Direct Single-Instanton Contributions

to Finite-Energy Sum Rules

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Abstract

Instanton contributions to pseudoscalar finite-energy sum rules are extracted from the explicit single-instanton contribution to the pseudoscalar Laplace sum rule in the instanton liquid model.
Finite energy sum rules in the pseudoscalar meson channels have been used by a number of researchers to obtain bounds on quark masses [1,2,3]. Substantial higher-order perturbative contributions to the pseudoscalar correlation function are both known [2,3,4] and controllable [5]. As emphasized in ref. 3, however, such calculations are understood to be subject to serious uncertainties from direct instanton contributions [6], which have not been sufficiently well-understood to be incorporated into even the most recent finite energy sum rule calculations [2]. In this note, we use the known contribution to Laplace sum rules in the instanton liquid model [7] to extract the direct single-instanton contribution to finite-energy sum rules:

\[ R^p_0(s) = \frac{1}{\pi} \int_0^\infty \text{Im} \left[ \pi^p(t) \right] e^{-st} dt \]

\[ = \left( \frac{4\pi^2 n_c \rho^2}{3m_s^2} \right) \frac{3\rho^2}{8\pi^2 s^3} e^{-\rho^2/2s} \left[ K_0 \left( \frac{\rho^2}{2s} \right) + K_1 \left( \frac{\rho^2}{2s} \right) \right] \]

\[ = \frac{3\rho^2}{8\pi^2 s^3} e^{-\rho^2/2s} \left[ K_0 \left( \frac{\rho^2}{2s} \right) + K_1 \left( \frac{\rho^2}{2s} \right) \right]. \quad (1) \]

In (1), \( 1/\rho \approx 600 \text{ MeV} \) is the instanton size, \( s \) is the Borel parameter \( (s \equiv 1/M^2) \) and \( \pi^p(q^2) \) denotes the correlator of appropriate light-quark pseudoscalar currents \( i\bar{q}\gamma_5 q \). In the instanton liquid model the quantity \( n_c \) parametrizes the instanton density and \( m_s \) is the self-consistent dynamical mass.

The finite energy sum rules we wish to obtain are

\[ F^p_k(s_0) \equiv \frac{1}{\pi} \int_0^{s_0} \text{Im} \left[ \pi^p(t) \right] t^k dt. \quad (2) \]

To evaluate the contributions to (2) in the instanton liquid model, recall that \( R^p_0(s) \) in (1) is itself a Laplace transform:

\[ R^p_0(s) = \mathcal{L} \left[ \frac{1}{\pi} \text{Im} \left( \pi^p(t) \right) \right]. \quad (3) \]
\[
\mathcal{L}[f(t)] \equiv \int_0^\infty f(t)e^{-st}dt.
\] (4)

From (2) and (3) we see that
\[
\frac{d}{dt} F_k^p(t) = \mathcal{L}^{-1} [R_0^p(s)] t^k.
\] (5)

Upon taking the Laplace transform of both sides of (5) and noting from (2) that
\[ F_k^p(0) = 0, \]
we obtain
\[
F_k^p(t) = \mathcal{L}^{-1} \left[ \frac{1}{s} \left( -\frac{d}{ds} \right)^k R_0^p(s) \right]
\] (6)

An explicit expression for \( F_k^p(t) \) can be obtained from the identity [8]
\[
\frac{1}{2s} e^{-1/2s} K_0(1/2s) = -\pi \int_0^\infty J_0(x)Y_0(x)e^{-sx^2}x \, dx
\]
\[ = \mathcal{L} \left[ -\frac{\pi}{2} J_0(\sqrt{t})Y_0(\sqrt{t}) \right].
\] (7)

We differentiate both sides of (7) with respect to \( s \), noting that \( K'_0(z) = -K_1(z) \) and that
\[
\frac{d}{ds} \mathcal{L}[f(t)] = \mathcal{L}[-tf(t)],
\]
in order to obtain the relation
\[
H_0(s) \equiv \frac{1}{(2s)^3} e^{-1/2s} [K_0(1/2s) + K_1(1/2s)] = \mathcal{L}[h(t)],
\] (8)

\[
h(t) = \frac{\pi}{4} t J_0(\sqrt{t})Y_0(\sqrt{t}) + \mathcal{L}^{-1} \left[ \frac{1}{2s} \mathcal{L} \left[ -\frac{\pi}{2} J_0(\sqrt{t})Y_0(\sqrt{t}) \right] \right]
\]
\[ = \frac{\pi}{4} t J_0(\sqrt{t})Y_0(\sqrt{t}) - \frac{\pi}{4} \int_0^t J_0(\sqrt{w})Y_0(\sqrt{w})dw,
\] (9)

where the integral in the final line above is a convolution of \( J_0(\sqrt{t})Y_0(\sqrt{t}) \) and \( 1/2 = \mathcal{L}^{-1}(1/2s) \). Comparing the top line of (8) with (3), we see that
\[
R_0^p(s) = \frac{3}{\pi^2 \rho^4} H_0(s/\rho^2)
\] (10)

