SECTORIALITY OF THE LAPLACIAN ON ASYMPTOTICALLY HYPERBOLIC SPACES

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Abstract. We prove that both the Laplacian on functions, and the Lichnerowicz Laplacian on symmetric 2-tensors with respect to asymptotically hyperbolic metrics, are sectorial maps in weighted Hölder spaces. As an application, the machinery of analytic semigroups then applies to yield well-posedness results for parabolic evolution equations in these spaces.

1. Introduction

An initial-value problem for a parabolic partial differential equation is said to be well-posed if there is short-time existence, uniqueness, and continuous dependence of the solution on the data in a given topology. Often well-posedness is proved by setting up a contraction mapping argument in appropriate function spaces. An elegant way to encode this argument involves rewriting the equation as an ordinary differential equation with values in a Banach space, and then applying semigroup methods to its linearization. This point of view has been extensively developed in the literature and robust results exist for linear, semilinear, quasilinear and fully nonlinear equations; see for example the monographs [1, 8].

In this paper, with a view to applications in geometric analysis, we prove a key estimate that allows us to prove well-posedness for heat-type equations on a complete noncompact manifold $M^n$ with geometry modeled on an asymptotically hyperbolic metric. We briefly describe the setting here, with details given in §2.2. We assume that $M$ is a smooth manifold that is the interior of a compact manifold with boundary $\overline{M} = M \cup \partial M$. Given a smooth conformal structure $(\partial M, [\hat{g}])$ on the boundary, we assume there is a collar neighbourhood of $\partial M$ in $\overline{M}$ and a diffeomorphism that identifies this collar with a product $[0,1)_\rho \times \partial M$, and we consider the model metric in the product neighbourhood given by

$$g_\ast = \frac{d\rho^2 + \hat{g}}{\rho^2}$$

This metric is smoothly conformally compact at $\rho = 0$ with conformal infinity $[\hat{g}]$. A calculation shows that all sectional curvatures of $g_\ast$ tend to $-1$ as $\rho \to 0^+$. This condition does not depend on the choice of representative in $[\hat{g}]$ since changing representatives leads
to a new model of the same form up to diffeomorphisms and error terms which vanish to higher order on the boundary.

Now use this model metric to define intrinsic weighted Hölder spaces of $p$-tensors, $\rho^\nu \mathcal{C}^{\ell,\alpha}(M, T^p M)$, where $\mathcal{C}^{\ell,\alpha}(M, T^p M)$ is the $(\ell, \alpha)$ little Hölder space, defined in Section 2.2. We say that $g$ is an asymptotically hyperbolic metric with conformal infinity $\hat{g}$ if there exists $\nu > 0$ and some $k \in \rho^\nu \mathcal{C}^{\ell,\alpha}(M, T^2 M)$, $\ell \geq 2$ so that in the product neighborhood

$$g = g_* + k.$$ 

Such a metric need not be smoothly conformally compact. These spaces have been introduced by Biquard [3] and we use the analysis of the corresponding geometric Laplace operators that appears in [4].

We motivate the analytic part of our result using the framework developed in [8]. We suppose that $D \subset X$ is a dense inclusion of Banach spaces and we suppose that an evolution equation for a quantity of interest $u$ is cast as an ordinary differential equation with values in $X$,

$$u'(t) = F(u(t)),$$

with $u : I \subset \mathbb{R} \rightarrow X$, where $I$ is a subinterval of the real line containing zero. Here $F : D \rightarrow X$ is a (possibly nonlinear) Fréchet differentiable map that satisfies a number of structural assumptions, the most important of which is that its linearization $L$ at a point $u_0 \in D$ be a sectorial operator $X \rightarrow X$. A linear operator $L : X \rightarrow X$ is sectorial if its resolvent set (i.e., the set of complex numbers $\lambda$ for which $\lambda I - L$ has bounded inverse) is contained in a certain sector $S_\omega$ of the complex plane with vertex at $\omega$, and if there is a constant $C > 0$ so that for all $\lambda$ in this sector, the resolvent estimate for the operator norm $\| \cdot \|_{L(X,X)}$

$$\|(\lambda I - L)^{-1}\|_{L(X,X)} \leq \frac{C}{|\lambda - \omega|}$$

holds. This hypothesis allows one to define a heat semigroup $e^{tL}$ via the functional calculus, upon which the subsequent theory of analytic semigroups is based. This theory leads to a wealth of well-posedness results. Many parabolic partial differential operators can be cast into this framework; again see [8].

Given that many evolution equations in geometry have linearizations involving geometric Laplacians, we need to be explicit about the function spaces in which we have sectoriality. Motivated primarily by evolution equations for functions and 2-tensors, we show here that the Laplacian on functions and a gauge-adjusted linearized Einstein operator on symmetric 2-tensors are sectorial in the little Hölder spaces defined above. Let $\Delta^g_L$ denote the Lichnerowicz Laplacian operator corresponding to a metric $g$ acting on symmetric 2-tensors, $S$. 
In particular we prove

**Theorem 1.** For an asymptotically hyperbolic metric \((M^n, g)\), and any integer \(\ell \geq 2\) and \(\alpha \in (0, 1)\), and \(\mu \in (0, n-1)\), there exists a \(\delta > 0\) and \(\lambda_0 > 0\) such that

1. on a sector of the form
   \[ S = \left\{ \lambda = re^{i\theta} \in \mathbb{C} : r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \right\}, \]

   \(\Delta^g\) on functions is sectorial in \(\rho^{\mu}\mathcal{G}_{\ell,\alpha}(M, \mathbb{R})\);

2. on a sector of the form
   \[ S + \lambda_0 = \left\{ \lambda + \lambda_0 \mid \lambda = re^{i\theta} \in \mathbb{C}, r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \right\}, \]

   \(\Delta^g - 2(n-1)\) on symmetric 2-tensors is sectorial in \(\rho^{\mu}\mathcal{G}_{\ell,\alpha}(M, S)\).

There are two main parts to the proof of this Theorem. The first part is to show that the resolvent set of the operator is confined to a suitable sector of the complex plane. In our setting, this follows from a Fredholm theorem of Biquard and Mazzeo and explains the range of weights \(\mu\) for which our result applies. The second part is to prove the resolvent estimate. Here we proceed by contradiction and perform a blowup analysis. With this theorem, one can then apply results in [8] to obtain well-posedness results for geometric PDE.

In a forthcoming paper we discuss an application of these results to the normalized Ricci flow.

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2. **Background**

2.1. **Analytic background.** In this section we define a sectorial operator between Banach spaces. For applications of such operators to evolution equations, see [1, 8]; for applications to the Ricci flow, see [2, 6].

For complex Banach spaces \(X, Y\) we let \(\mathcal{L}(X, Y)\) denote the set of bounded linear operators from \(X\) to \(Y\), with operator norm denoted by \(\| \cdot \|_{\mathcal{L}(X,Y)}\).

Consider a continuous inclusion of Banach spaces \(D \hookrightarrow X\), and a closed linear operator \(L : D \to X\). We define the spectrum of \(L\), \(\sigma_X(L)\), to be the set of complex numbers \(\lambda\) for which \((\lambda I - L)\) does not have a bounded inverse, and the resolvent set of \(L\) by \(\rho_X(L) := \mathbb{C} \setminus \sigma_X(L)\).

