Uncertainty quantification in traffic models via intrusive method

( Joint work with M. Herty)

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Innovative numerical methods for evolutionary partial differential equations and applications
Framework & Motivations
Framework

Mathematical modeling of traffic flow on a **single road**, by means of:

- a **microscopic** (agent-based) follow-the-leader model based on ODEs
- a **MACROSCOPIC** (fluid-dynamic) model based on conservation laws
- a **Mesoscopic** (gas-kinetic) model provides a statistical description
Uncertainty

Limitations for obtaining reliable traffic forecast

- highly nonlinear dynamics
- traffic is subjected to various sources of uncertainties
  - errors in the measurements
  - estimate the reaction time of cars and drivers

Possible approaches

- non intrusive methods
  - fixed number of samples using deterministic algorithms (i.e. Monte Carlo)
- intrusive methods
  - reformulate the problem and solve - only once - a (big) system of deterministic equations (i.e. Stochastic Galerkin)
## Stochastic Galerkin approach

- $\xi$ uncertainty described by a random variable $\omega$ on $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$
  - we are dealing with $u(t, x, \xi) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \to \mathbb{R}^d$
    i.e. $\partial_t u(t, x, \xi) + \partial_x f(u(t, x, \xi)) = 0$

- the generalized polynomial chaos gPC expansion\(^1\):
  - we discretize the probability space $\Omega$ and the stochastic quantities are represented by infinite series expansions:
    \[
    \phi(\xi) : \Omega \to \mathbb{R} \text{ orthonormal polynomials w.r.t. the inner product and }
    \{ \phi_i(\xi) \}_{i=0}^{\infty} \text{ is a basis of } L^2(\Omega, \mathbb{P}): 
    \]
    \[
    u(t, x, \xi) = \sum_{k=0}^{\infty} \hat{u}_k(t, x) \phi_k(\xi) \quad \text{where} \quad \hat{u}_k(t, x) = \int_{\Omega} u(t, x, \xi) \phi_k(\xi) \, d\mathbb{P}.
    \]

We can express the mean and variance of $u(t, x, \xi)$ as
\[
\mathbb{E}[u(t, x, \xi)] = \hat{u}_0(t, x) \quad \text{and} \quad \text{Var}[u(t, x, \xi)] = \sum_{k=1}^{\infty} \hat{u}_k^2(t, x).
\]

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\(^1\)P. Pettersson, G. Iaccarino, and J. Nordström, *Polynomial chaos methods for hyperbolic partial differential equations*, Springer International Publishing, 2015
Stochastic Galerkin approach

**Idea:** expand the stochastic quantities in truncated series and then project

\[
G_K[u](t, x, \xi) = \sum_{i=0}^{K} \hat{u}_i(t, x) \phi_i(\xi)
\]

For any fixed \((t, x)\), the expansion converges in the sense\(^2\)

\[
\|G_K[u](t, x, \cdot) - u(t, x, \cdot)\|_2 \to 0 \quad \text{for} \quad K \to \infty.
\]

Substituting the expansions in the evolution equations and applying the Galerkin projection lead to a deterministic system for the coefficients of the truncated series, due to the orthogonality of the basis functions, i.e.

\[
\langle \sum_{i=0}^{K} \hat{u}_i(t, x) \phi_i(\xi), \phi_j(\xi) \rangle = \hat{u}_j(t, x)
\]

\(^2\)R.H. Cameron, W.T Martin, *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*. Ann Math, 1947.
Traffic models with uncertainty
Microscopic traffic models

$N$ cars on a infinite road, overtaking not possible

$x_i(t)$ position of car $i$ at time $t$

$v_i(t)$ velocity of car $i$ at time $t$

$a_i(t)$ acceleration of car $k$ at time $t$

$X_1 < X_2 < \ldots < X_N$

Remark

Note that the $N$-th car (the leader) needs a special dynamic because has no one in front of him.
...in formulas

