LARGE SOLAR ANGLE AND SEESAW MECHANISM: A BOTTOM-UP PERSPECTIVE

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Abstract

In addition to the well established large atmospheric angle, a large solar angle is probably present in the leptonic sector. In the context of the see-saw and by means of a bottom-up approach, we explore which patterns for the Dirac and Majorana right-handed mass matrices provide two large mixings in a robust way and with the minimal amount of tuning. Three favourite patterns emerge, which have a suggestive physical interpretation in terms of the role played by right-handed neutrinos: in both solar and atmospheric sectors, either a single or a pseudo-Dirac pair of right-handed neutrinos dominates. Each pattern gives rise to specific relations among the neutrino mixing angles and mass differences, which lead to testable constraints on $U_{e3}$. The connection with the rate of LFV charged lepton decays is also addressed.
1 Introduction

Combined results [1] from SNO [2] and Super-Kamiokande [3] suggest that another large mixing angle is present in the leptonic sector, in addition to the well established large mixing angle involved in atmospheric oscillations [4]. CHOOZ data [5], on the contrary, constrain the third angle $\theta_{13}$ to be smaller than the Cabibbo angle. Such an interestingly different pattern of lepton mixings as compared to quark mixings should be interpreted as the result of some specific property characterizing the physics beyond the Standard Model.

The see-saw mechanism [6] has emerged as the most elegant explanation of the smallness of neutrino masses: $M_\nu = Y^T M^{-1} Y v^2$, $Y v$ and $M$ being respectively the Dirac and the Majorana mass matrices [1]. Triggered by such impressive experimental results, one would like to understand if the requirement of a large solar angle leads to some specific realization of the see-saw mechanisms. Then, one could look for theoretical implications - like those on the underlying flavour symmetry - and also for phenomenological implications - like the magnitude of $U_{e3}$ and the rate of lepton flavour violating (LFV) charged lepton decays [7, 8]. Many see-saw neutrino mass models have been proposed in the last years (for a review and a set of references see e.g. [9]), in particular in connection with flavour symmetries and grand unified theories. From the model building point of view, with an hierarchy between $m_\odot \equiv \sqrt{\Delta m^2_{\odot}}$ and $m_{\odot} \equiv \sqrt{\Delta m^2_{\odot}}$, it is much more challenging to end up with a large mixing than with a small one. This holds for the atmospheric angle, so that it holds a fortiori when requiring a large solar angle too. Indeed, either one imposes various tunings between the many elements of $Y$ and $M$ [10], or one has to deal with apparently intricate patterns for $Y$ and $M$ which could only be obtained from highly non-trivial flavour symmetries [11, 12, 13, 14, 15].

To overcome the unavoidably fragmentary picture which arises by considering the many specific models [2], one should investigate in a model independent way which essential features of $Y$ and $M$ can account for two large angles and two mass scales. In this paper we tackle this problem from a bottom-up point of view and we identify those $Y$ and $M$ patterns which give two large mixings because of their own structure with the minimal amount of tuning [3]. Since their structure is just the reflection of the different roles played by right-handed neutrinos, from the analysis of such 'natural' patterns it will turn out that few suggestive possibilities for such roles. As it will be discussed, interesting phenomenological implications also follow, in particular for the magnitude of $U_{e3}$.

At low energy and in the basis were charged leptons are diagonal, the neutrino sector is described in terms of nine physical quantities, while at high energy, that is above the scale where right handed neutrinos are integrated out, it is specified by eighteen. Casas and Ibarra [18] identify the hidden parameters of the see-saw with the six parameters of a complex orthogonal matrix acting on the left side of $L \equiv M^{-1/2} Y v$ and the three right handed neutrino mass eigenvalues. Of course, it is unrealistic at present to follow a complete bottom-up approach to reconstruct $Y$ and $M$ [19]. In fact, even if other observable quantities, like the rate of LFV processes [18, 20, 21], are sensitive to the hidden see-saw parameters (see e.g. [19, 22]), such observables – even if measured – provide upper limits, so that one is still far from having any direct access to the elements of $Y$ and $M$.

In order to fill this gap between the number of observables at low and high energy, one needs

1We are dealing here with the simplest version of the see-saw mechanism, obtained by just adding three right-handed neutrinos to the matter content of the Standard Model.
2For previous interesting attempts see [16].
3We will not consider the extended flavour democracy scenario (see e.g. [7] and references therein).
some 'theoretical salt'. The construction of specific models \[23\] according to the rules dictated by some flavour and/or GUT symmetry is a top-down approach to the problem of neutrino masses and mixings and the only ambiguities are related to the predictive power of the theory. In this sense, the neutrino oscillation data can test the different theoretical high energy models, but of course they cannot select a particular one.

In the bottom-up approach of this paper, we adopt instead the requirements of robustness and economy as additional ingredients. Indeed, our aim is to isolate those \{\nu, M\} which account for neutrino mixings and the hierarchy \(m^2_{\nu} \gg m^2_{\odot}\) in a natural way. In this spirit, we look for the structures of \(Y\) and \(M\) which are stable, in the sense that small perturbations in the parameters just induce small corrections in the physical observables. Thus, a large mixing angle cannot result from a tuning of the many elements of \(Y\) and \(M\), which would be presumably unstable, but rather from a simple relation between a couple of matrix elements. As another consequence, we exclude the nearly degenerate neutrino mass spectrum because the mixing angles are very unstable under radiative corrections \[24\]. The approach is also economical as it minimizes the number of the conditions on the parameters, while allowing most of the parameters to remain generic. One can derive some generic predictions in the framework of each pattern in the form of quantitative constraints between experimental observables. In this sense, we may decide if one generic pattern is plausible or disfavoured by present and future data. We carry out our analysis for the two viable scenarios for the neutrino mass spectrum, namely the hierarchical (Hi) and inverse hierarchical (iH) case.

We first apply the stability criteria to the analysis of the atmospheric angle. In Section 2, we display the patterns of \(M_{\nu}\) consistent with large atmospheric mixing and \(m^2_{\Theta} \gg m^2_{\odot}\) for these two scenarios. Then, we discuss how to obtain them with the least amount of tuning. In the (Hi) case, the naturalness criteria for the atmospheric sector are satisfied by the well known dominance hypothesis \[25, 12, 13\]: the mass scale of the heaviest light neutrino is essentially provided by only one right handed neutrino, and the fact that the angle is large is obtained by imposing that its Yukawa couplings to the \(\mu\) and \(\tau\) have comparable strength. The dominance hypothesis has interesting phenomenological \[21, 23\] and theoretical \[13\] consequences. Also for the (iH) case, as will be shown, it is possible to recognize a dominance mechanism which would be solidly realized by a pseudo-Dirac pair of right-handed neutrinos having a suitable hierarchy in their Yukawa couplings. Models for (iH) have been studied in particular in association with the breaking of \(L_e - L_{\mu} - L_{\tau}\) \[27, 28, 29, 30\].

Along this line of reasoning, the next step is to investigate which pattern of the sub-leading elements of \(M_{\nu}\) accounts for the presence of a large or maximal solar angle in the most natural way. This is done in Section 3. Dealing with sub-leading contributions is a delicate task in the presence of several small parameters. Moreover, one has to take into account the CP violating phases \[31\] potentially present in \(Y\). We carry out this program in a quantitative and systematic way. Of course some results are already known as empirical rules from many previous analyses, in particular from the model building point of view (see, e.g. \[23, 11, 12, 14\]). We perform the calculation in a systematic way to see explicitly how much these intuitive ideas are reliable and if there are important exceptions. Fortunately, a more quantitative reappraisal is already possible with reliable approximate expressions, where to recognize immediately the leading and the sub-leading contributions, keeping under control the order of magnitude of the negligible sub-subleading contributions. These expressions show that a large solar angle represents a strong constraint and suggest peculiar structures for \(Y\) and \(M\). This also explains why it is non trivial to build models which account for two large angles. It turns out that the quantitative analysis supports patterns which have a remarkably simple physical interpretation in terms of how right-handed neutrinos couple to each light neutrino flavour eigenstate.
We then describe in some detail these suggestive patterns. In the (Hi) case, as we have already mentioned, the atmospheric mass scale and mixing angle essentially arise from just one of the right-handed companions, so that the solar sector is directly linked to the properties of the other two right-handed neutrinos. Two basic possibilities automatically lead to a sufficiently large solar angle which we will refer to as double dominance and pseudo-Dirac for short. The first case presents also in the solar sector a mechanism analogous to the one at work for the atmospheric sector: one among the remaining two right-handed neutrinos is essentially responsible for the solar mass scale and the large solar angle is obtained when it couples with the same strength to the $e$ and to the $\mu$ and $\tau$ combination orthogonal to the heaviest neutrino eigenstate. To approach a maximal mixing would need a precise equality of the couplings. In the double dominance each light neutrino has its preferred right-handed companion. The second possibility for having a stable large solar angle, arises when the solar sector of the light neutrino mass matrix possesses a pseudo-Dirac structure. The solar angle approaches its maximal value when the Dirac structure is stressed. This naturally arises when the two remaining right-handed neutrinos possess a pseudo-Dirac structure and a suitable hierarchy of Yukawa couplings. In the (iH) case, the mass of the pseudo-Dirac pair is close to the atmospheric scale; $m_\odot$ being much smaller, the pseudo-Dirac structure has to be well approached. As a consequence, the deviation of the solar angle from $\pi/4$ has to be very tiny. The relation is so strong that only the LOW solution is really viable.

From the point of view of introducing the minimal amount of tuning, these three scenarios are preferred. They are all characterized by the dominance of either a single or a pseudo-Dirac pair of right-handed neutrinos in both the solar and atmospheric sectors. On the contrary, the simplest flavour symmetry, a plain $U(1)$ symmetry with all charge assignments of the same sign, would predict a democratic dominance pattern, with all right handed neutrinos contributing roughly the same to all light neutrino masses. Then, the above considerations show that the most natural structures from a bottom-up perspective are realized only by means of non-trivial flavour symmetries such as $U(1)$’s with holomorphic zeros from supersymmetry or non-abelian flavour symmetries. Alternatively, one could consider the above three possibilities as recipes for model building. Of course, deviations from these structure require introducing more correlations among the parameters.

The above natural patterns for $Y$ and $M$ are also interesting from a phenomenological point of view. Being quite constrained, it is straightforward to look for their predictions. In particular, those for $U_{e3}$ are remarkably different: in the (Hi) case, $U_{e3}$ could not escape detection if the experimental sensitivity reaches few percent, while in the (iH) case, on the contrary, only LOW is viable and $U_{e3}$ is predicted to be smaller than one per mille. All the three scenarios considered predict a rate for $\beta\beta0\nu$ too small to be observed. In Section 4 we discuss the connections with the LFV decays $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$, emphasising the strong dependence on the absolute scales of right-handed neutrino masses and on the other hidden see-saw parameters. On the contrary, sizeable electron and muon electric dipoles would provide an indication for a hierarchy in the right-handed neutrino spectrum.

In section 5 we show that the hidden parameters identified with the complex orthogonal matrix admit an interesting physical interpretation as a dominance matrix. Outlook and conclusions are presented in Section 6.
2 Accounting for the Atmospheric Neutrino Mixing Oscillations

2.1 The See-Saw Parameters

In the see-saw model \([6]\) the effective neutrino mass matrix is given in terms of the Dirac mass matrix \(Y\) and the Majorana mass matrix \(M\) for right-handed neutrinos as

\[
M_\nu = U^* \begin{pmatrix} m_1 & & \\
& m_2 & \\
& & m_3 \end{pmatrix} U^\dagger = Y^T Y \nu^2 M, \tag{1}
\]

where \(Y\) is defined in the basis where the charged leptons are the mass eigenstates \((e, \mu, \tau)\) and, for definiteness, we choose the basis of \(M\) eigenstates \((M_x, M_y, M_z > 0)\) for the right-handed neutrinos. Their ordering will be defined below. The complete determination of \(M_\nu\) would provide us with 9 parameters: 3 eigenvalues of \(M_\nu\), 3 angles and 3 phases of \(U\). Instead, the see-saw model has 21 parameters: 18 complex numbers of the matrix \(Y\) and 3 eigenvalues of \(M\). At the lepton flavour violation scale, in the framework of the see-saw model (i.e., the SM or the MSSM plus the right-handed neutrino masses), 3 phases in \(Y\) can be reabsorbed in the definition of the leptonic fields. There are 18-9=9 more parameters in the see-saw model than those to be measured in the matrix \(M_\nu\).

