Hecke algebras, modular categories and 3-manifolds quantum invariants

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Received 23 March 1998; in revised form 6 November 1998

Abstract

We construct modular categories from Hecke algebras at roots of unity. For a special choice of the framing parameter, we recover the Reshetikhin–Turaev invariants of closed 3-manifolds constructed from the quantum groups $\mathfrak{u}_q^sl(N)$ by Reshetikhin–Turaev and Turaev–Wenzl, and from skein theory by Yokota. The possibility of such a construction was suggested by Turaev, as a consequence of Schur–Weil duality. We then discuss the choice of the framing parameter. This leads, for any rank $N$ and level $K$, to a modular category $\mathcal{H}^{N,K}$ and a reduced invariant $\tau_{N,K}$. If $N$ and $K$ are coprime, then this invariant coincides with the known invariant $\tau_{PSU(N)}$ at level $K$. If $\gcd(N,K) = d > 1$, then we show that the reduced invariant admits spin or cohomological refinements, with a nice decomposition formula which extends a theorem of H. Murakami.

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Introduction

Our goal is to construct modular categories from the Hecke algebras of type $A$ at roots of unity, and to obtain certain reductions and refinements of them. Our main results are the following.

- We give a (reasonably self-contained) construction of modular categories underlying the known $SU(N)$ quantum invariants. As expected, at rank-level $(N, K)$, isomorphism classes of simple objects are indexed by the set of Young diagrams with at most $N - 1$ rows and $K$ columns, whose cardinality is $(N + K - 1)!/(N - 1)!K!$. The dimensions of the TQFT modules are given by the Verlinde formula.

- At rank-level $(N, K)$ we obtain a reduced modular category $\mathcal{H}^{N,K}$, and a reduced invariant $\tilde{\tau}_{N,K}$. Here the number of non-isomorphic simple objects is $d(N + K - 1)!/N!K!$, with $d = \gcd(N,K)$. Except for $\gcd(N,K) = 1$, in which case we recover the $PSU(N)$ invariant, this result seems to be new.

- We use a grading modulo $d = \gcd(N,K)$ to define invariants of 3-manifolds equipped with what we call a spin structure with coefficients modulo $d$, and we prove a decomposition
formula which extends a theorem of Murakami [30]. This refinement holds when \( d \) is even, and \( N/d, K/d \) are odd (the spin case); in the other cases, we obtain cohomological refinements.

The quantum invariants, predicted by Witten using Chern–Simons theory and path integrals, were first constructed by Reshetikhin and Turaev [34] and Turaev and Wenzl [37] using representation theory of quantum groups. The work of Turaev [36] shows that a key concept in the construction of these invariants, as well as in extending them to a Topological Quantum Field Theory (TQFT), is that of a modular category.

A modular category is a braided category with some additional algebraic features (duality, twist, a finite set of simple objects satisfying a domination property and a non-degeneracy axiom). The interest of this concept is that it provides a Topological Quantum Field Theory in dimension 3, and in particular, invariants of links and 3-manifolds. However, in particular examples, it is not easy to define precisely the modular category and to check the required properties. For the \( SU(N) \) invariants, first constructed using representation theory of \( U_q(sl(N)) \), it is known that underlying modular categories can be derived from the category of representations of the quantum group at roots of unity \([37, 26, 4]\). In Section 2, we will give an alternative elementary construction of modular categories producing the same invariants.\(^1\)

A skein theoretic construction of the \( SU(N) \) invariants was obtained by Yokota [42]; subsequent developments towards the associated TQFTs were made by Lickorish \([23, 24]\). Our main tool here will be this skein method combined with the structure of the Hecke algebra, which in our context is obtained as the Homflypt skein module of a cylinder \( D^2 \times [0, 1] \) with boundary points. This algebra has been intensively studied (see \([13]\) for a list of references). For our purpose, we emphasize the work of Jones \([17]\) and Wenzl \([40]\); see also \([15, 11]\). Our normalization and description of idempotents coincide with those of Aiston and Morton \([1, 3, 28]\).

Following the work of Kirby and Melvin for the \( SU(2) \) case \([20]\), Kohno and Takata have studied symmetry formulas for the \( SU(N) \) quantum invariants, and defined the \( PSU(N) \) invariants \([21, 22]\). This was used by Murakami in \([30]\). Our reductions and refinements formulas generalize these results.

Invariants of 3-manifolds from Hecke algebras were obtained by Wenzl in \([41]\); modular and semi-simple categories from unoriented link invariants (BCD case) are considered by Turaev and Wenzl in \([38]\).

Masbaum and Wenzl \([26]\) have proved the integrality of \( SU(N) \) quantum invariants at roots of unity of prime order, and they have shown that this follows from existence of integral modular categories. This is developed by Bruguïères in \([9]\).

1. Young idempotents and homflypt skein theory

1.1 The Homflypt functor

Let \( M \) be an oriented 3-manifold. We denote by \( \mathcal{H}(M) \) the \( k \)-module freely generated by isotopy classes of framed links in \( M \), quotiented by the Homflypt relations given in Fig. 1.

\(^1\) Two modular categories producing the same invariant should be equivalent. A complete treatment of this question has yet to be given.
Here \( k \) is an integral domain containing the invertible elements \( a, v, s \); we suppose moreover that \( s - s^{-1} \) is invertible in \( k \). (For \( L \neq \emptyset \), the third equality is a consequence of the others.)

1.1.1. Note about the framing

Here a framing is a trivialization of the normal bundle up to homotopy. This is equivalent to an orientation of the link together with a non-singular normal vector field up to homotopy. In the figures, a preferred convention using the plane gives the framing.

The Homflypt polynomial \([16, 32]\) gives an isomorphism
\[
\langle \cdots \rangle : \mathcal{H}(S^3) \to k
\]
\[
L \mapsto \langle L \rangle
\]
normalized by \( \langle \emptyset \rangle = 1 \).

An embedding \( j : M \hookrightarrow N \) gives a well-defined operator \( \mathcal{H}(j) : \mathcal{H}(M) \to \mathcal{H}(N) \). This makes \( \mathcal{H} \) into a functor from the category of 3-manifolds whose morphisms are isotopy classes of embeddings, to the category of \( k \)-modules.

To an oriented embedding of a disjoint union of solid tori
\[
g = \bigoplus_{i=1}^{m} g_i : \bigoplus_{i=1}^{m} D_i^2 \times S^1 \to S^3,
\]
is associated a multilinear map
\[
H(\ g) : \mathcal{H}(D^2 \times S^1)^{ \otimes m} \to \mathcal{H}(S^3) \approx k.
\]
This map only depends on the isotopy class of the framed link \( L = (L_1, \ldots, L_m) \) underlying \( g \). The image of \( x_1 \otimes \ldots \otimes x_m \) under this map is said to be obtained by cabling the components \( L_i \) with the skein elements \( x_i \), and is denoted by \( \langle L_1(x_1), \ldots, L_m(x_m) \rangle \) or \( \langle L(x_1, \ldots, x_m) \rangle \).

1.1.2. Relative Homflypt skein module

In the case where \( M \) has non-empty boundary, we may fix a finite set of points, \( l \), in the boundary of \( M \), equipped with a trivialization of the tangent bundle of \( \partial M \) at each point, and define the relative skein module \( \mathcal{H}(M, l) \). Here generators are links \( L \subset M \) with boundary \( l \), equipped with relative framing (extension to \( L \) of the trivialization fixed on \( l \), up to relative homotopy). Note that a point in \( l \) will be incoming or outgoing, depending on the orientation of the trivialization.
1.2. The Hecke category

The $k$-linear Hecke category $H$ is defined as follows. An object in this category is a disc $D^2$ equipped with a set of points with trivialization as above. If $\alpha = (D^2, l_0)$ and $\beta = (D^2, l_1)$ are two objects, the module $\text{Hom}_H(\alpha, \beta)$ is $H(D^2 \times [0; 1], l_0 \times 0 \sqcup l_1 \times 1)$. The notation $H(\alpha, \beta)$ and $H_\alpha$ will be used respectively for $\text{Hom}_H(\alpha, \beta)$ and $\text{End}_H(\alpha)$. For composition, we use the covariant notation

$$(f, g) \mapsto fg.$$ 

When we draw a figure to describe a morphism, the time parameter goes upwards, so that the morphism $fg$ is depicted with $g$ lying above $f$.

1.2.1. Note

The terminology Hecke category was introduced by Turaev in [35]. His definition gives a category equivalent to an unframed version of ours.

We will use the notation $\hat{f}$ for the closure in $H(D^2 \times S^1)$ of the morphism $f \in H_\alpha$. Let $U_0$ be a 0-framed unknot in $S^3$. We use the notation $\langle f \rangle$ for $\langle U_0(\hat{f}) \rangle$ (the quantum trace).

We will simply denote by $n$ the object formed by the $n$ points $-1 + (2j - 1)/n$, $j = 1, \ldots, n$, equipped with the standard trivialization.

For $\varepsilon = \pm 1$ let $j_\varepsilon : D^2 \hookrightarrow D^2$ be the embedding which sends $z$ to $\varepsilon/2 + Z/4$. By using $j = j_{-1} \sqcup j_1 : D^2 \sqcup D^2 \hookrightarrow D^2$, we make $H$ into a monoidal category. This category has a braiding and a twist operator. We can define in $H$ a duality rule so that we get a ribbon category ([36], see also [39]). We proceed as follows. To an object $\alpha = (D^2, l)$, we associate the object $\alpha^* = (D^2, -\overline{l})$, where $-\overline{l}$ is the set of points with trivialization obtained by applying to $l$ the differential of $z \mapsto -\overline{z}$, and define the morphisms $b_\alpha \in H(0, \alpha \otimes \alpha^*)$ and $d_\alpha \in H(\alpha^* \otimes \alpha, 0)$ accoding to the figure below (a copy of $l_\alpha$ is embedded along the framed arc).

$$
\begin{align*}
\alpha & \quad b_\alpha = \alpha \quad d_\alpha = \alpha \\
\end{align*}
$$

Our purpose is to discuss which modular categories arise from this ribbon Hecke category. For this, we need a finite set of simple objects with nice properties [36, p. 74].

1.3. Idempotents in the Hecke algebra

The algebra $H_n$ is isomorphic to the quotient of the algebra of the braid group $k[B_n]$ by the Homflypt relation

$$
\begin{align*}
\begin{array}{c}
a^{-1} \\
-a \\
\end{array} = (s - s^{-1})
\end{align*}
$$

The associativity isomorphisms will be omitted. They connect the corresponding points in the obvious way. Note that the object $\text{1}_n$ is defined up to these associativity isomorphisms, and is canonically isomorphic to the object $n$. 

\footnote{The associativity isomorphisms will be omitted. They connect the corresponding points in the obvious way. Note that the object $\text{1}_n$ is defined up to these associativity isomorphisms, and is canonically isomorphic to the object $n$.}
which is the Hecke algebra of type \( A_{n-1} \). An unframed version of this result was proved independently by Morton and Traczyk [29] and Turaev [35]. This algebra is known to be generically semi-simple. It is a deformation of the algebra of the symmetric group, and its structure can be obtained by extending the classical Young theory [8, 17, 40, 15, 11].

Recall that, to a partition of \( n \), \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1) \), \( \lambda_1 + \cdots + \lambda_p = n \), is associated a Young diagram of size \( |\lambda| = n \), which we denote also by \( \lambda \). This diagram has \( n \) cells indexed by \((i, j)\), \( 1 \leq i \leq p, 1 \leq j \leq \lambda_i \).

