Groups of Operators for Evolution Equations of Quantum Many-Particle Systems

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Abstract

The aim of this work is to study the properties of groups of operators for evolution equations of quantum many-particle systems, namely, the von Neumann hierarchy for correlation operators, the BBGKY hierarchy for marginal density operators and the dual BBGKY hierarchy for marginal observables. We show that the concept of cumulants (semi-invariants) of groups of operators for the von Neumann equations forms the basis of the expansions for one-parametric families of operators for evolution equations of infinitely many particles.

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1 Introduction

Recently we observe significant progress in the study of the evolution equations of quantum many-particle systems [1],[2]. In particular it is involved in such fundamental problem as the rigorous derivation of quantum kinetic equations [3]-[9].

As is well known, there are various possibilities to describe the evolution of quantum many-particle systems [1]. The sequence of the von Neumann equations for density operators [2],[11], the von Neumann hierarchy for correlation operators [23], the BBGKY hierarchy for marginal density operators [11] and the dual BBGKY hierarchy for marginal observables [1] give the equivalent approaches for the description of the evolution of finitely many particles. Papers [23]-[27] constructed the one-parametric families of operators that define solutions of the Cauchy problem for these evolution equations. It was established that the concept of cumulants (semi-invariants) of groups of operators for the von Neumann equations forms the basis of the one-parametric families of operators of various evolution equations of quantum systems of particles, in particular, the BBGKY hierarchy for infinitely many particles [23].

The aim of the paper is to investigate properties of groups of operators for evolution equations of quantum many-particle systems related with their cumulant structure on suitable Banach spaces. In the beginning we will formulate some necessary facts about the description of quantum many-particle systems.

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1.1 Group of Operators for the von Neumann Equation

Let a sequence \( f = (I, f_1, \ldots, f_n, \ldots) \) be an infinite sequence of self-adjoint operators \( f_n \) (\( I \) is a unit operator) defined on the Fock space \( \mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n \) over the Hilbert space \( \mathcal{H} \) (\( \mathcal{H}^0 = \mathbb{C} \)). Operators \( f_n \) defined in the \( n \)-particle Hilbert space \( \mathcal{H}_n = \mathcal{H}^\otimes n \) will be denoted by \( f_n(1, \ldots, n) \). For a system of identical particles obeying Maxwell-Boltzmann statistics, one has \( f_n(1, \ldots, n) = f_n(i_1, \ldots, i_n) \) if \( \{i_1, \ldots, i_n\} \in \{1, \ldots, n\} \).

Let \( \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{L}_n^1(\mathcal{H}_n) \) be the space of sequences \( f = (I, f_1, \ldots, f_n, \ldots) \) of trace class operators \( f_n = f_n(1, \ldots, n) \in \mathcal{L}_n^1(\mathcal{H}_n) \), satisfying the above-mentioned symmetry condition, equipped with the trace norm

\[
\|f\|_{\mathcal{L}_n^1(\mathcal{F}_\mathcal{H})} = \sum_{n=0}^{\infty} \alpha^n \|f_n\|_{\mathcal{L}_n^1(\mathcal{H}_n)} = \sum_{n=0}^{\infty} \alpha^n \text{Tr} f_1 \ldots f_n(1, \ldots, n),
\]

where \( \alpha > 1 \) is a real number, \( \text{Tr}_{1,\ldots,n} \) is the partial trace over \( 1, \ldots, n \) particles. We will denote by \( \mathcal{L}_{\alpha,0}^1 \) the everywhere dense set in \( \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) \) of finite sequences of degenerate operators \([14]\) with infinitely differentiable kernels with compact supports. We will also consider the space \( \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{L}_n^1(\mathcal{H}_n) \).

We note that the sequences of operators \( f_n \in \mathcal{L}_n^1(\mathcal{H}_n), n \geq 1 \), whose kernels are known as density matrices \([12]\) defined on the \( n \)-particle Hilbert space \( \mathcal{H}_n = \mathcal{H}^\otimes n = L^2(\mathbb{R}^{3n}) \), describe the states of a quantum system of non-fixed number of particles. The space \( \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) \) contains sequences of operators more general than those determining the states of systems.

The evolution of all possible states of quantum systems is described by the initial-value problem to the von Neumann equation \([10][11]\). A solution of such Cauchy problem is defined by the following one-parametric family of operators on \( \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) \)

\[
\mathbb{R}^1 \ni t \mapsto \mathcal{G}(-t)f := \mathcal{U}(-t)f\mathcal{U}^{-1}(-t),
\]

where \( f \in \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) \) and \( \mathcal{U}(t) = \bigoplus_{n=0}^{\infty} \mathcal{U}_n(t) \),

\[
\mathcal{U}_n(t) := e^{-\frac{i}{\hbar}tH_n},
\]

\[
\mathcal{U}_n^{-1}(t) := e^{\frac{i}{\hbar}tH_n},
\]

\( \mathcal{U}_0(t) = I \) is a unit operator. The Hamiltonian \( H = \bigoplus_{n=0}^{\infty} H_n \) in \([12]\) is a self-adjoint operator with domain \( \mathcal{D}(H) = \{\psi = \bigoplus_{n=0}^{\infty} \psi_n \in \mathcal{F}_\mathcal{H} \mid \psi_n \in \mathcal{D}(H_n) \subset \mathcal{H}_n, \sum_n \|H_n\psi_n\|^2 < \infty \} \subset \mathcal{F}_\mathcal{H} \).

Assume \( \mathcal{H} = L^2(\mathbb{R}^3) \) then an element \( \psi \in \mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^{3n}) \) is a sequence of functions \( \psi = (\psi_0, \psi_1(q_1), \ldots, \psi_n(q_1, \ldots, q_n), \ldots) \) such that \( \|\psi\|^2 = \|\psi_0\|^2 + \sum_{n=1}^{\infty} \int dq_1 \ldots dq_n |\psi_n(q_1, \ldots, q_n)|^2 < +\infty \). On the subspace of infinitely differentiable functions with compact supports \( \psi_n \in L^2_0(\mathbb{R}^{3n}) \subset L^2(\mathbb{R}^{3n}) \), \( n \)-particle Hamiltonian \( H_n \) acts according to the formula \( (H_0 = 0) \)

\[
H_n\psi_n = -\frac{\hbar^2}{2} \sum_{i=1}^{n} \Delta q_i \psi_n + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k}^{n} \Phi^{(k)}(q_{i_1}, \ldots, q_{i_k}) \psi_n.
\]

where \( \Phi^{(k)} \) is a \( k \)-body interaction potential satisfying Kato conditions \([13]\) and \( \hbar = 2\pi\hbar \) is a Planck constant.

We remark that the nature of notations \([1,2]\) used for unitary groups \( e^{\pm \frac{i}{\hbar}tH_n} \) is related to the correspondence principle between quantum and classical systems \([1]\) and is a consequence of the existence of two approaches to the description of the evolution of observables in the framework of observables or states.

