Weak Solutions to Monge–Ampère Type Equations on Compact Hermitian Manifold with Boundary

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Abstract
We prove the bounded subsolution theorem for the complex Monge–Ampère type equation, with the right-hand side being a positive Radon measure, on a compact Hermitian manifold with boundary.

Keywords Weak solutions · Monge–Ampère equations · Hermitian manifolds

1 Introduction
Let $(M, \omega)$ be a smooth compact $n$-dimensional Hermitian manifold with the non-empty boundary $\partial M$. In [24], we studied weak quasi-plurisubharmonic solutions of the Dirichlet problem for the complex Monge–Ampère equation. In this paper, we extend those results to the complex Monge–Ampère type equation where the right-hand side depends also on the solution.

Let $\mu$ be a positive Radon measure on $M = \overline{M} \setminus \partial M$. Suppose that $F(u, z) : \mathbb{R} \times M \to \mathbb{R}^+$ is a non-negative function. Let $\varphi \in C^0(\partial M)$. We consider the Dirichlet problem
\[ \begin{cases} u \in PSH(M, \omega) \cap L^\infty(M), \\ (\omega + dd^c u)^n = F(u, z)\mu, \\ u = \varphi \quad \text{on } \partial M. \end{cases} \quad (1.1) \]

When \( M \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \), the problem for plurisubharmonic functions (\( \omega \) is just the zero form) was studied by Bedford and Taylor [2] for \( \mu = dV_{2n} \) the Lebesgue measure, \( F \in C^0(\mathbb{R} \times \overline{M}) \) and \( F^{1/n} \) convex and non-decreasing in \( u \). Generalizations for weak solutions were done by Cegrell and Kołodziej [6, 8, 22] with more general function \( F(u, z) \) and a positive Radon measures \( \mu \). Classical smooth solutions were obtained in [5]. On the product of a compact Kähler manifold and an annulus this problem, with \( F \equiv 0 \), is the geodesic equation on the space of Kähler potentials of the manifold. For \( F(u, z) = e^{\lambda u} \) with \( \lambda \in \mathbb{R} \), one obtains the Kähler–Einstein metric equations as in Cheng and Yau [9], where the authors obtained complete Kähler–Einstein metrics, but then the potentials tend to infinity when the argument approaches the boundary. Berman [4] discovered that one may use the solution of a family of Monge–Ampère type equation to approximate various envelopes of quasi-plurisubharmonic functions. This approximation process is very useful in studying the regularity of global envelopes (see [10, 14, 28]).

On a compact Hermitian manifold with boundary, if \( F(u, z) = 1 \) and \( \mu \) is a positive Radon measure, we obtained weak solutions of the problem under the hypothesis that a subsolution exists, [24]. To deal with a general (non-Kähler) Hermitian metric \( \omega \), we needed to adapt a suitable comparison principle from [23]. Furthermore, we can no longer rely on assumption that the boundary is pseudoconvex. We bypassed this by employing the Perron envelope and the existence of a bounded subsolution. Here, we use similar strategy to obtain the following result.

**Theorem** (c.f. Theorem 3.2) Assume that \( F(u, z) \) is a bounded non-negative function which is continuous and non-decreasing in the first variable and \( \mu \)-measurable in the other one. Let \( \mu \) be a positive Radon measure which is locally dominated by Monge–Ampère measures of bounded plurisubharmonic functions. Then, the Dirichlet problem \((1.1)\) has a solution if and only if there is a bounded subsolution \( u \in PSH(M, \omega) \cap L^\infty(M) \) satisfying \( \lim_{z \to x} u(z) = \varphi(x) \) for every \( x \in \partial M \) and

\[(\omega + dd^c u)^n \geq F(u, z)\mu \quad \text{on } M.\]

In many cases of interest, the extra assumption on \( \mu \) (the local domination by Monge–Ampère measures of bounded plurisubharmonic functions) is always satisfied once the subsolution \( u \) exists, for example, \( F(t, z) = e^{\lambda t} \) with \( \lambda \in \mathbb{R} \) or \( \mu = \omega^n \). If \( F(t, z) = e^{\lambda t} \) with \( \lambda > 0 \), then we also have the uniqueness of the solution (Corollary 3.3).

**Organization** In Sect. 2, we study the problem \((1.1)\) in a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \). We give a partial generalization of a result in [8] and study the convergence in capacity in dimension \( n = 2 \). In Sect. 3, we prove the main theorem. We also give a sketch of the proof of Hölder continuity of solutions of the Laplace equation on compact Hermitian manifold with boundary in the appendix.
2 Monge–Ampère type equations and stability of solutions

In this section, we first study the problem \((1.1)\) in the special case \(M \equiv \Omega\) a bounded strictly pseudoconvex domain in \(\mathbb{C}^n\). This will provide the construction of the lift of quasi-plurisubharmonic function in the Perron envelope method for the manifold case.

Let us recall the following version of the comparison principle for a background Hermitian metric, see [23, Theorem 3.1].

**Lemma 2.1** Let \(\Omega\) be a bounded open set in \(\mathbb{C}^n\). Fix \(0 < \theta < 1\). Let \(u, v \in PSH(\Omega, \omega) \cap L^\infty(\Omega)\) be such that \(\lim\inf_{z \to \partial \Omega} (u - v) \geq 0\). Suppose that \(-s_0 = \sup_{\Omega} (v - u) > 0\) and \(\omega + dd^c v \geq \theta \omega\) in \(\Omega\). Then, for any \(0 < s < \theta_0 := \min\{\frac{\theta n}{16B}, |s_0|\}\),

\[
\int_{\{u < v + s_0 + s\}} (\omega + dd^c v)^n \leq \left(1 + \frac{sB}{\theta n} C_n\right) \int_{\{u < v + s_0 + s\}} (\omega + dd^c v)^n,
\]

where \(C_n\) is a dimensional constant and \(B > 0\) is a constant such that on \(\Omega\),

\[-B\omega^2 \leq 2n dd^c \omega \leq B\omega^2, \quad -B\omega^3 \leq 4n^2 d\omega \wedge d^c \omega \leq B\omega^3.\]

It gives a useful comparison principle for solutions of Monge–Ampère type equations. We shall use it frequently in the paper.

**Proposition 2.2** Let \(\Omega\) be a bounded open set in \(\mathbb{C}^n\). Let \(u, v \in PSH(\Omega, \omega) \cap L^\infty(\Omega)\) be such that \(\lim\inf_{z \to \partial \Omega} (u - v)(z) \geq 0\). Let \(F(t, z) : \mathbb{R} \times \Omega \to \mathbb{R}^+\) be a non-negative function which is non-decreasing in \(t\) and \(d\mu\)-measurable in \(z\). Suppose \(\mu \leq v\) as measures and

\[(\omega + dd^c u)^n = F(u, z)\mu, \quad (\omega + dd^c v)^n = F(v, z)v\]

in \(\Omega\). Then, \(u \geq v\) on \(\Omega\).

