the $r$ variable). Since (2.7) with $\rho = 1/4$ implies that

$$\int_{C^{(0)} \cap (D_v)_{1/16} \cap B_{1/4}(0)} \frac{|v(X) - \kappa (X) \frac{1}{\sigma} c(0)|^2 d\mathcal{H}^n(X)}{\text{dist}(X, D_v)^{5/2}} < \infty, \quad (2.15)$$

we have for $j = 1, \ldots, 4$ that

$$\int_{H_j \cap (D_v)_{1/16} \cap B_{1/4}(0)} \frac{|v(r\omega_j, y) - \kappa (r, y) \frac{1}{\sigma} c(0)|^2 d\mathcal{H}^n(r, y)}{\text{dist}(X, D_v)^{5/2}} < \infty. \quad (2.16)$$
Using the form of $\Psi$, (2.16) and the triangle inequality, we deduce that

$$\int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \frac{|\Sigma_{i=1}^{n-1} y^p b_p - \sum_{j=1}^{4} \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2}{\text{dist}(X, D_v)^{5/2}} d\mathcal{H}^n < \infty. \tag{2.17}$$

We deduce directly from (2.17) that

$$\lim_{\rho \downarrow 0^+} \rho^{-5/2} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \frac{|\Sigma_{p=1}^{n-1} y^p b_p - \sum_{j=1}^{4} \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2}{\text{dist}(X, D_v)^{5/2}} d\mathcal{H}^n = 0. \tag{2.18}$$

We claim that this means that each $b_p$ is in the subspace

$$T := \left\{ \sum_{j=1}^{4} c_j H_j : c \in \mathbb{R}^{1+k} \times \{0\}^{n-1} \right\}.$$

To see this, suppose for the sake of contradiction that

$$S := \text{span}\{b_1, \ldots, b_{n-1}\} \not\subseteq T \tag{2.19}$$

and write $L(r, y) = \Sigma_{p=1}^{n-1} y^p b_p$. Since $L$ does not depend on the $r$-variable, if $Y = (0, y) \in L^{-1}(T) \cap A$, then $(r \omega_j, y) \in L^{-1}(T)$ for all $r > 0$ and each $j = 1, \ldots, 4$. By assumption we have that $L^{-1}(T) \cap A \cap B_{1/8}(0) \neq A \cap B_{1/8}(0)$, whence $\dim_H(L^{-1}(T) \cap A) \leq n - 2$. Since $\mathcal{H}^{n-2}(D_v) = \infty$ (and we are in fact free to assume - by otherwise employing the argument of Theorem 2.2 - that $\mathcal{H}^{n-2}(D_v \cap B_{1/8}(0)) = \infty$), we can find some subset $\mathcal{D}' \subset D_v \cap B_{1/8}(0)$ with $\mathcal{H}^{n-2}((\mathcal{D'} \cap B_{1/8}(0)) > 0$ and $\text{dist}(\mathcal{D}', D_v) > 0$. Thus we know that $\delta := \inf_{(0, y) \in \mathcal{D}'} \text{dist}(L(r, y), T)$ is strictly positive. Lemma A.3 tells us that $\mathcal{H}^{n}((\mathcal{D'} \cap B_{1/4}(0)) \geq c \rho^2$ for some $c = c(\mathcal{D'}, n) > 0$. Moreover, for sufficiently small $\rho > 0$, we have that $\text{dist}(L(r, y), T) \geq \delta/2$ for all $(r, y) \in (\mathcal{D'} \cap B_{1/4}(0))$. Thus we can bound below by integrating only over $(\mathcal{D'} \cap B_{1/4}(0))$ to deduce that

$$\rho^{-5/2} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \frac{|\Sigma_{p=1}^{n-1} y^p b_p - \sum_{j=1}^{4} \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2}{\text{dist}(X, D_v)^{5/2}} d\mathcal{H}^n \geq c \rho^{-5/2}(\delta/2)^2 \rho^2 \geq c \rho^{-1/2} \rightarrow \infty$$

as $\rho \downarrow 0^+$, which is a contradiction. Therefore $S \subset T$ and the claim is proved.

So, we have that for each $p \in \{1, \ldots, n-1\}$, there is some $c_p \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ for which $b_p = \sum_{j=1}^{4} c_p \frac{1}{2} H^{(0)}_j$ and so

$$\lim_{\rho \downarrow 0^+} \rho^{-5/2} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \frac{|\sum_{p=1}^{n-1} y^p c_p - \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2}{\text{dist}(X, D_v)^{5/2}} d\mathcal{H}^n = 0. \tag{2.20}$$

Now observe that

$$\rho^{-5/4} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \sum_{j=1}^{4} |\sum_{p=1}^{n-1} y^p c_p - \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2 d\mathcal{H}^n = \rho^{-5/4} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \sum_{j=1}^{4} |\sum_{p=1}^{n-1} y^p c_p - \kappa(r, y) \frac{1}{2} H^{(0)}_j|^2 d\mathcal{H}^n \tag{2.21}$$

$$-\rho^{-5/4} \int_{C^{(0)} \cap (D_v)_{\rho \cap B_{1/4}(0)}} \sum_{j,k} \left(\sum_{p=1}^{n-1} y^p c_p - \kappa(x) \frac{1}{2} H^{(0)}_k\right) \cdot \left(\sum_{p=1}^{n-1} y^p c_p - \kappa(x) \frac{1}{2} H^{(0)}_j\right) d\mathcal{H}^n.$$
The first of these two terms tends to zero as $\rho \downarrow 0^+$ by (2.20). By Cauchy-Schwarz and the fact that $\|p_{H^k_n}\| \leq 1$, the second term is at most
\[
c \left( \rho^{-5/2} \int_{C(0) \cap (D_v)_{\rho} \cap B_{1/4}(0)} \sum_{j=1}^4 \left( \sum_{p=1}^n y^p c_p - \kappa \right)^2 dH^n \right)^{1/2}
\times \left( \int_{C(0) \cap (D_v)_{\rho} \cap B_{1/4}(0)} \left| \sum_{p=1}^n y^p c_p - \kappa \right|^2 dH^n \right)^{1/2}.
\]
The second of these two integrals is bounded and the first goes to zero by (2.20). We therefore conclude that the integral on the left-hand side of (2.21) goes to zero as $\rho \downarrow 0^+$ and this means that
\[(2.22) \lim_{\rho \downarrow 0^+} \rho^{-5/4} \int_{H^0(0) \cap (D_v)_{\rho} \cap B_{1/4}(0)} \left| \sum_{j=1}^4 \left( \sum_{p=1}^n y^p c_p - \kappa \right)^2 dH^n = 0 \right|
for each $j = 1, \ldots, 4$. Note also that it follows from (2.16) that
\[(2.23) \lim_{\rho \downarrow 0^+} \rho^{-5/4} \int_{H^0(0) \cap (D_v)_{\rho} \cap B_{1/4}(0)} |v(r\omega, y) - \kappa(r, y)|^2 dH^n = 0.
And so using this in conjunction with (2.22) and the triangle inequality, we easily get that
\[(2.24) \lim_{\rho \downarrow 0^+} \rho^{-5/4} \int_{H^0(0) \cap (D_v)_{\rho} \cap B_{1/4}(0)} |v(r\omega, y) - \Sigma_{p=1}^n y^p c_p|^2 dH^n = 0,
as claimed. □

We introduce one more piece of notation and terminology: Given $v \in \mathcal{B}(C(0))$ where $C(0) \in C_{n-1}$, $Y \in D_v$ and $\rho \in (0, 1]$ we say that $v \in \mathcal{H}(C(0))$ dehomogenizes $v$ in $B_{\rho}(Y)$ when
\[(2.25) \int_{C(0) \cap B_{\rho}(Y)} |v - \psi|^2 dH^n = \inf_{\psi \in \mathcal{H}(C(0))} \int_{C(0) \cap B_{\rho}(Y)} |v - l|^2 dH^n.
When $v$ satisfies
\[(2.26) \inf_{l \in \mathcal{H}(C(0))} \int_{C(0) \cap B_{\rho}(Y)} |v - l|^2 dH^n = \int_{C(0) \cap B_{\rho}(Y)} |v|^2 dH^n,
we say that $v$ is dehomogenized in $B_{\rho}(Y)$. It is straightforward (using orthogonal projection in $L^2(C(0) \cap B_{\rho}(Y), C(0)_{\perp})$) to prove the existence of dehomogenizers and one can see (from (2.11)) that it is equivalent to being $L^2$-orthogonal to the functions $(r\omega, y) \mapsto r\varphi(\omega)$ and $(r\omega, y) \mapsto y^p e_j^{C(0)}$ for $p = 1, \ldots, n-1$, $j = 1, \ldots, 1+k$. We are now in a position to categorize homogeneous degree one blow-ups in cases II and III.

**Lemma 2.5.** Fix a properly aligned cone $C(0) \in C_{n-1}$. Suppose that $v \in \mathcal{B}(C(0))$ is homogeneous degree one and that $\mathcal{H}^{n-2}(D_v) = \infty$. Then $v \in \mathcal{H}(C(0))$.

**Proof.** For any homogeneous degree one blow up $w$, we will write
\[S(w) = \{ Y \in A : w(X + Y) = w(X) \text{ for all } X \in C(0) \cap \{ r > 0 \} \cap B_1(0) \}.
It is easy to verify, using the homogeneity of $w$, that $S(w)$ is always a linear subspace of $A$. We will prove, by induction on $d$, the following statement: If $v \in \mathcal{B}(C(0))$ is homogeneous degree one with $\mathcal{H}^{n-2}(D_v) = \infty$ and has $\dim S(v) = n - d$, then $v \in \mathcal{H}(C(0))$. If $\dim S(v) = n - 1$, then $S(v) = A$, from which, using the homogeneity of $v$, we immediately deduce that $v \in \mathcal{H}(C(0))$. So, proceeding via the inductive hypothesis we now fix $d \geq 2$ and may assume that any homogeneous degree one blow up $w \in \mathcal{B}$ with $\mathcal{H}^{n-2}(D_w) = \infty$ and $\dim S(w) > n - d$ belongs to $\mathcal{H}(C(0))$. Now, for $Y \in D_v$
and \( \rho > 0 \) let \( \psi_{Y, \rho} \) be the function that dehomogenizes \( v \) in \( B_\rho(Y) \). Obviously we may assume that \( v \notin H(C(0)) \) (or else there is nothing to prove), so that \( v - \psi_{Y, \rho} \neq 0 \). And note that that since \( H^{n-2}(D_v) = \infty \), we have that \( D_v \setminus S(v) \neq \emptyset \). Now we proceed in a series of steps.

In steps 1. and 2. we prove that \( v|_{H_j^{(0)} \cap \{ r > 0 \}} \) is \( C^1 \) up to the axis, away from points of \( S(v) \). The argument is a modification of the proof of Proposition 4.2 in [Wic14].

**Step 1.** We claim that for any compact subset \( K \) of \( A \setminus S(v) \), there exists \( \epsilon = \epsilon(v, K) \in (0, 1) \) such that for any \( Y \in D_v \cap K \) and \( \rho \in (0, \epsilon) \),

\[
\int_{C(0) \cap (B_\rho(Y) \setminus B_{\rho/2}(Y))} R_{Y,j}^{2-n} \left| \frac{\partial((v - \psi_{Y,j}\frac{1}{\rho_Y})/R_Y - \psi_{Y,j})}{\partial R_Y} \right|^2 d\mathcal{H}^n \geq \epsilon \rho^{-n-2} \int_{C(0) \cap B_\rho(Y)} |v - \psi_{Y,j}\frac{1}{\rho_Y} - \psi_{Y,j}|^2 d\mathcal{H}^n.
\]

If this were false, there would exist a sequence of points \( \{Y_j\}_{j=1}^\infty \in D_v \cap K \) and radii \( \rho_j \downarrow 0^+ \) such that, writing \( \psi_j := \psi_{Y_j, \rho_j} + \psi_{Y,j}\frac{1}{\rho_Y} \), we have that \( v - \psi_j \) is not identically equal to zero for any \( j \) and

\[
\int_{C(0) \cap (B_{\rho_j}(Y_j) \setminus B_{\rho_j/2}(Y_j))} R_{Y,j}^{2-n} \left| \frac{\partial((v - \psi_j)/R_{Y_j})}{\partial R_{Y,j}} \right|^2 d\mathcal{H}^n < \epsilon_j \rho_j^{-n-2} \int_{C(0) \cap B_{\rho_j}(Y_j)} |v - \psi_j|^2 d\mathcal{H}^n,
\]

for some \( \epsilon_j \downarrow 0^+ \). By property (25II), we have that

\[
V_j := \|v - \psi_j\|^{-1}_{L^2(C(0) \cap B_1(0))} (v - \psi_j) \in \mathfrak{B}
\]

for each \( j \) and by property (25I), we also have that

\[
w_j := \tilde{V}_{Y_j, \rho_j} = \|V_j(Y_j + \rho_j(\cdot))\|^{-1}_{L^2(C(0) \cap B_1(0))} V_j(Y_j + \rho_j(\cdot)) \in \mathfrak{B}
\]

for each \( j \). One can check from (220) that \( w_j \) is dehomogenized in \( B_1(0) \). Then, using (25III), we have that there exists \( w \in \mathfrak{B} \) and a subsequence of \( \{j\} \) (which we pass to without changing notation) for which \( w_j \to w \) in \( C^2_{loc}(C(0) \cap \{ \text{dist}(\cdot, D_w) > 0 \} \cap B_1(0); C(0, \perp)) \). We can also check, using the uniform bound (1) of Lemma 2.1, that the convergence is in \( L^2((C(0) \cap B_1(0); C(0, \perp)) \), from which it is easy to see that \( w \) is dehomogenized in \( B_1(0) \). Assume also, by compactness of \( K \), that along this subsequence we have \( Y_j \to Y \in D_v \cap K \). Dividing (225) by \( \rho_j^{-n-2} \int_{C(0) \cap B_{\rho_j}(Y_j)} |v - \psi_j|^2 d\mathcal{H}^n \) and making the appropriate substitutions in the integrals we see that

\[
\int_{C(0) \cap (B_1(0) \setminus B_{1/2}(0))} R_{Y,j}^{2-n} \left| \frac{\partial(w_j/R)}{\partial R} \right|^2 d\mathcal{H}^n < \epsilon_j,
\]

which implies that

\[
\int_{C(0) \cap (B_1(0) \setminus B_{1/2}(0))} \left| \frac{\partial(w/R)}{\partial R} \right|^2 d\mathcal{H}^n = 0,
\]

which means that \( w \) is homogeneous degree one on \( C(0) \cap \{ r > 0 \} \cap (B_1(0) \setminus B_{1/2}(0)) \). And note that by unique continuation of harmonic functions, it is equal to its homogeneous degree one extension to \( C(0) \cap \{ r > 0 \} \cap B_1(0) \).
Let us now see that \( \dim S(w) > \dim S(v) \): Write \( \mu_j = \|V^j(Y_j + \rho_j(\cdot))\|_{L^2(C(0) \cap B_1(0))} \). For each \( X_0 \in C(0) \cap \{ r > 0 \} \cap B_1(0) \), sufficiently small \( \sigma > 0 \) and sufficiently large \( j \), we have:

\[
\sigma^{-n} \int_{C(0) \cap B_0(X_0)} w^j(X + Y) d\mathcal{H}^n(X) = \mu_j^{-1} \sigma^{-n} \int_{C(0) \cap B_0(X_0)} V^j(Y_j + \rho_j(X + Y)) d\mathcal{H}^n(X) = (1 + \rho_j) \mu_j^{-1} \sigma^{-n} \int_{C(0) \cap B_0(X_0)} V^j(Y_j + (1 + \rho_j)^{-1} \rho_j(Y - Y_j) + (1 + \rho_j)^{-1} \rho_j X) d\mathcal{H}^n(X) = (1 + \rho_j)^{n+1} \mu_j^{-1} \sigma^{-n} \int_{C(0) \cap B(1+\rho_j)^{-1} + (1+\rho_j)^{-1}(Y-Y_j+X_0)} V(Y_j + \rho_j X) d\mathcal{H}^n(X) = (1 + \rho_j)^{n+1} \mu_j^{-1} \sigma^{-n} \int_{C(0) \cap B(1+\rho_j)^{-1} + (1+\rho_j)^{-1}(Y-Y_j+X_0)} w^j(X) d\mathcal{H}^n(X).
\]

Letting \( j \to \infty \) and \( \sigma \downarrow 0^+ \), we conclude that \( w(X_0 + Y) = w(X_0) \) for every \( X_0 \in C(0) \cap \{ r > 0 \} \cap B_1(0) \), which implies that \( Y \in S(w) \). Since (as one can easily check) \( S(v) \subset S(w) \), we have that \( \dim S(w) \geq n - d + 1 \). Thus by the inductive hypothesis we have that \( w \in \mathcal{H}(C(0)) \). However, since \( w \) is dehomogenized in \( B_1(0) \) we deduce that \( w \equiv 0 \). But now, following the proof of Lemma 5.7 of \([\text{Wi98}]\), we deduce a contradiction: Since the \( w^j \) are uniformly bounded in \( B_{1/2}(0) \) (by Lemma 2.1) and \( w^j \to 0 \) uniformly on compact subsets of \( C(0) \cap \{ r > 0 \} \cap B_1(0) \), given any \( \eta > 0 \), we can choose \( j \) sufficiently large so that

\[
\int_{C(0) \cap B_1(0)} |w^j|^2 d\mathcal{H}^n \leq \eta.
\]

Since by construction we have that \( \int_{C(0) \cap B_1(0)} |w^j|^2 d\mathcal{H}^n = 1 \), this shows that for any \( \eta > 0 \), we have

\[
\int_{C(0) \cap B_1(0)} |w^j|^2 d\mathcal{H}^n \geq 1 - \eta
\]

for sufficiently large \( j \). But now, For \( r, s \in (1/8, 1/2) \) and \( \omega \in C(0) \cap \{ r > 0 \} \cap \partial B_1(0) \), we have

\[
|w^j(r \omega) / r - w^j(s \omega) / s| = \left| \int_s^r \frac{\partial (w^j(t \omega) / t)}{\partial t} dt \right| \leq \int_s^r \left| \frac{\partial (w^j(t \omega) / t)}{\partial t} \right| dt,
\]

and so by the triangle inequality, Cauchy-Schwarz and the fact that \( |r/s| \) is bounded we have

\[
|w^j(r \omega)|^2 \leq c \left( |w^j(s \omega)|^2 + \int_{1/4}^1 t^{n-1} \left| \frac{\partial (w^j(t \omega) / t)}{\partial t} \right|^2 dt \right)
\]

for some constant \( c = c(n) > 0 \). Now we integrate with respect to \( \omega \in C(0) \cap \{ r > 0 \} \cap \partial B_1(0) \). Then, we multiply by \( r^{n-1} \) and integrate with respect to \( r \) in \( (1/4, 1) \) and finally we multiply by \( s^{n-1} \) and integrate with respect to \( s \) in \( (1/4, 1/2) \) to give (using the coarea formula)

\[
\int_{C(0) \cap B_1(0) \setminus B_{1/4}(0)} |w^j|^2 d\mathcal{H}^n \leq c \left( \int_{C(0) \cap B_{1/2}(0) \setminus B_{1/4}(0)} |w^j|^2 d\mathcal{H}^n + \int_{C(0) \cap B_1(0) \setminus B_{1/4}(0)} \left| \frac{\partial (w^j / R)}{\partial R} \right|^2 d\mathcal{H}^n \right)
\]
for some $c = c(n) \geq 1$. Now we add $\int_{C(0) \cap B_{1/5}(0)} |w'|^2 dH^n$ to both sides and use \((2.31)\) and the fact that the final term in the above line tends to zero to deduce that

\[
\eta > \int_{C(0) \cap B_{1/2}(0)} |w'|^2 dH^n \geq c(n) > 0
\]

independently of $j$, which is a contradiction. Thus the proof of the claim is complete and the estimate \((2.27)\) indeed holds.

