TRINITY OF THE EISENSTEIN SERIES

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Abstract. We study the weight 2 parabolic/elliptic/hyperbolic Eisenstein series, which gives a harmonic/polar harmonic/locally harmonic Maass form, simultaneously. Furthermore, by means of the hyperbolic Eisenstein series, we can redefine the hyperbolic Rademacher symbol introduced by Duke-Imamoğlu-Tóth. The symbol is expressed explicitly in terms of continued fraction coefficients of the corresponding two real quadratic irrationals.

1. Introduction

For each non-identity element \( \gamma \neq \pm I \) of the modular group \( \Gamma := \text{SL}_2(\mathbb{Z}) \), we define the Eisenstein series

\[
E_\gamma(z,s) := \sum_{Q \sim Q_0' \gamma} \frac{\text{sgn}(Q') y^s}{Q'(z,1)^s}, \quad z = x + iy \in \mathbb{H}, \text{Re}(s) > 0.
\]

This series has different aspects depending on \(|\text{tr}(\gamma)| = 2\), \(|\text{tr}(\gamma)| < 2\), or \(|\text{tr}(\gamma)| > 2\), that is, \( \gamma \) is parabolic, elliptic, or hyperbolic. If \( \gamma \) is parabolic, the series gives the weight 2 classical Eisenstein series

\[
E_2(z,s) := \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{(cz+d)^2 |cz+d|^{2s}}.
\]

Then the limit value as \( s \to 0^+ \) becomes the weight 2 harmonic Maass form \( E_2^*(z) \). Similarly, for the elliptic case, Bringmann-Kane [7] essentially showed that the value \( E_\gamma(z,0) \), after the analytic continuation, gives a polar harmonic Maass form of weight 2. On the other hand, in the hyperbolic case, \( E_\gamma(z,s) \) should not be a modular form. Our first result puts the last case into the same framework as the above two cases.

Theorem 1.1. (Theorem 3.4) For a hyperbolic element \( \gamma \), the modified Eisenstein series \( \tilde{E}_\gamma(z,s) \) defined in (3.7) is a modular form of weight 2, and \( E_\gamma(z,s) = \text{sgn}(\text{tr}(\gamma)) \tilde{E}_\gamma(z,s) \) for \( \text{Im}(z) \gg 0 \). Furthermore, after the analytic continuation, the value \( \tilde{E}_\gamma(z,0) \) is given by

\[
\tilde{E}_\gamma(z,0) = 2 \int_{\gamma z_0}^{\gamma z_0} \left( \frac{j'(z)}{j(\tau) - j(z)} - E_2^*(z) \right) \frac{d\tau}{Q_\gamma(\tau,1)},
\]

which is a locally harmonic Maass form of weight 2.

We take a step forward in the hyperbolic case by comparing with the parabolic case. Recently, inspired by Ghys [15], Duke-Imamoğlu-Tóth [13] introduced the hyperbolic Rademacher symbol \( \Psi_\gamma(\sigma) \) to study the linking numbers of modular links defined from two hyperbolic elements \( \gamma \) and \( \sigma \). This is a hyperbolic analogue of the original Rademacher symbol \( \Psi(\sigma) \) introduced in [33]. In this article, we redefine \( \Psi_\gamma(\sigma) \) by means of the cycle (geodesic) integral, and give two types of explicit formulas. Recalling that the original \( \Psi(\sigma) \) for a hyperbolic element \( \sigma \) is given by

\[
\Psi(\sigma) = \int_{z_0}^{\sigma z_0} E_2^*(z)dz,
\]

it is natural to consider the cycle integral

\[
\int_{z_0}^{\sigma z_0} E_\gamma(z,s)dz
\]
at \( s = 0 \) for two hyperbolic elements \( \gamma, \sigma \). However, since \( E_\gamma(z, s) \) is not a modular form, this cycle integral depends on the choice of \( z_0 \). Here by combining with Katok’s work [23], we define
\[
\tilde{\Psi}_\gamma(\sigma) := -\text{sgn}(\sigma) \left| \lim_{m \to \infty} \left( \int_{\sigma^{m+1}z_0} E_\gamma(z, s)dz + \int_{\epsilon^{-1}\sigma^{m+1}z_1} E_{\epsilon^{-1}\gamma}(z, s)dz \right) \right|_{s=0}.
\]
Here \( z_0 \) is an arbitrary element on the geodesic \( S_\sigma \) and we put \( \epsilon = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and \( z_1 = -z_0 \). In addition \( \text{sgn}(\sigma) = \text{sgn}(c(a + d)) \) for \( \sigma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \). Then we have the following results.

**Theorem 1.2.** (Theorem 4.7 and Theorem 4.12) Let \( \gamma, \sigma \) be primitive hyperbolic elements such that \( \sigma \) and \( \sigma^{-1} \) are not in the conjugacy class of \( \gamma \). Then
\[
\tilde{\Psi}_\gamma(\sigma) = -\frac{\sqrt{D_\gamma}}{2\pi i} \sum_{g \in \Gamma \setminus \Gamma/\Gamma_{\sigma}} \frac{1}{S_\gamma \cap \Gamma \setminus S_{\sigma}}
\]
where \( D_\gamma = \text{tr}(\gamma)^2 - 4 \).

**Theorem 1.3.** (Theorem 5.6) Let \( \gamma, \sigma \) be of the form
\[
\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2m-1} & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix},
\]
where both \( \sigma \) and \( \sigma^{-1} \) are not in the conjugacy class of \( \gamma \). Then there exists an explicit function \( \psi_\gamma(\sigma) \in \mathbb{Z} \) with \( 0 \leq \psi_\gamma(\sigma) \leq 2mn \) such that
\[
\tilde{\Psi}_\gamma(\sigma) = -\frac{4\pi i}{\sqrt{D_\gamma}} \left( \sum_{0 \leq i < 2n} \sum_{0 \leq j < 2m} \min(a_i, b_j) - \psi_\gamma(\sigma) \right).
\]

Theorem 1.2 gives an interpretation of \( \tilde{\Psi}_\gamma(\sigma) \) as the intersection numbers of two closed geodesics, and Theorem 1.3 makes \( \tilde{\Psi}_\gamma(\sigma) \) computable. In particular, the computability was one problem in Duke-Imamoğlu-Tóth [13]. In the above theorems, we assume that \( \gamma, \sigma \) are primitive for simplicity, but we can remove this assumption. The condition between \( \gamma \) and \( \sigma \) is the same as that in [13]. Recently, Rickards [35] gave another explicit formula for the combination of the hyperbolic Rademacher symbols.

Finally, as a corollary of our argument, we give one answer to the open question given in [32] on weight 0 analogue of the Eisenstein series. For each non-identity element \( \gamma \in \Gamma \) again, we define the weight 0 Eisenstein series
\[
E_{0,\gamma}(z, s) := \sum_{Q' \sim Q_\gamma} \frac{y^s}{|Q'(z, 1)|^s}, \quad z \in \mathbb{H}, \text{Re}(s) > 1,
\]
without the sign function. This type of Eisenstein series is recently studied by von Pippich et al. [20, 21, 31, 32, 36], and meromorphically continued to the whole \( s \)-plane. As an analogue of the Kronecker limit formula, in the elliptic case, the leading term of \( E_{0,\gamma}(z, s) \) at \( s = 0 \) is already known. As for the hyperbolic case, we derive the following limit formula.

**Theorem 1.4.** (Theorem 6.1) Let \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) be a primitive hyperbolic element with \( c > 0 \) and \( a + d > 2 \). Then we have
\[
E_{0,\gamma}(z, s) = -\frac{1}{2} \left[ -\int_{\tau_0}^{\tau} \log |j(\tau) - j(\tau)| \frac{d\tau}{\Lambda_{\gamma}(\tau, 1)} + \sum_{g \in \Gamma \setminus \Gamma} \frac{\arcsin^2 \left( \frac{1}{\cosh(d_{\text{hyp}}(g\tau, \mathbb{H}_z))} \right)}{s^2 + O(s^3)} \right].
\]

The rest of the article is organized as follows. We begin by presenting some basics about continued fractions. Then in Section 3 we introduce the precise definitions of the Eisenstein series, and the proof of Theorem 1.1. In Section 4, we first recall the classical results on the original Rademacher symbol \( \Psi(\sigma) \), and Duke-Imamoğlu-Tóth’s work. After that, we study our \( \tilde{\Psi}_\gamma(\sigma) \) and show Theorem 1.2. By combining with the theory of continued fractions, we prove Theorem 1.3 in Section 5. Finally, Theorem 1.4 is given in Section 6. The backgrounds, details, and the precise definitions are described in each section.

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2. Preliminaries: Continued Fractions

We begin by reviewing the theory of continued fractions. For any real number \( w \in \mathbb{R} \setminus \mathbb{Q} \), we have the following unique expression

\[
w = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},
\]

where \( a_j \in \mathbb{Z} \) with \( a_j \geq 1 \) for \( j \geq 1 \). This expression is called the (simple) continued fraction expansion of \( w \). In contrast, a rational number has two finite continued fraction expansions since

\[
[a_0, a_1, \ldots, a_n + 1] = [a_0, a_1, \ldots, a_n, 1].
\]

An ancient result of Euler and Lagrange asserts that a continued fraction of \( w \in \mathbb{R} \setminus \mathbb{Q} \) is periodic if and only if it is a real quadratic irrational. The expansion then has the form

\[
[w, \ldots, w, a_{n+1}, a_{n+2}, \ldots] = [k_0, \ldots, k_{r-1}, a_0, \ldots, a_{n-1}, a_0, \ldots, a_{n-1}, \ldots].
\]

In particular, the expression is purely periodic precisely when \( w \) is reduced, that is,

\[
w > 1, -1 < w' < 0
\]

where \( w' \) is the Galois conjugate of \( w \). For any real quadratic irrational \( w = [k_0, \ldots, k_{r-1}, a_0, \ldots, a_{n-1}] \), we define \( \delta_w \in \text{SL}_2(\mathbb{Z}) \) by

\[
\delta_w := \begin{cases} \begin{pmatrix} k_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{r-1} & 1 \\ 1 & 0 \end{pmatrix} & \text{if } r \text{ is even} \\ \begin{pmatrix} k_0 & 0 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} k_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } r \text{ is odd} \end{cases}
\]

Then \( \delta_w^{-1}w \) is also a real quadratic irrational, and has a purely periodic expansion. For compatibility with the actions of \( \text{SL}_2(\mathbb{Z}) \) instead of \( \text{GL}_2(\mathbb{Z}) \), we can assume that \( 2r \) and \( 2n \) are minimal even integers such that \( w = [k_0, \ldots, k_{2r-1}, a_0, \ldots, a_{2n-1}] \) without loss of generality. For an example case of \( w = (3 + \sqrt{5})/2 \), its continued fraction expansion is given by \( w = [2, 1, [1], \overline{1}], \) not \( w = [2, 1] \) here.

Let \( w = [k_0, \ldots, k_{2r-1}, a_0, \ldots, a_{2n-1}] \) be a real quadratic irrational with minimal \( r, n \in \mathbb{Z} \), and \( \Gamma = \text{SL}_2(\mathbb{Z}) \). Now we consider the stabilizer subgroup

\[
\Gamma_w := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \gamma w = \frac{aw + b}{cw + d} = w \right\}.
\]

If we take \( \delta_w \in \Gamma \) as above, then the equation \( \delta_w^{-1}w \Gamma_w \delta_w = \Gamma_w \delta_w^{-1}w \) holds. For the reduced \( \delta_w^{-1}w \), it is well-known that

\[
\Gamma_{\delta_w^{-1}w} := \{ \pm \gamma_w \mid n \in \mathbb{Z} \},
\]

where

\[
\gamma_w := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma
\]

is defined from the period of \( w \). Thus we reach the following lemma.

**Lemma 2.1.** For a real quadratic irrational \( w = [k_0, \ldots, k_{2r-1}, a_0, \ldots, a_{2n-1}] \) with minimal \( r, n \in \mathbb{Z} \), we put

\[
\delta_w := \begin{pmatrix} k_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{2r-1} & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_w := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma.
\]

Then the stabilizer \( \Gamma_w \) is given by

\[
\Gamma_w = \left\{ \pm \delta_w \gamma_w^n \delta_w^{-1} \mid n \in \mathbb{Z} \right\}.
\]

In particular, the group \( \Gamma_w/\{ \pm 1 \} \) is an infinite cyclic group.

Moreover, from the easy facts that

\[
\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} [a_0, a_1, \ldots] = [k, a_0, a_1, \ldots] \quad \text{and} \quad \det \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} = -1,
\]

we have the following lemma.
Lemma 2.2. For a reduced real quadratic irrational \( w = [a_0, \ldots, a_{2n-1}] \) with a minimal even length \( 2n \in 2\mathbb{Z} \), the \( \Gamma \)-orbit of \( w \) is given by

\[
\Gamma w = \bigcup_{r=0}^{\infty} \bigcup_{0 \leq i < 2n, r \bmod 2} A_{r,i},
\]

where

\[
A_{r,i} = \{ [k_0, \ldots, k_{r-1}, a_i, a_{r+1}, a_0, \ldots, a_{2n-1}] \mid k_0 \in \mathbb{Z}, k_1, \ldots, k_{r-1} \in \mathbb{Z}_{>0}, k_{r-1} \neq a_{i-1} \}.
\]

Finally, we show a useful proposition on the continued fraction expansions of the Galois conjugate \( w' \).

To give it, we prepare an easy lemma.

Lemma 2.3. Let \( a, k \in \mathbb{Z}, b \in \mathbb{Z}_{\neq -1} \) be integers, and \( w \in \mathbb{R} \). Then we have

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} w = [-1, 1, w-1],
\]

\[
\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} [-1, 1, a-1, w] = [k-a, 1, 1, w-1],
\]

\[
\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} [b, 1, w-1] = \begin{cases} [k, b, 1, w-1] & \text{if } b > -1, \\ [k-1, 1, -b-2, w] & \text{if } b < -1. \end{cases}
\]

Proof. We here give a proof for the first formula. By the definition, we see that

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} w = 1 \frac{1}{w} = -1 + \frac{1}{w} = -1 + \frac{1}{1 + \frac{1}{w}} = [-1, 1, w-1].
\]

In a similar manner, we can obtain the remaining formulas. \( \square \)

By using this lemma, we can compute the continued fraction expansion of the Galois conjugate \( w' \) for \( w = [k_0, \ldots, k_{r-1}, a_0, \ldots, a_{2n-1}] \) with \( k_{r-1} \neq a_{2n-1} \). By the well-known fact [1, Lemma 1.28] that

\[
S\alpha' = -\frac{1}{\alpha'} = [a_{2n-1}, a_1, a_0]
\]

for a reduced real quadratic irrational \( \alpha = [a_0, a_1, \ldots, a_{2n-1}] \), we have

\[
w' = \begin{pmatrix} k_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{r-1} & 1 \\ 1 & 0 \end{pmatrix} S^{-1} [a_0, a_1, \ldots, a_{2n-1}].
\]

When \( r = 0 \), by the first formula of Lemma 2.3,

\[
w' = S^{-1} [a_{2n-1}, \ldots, a_0] = [-1, 1, a_{2n-1} - 1, a_{2n-2}, \ldots, a_0, a_{2n-1}].
\]

When \( r = 1 \), by the second formula of Lemma 2.3,

\[
w' = [k_0 - a_{2n-1} - 1, 1, a_{2n-2} - 1, a_{2n-3}, \ldots, a_0, a_{2n-1}, a_{2n-2}].
\]

When \( r = 2 \), by the third formula of Lemma 2.3,

\[
w' = [k_0, k_1 - a_{2n-1} - 1, 1, a_{2n-2} - 1, a_{2n-3}, \ldots, a_0, a_{2n-1}, a_{2n-2}]
\]

if \( k_1 > a_{2n-1} \), and

\[
w' = [k_0 - 1, 1, a_{2n-1} - k_1 - 1, 1, a_{2n-2}, \ldots, a_0, a_{2n-1}]
\]

if \( k_1 < a_{2n-1} \). Repeating the process for \( r \geq 3 \), we obtain the proposition.