**Definition 2.1.** An operator \(L : D \subset X \to X\) is sectorial in \(X\) if:

1. the resolvent set \(\rho_X(L)\) contains a sector, i.e., if there exist constants \(\omega \in \mathbb{R}\) and \(\theta \in (\pi/2, \pi)\) such that
   \[ \rho_X(L) \supset S_\omega := \{ \lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta \}, \]

   and
there exists a constant $C > 0$ so that for all $\lambda$ in this sector, the resolvent estimate holds, i.e., for $R_\lambda = R_\lambda(L) = (\lambda I - L)^{-1}$,

$$\|R_\lambda\|_{\mathcal{L}(X,X)} = \|((\lambda I - L)^{-1})\|_{\mathcal{L}(X,X)} \leq \frac{C}{|\lambda - \omega|}.$$  

In our proof of the resolvent estimate, we use the following lemma from [8].

**Lemma 2.2** (Proposition 2.1.11 of [8]). Let $X$ be a complex Banach space, and $L : D \subset X \to X$ a linear operator such that the resolvent set $\rho_X(L)$ contains a half-plane $\{\lambda \in \mathbb{C} : \text{Re } \lambda \geq \omega\}$ for some $\omega \in \mathbb{R}$. If there exists a constant $C > 0$ such that for all $\lambda$ in this half-plane,

$$\|\lambda(\lambda - L)^{-1}\|_{\mathcal{L}(X,X)} \leq C,$$

then $L$ is sectorial.

### 2.2. Geometric background.

In this section we define the class of asymptotically hyperbolic metrics of interest and refer the reader to [4] for further details. We begin by discussing the model metrics.

Let $M^n$ be the interior of a smooth compact manifold with boundary $\overline{M} = M \cup \partial M$. Recall that a smooth boundary defining function $\rho$ for $\partial M$ is a function $\rho \in \mathcal{C}^\infty(\overline{M})$ such that $\rho \geq 0$, $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on $\partial M$. Given a conformal class $[\hat{g}]$ of metrics on $\partial M$ and a smooth representative $\hat{g} \in [\hat{g}]$, it is always possible to find a neighbourhood $U$ of $\partial M$ in $\overline{M}$ and a diffeomorphism that identifies $U$ with $[0,1) \times \partial M$ where the metric may be written

$$g_* = \frac{d\rho^2 + \hat{g}}{\rho^2}.$$  

This definition is independent of the conformal representative chosen in the sense that changing representatives changes the diffeomorphism and leads to a metric of the same form up to higher order corrections in $\rho$. A computation shows that all sectional curvatures of such metrics approach $-1$ near $\partial M$. We assume that $g_*$ is extended into $M$ to remain a smooth and complete metric. These model metrics are then smoothly conformally compact since $\rho^2 g_*$ extends to a smooth metric on $\overline{M}$.

By shrinking the collar neighborhood $U$ if necessary, we may assume that it is covered by finitely many smooth boundary coordinate charts $\{O_j, (\rho, \theta^\alpha_j)\}$ where $\rho$ is the boundary defining function for $\partial M$ and $\theta^\alpha_j$, $\alpha = 1, \ldots, n-1$, are local coordinates on $\partial M$. These are extended into the interior of $\overline{M}$ by requiring them to be constant along the integral curves of the gradient of $\rho$ with respect to $\overline{g}_* := \rho^2 g_*$. Note that we often write $\theta^0 = \rho$ and use Latin indices for the range $0, 1, \ldots, n-1$ and Greek indices for the range $1, \ldots, n-1$.

We need several function spaces defined in terms of $g_*$. For any tensor bundle $E$ over $M$, let $\mathcal{C}_c^\infty(M, E)$ denote the space of compactly supported smooth tensors on $M$. For $u, v \in \mathcal{C}_c^\infty(M, E)$, define the $L^2$ pairing by

$$(u, v) := \int_M \langle u, v \rangle_{g_*} \, d\text{vol}_{g_*},$$
where $\langle u, v \rangle_{g_*}$ denotes the standard tensor contraction induced by the metric $g_*$, and define the associated norm by $\|u\|^2_{L^2} = (u, u)$. The space $L^2(M, E)$ is defined as the completion of $\mathcal{C}^\infty(M, E)$ with respect to the $L^2$-norm. We take the Hermitian $L^2$ pairing in working with complex vector bundles.

The $\mathcal{C}^\ell$ norm is defined by

$$\|u\|_{\mathcal{C}^\ell(M, E)} := \sum_{j=0}^\ell \sup_{p \in M} |(\nabla^{g_*})^j u|_{g_*}(p).$$

Before introducing the Hölder quotient, we note that by a path we mean any piecewise $\mathcal{C}^1$ map $\gamma : [0, 1] \to M$. The length of a path is given by

$$\text{len}(\gamma) = \int_0^1 |\gamma'(s)|_{g_*} ds.$$
We require some theory of elliptic operators on these manifolds. In [9, 10], Mazzeo constructs a pseudodifferential calculus, the 0-calculus, that contains the inverses of operators like $P^g$. We refer the reader to [4] for a concise treatment of the details; see also [3, 7] for other approaches.

Condition S1 in Definition 2.1 is a statement of where the resolvent $\lambda - P^g$ is invertible. In order to state the mapping properties of $P^g$ and study the asymptotics of solutions of associated equations, we need two simpler model operators. The first model operator is the normal operator of $P^g$, which in our setting is the same operator but with respect to the hyperbolic metric, so $P^h$ where $h$ is the hyperbolic metric on the open unit ball. Geometric Laplace operators as in equation (2.3) are said to be fully elliptic if $P^h$ is invertible as an unbounded operator in $L^2$ on hyperbolic space.

For the second model operator, given $s \in \mathbb{C}$ define the indicial operator $I_s(P^g) : E|_{\partial M} \rightarrow E|_{\partial M}$ by

$$I_s(P^g)\overline{\mu} = \rho^{-s}P^g(\rho^s\overline{\mu})|_{\partial M},$$

where $\overline{\mu}$ is a $C^2$ section of $E|_{\partial M}$ extended arbitrarily into $M$. It can be shown that $I_s(P^g)$ is a polynomial in $s$ and is independent of the choice of extension. We call any value of $s$ for which $I_s(P^g)$ vanishes an indicial root. For any operator $P^g$ of the form (2.3), the set of real parts of the indicial roots are symmetric about $\frac{n-1}{2}$. For such $P^g$ there is a largest positive constant $\Xi > 0$ such that the interval $\frac{n-1}{2} - \Xi < s < \frac{n-1}{2} + \Xi$ contains no other indicial roots; $\Xi$ is called the indicial radius of $P^g$. Recall that $P^g$ is Fredholm if $\dim \ker P^g$ and $\dim \ker(P^g)^*$ are finite dimensional and the range of $P^g$ is closed. The fundamental properties of $P^g$ are encoded in the following result.