**Step 2.** With $K$ as before and $Y \in D_v \cap K$, we have

\[
w := \frac{v + \kappa Y}{||v + \kappa Y||_{L^2(C(0) \cap B_1(0))}} \in B,
\]

And write $\phi_{Y, \rho}$ for the function that dehomogenizes $w$ on $B_\rho(Y)$. Since $\partial(\phi_{Y, \rho}/R)/\partial R \equiv 1$, \((2.27)\) now implies that

\[
\epsilon \rho^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_{Y, \rho}|^2 dH^n \leq \int_{C(0) \cap (B_\rho(Y) \backslash B_{\rho/2}(Y))} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n.
\]

Also, \((\text{B}6)\) applied to $||w - \phi_{Y, \rho}||_{L^2(C(0) \cap B_1(0))}$ tells us that

\[
\int_{C(0) \cap B_{\rho/2}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n \leq c \rho^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_{Y, \rho}|^2 dH^n.
\]

Combining these two inequalities we see that

\[
\epsilon \int_{C(0) \cap B_{\rho/2}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n \leq \int_{C(0) \cap (B_\rho(Y) \backslash B_{\rho/2}(Y))} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n.
\]

Adding $\int_{C(0) \cap B_{\rho/2}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n$ to both sides (‘hole-filling’) and dividing by $(1 + \epsilon)$ we get that

\[
\int_{C(0) \cap B_{\rho/2}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n \leq \eta \int_{C(0) \cap B_{\rho}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n.
\]

for some $\eta \in (0, 1)$. Then, by iterating this with $2^{-i} \rho$ in place of $\rho$ and using a standard argument to interpolate between these scales, we deduce that

\[
\int_{C(0) \cap B_{\rho}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n \leq c \left(\frac{\sigma}{\rho}\right) \mu \int_{C(0) \cap B_{\rho}(Y)} R_Y^{2-n} |\partial(w/\rho_Y)|^2 dH^n,
\]

for some $\mu = \mu(n, k, C(0), v, K) \in (0, 1)$, $c = c(n, k, C(0)) > 0$ and $0 < \sigma \leq \rho/2 \leq \epsilon/4$. Then using \((2.38)\) and \((\text{B}6)\) we deduce easily from this that

\[
\sigma^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_{Y, \rho}|^2 dH^n \leq c \epsilon^{-1} \left(\frac{\sigma}{\rho}\right) \mu \rho^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_{Y, \rho}|^2 dH^n
\]

for some $c = c(n, k, C(0)) > 0$ and $0 < \sigma \leq \rho/2 \leq \epsilon/8$. Using \((2.32)\) and the triangle inequality, it is then straightforward to check that there exists a single $\phi_Y \in \mathcal{H}(C(0))$ for which

\[
\sigma^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_Y|^2 dH^n \leq c \epsilon^{-1} \left(\frac{\sigma}{\rho}\right) \mu \rho^{-n-2} \int_{C(0) \cap B_{\rho}(Y)} |w - \phi_Y|^2 dH^n
\]
for all \( \sigma \in (0, \rho/2] \). Via the appropriate transformation in the integral, we rewrite (2.43) in terms of \( v \) and see that for each \( Y \in \mathcal{D}_v \cap K \), there is \( \varphi_Y \in \mathcal{H}(\mathcal{C}(0)) \) for which we have

\[
|v - \kappa_Y|^{c(0)} - \varphi_Y|^{2} d\mathcal{H}^n \leq \beta \left( \frac{\sigma}{\rho} \right)^2 \rho^{-n-2} \int_{\mathcal{C}(0) \cap B_{\rho}(Y)} |v - \kappa_Y|^{c(0)} - \varphi_Y|^{2} d\mathcal{H}^n,
\]

for all \( 0 < \sigma \leq \rho/2 \leq \epsilon/8 \). Thus the hypothesis (A.1) of Lemma A.1 is satisfied at points of \( K \cap \mathcal{D}_v \).

Using standard estimates for harmonic functions and Lemma A.3 to check (A.2) and by the fact that \( K \subset A \setminus S(v) \) was arbitrary, we use Lemma A.1 to complete the proof of the claim: Thus we have shown that

\[
v|_{\mathcal{H}_j^{(0)}} \in C^1((\mathcal{H}_j^{(0)} \setminus S(v)); \mathcal{H}_j^{(0)\perp}).
\]

for each \( j = 1, \ldots, 4 \). Remember, \( \mathcal{H}_j^{(0)} \) is closed.

**Step 3.** Let \( \mathcal{T}_v \) denote the set of points \( Y \in \mathcal{D}_v \cap B_{1/4}(0) \) for which

\[
\mathcal{H}^{n-1}(\mathcal{D}_v \cap B_{\eta}(Y)) > 0 \quad \text{for every } \eta > 0.
\]

We claim that for all \( Y \in \mathcal{T}_v \), we must have that

\[
\kappa_Y|_{\mathcal{H}_j^{(0)}} = \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}},
\]

with \( c_p \) as per (2.12) of Lemma 2.4. To see this, we use Lemma 2.4.

\[
\begin{aligned}
\rho^{-5/4} \int_{\mathcal{H}_j^{(0)} \cap (\mathcal{D}_v \mu \cap B_{1/4}(0))} |v(r \omega_j, y) - \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}}|^{2} d\mathcal{H}^n \\
\geq \rho^{-1} \int_{\mathcal{H}_j^{(0)} \cap (B_{1/4}(0) \times \mathcal{T}_v) \cap B_{1/4}(0)} |v(r \omega_j, y) - \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}}|^{2} d\mathcal{H}^n \\
\geq \rho^{-1} \int_0^\rho \int_{(B_{\rho}^{1+k}(0) \times \mathcal{T}_v) \cap B_{1/4}(0)} |v(r \omega_j, y) - \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}}|^{2} d\mathcal{H}^n dy dr.
\end{aligned}
\]

Lemma 2.4 implies that this goes to zero as \( \rho \to 0 \) and so by Lebesgue differentiation we conclude that

\[
\lim_{r \to 0} v(r \omega_j, y) = \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}}
\]

for \( \mathcal{H}^{n-1} \)-almost every \( y \in \mathcal{T}_v \), i.e. (2.47) holds at \( \mathcal{H}^{n-1} \)-almost every point of \( \mathcal{T}_v \). But \( v|_{\mathcal{H}_j^{(0)} \cap B_{1/4}(0)} \) is continuous along \( A \) and using the definition of \( \mathcal{T}_v \) (2.46), we can see that every point of \( \mathcal{T}_v \) is a limit point of a sequence along which (2.47) holds. Thus we have shown (2.47).

Now, if we are in case III, we can complete the proof from here because in this case, \( \mathcal{T}_v = \mathcal{D}_v \cap B_{1/4}(0) \). Thus for any \( j \in \{1, \ldots, 4\} \), the odd reflection of \( v(r \omega_j, y) - \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}} \) in the \( r \)-variable is entire, homogeneous degree one, harmonic and equal to zero on \( \{r = 0\} \). It follows that

\[
v(r \omega_j, y) = ra + \sum_{p=1}^{n-1} y^p c_p|_{\mathcal{H}_j^{(0)}}
\]

for some \( a \in \mathcal{H}_j^{(0)\perp} \), which proves exactly that \( \psi \in \mathcal{H}(\mathcal{C}(0)) \).

So, we may now assume that we are in case II and in steps 4. and 5. we will complete the proof in this case. For the rest of the proof we fix \( i \in \{1, 2\} \) and pass without changing notation to the ‘cross-section’ of \( v_i \), i.e. its restriction to the subspace of \( \mathcal{P}_i^{(0)} \) that orthogonally complements \( S(v) \subset \mathcal{P}_i^{(0)} \). The original blow-up is completely determined by its cross-section and the results we have thus far...
deduced hold for the cross-section (e.g. (2.47) holds and using translation invariance, the cross-section is still harmonic away from $D_v$). Therefore this essentially amounts to assuming that $S(v) = \{0\}$. Since we will be interested in working away from the origin, write $T_v = T_v \cap (B_{1/4}(0) \setminus B_{1/8}(0))$ and $A' = A \cap (B_{1/4}(0) \setminus B_{1/8}(0))$.

**Step 4.** Notice that for any fixed $P \in \{1, \ldots, n - 1\}$, we have (using (2.47) and taking derivatives on the set $\text{Int}(T_v')$ - the interior of $T_v'$ relative to $A'$) that

$$\tag{2.48} (D_p v_i)|_{T_v'} \equiv \frac{\partial}{\partial r} p_i^{(0)} \text{ on } \text{Int}(T_v').$$

By continuity of $D_p v_i$ on $A$, we get that this identity holds on all of $\overline{\text{Int}(T_v')}$. We now claim that $v_i$ is harmonic at points of $A' \setminus \text{Int}(T_v')$. Firstly, since step 2 shows that $v_i|_{\overline{P_i^{(0)}}}$ is Lipschitz, we have (by (2) of Lemma A.3) that the set $D_v \setminus T_v$ is removable for $v_i$ (because for each $Y \in D_v \setminus T_v$, there is $\eta > 0$ for which $H^{n-1}(D_v \cap B_\eta(Y)) = 0$).

So we must show that $T_v \setminus \text{Int}(T_v)$ is removable. Observe the general fact that

$$A' \setminus T_v' \supset T_v \setminus \text{Int}(T_v').$$

Thus for $Y = (0, y) \in T_v' \setminus \text{Int}(T_v')$, there exists a sequence of points $Y_m = (0, y_m) \in A' \setminus T_v'$ with $Y_m \to Y$. Write $P_i = |H_j^{(0)}| + |H_j^{(0)}|$. Since $v_i$ is $C^1$ at $Y_m$ for each $m$, we have

$$\lim_{r \to 0^+} \frac{\partial}{\partial r} v(r \omega_j, y_m) = \lim_{r \to 0^+} \frac{\partial}{\partial r} v(r \omega_j, y_m).$$

Letting $m \to \infty$ and using (2.45), we get that

$$\lim_{r \to 0^+} \frac{\partial}{\partial r} v(r \omega_j, y) = \lim_{r \to 0^+} \frac{\partial}{\partial r} v(r \omega_j, y).$$

Thus we get that the partial derivatives of $v_i$ all exist at $(0, y)$ and we can then deduce from (2.45) that $v_i$ is $C^1$ at $(0, y)$. By considering $v_i - \sum_{\alpha=1}^{n-1} y^p c_\alpha^p$ and using Lemma A.3 (the fact that the zero set of a $C^1$ harmonic function is removable), we conclude that $v_i$ is indeed harmonic at points of $A' \setminus \text{Int}(T_v')$.

**Step 5.** We now complete the proof in case II. Notice that - in light of the fact that $v_i|_{A' \setminus \{0\}}$ is $C^1$, (2.48) implies that $D_p v_i$ is continuous on $P_i^{(0)} \cap (B_{1/4}(0) \setminus B_{1/8}(0))$. Now, since $v_i$ is homogeneous degree one, we also know that $D_p v_i$ is homogeneous degree zero and therefore its restriction to the sphere $S := \partial B_{3/16}(0) \cap P_i^{(0)}$ is harmonic on $S \setminus \text{Int}(T_v')$. If the maximum and minimum of $D_p v_i|_S$ are both obtained on $\text{Int}(T_v') \cap S$, then $D_p v_i$ is constant on the sphere (equal to $c_\alpha^p$) and so by homogeneity, $D_p v_i$ is constant. If, on the other hand, $(D_p v_i)|_S$ has a local maxima or minima at $X_0 \in S \setminus \text{Int}(T_v')$ then (again using homogeneity), $D_p v_i$ has a local maxima (or minima) at $X_0$. But $D_p v_i$ is harmonic and so this too means that $D_p v_i$ is constant. Since $p \in \{1, \ldots, n - 1\}$ and $i \in \{1, 2\}$ were arbitrary, this means that $D_p v_i$ is constant for all $p = 1, \ldots, n - 1$ and $i = 1, 2$. Thus we indeed have that $v_i|_A = \sum_{p=1}^{n-1} y^p c_\alpha^p$ and we can complete the proof in the same way as before.

\[\square\]

**Theorem 2.6.** Fix a properly aligned cone $C^{(0)} \in C_{n-1}$. There exists $\bar{\theta}_1 = \bar{\theta}_1(n, k, C^{(0)}) \in (0, 1/8)$ and $\mu = \mu(n, k, C^{(0)}) \in (0, 1)$ such that the following is true. For any $v \in \mathcal{B}(C^{(0)})$, there exists $\psi \in \mathcal{H}(C^{(0)})$ and $\kappa \in \mathbf{R}^{1+k} \times \{0\}$ with

$$\sup_{C^{(0)} \cap B_1(0)} |\psi| + |\kappa| \leq c \int_{C^{(0)} \cap B_1(0)} |v|^2$$

\[\tag{2.49}\]
such that for any $\theta \in (0, \bar{\theta}_1)$ we have the estimate:

$$\theta^{-n-2} \int_{C(0) \cap B_\rho(0)} |v - \kappa Y c(0) - \psi|^2 d\mathcal{H}^n \leq c\theta^\mu \int_{C(0) \cap B_1(0)} |v|^2 d\mathcal{H}^n$$

for some $c = c(n, k, C(0)) > 0$.

Proof. By Lemma A.1, it suffices to verify the hypotheses of Lemma A.1. To do this we argue exactly as in Steps 1 and 2 of the proof of Lemma 2.5. That is, we first argue by contradiction to prove that there exists $\epsilon = \epsilon(n, k, C(0)) > 0$ such that for every $\rho \in (0, 1/2)$ and $Y \in D_v \cap B_{1/2}(0)$, there exists $\varphi_Y \in \mathcal{H}(C(0))$ such that

$$\int_{C(0) \cap (B_\rho(Y) \setminus B_{\rho/2}(Y))} R^{2-n} \left| \frac{\partial((v - \kappa Y c(0) - \varphi_Y)/R)}{\partial R} \right|^2 d\mathcal{H}^n$$

$$\geq \epsilon \rho^{-n-2} \int_{C(0) \cap B_\rho(0)} |v - \kappa Y c(0) - \varphi_Y|^2 d\mathcal{H}^n.$$

Then by the same ‘hole-filling’ argument as in Step 2. of the proof of Lemma 2.5, we get that there is $\bar{\theta} \in (0, 1)$ such that for every $Y \in D_v$, there is $\varphi_Y \in \mathcal{H}(C(0))$ and $\kappa_Y \in \mathbb{R}^{1+k} \times \{0\}^n$ such that

$$\sigma^{-n-2} \int_{C(0) \cap B_\sigma(Y)} |v - \kappa_Y c(0) - \varphi_Y|^2 d\mathcal{H}^n \leq \beta \left( \frac{\sigma}{\rho} \right)^\mu \rho^{-n-2} \int_{C(0) \cap B_\rho(Y)} |v - \kappa_Y c(0) - \varphi_Y|^2 d\mathcal{H}^n,$$

for all $0 < \sigma \leq \rho/2 \leq \gamma/2$, for some $\gamma > 0$, which verifies (A.1). We use Lemma A.3 to verify (A.2) and hence the existence of $\bar{\theta}_1$ is implied by the conclusions of Lemma A.1.

\[ \square \]

3. Gaps In The Top Density Part

In this section we analyse the structure of a minimal two-valued graph in regions in which there are no points of density greater than or equal to 2. First we introduce some more terminology.

Definitions. Let $V = V_f \subseteq \mathcal{V}$. For $X \in \text{spt} \|V\|$, the assignment of single-valued Lipschitz functions $f_i : B_\delta^0(pX) \times \{0\}^k \to \mathbb{R}^k$ for $i = 1, 2$ and some $\delta > 0$ such that $V([B_\delta^0(pX) \times \mathbb{R}^k] = |\text{graph } f_1| + |\text{graph } f_2|$ is called a labelling of $f$ in $B_\delta^0(pX)$. If $U \subseteq B_2(0) \times \mathbb{R}^k$ is such that $V|U = V_1 + V_2$, where for $i = 1, 2$, $V_i$ is a stationary, Lipschitz single-valued graph, then we say that $V$ decomposes in $U$.

Remarks: (1) Note that a stationary, Lipschitz single-valued graph is smooth away from a codimension four singular set. This follows from standard regularity theory for weak solutions of the Minimal Surface System: A homogeneous degree one Lipschitz weak solution $f : \mathbb{R}^d \to \mathbb{R}^k$ to the Minimal Surface System is necessarily linear if $d \in \{1, 2, 3\}$ (the $d = 3$ case follows from the main theorem of Bar80; for $d = 2$ one can argue by applying results of AA76 to the link of the graph and for $d = 1$ it follows from the definition of stationarity). Using Allard’s Regularity Theorem and the general stratification of the singular set (see 3.1 below), this implies that a general Lipschitz weak solution to the Minimal Surface System is $C^{1,\alpha}$ (and hence, by standard regularity theory for elliptic systems, smooth) away from a codimension four set. The well-known example of LO77 (see Example 2 in the introduction) shows that such a weak solution can indeed have singularities on a codimension four set. (2) It is easy to see that $V$ decomposes in any region which is free of points at which the values of $f$ coincide, i.e. free of points $X \in \text{spt } V$ for which $f_1(pX) = f_2(pX)$.

The main result of the section is the following:

**Theorem 3.1.** Suppose that $V \in \mathcal{V}$ is such that $\{Z \in B_2(0) \times \mathbb{R}^k : \Theta_V(Z) \geq 2\} = \emptyset$. Then $V$ decomposes in $(B_2^0(0) \times \mathbb{R}^k)$. 


We use without proof the following two facts: Firstly, for a stationary cone $C$, we write
\[ S(C) := \{ Z \in \mathbb{R}^{n+k} : \Theta_C(Z) = \Theta_C(0) \} \]
This is called the spine of $C$ and it is a linear subspace of $\mathbb{R}^{n+k}$. For any stationary varifold $V$, we write
\[ S_j := \{ X \in \text{sing } V : \dim S(C) \leq j \forall C \in \text{Var } (V, X) \} \]
and it is standard that
\[(3.1) \quad \dim_{\mathcal{H}} S_j \leq j.\]
Secondly, any tangent cone to a minimal two-valued graph is the graph of either a single or two-valued Lipschitz graph (and is of course also minimal). This is straightforward to check using the definition of tangent cones and the Arzela-Ascoli Theorem.

**Lemma 3.2.** Suppose that $V \in \mathcal{V}$ is such that $\{ Z \in B^k_2(0) \cap \mathbb{R}^k : \Theta_V(Z) \geq 2 \} = \emptyset$. Then $\dim_{\mathcal{H}} \text{sing } V \leq n - 3$.

**Proof.** Pick $X \in \text{sing } V$ and consider $C \in \text{Var } (V, X)$. Bearing in mind the remarks preceding the Lemma, the proof will be complete once we show that $\dim S(C) \leq n - 3$.

Suppose first that $C$ is a single-valued graph and assume for the sake of contradiction that $\dim S(C) \in \{ n, n-1, n-2 \}$. If $\dim S(C) = n$, then $C$ would be a multiplicity one plane. By Allard’s Regularity Theorem, this would mean that $X \in \text{reg } V$, which is a contradiction. If $\dim S(C) = \{ n-1, n-2 \}$, then we can write $C = C_0 \times \mathbb{R}^d$ where $d \in \{ n-1, n-2 \}$ and $C_0$ is the graph of a single-valued, Lipschitz, homogeneous degree one weak solution to the Minimal Surface System $g : \mathbb{R}^{d'} \to \mathbb{R}^k$, where $d' \in \{ 1, 2 \}$. In both of these cases, $g$ must be linear, which proves that $C$ is a multiplicity one plane and thus we derive a contradiction as before.