Using the rescaling relation \( G(s/\rho^2) = \rho^2 \mathcal{L}[g(\rho^2 t)] \) for \( G(s) = \mathcal{L}[g(t)] \), one can easily show via (3) and (10) that
\[
F_k^p(t) = \frac{3}{\pi^2 \rho^{4+2k}} \phi_k(\rho^2 t)
\] (11)
where
\[ \phi_k(t) = L^{-1} \left[ \frac{1}{s} \left( -\frac{d}{ds} \right)^k H_0(s) \right] = \int_0^t \tau^k h(\tau) d\tau. \] (12)

We find from substitution of (9) into (12) that
\[ \phi_k(t) = \frac{\pi}{4} \int_0^t d\tau \tau^k \left[ \tau J_0(\sqrt{\tau}) Y_0(\sqrt{\tau}) - \int_0^\tau dw J_0(\sqrt{w}) Y_0(\sqrt{w}) \right] = \frac{\pi}{4(k+1)} \int_0^t \left[ (k+2)\tau^{k+1} - t^{k+1} \right] J_0(\sqrt{\tau}) Y_0(\sqrt{\tau}) d\tau. \] (13)

Substitution of (13) into (11) yields a closed-form expression for the instanton contribution (2) to finite energy sum rules:
\[ F^p_k(s_0) = \frac{3}{4\pi(k+1)} \int_0^{s_0} \left[ (k+2)w^{k+1} - s_0^{k+1} \right] J_0(\rho\sqrt{w}) Y_0(\rho\sqrt{w}) dw. \] (14)

The appearance of the explicit \( s_0^{k+1} \) term in (14) reminiscent of perturbative contributions, raises the concern that the instanton and perturbative contributions might be comparable. A simplification of (14) addresses this question. Applying a change of variables in (14), using the identity
\[ \int x J_0(x) Y_0(x) dx = \frac{1}{2} x^2 \left[ J_0(x) Y_0(x) + J_1(x) Y_1(x) \right] \] (15)
and performing an integration by parts results in the expression
\[ F^p_k(s_0) = -\frac{3}{4\pi} \int_0^{s_0} w^{k+1} J_1(\rho\sqrt{w}) Y_1(\rho\sqrt{w}) dw \] (16)
which, by comparing the integrands, is easily seen to be smaller than the leading perturbative contribution. From comparison of (14) and (2) it is also possible to make the identification
\[ \frac{1}{\pi} Im \left[ \pi^p(w) \right]_{\text{inst}} = -\frac{3}{4\pi} w J_1(\rho\sqrt{w}) Y_1(\rho\sqrt{w}) \] (17)
Approximate expressions for the inverse Laplace transforms (13) in terms of elementary trigonometric functions may be obtained via asymptotic expansion methods in the complex plane. We rewrite (13) as follows:

\[
F_p^k(t) = \frac{1}{2\pi i} \int_C \left[ \frac{1}{s} \left( -\frac{d}{ds} \right)^k R_0^p(s) \right] e^{st} ds,
\]

with the contour \( C \) in the complex \( s \) plane [Fig. 1] being a vertical line on which \( \text{Re}(s) \) is a positive constant. We can distort \( C \) as indicated in Fig. 2. The arc contributions \( C_1 \) and \( C_2 \) vanish, because as \( |s| \to \infty \)

\[
R_0^p(s) \to \frac{1}{|s|^2},
\]

as is evident from (14). Consequently, \( F_p^k(t) \) can be expressed as an integral around the Hankel loop contour \( L \) given in Fig. 2.

To proceed further, we make use of the asymptotic expansion [9]

\[
K_0(z) + K_1(z) \sim \left( \frac{\pi}{2z} \right)^\frac{1}{2} e^{-z} \sum_{n=0}^{\infty} a_n z^{-n},
\]

\[
a_0 = 2, \quad a_1 = \frac{1}{4}, \quad a_2 = -\frac{3}{64}, \quad a_3 = \frac{15}{512}, \ldots
\]

in order to obtain the following result:

\[
F_p^k(t) \sim \sum_{n=0}^{\infty} b_n \frac{1}{2\pi i} \int_L \left[ \left( -\frac{d}{ds} \right)^k \left[ e^{-\rho^2/s s^{n-3+1/2}} \right] \right] ds
\]

where

\[
b_n \equiv \frac{3\rho^{1-2n}}{8\pi^{3/2}} 2^n a_n.
\]