First order:

\[
\begin{cases}
\dot{x}_i(t) = v_i(t) \\
v_i(t) = \begin{cases}
s\left(\frac{L}{x_{i+1}(t)-x_i(t)}\right) & i = 1, \ldots, N-1 \\
\bar{s} & i = N
\end{cases}
\end{cases}
\]

\(s(\Delta x)\) is a given velocity function

Second order model:

\[
\begin{cases}
\dot{x}_i(t) = v_i(t) \\
\dot{v}_i(t) = \begin{cases}
a(x_{i+1}(t), x_i(t), v_{i+1}(t), v_i(t)) & i = 1, \ldots, N-1 \\
\bar{a} & i = N
\end{cases}
\end{cases}
\]

where \(a = C \frac{v_{i+1}(t)-v_i(t)}{\Delta x_i^2(t)} + \frac{A}{t_r} \left(s\left(\frac{L}{\Delta x_i(t)}\right) - v_i(t)\right), C, A, t_r, L > 0\)
**Micro model with uncertainty**

**Uncertainty**: Estimation of the distance between two vehicles at initial time:

\[ x_{i+1}^0 - x_i^0 + \xi \]

\[ \rightarrow x_i(t, \xi) \approx \sum_{k=0}^{K} \hat{x}_{ik}(t) \phi_k(\xi) \]

First order:

\[
\begin{align*}
\dot{x}_i(t, \xi) &= v_i(t, \xi) \\
v_i(t, \xi) &= s \left( \frac{L}{x_{i+1}(t, \xi) - x_i(t, \xi)} \right) \\
v_N &= \bar{s}.
\end{align*}
\]

\[
\begin{align*}
\hat{x}_{ik} &= \hat{v}_{ik} \\
\hat{v}_{ik} &= \hat{s}_{ik} \left( \frac{L}{\Delta x_i} \right) \\
\hat{v}_N &= \bar{s} e_1
\end{align*}
\]

system of \( N \times (K + 1) \) equations

\[ \hat{s}_{ik} = \int_{\Omega} s \left( \frac{L}{x_{i+1} - x_i + \xi} \right) \Phi_k(\xi)p(\xi) d\xi, \text{ if } s \text{ is linear: } \hat{s}_{ik} \left( \frac{L}{\Delta x_i} \right) \approx s \left( \frac{L}{\Delta \hat{x}_{ik}} \right), \]

where \( \Delta \hat{x}_{ik} = \hat{x}_{i+1} - \hat{x}_{ik} \)
Micro model with uncertainty

Second order:

\[
\begin{align*}
\dot{x}_{ik}(t) &= \dot{v}_{ik}(t) \\
\dot{v}_{ik}(t) &= C \left( \mathcal{P}^{-2}(\Delta \ddot{x}_{ik}) \Delta \dot{v}_{ik} \right) + \frac{A}{t_r} \left( s_{ik} - \sum_{k=0}^{\infty} \dot{v}_{ik}(t) \right) \\
\dot{v}_N &= \bar{a}.
\end{align*}
\]

system of \(2N \times (K + 1)\) equations

- \(\mathcal{P}(\hat{u}) := \sum_{\ell=0}^{K} \hat{u}_{\ell} \mathcal{M}_{\ell}\) and \(\mathcal{M}_{\ell} := (\langle \phi_{\ell}, \phi_i \phi_j \rangle)_{i,j=0}^{K}\) is a symmetric matrix of dimension \((K + 1) \times (K + 1)\) for any fixed \(\ell \in \{0, \ldots, K\}\).
Kinetic traffic flow models

$v \in [0, V_M]$ is the velocity

$g(t, x, v)$ is the mass distribution function of traffic

$Q[g]$ models the car–to–car interactions

$\varepsilon > 0$ relaxation rate towards the equilibrium

BGK type models

\[
\partial_t g(t, x, v) + v \partial_x g(t, x, v) = \frac{1}{\varepsilon} Q[g](t, x, v), \quad g(0, x, v) = g_0(x, v)
\]

- $\int_0^{V_M} g_0(x, v) \, dv = \rho_0(x)$
- $Q[g] = M_g(v; \rho) - g$ is the linear operator of BGK$^a$ type
- $M_g(v; \rho)$ describes the distribution at the equilibrium (Maxwellian)