Suppose now that $C$ is a two-valued graph. If $\dim S(C) = n$, then $C$ must be a multiplicity two plane which implies that $\Theta_V(X) = 0$, but this is false by hypothesis. If $\dim S(C) = n - 1$ and we again write $C = C_0 \times \mathbb{R}^{n-1}$, then $C_0$ is the union of four rays meeting at a point. This means that $C_0$ and hence $C$ has density equal to two at the origin and hence $\Theta_V(X) = 0$, which is again a contradiction. Finally, suppose that $\dim S(C) = n - 2$ and write $C = C_0 \times \mathbb{R}^{n-2}$. Consider the link $M := C_0 \cap S^{2+k-1}$, which is a 1-dimensional stationary integral varifold in the sphere. Suppose that $\text{sing } M \neq \emptyset$ and pick $Y \in \text{sing } M$. The Allard-Almgren classification of stationary 1-varifolds ([AA76]) together with the fact that $C_0$ is a two-valued graph implies that a tangent cone $D \in \text{Var } (M, Y)$ is a union of 4 rays meeting at a point. This means that
\[(3.2) \quad 2 = \Theta_D^1(0) = \Theta_M^1(Y) = \Theta_{C_0}^1(Y) \leq \Theta_{C_0}^2(0) = \Theta_C^2(0) = \Theta_C^0(X),\]
which is a contradiction. Therefore $M$ is free of singular points and so must consist of a union of two disjoint great circles. We then deduce that $\Theta_{C_0}^2(0) = 2$ and therefore that $\Theta_V^0(X) = 2$. This is again a contradiction and we therefore have that $\dim S(C) \leq n - 3$, as required.

**Proof of Theorem 5.4.** Write $S := \text{sing } V \cap (B^n_1(0) \times \mathbb{R}^k)$. Crucially, since $\dim_{\mathcal{H}} pS \leq n - 3$, we have that $B_1 \setminus pS$ is simply connected (see e.g. the appendix to [SW10] for a proof that the complement of a set of $\mathcal{H}^{n-2}$ measure zero is simply connected).

Write $\Omega := B^n_1(0) \setminus pS$. By a somewhat delicate process, we construct two smooth functions $f_1, f_2 : \Omega \to \mathbb{R}^k$ which are solutions to the Minimal Surface System on $\Omega$ and such that graph $f \cap (\Omega \times \mathbb{R}^k)$ is the disjoint union graph $f_1 \cup \text{graph } f_2$.

First note that for any $x \in \Omega$, there exists $\eta_x > 0$ such that graph $f \cap (B^n_{\eta_x}(x) \times \mathbb{R}^k)$ is the disjoint union of two smooth graphs $G^x_0$ and $G^x_0$, say. Fix a point $x \in \Omega$ and write $G^x_0 = \text{graph } f_1$ and $G^x_0 = \text{graph } f_2$. For any other point $y \in \Omega$, since $\Omega$ is path-connected, we can find a simple continuous path $\gamma_y : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. 


By the compactness of $\gamma := \gamma([0,1])$, we have that

\begin{equation}
\gamma \subset B^n_{\eta_x}(x) \cup \bigcup_{i=1}^N B^n_{\eta_i}(z_i) \cup B^n_{\eta_y}(y)
\end{equation}

for some $z_i \in \gamma$ for $i = 1, ..., N$. Assume that $\gamma(t_i) = z_i$ where $t_1 < ... < t_N$. Since graph $f \cap ((B^n_{\eta_x}(x) \cup B^n_{\eta_i}(z_i)) \times \mathbb{R}^k)$ is embedded and consists of two connected components, there is a bijection $\Phi : \{a, b\} \rightarrow \{a, b\}$ so that $G^z_{\Phi(a)} \cup$ graph $f_1$ and $G^z_{\Phi(b)} \cup$ graph $f_2$ are disjoint, embedded, smooth submanifolds. Thus we can extend $f_1$ and $f_2$ to the domain $B^n_{\eta_x}(x) \cup B^n_{\eta_i}(z_i)$ in such a way that $f_1$ and $f_2$ are both smooth solutions to the Minimal Surface System.

We continue this process: Given smooth solutions to the minimal surface system $f_i : B^n_{\eta_x}(x) \cup (\bigcup_{i=1}^K B^n_{\eta_i}(z_i)) \rightarrow \mathbb{R}^k$, for which

\begin{equation}
\text{graph } f \cap \left[ B^n_{\eta_x}(x) \cup \bigcup_{i=1}^{K-1} B^n_{\eta_i}(z_i) \right] \times \mathbb{R}^k
\end{equation}

is the disjoint union graph $f_1 \cup$ graph $f_2$, the above procedure gives a labelling in $B^n_{\eta_K}(z_K)$ which suitably extends the domains of $f_i$ for $i = 1, 2$. When this process terminates, we have defined a labelling at $y$, i.e. without loss of generality we will assume that

\begin{equation}
\text{graph } f_1 \cap (B^n_{\eta_y}(y) \times \mathbb{R}^k) = G^y_a \quad \text{and} \quad \text{graph } f_2 \cap (B^n_{\eta_y}(y) \times \mathbb{R}^k) = G^y_b.
\end{equation}

Let $F$ denote the two-\textbf{R}^{n+k}-valued function $F(x) = (x, f(x))$ and notice that $f(\gamma)$ is embedded and is the disjoint union of two paths $\omega_1$ and $\omega_2$ in graph $f \subset \textbf{R}^{n+k}$ such that $\omega_i(0) = f_i(x)$ for $i = 1, 2$, say, and

\begin{equation}
\omega_1(1) \in G^y_a \quad \text{and} \quad \omega_2(1) \in G^y_b.
\end{equation}

We now see that the labelling produced in (3.3) is well-defined. Take another path $\gamma'$ connecting $x$ to $y$ along which the same process has been performed but assume for the sake of contradiction that by labelling along a finite sequence of balls covering $\gamma'$ in the manner described above, we obtain a different - i.e. the opposite, as there are only two - labelling of $f$ in $B^n_{\eta_y}(y)$. Note again that since $\gamma' \in \Omega$, the image $f(\gamma')$ is the disjoint union of two paths $\omega'_1$ and $\omega'_2$ in graph $f$. This time we have $\omega'_1(0) = f_1(x)$ as before, but

\begin{equation}
\omega'_1(1) \in G_b \quad \text{and} \quad \omega'_2(1) \in G_a.
\end{equation}

Consider now the loop $\Gamma := \gamma^{-1} \circ \gamma'$. By construction, we have that

\begin{equation}
F(\Gamma) = \omega^{-1} \circ \omega'_2 \circ \omega^{-1} \circ \omega'_1 \in \text{graph } f
\end{equation}

This is a loop in graph $f$. Since $\Omega$ is simply connected we can continuously contract $\Gamma$ while staying in $\Omega$, i.e. we have a continuous family $\{\Gamma_t\}_{0 \leq t \leq 1}$ of loops, all of which lie in $\Omega$ and such that $\Gamma_0 = \Gamma$ and $\Gamma_1$ is a single point $\{x_0\} \subset \Omega$. By the Lipschitz continuity of $f$ and the continuity of $\Gamma_t(s)$ in both variables (and the fact that a (two-valued) graph is simply connected), we get that $F(\Gamma_t)$ must also contract to a single point, but this means that $(x_0, f(x_0))$ is a single multiplicity two point, which means that the graph is not embedded at $(x_0, f(x_0))$. This contradiction implies that the labelling we described must in fact be well-defined. We can therefore define two functions $f_1$ and $f_2$ on the whole of $\Omega$ as claimed.

We extend $f_1$ and $f_2$ to the whole of $B^n_{\eta}(0)$ simply by noting that $\Omega$ is dense in $B_1$. Now we claim that the graphs of $f_1$ and $f_2$ are both stationary in $B^n_{\eta}(0) \times \mathbb{R}^k$ (what we know already is that they are stationary in $(B^n_{\eta}(0) \setminus \partial \mathcal{S}) \times \mathbb{R}^k$). We have that $V_{f_1}$ - the varifold associated to graph $f_1$ - is an integral $n$-varifold which is stationary away from $\mathcal{S}$. However, since $\mathcal{H}^{n-1}(\mathcal{S}) = 0$ and we have the volume growth bound

\begin{equation}
\|V\|(B^n_{\eta}(X)) \leq c\rho^n \quad \forall \ X \in \mathcal{S}
\end{equation}

for some $c > 0$. The proof is complete.
(which follows from the fact that $V_{f_1}$ is a Lipschitz graph), a standard cut-off argument implies that graph $f_1$ is stationary. The same holds for the varifold associated to graph $f_2$ and this completes the proof. □

4. Partial Graphical Representation and $L^2$ Estimates

Fix a properly aligned cone $C^{(0)} \in \mathbb{C}$ and write $A := A(C^{(0)})$. In this section we show that when a minimal two-valued Lipschitz graph $V$ is sufficiently close to the cone $C^{(0)}$, the part of $\text{spt} \parallel V \parallel$ that lies away from a small tube around the axis of $C^{(0)}$ can be represented as the graph of a smooth function $u$ defined on a domain in $C^{(0)}$. We also record important estimates that hold close to any point of density at least 2, which are based on the main $L^2$ estimates of [Sim93].

**Theorem 4.1.** Fix $\tau \in (0, 1)$. There exists $\epsilon = \epsilon(n, k, C^{(0)}, \tau) > 0$ such that the following is true. If $V \in \mathcal{V}$ and $C \in \mathbb{C}$ are such that:

1. $\|V\|(B_2^n(0) \times \mathbb{R}^k) \leq \|C^{(0)}\|(B_2^n(0) \times \mathbb{R}^k) + 1$

2. $Q_V(C) := \left( \int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \parallel C \parallel) d\|V\|(X) + \int_{(B_2^n(0) \times \mathbb{R}^k) \setminus \{r < 1/8\}} \text{dist}^2(X, \text{spt} \parallel V \parallel) d\|C\parallel(X) \right)^{1/2} < \epsilon$,

3. $d_H(\text{spt} \parallel C \parallel \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \parallel C^{(0)}\parallel \cap (B_2^n(0) \times \mathbb{R}^k)) < \epsilon$ then there exists a domain $U \subset C \cap \{r > 0\}$ and a function $\tilde{u} \in C^\infty(U \cap B_{15/8}(0); C^\perp)$ such that

$U_\tau := C \cap B_{15/8}(0) \cap \{r > \tau\} \subset U$,

and

$V \upharpoonright (B_{15/8}(0) \cap \{r > \tau\}) = \text{graph} \tilde{u} \upharpoonright (B_{15/8}(0) \cap \{r > \tau\})$

and such that for any $Z = (\xi, \eta) \in \text{spt} V \cap B_{1/2}(0)$ with $\Theta_V(X) \geq 2$ and any $\rho \in (0, 1/2]$, we have the estimates:

4. $|\xi|^2 + \int_{B_{\rho/2}(Z)} \sum_{j=1}^{m} |e^{i \varphi_{j}}| |d\|V\||^2 d\text{dist}^2(X, \text{spt} \parallel C \parallel) d\|V\||(X)$,

5. $\int_{B_{\rho/2}(Z)} \frac{\text{dist}^2(X, \text{spt} \parallel C \parallel)}{|X - Z|^{n-1/2}} d\|V\||(X) \leq c\rho^{-n/2} \int_{B_{\rho}(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \parallel C \parallel) d\|V\)||(X)$,

6. $\int_{\tilde{u} \cap B_{\rho/2}(Z)} \frac{|\tilde{u}(X) - (\xi, 0)^{\perp}|^2}{|X - Z|^{n+3/2}} d\mathcal{H}^n(X) \leq c\rho^{-n/3} \int_{B_{\rho}(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \parallel C_Z \parallel) d\|V\||(X)$,

and

7. $\int_{\tilde{u} \cap B_{\rho/2}(Z)} R_{Z}^{2-n} \left| \frac{\partial((\tilde{u} - (\xi, 0)^{\perp})/\partial R)}{|R_{Z}|} \right|^2 d\mathcal{H}^n(X) \leq c\rho^{-n/2} \int_{B_{\rho}(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \parallel C \parallel) d\|V\||(X)$,

for some $c = c(n, k, C^{(0)})$ where $C_Z = (\eta_{0}, -Z), C$ and $R_Z = R_Z(X) := |X - Z|$.
Proof. Suppose we have a sequence \( \{ \epsilon_j \}_{j=1}^{\infty} \) with \( \epsilon_j \downarrow 0^+ \) and \( \{ V_j \}_{j=1}^{\infty} \in V \) and \( \{ C_j \}_{j=1}^{\infty} \subset C \) satisfying (1) and (2) above with \( V_j, C_j \) and \( \epsilon_j \) in place of \( V, C \) and \( \epsilon \) respectively. We will argue that the conclusions of the theorem must hold at least along a subsequence, which - by arguing by contradiction - is enough to prove the theorem. Using the mass bound (1) above and the varifold compactness theorem there exists a subsequence \( \{ j' \} \) of \( \{ j \} \) (which we pass to without changing notation) and a stationary integral \( n \)-varifold \( D \) in \( B^2_2(0) \times \mathbb{R}^k \) for which \( V^j \to D \). We get from (1) and (2) above that \( \text{spt} \| D \| \subset \text{spt} \| C(0) \| \) and \( \text{spt} \| D \| \setminus \{ r < 1/8 \} = \text{spt} \| C(0) \| \setminus \{ r < 1/8 \} \) and since, by the constancy theorem ([Sim83, § 41]), \( D \) has constant integer multiplicity on each of the connected components of \( \text{spt} \| C(0) \| \setminus A \), the mass bound (1) (which is satisfied by \( D \)) implies that this multiplicity must in fact be one everywhere and hence that \( D = C(0) \).

So, for any \( X \in \text{spt} \| C(0) \| \cap B_2(0) \cap \{ r \geq 2\tau \} \), for sufficiently large \( j \) we may apply Allard's Regularity Theorem to \( V^j [B_2(0) \cap \{ \rho \}] \) to deduce that \( V^j [B_{2/2}(X) = \text{graph} u^X] \) for some smooth solution \( u^X \) to the minimal surface system defined on some open domain in \( \text{spt} \| C \| \cap \{ r > 0 \} \).

Since we may do this at each point of the compact set \( \text{spt} \| C(0) \| \cap B_{15/8}(0) \cap \{ r \geq 2 \tau \} \), using a simple covering argument and unique continuation of smooth solutions to the minimal surface system, we deduce that provided \( j \) is sufficiently large (depending on \( \tau \)), there exists a domain \( U_j \subset C \cap \{ r > 0 \} \) and a function \( \tilde{u}^j \in C^\infty(U_j \cap B_{15/8}(0); C^k) \) satisfying (4.1) and (4.2) and the estimate
\[
(4.7) \quad \sup_{U_j} r^{-1} |\tilde{u}^j| + \sup_{U_j} |D\tilde{u}^j| \leq 1.
\]

For the remainder of the proof we simply drop the index \( j \). And without changing notation let \( U \) be the maximal such domain on which such a function \( \tilde{u} \) can be defined satisfying the above estimate. Now we claim that:
\[
\int_{B_{15/8}(0) \setminus \text{graph} u} r^2(X)d\| V \| (X) + \int_{U \cap B_{15/8}(0)} r^2(X)|D\tilde{u}^j|^2dH^n(X) \leq c \int_{B^2_2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \| C \|)d\| V \| (X)
\]
for some constant \( c = c(n, k, C(0)) > 0 \) not depending on \( \tau \). To see this first write, for \( \rho \in (0, 1/2) \) and \( |y| < 1 \), \( A_\rho(y) = \{(r\omega, z) : |r - \rho/2|^2 + |z - y|^2 < \rho^2/16, \omega \in S^{n+k-1}\} \). Then the key claim which must be verified in order for the proof of Lemma 2.6 and hence Theorem 3.1 to carry over to the present setting is that for \( (\xi, \zeta) \in C \cap B_{3/2}(0) \cap \partial U \), we have the estimate
\[
(4.9) \quad c^{-1} |\xi|^{n+2} \leq \int_{A_\rho(0) \setminus \text{spt} \| C \|)d\| V \| (X)
\]
for some constant \( c = c(n, k, C(0)) > 0 \). This indeed holds because if the above inequality were to fail, i.e. if no such constant \( c \) existed, then arguing as we have done already in this proof, \( \text{spt} \| V \| \) would be expressible as a graph over \( C \) in some small neighbourhood of \( (\xi, \zeta) \), contradicting the maximality of \( U \). Thus the inequality holds for some \( c = c(n, k, C(0)) > 0 \). Now, using (4.9), the fact that
\[
(4.10) \quad \int_{B_{10/2}(0, \zeta)} r^2d\| V \| \leq c|\xi|^{n+2}, \quad c = c(n, k, C(0))
\]
and the fact that
\[
(4.11) \quad \int_{U \cap B_{10/2}(0, \zeta)} |D\tilde{u}^j|^2r^2dH^n \leq c|\xi|^{n+2}
\]
Lemma 5.1. Suppose excess $D \subset \mathbb{R}^n \in \mathcal{C}$ with $A := A(C^{(0)}) = \{0\} \times \mathbb{R}^n$. We define

$$\|V^j\|(B_2^m(0) \times \mathbb{R}^k) \leq \|C^{(0)}\|(B_2^m(0) \times \mathbb{R}^k) + 1$$

and such that

$$Q_{V^j}(C^j) := \left( \int_{B_2^m(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt } ||C^j||) d\|V^j\|(X) \right)^{1/2} < \epsilon_j,$$

$$d_H(D_j \cap B_2(0), D \cap B_2(0)) \to 0,$$

$$d_H(\text{spt } ||C^j|| \cap (B_2^m(0) \times \mathbb{R}^k), \text{spt } ||C^{(0)}|| \cap (B_2^m(0) \times \mathbb{R}^k)) \leq \epsilon_j,$$

such that if $D \cap B_1(0) \neq A(C^{(0)}) \cap B_1(0)$, then $C^j \in \mathcal{P}$ for all $j$.

5. Non-Concentration Estimates

In this section we define a distinguished subset $\mathcal{D}$ of $A$, which is to be thought of as the points on the axis of the base cone $C^{(0)}$ at which the good density points of $V$ accumulate as the excess $E$ at scale 1 becomes small. Then using (4.3) of Theorem 4.1 we prove that as $E$ becomes small, excess does not concentrate near points of $\mathcal{D}$. Suppose we have

**Hypotheses**

1. A properly aligned cone $C^{(0)} = C^{(0)}_0 \times \mathbb{R}^m \in \mathcal{C}$ such that $A := A(C^{(0)}) = \{0\} \times \mathbb{R}^n$.

2. A sequence $\{\epsilon_j\}_{j=1}^\infty$ of positive numbers such that $\epsilon_j \downarrow 0^+$.

3. A sequence of minimal two-valued graphs $\{V^j\}_{j=1}^\infty \in \mathcal{V}$ with $0 \in \text{spt } V^j,$ the mass bound

$$\|V^j\|(B_2^m(0) \times \mathbb{R}^k) \leq \|C^{(0)}\|(B_2^m(0) \times \mathbb{R}^k) + 1$$

such that

$$Q_{V^j}(C^j) := \left( \int_{B_2^m(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt } ||C^j||) d\|V^j\|(X) \right)^{1/2} < \epsilon_j,$$

$$d_H(D_j \cap B_2(0), D \cap B_2(0)) \to 0,$$

$$d_H(\text{spt } ||C^j|| \cap (B_2^m(0) \times \mathbb{R}^k), \text{spt } ||C^{(0)}|| \cap (B_2^m(0) \times \mathbb{R}^k)) \leq \epsilon_j,$$

such that if $\mathcal{D} \cap B_1(0) \neq A(C^{(0)}) \cap B_1(0)$, then $C^j \in \mathcal{P}$ for all $j$.