The properties of a one-parametric family \( \{\mathcal{G}(-t)\}_{t \in \mathbb{R}} \) of operators \([11]\) follow from the properties of groups \([1,2]\) described, for example, in \([12]\).
Proposition 1.1 ([10], [11]). On the space $\mathfrak{L}^1(\mathcal{F}_H)$ mapping (1.1) defines an isometric strongly continuous group, i.e. one is a $C_0$-group, which preserves positivity and self-adjointness of operators.

If $f \in \mathfrak{L}_0^1(\mathcal{F}_H) \subset \mathcal{D}(-\mathcal{N}) \subset \mathfrak{L}^1(\mathcal{F}_H)$ then in the sense of the norm convergence of the space $\mathfrak{L}^1(\mathcal{F}_H)$ there exists a limit that is determined the infinitesimal generator: $-\mathcal{N} = \bigoplus_{n=0}^{\infty} (-\mathcal{N}_n)$ of group (1.1)

$$\lim_{t \to 0} \frac{1}{t} (\mathcal{G}(t)f - f) = -\frac{i}{\hbar} (Hf - fH) := -\mathcal{N}f,$$

where $H = \bigoplus_{n=0}^{\infty} \mathcal{H}_{n}$ is the Hamiltonian [13] and the operator: $(-i/\hbar)(Hf - fH)$ is defined on the domain $\mathcal{D}(H) \subset \mathcal{F}_H$.

Group of operators (1.1) and infinitesimal generator (1.4) studied within the framework of kernels and symbols of the operators in [12] and for the Wigner representation in [19]. Some applications of evolution groups of quantum systems are considered in [18].

1.2 Group of Operators for the Heisenberg Equation

The adjoint to $\mathfrak{L}^1(\mathcal{F}_H)$ space is isometric to the space $\mathfrak{L}(\mathcal{F}_H)$ of sequences $g = (I, g_1, \ldots, g_n, \ldots)$ of bounded operators $g_n$ ($I$ is a unit operator) defined on the Hilbert space $\mathcal{H}_n$ satisfying symmetry property $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$ for $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n\}$ with an operator norm. The space $\mathfrak{L}(\mathcal{F}_H)$ is dual to the space $\mathfrak{L}^1(\mathcal{F}_H)$ with respect to the bilinear form

$$\langle g | f \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, n} g_n f_n,$$

(1.5)

We will also consider more general space $\mathfrak{L}_\gamma(\mathcal{F}_H)$ than $\mathfrak{L}(\mathcal{F}_H)$ with a norm

$$\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_H)} = \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)},$$

where $0 < \gamma < 1$ and $\|\|_{\mathfrak{L}(\mathcal{H}_n)}$ is an operator norm [14].

An observable of finitely many quantum particles is a sequence of self-adjoint operators from $\mathfrak{L}_\gamma(\mathcal{F}_H)$ and positive normalized continuous linear functional (1.5) on the space of observables is interpreted as its mean value [13]. The case of the unbounded observables can be reduced to the case under consideration [10].

The evolution of observables is described by the initial-value problem to the Heisenberg equation (the dual von Neumann equation) [10], [11]. On $\mathfrak{L}(\mathcal{F}_H)$ a solution of the Cauchy problem to the Heisenberg equation is defined the following one-parametric family of operators [10]

$$\mathbb{I}^1 \ni t \mapsto \mathcal{G}(t)g : = \mathcal{U}(t)g \mathcal{U}^{-1}(t),$$

(1.6)

where $\mathcal{U}(t) = \bigoplus_{n=0}^{\infty} \mathcal{U}_n(t)$ and the operators $\mathcal{U}_n(t)$ are defined by formulas (1.2). Mapping $\mathcal{G}(t)$ (1.6) is adjoint (dual) to $\mathcal{G}(-t)$ (1.1).

Proposition 1.2 ([10], [13]). On the space $\mathfrak{L}_\gamma(\mathcal{F}_H)$ mapping (1.6) defines an isometric $*$-weak continuous group, i.e. one is a $C_0^*$-group. This group preserves the self-adjointness of operators. The infinitesimal generator $\mathcal{N} = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$ of this group of operators is a closed operator for the $*$-weak topology and on its domain of the definition $\mathcal{D}(\mathcal{N}) \subset \mathfrak{L}_\gamma(\mathcal{F}_H)$ which is the everywhere dense set for the $*$-weak topology it is defined in the sense of the $*$-weak convergence of the space $\mathfrak{L}_\gamma(\mathcal{F}_H)$ by the formula

$$w^* \lim_{t \to 0} \frac{1}{t}(\mathcal{G}(t)g - g) = -\frac{i}{\hbar} (gH - Hg),$$

(1.7)

where $H = \bigoplus_{n=0}^{\infty} \mathcal{H}_{n}$ is the Hamiltonian [13] and the operator: $\mathcal{N}g = (-i/\hbar)(gH - Hg)$ is defined on the domain $\mathcal{D}(H) \subset \mathcal{F}_H$. 

3
1.3 Cumulants of Groups of Operators

Further to the group \(\{G(t)\}_{t \in \mathbb{R}}\) we will also consider more general mappings on the space \(\mathfrak{L}^1_0(\mathcal{F}_H)\). Namely let us expand the group \(G(t) = \bigoplus_{n=0}^{\infty} G_n(t)\) as following cluster expansions

\[
G_n(t, Y) = \sum_{P: Y = \bigcup_i X_i} \prod_{X_i \subseteq P} \mathcal{A}_{|X_i|}(t, X_i), \quad n = |Y| \geq 0,
\]

where the notations are similar to that in formula (1.8). The operator \(\mathcal{A}_n(t, Y)\) is defined by the formula (1.8), i.e.

\[
\lim_{t \to 0} \frac{1}{t} (\mathcal{A}_n(t, Y) - I) f_n(Y) = -\mathcal{N}_n(Y) f_n(Y),
\]

where for \(f_n \in \mathfrak{L}^1_0(\mathcal{H}_n)\) this limit exists in the sense of the norm convergence of the space \(\mathfrak{L}^1(\mathcal{H}_n)\).

The infinitesimal generator of the \(n\)-th order cumulant, \(n \geq 2\), is an operator \((-\mathcal{N}^{(n)}_{\text{int}})\) defining by \(n\)-body interaction potential (1.3). According to the equality

\[
\sum_{P: Y = \bigcup_i X_i} (-1)^{|P| - 1} (|P| - 1)! \mathcal{A}_{|X_i|}(-t, X_i) = \sum_{k=1}^{n} (-1)^{k-1} s(n, k)(k - 1)! = \delta_{n,1},
\]

where \(s(n, k)\) is the Stirling numbers of the second kind and \(\delta_{n,1}\) is a Kroneker symbol, for the \(n\)-th order cumulant, \(n \geq 2\), in the sense of a point-by-point convergence of the space \(\mathfrak{L}^1(\mathcal{H}_n)\) we have

\[
\lim_{t \to 0} \frac{1}{t} \sum_{X_i \subseteq P} \mathcal{A}_{|X_i|}(t, X_i) f_n(Y) = \sum_{X_i \subseteq P} (-1)^{|P| - 1} (|P| - 1)! \mathcal{N}_{|X_i|}(X_i) f_n(Y) = \sum_{X_i \subseteq P} (-1)^{|P| - 1} (|P| - 1)! \sum_{X_i \subseteq X_j} \sum_{X_i \subseteq X_l} (\mathcal{N}^{(k)}_{\text{int}}(i_1, \ldots, i_k)) f_n(Y).
\]

