Ph.D. thesis

On certain complex surface singularities

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Introduction

Preliminaries and main results

This thesis contains the results of two joint papers with András Némethi \[53, 54\] and the background related to them. Némethi’s program aims to create bridges between different areas of the singularity theory, e.g. to compare the topological and analytic/algebraic invariants of complex singularities. In our case the bridge is created between immersion theory (differential topology) and two rather distinct areas of local complex singularity theory (‘Milnor fibration package’ and ‘analytic stability package’.)

The main objects of our study are holomorphic germs \(\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)\). These are triples of locally convergent complex power series in two variables. We are mostly interested in two classes of them. The first class consists of the holomorphic germs singular only at the origin, i.e. \(\Phi|_{\mathbb{C}^2 \setminus \{0\}}\) is an immersion. The second class is a subset of the first one, it contains the so-called finitely \(A\)-determined (or \(A\)-finite) germs. By Mather–Gaffney criterion \[71\] these are the germs whose restrictions \(\Phi|_{\mathbb{C}^2 \setminus \{0\}}\) are stable immersions.

If \(\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)\) is singular only at the origin, one can associate with \(\Phi\) an immersion from the 3-sphere \(S^3\) to the 5-sphere \(S^5\) at the level of links. This immersion is the restriction of \(\Phi\) to a suitably chosen 3-sphere around the origin of \(\mathbb{C}^2\). If additionally \(\Phi\) is finitely \(A\)-determined, the associated immersion is stable, i.e. it has only regular simple and double values, with transverse intersection of the branches at each double value.

In our case the associated immersion plays a similar role as the link in the case of isolated singularities. As it is a \(C^\infty\) map, it allows more flexible deformations than the holomorphic germ \(\Phi\). Up to regular homotopy (which means deformation through immersions) the immersions \(f : S^3 \looparrowright S^5\) are completely classified by the integer valued invariant \(\Omega(f)\), the so-called Smale invariant of the immersion \(f\). Stephen Smale published his invariant in 1959 \[64\], and already two years later David Mumford in his seminal article \[50\] asked for the analytic/algebraic characterization of the Smale invariant of the immersion associated with a holomorphic germ \(\Phi\). The main result of \[53\] provides a complete answer to the question of Mumford, see Chapter 3. We identify the Smale invariant of the associated immersion with an analytic invariant of \(\Phi\), namely, with the number of
the cross caps of a stabilization of $\Phi$. This result implies various consequences both in singularity theory and immersion theory.

The image of a finitely $\mathcal{A}$-determined germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can also be defined as the zero set of certain germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, and provides a non-isolated hypersurface singularity $(X, 0) := (f^{-1}(0), 0) = (\text{im}(\Phi), 0) \subset (\mathbb{C}^3, 0)$. Indeed, $df$ vanishes along the set of the double values of $\Phi$. A very restricted class of non-isolated singularities occurs in this way. The normalization of $(X, 0)$ is smooth, in fact, its normalization map is $\Phi$, and all the transverse curves corresponding to the components of the singular locus of $(X, 0)$ have type $A_1$.

The Milnor fibre of $(X, 0) = f^{-1}(0) \subset (\mathbb{C}^3, 0)$ is defined as the $\delta$-level set $f^{-1}(\delta)$ of $f$ intersected with a ball $B_\epsilon^6$ with sufficiently small radius $\epsilon$, $0 < \delta \ll \epsilon$. It plays a central role in the study of local singularities. In the case of isolated singularities, the boundary of its Milnor fibre is diffeomorphic to the link of $(X, 0)$. András Némethi and Ágnes Szilárd in [55] present a general algorithm, which provides the boundary of the Milnor fibre for any non-isolated hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$, although it is rather technical and in concrete examples is rather computational. In [54] we present an independent algorithm providing the Milnor fibre boundary for our restricted class of non-isolated singularities, see Chapter 5. This algorithm uses directly the geometry of $(X, 0)$. Namely, it produces the Milnor fibre boundary as a surgery of $S^3$ along the double point locus of the immersion associated with $\Phi$. Technically the algorithm provides a plumbing graph of the Milnor fibre boundary by modifying a good embedded resolution graph of the double point locus of $\Phi$.

Summary and organization of the thesis

Immersions associated with holomorphic germs. The main purpose of Chapter 1 is to introduce Mond invariants $C(\Phi)$ and $T(\Phi)$ of holomorphic germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. $C(\Phi)$ is the number of cross cap points, $T(\Phi)$ is the number of triple values of a stabilization of $\Phi$, but each can be calculated as the codimension of a suitable ideal of the local ring as well, without stabilizing $\Phi$. Chapter 1 also serves as a collection of relations of these invariants with other concepts, which will appear in the subsequent chapters.

First we introduce the notion of germs and the finiteness property of them. We follow [20, 48]. The associated smooth map $S^{2n-1} \to S^{2p-1}$ is defined for finite germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, it slightly generalizes the notion of the associated immersion introduced in [53].

The theory of stability and finite determinacy of germs were devised by e.g. H. Whitney, J. N. Mather and C. T. C. Wall [71, 35, 36, 37, 38]. We introduce both concepts with respect to $\mathcal{A}$-equivalence (left-right equivalence). The discussion of the stability also
serves as a background for the singular Seifert surfaces, which are defined in Chapter 2 as stable smooth maps. We review theorems of Mather and Gaffney characterizing the stability and finite $A\!$-determinacy in terms of the $A$ and $A_e$ codimensions of a germ. The only stable multigerms of a map $\mathbb{C}^2 \to \mathbb{C}^3$ are regular simple points, regular double values with transverse intersection of the branches, regular triple values with regular intersection of the branches and simple Whitney umbrella (cross cap) points. The triple values and the Whitney umbrellas are isolated points.

The discussion of the Fitting ideals has two purposes. The defining equation $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ of the image of a finite germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be determined using Fitting ideals. Moreover, by this process one can calculate the equations of the multiple point spaces of $\Phi$ in the target as well. In particular, the number of the triple values of a stabilization can be determined with the help of Fitting ideals.

We introduce $C(\Phi)$ as the codimension of the ideal in the local ring $\mathcal{O}_{(\mathbb{C}^2, 0)}$ generated by the determinants of the $2 \times 2$ minors of the Jacobian matrix of $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ [43, 45]. Similarly, $T(\Phi)$ is the codimension of the second Fitting ideal associated with $\Phi$ in $\mathcal{O}_{(\mathbb{C}^3, 0)}$. If $\Phi$ is finitely $A\!$-determined, then both $C(\Phi)$ and $T(\Phi)$ are finite, and any stabilization of $\Phi$ has $C(\Phi)$ cross caps and $T(\Phi)$ triple values. The finiteness of $C(\Phi)$ is equivalent with the fact that $\Phi$ is singular only at the origin, which also means that the associated map $S^3 \to S^5$ is an immersion. Finite $A\!$-determinacy is equivalent with the fact that the associated map $S^3 \to S^5$ is a stable immersion. Mond [45] introduced a third invariant $N(\Phi)$ for corank–1 germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, such that the finite $A\!$-determinacy of $\Phi$ is equivalent with the finiteness of the three invariant $C(\Phi)$, $T(\Phi)$ and $N(\Phi)$. These invariants appear in the formulas expressing the image Milnor number (which is the second Betti number of the image of a stabilization of $\Phi$), and in the formulas comparing the Milnor numbers of the four double point spaces as well [47, 30, 34]. The structure of the double point spaces plays an important role in Chapter 5. The end of Chapter 1 is a short review about a reinterpretation of Mond’s invariants provided by W. L. Marar and J. J. Nuño-Ballesteros [32].

Chapter 2 provides an introduction to the Hirsch-Smale theory [16, 64], which transforms regular homotopy problems (differential topology) to homotopy theory (algebraic topology). The main purpose is to review the Hughes–Melvin definition of the integer valued Smale invariant of immersions $S^3 \leftrightarrow \mathbb{R}^5$ [17], and the Ekholm–Szűcs formulas expressing the Smale invariant in terms of the properties of singular Seifert surfaces [9].

Smale’s theorem is the generalization of the Whitney–Graustein theorem about plane curve immersions $S^1 \leftrightarrow \mathbb{R}^2$, and it is a special case of the Hirsch theorem, in fact, a special case of the $h$-principle of Gromov. We review Hirsch theorem and the construction of the
Smale invariant $\Omega(f)$ of an immersion $f : S^n \hookrightarrow \mathbb{R}^q$. $\Omega(f)$ is an element of an Abelian group depending on the dimensions $n$ and $q$, and two immersions $S^n \hookrightarrow \mathbb{R}^q$ are regular homotopic if and only if their Smale invariants are equal. The possibility of the ‘sphere eversion’ is a consequence of Smale’s theorem as well.

J. F. Hughes and P. M. Melvin [17] proved that there are embeddings $S^3 \hookrightarrow \mathbb{R}^5$ which are not regular homotopic to each other. They also express the Smale invariant of an embedding $S^3 \hookrightarrow \mathbb{R}^5$ with the signature of a Seifert surface, which is a 4-manifold in $\mathbb{R}^5$ whose boundary is the image of the embedding. T. Ekholm and A. Szűcs [9] generalized this result to arbitrary immersions $S^3 \hookrightarrow \mathbb{R}^5$ using singular Seifert surfaces. One contribution of their formulas is the invariant $L(f)$ of stable immersions $f : S^3 \hookrightarrow \mathbb{R}^5$ introduced by Ekholm in [7]. It has several slightly different definitions [7, 8, 9, 60]. We review these definitions and the proof of their equivalence.

The end of Chapter 2 is an outline of various related results, we mention here two of them. S. Kinjo defined immersions $S^4 \hookrightarrow \mathbb{R}^4$ associated with plumbing graphs of type $A$ and $D$. The point is that their Smale invariants agree with the Smale invariants of the immersions associated with the coverings $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ of the singularities of type $A$ and $D$, see below in the introduction. The $\mathbb{Z}_2$ valued cobordism invariant called ‘total twist’ of immersions $M^3 \hookrightarrow \mathbb{R}^5$ [18] is used later, in Chapter 5, to conclude that $C(\Phi)$ and the number of the non-trivially covered double point curve components of a finitely determined germ $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ have the same parity.

Chapter 3 includes our results published in [53]. The main theorem answers the question of Mumford mentioned above. Namely, $\Omega(\Phi|_{S^3}) = -C(\Phi)$ holds for holomorphic germs $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ singular only at the origin, where $\Omega(\Phi|_{S^3})$ is the Smale invariant of the immersion $\Phi|_{S^3} : S^3 \hookrightarrow S^5$ associated with $\Phi$. We refer to this result as the ‘main formula’. There are several consequences of this result.

Using the main formula we can provide explicit (‘algebraic’) representatives of each regular homotopy class of immersions $S^3 \hookrightarrow S^5$. This answers a question of Smale [64]. In contrast with the known $C^\infty$ constructions, these realizations are very simple polynomial maps, and the computation of the Smale invariant via $C(\Phi)$ is extremely simple.

The non-standard embeddings $S^3 \hookrightarrow \mathbb{R}^5$ of Hughes and Melvin [17] cannot be realized as the associated immersion of some $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$. Although this fact follows from a deep result of Mumford [50], using our main formula we provide a new proof for it. Moreover, the ‘topological vanishing’ $\Omega(\Phi|_{S^3}) = 0$ implies that $\Phi$ is the regular germ via our formula.

We prove the formula $\Omega(\Phi|_{S^3}) = -C(\Phi)$ in two steps. We introduce the newly defined ‘complex Smale invariant’ $\Omega_C(\Phi)$ of $\Phi$. It turns out that $\Omega_C(\Phi) = C(\Phi)$ and $\Omega_C(\Phi) =$
We use $C(\Phi)$ as the number of cross caps of a stabilization of $\Phi$, and we do not use its algebraic definition as the codimension of the ramification ideal. However, as a by-product of the calculation we provide a new proof for the theorem of Mond declaring the equivalence of the two definitions of $C$ in the case of corank–1 germs.

Note that the integer valued Smale invariant is well defined only up to sign. To determine the correct sign of the main formula, we fix generators of the corresponding infinite cyclic groups. The Hughes–Melvin and Ekholm–Szűcs formulas carry the sign ambiguity as well. We determine their correct sign with the help of the main formula and calculations of concrete examples. As a by-product of this procedure, we express the contributions of the Ekholm–Szűcs formulas in terms of $C(\Phi)$ and $T(\Phi)$ for a special singular Seifert surface, which is created from a holomorphic stabilization of $\Phi$ by stabilizing the complex Whitney umbrella points in $C^\infty$ sense. In particular, $L(\Phi|_{S^3}) = C(\Phi) - 3T(\Phi)$ holds for finitely determined germs $\Phi : (C^2, 0) \to (C^3, 0)$ and their associated stable immersions $\Phi|_{S^3} : S^3 \hookrightarrow S^5$.

It follows from our results that the analytic invariants $C(\Phi)$ and $T(\Phi)$ are $C^\infty$ invariants as well, moreover, $C(\Phi) - 3T(\Phi)$ is a topological invariant of $\Phi$.

**Boundary of the Milnor fibre.** Chapter 4 contains the definitions and some properties of the Milnor fibre and the resolution of surface singularities $(X, 0) = (f^{-1}(0), 0) \subset (C^3, 0)$. The plumbing construction and the embedded resolution of plane curve singularities are also summarised in Chapter 4.

The Milnor fibre of an isolated hypersurface surface singularity is well studied and it is rather well understood. It has the homotopy type of a bouquet of 2–spheres, it is an oriented smooth 4–manifold whose boundary is diffeomorphic with the link of the singular germ and also with the boundary of any resolution of the germ [42, 51, 52]. This boundary is a plumbed 3–manifold and one can take as a plumbing graph any of the resolution graphs. It is the basic bridge between the Milnor fibre and the resolution (both of them being complex analytic fillings of it), and this connection produces several nice formulas connecting the invariants of these fillings. Here primarily we think about formulas of Laufer [27] or Durfee [5] and their generalizations, see e.g. [70].

For non–isolated hypersurface singularities in $(C^3, 0)$ the situation is more complicated. First of all, the link of the germ is not smooth, hence the boundary of the Milnor fibre cannot be isomorphic with it. Moreover, a (any) resolution is in fact the resolution of the normalization (which might contain considerable less information than what one needs in order to recover the Milnor fibre $F$, or the Milnor fibre boundary $\partial F$), see e.g. [62, 55]. For example (see our case), it can happen that the normalization is smooth, while $\partial F$ is rather
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complicated. However, the boundary of the Milnor fibre is still a plumbed 3–manifold, and one expects that its plumbing graph codifies considerable information about the germ. \( \partial F \) can be obtained by surgery of two pieces: one of them is the boundary of the resolution of the normalization, the other one is related to the transverse singularities associated with the singular curves of the hypersurface singularity \([62, 55, 39]\). In particular, the boundary of the Milnor fibre plays the same crucial role as in the isolated singularity case (in fact, it is the unique object in this case, which might fulfil this role): it is the first step in the description of the Milnor fibre, and it is the bridge in the direction of the resolution and the transverse types of the components of the singular locus.

[55] presents a general algorithm, which provides the boundary of the Milnor fibre \( \partial F \) for any non–isolated hypersurface singularity in \((f^{-1}(0), 0) \subset (\mathbb{C}^3, 0)\). However, this algorithm uses (some information from) the embedded resolution of this pair, hence it is rather technical and in concrete examples is rather computational. Therefore, for particular families of singularities it is preferable to find more direct description of the plumbing graph of \( \partial F \) directly from the peculiar intrinsic geometry of the germ. For several examples in the literature see e.g. [55] (homogeneous singularities, cylinders of plane curves, \( f = zf'(x, y), f = f'(x^ay^b, z) \)), [63] (\( f = g(x, y) + zh(x, y) \)); or for other classes consult also [40] and [1].

