A CONNECTION BETWEEN THE UNCERTAINTY PRINCIPLES ON THE
REAL LINE AND ON THE CIRCLE

NILS BYRIAL ANDERSEN

Abstract. The purpose of this short note is to exhibit a new connection between the Heisenberg Uncertainty Principle on the line and the Breitenberger Uncertainty Principle on the circle, by considering the commutator of the multiplication and difference operators on Bernstein functions.

1. Introduction

Consider the Bernstein space $B^2_R$ defined as the subspace of functions $f$ in $L^2(\mathbb{R})$ whose (distributive) Fourier transform $\mathcal{F}f$ has support in the interval $[-R, R]$. Let $A_\delta$ denote the family of normalized backward difference operators

$$A_\delta f(z) = \frac{f(z) - f(z - \delta)}{\delta} \quad (f \in B^2_R, z \in \mathbb{C}),$$

for $\delta \in (0, 1]$. Let $\dot{B}^2_R = \{ f \in B^2_R : xf(x) \in L^2(\mathbb{R}) \}$ (note that $f \in \dot{B}^2_R \Rightarrow xf(x) \in B^2_R$), and let $B : \dot{B}^2_R \rightarrow B^2_R$ denote the multiplication operator

$$Bf(x) = xf(x) \quad (f \in \dot{B}^2_R, x \in \mathbb{R}).$$

Using an operator theoretic approach, see \cite{2, 3, 4, 5, 6, 9, 10, 11}, we get the uncertainty inequalities

$$\| (A_\delta - a)f \|_2 \| (B - b)f \|_2 \geq \frac{1}{2} |\langle f(\cdot - \delta), f \rangle|,$$

for $0 \neq f \in \dot{B}^2_R$, and all $a, b \in \mathbb{C}$.

At the limit $\delta \rightarrow 0$ (and $R$ arbitrary), we recover the Heisenberg Uncertainty Principle \cite{7} for functions on the line, and at the endpoint $\delta = 1$ (with $R = \pi$), we recover the Breitenberger Uncertainty Principle \cite{11} for functions on the circle, thus giving a new, and easy, link between the two Uncertainty Principles. Another connection between the two Uncertainty Principles was discussed in \cite{10}.

Finally, we show the equivalence of the Heisenberg Uncertainty Principle to another Uncertainty Principle on the circle \cite{11}, using the central difference operators

$$C_\delta f(z) = \frac{f(z + \delta) - f(z - \delta)}{2\delta} \quad (f \in B^2_R, z \in \mathbb{C}),$$

for $\delta \in (0, 1]$.

We do not discuss (asymptotical) optimal functions for the Uncertainty Principles here, but refer to the references for a discussion of this matter. For more references to the subject of this article, we refer to the recent monograph \cite{2}.

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2. Uncertainty Principles for Symmetric and Normal Operators

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. For $A$ and $B$ linear operators with domains $\mathcal{D}(A), \mathcal{D}(B)$ respectively, and range in $\mathcal{H}$, the (normalized) expectation value of the operator $A$ with respect to $f \in \mathcal{D}(A)$ is defined as

$$\tau_A(f) := \frac{\langle Af, f \rangle}{\langle f, f \rangle},$$

and the standard deviation, or variance, of the operator $A$ with respect to $f \in \mathcal{D}(A)$ is defined as

$$\sigma_A(f) := \| Af - \tau_A(f) f \| = \min_{a \in \mathbb{C}} \| (A - a) f \|.$$

We notice that $\tau_A(f)$ is the orthogonal projection of $Af$ onto $f$. The commutator of $A$ and $B$ is defined as $[A, B] := AB - BA$, with domain $\mathcal{D}(AB) \cap \mathcal{D}(BA)$.

From [6, Corollary 1], [11, Theorem 3.1] and [11, Corollary 3.3], we get the following uncertainty principle:

**Theorem 1.** If $A, B$ are symmetric or normal operators on a Hilbert space $\mathcal{H}$, then

$$\|(A - a)f\| \|(B - b)f\| \geq \sigma_A(f) \sigma_B(f) \geq \frac{1}{2} \|([A, B]f, f)\|,$$

for all nonzero $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, and all $a, b \in \mathbb{C}$.