Here for the operator \(\Phi^{(n)}\) from Hamiltonian (1.3) the operator \(\mathcal{N}^{(n)}_{\text{int}}\) is defined by the formula

\[
\mathcal{N}^{(n)}_{\text{int}} f_n := - \frac{i}{\hbar} (f_n \Phi^{(n)} - \Phi^{(n)} f_n).
\]

Summing coefficients before every operator \(\mathcal{N}^{(k)}_{\text{int}}\) we deduce

\[
\lim_{t \to 0} \frac{1}{t} \mathcal{A}_n(t, Y) f_n(Y) = -\mathcal{N}^{(n)}_{\text{int}}(Y) f_n(Y),
\]

Thus for \(f_n \in \mathfrak{L}^1_0(\mathcal{H}_n)\) the generator of the \(n\)-th order cumulant is defined by formula (1.12) in the sense of the norm convergence of the space \(\mathfrak{L}^1(\mathcal{H}_n)\).

The dual cumulants of the groups \(G(t) = \bigoplus_{n=0}^{\infty} G_n(t)\) (1.6) will be introduced in Section 4. For classical many-particle systems cumulants of evolution operators were introduced in [26, 27].
2 Group of Operators for the von Neumann Hierarchy

The evolution of correlations of quantum finitely many particles is described by the von Neumann hierarchy for the correlation operators [23]. It is an equivalent approach for the description of the evolution of all possible states of quantum many-particle systems in comparison with the approach formulated above.

In what follows we will use such abridged notations: \( Y \equiv (1, \ldots, n) \) and \( Y_P \equiv \left( X_1, \ldots, X_{|P|} \right) \) is a set whose elements are \(|P|\) mutually disjoint subsets \( X_i \subset Y \) of the partition \( P : Y = \bigcup_{i=1}^{|P|} X_i \). The \(|P|\)th-order cumulant [1.9] in this case we denote by \( \mathfrak{A}_n(t, Y_P) \).

A solution of an initial-value problem to the von Neumann hierarchy is defined by a one-parametric family of nonlinear operators [13] constructed in [23] with the following properties.

**Theorem 2.1.** If \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \), \( n \geq 1 \), then the one-parametric family of nonlinear operators

\[
\mathbb{R}^1 \ni t \mapsto \left( \mathfrak{A}_t(f) \right)_n(Y) := \sum_{P : Y = \bigcup_{i} X_i, \ |P| > 1} \mathfrak{A}_n(t, Y_P) \prod_{X_i \in P} f_{|X_i|}(X_i) 
\]

(2.1)

is a \( C_0 \)-group. On the subspace \( \mathcal{L}^1_0(\mathcal{H}_n) \subset \mathcal{L}^1(\mathcal{H}_n) \) the infinitesimal generator \( \mathfrak{R} \) of group (2.1) is defined by the operator

\[
\left( \mathfrak{R}(f) \right)_n(Y) := -\mathcal{N}_n(Y) f_n(Y) + \sum_{P : Y = \bigcup_{i} X_i, \ |P| > 1} \sum_{Z_1 \subset X_i, \ Z_1 \neq \emptyset} \cdots \sum_{Z_{|P|} \subset X_{|P|}, \ Z_{|P|} \neq \emptyset} \left( -\mathcal{N}_n \left( \sum_{i=1}^{|P|} |Z_i| \right) \right) \prod_{X_i \in P} f_{|X_i|}(X_i),
\]

(2.2)

where the notations are similar to that in (1.8) and the operator \( \mathcal{N}_n \) is defined by formula (1.11).

**Proof.** Mapping (2.1) is defined for \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \), \( n \geq 1 \), and the following inequality holds

\[
\| \left( \mathfrak{A}_t(f) \right)_n \|_{\mathcal{L}^1(\mathcal{H}_n)} \leq n!e^{2n+1} c^n,
\]

(2.3)

where \( c := \max_{P : Y = \bigcup_{i} X_i} \| f_{|X_i|}(X_i) \|_{\mathcal{L}^1(\mathcal{H}_n)} \). Indeed, since for \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \) the equality holds [23]

\[
\text{Tr}_{1,\ldots,n} |\mathcal{G}_n(-t) f_n| = \| f_n \|_{\mathcal{L}^1(\mathcal{H}_n)},
\]

we have

\[
\| \left( \mathfrak{A}_t(f) \right)_n \|_{\mathcal{L}^1(\mathcal{H}_n)} \leq \sum_{P : Y = \bigcup_{i} X_i, \ |P| > 1} \sum_{Z_1 \subset X_i, \ Z_1 \neq \emptyset} \cdots \sum_{Z_{|P|} \subset X_{|P|}, \ Z_{|P|} \neq \emptyset} \left( |P| - 1 \right)! \prod_{X_i \in P} \| f_{|X_i|} \|_{\mathcal{L}^1(\mathcal{H}_n)} \leq \sum_{P : Y = \bigcup_{i} X_i} c^{|P|} \sum_{k=1}^{|P|} s(|P|, k)(k-1)! \leq \sum_{P : Y = \bigcup_{i} X_i} c^{|P|} \sum_{k=1}^{|P|} k^{|P|-1} \leq n!e^{2n+1} c^n,
\]

where \( s(|P|, k) \) are the Stirling numbers of the second kind. That is, \( \left( \mathfrak{A}_t(f) \right)_n \in \mathcal{L}^1(\mathcal{H}_n) \) for arbitrary \( t \in \mathbb{R}^1 \) and \( n \geq 1 \).

The group property of a one-parametric family of nonlinear operators (2.1), i.e. \( \mathfrak{A}_{t_1} \left( \mathfrak{A}_{t_2}(f) \right) = \mathfrak{A}_{t_1+t_2}(f) \), was proved in [23].
The strong continuity property of the group \( \{ \mathfrak{A}_t \}_{t \in \mathbb{R}} \) over the parameter \( t \in \mathbb{R} \) is a consequence of the strong continuity of group \( \{ \mathfrak{L}_t \}_{t \in \mathbb{R}} \) of the von Neumann equation \([10]\). Indeed, according to identity \((1.10)\) the following equality holds:

\[
\sum_{P: Y = \bigcup_i X_i} \sum_{P': Y_P = \bigcup_k Z_k} (-1)^{|P'| - 1} (|P'| - 1)! \prod_{X_i \subset P} f_{X_i}(X_i) = f_n(Y),
\]

Therefore, for \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \subset \mathcal{L}^1(\mathcal{H}_n), n \geq 1 \), we have

\[
\lim_{t \to 0} \| (\mathfrak{A}_t(f))_n(Y) - f_n(Y) \|_{\mathcal{L}^1(\mathcal{H}_n)} \leq \sum_{P: Y = \bigcup_i X_i} \sum_{P': Y_P = \bigcup_k Z_k} (|P'| - 1)! \lim_{t \to 0} \| \left( \prod_{Z_k \subset P'} \mathfrak{G}_{|Z_k|}(-t, Z_k) - I \right) \prod_{X_i \subset P} f_{X_i}(X_i) \|_{\mathcal{L}^1(\mathcal{H}_n)}.
\]

