Fekete-Szegö Problem for a Subclass of Analytic Functions Associated with Chebyshev Polynomials

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ABSTRACT: In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for a certain subclass by means of Chebyshev polynomial expansions of analytic functions in $D$. Also, we solve Fekete-Szegő problem for functions in this subclass.

Key Words: Analytic and univalent functions, Subordination, Coefficient bounds, Chebyshev polynomial, Fekete-Szegő problem.

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1. Introduction

Let $A$ be the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $D = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Also, let $S$ be the subclass of $A$ consisting of the form (1.1) which are univalent in $D$.

Let $f$ and $g$ be analytic functions in $D$. We define that the function $f$ is subordinate to $g$ in $D$ and denoted by

$$f(z) \prec g(z) \quad (z \in D),$$

if there exists a Schwarz function $\omega$, which is analytic in $D$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in D$) such that

$$f(z) = g(\omega(z)) \quad (z \in D).$$

If $g$ is a univalent function in $D$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(D) \subset g(D).$$

Chebyshev polynomials play a considerable role in numerical analysis ([4], [8]). There are four kinds of Chebyshev polynomials. The first and second kinds of Chebyshev polynomials are defined by $T_n(t) = \cos n \varphi$ and $U_n(t) = \frac{\sin (n+1) \varphi}{\sin \varphi}$ ($-1 < t < 1$) where $n$ denotes the polynomial degree and $t = \cos \varphi$. For a brief history of Chebyshev polynomials of the first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [1]-[16].

Now, we define a subclass of analytic functions in $D$ with the following subordination condition:

Definition 1.1. A function $f \in A$ given by (1.1) is said to be in the class $N(\lambda, \beta, t)$ for $0 \leq \beta \leq \lambda \leq 1$ and $t \in \left( \frac{1}{2}, 1 \right]$ if the following subordination hold:

$$\frac{\lambda \beta z^2 f'''(z) + (2 \lambda \beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda \beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2} \quad (z \in D).$$

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We consider that if \( t = \cos \varphi \left( \frac{2}{3} < \varphi < \frac{\pi}{2} \right) \), then \( H(z, t) = \frac{1}{1 - 2 \cos \varphi z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n + 1) \varphi}{\sin \varphi} z^n \) \((z \in \mathbb{D})\). Thus, \( H(z, t) = 1 + 2 \cos \varphi z + (3 \cos^2 \varphi - \sin^2 \varphi) z^2 + \cdots \) \((z \in \mathbb{D})\).

So, according to [15], we write the Chebyshev polynomials of the second kind as following:

\[
H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in \mathbb{D}, -1 < t < 1)
\]

where \( U_{n-1}(t) = \frac{\sin(n \arccos t)}{\sqrt{1 - t^2}} \) \((n \in \mathbb{N})\) and we have \( U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t) \),

\[
U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \cdots .
\]

The Chebyshev polynomials \( T_n(t), t \in [-1, 1] \) of the first kind have the generating function of the form \( \sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \) \((z \in \mathbb{D})\).

There is the following connection by the Chebyshev polynomials of the first kind \( T_n(t) \) and the second kind \( U_n(t) \):

\[
\frac{dT_n(t)}{dt} = nU_{n-1}(t), \quad T_n(t) = U_n(t) - tU_{n-1}(t), \quad 2T_n(t) = U_n(t) - U_{n-2}(t).
\]

In 1933, Fekete and Szegö [6] obtained a sharp bound of the functional \(|a_3 - \mu a_2^2|\), with real \( \mu \) \((0 \leq \mu \leq 1)\) for a univalent function \( f \). Since then, the problem of finding the sharp bounds for this functional of any compact family of functions or \( f \in \mathcal{A} \) with any complex \( \mu \) is known as the classical Fekete-Szegö problem or inequality.

In this paper, we obtain initial coefficients \(|a_2|\) and \(|a_3|\) for subclass \( N(\lambda, \beta, t) \) by means of Chebyshev polynomials expansions of analytic functions in \( \mathbb{D} \). Also, we solve Fekete-Szegö problem for functions in this subclass.

### 2. Coefficient bounds for the function class \( N(\lambda, \beta, t) \)

We begin with the following result involving initial coefficient bounds for the function class \( N(\lambda, \beta, t) \).

