ON COSETS OF CENTRALIZER OF INVOLUTIONS IN ALTERNATING GROUPS

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Abstract. In this article, we provide counterexamples to a conjecture of M. Pellegrini and P. Shumyatsky which states that each coset of the centralizer of an involution in a finite non-abelian simple group $G$ contains an odd order element, unless $G = PSL(n, 2)$ for $n \geq 4$. More precisely, we show that the conjecture does not hold for the alternating group $A_{2n}$ for all $n \geq 4$.

1. Introduction

The subgroup of a group $G$ generated by the set of commutators $\{(a, b) = aba^{-1}b^{-1} \mid a, b \in G\}$ in $G$ is called the commutator subgroup of $G$, and is usually denoted by $[G, G]$ or $G'$. The subgroup $G'$ is normal in $G$, and therefore for a non-abelian simple group $G$, we have $G' = G$. It was proved by O. Ore in [Ore51] that any element in the alternating group $A_n$ ($n \geq 5$) is a single commutator. Further, he conjectured that any element of a finite non-abelian simple group is a single commutator. This conjecture, called the Ore’s conjecture, is now solved in the affirmative owing to the contributions of several mathematicians (see [LOST10]). We refer the readers to an excellent article by G. Malle (see [Mal14] and the references therein) on Ore’s conjecture.

For a finite group $G$, an element $g \in G$ is called a coprime commutator if $g = [x, y]$ where $x, y \in G$ are elements whose orders are coprime. The authors in [PS12] proved the following:

Theorem 1.1 (Theorem 1.1, [PS12]). Let $q > 3$ be a prime-power. Then every element of $PSL(2, q)$ is a coprime commutator.

In the same article, they conjecture that the above holds true for all finite non-abelian simple groups. The following result played a crucial role in the proof of the above theorem.

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Theorem 1.2 (Theorem 1.2, [PS12]). Let $q > 3$ be a prime-power. Then every coset of the centralizer of an involution in $\text{PSL}(2, q)$ has an odd order element.

In the light of the above result, they further conjectured the following:

Conjecture 1.3. [Conjecture 1.4, [PS12]] Each coset of the centralizer of an involution in a finite non-abelian simple group has an odd order element, unless $G = \text{PSL}(n, 2)$ for $n \geq 4$.

They showed that $G = \text{PSL}(n, 2)$ for $n \geq 4$ is indeed an exception to the above conjecture. In [Shu15], the author showed that every element in the alternating group $A_n$ ($n \geq 5$) is a coprime commutator by direct computations. Recently, G. Zini (see [Zin21]) has verified Conjecture 1.3 for the Suzuki groups $^2B_2(q)$ where $q = 2^{2n+1}$ ($n \geq 1$). Further, he uses this result to prove that every element in the simple group $^2B_2(q)$ is a coprime commutator.

In this article, we consider the alternating groups $A_{2n}$ ($n \geq 4$). For each of these groups, we exhibit an involution and a coset of the centralizer of that involution consisting entirely of even order elements. This is an infinite family of counterexamples to Conjecture 1.3. The main theorem is as follows:

Theorem 1.4. Let $N = 2^n$, $n \geq 3$. Let $x = (1 \ 2)(3 \ 4) \cdots (2^{n-1} - 1 \ 2^{n-1}) \in A_N$. Let $Z_{A_N}(x)$ denote the centralizer of $x$ in $A_N$. Let $y = (1 \ 2^n + 1)(2 \ 2^{n-1} + 2) \cdots (2^{n-1} \ 2^n) \in A_N$. Then each element in the coset $yZ_{A_N}(x)$ is a product of disjoint cycles, each of even length. In particular, each element in the coset $yZ_{A_N}(x)$ has even order.

Remark 1.5. A permutation $\pi$ has even order if and only if it contains an even length cycle in its disjoint cycle decomposition. Thus, in order to disprove the conjecture we just need to show that every element in $yZ_{A_N}(x)$ has at least one cycle of even length in its disjoint cycle decomposition. However, in the above theorem we prove a much stronger statement that each element in the coset $yZ_{A_N}(x)$ is a product of disjoint cycles, each of even length.

2. Proof of the main theorem

We fix some notations which will be used throughout the paper. Let $N = 2^n$, $n \geq 3$. Let $[N]$ denote the set $\{1, 2, \ldots, N\}$. We consider the alternating group $A_N$ on $[N]$. Let $x = (1 \ 2)(3 \ 4) \cdots (2^{n-1} - 1 \ 2^{n-1}) \in A_N$. Let $Z_{A_N}(x)$ be the centralizer of $x$ in $A_N$. Take $y = (1 \ 2^n + 1)(2 \ 2^{n-1} + 2) \cdots (2^{n-1} \ 2^n) \in A_N$. We divide $[N]$
into two disjoint subsets $A = \{1, 2, \ldots, 2^{n-1}\}$ and $B = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n\}$, each of size of $2^{n-1}$. We say that a permutation $\pi \in S_N$ fixes $A$ if for all $a \in A$, $\pi(a) \in A$. This immediately implies $\pi$ fixes $B$ as well.

We start with a general lemma.

**Lemma 2.1.** Let $z \in A_N$ be such that $z$ fixes $A$. Then $\tau = yz$ is a product of disjoint cycles each of even length.