**Proof** We may additionally assume that \(\lim\inf_{z \to \partial \Omega} (u - v) \geq 2a > 0\). The general case follows after replacing \(u\) by \(u + 2a\) and letting \(a \to 0\). Thus \(\{u < v\} \subset \Omega' \subset \subset \Omega\) for some open set \(\Omega'\). By subtracting from \(u, v\) a constant \(C\) and replacing \(F(t, z)\) by \(F(t + C, z)\), we may also assume that \(u, v \leq 0\). Arguing by contradiction, suppose that \(\{u < v\}\) was non-empty. Since \(\Omega\) is bounded, there exists a bounded, strictly plurisubharmonic function \(\rho \in C^2(\overline{\Omega})\) such that \(-C \leq \rho \leq 0\). Then one can first multiply \(\rho\) by small positive constant and then fix small positive constants \(\theta, \tau > 0\) such that the set \(\{u < (1 + \tau)^\frac{1}{\tau} v + \rho\} \subset \subset \Omega\) is non-empty and

\[dd^c \rho \geq 2\theta \omega, \quad 1 + \theta \geq (1 + \tau)^\frac{1}{\tau} .\]
Put $\hat{v} = (1 + \tau)^{1/2} v + \rho$. Then,

$$\omega^n_{\hat{v}} = \left( \omega + dd^c \rho + dd^c (1 + \tau)^{1/2} v \right)^n \geq \left[ (1 + 2\theta) \omega + dd^c (1 + \tau)^{1/2} v \right]^n \geq (1 + \tau) \omega^n_v. \tag{2.1}$$

Denote by $U(s)$ the set $\{ u < \hat{v} + s_0 + s \}$ where $-s_0 = \sup_{\Omega} (\hat{v} - u) > 0$. Then, for $0 < s < |s_0|$,

$$U(s) \subset \Omega, \quad \sup_{U(s)} \{ \hat{v} + s_0 + s - u \} = s.$$

It follows from Lemma 2.1 that for every $0 < s < \min\{\frac{\theta^n}{16B}, |s_0|\},$

$$0 < \int_{U(s)} \omega^n_{\hat{v}} \leq \left( 1 + \frac{sB}{\theta^n} C_n \right) \int_{U(s)} \omega^n_u,$$

where the first inequality holds because $\omega + dd^c \hat{v} \geq \theta \omega$. Note that $u < \hat{v}$ on $U(s)$. Hence, the monoticity of $F$ implies

$$\omega^n_u = F(u, z) \mu \leq F(u, z)v \leq F(\hat{v}, z)v \leq F(v, z)v = \omega^n_v.$$

Combining this and (2.1), we get

$$(1 + \tau) \int_{U(s)} \omega^n_{\hat{v}} \leq \left( 1 + \frac{sB}{\theta^n} C_n \right) \int_{U(s)} \omega^n_v.$$ 

Therefore, $0 < \tau \leq sBC_n/\theta^n$, which is impossible for $s > 0$ small enough. Hence, $u \geq v$ on $\Omega$ and the proof is completed. \hfill \Box

Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\varphi \in C^0(\partial \Omega)$ and $\mu$ be a positive Radon measure in $\Omega$. Suppose that $F(u, z) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^+$ is non-negative function which is continuous and non-decreasing in the first variable and $\mu$-measurable in the other one. Consider the following Dirichlet problem

$$\begin{cases}
  u \in PSH(\Omega, \omega) \cap L^\infty(\Omega), \\
  (\omega + dd^c u)^n = F(u, z) \mu, \\
  \lim_{z \rightarrow x} u(z) = \varphi(x) \quad \text{for } x \in \Omega.
\end{cases} \tag{2.2}$$

By Proposition 2.2, there is at most one solution to the problem. Furthermore, in the special case $\mu \equiv 0$, by [23, Theorem 4.2], there is a unique continuous (maximal) $\omega$-plurisubharmonic (or $\omega$-psh for short) function $h$ solving

$$(\omega + dd^c h)^n \equiv 0, \quad h = \varphi \quad \text{on } \partial \Omega. \tag{2.3}$$
We now study the existence of the solution. The Cegrell class $E_0(\Omega)$ is the set of all functions $v \in PSH(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\lim_{z \to \partial \Omega} v(z) = 0 \text{ and } \int_{\Omega} (dd^c v)^n < +\infty.$$ 

The following generalizes [22, Theorem 1.1], where the case $\omega = 0$ was treated.

**Theorem 2.3** Suppose that $F(t, z)$ is a bounded non-negative function which is continuous and non-decreasing in the first variable and $\mu$-measurable in the second one. Suppose that $d\mu \leq (dd^c v)^n$ for a function $v \in E_0(\Omega)$. Then, there exists a unique bounded $\omega$-psh function in $\Omega$ solving

$$(\omega + dd^c u)^n = F(u, z) d\mu,$$

$$\lim_{z \to x} u(z) = \varphi(x) \text{ for } x \in \partial \Omega.$$ 

**Proof** The proof follows the lines of [22, Theorem 1.1] (see also [6]), which applied Schauder’s fixed theorem. In the presence of the Hermitian background form, we need results from [24] to verify the hypothesis of that theorem.

We start with the following simple observation.

**Lemma 2.4** We may assume that $\mu$ has compact support in $\Omega$.

**Proof** Suppose that the problem is solvable for compactly supported measures. Let $\{\Omega_j\}_{j \geq 1}$ be an increasing exhausting sequence of open sets that are relatively compact in $\Omega$. Then we can find a sequence of functions: $u_j \in PSH(\Omega, \omega) \cap L^\infty(\Omega)$ with $\lim_{z \to x} u_j(z) = \varphi(x)$ solving

$$(\omega + dd^c u_j)^n = F(u_j, z) 1_{\Omega_j} d\mu.$$ 

By Proposition 2.2, this is a decreasing sequence and $v + h \leq u_j \leq h$ on $\overline{\Omega}$, where $h$ is the maximal function defined in (2.3). Put $u = \lim u_j$. By monotone convergence theorems [13] (see also [3]), we have that $\omega^u_{1_{\Omega_j}}$ converges weakly to $\omega^u_n$. Also $F(u_j, z) 1_{\Omega_j} d\mu$ converges weakly to $F(u, z) d\mu$ when $j \to \infty$. Thus, $u$ satisfies the equation $\omega^u_n = F(u, z) d\mu$. \qed

By Lemma 2.4, we assume that $\text{supp} \mu$ is compact in $\Omega$. Since $F$ is bounded, without loss of generality, we may assume that $0 \leq F \leq 1$, as we can rescale $\mu$ and $v$ by a positive constant. Let $h$ be a maximal $\omega$-psh in $\Omega$ as in (2.3). Then, the set

$$\mathcal{A} = \{u \in PSH(\Omega, \omega) : v + h \leq u \leq h\}$$ 

is a convex and bounded set in $L^1(\Omega)$ with respect to $L^1$-topology. Thus, it is a compact set. We define the map $T : \mathcal{A} \to \mathcal{A}$, where $T(u) = w$ is the solution of

$$w \in \mathcal{A}, \quad (\omega + dd^c w)^n = F(u, z) d\mu,$$ 

$\square$
where this solution is uniquely determined by [24, Theorem 3.1].

Next, we shall verify that $T$ is continuous. Let $\{u_j\}$ be a sequence in $\mathcal{A}$ such that $u_j \to u$ in $L^1(\Omega)$. Set $w = T(u)$ and $w_j = T(u_j)$. The continuity of $T$ will follow if we have

$$\lim \sup w_j \leq w \leq \lim \inf w_j. \quad (2.4)$$

Since $\mathcal{A}$ is compact in $L^1(\Omega)$, we may assume that $w_j$ converges to $w$ in $L^1(\Omega)$. In the arguments, we often pass to a subsequence which does not affect the proof. We also skip the renumbering of indices of those subsequences.

