ON PROPERTIES OF GEOMETRIC PREDUALS OF C^{k,\omega} SPACES

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Abstract. Let C^{k,\omega}_b(\mathbb{R}^n) be the Banach space of C^k functions on \mathbb{R}^n bounded together with all derivatives of order \leq k and with derivatives of order k having moduli of continuity majorated by c \cdot \omega, c \in \mathbb{R}_+, for some \omega \in C(\mathbb{R}_+). Let C^{k,\omega}_b(S) := C^{k,\omega}_b(\mathbb{R}^n)|_S be the trace space to a closed subset S \subset \mathbb{R}^n. The geometric predual G^{k,\omega}_b(S) of C^{k,\omega}_b(S) is the minimal closed subspace of the dual \left( C^{k,\omega}_b(\mathbb{R}^n) \right)^* containing evaluation functionals of points in S. We study geometric properties of spaces G^{k,\omega}_b(S) and their relations to the classical Whitney problems on the characterization of trace spaces of C^k functions on \mathbb{R}^n.

1. Formulation of Main Results

1.1. Geometric Preduals of C^{k,\omega} Spaces. In what follows we use the standard notation of Differential Analysis. In particular, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n denotes a multi-index and |

\begin{align}
|\alpha| := \sum_{i=1}^{n} \alpha_i. \quad 
\end{align}

Also, for x = (x_1, \ldots, x_n) \in \mathbb{R}^n,

\begin{align}
 x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i} \quad \text{and} \quad D^\alpha := \prod_{i=1}^{n} D_i^{\alpha_i}, \quad \text{where} \quad D_i := \frac{\partial}{\partial x_i}.
\end{align}

Let \omega be a nonnegative function on (0, \infty) (referred to as modulus of continuity) satisfying the following conditions:

(i) \omega(t) and \frac{t}{\omega(t)} are nondecreasing functions on (0, \infty);

(ii) \lim_{t \to 0^+} \omega(t) = 0.

Definition 1.1. C^{k,\omega}_b(\mathbb{R}^n) is the Banach subspace of functions f \in C^k(\mathbb{R}^n) with norm

\begin{align}
\|f\|_{C^{k,\omega}_b(\mathbb{R}^n)} := \max \left( \|f\|_{C^k_b(\mathbb{R}^n)}, |f|_{C^{k,\omega}_b(\mathbb{R}^n)} \right),
\end{align}

where

\begin{align}
\|f\|_{C^k_b(\mathbb{R}^n)} := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)|
\end{align}

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and

\[
|f|_{C^k_b(\mathbb{R}^n)} := \max_{|\alpha|=k} \left\{ \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|)} \right\}.
\]

Here $\| \cdot \|$ is the Euclidean norm of $\mathbb{R}^n$.

If $S \subset \mathbb{R}^n$ is a closed subset, then by $C^k_b(S)$ we denote the trace space of functions $g \in C^k_b(\mathbb{R}^n)|S$ equipped with the quotient norm

\[
\|g\|_{C^k_b(S)} := \inf \{ \|\tilde{g}\|_{C^k_b(\mathbb{R}^n)} : \tilde{g} \in C^k_b(\mathbb{R}^n), \tilde{g}|S = g \}.
\]

Let $\left( C^k_b(\mathbb{R}^n) \right)^* \subset C^k_b(\mathbb{R}^n)$ be the dual of $C^k_b(\mathbb{R}^n)$. Clearly, each evaluation functional $\delta^0_x$ at $x \in \mathbb{R}^n$ (i.e., $\delta^0_x(f) := f(x)$, $f \in C^k_b(\mathbb{R}^n)$) belongs to $\left( C^k_b(\mathbb{R}^n) \right)^*$ and has norm one. By $G^k_b(S) \subset \left( C^k_b(\mathbb{R}^n) \right)^*$ we denote the minimal closed subspace containing all $\delta^0_x$, $x \in S$.

**Theorem 1.2.** The restriction map to the set $\{ \delta^0_s : s \in S \} \subset G^k_b(S)$ determines an isometric isomorphism between the dual of $G^k_b(S)$ and $C^k_b(S)$.

In what follows, $C^k_b(S)$ will be referred to as the geometric predual of $C^k_b(S)$. In the present paper we study some properties of these spaces. The subject is closely related to the classical Whitney problems, see [W1, W2], asking about the characterization of trace spaces of $C^k$ functions on $\mathbb{R}^n$ (see survey [F3] and book [BB2] and references therein for recent developments in the area). Some of the main results of the theory can be reformulated in terms of certain geometric characteristics of spaces $G^k_b(S)$, see sections 1.2 and 1.3.

### 1.2. Finiteness Principle

For $C^k_b(S)$ this principle was conjectured by Yu. Brudnyi and P. Shvartsman in the 1980s in the following form (cf. [F3, p. 210]).

**Finiteness Principle.** To decide whether a given $f : S \to \mathbb{R}$, $S \subset \mathbb{R}^n$, extends to a function $F \in C^k_b(\mathbb{R}^n)$, it is enough to look at all restrictions $f|_{S'}$, where $S' \subset S$ is an arbitrary $d$-element subset. Here $d$ is an integer constant depending only on $k$ and $n$.

More precisely, if $f|_{S'}$ extends to a function $F^{S'} \in C^k_b(\mathbb{R}^n)$ of norm at most 1 (for each $S' \subset S$ with at most $d$ elements), then $f$ extends to a function $F \in C^k_b(\mathbb{R}^n)$, whose norm is bounded by a constant depending only on $k$ and $n$.

For $k = 0$ (the Lipschitz case) the McShane extension theorem [McS] implies the Finiteness Principle with the optimal constant $d = 2$. Also, for $n = 1$ the Finiteness Principle is valid with the optimal constant $d = k + 2$. The result is essentially due to Merrien [M].

In the multidimensional case the Finiteness Principle was proved by Yu. Brudnyi and Shvartsman for $k = 1$ with the optimal constant $d = 3 \cdot 2^{n-1}$, see [BS1]. In the early 2000s the Finiteness Principle was proved by C. Fefferman for all $k$ and $n$ for regular moduli of continuity $\omega$ (i.e., $\omega(1) = 1$), see [F1]. The upper bound for the constant $d$ in the Fefferman proof was reduced later to $d = 2^{k+n}$ by Bierstone and P. Milman [BM] and independently and by a different method by Shvartsman [S]. The obtained results (and the Finiteness Principle in general) admit the following reformulation in terms of geometric characteristics.
of closed unit balls $B_b^k(S)$ of $G_b^k(S)$. Specifically, let $B_b^k(S; m) \subset B_b^k(S)$, $m \in \mathbb{N}$, be the balanced closed convex hull of the union of all finite-dimensional balls $B_b^k(S') \subset G_b^k(S')$, $S' \subset S$, card $S' \leq m$.

**Theorem 1.3.** There exist constants $d \in \mathbb{N}$ and $c \in (1, \infty)$ such that

$$B_b^k(S; d) \subset B_b^k(S) \subset c \cdot B_b^k(S; d).$$

Here for $k = 0$, $d = 2$ (- optimal) and $c = 1$, for $n = 1$, $d = k + 2$ (- optimal) and $c$ depends on $k$ only, for $k = 1$, $d = 3 \cdot 2^{n-1}$ (- optimal) and $c$ depends on $k$ and $n$ only, and for $k \geq 2$, $d = 2^{(k+1)}$ and $c = \frac{\bar{c}}{\omega(1)}$, where $\bar{c}$ depends on $k$ and $n$ only.

1.3. **Complementability of Spaces $G_b^{k,\omega}(S)$.** We begin with a result describing bounded linear operators on $G_b^{k,\omega}(\mathbb{R}^n)$. To this end, for a Banach space $X$ by $C_b^{k,\omega}(\mathbb{R}^n; X)$ we denote the Banach space of $X$-valued $C^k$ functions on $\mathbb{R}^n$ with norm defined similarly to that of Definition 1.1 with absolute values replaced by norms $\| \cdot \|_X$ in $X$. Let $\mathcal{L}(X_1; X_2)$ stand for the Banach space of bounded linear operators between Banach spaces $X_1$ and $X_2$ equipped with the operator norm.

**Theorem 1.4.** The restriction map to the set $\{\delta_x^0 : x \in \mathbb{R}^n\} \subset G_b^{k,\omega}(\mathbb{R}^n)$ determines an isometric isomorphism between $\mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n); X)$ and $C_b^{k,\omega}(\mathbb{R}^n; X)$.

Let $q_S : C_b^{k,\omega}(\mathbb{R}^n) \rightarrow C_b^{k,\omega}(S)$ be the quotient map induced by the restriction of functions on $\mathbb{R}^n$ to $S$. A right inverse $T \in \mathcal{L}(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ for $q_S$ (i.e., $q_S \circ T = id$) is called a linear extension operator. The set of such operators is denoted by $Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$.

**Definition 1.5.** An operator $T \in Ext(C_b^{k,\omega}(S); C_b^{k,\omega}(\mathbb{R}^n))$ has depth $d \in \mathbb{N}$ if for all $x \in \mathbb{R}^n$ and $f \in C_b^{k,\omega}(S)$,

$$T f(x) = \sum_{i=1}^{d} \lambda_i f_i(x),$$

where $y_i \in S$ and $\lambda_i$ depend only on $x$.

Linear extension operators of finite depth exist. For $k = 0$ (the Lipschitz case) the Whitney-Glaeser linear extension operators $C_b^{0,\omega}(S) \rightarrow C_b^{0,\omega}(\mathbb{R}^n)$, see [Gl], have depth $d$ depending on $n$ only and norms bounded by a constant depending on $n$ only. In the 1990s bounded linear extension operators $C_b^{1,\omega}(S) \rightarrow C_b^{1,\omega}(\mathbb{R}^n)$ of depth $d$ depending on $n$ only with norms bounded by a constant depending on $n$ only were constructed by Yu. Brudnyi and Shvartsman [BS2]. Recently bounded linear extensions operators of depth $d$ depending on $k$ and $n$ only were constructed by Luli [Lu] for all spaces $C_b^{k,\omega}(S)$; their norms are bounded by $C^{C_{1,\omega}}$, where $C \in (1, \infty)$ is a constant depending on $k$ and $n$ only. (Earlier such extension operators were constructed for finite sets $S$ by C. Fefferman [F2, Th. 8].)

In the following result we identify $(G_b^{k,\omega}(S))^*$ with $C_b^{k,\omega}(S)$ by means of the isometric isomorphism of Theorem 1.2.
Theorem 1.6. For each \( T \in Ext(C_b^{k,\omega}(S);C_b^{k,\omega}(\mathbb{R}^n)) \) of finite depth there exists a bounded linear projection \( P : G_b^{k,\omega}(\mathbb{R}^n) \rightarrow G_b^{k,\omega}(S) \) whose adjoint \( P^* = T \).

Remark 1.7. It is easily seen that if \( T \) has depth \( d \) and is defined by (1.6), then
\[
p(x) := P(\delta^0_0) = \sum_{i=1}^d \lambda_i^2 \cdot \delta^0_{\mu_i} \quad \text{for all} \quad x \in \mathbb{R}^n.
\]
Moreover, \( p \in C_b^{k,\omega}(\mathbb{R}^n; G_b^{k,\omega}(S)) \) and has norm equal to \( \| T \| \) by Theorem 1.4.

1.4. Approximation Property. Recall that a Banach space \( X \) is said to have the approximation property, if, for every compact set \( K \subset X \) and every \( \varepsilon > 0 \), there exists an operator \( T : X \rightarrow X \) of finite rank so that \( \| Tx - x \| \leq \varepsilon \) for every \( x \in K \).

Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space \( H^\infty \) of bounded holomorphic functions on the open unit disk. The first example of a space which fails to have the approximation property was constructed by Enflo [E]. Since Enflo’s work several other examples of such spaces were constructed, for the references see, e.g., [L].

A Banach space has the \( \lambda \)-approximation property, \( 1 \leq \lambda < \infty \), if it has the approximation property with the approximating finite rank operators of norm \( \leq \lambda \). A Banach space is said to have the bounded approximation property, if it has the \( \lambda \)-approximation property for some \( \lambda \). If \( \lambda = 1 \), then the space is said to have the metric approximation property.

Every Banach spaces with a basis has the bounded approximation property. Also, it is known that the approximation property does not imply the bounded approximation property, see [LJ]. It was established by Pelczyński [P] that a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a separable Banach space with a basis.

Next, for Banach spaces \( X, Y \) by \( \mathcal{F}(X, Y) \subset \mathcal{L}(X, Y) \) we denote the subspace of linear bounded operators of finite rank \( X \rightarrow Y \). Let us consider the trace mapping \( V \) from the projective tensor product \( Y^* \hat{\otimes}_\pi X \rightarrow \mathcal{F}(X, Y)^* \) defined by
\[
(Vu)(T) = \text{trace}(Tu), \quad \text{where} \quad u \in Y^* \hat{\otimes}_\pi X, \quad T \in \mathcal{F}(X, Y),
\]
that is, if \( u = \sum_{n=1}^\infty y_n \otimes x_n \), then \( (Vu)(T) = \sum_{n=1}^\infty y_n(Tx_n) \).

It is easy to see that \( \| Vu \| \leq \| u \|_\pi \). The \( \lambda \)-bounded approximation property of \( X \) is equivalent to the fact that \( \| u \|_\pi \leq \lambda \| Vu \| \) for all Banach spaces \( Y \). This well-known result (see, e.g., [DE] page 193) is essentially due to Grothendieck [G].

Our result concerning spaces \( G_b^{k,\omega}(S) \) reads as follows.

