Supermartingale deflators in the absence of a numéraire

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Abstract
In this paper we study arbitrage theory of financial markets in the absence of a numéraire both in discrete and continuous time. In our main results, we provide a generalization of the classical equivalence between no unbounded profits with bounded risk and the existence of a supermartingale deflator. To obtain the desired results, we introduce a new approach based on disintegration of the underlying probability space into spaces where the market crashes at deterministic times.

Keywords Supermartingale deflator · Absence of a numéraire · NUPBR · Fundamental theorem of asset pricing · Arbitrage of the first kind

Mathematics Subject Classification 60G48 · 91B70 · 91G99

JEL classification C02

1 Introduction

Overview A nearly universal assumption in the arbitrage theory of financial markets is the existence of a numéraire, i.e., the existence of a strictly positive traded asset. For instance, this assumption underlies the celebrated fundamental theorems of Delbaen, Schachermayer, Kabanov, and Kardaras [3,6,9]. In practice, however, it is not always reasonable to make this assumption. Indeed, it may well happen in the presence of credit or systemic risk that all assets under consideration default in finite time. For instance, when a country defaults on its debt and issues a new currency, this devalues not only the domestic bond market but also
its numéraire. Markets with arbitrarily low negative interest rates are approximations of this situation. Moreover, a similar situation occurs in financial models where the assets within a defaultable market segment are quoted in terms of an index or market average, as can be reasonable in portfolio optimization or hedging. Motivated by these examples, the purpose of this work is to study arbitrage theory of financial markets in the absence of a numéraire.

Our main result is a generalization of the classical equivalence between no unbounded profits with bounded risk (NUPBR) and the existence of a supermartingale deflator [10]. NUPBR is a pivotal notion in arbitrage theory and a minimal requirement for reasonable financial models [2,3,7,9,10]. It also plays a fundamental role in defining path-wise stochastic integrals in model-free finance [11]. To put it into context, at least for markets with numéraire, Delbaen and Schachermayer's condition of no free lunch with vanishing risk (NFLVR) is equivalent to NUPBR together with no-arbitrage (NA) [2]. However, there are many reasonable models such as the three-dimensional Bessel process which satisfy NUPBR but violate NA. Moreover, NUPBR is all that is needed for ensuring that expected utility maximization is well defined, and the maximizer is precisely the desired supermartingale deflator [7]. We show that a similar result holds for markets without numéraire. Namely, in discrete time, NUPBR remains equivalent to the existence of a supermartingale deflator. In continuous time, this equivalence holds under an independence assumption on the time of the market crash. Without this independence assumption, the equivalence holds subject to additional boundedness conditions on the market, which can be rephrased equivalently as boundedness conditions on the deflator. It remains open to what extent these additional conditions are really necessary.

Arbitrage theory without numéraire   The construction of the supermartingale deflator in [10] via maximization of the expected log-utility presupposes the existence of a numéraire to ensure that the maximization problem is well-defined. This is not merely a shortcoming of the proof but turns out to be a fundamental problem, which requires several adaptations of classical definitions and arguments, as outlined next.

First, the notion of NUPBR is too weak. Recall that NUPBR is defined as boundedness in probability of the payoffs at the terminal time $T$. When there is a numéraire, this implies boundedness in probability of the payoffs at all intermediate times $t < T$, see [10]. However, in the absence of a numéraire, the payoffs at intermediate times may be unbounded in probability, as e.g. in Example 2.10, and this rules out the existence of a strictly positive supermartingale deflator.

Second, the notion of fork convexity (also known as switching property) is too weak. According to the classical definition, fork convexity allows an agent to switch from any given asset to any other strictly positive asset. However, markets without numéraire may not contain any strictly positive asset at all. In this case fork convexity is trivially satisfied. The correct modification is to allow the agent to switch to a new asset contingent on the new asset being positive at the given time and state of nature, as spelled out in Definition 2.4.

Third, the following argument, which is crucial for the construction of a deflator in [10, Theorem 2.3], breaks down: if the terminal payoff $X_T$ of an asset $X$ is optimal within the set of all terminal payoffs, then the payoff $X_t$ is optimal within the set of all payoffs at time $t$, for any intermediate time $t < T$. For example, this clearly does not hold on markets where all the terminal payoffs vanish identically. Additionally, some arguments in [10] concerning the regularization of generalized supermartingales break down because they also rely on the existence of a numéraire.

The time where the market crashes   Methodologically, this work relies heavily on an analysis of the first time $\tau$ where all assets in the market vanish or, more succinctly, the time
τ where the market crashes. Loosely speaking, one may partition the scenario space Ω into disjoint subsets Ω_τ where τ is constant and equal to t. On each slice Ω_τ, there exists a process which is strictly positive up to time t and therefore can serve as a numéraire. Thus, one obtains under the classical conditions of [10] a supermartingale deflator Z_t on each space Ω_τ endowed with the conditional probability measure. These local deflators Z'_t on Ω_τ can then be pasted into a global deflator Z on Ω.

This sketch can be turned rather directly into a rigorous proof if τ has countable support; see Theorem 2.18. Otherwise, the conditional probabilities (provided they exist) may be singular with respect to P, and consequently NUPBR on Ω does not entail NUPBR on Ω_τ. To overcome these issues, we discretize time into a finite dyadic grid of 2^n intervals and apply the above pasting method there. This produces a strictly positive supermartingale deflator on the grid. Passing to the limit n → ∞ while preserving the strict positivity is the most important and difficult part of the paper. This requires good lower bounds on the deflators or, equivalently, good upper bounds on the assets.

Previous literature
Previously, arbitrage theory for markets without numéraires has been studied only in finite discrete time by Tehranchi [12]. However, recently a related preprint of Bálint [1] on continuous-time markets, based on research independent of ours, has appeared. The philosophy in [1], as well as in the present paper, is as follows: one first localizes the market, then constructs local deflators on each localized piece, and finally pastes them together to get a global deflator. The essential difference is that the localization in [1] is performed in time, i.e., the terminal date T is approximated from below by a sequence of stopping times (T^n)_{n≥1} such that on each time horizon [0, T^n) the market contains a numéraire. In contrast, the methodology of this paper is to localize the market in “space” in the sense that the sample space Ω is partitioned into different parts such that there exists a numéraire under the corresponding conditional measure. Then it is intuitively clear that the approach in [1] works very well in continuous time but fails if the underlying processes are non-adapted to the given filtration; on the other hand, our techniques can handle non-adaptedness (in particular for finite discrete time markets) but need more technical assumptions to work in continuous time. Hence, we believe that both approaches can provide alternative and complementary perspectives for future research. Besides, we also give an explicit formula for constructing a deflator (see Theorems 2.14, 2.18 and 2.21), while the results in [1] are elegant but rather abstract.

Structure of the paper
This paper is organized in the following way. In Sect. 2 we introduce the setup, notations, and main results. In Sect. 3.1 we prove the first main result in finite discrete time. Note that Tehranchi [12] also proved a similar result, but our approach is quite different and provides an alternative perspective. In Sect. 3.2 we consider markets in continuous time and find an equivalence condition for the existence of a supermartingale deflator.
• We mean by a stochastic process \((X_t)_{t \in \mathbb{I}}\) simply a collection of \(\mathcal{F}\)-measurable random variables.
  
  • For any measure \(\mathbb{Q}\) on \((\Omega, \mathcal{F})\) we denote by \(L^0(\mathbb{Q})\) the set of all (equivalence classes of) random variables, which we endow with the metric which induces convergence in \(\mathbb{Q}\)-probability. Moreover, we denote by \(L^0_+(\mathbb{Q}) \subseteq L^0(\mathbb{Q})\) the set of nonnegative random variables and by \(L^0_{++,}(. \subseteq L^0_+(\mathbb{Q})\) the set of strictly positive random variables \(X\) in the sense that \(\mathbb{Q}[X > 0] = 1\).
  
  • We call a set \(\mathcal{C} \subseteq L^0(\mathbb{Q})\) to be \(\mathbb{Q}\)-bounded or bounded in \(L^0(\mathbb{Q})\) if it is bounded in probability with respect to \(\mathbb{Q}\), namely
  \[
  \lim_{M \to \infty} \sup_{X \in \mathcal{C}} \mathbb{Q}[|X| \geq M] = 0.
  \]

• Following [14] we say that a set \(\mathcal{C} \subseteq L^0_+(\mathbb{Q})\) is \(\mathbb{Q}\)-convex compact or convexly compact if it is convex, closed, and \(\mathbb{Q}\)-bounded.

• Following [10], we say that a stochastic process \((X_t)_{t \in [0,T]}\) defined on \([0, T]\) is \(\mathbb{Q}\)-cadlag if the mapping \([0, T] \ni t \mapsto X_t \in L^0(\mathbb{Q})\) is right-continuous and has left-limits.

**Definition 2.2** In the discrete-time setting, we call a collection of nonnegative processes, denoted by \(\mathcal{X}\), a **wealth process set** or **market** on \([0, 1, \ldots, T]\) if it satisfies the following two conditions:

(i) Each \(X \in \mathcal{X}\) satisfies \(X_0 = 1\),

(ii) for each \(X \in \mathcal{X}\) we have that \(X\) vanishes on the stochastic interval \(\tau^X, T\], where \(\tau^X := \inf\{t \in [0, 1, \ldots, T] | X_t = 0\}\) with the convention \(\inf \emptyset := \infty\).

In the continuous-time setting, we call a collection of nonnegative processes \(\mathcal{X}\) a **wealth process set** or **market** on \([0, T]\) if it satisfies the following two conditions:

(i) Each \(X \in \mathcal{X}\) has càdlàg paths and satisfies \(X_0 = 1\),

(ii) for each \(X \in \mathcal{X}\) we have that \(X\) vanishes on the stochastic interval \(\tau^X, T\], where \(\tau^X := \inf\{t \in [0, T] | X_t = 0 \text{ or } X_{t-} = 0\}\) with the convention \(\inf \emptyset := \infty\) for all \(X \in \mathcal{X}\).

Furthermore, a wealth process set \(\mathcal{X}\) is called \(\mathbb{F}\)-adapted if each \(X \in \mathcal{X}\) is an \(\mathbb{F}\)-adapted process.

**Definition 2.3** We call an element \(X^{num} \in \mathcal{X}\) a **numéraire** for the market \(\mathcal{X}\) if \(X^{num}_t\) is strictly positive for all time \(t\).

Our goal of this paper is to analyze markets which do not necessarily contain a numéraire, both in the case where the market \(\mathcal{X}\) is \(\mathbb{F}\)-adapted or not.

In the spirit of [10,13], we introduce a notion of generalized fork convexity for wealth process sets.

**Definition 2.4** We say that a wealth process set \(\mathcal{X}\) defined on \(\mathbb{I}\) satisfies the **generalized fork convexity** if the following two conditions hold:

(i) \(\mathcal{X}\) is convex, i.e., \(\lambda X^1 + (1 - \lambda)X^2 \in \mathcal{X}\) for any \(\lambda \in [0, 1]\), \(X^1, X^2\) in \(\mathcal{X}\),

(ii) for any \(X^1, X^2, X^3\) in \(\mathcal{X}\), \(s \in \mathbb{I}\), and \(\Lambda \in \mathbb{F}_s\), the process defined by

\[
X_t := X^1_t \mathbb{I}_{[t < s]} + \left[1_{\Lambda} \left(\frac{X^2_t}{X^3_t} X^1_t \mathbb{I}_{[X^2_t > 0]} + X^1_t \mathbb{I}_{[X^2_t = 0]}\right) + 1_{\Lambda^c} \left(\frac{X^3_t}{X^2_t} X^1_t \mathbb{I}_{[X^3_t > 0]} + X^1_t \mathbb{I}_{[X^3_t = 0]}\right)\right] \mathbb{I}_{[t \geq s]}, \quad t \in \mathbb{I};
\]

belongs to \(\mathcal{X}\).
In words, the generalized fork convexity means that the agent on this market will switch to another portfolio at time \( t \) only when the wealth process associated to the new portfolio has a positive value at this instant, otherwise she will keep her original position.

**Remark 2.5** We point out that our notion of fork convexity is slightly more general than the usual one introduced by Žitković [13] and also used in Karadaras [10], even if the market \( \mathcal{X} \) possesses a numéraire. More precisely, in the notion of Žitković [13], the switched portfolios \( X^2 \) and \( X^3 \) in (2.1) have to be strictly positive. Since in our work, we analyze markets which may not contain a numéraire, we believe that our slight generalization of fork convexity is the natural extension in that setting. To justify our notion, we observe that in the presence of a numéraire, the property for a market to satisfy NUPBR, meaning that the final value set \( C_T := \{ X_T : X \in \mathcal{X} \} \) is \( \mathbb{P} \)-bounded, does not depend on the choice of the definition of the fork convexity (between the one of Žitković [13] and ours). More precisely, we have in Lemma 2.6 the following result, whose proof we provide in the appendix:

**Lemma 2.6** Let \( \mathcal{X} \) be a market which is \( \mathbb{F} \)-adapted and contains a numéraire and assume that it is fork convex in the sense of Žitković [13]. Then the market \( \mathcal{X} \) satisfies the NUPBR condition if and only if its fork convex hull taken with respect to our notion (see Definition 2.4) satisfies the NUPBR condition.

In the spirit of [10], we introduce the notion of a (generalized) supermartingale deflator.

**Definition 2.7** We call a nonnegative stochastic process \((Y_t)_{t \in \mathbb{I}}\) a generalized supermartingale on \( \mathbb{I} \) if for all \( s, t \in \mathbb{I} \) with \( s \leq t \)
\[
\mathbb{E}[\frac{Y_t}{Y_s} \mid \mathcal{F}_s] \leq 1.
\]

**Definition 2.8** We call a nonnegative stochastic process \((Z_t)_{t \in \mathbb{I}}\) a generalized supermartingale deflator on \( \mathbb{I} \) for \( \mathcal{X} \) if \( Z_0 \leq 1 \) and \( ZX \) is a generalized supermartingale for all \( X \in \mathcal{X} \), i.e. for all \( s, t \in \mathbb{I} \) with \( s \leq t \)
\[
\mathbb{E}[\frac{Y_t}{X_t Z_t} \mid \mathcal{F}_s] \leq 1. \tag{2.2}
\]

Moreover, when the market is \( \mathbb{F} \)-adapted, we call \((Z_t)_{t \in \mathbb{I}}\) a supermartingale deflator if \((Z_t)_{t \in \mathbb{I}}\) is additionally \( \mathbb{F} \)-adapted.

**Remark 2.9** In the above Eq. (2.2) we apply the convention that \( 0/0 := 0 \). Thanks to the property (ii) of a wealth process set, we have \( \{ X_s = 0 \} \subseteq \{ X_t = 0 \} \) for \( s \leq t \), which ensures that the formulation (2.2) is well-defined. To rule out trivialities, we are interested in the existence of strictly positive (generalized) supermartingales.

In a market \( \mathcal{X} \) which is fork convex (in the sense of [13]) and possesses a numéraire, Kardaras has proven in [10, Theorem 2.3] the equivalence between \( \mathcal{X} \) satisfying the NUPBR condition and the existence of a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator. It is natural to ask the question if this equivalence also holds true for a market \( \mathcal{X} \) satisfying the (generalized) fork convexity property, but which does not possess a numéraire. It turns out that this equivalence fails when a numéraire is absent, as shown in the following example.

