Complex Periodic Mixed-Mode Oscillation Patterns in a Filippov System

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Abstract: The main task of this article is to study the patterns of mixed-mode oscillations and non-smooth behaviors in a Filippov system with external excitation. Different types of periodic spiral crossing mixed-mode oscillation patterns, i.e., “cusp-F”/“fold-F” oscillation, “cusp-F”/“two-fold/two-fold/fold-F” oscillation and “two-fold/fold-F” oscillation, are explored. Based on the analysis of the equilibrium and tangential singularities of the fast subsystem, spiral crossing oscillation around the tangential singularities is investigated. Meanwhile, by combining the fast and slow analysis methods, we can observe that the cusp, two-fold and fold-cusp singularities play an important role in generating all kinds of complex mixed-mode oscillations.

Keywords: mixed-mode oscillations; tangential singularity; spiral crossing oscillations; external excitation

MSC: 34C15; 34C05; 37G10; 37G18

1. Introduction

As a typical non-smooth dynamic system, the Filippov system reflected in the mathematical model can be expressed as discontinuous differential equations whose right-hand side is discontinuous [1]. The main motivation for studying the Filippov system comes from the fact that non-smooth factors in many practical engineering systems can be described by this kind of model, such as mechanical systems with friction [2], switched electronic systems [3], discontinuous control systems [4] and others [5,6]. Generally, a Filippov system always has a switching surface which connects two types of flows. When the trajectory touches the switching surface, the system is redefined, which can cause the qualitative changes in the system’s dynamics, such as boundary equilibrium bifurcations, multiple collision and non-smooth periodic orbit bifurcations [7–10]. Especially, the system may exhibit various types of special phenomena on the switching surface, such as sliding motion and fold, cusp, two-fold, fold-cusp tangential singularities [11,12].

On the other hand, many important practical engineering problems also involve coupling of different time scales [13–16]. This type of system may cause mixed-mode oscillations, which are formed by a relatively large excursion and nearly harmonic small amplitude oscillation during every evolution period. For example, Abdelouahab et al. [17] studied the existence of mixed-mode oscillations and canard oscillations in the neighborhoods of Hopf-like bifurcation points based on the global and local canard explosion search algorithm. Liu et al. [18] found that the folded surface and critical manifold both play an important role in the existence of mixed-mode oscillations at the folded saddle in the perturbed system. Ma et al. [19] explored the evolution mechanism of different mixed-mode oscillation patterns caused by the pitchfork bifurcation and related delay behaviors in a van der Pol–Duffing system with parameter excitation. Yu et al. [20] studied the singular Hopf bifurcation conditions and MMO behaviors in the parametric perturbed BVP system and investigated the mechanisms of two different types of MMOs using a generalized fast-slow analysis method. Chen et al. [21] explored multiple fast-slow motions, including
“periodic bursts with quasi-periodic spiking”, “torus/short transient” mixed mode oscillations, “pitchfork/long transient” periodic mixed mode oscillations, amplitude-modulated and irregular oscillations from numerical method. It can be seen that the behaviors of the mixed-mode oscillation can be produced by many factors, such as different kinds of bifurcation structures [22,23], delay behaviors [24,25], and hysteresis loops [26,27]. However, most of the results about mixed-mode oscillations are made for smooth systems. When the non-smooth vector field contains multiple time scale couplings, mixed-mode oscillations may also observed. For example, Simpson et al. [28] investigated a piecewise-smooth linear FitzHugh–Nagumo system and showed that the piecewise-smooth linear model may exhibit MMO more easily than the classical FitzHugh–Nagumo model which contains a cubic polynomial as the only nonlinear term. Wang et al. [29] found that the delayed C-bifurcation leads to different types of transitions between multiple attractors, and explained the mechanism of mixed-mode oscillations in a typical Chua’s system with external excitation and a piecewise resistor. Even so, up to now, the influences of a non-smooth vector field on the vibration of mixed modes are rarely studied.

