ON THE $\mathbb{A}^1$-DEGREE OF A WEYL COVER

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Abstract. The notion of $\mathbb{A}^1$-degree provides an arithmetic refinement of the usual notion of degree in algebraic geometry. In this note, we compute $\mathbb{A}^1$-degrees of certain finite covers $f: \mathbb{A}^n \to \mathbb{A}^n$ induced by quotients under actions of Weyl groups. We use knowledge of the cohomology ring of partial flag varieties as a key input in our proofs.

1. Introduction

We work over a field $K$, which is arbitrary unless stated otherwise. Associated to a finite morphism $f: \mathbb{A}^n \to \mathbb{A}^n$ of $K$-varieties, we have the usual notion of its degree, denoted by $\deg f$ and defined to be the degree of the induced extension of function fields. Refining this, $\mathbb{A}^1$-enumerative geometry provides a notion of an $\mathbb{A}^1$-degree, denoted by $\deg_{\mathbb{A}^1} f$, which is an element of the Grothendieck-Witt ring $GW(K)$. The Grothendieck-Witt ring is generated by symmetric bilinear forms on $K$-vector spaces up to isomorphism, and the usual degree $\deg f$ can be recovered by taking the rank of the bilinear form $\deg_{\mathbb{A}^1} f$.

If $K$ is algebraically closed, then the rank homomorphism $GW(K) \to \mathbb{Z}$ is an isomorphism, and $\deg_{\mathbb{A}^1} f$ contains no more information than $\deg f$. However, if $K = \mathbb{R}$, then the rank homomorphism $GW(\mathbb{R}) \to \mathbb{Z}$ has kernel isomorphic to $\mathbb{Z}$, reflecting the fact that $\deg_{\mathbb{A}^1} f$ also contains the data of the Brouwer degree of the underlying map of $\mathbb{R}$-manifolds. In general, $\deg_{\mathbb{A}^1} f$ can be viewed as an enrichment of $\deg f$ that contains interesting arithmetic data.

In this paper, we compute $\mathbb{A}^1$-degrees of quotient maps induced by Weyl groups. As a first example, one may consider the quotient map $\pi: \mathbb{A}^n \to \mathbb{A}^n/S_n \simeq \mathbb{A}^n$ of affine space by action of the symmetric group on the coordinates. The usual degree of $\pi$ is $\deg \pi = n!$, and it turns out that $\deg_{\mathbb{A}^1} \pi = \frac{n!}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle)$ for $n \geq 2$. This follows easily from the fact that $S_n$ contains a simple reflection, leading us to the following preliminary observation.

Proposition 1. Let $G$ be a finite group acting linearly on a finite-dimensional $K$-vector space $V$. If the ring $K[V]^G$ of $G$-invariants of $K[V]$ is a polynomial ring and $G$ contains a simple reflection, then the $\mathbb{A}^1$-degree of $\pi: \text{Spec } K[V] \to \text{Spec } K[V]^G$ is given by

$$\deg_{\mathbb{A}^1} \pi = \frac{\deg \pi}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle).$$

For instance, Proposition 1 applies to quotients of root spaces by Weyl groups when $K$ is of characteristic zero by the Chevalley–Shephard–Todd theorem (see [Che55, (A)]) or in arbitrary characteristic when the Weyl group is of type $A$ or $C$ (see [Dem73, Théorème]).

We can also compute $\mathbb{A}^1$-degrees in situations where Proposition 1 does not apply. For example, we will show that the $\mathbb{A}^1$-degree of the quotient map $\mathbb{A}^4/(S_2 \times S_2) \to \mathbb{A}^4/S_4$ is

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We give precise definitions of $GW(K)$ and $\deg_{\mathbb{A}^1} f$ in Section 2.
given by $4 \cdot \langle 1 \rangle + 2 \cdot \langle -1 \rangle$, so in particular, the $\mathbb{A}^1$-degree is no longer a multiple of $\langle 1 \rangle + \langle -1 \rangle$. Generalizing this example, we prove the following:

**Theorem 2.** Let $n_1, \ldots, n_r$ be positive integers satisfying $n = \sum_{i=1}^{r} n_i$. The $\mathbb{A}^1$-degree of the map $\pi: \mathbb{A}^n_K / \prod_{i=1}^{r} S_{n_i} \to \mathbb{A}^n_K / S_n$ is given by

$$\deg^A_1 \pi = \frac{\deg \pi - a}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle) + a \cdot \langle 1 \rangle$$

$$= \frac{1}{2} \left( \frac{n!}{\prod_{i=1}^{r} n_i!} + a \right) \cdot \langle 1 \rangle + \frac{1}{2} \left( \frac{n!}{\prod_{i=1}^{r} n_i!} - a \right) \cdot \langle -1 \rangle,$$

where $a = [\frac{n}{2}]! / \prod_{i=1}^{r} [\frac{n_i}{2}]!$ if at most one $n_i$ is odd and $a = 0$ otherwise.

