Müntz linear transforms of Brownian motion

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ABSTRACT

We consider a class of linear Volterra transforms and determine explicitly the kernels from the spaces that they reproduce. The kernels resemble Müntz-Legendre polynomials to which they are very related. A link between these self-reproducing kernels and reproducing kernel systems is outlined. As a consequence, some properties of Müntz kernels are, in fact, inherited from the corresponding kernel systems. This combined with the link to stationary processes is exploited to find a necessary and sufficient condition for existence of Müntz kernels of infinite order.

Keywords: Enlargement of filtrations; Goursat kernels; Gramian matrices; Müntz polynomials; self-reproducing kernels; reproducing Hilbert spaces; Volterra transform.

AMS 2000 subject classification: 26G15; 60J10.

1. Introduction

There has been a renewed interest in Müntz spaces particularly on topics related to Markov inequalities and approximation theory, see for example ([4], [5], [6]) and the references therein. In the meanwhile, Volterra transforms with non square integrable kernels, involving some functional spaces, provide interesting examples of non-canonical decompositions of the Brownian filtration. This motivated many studies on the topic, for instance see ([1], [2], [3], [7], [13], [14], [15], [16], [11], [25], [26], [27]). Our aim in this paper is to study those Volterra transforms involving Gaussian spaces which are generated from sequences of Müntz polynomials. For readings on Enlargement of filtrations, we refer to ([3], [20], [21], [22], [23]).

To be more precise, let us start by fixing our mathematical setting. Assume that \( \Lambda = \{\lambda_1, \lambda_2, \ldots\} \) is a sequence of reals such that

(1) \( \lambda_i > -1/2, \quad \lambda_i \neq \lambda_j, \quad i \neq j, \quad i = 1, 2, \ldots \). \]

Observe that (1) insures that the generalized Müntz polynomials \( f_i(x) := x^{\lambda_i} \) lie in \( L^2_{loc}(\mathbb{R}^+, dx) \) for \( i = 1, 2, \ldots \). For \( t > 0 \), let us introduce Müntz spaces which are defined as

(2) \( M_{n,t}(\Lambda) = \text{Span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}; x \in (0, t]\} \)
and

\[ M_t(\Lambda) = \lim_{n \to \infty} M_{n,t}(\Lambda). \]

An associated orthogonal system, known as Müntz-Legendre polynomials or system, is specified by

\[ L_k(x) = \sum_{j=1}^{k} c_{j,k} x^{\lambda_j}, \quad c_{j,k} = \prod_{i=1, i \neq j}^{k} \left( \lambda_i + \lambda_j + 1 \right) / \prod_{i=1}^{k} (\lambda_i - \lambda_j), \]

with \( L_k(1) = 1, k = 1, 2, \cdots n \). Recall the celebrated Müntz-Szasz theorem, see [5], which states that \( M_1(\Lambda) = L^2_{\text{loc}}((0,1], dx) \) if and only if

\[ \sum_{i=1}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty. \]

Now, let \((B_t, t \geq 0)\) to be a standard Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and denote by \(\{\mathcal{F}_{B_t}, t \geq 0\}\) the filtration it generates. Define the associated family of Müntz Gaussian spaces as

\[ \mathcal{H}_{n,t}(\Lambda, B) = \text{Span} \left\{ \int_0^t f_1(s) dB_s, \int_0^t f_2(s) dB_s, \cdots, \int_0^t f_n(s) dB_s \right\}, \]

for \( t > 0 \) and \( n < \infty \), and, whenever the a.s. limit exists, we set

\[ \mathcal{H}_{\infty,t}(\Lambda, B) = \lim_{n \to \infty} \mathcal{H}_{n,t}(\Lambda, B). \]

