Dispersion relations of strained as well as complex Lieb lattices

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(Dated: January 6, 2015)

Abstract

We investigate the dispersion relations of strained as well as complex Lieb lattices systematically based on the tight-binding method when the nearest-neighbor approximation is adopted. We find that edge states will no appear for strained Lieb lattices and $\mathcal{PT}$-symmetry Lieb lattice cannot be obtained.

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I. INTRODUCTION

Recently, edge states of different kind of lattices with certain boundaries attract a lot of attention from all over the world. Among the lattices, honeycomb lattices [1–6] and Lieb lattices [7–12] are quite involved. Photonic topological insulators based on the two kinds of lattices have been also reported [13–15]. The reason that such kind of lattices can exhibit exotic optical properties, is because there exist Dirac cones among bands in momentum space. Besides, edge states will lie in the bulk band gap and spatially localize on the boundaries of the strained lattices [1, 12, 16, 17], which is robust against defects due to the topological protection.

A Lieb lattice, which is a face-centered square lattice, has three lattice sites in each unit cell. As reported previously [7–10], it displays a triply denegerate diabolical point at which two conical bands and a flat band intersect. The flat band is a non-dispersive band and topological trivial [17], which corresponds to totally degenerate eigen-states. As the states are degenerate, they are strongly correlated [18], and lead to enhanced light-matter interaction [19]. In addition to Lieb lattices, flat bands are also reported in honeycomb lattices with inhomogeneous strains [20] or polarized momschromatic light irradiations [5], kagome lattices [21, 22], square lattices [18], $T_3$ optical lattices [23], etc. Similar to honeycomb lattices, Klein tunneling [8] and conical diffractoin [10] are also demonstrated in Lieb lattices. Different from honeycomb lattices, edge states of strained Lieb lattices are not reported systematically ever before to the best of our knowledge. Even though edge states of Lieb lattices are mentioned previously [9, 11, 12, 15, 17], they are not investigated in-depth.

It is demonstrated that if the sites of the honeycomb lattice exert gain and loss alternatively, the corresponding dispersion relation will be complex. However, the dispersion relation will be real again if the honeycomb lattice is deformed artificially. That’s to say, $\mathcal{PT}$-symmetry honeycomb lattices can be produced [24]. Similar to honeycomb lattices, Lieb lattices can be also made complex, which are not investigated ever before. Can complex Lieb lattices also exhibit $\mathcal{PT}$-symmetric properties? This paper will also answer this question.
II. RESULTS AND DISCUSSIONS

For investigating the edge states, two methods are mainly used. One way is the nearest-neighbor method, which study the system with exact diagonalisation of the cylinder [1]. Another way is to relate the presence of the edge states to the geometrical phase (Zak’s phase) of the bulk [25, 26]. During our theoretical analyses, nearest-neighbor tight-binding method is adopted, which is also true in Refs. [1, 17, 24].

A. Two-dimensional case

A Lieb lattice is displayed in Fig. 1(a), in which the lattice sites in a dashed square is a unit cell. We assume that the hopping among the lattice points only happens between the nearest-neighbor (NN) sites, as shown by the double-headed arrows. Therefore, the corresponding Hamiltonian can be written as

$$ H_{TB} = -t \sum_m \left[ (f_{r_m}^* g_{r_m + e_1} + f_{r_m}^* g_{r_m + e_2} + f_{r_m}^* h_{r_m + e_3} + f_{r_m}^* h_{r_m + e_4}) \right] + \text{h.c.}, $$

where $f_{r_m}^*$ is the creation operator on the $m$th lattice site, $r_m$ is the position of the $m$th lattice site, $t$ is the hopping strength, $e_1 = (-a/2, 0)$, $e_2 = (a/2, 0)$, $e_3 = (0, -a/2)$, and $e_4 = (0, a/2)$. In real and momentum spaces, we have the transform pairs

$$ f_k = \frac{1}{\sqrt{N}} \sum_m f_{r_m} \exp(i k \cdot r), \quad f_{r_m} = \frac{1}{\sqrt{N}} \sum_k f_k \exp(-i k \cdot r). $$

