SOME BOUNDS FOR RAMIFICATION OF $p^n$-TORSION SEMI-STABLE REPRESENTATIONS

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Abstract. Let $p$ be an odd prime, $K$ a finite extension of $\mathbb{Q}_p$, $G_K = \text{Gal}(\bar{K}/K)$ its absolute Galois group and $e = e(K/\mathbb{Q}_p)$ its absolute ramification index. Suppose that $T$ is a $p^n$-torsion representation of $G_K$ that is isomorphic to a quotient of $G_K$-stable $\mathbb{Z}_p$-lattices in a semi-stable representation with Hodge-Tate weights $\{0, \ldots, r\}$. We prove that there exists a constant $\mu$ depending only on $n$, $e$ and $r$ such that the upper numbering ramification group $G^{(\mu)}$ acts on $T$ trivially.

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1. Introduction

Let $p > 2$ be a prime number and $k$ a perfect field of characteristic $p$. We denote by $W = W(k)$ the ring of Witt vectors with coefficients in $k$. Fix $K$ a totally ramified extension of $W[1/p]$ of degree $e$ and $\bar{K}$ an algebraic closure of $K$. Fix $\pi \in \mathcal{O}_K$ an uniformizer and $(\pi_s)_{s \geq 0}$ a compatible system of $p^s$-th root of $\pi$. Set $G = \text{Gal}(\bar{K}/K)$ and for all non negative integer $s$, put $K_s = K(\pi_s)$ and $G_s = \text{Gal}(\bar{K}/K_s)$. Denote by $G^{(\mu)}$ and $G^{(\mu)}_s$ ($\mu \in \mathbb{R}$) the upper ramification filtration of $G$ and $G_s$, as defined in §1.1 of [9]. Note that conventions of loc. cit. differ by some shift with definition of [23], Chap. IV. Finally, let $v_K$ be the discrete valuation on $K$ normalized by $v_K(\pi) = 1$. It extends uniquely to a (not discrete) valuation on $\bar{K}$, that we denote again $v_K$.

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Consider $r$ a positive integer and $V$ a semi-stable representation of $G$ with Hodge-Tate weights in $\{0, 1, \ldots, r\}$. Let $T$ be the quotient of two $G$-stable $\mathbb{Z}_p$-lattices in $V$. It is a representation of $G$, which is killed by $p^n$ for some integer $n$. Denote by $\rho: G \to \text{Aut}_{\mathbb{Z}_p}(T)$ the associated group homomorphism and by $L$ (resp. $L_s$) the finite extension of $K$ (resp. $K_s$) defined by $\ker \rho$ (resp. $\ker \rho_{G_s}$). We will prove:

**Theorem 1.1.** Keeping previous notations, for any integer $s > n + \log_p (\frac{er}{p-1})$ and for all real number $\mu > \frac{ernp^n}{p-1}$, $G_s^(\mu)$ acts trivially on $T$.

**Remark 1.2.** Condition on $s$ implies $\frac{ernp^n}{p-1} < ep^s$. Hence one may always choose $\mu = ep^s$.

We also obtain a bound for the ramification of $L/K$:

**Theorem 1.3.** Write $\frac{mn}{p-1} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. Then:

1. If $\mu > 1 + e(n + \alpha) + \max(\frac{\alpha}{p^\alpha}, \frac{\beta}{p^\beta}),$ then $G^{(\mu)}$ acts trivially on $T$;
2. $\nu_K(\mathcal{D}_{L/K}) < 1 + e(n + \alpha + \beta) - \frac{1}{p^{\alpha + \beta}}$.

where $\mathcal{D}_{L/K}$ is the different of $L/K$.

Before this work, some partial results were already known in this direction. First, in [9] and [11], Fontaine uses Fontaine-Laffaille theory (developed in [8]) to get some bounds when $e = 1$, $n = 1$, $r < p - 1$ and $V$ is crystalline. In [1], Abrashkin follows Fontaine’s general ideas to extend the result to arbitrary $n$ (other restrictions remain the same). Later, with the extension by Breuil of Fontaine-Laffaille theory to semi-stable case (see [3]), it has been possible to achieve some cases where $V$ is not crystalline. Precisely in [4], Breuil obtains bounds for semi-stable representations that satisfies Griffith transversality when $n = 1$ and $er < p - 1$. Very recently in [14] and [15], Hattori proves a bound for all semi-stable representations with $r < p - 1$ ($e$ and $n$ are arbitrary here). All these bounds have the same shape

\[ e \left( n + \frac{r}{p-1} \right) + \text{cte} \]

with $0 \leq \text{cte} \leq 1$. Since $r$ is always assumed to be $< p - 1$, one can see that these bounds are better than ours. However, the most important feature of Theorem 1.3 is to be applicable for any $r$! Furthermore, one remark that bounds of Theorem 1.3 have a logarithmic dependance in $r$, which may be quite surprising after (1.0.1) (where the dependance seems to be linear\(^2\)). Actually, it is very plausible that, using analogous methods, one can improve Theorem 1.3 in order to fit with (1.0.1). Precisely, we conjecture the following.

**Conjecture 1.4.** Writing $\frac{mp}{p-1} = p^{\alpha'} \beta'$ with $\alpha' \in \mathbb{N}$ and $\frac{1}{p} < \beta' \leq 1$, we have:

1. If $\mu > 1 + e(n + \alpha') + \max(e\beta' - \frac{\alpha'}{p^{\alpha'}}, \frac{\beta'}{p^{\beta'}}),$ then $G^{(\mu)}$ acts trivially on $T$;
2. $\nu_K(\mathcal{D}_{L/K}) < 1 + e(n + \alpha' + \beta') - \frac{1}{p^{\alpha' + \beta'}}$.

We finally wonder if better bounds exist when $V$ is crystalline. It is actually the case when $e = 1$ and $r < p - 1$ by results of Fontaine and Abrashkin, but it is not clear to us how to extend this to a more general setting.

Let us now explain the general plan of our proof (and in the same time of the article). For this we introduce first further notations: let $K_{\infty} = \bigcup_{s=1}^{\infty} K_s$ and $G_{\infty} = \text{Gal}(\bar{K}/K_{\infty})$. By some works of Fontaine, Breuil and Kisin, we know that the restriction of $T$ to $G_{\infty}$ is described by some data of (semi-)linear algebra that we will call in the sequel "Kisin modules". Let us call it $\mathfrak{M}$. In the two following sections, we will show that the data of $\mathfrak{M}$ is enough to recover the whole action of $G_s$ on $T$ for $s > s_{\text{min}} := n - 1 + \log_p(nr)$.

---

1 See Proposition 9.2.2.2 of [2] for the statement
2 Of course, it does not mean anything since these bounds are valid under the assumption for $r < p - 1$, and certainly not for $r$ going to infinity.
3 In fact, these modules were first introduced by Breuil in [5] and [6]. However, we think that the terminology is not so bad since “Breuil modules” is already used for other things and “Kisin modules” were actually intensively studied by Kisin in [17] and [18].
More precisely, we first prove in section 2 (Theorem 2.5.5) that any Kisin module killed by \( p^n \) determines a canonical representation of \( G_s \) with \( s > s_{\text{min}} \) (and not only \( G_\infty \)). Note that this first step does not use any assumption of semi-stability: our result is valid for all representations (killed by \( p^n \)) coming from a Kisin module; no matter if it can be realized as a quotient of two lattices in a semi-stable representation. Then, in section 3, we show that the \( G_s \)-representation attached to \( \mathcal{M} \) coincides with \( T|_{G_s} \). At this level, let us mention an interesting corollary of the theory developed in these two sections:

**Theorem 1.5** (Corollary 3.3.5). Let \( V \) and \( V' \) be two semi-stable representations of \( G \). Let \( T \) (resp. \( T' \)) a quotient of two \( G \)-lattices in \( V \) (resp. \( V' \)) which is killed by \( p^n \). Then any morphism \( G_\infty \)-equivariant \( f : T \to T' \) is \( G_s \)-equivariant for all integer \( s > n - 1 + \log_p(nr) \).

Then, we conclude the proof of Theorem 1.1 using usual techniques developed by Fontaine in [9]. Using some kind of transitivity formulas, we then deduce Theorem 1.3. Finally, in the section 5, we begin a discussion about the possibility, given a torsion representation of \( G_\infty \), to write it as a quotient of two lattices in a \( Q_p \)-representation satisfying some properties (like being crystalline, semi-stable, with prescribed Hodge-Tate weights).

**Conventions.** For any \( Z \)-module \( M \), we always use \( M_n \) to denote \( M/p^nM \). If \( A \) be a ring, then \( M(A) \) will denote the ring of \( d \times d \)-matrices with coefficients in \( A \). We reserve \( \varphi \) to represent various Frobenius structures (except that \( \sigma \) stands for usual Frobenius on \( W(k) \)) and \( \varphi_M \) will denote the Frobenius on \( M \). But we always drop the subscript if no confusion arises.

Finally, if \( A \) is a ring equipped with a valuation \( v_A \) we will often set:

\[
\mathfrak{a}_A^{\geq v} = \{ x \in A / \sqrt{A}(x) \geq v \} \quad \text{and} \quad \mathfrak{a}_A^{> v} = \{ x \in A / \sqrt{A}(x) > v \}.
\]

**2. \( G_s \)-representation attached to a torsion Kisin module**

In this section, we prove that \( G_\infty \)-representation \( T|_{G_s}(\mathcal{M}) \) attached to Kisin modules \( \mathcal{M} \) killed by \( p^n \) can be naturally extended to a \( G_s \)-representation for all \( s > n - 1 + \log_p(nr) \) (and sometimes better).

**2.1. Definitions and basic properties of Kisin modules.** Recall the following notations: \( k \) is a perfect field, \( W = W(k) \), \( K \) is a totally ramified extension of \( W[1/p] \) of degree \( e \), \( \pi \) is a fixed uniformizer of \( K \). Recall also that we have fixed a positive integer \( r \). Define \( E(u) \) to be the minimal polynomial of \( \pi \) over \( W[1/p] \).

The base ring for Kisin modules is \( \mathcal{S} = W[[u]] \). It is endowed with a Frobenius map \( \varphi : \mathcal{S} \to \mathcal{S} \) defined by:

\[
\varphi \left( \sum_{i \geq 0} a_i u^i \right) = \sum_{i \geq 0} \sigma(a_i) u^{ei},
\]

where \( \sigma \) stands for usual Frobenius on \( W \). By definition, a free Kisin module (of height \( \leq r \)) is a \( \mathcal{S} \)-module \( \mathcal{M} \) free of finite rank equipped with a \( \varphi \)-semi-linear endomorphism \( \varphi_\mathcal{M} : \mathcal{M} \to \mathcal{M} \) such that the following condition holds:

\[
(2.1.1) \quad \text{the } \mathcal{S} \text{-submodule of } \mathcal{M} \text{ generated by } \varphi_\mathcal{M}(\mathcal{M}) \text{ contains } E(u)^e \mathcal{M}.
\]

We denote by \( \text{Mod}^{\varphi}_\mathcal{S} \) their category. Of course, a morphism of \( \text{Mod}^{\varphi}_\mathcal{S} \) is just a \( \mathcal{S} \)-linear map that commutes with Frobenius actions. In the sequel, if there is no risk of confusion, we will often write \( \varphi \) instead of \( \varphi_\mathcal{M} \).

There is also a notion of torsion Kisin modules of height \( \leq r \). They are modules \( \mathcal{M} \) over \( \mathcal{S} \) equipped with a \( \varphi \)-semi-linear map \( \varphi : \mathcal{M} \to \mathcal{M} \) such that:

- \( \mathcal{M} \) is finitely generated and killed by a power of \( p \);
- \( \mathcal{M} \) has no \( u \)-torsion;
- condition (2.1.1) holds.

Let us call \( \text{Mod}^{\varphi}_\mathcal{S}(\mathcal{M}) \) (resp. \( \text{Mod}^{\varphi}_\mathcal{S}(\mathcal{M}) \), resp. \( \text{Free}^{\varphi}_\mathcal{S}(\mathcal{M}) \)) the category of all torsion Kisin modules (resp. of torsion Kisin modules killed by \( p^n \), resp. torsion Kisin modules killed by \( p^n \) and free over \( \mathcal{S}_n = \mathcal{S}/p^n\mathcal{S} \)). Obviously \( \text{Free}^{\varphi}_\mathcal{S}(\mathcal{M}) \subset \text{Mod}^{\varphi}_\mathcal{S}(\mathcal{M}) \) and \( \bigcup_{n \geq 1} \text{Mod}^{\varphi}_\mathcal{S}(\mathcal{M}) = \text{Mod}^{\varphi}_\mathcal{S}(\mathcal{M}) \) (the union is increasing).

It is proved in Proposition 2.3.2 of [19] that torsion Kisin modules are exactly quotients of two
free Kisin modules of same rank. In particular every object in \( \text{Mod}^\varphi_r \) is a quotient of an object in \( \text{Free}^\varphi_r \). We finally note that \( \text{dévissages} \) with torsion Kisin modules are in general quite easy to achieve since if \( \mathfrak{M} \) is in \( \text{Mod}^\varphi_r \) then \( \mathfrak{M}(p) = \ker p|\mathfrak{M} \) and \( \mathfrak{M}/\mathfrak{M}(p) \) are respectively in \( \text{Mod}^\varphi_r \) and \( \text{Mod}^\varphi_r \) and we obviously have an exact sequence \( 0 \to \mathfrak{M}(p) \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}(p) \to 0 \) (see Proposition 2.3.2 in [19]).

2.2. Functors to Galois representations. We first need to define some period rings. Let \( R = \lim_{\leftarrow \pi} \mathcal{O}_k/p \) where transition maps are Frobenius. By definition an element \( x \in R \) is a sequence \( (x(0), x(1), \ldots) \) such that \( (x(s+1))^p = x(s) \). Fontaine proves in [12] that \( R \) is equipped with a valuation defined by \( v_R(x) = \lim_{s \to \infty} p^s v_K(x(s)) \) if \( x \neq 0 \). (In this case, \( x(s) \) does not vanish for \( s \) large enough and its valuation is then well defined; starting from this rank, the sequence \( p^s v_K(x(s)) \) is constant.) Note that \( k \) embeds naturally in \( R \) via \( \lambda \mapsto (\lambda(0), \lambda(1), \ldots) \) where \( \lambda(s) \) is the unique \( p^s \)-th root of \( \lambda \) in \( k \) (recall that \( k \) is assumed to be perfect). This embedding turns \( R \) into a \( k \)-algebra.