The proof of Theorem 2 involves applying the algorithm in [KW19, Section 2] together with knowledge of the cohomology ring of partial flag varieties of type $A$. Motivated by this, we extend Theorem 2 to apply to Weyl groups of other types as follows:

**Theorem 3.** Let $K$ be a field of characteristic 0. Let $G$ be a simple complex Lie group with root space $V/K$, and let $P \subset G$ be a parabolic subgroup. Let $W$ be the Weyl group of $G$, and let $W_P \subset W$ be the associated parabolic subgroup. Then the $\mathbb{A}^1$-degree of the map $\pi: \text{Spec} K[V]^W_P \to \text{Spec} K[V]^W$ is given by

$$\deg^A_1 \pi = \frac{\deg \pi - a}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle) + a \cdot \langle \alpha \rangle,$$

where $\alpha \in K^\times$, and $a$ is equal to the number of cosets $\omega \cdot W_P \in W/W_P$ for which $\omega^{-1} \omega_0 \omega \in W_P$ and $\omega_0 \in W$ is the longest word.

The element $\alpha$ in the statement of Theorem 3 depends on the choice of identifications of $\text{Spec}(K[V]^W)$ and $\text{Spec}(K[V]^W_P)$ with $\mathbb{A}^{\text{dim}(V)}$. Such identifications are equivalent to choosing generators of $\text{Spec}(K[V]^W)$ and $\text{Spec}(K[V]^W_P)$ as polynomial rings over $K$. In particular, scaling a generator of $\text{Spec}(K[V]^W)$ by $\alpha'$ scales $\deg^A_1 \pi$ by $(\alpha')^{-1}$, so there is always a choice of generators making $\alpha$ in Theorem 3 equal to 1.

In the type-A case (i.e., Theorem 2), we show taking the obvious choice of generators using elementary symmetric functions yields $\alpha = 1$. On the other hand, the number $a$ in the statement of Theorem 3 can be computed explicitly in all cases, as we demonstrate in the following result:

**Proposition 4.** We have $a = 0$ in Theorem 3 except in the following cases, tabulated according to the Dynkin diagrams of $G$ and of the quotient of $P$ by its unipotent radical $U(P)$:

| $G$     | $P/U(P)$                                      | $a$               |
|---------|-----------------------------------------------|-------------------|
| $A_n$   | $\prod_{i=1}^{r} A_{n_i}$ with $n = \sum_{i=1}^{r} n_i$ and $\#\{\text{odd } n_i\} \leq 1$ | $[\frac{n}{2}]! / \prod_{i=1}^{r} [\frac{n_i}{2}]!$ |
| $D_{2n+1}$ | $D_{2n}$                                     | $\frac{3}{2}$    |
| $E_6$   | $D_5$                                        | $3$               |
| $E_6$   | $D_4$                                        | $6$               |

Thus, for all pairs $(G, P)$ not tabulated above, the $\mathbb{A}^1$-degree of the map $\pi: \text{Spec} K[V]^W_P \to \text{Spec} K[V]^W$ is given by

$$\deg^A_1 \pi = \frac{\deg \pi}{2} \cdot (\langle 1 \rangle + \langle -1 \rangle).$$
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2. Background Material

Before we prove our results, we provide a brief exposition on Grothendieck-Witt rings and on the $\mathbb{A}^1$-degree in the case of finite maps between affine spaces. Strictly speaking, there is not (as of yet) a notion of $\mathbb{A}^1$-degree for maps of affine spaces in the literature on $\mathbb{A}^1$-enumerative geometry, which is largely concerned with maps of spheres. For maps of affine spaces, a notion of local $\mathbb{A}^1$-degree is defined in [KW19, Definition 11], and a suitable notion of (global) $\mathbb{A}^1$-degree will be defined in forthcoming work of Kass et al. (see [KLSW19]). In this section, we largely follow [KW19, Section 1], which gives an algorithm for computing the local $\mathbb{A}^1$-degree around a $K$-rational point in the source.

Because the foundations are still being written, our results can be interpreted as follows. We start with a finite map $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ given in Proposition 1, Theorem 2, or Theorem 3.

1. We compute the local $\mathbb{A}^1$-degree of $\pi$ at the origin.
2. For each $K$-point $q \in \mathbb{A}^n$ such that all the closed points of $\pi^{-1}(q)$ are $K$-rational, the sum of the local degrees at the closed points of $\pi^{-1}(q)$ agree with the local $\mathbb{A}^1$-degree of $\pi$ at the origin [KW19, Corollary 31]. We use this sum as a preliminary definition of $\mathbb{A}^1$-degree (Definition 13).
3. Our computation of the local $\mathbb{A}^1$-degree of $\pi$ at the origin will agree with the global notion of $\mathbb{A}^1$-degree in [KLSW19] as the global $\mathbb{A}^1$-degree will be able to be computed as a sum over local $\mathbb{A}^1$-degrees.

2.1. The Grothendieck-Witt Ring. We now recall the definition of the Grothendieck-Witt ring of $K$.

Definition 5. Denoted by $GW(K)$, the Grothendieck-Witt ring of $K$ is defined to be the group completion of the semi-ring (under the operations of direct sum and tensor product) of isomorphism classes of symmetric nondegenerate bilinear forms on finite-dimensional vector spaces valued in $K$.

In addition to the abstract definition of $GW(K)$ given in Definition 5, it is often useful to have an explicit presentation.

Definition 6. For $u \in K^\times$, define $\langle u \rangle \in GW(K)$ to be the class of the nondegenerate symmetric bilinear form that sends $(x, y) \in K^2$ to $u \cdot xy \in K$.