Plugging Eq. (2) into Eq. (1), we can obtain

$$ H_{TB} = -t \sum_k \left[ f_k^* g_k [\exp(-i k \cdot e_1) + \exp(-i k \cdot e_2)] \right] $$

$$ -t \sum_k f_k g_k^* [\exp(i k \cdot e_1) + \exp(i k \cdot e_2)] $$

$$ -t \sum_k f_k^* h_k [\exp(-i k \cdot e_3) + \exp(-i k \cdot e_4)] $$

$$ -t \sum_k f_k h_k^* [\exp(i k \cdot e_3) + \exp(i k \cdot e_4)]. $$

Equation (3) can be also rewritten as

$$ H_{TB} = \sum_k \begin{bmatrix} f_k^* \ g_k \ h_k^* \end{bmatrix} \mathcal{H} \begin{bmatrix} f_k \\
 g_k \\
 h_k \end{bmatrix} $$

$$ \mathcal{H} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{H}_{13} \\
 \mathcal{H}_{21} & \mathcal{H}_{22} & \mathcal{H}_{23} \\
 \mathcal{H}_{31} & \mathcal{H}_{32} & \mathcal{H}_{33} \end{bmatrix} $$

$$ \mathcal{H}_{ij} = \left(\begin{array}{ccc}
-t & 0 & 0 \\
0 & -t & 0 \\
0 & 0 & -t
\end{array}\right) $$

$$ \mathcal{H}_{12} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{2}{a} & 0 & 0
\end{array}\right) $$

$$ \mathcal{H}_{13} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{2}{a} & 0
\end{array}\right) $$

$$ \mathcal{H}_{23} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{2}{a}
\end{array}\right) $$

$$ \mathcal{H}_{31} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{2}{a}
\end{array}\right) $$

$$ \mathcal{H}_{32} = \left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{2}{a} & 0
\end{array}\right) $$

$$ \mathcal{H}_{33} = \left(\begin{array}{ccc}
\frac{2}{a} & 0 & \frac{2}{a}
\end{array}\right) $$

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with the Hamiltonian kernel

$$\mathcal{H} = -2t \begin{bmatrix} 0 & \cos \left( \frac{ak_x}{2} \right) & \cos \left( \frac{ak_y}{2} \right) \\ \cos \left( \frac{ak_x}{2} \right) & 0 & 0 \\ \cos \left( \frac{ak_y}{2} \right) & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (5)

Therefore, we can solve Eq. (5) for its eigenvalues

$$\beta_1 = 0,$$

$$\beta_{2,3} = \pm 2t \sqrt{\cos^2 \left( \frac{ak_x}{2} \right) + \cos^2 \left( \frac{ak_y}{2} \right)},$$  \hspace{1cm} (6)

which correspond to the dispersion relation. In Fig. 1(b), we display the dispersion relation in the first Brillouin zone with $a = 1$ (corresponding $k_{x,y} \in [-\pi, \pi]$). Note that we always take $a = 1$ throughout the paper. It is clear that there is a flat band and two symmetric conical bands about the flat band; such symmetry is known as the particle-hole symmetry [10–12]. The two conical bands posses a Dirac cone located at one corner of the first Brillouin zone and intersected by the flat band, as shown in Fig. 1(c), which is zoomed in of the region marked by an ellipse in Fig. 1(b).

![FIG. 1. (a) Lieb lattice. (b) Dispersion relation in the first Brillouin zone. (c) Zoomed in the region marked with an ellipse in (b).](image)

**B. Dispersion relations of strained Lieb lattices**

Honeycomb lattices can be strained with bearded, zigzag, bearded-zigzag, and armchair boundaries [1, 6], which lead to different dispersion relations. Similar to honeycomb lattices,
strained Lieb lattices can also have different boundaries, such as solid, pointy, solid-pointy, zigzag boundaries, etc. Therefore, different strained Lieb lattices may possess different dispersion relations. In this section, we will have a thorough discussion on the dispersion relations of strained Lieb lattices with different boundaries.

