Compact space and infrared behavior of the effective QCD

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Abstract

We aim to investigate the infrared regime in the effective theory of the new quadratic gauge in the physical compact space by defining it on 4-sphere. Abelian dominance is characterized by off-diagonal gluons acquiring dynamical masses and it hints at existence of confinement. We do observe Abelian dominance in the ghost condensed vacuum of this theory on 4-sphere. Along with this observation, we find an unusual result that mass of an off-diagonal gluon on 4-sphere is position dependent in this theory as a consequence of the curved geometry. This suggests that the curvature of 4-sphere does not change the infrared behaviour of present theory on 4-sphere from that of the theory with the same quadratic gauge in Euclidean spacetime as Abelian dominance is observed in Euclidean spacetime too while it has caused mass of an off-diagonal gluon on 4-sphere to be position dependent. The effective action in the confined phase on 4-sphere and that in the 4-dim Euclidean space are found to be identical in a profound outcome.

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I. INTRODUCTION

The studies of gauge theory on the compact manifold have been of great theoretical interest with physical relevance motivated by experimental circumstances, (see e.g., Ref. [1] and refs therein). Most such studies are on lattice and not analytical whereas we here attempt an analytical study. The 4-sphere is a compact space therefore, an effective theory on 4-sphere provides a model to study consequences of compactness of physical space on phenomena in QCD. Thus, the present theory on 4-sphere is purposed to be a model to investigate confinement in the finite physical volume analytically. The idea of formulating gauge theories on 4-sphere and hypersphere in general commences with a study of massless Euclidean QED (quantum electrodynamics) on a hypersphere in 5-dimensional Euclidean space[2]. It was argued there that compactification of space-time leads to infrared-finiteness of this theory. Thereafter manifestly $O(n)$-covariant formulation of gauge theories on a hypersphere have been considered at a few occasions in different contexts[3–6, 10–12]. For example, the manifestly $O(n)$-covariant formulation was reconstructed using conformal Killing vectors in Ref.[11]. To the best of our knowledge, non perturbative inspection in QCD on 4-sphere has never been carried out. We are required to introduce a suitable gauge-fixing in this theory to analyze the underlying field theoretical properties of a gauge theory. In the study of massless Euclidean QED on a hypersphere, the following gauge condition was deliberately chosen such that analysis retains a manifest $O(n)$-covariance[2],

$$(r_\alpha \partial_\beta - r_\beta \partial_\alpha) \bar{A}_\beta = \bar{A}_\alpha; \ r_\alpha \text{ is cartesian coordinate, } r_\alpha r_\alpha = 1, \alpha = 1, 2, ..., n.$$  \hspace{1cm} (1)

Here $\bar{A}$ is the gauge field on the hypersphere. With this condition, extension of the analysis to the non abelian gauge theory is straight forward. However, the gauge is inconsistent for gauge fixing procedure on sphere as it has got an extra free index. The gauge given in Eq.(2) rectifies the flaw since it does not have any free index and is equivalent of the gauge in Eq.(1)[10],

$$r_\alpha (r_\alpha \partial_\beta - r_\beta \partial_\alpha) \bar{A}_\beta = r_\alpha i M_{\alpha \beta} \bar{A}_\beta = 0.$$ \hspace{1cm} (2)

This gauge despite being equivalent to that in Eq.(1) has more similar form to $\partial_\mu A_\mu = 0$ than the former. One can fairly expect that compactification regularizes infrared divergences in QCD also. For this reason, QCD on 4-sphere $S^4$ would be more appropriate for analyzing
properties of QCD at a low-energy region. This provides us the motive to apply the proposed
gauge on $S^4$ as it has already shown nice low-energy attributes in the flat 4-dim space.