**Lemma 5.1.** Suppose $C^{(0)}$, $\{\epsilon_j\}_{j=1}^\infty$, $\{V^j\}_{j=1}^\infty$, $\mathcal{D}$, $\{C^j\}_{j=1}^\infty$ and $\{\tau_j\}_{j=1}^\infty$ satisfy Hypotheses $\dagger$. If $\mathcal{D} \neq A \cap B_2(0)$, then $C^{(0)} \in \mathcal{P}$. 

□
Proof. Consider $Z \in (A \setminus D) \cap B_2(0)$. By \eqref{5.2}, we know that there exists some $\delta = \delta(Z) > 0$ such that for sufficiently large $j$ we have

\begin{equation}
B_\delta(Z) \cap D_j = \emptyset.
\end{equation}

Thus we have a decomposition $V^j \mathbin{\mid} B_\delta(Z) = V^j_1 + V^j_2$ of $V^j$ in $B_\delta(Z)$ as per Lemma \ref{5.1}. Using \eqref{5.4} and varifold compactness, we may pass to another subsequence (depending on $Z$) along which $V^j_i$ converges in the sense of varifolds to a stationary integral $n$-varifold $W_i$ in $B_\delta(Z)$ for $i = 1, 2$. Since $V^j \mathbin{\mid} B_\delta(Z) \to C^{(0)} \mathbin{\mid} B_\delta(Z)$, we know that $W_1 + W_2 = C^{(0)} \mathbin{\mid} B_\delta(Z)$. By applying the Constancy Theorem (\cite{Sim88} § 41) on each of the connected components of $\mathop{\text{spt}} C^{(0)} \mathbin{\mid} \{ r > \delta \}$ for arbitrary $\delta > 0$, we deduce that $W_i$ has constant multiplicity along each of the half-planes that constitute $C^{(0)}$. Since $W_i$ is itself stationary in $B_\delta(Z)$ this implies that there must be a plane $P_i^{(0)}$ for which $W_i = P_i^{(0)} \mathbin{\mid} B_\delta(Z)$ whence $C^{(0)} \mathbin{\mid} B_\delta(Z) = (|P_1^{(0)}| + |P_2^{(0)}|) \mathbin{\mid} B_\delta(Z)$ whence $C^{(0)} = |P_1^{(0)}| + |P_2^{(0)}|$, because $C^{(0)}$ is a cylindrical cone.

The following Lemma is analogous to Corollary 3.2 of \cite{Sim93}.

Lemma 5.2. Suppose $C^{(0)}$, $\{ \epsilon_j \}_{j=1}^\infty \{ V^j \}_{j=1}^\infty \{ D \}, \{ C^{(j)} \}_{j=1}^\infty$ and $\{ \tau_j \}_{j=1}^\infty$ satisfy Hypotheses \dagger. For $\delta \in (0, 1/8)$, there exists $J = J(\delta) \in \mathbb{N}$ such that for all $j \geq J$, we have

\begin{equation}
\int_{(D)_{\delta} \cap B_1(0)} \mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X) \leq c \delta^{1/2} E_j^2;
\end{equation}

for some $c = c(n, k, C^{(0)}) > 0$.

Proof. Choose $j$ (depending on $\delta > 0$) such that $d_H(D_j \cap B_2(0), D \cap B_2(0)) < \delta$. Then for any $Y \in D$, there exists $Z = (\xi, \eta) \in B_\delta(Z) \cap D_j$ at which the estimates of Theorem \ref{4.1} hold. Therefore for $\delta < \rho \leq 1/4$,

\[
\rho^{-n+1/2} \int_{B_\rho(Y)} \mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X) \\
\leq \rho^{-n+1/2} \int_{B_{\rho+\delta}(Z)} \mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X) \\
\leq \frac{(\rho + \delta)^{-n-1/2}}{\rho^{-n-1/2}} \int_{B_{\rho+\delta}(Z)} \frac{\mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X)}{|X - Z|^{-n-1/2}} \\
\leq c(n, \alpha) \int_{B_{\rho+\delta}(Z)} \frac{\mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X)}{|X - Z|^{-n-1/2}} \\
\leq c(n, \alpha) E_j^2, \text{ using } \ref{4.1}. 
\]

Thus we have that for $Y \in D$, $\rho \in (\delta, 1/4]$ and sufficiently large $j$,

\begin{equation}
\rho^{-n+1/2} \int_{B_\rho(Y)} \mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X) \leq c E_j^2.
\end{equation}

Now we cover $(D)_{\rho/2} \cap B_1(0)$ by a finite collection of balls $\{ B_\rho(Y_i) \}_{i=1}^M$, where $Y_i \in D$ for each $i$, $M \leq c(n, k) \rho^{-m}$ and which can be partitioned into $c = c(n, k)$ pairwise disjoint sub-collections. Summing up the integrals gives

\begin{equation}
\rho^{-i+1/2} \int_{(D)_{\rho/2} \cap B_1(0)} \mathop{\text{dist}}^2(X, \mathop{\text{spt}} C^{(j)} \mathbin{\mid} d) \| V^j \|(X) \leq c E_j^2,
\end{equation}

which implies the result.
6. Constructing Blow-Ups

Suppose $C^{(0)}$, $\{\epsilon_j\}_{j=1}^{\infty}$, $\{V^j\}_{j=1}^{\infty}$, $D$, $\{C^j\}_{j=1}^{\infty}$, and $\{\tau_j\}_{j=1}^{\infty}$ satisfy Hypotheses †. Then possibly after passing to a subsequence we have the following:

\[(1_j) \quad \|V^j\|(B_2^n(0) \times \mathbb{R}^k) \leq \|C^{(0)}\|(B_2^n(0) \times \mathbb{R}^k) + 1/2 \text{ for all } j.\]

\[(2_j) \quad Q_{V^j}(C^j) := \left( \int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C^j||)d\|V^j\|(X) \right)^{1/2} < \epsilon_j,
\]

for all $j$.

\[(3_j) \quad d_H(\text{spt} \|C^j\| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \|C^{(0)}\| \cap (B_2^n(0) \times \mathbb{R}^k)) < \epsilon_j \text{ for all } j.\]

\[(4_j) \quad \text{By Lemma 3.1} \quad \text{For any } Y \in A \cap B_{15/8}(0) \text{ and } \rho \in (0, 1/8), \text{ such that } B_\rho(Y) \cap D_j = \emptyset, \text{ we have the decomposition } V^j|(B_\rho^n(Y) \times \mathbb{R}^k) = V^j_1 + V^j_2, \text{ where for } i = 1, 2, V^j_i \text{ is a minimal (Lipschitz) graph with } \dim_H(\text{sing } V^j_i) \leq n - 4.\]

\[(5_j) \quad \text{By Lemma 5.1 and [57], a closed set } D \subset A \text{ for which } d_H(D_j \cap \overline{B_2(0)}, D \cap \overline{B_2(0)}) \to 0 \text{ such that if } D \cap B_1(0) \neq A \cap B_1(0), \text{ then } C^{(0)} \in P \text{ and } C^j \in P \text{ for each } j.\]

\[(6_j) \quad \text{By Theorem 4.1 a sequence of functions } \tilde{u}^j \in C^\infty(\tilde{U}_j; C^{j_1}) \text{ where } \tilde{U}_j \supset C^j \cap \{r > \tau_j\} \cap B_{15/8}(0) \text{ and such that }\]

\[V^j \mid (B_{15/8}(0) \cap \{r > \tau_j\}) = |\text{graph } \tilde{u}^j| \mid (B_{15/8}(0) \cap \{r > \tau_j\})\]

\[(7_j) \quad \text{By Lemma 5.2 For any } \delta \in (0, 1/8), \text{ there exists } J(\delta) \in \mathbb{N} \text{ such that for } j \geq J:\]

\[(6.1) \quad \int_{(D)_j \cap B_1(0)} \text{dist}^2(X, \text{spt } \|C^j||)d\|V^j\|(X) \leq c\delta^{1/2}E_j^2,\]

\[(8_j) \quad \text{By (4.3), (4.5) and (4.6) of Theorem 4.1 For any } Z = (\xi, \eta) \in D_j \cap B_1(0) \text{ and } \rho \in (0, 1/2):\]

\[(6.2) \quad |\tilde{u}^j - \int_{B_\rho(Z)} m(e_{j+k}^+V^j)|^2d\|V^j\|(X) \leq c \int_{B_\rho(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt } \|C^j||)d\|V^j\|(X)\]

\[(6.3) \quad \int_{\tilde{U}_j \cap B_\rho(Z)} \frac{|\tilde{u}^j - (\xi, \eta)^{1/2}c_\rho|^2}{|X - Z|^n+3/2} \leq c\rho^{-n-3/2} \int_{B_\rho(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt } \|C^j||)d\|V^j\|(X),\]

and

\[(6.4) \quad \int_{\tilde{U}_j \cap B_\rho(Z)} \left( \frac{\partial((\tilde{u}^j - (\xi, \eta)^{1/2}c_\rho)/R_Z)}{|R_Z|} \right)^2 \leq c\rho^{-n-2} \int_{B_\rho(Z) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt } \|C^j||)d\|V^j\|(X),\]

where $R_Z = R_Z(X) := |X - Z|$.

We now construct the blow-up class $\mathcal{B}(C^{(0)})$.

6.1. No Gaps. Suppose first that $C^{(0)} \notin P$ and fix $\tau > 0$. It is easy to see that for sufficiently large $j$, we have a domain $\Omega^j$ which is such that $C^{(0)} \supset \Omega^j \supset C^{(0)} \cap B_{15/8}(0) \cap \{r > \tau\}$ and a function $\psi^j : \Omega^j \to C^{0} \perp$ which is affine on each connected component of its domain and is such
that \( C^j [(B_{15/8}(0) \cap \{ r > \tau \}) = \text{graph } \psi_j^i \mid (B_{15/8}(0) \cap \{ r > \tau \})]. \) For sufficiently large \( j \) we define \( u^j \in C^\infty(\mathbb{C}^0 \cap \{ r > \tau \} \cap B_{15/8}(0); \mathbb{C}^{(0)\perp}) \) by
\[
(6.5) \quad u^j(X) := \tilde{u}^j(X + \psi^j(X)).
\]

Since \( \tau > 0 \) is arbitrary, standard elliptic estimates then imply that there exists a harmonic function \( v \in C^2(\mathbb{C}^0 \cap \{ r > 0 \} \cap B_{15/8}(0); \mathbb{C}^{(0)\perp}) \) and a subsequence \( \{ j' \} \) of \( \{ j \} \) for which \( E_{j'}^{-1}u^j \to v \) as \( j' \to \infty \) in \( C^2(K) \) for each compact subset of \( \mathbb{C}^0 \cap \{ r > 0 \} \cap B_{15/8}(0). \)

Now, given any small \( \epsilon > 0, \) we can use the non-concentration estimate \((\tilde{t}_j)\) to deduce that for sufficiently small \( \delta \) and sufficiently large \( j \) depending on \( \delta \) we have
\[
(6.6) \quad \int_{\mathbb{C}^0 \cap \{ r < \delta \} \cap B_1(0)} |E_j^{-1}u^j|^2 d\mathcal{H}^n \leq \epsilon.
\]

Combining this with the uniform convergence of \( E_j^{-1}u^j \) to \( v \) on \( \mathbb{C}^0 \cap \{ r \geq \tau \} \cap B_1(0) \) we deduce that the convergence \( E_j^{-1}u^j \to v \) is also in \( L^2(\mathbb{C}^0 \cap B_1(0); \mathbb{C}^{(0)\perp}). \) Elliptic estimates also then tell us that after passing to a further subsequence we have smooth convergence of \( E_j^{-1}u^j \to v \) on \( \mathbb{C}^0 \cap \{ r > 0 \} \cap B_1(0). \)

6.2. Gaps. We suppose that \( D \cap B_1(0) \neq A \cap B_1(0); \) constructing the blow-up class in this case is a more involved process. By \((5.6)_1\) we have that \( \mathbb{C}^0 = |P_1^0| + |P_2^0| \in \mathcal{P} \) and \( \mathbb{C}^j = |P_1^j| + |P_2^j| \in \mathcal{P}. \)

Write \( d = d(X) := \text{dist}(X, D) \) (in the previous case, \( r \equiv d \)) and fix \( \tau > 0. \) Note that we can write \( P_i^j = \{ \text{graph } \psi_i^j \}, \) where \( \psi_i^j : P_i^0 \to P_i^{(0)\perp}. \) Using \((5.6)_1\) choose \( j \) sufficiently large (depending on \( \tau \)) such that \( d_H(D_j \cap B_2(0), D \cap B_2(0)) < \tau/2. \) For \( i = 1, 2, \) define \( u_i^j \in C^\infty(P_i^0 \cap B_{15/8}(0) \cap \{ r > \tau_j/2 \}; P_i^{(0)\perp}) \) by \( u_i^j(X) = \tilde{u_i^j}(X + \psi_i^j(X)) \) (where \( \tilde{u}^j \) is as in \((6.6)).\)

Consider a point \( Y \in A \cap B_{15/8}(0) \cap \{ d > \tau \}. \) By choice of \( j, \) \( B_{\tau/2}(Y) \cap D_j = \emptyset \) and so by \((4.4), \) we have the decomposition \( V^j \mid B_{\tau/2}(Y) = V^j_1 + V^j_2 \) for each \( j, \) where (without loss of generality), \( V^j_1 \to P_1^0 \mid B_{\tau/2}(Y) \) as \( j \to \infty, \) for \( i = 1, 2. \) Since \( P_i^j \to P_i^0, \) by arguing as in the proof of Lemma \( 5.1 \) for sufficiently large \( j \) we can use Allard’s Regularity Theorem to deduce that \( V^j_1 \mid B_{\tau/4}(Y) = |\text{graph } \tilde{u}_i^j| \mid B_{\tau/4}(Y) \) for some smooth function \( \tilde{u}_i^j : O_i^j \to P_i^{(0)\perp}, \) where \( O_i^j \) is a domain such that \( P_i^j \supset O_i^j \supset B_{\tau/4}(Y) \cap P_i^j. \) We then define \( u_i^j \) on a domain in \( P_i^0 \) that contains \( B_{\tau/4}(Y) \cap P_i^0 \) by \( u_i^j(Y) := \tilde{u}_i^j(X + \psi_i^j(X)). \) By covering \( \{ r < \tau/8 \} \cap \{ d > \tau \} \cap B_{15/8}(0) \) by a finite collection of balls \( \{ B_{\tau/4}(Y_p) \}_{p=1}^M \) where \( M \leq c(n, m, \tau) \) and performing this construction at each \( Y_p \) for \( p = 1, \ldots, M \) \( \text{for } i = 1, 2, \) we deduce that
\[
V^j \mid \{ \{ r < \tau/8 \} \cap \{ d > \tau \} \cap B_{15/8}(0) \}
= \sum_{i=1}^2 \{ \text{graph } u_i^j \} \mid \{ \{ r < \tau/8 \} \cap \{ d > \tau \} \cap B_{15/8}(0) \}.
\]

Now, by choosing \( j \) sufficiently large (so that \( \tau_j < \tau/8, \)) unique continuation of smooth solutions to the Minimal Surface System tell us that for \( i = 1, 2 \) and each \( j, \) there is a single smooth function \( u_i^j \) defined on a domain in \( P_i^0 \) that contains \( P_i^0 \cap \{ d > \tau \} \cap B_{15/8}(0) \) and which takes values in \( P_i^{(0)\perp} \) such that
\[
V^j \mid \{ \{ d > \tau \} \cap B_{15/8}(0) \}
= \sum_{i=1}^2 \{ \text{graph } u_i^j \} \mid \{ \{ d > \tau \} \cap B_{15/8}(0) \}.
\]
We now blow up $u^j_i$ off $P^j_i$ with respect to the excess $E_j$: i.e. Given $\varphi \in C^\infty_c (P^j_i(0) \cap \{ d > \tau \} \cap B_{15/8}(0); \mathbb{R})$, if $v^{j,\kappa}_i$ is one component function of $u^j_i$ then standard elliptic estimates give

$$
\left| \int_{P^j_i(0) \cap \{ d > 2\tau \} \cap B_{15/8}(0)} Du^{j,\kappa}_i \cdot D\varphi \right| \leq c(n, \tau) \left| \int_{P^j_i(0) \cap \{ d > \tau \} \cap B_{15/8}(0)} |v^{j,\kappa}_i|^2, \right.
$$

and we seek to bound this final integral by $c(n, k, \tau, C^{(0)}) E^2_j$. To do so, split it up into two parts.

Firstly, away from a small neighbourhood of the axis, $|v^{j,\kappa}_i| = \text{dist}(X + u^{j,\kappa}_i(X), C^j)$. But at a point $Y \in A \cap \{ d \geq \tau \} \cap B_{15/8}(0)$, we must use Lemma A.6 to deduce that for sufficiently large $j$ depending on $Y$ and $\tau$, we have

$$
\int_{P^j_i(0) \cap B_{r/2}(Y)} |v^{j,\kappa}_i|^2 \leq c(n, k, \tau, C^{(0)}) E^2_j.
$$

Then a simple covering argument gives that

$$
\int_{P^j_i(0) \cap \{ d > \tau \} \cap B_{15/8}(0)} |v^{j,\kappa}_i|^2 \leq c(n, k, \tau, C^{(0)}) E^2_j.
$$

Thus we can indeed bound the right-hand side of (6.7) by $c(n, k, \tau, C^{(0)}) E^2_j$. Since $\tau > 0$ is arbitrary, we therefore deduce that there exist harmonic functions $v_i \in C^2(P^j_i(0) \cap \{ d > 0 \} \cap B_{15/8}(0); P^j_i(0) ^\perp)$ for $i = 1, 2$ and a subsequence $\{ j' \}$ of $\{ j \}$ for which $E^{-1}_j u^{j'}_i \to v$ as $j' \to \infty$ in $C^2(K)$ for each compact subset of $P^j_i(0) \cap \{ d > 0 \} \cap B_{15/8}(0)$ for $i = 1, 2$. Also, given any small $\epsilon > 0$, we can use (6.1) to deduce that for sufficiently small $\delta$,

$$
\int_{(D)_\delta \cap B_1(0)} |E^{-1}_j u^{j'}_i|^2 dH^n \leq \epsilon
$$

and combine this with the uniform convergence of $E^{-1}_j u^{j'}_i$ to $v_i$ on $P^j_i(0) \cap \{ d > \tau \} \cap \overline{B_1(0)}$ to deduce that the convergence $E^{-1}_j u^{j'}_i \to v_i$ is also in $L^2(P^j_i(0) \cap B_1(0); P^j_i(0) ^\perp)$ for $i = 1, 2$. Elliptic estimates also then tell us that after passing to a further subsequence we have smooth convergence of $E^{-1}_j u^{j'}_i \to v_i$ on $P^j_i(0) \cap \{ d > 0 \} \cap B_1(0)$. We write $v : C^{(0)}(r > 0) \cap B_1(0) \to C^{(0) \perp}$ for the function such that $v|_{P^j_i(0) \cap B_1(0)} = v_i|_{r > 0}$.