In view of the the fact that group \( \{ \mathfrak{G}_n(-t) \}_{t \in \mathbb{R}} \) is a strong continuous group, which implies that, for mutually disjoint subsets \( X_i \subset Y \), if \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \subset \mathcal{L}^1(\mathcal{H}_n) \) in the sense of the norm convergence \( \mathcal{L}^1(\mathcal{H}_n) \) there exists the limit

\[
\lim_{t \to 0} \left( \prod_{Z_k \subset P'} \mathfrak{G}_{|Z_k|}(-t, Z_k) f_n - f_n \right) = 0.
\]

Thus if \( f \in \mathcal{L}^1(\mathcal{F}_n) \) we finally obtain

\[
\lim_{t \to 0} \| (\mathfrak{A}_t(f))_n - f_n \|_{\mathcal{L}^1(\mathcal{H}_n)} = 0.
\]

We now construct the infinitesimal generator \( \mathfrak{R} \) of group \( \{ \mathfrak{L}_t \}_{t \in \mathbb{R}} \). Taking into account that for \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \) equality \((1.1)\) holds, we differentiate the \(|P|\)th-order cumulant \( \mathfrak{A}_{|P|}(t, Y_P) \) for all \( \psi_n \in \mathcal{D}(\mathcal{H}_n) \subset \mathcal{H}_n \) in the sense of the point-by-point convergence. According to equality \((1.12)\) for \(|P| \geq 2\) we derive

\[
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{|P|}(t, Y_P) f_n \psi_n = \sum_{P: Y_P = \bigcup_k Z_k} (-1)^{|P'| - 1} (|P'| - 1)! \sum_{Z_k \subset P'} (-N_{\text{int}}(\sum_{i=1}^{|P|} |Z_i|)) f_n \psi_n,
\]

(2.4)

where \( \sum_{Z_j \subset X_j} \) is a sum over all subsets \( Z_j \subset X_j \) of the set \( X_j \). Then in view of equality \((2.4)\) for group \( \{ \mathfrak{L}_t \}_{t \in \mathbb{R}} \) we obtain

\[
\lim_{t \to 0} \frac{1}{t} \left( (\mathfrak{A}_t(f))_n - f_n \right) \psi_n = \lim_{t \to 0} \frac{1}{t} \left( \sum_{P: Y = \bigcup_i X_i} \mathfrak{A}_{|P|}(t, Y_P) \prod_{X_i \subset P} f_{X_i}(X_i) - f_n(Y) \right) \psi_n
\]

\[
= \lim_{t \to 0} \frac{1}{t} \left( \mathfrak{A}_1(t, Y) f_n - f_n \right) \psi_n + \sum_{P: Y = \bigcup_i X_i, |P| > 1} \lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{|P|}(t, Y_P) \prod_{X_i \subset P} f_{X_i}(X_i) \psi_n
\]

\[
= (-N_n f_n)(Y) \psi_n + \sum_{P: Y = \bigcup_i X_i, |P| > 1} \sum_{Z_1 \subset X_1, Z_1 \neq \emptyset} \ldots \sum_{Z_|P| \subset X_{|P|}, Z_{|P|} \neq \emptyset} (-N_{\text{int}}(\sum_{i=1}^{|P|} |Z_i|)) \prod_{X_i \subset P} f_{X_i}(X_i) \psi_n.
\]
Thus for \( f \in L^1_0(\mathcal{F}_\mathcal{H}) \subset D(\mathcal{H}) \subset L^1(\mathcal{F}_\mathcal{H}) \) in the sense of the norm convergence \( L^1(\mathcal{H}_n) \) we finally have
\[
\lim_{t \to 0} \frac{1}{t} \left( (A_t(f))_n - (f(n)) \right) = 0.
\]

We give an example which illustrates the structure of expansion (2.1). For the correlation operators \( f = (0, f_1(1), 0, \ldots) \) that is interpreted as satisfying the ”chaos” property [1], we have
\[
(A_t(f))_n = A_n(t, 1, \ldots, n) \prod_{i=1}^n f_1(i), \quad n \geq 1,
\]
i.e. if at the initial instant there are no correlations in a system, the correlations generated by the dynamics of a system are completely governed by cumulants of groups (1.1).

We now consider the structure of infinitesimal generator (2.2) for a two-body interaction potential
\[
(\mathcal{N}(f))_n(Y) = -\mathcal{N}_n(Y) f_n(Y) + \sum_{P: Y = X_1 \cup X_2} \sum_{i_1 \in \{X_1\}} \sum_{i_2 \in \{X_2\}} (-\mathcal{N}^{(2)}_{\text{int}}(i_1, i_2)) f_{|X_1|}(X_1) f_{|X_2|}(X_2),
\]
where the symbol \( \sum_{P: Y = X_1 \cup X_2} \) means summation over all partitions of the set \( Y \) into two nonempty parts \( X_1 \) and \( X_2 \) and the operator \( \mathcal{N}^{(2)}_{\text{int}} \) is defined by formula (1.11). For classical systems this generator is an equivalent notion of the generator of the Liouville hierarchy [20] formulated in [21].

### 3 Group of Operators for the Quantum BBGKY Hierarchy

The evolution of all possible states both finitely and infinitely many quantum particles is described by the initial-value problem to the BBGKY hierarchy for marginal density operators [11, 24]. For finitely many particles this hierarchy of equations is an equivalent to the von Neumann equation.

We will use notations from the previous section. Since \( Y_P \equiv (X_1, \ldots, X_{|P|}) \) then \( Y_1 \) is the set consisting of one element of the partition \( P \ (|P| = 1) \) of the set \( Y \equiv (1, \ldots, s) \). In this case for \( n \geq 0 \) the \((1 + n)\text{th-order cumulant of operators} \ (1.1) \ is defined by the formula
\[
\mathcal{A}_{1+n}(t, Y_1, X \setminus Y) := \sum_{P: Y_1 \setminus Y = P \setminus X_1} (-1)^{|P| - 1}(|P| - 1)! \prod_{X_i \in P} \mathcal{G}_{|X_i|}(-t, X_i), \quad (3.1)
\]
where \( \sum_P \) is the sum over all possible partitions \( P \) of the set \( \{Y_1, X \setminus Y\} = \{Y_1, s + 1, \ldots, s + n\} \) into \(|P| \) nonempty mutually disjoint subsets \( X_i \subset \{Y_1, X \setminus Y\} \).

On the space \( L^1_\alpha(\mathcal{F}_\mathcal{H}) \) a solution of the initial-value problem to the BBGKY hierarchy is defined by a one-parametric mapping [24] with the following properties.

