ANTICYCLOMOTOMIC $\mu$-INVARIANTS OF RESIDUALLY REDUCIBLE GALOIS REPRESENTATIONS

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Abstract. Let $E$ be an elliptic curve over an imaginary quadratic field $K$, and $p$ be an odd prime such that the residual representation $E[p]$ is reducible. The $\mu$-invariant of the fine Selmer group of $E$ over the anticyclotomic $\mathbb{Z}_p$-extension of $K$ is studied. We do not impose the Heegner hypothesis on $E$, thus allowing certain primes of bad reduction to decompose infinitely in the anticyclotomic $\mathbb{Z}_p$-extension. It is shown that the fine $\mu$-invariant vanishes if certain explicit conditions are satisfied. Further, a partial converse is proven.

1. Introduction

Iwasawa theory is the study of objects of arithmetic interest over infinite towers of number fields. K. Iwasawa conjectured that over the cyclotomic $\mathbb{Z}_p$-extension of a number field $F$, the $p$-primary part of the Hilbert class group is a finitely generated $\mathbb{Z}_p$-module (see [17]). This is known as Iwasawa’s $\mu = 0$ conjecture for the number field $F$. In [9], B. Ferrero and L. Washington proved this conjecture for all finite abelian extensions $F$ over of $\mathbb{Q}$. The Iwasawa theory of abelian varieties (in particular, elliptic curves) was initiated by B. Mazur in [25]. The main object of study is the $p$-primary Selmer group of an elliptic curve $E$ defined over a number field $F$, with good ordinary reduction at $p$. The Selmer group over a $\mathbb{Z}_p$-extension of $F$ is a cofinitely generated module over the Iwasawa algebra, $\Lambda$. In [11], R. Greenberg analyzed the algebraic structure of these Selmer groups. For elliptic curves $E$ over $\mathbb{Q}$, it is conjectured that if the residual representation on $E[p]$ is irreducible, then $\mu$-invariant of the $p$-primary Selmer group over the cyclotomic $\mathbb{Z}_p$-extension, denoted $\mu(E/\mathbb{Q}^{cyc})$, vanishes (see [12, Conjecture 1.11]).

The fine Selmer group is a subgroup of the classical Selmer group obtained by imposing vanishing conditions at primes above $p$. It had first been studied by K. Rubin [32] and B. Perrin-Riou [28, 29], under various guises. The analysis of the fine Selmer groups is an essential part of K. Kato’s seminal work on the Iwasawa Main Conjecture for elliptic curves and modular forms (see [19]). In recent years, their study has gained considerable momentum (see for instance [5, 35, 34, 18, 11]). J. Coates and R. Sujatha conjectured that the the fine Selmer group of $E$ over $F^{cyc}$ is $\Lambda$-cotorsion with associated $\mu$-invariant, $\mu^{fine}(E/F^{cyc})$, equal to zero [5, Conjecture A]. The formulation of this conjecture makes no hypothesis on the reduction type at primes above $p$ or the residual representation on $E[p]$. However, such a statement need not be true for the (classical) Selmer group even for elliptic curves over $\mathbb{Q}$ with good ordinary reduction at $p$. In fact, Mazur provided examples of elliptic curves $E/\mathbb{Q}$ for which the residual representation $E[p]$ is reducible and...
\( \mu(E/Q^{\text{cyc}}) \) is non-zero. There is a systematic approach towards finding examples of elliptic curves with positive \( \mu(E/F^{\text{cyc}}) \) (see [5, 7]).

On the other hand, the conjecture by Coates and Sujatha predicts a close relationship in the growth of ideal class groups and fine Selmer groups in cyclotomic \( \mathbb{Z}_p \)-extensions. Some evidence towards this is provided in [5, Theorem 3.4]. In particular, the result of Ferrero and Washington implies that for an elliptic curve over an abelian number field, \( \mu^{\text{fine}}(E/F^{\text{cyc}}) \) is zero when \( E[p] \) is reducible (see [5, Corollary 3.6]). Subsequently, the relation in the growth of ideal class groups and fine Selmer groups has been studied in more general settings (see [23, 20, 21, 22]).

In this paper, we study the relationship in the growth of fine Selmer groups and class groups in anticyclotomic \( \mathbb{Z}_p \)-extensions of imaginary quadratic fields.

Several authors have studied classical Selmer groups of elliptic curves (more generally, abelian varieties or modular forms) in anticyclotomic \( \mathbb{Z}_p \)-extensions of imaginary quadratic fields (see for example [33, 2, 30]). In line with the conjecture of Coates and Sujatha, for an elliptic curve defined over the imaginary quadratic field \( K \), its fine Selmer group over the anticyclotomic \( \mathbb{Z}_p \)-extension is expected to be \( \Lambda \)-cotorsion with \( \mu^{\text{fine}}(E/K^{\text{ac}}) = 0 \) (see [24, Conjecture B]). Most results in literature focus on the case when the residual representation is irreducible. In contrast, this paper primarily studies the case when the residual representation is reducible. The first main result (see Theorem 3.2) shows that if certain conditions are satisfied then the \( \mu \)-invariant of the \( p \)-primary fine Selmer group may be detected by a certain analogous fine Selmer group associated to the residual representation, \( E[p] \). Further, if \( E[p] \) is reducible then there are characters

\[
\varphi_1, \varphi_2 : \text{Gal}(\overline{K}/K) \to \text{GL}_1(\mathbb{F}_p)
\]

which fit into a short exact sequence

\[
0 \rightarrow \mathbb{F}_p(\varphi_1) \rightarrow E[p] \rightarrow \mathbb{F}_p(\varphi_2) \rightarrow 0.
\]

Note that \( \varphi_2 = \chi \varphi_1^{-1} \), where \( \chi \) denotes the mod-\( p \) cyclotomic character. This short exact sequence makes it possible to analyze the structure of the fine Selmer group associated to \( E[p] \). Next, we prove a partial converse to the above theorem (see Theorem 3.4). A key difference between the cyclotomic and anticyclotomic \( \mathbb{Z}_p \)-extensions is the following: in the cyclotomic extension, all primes are finitely decomposed, whereas in the anticyclotomic extension, there are infinitely many primes which split completely. For the aforementioned result of Coates and Sujatha in the cyclotomic extension case, this fact is crucially used (in the proofs of [3] Lemmas 3.2, 3.7, Theorem 3.4). The elliptic curve \( E \) is said to satisfy the Heegner hypothesis if all primes \( v \nmid p \) of \( K \) at which it has bad reduction are split completely in \( K \). By Fact 2.3, such a a prime number is finitely decomposed in the anticyclotomic \( \mathbb{Z}_p \)-extension, \( K^{\text{ac}} \). We work in a more general setting, where primes of bad reduction of \( E \) may be infinitely decomposed in \( K^{\text{ac}} \), and thereby rely on a different strategy (from that of Coates and Sujatha) to prove our results.

Two elliptic curves, \( E_1 \) and \( E_2 \), are said to be \( p \)-congruent if the \( p \)-torsion subgroups \( E_1[p] \) and \( E_2[p] \) are isomorphic as Galois modules. In [14], R. Greenberg and V. Vatsal studied the relation between cyclotomic invariants of elliptic curves over \( \mathbb{Q} \) which are \( p \)-congruent. In [13, Proposition 4.1.6], Greenberg reformulated the conjecture of Coates and Sujatha in terms of the vanishing of a second cohomology group with coefficients in \( E[p] \). From this, it is immediate
that if elliptic curves $E_1$, $E_2$ are $p$-congruent then $\mu_\text{fine}(E_1/\mathbb{Q}^{\text{cyc}}) = 0$ if and only if $\mu_\text{fine}(E_2/\mathbb{Q}^{\text{cyc}}) = 0$. In the same spirit, we prove a result for fine $\mu$-invariants over $K^{ac}$ (see Theorem 5.2), without imposing any hypothesis on the reducibility of the residual representations.