Theorem 1.8. (1) Spaces \( G_b^{k,\omega}(\mathbb{R}^n) \) have the \( \lambda \)-approximation property with
\[
\lambda = \lambda(k, n, \omega) := 1 + C \cdot \lim_{t \to \infty} \frac{1}{\omega(t)}, \quad \text{where} \quad C \text{ depends on } k \text{ and } n \text{ only}.
\]

(2) All the other spaces \( G_b^{k,\omega}(S) \) have the \( \lambda \)-approximation property with
\[
\lambda = C' \cdot \lambda(1, n, \omega), \quad \text{where} \quad C' \text{ is a constant depending on } n \text{ only}, \text{ if } k = 0, 1, \text{ and with }
\lambda = \frac{C'' \cdot \lambda(k, n, \omega)}{\omega(1)}, \quad \text{where} \quad C'' \text{ is a constant depending on } k \text{ and } n \text{ only}, \text{ if } k \geq 2.
If \( \lim_{t \to \infty} \omega(t) = \infty \), then (1) implies that the corresponding space \( G^{k,\omega}_b(\mathbb{R}^n) \) has the metric approximation property. In case \( \lim_{t \to \infty} \omega(t) < \infty \), one can define the new modulus of continuity \( \tilde{\omega} \) (cf. properties (i) and (ii) in its definition) by the formula
\[
\tilde{\omega}(t) = \max\{\omega(t), t\}, \quad t \in (0, \infty).
\]
It is easily seen that spaces \( C^{k,\omega}_b(\mathbb{R}^n) \) and \( C^{k,\tilde{\omega}}_b(\mathbb{R}^n) \) are isomorphic. Thus \( G^{k,\omega}_b(\mathbb{R}^n) \) is isomorphic to space \( G^{k,\tilde{\omega}}_b(\mathbb{R}^n) \) having the metric approximation property. However, the distortion of this isomorphism depends on \( \omega \). So, in general, it is not clear whether \( G^{k,\omega}_b(\mathbb{R}^n) \) itself has the metric approximation property.

Remark 1.9. It is not known, even for the case \( k = 0 \), whether all spaces \( C^{k,\omega}_b(\mathbb{R}^n) \) have the approximation property (for some results in this direction for \( k = 0 \) see, e.g., [K]).

At the end of this section we formulate a result describing the structure of operators in \( \mathcal{L}(G^{k,\omega}_b(\mathbb{R}^n); X) \), where \( X \) is a separable Banach space with the \( \lambda \)-approximation property. In particular, it can be applied to \( X = G^{k,\omega}_b(S) \) and \( \lambda := \lambda(S, k, n, \omega) \) the constant of the approximation property for \( G^{k,\omega}_b(S) \) of Theorem 1.8 (2).

Theorem 1.10. There exists the family of norm one vectors \( \{v_j\}_{j \in \mathbb{N}} \subset X \) and given \( H \in \mathcal{L}(G^{k,\omega}_b(\mathbb{R}^n); X) \) the family of functions \( \{h_j\}_{j \in \mathbb{N}} \subset C^{k,\omega}_b(\mathbb{R}^n) \) of norms \( \leq 32 \cdot \lambda^2 \cdot \|H\| \) such that for all \( x \in \mathbb{R}^n \), \( \alpha \in \mathbb{Z}^n_+, |\alpha| \leq k \),
\[
H(\delta^\alpha_x) = \sum_{j=1}^{\infty} D^\alpha h_j(x) \cdot v_j
\]
(convergence in \( X \)).
1.5. **Preduals of** $G_b^{k,\omega}(S)$ **Spaces.** Let $C_b^{k,\omega}({\mathbb R}^n)$ be the subspace of functions $f \in C_b^{k,\omega}({\mathbb R}^n)$ such that

(i) for all $\alpha \in \mathbb{Z}_+^n$, $0 \leq |\alpha| \leq k$,

$$\lim_{\|x\| \to \infty} D^\alpha f(x) = 0;$$

(ii) for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = k$,

$$\lim_{\|x-y\| \to 0} \frac{D^\alpha f(x) - D^\alpha f(y)}{\omega(\|x-y\|)} = 0.$$

It is easily seen that $C_b^{k,\omega}({\mathbb R}^n)$ equipped with the norm induced from $C_b^{k,\omega}({\mathbb R}^n)$ is a Banach space. By $C_b^{k,\omega}_0(S)$ we denote the trace of $C_b^{k,\omega}({\mathbb R}^n)$ to a closed subset $S \subset {\mathbb R}^n$ equipped with the trace norm.

If $\lim_{t \to 0^+} \frac{1}{\omega(t)} > 0$ (see condition (i) for $\omega$ in section 1.1), then clearly, the corresponding space $C_b^{k,\omega}_0(S)$ is trivial. Thus we may naturally assume that $\omega$ satisfies the condition

(1.10) $$\lim_{t \to 0^+} \frac{t}{\omega(t)} = 0.$$

In the sequel, the weak* topology of $C_b^{k,\omega}_0(S)$ is defined by means of functionals in $G_b^{k,\omega}(S) \subset (C_b^{k,\omega}({\mathbb R}^n))^\ast$. Convergence in the weak* topology is described in section 4.2.

**Theorem 1.12.** Suppose $\omega$ satisfies condition (1.10).

1. Space $(C_b^{k,\omega}({\mathbb R}^n))^\ast$ is isomorphic to $C_b^{k,\omega}({\mathbb R}^n)$, isometrically if $\lim_{t \to \infty} \omega(t) = \infty$.

2. If there exists a weak* continuous operator $T \in \text{Ext}(C_b^{k,\omega}_0(S); C_b^{k,\omega}({\mathbb R}^n))$ such that $T(C_b^{k,\omega}_0(S)) \subset C_b^{k,\omega}({\mathbb R}^n)$, then $(C_b^{k,\omega}_0(S))^\ast$ is isomorphic to $C_b^{k,\omega}(S)$.

From the first part of the theorem we obtain (for $\omega$ satisfying (1.10)):

**Corollary 1.13.** The space of $C^\infty$ functions with compact supports on ${\mathbb R}^n$ is dense in $C_b^{k,\omega}_0({\mathbb R}^n)$. In particular, all spaces $C_b^{k,\omega}_0(S)$ are separable.

It is not clear whether the condition of the second part of the theorem is valid for all spaces $C_b^{k,\omega}_0(S)$ with $\omega$ subject to (1.10). Here we describe a class of sets $S$ satisfying this condition. As before, by $P_{k,n}$ we denote the space of real polynomials on ${\mathbb R}^n$ of degree $k$, and by $Q_r(x) \subset {\mathbb R}^n$ the closed cube centered at $x$ of sidelength $2r$.

**Definition 1.14.** A point $x$ of a subset $S \subset {\mathbb R}^n$ is said to be weak $k$-Markov if

$$\lim_{r \to 0} \left\{ \sup_{p \in P_{k,n} \setminus \emptyset} \left( \frac{\sup_{Q_r(x)} |p|}{\sup_{Q_r(x) \cap S} |p|} \right) \right\} < \infty.$$

A closed set $S \subset {\mathbb R}^n$ is said to be weak $k$-Markov if it contains a dense subset of weak $k$-Markov points.
The class of weak $k$-Markov sets, denoted by $\text{Mar}^*_k(\mathbb{R}^n)$, was introduced and studied by Yu. Brudnyi and the author, see [BB1, B]. It contains, in particular, the closure of any open set, the Ahlfors $p$-regular compact subsets of $\mathbb{R}^n$ with $p > n - 1$, a wide class of fractals of arbitrary positive Hausdorff measure, direct products $\prod_{j=1}^{l} S_j$, where $S_j \in \text{Mar}^*_k(\mathbb{R}^n)$, $1 \leq j \leq l, n = \sum_{j=1}^{l} n_j$, and closures of unions of any combination of such sets. Solutions of the Whitney problems (see sections 1.2 and 1.3 above) for sets in $\text{Mar}^*_k(\mathbb{R}^n)$ are relatively simple, see [BB1].

We prove the following result.

**Theorem 1.15.** Let $S' \in \text{Mar}^*_k(\mathbb{R}^n)$ and $\omega$ satisfy (1.10). Suppose $H : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map such that

(a) the entries of its Jacobian matrix belong to $C_{b}^{k-1, \omega_0}(\mathbb{R}^n)$, where $\omega_0$ satisfies

(1.11) $\lim_{t \to 0^+} \frac{\omega_0(t)}{\omega(t)} = 0$

(b) the map $H|_{S'} : S' \to S = H(S')$ is a proper retraction.\(^1\)

Then the condition of Theorem 1.12 holds for $C_{b}^{k, \omega}(S)$. Thus $C_{b}^{k, \omega}(S)$ is isomorphic to $(C_{b}^{1, \omega}(S))^*$ and so $C_{b}^{k, \omega}(S)$ and $C_{b}^{1, \omega}(S)$ have the metric approximation property.

**Remark 1.16.** (1) In addition to weak $k$-Markov sets $S \subset \mathbb{R}^n$, Theorem 1.15 is valid, e.g., for a compact subset $S$ of a $C^{k+1}$-manifold $M \subset \mathbb{R}^n$ such that the base of the topology of $S$ consists of relatively open subsets of Hausdorff dimension $> \dim M - 1$. Indeed, in this case there exist tubular open neighbourhoods $U_M \subset V_M \subset \mathbb{R}^n$ of $M$ such that $\cl(U_M) \subset V_M$ together with a $C^{k+1}$ retraction $r : U_M \to M$. Then, due to the hypothesis for $S$, the base of topology of $S' := r^{-1}(S) \cap \cl(U_M)$ consists of relatively open subsets of Hausdorff dimension $> n - 1$ and so $S' \in \text{Mar}^*_p(\mathbb{R}^n)$ for all $p \in \mathbb{N}$, see, e.g., [B] page 536]. Moreover, it is easily seen that $r|_{S'}$ is the restriction to $S'$ of a map $H \in C_{b}^{k+1}(\mathbb{R}^n; \mathbb{R}^n)$. Decreasing $V_M$, if necessary, we may assume that $S'$ is compact, and so the triple $(H, S', S)$ satisfies the hypothesis of the theorem.

(2) Under conditions of Theorem 1.15 $C_{b}^{k, \omega}(S)$ is isomorphic to the second dual of $C_{b}^{1, \omega}(S)$.

2. **Proof of Theorem 1.2**

By $\delta_x^\alpha, x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n$, we denote the evaluation functional $D^\alpha|_x$. By definition each $\delta_x^\alpha, |\alpha| \leq k$, belongs to $(C_{b}^{k, \omega}(\mathbb{R}^n))^*$ and has norm $\leq 1$. Similarly, functionals $\frac{\delta^\alpha_x - \delta^\alpha_y}{\omega(|x-y|)}$, $|\alpha| = k, x, y \in \mathbb{R}^n, x \neq y$, belong to $(C_{b}^{k, \omega}(\mathbb{R}^n))^*$ and have norm $\leq 1$.

**Proposition 2.1.** The closed unit ball $B$ of $(C_{b}^{k, \omega}(\mathbb{R}^n))^*$ is the balanced weak* closed convex hull of the set $V$ of all functionals $\delta_x^\alpha, |\alpha| \leq k$, and $\frac{\delta^\alpha_x - \delta^\alpha_y}{\omega(|x-y|)}$, $|\alpha| = k, x, y \in \mathbb{R}^n, x \neq y$.

\(^1\)I.e., $S \subset S'$ and $H|_{S'}(x) = x$ for all $x \in S$, and for each compact $K \subset S$ its preimage $(H|_{S'})^{-1}(K)$ is compact.
Proof. Clearly, $V \subset B$ and therefore the required hull $\hat{V} \subset B$ as well. Assume, on the contrary, that $\hat{V} \neq B$. Then due to the Hahn-Banach theorem there exists an element $f \in C^k_b(\mathbb{R}^n)$ of norm one such that $\sup_{v \in \hat{V}} |v(f)| \leq c < 1$. Since $V \subset \hat{V}$, this implies 
$$\|f\|_{C^k_b(\mathbb{R}^n)} \leq c < 1,$$
a contradiction proving the result. \hfill \square

Let $X$ be the minimal closed subspace of $(C^k_b(\mathbb{R}^n))^*$ containing $V$.

**Proposition 2.2.** $X^*$ is isometrically isomorphic to $C^k_b(\mathbb{R}^n)$.

**Proof.** For $h \in X^*$ we set $H(x) := h(\delta^0_x)$, $x \in \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard orthonormal basis in $\mathbb{R}^n$. By the mean-value theorem for functions in $C^k_b(\mathbb{R}^n)$ we obtain, for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| < k$, $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{\delta^\alpha_{x+t e_i} - \delta^\alpha_x}{t} = \delta^\alpha_{x+e_i},$$

(convergence in $(C^k_b(\mathbb{R}^n))^*$). From here by induction we deduce easily that $H \in C^k(\mathbb{R}^n)$ and for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq k$, $x \in \mathbb{R}^n$,

$$h(\delta^0_x) = D^\alpha H(x).$$

This shows that $H \in C^k_b(\mathbb{R}^n)$ and $\|H\|_{C^k_b(\mathbb{R}^n)} \leq \|h\|_{X^*}$. Considering $H$ as the bounded linear functional on $(C^k_b(\mathbb{R}^n))^*$ we obtain that $H|_V = h|_V$. Thus, by the definition of $X$,

$$H|_X = h.$$

Since the unit ball of $X$ is $B \cap X$,

$$\|h\|_{X^*} \leq \|H\|_{C^k_b(\mathbb{R}^n)} \left( \leq \|h\|_{X^*} \right).$$

Hence, the correspondence $h \mapsto H$ determines an isometry $I : X^* \to C^k_b(\mathbb{R}^n)$. Since the restriction of each $H \in C^k_b(\mathbb{R}^n)$, regarded as the bounded linear functional on $(C^k_b(\mathbb{R}^n))^*$, to $X$ determines some $h \in X^*$, map $I$ is surjective.