1. Solid edge

The Lieb lattice with solid edges is as shown in Fig. 2(a), in which we mark off a unit cell with a square. For convenience, we only show one layer, i.e. \( n = 1 \). According to the NN approximation, the Hamiltonian can be written as

\[
H_{\text{straight}} = -t \sum_m \left[ f_m^* (f_m' + f_m) + f_m^* (f_m + f_m'' + f_m) \right] \\
- t \sum_m \left[ f_m^* (f_m + f_m^*) + f_m^* (f_m + f_m^*) \right] \\
- t \sum_m f_m^* (f_m + f_m + f_m'').
\]

(7)

In momentum space, Eq. (7) can be written as

\[
H_{TB} = \sum_k \begin{bmatrix} f_m^* \ f_m^* \ f_m^* \ f_m^* \ f_m^* \end{bmatrix} \begin{bmatrix} f_m \\ f_m \\ f_m \\ f_m \\ f_m \end{bmatrix} \begin{bmatrix} 0 \cos(k_{x}^2) \ 0 \ 0 \ 0 \\ \cos(k_{x}^2) \ 0 \ \frac{1}{2} \ 0 \ 0 \\ 0 \ \frac{1}{2} \ 0 \ 0 \ \frac{1}{2} \\ 0 \ 0 \ 0 \ 0 \ \cos(k_{x}^2) \\ 0 \ 0 \ \frac{1}{2} \ \cos(k_{x}^2) \ 0 \end{bmatrix}.
\]

(8)

in which the corresponding Hamiltonian kernel can be written as

\[
H = -2t \begin{bmatrix} 0 \ 0 \ \frac{1}{2} \ \cos(k_{x}^2) \\ 0 \ 0 \ 0 \ \frac{1}{2} \ \cos(k_{x}^2) \\ \frac{1}{2} \ \frac{1}{2} \ 0 \ 0 \ \cos(k_{x}^2) \ 0 \ \frac{1}{2} \ \cos(k_{x}^2) \ 0 \end{bmatrix}.
\]

(9)

It is clear that the Hamiltonian kernel is a 5 \times 5 matrix, and the size of the matrix for any \( n \) can be induced as \( (2 + 3n) \times (2 + 3n) \). In Figs. 2(b1) and 2(b2), we show the band structures of a 40-layer Lieb lattice with two solid edges in \( k_x \in [-\pi, \pi] \) and \( k_x \in [0, 2\pi] \),
respectively. Even though the band structure is similar to those of a bearded honeycomb lattice [1, 4, 6], there is no degenerated band (the band at $\beta = 0$ is from the flat band) in the energy spectrum. Clearly, there is no edge state for this kind of strained Lieb lattice.

![Diagram of strained one-layer Lieb lattice with solid edges. The sites in the square compose a unit cell. (b1) Corresponding band structure with $k_x \in [-\pi, \pi]$ and $n = 40$. (b2) Same as (b1) but with $k_x \in [0, 2\pi]$.

FIG. 2. (a) Strained one-layer Lieb lattice with solid edges. The sites in the square compose a unit cell. (b1) Corresponding band structure with $k_x \in [-\pi, \pi]$ and $n = 40$. (b2) Same as (b1) but with $k_x \in [0, 2\pi]$.

2. Pointy edge

A Lieb lattice with pointy edges is exhibited in Fig. 3(a), in which the unit cell is also marked by a square. The corresponding Hamiltonian in momentum space can be written as

$$H_{TB} = \sum_k \begin{bmatrix} f_{m1}^* & f_{m2}^* & f_{m3}^* & f_{m4}^* \end{bmatrix} \mathcal{H} \begin{bmatrix} f_{m1} \\ f_{m2} \\ f_{m3} \\ f_{m4} \end{bmatrix},$$ (10)

with

$$\mathcal{H} = -2t \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \cos \left( \frac{k_x}{2} \right) & 0 \\ \frac{1}{2} \cos \left( \frac{k_x}{2} \right) & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}. $$ (11)
The dispersion relation of this strained Lieb lattice can be obtained according to Eq. (11), and the results are displayed in Figs. 3(b1) and 3(b2), in which there is a band gap between the flat and the upper (bottom) bands. One can clearly see that there is still no edge state in the band gap.