The authors expect that the methods in this paper should generalize to residually reducible Galois representations arising from abelian varieties. The results shall however be more technical and the arguments more cumbersome in the higher dimensional setting. The authors choose a less general framework in which the inherent simplicity of the underlying ideas come across easily.

The paper is organized into six sections. Preliminary notions are discussed in §2. In §3 and §4 we state and prove the main results: we establish a criterion for the vanishing of the $\mu$-invariant of the fine Selmer group in the anticyclotomic $\mathbb{Z}_p$-extension. We also prove a partial converse to the above theorem. In §5 we compare the anticyclotomic $\mu$-invariant for two elliptic curves which are $p$-congruent. Finally, in §6 we list examples illustrating the results in this article.

2. Preliminaries

Throughout, let $p$ be an odd prime and $K$ be an imaginary quadratic field. Let $S_p$ be the set of primes of $K$ above $p$. Fix an elliptic curve $E$ over $K$. Set $G_K := \text{Gal}(\overline{K}/K)$ to be the absolute Galois group of $K$. Denote by $E[p^n]$ (resp. $E[p^\infty]$) the $p^n$-torsion (resp. $p$-primary torsion) subgroup of $E(\overline{K})$. The Tate module $T_p(E)$ is the inverse limit with respect to multiplication by $p$ maps,

$$T_p(E) := \lim_{\leftarrow} E[p^n].$$

The $\mathbb{Z}_p$-module $T_p(E)$ is free of rank 2 and the group of $\mathbb{Z}_p$-linear automorphisms of $T_p(E)$ is identified with $\text{GL}_2(\mathbb{Z}_p)$. The action of $G_K$ on $T_p(E)$ induces a continuous Galois representation, $\rho_E : G_K \to \text{GL}_2(\mathbb{Z}_p)$. Set $\overline{\rho}_E$ to denote the mod-$p$ reduction of $\rho_E$, as depicted

$$\begin{array}{ccc}
G_K & \xrightarrow{\rho_E} & \text{GL}_2(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
\overline{\rho}_E & \xrightarrow{} & \text{GL}_2(\mathbb{F}_p).
\end{array}$$

The residual representation $\overline{\rho}_E$ is induced by the action of $G_K$ on $E[p]$. Let $\mathcal{N}$ be the conductor of $E$ and $S$ the set of primes that divide $\mathcal{N}p$. It is known that $\rho_E$ is unramified at all primes $v \notin S$. Let $K_S$ denote the maximal algebraic extension of $K$ in which all primes $v \notin S$ are unramified and set $G_{K,S} := \text{Gal}(K_S/K)$. Throughout this paper, we primarily focus on the case when the Galois representation $\rho_E$ is residually reducible. However, some of the results will apply to the residually irreducible case as well. We shall be careful to make precise when the following hypothesis is required.

**Hypothesis 2.1.** Assume that the residual representation $\overline{\rho}_E$ is reducible, i.e. $E[p]$ contains a proper non-zero $G_K$-stable submodule.
This is indeed the case when \( E(K)[p] \neq 0 \). Though the condition is in fact far more general. In this setting, there are characters
\[
\varphi_1, \varphi_2 : G_{K,S} \to GL_1(\mathbb{F}_p)
\]
and a 1-cocycle
\[
\beta : G_{K,S} \to \mathbb{F}_p(\varphi_1 \varphi_2^{-1})
\]
such that
\[
\overline{\rho}_E \simeq \left( \begin{array}{c}
\varphi_1 \\
\varphi_2
\end{array} \right).
\]
The residual representation \( \overline{\rho}_E \) is said to be indecomposable if the cohomology class \( [\beta] \in H^1(G_{K,S}, \mathbb{F}_p(\varphi_1 \varphi_2^{-1})) \) is non-zero, and split otherwise. If \( \overline{\rho}_E \) is split, then \( \overline{\rho}_E \simeq \left( \begin{array}{c}
\varphi_1 \\
\varphi_2
\end{array} \right) \), and the characters \( \varphi_1 \) and \( \varphi_2 \) may be interchanged. If \( \overline{\rho}_E \) is indecomposable, then \( \varphi_1 \) is the unique character such that \( E[p] \) contains a Galois submodule isomorphic to \( \mathbb{F}_p(\varphi_1) \).

**Remark 2.2.** If \( E_\mathbb{Q} \) is an elliptic curve such that \( E[p] \) is reducible as a Gal(\( \mathbb{Q}/\mathbb{Q} \))-module then \( E \) has either good ordinary reduction or bad reduction at \( p \). By a result of Fontaine, if \( E_\mathbb{Q} \) is an elliptic curve with good supersingular reduction at \( p \), then \( \overline{\rho}_{E|G_{G_{\mathbb{Q}_p}}} \) is irreducible.

Suppose \( K \) is a \( \mathbb{Z}_p \)-extension of \( K \) which is Galois over \( \mathbb{Q} \). Then, there are exactly two cases to consider: either \( K \) is the cyclotomic \( \mathbb{Z}_p \)-extension or the anticyclotomic \( \mathbb{Z}_p \)-extension. Note that \( \Gamma_K := Gal(K/K) \) is an index two normal subgroup of Gal(\( \mathbb{Q}/\mathbb{Q} \)). The group Gal(\( \mathbb{Q}/\mathbb{Q} \)) acts on \( \Gamma_K \), as is explained. Let \( \tau \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) and \( x \in \Gamma_K \); choose a lift \( \tilde{\tau} \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) of \( \tau \) and set \( \tau \cdot x := \tilde{\tau} x \tilde{\tau}^{-1} \). Since \( \Gamma_K \) is abelian, \( \tau \cdot x \) does not depend on the choice of the lift \( \tilde{\tau} \). If the action of Gal(\( \mathbb{Q}/\mathbb{Q} \)) on \( \Gamma_K \) is via the trivial (resp. non-trivial) character, then \( K \) is the cyclotomic (resp. anticyclotomic) extension of \( K \). Thus, the cyclotomic extension is pro-cyclic and the anticyclotomic extension is pro-dihedral. We only consider the anticyclotomic extension in this article, and denote it by \( K^{ac} \). Set \( \Gamma \) to denote Gal(\( K^{ac}/K \)). For \( n \geq 0 \), the \( n \)-th layer is the unique number field \( K_n \) such that \( K \subseteq K_n \subseteq K^{ac} \) and \( [K_n : K] = p^n \). Note that \( K_n \) is Galois over \( \mathbb{Q} \) and its Galois group Gal(\( K_n/\mathbb{Q} \)) is (isomorphic to) the dihedral group of order \( 2p^n \).

For a set of primes \( S' \), set \( S'(K^{ac}) \) (resp. \( S'(K_n) \)) to denote the primes of \( K^{ac} \) (resp. \( K_n \)) which lie above some prime \( v \in S' \). For instance, \( v(K^{ac}) \) (resp. \( v(K_n) \)) will denote the primes \( \eta v \) of \( K^{ac} \) (resp. of \( K_n \)). The set of primes above a given prime of \( K \) in the cyclotomic \( \mathbb{Z}_p \)-extension \( K^{cyc} \) is finite. This is not the case for \( K^{ac} \); the following characterization is well known.

**Fact 2.3.** Let \( v \) be a prime of \( K \) and \( l \) the prime number such that \( v|l \). The set of primes \( v(K^{ac}) \) is finite if and only if \( l = p \) or \( l \) splits in \( K \).

