Quasi-periodic incompressible Euler flows in 3D

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Abstract. We prove the existence of time-quasi-periodic solutions of the incompressible Euler equation on the three-dimensional torus $\mathbb{T}^3$, with a small time-quasi-periodic external force. The solutions are perturbations of constant (Diophantine) vector fields, and they are constructed by means of normal forms and KAM techniques for reversible quasilinear PDEs.

Keywords: Fluid dynamics, Euler equation, vorticity formulation, KAM for PDEs, quasi-periodic solutions.

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1 Introduction

We consider the Euler equation for an incompressible fluid on the three-dimensional torus $\mathbb{T}^3$, $T := \mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{cases}
\partial_t U + U \cdot \nabla U + \nabla p = \varepsilon f(\omega t, x) \\
\text{div} U = 0
\end{cases}$$

(1.1)

where $\varepsilon \in (0, 1)$ is a small parameter, $\omega \in \mathbb{R}^s$ is a Diophantine $\nu$-dimensional vector, the external force $f$ belongs to $C^3(\mathbb{T}^s \times \mathbb{T}^3, \mathbb{R}^3)$ for some integer $q > 0$ large enough, $U = (U_1, U_2, U_3) : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ is the velocity field, and $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ is the pressure. We look for time-quasi-periodic solutions of (1.1), oscillating with time frequency $\omega$. This leads to solve the equation

$$\begin{cases}
\omega \cdot \partial_x u + U \cdot \nabla U + \nabla p = \varepsilon f(\varphi, x) \\
\text{div} U = 0
\end{cases}$$

(1.2)

where the unknown velocity field $U : \mathbb{T}^s \times \mathbb{T}^3 \to \mathbb{R}^3$ and the unknown pressure $p : \mathbb{T}^s \times \mathbb{T}^3 \to \mathbb{R}$ are functions of $(\varphi, x) \in \mathbb{T}^s \times \mathbb{T}^3$. We look for solutions which are small perturbation of a constant vector field $\zeta \in \mathbb{R}^3$, namely we look for solutions of the form

$$U(\varphi, x) = \zeta + u(\varphi, x) \quad \text{with} \quad \text{div} u = 0.$$  

Plugging this ansatz into the equation, one is led to solve

$$\begin{cases}
\omega \cdot \partial_x u + \zeta \cdot \nabla u + u \cdot \nabla u + \nabla p = \varepsilon f(\varphi, x) \\
\text{div} u = 0.
\end{cases}$$

(1.3)

We assume that the forcing term $f(\varphi, x)$ is an odd function of the pair $(\varphi, x)$, namely

$$f(\varphi, x) = -f(-\varphi, -x) \quad \forall (\varphi, x) \in \mathbb{T}^s \times \mathbb{T}^3.$$  

(1.4)

We look for solutions $(u, p)$ of (1.3) that are even functions of the pair $(\varphi, x)$, namely

$$u(\varphi, x) = u(-\varphi, -x), \quad p(\varphi, x) = p(-\varphi, -x) \quad \forall (\varphi, x) \in \mathbb{T}^s \times \mathbb{T}^3.$$  

(1.5)

For any real $s \geq 0$, we consider the Sobolev spaces of real scalar and vector-valued functions of $(\varphi, x)$

$$H^s = H^s(\mathbb{T}^{s+3}, \mathbb{R}^3) := \left\{u(\varphi, x) = \sum_{(\ell, j) \in \mathbb{Z}^{s+3}} \hat{u}(\ell, j)e^{i(\ell \varphi + j \cdot x)} : \|u\|_s := \left( \sum_{(\ell, j) \in \mathbb{Z}^{s+3}} |\hat{u}(\ell, j)|^2 \right)^{\frac{1}{2}} < \infty \right\}$$

$$H^s_0 := \left\{u \in H^s : \int_{\mathbb{T}^3} u(\varphi, x) \, dx = 0\right\}, \quad \langle \ell, j \rangle := \max\{1, |\ell|, |j|\}.$$  

(1.6)

We fix any bounded open set $\Omega \subseteq \mathbb{R}^s \times \mathbb{R}^3$ to which the parameters $(\omega, \zeta)$ belong. The main result of the paper is the following theorem.

**Theorem 1.1.** There exist $q = q(\nu) > 0$, $s = s(\nu) > 0$, such that for every forcing term $f \in C^q(\mathbb{T}^s \times \mathbb{T}^3, \mathbb{R}^3)$ satisfying (1.3) there exist $\varepsilon_0 = \varepsilon_0(f, \nu) \in (0, 1)$, $C = C(f, \nu) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the following holds. There exists a Borel set $\Omega_\varepsilon \subset \Omega$ of asymptotically full Lebesgue measure, i.e. $\lim_{\varepsilon \to 0} |\Omega \setminus \Omega_\varepsilon| = 0$, such that for every $(\omega, \zeta) \in \Omega_\varepsilon$ there exist $u = u(\dots, \omega, \zeta) \in H^s(\mathbb{T}^{s+3}, \mathbb{R}^3)$ and $p = p(\dots, \omega, \zeta) \in H^s(\mathbb{T}^{s+3}, \mathbb{R})$, even functions of the pair $(\varphi, x)$, that solve equation (1.3), with $\|u\|_s, \|p\|_s \leq C\varepsilon^b$ for some $b \in (0, 1)$.

Theorem 1.1 is deduced from Theorem 1.2 below, dealing with the vorticity formulation of the problem, which we now introduce.

As is well-known, if we take the divergence of the first equation in (1.3), we can determine the pressure $p$ in terms of the unknown $u$ and the forcing term $f$, namely we have

$$\Delta p + \text{div}(u \cdot \nabla u) = \varepsilon \text{div} f(\omega t, x),$$

(2.2)
whence

\[ p = \Delta^{-1} \left[ e \text{div} f(\omega t, x) - \text{div}(u \cdot \nabla u) \right]. \]  

(1.7)

If we consider the average in the space variable \( x \in T^3 \) of equation (1.3), we get the equation

\[ \omega \cdot \partial_x u_0(\varphi) = f_0(\varphi), \]

where \( u_0(\varphi) := \frac{1}{(2\pi)^3} \int_{T^3} u(\varphi, x) \, dx, \quad f_0(\varphi) := \frac{1}{(2\pi)^3} \int_{T^3} f(\varphi, x) \, dx. \]

This equation can be solved by assuming that the frequency vector \( \omega \) is Diophantine, i.e.

\[ |\omega \cdot \ell| \geq \frac{\gamma}{|\ell|}, \quad \forall \ell \in \mathbb{Z}^3 \setminus \{0\}, \]

because, by (1.4), \( f \) has zero average in \((\varphi, x)\). Hence, without loss of generality, we can assume that \( f \) has zero average in space, namely

\[ \int_{T^3} f(\varphi, x) \, dx = 0. \]  

(1.8)

We define the vorticity

\[ v := \nabla \times u := \text{curl} u := \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix}. \]

(1.9)

By taking the curl of equation (1.3) (recall that \( \nabla \times \nabla \Phi = 0 \) for any smooth scalar function \( \Phi \)), as is well known, one obtains the equation for the vorticity \( v(\varphi, x) \)

\[ \omega \cdot \partial_x v + \zeta \cdot \nabla v + u \cdot \nabla v - v \cdot \nabla u = \varepsilon F(\varphi, x), \quad F := \nabla \times f. \]

We construct an odd solution \( v(\varphi, x) \) of the latter equation and we shall prove that there are even functions \((u(\varphi, x), p(\varphi, x))\) which solves the equation (1.3). Since \( \text{div} u = 0 \), one has

\[ \text{curl} v = \text{curl}(\text{curl} u) = -\Delta u; \]

therefore, if the space average of \( u \) is zero, then

\[ u = (-\Delta)^{-1} \text{curl} v, \]

and one has the vorticity equations

\[
\begin{cases}
\omega \cdot \partial_x v + \zeta \cdot \nabla v + u \cdot \nabla v - v \cdot \nabla u = \varepsilon F(\varphi, x), \quad F := \nabla \times f, \\
u = \nabla \times [(-\Delta)^{-1} v],
\end{cases}
\]

(1.10)

where \((-\Delta)^{-1}\) is the Fourier multiplier of symbol \(|\xi|^{-2}\) for \( \xi \in \mathbb{Z}^3, \xi \neq 0 \), and zero for \( \xi = 0 \), namely

\[ u(x) = \sum_{\xi \in \mathbb{Z}^3} \hat{u}(\xi) e^{i\xi \cdot x} \Rightarrow (-\Delta)^{-1} u(x) = \sum_{\xi \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|\xi|^2} \hat{u}(\xi) e^{i\xi \cdot x}. \]

(1.11)

The divergence of \( v \) and the space average of \( v \) are both zero, because \( v \) is a curl. We will prove that if \( v \) solves (1.10) then \( u := \nabla \times (\Delta^{-1}v) \) solves (1.2).

We define the spaces of even/odd functions of the pair \((\varphi, x)\)

\[ X := \{ h \in L^2(T^{\nu+3}; \mathbb{R}^3) : h(\varphi, x) = h(-\varphi, -x) \}, \]

\[ Y := \{ h \in L^2(T^{\nu+3}; \mathbb{R}^3) : h(\varphi, x) = -h(-\varphi, -x) \} \]

and

\[ \Pi_0 h := \frac{1}{(2\pi)^3} \int_{T^3} h(\varphi, x) \, dx, \quad \Pi_0^\perp := \text{Id} - \Pi_0. \]  

(1.12)
We will prove that it is enough to look for smooth solutions \( v(\varphi, x) \in Y \) with zero average that solve the projected equation

\[
\Pi_0^\perp \left( \omega \cdot \partial_x v + \zeta \cdot \nabla v + u \cdot \nabla v - v \cdot \nabla u \right) = \varepsilon F(\varphi, x), \quad F := \nabla \times f, \quad u = \nabla \times \left[ \left( -\Delta \right)^{-1} v \right].
\]

(1.14)

It is convenient to replace \((-\Delta)^{-1}\) with an extension of it that is invertible also on functions with nonzero space average; with \( \Pi_0 \) defined by (1.13), we then define

\[
\Lambda := \Pi_0 - \Delta, \quad \Lambda^{-1} := \Pi_0 + \left( -\Delta \right)^{-1},
\]

(1.15)

so that \( \Lambda, \Lambda^{-1} \) are Fourier multipliers of symbols, respectively, \(|\xi|^2, |\xi|^{-2}\) for \( \xi \in \mathbb{Z}^3 \setminus \{0\} \), and 1 for \( \xi = 0 \). Thus \( \Lambda \Lambda^{-1} u = u \) for all periodic functions \( u \) (of course, introducing smooth cutoff functions, these symbols can be extended to all \( \mathbb{R}^3 \) by preserving the property \( \Lambda \Lambda^{-1} = \text{Id} \)).

We introduce the rescaling \( v = \tilde{\varepsilon} \tilde{v}, \tilde{\varphi} := \varepsilon^{1/2} \), and write (1.14) (with \((-\Delta)^{-1}\) replaced by \(\Lambda^{-1}\)) in terms of \( \varepsilon, \tilde{v} \), namely

\[
\Pi_0^\perp \left( \omega \cdot \partial_x \tilde{v} + \zeta \cdot \nabla \tilde{v} + \tilde{\varepsilon} \tilde{\varepsilon} \cdot \nabla \tilde{v} - \tilde{\varepsilon} \tilde{v} \cdot \nabla \tilde{u} \right) = \tilde{\varepsilon} F(\varphi, x), \quad \tilde{u} = \text{curl}(\Lambda^{-1} \tilde{v}).
\]

(1.16)

We will show the existence of solutions of (1.16) with a Nash-Moser approach, by finding zeros of the nonlinear operator (after dropping all the tilde)

\[
F : \mathcal{H}_0^{s+1}(\mathbb{T}^{d+3}, \mathbb{R}^3) \cap Y \rightarrow \mathcal{H}_0^s(\mathbb{T}^{d+3}, \mathbb{R}^3) \cap X
\]

defined by

\[
F(v) := \omega \cdot \partial_x v + \zeta \cdot \nabla v + \varepsilon \Pi_0 \left[ H(v) \cdot \nabla v - v \cdot \nabla H(v) - F(\varphi, x) \right], \quad H(v) := \text{curl}(\Lambda^{-1} v).
\]

(1.17)

We consider parameters \((\omega, \zeta)\) in a bounded open set \( \Omega \subseteq \mathbb{R}^d \times \mathbb{R}^3 \); we will use such parameters along the proof in order to impose appropriate non resonance conditions. Now we state precisely the main results of the paper.

**Theorem 1.2.** There exist \( q = q(\nu) > 0, s = s(\nu) > 0 \), such that for every forcing term \( f \in \mathcal{C}^q(\mathbb{T}^d \times \mathbb{T}^3, \mathbb{R}^3) \) satisfying (1.4) there exists \( \varepsilon_0 = \varepsilon_0(f, \nu) \in (0, 1) \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) the following holds. There exists a Borel set \( \Omega_\varepsilon \subseteq \Omega \) of asymptotically full Lebesgue measure, i.e. \( \lim_{\varepsilon \to 0} |\Omega \setminus \Omega_\varepsilon| = 0 \) such that for any \((\omega, \zeta) \in \Omega_\varepsilon \) there exists \( v(\omega, \zeta) \in \mathcal{H}_0^s(\mathbb{T}^{d+3}, \mathbb{R}^3) \), \( v = \text{odd}(\varphi, x) \) such that \( F(v) = 0 \). Moreover \( \sup_{(\omega, \zeta) \in \Omega_\varepsilon} \|v(\omega, \zeta)\| \leq C \varepsilon^{a} \), for some constants \( C = C(\nu) > 0 \) and \( a = a(\nu) \in (0, 1) \).

From Theorem 1.2 we will deduce (see Chapter 11) the existence of \( u(\varphi, x), p(\varphi, x) \) solving (1.3).

**Related literature: Euler equations.** The celebrated Euler equation, one of the most important mathematical models from fluid dynamics, has been extensively studied in the last century; we refer, for example, to the book [34] and the survey [22] for a rich introduction to the subject with a detailed overview of its vast literature and its interesting open problems.

Starting from the Seventies, the Cauchy problem for the incompressible Euler equation has been analyzed by many authors. Without even trying to be expository, we mention just few results (see e.g. [34] for much more references). The local-in-time existence of smooth solutions for the incompressible Euler equation has been established by Kato in [37, 38]. Concerning the global existence of smooth solutions, the situation is very different in the two-dimensional (2D) case and in the three dimensional (3D) one. Indeed, the Beale-Kato-Majda criterion [12] says that if a smooth solution of the Euler equation on a time interval \([0, T_0]\) cannot be continued to \(T_0\), then its vorticity \(v(t, x)\) satisfies \(\int_0^{T_0} \|v(t, \cdot)\|_{L^\infty} dt = \infty\); viceversa, a priori estimates on that vorticity integral implies the existence of global smooth solutions. In the 2D case, the vorticity is conserved along the fluid particle trajectories; this implies the identity \(\|v(t, \cdot)\|_{L^\infty} = \|v(0, \cdot)\|_{L^\infty}\) and, therefore, the existence of a unique smooth global solution (by Beale-Kato-Majda criterion). We also mention that for the 2D Euler equation Yudovich [40] proved the existence of global weak solutions when the initial vorticity is in the space \(L^2 \cap L^1 \), while ill-posedness for strong solutions of borderline Sobolev
regularity has been recently proved by Bourgain and Li [19]; we also refer to [19] for other references on ill-posedness in both 2D and 3D. We mention the very recent work of Elgindi [26] about singularity formation in finite time for solutions $C^{1,\alpha}_{t,x}$ with spatial decay.

In 3D, the vorticity equation contains an additional term, called “stretching term”. Whether smooth solutions of the 3D Euler (and also Navier-Stokes) equation blow up in finite time or exist globally in time is one of the most famous open problems in the theory of PDEs. Global weak solutions in 3D have been investigated starting form the Nineties. Schnirelman [45] constructed global and distributional solutions in $L^2_{t,x}$; De Lellis and Székelyhidi in [25] constructed global weak solutions in $L^\infty_{t,x}$ with compact support; then these results have been followed by several important improvements, see e.g. [20] and [31].

Regarding time quasi-periodic solutions of the Euler equation, to the best of our knowledge the only result in literature is the 2D one of Crouseilles and Faou [24], where the scalar vorticity formulation allows the construction of quasi-periodic solutions as localized travelling profiles of special form with compact support, avoiding the small divisors problem one usually encounters in KAM results. On the other hand, KAM techniques have been used by Khesin, Kuksin and Peralta-Salas in [32], [33] to obtain non-mixing results for the 3D Euler equation around steady flows.

In this paper, using normal form and KAM (Kolmogorov-Arnold-Moser) techniques, we construct smooth (strong) solutions of the forced 3D Euler equations that are quasi-periodic in time; as a consequence, in particular, they are global in time. This is one of the few KAM results for quasilinear PDEs in higher space-dimension.

Related literature: KAM for quasilinear PDEs. The existence of time-periodic and quasi-periodic solutions of PDEs (which is often referred to as “KAM for PDEs” theory) started in the late 1980s with the pioneering papers of Kuksin [39], Wayne [47] and Craig-Wayne [21]; we refer to the recent review [13] of Berti for a general presentation of the theory, of its protagonists and its state-of-the-art. Many PDEs arising from fluid dynamics are fully nonlinear or quasi-linear equations, namely equations where the nonlinear part contains as many derivatives as the linear part. The key idea in order to deal with these kind of PDEs has been introduced by Iooss, Plotnikov and Toland [30] in the problem of finding periodic solutions for the water waves equation. Using a Nash-Moser iteration to overcome the small divisors problem, their strategy to solve the linearized equation at any approximate solution relies on a normal form procedure based on pseudo-differential calculus. This approach to quasilinear and fully nonlinear PDEs with small divisors has been further developed in [2], [1] for periodic solutions, and it has been successfully combined with a KAM reducibility procedure to develop a general method for 1D problems for the construction of quasi-periodic solutions of quasilinear and fully nonlinear PDEs and for the analysis of the dynamics of linear PDEs with time-quasi-periodic unbounded potentials, see [4], [5], [17], [3], [7], [8]; see [13] for a more complete list of references. We recall that, in this context, a linear operator is said to be reducible if there exists a change of variables, bounded on Sobolev spaces, that conjugates it to a diagonal (or block-diagonal) operator. The extension of KAM reducibility results to higher space dimension $d > 1$ is a difficult matter. The first reducibility result in higher dimension has been obtained by Eliasson and Kuksin in [27] for the linear Schrödinger equation with a bounded analytic potential. Their proof is strongly based on the fact that the eigenvalues of the Laplacian on $\mathbb{T}^d$ are separated and hence some kind of second order Melnikov conditions (lower bounds on differences of the eigenvalues) can be imposed block-wise.

By extending the Craig-Wayne method [21], Bourgain [18] and then Berti-Bolle [14], [15], Berti-Corsi-Procesi [19] proved the existence of invariant tori for nonlinear wave (NLW) and Schrödinger (NLS) equations with bounded perturbations in higher space dimension. These results are based on a technique called multiscale analysis, which allows to solve the linearized equations arising in the Nash-Moser iterative procedure by imposing suitable lower bounds, called first order Melnikov conditions, on its eigenvalues. Extending KAM theory to PDEs with unbounded perturbations in higher space dimension is one of the open problems in the field. A natural strategy is to try to extend the normal form methods based on pseudo-differential calculus developed in 1D in [4], [5], [17], [3], [7], [8]. Up to now, this has been achieved only in few examples, namely the Kirchhoff equation [35], [23], the non-resonant transport equation [29], [11] and the quantum harmonic oscillator on $\mathbb{R}^d$ and Zoll manifolds [9], [10].
1.1 Description of the strategy

In the present paper, to overcome the small divisors problem, we construct invariant tori of the Euler equation by means of a Nash-Moser iteration. Therefore, the core of the paper is the analysis of the linearized operators (see [33]) arising in the Nash-Moser scheme, performed in Sections [11,1]. The strategy is somehow related to the one developed in [11], since the linearized Euler equation is a linear transport-like equation with a small, quasi-periodic in time perturbation of order one. On the other hand the procedure developed in [11] does not apply. The main reason is that in this case we deal with a vector transport operator of the form

\[ h = (h_1, h_2, h_3) \mapsto Lh := \omega \cdot \partial_x h + \zeta \cdot \nabla h + \varepsilon a(\varphi, x) \cdot \nabla h + \varepsilon R(\varphi) h \]  

(1.18)

where \( R(\varphi) = \{ \text{Op}(r_{ij}(\varphi, x, \xi)) \}_{i,j=1,2,3} \) is a 3 \( \times \) 3 matrix-valued pseudo-differential operator of order 0, whereas in [11] the transport operator to normalize is scalar. In order to invert the operator \( L \) in (1.18), we construct a normal form procedure which reduces the operator (1.18) to a 3 \( \times \) 3 block-diagonal operator of the form

\[ \omega \cdot \partial_x h + \zeta \cdot \nabla h + \text{Op}(Q(\xi)) h \]

where the 3 \( \times \) 3 matrix symbol \( Q(\xi) \in \text{Mat}_{3 \times 3} \) satisfies \( \| Q(\xi) \|_{\text{HS}} \lesssim \varepsilon(\xi)^{-1} \) where the norm \( \| \cdot \|_{\text{HS}} \) is the standard Hilbert-Schmidt norm of the matrices. Note that we obtain only a 3 \( \times \) 3 block-diagonalization. This is due to the fact that the unperturbed operator (which is (1.18) for \( \varepsilon = 0 \)) has eigenvalues of multiplicity 3, namely for every \( j \in \mathbb{Z}^d \setminus \{0\} \) the functions

\[ \left( \begin{array}{c} e^{ij} \cdot x \\ 0 \\ 0 \end{array} \right), \quad \left( \begin{array}{c} 0 \\ e^{ij} \cdot x \\ 0 \end{array} \right), \quad \left( \begin{array}{c} 0 \\ 0 \\ e^{ij} \cdot x \end{array} \right) \]

(1.19)

are orthogonal eigenfunctions in \( L^2(\mathbb{T}^d, C^3) \) corresponding to the eigenvalue \( i\zeta \cdot j \).

The fact that we deal with 3 \( \times \) 3 matrix-valued pseudo-differential operators is actually the main technical difficulty of the paper. The point is that, if we take two matrix-valued pseudo-differential operators \( A = \text{Op}(A(\varphi, x, \xi)), B = \text{Op}(B(\varphi, x, \xi)) \) of order \( m, m' \) respectively, it is not true (unlike for scalar pseudo-differential operators) that the commutator \([A, B]\) gains one derivative, namely it is not of order \( m + m' - 1 \). Indeed the principal symbol of the commutator is given by the commutator of two 3 \( \times \) 3 matrices \([A(\varphi, x, \xi), B(\varphi, x, \xi)]\), which, in general, is not zero. This difficulty appears at the normal form step which allows to eliminate the zeroth order term. In fact, at the highest order term (which is \( \omega \cdot \partial_x + (\zeta + \varepsilon a(\varphi, x)) \cdot \nabla \)) the linearized operator \( L \) acts in a diagonal way with respect to the three components \((h_1, h_2, h_3)\), whereas at the zeroth order term the dynamics on these components is strongly coupled. This implies that, in order to reduce to constant coefficients the zeroth order term of the linearized operator, we need to solve a variable coefficients homological equation, see the equation (1.21) below.

In the reduction of the zeroth order term we use the reversible structure of the Euler equation: working with functions with even/odd parity in the pair \((\varphi, x)\) eliminates some average terms that would be an obstruction to the reduction procedure. By reversibility, the reduction to constant coefficients of the zero order term corresponds to its complete cancellation.

Once the zeroth order term has been removed, in order to reduce to constant coefficients also the lower order terms (starting from the order \(-1\)), we use the fact that the perturbation to normalize is at least one-smoothing. The homological equations arising along the procedure have constant coefficients (see (1.22)) and the solutions are of the same order as the remainders we want to normalize: this implies that they gain derivatives. This gain of regularity replaces the gain of derivatives that, in the scalar case, is given by the gain of one derivative of commutators.

Now we describe in more details all the steps of our reduction procedure.

- **Reduction of the highest order term.** As already said, at the highest order term the operator \( L \) acts in a diagonal way on the three components \((h_1, h_2, h_3)\). Hence, in order to reduce to constant coefficients the highest order term, it is enough to diagonalize the transport operator

\[ T := \omega \cdot \partial_x + (\zeta + \varepsilon a(\varphi, x)) \cdot \nabla \].

(1.20)
This is the content of Proposition 4.1, whose proof follows [29]. The only difference is that here, since the vector field \( a(\varphi, x) \) has zero space average and zero divergence, we conjugate \( T \) to the operator \( \omega \cdot \partial_x + \zeta \cdot \nabla \), provided the vector \( (\omega, \zeta) \in \mathbb{R}^{7+3} \) is Diophantine (see (4.7)), whereas in [29] the operator (1.20) is conjugated to a constant coefficients operator of the form \( \omega \cdot \partial_x + M(\omega, \zeta) \cdot \nabla \) with constant vector field \( m(\omega, \zeta) = \zeta + O(\varepsilon) \).

Then in Lemma 4.6 we prove that the remainder \( \varepsilon R \) (see (4.18)) is conjugated, by means of the reversibility preserving invertible map \( A \) constructed in Proposition 4.1 to another reversible operator of order zero which has the form \( \text{Op}(R^{(1)}_0) + \text{Op}(R^{(1)}_1) \) where \( R^{(1)}_i \) is a matrix-valued symbol of order \( i \), \( i = 0, -1 \). We prove that the zeroth order term \( R^{(1)}_0 \) satisfies the symmetry condition \( R^{(1)}_0(\varphi, x, \xi) = R^{(1)}_0(\varphi, x, -\xi) \). This condition, together with the reversibility (which, for the symbol, becomes \( R^{(1)}_0(\varphi, x, \xi) = -R^{(1)}_0(-\varphi, -x, -\xi) \)), allows to perform the normal form step at the zeroth order term.

- **Reduction of the zeroth order term.** In order to eliminate the zeroth order term \( \text{Op}(R^{(1)}_0) \) from the operator \( L^{(1)} \) defined in (4.11), we conjugate such an operator by means of the transformation \( B = \text{Id} + \text{Op}(M(\varphi, x, \xi)) \) where the zeroth order symbol \( M(\varphi, x, \xi) \) has to satisfy the variable coefficients homological equation

\[
(\omega \cdot \partial_x + \zeta \cdot \nabla + R^{(1)}_0(\varphi, x, \xi)) M(\varphi, x, \xi) + R^{(1)}_0(\varphi, x, \xi) = 0. \tag{1.21}
\]

This equation is solved in Section 5.1. The main point is to transform the operator \( \omega \cdot \partial_x + \zeta \cdot \nabla + R^{(1)}_0(\varphi, x, \xi) \) (acting on the space of symbols \( S_{0,0}^{0,0} \) cf. Definition 2.5) into the operator \( \omega \cdot \partial_x + \zeta \cdot \nabla \), for all Diophantine vectors \( (\omega, \zeta) \in DC(\gamma, \tau) \). This is made by means of the iterative scheme of Lemma 5.3 using the property that the symmetry conditions (5.7) are preserved along the iteration. This means that the space-time average of the remainders is always zero and therefore there are no corrections to the normal form operator \( \omega \cdot \partial_x + \zeta \cdot \nabla \). We finally get the operator \( L^{(2)} \) in (5.22) which is a one-smoothing perturbation of the constant coefficients operator \( \omega \cdot \partial_x + \zeta \cdot \nabla \).

The fact that no correction to the constant vector field \( \omega \cdot \partial_x + \zeta \cdot \nabla \) comes from terms of order one and zero in the linearized operator is due to two different reasons: as observed above, the first order term gives no corrections because the coefficient \( a(\varphi, x) \) in (1.20) has zero space average and zero divergence, while the zeroth order term gives no corrections because Euler equation is reversible and we are working in the corresponding invariant subspace (namely, where the vorticity is odd in the pair \( (\varphi, x) \)).

- **Reduction of the lower order terms.** In Proposition 6.1 we construct a reversibility preserving transformation that conjugates the operator \( L^{(2)} \) in (5.22) to the operator \( L^{(3)} \) in (6.2), which is a regularizing perturbation of arbitrary negative order of the constant coefficients operator

\[
\omega \cdot \partial_x + \zeta \cdot \nabla + Q.
\]

Here \( Q \) is a \( 3 \times 3 \) block-diagonal operator of order \(-1\) of the form

\[
Q h(x) = \sum_{\xi \in \mathbb{Z}^3} Q(\xi) \hat{h}(\xi) e^{i \xi \cdot x}, \quad h \in L^2(T^3, \mathbb{R}^3)
\]

with

\[
Q(\xi) \in \text{Mat}_{3 \times 3}(\mathbb{C}), \quad \|Q(\xi)\|_{HS} \lesssim \varepsilon(\xi)^{-1}.
\]

This is proved iteratively in Lemma 6.3. In that Lemma, the homological equation we solve at each step is a constant coefficients equation of the form

\[
(\omega \cdot \partial_x + \zeta \cdot \nabla) M(\varphi, x, \xi) = R(\varphi, x, \xi) - \langle R \rangle_{\varphi, x}(\xi) \tag{1.22}
\]

where \( \langle R \rangle_{\varphi, x}(\xi) \) is the \((\varphi, x)\)-average of the symbol \( R \), see (6.13).
Since $R$ is a symbol of order $-n$ with $n \geq 1$, for $(\omega, \zeta)$ Diophantine, equation (1.22) has a solution $M$ which is a symbol of order $-n$, namely the same order as the one we want to normalize. At the $n$-th step, since the remainder that we normalize is of order $-(n+1)$, then also the solution of the equation (6.13) is of order $-(n+1)$. This allows to show that the new error term is of order $-(n + 2)$.

- **Reducibility.** To complete the reduction to constant coefficients of the linearized operator, the next step is the reducibility scheme of Section 8 in which we conjugate iteratively the operator $L_0$ in (7.3) to a $3 \times 3$, time independent block diagonal operator of the form $\omega \cdot \partial_\varphi + \zeta \cdot \nabla + Q_\infty$. Here $Q_\infty$ is a $3 \times 3$ block diagonal operator $\text{diag}_{j \in \mathbb{Z} \setminus \{0\}} (Q_\infty)^{ij}$ where the $3 \times 3$ matrices $(Q_\infty)^{ij}$ satisfy $\|((Q_\infty)^{ij})\|_{\text{HS}} \lesssim \varepsilon |j|^{-1}$ for all $j \in \mathbb{Z}^3 \setminus \{0\}$, see Lemma 8.3. Along the iterative KAM procedure, we need to solve the homological equation (8.13). In order to solve it (see Lemma 8.3), for any $(\ell, j, j') \in \mathbb{Z}^3 \times (\mathbb{Z}^3 \setminus \{0\}) \times (\mathbb{Z}^3 \setminus \{0\})$, $(\ell, j, j') \neq (0, j, j)$, we have to invert the linear operator

$$L(\ell, j, j') : \text{Mat}_3^{\times 3} \rightarrow \text{Mat}_3^{\times 3}, \quad M \mapsto i(\omega \cdot \ell + \zeta \cdot (j - j')) M + Q_j^i M - M Q_j^i$$

where $Q_j^i$ are time independent $3 \times 3$ matrices. Then we impose second order Melnikov non-resonance conditions with loss of derivatives both in time and in space, involving the invertibility of such kinds of operators and suitable estimates for their inverses, see 8.3. Note that, since the Hamiltonian structure of the Euler equations is not the standard constant one (see 12), the $3 \times 3$ blocks $Q_j^i$ are, in general, not self-adjoint. Hence, to verify that the set of parameters satisfying the required non-resonance conditions has a large Lebesgue measure, we need to control a sufficiently large number of derivatives with respect to the parameter $(\omega, \zeta)$. In particular we prove that the ninth derivative of the determinant of the $9 \times 9$ matrix representing $L(\ell, j, j')$ in (1.22) is big, in order to show that the resonant sets have small Lebesgue measure (see Lemmata 10.3, A.1). Since the $3 \times 3$ blocks could be not self-adjoint, we cannot deduce that our solutions are linearly stable.

As a conclusion of this introduction, we remark that the quadratic nonlinearity of the Euler equation is already in normal form (in the sense of homogeneity order, namely a Poincaré-Dulac normal form) because of the dispersion relation $\lambda(\xi) = i\zeta \cdot \xi$ of the unperturbed operator $\zeta \cdot \nabla$ is exactly linear. This means that the normal form approach, which in KAM theory is usually a very efficient way of extracting the first contribution to the frequency-amplitude relation from the nonlinearity of a PDE, for the Euler equation gives no improvement with respect to the equation itself. This makes it especially difficult to construct of quasi-periodic solutions for the autonomous (i.e. without the external forcing $f$) Euler equation.

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## 2 Norms and linear operators

In this section we collect some general definitions and known results concerning norms, pseudo-differential operators and matrix representation of operators which are used in the whole paper. Subsection 2.2 deals with some conjugacy properties of curl, $(-\Delta)^{-1}$ with changes of variables, required by the analysis of the Euler equation.