Theorem 7 ([EKM08, Theorem 4.7, p. 23]). The group $GW(K)$ is generated by the elements $\langle u \rangle$ with $u \in K^\times$, subject to the following relations:

1. $\langle u \cdot v^2 \rangle = \langle u \rangle$,
2. $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u + v) \rangle$ if $u + v \neq 0$. 

The second relation in Theorem 7 is easy to see. Let \( e_1 \) and \( e_2 \) be a basis of a rank 2 vector space, with a bilinear form represented by \( \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \) with respect to its basis. With respect to the new basis \( e_1 + e_2, be_1 - ae_2 \) the bilinear form is represented by \( \begin{pmatrix} u + v & 0 \\ 0 & uv(u + v) \end{pmatrix} \).

As a corollary of the second relation in Theorem 7, the following relation is well-known and important for us. We couldn’t find a proof so we include it here.

**Lemma 8.** For any \( u \in K^* \) we have \( \langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle \) as elements of \( GW(K) \).

**Proof.** We have the following equalities:

\[
\langle u \rangle + \langle -u \rangle = (\langle u \rangle + \langle 1 - u \rangle) + \langle -u \rangle - (\langle 1 - u \rangle)
\]

\[
= \langle 1 \rangle + ((u(1 - u)) + \langle -u \rangle) - (\langle 1 - u \rangle)
\]

\[
= \langle 1 \rangle + \langle -u^2 \rangle + \langle -u^2(u(1 - u)) \rangle - \langle 1 - u \rangle
\]

\[
= \langle 1 \rangle + \langle -1 \rangle + (\langle 1 - u \rangle) - \langle 1 - u \rangle.
\]

The form \( \langle 1 \rangle + \langle -1 \rangle \) in Lemma 8 is called the **hyperbolic form**. It is easy to see from the second relation in Theorem 7 that the product of the hyperbolic form with any element in \( GW(K) \) is an integral multiple of the hyperbolic form. Also, \( \langle 1 \rangle + \langle -1 \rangle \) is equivalent to the bilinear form \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in \( GK(K) \). This is easy to see in characteristic not 2, as the matrices \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) define equivalent bilinear forms. In characteristic 2, this is true because \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) are equivalent bilinear forms over \( \mathbb{F}_2 \).

For more information about Grothendieck-Witt rings, see [EKM08, Lam05, WW19].

### 2.2. Algorithm for Computing \( A^1 \)-degrees

In this subsection, we recall an algorithm from [KW19] for computing the \( A^1 \)-degree of a finite map between affine spaces when there exists a fiber whose closed points are \( K \)-rational. Let \( f : A^n \to A^n \) be a finite map, and let \( (f_1, \ldots, f_n) \) be its component functions.

**Definition 9.** The local algebra of \( f \) at \( K \)-point \( p = (a_1, \ldots, a_n) \in A^n \) is \( Q_p(f) := K[x_1, \ldots, x_n]_{m_p}/(f_1 - b_1, \ldots, f_n - b_n) \), where \( (b_1, \ldots, b_n) = f(p) \in A^n \) and \( m_p \) is the maximal ideal of \( p \). The distinguished socle element is \( E_p(f) := \text{det}(a_{ij}) \in Q(f) \) where \( a_{ij} \in K[x_1, \ldots, x_n] \) are polynomials such that \( f_i = \sum_j a_{ij} \cdot (x_j - a_j) \). We will denote \( Q_0(f) \) and \( E_0(f) \) as \( Q(f) \) and \( E(f) \) respectively.

**Definition 10.** To a linear functional \( \phi : Q_p(f) \to K \), we can associate a symmetric bilinear form \( \beta_\phi \) on \( Q_p(f) \) defined by \( \beta_\phi(a, b) = \phi(ab) \).

From [KW19, Main Theorem], we can compute the local \( A^1 \)-degree of \( f \) as the class of a bilinear form on \( Q(f) \) (viewed as a \( K \)-vector space) in the Grothendieck-Witt ring of \( K \).

**Theorem 11** ([KW19, Main Theorem]). The local \( A^1 \)-degree of \( f \) at a \( K \)-rational point \( p \) in the domain \( A^n \), denoted by \( \text{deg}_{A^1} f \), is given by the class of the bilinear form \( \beta_\phi \) in \( GW(K) \), where \( \phi \) is any linear functional sending the distinguished socle element \( E(f) \) to 1.
From this, one can compute the sum of the local $\mathbb{A}^1$-degrees in a fiber of $f$ whose closed points are all $K$-rational.

**Theorem 12** ([KW19, Corollary 31]). The sum

$$\sum_{p \in f^{-1}(q)} \deg_{p}^{\mathbb{A}^1} f$$

is independent of $q \in \mathbb{A}^n$, as $q$ varies over all $K$-rational points in $\mathbb{A}^n$ where the closed points in $f^{-1}(q)$ are $K$-rational.

In all the cases we consider, we choose $q = 0$ and $f^{-1}(0)$ is supported at the origin. In light of Theorem 12 and the forthcoming work of Kass et al. in [KLSW19], we make the following preliminary definition.