Indeed, there is a large enough value of \( n \) such that all primes of \( K_n \) above \( p \) are totally ramified in \( K^{ac} \), see the last three lines of [3, p. 2131]. A prime \( l \neq p \) which does not split in \( K \) must split completely in \( K^{ac} \), see the first paragraph of p. 2132 of loc. cit. Prime numbers \( l \) which split in \( K \) must be finitely decomposed in \( K^{ac} \), see for example, Corollary 1 of loc. cit.
Recall that \( \mathcal{N} \) is the conductor of \( E \). Denote by \( \overline{\mathcal{N}} \) the Artin conductor of \( p_E \). Note that \( \overline{\mathcal{N}} \) divides \( \mathcal{N} \), denote by \( (\mathcal{N}/\overline{\mathcal{N}}) \) the quotient. We make an assumption on a certain subset of primes in \( S \) which is described below.

**Definition 2.4.** Denote by \( \Sigma \subset S \) the set of primes \( v \nmid p \) at which all of the following conditions are satisfied.

1. \( v \mid (\mathcal{N}/\overline{\mathcal{N}}) \).
2. If \( p \geq 5 \) and \( \mu_p \subset K_v \), then \( E \) has split multiplicative reduction at \( v \).
3. If \( p = 3 \) and \( \mu_3 \subset K_v \), then \( E \) has split multiplicative reduction or additive reduction at \( v \).

**Hypothesis 2.5.** Let \( E \) be an elliptic curve over an imaginary quadratic field \( K \). Let \( v \in \Sigma \) and \( l \) be the prime number such that \( v \mid l \). Then \( l \) is split in \( K \).

**Definition 2.6.** Suppose that \( p_E \) is reducible and indecomposable. Let \( \Sigma(\varphi_2) \) be the set of primes \( v \in S \setminus S_{\varphi} \) such that \( \varphi_2 \mid G_{\varphi_v} \).

**Hypothesis 2.7.** Suppose that \( p_E \) is reducible and indecomposable. Let \( v \in \Sigma(\varphi_2) \) and \( l \) be the prime number in such that \( v \mid l \). Then \( l \) is split in \( K \).

**Remark 2.8.** The above hypotheses are all weaker than the Heegner hypothesis. Some of our results will require the above hypotheses; they will be assumed only when explicitly stated. For instance, Theorem 3.2 will require Hypothesis 2.5 (and Hypothesis 2.7 when \( p_E \) is indecomposable). On the other hand, Theorem 3.3 does not require these hypotheses.

We now introduce the fine Selmer group. At each prime \( v \in S \), set

\[
\mathcal{H}_v(K^{ac}, E[p^{\infty}]) := \prod_{\eta \in \varepsilon(K^{ac})} H^1(K^{ac}_{\eta}, E[p^{\infty}]).
\]

**Definition 2.9.** The fine Selmer group associated to \( E[p^{\infty}] \) is defined as follows

\[
\mathcal{R}_{p^{\infty}}(E/K^{ac}) := \ker \left\{ H^1(K_S/K^{ac}, E[p^{\infty}]) \longrightarrow \bigoplus_{v \in S} \mathcal{H}_v(K^{ac}, E[p^{\infty}]) \right\}.
\]

Recall that \( \Gamma := \text{Gal}(K^{ac}/K) \simeq \mathbb{Z}_p \). The Iwasawa algebra \( \Lambda \) is the completed group algebra \( \mathbb{Z}_p[\Gamma] := \lim_{\leftarrow n} \mathbb{Z}_p[\Gamma/\Gamma^{p^n}] \). After fixing a topological generator \( \gamma \) of \( \Gamma \), there is an isomorphism of rings \( \Lambda \cong \mathbb{Z}_p[X] \), by sending \( \gamma - 1 \) to the formal variable \( X \). The fine Selmer group is a cofinitely generated \( \Lambda \)-module.

The Pontryagin dual \( \mathcal{R}_{p^{\infty}}(E/K^{ac})^\vee \) is, up to pseudo-isomorphism, a finite direct sum of cyclic \( \Lambda \)-modules:

\[
\mathcal{R}_{p^{\infty}}(E/K^{ac})^\vee \cong \Lambda^r \oplus \bigoplus_{i} \Lambda/(\mu_i) \oplus \bigoplus_{j} \Lambda/(f_j(T)).
\]

Here, \( \mu_i > 0 \) and \( f_j(T) \) is a distinguished polynomial (i.e., a monic polynomial with non-leading coefficients divisible by \( p \)). Here, \( r \) is the \( \Lambda \)-corank of the fine Selmer group. The \( \mu \)-invariant of the fine Selmer group is defined as follows,

\[
\mu^\text{fine}(E/K^{ac}) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=0}^{s} \mu_i & \text{if } s > 0. \end{cases}
\]
The number of summands $s$ is a well-defined invariant and we refer to it as the \textit{$\mu$-multiplicity}. We now introduce the fine Selmer group associated to the residual representation on $E[p]$. At each prime $v$ of $K$, set
\[ \mathcal{H}_v(K_{ac}, E[p]) := \prod_{\eta \in \eta(K_{ac})} H^1(K_{ac}/\eta, E[p]). \]

**Definition 2.10.** Let $T$ be a finite set of primes containing $S$. The fine Selmer group associated to $E[p]$ and the set of primes $T$ is defined as follows
\[ \mathcal{R}^T(E[p]/K_{ac}) := \ker \left\{ H^1(K_T/K_{ac}, E[p]) \to \bigoplus_{v \in T} \mathcal{H}_v(K_{ac}, E[p]) \right\}. \]

Set $\mathcal{R}(E[p]/K_{ac}) = \mathcal{R}^S(E[p]/K_{ac})$; this is the mod-$p$ fine Selmer group.

Set $\Omega$ to denote the mod-$p$ Iwasawa algebra, $\Omega := \Lambda/(p) \simeq \mathbb{F}_p[X]$. Note that both $\mathcal{R}(E[p]/K_{ac})$ and $\mathcal{R}_{p^\infty}(E/K_{ac})[p]$ are $\Omega$-modules. Since $\mathcal{R}_{p^\infty}(E/K_{ac})$ is cofinitely generated over $\Lambda$, it follows that $\mathcal{R}_{p^\infty}(E/K_{ac})[p]$ is cofinitely generated over $\Omega$. The following is an easy consequence of the structure theory of $\Lambda$-modules.

**Lemma 2.11.** The $\Omega$-corank of $\mathcal{R}_{p^\infty}(E/K_{ac})[p]$ is equal to $r + s$, where $r$ and $s$ are defined above. In particular,
\[ \mathcal{R}_{p^\infty}(E/K_{ac}) \] is $\Lambda$-cotorsion with $\mu^\text{fine}(E/K_{ac}) = 0 \iff \mathcal{R}_{p^\infty}(E/K_{ac})[p]$ is finite.