The quadratic gauge has been extensively studied in various settings recently\(^{13-19}\) and
found to have substantial implications in non-perturbative sector of QCD. It is given as
follows

$$H^a[A^\mu(x)] = A^a_\mu(x)A^\mu_a(x) = f^a(x); \text{ for each } a,$$

where $f^a(x)$ is an arbitrary function of $x$. The Faddeev-Popov action in the flat space is
given as

$$\mathcal{L}_Q = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu_a} + \frac{\zeta}{2} F^{a2} + F^a A^a_\mu A^\mu_a - 2\bar{c}^a A^\mu_a (D_\mu c)^a,$$

where the field strength $F_{\mu\nu}^a = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g f^{abc} A^b_\mu(x)A^c_\nu(x)$, the $F^a$ are a set of
auxiliary fields, $c, \bar{c}$ are ghost and anti-ghost fields respectively, $\zeta$ is an arbitrary gauge fixing
parameter and $(D_\mu c)^a = \partial_\mu c^a - g f^{abc} A^b_\mu c^c$. The summation over indices $a, b$ and $c$ each runs
independently over 1 to $N^2 - 1$ in Eq.(4).

Now we present the plan of the paper. Next section reviews Yang-Mills theory on a
general hypersphere and the gauge fixing procedure for condition in Eq.(2) using BRST
invariance as the first principle. In section III however we limit our selves to 4-sphere in
order to render the study relevant to physical situation i.e., the compact space study of the
phenomenon under consideration. We develop the effective theory in the quadratic gauge on
4-sphere with the help of Faddeev-Popov procedure rather than BRST invariance as the first
principle and probe non trivial ghost vacua in this theory. In the last section, we conclude
this letter.

II. SU(N) QCD ON HYPERSPHERE

Here we review the manifestly $O(n)$ covariant formalism of QCD on $(n-1)$-dimensional
hypersphere, $S^{n-1}$. The sphere is embedded in the $n$-dimensional Euclidean space $\mathbb{R}^n$. We
consider a unit radius $S^{n-1}$ to bring neatness in mathematical expressions as physics does
not depend upon the size. So we have the constraint, $r_\alpha r_\alpha = 1$ with $r_\alpha$ being the cartesian
coordinate of the point on $S^{n-1}$ and $\alpha = 1, 2, ..., n$. Therefore, the coordinate $r_\alpha$ is dependent
variable since $r_\alpha = \pm \sqrt{1 - r_\mu r_\mu}$ where $\mu = 1, 2, ..., n - 1$. For QCD on hypersphere, the
propagation of fields remains transverse to radial direction. Spherical symmetry in the system preserves the angular momentum, therefore the fundamental operator that governs the dynamics is the angular momentum operator given as below

\[ M_{\alpha\beta} = -i (r_{\alpha} \partial_{\beta} - r_{\beta} \partial_{\alpha}); \quad \partial_{\beta} \equiv \frac{\partial}{\partial r_{\beta}}, \quad \alpha, \beta = 1, 2, ..., n. \]  

(5)

In particular,

\[ M_{\mu n} = -M_{n\mu} = ir_n \partial_{\mu}, \]  

(6)

since \( \partial_{r_n} = 0 \) by definition regardless of what it operates on as \( r_n \) is dependent variable, we shall come to this point again and relax this situation a bit without disturbing the physics. The operator \( M_{\alpha\beta} \) satisfies the following Lie algebra

\[ [M_{\alpha\beta}, M_{\gamma\eta}] = i(\delta_{\alpha\gamma} M_{\beta\eta} - \delta_{\beta\gamma} M_{\alpha\eta} - \delta_{\alpha\eta} M_{\beta\gamma} + \delta_{\beta\eta} M_{\alpha\gamma}). \]  

(7)

Now we introduce stereographic coordinates \( \{x_{\mu}, \mu = 1, 2, ..., n-1\} \) on the hyperplane \( \mathbb{R}^{n-1} \). They map the coordinates for the point on a sphere, \( r_\mu \) as

\[ r_\mu = \frac{2x_\mu}{1 + x^2}, \quad r_n = \frac{1 - x^2}{1 + x^2}, \quad x^2 \equiv x_\mu x_\mu. \]  

(8)

The gauge field on \( S^{n-1} \) is Lie algebra valued i.e., \( \bar{A}_{\alpha} = \bar{A}_{\alpha} T^a \), where \( T^a \) are generators of SU(N) group. The gauge field \( \bar{A} \) maps to the gauge field \( A \) on the stereographic hyperplane as follows[3]

\[ \bar{A}_\mu(r) = \frac{1 + x^2}{2} A_\mu(x) - x_\mu x_\nu A_\nu(x), \quad \bar{A}_n = -x_\mu A_\mu; \quad \mu, \nu = 1, 2, ..., n-1. \]  