Proposition 2.4. Let \( w \) be a real quadratic irrational of the form \( w = [k_0, \ldots, k_{r-1}, a_0, \ldots, a_{2n-1}] \) with \( k_{r-1} \neq a_{2n-1} \). Then its Galois conjugate \( w' \) has the following continued fraction expansion:

\[
w' = \begin{cases} [-1, 1, a_{2n-1} - 1, a_{2n-2}, \ldots, a_0, a_{2n-1}] & \text{if } r = 0, \\ [k_0 - a_{2n-1} - 1, 1, a_{2n-2} - 1, a_{2n-3}, \ldots, a_0, a_{2n-1}, a_{2n-2}] & \text{if } r = 1, \\ [k_0, \ldots, k_{r-2}, k_{r-1} - a_{2n-1} - 1, 1, a_{2n-2} - 1, a_{2n-3}, \ldots, a_0, a_{2n-1}, a_{2n-2}] & \text{if } r \geq 2, \\ [k_0, \ldots, k_{r-3}, k_{r-2} - 1, 1, a_{2n-1} - k_{r-1} - 1, a_{2n-2}, \ldots, a_0, a_{2n-1}, a_{2n-2}] & \text{if } r \geq 2, \end{cases}
\]

where the third line holds if \( k_{r-1} > a_{2n-1} \), and the fourth line holds if \( 0 < k_{r-1} < a_{2n-1} \). Moreover, if an inner entry becomes 0, then we regard as

\[
[\ldots, a_0, 0, b, \ldots] = [\ldots, a + b, \ldots].
\]
Example 2.5. Let \( w = [2, 1, 1, 4, 3, 2] = \frac{36 + 2\sqrt{39}}{19} \). Then by the fourth line of Proposition 2.4, we have
\[
 w' = [2 - 1, 1, 2 - 1, \frac{3}{4}, 1, 2] = [1, 1, 0, 3, 4, 1, 2].
\]
Now a 0-entry appears, so that this equals
\[
 w' = [1, 4, 4, 1, 2, 3] = \frac{36 - 2\sqrt{39}}{19}.
\]

3. Eisenstein Series of Weight 2

Let \( \mathbb{H} \) be the upper half-plane, and \( \Gamma \) := \( \text{SL}_2(\mathbb{Z}) \) the modular group. Let \( Q \) be the set of binary quadratic forms
\[
 Q(X, Y) = [A, B, C] = AX^2 + BXY + CY^2
\]
with integral coefficients \( A, B, C \in \mathbb{Z} \). The group \( \Gamma \) also acts on this set \( Q \) by
\[
 Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} (X, Y) := Q(aX + bY, cX + dY).
\]

Two quadratic forms \( Q \) and \( Q' \) are said to be \( \Gamma \)-equivalent, and we write \( Q \sim Q' \) if there is an element \( g \in \Gamma \) such that \( Q' = Q \circ g \). For each element \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), we define the corresponding quadratic form by
\[
 Q_\gamma(X, Y) := cX^2 + (d - a)XY - bY^2.
\]
The roots of the polynomial \( Q_\gamma(z, 1) \) are the fixed points of the action of \( \gamma \), and the equations
\[
 Q_{\gamma^{-1}}(z, 1) = (Q_\gamma \circ g)(z, 1) = j(g, z)^2 Q_\gamma(gz, 1)
\]
hold for any \( g \in \Gamma \). Here we put \( j(g, z) := cz + d \) for any \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and \( z \in \mathbb{C} \). For each quadratic form \( [A, B, C] \in Q \), we put \( \text{sgn}([A, B, C]) := \text{sgn}(A) \in \{-1, 0, 1\} \). Moreover, for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), we put \( \text{sgn}(\gamma) := \text{sgn}(c(a + d)) \in \{-1, 0, 1\} \), that is, \( \text{sgn}(\gamma) = \text{sgn}(Q_\gamma) \text{sgn}(\text{tr}(\gamma)) \). Throughout this article, we take the principal branch of log-function by \( \text{Im} \log z \in (-\pi, \pi] \). Under these notations, we introduce the following function.

Definition 3.1. Let \( \pm I \neq \gamma \in \Gamma \) be a non-identity element. For \( z \in \mathbb{H} \) and \( \text{Re}(s) > 0 \), we define the real analytic Eisenstein series
\[
 E_\gamma(z, s) := \sum_{Q \prec Q_\gamma} \frac{\text{sgn}(Q')}{|Q'(z, 1)|^s}.
\]

This series converges absolutely and locally uniformly for \( \text{Re}(s) > 0 \) and \( z \in \mathbb{H} \). Since \( Q_{-\gamma}(X, Y) = Q_{-1}(X, Y) = -Q_\gamma(X, Y) \), the relation \( E_{\gamma^{-1}}(z, s) = E_\gamma(z, s) \) holds. Our goal in this section is to analyze its modular behavior at \( s = 0 \) after the analytic continuation. Now we divide the study into three cases, (i) parabolic, (ii) elliptic, and (iii) hyperbolic.

3.1. Parabolic Case. This case is very well-known. A good reference for the basic theory of the modular group \( \text{SL}_2(\mathbb{Z}) \) is [28, Section 4.1]. An element \( \gamma \in \Gamma \) is said to be parabolic if \( \gamma \neq \pm I \) and \( |\text{tr}(\gamma)| = 2 \). The parabolic element \( \gamma \) has the unique fixed point \( w_\gamma \) in \( \mathbb{Q} \cup \{i\infty\} \), and we can take \( M_\gamma \in \Gamma \) such that \( M_\gamma i\infty = w_\gamma \). Since the stabilizer subgroup \( \Gamma_{\infty} := \{ g \in \Gamma \mid g i\infty = i\infty \} \) is given by \( \Gamma_{\infty} = \{ \pm T^n \mid n \in \mathbb{Z} \} \), we have the expression \( \gamma = \pm M_\gamma T^n M_\gamma^{-1} \) for some \( n \in \mathbb{Z} \). By these facts, we can easily check that \( E_\gamma(z, s) = E_{T^n}(z, s) \), so that our problem is reduced to the case of \( \gamma = T^n \) for a positive integer \( n > 0 \). Then the Eisenstein series is expressed as
\[
 E_{T^n}(z, s) = \sum_{g \in \Gamma_{\infty} \setminus \Gamma} \frac{\text{sgn}(Q_{g^{-1}T^n g}) y^s}{|Q_{g^{-1}T^n g}(z, 1)|^s}.
\]
For \( g = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \Gamma_{\infty} \setminus \Gamma \), we see that \( g^{-1}T^n g = \begin{pmatrix} 1+c n & d n^2 \\ -c^2 n & 1-c d n \end{pmatrix} \), that is, \( Q_{g^{-1}T^n g}(z, 1) = -n(cz + d)^2 = -nj(g, z)^2 \). This gives the expression
\[
 E_{T^n}(z, s) = \frac{1}{n^{s+1}} \sum_{g \in \Gamma_{\infty} \setminus \Gamma} \frac{\text{Im}(g z)^s}{j(g, z)^s}.
\]
By using Hecke’s trick (see [6, Proof of Lemma 6.2]) or the Fourier expansion, we get the analytic continuation of this function to \( s = 0 \). This gives a harmonic Maass form of weight 2 on \( \Gamma \) of the form

\[
\lim_{s \to 0^+} E_{rs}(z, s) = \frac{1}{m} \left( 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m) q^m - \frac{3}{ry} \right) = \frac{1}{m} E_2^*(z),
\]

where \( q := e^{2\pi i \tau} \) and \( \sigma_1(m) := \sum_{d \mid m} d \) is the divisor sum. Here a real analytic function \( f : \mathbb{H} \to \mathbb{C} \) is called a harmonic Maass form of weight \( k \) on \( \Gamma \) if

1. \( j(g, z)^{-k} f(gz) = f(z) \) for any \( g \in \Gamma \),
2. \( \Delta_k f(z) := \left( -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i k y \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \right) f(z) = 0 \),
3. \( f(x + iy) = O(y^k) \) as \( y \to \infty \) for some \( \alpha > 0 \).

### 3.2. Elliptic Case

An element \( \gamma \in \Gamma \) is said to be elliptic if \( | \text{tr}(\gamma) | < 2 \). The elliptic element \( \gamma \) has two fixed points \( w_+ \in \mathbb{H} \) and its complex conjugate. The remarkable fact is that any elliptic fixed point \( w_+ \) is \( \Gamma \)-equivalent to either \( i \) or \( \omega := e^{2\pi i/3} \), that is, \( gw_+ = i \) or \( \omega \) holds for some \( g \in \Gamma \). Moreover the stabilizer subgroup \( \Gamma_{w_+} \) for \( w_+ = i, \omega \) is a finite cyclic group given by \( \Gamma_i = \{ S^n : (0 \ 1 \ 
 1 \ 0) \mid 0 \leq n \leq 3 \} \) and \( \Gamma_\omega = \{ U^n : (1 \ -1 \ 0 \ 1)^n \mid 0 \leq n \leq 5 \} \). Therefore, in this case also, our problem is reduced to the cases of \( \gamma = S \) and \( U \).

For \( \gamma = S \), the Eisenstein series is expressed as

\[
E_S(z, \tau) = \frac{1}{|\Gamma|^2} \sum_{g \in \Gamma} \text{sgn}(Q_{g^{-1} Sg}) y^s \frac{\text{Im}(gz)^s}{|j(gz)^2(gz - \tau)(gz - \tau')|^{\sigma} |j(gz - \tau)(gz - \tau')|^{\sigma^*}}.
\]

Here \( \text{sgn}(Q_{g^{-1} Sg}) \) is always positive. In general, for any \( z, \tau \in \mathbb{H} \) which are \( \Gamma \)-inequivalent each other, we introduce a new function

\[
E(z, \tau, s) := \frac{\text{Im}(\tau)^{s+1} \text{Im}(gz)^s}{|j(gz)^2(gz - \tau)(gz - \tau')|^{\sigma} |j(gz - \tau)(gz - \tau')|^{\sigma^*}}, \quad \text{Re}(s) > 0.
\]

Then our Eisenstein series is expressed as \( E(z, \tau, s) = \text{Im}(w_+)^{-s-1} E(z, \tau, s)/|\Gamma_{w_+}| \) for \( \gamma = S, U \). In particular, this function \( E(z, \tau, s) \) satisfies the modular transformation laws

\[
j(gz)^{-2} E(gz, \tau, s) = E(z, \tau, s)
\]

and \( E(z, \tau, s) = E(z, \tau, s) \) for any \( g \in \Gamma \). This immediately follows from \( j(gh, z) = j(h, gz)j(h, z) \) and \( j(g, \tau)(z - g\tau) = j(g^{-1}, z)(z - \tau) \).

Here we give the analytic continuation of \( E(z, \tau, s) \) to \( s = 0 \) generally, which was essentially realized by Bringmann-Kane [7] and others [9] as follows. First, Bringmann-Kane considered a slightly different function

\[
\mathcal{P}(z, \tau, s) := \frac{\text{Im}(\tau)^{s+1} \text{Im}(gz)^s}{|j(gz)^2(gz - \tau)(gz - \tau')|^{\sigma} |j(gz - \tau)(gz - \tau')|^{\sigma^*}}, \quad \text{Re}(s) > 0.
\]

After a lengthy calculation, they derived the analytic continuation of \( \mathcal{P}(z, \tau, s) \) to \( s = 0 \), and the following explicit formula.

\[
\mathcal{P}(z, \tau, 0) = -2\pi \left( \frac{j'(z)}{j(\tau) - j(z)} - E_2^*(z) \right),
\]

where \( j(z) \) is the elliptic modular \( j \)-function defined by

\[
j(z) := \frac{\left( 1 + 240 \sum_{d=1}^{\infty} \frac{d^3 q^d}{1 - q^d} \right)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} = q^{-1} + 744 + 196884q + \cdots,
\]

and \( ' = (2\pi i)^{-1} d/dz \). This function (3.4) gives a polar harmonic Maass form of weight 2 on \( \Gamma \), which is a harmonic Maass form with poles in the upper half-plane. Next, we consider the difference for \( \text{Re}(s) > 0 \),

\[
E(z, \tau, s) - \text{Im}(z)^s \mathcal{P}(z, \tau, s)
\]

satisfies

\[
= \sum_{g \in \Gamma} j(gz)^2 j(gz)^{2s} (gz - \tau)(gz - \tau') (gz - \tau)^{-s} \left( \frac{gz - \tau}{gz - \tau'} - 1 \right).
\]
To estimate the last term, we show the following lemma.

Lemma 3.2. Let $K$ be a compact subset in $\text{Re}(s) > -2$. For a fixed $z, \tau \in \mathbb{H}$ and any $s \in K$, there exists a constant $C = C(\tau, K)$ such that

$$\left| \frac{gz - \tau}{\overline{gz - \tau}} \right|^s - 1 \leq C \frac{\text{Im}(gz)}{|gz - \tau|^2}.$$  

Proof. For any $z, \tau \in \mathbb{H}$, we easily get

$$\left| \frac{z - \tau}{z - \overline{\tau}} \right|^2 = 1 - \frac{4 \text{Im}(z) \text{Im}(\tau)}{|z - \overline{\tau}|^2} =: 1 - a(z).$$

For any $\varepsilon > 0$, the region for $a(z) \geq \varepsilon$ is a compact subset in $\mathbb{H}$. Thus we have $a(gz) < 1/2$ for all but finitely many $g \in \Gamma$ since $\Gamma$ acts properly discontinuously on $\mathbb{H}$. For such $g$, by applying Taylor’s theorem to $(1 - r)^{-s/2}$ at $r = 0$, we have

$$\left| \frac{gz - \tau}{\overline{gz - \tau}} \right|^s = (1 - a(gz))^{-s/2} = 1 + \frac{8}{2}(1 - c)^{-s/2 - 1}a(gz)$$

for some $0 < c < a(gz) < 1/2$. Therefore, for $\text{Re}(s) > -2$ we have

$$\left| \frac{gz - \tau}{\overline{gz - \tau}} \right|^s - 1 \leq |s|^{2\text{Re}(s)/2}a(gz).$$

This concludes the proof.  