**Proof.** Note that $(\mathcal{R}_{p^\infty}(E/K_{ac})[p])^\vee$ is isomorphic to $\mathcal{R}_{p^\infty}(E/K_{ac})^\vee/p$. Let $\Phi$ be a pseudo-isomorphism of $\Lambda$-modules, i.e. a homomorphism,
\[ \Phi : \mathcal{R}_{p^\infty}(E/K_{ac})^\vee \to \Lambda^r \oplus \left( \bigoplus_{i=1}^s \Lambda/(p^{i+1}) \right) \oplus \left( \bigoplus_{j=1}^t \Lambda/(f_j(T)) \right) \]
with finite kernel and cokernel. The mod-$p$ reduction $\overline{\Phi}$ is the following map
\[ \overline{\Phi} : (\mathcal{R}_{p^\infty}(E/K_{ac})[p])^\vee \to \Omega^{r+s} \oplus \left( \bigoplus_{j=1}^t \Omega/(\overline{f}_j(T)) \right), \]
where $\overline{f}_j(T)$ is the mod-$p$ reduction of $f_j(T)$. Since $f_j(T)$ is a distinguished polynomial, $\overline{f}_j(T) = T^{\deg f_j}$ and $\Omega/(\overline{f}_j(T))$ is finite. The result follows. \qed

3. **ACriterion for the Vanishing of the Fine $\mu$-invariant**

In this section, we establish a criterion for the vanishing of the fine $\mu$-invariant in the anticyclotomic $\mathbb{Z}_p$-extension. Before stating the result, let us introduce some notation. We shall not assume that $\overline{\rho}_E$ is reducible in this section, unless when explicitly stated. However, when it is reducible, recall that the characters $\varphi_1$ and $\varphi_2$ are such that $E[p]$ fits into a short exact sequence of Galois-modules,
\[ 0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0. \]

Let $K(\varphi_i)$ be an extension of $K$ fixed by $\ker \varphi_i$. For an algebraic extension $F$ of $K$, let $F(\varphi_i)$ be the composite $F \cdot K(\varphi_i)$. Let $\mathcal{L}_n^{(i)}$ be the maximal abelian unramified
$p$-extension of $K_n(\varphi_i)$ split at primes above $S(K_n(\varphi_i))$ (called the $p$-Hilbert $S$-class field extension). Class field theory prescribes a natural isomorphism

$$\text{Gal}(\mathcal{L}_n^{(i)}/K_n(\varphi_i)) \simeq \text{Cl}_S(K_n(\varphi_i))[\mathbb{Z}^\infty],$$

where $\text{Cl}_S(K_n(\varphi_i))[\mathbb{Z}^\infty]$ is the $p$-primary part of the $S$-class group of $K_n(\varphi_i)$.

Let $M$ be any $\mathbb{Z}_p$-module on which $\text{Gal}(K(\varphi_i)/K)$ acts by $\mathbb{Z}_p$-linear automorphisms. Let $\psi: \text{Gal}(K(\varphi_i)/K) \to \mathbb{F}_p^\times$ be a character, and consider its Teichmüller lift, $\tilde{\psi}: \text{Gal}(K(\varphi_i)/K) \to \mathbb{Z}_p^\times$. Set

$$M_\psi := \{x \in M | g \cdot x = \tilde{\psi}(g)x\}.$$

There is a unique field extension $\mathcal{E}^{(i)}_n$ of $K_n(\varphi_i)$ which is Galois over $K$ with the additional property that $\text{Gal}(\mathcal{E}^{(i)}_n/K_n(\varphi_i))$ is identified with $(\text{Cl}_S(K_n(\varphi_i))[\mathbb{Z}^\infty])_{\varphi_i}$.

Set $\mathcal{L}^{(i)} := \bigcup_n \mathcal{L}^{(i)}_n$ and $\mathcal{E}^{(i)} := \bigcup_n \mathcal{E}^{(i)}_n$. Further, setting $\mathcal{X}^{(i)} = \text{Gal}(\mathcal{L}^{(i)}/K^{ac}(\varphi_i))$, standard arguments show that $\mathcal{X}^{(i)}$ is a finitely generated torsion $\Lambda$-module. Note that $\mathcal{X}^{(i)}$ decomposes into a direct sum of $\Lambda$-submodules

$$\mathcal{X}^{(i)} = \bigoplus \mathcal{X}^{(i)}_\psi,$$

where $\psi$ ranges over the characters $\text{Gal}(K(\varphi_i)/K) \to \mathbb{F}_p^\times$. Recall that the $\psi$-eigenspace $\mathcal{X}^{(i)}_\psi$ is defined as follows

$$\mathcal{X}^{(i)}_\psi := \{x \in \mathcal{X}^{(i)} | g \cdot x = \tilde{\psi}(g)x\}.$$ 

Let $Y_i$ be the $\varphi_i$-eigenspace $\mathcal{X}^{(i)}_{\varphi_i}$ and $\tilde{Y}_i$ be the $\varphi_i$-component of the $p$-Hilbert class field extension $\mathcal{M}^{(i)}$ of $K^{ac}(\varphi_i)$. Denote their $\mu$-invariants (with respect to the $\mathbb{Z}_p$-extension $K^{ac}(\varphi)/K(\varphi_i)$) by $\mu(Y_i)$ and $\mu(\tilde{Y}_i)$, respectively. Note that $Y_i$ is the quotient of $\tilde{Y}_i$ where all the primes in $S(K^{ac}(\varphi_i))$ are split. We identify $Y_i$ with the Galois group $\text{Gal}(\mathcal{E}^{(i)}/K^{ac})$.

The next result follows from the work of H. Hida [16, Theorem 1] (see also [10, Theorem 1.1]) and Rubin [31, Theorem 4.1].

**Theorem 3.1.** Suppose that $p = v\overline{\sigma}$ in $K$. Let $G_v = \text{Gal}(K_v^*/K_v)$ and assume that $\varphi_1|G_v \neq 1$, $\overline{\varphi}_1|G_v$ (or equivalently, $\varphi_2|G_v \neq 1$, $\overline{\varphi}_2|G_v$). Then,

$$\mu(\tilde{Y}_1) = \mu(\tilde{Y}_2) = \mu(Y_1) = \mu(Y_2) = 0.$$

**Proof.** Since $\varphi_1 \varphi_2 = \overline{\varphi}_1$, the condition on $\varphi_1$ is equivalent to that on $\varphi_2$. Let $\varphi_i$ be either of the characters, $\varphi_1$ or $\varphi_2$. Let $\mathcal{Q}^{(i)}$ be the maximal abelian pro-$p$ extension of $K^{ac}(\varphi_i)$ which is unramified away from $v(K^{ac})$. Rubin identifies $\mathcal{Q}^{(i)}$ with an anticyclotomic $p$-adic $L$-function, and Hida (also T. Finis) proves that the $\mu$-invariant of such an anticyclotomic $p$-adic $L$-function vanishes. Combining their results, we deduce that $\mu(\mathcal{Q}^{(i)}) = 0$ for $i = 1, 2$. Since $\tilde{Y}_i$ is a quotient of $\mathcal{Q}^{(i)}$, it follows that $\mu(\tilde{Y}_i) = 0$ for $i = 1, 2$. Hence, $\mu(Y_i) = 0$ for $i = 1, 2$ as well. \qed

The most explicit examples of reducible Galois representations arise from elliptic curves with $p$-torsion points over the base field. In this setting, the conditions of Theorem 3.1 are not satisfied since $\{\varphi_1, \varphi_2\} = \{1, \overline{\varphi}\}$. We now state the main theorems of this article.

**Theorem 3.2.** Suppose the following conditions hold.
Corollary 3.5. Let $E$ be an elliptic curve defined over $K$ and $S$ be the set of primes dividing $Np$, where $N$ is the conductor of $E$. Assume that

1. $E(K)[p] \neq 0$.
2. $\mathcal{R}_{p}\mathcal{R}(E/K^{ac})$ is a cotorsion $\Lambda$-module with $\mu^{\text{fine}}(E/K^{ac}) = 0$.
3. The Heegner hypothesis is satisfied: for $v \in S \setminus S_p$, let $l$ be the prime number such that $v|l$, then $l$ splits in $K$.

Then the (classical) Iwasawa $\mu$-invariant, $\mu(K^{ac}/K) = 0$.