Chapter 5 contains the results of [54]. It provides an explicit construction producing the plumbing graph for the boundary of the Milnor fiber of a non-isolated hypersurface singularity \((X, 0)\) given by the image of a finitely determined complex analytic map germ \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \). One of the main ingredients is the link \( L = \{L_i\}_i \) in \( S^3 \) of the reduced double point curve \((D, 0) = \Phi^{-1}(\Sigma, 0) \subset (\mathbb{C}^2, 0)\) with irreducible components \( \{D_i\}_i \), where \((\Sigma, 0)\) is the reduced singular locus. It is equipped with a pairing of the components \( L_i \leftrightarrow L_{\sigma(i)} \) induced by the pairing of the double points. Then, the Milnor fiber boundary is constructed as a surgery of \( S^3 \) along \( L \) such that whenever \( i \neq \sigma(i) \) the tubular neighbourhoods of the paired components have to be glued together, while in the case \( i = \sigma(i) \) a special 3-manifold \( Y := S^1 \times S^1 \times I / \sim \) with torus boundary is glued along \( L_i \). The description of the gluing maps uses the newly defined invariants ‘vertical indices’ associated with the irreducible components of \( \Sigma \). Their relation with \( C(\Phi) \) is clarified whenever \( \Phi \) is a corank–1 germ and \( T(\Phi) \) vanishes. As a result, the explicit plumbing graph of the Milnor fiber boundary is constructed from a good embedded resolution graph of \( D \). The algorithm is also illustrated on several examples.
The main examples

**Finitely determined germs.** D. Mond’s list of simple germs [45, Table 1] contains finitely $\mathcal{A}$-determined holomorphic germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ with $\text{rk}(d\Phi_0) = 1$. These germs are organized in four families $S_{k-1}, B_k, C_k, H_k$, and there are two sporadic elements, the Whitney umbrella (cross cap) and $F_4$.

The associated immersions of $S_{k-1}$ provide representatives of all regular homotopy classes with negative Smale invariant, see Section 3.5.

The Whitney umbrella is stable as a holomorphic germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, see Example 1.2.16, but not in the $C^\infty$ sense, as a real germ $(\mathbb{R}^4, 0) \to (\mathbb{R}^6, 0)$. We present a $C^\infty$ stabilization of the complex Whitney umbrella in Section 3.7 and we calculate directly the contributions of the Ekholm–Szűcs formula.

In Section 5.5 we present a plumbing graph of the Milnor fibre boundary of the image of $\Phi$ for all members of Mond’s list.

The germs $\Phi$ of type $H_k$ are the only germs in the list with $T(\Phi) \neq 0$. In Section 1.3 we calculate $T(\Phi)$ and the equation of $(\text{im}(\Phi), 0) \subset (\mathbb{C}^3, 0)$ using Fitting ideals.

$\Phi(s, t) = (s^2, t^2, s^3 + t^3 + st)$ is a corank–2 germ from [31], that is, $\text{rk}(d\Phi_0) = 0$. We present the Milnor fibre boundary of its image, see Section 5.5.

**Quotient singularities.** The simple singularities $(X, 0) \subset (\mathbb{C}^3, 0)$ are the quotient singularities of type $A-D-E$, that is, $(X, 0) \cong (\mathbb{C}^2, 0)/G$ for a certain finite subgroup $G \subset GL(2, \mathbb{C})$. The covering map of a quotient singularity is a germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose image is $(X, 0)$. The components of $\Phi$ are the generators of the $G$-invariant algebra $O_G(\mathbb{C}^2, 0)$.

These germs $\Phi$ are singular only at the origin, but they are not finitely $\mathcal{A}$-determined germs, i.e. $\Phi|_{\mathbb{C}^2 \setminus \{0\}}$ is a nonstable immersion. In fact, if $|G| > 2$, then every point of $(X, 0)$ is at least triple value of $\Phi$, and for $|G| = 2$ the transversality of the branches does not hold.

In Subsection 1.4.3 we present $C(\Phi)$ of these germs $\Phi$, hence the Smale invariant of the associated immersions $S^3 \to S^3/G \hookrightarrow S^5$ follows from the main formula, see Section 3.5.

In Section 3.7 we present a holomorphic stabilization of the covering germ $\Phi$ of the $A_1$ singularity, and we calculate $L(\Phi|_{S^3})$ directly. Note that its associated immersion $S^3 \to \mathbb{RP}^3 \hookrightarrow S^5$ is regular homotopic with the immersions of the same structure (that is, a composition of a covering with an embedding) studied in several articles [41, 11, 7, 24], see the discussion in Subsection 2.3.2.
Notations and terminology

- \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \): the set of integers, rational, real, complex numbers.
- \( A \cong B \) means that the (oriented) smooth manifold \( A \) and \( B \) are (oriented) diffeomorphic.
- \( A \cong B \) means that the algebraic structures or bundles \( A \) and \( B \) are isomorphic, or denotes the analytic equivalence of the analytic spaces/germs \( A \) and \( B \).
- \( A \langle a, b \rangle \) means that the algebraic structure (module, vector space) \( A \) is generated by \( a \) and \( b \). \( \langle a, b \mid c, d \rangle \) denotes the group presented by generators \( a \) and \( b \) and relations \( c \) and \( d \).
- \( (a, b) \) denotes the ideal in the ring/algebra \( R \) generated by the elements \( a, b \in R \) (or denotes the pair of arbitrary elements \( a \) and \( b \) as well).
- \( \dim A \) denotes the dimension of the vector space or manifold \( A \) over \( \mathbb{R} \). Dimension over another field \( \mathbb{F} \) is denoted by \( \dim_{\mathbb{F}} \). The rank of a matrix/linear map \( M \) is denoted by \( \text{rk}(M) \).
- \( S^n \), resp. \( B^{n+1} \) denotes the (oriented) diffeomorphism type of the unit sphere, resp. the unit ball in \( \mathbb{R}^{n+1} \). The 2-disc is denoted by \( D^2 \). When it is important, we distinguish the embedded and the abstract spheres and balls in the notation.
- \([X, Y]\) denotes the set of homotopy classes of continuous maps between the topological spaces \( X \) and \( Y \).
- \( X \vee Y \) denotes the bouquet (wedge, one-point union) of the (pointed) topological spaces \( X \) and \( Y \).
- \( H_n(X, G) \), resp. \( H^n(X, G) \) denotes the homology groups, resp. cohomology groups of the topological space \( X \) with coefficient group \( G \).
- \( \pi_n(X) \) is the \( n \)-th homotopy group of the topological space \( X \).
- \( dg \ (dg_x) \) denotes the differential/tangent map/Jacobian matrix of the function/map/germ \( g \) (at the point \( x \)).
- We call a smooth (or analytic) germ \( g : (X, x) \rightarrow (Y, y) \) regular (resp. singular), if the rank of \( dg_x \) is equal (resp. is less than) the minimum of the dimensions of \( X \) and \( Y \).
- \( GL(n, \mathbb{F}) \) denotes the set of \( n \times n \) matrices over \( \mathbb{F} \) with determinant \( \neq 0 \). In the real case \( GL^+(n, \mathbb{R}) \) is the set of \( n \times n \) matrices with determinant \( > 0 \).
- \( O(n) \) (\( SO(n) \)), resp. \( U(n) \) (\( SU(n) \)) denotes the set of real orthogonal matrices (with determinant \( +1 \)), resp. complex unitary matrices (with determinant \( +1 \)).
- \( V_m(\mathbb{F}^q) \) (where \( \mathbb{F} \) denotes \( \mathbb{R} \) or \( \mathbb{C} \) is the Stiefel manifold that consists of linearly independent \( m \)-frames of \( \mathbb{F}^q \).
1.1. Finite complex germs and their restrictions at the level of links

1.1.1. Finite germs. The basic objects of the thesis are certain types of holomorphic map germs \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \). The concept of the germ is an important tool in the study of the local behaviour of the maps.

Two subsets \( X_1 \) and \( X_2 \) of the topological space \( X \) have the same germ at \( x_0 \in X_1 \setminus X_2 \), if there is a neighbourhood \( U \) of \( x_0 \) such that \( U \setminus X_1 = U \setminus X_2 \). The equivalence classes of this equivalence relation are called the germs of spaces at the point \( x_0 \) [20, Definition 3.4.1.]. The germs along a subset \( S_0 \) can be defined similarly.

Let \( (X, x_0) \) and \( (Y, y_0) \) be two germs of topological spaces. A germ of a continuous map \( f : (X, x_0) \to (Y, y_0) \) is defined as an equivalence class of maps \( f : U \to W \), with \( f(x_0) = y_0 \), where \( U \) and \( W \) are representatives of \( (X, x_0) \) and \( (Y, y_0) \) respectively. Two maps \( f_1 : U_1 \to W \) and \( f_2 : U_2 \to W \) are equivalent (they define the same germ) if they agree on an open neighbourhood \( V \subset U_1 \setminus U_2 \) of \( x_0 \) [20, Definition 3.4.6.]. Changing \( x_0 \) to a finite subset \( S_0 \subset X \) we get the notion of multi-germ. [48, 71]

We also can define smooth (resp. analytic, holomorphic) germs of maps, as germs of smooth (resp. analytic, holomorphic) maps. We study holomorphic germs \( (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) throughout this chapter, and germs of complex analytic spaces are studied in Chapter 4.

The notion of germ is particularly interesting in the complex analytic category, because of uniqueness of analytic continuation: if \( U_1 \) and \( U_2 \) are open sets in \( \mathbb{C}^n \) with \( U_1 \cap U_2 \) connected, and \( f_i : U_i \to \mathbb{C}^p \) are complex analytic maps which coincide on some open \( V \subset U_1 \cap U_2 \), they coincide on all of \( U_1 \cap U_2 \) [48].

The holomorphic germs \( (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) can be identified with the \( p \)-tuples of locally convergent power series in \( n \) variables with constant terms 0, i.e. with the elements of \( \mathfrak{m}_{(\mathbb{C}^n, 0)} \cdot \mathcal{O}^p_{(\mathbb{C}^n, 0)} \), where \( \mathfrak{m}_{(\mathbb{C}^n, 0)} \) is the unique maximal ideal in the local ring \( \mathcal{O}_{(\mathbb{C}^n, 0)} \).

A holomorphic germ \( \Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) is called finite, if \( \Phi^{-1}(0) \) is a finite set, that is, \( \Phi^{-1}(0) = \{0\} \) for a small enough representative of \( \Phi \) [20, Definition 3.4.7., Theorem 3.4.24].
Finiteness can be characterized by the local algebras. The germ \( \Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) induces an algebra homomorphism \( \Phi^* : \mathcal{O}_{(\mathbb{C}^p, 0)} \to \mathcal{O}_{(\mathbb{C}^n, 0)} \). In this way \( \mathcal{O}_{(\mathbb{C}^n, 0)} \) becomes a module over \( \mathcal{O}_{(\mathbb{C}^p, 0)} \), sometimes it is denoted by \( \Phi_* \mathcal{O}_{(\mathbb{C}^n, 0)} \) as well.

**Theorem 1.1.1** ([20] Theorem 3.4.24]). The following facts are equivalent:

\begin{enumerate}[(a)]
  \item \( \Phi \) is finite.
  \item \( \mathcal{O}_{(\mathbb{C}^n, 0)} \) is a finitely generated \( \mathcal{O}_{(\mathbb{C}^p, 0)} \)-module.
  \item \( \mathcal{O}_{(\mathbb{C}^n, 0)}/(\Phi^* \mathfrak{m}_{(\mathbb{C}^p, 0)}) \) is a finite dimensional \( \mathbb{C} \)-vector space.
\end{enumerate}

Note that \( (\Phi^* \mathfrak{m}_{(\mathbb{C}^p, 0)}) = \mathfrak{m}_{(\mathbb{C}^p, 0)} \cdot \mathcal{O}_{(\mathbb{C}^n, 0)} \) denotes the ideal generated by the coordinate functions of \( \Phi \).

A finite germ \( \Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) is generically 1 to 1 to its image, if there is a hypersurface \( (Y, 0) = f^{-1}(0) \subset (\mathbb{C}^p, 0) \) (where \( f : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0) \)) such that \( \Phi \) induces a bijection between \( \Phi^{-1}((\mathbb{C}^p, 0) \setminus (Y, 0)) \) and \( \mathrm{im}(\Phi) \setminus (Y, 0) \).

### 1.1.2. Restriction of \( \Phi \) at the level of links.

Here we present the discussion from [53] Section 2.1.] slightly generalized. This topic is closely related to the link of complex analytic spacegerms, which is discussed in Chapter 4.

If \( (X, 0) \) is a complex analytic germ with an isolated singularity \( 0 \in X \) then its link \( K \) can be defined as follows. Set a real analytic map \( \rho : X \to [0, \infty) \) such that \( \rho^{-1}(0) = \{0\} \). Then, for \( \varepsilon > 0 \) sufficiently small, \( K := \rho^{-1}(\varepsilon) \) is an oriented manifold, whose isotopy class (in \( X \setminus \{0\} \)) is independent of all the choices, cf. Lemma (2.2) and Proposition (2.5) of [29]. E.g., if \( (X, 0) \) is a subset of \( (\mathbb{C}^N, 0) \), then one can take the restriction of \( \rho(z) = |z|^2 \) (the norm of \( z \)). In this way, the link of \( (\mathbb{C}^N, 0) \) is the sphere \( S^{2n-1}_{\varepsilon} \). Nevertheless, the general definition is very convenient even if \( (X, 0) = (\mathbb{C}^N, 0) \).

Let \( \Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) be a finite holomorphic germ \( (n < p) \). Define \( \rho : (\mathbb{C}^n, 0) \to [0, \infty) \) by \( \rho(z) = |\Phi(z)|^2 \). Since \( \Phi^{-1}(0) = \{0\} \), \( \rho^{-1}(0) = \{0\} \) too.

**Lemma 1.1.2.** There exists an \( \varepsilon_0 > 0 \) sufficiently small such that \( \mathfrak{B}_\varepsilon := \Phi^{-1}(\{z : |z| \leq \varepsilon\}) \) is a non-metric \( \mathcal{C}^\infty \) closed ball around the origin of \( \mathbb{C}^n \) for any \( \varepsilon < \varepsilon_0 \). Its boundary, \( \Phi^{-1}(S^{2p-1}_\varepsilon) \) is canonically diffeomorphic to \( S^{2n-1} \). In fact, for \( \bar{\varepsilon} \) with \( 0 < \bar{\varepsilon} \ll \varepsilon \), any standard metric sphere \( S^{2n-1}_{\bar{\varepsilon}} \) sits in \( \mathfrak{B}_\varepsilon \), and it is isotopic with \( \Phi^{-1}(S^{2p-1}_\varepsilon) \) in \( \mathfrak{B}_\varepsilon \setminus \{0\} \).

In the sequel \( \Phi^{-1}(S^{2p-1}_\varepsilon) \) and \( S^{2n-1}_{\varepsilon} \subset \mathbb{C}^n \) will be identified. When it is important to differentiate them we will use the notation \( \mathcal{G} = \mathcal{G}^{2n-1} := \Phi^{-1}(S^{2p-1}_\varepsilon) \). We write also \( S^{2n-1} = S^{2n-1}_{\varepsilon} \) and \( S^{2p-1} = S^{2p-1}_{\varepsilon} \).