(9)

As we are in Euclidean frame we do not bother about upper and lower indices. The following transversality condition is inherent in the QCD on hypersphere as it results from innate stereographic projections in Eqs.(8),(9)

\[ r_\alpha \bar{A}_\alpha = 0 \Rightarrow r_\alpha \bar{A}_\alpha^a = 0, \quad \alpha = 1, 2, ..., n. \]  

(10)

It implies that the gluon \( \bar{A}^a \) lives on the tangent space of the \( S^{n-1} \). Therefore, the equivalence of gauges in Eqs.(1),(2) is apparent.

The structure of a gauge transformation encodes the effect of the spatial geometry on the effective theory. Therefore, the effective theory in the same gauge on different geometries
has different forms. Let us now turn to the gauge transformation of the gluon on $S^{n-1}$ as follows

$$\delta \bar{A}^a_\beta = r_\alpha (r_\alpha D_\beta - r_\beta D_\alpha) \epsilon^a = r_\alpha i \mathcal{M}_{\alpha \beta} \epsilon^a,$$  \hspace{1cm} (11)

where $\epsilon$ is a parameter of transformation, $D_\beta \epsilon^a = \partial_\beta \epsilon^a - gf^{abc} \bar{A}^b_\beta \epsilon^c$ is a usual covariant derivative of a parameter. We note that the operator in the bracket, $\mathcal{M}_{\alpha \beta}$ is a covariantized version of the angular momentum operator due to local symmetry. Contracting Eq.(11) with $r_\beta$ we get zero which confirms that the infinitesimal gauge transformation is always tangential to $S^{n-1}$. Thus, the transformed and original field both live on the same tangent space. The Eq.(11) can be simplified a bit as follows

$$\delta \bar{A}^a_\beta = (D_\beta - r_\alpha r_\beta D_\alpha) \epsilon^a, \quad \text{since} \quad r_\alpha r_\alpha = 1.$$  \hspace{1cm} (12)

The conventional field strength $\bar{F}^a_{\alpha \beta}$ of $\bar{A}$ as defined in the introduction is not $O(n)$ covariant under the transformation in Eq.(12) as we see from the expression below

$$\delta \bar{F}^a_{\alpha \beta} = f^{abc} \bar{F}^b_{\alpha \beta} \epsilon^c + r_\alpha \left( D_\beta + \frac{1}{r_n} \delta_{\beta n} \right) r_\mu \partial_\mu \epsilon^a - r_\beta \left( D_\alpha + \frac{1}{r_n} \delta_{\alpha n} \right) r_\mu \partial_\mu \epsilon^a.$$  \hspace{1cm} (13)

We notice that the last two terms are not manifestly covariant. To get rid of such terms, a rank-3 tensor is required. We can deduce the following from Eq.(13)

$$r_\gamma \delta \bar{F}^a_{\alpha \beta} + r_\beta \delta \bar{F}^a_{\gamma \alpha} + r_\alpha \delta \bar{F}^a_{\beta \gamma} = f^{abc} (r_\gamma \bar{F}^b_{\alpha \beta} + r_\beta \bar{F}^b_{\gamma \alpha} + r_\alpha \bar{F}^b_{\beta \gamma}) \epsilon^c,$$  \hspace{1cm} (14)

which is covariant manifestly. Therefore we define a rank-3 tensor as

$$\bar{F}^a_{\alpha \beta \gamma} = r_\gamma \bar{F}^a_{\alpha \beta} + r_\beta \bar{F}^a_{\gamma \alpha} + r_\alpha \bar{F}^a_{\beta \gamma}.$$  \hspace{1cm} (15)

With the field strength $\bar{F}^a_{\alpha \beta \gamma}$, the gauge invariant QCD action on $S^{n-1}$ is given by

$$S_{YM} = -\frac{1}{12} \int d\Omega \; \bar{F}^a_{\alpha \beta \gamma} \bar{F}^a_{\alpha \beta \gamma}.$$  \hspace{1cm} (16)