**Example 2.10** The following market satisfies NUPBR but does not admit any strictly positive generalized supermartingale deflator. In a continuous-time setting, let \( T = 1 \) and consider for each \( n \in \mathbb{N} \) the deterministic process \( X^n \) defined by
\[
X^n_t := \min\{1 + (2n - 2)t, 2n - 2nt\}, \quad t \in [0, 1]. \tag{2.3}
\]
In other words, for each \( n \in \mathbb{N} \) the process \( X^n \) is linear between \( 1 \) and \( n \) on the time interval \( [0, \frac{1}{2}] \) and linear between \( n \) and 0 on the time interval \( [\frac{1}{2}, 1] \). Let \( \mathcal{X} \) be the fork convex hull of all \( X^n \). This market satisfies NUPBR because the \( T \)-value set \( C_T = \{0\} \) is \( \mathbb{P} \)-bounded. However for each \( t \in (0, 1) \), we have by (2.3) that \( \sup_{n \in \mathbb{N}} X^n_t = \infty \), hence, as each \( X^n \) is deterministic, the \( t \)-value set \( C_t \) is not \( \mathbb{P} \)-bounded. This in turn contradicts the existence of a strictly positive generalized supermartingale deflator, which would enforce the \( \mathbb{P} \)-boundedness of \( C_t \) for all \( t \).

**Remark 2.11** As pointed out in [10], note that when considering a market possessing a numéraire which satisfies the fork convexity, the \( \mathbb{P} \)-boundedness of the final value set \( C_T := \{X_T : X \in \mathcal{X}\} \) is equivalent to the \( \mathbb{P} \)-boundedness of all the intermediate value set \( C_t := \{X_t : X \in \mathcal{X}\} \) for all \( t \). This equivalence may fail when there is no numéraire, as shown in the above Example 2.10.

The above discussions suggest to ask whether the existence of a strictly positive generalized supermartingale deflator is equivalent to the \( \mathbb{P} \)-boundedness of \( C_t := \{X_t : X \in \mathcal{X}\} \) for all \( t \). The latter property is the content of the following definition.

**Definition 2.12** A market \( \mathcal{X} \) satisfies the NUPBR\(_t\) condition at time \( t \) if the intermediate value set \( C_t := \{X_t : X \in \mathcal{X}\} \) is \( \mathbb{P} \)-bounded. In particular, the NUPBR\(_T\) condition for the final time \( T \) coincides with the classical NUPBR condition.

It turns out that in the discrete-time setting (see Theorem 2.14) as well as under some additional structure on the market (see Theorems 2.18 and 2.25) the equivalence indeed holds. Moreover, in the general setting for the continuous-time case, we provide in our main Theorem 2.21 a stronger condition than the \( \mathbb{P} \)-boundedness of all \( C_t := \{X_t : X \in \mathcal{X}\} \) and show that this condition is indeed equivalent to the existence of a strictly positive, (generalized) supermartingale deflator.

**Remark 2.13** At first glance, one could guess that the \( \mathbb{P} \)-boundedness of all \( C_t := \{X_t : X \in \mathcal{X}\} \) should always ensure the existence of a strictly positive generalized supermartingale deflator for the following reason. The \( \mathbb{P} \)-boundedness of all \( C_t \) ensures that each \( C_t \) is convexly compact, which in turn by [8, Theorem 1.1] ensures the existence of a maximal element \( \hat{f}_t \) with respect to the preference relation \( \preceq \) defined by \( f \preceq g \) if and only if \( \mathbb{E}_P[f/g] \leq 1 \) with the convention \( 0/0 = 0 \). However, note that compared to the classical case where a market contains a numéraire, see [10, Theorem 3.2], one cannot guarantee that the process \( (\hat{f}_t) \) is strictly positive and hence the process \( (1/\hat{f}_t) \) may not form a strictly positive generalized supermartingale deflator.

Instead, we will see later that for markets which do not possess a numéraire, the existence of a strictly positive generalized supermartingale deflator depends crucially on the behaviour of the process \( (\hat{f}_t)_{t \in \mathbb{I}} \) hitting zero. More precisely, we define the debut of \( (\hat{f}_t)_{t \in \mathbb{I}} \) at the origin:

\[
\tau = \inf \left\{ t \in \mathbb{I} : \hat{f}_t = 0 \right\},
\]

with the convention \( \inf \emptyset := \infty \). In view of the property that \( \mathbb{E}_P[1/\hat{f}_t] \leq 1 \) for all \( f \in C_t \), we indeed have that \( \{\hat{f}_t = 0\} \subseteq \{X_t = 0\} \) for all \( X_t \in \mathcal{X} \), which in turn implies that after time \( \tau \), the whole market becomes extinct, or in other words, the market \( \mathcal{X} \) only survives on \([0, \tau)\). Assume for the moment that \( \tau \) is measurable (we refer to Sect. 2.2 for the precise conditions) and denote by \( L(\tau) \) the distribution of \( \tau \) on \([0, T] \cup \{\infty\} \). One of the crucial observations in this paper is that the support \( L(\tau) \) determines conditions for the existence of a strictly positive generalized supermartingale deflator; we refer to Theorems 2.18, 2.21, and 2.25.
2.1 Main results in discrete-time

**Theorem 2.14** Let $\mathcal{X}$ be a market satisfying the generalized fork convexity property. Then the following two statements are equivalent:

(i) $\text{NUPBR}_t$ holds for every $t$, i.e., the set $\mathcal{C}_t := \{X_t : X \in \mathcal{X}\}$ is bounded in probability for every $t$.

(ii) There exists a strictly positive generalized supermartingale deflator.

If we assume in addition that the market is $\mathbb{F}$-adapted, then the following two statements are equivalent.

(i) $\text{NUPBR}_t$ holds for every $t$.

(ii) There exists a strictly positive supermartingale deflator.

The proof of Theorem 2.14 is provided in Sect. 3.1.

2.2 Main results in continuous-time

In this subsection, we provide our main results in the continuous-time setting. Let us first introduce the notion of a generalized numéraire.

**Definition 2.15** A process $\bar{X} \in \mathcal{X}$ is called a *generalized numéraire* if for every $t \in [0, T]$ and $X \in \mathcal{X}$, one has $\mathbb{P}\left(\{X_t > 0\} \cap \{\bar{X}_t = 0\}\right) = 0$.

Note that generalized numéraires are not required to be strictly positive. However, if a market possesses a numéraire, then the notions of generalized numéraire and (classical) numéraire coincide. Financially speaking, a generalized numéraire is an asset which can only default if the whole market defaults. As a possible example, one may consider a government bond of a country which has AAA sovereign credit rating. In the present continuous-time setting, we assume that a generalized numéraire exists.

**Assumption 2.16** The market $\mathcal{X}$ contains a generalized numéraire $\bar{X} \in \mathcal{X}$.

In addition, we impose the following standing condition on the filtration.

**Assumption 2.17** The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions, meaning that $\mathcal{F}$ is $\mathbb{P}$-complete, each $\mathcal{F}_t$ is $\mathbb{P}$-$\mathcal{F}$-complete, and $\mathbb{F}$ is right-continuous.

This standard assumption guarantees the existence of càdlàg versions of supermartingales; see [5, Theorem VI.4, p. 69]. Moreover, in the presence of a generalized numéraire $(\bar{X}_t)$, which by definition satisfies $\{\bar{X}_t = 0\} \subseteq \{X_t = 0\}$ $\mathbb{P}$-a.s. for all $X \in \mathcal{X}$, this assumption guarantees that the following debut $\tau$ is $\mathcal{F}$-measurable, see e.g. [4, Theorem III.44, p. 64], since $\mathcal{F}$ by assumption is $\mathbb{P}$-complete:

$$\tau := \inf\{t \in [0, T] : \bar{X}_t = 0\},$$

using the convention $\inf \emptyset := \infty$. This allows us to consider the distribution $\mathcal{L}(\tau)$ of $\tau$ on $[0, T] \cup \{\infty\}$ whose support turns out to determine the conditions for the existence of a strictly positive supermartingale deflator; see also Remark 2.13. We first start with the result stating that as long as $\mathcal{L}(\tau)$ is discrete, we obtain the desired equivalence between the existence of a strictly positive generalized supermartingale deflator and $\text{NUPBR}_t$ for all $t$, like in the discrete-time setting.
Theorem 2.18 Let the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})\) satisfy the usual conditions, let \(\mathcal{X}\) be a market satisfying the generalized fork convexity, assume that \(\mathcal{X}\) possesses a generalized numéraire \(\overline{X}\), and let \(\tau\) denote its debut at zero as in (2.4). If the support of \(\mathcal{L}(\tau)\) only consists of atoms, then the two following statements are equivalent.

(i) NUPBR\(_t\) holds for every \(t\), i.e., the set \(C_t := \{X_t : X \in \mathcal{X}\}\) is bounded in probability for every \(t\).

(ii) There exists a strictly positive, \(\mathbb{P}\)-càdlàg generalized supermartingale deflator.

If we assume in addition that the market is \(\mathbb{F}\)-adapted, then the following two statements are equivalent:

(i) NUPBR\(_t\) holds for every \(t\).

(ii) There exists a strictly positive, càdlàg supermartingale deflator.

Remark 2.19 Note that if \(\mathcal{X}\) contains a numéraire, then \(\mathcal{L}(\tau) = \delta_{\{\infty\}}\), i.e., the distribution of \(\tau\) is the Dirac measure at \(\infty\). In this case, the above Theorem 2.18 coincides with the classical result of Kardaras in [10, Theorem 2.3] but with respect to the fork convexity defined as in Definition 2.4, see also Lemma 2.6 and Remark 2.11.

The proof of Theorem 2.18 is similar to the one for Theorem 2.14 in the discrete-time case. Roughly speaking, the idea is to construct for each \(t\) in the support of \(\mathcal{L}(\tau)\) a strictly positive “local supermartingale deflator” under each \(\mathbb{P}[\cdot | \tau = t]\) in order to paste them into a global one. As we will see later, the difficulty of providing a characterization for the existence of a strictly positive generalized supermartingale deflator arises when the support of \(\mathcal{L}(\tau)\) contains an uncountable subset \(J \subseteq [0, T]\). At first glance, one would like to follow the same approach as for the case where the support of \(\mathcal{L}(\tau)\) only consists of atoms. More precisely, assume that regular conditional probabilities \(\mathbb{P}[\cdot | \tau = t]\) for \(t \in J\) exist and one could construct for each \(t \in J\) a strictly positive “local supermartingale deflator” under each \(\mathbb{P}[\cdot | \tau = t]\), in order to paste them into a global one. However, since \(J\) is uncountable, not all of these conditional probabilities are absolutely continuous with respect to \(\mathbb{P}\). Therefore, the condition that NUPBR\(_t\) holds for each \(s\) may fail with respect to some \(\mathbb{P}[\cdot | \tau = t]\), even if we impose it to hold with respect to \(\mathbb{P}\), and as a consequence one cannot construct strictly positive “local supermartingale deflectors” for those conditional probabilities.

To overcome this technical difficulty we introduce a stronger condition than that NUPBR\(_t\) for each \(t\). This stronger condition roughly speaking requires \(C_t\) to be bounded uniformly with respect to all conditional \(\mathbb{P}[\cdot | \tau \in (r, u) \cap J]\) for any \(r, u\) on a countable dense set. This condition effectively allows one to transfer the discrete-time argument to the general continuous-time setting and allows us to formulate in the following Theorem 2.21 a characterization of the existence of a strictly positive generalized supermartingale deflator. To this end, we introduce the following notation, where we recall our standing assumption in the continuous-time setting that the market \(\mathcal{X}\) possesses a generalized numéraire \(\overline{X}\), whose debut at zero is denoted by \(\tau\) as in (2.4).

Notation 2.20 Let the market \(\mathcal{X}\) possess a generalized numéraire \(\overline{X}\) with debut \(\tau\) and corresponding distribution \(\mathcal{L}(\tau)\). Then, from now on, we will use the following notation:

- \(A\) denotes the collection of all atoms in the support of \(\mathcal{L}(\tau)\);
- \(\mathcal{J} := \text{supp}(\mathcal{L}(\tau)) \setminus A\);
- for each \(r < u \in [0, T]\) denote by \(Q_{r,u} : \mathcal{F} \to [0, 1]\) the map
  \[
  Q_{r,u}[\cdot] := \begin{cases} 
  \mathbb{P}[\cdot | \tau \in (r, u) \cap \mathcal{J}] & \text{if } \mathbb{P}[\tau \in (r, u) \cap \mathcal{J}] > 0, \\
  0 & \text{else.}
  \end{cases}
  \]
Theorem 2.21 Using Notation 2.20, let the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfy the usual conditions, let $\mathcal{X}$ be an $\mathbb{F}$-adapted market satisfying the generalized fork convexity, and assume that $\mathcal{X}$ possesses a generalized numéraire, allowing for $\mathcal{J} \neq \emptyset$. Then the following statements (i) and (ii) are equivalent:

(i) The following two properties hold:

(a) $\text{NUPBR}_t$ holds for every $t \in [0, T]$.

(b) There exists a countable dense subset $\mathcal{D} \subseteq [0, T]$ containing $0$ and $T$ such that for every $t$ in $\mathcal{D}$

$$\limsup_{M \to \infty} \sup_{r,u \in \mathcal{D}, r > t} \sup_{X_t \in \mathcal{C}_t} \mathbb{Q}_{r,u}[X_t \geq M] = 0.$$  \quad (2.5)

(ii) The following two properties hold:

(a) There exists a strictly positive càdlàg supermartingale deflator $(Z_s)_{s \in [0, T]}$ for $\mathcal{X}$.

(b) There exist a countable dense subset $\mathcal{D} \subseteq [0, T]$ containing $\{0, T\}$ and a strictly positive process $(Z_t^\infty)_{t \in \mathcal{D}}$ with $Z_0^\infty \leq 1$ such that

- for all $s < t$ in $\mathcal{D}$ and for all $X \in \mathcal{X}$,

$$\mathbb{E}_P \left[ \frac{X_t Z_s^\infty}{Z_t^s} \right] \leq 1,$$

- for all $s < t$ in $\mathcal{D}$ and all $\mathbb{Q}_{r,u}$ with $t \leq r < u$ in $\mathcal{D}$,

$$\mathbb{E}_{\mathbb{Q}_{r,u}} \left[ \frac{X_t Z_s^\infty}{Z_t^s Z_u^\infty} \right] \leq 1,$$

- for all $t$ in $\mathcal{D}$,

$$\limsup_{M \to \infty} \sup_{r,u \in \mathcal{D}, r > t} \mathbb{Q}_{r,u} \left[ \frac{1}{Z_t^\infty} \geq M \right] = 0.$$  \quad (2.6)

Remark 2.22 The property $\mathcal{J} \neq \emptyset$ can only happen if the market does not possess a numéraire, since under presence of a numéraire $\text{supp}(\mathbb{L}(\tau)) = \{\infty\}$.

Remark 2.23 Conditions (2.5) and (2.6) admit the following financial interpretation. By definition, the conditional measure $\mathbb{Q}_{r,u}$ models the investment possibilities of an informed trader who knows that the market crashes between times $r$ and $u$. Thus, the uniform integrability Condition (2.5) means that such informational advantages do not aggregate into unbounded profit with bounded risk. Similarly, Condition (2.6) means that the deflator is bounded in probability uniformly with respect to all such insider information.

Remark 2.24 In the discrete case, we will see from the proof of Theorem 2.14 that the strictly positive generalized supermartingale deflator is not only defined with respect to $\mathbb{P}$ but also all the conditional measures $\mathbb{P}[\cdot | \tau = t]$ for $t \in \{0, 1, \ldots, T\} \cup \{\infty\}$, see Remark 3.7. In the continuous-time case the properties on $(Z_t^\infty)_{t \in \mathcal{D}}$ can be roughly seen as the analogue.

As discussed above in Remark 2.13, the key property we need of $\tau$ defined in (2.4) is that there exists a generalized numéraire which is strictly positive on $[0, \tau)$, whereas the market dies out on $[\tau, T]$. It turns out that if we can find a random time $\tilde{\tau}$ which possesses exactly this property and is independent of the market, then we obtain the equivalence between the existence of a strictly positive, càdlàg supermartingale deflator and the property $\text{NUPBR}_t$ for all $t$. 