This paper investigates the mixed-mode oscillations and non-smooth dynamical behaviors in a piecewise nonlinear system with external excitation, focusing on the effects of the tangential singularities on the mixed-mode oscillations. For this purpose, we continue to analyze a realistic model in the literature [30], focusing on the effect of the external excitation. The basic circuit model with the external excitation is presented in Figure 1, where $E > 0$ stands for the voltage source and $R$ represents a resistive load. The voltage $v_{out}$ across $R$ is the system output.

![Figure 1. The basic circuit with a periodic excitation.](image)

The structure of the paper is set up as follows. The differential equation model of the circuit is established and the stabilities of the equilibrium and tangential singularities conditions of the fast subsystem are given in Section 2. Then, three new mixed-mode oscillation patterns, i.e., “cusp-F$^-$/fold-F$^-$” oscillation, “cusp-F$^-$/two-fold/two-fold/fold-F$^-$” oscillation and “two-fold/fold-F$^-$” oscillation are reported and the associated evolution mechanism are presented in Section 3. In Section 4, we present a brief conclusion of the paper.

2. Hybrid Model

2.1. Mathematical Model

Considering the inductor current $i_L$ and the voltage $v_C$ as state variables, the circuit described in Figure 1 can be written as

$$L \frac{di_L}{d\tau} = (E - i_L r_L)(q_1 + q_2) - i_r D S q_1 - (V_D 0 + \frac{i_L v_C + v_C}{r_C + R}) q_2 + i_G \cos(\Omega \tau),$$

$$C \frac{dv_C}{d\tau} = \frac{R}{r_C + R} i_L q_2 - \frac{1}{r_C + R} v_C,$$

(1)
where \( q_1 \) and \( q_2 \) are either 0 or 1 and not simultaneously equal to 0 or 1, \( i_G > 0 \) is the excitation amplitude and \( \Omega = o(\epsilon) \) is the excitation frequency. The voltage output \( v_{out} \) is given by \( v_{C} \), i.e.,

\[
v_{out} = \frac{r_C R}{r_C + R} i_L q_2 + \frac{R}{r_C + R} i_C C.
\]

(2)

Considering that \( r_L \neq 0 \) and \( r_{DS} = 0, V_{D0} = 0, r_C = 0, \) thus \( v_{out} = v_C \) and the above model can be simplified as follows

\[
L \frac{di_L}{d\tau} = (E - i_L r_L) (q_1 + q_2) - v_C q_2 + i_G \cos(\frac{\omega}{\sqrt{LC}} t),
\]

\[
C \frac{dv_C}{d\tau} = i_L q_2 - \frac{v_C}{R}.
\]

(3)

By using the dimensionless transformation

\[
x = \frac{i_l}{E} \sqrt{\frac{L}{C}}, \quad y = \frac{v_C}{E}, \quad t = \frac{\tau}{\sqrt{LC}}, \quad \alpha = \frac{1}{R} \sqrt{\frac{L}{C}}, \quad \beta = r_L \sqrt{\frac{C}{L}}, \quad \omega = \sqrt{LC \Omega}, \quad A = i_G,
\]

the model (3) can be expressed in the form

\[
\dot{x} = (1 - \beta x)(q_1 + q_2) - yq_2 + A \cos(\omega t),
\]

\[
\dot{y} = xq_2 - ay,
\]

(4)

where the new parameters are \( \alpha > 0, \beta > 0, w > 0 \). According to Ponce and Pagano [30], a new differential equation \( \dot{z} = w(x - z) \) was introduced, and the sliding control scheme can be defined as

\[
q_2 = 1 - q_1 = \begin{cases} 0, & \text{if } h(x) > 0, \\ 1, & \text{if } h(x) < 0, \end{cases}
\]