By this lemma, the right-hand side of (3.5) is analytically continued to $s = 0$, and has a zero at $s = 0$. Thus we get the following result.

Theorem 3.3. Let $z, \tau \in \mathbb{H}$ be $\Gamma$-inequivalent elements. Then the function $E(z, \tau; s)$ is analytically continued to $s = 0$, and equals

$$\lim_{s \to 0^+} E(z, \tau; s) = -2\pi \left( \frac{j'(z)}{j(\tau) - j(z)} - E_2^*(z) \right),$$

which gives a polar harmonic Maass form of weight 2 on $\Gamma$ in $z$, whose poles are located on the $\Gamma$-orbit of $\tau$.

As an conclusion of this theorem, we immediately see that

$$\lim_{s \to 0^+} E_S(z, s) = \pi \left( \frac{j'(z)}{j(z) - 1728} + E_2^*(z) \right), \quad \lim_{s \to 0^+} E_U(z, s) = \frac{2\pi}{3\sqrt{3}} \left( \frac{j'(z)}{j(z)} + E_2^*(z) \right).$$

Finally, We note that Asai-Kaneko-Ninomiya [2] showed the Fourier expansion

$$j_m(z) = \sum_{m=0}^{\infty} j_m(\tau)q^m, \quad \text{Im}(z) > \text{Im}(\tau),$$

where $j_m(z)$ is the unique polynomial in $j(z)$ having a Fourier expansion of the form $q^{-m} + O(q)$. The set $\{j_m(z)\}_{m=0}^{\infty}$ consists a natural basis for the space $M_0^!(\Gamma)$ of weakly holomorphic modular forms of weight 0 on $\Gamma$, and form a Hecke system in the sense of [9]. Combining with the Fourier expansion of $E_2^*(z)$, we get the expansion

$$\frac{j'(z)}{j(\tau) - j(z)} - E_2^*(z) = \sum_{m=1}^{\infty} (j_m(\tau) + 24\sigma_1(m))q^m + \frac{3}{\pi y}, \quad \text{Im}(z) > \text{Im}(\tau).$$

Here the function $\sum_{m=1}^{\infty} (j_m(\tau) + 24\sigma_1(m))q^m$ is also a natural basis in the sense of Niebur’s Poincaré series. Actually, the Niebur Poincaré series $G_m(z, s)$ defined in [12, p.969] satisfies $G_m(z, 0) = j_m(z) + 24\sigma_1(m)$.  

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3.3. Hyperbolic Case. This case is slightly different from the above two cases. An element \( \gamma \in \Gamma \) is said to be hyperbolic if \( |\text{tr}(\gamma)| > 2 \). The hyperbolic element \( \gamma \) has two fixed points \( w_\gamma, w'_\gamma \), which are real quadratic irrationals and Galois conjugate each other. In this article, we always order them by \( w_\gamma > w'_\gamma \). We call \( \gamma \) primitive if \( \Gamma_{w_\gamma} = \{ \pm \gamma^n \mid n \in \mathbb{Z} \} \) holds. The different point is that the Eisenstein series \( E_\gamma(z, s) \) does not satisfy the modular transformation laws. Actually, this immediately follows from the expression

\[
E_\gamma(z, s) = \sum_{g \in \Gamma \setminus \mathbb{H}} \frac{\text{sgn}(Q_{g^{-1}\gamma g}) \text{Im}(z)^s}{Q_{g^{-1}\gamma g}(z, 1)|Q_{g^{-1}\gamma g}(z, 1)|^s},
\]

and the fact \( \text{sgn}(Q_{g^{-1}\gamma g}) \neq \text{sgn}(Q_{gh^{-1}\gamma gh}) \) in general. For more details, see Lemma 4.9 in the next section.

To recover the modularity of the Eisenstein series, we define the sign function \( \chi_\gamma(z) \) as follows. For a hyperbolic element \( \gamma \), we denote by \( S_\gamma \) the oriented geodesic in \( \mathbb{H} \) connecting two fixed points \( w_\gamma \) and \( w'_\gamma \). Here the orientation is clockwise if \( \text{sgn}(\gamma) = \text{sgn}(Q_{z, \gamma}) \text{sgn}(\text{tr}(\gamma)) > 0 \), and counter-clockwise if \( \text{sgn}(\gamma) < 0 \). Then the geodesic \( S_\gamma \) separates the upper half-plane as \( \mathbb{H} = A^+ \cup S_\gamma \cup A^- \), where \( A^- \) is the bounded region. The sign function \( \chi_\gamma(z) \) on \( \mathbb{H} \setminus S_\gamma \) is defined by

\[
\chi_\gamma(z) := \text{sgn}(\gamma) \times \begin{cases} +1 & \text{if } z \in A^+ \setminus S_\gamma, \\ -1 & \text{if } z \in A^- \setminus S_\gamma. \end{cases}
\]

This satisfies \( \chi_{g^{-1}\gamma g}(z) = \chi_g(z) \) for any \( g \in \Gamma \). Then we define the modified function

\[
\tilde{E}_\gamma(z, s) := \sum_{g \in \Gamma \setminus \mathbb{H}} \frac{\chi_{g^{-1}\gamma g}(z) \text{Im}(z)^s}{Q_{g^{-1}\gamma g}(z, 1)|Q_{g^{-1}\gamma g}(z, 1)|^s}, \quad \text{Re}(s) > 0
\]

\[(3.7)\]

This is defined on \( \mathbb{H} \setminus (\bigcup_{g \in \Gamma} S_{g^{-1}\gamma g}) \) and satisfies the modular transformation law \( j(g, z)^{-2} \tilde{E}_\gamma(gz, s) = \tilde{E}_\gamma(z) \) for any \( g \in \Gamma \). In particular, since \( \chi_{g^{-1}\gamma g}(z) = \text{sgn}(Q_{g^{-1}\gamma g}) \text{sgn}(\text{tr}(\gamma)) \) holds for \( z \in \mathbb{H} \) with large enough imaginary part, \( E_\gamma(z, s) = \text{sgn}(\text{tr}(\gamma)) \tilde{E}_\gamma(z, s) \) on the connected component \( \bigcap_{g \in \Gamma} A^+_{g^{-1}\gamma g} \). In this sense, we regard this function as a modular completion of the Eisenstein series \( E_\gamma(z, s) \).

The goal of this part is the following theorem paired with Theorem 3.3.

**Theorem 3.4.** Let \( \gamma \in \Gamma \) be a primitive hyperbolic element with \( \text{sgn}(Q_\gamma) > 0 \) and \( \text{tr}(\gamma) > 2 \). For \( z \in \mathbb{H} \setminus (\bigcup_{g \in \Gamma} S_{g^{-1}\gamma g}) \), the function \( \tilde{E}_\gamma(z, s) \) is analytically continued to \( s = 0 \), and equals

\[
\lim_{s \to 0^+} \tilde{E}_\gamma(z, s) = 2 \int_{\gamma_0}^{\gamma_0'} \left( \frac{j'(\tau)}{j(\tau) - j(z)} - E_2^*(z) \right) \frac{d\tau}{Q_2(\tau, 1)},
\]

which gives a locally harmonic Maass form of weight 2 on \( \Gamma \). Moreover, \( \tilde{E}_\gamma(z, s) = \tilde{E}_{\gamma^{-1}}(z, s) = -\tilde{E}_{-\gamma}(z, s) \) and

\[
\tilde{E}_{\gamma^n}(z, s) = \frac{1}{a_n} \tilde{E}_\gamma(z, s),
\]

hold for a positive integer \( n > 0 \), where the sequence \( \{a_n\}_{n=1}^\infty \) is defined by \( a_n = \text{tr}(\gamma)a_{n-1} - a_{n-2}, a_0 = 0, a_1 = 1 \).

Here a locally harmonic Maass form is a harmonic Maass form with jumping singularity on the net of geodesics in \( \mathbb{H} \), introduced by Bringmann-Kane-Kohnen [8]. Actually, the function in the above theorem is modular form of weight 2, annihilated by the action of \( \Delta_2 \), and has jumping singularities on geodesics.

To obtain the analytic continuation of \( \tilde{E}_\gamma(z, s) \) to \( s = 0 \), we again use the function \( E(z, \tau; s) \) defined in (3.3), and its cycle integral

\[
\int_{\gamma_0}^{\gamma_0'} E(z, \tau; s) \frac{d\tau}{Q_2(\tau, 1)}.
\]

Here the path of integration is in the geodesic \( S_\gamma \). Since \( E(z, \tau; s) \) is \( \Gamma \)-invariant in \( \tau \), this integral is independent of the choice of \( \gamma_0 \in S_\gamma \). This type of integral was essentially appeared in Hecke [18] as the constant term in a hyperbolic Fourier expansion, and studied by Kaneko [22], Duke-Imamoglu-Töth
First, we assume that $\gamma$ is primitive with $\text{sgn}(Q_\gamma) > 0$ and $\text{tr}(\gamma) > 2$. By the usual unfolding argument, for $\text{Re}(s) > 0$ we have

$$
\int_{\gamma_0}^{\gamma_0} E(z, \tau; s) \frac{d\tau}{Q_\gamma(\tau, 1)} = 2 \sum_{g \in \Gamma_w \setminus \Gamma_n \in \mathbb{Z}} \sum_{j} j(\gamma^ng, z)^2 (\gamma^ngz - \tau)(\gamma^ngz - \bar{\tau})(\gamma^ngz - \tau)(\gamma^ngz - \bar{\tau})^s Q_\gamma(\tau, 1)
$$

Then we obtain

$$
E_\gamma(z, s) + \frac{1}{\pi} \left( \frac{2}{\sqrt{D_\gamma}} \right) s \int_{\gamma_0}^{\gamma_0} E(z, \tau; s) \frac{d\tau}{Q_\gamma(\tau, 1)} = \sum_{g \in \Gamma_w \setminus \Gamma_n \in \mathbb{Z}} \chi_\gamma(gz) \text{Im}(gz)^s \left( 1 + \frac{2\chi_\gamma(gz)}{\pi} \int_{S_\gamma} \frac{\text{Im}(\tau)Q_\gamma(gz, 1)}{(gz - \tau)(gz - \bar{\tau})} d\tau \right) \frac{2\text{Im}(\tau)Q_\gamma(gz, 1)}{\sqrt{D_\gamma}(gz - \tau)(gz - \bar{\tau})} \frac{d\tau}{Q_\gamma(\tau, 1)},
$$

where we put $D_\gamma := \text{tr}(\gamma)^2 - 4$. The first step is to show the following lemma.

**Lemma 3.5.** The integral in (3.9) converges for $\text{Re}(s) > -1$, and at $s = 0$ we have

$$
\frac{2\chi_\gamma(gz)}{\pi} \int_{S_\gamma} \frac{\text{Im}(\tau)Q_\gamma(gz, 1)}{(gz - \tau)(gz - \bar{\tau})} d\tau \frac{d\tau}{Q_\gamma(\tau, 1)} = -1.
$$

**Proof.** Let $M_\gamma \in \text{SL}_2(\mathbb{R})$ be the scaling matrix defined by

$$
M_\gamma := \frac{1}{\sqrt{w_\gamma - w'_\gamma}} \begin{pmatrix} w_\gamma & w'_\gamma \\ 1 & 1 \end{pmatrix},
$$

which sends 0 and $i\infty$ to $w'_\gamma$ and $w_\gamma$, respectively. This diagonalizes $\gamma$ by

$$
M_\gamma^{-1} \gamma M_\gamma = \begin{pmatrix} j(\gamma, w_\gamma) & 0 \\ 0 & j(\gamma, w'_\gamma) \end{pmatrix} =: \begin{pmatrix} \xi_\gamma & 0 \\ 0 & \xi^{-1}_\gamma \end{pmatrix},
$$

By putting $\tau = M_\gamma \xi_\gamma$, the left-hand side of the statement equals

$$
\frac{2\chi_\gamma(gz)}{\pi} \int_0^{\infty} \frac{\text{Im}(M_\gamma \xi_\gamma)Q_\gamma(gz, 1)}{(gz - M_\gamma \xi_\gamma)(gz - M_\gamma(-i))} \left| \frac{2\text{Im}(M_\gamma \xi_\gamma)Q_\gamma(gz, 1)}{\sqrt{D_\gamma}(gz - M_\gamma \xi_\gamma)(gz - M_\gamma(-i))} \right|^s idt...\text{integrated over $0 < \tau < \infty$}.
$$

Since $Q_{M_\gamma^{-1}}(X, Y) = -\sqrt{D_\gamma}XY$, this becomes

$$
\frac{2^{s+1}}{\pi} \chi_{M_\gamma^{-1}}(M_\gamma^{-1}gz) \int_0^{\infty} \frac{tM_\gamma^{-1}gz}{(M_\gamma^{-1}gz - it)(M_\gamma^{-1}gz + it)} \left| \frac{tM_\gamma^{-1}gz}{(M_\gamma^{-1}gz - it)(M_\gamma^{-1}gz + it)} \right|^s dt
$$

This integral converges for $\text{Re}(s) > -1$, and at $s = 0$ this equals

$$
-\frac{\text{sgn}(\text{Re}(M_\gamma^{-1}gz))}{\pi i} \int_0^{\infty} \left( \frac{1}{t - iM_\gamma^{-1}gz} - \frac{1}{t + iM_\gamma^{-1}gz} \right) dt = -1.
$$

Here we use the equation $\chi_{M_\gamma^{-1}}(M_\gamma z) = \chi_\gamma(M_\gamma z) = -\text{sgn}(\text{Re}(z))$. □
On reflection of this lemma, the equation (3.9) is rewritten as

\[(3.12)\]

\[E_\gamma(z, s) + \frac{1}{\pi} \left( \frac{2}{\sqrt{D_\gamma}} \right)^s \int_{\tau_0}^{\gamma s} E(z, \tau; s) \frac{d\tau}{Q_\gamma(\tau, 1)} = \frac{2}{\pi} \sum_{g \in \Gamma_{w_\gamma} \setminus \Gamma} \text{Im}(g z)^s \int_{S_\gamma} \left( \frac{\text{Im}(\tau) Q_\gamma(g z, 1)}{\sqrt{D_\gamma (g z - \tau)(g z - \tau^*)}} \right)^s \left( 1 - \frac{2}{\sqrt{D_\gamma (g z - \tau)(g z - \tau^*)}} \right)^{-s} \frac{d\tau}{Q_\gamma(\tau, 1)}.\]

The next step is to evaluate the inner difference similar as Lemma 3.2. We denote by \(d_{\text{hyp}}\) the metric on the upper half-plane \(\mathbb{H}\). This metric is defined by

\[(3.13)\]

\[d_{\text{hyp}}(z, \tau) = \text{arcosh} \left( 1 + \frac{|z - \tau|^2}{2 \text{Im}(z) \text{Im}(\tau)} \right) = \text{arsinh} \left( \frac{|(z - \tau)(z - \tau^*)|}{2 \text{Im}(z) \text{Im}(\tau)} \right),\]

and satisfies \(d_{\text{hyp}}(g z, g \tau) = d_{\text{hyp}}(z, \tau)\) for any \(g \in \text{SL}_2(\mathbb{R})\). Then we have the following lemma.