**Notations.** In the whole paper, the notation $A \lesssim_{s, m, k_0, \alpha} B$ means that $A \leq C(s, m, k_0, \alpha)B$ for some constant $C(s, m, k_0, \alpha) > 0$ depending on the Sobolev index $s$, the constants $\alpha, m$ and the index $k_0$ which is the maximal number of derivatives with respect to the parameters $(\omega, \zeta)$ that we need to control along our proof. We always omit to write the dependence on $\nu$ which is the number of frequencies and $\tau$, which is the constant appearing in the non-resonance conditions (see for instance 8.2, 8.3). Starting from Section 3 we omit to write the dependence on $k_0$ since it is fixed as $k_0 := 11$ in 10.2. Hence, we write $\lesssim_{s, m, \alpha}$ instead of $\lesssim_{s, m, k_0, \alpha}$. We often write $u = \text{even}(\varphi, x)$ if $u \in X$ and $u = \text{odd}(\varphi, x)$ if $u \in Y$ (recall the Definition (1.12)).
2.1 Function spaces and pseudo differential operators

We denote by $| \cdot |$ the Euclidean norm of vectors and by $| \cdot |_{HS}$ the Euclidean (“Hilbert-Schmidt”) norm of matrices: if $v \in \mathbb{C}^n$ has components $v_j$, and $M \in \text{Mat}_{n \times m}(\mathbb{C})$ has entries $M_{j,k}$, then

$$|v|^2 := \sum_{j=1}^{n} |v_j|^2, \quad ||M||_{HS} := \sum_{1 \leq j \leq n, 1 \leq k \leq m} |M_{j,k}|^2. \quad (2.1)$$

Let $a : \mathbb{T}^n \times \mathbb{T}^3 \to E$, $a = a(\varphi, x)$, be a function taking values in the space of scalars, or vectors, or matrices, namely $E = \mathbb{C}^n$ or $E = \text{Mat}_{n \times m}(\mathbb{C})$. Then, for $s \in \mathbb{R}$, its Sobolev norm $||a||_s$ is defined as

$$||a||_s^2 := \sum_{(\ell,j) \in \mathbb{Z}^n \times \mathbb{Z}^3} (\ell,j)^{2s} |\hat{a}(\ell,j)|^2, \quad (\ell,j) := \max\{1, |\ell|, |j|\}, \quad (2.2)$$

where $\hat{a}(\ell,j) \in E$ (which are scalars, or vectors, or matrices) are the Fourier coefficients of $a(\varphi, x)$, namely

$$\hat{a}(\ell,j) := \frac{1}{(2\pi)^{n+3}} \int_{\mathbb{T}^n \times \mathbb{T}^3} a(\varphi,x) e^{-i(\ell \varphi + j x)} \, d\varphi \, dx,$$

and $|\hat{a}(\ell,j)|$ is their norm defined in (2.1). We denote

$$H^s := H^s_{\varphi,x} := H^s(\mathbb{T}^n \times \mathbb{T}^3) := H^s(\mathbb{T}^n \times \mathbb{T}^3, E) := \{u : \mathbb{T}^n \times \mathbb{T}^3 \to E, \ |u|_s < \infty\},$$

for $E = \mathbb{C}^n$ or $E = \text{Mat}_{n \times m}(\mathbb{C})$; we write, in short, $H^s$ both for vectors and for matrices.

In the paper we use Sobolev norms for (real or complex, scalar- or vector- or matrix-valued) functions $u(\varphi, x; \omega, \zeta)$, $(\varphi, x) \in \mathbb{T}^n \times \mathbb{T}^3$, depending on parameters $(\omega, \zeta) \in \mathbb{R}^{n+3}$ in a Lipschitz way together with their derivatives. We use the compact notation $\lambda := (\omega, \zeta)$ to collect the frequency $\omega$ and the depth $\zeta$ into one parameter vector.

Recall the standard multi-index notation: for $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we denote $|k| := k_1 + \ldots + k_n$ and $k! := k_1! \cdots k_n!$; for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, we denote the derivative $\partial^k_\lambda := \partial_{\lambda_1} \cdots \partial_{\lambda_n}$ and the monomial $\lambda^k := \lambda_1^{k_1} \cdots \lambda_n^{k_n}$. We fix

$$s_0 > (n+3) + k_0 + 2 \quad (2.3)$$

once and for all, and define the weighted Sobolev norms in the following way.

**Definition 2.1. (Weighted Sobolev norms)** Let $k_0 \geq 1$ be an integer, $\gamma \in (0,1]$, and $s \geq s_0$. Given a function $u : \mathbb{R}^{n+3} \to H^s(\mathbb{T}^n \times \mathbb{T}^3)$, $\lambda \mapsto u(\lambda) = u(\varphi, x; \lambda)$ that admits $k_0$ derivatives with respect to $\lambda$, we define its weighted Sobolev norm

$$||u||_{s;k_0,\gamma}^2 := \max_{\alpha \in \mathbb{N}^{n+3}} \sup_{|\alpha| \leq k_0} |\partial^\alpha_\lambda u(\lambda)||_{s-|\alpha|}.$$

For $u$ independent of $(\varphi, x)$, we simply denote by $|u|_{s;k_0,\gamma}$ the same norm.

For any $N > 0$, we define the smoothing operators (Fourier truncation)

$$(\Pi_N u)(\varphi, x) := \sum_{(\ell,j) \leq N} \hat{u}(\ell,j) e^{i(\ell \varphi + j x)}, \quad \Pi_N^\perp := \text{Id} - \Pi_N. \quad (2.4)$$

**Lemma 2.2. (Smoothing)** The smoothing operators $\Pi_N, \Pi_N^\perp$ satisfy the smoothing estimates

$$||\Pi_N u||_{s;k_0,\gamma} \leq N^a ||u||_{s-a;k_0,\gamma}, \quad 0 \leq a \leq s, \quad (2.5)$$

$$||\Pi_N^\perp u||_{s;k_0,\gamma} \leq N^{-a} ||u||_{s+a;k_0,\gamma}, \quad a \geq 0. \quad (2.6)$$
Lemma 2.3. (Product and composition) (i) For all \( s \geq s_0 \),
\[
\|\omega\|_{s-\gamma}^k \leq C(s, k_0) \|u\|_{s-\gamma}^k \|v\|_{s-\gamma}^k + C(s_0, k_0) \|u\|_{s-\gamma}^k \|v\|_{s-\gamma}^k.
\]  
(ii) Let \( \|\alpha\|_{s_0-\gamma}^k \leq \delta(s_0, k_0) \) small enough. Then the composition operator
\[
A : u \mapsto A u,
\]
\((Au)(\varphi, x) := u(\varphi, x + \alpha(\varphi, x))\), satisfies the following tame estimates: for all \( s \geq s_0 \),
\[
\|Au\|_{s-k-\gamma}^k \leq \|u\|_{s-k-\gamma}^k + \|\alpha\|_{s_0-\gamma}^k \|u\|_{s-\gamma}^k.
\]  
The function \( \alpha \), defined by the inverse diffeomorphism \( y = x + \alpha(\varphi, x) \) if and only if \( x = y + \alpha(\varphi, y) \), satisfies
\[
\|\alpha\|_{s-k-\gamma}^k \leq \|\alpha\|_{s_0-\gamma}^k.
\]
As a consequence
\[
\|A^{-1}u\|_{s-k-\gamma}^k \leq \|u\|_{s-k-\gamma}^k + \|\alpha\|_{s_0+k+1}^k \|u\|_{s-\gamma}^k.
\]  
(iii) Assume that \( \|\alpha\|_{s_0+k+1}^k \leq \delta(s_0, k_0) \) small enough. Then
\[
\|(A - \text{Id})h\|_{s-k-\gamma}^k \leq \|\alpha\|_{s_0+k+1}^k \|h\|_{s-\gamma}^k + \|\alpha\|_{s_0-\gamma}^k \|h\|_{s_0-\gamma}^k.
\]  
Similar estimates hold for the operators \( A^{-1}, (A^{-1})^* \).

Proof. Items (i), (ii) follows as in [17], taking into account Definition 2.1 (here \( \partial^\alpha u \), \( |\alpha| \leq k_0 \), is estimated in \( H^{s-|\alpha|} \), whereas in [17] and [3] they are estimated in \( H^s \)) and the definition of \( s_0 \) in [24]. We only prove (iii). A direct calculation shows that
\[
(A - \text{Id})h(\varphi, x) = \int_0^1 \nabla h(\varphi, x + t\alpha(\varphi, x)) \cdot \alpha(\varphi, x) dt.
\]
Then, the estimate on \( A - \text{Id} \) follows by applying the estimates (2.7), (2.9). The estimate for \( A^{-1} - \text{Id} \) can be proved similarly. Now we estimate the operator \( A^* - \text{Id} \). A direct calculation shows that
\[
A^* h(\varphi, y) = \det(\text{Id} + \nabla \alpha(\varphi, y)) h(\varphi, y + \alpha(\varphi, y)).
\]
Note that for \( \|\nabla \alpha\|_{L^\infty} \) small enough one has \( \det(\text{Id} + \nabla \alpha(\varphi, y)) \geq \frac{1}{2} \). This is guaranteed by the smallness assumption \( \|\alpha\|_{s_0+k+1}^k \leq \delta \), by the estimate (2.10) and by Sobolev embeddings. One writes
\[
(A^* - \text{Id})h(\varphi, y) = \left( \det(\text{Id} + \nabla \alpha(\varphi, y)) - 1 \right) A^{-1} h(\varphi, y) + (A^{-1} - \text{Id})h(\varphi, y).
\]  
By estimates (2.7), (2.10) together with the smallness assumption \( \|\alpha\|_{s_0+k+1}^k \leq \delta \) one deduces that
\[
\|\det(\text{Id} + \nabla \alpha(\varphi, y)) - 1\|_{s-k-\gamma}^k \leq \|\alpha\|_{s_0+k+1}^k.
\]
Therefore the claimed estimate for \( A^* - \text{Id} \) follows by (2.13), (2.7), (2.11). The estimate for \( (A^{-1})^* - \text{Id} \) can be proved similarly. \( \square \)

Let \( m : \mathbb{R}^{+3} \to \mathbb{R}^3 \), \((\omega, \zeta) \mapsto m(\omega, \zeta)\), be a \( k_0 \) times differentiable function satisfying \( |m - \zeta|_{s_0+k-\gamma} \leq \frac{1}{2} \), and define the set
\[
O(\gamma, r) := \left\{ (\omega, \zeta) \in \mathbb{R}^r \times \mathbb{R}^3 : |\omega \cdot \ell + m(\omega, \zeta) \cdot j| \geq \frac{\gamma}{|\ell, j|^2}, \forall (\ell, j) \in \mathbb{Z}^{r+3} \setminus \{0, 0\} \right\}.
\]  
The equation \((\omega \cdot \partial_\phi + m \cdot \nabla)v = u\), where \( u(\varphi, x) \) has zero average with respect to \((\varphi, x)\), has the periodic solution
\[
(\omega \cdot \partial_\phi + m \cdot \nabla)^{-1} u(\varphi, x) := \sum_{(\ell, j) \in \mathbb{Z}^{r+3} \setminus \{0, 0\}} \frac{\tilde{u}(\ell, j)}{i(\omega \cdot \ell + m \cdot j)} e^{i(\phi + j x)}.
\]  

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We define its extension to all \((\omega, \zeta) \in \mathbb{R}^\nu \times \mathbb{R}^3\) as

\[
(\omega \cdot \partial_x + m \cdot \nabla)^{-1} u(\varphi, x) := \sum_{(l, j) \in \mathbb{Z}^{\nu+3}} \frac{\chi((\omega \cdot \ell + m \cdot j)\gamma^{-1}(l, j)^\gamma)}{i(\omega \cdot \ell + m \cdot j)} \hat{u}(l, j) e^{i(\varphi + j \cdot x)},
\]

where \(\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\) is an even and positive cut-off function such that

\[
\chi(\xi) = \begin{cases} 
0 & \text{if } |\xi| \leq \frac{1}{3}, \\
1 & \text{if } |\xi| \geq \frac{2}{3},
\end{cases} \quad \partial_\xi \chi(\xi) > 0 \quad \forall \xi \in \left(\frac{1}{3}, \frac{2}{3}\right).
\]

Note that \((\omega \cdot \partial_x + m \cdot \nabla)^{-1} u = (\omega \cdot \partial_x + m \cdot \nabla)^{-1} u\) for all \((\omega, \zeta) \in \mathcal{O}(\gamma, \tau)\).

**Lemma 2.4. (Diophantine equation)** One has

\[
\| (\omega \cdot \partial_x + m \cdot \nabla)^{-1} u \|_{k_0, \gamma} \lesssim k_0 \gamma^{-1} \| u \|_{k_0, \gamma}, \quad \tau_0 := k_0 + \tau(k_0 + 1).
\]

**Definition 2.5. (Pseudo-differential operators and symbols)** Let \(m \in \mathbb{R}, s \geq s_0, \beta \in \mathbb{N}, n \in \mathbb{N}\). We say that an operator \(A = A(\varphi)\) is in the class \(\mathcal{OP}_S^{m, s, \beta}\) if there exists a function \(A : T^* \times T^3 \times \mathbb{R}^3 \to \text{Mat}_{n \times n}(\mathbb{C})\), \(A = A(\varphi, x, \xi)\), differentiable \(s\) times differentiable way on the parameters \(\lambda \in \mathbb{R}^{\nu+3}\), such that

\[
Au(x) = \text{Op}(A)u(x) = \sum_{\xi \in \mathbb{Z}^3} A(\varphi, x, \xi) \hat{u}(\xi) e^{i\varphi \cdot \xi} \quad \forall u \in \mathcal{C}^\infty(T^3, \mathbb{C}^n),
\]

and

\[
|A|_{m, s, \beta} := \sup_{|\alpha| \leq \beta} \sup_{\xi \in \mathbb{R}^3} \| \partial_\xi^\alpha A(\cdot, \xi) \|_{s, \beta} < \infty;
\]

in that case, we also say that \(A(\varphi, x, \xi)\) is in the class \(S_{m, s}^{m, s, \beta}\). The operator \(A\) is said to be a pseudo-differential operator of order \(m\), and the function \(A\) is its symbol.

If \(A = A(\lambda)\) depends in a \(k_0\) times differentiable way on the parameters \(\lambda = (\omega, \zeta) \in \mathbb{R}^{\nu+3}\), we define

\[
|A|^{k_0, \gamma}_{m, s, \beta} := \sup_{|\alpha| \leq \beta} \sup_{\xi \in \mathbb{R}^3} \| \partial_\xi^\alpha A(\cdot, \xi) \|_{s, \beta} < \infty;
\]

and

\[
|A|^{k_0, \gamma}_{m, s, \beta} = \max_{|k| \leq k_0} \sup_{\lambda \in \mathbb{R}^{\nu+3}} |\partial_\lambda^k A(\lambda)|_{m, s, \beta}.
\]

The values of \(n\) in Definition 2.3 that we need for our problem are \(n = 1\) (when both \(u\) and \(A\) are scalar functions) and \(n = 3\) (when the function \(u\) takes values in \(\mathbb{R}^3\) or \(\mathbb{C}^3\), and the symbol \(A(\varphi, x, \xi)\) is a \(3 \times 3\) matrix). Recall that, when \(A(\varphi, x, \xi)\) is a matrix, its Sobolev norm is given by (2.11)-(2.2).

In the rest of this section we assume, without explicitly writing it, that all functions and symbols depend in a \(k_0\) times differentiable way on the parameter \(\lambda \in \mathbb{R}^{\nu+3}\).

Given a symbol \(A = A(\varphi, x, \xi) \in S_{m, s}^{m, s, \beta}\), we define the averaged symbol \(\langle A \rangle_{\varphi, x}\) as

\[
\langle A \rangle_{\varphi, x}(\xi) := \frac{1}{(2\pi)^{\nu+3}} \int_{T^{\nu+3}} A(\varphi, x, \xi) \, d\varphi \, dx.
\]

One easily verifies that, for all \(s \geq 0\),

\[
\langle A \rangle_{\varphi, x} \in S_s^{m, s, \beta} \quad \text{and} \quad \text{Op}(\langle A \rangle_{\varphi, x}) \lesssim \text{Op}(A) \lesssim \text{Op}(A)
\]

Moreover if a symbol \(A = A(\varphi, x)\) is independent of \(\xi\), then the corresponding operator \(\text{Op}(A) \in \mathcal{OPS}_s^{m, s, \beta}\) is a multiplication operator, and

\[
\text{Op}(A) : h(\varphi, x) \mapsto A(\varphi, x)h(\varphi, x), \quad |\text{Op}(A)|^{k_0, \gamma}_{s, s, \beta} = |A|^{k_0, \gamma}_{s, s, \beta}.
\]

By the definition of the norm in (2.19), and using the interpolation estimate (2.21), it follows that if \(A = \text{Op}(A) \in \mathcal{OPS}_s^{m, s, \beta}, B = \text{Op}(B) \in \mathcal{OPS}_s^{m, s, \beta}, s \geq s_0\), then \(\text{Op}(AB) \in \mathcal{OPS}_s^{m+m', s, \beta}\) and

\[
|\text{Op}(AB)|^{k_0, \gamma}_{m+m', s, \beta} \lesssim \| A \|^{k_0, \gamma}_{m, s, \beta} \| B \|^{k_0, \gamma}_{m', s, \beta} + |A|^{k_0, \gamma}_{m, s, \beta} |B|^{k_0, \gamma}_{m', s, \beta}.
\]
Lemma 2.7. (Composition of pseudo-differential operators)

Let \( |m| \) where the remainder \( \| \Omega(A^n) \|_{0,s,\beta} \leq \left( C(s, \beta) \| \Omega(A) \|_{0,s,\beta} \right)^{n-1} \| \Omega(A) \|_{0,s,\beta} \) for some constant \( C(s, \beta) > 0 \).

Lemma 2.6. Let \( s \geq s_0 \) and \( \Lambda(\lambda) \in \mathcal{OPS}^{\lambda}_{s,0} \), \( u(\lambda) \in H^s(\mathbb{R}^+, \mathbb{R}^3) \) for all \( \lambda \in \mathbb{R}^{n/3} \). Then

\[
\| \Lambda u \|_{s,\alpha,\beta} \lesssim \| A_{0,s,0} \|_{s,\alpha,\beta} + \| \Lambda \|_{0,s,0} \| u \|_{s,\alpha,\beta}.
\]

Lemma 2.7. (Composition of pseudo-differential operators) Let \( s \geq s_0, m, m' \in \mathbb{R}, \beta \in \mathbb{N} \).

(i) Let \( A = \Omega(A) \in \mathcal{OPS}^m_{s,1}, B = \Omega(B) \in \mathcal{OPS}^{m'}_{s+|m|+\beta,\beta}. \) Then the composition \( AB \) belongs to \( \mathcal{OPS}^{m+m'}_{s,\beta} \), and

\[
|AB|_{m+m',s,\beta} \lesssim \| A \|_{m,s,\beta} \| B \|_{m',s+|m|+\beta,\beta} + \| A \|_{m,s,\beta} \| B \|_{m',s+|m|+\beta,\beta}.
\]

(ii) Let \( A = \Omega(A) \in \mathcal{OPS}^m_{s,1}, B = \Omega(B) \in \mathcal{OPS}^{m'}_{s+|m|+2,0}. \) Then

\[
AB = \Omega(A(\varphi, x, \xi)B(\varphi, x, \xi)) + R_{AB}, \quad R_{AB} \in \mathcal{OPS}^{m+m'-1}_{s,0},
\]

where the remainder \( R_{AB} \) satisfies

\[
|R_{AB}|_{m+m'-1,s,0} \lesssim \| A \|_{m,s,1} \| B \|_{m',s+|m|+2,0} + \| A \|_{m,s,0} \| B \|_{m',s+|m|+2,0}.
\]

Proof. See Lemma 2.13 in \([17]\).

For functions \( u, v : \mathbb{T}^3 \to \mathbb{R}^n, n = 1 \) or \( n = 3 \), we consider the \( L^2(\mathbb{T}^3, \mathbb{R}^n) \) scalar product

\[
\langle u, v \rangle_{L^2(\mathbb{T}^3, \mathbb{R}^n)} := \Pi_0 (u \cdot v)
\]

where \( \Pi_0 \) is the space average defined in \([13]\) and \( \langle \cdot, \cdot \rangle \) is the standard scalar product in \( \mathbb{R}^n \). The adjoint of an operator mapping \( H^s(\mathbb{T}^3 \times \mathbb{R}^3) \) into itself is considered with respect to the scalar product \( \langle \cdot, \cdot \rangle \).

Lemma 2.8 (Adjoint). Let \( m \in \mathbb{R}, s \geq s_0 \) and \( A = \Omega(A) \in \mathcal{OPS}^m_{s+|m|+0,0}. \) Then the adjoint operator \( \Omega(A)^* \in \mathcal{OPS}^m_{s,0} \) and \( \| \Omega(A)^* \|_{m,s,0} \lesssim \| \Omega(A) \|_{m,s+|m|+0,0}. \)

Proof. For a matrix symbol \( A = (A_{ik})_{i,k=1,2,3} \), one computes \( \Omega(A)^* = (\Omega(A_{ki})^*)_{i,k=1,2,3} \). Then we argue as in Lemma 2.16 \([17]\) to each operator \( \Omega(A_{ki})^* \).

Lemma 2.9 (Neumann series). Let \( s \geq s_0 \) and \( \Psi \in \mathcal{OPS}^m_{s,0}, m \geq 0. \) There exists \( \delta = \delta(s_0, k_0) \in (0,1) \)

small enough such that, if \( |\Psi|_{m,s,0} \leq \delta \), then \( \Phi = 1 + \Psi \) is invertible and \( |\Phi^{-1} - 1|_{m,s,0} \lesssim s_0 \| \Psi \|_{m,s,0}. \)

Proof. See Lemma 2.17 in \([17]\).

2.2 Some conjugations with changes of variables

In the next lemma we exploit some properties of some pseudo-differential operators conjugated by a change of variables. We will always assume the hypotheses of Lemma 2.3.

Lemma 2.10. Let \( S > s_0, \alpha(\cdot, \lambda) \in H^S, \lambda \in H^S, M(\cdot, \lambda) = (a_{ij}(\cdot, \lambda))_{i,j=1,2,3} \in H^S \) and \( \| \alpha \|_{s_0} \leq \delta \) for some \( \delta \in (0,1) \) small enough. Then \( A^{-1} \Omega(M)A \) is the multiplication operator by the \( 3 \times 3 \) matrix \( \tilde{M}(\varphi, y) = M(\varphi, y + \tilde{\alpha}(\varphi, y)). \) Moreover for any \( s_0 \leq s \leq S, \beta \in \mathbb{N} \)

\[
\| \Omega(M) \|_{s,0,\beta} \lesssim s_0 \| M \|_{s,0,\beta} + \| \alpha \|_{s_0} \| M \|_{s,0,\beta}.
\]

Proof. The lemma is a straightforward consequence of Lemma 2.3(ii) and the estimate \( \| M \|_{s,0,\beta} \leq \| M \|_{s,0} \).
Lemma 2.11. Let $S > s_0$, $\alpha(\cdot; \lambda) \in H^{S+1}$, $a(\cdot; \lambda) \in H^S$ and $\|a\|_{S_0+1}^{k_0, \gamma} \leq \delta$ for some $\delta \in (0, 1)$ small enough. Then for any $s_0 \leq s \leq S$, $\beta \in \mathbb{N}$, $A^{-1}a \cdot \nabla A = \text{Op}(M_N) \in \mathcal{OP}_S^{1, \beta}$ and $A^{-1}\text{curl} A = \text{Op}(M_{\text{curl}}) \in \mathcal{OP}_S^{1, \beta}$. The following estimates hold for any $s_0 \leq s \leq S$:

$$|A^{-1}a \cdot \nabla A|_{1,s,\beta}^{k_0, \gamma} \lesssim s_{k_0, \beta} 1 + \|a\|_{S+1}^{k_0, \gamma},$$

$$|A^{-1}\text{curl} A|_{1,s,\beta}^{k_0, \gamma} \lesssim s_{k_0, \beta} 1 + \|a\|_{S+1}^{k_0, \gamma}.$$

Furthermore, the symbols $M_N, M_{\text{curl}}$ satisfy the symmetry conditions

$$M_N(\varphi, x, \xi) = -M_N(\varphi, x, -\xi), \quad M_{\text{curl}}(\varphi, x, \xi) = -M_{\text{curl}}(\varphi, x, -\xi).$$

Proof. By recalling the definition of curl given in (1.9), it suffices only to analyze for any $i = 1, 2, 3$ the operator $M := A^{-1}\partial_i A$. A direct calculation shows that $M = \partial_i + A^{-1}[\partial_i, A] \cdot \nabla$. This immediately implies that $M = \text{Op}(M) \in \mathcal{OP}_S^{1, k}$ for any $k \in \mathbb{N}$ and $M(\varphi, x, \xi) = -M(\varphi, x, -\xi)$. Moreover, by applying the estimates (2.9), (2.10), (2.23) and the trivial fact that $|\partial_i|_{1,s,k} \leq 1$ for any $s,k$ one gets the estimate $|M|_{1,s,\beta}^{k_0, \gamma} \lesssim s_{k_0, \beta} 1 + \|a\|_{S+1}^{k_0, \gamma}$.

In the following we analyze the conjugation of the operator $A^{-1}$ by means of a change of variables, where we recall that $\Lambda := \Pi_0 - \Delta$. Note that the action of the operator $\Lambda$ on a function $u \in H^2(\mathbb{T}^3, \mathbb{R})$ is given by

$$\Lambda u(x) = \tilde{u}(0) + \sum_{\xi \in \mathbb{Z}^3 \backslash \{0\}} |\xi|^2 \tilde{u}(\xi)e^{ix\cdot \xi}. $$

We identify the operator $\Lambda$ with $\text{Op}(\lambda(\xi))$ where

$$\lambda \in C^\infty(\mathbb{R}^3, \mathbb{R}), \quad \lambda(\xi) = \lambda(-\xi), \quad \inf_{\xi \in \mathbb{R}^3} \lambda(\xi) > 0,$$

$$\lambda(\xi) = |\xi|^2 \quad \text{if} \quad |\xi| \geq 1 \quad \text{and} \quad \lambda(0) = 1. \quad (2.28)$$

The inverse of $\Lambda$ is computed explicitly on periodic functions as

$$\Lambda^{-1}u(x) = \tilde{u}(0) + \sum_{\xi \in \mathbb{Z}^3 \backslash \{0\}} \frac{1}{|\xi|^2} \tilde{u}(\xi)e^{ix\cdot \xi}. $$

Then for any $s, \beta \geq 0$

$$|\Lambda|_{2,s,\beta}, \quad |\Lambda^{-1}|_{-2,s,\beta} \lesssim 1. \quad (2.29)$$

We also identify the projector $\Pi_0$ with $\text{Op}(\chi_0(\xi))$ where $\chi_0 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\chi_0 \in C^\infty(\mathbb{R}^3, \mathbb{R}), \quad \chi_0(\xi) = \chi_0(-\xi), \quad 0 \leq \chi_0 \leq 1, \quad \chi_0(0) = 1, \quad \text{supp}(\chi_0) \subset \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{1}{2}\}. \quad (2.30)$$

Hence for any $m, \alpha, s \geq 0$

$$|\Pi_0|_{-m,s,\alpha} \lesssim m, \alpha. \quad (2.31)$$

Lemma 2.12. $S > s_0$, $\alpha(\cdot; \lambda) \in H^{S+2}$ and $\|a\|_{S_0+2}^{k_0, \gamma} \leq \delta$ for some $\delta = \delta(k_0, \nu) \in (0, 1)$ small enough. Then $P_{\Lambda} := A^{-1}A = \Lambda + P_2 + P_1$ where, for $s_0 \leq s \leq S$, $P_2 = \text{Op}(P_2) \in \mathcal{OP}_S^{2, 1}$, $P_1 \in \mathcal{OP}_S^{1, 0}$ satisfy the estimates

$$|P_2|_{2,s,\gamma}^{k_0, \gamma} \lesssim s_{k_0, \gamma} \|a\|_{S+2}^{k_0, \gamma},$$

Furthermore, the symbol $P_2$ satisfies the symmetry condition $P_2(\varphi, x, \xi) = P_2(\varphi, x, -\xi)$.

Proof. The assumption of the lemma allows to apply Lemma 2.3 on the change of variables. A direct calculation shows that $-A^{-1}\Delta A$ is an elliptic operator of the form

$$-A^{-1}\Delta A = -\Delta - \sum_{i,j=1}^3 a_{ij}(\varphi, x)\partial_{x_i}x_j + \sum_{i=1}^3 b_i(\varphi, x)\partial_{x_i}, \quad (2.32)$$
Lemma 2.13. Let $P$ inverse is of the form (2.8), one computes

$$\Pi_0 A = \Pi_0 + \Pi_0 (A - \Id)$$

and

$$\Pi_0 ((A - \Id) h) = \Pi_0 (g_0 h), \quad q_0(\varphi, x) := (A - \Id)^* [1] = \det (I + D\varphi(x)) - 1.$$  

Then using (2.23), Lemma 2.7(iii) and the trivial facts that $|\Pi_0|_{0, s, 0} \leq 1$, one obtains that $\Pi_0 (A - \Id) \in \mathcal{OPS}_{s, 1}^0$, with

$$|\Pi_0 (A - \Id)|_{0, s, 1} \lesssim s, k_0 \parallel \alpha \parallel_{s+1}^{k_0, \gamma}.$$  

Since $A^{-1} \Pi_0 = \Pi_0$, by (2.23), (2.24), one gets that

$$A^{-1} \Lambda A = \Lambda + \mathcal{P}_2 + \mathcal{P}_1,$$

so

$$\mathcal{P}_2 := -\sum_{i, j = 1}^{3} a_{ij}(\varphi, x) \partial_{x_i} x_j, \quad \mathcal{P}_1 := \sum_{i = 1}^{3} b_i(\varphi, x) \partial_{x_i} + \Pi_0 (A - \Id).$$

By (2.23), Lemma 2.7 using that $|\partial_{x_i} x_j|_{2, s, 1}, |\partial_{x_i} x_j|_{1, s, 1} \lesssim 1$ and the estimates (2.23), (2.25) one gets the claimed bounds on $\mathcal{P}_1$ and $\mathcal{P}_2$. Moreover by the formula (2.26), one deduces that the symbol of the operator $\mathcal{P}_2$ is $P_2(\varphi, x, \xi) = \sum_{i, j = 1}^{3} a_{ij}(\varphi, x) \xi_i \xi_j$, which is even with respect to the variable $\xi_i$.}

**Lemma 2.13.** Let $s > s_0$, $\alpha(\cdot; \lambda) \in H^{s+\mu}$, and $\parallel \alpha \parallel_{s_0 + \mu}^{k_0, \gamma} \leq \delta$ for some $\delta = \delta(s, k_0, \mu) > 0$ large enough. Then $\Lambda_\alpha := A^{-1} \Lambda A$ defined in Lemma 2.12 is invertible and its inverse is of the form

$$\mathcal{P}_A^{-1} = A^{-1} \Lambda^{-1} A = \mathcal{P}_{-2} + \mathcal{P}_{-3},$$

where $\mathcal{P}_{-2} = \text{Op}(\mathcal{P}_{-2}) \in \mathcal{OPS}_{s, 2}^{-2}$, $\mathcal{P}_{-3} \in \mathcal{OPS}_{s, 0}^{-3}$ satisfy the estimates

$$|\mathcal{P}_{-2}|_{0, s, 1}^{k_0, \gamma}, |\mathcal{P}_{-3}|_{0, s, 0}^{k_0, \gamma} \lesssim s, k_0 \parallel \alpha \parallel_{s+1}^{k_0, \gamma}.$$  

Furthermore, the symbol $\mathcal{P}_{-2}$ satisfies the symmetry condition $\mathcal{P}_{-2}(\varphi, x, \xi) = \mathcal{P}_{-2}(\varphi, x, -\xi)$.

**Proof.** By Lemma 2.12 we write the operator $\mathcal{P}_\lambda = \Lambda + \mathcal{P}_2 + \mathcal{P}_1$ as

$$\mathcal{P}_\lambda = \Lambda (\Id + \mathcal{F}), \quad \mathcal{F} := \Lambda^{-1} \mathcal{P}_2 + \Lambda^{-1} \mathcal{P}_1.$$  

Note that $\Lambda^{-1} \mathcal{P}_2$ is of order 0 and $\Lambda^{-1} \mathcal{P}_1$ is of order $-1$. Define

$$\mathcal{F}_0 := \text{Op}(\mathcal{F}_0(\varphi, x, \xi)), \quad \mathcal{F}_0(\varphi, x, \xi) : = \lambda(\xi)^{-1} P_2(\varphi, x, \xi), \quad \mathcal{F}_{-1} := \mathcal{F} - \mathcal{F}_0$$

(in fact, $\mathcal{F}_0 = \mathcal{P}_2 \Lambda^{-1}$). By applying Lemma 2.7 and the estimates of $\mathcal{P}_1, \mathcal{P}_2$ provided by Lemma 2.12 (recall also that $\Lambda \equiv \text{Op}(\lambda)$, see (2.26)) with $s$ replaced by $s + 2$, one obtains the bounds

$$|\mathcal{F}_0|_{0, s+2, 1}^{k_0, \gamma}, |\mathcal{F}_{-1}|_{-1, s+2, 1}^{k_0, \gamma} \lesssim s, k_0 \parallel \alpha \parallel_{s+1}^{k_0, \gamma}.$$  

for some constant $\mu > 0$. Moreover, since $\lambda(\xi)$ and the symbol $P_2(\varphi, x, \xi)$ are even functions of $\xi$, then also $\mathcal{F}_0(\varphi, x, \xi)$ is even in $\xi$. By Lemma 2.7 using (2.39) and the hypothesis that $\parallel \alpha \parallel_{s_0 + \mu}^{k_0, \gamma}$ is small enough, we deduce that $\Id + \mathcal{F}$ is invertible, and its inverse satisfies the estimate

$$|\Id - \mathcal{F}|^{-1}_{0, s, 0}^{k_0, \gamma} \lesssim s, k_0 \parallel \alpha \parallel_{s+1}^{k_0, \gamma}.$$  

By Lemma 2.7(ii) and (2.39) one has

$$\mathcal{F}_0 \text{Op} \left( \frac{1}{1 + \mathcal{F}_0} \right) = \text{Op} \left( \frac{\mathcal{F}_0}{1 + \mathcal{F}_0} \right) + \mathcal{R}_{\mathcal{F}_0}, \quad |\mathcal{R}_{\mathcal{F}_0}|_{0, s, 0}^{k_0, \gamma} \lesssim s \parallel \alpha \parallel_{s+1}^{k_0, \gamma}.$$  

(2.41)
Since $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{-1}$, one has
\[
(\text{Id} + \mathcal{F})\text{Op} \left( \frac{1}{1 + F_0} \right) = \text{Op} \left( \frac{1}{1 + F_0} \right) + \text{Op} \left( \frac{F_0}{1 + F_0} \right) + \mathcal{R}_{F_0} + \mathcal{F}_{-1}\text{Op} \left( \frac{1}{1 + F_0} \right)
= \text{Id} + \mathcal{R}_{F_0} + \mathcal{F}_{-1}\mathcal{G}, \quad \mathcal{G} := \text{Op} \left( \frac{1}{1 + F_0} \right),
\]
whence, applying $(\text{Id} + \mathcal{F})^{-1}$ from the left, we get
\[
\mathcal{G} = (\text{Id} + \mathcal{F})^{-1} - \mathcal{R}_{\mathcal{F}}, \quad \mathcal{R}_{\mathcal{F}} := -(\text{Id} + \mathcal{F})^{-1}(\mathcal{R}_{F_0} + \mathcal{F}_{-1}\mathcal{G}). \tag{2.42}
\]
Since $\frac{1}{1 + F_0} = \sum_{n \geq 0} (-1)^n F_0^n$, by estimates $(2.25)$, $(2.39)$, using the assumption that $\|\alpha\|_{k_0 + \mu} \leq \delta$ is small enough, one gets
\[
|\mathcal{G}|_{k_0,\gamma} \leq \sum_{n \geq 0} \|\alpha\|_{k_0 + \mu} = \delta.
\]
Then, recalling $(2.37)$ and using $(2.32)$ to substitute $(\text{Id} + \mathcal{F})^{-1}$, we get
\[
\mathcal{P}_{\lambda}^{-1} = (\text{Id} + \mathcal{F})^{-1}\lambda^{-1} = (\mathcal{G} + \mathcal{R}_{\mathcal{F}})\lambda^{-1} = \mathcal{P}_{-2} + \mathcal{P}_{-3}
\]
with
\[
\mathcal{P}_{-2} = \mathcal{G}\lambda^{-1} = \text{Op}(\mathcal{P}_{-2}), \quad \mathcal{P}_{-2}(\varphi, x, \xi) := \frac{1}{\lambda(\xi)(1 + F_0(\varphi, x, \xi))}, \quad \mathcal{P}_{-3} := \mathcal{R}_{\mathcal{F}}\lambda^{-1}.
\]
Since $F_0$ and $\lambda$ are even in $\xi$, then $\mathcal{P}_{-2}$ is also even in $\xi$. By $(2.40)$, $(2.41)$, $(2.42)$, $(2.43)$ we get $|\mathcal{R}_{\mathcal{F}}|_{1,0} \leq \sum_{k_0, k_0} \|\alpha\|_{k_0 + \gamma}$. Thus the estimates for $\mathcal{P}_{-2}, \mathcal{P}_{-3}$ follow by the composition estimates of Lemma 2.7 and by using $(2.25)$.

