CONSTRUCTION OF POLYNOMIAL PRESERVING COCHAIN EXTENSIONS BY BLENDING

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Abstract. A classical technique to construct polynomial preserving extensions of scalar functions defined on the boundary of an $n$ simplex to the interior is to use so-called rational blending functions. The purpose of this paper is to generalize the construction by blending to the de Rham complex. More precisely, we define polynomial preserving extensions which map traces of $k$ forms defined on the boundary of the simplex to $k$ forms defined in the interior. Furthermore, the extensions are cochain maps, i.e., they commute with the exterior derivative.

1. Introduction

In applications such as the finite element approximation of partial differential equations and in computer aided geometrical design problems, there often arises a need for a method for extending a piecewise smooth function given on the boundary of a domain to the entire domain, in particular when the domain is a hypercube or a simplex. There is a considerable literature on this subject, dating back to early work of Coons [11] in the context of computer aided design, while more mathematical oriented studies of such problems were initiated in [7, 10]. The constructions were often referred to as transfinite interpolation or blending function methods, since the extension is obtained by combining, or blending, the transfinite boundary data using rational or polynomial basis functions. In [21], the two dimensional scheme described in [21] was generalized to the case of tetrahedra, and with a brief discussion of its generalization to $n$ simplices. Alternative approaches, using polynomial rather than rational blending functions, were studied in [8, 19, 20, 23]. A summary of much of this early work can be found in [6].

More recently, the study of extension operators that preserve a polynomial structure of the boundary data have played a key role in the analysis of finite element methods of high polynomial order, cf. [1, 5, 9, 22]. In particular, the importance of such extensions that commute with the exterior derivative, i.e., cochain extensions, was illustrated by the analysis given in [13]. The results of these papers further...
motivated the theory developed in the series of three papers, \[14, 15, 16\], where polynomial preserving cochain extensions are constructed for the de Rham complex in three dimensions. An important additional property of these extensions is that they require only weak regularity of the boundary data to be well defined.

The purpose of this paper is to extend the method of blending to define polynomial preserving cochain extensions for differential forms of arbitrary order on \( n \) simplices. More specifically, for \( S_n = [x_0, x_1, \ldots, x_n] \subset \mathbb{R}^n \), an \( n \) dimensional simplex, we define extensions \( E^k_n \) which map piecewise smooth \( k \) forms defined on the boundary, \( \partial S_n \), to smooth forms on \( S_n \), such that they preserve polynomial structures and commute with the exterior derivative. For scalar-valued functions, or zero forms, the extensions presented here correspond to operators defined in \[21\], while the general construction for higher order forms appears to be new.

An outline of the paper is as follows. In Section 2, we introduce some basic notation and recall the construction of extensions by blending in the case of zero-forms, i.e., for scalar-valued functions. In particular, we verify that these extensions are polynomial preserving. We also present a summary of the main results of the paper and an application of the construction. In Section 3, we discuss the extension in the case of one-forms. The explicit construction in this basic case provides a motivation for the general construction for \( k \)-forms to follow. The discussion in Section 3 also motivates the construction of a family of order-reduction operators, presented in Section 4, which will play a key role in designing the coefficient operators, \( A^k_{I,J} \), that will be used to define the extensions \( E^k_n \). A precise definition of the coefficient operators \( A^k_{I,J} \) is given in Section 5 and three key properties of these operators are established. In Section 6, we then show that the operators \( E^k_n \) are polynomial preserving cochain extensions.

2. Preliminaries

2.1. Notation. We will use \([, \ldots, ,] \) to denote the simplex obtained by convex combination of the arguments. The simplex \( S_n = [x_0, x_1, \ldots, x_n] \subset \mathbb{R}^n \), \( n \geq 1 \), will be considered to be a fixed \( n \) simplex throughout this paper. The barycentric coordinate associated to the vertex \( x_i \) will be denoted by \( \lambda_i = \lambda_i(x) \), i.e., \( \lambda_i \) is a linear function on \( S_n \) satisfying \( \lambda_i(x_j) = \delta_{i,j} \) and such that

\[
x = \sum_{i=0}^{n} \lambda_i(x)x_i, \quad x \in S_n.
\]

Furthermore, the boundary of \( S_n \), \( \partial S_n \), consists of all \( x \in S_n \) such that \( \lambda_i(x) = 0 \) for at least one index \( i \in \{0, \ldots, n\} \). We will use \( \Lambda^k(\partial S_n) \) to denote the space of piecewise smooth \( k \) forms defined on \( \partial S_n \). More precisely, each element \( u \in \Lambda^k(\partial S_n) \) is smooth on each \((n-1)\) dimensional subsimplex of \( \partial S_n \), and with single-valued traces at the interfaces. Correspondingly, \( \Lambda^k(S_n) \) will denote the space of smooth \( k \) forms defined on \( S_n \) and \( \Lambda^k(S_n) \) will denote the space of forms in \( \Lambda^k(S_n) \) with vanishing trace on \( \partial S_n \). Our main goal is to construct extension operators \( E^k_n : \Lambda^k(\partial S_n) \to \Lambda^k(S_n), \ k = 0, 1, \ldots, n-1, \) with desired properties.
We will use $d = d^k : \Lambda^k(S_n) \to \Lambda^{k+1}(S_n)$ to denote the exterior derivative defined by
\[
du_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial v_j u_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}),
\]
where the hat symbol, e.g., $\hat{v}_j$, is used to indicate a suppressed argument and the vectors $v_j$ are elements of $\mathbb{R}^n$. We also use $u^1 \wedge u^2$ to denote the wedge product mapping a $j$ form $u^1$ and $k$ form $u^2$ into a $j+k$ form. A smooth map $F : S_n \to \partial S_n$ provides a pullback of a differential form from $\partial S_n$ to $S_n$, i.e., a map from $\Lambda^k(\partial S_n) \to \Lambda^k(S_n)$ given by
\[
(F^*u)_x(v_1, \ldots, v_k) = u_{F(x)}(DF_x(v_1), \ldots, DF_x(v_k)).
\]
The pullback respects exterior products and differentiation, i.e.,
\[
F^*(u^1 \wedge u^2) = F^*u^1 \wedge F^*u^2, \quad F^*(du) = d(F^*u).
\]
The pullback of the inclusion map of $\partial S_n$ into $S_n$ is the trace map, $\text{tr}_{\partial S_n}$, and we note that the spaces $\Lambda^k(\partial S_n)$ are precisely defined so that $\Lambda^k(\partial S_n) = \text{tr}_{\partial S_n} \Lambda^k(S_n)$.

For each integer $r > 0$, the polynomial subspaces $\mathcal{P}_r \Lambda^k(S_n)$ consist of all elements $u \in \Lambda^k(S_n)$ such that for fixed vectors $v_1, \ldots, v_k$, the function $u_x(v_1, \ldots, v_k)$, as a function of $x$, is an element of $\mathcal{P}_r(S_n)$, i.e., the space of polynomials of degree less than or equal to $r$ defined on $S_n$. The trimmed space $\mathcal{P}_r^- \Lambda^k(S_n)$ is the subspace consisting of all $u \in \mathcal{P}_r \Lambda^k(S_n)$ such that $u_x(x-a) \in \mathcal{P}_r \Lambda^{k-1}(S_n)$ for any fixed $a \in \mathbb{R}^n$. Here, the symbol $\cdot$ is used to denote contraction, i.e.,
\[
(u_x(x-a))_x(v_1, \ldots, v_{k-1}) = u_x(x-a, v_1, \ldots, v_{k-1}).
\]
The corresponding spaces on the boundary are defined by
\[
\mathcal{P}_r \Lambda^k(\partial S_n) = \text{tr}_{\partial S_n} \mathcal{P}_r \Lambda^k(S_n) \quad \text{and} \quad \mathcal{P}_r^- \Lambda^k(\partial S_n) = \text{tr}_{\partial S_n} \mathcal{P}_r^- \Lambda^k(S_n).
\]
We refer to [2, 3, 4] for more details on the polynomial and piecewise polynomial spaces of differential forms.