$^a$P. L. Bhatnagar, E. P. Gross, and M. Krook *A Model for Collision Processes in Gases I. Small Amplitude Processes in Charged and Neutral One-Component Systems*, Phys. Rev., 1954
Kinetic model with uncertainty

We are interested in the evolution of \( g(t, x, w, \xi) \):

\[
\frac{\partial_t g(t, x, w, \xi) + \partial_x ([w - h(\rho(\xi))]g(t, x, w, \xi))}{\varepsilon} = \frac{1}{\varepsilon} (M_g(w; \rho(\xi)) - g(t, x, w, \xi))
\]

\[ g(0, x, v, \xi) = g_0(x, v, \xi) \]

Spectral expansion and Galerkin projection (\( \sum_{i=0}^{K} \tilde{g}_i \phi_i(\xi) \)):

\[
\begin{align*}
\partial_t \tilde{g}_i(t, x, w) + \partial_x \left( (wI_d - P(h(\tilde{\rho}))) \tilde{g}(t, x, w) \right) &= \frac{1}{\varepsilon} \left( \tilde{M}_i(w; \tilde{\rho}) - \tilde{g}_i(t, x, w) \right) \\
\tilde{g}_i(0, x, w) &= \int_\Omega g_0(t, x, w, \xi) \phi_i(\xi) p_\Xi(\xi) d\xi
\end{align*}
\]

where \( \forall i = 0, \ldots, K \):

- \( (P(h(\tilde{\rho})) \tilde{g})_i = \sum_{j=0}^{K} \int_\Omega h \left( \sum_{\ell=0}^{K} \tilde{\rho}_\ell \phi_\ell(\xi) \right) \tilde{g}_j \phi_j(\xi) \phi_i(\xi) p_\Xi(\xi) d\xi, \)

- \( \tilde{M}_i(w; \tilde{\rho}(t, x)) = \int_\Omega M_g(w; \sum_{\ell=0}^{K} \tilde{\rho}_\ell(t, x) \phi_\ell(\xi)) \phi_i(\xi) p_\Xi dw d\xi, \)
Macroscopic traffic flow models

\( \rho(x, t) \) density of cars at point \( x \) and time \( t \)

\( v(x, t) \) velocity of cars at point \( x \) and time \( t \)

\( f(x, t) = \rho(x, t)v(x, t) \) flux of cars at point \( x \) and time \( t \)

**First order model: LWR**

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho V_{eq}(\rho)) &= 0, \quad x \in \mathbb{R}, \ t > 0 \\
\rho(0, x) &= \rho_0(x) \quad x \in \mathbb{R}
\end{align*}
\]

It is a hyperbolic conservation law where the velocity depends on the density and typically \( V_{eq}(\rho) = 1 - \rho \),
Macro with uncertainty: LWR

We are interested in the evolution of $\rho(t, x, \xi)$

$$\partial_t \rho(t, x, \xi) + \partial_x (\rho(t, x, \xi) V_{eq}(\rho(t, x, \xi))) = 0$$

$$\rho(0, x, \xi) = \rho_0(x, \xi)$$

Spectral expansion and Galerkin projection ($\sum_{i=0}^{K} \hat{\rho}_i \phi_i(\xi)$)

$$\begin{align*}
\partial_t \hat{\rho} + \partial_x \left( P(\hat{\rho}(t, x)) \hat{V}_{eq}(\hat{\rho}(t, x)) \right) &= \vec{0} \\
\hat{\rho}(0, x) &= \hat{\rho}_0
\end{align*}$$

with $\vec{0} = (0, \ldots, 0)^T$ vector of $K + 1$ components.