**Theorem 2.3** (Theorem 13 of [4]). Let $P^g$ be a fully elliptic Laplace operator as in (2.3). For any $\ell \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, the natural extension

$$P^g : \mathcal{E}^{\ell+2,\alpha} (M, E) \rightarrow \mathcal{E}^{\ell,\alpha} (M, E)$$

is Fredholm if and only if $|\mu - \frac{n-1}{2}| < \Xi$. Further, $P^g$ is an isomorphism on these spaces if and only if $\ker_{L^2}(P^g) = \{0\}$.

### 3. Sectoriality of the Laplacian

In this section we prove Theorem 1. In §3.1 we explain how to obtain the resolvent estimate on weighted spaces from an estimate for the operator conjugated by powers of the defining function on unweighted spaces. We also provide a key decomposition for the Laplace operator. In §3.2 we prove sectoriality for the Laplacian acting on functions. In §3.3 we prove sectoriality for the Lichnerowicz Laplacian on symmetric 2-tensors.

#### 3.1. The resolvent estimate.

As in §2.2, let $(M, g)$ be an asymptotically hyperbolic metric of the form

$$g = g_* + k = \frac{d\rho^2 + \tilde{g}}{\rho^2} + k,$$

(3.1)
with \( k \in \mathcal{C}_0^{\ell,\alpha}(M; \mathcal{S}) \), \( \nu > 0 \). Let \( \{\mathcal{O}_i, (\rho, \theta^\alpha_i)\} \) be a reference coordinate chart in the collar neighbourhood of the boundary. From [10] we know that the space of vector fields which vanish on the boundary, hereafter called 0-vector fields, are locally generated by

\[
\rho \partial_\rho, \quad \rho \partial_\theta^\alpha.
\]

In the calculations that follow we choose these vectors fields and their duals \( \frac{d\rho}{\rho}, \frac{d\theta^\alpha}{\rho} \) as the preferred bases for tensor bundles.

It is convenient to use an equivalent norm for Hölder spaces, defined in terms of the Möbius charts described in [7, Chapter 2]. Let \( B_r \) denote the open ball about \((1,0,\ldots,0)\) of (hyperbolic) radius \( r \) in the half-space model of hyperbolic space, \( \mathbb{U}^n = \{(x, y^1, \ldots, y^{n-1}) : x > 0, y^\alpha \in \mathbb{R}\} \). For \( p \in \mathbb{U} \) in the domain of a reference chart \( \mathcal{O} \) with coordinates \((\rho, \theta^\alpha)\), we denote \( p = (\rho_0, \theta^\alpha_0) \), and define a map \( \Phi_p : B_r \subset \mathbb{U}^n \to \mathcal{O} \subset M \) by

\[
\Phi_p(x, y^\alpha) = (\rho_0 x, \theta^\alpha_0 + \rho_0 y^\alpha).
\]

It is possible to cover \( M \) by a locally uniformly finite covering of the balls \( \{\Phi_p(B_1)\} \) near \( \partial M \), and finitely many other coordinate balls that cover \( M \setminus \mathcal{U} \), which we label in the same way for consistency. In sum we obtain a uniformly locally finite cover of \( M \) of the form \( \{\Phi_{\rho_i}(B_1)\} \). An equivalent norm for \( \mathcal{C}_0^{\ell,\alpha}(M, E) \) is given by

\[
\|u\|_{\ell,\alpha;\mu} = \sup_i \|\Phi_{\rho_i}^*(\rho^{-\mu}u)\|_{\mathcal{C}_0^{\ell,\alpha}(B_1)},
\]

where the norm on the right hand side is the Hölder norm of the components of \( u \) computed with respect to the background Euclidean metric on \( B_1 \). For the remainder of this section, we use (3.3) as the norm on Hölder spaces.

We now set up the proof of the resolvent estimate. For the remainder of this subsection we assume that \( P = \Delta + \mathcal{K} \) is a geometric Laplace operator on a tensor bundle \( E \), and that for \( D = \mathcal{C}_0^{\ell+2,\alpha}(M; E) \), \( X = \mathcal{C}_0^{\ell,\alpha}(M; E) \) for some value of \( \mu \), the sectoriality property S1 of Definition 2.1 has been established on some sector \( S_\omega \). We drop the superscript \( g \) on \( P \) to lighten the notation. We consider the case where \( \ell = 0 \); the details for \( \ell \in \mathbb{N} \) are similar.

To obtain the resolvent estimate between weighted spaces, it is convenient to instead work in unweighted spaces. To achieve this, conjugate the operator \( P \) by \( \rho^\mu \) to obtain \( \rho^{-\mu}P\rho^\mu \). For example, if \( \lambda \) is in the resolvent set of \( P \), then \( \lambda - P \) is invertible and if \( u \in \mathcal{C}_0^{0,\alpha} \) we may set

\[
w = (\lambda - P)^{-1}u.
\]

This means that \( u = (\lambda - P)w \), and since \( u, w \) are both in \( \mathcal{C}_0^{0,\alpha}(M; E) \) we may write \( u = \rho^\mu \overline{u} \) and \( w = \rho^\mu \overline{w} \) for \( \overline{u}, \overline{w} \) in \( \mathcal{C}_0^{0,\alpha}(M; E) \). Thus we obtain

\[
\overline{u} = (\lambda - \rho^{-\mu}P\rho^\mu)\overline{w}.
\]

Lemma 3.1. If there exists \( C > 0 \) so that for all \( \lambda \in S_\omega \) and for any \( v \in \mathcal{C}_0^{0,\alpha}(M; E) \),

\[
\|(\lambda - \rho^{-\mu}P\rho^\mu)^{-1}v\|_{0,\alpha} \leq \frac{C}{|\lambda - \omega|} \|v\|_{0,\alpha}.
\]
then for the same $C$ and $\lambda$ and any $u \in C^0_\mu(M;E)$,

$$
\| (\lambda - P)^{-1} u \|_{0,\alpha;\mu} \leq \frac{C}{|\lambda - \omega|} \| u \|_{0,\alpha;\mu}.
$$

**Proof.** Taking $u \in C^0_\mu(M;E)$ and $w = (\lambda - P)^{-1} u$, we calculate

$$
\| (\lambda - P)^{-1} u \|_{0,\alpha;\mu} = \| \rho^{-\mu}(\lambda - P)^{-1} u \|_{0,\alpha} = \| \rho^{-\mu}(\lambda - P)^{-1} \rho^\mu \rho^{-\mu} u \|_{0,\alpha} \leq \frac{C}{|\lambda - \omega|} \| \rho^{-\mu} u \|_{0,\alpha} = \frac{C}{|\lambda - \omega|} \| u \|_{0,\alpha;\mu}.
$$

\[\Box\]

It is also convenient to have an expression for the conjugated operator $\rho^{-\mu} P \rho^\mu$ near the boundary. Here this operator is well approximated by a Laplacian plus terms that are under control.