FIG. 3. Figure setup is as Fig. 2, but with pointy edges.

3. Solid-pointy edge

The Lieb lattice can be also strained with a hybrid boundaries, as shown in Fig. 4(a) – one edge is solid and the other is pointy. The corresponding band structure can be obtained as displayed in Figs. 4(b1) and 4(b2), which are similar to those displayed in Fig. 3. Therefore, there is still no edge states for this case.

C. Complex Lieb lattices

In the above sections, we theoretically investigate the dispersion relations of strained Lieb lattices with different edges. Similar to honeycomb waveguides [24], Lieb waveguides may be also made be complex. Since there are three sites in each unit cell of Lieb lattice, we assume one waveguide exhibits gain, and the other two loss.

The structure of complex Lieb waveguide lattices with alternating gain and loss is displayed in Fig. 5(a). Considering the arrangement in Fig. 1(a) and according to the tight-
binding model, the dynamics of the system can be described by

\[i\partial_z f_{rm} = -\Delta f_{rm} - i\gamma f_{rm} + t(c g_{r_{m+1}} + g_{r_{m+2}} + h_{r_{m+3}} + h_{r_{m+4}}),\]

\[i\partial_z g_{rm} = +\Delta g_{rm} + i\gamma g_{rm} + t(f_{r_{m+1}} + c f_{r_{m+2}}),\]

\[i\partial_z h_{rm} = +\Delta h_{rm} + i\gamma h_{rm} + t(f_{r_{m+3}} + f_{r_{m+4}}),\]

where \(\gamma\) describes the gain and loss of the waveguides, \(c\) is the deforming coefficient of the lattice, and \(\Delta\) is the detuning in the effective index between adjacent waveguides.

We introduce such solutions

\[f_{rm} = F \exp[i(\beta z + mk_x + nk_y)],\]

\[g_{rm} = G \exp[i(\beta z + mk_x + nk_y)],\]

\[h_{rm} = H \exp[i(\beta z + mk_x + nk_y)]\]

for Eq. (12). Here, \(F\), \(G\), and \(H\) are the amplitudes. Therefore, we obtain such eigenvalue problem

\[
\begin{bmatrix}
\Delta + i\gamma & -t[c \exp(i k_y) + \exp(-i k_y)] & -2t \cos(k_x/2) \\
-t[c \exp(-i k_y) + \exp(i k_y)] & -(\Delta + i\gamma) & 0 \\
-2t \cos(k_x/2) & 0 & -(\Delta + i\gamma)
\end{bmatrix}
\begin{bmatrix}
F \\
G \\
H
\end{bmatrix}
= \beta
\begin{bmatrix}
F \\
G \\
H
\end{bmatrix},
\]

FIG. 4. Figure setup is as Fig. 2, but with solid-pointy edge.
and the corresponding dispersion relation can be calculated as

\[
\beta_1 = -(\Delta + i\gamma),
\]

\[
\beta_{2,3} = \pm \sqrt{\Delta^2 - \gamma^2 + 2i\gamma\Delta + t^2(c-1)^2 + 4ct^2\cos^2\left(\frac{k_x}{2}\right) + 4t^2\cos^2\left(\frac{k_y}{2}\right)}.
\] (14)