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Theorem 2.25 Let the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfy the usual conditions, let \(\mathcal{X}\) be an \(\mathbb{F}\)-adapted market satisfying the generalized fork convexity, and suppose that there exists an \(\mathcal{F}\)-measurable random time \(\tilde{\tau} : \Omega \to (0, T] \cup \{\infty\}\) which satisfies the following two properties:

1. For each \(s < t\) the event \(\{\tilde{\tau} \in (s, t]\}\) is independent of \(\mathcal{F}_s\).
2. For each \(s < t\) such that \(\mathbb{P}[\tilde{\tau} \in (s, t]] > 0\), the market \(\mathcal{X}\) under \(\mathbb{P}[\cdot | \tilde{\tau} \in (s, t)]\) contains a numéraire until time \(s\) and all elements in \(\mathcal{X}\) vanish after time \(t\).

Then the following two statements are equivalent:

(i) NUPBR, holds for every \(t\).
(ii) There exists a strictly positive, càdlàg supermartingale deflator.

Remark 2.26 Condition (1) means that at each time \(s\), all the information given on the market modelled by \(\mathcal{F}_s\) does not provide any information when the market will crash in the future. This may be realistic in a situation where \(\tilde{\tau}\) models the appearance of an extreme event like a natural disaster.

3 Proof of the main results

We first provide auxiliary results which will be used frequently in the proof of our main results. The following lemma is well known, but we provide its proof for the sake of completeness.

Lemma 3.1 Let \(\mathbb{P}\) be any probability measure on a measurable space, let \(C\) be any subset in \(L^0_+ (\mathbb{P})\) which is bounded in \(\mathbb{P}\)-probability and let \(Q \ll \mathbb{P}\). Then the following holds true.

(i) The \(\mathbb{P}\)-closure \(cl_\mathbb{P}(C)\) of \(C\) is bounded in \(L^0_+ (\mathbb{P})\).
(ii) The \(\mathbb{P}\)-closure of the \(\mathbb{P}\)-solid hull \(sol_\mathbb{P}(C)\) of \(C\) is bounded in \(L^0_+ (\mathbb{P})\), where the \(\mathbb{P}\)-solid hull \(sol_\mathbb{P}(C)\) is defined by \(sol_\mathbb{P}(C) := \{g \in L^0_+ (\mathbb{P}) : g \leq f\ P\text{-a.s. for some } f \in C\}\).
(iii) The set \(C\) is bounded in \(Q\)-probability.

Proof (i) Let \(f \in cl_\mathbb{P}(C)\) and let \(f^n, n \in \mathbb{N}\), be a sequence in \(C\) such that \(f^n\) converges to \(f\) in \(\mathbb{P}\)-probability. By passing to a subsequence, we can also assume that the sequence convergences \(\mathbb{P}\)-a.s. For a given \(\varepsilon > 0\), since \(C\) is bounded in \(L^0_+ (\mathbb{P})\), there is an \(M > 0\) such that \(\mathbb{P}[f^n > M] \leq \varepsilon\) holds for all \(n\), which in turn implies by Fatou’s lemma that \(\mathbb{P}[f > M] \leq \lim inf_{n \to \infty} \mathbb{P}[f^n > M] \leq \varepsilon.\) Consequently, we conclude the \(\mathbb{P}\)-boundedness of \(cl_\mathbb{P}(C)\).

(ii) If \(C\) is bounded in \(L^0_+ (\mathbb{P})\), then by definition its \(\mathbb{P}\)-solid hull \(sol_\mathbb{P}(C)\) is also bounded in \(L^0_+ (\mathbb{P})\), and so by (i) is then its \(\mathbb{P}\)-closure.

(iii) Let \(Z \in L^1 (\mathbb{P})\) denote the Radon–Nikodym derivative of \(Q\) with respect to \(\mathbb{P}\). Then we have for any \(N, M > 0\) and \(X \in C\) that

\[
\mathbb{Q}[X \geq M] = \mathbb{E}_\mathbb{P}[Z 1_{\{X \geq M\}}] \\
= \mathbb{E}_\mathbb{P}[Z 1_{\{X \geq M\}} 1_{\{Z > N\}}] + \mathbb{E}_\mathbb{P}[Z 1_{\{X \geq M\}} 1_{\{Z \leq N\}}] \\
\leq \mathbb{E}_\mathbb{P}[Z 1_{\{Z > N\}}] + N \mathbb{P}[X \geq M].
\]

Therefore, using that by assumption \(C\) is bounded in \(\mathbb{P}\)-probability, we see that for all \(N > 0\)

\[
\lim_{M \to \infty} \sup_{X \in C} \mathbb{Q}[X \geq M] \leq \mathbb{E}_\mathbb{P}[Z 1_{\{Z > N\}}].
\]

Letting now \(N\) tend to infinity implies (iii), as \(Z \in L^1 (\mathbb{P})\). \(\square\)
A crucial tool for the proof of our main results is the notion of static deflators introduced by Kardaras [8].

**Lemma 3.2** Let \( C \subseteq L^0_+ \) be convexly compact. Let \( \leq \) be a binary relation on \( C \) such that \( f \leq g \) if and only if \( \mathbb{E}[f/g] \leq 1 \) with the convention \( 0/0 = 0 \). Then there exists a unique maximal element \( \hat{f} \in C \) with respect to this relation \( \leq \). In particular, it holds that \( \{ \hat{f} = 0 \} \subseteq \{ f = 0 \} \mathbb{P}\text{-a.s. for all } f \in C \).

**Proof** See the proof of [8, Theorem 1.1]. \( \square \)

Note that the above lemma does not ensure that \( \hat{f} \) is strictly positive. However this follows from the stronger assumption \( C \cap L^0_{++} \neq \emptyset \), as stated next.

**Lemma 3.3** (Theorem 3.2 in [10]) Let \( C \subseteq L^0_+ \) be convexly compact such that \( C \cap L^0_{++} \neq \emptyset \). Then there exists a unique \( \hat{f} \in C \cap L^0_{++} \) such that \( \mathbb{E}[f/\hat{f}] \leq 1 \) holds for all \( f \in C \), which is then called the static deflator.

The next proposition shows that the main result in Kardaras [10, Theorem 2.3] remains valid also when a fork-convex wealth process set \( \mathcal{X} \) does not contain a strictly positive process, but the closure \( \text{cl}_\mathbb{P}(C_T) \) of the final value set \( C_T \) contains a strictly positive random variable.

**Proposition 3.4** Let \( \mathcal{X} \) be a fork convex wealth process set such that \( \text{cl}_\mathbb{P}(C_T) \) contains a strictly positive random variable. Then the main result in Kardaras [10, Theorem 2.3] remains valid, i.e., NUPBR is equivalent to the existence of a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator.

**Proof** A careful inspection shows that the proof of [10, Theorem 2.3] remains valid also in this slightly more general case, provided we can show that \( \text{cl}_\mathbb{P}(C_T) \cap L^0_{++}(\mathbb{P}) \neq \emptyset \) ensures that \( \text{cl}_\mathbb{P}(C_T) \cap L^0_{++}(\mathbb{P}) \neq \emptyset \) for all \( t < T \). This is shown in the remaining part.

Suppose that \( g_T \in \text{cl}_\mathbb{P}(C_T) \) is strictly positive and let \( \xi^n, n \in \mathbb{N} \), be a sequence in \( \mathcal{X} \) such that \( \xi^n_T \) converges to \( g_T \) in probability. By passing to a subsequence we can even require the convergence to hold \( \mathbb{P} \)-almost surely. Since \( \mathbb{P}[g_T > 0] = 1 \), for each \( \varepsilon > 0 \) there is an \( \eta > 0 \) such that \( \mathbb{P}[g_T > \varepsilon] \geq 1 - \eta \) and \( \eta \) converges to 0 if \( \varepsilon \to 0 \). Moreover, by Egorov’s theorem, there exists a set \( \Gamma_T \in \mathcal{F} \) with \( \mathbb{P}[\Gamma_T] \geq 1 - \eta \) such that \( \xi^n_T \) converges to \( g_T \) uniformly on \( \Gamma_T \). Hence it holds that \( \mathbb{P}[\Gamma_T \cap \{ g_T > \varepsilon \}] \geq 1 - 2\eta \) and there exists an \( N \) such that for all \( n \geq N, \xi^n_T > \frac{\varepsilon}{2} > 0 \) on \( \Gamma_T \cap \{ g_T > \varepsilon \} \). Furthermore, thanks to the property (ii) of the wealth process set, we have \( \xi^n_T > 0 \) on \( \Gamma_T \cap \{ g_T > \varepsilon \} \) for all \( n \geq N \). Now let \( \hat{f}_t \in \text{cl}_\mathbb{P}(C_t) \) be the maximal element in the sense of Lemma 3.2. Then, we have

\[
\mathbb{P}[\hat{f}_t > 0] \geq \mathbb{P}[\xi^n_t > 0] \geq \mathbb{P}[\Gamma_T \cap \{ g_T > \varepsilon \}] \geq 1 - 2\eta.
\]

Since \( \varepsilon \) (and hence \( \eta \)) can be chosen arbitrarily small, we conclude that \( \mathbb{P}[\hat{f}_t > 0] = 1 \). \( \square \)

**Remark 3.5** Using Lemma 3.1, we see that Proposition 3.4 remains valid when replacing \( \mathbb{P} \) by any \( \mathbb{Q} \ll \mathbb{P} \).

The following lemma will be crucial to construct supermartingale deflators (i.e., \( \mathbb{P} \)-adapted ones).

**Lemma 3.6** Let \( \mathbb{Q} \) and \( \mathbb{P} \) be two probabilities such that \( \mathbb{Q} \ll \mathbb{P} \) on \( \mathcal{F} \). Let \( C \) be any \( \mathbb{P} \)-bounded subset in \( L^0(\mathcal{Q}, \mathcal{G}, \mathbb{P}) \) for some sub-\( \sigma \)-field \( \mathcal{G} \subseteq \mathcal{F} \). Then any \( \mathbb{Q} \)-random variable \( g \in \text{cl}_\mathbb{Q}(C) \) admits a \( \mathcal{G} \)-measurable \( \mathbb{Q} \)-version \( g' \) which is in the \( \mathbb{P} \)-closure of the \( \mathbb{P} \)-solid hull of \( C \).
Proof Since \( g \) is in the closure of \( \mathcal{C} \) inside \( L^0(\Omega) \), there exists a sequence \((g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C} \subseteq L^0(\Omega)\) such that \( g = \lim_{n \to \infty} g_n \) \( \mathbb{Q} \)-a.s. In addition, as \( \mathbb{Q} \ll \mathbb{P} \) and \( \mathcal{C} \) is bounded in \( \mathbb{P} \)-probability, it is also bounded in \( L^0(\mathbb{Q}) \) and consequently \( g \) is finite \( \mathbb{Q} \)-a.s.

Without loss of generality we can assume that each \( g_n \) is a \( \mathbb{P} \)-random variable in \( \mathcal{C} \) (i.e., we pick a representative of \( g_n \) such that its equivalence class modulo \( \mathbb{P} \)-null sets belongs to \( \mathcal{C} \subseteq L^0(\mathbb{P}) \)); in particular, every \( g_n \) is \( \mathcal{G} \)-measurable. Then, consider the set

\[
A := \{ \omega \in \Omega : g_n(\omega), n \in \mathbb{N}, \text{ is a Cauchy sequence in } \mathbb{R} \},
\]

which is \( \mathcal{G} \)-measurable. Since \( g_n \) converges to a \( \mathbb{R} \)-valued random variable \( g \) \( \mathbb{Q} \)-a.s., we have \( \mathbb{Q}[A] = 1 \). Now we define

\[
g' := \lim_{n \to \infty} g_n I_A = \lim_{n \to \infty} g_n I_A.
\]

Clearly, we have the \( \mathcal{G} \)-measurability of \( g' \) and \( \mathbb{Q}[g = g'] = 1 \). Hence, the equivalence class of \( g' \) modulo \( \mathbb{Q} \)-null set in \( L^0(\mathbb{Q}) \) is equal to \( g \). Moreover, by construction of \( g' \), we see that \( g' \) is in the \( \mathbb{P} \)-closure of the \( \mathbb{P} \)-solid hull of \( \mathcal{C} \). This completes the proof. \( \square \)

3.1 Proof of the main results in discrete-time

In this subsection we provide the proof of our main result Theorem 2.14 in the discrete-time setting. We first note that the most important tool in its proof is the concept of static deflators, see Lemma 3.2. To visualize its importance, suppose that the NUPBR condition holds for each \( t \). Then the \( \mathbb{P} \)-closure \( \text{cl}_{\mathbb{P}}(\mathcal{C}_t) \) of \( \mathcal{C}_t \) is convexly compact and we can pick for every \( t \) a maximal element \( \hat{f}_t \in \text{cl}_{\mathbb{P}}(\mathcal{C}_t) \). For this finite sequence \( \hat{f}_t, t \in \{0, 1, \ldots, T\} \), we define \( \tau \) as its first hitting time of 0, namely

\[
\tau := \inf \{ t \in \{0, 1, \ldots, T\} : \hat{f}_t = 0 \}, \tag{3.1}
\]

with the convention \( \inf \emptyset = \infty \). Observe that \( \tau \) is \( \mathcal{F} \)-measurable and for each \( t \) one has that

\[
\{ \tau = \infty \} = \{ \hat{f}_s > 0, \forall s \in \{0, 1, \ldots, T\} \},
\]

\[
\{ \tau = t \} = \{ \hat{f}_s > 0, \forall s < t, s \in \{0, 1, \ldots, T\} \} \cap \{ \hat{f}_t = 0 \}.
\]

It follows immediately that for all \( t \in \{0, 1, \ldots, T\} \cup \{\infty\}, \)
\[
\forall s < t, s \in \{0, 1, \ldots, T\}, \quad \mathbb{P}[\hat{f}_s > 0 \mid \tau = t] = 1. \tag{3.2}
\]

Moreover, in view of Lemma 3.2, we have \( \mathbb{P}[X_t = 0 \mid \tau = t] = 1 \) for all \( X \in \mathcal{X} \). As a consequence of the “non-rebound” property (ii) of a wealth process set, one then has for each \( t \in \{0, 1, \ldots, T\} \) that

\[
\forall r \geq t, r \in \{0, 1, \ldots, T\}, \forall X \in \mathcal{X}, \quad \mathbb{P}[X_r = 0 \mid \tau = t] = 1. \tag{3.3}
\]

We will heavily take use of these properties of \( \tau \) in the following proof of Theorem 2.14.

Proof of Theorem 2.14 We start with the well-known direction \( (ii) \Rightarrow (i) \), whose short proof we provide for the sake of completeness. Suppose that (ii) holds, i.e., there exists a strictly positive generalized supermartingale deflator \( Z \) for \( \mathcal{X} \). Then as \( Z_0 \leq 1 \) by definition and since \( X_0 = 1 \) for all \( X \in \mathcal{X} \), we see that (2.2) ensures that
\[
\mathbb{E}_\mathbb{P}[X_t Z_t] \leq 1. \tag{3.4}
\]
Therefore, Markov’s inequality implies the \( \mathbb{P} \)-boundedness of the set \( \{X_t Z_t : X_t \in C_t\} \), which means that for any \( \varepsilon > 0 \), there exists an \( M > 0 \) such that for all \( X_t \in C_t \), it holds that \( \mathbb{P}[X_t Z_t \geq M] \leq \varepsilon \). In addition, note that

\[
\mathbb{P}[X_t \geq M^2] = \mathbb{P}[X_t \geq M^2; Z_t \geq \frac{1}{M}] + \mathbb{P}[X_t \geq M^2; Z_t < \frac{1}{M}]
\leq \mathbb{P}[X_t Z_t \geq M] + \mathbb{P}[Z_t < \frac{1}{M}].
\]

Applying Markov’s inequality again together with (3.4) and the strict positivity of \( Z_t \) hence assures that we can pick \( M \) large enough such that both \( \mathbb{P}[X_t Z_t \geq M] \leq \varepsilon \) and \( \mathbb{P}[Z_t \leq \frac{1}{M}] \leq \varepsilon \) are satisfied. This in turn shows the \( \mathbb{P} \)-boundedness of the set \( C_t := \{X_t : X \in \mathcal{X}\} \) as desired.