**Lemma 3.2.** For $\ell \geq 2$, $0 < \alpha < 1$, $\nu > 0$, let $g = g_* + k$, with $k \in C^0_{\nu,\alpha}$, be an asymptotically hyperbolic metric. Let $P = \Delta g + K$ be a geometric Laplace operator on the tensor bundle $E$. For a section $u$ of $E$,

$$
\rho^{-\mu} P[\rho^\mu u] = (\rho \partial_\rho)^2 u + \tilde{g}^{\alpha \beta} (\rho \partial_{\theta^\alpha})(\rho \partial_{\theta^\beta}) u + Z_{\text{van}} u + Z_{\text{bdd}} u,
$$

where $Z_{\text{van}}$ is a second-order operator whose coefficients lie in at least $C^0_{\nu,\alpha}$ and $Z_{\text{bdd}}$ is a first-order operator with uniformly bounded coefficients lying in at least $C^0_{\nu,2}$.

**Proof.** We suppose that $u$ is a tensor of covariant rank $p$ and contravariant rank $q$. In the components relative to the frame described at the beginning of this section in 3.2, we write

$$
u^a u = u_{i_1 \cdots i_p}^{j_1 \cdots j_q} (\rho \partial_{\theta^a}) \otimes \cdots \otimes (\rho \partial_{\theta^q}) \otimes \frac{d\theta^{i_1}}{\rho} \otimes \cdots \otimes \frac{d\theta^{i_p}}{\rho}.\]

Thus

$$
\nabla^a u = u_{i_1 \cdots i_p,a}^{j_1 \cdots j_q} (\rho \partial_{\theta^a}) \otimes \cdots \otimes (\rho \partial_{\theta^q}) \otimes \frac{d\theta^{i_1}}{\rho} \otimes \cdots \otimes \frac{d\theta^{i_p}}{\rho} \otimes \frac{d\theta^a}{\rho},
$$

where the components of $\nabla^a u$ are given by

$$
u_{i_1 \cdots i_p,a}^{j_1 \cdots j_q} = \rho \partial_{\theta^a} u_{i_1 \cdots i_p}^{j_1 \cdots j_q} + \sum_{t=1}^q \Gamma_{i_1 \cdots i_p,a}^{j_1 \cdots j_q,s \cdots } + \sum_{t=1}^p \Gamma_{i_1 \cdots i_p,a}^{j_1 \cdots j_q} u_{i_1 \cdots s \cdots i_p,a,t}.\]

Here, $\Gamma$ denotes the Christoffel symbols of the Levi-Civita connection $\nabla^a$. As we do not need the precise form of this equation, we may express it abstractly as

$$
\nabla_a u = \rho \partial_{\theta^a} u + \Gamma * u,
$$
where $*$ denotes linear combinations of tensor contractions of $\Gamma$ with $u$ using $g$ and $g^{-1}$. Recall that $\Delta^g u = g^{ab} \nabla_b \nabla_a u$; thus, applying a covariant derivative to $\nabla_a u$ yields abstractly

$$\nabla_b \nabla_a u = \rho \partial_{\rho^b} (\rho \partial_{\rho^a} u + \Gamma \ast u) + \Gamma \ast (\rho \partial_{\rho^a} u + \Gamma \ast u)$$

$$= (\rho \partial_{\rho^b} (\rho \partial_{\rho^a} u + \Gamma \ast u) + \Gamma \ast (\rho \partial u) + (\rho \partial \Gamma) \ast u + \Gamma \ast \Gamma \ast u.$$ We arrive at the expression

$$g^{ab} \nabla_b \nabla_a u + Ku = g^{ab} (\rho \partial_{\rho^b} (\rho \partial_{\rho^a} u + \Gamma \ast u) + (\rho \partial u) + (\rho \partial \Gamma) \ast u + \Gamma \ast \Gamma \ast u,$$

where

$$Z_{\text{van}} u = (g^{ab} - g_{\ast}^{ab}) (\rho \partial_{\rho^b} (\rho \partial_{\rho^a} u + \Gamma \ast u),$$

and

$$Z_{\text{bdd}} u = \Gamma \ast (\rho \partial_{\rho^a} u) + (\rho \partial_{\rho^b} \Gamma) \ast u + \Gamma \ast \Gamma \ast u + Ku.$$

Regarding the second-order operator $Z_{\text{van}}$, since $g = g_{\ast} + k$, we may express $g^{-1}$ in terms of $g_{\ast}^{-1}$ and $k$ for $k$ with sufficiently small norm to ensure invertibility of $g$. Writing $g^{ab}$ for the components of $g^{-1}$ and $g_{\ast}^{ab}$ for the components of $g_{\ast}^{-1}$ relative to our choice of frame, one finds

$$g^{ab} = (g_{\ast} + k)^{ab} = g_{\ast}^{ab} - g_{\ast}^{al} g_{\ast}^{lm} k_{ml} + (g_{\ast} + k)^{bl} g_{\ast}^{om} g_{\ast}^{aq} k_{lp} k_{mq},$$

which shows that $g^{ab} - g_{\ast}^{ab}$ lies in $\mathcal{C}_{\nu}^{\ell, \alpha}$.

Regarding the first order operator $Z_{\text{bdd}}$, one sees from the formula for Christoffel symbols in terms of a local frame that $\Gamma$ involves one 0-derivative of the metric, and the 0th-order term $K$ depends on the curvature of $g$; thus, $Z_{\text{bdd}}$ involves at most two 0-derivatives of $g$. As $\ell \geq 2$, all of the terms of $Z_{\text{bdd}}$ are uniformly bounded on $M$.

In order to complete the proof we conjugate $P$ by $\rho^\mu$. Observe that $\rho \partial_{\rho}[\rho^\mu u] = \rho^\mu \rho \partial_{\rho} u + \mu \rho^\mu u$ and $(\rho \partial_{\rho})^2[\rho^\mu u] = \rho^\mu (\rho \partial_{\rho})^2 u + 2 \mu \rho^\mu \rho \partial_{\rho} u + \mu^2 \rho^\mu u$. Thus the conjugated operator has the required form with the new terms absorbed into $Z_{\text{bdd}}$. Since the application of 0-derivatives does not alter the power of vanishing in $\rho$, the $Z_{\text{bdd}}$ remains uniformly bounded on $M$. \hfill \Box

We are now ready to establish the resolvent estimate in a half-plane, presuming property S1 from Definition 2.1 holds. We present the argument for general tensors but suppress indices to lighten the notation.

**Proposition 3.3.** For $\ell \geq 2$, $0 < \alpha < 1$, $\nu > 0$, let $g = g_{\ast} + k$, with $k \in \mathcal{C}_{\nu}^{\ell, \alpha}$, be an asymptotically hyperbolic metric. Let $P = \Delta^g + K$ be a geometric Laplace operator on the tensor bundle $E$ whose resolvent set contains a sector $S_{\omega}$, as given in S1 in Definition 2.1, with $\omega > 0$. Then there exists $C > 0$ so that for all $\lambda \in \mathbb{C}$ such that $\text{Re} \, \lambda \geq \omega$, we have that for any $u \in \mathcal{C}_{0, \alpha}^{0, \alpha} (M; E)$,

$$\|\lambda (\lambda - \rho^{-\mu} P \rho^\mu)^{-1} u\|_{0, \alpha} \leq C \|u\|_{0, \alpha}.$$  

**Proof.** We proceed by contradiction. If the estimate were false, then for every $m \in \mathbb{N}$, there would exist $\lambda_m \in \mathbb{C}$ with $\text{Re} \, \lambda_m \geq \omega$ and $u_m \in \mathcal{C}_{0, \alpha}^{0, \alpha}$ where

$$\|\lambda_m (\lambda_m - \rho^{-\mu} P \rho^\mu)^{-1} u_m\|_{0, \alpha} > m \|u_m\|_{0, \alpha}.$$  

(3.5)

We remind the reader that we sometimes write $R_\lambda = R_\lambda (\rho^{-\mu} P \rho^\mu)$ for $(\lambda - \rho^{-\mu} P \rho^\mu)^{-1}$.  