Note: an arbitrary but consistent gPC expansion is required for $V_{eq}$, i.e. $V_{eq} = 1 - \rho$ leads to $\hat{V}_{eq}(\hat{\rho}(t, x)) = e_1 - \hat{\rho}$
Macroscopic models

Second order ARZ:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0, \quad x \in \mathbb{R}, \; t > 0 \\
\partial_t (v + h(\rho)) + v \partial_x (v + h(\rho)) &= \frac{1}{\tau} (V_{eq}(\rho) - v), \quad x \in \mathbb{R}, \; t > 0
\end{align*}
\]

in conservative form:

\[
\begin{align*}
\partial_t \rho + \partial_x (z - \rho h(\rho)) &= 0, \quad x \in \mathbb{R}, \; t > 0 \\
\partial_t z + \partial_x (\frac{z^2}{\rho} - z h(\rho)) &= \frac{\rho}{\tau} (V_{eq}(\rho) - v), \quad x \in \mathbb{R}, \; t > 0 \\
v(\rho, z) &= \frac{z}{\rho} - h(\rho)
\end{align*}
\]

- \( h(\rho) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is the hesitation function or traffic pressure law,
- \( \tau > 0 \) (reaction time) makes drivers tend to the equilibrium velocity. In the limit \( \tau \rightarrow 0 \) we recover a first order model where \( v = V_{eq} \)
- the system is strictly hyperbolic if \( \rho > 0 \).
Macro with uncertainty: ARZ

**Naive idea:** substitute the truncated expansions (gPC) into the random system and then use a Galerkin ansatz to project it, i.e. \( \hat{f}(\hat{\rho}(t, x)) = \langle f(\sum_{k}^{K} \hat{\rho}_k(t, x)\phi_k(\cdot)), \phi_i(\cdot) \rangle_{i=0,\ldots,K} \)

**BUT** here the Jacobian of the flux function consists of the projected entries of the deterministic Jacobian \( \Rightarrow \) not necessarily real eigenvalues and full set of eigenvectors \( \Rightarrow \) LOSS of hyperbolicity
gPC formulation for ARZ

To solve the problem:

- more assumptions on the basis functions and a change of variable, i.e. $\hat{z}^\rho$
- derive the ARZ from the BGK approximation

### gPC formulation for ARZ

$$\begin{align*}
\partial_t \hat{\rho}_i(t, x) + \partial_x \left[ \hat{z}_i(t, x) - (\mathcal{P}(\hat{\rho}(t, x))\hat{\rho}(t, x))_i \right] &= 0 \\
\partial_t \hat{z}_i(t, x) + \partial_x \left[ (\mathcal{P}(\hat{z}(t, x))\mathcal{P}^{-1}(\hat{\rho}(t, x))\hat{z}(t, x))_i - (\mathcal{P}(\hat{\rho}(t, x))\hat{z}(t, x))_i \right] &= \frac{1}{\tau} \left( (\mathcal{P}(V_{eq}(\hat{\rho}(t, x)))\hat{\rho}(t, x) + \mathcal{P}(h(\hat{\rho}(t, x)))\hat{\rho}(t, x))_i - \hat{z}_i(t, x) \right) \\
i &= 0, \ldots, K
\end{align*}$$

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$^3$S. Gerster, M. Herty, E. I., *Stability analysis of a hyperbolic stochastic Galerkin formulation for the Aw-Rascle-Zhang model with relaxation*, MBE, 2021.
From Micro to Macro

Theorem (E.I.)

Let $\xi$ be a random variable and be $N$ cars of fixed length $L$. Assume that $s\left(\frac{L}{\Delta x}\right) = v(\rho)$. Then the stochastic ODEs system

$$
\begin{aligned}
\dot{x}_i(t,\xi) &= v_i(t,\xi) \\
v_i(t,\xi) &= s\left(\frac{L}{x_{i+1}(t,\xi)-x_i(t,\xi)}\right) \\
v_N &= \bar{s}.
\end{aligned}
$$

converges to the stochastic LWR model

$$
\partial_t \rho(t,x,\xi) + \partial_x (\rho(t,x,\xi) V(\rho(t,x,\xi))) = 0
$$

$$
\rho(0,x,\xi) = \rho_0(x,\xi)
$$

for $L \to 0$ and $N \to \infty$.

Note: the same can be proven for the second order model.
From Kinetic to Macro

Theorem (M. Herty, E.I.)