Here $d\Omega$ is an invariant measure on $S^{n-1}$ which is given in terms of the coordinates $(r_\mu)$ as follows

$$d\Omega = \frac{1}{|r_n|} \prod_{\mu=1}^{n-1} dr_\mu.$$  \hspace{1cm} (17)
We can now analyze the effective theory of the condition in Eq. (2) using the BRST invariance [7–9] as the first principle [10]. In this gauge fixing procedure, we write the gauge fixing and ghost term to be added in Eq. (16) as a total BRST differential as shown below

\[ S_{GF} + S_{ghost} = \int d\Omega \delta \left[ \tilde{c}^a (r_\alpha i M_{\alpha \beta} \tilde{A}_\beta^a + \frac{\zeta}{2} \tilde{B}^a) \right]. \]  

(18)

The overbar indicates fields on sphere. The BRST transformations are given as below

\[
\begin{align*}
\delta \tilde{A}_\beta^a &= (D_\beta - r_\alpha r_\beta D_\alpha) \tilde{c}^a = r_\alpha i M_{\alpha \beta} \tilde{c}^a, \\
\delta \tilde{c}^a &= \frac{1}{2} f^{abc} \tilde{c}^b \tilde{c}^c, \\
\delta \tilde{c}^a &= \tilde{B}^a, \\
\delta \tilde{B}^a &= 0.
\end{align*}
\]

(19)

We note that only the gauge field has space index and thus only its BRST transformation changes on the hypersphere from that in the flat space. The BRST transformations of the rest are the same as those in the flat space. Nilpotency of the transformation assures the BRST invariance of the resulting effective theory. Using these transformations, we expand the Eq. (18) to get

\[ S_{GF} + S_{ghost} = \int d\Omega \left[ \frac{\zeta}{2} (\tilde{B}^a)^2 + \tilde{B}^a r_\alpha i M_{\alpha \beta} \tilde{A}_\beta^a - \tilde{c}^a r_\alpha i M_{\alpha \beta} r_\gamma i M_{\gamma \beta} \tilde{c}^a \right]. \]

(20)

We have now all that we need to move on to the main purpose of the paper.

### III. EFFECTIVE THEORY ON 4-SPHERE AND ITS NON PERTURBATIVE SECTOR

In refs. [13, 16], we have shown that Abelian dominance is observed in the effective quadratic gauge fixed theory in the flat space. Here we limit the discussion to 4 dimensions. We propose the effective theory in the same quadratic gauge on 4-sphere and address the issue of confinement in the compact space by analyzing its non perturbative regime. We derive the theory with Faddeev Popov (FP) method rather than BRST invariance as the first principle. The generalization to higher dimensions is obvious. The FP method relies on the FP operator which depends upon the choice of gauge fixing and gauge transformation. As mentioned earlier, the transformation is determined by underlying space hence, we must choose the transformation on 4-sphere (Eq. (11)) in deriving the required FP operator to
obtain the effective action in the quadratic gauge built on the $S^4$ which we get as in Eq.(24). We shall then investigate Abelian dominance by probing mass matrix for gluons on $S^4$ in the ghost condensed phase of this effective theory in which all the ghost condensates are identical. We also find that the necessity to realize this condensate identity within a physical mechanism consistent on the $S^4$ leads to mass of an off-diagonal gluon on $S^4$ to be position dependent. The quadratic gauge and the corresponding FP operator $\Delta_{FP}$ on $S^4$ are respectively as follows

\[ \bar{A}_a^\alpha(r) \bar{A}_\beta^\beta(r) = f^\alpha(r), \quad \beta = 1, 2, ..., 5; \text{ for each } a \text{ and}, \] (21)

\[ \Delta_{FP} = \det \left[ 2 \bar{A}_\beta^\beta \left( \partial_\beta \delta^{ab} - g f^{acb} \bar{A}_c^\alpha - r_\alpha r_\beta \left( \partial_\alpha \delta^{ab} - g f^{acb} \bar{A}_c^\alpha \right) \right) \right]. \] (22)