This completes the proof of the proposition. \hfill \square

Note that equation (2.12) shows that the minimal closed subspace $G^k_b(\mathbb{R}^n) \subset (C^k_b(\mathbb{R}^n))^*$ containing all $\delta^0_x$, $x \in \mathbb{R}^n$, coincides with $X$. Thus, $(G^k_b(\mathbb{R}^n))^*$ is isometrically isomorphic to $C^k_b(\mathbb{R}^n)$; this proves Theorem 1.2 for $S = \mathbb{R}^n$.

**Corollary 2.3.** The closed unit ball of $G^k_b(\mathbb{R}^n)$ is the balanced closed convex hull of the set $V$ of all functionals $\delta^\alpha_x$, $|\alpha| \leq k$, and $\frac{\delta^\alpha_x - \delta^\alpha_y}{\omega(\|x-y\|)}$, $|\alpha| = k$, $x, y \in \mathbb{R}^n$, $x \neq y$.

**Proof.** The closed unit ball of $G^k_b(\mathbb{R}^n)$ is $B \cap G^k_b(\mathbb{R}^n)$. Since the weak* topology of $(C^k_b(\mathbb{R}^n))^*$ induces the weak topology of $G^k_b(\mathbb{R}^n)$ and the weak closure of the balanced convex hull of $V$ coincides with the norm closure of this set, the result follows from Proposition 2.1. \hfill \square
Now, let us consider the case of general $S \subseteq \mathbb{R}^n$. Let $h \in (G^k_b(S))^*$. We set $H(x) := h(\delta^0_x)$, $x \in S$. Due to the Hahn-Banach theorem, there exists $\tilde{h} \in (G^k_b(\mathbb{R}^n))^*$ such that $\tilde{h}|_{G^k_b(S)} = h$ and $\|\tilde{h}\|_{(G^k_b(\mathbb{R}^n))^*} = \|h\|_{(G^k_b(S))^*}$. Let us define $\tilde{H}(x) = \tilde{h}(\delta^0_x)$, $x \in \mathbb{R}^n$. According to Proposition 2.2 $\tilde{H} \in C^k_b(\mathbb{R}^n)$ and $\|\tilde{H}\|_{C^k_b(\mathbb{R}^n)} = \|\tilde{h}\|_{(G^k_b(\mathbb{R}^n))^*}$. Moreover, $\tilde{H}|_S = H$. This implies that $H \in C^k_b(S)$ and has norm $\leq \|h\|_{(G^k_b(S))^*}$. Hence, the correspondence $h \mapsto H$ determines a bounded nonincreasing norm linear injection $I_S : (G^k_b(S))^* \to C^k_b(S)$. Let us show that $I_S$ is a surjective isometry. Indeed, for $H \in C^k_b(S)$ there exists $\tilde{H} \in C^k_b(\mathbb{R}^n)$ such that $\tilde{H}|_S = H$ and $\|\tilde{H}\|_{C^k_b(\mathbb{R}^n)} = \|H\|_{C^k_b(S)}$. In turn, due to Proposition 2.2 there exists $\tilde{h} \in (G^k_b(\mathbb{R}^n))^*$ such that $\tilde{H}(x) = \tilde{h}(\delta^0_x)$, $x \in \mathbb{R}^n$, and $\|\tilde{H}\|_{C^k_b(\mathbb{R}^n)} = \|\tilde{h}\|_{(G^k_b(\mathbb{R}^n))^*}$. We set $h := \tilde{h}|_{G^k_b(S)}$. Then $h \in (G^k_b(S))^*$ and $H(x) = h(\delta^0_x)$, $x \in S$, i.e., $I_S(h) = H$ and
\[
\left(\|h\|_{(G^k_b(S))^*} \geq \|I_S(h)\|_{C^k_b(S)} \geq \|h\|_{(G^k_b(S))^*}\right).
\]

The proof of Theorem 1.2 is complete.

3. Proofs of Theorems 1.3, 1.4, 1.6

3.1. Proof of Theorem 1.3.

Proof. According to the Finiteness Principle there exist constants $d \in \mathbb{N}$ and $c \in (1, \infty)$ such that for all $f \in C^k_b(S)$,
\[
\sup_{S' \subseteq S; \text{card} S' \leq d} \|f\|_{C^k_b(S')} \leq \|f\|_{C^k_b(S)} \leq c \cdot \left(\sup_{S' \subseteq S; \text{card} S' \leq d} \|f\|_{C^k_b(S')}\right).
\]

Here for $k = 0$, $d = 2$ (-optimal) and $c = 1$, see [McS], for $n = 1$, $d = k + 2$ (-optimal) and $c$ depends on $k$ only, see [M], for $k = 1$, $d = 3 \cdot 2^{n-1}$ (-optimal) and $c$ depends on $k$ and $n$ only, see [BS1], and for $k \geq 2$, $d = 2^{k+1}$ and $c = \frac{\bar{c}}{\omega(1)}$, where $\bar{c}$ depends on $k$ and $n$ only, see [FI1] and [BM], [S].

Considering $f$ as the bounded linear functional on $G^k_b(S)$, we get from (3.13) the required implications
\[
\left.B^k_{b,S}(d) \subset B^k_{b,S} \subset c \cdot B^k_{b,S}(d)\right).
\]

Indeed, suppose, on the contrary, that there exists $v \in B^k_{b,S} \setminus c \cdot B^k_{b,S}(d)$. Let $f \in C^k_b(S)$ be such that
\[
\sup_{c \cdot B^k_{b,S}(d)} |f| < |f(v)|.
\]
By the definition of $B^k_w(S; d)$ the left-hand side of the previous inequality coincides with $c \cdot \left( \sup_{S' \subset S, \text{card} S' \leq d} \| f \|_{C^k_w(S')} \right)$. Hence,

$$c \cdot \left( \sup_{S' \subset S, \text{card} S' \leq d} \| f \|_{C^k_w(S')} \right) < |f(v)| \leq \| f \|_{C^k_w(S)},$$

a contradiction with (3.13).

3.2. Proof of Theorem 1.4.

Proof. We set

$$(3.14) \quad r_X(F)(s) := F(\delta^0_s), \quad F \in \mathcal{L}(G^k_w(\mathbb{R}^n); X), \quad s \in \mathbb{R}^n.$$ 

Applying the arguments similar to those of Proposition 2.2 we obtain

$$r_X(F) \in C^k_w(\mathbb{R}^n; X) \quad \text{and} \quad \| r_X(F) \|_{C^k_w(\mathbb{R}^n; X)} \leq \| F \|_{\mathcal{L}(G^k_w(\mathbb{R}^n); X)}.$$ 

On the other hand, for each $\varphi \in X^*$, $|\varphi|_{X^*} = 1$, function $r_\mathbb{R}(\varphi \circ F) \in C^k_w(\mathbb{R}^n)$. So, since $r_\mathbb{R}(\varphi \circ F) = \varphi(r_X(F))$,

$$\| \varphi \circ F \|_{(G^k_w(\mathbb{R}^n))^*} = \| r_\mathbb{R}(\varphi \circ F) \|_{C^k_w(\mathbb{R}^n)} = \| \varphi(r_X(F)) \|_{C^k_w(\mathbb{R}^n)} \leq \| r_X(F) \|_{C^k_w(\mathbb{R}^n; X)}.$$

Taking supremum over all such $\varphi$ we get

$$\| F \|_{\mathcal{L}(G^k_w(\mathbb{R}^n); X)} \leq \| r_X(F) \|_{C^k_w(\mathbb{R}^n; X)} \leq \| F \|_{\mathcal{L}(C^k_w(\mathbb{R}^n); X)}.$$ 

This shows that $r_X : \mathcal{L}(G^k_w(\mathbb{R}^n); X) \to C^k_w(\mathbb{R}^n; X)$ is an isometry. Let us prove that it is surjective.

Since every finite subset of $\mathbb{R}^n$ is interpolating for $C^k_w(\mathbb{R}^n)$, the set of vectors $\delta^0_s \in G^k_w(\mathbb{R}^n)$, $s \in \mathbb{R}^n$, is linearly independent. Hence, each $f \in C^k_w(\mathbb{R}^n; X)$ determines a linear map $\hat{f} : \text{span}\{\delta^0_s : s \in \mathbb{R}^n\} \to X$,

$$\hat{f} \left( \sum_j c_j \delta^0_{s_j} \right) := \sum_j c_j f(s_j), \quad \sum_j c_j \delta^0_{s_j} \in \text{span}\{\delta^0_s : s \in \mathbb{R}^n\} \subset C^k_w(\mathbb{R}^n).$$

Next, for each $\varphi \in X^*$, $|\varphi|_{X^*} = 1$, function $\varphi \circ f \in C^k_w(\mathbb{R}^n)$ and

$$\| \varphi \circ f \|_{C^k_w(\mathbb{R}^n)} \leq \| f \|_{C^k_w(\mathbb{R}^n; X)}.$$ 

Since $r_\mathbb{R} : (G^k_w(\mathbb{R}^n))^* \to C^k_w(\mathbb{R}^n)$ is an isometric isomorphism, there exists $\ell_{\varphi \circ f} \in (G^k_w(\mathbb{R}^n))^*$ such that $r_\mathbb{R}(\ell_{\varphi \circ f}) = \varphi \circ f$. Clearly, $\ell_{\varphi \circ f}$ coincides with $\varphi \circ \hat{f}$ on $\text{span}\{\delta^0_s : s \in \mathbb{R}^n\}$ and for all $v \in C^k_w(\mathbb{R}^n)$,

$$|\ell_{\varphi \circ f}(v)| \leq \| \ell_{\varphi \circ f} \|_{(G^k_w(\mathbb{R}^n))^*} \cdot \| v \|_{C^k_w(\mathbb{R}^n)} = \| \varphi \circ f \|_{C^k_w(\mathbb{R}^n)} \cdot \| v \|_{C^k_w(\mathbb{R}^n)} \leq \| f \|_{C^k_w(\mathbb{R}^n; X)} \cdot \| v \|_{C^k_w(\mathbb{R}^n)}.$$
These imply that \( \hat{f} : \text{span}\{\delta^0_x : x \in \mathbb{R}^n\} \to X \) is a linear continuous operator of norm \( \leq \|f\|_{C^\infty_b(\mathbb{R}^n;X)} \). Hence, it extends to a bounded linear operator \( F : \text{cl}(\text{span}\{\delta^0_x : x \in \mathbb{R}^n\}) =: G^\infty_b(\mathbb{R}^n) \to X \) such that \( r_X(F) = f \). Thus, \( r_X(F) : L(G^\infty_b(\mathbb{R}^n); X) \to C^\infty_b(\mathbb{R}^n; X) \) is an isometric isomorphism.

The proof of the theorem is complete. □

3.3. Proof of Theorem 1.6.

Proof. Without loss of generality we may assume that \( T \) has depth \( d \) and is defined by (1.6). Let \( T : (C^\infty_b(\mathbb{R}^n))^* \to (C^\infty_b(S))^* \) be the adjoint of \( T \) and \( q^*_S : (C^\infty_b(S))^* \to (C^\infty_b(\mathbb{R}^n))^* \) the adjoint of the quotient map \( q_S : C^\infty_b(\mathbb{R}^n) \to C^\infty_b(S) \). Clearly, \( q^*_S \) is an isometric embedding which maps the closed subspace of \((C^\infty_b(S))^*\) generated by \( \delta \)-functionals of points in \( S \) isometrically onto \( C^\infty_b(S) \subset (C^\infty_b(\mathbb{R}^n))^* \). We define

\[ (3.15) \quad P := q^*_S \circ T^*. \]

By the definition of \( T_S \), for each \( \delta^0_x \in C^\infty_b(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \setminus S \), and \( f \in C^\infty_b(S) \) we have, for some \( y^i_x \in S \),

\[ (P\delta^0_x)(f) = \delta^0_x(Tq_S f) = \sum_{i=1}^d \lambda^i_x \cdot f(y^i_x) = \left( \sum_{i=1}^d \lambda^i_x \cdot \delta^0_y \right)(f). \]

Hence,

\[ (3.16) \quad P\delta^0_x = \sum_{i=1}^d \lambda^i_x \cdot \delta^0_{y^i_x} \quad \text{for all} \quad x \in \mathbb{R}^n \setminus S. \]

Since \( T \in \text{Ext}(C^\infty_b(S); C^\infty_b(\mathbb{R}^n)) \),

\[ (3.17) \quad P\delta^0_x = \delta^0_x \quad \text{for all} \quad x \in S. \]

Thus \( P \) maps \( C^\infty_b(\mathbb{R}^n) \) into \( C^\infty_b(S) \) and is identity on \( C^\infty_b(S) \). Hence, \( P \) is a bounded linear projection of norm \( \|P\| \leq \|Tq_S\| \leq \|T\| \).

Next, under the identification \((G^\infty_b(S))^* = C^\infty_b(S)\) for all closed \( S \subset \mathbb{R}^n \) (see Theorem 1.2), for all \( x \in \mathbb{R}^n \setminus S \), and \( f \in C^\infty_b(S) \) we have by (3.16)

\[ (P^*f)(\delta^0_x) = f(P\delta^0_x) = f \left( \sum_{i=1}^d \lambda^i_x \cdot \delta^0_{y^i_x} \right) = \sum_{i=1}^d \lambda^i_x \cdot f(y^i_x) = (Tf)(\delta^0_x). \]

The same identity is valid for \( x \in S \), cf. (3.17).

This implies that \( P^* = T \) and completes the proof of the theorem. □

4. Proofs of Theorems 1.8 and 1.10

Sections 4.1 and 4.2 contain auxiliary results used in the proof of Theorem 1.8.
4.1. **Jackson Theorem.** Recall that the *Jackson kernel* $J_N$ is the trigonometric polynomial of degree $2\tilde{N}$, where $\tilde{N} := \left\lfloor \frac{N}{2} \right\rfloor$, given by the formula

$$J_N(t) = \gamma_N \left( \frac{\sin \frac{\tilde{N}t}{2}}{\sin \frac{t}{2}} \right)^4,$$

where $\gamma_N$ is chosen so that $\int_{-\pi}^{\pi} J_N(t) \, dt = 1$.