If $J = \{j_0, j_1, \ldots, j_m\}$ is an ordered subset of $\{0, 1, \ldots, n\}$, but not necessarily increasingly ordered, we will refer to $J$ as an index set. We will use $\Gamma$ to denote the set of all increasing ordered subsets of $\{0, 1, \ldots, n\}$. In other words, if $J \in \Gamma$, then $J$ is an index set of the form
\[
J = \{j_0, j_1, \ldots, j_m\}, \quad \text{where} \ 0 \leq j_0 < j_1 \ldots < j_m \leq n.
\]
The number of elements in $J$ will be denoted $|J|$, and $\Gamma_m$ is the subset of $\Gamma$ consisting of all $J \in \Gamma$ with $|J| = m + 1$. Furthermore, for any $I \in \Gamma$, we let
\[
\Gamma_m(I) = \{J \in \Gamma_m : J \subset I\}.
\]
For an index set $J$, the complement of $J$ relative to the set $\{0, 1, \ldots, n\}$ will be denoted $J^c$ and $[x, J]$ will denote the simplex generated by the vertices $\{x_j\}_{j \in J}$, with orientation induced by the order of $J$. Furthermore, $\lambda_J = \lambda_J(x)$ will denote the corresponding sum of barycentric coordinates, i.e., $\lambda_J = \sum_{j \in J} \lambda_j$, and we will use $\phi_J$ to denote the associated Whitney form, given by
\[
\phi_J = \sum_{i=0}^{m} (-1)^i \lambda_{j_i} d\lambda_{j_0} \wedge \ldots \wedge d\hat{\lambda}_{j_i} \wedge \ldots \wedge d\lambda_{j_m}, \quad \text{if} \ J = \{j_0, j_1, \ldots, j_m\}.
\]
We note that if $J = \{j\} \in \Gamma_0$, then $\phi_J = \lambda_J = \lambda_j$. Furthermore, the set $\{\phi_J\}_{J \in \Gamma_k}$ is a basis for $P_1^k \Lambda^k(S_n)$. We will also use the notation

$$\delta \phi_J = \sum_{i=0}^{m} (-1)^i \phi_J(\hat{i}), \quad J = \{j_0, j_1, \ldots, j_m\},$$

where $J(\hat{i})$ refers to the set $J$, but with the index $j_i$ removed.

2.2. Scalar-valued functions. We next give a quick review of the extensions studied in [21] for scalar-valued functions, or zero forms, but in a slightly different notation. For each index set $I \in \Gamma_m$, $m \geq 1$, and $j \in \{0, \ldots, n\}$, we define the map $P_{I,j} : S_n \to \partial S_n$ by

$$P_{I,j}x = x + \sum_{i \in I} \lambda_i(x)(x_j - x_i).$$

In fact, $P_{I,j}$ is a projection onto the set $\{x \in S_n : \lambda_i(x) = 0, i \in I, i \neq j\}$. Note that if $j \in I$, then $P_{I,j} = P_{I,j'}$, where $I'$ represents the index set obtained from $I$ by removing $j$. In Figure 2.1, we show all the possible points $P_{I,j}x$, for $j \in I$, in the case when $n = 2$.

To define the extension $E_0^n : \Lambda^0(\partial S_n) \to \Lambda^0(S_n)$, we will utilize the corresponding pullbacks $P_{I,j}^*$, defined by $(P_{I,j}^*u)_x = u(P_{I,j}x)$. We define

$$E_0^n u = \frac{1}{n} \sum_{I \in \Gamma_m} \sum_{1 \leq m \leq n} (-1)^{m+1} \sum_{j \in I} \frac{\lambda_j}{\lambda_I} P_{I,j}^* u.$$

To see that $E_0^n$ is an extension, we need to check that $\text{tr}_{\partial S_n} E_0^n u = u$. Without loss of generality, consider a point $x \in \partial S_n$ such that $\lambda_0(x) = 0$. Then we have that

$$\frac{1}{n} \sum_{I \in \Gamma_m} \sum_{0 \in I} (\frac{\lambda_j}{\lambda_I} P_{I,j}^* u)_x = u_x.$$

In addition, if $I \in \Gamma$ such that $0 \notin I$, and $I' = \{0, I\}$ then

$$\left(\frac{\lambda_j}{\lambda_I} P_{I,j}^* u\right)_x = \left(\frac{\lambda_j}{\lambda_{I'}} P_{I',j}^* u\right)_x.$$
Therefore, the terms corresponding to \( I \) and \( I' \) cancel at the point \( x \) and we can conclude that \((E_n^0 u)_x = u_x\). This shows that \( E_n^0 \) is an extension.

Because of the rational factors in the definition of \( E_n^0 \), it is not entirely obvious that the operator preserves the polynomial structure of \( u \). However, we observe that the pullbacks \( P^*_{I,j} \) have the property that if \( u \in P_r \Lambda^0(\partial S_n) \), then \( P^*_{I,j} u \in P_r \Lambda^0(S_n) \). To show that the same property holds for the extension \( E_n^0 \), we fix an \( I \in \Gamma \) and consider the sum \( \sum_{j \in I} (\lambda_j/\lambda_I)P^*_{I,j} u \). If \( I \) is the maximal set \( I = \{0, \ldots, n\} \), then \( \lambda_I(x) \equiv 1 \) on \( S_n \), so the sum is simply \( \sum_{j=0}^n \lambda_j(x)u_{x,j} \), which is the linear function that interpolates \( u \) at the vertices. For any other \( I \), there is at least a vertex \( x_p \) such that \( p \notin I \), and

\[
\sum_{j \in I} (\lambda_j/\lambda_I)P^*_{I,j} u = \sum_{j \in I} (\lambda_j/\lambda_I)(P^*_{I,j} u \pm P^*_{I,p} u) = P^*_{I,p} u + \sum_{j \in I} (\lambda_j/\lambda_I)(P^*_{I,j} u - P^*_{I,p} u).
\]

The first term on the right hand side is polynomial preserving, while we have

\[
(2.1) \quad P^*_{I,j} u_x - P^*_{I,p} u_x = \int_0^1 \frac{d}{d\tau} u(1-\tau)P_{I,p}x + \tau P_{I,j}x \ d\tau = \lambda_I \int_0^1 (du(1-\tau)P_{I,p}x \tau P_{I,j}x \phi_p(x_j - x_p) \ d\tau,
\]

which has \( \lambda_I \) as a factor. Furthermore, the curve \( (1-\tau)P_{I,p}x + \tau P_{I,j}x \) for \( \tau \in [0,1] \) belongs to the set \( \{x \in S_n : \lambda_i(x) = 0, \ i \in I(j)\} \subset \partial S_n \). Therefore, we can conclude that \( E_n^0(P_r \Lambda^0(\partial S_n)) \subset P_r \Lambda^0(S_n) \).

### 2.3. The main result.

The discussion above shows that the operator \( E_n^0 : \Lambda^0(\partial S_n) \to \Lambda^0(S_n) \) is a polynomial preserving extension operator. The main result of this paper is to construct corresponding operators \( E_n^k : \Lambda^k(\partial S_n) \to \Lambda^k(S_n) \), \( 1 \leq k < n \), satisfying \( \text{tr}_{\partial S_n} E_n^k u = u \) and which are cochain extensions in the sense that the diagram

\[
\begin{array}{ccccccc}
\Lambda^0(\partial S_n) & \to & \Lambda^1(\partial S_n) & \to & \ldots & \to & \Lambda^{n-1}(\partial S_n) \\
E_n^0 \downarrow & & E_n^1 \downarrow & & \ldots & & E_n^{n-1} \downarrow \\
\Lambda^0(S_n) & \to & \Lambda^1(S_n) & \to & \ldots & \to & \Lambda^{n-1}(S_n),
\end{array}
\]

commutes. In other words, \( dE_{n}^{k-1} = E_{n}^{k} d \) for \( 1 \leq k < n \), and we will also show that \( dE_{n}^{n-1} = 0 \). Furthermore, the extensions \( E_n^k \) are polynomial preserving in the sense that

\[
E_n^k(P_r \Lambda^k(\partial S_n)) \subset P_r \Lambda^k(S_n) \quad \text{and} \quad E_n^k(P_r^{-} \Lambda^k(\partial S_n)) \subset P_r^{-} \Lambda^k(S_n),
\]

for \( 0 \leq k < n \) and \( r \geq 1 \).

All the operators \( E_n^k \), \( 0 \leq k < n \), that we construct will be rational functions of the form

\[
(2.2) \quad E_n^k u = \frac{1}{n} \sum_{I \in \Gamma_m} (-1)^{m+1} \sum_{j \in \Gamma_s(I)} \frac{\phi_j}{\lambda_j^{s+1}} \wedge \Lambda^k_{I,j} u,
\]
where the coefficient operators \( A^k_{I,J} \) map \( \Lambda^k(\partial S_n) \) to \( \Lambda^{k-s}(S_n) \) for \( J \in \Gamma_s(I) \), and such that \( A^k_{I,J} = P^*_a \) for \( J = \{j\} \in \Gamma_0 \). However, in contrast to what was done in the series of papers [14, 15, 16], we will not perform a careful discussion of bounds for the corresponding operator norms. It is clear that the extensions constructed by blending will not be well defined on spaces with weak regularity, such as \( L^2(\partial S_n) \).