**Definition 13.** The global $\mathbb{A}^1$-degree of $f$, denoted $\deg_{\mathbb{A}^1} f$ is defined to be

$$\deg_{\mathbb{A}^1} f := \sum_{p \in f^{-1}(q)} \deg_{p}^{\mathbb{A}^1} f$$

for any $K$-rational point $q \in \mathbb{A}^n$ such that the closed points in $f^{-1}(q)$ are $K$-rational.

Lastly, in the proof of Theorem 2 only, we will make use of the Jacobian element, which is defined as follows:

**Definition 14.** The Jacobian element is $J(f) := \det \left( \frac{\partial f_i}{\partial x_j} \right) \in Q(f)$.

The Jacobian element and distinguished socle element are related to each other by the equation $J(f) = (\dim_K Q(f)) \cdot E(f)$ [SS75, (4.7) Korollar], so $J(f)$ contains the same information as $E(f)$ if the characteristic of $K$ does not divide the dimension of $Q(f)$ as a $K$-vector space.

3. **Proofs of the Results**

We first note that for all of the maps $\pi: \mathbb{A}^n \to \mathbb{A}^n$ we consider, $\pi^{-1}(0)$ is supported at the origin. This is because the orbit of $0 \in \mathbb{A}^n$ under a linear group action is just the origin. In particular, this means $\deg_{\mathbb{A}^1} \pi$ can be evaluated using the definition of the local $\mathbb{A}^1$-degree in Theorem 11.

3.1. **Proof of Proposition 1.** Since $G$ contains a simple reflection $r$, the map $\pi$ factors through $\text{Spec}(K[V]^r)$:

$$\pi: \text{Spec}(K[V]) \to \text{Spec}(K[V]^r) \to \text{Spec}(K[V]^G).$$

It is easy to check (for example using [KW19, Section 1]) that

$$\deg_{\mathbb{A}^1} (\text{Spec}(K[V]) \to \text{Spec}(K[V]^r)) = \langle 1 \rangle + \langle -1 \rangle.$$

By the fact that local $\mathbb{A}^1$-degrees are multiplicative in compositions, we have that

$$\deg_{\mathbb{A}^1} \pi = (\langle 1 \rangle + \langle -1 \rangle) \cdot \deg_{\mathbb{A}^1} (\text{Spec}(K[V]^r) \to \text{Spec}(K[V]^G)).$$

It follows from the presentation of the Grothendieck-Witt ring in [EKM08, Theorem 4.7] that any product with the hyperbolic form $\langle 1 \rangle + \langle -1 \rangle$ is actually an integral multiple of the
hyperbolic form. Thus, there is some integer $N$ such that $\deg_{\mathcal{A}}^{h} \pi = N \cdot ((1) + (-1))$. Taking the rank of $\deg_{\mathcal{A}}^{h} \pi$, we find that $2N = \deg_{\mathcal{A}}^{h} \pi = \deg \pi$, which is the desired result. \qed

It turns out to be more efficient from an expository standpoint to prove Theorem 3 and Proposition 4 before Theorem 2, so we order the remaining proofs accordingly.

3.2. Proof of Theorem 3. Consider the algebra
\[ Q := Q(\pi) \simeq K[V]^{W_P}/(K[V]^{W})^{+}, \]
where for a graded ring $R$, we denote by $R^{+}$ its irrelevant ideal. Suppose that we can produce a $K$-linear functional $\phi: Q \to K$ sending the distinguished socle element $E \in Q$ to 1. Then because $\pi^{-1}(0) = \{0\}$, it follows from Theorem 11 that the symmetric bilinear form $\beta_{\phi}: Q \times Q \to K$ defined by $\beta_{\phi}(a_1, a_2) := \phi(a_1a_2)$ has the property that its class in $GW(K)$ is equal to $\deg_{\mathcal{A}}^{h} \pi$.

We now briefly sketch our idea for producing the desired functional $\phi$. The key observation is that $Q$ can be identified with the Chow ring of a certain partial flag variety, we can choose $\phi$ to be a certain scalar multiple of the integration map. Because the cohomology of this partial flag variety is spanned by Schubert varieties, and because each Schubert variety has a dual Schubert variety, this forces $\beta_{\phi}$ to be a direct sum of copies of the bilinear forms ($\alpha$) and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where the multiplicity of each form depends on how many Schubert varieties are cohomologically equivalent to their dual Schubert variety.

By [Bor53, Proposition 29.2(a)], the partial flag variety $F := G/P$ has cohomology\(^2\)
\[ H^{\bullet}(F, K) = K[V]^{W_P}/(K[V]^{W})^{+} = Q. \]

We might try to take the functional $\phi$ to be the integration map on $H^{\bullet}(F, K)$, but to make this work, we would need to verify that the distinguished socle element $E$, viewed as an element of $H^{\bullet}(F, K)$, integrates to 1. This also depends on the choice and ordering of polynomial generators of $K[V]^{W_P}$ and $K[V]^{W}$ providing isomorphisms $\text{Spec}(K[V]^{W_P}) \simeq \text{Spec}(K[V]^{W}) \simeq \mathbb{A}^{\dim K V}$. We verify this in the case where $G = \text{SL}_n$ in Section 3.4, where we used elementary symmetric functions as the generators in the invariant rings.