For a $2\pi$-periodic real function $f \in C(\mathbb{R})$ we set

$$\text{(4.18)} \quad (L_N f)(x) := \int_{-\pi}^{\pi} f(x - t) J_N(t) \, dt, \quad x \in \mathbb{R}.$$  

Then the classical *Jackson theorem* asserts (see, e.g., [T, Ch. V]): $L_N f$ is a real trigonometric polynomial of degree at most $N$ such that

$$\text{(4.19)} \quad \sup_{x \in (-\pi, \pi)} |f(x) - (L_N f)(x)| \leq c \omega(f, \frac{1}{N}),$$

for a numerical constant $c > 0$; here $\omega(f, \cdot)$ is the modulus of continuity of $f$.

4.2. **Convergence in the Weak* Topology of $C^{k,\omega}_{b}(\mathbb{R}^n)$.** In the proof of Theorem 1.8 we use the following result. As before, we equip $C^{k,\omega}_{b}(\mathbb{R}^n)$ with the weak* topology induced by means of functionals in $G^{k,\omega}_{b}(\mathbb{R}^n) \subset (C^{k,\omega}_{b}(\mathbb{R}^n))^*$.  

**Proposition 4.1.** A sequence $\{f_i\}_{i \in \mathbb{N}} \subset C^{k,\omega}_{b}(\mathbb{R}^n)$ weak* converges to $f \in C^{k,\omega}_{b}(\mathbb{R}^n)$ if and only if

(a) \quad \sup_{i \in \mathbb{N}} \|f_i\|_{C^{k,\omega}_{b}(\mathbb{R}^n)} < \infty;

(b) \quad \text{For all } \alpha \in \mathbb{Z}^n_+, 0 \leq |\alpha| \leq k, \text{ } x \in \mathbb{R}^n \quad \lim_{i \to \infty} D^\alpha f_i(x) = D^\alpha f(x).$

**Proof.** Without loss of generality we may assume that $f = 0$. If $\{f_i\}_{i \in \mathbb{N}}$ weak* converges to 0, then (a) follows from the Banach-Steinhaus theorem and (b) from the fact that each $\delta^\alpha_x \in G^{k,\omega}_{b}(\mathbb{R}^n)$.

Conversely, suppose that $\{f_i\}_{i \in \mathbb{N}} \subset C^{k,\omega}_{b}(\mathbb{R}^n)$ satisfies (a) and (b) with $f = 0$. Let $g \in G^{k,\omega}_{b}(\mathbb{R}^n)$. According to Corollary 2.3 given $\varepsilon > 0$ there exist $J \in \mathbb{N}$ and families $c_{j\alpha} \in \mathbb{R}$, $x_{j\alpha} \in \mathbb{R}^n$, $1 \leq j \leq J$, $\alpha \in \mathbb{Z}^n_+$, $0 \leq |\alpha| < k$, and $d_{j\alpha} \in \mathbb{R}$, $x_{j\alpha}, y_{j\alpha} \in \mathbb{R}^n$, $x_{j\alpha} \neq y_{j\alpha}$, $1 \leq j \leq J$, $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = k$, such that

$$g = \sum_{j, \alpha} c_{j\alpha} \delta^\alpha_{x_{j\alpha}} + \sum_{j, \alpha} d_{j\alpha} \frac{\delta^\alpha_{x_{j\alpha}} - \delta^\alpha_{y_{j\alpha}}}{\omega(\|x_{j\alpha} - y_{j\alpha}\|)} + g',$$
where
\[ \sum_{j,\alpha} |c_{j\alpha}| + \sum_{j,\alpha} |d_{j\alpha}| \leq \|g\|_{G^k_b(\mathbb{R}^n)} \quad \text{and} \quad \|g\|_{G^k_b(\mathbb{R}^n)} < \frac{\varepsilon}{2M}, \quad M := \sup_{i \in \mathbb{N}} \|f_i\|_{G^k_b(\mathbb{R}^n)}. \]

Further, due to condition (b), there exists \( I \in \mathbb{N} \) such that for all \( i \geq I, \)
\[ \left| f_i \left( \sum_{j,\alpha} c_{j\alpha} \delta_{x_j} + \sum_{j,\alpha} d_{j\alpha} \frac{\delta_{x_j} - \delta_{y_j}}{\omega(\|x_j - y_j\|)} \right) \right| < \frac{\varepsilon}{2}. \]

Also, for such \( i, \)
\[ |f_i(g')| \leq \|f_i\|_{C^k_b(\mathbb{R}^n)} \cdot \|g'\|_{G^k_b(\mathbb{R}^n)} < M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}. \]

Combining these inequalities we obtain for all such \( i: \)
\[ |f_i(g)| < \varepsilon. \]

This shows that \( \lim_{i \to \infty} f_i(g) = 0. \) Thus \( \{f_i\}_{i \in \mathbb{N}} \) weak* converges to 0, as required. \( \square \)

4.3. **Proof of Theorem 1.8 (1).** We set
\[ \mathbb{K}_N^n := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i| \leq N \}. \]

Let \( \rho : \mathbb{R}^n \to [0,1] \) be a fixed \( C^\infty \) function with support in the cube \( \mathbb{K}_1^n, \) equals one on the unit cube \( \mathbb{K}_2^n. \) For a natural number \( \ell \) we set \( \rho_\ell(x) := \rho(x/\ell), \) \( x \in \mathbb{R}^n. \) Then there exist constants \( c_{k,n} \) (depending on \( k \) and \( n \)) such that
\[ \sup_{x \in \mathbb{R}^n} |D^\alpha \rho_\ell(x)| \leq \frac{c_{k,n}}{\ell^{||\alpha||}} \quad \text{for all} \quad \alpha \in \mathbb{Z}_+^n \quad \text{with} \quad |\alpha| \leq k + 1. \]

Let \( f \in C^k_b(\mathbb{R}^n). \) We define a \( 8\ell \sqrt{n} \)-periodic in each variable function \( f_\ell \) on \( \mathbb{R}^n \) by
\[ f_\ell(v + x) = \rho_\ell(x) \cdot f(x), \quad v + x \in 8\ell \sqrt{n} \cdot \mathbb{Z}^n + \mathbb{K}_4^n \cdot \sqrt{n}. \]

Note that \( f_\ell \) coincides with \( f \) on the cube \( \mathbb{K}_4^n. \)

**Lemma 4.2.** There exists a constant \( C_\ell = C(\ell, k, n, \omega) \) (i.e., depending on \( \ell, k, n \) and \( \omega \)) such that
\[ \lim_{\ell \to \infty} C_\ell = 1 + c_{k,n} \cdot 4 \sqrt{n} \cdot (k + 1) \cdot \lim_{\ell \to \infty} \frac{1}{\omega(\ell)}, \]
and
\[ \|f_\ell\|_{C^k_b(\mathbb{R}^n)} \leq C_\ell \cdot \|f\|_{C^k_b(\mathbb{R}^n)}. \]

**Proof.** We use the standard multi-index notation. According to the general Leibniz rule, for \( \alpha \in \mathbb{Z}_+^n, |\alpha| \leq k, \)
\[ D^\alpha f_\ell = \sum_{\nu: \nu \leq \alpha} \binom{\alpha}{\nu} (D^\nu \rho_\ell) \cdot (D^{\alpha-\nu} f) \quad \text{on} \quad \mathbb{K}_4^n. \]
From here and (4.20) we get for $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \geq 1$,

$$\sup_{x \in \mathbb{R}^n} |D^\alpha f_\ell(x)| \leq \|f\|_{C^k_b(\mathbb{R}^n)} \cdot \left( \sum_{\nu: 0 < \nu \leq \alpha} \left( \frac{\alpha}{\nu} \cdot \frac{c_{k,n}}{|\ell|^{\nu}} + 1 \right) \right)$$

$$\leq \|f\|_{C^k_b(\mathbb{R}^n)} \cdot \left( c_{k,n} \cdot \left( \frac{1}{\ell} \cdot \frac{|\alpha|}{1} - 1 \right) + 1 \right) \leq \|f\|_{C^k_b(\mathbb{R}^n)} \cdot \left( \frac{1}{\ell} \cdot \frac{|\alpha| - 1}{1} \cdot \frac{c_{k,n} \cdot |\alpha|}{\ell} + 1 \right).$$

Hence,

$$\|f_\ell\|_{C^k_b(\mathbb{R}^n)} \leq \left( \frac{1}{\ell} \cdot \frac{\max\{k-1,0\}}{1} \cdot \frac{c_{k,n} \cdot k}{\ell} + 1 \right) \cdot \|f\|_{C^k_b(\mathbb{R}^n)} =: C_1(\ell, k, n) \cdot \|f\|_{C^k_b(\mathbb{R}^n)}.$$

Similarly, for $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = k$, and $x, y \in \mathbb{K}_{2\ell}^n$ using properties of $\omega$ we obtain (4.22)

$$|D^\alpha f_\ell(x) - D^\alpha f_\ell(y)| \leq \sum_{\nu: \nu \leq \alpha} \left( \frac{\alpha}{\nu} \cdot (|D^\nu \rho_\ell(x) - D^\nu \rho_\ell(y)| \cdot |D^{\alpha - \nu} f(x)| + |D^\nu \rho_\ell(y)| \cdot |D^{\alpha - \nu} f(x) - D^{\alpha - \nu} f(y)|) \right)$$

$$\leq \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \left( \sum_{\nu: 0 < \nu \leq \alpha} \left( \frac{\alpha}{\nu} \cdot c_{k,n} \cdot \left( \frac{||x - y||}{\ell^{\nu + 1}} + \frac{||x - y||}{\ell^{\nu}} \right) \right) \right) + \frac{c_{k,n} \cdot ||x - y||}{\ell} + \omega(||x - y||)$$

$$= \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \left( c_{k,n} \cdot \left( \frac{1}{\ell} \cdot \frac{k + 1}{1} - 1 \right) \cdot ||x - y|| + \omega(||x - y||) \right)$$

$$\leq \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y||) \cdot \left( c_{k,n} \cdot \left( \frac{1}{\ell} \cdot \frac{k + 1}{1} \cdot \frac{4\sqrt{n} \cdot (k + 1)}{\ell \cdot \omega(4\ell \sqrt{n})} + 1 \right) \right)$$

$$=: \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y||) \cdot C_2(\ell, k, n, \omega).$$

Observe that

(4.23) $$\lim_{\ell \to \infty} C_2(\ell, k, n, \omega) = 1 + c_{k,n} \cdot 4\sqrt{n} \cdot (k + 1) \cdot \lim_{\ell \to \infty} \frac{1}{\omega(\ell)}.$$}

Next, assume that $x, y \in \text{supp } f_\ell$. Since the case $x, y \in \mathbb{K}_{2\ell}^n$ was considered above, without loss of generality we may assume that $x \in \mathbb{K}_{2\ell}^n$ and $y \in v + \mathbb{K}_{2\ell}^n$ for some $v \in (8\ell \sqrt{n} \cdot \mathbb{Z}^n) \setminus \{0\}$. Then for $y' := y - v \in \mathbb{K}_{2\ell}^n$ we have $D^\alpha f_\ell(y') = D^\alpha f_\ell(y)$. Also,

$$||x - y|| = ||x - y' - v|| \geq ||v|| - ||x - y'|| \geq 8\ell \sqrt{n} - 4\ell \sqrt{n} = 4\ell \sqrt{n}.$$}

Therefore for $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = k$,

$$|D^\alpha f_\ell(x) - D^\alpha f_\ell(y)| = |D^\alpha f_\ell(x) - D^\alpha f_\ell(y')| \leq C_2(\ell, k, n, \omega) \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y'||)$$

$$\leq C_2(\ell, k, n, \omega) \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(4\ell \sqrt{n}) \leq C_2(\ell, k, n, \omega) \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y||).$$
Finally, in the case \( x \in \text{supp} \ f_\ell \cap \mathbb{K}^n_{2\ell} \), \( y \notin \text{supp} \ f_\ell \), there exists a point \( y' \in \mathbb{K}^n_{2\ell} \) lying on the interval joining \( x \) and \( y \) such that \( f_\ell(y') = 0 \). Since \( ||x - y'|| \leq ||x - y|| \) and inequality (4.22) is valid for \( x \) and \( y' \), a similar inequality is valid for \( x \) and \( y \). Hence, combining the considered cases we conclude that (cf. (1.4))

\[
|f_\ell|_{C^{k,\omega}(\mathbb{R}^n)} \leq C_2(\ell, k, n, \omega) \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)}.
\]

Therefore the inequality of the lemma is valid with

\[
C(\ell, k, n, \omega) := \max \{ C_1(\ell, k, n), C_2(\ell, k, n, \omega) \}, \quad \square
\]

We set

\[
(4.24) \quad \lambda := \frac{4\ell \sqrt{n}}{\pi}.
\]

For a natural number \( N \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we define

\[
(4.25) \quad (E_N f_\ell)(x) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f_\ell(x_1 - \lambda t_1, \ldots, x_n - \lambda t_n) J_N(t_1) \cdots J_N(t_n) \ dt_1 \cdots dt_n.
\]

**Lemma 4.3.** Function \( E_N f_\ell \) satisfies the following properties:

(a) \( (E_N f_\ell)(\lambda x), x \in \mathbb{R}^n \), is the trigonometric polynomial of degree at most \( N \) in each coordinate;

(b) \[
\|E_N f_\ell\|_{C^{k,\omega}(\mathbb{R}^n)} \leq C_\ell \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)};
\]

(c) There is a constant \( c_\lambda \), \( \lim_{N \to \infty} c_\lambda = 0 \), such that

\[
\|f_\ell - E_N f_\ell\|_{C^{k,\omega}(\mathbb{R}^n)} \leq c_\lambda \cdot \|f\|_{C^{k,\omega}(\mathbb{R}^n)}.
\]

**Proof.** (a) If we set \( f_\ell^\lambda(x) := f_\ell(\lambda x), x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), then

\[
E_N f_\ell(\lambda x) = (L_1 f_\ell^\lambda) \cdots (L_n f_\ell^\lambda)(x),
\]

where \( L_i^\lambda \) is the Jackson operator (4.18) acting on univariate functions in variable \( x_i, 1 \leq i \leq n \). This and the properties of \( L_N^\lambda \) give the required statement.