### 2.3 Matrix representation of linear operators

Let us consider a linear matrix operator $\mathcal{R} = (\mathcal{R}_{ik})_{i,k=1,2,3} : L^2(T^3, \mathbb{R}^3) \to L^2(T^3, \mathbb{R}^3)$ where $\mathcal{R}_{ik} : L^2(T^3, \mathbb{R}) \to L^2(T^3, \mathbb{R})$ for any $i, k = 1, 2, 3$. Such an operator can be represented as
\[
\mathcal{R}u(x) := \sum_{j, j' \in \mathbb{Z}^3} \mathcal{R}_{ij}^j [\hat{u}(j')] e^{ij'x}, \quad {\text{for}} \ u(x) = \sum_{j \in \mathbb{Z}^3} \hat{u}(j) e^{ijx}, \tag{2.44}
\]
where, for $j, j' \in \mathbb{Z}^3$, $\mathcal{R}_{ij}^j$ is the $3 \times 3$ matrix defined by
\[
\mathcal{R}_{ij}^j := \left( (\mathcal{R}_{ik})_{jj'}^j \right)_{i,k=1,2,3}, \quad (\mathcal{R}_{ik})_{jj'}^j := \frac{1}{(2\pi)^3} \int_{T^3} \mathcal{R}_{ik}[e^{ij'x}] e^{-ijx} dx. \tag{2.45}
\]
It is immediate to check that the matrix representation $(2.44)$ of the operator $\mathcal{R}$ is equivalent to the pseudo-differential representation $\mathcal{R} = \text{Op} (r)$, namely $\mathcal{R}u(x) = \sum_{\xi \in \mathbb{Z}^3} r(x, \xi) \hat{u}(\xi)e^{i\xi x}$, where the symbol $r(x, \xi)$, for $\xi = j' \in \mathbb{Z}^3$, is the $3 \times 3$ matrix given by
\[
r(x, j') = \sum_{j \in \mathbb{Z}^3} \mathcal{R}_{ij}^j e^{ij'x}. \tag{2.46}
\]
Thus $\mathcal{R}_{ij}^j$ is the Fourier coefficient $r(j, j')$ of frequency $j - j'$ of the function $x \mapsto r(x, j')$.

**Definition 2.14** (Block-diagonal operator). We say that an operator $\mathcal{R}$ as in $(2.44)$-$(2.45)$ is a $3 \times 3$ block-diagonal operator if $\mathcal{R}_{jj'} = 0$ for all $j, j' \in \mathbb{Z}^3$ with $j \neq j'$.

By $(2.40)$, $\mathcal{R}$ is a block-diagonal operator if and only if the symbol $r(x, j')$ does not depend on $x$.

We also consider smooth $\varphi$-dependent families of linear operators $\mathcal{T}_\varphi : B(L^2(T^3, \mathbb{R}^3)) \to B(L^2(T^3, \mathbb{R}^3))$, $\varphi \mapsto \mathcal{R}(\varphi)$, which we write in Fourier series with respect to $\varphi$ as
\[
\mathcal{R}(\varphi) = \sum_{\ell \in \mathbb{Z}^3} \hat{\mathcal{R}}(\ell) e^{i\ell \varphi}, \quad \hat{\mathcal{R}}(\ell) := \frac{1}{(2\pi)^3} \int_{T^3} \mathcal{R}(\varphi)e^{-i\ell \varphi} d\varphi, \quad \ell \in \mathbb{Z}^3.
\]
According to (2.45), for any \( \ell \in \mathbb{Z}^d \), the linear operator \( \tilde{R}(\ell) \in \mathcal{B}(L^2(\mathbb{T}^d, \mathbb{R}^3)) \) is identified with the matrix 
\[ (\tilde{R}(\ell))_{j,j'} \in \mathbb{C} \] 
where each entry \( \tilde{R}(\ell)_{j,j'} \) belongs to \( \text{Mat}_{3 \times 3}(\mathbb{C}) \). A map \( T^\nu \to \mathcal{B}(L^2(\mathbb{T}^d, \mathbb{C}^3)) \), \( \varphi \mapsto \mathcal{R}(\varphi) \) can be also regarded as a linear operator \( L^2(T^{\nu+3}, \mathbb{R}^3) \to L^2(T^{\nu+3}, \mathbb{R}^3) \) by
\[
\mathcal{R}u(\varphi, x) := \sum_{\ell, \ell' \in \mathbb{Z}^d} \tilde{R}(\ell - \ell') \hat{u}(\ell', \varphi) e^{i(\ell' \varphi + \ell' x)}, \quad \forall u \in L^2(T^{\nu+3}, \mathbb{C}^3). 
\] (2.47)
The representation (2.47) of the operator \( \mathcal{R} \) is also equivalent to the pseudo-differential representation \( \mathcal{R} = \text{Op}(r) \), where the symbol \( r(\varphi, x, \xi) \), for \( \xi = j' \in \mathbb{Z}^3 \), is
\[
r(\varphi, x, j') = \sum_{j \in \mathbb{Z}^3} R_{j}^{j'}(\varphi) e^{i(j-j') \cdot x},
\] (2.48)
which is, in fact, (2.46) with, in addition, the dependence on \( \varphi \). Similarly as above, \( R_{j}^{j'}(\varphi) \) is the Fourier coefficient \( \hat{r}(\varphi, j - j', \varphi) \) of frequency \( j - j' \) of the function \( x \mapsto r(\varphi, x, j') \). If we expand both \( \hat{u}(\varphi, j') \) and \( \hat{r}(\varphi, j - j', \varphi) \) in Fourier series also in the \( \varphi \) variable, we deduce that \( \tilde{R}(\ell - \ell') \) appearing in (2.47) is the Fourier coefficient of frequency \( (\ell - \ell', j - j') \) of the function \( (\varphi, x) \mapsto r(\varphi, x, j') \).

**Definition 2.15.** (Matrix decay norm) Let \( \mathcal{R} \) be an operator represented by the matrix in (2.41). For \( s \geq 0 \), we define its matrix decay norm
\[
|\mathcal{R}|_s := \sup_{|k| \leq k_0} \left( \sum_{j,j' \in \mathbb{Z}^{\nu+3}} \langle j, j' \rangle^{2s} \| \tilde{R}(\ell) \|_{\text{HS}}^2 \right)^{\frac{1}{2}},
\] (2.49)
If the operator \( \mathcal{R} = \mathcal{R}(\lambda) \) is \( k_0 \) times differentiable in \( \mathbb{R}^{\nu+3} \), we define for \( s \geq 0 \)
\[
|\mathcal{R}|_{k_0, s} := \sup_{|k| \leq k_0} \sup_{\lambda \in \mathbb{R}^{\nu+3}} |\tilde{R}(\lambda)|_{s-|k|}.
\] (2.50)

**Remark 2.16.** The definition of the norm \( |\mathcal{R}|_s \) in (2.49) is very similar to the one of the norm \( |\mathcal{R}|_{0,0} \) in (2.19): the only difference is that the sup in (2.49) is over \( j' \in \mathbb{Z}^3 \) (which is the natural choice when using matrices with row and column indices in \( \mathbb{Z}^3 \)), while the sup in (2.19) is over \( \xi \in \mathbb{R}^3 \) (which is the natural choice when derivatives of symbols with respect to \( \xi \) have to be considered to prove composition formulas like in Lemma 2.17(ii)). In fact, the norms \( |\cdot|_s \) and \( |\cdot|_{0,0} \) are equivalent for operators acting on periodic functions.

The norm \( |\cdot|_{k_0, s} \) is increasing, namely \( |\mathcal{R}|_{k_0, s} \leq |\mathcal{R}|_{k_0, s'} \) for all \( s \leq s' \). Moreover \( |\mathcal{R}|_{k_0, s} \leq |\mathcal{R}(D)|^m |\mathcal{R}|_{k_0, s} \) for all \( m \geq 0 \). We now state some standard properties of the decay norms that are needed for the reducibility scheme of Section 3.

**Lemma 2.17.** (i) Let \( s \geq s_0 \), \( |\mathcal{R}|_{k_0, s} \), \( \| u \|_{k_0, s} < \infty \). Then
\[
\| \mathcal{R} u \|_{k_0, s} \leq s_{k_0} |\mathcal{R}|_{k_0, s} \| u \|_{k_0, s} + |\mathcal{R}|_{k_0, s} \| u \|_{k_0, s}.
\]
(ii) Let \( s \geq s_0 \), \( |\mathcal{R}|_{k_0, s} \), \( \| Q \|_{k_0, s} < \infty \). Then
\[
\| \mathcal{R} Q \|_{k_0, s} \leq s_{k_0} |\mathcal{R}|_{k_0, s} \| Q \|_{s_0, s} + |\mathcal{R}|_{s_0, s} \| Q \|_{s_0, s}.
\]
(iii) Let \( s \geq s_0 \), \( |\mathcal{R}|_{k_0, s} < \infty \). Then there exists a constant \( C(s, k_0) > 0 \) such that, for any integer \( n \geq 1 \),
\[
|\mathcal{R}|_{k_0, s} \leq C(s, k_0)^{-1} (|\mathcal{R}|_{k_0, s})^{n-1} |\mathcal{R}|_{s_0, s}.
\]
(iv) Let \( s \geq s_0 \), \( |\mathcal{R}|_{k_0, s} < \infty \). Then there exists \( \delta(s, k_0) \in (0,1) \) small enough such that, if \( |\mathcal{R}|_{s_0, s} \leq \delta(s, k_0) \), then the map \( \Phi = \text{Id} + \mathcal{R} \) is invertible and the inverse satisfies the estimate
\[
|\Phi^{-1} - \text{Id}|_{s_0, s} \leq s_{k_0} |\mathcal{R}|_{s_0, s}.
\]
(v) Let \( s \geq s_0 \), \( |\mathcal{R}|_{k_0, s} < \infty \) and let \( Z \) be the \( 3 \times 3 \) block-diagonal operator defined by \( Z = \text{diag}_{j \in \mathbb{Z}^3} \tilde{R}(j) \). Then \( |Z|_{s_0, s} \leq |\mathcal{R}|_{s_0, s} \). As a consequence,
\[
|\mathcal{R}^j(0)|_{k_0, s} \leq |\mathcal{R}|_{k_0, s}.
\]
For $N > 0$, we define the operator $\Pi_N \mathcal{R}$ by means of its $3 \times 3$ block representation in the following way:

$$
(\Pi_N \mathcal{R})^i_j (\ell) := \begin{cases} 
\hat{\mathcal{R}}_j^i (\ell) & \text{if } |\ell|, |j - j'| \leq N, \\
0 & \text{otherwise}.
\end{cases}
$$

Moreover, $\Pi_N^t \mathcal{R} := \mathcal{R} - \Pi_N \mathcal{R}$. \hfill (2.51)

**Lemma 2.18.** For all $s, \alpha \geq 0$, one has $\|\Pi_N \mathcal{R}\|^k_{\alpha, s, 0} \leq N^{\alpha} \|\mathcal{R}\|^k_{0, \alpha, s}$ and $\|\Pi_N^t \mathcal{R}\|^k_{\alpha, s, 0} \leq N^{-\alpha} \|\mathcal{R}\|^k_{0, \alpha, s + \alpha}$.

By (2.47), (2.48), as observed in Remark 2.16 the decay norms \textbf{2.15} and the pseudo-differential norms \textbf{2.5} are strictly related; in the next lemma (whose proof is a simple check) we state a link between these norms.

**Lemma 2.19.** Let $s \geq s_0$, $\mathcal{R} \in \mathcal{OP}S_{s, 0}$. Then $|\mathcal{R}|^k_{\alpha, \gamma, s} \lesssim |\mathcal{R}|^k_{0, \gamma, s}$.

### 2.4 Real and reversible operators

Remember that, for any function $u(\varphi, x)$, $u \in X$ means $u = \text{even}(\varphi, x)$, and $u \in Y$ means $u = \text{odd}(\varphi, x)$.

**Definition 2.20.** (i) We say that a linear operator $\Phi$ is reversible if $\Phi : X \to Y$ and $\Phi : Y \to X$. We say that $\Phi$ is reversibility-preserving if $\Phi : X \to X$ and $\Phi : Y \to Y$.

(ii) We say that an operator $\Phi : L^2(T^3, \mathbb{R}^3) \to L^2(T^3, \mathbb{R}^3)$ is real if $\Phi(u) \in L^2(T^3, \mathbb{R}^3)$ for any $u \in L^2(T^3, \mathbb{R}^3)$.

**Lemma 2.21.** Let $A = \text{Op}(a) \in \mathcal{OP}S_{s, 0, \alpha}$. Then the following holds:

(i) $A$ is reversible if and only if $a(\varphi, x, \xi) = -a(-\varphi, -x, -\xi)$, namely $a = \text{even}(\varphi, x, \xi)$;

(ii) $A$ is reversibility-preserving if and only if $a(\varphi, x, \xi) = a(-\varphi, -x, -\xi)$, namely $a = \text{even}(\varphi, x, \xi)$.

(iii) $A$ is real if and only if $a(\varphi, x, \xi) = \bar{a}(\varphi, x, -\xi)$.

It can be convenient to reformulate real and reversibility properties of linear operators in terms of matrix representation provided in Section 2.3.

**Lemma 2.22.** A linear operator $\mathcal{R}$ is

(i) real if and only if $\hat{\mathcal{R}}^j_i (\ell) = \hat{\mathcal{R}}_{-j}^{-i} (-\ell)$ for all $\ell \in \mathbb{Z}^3$, $j, j' \in \mathbb{Z}^3$;

(ii) reversible if and only if $\hat{\mathcal{R}}^j_i (\ell) = -\hat{\mathcal{R}}_{-j}^{-i} (-\ell)$ for all $\ell \in \mathbb{Z}^3$, $j, j' \in \mathbb{Z}^3$;

(iii) reversibility-preserving if and only if $\hat{\mathcal{R}}^j_i (\ell) = \hat{\mathcal{R}}_{-j}^{-i} (-\ell)$ for all $\ell \in \mathbb{Z}^3$, $j, j' \in \mathbb{Z}^3$.

## 3 The linearized operator

In sections \textbf{3.8} we assume the following ansatz, which will be recursively verified along the Nash-Moser iteration. We assume that $v \in \mathcal{C}^\infty(\mathbb{T}^3 \times T^3, \mathbb{R}^3)$, $v = \text{odd}(\varphi, x)$, and

$$
\|v\|_{\mathcal{L}^{k_0}} \leq 1 \hfill (3.1)
$$

where $\mu_0 = \mu_0(\nu, \tau, k_0) > 0$ is large enough.

As we explained at the beginning of Section 2, from now on we omit to write the dependence on $k_0$ when we write $\mathcal{L}$, namely we write $\mathcal{L}_{s, m}^s \mathcal{L} \ldots$ instead of $\mathcal{L}_{s, m, k_0}^s \mathcal{L} \ldots$.

Given a function $f$ depending on $u$, we denote by

$$
\Delta_{12} f := f(u_1) - f(u_2). \hfill (3.2)
$$

We want to study the linearized operator $\mathcal{L} := \mathcal{F}'(v)$, where $\mathcal{F}(v)$ is defined in \textbf{1.17}. Using the identity $\Pi_0^t = \text{Id} - \Pi_0$, for all $s \geq s_0$ one has

$$
\mathcal{L} : H^{s+1}_0 \to H^s_0, \quad \mathcal{L} = \Pi_0^t (\omega \cdot \partial_\varphi + \zeta \cdot \nabla + \varepsilon a(\varphi, x) \cdot \nabla + \varepsilon \mathcal{R}(\varphi)) \Pi_0^t, \hfill (3.3)
$$

where $a(\varphi, x)$ is the vector field

$$
a(\varphi, x) := (\mathcal{U}(v))(\varphi, x) = \text{curl} \Lambda^{-1} v, \hfill (3.4)
$$

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\[ U := \text{curl } \Lambda^{-1} \text{ is defined in (1.17, 1.15)} \] (see also (2.28)), and \( R(\varphi) \) is a matrix-valued linear pseudodifferential operator of order zero, given by

\[
\begin{align*}
\mathcal{R} &= \mathcal{R}_0 + \mathcal{R}_1, \\
\mathcal{R}_0 h &= M_{\mathcal{U}}(\varphi, x) h - v \cdot \nabla h, \quad \mathcal{R}_1 h := M_v(\varphi, x) U(h),
\end{align*}
\]

(3.5)

where \( \mathcal{D}U(v) \) is the jacobian matrix of \( \mathcal{U}(v) = \text{curl } \Lambda^{-1}v \) and \( Dv \) is the jacobian matrix of \( v \). Note that \( \mathcal{R}_0 \) is an operator of order 0 and \( \mathcal{R}_1 \) is an operator of order \(-1\). Note that since \( v = \text{odd}(\varphi, x) \), then

\[
M_{\mathcal{U}} = \text{odd}(\varphi, x), \quad M_v = \text{even}(\varphi, x),
\]

(3.6)

We first analyze the operator \( \mathcal{L} \) without the projector \( \Pi_0^\frac{1}{2} \), namely we consider the operator

\[
\mathcal{L}^{(0)} := \omega \cdot \partial_\varphi + (\zeta + \varepsilon a(\varphi, x)) \cdot \nabla
\]

(3.7)

By the definitions (3.5) and using the product estimates (2.7), for \( \|v\|_{s+\mu}^{k_0, \gamma} \leq 1 \) the following tame estimates hold for any \( s \geq s_0 \):

\[
\|\varepsilon a(\varphi, x) \cdot \nabla h + \varepsilon R(\varphi) h\|_{s} \lesssim \varepsilon (\|h\|_{s+1}^{k_0, \gamma} + \|v\|_{s+\mu}^{k_0, \gamma} \|h\|_{s_0+1})
\]

(3.8)

\section{Reduction to constant coefficients of the highest order term}

First we state a Proposition which allows to reduce to constant coefficients the operator

\[
\mathcal{T} := \omega \cdot \partial_\varphi + (\zeta + \varepsilon a(\varphi, x)) \cdot \nabla
\]

(3.1)

where we recall that, by (3.3), one has that

\[
\Pi_0 a = 0 \quad \text{and} \quad \text{div}(a) = 0.
\]

(4.2)

\textbf{Proposition 4.1.} For any \( \gamma \in (0, 1) \), \( \varepsilon_0 > 0 \), \( S > s_0 \), \( \tau > 0 \) there exist \( \delta = \delta(S, k_0, \tau, \nu) \), \( \mu_0 = \mu_0(k_0, \tau, \nu) > 0 \) and \( \tau_1 = \tau_1(k_0, \tau, \nu) > 0 \) such that if (3.1) holds and

\[
N_0^{\tau_1} \varepsilon \gamma^{-1} \leq \delta,
\]

then the following holds. There exists an invertible diffeomorphism \( \mathbb{T}^3 \rightarrow \mathbb{T}^3 \), \( x \mapsto x + \alpha(\varphi, x; \omega, \zeta) \) with inverse \( y \mapsto y + \tilde{\alpha}(\varphi, y; \omega, \zeta) \), defined for all \( (\omega, \zeta) \in \mathbb{R}^{r+3} \), satisfying

\[
\|\alpha\|_{s}^{k_0, \gamma}, \|\tilde{\alpha}\|_{s}^{k_0, \gamma} \lesssim \varepsilon_0 N_0^{\tau} \varepsilon \gamma^{-1} \|v\|_{s+\mu}^{k_0, \gamma}, \quad \forall s_0 \leq s \leq S
\]

(4.4)

(with \( \tau_0 \) defined in (2.18)) such that, defining

\[
\mathcal{A}h(\varphi, x) := h(\varphi, x + \alpha(\varphi, x)) \quad \text{with inverse} \quad \mathcal{A}^{-1}h(\varphi, y) = h(\varphi, y + \tilde{\alpha}(\varphi, y)),
\]

one gets the conjugation

\[
\mathcal{A}^{-1} \mathcal{T} \mathcal{A} = \omega \cdot \partial_\varphi + \zeta \cdot \nabla
\]

(4.6)

for all \( (\omega, \zeta) \in \text{DC}(\gamma, \tau) \), where

\[
\text{DC}(\gamma, \tau) := \left\{ \lambda = (\omega, \zeta) \in \mathbb{R}^{r+3} : |\omega| \ell + |\zeta| j \geq \frac{\gamma}{\ell, j} \right\}, \quad \forall (\ell, j) \in \mathbb{Z}^{r+3} \setminus \{0, 0\}
\]

(4.7)

Furthermore \( \alpha, \tilde{\alpha} \) are odd(\( \varphi, x \)), and therefore \( \mathcal{A}, \mathcal{A}^{-1} \) are reversibility preserving maps. Such maps satisfy the tame estimates

\[
\|\mathcal{A}^{\pm 1} h\|_{s}^{k_0, \gamma} \lesssim_{\varepsilon} \|h\|_{s}^{k_0, \gamma} + \|v\|_{s+\mu}^{k_0, \gamma} \|h\|_{s_0+1}^{k_0, \gamma}, \quad \forall s_0 \leq s \leq S,
\]

(4.8)
Let $s_1 \geq s_0$ and assume that $v_1, v_2$ satisfy (3.1) with $\mu_0 \geq s_1 + \mu$. Then for any $\lambda = (\omega, \zeta) \in DC(\gamma, \tau)$ one has
\[ \|\Delta_1 a\|_{s_1}, \|\Delta_2 b\|_{s_1}, N_0^7 \|v_1 - v_2\|_{s_1+\mu}, \]
\[ \|\Delta_1 A^k h\|_{s_1}, \|\Delta_2 A^k h\|_{s_1}, N_0^7 \|v_1 - v_2\|_{s_1+\mu} \|h\|_{s_1+1}. \] (4.9)

**Remark 4.2.** In this section we only need that the constant $C_0$ in (4.7) is strictly bigger than one (see Lemma 4.7). In Lemma 10.4, we shall take the constant $C_0 \geq C(\tau)$ large enough.

In order to prove Proposition 4.1, we closely follow [29]. First we show the following iterative Lemma. We fix the constants
\[ N_0 > 0, \quad \chi := \frac{3}{2}, \quad N_{-1} := 1, \quad N_n := N_0^n, \quad n \geq 0, \]
\[ a := 3(\tau_0 + 1) + 2, \quad b := a + 1, \] (4.10)
and recall that $\tau_0$ is defined in Lemma 2.3.

**Lemma 4.3.** Let $\gamma \in (0, 1), S > s_0, \tau > 0$. Then there exist $\delta = \delta(S, k_0, \tau, \nu) \in (0, 1), N_0 = N_0(S, k_0, \tau, \nu) > 0, \mu = \mu(k_0, \tau, \nu) > 0$ such that if (3.1), (4.9) are fulfilled with $\mu_0 \geq s_0 + \mu$, then the following statements hold for all $n \geq 0$.

There exists a linear operator
\[ T_n := \omega \cdot \partial_x + m_n \cdot \nabla + a_n(\phi, x) \cdot \nabla \] (4.11)
defined for any $\lambda = (\omega, \zeta) \in \mathbb{R}^{k+3}$ such that
\[ \|a_n\|_{s_1, \gamma} \leq C_\gamma(s) N_{-1}^{-a} \|v\|_{s_1+\mu}, \quad \|a_n\|_{s_1, \gamma} \leq C_\gamma(s) N_{-1-\epsilon} \|v\|_{s_1+\mu}, \quad \forall 0 \leq s \leq S, \]
\[ m_n - \zeta \leq 0 \quad \text{and, if } n \geq 1, \quad |m_n - m_{n-1}| \leq 0. \] (4.12)
for some constant $C_\gamma(s) = C_\gamma(s, k_0, \tau) > 0$. If $n = 0$, define $\mathcal{O}_0^\gamma := \mathbb{R}^{k+3}$; if $n \geq 1$, define
\[ \mathcal{O}_n^\gamma := \left\{ (\omega, \zeta) \in \mathcal{O}_{n-1}^\gamma : \|\omega \cdot \ell + m_{n-1} \cdot j \| \geq \frac{\gamma}{(\ell, j)_\gamma} \right\}, \quad \forall (\ell, j) \in 2^{k+3} \setminus \{0, 0\}, \quad |(\ell, j)| \leq N_{n-1}. \] (4.13)

For $n \geq 1$, there exists an invertible diffeomorphism of the torus $\mathbb{T}^3 \to \mathbb{T}^3, x \mapsto x + \alpha_{n-1}(\phi, x)$ with inverse $\mathbb{T}^3 \to \mathbb{T}^3, y \mapsto y + \tilde{\alpha}_{n-1}(\phi, y)$ such that
\[ \|\alpha_{n-1}\|_{s_1, \gamma} \leq N_{n-1}^{-a} N_{n-2}^{-\epsilon} \|v\|_{s_1+\mu}, \quad \forall 0 \leq s \leq S, \]
\[ \|\alpha_{n-1}\|_{s_1, \gamma} \leq N_{n-1}^{-a} N_{n-2}^{-\epsilon} \|v\|_{s_1+\mu}, \quad \forall 0 \leq s \leq S. \] (4.14)
(with $\tau_0$ defined in (2.13)). The operator
\[ A_{n-1} : h(\phi, x) \mapsto h(\phi, x + \alpha_{n-1}(\phi, x)) \]
with inverse
\[ A_{n-1}^{-1} : h(\phi, x) \mapsto h(\phi, y + \tilde{\alpha}_{n-1}(\phi, y)) \]
satisfy, for any $\lambda = (\omega, \zeta) \in \mathcal{O}_n^\gamma$, the conjugation
\[ T_n = A_{n-1}^{-1} T_{n-1} A_{n-1}. \] (4.15)

Furthermore, $a_n = \text{even}(\phi, x), \alpha_{n-1}, \tilde{\alpha}_{n-1} = \text{odd}(\phi, x)$, implying that $T_n$ is a reversible operator and $A_{n-1}, A_{n-1}^{-1}$ are reversibility preserving operators.

**Proof.** Proof of the statement for $n = 0$. The claimed statements for $n = 0$ follows directly by defining $a_0 := c = c \varepsilon \varepsilon \curl A^{-1} v$, see (3.3).

Proof of the induction step. Now assume that the claimed properties hold for some $n \geq 0$ and let us prove them at the step $n + 1$. We look for a diffeomorphism of the torus $\mathbb{T}^3 \to \mathbb{T}^3, x \mapsto x + \alpha_n(\phi, x)$, with inverse given by $y \mapsto y + \tilde{\alpha}_n(\phi, y)$ such that defining
\[ A_n : h(\phi, x) \mapsto h(\phi, x + \alpha_n(\phi, x)), \quad A_n^{-1} : h(\phi, x) \mapsto h(\phi, x + \tilde{\alpha}_n(\phi, x)) \]
the operator $A_n^{-1}T_n A_n$ has the desired properties. One computes

$$A_n^{-1}T_n A_n = \omega \cdot \partial \varphi + \eta \nabla + A_n^{-1} \left[ \omega \cdot \partial \varphi \alpha_n + \eta \nabla \alpha_n + a_n + a_n \cdot \nabla \alpha_n \right] \nabla$$

$$= \omega \cdot \partial \varphi + \eta \nabla + A_n^{-1} \left[ \omega \cdot \partial \varphi \alpha_n + \eta \nabla \alpha_n + \Pi N, a_n \right] \nabla + a_{n+1} \cdot \nabla \tag{4.16}$$

where

$$a_{n+1} := A_n^{-1} f_n, \quad f_n := \Pi_{\mathbb{N}} a_n + a_n \cdot \nabla \alpha_n, \tag{4.17}$$

and the projectors $\Pi N, \Pi_{\mathbb{N}}$ are defined by (2.23). For any $(\omega, \zeta) \in O_\gamma^+$, we solve the homological equation

$$\omega \cdot \partial \varphi \alpha_n + \eta \nabla \alpha_n + \Pi N, a_n = \langle a_n \rangle_{\varphi, x} \tag{4.18}$$

(where $\langle a_n \rangle_{\varphi, x}$ is the average of $a_n$ in time and space), and, recalling (2.16), we extend its solution to the whole parameter space $(\omega, \zeta) \in \mathbb{R}^{r+3}$ by defining

$$a_n := \left( \omega \cdot \partial \varphi + \eta \nabla \right)^{-1} \left[ \langle a_n \rangle_{\varphi, x} - \Pi N, a_n \right]. \tag{4.19}$$

We define

$$T_{n+1} := \omega \cdot \partial \varphi + a_{n+1} \cdot \nabla + a_{n+1} \cdot \nabla \tag{4.20}$$

where

$$a_{n+1} := a_n + \langle a_n \rangle_{\varphi, x}. \tag{4.21}$$

We observe that $T_{n+1}$ is defined for all $(\omega, \zeta) \in \mathbb{R}^{r+3}$, and, for $(\omega, \zeta) \in O_\gamma^+$, one has $A_n^{-1} T_n A_n = T_{n+1}$. Clearly $a_n(\varphi, x; \omega, \zeta)$ is $C^\infty$ in $(\varphi, x)$ and $k_0$ times differentiable in $(\omega, \zeta) \in \mathbb{R}^{r+3}$. Furthermore, by Lemma 2.3 and by the smoothing property (2.36), for any $s \geq 0$, one has

$$\|\alpha_n k_0, \zeta \| \leq \gamma^{-1} \|\Pi N, a_n k_0, \zeta \| \leq N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \|, \tag{4.22}$$

$$\|\nabla \alpha_n k_0, \zeta \| \leq \gamma^{-1} \|\Pi N, a_n k_0, \zeta \| \leq N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \|. \tag{4.23}$$

The latter estimate, together with Lemma 2.3 and the induction estimates on (4.12) on $a_n$, imply that for any $s_0 \leq s \leq S$

$$\|\alpha_n k_0, \zeta \| \leq \gamma^{-1} \|a_n k_0, \zeta \| \leq N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \|, \tag{4.24}$$

which are the estimates (4.14) at the step $n+1$. Note that, using the definition of the constant $a$ in (4.10) and the ansatz (5.1), from (4.23), with $s = s_0$ one deduces that

$$\|\alpha_n k_0, \zeta \| \leq \gamma^{-1} \|a_n k_0, \zeta \| \leq \gamma^{-1} \|a_n k_0, \zeta \|. \tag{4.25}$$

Hence the smallness condition (4.3) (choosing $\tau_1 \geq \tau_0$), together with Lemma 2.3 and the estimate (4.22) leads to the estimate

$$\|A_n^{-1} h_{k_0, \zeta} \| \leq \|h_{k_0, \zeta} \| + N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \| h_{k_0, \zeta}, \quad \forall s_0 \leq s \leq S + b. \tag{4.26}$$

We now estimate the function $a_{n+1}$ defined in (4.17). First, we estimate $f_n$. By (4.10), (4.12) and using also the ansatz (5.1) and the smallness condition (4.3), one has that

$$N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \| \leq 1. \tag{4.27}$$

By (2.7), (2.23), (2.5), (2.12), (2.20) one has

$$\|f_n k_0, \zeta \| \leq \|a_n k_0, \zeta \|, \tag{4.28}$$

$$\|f_n k_0, \zeta \| \leq N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \| + N^\gamma_\zeta \gamma^{-1} \|a_n k_0, \zeta \| a_n k_0, \zeta, \quad \forall s_0 \leq s \leq b. \tag{4.29}$$

$$\|f_n k_0, \zeta \| \leq \|a_n k_0, \zeta \| \|a_n k_0, \zeta \| \leq \|a_n k_0, \zeta \|, \quad \forall s_0 \leq s \leq S. \tag{4.30}$$
Lemma 4.4. Let
\[ H \]
Hence (4.25)-(4.27) imply that, for any \( s \)
\[ a \]
reversibility preserving and therefore, by recalling (4.17),
\[ \tilde{\beta} \]
reversibility preserving, one has
We apply Lemma 4.3. Since \( \alpha \)
\[ \beta \]
estimate on \( a \)
\[ \alpha \]
Finally, by (4.19), since \( a_n = \text{even}(\varphi, x) \), then \( \alpha_n, \tilde{\alpha}_n = \text{odd}(\varphi, x) \). This implies that the maps \( A_n^{\pm 1} \) are reversibility preserving and therefore, by recalling (4.17), \( a_n + 1 = \text{even}(\varphi, x) \) and the claimed statement is proved.

We then define
\[ \tilde{A}_n := A_0 \circ A_1 \circ \ldots \circ A_n, \quad \text{with inverse} \quad \tilde{A}_n^{-1} = A_n^{-1} \circ A_{n-1}^{-1} \circ \ldots \circ A_1^{-1}. \] (4.29)

Lemma 4.4. Let \( S > s_0, \gamma \in (0, 1) \). Then there exist \( \delta = \delta(S, k_0, \tau, \nu) \in (0, 1), \mu = \mu(k_0, \tau, \nu) > 0 \) such that if (4.31) holds with \( \mu_0 < s_0 + \mu \) and if (4.33) holds, then the following properties hold.