Now, consider \( W(R) \) (resp. \( W_n(R) \)) the ring of Witt vectors (resp. truncated Witt vectors) with coefficients in \( R \). It is a \( W \)-algebra (resp. a \( W_n(k) \)-algebra). Moreover, since Frobenius is bijective on \( R, W_n(R) = W(R)/p^n W(R) \). Recall that we have fixed \( (\pi_s) \) a compatible sequence of \( p^s \)-roots of \( \pi \). It defines an element \( \pi \in R \) whose Teichmüller representative is denoted by \([\pi]\). We can then define an embedding \( \mathfrak{S} \hookrightarrow W(R), u \mapsto [u] \). For any positive integer \( n \), reducing modulo \( p^n \), we get a map \( \mathfrak{S}_n \leftarrow W_n(R) \) which remains injective. In the sequel, we will often still denote by \( u \) its image in \( W(R) \) and \( W_n(R) \). Let \( O_{\mathcal{E}} \) be the closure in \( W(Frac R) \) of \( \mathcal{S}[1/u] \) (for the \( p \)-adic topology). Define \( \mathcal{E} = Frac O_{\mathcal{E}} \), and \( \mathcal{E}^{ur} \) the \( p \)-adic completion of the maximal (algebraic) unramified extension of \( \mathcal{E} \) in \( W(Frac R)[1/p] \). Denote \( O_{\mathcal{E}^{ur}} \) its ring of integers and put \( \mathcal{E}^{ur} = W(R) \cap O_{\mathcal{E}^{ur}} \). Clearly \( \mathcal{E}^{ur} \) is subring of \( W(R) \) and one can check (see Proposition 2.2.1 of [19]) that it induces an embedding \( \mathcal{S}^{ur} \hookrightarrow \mathcal{E}^{ur}/p^n W^{ur} \hookrightarrow W_n(R) \). Remark finally that all previous rings are endowed with a Frobenius action.

Recall that \( G \) (resp. \( G_s \)) is the absolute Galois group of \( K \) (resp. \( K_s = K/\pi_s \)) and that \( G_{\infty} \) is the intersection of all \( G_s \). Denote by \( \text{Rep}_{\mathcal{E}}^{\text{free}}(G_{\infty}) \) (resp. \( \text{Rep}_{\mathcal{E}}^{\text{tor}}(G_{\infty}) \)) the category of free (resp. torsion) \( \mathcal{E} \)-representations of \( G_{\infty} \). We define functors \( T_{\mathcal{E}} : \text{Mod}^\varphi_r \rightarrow \text{Rep}_{\mathcal{E}}^{\text{free}}(G_{\infty}) \) and \( T_{\mathcal{E}^{ur}} : \text{Mod}^\varphi_r \rightarrow \text{Rep}_{\mathcal{E}^{ur}}^{\text{tor}}(G_{\infty}) \) by:

\[
T_{\mathcal{E}}(\mathfrak{M}) := \text{Hom}_{\mathcal{E}}(\mathfrak{M}, \mathcal{E}^{ur}) \quad \text{and} \quad T_{\mathcal{E}^{ur}}(\mathfrak{M}) := \text{Hom}_{\mathcal{E}^{ur}}(\mathfrak{M}, \mathcal{E}^{ur})
\]

where \( \text{Hom}_{\mathcal{E}} \) means that we take all \( \mathcal{E} \)-linear morphism that commutes with Frobenius. Note that \( T_{\mathcal{E}}(\mathfrak{M}) \) and \( T_{\mathcal{E}^{ur}}(\mathfrak{M}) \) are not representations of \( G \) because this group does not act trivially on \( \mathcal{E} \subset W(R) \). If \( n' \geq n \) then any object \( \mathfrak{M} \) of \( \text{Mod}^\varphi_r \) is obviously also in \( \text{Mod}^{\varphi_{n'}}_r \), and we have a canonical identification \( T_{\mathcal{E}^{ur}}(\mathfrak{M}) \simeq T_{\mathcal{E}^{ur}}(\mathfrak{M}_{n'}) \). This fact allows us to glue all functors \( T_{\mathcal{E}^{ur}} \) and define \( T_{\mathcal{E}^{ur}} : \text{Mod}^\varphi_r \rightarrow \text{Rep}_{\mathcal{E}^{ur}}^{\text{tor}}(G_{\infty}) \). An important result is the exactness of \( T_{\mathcal{E}^{ur}} \) (see Corollary 2.3.4 of [19]).

Lemma 2.2.1 (Fontaine). Let \( n \) be an integer and \( \mathfrak{M} \) be an object of \( \text{Mod}^\varphi_r \). The embedding \( \mathcal{S}^{ur} \hookrightarrow W_n(R) \) induces an isomorphism \( T_{\mathcal{E}}(\mathfrak{M}) \simeq \text{Hom}_{\mathcal{E}}(\mathfrak{M}, W_n(R)) \).

Proof. See Proposition B.1.8.3 of [10].

2.3. The modules \( J_{n,s}(\mathfrak{M}) \). Let \( n \) be an integer and \( \mathfrak{M} \) an object of \( \text{Mod}^\varphi_r \). For all non negative real number \( c \), we define \( a^{<c}_R = \{ x \in R/ v_R(x) > c \} \) and \( a^{>c}_R \) the ideal of \( W_n(R) \) generated by all \( [x] \) with \( x \in a^{<c}_R \) and, by the same way, \( a^{<c}_R \) and \( a^{>c}_R \). We have very explicit descriptions of these ideals:

Lemma 2.3.1. Let \( c \in \mathbb{R}^+ \). Then:

1. For all \( x_0, \ldots, x_{n-1} \in R \), \( (x_0, \ldots, x_{n-1}) \) is in \( [a^{c}] \) (resp \( [a^{<c}] \)) if and only if \( v_R(x_i) > p^c \) (resp. \( v_R(x_i) \geq p^c \)) for all \( i \);

2. If \( \gamma \in R \) has valuation \( c \), then \( [a^{>c}] \) is the principal ideal generated by \( [\gamma] \).

Proof. Easy with the formula \([z](x_0, \ldots, x_{n-1}) = (zx_0, z^p x_1, \ldots, z^{p^{n-1}} x_{n-1})\).
Since \([a,c]_R\) is stable under \(\varphi\) and \(G\)-action, the quotient \(W_n(R)/[a,c]_R\) inherits a Frobenius action and it makes sense to define:

\[
J_{n,c}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R)/[a,c]_R).
\]

It is endowed with an action of \(G\). Let's also denote \(J_{n,\infty} = \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R)) \simeq T_{\mathfrak{S},\alpha}(\mathfrak{M})\) (Lemma 2.2.1). Obviously, if \(c \leq \varepsilon \leq \infty\), reduction modulo \([a,c]_R\) defines a natural \(G\)-equivariant morphism \(\rho_{\varepsilon,c} : J_{n,c}(\mathfrak{M}) \rightarrow J_{n,c}(\mathfrak{M})\). If \(c \leq \varepsilon' \leq \infty\), we have \(\rho_{\varepsilon',c} = \rho_{\varepsilon,c} \circ \rho_{\varepsilon',\varepsilon}\).

**Lemma 2.3.2.** \(u\) is nilpotent in \(W_n[u]/E(u)^r\).

**Proof.** Since \(E(u)\) is an Eisenstein polynomial, the congruence \(E(u) \equiv a^r \pmod{p}\) holds in \(W[u]\). Hence \(E(u)^r \equiv u^{\varepsilon r} \pmod{p}\), which means that \(u^{\varepsilon r}\) is divisible by \(p\) in \(W[u]/E(u)^r\). It follows that \(p^n\) divides \(u^{\varepsilon r}\) in \(W[u]/E(u)^r\), i.e. \(u^{\varepsilon r}\) vanishes in \(W_n[u]/E(u)^r\).

Fix \(N\) a positive integer such that \(u^N = 0\) in \(W_n[u]/E(u)^r\). By previous proof one can take \(N = \varepsilon r\), but in many situations this exponent can be improved. In the following subsection, we will examine several examples. From now on, we put \(b = \frac{N}{p-1}\) and \(a = b + N = \frac{N}{p-1}\).

**Proposition 2.3.3.** The morphism \(\rho_{\infty,b} : T_{\mathfrak{S},\alpha}(\mathfrak{M}) \rightarrow J_{n,b}(\mathfrak{M})\) is injective and its image is \(\rho_{a,b}(J_{n,\alpha})(\mathfrak{M})\).

**Proof.** We first prove injectivity. Let \(f : \mathfrak{M} \rightarrow [a,b]_R\) be a \(\varphi\)-morphism. We want to show that \(f = 0\). First, remark that since \(\mathfrak{M}\) is finitely generated, values of \(f\) are in \([a,b]_R\) for some \(b' > b\). Let \(x \in \mathfrak{M}\). By definition of \(N\), \(u^N x\) belongs to \(E(u)^r \mathfrak{M}\). By condition (2.1.1) we write \(u^N x = \lambda_1 \varphi(x_1) + \cdots + \lambda_k \varphi(x_k)\). Applying \(f\), we get:

\[
u^N f(x) = \lambda_1 \varphi(f(x_1)) + \cdots + \lambda_k \varphi(f(x_k)) \in [a,b]_R^{\varepsilon r
}
\]

and then \(f(x) \in [a,b]_R^{\varepsilon r-N}\) (since \(u = [x]\)). Repeating the argument again and again, we see that \(f(\mathfrak{M}) \subset \bigcap_{b \geq b'} [a,b]_R^{\varepsilon r-N} W_n(R)\) where \((b_i)\) is the sequence defined by \(b_0 = b'\) and \(b_{i+1} = pb_i - N\). Now \(b' > \frac{N}{p-1}\) implies \(\lim b_i = \infty\). Injectivity follows.

Let's prove the second part of the proposition. Since \(\rho_{\infty,b}\) factors through \(\rho_{a,b}\) we certainly have \(\rho_{\infty,b}(J_{n,\infty}) \subset \rho_{a,b}(J_{n,\alpha})\). Conversely, we want to prove that if \(f : \mathfrak{M} \rightarrow W_n(R)/[a,a^{\varepsilon}]_R\) is a \(\varphi\)-morphism, then there exists a \(\varphi\)-morphism (necessarily unique) \(g : \mathfrak{M} \rightarrow W_n(R)\) such that \(g \equiv f \pmod{[a,b]_R}\). Assume first \(\mathfrak{M} \in \text{Free}_{\mathfrak{S},\alpha}\) and pick \(e_1, \ldots, e_d\) a basis of \(\mathfrak{M}\) over \(\mathfrak{S},\alpha\). Let \(A\) be a matrix with coefficients in \(\mathfrak{S}\) such that:

\[(\varphi(e_1) - e_1, \ldots, \varphi(e_d) - e_d) = (\alpha_1, \ldots, \alpha_d) A\]

and let \(X\) be a line vector with coefficients in \(W_n(R)\) that lifts \((f(e_1), \ldots, f(e_d))\).

The commutation of \(f\) and \(\varphi\) implies \(X A \equiv \varphi(X) \pmod{[a,a^{\varepsilon}]_R}\). Actually, the congruence holds in \([a,a^{\varepsilon}]_R\) for some \(a' > a\). For the rest of the proof, fix \(\alpha \in R\) some element of valuation \(a'\). By Lemma 2.3.1.2, \([a,a^{\varepsilon}]_R\) is the principal ideal generated by \([a]\). Therefore, one has \(X A - \varphi(X) = -[a] Q\) with coefficients of \(Q\) in \(W_n(R)\). We want to prove that there exists a matrix \(Y\) with coefficients in \([a,a^{\varepsilon}]_R\) such that \((X + Y) A = \varphi(X + Y)\). Let us search \(Y\) of the shape \([\beta] Z\) with \(\beta = \frac{1}{[a]}\) (which belongs to \(R\) because of valuators) and coefficients of \(Z\) in \(W_n(R)\). Our condition then becomes:

\[
[\beta] Z A = [\beta] \varphi(Z) + [a] Q
\]

Using condition (2.1.1) and \(u^N \in E(u)^r W_n[u]\), we find a matrix \(B\) (with coefficients in \(\mathfrak{S}\)) such that \(BA = u^N\). Multiplying (2.3.2) by \(B\) on the left and simplifying by \([a]\), we get the new equation:

\[
Z = [\gamma] \varphi(Z) B + Q B
\]

with \(\gamma = a^{p-1}/u^N\). Remark that \(v_R(\gamma) = a' (p-1) - N > 0\); hence \(\gamma \in R\). Now define a sequence \((Z_i)\) by \(Z_0 = 0\) and \(Z_{i+1} = [\gamma] \varphi(Z_i) B + Q B\). We have \(Z_{i+1} - Z_i = [\gamma] \varphi(Z_i - Z_{i-1}) B\). Since \(v_R(\gamma) > 0\), \(Z_{i+1} - Z_i\) goes to 0 for the \(u\)-adic topology (which is separate and complete on \(W_n(R)\)) when \(i\) goes to infinity. Hence \((Z_i)\) converges to a limit \(Z\) which is solution of (2.3.3).
Finally, if $\mathfrak{M}$ is just an object of $\text{Mod}_{\mathfrak{S}_a}^{\phi,r}$ consider $\mathfrak{M}' \in \text{Free}_{\mathfrak{S}_a}^{\phi,r}$ and a surjective map $f: \mathfrak{M}' \to \mathfrak{M}$. Then $\ker f$ is in $\text{Mod}_{\mathfrak{S}_a}^{\phi,r}$ and sits in the following diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & J_{n,a}(\mathfrak{M}) \\
\downarrow & & \downarrow \\
T_{\phi,r}(\mathfrak{M}) & \rightarrow & J_{n,b}(\mathfrak{M}) \\
\downarrow & & \downarrow \\
T_{\phi,r}(\mathfrak{M}') & \rightarrow & J_{n,a}(\mathfrak{M}') \\
\downarrow & & \downarrow \\
T_{\phi,r}(\ker f) & \rightarrow & J_{n,b}(\ker f)
\end{array}
$$

All columns are exact (by left exactness of $\text{Hom}$) and the map on last line is injective (by first part of proposition). An easy diagram chase then ends the proof.

**Remark 2.3.4.** In general, $\rho_{a,b}$ is not surjective (nor injective) even for $a$ and $b$ big enough. Counter examples are very easy to produce: for instance, $\mathfrak{M} = \mathfrak{S}_1$ equipped with $\varphi(\epsilon) = E(u)^r$ is convenient.

### 2.4. Brief discussion about sharpness of $N$.

Here we are interested in finding integers $N$ (as small as possible) such that $u^N = 0$ in $W_n[u]/E(u)^r$. As we have said before $N = e n$ is always convenient. If $n = 1$, it is obviously the best constant. However, it is not true anymore for bigger $n$: the three following lemmas could give better exponents in many cases. We do not know how to find the sharpest $N$ in general.

In this paragraph, we will denote by $[x]$ the smallest integer not less than $x$.