Claim: We may assume \( \lambda_m \to +\infty \) in the complex plane.

Indeed if this claim were false, then some subsequence of \( \lambda_m \) would converge to a finite point \( \lambda_* \) with \( \text{Re} \, \lambda_* \geq \omega \), which by assumption is in the resolvent set. The resolvent operator \( \lambda \mapsto R_\lambda(\rho^{-\mu} P \rho^\mu) \) is analytic in \( \lambda \) and thus continuous, so that \( \| R_{\lambda_m} \| \to \| R_{\lambda_*} \| \) in the operator norm; however, if a subsequence of \( \lambda_m \) were to converge, then estimate (3.5) would show that \( \| R_{\lambda_m} \| \to \infty \), so that \((\lambda_* - \rho^{-\mu} P \rho^\mu)\) could not have a bounded inverse, and so \( \lambda_* \) could not be in the resolvent set. This contradiction establishes the claim.

Now define

\[
(3.6) \quad w_m = \frac{R_{\lambda_m}(\rho^{-\mu} P \rho^\mu)u_m}{\| R_{\lambda_m}(\rho^{-\mu} P \rho^\mu)u_m \|_{0,\alpha}}.
\]

Observe that by virtue of this definition, each \( w_m \) satisfies an elliptic partial differential equation

\[
(3.7) \quad (\lambda_m - \rho^{-\mu} P \rho^\mu) w_m = \frac{u_m}{\| R_{\lambda_m}(\rho^{-\mu} P \rho^\mu)u_m \|_{0,\alpha}} = \frac{1}{\beta_m} u_m,
\]

where \( \beta_m := \| R_{\lambda_m}(\rho^{-\mu} P \rho^\mu)u_m \|_{0,\alpha} \).

By elliptic regularity, \( w_m \) is locally in \( \mathscr{C}^{2,\alpha} \). By construction, \( \| w_m \|_{0,\alpha} \equiv 1 \) for all \( m \), so there must exist a point \( p_m \in M \) such that on the ball of \( g \)-radius 1, \( B_m := B(p_m, 1) \), we have

\[
(3.8) \quad \| w_m \|_{0,\alpha; B_m} \geq \frac{1}{2}.
\]

In the topology of the compact manifold with boundary \( \overline{M} \), we may now pass to a convergent subsequence \( p_m \to p \). There are two model cases: Case i) \( p \in M \) (i.e., the subsequence accumulates in the interior of \( M \), and Case ii) \( p \in \partial M \) (i.e., where the subsequence accumulates at a particular point on the conformal boundary at infinity). In both cases we can construct a solution to a limiting PDE and derive a contradiction. We detail the case where \( p \in \partial M \) and refer the reader to our published work [2] for the interior case, which is similar to the case of a closed manifold.

Case ii): Assume \( p \in \partial M \). It is helpful to recall the decomposition of the metric \( g \) near the boundary, as specified in equation (3.1). We may without loss of generality assume that \( p \) lies in a reference chart \((\mathcal{O}, (\rho, \theta^\alpha))\) on \( \partial M \) which is centered at \( p \). In this chart, the metric takes the form

\[
g = \frac{d\rho^2 + \hat{g}_{\alpha\beta}(\theta)d\theta^\alpha d\theta^\beta}{\rho^2} + k,
\]

with \( p = (\rho, \theta^\alpha) = (0, 0) \) in these coordinates. We advance and relabel the subsequence so that that \( p_m \) lies in this chart.

We now consider several simplifying transformations that enable us to “zoom in” along \( p_m \) and derive a solution to a PDE that results in a contradiction. We begin by formulating the equation, and then we study the estimates for the solutions.
The zooming sequence of equations. Begin with the sequence of equations \((3.7)\). Using Lemma 3.2 this sequence takes the form

\[
\begin{align*}
\lambda_m - \left( (\rho \partial_\rho)^2 + \tilde{g}^{\beta \nu}(\rho, \theta^\alpha)\rho \partial_\beta \partial_\nu \right) + Z_{\text{van}} + Z_{\text{bdd}} \right) w_m = \beta_m^{-1} u_m. 
\end{align*}
\]

For each \(m\), we use the definition of Möbius charts to pull the equations \((3.9)\) back from \(B_m\) to a fixed ball \(B\). Under \(\Phi_{p_m} : B \to B_m\) given by \((\rho, \theta^\alpha) = (\rho_m x, \theta_m^\alpha + \rho_m y^\alpha) = \Phi_{p_m}(x, y^\alpha)\), we find that the derivatives \(\rho \partial_\rho, \rho \partial_\theta\) transform to \(x \partial_x, x \partial_y\) derivatives; thus we obtain

\[
\begin{align*}
\left[ \lambda_m - \left( (x \partial_x)^2 + \tilde{g}^{\beta \nu}(x, \theta_m^\alpha + \rho_m y^\alpha)x \partial_\beta \partial_\nu \right) + Z_{\text{van}} + Z_{\text{bdd}} \right] w_m = \beta_m^{-1} u_m. 
\end{align*}
\]

It is helpful to divide both sides by \(|\lambda_m|\),

\[
\begin{align*}
\left[ \frac{\lambda_m}{|\lambda_m|} - \frac{1}{|\lambda_m|} \left( (x \partial_x)^2 + \tilde{g}^{\beta \nu}(x, \theta_m^\alpha + \rho_m y^\alpha)x \partial_\beta \partial_\nu \right) 
\right. 
\end{align*}
\]

Next, we apply a coordinate change in the \(x\) variable given by \(x = e^{-t}\). This has the effect of mapping the center of \(B\) to \((0,0)\) so that the image of \(B\) contains a ball in \((t, y^\alpha)\) coordinates. Finally, for each equation, we dilate the variables to effectuate the zoom. Let \(t = |\lambda_m|^{-1/2} r\), and let \(y^\alpha = |\lambda_m|^{-1/2} s^\alpha\). Thus the image of a ball of fixed size in \((t, y^\alpha)\) variables is mapped to a ball \(\Omega_m\) in \((r, s^\alpha)\) coordinates of radius \(|\lambda_m|^{1/2}\) which tends to infinity in \(m\). Thus the sequence of equations is defined on an exhaustion of \(\mathbb{R}^n\).