We need the following result which is essentially due to Cegrell [7].

**Lemma 2.5** There is a subsequence of $\{F(u_j, z)\}_{j \geq 1}$ that converges to $F(u, z)$ in $L^1(d\mu)$.

**Proof of Lemma 2.5** Since $u_j$ is uniformly bounded, we may subtract a constant and assume that $u_j, u \leq 0$ for all $j \geq 1$. Then it follows from [24, Lemma 2.1] that $\lim \int \Omega u_j d\mu = \int \Omega u d\mu$. Since $\mu$ is dominated by capacity, it follows from [24, Corollary 2.2] that $u_j \to u$ in $L^1(d\mu)$. Passing to a subsequence, we have that $u_j$ converges to $u$ almost everywhere in $d\mu$. Since $F(t, z)$ is continuous in $t$, the sequence $F(u_j, z)$ converges $\mu$-a.e. to $F(u, z)$. By the Lebesgue-dominated convergence theorem, we get the conclusion. \qed

After passing to a subsequence, we may also assume that $u_j$ converges to $u$ almost everywhere in $d\mu$. Let us check the first inequality in (2.4). Define $\widehat{u}_k = \inf_{j \leq k} u_j$ and denote by $\widehat{w}_k \in \mathcal{A}$ the sequence of solutions to

$$(\omega + dd^c \widehat{w}_k)^n = F(\widehat{u}_k, z)d\mu.$$  

Since $(\omega + dd^c \widehat{w}_k)^n$ is increasing, by the comparison principle [23, Corollary 3.4], $\widehat{w}_k$ decreases to $\widehat{w} \in \mathcal{A}$. As $u_j$ converges to $u$, we have that $\widehat{u}_k$ increases to $u$ (almost everywhere in $d\mu$). Therefore,

$$(\omega + dd^c \widehat{w})^n = \lim_{k \to +\infty} (\omega + dd^c \widehat{w}_k)^n = \lim_{k \to +\infty} F(\widehat{u}_k, z)d\mu = F(u, z)d\mu,$$

where the first equality holds by the convergence theorem in [13] (see also [3]) and the last equality follows from Lemma 2.5 and the Lebesgue monotone convergence theorem. By the uniqueness of the solution, $w = \widehat{w}$. Note that $\widehat{w}_k \geq w_k$ because $(\omega + dd^c w_k)^n \geq (\omega + dd^c \widehat{w}_k)^n$. Therefore, $w = \lim \widehat{w}_k \geq \lim \sup w_k$.

Next we prove the second inequality in (2.4). By Hartogs’ lemma for $\omega$-psh functions, $u = (\lim \sup_{j \to +\infty} u_j)^*$. Define $\widehat{u}_k = (\sup_{j \geq k} u_j)^*$ and $\widehat{w}_k = T(\widehat{u}_k)$. Since $(\omega + dd^c \widehat{w}_k)^n = F(\widehat{u}_k, z)d\mu$ is decreasing, again by the comparison principle [23], one gets that the sequence $\widehat{w}_k$ is increasing to some $\widehat{w} \in \mathcal{A}$. Note also that

$$(\omega + dd^c w_k)^n = F(u_k, z)d\mu \leq F(\widehat{u}_k, z)d\mu = (\omega + dd^c \widehat{w}_k)^n.$$
Hence, \( \tilde{w}_k \leq w_k \). Furthermore, \( \tilde{w}_k \downarrow u \) (almost everywhere in \( d\mu \)). By the convergence theorem in [DK12] and Lemma 2.5, we infer

\[
(\omega + dd^c \tilde{w})^n = \lim_{k \to +\infty} F(\tilde{u}_k, z) d\mu = F(u, z) d\mu.
\]

As above by the uniqueness, \( w = \tilde{w} \). Then, \( w = \lim \tilde{w}_k \leq \lim \inf w_k \).

Thus, we conclude the continuity of \( T \). The Schauder theorem says that \( T \) has a fixed point \( T(u) = u \). This gives the existence of a solution to the Dirichlet problem. \( \square \)

**Remark 2.6** Thanks to the first lemma of the above proof, the theorem can be extended to the case of unbounded \( F \), e.g., \( F(u, z) = |u|^{-\alpha} \) with \( \alpha > 0 \), in the Dirichlet problem:

\[
\begin{cases}
  u \in PSH(\Omega) \cap L^\infty(\overline{\Omega}), \\
  (dd^c u)^n = |u|^{-\alpha} d\mu, \\
  u = 0 \quad \text{on } \partial\Omega.
\end{cases}
\]

Indeed, suppose that there exists a subsolution \( u \in PSH(\Omega) \cap L^\infty(\overline{\Omega}) \) such that \( u = 0 \) on \( \partial\Omega \) and \( (dd^c u)^n \geq |u|^{-\alpha} d\mu \). The proof above shows that for the sequence \( \Omega_j \uparrow \Omega \) in Lemma 2.4, we can find \( u_j \in PSH(\Omega) \cap L^\infty(\overline{\Omega}) \) such that

\[
(dd^c u_j)^n = |u_j|^{-\alpha} 1_{\Omega_j} d\mu, \quad u_j = 0 \quad \text{on } \partial\Omega.
\]

Since \( |u_j|^\alpha (dd^c u_j)^n \leq |u|^\alpha (dd^c u)^n \), by the comparison principle \( u \leq u_j \leq 0 \). Similarly, \( u_j \) is a uniformly bounded and decreasing sequence. Therefore, \( u = \lim_{j \to \infty} u_j \) is the unique solution to the Dirichlet problem.

Furthermore, one can obtain better regularity of the solution under stronger assumptions on \( \mu \). Let us consider \( \mu = f(z) dV_{2n} \) with \( f \in L^1(\Omega) \) and let \( \rho \) be the strictly plurisubharmonic defining function for \( \Omega \). Assume that \( |\rho|^{-\alpha} \in L^p(\Omega) \) for some \( p > 1 \) (e.g., when \( 0 < \alpha < 1 \) and \( f \in L^p(\Omega) \) is bounded near the boundary \( d\Omega \)). Then, we have

\[
|\rho|^{-\alpha} f dV_{2n} = (dd^c w)^n, \quad w = 0 \quad \text{on } d\Omega
\]

for some Hölder continuous plurisubharmonic function \( w \) (see [16]). Therefore,

\[
f dV_{2n} = |\rho|^\alpha (dd^c w)^n \leq |w + \rho|^\alpha [dd^c (w + \rho)]^n.
\]

Therefore, \( \rho + w \) is a Hölder continuous subsolution to the Dirichlet problem. So there is a unique bounded solution \( u \). Furthermore,

\[
|u|^{-\alpha} f dV_{2n} = (|\rho|/|u|)^\alpha |\rho|^{-\alpha} f dV_{2n}.
\]

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A simple use of Hölder inequality shows that the measure on the right-hand side is well dominated by capacity on every compact set of $\Omega$. Thanks to [24, Remark 4.5], we get that $u$ is continuous. Note that the Hölder continuity of $u$ is proved in [18].