**Lemma 2.4.1.** We have $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = ep^{n-1}[\frac{1}{p} - 1]$.

**Proof.** Just remark that $E(u)p^{n-1} \equiv u^{p^{n-1}} \pmod{p^n}$.

**Lemma 2.4.2.** Assume $E(u) = u^e - p$. Then $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = e(n + r - 1)$.

**Remark 2.4.3.** If $K/W[1/p]$ is tamely ramified, up to changing $K$ by an unramified extension, we can always select an uniformizer whose minimal polynomial is $E(u) = u^e - p$.

**Proof.** Up to performing the variables change $v = u^e$, one may assume $e = 1$. We then have an isomorphism $f : K[u]/E(u)^r \to K^r$, $P \mapsto (P(p), P'(p), \ldots, \frac{P^{(r-1)}(p)}{r!})$ whose inverse is given by $f^{-1}(x_0, \ldots, x_{r-1}) = x_0 + x_1(u - p) + \cdots + x_{r-1}(u - p)^{r-1}$. In particular $f(W_n[u]/E(u)^r) \supset W^r$. Moreover:

$$f(u^N) = \left(p^N, Np^{N-1}, \ldots, \binom{N}{r-1}p^{N-r+1}\right) \in p^{N-r+1}W^r = p^nW^r.$$

Conclusion follows.

**Lemma 2.4.4.** There exists a constant $c$ depending only on $K$ such that $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = en + c(r - 1)$.

**Proof.** The general plan of the proof is very similar to the previous one. We first consider the map $f : W[1/p][u]/E(u)^r \to K^r$, $P \mapsto (P(\pi), P'(\pi), \ldots, \frac{P^{(r-1)}(\pi)}{r!})$. It is $W[1/p]$-linear and injective. Since both sides are $W[1/p]$-vector spaces of dimension $er$, $f$ is an isomorphism. Denote by $\varpi \in W[1/p][u]/E(u)^r$ the preimage of $(\pi, 0, \ldots, 0)$. The inverse of $f$ is then given by the formula:

$$f^{-1}(x_0, \ldots, x_{r-1}) = X_0(\varpi) + X_1(\varpi)(u - \varpi) + \cdots + X_{r-1}(\varpi)(u - \varpi)^{r-1},$$

where $X_i$ are polynomials with coefficients in $W[1/p]$ such that $X_i(\pi) = x_i$. Second, we would like to bound below the “$p$-adic valuation” of $f^{-1}(x_0, \ldots, x_{r-1})$ when all $x_i$’s lies in $\mathcal{O}_K$. For that, we remark that $E(\varpi)$ is mapped to $0$ by $f$; hence it vanishes. Solving this equation by successive
approximations, we find that $\varpi$ can be written $P_0(u) + P_1(u)E(u) + \cdots + P_{r-1}(u)E(u)^{r-1}$ with $P_0(u) = u$ and:

$$E'(u)P_1(u) \equiv \frac{E(P_0(u) + P_1(u)E(u) + \cdots + P_{r-1}(u)E(u)^{r-1})}{E(u)^{r}} \pmod{E(u)}$$

where $P_i$ are uniquely determined modulo $E(u)^{r-1}$. Let $F(u) \in W[1/p][u]/E(u)$ be the inverse of $E'(u)$ and $v$ an integer such that $p^v F(u) \in W[u]/E(u) \simeq \mathcal{O}_K$. (Note that $v = \lfloor v_F(D_K/W[1/p]) \rfloor$ is convenient.) By induction we easily prove that $p^v P_1(u) \in W[u]/E(u)^{r-1}$, and then that $Q(\varpi) \in W[u]/E(u)^r$ for all $Q \in p^{(r-1)v}W[u]$. Consequently $f(W[u]/E(u)^r) \supset p^{(r-1)v}\mathcal{O}_K$. Finally, defining $c = ev + 1$ and $N = cn + c(r-1)$, we have:

$$f(u^N) = \left(\begin{array}{c} N \\ 1+n-r+1 \end{array}\right) \in \pi^{N-r+1}, \mathcal{O}_K \subset p^{(r-1)v+n}, \mathcal{O}_K$$

and we are done.

### 2.5. Some quotients of $W_n(R)$

The aim of this last subsection is to study the structure of quotients $W_n(R)/[a_R^{c}\mathcal{O}_K]$ that appears in the definition of $J_{n,c}$ (see formula (2.3.1)). It will allow us to derive interesting corollaries about the prolongation to a finite index subgroup of $G$ of the natural action of $G_{\infty}$ on $T_{\Delta_n}(\mathfrak{N})$.

For a non negative integer $s$, let us denote by $\theta_s$ the ring morphism $R \to \mathcal{O}_K/p, x = (x^{(0)}, x^{(1)}, \ldots) \mapsto x^{(s)}$. We emphasize that it is not $k$-linear: it induces a morphism of $k$-algebras between $R$ and $k \otimes_{k,s} \mathcal{O}_K/p$. For a non negative real number $c$, define:

$$a_R^{c} = \{x \in \bar{K}/v_K(x) > c\} \subset \mathcal{O}_K.$$

**Lemma 2.5.1.** Let $c$ be a positive real number. For any integer $s > \log_p(\frac{c}{p})$, the map $\theta_s$ induces a Galois equivariant isomorphism of $k$-algebras

$$R/a_{K}^{c} \to k \otimes_{k,s} \mathcal{O}_K/a_{K}^{c/p^s}.$$

**Proof.** The map is clearly surjective. It remains to show that $x = (x^{(0)}, x^{(1)}, \ldots)$ has valuation greater than $c$ if and only if $v_K(x^{(s)}) > \frac{c}{p^s}$, which follows directly from $\frac{c}{p} < c$.

**Proposition 2.5.2.** Let $c$ be a positive real number. For any $s > n - 1 + \log_p(\frac{c}{p})$, $\theta_s$ induces a Galois equivariant isomorphism of $W_n(k)$-algebras:

$$\frac{W_n(R)}{[a_R^{c}]} \to W_n(k) \otimes_{W_n(k),\sigma^s} \frac{W_n(\mathcal{O}_K/p)}{[a_{K}^{c/p^s}]}.$$

**Proof.** Since $\theta_s$ is surjective, the above map is also surjective. Let $x = (x_0, \ldots, x_{n-1}) \in W_n(R)$ and assume that $x^{(s)} = (x^{(0)}_s, \ldots, x^{(n-1)}_s)$ lies in $[a_{K}^{c/p^s}]$. By an analogue of Lemma 2.3.1.1, one obtain $v_K(x^{(s)}_i) > \frac{c}{p^{i+s}}$ for all $i$. Hence, $x^{(s)}_i$ is in $a_{K}^{c/p^s}$. Since $\log_p(\frac{c}{p}) = i + \log_p(\frac{c}{p}) < n - 1 + \log_p(\frac{c}{p}) < s$, we can apply Lemma 2.5.1 and deduce $x_i \in [a_{K}^{c/p^s}]$, i.e. $v_K(x_i) > c/p^s$ for all $i$. By Lemma 2.3.1.1, it follows that $x \in [a_{K}^{c}]$. Thus, the map of the proposition is injective and we are done.

Define increasing functions $s_0$ and $s_1$ by $s_0(c) = n - 1 + \log_p(\frac{c}{p})$ and $s_1(c) = n - 1 + \log_p(\frac{c(p-1)}{ep^{2}}) = s_0(c) + \log_p(1 - \frac{1}{p})$. Recall that we have defined $a = \frac{pN}{1+1}$ (where $N$ is an integer such that $u^{N} = 0$ in $W_n[u]/E(u)$) and set $s_{\min} = s_1(a) = n - 1 + \log_p(\frac{c}{e})$. If we choose $N = \kappa n$, we just have $s_{\min} = n - 1 + \log_p(\kappa n)$.

**Proposition 2.5.3.** Let $n$ be a positive integer and $\mathfrak{M} \in \text{Mod}_{\Delta_n}$. For any non negative integer $s > s_1(c)$, the natural action of $G_s$ on $W_n(R)$ turns $J_{n,c}(\mathfrak{M})$ into a $\mathbb{Z}_p[G_s]$-module. Furthermore, we have the following compatibilities:

- the action of $G_s$ is compatible with the usual action of $G_{\infty}$ on $J_{n,c}(\mathfrak{M})$;
- if $s' \geq s \geq s_1(c)$, actions of $G_{s'}$ and $G_s$ on $J_{n,c}$ are compatible each other;
- if $c' \geq c$ and $s \geq s_1(c')$, then $p_{c',c} : J_{n,c}(\mathfrak{M}) \to J_{n,c}(\mathfrak{M})$ is $G_s$-equivariant.
Proof. For the first statement, it is enough to show that $G_s$ acts trivially on $u \in \mathcal{W}_n(R)[a_p^{\log n}]$ for $s = 1 + [s_1(c)]$ (where $[\cdot]$ denotes the integer part). Put $s' = 1 + [s_0(c)]$. Since $0 \leq \log_p(p^{-s_0}) \leq 1$, we have $s' = s$ or $s' = s + 1$. By Proposition 2.5.2, $W_n(R)[a_p^{\log n}]$ is isomorphic to $W_n(O_K/p^s)[a_p^{\log n}]$. Hence we have to show that $\eta[\pi_s] - [\pi_s]$ belongs to $[a_p^{\log n}]$ for all $g \in G_s$. It is clear for $g \in G_{s'}$ (since the difference vanishes). It remains to consider the case where $s' = s + 1$ and $g \notin G_{s'} = G_{s+1}$. Then $g\pi_{s+1} = (1 + \eta)\pi_{s+1}$ where $(1 + \eta)$ is a primitive $p$-th root of unity. Let us compute $(g\pi_{s+1}, 0, \ldots, 0) - (\pi_{s+1}, 0, \ldots, 0) = (x_0, \ldots, x_{n-1})$ in $W_n(O_K)$. By writing phantom components, we get the following system:

\[
\begin{align*}
x_0 &= \eta \pi_{s+1} \\
x_0^p + px_1 &= 0 \\
\vdots \\
x_0^{p^{n-1}} + px_1^{p^{n-2}} + \cdots + p^{n-1}x_{n-1} &= 0
\end{align*}
\]

Using $v_K(\eta) = \frac{s}{p-1}$, we easily prove by induction on $i$ that $v_K(x_i) = \frac{s}{p-1} + \frac{1}{p^i-1}$. Thus $v_K(x_i) > \frac{s}{p^i-1}$ for all $i$ and $(x_0, \ldots, x_{n-1}) \in [a_p^{\log n}]$ as expected.

Second part of proposition (i.e. compatibilities) is obvious. \hfill \square

Remark 2.5.4. If $c \geq \frac{s_{\min}}{p-2}$, the bound $s_1(c)$ that appears in the Theorem can be replaced by $s_1(c-1)$. The proof is totally the same.

Theorem 2.5.5. For any $\mathfrak{M} \in \text{Mod}_{\mathcal{S}}^{c,r}$ and any integer $s > s_{\min}$, $T_{\mathfrak{M}}(\mathfrak{M})$ is canonically endowed with an action of $G_s$ (which prolongs the natural action of $G_{s-1}$).

Proof. Just combine Propositions 2.3.3 and Proposition 2.5.3. \hfill \square

Remark 2.5.6. Using Remark 2.5.4, it appears that we may replace $s_{\min} = s_1(a)$ by $s_1(a-1)$ in previous Theorem. However, it won’t be useful in the sequel since $s_{\min}$ is really needed in Theorem 3.3.4.

3. Torsion semi-stable Galois representations

In this section, we use the theory of $(\varphi, \hat{\Gamma})$-modules to define $\hat{J}_{n,a}(\mathfrak{M})$ attached to $p^n$-torsion semi-stable representation $T$. After establishing isomorphism (of $\mathbb{Z}_p[G_{s-1}]$-modules) between $\hat{J}_{n,a}(\mathfrak{M})$ and $J_{n,a}(\mathfrak{M})$, we will show that $J_{n,a}(\mathfrak{M}) \simeq T$ as $G_s$-modules with $s > s_{\min}$.

3.1. Torsion $(\varphi, \hat{\Gamma})$-modules. We refer readers to [13] for the definition and standard facts on semi-stable representations.

We first review some facts on $(\varphi, \hat{\Gamma})$-modules in [20] and extend them to $p^n$-torsion case. We denote by $S$ the $p$-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There is a unique continuous map (Frobenius) $\varphi : S \to S$ which extends the Frobenius on $\mathcal{S}$. Define a continuous $W(k)$-linear derivation $N : S \to S$ such that $N(u) = -u$.

Recall $R = \lim_{\rightarrow} O_K/p$. There is a unique surjective continuous map $\theta : W(R) \to \hat{O}_K$ which lifts the projection $R \to O_K/p$ onto the first factor in the inverse limit. We denote by $A_{\text{cris}}$ the $p$-adic completion of the divided power envelope of $W(R)$ with respect to $\text{Ker}(\theta)$. Recall that $[\pi] \in W(R)$ is the Teichmüller representative of $\pi = (\pi_n)_{n \geq 0} \in R$ and we embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via $u \mapsto [\pi]$. Since $\theta([\pi]) = \pi$, this embedding extends to an embedding $\mathcal{S} \hookrightarrow A_{\text{cris}}$, and $\theta|S$ is the $W(k)$-linear map $\pi : S \to O_K$ defined by sending $u$ to $\pi$. The embedding is compatible with Frobenius endomorphisms. As usual, we write $B_{\text{cris}}^n := A_{\text{cris}}[1/p]$.

For any field extension $F/q_p$, set $F_{\infty} := \bigcup_{n=1}^{\infty} F(q_p^n)$ with $q_p^n$ a primitive $p^n$-th root of unity. Note that $K_{\infty,q_p^n} := \bigcup_{n=1}^{\infty} K_{(\pi_n,q_p^n)}$ is Galois over $K$. Let $G_{q_p^n} := \text{Gal}(K_{\infty,q_p^n}/K_{p^n})$, $H_K := G_{q_p^n} \cap K_{\infty,q_p^n}$, and $\hat{G} := G_{q_p^n} \rtimes H_K$ and $G_{p^n} \simeq \mathbb{Z}_p(1)$. For any $g \in G$, write $\xi(g) = g([\pi])/$. Then $\xi(g)$ is a cocycle.
from $G$ to the group of units of $R^*$. In particular, fixing a topological generator $\tau$ of $G_{p^\infty}$, the fact that $\hat{G} = G_{p^\infty} \rtimes H_K$ implies that $\xi(\tau) = (\xi_s)_{s \geq 0} \in R^*$ with $\xi_s$ a primitive $p^s$-th root of unity. Therefore, $t := -\log([\xi(\tau)]) \in A_{\text{cris}}$ is well defined and for any $g \in G$, $g(t) = \chi(g)t$ where $\chi$ is the cyclotomic character. We reserve $\xi$ for $\xi(\tau)$.