If \( c \) is the cell of index \((i, j)\) in a Young diagram, its hook-length \( hl(c) \) and its content \( cn(c) \) are defined by

\[
hl(c) = \lambda_i + \lambda_j^* - i - j + 1, \quad cn(c) = j - i.
\]

Here \( \lambda^* \) is the transposed Young diagram and \( \lambda_j^* \) is the length of the \( j \)th column of \( \lambda \) (the \( j \)th row of \( \lambda^* \)).

For \( n \geq 0 \), the quantum integer \([n]\) and the quantum factorial \([n]!\) are defined by
\[
[n] = (s^n - s^{-n})/(s - s^{-1}) \quad \text{and} \quad [n]! = [n]_{j=1} [j].
\]

For a Young diagram \( \lambda \), we will use the notation \([hl(\lambda)]\) for the product over all cells of the quantum hook-lengths.

\[
[hl(\lambda)] = \prod_{\text{cells}} [hl(c)]
\]

1.3.1. Symmetrizers

Let \( \sigma_i \in H_n, i = 1, \ldots, n - 1 \), be represented by the standard generators of the braid group \( B_n \) (the strand numbered \( i \) crosses over the strand numbered \( i + 1 \)).

**Proposition 1.1.** If \([n]!\) is invertible in \( k \), then there exists a unique idempotent \( f_n \in H_n \) such that \( \forall \sigma_i f_n = f_n \sigma_i \), and a unique idempotent \( g_n \in H_n \) such that \( \forall \sigma_i g_n = -as^{-1}g_n = g_n \sigma_i \).

**Proof.** It is shown in [27, 3] that the deformation \( f_n \) of the Young symmetrizer, given below, satisfies the required condition.

\[
f_n = \frac{1}{[n]!} s^{-n(n-1)/2} \sum_{\pi \in S_n} (as^{-1})^{-l(\pi)} w_{\pi}.
\]

Here \( w_\pi \) is the positive braid associated with the permutation \( \pi \), and \( l(\pi) \) is the length of \( \pi \).

One can also construct \( f_n \) recursively using the formulas below [42].

\[
[2]f_2 = s^{-1} \quad \leftrightarrow \quad + a^{-1} \quad \leftrightarrow \\

[n + 1]f_{n+1} = - [n - 1] f_n \otimes 1_1 + [2][n](f_n \otimes 1_1)(1_{n-1} \otimes f_2) (f_n \otimes 1_1)
\]

(Here, \( 1_p \in H_p \) is the identity.)

Suppose now that \( f_n' \) satisfies the condition of the theorem. Then we have that \( f_n f_n' = f_n' = \zeta f_n \), and \( \zeta = 1 \) by idempotence.
For $g_n$ we can proceed similarly, either with the deformed antisymmetrizer

$$g_n = \frac{1}{[n]!} s^{n(n-1)/2} \sum_{\pi \in S_n} (-as)^{l(\pi)} w_{\pi},$$

or with the recursive formulas

$$g_1 = 1_1,$$
$$g_{n+1} = 1_1 \otimes g_n - \frac{[2][n]}{[n+1]} (1_1 \otimes g_n) (f_2 \otimes 1_{n-1}) (1_1 \otimes g_n). \quad \square$$

Note that $1_1 = f_2 + g_2$; this can be used to obtain more symmetric recursive formulas for $f_n$ and $g_n$. Our choice immediately gives the following lemma [42], which is useful in Yokota’s skein computations.

**Lemma 1.2.** For any two integers $p, q$ such that $[p + 1]!$ and $[q + 1]!$ are invertible, one has

$$[p + q] f_p \otimes g_q = [p + 1][q](1_p \otimes g_q) (f_{p+1} \otimes 1_{q-1}) (1_p \otimes g_q)$$
$$+ [p][q + 1] (f_p \otimes 1_q) (1_{p-1} \otimes g_{q+1}) (f_p \otimes 1_q).$$

The skein computations frequently use the absorbing property below whose proof follows the definition of the symmetrizers.

**Lemma 1.3.** Suppose that $[n]!$ is invertible. For any $m < n$, one has

(a) $(f_m \otimes 1_{n-m}) f_n = f_n (f_m \otimes 1_{n-m}) = f_n,$
(b) $(g_m \otimes 1_{n-m}) g_n = g_n (g_m \otimes 1_{n-m}) = g_n.$

### 1.3.2. Aiston–Morton description of Young symmetrizers.

For a Young diagram $\lambda$ of size $n$, we denote by $\square_{\lambda}$ the object of the category $H$ formed with one point for each cell $c$ of $\lambda$, equipped with the standard trivialization; if $c$ has index $(i, j)$ (ith row, and $j$th column), then the corresponding point in $D^2$ is $(i + j \sqrt{-1})/(n + 1)$. Suppose that $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1)$, and that $\lambda^\vee = (\lambda_p \geq \cdots \geq \lambda_1 \geq 1)$ is the transposed Young diagram.

Let $F_\lambda$ (resp. $G_\lambda$) be the element in $H_{\lambda^\vee}$ formed with one copy of $[\lambda_i]! f_{\lambda_i}$ along the row $i$, for $i = 1, \ldots, p$ (resp. one copy of $[\lambda_j]! g_{\lambda_j}$ along the column $j$, for $j = 1, \ldots, q$). Note that these expressions have no denominators.

In the proposition below, $<$ denotes the lexicographic ordering, and $\lambda$ and $\mu$ are Young diagrams with the same number of cells.

**Proposition 1.4.** (a) If $\mu < \lambda$, then $F_\lambda H(\square_\lambda, \square_\mu) G_\mu = 0$.
(b) If $\lambda < \mu$, then $G_\lambda H(\square_\lambda, \square_\mu) F_\mu = 0$.
(c) One has $F_\lambda H_{\lambda^\vee} G_\lambda = k F_\lambda G_\lambda$.
(d) One has $G_\lambda H_{\lambda^\vee} F_\lambda = k G_\lambda F_\lambda$.

This proposition is proved in [3, Section 4]. We outline here the proof given there.
Proof. (a) Every family of \(|\lambda|\) braids in \(H(\square_\lambda, \square_\mu)\), which induces all bijections between the cells of \(\lambda\) and the cells of \(\mu\), is a basis of \(H(\square_\lambda, \square_\mu)\). The hypothesis \(\mu < \lambda\) implies that any bijection between the cells of \(\lambda\) and the cells of \(\mu\) carries at least two cells in some row of \(\lambda\), to cells in the same column of \(\mu\). So we can find a basis of \(H(\square_\lambda, \square_\mu)\) represented by braids which connect in a separate cylinder two points in some row of \(\square_\lambda\) with two points in some column of \(\square_\mu\). One can deduce that \(F_\lambda H(\square_\lambda, \square_\mu)G_\mu = 0\). Assertion (b) is shown similarly.

(c) We say that a permutation \(\pi\) between the cells of \(\lambda\) does not separate, if some pair of cells in the same row is mapped to some pair of cells in the same column; other permutations are said to separate. Using that any separating permutation \(\pi\) of the cells can be written \(\pi = \pi_R \pi_C\) (permutations act on the left), where \(\pi_R\) (resp. \(\pi_C\)) preserves the rows (resp. the columns), we see that we can find a basis represented by braids \(b_n\) indexed by permutations such that

\[
\begin{align*}
F_\lambda b_\pi G_\mu &= 0 & \text{if } \pi \text{ does not separate} \\
F_\lambda b_\pi G_\mu &= \zeta_n F_\lambda G_\mu & \text{if } \pi \text{ separates.}
\end{align*}
\]

Statement (c) follows. Assertion (d) is shown similarly.

The argument used in the proof of (a) above shows the following.

Lemma 1.5. (a) If \(p > \lambda_1\), and for some object \(x\) containing \(|\lambda|\) points with positively oriented trivialization, \(x = x_1 \otimes f_p \otimes x_2 \in H_x\), then one has

\[x H(x, \square_\lambda) G_\lambda = 0 \quad \text{and} \quad G_\lambda H(\square_\lambda, x) x = 0.\]

(b) If \(p > \lambda_\gamma\), and for some object \(x\) containing \(|\lambda|\) points with positively oriented trivialization, \(x = x_1 \otimes g_p \otimes x_2 \in H_x\), then one has

\[F_\lambda H(\square_\lambda, x) x = 0 \quad \text{and} \quad x H(x, \square_\lambda) F_\lambda = 0.\]

Let \(\tilde{y}_\lambda = F_\lambda G_\lambda\). A consequence of Proposition 1.4 is that \(\tilde{y}_\lambda\) is a quasi-idempotent.

Proposition 1.6. One has \(\tilde{y}_\lambda^2 = [hl(\lambda)] \tilde{y}_\lambda\).

We note here that Yokota’s idempotents in [42] are based on \(G_\lambda F_\lambda G_\lambda\) rather than on \(F_\lambda G_\lambda\). This makes essentially no difference in the computation of the normalizing coefficient. The proposition above follows from the lemma below, which is a version of [42, Lemma 2.3] (see also [2]). Here \(1^n = (1, \ldots, 1)\) denotes the Young diagram with one column containing \(n\) cells.

Lemma 1.7 [Yokota’s lemma]. Let \(\lambda = 1^n + \mu\) be a Young diagram with \(n\) rows, then one has

(a) \(F_\lambda G_\lambda F_\lambda G_\lambda = \prod_{i=1}^n [\lambda_i + n - i] F_\lambda (1_{\square_\lambda \otimes G_\mu}) (1_{\square_\lambda \otimes F_\mu}) G_\lambda\),

(b) \(G_\lambda F_\lambda G_\lambda F_\lambda = \prod_{i=1}^n [\lambda_i + n - i] G_\lambda (1_{\square_\lambda \otimes F_\mu}) (1_{\square_\lambda \otimes G_\mu}) F_\lambda\).

Here the obvious isomorphism between \(\square_\lambda\) and \(\square_\lambda \otimes \square_\mu\) is omitted.

Proof. We will show the formula with generic parameters. In the following, the cells in the superscript indicate in which way the corresponding symmetrizer or antisymmetrizer is inserted.
We denote by $V_i$ the skein element obtained from $F_j$ by replacing, for $j = i + 1, \ldots, n$, the copy of $[\lambda_j]!f_{\lambda_j}$ by $[\lambda_j - 1]!f_{\lambda_j - 1}^{(j, \lambda)}$. Note that $V_n = F_\lambda$, and $V_0 = 1_{\lambda} \otimes F_\mu$. We will show the following lemma, which proves the required formulas.

**Lemma.** For $i = 1, \ldots, n$, one has

(a) $[n - i + 1]F_i G_\lambda V_i G_\lambda = [\lambda_i + n - i]F_i G_\lambda V_{i-1} G_\lambda$,
(b) $[n - i + 1]G_\lambda V_i G_\lambda = [\lambda_i + n - i]G_\lambda V_{i-1} G_\lambda F_\lambda$.