On the other hand, a lesson to be learned from the series mentioned above, see also [17, 18], is that some additional averaging technique has to be used to be able to construct such weak regularity extensions.

2.4. Polynomial complexes. As an example of a direct application of the extensions \( E^k_n \) constructed below, we will briefly consider polynomial complexes of differential forms on \( \mathbb{R}^n \), cf. [3, Section 3.5] and [12, Section 4.2]. For any \( r > 0 \), the full polynomial complex

\[
\mathbb{R} \to \mathcal{P}_r \Lambda^0 \xrightarrow{d} \mathcal{P}_r^{−1} \Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^{−n} \Lambda^n \to 0,
\]

and the trimmed polynomial complex

\[
\mathbb{R} \to \mathcal{P}^−_r \Lambda^0 \xrightarrow{d} \mathcal{P}^−_r \Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}^−_r \Lambda^n \to 0,
\]

are both exact. In fact, by combining the full polynomial spaces and the trimmed polynomial spaces, we can obtain \( 2^{n−1} \) different exact polynomial complexes of the form

\[
(\mathcal{P}^k \Lambda^k)_x = \int_0^1 \tau^{k−1}u(1−\tau)a+\tau x^j(x−a)\,d\tau,
\]

where \( a \in \mathbb{R}^n \) is fixed. More precisely, if the spaces \( \mathcal{P}^k \Lambda^{k−1} \) and \( \mathcal{P}^k \Lambda^k \) are related as in [12, 13], then the operator \( Q^k \) maps \( \mathcal{P}^k \Lambda^{k−1} \) to \( \mathcal{P}^k \Lambda^k \), and if \( u \in \mathcal{P}^k \Lambda^k \) satisfies \( du = 0 \), then \( u = dQ^k u \). In fact, any element \( u \) of the polynomial spaces \( \mathcal{P}^k \Lambda^k \) admits the representation

\[
u = dQ^k u + Q^{k+1} du.
\]

It also well-known that the corresponding polynomial complexes with boundary conditions are exact. More precisely, we consider complexes of the form

\[
\mathcal{P}^k \Lambda^0(S_n) \xrightarrow{d} \mathcal{P}^k \Lambda^1(S_n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}^k \Lambda^n(S_n) \to \mathbb{R},
\]

where the spaces \( \mathcal{P}^k \Lambda^k(S_n) \) are the restriction of the spaces \( \mathcal{P}^k \Lambda^k \) to \( S_n \), and with vanishing traces on \( \partial S_n \). The exactness can in this case be established by counting degrees of freedom, using the results of [3, Section 4] or [4, Section 5]. On the other hand, in this setting it is not straightforward to construct a simple operator \( Q^k \) which maps \( \mathcal{P}^k \Lambda_k(S_n) \) to \( \mathcal{P}^k \Lambda^{k−1}(S_n) \), and which has the property that \( u = dQ^k u \) if \( du = 0 \). For example, if we consider an operator of the form \( Q^k \) above, then it seems impossible to choose the point \( a \in S_n \) such that the vanishing trace condition is preserved. However, by using the extension operator \( E^k_n \) constructed in this paper, we can define \( \hat{Q}^k \) as

\[
\hat{Q}^k u = (I - E^k_n \circ \text{tr}_{\partial S})Q^k u,
\]
where the operator $Q^k$ is a Poincaré-type operator of the form above, with $a \in S_n$. It follows from the properties of $E_n^{k-1}$ and $Q^k$ that the operator $Q^k$ maps $P_r \Lambda^k(S_n)$ to $\mathcal{P}^{-1}_r \Lambda^{k-1}(S_n)$ and $\mathcal{P}_r \Lambda^{k}(S_n)$ to $\mathcal{P}_r \Lambda^{k-1}(S_n)$. Furthermore, we have

$$dQ^k u + \dot{Q}^{k+1} u = u - E_n^k \circ \text{tr}_{S_n} u = u, \quad u \in \Lambda^k(S_n).$$

3. The Case of One Forms

To motivate the general construction for $k$ forms, we first consider the construction for one forms. The operator $E_n^1$ must satisfy the commuting property, $E_n^1 du = dE_n^0 u$. The right hand side of this identity is known, and given by

$$dE_n^0 u = \frac{1}{n} \sum_{I \in \Gamma_m} (-1)^{m+1} \sum_{j \in I} \left[ d\left(\frac{\lambda_j}{\lambda_I}\right) P^*_{I,j}u + \frac{\lambda_j}{\lambda_I} P^*_{I,j} d\lambda^I u \right],$$

where we have used the fact that $P^*_{I,j}$ commutes with the exterior derivative. The goal is to write the complete right hand side in terms of $du$. For a fixed $j \in I$, we have

$$d\left(\frac{\lambda_j}{\lambda_I}\right) = \frac{1}{\lambda_I} [\lambda_j d\lambda_j - \lambda_j d\lambda_I] = \frac{1}{\lambda_I} \sum_{I,i \neq j} [\lambda_i d\lambda_I - \lambda_j d\lambda_I] = \sum_{i \in I, i \neq j} \frac{\phi_{i,j}}{\lambda_I},$$

Since $\phi_{j,i} = -\phi_{i,j}$, we then obtain

$$\sum_{j \in I} d\left(\frac{\lambda_j}{\lambda_I}\right) P^*_{I,j}u = \sum_{J \in \Gamma_1(I)} \frac{\phi_{J}}{\lambda_I} (\delta P^*_I u)_J,$$

where $(\delta P^*_I u)_J = P^*_{I,j} u - P^*_{I,i} u$ if $J = \{i, j\}$. Furthermore, as in (2.1), we obtain

$$(\delta P^*_I u)_J = \lambda_I(x) \int_0^1 (du)_{(1-\tau)P_{i,x} + \tau P_{j,x}} \cdot d\tau, \quad J = \{i, j\}.$$}

Since the above formula depends on $du$, rather than $u$, this leads to a possible definition of $E_n^1 u$ such that the desired commuting relation holds. More precisely, we can define

$$E_n^1 u = \frac{1}{n} \sum_{I \in \Gamma_m} (-1)^{m+1} \left[ \sum_{j \in I} \frac{\lambda_j}{\lambda_I} P^*_{I,j} u + \sum_{J \in \Gamma_1(I)} \frac{\phi_{J}}{\lambda_I} R^1_{I,J} u \right],$$

where the operator $R^1_{I,J}$ is defined by

$$(R^1_{I,J} u)_x = \lambda_I(x) \int_0^1 u_{(1-\tau)P_{i,x} + \tau P_{j,x}} \cdot d\tau$$

for any $I \in \Gamma$ and $J$ an index set of length 2. Alternatively, we have

$$(R^1_{I,J} u)_x = \int_{P_{i,x}}^{P_{j,x}} u, \quad J = \{i, j\},$$

where we have used differential form notation for writing the integral of a one-form $u$ over the one-dimensional space $[P_{i,x}, P_{j,x}]$. The problem with the definition of $E_n^1$ above is that the line $[P_{i,x}, P_{j,x}]$ will in general not belong to the boundary.
We will therefore replace the operator $R^1_{I,j}$ by an alternative operator, $A^1_{I,j}$, defined by

$$A^1_{I,j}u = \frac{1}{n-1} \sum_{p \notin J} [R^1_{I,(p,j)}u - R^1_{I,(p,i)}u], \quad J = \{i,j\}.$$  

This operator will still satisfy the key relation $A^1_{I,j}du = (\delta P^*_I u)_J$, and the line from $[P_{I,j}x, P_{I,j}x]$ will belong to the set $\lambda_i = 0$ if $J = \{i,j\}$ and $p \notin J$. As a consequence, the operator $A^1_{I,j}$ maps one forms defined on $\partial S_n$ to scalar functions defined on $S_n$, and the operator $E^1_n$, defined by

$$E^1_n = \frac{1}{n} \sum_{I \in \Gamma \cap \Gamma_n} (-1)^{m+1} \left[ \sum_{j \in I} \frac{\lambda_j}{\lambda_I} P^*_I u + \sum_{J \in \Gamma(I)} \phi_J \frac{\phi_J}{\lambda_J^2} A^1_{I,J} u \right],$$

will satisfy the commuting relation $dE^1_n u = E^1_n du$.