The above problem appeared in [2] and was more recently studied in [11] and [18, 19].

It is well known that the convergence of $\omega$-psh functions in $L^p(\Omega)$ does not imply the weak convergence of their associated Monge–Ampère operators. However, under the assumption of a uniform bound of the Monge–Amère measures, this is the case, see [8, Theorem 2.2]. We give a (partial) corresponding stability result for Hermitian background metrics.

**Theorem 2.7** Suppose $d\mu = (dd^c v)^n$ for some $v \in E_0(\Omega)$. Let $0 \leq f_j \leq 1$ be a sequence of $d\mu$–measurable functions such that $f_j d\mu$ converges weakly to $f d\mu$ as measures. Suppose that $u_j$ solves

\[
\begin{aligned}
&u_j \in PSH(\Omega, \omega) \cap L^\infty(\Omega), \\
&(\omega + dd^c u_j)^n = f_j d\mu, \\
&\lim_{z \to x} u_j(z) = \varphi(x) \text{ for } x \in d\Omega,
\end{aligned}
\]

for a continuous function $\varphi$.

Then, for $u = \lim u_j$ in $L^1(\Omega)$, we have $(\omega + dd^c u)^n = f d\mu$.

**Proof** Let $h$ be the maximal function defined in (2.3). Then by the comparison principle [23, Corollary 3.4], we have $h + v \leq u_j \leq h$. Thus, the sequence $\{u_j\}$ is uniformly bounded and this allows to apply [24, Lemma 3.5 c)] and conclude that there is a subsequence $\{u_{js}\}$ such that

\[
\lim_{j_s \to \infty} \int_\Omega |u_{js} - u|((\omega + dd^c u_{js})^n = 0,
\]

and moreover $\omega_{u_{js}}^n$ converges weakly to $\omega_u^n = f d\mu$, see [24, Lemma 3.6]. This gives the result. \(\square\)

In this statement, we would like to have also that a subsequence of $u_j$ converges to $u$ in capacity. Near the end of this section (Corollary 2.10), we are able to do this for $n = 2$. Recall that the Bedford-Taylor capacity for a Borel set $E \subset \Omega$ is defined by

\[
cap(E) = \cap(E, \Omega) = \sup \left\{ \int_E (dd^c v)^n : v \in PSH(\Omega), -1 \leq v \leq 0 \right\}.
\]

Note that there is another (natural) capacity associated with the metric $\omega$ which is equivalent to the one above (see [24, Lemma 5.6]).

**Proposition 2.8** Let $\{u_j\}$ be the sequence in Theorem 2.7 in the case of smooth $\varphi$. Then, $u_j$ converges to $u$ in capacity if and only if

\[
\lim_{j \to +\infty} \int_\Omega |u_j - u|\omega_{u_j}^k \wedge \omega^{n-k} = 0, \quad k = 0, \ldots, n.
\]
Proof Suppose that $u_j \to u$ in capacity, i.e., for a fixed $\varepsilon > 0$, we have

$$\lim_{j \to +\infty} \text{cap}(|u_j - u| > \varepsilon) = 0.$$ 

Let $g$ be a strictly plurisubharmonic function in a neighborhood of $\Omega$ such that $dd^c g \geq \omega$ and $g = -\varphi$ on $\partial \Omega$. Denoting $\hat{u}_j = u_j + g$, we have $\hat{u}_j \to \hat{u} = u + g$ in capacity. Let $\rho$ be a strictly plurisubharmonic defining function of $\Omega$ such that $dd^c \rho \geq \omega$ on $\Omega$. By [24, Lemma 3.3, Corollary 3.4],

$$\sup_{j \geq 1} \int_{\Omega} (dd^c \hat{u}_j)^k \wedge (dd^c \rho)^{n-k} \leq C$$

(2.8)

for some $C$. We can repeat the argument of [24, Lemma 2.3], with the sequence $\{w_j\}_{j \geq 1}$ in the place of $\{\hat{u}_j\}_{j \geq 1}$, to obtain

$$\lim_{j \to +\infty} \int \frac{|u_j - u| (dd^c \hat{u}_j)^k \wedge (dd^c \rho)^{n-k}}{n} = 0,$$

This gives the proof the necessary condition.

It remains to prove the other implication. Suppose that we have the limit (2.7) for all $k = 0, \ldots, n$. Note that

$$\text{cap}(|u_j - u| > \varepsilon) = \text{cap}(|\hat{u}_j - \hat{u}| > \varepsilon) \leq \text{cap}(\hat{u}_j - \hat{u} > \varepsilon) + \text{cap}(\hat{u} - \hat{u}_j > \varepsilon).$$

By Hartogs’ Lemma, it is easy to see that $\max\{\hat{u}_j, \hat{u}\} \to \hat{u}$ in capacity. Hence, $\lim_{j \to +\infty} \text{cap}(\hat{u}_j - \hat{u} > \varepsilon) = 0$. The remaining term is estimated as in [8, page 718] via the comparison principle for plurisubharmonic functions:

$$\text{cap}(\hat{u}_j < \hat{u} - \varepsilon) \leq \frac{2}{\varepsilon} \int_{|u_j - u - \varepsilon/2|} (dd^c \hat{u}_j)^n \leq \frac{2}{\varepsilon} \int_{\Omega} |u_j - u|(dd^c \hat{u}_j)^n.$$ 

Denote $\tau = dd^c g - \omega \leq C \omega$. Then, $dd^c \hat{u}_j = \omega_{u_j} + \tau$. It follows that

$$\int_{\Omega} |u_j - u|(dd^c \hat{u}_j)^n = \sum_{k=0}^{n} \binom{n}{k} \int_{\Omega} |u_j - u|\omega_{u_j}^k \wedge \tau^{n-k}$$

which goes to zero by the assumption. Thus, the convergence in capacity is proved by the previous inequality. \qed

Lemma 2.9 Suppose $n = 2$. Let $u, v \in PSH(\Omega, \omega) \cap L^\infty(\Omega)$ be such that $u \leq v$ in $\Omega$ and $u = v$ near $\partial \Omega$. Let $-1 \leq \rho \leq 0$ be a plurisubharmonic function in $\Omega$. Then,

$$\int_{\Omega} (v - u)^3 (dd^c \rho)^2 \leq 6 \int_{\Omega} (v - u)\omega_u^2 + C \int_{\Omega} (v - u)^2 \omega^2 + CE,$$
where $C$ is a uniform constant depending only on $\Omega$ and $\omega$ and

$$E = \left( \int_{\Omega} (v - u) \omega_u \wedge \omega \right)^{\frac{1}{2}} \left( \int_{\Omega} (v - u)^2 \omega^2 \right)^{\frac{1}{2}}.$$

**Proof** By quasi-continuity of plurisubharmonic and $\omega$-psh functions and the convergence theorems in [3] and [13], we may assume that all functions are smooth. Let us denote $h = v - u \geq 0$. Then, $dd^c h = \omega_v - \omega_u$ and $dh \wedge d^c h$ is a positive $(1,1)$-current. By integration by parts

$$\int_{\Omega} h^3 (dd^c \rho)^2 = \int_{\Omega} \rho dd^c h^3 \wedge dd^c \rho$$

$$= \int_{\Omega} \rho (3h^2 dd^c h + 6hdh \wedge d^c h) \wedge dd^c \rho$$

$$\leq 3 \| \rho \|_{\infty} \int_{\Omega} h^2 \omega_u \wedge dd^c \rho.$$ Using the integration by parts again

$$\int_{\Omega} h^2 \omega_u \wedge dh \wedge d^c \rho = \int_{\Omega} \rho dd^c (h^2 \omega_u).$$