Proof. Note that $E(K)[p] \neq 0$ when the residual representation is of the form $\overline{p}_E \simeq \left( \begin{array}{cc} 1 & * \\ \chi & \end{array} \right)$. The result follows from Theorems 3.4 and Proposition 4.3. \qed

Example 6.2 illustrates Corollary 3.5. Before proving the above results, a brief sketch of the method is provided. The Kummer sequence

\[
(3.1) \quad 0 \to E[p] \to E[p^{\infty}] \xrightarrow{x_p} E[p^{\infty}] \to 0
\]

induces a comparison map

\[
(3.2) \quad \Psi : \mathcal{R}(E[p]/K^{ac}) \to \mathcal{R}_{p^{\infty}}(E/K^{ac})[p].
\]

The main theorem is deduced from the following statements.

1. The kernel of $\Psi$ is finite. If Hypothesis 2.5 is satisfied, then $\text{cok} \Psi$ is also finite. This step does not require any assumption on $\overline{p}_E$.
2. Suppose that $\overline{p}_E$ is reducible. Further, if the conditions of Theorem 4.2 are satisfied, then the mod-$p$ fine Selmer group $\mathcal{R}(E[p]/K^{ac})$ is finite.
We now analyze the map $\Psi$ and deduce a criterion for the vanishing of the fine $\mu$-invariant. To show that $\text{cok } \Psi$ is finite when Hypothesis 2.5 is satisfied, we prove the following result.

**Lemma 3.6.** Let $v \in S \setminus \Sigma$ be a prime such that $v \nmid p$ and $\eta$ be a prime of $K^{ac}$ above $v$. Then, the group $E(K^{ac}_\eta)[p^\infty]$ is $p$-divisible.

**Proof.** Since $v \notin \Sigma$, (at least) one of the following conditions is satisfied:

1. $v \nmid (N \setminus N)$,
2. $p \geq 5$, $\mu_p \subset K_v$, and $E$ has non-split multiplicative reduction or additive reduction at $v$,
3. $p = 3$, $\mu_3 \subset K_v$, and $E$ has non-split multiplicative reduction at $v$.

The claim follows from the proof of [8 Lemma 4.1.2] and [15 Proposition 5.1(iii)].

For $v \in S$, let $h_v$ denote the natural map

$$h_v : H_v(K^{ac}, E[p]) \to H_v(K^{ac}, E[p^\infty])[p]$$

induced from (3.1). For each prime $\eta|v$ of $K^{ac}$, set $h_\eta$ to denote the natural map

$$h_\eta : H^1(K^{ac}_\eta, E[p]) \to H^1(K^{ac}_\eta, E[p^\infty])[p],$$

also induced from (3.1).

**Corollary 3.7.** If Hypothesis 2.5 is satisfied, then $\ker h_v$ is finite for $v \in S$.

**Proof.** The kernel of $h_v$ is the product of $\ker h_\eta$, as $\eta$ ranges over $v(K^{ac})$. By Kummer theory, $\ker h_\eta$ is isomorphic to $H^0(K^{ac}_\eta, E[p^\infty])/p$; hence, it is finite.

First, consider the case when $v \in \Sigma \cup S_p$. Let $l$ be the prime number for which $v|l$. By Hypothesis 2.5, if $v \in \Sigma$, then $l$ is split in $K$. Therefore, by Fact 2.3, the set of primes $v(K^{ac})$ is finite for $v \in \Sigma \cup S_p$. Since there are only finitely many primes $\eta \in v(K^{ac})$, $\ker h_v$ is finite in this case.

Next, consider the case when $l \in S \setminus (\Sigma \cup S_p)$. Lemma 3.6 asserts that the group $E(K^{ac}_\eta)[p^\infty]$ is $p$-divisible for all $\eta \in v(K^{ac})$. In this case,

$$\ker h_\eta = H^0(K^{ac}_\eta, E[p^\infty])/p = 0.$$

Therefore, $h_v$ is injective.

**Proposition 3.8.** The kernel of $\Psi$ is finite. If Hypothesis 2.5 is satisfied, then $\text{cok } \Psi$ is also finite.

**Proof.** The Kummer sequence (3.1) induces the commutative diagram:

$$
\begin{array}{ccccccc}
0 & \to & R(E[p]/K^{ac}) & \to & H^1(K_{S/K^{ac}}, E[p]) & \to & \text{im}(\Phi_E) & \to & 0 \\
\downarrow \psi & & \downarrow \eta & & \downarrow h & & \downarrow & \\
0 & \to & R_{p^\infty}(E/K^{ac})[p] & \to & H^1(K_{S/K^{ac}}, E[p^\infty])[p] & \to & \bigoplus_{v \in S} H_v(K^{ac}, E[p^\infty])[p].
\end{array}
$$

Here, $\Phi_E$ is the natural map

$$\Phi_E : H^1(K_{S/K^{ac}}, E[p]) \to \bigoplus_{v \in S} H_v(K^{ac}, E[p]).$$
Clearly, the map $g$ is surjective. The snake lemma yields an exact sequence,
\[(3.3) \quad 0 \to \ker \Psi \to \ker g \to \ker h \to \cok \Psi \to 0.\]

Since $\ker g \simeq H^0(K^\text{ac}, E[p_\infty])/\rho$, it is finite (with cardinality at most $p^2$). From (3.3), we deduce that $\ker \Psi$ is finite. It follows from Corollary 3.7 that $\ker h$ is finite. Therefore, $\cok \Psi$ is finite as well. \hfill \Box

**Remark 3.9.** The proof of the above proposition shows that $\ker h$ must be finite if $\mu^\text{fine}(E/K^\text{ac}) = 0$ (even if Hypothesis 2.5 is not satisfied). This point is of considerable interest to the authors, since there is no apparent reason to suggest why this should be true in general.

**Proposition 3.10.** Let $E/K$ be an elliptic curve. The following assertions hold.

1. If $\mathcal{R}_p^\text{ac}(E/K^\text{ac})$ is $\Lambda$-cotorsion with $\mu^\text{fine}(E/K^\text{ac}) = 0$, then $\mathcal{R}(E[p]/K^\text{ac})$ is finite.
2. Assume that Hypothesis 2.5 is satisfied. Then, the fine Selmer group is $\Lambda$-cotorsion with $\mu^\text{fine}(E/K^\text{ac}) = 0$ if and only if $\mathcal{R}(E[p]/K^\text{ac})$ is finite.

**Proof.** The assertion is a consequence of Lemma 2.11 and Proposition 3.8. \hfill \Box

Let $F = K(E[p])$ be the field extension of $K$ left fixed by the kernel of $\overline{\rho}_E$. Note that $F$ is a Galois extension of $K$ with Galois group isomorphic to the image of $\overline{\rho}_E$. In particular, if $\overline{\rho}_E$ is reducible, then it is a 2-step solvable extension of $K$. Set $F_\infty$ to denote the compositum $K^\text{ac} \cdot F$. In the remainder of this section, let $X_E^S$ be the $S$-Hilbert class field extension of $F_\infty$. Note that there is a natural action of $\text{Gal}(F/K)$ on $X_E^S$. The $\mu$-invariant of $X_E^S$ is taken with respect to the $\mathbb{Z}_p$-extension $F_\infty/F$. Recall that in this section, there is no hypothesis on (the reducibility) of the residual representation $\overline{\rho}_E$.

**Corollary 3.11.** Let $E/K$ be an elliptic curve such that

1. Hypothesis 2.5 is satisfied.
2. $\mu(X_E^S) = 0$.

Then, the fine $\mu$-invariant, $\mu^\text{fine}(E/K^\text{ac}) = 0$.