Therefore, we get the gauge fixing and the ghost terms as below

\[ S_{GFQ} + S_{ghostQ} = \int d\Omega \left[ \frac{\xi}{2} (B^a)^2 + B^a \bar{A}_\beta^\alpha \bar{A}_\beta^\beta - 2 \tilde{c}^a \bar{A}_\beta^\beta D_\beta \bar{c}^a + 2 \tilde{c}^a \bar{A}_\beta^\beta r_\alpha r_\beta D_\alpha \bar{c}^a \right], \] (23)

\[ d\Omega \text{ is angular measure for 4-sphere. The inherent transversality gets rid of the last ghost term in Eq.}(23), \text{ therefore it actually contains only the first three terms}, \]

\[ S_{GFQ} + S_{ghostQ} = \int d\Omega \left[ \frac{\xi}{2} (B^a)^2 + B^a \bar{A}_\beta^\alpha \bar{A}_\beta^\beta - 2 \tilde{c}^a \bar{A}_\beta^\beta D_\beta \bar{c}^a \right], \text{ since } r_\beta \bar{A}_\beta^\alpha = 0. \] (24)

The resulting action

\[ S_{eff} = S_{YM} + S_{GFQ} + S_{ghostQ}, \] (25)

is invariant under the same BRST transformations of Eq.(19). Therefore, the theory is FFBRST[20–22] compatible as it is BRST invariant which means that FFBRST technique can be applied to the current theory to connect it to a different effective theory e.g., Lorenz gauge fixed theory on 4-sphere. It is an interesting problem as FFBRST in the curved geometry is scarcely explored. The details are however subject of future study.

Before moving further, we like to find the equivalent of the quadratic gauge on the 4-sphere and see whether such equivalent gauge is of some interest. The transversality condition

\[ r_\alpha \bar{A}_\alpha^\alpha = 0, \quad \alpha = 1, 2, ..., 5, \] (26)

gives us the following identity upon differentiation[10]

\[ r_\alpha \partial_\beta \bar{A}_\alpha^\beta = -\bar{A}_\beta^\alpha + \frac{r_\beta}{r_5} \bar{A}_5^\alpha. \] (27)
Now,
\[
\bar{A}_\beta^a A_\beta^a = \bar{A}_\beta^a \left[ \bar{A}_\beta^a - r_\beta \frac{A_\beta^a}{r_5} \right]
\]
\[
= -\bar{A}_\beta^a r_\alpha \partial_\beta \bar{A}_\alpha^a \quad \text{since Eq. (27)}
\]
\[
= \bar{A}_\beta^a (r_\beta \partial_\alpha - r_\alpha \partial_\beta) \bar{A}_\alpha^a \quad \text{since } r_\alpha \bar{A}_\alpha^a = 0
\]
\[
= \bar{A}_\beta^a iM_{\beta\alpha} A_\alpha^a.
\] (28)

The Eq. (28) in general implies that \( iM_{\beta\alpha} A_\alpha^a = \bar{A}_\beta^a + r_\beta W^a \), where \( W^a \) is a function of fields \( \bar{A}_\alpha^a, B^a, \bar{c}^a, \bar{e}^a \). We note that the gauge in Eq. (1) emerges as a special case of equivalence with \( W^a = 0 \). The Eq. (28) also implies that vectors \( iM_{\beta\alpha} A_\alpha^a - A_\beta^a \) and \( A_\beta^a \) are mutually orthogonal. Therefore, the vector \( iM_{\beta\alpha} A_\alpha^a - A_\beta^a \) is directed radially.

A. Infrared regime of the theory

We focus on the ghost Lagrangian which is expanded as follows
\[
2\bar{c}^a \bar{A}_\beta^a D_\beta c^a = 2\bar{c}^a \bar{A}_\beta^a \partial_\beta c^a - 2gf^{abc}\bar{c}^a \bar{c}^c \bar{A}_\beta^b \bar{A}_\beta^c. \tag{29}
\]

In the desired ghost condensed phase, the first term on the right vanishes as we show in the last paragraph of the present section. The second term is suggestive of the mass matrix for gluons on \( S^4 \) as follows
\[
(M^2)_{ab}^{\text{dyn}} = 2g \sum_{c=1}^{N^2-1} f^{abc} \langle \bar{c}^a c^c \rangle. \tag{30}
\]

In the symmetric state, where all ghost-anti-ghost condensates are identical i.e.,
\[
\langle \bar{c}^1 c^1 \rangle = \ldots = \langle \bar{c}^1 c^{N^2-1} \rangle = \ldots = \langle \bar{c}^{N^2-1} c^1 \rangle = \ldots = \langle \bar{c}^{N^2-1} c^{N^2-1} \rangle = K', \tag{31}
\]
an interesting situation arises. The mass matrix in this state becomes
\[
(M^2)_{ab}^{\text{dyn}} = 2g \sum_{c=1}^{N^2-1} f^{abc} K', \tag{32}
\]

which is an anti-symmetric matrix due to the anti-symmetry of the structure constant \( f^{abc} \).