(5)

where \( x = (x, y, z), h(x) = y - y_r + k(x - z) \), and \( y_r > 1 \) is the normalized voltage and \( k > 0 \). It can be seen that the system with the above sliding mode scheme is obtained by connecting two vector fields

\[
F^+(x) = \begin{bmatrix} 1 - \beta x - y + \delta \\ x - ay \\ w(x - z) \end{bmatrix}, F^-(x) = \begin{bmatrix} 1 - \beta x + \delta \\ -ay \\ w(x - z) \end{bmatrix},
\]

(6)

according to the sign of \( h(x) \), where \( \delta = A \cos(\omega t) \). \( h \) defines the discontinuity manifold \( \Sigma = \{ x \in \mathbb{R}^3 : h(x) = 0 \} \), and dividing the whole state space into two regions: one is \( S^+ = \{ x \in \mathbb{R}^3 : h(x) > 0 \} \), and the other is \( S^- = \{ x \in \mathbb{R}^3 : h(x) < 0 \} \).

Since the excitation frequency \( \omega = O(\epsilon) \) is much less than the natural frequency, the extraneous excitation term evolves very slowly with the change of time, which indicates that the whole system has two time scales. Thus, the system can be thought of as the coupling of two subsystems, one is a slow subsystem, which is a piecewise-smooth dynamical system (6), and the other is a fast subsystem written as \( \delta = A \cos(\omega t) \). Furthermore, the main characteristics of the whole system is determined by the fast subsystem, while the slow subsystem plays a moderating role in the behaviors of the whole system. Therefore, we first study the stability and bifurcation dynamics of the fast subsystem, by considering \( \delta \) as a modulation parameter.

The equilibrium points for the vector fields \( F^+ \) and \( F^- \) are

\[
E^+ = (\frac{a(\delta + 1)}{\beta \alpha + 1}, \frac{\delta + 1}{\beta \alpha + 1}, \frac{a(\delta + 1)}{\beta \alpha + 1}), E^- = (\frac{1 + \delta}{\beta}, 0, \frac{1 + \delta}{\beta}),
\]
respectively. $E^+$ is a stable node (for $\beta > \alpha + 2$ or $\beta < \alpha - 2$) or stable focus (for $\alpha - 2 < \beta < \alpha + 2$) since the associated eigenvalues have a negative real part, namely

$$\left( -\frac{1}{2}(\alpha + \beta) - \frac{1}{2}\sqrt{(\alpha - \beta)^2 - 4}, -\frac{1}{2}(\alpha + \beta) + \frac{1}{2}\sqrt{(\alpha - \beta)^2 - 4}, w \right).$$

Because $h(E^+) = \frac{1+\delta}{\alpha\beta + 1} - y_r$, $E^+$ is an admissible stable equilibrium if $\frac{1+\delta}{\alpha\beta + 1} > y_r$, while if $\frac{1+\delta}{\alpha\beta + 1} = y_r$, $E^+$ is a boundary equilibrium, else $E^+$ is a virtual stable equilibrium.

$E^-$ is always a admissible stable node since the associated eigenvalues are real and negative, namely $(-\beta, -\alpha, -w)$, and $h(E^-) = - y_r < 0$.

2.2. Tangential Singularities

Since the fast subsystem in Equation (6) is a piecewise-smooth dynamical system, the type of contact between the smooth vector fields $F^\pm$ and the switching surface $\Sigma$ can be explained by the Lie derivatives $\mathcal{L}_{F^\pm}h(x) = \langle \nabla h, F^\pm \rangle$ where $\langle \cdot, \cdot \rangle$ and $\nabla h$ denote the canonical inner product and the gradient of switching boundary function $h$, respectively. The m-order Lie derivatives are defined as $\mathcal{L}_{F^\pm}^m h = \langle \nabla \mathcal{L}_{F^\pm}^{m-1} h, F \rangle, m=2,3,\cdots$.