(b) According to definition (4.25), for each \( \alpha \in \mathbb{Z}^n_+, |\alpha| \leq k, \)

\[
(4.26) \quad \sup_{x \in \mathbb{R}^n} |D^\alpha (E_N f_\ell)(x)| = \sup_{x \in \mathbb{R}^n} |(E_N D^\alpha f_\ell)(x)| \leq \|f_\ell\|_{C^{k}(\mathbb{R}^n)} \leq C_\ell \cdot \|f\|_{C^{k}(\mathbb{R}^n)}.
\]

In turn, if \( |\alpha| = k \) and \( x, y \in \mathbb{R}^n \), then

\[
|D^\alpha (E_N f_\ell)(x) - D^\alpha (E_N f_\ell)(y)| \leq |(E_N D^\alpha f_\ell)(x) - (E_N D^\alpha f_\ell)(y)| \leq |f_\ell|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y||) \leq C_\ell \cdot |f|_{C^{k,\omega}(\mathbb{R}^n)} \cdot \omega(||x - y||).
\]

Thus (4.26) and (4.27) give the required statement.

(c) For each \( \alpha \in \mathbb{Z}^n_+, |\alpha| \leq k - 1 \), using (4.25) and due to (4.19) one obtains
Proof.

(4.30) $L_{N,\ell}f := E_Nf_\ell.$

According to Lemma 4.3, $L_{N,\ell} : C^k_\omega(\mathbb{R}^n) \rightarrow C^k_\omega(\mathbb{R}^n)$ is a finite rank bounded linear operator of norm $\leq C_\ell$.

Lemma 4.4. Operators $L_{N,\ell}$ are weak* continuous.

Proof. Since $C^k_\omega(\mathbb{R}^n) = (C^k_\omega(\mathbb{R}^n))^*$ and $C^k_\omega(\mathbb{R}^n)$ is separable, $C^k_\omega(\mathbb{R}^n)$ equipped with the weak* topology is a Frechet space. Then $L_{N,\ell}$ is weak* continuous if and only if for each sequence $\{f_i\}_{i \in \mathbb{N}} \subset C^k_\omega(\mathbb{R}^n)$ weak* converging to $0 \in C^k_\omega(\mathbb{R}^n)$ the sequence $\{L_{N,\ell}f_i\}_{i \in \mathbb{N}}$ weak* converges to 0 as well. Note that such a sequence $\{f_i\}_{i \in \mathbb{N}}$ is bounded in $C^k_\omega(\mathbb{R}^n)$ due to the Banach-Steinhaus theorem. Then $\{L_{N,\ell}f_i\}_{i \in \mathbb{N}}$ is bounded as well and according to Proposition 4.1 we must prove only that

(4.31) $\lim_{i \to \infty} D_\alpha(L_{N,\ell}f_i(x)) = 0$ for all $\alpha \in \mathbb{Z}_+^n$, $0 \leq |\alpha| \leq k$, $x \in \mathbb{R}^n$.

Further, since $D_\alpha(L_{N,\ell}f_i(x)) = (E_ND_\alpha(f_i))(x)$ for such $\alpha$ and $x$, $\{D_\alpha(f_i)\}_{i \in \mathbb{N}}$ is a bounded sequence of continuous functions and $E_N$ is the convolution operator with the absolutely integrable kernel, to establish (4.31) it suffices to prove (due to the Lebesgue dominated convergence theorem) that

\[
\lim_{i \to \infty} D_\alpha(f_i)(x) = 0 \quad \text{for all} \quad \alpha \in \mathbb{Z}_+^n, \ 0 \leq |\alpha| \leq k, \ x \in \mathbb{R}^n.
\]

The latter follows directly from the definition of $(f_i)_\ell$, see (4.21), the general Leibniz rule and the fact that $D_\alpha f_i(x) \rightarrow 0$ as $i \to \infty$ for all the required $\alpha$ and $x$ (because $\{f_i\}_{i \in \mathbb{N}}$ weak* converges to 0). Thus we have proved that operators $L_{N,\ell}$ are weak* continuous. 

Lemma 4.4 implies that there exists a bounded operator of finite rank $H_{N,\ell}$ on $C^k_\omega(\mathbb{R}^n)$ whose adjoint $H_{N,\ell}^*$ coincides with $L_{N,\ell}$. 

Lemma 4.5. The sequence of finite rank bounded operators \( \{H_{N,N}\}_{N \in \mathbb{N}} \) converges pointwise to the identity operator on \( G_b^{k,\omega}(\mathbb{R}^n) \).

Proof. Let \( g \in G_b^{k,\omega}(\mathbb{R}^n) \). Due to Corollary \(^{23}\) given \( \varepsilon > 0 \) there exist \( J \in \mathbb{N} \) and families 
\[
c_j \in \mathbb{R}, \quad x_j \in \mathbb{R}^n, \quad 1 \leq j \leq J, \quad \alpha \in \mathbb{Z}_+^n, \quad 0 \leq |\alpha| < k, \quad \text{and} \quad d_{j\alpha} \in \mathbb{R}, \quad x_{j\alpha}, y_{j\alpha} \in \mathbb{R}^n, \quad x_{j\alpha} \neq y_{j\alpha}, \quad 1 \leq j \leq J, \quad \alpha \in \mathbb{Z}_+^n, \quad |\alpha| = k,
\]

such that
\[
g = \sum_{j,\alpha} c_{j\alpha} \delta^\alpha_{x_j} + \sum_{j,\alpha} d_{j\alpha} \frac{\delta^\alpha_{x_j} - \delta^\alpha_{y_j}}{\omega(\|x_{j\alpha} - y_{j\alpha}\|)} + g'' =: g' + g'',
\]

where
\[
\sum_{j,\alpha} |c_{j\alpha}| + \sum_{j,\alpha} |d_{j\alpha}| \leq \|g\|_{G_b^{k,\omega}(\mathbb{R}^n)} \quad \text{and} \quad \|g''\|_{G_b^{k,\omega}(\mathbb{R}^n)} \leq \frac{\varepsilon}{2(C_N + 1)},
\]

see Lemma 4.2 for the definition of \( C_N \).

Next, for each \( f \in C_b^{k,\omega}(\mathbb{R}^n) \), \( \|f\|_{C_b^{k,\omega}(\mathbb{R}^n)} = 1 \), we have by means of Lemma 4.3
\[
|f(H_{NN} g - g)| = \left| (L_{NN} f - f)(g) \right| \leq \left| (L_{NN} f - f)(g') \right| + \left| (L_{NN} f - f)(g'') \right|
\]
\[
< \left| (E_N f - f)(g') \right| + \|L_{NN} - \text{id}\| \cdot \frac{\varepsilon}{2(C_N + 1)} \leq \left| (E_N f - f)(g') \right| + \frac{\varepsilon}{2}.
\]

Let \( N_0 \in \mathbb{N} \) be so large that all points \( x_{j\alpha}, y_{j\alpha} \) as above belong to \( \mathbb{K}_{N_0} \). Since \( f_{N_0} = f \) on \( \mathbb{K}_{N_0} \), for all \( N \geq N_0 \),
\[
(E_N f - f)(g') = (E_N f - f)(g').
\]

Hence, due to Lemma \(^{4,3}(c)\) for \( z = x_{j\alpha} \) or \( y_{j\alpha} \),
\[
\left| (E_N f - f)(\delta^\alpha_z) \right| \leq \|E_N f - f\|_{C_b^k(\mathbb{R}^n)} \leq c_N.
\]

This implies that for all \( N \geq N_0 \)
\[
\left| (E_N f - f)(g') \right| \leq c_N \cdot \left( \sum_{j,\alpha} |c_{j\alpha}| + \sum_{j,\alpha} 2|d_{j\alpha}| \right) \cdot \max_{j,\alpha} \left\{ \frac{1}{\omega(\|x_{j\alpha} - y_{j\alpha}\|)} \right\}.
\]

Choose \( N_0' \geq N_0 \) so large that for all \( N \geq N_0' \) the right-hand side of the previous inequality is less than \( \frac{\varepsilon}{2} \). Then combining this with (4.32) we get for all \( N \geq N_0' \),
\[
\|H_{NN} g - g\|_{G_b^{k,\omega}(\mathbb{R}^n)} < \varepsilon.
\]

This shows that for all \( g \in G_b^{k,\omega}(\mathbb{R}^n) \)
\[
\lim_{N \to \infty} H_{NN} g = g
\]

which completes the proof of the lemma. \( \square \)

Let us finish the proof of the theorem for \( S = \mathbb{R}^n \). We set
\[
T_N := \left( 1 + c_{k,n} \cdot 4\sqrt{n} \cdot (k + 1) \cdot \lim_{t \to \infty} \frac{1}{\omega(t)} \right) \cdot \frac{H_{NN}}{C_N},
\]

(4.33)
see Lemma 4.2 for the definition of $C_N$. Since $\{C_N\}_{N \in \mathbb{N}}$ converges to the first factor in the definition of $T_N$, due to Lemma 4.5 $\{T_N\}_{N \in \mathbb{N}}$ is the sequence of operators of finite rank on $G_b^{k,\omega}(\mathbb{R}^n)$ of norm at most $\lambda := 1 + c_{k,n} \cdot 4\sqrt{n} \cdot (k + 1) \cdot \lim_{t \to \infty} (1/\omega(t))$ converging pointwise to the identity operator. In particular, this sequence converges uniformly to the identity operator on each compact subset of $G_b^{k,\omega}(\mathbb{R}^n)$. This shows that $G_b^{k,\omega}(\mathbb{R}^n)$ has the $\lambda$-approximation property with respect to the approximating sequence of operators $\{T_N\}_{N \in \mathbb{N}}$.

The proof of Theorem 1.8 for $S = \mathbb{R}^n$ is complete. $\square$

4.4. Proof of Theorem 1.8 (2).

Proof. In the case of $G_b^{k,\omega}(S)$, the required sequence of finite rank linear operators approximating the identity map is $\{PT_N|_{G_b^{k,\omega}(S)}\}_{N \in \mathbb{N}}$, where $T_N$ are linear operators defined by (4.33) and $P : G_b^{k,\omega}(\mathbb{R}^n) \to G_b^{k,\omega}(S)$ is the projection of Theorem 1.6. We have

$$\|PT_N|_{G_b^{k,\omega}(S)}\| \leq \|P\| \cdot \|T_N\| =: \|P\| \cdot \lambda(k,n,\omega).$$

Choosing here $P$ corresponding to the extension operators of papers [GI] ($k = 0$), [BS2] ($k = 1$) and [La] ($k \geq 2$) we obtain the required result.

The proof of Theorem 1.8 is complete. $\square$

4.5. Proof of Theorem 1.10.

Proof. Due to the result of Pełczyński [P] there are a separable Banach space $Y$ with a norm one monotone basis $\{b_j\}_{j \in \mathbb{N}}$, an isomorphic embedding $T : X \to Y$ with distortion $\|T\| \cdot \|T^{-1}\| \leq 4\lambda$, and a linear projection $P : Y \to T(X)$ with $\|P\| \leq 4\lambda$. For an operator $H \in \mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n); X)$ we define

$$\tilde{H} := T \cdot H \in \mathcal{L}(G_b^{k,\omega}(\mathbb{R}^n); Y).$$

Then for each $x \in \mathbb{R}^n$,

$$\tilde{H}(\delta_x^0) = \sum_{j=1}^{\infty} \tilde{h}_j(x) \cdot b_j$$

for some $\tilde{h}_j(x) \in \mathbb{R}$, $j \in \mathbb{N}$.

Further, consider the family of bounded linear functionals $\{b_j^*\}_{j \in \mathbb{N}} \subset Y^*$ such that $b_j^*(b_i) = \delta_{ij}$ (the Kronecker delta) for all $i,j \in \mathbb{N}$. As the basis $\{b_j\}_{j \in \mathbb{N}}$ is monotone, $\|b_j^*\| \leq 2$ for all $j \in \mathbb{N}$. Since $b_j^* \circ \tilde{H} \in (G_b^{k,\omega}(\mathbb{R}^n))^* = C_b^{k,\omega}(\mathbb{R}^n)$, the functions $\tilde{h}_j$, $\tilde{h}_j(x) := (b_j^* \circ \tilde{H})(\delta_x^0)$, $x \in \mathbb{R}^n$, belong to $C_b^{k,\omega}(\mathbb{R}^n)$ and

$$\|\tilde{h}_j\|_{C_b^{k,\omega}(\mathbb{R}^n)} \leq 2 \cdot \|T\| \cdot \|H\| \text{ for all } j \in \mathbb{N}.\tag{4.34}$$
In particular, \((b_j^* \circ \tilde{H})(\delta^\alpha_x) = D^\alpha (b_j^* \circ \tilde{H})(\delta^0_y) = D^\alpha \tilde{h}_j(x)\) for all \(\alpha \in \mathbb{Z}^n_+, |\alpha| \leq k, x \in \mathbb{R}^n, j \in \mathbb{N}\). This implies that for all such \(\alpha\) and \(x\),

\[
\tilde{H}(\delta^\alpha_x) = \sum_{j=1}^{\infty} D^\alpha \tilde{h}_j(x) \cdot b_j.
\]

Next, since the range of \(\tilde{H}\) is the subset of \(T(X)\),

\[H = T^{-1} \cdot P \cdot \tilde{H}.
\]

From here and (4.35) we obtain, for all \(\alpha \in \mathbb{Z}^n_+, |\alpha| \leq k, x \in \mathbb{R}^n\),

\[
H(\delta^\alpha_x) = \sum_{j=1}^{\infty} D^\alpha \tilde{h}_j(x) \cdot (T^{-1} \cdot P)(b_j)
\]

(convergence in \(X\)). Finally, we set

\[
h_j := \|(T^{-1} \cdot P)(b_j)\| \cdot \tilde{h} \quad \text{and} \quad v_j := \frac{(T^{-1} \cdot P)(b_j)}{\|(T^{-1} \cdot P)(b_j)\|}, \quad j \in \mathbb{N}.
\]

Then all \(v_j \in X\) are of norm one. In turn, all \(h_j \in C_b^{k,\omega}(\mathbb{R}^n)\) and due to (4.34), (4.37) and the properties of \(T\) and \(P\) for all \(j \in \mathbb{N}\),

\[
\|h_j\|_{C_b^{k,\omega}(\mathbb{R}^n)} \leq \|T^{-1}\| \cdot \|P\| \cdot \|\tilde{h}\|_{C_b^{k,\omega}(\mathbb{R}^n)} \leq 2 \cdot \|T\| \cdot \|T^{-1}\| \cdot \|P\| \cdot \|H\| \leq 32 \cdot \lambda^2 \cdot \|H\|.
\]

Moreover, by (4.36), for all \(\alpha \in \mathbb{Z}^n_+, |\alpha| \leq k, x \in \mathbb{R}^n\),

\[
H(\delta^\alpha_x) = \sum_{j=1}^{\infty} D^\alpha h_j(x) \cdot v_j,
\]

as required.