For any integer $n \geq 0$, let $t^{(n)} = t^{(n)}(\xi(\tau))$ where $n = (p-1)\xi(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. Define subrings $R_{K_n}$ and $\hat{R}$ of $B_{\text{cris}}^+$ as in §2.2, [20]:

$$R_{K_n} := \left\{ x = \sum_{i=0}^{\infty} f^i t^{(i)}, f_i \in S[1/p] \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\}$$

and $\hat{R} := W(R) \cap R_{K_n}$. Let $I_+R = \{ x \in R / \nu_p(x) > 0 \} = a_{\hat{R}}^{+\infty}$ be the maximal ideal of $R$. We have exact sequences

$$0 \to W_n(I_+R) \to W_n(R) \overset{\nu}{\to} W_n(\hat{k}) \to 0 \quad 0 \to W(I_+R) \to W(R) \overset{\nu}{\to} W(\hat{k}) \to 0$$

where $\nu_p$ are $\nu$ are induced by the composite $R \to \mathcal{O}_K/p \to \hat{k}$, the first map being the projection onto the first factor in the inverse limit. One can naturally extend $\nu$ to $\nu : B_{\text{cris}}^+ \to W(\hat{k})[1/p]$ (see the proof of Lemma 2.2.1 in [20]). For any subring $A$ of $B_{\text{cris}}^+$ (resp. $W_n(R)$), we write $I_+A = \text{Ker}(\nu) \cap A$ (resp. $I_+A = \text{Ker}(\nu_p) \cap A$) and $I_+ := I_+\hat{R}$. Now recall $M_0$ stands for $M/p^nM$.

**Lemma 3.1.1.** We have the following commutative diagram :

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & W_n(I_+R) & \longrightarrow & W_n(R) & \longrightarrow & W_n(\hat{k}) & \longrightarrow & 0 \\
0 & \longrightarrow & I_+ & \longrightarrow & \hat{R}_n & \longrightarrow & W_n(k) & \longrightarrow & 0
\end{array}
\] (3.1.1)

such that both rows are short exact and all vertical arrows are injective.

**Proof.** By Lemma 2.2.1 in [20], we have a commutative diagram of exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I_+W(R) & \longrightarrow & W(R) & \longrightarrow & W(\hat{k}) & \longrightarrow & 0 \\
0 & \longrightarrow & I_+ & \longrightarrow & \hat{R} & \longrightarrow & W(k) & \longrightarrow & 0
\end{array}
\]

Modulo $p^n$ and noting that $I_+W(R) = I_+B_{\text{cris}}^+ \cap W(R) = W(I_+R)$, we get

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & W_n(I_+R) & \longrightarrow & W_n(R) & \longrightarrow & W_n(\hat{k}) & \longrightarrow & 0 \\
0 & \longrightarrow & I_+ & \longrightarrow & \hat{R}_n & \longrightarrow & W_n(k) & \longrightarrow & 0
\end{array}
\]

Now it suffices to show that the bottom arrow is left exact and the last two vertical arrows are injective. The last one is obvious. To see the middle arrow is injective, it suffices to show that $(p^nW(R)) \cap \hat{R} = p^n\hat{R}$. Note that $\hat{R} = R_{K_n} \cap W(R)$. Let $x \in W(R)$ such that $p^n x \in \hat{R} \subset R_{K_n}$. Then $x \in W(R) \cap R_{K_n} = \hat{R}$. So $p^n x \in p^n\hat{R}$ and $(p^nW(R)) \cap \hat{R} = p^n\hat{R}$. To see the bottom is left exact, it suffices to show that $I_+ \cap p^n\hat{R} = p^nI_+$. But $I_+ = I_+ \cap R_{K_n} \cap \hat{R}$. Let $x \in \hat{R}$ such that $p^n x \in I_+ \cap p^n\hat{R}$. Then $x \in I_+ \cap R_{K_n}$. Thus $x \in I_+ \cap R_{K_n} \cap \hat{R} = I_+ \cap p^n\hat{R} = p^nI_+$. \hfill \square

As in Lemma 2.1.1 in [20], we see that $\hat{R}$ (resp. $\hat{R}_n$) is $\varphi$-stable and $G$-action on $\hat{R}$ factors through $\hat{G}$. Let $(\mathfrak{M}, \varphi)$ be a finite free or $p^n$-torsion Kisin module of height $\leq r$, set $\mathfrak{M} := \hat{R} \otimes_{\varphi, s} \mathfrak{M}$ and consider the following composite

\[
\begin{array}{cccccc}
\mathfrak{M} & \simeq & S \otimes_{\varphi} \mathfrak{M} & \to & S \otimes_{\varphi, s} \mathfrak{M} & \to & \hat{R} \otimes_{\varphi, s} \mathfrak{M} = \mathfrak{M}
\end{array}
\]

where the first map is $\phi \otimes \text{id}$. We claim that it is an injective (thus $\mathfrak{M}$ can be always regarded as a $\varphi(S)$-submodule of $\mathfrak{M}$). Indeed, by Lemma 3.1.1, we have $\varphi(S_n) \hookrightarrow \hat{R}_n \hookrightarrow W_n(R)$. Thus
the claim is clear if $\mathcal{M}$ is finite $\mathcal{S}$-free or $\mathcal{M}$ is finite $S$-free. For a general $\mathcal{M}$ which is killed by $p^n$, by the discussion in the end of §2.1, $\mathcal{M}$ can be written as a successive extension of finite free $\mathcal{S}_1$-modules. Therefore one can reduce the proof of the claim to the following lemma.

**Lemma 3.1.2.** The functor $\mathcal{M} \mapsto \hat{R} \otimes_{\mathcal{S}_1, \phi} \mathcal{M}$ (resp. $\mathcal{M} \mapsto W(R) \otimes_{\phi, \mathcal{S}_1} \mathcal{M}$) is an exact functor from the category of Kisin modules to the category of $\hat{R}$-modules (resp. $W(R)$-modules).

**Proof.** We only prove the exactness of the first functor, the proof for the second being totally the same. It suffices to prove that $\text{Tor}^1_{\hat{R}}(\mathcal{M}, \hat{R}) = 0$ for any Kisin module $\mathcal{M}$. Note that there exists finite free Kisin modules $L_1 \subset L_2$ such that $\mathcal{M} = L_2/L_1$ (cf discussion in the end of §2.1). Since $\hat{R} \mapsto W(R)$ is an integral domain and $\varphi : W(R) \to W(R)$ is injective, we see $\hat{R} \otimes_{\mathcal{S}_1, \phi} L_1 \to \hat{R} \otimes_{\mathcal{S}_1, \phi} L_2$ is injective. Thus $\text{Tor}^1_{\hat{R}}(\mathcal{M}, \hat{R}) = 0$. \[\square\]

Let $(\mathcal{M}, \varphi)$ be a Kisin module of height $\leq r$ and $\hat{\mathcal{M}} := \hat{R} \otimes_{\mathcal{S}, \varphi} \mathcal{M}$. Frobenius $\varphi$ on $\mathcal{M}$ can be extended to $\hat{\mathcal{M}}$ semi-linearly by $\varphi_{\hat{\mathcal{M}}}(a \otimes x) = \varphi_{\hat{\mathcal{M}}}(a) \otimes \varphi(x)$. Now we can make the following definition: a $(\varphi, \hat{G})$-module of height $\leq r$ is a triple $(\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G})$ where

1. $(\mathcal{M}, \varphi_{\mathcal{M}})$ is a Kisin module of height $\leq r$;
2. $\hat{G}$ is a $\hat{R}$-semi-linear $\hat{G}$-action on $\mathcal{M} = \hat{R} \otimes_{\mathcal{S}_1, \phi} \mathcal{M}$;
3. $\hat{G}$ commutes with $\varphi_{\hat{\mathcal{M}}}$ on $\mathcal{M}$, i.e. for any $g \in \hat{G}$, $g \varphi_{\hat{\mathcal{M}}} = \varphi_{\hat{\mathcal{M}}} g$;
4. regard $\mathcal{M}$ as a $(\varphi(\mathcal{S}))$-submodule in $\hat{\mathcal{M}}$, then $\mathcal{M} \subset \hat{\mathcal{M}}_{\hat{G}}$;
5. $\hat{G}$ acts on $W(k)$-module $M := \mathcal{M}/I_\mathcal{M}$ semi-linearly trivially.

A morphism $f : (\mathcal{M}, \varphi, \hat{G}) \to (\mathcal{M}', \varphi', \hat{G}')$ is a morphism $f : (\mathcal{M}, \varphi) \to (\mathcal{M}', \varphi')$ of Kisin modules such that $\hat{R} \otimes_{\mathcal{S}_1, \phi} f : \mathcal{M} \to \mathcal{M}'$ is $\hat{G}$-equivariant. If $\mathcal{M} = (\mathcal{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module, we will often abuse notations by denoting $\hat{\mathcal{M}}$ the underlying module $\hat{R} \otimes_{\mathcal{S}_1, \phi} \mathcal{M}$. A $(\varphi, \hat{G})$-module $\mathcal{M} := (\mathcal{M}, \varphi, \hat{G})$ is called finite free (resp. $p^n$-torsion) if $\mathcal{M}$ is finite $\mathcal{S}$-free (resp. $\mathcal{M}$ is killed by $p^n$).

Let $\mathcal{M} = (\mathcal{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. We can associate $\mathbb{Z}_p[\hat{G}]$-modules:

$$\hat{T}(\mathcal{M}) := \text{Hom}_{\hat{R}, \varphi}(\mathcal{M}, W(R))$$

if $\mathcal{M}$ is finite $\mathcal{S}$-free.

and

$$\hat{T}_n(\mathcal{M}) := \text{Hom}_{\hat{R}, \varphi}(\mathcal{M}, W_n(R))$$

if $\mathcal{M}$ is of $p^n$-torsion.

Here $\hat{G}$ acts on $\hat{T}(\mathcal{M})$ (resp. $\hat{T}_n(\mathcal{M})$) via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in \hat{G}$ and $f \in \hat{T}(\mathcal{M})$ (resp. $\hat{T}_n(\mathcal{M})$). For any $f \in T_\phi(\mathcal{M})$ (resp. $T_{\phi_n}(\mathcal{M})$), set $\theta(f) \in \text{Hom}_{\hat{R}}(\mathcal{M}, W(R))$ (resp. $\theta_n(f) \in \text{Hom}_{\hat{R}}(\mathcal{M}, W_n(R))$) via:

$$\theta(f)(a \otimes x) = a \varphi(f(x))$$

for any $a \in \hat{R}$, $x \in \mathcal{M}$.

It is routine to check that $\theta : T_\phi(\mathcal{M}) \to \hat{T}(\mathcal{M})$ (resp. $\theta_n : T_{\phi_n}(\mathcal{M}) \to \hat{T}_n(\mathcal{M})$) is well-defined.

Denote by $\text{Rep}_{\text{tor}}(G)$ the category of $G$-representations on finite type $\mathbb{Z}_p$-modules which are killed by some $p$-power, and $\text{Rep}_{\text{tor}}^{\text{ss}}(G)$ the full subcategory of torsion semi-stable representations with Hodge-Tate weights in $\{0, \ldots, r\}$ in the sense that there exist $G$-stable $\mathbb{Z}_p$-lattices $\Lambda_1 \subset \Lambda_2 \subset V$ such that $V$ is semi-stable with Hodge-Tate weights in $\{0, \ldots, r\}$ and $T \simeq \Lambda_2/\Lambda_1$ as $\mathbb{Z}_p[G]$-modules. The following is the main result of this subsection.

**Theorem 3.1.3.**

1. Let $\hat{\mathcal{M}} := (\mathcal{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. Then $\theta$ (resp. $\theta_n$) induces a natural isomorphism of $\mathbb{Z}_p[\hat{G}_\infty]$-modules $\theta : T_\phi(\mathcal{M}) \simeq \hat{T}(\mathcal{M})$ (resp. $\theta_n : T_{\phi_n}(\mathcal{M}) \simeq \hat{T}_n(\mathcal{M})$).

2. $\hat{T}$ induces an anti-equivalence between the category of finite free $(\varphi, \hat{G})$-modules of height $\leq r$ and the category of $G$-stable $\mathbb{Z}_p$-lattices in semi-stable representations with Hodge-Tate weights in $\{0, \ldots, r\}$.

3. For any $T \in \text{Rep}_{\text{tor}}^{\text{ss}, r}(G)$, there exists a torsion $(\varphi, \hat{G})$-modules $\hat{\mathcal{M}}$ such that $\hat{T}_n(\hat{\mathcal{M}}) \simeq T$ as $\mathbb{Z}_p[G]$-modules.
Proof. (1) If $\mathcal{M}$ is finite $\mathfrak{S}$-free then it has been proved in Theorem 2.3.1 in [20]. The proof of the $p^n$-torsion case is almost the same, except one need to check that $\mathcal{M}$ is a $\varphi(\mathfrak{S})$-submodule of $\mathcal{M}$ via (3.1.2), which has been proved below (3.1.2).

(2) See Theorem 2.3.1 in [20].

(3) Let $\Lambda_1 \subset \Lambda_2$ be $G$-stable $\mathbb{Z}_p$-lattices inside a semi-stable representation with Hodge-Tate weights in $\{0, \ldots, r\}$ such that $T \simeq \Lambda_2/\Lambda_1$ as $\mathbb{Z}_p[G]$-modules. By (2), there exists an injection of Kisin modules (resp. $(\varphi, \hat{G})$-modules) $i : \mathcal{L}_2 \hookrightarrow \mathcal{L}_1$ (resp. $i : \hat{\mathcal{L}}_2 \hookrightarrow \hat{\mathcal{L}}_1$) that corresponds the inclusion $\Lambda_1 \subset \Lambda_2$. Write $\mathcal{M} := \mathcal{L}_1/\mathcal{L}_2$ (resp. $\hat{\mathcal{M}} := \hat{\mathcal{L}}_1/\hat{\mathcal{L}}_2$). Apparently, there are a $\varphi$-action and a $G$-action on $\mathcal{M}$ induced from $\mathcal{L}_1$ and $\mathcal{L}_2$. We claim that $\mathcal{M} \simeq \hat{\mathcal{R}} \otimes_{\varphi, \hat{G}} \mathcal{M}$ as $\varphi$-modules and $(\mathcal{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-modules. To see these, tensor $\hat{\mathcal{R}}$ to the exact sequence $0 \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{M} \to 0$. By the proof of Lemma 3.1.2, we see that the sequence $0 \to \hat{\mathcal{L}}_2 \to \hat{\mathcal{L}}_1 \to \hat{\mathcal{R}} \otimes_{\varphi, \hat{G}} \mathcal{M} \to 0$ is still exact. Thus $\mathcal{M} \simeq \hat{\mathcal{R}} \otimes_{\varphi, \hat{G}} \mathcal{M}$ as $\varphi$-modules. Moreover, we have the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow \hat{\mathcal{R}} \otimes_{\varphi, \hat{G}} \mathcal{M} \longrightarrow 0 \\
\mathcal{M} \downarrow \quad \mathcal{M} \downarrow \\
0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow 0
\end{array}
\]

So $\varphi$-action and $\hat{G}$-action on $\mathcal{M}$ commutes, $H_K$ acts on $\mathcal{M}$ (as $\varphi(\mathfrak{S})$-submodule in (3.1.2)) trivially, and $G$ acts on $\mathcal{M}/I_z\mathcal{M}$ trivially. Thus $\mathcal{M} = (\mathcal{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module. Finally, to see that $\hat{T}_n(\mathcal{M}) \simeq T$ as $\mathbb{Z}_p[G]$-modules, it suffices to show that $\hat{T}_n(\mathcal{M}) \simeq \hat{T}(\mathcal{L}_2)/\hat{T}(\mathcal{L}_1)$ and we reduce the proof to the following Lemma. □

Lemma 3.1.4. Let $0 \to \hat{\mathcal{L}}_2 \to \hat{\mathcal{L}}_1 \to \hat{\mathcal{M}} \to 0$ be an exact sequence of $(\varphi, G)$-modules with $\hat{\mathcal{L}}_1$, $\hat{\mathcal{L}}_2$ finite free and $\hat{\mathcal{M}}$ killed by $p^n$. Then we have an exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to \hat{T}(\mathcal{L}_1) \to \hat{T}(\mathcal{L}_2) \to \hat{T}_n(\mathcal{M}) \to 0$.