**Definition 1.1.3.** The restriction \( \Phi_{|\mathcal{G}} : \mathcal{G} \to S^{2p-1} \) is the (smooth) map associated with \( \Phi \) at the level of links.
1.2. Finite determinacy and stability

1.2.1. Equivalence of the germs. Several equivalence relations of the germs are studied in the literature. We refer mainly to Wall’s survey paper [71] and the book of Mond and Nuño-Ballesteros [48] (which is still unpublished). The concept of stability and finite determinacy makes sense after fixing a certain equivalence relation. Geometrically $A$-equivalence is the most reasonable, that corresponds to the coordinate changes in the source and in the target of a germ. We use only $A$-equivalence in this thesis. However, the results and classification theorems connected with $A$-equivalence require the study of some of the other equivalences as well.

Most of the equivalences come from certain group actions on the set of germs, the equivalence classes are the orbits of the action. The group $L$ (resp. $R$) consists of the germs of local automorphism of the target (resp. the source), and acts on the germs by composing from the left (resp. from the right). The corresponding equivalence relation is the left (resp. right) equivalence of the germs. The direct product of these groups is denoted by $A$, its left-right action induces $A$-equivalence. The type of the local automorphisms depends on the category of the germs: these are local homeomorphisms for continuous germs, local diffeomorphisms for smooth germs, local analytic equivalences, biholomorphisms for real analytic, holomorphic germs. There are two other groups which are often studied, $C$ and $K$, the last one is contact equivalence.

In this thesis we use only $A$-equivalence, but in several different categories. For the study of holomorphic germs we usually use complex analytic $A$-equivalence, but some of our results can be interpreted in other categories as well: as it will turn out, some analytic invariants of the germs are invariant under topological (or smooth) $A$-action as well, see Subsection 3.1.6. The flexibility to use the smooth category is provided by the study of the link, since $\Phi|_{S}$ is a smooth map between smooth manifolds (spheres).

The singular Seifert surfaces discussed in Subsection 2.2.3 are defined as stable smooth maps. Thus we need $A$-equivalence of global maps as well: that corresponds to the action of global homeomorphisms (diffeomorphism, biholomorphism) of the source and the target. Moreover, the stability with respect to this equivalence (global $A$-equivalence) can be reduced to the local stability of the germs of the map, [48, 37].

The following general concepts can be found in [71, 48], here we present the main theorems to provide a stable background to our study. Most of them hold for $L$, $R$, $C$, $K$-equivalences too, in smooth and analytic categories as well. But they are meaningless in the continuous category, because there is no Taylor-expansion and the codimension of the orbits cannot be defined.
Note that most of the notions and invariants introduced in this chapter are $\mathcal{A}$-invariants. That is, if $\Phi_1$ and $\Phi_2$ are $\mathcal{A}$-equivalent germs (as complex analytic germs), then $\Phi_1$ is $\mathcal{A}$-stable (resp. finitely $\mathcal{A}$-determined) if and only if $\Phi_2$ is $\mathcal{A}$-stable (resp. finitely $\mathcal{A}$-determined), $C(\Phi_1) = C(\Phi_2)$, $T(\Phi_1) = T(\Phi_2)$, and similarly the invariants $N$, $\mu_I$, $J$, $\mu(D^2/S_2)$ (and the Milnor number of other double point spaces) agree for $\Phi_1$ and $\Phi_2$.

1.2.2. Unfolding and stability. $\mathcal{A}$-stability means that any small perturbation of a germ is $\mathcal{A}$-equivalent with the germ itself. To make this precise, we need the notion of unfoldings.

**Definition 1.2.1.** (a) An $r$-parameter unfolding of $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a germ

$$\tilde{\Phi} : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}^p \times \mathbb{C}^r, 0),$$

$$\tilde{\Phi}(u, v) = (\tilde{\Phi}_v(u), v),$$

such that $\tilde{\Phi}_0 = \Phi$.

(b) Two unfoldings $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ of $\Phi$ are $\mathcal{A}$-equivalent unfoldings, if there are diffeomorphism germs $\phi : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C}^n \times \mathbb{C}^r, 0)$ and $\psi : (\mathbb{C}^p \times \mathbb{C}^r, 0) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$, which are unfoldings of the identity of $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$, and

$$\tilde{\Phi}_2 = \psi \circ \tilde{\Phi}_1 \circ \phi.$$

(c) An unfolding of $\Phi$ is trivial if it is equivalent with the constant unfolding, $\tilde{\Phi}(u, v) = (\Phi(u), v)$.

If the unfolding $\tilde{\Phi}$ is fixed, then we will use the notation $\Phi_v$ instead of $\tilde{\Phi}_v$ for the deformation of $\Phi$ corresponding to the parameter value $v$.

An unfolding is versal if it induces all the unfoldings up to $\mathcal{A}$-equivalence:

**Definition 1.2.2.** (a) The pull-back of an $r$-parameter unfolding $\tilde{\Phi}$ of $\Phi$ with a germ $h : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ is the $q$-parameter unfolding $h^*(\tilde{\Phi}) : (\mathbb{C}^n \times \mathbb{C}^q, 0) \to (\mathbb{C}^p \times \mathbb{C}^q, 0)$,

$$h^*(\tilde{\Phi})(u, v) = (\tilde{\Phi}_{h(v)}(u), v).$$

(b) An $r$-parameter unfolding $\tilde{\Phi}$ of $\Phi$ is called versal unfolding, if any unfolding (with arbitrary number of parameters) is equivalent with $h^*(\tilde{\Phi})$ for a certain germ $h$.

**Definition 1.2.3.** A germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is stable (more precisely, $\mathcal{A}$-stable) if any unfolding of $\Phi$ is trivial.

The definitions of the unfolding and trivial unfolding generalize to multi-germs, replacing the base point 0 in the source with a finite set $S$. Then the stability of a multi-germ can be defined in the same way. Moreover the stability of the multi-germs can be reduced
to the stability of its germs. For details we refer to [48]. Here we present only the simplest case, which is used many times in this thesis. First we need a definition.

**Definition 1.2.4.** The subspaces $E_i$ of the vector space $E$ (of finite dimension) meet in general position (or have regular intersection) if $\sum_i \text{codim} E_i = \text{codim} \bigcap_i E_i$.

Note that for two subspaces regular intersection means transversality.

**Proposition 1.2.5.** A multigerm of regular germs (i.e., each branch is the germ of an immersion) is stable if and only if the intersection of the branches is regular.

**Remark 1.2.6.** By Theorem 4 in [37] (see also in [48]) the stability of proper smooth global maps is equivalent with the stability of its multi-germs. This fact shows that we do not need to introduce the notion of ‘global stability’. We call a smooth map stable if all its multi-germs of are stable.

### 1.2.3. Finite determinacy and codimension.

**Definition 1.2.7.** A holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely $\mathcal{A}$-determined if there is an integer $k$ such that whenever the $k$-jet ($k$-th Taylor polynomial) of a germ equals with the $k$-jet of $\Phi$, then the germ is $\mathcal{A}$-equivalent with $\Phi$.

Next we define the $\mathcal{A}$-codimension of $\Phi$, this is the codimension of the $\mathcal{A}$-orbit of $\Phi$ in the space of the germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We define the $\mathcal{A}_c$-codimension as well [71], pg. 485[. Stability and finite determinacy can be characterized by these invariants.

Let $\theta_{(\mathbb{C}^n, 0)}$ denote the set of the germs of vector fields on $(\mathbb{C}^n, 0)$, these are the germs of holomorphic sections of the tangent bundle $T\mathbb{C}^n \to \mathbb{C}^n$. $\theta(\Phi)$ is the space of germ of vector fields along $\Phi$, these are the germs of holomorphic sections of the pull-back tangent bundle $\Phi^*T\mathbb{C}^p \to \mathbb{C}^n$. Let $\theta_0(\Phi) \subset \theta(\Phi)$ denote the vector fields along $\Phi$ which are 0 at the origin, clearly $\theta_0(\Phi) = m_{(\mathbb{C}^n, 0)} \cdot \theta(\Phi)$. Note that $\theta_{(\mathbb{C}^n, 0)}$, $\theta(\Phi)$ and $\theta_0(\Phi)$ are $\mathcal{O}_{(\mathbb{C}^n, 0)}$-modules, in particular they are $\mathbb{C}$-vector spaces.

Furthermore, the following identifications hold by introducing local coordinates.

$$\theta(\Phi) \cong \mathcal{O}^p_{(\mathbb{C}^n, 0)}, \quad \theta_{(\mathbb{C}^n, 0)} \cong \mathcal{O}^n_{(\mathbb{C}^n, 0)}.$$

Define the $\mathbb{C}$-linear maps

$$t\Phi : \theta_{(\mathbb{C}^n, 0)} \to \theta(\Phi), \quad t\Phi(w) = d\Phi \circ w,$$

$$\omega\Phi : \theta_{(\mathbb{C}^n, 0)} \to \theta(\Phi), \quad \omega\Phi(w) = w \circ \Phi.$$

In local coordinates $t\Phi$ is the multiplication by the Jacobian of $\Phi$, while $\omega\Phi$ is the substitution of the components of $\Phi$. 
The tangent space $T\mathcal{A}\Phi$ of the $\mathcal{A}$-orbit of $\Phi$ is defined as

$$T\mathcal{A}\Phi = t\Phi(m_{(\mathbb{C}^n,0)} \cdot \theta_{(\mathbb{C}^n,0)}) + \omega\Phi(m_{(\mathbb{C}^n,0)} \cdot \theta_{(\mathbb{C}^p,0)}),$$

it is a subspace of $\theta_0(\Phi)$ \[71\] pg. 485., \[48\].

**Definition 1.2.8.** The $\mathcal{A}$-codimension of $\Phi$ is $\dim\mathbb{C}(\theta_0(\Phi)/T\mathcal{A}\Phi)$.

The extended tangent space $T\mathcal{A}_e\Phi$ is defined as

$$T\mathcal{A}_e\Phi = t\Phi(\theta_{(\mathbb{C}^n,0)}) + \omega\Phi(\theta_{(\mathbb{C}^p,0)}),$$

it is a subspace of $\theta(\Phi)$ \[71\], \[48\].

**Definition 1.2.9.** The $\mathcal{A}_e$-codimension of $\Phi$ is $\dim\mathbb{C}(\theta(\Phi)/T\mathcal{A}_e\Phi)$.

To motivate these definitions note that $\theta(\Phi)$ is equal to the ‘set of one-parameter infinitesimal deformations of $\Phi$’, that is

$$\theta(\Phi) = \left\{ \frac{\partial}{\partial t} \Phi_t|_{t=0} \mid (\Phi_t, t) \text{ is a one-parameter unfolding of } \Phi_0 = \Phi \right\},$$

and $\theta_0(\Phi)$ contains the elements of $\theta(\Phi)$ correspond to the unfoldings with $\Phi_t(0) = 0$. $\theta_0(\Phi)$ can be considered as the tangent space of the space of the germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ at $\Phi$.

$T\mathcal{A}\Phi$ consists of the vector fields along $\Phi$ which correspond to trivial infinitesimal deformations and are 0 at the origin, ie.

$$T\mathcal{A}\Phi = \left\{ \frac{\partial}{\partial t} (\psi_t \circ \Phi \circ \phi_t^{-1})|_{t=0} \mid \phi_t(0) = 0 , \psi_t(0) = 0 \right\},$$

where $\psi_t$ and $\phi_t$ are one-parameter families of germs of biholomorphisms in the source and in the target with $\phi_0 = \text{id}$, $\psi_0 = \text{id}$. Without the extra conditions $\phi_t(0) = 0$, $\psi_t(0) = 0$ the formula provides $T\mathcal{A}_e\Phi$.

The extended codimension is a more natural concept in many cases, see e.g. Theorem 1.2.10 below. However $T\mathcal{A}_e\Phi$ does not correspond to any group action or equivalence of the germs, since it allows the origin to move.

Stability and finite determinacy can be determined by the following characterization theorems by Mather, see Theorems 1.2., 3.4. and Proposition 4.5.2. in \[71\], or \[35,36,48\].

**Theorem 1.2.10.** Let $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ a holomorphic germ. Then the following are equivalent:

(a) $\Phi$ is stable.
(b) The $\mathcal{A}_e$-codimension of $\Phi$ is 0.
Property (b) is called ‘infinitesimal stability’. By Theorem 1.2.10 this is equivalent with stability. In the sense of Remark 1.2.6 ‘global stability’ is also equivalent with them.

**Theorem 1.2.11.** Let $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic germ. Then the following are equivalent:

(a) $\Phi$ is finitely $\mathcal{A}$-determined.
(b) The $\mathcal{A}_e$-codimension of $\Phi$ is finite.
(c) The $\mathcal{A}$-codimension of $\Phi$ is finite.
(d) $\Phi$ admits a versal unfolding.

The minimal number of parameters of a versal unfolding is exactly the $\mathcal{A}_e$-codimension of $\Phi$. A versal unfolding with minimal number of parameters is called *miniversal* unfolding. This is unique up to $\mathcal{A}$-equivalence of unfoldings. All the other versal unfoldings can be obtained as a trivial unfolding of the miniversal unfolding up to $\mathcal{A}$-equivalence of unfoldings.

Finally, by Mather-Gaffney criterion [71, Theorem 2.1.] finite determinacy is equivalent with isolated instability, which means each multi-germ of $\Phi|_{\mathbb{C}^n \setminus \{0\}}$ is stable in a sufficiently small representative of $\Phi$.

**Theorem 1.2.12 (Mather-Gaffney criterion [30, 71, 48]).** A finite holomorphic germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ ($n < p$) is finitely determined if and only if $\Phi$ has isolated instability at $0$.

Note that there is a more general version of Mather-Gaffney criterion, which agrees with our version for finite germs mapping $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$. Cf. [30, Theorem 2.11.], [71, Theorem 2.1.] and [48].

**Remark 1.2.13.** The concept of stability and finite determinacy can be defined similarly for $\mathcal{L}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$ equivalences too, and they determine important classes of germs. For example $\mathcal{L}$-stable (resp. $\mathcal{R}$-stable) germs are the germs of the immersions (resp. submersions), and a germ is finitely $\mathcal{C}$-determined if and only if it is finite. The codimension and the extended codimension can also be defined for the other groups, and the characterisation theorems 1.2.10, 1.2.11, 1.2.12 hold in the same form, if we replace $\mathcal{A}$ with any of the other groups.

Furthermore, all the concepts and theorems 1.2.10, 1.2.11 hold in smooth and real analytic category too. However Theorem 1.2.12 holds only for holomorphic germs. For real analytic germs the following criterion is useful.

**Proposition 1.2.14 ([71, Proposition 1.7.]).** For real analytic germs the following are equivalent:
(a) The germ is finitely determined as a smooth germ.
(b) The germ is finitely determined as a real analytic germ.
(c) The complexification of the germ is finitely determined as a holomorphic germ.

1.2.4. Examples.

Example 1.2.15 \((\mathbb{C}^n \to \mathbb{C})\). A germ \(\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) is stable if and only if it is regular (submersion) or it is Morse type, i.e. equivalent with \((x_i)_i \mapsto \sum_{i=1}^{n} x_i^2\).

A germ \(\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) with isolated singularity at 0 (which means \(\Phi\) is submersion outside the origin) is finitely determined. See [20, Theorem 9.1.3.] or [48].

Example 1.2.16 (Whitney-umbrella). The \(\mathcal{A}\)-equivalence class of) the germ \(\Phi(s, t) = (s, t^2, st)\) is called Whitney umbrella (or cross cap, pinch point), see Figure 1.1. Another often used form is \(\Psi(s, t) = (s, t^2, t(s+t^2))\), see Example 5.1. in [32], which is summarized in paragraph 1.5.2 and \(S_0\) from Mond’s simple germ list in [45]. \(\Phi\) and \(\Psi\) are \(\mathcal{A}\)-equivalent, since \(\Psi = \psi \circ \Phi \circ \phi\) holds with the germs of diffeomorphisms \(\phi(s, t) = (s + t^2, t)\) and \(\psi(x, y, z) = (x - y, y, z)\).