Theorem 1 can be used to prove the Heisenberg Uncertainty Principle on the real line (with $\mathcal{H} = L^2(\mathbb{R})$, $Af(x) = xf(x)$, $Bf(x) = if'(x)$), and the Breitenberger Uncertainty Principle on the circle (with $\mathcal{H} = L^2(\pi, \pi)$, $Af(x) = e^{ix}f(x)$, $Bf(x) = if'(x)$), see the references.

3. The Bernstein Spaces $B^2_R$

Let $R > 0$. Recall the definition of the Bernstein spaces $B^2_R$ from the introduction. By the classical Paley–Wiener theorem, $B^2_R$ can be identified with the space of entire functions $f$ on $\mathbb{C}$ of exponential type $R$, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. It is well-known that $B^2_R$ is a Hilbert space equipped with the $L^2(\mathbb{R})$ norm $\|f\|_2$ (of the restriction of $f$ to the real line), which is invariant under differentiation, and the Bernstein inequality $\|f'\|_2 \leq R \|f\|_2$ holds, for all $f \in B^2_R$. See also [8, Lecture 20] for general results concerning $L^p$-Bernstein spaces.

Let $\text{sinc}(z) = \sin(\pi z)/\pi z$. Let $l^2(\mathbb{Z})$ denote the space of square-summable sequences defined on the integers $\mathbb{Z}$. Then $B^2_{\pi}$ and $l^2(\mathbb{Z})$ are isomorphic, with proportional norms, and the isomorphism is given by the Whittaker–Kotel’nikov–Shannon Sampling Formula

$$f(z) = \sum_{n \in \mathbb{Z}} a_k \text{sinc}(z - n), \quad \{a_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}),$$

which converges in $L^2$ to a function $f \in B^2_{\pi}$, given as the unique solution of the interpolation problem $f(k) = a_k$, $k \in \mathbb{Z}$. Conversely, for any function $f \in B^2_{\pi}$, the sequence $\{f(k)\}_{k \in \mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$.

4. Uncertainty Principles for Bernstein Spaces

Consider the family of difference operators $A_\delta : B^2_R \to B^2_R$, with $\delta \in (0, 1]$, from the introduction. For $\delta = 1$, this is the usual backward difference operator $\partial f(z) = A_1(f) = f(z) - f(z - 1)$. The adjoint operator $A^*_\delta : B^2_R \to B^2_R$, is given by $A^*_\delta f(z) = (f(z) - f(z + \delta))/\delta$, and since $A_\delta A^*_\delta = A^*_\delta A_\delta$, we see that $A$ is a normal operator. The multiplication operator $B$ is obviously a symmetric operator.

A small computation yields that

$$[A_\delta, B]f(x) = f(x - \delta) \quad (f \in B^2_R, \ x \in \mathbb{R}).$$

From Theorem 1, we thus have
Theorem 2. Let $\delta \in (0, 1]$. Let $0 \neq f \in \dot{B}_R^2$. Then
\[ \| (A_\delta - a) f \|_2 \| (B - b) f \|_2 \geq \sigma_{A_\delta}(f) \sigma_B(f) \geq \frac{1}{2} | \langle f'(-\delta), f \rangle |, \]
fors all $a, b \in \mathbb{C}$.

Let $\{f(n)\}_{n \in \mathbb{Z}}$ be a sequence in $l^2(\mathbb{Z})$, and denote by $\hat{f} \in L^2(-\pi, \pi)$ the Fourier inverse of $f$. The Fourier series corresponding to the function $e^{i\theta} \hat{f}$ is the sequence $\{f(n-1)\}$. The Fourier inverse of $xf(x) \in B^2_R$, or $\{nf(n)\} \in l^2(\mathbb{Z})$, is $\frac{i}{\pi} \hat{f}$. Let $\delta = 1$ and $R = \pi$, then Theorem 2 yields

Corollary 3. The Breitenberger Uncertainty Principle for functions on the circle. Let $f \in \dot{B}_R^2$. Then
\[ \| (e^{i\theta} - a) \hat{f} \|_2 \left\| \left( \frac{d}{d\theta} - b \right) \hat{f} \right\|_2 \geq \frac{1}{2} \left| \langle e^{i\theta} \hat{f}, \hat{f} \rangle \right|, \]
fors all $a, b \in \mathbb{C}$.