We first develop $G_\lambda V_i G_\lambda$, using the Lemmas 1.2 and 1.3.

$$G_\lambda V_i G_\lambda = [\lambda_i] G_\lambda g_n^{(i, 1)} \cdots g_{n-i+1}^{(i, 1)} f_{\lambda_i}^{(i, 1)} \cdots g_{n-i}^{(i, \lambda_i)} g_{n-i+1}^{(i, 1)} V_{i-1} G_\lambda$$

$$= \left[\frac{\lambda_i + n - i}{n - i + 1}\right] G_i V_{i-1} G_\lambda + \text{coeff} \times G_\lambda f_{\lambda_i-1}^{(i, 2)} \cdots g_{n-i+2}^{(i, 1)} g_{n-i+1}^{(i, \lambda_i)} V_{i-1} G_\lambda.$$

We have to show that the second term is zero. We expand the copy of $f_{\lambda_i-1}$ as a linear sum of braids so that we have only to consider

$$\eta_i = G_\lambda \sigma^{(i, 2)} \cdots (i, \lambda_i) g_n^{(i, 2)} g_{n-i+1}^{(i, 1)} \cdots V_{i-1} G_\lambda$$

where $\sigma^{(i, 2)} \cdots (i, \lambda_i)$ is represented by a braiding of the strings corresponding to the superscript. For $v = 1, \ldots, i$, let $\eta_{iv}$ be defined by

$$\eta_{iv} = G_\lambda \sigma^{(i, 2)} \cdots (i, \lambda_i) g_n^{(i, 2)} g_{n-v+1}^{(i, 1)} g_{n-v+2}^{(i, \lambda_i)} V_{i-1} G_\lambda.$$

We have, for $v = 1, \ldots, i - 1$,

$$\eta_{i, v+1} = G_\lambda \sigma^{(i, 2)} \cdots (i, \lambda_i) g_{n-v+1}^{(i, 1)} g_{n-v+2}^{(i, 1)} \cdots V_{i-1} G_\lambda,$$

The recursive formula for the antisymmetrizers shows that we can write $\eta_{i, v+1}$ as a linear combination of $\eta_{i, v}$ and $\xi_{i, v}$, where

$$\xi_{iv} = G_\lambda \tau^{(v, 1), (i, 2)} \tau^{(v, 1), (i, 2)} V_{i-1} G_\lambda,$$

and $\tau^{(v, 1), (i, 2)}$ is represented by a braiding of the two strings indicated by the superscript, inserted between the two first columns. Moreover we can choose $\tau^{(v, 1), (i, 2)}$ so that in $\xi_{iv}$ the copy of the symmetrizer $f_{\lambda_i}$ is directly joined to $g_{\lambda_i} \otimes \cdots \otimes g_{\lambda_i}^{-1}$, so that $\xi_{iv}$ is zero. (Lemma 1.5 can be applied.)

This is the key argument in Yokota's proof (a moment thought and some drawings may be useful). Note that we have $\eta_{ii} = \eta_i$. The conclusion follows from the lemma below.

**Lemma.** For $v = 1, \ldots, i$, one has $F_{\lambda} \eta_{iv} = \eta_{iv} F_{\lambda} = 0$.

This lemma is obtained recursively. The case $v = 1$ comes from Lemma 1.5.

If $[hl(\lambda)]$ is invertible, we define the idempotent $y_\lambda$ by

$$y_\lambda = [hl(\lambda)]^{-1}.$$

Suppose that $\lambda$ and $\mu$ are Young diagrams with the same number of cells, such that $[hl(\lambda)]$ and $[hl(\mu)]$ are invertible. From Proposition 1.4, we have
Proposition 1.8. (a) If \( \mu \neq \lambda \), then \( y_\lambda H(\Box_\lambda, \Box_\mu)y_\mu = 0 \).
(b) One has that \( y_\lambda H\Box_\lambda y_\lambda = ky_\lambda \).

Let \( \lambda \subset \mu \) be two Young diagrams, the complement of \( \lambda \) in \( \mu \) is called a skew Young diagram and is denoted by \( \mu/\lambda \). As above we can define the object \( \Box_{\mu/\lambda} \) in the category \( H \) with one point for each cell. The following is proven in the same way as Proposition 1.6.

Lemma 1.9. Let \( \lambda \subset \mu \) be two Young diagrams, and let \( x \in H_{\Box_{\mu/\lambda}} \). One has that
\[
G_\mu(y_\lambda \otimes x)F_\mu = [hl(\lambda)]G_\mu(1_{\Box_\lambda} \otimes x)F_\mu.
\]

Corollary 1.10 (Absorbing property). Let \( \lambda \subset \mu \) be two Young diagrams, and suppose that \([hl(\mu)]\) is invertible. One has
\[
y_\mu(y_\lambda \otimes 1_{\Box_{\lambda}})y_\mu = y_\mu.
\]

We will need the following formulas.

Proposition 1.11. (a) Suppose that \( \lambda \subset \mu \) are two Young diagrams such that the skew diagram \( \mu/\lambda \) has only one cell \( c \) (\( |\mu| = |\lambda| + 1 \)).

\[
y_\mu = a^{2|\lambda|}S^{2\text{cn}(c)}y_\mu
\]

(b) (Framing coefficient).

\[
y_\lambda = a^{(|\lambda|^2v^{-|\lambda|}S^{2\sum_{\text{cells}} \text{cn}(c)})}y_\lambda
\]

Statements (a) and (b) can be deduced from each other. Statement (b) is Theorem 5.5 of [3]; the proof given there is an elementary skein calculation which contains statement (a). The framing coefficients (b) were first computed by Wenzl [41, Lemma 3.2.1]. The next sub-section shows that \( y_\lambda \) belongs to the simple component of \( H_n \) indexed by \( \lambda \). Using this, the proposition above follows from Wenzl’s result.
1.4. Structure of the generic Hecke algebra and path idempotents

Here we suppose that \( n \) is fixed, and that, in the domain \( k \), \( [j] \) is invertible for every \( j \leq n \), so that all the idempotents \( y_j \) exist in \( H_n \).

A standard tableau \( t \) with shape a Young diagram \( \lambda = \lambda(t) \) is a labelling of the cells, with the integers 1 to \( n \), which is increasing along rows and columns. We denote by \( t' \) the tableau obtained by removing the cell numbered by \( n \). We define \( a_t \in H(n, \Box_\lambda) \) and \( b_t \in H(\Box_\lambda, n) \) by:

\[
\begin{align*}
\lambda_1 &= \beta_1 = 1_1 \\
\lambda_t &= (\lambda_t \otimes 1_1)Q_t y_{\lambda_t} \\
\beta_t &= y_{\lambda_t}Q_t^{-1}(\beta_t \otimes 1_1).
\end{align*}
\]

Here \( Q_t \in H(\Box_{\lambda(t')} \otimes 1, \Box_\lambda) \) is composed of two isomorphisms, using the intermediate object obtained from \( \Box_{\lambda(t')} \otimes 1 \) by moving the added point to its place in \( \Box_\lambda \); these two isomorphisms are obtained by connecting the corresponding points in the obvious way.

Note that \( \beta_t \lambda_t = 0 \) if \( t \neq \tau \), and \( \beta_t \lambda_t = y_{\lambda(t')} \).

**Theorem 1.12.** (a) The family \( \lambda \beta_t \), for all standard tableaux \( t, \tau \) such that \( \lambda(t) = \lambda(\tau) \) forms a basis for \( H_n \).

(b) There exists an algebra isomorphism

\[
\bigoplus_{|\lambda|=n} \mathcal{M}_d(\lambda, k) \cong H_n,
\]

where \( d_\lambda \) is the number of standard tableaux with shape \( \lambda \), and \( \mathcal{M}_d(\lambda, k) \) is the algebra of \( d_\lambda \times d_\lambda \) matrices with coefficients in \( k \).

**Proof.** We have (here \( \delta \) is the Kronecker delta)

\[
\lambda_t \beta_t \lambda_\sigma \beta_\sigma = \delta_{t=\sigma} \lambda_t \beta_\sigma.
\]

This shows independence. Moreover the number of vectors is equal to the dimension. This shows the proposition over the field of quotients of the domain \( k \). The above formula gives the coordinate forms and shows that the result (a) is valid over \( k \).

We can index the entries of the matrices in \( \mathcal{M}_d(\lambda, k) \) with the tableaux of shape \( \lambda \). The isomorphism in (b) is then obtained by sending the elementary matrix \( E_t \) to \( \lambda_t \beta_t \). □

The \( \lambda_t \beta_t \) are matrix units in the sense of Ram and Wenzl [33]. The isomorphism in (b) gives the semi-simple decomposition of \( H_n \). The simple components are indexed by Young diagrams. The component indexed by \( \lambda \) is the two-sided ideal generated by \( y_{\lambda} \); its rank is \( d_\lambda^2 \). The minimal central idempotents are the \( z_\lambda = \sum_{\lambda(t)=\lambda} \lambda_t \beta_t \).

By using the \( H_{n-1} \)-module structure of \( H_n \), it can be shown recursively that our indexation of the simple components coincides with Wenzl’s. The diagonal elements \( p_t = \lambda_t \beta_t \) are the path idempotents described in [40]; this can be seen from Wenzl formula [40, (2.7)].
Theorem 1.13 (Branching formula).

\[ y_\lambda \otimes 1_1 = \sum_{\mu : \lambda \subseteq \mu, |\mu| = |\lambda| + 1} (y_\lambda \otimes 1_1) y_\mu (y_\lambda \otimes 1_1) \]

We have omitted in this formula the standard isomorphism between \( \Box \lambda \otimes 1 \) and \( \Box_\mu \).

This branching formula for the path idempotents is given by Wenzl in [40]. A similar formula is given by Yokota [42, Proposition 2.11], with a proof using skein calculus.

Proof. Let \( \zeta \in H(\Box_\lambda, |\lambda|) \) be an isomorphism. We have

\[ y_\lambda \otimes 1_1 = \sum_{\mu} ((y_\lambda \zeta) \otimes 1_1) z_\mu (\zeta^{-1} y_\lambda \otimes 1_1) \]

\[ = \sum_{\mu} \sum_{\lambda(t) = \mu} ((y_\lambda \zeta) \otimes 1_1) p_t (\zeta^{-1} y_\lambda \otimes 1_1) \]

\[ = \sum_{\mu} \sum_{\lambda(t) = \mu} ((y_\lambda \cdots y_{\lambda(t')}) \otimes 1_1) y_\mu ((y_{\lambda(t')} \cdots y_\lambda) \otimes 1_1). \]

In the above, only those tableaux \( t \) with \( \lambda(t') = \lambda \) contribute, and all these contributions are proportional to \( (y_\lambda \otimes 1_1) y_\mu (y_\lambda \otimes 1_1) \). Hence we obtain coefficients \( \xi_{\mu} \) such that

\[ y_\lambda \otimes 1_1 = \sum_{|\mu| = |\lambda| + 1} \xi_{\mu} (y_\lambda \otimes 1_1) y_\mu (y_\lambda \otimes 1_1). \]

For each diagram \( v \) such that \( \lambda \subseteq v \) and \( |v| = |\lambda| + 1 \), we get \( \xi_v = 1 \) by writing \( y_\lambda(y_\lambda \otimes 1_1) \).

\[ \langle y_\lambda \rangle = \prod_{\text{cells}} \frac{v^{-1} s^{cn(c)} - vs^{-cn(c)}}{s^{bl(c)} - s^{-bl(c)}} \]

Recall that \( \langle y_\lambda \rangle \) is the Homflypt polynomial of a 0-framed unknot cabled with the closure \( \hat{y}_\lambda \) of \( y_\lambda \) in \( D^2 \times S^1 \). We will denote \( \langle y_\lambda \rangle \) simply by \( \langle \lambda \rangle \), and call it the quantum dimension of \( \lambda \).

The assertion above can be proven by a skein calculation (see Proposition 2.4 in [42] or [2]). An alternative proof [40] is to use the Young algebra. From branching formula (1.13), we can deduce that \( \hat{y}_\lambda \) is the \( \lambda \)-indexed Schur polynomial in the \( \hat{g}_k \); hence the general formula follows from

\[ \langle g_k \rangle = \prod_{i=1}^{k} \frac{v^{-1} s^{1-i} - vs^{1-i}}{s^i - s^{-i}}, \]

which can be proven by using the recursive formula for \( g_k \).