(i) \[ \tilde{A}_n(h(x, y)) = h(x + \beta_n(y), x), \quad \tilde{A}_n^{-1}(h(x, y)) = h(x + \tilde{\beta}_n(y), x), \]
where, for any \( s_0 < s < S \),
\[ \beta_n(0) \leq k_0 \gamma, \quad \tilde{\beta}_n(0) \leq k_0 \gamma, \]
\[ \beta_n - \beta_n^{-1}(0) \leq k_0 \gamma, \quad \tilde{\beta}_n - \tilde{\beta}_n^{-1}(0) \leq k_0 \gamma, \]
\[ n \geq 1. \]
As a consequence,
\[ \beta_n(0) \leq k_0 \gamma, \quad \beta_n^{-1}(0) \leq k_0 \gamma, \quad \forall s_0 < s < S. \] (4.30)

Furthermore, \( \beta_n, \tilde{\beta}_n = \text{odd}(\varphi, x) \).

(ii) For any \( s_0 < s < S \), the sequence \( (\beta_n)_{n \in \mathbb{N}} \) (resp. \( (\tilde{\beta}_n)_{n \in \mathbb{N}} \)) is a Cauchy sequence with respect to the norm \( \| \beta \|_s \) and it converges to some limit \( \alpha \) (resp. \( \tilde{\alpha} \)). Furthermore \( \alpha, \tilde{\alpha} = \text{odd}(\varphi, x) \) and, for any \( s_0 < s < S, \)
\[ n \geq 1, \]
\[ \beta_n - \beta_n^{-1}(0) \leq k_0 \gamma, \quad \tilde{\beta}_n - \tilde{\beta}_n^{-1}(0) \leq k_0 \gamma, \]
\[ \| \alpha - \beta_n \|_s \leq k_0 \gamma, \quad \| \alpha - \tilde{\beta}_n \|_s \leq k_0 \gamma, \]
\[ n \geq 1. \] (4.32)

(iii) Define
\[ A \beta h(\varphi, x) := h(\varphi, x + \alpha(\varphi, x)), \quad \text{with inverse} \quad A^{-1} h(\varphi, y) = h(\varphi, y + \alpha(\varphi, y)). \] (4.33)
Then, for any \( s_0 < s < S, \) \( A_n^{\pm 1} \) converges pointwise in \( H^s \) to \( A^{\pm 1} \), namely
\[ \lim_{n \to +\infty} |A^{\pm 1} h - A_n^{\pm 1} h|_s = 0 \]
for any \( h \in H^s \).

Proof. Proof of (i). We prove the Lemma arguing by induction. For \( n = 0 \), one has that \( A_0^{\pm 1} = A_0^{\pm 1} \) and we set \( \beta_0 := \alpha_0, \tilde{\beta}_0 := \alpha_0 \). Then the first estimate (4.30) follows by (4.14) (applied with \( n = 1 \)). The second statement in (4.30) for \( n = 0 \) is empty. Now assume that the claimed statement holds for some \( n \geq 0 \) and let us prove it at the step \( n + 1 \). We prove the claimed statement for \( \tilde{A}_{n+1} \) since the proof for the map \( A_{n+1}^{\pm 1} \) is similar. Using that \( \tilde{A}_{n+1} := \tilde{A}_n \circ \tilde{A}_{n+1} \), one computes that
\[ \tilde{A}_{n+1}(h(\varphi, x)) = h(\varphi, x + \beta_{n+1}(\varphi, x)), \quad \beta_{n+1} := \beta_n + \tilde{A}_n[\alpha_{n+1}]. \] (4.34)
We apply Lemma 4.3. Since \( \alpha_{n+1} = \text{odd}(\varphi, x) \) and, by the induction hypothesis \( \beta_n = \text{odd}(\varphi, x) \), \( \tilde{A}_n \) is reversibility preserving, one has \( \beta_{n+1} = \text{odd}(\varphi, x) \). By the induction estimate (4.31) for \( s = s_0 \), using the
ansatz (3.1), \( \| \beta_n \|_{s_0} \| \leq N_0 \varepsilon \gamma^{-1} \). Then, by the smallness condition (4.3), we can apply Lemma 2.3 and (4.12), (3.1), (4.31), (4.3), obtaining that, for any \( s_0 \leq s \leq S \),
\[
\| \beta_{n+1} - \beta_n \|_s \leq \| \alpha_{n+1} \|_s \| \beta_n \|_s + \| \beta_n \|_s \| \alpha_{n+1} \|_s \leq \| N_{n+1} N_n^{-\alpha} \varepsilon \gamma^{-1} \| v \|_{s+\gamma},
\]
which is (4.30) at the step \( n + 1 \). The estimate (4.31) at the step \( n + 1 \) follows by using a telescoping argument, since the series \( \sum_{n \geq 0} N_n N_n^{-\alpha} < \infty \).

**Proof of (ii).** It follows by item (i), using the estimate (4.30) and a telescoping argument.

**Proof of (iii).** It follows by item (ii), using the same arguments of the proof of Lemma B6-(i) in [6]: given \( h \in H^s, \varepsilon_1 > 0 \), there exists \( N > 0 \) (sufficiently large, depending on \( \varepsilon_1, s, h \)) such that \( h_1 := \Pi_N h \) satisfies \( \| \tilde{A} h_1 \|_s \leq \varepsilon_1 / 4, \| \tilde{A}_n h_1 \|_s \leq \varepsilon_1 / 4 \) uniformly in \( n \) (bound (4.31) is uniform in \( n \)). On the other hand, \( h_0 := \Pi_N h \) satisfies
\[
(\| A - \tilde{A}_n \| h_0) \leq \int_0^1 \frac{d\theta}{\theta} h_0(\varphi, x + \beta_n(\varphi, x) + \theta(\alpha(x, x) - \beta_n(\varphi, x))) d\theta \\
\leq \| \nabla h_0 \|_s \| \alpha - \beta_n \|_s + \| \nabla h_0 \|_s \| \alpha - \beta_n \|_s \leq \varepsilon_1 / 2
\]
for all \( n \geq n_0 \), for some \( n_0 \) depending on \( \varepsilon_1, s, h \).

**Lemma 4.5.** (i) The sequence \( (m_n)_{n \in \mathbb{N}} \) satisfies
\[
\sup_{(\omega, \zeta) \in \Omega_n} |m_n(\omega, \zeta) - \zeta| \leq \varepsilon N_n^{-\alpha}.
\]

(ii) The following inclusion holds: \( DC(\gamma, \tau) \subseteq \cap_{n \geq 0} \Omega_n \) (recall the definitions (4.4), (4.13)).

**Proof.** **Proof of (i).** By recalling the definition (4.28), using (4.11), (4.15), one obtains
\[
\omega \cdot \partial_\varphi + m_n \cdot \nabla + a_n \cdot \nabla = T_n = \tilde{A}^{-1}_n T_0 \tilde{A}_n, \quad \forall (\omega, \zeta) \in \Omega_n
\]
and by Lemma 1.24 (ii) one computes explicitly
\[
\tilde{A}^{-1}_n T_0 \tilde{A}_n = \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \tilde{A}^{-1}_n (\omega \cdot \partial_\varphi \beta_n + \zeta \cdot \nabla \beta_n + \varepsilon a + \varepsilon a \cdot \nabla \beta_n) \cdot \nabla.
\]
Since \( \tilde{A}_n \) is a change of variable, one has \( \tilde{A}_n [c] = c \) for all constant \( c \in \mathbb{R} \). Hence, by (4.36), (4.37), one obtains the identity
\[
m_n + \tilde{A}_n \beta_n = \zeta + \omega \cdot \partial_\varphi \beta_n + \zeta \cdot \nabla \beta_n + \varepsilon a + \varepsilon a \cdot \nabla \beta_n\) .
\]
Note that
\[
\int_{T^{t+3}} (\omega \cdot \partial_\varphi \beta_n + \zeta \cdot \nabla \beta_n) d\varphi dx = 0, \quad \int_{T^{t+3}} a(\varphi, x) d\varphi dx = 0,
\]
\[
\int_{T^{t+3}} a \cdot \nabla \beta_n d\varphi dx = -\int_{T^{t+3}} \text{div}(a) \beta_n d\varphi dx = 0.
\]
Taking the space-time average of the equation (4.38), and using (4.39), we deduce that
\[
m_n - \zeta = -\langle \tilde{A}_n \| a, \varphi, x \rangle \forall (\omega, \zeta) \in \Omega_n\) .
\]
The claimed estimate then follows by Lemma 2.3 applying (4.12), (4.31), (3.1), (4.3).

**Proof of (ii).** We prove the claimed inclusion by induction, i.e. we show that \( DC(\gamma, \tau) \subseteq \Omega_n \) for any \( n \geq 0 \). For \( n = 0 \), the inclusion holds since \( \Omega_0 \) := \( \mathbb{R}^{t+3} \) and \( DC(\gamma, \tau) \subseteq \mathbb{R}^{t+3} \) (see (4.7)). Now assume that \( DC(\gamma, \tau) \subseteq \Omega_n \) for some \( n \geq 0 \) and let us prove that \( DC(\gamma, \tau) \subseteq \Omega_{n+1} \). Let \( (\omega, \zeta) \in DC(\gamma, \tau) \). By the induction hypothesis, \( (\omega, \zeta) \) belongs to \( \Omega_n \). Therefore, by item (i), one has \( |m_n(\omega, \zeta) - \zeta| \leq \varepsilon N_n^{-\alpha} \). Hence, for all \( (\ell, j) \in \mathbb{Z}^{t+3} \setminus \{(0, 0)\} \), \( (\ell, j) \leq N_n \), one has
\[
|\omega \cdot \ell + m_n(\omega, \zeta) \cdot j| \geq |\omega \cdot \ell + \zeta \cdot j| - |m_n(\omega, \zeta) - \zeta \cdot j| \geq \frac{C_0 \gamma}{(\ell, j)^{s+\gamma}} - \varepsilon CN_n N_n^{-\alpha} \geq \frac{\gamma}{(\ell, j)^{s+\gamma}},
\]
provided $CN_n^{1+\tau} N_{n-1}^\alpha \varepsilon^{-1} \leq C_0 - 1$. This holds for all $n \geq 0$ provided
\[ CN_n^{1+\tau} \varepsilon^{-1} \leq C_0 - 1. \] (4.41)
Condition (4.41) is fulfilled by taking $C_0 \geq 2$, using (4.11) and the smallness condition (4.3). Thus, by the
definition of $C_n^{\tau+1}$ (see (4.12)), one has that $(\omega, \zeta) \in C_n^{\tau+1}$, and the proof is concluded.

**Proof of Proposition 4.2.** For any $(\omega, \zeta) \in DC(\gamma, \tau)$, by Lemma 1.5, $m_n \rightarrow \zeta$ as $n \rightarrow \infty$. By (4.31), (4.12) and Lemma 2.8 one has $\|\tilde{A}_{n-1}a_n\|_{s_0} \leq \|a_n\|_{s_0} \rightarrow 0$ Also,
\[ \|\partial_\alpha \beta_{n-1} - \partial_\alpha \alpha\|_{s_0}, \|\nabla \beta_{n-1} - \nabla \alpha\|_{s_0} \leq \|\beta_{n-1} - \alpha\|_{s_0+1} \rightarrow 0 \quad (n \rightarrow \infty). \]

Hence, passing to the limit in norm $\| \|$ in the identity (4.38), we obtain the identity
\[ \omega \cdot \partial_\alpha \alpha + \zeta \cdot \nabla \alpha + \varepsilon a(\varphi, x) + \varepsilon a(\varphi, x) \cdot \nabla \alpha = 0 \] (4.42)
in $H_{s_0}(\mathbb{T}^{\nu+3})$, and therefore pointwise for all $(\varphi, x) \in \mathbb{T}^{\nu+3}$, for any $(\omega, \zeta) \in DC(\gamma, \tau)$. As a consequence,
\[ A^{-1} T A = \omega \cdot \partial_\alpha + \zeta \cdot \nabla + \{ A^{-1} (\omega \cdot \partial_\alpha + \zeta \cdot \nabla + \varepsilon a(\varphi, x) + \varepsilon a(\varphi, x) \cdot \nabla \alpha) \} \cdot \nabla = \omega \cdot \partial_\alpha + \zeta \cdot \nabla \]
for all $(\omega, \zeta) \in DC(\gamma, \tau)$, which is (4.6). The estimates (4.4), (4.8) follow from the estimates (4.82) and by Lemma 2.3

It remains only to prove the estimate (4.9). Let $a_i := a_i(v_i)$, $i = 1, 2$ satisfy (4.12) and assume that, for $s_1 > s_0$, $v_1, v_2$ satisfy (3.1) with $\mu_0 \geq s_1 + \mu$. Let $a_i$ and $A_i, i = 1, 2$, be the corresponding function and operator given by Lemma 4.4(ii), (iii). Then, by (4.42) and (4.6), for $i = 1, 2$ one has
\[ A_i^{-1} T_i A_i = L_0, \quad T_i(a_i) + a_i = 0 \quad \forall (\omega, \zeta) \in DC(\gamma, \tau), \] (4.43)
where $L_0 := \omega \cdot \partial_\alpha + \zeta \cdot \nabla$ and $T_i := L_0 + \varepsilon a_i(\varphi, x) \cdot \nabla$. Hence
\[ T_i(a_1 - a_2) + f = 0, \quad f := (a_1 - a_2) \cdot \nabla a_2 + a_1 - a_2. \]

By (4.43) one has $T_i = A_i L_0 A_i^{-1}$, and therefore
\[ L_0 A_i^{-1}(a_1 - a_2) + A_i^{-1}(f) = 0. \]
Since $\langle A_i^{-1}(f) \rangle_{\varphi, x} = -\langle L_0 A_i^{-1} (a_1 - a_2) \rangle_{\varphi, x} = 0$, $(\omega, \zeta) \in DC(\gamma, \tau)$ and $a_1 - a_2, A_i^{-1}(a_1 - a_2) = \text{odd}(\varphi, x)$ (the operator $L_0$ has only the trivial kernel, restricted to the space of odd functions in $(\varphi, x)$), one has
\[ a_1 - a_2 = -A_i L_0^{-1} A_i^{-1}(f), \quad \forall (\omega, \zeta) \in DC(\gamma, \tau). \]
Then, using that $\|v_i\|_{s_1 + \mu} \leq 1$, by applying (4.8.3), (4.4) and the product estimate (2.7), recalling also that by (3.4), $a_1 = \text{cURL} A_i^{-1} v_i$, $i = 1, 2$, one gets the estimate (4.9) for $\Delta_2 \alpha$. The corresponding estimate for $\Delta_2 \alpha$ can be done by using that $a_i = -A_i^{-1} (a_i)$ and using the mean value theorem. Finally, the estimate for $\Delta_2 A^{1,1}$, $\Delta_2 A^\nu$ follow by using the estimate for $\Delta_2 \alpha$ and $\Delta_2 \alpha$, using the mean value theorem and Lemma 2.3.

In the next Lemma we exploit the conjugation of the operator $\mathcal{L}_0$ defined in (3.7) by means of the map $\mathcal{A}$ constructed in Proposition 4.1. With a slight abuse of notations we denote with the same letter $\mathcal{A}$ the operator acting on $H^s(\mathbb{T}^\nu \times \mathbb{T}^3, \mathbb{R})$ and acting on $H^s(\mathbb{T}^\nu \times \mathbb{T}^3, \mathbb{R})$. The action on the spaces $H^s(\mathbb{T}^\nu \times \mathbb{T}^3, \mathbb{R})$ is given by
\[ \mathcal{A} h = (\mathcal{A} h_1, \mathcal{A} h_2, \mathcal{A} h_3), \quad \forall h = (h_1, h_2, h_3) \in H^s(\mathbb{T}^\nu \times \mathbb{T}^3, \mathbb{R}^3). \]

**Lemma 4.6.** For any $S > s_0, \gamma \in (0, 1), \tau > 0$ there exists $\delta = \delta(S, k_0, \tau, \nu) \in (0, 1), \mu = \mu(k_0, \tau, \nu) > 0$ such that if (3.1) holds and $N_0^\gamma \varepsilon^{-1} \leq \delta, for any $(\omega, \zeta) \in DC(\gamma, \tau)$ (see (3.7)) one has
\[ \mathcal{L}^{(1)} := A^{-1} \mathcal{L}_0 A = \omega \cdot \partial_\alpha + \zeta \cdot \nabla + \mathcal{R}_0^{(1)} + \mathcal{R}_{-1}^{(1)} \] (4.44)
where $\mathcal{R}_0^{(1)} = \text{Op}(R_0^{(1)}) \in \mathcal{OPS}_{S,1}^0$, $\mathcal{R}_{-1}^{(1)} = \text{Op}(R_{-1}^{(1)}) \in \mathcal{OPS}_{S,0}^{-1}$ and
\[
|\mathcal{R}_0^{(1)}|_{k_0,0}^{\delta_0/2} \leq s \epsilon \|v\|_{s+2}, \quad \forall s_0 \leq s \leq S.
\]
(4.45)
Moreover $\mathcal{L}^{(1)}$, $\mathcal{R}_0^{(1)}$, $\mathcal{R}_{-1}^{(1)}$ are real and reversible operators and the symbol of the zero-th order operator $\mathcal{R}_0^{(1)}$ satisfies the symmetry condition
\[
R_0^{(1)}(\varphi, x, \xi) = R_0^{(1)}(\varphi, x, -\xi).
\]
(4.46)
Let $s_1 \geq s_0$ and assume that $v_1, v_2$ satisfy (3.1) with $\mu_0 \geq s_1 + \mu$. Then, for any $\lambda = (\omega, \zeta) \in DC(\gamma, \tau)$,
\[
|\Delta_{12}\mathcal{R}_0^{(1)}|_{0,s_1,1}, |\Delta_{12}\mathcal{R}_{-1}^{(1)}|_{1,s_1,0} \leq s_1 \epsilon \|v_1 - v_2\|_{s_1 + \mu}.
\]
(4.47)
Proof. By Proposition 4.1 using the formula (3.3), one has that
\[
\mathcal{L}^{(1)} := \mathcal{A}^{-1}\mathcal{L}^{(0)}\mathcal{A} = \varpi \cdot \partial_{\varphi} + \varpi \cdot \nabla + \epsilon \mathcal{A}^{-1}\mathcal{R}_0\mathcal{A} + \epsilon \mathcal{A}^{-1}\mathcal{R}_{-1}\mathcal{A}.
\]
We analyze separately the terms $\mathcal{A}^{-1}\mathcal{R}_0\mathcal{A}$ and $\mathcal{A}^{-1}\mathcal{R}_{-1}\mathcal{A}$.
Analysis of $\mathcal{A}^{-1}\mathcal{R}_0\mathcal{A}$. By the formula (3.3), we recall that
\[
\mathcal{R}_0 := M_\mathcal{U}(\varphi, x) = v \cdot \nabla \mathcal{U}
\]
(4.48)
where $\mathcal{U} := \text{curl} \Lambda^{-1}$ is defined in (1.17) and we denote by $M_\mathcal{U}(\varphi, x)$ the multiplication operator by the $3 \times 3$ matrix $M_\mathcal{U}(\varphi, x)$. By the definition of $M_\mathcal{U}$ it is straightforward to verify that
\[
\|M_\mathcal{U}\|^{k_0,0} \lesssim \|v\|^{k_0,0}, \quad \forall s \geq s_0.
\]
(4.49)
Hence, by Lemma 2.10, by the estimates (4.50), (4.53) and using (3.3) and $N_0^{\gamma_1} \leq 1$, one gets that $\mathcal{A}^{-1}\text{Op}(M_\mathcal{U})\mathcal{A} = \text{Op}(M_\mathcal{U}(\varphi, x))$ is a multiplication operator with
\[
\|\text{Op}(\tilde{M})|_{0,s,0}^{k_0,0} \lesssim \|v\|^{k_0,0}, \quad \forall s \geq s_0.
\]
(4.50)
Since $M_\mathcal{U}$, $\bar{\sigma}$ is odd$(\varphi, x)$ then also $\tilde{M}_\mathcal{U}$ is odd$(\varphi, x)$. We now study the conjugation $\mathcal{A}^{-1}v \cdot \nabla \mathcal{U}$. Since $\mathcal{U} = \text{curl} \Lambda^{-1}$, we write
\[
\mathcal{A}^{-1}v \cdot \nabla \mathcal{U} = (\mathcal{A}^{-1}v \cdot \nabla \mathcal{A})(\mathcal{A}^{-1}\text{curl} \Lambda)(\mathcal{A}^{-1}\Lambda^{-1}\mathcal{A}).
\]
This formula, together with Lemmata 2.11, 2.12, the estimates (4.3), the ansatz (3.3) and the bound $N_0^\gamma \leq 1$, imply that
\[
\mathcal{A}^{-1}v \cdot \nabla \mathcal{U} = \text{Op}(M_\mathcal{V})\text{Op}(M_{\text{curl}})\mathcal{P}_{-2} + \text{Op}(M_\mathcal{V})\text{Op}(M_{\text{curl}})\mathcal{P}_{-3}
\]
with $\mathcal{P}_{-2} = \text{Op}(P_{-2})$,
\[
M_\mathcal{V}(\varphi, x, \xi) = -M_\mathcal{V}(\varphi, x, -\xi), \quad M_{\text{curl}}(\varphi, x, \xi) = -M_{\text{curl}}(\varphi, x, -\xi),
\]
\[
P_{-2}(\varphi, x, \xi) = P_{-2}(\varphi, x, -\xi)
\]
(4.51)
and
\[
M_\mathcal{V}, M_{\text{curl}} \in \mathcal{S}_{S,1}^1, \quad \mathcal{P}_{-2} \in \mathcal{OPS}_{S,1}^2, \quad \mathcal{P}_{-3} \in \mathcal{OPS}_{S,1}^3.
\]
(4.52)
\[
\|\text{Op}(M_\mathcal{V})|_{1,s,1}^{k_0,0} \lesssim \|v\|^{k_0,0},
\]
\[
\|\text{Op}(M_{\text{curl}})|_{1,s,1}^{k_0,0} \lesssim \|v\|^{k_0,0}, \quad \|\mathcal{P}_{-2}|_{k_0,0}^{k_0,0}, \quad \|\mathcal{P}_{-3}|_{k_0,0}^{k_0,0}, \quad 0 \leq s \leq S.
\]
(4.53)
for any $s_0 \leq s \leq S$. Hence, by (4.48), (4.50), (4.52), (4.53), by applying also (2.28), (2.29), Lemma 2.7 and the ansatz (5.1) with $\mu_0 > 0$ large enough, one obtains that
\[
\mathcal{A}^{-1}\mathcal{R}_0\mathcal{A} = \text{Op}(R_{0,0}) + \text{Op}(R_{0,-1}), \quad R_{0,0} \in \mathcal{S}_{S,1}^0, \quad R_{0,-1} \in \mathcal{S}_{S,0}^{-1}
\]
(4.54)
\[
R_{0,0}(\varphi, x, \xi) = \tilde{M}_\mathcal{U}(\varphi, x) - M_\mathcal{V}(\varphi, x, \xi)M_{\text{curl}}(\varphi, x, \xi)P_{-2}(\varphi, x, \xi),
\]
\[
|\text{Op}(R_{0,0})|_{0,s,1}^{k_0,0} \lesssim \|v\|^{k_0,0}, \quad \|\mathcal{P}_{-3}|_{k_0,0}^{k_0,0}, \quad \forall s_0 \leq s \leq S.
\]
Moreover (4.52) implies that

\[ R_{0,0}(\varphi, x, \xi) = R_{0,0}(\varphi, x, -\xi) \]  

(4.55)

and, since \( A \) is reversibility preserving and \( R_0 \) is reversible, then \( \text{Op}(R_{0,0}), \text{Op}(R_{0,-1}) \) are reversible.

**Analysis of \( A^{-1}R_{-1}A \).** By (3.53), one writes

\[ A^{-1}R_{-1}A = (A^{-1}\text{Op}(M_0)A)(A^{-1}\text{curl}A)(A^{-1}A^{-1}) \cdot \]

Then using that \( \|M_0\|_{s}^{k_0,\gamma} \lesssim \|v\|^{k_0,\gamma}_{s+1} \), by applying Lemmata 2.10, 2.11, 2.13, the composition Lemma 2.7, the estimate (4.4), the ansatz (3.1) and (4.54), (4.55), (4.56). The estimates (4.47) can be proved by similar arguments.

\[ \mathcal{R}_0^{(1)} := \varepsilon \text{Op}(R_{0,0}), \quad \mathcal{R}_{-1}^{(1)} := \varepsilon \text{Op}(R_{0,-1}) + \varepsilon A^{-1}R_{-1}A \]

using (4.54), (4.55), (4.56). The estimates (4.47) can be proved by similar arguments. \( \square \)

### 5 Elimination of the zero-th order term via a variable coefficients homological equation

In this section, our aim is to construct a transformation of the form \( \mathcal{B} = \text{Id} + \mathcal{M} \), with \( \mathcal{M} = \text{Op}(\mathcal{M}(\varphi, x, \xi)) \) of order 0, in such a way that the transformed operator \( \mathcal{B}^{-1}\mathcal{L}(1)^{\mathcal{B}} \) is a one-smoothing perturbation of the operator \( \omega \cdot \partial_x + \zeta \cdot \nabla \). This means that we look for \( \mathcal{M} \) that completely eliminates the zero-th order term \( \mathcal{R}_0^{(1)} \) from the operator \( \mathcal{L}(1) \) in (4.44). The main technical issue here is that we deal with *matrix-valued* pseudo-differential operators, therefore the commutator \( [\mathcal{R}_0^{(1)}, \mathcal{M}] \) does not gain derivatives (unlike commutators of *scalar* pseudo-differential operators do), so that \( \mathcal{L}_0^{(1)}, \mathcal{M} \) is still an operator of order 0. This implies that, in order to remove the zero-th order term, we have to solve a *variable coefficients* homological equation, which is an equation in the unknown \( \mathcal{M}(\varphi, x, \xi) \) of the form

\[
\left( \omega \cdot \partial_x + \zeta \cdot \nabla + R_0^{(1)}(\varphi, x, \xi) \right) \mathcal{M}(\varphi, x, \xi) + R_0^{(1)}(\varphi, x, \xi) = 0.
\]

### 5.1 The homological equation at the zero-th order term

In order to simplify notations in this section, we set \( V(\varphi, x, \xi) := R_0^{(1)}(\varphi, x, \xi) \). To deal with the equation (5.1), the first step is to diagonalize the linear operator

\[
\mathcal{P} := \omega \cdot \partial_x + \zeta \cdot \nabla + V(\varphi, x, \xi)
\]

acting on the space of matrix symbols \( S_{s+1,0}^0 \) for any \( s_0 \leq s \leq S \). The action of the operator \( \mathcal{P} \) is given by

\[
\mathcal{P} : S_{s+1,0}^0 \to S_{s,0}^0,
\]

\[
A(\varphi, x, \xi) \mapsto \omega \cdot \partial_x A(\varphi, x, \xi) + \zeta \cdot \nabla A(\varphi, x, \xi) + V(\varphi, x, \xi) A(\varphi, x, \xi).
\]

(5.2)

To develop the reducibility scheme, we use the pseudo differential norm

\[
|V|_{s}^{k_0,\gamma} := |\text{Op}(V)|_{s}^{k_0,\gamma} = \max_{|\beta| \leq k_0} \|\partial_x^\beta V(\cdot, \xi ; \lambda)\|_s,
\]

see Definition (2.5). By Lemma 2.2 and by the estimate (2.7), one easily gets the following properties of the norm \( | \cdot |_{s}^{k_0,\gamma} \).
Lemma 5.1. (i) Let $s \geq s_0$ and $V, B \in \mathcal{S}^0_{s_0}$. Then
\[
|VB|_{s_0}^{k_0, \gamma} \lesssim_s |V|_{s_0}^{k_0, \gamma} |B|_{s_0}^{k_0, \gamma} + |V|_{s_0}^{k_0, \gamma} |B|_{s_0}^{k_0, \gamma}.
\]
(ii) Let $N > 0$, $s \geq 0$. Then
\[
|\Pi N V|_{s_0}^{k_0, \gamma} \leq N^\alpha |V|_{s_0}^{k_0, \gamma}, \quad 0 \leq \alpha \leq s, \quad |\Pi N V|_{s_0}^{k_0, \gamma} \leq N^{-\alpha} |V|_{s_0}^{k_0, \gamma}, \quad \alpha \geq 0.
\]

Proposition 5.2. Let $S > s_0$, $\tau > 0$, $\gamma \in (0, 1)$. Then there exists $\mu = \mu(\tau, \nu, k_0) > 0$ and $\delta = \delta(S, k_0, \tau, \nu) \in (0, 1)$ small enough such that if $\mathbf{3.11}, \mathbf{3.12}$ hold with $\mu_0 \geq s_0 + \mu$ and $\tau_1 = \tau_1(k_0, \tau, \nu) > 0$ large enough, then the following properties hold.

(i) There exists a $k_0$ times differentiable matrix valued symbol $\Phi_\infty(\omega, \zeta) \in \mathcal{S}^0_{S, 0}$ such that, for any $(\omega, \zeta) \in DC(\gamma, \tau)$ (see (4.7)), one has
\[
\Phi_\infty^{-1} \mathcal{P} \Phi_\infty = \omega \cdot \partial_\rho + \zeta \cdot \nabla,
\]
namely $\Phi_\infty^{-1} \mathcal{P} \Phi_\infty = \omega \cdot \partial_\rho A + \zeta \cdot \nabla A$ for all symbols $A(\phi, x, \xi)$.

Moreover, $\Phi_\infty^{\pm 1} = \text{even}(\phi, x, \xi)$, $\Phi_\infty(\phi, x, \xi)^{\pm 1} = \text{even}(\xi)$ and the following estimates hold:
\[
|\Phi_\infty^{\pm 1} - \text{Id}|_{s_0}^{k_0, \gamma} \lesssim_s N_\infty^{\alpha \gamma - 1} |V|_{s_0}^{k_0, \gamma}, \quad \forall s_0 \leq s \leq S.
\]

(ii) There exists $M \in \mathcal{S}^0_{S, 0}$, $M = \text{even}(\phi, x, \xi)$, that solves the equation
\[
(\omega \cdot \partial_\rho + \zeta \cdot \nabla) M(\phi, x, \xi) + V(\phi, x, \xi) M(\phi, x, \xi) + V(\phi, x, \xi) = 0,
\]
with
\[
|M|_{s_0}^{k_0, \gamma} \lesssim_s \epsilon^{-1} |\nu|_{s_0}^{k_0, \gamma}, \quad s_0 \leq s \leq S.
\]

(iii) Let $s_1 \geq s_0$ and let $u_1, u_2$ satisfy (3.1) with $\mu_0 \geq s_1 + \mu$. Then, for any $(\omega, \zeta) \in DC(\gamma, \tau)$, one has
\[
|\Delta_{12} \Phi_\infty^{\pm 1}|_{s_1} \lesssim_s N_\infty^{\alpha \gamma - 1} |V_1 - V_2|_{s_1 + \mu}, \quad |\Delta_{12} M|_{s_1} \lesssim_s \epsilon^{-1} |V_1 - V_2|_{s_1 + \mu}.
\]

To prove Proposition 5.2 we fix the constants
\[
N_{-1} := 1, \quad N_0, \tau > 0, \quad N_n := N_0^{(3/2)^n}, \quad \tau_0 := k_0 + \tau(k_0 + 1), \quad a := 3\tau_0 + 1, \quad b := a + 1
\]
and prove the following Proposition first.

Proposition 5.3. Let $S > s_0$, $\tau > 0$, $\gamma \in (0, 1)$. Then there exists $N_0 = N_0(S, k_0, \tau, \nu) > 0$, $\mu = \mu(k_0, \tau, \nu) > 0$ large enough, $\delta = \delta(S, k_0, \tau, \nu) \in (0, 1)$ small enough, and $C_\nu(s) > 0$, $s_0 \leq s \leq S$, such that if $\mathbf{3.11}, \mathbf{3.12}$ hold (for $\tau_1$ possibly larger) with $\mu_0 \geq s_0 + \mu$, the following statements hold for all $n \geq 0$.

\textbf{(S1)} There exists a linear operator (acting on symbols)
\[
P_n := \omega \cdot \partial_\rho + \zeta \cdot \nabla + V_n(\phi, x, \xi) : \mathcal{S}_{s+1, 0}^0 \to \mathcal{S}_{s_0}^0, \quad s_0 \leq s \leq S
\]
with
\[
V_n = \text{odd}(\phi, x, \xi), \quad V_n(\phi, x, \cdot) = \text{even}(\xi)
\]
and
\[
|V_n|_{s_0}^{k_0, \gamma} \leq C_\nu(s) N_{n_0-1}^{\gamma - 1} |\nu|_{s_0 + \mu}^{k_0, \gamma}, \quad |V_n|_{s_0 + \mu}^{k_0, \gamma} \leq C_\nu(s) N_{n_0-1}^{\gamma - 1} |\nu|_{s_0 + \mu}^{k_0, \gamma}
\]
for some constant $C_\nu(s) = C_\nu(s, k_0, \tau) > 0$. If $n \geq 1$, then there exists $\Psi_n - 1 \in \mathcal{S}_{s_0}^0, \forall s \geq s_0$, $\Psi_n - 1 = \text{even}(\phi, x, \xi), \Psi_{n_0}(-1)(\phi, x, \cdot) = \text{even}(\xi), \Psi_{n_0}(-1)(\phi, x, \xi) = \text{even}(\xi)$, such that
\[
|\Psi_{n-1}|_{s}^{k_0, \gamma} \lesssim N_{n-1}^{\gamma - 1} |V_{n-1}|_{s}^{k_0, \gamma}, \quad \forall s \geq s_0
\]
and $\Phi_n := \text{Id} + \Psi_n$ is invertible and, for any $\lambda = (\omega, \zeta) \in DC(\gamma, \tau)$, satisfies
\[
P_n = \Phi_{n-1}^{-1} P_{n-1} \Phi_{n-1}.
\]
(S2)_n Let s_1 \geq s_0 and assume that u_1, u_2 satisfy the ansatz (3.1) with μ_0 \geq s_1 + μ. Then, for any (ω, ζ) \in DC(γ, τ), one has

|\Delta_{12} V_n|_{s_1} \leq N_{n-3}^{-1} c \| u_1 - u_2 \|_{s_1+μ},
|\Delta_{12} V_n|_{s_1+b} \leq N_{n-1}^{-1} c \| u_1 - u_2 \|_{s_1+μ},

and for n \geq 1

|\Delta_{12} \Psi_{n-1}|_{s_1} \leq N_{n-1}^{\tau_1} N_{n-2}^{-2} c \| u_1 - u_2 \|_{s_1+μ},
|\Delta_{12} \Psi_{n-1}|_{s_1+b} \leq N_{n-1}^{\tau_1} N_{n-2}^{-2} c \| u_1 - u_2 \|_{s_1+μ}.