For \( n \in \mathbb{N} \) and \( t > 0 \) fixed, the covariance matrix \( m^n_t \) of \( \int_0^t f(s) dB_s \), where \( f = (f_1, f_2, \cdots, f_n)^* \) and \(^*\) stands for the transpose operation, has an inverse which we denote by \( \alpha(t) \). Put \( \phi(. \cdot) = \alpha(\cdot \cdot \cdot \cdot) \cdot f(\cdot \cdot \cdot) \). Following [14], the kernel \( k_n(t, s) = \phi^*(t \cdot f(s), 0 < s \leq t < +\infty \), is a Goursat-Volterra kernel of order \( n \). That is, the Wiener measure is preserved by the Volterra transform \( T_n \) which is defined by

\[ T_n : C (\mathbb{R}^+, \mathbb{R}) \to C (\mathbb{R}^+, \mathbb{R}), \]

\[ (B_t, t \geq 0) \to \left( B_t - \int_0^t \int_0^n k_n(u, v) dB(v) du, t \geq 0 \right) \]

and the orthogonal decomposition

\[ \mathcal{F}_t^B = \mathcal{F}^{T_n(B)}_t \otimes \sigma (\mathcal{H}_{n,t}(\Lambda, B)) \]

holds for any \( t \geq 0 \). Here, by \( \mathcal{F} \otimes \mathcal{G} \) we mean \( \mathcal{F} \vee \mathcal{G} \) with independence between \( \mathcal{F} \) and \( \mathcal{G} \). Observe that the filtration \( \mathcal{F}^{T_n(B)}_t, s < t \) coincides, up to null sets, with the filtration of an \( f \)-generalized bridge, over the interval \([0, t]\), whose law can be constructed, for instance, by desintegrating the Wiener measure along \( \mathcal{H}_{n,t}(\Lambda, B) \), see [1]. By construction, the above family of kernels satisfy some interesting properties.
which we shall now recall. We have \( k_n(u, v) = 0 \), for \( u \leq v < \infty \), and for \( n < \infty \) the integrability condition

\[
\int_0^t \left( \int_0^u k_n^2(u, v) \, dv \right)^{\frac{1}{2}} \, du < \infty, \quad t > 0
\]

holds true. Furthermore, \( k_n \) satisfies the self-reproducing property

\[
k_n(t, s) = \int_0^s k_n(t, u)k_n(s, u) \, du, \quad 0 < s \leq t.
\]

In our setting, following [14], but also [27], the kernel \( k_n \) takes the form

\[
k_n(t, s) = t^{-1} \sum_{j=1}^n a_{j,n} (s/t) \lambda_j, \quad 0 < t < \infty,
\]

where the sequence of reals \( a_{1,n}, a_{2,n}, \ldots, a_{n,n} \) is uniquely determined as a solution to a system of linear equations. The system which given by equation (13) below was first discovered by P. Lévy, see [25], when he claimed that for any integer \( n \) there exists a polynomial \( F_n(.) \) of degree \( n \) such that the process \( \left( \int_0^t F_n(u/t) \, dB_u, t > 0 \right) \) is a Brownian motion having a filtration which is strictly smaller than that of the original Brownian motion \( B \). This example was further studied [7]. After solving (13), we realized that the expressions of \( (k_n, n = 1, 2, \cdots) \) resemble those of Müntz-Legendre polynomials. We explain the link by a connection between reproducing kernel systems and self-reproducing kernels. The explicit formula for the inverse of the Cauchy matrix with entries

\[
(m_i^j)^n = \frac{1}{\lambda_i + \lambda_j + 1}, \quad i, j = 1, 2, \cdots, n,
\]

is known and may be found in [12]. However, here, we propose another method to compute it and thus resolve the aforementioned linear system. Next, Müntz kernels are homogeneous of degree \(-1\) in the sense that \( k_n(\alpha t, \alpha s) = \alpha^{-1} k_n(t, s) \) for \( \alpha > 0 \). As a consequence, the associated Volterra transforms have a close connection to a class of stationary Gaussian processes through an isometry \( U \) which maps \( L^2(\mathbb{R}_+, dx) \) into \( L^2(\mathbb{R}, dx) \). That is to say that the process \( (U \circ T_n(B)_t, t \geq 0) \) has the moving average representation

\[
U \circ T_n(B)_t = \int_{-\infty}^t \eta_n(t - r) \, d\beta_r,
\]

where \( \beta \) is a Brownian motion indexed by \( \mathbb{R} \), and \( \eta_n \) has the Fourier transform

\[
\hat{\eta}_n(\xi) = \frac{2}{\pi} \frac{1}{1 - 2i\xi} \prod_{j=1}^n \frac{\xi - i(\frac{1}{2} + \lambda_j)}{\xi + i(\frac{1}{2} + \lambda_j)}.
\]

Applying then the characterization given in [24], a glance at (11) allows to detect an inner factor which implies, as expected, that the representation (10) is not canonical.