**Definition.** Corresponding to $C^{(0)}$, $\{ \epsilon_j \}_{j=1}^\infty$, $\{ \tau_j \}_{j=1}^\infty$, $\{ V^j \}_{j=1}^\infty$, and $\{ V^j_i \}_{j=1}^\infty$ satisfying Hypotheses †, a function $v \in L^2((C^{(0)} \cap B_1(0), C^{(0) \perp}) \cap C^\infty((C^{(0)} \cap \{ r > 0 \} \cap B_1(0); C^{(0) \perp})$ constructed in this way is called a blow-up of the sequence $V^j$ off $C^j$ relative to $C^{(0)}$. We define $\mathcal{B}(C^{(0)})$ to be the class of all blow-ups relative to $C^{(0)}$.

### 7. Properties of Blow-Ups

Now we prove that the class $\mathcal{B}(C^{(0)})$ defined above is indeed a proper blow-up class in the sense that it satisfies the properties listed at the start of Section 2.

**Theorem 7.1.** Given a properly aligned cone $C^{(0)} \in \mathcal{C}$, the class $\mathcal{B}(C^{(0)})$ is a proper blow-up class.

The rest of this section consists of the proof of this theorem. First note that it is clear from the construction that (B1) and (B2) hold.

**Proof of (B3).** For $v \in \mathcal{B}(C^{(0)})$, let $D_j$ be as per (41). If $D_j = \emptyset$ for all sufficiently large $j$, then set $D = \emptyset$. Otherwise, we pass to a subsequence for which $D_j \neq \emptyset$ and since (by upper semicontinuity of $\Theta_{V^j}$) $D_j \cap B_2(0)$ is closed for each $j$, the sequential compactness of the Hausdorff metric on the space of closed subsets of a compact space means that we can pass to another subsequence along
which $D_j \cap \overline{B_2(0)}$ converges in the Hausdorff metric to a closed subset $D \subset \overline{B_2(0)}$. The (joint) upper semicontinuity of $\Theta_{V_j}(\cdot)$ (with respect to both varifold convergence and the spatial variable) shows that $D \subset A$. Setting $D_0 := D$, we see that (7.3) follows immediately from the construction of the blow up and Lemma 5.1.

Proof of (7.4) The argument here is exactly as in pages 635-639 of [Sim93]. Pick $\zeta \in C_c^\infty(B_{5/8}(0))$ such that $\zeta(x, y) = \zeta(|x|, y)$ and with the property that the radial partial derivative $\partial \zeta / \partial r$ vanishes on the neighbourhood $\{|x| < \tau\}$ of the axis, for some fixed $\tau$. Notice then in particular that

$$D_\rho \zeta(|x|, y) = 0$$

for $p = 1, ..., l + k$ and $|x| < \tau$. Plugging in the vector field $\Phi(X) = \zeta e_p$ to the first variation formula (1.1)), we deduce that the coordinate function $x_p$ is weakly harmonic on $V$ with respect to the intrinsic gradient operator on the varifold, i.e. we have that

$$0 = \int_{\text{spt } V_j \cap B_{5/8}(0)} \nabla V_j x_p \cdot \nabla V_j \zeta \, d \|V_j\|(X)$$

for any $p \in \{1, ..., l + k\}$. We then estimate this expression, starting with the part lying close to the axis:

$$\left| \int_{\text{spt } V_j \cap \{|x|<\tau\} \cap B_{5/8}(0)} \nabla V_j x_p \cdot \nabla V_j \zeta \, d \|V_j\|(X) \right|$$

$$= \left| \int_{\text{spt } V_j \cap \{|x|<\tau\} \cap B_{5/8}(0)} e_p \cdot \nabla V_j \zeta \, d \|V_j\|(X) \right|$$

$$= \left| \int_{\text{spt } V_j \cap \{|x|<\tau\} \cap B_{5/8}(0)} e_p \cdot D \zeta^\perp_{V_j} \, d \|V_j\|(X) \right|$$

$$= \left| \int_{\text{spt } V_j \cap \{|x|<\tau\} \cap B_{5/8}(0)} e_p \cdot \left( D \zeta - D \zeta^\perp_{V_j} \right) \, d \|V_j\|(X) \right|$$

$$\leq \sup |D \zeta| \sum_{q=k+2}^{n+k} \int_{\text{spt } V_j \cap \{|x|<\tau\} \cap B_{5/8}(0)} |e_q^\perp_{V_j}| \, d \|V_j\|(X).$$

Using Cauchy-Schwarz we get that this is at most

$$c \sup |D \zeta| (\|V_j\| (\{|x| < \tau\} \cap B_{5/8}(0)))^{1/2} \left( \int_{\{|x|<\tau\}} \sum_{q=k+2}^{n+k} |e_q^\perp_{V_j}|^2 \, d \|V_j\|(X) \right)^{1/2}$$

for some constant $c = c(n)$. By the fact that $\|V_j\| (\{|x| < \tau\} \cap B_{5/8}(0)) \leq c\tau$ for sufficiently large $j$ (because $V_j \to C^{(0)}$) and the estimate (4.3) of Theorem 4.1 this is at most

(7.4) $c \sup |D \zeta| \sqrt{\eta} E_j.$

Away from the axis, i.e. in $\{|x| \geq \tau\} \cap B_{5/8}(0)$, the support of $V_j$ is the union of four smooth sheets each of which is a graph over one of the half-planes which make up $C^j$. Thus away from the axis, the situation is the same as the multiplicity one case and the remainder of the proof can be completed exactly as is done in the proof of Lemma 1 (pages 636 - 639) of [Sim93].
Proof of (285): Firstly, if \( v \in \mathcal{B}(C(0)) \) is not identically zero, then for any \( Y \in A \cap B_1(0) \) and \( \rho \in (0,1/2(1-|Y|)] \), we have that \( \tilde{v}_{Y,\rho} \) is a blow-up of \( (\eta_{Y,\rho})^*V \) off \( C^j \) relative to \( C(0) \), which shows (285).

For (285II): If \( C(0) \in C_{n-2} \), then notice that for sufficiently large \( j \), \( C^j \in \mathcal{P} \) (this is because as one can easily check - the Hausdorff limit of \( \{A(C^j)\}_{j=1}^\infty \) is contained in \( A(C(0)) \) and therefore \( \dim(A(C^j)) \leq \dim(A(C(0))) \)). Then, for \( \psi : C(0) \cap \{r > 0\} \to C(0)^\perp \) such that \( \psi|_{\mathcal{P}_j} \) is affine, we can define a new sequence of cones \( \hat{C}^j \in \mathcal{P} \) by letting \( \hat{C}^j \) be the unique element of \( C \) which contains the graph of \( \psi^j + E_j\psi \), where \( \psi^j \) is the function that graphically represents \( C^j \) over \( C(0) \). Then, \( \|v - \psi\|_{L^2(C(0) \cap B_1(0))}^{-1}(v - \psi) \) is a blow-up of \( V^j \) off \( \hat{C}^j \) relative to \( C(0) \).

Now, if \( C(0) \in C_{n-1}, (\xi, 0) \in \mathbb{R}^{1+k} \times \{0\} \) and \( \psi \in \mathcal{H}(C(0)) \), first we replace the sequence \( \{V^j\}_{j=1}^\infty \) with \( \{(\tau_{(E,j,\xi,0)}), V^j\}_{j=1}^\infty \) and then define \( \hat{C}^j \in C \) in the same way as before. That is, we let \( \hat{C}^j \) be the unique element of \( C \) containing graph(\( \psi^j + E_j\psi \)), where \( \psi^j \) is the function that graphically represents \( C^j \) over \( C(0) \) (at least away from a small neighbourhood of the axis). The new sequences of varifolds and cones satisfy Hypotheses \( \dagger \) and \( \|v - (\xi, 0, 0)^\perp c(0) - \psi\|_{L^2(C(0) \cap B_1(0))}^{-1}(v - (\xi, 0, 0)^\perp c(0) - \psi) \) is a blow-up of \( (\tau_{(E,j,\xi,0)}), V^j \) off \( \hat{C}^j \) relative to \( C(0) \).

To see (285III), suppose that for each \( j \), \( v_j \) is the blow-up of \( \{V^j_{\rho_{j}}\}_{j=1}^\infty \) (off \( \{C^j_{\rho_{j}}\}_{j=1}^\infty \), say) and write \( E_{\rho_{j}} \) for the excess of \( V^j_{\rho_{j}} \) (as per (5.4)). Then, for each \( j \), notice that we can choose \( p_j \) such that \( \{p_j\}_{j=1}^\infty \) is strictly increasing and such that

\[
(7.5) \quad \|(E_{\rho_{j}})^{-1}u_{p_j} - v_j\|_{L^2(C(0) \cap B_1(0))} < j^{-1},
\]

where \( u_{p_j} \) is the function that represents \( V^j_{\rho_{j}} \) as a graph over \( C(0) \) as per (7.1). That this is possible is clear from the construction of the blow-up. We then select a further subsequence of the \( \{V^j_{\rho_{j}}\}_{j=1}^\infty \) to ensure that \( E_{\rho_j} \to 0 \) as \( j \to \infty \). With \( D_j' = \{X \in B_2(0) : \Theta_{v_j}(X) \geq 2\} \), we construct \( D \) as in the proof of (283) for which \( D_j' \cap \overline{B_2(0)} \) converges in the Hausdorff metric to \( D \subset \overline{B_2(0)} \). Then we choose \( \epsilon_j \) and \( \tau_j \) such that \( C(0), \{\epsilon_j\}_{j=1}^\infty, \{V^j_{\rho_{j}}\}_{j=1}^\infty, D, \{C^j_{\rho_{j}}\}_{j=1}^\infty \) satisfy Hypotheses \( \dagger \) and therefore we can define \( v \) to be a blow-up of \( V^j_{\rho_{j}} \) off \( C^j_{\rho_{j}} \). Then using (7.5), elliptic estimates, the Arzela-Ascoli theorem, a compact exhaustion and another diagonalisation, we deduce that along a further subsequence, \( v_j, v \to v \) locally in \( C^2 \) as required.

Proof of (286): Let \( Y \in D \cap B_{1/2}(0) \) and suppose that \( Y_j = (\xi_j, \eta_j) \in D_j \) is such that \( Y_j \to Y \) as \( j \to \infty \). We have from (6.3) that

\[
(7.6) \quad \int_{\hat{E}_j \cap B_{1/2}(Y_j)} \frac{|\tilde{\nu}_j - (\xi_j, 0)^{\perp} c_j|}{|X - Y_j|^{n+3/2}} dH^n(X) \leq c \rho^{-n-3/2} \int_{B_{\rho}^3(Y_j) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C^j_X\|)d||V_j\|\,(X),
\]

for \( c = c(C(0), \alpha) \) and \( \rho \in (0, 1/2) \). Note that the sequence \( E_j^{-1}(\xi_j, 0)^{\perp} c_j \) is bounded by (6.2) and therefore has a convergent subsequence, which we denote \( \kappa_Y^{\perp} C_j \). We get directly from (7.6) that the limit \( \kappa_Y^{\perp} C_j \) indeed depends only on \( Y \) and that

\[
(7.7) \quad |\kappa_Y^{\perp} C_j| \leq c = c(n, k, C(0)).
\]

And, by passing to further subsequences, we may indeed assume that there is a single vector \( \kappa_Y \in \mathbb{R}^{1+k} \times \{0\} \) for which \( \kappa_Y^{\perp} C_j \) is indeed the image of \( \kappa_Y \) under the orthogonal projection onto \( (T_Y C^j)^{\perp} \). Then, by replacing the variable of integration \( X \) by \( X + \psi_j^j(X) \), dividing by \( E_j^2 \), using the strong \( L^2 \) convergence of \( E_j^{-1}\nu_j \) to \( v \), the \( C_{\text{loc}}^2 \) convergence of \( \psi_j \) to 0, the non-concentration
In a similar way to before we must write the integral over a domain in $\mathbb{C}(0)$. For the final estimate, let $Y \in \mathcal{D} \cap B_{1/2}(0)$ and suppose again that $Y_j = (\xi_j, \eta_j) \in \mathcal{D}$ is such that $Y_j \to Y$ as $j \to \infty$. We have from (6.14) that

$$
\int_{U_j \cap B_{1/2}(Y_j)} (R_x)^{2-n} \left| \frac{\partial}{\partial R_x} \left( \frac{(\tilde{w}_j - (\xi_j, 0))^{+ c_j}}{R_y} \right) \right|^2 d\mathcal{H}^n 
\leq c \rho^{-n-2} \int_{B_{\rho}(Y_j) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} ||C_j||) d\|V\|(X).
$$

In a similar way to before we must write the integral over a domain in $\mathbb{C}(0)$, divide by $E_j^2$ and carefully let $j \to \infty$ to get the result (here we must in addition use the smooth convergence of $E_j^{-1} w^j$ away from the axis). This proves the main estimate of (26II).

This completes the proof that the blow-up class $\mathfrak{B}(\mathbb{C}(0))$ is proper.

## 8. Excess Improvement

For a fixed $\mathbb{C}(0) \in \mathcal{C}$, we say that $V \in \mathcal{V}$, $\mathbb{C}(0), \mathbb{C} \in \mathcal{C}$ and $\epsilon > 0$ satisfy Hypotheses ($\star$) if the following holds.

1. $\|V\|(B_2^n(0) \times \mathbb{R}^k) \leq \|\mathbb{C}(0)\|(B_2^n(0) \times \mathbb{R}^k) + 1$

Let $Q_V(C) := \left( \int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} ||C||) d\|V\|(X) \right)^{1/2}$.

2. $\delta_0 < \epsilon_0$

Let $\mathbb{C}(0) \in \mathcal{C}$ be properly aligned. There exists $\dot{\theta} = \dot{\theta}(n, k, \mathbb{C}(0)) \in (0, 1)$ such that for any $\dot{\theta} \in (0, \dot{\theta})$, there are $\epsilon_0 = \epsilon_0(n, k, \dot{\theta}, \mathbb{C}(0)) > 0$ and $\delta_0 = \delta_0(n, k, \mathbb{C}(0), \dot{\theta}) > 0$ such that the following is true: If $V \in \mathcal{V}$, $\mathbb{C}(0), \mathbb{C} \in \mathcal{C}$ and $\epsilon_0$ satisfy Hypotheses ($\star$) then either

(A) There exists and $Y \in A(\mathbb{C}(0)) \cap B_1(0)$ with

$$
B_{\delta_0}(Y) \cap \{X : \Theta_V(X) \geq 2\} = \emptyset.
$$

Or,

(B) There exists $\mu = \mu(n, k, \mathbb{C}(0)) \in (0, 1)$, $\gamma = \gamma(n, k, \mathbb{C}(0), \dot{\theta}) \geq 1$, $\mathbb{C}' \in \mathcal{C}$ and an orthogonal rotation $\Gamma$ of $\mathbb{R}^{n+k}$ with

(a) $A(\mathbb{C}') \subset A(\mathbb{C}(0))$ (where if $A(\mathbb{C}') = \emptyset$, we deem this to be vacuously true)

(b) $|\Gamma - \text{id}_{\mathbb{R}^{n+k}}| \leq \gamma Q_V(C),$

(c) $d_H(\text{spt} ||\mathbb{C}'|| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} ||\mathbb{C}(0)|| \cap (B_2^n(0) \times \mathbb{R}^k)) \leq \gamma \hat{Q}_V(C)$ and

(d) $\theta^{-n-2} \int_{B_\rho^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} ||\Gamma \mathbb{C}'||) d\|V\|(X)$.
\begin{equation}
\theta^{-n-2} \int_{\Gamma((B_{\theta}^n(0) \times \mathbb{R}^k) \setminus \{r < \theta/8\})} \text{dist}^2(X, \text{spt} \|V\|) d\|\gamma\|, C' \|\text{spt}^\perp(X) < c\theta^{2n} Q^n_{\mathbb{R}}(C),
\end{equation}

for some \(c_1 = c(n, k, C(0)) > 0\).

\textbf{Proof.} To establish the Lemma we take arbitrary sequences \(\{\delta_j\}_{j=1}^{\infty}\) and \(\{\varepsilon_j\}_{j=1}^{\infty}\) of positive numbers with \(\delta_j, \varepsilon_j \downarrow 0^+\) as \(j \to \infty\) and arbitrary sequences \(C_j, V_j\) satisfying (1)-(3) and prove that the conclusions of the Lemma hold along a subsequence \(j\). By arguing by contradiction, this is sufficient to prove the Lemma. If \(A(C(0)) = \emptyset\), then for sufficiently large \(j\), \(\mathcal{D}_j = \emptyset\) and so \(V_j\) decomposes in \(B_{\theta}^n(0) \times \mathbb{R}^k\) into two single-valued Lipschitz graphs. Then the content of the Lemma reduces to two separate applications of the main excess improvement Lemma of Allard’s Regularity Theorem. Thus we assume that \(A(C(0)) \neq \emptyset\).

First notice that we are free to assume that there is a subsequence along which (A) of the statement fails (else there is nothing more to prove), so pass to this subsequence. Thus we have that for every point \(Y \in A(C(0)) \cap B_1(0)\),

\begin{equation}
B_{\delta_j}(Z) \cap \{X : \Theta_{V_j}(X) \geq 2\} \neq \emptyset
\end{equation}

for sufficiently large \(j\). Observing (as was done in the previous section) that \(\mathcal{D}_j \cap \overline{B_{\theta}^n(0)}\) is closed, the sequential compactness of the Hausdorff metric on the space of closed subsets of a compact space means that there exists a closed subset \(\mathcal{D} \subset A \cap \overline{B_{\theta}^n(0)}\) such that (along a further subsequence which we pass to without changing notation), \(d_H(\mathcal{D}_j \cap \overline{B_{\theta}^n(0)}, \mathcal{D} \cap \overline{B_{\theta}^n(0)}) \to 0\). Now we can choose \(\tau_j \to 0\) such that \(C(0), \{\delta_j\}_{j=1}^{\infty}, \{V_j\}_{j=1}^{\infty}, \mathcal{D}, \{C_j\}_{j=1}^{\infty}\) and \(\{\tau_j\}_{j=1}^{\infty}\) satisfy Hypotheses \(\dagger\) and we get from (8.3) that for any blow-up \(v\) of \(V_j\) relative to \(C(0)\), we have \(D_u \cap B_1(0) = A(C(0)) \cap B_1(0)\).

Suppose first that \(C^{(0)} \in \mathcal{C}_{n-2}\) and let \(\psi\) be as in (2.6). Consider now the sequence \(\mathcal{C}_j \in \mathcal{C}\) (defined in the proof of (9.85) in Section 7) for which \(\|v - \psi\|_{L^2(C(0) \cap B_1(0))} > \|v - \psi\|\) is a blow up of \(V_j\) off \(\hat{C}_j\) (that is we let \(\hat{C}_j\) be the unique element of \(\mathcal{C}\) containing graph\(\hat{\psi}_j + E_j\hat{\psi}\), where \(\hat{\psi}_j\) is the function that represents \(C_j\) as a graph over \(C(0)\).