Let us define the matrix

\[
L \equiv M^{-1/2} Y = \begin{pmatrix} x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
z_1 & z_2 & z_3 \end{pmatrix}, \tag{2}
\]

such that \(M_\nu = L^T L\) and the bases are defined as above (3 phases in the 9 complex matrix elements are unphysical at low energies). The see-saw mechanism involves \(L\) and the \(M\) eigenvalues. If CP is conserved in the lepton sector of the whole theory (i.e., at energies above the scale where right-handed neutrinos decouple), the phases of the fields can be chosen to have \(Y\), \(M\) and \(M_\nu\) to be real matrices. Negative \(M\) (\(M_\nu\)) eigenvalues correspond to CP odd right-handed (left-handed) neutrino eigenstates. It is worth stressing the following property: \(M\) and \(M_\nu\) possess the same number of negative eigenvalues. Actually, we shall prefer here the convention where \(M\) and \(M_\nu\) eigenvalues are all positive, and multiply the lines in \(Y\) and \(L\) that correspond to a CP odd right-handed neutrino state by a phase \(i\). In general however, the phases may be important and the \(L\) matrix elements are complex, while \(M_x, M_y, M_z\) are positive.

As noticed by Casas and Ibarra \([18]\), extracting \(L\) from \(M_\nu\) suffers from the ambiguity

\[
M_\nu = L^T L = (RL)^T (RL), \tag{3}
\]

where \(R\) is any complex orthogonal matrix, i.e., any element of \(O(3, \mathbb{C})\). More explicitly, one has

\[
M_{\nu ij} = x_i x_j + y_i y_j + z_i z_j, \tag{4}
\]

which is clearly invariant under

\[
\begin{pmatrix} x_i \\
y_i \\
z_i \end{pmatrix} \rightarrow R \begin{pmatrix} x_i \\
y_i \\
z_i \end{pmatrix}, \quad i = 1, 2, 3, \tag{5}
\]
where $R$ can be parameterized as

$$R = e^{\alpha J_x} e^{\beta J_y} e^{\gamma J_z},$$

with $\alpha, \beta, \gamma$ three complex numbers and $J_{x,y,z}$ are the $O(3)$ generators. Therefore the solutions $L$ of eq. (3) can be written as

$$L = \tilde{R} \begin{pmatrix} 0 & 0 & \tilde{x}_3 \\ 0 & \tilde{y}_2 & \tilde{y}_3 \\ \tilde{z}_1 & \tilde{z}_2 & \tilde{z}_3 \end{pmatrix},$$

where $\tilde{R} \in O(3, \mathbb{C})$ remains arbitrary and the matrix elements $\tilde{x}_3, \ldots, \tilde{z}_3$ (among which 3 are real, the other 3 being complex) can be determined from $\mathcal{M}_\nu$, up to sign ambiguities. Of course, both $\tilde{R}$ and the eigenvalues of $M$ are physical in the framework of the see-saw model, but other experiments are needed to access them, as we also discuss below.

In some cases discussed below, the matrix $R$ defined by Casas and Ibarra (which should not be confused with $\tilde{R}$) can be interpreted as a dominance matrix, which associates each light neutrino mass eigenvalue $m_i$ with a given combination of right-handed neutrino masses. In general however, its interpretation is not transparent because it is not unitary. For this reason, we prefer to work with (2) rather than with (3) and (6).

### 2.2 See-Saw Realizations of Mass Hierarchies and Large Mixings

Our aim in this section is to identify the patterns for the see-saw parameters – more exactly, the patterns for $L$, since we do not assume any a priori hierarchy for the right-handed neutrino masses – that fit in the most natural way the experimental results. Because of the surplus in the parameters, one must adopt a simplifying assumption and it looks reasonable to assume each feature of $\mathcal{M}_\nu$ to be due to a simple relationship in $L$, such as two parameters being almost equal or very different, rather than to assume an elaborate conspiracy between the many elements of a matrix. We also apply the same considerations in linking the properties of $L$ to those of $Y$ and $M$. We first sketch the two patterns that allow to minimize the tuning of the seesaw parameters.

The first feature is the hierarchy between $m_{10}^2$ and $m_{10}^2$. A way to account for one eigenvalue being much larger than another, is to assume a dominance mechanism [25, 12, 13], in which one right-handed neutrino, associated to a line in $L$ which we take to be $z$, gives the dominant contribution to the larger eigenvalue $m_i$ of $M_\nu$. Then

$$m_i = \sum_j ((LU)_{ji})^2 \approx \left( \sum_k U_{ki} \tilde{z}_k \right)^2 \approx \sum_k (z_k)^2 \equiv z^2,$$

and (phases are unimportant here) the corresponding eigenstate is approximately $z_j \nu^j/z$, so that

$$U_{ei} \sim z_1/z, \quad U_{\mu i} \sim z_2/z, \quad U_{\tau i} \sim z_3/z.$$
The second feature of neutrino oscillations is the favoured large mixing angle solar solutions. A large mixing angle is naturally obtained with a pseudo-Dirac structure of $M_{\nu}$, i.e. when the dominant entries are off-diagonal (as discussed below, this cannot be applied to the atmospheric mixing angle if almost degenerate neutrinos are excluded). In this case at least two mass eigenstates are very close in mass, with CP-violating Majorana phases differing by $\pi/2$ (opposite CP parities in the CP-conserving case), and strongly mixed. Such a structure requires strong correlations between the couplings of at least two right-handed neutrinos. The most natural origin for these correlations is the existence of a pseudo-Dirac pair of right-handed neutrinos – recall that, in the CP-conserving case, there should be as many right-handed neutrinos with negative CP parities as light neutrinos with negative CP parities.

We concentrate on the $2 \times 2$ pseudo-Dirac sector of the mass matrices. Then there is a basis in which $M$ takes the form:

$$
\begin{pmatrix}
0 & \bar{M} \\
\bar{M} & 0
\end{pmatrix} \quad (+ \text{ small corrections})
$$

where we have chosen the pseudo-Dirac pair of right-handed neutrinos to correspond to two lines in $L$, e.g., $x$ and $y$. Neglecting the smaller contributions (deviations from the pseudo-Dirac pattern) one gets for the $L$ matrix elements:

$$
y_i = \frac{(\bar{Y}_{2i} + \bar{Y}_{1i})}{\sqrt{2\bar{M}}} v \quad \text{and} \quad x_i = \frac{i(\bar{Y}_{2i} - \bar{Y}_{1i})}{\sqrt{2\bar{M}}} v,$$

where the Yukawa matrix $\bar{Y}$ is in the basis defined by (10). Eq. (11) leads to a pseudo-Dirac structure for $M_{\nu}$, with $m_1 \simeq -m_2$, provided the hierarchy among $\bar{Y}$ couplings is as follows:

$$
\bar{Y}_{21} \gg \bar{Y}_{11} \quad \text{and} \quad \bar{Y}_{12} \gg \bar{Y}_{22} \quad \text{(or} \ \bar{Y}_{2j} \equiv \bar{Y}_{1j})
$$

so that $|y_1 + ix_1| \ll |y_2 + ix_2|$ and $|y_2 - ix_2| \ll |y_1 - ix_1|$ or vice versa.

The diagonalization of the resulting $M_{\nu}$ yields a large mixing angle which becomes closer to $\pi/4$ as the pseudo-Dirac configuration is better approached. Therefore, a close to maximal mixing angle in neutrino oscillations (as indicated by the solar data in the LOW region) suggests a pseudo-Dirac structure in $M$ and in $M_{\nu}$, which requires in turn that some particular Yukawa couplings are predominant. However this option cannot be advocated for the atmospheric oscillations, because it is not possible to accommodate solar neutrino oscillations within a 3-neutrino scheme with $m_2 \simeq -m_3 > m_\odot$.

As already stressed, an almost degenerate mass spectrum – perhaps the most obvious explanation for the large mixing angles – is not yet excluded by experiments. But it is generically disfavoured because the large mixing angles become strongly scale dependent, unless some compensation mechanism is introduced [24].

### 2.3 Patterns for $M_{\nu}$

Once solutions with three almost degenerate eigenvalues are excluded as unstable under the RGE evolution, the largest $m_i$ eigenvalue must be $O(m_\odot)$. Furthermore, the matrix $M_{\nu}$ must be consistent with the experimental facts:

a) $m_\odot^2 = m_2^2 - m_1^2 \ll m_\odot^2$, namely the hierarchy between solar and atmospheric neutrino mass differences;

b) large atmospheric mixing angle $\theta_{23} \approx \pi/4$ and good evidence for large solar mixing angle $\theta_{12} \approx \pi/4$
as well;
c) the CHOOZ constraint $|U_{e3}| < 0.2$.

There are basically two patterns for $M_\nu$ that are consistent with these properties (the choice of the ordering $i = 1, 2, 3$ of the $m_i$ is suitable for the standard conventions), which we now discuss in turn.

### 2.3.1 Hierarchical pattern: $m_3 \simeq m_\oplus \gg m_1, m_2$

The properties a), b), c) lead to a pattern for $M_\nu$ consistent with the assumption of the dominance of the atmospheric oscillations by one right-handed neutrino, namely (the intervals at 90% C.L. are taken from Ref. [34]),

$$ M_{\nu ij} = z_i z_j (1 + \mathcal{O}(\rho)) $$

with

$$ \sum_{i=1}^{3} z_i^2 \simeq m_\oplus = (4 - 8) \times 10^{-2} \text{eV} $$

$$ \frac{z_2}{z_3} \simeq \tan \theta_{23} = (0.6 - 1.7) $$

$$ |z_1/z_3| \lesssim \tan \theta_{13} < 0.2 $$

and $\mathcal{O}(\rho)$ denotes a matrix whose entries are at most of order $\rho$, with $\rho = m_2/m_3$. Thus this pattern is characterized by dominant matrix elements of order $m_\oplus$ in the $(2,3)$ submatrix of $M_\nu$, so that $z_2 \simeq z_3 \approx \sqrt{m_\oplus/2}$, while the other are smaller since $z_1 \lesssim \sqrt{m_\oplus/2} \tan \theta_{13}$. This means that one right-handed neutrino dominates the atmospheric oscillations. According to the values of $x_i, y_i$, there are several possibilities to obtain $m_\odot$ and $\theta_{12}$ from the other two right-handed neutrinos, as discussed in the next section.

In a bottom-up approach, it is convenient to redefine $L$ by a rotation

$$ (x_2, x_3) \rightarrow (x_-, x_+) = (\cos \theta_0 x_2 - \sin \theta_0 x_3, \cos \theta_0 x_3 + \sin \theta_0 x_2) \ , $$

where $\theta_0 = \arctan(z_2/z_3)$, and analogously for $(y_2, y_3) \rightarrow (y_-, y_+)$, $(z_2, z_3) \rightarrow (z_-, z_+)$. Since $\theta_0 \simeq \theta_{23}$, the (+) components are approximately aligned with the heaviest neutrino mass eigenstate $\nu_3$, while the (−) components are orthogonal. After this rotation one has $z_- = 0$ and $z_+ \simeq \sqrt{m_\oplus}$, which implies

$$ M_{\nu - -}, \ M_{\nu + -} \ll M_{\nu + +} \simeq m_\oplus \ . $$

The atmospheric angle $\theta_{23}$ differs from $\theta_0$ by a small angle that can be treated perturbatively as we do in the next section. In the rotated basis,

$$ L = \begin{pmatrix} x_1 & x_- & x_+ \\ y_1 & y_- & y_+ \\ z_1 & 0 & z_+ \end{pmatrix} \ , $$

with $z_+ \simeq \sqrt{m_\oplus}$, and all other entries are smaller than $\sqrt{m_\oplus}$: $x_\pm, y_\pm \ll \sqrt{m_\oplus}$ expresses the dominance condition, while $x_1, y_1, z_1 \ll \sqrt{m_\oplus}$ reflects the smallness of $\tan \theta_{13}$. How small they should be depends on the favoured solar neutrino solution and on the value of $\tan \theta_{13}$.