1.5. The \( \mathcal{C} \)-completed Hecke category

Suppose \( \mathcal{C} \) is a set of Young diagrams \( \lambda \) such that the Young idempotents \( y_\lambda \) exist. The \( \mathcal{C} \)-completed Hecke category \( H^\mathcal{C} \) is defined as follows.
An object in this category is a disc $D^2$ equipped with a finite set of points with trivialization as before labelled with Young diagrams in $\mathcal{C}$. If $x = (D^2, l)$ is such an object, its expansion $E(x) = (D^2, E(l))$ is obtained by embedding the object $\Box_z$ in a neighbourhood of $l_z$ according to the trivialization. The tensor product $y_{\chi^{(1)}} \otimes \cdots \otimes y_{\chi^{(m)}}$ defines an idempotent $\pi_x \in H_{E(x)}$. The module $H^x(x, \beta)$ is defined by

$$H^x(x, \beta) = \pi_x H(E(x), E(\beta))\pi_x.$$ 

The duality extends to the category $H^c$ in a natural way, and we again have a ribbon category. We denote simply by $\lambda$ the object of $H^c$ which is a disc $D^2$ with the origin labelled by $\lambda$.

1.6. Homflypt calculus using ribbon graphs

Following Turaev ([36, I.2), we can define the category $Rib_H$ of ribbon graphs over $H$, and a canonical functor $F_H$: $Rib_H \to H$. A coloured ribbon graph gives a morphism in the category $H$ (the functor $F_H$ will be implicit). This can be described as follows. For each band in the graph, coloured with $x$, embed (using the framing of the band) a copy of $1_x$, the identity of $x$; for each loop coloured with $x$ embed a copy of $\hat{1}_x$, the closure of $1_x$; for each coupon coloured with the morphism $f$, embed a copy of $f$.

We can proceed similarly with the $\mathcal{C}$-completed Hecke category $H^c$.

2. The modular categories $H^{SU(N,K)}$ and $H^{PSU(N,K)}$

2.1. Roots of unity

In this section, we suppose that $s$ is a primitive $2(N + K)$th root of unity, and that $v = s^{-N}$ (rank $N$ and level $K$, $N \geq 2$ and $K \geq 1$). We suppose moreover that $N + K$ is invertible in $k$. A consequence is that $[n]$ is invertible for $n < N + K$. This can be seen as follows. We have that

$$[n] = s^{-n+1} \prod_{1 < j | n} \phi_j(s^2).$$

Here $\phi_j \in \mathbb{Z}[X]$ is the $j$-indexed cyclotomic polynomial. The required invertibility is a consequence of the following lemma. Here we suppose $p \geq 2$.

Lemma 2.1. If $j \not\equiv p \mathbb{Z}$, then $\phi_j$ divides $p$ in $\mathbb{Z}[X]/\phi_p$.

Proof. Let $d = \gcd(p, j)$. In $\mathbb{Z}[X]$, one has a relation $U(X^j - 1) + V(X^p - 1) = X^d - 1$. If $j > d$, this implies a relation $U_1 \phi_j + V_1 \phi_p = 1$, hence $\phi_j$ is invertible in $\mathbb{Z}[X]/\phi_p$. If $j = d$ then using the derivative of $X^p - 1 = \phi_p T$, we get $p \equiv X^T \phi_j \mod \phi_p$. \[\square\]

We observe that the idempotent $y_{\lambda}$ exists for every Young diagram $\lambda$ with $\lambda_1 + \lambda_\gamma \leq N + K$. We denote by $\mathcal{C}_{N,K}$ the set of these Young diagrams and consider the category $H^{SU(N,K,a)}$ obtained from
the $\mathcal{C}_{N,K}$-completed Hecke category, $H^{C_{N,K}}$, by applying the following purification procedure [36, p. 504].

**Definition.** A morphism $f \in H^{C_{N,K}}(x, \beta)$ is negligible iff

$$\forall g \in H^{C_{N,K}}(\beta, z), \quad \langle fg \rangle = 0$$

Objects of $H^{(N,K,a)}$ are those of $H^{C_{N,K}}$. The module $H^{(N,K,a)}(x, \beta)$ is the quotient of $H^{C_{N,K}}(x, \beta)$ by the submodule of negligible morphisms. One can verify that composition and tensor product are well defined on the quotient. The negligible morphisms give local relations in Homflypt modules. We denote by $\mathcal{H}^{(N,K,a)}(M)$, the quotient of $\mathcal{H}^{C_{N,K}}(M)$ by these relations. Note that for $M = S^3$, no new relation appears. In the case $M = D^2 \times S^1$, the algebra structure is well defined on the quotient. The multilinear form $\langle L(\ldots) \rangle$ is well defined on $\mathcal{H}^{(N,K,a)}(D^2 \times S^1)$. More generally, ribbon graphs may be coloured using the category $H^{(N,K,a)}$.

For a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathcal{C}_{N,K}$, the identity morphism $1_\lambda$ is negligible if and only if the quantum dimension $\langle \lambda \rangle$ is zero. This will be the case if and only if $K < \lambda_1$ or $N < \lambda_1'$. We will use the following sets of Young diagrams:

$$T_{N,K} = \{ (\lambda_1, \ldots, \lambda_p) : \lambda_1 \leq K \quad \text{and} \quad p \leq N \}$$

$$\Gamma_{N,K} = \{ (\lambda_1, \ldots, \lambda_p) : \lambda_1 \leq K \quad \text{and} \quad p < N \}.$$ Denote by $1^N$ (resp. $(K)$) the diagram, with one column containing $N$ cells (resp. one row containing $K$ cells). The proposition below shows that $1^N$ (resp. $(K)$) is practically influential. It will also explain our choice of the framing parameter $a$.

**Proposition 2.2.** The following identities hold in the category $H^{(N,K,a)}$.

\[
\begin{array}{ccc}
\begin{array}{c}
g_N \\downarrow \quad \downarrow \\
\end{array} & = & a^{2N} s^2 \\
\begin{array}{c}
g_N \\downarrow \quad \downarrow \\
\end{array} & & \begin{array}{c}
g_N \\downarrow \quad \downarrow \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
f_K \\downarrow \quad \downarrow \\
\end{array} & = & a^{2K} s^{-2} \\
\begin{array}{c}
f_K \\downarrow \quad \downarrow \\
\end{array} & & \begin{array}{c}
f_K \\downarrow \quad \downarrow \\
\end{array}
\end{array}
\]

**Proof.** We justify the first equality. The proof of the second one is similar. The branching formula 1.13 gives ($g_{N+1}$ is negligible)

$$g_N \otimes 1_1 = (g_N \otimes 1_1) y_{(2,1^{s-1})} (g_N \otimes 1_1).$$

The formula comes from Proposition 1.11. \qed
Observe that (at rank \( N \), level \( K \)), we have \( \langle g_N \rangle = 1 \), and that the framing coefficient for \( g_N \) is \( (a^N s)^N \). Using this we obtain the following corollary.

**Corollary 2.3.** Suppose that \( a^N s = 1 \). Then the morphism represented by a coloured ribbon graph is not changed

- if some band coloured with the object \( 1^N \) is twisted or moved across any other band,
- or if some loop coloured with the object \( 1^N \) is removed.

Until the end of this section, we suppose that \( a^N s = 1 \). If \( \lambda \) is a Young diagram in \( \Gamma_{N,K} \), we denote by \( \lambda^* \) the skew diagram \( \mu / \lambda \), with \( \mu = \lambda^N \), up to a rotation in the plane, \( \lambda^* \) is a Young diagram in \( \Gamma_{N,K} \).

**Lemma 2.4.** (a) For any \( \lambda \in \bar{\Gamma}_{N,K} \), one has

\[
\langle \lambda^* \rangle = \langle \lambda \rangle.
\]

(b) For any \( \lambda \in \bar{\Gamma}_{N,K} \) of the form \( \lambda = 1^N + v \), one has

\[
\langle L(\hat{y}_2, \ldots) \rangle = \langle L(\hat{y}_v, \ldots) \rangle.
\]

In particular, for \( j \leq K \), one has

\[
\langle j^N \rangle = 1.
\]

**Proof.** If \( v = s^{-N} \) (rank \( N \)), the quantum dimension is given by

\[
\langle \lambda \rangle = \prod_{\text{cells}} \frac{[N + cn(c)]}{[hl(c)]}. \quad \square
\]

**Lemma 2.5.** For any \( \lambda \in \bar{\Gamma}_{N,K} \), we have

\[
\begin{array}{c}
\lambda_1^N \\
\lambda^*
\end{array}
\quad = \quad \frac{1}{\langle \lambda \rangle} y_\lambda .
\]

**Proof.** From the definition of the idempotents, we see that the left-hand side is proportional to \( y_2 \). The coefficient is obtained by comparing the quantum traces. By using the absorbing Property 1.10, the trace of the left-hand side is \( \langle \lambda_1^N \rangle = 1 \). \( \square \)

### 2.1.1. Reversing orientation

Let \( L \) be a framed link, and let \( L' \) be the framed link obtained from \( L \) by reversing the orientation of the first component (i.e. by changing the sign of the second vector in the trivialization of the normal bundle).
Proposition 2.6. For any \( \lambda \in \Gamma_{N, K} \), one has
\[
\langle L(\hat{y}_\lambda, \ldots) \rangle = \langle L(\hat{y}_{\lambda^*}, \ldots) \rangle.
\]

Proof. Using Lemma 2.5, we introduce somewhere the idempotent associated with the diagram \( \lambda_1^N \). Then we can cut the band coloured with \( \lambda_1^N \), move one of the ends along the first component \( L_1 \) of \( L \) and glue it back again, so that the band coloured with \( \lambda^* \) goes along \( L_1 \) with the reverse orientation. By Corollary 2.3 this does not change the evaluated Homflypt polynomial. Using Lemma 2.5 again, we get the result. \( \square \)

The above suggests that we could build an isomorphism between the trivial object 0 and 1 \( N \), and also between the dual of \( \lambda \) (denoted by \( \lambda^* \)) and \( \lambda^* \) (note the difference between * and \( \ast \)). In order to do that, we will add some morphisms.

We saw that we can define morphisms in the Hecke category, and more generally skein elements in Homflypt skein modules by using coloured ribbon graphs. We extend the \( C_{N, K} \)-completed Hecke category and the Homflypt skein theory by allowing ribbon graphs with incoming or outgoing vertices, coloured with \( r = 1^N \). Together with the relation of isotopy rel. boundary, the Homflypt relations, and the negligible morphisms, we add the relation given by gluing an incoming \( r \)-coloured vertex with an outgoing one. (Note that the half twist is not trivial; hence the orientations of the glued ends of the ribbon edges must be respected.)

We will denote by \( \overline{\text{H}}^{(N, K, a)} \) the category whose objects are those of \( \text{H}^{(N, K, a)} \) and whose morphisms are defined using the extended Homflypt skein theory above, and by \( \overline{\text{H}}^{(N, K, a)} \) the extended Homflypt skein functor.

From Corollary 2.3, we deduce that the Homflypt polynomial extends to an isomorphism \( \overline{\text{H}}^{(N, K, a)}(S^3) \approx k \).

In the category \( \overline{\text{H}}^{(N, K, a)} \), an object \( \lambda \in \Gamma_{N, K} \) is still simple, and its dual \( \lambda^* \) is isomorphic to \( \lambda^* \).

