**L-HOMOLOGIES OF DOUBLE COMPLEXES**

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**Abstract.** The notion of \(L\)-homologies (of double complexes) as proposed in this paper extends the notion of classical horizontal and vertical homologies, along with two other new homologies introduced in the homological diagram lemma called salamander lemma. We enumerate all \(L\)-homologies associated with an object of a double complex and provide new examples of exact sequences. We describe a classification problem of these exact sequences. We study two poset structures on these \(L\)-homologies; one of them determines the trivialities of horizontal and vertical homologies of an object in terms of other \(L\)-homologies of that object, whereas the second structure shows the significance of the two homologies introduced in salamander lemma. Finally, we prove the existence of a faithful amnestic Grothendieck fibration from the category of \(L\)-homologies to a category consisting of objects and morphisms of a given double complex.

1. Introduction

In studying double complexes, we come across with horizontal and vertical homologies associated with objects of double complexes. It is in [1], where two new homologies have been introduced to formulate a new homological diagram lemma for abelian categories. We observe that all these four types of homologies are of the form \(U/V\), where \(V \leq U\) follows from the properties of double complexes, and \(U, V\) are joins/meets of kernels and images respectively. Given an object \(A\) of a double complex, we notice that the images of all morphisms with codomain \(A\) and kernels of all morphisms with domain \(A\) form two lattices. Based on these observations, we define an \(L\)-complex and hence an \(L\)-homology \(U/V\) (see Definition 3.2) having \(U\) and \(V\) as elements of the above mentioned lattices. We obtain a complete list of such \(L\)-homologies associated with an object of a double complex (see Proposition 3.3).

These \(L\)-homologies are accompanied with two partial order relations. One of these, (see Definition 4.1) on the set of \(L\)-homologies of an object, forms a join-semilattice with a top element (see Theorem 4.2) and trivialities of other \(L\)-homologies imply trivial horizontal and vertical homologies of that object (see Corollary 4.3). The second relation is based on canonical morphisms (in the sense of Proposition 2.1) between \(L\)-homologies, and the set of \(L\)-homologies of an object under this relation forms a bounded poset. Using a sufficient condition of exactness of a sequence of subquotients, we give several examples of exact sequences of \(L\)-homologies formulating in terms of this second partial order relation (see Proposition 4.10) and discuss about the classification problem of those exact sequences.

Finally, for a given double complex, we construct a category of \(L\)-homologies of objects of the double complex and define a functor from it to a category whose objects and morphisms are the same as that of the given double complex along with the identity morphisms. This functor is a

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Grothendieck fibration in the sense of [5]. It is also faithful, amnestic and has a left adjoint and a right adjoint.

The key idea of this paper is the notion of $L$-homologies, which generalizes the existing notion of homologies of double complexes and gives a way to construct exact sequences. The structures on these $L$-homologies help to understand inter-relation between them.

2. PRELIMINARIES

In this section, we recall some of the definitions and properties related to subobjects and subquotients in abelian categories. For a fuller treatment we refer the reader to [2], [3], [8], [9] and [7].

Let $\mathcal{A}$ be an well-powered abelian category. The image, written as $\text{Im} f$, of a morphism $f: A \to B$ in $\mathcal{A}$ is the smallest subobject of $B$ such that $f$ factors through the representing monomorphism $m_{\text{Im} f}: \text{Im} f \to B$. Every morphism $f: A \to B$ in $\mathcal{A}$ has an epi-mono factorization: $A \xrightarrow{e} \text{Im} f \xrightarrow{m} B$. A sequence of composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called exact at $B$ if $\text{Im} f = \text{Ker} g$.

