Exceptional Points from the Hamiltonian of a hybrid physical system: Squeezing and anti-Squeezing

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Abstract. We study the appearance of Exceptional Points in a hybrid system composed of a superconducting flux-qubit and an ensemble of nitrogen-vacancy colour centres in diamond. We discuss the possibility of controlling the generation of Exceptional Points, by the analysis of the model space parameters. One of the characteristic features of the presence of Exceptional Points, is the departure from the exponential decay behaviour of the observables as a function of time. We study the time evolution of different initial states, in the presence of the hybrid system, by computing the reduced density matrix of each subsystem. We present the results we have obtained for the steady behaviour of different observables. We analyse the appearance of Squeezed Spin States and anti-Squeezed Spin States.

1 Introduction

Quantum information processing [1,2] can involve the interplay of atoms in optical traps [3,4], superconducting circuits [5,6], nuclear spins [7] and defects in a crystal lattice [8], among other physical components. Usually, quantum coherence of these localized systems is degraded by the interaction with the environment [9,10]. It can be differentiated two to include the interaction with the environment [11,12]. In one case, the environment consists of microscopic degrees of freedom and it is described by an energy continuum [13–15]. While, in the other case, it can be modelled by some macroscopic approximation of these microscopic degrees of freedom [16–22]. In a recent work [12], these two types of interactions were compared and combined to describe the essentials of a tight-binding model, which could be implemented by an optical lattice with a defect scattering centre. The authors of [12] have characterized in detail the spectrum of the system, both the discrete and the scattering states, in terms of the coupling parameters.

In the last years, in different pioneering experiments [18–20,23–26], as a result of the coupling of the physical system with its environment, the occurrence of dynamical breaking of symmetry has been observed. As a function of time, the relevant physical observables obey an oscillatory pattern in a region of the space of control parameters and an overdamped behaviour in another. The reported results are consistent with the characteristic features of a critical phase transition. It was the authors of [23], who after studying spin swapping in atomic systems, have labelled the observed transition as a Dynamical Phase Transition (DPT) [23]. This phenomenon was also reported in other experimental contexts in earlier works [18–20,25]. It is worthy of mention the study of resonance trapping in microwave cavities [25] and the analysis of einselection [18–20] in mesoscopic quantum dots. In both works long- and short-lived states interfere among themselves, and the so-called bound states in the continuum [27,28] are generated. It should be mentioned that the problem of long-lived states dates from the beginning of the development of quantum mechanics [29,30].

The dynamics of open physical systems can be described in the framework of non-hermitian Hamiltonians [11,13–15,31–42]. When non-hermitian Hamiltonians are studied in terms of the space of model parameters, regions with different symmetry emerge depending on the spectrum. The boundary of the regions is formed by points for which two or more eigenvalues degenerate into one and the corresponding eigenvectors become parallel. These points are known as Exceptional Points (EPs). Non-hermitian Hamiltonians arise naturally within the formalism of Feshbach [11,13,14,31]. The starting point of this approach is to separate the space of the system under study into two subspaces, one containing the central physical system, and the other related to the extended environment. Though the Hamiltonian of the whole system is a hermitian operator, the solution to the problem in the localized region can be given in terms of an effective non-hermitian
Hamiltonian. That is, the matrix elements, obtained by projecting out the degrees of freedom associated with the environment, can take complex values [11,13,14,31].

It was in 1998, with the work due to Bender and Boettcher [43,44] that the class of hamiltonians invariant under Parity-Time Reversal (PT) symmetry, have gained attention. In general, a PT-symmetric system represents a particular macroscopic approximation of a true microscopic open system. From then to nowadays, they have been useful in the understanding of different physical problems, i.e. microwave cavities [45], atomic diffusion [46], electronic circuits [47], optical waveguide arrays [48], quantum critical phenomena [49–52].

Recently, some experiments were performed in the vicinity of EP. As an example, let us mention the experiments reported in [53] and [54]. The authors of [53] have studied, by using quantum tomography techniques, a superconducting qubit with dissipation close to its EP [53]. They have found that, due to dissipation, the system stabilizes the initial state within the formalism of Green Matrix.

The contribution of the superconducting flux-qubit to the Hamiltonian, $H_{Fq}$ of equation (1), can be written in the basis of clockwise and anticlockwise qubit-persistent-currents [58,62,80–83] as

$$H_{Fq} = \frac{1}{2} (\Delta s_x + \epsilon s_z),$$  \hspace{1cm} (2)

in terms of the persistent current in the qubit, $I_p$, the external flux threading the qubit loop, $\Phi_{ex}$, the flux quantum, $\Phi_0 = 1/(2e)$, and the tunnel splitting, $\Delta$, the bias energy is given by $\epsilon = 2I_p(\Phi_{ex} - 3\Phi_0)/2$. The Pauli spin-1/2 operators of the qubit are denoted by $\{s_x, s_y, s_z\}$.

The term $H_S$ of equation (1) models ensemble of NV− colour centres in diamond. It can be written in terms of total collective spin-operators of the NV ensemble, $\{S_x, S_y, S_z\}$. It reads

$$H_S = D S_x^2 + E (S_y^2 - S_z^2).$$  \hspace{1cm} (3)

The first term of equation (3) corresponds to a one-axis twisting interaction [84], and it is related to the zero-field splitting $D = 2.878$ GHz. This term usually leads to a squeezing pattern. The second term is associated to the strain-induced splitting of the ensemble [60]. It can be thought as a Lipkin interaction [85–88] with strength $E$.