**Theorem 3.1.** If \( f \in L^1_\alpha(\mathcal{F}_\mathcal{H}) \) and \( \alpha > e \), then the one-parametric mapping
\[
\mathbb{R}^1 \ni t \mapsto (U(t) f)(s) := \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{T}_{r+1, \ldots, s+n} \mathcal{A}_{1+n}(t, Y_1, X \setminus Y) f_{s+n}(X) \quad (3.2)
\]
is a \( C_0 \)-group. On the subspace \( L^1_{\alpha, 0} \subset L^1_\alpha(\mathcal{F}_\mathcal{H}) \) the infinitesimal generator \( \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathcal{B}_n \ of \ group \ (3.2) \) is defined by the operator \((s + 1)\)
\[
(\mathcal{B} f)(s) := -\mathcal{N}_s(Y) f_s(Y) + \sum_{k=1}^{s} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k} \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{T}_{r+1, \ldots, s+n} \mathcal{N}^{(k+n)}_{\text{int}}(i_1, \ldots, i_k, X \setminus Y) f_{s+n}(X), \quad (3.3)
\]
where on \( L^1_{\alpha}(\mathcal{H}_{s+n}) \subset L^1(\mathcal{H}_{s+n}) \) the operator \( \mathcal{N}^{(k+n)}_{\text{int}} \) is defined by formula (1.11).
where the operator $L$ is defined on $\mathcal{H}$ according to equality (2.4) in the framework of kernels of operators $f_s$ $(s$-particle density density matrix or marginal

Proof. If $f \in \mathcal{L}^1_\alpha(\mathcal{F}_\mathcal{H})$ mapping (3.2) is defined provided that $\alpha > e$ and the following estimate holds

$$
\|U(t)f\|_{\mathcal{L}^1_\alpha(\mathcal{F}_\mathcal{H})} \leq c_\alpha \|f\|_{\mathcal{L}^1_\alpha(\mathcal{F}_\mathcal{H})},
$$

where $c_\alpha = e^2 (1 - \frac{e}{\alpha})^{-1}$ is a constant. Similar to (2.3) this estimate comes out from the inequality for cumulant (3.1)

$$
\|\mathfrak{A}_{1+n}(t)f_{s+n}\|\mathcal{L}^1(\mathcal{H}_{s+n}) \leq n!e^{n+2} \|f_{s+n}\|\mathcal{L}^1(\mathcal{H}_{s+n}^*).
$$

The strong continuity property of the group $U(t)$ over the parameter $t \in \mathbb{R}$ is a consequence of the strong continuity of group (1.1) of the von Neumann equation.

We now construct an infinitesimal generator of group (3.2). Taking into account that for $f_n \in \mathcal{L}^1_0(\mathcal{H}_n)$ equality (1.4) holds, we differentiate the expression of cumulant (3.1) in the sense of the point-by-point convergence. According to equality (2.4) for $n \geq 1$, we derive

$$
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{1+n}(t,Y_1,X\backslash Y)f_{s+n}\psi_{s+n} = \sum_{Z \subset Y, Z \neq \emptyset} (-\mathcal{N}^{(\mathcal{H})}_{\int}(s+n))(Z,s+1,\ldots,s+n)f_{s+n}\psi_{s+n} =
$$

$$
\sum_{|Y|=k} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k=1} (\mathcal{N}^{(\mathcal{H})}_{\int}(i_1,\ldots,i_k,X\backslash Y)f_{s+n}\psi_{s+n},
$$

where $\sum_{Z \subset X}$ is a sum over all subsets $Z \subset X$ of the set $X$. Then taking into account formula (1.4) for $n = 1$ and equality (3.1) for group (3.2) we obtain

$$
\lim_{t \to 0} \frac{1}{t} \left((U(t)f) - f_s\right)\psi_s =
$$

$$
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{1}(t,Y)f_s - f_s)\psi_s + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \lim_{t \to 0} \frac{1}{t} \mathfrak{A}_{1+n}(t,Y_1,X\backslash Y)f_{s+n}\psi_{s+n} =
$$

$$
\mathcal{N}_s f_s \psi_s + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \sum_{k=1}^{s} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k=1} (\mathcal{N}^{(\mathcal{H})}_{\int}(i_1,\ldots,i_k,X\backslash Y))\psi_s.
$$

Thus, if $\mathcal{L}^1{\alpha,0} \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{L}^1_\alpha(\mathcal{F}_\mathcal{H})$ we finally have in the sense of the norm convergence

$$
\lim_{t \to 0} \frac{1}{t} \left((U(t)f) - f\right)\mathcal{L}^1_\alpha(\mathcal{F}_\mathcal{H}) = 0,
$$

where the operator $\mathcal{B}$ on $\mathcal{L}^1_\alpha,0$ is given by formula (3.3).

We now give an example which illustrates the structure of infinitesimal generator (3.3). In the case of two-body interaction potential (1.3) operator (3.3) has the form $(s \geq 1)$

$$
(\mathcal{B}f)_s(Y) = -\mathcal{N}_s(Y)f_s(Y) + \sum_{i=1}^{s} \text{Tr}_{s+1}(-\mathcal{N}^{(2)}_{\int}(i,s+1)f_{s+1}(Y,s+1),
$$

where the operator $\mathcal{N}^{(2)}_{\int}$ is defined on $\mathcal{L}^1_\alpha(\mathcal{H}_{s+1}) \subset \mathcal{L}^1_\alpha(\mathcal{H}_{s+1})$ by formula (1.11) for $n = 2$. For $\mathcal{H} = L^2(\mathbb{R}^3)$ in the framework of kernels of operators $f_s$ $(s$-particle density density matrix or marginal
The independent from the variable recurrence relations (1.8) with respect to the 1st and trace class operators $f_n$ then if
\[ \langle \mathcal{A}_s \rangle (q_1, \ldots, q_s; q'_1, \ldots, q'_s) = \]
\[ -\frac{i}{\hbar} \left( -\frac{\hbar^2}{2} \sum_{i=1}^{s} (\Delta q_i - \Delta q'_i) + \sum_{i<j=1}^{s} (\Phi^{(2)}(q_i - q_j) - \Phi^{(2)}(q'_i - q'_j)) \right) f_s(q_1, \ldots, q_s; q'_1, \ldots, q'_s) - \]
\[ \frac{i}{\hbar} \sum_{i=1}^{s} \int dq_{s+1} (\Phi^{(2)}(q_i - q_{s+1}) - \Phi^{(2)}(q'_i - q_{s+1})) f_{s+1}(q_1, \ldots, q_s, q_{s+1}; q'_1, \ldots, q'_s, q_{s+1}). \]