A natural question is to know whether there exist Müntz transforms of infinite orders. A partial answer is given in [15] and [16], where the authors established their existence. In particular, for an infinite sequence \( \Lambda \) the condition \( \sup \lambda_j < \infty \) is a necessary one. Furthermore, if \( 0 < \lambda_1 < \lambda_2 < \cdots \) then there exists no corresponding
Müntz transform associated to $\Lambda$. Also, when there is convergence, by letting $n \to \infty$ in (11) we obtain the class of kernels constructed in Theorem 2 of [15]. We conclude, in this paper, that the condition $\sum_{j=1}^{\infty} (2\lambda_j + 1) < \infty$ is necessary and sufficient for the existence of Müntz transforms with infinite dimensional orthogonal complement.

2. Müntz Gaussian Hilbert spaces and transforms

Let $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ be a sequence of reals satisfying condition (1). An element of span$\{x^{\lambda_1}, \cdots, x^{\lambda_n}; (0, t]\}$, for some $n \in \mathbb{N}$ and $t > 0$, is called a Müntz polynomial. These generalized polynomials span the spaces introduced in formulae (2) and (3) and the condition $-1/2 < \lambda_i, i = 1, 2, \ldots$, insures that the above polynomials belong to $L^2((0, t])$ for all $0 < t < \infty$. Next, to the linear spaces $M_{n,t}(\Lambda)$ and $M_t(\Lambda)$ we associate, respectively, the families of Müntz Gaussian spaces $(H_{n,t}(\Lambda, B), t \geq 0)$ and $(H_{\infty,t}(\Lambda, B), t \geq 0)$ defined by (6) and (7). Recall that the first Wiener chaos of $B$ on $(0, t]$, for some fixed $t \geq 0$, is specified by

$$H_t(B) = \left\{ \int_0^t f(u) dB_u : f \in L^2((0, t]) \right\}.$$ 

Clearly, the orthogonal complements of the spaces of $H_{n,t}(\Lambda, B)$ and $H_t(\Lambda, B)$, in $H_t(B)$, are respectively given by

$$H^\perp_{n,t}(\Lambda, B) = \left\{ \int_0^t f(u) dB_u : f \in L^2((0, t]), \int_0^t f(s)p(s) ds = 0, p \in M_{n,t}(\Lambda) \right\}$$

and

$$H^\perp_t(\Lambda, B) = \left\{ \int_0^t f(u) dB_u : f \in L^2((0, t]), \int_0^t f(s)p(s) ds = 0, p \in M_t(\Lambda) \right\}.$$ 

Observe that $M_t(\Lambda) = L^2((0, t], dx)$ for any $0 < t < \infty$ if and only $M_t(\Lambda) = L^2((0, 1], dx)$. Furthermore, that happens if and only if (5) holds, see [5]. It follows that in the total case, $H(\Lambda, B)$ is total in $H_t(B)$ which gives that $H^\perp(\Lambda, B) = \{0\}$. Otherwise, $H^\perp(\Lambda, B)$ is a nontrivial set. We are now ready to state the following result and, for the sake of completeness, a fool proof is given.