It is well known that the content of paramagnetic impurities in diamond affects the coherence of the ensemble of NV−-colour-centres [58,89–93]. Among them, one of the simplest magnetic impurities is formed by neutral nitrogen atoms, P1 centres. The decoherence of the NV ensemble, in the presence of P1 centres in the crystal, depends on the concentration of P1 centres [92,93]. We shall model the effect of P1 centres in the ensemble by replacing the coupling interaction of the NV centres with the SFQ [58], $(g/2) s_z S_z$, by an asymmetric interaction of the form

$$H_{int-qs} = 2g s_z (\alpha S_+ + S_-)$$

$$= 2g s_z ((1 + \alpha)S_x + i(\alpha - 1)S_y).$$  \hspace{1cm} (4)

A similar approach have been reported [94–96].

The transformation

$$\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}.$$  \hspace{1cm} (5)

Allows us to diagonalize the flux-qubit sector, $H_{Fq}$, of the Hamiltonian equation (1). Notice that, the transformation between both sets of operators, $\{s_x, s_y, s_z\}$ and $\{\sigma_x, \sigma_y, \sigma_z\}$, preserves the commutation relations of the $su(2)$ algebra. In terms of the new operators $H_{Fq}$ can be written as

$$H_1 = \frac{1}{2} E_{qb} \sigma_z,$$  \hspace{1cm} (6)

with $\cos \beta = \epsilon/E_{qb}$, $\sin \beta = -\Delta/E_{qb}$, and $E_{qb} = \sqrt{\epsilon^2 + \Delta^2}$. Similarly, $H_{int-qs}$ can be written as

$$H_2 = 2g (\cos \beta \sigma_z + \sin \beta \sigma_x) (\alpha S_+ + S_-).$$  \hspace{1cm} (7)

Finally, we shall rewrite the Hamiltonian of the physical system as

$$H = H_1 + H_2 + H_S.$$  \hspace{1cm} (8)

In general, this Hamiltonian is a non-hermitian operator, except if $\alpha = 1$. We can aim to diagonalize $H$ in the
product basis
\[ |k_{qb}, N_S, k_S\rangle = |k_{qb}\rangle \otimes |N_S, k_S\rangle, \]
\[ |N_S, k_S\rangle = N_S S^{k_S}_+ |0\rangle_S, \]
\[ |k_{qb}\rangle = N_{qb} \sigma^k_{qb} |0\rangle_{qb}, \]
(9)
where \( S^\pm = S_x \pm iS_y \) and \( \sigma^\pm = \sigma_x \pm i\sigma_y, \). The label \( k_S \) can run from 0 to the total number of NV centres of the system, \( N \). While \( k_{qb,j} \) takes the values 0 or 1. \( \{ k_{qb,j}\} \) represents each of the arrays of \( N_{qb} \) superconducting qubits. \( N_S \) and \( N_{qb} \) are normalization factors.

The Hamiltonian of equation (1) is invariant under parity symmetry but not under time-reversal symmetry, consequently, it is not invariant under \( PT \) symmetry. This can be seen from the transformation properties of the spin operators \([97]\). Let us briefly review the corresponding transformations. Under spatial reflection, \( P \), the spin operator transforms as
\[ P \, S \, P^{-1} = S. \]
On the other hand, time-reversal operation is an antiunitary operator \( T = U \, K \), with \( U \) an unitary operator and \( K \) the complex conjugation operator \([97]\). Consequently, under time reversal, the spin operator transforms as
\[ T \, S \, T^{-1} = -S, \]
and \( i \to -i \). The \( T \)-symmetry is broken by the first term of \( H \) of equation (8). Notice that the states of the basis of equation (9), can be classified according to its transformation under parity
\[ P | k_{qb}, N_S, k_S\rangle = (-1)^{k_S+k_{qb}} | k_{qb}, N_S, k_S\rangle. \]
(10)
Nevertheless, \( H \) and \( H^\dagger \) are quasi-hermitian operators. This can be proved straightforward by observing that \( H^\dagger = H^T \) so that they are isospectral Hamiltonians
\[ H = \tilde{P} J \tilde{P}^{-1}, \]
\[ H^\dagger = \tilde{P} J \tilde{P}^{-1}. \]
(11)
In general, \( J \) is a Jordan matrix, while \( \tilde{P} \) and \( \tilde{P} \) are the matrices of the generalized eigenvectors of \( H \) and \( H^T \), respectively. Finally, the symmetry operator \( S \), such that \( H = SH^T S^{-1} \), is given by \( S = \tilde{P} \tilde{P}^{-1} \).

The spectrum of the Hamiltonian can consist of real eigenenergies or complex-conjugate pairs eigenenergies, depending on the values of the parameters in the space \( \{ \epsilon, \Delta, D, E, g, \alpha \} \). Also, for particular values of the parameters, exceptional points are observed. The characteristics of the spectrum of the Hamiltonian of equation (1) determine the time evolution of an initially prepared. For real spectrum, the mean value of the observables will display a pattern of collapses and revivals. Meanwhile, in the regimen of complex-conjugate pair spectrum, the observables will show the behaviour of systems with gain-loss balance \([98]\). Besides, in the case of exceptional points, the time evolution of the mean values of observable will present departures from the usual exponential decay behaviour. We shall vary the values of \( \{ g/E, \alpha \} \) to control the generation of which EPs.

### 2.1 Time evolution

The dynamics generated by the Hamiltonian of equation (8) can be captured by introducing the non-Hermitian prescriptions given in \([99,100]\). Alternatively, we can use the Green operator formalism \([24,101]\). We shall refresh the essentials of it.