To see that the operator $E^1_n$, defined by (3.2), is an extension, we have to show that $\text{tr}_{\partial S_n} E^1_n u = u$. This follows by essentially the same argument as for zero forms, (cf. also the proof of Theorem 6.1 below). As in the case of zero forms, we will have

$$\text{tr}_{\lambda_0} = \frac{1}{n} \sum_{I \in \Gamma \cap \Gamma_n} (-1)^{m+1} \sum_{j \in I} \frac{\lambda_j}{\lambda_I} P^*_I u = \text{tr}_{\lambda_0} u = 0.$$

To complete the argument, it will therefore be enough to show that

$$\text{tr}_{\lambda_0} \left[ \sum_{I \in \Gamma \cap \Gamma_n} (-1)^{m+1} \sum_{J \in \Gamma(I)} \phi_J \frac{\phi_J}{\lambda_J^2} A^1_{I,J} u \right] = 0.$$  

If $0 \in J$, then $\text{tr}_{\lambda_0} \phi_J = 0$. On the other hand, if $0 \notin J$, then we can conclude from (3.1) that for any $I \in \Gamma$ such that $0 \notin I$ and $J \in \Gamma(I)$,

$$\text{tr}_{\lambda_0} (A^1_{I,j} u - A^1_{I,j} u) = 0, \quad J = \{0,I\}.$$  

Therefore, the terms corresponding to $I$ and $I'$ on the left hand side of (3.3) cancel. We can therefore conclude that the identity (3.3) holds, and as a consequence, $E^1_n$ is an extension.

The fact that the operator $E^1_n$ maps piecewise smooth one-forms defined on $\partial S_n$ to smooth one-forms defined on $S_n$ and also preserves the polynomial structure is not at all obvious, since the operator $R^k_{I,j}$ and hence the operator $A^1_{I,j}$ has one factor of $\lambda_I$, but not two. However, we will show in Section 6 using an alternative representation of the operator $E^1_n$, that $E^1_n$ does, in fact, have these properties.

4. THE OPERATORS $R^k_{I,j}$ AND THEIR PROPERTIES

To develop a formula for $E^1_n$ of the form (2.2), in the general case, we will first consider how to generalize the operators $R^k_{I,j}$, introduced above, to the case when $k > 1$ and $J$ is a more general index set. We recall that when $J$ is an index set, the set $[x,J]$ is the simplex generated by the vertices $\{x_j\}_{j \in J}$. For any $x \in S_n$ and index set $I \in \Gamma$, the corresponding simplex $[P_{I,j}x]$ is the convex combinations of
the points \( \{ P_{I,j}x \}_{j \in J} \). In general, the simplex \([P_{I,j}x]\) will not be a subset of \( \partial S_n \), cf. Figure 2.1. However, if \( I \cap J^c \) is nonempty, where we recall that \( J^c \) is the complement of \( J \), we will indeed have \([P_{I,j}x] \subset \partial S_n \), since any \( \lambda_i \), with \( i \in I \cap J^c \), will be identically zero on \([P_{I,j}x]\).

Key tools for the construction of the operators \( R_{I,j}^k \) are the maps \( F_I : S_n \times \partial S_n \to S_n \) given by

\[
F_I(x, y) = x + \sum_{i \in I} \lambda_i(x)(y - x_i),
\]

where \( I \in \Gamma \). We observe that \( F_I(x, x_j) = P_{I,j}x \). Furthermore, if we restrict the domain of the map \( F_I \) to \( S_n \times [x_j] \), where \( I \cap J^c \) is nonempty, then the range is a subset of \( \partial S_n \). The pullback \( F_I^* \), is a map

\[
F_I^* : \Lambda^k(S_n) \to \Lambda^k(S_n \times \partial S_n).
\]

In the discussion below, we will be interested in operators of the form \( \text{tr}_{S_n \times [x_j]} \circ F_I^* \), mapping \( \Lambda^k(S_n) \) to \( \Lambda^k(S_n \times [x_j]) \). In particular, it follows from the discussion above that if \( I \cap J^c \) is nonempty, then this operator maps \( \Lambda^k(\partial S_n) \) to \( \Lambda^k(S_n \times [x_j]) \).

A space of \( k \)-forms on a product space can be expressed using the tensor product \( \otimes \) as

\[
\Lambda^k(S_n \times \partial S_n) = \sum_{s=0}^{k} \Lambda^{k-s}(S_n) \otimes \Lambda^s(\partial S_n).
\]

In other words, elements \( U \in \Lambda^{k-s}(S_n) \otimes \Lambda^s(\partial S_n) \) can be written as a sum of terms of the form

\[
a(x, y) dx^{k-s} \otimes dy^s,
\]

where \( dx^{k-s} \) and \( dy^s \) run over bases in \( \text{Alt}^{k-s}(S_n) \) and \( \text{Alt}^s(\partial S_n) \), respectively, and where \( a \) is a scalar function on \( S_n \times \partial S_n \). Here \( \text{Alt}^k \) is the corresponding space of algebraic \( k \) forms. Furthermore, for each \( s, 0 \leq s \leq k \), there is a canonical map \( \Pi_s : \Lambda^k(S_n \times \partial S_n) \to \Lambda^{k-s}(S_n) \otimes \Lambda^s(\partial S_n) \) such that

\[
U = \sum_{s=0}^{k} \Pi_s U, \quad U \in \Lambda^k(S_n \times \partial S_n).
\]

The functions \( \Pi_s F_I^* u \in \Lambda^{k-s}(S_n) \otimes \Lambda^s(\partial S_n) \) can be identified as

\[
(\Pi_s F_I^* u)_{x,y}(v_1, \ldots, v_{k-s}, t_1, \ldots, t_s)
\]

\[
= u_{F_I(x,y)}(D_x F_I v_1, \ldots, D_x F_I v_{k-s}, D_y F_I t_1, \ldots, D_y F_I t_s),
\]

where the tangent vectors \( v_i \in T(S_n) \) and \( t_i \in T_y(\partial S_n) \). For the special function \( F_I \) in our case, we have \( D_y F_I = \lambda_I(x) I \), while

\[
D_x F_I = D_x x + \sum_{i \in I} (y - x_i)d_x \lambda_i = \sum_{i \in I^c} (x_i - y)d_x \lambda_i,
\]

where \( D_x x \) is the identity matrix.

The basic commuting property for pull-backs, namely \( dF^* = F^* d \), can be expressed in the present setting as

\[
(4.1) \quad \Pi_s F_I^* d u = \Pi_s d F_I^* u = d_S \Pi_s F_{I,j}^* u - (-1)^{k-s} d_S \Pi_{s-1} F_I^* u, \quad u \in \Lambda^k(S_n),
\]
where $0 \leq s \leq k + 1$, and where $d_S$ and $d_{\partial S}$ denote the exterior derivative with respect to the spaces $S_n$ and $\partial S_n$, respectively.

Let $I \in \Gamma$ and $J$ an index set with $|J| = s + 1$, $0 \leq s \leq k \leq n$. We introduce a family of operators $R^k_{I,J}$, mapping $\Lambda^k(S_n)$ to $\Lambda^{k-s}(S_n)$, defined by

$$(4.2) \quad (R^k_{I,J} u)_x = \int_{[x_j]} (\Pi_\star F^*_I u)_x.$$  

For $s > k$, we define $R^k_{I,J}$ to be zero. If $v_1, \ldots, v_{k-s}$ are vectors in $\mathbb{R}^{n+1}$ and $t_1, \ldots, t_s$ is any orthonormal basis for the tangent space $T[x_j] = T[P_{I,J} x]$, then

$$(R^k_{I,J} u)_x(v_1, \ldots, v_{k-s}) = \lambda_I(x)^s \int_{[x_j]} (u_{F_I(x,y)} D_x F_I v_1 \ldots D_x F_I v_{k-s}) (t_1, \ldots, t_s) dy.$$  

Note that since $u$ is a $k$-form, $u_{F_I(x,y)} D_x F_I v_1 \ldots D_x F_I v_{k-s}$ is an $s$ form with respect to $y$, which we can then integrate over the $s$ dimensional space $[x_j]$. In the final formula, we see that the integral to be evaluated is an integral over $[x_j]$ for a fixed $x$ and vectors $v_1, \ldots, v_{k-s}$.