**Proof.** By inflation-restriction, we have
\[0 \to H^1(\text{Gal}(F_\infty/K^\text{ac}), E[p]) \to H^1(K^\text{ac}, E[p]) \to \text{Hom}(F, E[p])^{\text{Gal}(F/K)}.\]

This induces the following map with finite kernel
\[(3.4) \quad \mathcal{R}(E[p]/K^\text{ac}) \to \text{Hom}(X_E^S/p, E[p])^{\text{Gal}(F/K)}.\]

Since $\mu(X_E^S) = 0$, the group $X_E^S/p$ is finite. By (3.4), the mod-$p$ fine Selmer group $\mathcal{R}(E[p]/K^\text{ac})$ is finite. The result follows from Proposition 3.10. \hfill \Box

### 4. Finiteness of the Residual Fine Selmer Group

Throughout this section $\overline{\rho}_E$ is assumed to be reducible. Therefore, $E[p]$ fits into the short exact sequence,
\[0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0.\]

This allows the analysis of the algebraic structure of the mod-$p$ fine Selmer group $\mathcal{R}(E[p]/K^\text{ac})$ in terms of the mod-$p$ fine Selmer groups $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^\text{ac})$ for $i = 1, 2$ (definitions given below). This completes the proof of Theorems 3.2 and 3.4.
**Definition 4.1.** Let $S$ be the set of primes dividing $N_p$, where $N$ is the conductor of $E$. The fine Selmer group $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac})$ is defined as follows

$$\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac}) := \ker \left\{ H^1(K_S/K^{ac}, \mathbb{F}_p(\varphi_i)) \to \prod_{\eta \in S(K^{ac})} H^1(K_{\eta}^{ac}, \mathbb{F}_p(\varphi_i)) \right\}. $$

The dependence on $S$ is suppressed in the notation.

**Proposition 4.2.** Suppose that $\rho_E$ is reducible and let $\varphi_i$ be one of the characters on the diagonal of $\rho_E$. Then, $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac})$ is finite if and only if $\mu(Y_i) = 0$.

*Proof.* Set $\Delta := \text{Gal}(K(\varphi_i)/K)$. Since the order of $\Delta$ is coprime to $p$, \(\text{Gal}(K^{ac}(\varphi_i)/K) \simeq \Delta \times \text{Gal}(K^{ac}/K)\).

The group $\text{Gal}(K^{ac}(\varphi_i)/K)$ acts on $\mathcal{X}^{(i)}$. Let $g \in \text{Gal}(K^{ac}(\varphi_i)/K)$ and $x \in \mathcal{X}^{(i)}$. Choose a lift $\tilde{g} \in \text{Gal}(\mathcal{L}^{(i)}/K)$ and set $g \cdot x := \tilde{g}x\tilde{g}^{-1}$. Since $\mathcal{X}^{(i)}$ is abelian, $g \cdot x$ is independent of the choice of lift, $\tilde{g}$. This induces the action of $\Delta$ on $\mathcal{X}^{(i)}$. There is a natural isomorphism

$$\text{Hom}(Y_i, \mathbb{F}_p) \simeq \text{Hom}(\mathcal{X}^{(i)}, \mathbb{F}_p(\varphi_i))^\Delta. $$

Since the order of $\text{Gal}(K^{ac}(\varphi_i)/K^{ac})$ is prime to $p$, it follows (from restriction-corestriction) that the cohomology group $H^j(\text{Gal}(K^{ac}(\varphi_i)/K^{ac}), \mathbb{F}_p(\varphi_i)) = 0$ for $j > 0$. It follows from the inflation-restriction sequence that

$$H^1(K_S/K^{ac}, \mathbb{F}_p(\varphi_i)) \cong \text{Hom}(\text{Gal}(K_S/K^{ac}(\varphi_i)), \mathbb{F}_p(\varphi_i))^\Delta, $$

where $\Delta$ is identified with $\text{Gal}(K^{ac}(\varphi_i)/K^{ac})$. This induces an isomorphism

$$\iota : \mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac}) \cong \text{Hom}(Y_i/pY_i, \mathbb{F}_p). $$

It is an easy consequence of the structure theory of $\Lambda$-modules (see the argument in the proof of Lemma 2.11) that the $\mu$-invariant of $Y_i$ is zero if and only if $Y_i/pY_i$ is finite. The result follows. \(\square\)

**Proposition 4.3.** Suppose that $\rho_E$ is reducible and $\varphi_i$ is one of the characters on the diagonal. Suppose that the following assumptions hold.

1. $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac})$ is finite.
2. The Heegner hypothesis is satisfied: let $v \in S \setminus S_p$ and $l$ be the prime number such that $v|l$, then $l$ splits in $K$.

Then, we have that $\mu(\tilde{Y}_i) = 0$.

This result is used in the proof of Corollary 3.5 where it is further assumed that $\varphi_1 = 1$. In this special case, the assertion is that the (classical) Iwasawa invariant $\mu(K^{ac}/K)$ vanishes.

*Proof.* Recall that the Galois extensions $\mathcal{M}^{(i)}$ and $\mathcal{X}^{(i)}$ of $K^{ac}(\varphi_i)$ are defined on p. 7. Consider the natural action of $\Delta := \text{Gal}(K(\varphi_i)/K)$ on $\mathcal{M}^{(i)}$ (resp. $\mathcal{X}^{(i)}$) and recall that $\tilde{Y}_i$ (resp. $Y_i$) denotes $\varphi_i$-component of $\mathcal{M}^{(i)}$ (resp. $\mathcal{X}^{(i)}$) with respect to this action. At each prime $\eta$ of $K^{ac}$, set $H^1_{\text{nr}}(K_{\eta}^{ac}, \mathbb{F}_p(\varphi_i))$ to denote the subspace of unramified cohomology classes in $H^1(K_{\eta}^{ac}, \mathbb{F}_p(\varphi_i))$. Set $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{ac})$ to consist of cohomology classes $f \in H^1(K_S/K^{ac}, \mathbb{F}_p(\varphi_i))$ which are unramified at every
prime $\eta \in S(K^{ac})$. By an application of inflation-restriction (see the argument in the proof of Proposition \[ \ref{12} \]),

$$\mathcal{R}'(\mathbb{F}_p(\varphi_i) / K^{ac}) \simeq \text{Hom}(\mathcal{M}^{(i)}/p, \mathbb{F}_p(\varphi_i)) \Delta \simeq \text{Hom}(\tilde{Y}_i/p, \mathbb{F}_p),$$

$$\mathcal{R}(\mathbb{F}_p(\varphi_i) / K^{ac}) \simeq \text{Hom}(\mathcal{X}^{(i)}/p, \mathbb{F}_p(\varphi_i)) \Delta \simeq \text{Hom}(Y_i/p, \mathbb{F}_p).$$

The groups fit into an exact sequence,

$$0 \to \mathcal{R}(\mathbb{F}_p(\varphi_i) / K^{ac}) \to \text{Hom}(\tilde{Y}_i/p, \mathbb{F}_p(\varphi_i)) \to \prod_{\eta \in S(K^{ac})} H^1_{nr}(K^{ac}, \mathbb{F}_p(\varphi_i)).$$

By Fact \[ \ref{2.3} \] the set $S(K^{ac})$ is finite. Hence, the group on the right is finite. We deduce that $\tilde{Y}_i/p$ is finite, therefore the $\mu$-invariant of $\tilde{Y}_i$ is zero. □

**Proof of Theorem \[ \ref{3.2} \]** By Proposition \[ \ref{11} \] it suffices to show that $\mathcal{R}(E[p]/K^{ac})$ is finite. The short exact sequence

$$0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0,$$

induces the long exact sequence

$$\cdots \to H^0(K^{ac}, \mathbb{F}_p(\varphi_2)) \xrightarrow{\delta^0} H^1(K^{ac}, \mathbb{F}_p(\varphi_1)) \to H^1(K^{ac}, E[p]) \xrightarrow{} \cdots.$$

Set $S_{ac} := S(K^{ac})$ and

$$U := \prod_{\eta \in S_{ac}} \text{image } \delta^0_\eta \subset \prod_{\eta \in S_{ac}} H^1_{nr}(K^{ac}, \mathbb{F}_p(\varphi_1)).$$

If $\overline{\rho}_E$ is split, then $\delta^0_\eta$ is the 0 map; hence, $U = 0$.