Now, let us prove the implication (i) \( \Rightarrow \) (ii) and hence assume that NUPBR\(_{\mathcal{T}}\) holds for each \( t \). Then every \( \text{cl}_\mathbb{P}(C_t) \) is convexly compact, which by Lemma 3.2 allows us to choose a sequence of maximal elements \( \hat{f}_t \in \text{cl}_\mathbb{P}(C_t), t \in \{0, 1, \ldots, T\} \), and can define the random time \( \tau \) as in (3.1). Without loss of generality, we may assume that all \( \{\tau = t\}, t \in \{0, 1, \ldots, T\} \), have positive \( \mathbb{P} \)-measure (otherwise, we consider the subset \( \{t \in \{0, 1, \ldots, T\} \cup \{\infty\} : \mathbb{P}[\tau = t] > 0\} \)), and we introduce the notion \( Q_t[\cdot] \) to denote the conditional probability \( \mathbb{P}[\cdot | \tau = t] \) for \( t \in \{0, 1, \ldots, T\} \cup \{\infty\} \). We divide the proof of the implication (i) \( \Rightarrow \) (ii) into several steps.

**Step 1** (Local supermartingale deflators). In this step we will show that for every \( t \in \{0, 1, \ldots, T\} \cup \{\infty\} \) there is a strictly positive generalized supermartingale deflator \( Y_t^T \) with respect to the conditional probability \( Q_t \). First, let \( t = \infty \). Since \( Q_\infty \ll \mathbb{P} \), the NUPBR\(_{\mathcal{T}}\) condition also holds with respect to \( Q_\infty \). In particular, by Lemma 3.1 the set \( \text{cl}_Q(Q_\infty(C_t)) \) is convexly compact as a subset in \( L^1(\mathbb{Q}_\infty) \) for each \( s \). For every \( s \in \{0, 1, \ldots, T\} \) we therefore obtain by Lemma 3.2 that there exists a unique maximal element \( \hat{f}_s^Q \in \text{cl}_Q(C_s) \). Moreover, in view of (3.2), there is a maximal element \( \hat{f}_s^P \in \text{cl}_P(C_s) \) such that \( \mathbb{P}[\hat{f}_s^P > 0 | \tau = \infty] = 1 \). Since \( \text{cl}_P(C_s) \subseteq \text{cl}_Q(Q_\infty(C_t)) \), we obtain by uniqueness of the maximal element that \( \hat{f}_s^P = \hat{f}_s^Q \) \( Q_\infty \)-a.s., which we denote by \( \hat{f}_s \), and which satisfies that \( Q_\infty[\hat{f}_s > 0] = 1 \). In other words, under the conditional measure \( Q_\infty \), the fork convex market \( \mathcal{X} \) contains a numéraire and therefore, by the classical result proved by Kardaras in [10, Theorem 2.3] together with Proposition 3.4 and Remark 3.5, there exists a generalized supermartingale deflator \( Y_\infty^\mathcal{X} \) with the following properties: \( Y_\infty^\mathcal{X} \) is strictly positive with respect to \( Q_\infty \), \( Y_0^\mathcal{X} \leq 1 \), and for all \( X \in \mathcal{X}, s < r \in \{0, 1, \ldots, T\} \),

\[
\mathbb{E}_{Q_\infty}\left[\frac{X_r Y_r^\mathcal{X}}{X_s Y_s^\mathcal{X}} | \mathcal{F}_s\right] \leq 1.
\]

Next we consider \( t = T \). Again, from (3.2) we can conclude that strictly before time \( T \) the market \( \mathcal{X} \) satisfies the NUPBR\(_{\mathcal{T}}\) condition for all \( t \in \{0, 1, \ldots, T - 1\} \) and contains a numéraire with respect to the conditional measure \( Q_T \). Therefore, there exists a strictly positive generalized supermartingale deflator \( Y_T^T \) defined on \( \{0, 1, \ldots, T - 1\} \) with respect to \( Q_T \). Furthermore, in view of (3.3), we have \( \mathbb{Q}_T[X_T = 0] = 1 \) for all \( X \in \mathcal{X} \), which implies that \( \mathbb{E}_{Q_T}\left[\frac{X_T Y_T^\mathcal{X}}{X_T Y_T^\mathcal{X}} | \mathcal{F}_T\right] = 0 \leq 1 \) for all \( s < T \). Hence, we can extend \( Y_T^T \) defined on \( \{0, 1, \ldots, T - 1\} \) to \( \{0, 1, \ldots, T\} \) by setting \( Y_T^T := 1 \), and the resulting process \( Y_s^T \) is a strictly positive generalized supermartingale deflator for \( \mathcal{X} \) with respect to \( Q_T \). We continue backwards for all \( t \in \{0, 1, \ldots, T\} \) with this procedure. Then for every \( t \in \{0, 1, \ldots, T\} \) we get a strictly positive generalized supermartingale deflator \( Y_t^T \) with respect to \( Q_t \) such that for all \( s \geq t, Y_s^T = 1 \) whereas for all \( s < t, Y_s^T \) is constructed as in the classical case by Kardaras [10, Theorem 2.3], but under the measure \( Q_t \). In particular, for \( s < t \), we know from [10, Theorem 2.3] that \( Y_s^T \) can be obtained.
by the relation that
\[ \frac{1}{Y_t} \in \text{cl}\Q_t(C_s) \] is the \textit{static deflator} (cf. Lemma 3.3) in \text{cl}\Q_t(C_s) w.r.t. \Q_t. \hspace{1cm} (3.6)

Of course, the supermartingale deflator property indeed gives us for any \( s < r \) and \( X \in \mathcal{X} \) that
\[ \E_{\Q_t}[\frac{X}{X_s Y_{r,t}} | \mathcal{F}_s] \leq 1. \hspace{1cm} (3.7) \]

\textit{Step 2} (Pasting local deflators to a global one). In this step, we glue all \((Y^t_s)_{s \in \{0, 1, \ldots, T\}}\), \( t \in \{0, 1, \ldots, T\} \cup \{\infty\} \), together to obtain a strictly positive generalized supermartingale deflator for \( \mathcal{X} \) under \( \P \). More precisely, we define the process \((Z_s)_{s \in \{0, 1, \ldots, T\}}\) by setting for each \( s \)
\[ Z_s := \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} Y^t_s 1_{\{\tau = t\}}, \quad s \in \{0, 1, \ldots, T\}. \hspace{1cm} (3.8) \]

Then, for all \( s < r \in \{0, 1, \ldots, T\} \), \( X \in \mathcal{X} \), and \( A_s \in \mathcal{F}_s \), by (3.5), (3.7), (3.8), and the definition of \( \Q_t[\cdot] := \P[\cdot \mid \tau = t] \), we deduce that
\[ \E_\P \left[ \frac{X}{X_s Z_t} I_{A_s} \right] = \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \E_{\Q_t} \left[ \frac{X}{X_s Z_t} I_{A_s} \right] \P[\tau = t] \\
= \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \E_{\Q_t} \left[ \frac{X}{X_s Y^t} I_{A_s} \right] \P[\tau = t] \\
\leq \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \Q_t[A_s] \P[\tau = t] \\
= \P[A_s], \]
which implies that \( \E_\P \left[ \frac{X}{X_s Z_t} | \mathcal{F}_s \right] \leq 1. \) Furthermore, since by construction each \( Y^t \) is strictly positive under \( \Q_t \) for all \( t \in \{0, 1, \ldots, T\} \cup \{\infty\} \), using a similar argument as above we can see that for all \( s \in \{0, 1, \ldots, T\} \),
\[ \P[Z_s > 0] = \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \Q_t[Z_s > 0] \P[\tau = t] \\
= \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \Q_t[Y_s^t > 0] \P[\tau = t] \\
= \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \P[\tau = t] = 1. \]

Hence, we conclude that the process \((Z_s)_{s \in \{0, 1, \ldots, T\}}\) defined in (3.8) is indeed a strictly positive generalized supermartingale deflator for \( \mathcal{X} \) under \( \P \). This finishes the proof for the non \( \mathcal{F} \)-adapted case.

\textit{Step 3} (Supermartingale deflator (i.e., \( \mathcal{F} \)-adapted one) for \( \mathcal{F} \)-adapted market). It remains to show that one can construct a supermartingale deflator (i.e., an \( \mathcal{F} \)-adapted one) if the market is \( \mathcal{F} \)-adapted. Using the notations introduced before, we know from (3.6) that for any \( t \in \{0, 1, \ldots, T\} \) there exists a strictly positive generalized supermartingale deflator \((Y^t_s)_{s \in \{0, 1, \ldots, T\}}\) with respect to the conditional measure \( \Q_t \) which satisfies for all \( s < t \) that \( \frac{1}{Y^t_s} \in \text{cl}\Q_t(C_s) \), whereas \( Y^t_s = 1 \) for all \( s \geq t \). Now, for any \( s < t \), since the market is \( \mathcal{F} \)-adapted, we can apply Lemma 3.6 with \( \Q := \Q_t \) and \( g := \frac{1}{Y^t_s} \) to find an \( \mathcal{F}_s \)-measurable...
\( \mathbb{Q}_t \)-version of \( \frac{1}{\mathbb{P}_t} \), which we still denote \( \frac{1}{\mathbb{P}_t} \) for the ease of notation. From this construction, we obtain an \( \mathbb{F} \)-adapted strictly positive generalized supermartingale deflator with respect to \( \mathbb{Q}_t \) which we again denote by \( (Y^t_i)_{t \in [0, 1, \ldots, T]} \) for the ease of notation. Consequently, the generalized supermartingale property (3.7) can be rewritten as the standard supermartingale property, namely for all \( X \in \mathcal{X} \) and \( r \geq s \)

\[
\mathbb{E}_{\mathbb{Q}_r}[X_r Y^t_r | \mathcal{F}_s] \leq X_s Y^t_s,
\]
or, equivalently, for all \( A_s \in \mathcal{F}_s \),

\[
\mathbb{E}_{\mathbb{Q}_r}[X_r Y^t_r 1_{A_s}] \leq \mathbb{E}_{\mathbb{Q}_r}[X_s Y^t_s 1_{A_s}].
\]

Now, let \((Z_s)_{s \in [0, 1, \ldots, T]}\) be the process defined in (3.8) and let \((\tilde{Z}_s)_{s \in [0, 1, \ldots, T]}\) be the process defined by setting \( \tilde{Z}_s := \mathbb{E}_P[Z_s | \mathcal{F}_s] \) for each \( s \in \{0, 1, \ldots, T\} \). Then we get that

\[
\mathbb{E}_P[X_r \tilde{Z}_r 1_{A_s}] = \mathbb{E}_P[X_r Z_r 1_{A_s}] = \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \mathbb{E}_{\mathbb{Q}_r}[X_r Y^t_r 1_{A_s}] \mathbb{P}[\tau = t] \\
\leq \sum_{t \in \{0, 1, \ldots, T\} \cup \{\infty\}} \mathbb{E}_{\mathbb{Q}_r}[X_s Y^t_s 1_{A_s}] \mathbb{P}[\tau = t] \\
= \mathbb{E}_P[X_s Z_s 1_{A_s}] = \mathbb{E}_P[X_s \tilde{Z}_s 1_{A_s}].
\]

Since by definition every \( \tilde{Z}_s \) is \( \mathcal{F}_s \)-measurable, the above inequality indeed shows that \((\tilde{Z}_s)_{s \in [0, 1, \ldots, T]}\) is a strictly positive supermartingale deflator for \( \mathcal{X} \) under \( \mathbb{P} \). This completes the proof of Theorem 2.14.

\[\square\]

**Remark 3.7** The construction of the generalized supermartingale deflator \((Z_s)_{s \in [0, 1, \ldots, T]}\) in the proof of Theorem 2.14, see (3.8), shows that \((Z_s)_{s \in [0, 1, \ldots, T]}\) is not only a strictly positive generalized supermartingale deflator under \( \mathbb{P} \), but also simultaneously under all \( \mathbb{Q}_t[\cdot] := \mathbb{P}[\cdot | \tau = t] \) for \( t \in \{0, 1, \ldots, T\} \cup \{\infty\} \). This observation turns out to be important in the proof of Theorem 2.21.

### 3.2 Proof of the main results in continuous time

#### 3.2.1 Proof of Theorem 2.18

In this subsubsection we provide the proof of Theorem 2.18, which is essentially the same as the one of Theorem 2.14.

**Proof of Theorem 2.18** First, note that the implication (ii) \(\Rightarrow\) (i) follows by the same argument as, e.g., in the proof of Theorem 2.14. Hence we now show that the implication (i) \(\Rightarrow\) (ii) holds.

To that end, for each \( t \in \mathcal{A} \) we denote \( \mathbb{Q}_t[\cdot] := \mathbb{P}[\cdot | \tau = t] \). Since (i) ensures that the NUPBR\(_s\) condition holds for each \( s \in [0, T) \) under the measure \( \mathbb{P} \), and as all \( \mathbb{Q}_t, t \in \mathcal{A} \), are absolutely continuous with respect to \( \mathbb{P} \), the market \( \mathcal{X} \) satisfies the NUPBR\(_s\) condition for each \( s \in [0, T] \) also under every \( \mathbb{Q}_t, t \in \mathcal{A} \). Moreover, since \( \{\tau = t\} = \{X_s > 0, \forall s < t \} \cap \{X_t = 0\} \), we have similarly to the discrete-time case (3.2) and (3.3), that for every \( t \in \mathcal{A} \),

\[
\forall s < t, s \in [0, T], \quad \mathbb{Q}_t[X_s > 0] = 1,
\]

\[\square\]
and
\[
\forall s \geq t, s \in [0, T], \forall X \in \mathcal{X}, \quad \mathbb{Q}_t[X_s = 0] = 1. \tag{3.10}
\]

Next, for each \( t \in \mathcal{A} \cap [0, T] \), in view of (3.9) and (3.10) we have

(i) The market \( \mathcal{X} \) contains a numéraire strictly before time \( t \) with respect to \( \mathbb{Q}_t \);
(ii) For all \( X \in \mathcal{X} \) and \( r \geq t \), \( \mathbb{Q}_t[X_r = 0] = 1 \).