The set of subobjects of an object $A$ in $\mathcal{A}$ forms a bounded modular lattice $(\text{Sub}(A), \subseteq, \lor, \land, \bot, \top)$, where for two subobjects $S$ and $S'$ of $A$, $S \subseteq S'$ if there exists a monomorphism $\phi: S \to S'$ such that the monomorphism $m_{\phi}: S \to A$ factors through the monomorphism $m_{S^\prime}: S^\prime \to A$ by $\phi$, i.e. $m_S = m_{S^\prime} \phi$, the union $S \lor S'$ of $S$ and $S'$ is the image of the morphism $S \sqcup S' \to A$, the intersection $S \land S'$ of $S$ and $S'$ is the pullback of the morphisms $m_S: S \to A$ and $m_{S^\prime}: S^\prime \to A$, the smallest subobject (or the bottom element) $\bot$ of Sub($A$) is the monomorphism $0 \to A$, the biggest subobject (or the top element) $\top$ of Sub($A$) is the monomorphism $A \to A$, and the modularity law means that for any three subobjects $X, Y, Z$ of $A$, and $X \subseteq Z$ implies $X \lor (Y \land Z) = (X \lor Y) \land Z$.

Every morphism $f: A \to B$ in $\mathcal{A}$ induces an adjunction between Sub($A$) and Sub($B$) as follows:

$$
\text{Sub}(A) \xrightarrow{f} \text{Sub}(B),
$$

where, the direct image functor $f$ is defined on a subobject $S$ of $A$ as the image of the composite $S \to A \to B$, whereas for a subobject $T$ of $B$ the inverse image functor is defined as the pullback of the morphisms $m_T: T \to B$ and $f: A \to B$. From the functoriality, we have $f(S) \subseteq f(S')$ and $f^{-1}(T) \subseteq f^{-1}(T')$ whenever $S \subseteq S'$ and $T \subseteq T'$ for subobjects $S, S'$ of $A$ and subobjects $T, T'$ of $B$. For a subobject $S$ of $A$ and a subobject $T$ of $B$, by adjunction we obtain $f(S) \subseteq T$ if and only if $S \subseteq f^{-1}(T)$. For subobjects $S, S'$ of $A$ and subobjects $T, T'$ of $B$, we have $f(S \lor S') = f(S) \lor f(S')$ and $f^{-1}(T \land T') = f^{-1}(T) \land f^{-1}(T')$. The direct and inverse image functors are related to each other as $f^{-1}(f(S)) = S \lor \text{Ker} f$ and $f(f^{-1}(T)) = T \land \text{Im} f$, where $S$ is a subobject of $A$ and $T$ is a subobject of $B$.

The following proposition is about existence of canonical morphisms and about holding certain commutative diagrams between subquotients. The hypothesis and the conclusions of this proposition is going to play a major role for rest of our work.
Proposition 2.1. In the diagram below:

![Diagram](image_url)

(a) let \( e: A \to B \) be a morphism in \( \mathbb{A} \), and let \( U, P \) be subobjects of \( A \) with \( P \leq U \). Let \( V, Q \) be subobjects of \( B \) with \( Q \leq V \). If \( e(P) \leq Q \) and \( U \leq e^{-1}(V) \) then there are canonical morphisms \( e': U \to V \) and \( e'': U/P \to V/Q \) such that \( t_v e' = e_U \) and \( e'' \pi_P = \pi_Q e' \).

(b) Let \( P \leq U \) be subobjects of \( A \), \( Q \leq V \) be subobjects of \( B \), \( R \leq W \) be subobjects of \( C \), \( S \leq X \) be subobjects of \( X \). Let \( e(P) \leq Q \), \( U \leq e^{-1}(V) \); \( e(Q) \leq R \), \( V \leq f^{-1}(W) \); \( g(P) \leq S \), \( U \leq g^{-1}(X) \); \( h(S) \leq R \), \( X \leq h^{-1}(W) \), and \( fe = hg \). Then there are canonical morphisms \( e', f', g', h' \) and \( e'', f'', g'', h'' \) such that \( f''e' = h'g' \) and \( f''e'' = h''g'' \).