The resulting mass matrix has \( N(N-1) \) non-zero eigenvalues only and thus has nullity \( N-1 \). The non-zero eigenvalues occur in conjugate pairs. This implies \( N(N-1) \) off-diagonal gluons on sphere have become massive with \( M_{\text{gluon}} = \frac{1}{\sqrt{2}}(1 \pm i)m \) and \( N-1 \) diagonal gluons remain
massless. Thus in this description, the positive real part i.e., \( \text{Re}(M_{\text{gluon}}) > 0 \), of the off-diagonal gluon mass makes interactions short-ranged. Only the diagonal gluons mediate interactions in the IR limit which are long range, strongly indicating Abelian dominance on the sphere. Abelian dominance is observed in the effective quadratic gauge fixed theory in the Euclidean space too[13, 16]. Thus, non perturbative behaviour of the quadratic gauge fixed theory on \( \mathbb{S}^4 \) and in \( \mathbb{R}^4 \) is similar. The form similarity of ghost Lagrangian on \( \mathbb{S}^4 \) and in \( \mathbb{R}^4 \) (See Eqs.(24) and (4)) is particular to this gauge only. The ghost Lagrangian of Lorenz gauge, for example, on \( \mathbb{S}^4 \) and in \( \mathbb{R}^4 \) are not form similar.

It is imperative to realize the identity in Eq.(31) within the consistent mechanism on 4-sphere for rendering the non perturbative consequence of the theory physically relevant. The realization of the identity within the physical mechanism is not possible through the Coleman Weinberg approach on the curved spacetime unlike in the flat spacetime. There is only one way to demonstrate the identity in the Eq.(31) in a physically consistent manner on the curved spacetime and that is to use two known results. The first is the mapping between ghosts on sphere and corresponding ghosts in the flat space [10]

\[
\hat{c}^a(r) = \frac{1 + x^2}{2} \tilde{c}^a(x), \quad \bar{c}^a(r) = \frac{1 + x^2}{2} c^a(x),
\]

(33)

where \( \tilde{c}(x) \), \( c(x \) are anti-ghost and ghost fields respectively in the Euclidean space. Second is the following identity for ghosts in the Euclidean spacetime \( \mathbb{R}^4 \) whose realization within the Coleman Weinberg mechanism is given in [13, 16],

\[
\langle \hat{c}^1 \bar{c}^1 \rangle = \ldots = \langle \hat{c}^1 c^{N^2-1} \rangle = \ldots = \langle c^{N^2-1} \bar{c}^1 \rangle = \ldots = \langle \hat{c}^{N^2-1} \bar{c}^{N^2-1} \rangle = K,
\]

(34)

where \( K \) is a constant [13]. Putting Eq.(33) in Eq.(34), we get the identity in Eq.(31),

\[
\langle \hat{c}^1 \bar{c}^1 \rangle = \ldots = \langle \hat{c}^1 c^{N^2-1} \rangle = \ldots = \langle \hat{c}^{N^2-1} \bar{c}^1 \rangle = \ldots = \langle \hat{c}^{N^2-1} c^{N^2-1} \rangle = (\frac{1 + x^2}{2})^2 K = K'.
\]

(35)

Therefore, mass matrix in Eq.(32) becomes

\[
(M^2)^{ab}_{\text{dyn}} = 2g \sum_{c=1}^{N^2-1} f^{abc} K' = 2g \left[ \frac{1 + x^2}{2} \right]^2 \left( \sum_{c=1}^{N^2-1} f^{abc} K \right).
\]

(36)