The Whitney umbrella is stable as a holomorphic germ \((\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\), and also as real analytic or smooth germ \((\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\), cf. 1.2.12. Its stability is very important throughout the thesis, hence we present here a proof of this fact, by calculating the \(\mathcal{A}\)-codimension of \(\Phi\) (and then referring to Theorem 1.2.10).

For an arbitrary germ of a vector field

\[
\omega \Phi(w)(s, t) = \begin{pmatrix} w_1(s, t^2, st) \\ w_2(s, t^2, st) \\ w_3(s, t^2, st) \end{pmatrix} \in \theta(\Phi),
\]

one has

\[
d\Phi(s, t) = \begin{pmatrix} 1 & 0 \\ 0 & 2t \\ t & s \end{pmatrix},
\]

thus \(\omega \Phi(\theta_{(\mathbb{C}^3, 0)})\) consists of germs of vector fields whose components are power series of \(s, t^2\) and \(st\). It is clear that the only monomials which cannot appear in the components are the odd powers of \(t\). The Jacobian of \(\Phi\) is
thus for a germ of a vector field

\[ v(s, t) = \begin{pmatrix} v_1(s, t) \\ v_2(s, t) \end{pmatrix} \in \theta({\mathbb C}^2, 0), \]

\[ t\Phi(v)(s, t) = d\Phi \cdot v = \begin{pmatrix} v_1(s, t) \\ 2tv_2(s, t) \\ tv_1(s, t) + sv_2(s, t) \end{pmatrix} \in \theta(\Phi). \]

Therefore \( t\Phi(\theta({\mathbb C}^2, 0)) \) contains the vector fields whose coordinate functions are odd powers of \( t \). One can conclude that

\[ T\mathcal{A}_e \Phi = t\Phi(\theta({\mathbb C}^2, 0)) + \omega\Phi(\theta({\mathbb C}^3, 0)) = \begin{pmatrix} \mathcal{O}_{(\mathbb C^2, 0)} \\ \mathcal{O}_{(\mathbb C^2, 0)} \\ \mathcal{O}_{(\mathbb C^2, 0)} \end{pmatrix} = \theta(\Phi), \]

thus the \( \mathcal{A}_e \)-codimension of \( \Phi \) is 0, which means \( \Phi \) is stable.

**Example 1.2.17** \((\mathbb{C}^2 \to \mathbb{C}^3)\). (a) A germ \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) is stable if and only if it is

(1) an immersion (given by \( \Phi(s, t) = (s, t, 0) \) up to \( \mathcal{A} \)-equivalence), or

(2) a Whitney umbrella.

(b) The stable multi-germs of a map from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \) are the

(1) regular single values (with 1 preimage in which the germ of the map is an immersion),

(2) regular double values with transverse intersection,

(3) regular triple values with regular intersection,

(4) single Whitney umbrella points.

(c) A germ \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) is finitely \( \mathcal{A} \)-determined if and only if (a small enough representative of) \( \Phi|_{\mathbb{C}^2 \setminus \{0\}} \) is a stable immersion with only double values. That is, all the multi-germs of \( \Phi|_{\mathbb{C}^2 \setminus \{0\}} \) have type (1) or (2) from the list of the stable multi-germs in part (b).

Part (c) follows from Theorem 1.2.12 and from the fact, that isolated stable multi-germs ((3) and (4) from the list in (b)) can be omitted with the choice of a small enough representative of \( \Phi \).

**Example 1.2.18** \((\mathbb{R}^2 \to \mathbb{R}^3)\). The stable germs and multigerms of the real analytic or smooth maps from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) are formally the same as the complex ones. That is, part (a) and (b) of 1.2.17 hold word for word if we change \( \mathbb{C} \) to \( \mathbb{R} \). This follows from the fact that the calculation of the \( \mathcal{A}_e \)-codimension is a formal algebraic procedure, cf. Example 1.2.16.
1. DEFORMATION OF COMPLEX MAP GERMS

Figure 1.1. Stable multigerms of a maps $\mathbb{R}^2 \to \mathbb{R}^3$: simple values, double values, isolated triple value and Whitney umbrella.

However part (c) of Example 1.2.17 is not valid for real germs, only in one direction: a finitely $\mathcal{A}$-determined real analytic or smooth germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is stable immersion outside the origin. This follows directly from Proposition 1.2.14 and part (c) of Example 1.2.17. On the other hand, $\Phi$ is an immersion outside the origin if and only if the Jacobian ideal of $\Phi$ is concentrated to the origin. It can happen that the real zero set of the Jacobian ideal consists only the origin, but its complex zero set is bigger: such a germ is an immersion outside the origin as a real germ, but not as a complex germ, hence it is not finitely determined. Finitely determined real germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ are studied for example in [33].

Example 1.2.19 ($\mathbb{R}^4 \to \mathbb{R}^6$). A holomorphic map from $\mathbb{C}^2$ to $\mathbb{C}^3$ can be regarded as a smooth (real analytic) map from $\mathbb{R}^4$ to $\mathbb{R}^6$. However the stability of the complex map does not imply the stability of the real map, because the complex Whitney umbrella is not stable as a germ from $(\mathbb{R}^4, 0)$ to $(\mathbb{R}^6, 0)$. The list of the stable multi-germs of a map
from $\mathbb{R}^4$ to $\mathbb{R}^6$ consists of regular simple, double and triple values with regular intersection (cf. 1.2.17), and – instead of the isolated complex Whitney umbrella points – it contains the \textit{fold points} (whose normal form is $(s, t_1, t_2, u) \mapsto (s^2, st_1, st_2, t_1, t_2, u)$), which form a submanifold of dimension 1. In Section 3.7 we present a real stabilization of the complex Whitney umbrella.

1.3. Fitting ideals

1.3.1. Presentation matrix. The equation of the image and the multiple point spaces in the target can be determined by Fitting ideals. Here we present the general procedure and some examples. For more details and the properties of these ideals we refer to [48, 49].

Let $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finite holomorphic germ and $p = n + 1$. Then $\Phi$ has a well-defined image, $(\Phi(\mathbb{C}^n, 0)$ is an analytic set in $(\mathbb{C}^p, 0)$. The ideal of the functions vanishing on the image of $\Phi$ can be determined by Fitting ideal method.

By Theorem 1.1.1 $\mathcal{O}(\mathbb{C}^n, 0)$ is a finitely generated $\mathcal{O}(\mathbb{C}^p, 0)$-module, see also [48, 49], we denote it by $\Phi_* \mathcal{O}(\mathbb{C}^n, 0)$. A choice of a set of generators induces an epimorphism $\mathcal{O}_M(\mathbb{C}^p, 0) \to \Phi_* \mathcal{O}(\mathbb{C}^n, 0)$. Its kernel is also finitely generated, because $\mathcal{O}(\mathbb{C}^p, 0)$ is Noetherian. That can be summarised in the exact sequence

$$\mathcal{O}_N(\mathbb{C}^p, 0) \to \mathcal{O}_M(\mathbb{C}^p, 0) \to \Phi_* \mathcal{O}(\mathbb{C}^n, 0) \to 0.$$  

We call the homomorphism $\lambda : \mathcal{O}_N(\mathbb{C}^p, 0) \to \Phi_* \mathcal{O}_M(\mathbb{C}^p, 0)$ the presentation of $\mathcal{O}(\mathbb{C}^n, 0)$, its matrix is the presentation matrix of $\Phi$ (denoted also by $\lambda$). Each column of $\lambda$ expresses a relation between the generators, and they together generate all the relations.

$\lambda$ may be chosen injective, and this forces $N = M$ ($\lambda$ is a square matrix) [48].

**Definition 1.3.1.** The $i$-th Fitting ideal $\mathcal{F}_i(\Phi_* \mathcal{O}(\mathbb{C}^n, 0))$ of $\Phi$ is the ideal in $\mathcal{O}(\mathbb{C}^p, 0)$ generated by the determinants of the $(N - i) \times (N - i)$ minors of the presentation matrix $\lambda$.

It can be shown that the Fitting ideals do not depend on the choice of the presentation $\lambda$.

$\mathcal{F}_i(\Phi_* \mathcal{O}(\mathbb{C}^n, 0))$ provides an analytic structure of the $(i + 1)$-fold multiple value set of $\Phi$.

**Theorem 1.3.2.** The zero set of $\mathcal{F}_i(\Phi_* \mathcal{O}(\mathbb{C}^n, 0))$ consists of points in $(\mathbb{C}^p, 0)$ with at least $i + 1$ preimages counting with multiplicity.

Especially, for $i = 0$ the Fitting ideal provides an analytic structure of the image of $\Phi$. 

Theorem 1.3.3. The zero set of \( \mathcal{F}_0(\Phi, \mathcal{O}_{\mathbb{C}^n, 0}) \) is the image of \( \Phi \). In other words, \( \Phi(\mathbb{C}^n, 0) = \det(\lambda)^{-1}(0) \).

Although \( \det(\lambda) \) is not necessarily a reduced equation, its geometrical meaning is very natural, as we will see in Example 1.3.5.

A presentation can be found by the following procedure. Let \( (x_i)_{i=1}^n \) denote the coordinates of the point \( \Phi \) in the source, and let \( (y_i)_{i=1}^{n+1} \) denote the coordinates in the target. \( \Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_{n+1}(x)) \), that is, \( \Phi_j = \Phi^*(y_j) \in \mathcal{O}_{\mathbb{C}^n, 0} \).

1. Define \( \bar{\Phi} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) as \( \bar{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)) \). Suppose that \( \bar{\Phi} \) is finite. This always can be obtained by a linear change of the coordinates.
2. Find generators \( g_1, \ldots, g_N \) of \( \bar{\Phi} \mathcal{O}_{\mathbb{C}^n, 0} \), i.e. \( (g_i)_{i=1}^N \) generates \( \mathcal{O}_{\mathbb{C}^n, 0} \) as a module over \( \mathcal{O}_{\mathbb{C}^n, 0} \) via \( \bar{\Phi} \). By Nakayama lemma that happens if and only if the classes of \( g_i \) form a basis of the \( \mathbb{C} \)-vector space \( \mathcal{O}_{\mathbb{C}^n, 0}/(\bar{\Phi}^m \mathcal{O}_{\mathbb{C}^n, 0}) = \mathcal{O}_{\mathbb{C}^n, 0}/(\Phi_1, \ldots, \Phi_n) \), which is finite dimensional since \( \bar{\Phi} \) is finite. Note that one of the generators must be a unit, we take \( g_1 = 1 \).
3. Find the coefficients \( \tilde{\lambda}_{ij} \in \mathcal{O}_{\mathbb{C}^n, 0} \) (in the target) to express \( g_j \Phi_{n+1} = \sum_{i=1}^N \bar{\Phi}^i(\tilde{\lambda}_{ij}) g_i \) for all \( i, j = 1, \ldots, N \).
4. Define \( \lambda_{ij} = \tilde{\lambda}_{ij} - \delta_{ij}y_{n+1} \), where \( \delta_{ij} \) is 0 for \( i \neq j \) and 1 for \( i = j \). Then \( \lambda = (\lambda_{ij})_{i,j=1}^N \) is a presentation matrix of \( \Phi \mathcal{O}_{\mathbb{C}^n, 0} \).

1.3.2. Examples.

Example 1.3.4 (\( H_k, k \geq 1 \)). In this case \( \Phi(s, t) = (s, t^2, st + t^{3k-1}) \). This is a simple germ from Mond’s list \([45]\), in fact, the first one, for which finding the equation of the image is not trivial. Here we provide it as the determinant of a presentation matrix. We use coordinates \( (s, t) \) in the source and \( (x, y, z) \) in the target. \( \bar{\Phi}(s, t) = (s, t^2) \) is also finite, since \( \bar{\Phi}^{-1}(0, 0) = (0, 0) \). \( \mathcal{O}_2/(s, t^2) = \mathbb{C}^3(1, t, t^2) \), thus the module \( \bar{\Phi}_*(\mathcal{O}_2) \) is generated by \( (1, t, t^2) \). Then

\[
1 \cdot (st + t^{3k-1}) = 0 \cdot 1 + s \cdot t + (t^3)^{k-1} \cdot t^2 = 0 \cdot 1 + x \cdot t + y^{k-1} \cdot t^2,
\]

\[
t \cdot (st + t^{3k-1}) = (t^3)^k \cdot 1 + 0 \cdot t + s \cdot t^2 = y^k \cdot 1 + 0 \cdot t + x \cdot t^2,
\]

\[
t^2 \cdot (st + t^{3k-1}) = st^3 \cdot 1 + (t^3)^k \cdot t + 0 \cdot t^2 = xy \cdot 1 + y^k \cdot t + 0 \cdot t^2.
\]

Thus

\[
\lambda = \begin{pmatrix}
-z & y^k & xy \\
x & -z & y^k \\
y^{k-1} & x & -z
\end{pmatrix}
\]

is a presentation matrix of \( \Phi_* \mathcal{O}_{\mathbb{C}^2, 0} \).
The equation of the image is \( f(x, y, z) = \det(\lambda) = y^{3k-1} - z^3 + x^3y + 3xyz^{k-1} = 0 \).

\( \mathcal{F}_0(\Phi_*\mathcal{O}_{(C^2, 0)}) \) is the ideal generated by \( f \). The determinants of the 2\( \times \)2 minors of \( \lambda \) generate the first Fitting ideal, \( \mathcal{F}_1(\Phi_*\mathcal{O}_{(C^2, 0)}) \). Its zero set consists of the double values of \( \Phi \). Finally, \( \mathcal{F}_2(\Phi_*\mathcal{O}_{(C^2, 0)}) \) is generated by the 1\( \times \)1 minors of \( \lambda \), thus \( \mathcal{F}_2(\Phi_*\mathcal{O}_{(C^2, 0)}) = (x, z, y^{k-1}) \).

Example 1.3.5 (Singularities of type \( A \)). These are quotient singularities of the form \( (X, 0) = (C^2, 0)/\mathbb{Z}_k \), where \( \mathbb{Z}_k = \{ \xi \in \mathbb{C} \mid \xi^k = 1 \} \) denotes the cyclic group of order \( k \), and the action is \( \xi \ast (s, t) = (\xi s, \xi^{-1}t) \) for \( \xi \in \mathbb{Z}_k \). \( (X, 0) \) is the image of a map \( \Phi \), whose components are the generators of the invariant algebra \( \mathcal{O}_{(C^2, 0)}^{\mathbb{Z}_k} \), see [65, page 95], namely \( \Phi(s, t) = (s^k, t^k, st) \). For \( k = 2 \)

\[
\lambda = \begin{pmatrix}
-z & 0 & 0 & xy \\
0 & -z & y & 0 \\
0 & x & -z & 0 \\
1 & 0 & 0 & -z
\end{pmatrix}
\]

is a presentation matrix of \( \Phi_*\mathcal{O}_{(C^2, 0)} \). \( f(x, y, z) = \det(\lambda) = (z^2 - xy)^2 \) is a not reduced equation of the image of \( \Phi \): it reflects to the fact that \( \Phi \) is a double cover of its image outside the origin. \( \mathcal{F}_1(\Phi_*\mathcal{O}_{(C^2, 0)}) \), the ideal generated by the determinants of the 3\( \times \)3 minors of \( \lambda \) equals to the ideal generated by \( z^2 - xy \), corresponding to the fact that the whole image of \( \Phi \) consists of double values. \( \mathcal{F}_2(\Phi_*\mathcal{O}_{(C^2, 0)}) = (x, y, z) = \mathfrak{m}_{(C^3, 0)} \) is a codimension 1 ideal in \( \mathcal{O}_{(C^3, 0)} \), that means any stable deformation of \( \Phi \) has 1 triple value, see [1.4.2] and Figure [1.3].