In the special case when $s = 0$, i.e., when the simplex $[x_j]$ degenerates to a vertex $x_j$, the operator $R^k_{I,J}$ is interpreted as $P^*_I J$. When $s = 1$ and $J = \{i, j\}$, we can utilize the parameterization $y = (1 - \tau)x_i + \tau x_j$, $\tau \in [0, 1]$ of $[x_j]$ to verify that the definition of $R^k_{I,J}$ given in Section 3 corresponds exactly to the definition given by (4.2). If $I$ and $J$ are related such that $I \cap J^c$ is nonempty, then $R^k_{I,J} u$ will only depend on $\text{tr}_{\partial S_n} u$. Furthermore, if $i \in I \cap J^c$, then all the vectors $D_x F_I v_1, \ldots, D_x F_I v_{k-s}$ and $t_1, \ldots, t_s$ will belong to the tangent space of the boundary simplex $\{x \in S_n : \lambda_I(x) = 0\}$, a space of dimension $< n$. Hence, for $k = n$, it follows that all operators of the form $R^n_{I,J}$ are identically zero in this case.

We will summarize the key properties of the operator $R^k_{I,J}$ in the three lemmas given below.

**Lemma 4.1.** Let $I \in \Gamma$ and assume that $j \in \{0, \ldots, n\}$ is such that $j \notin I$. Then for all index sets $J$,

$$\text{tr}_{\lambda_j = 0} R^k_{I,J} u = \text{tr}_{\lambda_j = 0} R^k_{I', J} u,$$

where $I' \in \Gamma$ is equal to $\{j, I\}$ up to a reordering. Moreover, if $I \cap J^c$ is nonempty, then $R^k_{I,J} u$ only depends on $\text{tr}_{\partial S_n} u$, and the operators $R^k_{I,J}$ are all identically zero.

**Proof.** The properties obtained when $I \cap J^c$ is nonempty are already observed above. Furthermore, we have for all $y \in \partial S_n$

$$F_I(x, y) = F_{I'}(x, y), \quad \{x \in S_n : \lambda_J(x) = 0\}.$$

As a consequence,

$$\text{tr}_{\lambda_j = 0} F_I(\cdot, y)_u = \text{tr}_{\lambda_j = 0} F_{I'}(\cdot, y)_u, \quad y \in \partial S_n.$$

The desired result follows directly from the definition of the operators $R^k_{I,J}$. □
It follows directly from the definition of the operators $R_{I,J}^k$, and the smoothness of the functions $F_I(\cdot , y)$ on $S_n$ for any fixed $y \in \partial S_n$, that the operator $\lambda_I^{-s} R_{I,J}^k$ maps $\Lambda^k(S_n)$ to $\Lambda^{k-s}(S_n)$. The corresponding result in the polynomial case is given below.

**Lemma 4.2.** Let $I \in \Gamma$ and $J$ an index set with $|J| = s + 1$.

i) If $u \in \mathcal{P}_r \Lambda^k(S_n)$, then $\lambda_I^{-s} R_{I,J}^k u \in \mathcal{P}_r \Lambda^{k-s}(S_n)$;

ii) If $u \in \mathcal{P}_- \Lambda^k(S_n)$ then $\lambda_I^{-s} R_{I,J}^k u \in \mathcal{P}_- \Lambda^{k-s}(S_n)$.

Furthermore, if $I \cap J^c$ is nonempty, then the assumptions in the two cases can be reduced to the trace conditions $u \in \mathcal{P}_r \Lambda^k(\partial S_n)$ and $u \in \mathcal{P}_- \Lambda^k(\partial S_n)$, respectively.

**Proof.** Recall that

$$\lambda_I(x)^{-s} (R_{I,J}^k u)(x)(v_1, \ldots, v_{k-s}) = \int_{[x,j]} (u_{F_I(x,y)} D_x F_I v_1, \ldots D_x F_I v_{k-s})(t_1, \ldots, t_s) dy,$$

where $t_1, \ldots, t_s$ is any orthonormal basis for the tangent space $T[x,j]$. Since $F_I$ is linear in $x$, the integrand preserves the polynomial degree of $u$ for each fixed $y \in [x,j]$. Since the same will be true for the integral with respect to $y$, the first part of the lemma follows.

To show the $\mathcal{P}_-^r$ spaces are also preserved, we look at $\lambda_I^{-s} R_{I,J}^k u \cdot (x - x_j)$, where $j \in J$. In fact, we choose $j \in I \cap J$ if this set is nonempty. It then follows that

$$DF_x(x - x_j) = \sum_{\ell \in I^c} (x_\ell - y) \lambda_{I}(x) = F_I(x, y) - y,$$

which gives

$$\lambda_I(x)^{-s} (R_{I,J}^k u \cdot (x - x_j)) = \int_{[x,j]} (\Pi_s F_I^s u') x,$$

where for each fixed $y$, we have $u'_y = u \cdot (x - x_j)$. In other words, $R_{I,J}^k u \cdot (x - x_j)$ satisfies the same definition as $R_{I,J}^k u$, but with $u$ replaced by $u'$. However, if $u \in \mathcal{P}_- \Lambda^k(S_n)$, then $u' \in \mathcal{P}_r \Lambda^k(S_n)$ for each $y$, and hence the same argument as above shows that $\lambda_I^{-s} R_{I,J}^k u \cdot (x - x_j) \in \mathcal{P}_r \Lambda^k(S_n)$. Alternatively, if $I \cap J$ is empty, we obtain

$$DF_x(x - x_j) = x - x_j + \sum_{\ell \in I} (y - x_\ell) \lambda_{I}(x) = F_I(x, y) - x_j.$$

Also in this case, we obtain an expression of the form (4.3), but now with $u' = u \cdot (x - x_j)$. As above, we again can conclude that $\lambda_I^{-s} R_{I,J}^k u \cdot (x - x_j) \in \mathcal{P}_r \Lambda^k(S_n)$ if $u \in \mathcal{P}_- \Lambda^k(S_n)$. Hence, the second part of lemma has been established. The final conclusion, related to the assumption $I \cap J^c$ nonempty, again follows from the fact that $R_{I,J}^k u$ only depends on $\text{tr}_\partial S_n u$ in this case. \[\square\]}
For \( I \in \Gamma \) and index sets \( J \) with \(|J| = s + 1\), we define
\[
(\delta R^k_{I}u)_J = \sum_{i=0}^{s} (-1)^i R^k_{I,J(\hat{i})},
\]
The following relation will be a key tool to show that the extensions \(E^k_n\) are cochain maps.

**Proposition 4.3.** Let \( I \in \Gamma \) and \( J \) an index set with \(|J| = s + 1\), where \(0 \leq s \leq k + 1\). The operators \( R^k_{I,J} \) satisfy the relations
\[
d(R^k_{I,J}u) = R^{k+1}_{I,J}du + (-1)^{k-s}(\delta R^k_{I}u)_J, \quad u \in \Lambda^k(S).
\]

**Proof.** By applying (4.1), we get
\[
dR^k_{I,J}u = \int_{\partial J} d_S \Pi_s F^*_I u = \int_{\partial J} [\Pi_s F^*_I du + (-1)^{k-s}d_{[x,J]}\Pi_{s-1}F^*_I u]
\]
\[
= R^{k+1}_{I,J}du + (-1)^{k-s} \int_{\partial [x,J]} \Pi_{s-1}F^*_I u,
\]
where we have used Stokes theorem for the last equality. The proof of the proposition is completed by observing that
\[
\int_{\partial [x,J]} \Pi_{s-1}F^*_I u = (\delta R^k_{I}u)_J.
\]
\]

5. The operators \( A_{I,J}^k \) and their properties

In this section, we define the coefficient operators \( A_{I,J}^k \) for \( I \in \Gamma_m, 1 \leq m \leq n \), and \( J \in \Gamma_s(I), 0 \leq s \leq k \leq n - 1 \), as operators mapping \( \Lambda^k(\partial S_n) \) to \( \Lambda^{k-s}(S_n) \).

As we saw already in Section 3, the simplexes \([P_{I,J,x}]\), defined as all convex combinations of the points \( P_{I,j,x}, j \in J \), will in general not be a subset of \( \partial S_n \), unless \( I \cap J^c \) is nonempty. As a consequence, the operators \( R^k_{I,J} \) are not well defined for functions in \( \Lambda^k(\partial S_n) \) unless this condition holds. We will therefore define the operators \( A_{I,J}^k \) as linear combinations of these order reduction operators in such a way that they are well defined for functions in \( \Lambda^k(\partial S_n) \). We have already defined the operators \( A_{I,J}^k \) for \( J = \{j\} \in \Gamma_s(I) \) by \( A_{I,J}^k u = R^k_{I,J}u = P^*_I u \), and in the case \( J \in \Gamma_1(I) \) by (3.1). The general definition given below will generalize the definitions given in these special cases.