**Lemma 3.6.** Let \(\gamma \in \Gamma\) be an hyperbolic element, and \(D_\gamma := \text{tr}(\gamma)^2 - 4\) the discriminant of \(Q_\gamma(X, Y)\). Then we have

\[|Q_\gamma(z, 1)| = \sqrt{D_\gamma} \text{Im}(z) \cosh(d_{\text{hyp}}(z, S_\gamma)),\]

where \(S_\gamma\) is the geodesic in \(\mathbb{H}\) connecting two fixed points \(w_\gamma > w'_\gamma\) of \(\gamma\).

**Proof.** This lemma is described in [36, Lemma 2.5.4]. We now review his proof. By [4, Theorem 7.9.1 (ii)], we have

\[
\cosh(d_{\text{hyp}}(z, S_\gamma)) = \cosh(d_{\text{hyp}}(M_\gamma^{-1} z, \mathcal{I})) = \frac{1}{\sin \text{arg}(M_\gamma^{-1} z)} = \frac{|M_\gamma^{-1} z|}{\text{Im}(M_\gamma^{-1} z)} = \frac{|Q_{M_\gamma^{-1}}(M_\gamma^{-1} z, 1)|}{\sqrt{D_\gamma} \text{Im}(M_\gamma^{-1} z)},
\]

where \(\mathcal{I}\) is the \(y\)-axis. \(\square\)

By this lemma, we get

\[
\frac{2}{\sqrt{D_\gamma}} \left( \frac{\text{Im}(\tau) Q_\gamma(g z, 1)}{\sqrt{D_\gamma (g z - \tau)(g z - \tau^*)}} \right) = \frac{\cosh(d_{\text{hyp}}(g z, S_\gamma))}{\sinh(d_{\text{hyp}}(g z, \tau))}.
\]

Since \(\tau \in S_\gamma\), this function is bounded by \(1/\tanh(d_{\text{hyp}}(g z, S_\gamma))\). Furthermore, by a similar argument as in the proof of Lemma 3.2, since the geodesic segment \(\gamma_0 \tau \gamma_0^{-1}\) is compact, we have

\[0 < b(g z) := \frac{\cosh(d_{\text{hyp}}(g z, S_\gamma))}{\sinh(d_{\text{hyp}}(g z, \tau))} - 1 < \frac{1}{\tanh(d_{\text{hyp}}(g z, S_\gamma))} - 1 < \frac{1}{2}\]

for all but finitely many \(g \in \Gamma_{w_\gamma} \setminus \Gamma\). Therefore,

\[
1 - \frac{2}{\sqrt{D_\gamma}} \left( \frac{\text{Im}(\tau) Q_\gamma(g z, 1)}{\sqrt{D_\gamma (g z - \tau)(g z - \tau^*)}} \right)^{-s} \approx |1 + b(g z)|^{-s} - 1 \ll b(g z) < \frac{1}{\sinh(d_{\text{hyp}}(g z, \tau))}.
\]

This gives the analytic continuation of (3.12) to \(s = 0\), and shows that the right-hand side of (3.12) has a zero at \(s = 0\). Combining with Theorem 3.3, we get the first part of Theorem 3.4.

Finally we show the second part of Theorem 3.4. For a primitive \(\gamma\) with \(\text{sgn}(Q_\gamma) > 0\) and \(\text{tr}(\gamma) > 2\), and a positive integer \(n > 1\), we want to consider the function

\[
\tilde{E}_{\gamma n}(z, s) = \sum_{g \in \Gamma_{w_\gamma} \setminus \Gamma} \frac{\chi_{\gamma n}(g z) \text{Im}(z)^s}{j(g z)^2 Q_{\gamma n}(g z, 1)|Q_{\gamma n}(g z, 1)|^s}.
\]
Here $\gamma^n$ has the same fixed points as $\gamma$, so that $w_{\gamma^n} = w_{\gamma}$ and $\chi_{\gamma^n}(z) = \chi_{\gamma}(z)$ hold. By (3.11), we see that

$$Q_{\gamma^n}(z, 1) = j(M_\gamma^{-1}, z)^2Q_{M_\gamma^{-1}\gamma^n M_\gamma}(M_\gamma^{-1}z, 1) = j(M_\gamma^{-1}, z)^2(\xi_\gamma^n - \xi_\gamma^n)M_\gamma^{-1}z$$

$$= \frac{\xi_\gamma^n - \xi_\gamma^n}{\xi_\gamma - \xi_\gamma} Q_{\gamma}(z, 1).$$

This coefficient satisfies the recurrence relation $a_n = \operatorname{tr}(\gamma)a_{n-1} - a_{n-2}, a_0 = 0, a_1 = 1$. This concludes the proof.

4. Hyperbolic Rademacher Symbol

4.1. Classical Rademacher Symbol. Let $\Delta(z)$ be the discriminant function defined by $\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}$. This function is a weight 12 cusp form on $\Gamma := \text{SL}_2(\mathbb{Z})$. We now consider the function

$$\delta(z) = 2\pi iz - 24 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{mz}}{mn}.$$ 

Since this $\delta(z)$ satisfies $e^{\delta(z)} = \Delta(z)$, there exists an integer-valued function $\Phi : \Gamma \to \mathbb{Z}$ such that

(4.1) $$\delta(\sigma z) - \delta(z) = 6 \log(-(cz + d)^2) + 2\pi i \Phi(\sigma)$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Here $z$ is in the principal branch $\text{Im} \log z \in (-\pi, \pi]$. This function is so-called the Dedekind symbol, originally studied by Dedekind [11]. Let $s(a, c) := \sum_{k=1}^{\infty} \left( \frac{k}{c} \right) \left( \frac{ak}{c} \right)$ be the Dedekind sum with $(x) := x - \lfloor x \rfloor - 1/2$ if $x \in \mathbb{R} \setminus \mathbb{Z}$ and zero otherwise, then the Dedekind symbol $\Phi(\sigma)$ is expressed as

$$\Phi(\sigma) = \begin{cases} 
\frac{a + d}{b} - 12 \text{sgn}(c) \cdot s(a, |c|) & \text{if } c \neq 0, \\
\frac{b}{d} - 3 & \text{if } c = 0.
\end{cases}$$

Later, Rademacher [33, Satz 7] (or see [34]) introduced the slightly modified function so-called the Rademacher symbol

$$\Psi(\sigma) := \begin{cases} 
\frac{a + d}{b} - 12 \text{sgn}(c) \cdot s(a, |c|) - 3 \text{sgn}(c(a + d)) & \text{if } c \neq 0, \\
\frac{b}{d} - 3 & \text{if } c = 0
\end{cases}$$

to be a class invariant, that is,

$$\Psi(\sigma) = \Psi(-\sigma) = -\Psi(\sigma^{-1}) = \Psi(g^{-1}\sigma g)$$

holds for any $g \in \Gamma$.

Remark 4.1. In [34, Eq. (59)], the Dedekind symbol $\Phi(\sigma)$ for $c = 0$ is given by $b/d$ without $-3$. This happens because they used the expression

$$6 \log(-(cz + d)^2) = 12 \text{sgn}(c)^2 \log \left( \frac{cz + d}{i \text{sgn}(c)} \right), \quad z \in \mathbb{H}$$

and regarded as 0 if $c = 0$. The merit of their notation is to represent $\Psi(\sigma) = \Phi(\sigma) - 3 \text{sgn}(c(a + d))$ simply.

In particular, for a hyperbolic element $\sigma$, the Rademacher symbol $\Psi(\sigma)$ has another characterization. Let $E_2^*(z)$ be the harmonic Maass form given in (3.2). Meyer [27] (or see [37]) derived the following expression in terms of the weight 2 cycle integral.

Theorem 4.2. [37, Section III] Let $\sigma \in \Gamma$ be a hyperbolic element. Then the Rademacher symbol $\Psi(\sigma)$ is given by

$$\Psi(\sigma) = \int_{z_0}^\sigma E_2^*(z)dz,$$

where the path of integration is in the geodesic $S_\sigma$.

This expression is independent of the choice of $z_0 \in S_\sigma$. Moreover, Lang [25, (2.17)] and Zagier [37, Section V, Lemma] established a simple explicit formula.
Theorem 4.3. [25, 37] Let $\sigma \in \Gamma$ be a hyperbolic element of the form
\[
\sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix}.
\]
Then we have
\[
\Psi(\sigma) = \sum_{j=0}^{2m-1} (-1)^j b_j.
\]

4.2. A Hyperbolic Analogue. In 2017, Duke-Imamoğlu-Tóth [13] introduced a hyperbolic analogue of the Rademacher symbol inspired by Ghys’ work [15]. In this part, we recall their work.

Let $j_m(z)$ be the modular function of the form $j_m(z) = q^{-m} + O(q)$ defined in (3.6). For a primitive hyperbolic element $\gamma \in \Gamma$ with $\text{sgn}(Q_\gamma) > 0$ and $\text{tr}(\gamma) > 2$, we consider the cycle integral
\[
\val_m(\gamma) := -\int_{z_0}^{\gamma z_0} j_m(z) \sqrt{D_\gamma} \frac{dz}{Q_\gamma(z,1)},
\]
where $D_\gamma := \text{tr}(\gamma)^2 - 4$ is the discriminant of $Q_\gamma(X,Y)$ defined in (3.1). In particular, since $j_m(z)$ is holomorphic in $\mathbb{H}$, the value is independent of the choice of $z_0 \in \mathbb{H}$ and the path of integration. The notation \(\text{val}\) was introduced by Kaneko in [22] to represent the normalized cycle integral $\val_m(\gamma)/\val_0(\gamma)$, where $\val_0(\gamma) = 2 \log \xi_\gamma$ is the length of the closed geodesic $\Gamma_{w_\gamma} \setminus S_\gamma$. Then the generating function
\[
F_\gamma(z) := \sum_{m=0}^{\infty} \val_m(\gamma) q^m, \quad z \in \mathbb{H}
\]
gives a weight 2 rational cocycle $r_\gamma : \Gamma \times \mathbb{H} \to \mathbb{C}$ by
\[
r_\gamma(\sigma, z) := j(\sigma, z)^{-1} F_\gamma(\sigma z) - F_\gamma(z),
\]
whose poles are located at real quadratic irrationals. For this, there uniquely exists the weight 0 cocycle $R_\gamma(\sigma, z)$ satisfying $\frac{1}{\gamma} R_\gamma(\sigma, z) = r_\gamma(\sigma, z)$ since both of the generators $T, S$ of $\Gamma$ are torsion elements. Let $L_\gamma(s, \alpha)$ be the Dirichlet series defined by
\[
L_\gamma(s, \alpha) := \sum_{m=1}^{\infty} \frac{\val_m(\gamma) e^{2\pi im\alpha}}{m^s}
\]
with a rational number $\alpha \in \mathbb{Q}$. This series converges for $\text{Re}(s) \gg 0$, and has the meromorphic continuation to $s > 0$ (see [13, Theorem 4.1]). Then the cocycle $R_\gamma(\sigma, z)$ for $\sigma = (a\ b\ c\ d) \in \Gamma$ with $c \neq 0$ is given by
\[
R_\gamma(\sigma, z) = \sum_{g \in \Gamma_{w_\gamma} \setminus \Gamma} \left[ \log(z - w_{g^{-1}g}) - \log(z - w'_{g^{-1}g}) \right] + \frac{1}{2\pi i} L_\gamma(1, a/c) + \val_0(\gamma) \cdot \frac{a + d}{c},
\]
where $w_{g^{-1}g} > w'_{g^{-1}g}$ are the fixed points of $g^{-1}g$. As remarked in [13, Remark 3.1], the summation in the right-hand side is finite sum. Under these notations, they defined the hyperbolic analogues of the Dedekind symbol and the Rademacher symbol.

Definition 4.4. [13] Let $\gamma \in \Gamma$ be a primitive hyperbolic element with $\text{sgn}(Q_\gamma) > 0$ and $\text{tr}(\gamma) > 2$. Then the hyperbolic Dedekind symbol $\Phi_\gamma(\sigma)$ is defined by
\[
\Phi_\gamma(\sigma) := \frac{2}{\pi} \lim_{y \to \infty} \text{Im} \ R_\gamma(\sigma, iy) = -\frac{1}{\pi^2} \text{Re} \ L_\gamma(1, \sigma i \infty).
\]
In addition, for another primitive hyperbolic element $\sigma$ with $\text{tr}(\sigma) > 2$ such that both $\sigma$ and $\sigma^{-1}$ are not in the conjugacy class of $\gamma$, the hyperbolic Rademacher symbol $\Psi_\gamma(\sigma)$ is defined by
\[
\Psi_\gamma(\sigma) := \lim_{n \to \infty} \frac{\Phi_\gamma(\sigma^n)}{n}.
\]
Then they [13, Theorem 3] established a beautiful connection between $\Psi_\gamma(\sigma)$ and the linking number of two modular knots defined from $\gamma$ and $\sigma$. We here note the interpretation of the hyperbolic Dedekind symbol $\Phi_\gamma(\sigma)$. 

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**Theorem 4.5.** [13, Theorem 5.2] For any \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) with \( c \neq 0 \), we have \( \Phi_\gamma(\sigma^{-1}) = \Phi_\gamma(\sigma) \) and
\[
\Phi_\gamma(\sigma) = -\sum_{g \in \Gamma_\\sigma \setminus \Gamma} \frac{\text{sgn}(Q_g \gamma \sigma g)}{Q_g \gamma \sigma g(z, 1)} \left| Q_g \gamma \sigma g(z, 1) \right|^s.
\]

**Remark 4.6.** For the weight 0 rational cocycle \( R(\sigma, z) := \delta(\sigma z) - \delta(z) \) given in (4.1), we can get a similar characterization for \( \Phi(\sigma) \) as
\[
\Phi(\sigma) = \frac{1}{2\pi i} \lim_{y \to \infty} \text{Im} R(\sigma, iy).
\]
Moreover Rademacher showed in [33, Satz 9], for any non-elliptic element \( \sigma \) and any integer \( n \in \mathbb{Z} \), \( \Psi(\sigma^n) = n\Psi(\sigma) \). Thus we can obtain the expression
\[
\Psi(\sigma) = \lim_{n \to \infty} \frac{\Phi(\sigma^n)}{n},
\]
(see also Barge-Ghys [3]). In this sense, the above definitions of \( \Phi_\gamma(\sigma) \) and \( \Psi_\gamma(\sigma) \) are suitable analogues of the classical objects.

### 4.3. Integrals along Geodesics.
Our aim of this section is to redefine the hyperbolic Rademacher symbol \( \Psi_\gamma(\sigma) \) in a similar way as Theorem 4.2, and give an interpretation as intersection numbers of closed geodesics. After that, in Section 5, we derive a similar explicit formula for \( \Psi_\gamma(\sigma) \) as Theorem 4.3. This explicit formula makes the symbol \( \Psi_\gamma(\sigma) \) computable.