The proof of Theorem 1.10 is complete. \(\square\)

5. PROOFS OF THEOREM 1.12 AND COROLLARY 1.13

5.1. Proof of Theorem 1.12 (1).

Proof. Let \(\Lambda_{n,k} := \{\alpha \in \mathbb{Z}^n_+ : |\alpha| \leq k\}\). We set

\[
M_{n,k} := (\Lambda_{n,k} \times \mathbb{R}^n) \cup (\{\Lambda_{n,k} \setminus \Lambda_{n,k-1}\} \times (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n),
\]

where \(\Delta_n := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}\).

Space \(M_{n,k}\) has the natural structure of a \(C^\infty\) manifold, in particular, it is a locally compact Hausdorff space. By \(C_b(M_{n,k})\) we denote the Banach space of bounded continuous functions on \(M_{n,k}\) equipped with supremum norm. Let us define a linear map \(I : C_b^{k,\omega}(\mathbb{R}^n) \to C_b(M_{n,k})\) by the formula
(5.39) \[ I(f)(m) = \begin{cases} D^\alpha f(x) & \text{if } m = (\alpha, x) \in \Lambda_{n,k} \times \mathbb{R}^n \\ \frac{D^\alpha f(x) - D^\alpha f(y)}{\omega(\|x - y\|)} & \text{if } m = (\alpha, (x, y)) \in (\Lambda_{n,k} \setminus \Lambda_{n,k-1}) \times (\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta_n), \end{cases} \]

**Proposition 5.1.** \( I \) is a linear isometric embedding.

**Proof.** The statement follows straightforwardly from the definitions of the involved spaces. \( \square \)

Since \( I(C_b^k(\mathbb{R}^n)) \) is a closed subspace of \( C_b(M_{n,k}) \), the Hahn-Banach theorem implies that the adjoint map

(5.40) \[ I^* : (C_b(M_{n,k}))^* \to (C_b^k(\mathbb{R}^n))^* \]

of \( I \) is surjective of norm one.

Similarly, \( I \) maps \( C_b^k(\mathbb{R}^n) \) isometrically into the Banach subspace \( C_b(M_{n,k}) \subset C_b(M_{n,k}) \) of continuous functions on \( M_{n,k} \) vanishing at infinity. Thus the adjoint of \( I_0 := I|_{C_b^k(\mathbb{R}^n)} \) is the surjective map of norm one

(5.41) \[ I_0^* : (C_0(M_{n,k}))^* \to (C_0^k(\mathbb{R}^n))^*. \]

According to the Riesz representation theorem (see, e.g., [DS]), \( (C_0(M_{n,k}))^* \) is isometrically isomorphic to the space of countably additive regular Borel measures on \( M_{n,k} \) with the norm being the total variation of measure. In what follows we identify these two spaces.

In the proof we use the following result.

**Proposition 5.2.** If \( \omega \) satisfies condition (1.10), then \( C_b^k(\mathbb{R}^n) \) is weak* dense in \( C_b^k(\mathbb{R}^n) \).

**Proof.** Let \( \{L_{NN}\}_{N \in \mathbb{N}} \) be finite rank bounded linear operators on \( C_b^k(\mathbb{R}^n) \) defined by (4.30). According to Lemma 4.5, for each \( f \in C_b^k(\mathbb{R}^n) \) the sequence \( \{L_{NN}f\}_{N \in \mathbb{N}} \) weak* converges to \( f \). Moreover, each \( L_{NN}f \in C^\infty(\mathbb{R}^n) \), cf. Lemma 4.1(a). We set

(5.42) \[ \hat{f}_N := \rho_N \cdot L_{NN}f, \]

see section 4.3. Then \( \hat{f}_N \) is a \( C^\infty \) function with compact support on \( \mathbb{R}^n \) satisfying, due to Lemmas 4.2 and 4.3(b), the inequality

(5.43) \[ \|\hat{f}_N\|_{C_b^k(\mathbb{R}^n)} \leq C_N^2 \|f\|_{C_b^k(\mathbb{R}^n)}, \]

where \( \lim_{N \to \infty} C_N = 1 + c_{k,n} \cdot 4\sqrt{n} \cdot (k + 1) \cdot \lim_{t \to \infty} \frac{1}{\omega(t)} \).

Clearly, sequence \( \{D^\alpha \hat{f}_N\}_{N \in \mathbb{N}} \) converges pointwise to \( D^\alpha f \) for all \( f \in C_b^k(\mathbb{R}^n) \) and all \( \alpha \in \mathbb{Z}_+^n, |\alpha| \leq k \). Also, due to condition (1.10) all \( \hat{f}_N \in C_b^k(\mathbb{R}^n) \). Hence, according to Proposition 4.4, sequence \( \{\hat{f}_N\}_{N \in \mathbb{N}} \) weak* converges to \( f \). This shows that \( C_b^k(\mathbb{R}^n) \) is weak* dense in \( C_b^k(\mathbb{R}^n) \). \( \square \)

Next, let \( i^* : (C_b^k(\mathbb{R}^n))^* \to (C_0^k(\mathbb{R}^n))^* \) be the linear surjective map of norm one

adjoint to the isometrical embedding \( i : C_0^k(\mathbb{R}^n) \hookrightarrow C_b^k(\mathbb{R}^n). \)
Corollary 5.3. Restriction of \( i^* \) to \( G_b^{k,\omega}(\mathbb{R}^n) \) is injective.

Proof. Proposition 5.2 implies that functions in \( C_0^{k,\omega}(\mathbb{R}^n) \) regarded as linear functionals on \( G_b^{k,\omega}(\mathbb{R}^n) \) separate the points of \( G_b^{k,\omega}(\mathbb{R}^n) \) (see (5.45)). If \( i^*(v) = 0 \) for some \( v \in G_b^{k,\omega}(\mathbb{R}^n) \), then

\[
0 = (i^*(v))(f) = f(v) \quad \text{for all} \quad f \in C_0^{k,\omega}(\mathbb{R}^n).
\]

Hence, \( v = 0 \). \( \square \)

We set

\[
\tilde{\delta}_x^\alpha := i^*(\delta_x^\alpha), \quad |\alpha| \leq k, \quad x \in \mathbb{R}^n.
\]

By definition, maps \( \phi_\alpha : \mathbb{R}^n \to (C_b^{k,\omega}(\mathbb{R}^n))^* \), \( x \mapsto \tilde{\delta}_x^\alpha \), \( |\alpha| \leq k \), are continuous and bounded and so are the maps \( i^* \circ \phi_\alpha : \mathbb{R}^n \to (C_0^{k,\omega}(\mathbb{R}^n))^* \).

Proposition 5.4. The range of \( I_0^* \) coincides with \( i^*(G_b^{k,\omega}(\mathbb{R}^n)) \).

Proof. Let \( \mu \in (C_0(M_{n,k}))^* \) be a countably additive regular Borel measures on \( M_{n,k} \). We set, for all admissible \( \alpha \) and all Borel measurable sets \( U \subset M_{n,k} \),

\[
\mu_1^\alpha(U) = \mu(U \cap \{\alpha\} \times \mathbb{R}^n) \quad \text{and} \quad \mu_2^\alpha(U) = \mu(U \cap \{\alpha\} \times ((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n)).
\]

Then \( \mu = \sum_{\alpha,j} \mu_j^\alpha \).

Let us show that each \( I_0^*(\mu_j^\alpha) \) belongs to \( i^*(G_b^{k,\omega}(\mathbb{R}^n)) \). Indeed, for \( j = 1 \) consider the Bochner integral

\[
J(\mu_1^\alpha) := \int_{x \in \mathbb{R}^n} i^*(\phi_\alpha(x)) \, d\mu_1^\alpha(x) = i^* \left( \int_{x \in \mathbb{R}^n} \phi_\alpha(x) \, d\mu_1^\alpha(x) \right).
\]

Since \( \phi_\alpha \) is continuous and bounded, the above integral is well-defined and its value is an element of \( i^*(G_b^{k,\omega}(\mathbb{R}^n)) \). By the definition of the Bochner integral, for each \( f \in C_0^{k,\omega}(\mathbb{R}^n) \),

\[
(J(\mu_1^\alpha))(f) = \int_{x \in \mathbb{R}^n} i^*(\phi_\alpha(x))(f) \, d\mu_1^\alpha(x) = \int_{x \in \mathbb{R}^n} D^\alpha f(x) \, d\mu_1^\alpha(x) =: (I_0^*(\mu_1^\alpha))(f).
\]

Hence,

\[
I_0^*(\mu_1^\alpha) = J(\mu_1^\alpha) \in i^*(G_b^{k,\omega}(\mathbb{R}^n)).
\]

Similarly, for \( \alpha \in \Lambda_{n,k} \setminus \Lambda_{n,k-1} \) we define

\[
J(\mu_2^\alpha) := \int_{z = (x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n} i^*(\phi_\alpha(x) - i^*(\phi_\alpha(y))) \, \omega(||x - y||) \, d\mu_2^\alpha(z).
\]

Then, as before, we obtain that

\[
I_0^*(\mu_j^\alpha) = J(\mu_j^\alpha) \in i^*(G_b^{k,\omega}(\mathbb{R}^n)).
\]

Thus we have established that the range of \( I_0^* \) is a subset of \( i^*(G_b^{k,\omega}(\mathbb{R}^n)) \). Since the map \( I_0^* \) is surjective and its range contains all \( \tilde{\delta}_x^\alpha \), it must contain \( i^*(G_b^{k,\omega}(\mathbb{R}^n)) \) as well.

This completes the proof of the proposition. \( \square \)
In particular, we obtain that \((C^k_0(\mathbb{R}^n))^* = i^*(G^k_0(\mathbb{R}^n))\), i.e., by the inverse mapping theorem \(i^*\) restricted to \(G^k_0\) maps it isomorphically onto \((C^k_0(\mathbb{R}^n))^*\).

Let us show that if \(\lim_{t \to \infty} \omega(t) = \infty\), then \(i^*\) is an isometry. Assume, on the contrary, that for some \(v \in G^k_0(\mathbb{R}^n)\),

\[
(5.46) \quad \|i^*(v)\|_{(C^k_0(\mathbb{R}^n))^*} < \|v\|_{G^k_0(\mathbb{R}^n)}.
\]

Let \(f \in C^k_0(\mathbb{R}^n)\), \(\|f\|_{C^k_0(\mathbb{R}^n)} = 1\), be such that

\[
v(f) = \|v\|_{G^k_0(\mathbb{R}^n)}.
\]

Let \(\{f_N\}_{N \in \mathbb{N}} \subset C^k_0(\mathbb{R}^n)\), \(\|f_N\|_{C^k_0(\mathbb{R}^n)} \leq C^*_N, N \in \mathbb{N}\), be the sequence of Proposition 5.2 weak* converging to \(f\). Observe that \(\lim_{N \to \infty} C_N = 1\) due to the above condition for \(\omega\).

Then from (5.46) and (5.43) we obtain

\[
\|v\|_{G^k_0(\mathbb{R}^n)} = v(f) = \lim_{N \to \infty} v(f_N) = \lim_{N \to \infty} (i^*(v))(f_N)
\]

\[
\leq \lim_{N \to \infty} (i^*(v))(f_N) = \lim_{N \to \infty} \|f\|_{G^k_0(\mathbb{R}^n)} \leq \|i^*(v)\|_{(C^k_0(\mathbb{R}^n))^*} < \|v\|_{G^k_0(\mathbb{R}^n)},
\]

a contradiction proving that \(i^*\) is an isometry.

The proof of Theorem 1.12 (1) is complete. \(\square\)

5.2. Proof of Theorem 1.12 (2).

Proof. By the hypotheses of the theorem there exists a weak* continuous operator \(T \in \text{Ext}(C^k_0(S);C^k_0(\mathbb{R}^n))\) such that \(T(C^k_0(S)) \subset C^k_0(\mathbb{R}^n)\). This implies that there is a bounded linear projection of the geometric preduals of the corresponding spaces \(P : G^k_0(\mathbb{R}^n) \to C^k_0(S)\) such that \(P^* = T\). Let \(q_S : G^k_0(\mathbb{R}^n) \to C^k_0(S)\) and \(q_S0 : C^k_0(\mathbb{R}^n) \to C^k_0(S)\) denote the quotient maps induced by restrictions of functions on \(\mathbb{R}^n\) to \(S\). Finally, let \(i : C^k_0(\mathbb{R}^n) \to C^k_0(\mathbb{R}^n)\) and \(i_S : C^k_0(S) \to C^k_0(S)\) be the bounded linear maps corresponding to inclusions of the spaces. Note that \(i\) is an isometric embedding and \(i_S\) is injective of norm \(\leq 1\).