The integrals in (22) can be evaluated through explicit use of Schl"afli’s integral [10] over the Hankel contour \( L \):

\[
J_v(z) = \frac{1}{2\pi i} \int_L w^{-v-1} e^{z(w-1/w)/2} dw,
\]
valid for $Re(z) > 0$. Correspondence between (22) and (24) is obtained by letting $w = (\sqrt{t}/\rho)s$, $z = 2\rho\sqrt{t}$, in which case we find for $k = 0$ that

$$F_0(t) \sim \sum_{n} b_n \left( \frac{\rho}{\sqrt{t}} \right)^{n-5/2} \frac{1}{2\pi i} \int_{L} w^{n-7/2} e^{\rho\sqrt{t}(w-1/w)} dw$$

$$= \sum_{n} b_n \left( \frac{\rho}{\sqrt{t}} \right)^{n-5/2} J_{5/2-n}(2\rho\sqrt{t}).$$

Higher sum-rule moments can be obtained via explicit differentiation with respect to $s$ in the integrand of (22); e.g.,

$$F^p_1(t) \sim \sum_{n=0} b_n \left\{ \left( \frac{5}{2} - n \right) \frac{1}{2\pi i} \int_{L} e^{st-\rho^2/s} s^{n-9/2} ds - \rho^2 \frac{1}{2\pi i} \int_{L} e^{st-\rho^2/s} s^{n-11/2} ds \right\}$$

$$= \sum_{n=0} b_n \left\{ \left( \frac{5}{2} - n \right) \left( \frac{\rho}{\sqrt{t}} \right)^{n-4} J_{2-n}(2\rho\sqrt{t}) - \rho^2 \left( \frac{\rho}{\sqrt{t}} \right)^{n-4} J_{2-n}(2\rho\sqrt{t}) \right\}.$$  

Finally, we note that Bessel functions of half-integer order can be expressed in terms of elementary trigonometric functions. We find from (25) that

$$F_0(s_0) = \frac{3}{4\pi^2 \rho^4} \left\{ \sin(2\rho\sqrt{s_0}) \left[ -\rho^2 s_0 + \frac{25}{32} + O\left( \frac{1}{\rho^2 s_0} \right) \right] \right.$$  

$$+ \cos(2\rho\sqrt{s_0}) \left[ -\frac{7\rho s_0^{1/2}}{4} + \frac{15}{128\rho s_0^{1/2}} + O\left( \frac{1}{\rho^2 s_0^{3/2}} \right) \right] \right\}.$$  

For $\rho^2 s_0 > 2$ this approximate expression differs from (14) with $k = 0$ by less than 5%.

Given an instanton size $1/\rho \approx 600$ MeV, eq. (27) is seen to oscillate slowly as $s_0$ increases past $1$ GeV$^2$, going from positive to negative as $s_0$ increases past $2.9$ GeV$^2$. Since the purely-perturbative contribution is also positive and quadratic in $s_0$ [1,2], we see the effect of instanton contributions is to enhance the size of field-theoretic contributions to $F^p_0$ at low $s_0$, but to diminish somewhat the magnitude of field-theoretic contributions for values of the continuum threshold chosen to be above $2.9$ GeV$^2$. The corresponding expression for $F_1$ in
terms of elementary trigonometric functions can be obtained from (26):

\[ F_1(s_0) = \frac{3}{8\pi^2\rho^6} \left\{ \sin(2\rho\sqrt{s_0}) \left[ -2\rho^4 s_0^2 + \frac{129}{16} \rho^2 s_0 + \mathcal{O}(1) \right] \\
+ \cos(2\rho\sqrt{s_0}) \left[ -\frac{11}{2} \rho^3 s_0^{3/2} + \frac{531}{64} \rho s_0^{1/2} + \mathcal{O}\left(\frac{1}{\rho\sqrt{s_0}}\right) \right] \right\} \quad . \quad (28) \]

Once again, the leading instanton contribution to \( F_1 \) is seen to be lower-degree in \( s_0 \) than the \( \mathcal{O}(s_0^3) \) purely-perturbative contribution. As a final comment, it should be noted that for detailed phenomenological work, all the FESRs require inclusion of an overall renormalization-group factor which is identical for the (leading) perturbative and instanton contributions.

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Figure Captions

Fig. 1: The contour $C$ characterizing the inverse Laplace-transform contour integral.

Fig. 2: Distortion of $C$ into the sum of infinite-radius arc contributions $C_{1,2}$ and the Hankel loop contour $L$. 
Fig. 2