**Proof.** Let $z \in A_N$ be as in the statement of the lemma. We note that $y(a) \in B$ for all $a \in A$, and $y(b) \in A$ for all $b \in B$. The same holds for $\tau$ since $z$ fixes $A$ and $B$. Let $a_1$ be any element of $A$. Then $\tau(a_1) = b_1$ for some $b_1 \in B$. We know that $\tau(b_1) \in A$. If $\tau(b_1) = a_1$, then the cycle containing $a_1$ in the disjoint cycle decomposition of $\tau$ is $(a_1, b_1)$ which is of even length and we are done. If not, then $\tau(b_1) = a_2$ for some $a_2 \in A \setminus \{a_1\}$. Then $\tau(a_2) = b_2$ for some $b_2 \in B \setminus \{b_1\}$. If $\tau(b_2) = a_1$, then the cycle containing $a_1$ is $(a_1 b_1 a_2 b_2)$ which is of even length and we are done. If not, we proceed as before and after a finite number of steps, the cycle containing $a_1$ in the disjoint cycle decomposition of $\tau$ is of the form $(a_1 b_1 \cdots a_k b_k)$ for some $k, 1 \leq k \leq 2^{n-1}$, which is of even length. Similarly for any $b \in B$, the cycle containing $b$ in the disjoint cycle decomposition of $\tau$ is of even length. This completes the proof. \hfill $\Box$

**Proof of Theorem 1.4.** Let $z \in Z_{A_N}(x)$. We claim that $z$ fixes $A$. Suppose, on the contrary, that $z$ does not fix $A$. Then there exists $a \in A$ such that $z(a) = b$ for some $b \in B$. Then,

$$z^{-1}(b) = a \implies xz^{-1}(b) = a' \text{ for some } a' \in A \setminus \{a\}$$

$$\implies zxz^{-1}(b) = z(a') \implies x(b) = z(a') \implies b = z(a').$$

This is a contradiction as $a' \neq a$ and $z(a) = b$. This proves our claim that $z$ fixes $A$. The proof of the theorem now follows from Lemma 2.1. \hfill $\Box$

**Remark 2.2.** Both Lemma 2.1 and Theorem 1.4 remains true if we replace $A_N$ by $S_N$, the symmetric group on $[N]$.

**Remark 2.3.** Note that $A_8 \cong PSL(4, 2)$ which already appears in Conjecture 1.3 as an exception. For $n \geq 4$, $A_{2n}$ is not isomorphic to any of the simple groups $PSL(m, 2)$ for $m \geq 4$ and hence is a counterexample to the conjecture.

We write an example for the purpose of clarity. Before proceeding with the example we fix a notation. Given $n \in \mathbb{N}$, we denote the cycle type of a permutation
\(\pi \in S_n\) as \(1^{c_1}2^{c_2} \cdots\), where \(c_i\) denotes the number of \(i\)-cycles in the disjoint cycle decomposition of \(\pi\). Clearly, \(1^{c_1}2^{c_2} \cdots\) satisfies \(\sum_i ic_i = n\), whence \(1^{c_1}2^{c_2} \cdots\) is a partition of \(n\). We know that the cycle type of a permutation \(\pi \in S_n\) determines \(\pi\) up to conjugacy in \(S_n\).

**Example 2.4.** For \(N = 2^3 = 8\), we have the following: \(x = (1 \ 2)(3 \ 4) \in A_8\) and \(y = (1 \ 5)(2 \ 6)(3 \ 7)(4 \ 8) \in A_8\). Then, the centralizer is given by

\[Z_{A_8}(x) = \langle (1 \ 3)(2 \ 4), (1 \ 2)(3 \ 4), (1 \ 2)(5, 8), (1 \ 2)(6 \ 8), (1 \ 2)(7 \ 8) \rangle.\]

We have \(|Z_{A_8}(x)| = 96\). We used GAP to get the list of all 96 elements in the coset \(yZ_{A_8}(x)\). The cycle type of the elements occurring in \(yZ_{A_8}(x)\) belongs to the set \(\{2^4, 2^16^1, 4^2\}\) of partitions of 8. We conclude that the elements in \(yZ_{A_8}(x)\) have orders 2, 4, and 6.

For \(N = 2^4 = 16\), we have the following: \(x = (1 \ 2)(3 \ 4)(5 \ 6)(7 \ 8) \in A_{16}\) and \(y = (1 \ 9)(2 \ 10)(3 \ 11)(4 \ 12)(5 \ 13) (6 \ 14)(7 \ 15)(8 \ 16) \in A_{16}\). We have \(|Z_{A_{16}}(x)| = 7741440\). Using GAP we see that the cycle type of the elements occurring in \(yZ_{A_{16}}(x)\) belongs to the set

\[X = \{2^8, 2^44^2, 2^26^2, 2^61^1, 2^24^18^1, 2^310^1, 6^110^1, 4^112^1, 2^114^1, 2^42^61, 8^2, 4^4\},\]

of partitions of 16. It immediately follows that the orders of elements in \(yZ_{A_{16}}(x)\) belongs to the set \(\{2, 4, 6, 8, 10, 12, 14, 30\}\).

We conclude this article with a final remark.

**Remark 2.5.** The involution \(y \in A_N\) in Theorem 1.4 is chosen such that it has no fixed points. Such a choice is possible only when \(N\) is of the form \(2^n\) for \(n \geq 3\). GAP computations strongly indicate that for the family of alternating groups \(A_n\), these might well be the only exceptions to Conjecture 1.3.

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