Noting that for sufficiently large \(j\) there is a constant \(c = c(n, k, C(0)) > 0\) such that

\begin{equation}
\text{spt} \|V_j\| \cap ((B_{\theta}^n(0) \times \mathbb{R}^k) \setminus \{r > \delta_j\} \cap B_{\theta}(0)),
\end{equation}

and therefore with \(c\) as above, choosing \(\hat{\theta}\) such that \(\hat{\theta} < 1/8\), we have, using the decay of blow-ups (2.6), the non-concentration estimate (6.1) and the strong \(L^2\) convergence to the blow-up away from \(D_{v-\psi}\), that

\begin{equation}
\theta^{-n-2} \int_{B_{\theta}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|\hat{C}_j\|) d\|\text{spt}^\perp(X) < c\theta^{2n} \int_{B_{\theta}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C_j\|) d\|V_j\|,(X)
\end{equation}

for some \(c = c(n, k, C(0)) > 0\) and for sufficiently large \(j\). Using (8.5), (4.2) and (4.7), we deduce also that

\begin{equation}
\theta^{-n-2} \int_{(B_{\theta}^n(0) \times \mathbb{R}^k) \setminus \{r < \theta/8\}} \text{dist}^2(X, \text{spt} \|V_j\|) d\|\hat{C}_j\|(X)
\end{equation}

for some constant \(c = c(n, k, C(0)) > 0\). Now let \(\Gamma_j\) be the orthogonal rotation of \(\mathbb{R}^{n+k}\) which minimizes \(\|\Gamma_j - \text{id}_{\mathbb{R}^{n+k}}\|\) subject to \(A((\mathbb{R}^{n+k})^\perp \hat{C}_j) \subset A(C(0))\). Thus we set \(C_j = (\mathbb{R}^{n+k})^\perp \hat{C}_j\) so that (a) and (d) hold with \(C_j\) and \(V_j\) in place of \(C\) and \(V\) in place of \(\Gamma\). It is immediate from the construction and the estimate of Lemma (2.2) that there exists \(\gamma \geq 1\) verifying (b) and (c). Thus indeed (B) holds for sufficiently large \(j\), which completes the proof of the Lemma in this case.
When $C^{(0)} \in C_{n-1}$, the proof is completed in an analogous manner, with Lemma 2.6 replacing the use of Lemma 2.2. In this case we must choose $\theta$ such that $c\theta < \theta_1$ (where $c$ is as in (8.4)) and $\theta_1$ is as in the statement of Lemma 2.6.

The next Lemma is a technical point which allows us to only consider blow-ups off sequences in $\mathcal{P}$ whenever $C^{(0)} \in \mathcal{P}$. This is an assumption which has been in place since [51] but we show now that it indeed suffices to consider only this scenario. We use the notation $\mathcal{Q}_V(C)$ as in the previous lemma.

**Lemma 8.2.** Fix a properly aligned cone $C^{(0)} \in \mathcal{P}$ and let $\delta > 0$. There exists $\epsilon_1 = \epsilon_1(n, k, C^{(0)}, \delta) > 0$ and $\eta = \eta(n, k, C^{(0)}, \delta) > 0$ such that the following is true. If $V \in \mathcal{V}$, $C^{(0)} \in \mathcal{C}$ and $\epsilon \in (0, \epsilon_1)$ satisfy Hypotheses (⋆) and there exists $Y \in A(C^{(0)}) \cap B_1(0)$ for which

$$
B_\delta(Y) \cap \{X : \Theta_V(X) \geq 2\} = \emptyset,
$$

Then

$$
\int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{C}) d\|V\|(X) \geq \eta \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V\|(X),
$$

where $\mathcal{P} \in \mathcal{P}$ is defined such that

$$
\int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V\|(X) \leq (3/2) \inf_{\mathcal{P} \in \mathcal{P}} \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V\|(X).
$$

**Proof.** If the lemma is false then there is some fixed $\delta > 0$, sequences of numbers $\{\epsilon_j\}_{j=1}^\infty$, $\{\eta_j\}_{j=1}^\infty$ with $\epsilon_j, \eta_j \downarrow 0^+$, points $Y_j \in A(C^{(0)}) \cap B_1(0)$ and $\{C_j\}_{j=1}^\infty \subset \mathcal{C}$, $\{V_j\}_{j=1}^\infty \subset \mathcal{V}$ satisfying (1)-(4) with $V_j$, $C_j$, $\epsilon_j$ and $Y_j$ in place of $V$, $C$, $\epsilon$ and $Y$ respectively but for which

$$
\int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{C}) d\|V_j\|(X) < \eta_j \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X)
$$

for all $j$ and some $\mathcal{P} \in \mathcal{P}$ with

$$
\int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X) \leq (3/2) \inf_{\mathcal{P} \in \mathcal{P}} \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X).
$$

From the definition of $\mathcal{P}^j$, the fact that $C^{(0)} \in \mathcal{P}$ and (2) of the present lemma, we see that $\mathcal{P}^j \to C^{(0)}$. After first passing to a subsequence for which $Y_j \to Y \in A \cap B_1(0)$, let $v$ be a blow-up of $V_j$ off $\mathcal{P}$. On the other hand, notice that by (8.10) and the triangle inequality, we have that

$$
\int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X) \leq c \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X),
$$

for some absolute constant $c > 0$. Let $\phi^j$ be the function that represents $C^j$ as a graph over $\mathcal{P}^j$ and let $\tilde{\phi}^j$ be the function that represents $\mathcal{P}^j$ as a graph over $C^{(0)}$ (away from a small neighbourhood of $A(C^{(0)})$). Define

$$
\overline{E}_j^2 := \int_{B_\delta^2(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \mathcal{P}) d\|V_j\|(X).
$$

Since $\phi^j$ is locally affine, the $L^2$ bounds give bounds on the first derivatives and thus we can deduce that along a subsequence $E_j^{-1} \phi^j(X + \tilde{\phi}^j(X))$ converges locally uniformly in $C^{(0)} \cap \{r > 0\} \cap B_2(0)$ to some function $\phi$, say. Dividing (8.13) by $\overline{E}_j^2$ and letting $j \to \infty$ shows in fact that $\phi = v$.

Now let us see that $\phi \neq 0$: From (8.7) and Lemma 5.1 we have that $B_{\delta/2}(Y) \cap \mathcal{D}_v = \emptyset$ and so we see from (233) that graph $v$ is a pair of planes. Notice again now that by a pointwise triangle
inequality we have that
\[
\int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|P^j\|) d\|V^j\|(X)
\]
\[\leq \int_{B_2^n(0) \times \mathbb{R}^k} 2\text{dist}^2(X, \text{spt} \|C^j\|) d\|V^j\|(X)
\]
\[+ cd_1^2 \text{(spt} \|P^j\| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \|C^j\| \cap (B_2^n(0) \times \mathbb{R}^k))
\]
where \(c\) is a positive absolute constant, from which, using (8.10), we get that
\[
0 < c \leq \bar{E}_j^{-2} d_\mathcal{H}(\text{spt} \|P^j\| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \|C^j\| \cap (B_2^n(0) \times \mathbb{R}^k)).
\]
This implies easily that
\[
\|\phi\|_{L^2(B_1(0) \cap C(0))} \geq c > 0.
\]
But now, if we write \(\bar{\Psi}^j\) for the unique pair of planes containing \(\text{graph}(\bar{\psi}^j + \bar{E}_j \phi)\), we have (since \(\phi = v\)) that
\[
\bar{E}_j^{-2} \int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|\bar{\Psi}^j\|) d\|V^j\|(X) \to 0
\]
as \(j \to \infty\) and yet by virtue of (8.15),
\[
\bar{E}_j^{-2} d_\mathcal{H}(\text{spt} \|\bar{\Psi}^j\| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \|P^j\| \cap (B_2^n(0) \times \mathbb{R}^k)) \geq c > 0
\]
for some \(c\). Thus for sufficiently large \(j\), we contradict (8.11) and this completes the proof of the Lemma.

\section{Main Excess Improvement Lemma.}

\begin{lemma}
Let \(C(0) \in C\) be properly aligned. There exists \(\bar{\theta} = \bar{\theta}(n, k, C(0)) \in (0, 1)\) such that for any \(\theta \in (0, \bar{\theta})\), there is \(c_0 = c_0(n, k, \theta, C(0)) > 0\) such that the following is true: If \(V \in \mathcal{V}, C(0), C \in C\) and \(c_0 > 0\) satisfy Hypotheses (\(\ast\)) then there exists \(\mu = \mu(n, k, C(0)) \in (0, 1), \gamma = \gamma(n, k, C(0), \bar{\theta}) \geq 1, C' \in C\) and an orthogonal rotation \(\Gamma\) of \(\mathbb{R}^{n+k}\) with
\[
\begin{align*}
&1 \quad A(C') \subset A(C(0)) \quad (\text{where if } A(C') = \emptyset, \text{we deem this to be vacuously true}) \\
&2 \quad |\Gamma - 1|_{\mathbb{R}^{n+k}} \leq \gamma Q_V(C), \\
&3 \quad d_\mathcal{H}(\text{spt} \|C'\| \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt} \|C(0)\| \cap (B_2^n(0) \times \mathbb{R}^k)) \leq \gamma Q_V(C) \quad \text{and} \\
&4 \quad \theta^{-n-2} \int_{B_2^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|\Gamma_* C'\|) d\|V\|(X)
\end{align*}
\]
\[
\begin{align*}
&+ \theta^{-n-2} \int_{\Gamma((B_3^n(0) \times \mathbb{R}^k) \setminus \{r < \theta/8\})} \text{dist}^2(X, \text{spt} \|V\|) d\|\Gamma_* C'\|(X) < c \theta^{2\mu} Q_V^2(C),
\end{align*}
\]
for some \(c_1 = c_1(n, k, C(0)) > 0\).
\end{lemma}

\textit{Proof.} To establish the Lemma we take an arbitrary sequence \(\{\epsilon_j\}_{j=1}^\infty\) of positive numbers with \(\epsilon_j \downarrow 0^+\) as \(j \to \infty\) and arbitrary sequences \(C^j \in C\) and \(V^j \in \mathcal{V}\) satisfying (1)-(3) of the statement of the Lemma and prove that the conclusions of the Lemma hold along a subsequence \(\{j'\}\) of \(\{j\}\).

As per the proof of Lemma \textit{S.1}, we assume that \(A(C(0)) \neq \emptyset\). If \(A\) of Lemma \textit{S.1} fails for all sufficiently large \(j\), then by applying Lemma \textit{S.1} we see that \(B\) holds for all sufficiently large \(j\), which is precisely the main conclusions of Lemma \textit{S.3}\. Thus we may assume that there is a subsequence along which \(A\) of Lemma \textit{S.1} holds. Pass to this subsequence. Thus for each \(j\), we have \(Y^j_j \in A(C(0)) \cap B_1(0)\) for which
\[
B_{\delta_0}(Y^j_j) \cap \{X : \Theta_{V^j}(X) \geq 2\} = \emptyset,
\]
where \(\Theta_{V^j}(X)
\]
where \( \delta_0 > 0 \) is from Lemma \([8.1]\). By compactness of \( A(C^{(0)}) \cap B_1(0), \ Y_j \to Y \in A(C^{(0)}) \cap B_1(0) \) and it is easy to see that for sufficiently large \( j \),
\begin{equation}
B_{\delta_0/2}(Y) \cap \{ \mathcal{X} : \Theta_{V_j}(X) \geq 2 \} = \emptyset.
\end{equation}
This shows that for every blow-up \( v \) of \( V_j \) relative to \( C^{(0)} \), we have \( D_v \cap B_1(0) \neq A \cap B_1(0) \) and therefore \( C^{(0)} \in \mathcal{P} \). Now we apply Lemma \([8.2]\) with \( \delta = \delta_0/2 \) to deduce that there is some \( \eta = \eta(n, k, C^{(0)}, \theta) > 0 \) such that for sufficiently large \( j \),
\begin{equation}
\int_{B_{\delta_0}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt}(C^j)) d\nu_j \geq \eta \int_{B_{\delta_0}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt}(P^j)) d\nu_j(X),
\end{equation}
where \( P^j \) is defined via \([8.5]\) with \( V_j \) in place of \( V \). Now we can indeed work in the proper blow-up class \( \mathfrak{B}(C^{(0)}) \) and complete the proof exactly as in Lemma \([8.1]\) noting that it is indeed sufficient, in light of \([8.10]\), to prove, as we do, that
\begin{equation}
\theta^{-n-2} \int_{B_{\delta_0}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt}(C^j)) d\nu_j \leq c\theta^\mu \int_{B_{\delta_0}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt}(P^j)) d\nu_j(X)
\end{equation}
for sufficiently large \( j \), instead of the same inequality but with \( C^j \) on the right-hand side. To see the final claim of the Lemma note that when \( C^{(0)} \) and \( C \) are both in \( C_{n-2} \), we have (from Lemma \([2.2]\)) that \( C^j \) is always a pair of planes. It is straightforward to check that for sufficiently large \( j \) (depending on \( C^{(0)} \)), we have \( \dim A(C^j) \leq \dim A(C^{(0)}) \leq n - 2 \).

9. Main \( \epsilon \)-Regularity Theorems

The proofs of Theorems \([1]\) and \([2]\) are very similar. We give the proof of Theorem \([1]\) and describe afterwards the minor changes necessary to prove Theorem \([2]\).

9.1. Proof of Theorem \([1]\). Let \( c, \mu, \bar{\theta} \) be such as they are in the statement of Lemma \([8.3]\) and choose \( \theta \in (0, \bar{\theta}) \) such that \( c\theta^{2\mu} \leq 1/2 \). We claim that by iterating Lemma \([8.3]\) we can produce a sequence \( \{C^{(j)}\}_{j=1}^{\infty} \in C_{n-2} \) with (either \( A(C^{(j)}) = \emptyset \) or \( A(C^{(j)}) \subset C^{(0)} \)) and a sequence \( \{\Gamma_j\}_{j=1}^{\infty} \) of orthogonal rotations of \( \mathbb{R}^{n+k} \) satisfying
\begin{align}
(9.1) & \quad |\Gamma_j - \Gamma_j^{-1}| \leq c2^{-1} Q_V(C^{(0)}), \\
(9.2) & \quad d_H(\text{spt}(C^{(j)})) \cap (B_2^n(0) \times \mathbb{R}^k), \text{spt}(C^{(j-1)}) \cap (B_2^n(0) \times \mathbb{R}^k) \leq c2^{-1} Q_V(C^{(0)}), \\
(9.3) & \quad \theta^{-j(n+2)} \int_{B_{\theta^j}^n(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt}(\Gamma_j^* C^{(j)})) d\nu_j \leq 2^{-j} Q_V(C^{(0)}), \\
(9.4) & \quad \theta^{-j(n+2)} \int_{\Gamma_j^* (\{B_{\theta^j}^n(0) \times \mathbb{R}^k\}) \{r < \theta^j/8\}} \text{dist}^2(X, \text{spt}(V )) d\nu_j \leq 2^{-j} Q_V(C^{(0)}),
\end{align}
for some \( c = c(n, k, \theta, C^{(0)}) > 0 \) and for all \( j \geq 1 \). To prove this claim, we construct the sequence inductively:

Let \( \gamma \) and \( \epsilon_0 \) be as in Lemma \([8.3]\). By choosing \( \epsilon \) in Theorem \([1]\) sufficiently small and applying Lemma \([8.3]\) in place of \( C \) we produce a new varifold \( C^{(1)} \in C_{n-2} \) and \( \Gamma^1 \) which, by our choice of \( \theta \) and the conclusions of Lemma \([8.3]\) show that \([9.1]-[9.3]\) hold with \( j = 1 \).

Now suppose we have constructed \( \{C^{(j)}\}_{j=1}^{j} \) and \( \Gamma^j \) satisfying \([9.1]-[9.4]\). By choice of \( \epsilon \), we insist that \( \epsilon \gamma(1 + \frac{1}{2} + \frac{1}{2^2} + ...) < \epsilon_0 \). To complete the inductive step, note that \([9.2] - [9.3]\) imply that the hypotheses of Lemma \([8.3]\) are satisfied with \( C^{(j)} \) in place of \( C \) and with \( (\Gamma^j)^{-1} \) in place of \( V \). Applying the Lemma produces a new varifold \( C^{(j+1)} \) and a rigid motion \( \Gamma^{j+1} \) which are easily seen to satisfy \([9.1]-[9.4]\) and this shows that we indeed have the sequence as claimed.
Note that by choosing $\epsilon$ sufficiently small such that for any $Z \in \text{spt} \|V\| \cap B_{3/4}(0)$, we can repeat the proof of the claim but starting with $(\eta_{Z,3/16})_* V$ in place of $V$. Then the estimate (9.12) implies that for each $Z \in \text{spt} \|V\| \cap B_{3/4}(0)$, the sequence $\{C_{Z}^{(j)}\}_{j=1}^\infty$ whose existence is asserted by the claim converges to some $C_Z := \lim_{j \to \infty} C_{Z}^{(j)} \in C_{n-2}$ and that there exists some orthogonal rotation $\Gamma_Z$ and $\alpha = \alpha(n, k, C^{(0)}) \in (0, 1)$ for which (writing $V^Z := (\eta_{Z,3/16})_* V$) we have

\[
\begin{align*}
|\Gamma_Z - \text{id}_{\mathbb{R}^{n+k}}| &\leq c Q_{V^Z}(C^{(0)}) \\
d_H(\text{spt} \|C_Z\| \cap (B_2^*(0) \times \mathbb{R}^k), \text{spt} \|C^{(0)}\| \cap (B_2^*(0) \times \mathbb{R}^k)) &\leq c Q_{V^Z}(C^{(0)}) \\
\rho^{-n+2} \int_{\Gamma_Z((B_\rho^*(0) \times \mathbb{R}^k) \setminus \{r < \rho/8\})} \text{dist}^2(X, \text{spt} \|V^Z\|) d\|\text{dist}\|(\Gamma_Z)_* C_Z(X) &\leq \rho^{2\alpha} Q_{V^Z}(C^{(0)}) \\
\text{and} \\
\rho^{-n+2} \int_{B_\rho^*(0) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C_Z\|) d\|\text{dist}\|(X) &\leq \rho^{2\alpha} Q_{V^Z}(C^{(0)})
\end{align*}
\]

and

\[
\rho^{-n+2} \int_{B_{\rho}^*(X_1) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C_{X_2}\|) d\|\text{dist}\|(X)
\]

\[
\leq \rho^{-n+2} \int_{B_{\rho}^*(X_2) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C_{X_2}\|) d\|\text{dist}\|(X)
\]

\[
\leq \alpha \rho^{2\alpha} Q_{V}(C^{(0)})
\]

and similarly

\[
\sigma^{-n+2} \int_{\Gamma_{X_2}((B_\rho^*(0) \times \mathbb{R}^k) \setminus \{r < 1/8\})} \text{dist}^2(X, \text{spt} \|\eta_{X_1, \sigma/2}\|, \|V\|) d\|\text{dist}\|(\Gamma_{X_2})_* C_{X_2}\|(X) \leq \sigma \rho^{2\alpha} Q_{V}(C^{(0)}).
\]

Combining (9.10) and (9.12) means that we can again iterate Lemma 1 starting with $(\eta_{X_1, \sigma})_* V$ in place of $V$ and $(\Gamma_{X_2})_* C_{X_2}$ in place of $C$ to deduce the existence of some $C'_{X_1} \in C_{n-2}$ for which

\[
(\rho \sigma)^{-n+2} \int_{B_{\rho \sigma}^*(X_1) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|C'_{X_1}\|) d\|\text{dist}\|(X)
\]

\[
\leq c \rho^{\sigma} \sigma^{-n+2} \int_{B_{\rho \sigma}^*(X_1) \times \mathbb{R}^k} \text{dist}^2(X, \text{spt} \|\Gamma_{X_2})_* C_{X_2}\|) d\|\text{dist}\|(X)
\]

for all $\rho \in (0, \theta)$. But now (9.9) and (9.13) show that in fact $C'_{X_1} = (\Gamma_{X_2})_* C_{X_1}$. Thus this iteration argument implies (using also (9.13) and (9.10)) that

\[
d_H(\text{spt} \|(\Gamma_{X_2})_* C_{X_1}\| \cap B_1(0), \text{spt} \|\Gamma_{X_2})_* C_{X_2}\| \cap B_1(0)) \leq c \|p_{X_1} - p_{X_2}\|^{2\alpha} Q_{V}(C^{(0)}),
\]
which indeed implies that \( f \) is \( C^{1,\alpha} \). Moreover, if both \( X_1, X_2 \in \text{sing} \, V \), we have that

\[
\|\Gamma_{X_1} - \Gamma_{X_2}\| \leq c(n, k, C^{(0)})^2 \|Q_V(C^{(0)})\|.
\]

Since every tangent cone to \( V \) in \( B^n_{5/8}(0) \times \mathbb{R}^k \) is in \( C_{n-2} \), we know that \( f \) does not have any branch points in \( B^n_{5/8}(0) \times \mathbb{R}^k \), which implies that \( f|_{B^n_{5/8}(0)} = \text{graph} \, f_1 \cup \text{graph} \, f_2 \), where for \( i = 1, 2 \), \( f_i : B^n_{5/8}(0) \to \mathbb{R}^k \) is a \( C^{1,\alpha} \) weak solution to the Minimal Surface System (and \( M_i := \text{graph} \, f_i \) is an embedded submanifold). Standard elliptic regularity then shows that \( M_i \cap B^n_{1/2}(0) \) is in fact a smooth minimal submanifold for \( i = 1, 2 \). It is then clear that \( \text{sing} \, V = M_1 \cap M_2 \).