Theorem 2.1. Let $\Lambda$ be a sequence of reals satisfying condition (1). For a fixed $n < \infty$, the kernel $k_n(\cdot, \cdot)$ which is defined by $k_n(t, s) = t^{-1}k_n(s/t)$ for $0 < s \leq t < \infty$ and $k_n(t, s) = 0$ otherwise, with

$$(12) \quad k_n(s) = \sum_{j=1}^{n} a_{j,n} s^{\lambda_j}, \quad a_{j,n} = \prod_{i=1, i \neq j}^{n} (\lambda_i + \lambda_j + 1) \prod_{i=1, i \neq j}^{n} (\lambda_j - \lambda_i), \quad j = 1, \ldots, n,$$

is a Goursat-Volterra kernel of order $n$. Moreover, for all $t > 0$ the orthogonal complement of $H_t(T_n(B))$ in $H_t(B)$ is $H_{n,t}(\Lambda, B)$, where $T_n$ is the Volterra transform associated to the kernel $k_n$. 

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Proof. Following [8], $T_n$ preserves the Wiener measure if and only if $k_n$ satisfies (9). But, this is equivalent to saying that $(a_{i,n}, i = 1, 2, \ldots, n)$ solves the linear system

$$
\sum_{i=1}^{n} \frac{a_{i,n}}{\lambda_i + \lambda_j + 1} = 1, \quad j = 1, \ldots, n,
$$

which has a unique solution. To find it, consider the $n$-degree polynomial

$$
p_n(x) = \prod_{j=1}^{n} (x + \lambda_j + 1) - \sum_{i=1}^{n} a_{i,n} \prod_{j=1, j \neq i}^{n} (x + \lambda_j + 1)
$$

which, of course, has at most $n$ roots. But $p_n(x) = 0$ is equivalent to

$$
\sum_{i=1}^{n} a_{i,n} x + \lambda_i + 1 = 1, \quad j = 1, \ldots, n.
$$

To check this, write

$$
\int_0^t \left( \int_0^u k_n^2(u, v) \, dv \right)^{1/2} \, du = \int_0^t \left( \int_0^1 k_n^2(u, ur) \, dr \right)^{1/2} \, du
$$

$$
= 2\sqrt{t} \left( \int_0^1 k_n^2(r) \, dr \right)^{1/2}
$$

$$
= 2\sqrt{tk_n^{1/2}(1)} < +\infty,
$$

where we used the homogeneity and self-reproducing properties of $k_n$. It remains to identify $H_t(B) \setminus H_t(T_n(B))$ for an arbitrarily $t > 0$. But, this amounts to solving the integral equation

$$
f(s) = \int_0^s k_n(s, u) f(u) \, du
$$

which should hold for all $s > 0$. This is easily seen, by differentiation, to be equivalent to an ordinary linear differential equation of degree $n$. The functions $s \rightarrow s^{\lambda_j}$, $j = 1, \ldots, n$, being $n$ linearly independent solutions, we conclude that $H_{n,t}(\Lambda, B)$ is the orthogonal complement of $H_t(T_n(B))$ in $H_t(B)$ as required. \hfill \square

**Remark 2.1.** Another way to solve (13) is to use the obvious decomposition

$$
\prod_{j=1}^{n} \frac{x - \lambda_j}{x + \lambda_j + 1} = 1 - \sum_{j=1}^{n} \frac{a_{j,n}}{x + \lambda_j + 1}, \quad x \neq -\lambda_j - 1, \quad j = 1, 2, \ldots, n.
$$

This is not to confuse with the decomposition

$$
\prod_{j=1}^{n} \frac{1}{\lambda_j - x} = \sum_{j=1}^{n} \frac{b_{j,n}}{\lambda_j - x}, \quad b_{j,n} = \prod_{i=1, i \neq j}^{n} \frac{1}{\lambda_i - \lambda_j}, \quad x \neq \lambda_j, \quad j = 1, 2, \ldots, n.
$$
Remark 2.2. For a fixed $t > 0$, $\mathcal{F}^{T_n(B)}_t$ coincides, up to null sets, with $\sigma\{B^{(br)}_u, u \leq t\}$, where $(B^{(br)}_u, u \leq t)$ is a generalized bridge over the interval $[0, t]$. This can be constructed as

$$B^{(br)}_u = B_u - \psi^*_t(u) \cdot \int_0^t f(s) dB_s$$

for $0 < u < t$, where

$$\psi_t(u) = \alpha^n_t \cdot \int_0^u f(r) dr.$$ 

$T_n$ is not injective as $T_n(B^{(br)}) = T_n(B)$ over the life time interval $[0, t]$. Furthermore, (14) represents a noncanonical decomposition of the generalized bridge, see [1] for more details.