Associated to the Hamiltonian \( H \), we can introduce the Green operator, \( G \). It reads
\[ G(\omega) = (\omega I - H)^{-1}. \]
(12)
The time evolution of the system is related to the Fourier Transform of the Green operator, \( F \)
\[ F(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G(\omega)e^{-i\omega t}. \]
(13)
We can introduce the transition matrix \( P \) as
\[ P(t) = \mathcal{N}^2(t) \, F(t) \, F(t)^\dagger, \]
(14)
with \( \mathcal{N}(t) = (\text{Tr}(F^\dagger(t) \, F(t)))^{-1/2} \).

The transition probabilities, between the states of equation (9), are given by the matrix elements of \( P \).

In the basis of equation (9), a general normalized initial state can be written as
\[ \langle I | = (c_1, c_2, ..., c_n)^T, \]
(15)
the coefficient \( c_l \) is the contribution of the a generic state \( |l\rangle \) of the basis of equation (9), being \( n \) dimension of the space \( (n = 2(N + 1) \) for a system consisting of a SFQ and an ensemble of N NVs). The normalization condition, \( \langle I | I \rangle = 1 \), implies that
\[ \sum_{k=1}^{n} |c_k|^2 = 1. \]
(16)

The state \( |I\rangle \) of equation (15) evolves in time as
\[ |I(t)\rangle = \mathcal{N}(t) \, F|I(0)\rangle, \]
(17)
the mean value of an operator \( \hat{A} \), as a function of time, can be computed as \( \langle \hat{A}(t) \rangle = \langle I(t)|\hat{A}|I(t)\rangle \).

The initial state of the physical system is given by
\[ |I(0)\rangle = |I\rangle_{qb} \otimes |I\rangle_{NV}. \]
(18)
We shall assume that the initial state of the NVs, \( |I\rangle_{NV} \), is prepared as coherent spin-states of the form
\[ |I\rangle_{NV} = N_{S} \, e^{i\phi_{S}} |S_{+}\rangle > 0, \]
with \( z_{S} = -e^{-i\phi_{S}} \tan(\theta_{S}/2) \), where the angles \( (\theta_{S}, \phi_{S}) \) define the direction \( n_{S} = (\sin \theta_{S} \cos \phi_{S}, \sin \theta_{S} \sin \phi_{S}, \cos \theta_{S}) \), such that \( S \cdot n_{S} |I\rangle_{NV} = -S |I\rangle_{NV} \), with \( S = N/2 \).

While we shall consider a particular state of the form \( |k_{qb}\rangle \) (Eq. (9)) for the initial state of the superconducting qubits, \( |I\rangle_{qb} \).
2.2 Physical observables

We shall discuss the effect of non-hermitian term $H_{int - qs}$, equation (4) upon physical observables.

The survival probability, $p(t)$, of a given initial state as a function of time is computed as

$$p(t) = |\langle I(0)|I(t)\rangle|^2,$$  \hspace{1cm} (20)

where $|I(0)\rangle$ is the state at which the system has been initially prepared.

Another observable of the system is the squeezing spin parameter. This parameter accounts for the decrease of the fluctuations in one of the components of the total spin, bellow the quantum limit, while the fluctuations of the other components of the total spin increase. There is not a unique way to characterize uncertainty relations of complementary operators \cite{103,104}. We choose to use a squeezing parameter invariant under rotations \cite{105}. We shall adopt the definition given by Kitagawa and Ueda \cite{84}. To do so, we shall introduce a set of orthogonal axes $\{n_x, n_y, n_z\}$. We shall take the unitary vector $n_x$ along the direction of the total spin $J$. Next, we shall look for the minimum value of the fluctuation of the component of the spin in the plane perpendicular to $n_x$, so to fix the direction $n_x$. The Heisenberg Uncertainty Relation for the angular momentum operator is

$$\Delta^2 J_x \cdot \Delta^2 J_y \geq \frac{1}{4} |\langle J_x \rangle|^2,$$  \hspace{1cm} (21)

with $\Delta^2 J_k = \langle J_k^2 \rangle - \langle J_k \rangle^2$.

From them, the squeezing parameters \cite{84} are defined as

$$\zeta_x^2 = 2\Delta^2 J_x, \quad \zeta_y^2 = 2\Delta^2 J_y.$$  \hspace{1cm} (22)

We stay that a state is squeezed in the $x'$-direction if $\zeta_x^2 < 1$.

To complement our results, we shall compute the discrete SU(2) Wigner distribution function \cite{63,64,106–109}. The non-classical properties of a collective system of spins can be described in phase space by the quasiprobability function on the sphere $(\theta, \phi) \in S_2$ introduced by Stratonovich \cite{106}. Following \cite{106}, we define the Wigner operator as

$$\hat{w}(\theta, \phi) = \frac{2\sqrt{\pi}}{2J + 1} \sum_{L=0}^{2J} \sum_{M=-L}^{L} Y_{LM}^* (\theta, \phi) \hat{T}_L^\dagger.$$  \hspace{1cm} (23)

where $Y_{LM}(\theta, \phi)$ is the spherical harmonics function. The irreducible tensor operator, $\hat{T}_L^\dagger$, is written as

$$\hat{T}_L^\dagger = \frac{\sqrt{2L + 1}}{\sqrt{\Gamma(2L + 1)}} \sum_{m, m' = -J}^{< J m LM | J m' \rangle \langle J m |} Y_{LM}(\theta, \phi) \hat{T}_L^\dagger,$$  \hspace{1cm} (24)

and it represents all the possibilities to have collective angular momentum $J$, from a state with angular momentum $J$ coupled to a state of angular momentum $L$. In equation (24), $\langle J m L M \rangle$ are the Clebsch-Gordan coefficients accounting for the coupling $J$ and $L (0 \leq L \leq 2J)$, to total angular momentum $J$.