(c) Let \( A \xrightarrow{e} B \xrightarrow{f} C \) be composable morphisms in \( \mathbb{A} \). If \( U, P \) are subobjects of \( A \) with \( P \leq U \); \( V, Q \) are subobjects of \( B \) with \( Q \leq V \); \( W, R \) are subobjects of \( C \) with \( R \leq W \); and \( e(P) \leq Q \), \( U \leq e^{-1}(V) \); \( f(Q) \leq R \), \( V \leq f^{-1}(W) \); \( g(P) \leq S \), \( U \leq g^{-1}(X) \), then the sequence \( U/P \to V/Q \to W/R \) is exact if

\[
(e(U) \vee Q) = f^{-1}(R) \wedge V.
\]

Proof. (a) Since \( U \leq e^{-1}(V) \) implies \( e(U) \leq V \), by the universal property of \( t_U \), there exists a unique morphism \( e': U \to V \) such that \( t_v e' = e_U \), and since \( e(P) \leq Q \), by the universal property of \( \pi_Q \), there exists a unique morphism \( e'': U/P \to V/Q \) such that \( e'' \pi_P = \pi_Q e' \).

(b) Applying the argument of (a) on the hypothesis, we obtain the canonical morphisms \( e', f', g', h', e'', f'', g'', h'' \), and the identities:

\[
\begin{align*}
t_v e' &= e_U, \quad e'' \pi_P = \pi_Q e'; \\
t_w f' &= f t_v, \quad f'' \pi_Q = \pi_R f'; \\
&\quad i_{\pi_S} h' = h t_x, \quad h'' \pi_S = \pi_R h'; \\
&\quad i_{\pi_S} g' = g t_U, \quad g'' \pi_P = \pi_S g'.
\end{align*}
\]

To obtain the identity \( f''e' = h'g' \), we notice that

\[
t_w f' e' = f t_v e' = f e t_u = h g t_u = h t_x g' = t_w h' g',
\]

and since \( t_w \) is a monomorphism, we have the desired identity. Finally, to obtain \( f''e'' = h''g'' \), we observe that

\[
\begin{align*}
f'' e'' &= e'' \pi_P = f'' \pi_Q e' = \pi_P f' e' = \pi_R h' g' = h'' \pi_S g' = h'' \pi_P,
\end{align*}
\]

and since \( \pi_P \) is an epimorphism, we have the desired commutativity.

(c) The existence of the canonical morphisms \( U/P \to V/Q \) and \( V/Q \to W/R \) follow from (b). To have the condition (1), we observe that the image of the morphism \( U/P \to V/Q \) is \( (e(U) \vee Q)/Q \), while the kernel of the morphism \( V/Q \to W/R \) is \( (f^{-1}(R) \wedge V)/Q \). \( \square \)
3. *L*-HOMOLOGIES

Let us consider a portion of a double complex with objects and morphisms as shown in the diagram below:

```
   x   w
      ↘   ↘
   m   C
     ↘   ↘
   G
```

where \( w = mx \), \( p = ca \), \( r = ec \), \( q = ge \), \( t = bv \). For every object \( A \) of (2), consider the diagram of canonical monomorphisms induced by the double complex structure:

```
          Ker f
          ↘   ↘
         Kerq
        ↘   ↘
Kere   Kerf
```

Notation. We denote the lattice generated by the subobjects \( \text{Kere}, \text{Kerf}, \text{Kerq} \) by \( \text{L}_\text{Ker} \), while \( \text{L}_\text{Im} \) denotes the lattice generated by the subobjects \( \text{Imc}, \text{Imd}, \text{Imp} \).

**Definition 3.1.** An *L*-complex associated with an object \( A \) of a double complex (2) is an ordered pair of sets \( (M_c, M_d) \) of morphisms such that (i) \( A \) is the codomain for each \( f \in M_c \), (ii) \( A \) is the domain for each \( g \in M_d \), and for all \( f \in M_c, g \in M_d, \text{Im} f \leq \text{Ker} g \).

**Definition 3.2.** An *L*-homology of an object \( A \) of a double complex (2) is a triple \((U, V, U \rightarrow U/V)\) such that \( U \in \text{L}_\text{Ker}, V \in \text{L}_\text{Im}, \text{V} \leq \text{U} \). We say an *L*-homology \( U/V \) is trivial if \( U/V \) is a zero object.