**Theorem 2.1.** Let the function \( f(z) \) given by (1.1) be in the class \( N(\lambda, \beta, t) \). Then

\[
|a_2| \leq \frac{2t}{2\lambda\beta + \lambda - \beta + 1}
\]

and

\[
|a_3| \leq \frac{8t^2 - 1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}.
\]

**Proof.** Let \( f \in N(\lambda, \beta, t) \). From (1.2), we have

\[
\frac{\lambda \beta z^3 f'''(z) + (2\lambda \beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda \beta z^2 f''(z) + (1 - \lambda + \beta)zf'(z) + (1 - \lambda + \beta)f(z)} = 1 + U_1(t)p(z) + U_2(t)p^2(z) + \cdots
\]

for some analytic functions

\[
p(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \quad (z \in \mathbb{D}),
\]

such that \( p(0) = 0, |p(z)| < 1 \) \((z \in \mathbb{D})\). Then, for all \( j \in \mathbb{N} \),

\[
|c_j| \leq 1
\]

and for all \( \mu \in \mathbb{R} \)

\[
|c_2 - \mu c_1^2| \leq \max \{1, |\mu|\}.
\]

It follows from (2.3) that

\[
\frac{\lambda \beta z^3 f'''(z) + (2\lambda \beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda \beta z^2 f''(z) + (1 - \lambda + \beta)zf'(z) + (1 - \lambda + \beta)f(z)} = 1 + U_1(t)c_1z + \left[ U_1(t)c_2 + U_2(t)c_1^2 \right] z^2 + \cdots .
\]
It follows from (2.7) that
\[(2\lambda \beta + \lambda - \beta + 1) a_2 = U_1 (t) c_1, \tag{2.8}\]
and
\[2(6\lambda \beta + 2\lambda - 2\beta + 1) a_3 - (2\lambda \beta + \lambda - \beta + 1)^2 a_2^2 = U_1 (t) c_2 + U_2 (t) c_1^2. \tag{2.9}\]
From (1.3), (2.8) and (2.5), we have
\[|a_2| \leq \frac{2t}{2\lambda \beta + \lambda - \beta + 1}. \tag{2.10}\]
By using (1.3) and (2.5) in (2.9), we obtain
\[|a_3| \leq \frac{8t^2 - 1}{2(6\lambda \beta + 2\lambda - 2\beta + 1)}. \tag{2.11}\]
which completes the proof of Theorem 2.1.

For \(\lambda = 1\) in Theorem 2.1, we obtain the following corollary.

**Corollary 2.2.** Let the function \(f(z)\) given by (1.1) be in the class \(N(1, \beta, t)\). Then
\[|a_2| \leq \frac{2t}{\beta + 2}\]
and
\[|a_3| \leq \frac{8t^2 - 1}{2(4\beta + 3)}.\]

If we choose \(\beta = 0\) in Theorem 2.1, we get the following corollary.

**Corollary 2.3.** Let the function \(f(z)\) given by (1.1) be in the class \(N(\lambda, 0, t)\). Then
\[|a_2| \leq \frac{2t}{\lambda + 1}\]
and
\[|a_3| \leq \frac{8t^2 - 1}{2(2\lambda + 1)}.\]

For \(\beta = \lambda\) in Theorem 2.1, we obtain the following corollary.

**Corollary 2.4.** Let the function \(f(z)\) given by (1.1) be in the class \(N(\beta, t)\). Then
\[|a_2| \leq \frac{2t}{2\beta^2 + 1}\]
and
\[|a_3| \leq \frac{8t^2 - 1}{2(6\beta^2 + 1)}.\]

**Remark 2.5.** For \(\beta = 0\) and \(\lambda = 1\) in Theorem 2.1, we obtain result of Dziok et al. [5, Theorem 6].
3. Fekete-Szegö inequality for the function class \( N(\lambda, \beta, t) \)

Now, we find the sharp bounds of Fekete-Szegö functional \(|a_3 - \mu a_2^2|\) defined for \( N(\lambda, \beta, t) \).

**Theorem 3.1.** Let the function \( f(z) \) given by (1.1) be in the class \( N(\lambda, \beta, t) \). Then for some \( \mu \in \mathbb{R} \),

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{6\lambda \beta + 2\lambda - 2\beta + 1}{6\lambda \beta + 2\lambda - 2\beta + 1} & \mu \in [\mu_1, \mu_2], \\
\frac{8t^2 - 1}{2t} - \mu \frac{4t(6\lambda \beta + 2\lambda - 2\beta + 1)}{2(2\lambda + \lambda - \beta + 1)^2} & \mu \notin [\mu_1, \mu_2],
\end{array} \right.
\]  

(3.1)

where \( \mu_1 = \frac{(8t^2 - 2t - 1)(2\lambda \beta + \lambda - \beta + 1)^2}{8t^2(6\lambda \beta + 2\lambda - 2\beta + 1)} \) and \( \mu_2 = \frac{(8t^2 + 2t - 1)(2\lambda \beta + \lambda - \beta + 1)^2}{8t^2(6\lambda \beta + 2\lambda - 2\beta + 1)} \).