Proof. Proof of (S1)_0 - (S2)_0. It follows by Lemma 4.6 by setting V_0 = V := \kappa_0^{(1)}.

Proof of (S1)_{n+1}. Arguing by induction, at the n-th step, we have the operator \mathcal{P}_n in (5.5), whose remainder V_n satisfies the estimate (5.8). Let Φ_n(φ, x, ζ) = \text{Id} + Ψ_n(φ, x, ζ), where the matrix valued symbol T^* × T^3 × R^3 \to \text{Mat}_{3,3}(C), (φ, x, ζ) \mapsto Ψ_n(φ, x, ζ) has to be determined. One has

\mathcal{P}_n(Φ_n A) = Φ_n(ω \cdot \partial_φ A + ζ \cdot \nabla A) + (ω \cdot \partial_φ Ψ_n + ζ \cdot \nabla Ψ_n) A + (Π_{n} V_n) A + (Π_{n}^1 V_n) A + V_n Ψ_n A

for all symbols A = A(φ, x, ζ), namely

\mathcal{P}_n Φ_n = Φ_n(ω \cdot \partial_φ + ζ \cdot \nabla) (ω \cdot \partial_φ + ζ \cdot \nabla) Ψ_n + Π_{n} V_n + Π_{n}^1 V_n + V_n Ψ_n

(5.11)

where, to simplify notations, we do not write explicit dependence on (φ, x, ζ). The symmetry conditions (5.7) imply that V_n(·, ζ) = odd(φ, x), therefore, for any (ω, ζ) \in C^\infty, we can solve the homological equation

(ω \cdot \partial_φ + ζ \cdot \nabla)Ψ_n(φ, x, ζ) + Π_{n} V_n(φ, x, ζ) = 0

(5.12)

by defining

Ψ_n(φ, x, ζ) := -(ω \cdot \partial_φ + ζ \cdot \nabla)_{	ext{ex}} Π_{n} V_n(φ, x, ζ),

(5.13)

where we recall (2.10). Since V_n(·, ζ) = odd(φ, x) and V_n(φ, x, ·) = even(ζ), it is easy to verify that Ψ_n(·, ζ) = even(φ, x), Ψ_n(φ, x, ·) = even(ζ). By Lemmata (5.7), one immediately gets the estimates

|Ψ_n|^k \cdot γ \leq N_{n}^{\tau_0} \gamma^{-1} |Ψ_n|^k \cdot γ, \quad ∀s \geq s_0

(5.14)

which is (5.8) at the step n + 1. The estimate (5.14), together with (5.8) and (3.1), imply that for s = s_0

|Ψ_n|^k \cdot γ \leq N_{n}^{\tau_0} N_{n-2}^{-2} c \gamma^{-1}.

(5.15)

Hence, by taking ε\gamma^{-1} small enough and recalling (5.5), the matrix Φ_n(φ, x, ζ) = \text{Id} + Ψ_n(φ, x, ζ) is invertible by standard Neumann series, and

|Φ_n^{±1} - \text{Id}|^k \cdot γ \leq |Ψ_n|^k \cdot γ, \quad ∀s \geq s_0

(5.16)

Hence (5.11), (5.12) imply that

\mathcal{P}_{n+1} := \Phi_n^{-1} \mathcal{P}_n Φ_n = \omega \cdot \partial_φ + ζ \cdot \nabla + V_{n+1}(φ, x, ζ),
V_{n+1} := Π_{n}^1 V_n + (\Phi_n^{-1} - \text{Id}) Π_{n}^1 V_n + Φ_n^{-1} V_n Ψ_n.

(5.17)

Since V_n = odd(φ, x, ζ), V_n(φ, x, ·) = even(ζ) and Ψ_n = even(φ, x, ζ), Ψ_n(φ, x, ·) = even(ζ), it follows that V_{n+1} has the same parities as V_n, namely V_{n+1} = odd(φ, x, ζ), V_{n+1}(φ, x, ·) = even(ζ).

It only remains to show the estimates (5.8) at the step n + 1. By Lemma (5.1) and the estimates (5.14) - (5.16), for any s_0 \leq s \leq S one has

|V_{n+1}|^k \cdot γ \leq N_{n}^{\tau_0} |V_n|^k \cdot γ + C(s) N_{n}^{\tau_0} \gamma^{-1} |V_n|^k \cdot γ |V_n|^k \cdot γ,
|V_{n+1}^1|^k \cdot γ \leq |V_n|^k \cdot γ + (1 + C(s) N_{n}^{\tau_0} \gamma^{-1} |V_n|^k \cdot γ).

(5.18)

Then by the induction estimates on V_n, by the choice of a, b in (5.5), and by taking ε\gamma^{-1} \leq δ(S) small enough and N_0 = N_0(S) > 0 large enough, one gets the estimate (5.8) at the step n + 1. The estimates in (S2)_{n+1} can be proved arguing similarly. □
For $n \geq 1$, let
\[\hat{\Phi}_n := \Phi_0 \Phi_1 \cdots \Phi_n.\]  

**Lemma 5.4.** (i) For any $s_0 \leq s \leq S$, the sequence $\hat{\Phi}_n^\pm$ converges in the norm $\| \cdot \|_{s,0}$ to some limit $\Phi^\pm$ with $\Phi^\pm - \text{Id} = \text{even}(\phi, \xi)$, $\mathcal{M}_s^\pm - \text{Id} = \text{even}(\xi)$. Moreover
\[|\Phi^\pm - \text{Id}|_{s} \lesssim s N_0^\pm \varepsilon^{-1} \|v\|_{s,0}, \quad \forall s_0 \leq s \leq S.\]

(ii) Let $s_1 \geq s_0$ and assume that $v_1, v_2$ satisfy (5.1) with $\mu_0 \geq s_1 + \mu$. Then for any $\lambda \in DC(\gamma, \tau)$,
\[|\Delta_2 \hat{\Phi}_{\infty}^\pm|_{s_1} \lesssim s_1 N_0^\pm \varepsilon^{-1} \|v_1 - v_2\|_{s_1 + \mu}.\]

**Proof.** The proof of (i) is standard and it can be done as the one of Corollary 4.1 in [4]. The proof of (ii) follows by similar arguments, using Proposition 5.3 (S2). \qed

**Proof of Proposition 5.2.** Item (i) follows in a straightforward way by Proposition 5.3 and Lemma 5.4. Now we prove item (ii). By Proposition 5.2 (i), one is led to solve the equation
\[\omega \cdot \partial_\phi + \zeta \cdot \nabla)(\Phi_\infty(\phi, \xi))^{-1}M(\phi, \xi) = -\Phi_\infty(\phi, \xi)^{-1}V(\phi, \xi).\]  

Since $V = \text{odd}(\phi, \xi)$, $\Phi_\infty^\pm = \text{even}(\phi, \xi)$, $\Phi_\infty^\pm = \text{even}(\xi)$, one deduces that
\[\Phi_\infty^\pm = \text{even}(\phi, \xi), \quad V(\xi, \phi) = \text{odd}(\phi, \xi).\]

Hence $\Phi_\infty^{-1}V(\cdot, \xi) = \text{odd}(\phi, \xi)$, and therefore its average in $\phi$ is zero. As a consequence, the equation (5.20) is solved for any $\omega, \zeta \in DC(\gamma, \tau)$ by setting
\[M(\phi, \xi) := -\Phi_\infty(\phi, \xi)(\omega \cdot \partial_\phi + \zeta \cdot \nabla)\text{ext}[\Phi_\infty^{-1}V](\phi, \xi).\]

One easily verifies that $M = \text{even}(\phi, \xi)$. The claimed estimates on $M$, $\Phi_\infty^\pm$ follow by Lemmata 2.4, 3.1 (i), 5.4 and by the estimate (4.45) (recall that $V := R_0^{(1)}$). \qed

### 5.2 Elimination of the zero-th order term

**Lemma 5.5.** Let $S > s_0$, $\gamma \in (0, 1)$, $\tau > 0$. Then there exists $\mu = \mu(S, k_0, \tau, \nu) > 0$, $\tau_1 = \tau_1(k_0, \tau, \nu) > 0$ and $\delta = \delta(S, k_0, \tau, \nu) \in (0, 1)$ such that if (3.1), (4.3) are fulfilled with $\mu_0 \geq s_0 + \mu$, then the following properties hold. There exist a real and reversibility preserving map $M = \text{Op}(M) \in \mathcal{OPS}_S^{1,0}$ such that the map $B := \text{Id} + M$ is invertible, with
\[|M|_{s_0, \gamma}^\pm, |B^{-1} - \text{Id}|_{s_0, \gamma}^\pm, |M^\pm|_{s_0, \gamma}^\pm \lesssim s \varepsilon^{-1} \|v\|_{s_0, \gamma}, \quad \forall s_0 \leq s \leq S,\]  

and a real and reversible operator $L^{(2)}$ of the form
\[L^{(2)} := \omega \cdot \partial_\phi + \zeta \cdot \nabla + R^{(2)},\]  

defined for all $(\omega, \zeta) \in \mathbb{R}^{r+3}$, with $R^{(2)} \in \mathcal{OPS}_S^{1,0}$ and
\[|R^{(2)}|_{s_0, \gamma}^\pm \lesssim s \varepsilon \|v\|_{s_0, \gamma} \quad \forall s_0 \leq s \leq S,\]  

such that, for all $(\omega, \zeta) \in DC(\gamma, \tau)$, the operator $L^{(1)}$ defined in (4.44) is conjugated to $L^{(2)}$, namely
\[L^{(2)} = B^{-1} L^{(1)} B.\]

Let $s_1 \geq s_0$ and assume that $v_1, v_2$ satisfy (3.1) with $\mu_0 \geq s_1 + \mu$. Then, for all $\lambda = (\omega, \zeta) \in DC(\gamma, \tau)$,
\[|\Delta_2 B^{\pm 1}|_{s_0, s_1}^\pm, |\Delta_2 B^{\pm 1}|_{s_0, s_1}^\pm \lesssim s_1 \varepsilon \gamma^{-1} \|v_1 - v_2\|_{s_1 + \mu},\]
\[|\Delta_2 R^{(2)}|_{s_0, s_1}^\pm \lesssim s_1 \varepsilon \|v_1 - v_2\|_{s_1 + \mu}.\]  

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Proof. To eliminate the term of order 0 from the operator $L^{(1)}$ in (4.41), we look for a transformation

$$B = \text{Id} + M, \quad M = \text{Op}(M(\varphi, x, \xi)) \in \text{OPS}_{S,0}^0,$$

where the symbol $M(\varphi, x, \xi)$ has to be determined. By Lemma 2.7, one has

$$\mathcal{R}^{(1)}_0 M = \text{Op}(R^{(1)}_0(\varphi, x, \xi)M(\varphi, x, \xi)) + \mathcal{R}^{(1)}_0 M,$$

where the remainder $\mathcal{R}^{(1)}_0 M$ is an operator of order $-1$. Hence

$$L^{(1)} B = B(\omega \cdot \partial_\varphi + \zeta \cdot \nabla) + \text{Op}(\omega \cdot \partial_\varphi + \zeta \cdot \nabla) M + R^{(1)}_0 M + R^{(1)}_0 + \mathcal{R}^{(1)}_0 M.$$

Note that $\mathcal{R}^{(1)}_0 B$ is also an operator of order $-1$. By Proposition 5.2 applied with $V := R^{(1)}_0$, there exists a symbol $M(\varphi, x, \xi) \in S^0_{S,0}$, even($\varphi, x, \xi$), that solves the equation (5.1) and satisfies the bounds (5.21).

By (5.27), (5.1) one deduces formula (5.22) with

$$\mathcal{R}^{(2)} := B^{-1} \mathcal{R}^{(1)}_0 M + B^{-1} \mathcal{R}^{-1}_0 B.$$

The bounds for $M^*$ and $B^{-1} - \text{Id}$ follow by Lemmata 2.8, 2.9. The estimate of $|\mathcal{R}^{(2)}|_{-1,s,0}^{k_0,\gamma}$ then follows by Lemma 2.7 and by estimates (5.21), (5.22), (3.1), (4.3). Since $M$ is even($\varphi, x, \xi$), the maps $B^{\pm 1}$ are reversibility preserving maps, hence since $L^{(1)}$ is reversible then also $L^{(2)}$ is reversible. The estimates (5.21) can be proved arguing similarly.

6 Reduction to constant coefficients of the lower order terms

In this section we reduce the operator $L^{(2)}$ in (5.22) to constant coefficients up to an arbitrarily regularizing remainder. More precisely, we prove the following Proposition.

**Proposition 6.1.** Let $S > s_0$, $M \in \mathbb{N}$, $\gamma \in (0, 1)$, $\tau > 0$. Then there exists $\mu = \mu(M, k_0, \tau, \nu) > 0$, $\delta = \delta(S, M, k_0, \tau, \nu) > 0$ such that if (3.1), (4.3) are fullfilled with $\mu_0 \geq s_0 + \mu$, then the following holds.

There exists a real and reversibility preserving, invertible map $V$ satisfying

$$|V|^{k_0} - \text{Id}^{k_0,\gamma}_{-1,s,0}, |V^*| - \text{Id}^{k_0,\gamma}_{-1,s,0} \lesssim_{s, M} 1 \|v\|^{k_0,\gamma}_{s+\mu}, \quad \forall s \leq S$$

and a linear operator $L^{(3)}$ defined by

$$L^{(3)} := \omega \cdot \partial_\varphi + \zeta \cdot \nabla + Q + R^{(3)}$$

where $Q = \text{Op}(Q(\xi)) \in \text{OPS}^{\delta}_{-1,0}$ is an operator whose $3 \times 3$ matrix symbol is independent of $(\varphi, x)$, $R^{(3)}$ belongs to $\text{OPS}^{\delta}_{-1,0}$, and

$$|Q|^{k_0,\gamma}_{-1,s,0}, |R^{(3)}|^{k_0,\gamma}_{-1,M,s,0} \lesssim_{s, M} 1\|v\|^{k_0,\gamma}_{s+\mu}, \quad \forall s \leq S.$$

The operators $L^{(3)}$, $Q$, $R^{(3)}$ are real and reversible operators defined for all the values of the parameters $(\omega, \zeta) \in \mathbb{R}^{\nu+3}$. For any $(\omega, \zeta) \in DC(\gamma, \tau)$, one has $V^{-1} L^{(2)} V = L^{(3)}$.

Let $s_1 \geq s_0$ and let $v_1, v_2$ satisfy (5.1) with $\mu_0 \geq s_1 + \mu$. Then for any $(\omega, \zeta) \in DC(\gamma, \tau)$ one has

$$|\Delta_{12}^2 V|^{k_0}_{s_1,0}, |\Delta_{12}^2 V^*|^{k_0}_{s_1,0} \lesssim_{s_1, M} 1 \|v_1 - v_2\|_{s_1+\mu},$$

$$|\Delta_{12} Q|^{k_0}_{-1,s_1,0}, |\Delta_{12} R^{(3)}|^{k_0}_{-1,M,s_1,0} \lesssim_{s_1, M} 1\|v_1 - v_2\|_{s_1+\mu}.$$

Proposition 6.1 follows by the following iterative lemma.
Lemma 6.2. Let $S > s_0$, $M \in \mathbb{N}$, $\gamma \in (0,1)$, $\tau > 0$. Then there exists $\mu_M = \mu(M,k_0,\tau,\nu) > 0$, $\delta = \delta(S,M,k_0,\tau,\nu) \in (0,1)$ and $\mu_1 < \mu_2 < \ldots < \mu_M$ such that if $\{1.1\}$, $\{1.3\}$ are fulfilled with $\mu_0 \geq s_0 + \mu_M$, then the following holds. For any $n = 0, \ldots, M - 1$, there exists a linear operator $\mathcal{L}_n^{(2)}$ of the form

$$\mathcal{L}_n^{(2)} := \omega \cdot \partial_\phi + \zeta \cdot \nabla + \mathcal{Z}_n + \mathcal{R}_n^{(2)}$$

where $\mathcal{Z}_n = \text{Op}(\mathcal{Z}_n(\xi)) \in \mathcal{OPS}^{-1}_{S,0}$ and $\mathcal{R}_n^{(2)} = \text{Op}(\mathcal{R}_n^{(2)}(\varphi,x,\xi)) \in \mathcal{OPS}^{-(n+1)}_{S,0}$ are defined for all the values of the parameters $(\omega, \zeta) \in \mathbb{R}^{r+3}$ and satisfy

$$|\mathcal{Z}_n|_{-1,0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v\|_{s+\mu_n}^{k_0,\gamma} \forall s_0 \leq s \leq S,$$

$$|\mathcal{R}_n^{(2)}|_{-(n+1),0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v\|_{s+\mu_n}^{k_0,\gamma} \forall s_0 \leq s \leq S. \quad (6.6)$$

The operators $\mathcal{L}_n^{(2)}, \mathcal{Z}_n, \mathcal{R}_n^{(2)}$ are real and reversible. For any $n = 1, \ldots, M$ there is a reversibility preserving, invertible map $\mathcal{T}_n$ defined for any $(\omega, \zeta) \in \mathbb{R}^{r+3}$, satisfying the estimate

$$|\mathcal{T}_n^{\pm 1} - \text{Id}|_{-s,n,0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v\|_{s+\mu_n}^{k_0,\gamma} \forall s_0 \leq s \leq S \quad (6.7)$$

and

$$\mathcal{L}_n^{(2)} = \mathcal{T}_n^{-1}\mathcal{L}_{n-1}^{(2)}\mathcal{T}_n \forall (\omega, \zeta) \in DC(\gamma, \tau). \quad (6.8)$$

Let $s_1 \geq s_0$ and assume that $v_1, v_2$ satisfy $\{6.1\}$ with $\mu_0 \geq s_1 + \mu_M$. Then, for any $\lambda = (\omega, \zeta) \in DC(\gamma, \tau)$,

$$|\Delta_{12} \mathcal{T}_n^{\pm 1}|_{-n,s,0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v_1 - v_2\|_{s+\mu_n} \quad (6.9)$$

Proof. We prove the lemma arguing by induction. For $n = 0$ the claimed statement follows by Lemma $\{5.5\}$ by defining $\mathcal{L}_0^{(2)} := \mathcal{L}_0^{(2)}, \mathcal{Z}_0 = 0, \mathcal{R}_0^{(2)} := \mathcal{R}_0^{(2)}$.

We assume that the claimed statement holds for some $n \in \{0, \ldots, M - 1\}$ and we prove it at the step $n + 1$. Let us consider a transformation $\mathcal{T}_{n+1} = \text{Id} + \mathcal{M}_{n+1}$ where $\mathcal{M}_{n+1} = \text{Op}(\mathcal{M}_{n+1}(\varphi,x,\xi))$ is an operator of order $-(n+1)$ which has to be determined. One computes

$$\mathcal{L}_n^{(2)}\mathcal{T}_{n+1} = \mathcal{T}_{n+1}(\omega \cdot \partial_\phi + \zeta \cdot \nabla) + \mathcal{Z}_n + \text{Op}(\omega \cdot \partial_\phi + \zeta \cdot \nabla)\mathcal{M}_{n+1} + \mathcal{R}_n^{(2)} + \mathcal{Z}_n\mathcal{M}_{n+1} + \mathcal{R}_n^{(2)}\mathcal{M}_{n+1}. \quad (6.10)$$

We define the symbol $\mathcal{M}_{n+1}(\varphi,x,\xi)$ as

$$\mathcal{M}_{n+1} := (\omega \cdot \partial_\phi + \zeta \cdot \nabla)^{-1}\big[\text{ext}_{1}(\mathcal{R}_n^{(2)}(\varphi,x)) - \mathcal{R}_n^{(2)}\big], \quad (6.11)$$

which is defined for any $(\omega, \zeta) \in \mathbb{R}^{r+3}$ (recall $\{24.16\}$). Clearly $\mathcal{M}_{n+1}$ is an operator of the same order as $\mathcal{R}_n^{(2)}$, namely $\mathcal{M}_{n+1} \in \mathcal{OPS}^{-(n+1)}_{S,0}$, and, by Lemma $\{2.4\}$ and the induction estimate $\{6.6\}$, one verifies that

$$|\mathcal{M}_{n+1}|_{-(n+1),s,0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v\|_{s+\mu_n+\tau_0}, \quad \forall s_0 \leq s \leq S. \quad (6.12)$$

Furthermore, for any $(\omega, \zeta) \in DC(\gamma, \tau)$, $\mathcal{M}_{n+1}(\varphi,x,\xi)$ solves the homological equation

$$(\omega \cdot \partial_\phi + \zeta \cdot \nabla)\mathcal{M}_{n+1} + \mathcal{R}_n^{(2)} = (\mathcal{R}_n^{(2)})(\varphi,x). \quad (6.13)$$

Using the ansatz $\{3.1\}$ with $\mu_0 \geq s_0 + \mu_n + \tau_0$, one gets $|\mathcal{M}_{n+1}|_{-(n+1),s,0}^{0,\gamma} \lesssim \varepsilon \gamma^{-1}$, hence by taking $\varepsilon \gamma^{-1}$ small enough, one can apply Lemma $\{2.9\}$ obtaining that $\mathcal{T}_{n+1} = \text{Id} + \mathcal{M}_{n+1}$ is invertible with inverse $\mathcal{T}_{n+1}^{-1}$ satisfying the estimate

$$|\mathcal{T}_{n+1}^{-1} - \text{Id}|_{-s,n,0}^{0,\gamma} \lesssim_{s,n} \varepsilon \gamma^{-1} \|v\|_{s+\mu_n+\tau_0}; \quad \forall s_0 \leq s \leq S. \quad (6.14)$$

Hence the estimate $\{6.7\}$ at the step $n + 1$ holds by taking $\mu_{n+1} \geq \mu_n + \tau_0$. By $\{6.10\}, \quad \{6.13\}$ we then get, for any $(\omega, \zeta) \in DC(\gamma, \tau)$, the conjugation $\{6.8\}$ at the step $n + 1$ where $\mathcal{L}_n^{(2)}$ has the form $\{6.5\}$, with

$$\mathcal{Z}_{n+1} := \mathcal{Z}_n + \text{Op}(\mathcal{R}_n^{(2)}(\varphi,x)), \quad \mathcal{R}_n^{(2)} := (\mathcal{T}_{n+1}^{-1} - \text{Id})\mathcal{Z}_{n+1} + \mathcal{T}_{n+1}^{-1}(\mathcal{Z}_n\mathcal{M}_{n+1} + \mathcal{R}_n^{(2)}\mathcal{M}_{n+1}). \quad (6.15)$$

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Since $\mathcal{M}_{n+1}, T_{n+1}^{\pm}, R_{n}^{(2)}$ are defined for all $(\omega, \zeta) \in \mathbb{R}^{v+3}$, then also $R_{n+1}^{(2)}$ is defined for all $(\omega, \zeta) \in \mathbb{R}^{v+3}$. Since $Z_{n}$ is of order $-1$ and $R_{n}^{(2)}, M_{n+1}$ are of order $-(n+1)$, one gets that $Z_{n}M_{n+1} + R_{n}^{(2)}M_{n+1}$ is of order $-(n+2)$. The estimate (6.10) at the step $n+1$ for the operator $Z_{n+1}$ follows by the induction estimate on $Z_{n}, R_{n}^{(2)}$ and by the property (2.22). The estimate for the operator $R_{n+1}^{(2)}$ follows by (6.12) , (6.14), the estimates for $Z_{n+1}$ and $Z_{n}, R_{n}^{(2)}$ and using Lemma 2.7 (i). Furthermore, since $R_{n}^{(2)}$ and $Z_{n}$ are reversible, then by (6.13) one verifies that $M_{n+1} = \text{Op}(M_{n+1})$ is reversibility preserving and therefore, by (6.14), $Z_{n+1}$ and $R_{n+1}^{(2)}$ are reversible operators. The estimates (6.9) can be proved by similar arguments. □

**Proof of Proposition 6.1.** We define
\[
L^{(3)} := L^{(2)}_{M-1}, \quad Q := Z_{M-1}, \quad R^{(3)} := R^{(2)}_{M-1}, \quad V := T_{n} \circ \ldots \circ T_{M-1}.
\]

Then, in order to deduce the claimed properties of $Z$ and $R^{(3)}$, it suffices to apply Lemma 6.2 for $n = M - 1$. The estimate (6.1) then follows by the estimate (6.7) on $T_{n}, n = 1, \ldots, M - 1$, using the composition Lemma 2.7 (i) and Lemma 2.8 (to estimate the adjoint operator $V^{*}$). The estimates (6.4) follow similarly.

7 Conjugation of the operator $\mathcal{L}$

In this section we go back to the operator $\mathcal{L}$ in (3.3) which contains the projector $\Pi_{0}^{\perp}$, see (1.13). We define
\[
\mathcal{E} := A B V \quad \text{and} \quad \mathcal{E}_{\perp} := \Pi_{0}^{\perp} \mathcal{E} \Pi_{0}^{\perp}
\]

We also recall that $H_{0}^{\perp}$ is defined in (1.6).

**Lemma 7.1.** Let $S > s_{0}, M \in \mathbb{N}, \tau > 0, \gamma \in (0, 1)$. Then there exists $\delta = \delta(S, M, k_{0}, \tau) \in (0, 1)$ small enough and $\mu_{M} := \mu(M, k_{0}, \tau, \nu) > 0$ such that if (6.1), (6.4) are fulfilled with $\mu_{0} \geq s_{0} + \mu_{M}$, the following properties hold.

(i) For any $s_{0} \leq s \leq S$, the map $\mathcal{E}_{\perp} : H_{0}^{\perp} \to H_{0}^{\perp}$ is invertible and satisfies the tame estimates
\[
\left\| \mathcal{E}_{\perp}^{-1} h \right\|_{s_{0}, M}^{k_{0}, \gamma} \leq \left\| h \right\|_{s_{0}, M}^{k_{0}, \gamma} + \left\| \left( \mathcal{E}_{\perp}^{-1} \right)^{\perp} h \right\|_{s_{0}, M}^{k_{0}, \gamma}, \quad \forall s_{0} \leq s \leq S.
\]

Furthermore $\mathcal{E}_{\perp}$ and $\mathcal{E}_{\perp}^{-1}$ are real and reversibility preserving.

(ii) For any $(\omega, \zeta) \in \text{DC}^{(3)}(\gamma, \tau)$, the map $\mathcal{E}_{\perp}$ conjugates the operator $\mathcal{L}$ in (3.3) to an operator $\mathcal{L}_{0} : H_{0}^{s+1} \to H_{0}^{s+1}$ for any $s_{0} \leq s \leq S$, defined by
\[
\mathcal{L}_{0} := \mathcal{E}_{\perp}^{-1} \mathcal{L} \mathcal{E}_{\perp} = \omega \cdot \partial_{\phi} + \zeta \cdot \nabla + Q_{0} + R_{0}
\]

where $Q_{0}$ is a $3 \times 3$ time independent block-diagonal operator (see Definition 2.14) defined for any $(\omega, \zeta) \in \mathbb{R}^{v+3}$, represented by the matrix $Q_{0} = \text{diag}_{j \in \mathbb{Z}^{\gamma}\backslash\{0\}}(Q_{0})_{j}$, satisfying
\[
\sup_{j \in \mathbb{Z}^{\gamma}\backslash\{0\}} \left\| (Q_{0})_{j} \right\|_{\text{HS}} \leq \mu_{0} \left\| v \right\|_{s_{0} + \mu_{M}}^{k_{0}, \gamma}
\]

and $R_{0}$ is a linear operator defined for any $(\omega, \zeta) \in \mathbb{R}^{v+3}$ satisfying the estimate
\[
\left\| R_{0}(D)^{M} \right\|_{s_{0}, M}^{k_{0}, \gamma} \leq \mu_{0} \left\| v \right\|_{s_{0} + \mu_{M}}^{k_{0}, \gamma}, \quad \forall s_{0} \leq s \leq S
\]

where the decay norm $\left\| \cdot \right\|_{s_{0}, \gamma}$ is defined in Definition 2.15 and $(D)^{M}$ is the Fourier multiplier of symbol $\langle \xi \rangle^{M} = (1 + \left| \xi \right|^{2})^{M/2}$. Furthermore, $\mathcal{L}_{0}, Q_{0}, R_{0}$ are real and reversible.

(iii) Let $s_{1} \geq s_{0}$ and assume that $v_{1}, v_{2}$ satisfy (3.1) with $\mu_{0} \geq s_{1} + \mu_{M}$. Then for any $(\omega, \zeta) \in \text{DC}^{(3)}(\gamma, \tau)$, one has
\[
\left\| \Delta_{12} \mathcal{E}_{\perp}^{-1} h \right\|_{s_{1}, M} \leq \left\| v_{1} - v_{2} \right\|_{s_{1} + \mu_{M}} \left\| h \right\|_{s_{1} + 1},
\]

\[
\left\| \Delta_{12} R_{0}(D)^{M} \right\|_{s_{1}, M}^{k_{0}, \gamma} \leq \left\| v_{1} - v_{2} \right\|_{s_{1} + \mu_{M}}, \quad \text{sup}_{j \in \mathbb{Z}^{\gamma}\backslash\{0\}} \left\| (\Delta_{12} Q_{0})_{j} \right\|_{\text{HS}} \leq \mu_{0} \left\| v_{1} - v_{2} \right\|_{s_{1} + \mu_{M}}.
\]

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Proof. To simplify notations, in this proof we write $\| \cdot \|_s$ instead of $\| \cdot \|^{b_0}$. Also, given any operator $A$ mapping $H^s(\mathbb{T}^{d+3}, \mathbb{R}^3)$ into itself, we denote by $A[1] = A[1](\varphi, x)$ the $3 \times 3$ matrix of entries
\[
(A[1])_{j,k} := A_{j,k}[1] = \langle A(e_k), e_j \rangle_{\mathbb{R}^3}, \quad j, k = 1, 2, 3, \tag{7.7}
\]
where $e_j$ is the $j$-th vector of the canonical basis of $\mathbb{R}^3$; we denote by $\Pi_0 A[1] = (\Pi_0 A[1])(\varphi)$ the $3 \times 3$ matrix of entries $\Pi_0 A_{j,k}[1](\varphi)$. Note that, if $A = \text{Op}(a(\varphi, x)), \Pi_0 A[1] = a(\varphi, x, 0)$.

Proof of (i). By Proposition 4.1 Lemma 4.6 [5,5] Proposition 6.1 using also Lemma 2.6 the operator $\mathcal{E}$ is invertible, $\mathcal{E}, \mathcal{E}^\dagger$ are real and reversibility preserving and, for $s_0 \leq s \leq S$,
\[
\begin{align*}
\| \mathcal{E}^\dagger h \|_s & \lesssim_{s, M} \| h \|_s + \| v \|_{s+\mu_M} \| h \|_{s_0}, \\
\| (\mathcal{E}^\dagger - \text{Id})h \|_s & \lesssim_{s, M} N^0_\mu \varepsilon \gamma^{-1} (\| v \|_{s_0+\mu_M} \| h \|_{s+1} + \| v \|_{s+\mu_M} \| h \|_{s_0+1}), \tag{7.8}
\end{align*}
\]
for some $\mu_M > 0$. For any $u \in H^s(\mathbb{T}^{d+3}, \mathbb{R}^3)$, we split $u = \Pi^0 u + \Pi_0 u$, where $\Pi^0 u \in H^s_0(\mathbb{T}^{d+3}, \mathbb{R}^3)$ and $\Pi_0 u \in H^s := H^s(\mathbb{T}^{d+3}, \mathbb{R}^3)$ (recall that $(\Pi_0 u)(\varphi)$ is the space average of $u$; it is a function of $\varphi$, independent of $x$). Hence the operator $\mathcal{E}$ is decomposed accordingly into
\[
\mathcal{E} = \begin{pmatrix} \Pi_0 \mathcal{E} \Pi_0 & \Pi_0 \mathcal{E} \Pi^0 \\ \Pi^0 \mathcal{E} \Pi_0 & \Pi^0 \mathcal{E} \Pi^0 \end{pmatrix}.
\]

**Step 1: Invertibility of $\Pi_0 \mathcal{E} \Pi_0$.** We write $\Pi_0 \mathcal{E} \Pi_0 = \Pi_0 + \mathcal{R}_\mathcal{E}$ with $\mathcal{R}_\mathcal{E} := \Pi_0 (\mathcal{E} - \text{Id}) \Pi_0$. For any $f \in H^s_0(\mathbb{T}^{d+3}, \mathbb{R}^3)$, namely $f = f(\varphi)$ independent of $x$, one has
\[
(\mathcal{E} f)(\varphi, x) = (\mathcal{E}[1])(\varphi, x) \cdot f(\varphi) 
\]
where $\ast$ is the matrix product of the matrix $\mathcal{E}[1]$ by the vector $f$. In fact, $\mathcal{B} \mathcal{V}$ is a pseudo-differential operator of matrix symbol, say, $P(\varphi, x, \xi)$; therefore $\mathcal{B} \mathcal{V} f = P(\varphi, x, 0) f(\varphi) = (\mathcal{B} \mathcal{V})[1](\varphi, x) f(\varphi)$; and then $A(\mathcal{B} \mathcal{V})[1](\varphi, x) f(\varphi) = (\mathcal{B} \mathcal{V})[1](\varphi, x + \alpha(\varphi, x)) f(\varphi)$ because $A$ is a change of the $x$ variable. Hence
\[
\mathcal{R}_\mathcal{E} \Pi_0 h = \Pi_0 (\mathcal{E} - \text{Id}) \Pi_0 h = (\Pi_0 (\mathcal{E} - \text{Id}) [1]) h(\varphi) = (\Pi_0 h)(\varphi)
\]
because $(\Pi_0 h)(\varphi)$ is independent of $x$. Therefore, by the estimate (7.5), using the ansatz (3.1) and the product estimate (2.7), one obtains that
\[
\| \mathcal{R}_\mathcal{E} \Pi_0 h \|_s \lesssim_{s, M} N^0_\mu \varepsilon \gamma^{-1} (\| v \|_{s_0+\mu_M} \| \Pi_0 h \|_s + \| v \|_{s+\mu_M} \| \Pi_0^0 h \|_{s_0}), \quad \forall s_0 \leq s \leq S. \tag{7.10}
\]
By Neumann series, using the smallness condition (4.9), one then gets that $\Pi_0 \mathcal{E} \Pi_0 : H^s_0 \to H^s_0$ is invertible and
\[
\| (\Pi_0 \mathcal{E} \Pi_0)^{-1} \|_s \lesssim_{s, M} \| \Pi_0 h \|_s + \| v \|_{s+\mu_M} \| \Pi_0^0 h \|_{s_0}, \quad \forall s_0 \leq s \leq S. \tag{7.11}
\]

**Step 2: The operator $M := \Pi^0_\theta \mathcal{E} \Pi^0_\theta - (\Pi^0_\theta \mathcal{E} \Pi_0)(\Pi_0 \mathcal{E} \Pi_0)^{-1}(\Pi_0 \mathcal{E} \Pi^0_\theta)$ is invertible, and
\[
\| M^{-1} z \|_s \lesssim_{s, \mu} \| z \|_s + \| v \|_{s+\mu_M} \| z \|_{s_0}, \quad \forall s_0 \leq s \leq S, \quad \forall z \in H^s_0. \tag{7.13}
\]
Let us prove the invertibility of $M$. Since $\mathcal{E} : H^s \to H^s$ is invertible, given any $z \in H^s_0$ there exists a unique $h \in H^s$ such that $\mathcal{E} h = z$. Since $(\Pi_0 \mathcal{E} \Pi_0) : H^s_0 \to H^s_0$ is invertible, one has $\mathcal{E} h = z$ if and only if
\[
\begin{pmatrix} \Pi_0 \mathcal{E} \Pi_0 \\ \Pi^0_\theta \mathcal{E} \Pi_0 \\ \Pi^0_\theta \mathcal{E} \Pi^0_\theta \end{pmatrix} \begin{pmatrix} \Pi_0 h \\ \Pi^0_\theta h \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix} \iff \Pi^0_\theta h = z.
\]
Using (7.3),
\[
\| M^{-1} z \|_s = \| \Pi^0_\theta h \|_s \leq \| h \|_s = \| \mathcal{E}^{-1} z \|_s \lesssim_{s, M} \| z \|_s + \| v \|_{s+\mu_M} \| z \|_{s_0},
\]
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and \((7.13)\) is proved.