For now, let $\alpha \in K^{\times}$ be such that $\frac{1}{\alpha}$ is the integral of $E$, and let $\phi$ be such that $\frac{1}{\alpha} \cdot \phi$ is the integration map. We now compute the intersection pairing $\beta_{\phi}$ on $Q$. To do this, we use the following three facts (see [Buc05, Section 2.1]):

1. The cohomology $H^{\bullet}(F, K)$ of $F$ has a basis given by the classes of the Schubert varieties;
2. Schubert varieties are indexed by cosets of $W/W_P$; and
3. the basis of Schubert varieties has a dual basis under the integration pairing, also given by Schubert varieties. The Schubert variety dual to the Schubert variety associated to the coset $\omega W_P$ is given by the coset $\omega_0 \omega W_P$, where $\omega_0 \in W$ is the longest word.

It follows that the matrix of $\beta_{\phi}$ with respect to the basis of Schubert classes is block-diagonal, where the blocks are of two types: ($\alpha$) arising from self-dual Schubert classes, and $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$

\(^2\)Note that our partial flag variety is defined over $\mathbb{C}$, but we take its cohomology with coefficients in $K$.\]
arising from all other dual pairs. Note that the class of \((\alpha)\) in \(GW(K)\) is given by \(\langle \alpha \rangle\) and that Lemma \ref{lem:partition} implies that the class of \(\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\) is given by \(\langle 1 \rangle + \langle -1 \rangle\).

Let \(a\) be the number of self-dual Schubert classes. Then the number of other dual pairs of Schubert classes is simply given by \(\frac{1}{2}(\dim_K Q - a) = \frac{1}{2}(\deg \pi - a)\). The theorem now follows upon observing that \(a\) is equal to the number of cosets \(\omega \cdot W_P\) such that \(\omega_0 \omega\) belongs to the same coset, which is equivalent to saying that \(\omega^{-1} \omega_0 \omega \in W_P\).

\[\text{3.3. Proof of Proposition 4.}\] Suppose that the Dynkin diagram of \(G\) is not any one of \(A_n, D_n\) for \(n\) odd, or \(E_6\). Then the longest word \(\omega_0\) is in the center of \(W\) ([Bou81, Planches I-IX]), and the support of \(\omega_0\) is full (in the sense that one requires every generator of \(W\) to express \(\omega_0\)). It follows that \(\omega^{-1} \omega_0 \omega = \omega_0\) is not contained in any parabolic subgroup of \(W\), so we must have that \(a = 0\). We treat the remaining cases separately as follows.

\[\text{3.3.1. The} A_n \text{ Case.}\] In this case, the Weyl group of \(G\) is \(W = S_n\), and any parabolic subgroup \(W_P \subset W\) is of the form \(W_P = \prod_{i=1}^r S_{n_i}\), where \(n = \sum_{i=1}^r n_i\). The longest word \(\omega_0 \in S_n\) is the permutation that sends \(i\) to \(n - i\) for every \(i\). Recall that the number \(a\) of self-dual Schubert classes is equal to the number of cosets \(\omega \cdot \prod_{i=1}^r S_{n_i}\) such that \(\omega_0 \omega\) belongs to the same coset, which is further equal to the number of elements in the set \(P\) of partitions of \(\{1, \ldots, n\}\) into blocks \(B_1, \ldots, B_r\) (not necessarily contiguous) of sizes \(n_1, \ldots, n_r\) such that swapping \(n - i\) for each \(i\) preserves those blocks. If all of the blocks are of even size, then \(#P\) is equal to the number of partitions of \(\{1, \ldots, \frac{n}{2}\}\) into blocks of size \(\frac{n}{2}\). If some block has odd size, then that block must contain \(\frac{n+1}{2}\) (in particular, \(n\) must be odd) and must therefore be the only block of odd size. Thus, if there is a single block of odd size that contains \(\frac{n+1}{2}\), then \(#P\) is equal to the number of partitions of \(\{1, \ldots, \frac{n-1}{2}\}\) into blocks of size \(\left\lfloor \frac{n}{2} \right\rfloor\), and \(#P = 0\) otherwise. So \(a = #P = \frac{1}{2}! / \prod_{i=1}^r \left\lfloor \frac{n}{2} \right\rfloor\) if at most one \(n_i\) odd and \(a = #P = 0\) otherwise, as desired.

\[\text{3.3.2. The} D_n \text{ case.}\] In this case, the Weyl group of \(G\) has presentation

\[W = \langle r_1, \ldots, r_n : (r_i r_j)^{m_{ij}} = 1 \rangle,\]

where the \(m_{ij}\) are defined by

\[m_{ij} = \begin{cases} 
1, & \text{if } i = j \\
2, & \text{if } (i, j) = (1, 2) \text{ or } |i - j| > 1 \text{ for } (i, j) \neq (1, 3), (3, 1) \\
3, & \text{if } |i - j| = 1 \text{ for } i, j \geq 2.
\end{cases}\]