In Section 3, we studied the Eisenstein series
\[
E_\gamma(z, s) := \sum_{g \in \Gamma_\\sigma \setminus \Gamma} \frac{\text{sgn}(Q_g \gamma \sigma g)}{Q_g \gamma \sigma g(z, 1)} |Q_g \gamma \sigma g(z, 1)|^s.
\]
For \( \gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), the limit of \( E_T(z, s) \) to \( s \to 0^+ \) gives the harmonic Maass form \( E_2(z) \), and Theorem 4.2 asserts that its cycle integral
\[
\int_{z_0}^{z_0} \lim_{s \to 0^+} E_T(z, s)dz
\]
gives the Rademacher symbol. Similarly, we want to consider the cycle integral of \( E_\gamma(z, s) \) for a hyperbolic element \( \gamma \). However, since the function \( E_\gamma(z, s) \) is not modular form, the integral depends on the choice of \( z_0 \in S_\gamma \). Therefore, we consider the following limit instead. In addition, we take a sum of cycle integrals as in Katok [23] here.

Let \( \gamma, \sigma \) be primitive hyperbolic elements such that both \( \sigma \) and \( \sigma^{-1} \) are not in the conjugacy class of \( \gamma \). Then for any \( z_0 \in S_\gamma \) and a positive integer \( n > 0 \), we define
\[
(4.2) \quad \tilde{\Psi}_\gamma(\sigma^n) := -\text{sgn}(\sigma) \left[ \lim_{s \to \infty} \left( \int_{\gamma = z_0}^{\gamma = z_0} E_\gamma(z, s)dz + \int_{\epsilon = -1}^{\epsilon = +1} E_{\gamma^{-1}, \sigma}(z, s)dz \right) \right]_{s=0},
\]
where \( \epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), \( z_0 = -z_0 \in S_{\gamma^{-1}} \), and the path of integrations are in \( S_\gamma, S_{\gamma^{-1}} \), respectively. As we show later, the function in the bracket does converge at \( s = 0 \). Our first result is the following.

**Theorem 4.7.** Let \( \gamma, \sigma \) be as above. The hyperbolic Rademacher symbol \( \tilde{\Psi}_\gamma(\sigma) \) is given by
\[
\tilde{\Psi}_\gamma(\sigma) = -\frac{2\pi i}{\sqrt{D_\gamma}} \sum_{g \in \Gamma_\\sigma \setminus \Gamma / \Gamma_\\sigma \text{ intersects with } S_\gamma} 1.
\]
In particular, this symbol satisfies
\[
(4.3) \quad \tilde{\Psi}_\gamma(\sigma) = \tilde{\Psi}_\gamma(-\sigma) = \tilde{\Psi}_\gamma(\sigma^{-1}) = \tilde{\Psi}_\gamma(g^{-1}\sigma g)
\]
for any \( g \in \Gamma \), and \( \tilde{\Psi}_\gamma(\sigma^n) = n\tilde{\Psi}_\gamma(\sigma) \) for any positive integer \( n > 0 \).

Before proving the theorem, we prepare two lemmas.

**Lemma 4.8.** For hyperbolic elements \( \gamma, \sigma \), we have
\[
\lim_{n \to \pm\infty} \text{sgn}(\sigma^{-n}\gamma\sigma^n) = \text{sgn}(M_{\sigma^{-1}}\gamma M_\sigma) \times \left\{ \begin{array}{ll} \pm \text{sgn}(\sigma) & \text{if } S_\gamma \text{ intersects with } S_\sigma, \\ 1 & \text{otherwise}. \end{array} \right.
\]
Proof. By the definition of the orientation of $S_\gamma$, conversely, $\text{sgn}(\gamma) = \text{sgn}(Q_\gamma) \text{sgn}(\text{tr}(\gamma)) > 0$ holds if the cusp $i\infty$ is the left-hand side seen from the geodesic $S_\gamma$, and $\text{sgn}(\gamma) < 0$ if $i\infty$ is the right-hand side. By the action of the scaling matrix $M_\sigma^{-1}$ given in (3.10), the geodesic $S_\gamma$ is moved to the imaginary axis oriented from 0 to $i\infty$ if $\text{sgn}(\sigma) > 0$, and opposite otherwise. In particular, the sequence $\{M_\sigma^{-1}\sigma n i\infty = -\xi_\sigma^2 | n \in \mathbb{Z}\}$ is on the negative real line.

Two points $i\infty$ and $M_\sigma^{-1}i\infty = -1$ are in the same side from the geodesic $M_\sigma^{-1}S_\gamma - n_\gamma g_\sigma^\infty$ if and only if $M_\sigma^{-1}S_\gamma - n_\gamma g_\sigma^\infty$ does not straddle the point $-1$. This means that

$$\text{sgn}(\sigma^n g_\sigma^\infty) = \text{sgn}(M_\sigma^{-1}\sigma^n g_\sigma M_\sigma) \times \begin{cases} -1 & \text{if } M_\sigma^{-1}S_\gamma - n_\gamma g_\sigma \text{ intersects with } x = -1, \\ 1 & \text{otherwise,} \end{cases}$$

$$= \text{sgn}(M_\sigma^{-1}\gamma M_\sigma) \times \begin{cases} -1 & \text{if } \xi_\sigma^{-2n}S_\gamma^{-1}g_\sigma \text{ intersects with } x = -1, \\ 1 & \text{otherwise.} \end{cases}$$

Here we recall that $\xi_\sigma^2 > 1$ if $\text{sgn}(\sigma) > 0$ and $0 < \xi_\sigma^2 < 1$ if $\text{sgn}(\sigma) < 0$. When $S_\gamma g_\sigma^{-1}g_\sigma M_\sigma$ does not intersect with the imaginary axis, that is, $S_\gamma$ does not intersect with $S_\sigma$, the geodesic $\xi_\sigma^{-2n}S_\gamma^{-1}g_\sigma M_\sigma$ does not intersect with $x = -1$ for a large $|n|$. Similarly we can check the case when $S_\gamma$ intersects with $S_\sigma$. $\square$

Lemma 4.9. Let $\gamma$ be a hyperbolic element. Then for any $\sigma \in \Gamma$, we have

$$j(\sigma, z)^{-2}E_\gamma(\sigma z, s) = E_\gamma(z, s) - 2 \sum_{g \in \Gamma \setminus \Gamma, \gamma \sigma} \frac{\text{sgn}(Q_{g^{-1}g}) \text{Im}(z)^s}{Q_{g^{-1}g}(z, 1)|Q_{g^{-1}g}(z, 1)|^s}.$$

Proof. By the definition, we have

$$j(\sigma, z)^{-2}E_\gamma(\sigma z, s) = \sum_{g \in \Gamma \setminus \Gamma, \gamma \sigma} \frac{\text{sgn}(Q_{g^{-1}g}) \text{Im}(z)^s}{Q_{g^{-1}g}(z, 1)|Q_{g^{-1}g}(z, 1)|^s}.$$
which converges to 0 as \( m \to \infty \). Thus in this case, we have

\[
\lim_{m \to \infty} \int_{\sigma_m} E_{\gamma}(z,s)dz = \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma_{\omega_{\gamma}}} \int_{0}^{\infty} \frac{-\text{sgn}(Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}})y^s dz}{Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)}|Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)|^s} + \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma_{\omega_{\gamma}}} \int_{0}^{\infty} \frac{\text{sgn}(Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}})y^s dz}{Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)}|Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)|^s}.
\]

In the case of \( \text{sgn}(\sigma) < 0 \), we also get the same equation with the minus sign.

As for the second half of the right-hand side of (4.2), by using easy facts

\[
e^{-1} \Gamma \Gamma = \Gamma, \quad \text{sgn}(Q_{\epsilon-\epsilon_{\gamma}}) = -\text{sgn}(Q_{\gamma}), \quad Q_{\epsilon-\epsilon_{\gamma}}(z,1) = -Q_{\gamma}(z,1),
\]

and repeating the same argument as above, we can get a similar expression. Combining them, we have

\[
\tilde{\Psi}_{\gamma}(\sigma) = \left[ \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma_{\omega_{\gamma}}} \int_{0}^{\infty} \frac{-\text{sgn}(Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}})|y|^s dz}{Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)}|Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)|^s} + \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma_{\omega_{\gamma}}} \int_{-\infty}^{\infty} \frac{\text{sgn}(Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}})|y|^s dz}{Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)}|Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)|^s} \right]_{s=0}.
\]

By Cauchy’s theorem, all terms in the second summation at \( s = 0 \) vanish. Since the first summation is finite sum, the summations converges at \( s = 0 \), and equals

\[
\tilde{\Psi}_{\gamma}(\sigma) = \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma_{\omega_{\gamma}}} \int_{0}^{\infty} \frac{-\text{sgn}(Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}})}{Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)}|Q_{M^{\alpha}_{\gamma}g^{\alpha}_{\gamma}M_{\gamma}(z,1)|^s} dz,
\]

The last equality follows from the residue theorem. Thus we obtain the first part of Theorem 4.7. The equations (4.3) immediately follows from this expression of \( \tilde{\Psi}_{\gamma}(\sigma) \), and the relation \( \tilde{\Psi}_{\gamma}(\sigma) = \pi \Psi_{\gamma}(\sigma) \) follows from the expression in (4.2) and \( \text{sgn}(\sigma) = \text{sgn}(\sigma) \). This concludes the proof of Theorem 4.7.

We next show the coincidence of our definition and Duke-Iamamoğlu-Tóth’s. To do this, we realize the hyperbolic Dedekind symbol \( \Phi_{\gamma}(\sigma) \) as the integral of \( E_{\gamma}(z,s) \) along the infinite geodesic from \( \sigma i\infty \) to \( i\infty \). For any \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \) with \( c \neq 0 \) and \( \text{Re}(s) > 0 \), by a similar argument, we have

\[
\int_{\sigma i\infty}^{i\infty} E_{\gamma}(z,s)dz + \int_{i\infty}^{-i\infty} E_{\gamma-\epsilon_{\gamma}}(z,s)dz = \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma} \int_{I_{\gamma}} \frac{\text{sgn}(Q_{g^{-\gamma}(z,1)})|\text{Im}(z)|^s dz}{Q_{g^{-\gamma}(z,1)}|Q_{g^{-\gamma}(z,1)|^s},
\]

where the path of integration \( I_{\gamma} \) is the vertical line from \( -i\infty \) to \( i\infty \) passing through the rational number \( \sigma i\infty \). As remarked in [13, Remark 3.1], the number of \( g \in \Gamma_{\omega_\gamma} \setminus \Gamma \) such that \( I_{\gamma} \) intersects with the geodesic \( S_{g^{-\gamma}} \) is finite. By the residue theorem again, this converges at \( s = 0 \), and equals

\[
\tilde{\Phi}_{\gamma}(\sigma) := \left[ \int_{\sigma i\infty}^{i\infty} E_{\gamma}(z,s)dz + \int_{i\infty}^{-i\infty} E_{\gamma-\epsilon_{\gamma}}(z,s)dz \right]_{s=0} = \sum_{g \in \Gamma_{\omega_\gamma} \setminus \Gamma} \int_{I_{\gamma}} \frac{\text{sgn}(Q_{g^{-\gamma}(z,1)})}{Q_{g^{-\gamma}(z,1)}} dz = \frac{2\pi i}{\sqrt{D_{\gamma}}} \sum_{w_{\gamma}^{-\gamma} < \sigma i\infty < w_{\gamma}^{-\gamma}} 1.
\]

This is the definition of \( \tilde{\Phi}_{\gamma}(\sigma) \). In addition, we put \( \tilde{\Phi}_{\gamma}(\sigma) = 0 \) naturally for \( \sigma \in \Gamma_{\infty} \). Comparing with Theorem 4.5, this \( \tilde{\Phi}_{\gamma}(\sigma) \) satisfies

\[
\tilde{\Phi}_{\gamma}(\sigma) = \frac{2\pi i}{\sqrt{D_{\gamma}}} \Phi_{\gamma}(\sigma),
\]

where \( \Phi_{\gamma}(\sigma) \) is the value at \( s = 0 \) of the function \( \Phi_{\gamma}(\sigma) \).
that is, this gives the hyperbolic Dedekind symbol.

Furthermore, we obtain the following relations.

**Proposition 4.10.** Let \( \gamma \in \Gamma \) be a primitive hyperbolic element. For any \( \sigma_1, \sigma_2 \in \Gamma \), we have

\[
(4.5) \quad \tilde{\Phi}_\gamma(\sigma_1 \sigma_2) = \tilde{\Phi}_\gamma(\sigma_1) + \tilde{\Phi}_\gamma(\sigma_2) + \frac{4\pi i}{\sqrt{D_\gamma}} \sum_{\gamma \in \Gamma \setminus \Gamma} \frac{1}{w_{\sigma_1^{-1} \gamma} < \sigma_1^{-1} \gamma < \infty}.
\]

**Proof.** By a direct calculation, we have

\[
\int_{\sigma_1 \gamma < \infty} E_\gamma(z, s) \frac{dz}{j(\sigma_1, z)^2} = \int_{\sigma_2 \gamma < \infty} E_\gamma(z, s) \frac{dz}{j(\sigma_2, z)^2} + \int_{\sigma_2 \gamma < \infty} E_\gamma(z, s) \frac{dz}{j(\sigma_1, z)^2}.
\]

By Lemma 4.9, the second integral is equal to

\[
\int_{\sigma_2 \gamma < \infty} E_\gamma(z, s) \frac{dz}{j(\sigma_1, z)^2} = \int_{\sigma_2 \gamma < \infty} E_\gamma(z, s) \frac{dz}{j(\sigma_2, z)^2} + \sum_{\gamma \in \Gamma \setminus \Gamma} \int_{\sigma_2 \gamma < \infty} \text{sgn}(Q_{\gamma^{-1} \gamma}) \text{Im}(z)^s dz.
\]

Similarly we can compute the integration from \( e^{-1} \sigma_1 \sigma_2 \in \infty \) to \( i \infty \). Thus we obtain

\[
\tilde{\Phi}_\gamma(\sigma_1 \sigma_2) = \tilde{\Phi}_\gamma(\sigma_1) + \tilde{\Phi}_\gamma(\sigma_2) - 2 \sum_{\gamma \in \Gamma \setminus \Gamma} \int_{\sigma_2 \gamma < \infty} \frac{\text{sgn}(Q_{\gamma^{-1} \gamma}) dz}{Q_{\gamma^{-1} \gamma}(z, 1)}.
\]

which equals (4.5).

From this proposition, we get some basic relations.

**Corollary 4.11.** Let \( \gamma \) be a primitive hyperbolic element. For any \( \sigma \in \Gamma \), we have

\[
(4.6) \quad \tilde{\Phi}_\gamma(\sigma^{-1}) = \tilde{\Phi}_\gamma(\sigma),
\]

\[
(4.7) \quad \tilde{\Phi}_\gamma(T \sigma) = \tilde{\Phi}_\gamma(\sigma),
\]

\[
(4.8) \quad \tilde{\Phi}_\gamma(S \sigma) = \tilde{\Phi}_\gamma(\sigma) + \tilde{\Phi}_\gamma(S) + \frac{4\pi i}{\sqrt{D_\gamma}} \sum_{\gamma \in \Gamma \setminus \Gamma} \frac{1}{w_{\sigma^{-1} \gamma} < \sigma^{-1} \gamma < \infty}.
\]

**Proof.** Applying Proposition 4.10 to \( \sigma_1 = \sigma^{-1}, \sigma_2 = \sigma \), we have

\[
0 = \tilde{\Phi}_\gamma(I) = \tilde{\Phi}_\gamma(\sigma^{-1}) + \tilde{\Phi}_\gamma(\sigma) + \frac{4\pi i}{\sqrt{D_\gamma}} \sum_{\gamma \in \Gamma \setminus \Gamma} \frac{1}{w_{\sigma^{-1} \gamma} < \sigma^{-1} \gamma < \infty}.
\]

This implies (4.6). The equations (4.7), (4.8) also immediately follows from the above proposition.