From the branching formula (Theorem 1.13), we can see that the simple objects in \( \Gamma_{N, K} \) dominate the category \( \overline{\text{H}}^{(N, K, a)} \). This gives us all the defining properties of a modular category except the non-degeneracy axiom. We say that \( \overline{\text{H}}^{(N, K, a)} \) is a pre-modular category.

Note that distinct objects \( \hat{\lambda}, \mu \in \Gamma_{N, K} \) are not isomorphic, but this does not imply the non-degeneracy axiom.

2.2. The handle slide condition

We say that \( \Omega \in \mathcal{H}^{(N, K, a)}(D^2 \times S^1) \) satisfies the Kirby condition if
\[
\begin{align*}
\forall x & \in \mathcal{H}^{(N, K, a)}(D^2 \times S^1) \quad \langle H_1(x, \Omega) \rangle = \langle U_0(x) \rangle \langle U_1(\Omega) \rangle, \\
\langle U_1(\Omega) \rangle & \text{ is invertible.}
\end{align*}
\]

Here, for \( \varepsilon \in \{-1, 0, 1\} \), we denote by \( U_\varepsilon \) the unknot with framing \( \varepsilon \), and by \( H_\varepsilon \) the Hopf link with linking number one and both components having framing \( \varepsilon \).

A solution of the above is essentially unique. In [6], we discussed this condition in the context of a formal skein theory.
A framed link $L$ determines by surgery a 3-manifold denoted by $S^3(L)$ (every closed oriented 3-manifold can be obtained in this way). As a consequence of Kirby’s theorem [19], if a solution $\Omega$ exists, then an appropriate normalization of $\langle L(\Omega, \ldots, \Omega) \rangle$ is an invariant of the surgered manifold $M = S^3(L)$. It is convenient to choose a solution $\omega$ of (K) such that $\langle U_1(\omega) \rangle \langle U_{-1}(\omega) \rangle = 1$. (It may be necessary to extend $k$ by adding a square root.) An appropriate normalization is then

$$\tau(M) = \langle U_1(\omega) \rangle^{-\sigma(L)} \langle L(\omega, \ldots, \omega) \rangle.$$

Here $\sigma(L)$ is the signature of the linking matrix, i.e. $\sigma(L) = b_+ - b_-$, where $b_+$ (resp. $b_-$) is the number of positive (resp. negative) eigenvalues of the linking matrix $B_L$ associated with $L$.

Recall that we have denoted by $\Gamma_{N,K}$ the set of Young diagrams with at most $N - 1$ rows and $K$ columns (the empty diagram is included). We set

$$\Omega = \sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle \Omega_{\lambda}.$$ 

Following Yokota [42], we can show the sliding property. From this sliding property and Proposition 2.6 we can deduce the first part of the Kirby condition (the handle slide).

**Proposition 2.7** (Sliding property). *The Homflypt polynomial of a link in $S^3$, which has one of its components cabled with the skein element $\Omega$, satisfies the equality in Fig. 2.*

In this figure, the dashed curve means that the component cabled with $\Omega$ may be non-trivially embedded in the sphere; the bracket notation for the Homflypt polynomial is omitted.

**Proof.** Using the definition of $\Omega$, the branching formula, and Lemma 2.5.

$$lhs = \sum_{\lambda} \langle \lambda \rangle \sum_{\mu \in \mu \mid \mu = |\lambda| + 1} \langle \mu \rangle \langle A(\lambda, \mu) \rangle,$$

where $A(\lambda, \mu)$ is the skein element represented in Fig. 3. Note that in the above, we can forget any $\mu$ with $\mu_1 = K + 1$, because in this case the corresponding skein element contains a negligible morphism. We can cut the band coloured with $\mu_1^*$, move one of the ends along the closed component and glue it back again, so that the band coloured with $\mu^*$ goes along this closed
component with the reverse orientation. Then we apply Lemma 2.5 and the branching formula again. The result follows, using Lemma 2.4. Note that \((\lambda, \mu) \mapsto (\mu_1^N / \mu, \mu_1^N / \lambda)\) defines an involution on the indexation set

\[ \{ (\lambda, \mu) : \lambda \in \Gamma_{N,K}, \, \lambda \subset \mu, |\mu/\lambda| = 1 \text{ and } \mu_1 \leq K \} . \]

2.3. The invariant \(s^{SU(N,K)}\), and the modular category \(H^{SU(N,K)}\)

Until the end of Section 2, we will consider the choice of the framing parameter \(a = s^{-1/N}\). More precisely, \(a\) is a primitive \(2N(K + N)\)th root of unity, and \(s = a^{-N}\).

For this choice of parameter \(a\), we denote the category \(H^{(N,K,a)}\) by \(H^{SU(N,K)}\), and the corresponding skein functor by \(\mathcal{F}^{SU(N,K)}\).

Lemma 2.8.

\[
\langle U_1(\Omega) \rangle \langle U_{-1}(\Omega) \rangle = \langle \Omega \rangle = (-1)^{N(N-1)/2} \frac{N(N+K)^{N-1}}{\prod_{j=1}^{N} (s^j - s^{-j})^{2(N-1)}}.
\]

Proof. The first equality comes from the sliding property and Lemma 2.9 below. For the second equality, we follow Erlijman’s computation in [12].

\[
\langle \lambda \rangle = \frac{\prod [N + cn]}{\prod [h^l]} = s^{-(N-1)|\lambda|} \mathcal{S}_\lambda(1, s^2, \ldots, s^{2(N-1)}).
\]

Here \(\mathcal{S}_\lambda\) is the \(\lambda\)-indexed Schur symmetric polynomial ([25, Chapter 1]. We get

\[
\langle \lambda \rangle^2 = \frac{a_\rho + 2 \vec{a}_\rho + \lambda}{a_\rho \vec{a}_\rho},
\]

with \(\rho = (N - 1, N - 2, \ldots, 0)\) and, for \(l = (l_1, \ldots, l_N)\),

\[
a_l = \det(s^{2(l-1)^t})_{1 \leq i, j \leq N}.
\]
We have
\[ a_\rho \tilde{a}_\rho = \prod_{i < j} (s^{2j} - s^{2i})(s^{-2j} - s^{-2i}) = (-1)^{N(N-1)/2} \prod_{v=1}^{N-1} (s^v - s^{-v})^{2(N-v)} \]

\[ \sum_{\lambda \in \Gamma_{N,K}} a_{\rho + \lambda} \tilde{a}_{\rho + \lambda} = \sum_{N+K > l_1 > \ldots > l_k = 0} \det(s^{2(i-1)l_i}) \det(s^{-2(i-1)l_i}) \]

\[ \sum_{\lambda \in \Gamma_{N,K}} a_{\rho + \lambda} \tilde{a}_{\rho + \lambda} = \frac{1}{(N-1)!} \sum_{0 \leq l_1, \ldots, l_{N-k} < N+K} \sum_{\pi, \pi' \in S_N} e_\pi e_{\pi'} \prod_{i=1}^{N-1} s^{2l_i(\pi(i) - \pi'(i))}. \]

Here, we have symmetrized the index set for \( l \) (and added terms which are zero), and we expanded the determinants.

\[ \sum_{\lambda \in \Gamma_{N,K}} a_{\rho + \lambda} \tilde{a}_{\rho + \lambda} = \frac{1}{(N-1)!} \sum_{\pi, \pi' \in S_N} \sum_{\pi \neq \pi'} e_\pi e_{\pi'} \prod_{i=1}^{N-1} s^{2l_i(\pi(i) - \pi'(i))}. \]

For \( \pi \neq \pi' \), a zero term appears, and the \( N! \) remaining terms are all equal. We get

\[ \sum_{\lambda \in \Gamma_{N,K}} a_{\rho + \lambda} \tilde{a}_{\rho + \lambda} = N(N+K)^{N-1}. \]

Our formula follows.

**Lemma 2.9.** If \( \lambda \neq \emptyset \), then the following morphism is zero in \( H^{SU(N,K)} \).

\[ \Omega \]

\[ \lambda \]

**Proof.** By the sliding property, for any Young diagram \( \mu \) such that \( \lambda \subset \mu \), and \( |\mu/\lambda| = 1 \), we have

By Proposition 1.11, if the colour \( \lambda \) is not killed by inserting a 0-framed meridian cabled with \( \Omega \), then \( a^{2|\lambda|} \tilde{a}^{2\det(\mu/\lambda)} = 1 \) for any \( \mu \) as above with \( \langle \mu \rangle \neq 0 \). If two such \( \mu \) exist, with added cells of respective indices \((i, j)\) and \((i', j')\), then \( s^{2(j-j'+i-i')} = 1 \) and \(- (N + K) < j' - j + i - i' < N + K\).
This implies \((i, j) = (i', j')\). The remaining diagrams are those with \(K\) columns and \(n\) rows, \(n < N\). In this case we get \(a^{2nK - 2n} = a^{2n(K + N)} = 1\). The order of \(a\) implies \(n = 0\). \(\square\)

We suppose that \(N(N + K)\) is invertible. We set \(\eta^{-2} = \langle \Omega \rangle\) (a quadratic extension of \(k\) may be needed), \(\omega = \eta\Omega\), \(\Delta = \langle U_1(\omega) \rangle\).

**Theorem 2.10.** There exists an invariant of compact oriented 3-manifolds defined on a surgery presentation by the following formula:

\[
\tau^{SU(N, K)}(S^3(L)) = A^{-\sigma(L)}\langle L(\omega, \ldots, \omega) \rangle.
\]

This invariant can be extended to manifolds with (coloured) links by the formula

\[
\tau^{SU(N, K)}(S^3(L), K) = A^{-\sigma(L)}\langle L(\omega, \ldots, \omega) \cup K \rangle.
\]

The above is a consequence of the Kirby theorem. Using [36], the following constructs the associated TQFT (and proves again the theorem above).

**Theorem 2.11.** The category \(H^{SU(N, K)}\) is a modular category with \(\Gamma_{N, K}\) as a representative set of isomorphism classes of simple objects.

**Proof.** We have to check invertibility of the \(S\) matrix whose entries, indexed by \(\Gamma_{N, K}\), are the evaluations of the Homflypt polynomial for a Hopf link \(H_0\) (0-framed, and with linking \(+1\)), whose components are coloured with the corresponding indices. The lemma below gives the result. Here the entries of the matrix \(S\) are the evaluations of the Homflypt polynomial for the Hopf link \(H_0\) (0-framed, and with linking \(-1\)). \(\square\)

**Lemma 2.12.** One hase \((I\) is the unit matrix\)

\[
SS = \langle \Omega \rangle I.
\]

**Proof.** The \((\lambda, \mu)\)-indexed entry of the matrix \(SS\) can be written

\[
u_{\lambda\mu} = \langle H_0(\gamma_{\lambda\mu}^*, \Omega) \rangle.
\]

We write \(u_{\lambda\mu}\) as the quantum trace of the morphism \(\gamma_{\lambda\mu}\) represented by a \(\lambda \otimes \mu^*\)-coloured band together with an \(\Omega\)-cabled meridian. Using that the objects in \(\Gamma_{N, K}\) dominate, and Lemma 2.9, we obtain that there exists a finite family \((x_i, \beta_i), x_i \in H^{SU(N, K)}(\lambda \otimes \mu^*, 0), \beta_i \in H^{SU(N, K)}(0, \lambda \otimes \mu^*),\) such that

\[
\gamma_{\lambda\mu} = \langle \Omega \rangle \sum \beta_i.
\]

If \(\lambda\) and \(\mu\) are distinct in \(\Gamma_{N, K}\), then the two modules above are zero, hence we have \(u_{\lambda\mu} = 0\).