Motivated by Remark 2.2, we return to the family of matrices $m^n$ and compute their inverses. In fact, $\alpha^n_1$ is computed explicitly in [12]. However, we provide here another method to determine it.

Corollary 2.1. We have $\phi_i(t) = a_{i,n} t^{-\lambda_i - 1}$, $i = 1, 2, \cdots, n$. Furthermore, the entries of $\alpha_t$ are given by $(\alpha_t)_{i,j} = a_{i,n} a_{j,n} (\lambda_i + \lambda_j + 1)^{-1} t^{-\lambda_i - \lambda_j - 1}$.

Proof. The entries of $\phi(t)$ are identified form the expression of $k_n$ given in Theorem 2.1. Next, from [2], we know that $(\alpha_t(t), t > 0)$ is given, in terms of $\phi$, by

$$\alpha_t(t) = \int_t^\infty \phi(u) \cdot \phi^*(u) du + \alpha_\infty, \quad \phi(t) = \alpha_t \cdot f(t), \quad 0 < t < \infty.$$ 

$\alpha_\infty \equiv 0$ because $s^{\lambda_1}, s^{\lambda_2}, \cdots, s^{\lambda_n}$ are not square integrable. Plugging in the vector $\phi$ we obtain $\alpha_t$. \qed

Müntz kernels can be expressed in terms of Müntz-Legendre polynomials which form an orthogonal basis of $M_{n,1}(\Lambda)$. These are obtained by Gram-Schmidt procedure and are specifically given by (4). For this and related topics, we refer to [5] and [6].

Proposition 2.1. For $\Lambda$ satisfying (1) with $n < \infty$, the following assertions hold true. 1) We have $k_n(x) = \sum_{j=1}^n (1 + 2\lambda_j) L_j(x)$. Consequently, we have

$$k_n(x) = x^{-\lambda_n} \frac{\partial}{\partial x} \left(x^{\lambda_n+1} L_n(x)\right)$$

or, equivalently, $L_n(x) = x^{-\lambda_n-1} \int_0^x s^{\lambda_n} k_n(s) ds$. Note that unlike Müntz-Legendre polynomials, the kernel $k_n$ does not depend of the order of $\lambda_1, \lambda_2, \cdots, \lambda_n$. 2) The sequence $k$ satisfies the integro-difference equation $k_n(x) = k_{n-1}(x) + (2\lambda_n + 1)x^{\lambda_n}(1 - \int_0^1 u^{-\lambda_n-1} k_{n-1}(u) du)$.

Proof. 1) We have

$$k_n(x) - k_{n-1}(x) = a_{n,n} + \sum_{j=1}^{n-1} (a_{j,n} - a_{j,n-1}) x^{\lambda_j}.$$
From
\[(1 + 2\lambda_n)c_{n,n} = a_{n,n}\]
and
\[(1 + 2\lambda_n)c_{i,n} = a_{j,n} - a_{j,n-1}.\]
we deduce the first asserted formula. As a by-product formula, we note that
\[(\lambda_j + \lambda_n + 1)c_{j,n} = a_{j,n}, \quad j \leq n.\]
The second assertion is easily obtained by integration. 2) We quote, from [6], the recurrence formula
\[(x^{\lambda_n + \lambda_{n-1} + 1}(x^{-\lambda_n} L_n(x)))' = (x^{\lambda_{n-1} + 1} L_{n-1}(x))'.\]
Combining with 1) and simplifying yields
\[(x^{-\lambda_n} L_n(x))' = x^{-\lambda_n - 1} k_n(x) - (2\lambda_n + 1)x^{-2\lambda_n - 2}\int s^{\lambda_n} k_n(s) ds.\]
Differentiating, we find
\[-\lambda_n k_n(x) + x k'_n(x) = (\lambda_n + 1)k_{n-1} + x k'_{n-1}.\]
This is nothing but a differential form of the integro-difference equation. It remains to use 1) on the form
\[k_n(x) = k_{n-1}(x) + (1 + 2\lambda_n) L_n(x)\]
and the fact that \(L_n(1) = 1\) to conclude.

\[\square\]

Our aim now is to outline a connection between self-reproducing kernels and the classical kernel systems.