Next, consider the case when $\overline{\rho}_E$ is indecomposable. When $v \in S \setminus (\Sigma(\varphi_2) \cup S_p)$, note that $H^0(K_v, \mathbb{F}_p(\varphi_2)) = 0$. Let $\eta \in v(K^{ac})$. Since $K^{ac}_\eta/K_v$ is a $p$-extension, for all $v \in S \setminus (\Sigma(\varphi_2) \cup S_p)$, the product $\prod_{\eta \in v(K^{ac})} H^0(K^{ac}_\eta, \mathbb{F}_p(\varphi_2)) = 0$. On the other hand, for all primes $v \in \Sigma(\varphi_2) \cup S_p$, it follows from Hypothesis \[ \ref{2.7} \] and Fact \[ \ref{2.3} \] that there are only finitely many primes $\eta$ in $v(K^{ac})$. Therefore, the assumptions imply that $U$ is finite.

Consider the diagram

$$\begin{array}{cccccc}
H^1(K_S/K^{ac}, \mathbb{F}_p(\varphi_1)) & \longrightarrow & H^1(K_S/K^{ac}, E[p]) & \longrightarrow & H^1(K_S/K^{ac}, \mathbb{F}_p(\varphi_2)) \\
\downarrow & & & & \downarrow \\
0 & \longrightarrow & \left( \prod_{\eta \in S_{ac}} H^1(K^{ac}_\eta, \mathbb{F}_p(\varphi_1)) \right)/U & \longrightarrow & \prod_{\eta \in S_{ac}} H^1(K^{ac}_\eta, E[p]) & \longrightarrow & \prod_{\eta \in S_{ac}} H^1(K^{ac}_\eta, \mathbb{F}_p(\varphi_2)) \\
\end{array}$$

and the associated short exact sequence

$$\text{ker } \alpha \to \mathcal{R}(E[p]/K^{ac}) \to \mathcal{R}(\mathbb{F}_p(\varphi_2)/K^{ac}).$$

The group $\text{ker } \alpha$ fits into an exact sequence

$$0 \to \mathcal{R}(\mathbb{F}_p(\varphi_1)/K^{ac}) \to \text{ker } \alpha \to U.$$

Proposition \[ \ref{12} \] asserts that $\mathcal{R}(\mathbb{F}_p(\varphi_i), K^{ac})$ are finite for $i = 1, 2$. This requires the assumption that $\mu(Y_i) = 0$ for $i = 1, 2$. It follows from \[ \ref{11} \] and \[ \ref{12} \] that $\mathcal{R}(E[p]/K^{ac})$ is also finite. By Lemma \[ \ref{2.11} \] we have that $\mathcal{R}_{p=\infty}(E/K^{ac})$ is $\Lambda$-cotorsion with $\mu^{\text{fine}}(E/K^{ac}) = 0$. This completes the proof. □
Proof of Theorem 5.2. Refer to the argument in the proof of Theorem 3.2. Lemma 2.11 asserts that if \( R_{p\infty}(E/K^{ac}) \) is \( \Lambda \)-torsion with \( \mu^{\text{fine}}(E/K^{ac}) = 0 \), then \( R(E[p]/K^{ac}) \) is finite. Refer to (4.1). Since \( H^0(K_S/K^{ac}, \mathbb{F}_p(\varphi_1)) \) is finite, finiteness of \( R(E[p]/K^{ac}) \) implies that \( \ker \alpha \) is finite. By (1.2), it is seen that \( R(\mathbb{F}_p(\varphi_1)/K^{ac}) \) is finite. Therefore, by Proposition 1.2, it follows that \( \mu(Y_1) = 0 \). This completes the proof in the case when \( \overline{\rho}_E \) is indecomposable. When \( \overline{\rho}_E \) is split, the proof is completed by interchanging the roles of \( \varphi_1 \) and \( \varphi_2 \).

\[ \square \]

5. **Congruent Galois representations**

In this section, we consider two elliptic curves \( E_1 \) and \( E_2 \) defined over \( K \) which are \( p \)-congruent, i.e. their residual representations are isomorphic,

\[ \overline{\rho}_{E_1} \simeq \overline{\rho}_{E_2}. \]

We make no assumption on the reducibility of the residual representations. However, it is assumed that both \( E_1 \) and \( E_2 \) satisfy the Heegner hypothesis.

**Hypothesis 5.1** (Heegner hypothesis). Let \( E/K \) be an elliptic curve with conductor \( \mathcal{N} \), and \( S \) be the set of primes dividing \( \mathcal{N}p \). Let \( v \in S \) and \( l \) be the prime number such that \( v|l \). Then, \( l \) splits in \( K \) if \( l \neq p \).

We prove the following main result in this section.

**Theorem 5.2.** Let \( E_1 \) and \( E_2 \) be elliptic curves over \( K \). Suppose that the following assumptions hold.

1. Both \( E_1 \) and \( E_2 \) satisfy the Heegner hypothesis.
2. The residual Galois representations, \( \overline{\rho}_{E_1} \) and \( \overline{\rho}_{E_2} \), are isomorphic.

Then,

\[ R_{p\infty}(E_1/K^{ac}) \text{ is } \Lambda\text{-torsion with } \mu^{\text{fine}}(E_1/K^{ac}) = 0 \]

\[ \iff R_{p\infty}(E_2/K^{ac}) \text{ is } \Lambda\text{-torsion with } \mu^{\text{fine}}(E_2/K^{ac}) = 0. \]

The key result used in the proof of the above theorem will be Proposition 3.10 which states that

\[ R_{p\infty}(E_i/K^{ac}) \text{ is } \Lambda\text{-torsion with } \mu^{\text{fine}}(E_i/K^{ac}) = 0 \iff R(E_i[p]/K^{ac}) \text{ is finite}. \]

In view of Lemma 2.11 it suffices to prove the following result.

**Proposition 5.3.** Let \( E_1 \) and \( E_2 \) be elliptic curves satisfying the conditions of Theorem 5.2. Then \( R(E_1[p]/K^{ac}) \) is finite if and only if \( R(E_2[p]/K^{ac}) \) is finite.