**Step 3:** Estimates of \(\Pi_0^b \mathcal{E} \Pi_0\), \(\Pi_0 \mathcal{E} \Pi_0^b\). By \((7.30)\), since \((\Pi_0 h)(\varphi)\) is independent of \(x\), one has

\[\Pi_0^b \mathcal{E} \Pi_0 h = \Pi_0^b (\mathcal{E} - \text{Id}) \Pi_0 h = (\Pi_0^b (\mathcal{E} - \text{Id})[1])(\varphi, x) \cdot (\Pi_0 h)(\varphi).\]

Also, by an explicit calculation,

\[\Pi_0 \mathcal{E} \Pi_0^b h = \Pi_0 (\mathcal{E} - \text{Id}) \Pi_0^b h = \Pi_0 (G \cdot \Pi_0^b h)\]

where “\(\cdot\)” is the “row by column” product, and \(G = G(\varphi, x)\) is the matrix with entries

\[G_{j,k}(\varphi, x) = (\mathcal{E}_{j,k} - \text{Id}_{j,k})^* [1](\varphi, x) = (\mathcal{E}^* - \text{Id})_{k,j} [1](\varphi, x), \quad j,k = 1,2,3. \tag{7.14}\]

The estimate \((7.8)\), the ansatz \((5.1)\), and the product estimate \((2.7)\) imply that

\[\|\Pi_0^b \mathcal{E} \Pi_0 h\|_{s,M} \lesssim s,M \varepsilon_0 \varepsilon_{\gamma}^{-1} \left( \|v\|_{s_0 + \mu_M} \|h\|_{s} + \|v\|_{s + \mu_M} \|h\|_{s_0} \right), \quad \forall s_0 \leq s \leq S, \tag{7.15}\]

for some \(\mu_M > 0\) large enough.

**Step 4:** Invertibility of \(\Pi_0^b \mathcal{E} \Pi_0\). By \((7.12)\) we have

\[\Pi_0^b \mathcal{E} \Pi_0^b = M + (\Pi_0^b \mathcal{E} \Pi_0) (\Pi_0 \mathcal{E} \Pi_0)^{-1} (\Pi_0 \mathcal{E} \Pi_0^b) = M (\Pi_0^b + S), \quad S := M^{-1} (\Pi_0^b \mathcal{E} \Pi_0) (\Pi_0 \mathcal{E} \Pi_0)^{-1} (\Pi_0 \mathcal{E} \Pi_0^b).\]

By \((7.11)\), \((7.13)\), \((7.15)\), we deduce that

\[\|S h\|_{s,M} \lesssim s,M \varepsilon_0 \varepsilon_{\gamma}^{-1} \left( \|v\|_{s_0 + \mu_M} \|h\|_{s} + \|v\|_{s + \mu_M} \|h\|_{s_0} \right), \quad \forall s_0 \leq s \leq S.\]

Therefore, by Neumann series, \(\Pi_0^b \mathcal{E} \Pi_0^b : H^s_0 \to H^s_0, \quad s_0 \leq s \leq S\) is invertible and it satisfies \((7.2)\).

**Proof of \((ii)\).** By Proposition \((7.1)\), Lemmata \((4.6)\) \((6.3)\), Proposition \((6.7)\) and recalling \((6.3)\), \((6.7)\), using that \(\Pi_0 \mathcal{E} \Pi_0^b = \Pi_0 (\mathcal{E} - \text{Id}) \Pi_0^b\) and \(\Pi_0 \mathcal{E} \Pi_0 = \Pi_0 (\mathcal{E} - \text{Id}) \Pi_0\), one obtains

\[\mathcal{L} \mathcal{E}_\perp = \Pi_0^b \mathcal{L}^{(0)} \mathcal{E} \Pi_0^b = \Pi_0^b \mathcal{L}^{(0)} (\mathcal{E} - \text{Id}) \Pi_0^b = \Pi_0^b \mathcal{E} \mathcal{E}^{(3)} \Pi_0^b - \Pi_0^b \mathcal{L}^{(0)} (\mathcal{E} - \text{Id}) \Pi_0^b = \mathcal{E}_\perp (\Pi_0^b \mathcal{L}^{(3)} \Pi_0^b) + \Pi_0^b (\mathcal{E} - \text{Id}) \Pi_0 \mathcal{L}^{(3)} \Pi_0^b - \Pi_0^b \mathcal{L}^{(0)} (\mathcal{E} - \text{Id}) \Pi_0^b. \tag{7.16}\]

Hence, recalling \((5.2)\), one obtains the conjugation \((7.3)\) with

\[
Q_0 := \Pi_0^b \mathcal{Q} \Pi_0^b, \quad R_0 := \Pi_0^b \mathcal{R}^{(3)} \Pi_0^b + A_1 - A_2, \quad A_1 := \mathcal{E}_\perp^{-1} \Pi_0^b (\mathcal{E} - \text{Id}) \Pi_0 \mathcal{L}^{(3)} \Pi_0^b, \quad A_2 := \mathcal{E}_\perp^{-1} \Pi_0^b \mathcal{L}^{(0)} (\mathcal{E} - \text{Id}) \Pi_0^b. \tag{7.17}\]

Since \(Q = \text{Op}(Q)\) has a symbol independent of \((\varphi, x)\), the operator \(Q_0 = \Pi_0^b \mathcal{Q} \Pi_0^b = \text{diag}_{j \in \mathbb{Z}^3 \setminus \{0\}} (Q_0)_j^2\) is a \(3 \times 3\) block diagonal operator (see Definition \((2.11)\)) with \((Q_0)_j^2 = (Q)_j^2 \in \text{Mat}_{3 \times 3}(\mathbb{C})\) for any \(j \in \mathbb{Z}^3 \setminus \{0\}\).

The estimate \((7.4)\) holds by \((6.3)\), by Remark \((2.19)\) and Lemma \((2.14)(v)\).

It only remains to prove the estimate \((7.5)\). First, we estimate the term \(\Pi_0^b \mathcal{R}^{(3)} \Pi_0^b\) in \((7.17)\). For any \(s_0 \leq s \leq S\), using Lemma \((2.19)\) and \((6.3)\), one has

\[\|\Pi_0^b \mathcal{R}^{(3)} \Pi_0^b \mathcal{E}_\perp^M_{s,M} \|_{s,M} \lesssim |\mathcal{R}^{(3)} \mathcal{E}_\perp |_{s,M} \|v\|_{s + \mu_M}. \tag{7.18}\]

Since \(\Pi_0 \omega_\varphi \cdot \partial_x \Pi_0^b\), \(\Pi_0 \zeta_\varphi \cdot \nabla\) and \(\Pi_0 Q \Pi_0^b\) are all zero, recalling \((6.2)\) one has \(\Pi_0 \mathcal{L}^{(3)} \Pi_0^b = \Pi_0 \mathcal{R}^{(3)} \Pi_0^b\). By \((7.9)\), we get

\[A_1 (D^M h) = \mathcal{E}_\perp^{-1} \Pi_0^b (\mathcal{E} - \text{Id}) \Pi_0 \mathcal{L}^{(3)} \Pi_0^b (D^M h) = (\mathcal{E}_\perp^{-1} \Pi_0^b \mathcal{E}[1])(\varphi, x) \cdot (\Pi_0 \mathcal{R}^{(3)} \Pi_0^b (D^M h))(\varphi).\]
Note that $R^{(3)}(D)^M$ is estimated in (7.18). Also, 

$$\Pi_0(\mathcal{E} - \text{Id})\Pi_0^+(D)^M h = \Pi_0(G \cdot \Pi_0^+(D)^M h) = \Pi_0[(D^M G) \cdot \Pi_0^+] h$$

with $G$ defined in (7.14). Recalling (5.3), (5.5), since $\Pi_0 = 0$, one has $\Pi_0^+ \mathcal{L}(0)\Pi_0 = \varepsilon M h(x)\Pi_0$. Hence 

$$A_2(D)^M h = \mathcal{E}_0^+ \Pi_0^+(\mathcal{E} - \text{Id})\Pi_0^+(D)^M h = \varepsilon(\mathcal{E}_0^+ M h)\cdot \Pi_0[(D^M G) \cdot \Pi_0^+] h].$$

Applying Lemma 2.19 (to bound $\|s\|$ with $\|s\|_{0,s,0}$), Lemmata 2.6 and 2.7 (to estimate composition of operators and their action on functions), bounds (2.23) (for the $\|s\|_{0,s,0}$ norm of any multiplicative matrix), (7.18) (for $|R^{(3)}(D)^M|_{0,s,0}$), (4.3) (for $\|M h\|_{s}$), (7.8) (for $\mathcal{E}$, $\mathcal{E}^*$), (7.2) (for $\mathcal{E}_0^+$), together with the assumptions (4.1), (4.3) and the trivial estimates $\Pi_0|_{0,s,0}, \Pi_0^+|_{0,s,0} \leq 1$, we obtain that 

$$|A_1(D)^M|_{s}^{k_0,\gamma}, |A_2(D)^M|_{s}^{k_0,\gamma} \lesssim_{s,M} \varepsilon|v|^{k_0,\gamma}.$$ 

This proves (7.5). The proof of the item (iii) follows by similar arguments. \qed

8 Reducibility and inversion

In this section we perform a reducibility scheme for the operator $\mathcal{L}_0$ given in Lemma 7.1. We recall, as observed in Section 2.3, that any linear operator can be described both by a matrix representation and by a pseudo-differential representation with symbol, see (2.4)-(2.48). In this section we consider transformations of $H^s_0$, the space of functions in $H^s$ having zero average in the space variable $x$ (see (1.6)).

We introduce some further notations. Given a matrix $A \in \text{Mat}_{3 \times 3}(\mathbb{C})$, we define the linear operators 

$$M_L(A) : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C}), \quad B \mapsto AB,$$

$$M_R(A) : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C}), \quad B \mapsto BA. \quad (8.1)$$

Since $\|AB\|_{HS}, \|BA\|_{HS} \leq \|A\|_{HS}\|B\|_{HS}$, one has 

$$\|M_L(A)\|_{op}, \|M_R(A)\|_{op} \leq \|A\|_{HS} \quad (8.2)$$

where $\|\|_{op}$ denotes the standard operator norm ($\|\|_{HS}$ is defined in (2.4)). Given $\tau, k_0, N_0 > 0$, we fix the constants 

$$\tau_0 := (k_0 + 1)\tau + k_0, \quad M := [2\tau_0 + 1], \quad \mathbf{a} := \max\left\{6\tau_0 + 1, \frac{3}{2}(\tau + 2\tau^2) + 1\right\}, \quad \mathbf{b} := \mathbf{a} + 1,$$

$$\mu(b) := \mu_M + \mathbf{b}, \quad N_{-1} := 1, \quad N_n := N_{n}^{\mu_M}, \quad n \geq 0, \quad \chi := 3/2, \quad (8.3)$$

where $[2\tau_0]$ is the integer part of $2\tau_0$ and $\mu_M$ is the constant appearing in Lemma 7.1. Note that Lemma 7.1 holds for any $M \in \mathbb{N}$ in (8.3) we fix its value so that $M \geq 2\tau_0$.

Remark 8.1 (Choice of the constants). The conditions $\mathbf{a} \geq 6\tau_0 + 1, \mathbf{b} \geq \mathbf{a} + 1$ in (8.3) are used to show the convergence in the iterative estimates (8.32). We also require that $\mathbf{a} \geq \frac{3}{2}(\tau + 2\tau^2) + 1$, for the measure estimates of Section 10 see Lemma 10.5.

Proposition 8.2 (Reducibility). Let $\gamma \in (0,1)$, $\tau > 0$, $S > s_0$ and assume (8.4) with $\mu_0 \geq s_0 + \mu(b)$. Then there exists $N_0 = N_0(S,k_0,\tau) > 0, \tau_2 = \tau_2(k_0,\tau,\nu) > 0$ large enough and $\delta = \delta(S,k_0,\tau) \in (0,1)$ small enough such that if 

$$N_0^2 \varepsilon \gamma^{-1} \leq \delta, \quad (8.4)$$

then for any integer $n \geq 0$ the following statements hold.

(S1)$_n$ There exists a real and reversible operator 

$$\mathcal{L}_n := \omega \cdot \partial_x + N_n + \mathcal{R}_n : H^{s+1}_0 \to H^s_0, \quad \forall s_0 \leq s \leq S$$

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defined for any \( \lambda = (\omega, \zeta) \in \mathbb{R}^{r+3} \) and \( k_0 \) times differentiable in \( \lambda = (\omega, \zeta) \in \mathbb{R}^{r+3} \) with the following properties. The operator \( \mathcal{N}_n \) is a 3 \( \times \) 3 block-diagonal operator (see Definition 2.14) defined by

\[
\mathcal{N}_n := \zeta \cdot \nabla + q_n, \quad q_n = \text{diag}_{j \in \mathbb{Z}^3 \setminus \{0\}} (Q_n)_{\gamma}, \quad (Q_n)_{\gamma} \in \text{Mat}_{3 \times 3}(\mathbb{C}),
\]

\[
\| (Q_n)_{\gamma} \|_{\text{HS}} \lesssim \varepsilon |j|^{-1} \| v \|_{S_n + \mu(b)}^{k_0 - \gamma}, \quad \| (Q_n)_{\gamma} - (Q_0)_{\gamma} \|_{\text{HS}} \lesssim \epsilon |j|^{-M} \| v \|_{S_n + \mu(b)}, \quad \forall j \in \mathbb{Z}^3 \setminus \{0\}. \tag{8.5}
\]

If \( n \geq 1 \), for all \( j \in \mathbb{Z}^3 \setminus \{0\} \) one has

\[
\| (Q_n)_{\gamma} - (Q_n - 1)_{\gamma} \|_{\text{HS}} \lesssim k_0 \varepsilon N_n^{-a} |j|^M \| v \|_{S_n + \mu(b)}. \tag{8.6}
\]

If \( n \geq 0 \), the operator \( R_n \) satisfies, for any \( s_0 \leq s \leq S \), the estimates

\[
\| R_n (D)_{s_0} \|_{\text{HS}} \leq C_{s} N_{n-1}^{-a} \| v \|_{S_n + \mu(b)}, \quad \| R_n (D)_{s_0 + b} \| \leq C_{s} N_{n-1} \| v \|_{S_n + \mu(b)} \tag{8.7}
\]

for some constant \( C_{s} = C_{s} (s, k_0, \tau) > 0 \) large enough. For any \( \ell \in \mathbb{Z}^3 \), \( j, j' \in \mathbb{Z}^3 \setminus \{0\} \), define the linear operator \( L_n (\ell, j, j') = L_n (\ell, j, j'; \lambda, \nu) = L_n (\ell, j, j'; \lambda) : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C}) \) by

\[
L_n (\ell, j, j') := i \omega \cdot \ell \text{Id} + M (N_n)_{\gamma} - M R_n (N_n)_{\gamma}. \tag{8.8}
\]

Note that, by \((8.5)\), it is \((N_n)_{\gamma} = i (\zeta \cdot j) I + (Q_n)_{\gamma} \in \text{Mat}_{3 \times 3}(\mathbb{C}) \), where \( I \in \text{Mat}_{3 \times 3} \) is the identity matrix.

If \( n = 0 \), we define \( \Omega_0 := DC(\gamma, \tau) \); if \( n \geq 1 \), we define

\[
\Omega_n := \left\{ \lambda = (\omega, \zeta) \in \Omega_{n-1} : \text{L}_n^{-1}(\ell, j, j') \text{ is invertible and } \| L_n^{-1}(\ell, j, j') \|_{\sup} \leq \left( \frac{\ell}{\gamma} \right)^{ |j| + |j'| } \right\}. \tag{8.9}
\]

If \( n \geq 1 \), there exists an invertible, real and reversibility preserving map \( \Psi_{n-1} = \text{Id} + \psi_{n-1} \), \( k_0 \) times differentiable in \( \lambda = (\omega, \zeta) \in \mathbb{R}^{r+3} \), such that, for any \( \lambda \in \Omega_n \),

\[
\lambda = \Phi_{n-1}^{-1} \lambda_n \Phi_{n-1}. \tag{8.10}
\]

Moreover, for any \( s_0 \leq s \leq S \), the map \( \psi_{n-1} \) satisfies the estimates

\[
\| \psi_{n-1} \|_{s_0} \lesssim \varepsilon N_n^{-a} \| v \|_{S_n + \mu(b)}, \quad \| (D)^{-M} \psi_{n-1} (D)_{s_0} \|_{s_0 + b} \lesssim \varepsilon N_n^{-a} \| v \|_{S_n + \mu(b)} \tag{8.11}
\]

\((S2)_n\) Assume that \( v_1, v_2 \) satisfy \((\ref{S3})\) with \( \mu_0 \geq s_0 + \mu(b) \). Then for any \( \lambda = (\omega, \zeta) \in \Omega_n^\gamma (v_1) \cap \Omega_n^\gamma (v_2) \) with \( \gamma_1, \gamma_2 \in [\gamma/2, 2\gamma] \), the following estimates hold:

\[
\| \Delta_1 R_n (D)_{s_0} \|_{s_0} \lesssim \varepsilon N_n^{-a} \| v_1 - v_2 \|_{s_0 + \mu(b)}, \quad \| \Delta_2 R_n (D)_{s_0 + b} \|_{s_0 + b} \lesssim \varepsilon N_n^{-a} \| v_1 - v_2 \|_{s_0 + \mu(b)} . \tag{8.12}
\]

Furthermore, if \( n \geq 1 \), for any \( j \in \mathbb{Z}^3 \setminus \{0\} \) one has

\[
\| \Delta_1 (Q_n - Q_n)_{\gamma} \|_{\text{HS}} \lesssim \varepsilon j^{-M} N_n^{-a} \| v_1 - v_2 \|_{s_0 + \mu(b)}, \quad \| \Delta_2 (Q_n)_{\gamma} \|_{\text{HS}} \lesssim \varepsilon j^{-1} \| v_1 - v_2 \|_{s_0 + \mu(b)}. \tag{8.13}
\]

\((S3)_n\) Let \( v_1, v_2 \) as in \((S2)_n\) and \( 0 < \rho \leq \gamma/2 \). Then

\[
N_n^{-a} \| v_1 - v_2 \|_{s_0 + \mu(b)} \leq \rho \implies \Omega_n^\gamma (v_1) \subseteq \Omega_n^{-\rho} (v_2). \tag{8.14}
\]

**Proof.** **Proof of \((S1)_n\).** The claimed properties follow directly from Lemma \((\ref{S1})\) and the definition of \( \Omega_n^\gamma \).

**Proof of \((S1)_{n+1}, (S2)_{n+1}\).** We only prove \((S1)_{n+1}\); the properties in \((S2)_{n+1}\) follow similarly.
Assume the claimed properties \((S1)_n\) holds for some \(n \geq 0\) and let us prove them at the step \(n + 1\). Let \(\Psi_n = \text{Id} + \psi_n\) where \(\psi_n\) is an operator to determine. We compute

\[
L_n \Psi_n = \Phi_n(\omega \cdot \partial_\nu + N_n) + \omega \cdot \partial_\nu \Psi_n + [N_n, \Psi_n] + \Pi_{N_n} R_n + \Pi_{\hat{N}} R_n + R_n \psi_n \tag{8.14}
\]

where \(N_n := \zeta \cdot \nabla \text{Id} + Q_n\); recall definition \((2.51)\) of the projectors \(\Pi_{N}, \Pi_{\hat{N}}\). Our purpose is to find a map \(\Psi_n\) solving the homological equation

\[
\omega \cdot \partial_\nu \Psi_n + [N_n, \Psi_n] + \Pi_{N_n} R_n = Z_n \tag{8.15}
\]

where \(Z_n\) is the \(3 \times 3\) block-diagonal operator

\[
Z_n := \text{diag}_{j \in \mathbb{Z}^3 \setminus \{0\}} [\hat{R}_n(0)]^j.
\]

**Lemma 8.3.** There exists a reversibility-preserving operator \(\Psi_n\) defined for all the values of the parameters \(\lambda = (\omega, \zeta) \in \mathbb{R}^{s+3}\) which satisfies, for any \(s_0 \leq s \leq s + b\), the estimates

\[
|\Psi_n|^{k_0 \gamma}, \quad |(D)^{-M} \Psi_n(D)^M|^{k_0 \gamma} \lesssim N_n^{4_0 \gamma} R_n(D)^M|^{k_0 \gamma} \tag{8.17}
\]

and solves, for any \(\lambda = (\omega, \zeta) \in \Omega^2_{n+1}\), the homological equation \((8.15)\).

**Proof.** To simplify notations, in this proof we drop the index \(n\) and we write \(+\) instead of \(+\). By using the matrix \(3 \times 3\) block representation of linear operators provided in Section \(2.3\) and recalling \((2.51)\), the homological equation \((8.15)\) is equivalent to solve, for all \(\ell \in \mathbb{Z}^v, j, j' \in \mathbb{Z}^3 \setminus \{0\},\)

\[
L(\ell, j, j') \hat{\Psi}^j_j(\ell) + \Pi_N \hat{R}^j_j(\ell) = \hat{Z}^j_j(\ell), \tag{8.18}
\]

where we recall that \(L(\ell, j, j')\) is defined by

\[
L(\ell, j, j') : \text{Mat}_{3 \times 3}(\mathbb{C}) \rightarrow \text{Mat}_{3 \times 3}(\mathbb{C}), \quad B \mapsto \bar{i}(\omega \cdot \ell + \zeta \cdot (j - j')) B + Q^j_j B - B Q^j_j.
\]

Since \(\text{Mat}_{3 \times 3}(\mathbb{C})\) is a vector space over \(\mathbb{C}\) of dimension 9, \(L(\ell, j, j')\) can be represented by a matrix in \(\text{Mat}_{9 \times 9}(\mathbb{C})\). By \((8.5)\) and \((3.1)\), one has

\[
\|Q^j_j\|_{\text{HS}} + \|Q^j_j\|_{\text{HS}}^{k_0 \gamma} \lesssim k_0 \frac{\varepsilon}{|j|} + \frac{\varepsilon}{|j'|}.
\]

For any fixed \((\ell, j, j')\), for all \(\lambda = (\omega, \zeta) \in \mathbb{R}^{s+3}\), the scalar product \(f(\lambda) := \omega \cdot \ell + \zeta \cdot (j - j')\) satisfies \(\partial_\tau \zeta f(\lambda) = \ell_k, \partial_\nu f(\lambda) = (j - j)_k, \) and \(\partial_\nu^\beta f(\lambda) = 0\) for all multi-indices \(\beta\) of length \(|\beta| \geq 2\). Thus

\[
|\gamma|^{\beta} |\partial_\nu^\beta (\omega \cdot \ell + \zeta \cdot (j - j'))| \leq \gamma|\ell - j'| \quad \forall \lambda \in \mathbb{R}^{s+3}, |\beta| \geq 1. \tag{8.20}
\]

Since \(\varepsilon \leq \gamma\) (see \((8.4)\)), the estimates \((8.19)\), \((8.20)\) imply that

\[
\gamma|\beta| \|\partial_\nu^\beta L(\ell, j, j'; \lambda)\|_{\text{op}} \lesssim \gamma|\ell - j'| \quad \forall \lambda \in \mathbb{R}^{s+3}, 1 \leq |\beta| \leq k_0.
\]

For each \((\ell, j, j')\), we consider the set \(\mathcal{A} := \mathcal{A}(\ell, j, j')\) of the parameters \(\lambda = (\omega, \zeta)\) such that \(L(\ell, j, j'; \lambda)\) is invertible, and

\[
\mathcal{K} := \mathcal{K}(\ell, j, j') := \left\{ \lambda \in \mathbb{R}^{s+3} : L(\ell, j, j'; \lambda) \text{ is invertible and } \|L(\ell, j, j'; \lambda)^{-1}\|_{\text{op}} \leq \frac{(\ell)^T|j|^T|j'|^T}{\gamma} \right\}. \tag{8.22}
\]

We observe that \(\mathcal{K} \subseteq \mathcal{A} \subseteq \mathbb{R}^{s+3}\), \(\mathcal{A}\) is open, and \(\mathcal{K}\) is closed. The map \(\mathcal{A} \rightarrow \text{Mat}_{9 \times 9}(\mathbb{C}), \lambda \mapsto L(\ell, j, j'; \lambda)^{-1}\) is \(k_0\) times differentiable on \(\mathcal{A}\). By induction, it is not difficult to prove that, for any multi-index \(\beta\) of positive length, the derivative \(\partial_\nu^\beta(L(\ell, j, j'; \lambda)^{-1})\) is a sum of terms of the form \(L^{-1}(\partial_\nu^\beta L)L^{-1} \cdots L^{-1}(\partial_\nu^\beta L)L^{-1}\) where \(1 \leq m \leq |\beta|\) and \(\beta_1, \ldots, \beta_m\) are nonzero multi-indices with \(\beta_1 + \ldots + \beta_m = \beta\) (\(L\) briefly denotes
By Whitney extension theorem (see, e.g., Appendix B in [3] where the weights \( \gamma^{[\beta]} \) are also considered), the function \( \mathcal{K} \to \text{Mat}_{n \times \ell} (\mathbb{C}), \lambda \mapsto L(\ell, j, j'; \lambda)^{-1} \) admits an extension to \( \mathbb{R}^{\nu+3} \), which we denote by \( L(\ell, j, j'; \lambda)^{-1}_{\text{ext}} \), satisfying
\[
\|L(\ell, j, j'; \lambda)^{-1}_{\text{ext}}\|_{\text{op}} \lesssim \gamma^{-1} \langle \ell \rangle^{\tau_0} |j|^{\tau_0} |j'|^{\tau_0}.
\]

By (8.23), the estimate (8.23) also holds for \( \beta = 0, \lambda \in \mathcal{K} \). Hence
\[
\sup_{0 \leq |\beta| \leq k_0} \sup_{\lambda \in \mathcal{K}} \gamma^{[\beta]} \|\partial^{[\beta]}_\lambda L(\ell, j, j'; \lambda)^{-1}_{\text{ext}}\|_{\text{op}} \lesssim \gamma^{-1} \langle \ell \rangle^{\tau_0} |j|^{\tau_0} |j'|^{\tau_0}.
\]

To estimate the decay norm of the operator \( D^{-M} \Psi \langle D \rangle^M \) we need to estimate \( \langle j \rangle^{-M} \|\partial^{[\beta]}_\lambda \hat{\Psi}'(\ell)\|_{\text{HS}} \langle j \rangle^{M} \).

By triangular inequality, since \( |j' - j| \leq N \), one has
\[
|j'|^{\tau_0} \leq (|j' - j| + |j|)^{\tau_0} \leq (N + |j|)^{\tau_0} \lesssim (N |j|)^{\tau_0}
\]
and therefore, since also \( |\ell| \leq N \),
\[
\langle j \rangle^{-M} \langle \ell \rangle^{\tau_0} |j|^{\tau_0} \langle j' \rangle^{M} \lesssim \langle j \rangle^{-M} N^{\tau_0} |j|^{\tau_0} (N |j|)^{\tau_0} \langle j' \rangle^{M} \lesssim N^{2\tau_0} \langle j' \rangle^{M}
\]
because, by (8.3), \( \nu \leq M \). Hence, multiplying (8.26) by \( \langle j \rangle^{-M} \langle j' \rangle^{M} \), using (8.24), and recalling Definition 2.15, we obtain the estimate (8.17) for \( D^{-M} \Psi \langle D \rangle^M \). The estimate for \( \Psi \) can be proved similarly.

From (8.3) and (7.1) one obtains that, for any \( s_0 \leq s \leq S \),
\[
|\Psi_n|_{s_0}^{k_0, \gamma}, \ |\langle D \rangle^{-M} \Psi_n \langle D \rangle^M|_{s_0}^{k_0, \gamma} \lesssim N^{2\tau_0-1} |\mathcal{R}_n \langle D \rangle^M|_{s_0}^{k_0, \gamma} \lesssim N^{2\tau_0} N_{-1} \epsilon^{\gamma-1} \|v|_{s_0 + \mu(b)}^{k_0, \gamma},
\]
\[
|\Psi_n|_{s_0 + b}^{k_0, \gamma}, \ |\langle D \rangle^{-M} \Psi_n \langle D \rangle^M|_{s_0 + b}^{k_0, \gamma} \lesssim N^{2\tau_0} N_{-1} \epsilon^{\gamma-1} \|v|_{s_0 + \mu(b)}^{k_0, \gamma},
\]
which are the estimates (8.11) at the step \( n + 1 \). Using the ansatz (3.1) with \( \mu = \mu(b) \), by (8.3), one has
\[
|\Psi_n|_{s_0}^{k_0, \gamma} \lesssim N^{2\tau_0-1} |\mathcal{R}_n \langle D \rangle^M|_{s_0}^{k_0, \gamma} \lesssim N^{2\tau_0} N_{-1} \epsilon^{\gamma-1} \lesssim N^{2\tau_0 \epsilon^{\gamma-1}}
\]
Then for \( \epsilon^{\gamma-1} \) small enough, by Lemma 2.17 (iv), \( \Phi_n = \text{Id} + \Psi_n \) is invertible and, for any \( s_0 \leq s \leq S \),
\[
|\Phi_n^{-1} - \text{Id}|_{s_0}^{k_0, \gamma} \lesssim s |\Psi_n|_{s_0}^{k_0, \gamma}, \ |\Phi_n^{-1} - \text{Id}|_{s_0 + b}^{k_0, \gamma} \lesssim s |\Psi_n|_{s_0 + b}^{k_0, \gamma}.
\]
We define
\[ \mathcal{L}_{n+1} := \omega \cdot \partial_p + \mathcal{N}_{n+1} + \mathcal{R}_{n+1}, \quad \mathcal{N}_{n+1} := \zeta \cdot \nabla \text{Id} + \mathcal{Q}_{n+1}, \]
\[ \mathcal{Q}_{n+1} := \mathcal{Q}_n + Z_n, \quad \mathcal{R}_{n+1} := (\Phi_n^{-1} - \text{Id}) Z_n + \Phi_n^{-1}(\Pi_{\mathcal{N}_n} R_n + \mathcal{R}_n \Psi_n). \]  
(8.31)

All the operators in (8.31) are defined for any \( \lambda = (\omega, \zeta) \in \mathbb{R}^{r+3} \). Since \( \Psi_n, \Phi_n, \Phi^{-1}_n \) are reversibility preserving and \( \mathcal{N}_n, \mathcal{R}_n \) are reversible operators, one gets that \( \mathcal{N}_{n+1}, \mathcal{R}_{n+1} \) are reversible operators. Moreover, by (8.14), (8.15), for \( (\omega, \zeta) \in \Omega^\gamma_{n+1} \) one has the identity \( \Phi^{-1}_n L_n \Phi_n = L_{n+1} \), which is (8.10) at the step \( n+1 \).

For any \( j \in \mathbb{Z}^3 \setminus \{0\} \), recalling the definition (8.19) of \( Z_n \),
\[ \| (\mathcal{N}_{n+1})_j \|_{\mathcal{H}^{k_0, \gamma}} = \| (\mathcal{Q}_{n+1})_j \|_{\mathcal{H}^{k_0, \gamma}} \leq \| \hat{R}_n(0)_j \|_{\mathcal{H}^S} \lesssim k_0 | \mathcal{R}_n(D)^M \mathcal{L}_{n_0}^{\gamma}(j)^{-M}. \]

Then the estimate (8.7) implies the estimate (8.6) at the step \( n+1 \). The estimate (8.5) at the step \( n+1 \) follows, as usual, by a telescoping argument, using the fact that \( \sum_{n \geq 0} N_{n-1}^a \) is convergent since \( a > 0 \) (see (8.3)). Now we prove the estimates (8.7) at the step \( n+1 \). By (8.31), one has
\[ \mathcal{R}_{n+1}(D)^M = (\Phi_n^{-1} - \text{Id}) Z_n(D)^M + \Phi_n^{-1}(\Pi_{\mathcal{N}_n} R_n(D)^M) + \Phi_n^{-1}(R_n(D)^M)(D)^{-M} \Psi_n(D)^M. \]

By the estimates (8.28), (8.29), (8.30), by applying Lemma 2.17(ii, v), Lemma 2.18 the smallness condition (8.3) and the induction estimate (8.7), for any \( s_0 \leq s \leq S \) we get
\[
\begin{align*}
| \mathcal{R}_{n+1}(D)^M |_{L^1_{s_0} \mathcal{H}^{k_0, \gamma}} & \lesssim N_{n-b} \mathcal{R}_n(D)^M |_{L^1_{s+b} \mathcal{H}^{k_0, \gamma}} + N_{n-2s} \mathcal{R}_n(D)^M |_{L^1_{s+b} \mathcal{H}^{k_0, \gamma}}, \\
| \mathcal{R}_{n+1}(D)^M |_{L^1_{s_0} \mathcal{H}^{k_0, \gamma}} & \lesssim | \mathcal{R}_n(D)^M |_{L^1_{s+b} \mathcal{H}^{k_0, \gamma}}.
\end{align*}
\]  
(8.32)

Using the definition of the constants \( a, b \) in (8.3) and the smallness condition (8.4), taking \( N_0 = N_0(S, k_0, \tau) > 0 \) large enough, one gets the estimate (8.7) at the step \( n+1 \).