As a consequence, following the same line of arguments as in the discrete-time case in Theorem 2.14, there exists a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator \( Y^t \) associated with \( \mathbb{Q}_t \), such that (cf. (3.6))
\[
\forall s < t, \quad Y^t_s = 1/\hat{f}^t_s, \tag{3.11}
\]
where \( \hat{f}^t_s \in \text{cl}_{\mathbb{Q}_t}(\mathcal{C}_s) \) is the “static deflator” (cf. Lemma 3.3) in \( \text{cl}_{\mathbb{Q}_t}(\mathcal{C}_s) \) with respect to \( \mathbb{Q}_t \), and
\[
\forall s \geq t, \quad Y^t_s = 1. \tag{3.12}
\]

Furthermore, if \( \infty \in \mathcal{A} \), then by (3.9) we know that under the conditional measure \( \mathbb{Q}_\infty \) the market \( \mathcal{X} \) contains a numéraire, and hence by [10, Theorem 2.3] there exists with respect to \( \mathbb{Q}_\infty \) a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator \( Y^\infty \). Now we define \( Z_s := \sum_{t \in \mathcal{A}} Y^t_s \mathbb{1}_{\{\tau = t\}} \) for \( s \in [0, T] \). We claim that \( (Z_s)_{s \in [0, T]} \) is a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator. Indeed, since for all \( t \in \mathcal{A} \), \( Y^t_s \) is strictly positive with respect to \( \mathbb{Q}_t \), we have for any \( s \in [0, T] \) that
\[
\mathbb{P}[Z_s > 0] = \sum_{t \in \mathcal{A}} \mathbb{Q}_t[Z_s > 0] \mathbb{P}[\tau = t] \\
= \sum_{t \in \mathcal{A}} \mathbb{Q}_t[Y^t_s > 0] \mathbb{P}[\tau = t] \\
= \sum_{t \in \mathcal{A}} \mathbb{P}[\tau = t] = 1,
\]

Using the same argument we can also show that \( Z \) is \( \mathbb{P} \)-càdlàg and that for all \( r < s \) in \([0, 1] \), \( X \in \mathcal{X} \), and \( A_s \in \mathcal{F}_s \),
\[
\mathbb{E}_\mathbb{P}\left[\frac{X_t Z_s}{X_s Z_s} \mathbb{1}_{A_s}\right] \leq \mathbb{P}[A_s],
\]
see also Step 2 in the proof of Theorem 2.14. This ensures that \( Z \) is indeed a strictly positive, \( \mathbb{P} \)-càdlàg generalized supermartingale deflator for \( \mathcal{X} \) under \( \mathbb{P} \). This finishes the proof for the non \( \mathbb{F} \)-adapted case.

For the case where we additionally assume that the market is \( \mathbb{F} \)-adapted, it remains to show that we can construct a strictly positive, càdlàg (and not only \( \mathbb{P} \)-càdlàg) supermartingale deflator (i.e., \( \mathbb{F} \)-adapted one). To that end, using the notations introduced before, we can now, since the market is \( \mathbb{F} \)-adapted, apply Lemma 3.6 to make sure that each \( Y^t_s \) defined in (3.11) is \( \mathcal{F}_s \)-measurable so that \( (Y^t_s)_{s \in [0, T]} \) is a strictly positive, \( \mathbb{P} \)-càdlàg supermartingale deflator with respect to \( \mathbb{Q}_t \). Consequently, the process \( Z_s = \sum_{t \in \mathcal{A}} Y^t_s \mathbb{1}_{\{\tau = t\}}, s \in [0, T] \), is a strictly positive, \( \mathbb{P} \)-càdlàg supermartingale deflator with respect to all \( \mathbb{Q}_t \). Consider the process \( \tilde{Z}_s := \mathbb{E}_\mathbb{P}[Z_s | \mathcal{F}_s], s \in [0, T] \). Then proceeding as in the proof of Theorem 2.14 we can show that \( X \tilde{Z} \) satisfies the (usual) supermartingale property under \( \mathbb{P} \) for all \( X \in \mathcal{X} \). Therefore, it remains to show that \( \tilde{Z} \) admits a càdlàg and \( \mathbb{F} \)-adapted \( \mathbb{P} \)-version. To that end, let \( \overline{X} \in \mathcal{X} \) be the generalized numéraire. Since \( \overline{X} \tilde{Z} \) is a nonnegative \( \mathbb{P} \)-supermartingale,
Fatou’s lemma together with the supermartingale property guarantees the right-continuity of the map \( s \mapsto \mathbb{E}_P [\tilde{X}_s \tilde{Z}_s] \). As \( P \) satisfies the usual conditions, a classical result from probability theory (see, e.g., [5, Theorem VI.4, p. 69]) hence ensures that there exists a càdlàg \( P \)-version of \( \tilde{X} \tilde{Z} \), say, \( S = (S_s)_{s \in [0, T]} \). Now we define a new strictly positive process \( Z' = (Z'_s)_{s \in [0, T]} \) by
\[
Z'_s := \frac{S_s}{\tilde{X}_s} \mathbb{1}_{\{\tilde{X}_s > 0\}} + \mathbb{1}_{\{\tilde{X}_s = 0\}}, \quad s \in [0, T].
\]
(3.13)

In view of the property (ii) of a wealth process in Definition 2.2, \( \tilde{X}_s(\omega) = 0 \) implies that \( \tilde{X}_s(\omega) = 0 \) for all \( r \geq s \), which in turn implies the càdlàg property of \( s \mapsto \mathbb{1}_{\{\tilde{X}_s = 0\}} \). Therefore, since \( S \) and \( \tilde{X} \) all have càdlàg paths, also \( Z' \) has càdlàg paths. Finally, for any \( X \in \mathcal{X} \), as \( \{X_t > 0\} \subseteq \{\tilde{X}_t > 0\} \) holds \( P \)-a.s., we have \( P \)-a.s. that
\[
X_t Z'_t = X_t \frac{S_t}{\tilde{X}_t} \mathbb{1}_{\{\tilde{X}_t > 0\}} = X_t \tilde{Z}_t \mathbb{1}_{\{X_t > 0\}} = X_t \hat{Z}_t.
\]
This together with the supermartingale deflator property of \( \tilde{Z} \) shows that \( Z' \) is indeed a strictly positive, càdlàg supermartingale deflator for \( \mathcal{X} \).

**3.2.2 Proof of Theorem 2.21**

In this subsubsection we provide the proof of Theorem 2.21. To that end, let us first argue why without loss of generality we may assume in its proof that both
\[
\mathcal{A} = \emptyset \quad (3.14)
\]
and \( J = \text{supp}(\mathcal{L}(\tau)) = [0, T] \), or equivalently,
\[
\forall \text{open interval } J \subseteq [0, T], \quad \mathcal{L}(\tau)[J] > 0. \quad (3.15)
\]
Indeed, recall that by assumption \( J = \text{supp}(\mathcal{L}(\tau)) \setminus \mathcal{A} \) is not empty. Moreover, from the proof of Theorem 2.18 we have seen that for each \( t \in \mathcal{A} \) one can find a strictly positive, \( (P-) \) càdlàg (generalized) supermartingale deflator \( Y_t \) for \( \mathcal{X} \) with respect to \( P[\cdot| \tau = t] \). Then by pasting them together we can obtain a process \( Z^\mathcal{A} \) defined by
\[
Z^\mathcal{A}_s := \sum_{t \in \mathcal{A}} Y^t_t \mathbb{1}_{\{\tau = t\}}, \quad s \in [0, T],
\]
such that \( Z^\mathcal{A} \) it is a strictly positive, \( (P-) \) càdlàg (generalized) supermartingale deflator for \( \mathcal{X} \) under the conditional measure \( P[\cdot| \tau \in \mathcal{A}] \). Now, denote by \( \mathcal{J}_1, \ldots, \mathcal{J}_n \) the connected component of \( J \). If we can find for each \( \mathcal{J}_i, i := 1, \ldots, n \), a strictly positive, \( (P-) \) càdlàg (generalized) supermartingale deflator \( Z^{\mathcal{J}_i} \) under the conditional measure \( P[\cdot| \tau \in \mathcal{J}_i] \), then by the same pasting arguments as in the discrete case the process
\[
Z_s := Z^\mathcal{A}_s \mathbb{1}_{\mathcal{A}}(s) + \sum_{i=1}^n Z^{\mathcal{J}_i}_s \mathbb{1}_{\mathcal{J}_i}(s), \quad s \in [0, T],
\]
will be a strictly positive, \( (P-) \) càdlàg (generalized) supermartingale deflator with respect to \( P \). Hence, in order to keep the notation short, we may indeed without loss of generality assume for the rest of this subsubsection that (3.14) and (3.15) hold.

Let us first prove the simpler direction (ii) \( \Rightarrow \) (i) in Theorem 2.21.
Proof of (ii) \(\Rightarrow\) (i) in Theorem 2.21 with (3.14) and (3.15) By the same argument as, e.g., in the proof of Theorem 2.14, we see that the existence of a strictly positive (generalized) supermartingale deflator \(Z\) implies that NUPBR\(_r\) holds for each \(s \in [0, T]\).

Now to see that also (i)(b) holds, note that by assumption there exist a countable dense subset \(D \subseteq [0, T] \) containing 0 and T and a strictly positive process \(Z^\infty\) defined on \(D\) with \(Z^\infty_0 \leq 1\) such that for all \(r < u \in D\)

\[
E_{Q_{r,u}} \left[ \frac{X_t Z^\infty_t}{X_0 Z^\infty_0} \right] \leq 1
\]

for all \(X \in \mathcal{X}\) under \(Q_{r,u}\), where we recall that \(Q_{r,u}[\cdot] = \mathbb{P}[\cdot | \tau \in (r, u)]\), see Notation 2.20. Then, since we have \(X_0 = 1\), see Definition 2.2, the above inequality implies that \(E_{Q_{r,u}}[X_t Z^\infty_t] \leq 1\). Hence, for any real number \(M > 0\), the Markov inequality implies that \(Q_{r,u}[X_t Z^\infty_t \geq M] \leq 1/M\). Consequently, we have for all \(X \in \mathcal{X}, M > 0\),

\[
Q_{r,u}[X_t \geq M] \leq Q_{r,u}[X_t Z^\infty_t \geq \sqrt{M}] + Q_{r,u}\left[ \frac{1}{Z^\infty_t} \geq \sqrt{M} \right]
\]

\[
\leq \frac{1}{\sqrt{M}} + Q_{r,u}\left[ \frac{1}{Z^\infty_t} \geq \sqrt{M} \right].
\]

Now, invoking (2.6), for any given \(\varepsilon > 0\) we can find \(M\) large enough such that \(Q_{r,u}\left[ \frac{1}{Z^\infty_t} \geq \sqrt{M} \right] < \varepsilon\) holds uniformly over all \(X \in \mathcal{X}\) and \(u > r \geq t \in D\). Hence, by combining all above estimates we can derive that

\[
\limsup_{M \to \infty} \sup_{r,u \in D, u > r \geq t} \sup_{X_t \in \mathcal{C}_t} Q_{r,u}[X_t \geq M] = 0,
\]

which is exactly the desired equation (2.5). This completes the proof of (ii) \(\Rightarrow\) (i) in Theorem 2.21.

The most technical part of this paper is to prove the implication (i) \(\Rightarrow\) (ii) in Theorem 2.21. For the ease of notation, we will without loss of generality assume that the set \(D\) in (i) of Theorem 2.21 satisfies the following:

\(D\) is the set of all dyadic numbers in \([0, T]\).

Moreover, we denote by \(D_k := \{i 2^k : i = 0, 1, \ldots, 2^k\}\) the collection of all \(k\)-th dyadic numbers in \([0, T]\).

Indeed, a careful look through the proof shows that the only property we actually use from the set of dyadic numbers \(D\) is that it is dense and that there exists an increasing sequence of finite sets \(D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots\) with \(\{0, T\} \subseteq D_i\) for each \(i\) satisfying \(D = \bigcup_{k \in \mathbb{N}} D_k\), which of course can be constructed for any countable dense subset \(D\) which contains 0 and \(T\).

Then, for every fixed \(k \in \mathbb{N}\) and \(r \in D_k \setminus \{T\}\), we use \(Q^k_r[\cdot]\) to denote the conditional probability

\[
Q^k_r[\cdot] := \mathbb{P}[\cdot | \tau \in (r, r + \frac{T}{2^k})].
\]

Moreover, a crucial object in the proof will be for each \(t \in D\) the set

\[
B_t := \{k \in \mathbb{N} : t \in D_k\}.
\]

Then, note under Assumptions (3.14) and (3.15), Eq. (2.5) implies that for all \(t \in D\),

\[
\limsup_{M \to \infty} \sup_{k \in B_t} \sup_{r \in D_k, r \geq t} \sup_{X_t \in \mathcal{C}_t} Q^k_r[X_t \geq M] = 0.
\]
With these preparations, we are now able to prove the implications (i) \(\Rightarrow\) (ii) in Theorem 2.21. Due to its technicality and length, we will divide the proof of the implication (i) \(\Rightarrow\) (ii) in Theorem 2.21 into five steps.

**Proof of (i) \(\Rightarrow\) (ii) in Theorem 2.21 with (3.14) and (3.15)** First we note that the assumption (3.14) implies that \(\mathbb{P}[\tau = \infty] = 0\), which means that \(\mathbb{P}[X_T = 0] = 1\). This in turn implies that \(\mathbb{P}[X_T = 0] = 1\) for all \(X \in \mathcal{X}\), meaning that the market has died out at the terminal time \(T\). Hence, it suffices to consider the market on the time interval \([0, T]\).

**Step 1 (Local deflators on \(D_k\)).** For each \(k\), denote by \(\mathcal{X}^k := \{(X_t)_{t \in D_k} : X \in \mathcal{X}\}\) the restriction of the market to the \(k\)-dyadic grid. To show that there exists for each \(k\) a deflator \((Z^k_s)_{s \in D_k}\) on \(\mathcal{X}^k\) with respect to each \(Q^k_r, r \in D_k \setminus \{T\}\), we follow the proof of Theorem 2.14. For a fixed \(k \in D_k\) and an \(r \in D_k \setminus \{T\}\), we see from the definition of \(\tau\) in (2.4) and the definition of the conditional measure \(Q^k_r\) in (3.16) that

(i) The market \(\mathcal{X}\) contains a numéraire on \([0, r]\) under the conditional measure \(Q^k_r;
(ii) For all \(X \in \mathcal{X}\) and \(u \geq r + \frac{T}{2^k}\), \(Q^k_r[X_u = 0] = 1\).

Since by assumption (i) NUPBR, holds for all \(r\), we can obtain as in the discrete-time case (see Theorem 2.14) a strictly positive generalized supermartingale deflator \(Y^{(k, r)}_s, s \in D_k\), for \(\mathcal{X}^k\) with respect to \(Q^k_r\). In particular, invoking the explicit construction of such \(Y^{(k, r)}\), see (3.6) or (3.11) and (3.12), we know that \((Y^{(k, r)}_s)_{s \in D_k}\) can be formulated as

\[
Y^{(k, r)}_s = \frac{1}{f^{(k, r)}_s} 1\{s \leq r\} + 1\{s > r\}, \quad s \in D_k,
\]

(3.19)

where \(f^{(k, r)}_s \in \text{cl}_{Q^k_r}(C_r)\) is the static deflator in \(\text{cl}_{Q^k_r}(C_r)\) with respect to \(Q^k_r\) (cf. Lemma 3.3).

Now we can paste these “local deflators” over all \(r \in D_k \setminus \{T\}\) as in the discrete case before. More precisely, using (3.19), we define the process \((Z^k_s)_{s \in D_k}\) by setting for each \(s \in D_k\)

\[
Z^k_s := \sum_{r \in D_k, r < T} Y^{(k, r)}_s 1\{\tau \in [r, r + \frac{T}{2^k}]\} = \sum_{r \in D_k, s \leq r < T} \frac{1}{f^{(k, r)}_s} 1\{\tau \in (r, r + \frac{T}{2^k}]\} + 1\{\tau \leq s\}.
\]

(3.20)

Following the arguments of Theorem 2.14 and Remark 3.7, we conclude that \((Z^k_s)_{s \in D_k}\) is a strictly positive generalized supermartingale deflator for \(\mathcal{X}^k\) with respect to \(\mathbb{P}\) and all \(Q^k_r, r \in D_k \setminus \{T\}\). This means for all \(r \in D_k \setminus \{T\}\), \(s < t \in D_k, A_s \in \mathcal{F}_s\), it holds that

\[
\mathbb{E}_{\mathbb{P}}[\frac{X_t Z^k_t}{X_s Z^k_s} 1_{A_s}] \leq \mathbb{P}[A_s], \quad \mathbb{E}_{Q^k_r}[\frac{X_t Z^k_t}{X_s Z^k_s} 1_{A_s}] \leq Q^k_r(A_s),
\]

(3.21)

which finishes Step 1 of the proof.