The point $x \in \Sigma$ is called tangential singularity (i.e., the orbit from $x$ is tangent to $\Sigma$) if $\mathcal{L}_{F^+} h(x) \cdot \mathcal{L}_{F^-} h(x) = 0$. A point $x \in \Sigma$ is called double tangency point (i.e., the trajectory of the smooth vector fields $F^\pm$ from x is both tangent to $\Sigma$) if $\mathcal{L}_{F^+} h(x) = \mathcal{L}_{F^-} h(x) = 0$. The tangential sets corresponding to $F^\pm$ are given by the space lines:

$$T^+_\delta = \{ x : h(x) = 0, \mathcal{L}_{F^+} h(x) = 0 \}$$

$$= \{ (x,y,z) : x = -\frac{(-w+k\alpha + \beta)}{1+k\beta}, y = \frac{(w-k\alpha+\beta)}{1+k\beta}, z = \frac{(-k^2+k\alpha+\beta+1-kw)}{k(-1+k\beta)} \}$$

$$T^-_\delta = \{ x : h(x) = 0, \mathcal{L}_{F^-} h(x) = 0 \}$$

$$= \{ x : x = \frac{(w-\alpha) y}{k\beta} + \frac{-w - k + k\delta}{k\beta}, y = y, z = \frac{(-\alpha + \beta + w)}{k\beta} \} + \frac{k + k\delta - wr - \beta r}{k\beta}. \}$$

It is well known that tangential singularities are important for the understanding of dynamical behaviors at a switching boundary and they form the boundaries dividing the switching surface $\Sigma$ into a crossing region and a sliding/escaping region:

Crossing regions are defined by $\Sigma^{c+} = \{ x \in \Sigma : \mathcal{L}_{F^+} h(x) > 0, \mathcal{L}_{F^-} h(x) > 0 \}$ and $\Sigma^{c-} = \{ x \in \Sigma : \mathcal{L}_{F^+} h(x) < 0, \mathcal{L}_{F^-} h(x) < 0 \}$;

The sliding region is defined by $\Sigma^s = \{ x \in \Sigma : \mathcal{L}_{F^+} h(x) < 0, \mathcal{L}_{F^-} h(x) > 0 \}$;

The escaping region is defined by $\Sigma^e = \{ x \in \Sigma : \mathcal{L}_{F^+} h(x) > 0, \mathcal{L}_{F^-} h(x) < 0 \}$.

In the 3-dimensional dynamical system, two important types of generic tangential singularities that are encountered on smooth portions of $\Sigma$ are as follows:

A point $x$ is a fold point about the smooth vector field $F^+$ if $x \in \Sigma, \mathcal{L}_{F^+} h(x) = 0$, while $\mathcal{L}_{F^-} h(x) \neq 0$, and the gradient vectors of $h(x)$ and $\mathcal{L}_{F^-} h(x)$ are linearly independent. Moreover, $x$ is a cusp point with respect to the vector field $F^+$ if $x \in \Sigma, \mathcal{L}_{F^+} h(x) = \mathcal{L}_{F^-} h(x) = 0$, while $\mathcal{L}_{F^-} h(x) \neq 0$, and the gradient vectors of $h(x)$, $\mathcal{L}_{F^+} h(x)$ and $\mathcal{L}_{F^-} h(x)$ are linearly independent [31].

With the same method, fold and cusp point related to the smooth vector field $F^-$ can also be defined. Moreover, it is possible for a point $x \in \Sigma$ to be a double tangency point.
When $x \in \Sigma$ is a fold point, the cusp point with respect to the vector fields $F^\pm$, $x$ is called a two fold, two-cusp singularity, respectively. If $x \in \Sigma$ is a fold singularity with respect to one vector field and a cusp singularity with respect to the other one, we call $x$ as a fold-cusp singularity [32].

The next result summarizes the conditions of the tangential singularities which will be covered in this paper according to the parameter $\beta, w, \delta$ and fixing $y_e = 2.0, k = 1.0, \alpha = 1.0$.

