In an Attempt to Introduce Long-range Interactions into Small-world Networks

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Distinguishing the long-range bonds with the regular ones, the critical temperature of the spin-lattice Guassian model built on two typical Small-world Networks (SWNs) is studied. The results show much difference from the classical case, and thus may induce some more accurate discussion on the critical properties of the spin-lattice systems combined with the SWNs.

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I. INTRODUCTION

Since the precursive study of D.J.Watts and H.E. Strogatz on Small-world Network (SWN)[1], there have been enormous research activities concerning the critical properties of some spin-lattice models built on them[2-13]. Among the various model systems, the Guassian model[14] is always chosen as a special one to give detailed investigation on some important problems because of its relative simplicity in mathematics. In the previous studies, due to the complexity of the problem itself and the often formidable mathematical task, two spins connected by a long-range bond are usually supposed to act in the same way as those connected by a regular one. In other words, their interaction contributes to the system hamiltonian to the same extent and they obey the redistribution mechanism. In short, there is no difference between a long-range bond and a regular one. However, one may be curious about the extent to which the features of phase transition will be different if one chooses certain physical quantities such as \( J \) in the Guassian model to be different. Also prompted by the recent studies of the critical properties of the system under long-range interactions[13,16], it seems quite reasonable and meaningful to make such an attempt on the SWN, which is always looked as the model systems for the networks in reality.

The present study is thus aimed to make an investigation on the features of the critical temperature of the Guassian model built on SWNs caused by the distance dependent power-law decaying long-range interaction (\( J \propto r^{-\alpha} \)). It is reported in such a stepwise sequence: Sec.II presents our definition of the spin-lattice Guassian model on SWNs and the certification of its feasibility to be categorized into the two main groups of SWNs being studied. The succeeding one embodies the thematic study of how the critical temperature is affected by the long-range interactions. Sec.IV gives the summarization with some discussions.

II. SPIN-LATTICE GUASSIAN MODEL BUILT ON SMALL-WORLD NETWORKS

Following the first prototype of Small-world Network and a number of other variants[1,11,17,18], we define the spin-lattice Guassian model built on SWNs in such a little different way: starting with an initial \( D \)-dimensional network of \( N \) spins with periodic boundary condition, each spin is linked to its 2\( kD \) nearest neighbors (we choose \( k = 1 \) in this paper). Then (1) without changing the original bonds, but with probability \( p_1 \), each pair of non-nearest neighboring spins is additionally connected by a “shortcut”(As a model system for networks in reality, we suppose the number of shortcut to be much smaller than \( N \). Practically we require \( NP_1 \ll 1 \)); (2) the spins are visited one after another, and each bond connecting a spin to one of its \( k \) nearest neighbors is reconnected with probability \( p_2 \) to another randomly chosen one. In the sequel, by comparing some of the results we got with those got in other studies, we will show how the modelling systems defined above can be categorized into the typical two main groups of SWNs. Let us begin with the classical Guassian effective Hamiltonian

\[
H = \frac{1}{2} \sum_{mn} K(r_{mn})\sigma_m\sigma_n - \frac{b}{2}\sum_m \sigma_m^2, \quad (1)
\]

where \( K(r_{mn}) = J(r_{mn})/k_BT \) is the reduced distance dependent interaction between a spin pair. The spins can take any real value between \((-\infty, +\infty)\), and the probability of finding a given spin between \( \sigma_m \) and \( \sigma_m + d\sigma_m \) is assumed to be the Guassian-type distribution \( p(\sigma_m) d\sigma_m \propto \exp(-b/2)\sigma_m^2 d\sigma_m \). \( b \) is the Guassian distribution constant. \( k_B \) the Boltzman constant and \( T \) the thermodynamic temperature.

Generally the critical point of a Guassian system is determined by the singularity of the free energy

\[
F = \frac{1}{2}k_BT \lim_{N \to \infty} N^{-1} \sum_q \ln(b - K(q)) + T \cdot C, \quad (2)
\]

where \( C \) is a constant. However, since \( K(q) \) here must

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not take any value greater than \( b \), one can always obtain the critical point by

\[ K_{\text{max}}(\mathbf{q}) = b. \]

(3)

here \( K_{\text{max}}(\mathbf{q}) \) is the Fourier transform of \( K(r_{mn}) \)

\[ K(\mathbf{q}) = \sum_{r_{mn}} K(r_{mn}) e^{i\mathbf{q} \cdot \mathbf{r}_{mn}}. \]

(4)

Supposing firstly there is no difference between along-range bond and a regular one, that is, \( K(\mathbf{q}) \) is a constant \( K \) independent of \( r_{mn} \). Eq.(4) then becomes

\[ K(\mathbf{q}) = K \sum_{r_{mn}} e^{i\mathbf{q} \cdot \mathbf{r}_{mn}}. \]

(5)

For the one-dimensional Guassian chain constructed by the shortcut-adding operation introduced at the beginning of this section, one can easily find \( r_{mn} = \pm M a \). where \( M \) is the minimum number of the intervals between any two spins. See Fig.1 (a) for an example, the value of \( M \) where \( B \) and that between nodes \( C \) is 2. Thus \( M \) can take any integral value between \([1, (N - 1)/2]\). where \( M = 1 \) correspond to the case of no “shortcut” has been added into the network. \( a \) is the lattice constant.