In general, we can define the Wigner function of the operator $A$, as $W_A(\theta, \phi) = \text{Tr}(\hat{w}(\theta, \phi))$. When $A$ is the density matrix operator, $\rho$, we obtain the discrete SU(2) Wigner function, $W(\theta, \phi)$. That is

$$W(\theta, \phi) = \text{Tr}(\rho \hat{w}(\theta, \phi)).$$  \hspace{1cm} (25)

From the previous definitions, the mean value of the operator $A$, $\text{Tr}(A \rho)$ can be computed as \cite{106–108}

$$\text{Tr}(A \rho) = \frac{(2J + 1)}{4\pi} \int d\phi \sin(\theta) d\theta W_A(\theta, \phi) W(\theta, \phi).$$  \hspace{1cm} (26)

In our case, we shall take $\rho$ as the reduced density matrix of the ensemble of NVs, $J = N/2$, being $N$ the number of NVs of the ensemble.

3 Results and discussion

We shall consider the case of two NV$^-$ colour centres in interaction with a SFQ. As experimentally, the values of $\epsilon$ and $\Delta$ can be controlled independently by using different external magnetic flux \cite{58}, we have fixed the value of $\epsilon = 0$, so that $E_{qh} = \Delta$. For the present case, the model space is divided into two independent blocks, according to the transformation of the vectors of the basis under parity. Ordering the basis as $\{ |k_q, k_S \rangle \} = \{ |0, 0 \rangle, |1, 1 \rangle, |0, 2 \rangle, |1, 0 \rangle, |0, 1 \rangle, |1, 2 \rangle \}$, the Hamiltonian can be written as

$$H = \begin{pmatrix}
D - \frac{\Delta}{4} & \frac{\gamma}{\alpha} & 0 & 0 & 0 \\
\frac{\alpha}{\gamma} & D - \frac{\Delta}{4} & 0 & 0 & 0 \\
0 & 0 & D + \frac{\Delta}{4} & \frac{\gamma}{\alpha} & E \\
0 & 0 & \frac{\gamma}{\alpha} & -\frac{\Delta}{4} & \frac{\gamma}{\alpha} \\
0 & 0 & E & \frac{\gamma}{\alpha} & D + \frac{\Delta}{4}
\end{pmatrix}.$$  \hspace{1cm} (27)

with $\frac{\gamma}{\alpha} = -\sqrt{2}g$. In terms of the parameters of the model, $\{\Delta, D, E, g, \alpha\}$, the eigenenergies of the Hamiltonian of equation (8) are given by

$$E_{1\pm} = \frac{1}{2} D + E \left( d_{\pm} - 2 \left( e^{-i\phi} \frac{C\pm}{|R\pm|} + e^{i\phi} |R\pm| \right) \right),$$
$$E_{2\pm} = \frac{1}{2} D + E \left( d_{\pm} + 2 \left( e^{-i(\frac{3}{2} + \phi}) \frac{C\pm}{|R\pm|} + e^{i(\frac{3}{2} - \phi}) |R\pm| \right) \right),$$
$$E_{3\pm} = \frac{1}{2} D + E \left( d_{\pm} + 2 \left( e^{-i\phi} \frac{C\pm}{|R\pm|} + e^{-i\phi} |R\pm| \right) \right),$$  \hspace{1cm} (28)

with

$$d_{\pm} = (1 \pm \delta) \frac{D}{E},$$
$$A_\pm = -27\gamma^2 (1 + \alpha^2) + d_{\pm}^2 - 9(1 - 2\gamma^2 \alpha)d_{\pm},$$
$$B_\pm = (3 + 4\gamma^2 \alpha) + d_{\pm}^2,$$
$$R_\pm = (A_{\pm} + \sqrt{B_{\pm}})^{1/3} = |R_\pm| e^{i\vartheta},$$  \hspace{1cm} (29)

given in terms of the adimensional parameters $\delta = \Delta/(2D)$ and $\gamma = g/E$. 
For the NV$^-$-colour-centres, we have taken $D = 2.88$ [GHz] and $E = 0.026$ [GHz].

In Figure 1, we present the position of EPs in the plane ($\alpha$, $\gamma$). The results correspond to the coalescence of the two first eigenvalues, $E_{1-}$ and $E_{2-}$. From left to right, the curves correspond to different values of the adimensional parameter $d_-$. We have considered the cases $d_- = 1, 0.5, -1.5, -3, -6, -9$, respectively. We observe a regular pattern for the presence of EPs.

As an example, in Figures 2 and 3, we display the real and the imaginary component of the eigenvalues $E_{1-}$ and $E_{2-}$ as a function of the parameters $\gamma$ and $\alpha$, for the case $d_- = 0$, which correspond to fixing $\Delta$, of equation (2), to the value $\Delta = 2D$. As pointed before, there is a region for which both eigenenergies coalesce to the same value, Exceptional Points. At those points the Hamiltonian is non-diagonalizable.