The double tangency point $x_D$ is given by $T_0^+ \cap T_0^-$, i.e.,

$$x_D = \left\{ \begin{array}{l} -\delta + 2w - 1, -\delta + 2w - 1, 1 - (\delta + w + \beta) \\ 0, \beta - 1 + \delta, 0 \end{array} \right\}$$

(7)

The point $x_D$ is a two-cusp singularity if $\beta = -w + 1$ and $\delta = 3 - 2w$ or a fold-cusp singularity if $\beta \neq -w + 1$ and $\delta = \frac{-2w^2 - 3w - 2\beta^2 w + \beta w - \beta + 1}{\beta w + 1 - \beta - w}$ (fold for $F^+$, cusp for $F^-$) or $\delta = 1 + 2\beta$ (cusp for $F^+$, fold for $F^-$). In the other case, the point $x_D$ is a two-fold singularity.

A straightforward calculation shows that the point $x_{CF^+} = (x_{CF^+}, y_{CF^+}, z_{CF^+})$ with

$$x_{CF^+} = \frac{3 - 4w - \beta + \beta w\delta - \beta \delta - \beta w - 2w\delta + 2w^2 + 3\delta}{3 - \beta^2 w - 3w - 2\beta w + \beta^2 w + w^2},$$

$$y_{CF^+} = \frac{3 + 2\beta^2 w - 5w - \beta + 2w\beta - \beta \delta - 2w - \beta w - \delta + 3\delta}{3 - \beta^2 w - 3w - 2\beta w + \beta^2 w + w^2},$$

$$z_{CF^+} = \frac{2\beta^2 - 2w\delta - \beta w - 6\beta + \beta w - 3w - 3w + 2w^2 + 6\delta}{3 - \beta^2 w - 3w - 2\beta w + \beta^2 w + w^2},$$

(8)

and the point $x_{CF^-} = (x_{CF^-}, y_{CF^-}, z_{CF^-})$ with

$$x_{CF^-} = \frac{2w - 1 + \beta \delta + \beta - \delta}{(\beta - 1)\beta},$$

$$y_{CF^-} = \frac{2\beta w}{\beta w + 1 - \beta - w},$$

$$z_{CF^-} = \frac{3 \beta w + 1 - 3\beta - 3w + 2\beta^2 w + \beta w - 6\beta - \beta \delta - \beta w + \delta + 2w^2}{\beta (\beta w + 1 - \beta - w)},$$

(9)

are cusp points related to $F^+$ and $F^-$, respectively, due to the fact that $\mathcal{L}^3_{F^+}, h(x_{CF^+}) = 0$, $\mathcal{L}^3_{F^-}, h(x_{CF^-}) = 0$, $\mathcal{L}^3_{F^+}, h(x_{CF^-}) = -2\beta w \neq 0$.

3. Mixed-Mode Oscillation and Its Mechanism

Based on the results of the analysis of the equilibria and the tangential singularities of Equation (6), we find that mixed-mode oscillations are obtained when the whole system undergoes a transformation between the fast system and slow system connected by the different types of tangential points on the switched surface.

In the following discussion, the parameters $\alpha = 1.0, w = 0.01, \beta = 0.32, y_e = 2.0, k = 1.0$ are always fixed and the excited frequency is fixed at $\omega = 0.01$. We study the evolution of the mixed-mode oscillation dynamics and the associated mechanism of the non-smooth behaviors at the switching boundary when the amplitude $A$ is changed.

Note that $E^+$ is an admissible stable focus if $\delta > 1.64$, a boundary stable focus if $\delta = 1.64$ or a virtual stable focus if $\delta < 1.64$, while $E^-$ is always a admissible stable node for the fixed parameters. A typical trajectory of the fast system is shown in Figure 2 for $\delta = 0.54$. The trajectory locally wraps around the singularity until the trajectory in the open region $S^-$ meets with or is near to the tangential sets $T_{0.54}$ and then moves to the admissible stable node $E^-$. 
Figure 2. $E^+$ is a virtual stable focus; $E^-$ is an admissible stable node. A typical orbit of the fast subsystem is shown: trajectories locally wrap around the tangential sets for $\delta = 0.54$.