In the general case, we define the operators \( A_{I,J}^k \) of the form
\[
A_{I,J}^k = c_{s,n}^k [(n-s)R^k_{I,J} - \sum_{p \in J^c} (\delta R^k_{I})_{(p,J)}],
\]
where \( c_{s,n}^k \) are constants to be specified below. Here
\[
(\delta R^k_{I})_{(p,J)} = R^k_{I,J} - \sum_{i=0}^{s} (-1)^i R^k_{I,(p,J(\hat{i}))},
\]
and the index set \( \{ p, J(\hat{i}) \} \) is given by
\[
\{ p, J(\hat{i}) \} = \{ p, j_0, \ldots, j_{i-1}, j_{i+1}, \ldots, j_s \} \quad \text{if} \quad J = \{ j_0, j_1, \ldots, j_s \}.
\]
Alternatively, we can express the operator \( A_{I,J}^k \) as
\[
(5.2) \quad A_{I,J}^k = c_{s,n}^k \sum_{p \in J'} \sum_{i=0}^s (-1)^i R_{I,J}^{k}.
\]
The constants \( c_{s,n}^k \) are given by
\[
 c_{s,n}^k = \frac{(-1)^{1+ks}(s!)^2}{(n-1)\cdots(n-s)}, \quad 1 \leq s \leq n-1, \quad \text{and} \quad c_{0,n}^k = -1.
\]
In fact, the key relations we will use below are that these constants satisfy
\[
(5.3) \quad \frac{c_{s,n}^{k+1}}{c_{s,n}^k} = (-1)^s \quad \text{and} \quad \frac{c_{s,n}^{k+1}}{c_{s-1,n}^k} = \frac{s^2}{(n-s)}(-1)^k,
\]
which can be easily checked. Note that when \( s = k = 1 \) and \( n > 1 \), we have \( c_{1,n}^1 = 1/(n-1) \), and as consequence, we see that \( 5.2 \) generalizes the definition given in (3.1). Furthermore, \( A_{I,J}^k = 0 \) for \( J \in \Gamma_s(I), s > k \).

We next establish three key properties of the operator \( A_{I,J}^k \), using analogous properties established for the operator \( R_{I,J}^k \).

**Lemma 5.1.** The operators \( A_{I,J}^k u \) only depend on the boundary traces of \( u \) and satisfy for any \( J \in \Gamma(I) \) and \( j \in \Gamma^c \),
\[
(5.4) \quad \text{tr}_{\lambda_j = 0} A_{I,J}^k u = \text{tr}_{\lambda_j = 0} A_{I,J}^k u,
\]
where \( \Gamma' \in \Gamma \) is equal to \( \{ j, I \} \) up to a possible reordering. Furthermore, the operators \( A_{I,J}^n \) are all identically zero.

**Proof.** For each \( i \in J \subset \Gamma_s(I) \), we have that \( i \in I \cap \{ p, J(\hat{i}) \}^c \). As a consequence, \( A_{I,J}^k u \) only depends on \( \text{tr}_{\partial S_n} u \). The rest of the results follow directly from the corresponding properties of the operators \( R_{I,J}^k \) given in Lemma 4.1. \( \square \)

**Lemma 5.2.** Let \( I \in \Gamma \) and \( J \in \Gamma_s(I) \). Then the operator \( \lambda_i^{-s} A_{I,J}^k u \) maps the spaces \( \Lambda^k(\partial S_n) \rightarrow \Lambda^{k-s}(S_n), \quad \mathcal{P}_r \Lambda^k(\partial S_n) \rightarrow \mathcal{P}_r \Lambda^{k-s}(S_n), \quad \text{and} \quad \mathcal{P}_r^0 \Lambda^k(\partial S_n) \rightarrow \mathcal{P}_r^0 \Lambda^{k-s}(S_n). \)

**Proof.** Since \( A_{I,J}^k u \) is a linear combination of operators of the form \( R_{I,(p,J(\hat{i}))} u \), for which \( I \cap \{ p, J(\hat{i}) \}^c \) is nonempty, the lemma follows directly from corresponding properties of the operators \( R_{I,J}^k \), cf. Lemma 4.2. \( \square \)

**Proposition 5.3.** Let \( I \in \Gamma \) and \( J \in \Gamma_s(I) \). The operators \( A_{I,J}^k u \) satisfy the relations
\[
(5.5) \quad A_{I,J}^{k+1} du = (-1)^s d(A_{I,J}^k u) + s(\delta A_{I,J}^k u), \quad 0 \leq s \leq k + 1, \quad 0 \leq k \leq n - 1.
\]
Proof. For \( s = 0 \), the relation \( dA_{I,J}^k u = A_{I,J}^k du \) follows from the corresponding property of the pullbacks \( P_{I,J}^r \). Next we show that for \( s \geq 1 \) we have
\[
(\delta A_{I,J}^k u)_J = -sc_{s-1,n}^k(\delta R_{I,J}^k u)_J.
\]
From (5.1), it follows that
\[
W_J = \sum_{p \in J^c} (\delta R_{I,J}^k u)_{(p,J)},
\]
where we have used the fact that for each fixed \( p \) and \( k \) fixed. However,
\[
(\delta W)_J = \sum_{a = 0}^s (-1)^a \left[ (\delta R_{I,J}^k u)_{(a,J(a))} + \sum_{p \in J^c} (\delta R_{I,J}^k u)_{(p,J(a))} \right]
\]
\[
= (s + 1)(\delta R_{I,J}^k u)_J + \sum_{p \in J^c} \sum_{a = 0}^s (-1)^a R_{I,J}^k (a,J(a)) u - \sum_{p \in J^c} ((\delta \circ \delta) R_{I,J}^k) u_J
\]
\[
= (n + 1)(\delta R_{I,J}^k u)_J,
\]
where we have used the fact that for each fixed \( p \),
\[
((\delta \circ \delta) R_{I,J}^k u)_J = \sum_{a = 0}^s (-1)^a \left[ \sum_{i = a}^{a-1} (-1)^i R_{I,J}^k (a,J(a)) - \sum_{i = a+1}^s (-1)^i R_{I,J}^k (p,J(a)) \right] = 0.
\]
This implies (5.6). For \( s \geq 1 \), it follows from Proposition (5.3), (5.5), (5.6), and the property \( \delta \circ \delta = 0 \) that
\[
d(A_{I,J}^k u) = c_{a,n}^k [(n-s) R_{I,J}^k u - \sum_{p \in J^c} (\delta R_{I,J}^k u)_{(p,J)}]
\]
\[
= c_{a,n}^k [(n-s) R_{I,J}^k u - \sum_{p \in J^c} (\delta R_{I,J}^k u)_{(p,J)}] + c_{s,n}^k (n-s)(-1)^{k-s} (\delta R_{I,J}^k u)_J
\]
\[
= \frac{c_{a,n}^k}{c_{a,n}^k} A_{I,J}^{k+1} du + c_{s,n}^k \frac{(n-s)}{sc_{s-1,n}^k} (-1)^{k-s+1}(\delta A_{I,J}^k u)_J
\]
\[
= (-1)^s A_{I,J}^{k+1} du + s(-1)^{s+1}(\delta A_{I,J}^k u)_J.
\]
This completes the proof.

\[
6. \text{ The operator } E_n^k
\]

In this final section, we prove that the operators \( E_n^k \), defined by (2.2), are polynomial preserving cochain extensions.