**Step 2 (Extension from \(D_k\) to \(D\)).** In Step 1, we have obtained a sequence \((Z^k)_{k \in \mathbb{N}}\) such that for each \(k\), \(Z^k\) is a strictly positive generalized supermartingale deflator for \(\mathcal{X}^k\) under \(\mathbb{P}\) and under all \(Q^k_r, r \in D_k \setminus \{T\}\). Like for the discrete-case we did not need the uniform boundedness assumption (2.5) in the construction of these \(Z^k\). In Step 2, we want to construct a process \(Z^D\) defined on \(D\) based on \((Z^k)_{k \in \mathbb{N}}\) such that \(Z^D\) satisfies certain strictly positive generalized supermartingale deflator properties for \(\mathcal{X}^D := \{(X_t)_{t \in D} : X \in \mathcal{X}\}\) with respect to \(\mathbb{P}\) and all \(Q^k_r, r \in D_k \setminus \{T\}, k \in \mathbb{N}\), cf. Remark 2.24. To achieve this goal, the assumption (2.5) is crucial. To that end, let us first start with some simple observations. Denote for each \(k \in \mathbb{N}\) and \(r \in D_k \setminus \{T\}\) the set

\[
A^k_r := \{\tau \in (r, r + \frac{T}{2^k}]\}.
\]
Now, let $t \in \mathcal{D}$ and consider $k \in B_t$, where $B_t$ is defined in (3.17). For every $r \geq t$ these $A_r^k, r \in \mathcal{D}_k \setminus \{T\}$, are pairwise disjoint, and we also have that $\bigcup_{r \geq t, r \in \mathcal{D}_k \setminus \{T\}} A_r^k = \{\tau > t\}$ is disjoint from the set $\{\tau \leq t\}$. Moreover, for $r \in \mathcal{D}_k \setminus \{T\}$, we have that

$$A_r^{k+1} \cup A_r^{k+1}\frac{t}{2^{r+T}} = A_r^k. \quad (3.22)$$

**Step 2.1 (Boundedness of convex combinations of $1/Z^k$).** For each $t \in \mathcal{D}$, we claim that the condition (2.5) (or more precisely condition (3.18)) implies that the convex hull of $\frac{1}{Z_t^k}$, $k \in B_t$, denoted by

$$K_t := \text{conv}\left\{ \frac{1}{Z_t^k} : k \in B_t \right\} \quad (3.23)$$

is $\mathbb{P}$-bounded. Note that since every ($Z^k$) is a strictly positive generalized supermartingale deflator for $\mathcal{X}^k$ under $\mathbb{P}$ and $t \in \mathcal{D}_k$, we have by definition that $Z_t^k > 0$ holds $\mathbb{P}$-a.s.; in particular, $\frac{1}{Z_t^k}$ is well defined.

Now, in view of (3.20) and the disjointness relation between $A_r^k, r \geq t$, and $\{\tau \leq t\}$, we see that

$$\frac{1}{Z_t^k} = \sum_{r \in \mathcal{D}_k, t \leq r < T} \hat{f}_t^{(k,r)} \mathbb{1}_{A_r^k} + \mathbb{1}_{\{\tau \leq t\}}, \quad (3.24)$$

where $\hat{f}_t^{(k,r)} \in \text{cl}_{\mathbb{Q}_t^k}(C_t)$ is the static deflator in $\text{cl}_{\mathbb{Q}_t^k}(C_t)$ with respect to $\mathbb{Q}_t^k$ (cf. Lemma 3.3). Now we fix an $\varepsilon > 0$ and consider $\lambda \frac{1}{Z_t^k} + (1 - \lambda) \frac{1}{Z_t^{k+1}}$ for any $\lambda \in [0, 1]$. By (3.24) and (3.22) we have

$$\lambda \frac{1}{Z_t^k} + (1 - \lambda) \frac{1}{Z_t^{k+1}} = \sum_{r \in \mathcal{D}_k, t \leq r < T} \left( \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r)} \right) \mathbb{1}_{A_r^{k+1}} \quad (3.25)$$

$$+ \sum_{r \in \mathcal{D}_k, t \leq r < T} \left( \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r+\frac{T}{2^{r+T}})} \right) \mathbb{1}_{A_r^{k+1}\frac{t}{2^{r+T}}} + \mathbb{1}_{\{\tau \leq t\}}.$$

Moreover, since $\mathbb{Q}_{t+1}^{k-1} \ll \mathbb{Q}_t^k$ and $\mathbb{Q}_{r+\frac{T}{2^{r+T}}}^{k+1} \ll \mathbb{Q}_r^k$ holds for all $r \in \mathcal{D}_k \setminus \{T\}$, and $C_t$ is convex, we have

$$\lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r)} \in \text{cl}_{\mathbb{Q}_t^{k+1}}(C_t),$$

$$\lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r+\frac{T}{2^{r+T}})} \in \text{cl}_{\mathbb{Q}_r^{k+1}\frac{t}{2^{r+T}}}(C_t),$$

which together with Lemma 3.1 allows us to use the uniform boundedness condition (3.18) to find an $M > 1$ independent of $k \in B_t$, $r \in \mathcal{D}_k$, and $\lambda \in [0, 1]$ such that both

$$\mathbb{Q}_{t+1}^{k+1}\left[ \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r)} \geq M \right] \leq \varepsilon,$$

$$\mathbb{Q}_{r+\frac{T}{2^{r+T}}}^{k+1}\left[ \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r+\frac{T}{2^{r+T}})} \geq M \right] \leq \varepsilon. \quad (3.26)$$
Now, denote
\[
\begin{align*}
    h_t^{(k,1,r)} &:= \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k,1,r)}, \\
    h_t^{(k+1,r+\frac{r}{2^{k+1}})} &:= \lambda \hat{f}_t^{(k,r)} + (1 - \lambda) \hat{f}_t^{(k+1,r+\frac{r}{2^{k+1}})}. 
\end{align*}
\]
(3.27)

Then (3.25) and (3.26) together with the observation that on \(\{\tau \leq t\}\) one has \(Z_t^k = Z_t^{k+1} = 1\), ensures for such \(M > 1\) that for all \(k \in B_t\)
\[
\begin{align*}
    \mathbb{P}\left[ \frac{1}{Z_t^k} + (1 - \lambda) \frac{1}{Z_t^{k+1}} \geq M \right] &= \mathbb{P}\left[ \frac{1}{Z_t^k} + (1 - \lambda) \frac{1}{Z_t^{k+1}} \geq M \right] \cap \{\tau > t\} \\
    &\leq \sum_{r \in D_k, t \leq r < T} Q_r^{k+1}\left[ h_t^{(k,1,r)} \geq M \right] \mathbb{P}\left[ A_t^{k+1} \right] \\
    &+ \sum_{r \in D_k, t \leq r < T} Q_r^{k+1}\left[ h_t^{(k+1,r+\frac{r}{2^{k+1}})} \geq M \right] \mathbb{P}\left[ A_t^{k+1} \right] \\
    &\leq \varepsilon \sum_{r \in D_k, t \leq r < T} \left( \mathbb{P}\left[ A_t^{k+1} \right] + \mathbb{P}\left[ A_t^{k+1} \right] \right) \\
    &\leq \varepsilon, 
\end{align*}
\]

where for the last inequality we used that
\[
\sum_{r \in D_k, t \leq r < T} \left( \mathbb{P}\left[ A_t^{k+1} \right] + \mathbb{P}\left[ A_t^{k+1} \right] \right) = \mathbb{P}\left[ \bigcup_{r \geq t, r \in D_k \setminus \{T\}} (A_t^{k+1} \cup A_t^{k+1}) \right] \leq 1.
\]

Since \(M > 1\) was independent of \(k \in B_t, r \in D_k\), and \(\lambda \in [0, 1]\), thanks to the uniform boundedness condition (3.18), we can show the same result for any convex combination of \(\frac{1}{Z_t^k}\) for \(k \in B_t\), which gives the \(\mathbb{P}\)-boundedness of the convex hull \(K_t\) defined in (3.23).

Step 2.2 (An application of Komlos lemma). Let \(t \in D\) and consider the sequence of nonnegative random variable \((Z_t^k)_{k \in B_t}\), where \(Z^k\) is the strictly positive generalized supermartingale deflator for \(X^k\) constructed in Step 1. We first claim that that the convex set \(\text{conv}\{Z_t^k : k \in B_t\}\) is \(\mathbb{P}\)-bounded. Indeed, for all \(k \in B_t\), using the generalized supermartingale property (with respect to \(\mathbb{P}\)), see (3.21), we have
\[
\mathbb{E}_\mathbb{P}\left[ X_t Z_t^k \right] \leq 1.
\]

Moreover, when considering any convex combinations \(Y_t := \lambda_1 Z_t^{k_1} + \cdots + \lambda_l Z_t^{k_l}\) of \(Z_t^k\)'s for \(k_1, \ldots, k_l \in B_t\), we also have that
\[
\mathbb{E}_\mathbb{P}\left[ X_t, Y_t \right] \leq 1. \quad (3.28)
\]

This in turn implies that the convex hull \(\text{conv}\{Z_t^k : k \in B_t\}\) of all \((Z_t^k)_{k \in B_t}\) is \(\mathbb{P}\)-bounded on \(\{X_t > 0\}\). On the other hand, from (3.20) we see that \(Z_t^1 = 1\) on \(\{\tau \leq t\}\) for all \(k \in B_t\). Moreover, \(\{\overline{X}_t = 0\} \subseteq \{\tau \leq t\}\) holds as \(\tau\) is the first hitting time of \(\overline{X}\) at 0. Therefore, we have \(Z_t^1 = 1\) on \(\{\overline{X}_t = 0\}\) for all \(k \in B_t\) which ensures that also \(Y_t = 1\) on \(\{\overline{X}_t = 0\}\) for any convex combination \(Y_t := \lambda_1 Z_t^{k_1} + \cdots + \lambda_l Z_t^{k_l}\) of \(Z_t^k\)'s for \(k_1, \ldots, k_l \in B_t\). This means that \(\text{conv}\{Z_t^k : k \in B_t\} = 1\) on \(\{\overline{X}_t = 0\}\), in particular it is \(\mathbb{P}\)-bounded also on \(\{\overline{X}_t = 0\}\). Combining the two facts above together we can conclude that the convex set \(\text{conv}\{Z_t^k : k \in B_t\}\) is indeed \(\mathbb{P}\)-bounded. Now we can apply Komlos lemma for nonnegative random variables, see [6, Appendix], to the sequence \((Z_t^k)_{k \in B_t}\) to get a sequence of forward convex combinations of \(Z_t^k, k \in B_t\), which is denoted by \(\text{fconv}Z_t^k, k \in B_t\), such that
that converges to \((3.23)\), is also \((\exists)\) strictly positive, finite valued random variable \(Z\) such that it converges to a strictly positive, finite valued random variable \((\text{and indices})\) such that both sequences of forward convex combinations with respect to \(P\) are bounded. Then, since \(1/Z_{i \rightarrow \infty} < \infty\) or equivalently, \(Z_{i} > 0\) \(P\)-a.s.

We point out that in the above convergent sequence \(\text{fconv} Z_{i}^{k} \), where

\[
\text{fconv} Z_{i}^{k} = \lambda^{k}_{1} Z_{i}^{1} + \ldots + \lambda^{k}_{j} Z_{i}^{j+l-1},
\]

these convex weights \((\lambda^{k}_{j})_{j=1, \ldots, l}\) and the indices \(l \in \mathbb{N}\), may depend on \(t \in \mathcal{D}\), and hence may vary when doing the above procedure separately for each \(t \in \mathcal{D}\). However, it turns out in the later part of the proof that we would like to have for each \(s, t \in \mathcal{D}\) joint convex weights \((\text{and indices})\) such that both sequences of forward convex combinations with respect to \((Z_{i}^{k})\) and \((Z_{i}^{l})\) converge.

To see this, we first consider \(\mathcal{D}_{0} := \{0, T\}\). By Komlos lemma, we can find a sequence of forward convex combinations of \(Z_{i}^{0} \), \(k \in \mathbb{N}\), which now is denoted by \((\text{fconv} Z_{i}^{0})_{k \in \mathbb{N}}\), such that it converges to a strictly positive, finite valued random variable \(Z_{0}^{\infty}\) \(P\)-a.s.. Then we apply Komlos lemma to the sequence \(\text{fconv} Z_{i}^{0}\), where \((\text{fconv} Z_{i}^{0})_{k \in \mathbb{N}}\) possesses the same forward convex combination form as in the sequence \((\text{fconv} Z_{i}^{0})_{k \in \mathbb{N}}\), but with \(Z_{i}^{k}\) replaced by \(Z_{i}^{k}\), to get a convergent sequence \((\text{fconv} Z_{i}^{k})_{k \in \mathbb{N}}\) which consists of forward convex combinations of \((\text{fconv} Z_{i}^{k})_{k \in \mathbb{N}}\), and converges to a strictly positive, finite valued random variable \(Z_{i}^{\infty}\) \(P\)-a.s.. Note that when using the convex combinations appeared in \((\text{fconv} Z_{i}^{k})_{k \in \mathbb{N}}\), the sequence \((\text{fconv} Z_{i}^{k})_{k \in \mathbb{N}}\) still converges to \(Z_{0}^{\infty}\) \(P\)-a.s..
we indeed obtained by the above procedure the desired property that
\[
\forall s, t \in \mathcal{D}, \lim_{k \to \infty} (\text{fconv}_{p(s,t)} Z^k_t) = Z^\infty_t \quad \text{and} \quad \lim_{k \to \infty} (\text{fconv}_{p(s,t)} Z^k_s) = Z^\infty_s. \tag{3.29}
\]

We finish Step 2 by remarking that as each \( Z^k \) is a generalized supermartingale deflator for \( \mathcal{X}^k \) with respect to \( \mathbb{P} \), we have by definition that \( Z^k_0 \leq 1 \) for all \( k \in \mathbb{N} \). This, in turn, ensures that \( Z^\infty_0 \leq 1 \) holds as well.

**Step 3** (Generalized supermartingale property (GSP) on \( \mathcal{D} \)).

**Step 3.1** (GSP on \( \mathcal{D} \) for \( \mathbb{P} \)-expectations). Our goal in Step 3.1 is to show that for all \( s < t \) in \( \mathcal{D} \) and for all \( X \in \mathcal{X} \),
\[
\mathbb{E}_\mathbb{P} \left[ \frac{X_t Z^{k+1}_s}{X_s Z^k_t} \right] \leq 1. \tag{3.30}
\]

To that end, let \( t > s \) in \( \mathcal{D} \), \( k \in B \cap B_s \) (which means that \( s, t \in \mathcal{D}_k \)) be fixed. We first claim that for any \( X \in \mathcal{X} \) and \( l \geq 1 \),
\[
\mathbb{E}_\mathbb{P} \left[ \frac{X_t Z^{k+1}_s}{X_s Z^k_t} 1_{\{ \tau > t \}} \right] \leq 1. \tag{3.31}
\]

We first consider the case \( l = 1 \). Note that on the event \( \{ \tau \leq t \} \) we have \( X_t = 0 \) and hence by Remark 2.9, it remains to show that
\[
\mathbb{E}_\mathbb{P} \left[ \frac{X_t Z^{k+1}_s}{X_s Z^k_t} 1_{\{ \tau > t \}} \right] \leq 1. \tag{3.32}
\]

To that end, in view of the formula (3.24) for \( Z^k \) and \( Z^{k+1} \) we have
\[
\frac{X_t Z^{k+1}_s}{X_s Z^k_t} 1_{\{ \tau > t \}} = \sum_{r \in \mathcal{D}_k, t \leq r < T} \left( \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k),r}_s} \mathbb{1}_{A^{k+1}_r} + \frac{X_t/\hat{f}^{(k+1),+\frac{T}{2k+1}}_t}{X_s/\hat{f}^{(k),+\frac{T}{2k+1}}_s} \mathbb{1}_{A^{k+1}_r} \right), \tag{3.33}
\]

where \( \hat{f}^{(k),r}_t \) and \( \hat{f}^{(k+1),+\frac{T}{2k+1}} \) are the corresponding static deflator for the closure of \( \mathcal{C}_t \) under the measure \( \mathbb{Q}^k \), \( \mathbb{Q}^{k+1} \), and \( \mathbb{Q}^{k+1}_{r+\frac{T}{2k+1}} \), respectively; the same holds when replacing \( t \) by \( s \). Moreover, by applying Lemma 3.6, we choose each \( \hat{f}^{(j,q),r}_t \), for \( j = k, k + 1, q = r, r + \frac{T}{2k+1} \), above to be \( \mathcal{F}_r \) and \( \mathcal{F}_s \)-measurable, respectively.