**Proof.** Let \( \mathcal{N}_i \) be the conductor of \( E_i \) and \( S_i \) be the set of primes dividing \( \mathcal{N}_ip \). Denote by \( T \) the union \( S_1 \cup S_2 \). Since \( E_1[p] \) and \( E_2[p] \) are isomorphic as Galois modules, the mod-\( p \) fine Selmer groups are isomorphic, i.e.

\[ R^T(E_1[p]/K^{ac}) \simeq R^T(E_2[p]/K^{ac}) \]

(recall \( R^T(E_i[p]/K^{ac}) \) from Definition 2.10). Let \( v \in T \) and \( \eta \in v(K^{ac}) \). Denote by \( G_\eta \) the absolute Galois group of \( K^{ac}_\eta \) and by \( I_\eta \subset G_\eta \) its inertia subgroup. Set \( H^1_{\text{nr}}(K^{ac}_\eta, E_i[p]) \) to denote the kernel of the restriction map,

\[ \text{res} : H^1(K^{ac}_\eta, E_i[p]) \rightarrow H^1(I_\eta, E_i[p])^{G_\eta}/I_\eta. \]
It follows from inflation-restriction that $H^1_{nr}(K_{\eta}^{ac},E_i[p])$ can be identified with $H^1(G_{\eta}/I_{\eta},E_i[p]^{I_{\eta}})$. Since $G_{\eta}/I_{\eta} \simeq \hat{\mathbb{Z}}$, it follows that (see [26 Proposition 1.7.7])

$$H^1(G_{\eta}/I_{\eta},E_i[p]^{I_{\eta}}) \simeq H_0(G_{\eta}/I_{\eta},E_i[p]^{I_{\eta}})$$

and hence is finite. By the Heegner hypothesis and Fact 22, the set $v(K^{ac})$ is finite for $v \in T$. Let $\mathcal{H}_v^{nr}(K^{ac},E[p]) \subseteq \mathcal{H}_v(K^{ac},E[p])$ denote the sum,

$$\mathcal{H}_v^{nr}(K^{ac},E_i[p]) := \bigoplus_{\eta \in v(K^{ac})} H^1_{nr}(K_{\eta}^{ac},E_i[p]).$$

It follows from the above discussion that $\mathcal{H}_v^{nr}(K^{ac},E_i[p])$ is finite for $v \in T$.

There is a natural exact sequence,

$$0 \rightarrow \mathcal{R}^T(E_i[p]/K^{ac}) \rightarrow \mathcal{R}(E_i[p]/K^{ac}) \rightarrow \bigoplus_{v \in T \setminus S_i} \mathcal{H}_v^{nr}(K^{ac},E_i[p]).$$

Since the last term in the above sequence is finite, it follows that $\mathcal{R}(E_i[p]/K^{ac})$ is finite if and only if $\mathcal{R}^T(E_i[p]/K^{ac})$ is finite. The result follows from this. \hfill \Box

**Proof of Theorem 5.2** Note that the Heegner hypothesis implies Hypothesis 25. Therefore, Proposition 5.10 applies in this setting and asserts that

$$\mathcal{R}_p(\Lambda/K^{ac})$$

is $\Lambda$-cotorsion with $\mu_{fine}(\Lambda/K^{ac}) = 0 \iff \mathcal{R}(E_i[p]/K^{ac})$ is finite.

On the other hand, Proposition 5.3 states that

$$\mathcal{R}(E_i[p]/K^{ac})$$

is finite $\iff \mathcal{R}(E_2[p]/K^{ac})$ is finite.

The result follows. \hfill \Box

### 6. Examples

In this section, we list some examples to illustrate the results in the article.

**Example 6.1.** We begin with an example which illustrates Theorem 5.2. In this example, $p = 3$ and $\tilde{E}$ is the elliptic curve 81.1-CMa1 over $K = \mathbb{Q}(\sqrt{-3})$. This is the elliptic curve with Weierstrass equation $y^2 + y = \frac{x^3}{3}$ base changed to $K$. It has complex multiplication and its endomorphism ring is $\mathbb{Z}[\frac{1 + \sqrt{-3}}{2}]$. The group $E(K)[3]$ is the full group $E[3]$ (and isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$). In particular, residual representation $\tilde{\rho}_E$ is the trivial representation $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, $\tilde{\rho}_E$ is split, $\varphi_1 = \varphi_2 = 1$, and $Y_1$ and $Y_2$ are both the 3-Hilbert class field extension of $K^{ac}$. The prime $p = 3$ is a prime of additive reduction and 3 is ramified in $K$. The conductor of $E$ is $N = 9\mathcal{O}_K$ and in particular is not divisible by any primes $v \nmid 3$. It follows that $\Sigma = \emptyset$ and Hypothesis 25 is satisfied. This unique prime above (3) is totally ramified in the anticyclotomic $\mathbb{Z}_3$-extension of $K$. Since the class number of $K$ is 1, we know that $\mu(K^{ac}/K) = 0$ [27 Proposition 4.2]. It follows from Theorem 3.2 that $\mathcal{R}_3(\Lambda/K^{ac})$ is a cotorsion $\Lambda$-module with $\mu_{fine}(\Lambda/K^{ac}) = 0$ (at $p = 3$). In other words, $\mathcal{R}_3(\Lambda/K^{ac})$ is a cofinitely generated $\mathbb{Z}_3$-module.

**Example 6.2.** This example demonstrates Corollary 3.5. Let $K = \mathbb{Q}(\sqrt{-10})$, $p = 5$, and $E$ be the elliptic curve 11a1 over $\mathbb{Q}$ defined by the Weierstrass equation $y^2 + y = x^3 - x^2 - 10x - 20$. The 5-torsion group $E(\mathbb{Q})[5]$, is equal to $\mathbb{Z}/5\mathbb{Z}$. In particular, $E(K)[5] \neq 0$. Assume that $\mathcal{R}_5(\Lambda/K^{ac})$ is a cotorsion $\Lambda$-module.
The set $S$ consists of the primes above 11 in $K$. Since 11 splits in $K$, it follows that condition 3 of Corollary 3.5 is satisfied. Corollary 3.5 asserts that if the $\mu$-invariant of the fine Selmer group $R_p^\infty(E/K^{ac})$ is zero, then the Iwasawa $\mu$-invariant of the 5-Hilbert class field extension of $K^{ac}/K$ is zero as well.

**Example 6.3.** This example demonstrates an application of Theorem 5.2. Consider the elliptic curves $E_1 = 201a1$ and $E_2 = 469a1$. Both elliptic curves have rank 1 with good ordinary reduction at the prime $p = 5$. Further, there is an isomorphism of $G_{\mathbb{Q}}$-modules, $E_1[5] \simeq E_2[5]$. Note that $E_1$ has bad reduction at the primes 3, 67 and $E_2$ has bad reduction at the primes 7, 67. Set $M$ to be the product of primes of bad reduction $3 \cdot 7 \cdot 67$. The primes of bad reduction of $E_1$, $E_2$ split in infinitely many imaginary quadratic number field $K$ of the form $\mathbb{Q}(\sqrt{-Mk+1})$ where $k \in \mathbb{Z}_{\geq 1}$, i.e. the Heegner hypothesis is satisfied. Theorem 5.2 applies; hence

$$R_p^\infty(E_1/K^{ac}) \text{ is } \Lambda\text{-cotorsion with } \mu^\text{fine}(E_1/K^{ac}) = 0$$

$$\iff R_p^\infty(E_2/K^{ac}) \text{ is } \Lambda\text{-cotorsion with } \mu^\text{fine}(E_2/K^{ac}) = 0.$$ 

**Example 6.4.** Through this final example we demonstrate that Theorem 5.2 can be used to relate $\mu$-invariants of isogenous elliptic curves over $K^{ac}$. Consider the isogenous elliptic curves $E_1 = 17a1$ and $E_2 = 17a2$. Both elliptic curves have rank 0 and good supersingular reduction at the prime $p = 11$. Further, there is an isomorphism of $G_{\mathbb{Q}}$-modules, $E_1[11] \simeq E_2[11]$. Consider the imaginary quadratic number field, $K = \mathbb{Q}(\sqrt{-8})$. Note that the prime 17 splits in $K$, i.e. both $E_1$ and $E_2$ satisfy the Heegner hypothesis. It is known that $\mu^\text{fine}(E_1/K^{ac}) = 0$ (see Table in §4). It follows from Theorem 5.2 that $\mu^\text{fine}(E_2/K^{ac}) = 0$, as well.

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