Finally, we check the relation between \( \tilde{\Psi}_\gamma(\sigma) \) and \( \tilde{\Psi}_\gamma(\sigma) \).

**Theorem 4.12.** Let \( \gamma, \sigma \) be primitive hyperbolic elements such that both \( \sigma \) and \( \sigma^{-1} \) are not in the conjugacy class of \( \gamma \). Then we have

\[
\tilde{\Psi}_\gamma(\sigma) = \lim_{n \to \infty} \frac{\tilde{\Phi}_\gamma(\sigma^n)}{n} \quad \text{i.e.} \quad \tilde{\Psi}_\gamma(\sigma) = \frac{2\pi i}{\sqrt{D_\gamma}} \Psi_\gamma(\sigma).
\]

To prove this theorem, we compute the hyperbolic Dedekind symbol \( \tilde{\Phi}_\gamma(\sigma) \) in two ways. First, by Proposition 4.10,

\[
\tilde{\Phi}_\gamma(\sigma^n) = n \tilde{\Phi}_\gamma(\sigma) + \frac{4\pi i}{\sqrt{D_\gamma}} \sum_{k=1}^{n-1} \sum_{\gamma \in \Gamma \setminus \Gamma} \frac{1}{w_{\sigma^{-k} \gamma} < \sigma^{-k} \gamma < \infty}.
\]
This implies that
\[
\lim_{n \to \infty} \frac{\tilde{\Phi}_\gamma(\sigma^n)}{n} = \tilde{\Phi}_\gamma(\sigma) + \sum_{g \in \Gamma \setminus \Gamma} \frac{4\pi i}{D_\gamma} \prod_{\gamma \in S_{g^{-1}g}} \prod_{w_{\gamma^{-1}g} < \sigma^{-}\infty, \sigma \infty < w_{\gamma g}} 1,
\]
where $\sigma^{-}\infty = w'_\sigma$ if $\text{sgn}(\sigma) > 0$, and $w_\sigma$ if $\text{sgn}(\sigma) < 0$.

On the other hand, the geodesics intersecting with $\mathcal{I}_\sigma$ are divided into two cases.

**Lemma 4.13.** Let $\sigma$ be a primitive hyperbolic element. Then we have two equalities
\[
\begin{align*}
&\{g \in \Gamma \setminus \Gamma \mid S_{g^{-1}g} \text{ intersects with } \mathcal{I}_\sigma \text{ and } S_\sigma\} = \{g \in \Gamma \setminus \Gamma \mid S_{g^{-1}g} \text{ intersects with } S_\sigma\}, \\
&2\{g \in \Gamma \setminus \Gamma \mid S_{g^{-1}g} \text{ straddles } \sigma(\infty, \sigma^{-\infty})\} = \{g \in \Gamma \setminus \Gamma \mid S_{g^{-1}g} \text{ does not intersect with } S_\sigma \text{ but } \mathcal{I}_\sigma\}.
\end{align*}
\]

**Proof.** Let $S$ be a geodesic intersecting with $\mathcal{I}_\sigma$ and $S_\sigma$. Then $\sigma^n S$ intersects with $\mathcal{I}_\sigma$ if and only if $n = 0$, which concludes the first equation. Next let $S'$ be a geodesic straddling $\sigma(\infty, \sigma^{-\infty})$. We assume that one endpoint of $S'$ is in between $\sigma^{-k+1}\infty$ and $\sigma^{-k}\infty$ for $k > 0$. Then $\sigma^n S'$ does not straddle $\sigma^{-\infty}\infty$ and not intersect with $S_\sigma$. \qed

From this lemma, we see that
\[
\tilde{\Psi}_\gamma(\sigma) = -\frac{2\pi i}{D_\gamma} \left[ \sum_{g \in \Gamma \setminus \Gamma \mid \text{intersects with } S_\sigma} 1 + 2 \sum_{g \in \Gamma \setminus \Gamma \mid \text{does not intersect with } S_\sigma} 1 \right] - \frac{4\pi i}{D_\gamma} \sum_{w_{\gamma^{-1}g} < \sigma^{-}\infty, \sigma \infty < w_{\gamma g}} 1.
\]

The equations (4.9) and (4.10) concludes the proof of Theorem 4.12.

Finally we note that the equation (4.9) gives another relation between $\tilde{\Phi}_\gamma(\sigma)$ and $\tilde{\Psi}_\gamma(\sigma)$.

**Corollary 4.14.** Let $\gamma, \sigma$ be as in Theorem 4.12. Then we have
\[
\tilde{\Psi}_\gamma(\sigma) = \tilde{\Phi}_\gamma(\sigma) + \frac{4\pi i}{D_\gamma} \sum_{g \in \Gamma \setminus \Gamma \mid w_{\gamma^{-1}g} < \sigma^{-}\infty, \sigma \infty < w_{\gamma g}} 1
\]
\[
= -\frac{2\pi i}{D_\gamma} \left[ \sum_{g \in \Gamma \setminus \Gamma \mid w_{\gamma^{-1}g} < \sigma^{-}\infty, \sigma \infty < w_{\gamma g}} 1 - 2 \sum_{g \in \Gamma \setminus \Gamma \mid w_{\gamma^{-1}g} < \sigma \infty, \sigma^{-\infty} < w_{\gamma g}} 1 \right].
\]

**Remark 4.15.** This is a hyperbolic analogue of the equation in Remark 4.1. In particular, this formula makes $\tilde{\Psi}_\gamma(\sigma)$ computable. Actually, both sums are finite, and we can easily make a list of all $g \in \Gamma \setminus \Gamma$ such that $w_{\gamma^{-1}g} < \sigma \infty < w_{\gamma g}$ by using Mathematica.

**Remark 4.16.** As one expected approach, we can consider the cycle integral
\[
\lim_{s \to \sigma} \left( \int_{z_0}^{\sigma z_0} \tilde{\mathcal{E}}_\gamma(z, s) dz + \int_{z_1}^{e^{-1}\sigma z_1} \tilde{\mathcal{E}}_{e^{-1}\gamma z}(z, s) dz \right)
\]
instead of (4.2). Since $\tilde{\mathcal{E}}_\gamma(z, s)$ is a modular form, this value is independent of the choice of $z_0 \in S_\sigma$. By a direct calculation, this converges to 0.

5. **Explicit Formulas**

In this section, we give an explicit formula for the hyperbolic Rademacher symbol $\tilde{\Phi}_\gamma(\sigma)$ as an analogue of Theorem 4.3. By Theorem 4.7, the problem is reduced to that for primitive elements
\[
\gamma = \begin{pmatrix} a_0 & 1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_{2n} & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m} & 1 \\ 1 & 0 \end{pmatrix},
\]
where both $\sigma$ and $\sigma^{-1}$ are not $\Gamma$-equivalent to $\gamma$. The goal of this section is to prove Theorem 5.6.
First, we divide the formula (4.4) for the hyperbolic Dedekind symbol \( \tilde{\Phi}_\gamma(\sigma) \) into the sum of another counting functions. For a rational number \( x \in \mathbb{Q} \), we put

\[
e_\gamma(x) := \tilde{\Phi}_\gamma(S) + \frac{4\pi i}{\sqrt{D_\gamma}} \sum_{g \in \Gamma_{w_\gamma}} 1.
\]

Then Corollary 4.11 implies that

\[
(5.2) \quad \tilde{\Phi}_\gamma(T\sigma) = \tilde{\Phi}_\gamma(\sigma), \quad \tilde{\Phi}_\gamma(S\sigma) = \tilde{\Phi}_\gamma(\sigma) + e_\gamma(\sigma i \infty).
\]

**Proposition 5.1.** For the hyperbolic element \( \sigma \) of the form

\[
\sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix},
\]

we have

\[
\tilde{\Phi}_\gamma(\sigma) = \sum_{j=0}^{2m-1} e_\gamma \left( (-1)^j [b_j, b_{j+1}, \ldots, b_{2m-1}] \right) + \tilde{\Phi}_\gamma(S).
\]

**Proof.** By the equation

\[
\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T^a S T^{-b} S^{-1},
\]

the element \( \sigma \) is expressed as \( \sigma = T^{b_0} S T^{-b_1} S^{-1} \cdots T^{-b_{2m-1}} S^{-1} \). From (5.2), we see that

\[
\tilde{\Phi}_\gamma(\sigma) = \tilde{\Phi}_\gamma(T^{-b_1} S^{-1} T b_2 S S^{-1} \cdots T^{-b_{2m-1}} S^{-1}) + e_\gamma([-b_1, \ldots, b_{2m-1}]).
\]

From (5.2) again, and the facts \( S^{-1} = -S \) and \( \tilde{\Phi}_\gamma(-\sigma) = \tilde{\Phi}_\gamma(\sigma) \), we have

\[
\tilde{\Phi}_\gamma(T^{-b_1} S^{-1} \cdots T^{-b_{2m-1}} S^{-1}) = \tilde{\Phi}_\gamma(T^{b_0} S \cdots S T^{-b_{2m-1}} S^{-1}) + e_\gamma([b_2, b_3, \ldots, b_{2m-1}]).
\]

Repeating this process concludes the proof. \(\square\)

By virtue of this proposition, our problem is reduced to getting an explicit formula for \( e_\gamma(x) \) for \( x = -[b_{2\ell-1}, b_{2m-1}] = [-b_{2\ell-1} - 1, b_{2\ell} - 1, b_{2\ell+1}, \ldots, b_{2m-1}] \) for \( \ell = 1, \ldots, m \) and \( x = [b_{2\ell}, \ldots, b_{2m-1}] \) for \( \ell = 1, \ldots, m - 1 \). To this end, it suffices to count the number

\[
(5.3) \quad \sum_{g \in \Gamma_{w_\gamma} \setminus \Gamma} 1.
\]

Since \( w_\gamma = [a_0, \ldots, a_{2n-1}] \) is a reduced real quadratic irrational, Lemma 2.2 implies

\[
\Gamma w = \bigcup_{r=0}^{\infty} \bigcup_{0 \leq i < 2n} \bigcup_{i \equiv r \ (2)} A_{r,i},
\]

where

\[
A_{r,i} = \{ [k_0, \ldots, k_r-1, a_i, a_{i+1}, \ldots, a_{2n-1}, a_0, \ldots, a_{i-1}] \mid k_0 \in \mathbb{Z}, k_1, \ldots, k_r-1 \in \mathbb{Z}_{>0}, k_{r-1} \neq a_{i-1} \}.
\]

Thus the number (5.3) is divided as

\[
\sum_{g \in \Gamma_{w_\gamma} \setminus \Gamma} 1 = \sum_{r=0}^{\infty} \sum_{0 \leq i < 2n} \sum_{i \equiv r \ (2)} \# \{ v \in A_{r,i} \mid v' < 0, x < v \text{ or } v < 0, x < v' \}
\]

\[
\therefore \sum_{r=0}^{\infty} \sum_{0 \leq i < 2n} \sum_{i \equiv r \ (2)} N_{r,i}(x),
\]

where \( v' \) is the Galois conjugate of \( v \).
Lemma 5.2. Let $\gamma, \sigma$ be as in (5.1), and $\ell = 1, \ldots, m$. Then $e_\gamma(x)$ for $x = [-b_{2\ell-1}, \ldots, b_{2m-1}] = \{-b_{2\ell-1}-1, b_{2\ell-1}+1, \ldots, b_{2m-1}\}$ is given by

$$e_\gamma(x) = -\frac{4\pi i}{\sqrt{D}} \left[ \sum_{0 \leq i < 2n} \min(a_i, b_{2\ell-1}) - \sum_{0 \leq k < n} \left( \delta(a_{2k} \geq b_{2\ell-1}) \delta([b_{2\ell}, \ldots, b_{2m-1}] \geq [a_{2k-1}, a_{2k-2}, \ldots, a_{2k+2\ell-2m}]) + \delta(a_{2k-1} \geq b_{2\ell-1}) \delta([b_{2\ell}, \ldots, b_{2m-1}] \geq [a_{2k}, a_{2k+1}, \ldots, a_{2k+2\ell-2m}]) \right) \right],$$

where we put $\delta(0) = 1$, and $a_i = a_i \pmod{2n}$.

Proof. First we count the number of $N_{r,i}(x)$.

(i) The case of $r = 0$. For $v = [a_{2k}, \ldots, a_{2k+2n}]$, its Galois conjugate is

$$v' = [-1, 1, a_{2k-1}-1, a_{2k-2}, \ldots, a_{2k-2n-1}],$$

by Proposition 2.4. Since $x < -1$, the inequality $v' < x < 0$ does never hold. Thus $N_{0,2k} = 0$ for any $0 \leq k < n$.