Finally, suppose that there exists \( Y \in A(C^{(0)}) \) for which \( (\mathbb{R}^{2+k} \times \{Y\}) \cap \text{sing} \, V \cap B^n_{1/2}(0) \) contains more than one point: Pick such a \( Y \) and let \( Z_1, Z_2 \) be two distinct such points. Write \( r := |Z_1 - Z_2| \). As before, by initial choice of \( \epsilon \), we can ensure that the hypotheses of Lemma 8.3 are satisfied for all such \( Z_1, Z_2 \) with \( (nZ_1, r/2)_r \) in place of \( V \) and \( C_{Z_2} \) in place of \( C \). In particular, the hypotheses of Theorem 4.1 are satisfied. But we now contradict the first term of the estimate (4.3) because by construction \( (r/2)^{-1}(Z_2 - Z_1) \in \text{sing} \, V, \) where \( (r/2)^{-1}(Z_2 - Z_1) = (\xi, 0) \) with \( |(\xi, 0)| = 1/2 \). Thus indeed \( \text{sing} \, V \cap B^n_{1/2}(0) \) is graphical over \( A(C^{(0)}) \cap B^n_{1/2}(0) \) and (9.14) gives the claimed \( C^{1,\alpha} \) estimate. This completes the proof of the theorem. \( \Box \)

**Proof of Theorem 2** The proof follows exactly the same lines as the proof of Theorem 2, except that one cannot deduce that the two-valued graph is \( C^{1,\alpha} \). The existence and regularity of \( \varphi \) is handled in the same way. In fact, we observe that \( B^n_{3/4}(0) \setminus p(\text{graph} \, \varphi) \) is the disjoint union of two simply connected components \( \Omega_1 \) and \( \Omega_2 \), say. Since \( \text{sing} \, V \subset \text{graph} \, \varphi \), we have that for any \( x \in \Omega_i \), \( f(x) \) consists of two distinct regular points of \( \text{spt} \, V \) and therefore \( \text{graph} \, f \cap (\Omega_i \times \mathbb{R}^k) \) for \( i = 1, 2 \) consists of two separate smooth minimal submanifolds. Thus we can in easily deduce that \( \text{spt} \, V \setminus \text{graph} \, \varphi \) consists of four minimal submanifolds. \( \Box \)

### 10. Proof of Theorem 4

In light of the standard stratification of the singular set of a minimal submanifold (see (3.1)), Theorem 4 follows directly from the following Lemma.

**Lemma 10.1.** Let \( V \in V \). For any multiplicity two point \( X \in \text{sing} \, V \) and \( C \in \text{Var} \, \text{Tan}(V, X) \):

- If \( \dim S(C) = n - 2 \), then \( C \in C_{n-2} \).
- If \( \dim S(C) = n - 1 \), then \( C \in C_{n-1} \).

**Proof.** When \( \dim S(C) = n - 2 \), the conclusion is in fact trivial after one observes that every cone \( C \) is itself a Lipschitz minimal two-valued graph. Thus the cross section is a one-dimensional such graph with no spine and therefore must be the union of four rays meeting at the origin.

When \( \dim S(C) = n - 2 \), by looking at the cross-section, the problem is immediately reduced to that of showing that a two-dimensional Lipschitz minimal two-valued graphical cone \( C_0 \) with trivial spine must be a pair of planes meeting only at the origin. Suppose then that \( C_0 \) is the varifold associated to the graph of the two-valued function \( f : \mathbb{R}^2 \to \mathbb{R}^k \). We will analyse the link \( \Sigma := \text{spt} \, C_0 \cap S^{1+k} \), which is defines a one-dimensional stationary varifold in the sphere \( S^{1+k} \). If the link does not contain any singularities, then (by the Allard-Almgren classification of one-dimensional stationary varifolds in Riemannian manifolds - [AA76]) it is the disjoint union of two great circles, in which case \( C_0 \) is a pair of planes meeting only at the origin and we are done. Thus we may assume that \( \Sigma \) has at least one singular point. In fact we show that this leads to a contradiction.
Lemma in full; it is similar to Lemma 4.3 of [Wic14].

Some General Regularity Lemmas.

A.1. decay of Lemmas that are analogous to the well-known characterizations of Hölder regularity in terms of \(x\),

\[
\text{Notice that for every } \Box \text{ singularities and this completes the proof.}
\]

Now fix a singular point \(X\); write \(X^{(0)}\) for the sphere metric, we have that\(\delta\) also.

\[
(X,\text{sing } \Sigma) \text{ consists of two (possibly coinciding) points. For notational ease we define the following two-valued function: For } (x,0) \in S^1 \times \{0\}, \text{ let } \tilde{f}(x,0) = S_x^k \cap \Sigma. \text{ We also write } \tilde{p} \text{ for the 'projection' which sends } S_x^k \text{ to } (x,0).
\]

Note that every singular point of \(\Sigma\) is a multiplicity two point of \(V\). The work of [AA76] gives us a good description of the singularities: For each point \(X \in \text{sing } \Sigma\), there is a \(\delta\) such that, writing \(d_S\) for the sphere metric, we have that

\[
(10.1) \quad \Sigma \cap \{d_S(\cdot,X) < \delta\} = \bigcup_{i=1}^{4} \{\gamma_i^X(s) : s \in [0,t_\delta]\}
\]

where for \(i = 1,\ldots,4\), \(\gamma_i^X : [0,1] \to S^{1+k}\) are geodesics with \(\gamma_i^X(0) = X\) and such that

\[
(10.2) \quad \sum_{i=1}^{4} \dot{\gamma}_i^X(0) = 0.
\]

Note that we can actually take \(\delta\) be the distance to the nearest singular point, \(i.e.

\[
(10.3) \quad \delta = \text{dist}(X,\text{sing } \Sigma \setminus \{X\}),
\]

where this distance is computed in the sphere metric.

Now fix a singular point \(X_0 \in \text{sing } \Sigma\). Let \(X_1\) denote a singular point at distance \(\delta\) from \(X_0\) and write \(X_1 = \gamma_1^{X_0}(t_\delta)\) (where \(\delta\) and \(t_\delta\) are as in (10.1)). Write \(x_i := \tilde{p}X_i\) for \(i = 0,1\) and write \(S^1 \times \{0\} = [-\pi,\pi] \times \{0\}\) in such a way that \(x_0 = 0\) and \(x_1 > 0\). Consider

\[
(10.4) \quad R := \tilde{f}(\{(x,0) : 0 < x < x_1\}) \setminus \gamma_1^{X_0}((0,t_\delta)).
\]

Notice that for every \(x \in (0,x_1)\), \(R \cap S_x^k\) is a single point. Thus in fact \(R = \gamma_j^{X_0}((0,t'))\) for some \(j \in \{2,3,4\}\), where

\[
(10.5) \quad t' := \inf_{t \in [0,1]} \gamma_j^{X_0}(t) \in S_{x_1}^k.
\]

Assume without loss of generality that \(j = 2\). Since \(X_1\) is a singular point, it is a multiplicity two point. This means that \(\Sigma \cap S_{x_1}^k\) is a single point and therefore that \(\gamma_2^{X_0}(t') = X_1\). However, observe that the great circles of which \(\gamma_i^{X_0}\) for \(i = 1,2\) are segments can only possibly meet at two antipodal points. Since they meet at \(X_0\), we deduce that they do in fact meet at \(-X_0\) and therefore that \(X_1 = -X_0\). This means \(\delta = \text{diam } S^{1+k}\) which implies that \(\Sigma\) is the union of four half-great-circles meeting only at the points \(X_0\) and \(-X_0\). We deduce that \(C_0\) is four half-planes meeting along a line, which means that \(\dim S(C) = n - 1\). This contradiction shows that \(\Sigma\) could not have had any singularities and this completes the proof.

\[
\Box
\]

A.1. Some General Regularity Lemmas. In section 2, we use two Campanato-type regularity Lemmas that are analogous to the well-known characterizations of Hölder regularity in terms of decay of \(L^2\) quantities but are adapted specifically to context at hand. We prove only the first such Lemma in full; it is similar to Lemma 4.3 of [Wic14].
**Lemma A.1.** Let $C = \sum_{j=1}^{4} |H_j| \in C_{n-1}$ be properly aligned and let $D \subset A$ be closed. Suppose there are numbers $\beta, \beta_1, \beta_2 \in (0, \infty)$, $\mu \in (0, 1)$ and $\epsilon \in (0, 1/4)$ such that the following hold:

Either $D = A$, $w \in C^2(C \cap \{r > 0\} \cap B^n_1(0); C^\perp)$ and we have:

1) For each $z \in D \cap B_{3/4}(0)$, there is a function $l_z : C \to C^\perp$ of the form $l_z = \psi_z + \kappa_z$ where $\psi_z \in (H2)$ with $\sup_{C \cap \{r > 0\} \cap B^n_1(0)} |l_z| \leq \beta$ and $\kappa_z \in R^{1+k} \times \{0\}^m$ with $|\kappa_z| \leq c\beta$ such that

\[
\sigma^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_z|^2 \leq \beta_1 \left( \frac{\sigma}{\rho} \right)^\mu \rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_z|^2
\]

for all $0 < \sigma \leq \rho/2 \leq \epsilon/2$.

Or $D \neq A$, $C = \sum_{i=1}^{2} |P_i| \in P$, $w \in C^2(C \cap \{\text{dist}(z, D) > 0\} \cap B^n_1(0); C^\perp)$, (1) holds at each point $z \in D \cap B_{3/4}(0)$ and we have

2) For each $z \in (A \setminus D) \cap B_{3/4}(0)$, there is $l_z : C \to C^\perp$ with $l_z|_{P_i}$ affine for $i = 1, 2$ such that

\[
\sigma^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_z|^2 \leq \beta_2 \left( \frac{\sigma}{\rho} \right)^\mu \rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_z|^2
\]

for each $l \in (H2)$ with $\sup_{C \cap \{r > 0\} \cap B^n_1(0)} |l| \leq \beta$ and $\kappa \in R^{1+k} \times \{0\}^m$ with $|\kappa| \leq c\beta$ and all $0 < \sigma \leq \rho/2 < \frac{1}{2} \min\{1/4, \text{dist}(z, D)\}$.

Then for each $j = 1, \ldots, 4$, we have that $w|_{H_j \cap \{r > 0\} \cap B_{1/2}(0)}$ extends continuously up to $\{r = 0\} \cap B_{1/2}(0)$ and is in $C^{1,\lambda}(H_j \cap \{r \geq 0\} \cap B_{1/2}(0), C^\perp)$ for some $\lambda = \lambda(n, k, \beta_1, \beta_2, \epsilon, \mu, \rho, \sigma, \epsilon) \in (0, 1)$ with

\[
\sup_{H_j \cap \{r \geq 0\} \cap B_{1/2}(0)} (|w| + |Dw|) + \sup_{x, y \in H_j \cap \{r \geq 0\} \cap B_{1/2}(0)} \frac{|Dw(x) - Dw(y)|}{|x - y|^\lambda} \leq c \left( \beta^2 + \int_{C \cap B_{\rho}(0)} |w|^2 \right)^{1/2}
\]

where $c = c(n, k, \beta_1, \beta_2) \in (0, \infty)$. The proof shows that there is $\epsilon' = \epsilon'(n, k, \beta_1, \beta_2, \epsilon, \mu) > 0$ such that at each point $z \in A \cap B_{1/2}(0)$, there is $l_z \in (H2)$ with $\sup_{C \cap \{r > 0\} \cap B^n_1(0)} |l_z| \leq \beta$ and $\kappa_z \in R^{1+k} \times \{0\}^m$ with $|\kappa_z| \leq c\beta$ such that

\[
\sigma^{-n-2} \int_{C \cap B_{\rho}(z)} |w - \kappa_z - l_z|^2 \leq \beta_1 \left( \frac{\sigma}{\rho} \right)^\mu \rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - \kappa_z - l_z|^2
\]

for all $0 < \sigma \leq \rho/2 \leq \epsilon'/2$.

**Proof.** Consider an arbitrary point $z \in (A \setminus D) \cap B_{3/4}(0)$ and let $\rho \in (0, 1/16]$. Let $Y \in D$ be such that $|z - Y| = \text{dist}(z, D)$. For $\gamma = \gamma(n, k) \in (0, 1/16]$ such that $\text{dist}(D, z) < \gamma \rho$, we have

\[
(\gamma \rho)^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_Y|^2 dH^n
\]

\[
\leq 2^{n+2} (\gamma \rho + |z - Y|)^{-n-2} \int_{C \cap B_{\rho+|z-Y|(Y)}} |w - l_Y|^2 dH^n
\]

\[
\leq 2^{n+2} \beta_1 \left( \frac{\gamma \rho + |z - Y|}{\rho - |z - Y|} \right)^\mu \left( \rho - |z - Y| \right)^{-n-2} \int_{C \cap B_{\rho-|z-Y|(Y)}} |w - l_Y|^2 dH^n
\]

\[
\leq 4^{n+2} \beta_1 \left( \frac{2 \gamma}{1 - \gamma} \right)^\mu \rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_Y|^2 dH^n.
\]
Choose $\gamma$ such that $4^{n+2}\beta_1\left(\frac{2\rho}{\mu}\right)^{\mu} < 1/4$. Thus we have that for any $z \in (A \setminus D) \cap B_{3/4}(0)$ and $\rho \in (0,1/16]$ for which $\rho \gamma > \text{dist}(z,D)$:

\((\gamma \rho)^{-n-2} \int_{C \cap B_{\gamma \rho}(z)} |w - l_Y|^2 d\mathcal{H}^n \leq 4^{-1} \rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_Y|^2 d\mathcal{H}^n.\)

On the other hand if $\text{dist}(z_1,A) \geq \gamma \rho$, we get from (A.2) that

\((\sigma \gamma \rho)^{-n-2} \int_{C \cap B_{\sigma \gamma \rho}(z)} |w - l_z|^2 \leq \beta_2 \sigma^\mu (\gamma \rho)^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l|^2\)

for each $\sigma \in (0,1/4]$ and any $l = \psi + \kappa c$ with $\sup_{C \cap (\sigma > 0) \cap B_{\gamma}(0)} |\psi| \leq \beta$ and $\kappa \in \mathbb{R}^{1+k} \times \{0\}^m$ with $|\kappa| \leq c\beta$. By iterating inequality (A.5) with $\rho = \gamma^j$ for $j = 1, 2, \ldots$ and using inequality (A.6), we see that for each $z \in (A \setminus D) \cap B_{3/4}(0)$, there is an integer $j_* \geq 1$ such that $\gamma^{j_*+1} < \text{dist}(z_1,A) \leq \gamma^{j_*}$ and $Y \in D$ such that $|z - Y| = \text{dist}(z,D)$ for which

\((\sigma \gamma^{j_*+1})^{-n-2} \int_{C \cap B_{\sigma \gamma^{j_*+1}}}(z) |w - l_z|^2 \leq \beta_2 \sigma^\mu (\gamma^{j_*+1})^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l|^2\)

for any $\sigma \in (0,1/4]$ and (using (A.5))

\((\gamma^j)^{-n-2} \int_{C \cap B_{\gamma^j}(z)} |w - l_Y|^2 d\mathcal{H}^n \leq 4^{-1} (\gamma^{j-1})^{-n-2} \int_{C \cap B_{\gamma^{j-1}}(z)} |w - l_Y|^2 d\mathcal{H}^n \leq 4^{-j} \gamma^{-n-2} \int_{C \cap B_{\gamma}(z)} |w - l_Y|^2 d\mathcal{H}^n\)

(A.8)

for $j \in \{1, \ldots, j_*\}$. Thus if $j_* \geq 1$, we get from using the triangle inequality, followed by (A.8) with $j = j_*$ and (A.9) with $\sigma = 1/2$:

\[\frac{1}{2} \gamma^{j_*+1} \int_{C \cap B_{\gamma^{j_*+1}}(z)} |l_z - l_Y|^2 \leq \frac{1}{2} \gamma^{j_*+1} \int_{C \cap B_{\gamma^{j_*+1}/2}(z)} |w - l_z|^2 d\mathcal{H}^n + \frac{1}{2} \gamma^{j_*+1} \int_{C \cap B_{\gamma^{j_*+1}/2}(z)} |w - l_Y|^2 d\mathcal{H}^n \leq c 4^{-(j-1)} \gamma^{-n-2} \int_{C \cap B_{\gamma}(z)} |w - l_Y|^2 d\mathcal{H}^n\]

for some constant $c = c(n,k,\gamma) \in (0,\infty)$. In particular, this implies that

\((\gamma^j)^{-n-2} \int_{C \cap B_{\gamma^j}(z)} |l_z - l_Y|^2 \leq c 4^{-j} \int_{C \cap B_{\gamma}(z)} |w - l_Y|^2 d\mathcal{H}^n.\)

Then, using (A.8) and the triangle inequality again we can deduce that for $j = 1, \ldots, j_*$, we have

\(\gamma^j)^{-n-2} \int_{C \cap B_{\gamma^j}(z)} |w - l_z|^2 d\mathcal{H}^n \leq c 4^{-(j-1)} \gamma^{-n-2} \int_{C \cap B_{\gamma}(z)} |w - l_Y|^2 d\mathcal{H}^n,\)

(A.10)

where $c = c(n,k,\gamma) \in (0,\infty)$.

Now, using (A.7), (A.8), (A.9) and the fact that $|\kappa_Y| \leq \beta$, for any $z \in C \cap B_{3/4}(0)$ we have for $z \in (A \setminus D) \cap B_{3/4}(0)$:

\(\rho^{-n-2} \int_{C \cap B_{\rho}(z)} |w - l_z|^2 \leq c \rho^{2\lambda} \left( \int_{C \cap B_{1}(0)} |w|^2 d\mathcal{H}^n + \beta^2 \right)\)
for all $\rho \in (0, \gamma]$, where $c = c(n, k, \gamma) \in (0, \infty)$ by considering the alternatives (i) $4\rho \leq \gamma^{j*+1}$ (in which case $\rho = \sigma^j \gamma^{j+1}$ for some $\sigma \in (0, 1/4]$ and we may use (A.7) and (A.8) with $j = j*$, provided $\gamma$ is chosen so that $\gamma^{j*} < 1/4$) or (ii) $\gamma^{j+1} < 4\rho \leq \gamma^j$ for some $j \in \{1, \ldots, j_*\}$ (in which case we may use (A.9)). In view of (A.11) and (A.1), we have shown (A.4), from which it is standard that the conclusions of the Lemma follow. $\Box$

This next Lemma is very similar but instead characterizes $C^{0, \lambda}$ regularity. The proof follows the same argument as before with minor modifications.

