Measurable cardinals and choiceless axioms
There is no elementary $j: V \rightarrow V$.

Reinhardt cardinal

Theorem: There is no $j: V \rightarrow V$.

Requires AC. Is a Reinhardt cardinal consistent if AC is dropped? $+$DC. Yes.

Q. Is $\text{NBG + Reinhardt}$ consistency-wise stronger than $\text{ZFC + I}_0$?

$\jmath: L(V_{\aleph_1}) \rightarrow L(V_{\text{Rein}})$ \quad \text{crit}(\jmath) < \omega_1$

$
j: V_{\omega_1} \rightarrow V_{\omega_1}$ $\Sigma_0$

$
j: V_{\omega_1} \rightarrow V_{\omega_1}$ $\Sigma_1$

$
j: V_{\omega_1} \rightarrow V_{\omega_1}$ $\Sigma_2$
$\exists z$

$T_0$ is equivalent w/

for all $T \in V_{\text{HF}}$ master coder $j : (V_{j+1}, T) \rightarrow (V_{j+1}, T)$

(Kunen) There is no $j : V_{\text{HF}} \rightarrow V_{\text{HF}}$

(Schüttenberg) The following are equivalent:

1. ZFC + $T_0$
2. ZF + $\lambda$-DC + $j : V_{\text{HF}} \rightarrow V_{\text{HF}}$

$j : L(V_{\text{HF}}) \rightarrow L(V_{\text{HF}})$

$\text{crit}(j) < \lambda$

$L(V_{\text{HF}}) \models \Gamma \vdash V_{\text{HF}} \models L(V_{\text{HF}})$
\[ L(Y_{\alpha+1}) \]

\[ V/ \]

\[ M = j: V_{\alpha+2} \rightarrow V_{\alpha+2} \]

Fact: \( L(Y_{\alpha+1}) \) is a forcing extension.

Q: \( L(C[R]) \) is not ??

\[ \mathbf{AD} L(C[R]) \]

Jensen: Exactly one of the following holds:

1. \( V \) is close to \( L \): for all singular \( \aleph \)
   
   \[ \aleph \text{ is singular in } L \text{ and } 2^{\aleph} = \aleph \]

2. \( V \) is far from \( L \): every cardinal is inaccessible in \( L \).

Wadham: If there is an extendible \( \kappa \)
one holds

(1) $V$ is close to $HOD$; for all singular $\lambda \geq \kappa$

(2) $\lambda$ is singular in $HOD$ and $\lambda^{+} \cap HOD = \lambda^{+}$

(3) $V$ is far from $HOD$; for all regular $\delta \geq \kappa$,

$\delta$ is measurable in $HOD$.

$\omega$-strongly measurable

\[ j : HOD(\kappa_{\omega}) \rightarrow HOD(\kappa_{\omega+1}) \]

$HOD$ conjecture. Large cardinals imply

$V$ is close to $HOD$.

Theorem (Woodin) If the $HOD$ conjecture is true, then a Reinhardt cardinal

- proper class of $\kappa$-alliances is inconsistent.

Rank Berkeley cardinals.
Def. \( \lambda \) is rank Berkeley if for all \( \alpha < \lambda \leq \beta \), there is \( j: V^\beta \rightarrow V^\beta \) with \( \text{crit}(j) = \alpha \) (and \( \text{crit}(j) < \lambda \)).

Open: Is a Reinhardt equiconsistent w/ a rank Berkeley?

Theorem (Cutolo). If \( \lambda \) is a singular Berkeley limit of extendibles, then \( \lambda^+ \) is measurable.

Crit.: it is consistent w/ \( \text{ZF} \) that every uncountable cardinal is singular.

Q (fisher?): Is it consistent w/ \( \lambda \) being rank Berkeley that every cardinal above \( \lambda \) is singular?
Thus it is consistent w/ $\text{CF}^{\omega_1}$ that $\kappa$ is a

Reinhardt

Theorem 1. If $\kappa$ is a rank

then there is a proper class of

regular $\delta$, for all suff

large $\delta$, $\text{cf}^{\omega_1}(\delta) = \kappa$.

for all $\gamma$, for all suff

large regular $\delta$, the club filter

on $\delta$ is $\gamma$-complete.

$\gamma \rightarrow \delta \rightarrow \omega$

(rank Reinhardt)

Theorem. For a club $\mathcal{C}$ of

the club filter on $\delta$ or $\delta^+$


is J-complete.

If F is a filter on X, a set A \subseteq X is a \textit{atom} of F if

\[ \forall A: S \in F \]

is an ultrafilter.

(AD) \[ F = \text{club filter on } \omega_1 \]

\[ \{ \alpha < \omega_1 : cF(\alpha) = \omega_1 \} \] is an atom

\[ \{ \alpha < \omega_1 : cF(\alpha) = \omega \} \] is an atom

A filter is \textit{atomic} if every positive set contains an atom.

Theorem: If there is a \textit{rant Berkeley},

\[ \text{on large regular cardinals,} \]
for $\mathfrak{c}$, if there's a club Berkely, then there is a club of $\mathfrak{c}$ such that $\mathfrak{c}$ or $\mathfrak{c}^+$ is measurable.