Notation. For simplicity, we use the notation \( U/V \) to denote an *L*-homology instead of the triple as mentioned in Definition 3.2. We denote an *L*-homology and the set of *L*-homologies of an object \( A \) of a double complex \( D \) by \( H_A \) and \( \text{LH}_A \) respectively. The set of all *L*-homologies of \( D \) will be denoted by \( \text{LH}_D \).

**Proposition 3.3.** For every object \( A \) of a double complex (2), there are eighteen *L*-homologies associated with fourteen *L*-complexes as described in the Table 1 below:
Table 1: L-complexes and L-homologies
Proof. From the diagram (3) we notice that in order to satisfy condition \( V \leq U \) of the Definition 3.2 there are eighteen possibilities of \( U/V \) as described in the Table 2 below:

| \( V \) | \( U \) | # |
|---|---|---|
| \( \text{Imd} \) | \( \text{Ker}_e, \text{Ker}_q, \text{Ker}_e \vee \text{Ker}_f \) | 3 |
| \( \text{Imc} \) | \( \text{Ker}_f, \text{Ker}_q, \text{Ker}_e \vee \text{Ker}_f \) | 3 |
| \( \text{Imp} \) | \( \text{Ker}_e, \text{Ker}_f, \text{Ker}_q, \text{Ker}_e \vee \text{Ker}_f, \text{Ker}_e \wedge \text{Ker}_f \) | 5 |
| \( \text{Imc} \vee \text{Imd} \) | \( \text{Ker}_e, \text{Ker}_f, \text{Ker}_q, \text{Ker}_e \vee \text{Ker}_f, \text{Ker}_e \wedge \text{Ker}_f \) | 2 |
| \( \text{Imc} \wedge \text{Imd} \) | \( \text{Ker}_e, \text{Ker}_f, \text{Ker}_q, \text{Ker}_e \vee \text{Ker}_f, \text{Ker}_e \wedge \text{Ker}_f \) | 5 |

Table 2: \( L \)-complexes and \( L \)-homologies

Remark 3.4. The \( L \)-homologies \( A_* \) and \( ^*A \) were introduced in \([1]\) (with notation \( A_\square \) and \( ^\square A \) respectively). For the rest of the \( L \)-homologies, we have used the following notation convention: \( A_h, A_v, \) and \( A_d \) respectively denote the horizontal, vertical, and diagonal \( L \)-homologies. Whenever \( h, v, \) and \( d \) appear as subscripts or superscripts (along with other symbols as superscripts or subscripts respectively), then they mean the denominator or numerator expressions of the \( A_h, A_v, \) and \( A_d \) respectively. The same holds with respect to subscript or superscript \( * \), whereas \( \vee \) and \( \wedge \) respectively mean the usual joins and meets.

Proposition 3.5. For a given double complex, the \( L \)-homologies \( X_d \) of objects \( X \) of the double complex induce a double complex.

Proof. Consider the double complex (2). First, we show that \( A_d \rightarrow B_d \rightarrow E_d \) is a horizontal complex. The proof of the vertical complex is similar. The canonical morphism \( A_d \rightarrow B_d \) (and similarly \( B_d \rightarrow E_d \)) exists because \( e(\text{Ker}_q) = (\text{Imc} \wedge \text{Ker}_g) \leq \text{Ker}_s \) and \( e(\text{Imp}) \leq e(\text{Imc}) = \text{Imr} \). To have the complex, we notice that

\[
e(\text{Ker}_q) \vee \text{Imr} = (\text{Imc} \wedge \text{Ker}_g) \vee \text{Imr} = \text{Imc} \wedge \text{Ker}_g \leq \text{Ker}_b \wedge \text{Ker}_s \leq (\text{Imr} \vee \text{Ker}_b) \wedge \text{Ker}_s.
\]

In order to obtain a double complex, what remains is to show the commutativity of a square having vertices as \( U_d, V_d, W_d, \) and \( X_d \). By the above argument once we have the canonical morphisms \( U_d \rightarrow V_d \rightarrow W_d \) and \( U_d \rightarrow X_d \rightarrow W_d \), the commutativity follows from the Proposition 2.1(a). \(

In next section, we are going to describe two poset structures of \( L \)-homologies and the category of \( L \)-homologies.