**Proposition 2.2.** For each fixed \(t > 0\), the kernel system associated to \(M_{n,t}(\Lambda)\) is given by \(g_{n,t}(u,v) = \frac{1}{t} \sum_{i=1}^{n}(1 + 2\lambda_i)L_i(\frac{u}{t}) L_i(\frac{v}{t})\) for \(0 < u, v \leq t\). Letting \(u \to t\) we get that \(k_n(t,s) = g_{n,t}(t,s) = \frac{1}{t} \sum_{i=1}^{n}(1 + 2\lambda_i)L_i(\frac{s}{t})\) for \(0 < s \leq t < \infty\).

**Proof.** The kernel system is given by \(g_{n,t}(u,v) = \sum_{k=1}^{n} q_{k,t}(u) q_{k,t}(v)\) where \((q_{k,t}(v), n = 1, \ldots, n)\) is an orthonormal sequence that generates \(M_{n,t}(\Lambda)\). This is a reproducing kernel in the sense that, for any \(Q_t \in M_{n,t}^\Lambda\), we have \(Q_t(u) = \int_0^t g_{n,t}(u,v) Q_t(v) dv\). Exploiting homogeneity, we easily check that the sequence \(q_{m,t}(x), x \in [0,t]; j = 1,2, \ldots, n\) defined by \(q_{m,t}(u) := \sum_{k=1}^{m} c_{k,m}(t) w^{\lambda_k} = \frac{1}{\sqrt{t}} \tilde{L}_m(u/t)\) satisfies the requirements. We conclude using continuity and the fact that \(L_n(1) = 1\). \(\square\)

**Remark 2.3.** Observe that, although the self-reproducing kernel \(k_n\) associated to \(M_{n,t}(\Lambda)\) is unique, the kernel may appear on different forms. In fact, as was observed in [14], if instead of \((f_1, f_2, \ldots, f_n)\) we work with any generating vector of \(M_{n,t}(\Lambda)\) then the resulting self-reproducing kernel will be the same. This property known as the kernel trick is, in fact, inherited from the reproducing kernels.

**Remark 2.4.** Assuming that \(k_n\) converges as \(n \to \infty\), then it is clear from \(k_n(1) = k_{n-1}(1) + (1 + 2\lambda_n)\) that we must have \(1 + 2\lambda_n \to 0\). This convergence problem is our focus in the next section.
3. INFINITE DIMENSIONAL KERNELS

We discuss here a question tackled in [16]. That consists on determining a necessary and sufficient condition for the existence of an infinite dimensional kernel associated to a sequence \( \Lambda \) satisfying condition (1). This turns out to be equivalent to having the pointwise convergence of \( k_n(\cdot) \) as \( n \to \infty \).

**Proposition 3.1.** For \( n < \infty \) the kernel \( k_n(t \lor s, t \land s) \) is positive definite. It converges as \( n \to \infty \) to a positive definite function if and only if \( k_n(t) \) converges in the \( L^2((0,1), dx) \) mean, and hence pointwise on \((0,1] \); this happens if and only if

\[
\sum_{i=1}^{n} (1 + 2\lambda_i) < \infty.
\]

**Proof.** Using (9) we see that \( k_n(t \lor s, t \land s) \) on \((0,1] \times (0,1] \) is the covariance function of the Gaussian process \( \left( \int_{0}^{t} k_n(t, r) dB_r, t \leq 1 \right) \) which gives the first assertion. Next, for \( n > m \) positive integers, we can write

\[
\int_{0}^{1} (k_n(x) - k_m(x))^2 dx = \sum_{i=m+1}^{n} (1 + 2\lambda_i) \sum_{j=m+1}^{n} (1 + 2\lambda_i) \int_{0}^{1} L_i(r) L_j(r) dr
\]

where we used \( \int_{0}^{1} L_i^2(r) dr = 1/(1 + 2\lambda_i) \). Thus, \( k_n \) is a Cauchy sequence if and only if (16) holds. \( \square \)