(ii) The case of $r = 1$. For $v = [k_0, a_{2k+1}, \ldots, a_{2k+2n}]$ with $k_0 \neq a_{2k}$, we have

$$v' = [k_0 - a_{2k - 1}, 1, a_{2k-1} - 1, a_{2k-2}, \ldots, a_{2k-2n-1}].$$

Now we consider the inequality $v' < x < 0$ for $v > 0$, the number $k_0$ should be $k_0 \geq 0$. The inequality $v' < x$ holds if and only if $k_0 - a_{2k} - 1 < b_{2\ell-1} - 1$, or $k_0 = a_{2k} - b_{2\ell-1}$ and $[b_{2\ell}, \ldots, b_{2m-1}] \geq [a_{2k-1}, a_{2k-2}, \ldots, a_{2k+2\ell-2m}]$. Thus the number of such $k_0$ is given by

$$N_{1,2k+1} = a_{2k} - \min(a_{2k}, b_{2\ell-1}) + \delta(a_{2k} \geq b_{2\ell-1}) \delta([b_{2\ell}, \ldots, b_{2m-1}] \geq [a_{2k-1}, a_{2k-2}, \ldots, a_{2k+2\ell-2m}]).$$

(iii) The case of $r = 2$. For $v = [k_0, k_1, a_{2k+1}, \ldots, a_{2k+2n}]$ with $k_1 \neq a_{2k+1}$, its Galois conjugate is given by

$$v' = \begin{cases} [k_0, \ldots, k_0 + 1, \ldots] & \text{if } k_1 > a_{2k+1} + 1, \\
[k_0 + a_{2k-1} - 1, a_{2k-2}, \ldots, a_{2k-2n-1}] & \text{if } k_1 = a_{2k+1} + 1, a_{2k} > 1, \\
[k_0 + a_{2k-1} - 1, a_{2k-2}, \ldots, a_{2k-2n-1}] & \text{if } k_1 = a_{2k+1} + 1, a_{2k} = 1, \\
[k_0 + 1, \ldots] & \text{if } 0 < k_1 < a_{2k+1}. \end{cases}$$

Since $|x - 0| > 1$, the contribution comes only from the third case. Then we consider $v < x < 0 < v'$. From $v' > 0$, we have $k_0 \geq -a_{2k-1} - 1$. On the other hand, from $v < x$, we get $k_0 < -b_{2\ell-1} - 1$, or $k_0 = -b_{2\ell-1} - 1$ and $[a_{2k+1} + 1, a_{2k+2}, \ldots, a_{2k+2n}] \geq [1, b_{2\ell-1}, b_{2\ell+1}, \ldots, b_{2m-1}]$. If $|b_{2\ell} > 1$, then the last inequality always holds, but for $b_{2\ell} = 1$, this is equivalent to $[a_{2k+1}, a_{2k+2}, \ldots, a_{2k+2\ell-2m}] \geq [b_{2\ell+1}, \ldots, b_{2m-1}]$. Thus we have

$$N_{2,2k+2} = \delta(a_{2k} = 1) \left[ a_{2k-1} - \min(a_{2k-1}, b_{2\ell-1}) + \delta(a_{2k-1} \geq b_{2\ell-1}) \delta(b_{2\ell} > 1) + \delta(b_{2\ell} = 1) \delta([a_{2k+1}, a_{2k+2}, \ldots, a_{2k+2\ell-2m}] \geq [b_{2\ell+1}, \ldots, b_{2m-1}]) \right].$$

(iv) The case of $r = 3$. For $v = [k_0, k_1, k_2, a_{2k+1}, \ldots, a_{2k+2n}]$ with $k_2 \neq a_{2k}$, its Galois conjugate is given by

$$v' = \begin{cases} [k_0, \ldots, k_0 + 1, \ldots] & \text{if } k_2 > a_{2k}, \\
[k_0, \ldots] & \text{if } 0 < k_2 < a_{2k}, k_1 > 1, \\
[k_0 + a_{2k-1} - 1, a_{2k-2}, \ldots, a_{2k-2n-1}] & \text{if } 0 < k_2 = a_{2k} - 1, k_1 = 1, \\
[k_0 + 1, \ldots] & \text{if } 0 < k_2 < a_{2k} - 1, k_1 = 1. \end{cases}$$

In this case also, the contribution comes only from the third case. Then we consider $v < x < 0 < v'$. From $v' > 0$, the number $k_0$ should be $k_0 \geq -a_{2k-1} - 1$. The inequality $v < x$ is equivalent to $k_0 < -b_{2\ell-1} - 1,$
or \( k_0 = -b_{2\ell - 1} - 1 \) and \([b_2, \ldots, b_{2m-1}] \geq [a_{2k}, a_{2k+1}, \ldots, a_{2k-2\ell+2m-1}]\). Thus we have

\[
N_{3,2k+1} = \delta(a_{2k} > 1) \left( a_{2k-1} - \min(a_{2k-1}, b_{2\ell-1}) + \delta(a_{2k-1} \geq b_{2\ell-1}) \delta([b_2, \ldots, b_{2m-1}] \geq [a_{2k}, a_{2k+1}, \ldots, a_{2k-2\ell+2m-1}]) \right).
\]

For \( r \geq 4 \), we easily see that the number \( N_{r,i} \) always becomes 0. Combining all results, we get

\[
\sum_{g \in \Gamma \setminus \Gamma_{\gamma}} \sum_{w'_{-1,\gamma} < 0, 2 < w'_{-1,\gamma}} \sum_{k=1}^{n} \left[ a_{2k-1} + a_{2k} - \min(a_{2k-1}, b_{2\ell-1}) - \min(a_{2k}, b_{2\ell-1}) + \delta(a_{2k-1} \geq b_{2\ell-1}) \delta([b_2, \ldots, b_{2m-1}] \geq [a_{2k}, a_{2k+1}, \ldots, a_{2k-2\ell+2m-1}]) + \delta(a_{2k} \geq b_{2\ell-1}) \delta([b_2, \ldots, b_{2m-1}] \geq [a_{2k-1}, a_{2k-2}, \ldots, a_{2k+2\ell-2m}]) \right].
\]

To conclude the proof, we next compute \( \tilde{\Phi}_y(S) \) in a similar way. By (4.4), \( \tilde{\Phi}_y(S) \) is expressed as

\[
\tilde{\Phi}_y(S) = -\frac{2\pi i}{\sqrt{D_y}} \sum_{g \in \Gamma \setminus \Gamma} \sum_{w'_{-1,\gamma} < 0, 2 < w'_{-1,\gamma}} \sum_{k=1}^{n} \left( a_{2k-1} + a_{2k} \right).
\]

By repeating the same argument as above for \( x = 0 \), we obtain (i) \( N_{0,2k} = 1 \), (ii) \( N_{1,2k+1} = a_{2k} \), (iii) \( N_{2,2k+2} = \delta(a_{2k} > 1) + \delta(a_{2k} = 1)(a_{2k-1} + 1) + a_{2k+1} - 1 \), (iv) \( N_{3,2k+1} = \delta(a_{2k} > 1)(a_{2k-1} + 1) + a_{2k} - 2 + \delta(a_{2k} = 1) \). Therefore, we get

\[
(5.7) \quad \tilde{\Phi}_y(S) = -\frac{4\pi i}{\sqrt{D_y}} \sum_{0 \leq i < 2n} a_i.
\]

This concludes the proof.

\[\square\]

**Corollary 5.3.** For the hyperbolic element \( \gamma \) of the form

\[
\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix},
\]

we have

\[
\tilde{\Phi}_y(S) = -\frac{4\pi i}{\sqrt{D_y}} \sum_{0 \leq i < 2n} a_i,
\]

that is,

\[
\sum_{g \in \Gamma \setminus \Gamma} \sum_{w'_{-1,\gamma} < 0, 2 < w'_{-1,\gamma}} \sum_{k=1}^{n} \left[ a_{2k-1} + a_{2k} \right] = 2 \sum_{0 \leq i < 2n} a_i.
\]

**Remark 5.4.** The corollary is an interesting analogue of the classical Rademacher symbol in the sense of Theorem 4.3. In particular, this gives the number of simple quadratic forms in the \( \Gamma \)-equivalent class of \( Q_\gamma \). Here \( Q(X,Y) = AX^2 + BXY + CY^2 \) is said to be simple if \( AC < 0 \) holds, which is studied by Choie-Zagier [10].

Similarly, we get a counting formula of \( e_\gamma(x) \) for \( x = [b_2, b_{2\ell+1}, \ldots, b_{2m-1}] \) as follows

**Lemma 5.5.** Let \( \gamma, \sigma \) be as in (5.1), and \( \ell = 1, \ldots, m - 1 \). Then \( e_\gamma(x) \) for \( x = [b_2, \ldots, b_{2m-1}] \) is given by

\[
e_\gamma(x) = -\frac{4\pi i}{\sqrt{D_y}} \left[ \sum_{0 \leq i < 2n} \min(a_i, b_{2\ell}) \right.
-
\sum_{0 \leq k < n} \left( \delta(a_{2k} \geq b_{2\ell}) \delta([b_{2\ell+1}, \ldots, b_{2m-1}] \geq [a_{2k+1}, a_{2k+2}, \ldots, a_{2k-2\ell+2m-1}]) + \delta(a_{2k-1} \geq b_{2\ell}) \delta([b_{2\ell+1}, \ldots, b_{2m-1}] \geq [a_{2k-2}, a_{2k-3}, \ldots, a_{2k+2\ell-2m}]) \right].
\]

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Combining Proposition 5.1 with Lemma 5.2 and Lemma 5.5, we get an explicit formula for \( \tilde{\Psi}_\gamma(\sigma) \). Furthermore, by using Theorem 4.12,

\[
\tilde{\Psi}_\gamma(\sigma) = \lim_{n \to \infty} \tilde{\Phi}_\gamma(\sigma^n),
\]

we get the following formula for the hyperbolic Rademacher symbol \( \tilde{\Psi}_\gamma(\sigma) \).

**Theorem 5.6.** Let \( \gamma, \sigma \) be as in (5.1). Then we have

\[
\tilde{\Psi}_\gamma(\sigma) = -\frac{4\pi i}{\sqrt{D_\gamma}} \sum_{0 \leq i < 2n} \min(a_i, b_j) - \sum_{0 \leq k < n} 0 \leq \ell < m \left( \delta(a_{2k} \geq b_{2\ell - 1}) \delta([b_{2\ell - 1}, b_{2\ell + 1}]) \right)
\]

\[
\sum_{0 \leq k < n} 0 \leq \ell < m \left( \delta(a_{2k} - 1 \geq b_{2\ell - 1}) \delta([b_{2\ell - 1}, b_{2\ell + 1}]) \right)
\]

\[
\sum_{0 \leq k < n} 0 \leq \ell < m \left( \delta(a_{2k} \geq b_{2\ell}) \delta([b_{2\ell + 1}, b_{2\ell + 2}]) \right)
\]

\[
\sum_{0 \leq k < n} 0 \leq \ell < m \left( \delta(a_{2k} - 1 \geq b_{2\ell}) \delta([b_{2\ell + 1}, b_{2\ell + 2}]) \right).
\]

Here we put \( a_i = a_i \pmod{2n}, b_j = b_j \pmod{2m} \).

### 6. Weight 0 Analogues of the Eisenstein Series

In this last section, we note one remarkable result on the weight 0 analogue of the Eisenstein series. Similar as in Section 3, for each non-identity element \( \gamma \in \Gamma \), we define the real analytic Eisenstein series

\[
E_{0,\gamma}(z, s) := \sum_{Q \sim Q_\gamma} \frac{y^s}{|Q(z, 1)|^s} = \sum_{g \in \Gamma \setminus \Gamma_\infty} \frac{y^s}{\|Q^{-1} g(z, 1)\|^s}, \quad \text{Re}(s) > 1,
\]

without the sign function. This series converges absolutely and locally uniformly for \( \text{Re}(s) > 1 \) and \( z \in \mathbb{H} \), and satisfies \( E_{0,\gamma}(g z, s) = E_{0,\gamma}(z, s) \) for any \( g \in \Gamma \). As in Section 3.1, for \( \gamma = T = \left( \frac{1}{0} \frac{1}{1} \right) \), this is the classical Eisenstein series

\[
E_{0,T}(z, s) = \sum_{g \in \Gamma_\infty \setminus \Gamma} \text{Im}(g(z))^s = \frac{1}{2} \sum_{(c,d)=1} y^s |cz + d|^{2s}.
\]

On the other hand, for elliptic or hyperbolic elements, we have

\[
E_{0,\gamma}(z, s) = \begin{cases} |D_\gamma|^{-s/2} \sum_{g \in \Gamma \setminus \Gamma_\infty} \frac{1}{\sinh(d_{hyp}(g(z, w_\gamma)))^s} & \text{if } \gamma \text{ is elliptic}, \\ D^{-s/2} \sum_{g \in \Gamma_\infty \setminus \Gamma} \frac{1}{\cosh(d_{hyp}(g(z, S_\gamma)))^s} & \text{if } \gamma \text{ is hyperbolic}, \end{cases}
\]

which immediately follows from (3.13) and Lemma 3.6. The hyperbolic case was originally studied by Petersson [30] and Kudla-Millson [24] for weight 2. More recently, Jorgenson-Kramer-von Pippich, and so on [20, 21, 31, 32, 36] studied the above weight 0 analogues.

Our goal in this section is getting the limit formulas for these Eisenstein series. In the classical case, the Eisenstein series \( E_{0,T}(z, s) \) is meromorphically continued to the whole \( s \)-plane, and we know that

\[
E_{0,T}(z, s) = 1 + \log(\eta(z)|^4) \cdot s + O(s^2)
\]

(6.1)

\[
= \frac{3\pi}{s - 1} - \frac{3\pi}{s} \log(\eta(z)|^4) + \frac{6}{\pi} \left( \gamma - \log 2 - \frac{6'(2)}{\pi^2} \right) + O(s - 1).
\]

This is so-called the Kronecker limit formula, where \( \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) is the Dedekind eta function, \( \gamma \) is the Euler constant, and \( \zeta(s) \) is the Riemann zeta function. (We hope there is no confusion between the notation \( \gamma \) used for a matrix and the Euler constant.) In the elliptic case, von Pippich [31] gave the meromorphic continuation of the elliptic Eisenstein series \( E_{0,\gamma}(z, s) \) for \( \gamma = S = \left( \frac{1}{1} \frac{-1}{0} \right), U = \left( \frac{1}{0} \frac{-1}{1} \right) \), and established the limit formula

\[
E_{0,\gamma}(z, s) = -\frac{2}{\Gamma_{w_\gamma}} \log |j(z) - j(w_\gamma)| \cdot s + O(s^2).
\]
Here the number \(|\Gamma_w\)| is given by 4 or 6 according as \(\gamma = S\) or \(U\). On the other hand, for a hyperbolic element \(\gamma\), Jorgenson-Kramer-von Pippich [20] derived the meromorphic continuation of the hyperbolic Eisenstein series \(E_0,\gamma(z, s)\) by using the spectral expansion. According to their results, the function \(E_0,\gamma(z, s)\) has a double zero at \(s = 0\). Moreover, von Pippich-Schwagenscheidt-Völz [32, Remark 5.7] described that it is an interesting problem to investigate the second order coefficient of \(E_0,\gamma(z, s)\) at \(s = 0\).

Our following result provides an answer to this problem. Similar as in Theorem 3.4, we can reduce the problem to the case that \(\gamma\) is a primitive hyperbolic element with \(\sgn(Q_\gamma) > 0\) and \(\text{tr}(\gamma) > 2\).

**Theorem 6.1.** Let \(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\) be a primitive hyperbolic element with \(\sgn(Q_\gamma) = c > 0\) and \(\text{tr}(\gamma) = a + d > 2\). Then the function \(E_0,\gamma(z, s)\) is analytically continued to \(s = 0\), and equals

\[
E_0,\gamma(z, s) = -\frac{1}{2} \left[ -\int_{\gamma_0} \log |j(z) - j(\tau)| \frac{\sqrt{D} d\tau}{Q_\gamma(\tau, 1)} + \sum_{g \in \Gamma_\gamma(\omega) \setminus \Gamma} \arcsin^2 \left( \frac{1}{\cosh(d_{hyp}(g z, S_\gamma))} \right) \right] s^2 + O(s^3),
\]

where the path of the integration is on the geodesic \(S_\gamma\).