3.1. Cusp-$F^-$/Fold-$F^-$ Periodic Spiral Crossing Oscillation

As shown in Figure 3a, a periodic mixed-mode oscillation can be obtained when the amplitude is fixed at $A = 0.92$. It is seen that the periodic mixed-mode oscillation can be divided into two parts (seen in Figure 3b), i.e., the spiral crossing oscillation and the periodic oscillation which are connected by the tangential singularities.

Figure 3. A typical mixed-mode oscillation pattern for $A = 0.92$. (a) Phase portrait; (b) time series of the mixed-mode oscillation.

To explain the mechanism of this mixed-mode oscillation, Figure 4a shows the overlap of the phase portraits of different amplitudes, while Figure 4b shows the transformed phase portrait and tangential singularities with the variation of the parameter $\delta$. The green line in Figure 4b refers to the two-fold singularities, the black line denotes the cusp singularities with respect to the vector field $F^-$, while the black point corresponds to the fold-cusp singularity.

As is shown in Figure 4a, the limit cycle (the gray orbit) with $A = 0.91$ is completely in the open region $S^-$ and does not meet the switching boundary $\Sigma$ when it oscillates around the admissible stable node $E^-$ in counter-clockwise direction. When $A = 0.92$, the limit cycle (the blue orbit) contact with the switching surface $\Sigma$ at $P = (4.7357, 0.0079, 2.7357)$ with $\delta = 0.53925$. Based on Equation (9), the point $P$ is the cusp singularity with respect to $F^-$ (seen in Figure 4b), which means that the trajectory in the open region $S^-$ is tangent to the switching boundary $\Sigma$ at $P$ and then crosses through the switching surface $\Sigma$ to the open region $S^+$ governed by the vector field $F^+$. Since $E^+$ is the virtual stable focus under the parameter conditions, the trajectory inevitably contacts with the switching surface $\Sigma$ at
the point \( P_1 \) in \( \Sigma_c^- \) when it scrolls down to the stable focus \( E^+ \), then returns to the open region \( S^- \) governed by the the admissible stable node \( E^- \).

In the process of moving from the the point \( P_1 \) to the stable \( E^- \), the trajectory may contact with the switching boundary again at \( P_2 \) in the crossing region \( \Sigma_c^- \), which may cause the trajectory to scroll down to the virtual stable focus. In this way, the trajectory may spiral around the switching surface \( \Sigma \) from the cusp singularity \( P \) until the trajectory in the open region \( S^- \) meets with or is near to the tangential sets \( T_{\delta} \), causing the trajectory to return back to the point \( P \) along the limit cycle in open region \( S^- \). We can refer to such mixed-oscillation formation as the cusp-F\(^-\)/fold-F\(^-\) periodic spiral crossing oscillation.

Figure 4. (a) Stable limit cycles with \( A = 0.91 \) (the gray orbit) and \( A = 0.92 \) (the blue orbit); (b) overlap of the tangential singularities branches and transformed phase portrait on the \((\delta,x,z)\).

3.2. Cusp-F\(^-\)/Two-Fold/Two-Fold/Fold F\(^-\) Periodic Spiral Crossing Oscillation

When \( A \) increases from 0.92, the mixed-mode oscillation obtained in Equation (6) may exhibit some interesting behaviors. For example, Figure 5 shows a group of mixed-mode oscillation patterns in Equation (6) with increasing values of \( A \) for fixed \( A = 1.35 \) and \( A = 1.7 \). It can be seen that the spiral crossing oscillation in the mixed-mode oscillation (see Figure 5a) may be divided into two parts, i.e., the left and right spiral crossing oscillation parts (see Figure 5b) with the increase of the parameter \( A \). The corresponding phase portraits are shown in Figure 5c,d. We may find that the left trajectories may spiral around the boundary \( \Sigma_s^- \), while the right trajectories may spiral around the boundary \( \Sigma_e^- \).