For arbitrary \( k \) the equation given by \( \det(\lambda) \) is the \( k \)-th power a reduced germ, reflecting to the \( k \)-fold covering outside the origin.

Example 1.3.6 (Cuspidal edge). Consider \( \Phi(s, t) = (s, t^2, t^3) \). The presentation matrix is

\[
\lambda = \begin{pmatrix}
-z & y^2 \\
y & -z
\end{pmatrix}
\]

\( \mathcal{F}_1(\Phi_*\mathcal{O}_{(C^2, 0)}) = (y, z) \). However \( \Phi \) does not have ordinary double values, the cuspidal edge \( \{ y = 0, z = 0 \} \) lies in the closure of the double values of a stable deformation of \( \Phi \).
that appear. In a lucky situation their number does not depend on the choice of the stabilization and it can be calculated by algebraic methods, that is, without stabilizing the germ. This is the procedure how the invariants $C$ and $T$ can be introduced.

A 1-parameter unfolding is a \textit{stabilization} (stable deformation) of the germ $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$, if $\Phi_v$ is stable for all parameter values $v \neq 0$. However, $\Phi_v$ is not a germ: to make the definition correct, we have to fix a representative.

Consider a 1-parameter unfolding $\tilde{\Phi}(u,v) = (\Phi_v(u), v)$, and fix small enough neighbourhoods $W \simeq B^{2p} \subset \mathbb{C}^p$ and $V \simeq D^2_\delta \subset \mathbb{C}$. Let $U = \tilde{\Phi}^{-1}(W \times V)$ and $U_v = \{u \in \mathbb{C}^n \mid \Phi(u,v) \in W\}$ for each $v \in V$. Then $\tilde{\Phi}$ is a stabilization of $\Phi$, if for every parameter value $v \in V$ each multi-germ of the map $\Phi_v : U_v \rightarrow W$ is stable. For a fixed parameter value $v \in V$, we call the map $\Phi_v$ also a stabilization of $\Phi$ – hopefully without any confusion.

Stabilization always exists, if $(n,p)$ are ‘nice dimensions’ in the sense of Mather [38, 48, 71], i.e. if stable maps are dense (in the $C^\infty$ sense). Note that in these dimensions stable maps are sometimes called \textit{generic}.

Take $p = n + 1$. If $\Phi$ is finitely determined, then it admits a versal unfolding $\tilde{\Phi}$ with $r$ parameters. Thus each stabilization of $\Phi$ is equivalent with $h^*(\tilde{\Phi})$ with some germ $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^r, 0)$. With the minimal number of parameters (i.e. $r$ equals to the $\mathscr{A}_e$-codimension of $\Phi$) the versal unfolding in unique, up to isomorphism.

Let us fix a small enough representative of the miniversal unfolding. The \textit{bifurcation set} consists of the points $v$ in the parameter space $(\mathbb{C}^r, 0)$ for which $\Phi_v$ is not stable. The bifurcation set is an analytic subvariety, see [48], indeed proper subvariety if $(n, n+1)$ are nice dimensions, which implies that the complement of it is connected in $(\mathbb{C}^r, 0)$. Thus if $\Phi_{v_0}$ and $\Phi_{v_1}$ are stable, then there is real path $v : [0,1] \rightarrow \mathbb{C}^r$ connecting $v_0$ and $v_1$ such that $\Phi_{vt}$ is stable for all $t$. In addition, the projection to the parameter space produces a locally trivial fibration of the image of the versal unfolding $\tilde{\Phi}$ over the complement of the bifurcation set. The fibres are the images of the stabilizations corresponding to different parameter values. See for details [47, 30, 61].

The image of a stabilization is called the \textit{disentanglement} of $\Phi$.

\textbf{Proposition 1.4.1 ([47 Lemma 1.3.])}. \textit{The topology of the disentanglement of a finitely $\mathscr{A}$-determined germ $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ is independent of the chosen stabilization.}

This theorem implies the well-definedness of the number of isolated stable multi-germs in each type. Especially, in the $(2,3)$ nice dimensions:
Theorem 1.4.2. Let \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be a finitely determined germ. Then the number of Whitney umbrellas and triple values are independent of the stabilization of \( \Phi \).

1.4.2. The invariants \( C \) and \( T \). Here we summarize the main properties of Mond’s invariants \( C \) and \( T \). For proofs and details we refer to [43, 44, 45, 49, 48].

Let \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be a holomorphic germ. Let \( M_j : \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \to \mathbb{C} \) denote the determinants of the three \( 2 \times 2 \) minors \((j = 1, 2, 3)\). Let \( J(\Phi) \) be the ideal of the local ring \( \mathcal{O}_{(\mathbb{C}^2, 0)} \) generated by the elements \( M_j \circ d\Phi \), where \( d\Phi \) is the Jacobian matrix of \( \Phi \). \( J(\Phi) \) is called the Jacobian ideal or the ramification ideal of \( \Phi \). Define

\[
C(\Phi) = \dim \mathbb{C} \mathcal{O}_{(\mathbb{C}^2, 0)} / J(\Phi) \quad \text{and} \quad T(\Phi) = \dim \mathbb{C} \mathcal{F}_2(\Phi, \mathcal{O}_{(\mathbb{C}^2, 0)}).
\]

Theorem 1.4.3. Let \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be a finitely determined germ. Then \( C(\Phi) \) and \( T(\Phi) \) are finite and any stabilization of \( \Phi \) has \( C(\Phi) \) Whitney umbrella points and \( T(\Phi) \) triple values.

The key tool for the proof of 1.4.3 is the so-called Cohen-Macaulay property of the ideals \( J(\Phi) \) and \( \mathcal{F}_2(\Phi, \mathcal{O}_{(\mathbb{C}^2, 0)}) \) in the case of finitely determined germs \((\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\). Cohen-Macaulay property implies a good behaviour under deformation, in our cases that means the conservation of the codimensions. E.g.

\[
\dim \mathbb{C} \mathcal{O}_{(\mathbb{C}^2, 0)} / J(\Phi) = \sum_{x \in U_v} \dim \mathbb{C} \mathcal{O}_{(\mathbb{C}^2, x)} / J(\Phi_{v,x})
\]

holds for a (not necessarily stable) perturbation \( \Phi_v : U_v \to W \) of \( \Phi \). Here \( \Phi_{v,x} \) denotes the germ of \( \Phi_v \) at \( x \). Then it is enough to check that \( \dim \mathbb{C} (\mathcal{O}_{(\mathbb{C}^2, x)} / J(\Phi_{v,x})) = 1 \) if \( \Phi_{v,x} \) is \( \mathcal{A} \)-equivalent with a Whitney-umbrella and 0 for regular germs, hence the sum counts the Whitney umbrellas of a stabilization. Similar discussion holds for \( \mathcal{F}_2(\Phi, \mathcal{O}_{(\mathbb{C}^2, 0)}) \). For details we refer to [45, 49, 48].

In Remark 3.3.6 we give an independent proof for the equality of \( C(\Phi) \) and the number of Whitney umbrella points of a stabilization in the case of corank–1 map germs, cf. Example 1.4.10.

In [45] Mond introduced a third invariant \( N(\Phi) \) for corank–1 germs, which ‘measures, in some sense, the non-transverse self-intersection concentrated at the origin’. Geometric interpretation of \( N(\Phi) \) is not yet known for the author of the present thesis. Later \( N \) was replaced with more intuitive invariants, see theorems 1.5.3 and 1.5.6. Here we do not present the definition of \( N \).

We summarize some equivalent characterizations of finite determinacy of germs \((\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\). See [45, 33].
Theorem 1.4.4. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ. Then the following are equivalent:

(a) $\Phi$ is finitely $\mathscr{A}$-determined.
(b) $\Phi|_{\mathbb{C}^2 \setminus \{0\}}$ is a stable immersion, cf. Theorem 1.2.11, Example 1.2.17.
(c) The associated map of spheres $\Phi|_{\mathbb{S}^3} : \mathbb{S}^3 \to S^5$ (cf. Definition 1.1.3) is a stable immersion, that is, it has only single values and double values with transverse intersection.

Theorem 1.4.5. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a corank–1 germ. Then $\Phi$ is finitely $\mathscr{A}$-determined if and only if $C(\Phi)$, $T(\Phi)$ and $N(\Phi)$ are finite.

On the other hand the finiteness of any of the three invariants has also a geometrical interpretation. E.g. the following holds for $C$. (Recall that a stabilization is defined in a sufficiently small neighbourhood of the origin.)

Theorem 1.4.6. Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic germ. Then the following are equivalent:

(a) $C(\Phi)$ is finite.
(b) $\Phi$ is singular only at the origin, that is, $\operatorname{rk}(d\Phi_x) < 2$ implies $x = 0$. In other words $\Phi|_{\mathbb{C}^2 \setminus \{0\}}$ is an immersion.
(c) The associated map of spheres $\Phi|_{\mathbb{S}^3} : \mathbb{S}^3 \to S^5$ is an immersion.

If these claims hold, then any stabilization of $\Phi$ has $C(\Phi)$ Whitney umbrella points. Cf. [44, 45].

If $\Phi^{-1}(0) \neq \{0\}$, then $\Phi^{-1}(0)$ has positive dimension, and along $\Phi^{-1}(0)$ the rank of $d\Phi$ is < 2. Hence corollary below follows from part (a) and (d) of Theorem 1.4.6.

Corollary 1.4.7. If $C(\Phi)$ is finite, then $\Phi$ is a finite germ.

Germs with finite $C$, or equivalently, germs singular only at the origin are our basic objects in Chapter 3.

Remark 1.4.8. $C(\Phi)$ can be defined also in higher dimensions, for germs $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$, where the higher dimensional Whitney umbrellas are isolated points. These germs are studied in [22].

Proposition 1.4.9. If $T(\Phi)$ is finite, then any stabilization of $\Phi$ has the same number of triple points, and this number is $T(\Phi)$ (cf. [49, 44]).

By Theorem 1.4.6 in the case of finite germs the failure of the finiteness of $C(\Phi)$ corresponds to the existense of a cuspidal edge, that is, a line of singular points. In general any of the following three properties implies the failure of the finite determinacy of a germ
1.4. INVARIANTS OF A STABILIZATION

$\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$: (a) there is a line of singular points, (b) there is a line of triple points, (c) there is a line of non-transverse self-intersection.

1.4.3. Examples.

Example 1.4.10 (Finitely determined corank–1 germs). Let $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ be a finitely determined germ with $\text{rk}(d\Phi_0) = 1$. Such a germ is $\mathcal{A}$-equivalent with a germ of the form $\Phi(s, t) = (s, p(s, t), q(s, t))$. Then the Jacobian ideal is generated by $\partial_t p$ and $\partial_t q$, thus its codimension $C(\Phi)$ is the intersection multiplicity of the plane curves $\partial_t p = 0$ and $\partial_t q = 0$ at the origin.

Several concrete examples can be found e.g. in the list of Mond containing the simple germs. For example the family $S_{k-1}$ given by $\Phi(s, t) = (s, t^2, t^3 + s^k t)$ has $C(\Phi) = k$. Whitney umbrellas in a stabilization, see Figure 1.2 for $k = 2$. For $k = 1$ the germ itself is $\mathcal{A}$-equivalent with the Whitney umbrella.

Example 1.4.11 ($\Sigma^{1,0}$ type germs). Assume that $\Phi(s, t) = (s, t^2, td(s, t))$, where $d(s, t) = g(s, t^2)$ for some germ $g$, such that $d(s, t)$ is not divisible by $t$. The ($\mathcal{A}$-equivalence classes of) these germs are often labeled by the Boardman symbol $\Sigma^{1,0}$, this notation refers to the classification of the germs up to their 2-jet, see [33, 43]. They are exactly the finitely determined corank–1 map germs with no triple points in their stabilization, that is, with $T(\Phi) = 0$, see [43, 33] for details.

Example 1.4.12 (A corank–2 germ). There are also finitely determined germs $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ with $\text{rk}(d\Phi_0) = 0$. Several examples can be found in [31]. A concrete one is $\Phi(s, t) = (s^2, t^2, s^3 + t^3 + st)$. For this germ $C(\Phi) = 3$ and $T(\Phi) = 1$.

Example 1.4.13 (Cuspidal edge). Let $\Phi(s, t) = (s, t^2, t^3)$ (cf. Example 1.4.11). Then $C(\Phi) = \infty$, $\Phi$ is not finitely determined, indeed it is not an immersion outside the origin.

Example 1.4.14 (Singularities of type $A$). As in Example 1.3.5, consider the $k$-fold cover of the $A_{k-1}$ singularity, $\Phi(s, t) = (s^k, t^k, st)$. Then $J(\Phi) = (s^k, t^k, s^{k-1} t^{k-1})$ and $C(\Phi) = k^2 - 1$, cf. Figure 1.3 for $k = 2$. Outside the origin $\Phi$ is an immersion, but not a stable immersion. In fact, the whole image of $\Phi$ consists of at least 2-fold multiple values, thus the intersection of the branches is not transverse. Thus $\Phi$ is not finitely determined. For $k = 2$, $T(\Phi) = 1$, cf. Example 1.3.5. However $T(\Phi) = \infty$ for $k \geq 3$, since the whole image consists of at least 3-fold multiple values.

The covering maps of the quotient singularities of type $D$ and $E_6$, $E_7$ and $E_8$ are also regular outside the origin, thus $C(\Phi)$ is finite for them. However they are not finitely determined, from the same reason that was explained for type $A$. These germs provide us
interesting examples later in Chapter 3. We present here a possible computation of $C(\Phi)$ for these germs.

**Example 1.4.15 (Singularities of type $D$).** These are the quotient singularities of form $(\mathbb{C}^2,0)/D_n$ where $D_n$ denotes the binary dihedral group, [65, page 89]. $\Phi(s,t) = (s^2t^2, s^{2n}+t^{2n}, st(s^{2n}-t^{2n}))$ [65, page 95]. By a computation $J(\Phi) = (st(s^{2n}-t^{2n}), s^2t^2(s^{2n}+t^{2n}), (s^{2n}-t^{2n})^2 - 4ns^{2n}t^{2n})$. In singularity theory the quotient is the $D_{n-2}$-singularity.

A possible computation of $\dim \mathbb{C} \left( \mathcal{O}_{(\mathbb{C}^2,0)}/J(\Phi) \right)$ is based on the following facts.

**Lemma 1.4.16.** (a) Take $f_1, f_2, h \in \mathcal{O}_{\mathbb{C}^2,0}$ such that $f_1f_2$ and $h$ are coprimes. Then one has the following exact sequence:

$$0 \rightarrow \mathcal{O}_{(\mathbb{C}^2,0)}/(f_2, h) \rightarrow \mathcal{O}_{(\mathbb{C}^2,0)}/(f_1f_2, h) \rightarrow \mathcal{O}_{(\mathbb{C}^2,0)}/(f_1, h) \rightarrow 0 .$$

(b) Take $f_1, f_2, g, h \in \mathcal{O}_{(\mathbb{C}^2,0)}$ such that the ideal $(f_1f_2, g, h)$ has finite codimension, and $h = f_1h'$ for some $h' \in \mathcal{O}_{(\mathbb{C}^2,0)}$. Then one has the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{C}^2,0}/(f_2, g, h') \rightarrow \mathcal{O}_{(\mathbb{C}^2,0)}/(f_1f_2, g, h) \rightarrow \mathcal{O}_{(\mathbb{C}^2,0)}/(f_1, g) \rightarrow 0 .$$

**Proof.** Part (a) is well-known as the additivity property of the local intersection number of plane curves, see e.g. [12]. The proof of part (b) is similar. \hfill $\square$

Using these lemmas the codimension of $J(\Phi)$ of the $D_{n-2}$ singularity can be calculated, and it is $C(\Phi) = 4n^2 + 12n - 1$.