4. Structures of \( L \)-Homologies

Definition 4.1. Let \( H_A = X/Y \) and \( H'_A = U/V \) be two elements of \( LH_A \). We define a relation \( \rightarrow \) on \( LH_A \) as \( H_A \rightarrow H'_A \) if \( X \leq U \) and \( V \leq Y \). We define the join of \( H_A \) and \( H'_A \) as \( H_A \vee H'_A = (X \vee U)/(Y \wedge V) \).

Theorem 4.2. For each object \( A \) of a double complex \( \mathcal{D} \), the pair \( (LH_A, \rightarrow) \) forms a join-semilattice with \( A_d \) as the top element.
Proof. The partial order relation $\leq$, and the definition of the relation $\rightarrow$ shows that $(\text{LH}_{A}, \rightarrow)$ is a poset. From the inclusion diagram (3) we notice that for $X, U \in \text{Lker}$ and $Y, V \in \text{Lim}$ we have $Y \vee V \leq X \vee U$. Therefore, the join of two elements of $\text{LH}_{A}$ exists. We observe that $U \leq \text{Ker}q$ and $\text{Lim}p \leq V$ for any double complex homology $U/V \in \text{LH}_{A}$, i.e. $A_{d}$ is the top element of $(\text{LH}_{A}, \rightarrow)$ as asserted.

**Remark 4.4.** The meet $X/Y \wedge U/V = (X \wedge U)/(Y \vee V)$ of two elements $X/Y$ and $U/V$ of $\text{LH}_{D}$ may not exist. For example, the meet of $A_{h}$ and $A_{v}$ does not exist because $\text{Im}c \vee \text{Im}d \leq \text{Ker}e \wedge \text{Ker}f$.

**Definition 4.5.** Let $f : A \rightarrow B$ be a morphism of $\mathfrak{D}$ and $H_{A}, H_{B}$ be elements of $\text{LH}_{A}$ and $\text{LH}_{B}$ respectively. We define a relation $\preceq$ on $\text{LH}_{D}$ as $H_{A} \preceq H_{B}$ if there exists a canonical morphism $\phi : H_{A} \rightarrow H_{B}$ in the sense of Proposition 2.1.

Given an object $A$ of $\mathfrak{D}$, we observe that the relation $\preceq$ in Definition 4.1 and the relation $\leq$ in Definition 4.5 are inter-related as follows.

**Proposition 4.6.** Given L-homologies $H_{A} = X/Y$ and $H_{A}^{l} = U/V$ of an object $A$ we have $H_{A} \leftrightarrow H_{A}^{l} \iff H_{A} \leq H_{A}^{l}$ if and only if $Y = V$.

**Theorem 4.7.** For each object $A$ of the double complex (2), the pair $(\text{LH}_{A}, \preceq)$ forms a poset with bottom and top elements respectively as $A$ and $A_{v}$.

**Proof.** The fact that the relation $\preceq$ is partially ordered, follows from the Proposition 2.1(a) and Definition 4.5. To find the top and bottom elements, let $U/V \in \text{LH}_{A}$. From the inclusion diagram (5), we obtain the following inclusions:

$$Ker e \wedge Ker f \leq U \leq Ker q, \quad \text{Im}p \leq V \leq \text{Im}c \vee \text{Im}d,$$

and by applying Proposition 2.1(a), we have

$$(Ker e \wedge Ker f)/\text{Im}p \leq U/V \leq Ker q/(\text{Im}c \vee \text{Im}d).$$

In other words, $A$ and $A_{v}$ are respectively the bottom and the top elements of the poset $(\text{LH}_{A}, \preceq)$ as asserted.

**Proposition 4.8.** Given a morphism $e : A \rightarrow B$ of (2), the trivialities of $A_{h}$ and $B_{h}$ imply an isomorphism between the top element $A_{v}$ of $(\text{LH}_{A}, \preceq)$ and the bottom element $B$ of $(\text{LH}_{B}, \leq)$.