Here,

$$\rho dd^c (h^2 \omega_u) = \rho \left[ dd^c h^2 \wedge \omega_u + 2dh^2 \wedge d^c \omega + h^2 dd^c \omega \right]$$

$$= \rho \left[ 2hdd^c h \wedge \omega_u + 2dh \wedge d^c h \wedge \omega_u + 2dh^2 \wedge d^c \omega + h^2 dd^c \omega \right]$$

$$\leq 2|\rho|h \omega_u^2 + 4|\rho| h dh \wedge d^c \omega + \rho h^2 dd^c \omega.$$ because $\rho$ is negative. Using $-C \omega^2 \leq dd^c \omega \leq C \omega^2$ and $-1 \leq \rho \leq 0$, we have

$$\int_{\Omega} h^2 \omega_u \wedge dh \wedge d^c \rho \leq 2 \int_{\Omega} h \omega_u^2 + 4 \left| \int_{\Omega} \rho dh \wedge d^c \omega \right| + C \int_{\Omega} h^2 \omega^2.$$ Thus, to complete the proof, we need to estimate the middle integral on the right-hand side. In fact, using [25, Proposition 1.4], we have

$$\left| \int_{\Omega} \rho dh \wedge d^c \omega \right| \leq C \left( \int_{\Omega} dh \wedge d^c h \wedge \omega \right)^{\frac{1}{2}} \left( \int_{\Omega} |\rho|^2 h^2 \omega^2 \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_{\Omega} dh \wedge d^c h \wedge \omega \right)^{\frac{1}{2}} \left( \int_{\Omega} h^2 \omega^2 \right)^{\frac{1}{2}}.$$
Note that $2dh \wedge d^c h = dd^c h^2 - 2hdd^c h \leq dd^c h^2 + 2h\omega_u$. Therefore,

$$2 \int_{\Omega} dh \wedge d^c h \wedge \omega \leq \int_{\Omega} (dd^c h^2 + 2h\omega_u) \wedge \omega.$$ 

By integration by parts

$$\int_{\Omega} dd^c h^2 \wedge \omega = \int_{\Omega} h^2 dd^c \omega^2 \leq C \int_{\Omega} h^2 \omega^2.$$ 

Combining these inequalities, we get

$$\left| \int_{\Omega} \rho h dh \wedge d^c \omega \right| \leq C \left( \int_{\Omega} h^2 \omega^2 + h\omega_u \wedge \omega \right)^{\frac{1}{2}} \left( \int_{\Omega} h^2 \omega^2 \right)^{\frac{1}{2}},$$

where we used an elementary inequality $(x + y)^{\frac{1}{2}} \leq x^{\frac{1}{2}} + y^{\frac{1}{2}}$ with $x, y \geq 0$ in the second inequality. This completes the proof.

**Corollary 2.10** If $n = 2$, then there is a subsequence $\{u_{js}\}$ of $\{u_j\}$ in Theorem 2.7 that converges to $u$ in capacity.

**Proof** Fix $a > 0$. Let us denote $w_s = \max\{u_{js}, u - 1/s\}$. By Hartogs’ Lemma $w_s \to u$ in capacity. Moreover, $\{|u-u_j| > 2a\} \subset \{|u-w_s| > a\} \cup \{|w_s-u_{js}| > a\}$. Therefore, it is enough to show that $\text{cap}(\{|w_s-u_{js}| > a\}) \to 0$ as $s \to +\infty$. By definition, we have $w_s = \max\{u_{js}, u - 1/s\} \geq u_{js}$. Then, we need to show

$$\text{cap}(\{|u_{js}<w_s-a\}) \to 0 \text{ as } s \to +\infty.$$ 

To this end, by Lemma 2.9,

$$\left( \frac{a}{2} \right)^3 \text{cap}(\{|u_{js}<w_s-a\}) \leq \int_{\Omega} (w_s - u_{js})\omega_{u_{js}}^2 + C \int_{\Omega} (w_s - u_{js})^2 \omega^2 + CE_s,$$

where

$$E_s = \left( \int_{\Omega} (w_s - u_{js})\omega_{u_{js}} \wedge \omega \right)^{\frac{1}{2}} \left( \int_{\Omega} (w_s - u_{js})^2 \omega^2 \right)^{\frac{1}{2}}.$$

By the assumption $u_j \to u$ in $L^1(\Omega)$ (they are uniformly bounded) and (2.5), the first and second integrals on the right-hand side go to zero as $s$ goes to infinity. Note that the first factor of $E_s$ is uniformly bounded by (2.8). Hence, the last term $E_s$ also goes to zero as $s \to +\infty$. Therefore, the conclusion follows. \hfill \Box
Proposition 2.8 and Corollary 2.10 have their analogues on a compact Hermitian manifold without boundary. They allow, for example, to provide another proof of the following result of Guedj and Lu [17, Proposition 3.4].

Lemma 2.11 Let \((X, \omega)\) be a compact \(n\)-dimensional Hermitian manifold (without boundary). Then for any \(A > 0\),

\[
\inf \left\{ \int_X (\omega + dd^c v)^n : v \in PSH(X, \omega), -A \leq v \leq 0 \right\} > 0.
\]

Proof By replacing \(\omega\) with \(\omega/A\) and \(v\) with \(v/A\), we may assume that \(A = 1\). We argue by contradiction. Suppose that there was a sequence \(\{u_j\}_{j \geq 1} \subset \text{PSH}(X, \omega)\) such that \(-1 \leq u_j \leq 0\) and

\[
\lim_{j \to \infty} \int_X \omega_{u_j}^n = 0, \quad \sup_X u_j = 0.
\]

By passing to a subsequence, we assume that \(u_j \to u\) in \(L^1(X)\) and \(u_j \to u\) a.e, where

\[
u = (\limsup_{j \to \infty} u_j)^* = \lim_{j \to \infty} (\sup_{\ell \geq j} u_{\ell})^*.
\]

Then, \(-1 \leq u \leq 0\) and \(u \in \text{PSH}(X, \omega)\). We will show that there exists a subsequence \(\{u_{js}\}\) of \(\{u_j\}\) such that \(\omega_{u_{js}}^n\) converges weakly to \(\omega_u^n\). Indeed, set

\[
w_j = \{u_j, u - 1/j\}.
\]

By the Hartogs lemma, \(w_j\) converges to \(u\) in capacity. Therefore, by the convergence theorem in [BT82] and [DK12] \(\lim_{j \to \infty} \omega_{w_j}^n = \omega_u^n\). Next, we will show that

\[
\int_X |u_j - u|(\omega + dd^c u_j)^n \to 0 \quad \text{and} \quad \int_X |u_j - u|(\omega + dd^c w_j)^n \to 0
\]
as \(j \to +\infty\). The first statement holds because \(\int_X |u_j - u|\omega_{u_j}^n \leq 2 \int_X \omega_{u_j}^n \to 0\) by the assumption. The second convergence also holds because \(w_j \to u\) in capacity and so the proofs of Lemma 2.1, Corollary 2.2, Lemma 2.3 in [24] can be applied because all considered functions are uniformly bounded and \(X\) is compact.