This pattern which is quite suggestive of the properties a), b), c) is not obtained in straightforward applications of abelian flavour symmetries to the seesaw mechanism [32]. But it can be implemented through the use of holomorphic zeros or non-abelian flavour symmetries [13, 14, 15].
2.3.2 Inverted hierarchy pattern: $m_1, m_2 \simeq m_@ \gg m_3$

Since $m_2^2 - m_1^2 = m_@^2 \ll m_3^2$, $m_1 \simeq \pm m_2$, but only $m_1 \simeq -m_2$ is consistent with the stability of the large mixing angle $\theta_{12}$ under radiative corrections. We shall therefore focus on the case where the mass eigenstates $\nu_1$ and $\nu_2$ form an approximate pseudo-Dirac pair. In this case the structure of the light neutrino mass matrix indicated by the data – leaving aside the now disfavoured small angle MSW solution – is the following:

$$M_{\nu 11} \sim m_@ \max \left( \frac{1 - \tan^2 \theta_{12}}{2}, \frac{m_3^2}{m_@^2}, \frac{U_{e 3}^2}{m_@^2} \right),$$

$$M_{\nu 22, 33, 23} \sim m_@ \max \left( \frac{1 - \tan^2 \theta_{12}}{2}, \frac{U_{e 3}^2}{m_@^2}, \frac{m_3^2}{m_@^2} \right),$$

$$M_{\nu 13} \sim M_{\nu 12} \sim m_@ \quad \frac{M_{\nu 13}}{M_{\nu 12}} \simeq - \tan \theta_{23}. \quad (18)$$

Thus $M_{\nu 12}$ and $M_{\nu 13}$ essentially determine the atmospheric neutrino parameters. The other matrix elements are related to the solar neutrino parameters and are much smaller.

Let us identify the patterns of $L$ that are compatible with those requirements. The more natural way to implement the mass hierarchy between the pseudo-Dirac pair and $\nu_3$ is to associate the first to two right handed neutrinos (in the sense that $m_@$ it is dominated by the $z_i$ and $y_i$) while $m_3$ is dominated by the $x_i$. Neglecting the contribution of the $x_i$ to the entries of $M_{\nu}$, Eqs. $(18)$ can be rewritten as:

$$z_1 z_3 + y_1 y_3 \simeq m_@ \sin \theta_{23},$$

$$z_1 z_2 + y_1 y_2 \simeq - m_@ \cos \theta_{23},$$

$$|z_i z_j + y_i y_j| \ll m_@ \quad \text{for} \ i = j = 1, \text{ or } i, j \neq 1. \quad (19)$$

and also $|x_i x_j| \ll m_@$. This is precisely the pattern obtained in Section (2.3) where it is shown that the simplest realization of a pseudo-Dirac mass structure $(15)$ is when $M_y$ and $M_z$ form a pseudo-Dirac pair with some hierarchy in the Yukawa couplings.

If CP is an exact symmetry, we can choose the lepton phases such that $M_{\nu}$ is a real matrix so that each line in $L$ must have matrix elements that are all real or all purely imaginary. Therefore the $z_i$ ($i = 1, 2, 3$) can be taken to be real and the $y_i$ ($i = 1, 2, 3$) to be imaginary in order to generate the opposite signs of the masses $m_1$ and $m_2$, also in correspondence with the opposite CP parities of $M_y$ and $M_z$ (the effects of CP violating phases will be discussed in more detail later).

In this inverse hierarchy scenario, it is useful to change to the new variables:

$$u_i = \frac{z_i + iy_i}{\sqrt{2}} \quad v_i = \frac{z_i - iy_i}{\sqrt{2}} \quad (20)$$

so that $M_{\nu ij} = u_i v_j + u_j v_i + x_i x_j$. The condition $(12)$ reads:

$$v_1 \gg v_{2,3} \quad u_{2,3} \gg u_1. \quad (21)$$

As for the hierarchical spectrum one defines ± components by the rotation

$$(x_2, x_3) \rightarrow (x_+, x_-) = (\cos \theta_0 x_2 - \sin \theta_0 x_3, \cos \theta_0 x_3 + \sin \theta_0 x_2) \quad (22)$$
with \( \theta_0 = \arctan(-u_3/u_2) \). After this rotation one has \( u_- = 0 \) and \( v_1 u_+ \simeq m_@ \), which implies
\[ M_{\nu^+} \sim M_{\nu^+} \sim v_1 u_+ \simeq m_@ . \] (23)

The atmospheric angle \( \theta_{23} \) differs from \( \theta_0 \) by a small angle that can be treated perturbatively as we do in the next section. In the rotated basis,
\[ L = \begin{pmatrix} x_1 & x_+ & x_- \\ u_1 & u_+ & 0 \\ v_1 & v_+ & v_- \end{pmatrix} , \] (24)

The symmetry of the solutions under \( y_i \to -y_i \) corresponds here to \( u_i \leftrightarrow v_i \).

There are three possibilities consistent with (23): 1) \( |v_1| > \sqrt{m_@} > |u_+| \), 2) \( |u_+| > \sqrt{m_@} > |v_1| \), 3) \( |u_+| \sim |v_1| \sim \sqrt{m_@} \). Since these three kinds of inverted hierarchy solutions correspond to the same pattern for \( M_\nu \), they should be connected by orthogonal matrices \( R \). If we use the parameterization (7) with \( \tilde{R} = 1 \), one obtains from the conditions (13) the solutions of type 2). A boost, i.e. a rotation by an imaginary angle \( i\lambda \) in the \((y, z)\) sector transforms the variables \((u, v)\) as
\[ u_i \to e^{\lambda} u_i \quad v_i \to e^{-\lambda} v_i \quad u_i v_j \to u_i v_j . \] (25)

When \( \lambda \) is large enough, a solution of type 1) is transformed into a solution of type 2). In between, the solutions 3) can be obtained. These orthogonal transformations (23) parameterize a continuum of equivalent mathematical solutions with quite different physics contents (see Section 3). Notice that in each case \( M \) is defined to be diagonal and does not commute with \( R \).

The elements of the matrix \( L \) are naturally bounded by \( O(v/M_i^{1/2}) \), as far as the Yukawa couplings are perturbative at the scale of lepton number violation. Therefore, if e.g. some \( y_i \sim \sqrt{m_@} \), one has \( M_y \lesssim v^2/m_@ \sim 5 \times 10^{14} \text{ GeV} \), and analogously for the \( x_i \) and \( z_i \). In the hierarchical case, one has (with our convention that \( m_@ \) is dominated by \( M_z \)) \( M_z \lesssim 5 \times 10^{14} \text{ GeV} \), while \( M_x \) and \( M_y \) remain free. In the inverted hierarchy case, there are two possibilities: for the case 3), \( M_y \) and \( M_z \) are bounded by \( 5 \times 10^{14} \text{ GeV} \), while in cases 1) and 2) the upper bound is lower since either \( v_1 \) or \( u_+ \) is larger than \( \sqrt{m_@} \).

Now that the large scale \( m_@ \) has been identified, the oscillation parameters \( \theta_{13}, \theta_{12}, \theta_{23} - \theta_0 \) and \( m_\odot \) can be determined by an expansion in these small parameters that are constrained by experimental data in turn. This is discussed in the next Section, where CP violating phases are also taken into account.

3 Origin of The Large Solar Angle

As discussed in the previous Section, only two patterns for \( M_\nu \) are consistent with the requirement of a large atmospheric mixing and \( m_@^2 >> m_{\odot}^2 \). Along this line of reasoning, the next step is to investigate whether the requirement of a large - or possibly maximal - solar angle can give further informations on the sub-leading terms of those two patterns. It is important to take into account the CP violating phases potentially present in \( Y \) and so in \( L \). In the following we carry out this program in a quantitative and systematic way. Of course some results are already known as empirical rules from many previous analyses. Fortunately, a more quantitative reappraisal is already possible with reliable approximate expressions.
In this section we adopt the following notation:

\[ A \approx B \iff A = B \left(1 + \mathcal{O} \left(U_{e3}^2, m_{\odot}/m_{\odot}\right)\right), \tag{26} \]

which relates the neglected contributions to the physically small parameters.

We are seeking for sufficiently well approximated expressions for the observables contained in \( \mathcal{M}_\nu \), in particular for the three mixing angles and the CP violating phases of the MNS matrix \( U \), which can be written in the form

\[ U = e^{i\alpha} \mathcal{W} R(\theta_{23}) R(\theta_{13}) \text{diag}(1, e^{-i\chi}, 1) R(\theta_{12}) \mathcal{V}. \tag{27} \]

where \( \mathcal{W} \) and \( \mathcal{V} \) are diagonal matrices of \( SU(3) \). However, in the interaction basis where charged lepton masses are diagonal, there is still the freedom of phase redefinition for each one of the leptons \( e, \mu, \tau \); at the level of the matrix \( L \) this amounts to the freedom of multiplying each column by an arbitrary phase.

Therefore \( \alpha \) and the phases in \( \mathcal{W} \) are not physical as they can be absorbed in the \( e, \mu, \tau \) (on the contrary, the two Majorana phases in \( \mathcal{V} \) and the CKM-like phase \( \chi \), often denoted by \( \delta \) in the literature, are physical; in particular \( \chi \) could be measured in oscillations at a neutrino factory). Once this is done, it is convenient to define the matrix

\[ \ell \equiv LR(\theta_{23}) R(\theta_{13}). \tag{28} \]

Indeed the effective light neutrino mass matrix reduces, in the basis defined by \( \ell \), to

\[ m = \ell^T \ell = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{29} \]

and the solar neutrino parameters \( m_\odot^2 \) and \( \theta_{12} \), as well as the CP-violating phases \( \chi \) and \( \mathcal{V} \), can be obtained by diagonalizing a \( 2 \times 2 \) matrix.

The CKM-like phase \( \chi \) is determined by the condition

\[ |m_{22}| \sin(\alpha_2 - \chi) = |m_{11}| \sin(\alpha_1 + \chi), \tag{30} \]

and the solar angle is given by

\[ \tan 2\theta_{12} = \frac{2|m_{12}|}{|m_{22} \cos(\alpha_2 - \chi) - |m_{11}| \cos(\alpha_1 + \chi)|}, \tag{31} \]

where \( \alpha_2 = \arg(m_{22}) - \arg(m_{12}) \) and \( \alpha_1 = \arg(m_{11}) - \arg(m_{12}) \). Eq. (30) has two solutions which differ by \( \pi \), and correspondingly \( \tan 2\theta_{12} \) can take both signs. However the two solutions are physically equivalent, as expected, since for \( 0 \leq \theta_{12} \leq \pi/2 \) they are related by exchanging \( m_1 \) and \( m_2 \). Thus, it is not restrictive to take \( m_\odot^2 \equiv \Delta m_{31}^2 \equiv m_2^2 - m_1^2 > 0 \), so that \( |m_{22}| \geq |m_{11}| \) corresponds to \( 0 \leq \theta_{12} < \pi/4 \) while \( |m_{22}| < |m_{11}| \) corresponds to \( \pi/4 < \theta_{12} \leq \pi/2 \) (“dark side”).

Eq. (30) is graphically solved for \( \chi \) by the construction of the two triangles in Fig. 1. The other phase dependent quantities discussed in this section, in particular the denominator of (31), are also shown in the figure. The CP-conserving case corresponds to a flat triangle. One can easily see that, for fixed \( |m_{11}| \) and \( |m_{22}| \), the maximal (resp. minimal) cancelation in the denominator of Eq. (31) is
Figure 1: The triangle defined by the condition on $\chi$ eq. (30): $AB = A'B = A''B = |m_{11}|$, $BD = |m_{22}|$. The denominator of eq. (31) is $A'D = |m_{22}| \cos(\alpha_2 - \chi) - |m_{11}| \cos(\alpha_1 + \chi) > A''D$, where $A''D = A'''D = |m_{22}| - |m_{11}|$. Thus, phases work against the large mixing.

obtained in the CP-conserving case with $m_{11}m_{22} > 0$ (resp. $m_{11}m_{22} < 0$). Thus the general effect of the CP-violating phases is to reduce the solar mixing angle with respect to the CP-conserving case with $m_{11}m_{22} > 0$.