**Proof.** The existence of the connecting morphism $A_{v} \rightarrow B$ follows from Proposition 2.1(a). Since $B_{h} = 1$, we obtain $e(Ker q) = \text{Im}e \wedge Ker g = Ker b \wedge Ker g$ whereas because of $A_{h} = 1$, we get $e^{-1}(\text{Im}r) = \text{Im}c \vee Ker e = \text{Im}c \vee \text{Im}d$. This proves the required isomorphism.

**Remark 4.9.** In [1], the Proposition 4.8 has been derived as a corollary of the salamander lemma.
4.1. **Classification problem of exact sequences of $L$-homologies.** For a given double complex $\mathfrak{D}$ and a composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathfrak{D}$, we observe that there are four possible types of exact sequences involving $L$-homologies of $X$, $Y$, and $Z$:

\begin{itemize}
  \item[(I)] $H_X \cong H_X \leq H_X$, (II) $H_X \leq H_X \leq H_Y$, (III) $H_X \leq H_Y \leq H_Y$, (IV) $H_X \leq H_Y \leq H_Z$.
\end{itemize}

The Proposition 4.10 below gives examples of exact sequences of $L$-homologies of the types (II) $H_X \leq H_X \leq H_Y$ and (III) $H_X \leq H_Y \leq H_Y$, where $X$ and $Y$ are two different objects of a double complex $\mathfrak{D}$.

**Proposition 4.10.** In the double complex (2), the following sequences are exact:

II. (a) $A_h \leq A_v \leq A_w$, (b) $b^A_d \leq d^A_v \leq v^B_d$, (c) $b^A_d \leq A_v \leq v^B_d$; (d) $C_v \leq C_s \leq C_d$.

III. (a) $A_\alpha \leq A_h \leq A_v$, (b) $C_s \leq A_v \leq A_d$; (c) $C_v \leq b^A_d \leq d^A_v$; (d) $A_s \leq B_v \leq B_s$.

\begin{proof}
To prove the existence of a connecting morphism of a sequence, we apply Proposition 2.11(a), whereas for the exactness we verify the condition (1) of the Proposition 2.1(c). To show the working scheme, here we only prove the exactness of II(a) and III(a).

II(a): For the left hand side of (1), we have $\text{Ker}(e) \cap \text{Im}(e) = \text{Ker}(e) \cap \text{Im}(e)$, whereas for the right hand side of (1), we get $e^{-1}(\text{Im}(e) \cap \text{Ker}(q)) = e^{-1}(e(\text{Im}(e) \cap \text{Ker}(q))) \cap \text{Ker}(q) = \text{Ker}(e) \cap \text{Im}(e)$.

III(a): For the left hand side of (1), we have $c(\text{Ker}(e) \cap \text{Im}(e)) = (\text{Ker}(e) \cap \text{Im}(e)) \cap \text{Im}(c)$, and for the right hand side of (1), we get $(\text{Im}(e) \cap \text{Im}(c)) \cap \text{Ker}(c) = (\text{Im}(e) \cap \text{Im}(c)) \cap \text{Im}(c)$ (by modularity).
\end{proof}

**Remark 4.11.** The examples II(a) and III(a) are the segments of the salamander lemma as proposed in [1].

**Definition 4.13.** Given a double complex $\mathfrak{D}$, we define

(i) a category $\text{Com}_\mathfrak{D}$ consisting objects of $\mathfrak{D}$ and morphisms of $\mathfrak{D}$ along with identity morphisms of each object of $\mathfrak{D}$;

(ii) a category $\text{Ilg}_\mathfrak{D}$ having objects as $L$-homologies of $\mathfrak{D}$ and morphisms as canonical morphisms between $L$-homologies (in the sense of Proposition 2.11(a)).