Thus, again by [24, Lemma 3.6], there exists a subsequence \(u_{js}\) such that \(\omega_{u_{js}}^n\) converges weakly to \(\omega_u^n\). In particular, \(\int_X \omega_{u_{js}}^n = \lim_{j \to +\infty} \int_X \omega_{u_{js}}^n = 0\). Hence, \(\omega_u^n \equiv 0\) on \(X\). This is a contradiction with the fact that the Monge–Ampère mass of a bounded \(\omega\)-psh function is always positive [23, Remark 5.7].

\(\square\)
3 The Dirichlet Problem

In this section, we solve the Dirichlet problem for Monge–Ampère type equation under the existence of a bounded subsolution. Let $\mu$ be a positive Radon measure on $M = \overline{M} \setminus \partial M$. Let $\varphi \in C^0(\partial M)$. Let $F(t, z) : \mathbb{R} \times M \to \mathbb{R}^+$ be a non-negative function which is non-decreasing in $t$ and $d\mu$-measurable in $z$. We consider the Dirichlet problem

$\begin{cases}
u \in \text{PSH}(M, \omega) \cap L^\infty(M), \\
(\omega + dd^c\nu)^n = F(u, z)\mu, \\
\lim_{z \to x} u(z) = \varphi(x) & \text{for } x \in \partial M.
\end{cases}$  

(3.1)

A necessary condition to solve the Dirichlet problem is the existence of a subsolution, i.e., a function $u \in \text{PSH}(M, \omega)$ satisfying $\lim_{z \to x} u(z) = \varphi(x)$ for $x \in \partial M$ and

$$(\omega + dd^c u)^n \geq F(u, z)\mu.$$  

(3.2)

We say that $\mu$ is locally dominated by Monge–Ampère measures of bounded plurisubharmonic functions if for each $p \in M$, there exists a coordinate ball $B \subset \subset M$ centered at $p$ and $v \in \text{PSH}(B) \cap L^\infty(B)$ such that

$$\mu|_B \leq (dd^c v)^n.$$  

(3.3)

By the subsolution theorem from [21] on a smaller ball $B' \subset \subset B$, we can choose $v$ in the Cegrell class $E_0(B')$, i.e., $\lim_{z \to \partial B'} v(z) = 0$ and $\int_{B'} (dd^c v)^n < +\infty$.

**Remark 3.1** The latter property often follows from the existence of a subsolution $u$. For example, in geometrically interesting cases: if either $F(u, z) = e^{\lambda u}$ with $\lambda \in \mathbb{R}$, or if $\mu$ is the volume form.

Our main result is as follows.

**Theorem 3.2** Suppose that $F(t, z)$ is a bounded non-negative function which is continuous and non-decreasing in the first variable and $\mu$-measurable in the second one. Let $\mu$ be a positive Radon measure satisfying (3.3). Suppose that there exists a bounded subsolution $u$ as in (3.2). Then there exists a bounded solution to the Dirichlet problem (3.1).

**Proof** We use the Perron envelope method as in [24, Theorem 1.2]. Let $B(\varphi, \mu)$ be the set

$$\left\{ w \in \text{PSH}(M, \omega) \cap L^\infty(M) : (\omega + dd^c w)^n \geq F(w, z)\mu, w^*|_{\partial M} \leq \varphi \right\},$$

(3.4)

where $w^*(x) = \limsup_{z \to x} w(z)$ for every $x \in \partial M$. Then, $B(\varphi, \mu)$ is non-empty as it contains $u$. Let $u_0 \in C^0(\overline{M})$ be a $\omega$-subharmonic solution to

$$(\omega + dd^c u_0) \wedge \omega^{n-1} = 0, \quad u_0 = \varphi \quad \text{on } \partial M.$$
(see, e.g., Corollary 4.1). By the comparison principle for $\omega$-subharmonic functions, we have
\begin{equation}
v \leq u_0 \quad \text{for every } v \in \mathcal{B}(\varphi, \mu). \tag{3.5}
\end{equation}

Thus, the function
\[ u(z) = \sup \{ v(z) : v \in \mathcal{B}(\varphi, \mu) \} \]
is well defined. We know that $u^* \in PSH(M, \omega) \cap L^\infty(M)$ and $u = u^*$ almost everywhere, outside a pluripolar set. Moreover, if $v_1, v_2 \in \mathcal{B}(\varphi, \mu)$, then so is $\max\{v_1, v_2\}$. Indeed, by an inequality of Demailly [12], we have
\begin{align*}
(\omega + dd^c \max\{v_1, v_2\})^n &\geq 1_{\{v_1 > v_2\}}(\omega + dd^c v_1)^n + 1_{\{v_1 \leq v_2\}}(\omega + dd^c v_2)^n \\
&\geq 1_{\{v_1 > v_2\}} F(v_1, z) \mu + 1_{\{v_1 \leq v_2\}} F(v_2, z) \mu \\
&= F(\max\{v_1, v_2\}, z) \mu.
\end{align*}

By this property and Choquet’s lemma, we can write $u = \lim_{j \to +\infty} u_j$, where $\{u_j\}_{j \geq 1} \subset \mathcal{B}(\varphi, \mu)$ is an increasing sequence. Therefore,
\begin{align*}
(\omega + dd^c u^*)^n &= \lim_{j \to +\infty} (\omega + dd^c u_j)^n, \\
&\geq \lim_{j \to +\infty} F(u_j, z) \mu \\
&= F(u, z) \mu = F(u^*, z) \mu,
\end{align*}
where the last equality follows the fact that $\mu$ does not charge pluripolar sets. Thus, $u = u^* \in \mathcal{B}(\varphi, \mu)$ is $\omega$-psh in $M$. It also follows from the definition and (3.5) that $u \leq u \leq u_0$. Hence, $u = \varphi$ is continuous on $\partial M$.

It remains to show that $\omega_u^n = F(u, z) \mu$ in $M$. To see this, let $B \subset \subset M$ be a coordinate ball in $M$. Following the same argument as in [24, Lemma 3.7], given the local solvability of the Dirichlet problem (Theorem 2.3), there exists $\tilde{u} \in \mathcal{B}(\varphi, \mu)$ such that $u \leq \tilde{u}$ and $(\omega + dd^c \tilde{u})^n = F(\tilde{u}, z) \mu$ in this small coordinate ball. By definition of $u$ we must have $\tilde{u} = u$ in $B$. In other words, $\omega_u^n = F(u, z) \mu$. Since $B$ is arbitrary, this proves our claim. $\square$

For $F(t, z) = e^{\lambda t}$ with $\lambda > 0$, we obtain a stronger statement.