We can rewrite this in terms of the Fourier coefficients $\{f(n)\}$,
\[ \left( \sum_{n \in \mathbb{Z}} |f(n-1) - af(n)|^2 \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} |(n-b)f(n)|^2 \right)^{\frac{1}{2}} \geq \frac{1}{2} \left| \sum_{n \in \mathbb{Z}} f(n-1)f(n) \right|, \]
which holds for all square-summable sequences $\{f(n)\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$.

Let $\delta \to 0$, then Theorem 2 yields

Corollary 4. The Heisenberg Uncertainty Principle for functions on the line. Let $f \in B^2_R$, for some $R > 0$. Then
\[ \left\| \left( \frac{d}{dx} - a \right) f \right\|_2 \left\| (x-b)f \right\|_2 \geq \frac{1}{2} \left| \langle f, f' \rangle \right|, \]
fors all $a, b \in \mathbb{C}$.

The inequality holds for $f \in \dot{B}_R^2$ by Theorem 2 and easily extends to all $f \in B^2_R$. A standard density argument furthermore extends the inequality to all functions $f \in L^2(\mathbb{R})$.

Finally, let us consider the central difference operators $C_\delta$. Since
\[ [C_\delta, B]f(x) = \frac{f(x+\delta) + f(x-\delta)}{2} \quad (f \in \dot{B}_R^2, x \in \mathbb{R}), \]
Theorem 1 gives

Theorem 5. Let $\delta \in (0, 1]$. Let $0 \neq f \in \dot{B}_R^2$. Then
\[ \| (C_\delta - a)f \|_2 \| (B - b)f \|_2 \geq \sigma_{C_\delta}(f) \sigma_B(f) \geq \frac{1}{2} \left| \langle f'(-\delta), f \rangle \right|, \]
for all $a, b \in \mathbb{C}$.

In the limit $\delta \to 0$, Theorem 5 yields the Heisenberg Uncertainty Principle as before. So let $\delta = 1$ and $R = \pi$, then,

Corollary 6. Let $f \in \dot{B}_R^2$. Then
\[ \left\| (\sin(\theta) - a) \hat{f} \right\|_2 \left\| \left( \frac{d}{d\theta} - b \right) \hat{f} \right\|_2 \geq \frac{1}{2} \left| \langle \cos(\theta) \hat{f}, \hat{f} \rangle \right|, \]
for all $a, b \in \mathbb{C}$.
5. Final remarks

Normally, when we discuss Uncertainty Principles mathematically, we say that \( f \) or \( \mathcal{F}f \) cannot be localized at the same time. Here, we assume that \( \mathcal{F}f \) is localized as \( \text{supp} \mathcal{F}f \subset [-R, R] \), or \( f \in B^2_R \), which is another reason why it may be interesting to look at \( B^2_R \).

The Bernstein inequality \( \|f'\|_2 \leq R \|f\|_2 \), together with the Heisenberg Uncertainty Principle, also yields the following inequality, for \( 0 \neq f \in B^2_R \), and \( a \in \mathbb{C} \),

\[
\frac{1}{2} \|f\|_2^2 \leq \|f'\|_2 \|f(a)\|_2 \leq R \|f\|_2 \|f(a)\|_2,
\]

or

\[
\|f(a)\|_2 \geq \frac{\|f\|_2}{2R},
\]

which indeed supports the claim that localization of the frequency, i.e., \( R \) small, implies indeterminacy of the position.

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Department of Mathematics, Aarhus University, Ny Munkegade 118, Building 1530, DK-8000 Aarhus C, Denmark

E-mail address: byrial@imf.au.dk