Since \( K \) is constant of Eq.(34), eigenvalues of the matrix in the bracket in Eq.(36) are constants. We denote them by \( m_a^* \), \( a \) is a group index. Here, it is clear that \( m_a^* = 0 \) when \( a \) is a diagonal index. Hence by diagonalizing Eq.(36), we get the following expression for
the square of mass of a gluon on \( S^4 \), \( M_a^2 \)

\[
M_a^2 = 2g \left[ \frac{1 + x^2}{2} \right]^2 m^* a^2 = (1 + r_5)^{-2} m_a^2, \quad m_a^2 \equiv 2gm^* a^2.
\]  

(37)

Non zero \( m_a^2 \) and thus \( M_a^2 \) are purely imaginary numbers. \( M_a^2 = m_a^2 = 0 \) for diagonal gluons. It is further important to note a point about \( m_a^2 \). The fact that \( K \) in the bracket in Eq.(36) corresponds to ghosts in Euclidean space as given in Eq.(34) and that the quadratic term in gluons on \( S^4 \) relates to quadratic term in gluons in \( \mathbb{R}^4 \) only as shown below in Eq.(38) clearly establishes that \( m_a^2 \) in Eq.(37) are nothing but squares of masses of gluons \( A(x) \) in the flat space \( \mathbb{R}^4 \),

\[
\bar{A}_\beta A^\beta (r) = \left( \frac{1 + x^2}{2} \right)^2 A^a A^b (x).
\]

It is easy to prove the above identity from the mapping in Eq.(9). The Eq.(37) suggests that mass of an off-diagonal gluon on the 4-sphere is position dependent which reflects the consequence of the curved spatial geometry on mass in this theory. Specifically, mass of an off-diagonal gluon remains same on a horizontal cross section of the sphere \( S^4 \), which is a ‘circle’ \( S^3 \) whereas it varies on such parallel cross sections. Thus, the curvature influences mass of an off-diagonal gluon on \( S^4 \) without altering the infrared behaviour of this theory on \( S^4 \) from that of the quadratic gauge fixed theory in the \( \mathbb{R}^4 \).

Now we show that the term \( \hat{c}^a \bar{A}^b A^a \partial_r c^a \) in Eq.(29) disappears in the ghost condensed phase. We note that \( \langle \hat{c}^a \partial_r c^a \rangle = 0 \) \[13\] in the flat space since \( \langle \hat{c}^a c^c \rangle = \text{const.} \) for all \( x \). Therefore, using the mappings, we get

\[
\langle \hat{c}^a \partial_r c^a \rangle = \frac{8}{(1 + x^2)^3} \langle \hat{c}^a K^\mu_{\alpha} \partial_{\alpha} \left[ \frac{2}{(1 + x^2)} c^a \right] \rangle = 0,
\]

where \( K_{\mu \alpha} \) is a killing vector,

\[
K_{\mu \nu} = \frac{1 + x^2}{2} \delta_{\mu \nu} - x_\mu x_\nu, \quad K_{\mu 5} = -x_\mu.
\]

In the usual scenario \( \partial_{r_5} \equiv 0 \) regardless of what it operates on, which we now relax a bit and apply as an only proposition the following

\[
\frac{d}{dr_5} r_5 = -1.
\]

(40)

The algebra in Eq.(7) is almost retained under this proposition, i.e.,

\[
[M_{\mu \nu}, M_{\gamma \eta}] = i(\delta_{\mu \gamma} M_{\nu \eta} - \delta_{\nu \gamma} M_{\mu \eta} - \delta_{\mu \eta} M_{\nu \gamma} + \delta_{\nu \eta} M_{\mu \gamma});
\]

\[
\gamma, \eta = 1, ..., 5, \mu, \nu = 1, ..., 4,
\]

\[
\text{for } \mu \neq \nu.
\]
\[ [M_{\mu 5}, M_{\lambda 5}] = 0. \] (42)

The only change introduced by the conjecture is that the commutator vanishes in the Eq. (42) which simplifies an algebra marginally, the rest in the Eq. (41) remain unchanged. Thus, the conjecture applied does not disturb the underlying physics of the theory. Using this proposition in Eq. (39), we can verify that

\[ \langle \tilde{c}^a K^{\mu \alpha} \partial_{\mu} \left[ \frac{2}{(1 + r_5^2)} c^a \right] \rangle = \langle \tilde{c}^a K^{\mu \alpha} \partial_{\mu} \left[ (1 + r_5^2) c^a \right] \rangle = 0 \]

\[ \Rightarrow K^{\mu \alpha} \langle \tilde{c}^a \partial_{\mu} c^a \rangle = 0 \text{ as } K^{\mu \alpha} r_\alpha = 0 \]

\[ \Rightarrow \langle \tilde{c}^a \partial_{\mu} c^a \rangle = 0. \] (43)