**Corollary 3.3** Let $\lambda > 0$. There exists a unique solution to the Dirichlet problem
\begin{align*}
\begin{cases}
u \in PSH(M, \omega) \cap L^\infty(M), \\
\omega_u^n = e^{\lambda u} \mu, \\
\lim_{z \to \partial x} u(z) = \varphi(x) \quad \text{for } x \in \partial M.
\end{cases}
\end{align*}
if and only if there exists a bounded subsolution.
Moreover, if the subsolution is Hölder continuous, so is the solution.
**Proof** The uniqueness of the solution under the hypothesis of the existence of a sub-
solution follows by the same arguments as in [25, Lemma 2.3]. Next, to prove the
equivalence, we only need to verify the local domination by Monge–Ampère measures
of bounded plurisubharmonic functions for \( \mu \) (see condition (3.3)). This is straight-
forward.

For the second conclusion, we first observe as in [24, Lemma 6.5] that \( u \) is Hölder
continuous on the boundary \( \partial M \). Since \( u \) is bounded,

\[
\omega^n_u = e^{\lambda u} \mu \leq C (\omega + dd^c u)^n.
\]

Thus, the Hölder continuity of \( u \) follows from the proof of [24, Theorem 1.4]. \( \square \)

Thanks to this, we can also obtain the solution of the Monge–Ampère equation as
the limit of the solutions of the Monge–Ampère type equations.

**Corollary 3.4** Suppose that there exists a function \( u \in PSH(M, \omega) \cap L^\infty(M) \)
which satisfies:

\[
\lim_{z \to x} u(z) = \varphi(x) \text{ for } x \in \partial M \text{ and } (\omega + dd^c u)^n \geq \mu \text{ in } M.
\]

Then, the sequence of solutions

\[
(\omega + dd^c u_\lambda)^n = e^{\lambda u} \mu \text{ with } \lambda > 0
\]

converges to a solution \( u \) of \( \omega^n_u = \mu \) and \( u = \varphi - \sup_M u \) on \( \partial M \) as \( \lambda \to 0 \).

**Proof** Let \( b = \sup_M u \). Then, \( v := u - b \leq 0 \) on \( M \). For every \( \lambda > 0 \), the function \( v \)
satisfies \( v = \varphi - b \) on \( \partial M \) and \( (\omega + dd^c v)^n \geq e^{\lambda v} \mu \) in \( M \). Applying Theorem 3.2,
we obtain the family of solutions \( \{u_\lambda\}_{0 < \lambda \leq 1} \) of

\[
(\omega + dd^c u_\lambda)^n = e^{\lambda u} \mu, \quad u_\lambda = \varphi - b \text{ on } \partial M.
\]

By the domination principle, the family is increasing in \( \lambda > 0 \) and \( u_\lambda \geq v \) for every
\( 0 < \lambda \leq 1 \). Set \( u = \lim_{\lambda \to 0} u_\lambda \). Then, \( u + b \) is a solution to \( \omega^n_u = \mu \) in \( M \) and \( u = \varphi \)
on \( \partial M \). \( \square \)

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**Appendix**

In the proofs in this paper as well as in [24], we use the existence and regularity of
solutions of the linear elliptic equation on a manifold with boundary. Those statements
are known, for instance, as consequences of general results for harmonic maps [20, Theorem 6] (see also [27, Theorem 5.3]). However, since the case of Hölder continuous boundary data seems not to be available in literature, we include the proof here for the sake of completeness.

Let \((M, \omega)\) be a compact \(n\)-dimensional Hermitian manifold with non-empty boundary \(\partial M\). Then \(M = M \cup \partial M\), where \(M\) is a (open) Hermitian manifold. Suppose that in local coordinate, we have

\[
\omega = \sqrt{-1}g_{i\bar{j}}(z)dz^i \wedge d\bar{z}^j.
\]

Define \(\Delta_g = g^{j\bar{i}}\partial_i \partial_{\bar{j}}\) be the Laplace operator associated to \(\omega\) and denote by \(\text{dist}(z, w)\) the distance function induced by \(\omega\).

**Proposition 4.1** Let \(\varphi \in C^0(\partial M)\). Then, there exists a unique continuous solution to

\[
(\omega + dd^c u) \wedge \omega^{n-1} = 0 \text{ in } M, \quad u = \varphi \text{ on } \partial M.
\]

Moreover, if \(\varphi\) is Hölder continuous on \(\partial M\), then so is the solution \(u\).

Since

\[
\Delta_g u = \text{tr}_\omega u = \frac{n dd^c u \wedge \omega^{n-1}}{\omega^n},
\]

we can separate the equation into two problems \(\Delta_g u_1 = 0\) in \(M\) with \(u_1 = \varphi\) on \(\partial M\) and \(\Delta_g u_2 = -n\) in \(M\) with \(u_2 = 0\) on \(\partial M\). The latter solution \(u_2\) is smooth by the classical PDEs.

We are thus reduced to proving the following.

**Proposition 4.2** Let \(\varphi \in C^0(\partial M)\). Then, there exists a unique continuous solution to

\[
\Delta_g u = 0 \text{ in } M, \quad u = \varphi \text{ on } \partial M.
\] (4.1)

Moreover, if \(\varphi\) is Hölder continuous on \(\partial M\), then so is \(u\).

The existence of continuous solutions follows exactly as in [15, pp. 24–25] by using the Perron envelope

\[
S_\varphi = \{v \in SH_\omega(M) \cap C^0(M) : v_{|\partial M} \leq \varphi\},
\]

where \(SH_\omega(M)\) is the set of all \(\Delta_g\)-subharmonic functions in \(M\). The function

\[
u(x) = \sup_{v \in S_\varphi} v(x) \quad \text{for } x \in M
\]

is the solution to (4.1) by the Perron method using harmonic liftings.
Lemma 4.3 Let $v \in S_\varphi$. Then there exists a function $\tilde{v}$ called a lift of $v$ in $B$ such that $\tilde{v} \geq v$ on $\overline{M}$ and satisfying $\Delta_g \tilde{v} = 0$ in $B$ and $\tilde{v} = v$ on $\partial B$.

The boundary condition is satisfied since using the regularity of the boundary of the domain, one can easily construct, as in [15], first the local barriers, and then the global one.

We now assume further that $\varphi$ belongs to $C^{0, \alpha}(\overline{M})$ with $0 < \alpha < 1$. Then, we wish to show that the solution also belongs to a Hölder space.

First, we will construct a Hölder continuous local barrier similarly as in [1, Theorem 6.2] on the coordinate half-ball at each boundary point.