In [11,24] for quantum and in the book [1] for classical systems of particles with a two-body interaction potential, an equivalent representation for group (3.2) was used, namely for the case under consideration the group $U(t)$ has the following representation
\[ U(t) = \mathcal{G}(-t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ a, \mathcal{G}(-t) \right] \ldots \right] = e^a \mathcal{G}(-t) e^{-a}, \quad (3.6) \]
where $[\ldots]$ is a commutator, the operator $a$ (an analog of the annihilation operator) is defined on the space $\mathcal{L}^1_\alpha(\mathcal{F}_H)$ by the formula
\[ (af)_s(Y) := \text{Tr}_{s+1} f_{s+1}(Y, s + 1), \quad (3.7) \]
the operators $e^{\pm a}$ are defined on the space $\mathcal{L}^1_\alpha(\mathcal{F}_H)$ by the expansions
\[ (e^{\pm a})_s(Y) = \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} \text{Tr}_{s+1,\ldots,s+n} f_{s+n}(Y, s + 1, \ldots, s + n), \quad s \geq 1. \]

and the group of operators $\mathcal{G}(-t)$ is defined by expression (1.1). Representation (3.6) is true in consequence of definition (3.7) of the operator $a$ and the validity of an equality for every $s \geq 1$
\[ \left( \left[ a, \ldots, \left[ a, \mathcal{G}(-t) \right] \ldots \right] f_s \right)_s(Y) = \text{Tr}_{s+1,\ldots,s+n} \mathcal{A}_{1+n}(t, Y, X \setminus Y) f_{s+n}(X), \]
where the notations are similar to that in (3.1). For example, since for an isometric group $\mathcal{G}(-t)$ and trace class operators $f$ the following equality holds
\[ \text{Tr}_{s+1} \mathcal{G}_1(-t, s + 1) f_{s+1}(Y, s + 1) = \text{Tr}_{s+1} f_{s+1}(Y, s + 1). \quad (3.8) \]
then if $n = 1$ we have
\[ \left( \left[ a, \mathcal{G}(-t) \right] f_s \right)_s(Y) = \text{Tr}_{s+1} \left( \mathcal{G}_{s+1}(-t, Y, s + 1) f_{s+1}(Y, s + 1) - \mathcal{G}_s(-t, Y) f_{s+1}(Y, s + 1) \right) = \text{Tr}_{s+1} \mathcal{A}_2(t, Y, s + 1) f_{s+1}(Y, s + 1). \]

We note that cluster expansions (1.8) can be put into basis of all possible representations of a group of operators for the quantum BBGKY hierarchy. In fact representation (3.6) we obtain solving recurrence relations (1.8) with respect to the 1st-order cumulants for the separation terms which are independent from the variable $Y$
\[ \mathcal{A}_{1+n}(t, Y, X \setminus Y) = \sum_{Z \subset X \setminus Y} \mathcal{A}_1(t, Y \cup Z) \sum_{P : (X \setminus Y) \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathcal{A}_1(t, X_i), \]
\[ 9 \]
where $\sum_{Z \subseteq X \setminus Y}$ is a sum over all subsets $Z \subset X \setminus Y$ of the set $X \setminus Y$. Then taking into account equality (3.8) and

$$\sum_{P: (X \setminus Y) \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! = (-1)^{|(X \setminus Y) \setminus Z|}, \tag{3.9}$$

for the group $U(t)$ we have

$$(U(t)f)_s(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \sum_{Z \subseteq X \setminus Y} (-1)^{|(X \setminus Y) \setminus Z|} G_{Y \cup Z}(-t,Y \cup Z) f_{|X|}(X).$$

Thus as a result of the symmetry property and definition (3.7) of the operator $a$ we derive (3.6)

$$U(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} a^{n-k} G(-t)a^k = e^a G(-t)e^{-a}. \tag{3.6}$$

We can obtain one more representation of the group $U(t)$ solving recurrence relations (1.8) with respect to the 1st-order and 2nd-order cumulants. In fact if $n \geq 1$ we get

$$\mathfrak{A}_{1+n}(t,Y_1,X \setminus Y) = \sum_{Z \subseteq X \setminus Y, Z \neq \emptyset} \mathfrak{A}_s(t,Y,Z) \sum_{P: (X \setminus Y) \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathfrak{A}_1(t,X_i). \tag{3.7}$$

Then taking into account equalities (3.8) and (3.9) we derive

$$(U(t)f)_s(Y) = \mathfrak{A}_1(t,Y)f_s(Y) + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \sum_{Z \subseteq X \setminus Y, Z \neq \emptyset} (-1)^{|(X \setminus Y) \setminus Z|} \mathfrak{A}_2(t,Y,Z) f_{s+n}(X). \tag{3.10}$$

An infinitesimal generator of a group of operators (3.6) has the following form

$$\mathfrak{B} = -\mathcal{N} + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ a, \ldots, [a, (-\mathcal{N})] \ldots \right] = e^a(-\mathcal{N})e^{-a}. \tag{3.10}$$

Representation (3.10) is true in consequence of definition (3.7) of the operator $a$ and the validity of equalities

$$([a, \ldots, [a, (-\mathcal{N})] \ldots] f)_s(Y) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \left( a^{n-k}(-\mathcal{N})a^k f \right)_s(Y) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{i_1 < \ldots < i_{n-k} = s+1} \text{Tr}_{s+1,\ldots,s+n}(-\mathcal{N}_{s+n-k}(Y,i_1,\ldots,i_{n-k})) f_{s+n}(X).$$

For a system of particles interacting through a two-body potential this equality reduces to the following one

$$([\mathcal{N}, a] f)_s(Y) = \sum_{i=1}^{s} \text{Tr}_{s+1}(-\mathcal{N}_{\text{int}}^{(2)})(i,s+1)f_{s+1}(Y,s+1),$$

that is true in view of the equality: $\text{Tr}_{s+1}\mathcal{N}_1(s+1)f_{s+1}(Y,s+1) = 0$. 

10
4 Group of Operators for the Quantum Dual BBGKY Hierarchy

The evolution of marginal observables of both finitely and infinitely many quantum particles is described by the initial-value problem to the dual BBGKY hierarchy. This hierarchy of equations is dual to the quantum BBGKY hierarchy in the sense of bilinear form (1.5) and for finitely many particles one is an equivalent to the Heisenberg equation (the dual von Neumann equation). For systems of classical particles the dual BBGKY hierarchy was examined in [1],[25],[26].

In this section we will use such abridged notations: $Y \equiv (1, \ldots ,s)$, $X \equiv Y \setminus \{j_1, \ldots ,j_{s-n}\}$. According to notations of section 2, the set $(Y \setminus X)_1$ consists of one element $Y \setminus X = (j_1, \ldots ,j_{s-n})$, i.e. the set $(j_1, \ldots ,j_{s-n})$ is connected subset of the partition $P$ $(|P| = 1)$. In the case under consideration the dual cumulants $\mathfrak{A}^+(t, (Y \setminus X)_1, X)$ are defined by the formula

$$\mathfrak{A}^+(t, (Y \setminus X)_1, X) := \sum_{P, \{(Y \setminus X)_1, X\} = \bigcup X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} G_{X_i}(t, X_i),$$

(4.1)

where $\sum_P$ is the sum over all possible partitions $P$ of the set $\{(Y \setminus X)_1, j_1, \ldots ,j_{s-n}\}$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset \{(Y \setminus X)_1, X\}$.

On the space $\mathfrak{L}_\gamma(F_H)$ a solution of the initial-value problem to the dual BBGKY hierarchy is defined by a one-parametric mapping (the adjoint mapping to (3.2) in the sense of bilinear form (1.5)) with the following properties.