Now Eq.(5) can be expressed as

\[ K(\mathbf{q}) = 2K \cos(qa) + 2Kp_1 \cos(2qa) + \cdots + 2Kp_1 \cos((N - 1)qa/2). \]

(6)

setting \( q = 0 \), \( K(\mathbf{q}) \) then reaches its maximum

\[ K_{\text{max}}(\mathbf{q}) = K(2 + (N - 3)p_1). \]

(7)

By similar analysis this result can be extended to systems with higher dimensionality as

\[ K_{\text{max}}(\mathbf{q}) = K(2d + (N - 3)p_1), \]

(8)

where \( d \) is the spatial dimension. Then the general expression of the critical point is obtained to be

\[ K_C = \frac{b}{2d + (N - 3)p_1}. \]

(9)

Where \( K_C \) is the value of \( K \) when the system reach its critical point.

In order to determine the critical point of the Guassian system constructed by the second kind of operation, we should take into account two important factors. Firstly, the bond-rewiring operation haven’t altered the total number of bonds in the network but it will change the coordination number of some certain nodes. Secondly, the rewiring operation makes the structure of the network rather irregular. For a convenience aim, we suppose the bonds rewired are reconnected to the modes distributing symmetrically around a reference one \( m \). At the same time, notice that the coordination number of the lattice is still 2 on average although the neighbors of some nodes can be greater or smaller than it. In this mean-field view, we get

\[ K(\mathbf{q}) = 2K(1 - p_2) \cos(qa) + 2Kp_2 \cos(qMa). \]

(10)

where \( M \) ranges from 2 to \((N - 1)/2\). Thus we obtain the critical point: \( K_C = b/2 \), and in case of higher dimensionality

\[ K_C = \frac{b}{2d}. \]

(11)

Seen from Eqs.(9) and (11), the critical point of the Guassian system built on the two kinds of networks we defined inclines to the same values as those got in earlier studies[11]. Therefore, we can undoubtedly categorize the networks we defined to the two main groups of SWNs–the adding-type SWN [11, 17, 18] and the rewiring-type SWN[1, 11] respectively.

III. CRITICAL TEMPERATURE OF THE SYSTEM UNDER LONG-RANGE INTERACTIONS

The distance dependent power-law decaying long-range interaction ubiquitously exists in reality and is relatively easy to treat, so with an aim to give the study rather qualitatively than quantificationally, the choose of such a form of interaction is naturally. From the point of view of theoretical physics, this means to set

\[ J(r_{mn}) = J \cdot \left( \frac{r_{mn}}{a} \right)^{-\alpha}, \]

(12)

where \( \alpha > 0 \) is the decaying rate constant.

Now let’s go on to see how and to what extent the long-range interactions affect the critical temperature of the
Fourier transform. This is in accordance with the results reported in earlier studies. But it does not always make sense. If we take $\alpha > 1.75(<1.75)$ the critical temperature decreases (increases) respectively, and when $\alpha \rightarrow \infty$ it tends to be a $p_2$ dependent constant. This shows that practically the critical temperature of the Gaussian system on the rewiring-type SWNs will be altered by the long-range interactions. It may be a result of the totally different role played by the long-range bonds.

IV. SUMMARY AND DISCUSSION

In this paper, we made an attempt to study the critical temperature of the spin-lattice Gaussian model built on the two groups of typical Small-world Networks affected by the power-law decaying long-range interaction. The results we got qualitatively show the new features of the critical temperature caused by the long-range interactions: On the adding-type SWN, the present of the long-range bonds increased the contact of a spin with the system, assisted the system to behave as a whole, and thus increased the critical temperature of the system to some extent. While on the rewiring-type SWN, the long-range interactions partly replaced the nearest-neighbor coupling, the change of the critical temperature then mostly depends on the competition of such
two kinds of interactions. Anyone’s superiority will lead to the increase or decrease of the critical temperature. If they contribute to the system’s Hamiltonian equally in extent, the critical point will remain as the classical result $K_C = b/2$.

We have to admit that the present study is not strictly accurate in calculating the critical temperature, but it indeed satisfied the intention to give a qualitatively analysis on it. We hope that further studies on the critical properties of spin-lattice models built on SWNs will continue to reveal interesting topics of the widely existing critical phenomena combined with the Small-world Networks.

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