Let \( \tilde{g}_i \) be a strong solution for the kinetic model for \( i = 0, \ldots, K \). Under some technical assumptions, the first and the second moment of \( \tilde{g}_i \), \((\tilde{\rho}, \tilde{z})\), formally fulfill pointwise in \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\) and for all \( i = 0, \ldots, K \) the second–order traffic flow model

\[
\begin{align*}
\partial_t \tilde{\rho}_i(t, x) + \partial_x \left[ \tilde{z}_i(t, x) - (\mathcal{P}(\tilde{\rho}(t, x))\tilde{\rho}(t, x))_i \right] &= 0 \\
\partial_t \tilde{z}_i(t, x) + \partial_x \left[ (\mathcal{P}(\tilde{z}(t, x))\mathcal{P}^{-1}(\tilde{\rho}(t, x))\tilde{z}(t, x))_i - (\mathcal{P}(\tilde{\rho}(t, x))\tilde{z}(t, x))_i \right] &= \\
&= \frac{1}{\epsilon} \left( (\mathcal{P}(V_{eq}(\tilde{\rho}(t, x)))\tilde{\rho}(t, x) + \mathcal{P}(h(\tilde{\rho}(t, x))))\tilde{\rho}(t, x) \right)_i - \tilde{z}_i(t, x) \\
\tilde{\rho}_i(0, x) &= \int_W \tilde{g}_{0,i}(t, x, w)dw, \quad \tilde{z}_i(0, x) = \int_W w \tilde{g}_{0,i}(t, x, w)dw.
\end{align*}
\]

Moreover, the system is hyperbolic for \( \tilde{\rho}_i > 0 \) and the solution is also a solution of the stochastic ARZ model.

\(^a\)S. Gerster, M. Herty, E. I., *Stability analysis of a hyperbolic stochastic Galerkin formulation for the Aw-Rascle-Zhang model with relaxation*, MBE, 2021.
Diffusion coefficient

Starting from

\[ \partial_t g(t, x, w, \xi) + \partial_x \left[ (w - h(\rho(t, x, \xi))) g(t, x, w, \xi) \right] = \frac{1}{\varepsilon} \left( M_g(w; \rho) - g(t, x, w, \xi) \right) \]

- assume \( \varepsilon > 0 \): small but positive.
- perform a first-order Chapman Enskog approximation
  \[ g(t, x, w, \xi) = M_g(w; \rho(t, x, \xi)) + \varepsilon g_1(t, x, w, \xi) \]
- obtain an advection-diffusion equation \(^4\).
  \[ \partial_t \rho + \partial_x (\rho V_{eq}(\rho)) = \varepsilon \partial_x (\mu(\rho) \partial_x \rho), \quad \rho = \rho(t, x, \xi), \]
  \[ \mu(\rho) = \left( -\partial_\rho Q_{eq}(\rho)^2 - \partial_\rho h(\rho) \partial_\rho Q_{eq}(\rho) \rho + Q_{eq}(\rho) \partial_\rho h(\rho) \right) + \int_V v^2 \partial_\rho M_f(v, \rho) dv \]

Tool for studying possible instabilities

\[ \mathbb{P}_{t,x}(\mu \leq 0) := \int_\Omega H(-\mu(\rho(t, x, \xi))) p\Xi(\xi) d\xi. \]

\(^4\text{M. Herty, G. Puppo, S. Roncoroni, G. Visconti, The BGK approximation of kinetic models for traffic, Kinetic & Related Models, 2020.}\)
Numerics
Numerical settings

- $\xi \sim U(0, 1)$,
- Basis choice: Haar basis

\[ \psi(\xi) := \begin{cases} 
1 & \text{if } 0 \leq \xi < \frac{1}{2}, \\
-1 & \text{if } \frac{1}{2} \leq \xi < 1, \\
0 & \text{else.} 
\end{cases} \]

Using a lexicographical order we identify the gPC basis

$\phi_0 = 1$, $\phi_1 = \psi$, $\phi_2 = \psi_{1,0}$, $\phi_3 = \psi_{1,1}$, \ldots