Theorem 4.1. If $g \in \mathfrak{L}_\gamma(F_H)$ and $\gamma < e^{-1}$, then the one-parametric mapping

$$\mathbb{R}^1 \ni t \mapsto (U^+(t)g)_s(Y) := \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n} = 1} \mathfrak{A}^+(t, (Y \setminus X)_1, X) g_{s-n}(Y \setminus X), \quad s \geq 1$$

(4.2)

is a $C^*_0$-group. The infinitesimal generator $\mathfrak{B}^+ = \bigoplus_{n=0}^{\infty} \mathfrak{B}^+_n$ of this group of operators is a closed operator for the $*$-weak topology and on the domain of the definition $D(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(F_H)$ which is the everywhere dense set for the $*$-weak topology of the space $\mathfrak{L}_\gamma(F_H)$ it is defined by the operator

$$(\mathfrak{B}^+g)_s(Y) := \mathcal{N}_s(Y)g_s(Y) + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k = 1} \mathcal{N}^{(k)}_{\text{int}}(j_1, \ldots ,j_k) g_{s-n}(Y \setminus \{j_1, \ldots ,j_n\}),$$

(4.3)

where the operator $\mathcal{N}^{(k)}_{\text{int}}$ is given by formula (1.11).

Proof. If $g \in \mathfrak{L}_\gamma(F_H)$ mapping (4.2) is defined provided that $\gamma < e^{-1}$ and the following estimate holds

$$\|U^+(t)g\|_{\mathfrak{L}_\gamma(F_H)} \leq e^2(1 - \gamma e)^{-1}\|g\|_{\mathfrak{L}_\gamma(F_H)}.$$
where \( s(n + 1, k) \) is the Stirling numbers of the second kind.

On the space \( \mathcal{L}_\gamma(\mathcal{F}_H) \) the *-weak continuity property of the group \( U^+(t) \) over the parameter \( t \in \mathbb{R}^1 \) is a consequence of the *-weak continuity of group \( (1.6) \) of the Heisenberg equation \( (10) \).

To construct an infinitesimal generator of the group \( \{U^+(t)\}_{t \in \mathbb{R}} \) we firstly differentiate the \( n \)th-term of expansion \( (4.2) \) in the sense of the point-by-point convergence of the space \( \mathcal{L}_\gamma \). If \( g \in \mathcal{D}(N) \subset \mathcal{L}_\gamma(\mathcal{F}_H) \) similar to equality \( (1.12) \) for \((1 + n)\)th-order dual cumulant \( (4.1), n \geq 1 \), we derive

\[
\lim_{t \to 0} \frac{1}{t} \mathfrak{A}^+_{1+n}(t, (Y \setminus X)_1, X) g_{s-n}(Y \setminus X) \psi_s = \sum_{Z \subset Y \setminus X, Z \neq \emptyset} \mathcal{N}^{|Z|+n}(Z, X) g_{s-n}(Y \setminus X) \psi_s =
\]

\[
\sum_{k=1}^{s-n} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k \in \{j_1, \ldots, j_{s-n}\}} \mathcal{N}^{(k+n)}(i_1, \ldots, i_k, X) g_{s-n}(Y \setminus X) \psi_s.
\]

Then according to equalities \( (1.7) \) and \( (4.4) \) for group \( (4.2) \) we obtain

\[
\lim_{t \to 0} \frac{1}{t} (\mathcal{A}^+_s(t) g_s - g_s) \psi_s = \lim_{t \to 0} \frac{1}{t} \mathfrak{A}^+_{s+1}(t, (Y \setminus X)_1, X) g_{s-n}(Y \setminus X) \psi_s =
\]

\[
\mathcal{N}_s g_s \psi_s + \sum_{n=1}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k=1}^{s} \mathcal{N}^{(k)}(j_1, \ldots, j_k) g_{s-n}(Y \setminus \{j_1, \ldots, j_n\}) \psi_s,
\]

where we used the identity

\[
\sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n=1}^{s} g_{s-n}(Y \setminus \{j_1, \ldots, j_n\}) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n=1}^{s} g_n(j_1, \ldots, j_n)
\]

which is valid in view of the Maxwell-Boltzmann statistics symmetry property.

Thus if \( g \in \mathcal{D}(\mathfrak{B}^+) \subset \mathcal{L}_\gamma(\mathcal{F}_H) \) in the sense of the *-weak convergence of the space \( \mathcal{L}_\gamma(\mathcal{F}_H) \) we finally have

\[
w^* - \lim_{t \to 0} \left( \frac{1}{t} (U^+(t) g - g) - \mathfrak{B}^+ g \right) = 0,
\]

where the generator \( \mathfrak{B}^+ = \bigoplus_{n=0}^{\infty} \mathfrak{B}^+_n \) of group \( (4.2) \) is given by formula \( (4.3) \) (the dual operator to generator \( (3.3) \)).

We now give examples of expansion \( (4.2) \) and infinitesimal generator \( (4.3) \). The sequence \( g = (0, g_1(1), 0, \ldots) \) corresponds to the additive-type observable \[25\] and in this case expansion \( (4.2) \) for the group \( U^+(t) \) get the form

\[
(U^+(t) g_s)(Y) \mathfrak{A}^+_s(t, 1, \ldots, s) \sum_{j=1}^{s} g_1(j), \quad s \geq 1.
\]

In the case of two-body interaction potential \( (1.3) \) operator \( (4.3) \) has the form

\[
(\mathfrak{B}^+ g_s)(Y) = \mathcal{N}_s(Y) g_s(Y) + \sum_{j_1 \neq j_2=1}^{s} \mathcal{N}^{(2)}_{|j_1|+j_2}(j_1, j_2) g_{s-1}(Y \setminus \{j_1\}), \quad s \geq 1,
\]

\[12\]
where the operator $\mathcal{L}_{int}^{(2)}$ is defined by formula (1.11) for $n = 2$. If $\mathcal{H} = L^2(\mathbb{R}^3)$ in terms of kernels of operators $g_s$, $s \geq 1$, for expression (4.6) we have

\[
(\mathfrak{D}^+ g)_s(q_1, \ldots, q_s; q'_1, \ldots, q'_s) = -\frac{i}{\hbar} \left( -\frac{1}{2} \sum_{i=1}^s (-\Delta q_i + \Delta q'_i) + \sum_{1 \leq i < j}^s \left( \Phi^{(2)}(q'_i - q'_j) - \Phi^{(2)}(q_i - q_j) \right) \right) g_s(q_1, \ldots, q_s; q'_1, \ldots, q'_s).
\]

where $(q_1, \ldots, q_s) \equiv (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_s)$. This expression for a system of classical particles is defined as a generator of the dual BBGKY hierarchy stated in [1],[25].

In the paper [25] for classical systems of particles an equivalent representation of group (4.2) was used, namely for the case under consideration group the $U^+(t)$ has the following representation

\[
U^+(t) = \mathcal{G}(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ \ldots \left[ \mathcal{G}(t), \mathfrak{a}^+, \ldots, \mathfrak{a}^+ \right] \right] = e^{-\mathfrak{a}^+ \mathcal{G}(t)} e^{\mathfrak{a}^+},
\]

(4.7)

where $\left[ \ldots \right]$ is a commutator, the operator $\mathfrak{a}^+$ (an analog of the creation operator) is defined on the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ by the formula

\[
(\mathfrak{a}^+ g)_s(Y) := \sum_{j=1}^s g_{s-1}(Y \setminus \{j\}),
\]

(4.8)

the operators $e^{\pm \mathfrak{a}^+}$ are defined on the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ by the expansions

\[
(e^{\pm \mathfrak{a}^+} g)_s(Y) = \sum_{n=0}^s \frac{\left( \pm 1 \right)^n}{n!} \sum_{j_1 \neq \ldots \neq j_n=1}^s g_{s-n}(Y \setminus \{j_1, \ldots, j_n\}), \quad s \geq 1.
\]

and the group of operators $\mathcal{G}(t)$ is defined by expression (1.6).