Lemma 5.5. \(T_0 := T| C^k_0(S) : C^k_0(S) \to C^k_0(\mathbb{R}^n)\) is a bounded linear map between Banach spaces.

Proof. For \(f \in C^k_0(S)\) we have

\[
\|T_0f\|_{C^k_0(\mathbb{R}^n)} = \|(T \circ i_S)(f)\|_{C^k_0(\mathbb{R}^n)} \leq \|T\| \cdot \|i_S\|_{C^k_0(S)} \leq \|T\| \cdot \|f\|_{C^k_0(S)},
\]

as required. \(\square\)

Now, we have the following two commutative diagrams of adjoints of the above bounded linear maps (one corresponding to upward arrows and another one to downward arrows):

\[
(5.47) \quad \begin{array}{cccc}
(C^k_0(\mathbb{R}^n))^* & \xrightarrow{i^*} & (C^k_0(\mathbb{R}^n))^* \\
q_S^* \uparrow & & \downarrow T^* & \downarrow q_S^*
\end{array}
\]

\[
(C^k_0(S))^* \xrightarrow{i_S^*} (C^k_0(S))^*.
\]
Here $T^* \circ q_S^* = (q_S \circ T)^*$ is id and $T_n^* \circ q_S^* = (q_S \circ T_0)^*$ is id, maps $q_S^*$ and $q_S^*$ are isometric embeddings and map $i^*$ is surjective.

Note that $i^*|_{G_{b}^{k,\omega}(\mathbb{R}^n)} : G_{b}^{k,\omega}(\mathbb{R}^n) \rightarrow (C_0^{k,\omega}(\mathbb{R}^n))^*$ is an isomorphism by the first part of the theorem. Also, by the definition of $P$ (see (3.15) in section 3.3 above),

$$q_S^* \circ (T^*|_{G_{b}^{k,\omega}(\mathbb{R}^n)}) = P.$$ \hfill (5.48)

Let us show that the map

$$I := i_S^* \circ (T^*|_{G_{b}^{k,\omega}(S)}) : G_{b}^{k,\omega}(S) \rightarrow (C_0^{k,\omega}(S))^*$$

is an isomorphism.

(a) Injectivity of $I$: If $I(v) = 0$ for some $v \in G_{b}^{k,\omega}(S)$, then by the commutativity of (5.47) and by (5.48),

$$0 = (q_S^* \circ i_S^*)(T^*v) = (i^* \circ q_S^*)(T^*v) = i^*(Pv) = i^*(v).$$

Since $i^*$ is injective, the latter implies that $v = 0$, i.e., $I$ is an injection.

(b) Surjectivity of $I$: Let $v \in (C_0^{k,\omega}(S))^*$. Since $T_0^*$ is surjective, there exists $v_1 \in (C_0^{k,\omega}(\mathbb{R}^n))^*$ such that $T_0^*(v_1) = v$. Further, since $i^*|_{G_{b}^{k,\omega}(\mathbb{R}^n)} : G_{b}^{k,\omega}(\mathbb{R}^n) \rightarrow (C_0^{k,\omega}(\mathbb{R}^n))^*$ is an isomorphism, there exists $v_2 \in G_{b}^{k,\omega}(\mathbb{R}^n)$ such that $i^*(v_2) = v_1$. Now, by the commutativity of (5.47),

$$v = (T_0^* \circ i^*)(v_2) = (i_S^* \circ T^*)(v_2) = (i_S^* \circ (T^* \circ q_S^*) \circ T^*)(v_2) = (i_S^* \circ T^*)(Pv_2) = I(Pv_2),$$

i.e., $I$ is a surjection.

So $I$ is a bijection and therefore by the inverse mapping theorem it is an isomorphism.

This completes the proof of the second part of Theorem 1.12. \hfill \square

5.3. \textbf{Proof of Corollary 1.13}.

\textit{Proof.} Let $X \subset C_0^{k,\omega}(\mathbb{R}^n)$ be the closure of the space of $C^\infty$ functions with compact supports on $\mathbb{R}^n$. Assume, on the contrary, that there exists $f \in C_0^{k,\omega}(\mathbb{R}^n) \setminus X$. Then there exists a functional $\lambda \in (C_0^{k,\omega}(\mathbb{R}^n))^*$ such that $\lambda|_X = 0$ and $\lambda(f) = 1$.

Let $i^* : (C_0^{k,\omega}(\mathbb{R}^n))^* \rightarrow (C_0^{k,\omega}(\mathbb{R}^n))^*$ be the adjoint of the isometrical embedding $i : C_0^{k,\omega}(\mathbb{R}^n) \hookrightarrow C_{b}^{k,\omega}(\mathbb{R}^n)$. According to the arguments of the proof of Theorem 1.12, $i^*|_{G_b^{k,\omega}(\mathbb{R}^n)} : G_{b}^{k,\omega}(\mathbb{R}^n) \rightarrow (C_0^{k,\omega}(\mathbb{R}^n))^*$ is an isomorphism. Hence, for $\tilde{\lambda} := (i^*|_{G_b^{k,\omega}(\mathbb{R}^n)})^{-1}(\lambda)$ we have $g(\tilde{\lambda}) = 0$ for all $g \in X$ and $f(\tilde{\lambda}) = 1$. Observe that $X$ is weak* dense in $C_0^{k,\omega}(\mathbb{R}^n)$ (see the proof of Proposition 5.2). Thus $X$ separates the points of $G_{b}^{k,\omega}(\mathbb{R}^n)$. Since $g(\tilde{\lambda}) = 0$ for all $g \in X$, the latter implies that $\tilde{\lambda} = 0$, a contradiction with $f(\tilde{\lambda}) = 1$.

This shows that $X = C_0^{k,\omega}(\mathbb{R}^n)$.

Clearly, $X$ is separable (it contains, e.g., the dense countable set of functions of the form $p \cdot \rho_N$, $N \in \mathbb{N}$, where $p$ are polynomials with rational coefficients and $\{\rho_N\}_{N \in \mathbb{N}}$ is a fixed sequence of $C^\infty$ cut-off functions weak* converging in $C_{b}^{k,\omega}(\mathbb{R}^n)$ to the constant function $f = 1$).
This completes the proof of the corollary.

\[\square\]

6. Proof of Theorem 1.15

6.1. Proof of Theorem 1.15 for Weak $k$-Markov Sets. First, we recall some results proved in [BB1, BB2, B].

1. If $S \in \text{Mar}_k^*(\mathbb{R}^n)$, then a function $f \in C^k_b(\mathbb{R}^n)$ has derivatives of order $\leq k$ at each weak $k$-Markov point $x \in S$, i.e., there exists a (unique) polynomial $T^k_x(f) \in \mathcal{P}_{k,n}$ such that

\[
\lim_{y \to x} \frac{|f(y) - T^k_x(f)(y)|}{\|y - x\|^k} = 0.
\]

If $T^k_x(f)(z) := \sum_{|\alpha| \leq k} c_{\alpha} (z - x)^\alpha$, $\alpha \in \mathbb{Z}_+^n$, then $c_\alpha$ is called the partial derivative of order $|\alpha|$ at $x$ and is denoted as $D^k_x f(x)$.

2. If $\tilde{f} \in C^{k,\omega}_{\text{b}}(\mathbb{R}^n)$ is such that $\tilde{f}|_S = f$, then the Taylor polynomial $T^k_x(\tilde{f})$ of order $k$ of $\tilde{f}$ at $x$ coincides with $T^k_x(f)$.

3. The analog of the classical Whitney-Glaeser theorem holds:

A function $f \in C(S)$ belongs to $C^k_b(\mathbb{R}^n)$ if and only if it has derivatives of order $\leq k$ at each weak $k$-Markov point $x \in S$ and there exists a constant $\lambda > 0$ such that for all weak $k$-Markov points $x, y \in S$, $z \in \{x, y\}$

\[
\max_{|\alpha| \leq k} |D^k_y f(x)| \leq \lambda
\]

and

\[
(6.49) \quad \max_{|\alpha| \leq k} \frac{|D^k_y (T^k_x(f) - T^k_y(f))(z)|}{\|x - y\|^{k - |\alpha|}} \leq \lambda \cdot \omega(\|x - y\|).
\]

Moreover,

\[
\|f\|_{C^{k,\omega}(S)} \approx \inf \lambda
\]

with constants of equivalence depending only on $k$ and $n$.

4. There exists a bounded linear extension operator $T : C^k_b(\mathbb{R}^n) \to C^k_b(\mathbb{R}^n)$ of finite depths

\[
(Tf)(x) := \begin{cases}
\sum_{i=1}^{\infty} \lambda_i(x) f(x_i) & \text{if } x \in \mathbb{R}^n \setminus S \\
f(x) & \text{if } x \in S,
\end{cases}
\]

where all $\lambda_i \in C^\infty(\mathbb{R}^n)$ and have compact supports in $\mathbb{R}^n \setminus S$, all $x_i \in S$ and for each $x \in \mathbb{R}^n$ the number of nonzero terms in the above sum is at most $(n+k) \cdot w$, where $w$ is the order of the Whitney cover of $\mathbb{R}^n \setminus S$.

The construction of $T$ repeats that of the Whitney-Glaeser extension operator [G], where instead of jets $T^k_x(f)$ of $f \in C^k_b(\mathbb{R}^n)$ at weak $k$-Markov points $x \in S$ (forming a dense subset of $S$) one uses polynomials of degree $k$ interpolating $f$ on certain subsets of cardinality $(n+k)$ close to $x$. In particular, as in the case of the Whitney-Glaeser extension
operator, we obtain that $T \in Ext(C^{k,\omega}_{b}(S); C^{k,\omega}_{b}(\mathbb{R}^n))$ for all moduli of continuity $\omega$. Also, by the construction, if $f \in C^{k,\omega}_{b}(S)$ is the restriction to $S$ of a $C^\infty$ function with compact support on $\mathbb{R}^n$, then $Tf \in C^{k,\omega}_{b}(\mathbb{R}^n)$ has compact supports in all closed $\delta$-neighbourhoods of $S$ (i.e., sets $[S]_{\delta} := \{ x \in \mathbb{R}^n : \| x - y \| \leq \delta \}$, $\delta > 0$).

**Proof of Theorem 1.13** for $S = \text{Mar}_t^1(\mathbb{R}^n)$. Let $\rho \in C^\infty(\mathbb{R}^n)$, $0 \leq \rho \leq 1$, be such that $\rho|_S = 1$, $\rho|_{\mathbb{R}^n \setminus [S]} = 0$ and for some $C_{k,n} \in \mathbb{R}^+$ (depending on $k$ and $n$ only)

$$\sup_{x \in \mathbb{R}^n} |D^n \rho(x)| \leq C_{k,n} \quad \text{for all} \quad \alpha \in \mathbb{Z}^n_+.$$  

(E.g., such $\rho$ can be obtained by the convolution of the indicator function of $[S]_2$ with a fixed radial $C^\infty$ function with support in the unit Euclidean ball of $\mathbb{R}^n$ and with $L_1(\mathbb{R}^n)$ norm one.) We define a new extension operator by the formula

$$\tilde{T}f = \rho \cdot Tf, \quad f \in C^{k,\omega}_{b}(S).$$

**Lemma 6.1.** Operator $\tilde{T} \in Ext(C^{k,\omega}_{b}(S); C^{k,\omega}_{b}(\mathbb{R}^n))$ for all moduli of continuity $\omega$.

**Proof.** We equip $C^{k,\omega}_{b}(\mathbb{R}^n)$ with equivalent norm

$$\| f \|_{C^{k,\omega}_{b}(\mathbb{R}^n)} := \max \left\{ \| f \|_{C^{k,\omega}_{b}(\mathbb{R}^n)}, |f|'_{C^{k,\omega}_{b}(\mathbb{R}^n)} \right\},$$

where $|f|'_{C^{k,\omega}_{b}(\mathbb{R}^n)}$ is defined similarly to $|f|_{C^{k,\omega}_{b}(\mathbb{R}^n)}$ but with the supremum taken over all $x \neq y$ such that $\| x - y \| \leq 1$, see (1.12)–(1.4). (Note that the constants of equivalence between these two norms depend on $\omega$.) Now, using word-by-word the arguments of Lemma 4.2 with $\rho_\ell$ replaced by $\rho$, $\ell$ replaced by 1, and $c_{k,n}$ replaced by $C_{k,n}$ we obtain for some constant $C = C(k,n,\omega)$ and all $h \in C^{k,\omega}_{b}(\mathbb{R}^n)$,

$$\| \rho \cdot h \|_{C^{k,\omega}_{b}(\mathbb{R}^n)} \leq C \cdot \| h \|_{C^{k,\omega}_{b}(\mathbb{R}^n)}.$$  

Since $T \in Ext(C^{k,\omega}_{b}(S); C^{k,\omega}_{b}(\mathbb{R}^n))$ for all moduli of continuity $\omega$, inequality (6.52) implies the required statement. \hfill \Box

Clearly, $\tilde{T}$ is of finite depth. Moreover, if $f \in C^{k,\omega}_{b}(S)$ is the restriction to $S$ of a $C^\infty$ function with compact support on $\mathbb{R}^n$, then $\tilde{T}f \in C^{k,\omega}_{b}(\mathbb{R}^n)$ and has compact support on $\mathbb{R}^n$ due to the properties of operator $T$, (see part (4) above). Finally, since the set of all $C^{k,1}_{b}(S)$ functions (i.e., for this space $\omega(t) := t$, $t \in \mathbb{R}^+$) with compact supports on $S$ is dense in $C^{k,\omega}_{0}(S)$ (because $\omega$ satisfies condition (1.10), see Corollary 1.13, the preceding property of $\tilde{T}$ and Lemma 6.1 imply that $\tilde{T}(C^{k,\omega}_{0}(S)) \subset C^{k,\omega}_{0}(\mathbb{R}^n)$. Therefore $\tilde{T}$ satisfies the hypotheses of Theorem 1.12 (2) (weak* continuity of $\tilde{T}$ follows from Theorem 1.6). This implies the required statement: $(C^{k,\omega}_{0}(S))^*$ is isomorphic to $C^{k,\omega}_{b}(S)$ for all $\omega$ satisfying (1.10) and all weak $k$-Markov sets $S$.