As for $m_\odot^2$, one has one phase independent relation,

$$m_\odot^2 \cos 2\theta_{12} = |m_{22}|^2 - |m_{11}|^2$$

and another one is

$$m_\odot^2 \sin 2\theta_{12} = 2|m_{12}|(|m_{22}| \cos(\alpha_2 - \chi) + |m_{11}| \cos(\alpha_1 + \chi))$$

which basically gives $m_\odot^2$ for near maximal mixing.

Two effects can conspire to give a large mixing angle: $|m_{12}| \gtrsim |m_{11}|, |m_{22}|$ and some amount of cancelation in the denominator of Eq. (31), with $|m_{22}| \cos(\alpha_2 - \chi) \sim |m_{11}| \cos(\alpha_1 + \chi)$. If we combine (31) and (33) into

$$\frac{m_\odot^2 \tan^2 \theta_{12}}{|m_{12}|^2 |1 - \tan^4 \theta_{12}|} = \frac{|m_{22}| \cos(\alpha_2 - \chi) + |m_{11}| \cos(\alpha_1 + \chi)}{|m_{22}| \cos(\alpha_2 - \chi) - |m_{11}| \cos(\alpha_1 + \chi)} \gtrsim 1,$$

one has a criterion to decide which mechanism is preferred. Indeed, if the denominator in (31) is made small by cancelation, the ratios in (34) are expected to be much large. Instead, if both terms in this denominator are small, these ratios are $O(1)$. This criterion will be applied to different possible seesaw patterns here below, by first obtaining also constraints on $|m_{12}|$.
3.1 Hierarchical Spectrum

Let us first consider the case of hierarchical neutrinos. As already discussed, the associated pattern of \( M_\nu \) suggests that there are two dominant elements in \( L \), which we are free to place in the third row of \( L \) – namely \( z_{23} \). The freedom linked to the phases \( \mathcal{W} \) can be used to ensure that \( \theta_{23} \) and \( \theta_{13} \) are real. The expressions for these angles can be written as

\[
\tan \theta_{23} \approx \frac{z_2}{z_3} + \frac{y_1 y_1^* + x_1 x_1^*}{z_3^2}, \tag{35}
\]

\[
\tan \theta_{13} \approx \frac{z_1}{z_+^2} + \frac{y_1 y_1^* + x_1 x_1^*}{z_+^2}, \tag{36}
\]

where the variables \( (x_\pm, y_\pm, z_\pm) \) are defined in (33), \( y_+ = y_1 + (z_1/z_+)y_1 \) and \( y_1 = y_1 - (z_1/z_+)y_1 \), and analogously for \( x_+ \), \( x_1 \). Therefore the two phases in \( \mathcal{W} \) are chosen to eliminate the overall phases in these two expressions, which are independent of the phase \( e^{i\alpha} \).

After performing these rotations in this approximation, one gets

\[
\ell = LR(\theta_{23})R(\theta_{13}) \approx \begin{pmatrix}
\bar{x}_1 & x_1^- & \bar{x}_+ \\
\bar{y}_1 & y_1^- & \bar{y}_+
\end{pmatrix}, \tag{37}
\]

where \( z^2 \equiv \sum_i z_i^2 \approx m_\odot \). Then \( m \) reads

\[
m \approx \begin{pmatrix}
\bar{y}_1^2 + \bar{x}_1^2 & \bar{y}_1 y_1^- + \bar{x}_1 x_1^- & 0 \\
0 & \bar{y}_1^2 + x_1^- & 0 \\
0 & 0 & z^2 + \bar{y}_+^2 + \bar{x}_+^2
\end{pmatrix}. \tag{38}
\]

By replacing these matrix elements in (30),(31), (32) and (33), one obtains the expressions for \( \tan^2\theta_{12} \) and \( m_\odot \). In the following, we shall describe separately the two mechanisms that can lead to a large atmospheric mixing angle: cancelation in the denominator of Eq. (31) and \( |m_{12}| \gg |m_{11}|, |m_{22}| \). One can see from Eq. (38) that the latter can be obtained only at the price of a cancelation between \( \bar{x}_1^2 \) and \( \bar{y}_1^2 \), and/or \( x_1^- \) and \( y_1^- \); in this case \( \nu_1 \) and \( \nu_2 \) form a pseudo-Dirac pair.

As far as \( |m_{12}| = \bar{x}_1 x_- + \bar{y}_1 y_- \) is concerned, one can use the fact that, generically, i.e. barring a close correlation between the z’s and y’s, one has \( |y_+| \sim |y_-| \sim \max(|y_2|,|y_3|) \), and analogous relations for the x components. Now from (38) one gets the bound

\[
|m_{12}| \sim |\bar{x}_1 x_- + \bar{y}_1 y_-| \lesssim |U_{e3}|m_\odot \tag{39}
\]

that can be used in (34) to test the large mixing mechanism in hierarchical models as discussed below in more detail.

If \( m_\odot \) is dominated by one right-handed neutrino and the solar mixing angle is due to some cancelation between the two terms in the denominator of Eq. (31), there are two situations as far as the two remaining neutrino mass eigenvalues are concerned. They can exhibit some hierarchy, with \( m_2 \simeq m_\odot \gg m_1 \), or they can be comparable in magnitude (but not much larger than \( m_\odot \) to keep large mixing angles stable under the RGE evolution). Let us discuss the first possibility in detail and then comment on the second one.
3.1.1 Double Dominance

In the presence of the large solar angle, the most natural realization of a hierarchy between \( m_2 \) and \( m_1 \) involves a dominance mechanism analogous to the one assumed in the atmospheric sector. Indeed, if \( \tilde{x}_1 \ll \tilde{y}_1 \), \( x_- \ll y_- \), the solar neutrino scale is dominated by the right-handed neutrino with mass \( M_y \), namely \( m_\odot \simeq m_2 \approx |y_1^2 + y_-^2| \) (the choice of \( x \) negligible with respect to \( y \) is not restrictive: \( \tilde{y}_1 \ll \tilde{x}_1 \), \( y_- \ll x_- \) corresponds just to the exchange \( M_x \leftrightarrow M_y \)). When \( \tilde{x}_1, x_- \) are much smaller than \( \tilde{y}_1, y_- \) the formulae above simplify considerably also because sines and cosines factorize out. Indeed, it turns out from the condition (30) that \( \alpha_2 - \chi \approx - (\alpha_1 + \chi) = 0 \). The CP violating phase \( \chi \) is

\[
\chi \approx \arg(y_-) - \arg(\tilde{y}_1).
\]

The expressions for the solar angle and the solar mass scale become simply

\[
\tan \theta_{12} \approx r, \quad m_\odot \approx |y_-|^2 (1 + r^2), \quad r \equiv \frac{|\tilde{y}_1|}{y_-}.
\]

Then, as was the case for the atmospheric angle, a large solar angle results if the ratio of two couplings, \( \tilde{y}_1/y_- \), is of order one. So, the prediction for a large solar angle is stable in the sense specified in the introduction. Together with the assumed dominance in \( m_\odot \), this corresponds to a double-dominance pattern (or triple-dominance pattern, since the lightest mass eigenvalue, \( m_1 \), is then dominated by the third right-handed neutrino). LMA requires \( 0.5 < r < 0.9 \) and \( |y_-|^2 = (3-8) \times 10^{-3} \) eV, while LOW requires \( 0.7 < r < 1.1 \) and \( |y_-|^2 = (1-3) \times 10^{-4} \) eV (using the 99% C.L. intervals of Ref. 33). A maximal solar angle would require a tuning of \( r \) to 1. Finally the Majorana phase in \( V \) associated with \( m_2 \) is \( \lambda_2 \equiv \arg(\mathcal{V}_{22}) \approx \arg(\tilde{y}_1) \). In the limit \( m_1 \to 0 \), there is no other CP-violating phase; in general however \( U \) contains a third CP-violating phase associated with \( m_1 \), \( \lambda_1 \equiv \arg(\mathcal{V}_{11}) \) (in our phase conventions, \( \arg(\mathcal{V}_{33}) = 0 \)).

A very constraining conditions holds for the double dominance pattern which follows from the (34), (39) and (41)

\[
|U_{e3}| \gtrsim \frac{1}{2} \sin 2\theta_{12} \frac{m_\odot}{m_\odot}.
\]

For LMA this means that \( U_{e3} \) should be within one order of magnitude from its present experimental limit, while it can be much smaller for LOW, as shown in Figure 4 a).

We finally note, from the assumed perturbativity property of the Yukawa couplings, that \( M_z \lesssim 5 \times 10^{14} \) GeV, while \( M_y \lesssim \frac{m_a}{m_\odot} \times 5 \times 10^{14} \) GeV. For the LOW solution \( M_y \) can lie above the unification scale (when the corresponding Yukawa couplings are of order one).

The double dominance pattern can also accommodate a small solar angle, with \( \tilde{y}_1 \ll y_- \) instead of \( \tilde{y}_1 \approx y_- \). For the (now strongly disfavoured by the data) SMA solution, one needs \( 0.01 < r < 0.03 \) and \( |y_-|^2 = (2-3) \times 10^{-3} \) eV. The CHOOZ angle is constrained to be small; more precisely, it lies in the following range:

\[
\tan \theta_{12} \frac{m_\odot}{m_\odot} \lesssim |U_{e3}| \lesssim \tan \theta_{12}.
\]

Thus \( |U_{e3}| \) is at least one order of magnitude smaller than its present experimental limit; this is to be contrasted with the large angle solutions for which there is no theoretical upper bound on \( |U_{e3}| \).
3.1.2 Pseudo-Dirac Solar Neutrinos

The alternative option for generating a large solar angle is to assume a pseudo-Dirac structure with $|m_{11}|, |m_{22}| \ll |m_{12}|$, i.e.

$$|\bar{y}^2_1 + \bar{x}^2_1|, |y^2_2 + x^2_2| \ll |\bar{y}_1 y^-_1 + \bar{x}_1 x^-_1| \equiv \bar{m}.$$ \hspace{1cm} (44)

This pattern leads to a strongly mixed pseudo-Dirac pair of neutrinos with masses $m_2 \simeq -m_1 \simeq \bar{m}$; $|U_{e3}| \bar{m} \gtrsim m_\odot$. As already discussed in Section 2, the conditions (44) require cancelations between the $y$’s and the $x$’s, which is most easily implemented with a pair of pseudo-Dirac right-handed neutrinos, with $M_y \simeq -M_x \simeq M$, together with some hierarchy among the Yukawa couplings.

Since a large angle is an automatic consequence of the smallness of $|m_{11}|, |m_{22}|$ implied by (44), no particular correlation between them is required, and the ratios in the test relation (34) are $O(1)$. Replacing $|m_{12}|$ by its upper bound $|U_{e3}| \bar{m}$ one finds the constraint

$$\frac{m^2_2}{m^2_\odot} \frac{\tan^2 \theta_{12}}{|1 - \tan^4 \theta_{12}|} \lesssim |U_{e3}|^2.$$ \hspace{1cm} (45)

As shown in fig. 2 b), in the LMA region $|U_{e3}|$ should be very close to its experimental bound and saturate the lower bound in Eq. (43). There is more room for the LOW solution. The lower bound on $|U_{e3}|$ in Eq. (43) depends on how $\theta_{12}$ is close to $\pi/4$; the smaller $1 - \tan^2 \theta_{12}$, the larger $|U_{e3}|$. 

Figure 2: Lower limits on $U_{e3}$ for double dominance a) and pseudo-Dirac b). To sketch the regions of the LMA and LOW solutions we have shown in blue the 95% C.L. contours of the first reference of [1]. The lower (upper) curve displayed for each color corresponds to $m^2_\odot = 2 \times 10^{-3}$ eV$^2$. 