The generator \(r_k\) of \(W\) correspond to the node \(k\) of the Dynkin diagram of \(D_n\) labeled below.

```
1
  \\
/ \ 3 \  \  \  \  \ 
2
```

```
The length $\ell(\omega)$ of $\omega \in W$ is the length of the shortest expression of $\omega$ as a product of the generators $r_k$. The unique longest element $\omega_0$ is an involution satisfying $\ell(\omega_0) = n^2 - n$, and when $n$ is odd, $\omega_0$ acts by conjugation on the generators as follows: $\omega_0 r_i^{-1} = r_i$, for $i \geq 3$ ([Bou81, Planche IV]). The (proper) parabolic subgroups of $W$ are precisely those subgroups $W_I = \langle r_i : i \in I \rangle$, where $I \subseteq \{1, \ldots, n\}$ is any subset. A parabolic subgroup $W_I$ is said to be maximal if $\# I = n - 1$.

Let $n \geq 5$ be odd. We claim that if $\omega^{-1} \omega_0 \omega \in W_I$ for some $\omega \in W$ and proper parabolic subgroup $W_I$, then $I = \{1, \ldots, n-1\}$. Note that for such an $\omega$, any element $\omega' \in Z \omega$ also satisfies $(\omega')^{-1} \omega_0 \omega' \in W_I$, where $Z$ is the centralizer of $\omega_0$ in $W$. Let

$$\sigma_{i,k} = \begin{cases} \prod_{j=1}^{k} r_j, & \text{if } i \leq k \\ 1, & \text{if } i > k \end{cases}$$

The following table shows how to reduce the length of a coset representative of $Z \omega$ as above by left-multiplication with elements of $Z$. In each row, the leftmost entry is a possible starting segment $b$ for $\omega$ expressed as a word $\omega = b \cdot c$, the middle entry is a re-expression of the $b$ that is more convenient for the purpose of length reduction, and the rightmost entry is a shortened segment $b' \in Z b$ with $\ell(b') < \ell(b)$. Because $\{r_1 r_2\} \cup \{r_i : i \geq 3\} \subset Z$ and because $\{r_i : i \geq 4\}$ is contained in the centralizer of $r_1$, it is sufficient to consider starting segments $b$ that begin with $r_1 r_3$.

| Starting Segment | Re-expression of Starting Segment | Shortened Segment | Conditions |
|------------------|----------------------------------|------------------|------------|
| $r_1 r_3 r_1$   | $r_3 \cdot r_1 r_3$              | $r_1 r_3$        | n/a        |
| $r_1 r_3 r_2$   | $(r_1 r_3 r_1)^3 \cdot r_3 r_1 r_3$ | $r_1 r_3$        | n/a        |
| $r_1 \sigma_{3,k} r_j$ | $(r_1 r_3 r_1)^3 \cdot \sigma_{4,k}$ | $r_1 \sigma_{3,k}$ | $1 \leq j \leq 2$ |
| $r_1 \sigma_{3,k} r_j$ | $r_1 r_3 \sigma_{4,j+1} r_j r_3 \sigma_{j+2,k} = r_3 r_1 \sigma_{3,k}$ | $r_1 \sigma_{3,k}$ | $3 \leq j \leq k - 1$ |
| $r_1 r_3 r_j$   | $r_j \cdot r_1 r_3$              | $r_1 r_3$        | $j > 4$    |

For example, the reduction in row 4 of the table is justified as follows: the defining relations of $W$ imply that $\sigma_{j+2,k} r_j = r_j \sigma_{j+2,k}$, and that $r_j r_j + 1 r_j = r_j + 1 r_j r_j + 1$.

Since each row of the table constitutes a reduction in length, we have shown that if the conjugacy class of $\omega_0$ meets $W_I$, then there is an element

$$\omega \in S := \{r_1 \sigma_{3,k} : 3 \leq k \leq n\}$$

such that $\omega^{-1} \omega_0 \omega \in W_I$. The length of the longest element of $S$ is $n - 1$, so we deduce that

$$\ell(\omega^{-1} \omega_0 \omega) \geq \ell(\omega_0) - 2 \cdot \ell(\omega) \geq (n^2 - n) - 2 \cdot (n - 1) = n^2 - 3n + 2.$$ 

To finish the proof of the claim, it is enough to see that $\ell_k < n^2 - 3n + 2$, where $k < n$ and $\ell_k$ is the length of the longest element of the (unique) maximal parabolic subgroup not containing $r_k$. The maximal lengths in a Weyl group of type $A_r$ or $D_r$ are $\left(\frac{r+1}{2}\right)$ and $r^2 - r$, respectively ([Bou81, Planches I and IV]). Using this fact together with the additivity of maximal lengths in products of Coxeter groups, we find that

$$\ell_k = \begin{cases} \binom{n}{2}, & \text{if } 1 \leq k \leq 2 \\ (k-1)^2 - (k-1) + \binom{n-k+1}{2}, & \text{if } 2 < k < n \end{cases}$$

It is easy to check in each case that $f(n, k) := n^2 - 3n + 2 - \ell_k > 0$ when $n \geq 5$. For example, in the case $2 < k < n$, one readily checks that $f(n, k)$ is minimal when $k = n - 1$, in which case $f(n, n-1) = 4n - 10 > 0$. Thus we have proven the claim.
Finally, in the case $I = \{1, \ldots, n - 1\}$, let $P_I \subset G = \text{SO}(2n)$ be an associated parabolic subgroup. We can realize the flag variety $G/P_I$ as a smooth quadric hypersurface $X$ of dimension $2n - 2$. Indeed, under the obvious transitive action of $G$ on such a quadric, the subgroup $M = \text{SO}(2n - 2) \subset P_I \subset G$ (embedded in the standard way by acting as the identity on the last two coordinates) stabilizes a point $p \in X$. Since the stabilizer $M' \subset G$ of $p$ is parabolic and contains $M$, it follows by inspecting its Dynkin diagram that $M' = P_I$. By [Rei72, Proof of Theorem 1.13], we have that

$$H^{n-1}(X, \mathbb{Z}) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2,$$

where the $L_i$ are classes of linear subspaces on $X$ satisfying $L_1^2 = L_2^2 = 1$ and $L_1 \cdot L_2 = 0$. Thus, there are exactly two self-dual classes, as desired.

3.3.3. The $E_6$ Case. This case can be verified using the following sage code. The first block of code computes the number of elements $\omega \in W$ such that $\omega^{-1}\omega_0\omega \in W_P$ in the cases where $P$ is a maximal parabolic subgroup.