In what follows, we shall study the time evolution of the system. We shall present the numerical results that we have calculated for a particular value of the coupling interaction constant among the NV$^-$-colour-centres and the SFQ, $g = 0.02$ [GHz]. In Figure 4, we present the behaviour of the complex eigenvalues of the Hamiltonian of equation (8), in units of $E$, as a function of the asymmetry parameter $\alpha$. It can be observed two exceptional points, $\alpha \approx 0.94$ and $\alpha \approx 1.24$. Notice that for $\alpha = 1$ the Hamiltonian $\hat{H}$ is a hermitian operator.

The matrix elements of $\hat{P}$, of equation (14), give us information about the transition probabilities between the vectors of the basis of equation (9). They determine the evolution in time of the physical observables. In Figure 5, we display the results we have obtained for the matrix elements of $\hat{P}$, of equation (14), as a function of the asymmetry parameter $\alpha$, for the set of parameters of Figure 4. Vertical dotted-lines are drawn at the values of $\alpha$ corresponding to the exceptional points ($\alpha_1 \approx 0.94$ and $\alpha_2 \approx 1.24$). For values of $\alpha$ in the range ($\alpha_1$, $\alpha_2$) the spectrum is real, as it can be seen from Figure 4. The transition probability curves plotted in this region were computed by making the temporary average in two periods of time. Outside this interval, we have performed the calculation at $t = 6$ [$\mu$ sec], when the system has reached the stationary regime. In panels (a) and (b) we show the diagonal elements of $\hat{P}$, while in panels (c) and (d) we present the non-diagonal elements of $\hat{P}$. For $\alpha < 0.94$ and $\alpha > 1.24$, the stationary regime is dominated by the
transition of the states with positive parity. For values of $\alpha$ corresponding to real spectrum ($0.94 < \alpha < 1.24$), all the states have non-zero diagonal entries, though the diagonal elements are dominated by the even states of the basis. In panel (a) solid-, dashed-, dotted-lines correspond to $P_{11}(t)$, $P_{22}(t)$, and $P_{33}(t)$, respectively. In panel (c) $P_{12}(t)$, $P_{13}(t)$ and $P_{23}(t)$ are presented by solid-, dashed-, dotted-lines, respectively. In panel (b) we plot $P_{44}(t)$, $P_{55}(t)$ and $P_{66}(t)$, while in panel (d) we plot $P_{ij}(t)$, for $i, j = 4, 5, 6$. Notice also that there is a particular value of $\alpha$ for which all non-zero transitions have the same probability, $\alpha \approx 0.824$, we shall show that at this point the steady-state behaves as a Schrödinger-cat state.

The study of the behaviour of the transition probabilities as a function of time, for different values of $\alpha$, are presented in Figure 6. In panels (a), (c) and (e) we show the results of the diagonal elements of $P(t)$ as a function of the time, while in panels (b), (d) and (f), the non-diagonal elements of $P(t)$ are displayed. For panels (a) and (b) we have chosen $\alpha = 0.6$, for panels (c) and (d) $\alpha = 1.1$, and for panels (e) and (f) $\alpha = 1.3$, respectively. When the spectrum of $H$ has complex-conjugate pair eigenvalues, it can be observed that the matrix elements of $P(t)$ show a non-exponential behaviour, which is the characteristic of systems with gain-loss balance. Meanwhile, for values of $\alpha$ for which the spectrum of $H$ consists only of real eigenenergies, we can observe an oscillatory pattern.

The results we have obtained for the Survival Probability, $p(t)$ of equation (20), as a function of $\alpha$, are displayed in Figure 7. The parameters are those of the previous figures. Initially, the SFQ is prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (0, 0)$, solid line, $(\pi/4, 0)$, dashed line, $(\pi/2, 0)$, dashed-dotted-line, and $(\pi, 0)$, dotted line, respectively. As in the previous figure, for values of $\alpha$ in the range $(0.94, 1.24)$ the curves were computed by making the temporary average in two periods of time. For $\alpha < 0.94$ and $\alpha > 1.24$ the results presented have been computed at $t = 6 [\mu \text{sec}]$. In panels (a) and (c) we show the behaviour of the transition probabilities for the even states, while in panels (b) and (d) we show the transition probabilities for the odd states. In panel (a) solid-, dashed-, dotted-lines correspond to $P_{11}(t)$, $P_{22}(t)$, and $P_{33}(t)$, respectively. In panel (c) $P_{12}(t)$, $P_{13}(t)$ and $P_{23}(t)$ are presented by solid-, dashed-, dotted-lines, respectively. In panel (b) we plot $P_{44}(t)$, $P_{55}(t)$ and $P_{66}(t)$, while in panel (d) we plot $P_{ij}(t)$, for $i, j = 4, 5, 6$. Notice also that there is a particular value of $\alpha$ for which all non-zero transitions have the same probability, $\alpha \approx 0.824$, we shall show that at this point the steady-state behaves as a Schrödinger-cat state.

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The results we have obtained for the Survival Probability, $p(t)$ of equation (20), as a function of $\alpha$, are displayed in Figure 7. The parameters are those of the previous figures. Initially, the SFQ is prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (0, 0)$, solid line, $(\pi/4, 0)$, dashed line, $(\pi/2, 0)$, dashed-dotted-line, and $(\pi, 0)$, dotted line, respectively. As in the previous figure, for values of $\alpha$ in the range $(0.94, 1.24)$ the curves were computed by making the temporary average in two periods of time. For $\alpha < 0.94$ and $\alpha > 1.24$ the results presented have been computed at $t = 6 [\mu \text{sec}]$. In panels (a) and (c) we show the behaviour of the transition probabilities for the even states, while in panels (b) and (d) we show the transition probabilities for the odd states. In panel (a) solid-, dashed-, dotted-lines correspond to $P_{11}(t)$, $P_{22}(t)$, and $P_{33}(t)$, respectively. In panel (c) $P_{12}(t)$, $P_{13}(t)$ and $P_{23}(t)$ are presented by solid-, dashed-, dotted-lines, respectively. In panel (b) we plot $P_{44}(t)$, $P_{55}(t)$ and $P_{66}(t)$, while in panel (d) we plot $P_{ij}(t)$, for $i, j = 4, 5, 6$.
the Mean Value of the Spin operator of the NV$^-$-colour-centres does not depend on the initial state. It should be noticed the existence of a value of $\alpha$ for which $\langle S_{NV} \rangle = 0$, $\alpha = 0.82402$.