A chain rule computation shows that \(x \partial_x\) transforms to \(-|\lambda_m|^{-1/2} \partial_r\), and \(x \partial_y^\alpha\) transforms to \(|\lambda_m|^{1/2} e^{-|\lambda_m|^{-1/2} r} \partial s^\alpha\). We now consider what happens if we convert equation \((3.10)\) to these variables. Since \(Z_{\text{bdd}}\) contains at most first order derivatives with bounded coefficients, there remains a prefactor of \(\frac{1}{|\lambda_m|^{1/2}}\) multiplying these terms after the change of variables. Thus this operator tends to zero uniformly as \(m \to \infty\). On the other hand the second-order terms in equation \((3.10)\) and in \(Z_{\text{van}}\) absorb the \(|\lambda_m|\) upon conversion and we obtain

\[
\begin{align*}
\left[ \lambda_m - \left( (\partial_r)^2 + \tilde{g}^{\beta \nu}(\rho, \theta^\alpha)\partial_\beta \partial_\nu \right) 
\right. 
\end{align*}
\]

\[
\begin{align*}
\left. + Z_{\text{van}} + \frac{1}{|\lambda_m|^{1/2}} Z_{\text{bdd}} \right] \tilde{w}_m = \tilde{u}_m
\end{align*}
\]
where we have written
\[
\tilde{w}_m(r, s) = w_m \left( \rho_m e^{-r/|\lambda_m|^{1/2}}, \theta_m^\alpha + \rho_m \frac{s^\alpha}{|\lambda_m|^{1/2}} \right),
\]
\[
\tilde{u}_m(r, s) = \frac{1}{|\lambda_m|^{\alpha/2}} u_m \left( \rho_m e^{-r/|\lambda_m|^{1/2}}, \theta_m^\alpha + \rho_m \frac{s^\alpha}{|\lambda_m|^{1/2}} \right),
\]
and
\[
Z_{\text{van}}' = (e^{-2r/|\lambda_m|} - 1) \partial_{s^\alpha} \partial_{s^\nu} + (g^{ab} - \tilde{g}^{ab}) \partial_{s^a} \partial_{s^b},
\]
where \(\partial_{s^\alpha}\) denotes either \(\partial_r\) or \(e^{-r/|\lambda_m|} \partial_{s^\alpha}\). Regarding \(Z_{\text{van}}'\), a MacLaurin expansion in \(r\) for the first term, and the fact that the asymptotically hyperbolic perturbation tensor \(k\) has the fall off rate \(k = O(\rho^\nu)\) in the second term, shows that the coefficients of \(Z_{\text{van}}'\) tend to zero as \(m \to \infty\).

**Estimates for the solutions.** With the sequence of equations (3.11) in hand, we now study estimates for the tensor fields \(\tilde{w}_m\) and \(\tilde{u}_m\). Recall the definition of \(w_m\) in (3.6) and the estimate (3.8). From the form of the Hölder norm given in (3.3), we find that \(1/2 \leq \|w_m\|_{0, \alpha; B, \text{Euc}} \leq 1\) in the Euclidean norm in \((x, y^\alpha)\) coordinates on \(B\). Thus

\[
\frac{1}{2} \leq \sup_{(x,y) \in B} |w_m(x, y)| + \sup_{(x,y) \neq (x', y') \in B} \frac{|w_m(x, y) - w_m(x', y')|}{|x - x'|^\alpha + |y - y'|^\alpha} \leq 1.
\]

We now transfer the estimates for \(w_m\) and \(u_m\) to \(\tilde{w}_m\) and \(\tilde{u}_m\):

\[
|\tilde{w}_m(r, s) - \tilde{w}_m(r', s')| = |w_m \left( e^{-r/|\lambda_m|}, \frac{s}{|\lambda_m|^{1/2}} \right) - w_m \left( e^{-r'/|\lambda_m|}, \frac{s'}{|\lambda_m|^{1/2}} \right)|
\]
\[
\leq \|w_m\|_{0, \alpha; B} \left( |e^{-r/|\lambda_m|} - e^{-r'/|\lambda_m|}| + \frac{1}{|\lambda_m|^{\alpha/2}} |s - s'|^\alpha \right)
\]
\[
\leq C \frac{1}{|\lambda_m|^{\alpha/2}} \left( |r - r'|^{\alpha} + |s - s'|^{\alpha} \right),
\]
where we have used a mean-value estimate on \(e^{-t}\) on the image of \(B\) in these coordinates and we have used the uniform bound on the \(\|w_m\|_{0, \alpha}\). Thus we conclude that the Hölder quotient of \(\tilde{w}_m\) in \((r, s)\) coordinates satisfies

\[
|\tilde{w}_m|_{0, \alpha} \leq \frac{C}{|\lambda_m|^{\alpha/2}}.
\]

Therefore as \(m \to \infty\), the \(|\tilde{w}_m|_{0, \alpha} \to 0\) and the Hölder norm is increasingly dominated by the sup-norm. In particular, \(\tilde{w}_m\) are uniformly bounded in the Hölder norm and eventually uniformly bounded from below by a positive constant.
Arguing analogously for the $\widetilde{u}_m$, observe that it follows from estimate (3.5) that we have
\[ |\lambda_m|^{-1} \beta_m^{-1} \|u_m\|_{0,\alpha} \leq \frac{1}{m}. \]
Thus, similar considerations as above imply that $\widetilde{u}_m \to 0$ uniformly in $C^{0,\alpha}$ on compact subsets of $\mathbb{R}^n$.

**Limiting equation.** Having completed these preliminaries, we now proceed to construct a limiting equation from (3.11). Note that throughout this argument we pass to subsequences and relabel the original sequence instead of using the unwieldy subsequence notation $\lambda_{n_k}$.

For the purposes of this argument, we introduce the operator $\Box := \partial^2_r + \tilde{g}^{\beta\nu}(0,0)\partial_s^\beta \partial_s^\nu$ on $\mathbb{R}^n$ in $(r,s)$ coordinates. Note that this is a constant coefficient elliptic operator.

Consider a subsequence $\lambda_m \in \mathbb{C}$ so that $|\lambda_m|/|\lambda_m|$ converges to a unit modulus complex number $\lambda^*$ with $\text{Re} \lambda^* \geq \omega$. Using local elliptic estimates and the Arzela-Ascoli lemma, on each ball of integer radius (with respect to $(r,s)$ coordinates) in $\mathbb{R}^n$ we may obtain a subsequence of $\widetilde{w}_m$ that converges to a $C^2$ solution to $(\lambda^* - \Box)\widetilde{w} = 0$ on that ball. Then applying a diagonal extraction argument, we conclude that there exists a bounded $C^2$ tensor field $\widetilde{w}$ on $\mathbb{R}^n$ whose components each satisfy
\[ (\lambda^* - \Box)\widetilde{w} = 0. \]
(3.13)

Since the $\widetilde{w}_m$ are uniformly bounded from below, at least one component of $\widetilde{w}$ is nonzero. But the only bounded solution to the PDE (3.13) is $\widetilde{w} = 0$, as can be seen by applying the Fourier transform (in the sense of distributions) to equation (3.13), and observing that the principal symbol is non-vanishing for $\text{Re} \lambda^* \geq \omega > 0$. This contradiction concludes the proof of Proposition 3.3. □

Proposition 3.3 and Lemma 2.2 establish the resolvent estimate in some sector, given property S1 of Definition 2.1. In the following subsections we consider specific cases in which we know that in fact property S1 holds.