Proof. Let $m$ be an integer not less than $n$. Consider the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow p^m \mathcal{L}_2 \longrightarrow p^m \mathcal{L}_1 \longrightarrow \mathcal{M} \longrightarrow 0 \\
\mathcal{M} \downarrow \quad \mathcal{M} \downarrow \\
0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow 0
\end{array}
\]

where the last vertical map is $p^m = 0$. By Snake lemma, we have an exact sequence

\[(3.1.4) \quad 0 \to \mathcal{M} \to (\mathcal{L}_2)_m \to (\mathcal{L}_1)_m \to \mathcal{M} \to 0.\]

Then we get a sequence of $\mathbb{Z}_p[G]$-modules

\[(3.1.5) \quad 0 \to \hat{T}_n(\mathcal{M}) \to \hat{T}_m((\mathcal{L}_1)_m) \to \hat{T}_m((\mathcal{L}_2)_m) \to \hat{T}_n(\mathcal{M}) \to 0.\]

Since $\hat{T}_m((\mathcal{L}_i)_m) \simeq (\hat{T}(\mathcal{L}_i))_m$ for $i = 1, 2$, it suffices to show that the above sequence is exact. But the underlying Kisin modules of the exact sequence (3.1.4) is exact. Since $T_\mathfrak{S}$ is exact, we get an exact sequence

\[0 \to T_\mathfrak{S}_m(\mathcal{M}) \to T_\mathfrak{S}_m((\mathcal{L}_1)_m) \to T_\mathfrak{S}_m((\mathcal{L}_2)_m) \to T_\mathfrak{S}_m(\mathcal{M}) \to 0.\]

Now the exactness of (3.1.5) follows from Theorem 3.1.3.(1). □

Remark 3.1.5. For a fixed $T \in \text{Rep}_{\text{ss}, G}^G(G)$, it may exist two different $(\varphi, G)$-modules $\hat{\mathcal{M}}, \tilde{\mathcal{M}}$ such that $\hat{T}_n(\mathcal{M}) \simeq \hat{T}_n(\tilde{\mathcal{M}}) \simeq T$. The classical example of this is that $T = \mathbb{Z}/p\mathbb{Z}$ with the trivial $G$-action and $K = \mathbb{Q}_p(\zeta_p)$. 


3.2. $G_s$-action on $\hat{T}(\mathfrak{L})$. Let $T \in \text{Rep}_{\text{tor}}(G)$ be a $p^n$-torsion representation, and $T \simeq \Lambda'/\Lambda$ where $\Lambda \subseteq \Lambda'$ are $G$-stable $\mathbb{Z}_p$-lattices in a semi-stable representation $V$ with Hodge-Tate weights in $\{0,\ldots,r\}$. By Theorem 3.1.3, there exists $(\varphi, \mathfrak{G})$-modules $\mathfrak{L}' \hookrightarrow \mathfrak{L}$ such that $\bar{T}(\mathfrak{L}) \hookrightarrow \bar{T}(\mathfrak{L'})$ corresponds to the injection $\Lambda \subseteq \Lambda'$ and $\bar{T}(\Omega(\mathfrak{L})) \simeq T$ where $\mathfrak{M} := \mathfrak{L}/\mathfrak{L}'$. Now write $\mathfrak{L}$, $\mathfrak{L}'$, $\mathfrak{M}$ the underlying Kisin modules for $\mathfrak{L}$, $\mathfrak{L}'$, $\mathfrak{M}$ respectively. Set $\mathcal{D} := S[1/p] \otimes_{\varphi, \mathfrak{G}} \mathfrak{L}$ and recall $\mathfrak{g}(g) := \frac{d(g)}{\mathfrak{g}}$ for any $g \in G$. §3.2 of [20] explains that there exists an unique $W(k)$-linear differential operator $N : \mathcal{D} \to \mathcal{D}$ over $\mathbb{N}(u) = -u$ such that $G$ acts on $B^+_\text{cris} \otimes \mathcal{D} \simeq B^+_{\text{cris}} \otimes_{\mathbb{R}} \mathfrak{L}$ via

\[
(3.2.1) \quad g(a \otimes x) = \sum_{i=0}^{\infty} g(a_i) \gamma_i(-\log(\mathfrak{g}(g[i]))) \otimes N^i(x), \quad \text{for any } a \in B^+_{\text{cris}}, \ x \in \mathcal{D}.
\]

In particular, recall $t := -\log(\mathfrak{g}(\mathfrak{l}))$ with $\mathfrak{g} = \mathfrak{g}(\tau)$ and $\tau$ is a fixed generator in $G_p$. For any $x \in \mathfrak{L}$, we have $\tau(x) = \sum_{i=0}^{\infty} \gamma_i(t) \otimes N^i(x)$. Let $A \subset B^+_{\text{cris}}$ be a $\varphi$-stable subring. Set

\[
I^m A = \{a \in A | \varphi^n(a) \in A \cap \text{Fil}^m B^+_{\text{cris}}, \text{ for all } n \geq 0\}.
\]

By proposition 5.1.3 in [12], $I^m W(R)$ is generated by $(\mathfrak{l} - 1)^m$ and $v_p(\mathfrak{l} - 1) = \frac{p-1}{p^m}$. Now, define $s_2(c) := n - 1 + \log_p\left(\frac{(p-1)c}{e}\right) = s_1(c) + 1$. We have the following lemma:

**Lemma 3.2.1.** For any $s > s_2(c), g \in G_s$, and $x \in \mathfrak{L}$

\[
g(x) - x = \left([a^s_{\mathfrak{R}}] + p^n W(R)\right) \otimes_{\mathbb{R}} \mathfrak{L}
\]

where, in a slight abuse of notations, we still denote by $[a^s_{\mathfrak{R}}]$ the ideal of $W(R)$ generated by all $[x]$ with $x \in a^s_{\mathfrak{R}}$.

**Proof.** Note that the $G$-action on $\mathfrak{L}$ factors through $\hat{G}$. So it suffices to consider the action of $\hat{G}_s$, which is the image of $G_s$ in $\hat{G}$. By Lemma 5.1.2 of [21] applied to $K_s$, we see that $G_s := G_{s,p,\infty} \rtimes H_K$, where $G_{s,p,\infty} := \text{Gal}(K_{\infty,p}/K_s)$. Note that $H_K$ acts on $\mathfrak{L}$ trivially and $G_{s,p,\infty}$ is topologically generated by $\tau^p$. Thus it suffices to prove the proposition for $g = \tau^p$. Writing

\[
(\tau - 1)^{i}(x) \in \left([a^s_{\mathfrak{R}}] + p^n v_p(i) W(R)\right) \otimes_{\mathbb{R}} \mathfrak{L}
\]

for all integer $i$ such that $v_p(i) > s - n$, i.e. $v_p(i) \geq s - n + 1$.

Using formula (3.2.1), an easy induction on $l$ shows that

\[
(3.2.2) \quad (\tau - 1)^{i}(x) = \sum_{m=0}^{\infty} \left(\sum_{i_1 + \ldots + i_l = m, i_k \geq 1} \frac{m!}{i_1! \ldots i_l!} \gamma_m(t) \otimes N^m(x)\right)
\]

for any $l \geq 0$ and $x \in \mathcal{D}$. In particular, $(\tau - 1)^{i}(x) \in (I^m B^+_{\text{cris}}(B^+_{\text{cris}} \otimes S \mathcal{D}))$. Since $x \in \mathfrak{L}$ and $W(R) \otimes_{\mathbb{R}} \mathfrak{L}$ is $G$-stable, we get $(\tau - 1)^{i}(x) \in (I^m W(R)(W(R) \otimes_{\mathbb{R}} \mathfrak{L}))$. So it suffices to show that $(\mathfrak{l} - 1)^{i} \in [a^s_{\mathfrak{R}}] + p^n v_p(i) W(R)$ for any $i$ satisfying $v_p(i) \geq s - n + 1$. Write $i = p^{v+n-s-1} m$ with $v \geq 0$ and $p \nmid m$. From $v_p(\mathfrak{l} - 1) = \frac{p-1}{p^m}$, it follows that $(\mathfrak{l} - 1)^{p^{v+n-s-1}} \in [a^s_{\mathfrak{R}}] + p^n W(R)$. By induction (on $v$), we easily find $(\mathfrak{l} - 1)^{p^{v+n-s-1}} \in [a^{s+n}_{\mathfrak{R}}] + p^n W(R)$, which is exactly the expected result.

\[\square\]

3.3. **Comparison between $J_{n,c}(\mathfrak{M})$ and $J_{n,c}(\mathfrak{M})$.** Let $T$ be a $p^n$-torsion semi-stable representation and $\mathfrak{M}$ an attached $(\varphi, \mathfrak{G})$-module via Theorem 3.1.3.(3). We have the following definitions and results similar to §2. For any non negative real number $c$, set

\[
J_{n,c}(\mathfrak{M}) := \text{Hom}_{\mathfrak{M}, G_p}([\mathfrak{M}], W_n(R)/[a^s_{\mathfrak{R}}]).
\]

and $J_{n,\infty}(\mathfrak{M}) := \hat{T}(\mathfrak{M})$. 

For any $c \leq \infty$, $\hat{J}_{n,c}(\hat{M})$ is a $\mathbb{Z}_p[G]$-module and, for any $c \leq c' \leq \infty$, the canonical projection induces a map $\hat{\rho}_{c,c'}: \hat{J}_{n,c}(\hat{M}) \to \hat{J}_{n,c'}(\hat{M})$. Moreover, to each $f \in J_{n,c}(\mathfrak{M})$, one can attach a morphism $\theta_{n,c}(f) \in \hat{J}_{n,c}(\mathfrak{M})$ defined by:

$$
(\forall a \in \hat{R}) (\forall x \in \mathfrak{M}) \quad \theta_{n,c}(f)(\alpha \otimes x) = \alpha \varphi(f(x)).
$$

**Proposition 3.3.1.** For any non negative integer $s > s_2(c) = n-1+\log_p(\frac{\mu-1}{c})$, $\theta_{n,c}: J_{n,c}(\mathfrak{M}) \to \hat{J}_{n,c}(\mathfrak{M})$ is an isomorphism of $\mathbb{Z}_p[G_s]$-modules.

**Remark 3.3.2.** Since $s_2(c) = s_1(c) + 1 \geq s_1(c)$, Proposition 2.5.3 shows that $J_{n,c}(\mathfrak{M})$ is endowed with an action of $G_s$. Hence it makes sense to claim that $\theta_{n,c}$ is $G_s$-equivariant. We do not know if Proposition 3.3.1 remains true under the smaller assumption "$s > s_1(c)$": we conjecture that it is false but we do not know any counter-example.

**Proof.** It is routine to check that $\theta_{n,c}(f)$ is well defined and preserves Frobenius. Hence $\theta_{n,c}$ is also well defined. Let’s first prove that it is bijective. Remark that $\varphi: W_n(R)/[a_R^{\geq c}] \cong W_n(R)/[a_R^{\geq pc}]$ is an isomorphism. It follows easily that $\theta_{n,c}$ is injective. For any $f \in \hat{J}_{n,c}(\mathfrak{M})$, set $f' := f|_{\mathfrak{M}}$ (recall that we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathfrak{M}$ via (3.1.2)). Then $f': \mathfrak{M} \to W_n(R)/[a_R^{\geq pc}]$ is $\varphi(\mathfrak{S})$-linear and is compatible with Frobenius. Since $\varphi: W_n(R)/[a_R^{\geq pc}] \cong W_n(R)/[a_R^{\geq pc}]$, we can set $f = \varphi^{-1}(f') : \mathfrak{M} \to W_n(R)/[a_R^{\geq pc}]$. It is finally easy to check that $f$ belongs to $J_{n,c}(\mathfrak{M})$ and that $\theta_{n,c}(f) = f$. Hence $\theta_{n,c}$ is surjective, as required.

It remains to prove that $\theta_{n,c}$ is $G_s$-equivariant. Let $g \in G_s$, $\alpha \in \hat{R}$ and $x \in \mathfrak{M}$. Expanding the definitions, we get $g(\theta_{n,c}(f)(\alpha \otimes x)) = \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x)))$. Moreover Lemma 3.2.1 shows that $g^{-1}(1 \otimes x)$ is congruent to $1 \otimes x$ modulo $[a_R^{\geq pc}]$ and hence that these two terms have same image under $\theta_{n,c}(f)$. Thus:

$$
\begin{align*}
\theta_{n,c}(f)(\alpha \otimes x) &= \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x))) = \alpha g(\theta_{n,c}(f)(1 \otimes x)) \\
&= \alpha g(\varphi(f(x))) = \alpha \varphi(g(f(x))) = \theta_{n,c}(g(f))(\alpha \otimes x)
\end{align*}
$$

and equivariance is proved. \hfill $\Box$

Recall that we have fixed an integer $N$ such that $u^N = 0$ in $W[u]/(u)^r$ and defined $b = \frac{N}{p-1}$ and $a = \frac{bN}{p}$. Combining Propositions 2.3.3 and 3.3.1, we directly get the following.

**Corollary 3.3.3.** The morphism $\hat{\rho}_{\infty,b}: \hat{T}_{\Theta_n}(\mathfrak{M}) \to \hat{J}_{n,b}(\mathfrak{M})$ is injective and its image is $\hat{\rho}_{a,b}(\hat{J}_{n,a}(\mathfrak{M}))$.

**Theorem 3.3.4.** With previous notations, the map $\theta_n: T_{\Theta_n}(\mathfrak{M}) \cong \hat{T}_n(\mathfrak{M}) \cong T$ is an isomorphism of $\mathbb{Z}_p[G_s]$-modules for all integer $s > s_{\min} = n-1+\log_p(\frac{N}{p-1})$.