Representation (4.7) is true in consequence of definition (4.8) of the operator $\mathfrak{a}^+$ and the validity of the equality

\[
\left( \frac{1}{n!} \left[ \ldots \left[ \mathcal{G}(t), \mathfrak{a}^+, \ldots, \mathfrak{a}^+ \right] \right] \right)_s(Y) = \sum_{j_1 \neq \ldots \neq j_{s-n}=1} \mathfrak{g}_{1+n}^+(t, (Y \setminus X)_1, X) g_{s-n}(Y \setminus X), \quad s \geq 1,
\]

(4.9)

where $X \equiv Y \setminus \{j_1, \ldots, j_{s-n}\}$ and $Y \setminus X = (j_1, \ldots, j_{s-n})$. For example, if $n = 1$ we have

\[
\left( \left[ \mathcal{G}(t), \mathfrak{a}^+ \right] \right)_s(Y) = \sum_{j=1}^s \left( \mathcal{G}_s(t, Y) - \mathcal{G}_{s-1}(t, Y \setminus \{j\}) \right) g_{s-1}(Y \setminus \{j\}) = \sum_{j=1}^s \mathfrak{g}_{1}^+(t, (Y \setminus \{j\})_1, j) g_{s-1}(Y \setminus \{j\}).
\]
and since identity (4.5) holds we obtain equality (4.9).

We can obtain one more representation of the group $U^+(t)$, if we express the dual cumulants $\mathfrak{A}^+_{1+n}(t)$, $n \geq 1$, of groups (1.6) with respect to the 1st-order and 2nd-order dual cumulants. In fact it holds

$$
\mathfrak{A}^+_{1+n}(t, (Y \setminus X)_1, X) = \sum_{Z \subseteq X, Z \neq \emptyset} \mathfrak{A}^+_Z(t, Y \setminus X, Z) \sum_{P: X \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathfrak{A}^+_1(t, X_i),
$$

where $\sum_{Z \subseteq X, Z \neq \emptyset}$ is a sum over all nonempty subsets $Z \subseteq X$ of the set $X$. Then taking into account an identity

$$
\sum_{P: X \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! \prod_{i=1}^{|P|} \mathfrak{A}^+_i(t, X_i) g_{s-n}(Y \setminus X) = \sum_{P: X \setminus Z = \bigcup_i X_i} (-1)^{|P|} |P|! g_{s-n}(Y \setminus X)
$$

and equality (3.9) we get the following representation of the group $U^+(t)$

$$(U^+(t)g)_{s}(Y) = \mathfrak{A}^+_1(t, Y) g_{s}(Y) + \sum_{n=1}^{\infty} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n}=1} (-1)^{|X \setminus Z|} \mathfrak{A}^+_2(t, Y \setminus X, Z) g_{s-n}(Y \setminus X).$$

An infinitesimal generator of group (4.7) has the following form

$$
\mathfrak{B}^+ = \mathcal{N} + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ \ldots \left[ \mathcal{N}, \mathfrak{a}^+, \ldots, \mathfrak{a}^+ \right] \right] = e^{-a^+} \mathcal{N} e^{a^+}. \quad (4.10)
$$

Representation (4.10) is true in consequence of definition (4.8) of the operator $a^+$ and the validity of an identity

$$(\left[ \ldots \left[ \mathcal{N}, \mathfrak{a}^+, \ldots, \mathfrak{a}^+ \right] \right] g)_{s}(Y) = \sum_{k=n+1}^{\infty} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_{k}=1} \mathcal{N}^{(k)}_{\text{int}}(j_1, \ldots, j_k) g_{s-n}(Y \setminus \{j_1, \ldots, j_n\}),$$

which in the case of a two-body interaction potential reduces to the following one

$$(\left[ \mathcal{N}, \mathfrak{a}^+ \right] g)_{s}(Y) = \sum_{j_1 \neq j_2=1}^{s} \mathcal{N}^{(2)}_{\text{int}}(j_1, j_2) g_{s-1}(Y \setminus \{j_1\}).$$

5 Conclusion

The concept of cumulants (1.9) of groups (1.1) of the von Neumann equations forms the basis of group expansions for quantum evolution equations, namely, the von Neumann hierarchy for correlation operators [23], as well as the BBGKY hierarchy for $s$-particle density operators [24] and the dual BBGKY hierarchy [26]. In the case of quantum systems of particles obeying Fermi or Bose statistics groups (2.1), (3.2) and (4.2) have different structures. The analysis of these cases will be given in a separate paper.

We have stated the properties of groups (2.1) and (4.2) on the space $\mathfrak{A}_{\alpha}(\mathcal{F}_H)$ and dual group (4.2) on $\mathfrak{L}_{\gamma}(\mathcal{F}_H)$. To describe the evolution of infinitely many particles [1] it is necessary to define the
one-parametric family of operators (3.2) on more general spaces than $L^1_\alpha(\mathcal{F}_H)$, for example, on the space of sequences of bounded operators containing the equilibrium states [22]. For dual group (4.2) the problem lies in the definition of functional (1.5) for operators from the corresponding spaces. In both these cases every term of the corresponding expansions contains the divergent traces [1],[24],[25] and the analysis of such a question for quantum systems remains an open problem.

On the space $L^\gamma(\mathcal{F}_H)$ one-parametric mapping (4.2) is not a strong continuous group. The group $\{U^+(t)\}_{t \in \mathbb{R}}$ of operators (4.2) defined on the space $L^\gamma(\mathcal{F}_H)$ is dual to the strong continuous group $\{U(t)\}_{t \in \mathbb{R}}$ of operators (3.2) for the BBGKY hierarchy defined on the space $L^1_\alpha(\mathcal{F}_H)$ and the fact that one is a $C^*_0$-group follows also from general theorems about properties of the dual semigroups [13],[17].

As mention above the group $\{G(-t)\}_{t \in \mathbb{R}}$ of operators (1.1) preserves positivity [16],[18]. The same property must be valid for the group $\{U(t)\}_{t \in \mathbb{R}}$ of operators (3.2) for the BBGKY hierarchy, but how to prove this property one is an open problem.

We have constructed infinitesimal generators (2.2), (3.3) on the subspace $L^1_{\alpha,0} \subset L^1_\alpha(\mathcal{F}_H)$ and generator (4.3) on $\mathcal{D}(\mathcal{B}^+) \subset L^\gamma(\mathcal{F}_H)$. The question of how to define the domains of the definition $\mathcal{D}(\mathfrak{N}), \mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{B}^+)$ of corresponding generators (2.2), (3.3) and (4.3) is an open problem [14],[18].

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