**Example 1.4.17 (Weighted homogeneous germs and $E_6$, $E_7$ and $E_8$).** Assume that the three components of $\Phi$ are weighted homogeneous of weights $w_1$ and $w_2$ and degree $d_1$, $d_2$ and $d_3$. Then, cf. [46], $C(\Phi) = \{d_1d_2 + d_2d_3 + d_3d_1 - (w_1 + w_2)(d_1 + d_2 + d_3 - w_1 - w_2) - w_1w_2 \}/w_1w_2$. Mond proved this identity for finitely $\mathscr{A}$-determined germs, but the same proof works for germs with finite $C(\Phi)$.

For example, if $\Phi : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^2,0)/G \hookrightarrow (\mathbb{C}^3,0)$ is as in Example 3.1.4, then all three components are homogeneous ($w_1 = w_2 = 1$). In the case of $A_{k-1}$ and $D_{n+2}$ the degrees are $(k,k,2)$ and $(4,2n,2n+2)$ respectively. Hence the values $C(\Phi)$ from Examples 1.4.14 and 1.4.15 follow in this way as well.

For $E_6$, $E_7$ and $E_8$ singularities the degrees are $(6,8,12)$, $(8,12,18)$ and $(12,20,30)$ respectively, see [65] 4.5.3–4.5.5, hence the corresponding values $C(\Phi)$ are 167, 383, 1079.
1.4. INVARIANTS OF A STABILIZATION

Figure 1.2. The real germs $S^\frac{s}{t}$, $\Phi^\frac{s}{t}(s,t) = (s,t^2,t^3 \mp s^2 t)$, and their stabilizations $\Phi^\frac{v}{t}(s,t) = (s,t^2,t^3 \mp s^2 t \pm vt)$, $v \geq 0$, with two Whitney umbrellas.

Figure 1.3. The real version of $A_1$ is the double covering $\Phi(s,t) = (s^2,t^2,st)$ of the cone $\{xy = z^2, \ x, y \geq 0\}$. The picture shows its stabilization $\Phi_v(s,t) = (s(s-v),t(t-v),st)$ with the three Whitney umbrellas and one triple value.
1.4.4. Disentanglement and the image Milnor number. Recall for finitely determined germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ the topology of the disentanglement (that is, the image of a stabilization) does not depend on the choice of the stabilization, cf. Proposition 1.4.1.

Note that the disentanglement is a stratified space, each stratum is a connected ‘isosingular locus’ of stable multi-germs.

In many cases the disentanglement plays the role of the Milnor fibre of isolated singularities, for example it has the following property, analogously with Theorem 4.1.10. (In fact, the proof uses 4.1.10, see [47]).

**Theorem 1.4.18 ([47, Theorem 1.4.]).** The disentanglement of a finitely determined germ $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is homotopy equivalent with a bouquet of $n$-spheres, i.e. with $\bigvee_{i=1}^{\mu_I(\Phi)} S^n$.

**Definition 1.4.19.** The number of the spheres in the bouquet (that is, the $n$-th Betti number of the disentanglement) $\mu_I(\Phi)$ is called the image Milnor number of $\Phi$.

**Theorem 1.4.20 ([30, 3.2.-3.4.], [47]).** For finitely determined germs $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$

$$\mu_I(\Phi) = \frac{1}{2}(4T(\Phi) - C(\Phi) - \mu(D(\Phi)) + 1)$$

holds, where $\mu(D(\Phi))$ is the Milnor number of the double point curve $D(\Phi) \subset \mathbb{C}^2$, see paragraph 1.5.1.

**Theorem 1.4.21 ([21, 47]).** For a finitely determined germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, the image Milnor number $\mu_I(\Phi)$ is bigger or equal than the $\mathcal{A}_e$-codimension of $\Phi$. If $\Phi$ is weighted homogeneous, then equality holds.

For finitely determined germs $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $n \geq 3$, the statement of Theorem 1.4.21 is Mond’s conjecture.

We use the disentanglement as a complex singular Seifert surface of the immersion $\Phi|_{\mathbb{S}^3} : \mathbb{S}^3 \hookrightarrow \mathbb{S}^5$. See Section 3.6.

1.5. Further invariants

1.5.1. The double point structure. The possible algebraic descriptions of the multiple point spaces of complex germs are studied in general in [58, 30, 45, 48]. Here we summarize the double point structure of germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The Milnor numbers of the four double point curves are related to each other through the invariants $C, T$ and $N$. Indeed, $N$ can be replaced by one of this Milnor numbers. In this paragraph we refer mainly to [34, 30].
The double point spaces of a finite germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be summarized in the following diagram (see [34, 30]).

\[
\begin{array}{ccc}
D^2(\Phi) & \to & D^2(\Phi)/S_2 \\
\downarrow & & \downarrow \\
D(\Phi) & \to & \Phi(D(\Phi))
\end{array}
\]

(1.5.1)

$D^2(\Phi) \subset (\mathbb{C}^2, 0) \times (\mathbb{C}^2, 0)$ is the lifting of the double point space of $\Phi$. A general algorithm to determine it can be found in [34, 45], here we sketch it in our case. We use the following notation: the coordinates of the point $(x, x') \in \mathbb{C}^k \times \mathbb{C}^k$ are $x_i, x'_i$ ($i = 1, 2$),

The coordinate functions of $\Phi$ are $\Phi_i, i = 1, 2, 3$.

Let $I_k$ denote the ideal of the diagonal in $(\mathbb{C}^k, 0) \times (\mathbb{C}^k, 0)$, that is $I_k = (x_i - x'_i)_{i=1,\ldots,k}$. Then $(\Phi \times \Phi)^*(I_3) \subset I_2$, thus there are $\alpha_{ij} \in \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^2)$ such that

$$
\Phi_i(x) - \Phi_i(x') = \sum_{j=1}^2 \alpha_{ij}(x, x')(x_j - x'_j)
$$

holds for $i = 1, 2, 3$. Then $D^2(\Phi) \in (\mathbb{C}^2, 0) \times (\mathbb{C}^2, 0)$ is the zero set of the ideal

(1.5.2) \hspace{1cm} (\Phi \times \Phi)^*(I_3) + R_2(\alpha),

where $R_2(\alpha) \subset \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^2)$ is the ideal generated by the determinants of the $2 \times 2$ minors of the matrix $\alpha = (\alpha_{ij})$.

Note that $\alpha(x, x) = d\Phi(x)$. Hence $D^2(\Phi)$ contains the ordinary double points $(x, x')$, that is $x \neq x'$ and $\Phi(x) = \Phi(x')$, as well as the points $(x, x)$ of the diagonal, where $x$ is a singular point of $\Phi$.

There are several other methods to determine $D^2(\Phi)$ which work in special cases. For instance $(\Phi \times \Phi)^*(I_3) : I_2$ provides the same ideal as (1.5.2) for finitely determined germs, see [58].

$D(\Phi) \subset (\mathbb{C}^2, 0)$ is the double point space in the source. It is the image of the projection $p_1 : (\mathbb{C}^2, 0) \times (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ restricted to $D^2(\Phi)$. The ideal of $D(\Phi)$ can be given as the 0-th Fitting ideal of the projection $p_1 : D^2(\Phi) \to (\mathbb{C}^2, 0)$.

Finite determinacy can be characterized by the help of the curves $D^2$ and $D$ as follows.

**Theorem 1.5.1** ([34, Theorems 2.4, 3.4, 3.5]). Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ a finite and generically 1 to 1 complex germ. The following are equivalent:

(a) $\Phi$ is finitely $\mathscr{A}$-determined.

(b) $D^2(\Phi)$ is a germ of reduced curve and $p_1 : D^2(\Phi) \to (\mathbb{C}^2, 0)$ is generically 1 to 1.

(c) $D(\Phi)$ is a germ of reduced curve.

(d) The Milnor number $\mu(D(\Phi))$ of $D(\Phi)$ is finite. (See Subsection 4.1.2 for definition).
Note that if $p_1 : D^2(\Phi) \to (\mathbb{C}^2, 0)$ is generically 1 to 1, then it is an embedding for a choice of a small enough representative. $\mu(D(\Phi))$ is called the Mond number of $\Phi$.

If $\Phi$ is finitely determined, there is a useful method to determine the reduced equation $d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ of $D(\Phi)$. Let $f = \det \lambda : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the equation of the image of $\Phi$. Then $d = \Phi^*(\partial_i f)/M_i$ is the same for all $i = 1, 2, 3$, where $M_i$ is the determinant of the $2 \times 2$ minor of $d\Phi$ removed the $i$-th row, and $\partial_i f$ is the $i$th partial derivative of $f$. See [59, 3].

$D^2(\Phi)/S_2$ is the quotient of $D^2(\Phi)$ by the $S_2$-action $(x, x') \mapsto (x', x)$ on $\mathbb{C}^2 \times \mathbb{C}^2$. $D^2(\Phi)/S_2$ embeds as a germ of analytic subspace of $(\mathbb{C}^2, 0) \times (\mathbb{C}^2, 0)/S_2$. If $\Phi$ is a corank–1 germ, then $D^2(\Phi)$ and also $D^2(\Phi)/S_2$ can be embedded to $\mathbb{C}^3$.

The space of double values $\Phi(D(\Phi)) \subset (\mathbb{C}^3, 0)$ is the zero set of the first Fitting ideal $\mathcal{F}_1(\Phi, \langle O(\mathbb{C}^2, 0) \rangle)$.

The four double point spaces can be defined also for maps, specially for a stabilization $\Phi_v : U_v \to W$ of $\Phi$. Then $D^2(\Phi_v)$ is a smooth curve in $U_v \times U_v$, and the projection $p_1 : D^2(\Phi_v) \to U_v$ is a stable immersion with transverse double values in the triple points of $\Phi_v$. $D^2(\Phi_v)/S_2$ is also a smooth curve, the factorization by $S_2$ is a double branched covering, ramifies at the (lifting of the) Whitney umbrella points. $\Phi_v \circ p_1 : D^2(\Phi_v) \to W$ is an immersion with triple values in the triple values of $\Phi_v$.

If $\Phi$ is a corank–1 finitely determined germ, then all the curves $D^2(\Phi), D(\Phi), D^2(\Phi)/S_2,$ $\Phi(D^2(\Phi))$ are isolated complete intersection (ICIS) (cf. Chapter [4]). However the Milnor number of these curves can be defined also for corank–2 germs, see [34]. The Milnor numbers relate to each other by the following formulas, according to the description of the double point curves of a stabilization.

**Theorem 1.5.2 ([34 Theorem 4.3.]).** If $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a finitely determined germ, then

$$\mu(D(\Phi)) = \mu(D^2(\Phi)) + 6T(\Phi),$$

$$\mu(D^2(\Phi)) = 2\mu(D(\Phi))/S_2 + C(\Phi) - 1,$$

$$\mu(D^2(\Phi)) = 2\mu(D^2(\Phi)) + C(\Phi) - 2T(\Phi) - 1.$$
transverse object. In [32] this object is the transverse slice of $\Phi$ defined below, while in Chapter 3 that is the immersion $\Phi|_{\mathbb{G}^3} : \mathbb{G}^3 \hookrightarrow S^5$ associated with $\Phi$.

The idea is the following. After change of coordinates the germ $\Phi$ can be considered as a 1-parameter unfolding of a finitely determined plane curve, called transverse slice. Then invariants of $\Phi$ can be obtained from the parameter space of the versal unfolding of the transverse slice.

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finite, generically 1 to 1, corank–1 holomorphic map germ. Up to $\mathcal{A}$-equivalence it can be written in a form $\Phi(s, t) = (s, p(s, t), q(s, t))$, such that $\{x = 0\} \cap \Phi(D(\Phi)) = \{0\}$ and $\{x = 0\} \cap C_0(\Phi(D(\Phi))) = \{0\}$, where $C_0(\Phi(D(\Phi)))$ denotes the Zariski tangent cone of $\Phi(D(\Phi))$.

Introduce the notation $\gamma_s(t) = (p(s, t), q(s, t))$. Then $\Phi(s, t) = (s, \gamma_s(t))$ is a 1-parameter unfolding of the curve $\gamma_0 : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, called transverse slice. The transverse slice is a finitely $\mathcal{A}$-determined germ of plane curve. Its finite determinacy follows from the transversality conditions described in the previous paragraph.

Note that the transverse slice is not unique, its analytic type depends on the choice of the coordinates, but its topological type does not. Its $\delta$-invariant $\delta(\gamma_0) = \dim_{\mathbb{C}}(\mathcal{O}(\mathbb{C}, 0)/\gamma_0^* \mathcal{O}(\mathbb{C}^2, 0)) = \mu(\gamma_0)/2$ is equal to the multiplicity of $\Phi(D(\Phi))$ at 0.

**Proposition 1.5.4 ([32] 3.1.).** $\Phi(s, t)$ is finitely $\mathcal{A}$-determined if and only if $\Phi(s, t) = (s, \gamma_s(t))$ is a stabilization of $\gamma_0$.

Consider the miniversal deformation $(a, \gamma_a(t))$ of $\gamma$ where $a = (a_1, \ldots, a_r) \in \mathbb{C}^r$. Then $\Phi(s, t) = (s, \gamma_s(t))$ can be pulled back from the versal deformation by a curve $a : (\mathbb{C}, 0) \to (\mathbb{C}^r, 0)$. If $\Phi$ is finitely determined, then $a$ intersects the bifurcation set

$$\mathcal{B} = \{a \in \mathbb{C}^r \mid \gamma_a \text{ is not stable}\}$$

only at the origin.

$\mathcal{B}$ is a hypersurface in $\mathbb{C}^r$, its reduced equation decomposes as $g = g_1g_2g_3$ corresponding to $\mathcal{A}$-codimension–1 multi-germs of plane curves. $\gamma_a = (t \mapsto \gamma_a(t))$ is a stable plane curve (that is, self-transverse immersion) if $g(a) \neq 0$. For $g_1(a) = 0$ the curve $\gamma_a$ has a cusp, for $g_2(a) = 0$ the curve $\gamma_a$ has a triple point, for $g_3(a) = 0$ the curve $\gamma_a$ has a tacnode, that is, self-tangent double point. In other words, the three components of $\mathcal{B}$ correspond to the three Reidemeister moves of a 1-parameter deformation of a plane curve.

**Theorem 1.5.5 ([32] 3.7–3.10.).** Let the finitely determined corank–1 germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be the stabilization of its transverse slice $\gamma_0$ pulled back from the miniversal unfolding by the curve $a : (\mathbb{C}, 0) \to (\mathbb{C}^r, 0)$. Then

$$C(\Phi) = \text{ord}(g_1 \circ a) \text{ and } T(\Phi) = \text{ord}(g_2 \circ a).$$
The order of \( g_i \circ \mathbf{a} \) is the intersection multiplicity of the curves \( g_i \) and \( \mathbf{a} \), that is, the number of the transverse intersection points of \( \{ g_i = 0 \} \) and a perturbation of \( \mathbf{a} \). In each intersection point \( a \) the corresponding curve \( \gamma_a \) has a cusp (resp. triple point). On the other hand, since \( \mathbf{a} \) describes a 1-parameter stable unfolding of \( \gamma_0 \), which is \( \Phi \), a 1-parameter perturbation of \( \mathbf{a} \) is a 2-parameter unfolding of \( \gamma_0 \), that is, a 1-parameter unfolding of \( \Phi \). The cusps and the triple points of \( \gamma_a \) correspond to the Whitney umbrellas and the triple values of the unfolding of \( \Phi \).