Since a large angle is an automatic consequence of the smallness of $|m_{11}|, |m_{22}|$ implied by (44), no particular correlation between them is required, and the ratios in the test relation (34) are $O(1)$. Replacing $|m_{12}|$ by its upper bound $|U_{e3}| \bar{m}$ one finds the constraint

$$\frac{m^2_2}{m^2_\odot} \frac{\tan^2 \theta_{12}}{|1 - \tan^4 \theta_{12}|} \lesssim |U_{e3}|^2.$$ \hspace{1cm} (45)

As shown in fig. 2 b), in the LMA region $|U_{e3}|$ should be very close to its experimental bound and saturate the lower bound in Eq. (43). There is more room for the LOW solution. The lower bound on $|U_{e3}|$ in Eq. (43) depends on how $\theta_{12}$ is close to $\pi/4$; the smaller $1 - \tan^2 \theta_{12}$, the larger $|U_{e3}|$. 

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From the perturbativity property of the Yukawa couplings one can derive an upper bound on the right-handed pseudo-Dirac neutrino mass, \( M \leq \frac{m_\alpha}{m} \times 5 \times 10^{14} \) GeV. Since \( \frac{m_\alpha}{m} \gtrsim |U_{e3}|^{-1} \), the pseudo-Dirac right-handed neutrino pair could be heavier than the third right-handed neutrino, whose mass satisfies \( M \leq 5 \times 10^{14} \) GeV.

### 3.1.3 Two-scale spectrum: \( m_1 \sim m_2 \)

The double dominance mechanism gives a fully hierarchical mass spectrum, \( m_1 \ll m_2 \ll m_3 \). However, \( m_1 \sim m_2 \) is in principle a viable option too. This corresponds to the case where no single right-handed neutrino dominates in the left upper 2 \( \times \) 2 submatrix in Eq. (13). Then, full Eqs. (30) to (32) must be used, and there is no simple correspondence between the solar neutrino parameters and the couplings of one right-handed neutrino as in the case of double dominance. The large solar angle requires \( |m_{22}| \cos(\alpha_2 - \chi) \sim |m_{11}| \cos(\alpha_1 + \chi) \) and \( |m_{12}| \sim |m_{11}|, |m_{22}| \), (which is obtained for (i) \( \bar{x}_1 \lesssim \bar{y}_1 \sim y_- \) if \( x_- \ll y_- \); (ii) \( y_1 \lesssim x_1 \sim x_- \) if \( x_- \ll x_- \); (iii) \( \bar{x}_1 \lesssim \bar{y}_1 \sim y_- \) or \( \bar{y}_1 \lesssim \bar{x}_1 \sim x_- \) if \( x_- \sim y_- \) and the solar neutrino scale is approximately given by \( m_\odot \sim |y_e^2 + x^2| \cos(\alpha_2 - \chi) + |\bar{y}_1^2 + \bar{x}_1^2| \cos(\alpha_1 + \chi) \). Thus, sensible variations in the magnitude of \( \theta_{12} \) and \( m_\odot \) are expected by slightly perturbing many parameters entering the previous expressions, in particular the phases. Indeed, as already mentioned, the mass work against a large solar angle (they are not dangerous in the double dominance case because the cosine in the denominator of Eq. (13) factorize out). Thus, the two-scale spectrum cannot arise from the see-saw in an economical and robust way. The CHOOZ angle satisfies again the same lower bound as in the double dominance case: \( |U_{e3}| \gtrsim O(m_\odot/m_\odot) \). However, due to the large number of parameters involved, this bound is less strict than Eq. (12). For completeness, note that the SMA solution would be accommodated with (i) \( \bar{y}_1 \ll y_- \) and \( x_1 \ll y_- \) if \( x_- \ll y_- \); (ii) \( \bar{x}_1 \ll x_- \) and \( y_1 < x_- \) if \( y_- \ll x_- \); (iii) \( \bar{x}_1 \ll x_- \) and \( y_1 \ll y_- \) if \( x_- \sim y_- \). The solar neutrino scale is then given by \( m_\odot \approx |y_e^2 + x_1^2|^2 - |\bar{y}_1^2 + \bar{x}_1^2|^2 \), and the CHOOZ angle lies in the same range as in the double dominance case if both \( \bar{x}_1, \bar{y}_1 \ll m(x_-, y_-) \).

### 3.2 Inverted Hierarchical Spectrum

Let us now discuss the alternative case of an inverse hierarchy in the neutrino mass spectrum. We shall work with the variables \( (x_i, u_i, v_i) \), introduced in section 2.3 such that \( L \) is given by (24). Up to irrelevant phase redefinitions, the implementation of the atmospheric neutrino oscillations require \( v_1 u_+ \approx m_{\odot} \), with the other components much smaller. As before, the unphysical phases \( \psi \) can be chosen so that \( \theta_{23} \) and \( \theta_{13} \) are real. Expanding in the smaller matrix elements of \( L \), one obtains the following expressions for these angles:

\[
\tan \theta_{23} \approx -\frac{u_3}{u_2} - \left( \frac{u_2^2}{u_3^2} \right) \frac{\bar{v}_1 \bar{u}_- + \bar{x}_1 x_-}{\bar{v}_1 u_+} \approx -\frac{u_3}{u_2} \left( \frac{u_2^2}{u_3^2} \right) \frac{-v_-^2 + x_1 x_-}{m_{\odot}},
\]

\[
\tan \theta_{13} \approx -\frac{v_-}{v_1} + \left( \frac{v_1^2}{v_2^2} \right) \frac{u_- v_+ + \bar{x}_- x_+}{\bar{v}_1 u_+} \approx -\frac{v_-}{v_1} \left( 1 + \frac{u_1 v_+}{m_{\odot}} \right) + \frac{x_- x_+}{m_{\odot}},
\]

where the variables \( (x_-, v_-, u_+) \) are defined in (13), with \( \theta_0 = \arctan(-u_3/u_2) \), \( u_- = 0 \), and

\[
\bar{x}_1 \bar{v}_1 = (v_1 x_1 + v_- x_-), \quad \bar{x}_- \bar{v}_1 = (v_1 x_1 - v_- x_-), \quad \bar{u}_1 \bar{v}_1 = v_1 u_1, \quad u_- \bar{v}_1 = -v_- u_1, \quad \bar{v}_1 = 0, \quad v_1^2 = v_1^2 + v_-^2, \quad u_+ \bar{v}_1 \approx m_{\odot}.
\]
In this approximation, the matrix $m$ defined in (29) reads

$$
m \approx \begin{pmatrix}
2u_1v_1 + \bar{x}_1^2 & \bar{v}_1u_+ + \bar{u}_1v_+ + \bar{x}_1x_+ & 0 \\
\bar{v}_1u_+ + \bar{u}_1v_+ + \bar{x}_1x_+ & 2u_+v_+ + x_+^2 & 0 \\
0 & 0 & \bar{x}_2^2
\end{pmatrix}.
$$

(49)

By replacing these matrix elements in (30), (31), (32) and (33), one obtains the expressions for $\tan^2 \theta_{12}$ and $m_\odot$.

Since the matrix $m$ and the oscillation parameters discussed in this section are invariant under the boost given in (25), it is worth defining the rapidity $\lambda = \frac{1}{2} \log(\frac{u_+}{v_1})$, such that

$$u_+ \simeq e^\lambda \sqrt{m_\odot}, \quad v_1 \simeq e^{-\lambda} \sqrt{m_\odot}.
$$

(50)

while the parameters $(u_1/u_+), (v_+/v_1), (v_-/v_1)$, are $\lambda$-independent. They appear in the approximate expressions:

$$\frac{m_\odot^2}{2m_\odot^2} \simeq 2\frac{u_1}{u_+} + 2\frac{v_+}{v_1} + \frac{x_+^2}{m_\odot},
$$

$$\tan^2 \theta_{12} - 1 \simeq 2\frac{u_1}{u_+} - 2\frac{v_+}{v_1} + \frac{x_+^2}{m_\odot},
$$

(51)

where the phase dependence discussed in Section 3 has been omitted for simplicity.

The important feature here is the relation $|m_{12}| \approx v_1 u_+ \approx m_\odot$. Since in this inverse hierarchy scenario, the large solar neutrino mixing is related to the pseudo-Dirac structure, we preclude a strong cancelation in the right-hand side of the relation (34), namely a precise cancelation between the matrix elements $m_{11}$ and $m_{22}$ of (49). In particular, the two expressions in (51) must be of the same order. Thus, we derive from (34), by replacing $|m_{12}|$ by $m_\odot$, the constraint

$$\frac{m_\odot^2}{m_\odot^2} \sim \frac{(1 - \tan^4 \theta_{12})}{\tan^2 \theta_{12}}.
$$

(52)

The comparison with the fits to neutrino experiments is displayed in Fig. 3. Therefore the inverse hierarchy pattern is strongly disfavoured for LMA unless we complement the pseudo-Dirac pattern with a fine tuning of $m_{11}$ and $m_{22}$. As for the LOW solution one needs the solar mixing angle to be very close to its maximal value, namely, $|1 - \tan^2 \theta_{12}| < 10^{-4}$.

Generically, i.e. without fortuitous cancelations, one expects that these strong upper limits on $m_{11}$ and $m_{22}$ will translate into correspondingly small values for $|v_+/v_1|, |u_1/u_+|$ and $|x_1^2/m_\odot, |x_+^2|/m_\odot$. As already argued, we assume the natural relations $|v_-| \sim |v_+|$ and $|x_-| \sim |x_+|$, to obtain a limit

$$|U_{e3}| \lesssim \frac{m_\odot^2}{2m_\odot^2}
$$

(53)

which is below the present experimental limits for LMA and well below it for LOW. Notice that this conclusion is quite independent of the tuning to get $m_\odot$ discussed above.

Actually, one can distinguish two typical mechanisms to implement the two small scales $m_\odot$ and $m_3 \approx \bar{x}_2^2$ as well as the small deviation from maximal solar mixing angle $\theta_{12}$:
Figure 3: The curve determined by Eq. (52). The blue regions are like those of Fig. 2.

1) **Double dominance by a pseudo-Dirac pair**, so that $m_\odot > m_3$: when the contributions of the terms of the form $x_i x_j$ can be neglected in (46), (47) and (51). Both the atmospheric and the solar neutrino oscillations are basically controlled by the sector $(u, v)$, hence $m_\odot$ and $\theta_{12}$ are dominated by the right-handed pseudo-Dirac pair contributions. In this case, the ratios $(u_1/u_+), (v_+ / v_1), (v_-/v_1)$, are approximately determined by (47) and (51).

2) Single dominance of $m_\odot$, with $m_\odot \sim m_3$: when the contributions of the terms of the form $u_i v_j$ can be neglected in (46), (47) and (49) (but for $u_+ v_1$), so that the terms $x_i x_j$ control the solar parameters $m_\odot$ and the deviation of $\theta_{12}$ from $\pi/4$, with a dominance by the third right-handed neutrino. Here, $x_1, x_-, x_+$, are approximately fixed by (47) and (51).

As we now turn to discuss, charged lepton flavour violating (CLFV) decays can provide some discrimination between these different possible patterns.

## 4 Charged lepton flavour violating radiative decays

In the previous section, we identified the patterns for $L$ that reproduce in the most natural way, from a bottom-up point of view, the experimental results. Of course, neither the absolute scale nor the hierarchy of right-handed neutrino masses can be deduced from oscillation data. Other experiments are needed to gain further insights; in particular, searches for flavour-violating charged lepton decays such as $\tau \to \mu \gamma$ and $\mu \to e \gamma$ offer the possibility to test another combination of $Y$ and $M$ than the one associated with oscillations.
It is well known that the flavour structure of soft terms in the slepton sector can lead to strong violations of lepton flavour in supersymmetric extensions of the Standard Model. Even if one assumes that the mechanism of supersymmetry breaking is flavour-blind, with $m^2_{L_{ij}} = m^2_{\tilde{e}_{ij}} = m^2_{\tilde{\nu}_{ij}}$ and $A^e_{ij} = A^\nu Y_{Y_{ij}}$ at some scale $M_U \sim M_{Pl}$, the soft terms can receive additional flavour-dependent contributions from various sources, which may lead to an observable rate for processes such as $\tau \to \mu \gamma$ and $\mu \to e \gamma$. Among the possible contributions are the radiative corrections induced by the right-handed neutrino couplings $Y_{ki}$, radiative corrections in the context of grand unification and other possible contributions from unknown physics between $M_{GUT}$ and $M_{Pl}$.