Now, $G^{k,\omega}_{b}(S)$ has the metric approximation property due to the Grothendieck result [G, Ch.I] (formulated before Remark 1.9 of section 1.4 above) because this space has the
approximation property by Theorem 1.8. Also, $C^k_0(S)$ has the metric approximation property because its dual has it, see, e.g., [C, Th. 3.10].

The proof of the theorem for $S \in \text{Mar}_k^*(\mathbb{R}^n)$ is complete. □

6.2. Proof of Theorem 1.15 in the General Case.

Proof. We require some auxiliary results.

Let $\tilde{w}$ be the modulus of continuity satisfying

$$
\lim_{t \to 0^+} \frac{\omega_0(t)}{\tilde{w}(t)} < \infty.
$$

Lemma 6.2. The restriction of the pullback map $H^*: C^k_b(\mathbb{R}^n) \to C^k_b(\mathbb{R}^n)$, $H^* f := f \circ H$, to $C^k_b(\mathbb{R}^n)$ belongs to $\mathcal{L}(C^k_b(\mathbb{R}^n); C^k_b(\mathbb{R}^n))$.

Proof. We set $H = (h_1, \ldots, h_n)$. Then by the hypothesis (a) of the theorem all $D^j h_j \in C^{k-1,\omega_0}_{\text{loc}}(\mathbb{R}^n)$, where $\omega_0$ satisfies (1.11).

Let $f \in C^k_b(\mathbb{R}^n)$. Then for each $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq k$, by the Faà di Bruno formula, see, e.g., [CS], we obtain

$$
(D^\alpha (f \circ H)) (x) = \sum_{0 < |\beta| \leq |\alpha|} D^\beta f(H(x)) \cdot P_\lambda \left( [D^\beta H(x)]_{0 < |\beta| \leq |\alpha|} \right);
$$

where $P_\lambda \left( [D^\beta H(x)]_{0 < |\beta| \leq |\alpha|} \right)$ are polynomials of degrees $\leq |\alpha|$ without constant terms with coefficients in $\mathbb{Z}_+$ bounded by a constant depending on $k$ and $n$ only in variables $D^\beta h_j$, $0 < |\beta| \leq |\alpha|$, $1 \leq j \leq n$. Since clearly $C^{0,\omega_0}_b(\mathbb{R}^n)$ is a Banach algebra with respect to the pointwise multiplication of functions, to prove the lemma it suffices to check that all $D^\beta f(H(\cdot))$ and $D^\beta h_j$ belong to $C^{0,\omega_0}_b(\mathbb{R}^n)$. For $|\beta| = k$ this is true because $D^\beta h_j \in C^{0,\omega_0}_{\text{loc}}(\mathbb{R}^n) \subset C^{0,\omega_0}_b(\mathbb{R}^n)$ by the definition of $H$ and by condition (6.53), while for $1 < |\beta| \leq k - 1$ because $D^\beta h_j \in C^{0,1}_{\text{loc}}(\mathbb{R}^n)$ which is continuously embedded into $C^{0,\omega_0}_b(\mathbb{R}^n)$.

Similarly, for $D^\lambda f(H(\cdot))$ with $1 \leq |\alpha| \leq k - 1$ this is true because of the continuous embedding $C^{0,1}_{\text{loc}}(\mathbb{R}^n) \hookrightarrow C^{0,\omega_0}_b(\mathbb{R}^n)$ and because $H$ is Lipschitz, while for $|\lambda| = k$ by the definition of $f$ and the fact that $H$ is Lipschitz.

Using this lemma we prove the following result.

Lemma 6.3. The operator $(H|_{S'})^*: C^k_{\tilde{b}}(S) \to C^k_{\tilde{b}}(S')$, $(H|_{S'})^* f := f \circ H|_{S'}$, is well-defined and belongs to $\mathcal{L}(C^k_{\tilde{b}}(S); C^k_{\tilde{b}}(S'))$. Moreover, it is weak* continuous.\footnote{Here the weak* topologies are defined by means of functionals in $G^k_{\tilde{b}}(\tilde{S})$, where $\tilde{S}$ stands for $S'$ or $S$.}

Proof. Let $\tilde{f} \in C^k_{\tilde{b}}(\mathbb{R}^n)$ be such that $\tilde{f}|_{S} = f$ and $\|\tilde{f}\|_{C^k_{\tilde{b}}(\mathbb{R}^n)} = \|f\|_{C^k_{\tilde{b}}(S)}$. Then by Lemma 6.2 we have

$$
(H|_{S'})^* f = f \circ H|_{S'} = (\tilde{f} \circ H)|_{S'} = (H^* \tilde{f})|_{S'} \in C^k_{\tilde{b}}(S') \quad \text{and}
$$

$$
\|(H|_{S'})^* f\|_{C^k_{\tilde{b}}(S')} \leq \|H^*\| \cdot \|\tilde{f}\|_{C^k_{\tilde{b}}(\mathbb{R}^n)} = \|H^*\| \cdot \|f\|_{C^k_{\tilde{b}}(S)}.
$$
This shows that the operator \( (H|_{S'})^* : C^k_{b,\tilde{\omega}}(S) \to C^k_{b,\omega}(S') \) is well-defined and belongs to \( L(C^k_{b,\tilde{\omega}}(S); C^k_{b,\omega}(S')) \).

Further, the fact that the operator \( H^* : C^k_{b,\tilde{\omega}}(\mathbb{R}^n) \to C^k_{b,\omega}(\mathbb{R}^n) \) is weak* continuous follows straightforwardly from Proposition 4.1, Lemma 6.2 and the the Faà di Bruno formula \([\text{BB}1]\). Let \( T \in \text{Ext}(C^k_{b,\tilde{\omega}}(S); C^k_{b,\omega}(\mathbb{R}^n)) \) be the extension operator of finite depth (see section 1.3) and \( q_{S'} : C^k_{b,\tilde{\omega}}(\mathbb{R}^n) \to C^k_{b,\omega}(S') \) be the quotient map induced by restrictions of functions on \( \mathbb{R}^n \) to \( S' \). Then clearly, for all \( f \in C^k_{b,\tilde{\omega}}(S) \),

\[
(H|_{S'})^* f = f \circ H|_{S'} = ((T f) \circ H)|_{S'} = (q_{S'} \circ H^* \circ T) f.
\]

Therefore \((H|_{S'})^* = q_{S'} \circ H^* \circ T\). Here the operator \( T \) is weak* continuous by Theorem 1.6 and the operator \( q_{S'} \) is weak* continuous because it is adjoint of the isometric embedding \( G^k_{b,\omega}(S') \hookrightarrow G^k_{b,\tilde{\omega}}(\mathbb{R}^n) \). This implies that the operator \( (H|_{S'})^* \) is weak* continuous as well.

We are ready to prove Theorem 1.15

Let \( \tilde{T} \in \text{Ext}(C^k_{b,\tilde{\omega}}(S'); C^k_{b,\omega}(\mathbb{R}^n)) \) be the extension operator of the first part of Theorem 1.15 see \([\text{BB}1]\). We set (for \( \tilde{\omega} := \omega \))

\[
E := \tilde{T} \circ (H|_{S'}). \tag{6.55}
\]

Lemma 6.4. Operator \( E \in \text{Ext}(C^k_{b,\tilde{\omega}}(S); C^k_{b,\omega}(\mathbb{R}^n)) \), is weak* continuous and maps \( C^k_{0,\omega}(S) \) in \( C^k_{0,\omega}(S') \).

Proof. The first two statements follow from the hypotheses of the theorem, Lemma 6.3 and the fact that \( \tilde{T} \) is weak* continuous. So let us check the last statement.

Let \( f \in C^k_{0,\omega}(S) \) be the restriction of a \( C^\infty \) function with compact support on \( \mathbb{R}^n \). Since \( H|_{S'} : S' \to S \) is a proper map (by hypothesis (b) of the theorem), \( (H|_{S'})^* f \in C^k_{b,\tilde{\omega}}(S') \) has compact support. Moreover, since \( f \in C^k_{b,\tilde{\omega}}(S) \), Lemma 6.3 applied to \( \tilde{\omega} = \omega_o \) implies that \( (H|_{S'})^* f \in C^k_{b,\omega_o}(S') \). Finally, since \( \tilde{T} \in \text{Ext}(C^k_{b,\omega_o}(S); C^k_{b,\omega_o}(\mathbb{R}^n)) \) as well, \( E f \in C^k_{b,\omega_o}(\mathbb{R}^n) \) and has compact support (because \( (H|_{S'})^* f \) has it). Due to condition \([\text{BB}2]\) for \( \omega_o \) we obtain from here that \( E f \in C^k_{0,\omega}(\mathbb{R}^n) \). Since the set of such functions \( f \) is dense in \( C^k_{0,\omega}(S) \) (see Corollary 1.13), \( E \) maps \( C^k_{0,\omega}(S) \) in \( C^k_{0,\omega}(\mathbb{R}^n) \), as required. \( \square \)

Now the result of the theorem follows from Lemma 6.4 and Theorem 1.12 (2); that is, \( G^k_{b,\omega}(S) \) is isomorphic to \( (C^k_{0,\omega}(S))^* \) and so \( G^k_{b,\omega}(S) \) and \( C^k_{0,\omega}(S) \) have the metric approximation property (see the argument at the end of section 6.1 above).

The proof of the theorem is complete. \( \square \)

References

[BB1] A. Brudnyi and Yu. Brudnyi, Traces of functions of the class \( C^k \) on weakly Markov subsets of \( \mathbb{R}^n \), St. Petersbg Math. J. 23 (2012), no. 1, 39–56.

[BB2] A. Brudnyi and Yu. Brudnyi, Methods of Geometric Analysis in extension and trace problems, Volume II, Monographs in Mathematics, Vol. 102, 103, Springer, Basel, 2012.
ON PROPERTIES OF GEOMETRIC PREDUALS OF $C^{k,\omega}$ SPACES

[B] A. Brudnyi, Differential calculus on topological spaces with weak Markov structure I, Rev. Mat. Iberoam. 31 (2015), no. 2, 531–574.

[BM] E. Bierstone and P. Milman, $\mathcal{C}^m$ norms of finite sets and $\mathcal{C}^m$ extension criteria, Duke Math. J. 137 (2007), 118–134.

[BS1] Y. Brudnyi and P. Shvartsman, Generalizations of Whitney’s extension theorem, Int. Math. Research Notices 3 (1994), 129–139.

[BS2] Y. Brudnyi and P. Shvartsman, The Whitney problem of existence of a linear extension operator, J. Geom. Anal. 7 (4) (1997) 515–574.

[C] P. G. Casazza, Approximation properties, In: W.B. Johnson and J. Lindenstrauss (eds.), Handbook of the geometry of Banach spaces, Vol. 1 (2001), 271–316.

[CS] G. M. Constantine and T. H. Savits, A multivariate Faà di Bruno formula with applications, Trans. Amer. Math. Soc. 348, No. 2 (1996), 503–520.

[DF] A. Defant and K. Floret, Tensor norms and operator ideals, Mathematics Studies 176, 1993.

[DS] N. Dunford and J. T. Schwartz, Linear operators: General theory, Interscience Publishers, 1958.

[E] P. Enflo, A counterexample to the approximation property in Banach spaces, Acta Math. 130 (1973), 309–317.

[F1] C. Fefferman, A generalized sharp Whitney theorem for jets, Revista Matematica Iberoamericana 21, No. 2 (2005), 577–688.

[F2] C. Fefferman, Extension of $C^{m,\omega}$-smooth functions by linear operators, Rev. Mat. Iberoamericana 25 (1) (2009) 1–48.

[F3] C. Fefferman, Whitney’s extension problems and interpolation of data, Bull. Amer. Math. Soc. 46 (2) (2009), 207–220.

[FJ] T. Figiel and W. B. Johnson, The approximation property does not imply the bounded approximation property, Proc. Amer. Math. Soc. 41 (1) (1973), 197–199.

[G] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16, 1955.

[Gl] G. Glaeser, Étude de quelques algèbres Tayloriennes, J. d’Analyse Math. 6 (1958), 1–125.

[K] N. J. Kalton, Spaces of Lipschitz and Holder functions and their applications, Collect. Math. 55 (2004), 171–217.

[L] J. Lindenstrauss, Some open problems in Banach space theory, Séminaire Choquet 18 (1975), 1–9.

[Lu] G. K. Luli, $C^{m,\omega}$ extension by bounded-depth linear operators, Adv. Math. 224 (2010), 1927–2021.

[M] J. Merrien, Prolongateurs de fonctions différentiables d’une variable réelle, J. Math. Pures Appl. (9) 45 (1966), 291–309.

[McS] E. McShane, Extension of range of function, Bull. Amer. Math. Soc. 40 (1934), 837–842.

[P] A. Pelczyński, Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis, Studia Math. 40 (1971), 239–243.

[S] P. Shvartsman, The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces, Trans. Amer. Math. Soc. 360 (2008), 5529–5550.

[T] A. F. Timan, Theory of approximation of functions of a real variable, Dover Publications, New York, 1994.

[W1] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89.

[W2] H. Whitney, Differentiable functions defined in closed sets. I, Trans. Amer. Math. Soc. 36 (1934), no. 2, 369–387.

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