The third invariant

\[ J(\Phi) := \text{ord}(g_3 \circ \mathbf{a}) \]

is the number of values \( a \in \mathbb{C}^r \) of a perturbation of \( \mathbf{a} \), such that \( \gamma_a \) has a tacnode.

The following relations hold to the new invariants.

**Theorem 1.5.6 ([32 3.8. and p. 1388]).** Let the finite, generically 1 to 1, corank-1 germ \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) be the 1-parameter unfolding of its transverse slice \( \gamma_0 \) pulled back from the miniversal unfolding by the curve \( \mathbf{a} : (\mathbb{C}, 0) \to (\mathbb{C}^r, 0) \). Then

(a) \( \Phi \) is finitely determined if and only if \( C(\Phi) \), \( T(\Phi) \) and \( J(\Phi) \) are finite (defined as \( \text{ord}(g_i \circ \mathbf{a}) \)).

(b) \( J(\Phi) = \delta(\gamma_0) + N(\Phi)/2 - 1 \).
CHAPTER 2

Immersions of spheres to Euclidean spaces

2.1. Hirsch–Smale theory

2.1.1. Hirsch theorem. This chapter contains the basic theorems of Hirsch and Smale, and the formulae of Hughes–Melvin [17] and Ekholm–Szűcs [9] corresponding to the immersions of the 3-sphere to \( \mathbb{R}^5 \). The last section contains other results and applications, which are related to our study presented in Chapter 3.

Hirsch–Smale theory [16, 64] transforms regular homotopy problems (differential topology) to homotopy theory (algebraic topology).

Let \( M^m \) and \( Q^q \) be smooth manifolds of dimension \( m \) and \( q \) respectively. A smooth map \( f : M \to Q \) is an immersion, if at every point \( x \in M \) the differential \( df_x : T_x M \to T_{f(x)} Q \) is injective (monomorphism). In other words, immersions are ‘local embeddings’. Two immersions \( f, g : M \to Q \) are regular homotopic if they can be deformed to each other through immersions, that is, if there exists a smooth map \( H : M \times [0, 1] \to Q \) such that \( H_0 = f, H_1 = g \) and \( H_t : M \to Q \) is an immersion for all \( t \in [0, 1] \). Regular homotopy is an equivalence relation on the set of immersions from \( M \) to \( Q \). An injective immersion of a compact manifold \( M \) is called embedding.

We use the following notations. \( f : M \hookrightarrow Q \) denotes an immersion from \( M \) to \( Q \), and \( f : M \hookrightarrow Q \) denotes an embedding. \( \text{Imm}(M, Q) \) is the set of regular homotopy classes of the immersions from \( M \) to \( Q \), and its subset \( \text{Emb}(M, Q) \) consists of regular homotopy classes which admit embedding representatives.

The differential of an immersion is a fibrewise injective linear bundle map \( df : TM \to TQ \), say bundle monomorphism. We call two bundle monomorphisms homotopic if they are homotopic through bundle monomorphisms.

**Theorem 2.1.1** (Hirsch [16]). If \( m < q \), the differential \( f \mapsto df \) induces a bijection between \( \text{Imm}(M^m, Q^q) \) and the homotopy classes of the bundle monomorphisms.

If \( Q = \mathbb{R}^q \), a more concrete identification can be done. For an immersion \( f : M \hookrightarrow \mathbb{R}^q \) the bundle \( \mathcal{E}^q := \{ f^* T\mathbb{R}^q \to M \} \) is trivial, moreover for different immersions the corresponding bundles are naturally isomorphic. Let \( \text{HOM}(TM, \mathcal{E}^q) \to M \) be the bundle of bundle homomorphisms from \( TM \) to \( \mathcal{E}^q \) (over the identity map of \( M \)), and its subbundle
MONO\((TM, \mathcal{E}^q) \to M\) consists of bundle monomorphisms. \(\text{HOM}(TM, \mathcal{E}^q) \to M\) is a vector bundle with fibre \(\text{Hom}(\mathbb{R}^m, \mathbb{R}^q)\). MONO\((TM, \mathcal{E}^q) \to M\) is a locally trivial bundle whose fibre is the Stiefel-manifold \(V_m(\mathbb{R}^q) \subset \text{Hom}(\mathbb{R}^m, \mathbb{R}^q)\) consists of monomorphisms from \(\mathbb{R}^m\) to \(\mathbb{R}^q\), or in other words, the linearly independent \(m\)-frames of \(\mathbb{R}^q\). (The compact Stiefel manifold consisting of orthonormal \(m\)-frames of \(\mathbb{R}^q\) is homotopy equivalent with \(V_m(\mathbb{R}^q)\) via its natural embedding. We do not distinguish them in the notation.)

**Theorem 2.1.2** (Hirsch [16]). (a) If \(m < q\), the differential \(f \mapsto df\) induces a bijection between \(\text{Imm}(M^m, \mathbb{R}^q)\) and the homotopy classes of the continuous sections of \(\text{MONO}(TM, \mathcal{E}^q) \to M\).

(b) If additionally \(M\) is parallelizable (that is, \(TM\) is trivial), then a chosen trivialization of \(TM\) provides a bijection between \(\text{Imm}(M, \mathbb{R}^q)\) and \([M, V_m(\mathbb{R}^q)]\).

Part (a) of Theorem 2.1.2 does not hold in general for \(\text{Imm}(M, Q)\), not even if all the immersions from \(M\) to \(Q\) are homotopic and \(f^*TQ \to M\) is a trivial bundle, cf. Example 2.1.13

**2.1.2. Smale invariant.** Although in the case of spheres the tangent bundle is not trivial in general, the homotopy classes of the sections in part (a) of Theorem 2.1.2 can be replaced by homotopy classes of continuous maps, as in part (b).

**Theorem 2.1.3** (Smale [64]). There is a bijection

\[\text{Imm}(S^n, \mathbb{R}^q) \leftrightarrow \pi_n(V_n(\mathbb{R}^q)).\]

The bijection is induced by the map \(f \mapsto \Omega(f)\), where \(\Omega(f) \in \pi_n(V_n(\mathbb{R}^q))\) is called the Smale invariant of the immersion \(f : S^n \subset \mathbb{R}^q\). (The relation between the bijections in Theorem 2.1.3 and in Theorem 2.1.2(b) is clarified in Remark 2.1.7)

Here we present the construction of the Smale invariant, cf. [18] p. 159, [7] 3.2.]. Let \(S^n\) be the unit sphere in \(\mathbb{R}^{n+1}\), that is \(S^n = \{x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma_{i=0}^n x_i^2 = 1\}\), and define \(S^n_+ = \{x \in S^n \mid x_0 \geq 0\}\) and \(S^n_- = \{x \in S^n \mid x_0 \leq 0\}\). Let \(U_\pm\) be an open neighbourhood of \(S^n_\pm\) in \(S^n\). Fix a trivialization \(b_\pm\) of the tangent bundle \(TU_\pm\).

By a regular homotopy it can be achieved that \(f : S^n \subset \mathbb{R}^q\) agrees with the standard embedding \(i : S^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^q\) on \(U_\pm\). We define a continuous map \(\omega(f) : S^n \to V_n(\mathbb{R}^q)\) as

\[\omega(f)(x) = \begin{cases} df_x(b_+(x)) & \text{if } x \in S^n_+, \\ dt_{\tau(x)}(b_+(\tau(x))) & \text{if } x \in S^n_- \end{cases}\]

where \(\tau(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)\) is the reflection in the hyperplane \(\{x_0 = 0\} \subset \mathbb{R}^{n+1}\) restricted to \(S^n \subset \mathbb{R}^{n+1}\). Then \(\Omega(f) := [\omega(f)] \in \pi_n(V_n(\mathbb{R}^q))\).
Remark 2.1.4. If $G$ is a connected Lie group, or a quotient of it by a closed connected subgroup, then $\pi_n(G)$ can be identified with the homotopy classes of the continuous maps $f : S^n \to G$ without any base point. This fact can be applied to $V_n(\mathbb{R}^q) = SO(q)/SO(n-q)$, cf. [66, 19]. Furthermore, for Lie groups, the group operation of $\pi_n(G)$ agrees with that induced by the pointwise multiplication in $G$; cf. [66, p. 88 and 89].

Then Theorem 2.1.3 can be reformulated as follows.

Theorem 2.1.5 (Smale [64]). For an immersion $f : S^n \to \mathbb{R}^q$, the Smale invariant $\Omega(f) \in \pi_n(V_n(\mathbb{R}^q))$ is independent of the choices, it is determined by $f$. Furthermore, two immersions $f$ and $g$ are regular homotopic if and only if $\Omega(f) = \Omega(g)$, and each element of $\pi_n(V_n(\mathbb{R}^q))$ arises as $\Omega(f)$ of some immersion $f$.

Remark 2.1.6. $\text{Imm}(S^n, \mathbb{R}^q)$ is a group with respect to the operation 'connected sum'. The connected sum $f \# g$ of the immersions $f$ and $g$ is the induced immersion from $(S^n \setminus \text{int}(B^n)) \cup_{S^{n-1}} (S^n \setminus \text{int}(B^n)) \cong S^n$ to $\mathbb{R}^q$ [18, p. 164], [7, 3.4]. The Smale invariant provides a group isomorphism $\text{Imm}(S^n, \mathbb{R}^q) \cong \pi_n(V_n(\mathbb{R}^q))$. In particular, in $\text{Imm}(S^n, \mathbb{R}^q)$ the identity element is the standard embedding, thus its Smale invariant is the identity element of $\pi_n(V_n(\mathbb{R}^q))$. See [18, 7].

Remark 2.1.7. In some dimensions $S^n$ is parallelizable, for instance if $n = 1$ or 3. Then by part (b) of Theorem 2.1.2 there is a bijection $\Theta : \text{Imm}(S^n, \mathbb{R}^q) \to \pi_n(V_n(\mathbb{R}^q))$, but it depends on the choice of the trivialization of $TS^n$. Hence it does not agree with the Smale invariant in general. The relation between them is $\Omega(f) = \Theta(f) - \Theta(\iota)$, where $\iota$ is the standard embedding. Cf. [7] and Example 2.1.12.

Question 2.1.8. Smale asked in [64] to characterize the regular homotopy classes in terms of the geometry of the immersion, and give explicit representatives of each regular homotopy class.

Several results appeared in this topic, see for instance examples 2.1.9, 2.1.11 and [18, 7, 24]. Our results also fit to this program, cf. [53] or Chapter 3.

2.1.3. Examples.

Example 2.1.9 (Eversion of the Sphere). The Smale invariant $\Omega(f)$ of an immersion $f : S^2 \to \mathbb{R}^3$ sits in $\pi_2(V_2(\mathbb{R}^3)) = \pi_2(SO(3)) = 0$. Thus any immersion $f : S^2 \to \mathbb{R}^3$ is regular homotopic with the standard embedding $\iota_{st} : S^2 \to \mathbb{R}^3$. For instance the standard embedding composed with the hyperplane reflection $(x, y, z) \to (-x, y, z)$ is regular homotopic with the standard embedding. A regular homotopy between them is an eversion of the sphere.
2. IMMERSIONS OF SPHERES TO EUCLIDEAN SPACES

Example 2.1.10 ($S^n \hookrightarrow \mathbb{R}^q$, $2n < q$). The Stiefel-manifold $V_n(\mathbb{R}^q)$ is $(q - n - 1)$-connected, cf. [19, 66]. Hence for $2n < q$ any immersion from $S^n$ to $\mathbb{R}^q$ is regular homotopic with the standard embedding.

Example 2.1.11 ($S^n \hookrightarrow \mathbb{R}^{2n}$). For an immersion $f : S^n \hookrightarrow \mathbb{R}^{2n}$

$$
\Omega(f) \in \pi_n(V_n(\mathbb{R}^{2n})) = \begin{cases} 
\mathbb{Z} & \text{if } n \text{ is even or } n = 1, \\
\mathbb{Z}_2 & \text{if } n \text{ is odd and } n \neq 1,
\end{cases}
$$

cf. [19, 66]. By a regular homotopy $f$ can be deformed to a stable immersion, which has self-transverse double values. If $n$ is even, it is possible to associate a sign to each double value, and $\Omega(f)$ is the algebraic number of the double values. For odd $n$ ($n \neq 1$) the Smale invariant agrees with the algebraic number of double values modulo 2. The exceptional case $n = 1$ is discussed in the next example.

Example 2.1.12 ($S^1 \hookrightarrow \mathbb{R}^2$). The Smale invariant of a plane curve immersion is an element of $\pi_1(V_1(\mathbb{R}^2)) = \pi_1(S^1) = \mathbb{Z}$. By the result of Whitney [72], the winding number is a complete regular homotopy invariant. For an immersion $f : S^1 \hookrightarrow \mathbb{R}^2$ the winding number $w(f)$ is the degree of $f'/|f'| : S^1 \rightarrow S^1$. Thus the winding number is the invariant according to part (b) of Theorem 2.1.2 but it is not the Smale invariant, cf. Remark 2.1.7. However, by fixing a base point and a support line with a convention for the starting of the curve at the base point, it is possible to associate a sign to each double value of the immersion, cf. [72]. Then $\Omega(f) \in \mathbb{Z}$ is the algebraic number of the double values, but its sign depends on the choice of the convention. By Remark 2.1.7 $\Omega(f) = \pm w(f) \pm 1$. The sign ambiguity can be avoided by fixing the conventions.

The examples 2.1.13 and 2.1.14 are results of a discussion with Tamás Terpai.

Example 2.1.13 ($S^1 \hookrightarrow S^2$). By Hirsch’s Theorem 2.1.1 Imm($S^1, S^2$) is in bijection with the bundle monomorphisms $TS^1 \rightarrow TS^2$ up to homotopy. Such a monomorphism associates to each point of $S^1$ a nonzero tangent vector of $S^2$, hence up to homotopy it can be considered as a map from $S^1$ to $SO(3)$. Thus Imm($S^1, S^2$) is in bijection with $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Even though $f^*TS^2 \rightarrow S^1$ is a trivial bundle for all $f : S^1 \hookrightarrow S^2$ immersions, part (a) of Theorem 2.1.2 cannot be applied. Indeed, the sections of $\text{MONO}(TS^1, \mathcal{E}^2) \rightarrow S^1$ up to homotopy are in bijection with $\mathbb{Z}$.

Example 2.1.14 ($S^n \hookrightarrow S^q$). The inclusion $\mathbb{R}^q \subset S^q = \mathbb{R}^q \cup \{P\}$ (where $P$ is any point of $S^q$) induces a map from Imm($S^n, \mathbb{R}^q$) to Imm($S^n, S^q$). This map is surjective, since every regular homotopy classes can be represented by an immersion avoiding the infinity. If $q \geq n + 2$, then the map is also injective, because the regular homotopy also can be
chosen such that it avoids infinity. Hence $\text{Imm}(S^n, S^q)$ can be identified with $\text{Imm}(S^n, \mathbb{R}^q)$ for $q \geq n + 2$.

The map from $\text{Imm}(S^n, \mathbb{R}^{n+1})$ to $\text{Imm}(S^n, S^{n+1})$ is surjective but not injective in general. Let $f$ and $g$ be immersions from $S^n$ to $\mathbb{R}^{n+1} \subset S^{n+1}$, and let $H : S^n \times I \rightarrow S^{n+1}$ be a regular homotopy between $f$ and $g$ through the infinity. It can be assumed that infinity is a regular value of $H$, and the algebraic number of its preimages is $\# H^{-1}(P) = d$. Then $g$ is regular homotopic with $f \# d \cdot \tau$ in $\mathbb{R}^{n+1}$, where $\tau : S^n \leftrightarrow \mathbb{R}^{n+1}$ is the hyperplane reflection of $\mathbb{R}^{n+1}$ restricted to $S^n \subset \mathbb{R}^{n+1}$, cf. Example 2.1.11, and # denotes the connected sum of two immersions, cf. [18, 7].