Figure 4: Upper bound on $(m^2_{\tilde{\nu}})^{\mu \tau}$ from the present experimental limit $BR(\tau \to \mu \gamma) < 1.1 \times 10^{-6}$ as a function of the mean sneutrino mass $m_{\tilde{\nu}}$ and of the $SU(2)_L$ gaugino soft mass $M_2$, for $\tan \beta = 10$, $A_0 = 0$ and sign($\mu$) = +. The numbers in brackets correspond to an improvement by a factor $10^3$ in the experimental limit on $BR(\tau \to \mu \gamma)$. The dark gray region is excluded because the lighter stau would here be the LSP, while the light grey region corresponds to $m^2_{\tilde{\nu}} < 0$. The region below the dash-dotted line could be explored by searching for LFV slepton decays at future colliders.

The running of the slepton masses from $M_U$ to the scale where right-handed neutrinos decouple, due to loop corrections involving the Yukawa coupling matrix $Y$, induces flavour non-diagonal terms.

---

For instance it is well known that in $SU(5)$ an additional source of LFV are the Yukawa couplings of the colored triplet.
Figure 5: Upper bound on $\frac{(m_{\tilde{\nu}}^2)_{ij}}{m_{\tilde{\nu}_0}^2}$ from the present experimental limit $BR(\mu \to e\gamma) < 1.2 \times 10^{-11}$ as a function of $m_{\tilde{\nu}}$ and $M_2$, for $\tan \beta = 10$, $A_0 = 0$ and sign($\mu$) = +. The numbers in brackets correspond to an improvement by a factor $10^3$ in the experimental limit on $BR(\mu \to e\gamma)$. The regions in dark grey and light grey are defined as in fig. 1.

in the slepton mass matrices which are proportional to the off-diagonal entries of

$$C = Y^\dagger \ln (M_U/M) Y.$$ (54)

These flavour-dependent soft terms induce LFV charged leptons radiative decays $l_i \to l_j \gamma$ via loops of charginos/neutralinos and sleptons. The corresponding amplitudes have been fully computed, in the mass insertion approximation, in Ref. [39]. For moderate and large values of $\tan \beta$, the dominant contribution arises from loops where charginos and sneutrinos circulate, with an insertion of the off-diagonal element of the sneutrino mass matrix $m_{\tilde{\nu}_i}^{2}$. This yields a branching ratio

$$BR(l_i \to l_j \gamma) \simeq \frac{\alpha^3}{G_{F}^2} f(M_2, \mu, m_{\tilde{\nu}}) |m_{\tilde{L}_{ji}}^2| \tan^2 \beta,$$ (55)

where $f$ is a function of the $SU(2)_L$ gaugino mass parameter $M_2$, the supersymmetric Higgs mass $\mu$ and the mean sneutrino mass $m_{\tilde{\nu}}$. Thus the dependence of $BR(l_i \to l_j \gamma)$ on the seesaw parameters
is encoded in the off-diagonal entries of the matrix \( C \); this remains true when the subdominant contributions are taken into account.

In figs. 4 and 5, we show the upper limits on \( (m_\nu^2)_{\mu\tau}/m_\nu^2 \) and \( (m_\nu^2)_{\nu\tau}/m_\nu^2 \) that can be inferred from the present bounds on the branching ratios for \( \tau \to \mu\gamma \) and \( \mu \to e\gamma \), respectively. The bounds are displayed in the \((m_\nu, M_2)\) plane, the variables which they are mostly sensitive to, for \( \tan \beta = 10 \), \( A_0 = 0 \) and sign \((\mu) = + \). However, since \( \text{BR}(l_i \to l_j\gamma) \) scales as \( \tan^2 \beta \) for moderate and large values of \( \tan \beta \), the upper limits associated with other values of \( \tan \beta \) can be obtained upon multiplication by \((10/\tan \beta)\). We also give in brackets the limits corresponding to an improvement by three orders of magnitude in the sensitivity to the branching ratios. Such an improvement is indeed expected for \( \mu \to e\gamma \) [40]. Prospects for \( \tau \to \mu\gamma \) are currently less optimistic, but searches for the LFV decay \( \chi_i^0 \to \chi_j^0\mu\tau \) at future colliders could provide limits on \( (m_\nu^2)_{\mu\tau}/m_\nu^2 \) of that order of magnitude in the region of the \((m_\nu, M_2)\) plane indicated on figs. 4 and 5 [38]. We do not consider the process \( \tau \to e\gamma \), which is much less constrained than \( \mu \to e\gamma \) by experiment, and hence does not provide additional information on the seesaw parameters.

In the framework of mSUGRA the coefficients \( C_{ij} \) are related to the \( (m_\nu^2)_{\mu\tau}/m_\nu^2 \) by

\[
m_{L_{ij}}^2 \simeq -\frac{3m_0^2 + A_0^2}{8\pi^2} C_{ij}.
\]  

The limits in figs. 4 and 5 corresponds to those in figs. 4 and 5. The latter are taken from [21] where a more complete discussed can be found.

In view of the following discussion, it is useful to express the quantity \( C_{ji} \) in the parametrization (4):

\[
C_{ji} = \frac{z_j z_i}{v^2} M_{\bar{\Theta}} \ln \left( \frac{M_U}{M_\Theta} \right) + \frac{y_j y_i}{z_j^2} \frac{M_y}{M_\Theta} \ln \left( \frac{M_U}{M_y} \right) + \frac{x_j x_i}{z_j^2} \frac{M_x}{M_\Theta} \ln \left( \frac{M_U}{M_x} \right).
\]  

At this stage, nothing has been assumed about the structure of \( L \), and Eq. (57) is completely general. We shall neglect CP violation in the following, and therefore replace \( z_j^* z_i \) by \( z_j z_i \), etc, with an additional − sign if the corresponding right-handed neutrino has a negative CP eigenvalue.

### 4.1 Hierarchical mass spectrum

In the case of a hierarchical light neutrino mass spectrum, \( m_\Theta \approx z_+^2 \), and it is convenient to rewrite Eq. (57) as

\[
C_{ji} = \frac{z_j z_i}{z_+^2} \frac{M_{\bar{\Theta}}}{M_\Theta} \ln \left( \frac{M_U}{M_\Theta} \right) + \frac{y_j y_i}{z_+^2} \frac{M_y}{M_\Theta} \ln \left( \frac{M_U}{M_y} \right) + \frac{x_j x_i}{z_+^2} \frac{M_x}{M_\Theta} \ln \left( \frac{M_U}{M_x} \right),
\]  

where \( M_\Theta \equiv v^2/M_\Theta \approx 5 \times 10^{14} \text{ GeV} \). The dominance hypothesis, together with the assumed perturbativity of Yukawa couplings, implies \( M_x \lesssim M_\Theta \). In the case of double dominance, one also has \( M_y \lesssim M_\Theta \equiv v^2/M_\Theta \) (with typical values \( M_\Theta \approx 5 \times 10^{15} \text{ GeV} \) for LMA, and \( M_\Theta \approx 10^{17} \text{ GeV} \) for LOW), and \( M_x \lesssim M_1 \equiv v^2/m_1 \) \( (M_1 \gg M_\Theta) \). Since \( m_1 \) is not known, the constraint on \( M_x \) is not very informative; however if one gives up the dominance in the solar neutrino sector, one has both \( M_x \lesssim M_\Theta \) and \( M_y \lesssim M_\Theta \). In the pseudo-Dirac case, Eq. (57) further simplifies to

\[
C_{ji} \simeq \frac{z_j z_i}{z_+^2} \frac{M_{\bar{\Theta}}}{M_\Theta} \ln \left( \frac{M_U}{M_\Theta} \right) + \frac{y_j y_i}{z_+^2} \frac{M_y}{M_\Theta} \ln \left( \frac{M_U}{M_y} \right).
\]  

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Figure 6: Upper bound on $C_{23} = (Y^\dagger \ln(\frac{M_U}{M_R}) Y)_{23}$ from the present experimental limit $BR(\tau \to \mu \gamma) < 1.1 \times 10^{-6}$ as a function of the mean sneutrino mass $m_{\tilde{\nu}}$ and of the $SU(2)_L$ gaugino soft mass $M_2$, for $\tan\beta = 10$, $A_0 = 0$ and $\text{sign}(\mu) = +$. The numbers in brackets correspond to an improvement by a factor $10^3$ in the experimental limit on $BR(\tau \to \mu \gamma)$. The dark gray region is excluded because the lighter stau would here be the LSP, while the light grey region corresponds to $m_0^2 < 0$. The region below the dash-dotted line could be explored by searching for LFV slepton decays at future colliders [38].

Note that the $\ln(M_U/M)$ term in Eq. (59) is proportional to $y_jy_i - x_jx_i$, not to $y_jy_i + x_jx_i$. This is important since cancellations may occur in one of the two combinations, depending on how the pseudo-Dirac structure is realized. More precisely, one may have $|y_1y_2 - x_1x_2| \ll |y_1y_2 + x_1x_2|$ if cancellations occur both in $m_{11}$ and $m_{22}$ (implying a suppression of the contribution of the pseudo-Dirac right-handed neutrino pair to $C_{12}$), and $|y_2y_3 + x_2x_3| \ll |y_2y_3 - x_2x_3|$ if cancellations occur in $m_{22}$. Furthermore one has $\bar{M} \lesssim v^2/\bar{m} \approx \sqrt{2}|1 - \tan^2 \theta_{12}| M_o$. Note however that the actual upper bound on $\bar{M}$ can be smaller, even in the presence of Yukawa couplings of order one. In fact $\bar{M} \sim \sqrt{2}|1 - \tan^2 \theta_{12}| M_o$ requires, in addition to Yukawa couplings of order one, both cancellations in $m_{11}$ and $m_{22}$, and $|y_1| \sim |y_-|$.
Figure 7: Upper bound on $C_{12} = (Y^\dagger \ln(M_U/M_R)Y)_{12}$ from the present experimental limit $BR(\mu \to e\gamma) < 1.2 \times 10^{-11}$ as a function of $m_\nu$ and $M_2$, for $\tan\beta = 10$, $A_0 = 0$ and $\text{sign}(\mu) = +$. The numbers in brackets correspond to an improvement by a factor $10^3$ in the experimental limit on $BR(\mu \to e\gamma)$. The regions in dark grey and light grey are defined as in fig. 1.

4.1.1 $\tau \to \mu \gamma$

In the double dominance case, the coefficient associated with the decay $\tau \to \mu \gamma$ is

$$C_{23} \simeq \frac{1}{2} \sin 2\theta_{23} \frac{M_z}{M_\odot} \ln \left( \frac{M_U}{M_z} \right) + \frac{y_2 y_3}{z_+^2} \frac{M_y}{M_\odot} \ln \left( \frac{M_U}{M_y} \right) + \frac{x_2 x_3}{z_+^2} \frac{M_x}{M_\odot} \ln \left( \frac{M_U}{M_x} \right).$$

(60)

Although, from Eq. (41), $|y_2 y_3| \lesssim m_\odot$ and $|x_2 x_3| \ll m_\odot$, the second and third terms in Eq. (60) can be as large as $\ln(M_U/M_\odot)$ and $\ln(M_U/M_1)$, respectively, if $M_y$ and $M_x$ are close to their respective upper bounds. In the pseudo-Dirac case:

$$C_{23} \simeq \frac{1}{2} \sin 2\theta_{23} \frac{M_z}{M_\odot} \ln \left( \frac{M_U}{M_z} \right) + \frac{y_2 y_3 - x_2 x_3}{z_+^2} \frac{\bar{M}}{M_\odot} \ln \left( \frac{M_U}{M} \right).$$

(61)

In general one expects the contribution of the pseudo-Dirac pair of right-handed neutrinos to be small, since from Eq. (41) $|y_2 y_3 + x_2 x_3| \ll m_\odot/\sqrt{2} |1 - \tan^2 \theta_{12}|$. Still it can be of order one when cancellations occur in $m_{22}$ due to both $y_3 \simeq \pm i x_3$ and $y_2 \simeq \pm i x_2$, since in this case $|y_2 y_3 - x_2 x_3| \gg |y_2 y_3 + x_2 x_3|$. 