Therefore, the term \[ \langle \tilde{c}^a \tilde{A}^{\beta a} \partial_{\beta} c^a \rangle = 0. \] Hence, the effective action in the ghost condensed phase now becomes

\[ S_{eff} = S_{YM} + \int d\Omega \left[ \frac{\xi}{2} (\tilde{B}^a)^2 + \tilde{B}^a \tilde{A}^a_{\beta} \tilde{A}_{\beta}^a + M_{\alpha}^2 \tilde{A}_{\beta}^a \tilde{A}_{\beta}^a \right]. \] (44)

### IV. EQUIVALENCE OF ACTIONS IN THE CONFINED PHASE

The result in 4 dimensions is even more profound and interesting in the following sense. Not just that the infrared behaviour of the theory with the quadratic gauge on the \( S^4 \) and in the \( R^4 \) is similar (Abelian dominance is observed in both spaces), the effective actions on the \( S^4 \) and in the \( R^4 \) in the corresponding ghost condensed phases also turn out to be identical as we see now. To prove it, we need to know some relations of quantities on sphere with corresponding quantities in the Euclidean space which are as follows \[ 11 \]

\[ d\Omega = \left( \frac{1 + x^2}{2} \right)^{-4} dx, \]

\[ \tilde{F}_{a\beta\gamma} \tilde{F}_{a\beta\gamma}^a = 3 \left( \frac{1 + x^2}{2} \right)^4 F^{\mu\nu a} F_{\mu\nu}^a, \] (45)

\[ \tilde{A}_{\beta}^a \tilde{A}_\beta^a (r) = \left( \frac{1 + x^2}{2} \right)^2 A^a_\mu A^b_\mu (x), \]

\[ \tilde{B}^a (r) = \left( \frac{1 + x^2}{2} \right)^2 B^a (x). \]

The first three of above relations can all be derived from the mappings given earlier whereas last one is intuitive from the dimensionality of fields. We keep in mind the relation between masses on sphere and Euclidean space in Eq. (37). Putting these relations along with Eq. (37)
in the effective action in the confined phase on $S^4$ in Eq. (44), we see that conformal factor, $\frac{1+x^2}{2}$ exactly cancels and we get the following term

$$S_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a + \frac{\xi}{2} (B^a)^2 + B^a A^a_\mu A^a_\mu + m^2_\alpha A^a_\mu A^a_\mu \right]. \quad (46)$$

The action on the right side is 4-dim Euclidean action in the ghost condensed phase [13]. The last term of this action reaffirms that $m^2_\alpha$ is square of mass of a gluon in the flat space. Thus, it is proved that the effective action in the confined phase on $S^4$ and that in the 4-dim Euclidean space are identical. The equality above is particular to 4 dimensions only.

V. CONCLUSION

Infrared behavior of QCD on 4-sphere and on a hypersphere in general has never been examined. We investigated the infrared regime in the quadratic gauge fixed theory in the physical compact space by defining it on 4-sphere. We observed that the inherent transversality condition plays a crucial role which gets rid of the undesired ghost term. One of the remaining ghost terms leads to Abelian dominance on $S^4$ in the ghost condensed phase which signals the presence of confinement on $S^4$ and in the general compact space. The compactness of $S^4$ does not affect the infrared regime in the present theory on $S^4$ in Eq. (24) from that in the quadratic gauge fixed theory in $\mathbb{R}^4$ but it influences mass of an off-diagonal gluon on $S^4$ to become position dependent in a way that on the particular horizontal cross section of $S^4$, the $S^3$ mass remains same and it varies on parallel horizontal cross sections. The equivalence between the effective actions in the confined phases on $S^4$ and in the 4-dim Euclidean space was proved at the end. The theory is also FFBRST compatible as it is BRST invariant. This will be particularly interesting area of further study as FFBRST has not been explored in the curved geometry.

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