To establish our next result, we need to stress out a connection to stationary processes which works thanks to the homogeneity property of our kernels. Introduce the isometry \( U : L^2(\mathbb{R}^+, dx) \to L^2(\mathbb{R}, dx) \) defined by \( U(f)(t) = e^{-t/2} f(e^t) \), \( t \geq 0 \), and set \( T_n = U \circ T_n \). For \( f \in L^1(\mathbb{R}, dx) \), we define its Fourier transform \( \hat{f}(\xi) = \frac{2}{\pi} \int_{\mathbb{R}} e^{i\xi t} f(t) dt \).

**Theorem 3.1.** Let \( \Lambda \) be a sequence of reals satisfying (1). Then, for each fixed \( n \in \mathbb{N} \), the process \( (T_n(B)_t, t \in \mathbb{R}) \) has the moving average representation \( T_n(B)_t = \int_{-\infty}^{t} \eta_n(t-r) d\beta_r \) where \( \beta \) is a Brownian motion indexed by \( \mathbb{R} \) and \( \eta_n \) is uniquely characterized by its Fourier transform

\[
\hat{\eta}_n(\xi) = \frac{2}{\pi} \frac{1}{1 - 2i\xi} \prod_{j=1}^{n} \frac{\xi - i(\frac{1}{2} + \lambda_j)}{\xi + i(\frac{1}{2} + \lambda_j)}.
\]

Consequently, there exists an infinite order Müntz kernel, associated to \( \Lambda \), if and only if \( \sum_{j=1}^{\infty} (2\lambda_j + 1) < \infty \).
Proof. We use the homogeneity property of $k_n$ to write

\[
T_n(B)_t = B - \int_0^t \int_0^u k_n(u, v) \, dB_v \, du
\]

\[
= \int_0^t \left( 1 - \int_v^t k_n(u, v) \, du \right) \, dB_v
\]

\[
= \int_0^t \left( 1 - \int_1^{t/v} k_n(vr, v) \, dr \right) \, dB_v
\]

\[
= \int_0^t \rho_n(t/v) \, dB_v, \quad t \geq 0,
\]

where

\[
\rho_n(x) = 1 - \int_1^x k_n(r, 1) \, dr.
\]

Thus, with $\eta_n(t) = 1_{\{t>0\}} U \circ \rho_n(t)$, we have the moving average representation

\[
\mathcal{T}_n(B)_t = \int_{-\infty}^t \eta_n(t-r) \, d\beta_r
\]

where $\beta$ is a Brownian motion indexed by $\mathbb{R}$. The Fourier transform of $\eta_n$ is easily found to be given by (17). Clearly, if such a kernel $k_\infty$ exists then we must have $k_n \rightarrow k_\infty$. That can happen if and only if $\hat{\eta}_n(\xi) \rightarrow \hat{\eta}_\infty(\xi)$ say. But the infinite product converges if and only if $\sum_{j=1}^{\infty} \frac{(2\lambda_j+1)}{(2\lambda_j+1)^2+1} < \infty$, see also Remark 3.2 below for more details, which gives our result.

\[\square\]

Remark 3.1. For $n \in \mathbb{N}$, $T_n$ preserves the Wiener measure if and only if $T_n(B)$ is a Brownian motion or an Ornstein-Uhlenbeck process. A necessary and sufficient condition for this to happen is that the spectral measure of $T_n(B)$ with respect to the Lebesgue measure is $(2/\pi)(1 + 4x^2)^{-1}$ on $\mathbb{R}$, see Lemma 1 in [23]. Our expression (17) agrees with this criterion.