**Fig. 6.** Behaviour of the non-zero entries of the matrix $P(t)$ of equation (14), as a function of the time, for the parameters of Figure 5. The results presented have been computed at $\alpha = 0.6$ (Panels (a) and (b)), $\alpha = 1.1$ (Panels (c) and (d)), and $\alpha = 1.3$ (Panels (e) and (f)), respectively. In Panels (a), (c) and (e), the behaviour of the diagonal elements of $P(t)$ are shown. Solid-, dashed-, dotted-lines, respectively.

**Fig. 7.** Survival Probability, $p(t)$ of equation (20), as a function of $\alpha$. The parameters are those of the previous figures. Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (0, 0)$ (solid line), $(\pi/4, 0)$ (dashed line), $(\pi/2, 0)$ (dashed-dotted-line), and $(\pi, 0)$ (dotted line), respectively. For values of $\alpha$ in the range $(0.94, 1.24)$ we plot the minimum value of $\xi^2_s$ in two periods of time. For $\alpha < 0.94$ and $\alpha > 1.24$ the results presented have been computed at $t = 6 \ [\mu \text{sec}]$.

**Fig. 8.** Mean Value of the Spin of the NV$^-$-colour-centres, $\langle S_{NV} \rangle$, as a function of $\alpha$. The parameters are those of the previous figures. Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (0, 0)$ (solid line), $(\pi/4, 0)$ (dashed line), $(\pi/2, 0)$ (dashed-dotted-line), and $(\pi, 0)$ (dotted line), respectively. For values of $\alpha$ in the range $(0.94, 1.24)$ we plot the average value in two periods of time. For $\alpha < 0.94$ and $\alpha > 1.24$ the results presented have been computed at $t = 6 \ [\mu \text{sec}]$.

**Fig. 9.** Squeezing Parameter, $\xi^2_s$, for the NV$^-$-colour-centres, as a function of $\alpha$. The parameters are those of the previous figures. Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (0, 0)$ (solid line), $(\pi/4, 0)$ (dashed line), $(\pi/2, 0)$ (dashed-dotted-line), and $(\pi, 0)$ (dotted line), respectively. For values of $\alpha$ in the range $(0.94, 1.24)$ we plot the minimum value of $\xi^2_s$ in two periods of time. For $\alpha < 0.94$ and $\alpha > 1.24$ the results presented have been computed at $t = 6 \ [\mu \text{sec}]$.
This figure is in agreement with the previous one. At the value of $\alpha = 0.82402$, for which $\langle S_{NV} \rangle = 0$, the value of the squeezing parameter diverges, the system is maximally anti-squeezed. In the region of real spectrum, the system shows an oscillatory pattern, with the minimum values shown in the curve.

We shall discuss in what follows the behaviour of the Squeezing parameter, the mean value of the total spin and the survival probability, as a function of the time, for different values of the asymmetry parameter, $\alpha$.

In Figures 10 and 11, we present the results we have obtained for the Squeezing Parameter, $\xi_{x}$, of equation (20), and for the Survival Probability, $p(t)$ of equation (22), as a function of the time. The parameters are those of the previous figures. Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_{0}, \phi_{0}) = (\pi/2, 0)$. The curves of Figure 10 were computed at $\alpha = 0.6$, 1.1 and 1.3, while the curves of Figure 11 were computed at the values of the asymmetry parameter $\alpha$ corresponding to the two EPs, $\alpha = 0.9424$ and 1.24556. From the figures, it can be observed that at the early stages of the time the state evolves as a squeezed state. This pattern is preserved at values of $\alpha$ corresponding to the regime of real spectrum, see Figure 9. For other values of the asymmetry parameter, the steady-state is an anti-squeezed state. Concerning the Survival Probability, $p(t)$, the results shown in Figures 10 and 11, can be understood in terms of the behaviour of the matrix elements of $P(t)$, which have been presented in Figure 5. The states with the largest probability, are the state $|k_{gb}, k_{NV}\rangle = |0, 2\rangle$ and $|0, 0\rangle$, which are present in the initial state of Figures 10 and 11. Consequently, we can use the information of Figure 5 to prepare initial states robust against decoherence. The results of Figure 11 support the idea that at the EPs the initial state evolves into a non-trivial state as reported in [53].

In both figures, it can be observed that the Survival Probability does not obey an exponential decay law. After some algebra, it can be shown that at the Exceptional Points the Survival Probability can be written as

$$p(t) = (a_{2} t^{2} - t) (a_{11} \sin(\omega_{1} t) + a_{12} \sin(\omega_{2} t)) + a_{01} \cos(\omega_{1} t) + a_{02} \cos(\omega_{2} t) + a_{03} \cos(\omega_{3} t) + a_{0} / (b_{2} t^{2} - b_{1} \sin(\omega_{1} t) + b_{01} \cos(\omega_{1} t) + b_{0}) ,$$

with $\omega_{1} = E_{1-} - E_{3-} = -0.080$ [MGz], $\omega_{2} = E_{1-} - E_{1+} = 2.853$ [MGz], $\omega_{3} = E_{1-} - E_{3+} = 2.934$ [MGz]. The coefficients of the expression of equation (30) are given in Table 1, for an initial state with the qubit in the ground state and a the NVs in a coherent state with $\theta_{0} = \pi/2$ and $\phi_{0} = 0$.