**Proof of (S3)_{n+1}.** Assume that we have proved the claimed statement for some \( n \in \mathbb{N} \) and let us prove it at the step \( n+1 \). Let \( \lambda = (\omega, \zeta) \in \Omega^\gamma_{n+1}(v_1) \). By the definition of the sets \( \Omega^\gamma_n \) (see (8.9)) and using the induction hypothesis, one has that
\[ \Omega^\gamma_{n+1}(v_1) \subseteq \Omega^\gamma_n(v_1) \subseteq \Omega^\gamma_n(v_2). \]
(8.33)

The property (S2)_n, together with (S2)_n, implies that
\[ \| \Delta_{12}(Q_n)_j \|_{\mathcal{H}^1} \lesssim \epsilon |j|^{-1} \| v_1 - v_2 \|_{s_0 + \mu(b)}, \quad \lambda = (\omega, \zeta) \in \Omega^\gamma_{n+1}(v_1). \]  
(8.34)

Let \( \lambda \in \Omega^\gamma_{n+1}(v_1) \). To prove the claimed inclusion, we have to show that for any \( (\ell, j, j') \) with \( \ell \in \mathbb{Z}^3 \setminus \{0\}, (\ell, j, j') \neq (0, j, j), |\ell|, |j - j'| \leq N_n \), the linear operator
\[ L_n(\ell, j, j' ; v_1) \equiv L_n(\ell, j, j'; \lambda, v_2(\lambda)) : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C}) \]
(see (8.8)) is invertible and \( \| L_n(\ell, j, j'; v_2) \|_{op} \leq (\gamma - \rho)^{-1} |\ell|^r |j|^r |j'|^r \). We distinguish two cases.

**Case 1:** \( \min\{|j|, |j'|\} \geq N_n^\rho \). By recalling (8.5) and the definitions (8.1), we write
\[
\begin{align*}
L_n(\ell, j, j' ; v_2) & = \Omega(\ell, j, j') + \Delta_n(j, j' ; v_2), \\
\Omega(\ell, j, j') & := i(\omega \cdot \ell + \zeta \cdot (j - j')) \text{Id}, \quad \Delta_n(j, j' ; v_2) := M_L((Q_n)_j) - M_R((Q_n)_j) .
\end{align*}
\]  
(8.35)

Since \( \lambda = (\omega, \zeta) \in DC(\gamma, \tau) \) (recall (8.3)), the operator \( \Omega(\ell, j, j') : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C}) \) is invertible and satisfies the estimate
\[ \| \Omega(\ell, j, j') \|_{op} \leq \frac{|\ell| \cdot |j - j'|^r}{\epsilon_0 \gamma}. \]
(8.36)

Furthermore, by the estimates (8.3), one has that
\[ \| \Delta_n(j, j' ; v_2) \|_{op} \lesssim \frac{\epsilon}{\min\{|j|, |j'|\}}. \]  
(8.37)
Since $\langle \ell, j - j' \rangle \lesssim N_n$ and $\min\{|j|, |j'|\} \geq N_n^\gamma$, the estimates (8.36), (8.37) immediately imply that
\[
\|\Omega(\ell, j, j')^{-1} \Delta_n(j, j'; v_2)\|_{\text{op}} \lesssim \varepsilon \gamma^{-1}.
\]
Hence, for $\varepsilon \gamma^{-1}$ small enough, the operator $L_n(\ell, j, j'; v_2)$ is invertible by Neumann series and
\[
\|L_n(\ell, j, j'; v_2)^{-1}\|_{\text{op}} \leq 2\|\Omega(\ell, j, j')^{-1}\|_{\text{op}} \leq \frac{(\ell^t |j|^t |j'|^t)}{\varepsilon \gamma^{-1} - \rho}.
\]
(8.38)
by taking $C_0 > 0$ large enough.

**Case 2:** $\min\{|j|, |j'|\} \leq N_n^\gamma$. By recalling (8.5) and the definitions (8.1), we write
\[
L_n(\ell, j, j'; v_2) = L_n(\ell, j, j'; v_1) + \Gamma_n(j, j'),
\]
\[
\Gamma_n(j, j') := -M_L((\Delta_{12} Q_n)^{j}_j) + M_R((\Delta_{12} Q_n)^{j}_j).
\]
(8.39)
By (8.31), using the property (8.2), one gets
\[
\|\Gamma_n(j, j')\|_{\text{op}} \lesssim \varepsilon \|v_1 - v_2\|_{\mu + \rho(b)}.
\]
(8.40)
Furthermore, since this case we have $|\ell|, |j - j'| \leq N_n$ and $\min\{|j|, |j'|\} \leq N_n^\gamma$ one has that
\[
(\ell^t |j|^t |j'|^t) \lesssim N_n^{2 \gamma + 2}.
\]
Hence, for $\lambda = (\omega, \zeta) \in \Omega_{n+1}(v_1)$, one has $\|L_n(\ell, j, j'; v_1)^{-1}\|_{\text{op}} \lesssim N_n^{2 \gamma + 2} \gamma^{-1}$, which, together with (8.40), implies that
\[
\|L_n(\ell, j, j'; v_1)^{-1} \Gamma_n(j, j')\|_{\text{op}} \leq C N_n^{2 \gamma + 2} \gamma^{-1} \|v_1 - v_2\|_{\mu + \rho(b)}.
\]
(8.41)
for some $C > 0$. If $C N_n^{2 \gamma + 2} \gamma^{-1} \|v_1 - v_2\|_{\mu + \rho(b)} \leq \rho \gamma^{-1}$, then, by Neumann series, $L_n(\ell, j, j'; v_2)$ is invertible and
\[
\|L_n(\ell, j, j'; v_2)^{-1}\|_{\text{op}} \leq \frac{(\ell^t |j|^t |j'|^t)}{\gamma - \rho}.
\]
(8.42)
Thus, (8.38), (8.42) imply that $\lambda = (\omega, \zeta) \in \Omega_{n+1}(v_2)$, and the proof is concluded. □

### 8.1 Convergence

**Lemma 8.4.** For any $j \in \mathbb{Z}^3 \setminus \{0\}$, the sequence $(\mathcal{N}_n)^j_j = i \zeta \cdot j \mathbf{1} + (\mathcal{Q}_n)^j_j, n \in \mathbb{N}$, converges in the norm $\|\cdot\|_{\text{HS}_\gamma}$ to some limit
\[
(\mathcal{N}_\infty)^j_j = i \zeta \cdot j \mathbf{1} + (\mathcal{Q}_\infty)^j_j \in \text{Mat}_{3 \times 3}(\mathbb{C}),
\]
and
\[
\|(\mathcal{N}_n)^j_j - (\mathcal{N}_\infty)^j_j\|_{\text{HS}_\gamma} \lesssim \varepsilon \|v\|_{\mu + \rho(b)}^b \gamma \lesssim \varepsilon \|v\|_{\mu + \rho(b)}^b \gamma \lesssim \varepsilon \|v\|_{\mu + \rho(b)}^b \gamma.
\]
(8.43)
Moreover the $3 \times 3$ block diagonal operator $\mathcal{Q}_\infty := \text{diag}_{j \in \mathbb{Z}^3 \setminus \{0\}}(\mathcal{Q}_\infty)^j_j$ is real and reversible.

**Proof.** The lemma follows in a standard way by the estimates (8.35), (8.6) and a telescoping argument. □

Now we define
\[
\Phi_n := \Phi_0 \circ \Phi_1 \circ \ldots \circ \Phi_n, \quad n \in \mathbb{N}.
\]
(8.45)
**Lemma 8.5.** Let $S > s_0$ and assume (8.4) and (8.1) with $\mu = \mu(b)$. For any $s_0 \leq s \leq S$, the sequence $\Phi_n, n \in \mathbb{N}$, converges in norm $\|\cdot\|_{\text{HS}_\gamma}$ to a real, reversibility-preserving and invertible map $\Phi_\infty$, with
\[
\|\Phi_{\infty}^{\pm 1} - \Phi_n^{\pm 1}\|_{\text{HS}_\gamma} \lesssim S N_n^{2} N_n^{2} \varepsilon \gamma^{-1} \|v\|_{\mu + \rho(b)}^b, \quad \|\Phi_{\infty}^{\pm 1} - \text{Id}\|_{\text{HS}_\gamma} \lesssim S N_n^{2} \varepsilon \gamma^{-1} \|v\|_{\mu + \rho(b)}^b.
\]
Proof. The lemma follows by \((8.11), \ (8.45)\), arguing as in Corollary 4.1 in [4].

For any \((\ell, j, j')\), \(\ell \in \mathbb{Z}^+, j, j' \in \mathbb{Z}^3 \setminus \{0\}\), we define
\[
L_\infty(\ell, j, j') := i\omega \cdot \ell + M_L((N_\infty)_j^{j'}) - M_R((N_\infty)_j^{j'})
\]  
and the set \(\Omega_\infty = \Omega_\infty(v)\) as
\[
\Omega_\infty := \left\{ \lambda \in DC(\gamma, \tau) : L_\infty(\ell, j, j'; \lambda) \text{ is invertible and } \|L_\infty(\ell, j, j'; \lambda)^{-1}\|_{\text{op}} \leq \frac{\langle \ell \rangle |j| |j'|^\tau}{2\gamma} \right\}
\]  
(8.46)

Lemma 8.6. One has \(\Omega_\infty \subseteq \cap_{n \geq 0} \Omega_n\).

Proof. We prove by induction that \(\Omega_\infty \subseteq \Omega_n\) for any integer \(n \geq 0\). For \(n = 0\) the statement is trivial since \(\Omega_0 := DC(\gamma, \tau)\) (see Proposition 8.2). Now assume that \(\Omega_\infty \subseteq \Omega_n\) for some \(n \geq 0\) and let us show that \(\Omega_\infty \subseteq \Omega_n\). Let \(\lambda \in \Omega_\infty\), \(\ell \in \mathbb{Z}^+, j, j' \in \mathbb{Z}^3 \setminus \{0\}\), \((\ell, j, j') \neq (0, j, j)\) and \(|\ell|, |j - j'| \leq N_n\). By using \((8.5), (8.8), (8.11)\), we write
\[
L_n(\ell, j, j') = L_\infty(\ell, j, j') + L_n(\ell, j, j') - L_\infty(\ell, j, j') = L_\infty(\ell, j, j')(\text{Id} + S_n(\ell, j, j'))
\]
\[
S_n(\ell, j, j') := L_\infty(\ell, j, j')^{-1} \left( M_L((Q_n - Q_\infty)_j^{j'}) - M_R((Q_n - Q_\infty)_j^{j'}) \right) .
\]  
(8.48)

By \((8.3), (8.44), (8.45)\), using \((3.1)\), one obtains that
\[
\|S_n(\ell, j, j')\|_{\text{op}} \lesssim \varepsilon^{-1} \left( \frac{\langle \ell \rangle |j| |j'|^\tau}{|j|^\tau} + \frac{\langle \ell \rangle |j| |j'|^\tau}{|j'|^\tau} \right) N_n^{-a}.
\]
By the triangular inequality, using that \(|\ell|, |j - j'| \leq N_n\) and recalling that by \((8.8), M > 2\tau\), one obtains the bound
\[
\frac{\langle \ell \rangle |j| |j'|^\tau}{|j|^\tau} \lesssim N_n^{2\tau}
\]
implying that (see \((8.3)\))
\[
\|S_n(\ell, j, j')\|_{\text{op}} \lesssim \varepsilon^{-1} N_n^{2\tau} N_n^{-a} \lesssim \varepsilon^{-1} N_n^{2\tau} .
\]  
(8.50)
Hence by \((8.48), (8.50)\), taking \(\varepsilon^{-1}\) small enough, the operator \(L_n(\ell, j, j')\) is invertible by Neumann series, with \(\|L_n(\ell, j, j'; \lambda)^{-1}\|_{\text{op}} \leq \gamma^{-1} \langle \ell \rangle |j| |j'|^\tau\). This shows that \(\lambda \in \Omega_n\), and the proof is complete.

Lemma 8.7. For any \(\lambda \in \Omega_\infty\), one has
\[
\Phi^{-1}_\infty L_0 \Phi_\infty = L_\infty := \omega \cdot \partial_x + N_\infty
\]
where the operator \(L_0\) is given in \((7.3)\) and the \(3 \times 3\) block-diagonal operator \(N_\infty\) in Lemma 8.7. Furthermore, the operator \(L_\infty\) is real and reversible.

Proof. By Lemma 8.6 and Proposition 8.2 one has \(\Phi^{-1}_n L_0 \Phi_n = \omega \cdot \partial_x + N_n + R_n\) for all \(n \geq 0\). The claimed statement then follows by passing to the limit as \(n \to \infty\), by using \((8.7)\) and Lemmata 8.4 and 8.5. 

8.2 Inversion of the operator \(L\).

By Lemmata 7.1, 8.7 on the set \(\Omega_n\), one has
\[
L = W_\infty L_0 W_\infty^{-1}, \quad W_\infty := \mathcal{E}_\perp \Phi_\infty .
\]  
(8.51)

Lemma 8.8. Let \(S > s_0, \tau > 0, \gamma \in (0, 1)\) and assume \((3.1)\) with \(\mu = \mu(b)\) (see \((8.3)\)). Then there exists \(\delta = \delta(S, k_0, \tau) \in (0, 1)\) small enough such that, if \(\varepsilon^{-1} \leq \delta\), then the maps \(W_\infty^{k_0} : H_0^S \to H_0^S\), \(s_0 \leq s \leq S\) are real and reversibility preserving and they satisfy the estimate
\[
\|W_\infty^{k_0} h\|_{s, \gamma} \lesssim_s \|h\|_{s_0, \gamma} + \|h\|_{s_0, \mu(b)} \|h\|_{s_0, \gamma} \quad \forall s_0 \leq s \leq S .
\]
We define the set $\Lambda_\infty := \{ \lambda = (\omega, \zeta) \in \mathbb{R}^{v+3} : \| (\omega \cdot \ell \text{Id} + (N_\infty)^j)^{-1} \|_{op} \leq \frac{\| \ell \| |j|}{2 \gamma} \forall (\ell, j) \in \mathbb{Z}^v \times (\mathbb{Z}^3 \setminus \{0\}) \}$.

In the next Lemma we construct a right inverse for the operator $\mathcal{L}$.

**Proof.** Let $\lambda = (\omega, \zeta) \in \Lambda_\infty$. Then we obtain a solution $g(\lambda) := \mathcal{L}^{-1}(\lambda)h(\lambda) \in H^s_0 \cap Y$ of the equation $\mathcal{L}_\infty \lambda = h$. Furthermore, the inverse operator admits an extension to the whole parameter space $\mathbb{R}^{v+3}$ (which we denote in the same way) satisfying the tame estimate

$$\| \mathcal{L}_\infty^{-1} h \|_{k_0, \gamma} \lesssim \gamma^{-1} \| h \|_{k_0, \gamma}.$$ 

**Proof.** Let $\lambda = (\omega, \zeta) \in \Lambda_\infty$. Then the equation $\mathcal{L}_\infty g = h$ admits the solution $g(x, y) = \sum_{(\ell, j) \in \mathbb{Z}^v \times (\mathbb{Z}^3 \setminus \{0\})} B(\ell, j) \hat{h}(\ell, j) e^{i(\ell \cdot \varphi + j \cdot x)},$ 

$$B(\ell, j) := (\omega \cdot \ell \text{Id} + (N_\infty)^j)^{-1} \in \text{Mat}_{3 \times 3}(\mathbb{C}), \quad (\ell, j) \in \mathbb{Z}^v \times (\mathbb{Z}^3 \setminus \{0\}).$$

Arguing as in the proof of Lemma 8.9 (see in particular [8.23]–[8.24]) one constructs an extension of $B(\ell, j; \lambda)$ that we denote by $(B(\ell, j; \lambda))_{ext}$ defined for any $\lambda \in \mathbb{R}^{v+3}$ and satisfying the estimate

$$\| (B(\ell, j))_{ext} \|_{k_0, \gamma} \lesssim \gamma^{-1} |\ell|^{-\sigma} \| h \|_{k_0, \gamma}.$$ 

The claimed estimate then follows by estimating $|\partial_{\lambda}^\beta [(B(\ell, j))_{ext} \hat{h}(\ell, j)]|$ for any $\beta \in \mathbb{N}^{v+3}$, $|\beta| \leq k_0$, arguing as in [8.26].

**Proposition 8.10.** Let $S > s_0, \tau > 0, \gamma \in (0, 1)$. Then there exists $T = T(k_0, \tau, \gamma) > \mu(\nu)$ (see [8.3]), $\delta = \delta(S, k_0, \tau) \in (0, 1)$ such that if [8.11] holds with $\mu_0 \geq s_0 + T$ and $\varepsilon^{-1} \leq \delta$, for any $\lambda = (\omega, \zeta) \in \Omega_\infty \cap \Lambda_\infty$, any $s_0 \leq s \leq S$, any $h(\lambda) \in H^{s+\tau}_0 \cap X$ there exists a solution $g := \mathcal{L}^{-1} h \in H^s_0 \cap Y$ of the equation $\mathcal{L} g = h$. Moreover $\mathcal{L}^{-1}$ admits an extension to the whole parameter space $\mathbb{R}^{v+3}$ (which we denote in the same way) that satisfies the tame estimates

$$\| \mathcal{L}^{-1} h \|_{k_0, \gamma} \lesssim \gamma^{-1} (\| h \|_{s+\tau, \gamma} + \| v \|_{k_0, \gamma} \| h \|_{s+\tau, \gamma}), \quad s_0 \leq s \leq S.$$ 

**Proof.** The proposition is a straightforward consequence of formula (8.51) and Lemmata 8.8, 8.9.

## 9 The Nash-Moser iteration

In this section we construct the solution of the equation $\mathcal{F}(v) = 0$ (see [1.11]) by means of a Nash Moser nonlinear iteration. We denote by $\Pi_n$ the orthogonal projector $\Pi_{N_n}$ (see [2.4]) on the finite dimensional space

$$\mathcal{H}_n := \{ v \in C^0(\mathbb{T}^{v+3}, \mathbb{R}^3) : \ v = \Pi_n v \},$$

and $\Pi_n^\perp := \text{Id} - \Pi_n$. The projectors $\Pi_n, \Pi_n^\perp$ satisfy the usual smoothing properties in Lemma 2.2 namely

$$\| \Pi_n v \|_{k_0, \gamma} \leq N_n v \| v \|_{s+\gamma}, \quad \| \Pi_n^\perp v \|_{k_0, \gamma} \leq N_{n-1} v \| v \|_{s+\gamma}, \quad s, b \geq 0.$$ 

We define the constants

$$\kappa := \frac{2}{3} + 7, \quad \alpha := \max \{ 6 + 3, 2 + 2 \gamma + 2 \gamma + 2 \gamma + 2 \gamma + 2 \gamma \}, \quad \beta := \frac{2}{3} + 3 + 3 \gamma + 3 \gamma + 3 \gamma + 3 \gamma \quad \gamma \in (0, 1),$$

where $T = T(k_0, \tau, \gamma) > 0$ is given in Proposition 8.10 and $\tau_0$ is defined in [8.3]. For $N_0 > 0$ we define $N_n := N_0^\gamma, \chi := 3/2, N_{-1} := 1$. We also fix the regularity of the forcing term $f$ in (1.3) as

$$f \in C^q(\mathbb{T}^{v+3} \times \mathbb{R}^3), \quad q := s_0 + b_1 + 1.$$
Remark 9.1 (Choice of the constants). The conditions \( \kappa \geq 6\tau + 7, a_1 \geq 6\tau + 13, b_1 \geq 3\tau + a_1 + 3 \) in (9.2) allow to prove the induction estimates in \((P2)_n, (P3)_n\) in Proposition 9.2. The extra conditions \( a_1 \geq \max\{3(\tau + \gamma + 1), 3\tau + 2\} \) are used in Section 14 for the measure estimates (Lemma 10.2).

**Proposition 9.2. (Nash-Moser)** Let \( \tau > 0 \), and let \( \Omega \) be a bounded open subset of \( \mathbb{R}^{\nu + 3} \). There exist \( \delta \in (0, 1), C_* > 0, \tau(k_0, \tau, \nu) > 0 \) such that if

\[
N_0^\tau \leq \delta, \quad N_0 := \gamma^{-1}
\]

then the following properties hold for all \( n \geq 0 \).

\((P1)_n\) There exists \( y_0 : \mathbb{R}^{\nu + 3} \to H^{\nu + b_1} \cap X \), with \( y_0 := F(v_0) \), satisfying

\[
\| y_n \|_{L^\infty} \leq C_* N_n^{-a_1}\n.
\]

There exists \( v_n : \mathbb{R}^{\nu + 3} \to \mathcal{H}_{n-1} \cap Y \), with \( v_0 := 0 \), satisfying

\[
\| v_n \|_{L^\infty} \leq 1.
\]

If \( n \geq 1 \), the difference \( h_n := v_n - v_{n-1} \) satisfies \( \| h_1 \|_{L^\infty} \leq \varepsilon^{-1} \) and

\[
\| h_n \|_{L^\infty} \leq N_n^{-\varepsilon^{-1}}.
\]

\((P2)_n\) If \( n = 0 \) we define \( G_0 := \Omega \). If \( n \geq 1 \), we define

\[
G_{n+1} := G_n \cap (\Omega^\gamma_n(v_n) \cap \Lambda^\gamma_n(v_n)),
\]

where \( \gamma_n := \gamma(1 + 2^{-n}) \) and the sets \( \Omega^\gamma_n(v_n), \Lambda^\gamma_n(v_n) \) are defined in (9.3), (8.47). The function \( F(v_n) \) is defined for all \( \lambda \in \mathbb{R}^{\nu + 3} \) and, for every \( \lambda \in G_n \), it coincides with \( y_n \).

\((P3)_n\) One has \( y_0 := \| y_n \|_{L^\infty} + \varepsilon(\| v_n \|_{L^\infty} + 1) \leq C_* N_n^\varepsilon \).

**Proof.** To use the norms \( \| \|_{L^\infty} \) (where the sup is over \( \lambda = (\omega, \zeta) \in \mathbb{R}^{\nu + 3} \), see Definition 2.1) and to employ the assumption that the set \( \tilde{\Omega} \subset \mathbb{R}^{\nu + 3} \) is bounded, we fix a \( C^\infty \) function \( \rho : \mathbb{R}^{\nu + 3} \to \mathbb{R} \) with compact support such that \( \rho(\lambda) = 1 \) for all \( \lambda \in (\omega, \zeta) \) in a neighborhood of the closure of \( \Omega \). Thus

\[
\rho v \|_{L^\infty} \leq \rho(\lambda) (\omega \cdot \partial_x v + \zeta \cdot \nabla v) \|_{L^\infty} \leq \rho(x) \|_{L^\infty},
\]

for all \( s \geq 0 \), all \( v = v(\varphi, x, \lambda) \).

**Proof of \((P1, 2, 3)_0\).** By (1.17), \( y_0 = F(0) = \varepsilon F(\varphi, x), \| F(0) \|_{L^\infty} \leq \varepsilon \| F \|_s \leq \varepsilon \| f \|_{s+1} \rangle \), then take \( C_* \geq 1 + \| f \|_{s+1+1} \) (in (9.3), we have fixed \( q = s_0 + b_1 + 1 \)).

Assume that \((P1, 2, 3)_n \), hold for some \( n \geq 0 \), and prove \((P1, 2, 3)_{n+1}\). By (P1)_n, one has \( \| v_n \|_{L^\infty} \leq 1 \), and the assumption (9.4) implies the smallness condition \( \varepsilon^{-1} \leq \delta \) of Proposition 8.10 by taking \( \tau(k_0, \tau, \nu) \) large enough and \( S = s_0 + b_1 \). Then Proposition 8.10 applies to the linearized operator

\[
\mathcal{L}_n := dF(v_n).
\]

This implies that there exists a linear operator \( \mathcal{L}_n^{-1} \), defined for any \( \lambda \in \mathbb{R}^{\nu + 3} \), which satisfies the tame estimate

\[
\| \mathcal{L}_n^{-1} h \|_{L^\infty} \leq \gamma^{-1} \| h \|_{L^\infty},
\]

such that for all \( \lambda \in G_{n+1} = G_n \cap \Omega^\gamma_n(v_n) \cap \Lambda^\gamma_n(v_n) \) one has \( \mathcal{L}_n \mathcal{L}_n^{-1} = \text{Id} \) (using the bound \( \gamma \leq \gamma(1 + 2^{-n}) \in [\gamma, 2\gamma] \)). Specializing (9.11) for \( s = s_0 \), using (9.6), one has

\[
\| \mathcal{L}_n^{-1} h \|_{L^\infty} \leq \gamma^{-1} \| h \|_{L^\infty}.
\]
We define the successive approximation
\[ v_{n+1} := v_n + h_{n+1}, \quad h_{n+1} := -\Pi_n \mathcal{L}_n^{-1} \Pi_n y_n \in \mathcal{H}, \tag{9.13} \]
and Taylor’s remainder \( Q_n := \mathcal{F}(v_{n+1}) - \mathcal{F}(v_n) - \mathcal{L}_n h_{n+1} \). By the definition of \( h_{n+1} \) in (9.13), splitting \( \mathcal{F}(v_n) = \Pi_n \mathcal{F}(v_n) + \Pi_n^\perp \mathcal{F}(v_n) \) and \( \mathcal{L}_n \Pi_n = \mathcal{L}_n - \mathcal{L}_n \Pi_n^\perp \), and multiplying each term by the factor \( \rho(\lambda) \), we calculate
\[ \rho \mathcal{F}(v_{n+1}) = \rho(\mathcal{F}(v_n) + \mathcal{L}_n h_{n+1} + Q_n) = y_{n+1} + z_{n+1} \tag{9.14} \]
where
\[ y_{n+1} := \rho \Pi_n^\perp \mathcal{F}(v_n) + \rho \mathcal{L}_n \Pi_n^\perp \mathcal{L}_n \Pi_n y_n + \rho Q_n, \tag{9.15} \]
and
\[ z_{n+1} := \rho \Pi_n [\mathcal{F}(v_n) - y_n] + \rho (\Id - \mathcal{L}_n \Pi_n^\perp) \Pi_n y_n. \]

Note that \( h_{n+1}, v_{n+1}, y_{n+1}, z_{n+1} \) are defined for all \( \lambda \in \mathbb{R}^{r+3} \), and \( k_0 \) times differentiable in \( \lambda \in \mathbb{R}^{r+3} \).

Let us estimate \( y_{n+1} \). Since \( v_n \in \mathcal{H}_{n-1} \), one has \( \Pi_n^\perp (\omega \cdot \partial_\omega v_n + \zeta \cdot \nabla v_n) = 0 \). Thus, by the definition of \( \mathcal{F} \) in (1.17), the product estimate (2.7), the smoothing property (9.1) (since \( v_n \in \mathcal{H}_{n-1} \)), and the induction estimate (9.6) (since \( \|v_n\|_{s_0+1} \leq \|v_n\|_{s_0} \leq 1 \)), and the assumption \( \|F\|_{s_0+1} \leq 1 \), one has
\[ \|\Pi_n^\perp \mathcal{F}(v_n)\|_{s_0} \leq \varepsilon (\|v_n\|_{s_0})^2 + \|F\|_{s_0+1} \leq \varepsilon N_{n-1} (\|v_n\|_{s_0+1}^2 + 1), \]
\[ \|\Pi_n^\perp \mathcal{F}(v_n)\|_{s_0} \leq \varepsilon N_{n-1} (\|v_n\|_{s_0+1}^2 + 1), \]
\[ \|\Pi_n^\perp \mathcal{F}(v_n)\|_{s_0} \leq \varepsilon N_{n-1} (\|v_n\|_{s_0+1}^2 + 1). \]
By (9.17), using (9.1), (9.11), \( \gamma^{-1} = N_0 \leq N_1 \), and (9.6), we estimate
\[ \|\rho \mathcal{L}_n \Pi_n^\perp \mathcal{L}_n \Pi_n y_n\|_{s_0} \leq \|\Pi_n^\perp \mathcal{L}_n \Pi_n y_n\|_{s_0} \leq N_{n-1} (\|v_n\|_{s_0+1}^2 + 1), \]
\[ \|\rho \mathcal{L}_n \Pi_n^\perp \mathcal{L}_n \Pi_n y_n\|_{s_0} \leq N_{n-1} (\|v_n\|_{s_0+1}^2 + 1), \]
\[ \|\rho \mathcal{L}_n \Pi_n^\perp \mathcal{L}_n \Pi_n y_n\|_{s_0} \leq N_{n-1} (\|v_n\|_{s_0+1}^2 + 1). \]
Since the non linear part of \( \mathcal{F} \) is quadratic (see (1.17)), \( Q_n \) is a quadratic operator of \( h_{n+1} \), independent of \( v_n \), with
\[ \|Q_n\|_{s_0} \leq \varepsilon \|h_{n+1}\|_{s_0}^{s_0} \|h_{n+1}\|_{s_0}^{s_0}, \quad s \geq s_0. \tag{9.19} \]
To estimate \( h_{n+1} \) we use (9.13), (9.11), (9.1), (9.6), and obtain
\[ \|h_{n+1}\|_{s_0} \leq N_{n+1}^{1+\frac{s_0}{2}} \|y_n\|_{s_0}, \quad \|h_{n+1}\|_{s_0} \leq N_{n+1}^{1+\frac{s_0}{2}} \|y_n\|_{s_0}, \tag{9.20} \]
By (9.19), (9.20), (9.1), we get
\[ \|Q_n\|_{s_0} \leq \varepsilon N_{n+1}^{1+\frac{s_0}{2}} \|y_n\|_{s_0}, \quad \|Q_n\|_{s_0} \leq \varepsilon N_{n+1}^{1+\frac{s_0}{2}} \|y_n\|_{s_0}, \tag{9.21} \]
By the definition (9.13) of \( y_{n+1} \), the first property in (9.9), the estimates (9.16), (9.18), (9.21), and the bound \( \|y_n\|_{s_0} \leq \varepsilon \) (which follows from (9.6)), we obtain
\[ \|y_{n+1}\|_{s_0} \leq N_{n+1}^{1+\frac{s_0}{2}} \|q_n\|, \quad \|y_{n+1}\|_{s_0} \leq N_{n+1}^{1+\frac{s_0}{2}} \|q_n\|, \tag{9.22} \]
where \( q_n \) is defined in (P3)_n.

**Proof of (P3)_{n+1}**. By the inductive assumption (P3)_n, \( \varepsilon \|v_n\|_{s_0+1} \leq q_n \). Hence, by (9.13), (9.20), and \( \varepsilon^{-1} \leq 1 \), one has
\[ \varepsilon \|v_{n+1}\|_{s_0+1} \leq \varepsilon \|v_n\|_{s_0+1} + \varepsilon \|h_{n+1}\|_{s_0+1} \leq N_{n+1}^{1+\frac{s_0}{2}} q_n. \tag{9.23} \]
Thus (9.22), (9.23) give
\[ q_{n+1} \leq C N_{n+1}^{1+\frac{s_0}{2}} q_n. \tag{9.24} \]
for some constant $C$. The estimate (9.24) and (P3), imply (P3)$_{n+1}$ for $\kappa > 6 + 6\tau$ and $N_0$ sufficiently large (depending on $\kappa, \tau$ and $C$ in (9.24)).

**Proof of (P1)$_{n+1}$.** The estimate (9.22) and the inductive assumption (9.5) imply (9.5) at the step $n + 1$ for $b_1 > 2 + 2\tau + \kappa + a_1$, $a_1 \geq 9 + 6\tau$, $N_0$ sufficiently large (depending on $a_1, b_1, \tau, \kappa$ and on the implicit constant in (9.22)) and $\varepsilon$ sufficiently small (depending also on $N_0$). By (9.20), (9.1), (9.5) we deduce (9.7) at the step $n + 1$ and, by telescoping series, (9.6) at the step $n + 1$. The estimate for $h_1$ follows from (9.13) using that $v_0 = 0$, $y_0 = F(v_0) = \varepsilon F(\varphi, x)$, and $\|F\|_{n+1} \leq \|F\|_{n+b_1} \leq 1$.

**Proof of (P2)$_{n+1}$.** For $\lambda \in \mathcal{G}_{n+1}$ one has $\rho(\lambda) = 1$, $L_n(L^{-1}_n = \text{Id}$ and, by inductive assumption, $F(v_n) = y_n$. Hence $z_{n+1}$ in (9.13) is zero, and, by (9.14), $F(v_{n+1}) = y_{n+1}$.

## 10 Measure estimate

In this section we prove that the set
\[
\mathcal{G}_\infty := \cap_{n \geq 0} \mathcal{G}_n
\] (10.1)
has large Lebesgue measure. We estimate the measure of its complement $\Omega \setminus \mathcal{G}_\infty$. The main result of this section is the following proposition.

**Proposition 10.1.** Let
\[
\tau := 9\max\{\nu, 3\} + 1, \quad k_0 := 11.
\] (10.2)
Then $|\Omega \setminus \mathcal{G}_\infty| \leq \gamma$.

The rest of this section is devoted to the proof of Proposition 10.1. By the definition (10.1), one has
\[
\Omega \setminus \mathcal{G}_\infty \subseteq \cup_{n \geq 0} (\mathcal{G}_n \setminus \mathcal{G}_{n+1}),
\] (10.3)
hence it is enough to estimate the measure of $\mathcal{G}_n \setminus \mathcal{G}_{n+1}$. By (9.8) and using elementary properties of set theory, one has that
\[
\mathcal{G}_n \setminus \mathcal{G}_{n+1} \subseteq (\mathcal{G}_n \setminus \Omega^\infty_n(v_n)) \cup (\mathcal{G}_n \setminus \Lambda^\infty_n(v_n)).
\] (10.4)
Note that $\mathcal{G}_{n+1} \subseteq DC(\gamma_n, \tau)$ for any $n \geq 0$ because $\mathcal{G}_{n+1} \subseteq \Omega^\infty_n(v_n) \subseteq DC(\gamma_n, \tau)$ by (9.3), (8.47).