**Proof.** Let \( f \in N(\lambda, \beta, t) \). By using (2.8) and (2.9) for some \( \mu \in \mathbb{R} \), we have

\[
|a_3 - \mu a_2^2| = \frac{U_1(t)}{2(6\lambda \beta + 2\lambda - 2\beta + 1)} c_2 + \left\{ \frac{U_2(t)}{U_1(t)} U_1(t) - 2\mu \frac{(6\lambda \beta + 2\lambda - 2\beta + 1) U_1(t)}{(2\lambda \beta + \lambda - \beta + 1)^2} \right\} c_1^2.
\]  

(3.2)

Then, in view of (2.6), we conclude that

\[
|a_3 - \mu a_2^2| \leq \frac{U_1(t)}{2(6\lambda \beta + 2\lambda - 2\beta + 1)} \max \left\{ 1, \left| \frac{U_2(t)}{U_1(t)} \right| U_1(t) - 2\mu \frac{(6\lambda \beta + 2\lambda - 2\beta + 1) U_1(t)}{(2\lambda \beta + \lambda - \beta + 1)^2} \right\}.
\]  

(3.3)

Finally, by using (1.3) in (3.3), we get

\[
|a_3 - \mu a_2^2| \leq \frac{t}{6\lambda \beta + 2\lambda - 2\beta + 1} \max \left\{ 1, \left| \frac{8t^2 - 1}{2t} - 4\mu \frac{(6\lambda \beta + 2\lambda - 2\beta + 1) t}{2(2\lambda + \lambda - \beta + 1)^2} \right| \right\}.
\]

Because \( t > 0 \), we obtain

\[
\left| \frac{8t^2 - 1}{2t} - 4\mu \frac{(6\lambda \beta + 2\lambda - 2\beta + 1) t}{2(2\lambda + \lambda - \beta + 1)^2} \right| \leq 1
\]

\[
\Leftrightarrow \left\{ \frac{(8t^2 - 2t - 1)(2\lambda \beta - \lambda - \beta + 1)^2}{8t^2(6\lambda \beta + 2\lambda - 2\beta + 1)} \right\} \leq \mu \leq \frac{(8t^2 + 2t - 1)(2\lambda \beta + \lambda - \beta + 1)^2}{8t^2(6\lambda \beta + 2\lambda - 2\beta + 1)}
\]

\[
\Leftrightarrow \mu_1 \leq \mu \leq \mu_2.
\]

This proves Theorem 3.1. \( \Box \)

For \( \lambda = 1 \) in Theorem 3.1, we obtain the following corollary.

**Corollary 3.2.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathbb{N}(1, \beta, t) \). Then for some \( \mu \in \mathbb{R} \),

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{4\beta + 3}{4\beta + 3} & \mu \in [\mu_1, \mu_2], \\
\frac{8t^2 - 1}{2t} - \mu \frac{4t(4\beta + 3)}{(\beta + 2)^2} & \mu \notin [\mu_1, \mu_2],
\end{array} \right.
\]  

where \( \mu_1 = \frac{(8t^2 - 2t - 1)(\beta + 2)^2}{8t^2(4\beta + 3)} \) and \( \mu_2 = \frac{(8t^2 + 2t - 1)(\beta + 2)^2}{8t^2(4\beta + 3)} \).

If we choose \( \beta = 0 \) in Theorem 3.1, we get the following corollary.

**Corollary 3.3.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathbb{N}(\lambda, 0, t) \). Then for some \( \mu \in \mathbb{R} \),

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{2\lambda + 1}{2\lambda + 1} & \mu \in [\mu_1, \mu_2], \\
\frac{8t^2 - 1}{2t} - \mu \frac{4t(2\lambda + 1)}{(\lambda + 1)^2} & \mu \notin [\mu_1, \mu_2],
\end{array} \right.
\]  

where \( \mu_1 = \frac{(8t^2 - 2t - 1)(\lambda + 1)^2}{8t^2(2\lambda + 1)} \) and \( \mu_2 = \frac{(8t^2 + 2t - 1)(\lambda + 1)^2}{8t^2(2\lambda + 1)} \).
For $\beta = \lambda$ in Theorem 3.1, we obtain the following corollary.

**Corollary 3.4.** Let the function $f(z)$ given by (1.1) be in the class $N(\beta, t)$. Then for some $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{6\beta^2+1}, & \mu \in [\mu_1, \mu_2], \\
\frac{8t^2-1}{2t} - \mu \frac{4t(6\beta^2+1)}{(2\beta^2+1)^2}, & \mu \notin [\mu_1, \mu_2], \end{cases}$$

where $\mu_1 = \frac{(8t^2-2t-1)(2\beta^2+1)^2}{8t(6\beta^2+1)}$ and $\mu_2 = \frac{(8t^2+2t-1)(2\beta^2+1)^2}{8t(6\beta^2+1)}$.

**Remark 3.5.** For $\beta = 0$ in Theorem 3.1, we obtain result of Mustafa and Akbulut [10].

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