We have included a detailed quantification of the departure from the exponential decay behaviour as Supplementary Material [110].
Table 1. Coefficients of equation (30) for an initial state with the qubit in the ground state and a the NVs in a coherent state, with $\theta_0 = \pi/2$ and $\phi_0 = 0$. The results of the first column are obtained at the Exceptional Point with $\alpha = 0.94043$, while the second column has been computed at $\alpha = 1.24556$. The rest of the parameters are those of Figure 11.

| Coefficient | EP1  | EP2  |
|-------------|------|------|
| $a_0$       | 0.388| 0.385|
| $a_{01}$    | 0.112| 0.115|
| $a_{02}$    | 0.170| 0.180|
| $a_{03}$    | 0.330| 0.320|
| $a_{11}$    | 0.111| 0.495|
| $a_{12}$    | 0.169| 0.772|
| $a_2$       | 0.029| 0.597|
| $b_0$       | 1.000| 1.001|
| $b_{01}$    | −0.000| −0.001|
| $b_1$       | 0.008| 0.136|
| $b_2$       | 0.327| 6.076|

3.1 Schrödinger-cat states

The construction of Schrödinger spin cat states (SSCS) was proposed by Agarwal and co-workers in [67]. Essentially, SSCS are generated by the superposition of two coherent spin states (CSS). More recently, different experiments have been proposed to generate steady SSCS from dissipative systems [68–72].

Let us consider a particular superposition of CSS, namely

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|I(\theta, \phi)\rangle + |I(\pi - \theta, \phi + \pi)\rangle), \quad (31)$$

being $\langle I(\theta, \phi)|I(\pi - \theta, \phi + \pi)\rangle = 0$ [111]. It can be proved that for this superposition of CSS states, $\langle \Psi |\Psi\rangle = 0$. It can be observed in Figure 8 that different initial states evolve, under the action of the Hamiltonian of equation (8), to a steady-state with mean value of the total spin of the NVs equals zero, $\langle S_{NV} \rangle = 0$. This might be interpreted as a sign of the presence of Schrödinger-cat states. To confirm this assumption we shall analyse the discrete Wigner function for this state. In the space of model parameters, this point corresponds to the value $\alpha = 0.82402$. As it can be observed in Figure 4, all the probabilities $P_{ij}$, in the stationary regime, are equally distributed.

First, let us discuss the behaviour of the mean value of the total spin and of the squeezing parameter for $\alpha = 0.82402$. In Figure 12, panels (a), (c) and (e), we show the results obtained for the Mean Value of the Total Spin of the ensemble of NVs of colour centres, $\langle S_{NV} \rangle$, as a function of the time. The Squeezing Parameter, $\xi^2(t)$, for a given Initial State as a function of the time is displayed in panels (b), (d) and (f). We have considered different initial states. We have adopted an Initial ground state for the superconducting qubit. We assume that the NV$^-$-colour-centres are prepared in a coherent state, with $(\theta_0, \phi_0) = (0, 0)$ for panels (a) and (b), $(\pi/2, 0)$ for panels (c) and (d), and $(\pi, 0)$ for panels (e) and (f), respectively.

From the figure, it can be observed that initial state of the system evolves into a steady-state with the mean value of the total spin of the NVs equals zero, $\langle S_{NV} \rangle = 0$, independent of the initial state chosen. At short times, the spin dynamics is caused by the spin-spin interaction of the ensemble of NV$^-$-colour-centres, namely the OAT term of $H$. The $H_{OAT}$ of $H$ is responsible for the squeezing pattern observed. At longer times, the growth of spin correlations causes both the depolarization observed in panels (a)–(c) and the increase of the squeezing parameter [65].

In Figures 13–15, we plot the Discrete SU(2) Wigner function for the NV$^-$-colour-centres [63,64,106–109], $W(\theta, \phi)$ of equation (25). In the left panels, we display the behaviour of $W(\theta, \phi)$ for the initial state and in the right ones for the steady-state. Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (\pi/2, 0), (\pi/4, 0)$ and $(0, 0)$, Figures 13–15, respectively. To obtain the Discrete SU(2) Wigner function for the NVs, we have computed the reduced density matrix for both subsystems. It can be observed that the steady-state, is independent of the preparation of the initial state. The SU(2) Wigner function is composed by two Wigner functions, namely one corresponding to the superposition of two coherent states, $|I(\pi/2, \phi)\rangle + |I(\pi/2, \phi + \pi)\rangle$, with $\phi \approx \pi/2$ and a pure state, $S_+\langle 0\rangle$. Thus, the steady-state is a Schrödinger spin cat-like state.
Initially, the superconducting qubits are prepared in its ground state, and the NVs in a coherent state with $(\theta_0, \phi_0) = (\pi/2, 0)$.