```
INPUT:
E6=WeylGroup(['E', 6]);
w0=E6.w0;
i=0;
for w in E6:
    if len((w.inverse()*w0*w).coset_representative([2,3,4,5,6]).reduced_word())==0:
        i=i+1;
        print i;
i=0;
for w in E6:
    if len((w.inverse()*w0*w).coset_representative([1,3,4,5,6]).reduced_word())==0:
        i=i+1;
        print i;
i=0;
for w in E6:
    if len((w.inverse()*w0*w).coset_representative([1,2,4,5,6]).reduced_word())==0:
        i=i+1;
        print i;
i = 0;
for w in E6:
    if len((w.inverse()*w0*w).coset_representative([1,2,3,5,6]).reduced_word())==0:
        i=i+1;
        print i;

OUTPUT:
5760
0
0
0
```

The above code shows that the only maximal parabolic subgroups $P$ that gives rise to a nonzero number of self-dual elements are the ones where the Dynkin diagram of $P/U(P)$ is $D_5$, which can be obtained by deleting the node labeled 1 in the Dynkin diagram of $E_6$ (as illustrated below), or by deleting the node labeled 6.
In this case, the desired number of self-dual elements is given by
\[ \# \{ \omega \in W : \omega^{-1}\omega_0\omega \in W_P \} / \# W_P = \frac{5760}{1920} = 3. \]

For the smaller parabolic subgroups, it suffices to consider only those $P$ that are properly
contained in a maximal parabolic subgroup that gives rise to a non-zero number of self-dual elements. The only such $P$ has the property that the Dynkin diagram of $P/U(P)$ is $D_4$ and is obtained by deleting the nodes labeled 1 and 6 from the Dynkin diagram of $E_6$, as illustrated below:

```
 2
 /\  \\
 1 3 4 5 6
```

The second block of code handles this case:
```
INPUT:
i=0;
for w in E6:
    if len((w.inverse()*w0*w).coset_representative([2,3,4,5]).reduced_word())==0:
        i=i+1;
print i;
```