**Lemma A.2.** Let $C \subset C$ be properly aligned and let $D \subset A$ be a closed. Let $w \in C^2(C \cap \{r > 0\} \cap B_1^n(0); C^\perp)$ Suppose there are numbers $\beta, \beta_1, \beta_2 \in (0, \infty)$, $\mu \in (0, 1)$ and $\epsilon \in (0, 1/4)$ such that the following hold:

Either $D = A$, $w \in C^2(C \cap \{r > 0\} \cap B_1^n(0); C^\perp)$ and we have:

1) For each $z \in D \cap B_{3/4}(0)$, there is $\kappa_z \in \mathbb{R}^{1+k} \times \{0\}^{n-1}$ with $|\kappa_z| \leq \beta$ such that

$$\sigma^{-n} \int_{C \cap B_{\sigma}(z)} |w - \kappa_z|^2 \leq \beta_1 \left(\frac{\sigma}{\rho}\right)^{\mu} \rho^{-n} \int_{C \cap B_{\sigma}(z)} |w - \kappa_z|^2$$

for all $0 < \sigma \leq \rho/2 \leq \epsilon/2$.

Or $D \neq A$, $C = \sum_{i=1}^{2} |P_i| \in C$, $w \in C^2(C \cap \{\text{dist}(. , D) > 0\} \cap B_1^n(0); C^\perp)$, (1) holds at each point $z \in D \cap B_{3/4}(0)$ and we have

2) For each $z \in (A \setminus D) \cap B_{3/4}(0)$, there is $\kappa_{z,i} \in \mathbb{P}_i^\perp$ for $i = 1, 2$ such that

$$\sum_{i=1}^{2} \sigma^{-n} \int_{P_i \cap \partial \sigma(z)} |w - \kappa_{z,i}|^2 \leq \sum_{i=1}^{2} \beta_2 \left(\frac{\sigma}{\rho}\right)^{\mu} \rho^{-n} \int_{P_i \cap \partial \sigma(z)} |w - \kappa_{z,i}|^2$$

for all $\kappa_i \in \mathbb{P}_i^\perp$ and all $0 < \sigma \leq \rho/2 < \frac{1}{2} \min\{1/4, \text{dist}(z, D)\}$.

Then for each $j = 1, \ldots, 4$ we have that $w|_{H_j \cap \{r > 0\} \cap B_{1/2}(0)}$ extends continuously up to $\{r = 0\} \cap B_{1/2}(0)$ and is in $C^{0, \lambda}(H_j \cap \{r \geq 0\} \cap B_{1/2}(0), C^\perp)$ for some $\lambda = \lambda(n, k, \beta_1, \beta_2, \epsilon, \mu \in (0, 1))$ with

$$\sup_{H_j \cap \{r \geq 0\} \cap B_{1/2}(0)} |w| + \sup_{x,y \in H_j \cap \{r \geq 0\} \cap B_{1/2}(0)} \frac{|w(x) - w(y)|}{|x-y|^\lambda} \leq c \left(\beta^2 + \int_{C \cap B_{1}(0)} |w|^2 \right)^{1/2}$$

where $c = c(n, k, \beta_1, \beta_2) \in (0, \infty)$.

Notice that by using standard derivative estimates for harmonic functions, one can show that a version of (A.2) holds for harmonic blow ups, but with any affine function appearing on the right hand side. That is we have

$$\sigma^{-n-2} \int_{B_{\sigma}(0)} |v - (v(0) + Dv(0) \cdot x)|^2 \leq c(n) \left(\frac{\sigma}{\rho}\right)^{2} \rho^{-n-2} \int_{B_{\rho}(0)} |v - L|^2,$$

for any affine $L : B_1^n(0) \to \mathbb{R}$. In verifying hypothesis (A.2) of Lemma A.1 for harmonic blow-ups, we will use the following Lemma, the proof of which follows similar lines of the proof of Lemma S.2.

**Lemma A.3.** Let $v : B_1^n(0) \to \mathbb{R}$ be harmonic with $\int_{B_1^n(0)} |v|^2 \leq 1$. There is a constant $\eta = \eta(n, k) > 0$ which is such that

$$\eta \int_{B_1^n(0)} |v(x) - (v(0) + Dv(0) \cdot x)|^2 dx \leq \int_{B_1^n(0)} |v(x) - l(x)|^2 dx$$
for every function \( l : \mathbb{R}^n(0) \to \mathbb{R} \) whose graph is the union of two half-planes meeting along an \((n - 1)\)-dimensional axis.

**Proof.** If the lemma is false then there is some sequence \( v^j : B^n_1(0) \to \mathbb{R} \) of harmonic functions, \( \{v^j\}_{j=1}^\infty \) of functions of the same form as \( l \) in the statement of the Lemma and a sequence of numbers \( \{\eta_j\}_{j=1}^\infty \) with \( \eta_j \downarrow 0^+ \) for which

\[
\eta_j \int_{B^n_1(0)} |v^j(x) - (v^j(0) + Dv^j(0) \cdot x)|^2 \, dx > \int_{B^n_1(0)} |v^j(x) - l(x)|^2 \, dx
\]

for all \( j \). Write \( L_j(x) := v^j(0) + Dv^j(0) \cdot x \). Standard derivative estimates and compactness theorems mean that there exists harmonic \( w : B^n_1(0) \to \mathbb{R} \) with

\[
\left( \int_{B^n_1(0)} |v^j - L_j|^2 \right)^{-1} (v^j - L_j) \to w
\]

in \( C^2_{loc}(B^n_1(0)) \). On the other hand, using (A.17) and using the special form of \( l^j \) we have estimates which are sufficient to assert that there exists \( \tilde{l} \) of the same form as \( l^j \) for which

\[
\left( \int_{B^n_1(0)} |v^j - L_j|^2 \right)^{-1} (l^j - L_j) \to \tilde{l}
\]

uniformly. Now, dividing (A.17) by \( \int_{B^n_1(0)} |v^j - L_j|^2 \) and letting \( j \to \infty \) shows that in fact \( \tilde{l} = w \).

Since \( w \) is smooth, this tells us that \( \tilde{l} \) is in fact an affine function on \( B^n_1(0) \). By a pointwise triangle inequality we have that

\[
\int_{B^n_1(0)} |v^j - L_j|^2 \leq 2 \int_{B^n_1(0)} |v^j|^2 + 2 \int_{B^n_1(0)} |l^j - L_j|^2,
\]

from which, using (A.17), we get that

\[
0 < c \leq \left( \int_{B^n_1(0)} |v^j - L_j|^2 \right)^{-1} \int_{B^n_1(0)} |l^j - L_j|^2,
\]

which implies easily that \( \tilde{l} \neq 0 \). But now, if we write

\[
\tilde{l} := \tilde{l} + L_j,
\]

we have (from [A.18], [A.19] and the fact that \( \tilde{l} = w \)) that \( \int_{B^n_1(0)} |v^j - \tilde{l}|^2 \to 0 \) as \( j \to \infty \) and yet \( \int_{B^n_1(0)} |\tilde{l} - L_j|^2 \geq c > 0 \) for some \( c \). Thus for sufficiently large \( j \), we contradict (A.15) and this completes the proof of the Lemma. \( \square \)

### A.2. Removability Theorems for Harmonic Functions

In section 2, we make use of some general results about removable sets for harmonic functions.

**Lemma A.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open, bounded domain and let \( D \subset \Omega \) be closed. Suppose that \( h : \Omega \setminus D \to \mathbb{R} \) is harmonic.

1. If \( h \) is bounded and \( H^{n-2}(D) < \infty \), then \( h \) can be extended to a harmonic function on the whole of \( \Omega \).
2. If \( h \) is Lipschitz and \( H^{n-1}(D) = 0 \), then \( h \) can be extended to a harmonic function on the whole of \( \Omega \).
3. If \( h \) is \( C^1 \) and \( D \subset \{h = 0\} \), then \( h \) can be extended to a harmonic function on the whole of \( \Omega \).
Proof. These statements are well-known to the experts and their proofs will not be reproduced here. The first is classical (but it is proved for general second order linear elliptic operators in [Ser64]). The second statement is easy to prove given that the hypothesis on $D$ imply that it has zero 1-capacity (for a detailed introduction to $\rho$-capacity, see [EC92]). The third is proved in [Kra83] and is also true for more general elliptic operators (see e.g. [JL05]). □

A.3. A Technical Lemma Concerning The Size of $\rho$-Neighbourhoods of Small Sets.

Lemma A.5. Let $D$ be a non-empty subset of the unit ball $B_1^n(0) \subset \mathbb{R}^n$ and suppose that $\mathcal{H}^{n-2}(D) > 0$. Then there is a constant $c = c(n, D) > 0$ such that $\mathcal{H}^n((D)_\rho) \geq c\rho^2$ for all $\rho > 0$.

Proof. We prove this by induction on dimension. For $n = 2$, we have that $(D)_\rho \supset B^2_\rho(x_0)$ for some $x_0$ (since $D$ is non-empty) and therefore it is clear that $\mathcal{H}^n((D)_\rho) \geq c\rho^2$, which checks the claim. For general $n \geq 3$, we first perform a bi-Lipschitz homeomorphism $B_1^n(0) \to B_1^{n-1}(0) \times (-1, 1)$ which maps $D$ to $\tilde{D}$ and we write

$$P_h := \{(x, y) \in B_1^{n-1}(0) \times (-1, 1) : y = h\}.$$  

Since $\tilde{D}$ is non-empty and the homeomorphism is Lipschitz we have that $\mathcal{H}^{n-2}(\tilde{D}) > 0$. Then the (‘rough’) coarea formula implies that

$$0 < \mathcal{H}^{n-2}(\tilde{D}) = \int_{-1}^1 \mathcal{H}^{n-3}(\tilde{D} \cap P_h)dh.$$  

This means that there exists some $A \subset (-1, 1)$ and $\epsilon > 0$ with $\mathcal{L}^1(A) > 0$ and for which

$$\mathcal{H}^{n-3}(\tilde{D} \cap P_h) > \epsilon$$  

for all $h \in A$. Since $P_h$ is a copy of $\mathbb{R}^{n-1}$ we may use the inductive hypothesis to deduce that

$$\mathcal{H}^{n-1}((\tilde{D})_\rho \cap P_h) \geq c(n, \epsilon)\rho^2$$  

for all $h \in A$. Therefore, by the coarea formula again:

$$\mathcal{H}^n((\tilde{D})_\rho) = \int_{-1}^1 \mathcal{H}^{n-1}((\tilde{D}_\rho \cap P_h))dh \geq \mathcal{L}^1(A)c(n, \epsilon)\rho^2.$$  

Applying the inverse of the bi-Lipschitz homeomorphism completes the proof. □

A.4. Bounding Excess from a Plane by the Excess from a Pair of Planes.

Lemma A.6. Let $u : B_1^n(0) \to \mathbb{R}^k$ be a smooth function. Suppose that $Q$ is an $n$-dimensional subspace of $\mathbb{R}^{n+k}$ distinct from $P := \mathbb{R}^n \times \{0\}^k$. There exists $\epsilon = \epsilon(n, k, Q) > 0$ and $c = c(n, k, Q) > 0$ such that if $\|u\|_{C^1(B_1^n(0))} < \epsilon$, then

$$\int_{\text{graph } u} \text{dist}^2(X, P)d\mathcal{H}^n(X) \leq c \int_{\text{graph } u} \text{dist}^2(X, P \cup Q)d\mathcal{H}^n(X).$$  

Proof. Write $A = A_Q := \{X \in \mathbb{R}^{n+k} : \text{dist}(X, P \cup Q) = \text{dist}(X, Q)\}$. If $\text{graph } u \cap A = \emptyset$, then the inequality holds trivially with $c = 1$. So assume that $\text{graph } u \cap A \neq \emptyset$ and observe that by the definition of $A$, it suffices to prove the inequality

$$\int_{\text{graph } u \cap A} \text{dist}^2(X, P)d\mathcal{H}^n(X) \leq c \int_{\text{graph } u \cap A} \text{dist}^2(X, Q)d\mathcal{H}^n(X),$$  

for some $c = c(n, k, Q) > 0$. Assume for the sake of contradiction that (A.29) does not hold. This means that there exists a sequence of numbers $\{\epsilon_j\}_{j=1}^\infty$ with $\epsilon_j \downarrow 0^+$ and $u^j : B_1^n(0) \to \mathbb{R}^k$ with $\|u^j\|_{C^1(B_1^n(0))} < \epsilon_j$ for which

$$\int_{\text{graph } u^j \cap A} \text{dist}^2(X, P)d\mathcal{H}^n(X) > j \int_{\text{graph } u^j \cap A} \text{dist}^2(X, Q)d\mathcal{H}^n(X)$$  

for all sufficiently large $j$. However, using the definition of $A_Q$ and the fact that $\text{dist}(X, P \cup Q) = \text{dist}(X, Q)$ for all $X$ in $\text{graph } u^j \cap A$, we have

$$\int_{\text{graph } u^j \cap A} \text{dist}^2(X, P)d\mathcal{H}^n(X) = 0$$  

and

$$\int_{\text{graph } u^j \cap A} \text{dist}^2(X, Q)d\mathcal{H}^n(X) \leq c \int_{\text{graph } u^j \cap A} \text{dist}^2(X, Q)d\mathcal{H}^n(X),$$  

which is a contradiction. □
for each \( j = 1, 2, \ldots \). Write \( \lambda_j := H^n(\text{graph } v^j \cap A)^{1/n} > 0 \). We dilate \( \mathbb{R}^{n+k} \) by \( \eta_{0,\lambda_j} : X \mapsto \frac{1}{\lambda_j}X \), which suggests defining \( v^j(x) := \frac{1}{\lambda_j}u^j(\lambda_jx) \) on \( B^n_{1/\lambda_j}(0) \) (so that graph \( v^j = \eta_{0,\lambda_j}(\text{graph } u^j) \)). Noting that \( P, Q \) and therefore \( A \) are invariant under dilation, we get that
\[
H^n(\text{graph } v^j \cap A) = 1
\]
for all \( j \) and that
\[
\sup_{B^n_{1/\lambda_j}(0)} |Dv_j(x)| = \sup_{B^n_{1/\lambda_j}(0)} |Du_j(\lambda_jx)| = \sup_{B^n_1(0)} |Du_j(x)| \leq \epsilon_j \to 0
\]
as \( j \to \infty \). Using both (A.31) and (A.32) we can then check that the sequence \( \{v^j\}_{j=1}^\infty \) is uniformly bounded on compact subsets of \( \mathbb{R}^n \). Since (A.32) implies that the sequence is uniformly bounded, Arzela-Ascoli implies that there exists a subsequence \( \{j'\} \) of \( \{j\} \) and a Lipschitz function \( v : \mathbb{R}^n \to \mathbb{R}^k \) for which \( v^{j'} \to v \) uniformly on compact subsets of \( \mathbb{R}^n \). In fact, we get from (A.32) that \( Du \equiv 0 \) and so \( v \) is constant. Moreover, there exists a fixed radius \( R > 0 \) for which \( \text{proj}_P(\text{graph } v^j \cap A) \subset B^n_R(0) \) for all \( j \). Therefore, by making appropriate substitutions in the integrands in (A.30) we get that
\[
\int_{\text{graph } v^j \cap A} \text{dist}^2(X, P) dH^n(X) > j \int_{\text{graph } v^j \cap A} \text{dist}^2(X, Q) dH^n(X).
\]
However, since the uniform limit \( v \) is constant and satisfies \( H^n(\text{graph } v \cap A) = 1 \), we know that the left-hand side is bounded above independent of \( j \) and the right-hand side below. By taking limits in \( j \), this leads to a contradiction. Explicitly, there are \( c, C \in (0, \infty) \), depending only on \( n, k \) and \( Q \) such that
\[
0 < c < \int_{\text{graph } v^j \cap A} \text{dist}^2(X, Q) dH^n(X)
\]
\[
< \frac{1}{j} \int_{\text{graph } v^j \cap A} \text{dist}^2(X, P) dH^n(X)
\]
\[
\leq \frac{1}{j} C \to 0,
\]
which is impossible. Thus indeed (A.28) holds for sufficiently small \( \epsilon > 0 \) and for some \( c = c(n, k, Q) \in (0, \infty) \).

\[\square\]

**References**

[AA76] WK Allard and FJ Almgren, *The structure of stationary one dimensional varifolds with positive density*, Inventiones mathematicae 34 (1976), no. 2, 83–97.

[All72] W.K. Allard, *On the first variation of a varifold*, The Annals of Mathematics 95 (1972), no. 3, 417–491.

[Alm00] F.J. Almgren, *Almgren’s big regularity paper: Q-valued functions minimizing dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension 2*, vol. 1, World Scientific Pub Co Inc, 2000.

[Bar80] J.L.M. Barbosa, *An extrinsic rigidity theorem for minimal immersions from s2 into sn*, J. Differential Geometry 14 (1980), no. 3, 355–368.

[EG92] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC, 1992.

[Hug14] S. T. Hughes, *Stationary, two-valued lipschitz graphs are \( C^{1,\alpha} \)*, preprint (2014).

[JL05] Petri Juutinen and Peter Lindqvist, *Removability of a level set for solutions of quasilinear equations*, Communications in Partial Differential Equations 30 (2005), no. 3, 305–321.

[Kra83] J Král, *Some extension results concerning harmonic functions*, Journal of the London Mathematical Society 2 (1983), no. 1, 62–70.

[KW13] Brian Krummel and Neshan Wickramasekera, *Fine properties of branch point singularities: Two-valued minimal graphs*.

[LO77] HB Lawson and R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Mathematica 139 (1977), no. 1, 1–17.
[Ser64] James Serrin, *Removable singularities of solutions of elliptic equations*, Archive for Rational Mechanics and Analysis **17** (1964), no. 1, 67–78.

[Sim83] L. Simon, *Lectures on geometric measure theory*, volume 3 of proceedings of the centre for mathematical analysis, australian national university, Australian National University Centre for Mathematical Analysis, Canberra (1983), 6.

[Sim93] ______, *Cylindrical tangent cones and the singular set of minimal submanifolds*, J. Differential Geom. **38** (1993), no. 3, 585–652.

[Sim96] ______, *Theorems on regularity and singularity of energy minimizing maps: based on lecture notes by norbert hungerbühler*, Birkhauser, 1996.

[SW10] L. Simon and N. Wickramasekera, *A frequency function and singular set bounds for branched minimal immersions*, Arxiv preprint arXiv:1012.5028 (2010).

[Whi97] Brian White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*, Journal fur die Reine und Angewandte Mathematik **488** (1997), 1–36.

[Wic] N. Wickramasekera, *A stratification theorem for stable codimension 1 rectifiable currents near points of density $\geq 3$*, Manuscript in Preparation.

[Wic04] ______, *A rigidity theorem for stable minimal hypercones*, Journal of Differential Geometry **68** (2004), no. 3, 433–514.

[Wic08] ______, *A regularity and compactness theory for immersed stable minimal hypersurfaces of multiplicity at most 2*, Journal of differential geometry **80** (2008), no. 1, 79.

[Wic14] ______, *A general regularity theory for stable codimension 1 integral varifolds*, Annals of Mathematics, To Appear (2014).

*E-mail address: S.T.Hughes@maths.cam.ac.uk*