In Figure 16, we present the points in the plane $(\alpha, \gamma)$, for which the steady-state behaves as a Schrödinger-cat states, for the case $d_\perp = 0$. As mentioned before, Schrödinger spin cat states (SSCS) can be generated in hybrid dissipative systems [69,71,73]. If we look for SSCS with $\langle S \rangle = 0$, we can use the study of Exceptional Points to distinguished different regions in the model parameter space. Then, by analysing the transition probabilities of the different states, we can detect points with equally distributed probabilities, which are candidates to generate SSCS states. These SSCS are similar to the states presented in [68,69]. Thus, they are potential candidates to quantum-enhanced measurements [68–72].

3.2 Hybrid system with N NV−-colour-centres in diamond.

To analyse qualitatively the effect of increasing the number of NV−-colour-centres in diamond, we can transform the Hamiltonian of equation (1) by applying a Holstein-Primakoff boson mapping [112]. That is

$$S_+ = b_+ \sqrt{N - b_+ b_-}, \quad S_-^\dagger = S_+^\dagger, \quad S_z = b_+ b_- - \frac{N}{2}, \quad (32)$$

with $[b, b^\dagger] = 1$. To leading order in the boson mapping of equation (32), the Hamiltonian reads...
The approximation $S_\pm \approx b_\pm \sqrt{N}$ is justified [113] for a system consisting of a large number of spins and only a few accessible spin excitations.

From the form $H_B$, it can be inferred that the adimensional parameter $\gamma$ must be scaled as $\gamma \rightarrow \gamma_N = \gamma / \sqrt{N}$.

We shall take into account the previous re-scaling of the coupling constants to analyse the exact results, which have been computed from the Hamiltonian of equation (1). In Figure 17, we present the contour plots for the appearance of EPs and of steady Schrödinger-cat states in the plane $(\alpha_N, N)$, with $\alpha_N = \alpha / \sqrt{N}$. The solid-line corresponds to the contour plot of EPs, while contour plot for steady Schrödinger-cat states are displayed in dashed-line. We have considered the case $d_- = 0$, with $D = 2.88$ [GHz] and $E = 0.026$ [GHz]. In panel (a) and (b) we plot results for $g = 0.02$ [GeV] ($\gamma_N = 1/(13\sqrt{N})$) and $g = 0.052$ [GeV] ($\gamma_N = 2/\sqrt{N}$), respectively. Though we plot continuous lines in Figure 17, they are meant to guide the eyes. They are valid for the case of even number of particles. For the present set of parameters, and in the range $0 \leq \alpha_N \leq 1$, we have not observed EPs in the case of an odd number of NVs (we have checked that for other sets of parameters, $d_- \neq 0$ and larger values of $E$, EPs are present for an odd number of particles, i.e. for $N = 3$ we have observed the presence of EPs at values of $\Delta = D/4$ [GHz], $E = 1.8$ [GHz], $-2.0 \leq g \leq 2.0$ [GHz] and $0.4 \leq \alpha \leq 1.5$). From the curves, it can be observed the presence of a regular pattern of Exceptional Points and a regular pattern for steady Schrödinger-cat states. As reported in [69], Schrödinger spin cat states with a large number of particles favour the achievement of the uncertainty Heisenberg limit.

4 Conclusions

In this work, we have studied the time evolution of a hybrid system consisting of NV–colour-centres in diamond in interaction with a superconducting flux qubit. We have modelled the dynamics of the system through a non-hermitian Hamiltonian, to take into account the effect of the environment on the ensemble of NV–centres in diamond. Though the Hamiltonian does not preserve $\mathcal{PT}$-symmetry, the spectrum consists of real eigenvalues or complex-conjugate pair eigenvalues, and it shows the characteristics features of a system with gain-loss balance [98]. We observed a regular pattern of Exceptional Points, as a function of the parameters of the model. At these points, the initial state evolves into a non-trivial steady-state. The study of the matrix elements of the Fourier Transform of the Green Matrix provides information on the transition probabilities of the states of the original base as a function of time. Thus, we can prepare robust initial states by combining the states of the base of equation (9) which show large transition probabilities at long intervals of time. The Survival Probability can be analysed to account for this effect. At Exceptional Points, the Survival Probability increases considerably, depending on the initial state adopted. It is observed that in the regime of real spectrum, the initial state evolves in time showing a periodical pattern of collapses and revivals. In this regime, the states are periodically squeezed. While in the regime of complex-conjugate pair spectrum the steady-state is not a squeezed state, and anti-squeezing is observed. At short times, the spin dynamics is caused by the spin-spin interaction of the ensemble of NV–colour-centres, namely the OAT term of $H$. The $H_{\alpha \beta \gamma}$ of $H$ is responsible for the squeezing pattern observed. At longer times, the growth of spin correlations causes both the depolarization observed and the increase of the squeezing parameter.

For certain values of model space, the anti-squeezed steady-state has mean value of the total spin of the
NVs equals zero, $\langle S_{NV} \rangle = 0$, independent of the initial state chosen. At these values, we have shown the presence of Schrödinger spin cat states, that is states which are a superposition of two coherent spin states. We have extended the previous analysis to systems with a larger number of NV$^-$-centre in diamond. We have rescaled the adimensional constant $\gamma$ as $\gamma/\sqrt{N}$, and we have found a regular pattern of Exceptional Points and a regular pattern of steady Schrödinger spin cat states in the plane $(\alpha_N, N)$. The regular generation of SSCS with large number of NV$^-$-centres in diamond might have potential applications in the field of quantum-metrology [68–72].

Work is in progress concerning the analysis of hybrid systems with more than one superconducting flux qubit, interacting with an ensemble of NV$^-$-colour-centres in diamond [62].

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**Author contribution statement**

All the authors were involved in the preparation of the manuscript. All authors have read and approved the final manuscript.

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