The phenomenon can be also understood by the analysis of the contact between the orbit and the tangential singularities. As shown in Figure 6, when the parameter \( A \) increases from 0.92 to 1.437, the trajectories of the cusp-F\(^-\)/fold-F\(^-\) periodic spiral crossing oscillation may get to the two-fold singularity at \( \delta = 1.437 \), which imply that spiral crossing trajectories may split into two parts, i.e., the left trajectories spiraled around the boundary \( \Sigma_s^- \), and the right trajectories spiraled around the boundary \( \Sigma_e^- \), connected by two two-fold points (see in Figure 6b). We can refer to such mixed oscillation formation as the cusp-F\(^-\)/two-fold/two-fold/fold F\(^-\) periodic spiral crossing oscillation.
Figure 5. Tangential singularities-induced mixed-mode oscillation patterns in Equation (6), where $A\cos(\omega t)$ (the dashed line) is overlayed to give a clear view that the frequency of the periodic mixed-mode oscillation is equal to $\omega$. (a,c) $A = 1.35$; (b,d) $A = 1.7$.

Figure 6. Overlap of the tangential singularities branches and transformed phase portrait on the $(\delta, x, z)$. (a) $A = 1.35$; (b) $A = 1.7$.

3.3. Two-Fold/Fold-F$^-$ Periodic Spiral Crossing Oscillation

By a further increase of parameter $A$, the left spiral crossing oscillation around the boundary $\Sigma^s$ may gradually disappear, and only the right spiral crossing oscillation is left, which still wraps around the boundary $\Sigma^r$, as shown in Figure 7a for $A = 2.80$. The corresponding phase portrait is presented in Figure 7b.

The mechanism analysis is obtained by the overlap of the phase diagram on the space of $(\delta, x, z)$ with the tangential singularities with the change of the parameter $\delta$, as presented in Figure 7c. The mechanism can be explained simply. When $A$ increases through 2.25, the point $P$ where the limit cycle of the vector $F^-$ intersects the boundary $\Sigma$ may pass through the fold-cusp singularity (fold respect to $F^+$ and cusp respect to $F^-$, shown in Figure 7c) along the cusp $F^-$ singularity at $\delta = -0.8737$, causing the trajectory to pass through the boundary $\Sigma$ and experience a sharp turn down to the two-fold point. The trajectories of the cusp-$F^-$/two-fold/two-fold/fold $F^-$ periodic spiral crossing oscillation may become unstable at $A = 2.25$, causing the left spiral crossing oscillation part to disappear (shown in Figure 5d) and evolve to a new mixed-mode oscillation pattern seen in Figure 7b. We can
refer to such mixed oscillation formation as the two-fold/fold-$F^-$ periodic spiral crossing oscillation.

![Figure 7. A typical mixed-mode oscillation pattern for $A = 2.80$. (a) Time history; (b) phase portrait; (c) transformed phase portrait.](image)

4. Conclusions

This article studies mixed-mode oscillation dynamics in a Filippov system with external excitation. When the amplitude of the excitation is changed, three new mixed-mode oscillation patterns, i.e., “cusp-$F^-$/fold-$F^-$” oscillation, “cusp-$F^-$/two-fold/two-fold/fold-$F^-$” oscillation and “two-fold/fold-$F^-$” oscillation are first reported. By regarding the excitation term as a bifurcation parameter, the stabilities of the (admissible and boundary) equilibrium and the conditions of different types of tangential points, such as cusp, two-fold and fold-cusp singularity, are explored. With the decrease of the excitation amplitude, when the excitation term passes through the two-fold point, the periodic spiral crossing oscillation may become unstable and a periodic oscillation with two (left and right) spiral crossing trajectories is created. When the excitation term passes through the fold-cusp point, the left spiral crossing trajectory of the periodic oscillation may suddenly disappear and only the right spiral crossing trajectory is left. Besides, the results proposed in this paper are advantageous to understand the mixed-mode oscillation in non-smooth dynamical systems. Our further work will focus on the effect of the different behaviors of mixed-mode oscillation caused by various switched scheme and the potential applications on Filippov system.

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