**Definition 4.14.** We define a functor $\mathcal{F}: \text{Ilg}_\mathfrak{D} \rightarrow \text{Com}_\mathfrak{D}$ as follows:

$$
\mathcal{F}(H_A) = A; \quad \mathcal{F}(H_A \rightarrow H_B) = \begin{cases} 1_A, & A = B, \\ A \rightarrow B, & \text{otherwise}. \end{cases}
$$

For every object $A$ of $\text{Com}_\mathfrak{D}$, the fibre of $\mathcal{F}$ at $A$, denoted by $\text{Fib}(A)$, is the subcategory of $\text{Ilg}_\mathfrak{D}$ consisting of those objects and morphisms in $\text{Ilg}_\mathfrak{D}$ which, by $\mathcal{F}$, are mapped to $A$ and $1_A$, respectively.

**Remark 4.15.** We notice that $\text{Fib}(A)$ is nothing but the set $\text{LH}_A$ of $L$-homologies of the object $A$. 
Definition 4.16. A functor $G : A \to X$ is said to be a Grothendieck fibration, if for every object $S$ in $A$, and every morphism $\phi : X \to G(S)$ in $X$, there exists a morphism $f : A \to S$ in $A$ such that:

(a) $G(f) = \phi$ (and in particular $G(A) = X$);

(b) given a morphism $g : B \to S$ in $A$ and a morphism $\psi : G(B) \to X$ in $X$ with $G(g) = \phi \psi$, there exists a unique morphism $h : B \to A$ with $G(h) = \psi$ and $fh = g$.

Theorem 4.17. The functor $F$ in (4.14), is a faithful amnestic Grothendieck fibration. Furthermore, $F$ has a left adjoint and a right adjoint:

$$
x \quad \Rightarrow \quad F \\ 
I \quad \downarrow \quad \downarrow \quad T \\
\text{Com}_D \quad \leftarrow \quad \frac{\text{LHom}_D}{\text{hlg}_D} \\
\text{hlg}_D \\
$$

where to each object $A$ of $\text{Com}_D$, the functors $I$ and $T$ assign respectively the bottom and top elements of $\text{Fib}(A)$.

Proof. In the category $\text{hlg}_D$, we notice that whenever a morphism exists between any two $L$-homologies, it is unique. Therefore, the functor $F$ obviously satisfy the property: $(F(f) = F(g)) \Rightarrow (f = g)$, for any two parallel morphisms $f$ and $g$ in $\text{hlg}_D$, i.e. $F$ is faithful, as asserted.

Let $f : U/V \to U'/V'$ be an isomorphism in $\text{hlg}_D$ such that $F(f) = 1_A$. By definition of $F$, this implies $U/V$ and $U'/V'$ are in $\text{Fib}(A)$, and isomorphism forces $f = 1_{U/V}$, i.e. $F$ is amnestic.

To prove $F$ is a Grothendieck fibration, let us consider the following two cases to verify the conditions of the Definition 4.16:

- $\phi$ an identity morphism: We take $f$ as $1_S$, and $h$ will be the unique canonical morphism $g$ itself.
- $\phi$ is not an identity morphism: We take $A$ as $\star X$ and the morphism $f$ is the (unique) composite $\star X \to X \to \star S \to S$. Since for any morphism $P \to Q$ in $\text{Com}_D$ there exists a unique canonical morphism $P \to \star Q$ in $\text{hlg}_D$, the existence of the unique morphism $h$ with the desired property follows.

We prove the adjunction $I \dashv F$, and the proof of $F \dashv T$ is similar. To prove $I \dashv F$, it is sufficient to prove the map

$$
\phi_{X,H_A} : \text{hom}_{\text{hlg}_D}(\star X,H_A) \to \text{hom}_{\text{Com}_D}(X,A)
$$

is a bijection for each object $H_A$ in $\text{hlg}_D$. But the bijection follows from the fact that for each $H_A$, there is a unique canonical morphism $f : \star X \to H_A$ and its image $\phi_{X,H_A}(f)$ (by definition of $F$) is the map $X \to A$ of $\text{Com}_D$.

Remark 4.18. The functor $F$ is an example of a $Z$. Janelidze form in the sense of [6]. For further applications of Janelidze forms, see [4] and the references given there.

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