3.2. **The Laplacian on functions.** We begin by considering the covariant Laplacian acting on functions, $\Delta^g u = g^{ab} \nabla^a \nabla^b u$, thought of as an unbounded operator on $L^2(M)$ with domain $D(\Delta^g) = H^2(M)$.

This operator is a real, formally self-adjoint and closed operator. If $h$ is the hyperbolic metric on the unit ball, a result of McKean [13] implies that the whole $L^2$ spectrum of the Laplacian $\Delta^h$ is contained in the ray $(-\infty, -\frac{(n-1)^2}{4}]$. If $g$ is asymptotically hyperbolic, a result of Mazzeo [12] implies that the spectrum may additionally contain finitely many eigenvalues in the interval $\left( -\frac{(n-1)^2}{4}, 0 \right)$. Thus $\lambda - \Delta^g$ is fully elliptic away from $\lambda \in (-\infty, 0)$.

For some weight $\mu$, to be specified below, let $D = \mathcal{C}^{\ell+2,\alpha}_\mu (M)$ and $X = \mathcal{E}^{\ell,\alpha}_\mu (M)$. We aim to to prove that $\Delta^g$ is sectorial on some sector with vertex $\omega = 0$. We use Theorem 2.3 to study the action of $\Delta^g$ on weighted little Hölder spaces, and thus we require the indicial operator of equation (2.4). A computation using this equation and equation (3.1)
shows that for a $C^2$ function $u$ on $\partial M$ extended arbitrarily into $M$,
\[ I_s(\Delta g)\pi = (s^2 - (n - 1)s)\pi. \]
This shows that the indicial roots are $s = 0$ and $s = n - 1$ and that the indicial radius is $\Xi = \frac{n-1}{2}$. Thus Theorem 2.3 implies that
\[ \Delta^g : \mathcal{E}^{\ell+2,\alpha}_\mu (M) \to \mathcal{E}^{\ell,\alpha}_\mu (M) \]
is Fredholm for $\mu \in (0, n - 1)$, with kernel equal to the $L^2$-kernel of $\Delta^g$.

Now consider the effect of adding a spectral parameter $\lambda$. The indicial operator becomes
\[ I_s(\lambda - \Delta g)\pi = (-s^2 + (n - 1)s + \lambda)\pi, \]
with corresponding indicial roots $s = \frac{n-1}{2} \pm \frac{1}{2} \sqrt{(n-1)^2 + 4\lambda}$. While we have a complex spectral parameter in mind, notice that if we momentarily think of $\lambda$ as real-valued and let $\lambda \downarrow \left( \frac{n-1}{2} \right)^2$, the range of weights for which we have a Fredholm result shrinks to zero. In a similar way, certain complex-valued $\lambda$ cause the Fredholm range to shrink. Thus given a fixed $\mu \in (0, n - 1)$ we must identify an appropriate set of $\lambda$ in the complex plane on which we obtain a Fredholm result. Given our intended application to sectorial operators, we then show that one such set is a sector.

To assist our calculation, we use the following Lemma. Note that the proof is a straightforward calculation using the polar representation of a complex number.

**Lemma 3.4.** For a real number $A > 0$ and for any $\varepsilon > 0$, there exists a $\delta > 0$ so that if $\lambda$ belongs to the sector
\[ S = \left\{ \lambda = re^{i\theta} \in \mathbb{C} : r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \right\} \]
then
\[ \Re \sqrt{A^2 + 4\lambda} > (1 - \varepsilon)A. \]

We then have the following proposition:

**Proposition 3.5.** For an asymptotically hyperbolic metric $(M^n, g)$, and any integer $\ell \geq 0$ and $\alpha \in (0, 1)$, given $\mu \in (0, n - 1)$ there exists a $\delta > 0$ so that for any $\lambda$ in the sector
\[ S = \left\{ \lambda = re^{i\theta} \in \mathbb{C} : r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \right\}, \]
the corresponding operator
\[ \lambda - \Delta^g : \mathcal{E}^{\ell+2,\alpha}_\mu (M; \mathbb{R}) \to \mathcal{E}^{\ell,\alpha}_\mu (M; \mathbb{R}) \]
is a Fredholm map. Moreover the kernel of $\lambda - \Delta^g$ on $\mathcal{E}^{\ell+2,\alpha}_\mu (M; \mathbb{R})$ coincides with its $L^2$ kernel.

**Proof.** Given $\mu \in (0, n - 1)$, choose $\varepsilon > 0$ so small that $\frac{n-1}{2} - \varepsilon < \mu < n - 1 - \frac{n-1}{2} - \varepsilon$. By Lemma 3.4 with $A = n - 1$, there exists a $\delta > 0$ so that for any $\lambda \in S$, $\Re \sqrt{(n-1)^2 + 4\lambda} > (1 - \varepsilon)(n-1)$. Thus $\mu$ lies in the Fredholm range of Theorem 2.3, and the result follows. \(\square\)
Combining this proposition with our observations above confining the \( L^2 \) spectrum of the Laplacian to a certain ray of the negative real line yields the following:

**Corollary 3.6.** Given the hypotheses of Proposition 3.5, then for \( \lambda \in S \),

\[
\lambda - \Delta^g : \mathcal{C}^\ell,\alpha^\mu(M; \mathbb{R}) \to \mathcal{C}^\ell,\alpha^\mu(M; \mathbb{R})
\]

is an isomorphism.

**Proof.** We check that the \( L^2 \) kernel of \( \lambda - \Delta^g \) is empty for \( \lambda \in S \). But if \( (\lambda - \Delta^g)v = 0 \), then \( v \) is an \( L^2 \)-eigenfunction of \( \Delta^g \) with eigenvalue \( \lambda \), which is impossible since \( S \cap (-\infty, 0] = \emptyset \).

\( \square \)

We conclude that the resolvent set of \( \Delta^g \) on little Hölder spaces contains a sector; consequently, we have verified condition S1 of sectoriality in Definition 2.1. In what follows we may always assume that the closure of the sector \( S \) still lies in the resolvent set; if not, we take slightly smaller \( \delta \) and use the fact that 0 is not in the spectrum of \( \Delta^g \).

We now are ready to prove Theorem 1 for the Laplacian on functions.

**Theorem 1 (part 1)** For an asymptotically hyperbolic metric \((M^n, g)\), and any integer \( \ell \geq 0 \) and \( \alpha \in (0, 1) \), given \( \mu \in (0, n - 1) \), there exists a \( \delta > 0 \) and a sector of the form

\[
S = \{ \lambda = re^{i\theta} \in \mathbb{C} : r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \}
\]

such that \( \Delta^g \) is sectorial in \( \mathcal{C}^\ell,\alpha^\mu \).