If \(\lambda = \mu\), then this two modules are generated by the duality morphisms. This gives

\[
\sum \beta_i = \zeta d_{\delta^*} b_2.
\]

We obtain \(\zeta = 1/\langle \lambda \rangle\) from the product \(b_{\delta^*} d_{\lambda^*} = \langle \lambda \rangle \langle \Omega \rangle\). We can conclude that \(u_{\lambda\lambda} = \langle \Omega \rangle\). \(\square\)

As already stated, using Turaev’s work, modularity gives the TQFT. The universal construction of [7] could also be applied here. The normalized invariant of connected closed 3-manifolds \(M\) equipped with \(p_1\)-structure (or 2-framing) \(x\) is defined by

\[
Z(M, x) = \eta \kappa^{-\sigma(x)} \epsilon^{SU(N, K)}(M),
\]
with \( k^3 = \Delta \). We obtain a TQFT functor \((V, Z)\) on the cobordism category \( C^2_{p_1} \) of \( p_1 \)-surfaces and (equivalence classes of) cobordisms.

Unfortunately the descriptions of structured surfaces in [36, 7] are not the same; however, this has essentially no influence on the description of the TQFT modules.

The vectors \( Z(\mathbf{D}^2 \times S^1, \hat{y}_\lambda), \lambda \in \Gamma_{N, K} \) form a basis for the TQFT module \( V(S^1 \times S^1) \). Moreover this basis is orthonormal with respect to the natural hermitian form on this module.

### 2.3.1. The fusion algebra and Verlinde dimension formula

The algebra structure on the skein module of the solid torus induces the fusion algebra structure on \( V(S^1 \times S^1) \). From modularity, we get the structure constants

\[
y_{\lambda} y_{\mu} = \sum_{v} c_{\lambda \mu}^{v} y_{v},
\]

with \( c_{\lambda \mu}^{v} \) equal to the rank of the module associated to a sphere, with two incoming points and one outgoing point coloured, respectively, by \( \lambda, \mu \) and \( v \). A combinatorial description for these ranks should be obtainable using [14].

For a genus \( g \) closed surface \( \Sigma_{g} \), we can compute the dimension \( d_{g} \) of the TQFT module \( V(\Sigma_{g}) \), which is equal to the invariant \( Z(\Sigma_{g} \times S^1) \). The result is given by the Verlinde formula.

\[
d_{g} = ((N + K)(N-1))^{g-1} \sum_{N+K > l_1 > l_2 > \ldots > l_g = 0} \prod_{1 \leq i < j \leq N} \left( \frac{-1}{(s_j - s_l - s_l)^{2}} \right)^{g-1}.
\]

### 2.4. The invariant \( \tau_{PSU(N, K)} \)

The algebra \( \mathcal{A}_{PSU(N, K)}(\mathbf{D}^2 \times S^1) \) is \( N \)-graded. Set \( \Gamma_{N, K}^{0} = \{ \lambda \in \Gamma_{N, K}, |\lambda| \equiv 0 \mod N \} \).

We can see that \( \Omega_{0} = \sum_{\lambda \in \Gamma_{N, K}^{0}} \langle \lambda \rangle \hat{y}_{\lambda} \) satisfies the handle slide condition (an arc with a 0-graded colour can slide over a component cabled with \( \Omega_{0} \)). We have the following lemma. We give no proof in this section; they can be adapted from those in Section 4, where the graded case is treated with more details.

**Lemma 2.13.** If \( d = \gcd(N, K) \) is even, and \( N' = N/d, K' = K/d \) are both odd, then

\[
\langle U_{1} \Omega_{0} \rangle = 0.
\]

In all other cases one has that

\[
\langle U_{1} \Omega_{0} \rangle \langle U_{-1} \Omega_{0} \rangle = d \langle \Omega_{0} \rangle = (-1)^{N(N-1)/2} \frac{d(N + K)^{N-1}}{\prod_{j=1}^{N-1}(s_j - s_j - s_j)^{2(N-1)}}.
\]

We say that the rank-level \( (N, K) \) is spin if \( d = \gcd(N, K) \) is even, and \( N' = N/d, K' = K/d \) are both odd. The terminology will be justified in Section 4.

We suppose now that the rank-level \( (N, K) \) is not spin, and that \( N + K \) is invertible.

We set \( \eta_{0} = d \langle \Omega_{0} \rangle \) (we extend \( k \) if necessary), \( \omega_{0} = \eta_{0} \Omega_{0}, \Delta_{0} = \langle U_{1} \omega_{0} \rangle \).
Theorem 2.14. There exists an invariant of compact oriented 3-manifolds defined on a surgery presentation by the following formula:

\[ \tau^{PSU(N,K)}(S^3(L)) = \Delta_0^{-\sigma(L)}(L(\omega_0, \ldots, \omega_0)) \]

For \( gcd(N, K) = 1 \) this invariant and an underlying modular category obtained from the quantum group are known [26].

Remark. There exists a refined invariant \( \tau^{SU(N,K)}(M, c) \), with \( c \) a cohomology class in \( H^1(M, \mathbb{Z}/gcd(N, K)) \) such that \( \tau^{PSU(N,K)}(M) = \tau^{SU(N,K)}(M, 0) \) (see Section 4).

2.4.1. The modular category

The objects of the category \( H^{SU(N,K)} \) are graded by the algebraic number of points in their expansion (signed with the orientation). Let \( H^{PSU(N,K)} \) be the full subcategory of \( H^{SU(N,K)} \), whose objects are zero graded modulo \( N \). We can show (we give no details).

Theorem 2.15. For \( gcd(N, K) = 1 \), the category \( H^{PSU(N,K)} \) is a modular category with \( \Gamma_{N,K}^0 \) as a representative set of isomorphism classes of simple objects.

In the case \( gcd(N, K) > 1 \), the category \( H^{PSU(N,K)} \) is not a modular category. In [10], Bruguieres obtains a modularization by using a general procedure developed there.

3. The modular category \( \hat{H}^{N,K} \) and the invariant \( \hat{\tau}_{N,K} \)

In this section, we work at rank \( N \) as before (i.e. \( v = s^{-N} \)), but level \( K \) will mean that, in the integral domain \( k \),

- \( s \) has order \( 2(N + K) \) if \( N + K \) is even,
- \( s \) has order \( N + K \) if \( N + K \) is odd.

In both cases \( s^2 \) has order \( N + K \); we have \( v = \varepsilon s^K \), with \( \varepsilon = (-1)^{N + K + 1} \). Let \( d = gcd(N, K) \), \( N = dN', K = dK' \). Motivated by the formulas in Proposition 1.11, we would like to fix the framing parameter \( a \) in such a way that the order of the multiplicative subgroup generated by \( a^N s \) and \( a^K s^{-1} \) is as small as possible. Note that this order is at least \( d \) in the odd case, and \( 2d \) in the even case. We show that this lower bound can be realized.

To simplify the discussion we suppose in the case \( N + K \) even that \( N' \) is odd; we set \( d = \varepsilon \beta \) with \( gcd(\varepsilon, 2K') = gcd(\beta, N') = gcd(\varepsilon, \beta) = 1 \). Recall that \( \varepsilon = (-1)^{N + K + 1} \).

Lemma 3.1. We can choose the framing parameter \( a \) so that \( (a^N s)^{\varepsilon} = 1 \) and \( (a^K s^{-1})^{\beta} = \varepsilon \).

(It may be necessary to extend the scalars.)

Proof. We have \( gcd(\varepsilon N', \beta K') = 1 \), so we can write a Bezout identity \( A\varepsilon N' + B\beta K' = 1 \). The required condition is then equivalent to \( A\varepsilon = B\beta s^{-\varepsilon A + \beta B} \). (The above shows which extension is needed.) \( \Box \)
Remark. If $N + K$ and $N'$ are both even, then we set $d = z\beta$, with $\gcd(z, K') = \gcd(\beta, N') = \gcd(z, \beta) = 1$, and in the above lemma we require that $(a^N)^2 = -1$ and $(a^{K'N^{-1}})^\beta = 1$. Following [22], we can show a level-rank duality formula, and recover this case by exchanging $N'$ and $K$.

In this section, and in Section 4, we suppose that the framing parameter $a$ satisfies the condition of Lemma 3.1. Recall that the category $H^{(N,K,a)}$ is the quotient of the $\mathcal{C}^{N,K}$-extended Hecke category by negligible morphisms. We define the category $\tilde{H}^{N,K}$ as follows. Objects are those of $H^{(N,K,a)}$, and the modules of morphisms are obtained from those of $H^{(N,K,a)}$ by adding as generators coloured ribbon graphs in which incoming or outgoing 1-valent vertices coloured with $(1)^{\otimes x}$ or $(K)^{\otimes \beta}$ are allowed, and quotienting by

- the relations given by gluing an incoming $(1)^{\otimes x}$-coloured (resp. $(K)^{\otimes \beta}$-coloured) vertex with an outgoing one,
- the relations given by introducing a copy of $y_N^x$, viewed as a coloured graph with one coupon, $eta K'$ bands coloured by $(1)^{\otimes x}$ above, and $zN'$ bands coloured by $(K)^{\otimes \beta}$ below.

The corresponding skein functor is denoted by $\tilde{H}^{N,K}$.

Exercise. Show that the Homflypt invariant extends to an isomorphism $\tilde{H}^{N,K}(S^3) \simeq k$. Set $\hat{I}_{N,K} = \{(1)^{\otimes i} \otimes \lambda, \ 0 \leq i < x$ and $\lambda \in \Gamma_{N,K}\}$. The category $\tilde{H}^{N,K}$ is a ribbon category, and $\hat{I}_{N,K}$ is a finite set of dominating simple objects. We have that $\tilde{H}^{N,K}$ is a pre-modular category. This uses the involution $\ast$ on the set $\hat{I}_{N,K}$ defined as follows.

For $x = (1)^{\otimes i} \otimes \lambda \in \hat{I}_{N,K}$, we set $x^\ast = (1)^{\otimes i'} \otimes \lambda^\ast$, where $i' \in \{0, \ldots, x - 1\}$ is such that the number of points in $x \otimes x^\ast$ is a multiple of $N\lambda$.

We can now proceed similarly as we did in Section 2.

Proposition 3.2. Let $L$ be a framed link in the 3-sphere, and let $L'$ be the link obtained from $L$ by reversing the orientation of the first component. Then for any $x \in \hat{I}_{N,K}$, one has

$$\langle L(x, \ldots) \rangle = \langle L'(x^\ast, \ldots) \rangle.$$  

Set

$$\Omega = \sum_{x \in \hat{I}_{N,K}} \langle x \rangle \hat{I}_x = \sum_{i=0}^{x-1} \hat{g}_i \Omega.$$  

where $\Omega$ is defined as before by

$$\Omega = \sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle \hat{y}_{\lambda}.$$  

Proposition 3.3 (Sliding property). The Homflypt polynomial of a link in $S^3$, which has one of its components cabled with the skein element $\Omega$, satisfies the equality in Fig. 2 (with $\Omega$ replaced by $\hat{\Omega}$).