Remark 3.2. We recall that for $p_n > 0$, $n \in \mathbb{N}$, the condition for the convergence of the infinite product

\[
\prod_{n=1}^{\infty} \frac{p_n + i\xi |p_n - 1|}{p_n - i\xi |p_n - 1|}
\]

is $\sum_{n=1}^{\infty} \frac{p_n}{p_n^2+1} < \infty$. This well-known result is found, for instance, in Theorem 2 of [15]. Notice that this the same as Müntz-Szasz condition (5) with $p_n = 1 + 2\lambda_n$.

Kernels of infinite orders take complicated forms which may involve special functions. To illustrate that, we shall now give a family of examples.
Proposition 3.2. Let \( r \) be such that \( 2(r - 1) \in \mathbb{N} \). Then the kernel

\[
k_{\infty}(t, s) = \frac{1}{t} \sum_{k=1}^{\infty} \frac{r}{k^{r+1}} \frac{(1)^{k+1}}{k!} \prod_{j=1}^{2r-1} \Gamma(-k\omega_j^r)^{(-1)^j} (s/t)^{j-r}
\]

where \( \omega_r = \exp(\pi i/r) \), is of infinite order. Furthermore, the moving average representation of Theorem 3.1, with \( \hat{\eta}_n(\xi) = -\frac{2}{r}(1 - 2i\xi)^{-1} \prod_{j=1}^{2r} \Gamma((i\xi)^{1/r} \omega_j^r)^{(-1)^j} \), holds.

Proof. We need two formulas found in [4], pp. 6-7. These state that

\[
\prod_{j=1, j \neq k} \frac{j^r - k^r}{j^r + k^r} = (-1)^{k+1} \frac{2k(k!)}{r} \prod_{j=1}^{2r-1} \Gamma(-k\omega_j^r)^{(-1)^j}
\]

where \( \omega = \exp(\pi i/r) \), and

\[
\prod_{j \geq 1} \frac{j^r - z^r}{j^r + z^r} = -\prod_{j=1}^{2r} \Gamma(-z\omega_j^r)^{(-1)^j}
\]

Thus, we can write

\[
a_{k,n} = \frac{2}{kr} \prod_{j=1, j \neq k} \frac{j^r + k^r}{j^r - k^r} = \frac{2}{kr} \left\{ (-1)^{k+1} \frac{2k(k!)}{r} \prod_{j=1}^{2r-1} \Gamma(-k\omega_j^r)^{(-1)^j} \right\}^{-1}
\]

as \( n \to \infty \). We also have

\[
\hat{\eta}_n(\xi) = \frac{2}{\pi} \frac{1}{1 - 2i\xi} \prod_{j=1}^{n} \frac{\xi - i\left(\frac{1}{2} + \lambda_j\right)}{\xi + i\left(\frac{1}{2} + \lambda_j\right)}
\]

\[
= \frac{2}{\pi} \frac{1}{1 - 2i\xi} \prod_{k=1}^{\infty} \frac{k^r + i/\xi}{k^r - i/\xi}.
\]

This is evaluated using the second formula quoted above. \( \square \)

Remark 3.3. For \( n \in \mathbb{N} \cup \{+\infty\} \), \( k_n \) and \( T_n \) as above, introduce the notations

\( T_n^{(0)} = Id \), \( T_n^{(1)} = T_n \) and \( T_n^{(m)} = T_n^{(m-1)} \circ T_n \), for \( m \geq 2 \), where \( \circ \) stands for the composition rule, for the iterated transforms. Since we are in the homogeneous case, we can apply results of [23] to conclude that M"untz transforms of any order are strongly mixing and a fortiori ergodic. Moreover, the orthogonal decomposition

\[
\mathcal{F}_t^B = \bigotimes_{k=1}^{\infty} \sigma \left( \int_0^t u^{\lambda_j} dT_n^{(k)}(B)_u, 1 \leq j \leq n \right)
\]

holds true, for \( n \in \mathbb{N} \cup \{+\infty\} \).

Acknowledgment: The authors thank Y. Hibino for pointing out an erroneous argument in the previous draft of this paper. The second author is greatly indebted to the National Science Council Taiwan for the research grant NSC 94-2115-M-009-019-.
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