**Theorem 6.1.** The operators \( E_n^k \), defined by (2.2), are extension operators.

**Proof.** We first note since the coefficients \( A_{I,J}^k u \) only depend on the boundary traces of \( u \), the same is true of \( E_n^k u \). To establish the extension property, we will, without loss of generality, show that \( \text{tr}_{\lambda_0 = 0} E_n^k u = \text{tr}_{\lambda_0 = 0} u \). In fact, the primal operator \( E_{n,0}^k \), given by
\[
E_{n,0}^k u = \frac{1}{n} \sum_{l \in I_m} \sum_{1 \leq m \leq n} \frac{\lambda_j}{\lambda_l} P_{I,J}^r u,
\]
is already an extension. To see this we can argue as we have done above for scalar valued functions and one-forms. The maps \( P_{I, j} \), where \( I \in \Gamma_1 \) with \( 0, j \in I \) and \( j \neq 0 \) are projections onto \( \{ x : \lambda_0(x) = 0 \} \). As a consequence,

\[
\text{tr}_{\lambda_0 = 0} \left[ \frac{1}{n} \sum_{l \in \Gamma_m} \sum_{0 \leq I} P_{I, l} \right] u = \text{tr}_{\lambda_0 = 0} u,
\]

while the rest of the terms of \( E_n^k u \) give no additional contribution to the trace due to the cancellation of terms corresponding to \( I \) and \( I' = \{ 0, I \} \), where \( 0 \notin I \). So to complete the proof, we need to show that

\[
\text{tr}_{\lambda_0 = 0} \left[ \frac{1}{n} \sum_{l \in \Gamma_m} \sum_{0 \leq I} ( -1 )^{m+1} \sum_{J \in \Gamma_s(I)} \frac{\phi_j}{\lambda^{s+1}} \wedge A_{I, j}^k \right] = 0.
\]

However, for terms corresponding to pairs \((I, J)\), with \( 0 \in J \), we have \( \text{tr}_{\lambda_0 = 0} \phi_j = 0 \), while if \( 0 \notin J \), the trace of the terms corresponding to the pairs \((I, J)\) and \((\{0, I\}, J)\) will cancel due to the trace property \((5.4)\) of the coefficients.

**Theorem 6.2.** The extensions \( E^k_n \) are cochain maps, i.e., they satisfy \( dE^k_n = E^{k+1}_n d \) for \( 0 \leq k \leq n - 2 \). In addition, \( dE^0_n u = 0 \).

**Proof.** We first observe that for \( I \in \Gamma \) and \( J \in \Gamma_s(I) \), we have

\[(6.1) \quad d(\frac{\phi_j}{\lambda^{s+1}}) = (s + 1) \sum_{i \in I \setminus J} \frac{\phi_{i, j}}{\lambda^{s+2}}.\]

In particular, the right hand side is zero if \( J = I \). By the Leibniz rule, we have

\[
dE^k_n u = \frac{1}{n} \sum_{I \in \Gamma_m} ( -1 )^{m+1} \sum_{J \in \Gamma_s(I)} d \left[ \frac{\phi_j}{\lambda^{s+1}} \wedge A_{I, j}^k \right] u
\]

\[
= \frac{1}{n} \sum_{I \in \Gamma_m} ( -1 )^{m+1} \sum_{s = 0}^k \sum_{J \in \Gamma_s(I)} \left[ d \left( \frac{\phi_j}{\lambda^{s+1}} \right) \wedge (A_{I, j}^k u) + ( -1 )^s \frac{\phi_j}{\lambda^{s+1}} d( A_{I, j}^k u ) \right].
\]

However, by using \((6.1)\), we obtain for each fixed \( I \in \Gamma \),

\[
\sum_{s = 0}^k \sum_{J \in \Gamma_s(I)} d \left( \frac{\phi_j}{\lambda^{s+1}} \right) \wedge (A_{I, j}^k u) = \sum_{s = 0}^k (s + 1) \sum_{J \in \Gamma_s(I)} \sum_{i \in I \setminus J} \frac{\phi_{i, j}}{\lambda^{s+2}} \wedge (A_{I, j}^k u)
\]

\[
= \sum_{s = 0}^k (s + 1) \sum_{J \in \Gamma_s(I)} \sum_{i = 0}^{s+1} ( -1 )^i A_{I, j}^k u
\]

\[
= \sum_{s = 0}^{k+1} \sum_{J \in \Gamma_s(I)} \sum_{i = 0}^{s+1} ( -1 )^i A_{I, j}^k u.
\]

Combining these results, noting that \( A_{I, j}^k u = 0 \) for \( J \in \Gamma_{k+1}(I) \), and using \((5.5)\), we get that

\[
dE^k_n u = E^{k+1}_n du, \quad 0 \leq k \leq n - 2, \quad \text{and} \quad dE^0_n u = 0.
\]

This completes the proof.
The final result we need to prove is that the extensions $E^k_n$ preserve smoothness and polynomial properties. The operator $E^k_n$ can be expressed as

$$E^k_n u = \frac{1}{n} \sum_{1 \leq m \leq n} (-1)^{m+1} E^{k(I)}_n u,$$

where each operator $E^{k(I)}_n$ is given by

$$E^{k(I)}_n = \sum_{J \in \Gamma_s(I)} \frac{\phi_J}{\lambda^{I+1}} \wedge A^k_{I,J} u.$$

We will show below that each operator $E^{k(I)}_n$ preserves smoothness and polynomial properties. It is worth noting that it follows from Lemma 5.2 that the operators $E^{k(I)}_n$ map $\Lambda^k(\partial S^n)$ to $\Lambda^{k-1}(S^n)$, and also

$$\mathcal{P}_r \Lambda^k(\partial S^n) \rightarrow \Lambda^{k-r}(S^n), \quad \text{and} \quad \mathcal{P}_{r+1} \Lambda^k(\partial S^n) \rightarrow \Lambda^{k-r+1}(S^n).$$

In fact, to obtain the result for the trimmed spaces, we also need to use the wedge product property for these spaces, for example expressed by formula (3.16) of [3].

Theorem 6.3. The extension operator $E^k_n$ maps the spaces $\Lambda^k(\partial S^n)$ to $\Lambda^{k-1}(S^n)$, $\mathcal{P}_r \Lambda^k(\partial S^n) \rightarrow \mathcal{P}_r \Lambda^{k}(S^n)$, and $\mathcal{P}_{r+1} \Lambda^k(\partial S^n) \rightarrow \mathcal{P}_{r+1} \Lambda^{k}(S^n)$.

The proof of this result will utilize the following alternative representation of the operators $E^{k(I)}_n$.

Lemma 6.4. The operators $E^{k(I)}_n$ admit the representation

$$E^{k(I)}_n u = \frac{1}{m+1} \left[ \sum_{J \in I} P^{*}_{I,J} u + dQ^k_n u + Q^{k+1}_n du \right],$$

where the operator $Q^k_n = Q^{k(I)}_n$ is given by

$$Q^k_n u = \sum_{J = \Gamma_s(I)} \frac{1}{s} \frac{(\delta \phi)_J}{\lambda^{I+1}} \wedge A^k_{I,J} u.$$

Since the derivation of the alternative representation of the operators $E^k_n$ is slightly technical, we will first use the representation (6.3) to prove Theorem 6.3.

Proof. (of Theorem 6.3) It is enough to show the desired mapping properties for each operator $E^{k(I)}_n$. From the result of Lemma 5.2 it easily follows that the operator $Q^k_n(I)$ maps $\Lambda^k(\partial S^n)$ to $\Lambda^{k-1}(S^n)$ and $\mathcal{P}_r \Lambda^k(\partial S^n)$ to $\mathcal{P}_{r+1} \Lambda^{k-1}(S^n)$. By combining this with the fact that the operators $P_{I,J}$ are linear, the desired result in the smooth case and the full polynomial case follows. Furthermore, since
Considered fixed. We first study the primal operator $E^n(I)$ maps $P_r^k(\partial S_n)$ into

$$P_r^k(S_n) \cap \lambda_i^{-1} P_{r+1}^k(S_n).$$

However, this space is identical to $P_r^k(S_n)$). To see this, let $u \in P_{r+1}^k(S_n)$ such that $\lambda_i^{-1} u \in P_r^k(S_n)$. It follows from the definition of the trimmed spaces that for any $x_j \in \Delta_0(S_n)$

$$u, (x - x_j) \in P_{r+1}^k(S_n).$$

But since $\lambda_i^{-1} u$ is also in $P_r^k(S_n)$ we have

$$\lambda_i^{-1} u, (x - x_j) = (\lambda_i^{-1} u, (x - x_j) \in P_{r+1}^k(S_n).$$

In other words, the polynomial form $u, (x - x_j)$ has $\lambda_i$ as a linear factor, and as a consequence, $\lambda_i^{-1} u, (x - x_j)$ is also a polynomial form, which then must be in $P_r^k(S_n)$. This implies that $\lambda_i^{-1} u \in P_r^k(S_n)$. This shows that the space given by (6.4) is included in $P_r^k(S_n)$, and the opposite inclusion is straightforward to check.