If $M_z$ is the heaviest right-handed neutrino mass, Eqs. (31) and (32) reduce to $C_{23} \approx \frac{1}{2} \sin \theta_{23} \frac{M_z}{M_\odot} \ln(M_U/M_z)$; in this case the observation of the process $\tau \to \mu \gamma$ would amount to a measurement of $M_z$ as a function of $M_2$ and $m_\nu$ – assuming no other contribution to the LFV slepton masses, e.g. from the supersymmetry breaking mechanism itself. In general, however, $M_z$ can be smaller than $M_\nu$ or $M_y$ (resp. $\tilde{M}$), and the second or third term may give the dominant contribution to $C_{23}$. Hence a measurement of $\text{BR}(\tau \to \mu \gamma)$ can only provide an upper bound on $M_z$. A small branching ratio, $|C_{23}| \ll \sin \theta_{23} \ln(M_U/M_\odot)$, would point towards two alternative scenarios [21]: (i) a scenario characterized by small Yukawa couplings in the neutrino sector and a “low” lepton number breaking scale, with both $M_z \ll M_\odot$ and $M_x, M_y \ll M_\odot$ (resp. $\tilde{M} \ll \sqrt{2}|1 - \tan^2 \theta_{12}| M_\odot$); (ii) a scenario characterized by the dominance of a lighter right-handed neutrino in the atmospheric sector and a “high” lepton number breaking scale, with $M_z \ll M_\odot$ and $M_y \sim M_\odot$ or $M_z \gtrsim M_\odot$ (resp. $\tilde{M}$ close to $\sqrt{2}|1 - \tan^2 \theta_{12}| M_\odot$), the contribution of $M_y$ (resp. $\tilde{M}$) to $C_{23}$ being suppressed by $|x_2 x_3|, |y_2 y_3| \ll m_\odot$ (resp. $|y_2 y_3 - x_2 x_3| \ll m_\odot/\sqrt{2}|1 - \tan^2 \theta_{12}|$). In the latter, the $B-L$ symmetry is broken close to – or even above – the scale where the MSSM gauge couplings unify (i.e. $2 \times 10^{16}$ GeV), which favours grand unified theories based on $SO(10)$ and larger gauge groups. On the contrary the former appears to disfavour such theories.

4.1.2 $\mu \to e\gamma$

Let us first discuss the LMA and LOW solutions, which are strongly favoured over SMA by the solar neutrino data. In the double dominance case, the coefficient associated with the decay $\mu \to e\gamma$ is

$$C_{12} \approx \frac{z_1}{z_+} \sin \theta_{23} \frac{M_z}{M_\odot} \ln \left( \frac{M_U}{M_z} \right) + \frac{y_1 y_2}{z_+} \frac{M_y}{M_\odot} \ln \left( \frac{M_U}{M_y} \right) + \frac{x_1 x_2}{z_+} \frac{M_x}{M_\odot} \ln \left( \frac{M_U}{M_x} \right). \quad (62)$$

In the pseudo-Dirac case,

$$C_{12} \approx \frac{z_1}{z_+} \sin \theta_{23} \frac{M_z}{M_\odot} \ln \left( \frac{M_U}{M_z} \right) + \frac{y_1 y_2 - x_1 x_2}{z_+^2} \frac{\tilde{M}}{M_\odot} \ln \left( \frac{M_U}{\tilde{M}} \right). \quad (63)$$

One can show that the contribution of the pseudo-Dirac pair of right-handed neutrinos is always suppressed relative to $\ln(M_U/M_\odot \sqrt{2}|1 - \tan^2 \theta_{12}|)$, due to either $|y_1 y_2 - x_1 x_2| \ll m_\odot/\sqrt{2}|1 - \tan^2 \theta_{12}|$ or $\tilde{M} \ll \sqrt{2}|1 - \tan^2 \theta_{12}| M_\odot$, while in the double dominance case $M_y$ or $M_z$ could in principle give a contribution of order one to $C_{12}$.

Expectations for $\text{BR}(\mu \to e\gamma)$ are very model-dependent. It is useful for the discussion to distinguish between two regimes differing by the value of $|U_{e3}|$. In the “large $|U_{e3}|$” regime, characterized by $2|U_{e3}| \gg \sin \theta_{12} m_\odot/m_\odot$ in the double dominance case (resp. $|U_{e3}| \gg m_\odot/\sqrt{2}|1 - \tan^2 \theta_{12}| m_\odot$ in the pseudo-Dirac case, where this regime exists only for the LOW solution), the first term in Eqs. (62) and (63) yields a $M_z$-dependent lower bound on $|C_{12}|$,

$$|C_{12}| \gtrsim \sin \theta_{23} \tan \theta_{13} \frac{M_z}{M_\odot} \ln \left( \frac{M_U}{M_z} \right). \quad (64)$$

Eq. (64) can be used to convert an experimental limit on $\text{BR}(\mu \to e\gamma)$ into an upper bound on $M_z$ as a function of $M_2$ and $m_\nu$ – assuming $|U_{e3}|$ is known. Actually the present limit on $\text{BR}(\mu \to e\gamma)$ already provides strong constraints on this regime; in particular a large $M_z$, $M_z \sim M_\odot$, is already excluded over a large portion of the $(m_\nu, M_2)$ plane, especially for LMA. The contribution of the other right-handed neutrinos also yields non-trivial constraints. In the double dominance case, $|C_{12}| \ll 1$ requires
\[ |y_1 y_2| \ll \sin 2 \theta_{12} m_\odot / 2 \text{ and/or } M_y \ll M_\odot \] (there are no analogous constraints for \(|x_1 x_2|\) and \(M_x \) since \(m_1 \) is unknown). In the pseudo-Dirac case, \(|C_{12}| \ll 1 \) requires \(|y_1 y_2 - x_1 x_2| \ll m_\odot / \sqrt{2} |1 - \tan^2 \theta_{12}| \) and/or \(M \ll \sqrt{2} |1 - \tan^2 \theta_{12}| M_\odot \), which is easily obtained due to the pseudo-Dirac structure. With the expected improvement by three orders of magnitude in the experimental limit on \(\text{BR}(\mu \to e\gamma)\), those constraints should become rather stringent – unless \(\mu \to e\gamma\) is observed.

Note finally that if atmospheric neutrino oscillations are dominated by the heaviest right-handed neutrino mass (i.e. \(M_x, M_y \leq M_2\), resp. \(\bar{M} \leq M_2\)), the contribution of \(M_z\) dominates both in \(\text{BR}(\tau \to \mu\gamma)\) and in \(\text{BR}(\mu \to e\gamma)\). This leads to the prediction

\[
\frac{|C_{12}|}{|C_{23}|} \approx \frac{\tan \theta_{13}}{\cos \theta_{23}} .
\] (65)

While Eq. (65) remains valid over a large region of the seesaw parameter space with \(M_x > M_z\) or \(M_y > M_z\) (resp. \(\bar{M} > M_z\)), a deviation from this relation is possible only if \(M_z\) is not the largest right-handed neutrino mass. In this case one may have \(|C_{12}| \gg |U_{e3}| \ln (M_U / M_\odot) \approx 8 |U_{e3}|\) (which is not excluded for the LOW solution yet), hence an observable rate for \(\mu \to e\gamma\).

The “small \(|U_{e3}|\)” regime, characterized by \(2|U_{e3}| \sim \sin 2 \theta_{12} m_\odot / m_\odot\) in the double dominance case (resp. \(|U_{e3}| \sim m_\odot / \sqrt{2} |1 - \tan^2 \theta_{12}| m_\odot\) in the pseudo-Dirac case), differs from the large \(|U_{e3}|\) regime by the absence of a lower bound on \(|C_{12}|\): here \(z_1 \to 0\) is perfectly compatible with the existing data. Thus \(\text{BR}(\mu \to e\gamma)\) can be extremely small even though \(M_z \sim M_\odot\) (i.e. even though \(\text{BR}(\tau \to \mu\gamma)\) is large), unlike in the previous regime. However \(|C_{12}| \gg |U_{e3}| \ln (M_U / M_\odot)\) is also a possibility when \(M_x\) or \(M_y \gg M_\odot \gtrsim M_z\) (resp. \(\bar{M} \gg M_\odot \gtrsim M_z\)), and \(\mu \to e\gamma\) should be observed at PSI in this case.

The discussion of the (strongly disfavoured) SMA solution follows the same lines. Again one can distinguish between two extreme regimes. In the “large \(|U_{e3}|\)” regime, where \(|U_{e3}| \sim \tan \theta_{12}\), the lower bound (64) holds like for the large angle solutions, as well as Eq. (65) in a large portion of the seesaw parameter space including the case where \(M_z\) is the largest right-handed neutrino mass. Note however that \(\text{BR}(\mu \to e\gamma)\) cannot be as large as in the LMA or LOW case, due to the upper bound \(|C_{12}| \lesssim \tan \theta_{12} \ln (M_U / M_\odot)\) (to be compared with \(|C_{12}| \lesssim \ln (M_U / M_\odot)\) for LMA/LOW). In the “small \(|U_{e3}|\)” regime, where \(|U_{e3}| \sim \tan \theta_{12} m_\odot / m_\odot\), \(\text{BR}(\mu \to e\gamma)\) can be extremely small without conflicting with existing data. Again the observation of \(\mu \to e\gamma\) with \(|C_{12}| \gg |U_{e3}| \ln (M_U / M_\odot)\) is possible if one of the two lightest right-handed neutrinos dominates in the atmospheric sector, i.e. if \(M_x\) or \(M_y \gg M_\odot \gtrsim M_z\).

4.2 Inverted Hierarchy Mass Spectrum

When the neutrino mass matrix is endowed with an approximate pseudo-Dirac pattern as required by an inverted hierarchy in its eigenvalues, it is convenient to rewrite the coefficients \(C_{ij}\) given by (55) in terms of the variables defined in Section 4.2 as follows:

\[
C_{ji} = \left( \frac{u^* \bar{u}_i}{u^*_i} \right) \frac{\bar{M}}{M_\odot} \ln \left( \frac{M_U}{M} \right) + \frac{x_i^* x_i}{u_i v_i} \frac{M_x}{M_\odot} \ln \left( \frac{M_U}{M_x} \right)
\]

\[
= \left( \frac{u^* \bar{u}_i}{u^*_i} e^{2 \lambda} + \frac{v^* v_i}{v^*_i} e^{-2 \lambda} \right) \frac{\bar{M}}{M_\odot} \ln \left( \frac{M_U}{M} \right) + \frac{x_i^* x_i}{m_\odot} \frac{M_x}{M_\odot} \ln \left( \frac{M_U}{M_x} \right) ,
\] (66)

Since, as discussed, an approximate pseudo-Dirac texture in the neutrino mass matrix is more naturally realized by a corresponding pseudo-Dirac pair of right-handed neutrinos, we have associated the masses
\[ \pm \tilde{M} \] to this pair. The dependence on the rapidity \( \lambda \) shows the dependence on the boosts \([23]\) and displays an example of the dependence of the \( C_{ij} \) on the orthogonal transformations discussed in Section 2.1.

The Yukawa coupling associated to the largest entry in the matrix \( L \), which in this case must be either \( u_+ \) or \( v_1 \), is perturbative if \( M \lesssim e^{-2|\lambda|} M_{\bar{\Theta}} \). The best limit on \( M_x \) is obtained when the \( x_i \) contributions saturate the quantities given by \([17]\) and \([21]\): \( M_x \lesssim (m_{\xi}/2U_{e3}^2 m_{\bar{\Theta}}^2) M_{\bar{\Theta}} \approx M_{\bar{\Theta}}/|U_{e3}| \), where the last bound relies on \([33]\), so that \( M_x \) lies well above \( M_{\bar{\Theta}} \).