**Proposition 10.2.** For any $n \geq 0$, the following estimates hold.
(i) $\mathcal{G}_n \setminus \Omega^\infty_n(v_n) \lesssim \gamma$ and, for any $n \geq 1$, $|\mathcal{G}_n \setminus \Omega^\infty_n(v_n)| \lesssim \gamma N_n^{-\frac{1}{2}}$.
(ii) $\mathcal{G}_n \setminus \Lambda^\infty_n(v_n) \lesssim \gamma$ and for any $n \geq 1$, $|\mathcal{G}_n \setminus \Lambda^\infty_n(v_n)| \lesssim \gamma N_n^{-\frac{1}{2}}$.

As a consequence $|\mathcal{G}_0 \setminus \mathcal{G}_1| \lesssim \gamma$ and for any $n \geq 1$, $|\mathcal{G}_n \setminus \mathcal{G}_{n+1}| \lesssim \gamma N_n^{-\frac{1}{2}}$.

Propositions 10.1 and 10.2 are proved at the end of this section. Now we estimate the measure of the set $\mathcal{G}_n \setminus \Omega^\infty_n(v_n)$. The estimate of the measure of $\mathcal{G}_n \setminus \Lambda^\infty_n(v_n)$ can be done arguing similarly (it is actually even easier). By the definitions (9.8), (8.47), one gets that
\[
\mathcal{G}_n \setminus \Omega^\infty_n(v_n) \subseteq \bigcup_{(\ell, j, j') \in \mathcal{I}} \mathcal{R}_{\ell, j, j'}(v_n)
\] (10.5)
where
\[
\mathcal{I} := \{ (\ell, j, j') \in \mathbb{Z}^3 \times (\mathbb{Z}^3 \setminus \{0\}) \times (\mathbb{Z}^3 \setminus \{0\}) : (\ell, j, j') \neq (0, j, j) \}
\] (10.6)
and
\[
\mathcal{R}_{\ell, j, j'}(v_n) := \{ \lambda = (\omega, \zeta) \in \mathcal{G}_n : L_{\infty}(\ell, j, j'; \lambda, v_n(\lambda)) \text{ is not invertible, or it is invertible and} \}
\[
\|L_{\infty}(\ell, j, j'; \lambda, v_n(\lambda))^{-1}\|_{op} > \frac{(\ell)^{-1}}{2m} \}
\] (10.7)
In the next lemma, we estimate the measure of the resonant sets $\mathcal{R}_{\ell, j, j'}(v_n)$. 

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Lemma 10.3. For any $n \geq 0$, one has that $|R_{\ell j'}(v_n)| \lesssim \gamma_{\beta, 9}^{\gamma}(\ell, j - j')^{\gamma} \rho^{3} j^{\gamma} j'^{\gamma} |j|^{-\gamma}$.  

Proof. Recalling (10.8), (10.4), for $(\ell, j, j') \in I$ (see (10.6)), we write $L_\infty(\ell, j, j'; \lambda) \equiv L_\infty(\ell, j, j'; \lambda, v_n(\lambda))$, $\lambda = (\omega, \zeta) \in \Omega$ as

$$L_\infty(\ell, j, j'; \lambda) = \lambda \cdot k \text{Id} + Q(\lambda) : \text{Mat}_{3 \times 3}(\mathbb{C}) \to \text{Mat}_{3 \times 3}(\mathbb{C})$$

where

$$k := (\ell, j - j'), \quad \lambda = (\omega, \zeta) \in \Omega, \quad Q(\lambda) := ML(\langle Q(\lambda, v_n(\lambda)) \rangle_j' - MR(\langle Q(\lambda, v_n(\lambda)) \rangle_j').$$

(10.8)

Since $(\ell, j, j') \in I$, one has $k = (\ell, j - j') \neq (0, 0)$. Recalling the definition (10.7), the resonant set $R_{\ell j'}(v_n)$ is equal to the set $R_A$ defined in (10.2), with $d := 9$, $A := L_\infty(\ell, j, j')$, $\eta = \frac{|\ell| |j| j'| \gamma}{2 \gamma_n}$, and $k_0 := d + 2 = 11$.

By the definitions (10.8) and the estimates (8.2), (8.4), (8.4), one obtains that $\|Q\|_{\text{op}} \lesssim \varepsilon$, hence, by the smallness condition (9.4) (by choosing $\tau$ large enough) one can apply Lemma A.1 obtaining that $|R_{\ell j'}(v_n)| \lesssim \gamma_{\beta, 9}^{\gamma}(\ell, j - j')^{\gamma} \rho^{3} j^{\gamma} j'^{\gamma} |j|^{-\gamma}$, then $\gamma_n \lesssim \gamma$, and we get the claimed estimate. $\square$

Lemma 10.4. Let $(\ell, j, j') \in I$, $(\ell, j - j') \leq N_n$, $\min \{j, |j'| \} \geq N_n^{\gamma}$. Then $R_{\ell j'}(v_n) = \emptyset$.

Proof. Let $\lambda = (\omega, \zeta) \in G_n$. Recalling (10.8), (10.4), one writes

$$L_\infty(\ell, j, j'; v_n) = \Omega(\ell, j, j') \text{Id} + \Delta_n(\ell, j, j'), \quad \Omega(\ell, j, j') := (\omega \cdot \ell + \zeta \cdot (j - j')) \text{Id},$$

$$\Delta_n(\ell, j, j') := \Omega(\ell, j, j')^{-1} \left( ML(\langle Q(\lambda, v_n(\lambda)) \rangle_j' - MR(\langle Q(\lambda, v_n(\lambda)) \rangle_j').$$

Note that since $\lambda \in DC(\gamma_n, \tau)$ (see (4.7)) one has that $\Omega(\ell, j, j')$ is invertible and

$$\|\Omega(\ell, j, j')^{-1}\|_{\text{op}} \leq \frac{(\ell, j - j')^{\gamma}}{C_0 \gamma_n} \lesssim \frac{N_n^{\gamma}}{C_0 \gamma_n}.$$ 

(10.9)

The latter estimate, together with (8.2), (8.4), (9.6), since $\min \{j, |j'| \} \geq N_n^{\gamma}$ and $\gamma_n = \gamma(1 + 2^{-n})$, implies that

$$\|\Delta_n(\ell, j, j')\|_{\text{op}} \leq C \varepsilon \gamma^{-1}$$

(10.10)

for some constant $C > 0$. Then for $\varepsilon \gamma^{-1}$ small enough, by Neumann series, $L_\infty(\ell, j, j'; v_n)$ is invertible and

$$\|L_\infty(\ell, j, j'; v_n)^{-1}\|_{\text{op}} \leq \frac{2(\ell, j - j')^{\gamma}}{C_0 \gamma_n} \lesssim \frac{2C(\tau)(\ell)^{\gamma} |j|^{\gamma} |j'|^{\gamma}}{C_0 \gamma_n} \leq \frac{(\ell)^{\gamma} |j|^{\gamma} |j'|^{\gamma}}{2 \gamma_n}$$

by choosing $C_0 = C(\tau) > 0$ large enough. This implies that $R_{\ell j'}(v_n) = \emptyset$. $\square$

Lemma 10.5. Let $(\ell, j, j') \in I$, $(\ell, j - j') \leq N_n$, $\min \{j, |j'| \} \leq N_n^{\gamma}$, $n \geq 1$. Then

$$R_{\ell j'}(v_n) \subseteq R_{\ell j'}(v_{n-1}).$$

Proof. We split the proof in two steps.

Step 1. We show that for $(\ell, j, j') \in I$, $|\ell|, |j - j'| \leq N_n$, for any $\lambda \in G_n$ one has

$$\|L_\infty(\ell, j, j'; v_n) - L_\infty(\ell, j, j'; v_{n-1})\|_{\text{op}} \lesssim N_n^{\gamma} N_{n-1}^{-a} \varepsilon \gamma^{-1} + \varepsilon N_n^{-a} (10.11)$$

The constant $a$ is defined in (8.3). By (10.8), (10.4), by the property (8.2), one computes

$$\|L_\infty(\ell, j, j'; v_n) - L_\infty(\ell, j, j'; v_{n-1})\|_{\text{op}} \lesssim \sup_{j \in \mathbb{Z} \setminus \{0\}} \|Q(\lambda, v_n) - Q(\lambda, v_{n-1})\|_{\text{op}}.$$ 

(10.12)

By the inductive definition of the sets $G_n$ in (9.8), recalling the definitions (8.4), (8.47) and Lemma 8.6 one gets the inclusion

$$G_n \subseteq \Omega_n^{\gamma_{n-1}}(v_{n-1}).$$

(10.13)
We apply Proposition \(S.2\{(S3)\}_n\) with \(v_1 \equiv v_{n-1}, v_2 \equiv v_n, \rho \equiv \gamma_{n-1} - \gamma_n = \gamma 2^{-n}\). By (10.7), one has

\[
N_0^{-1} \varepsilon \|v_n - v_{n-1}\|_{s_0 + \tau} \leq CN_{-1}^{r_0 + 2\tau} |N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1} \leq \gamma_n - \gamma_n
\]  

(10.14)

for some constant \(C > 0\), since

\[
C N_{-1}^{r_0 + 2\tau} |N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1} \leq 1, \quad n \geq 1
\]

by recalling that \(\tau > \mu(b), a_1 > 3\pi + \frac{3}{2}(\tau + 1)\) (see (S3), Proposition 8.10, and (9.2)) and by the smallness condition (9.4), by choosing \(\tau\) large enough. The estimate (10.14) implies that Proposition \(S.2\{(S3)\}_n\) applies, implying that \(G_n \subseteq \Omega_n^{\gamma_n} (v_{n-1}) \subseteq \Omega_n^\gamma (v_n)\) (recall also (10.13)). Then, we can apply the estimate (8.13) of Proposition \(S.2\{(S2)\}_n\) (with \(v_1 \equiv v_{n-1}, v_2 \equiv v_n, \gamma_1 \equiv \gamma_n, \gamma_2 \equiv \gamma_n\)), implying that

\[
\sup_{j \in \mathbb{Z} \setminus \{0\}} \| (Q_n(v_n) - Q_n(v_{n-1}))^j \|_{HS} \lesssim \varepsilon \|v_n - v_{n-1}\|_{s_0 + \tau} \lesssim N_{-1}^{2\tau} N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1} \lambda \in G_n.
\]  

(10.15)

Hence, by triangular inequality and by (8.44), (10.14), (9.6), one has

\[
\sup_{j \in \mathbb{Z} \setminus \{0\}} \| (Q_{\infty}(v_n) - Q_{\infty}(v_{n-1}))^j \|_{HS} \leq \sup_{j \in \mathbb{Z} \setminus \{0\}} \| (Q_{\infty}(v_n) - Q_n(v_n))^j \|_{HS} + \sup_{j \in \mathbb{Z} \setminus \{0\}} \| (Q_n(v_n) - Q_n(v_{n-1}))^j \|_{HS}
\]

\[
+ \sup_{j \in \mathbb{Z} \setminus \{0\}} \| (Q_n(v_{n-1}) - Q_{\infty}(v_{n-1}))^j \|_{HS} \lesssim N_{-1}^{2\tau} N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1} + \varepsilon N_{-1}^{-a_1}.
\]

(10.16)

The estimate (10.11) then follows by (10.12), (10.16).

**Step 2.** Let \(\lambda \in G_n\) and \((\ell, j, j') \leq N_n, \min \{|j|, |j'|| \leq N_n^\tau, n \geq 1\). We write

\[
L_{\infty}(\ell, j, j'; v_n) = L_{\infty}(\ell, j, j'; v_{n-1}) (\text{Id} + \Delta_n(\ell, j, j'))
\]

\[
\Delta_n(\ell, j, j') := L_{\infty}(\ell, j, j'; v_{n-1})^{-1} \left( L_{\infty}(\ell, j, j'; v_n) - L_{\infty}(\ell, j, j'; v_{n-1}) \right).
\]

(10.17)

Since \(|j - j'| \leq N_n\) and \(\min \{|j|, |j'|| \leq N_n^\tau\), by triangular inequality one has \(\max \{|j|, |j'|| \leq N_n^\tau\). Therefore, using that \(\lambda \in G_n\) and the estimate (10.11), one gets

\[
\| \Delta_n(\ell, j, j') \|_{op} \lesssim N_{-1}^{2\tau} |N_{-1}^{2\tau} N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1} + N_{-1}^{2\tau} N_{-2}^{-a_1} \varepsilon 2^{-2} \gamma^{-1}.
\]  

(10.18)

Hence, by the smallness condition (9.4), using that \(a_1 > 3\pi + \frac{3}{2}(\tau + 2\varepsilon)\) (see (9.2)), \(a > \frac{7}{2}(\tau + 2\varepsilon)\) (see (S3)), by (10.17), (10.18), one gets that \(\| \Delta_n(\ell, j, j') \|_{op} \leq 2^{-n}\), so that \(L_{\infty}(\ell, j, j'; v_n)\) is invertible and \(\|L_{\infty}(\ell, j, j'; v_{n-1})^{-1} \|_{op} \leq (\ell)^{\alpha} |j|^{\beta} |j'|^{\gamma} \frac{1}{2^{\tau\alpha}}\), implying the claimed inclusion. □

**Lemma 10.6.** For any \(n \geq 1\), the following inclusion holds:

\[
G_n \setminus \Omega_n^{\gamma_n}(v_n) \subseteq \bigcup_{(\ell, j, j') \in \mathcal{I}} \mathcal{R}_{\ell jj'}(v_n).
\]  

(10.19)

**Proof.** Let \(n \geq 1\). By the definition (10.7), \(\mathcal{R}_{\ell jj'}(v_n) \subseteq G_n\). By Lemmata (10.4, 10.5) for any \((\ell, j, j') \in \mathcal{I}, (\ell, j, j') \leq N_n, \min \{|j|, |j'|| \geq N_n^\tau\) then \(\mathcal{R}_{\ell jj'}(v_n) = \emptyset\) and if \(\min \{|j|, |j'|| \leq N_n^\tau\) then \(\mathcal{R}_{\ell jj'}(v_n) \subseteq \mathcal{R}_{\ell jj'}(v_{n-1})\). On the other hand \(G_n \cap \mathcal{R}_{\ell jj'}(v_{n-1}) = \emptyset\) (see (9.8)), therefore (10.19) holds. □

By the inclusion (10.5) and by Lemma 10.3 one gets that

\[
|G_0 \setminus \Omega_n^{\gamma}(v_0) | \lesssim \sum_{(\ell, j, j') \in \mathcal{I}} \| \mathcal{R}_{\ell jj'}(v_0) \| \lesssim \gamma \sum_{(\ell, j, j') \in \mathcal{I} \times \Omega_n^{\gamma}(v_0)} \frac{1}{(\ell)^{\alpha} |j|^{\beta} |j'|^{\gamma}} \lesssim \gamma.
\]  

(10.20)

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Now let \( n \geq 1 \). By the inclusion (10.19) and by Lemma 10.3 one has
\[
\left| C_n \setminus \Omega_n^{\infty}(v_n) \right| \leq \sum_{(i,j,j') \in I} \left| R_{i,j,j'}(v_n) \right| \leq \gamma \sum_{(i,j,j') \in I} \frac{1}{\langle i, j - j' \rangle} \leq \gamma N_n^{-\frac{9}{2}}. \tag{10.21}
\]

**Proof of Propositions 10.1, 10.2** Proposition 10.2(i) follows by recalling formula (10.2) and by applying the estimates (10.20), (10.21). Item (ii) can be proved by similar arguments. The estimates on \( C_n \setminus C_{n+1} \) follow by recalling formula (10.13) and by applying items (i), (ii).

Proposition 10.1 follows by applying Proposition 10.2, using the inclusion (10.3) and the fact that the series \( \sum_{n \geq 1} N_n^{-1/9} \) converges.

**11 Proof of Theorems 1.2, 1.1**

**Proof of Theorem 1.2** Fix \( \gamma := \varepsilon^c \) with \( 0 < c < \frac{1}{3} \) where \( \Omega \) is the constant appearing in the smallness condition (9.3). Note that, since \( k_0 \) and \( r \) have been fixed in (10.2), the constant \( \Omega = \|v\|_{L^2} \) only depends on the number of frequencies \( \nu \). Then \( N_n^{-\frac{9}{2}} = \gamma^{-\frac{9}{2}} \varepsilon^{-\frac{1}{2}} \gamma^{-\frac{9}{2}} \), therefore the smallness condition (9.3) is fulfilled for \( \varepsilon \) small enough. By Proposition 10.2(P1), using a telescoping argument, the sequence \( (v_n)_{n \geq 0} \) converges to \( v_{\infty} \in H^{s+1,1} \cap Y \) with respect to the norm \( \| \cdot \|_{s+1,1} \) and
\[
\|v_{\infty}\|_{s+1,1} \lesssim \varepsilon^{-1}, \quad \|v_{\infty} - v_n\|_{s+1,1} \lesssim \varepsilon^{-1} N_{n-1}^{-2}. \tag{11.1}
\]

By recalling (10.1) and Proposition 10.2(P2), for any \( \lambda \in G_{\infty}, F(v_n) \to 0 \) as \( n \to \infty \), therefore, the estimate (11.1) implies that \( F(v_{\infty}) = 0 \) for any \( \lambda \in G_{\infty} \). By setting \( \Omega_c := G_{\infty} \), by applying Proposition 10.1 and using that \( \gamma = \varepsilon^c \), one gets that \( \lim_{\lambda \to 0} \|\Omega_c\| = |\Omega| \) and hence the proof is concluded.

**Proof of Theorem 1.1** Let \( v \in H^3 \) be the solution of the equation \( F(v) = 0 \) provided by Theorem 1.2 where \( F \) is defined in (11.4). Let \( u := \mathcal{U}(v) = \text{curl}(\Lambda^{-1}v) \). One has \( \text{div} u = 0 \), \( \text{div} F = \text{div} (\text{curl} f) = 0 \) (the divergence of any curl is zero). This implies that \( \text{div} F = 0 \) provided by Theorem 1.2, \( \|u\|_{s+1,1} \leq 1 \), for \( \varepsilon \) small enough we can apply Proposition 4.1 to the transport operator
\[
T = \omega \cdot \partial_x + \zeta \cdot \nabla + \varepsilon u(x) \cdot \nabla
\]
and obtain that there exists a reversibility preserving, invertible map \( A \) of the form (4.5) such that
\[
A^{-1}(\omega \cdot \partial_x + \zeta \cdot \nabla + \varepsilon u \cdot \nabla) = A = \omega \cdot \partial_x + \zeta \cdot \nabla, \quad \forall (\omega, \zeta) \in \Omega_c \subseteq DC(\gamma, \tau).
\]
Hence, by (11.2), \( \omega \cdot \partial_x + \zeta \cdot \nabla ) A^{-1} (\text{div} v) = 0 \). Since \( (\omega, \zeta) \) is Diophantine, the kernel of \( \omega \cdot \partial_x + \zeta \cdot \nabla \) is given by the constants, so that \( \mathcal{A}^{-1} (\text{div} v) \) is a constant, say \( \mathcal{A}^{-1} (\text{div} v) = c \). This implies that \( \text{div} v = c \), because \( \mathcal{A}^{-1} \) is a change of variable. By periodicity, the space average of any divergence is zero, and therefore \( \text{div} v = c = 0 \).

Since \( \text{div} u = 0 \), \( \text{div} v = 0 \), integrating by parts one deduces that \( u \cdot \nabla v \) and \( v \cdot \nabla u \) have both zero average in the space variable \( x \in T^1 \). Also \( F \) has zero average in \( x \) (because \( F \) is a curl). This implies that \( \Pi^0 (u \cdot \nabla v - v \cdot \nabla u - F) = u \cdot \nabla v - v \cdot \nabla u - F \), and then from the equation \( F(v) = 0 \) we deduce that
\[
\omega \cdot \partial_x v + \zeta \cdot \nabla v + \varepsilon (u \cdot \nabla v - v \cdot \nabla u - F) = 0.
\]
Moreover, by Theorem 1.2 \( v \in H^3 \) has zero space average, therefore \( \Lambda^{-1} v = (\Delta)^{-1} v \) (see (1.15)) and \( u = \text{curl} (\Delta)^{-1} v \). Restore the tilde that have been removed when passing from (1.16) to (1.17), and consider \( v = \tilde{v} \). Then \( v \) solves (1.10) and \( \|v\|_s = \|\tilde{v}\|_s \leq C \tilde{v}^{1+a} = C \tilde{v}^b, \quad b := (1+a)/2 \).

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Now we show that there exists a pressure $p(\varphi, x)$ such that $(u, p)$ solves (1.3). Recall the general formula $\text{curl}(\text{curl} u) = -\Delta u + \nabla(\text{div } u)$. Then $\text{curl } u = \text{curl}(\text{curl}(-\Delta)^{-1}v) = v$ because $\text{div } v = 0$ and $\Pi_0 v = 0$.

Then a direct computation shows that

$$\text{curl}(u \cdot \nabla u) = u \cdot \nabla v - v \cdot \nabla u.$$ 

Since $\text{curl}(\omega \cdot \partial_x u) = \omega \cdot \partial_x v$, $\text{curl}(\zeta \cdot \nabla u) = \zeta \cdot \nabla v$ and $F = \text{curl } f$, the equation (1.10) can be rewritten as

$$\text{curl}(\Gamma) = 0 \quad \text{where} \quad \Gamma := \omega \cdot \partial_x u + \zeta \cdot \nabla u + \varepsilon u \cdot \nabla u - \varepsilon f.$$ 

Then $\Gamma$ is a smooth irrotational vector field. We observe that $\Gamma$ has zero average. Indeed $\Pi \circ \partial_x u$ is a curl, $\Pi \circ f = 0$ (by assumption), and, integrating by parts, $\Pi \circ (\partial_x u) = -\Pi_0((\text{div } u) = 0$ (because $\text{div } u = 0$), whence we deduce that $\Pi \circ \partial_x u = 0$. Then $p := (-\Delta)^{-1} \text{div } \Gamma$ satisfies $\nabla p = \Gamma$, namely

$$\omega \cdot \partial_x u + \zeta \cdot \nabla u + \varepsilon u \cdot \nabla u - \varepsilon f = -\nabla p.$$ 

Hence $u, p$ solve the equation (1.2) and $\|u\|_s, \|p\|_s \lesssim_s \varepsilon^6$.

### A Appendix

In this appendix we prove a lemma which allows to provide the measure estimate of the resonant sets defined in [10, 7]. Let $\mathcal{H}$ be a Hilbert space of dimension $d$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and let $\{\varphi_1, \ldots, \varphi_d\}$ be an orthonormal basis of $\mathcal{H}$. We consider the set $\mathcal{B}(\mathcal{H})$ of linear maps $B : \mathcal{H} \to \mathcal{H}$ with the operator norm $\|\|_{\text{op}}$. Given $B \in \mathcal{B}(\mathcal{H})$, we identify it with its matrix representation $(B^j)_j, j = 1, \ldots, d$ where $B^j := \langle B \varphi_i, \varphi_j \rangle_{\mathcal{H}}$.

Let

$$A(\lambda) = \lambda \cdot k \text{Id}_\mathcal{H} + Q(\lambda) \quad \text{with} \quad k \in \mathbb{Z}^d \setminus \{0\},$$

where the map $\mathbb{R}^p \to \mathcal{B}(\mathcal{H})$, $\lambda \mapsto Q(\lambda)$ is $k_0$ times differentiable. Let $\Omega \subseteq \mathbb{R}^p$ be a bounded, open set and $\eta > 0$. In the lemma below, we provide a measure estimate of the set

$$\mathcal{R}_A := \left\{ \lambda \in \Omega : A(\lambda) \text{ is not invertible or it is invertible and } \|A(\lambda)^{-1}\|_{\text{op}} > \eta \right\}. \quad (A.2)$$

A similar version of the following Lemma can be found in [13]. For sake of completeness we insert our own proof.

**Lemma A.1.** Let $k_0, d \in \mathbb{N}$ with $k_0 \geq d + 2$, $\gamma \in (0, 1)$. Then there exists $\delta_0 \in (0, 1)$ such that if

$$\|Q\|_{\text{op}} \lesssim_\delta \delta^{-k_0} \lesssim_\delta 0,$$ \quad (A.3)

then the Lebesgue measure of the set $\mathcal{R}_A$ satisfies $|\mathcal{R}_A| \lesssim (|k|\eta)^{-\frac{1}{2}}$.

**Proof.** First, if $A$ is invertible, one has the elementary inequality $\|A^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq C_0(d)\|A\|_{\mathcal{B}(\mathcal{H})}^{d-1} \det A^{-1}$, for some $C_0(d) > 0$. By (A.1), (A.3), since $\delta \leq 1 \leq |k|$, one has $\|A\|_{\mathcal{B}(\mathcal{H})} \leq C_1 |k|^{d-1}$ for some $C_1$ depending on $d, \Omega$. Then $\|A^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq C_2 |k|^{d-1} \det A^{-1}$ for some $C_2 > 0$. Hence

$$\mathcal{R}_A \subseteq \bar{\mathcal{R}}_A \quad \text{where} \quad \bar{\mathcal{R}}_A := \left\{ \lambda \in \Omega : |\det A(\lambda)| < \frac{C_2 |k|^{d-1}}{\eta} \right\}. \quad (A.4)$$

By using (A.1), (A.3), a direct calculation shows that

$$\det A(\lambda) = (\lambda \cdot k)^d + r_{d-1}(\lambda)(\lambda \cdot k)^{d-1} + \ldots + r_1(\lambda)(\lambda \cdot k) + r_0(\lambda)$$

where for any $n = 0, \ldots, d - 1$ the maps $\lambda \mapsto r_n(\lambda)$ are $k_0$ times differentiable and

$$|r_n|_{k_0, \gamma} \lesssim_{k_0, d} \delta, \quad \forall n = 0, \ldots, d - 1. \quad (A.5)$$
Now let
\[ \lambda = \frac{k}{|k|} s + v, \quad v \cdot k = 0, \quad \Gamma(s) := \det \left( A \left( \frac{k}{|k|} s + v \right) \right), \]
(A.7)
\[ q_n(s) := r_n \left( \frac{k}{|k|} s + v \right), \quad n = 0, \ldots, d - 1. \]
By (A.5), (A.7), using that \( \lambda \cdot k = |k| s \), one has
\[ \Gamma(s) = |k|^d (s^d + q r(s)), \quad q r(s) := \frac{q_{d-1}(s)s^{d-1}}{|k|} + \ldots + \frac{q_{1}(s)s}{|k|^{d-1}} + \frac{q_{0}(s)}{|k|^d}. \]
By (A.6), (A.7), one gets
\[ |q r^{k_0, \gamma} \lesssim_{k_0, d} \delta, \]
implying that
\[ \left| \left\{ s : \lambda = \frac{k}{|k|} s + v \in \Omega, \quad |\Gamma(s)| < C \left( \frac{d}{|k|} \right)^{d-1} \right\} \right| \lesssim_{d} \left( \frac{1}{|k| \eta} \right)^{\frac{d}{2}}. \]
By a Fubini argument one gets that \( |\tilde{R}_A| \lesssim (|k| \eta)^{-\frac{d}{2}} \) and the claimed statement follows by recalling (A.4).

References
[1] T. Alazard, P. Baldi, Gravity capillary standing water waves, Arch. Ration. Mech. Anal. 217 (2015), no. 3, 741–830.
[2] P. Baldi, Periodic solutions of fully nonlinear autonomous equations of Benjamin-Ono type, Ann. Inst. H. Poincaré (C) Anal. Non Linéaire 30 (2013), no. 1, 33–77.
[3] P. Baldi P., M. Berti, E. Haus, R. Montalto, Time quasi-periodic gravity water waves in finite depth, Inventiones Math. 214 (2), 739–911, 2018.
[4] P. Baldi, M. Berti, R. Montalto, KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation, Math. Annalen 359, 471–536, 2014.
[5] P. Baldi, M. Berti, R. Montalto, KAM for autonomous quasi-linear perturbations of KdV, Ann. Inst. H. Poincaré Analyse Non. Lin. 33, no. 6, 1589–1638, 2016.
[6] P. Baldi, E. Haus, R. Montalto. Controllability of quasi-linear Hamiltonian NLS equations, J. Differential Equations 264 (2018), 1786-1840.
[7] D. Bambusi. Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II. Comm. Math. Phys. 353(1):353–378, 2017.
[8] D. Bambusi. Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, I. Trans. Amer. Math. Soc. 370(3):1823–1865, 2018.
[9] D. Bambusi, B. Grebert, A. Maspero, D. Robert. Growth of Sobolev norms for abstract linear Schrödinger Equations. To appear in JEMS. Preprint arXiv:1706.09708 2017.
[10] D. Bambusi, B. Grebert, A. Maspero, and D. Robert. Reducibility of the quantum Harmonic oscillator in d-dimensions with polynomial time dependent perturbation. Analysis and PDEs, 11(3):775–799, 2018.
[11] D. Bambusi, B. Langella, R. Montalto. Reducibility of non-resonant transport equation on \( \mathbb{T}^d \) with unbounded perturbations. Ann. I. H. Poincaré 20, 1893–1929 (2019). https://doi.org/10.1007/s00029-019-00795-2.
[12] J.T. Beale, T. Kato, A. Majda. Remarks on the breakdown of smooth solutions for the 3D Euler equation. Commun. Math. Phys. 94, 61-66, 1984.

[13] M. Berti, KAM for PDEs, Boll. Unione Mat. Ital. (2016) 9:115-142.

[14] M. Berti M. P. Bolle. Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential. Eur. Jour. Math. 15, 229-286 (2013).

[15] M. Berti, P. Bolle. Sobolev quasi-periodic solutions of multidimensional wave equations with a multiplicative potential. Nonlinearity 25 (9), pp. 2579–2613, 2012.

[16] M. Berti, L. Corsi, M. Procesi. An abstract Nash-Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds. Comm. Math. Phys., vol. 334, n. 3, pp. 1413-1454, 2015.

[17] M. Berti, R. Montalto, Quasi-periodic standing wave solutions of gravity capillary standing water waves, Memoirs of the Amer. Math. Society. Vol. 263, Number 1273, 2019.

[18] J. Bourgain, Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations, Ann. of Math. 148, 363–439 (1998).

[19] J. Bourgain, D. Li, Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces, Invent. math. 201 (2015), 97–157.

[20] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr, V. Vicol, Onsager’s conjecture for admissible weak solutions, Comm. Pure Appl. Math. 72 (2019), no.2, 229–274.

[21] Walter Craig and C. Eugene Wayne. Newton’s method and periodic solutions of nonlinear wave equations. Comm. Pure Appl. Math., 46 (11): 1409–1498, 1993.

[22] P. Constantin, On the Euler equations of incompressible fluids, Bull. Amer. Math. Soc. N.S. 44 (2007), 4, 603–621.

[23] L. Corsi, R. Montalto. Quasi-periodic solutions for the forced Kirchhoff equation on $\mathbb{T}^d$. Nonlinearity 31,5075-5109 (2018). https://doi.org/10.1088/1361-6544/aad6fe

[24] N. Crouseilles, E. Faou, Quasi-periodic solutions of the 2D Euler equation, Asymptot. Anal. 81 (2013), no.1, 31–34.

[25] C. De Lellis, L. Székelyhidi. The Euler equations as a differential inclusion. Ann. of Math. (2) 170 (3), 1417–1436, 2009.

[26] T. Elgindi. Finite-Time Singularity Formation for $C^{1,\alpha}$ Solutions to the Incompressible Euler Equations on $\mathbb{R}^3$, arXiv:1904.04795.

[27] H.L. Eliasson and S.B. Kuksin. On reducibility of Schrödinger equations with quasiperiodic in time potentials. Comm. Math. Phys., 286(1):125–135, 2009.

[28] R. Feola, M. Procesi, Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations, J. Diff. Eq., 259, no. 7, 3389–3447, 2015.

[29] R. Feola, F. Giuliani, R. Montalto, M. Procesi. Reducibility of first order linear operators on tori via Moser’s theorem. Journal of Functional Analysis, 276 (2019) 932970.

[30] G. Iooss, P. I. Plotnikov, and J. F. Toland. Standing waves on an infinitely deep perfect fluid under gravity. Arch. Ration. Mech. Anal., 177(3):367–478, 2005.

[31] P. Isett, On the Endpoint Regularity in Onsager’s Conjecture, arXiv:1706.01549.

[32] B. Khesin, S. Kuksin, D. Peralta-Salas, KAM theory and the 3D Euler equation, Adv. Math. 267 (2014), 498–522.
[33] B. Khesin, S. Kuksin, D. Peralta-Salas, Global, local and dense non-mixing of the 3D Euler equation. arXiv:1911.04363

[34] A.J. Majda, A.L. Bertozzi, Vorticity and incompressible Flow. Cambridge texts in applied Mathematics, 2007.

[35] R. Montalto. A reducibility result for a class of linear wave equations on \( T^d \). Int. Math. Res. Notices, Vol. 2019, No. 6, pp. 1788–1862 doi: 10.1093/imrn/rnx167.

[36] J. Liu, X. Yuan, A KAM Theorem for Hamiltonian Partial Differential Equations with Unbounded Perturbations, Comm. Math. Phys, 307 (3) , 629–673, 2011.

[37] T. Kato. Nonstationary flows of viscous and ideal fluids in \( \mathbb{R}^3 \). J. Funct. Anal. 9, 269-305, 1972.

[38] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. 58(3), 181-205, 1975.

[39] S. B. Kuksin. Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum. Funct. Anal. Appl. 21:192–205, 1987.

[40] S. Kuksin, A KAM theorem for equations of the Korteweg-de Vries type, Rev. Math. Phys., 10, 3, 1–64, 1998.

[41] S. Kuksin, Analysis of Hamiltonian PDEs, Oxford University Press, 2000.

[42] P.J. Olver, A nonlinear Hamiltonian structure for the Euler equations, J. Math. Anal. Appl. 89 (1982), 233-250.

[43] C. Procesi, M. Procesi Reducible quasi-periodic solutions for the nonlinear Schrödinger equation. Boll. Unione Mat. Ital. (9) 189-236, 2016.

[44] H. Rüssmann, Invariant tori in non-degenerate nearly integrable Hamiltonian systems. Regul. Chaotic Dyn. 6(2), 119–204 (2001)

[45] A. Shnirelman. Weak solutions with decreasing energy of incompressible Euler equations. Comm. Math. Phys. 210, 541–603, 2000. MR 2002g:76009 Zbl 1011.35107

[46] V.I. Yudovich. Non-stationary flow of an incompressible liquid. Vychisl. Mat. Mat. Fiz. 3, 1032-1066, 1963.

[47] C. Eugene Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Comm. Math. Phys., 127(3):479-528, 1990.

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