- $\Delta x = 2 \cdot 10^{-2}$ on the space interval $[0, 2]$,
- $T_f = 1$ and $\Delta t$ fulfills the CFL condition,
- $h(\rho) = \rho$,
- $V_{eq}(\rho) = 1 - \rho$. 
Uncertainty quantification in hierarchical vehicular flow models
E. Iacomini | Università di Ferrara
Numerical settings

Initial data

Rarefaction wave:

\[
\rho(x, 0, \xi) = \begin{cases} 
0.55 + 0.3\xi & \text{for } x < 1, \\
0.3 & \text{for } x > 1,
\end{cases} \quad v(x, 0, \xi) = \begin{cases} 
0.2 & \text{for } x < 1, \\
0.7 & \text{for } x > 1.
\end{cases}
\]

Clever idea

We compute offline in a precomputation step the entries of the matrices \( P(\cdot) \) and the tensor \( \mathcal{M} \) \( \Rightarrow \) not computationally expensive.
Numerical convergence in $K$

**Mean of $\rho(\xi)$**

- $K=4$
- $K=8$
- $K=32$
- $K=64$

**Variance of $\rho(\xi)$**

- $K=4$
- $K=8$
- $K=32$
- $K=64$
Application: detect high risk regions

\[ P_{t,x}(\mu \leq 0) := \int_{\Omega} H(-\mu(\rho(t, x, \xi)))p_{\Xi}(\xi) d\xi. \]
Conclusion and future perspectives

Recap

- Uncertainty is introduced in traffic flow models to improve traffic forecast.

- Micro, kinetic and macroscopic scales are investigated and the convergence to the latter one is shown. Moreover the obtained formulation preserves hyperbolicity.

- The stability analysis is performed and the diffusion coefficient is studied.

- Numerical simulations illustrate the theoretical results.

What’s next

- Use real data to estimate $\xi$.

- Study the uncertainty in the non-local case.

- Study the uncertainty via "efficient" data-friendly non-intrusive methods.
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Thank you for your kind attention!

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Numerical scheme

Idea: employ the local Lax Friedrichs scheme to solve SG-ARZ combined with an IMEX scheme\(^1\) in the inhomogeneous case. The source term is treated implicitly, due to the stiffness while the convective term is treated explicitly for \(l = 0, \ldots, K, \ j = 0, \ldots, N:\)

\[
\begin{align*}
\bar{\rho}^{n+1}_{j,l} &= \bar{\rho}^{(1)}_{j,l} - \frac{\Delta t}{\Delta x} \left( F_j + \frac{1}{2} \left( \bar{\rho}^{(1)}, \bar{z}^{(1)} \right) - F_{j-1} - \frac{1}{2} \left( \bar{\rho}^{(1)}, \bar{z}^{(1)} \right) \right) \\
\bar{z}^{n+1}_{j,l} &= \bar{z}^{(1)}_{j,l} - \frac{\Delta t}{\Delta x} \left( F_j + \frac{1}{2} \left( \bar{\rho}^{(1)}, \bar{z}^{(1)} \right) - F_{j-1} - \frac{1}{2} \left( \bar{\rho}^{(1)}, \bar{z}^{(1)} \right) \right)
\end{align*}
\]

Implicit step

\[
\begin{align*}
\bar{\rho}^{(1)}_{j,l} &= \bar{\rho}^n_{j,l} \\
\bar{z}^{(1)}_{j,l} &= \frac{\tau}{\tau + \Delta t} \bar{z}^n_{j,l} + \frac{\Delta t}{\tau + \Delta t} \left( \mathcal{P}(\bar{\rho}^n) \hat{V}_{eq}^n + \mathcal{P}(\bar{\rho}^n) \bar{\rho}^n \right)
\end{align*}
\]

where \(F_j \pm \frac{1}{2}\) is the numerical flux of the local Lax Friedrichs scheme.