**Proof.** We already know that $\theta_n$ is bijective (Theorem 3.1.3.(1)). Now, consider the following commutative diagram:

$$
\begin{array}{ccc}
T_{\Theta_n}(\mathfrak{M}) & \xrightarrow{\theta_n} & \hat{T}_n(\mathfrak{M}) \\
\rho_{\infty,a} \downarrow & & \downarrow \hat{\rho}_{\infty,b} \\
J_{n,a}(\mathfrak{M}) & \cong & \hat{J}_{n,b}(\mathfrak{M}) \xrightarrow{\theta_{n,b}} \hat{J}_{n,b}(\mathfrak{M})
\end{array}
$$

Note that $s_{\min} = s_1(a) = s_2(b)$. Thus by definition of $G_s$-action on $T_{\Theta_n}(\mathfrak{M})$ (resp. by Proposition 2.5.3, resp. by Proposition 3.3.1), $\rho_{\infty,a}$ (resp. $\rho_{\infty,b}$ resp. $\theta_{n,b}$) is $G_s$-equivariant. Since $\hat{\rho}_{\infty,b}$ is injective (Corollary 3.3.3) and $G_s$-equivariant, we deduce that $\theta_n$ is also $G_s$-equivariant as claimed. \hfill $\Box$

We end this section by giving a proof of Theorem 1.5 of introduction. For convenience of the reader, we first recall its statement:

**Corollary 3.3.5 (Theorem 1.5).** Let $V$ and $V'$ be two semi-stable representations of $G$. Let $T$ (resp. $T'$) a quotient of two $G$-lattices in $V$ (resp. $V'$) which is killed by $p^s$. Then any morphism $G_{\infty}$-equivariant $f: T \to T'$ is $G_s$-equivariant for all integer $s > n-1+\log_p(nr)$.
Proof. Consider $\mathfrak{M}$ (resp. $\mathfrak{M}'$) some Kisin module such that $T_{\overline{\Theta}_n}(\mathfrak{M}) = T$ (resp. $T_{\overline{\Theta}_n}(\mathfrak{M}') = T'$). We may assume that $\mathfrak{M}$ and $\mathfrak{M}'$ are maximal in the sense of [7]. Then by Corollary 3.3.10 of loc. cit., $f$ comes from a morphism $g : \mathfrak{M}' \to \mathfrak{M}$. Using Theorem 3.3.4, one easily see that $T_{\overline{\Theta}_n}(g) = f$ is $G_s$-equivariant.

4. Ramification bound

In this section, we give proofs of Theorems 1.1 and 1.3 based on above preparations. Our strategy is similar to those in [1], [9], [14] and [15]. Let $n$ be a positive integer. Recall that we have defined several constants, that are:

- $N$ is an integer such that $u^N = 0$ in $W_n[u]/E(u)$ (recall also that one may choose $N = ern$);
- $b = \frac{N}{N - 1}$ and $a = \frac{nN}{N - 1}$;
- $s_0(a) = n - 1 + \log p \left( \frac{N}{N - 1} \right)$;
- $s_{\text{min}} = s_1(a) = s_2(b) = n - 1 + \log p \left( \frac{N}{N - 1} \right)$.

Note that if we have chosen $N = ern$, then $s_0(a)$ is nothing but the minority of $s$ that appear in Theorem 1.1. Let $T = \Lambda/\Lambda'$ be a quotient of two lattices in a semi-stable representation and assume that $T$ is killed by $p^n$. Since we have a surjective map $\Lambda/p^n\Lambda \to T$, it is enough to bound ramification for $\Lambda/p^n\Lambda$. Hence, without loss of generality, we will assume that $T$ is free of $\mathbb{Z}/p^n\mathbb{Z}$. By Theorem 3.1.3, there exists a $(\varphi, \hat{G})$-module $\mathfrak{M}$ such that $T_{\overline{\Theta}_n}(\mathfrak{M}) = T$. With our extra assumption, $\mathfrak{M}$ is finite free over $\mathfrak{S}_n$.

From now on, we fix an integer $s > s_0(a)$. Remark that $s_0(a) > s_{\text{min}}$ so that we also have $s > s_{\text{min}}$. Hence theory developed in previous sections applies. In particular, by Propositions 2.5.2 and 2.5.3, for all $c \in [0, ep^{-n+1}]$, we have a $G_s$-equivariant isomorphism

$$J_{n,c}(\mathfrak{M}) \simeq J_{n,c}^{(s)}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi} \left( \mathfrak{M}, W_n(k) \otimes_{W(n,k), \sigma'} \frac{W_n(O_{K}/p)}{[a_E^{c/p'}]} \right)$$

where the structure of $\mathfrak{S}$-module on $W_n(O_{K}/p)$ is given by $u \mapsto 1 \otimes \pi_u$. Moreover, by Corollary 3.3.3 and Theorem 3.3.4

$$T|_{G_s} \simeq \text{im} \rho_{a,b} : J_{n,a}(\mathfrak{M}) \to J_{n,b}(\mathfrak{M})$$

Define $L$ to be the splitting field of $T$, that is, $L = (\hat{K})^{\text{Ker}(\rho)}$, where $\rho : G_K \to GL_{Z_p}(T)$ the attached group homomorphism. Set $L_s = K_s L$.

4.1. The sets $J_{n,c}^{(s), E}(\mathfrak{M})$. Let $E$ be an algebraic extension of $K_s$ inside $\hat{K}$. By restriction, the valuation $v_K$ induces a valuation on $E$ and one may define, for all non negative real number $c$, $\hat{a}_E^c = \{ x \in E / v_K(x) \geq c \}$ and $\hat{a}_E^{c'} = \{ x \in E / v_K(x) > c \}$. If $c$ belongs to the interval $[0, ep^{-n+1}]$, we put

$$J_{n,c}^{(s), E}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi} \left( \mathfrak{M}, W_n(k) \otimes_{W(n,k), \sigma'} \frac{W_n(O_{K}/p)}{[a_E^{c/p'}]} \right).$$

They are $\mathbb{Z}_p$-modules, and if $E/K$ is Galois, they are endowed with an action of $G_s$. As usual, if $0 \leq c \leq c' \leq ep^{-n+1}$, we have a natural morphism $\rho_{c,c'}^{(s), E} : J_{n,c}^{(s), E}(\mathfrak{M}) \to J_{n,c'}^{(s), E}(\mathfrak{M})$. Apparently $J_{n,c}^{(s), E}(\mathfrak{M})$ injects $J_{n,c}^{(s), K}(\mathfrak{M}) = J_{n,c}^{(s), \mathfrak{M}}$.

The aim of this subsection is to show the following theorem.

Theorem 4.1.1. Notations as above. The natural injection $\rho_{a,b}^{(s), E}(J_{n,a}^{(s), E}(\mathfrak{M})) \subset \rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is bijective if and only if $L_s \subset E$.

Remark 4.1.2. By (4.0.3), $\rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is canonically isomorphic to $T$ as a $\mathbb{Z}_p[G_s]$-module.

In order to achieve the proof, we will need to lift $J_{n,c}^{(s), E}(\mathfrak{M})$ at $O_E$-level. We begin by defining a map $\varphi : W_n(O_E) \to W_n(O_E)$ by

$$\varphi(z) := (z_0^p, \ldots, z_{n-1}^p).$$
Note that $\varphi$ is not a ring homomorphism. Nevertheless one easily check that $\varphi([\lambda]z) = [\lambda^p]\varphi(z)$ for $\lambda \in \mathcal{O}_E$ and $z \in W_n(\mathcal{O}_E)$ and $\varphi$ is $G$-equivariant.

Remark 4.1.3. If $A$ is any ring, one can always define Frobenius $\phi : W(A) \to W(A)$ by $w_m(\phi(x)) = w_{m+1}(x)$, $\forall x \in W(A)$, where $w_n(x)$ is the $m$-th ghost component of $x$. Then $\phi$ can be proved to be a ring homomorphism (see p.14 in [16]). Unfortunately, such Frobenius does not preserve the kernel of natural projection $W(A) \to W_n(A)$ unless $A$ has characteristic $p$. Hence it is not well-defined on $W_n(A)$ if $A$ has characteristic 0.

Recall now that we have assumed that $\mathfrak{M}$ is finite $\mathcal{S}_n$-free. Select a basis $(e_1, \ldots, e_d)$ of $\mathfrak{M}$ and write $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ with $A \in \mathcal{M}(\mathfrak{S}_n)$. As discussed in §2.3, there exists $B \in \mathcal{M}(\mathfrak{S}_n)$ such that $AB = u^N1$. Let $\tilde{A}$ and $\tilde{B}$ be matrices in $\mathcal{M}(W_n(\mathcal{O}_K))$ that respectively lifts images of $A$ and $B$ under the ring homomorphism $\mathfrak{S}_n \to W_n(\mathcal{O}_K)$, $u \mapsto \pi_s$, $\lambda \mapsto \sigma^{-s}(\lambda)$ ($\lambda \in W_n(k)$). Apparently, $\tilde{AB} \equiv [\pi_s]^N$ (mod $W_n(p\mathcal{O}_K)$). Hence, using condition on $s$, one prove that there exists a matrix $R$ with coefficients in $W_n[p\mathcal{O}_K]$ such that $\tilde{AB} = [\pi_s]^N(I + R)$ (where $I$ is the identity matrix). Noting that $I + R$ is invertible, one get $\tilde{AB}(I + R)^{-1} = [\pi_s]^N1$. Hence, up to replacing $\tilde{B}$ by $\tilde{B}(I + R)^{-1}$, one may assume that $\tilde{AB} = [\pi_s]^N1$. Finally define a set

$$J_n(s,E)(\mathfrak{M}) := \{ (\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)/d / (\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_d)) = (\tilde{x}_1, \ldots, \tilde{x}_d)A \}.$$ 

The natural projection $W_n(\mathcal{O}_E) \to W_n(\mathcal{O}_E/p) \to W_n(\mathcal{O}_E/p)/[\mathfrak{A}_E^{p}\mathfrak{p}]$ induces a map $\tilde{p}_b(s,E) : J_n(s,E)(\mathfrak{M}) \to J_{n,c}(E)(\mathfrak{M})$.

**Lemma 4.1.4.** $\tilde{p}_b(s,E)$ is injective and its image is $\tilde{p}_{a,b} (J_{n,a}(E)(\mathfrak{M}))$.

**Proof.** During the proof, if $z$ is any element in $W_n(\mathcal{O}_E)$, we will denote by $z^{(i)} \in \mathcal{O}_E$ its $i$-th component. By the same way, we define $Z^{(i)}$ for a matrix $Z$ with entries in $W_n(\mathcal{O}_E)$. Also, if $Z$ is a matrix with entries in $\mathcal{O}_E$, we will denote by $v_K(Z)$ the smallest valuation of coefficients of $Z$.

We first show $\tilde{p}_b(s,E)$ is an injection. Assume that $X$ and $Y$ are in $J_n(s,E)(\mathfrak{M})$ such that $\tilde{p}_b(s,E)(X) = \tilde{p}_b(s,E)(Y) = 0$. Then $Z = X - Y \in [\mathfrak{A}_E^{\mathfrak{p}}] + W_n(p\mathcal{O}_E) = [\mathfrak{A}_E^{\mathfrak{p}}]$. Note that $Z = 0$. By contradiction that it is false and consider $m$ the smallest number such that $Z^{(m)} \neq 0$. Define $W := ZA = \varphi(X) - \varphi(Y) = \varphi(Y + Z) - \varphi(Y)$. Easy computations show that $W^{(i)} = 0$ for $i < m$ and

$$W^{(m)} = \sum_{i=1}^{p} \binom{p}{i} (Y^{(m)})^{p-i} (Z^{(m)})^i,$$

where the multiplication is computed component by component. If $1 \leq i < p$, we have

$$v_K \left( \binom{p}{i} (Y^{(m)})^{p-i} (Z^{(m)})^i \right) \geq c + v_K(Z^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$$

and, using $v_K(Z^{(m)}) > bp^{m-1-s}$ (recall that $Z \in [\mathfrak{A}_E^{\mathfrak{p}}]$), we find

$$v_K((Z^{(m)})^p) > (p-1)bp^{m-1-s} + v_K(Z^{(m)}) = Np^{m-1-s} + v_K(Z^{(m)}).$$

Hence each term in RHS of (4.1.1) has valuation greater than $Np^{m-1-s} + v_K(Z^{(m)})$. So $v_K(W^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$. On the other hand, comparing the $m$-th component of $WA = [\pi_s]^N Z$, we get $v_K(W^{(m)}) \leq Np^{m-1-s} + v_K(Z^{(m)})$. This is a contradiction and injectivity follows.

Let us now prove the second statement. Remark first that for all $c \in [0, ep^{s-n+1}]$, we have $W_n(\mathcal{O}_E/p)/[\mathfrak{A}_E^{c/p}] \simeq W_n(\mathcal{O}_E)/[\mathfrak{A}_E^{c/p}]$ and hence that $J_{n,c}(E)(\mathfrak{M})$ can be identified with

$$\{(\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)/d / (\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_d)) = (\tilde{x}_1, \ldots, \tilde{x}_d)A \mod [\mathfrak{A}_E^{c/p}] \}$$

modulo $[\mathfrak{A}_E^{c/p}]$.

Let $X = (\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)/d$ be an solution as above. We have equation $\varphi(X) = XA + Q'$ with coefficients of $Q'$ in $[\mathfrak{A}_E^{c/p}]$. Actually, the congruence holds in $[\mathfrak{A}_E^{2c/p}]$ for some $a'$ satisfying $\frac{c}{p} > a' > \frac{c}{p}$. Note that $\frac{c}{p} > a'$ implies that $W_n(p\mathcal{O}_E) \subset [\mathfrak{A}_E^{2c/p}]$. For the rest of the proof, fix $\alpha \in \mathcal{O}_E$ some element of valuation $a'$. By the similar argument as in Lemma 2.3.1.1(2), $[\mathfrak{A}_E^{2c/p}]$ is the principal ideal generated by $[\alpha]$. Therefore, one have $\varphi(X) - XA = [\alpha]Q$ with the coefficients of $Q$ in $W_n(\mathcal{O}_E)$. We want to prove that there exists a matrix $Y$ with coefficients in
\[ \alpha^b_{E^r} \] such that \((X + Y)\tilde{A} = \varphi(X + Y)\). Let us search for a solution of the shape \([X, Y] = \varphi(X + [\beta]Z)\). Multiplying \(\tilde{B}\) on both sides and noting that the non-zero divisor \(\varphi\) of \(W_n(\{E\})\), we need to prove that the following equation has a (necessarily unique) solution:

\[(4.1.2) \quad [\pi_s]N X + [\pi_s]N [\beta]Z = \varphi(X + [\beta]Z)\tilde{B}.
\]

Let us prove by induction on \(n\). If \(n = 0\), set \(Z_0 = 0\) and \(Z_{t+1} = \pi_s^{-N\beta^{-1}}(\varphi(X + \beta Z_t)\tilde{B} - \pi_s^N X)\).