It remains to establish the alternative representation (6.3) for the operators $E^n_k(I)$. Throughout the discussion below, the index set $I \in \Gamma_m$, $1 \leq m \leq n$, will be considered fixed. We first study the primal operator $E^n_k = E^n_{n,0}(I)$ given by

$$E^n_{n,0} u = \sum_{j \in \Gamma_0(I)} \phi_j^{(I)} \Lambda^{I,0}_i u = \sum_{i \in I} \Lambda_i \Lambda^{I,0}_i u.$$

In particular, for $k = 0$ we have $E^n_0 = E^n_0(I)$.

**Lemma 6.5.** For $I \in \Gamma_m$, the operator $E^n_k = E^n_{n,0}(I)$ has a representation of the form

$$E^n_{n,0} u = \frac{1}{m+1} \left[ \sum_{j \in I} P_{I,0}^* u + \lambda_i^{-1} \sum_{j \in \Gamma_1(I)} (\delta \phi)_j \Lambda^{I,0}_i u \right].$$

**Proof.**

$$\lambda_i^{-1} \sum_{i \in I} \Lambda_i P_{I,0}^* u = \frac{1}{m+1} \sum_{j \in I} \left[ P_{I,0}^* u + \lambda_i^{-1} \sum_{i \in I} \Lambda_i (P_{I,0}^* u - P_{I,0}^* u) \right]$$

$$= \frac{1}{m+1} \left[ \sum_{j \in I} P_{I,0}^* u + \lambda_i^{-1} \sum_{j \in I} \sum_{i \in I} (\lambda_i - \lambda_j) (P_{I,0}^* u - P_{I,0}^* u) \right]$$

$$= \frac{1}{m+1} \left[ \sum_{j \in I} P_{I,0}^* u + \lambda_i^{-1} \sum_{j \in \Gamma_1(I)} (\delta \phi)_j \Lambda^{I,0}_i u \right].$$

The representation for $E^n_{n,0}$ follows immediately. 

To establish an analogous result more generally, we will also need the following preliminary result.
Lemma 6.6. For \( I \in \Gamma_m \) and \( 0 \leq s \leq m \), the following identity holds.

\[
\sum_{J \in \Gamma_s(I)} d\left( \frac{(\delta \phi)_J}{\lambda_J^s} \right) \wedge A_{I,J}^u + \frac{s}{\lambda_{s+1}} \sum_{J \in \Gamma_{s+1}(I)} (\delta \phi)_J \wedge (\delta A_{I,J}^u)_J = \frac{(m+1)s}{\lambda_{s+1}} \sum_{J \in \Gamma_s(I)} \phi_J \wedge A_{I,J}^u.
\]

Proof. If \( J \in \Gamma_{s-1}(I) \), we get from (6.1) that

\[
d\left( \frac{(\delta \phi)_J}{\lambda_J^s} \right) = s \sum_{p \in I \setminus J} \frac{\phi_p}{\lambda_J^{s+1}}.
\]

As a further consequence, we obtain that if \( J \in \Gamma_s(I) \), then

\[
d\left( \frac{(\delta \phi)_J}{\lambda_J^s} \right) = \frac{s(s+1)}{\lambda_J^{s+1}} \phi_J + \frac{s}{\lambda_J^{s+1}} \sum_{i=0}^s (-1)^i \sum_{p \in I \setminus J} \phi_{p,J(\hat{i})}.
\]

If we wedge the second term with \( A_{I,J}^u \) and sum over \( J \in \Gamma_s(I) \), we obtain

\[
\sum_{J \in \Gamma_s(I)} \sum_{i=0}^s (-1)^i \sum_{p \in I \setminus J} \phi_{p,J(\hat{i})} \wedge A_{I,J}^u = - \sum_{J \in \Gamma_{s+1}(I)} \sum_{i=0}^{s+1} (-1)^{s+i} \phi_{J(\hat{i})} \wedge A_{I,J(\hat{i})} u = - \sum_{J \in \Gamma_{s+1}(I)} (\delta \phi)_J \wedge (\delta A_{I,J}^u)_J + (m - s) \sum_{J \in \Gamma_s(I)} \phi_J \wedge A_{I,J}^u.
\]

The desired result follows by collecting terms.

To establish Lemma 6.3, we will make use of the following operators,

\[
E_{n,\ell}^k = E_{n,\ell}^k(I) = \sum_{J \in \Gamma_s(I)} \frac{\phi_J}{\lambda_J^{s+1}} \wedge A_{I,J}^u, \quad 0 \leq \ell \leq k.
\]

In particular, we note that \( E_{n,\ell}^k(I) = E_n^k(I) \).

Proof. (of Lemma 6.3) We will prove by induction that the operator \( E_{n,\ell}^k \) admits a representation of the form

\[
(6.6) \quad E_{n,\ell}^k u = \frac{1}{m+1} \left[ \sum_{J \in \ell} P_{I,J}^u + \sum_{J \in \Gamma_{s+1}(I)} \frac{(\delta \phi)_J}{\lambda_J^{s+1}} \wedge (\delta A_{I,J}^u)_J \right. + dQ_{n,\ell}^k u + Q_{n,\ell}^{k+1} du,
\]

where,

\[
Q_{n,\ell}^k = \sum_{J \in \Gamma_s(I)} \frac{1}{s} \frac{(\delta \phi)_J}{\lambda_J^{s+1}} \wedge A_{I,J}^u,
\]
such that $Q_{n,k}^k = Q_n^k$ and $Q_{n,0}^k = 0$. Note that when $\ell = 0$, (6.6) is exactly the identity (6.5), while for $l = k$, $A_{I,J}^{k+1} du = (k+1)(\delta A_I^k u)_J$. This gives

$$Q_{n,k}^{k+1} du + \sum_{J \in \Gamma_{k+1}} (\delta \phi)_J^k \wedge (\delta A_{I,J}^k u)_J = Q_{n,k}^{k+1} du.$$

Therefore, the representation formula (6.3) follows from (6.6) for $l = k$.

It remains to carry out the induction argument to establish (6.6). If we assume that (6.6) holds for $\ell - 1$, then

$$(m+1)E_{n,\ell}^k u - \sum_{j \in I} P_{I,j}^* u = dQ_{n,\ell-1}^k u + Q_{n,\ell-1}^{k+1} du$$

$$+ \sum_{J \in \Gamma_{\ell}(I)} \left[ (m+1) \frac{\phi_J^k}{\lambda_{I}^{\ell+1}} \wedge A_{I,J}^{k+1} u + \frac{(\delta \phi)_J^k}{\lambda_{I}^\ell} \wedge (\delta A_{I,J}^{k+1} u)_J \right]$$

$$= dQ_{n,\ell-1}^k u + Q_{n,\ell-1}^{k+1} du$$

$$+ \sum_{J \in \Gamma_{\ell+1}(I)} \left[ (m+1) \frac{\phi_J^k}{\lambda_{I}^{\ell+1}} \wedge A_{I,J}^{k+1} u + (-1)^{(\ell-1)} \frac{1}{\ell} \frac{(\delta \phi)_J^k}{\lambda_{I}^\ell} \wedge dA_{I,J}^{k+1} u \right],$$

where we have used the relation (5.5). Now from Lemma 6.6 we get that

$$d(Q_{n,\ell}^k - Q_{n,\ell-1}^k) = \sum_{J \in \Gamma_{\ell+1}(I)} \frac{1}{\ell} d \left[ \frac{(\delta \phi)_J^k}{\lambda_{I}^{\ell+1}} \wedge A_{I,J}^{k+1} u \right]$$

$$= \sum_{J \in \Gamma_{\ell+1}(I)} \frac{1}{\ell} \left[ (-1)^{(\ell-1)} \frac{(\delta \phi)_J^k}{\lambda_{I}^\ell} \wedge dA_{I,J}^{k+1} u + (m+1) \frac{\phi_J^k}{\lambda_{I}^{\ell+1}} \wedge A_{I,J}^{k+1} u \right]$$

$$- \sum_{J \in \Gamma_{\ell+1}(I)} \frac{(\delta \phi)_J^k}{\lambda_{I}^{\ell+1}} \wedge (\delta A_{I,J}^{k+1} u)_J.$$

Combining these results, we get

$$(m+1)E_{n,\ell}^k u - \sum_{j \in I} P_{I,j}^* u = dQ_{n,\ell}^k u + Q_{n,\ell}^{k+1} du + \sum_{J \in \Gamma_{\ell+1}(I)} \frac{(\delta \phi)_J^k}{\lambda_{I}^{\ell+1}} \wedge (\delta A_{I,J}^{k+1} u)_J,$$

which completes the induction argument, and the proof of the lemma. □

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