\(^1\)L. Pareschi and G. Russo, *Implicit–explicit runge–kutta schemes and applications to hyperbolic systems with relaxation*, Journal of Scientific computing, 2005
# Hyperbolic formulation

## Assumptions on basis functions

A1) The precomputed matrices $M_\ell$ and $M_k$ commute for $\ell, k = 0, \ldots, K$.

A2) There is an eigenvalue decomposition $P(\hat{u}) = VD(\hat{u})V^T$ with constant eigenvectors.

A3) The matrices $P(\hat{u})$ and $P(\hat{y})$ commute for all $\hat{u}, \hat{y} \in \mathbb{R}^{K+1}$.

## SG hyperbolic preserving formulation

Moreover, assuming $h(\rho) = \rho^\gamma, \gamma = \{1, 2\}$, so $\hat{h}(\hat{\rho}) = P^{\gamma-1}(\hat{\rho})\hat{\rho}$, and its Jacobian of the form $\hat{h}'(\hat{\rho}) = VD_{\hat{h}'}(\hat{\rho})V^T$, we get

\[
\begin{cases}
\partial_t \hat{\rho} + \partial_x \left( \hat{z} - P(\hat{\rho})\hat{h}(\hat{\rho}) \right) = \vec{0}
\partial_t \hat{z} + \partial_x \left( P(\hat{z})P^{-1}(\hat{\rho})\hat{z} - P(\hat{z})\hat{h}(\hat{\rho}) \right) = \vec{0}.
\end{cases}
\]
Main result

Theorem 1

Let a gPC expansion with the properties (A1) – (A3), a stochastic Galerkin formulation of a hesitation function $\hat{h}(\hat{\rho})$ and a Galerkin formulation of an equilibrium velocity $\hat{V}_{eq}(\hat{\rho})$ be given. Assume further a Jacobian of the hesitation function

$$
\hat{h}'(\hat{\rho}) = D_{\hat{\rho}} \hat{h}(\hat{\rho}) = V D_{h'}(\hat{\rho}) V^T
$$

with constant eigenvectors.

Then, for smooth solutions (??) and (??) are equivalent and strongly hyperbolic. The characteristic speeds are

$$
\hat{\lambda}_1(\hat{\rho}, \hat{z}) = D(\hat{v}(\hat{\rho}, \hat{z})) - D_{h'}(\hat{\rho}) D(\hat{\rho}) \quad \text{and} \quad \hat{\lambda}_2(\hat{\rho}, \hat{z}) = D(\hat{v}(\hat{\rho}, \hat{z}))
$$

for $\hat{v}(\hat{\rho}, \hat{z}) = P^{-1}(\hat{\rho}) \hat{z} - \hat{h}(\hat{\rho})$, where $D(\hat{v})$ denote the eigenvalues of the matrix $P(\hat{v})$. 
Stability analysis

Theorem 2

Under the same assumptions of the previous Theorem, the first-order correction to the local equilibrium approximation reads

\[
\partial_t \hat{\rho} + \partial_x \hat{f}_{eq}(\hat{\rho}) = \tau \partial_x (\hat{\mu}(\hat{\rho}) \partial_x \hat{\rho}), \quad \hat{f}_{eq}(\hat{\rho}) = \hat{\rho} \ast \hat{V}_{eq}(\hat{\rho})
\]

\[
\hat{\mu}(\hat{\rho}) = V \left[ D(\hat{\rho})^2 D_{V_{eq}}(\hat{\rho}) \left( D_{V_{eq}}(\hat{\rho}) + D_{h}(\hat{\rho}) \right) \right] V^T.
\]

Furthermore, it is dissipative if and only if the sub-characteristic condition

\[
\hat{\lambda}_1(\hat{\rho}, \hat{z}) \leq \hat{f}'_{eq}(\rho) \leq \hat{\lambda}_2(\hat{\rho}, \hat{z})
\]

holds on \( \hat{z} = \hat{\rho} \ast \left( \hat{V}_{eq}(\hat{\rho}) + \hat{h}(\hat{\rho}) \right) \) with \( D_{V_{eq}}(\hat{\rho}) < \bar{\rho} \).