To see that \(Z_{t+1}\) is in \(\{E\}\), note that

\[
\varphi(X + \beta Z_t)\tilde{B} - \pi_s^N X = (X + \beta Z_t)\tilde{B} - \pi_s^N X = (\varphi(X)\tilde{B} - \pi_s^N X) + \sum_{i=1}^{p-1} \left( \begin{array}{c} p \\ i \end{array} \right) X^{p-i}(\beta Z_t)^i \tilde{B} + \beta^p(Z_t)^p \tilde{B}.
\]

Since \(\varphi(X)\tilde{B} - \pi_s^N X = \alpha Q\tilde{B} + \nu_K([\pi_s]^N \beta) \leq v_K(\alpha) \leq v_K(p)\) and \((p-1)\nu_K(\beta) \geq v_K([\pi_s]^N)\), we see that \(Z_{t+1}\) is in \(\{E\}\). Note that

\[
Z_{t+1} - Z_t = \pi_s^{-N\beta^{-1}}(\varphi(X + \beta Z_t) - \varphi(X + \beta Z_{t-1}))\tilde{B}
\]

Since \(v_K(p) \geq v_K([\pi_s]^N \beta)\) and \((p-1)\nu_K(\beta) > v_K([\pi_s]^N)\), we see that \(v_K(Z_{t+1} - Z_t) = \gamma + v_K(Z_{t-1} - Z_{t-1})\), where \(\gamma = \min(v_K(\beta), v_K(\beta^p - \pi_s^{-N})) > 0\). Hence \(Z_t\) converge to \(Z\) and we solve the equation (4.1.2) for \(n=1\).

Now assume that equation (4.1.2) has a solution for \(n \leq m - 1\), consider the \(n = m\) case. Recall that \(z^{(i)} \in \{E\}\) represents the \(i\)-th component of \(z \in W_m(\{E\})\). Set \(Z_0 = (Z_0^{(0)}, \ldots, Z_0^{(m-1)})\) where \(Z_0^{(m-1)} = 0\) and \((Z_0^{(0)}, \ldots, Z_0^{(m-2)})\) is the solution of (4.1.2) in \(n = m - 1\) case. Now set \(Z_{t+1} = [\pi_s]^{-N}[\beta]^{-1}(\varphi(X + [\beta]Z_t)\tilde{B} - [\pi_s]X)\). Since \((Z_0^{(0)}, \ldots, Z_0^{(m-1)})\) is the solution of (4.1.2) in \(n = m - 1\) case, we see that \(Z_t^{(i)} = Z_t^{(i)}\) for all \(l = 0, \ldots, m - 2\). Now it suffices to check that \(Z_t^{(i)}\) has coefficients in \(W_m(\{E\})\) and \(Z_t^{(i)}\) converges.

Since \(\varphi(X + [\beta]Z_t) = \varphi(X) + \varphi([\beta]Z_t)\in W_m(\{E\}/p)\), we have \(\varphi(X + [\beta]Z_t) = \varphi(X) + \varphi([\beta]Z_t) + C'\) with coefficients of \(C'\) in \(W_m(p\{E\})\). Since \(W_m(p\{E\}) \subset \alpha W_m(\{E\})\), we can write \(C' = \alpha C\) with coefficients of \(C\) in \(W_m(\{E\})\). Hence \(\varphi(X + [\beta]Z_t)\tilde{B} - [\pi_s]X = \varphi(X)\tilde{B} - [\pi_s]X + [\beta]p\varphi(Z_t)\tilde{B} + [\alpha]CB = \varphi(X)\tilde{B} + [\beta]^p\varphi(Z_t)\tilde{B} + \alpha CB\). Since \((p-1)\nu_K(\beta) > v_K([\pi_s]^N)\), we see that \(Z_{t+1}\) is well defined. Now \([\pi_s]N^N[\beta](Z_{t+1} - Z_t) = (\varphi(X + [\beta]Z_t) - \varphi(X + [\beta]Z_{t-1}))\tilde{B} = W_t\tilde{B}\), where

\[
W = \varphi(X + [\beta]Z_{t-1} - \varphi(X + [\beta]Z_{t-1})
\]

with \(V = X + [\beta]Z_{t-1}\). Since \(Z_t^{(l)} = Z_t^{(l)}\) for all \(l\) and \(0 \leq l \leq p - 2, W_t^{(l)} = 0\) for \(l = 0, \ldots, m - 2\) and

\[
W_t^{(m-1)} = \sum_{i=1}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (\varphi(V^{(m-1)})^{(i)}(\beta p^{m-1}(Z_t^{(m-1)} - Z_{t-1}^{(m-1)}))^{i}.
\]

Hence \(v_K([\pi_s]^N\beta)^{p^{m-1}} + v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)}) \geq v_K(\beta^p) + v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)})\) if \(i = p\) and

\[
v_K([\pi_s]^N\beta)^{p^{m-1}} + v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)}) \geq v_K(p) + v_K(\beta) + v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)})
\]

if \(1 \leq i \leq p - 1\). Since \((p-1)\nu_K(\beta) > v_K([\pi_s]^N)\) and \(v_K([\pi_s]^N\beta)^{p^{m-1}} \leq v_K(p)\), we get

\[
v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)}) \geq \gamma + v_K(Z_t^{(m-1)} - Z_{t-1}^{(m-1)}),
\]

where \(\gamma = \min(v_K(\beta), v_K(\beta^p - \pi_s^{-N}))\). Hence \(Z_t\) converges and we are done.

Proof of Theorem 4.1.1. We have \(G_1\)-equivariant bijections of sets:

\[
\bar{j}_n^{(i)}(\mathfrak{M}) \simeq \rho_{a,b}^{(i)}(\mathfrak{M}) \quad \text{by Lemma 4.1.4 applied with } E = \tilde{K}
\]

\[
\simeq \rho_{a,b}(\mathfrak{M}) \quad \text{by Formula (4.0.2)}
\]

\[
\simeq T_{G_1} \quad \text{by Proposition 2.3.3 and Theorem 3.3.4}.
\]
Taking fixed points under $\Gal(\bar{K}/E)$, we get a natural bijection $\tilde{J}_n^{(s),E}(\mathfrak{M}) \simeq T_{\Gal(\bar{K}/E)}$. Hence again by Lemma 4.1.4, $T_{\Gal(\bar{K}/E)} \simeq f_{a,b}^{(s),E}(J_{n,h}^{(s),E}(\mathfrak{M}))$, from which the theorem is easily deduced.

\section{Proof of Theorem 1.1.} Recall that $L_n = K_n^L$ with $L$ the splitting field of $T$. Now we are ready to bound the ramification of $L$. To do this, we need to recall the property $(F_m^{E/N})$ described by Fontaine (Proposition 1.5, [9]). First, in order to fix notations, we would like to recall some definitions about ramification filtration.

Let $F_1/F_0$ be a Galois extension of $p$-adic fields, with Galois group $G$. For all non negative real number $\lambda$, we define a normal subgroup $G(\lambda)$ of $G$ by

$$G(\lambda) = \{ \sigma \in G / v_{F_1}(\sigma(x) - x) \geq \lambda, \forall x \in \mathcal{O}_{F_1} \}$$

where $v_{F_1}$ is the valuation normalized by $v_{F_1}(F_1^*) = \mathbb{Z}$ and $\mathcal{O}_{F_1}$ is the ring of integers of $F_1$.

We underline that we use here conventions of [9] and that they differ by a shift with conventions of [23], Chap. IV. By definition $G(\lambda)$ is called the lower ramification filtration of $G$. Now, let $\varphi_{F_1/F_0} : [0, +\infty[ \to [0, +\infty]$ be the function defined by

$$\varphi_{F_1/F_0}(\lambda) := \int_0^\lambda \frac{\Card G(t)}{\Card G(1)} dt.$$

It is increasing, continuous, concave, piecewise affine and bijective. Let $\psi_{F_1/F_0}$ denote its inverse and set $G^{(u)} = G(\psi_{F_1/F_0}(a))$: it is the upper ramification filtration. Finally call $\lambda_{F_1/F_0}$ (resp. $\mu_{F_1/F_0}$) the last break in the lower (resp. upper) ramification filtration of $G$, that is the infimum of $\lambda$ (resp. $\mu$) such that $G(\lambda) = 1$ (resp. $G^{(u)} = 1$). Obviously $\mu_{F_1/F_0} = \varphi_{F_1/F_0}(\lambda_{F_1/F_0})$.

We refer to [23], Chap. IV for basic properties of these filtrations, and especially for Herbrand’s theorem that allows us to extend upper ramification filtration to infinite algebraic extension. In particular, $G_K$ is itself filtered by normal closed subgroups $G^{(u)}_K$. Note that $\mu_{F_1/F_0} = \inf \{ \mu \in \mathbb{R}^+ / G^{(u)} \subset G_{F_1} \}$ where $G_{F_0}$ and $G_{F_1}$ denote the absolute Galois groups of $F_0$ and $F_1$ respectively.

**Proposition 4.2.1 (Yoshida).** Let $F_1$ and $F_0$ be finite extensions of $K$ with $F_0 \subset F_1 \subset \bar{K}$ and $F_1$ is Galois. For any positive real number $m$, consider the following property

$$P_m^{F_1/F_0} : \begin{cases} \text{For any algebraic extension } E \text{ over } F_0, \\
\text{if there exists an } \mathcal{O}_{F_0}-\text{algebra homomorphism } \mathcal{O}_{F_1} \to \mathcal{O}_E/a_E^{-m}, \\
\text{then there exists a } F_0-\text{extension } F_1 \hookrightarrow E. \end{cases}$$

Let $e_{F_0/K}$ denote the ramification index of $F_0/K$. Then

$$\frac{P_m^{F_1/F_0}}{e_{F_0/K}} = \inf \{ m \in \mathbb{R}^+ / \text{the property } (P_m^{F_1/F_0}) \text{ holds} \}.$$

**Proof.** See Proposition 5.6 of [15].

We will also need the following corollary:

**Corollary 4.2.2.** If $(P_m^{F_1/F_0})$ holds for a positive real number $m$, then $v_K(D_{F_1/F_0}) < m$.

**Proof.** If $F_1/F_0$ is unramified, $v_K(D_{F_1/F_0}) = 0$ and the corollary is obvious. If not, Proposition 1.3 of [9] shows that $e_{F_0/K}v_K(D_{F_1/F_0}) < \mu_{F_1/F_0}$, and we are done.

We claim that $(P_m^{L_n/K})$ holds for $m = ap^{n-1-\varepsilon}$. To see this, pick $f : \mathcal{O}_{L_n} \to \mathcal{O}_E/a_E^{-m}$ an $\mathcal{O}_{K_n}$-algebra homomorphism. Obviously, for any real number $c \in [0, m]$, $f$ induces a map $f_c : \mathcal{O}_{L_n}/a_{L_n}^{c} \to \mathcal{O}_E/a_E^{-c}$.

**Lemma 4.2.3.**

1. For any $c \leq m$, $f_c$ is injective.
2. For any $c \leq a$, $f_{c+p^{n-1-\varepsilon}}$ induces an injection

$$W_n(\mathcal{O}_{L_n}/p)/[a_{L_n}^{\geq c/p^n}] \hookrightarrow W_n(\mathcal{O}_E/p)/[a_E^{\geq c/p^n}].$$
Proof. (1) It is the same as the proof of Lemma 4.4 of [14].

(2) Using an analogue of 2.3.1(1), one easily prove that natural projections $\mathcal{O}_{L_i}/p \to \mathcal{O}_{L_i}/a_{L_i}^{\geq p^{n-1-s}}$ and $\mathcal{O}_E/p \to \mathcal{O}_E/e_{L}^{\geq p^{n-1-s}}$ induce isomorphisms

$$W_n(\mathcal{O}_{L_i}/p)/[a_{L_i}^{\geq p^s}] \simeq W_n(\mathcal{O}_{L_i}/a_{L_i}^{\geq p^{n-1-s}})/[a_{L_i}^{\geq p^s}]$$

$$W_n(\mathcal{O}_E/p)/[a_{E}^{\geq p^s}] \simeq W_n(\mathcal{O}_E/a_{E}^{\geq p^{n-1-s}})/[a_{E}^{\geq p^s}].$$

Hence $f_{cp^{-1-s}}$ indeed induces a map

$$W_n(\mathcal{O}_{L_i}/a_{L_i}^{\geq p^{n-1-s}})/[a_{L_i}^{\geq p^s}] \to W_n(\mathcal{O}_E/a_{E}^{\geq p^{n-1-s}})/[a_{E}^{\geq p^s}]$$

and checking injectivity is now straightforward using (1).

Thus, we get injections:

$$\rho_{b,a}(L_{n,a}(\mathcal{M})) \hookrightarrow \rho_{b,a}(E_{n,a}(\mathcal{M})) \hookrightarrow \rho_{b,a}(J_{n,a}(\mathcal{M})) \simeq T$$

the first one being induced by $f$ (which is obviously compatible with Frobenius since it is a ring homomorphism). By Theorem 4.1.1, LHS is isomorphic to $T$. The composite map is then an injective endomorphism of $T$. Consequently, it is an isomorphism because $T$ is finite. It follows that $\rho_{b,a}(J_{n,a}(\mathcal{M}))$ is bijective and then, applying again Theorem 4.1.1, we get $L_s < E$. Property $P_{L_s/K_{s-1}}$ is proved.

By Proposition 4.2.1, one then get $\mu_{K_s/K_{s-1}} \leq e_{K_s/K_{s-1}} = \frac{Np^s}{p-1}$. Taking $N = ern$, one obtain Theorem 1.1. (Recall that $ern$ is not in general the best value one can choose (expect for $n = 1$). See §2.4 for a discussion about this.)

4.3. Proof of Theorem 1.3. Consider $\alpha$ and $\beta$ such that $\frac{N}{ep^s} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{2} < \beta \leq 1$. (If $N = ern$, then $\alpha$ and $\beta$ are those of Theorem 1.3). From now on, we fix $s = n + \alpha$. One certainly have that $s \geq s_0(\alpha) = n + \log_p\left(\frac{N}{ep^s}\right)$ as it was assumed at the beginning of this section.