Measurable cardinals,

Theorem. (R. Berkley) For a club class of cardinals $\mathfrak{c}$, every $\mathfrak{c}$-complete filter on an ordinal extends to a $\mathfrak{c}$-complete ultrafilter.

Theorem (Kunen) Under $AD + DC$, every $\omega_1$-complete filter on $\omega_1$ extends to an $\omega_1$-complete ultrafilter.
Ketonen order.

If \( \alpha \) is an ordinal and \( \gamma \) is a set, \( \mathcal{B}_\gamma \) is \( \gamma \)-complete u.p.s on \( \gamma \).

Ketonen order is an order on u.p.s on ordinals.

Fix an ordinal \( \delta \).

A function \( f : \mathcal{P}(\delta) \to \mathcal{P}(\delta) \) is ketonen if

1. \( f \) is Lipschitz
   - if \( x \subseteq \delta \) and \( \alpha \subseteq \delta \),
   - \( f(x) \cap \alpha \) depends only on \( x \cap \alpha \).

2. If \( W \in \mathcal{B}_\delta(\delta) \), then \( f^{-1}[W] \in \mathcal{B}_\delta(\delta) \)

   \( x \in U \mapsto f(x) \in W \subseteq \mathcal{B}_\delta(\delta) \)
Ketenen reducibility: \( U \leq^k W \)
of \( f \) Ketenen \( f : P(\mathcal{A}) \rightarrow P(\mathcal{B}) \)
s.t. \( f^{-1}[W] = U \).

Provable wellfounded (DC)
(ZFC)
Ultrapower Axiom \( f \longrightarrow \) for all \( f \), \( \beta_\nu(\mathcal{E}) \) is linearly ordered by Ketenen reducibility.

Open Does TD imply linearity of semi-linearity?

Rank Berkeley corners imply "semi-linearity" of the Ketenen order.

Theorem: If \( R \) is rank Berkeley, then \( U \) for any \( \mathcal{A} \)
then for some \( \kappa \), the relief order on \( B_\kappa (G) \) is almost linear: every level has cardinality \( \leq \chi \) and every set of in comparables has size \( \leq \xi \).

**Pseudo large cardinals**

**DEF**: \( \kappa \) is \((\kappa, \infty)\)-supercompact if for all \( \lambda \geq \kappa \), there is \( \lambda < \kappa \) and \( \pi : \mathcal{V}_\lambda \rightarrow \mathcal{V}_\lambda \) s.t. \( \pi (\mathcal{V}) \).

\( \kappa \) is almost supercompact if it is \((\kappa, \infty)\)-supercompact for all \( \lambda < \kappa \).

**P.2**
Theorem. (Wellordered collection)
If \( \kappa \) is almost supercompact, then for all \( \gamma \), if \( \langle A_\alpha : \alpha < \gamma \rangle \) is a sequence of nonempty sets, then there is a set \( \xi \), s.t. \( \bigcup A_\alpha \neq \xi \) for all \( \alpha < \gamma \) and \( \xi \) is the surjective image of \( V_\kappa \).

Fact. If there is a rank Berkeley then there is a proper class of almost supercompacts.
Cor. If $\alpha$ is countable and $\gamma \geq \alpha$, then $\gamma^+$ has cofinality at least $\xi$.

Proof. Suppose not. Then take $\gamma < \xi$ and $(\gamma_\xi : \xi < \eta)$ converging to $\gamma^+$.

By wellordered collection, there is a set $\xi$ that is the surjective image of $\nu$ and for each $\xi \in \xi$, there is a wellorder of $\gamma$ in $\xi$ of ordertype $\alpha_\xi^+$. 

\[
\sup(\alpha_\xi^+) = \sup \{ \text{rank}(\eta) : \xi \in \xi \}
\]

\[
f : \gamma \times \xi \rightarrow \gamma^+
\]

\[
f(\gamma, \xi) = \text{rank}_\xi(\alpha)
\]
\[ \text{ran}(f) = \mathbb{R}^+ \]

**Proof of wellordersed collection lemma.**

Suppose for all \( B < \gamma \), the wellordered collection lemma holds.

Fix \( \langle A_\beta : \beta < \gamma \rangle \). For each \( \beta < \gamma \), let \( B_\beta = \{ \gamma : \text{dom}(g) = V_\beta \} \), \( \text{ran}(g) \cap A_\beta \) for all \( \gamma < \beta \)

Now let \( j : V_\lambda \rightarrow V_{\lambda + 1} \) be elementary \( \lambda < \alpha \) and \( j(\gamma) = \eta \). Consider \( \text{ran}(j) \) is cofinal in \( \gamma \), so for cofinally many \( \beta < \gamma \), there's some \( g \in B_\gamma \cap \text{ran}(j) \).

(if \( B_{\text{ran}(j)} \))
\[ \text{ran}(j) \leq V_A < V_k \]

Let: \[ \sigma = \{ \text{ran}(f) : f \in B_B \cap \text{ran}(j) \} \]

For any \( B \in \text{ran}(j) \), for all \( E' \subset B \), \( \sigma \cap A_{E'} \) is nonempty

Define \( h : V_k \times V_f \rightarrow \sigma \)

\[ h(x, f) = j(f)(x) \]