Now, for a fixed \( r \in \mathcal{D}_k \) with \( r \geq t \) we observe that
\[
\mathbb{E}_\mathbb{P} \left[ \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k),r}_s} 1_{A^{k+1}_r} \right] = \mathbb{E}_{\mathbb{Q}^{k+1}} \left[ \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k),r}_s} \right] \mathbb{P}[A^{k+1}_r] = \mathbb{E}_{\mathbb{Q}^{k+1}} \left[ \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k),r}_s} \right] \mathbb{P}[A^{k+1}_r] \]

Recall that in (3.21) we have shown that \( X/\hat{f}^{(k+1),r}_t \) is a generalized supermartingale with respect to \( \mathbb{Q}^{k+1}_r \), i.e.,
\[
\mathbb{E}_{\mathbb{Q}^{k+1}_r} \left[ \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k+1),r}_s} \right] \leq 1. \tag{3.34}
\]

So, as \( \frac{\hat{f}^{(k),r}_s}{\hat{f}^{(k+1),+\frac{T}{2k+1}}} \) is \( \mathcal{F}_s \)-measurable, we have
\[
\mathbb{E}_{\mathbb{Q}^{k+1}_r} \left[ \frac{X_t/\hat{f}^{(k+1),r}_t}{X_s/\hat{f}^{(k+1),+\frac{T}{2k+1}}} \right] \leq \mathbb{E}_{\mathbb{Q}^{k+1}_r} \left[ \frac{\hat{f}^{(k),r}_s}{\hat{f}^{(k+1),+\frac{T}{2k+1}}} \right]. \tag{3.35}
\]
Also by recalling that a $Q^k_{r+1}$-version of $f_s^{(k+1,r)}$ is the static deflator for $C_{l_s}^{(k+1)}(C_s)$ with respect to $Q^k_r$, and that a $Q^k_r$-version of $f_s^{(k,r)}$ satisfies that $f_s^{(k,r)} \in \text{cl}_{Q^k_r}(C_s) \subseteq \text{cl}_{Q^k_{r+1}}(C_s)$, using that $Q^k_{r+1} \ll Q^k_r$, we obtain that
\[
\mathbb{E}_{Q^k_{r+1}}[\frac{f_s^{(k,r)}}{f_s^{(k+1,r)}}] \leq 1.
\]

Hence, from the above estimates we can conclude that
\[
\mathbb{E}_P\left[\frac{X_t}{f_s^{(k+1,r)}} \frac{f_s^{(k+1,r)}}{f_s^{(k+1,r)}} \mathbb{1}_{A^{k+1}_t} \right] \leq P[A^{k+1}_t].
\]

Moreover, by replacing $r$ with $r + \frac{T}{2^{k+1}}$, we obtain with the same arguments that
\[
\mathbb{E}_P\left[\frac{X_t}{f_s^{(k+1,r+\frac{T}{2^{k+1}})}} \frac{f_s^{(k+1,r+\frac{T}{2^{k+1}})}}{f_s^{(k+1,r+\frac{T}{2^{k+1}})}} \mathbb{1}_{A^{k+1}_t} \right] \leq P[A^{k+1}_{r+\frac{T}{2^{k+1}}}].
\]

Combining the above two bounds together with (3.33) implies that
\[
\mathbb{E}_P\left[\frac{X_t Z_{t+1}^{k+1}}{X_t Z_{T_s}^k} \mathbb{1}_{\{\tau > t\}} \right] \leq \sum_{r \in D, t \leq r < T} \left(P[A^{k+1}_t] + P[A^{k+1}_{r+\frac{T}{2^{k+1}}}]ight)
= \sum_{r \in D, t \leq r < T} P[A^k_r]
\leq 1,
\]
as we claimed in (3.32). Finally, for $l \geq 1$, we can adapt the above proof using the disjoint decomposition of $A^k_r$ into $A^{k+l}_r \cup A^{k+l}_{r+\frac{T}{2^{k+1}}} \cup \cdots \cup A^{k+l}_{r+\frac{l-1}{2^{k+1}}T}$. Hence the claim is proved.

Now for given $s < t$ in $\mathcal{D}$ we fix $k_0 \in B_t \cap B_s$. Then for every $k \geq k_0$, the bound (3.31) implies that
\[
\mathbb{E}_P\left[\frac{X_t Z_{t+1}^k}{X_t Z_{T_s}^{k_0}} \mathbb{1}_{\{\tau > t\}} \right] \leq 1.
\]

Letting $k \to \infty$, we obtain by Fatou’s lemma that $\mathbb{E}_P\left[\frac{X_t Z_{t+1}^\infty}{X_t Z_{T_s}^{k_0}} \right] \leq 1$. Then, by using the convexity of the function $(0, \infty) \ni x \mapsto 1/x$, we also have that
\[
\mathbb{E}_P\left[\frac{X_t Z_{t+1}^\infty}{X_t Z_{T_s}^{k_0}} \mathbb{1}_{\{\tau > t\}} \right] \leq 1.
\]

Finally, by letting $k_0 \to \infty$ we conclude that indeed for all $s < t$ in $\mathcal{D}$, for all $X \in \mathcal{X}$,
\[
\mathbb{E}_P\left[\frac{X_t Z_{t+1}^\infty}{X_t Z_{T_s}^\infty} \right] \leq 1. \tag{3.36}
\]

Step 3.2 (GSP on $\mathcal{D}$ for conditional expectations). In this step we claim that for any $s < t$ in $\mathcal{D}$ and any $X \in \mathcal{X}$ the bound (3.36) also holds true for any conditional probability $Q^k_q$, $k_0 \in B_t \cap B_s$, $q \in \mathcal{D}_{k_0} \setminus \{T\}$, namely that
\[
\mathbb{E}_{Q^k_q}\left[\frac{X_t Z_{t+1}^\infty}{X_t Z_{T_s}^\infty} \right] \leq 1. \tag{3.37}
\]

Indeed, to see this, observe first that if $q \leq t$, then $X_t = 0$ under the measure $Q^k_q$, $k \in \mathcal{B}_t \cap \mathcal{B}_s$, $q \in \mathcal{D}_{k_0} \setminus \{T\}$, denoted by $f_{s}^{(k+1,r)}$ and therefore using Remark 2.9 we see that (3.37) indeed holds in

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that case. It hence remains to consider the case where \( q > t \). Following the same line as in Step 3.1, we first consider the quantity \( \mathbb{E}_{\mathcal{Q}_{q}^{k_0}} \left[ \frac{X_r Z_{r+1}^{k+1}}{X_s Z_{s}^{k}} \right] \) for some \( k \geq k_0 \) and let

\[
B_{q,k_0,k} := \left\{ r \in \mathcal{D}_k : r \in (q, q + \frac{T}{2^{k_0}}) \right\}.
\]

In view of (3.33), we can use (3.34) derived in the last step together with the disjoint decomposition \( A_{q}^{k_0} = \bigcup_{r \in B_{q,k_0,k}} \left[ A_{r}^{k+1} \cup A_{+r+T}^{k+1} \right] \) to check that

\[
\mathbb{E}_{\mathcal{Q}_{q}^{k_0}} \left[ \frac{X_r Z_{r+1}^{k+1}}{X_s Z_{s}^{k}} \right] = \sum_{r \in B_{q,k_0,k}} \left( \mathbb{E}_{\mathcal{Q}_{q}^{k_0}} \left[ \frac{X_r Z_{r+1}^{k+1}}{X_s Z_{s}^{k}} \right] \mathbb{P}[A_{r}^{k+1}] + \mathbb{E}_{\mathcal{Q}_{q}^{k_0}} \left[ \frac{X_r Z_{r+1}^{k+1}}{X_s Z_{s}^{k}} \right] \mathbb{P}[A_{r+T}^{k+1}] \right).
\]

Finally, we can apply the same arguments used at the end of Step 3.1 to see first for each fixed \( k \geq k_0 \) that \( \mathbb{E}_{\mathcal{Q}_{q}^{k_0}} \left[ \frac{X_r Z_{r+1}^{k+1}}{X_s Z_{s}^{k}} \right] \leq 1 \), and then by considering forward convex combinations of denominators \( Z_{s}^{k} \) that indeed the desired inequality (3.37) holds.

Step 4 (Supermartingale property).

Step 4.1 (Supermartingale property on \( \mathcal{D} \)). Recall that by construction, see (3.21), we have for every \( k \in \mathbb{N} \) that the process \( (Z_{s}^{k})_{t \in \mathcal{D}_k} \) is a strictly positive generalized supermartingale deflator for \( \mathcal{X}^{k} \) with respect to \( \mathbb{P} \). Consequently, by the same arguments as in the proof of Theorem 2.14, we see that the process

\[
\tilde{Z}_{s}^{k} := \mathbb{E}_{\mathbb{P}} \left[ Z_{s}^{k} \mid \mathcal{F}_{t} \right], \quad t \in \mathcal{D}_k,
\]

is a strictly positive supermartingale deflator for \( \mathcal{X}^{k} \) under the measure \( \mathbb{P} \). In particular, it holds for all \( s < t \in \mathcal{D}_k \) and \( X \in \mathcal{X} \) that

\[
\mathbb{E}_{\mathbb{P}} \left[ X_t \tilde{Z}_{s}^{k} \mid \mathcal{F}_{s} \right] \leq X_s \tilde{Z}_{s}^{k}.
\]

Now fix \( s < t \) in \( \mathcal{D} \). Then, observe that the above supermartingale property implies for any \( k \in B_s \cap B_t \) that

\[
\mathbb{E}_{\mathbb{P}} \left[ X_t \text{fconv}_{p(s,t)} \tilde{Z}_{s}^{k} \mid \mathcal{F}_{s} \right] \leq X_s \text{fconv}_{p(s,t)} \tilde{Z}_{s}^{k}.
\]

In addition, note that since the two sequences

\[
(\text{fconv}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t} \quad \text{and} \quad (\text{fconv}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t}
\]

consists of nonnegative random variables, we can find by Komlos lemma subsequences of forward convex combinations consisting of members from \( (\text{fconv}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t} \) and \( (\text{fconv}^{(2)}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t} \) which we denote by

\[
(\text{fconv}^{(2)}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t} \quad \text{and} \quad (\text{fconv}^{(2)}_{p(s,t)} \tilde{Z}_{s}^{k})_{k \in B_s \cap B_t}.
\]
such that they converge to some nonnegative random variables $\tilde{Z}_t^\infty$ and $\tilde{Z}_t^\infty$ $\mathbb{P}$-a.s., respectively. Moreover, note that by Fatou’s lemma and (3.39), the supermartingale property is preserved by this limiting procedure, as

$$\mathbb{E}_\mathbb{P}\left[X_t \tilde{Z}_t^\infty \bigg| F_s\right] \leq \liminf_{k \to \infty} \mathbb{E}_\mathbb{P}\left[X_t \text{fconv}_{p(s, t)}^{(2)} \tilde{Z}_t^k \bigg| F_s\right]$$

$$\leq \liminf_{k \to \infty} X_s \text{fconv}_{p(s, t)}^{(2)} \tilde{Z}_s^k$$

$$= X_s \tilde{Z}_s^\infty.$$ 

Therefore, as $s < t \in \mathcal{D}$ was arbitrary, it remains to show that $\tilde{Z}_s^\infty$ and $\tilde{Z}_t^\infty$ are finite and strictly positive to conclude that $\tilde{Z}_t^\infty$ is a strictly positive supermartingale deflator for $\mathcal{X}^D$ on $\mathcal{D}$. We focus on time $t$ as for $s < t$ the argument is the same.

To see this, observe that since $\tilde{Z}_t^k := \mathbb{E}_\mathbb{P}[Z_t^k \big| F_t]$, we have that

$$\text{fconv}_{p(s, t)}^{(2)} \tilde{Z}_t^k = \mathbb{E}_\mathbb{P}\left[\text{fconv}_{p(s, t)}^{(2)} Z_t^k \bigg| F_t\right].$$

Moreover, recall that in (3.29) of Step 2.2 we have found $Z_t^\infty$ which is finite and strictly positive such that $\mathbb{P}$-a.s.,

$$Z_t^\infty = \lim_{k \to \infty} \text{fconv}_{p(s, t)}^{(2)} Z_t^k.$$ 

This, as the $\mathbb{P}$-almost surely type convergence is preserved by forward convex combinations, ensures that $\mathbb{P}$-a.s., also

$$Z_t^\infty = \lim_{k \to \infty} \text{fconv}_{p(s, t)}^{(2)} Z_t^k.$$ 

As a consequence, we have by Fatou’s lemma that

$$\mathbb{E}_\mathbb{P}\left[Z_t^\infty \bigg| F_t\right] = \mathbb{E}_\mathbb{P}\left[\lim_{k \to \infty} \text{fconv}_{p(s, t)}^{(2)} Z_t^k \bigg| F_t\right]$$

$$\leq \liminf_{k \to \infty} \mathbb{E}_\mathbb{P}\left[\text{fconv}_{p(s, t)}^{(2)} Z_t^k \bigg| F_t\right]$$

$$= \liminf_{k \to \infty} \text{fconv}_{p(s, t)}^{(2)} \tilde{Z}_t^k$$

$$= \tilde{Z}_t^\infty.$$ 

Since from Step 2.2 we know that $Z_t^\infty$ is strictly positive and so is its conditional expectation $\mathbb{E}_\mathbb{P}[Z_t^\infty \big| F_t]$, we can conclude thanks to the above inequality that indeed $\tilde{Z}_t^\infty$ is also strictly positive. Finally, to see that $\tilde{Z}_t$ is finite, note that on $\{X_t = 0\} \subseteq \{\tau \leq t\}$, we have $Z_t^k = 1$ for each $k$, see (3.20), which ensures that both $Z_t^\infty = 1$ and $\tilde{Z}_t^\infty = 1$ on $\{X_t = 0\}$. Moreover, since $\mathbb{E}_\mathbb{P}[X_t \tilde{Z}_t^\infty] \leq 1$ holds, we have $\tilde{Z}_t^\infty < \infty$ also on $\{X_t > 0\}$. This in turn shows that indeed $\tilde{Z}_t$ is finite and we can conclude that indeed $(\tilde{Z}_t^\infty)_{t \in \mathcal{D}}$ is a strictly positive supermartingale deflator for $\mathcal{X}^D := \{(X_t)_{t \in \mathcal{D}} : X \in \mathcal{X}\}$ with respect to $\mathbb{P}$.