Lemma 4.4 Suppose the origin $0 \in \partial M \cap B(0, R)$, $\rho$ is the defining function of $\partial M \cap B(0, R)$ in the coordinate ball $B(0, R)$ and $U_R = \{z \in B(0, R) : \rho(z) \leq 0\}$ is the coordinate half-ball centered at 0. Denote $\|\varphi\|_\alpha = c_1$ for the Hölder norm of $\varphi$ on $\partial M$. Let $0 < \tau \leq \alpha < 1$. Then, there exists a constant $k = k(\varphi, U_R)$ and a neighborhood $W$ of 0 such that the function

$$v(z) = k|\rho|^\tau(z) + c_1|z|^{\alpha} + \varphi(0)$$

is $\Delta_g$-superharmonic in $W \cap U_R$. Moreover,

$$v(0) = \varphi(0), \quad v(x) \geq \varphi(x) \quad \text{for every } x \in \partial M \cap B(0, R).$$

Proof We compute in $B(0, R)$,

$$dd^c|\rho|^\tau = -\tau|\rho|^\tau-1dd^c\rho - \tau(1-\tau)|\rho|^{\tau-2}d\rho \land d^c\rho,$$

and for $\alpha' = \alpha/2$,

$$dd^c|z|^{2\alpha'} = \alpha'|z|^{2(\alpha'-1)}dd^c|z|^2 - \alpha'(\alpha'-1)|z|^{2(\alpha'-2)}d|z|^2 \land d^c|z|^2.$$

Hence,

$$dd^c v(z) \land \omega^{n-1}/\omega^n \leq -k\tau(1-\tau)|\rho|^{\tau-2}|\nabla \rho|^2 + \frac{c_1\alpha}{2}|z|^{\alpha-2}.$$

Furthermore, $|\rho(z)| = |\rho(z) - \rho(0)| \leq c_2|z|$ for every $z \in \overline{B(0, R)}$. Since $\tau - 2 < 0$, it implies that

$$\frac{1}{n} \Delta_g v(z) \leq -c_3k|z|^{\tau-2}|\nabla \rho|^2 + \frac{c_1\alpha}{2}|z|^{\alpha-2}$$

$$= |z|^{\alpha-2} \left( c_1\alpha/2 - c_3k|z|^{\tau-\alpha}|\nabla \rho|^2 \right)$$

$$\leq |z|^{\alpha-2} \left( c_1\alpha - c_3kR^{\tau-\alpha}|\nabla \rho|^2 \right),$$

where $c_3 = \tau (1-\tau)c_2^{\tau-2}$. We used the fact $\tau \leq \alpha$ for the last inequality. Since $\rho$ is the defining function for $\partial M \cap B(0, R)$, it follows that $|\nabla \rho| > \varepsilon_0 > 0$ in a neighborhood
\[ W \] of 0. Then, we can choose large \( k > 0 \), independent of the boundary point 0, so that \( \Delta_g v \leq 0 \). Finally, the remaining properties of \( v \) hold because \( \rho(x) = 0 \) for every \( x \in \partial M \cap B(0, R) \).

**Proposition 4.5** There exists a function \( \overline{u}(x) \in C^{0,\alpha}(\overline{M}) \) that is \( \Delta_g \)-superharmonic in \( M \) and \( \overline{u}(\xi) = \varphi(\xi) \) for \( \xi \in \partial M \).

**Proof** We first show that at each point \( \xi \in \partial M \), there exist a \( \Delta_g \)-superharmonic function \( v_\xi \) in \( M \) such that \( v_\xi \in C^{0,\alpha}(\overline{M}) \) and

\[
\begin{align*}
\varepsilon(x) &\geq \varphi(x) \quad \text{on } \partial M, \\
v_\xi(\xi) &= \varphi(\xi), \\
\|v_\xi\|_{\alpha} &\leq C\|\varphi\|_{2\alpha},
\end{align*}
\]

where \( \| \cdot \|_{\alpha} \) denotes the \( \alpha \)-Hölder norm of the function and the constant \( C \) depends only on \( \overline{M} \) and the metric \( \omega \).

Without loss of generality, we may assume \( \xi \) is the origin and \( \varphi(0) = 0 \). By Lemma 4.4 for the boundary point \( 0 \in \partial M \) there exists a function \( v = k|\rho|^\alpha + c_1|z|^\alpha \) and a neighborhood \( W \) of 0 such that \( v \) satisfies (4.2), (4.3) and (4.4) on \( W \cap U_R \). Note that the constants \( k, c_1 \) are independent of the boundary points. We can extend this function to a global one as follows. Set \( k_1 = \sup_{\partial M} \varphi + 1 \). Then, for large \( k_2 \geq 1 \) (depending on \( k_1 \) and \( R \), but independent of the boundary point), the function

\[ \widehat{v} = \min\{k_1, k_2v\} \]

is \( \Delta_g \)-superharmonic on \( M \) and satisfies the list of required properties.

Now let us define

\[ \overline{u} = \inf\{v_\xi : \xi \in \partial M\}. \]

From \( |v_\xi(z) - v_\xi(w)| \leq C[\text{dist}(z, w)]^\alpha \), we deduce that \( |\overline{u}(z) - \overline{u}(w)| \leq C[\text{dist}(z, w)]^\alpha \). Thus, \( \overline{u} \) is Hölder continuous \( \Delta_g \)-superharmonic, and clearly \( \overline{u}(\xi) = \varphi(\xi) \) for every \( \xi \in \partial M \). This completes the proof. \( \square \)

By the similar argument, we can find a global Hölder continuous \( \Delta_g \)-subharmonic barrier \( \underline{u} \in C^{0,\alpha}(\overline{M}) \) such that

\[ \underline{u}(\xi) = \varphi(\xi) \quad \text{for } \xi \in \partial M. \]

Hence, by the maximum principle, \( u \leq \underline{u} \leq \overline{u} \) on \( \overline{M} \). Consequently, we get

\[
|u(x) - u(\xi)| \leq C[\text{dist}(x, \xi)]^\alpha \quad \text{for every } x \in \overline{M} \text{ and } \xi \in \partial M.
\]

Now we are going to show the global Hölder continuity of the solution.
Lemma 4.6 There exists a constant $C = C(\varphi, M, \omega)$ such that
\[ |u(x) - u(y)| \leq C[\text{dist}(x, y)]^\alpha \quad \text{for every } x, y \in M. \tag{4.7} \]

Proof By maximum principle, we get $\inf_{\partial M} \varphi \leq u \leq \sup_{\partial M} \varphi$. Denote $d_x = \text{dist}(x, \partial M)$ and $d_y = \text{dist}(y, \partial M)$. Suppose that $d_y \leq d_x$, and take $x_0, y_0 \in \partial M$ such that $\text{dist}(x, x_0) = d_x$ and $\text{dist}(y, y_0) = d_y$.

Case 1 $\text{dist}(x, y) \leq d_x/2$. Since $\Delta_\varphi$ is uniformly elliptic, we have the interior Hölder estimates (see Corollary 9.24 and Lemma 8.23 in [15])
\[ \|u\|_{C^{0,\alpha}(B_{R/2})} \leq CR^{-\alpha}\|u\|_{L^\infty(B_R)}, \]
where $B_R \subset M$ is a coordinate ball of small radius $0 < R < 1$. Since $y \in \overline{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset M$. Applying the interior inequality to $u - u(x_0)$ in $B_{d_x}(x)$, we get
\[ d_x^\alpha \frac{|u(x) - u(y)|}{[\text{dist}(x, y)]^\alpha} \leq C\|u - u(x_0)\|_{L^\infty(B_{d_x}(x))}. \]
By (4.6), the right-hand side is less than $C d_x^\alpha$. It follows that
\[ |u(x) - u(y)| \leq C[\text{dist}(x, y)]^\alpha. \]

Case 2 $d_y \leq d_x \leq 2 \text{dist}(x, y)$. Then,
\[ |u(x) - u(y)| \leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y) - u(y_0)| \leq C(d_x^\alpha + [\text{dist}(x_0, y_0)]^\alpha + d_y^\alpha) \leq C[\text{dist}(x, y)]^\alpha \]
since $\text{dist}(x_0, y_0) \leq d_x + \text{dist}(x, y) + d_y \leq 5 \text{dist}(x, y)$. \hfill \Box

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