```
OUTPUT:
i152
```

In this case, the desired number of self-dual elements is given by
\[ \# \{ \omega \in W : \omega^{-1}\omega_0\omega \in W_P \} / \# W_P = \frac{1152}{192} = 6. \]

This completes the proof of Proposition 4. \qed

3.4. Proof of Theorem 2. The idea is to use the same strategy as in the proof of Theorem 3. For convenience, let $m_i = \sum_{j=1}^i n_j$. Consider the partial flag variety $F := F(m_1, \ldots, m_r)$ parametrizing flags of $\mathbb{C}$-vector spaces $0 \subset V_1 \subset \cdots \subset V_r = \mathbb{C}^n$ where $V_i$ has dimension $m_i$. By [Bor53, Proposition 31.1], the integral cohomology ring of $F$ is given by
\[ H^\bullet(F, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, \ldots, x_n] \Pi_{i=1}^r S_{n_i}}{(\mathbb{Z}[x_1, \ldots, x_n] S_{n_i})^+}. \]

For any field $K$ (regardless of characteristic), we have that
\[ Q := Q(\pi) = \frac{K[x_1, \ldots, x_n] \Pi_{i=1}^r S_{n_i}}{(K[x_1, \ldots, x_n] S_{n_i})^+} = H^\bullet(F, K). \]

We want to take the functional $\phi$ to be the integration map on $H^\bullet(F, K)$, so we need to verify that the distinguished socle element $E := E(\pi)$, viewed as an element of $H^\bullet(F, K)$,
integrates to 1. To do this, consider the element \( \tilde{E} \in \mathbb{Z}[x_1, \ldots, x_n] \) defined by the formula for the distinguished socle element in Definition 9. Viewing \( \tilde{E} \) as an element of \( H^*(F, \mathbb{Z}) \) via the identification (1), it is easy to see that the image of \( \tilde{E} \) under the map \( H^*(F, \mathbb{Z}) \to H^*(F, K) \) is equal to \( E \). It now suffices to show that \( \tilde{E} \) is equal to the class of a point in \( H^*(F, \mathbb{Z}) \).

Notice that \( \tilde{E} \in H^\text{top}(F, \mathbb{Z}) \) and that \( H^\text{top}(F, \mathbb{Z}) \cong \mathbb{Z} \). By [SS75, proof of Korollar 4.7] (see also [KW19, proof of Lemma 4]), \( E \) is nonzero independent of \( K \), so we can vary \( K = \mathbb{F}_p \) over all primes \( p \) to see that the image of \( \tilde{E} \) in \( H^\text{top}(F, \mathbb{F}_p) \) must be nonzero for each prime \( p \). It follows that \( \tilde{E} \) is a generator of \( H^\text{top}(F, \mathbb{Z}) \cong \mathbb{Z} \) and therefore agrees with the class of a point up to sign.

To determine the sign, it suffices to compute the sign of the Jacobian element \( J := J(\pi) \), taking \( K = \mathbb{Q} \). We first consider the case where \( n_i = 1 \) for every \( i \). In this case, the Jacobian element is \( J = \prod_{1 \leq i < j \leq n} (x_i - x_j) \) by [LP02, Equation (1)]; notice that \( J \) is a Vandermonde determinant and can be expressed using the Leibniz formula as

\[
J = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{n-i}.
\]

On the other hand, the class of a point in \( F \) is by definition given by \( \prod_{i=1}^{n} x_i^{n-i} \) (see [BJS93, Section 1]). For any \( \sigma \in S_n \), we have that

\[
\sigma \cdot \prod_{i=1}^{n} x_i^{n-i} = \prod_{i=1}^{n} x_{\sigma(i)}^{n-i} = \text{sign}(\sigma) \cdot \prod_{i=1}^{n} x_i^{n-i}.
\]

It follows from combining (2) and (3) that \( J = n! \prod_{i=1}^{n} x_i^{n-i} \in H^\text{top}(F, \mathbb{Z}) \), so the signs agree.

We next consider the general case where not every \( n_i \) is equal to 1. Consider the composition of maps

\[
\text{Spec}(\mathbb{Q}[x_1, \ldots, x_n]) \to \text{Spec}(\mathbb{Q}[x_1, \ldots, x_n] \prod_{i=1}^{n} S_i) \to \text{Spec}(\mathbb{Q}[x_1, \ldots, x_n]^{S_n})
\]

The Jacobian element of the first map in (4) is the product of the Jacobian elements of the maps \( \text{Spec}(\mathbb{Q}[x_{m_{k-1}+1}, \ldots, x_{m_k}] \to \text{Spec}(\mathbb{Q}[x_{m_{k-1}+1}, \ldots, x_{m_k}]^{S_{m_k}}) \) over \( 1 \leq k \leq r \). It then follows from the Chain Rule that the Jacobian element of \( \text{Spec}(\mathbb{Q}[x_1, \ldots, x_n] \prod_{i=1}^{n} S_i) \to \text{Spec}(\mathbb{Q}[x_1, \ldots, x_n]^{S_n}) \) is

\[
J = \prod_{1 \leq i < j \leq n} (x_i - x_j) \bigg/ \prod_{k=1}^{r} \prod_{1 \leq i < j \leq m_{k-1}+1} (x_i - x_j) \prod_{1 \leq k < t \leq r} \prod_{1 \leq i \leq j \leq n} (x_{m_{k-1}+i} - x_{m_{t-1}+j}).
\]

In words, this Jacobian element takes the same form as the product of differences \( \prod_{1 \leq i < j \leq n} (x_i - x_j) \), but instead of taking all pairwise differences \( x_i - x_j \) for \( i < j \), we instead take the pairs \( i < j \) such that \( i \) and \( j \) are from different blocks, where we partition \( \{1, \ldots, n\} \) into contiguous blocks of size \( n_1, \ldots, n_r \). Now, we want to compare the Jacobian element with the class of a point in \( F \). As before, we visibly see that swapping two variables from different blocks switches the sign and swapping two variables from the same block preserves the Jacobian element. Therefore, the same is true for the formula for the class of a point in \( F \).
By [Buc05, Section 2.1], the class of a point in $F$ is the Schubert polynomial associated to the permutation
\begin{equation}
m_{r-1} + 1, \ldots, m_r, m_{r-2} + 1, \ldots, m_{r-1}, \ldots, 1, \ldots, m_1
\end{equation}
of the list $1, \ldots, n$. In words, the permutation (6) takes the numbers $1, \ldots, n$, splits them up into contiguous blocks of size $n_1, \ldots, n_r$ and reverses the order of the blocks (keeping the order within each block fixed). By [BJS93, Block decomposition formula], the Schubert polynomial associated to (6) is given by
\[ J = \frac{n!}{\prod_{i=1}^{r} n_i!} \prod_{i=1}^{r} \left( \prod_{j=1}^{n_i} x_j \right) \sum_{k=i+1}^{r} n_k \]
Expanding out (5) and keeping track of the signs, we find that
\[ J = \frac{n!}{\prod_{i=1}^{r} n_i!} \prod_{i=1}^{r} \left( \prod_{j=1}^{n_i} x_j \right) \sum_{k=i+1}^{r} n_k \]
so the signs agree.
We deduce that $\alpha = 1$, so the theorem now follows from Theorem 3 and Proposition 4. □

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