There is an action of $\mathbb{Z}/N$ on the set $\hat{I}_{N,K}$ defined as follows. The generator of $\mathbb{Z}/N$ acts by

$$(1)^{\otimes i} \otimes \lambda \mapsto (1)^{\otimes (i' + \lambda_N - 1)} \otimes ((K, \lambda) - \lambda_N^{N-1}).$$
Lemma 3.4. The proofs of the following are similar to those in Section 2.3.

where \( i' \in \{0, \ldots, z - 1\} \) is congruent to \( i + \lambda_{N-1} \) modulo \( z \). The idea is that we add to the diagram a row with \( K \) cells, and then each column which has \( N \) cells is replaced by an added copy of \( 1^N \).

Restricting this action to the group generated by \( \beta \), we obtain an action of \( \mathbb{Z}/zN' \). By considering the degree mod \( zN' \), we can see that this action is free. We denote by \( \tilde{\Gamma}_{N,K} \) a subset of \( \tilde{\Gamma}_{N,K} \), which is a representative set for the orbits. We set

\[
\tilde{\Omega} = \sum_{u \in \tilde{\Gamma}_{N,K}} \langle u \rangle \tilde{1}_u
\]

Cabling \( \tilde{\Omega} \) gives the same result as cabling \( N'z\tilde{\Omega} \), so that \( \tilde{\Omega} \) also satisfies the sliding property. The proofs of the following are similar to those in Section 2.3.

Lemma 3.4.

\[
\langle U_1(\tilde{\Omega}) \rangle \langle U_{-1}(\tilde{\Omega}) \rangle = \langle \tilde{\Omega} \rangle = (-1)^{(N-1)/2} \frac{d(N + K)^{N-1}}{\prod_{j=1}^{N-1} (s^j - s^{-j})^2(N-j)}.
\]

We suppose that \((N + K)\) is invertible. We set \( \bar{\eta}^{-2} = \langle \tilde{\Omega} \rangle \) (we extend \( k \) if necessary), \( \tilde{\omega} = \bar{\eta} \tilde{\Omega} \), \( \delta = \langle U_1(\tilde{\delta}) \rangle \).

Theorem 3.5. (a) There exists an invariant of compact oriented 3-manifolds defined on a surgery presentation by the following formula:

\[
\tilde{\tau}_{N,K}(S^3(L)) = \delta^{-\sigma(L)} \langle L(\tilde{\omega}, \ldots, \tilde{\omega}) \rangle
\]

(b) The category \( \tilde{\mathcal{H}}^{N,K} \) is a modular category with \( \tilde{\Gamma}_{N,K} \) as a representative set of isomorphism classes of simple objects.

Now we give the relation between the invariant \( \tau^{SU(N,K)} \) and \( \tilde{\tau}_{N,K} \) in the following reduction theorem. Here \( \tau^{U(1)}(M, \zeta) \) is a version of the invariant derived from linking matrices in [31] (\( U(1) \) invariant) for a root of unity \( \zeta \), whose order is \( N' \) (resp. \( 2N' \)) if \( N' \) is odd (resp. if \( N' \) is even).

Theorem 3.6 (Reduction formula). For every manifold \( M \), one has

\[
\tau^{SU(N,K)}(M) = \tau^{U(1)}(M, \zeta) \tilde{\tau}_{N,K}(M).
\]

We will give the proof in Section 5. We write this equality in the ring \( k \) where the invariant \( \tau^{SU(N,K)} \) has been defined. The definition of the reduced invariant \( \tilde{\tau}_{N,K} \) requires a choice of the parameters in \( k \); we denote by \( (\tilde{s}, \tilde{v}, \tilde{a}) \) this choice.

If \( d \) is even, then \( \tilde{s} = s, \tilde{v} = v \) and \( \zeta = (a^k s^{-1})^K \beta^j \) (the case \( N' \) even is not excluded).

If \( d \) is odd, then \( \tilde{s} = -s, \tilde{v} = -v \) and \( \zeta = (-a)^k s^{-1} \beta^j \).

The \( U(1) \) invariant in the formula above is then defined by

\[
\tau^{U(1)}(S^3(L), \zeta) = \left( \frac{\Lambda}{\delta} \right)^{-\sigma(L)} \left( \frac{\eta}{\bar{\eta}} \right)^m \sum_{j \in (\mathbb{Z}/N')^r} \zeta^j B_{\eta, j}.
\]
4. Refined invariants

4.1. Spin structures modulo an even integer

In [30], Murakami stated a decomposition formula for the invariant $\tau^{SU(N,K)}$ using some spin type structures (see Remark 2.7 in his paper). He observed that for $N = 2$ these are spin structures, and the corresponding refinements were studied in [20, 5]. For $N > 2$ he only gave a combinatorial description of the structures, and asked for a topological interpretation. We recall here the topological definition for these structures which we gave in [6].

Suppose $d$ is an even integer. Then there exists, up to homotopy, a unique non-trivial map $g : BSO \to K(\mathbb{Z}/d, 2)$. Define the fibration

$$\pi_d : BSpin(\mathbb{Z}/d) \to BSO$$

to be the pull-back, using $g$, of the path fibration over $K(\mathbb{Z}/d, 2)$. The space $BSpin(\mathbb{Z}/d)$ is a classifying space for the non-trivial central extension of the Lie group $SO$ by $\mathbb{Z}/d$, which we denote by $Spin(\mathbb{Z}/d)$.

Now we can use the fibration $\pi_d$ to define structures [St]. Let $\gamma_{Spin(\mathbb{Z}/d)} = \pi_d^*(\gamma_{SO})$ be the pull-back of the canonical vector bundle over $BSO$.

**Definition.** A spin structure with mod $d$ coefficients on a manifold $M$ is a homotopy class of fibre maps from the stable tangent bundle $\tau_M$ to $\gamma_{Spin(\mathbb{Z}/d)}$.

If non-empty, the set of these structures, denoted by $Spin(M; \mathbb{Z}/d)$, is affinely isomorphic to $H^1(M; \mathbb{Z}/d)$, by obstruction theory. Moreover the obstruction for existence is a class $w_2(M; \mathbb{Z}/d) \in H^2(M; \mathbb{Z}/d)$, which is the image of the Stiefel–Whitney class $w_2(M)$ by the homomorphism induced by the inclusion of coefficients $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/d$. As the Stiefel–Whitney class $w_2(M)$ is zero for every compact oriented 3-manifold, this shows that spin structures modulo $d$ exist on every closed oriented 3-manifold $M$. The following theorem gives a combinatorial description for these structures. Recall that a surgered manifold $M = S^3(L)$ is the boundary of a 4-manifold $W_L$ called the trace of the surgery. To each $\sigma \in Spin(M; \mathbb{Z}/d)$ is associated a relative obstruction $w_2(\sigma; \mathbb{Z}/d)$ in $H^2(W_L, M; \mathbb{Z}/d)$. The group $H^2(W_L, M; \mathbb{Z}/d)$ is free of rank $m = nL$. Taking the coordinates of the relative obstruction we get a map $\psi_L : Spin(M; \mathbb{Z}/d) \to (\mathbb{Z}/d)^m$.

**Theorem 4.1.** The map $\psi_L$ is injective, and its image is the set of those $(c_1, \ldots, c_m)$ which are solutions of the following $(\mathbb{Z}/d)$-characteristic equation

$$B_L \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \frac{d}{2} \begin{pmatrix} b_{11} \\ \vdots \\ b_{mm} \end{pmatrix} \pmod{d}.$$

Here the $b_{ii}$ are the diagonal values of the linking matrix $B_L$.

**Proof.** First we compute the absolute obstruction $w_2(W_L; \mathbb{Z}/d) = \xi_{\ast}(w_2(W_L))$, where $\xi_{\ast}$ is induced by the morphism of coefficients $\xi : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/d$. If $x$ is an integral 2-cycle in $W_L$ with self-intersection
$x \cdot x$ and $[x]_e$ denotes its homology class modulo an integer $v$, $w_2(W_L) \in H^2(W_L; \mathbb{Z}/2)$ is determined by the equation
\[ \forall x \langle w_2(W_L), [x]_e \rangle = x \cdot x \pmod{2} \]
Hence $w_2(W_L; \mathbb{Z}/d) \in H^2(W_L; \mathbb{Z}/d)$ is determined by
\[ \forall x \langle w_2(W_L; \mathbb{Z}/d), [x]_e \rangle = \xi(x) \cdot x = \frac{d}{2} x \cdot x \pmod{d}. \]

Now by functoriality, the relative obstruction lives in the inverse image of the absolute one under the map induced by inclusion $H^2(W_L, M; \mathbb{Z}/d) \to H^2(W_L; \mathbb{Z}/d)$. Using the affine structure over $H^1(M; \mathbb{Z}/d)$, we obtain an affine bijection between Spin$(M; \mathbb{Z}/d)$ and this inverse image. Whence we have the lemma by writing the equation above using coordinates. 

There is a formula for the bijection $\psi_{L,L'}$ corresponding to a Kirby move. Using the $\mathbb{Z}/d$-characteristic equation we see that the coefficient for a trivial component with framing $\pm 1$ is $d/2$. For the usual positive Fenn–Rourke move, the formula is
\[ \psi_{L,L'}(c_1, \ldots, c_{m-1}, d/2) = (c_1, \ldots, c_{m-1}, c'_m) \]
with
\[ c'_m = d/2 - \sum_i b_{im} c_i. \]
Here $b_{im}$ is the $(i, m)$-indexed coefficient of the matrix $B_{L'}$.

### 4.2. Spin refinements

We consider here the reduced theory of Section 3 in the spin case. Recall that this means
\[ d = \gcd(N, K) \]
\[ N' = N/d \text{ and } K' = K/d \]
are odd.

We decompose the skein element $\tilde{\omega} = \sum_i \tilde{\omega}_i$ according to the $\mathbb{Z}/d$-grading of the algebra $\mathcal{H}^{N,K}(D^2 \times S^1) = \bigoplus \mathcal{H}^{N,K}_\sigma(D^2 \times S^1)$.

**Theorem 4.2.** Provided $c = (c_1, \ldots, c_m)$ satisfies the modulo $d$ characteristic condition, the formula
\[ \tilde{\tau}_{N,K}^{\text{spin}}(M, \sigma) = \delta^{-\sigma(L)} \langle L(\tilde{\omega}_{c_1}, \ldots, \tilde{\omega}_{c_m}) \rangle \]
defines an invariant of the surgered manifold $M = S^3(L)$ equipped with the modulo $d$ spin structure $\sigma = \psi_{L}^{-1}(c_1, \ldots, c_m)$. Moreover,
\[ \forall M \tilde{\tau}_{N,K}(M) = \sum_{\sigma \in \text{Spin}(M; \mathbb{Z}/d)} \tilde{\tau}_{N,K}^{\text{spin}}(M, \sigma). \]

**Proof.** The following graded version of the sliding property can be derived from the proof of Proposition 2.7.
Fig. 4. Graded sliding property.

**Lemma 4.3 (Graded sliding property).** The Homflypt polynomial of a link in $S^3$, which has one of its components cabled with the skein element $\omega_\nu$, satisfies the equality in Fig. 4.

Using this sliding property, we get
\[ \forall \nu \forall x \in \mathcal{F}_v \times (D^2 \times S^1) \langle H_1(x, \omega_{d/2-\nu}) \rangle = \langle U_0(x_\nu) \rangle \langle U_1(\omega_{d/2}) \rangle. \]
This shows that the spin handle slide condition is satisfied.

We can deduce from Lemma 4.5 below that $\langle U_d(\omega_{d/2}) \rangle = \langle U_d(\omega) \rangle$ is invertible, and hence we have that the invariant $\tau^{\text{spin}}_{N,K}$ is well defined.