It is very easy now to bound valuation of $e_{L_s/K}$. We just write:

$$v_K(D_{L_s/K}) = 1 + es - \frac{1}{p^s} + e\left(1 + es - \frac{1}{p^s} + ap^{n-1-s}\right) = 1 + es - \frac{1}{p^s} + cp^{n-s}\beta = 1 + e(n + \alpha + \beta) - \frac{1}{p^s}$$

where the inequality $v_K(D_{L_s/K}) < ap^{n-1-s}$ follows from Corollary 4.2.2 and the fact that $\rho_{b,a}(J_{n,a}(\mathcal{M}))$ holds as it was seen before. Now, since $L$ is a subextension of $L_s$, we have $v_K(D_{L/K}) \leq v_K(D_{L_s/K})$ and the previous bound works also for $v_K(D_{L_s/K})$. Taking $N = ern$, we get Theorem 1.3.(2).

To bound $u_{K_s/K}$, we first need to extend the definition of $\varphi_{F_1/F_0}$ and $\psi_{F_1/F_0}$ to arbitrary finite extensions $F_1/F_0$ (non necessarily Galois). There are several standard ways to do this. For example, following [24], §1.2.1, one can put

$$\psi_{F_1/F_0}(\mu) = \int_0^\mu [G_{F_1}^{(1)} : G_{F_0}^{(1)}] dt$$

(whence $G_{F_1}$ and $G_{F_0}$ stands for absolute Galois groups of $F_1$ and $F_0$ respectively) and remark that this formula agrees with the previous definition when $F_1/F_0$ is Galois. Set also $\varphi_{F_1/F_0} = (\psi_{F_1/F_0})^{-1}$. We have an usual transitivity formula: if $F_0 \subset F_1 \subset F_2$ are finite extensions of $K$, then $\varphi_{F_2/F_0} = \varphi_{F_1/F_0} \circ \varphi_{F_1/F_2}$.

Lemma 4.3.1. Let $F_0 \subset F_1$ be two finite extension of $K$. Then

$$G_{F_1}^{(\mu)} = G_{F_1} \cap G_{F_0}^{(\varphi_{F_1/F_0}(\mu))}$$

for all $\mu \geq 0$.

Proof. Let $N$ be a normal extension of $F_0$, with $F_1 \subset N$. For all $\lambda \geq 0$, one have $\text{Gal}(N/F_1)^{(\lambda)} = \text{Gal}(N/F_1) \cap \text{Gal}(N/F_0)^{(\lambda)}$, that is $\text{Gal}(N/F_1)^{(\varphi_{N/F_1}(\lambda))} = \text{Gal}(N/F_1) \cap \text{Gal}(N/F_0)^{(\varphi_{N/F_0}(\lambda))}$. Putting $\mu = \varphi_{N/F_1}(\lambda)$ and using transitivity formula, one get $\text{Gal}(N/F_1)^{(\mu)} = \text{Gal}(N/F_1) \cap \text{Gal}(N/F_0)^{(\varphi_{F_1/F_0}(\mu))}$. Taking projective limit over all Galois extensions $N$, we get the desired property. \qed
By §4.2, we know that $G_{s}^{(p)} \subset G_{L}$ for all $\mu > \frac{Np^{n}}{p-1}$. Applying previous Lemma, we have $G_{s} \cap G_{K}^{(p_{s}, K)}$ also lies in $G_{L}$. Consequently

$$\mu_{L/K} \leq \mu_{L_{s}/K} \leq \max \left( \mu_{K_{s}/K}, \varphi_{K_{s}/K} \left( \frac{Np^{n}}{p-1} \right) \right).$$

By Remark 5.5 of [15], we know that $\mu_{t} := \mu_{K_{t}/K} = 1 + e(t + \frac{1}{p-1})$ for all $t \geq 1$. Using that subextensions of $K_{s}$ are exactly the $K_{t}$'s for $0 \leq t \leq s$, one easily see that $\varphi_{K_{s}/K}$ has the following shape

$$\begin{align*}
\varphi_{K_{s}/K}(\lambda) & \leq \mu_{s} + \frac{\lambda - \lambda_{s}}{p^{n}} = 1 + es - \frac{1}{p^{s}} + \frac{\lambda}{p^{s}}. \\
\end{align*}$$

Finally, taking $N = ern$ and using (4.3.1), one obtain Theorem 1.3.(1) (remember $s = n + \alpha$ and $\frac{t^{n}}{p^{n}} = p^{\beta}$).

5. Some results and questions about lifts

In this last section, we discuss some ideas about possible converses for Theorem 1.3. Precisely, we wonder when a given torsion representation of $G_{K}$ can be realized as a quotient of two lattices in a semi-stable (or even crystalline) representation, eventually with prescribed Hodge-Tate weights. Denote by $\text{Rep}_{G}(G_{K})$ (resp. $\text{Rep}_{\text{tor}}(G_{K})$, resp. $\text{Rep}_{p}^{*}(G_{K})$) the category of all $\mathbb{Z}_{p}$-representations of $G_{K}$ that are finitely generated and free (resp. killed by a power of $p$, resp. killed by $p^{n}$) as a $\mathbb{Z}_{p}$-module. For any full subcategory $\mathcal{C}$ of $\text{Rep}_{p}^{*}(G_{K})$, one can always raise the following question

**Question 5.1.** For any $T \in \text{Rep}_{\text{tor}}(G_{K})$ (resp. $T \in \text{Rep}_{p}^{*}(G_{K})$), does there exist $\Lambda$ and $\Lambda'$ in $\mathcal{C}$ such that $T \simeq \Lambda / \Lambda'$?

Obviously if $\mathcal{C}$ is stable under subobject (which will in general be true in interesting examples), it is enough to find $L$ together with a surjective $G_{K}$-equivariant morphism $\Lambda \to T$. In the sequel, we will call a lift such a morphism $\Lambda \to T$. If $\mathcal{C}$ is moreover stable by direct sum, the problem can be further reduced as follows.

**Proposition 5.2.** Assume that $\mathcal{C}$ is stable under subobjects and direct sums. Assume also that any $T \in \text{Rep}_{p}^{*}(G_{K})$ admits a lift $\Lambda \in \mathcal{C}$. Then the answer to Question 5.1 is “yes”.

**Proof.** We make an induction on $n$. The case $n = 1$ is obvious. Now assume the statement is valid for $m \leq n - 1$. Let $T$ be a representation killed by $p^{n}$. Then we have an exact sequence $0 \to T' \to T \to T'' \to 0$, where $T' = p^{n-1}T$ and $T'' = T'/T'$. Since $T''$ is killed by $p^{n-1}$, by induction, there exists an $\Lambda \in \mathcal{C}$ that lifts $T''$. Denote the surjections $\Lambda \to T''$ and $T \to T''$ by $f$ and $g$ respectively. Set $M := T \times T'' \Lambda = \{(x, y) \in T \times \Lambda / g(x) = f(y)\}$. Then we have an
exact sequence $0 \to T' \to M \to \Lambda \to 0$. Since $\Lambda$ is free over $\mathbb{Z}_p$, the sequence is split as $\mathbb{Z}_p$-module. In particular $pM \cong p\Lambda \oplus pT' = p\Lambda$ is finite free over $\mathbb{Z}_p$. Now we have exact sequence $0 \to pM \to M \to M' \to 0$ with $M' = M/pM$. Since $M/pM$ is killed by $p$, there exists an $\Lambda' \in C$ such that $\Lambda'$ lifts $M/pM$. Set $N := M \times_{M'} \Lambda'$. It sits in the exact sequence $0 \to pM \to N \to \Lambda' \to 0$, and since $pM$ and $\Lambda'$ are both finite free, $N$ is also. Note that $N$ is a lift of $M$ hence a lift of $T$. Now it remains to show that $N$ is in $C$. To see this, note that $N := M \times_{M'} \Lambda' \subset M \times \Lambda'$. Then $pN \subset (pM) \times p\Lambda'$. But $pM \cong p\Lambda \in C$. Hence $pN \subset p\Lambda \times p\Lambda'$ belongs to $C$. 

We also have a kind of descent property:

**Proposition 5.3.** Assume that the answer of question 5.1 is “yes”, for the category $C = \mathcal{C}_K$. 

Let $L/K$ be a finite extension. Denote by $\mathcal{C}_L$ the category whose objects are subrepresentations of restrictions to $G_L$ of objects in $\mathcal{C}_K$. Then, for any $T \in \text{Rep}_{\text{tor}}(G_L)$ (resp. $T \in \text{Rep}_{\mathbb{Z}/p^{n}\mathbb{Z}}(G_L)$), there exist $\Lambda$ and $\Lambda'$ in $\mathcal{C}_L$ such that $T \simeq \Lambda/\Lambda'$.

**Proof.** By a previous remark, it is enough to show that $T$ admits a lift in $C$. Let $T_0 := \text{Ind}_{G_K}^{G_L}(T)$. By assumption, there exists a lift $f : \Lambda_0 \to T_0$ with $\Lambda_0 \in \mathcal{C}_K$. Consider the $\mathbb{Z}_p$-linear map $pr : \mathbb{Z}_p[G_K] \to \mathbb{Z}_p[G_L]$ sending $g \in G_L$ to itself, and $g \in G_K$, $g \notin G_L$ to 0; it is surjective and $G_L$-equivariant for actions on both sides. Tensoring $pr$ by $T$, one gets a $G_L$-equivariant surjective morphism $T_0 \to T$ which, composed with $f$, gives the desired lift. 

Nevertheless, of course, the answer to Question 5.1 is in general negative. For instance, we have the following theorem that can be seen as a consequence of ramification bounds obtained in this paper.

**Theorem 5.4.** For any $r > 0$, answer to Question 5.1 is “no” if $C$ is the category of lattices in semi-stable representations with Hodge-Tate weights in $\{0, \ldots, r\}$.

**Proof.** There are several ways to prove this theorem. Below, we give two different methods.

The first one is based on results shown in this paper. Select a Galois extension $F/K$ which has very large ramification and let $T$ be the regular representation with $\mathbb{Z}/p^n\mathbb{Z}$-coefficients of $\text{Gal}(F/K)$. Then the splitting field of $T$ is $F$, and Theorem 1.3 shows that $T$ cannot in general be lifted a semi-stable representation with Hodge-Tate weights in $\{0, \ldots, r\}$.

The second proof we would like to give uses the main result of [19] which states that a finite free $\mathbb{Z}_p$-representation $\Lambda$ of $G_K$ is a lattice in a crystalline (resp. semi-stable) representation with Hodge-Tate weights $\{0, \ldots, r\}$ if and only if $\Lambda/p^n\Lambda$ is a quotient of two such lattices for any $n$. Therefore, starting from a representation $\Lambda$ such that $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is not semi-stable, there must exist an integer $n$ such that $\Lambda/p^n\Lambda$ gives a counter-example to Question 5.1 (with the category $C$ of the theorem).

Unfortunately, the above proof does not help us to solve the following more interesting question:

**Question 5.5.** Has Question 5.1 a positive answer when $C$ is the category of all lattices in semi-stable representations?

In fact, to check the above question, it suffices to look at representations killed by $p$ (by Proposition 5.2) and we may assume that $K = \mathbb{Q}_p$ (by Proposition 5.3). Here are some partial results in favor of a positive answer to Question 5.5.

**Proposition 5.6.** Let $T$ be a torsion representation of $G_{\infty}$. Then $T$ is a quotient of two representations arising from finite free Kisin modules.

**Proof.** By a similar argument as in proof of Proposition 5.2, we may assume that $T$ is killed by $p$. Let $M$ be the étale $\varphi$-module over $k((u))$ attached to $T$ (see for instance [10], A. 3). Since any torsion Kisin module can be written as a quotient of two free Kisin modules, it is enough to show that $M$ admits a submodule $\mathcal{M}$ which is a Kisin module of height $r$. Let $(e_1, \ldots, e_d)$ be a basis of $M$ and $A$ be the matrix with coefficients in $k((u))$ such that 

$$(\varphi(e_1), \ldots, \varphi(e_d)) = (e_1, \ldots, e_d)A.$$
Since changing all $e_i$’s in $u e_i$ changes $A$ in $u^{p-1} A$, one may assume that $A$ has coefficients in $k[[u]]$. Furthermore, the étaleness of Frobenius on $M$ exactly means that $A$ is invertible in $k((u))$. Hence det $A$ does not vanish. Finally, we choose $r$ such that det $A$ divides $u^{cr}$ (that is $r \geq \frac{1}{2} \mathrm{val}_p(\text{det } A)$) and we are done.

**Theorem 5.7.** Any tamely ramified $\mathbb{F}_p$-representation of $G_K$ can be written as a quotient of two lattices in a crystalline representation with Hodge-Tate weights between 0 and $1 + E(\frac{1}{p_e})$.

**Remark 5.8.** In particular, the answer to Question 5.5 is yes if $T$ is tamely ramified and killed by $p$. Proof. In a preliminary version of this paper, the authors gave a proof based on some computations in $p$-adic Hodge theory, making in particular an intensive use of results of [18]. The following simpler argument is due to an anonymous referee.

Put $r = 1 + E(\frac{1}{p_e})$ and denote by $I$ the inertia subgroup of $G_K$. Let $T$ be a tamely ramified representation of $G_K$ killed by $p$. Since the tame inertia group is procyclic of order prime to $p$, $T/I$ splits as a direct sum of irreducible representations. By [23], §1.7, every irreducible representation of $I$ is isomorphic to

$$\mathbb{F}_p \left( \theta_0^{n_0} \theta_1^{n_1} \cdots \theta_{d-1}^{n_{d-1}} \right)$$

where $\theta_i$'s are fundamental inertia character of level $d$ (see loc. cit.) and $n_i$'s are some integers in $\{0, \ldots, p-1\}$. Hence, $T/I$ can be written as a tensor product of at most $r$ irreducible representations $T_i$ of $I$ whose tame inertia weights are between 0 and $e$. By a classical result of Raynaud (see [22]), all $T_i$ come from finite flat group schemes. Using the fact that any finite flat group scheme can be embedded in a $p$-divisible group, one construct a crystalline lift of $T_i$ with Hodge-Tate weights in $\{0, 1\}$. Taking the tensor product of all these lifts, one get a crystalline representation $\Lambda$ with Hodge-Tate weights between 0 and $r$ together with a surjective $I$-equivariant morphism $f : \Lambda \to T$ (which certainly factors through $\mathbb{L}/p\mathbb{L}$). Since $\Lambda/p\Lambda$ and $T$ are finite dimensional over $\mathbb{F}_p$, they are finite and $f$ is $G_K$-equivariant for a finite Galois unramified extension $K'/K$. Consider the map

$$\text{Ind}_{G_{K'}}^{G_K} \Lambda = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} \Lambda \to T, \quad [\sigma] \otimes x \mapsto \sigma f(x).$$

It is apparently $G_K$-equivariant and surjective: it is a lift of $T$. Furthermore, the restriction of $\text{Ind}_{G_{K'}}^{G_K} \Lambda$ to $G_{K'}$ is a direct sum of copies of $\Lambda$, and hence is crystalline with Hodge-Tate weights in $\{0, \ldots, r\}$. Since $K'/K$ is unramified, also is $\text{Ind}_{G_{K'}}^{G_K} \Lambda$ and we are done.

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