Step 4.2 (Supermartingale property on $[0, T]$). The extend $(\tilde{Z}_t^\infty)_{t \in \mathcal{D}}$ from $\mathcal{D}$ to $[0, T]$ note that from the above step, we know that $S := \overline{X} \overline{Z}^\infty$ is nonnegative supermartingale on $\mathcal{D}$, hence by classical results, see for example [5, Theorem VI.2, p. 67], we can extend $S$ from $\mathcal{D}$ to $[0, T]$. Next, we can apply the same argument as in the proof of Theorem 2.18, see (3.13), to indeed obtain a strictly positive adapted càdlàg process $(Z_t)_{t \in [0, T]}$ such that $XZ$ is a supermartingale under $\mathbb{P}$ for all $X \in \mathcal{X}$.

Step 5 (A uniform bound for $1/Z_t^\infty$). In the last step we show the uniform bound (2.6) for $1/Z_t^\infty$ (for the dyadic case, the general case goes analogously), namely that for each $t \in \mathcal{D}$ we
have that
\[
\lim_{M \to \infty} \sup_{k_0 \in B_t} \sup_{r \in \mathcal{D}_{k_0}} Q_r^k \left[ \frac{1}{Z_t^r} \geq M \right] = 0. \tag{3.40}
\]

To that end, fix \( t \in \mathcal{D} \) and let \( k_0 \in B_t, r \in \mathcal{D}_{k_0} \). Now, let \( \{f_{\text{conv}}t Z_t^k\}_{k \in B_t} \) be any sequence of forward convex combinations which converges to \( Z_t^\infty \) \( \mathbb{P} \)-a.s., see also Step 2.2 for its existence, and denote
\[
f_{\text{conv}}t Z_t^k := \lambda_1 Z_t^k + \ldots + \lambda_{l+1} Z_t^{k+l}
\]
where \( \lambda_1, \ldots, \lambda_{l+1} \) are the corresponding convex weights. Invoking the concrete formula (3.20) of each \( Z_t^k \) and using a similar argument as in Step 2.1, see (3.25) and (3.27), but with respect to the following partition \( A_r^{k_0} = \bigcup_{i=0}^{2^{k-k_0}-1} \bigcup_{j=0}^{(2^l-1)} A^{k+l}_{r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77}} \) of \( A_r^{k_0} \), we see that
\[
\frac{1}{f_{\text{conv}}t Z_t^k} = \sum_{i=0}^{2^{k-k_0}-1} \sum_{j=0}^{2^l-1} h_t \left( k+l, r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77} \right),
\]
where every \( h_t \left( k+l, r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77} \right) \) belongs to \( \text{cl} Q_r^{k+l} \left[ f_{\text{conv}}t Z_t^k \right] \). Therefore, we deduce from the above identity that for any \( M > 0 \)
\[
Q_r^{k_0} \left[ \frac{1}{f_{\text{conv}}t Z_t^k} \geq M \right] = \mathbb{P} \left[ \frac{1}{f_{\text{conv}}t Z_t^k} \geq M ; A_r^{k_0} \right] \frac{1}{\mathbb{P}[A_r^{k_0}]} = \sum_{i=0}^{2^{k-k_0}-1} \sum_{j=0}^{2^l-1} \mathbb{P} \left[ A^{k+l}_{r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77}} \geq M \right] \frac{\mathbb{P}[A^{k+l}_{r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77}}]}{\mathbb{P}[A_r^{k_0}]}.
\]
Now, the uniform boundedness property (2.5) ensures for any given \( \varepsilon > 0 \) that there exists an \( M > 0 \) such that for all \( j, k, l, r \) we have that
\[
Q_r^{k+l} \left[ h_t \left( k+l, r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77} \right) \geq M \right] \leq \varepsilon.
\]
Combining this with the above equation ensures that for any such \( M > 0 \)
\[
Q_r^{k_0} \left[ \frac{1}{f_{\text{conv}}t Z_t^k} \geq M \right] \leq \varepsilon \sum_{i=0}^{2^{k-k_0}-1} \sum_{j=0}^{2^l-1} \frac{\mathbb{P}[A^{k+l}_{r + \frac{jT}{2^k+1} + \frac{jT}{2^k+77}}]}{\mathbb{P}[A_r^{k_0}]},
\]
By passing to the limit, we hence also get \( Q_r^{k_0} \left[ \frac{1}{Z_t^r} \geq M \right] \leq \varepsilon \), which in turn indeed implies (3.40) and finishes the proof.

\[\Box\]

**Remark 3.8** Compared to supermartingale deflators, one cannot expect that the convex combination of two generalized supermartingale deflators is again a generalized supermartingale deflator. Therefore, we could not directly construct \( Z_t^\infty \) from convex combinations of generalized supermartingale deflators on \( \mathcal{D}_k \) to obtain a generalized supermartingale deflator on \( \mathcal{D} \).
3.2.3 Proof of Theorem 2.25

In this subsubsection we provide the proof of Theorem 2.25 and hence all the corresponding assumptions are in force.

Proof The implication (ii) ⇒ (i) follows by the same argument as in the proofs presented before, hence it remains to show the implication (i) ⇒ (ii). To that end, let $\mathcal{D}$ consists of all the dyadic numbers on $[0, T]$, for each $k \in \mathbb{N}$ let $\mathcal{D}_k := \{iT/2^k : i := 0, 1, \ldots, 2^k \}$ be the collection of all $k$-th dyadic numbers in $[0, T]$, and let $\mathcal{X}^k := \{(X_t)_{t \in \mathcal{D}_k} : X \in \mathcal{X}\}$ be the restriction of the market to the $k$-dyadic grid. In the spirit of the last subsubsection, we define for any $r \in \mathcal{D}_k \setminus \{T\}$ here

$$Q^k_r[\cdot] := \mathbb{P}[\cdot | \tilde{\tau} \in (r, r + \frac{T}{2^k})].$$

Now observe first that thanks to the property of $\mathcal{E}$, the market $\mathcal{X}$ contains a numéraire under $Q^k_r$ until time $r$ and all elements in $\mathcal{X}$ vanish after time $r + \frac{T}{2^k}$. Hence, using exactly the same argument as in the proof of Step 1 in Theorem 2.21, we know that for each $k$ the process $(Z^k_t)_{t \in \mathcal{D}_k}$ defined by

$$Z^k_t = \sum_{r \in \mathcal{D}_k, t \leq r < T} \frac{1}{f_t^{(k,r)}} \mathbb{1}_{\{\tilde{\tau} \in (r, r + \frac{T}{2^k})\}} + \mathbb{1}_{\{\tilde{\tau} \leq t\}}, \quad t \in \mathcal{D}_k,$$

is a strictly positive generalized supermartingale deflator for $\mathcal{X}^k$ defined on $\mathcal{D}_k$, where by using Lemma 3.6 we pick $f_t^{(k,r)}$ to be an $\mathcal{F}_t$-measurable $Q^k_r$-version of the static deflator of $\text{cl}_{Q^k_r}(C_t)$. We claim that for each $t \in \mathcal{D}$ the set

$$\mathcal{K}_t := \text{conv}\left\{ \frac{1}{\mathbb{E}[Z^k_t | \mathcal{F}_t]} : k \in B_t \right\}$$

(recall that $B_t := \{k \in \mathbb{N} : t \in \mathcal{D}_k\}$) is $\mathbb{P}$-bounded.

Indeed, to see this, note that for each $k \in B_t$ the identity

$$\frac{1}{Z^k_t} = \sum_{r \in \mathcal{D}_k, t \leq r < T} f_t^{(k,r)} \mathbb{1}_{\{\tilde{\tau} \in (r, r + \frac{T}{2^k})\}} + \mathbb{1}_{\{\tilde{\tau} \leq t\}},$$

Jensen’s inequality, and the $\mathcal{F}_t$-measurability of every $f_t^{(k,r)}$ imply that

$$\frac{1}{\mathbb{E}[Z^k_t | \mathcal{F}_t]} \leq \mathbb{E}\left[\frac{1}{Z^k_t} \bigg| \mathcal{F}_t\right] \leq \sum_{r \in \mathcal{D}_k, t \leq r < T} f_t^{(k,r)} \mathbb{E}\left[\mathbb{1}_{\{\tilde{\tau} \in (r, r + \frac{T}{2^k})\}} \bigg| \mathcal{F}_t\right] + 1.$$

In addition, as by assumption the event $\{\tilde{\tau} \in (r, r + \frac{T}{2^k})\}$ is independent of $\mathcal{F}_r$, we have for all $r \in \mathcal{D}_k \setminus \{T\}$ with $r \geq t$ that

$$\mathbb{E}\left[\mathbb{1}_{\{\tilde{\tau} \in (r, r + \frac{T}{2^k})\}} \bigg| \mathcal{F}_t\right] = \mathbb{P}\left[\tilde{\tau} \in (r, r + \frac{T}{2^k})\right],$$

which implies that

$$\frac{1}{\mathbb{E}[Z^k_t | \mathcal{F}_t]} \leq \sum_{r \in \mathcal{D}_k, t \leq r < T} f_t^{(k,r)} \mathbb{P}\left[\tilde{\tau} \in (r, r + \frac{T}{2^k})\right] + 1. \quad (3.41)$$
Moreover, note that from Lemma 3.6 we know that \( \tilde{f}(k, r, t) \) can be taken from the \( \mathbb{P} \)-closure of the \( \mathbb{P} \)-solid hull of \( C_t \). Furthermore, since \( C_t \) is convex, also its \( \mathbb{P} \)-solid hull and the \( \mathbb{P} \)-closure of its \( \mathbb{P} \)-solid hull are convex. This together with the estimate \( \sum_{r \in D_k, t \leq r < T} \mathbb{P}[\tilde{\tau} \in (r, r + \frac{T}{2^n})] \leq \mathbb{P}[\tilde{\tau} > t] \leq 1 \) shows that for each \( k \in B_t \) the term

\[
\sum_{r \in D_k, t \leq r < T} \tilde{f}(k, r, t) \mathbb{P}[\tilde{\tau} \in (r, r + \frac{T}{2^n})]
\]

belongs to the \( \mathbb{P} \)-closure of the \( \mathbb{P} \)-solid hull of \( C_t \). Moreover, recall that the \( \mathbb{P} \)-closure of the \( \mathbb{P} \)-solid hull of \( C_t \) is \( \mathbb{P} \)-bounded thanks to the \( \mathbb{P} \)-boundedness \( C_t \), see Lemma 3.1, as by assumption (i) NUPBR holds for each \( t \). Therefore, we can conclude from (3.41) that indeed, the set

\[
K_t := \text{conv} \left\{ \frac{1}{\mathbb{E}[Z_k^t | F_t]} : k \in B_t \right\}
\]

is \( \mathbb{P} \)-bounded.

Next for each \( t \in D \) we apply Komlos lemma for the sequence \( \mathbb{E}[Z_k^t | F_t], k \in B_t \), to obtain a nonnegative limit \( Z_t \). Note that by the same argument as at the beginning of Step 2.2 in the proof of Theorem 2.21, we see that the sequence \( \mathbb{E}[Z_k^t | F_t], k \in B_t \), is \( \mathbb{P} \)-bounded, and hence \( Z_t < \infty \mathbb{P} \)-a.s.. Moreover, the above derived \( \mathbb{P} \)-boundedness of \( K_t \) ensures that \( \frac{1}{Z_t} \) is also finite \( \mathbb{P} \)-a.s., which means that \( Z_t \) is strictly positive. Since for every \( k \), we have that \( \mathbb{E}[Z_k^t | F_t], t \in D_k \), is a strictly positive supermartingale deflator, so is any forward convex combination. This in turn ensures that \( (Z_t)_{t \in D} \) is a strictly positive supermartingale deflator for \( \mathcal{X}^D \) under \( \mathbb{P} \) on all dyadic numbers. Then, the same argument as in Step 4.2 in the proof of Theorem 2.21 allows us to extend \( (Z_t) \) from \( D \) to \([0, T]\) such that it becomes a strictly positive, càdlàg supermartingale deflator on \([0, T]\). \( \square \)

4 Appendix

Let use provide the proof of the statement in Remark 2.5 and Lemma 2.6 that in the presence of a numéraire, the property for a market to satisfy NUPBR does not depend on the choice of the definition of the fork-convexity (between the one of Žitković [13] and ours).

**Proof of Lemma 2.6** First, if the fork convex hull of the market \( \mathcal{X} \) satisfies NUPBR, then so does \( \mathcal{X} \) being a subset of its hull. On the other hand, suppose that \( \mathcal{X} \) satisfies the NUPBR condition. We need to show that the fork-convex hull taken with respect to our notion also satisfies NUPBR. Note that by classical arguments (see, e.g., the beginning of the proof of Theorem 2.14) it suffices to prove the existence a strictly positive (generalized) supermartingale deflator for \( \mathcal{X}^D \) to guarantee that it satisfies NUPBR.

To that end, observe that since by assumption \( \mathcal{X} \) is \( \mathbb{F} \)-adapted and is fork convex in the sense of Žitković [13] which is also used in Kardaras [10], we can apply his result [10, Theorem 2.3], to guarantee the existence of a strictly positive supermartingale deflator \( Z \) for \( \mathcal{X} \). We claim that \( Z \) is also a strictly positive supermartingale deflator with respect to the fork-convex hull of \( \mathcal{X} \). To see this, let \( X^1, X^2, X^3 \in \mathcal{X} \) and let \( X \) be defined as in (2.1). Then
we have for $t \geq s$ and $A \in \mathcal{F}_s$ that
\[
\mathbb{E}_P\left[\frac{X_t Z_t}{X_s Z_s} \mid \mathcal{F}_s\right] = \mathbb{E}_P\left[\frac{\mathbb{1}_A \left(\frac{X_t^2}{X_s^2} X_t^1 \mathbb{1}_{\{X_s^2>0\}} + X_t^1 \mathbb{1}_{\{X_s^2=0\}}\right)}{X_s^1 Z_s} \mid \mathcal{F}_s\right] \\
+ \mathbb{E}_P\left[\frac{\mathbb{1}_{A^c} \left(\frac{X_t^2}{X_s^2} X_t^1 \mathbb{1}_{\{X_s^2>0\}} + X_t^1 \mathbb{1}_{\{X_s^2=0\}}\right)}{X_s^1 Z_s} \mid \mathcal{F}_s\right].
\]

For the first term on the right-hand-side of the above equation, we can use the facts that $X^i Z$ are generalized supermartingales for $i = 1, 2$, and that $\{X_s^2 \geq 0\}, \{X_s^2 = 0\}$ are $\mathcal{F}_s$-measurable to obtain that
\[
\mathbb{E}_P\left[\frac{\mathbb{1}_A \left(\frac{X_t^2}{X_s^2} X_t^1 \mathbb{1}_{\{X_s^2>0\}} + X_t^1 \mathbb{1}_{\{X_s^2=0\}}\right)}{X_s^1 Z_s} \mid \mathcal{F}_s\right] = \mathbb{E}_P\left[\frac{X_t^2 Z_t}{X_s^2 Z_s} \mid \mathcal{F}_s\right] \mathbb{1}_A \mathbb{1}_{\{X_s^2>0\}} + \mathbb{E}_P\left[\frac{X_t^1 Z_t}{X_s^1 Z_s} \mid \mathcal{F}_s\right] \mathbb{1}_A \mathbb{1}_{\{X_s^2=0\}} \\
\leq \mathbb{1}_A \mathbb{1}_{\{X_s^2>0\}} + \mathbb{1}_A \mathbb{1}_{\{X_s^2=0\}} = \mathbb{1}_A.
\]

Similarly, we can show that the second term on the right-hand-side of the above equation is bounded by $\mathbb{1}_{A^c}$, and hence we get
\[
\mathbb{E}_P\left[\frac{X_t Z_t}{X_s Z_s} \mid \mathcal{F}_s\right] \leq 1.
\]

This in turn, ensures that $Z$ is indeed a strictly positive supermartingale deflator for the fork convex hull of $\mathcal{X}$, which is sufficient to guarantee that the fork convex hull of $\mathcal{X}$ satisfies NUPBR. For more details we refer readers to the proof of [13, Proposition 3].

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