Here is what we do. In order to prove this theorem, we first generalize von Pippich’s result. More precisely, for any \(z, \tau \in \mathbb{H}\) which are not \(\Gamma\)-equivalent, we introduce the function

\[
P(z, \tau; s) := \sum_{g \in \Gamma} \frac{1}{\sinh(d_{hyp}(g z, \tau))^s}\quad \Re(s) > 1,
\]

and show that

\[
(6.2)\quad P(z, \tau; s) = -2 \log |j(z) - j(\tau)| \cdot s + O(s^2)
\]

after the meromorphic continuation to \(s = 0\). Actually this limit formula follows from the results of Gross-Zagier [17] and Fay [14]. The proofs of (6.2) and Theorem 6.1 are described by means of the intermediate function

\[
Q(z, \tau; s) := \sum_{g \in \Gamma} \frac{1}{\cosh(d_{hyp}(g z, \tau))^s}\quad \Re(s) > 1.
\]

To make this precise, we now recall the results of Gross-Zagier and Fay. For \(\Re(s) > 1\) and any \(\Gamma\)-invariant \(z, \tau \in \mathbb{H}\), we consider the automorphic Green function defined by

\[
G(z, \tau; s) := \frac{-2 \Gamma(s)^2}{\Gamma(2 s)} \sum_{g \in \Gamma} \left( \frac{1 + \cosh(d_{hyp}(g z, \tau))}{1 + \cosh(d_{hyp}(g z, \tau))} \right)^{-s} 2F_1 \left( 1, 2s; 2s; \frac{2}{1 + \cosh(d_{hyp}(g z, \tau))} \right),
\]

where \(2F_1(a, b; c; z)\) is Gauss’ hypergeometric function given by

\[
2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}
\]

with the Pochhammer symbol \((x)_k := x(x + 1) \cdots (x + k - 1)\). We note that Gross-Zagier’s definition given in [17] equals \(G(z, \tau; s)/2\), and Fay’s definition in [14] equals \(G(z, \tau; s)/8\pi\).

**Proposition 6.2.** [17, Proposition 5.1] For above \(z, \tau \in \mathbb{H}\), the function \(G(z, \tau; s)\) has the meromorphic continuation to the whole \(s\)-space, and we have

\[
\lim_{s \to 1} \left( G(z, \tau; s) + 8\pi E_{0,T}(z, s) + 8\pi E_{0,T}(\tau, s) - 8\pi \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(s - \frac{1}{2}\right) \zeta(2s - 1)}{\zeta(2s)} \right) = 4 \log |j(z) - j(\tau)| + 48.
\]

By combining this proposition and the Kronecker limit formula (6.1), we have

\[
G(z, \tau; s) = -\frac{24}{s - 1} + f(z, \tau) + O(s - 1),
\]

where \(\tau = u + iv\) and

\[
f(z, \tau) = 4 \log |j(z) - j(\tau)| + 24 \log |\eta(z)|^4 + 24 \log |\eta(\tau)|^4 - 48 \left( -\log 2 - 6\zeta(2) \pi^2 - 1 \right).
\]

**Proposition 6.3.** [14, (79)] For above \(z, \tau \in \mathbb{H}\), the function \(G(z, \tau; s)\) satisfies the functional equation

\[
G(z, \tau; s) - G(z, \tau; 1 - s) = \frac{8\pi}{1 - 2s} E_{0,T}(z, 1 - s) E_{0,T}(\tau, s).
\]
By this functional equation, we obtain the Taylor expansion at \( s = 0 \),
\[
G(z, \tau; s) = 4 \log |j(z) - j(w)| + O(s).
\]

To show (6.2), we consider the intermediate function \( Q(z, \tau; s) \). On this function, we have the following two lemmas, which are described in von Pippich’s preprint [31].

**Lemma 6.4.** For \( \Gamma \)-inequivalent \( z, \tau \in \mathbb{H} \) and \( \operatorname{Re}(s) > 1 \), we have
\[
P(z, \tau; s) = \sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{k!} Q(z, \tau; s + 2k),
\]
which converges absolutely and locally uniformly for \( \operatorname{Re}(s) > 1 \) and \( z \in \mathbb{H} \setminus \Gamma \tau \).

**Proof.** We first check the convergence of the right-hand side. For fixed \( \tau \in \mathbb{H} \) and \( s =: \operatorname{Re}(s) > 1 \), we have
\[
\left| \sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{k!} Q(z, \tau; s + 2k) \right| \leq \sum_{k=0}^{\infty} \frac{|(\frac{z}{k})|}{k!} \sum_{g \in \Gamma} \frac{1}{\cosh(d_{\text{hyp}}(g z, \tau))^s}.
\]

By the trivial bound \(|\Gamma(s)| \leq \Gamma(\sigma)|s| = (s/2 + k)/\Gamma(s/2) \leq \Gamma(s/2 + k)/\Gamma(s/2)|s| \). Also we get the bound \( \cosh(d_{\text{hyp}}(g z, \tau)) \geq C > 1 \) for any compact subset \( z \in K \subset \mathbb{H} \setminus \Gamma \tau \), where the constant \( C \) depends only on \( K \) and \( \tau \). Thus we obtain
\[
\left| \sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{k!} Q(z, \tau; s + 2k) \right| \leq \frac{\Gamma(\frac{z}{2})}{\Gamma^{(\frac{z}{2})}} \sum_{g \in \Gamma} \frac{1}{\cosh(d_{\text{hyp}}(g z, \tau))^s} \sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{k!} \frac{1}{C^2k} = \frac{\Gamma(\frac{z}{2})}{\Gamma^{(\frac{z}{2})}} \left( 1 - \frac{1}{C^2} \right)^{-\frac{z}{2}} Q(z, \tau; s) < \infty.
\]

Similarly for a fixed \( z \in \mathbb{H} \setminus \Gamma \tau \), we can show the locally uniformly convergence of this series for \( \operatorname{Re}(s) > 1 \). Then we can change the order of sums in the right-hand side. Finally the equation
\[
\sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{k!} \frac{1}{\cosh(d_{\text{hyp}}(g z, \tau))^s} = \frac{1}{\sinh(d_{\text{hyp}}(g z, \tau))^s}
\]
concludes the proof. \( \square \)

**Lemma 6.5.** For \( \Gamma \)-inequivalent \( z, \tau \in \mathbb{H} \) and \( \operatorname{Re}(s) > 1 \), we have
\[
G(z, \tau; s) = -\frac{2^s \Gamma(s)^2}{\Gamma(2s)} \sum_{k=0}^{\infty} \frac{(\frac{z}{k})}{(s + \frac{1}{2})_k} Q(z, \tau; s + 2k).
\]

**Proof.** This follows from the formula given in [16, 9.134.1],
\[
2F_1(\alpha, \beta; 2\beta; z) = \left( 1 - \frac{z}{2} \right)^{-\alpha} 2F_1 \left( \frac{\alpha}{2} + \frac{\alpha + 1}{2}, \beta + \frac{1}{2}; \frac{z}{2} - z \right).
\]

The details are similar as in the proof of Lemma 6.4. \( \square \)

By using the lemmas, we easily see that the right-hand side of the function
\[
P(z, \tau; s) + \frac{\Gamma(2s)}{2 \cdot \Gamma(s)^2} G(z, \tau; s) = \sum_{k=1}^{\infty} \frac{(\frac{z}{k})}{k!} \left( 1 - \frac{(s + \frac{1}{2})_k}{(s + 1)_k} \right) Q(z, \tau; s + 2k)
\]
converges in \( \operatorname{Re}(s) > -1/2 \), and has a double zero at \( s = 0 \). From (6.3), we obtain (6.2).

Next we consider the relation between the intermediate function \( Q(z, \tau; s) \) and the Eisenstein series \( E_{0, \gamma}(z, s) \) for a primitive \( \gamma \in \Gamma \) with \( \operatorname{sgn}(Q_\gamma) > 0 \) and \( \tau(\gamma) > 2 \). Similar as (3.8) in the proof of Theorem 3.4, we consider the cycle integral
\[
\int_{\gamma T_0}^{T_0} Q(z, \tau; s) \frac{d\tau}{Q_\gamma(\tau, 1)}.
\]
Here the path of integration is in the geodesic $S_{\gamma}$. Since $Q(z,\tau; s)$ is $\Gamma$-invariant in $\tau$, this integral is independent of the choice of $\tau_0 \in S_{\gamma}$. By the unfolding argument, for $\text{Re}(s) > 1$,
\[
\int_{\tau_0}^{\tau_0} Q(z,\tau; s) \frac{d\tau}{Q_{\gamma}(\tau, 1)} = 2 \int_{\tau_0}^{\tau_0} \sum_{g \in G_{\gamma} \setminus \Gamma} \sum_{n \in \mathbb{Z}} \frac{1}{\cosh(d_{\text{hyp}}(\gamma^n g z, \tau))^s} \frac{d\tau}{Q_{\gamma}(\tau, 1)}
\]
\[
= 2 \sum_{g \in G_{\gamma} \setminus \Gamma} \int_{S_{\gamma}} \frac{1}{\cosh(d_{\text{hyp}}(g z, \tau))^s} \frac{d\tau}{Q_{\gamma}(\tau, 1)},
\]
where $S_{\gamma}$ is oriented from $w'_\gamma$ to $w_\gamma$. By [4, Theorem 7.11.1], we have
\[
\cosh(d_{\text{hyp}}(g z, \tau)) = \cosh(d_{\text{hyp}}(M_\gamma \cdot i|M_\gamma^{-1} g z|, \tau)) \cosh(d_{\text{hyp}}(g z, S_{\gamma})),
\]
where $M_\gamma$ is the scaling matrix given in (3.10). For any $s \in \mathbb{H}$, we see that
\[
\int_{S_{\gamma}} \frac{1}{\cosh(d_{\text{hyp}}(M_\gamma i|\beta|, \tau)))^s} Q_{\gamma}(\tau, 1) = -\frac{1}{\sqrt{D_{\gamma}}} \int_{-\infty}^{\infty} \frac{dr}{\cosh(r - \log |\beta|)^s}
\]
\[
= -\frac{1}{\sqrt{D_{\gamma}}} \int_{-\infty}^{\infty} \frac{dr}{\cosh(r)^s}.
\]
From the formula [26, p.10]
\[
\int_{0}^{\infty} \sinh(t)^\alpha \cosh(t)^{-\beta} dt = \frac{\Gamma\left(\frac{\alpha + 1}{2}\right) \Gamma\left(\frac{\beta - \alpha}{2}\right)}{2\Gamma\left(\frac{\beta + 1}{2}\right)}, \quad (\text{Re}(\beta - \alpha) > 0, \text{Re}(\alpha) > -1),
\]
this integral equals
\[
\int_{S_{\gamma}} \frac{1}{\cosh(d_{\text{hyp}}(M_\gamma i|\beta|, \tau)))^s} Q_{\gamma}(\tau, 1) = -\frac{1}{\sqrt{D_{\gamma}}} \frac{1}{\Gamma\left(\frac{\beta - \alpha}{2}\right)}
\]
Therefore for $\text{Re}(s) > 1$, we obtain that
\[
\int_{\tau_0}^{\tau_0} Q(z,\tau; s) \frac{d\tau}{Q_{\gamma}(\tau, 1)} = \frac{2}{\sqrt{D_{\gamma}}} \frac{\sqrt{\pi} \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s + 1}{2}\right)} \sum_{g \in G_{\gamma} \setminus \Gamma} \frac{1}{\cosh(d_{\text{hyp}}(g z, S_{\gamma})))^s}
\]
\[
= \frac{2}{\sqrt{D_{\gamma}}} \frac{\sqrt{\pi} \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s + 1}{2}\right)} E_{0,\gamma}(z, s).
\]

**Theorem 6.6.** Let $\gamma$ be a primitive hyperbolic element with $\text{sgn}(Q_\gamma) > 0$ and $\text{tr}(\gamma) > 2$. For $\text{Re}(s) > 1$, we have
\[
E_{0,\gamma}(z, s) = -\frac{\Gamma\left(\frac{s + 1}{2}\right)}{2D_{\gamma} \sqrt{\pi} \Gamma\left(\frac{s}{2}\right)} \int_{\tau_0}^{\tau_0} Q(z,\tau; s) \frac{d\tau}{Q_{\gamma}(\tau, 1)}.
\]

By using these results, we now prove Theorem 6.1. For $\text{Re}(s) > 1$, we have
\[
Q(z,\tau; s) = P(z,\tau; s) - \sum_{k=1}^{\infty} \frac{(\gamma)}{k!} k Q(z,\tau; s + 2k)
\]
by Lemma 6.4. By plugging this in Theorem 6.6,
\[
E_{0,\gamma}(z, s) = -\frac{\Gamma\left(\frac{s + 1}{2}\right)}{2D_{\gamma} \sqrt{\pi} \Gamma\left(\frac{s}{2}\right)} \int_{\tau_0}^{\tau_0} \left[ P(z,\tau; s) - \sum_{k=1}^{\infty} \frac{(\gamma)}{k!} k Q(z,\tau; s + 2k) \right] \frac{d\tau}{Q_{\gamma}(\tau, 1)}
\]
Since the sum in the integral converges absolutely and locally uniformly, the termwise integration is legitimate when $z$ is not on the net of geodesics $S_{\gamma^{-1}}$. Then the expression
\[
E_{0,\gamma}(z, s) = -\frac{\Gamma\left(\frac{s + 1}{2}\right)}{2D_{\gamma} \sqrt{\pi} \Gamma\left(\frac{s}{2}\right)} \int_{\tau_0}^{\tau_0} P(z,\tau; s) \frac{d\tau}{Q_{\gamma}(\tau, 1)} - \sum_{k=1}^{\infty} \frac{(\gamma)}{k!} \int_{\tau_0}^{\tau_0} Q(z,\tau; s + 2k) \frac{d\tau}{Q_{\gamma}(\tau, 1)}
\]
\[
= -\frac{\Gamma\left(\frac{s + 1}{2}\right)}{2D_{\gamma} \sqrt{\pi} \Gamma\left(\frac{s}{2}\right)} \int_{\tau_0}^{\tau_0} P(z,\tau; s) \frac{d\tau}{Q_{\gamma}(\tau, 1)} - \sum_{k=1}^{\infty} \frac{D_{\gamma}^2 (\gamma)}{k!} E_{0,\gamma}(z, s + 2k)
\]
gives the meromorphic continuation of $E_1(z, s)$ to $s = 0$. Actually, the infinite sum in the right-hand side converges absolutely and locally uniformly for $\text{Re}(s) > -1$. At $s = 0$ we have the Taylor expansion of the form

$$E_1,\gamma(z, s) = \left[ \frac{\sqrt{D_\gamma}}{2} \int_{\gamma_0}^{\gamma_0 + \infty} \log |j(z) - j(\tau)| \frac{d\tau}{Q_\gamma(\tau, 1)} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{\sqrt{\pi D_\gamma \Gamma(k)}}{k \Gamma(k + \frac{1}{2})} E_{0,\gamma}(z, 2k) \right] s^2 + O(s^3).$$

For the second term, we easily see that

$$\frac{1}{4} \sum_{k=1}^{\infty} \frac{\sqrt{\pi D_\gamma \Gamma(k)}}{k \Gamma(k + \frac{1}{2})} E_{0,\gamma}(z, 2k) = \frac{1}{4} \sum_{k=1}^{\infty} \frac{\sqrt{\pi \Gamma(k)}}{k \Gamma(k + \frac{1}{2})} \sum_{\gamma \in \Gamma \backslash \Gamma_0} \frac{1}{\cosh(d_{\text{hyp}}(gz, S_\gamma))^{4k}} = \frac{1}{2} \sum_{\gamma \in \Gamma \backslash \Gamma} \arcsin^2 \left( \frac{1}{\cosh(d_{\text{hyp}}(gz, S_\gamma))} \right).$$

Therefore we finally obtain Theorem 6.1.

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