### 4.2.1 \( \tau \to \mu \gamma \)

The coefficient associated to the \( \tau \to \mu \gamma \) decay is

\[
C_{23} \approx -\frac{1}{2} \sin 2\theta_{23} e^{2\lambda} \frac{M}{M_{\bar{\Theta}}} \ln \left( \frac{M_U}{M} \right) + \frac{x_+^2 + x_-^2}{2 m_{\bar{\Theta}}} \sin 2(\theta_{23} + \gamma) \frac{M_x}{M_{\bar{\Theta}}} \ln \left( \frac{M_U}{M_x} \right),
\]

where \( \tan \gamma = x_+/x_- \). Since \( \sin 2\theta_{23} \) is large, the first term gives a contribution \( \mathcal{O}(1) \) when \( \tilde{M} \) approaches its bound \( e^{-2|\lambda|} M_{\bar{\Theta}} \), which could be much below the gauge coupling unification scale if \( \lambda > 1 \). On the contrary, this contribution is suppressed by a factor \( e^{-2|\lambda|} \) for \( \lambda < 0 \). As for the second term, it is strongly suppressed by the bounds on the \( x_i \) factors, such that \( (x_+^2 + x_-^2)/2 m_{\bar{\Theta}} \) is expected to be \( \mathcal{O}(m_{\xi}^2/2 m_{\bar{\Theta}}^2) \). This suppression can be compensated if the mass \( M_x \) gets close to the bounds imposed by the Yukawa coupling perturbativity condition. For LMA the bound on \( M_x \) lies near the gauge coupling unification scale, but for the more appropriate LOW solution it is around the Planck scale, which is clearly nonsense. Therefore, some suppression is expected on quite general grounds for this term.

### 4.2.2 \( \mu \to e \gamma \)

The coefficient \( C_{12} \) corresponding to the \( \mu \to e \gamma \) decay is

\[
C_{12} \approx \left[ \frac{u_+ e^{2\lambda} + v_+/v_1 e^{-2\lambda}}{u_+} \cos \theta_{23} + \frac{v_+/v_1 e^{-2\lambda} \sin \theta_{23}}{v_1} \right] \frac{M}{M_{\bar{\Theta}}} \ln \left( \frac{M_U}{M} \right) + \frac{x_1 \sqrt{x_+^2 + x_-^2}}{m_{\bar{\Theta}} \sin(\theta_{23} + \gamma)} \frac{M_x}{M_{\bar{\Theta}}} \ln \left( \frac{M_U}{M_x} \right),
\]

For simplicity, let us separately consider the two typical situations previously identified in Section 3.2:

1) **Double dominance by a pseudo-Dirac pair;** replacing the ratios \((u_+/u_+), (v_+/v_1), (v_-/v_1)\), by the expressions obtained from \([17]\) and \([51]\), yields

\[
C_{12} \approx \left[ \frac{m_{\xi}^2}{4 m_{\bar{\Theta}}^2} \cosh 2\lambda + (\tan \theta_{12} - 1) \sinh 2\lambda \right] \cos \theta_{23} - e^{-2\lambda} \tan \theta_{13} \sin \theta_{23} \left[ \frac{M}{M_{\bar{\Theta}}} \ln \left( \frac{M_U}{M} \right) \right] \approx \mathcal{O} \left( \frac{m_{\xi}^2}{4 m_{\bar{\Theta}}^2} \right) \ln \left( \frac{M_U}{M} \right) \lesssim (8 + 2|\lambda|) \frac{m_{\xi}^2}{4 m_{\bar{\Theta}}^2}
\]

(69)
where the last evaluations corresponds to the generic case - i.e., to a natural pattern in the sense adopted in this paper - and \( \lambda > 0 \). This limit is very small for the LOW case, the one most relevant here. Notice that in this case, the approximate relation holds: \( C_{12}/C_{23} \sim O\left(m_\nu^2/2m^2_\Theta\right) \). The upper limit on \( C_{12} \) can be saturated only if \( M \) is close to its bound which implies a relatively large \( \tau \rightarrow \mu \gamma \) decay. Of course, this contribution can be overcome by the other one if \( M_x \) is very large.

2) Single dominance of \( m_\nu \); then the components \( x_i \) can be evaluated from (17) and (21) in terms of \( m_\nu \), \((\tan^2 \theta_{12} - 1), U_{e3} \). However, the generic relation: \( x_1 x_+ \sim x_1 x_+ \gtrsim (x_1^2 + x_2^2)/2 \approx m_\nu^2/4m_\Theta \) leads to the estimate

\[
C_{12} \sim O\left(\frac{m_\nu^2}{4m^2_\Theta}\right) \frac{M_x}{M_\Theta} \ln \left(\frac{M_U}{M_x}\right). \tag{70}
\]

Therefore, in this case, \( C_{12} \) could be of \( O(1) \) only if \( M_x \) lies close to its bound \( M_\Theta/|U_{e3}| \). However, for the more suitable LOW solution, (18) implies that \( M_x \) is too close (or above) \( M_U \) for the result to be sensible. By assuming that the contributions from the \( x_i \) dominate in both \( C_{12} \) and \( C_{23} \) their ratio is given by: \( C_{12}/C_{23} \approx x_1/\sqrt{x_1^2 + x_2^2 \cos(\theta_{23} + \gamma)} \), which is less suppressed than in the previous case and just tells us that the right-handed neutrino with mass \( M_x \) is more coupled to \( e \) than to \( \mu \) or \( \tau \). This is not possible in the previous case of double dominance by a pseudo-Dirac pair because this ratio is suppressed by the conditions to implement a pseudo-Dirac pair. Of course the important prediction from the experimental viewpoint is for each coefficient.

5 On the Physical Interpretation of \( R \)

The ambiguities in the extraction of the see-saw parameters from the measurements of \( \mathcal{M}_\nu \) can be expressed [13] in terms of the complex orthogonal matrix \( R \) defined by:

\[
R \begin{pmatrix} \sqrt{m_1} \\ \sqrt{m_2} \\ \sqrt{m_3} \end{pmatrix} \begin{pmatrix} U_1^I \end{pmatrix} = M^{-1/2} Y v = L \tag{71}
\]

The matrices are defined as in the previous sections. The nice property of this choice of \( R \) is its immediate interpretation as the dominance matrix in the sense that:

(i) \( R \) is an orthogonal transformation from the basis of the left-handed leptons mass eigenstates to the one of the right-handed neutrino mass eigenstates;

(ii) if and only if an eigenvalue \( m_i \) of \( \mathcal{M}_\nu \) is dominated - in the sense already given before - by one right-handed neutrino eigenstate \( \nu_j \), then \( |R_{ij}| \approx 1 \);

(iii) if a light pseudo-Dirac pair is dominated by a heavy pseudo-Dirac pair, then the corresponding \( 2 \times 2 \) sector in \( R \) is a boost.

It is instructive to illustrate this property by a simple example, which in a sense also motivates the double dominance assumption in Section 3. Consider a hierarchical neutrino mass spectrum, \( m_3 \approx m_\Theta \gg m_2 \approx m_\nu \gg m_1 \). For the sake of the example, we fix the other 9 parameters (all phases are neglected) as follows: (a) The \( Y \) eigenvalues are very hierarchical with \( y_3 \gg y_2 \gg y_1 \); (b) bimaximal mixing angles in \( U \); (c) the \( V_\nu \) matrix defined, as usual, by the diagonalization \( Y = V_\nu^I \text{diag}(y_1, y_2, y_3) V_\nu \), is very small or very close to the deviations of \( U \) from the bimaximal case. This last assumption is interesting because it selects the case where the charged lepton flavour violations are very small, while the mixings are maximal in \( \mathcal{M}_\nu \). Then we get the approximate expressions for
the $M$ eigenvalues:

$$M_1 = \frac{2y_1^2}{m_\odot}, \quad M_2 = \frac{2y_2^2}{m_\oplus}, \quad M_3 = \frac{y_3^2}{4m_1},$$

(72)

where $M_3$ is much larger than $M_2 \ll M_\oplus$, while $M_1 \ll M_\odot$ depends on the hierarchy in the $Y$ eigenvalues. Notice that our assumption of hierarchical spectrum for the left-handed neutrinos and small charged lepton flavour violations led to a pattern of $M$ eigenvalues of the kind discussed in [21].

As for the dominance matrix $R$, which in this case is a rotation matrix, we get

$$R \approx \begin{pmatrix}
\sqrt{\frac{m_1}{m_\odot}} & 1 & \sqrt{\frac{m_\odot}{m_\oplus}} \\
-3\sqrt{\frac{m_1}{2m_\oplus}} & -\sqrt{\frac{m_\odot}{m_\oplus}} & 1 \\
1 & -\sqrt{\frac{m_1}{m_\odot}} & \sqrt{\frac{2m_1}{m_\oplus}}
\end{pmatrix}. \tag{73}
$$

The entries (close to) 1 in $R$ clearly indicate that $m_\oplus$ is dominated by $N_2$, $m_\odot$ by $N_1$ and, of course, $m_1$ by $N_3$. In comparing with the discussion in Ref. [21], one should take into account the different ordering of the right-handed neutrinos (by increasing masses) therein. It is worth stressing that our assumptions of hierarchical light neutrinos and $Y$ eigenvalues yielded the double dominance pattern in (72). Analogous exercises show how the dominance of a pseudo-Dirac pair displays in $R$.

As a second, more familiar example, we consider the class of models already mentioned, in which the fermion mass matrices are explained by an abelian flavour symmetry with only positive charges for all leptons. It is well known that in this case $\mathcal{M}_\nu$ is independent of the right-handed neutrino charges and the magnitude of its matrix elements $(\mathcal{M}_\nu)_{ij}$ are of $O(\epsilon_i \epsilon_j)$. It is easily seen that, in this framework, the $R$ matrix elements come out all of $O(1)$, characterizing the fact that all heavy neutrino states are comparably contributing to all light neutrino mass eigenvalues.

6 Outlook and Conclusions

To have a glimpse at the large solar mixing angle from a bottom-up perspective, we adopt the requirements of robustness and economy and we study the patterns for $Y$ and $M$ which, in this sense, naturally account for neutrino phenomenology. A quantitative appraisal of the structure of subleading terms and of the effect of CP violating phases, is done by means of reliable approximate expressions. Three scenarios, which admit a suggestive physical interpretation in terms of the role played by right-handed neutrinos, are favoured: single dominance in the atmospheric sector accompanied by single or pseudo-Dirac dominance in the solar sector, in the case of a hierarchical neutrino mass spectrum; pseudo-Dirac dominance in the atmospheric sector in the case of an inverted hierarchical spectrum.

The phenomenology of the three scenarios is quite rich. In the hierarchical case, experimentally relevant lower bounds for $U_{e3}$ are derived. In the inverted hierarchical case, on the contrary, upper bounds of the order of permille are obtained. All three scenarios predict a rate for $\beta\beta 0\nu$ too small to be observed (for a recent exhaustive analysis see Ref. [41]). Connections with the rate of lepton flavour violating decays $\tau \to \mu\gamma$ and $\mu \to e\gamma$ have also been discussed. These rates are linked to the nine hidden parameters of the see-saw. The strong dependence on the three right-handed eigenvalues is manifest, but also the parameters belonging to the $R$ matrix play an important role. For instance,
in the pseudo-Dirac dominance scenario, there is a very strong dependence on the ‘rapidity’ defined in Eq. (25), which represents the magnitude of one of the boosts contained in $R$.

It is worth emphasizing that, while being ‘minimalist’ from the bottom-up perspective, these favourite scenarios are inconsistent with a simple flavour symmetry like a $U(1)$ with only positive charges for leptons. Indeed, they suggest the presence of much richer flavour symmetries, like $U(1)$’s with holomorphic zeros from supersymmetry [13, 14] or non-abelian flavour symmetries [15]. Finally, we also stress that the matrix $R$ [18] admits a suggestive physical interpretation. Namely, it encodes the informations on the dominance mechanism at work.

**Note added in proof:** As stressed in the Introduction, the different dominance scenarios analysed in this paper from a bottom-up perspective have been discussed several times in the literature, mostly from a top-down viewpoint. The dominance of one right-handed neutrino first appeared in Ref. [25] and both the single and double dominance mechanisms consistent with large mixing angles were considered in Refs. [12, 13]. The dominance of a pseudo-Dirac pair in the solar sector was previously displayed in Refs. [28, 12]. Pseudo-Dirac dominance in the atmospheric sector has been recently discussed in Refs. [30].

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