Clearly, Eq. (14) is same as Eq. (6) if \(\Delta = \gamma = 0\) and \(c = 1\), so that the dispersion relation is as in Fig. 1(b). If \(\gamma = 0\) and \(c = 1\), the eigenvalues are all real and the dispersion relations are displayed in Fig. 5(b) (\(\Delta = 0.2\) for this case). In this case, the triply degenerate diabolic point disappears. In stead, a band gap appears between one conical band and the flat band. While if \(\Delta = 0\), \(c = 1\) and \(\gamma \neq 0\), the eigenvalues are complex. In Figs. 5(c) and 5(d), we show the real and imaginary parts of the dispersion relations with \(\gamma = 0.5\), which are quite similar to those of complex honeycomb lattices [24]. As demonstrated previously [24], lattice deformation may lead to completely real eigenvalues even though the lattice is complex. In other word, there exists \(\mathcal{PT}\)-symmetry lattices. Here, we also deform the complex Lieb lattice and investigate the corresponding dispersion relations. According to Eq. (14), when \(\Delta = 0\) and if \(0 < c \leq (1 - \gamma/t)\) or \(c \geq (1 + \gamma/t)\), \(\beta_{2,3}\) will be real. However, the flat band \(\beta_1\) is always complex, because it cannot be adjusted by the deforming coefficient \(c\). Since the eigenvalues are always complex, there does not exist \(\mathcal{PT}\)-symmetry Lieb lattice. Interestingly, the triply degenerate diabolic point will reappear in the real parts of the dispersion relation if \(c = 1 - \gamma/t\) or \(c = 1 + \gamma/t\). For other cases with \(0 < c < (1 - \gamma/t)\) or \(c > (1 + \gamma/t)\), there is no degenerate diabolic point. No matter what value of \(c\), the particle-hole symmetry holds for true in real part of the dispersion relation if \(\Delta = 0\).

### III. CONCLUSION

In summary, we have investigated the dispersion relations of strained and complex Lieb lattices. We find that there is always no edge state for strained Lieb lattices. As to complex Lieb lattices, \(\mathcal{PT}\)-symmetric Lieb lattices cannot be obtained, because the flat band is not affected by the deforming coefficient and always complex, even though the imaginary eigenvalues of the top and bottom bands can be waived. This investigation will help people understand the dispersion relations of strained Lieb lattices with different edges and complex Lieb lattices, and guide people choose right structures to fabricate topological insulators.
FIG. 5. (a) Sketch of complex Lieb lattice. The red waveguides exhibit gain, while the other two kinds exhibit loss. (b1) Dispersion relation with $\Delta = 0.2$ and $\gamma = 0$. (b2) A magnified view of the dispersion relation around a corner of the first Brillouin zone. (c) and (d) Figure setup is as (b), but for the real and imaginary parts with $\Delta = 0$ and $\gamma = 0.5$, respectively. For all the cases, $c = 1$.

ACKNOWLEDGMENTS

This work was supported by the 973 Program (2012CB921804), KSTIT of Shaanxi province (2014KCT-10), NSFC (11474228, 61308015, 61205112), CPSF (2014T70923, 2012M521773), and the Qatar National Research Fund NPRP 6-021-1-005 project.

[1] M. Kohmoto and Y. Hasegawa, “Zero modes and edge states of the honeycomb lattice,” Phys. Rev. B 76, 205402 (2007).

[2] O. Bahat-Treidel, O. Peleg, M. Grobman, N. Shapira, M. Segev, and T. Pereg-Barnea, “Klein tunneling in deformed honeycomb lattices,” Phys. Rev. Lett. 104, 063901 (2010).

[3] K. K. Gomes, W. Mar, W. Ko, F. Guinea, and H. C. Manoharan, “Designer Dirac fermions and topological phases in molecular graphene,” Nature 483, 306–310 (2012).
[4] M. C. Rechtsman, Y. Plotnik, J. M. Zeuner, D. Song, Z. Chen, A. Szameit, and M. Segev, “Topological creation and destruction of edge states in photonic graphene,” Phys. Rev. Lett. 111, 103901 (2013).

[5] A. Crespi, G. Corrielli, G. D. Valle, R. Osellame, and S. Longhi, “Dynamic band collapse in photonic graphene,” New J. Phys. 15, 013012 (2013).

[6] Y. Plotnik, M. C. Rechtsman, D. Song, M. Heinrich, J. M. Zeuner, S. Nolte, Y. Lumer, N. Malkova, J. Xu, A. Szameit, Z. C. Chen, and M. Segev, “Observation of unconventional edge states in ‘photonic graphene’,” Nat. Mater. 13, 57–62 (2014).