Now we show the decomposition formula. We can write
\[ \langle L(\omega, \ldots, \omega) \rangle = \sum_{c \in \mathcal{L}[d]} \langle L(\omega_c, \ldots, \omega_c) \rangle. \]
The result is contained in Lemma 4.4.

**Lemma 4.4.** If $c$ does not satisfy the modulo $d$ characteristic condition, then
\[ \langle L(\omega_c, \ldots, \omega_c) \rangle = 0. \]

**Proof.** Let $L = (L_1, \ldots, L_m)$. Up to a permutation of the components, we have to show that $\langle L(\omega_{c}, \ldots, \omega_{c}) \rangle = 0$, if $\sum_{j=1}^m b_j c_j \not= (d/2)b_1 \mod d$. The proof is in three steps.

If $L_1$ is a 1-framed unknot, then, using the sliding property, we get the result from Lemma 4.5.

If $L_1$ is an unknot with any framing, we can add to the link a trivial $\pm 1$-framed unknot cabled with $\omega_{d/2}$; by Lemma 4.5 this amounts to multiplying by an invertible element. We then use the sliding property to move this unknot around $L_1$, and this adds $\pm 1$ to the framing. This allows us to reduce the problem to the preceding case.

In the general case, the component $L_1$ can be unknotted by changing some crossings, and inserting a $\pm 1$-framed unknot cabled with $\omega_{d/2}$ around the crossing, in such a way that its linking number with $L_1$ is zero. As above, this amounts to multiplying by an invertible element, whence we have the result from the preceding case. \(\square\)
Lemma 4.5 (a) One has $\langle U_1 \tilde{\omega}_{d/2} \rangle \langle U_{-1}(\tilde{\omega}_{d/2}) \rangle = 1$.
(b) For $\varepsilon = \pm 1$, and $v \neq d/2$, one has $\langle U_\varepsilon(\tilde{\omega}_v) \rangle = 0$.

Proof. We compute $\langle U_1 \tilde{\omega}_v \rangle \langle U_{-1}(\tilde{\omega}_v) \rangle$. (Note that $[U_1(\tilde{\omega}_v)]$ and $\langle U_{-1}(\tilde{\omega}_v) \rangle$ are conjugate.) The graded sliding property shows that this is equal to the Homflypt invariant of the following coloured

link.

Using the vanishing Lemma 4.6 below, whose proof is adapted from the one given in Lemma 2.9, we obtain

$$\langle U_1 \tilde{\omega}_v \rangle \langle U_{-1}(\tilde{\omega}_v) \rangle = \tilde{n} \sum_{i=0}^{z-1} (a^N s)^{2vi} \sum_{j=0}^{\beta-1} (-1)^j (a^k s^{-1})^{2vj} \langle \tilde{\omega}_v \rangle.$$

If $\langle U_1(\tilde{\omega}_v) \rangle \langle U_{-1}(\tilde{\omega}_v) \rangle$ is not zero, then we have $v \equiv 0 \mod z$, and $v \equiv (\beta/2) \mod \beta$. This gives assertion (b), moreover $\langle U_1(\tilde{\omega}_{d/2}) \rangle \langle U_{-1}(\tilde{\omega}_{d/2}) \rangle = \tilde{n} d \langle \tilde{\omega}_{d/2} \rangle = 1$. \qed

Lemma 4.6. If $\lambda$ is a Young diagram in $I_{N,K}\backslash\{K^j, 0 \leq j < N\}$, then, for any $v \in \mathbb{Z}/d$, the following morphism is zero in $\hat{H}^{N,K}$.

4.3. Cohomological refinements

If the rank-level is not spin, we can proceed similarly. This time we have that $\langle U_1(\tilde{\omega}_v) \rangle = 0$, unless $v = 0 \mod d$.

Theorem 4.7. Provided $c = (c_1, \ldots, c_m)$ is in the kernel of the linking matrix modulo $d$, the formula

$$\tilde{\tau}^{\text{coh}}_{c_{N,K}}(M, \sigma) = \delta^{c_{12}(L)} \langle L(\tilde{\omega}_{c_1}, \ldots, \tilde{\omega}_{c_m}) \rangle$$
defines an invariant of the surgered manifold \( M = S^3(L) \) equipped with the cohomological class \( \sigma \in H^1(M, \mathbb{Z}/d) \) corresponding to \( c \). Moreover,

\[
\forall M \quad \tau_{N,K}^c(M) = \sum_{\sigma \in H^1(M, \mathbb{Z}/d)} \tau_{N,K}^{\text{coh}}(M, \sigma).
\]

5. Proof of the reduction formula

We show the reduction formula (Theorem 3.6) in the spin case. We proceed as follows. We use the modulo \( \beta \) grading to construct, as we did for \( \tau_{N,K} \), a spin refinement of the invariant \( \tau_{SU(N,K)}^{\text{spin}} \) satisfying the decomposition formula

\[
\tau_{SU(N,K)}^{\text{spin}}(M) = \sum_{\sigma \in \text{Spin}(M, \mathbb{Z}/\beta)} \tau_{SU(N,K)}^{\text{spin}}(M, \sigma).
\]

The same can be done with the reduced invariant \( \bar{\tau}_{N,K} \). Note that here we consider the modulo \( \beta \) grading, so that the decomposition formula below is not that of the preceding section if \( \beta \neq 1 \).

\[
\bar{\tau}_{N,K}(M) = \sum_{\sigma \in \text{Spin}(M, \mathbb{Z}/\beta)} \bar{\tau}_{N,K}(M, \sigma).
\]

We will prove the reduction theorem for the spin invariants. We need to specify some notation. We will use \( \langle \rangle \) for the Homflypt invariant evaluated at \( a \), and \( \langle \tilde{y} \rangle \) for the Homflypt invariant evaluated at \( \tilde{a} \). Note that the framing parameter does not appear in the coefficients of \( \hat{\Omega} \), so that we can use it for cabling and then evaluate the Homflypt invariant at \( a \) or \( \tilde{a} \).

We decompose the skein element \( \hat{\Omega} \) according to the \( Na \)-grading (resp. the \( \beta \)-grading).

\[
\hat{\Omega} = \sum_{\xi = 0}^{aN - 1} \hat{\Omega}^{(\xi)} \quad (\text{resp.} \quad \hat{\Omega} = \sum_{\nu = 0}^{\beta - 1} \hat{\Omega}_{\nu}).
\]

Formulas for the modulo \( \beta \) spin invariants are then

\[
\tau_{SU(N,K)}^{\text{spin}}(S^3(L), \sigma_c) = \Delta^{-\sigma(L)(\chi^{-1}\eta)^m} \langle L(\hat{\Omega}_{c_1}, \ldots, \hat{\Omega}_{c_m}) \rangle
\]

\[
\tau_{N,K}^{\text{spin}}(S^3(L), \sigma_c) = \delta^{-\sigma(L)(\chi^{-1}N\eta)^m} \langle L(\hat{\Omega}_{c_1}, \ldots, \hat{\Omega}_{c_m}) \rangle^\sim.
\]

Here \( c = (c_1, \ldots, c_m) \) satisfies the modulo \( \beta \) characteristic condition, and \( \sigma_c \) is the modulo \( \beta \) spin structure corresponding to \( c \). We can choose \( L \) so that the linking matrix is even. (This can be considered to be a consequence of the nullity of the cobordism group \( \Omega_3^{\text{spin}} \).) In this case \( c \) is modulo \( \beta \) characteristic if and only if \( c \) is in the kernel of the linking matrix modulo \( \beta \).

**Lemma 5.1.** We can find \( \bar{\xi}_1, \ldots, \bar{\xi}_m \in \mathbb{Z}/\alpha N \) such that \( \bar{\xi}_i \equiv c_i \mod \beta \), for \( i = 1, \ldots, m \) and \( \bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m) \) represents an element in the kernel of the linking matrix modulo \( \alpha N \).

**Proof.** Denote by \( B \) the linking matrix. The Bockstein operator

\[
\text{Ker}(B \otimes \mathbb{Z}/\beta) \to \text{coker}(B \otimes \mathbb{Z}/\alpha N).
\]
associated with the exact sequence
\[ 0 \to \mathbb{Z}/z^2N' \to \mathbb{Z}/zN \to \mathbb{Z}/\beta \to 0 \]
is zero since gcd(\(\beta, z^2N'\)) = 1. Hence we have that the map \(\text{Ker}(B \otimes \mathbb{Z}/zN) \to \text{Ker}(B \otimes \mathbb{Z}/\beta)\) is onto. \(\square\)

In the formula for \(\tau^{SU(N,K)}(S^3(L), \sigma_i)\) and \(\tilde{\xi}_{N,K}(S^3(L), \sigma_c)\) we can replace \(\dot{\Omega}_{c_i}\) by \(\sum_{j=0}^{3N'-1} y_k^j \hat{\Omega}(\bar{\zeta})\) (with \(\bar{\zeta}\) fixed as in the lemma above). Using the braiding and framing coefficients for \(y_{K',\rho}\), we obtain the formula
\[
\langle L(\dot{\Omega}_{c_1}, \ldots, \dot{\Omega}_{c_n}) \rangle = \sum_{j \in \mathbb{Z}/z^2N'} \zeta^t_j B_{ij} \langle L(\dot{\Omega}(\bar{\zeta})_i, \ldots, \dot{\Omega}(\bar{\zeta})_j) \rangle
\]
\[
\langle L(\dot{\Omega}_{c_1}, \ldots, \dot{\Omega}_{c_n}) \rangle = \chi^2 m \sum_{j \in \mathbb{Z}/N'} \zeta^t_j B_{ij} \langle L(\dot{\Omega}(\bar{\zeta})_i, \ldots, \dot{\Omega}(\bar{\zeta})_j) \rangle.
\]
We also have
\[
\langle L(\dot{\Omega}_{c_1}, \ldots, \dot{\Omega}_{c_n}) \rangle = \langle L(\dot{\Omega}(\bar{\zeta})_1, \ldots, \dot{\Omega}(\bar{\zeta})_m) \rangle^\gamma.
\]
For any framed link \(K\), we have \(\langle K \rangle = \langle a/\bar{a} \rangle^K \langle K \rangle^\gamma\). Using that \((a/\bar{a})^{3N} = 1\), we see that
\[
\langle L(\dot{\Omega}(\bar{\zeta})_1, \ldots, \dot{\Omega}(\bar{\zeta})_m) \rangle = \langle L(\dot{\Omega}(\bar{\zeta})_1, \ldots, \dot{\Omega}(\bar{\zeta})_m) \rangle^\gamma.
\]
We deduce the required formula with the following normalization of the \(U(1)\) invariant at the \(N'\)th root of unity \(\bar{\zeta}\).
\[
\tau^{U(1)}(S^3(L), \bar{\zeta}) = \left(\frac{\Delta}{\bar{\delta}}\right)^{-\sigma(L)} \left(\frac{\eta}{\bar{\eta}}\right)^m \sum_{j \in \mathbb{Z}/N'} \zeta^t_j B_{ij}.
\]
Note that \(g = (\Delta/\bar{\delta})(\eta/\bar{\eta})\) is a Gauss sum, whose square modulus is \(gg = (\eta/\bar{\eta})^2 = N'\).

The other cases are obtained similarly. When \(N + K\) is odd it is useful to note first that in the defining expression for \(\tau^{SU(N,K)}\) we can evaluate the Homflypt invariant at \((-a, -s, -v)\) as well as at \((a, s, v)\).

Acknowledgments

The author wishes to thank A. Beliakova, A. Bruguières, N. Habegger G. Masbaum, V. Turaev and P. Vogel for useful discussions and suggestions.

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