**Proof.** By Corollary 3.6, property S1 of sectoriality holds (see Definition 2.1). Applying Lemmas 2.2 and 3.1, and Proposition 3.3 we obtain the resolvent estimate S2 in this sector. Thus \( \Delta^g \) is sectorial in these weighted little Hölder spaces. \( \square \)

3.3. The Lichnerowicz Laplacian on symmetric 2-tensors. We now consider the operator

\[
L^g = \Delta^g_L - 2(n - 1),
\]

where \( \Delta^g_L \) is the Lichnerowicz Laplacian acting on symmetric 2-tensors given by

\[
\Delta^g_L u_{ij} = g^{kl} \nabla^g_k \nabla^g_l u_{ij} + 2 R^k_{ij} u_{kl} - R c^k_i u_{kj} - R c^k_j u_{ik}.
\]

Recall that \( \mathcal{S} \) denotes the bundle of symmetric 2-tensors. By Lemma 4.9 of [7], \( L^g \) defines an unbounded, closed, and self-adjoint operator on \( L^2(M; \mathcal{S}) \) with domain \( D(L^g) = H^2(M, \mathcal{S}) \). Note that if \( g = h \) is the model hyperbolic metric, then \( L^h = h^{kl} \nabla^h_k \nabla^h_l u_{ij} + 2 R^k_{ij} u_{kl} \), and by Lemma 14 of [4], \( \lambda - L^h \) is fully elliptic if \( \lambda \notin (-\infty, -\frac{(n-1)^2}{4}] \).

To understand the location of the essential spectrum, i.e., the set on which \( L^g \) is not Fredholm, one again needs to compute the indicial operator. We only summarize this computation, which can be found in sources [5, Lemma 2.9] and [11, Proposition 3]². The

²There is an apparent discrepancy between these references that traces back to the choice of basis for the bundle of symmetric 2-tensors; see the discussion on page 388 of [11] for a clear explanation. Note that our convention from page 7 most closely resembles [11]. Additionally we take \( \text{dim} M = n \) instead of \( n + 1 \).
action of $L^g$ on $S$ respects the decomposition of the bundle of symmetric 2-tensors into their traces, $\mathcal{G}$, and their trace-free parts, $S_0$:

$$S = \mathcal{G} \oplus S_0.$$ 

Further, the bundle $S_0$ of trace-free symmetric 2-tensors splits into three irreducible summands $\mathcal{V}_i$, corresponding to the normal $\mathcal{V}_1$, mixed $\mathcal{V}_2$, and purely tangential $\mathcal{V}_3$ components near the boundary relative to the decomposition given in equation (3.1). For each of these sub-bundles one computes an indicial operator. It turns out that the smallest interval centred about $\frac{n-1}{2}$ between the roots of these polynomials is always given by the indicial polynomial stemming from the action of $L^g$ on the bundle $\mathcal{V}_3$ of purely tangential trace-free tensor fields. Indeed, the indicial operator on $\mathcal{V}_3$ is

$$I_s(L^g|\mathcal{V}_3)\pi = (s^2 - (n-1)s)\pi,$$

where $\pi$ is a $\mathcal{C}^2$ section of $S|_{\partial M}$ extended arbitrarily into $M$. Thus the indicial roots are again $s = 0$ and $s = n-1$ and the indicial radius is $\Xi = \frac{n-1}{2}$. Theorem 2.3 implies that $L^g : \mathcal{C}^{\ell+2,\alpha}(M;S) \rightarrow \mathcal{C}^{\ell,\alpha}(M;S)$ is Fredholm for $\mu \in (0, n-1)$, with kernel equal to its $L^2$-kernel.

We again add a complex eigenvalue parameter $\lambda$. The next proposition documents the indicial roots of $\lambda - L^g$. Note that this proposition and proof is essentially a transcription of Proposition 3 of [11] to our setting; however, the operator $L^g$ differs by a sign according to our convention and we take $\dim M = n$ instead of $n+1$.

**Proposition 3.7.** For an asymptotically hyperbolic manifold $(M^n, g)$, the set of indicial roots of $\lambda - L^g$ corresponds to the pairs

$$\zeta_1^\pm = \frac{1}{2} \left( n - 1 \pm \sqrt{(n-1)^2 + 8(n-1) + 4\lambda} \right),$$

$$\zeta_2^\pm = \frac{1}{2} \left( n - 1 \pm \sqrt{(n-1)^2 + 4(n-1) + 4 + 4\lambda} \right),$$

$$\zeta_3^\pm = \frac{1}{2} \left( n - 1 \pm \sqrt{(n-1)^2 + 4\lambda} \right),$$

where $\zeta_i^\pm$ are the indicial roots of the operator on the sub-bundle $\mathcal{V}_i \subset S_0$ defined above, and the indicial roots of $\lambda - L^g$ on $\mathcal{G}$ are also given by $\zeta_1^\pm$.

Combining Lemma 3.4, Proposition 3.7, and Theorem 2.3 yields

**Proposition 3.8.** For an asymptotically hyperbolic metric $(M^n, g)$, and any integer $\ell \geq 0$, given $\mu \in (0, n-1)$, there exists a $\delta > 0$ so that for any $\lambda$ in the sector

$$S = \left\{ \lambda = re^{i\theta} \in \mathbb{C} : r > 0, \theta \in \left( -\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta \right) \right\}$$

the operator

$$\lambda - L^g : \mathcal{C}^{\ell+2,\alpha}(M;S) \rightarrow \mathcal{C}^{\ell,\alpha}(M;S)$$

is a Fredholm map. Moreover the kernel of $\lambda - L^g$ on $\mathcal{C}^{\ell+2,\alpha}(M;S)$ coincides with its $L^2$ kernel.
Consequently, in order to study the location of the spectrum of $L^g$ on weighted little Hölder spaces, we need to better understand the $L^2$ kernel. In general as a real and self-adjoint operator, $L^g$ has its spectrum confined to the real line, with the possibility of finitely many eigenvalues outside of $(-\infty, -(n-1)^2/4]$. So $\lambda - L^g$ has no $L^2$ kernel if $\lambda$ has nonzero imaginary part. Define

$$\lambda_0 = \sup_{v \in \mathcal{C}_c^\infty(M,S)} \frac{(v, L^g v)}{\|v\|^2_2}.$$  

(3.14)

It follows that $\lambda - L^g$ has no $L^2$ kernel on the translated sector $S + \lambda_0$, and so condition $S1$ of sectoriality Definition 2.1 holds on $S + \lambda_0$.

We are ready to finish the proof of the second part of Theorem 1, restated here:

**Theorem 1 (part 2)** For an asymptotically hyperbolic metric $(M^n, g)$, and any integer $\ell \geq 0$ and $\alpha \in (0,1)$, given $\mu \in (0, n-1)$, there exists a $\delta > 0$ and a sector of the form $S + \lambda_0$ so that $\Delta^2 - 2(n-1)$ on symmetric 2-tensors is sectorial in $\mathcal{C}^{\ell, \alpha}_{\mu}$.

**Proof.** The remarks immediately above show that property $S1$ for sectoriality holds on $S + \lambda_0$. We may assume the vertex $\omega$ is strictly positive by further translation of the sector, if necessary. Applying Lemma 3.1 and Proposition 3.3 we obtain the resolvent estimate $S2$ in this sector. Thus $L^g$ is sectorial in these weighted little Hölder spaces. \square

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