[7] V. Apaja, M. Hyrkä, and M. Manninen, “Flat bands, dirac cones, and atom dynamics in an optical lattice,” Phys. Rev. A 82, 041402 (2010).

[8] R. Shen, L. B. Shao, B. Wang, and D. Y. Xing, “Single dirac cone with a flat band touching on line-centered-square optical lattices,” Phys. Rev. B 81, 041410 (2010).

[9] N. Goldman, D. F. Urban, and D. Bercioux, “Topological phases for fermionic cold atoms on the lieb lattice,” Phys. Rev. A 83, 063601 (2011).

[10] D. Leykam, O. Bahat-Treidel, and A. S. Desyatnikov, “Pseudospin and nonlinear conical diffraction in Lieb lattices,” Phys. Rev. A 86, 031805 (2012).

[11] M. Nitä, B. Ostahie, and A. Aldea, “Spectral and transport properties of the two-dimensional Lieb lattice,” Phys. Rev. B 87, 125428 (2013).

[12] D. Guzmán-Silva, C. Mejía-Cortés, M. A. Bandres, M. C. Rechtsman, S. Weimann, S. Nolte, M. Segev, A. Szameit, and R. A. Vicencio, “Experimental observation of bulk and edge transport in photonic Lieb lattices,” New J. Phys. 16, 063061 (2014).

[13] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, “Photonic Floquet topological insulators,” Nature 496, 196–200 (2013).

[14] G. Q. Liang and Y. D. Chong, “Optical resonator analog of a two-dimensional topological insulator,” Phys. Rev. Lett. 110, 203904 (2013).

[15] M. A. Bandres, M. Rechtsman, A. Szameit, and M. Segev, “Lieb photonic topological insulator,” in “CLEO: 2014,” (Optical Society of America, 2014), p. FF2D.3.

[16] K.-I. Imura, A. Yamakage, S. Mao, A. Hotta, and Y. Kuramoto, “Zigzag edge modes in a $Z_2$ topological insulator: Reentrance and completely flat spectrum,” Phys. Rev. B 82, 085118 (2010).
[17] C. Weeks and M. Franz, “Topological insulators on the lieb and perovskite lattices,” Phys. Rev. B 82, 085310 (2010).

[18] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, “Nearly flatbands with nontrivial topology,” Phys. Rev. Lett. 106, 236803 (2011).

[19] Z. Wang, Y. Chong, J. D. Joannopoulos, and M. Soljacic, “Observation of unidirectional backscattering-immune topological electromagnetic states,” Nature 461, 772–775 (2009).

[20] M. C. Rechtsman, J. M. Zeuner, A. Tünnermann, S. Nolte, M. Segev, and A. Szameit, “Strain-induced pseudomagnetic field and photonic Landau levels in dielectric structures,” Nat. Photon. 7, 153–158 (2013).

[21] S. Deng, A. Simon, and J. Köhler, “The origin of a flat band,” J. Solid State Chem. 176, 412–416 (2003).

[22] S. Miyahara, K. Kubo, H. Ono, Y. Shimomura, and N. Furukawa, “Flat-bands on partial line graphs -systematic method for generating flat-band lattice structures-,” J. Phys. Soc. Jap. 74, 1918–1921 (2005).

[23] D. Bercioux, D. F. Urban, H. Grabert, and W. Häusler, “Massless dirac-weyl fermions in a $T^3$ optical lattice,” Phys. Rev. A 80, 063603 (2009).

[24] A. Szameit, M. C. Rechtsman, O. Bahat-Treidel, and M. Segev, “$\mathcal{PT}$-symmetry in honeycomb photonic lattices,” Phys. Rev. A 84, 021806 (2011).

[25] S. Ryu and Y. Hatsugai, “Topological origin of zero-energy edge states in particle-hole symmetric systems,” Phys. Rev. Lett. 89, 077002 (2002).

[26] P. Delplace, D. Ullmo, and G. Montambaux, “Zak phase and the existence of edge states in graphene,” Phys. Rev. B 84, 195452 (2011).