The normal degree $D(f)$, i.e. the degree of the Gauss map of $f$ is a regular homotopy invariant for immersions $f : S^n \hookrightarrow \mathbb{R}^{n+1}$ with the properties $D(\tau) = (-1)^n$ and $D(f \# g) = D(f) + D(g) - 1$, see [41]. Hence for odd $n$ a regular homotopy through $P \in S^{n+1}$ changes the normal degree with a multiple of 2, and so changes the regular homotopy class of the immersion $f : S^n \hookrightarrow \mathbb{R}^{n+1}$. Note that $D(f)$ is the winding number $w(f)$ for $f : S^1 \hookrightarrow \mathbb{R}^2$, cf. examples 2.1.12 and 2.1.13. Compare also with Hughes formula (2.3.2) for immersions from $S^3$ to $\mathbb{R}^4$, where $D(f) - 1$ is the first component of $\Omega(f)$, and Example 2.1.9 in which case $D(\tau) = 1$.

## 2.2. Immersions of $S^3$ to $\mathbb{R}^5$

2.2.1. Immersions and embeddings. Hughes and Melvin [17] proved that there are embeddings of $S^3$ to $\mathbb{R}^5$ which are not regular homotopic to each other, thus it is impossible to express the Smale invariant in terms of the double point set (like e.g. in Example 2.1.11). They expressed the Smale invariant of an embedding with the help of a Seifert surface.

For an immersion $f : S^3 \hookrightarrow \mathbb{R}^5$ the Smale invariant is in $\pi_3(V_3(\mathbb{R}^5))$, and $\pi_3(V_3(\mathbb{R}^5)) \cong \pi_3(SO(5)) \cong \pi_3(SO) \cong \mathbb{Z}$. The isomorphisms are induced by the $SO(2)$-fibration $SO(5) \to V_3(\mathbb{R}^5)$ and the standard embedding $SO(5) \hookrightarrow SO$, cf. [19, 66]. The isomorphism of $\pi_3(SO)$ and $\mathbb{Z}$ is a priori non–canonical, but they can be identified with a choice of a generator in $\pi_3(SO)$. It is described in Section 3.2.

**Theorem 2.2.1** (Hughes, Melvin [17]). $\text{Emb}(S^3, \mathbb{R}^5) \cong 24 \cdot \mathbb{Z} \subset \text{Imm}(S^3, \mathbb{R}^5) \cong \mathbb{Z}$.

Furthermore, $\Omega(f)$ of an embedding $f : S^3 \hookrightarrow \mathbb{R}^5$ can be expressed by the signature of a Seifert surface.

**Theorem 2.2.2** (Hughes, Melvin [17]). Let $f : S^3 \hookrightarrow \mathbb{R}^5$ be an embedding and $\tilde{f} : M^4 \hookrightarrow \mathbb{R}^5$ be a Seifert surface of $f$, i.e. $M^4$ is a compact oriented 4-manifold with
boundary $\partial M^4 = S^3$ and $\tilde{f}$ is an embedding such that $\tilde{f}|_{\partial M^4} = f$. Let $\sigma(M^4)$ be the signature of $M^4$. Then

$$\Omega(f) = \pm \frac{3}{2} \sigma(M^4).$$

Theorems 2.2.1 and 2.2.2 can be summarized on the following diagram:

$$f \quad \mapsto \quad \Omega(f)$$

$$\text{Imm}(S^3, \mathbb{R}^5) \quad \rightarrow \quad \mathbb{Z}$$

(2.2.2)

$$\text{Emb}(S^3, \mathbb{R}^5) \quad \rightarrow \quad 24 \cdot \mathbb{Z}$$

$$f \quad \mapsto \quad \Omega(f) = \pm \frac{3}{2} \sigma(M^4)$$

Remark 2.2.3. Hughes and Melvin proved a more general result for embeddings from $S^n$ to $S^{n+2}$. An alternative proof for Theorems 2.2.1 and 2.2.2 can be found in [67].

Remark 2.2.4. The sign ambiguity of the formula (2.2.1) is caused by the sign ambiguity of the Smale invariant as a $\mathbb{Z}$-invariant, more precisely, by the identification of the target of $\Omega$ with $\mathbb{Z}$, and by the nature of the proof of (2.2.1) as well. One of the goals of Chapter 3 is to indicate the correct sign in this and other Smale invariant formulae in this section, whenever the Smale invariant is replaced by the sign-refined Smale invariant, see Subsection 3.2.1.

For the proof of Theorems 2.2.1 and 2.2.2 Hughes and Melvin introduced an alternative definition of the Smale invariant of immersions $f : S^3 \looparrowright \mathbb{R}^5$. We review their construction here, since we use this definition of the Smale invariant throughout Chapter 3. Note that Szűcs used another construction in [67].

Let $U$ be a tubular neighbourhood of the standard $S^3 \subset \mathbb{R}^5$, and let $F : U \looparrowright \mathbb{R}^5$ be an orientation preserving immersion extending $f$, i.e. $F|_{S^3} = f$. Let $TU$ be the tangent bundle of $U$. It inherits a global trivialization from the natural trivialization of $T\mathbb{R}^5$. In particular, there is a map (the Jacobian matrix)

$$dF|_U : U \rightarrow GL^+(5, \mathbb{R}).$$

Its homotopy class is the Smale invariant of $f$:

$$\Omega(f) = [dF|_{S^3}] \in \pi_3(SO(5))$$

via the homotopy equivalence induced by the inclusion $SO(5) \subset GL^+(5, \mathbb{R})$. (The base point is irrelevant in $\pi_3(SO(5))$, cf. Remark 2.1.4).

Proposition 2.2.5. $\Omega(f)$ does not depend on the choice of $U$ and $F$, it depends only on the regular homotopy class of $f$, and $\Omega : \text{Imm}(S^3, \mathbb{R}^5) \to \pi_3(SO(5))$ is a bijection.
Indeed, Smale proved that his original invariant gives a bijection between \( \text{Imm}(S^3, \mathbb{R}^5) \) and \( \pi_3(V_3(\mathbb{R}^5)) \), cf. Subsection 2.1.2 and [64]. Hughes and Melvin proved that their alternative definition (2.2.3) of the Smale invariant does not depend on the choice of \( F \) and agrees with the original Smale invariant through the natural group isomorphism \( \pi_3(SO(5)) \to \pi_3(V_3(\mathbb{R}^5)) \).

2.2.2. Ekholm’s Vassiliev invariant \( L \). In analogy with Arnold’s invariants for plane curves \( S^1 \to \mathbb{R}^2 \), Ekholm [7] introduced the invariants \( J \) and \( L \) (denoted by \( \text{lk} \) originally) of stable immersions from \( S^3 \) to \( \mathbb{R}^5 \). Here we describe \( L \) in detail. It appears in Subsection 2.2.3 as a contribution in the Ekholm–Szűcs formulas, and for immersions associated with holomorphic germs we express \( L \) in terms of \( C \) and \( T \), see Subsection 3.1.5.

A stable immersion of \( S^3 \) to \( \mathbb{R}^5 \) has only regular simple and double values with transverse intersection of the branches at the double values. By [7, Proposition 5.2.2.] there are two types of codimension–1 immersions, i.e. immersions which can appear in a stable regular homotopy between two stable immersions:

(a) immersion with one triple value (with regular intersection of the branches) and

(b) immersion with self tangency at one point (that is, \( f(x) = f(y) = q \in S^5 \) is a double value of \( f : S^3 \to \mathbb{R}^5 \), and \( \dim(df_x(T_xS^3) + df_y(T_yS^3)) = 4 \) holds instead of transversality).

Ekholm’s invariants \( J \) and \( L \) are constant along regular homotopies through stable immersions. \( J \) changes by \( \pm 1 \) when the regular homotopy steps through an immersion with self tangency and does not change under triple point moves. \( L \) changes by \( \pm 3 \) under triple point moves and does not change when stepping through immersion with self tangency [7, Theorem 2]. \( J(f) \) is equal to the number of the double point curve components of \( f \) in the target (that is, the number of components of \( f(\gamma) \), see below).

The invariant \( L(f) \) of a stable immersion \( f : S^3 \to \mathbb{R}^5 \) measures the linking of a shifted copy of the double values with the image of \( f \). There are different versions for the definition, see below. Here we review these definitions and their equivalence in the simplest case, for immersions \( S^3 \to \mathbb{R}^5 \), although originally they were introduced in [7, 8, 9, 60] for different levels of generalizations (for other manifolds, higher dimensions).

Let \( f : S^3 \to \mathbb{R}^5 \) be a stable immersion, and let \( \gamma \subset S^3 \) be the double point locus of \( f \), i.e. \( \gamma = \{ p \in S^3 \mid \exists p' \in S^3 : p \neq p' \text{ and } f(p) = f(p') \} \). \( \gamma \) is a closed 1-manifold, and \( f|_{\gamma} : \gamma \to f(\gamma) \) is a 2-fold covering. \( \gamma \) is endowed with an involution \( \iota : \gamma \to \gamma \) such that \( f(p) = f(\iota(p)) \) for all \( p \in \gamma \).

The first definition is from [7, 6.2.]. Let \( v \) be a normal vector field of \( \gamma \subset S^3 \) such that \([\gamma] = 0 \) in \( H_1(S^3 \setminus \gamma, \mathbb{Z}) \). Such a vector field \( v \) is unique up to homotopy, and for
instance each component of a Seifert framing provides such a vector field. If \( \tilde{\gamma} \subset S^3 \) is the result of pushing \( \gamma \) slightly along \( v \), then the linking number \( \text{lk}_{S^3}(\gamma, \tilde{\gamma}) \) equals to 0. Let \( q = f(p) = f(\nu(p)) \) be a double value of \( f \). Then \( w(q) = df_p(v(p)) + df_{\nu(p)}(v(\nu(p))) \) defines a normal vector field \( w \) along \( f(\gamma) \). Let \( f(\gamma) \subset \mathbb{R}^5 \) be the result of pushing \( f(\gamma) \) slightly along \( w \), then \( \bar{f}(\gamma) \) and \( f(S^3) \) are disjoint. The invariant is the linking number

\[
L_1(f) := \text{lk}_{\mathbb{R}^5}(\bar{f}(\gamma), f(S^3))
\]

(or equivalently, \( L_1(f) = [\bar{f}(\gamma)] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z} \)). Note that Ekholm used an other notation: \( L_1(f) \) is denoted by \( \text{lk}(f) \), and \( L(f) \) is defined as \( [\text{lk}(f)/3] \) in \( \mathbb{7} \) 2.2., 6.2.]

The second definition is in \( \mathbb{9} \) Definition 11., \( \mathbb{60} \) Definition 2.2.]. The normal bundle \( \nu(f) \) of \( f \) is trivial, since the oriented rank–2 vector bundles over \( S^3 \) are classified by \( \pi_2(SO(2)) = 0 \). Any two trivializations are homotopic, since their difference represents an element in \( \pi_3(SO(2)) = 0 \). Let \( (v_1, v_2) \) be the homotopically unique normal framing of \( f \), and in a double value \( q = f(p) = f(\nu(p)) \) define \( u(q) = v_1(p) + v_1(\nu(p)) \). \( u \) is a normal vector field along \( f(\gamma) \), and let \( \bar{f}(\gamma) \subset \mathbb{R}^5 \) be the result of pushing \( f(\gamma) \) slightly along \( u \). Then \( \bar{f}(\gamma) \) and \( f(S^3) \) are disjoint. The invariant is the linking number (or equivalently, the homology class)

\[
L_2(f) := \text{lk}_{\mathbb{R}^5}(\bar{f}(\gamma), f(S^3)) = [\bar{f}(\gamma)] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z}.
\]

Note that the framing \( (v_1, v_2) \) can be replaced by an arbitrary nonzero normal vector field \( v \) of \( f \), since it can be extended to a framing whose first component is \( v \).

The third definition is in \( \mathbb{9} \) Definition 4.], see also \( \mathbb{8} \) 4.5., 4.6.]. Let \( v \) be a nonzero normal vector field of \( f \) over \( \gamma \), that is, a nowhere zero section of \( \nu(f)|_\gamma \). Let \( [v] \) be the homology class represented by \( v \) in \( H_1(E_0(\nu(f)), \mathbb{Z}) \cong \mathbb{Z} \), where \( E_0(\nu(f)) \) denotes the total space of the bundle of nonzero normal vectors of \( f \). Let \( u_v(q) = v(p) + v(\nu(p)) \) be the value of the vector field \( u_v \) along \( f(\gamma) \) at the point \( q = f(p) = f(\nu(p)) \). Let \( \bar{f}(\gamma)^{(v)} \) be the result of pushing \( f(\gamma) \) slightly along \( u_v \), then \( \bar{f}(\gamma)^{(v)} \) and \( f(S^3) \) are disjoint. The invariant is

\[
L_v(f) := \text{lk}_{\mathbb{R}^5}(\bar{f}(\gamma)^{(v)}, f(S^3)) - [v] = [\bar{f}(\gamma)^{(v)}] - [v],
\]

where \( [\bar{f}(\gamma)^{(v)}] \in H_1(\mathbb{R}^5 \setminus f(S^3), \mathbb{Z}) \cong \mathbb{Z} \).

By \( \mathbb{8} \) Lemma 4.15.] \( L_v(f) \) is well-defined, that is, \( L_v(f) \) does not depend on the choice of the normal field \( v \). Moreover, if \( v \) is the restriction of a (global) normal vector field of \( f \) to \( \gamma \), then \( [v] = 0 \). Indeed, the restriction of the normal field of \( f \) to a Seifert surface \( H \) of \( \gamma \) results a surface \( \overline{H} \subset E_0(\nu(f)) \), whose boundary is the image of \( v : \gamma \to E_0(\nu(f)) \). Hence \( L_v(f) = L_2(f) \).
The invariants \( L_1, L_2 \) (and so \( L_v \)) are equal to each other. This fact follows from the basic property of \( L_1 \) and \( L_2 \) proved in [7, 8], see Proposition 2.2.6 below. The proof of this proposition is a result of a discussion with András Szűcs.

**Proposition 2.2.6.** (a) The three definitions are equivalent, i.e. \( \pm L_1(f) = L_2(f) = L_v(f) \). Let us denote \( L_1(f) \) by \( L(f) \).

(b) \( L(f) \) is an invariant of stable immersions. It changes by \( \pm 3 \) under triple point moves and does not change under self tangency moves. In other words: if \( f \) and \( g \) are regular homotopic stable immersions, \( h : S^3 \times [0,1] \to \mathbb{R}^5 \) is a stable regular homotopy between them, then \( \pm (L(f) - L(g)) \) is equal to the algebraic number of triple values of the map \( H : S^3 \times [0,1] \to \mathbb{R}^5 \times [0,1], H(x,t) = (h(x,t),t) \).

**Proof.** Part (b) is proved independently for \( L_1 \) [7 Lemma 6.2.1.] and for \( L_2 = L_v \) [8 Theorem 1.]. Using them we prove part (a) as follows.

Since \( L_1 \) and \( L_2 \) changes in the same (or opposite) way along a regular homotopy, \( L_1 \pm L_2 \) is a regular homotopy invariant. Moreover \( L_1 \) and \( L_2 \) are additive under connected sum, see [8 Lemma 5.2., Proposition 5.4.], [7 6.5.]. It follows that \( L_1 \pm L_2 \) defines a homomorphism from \( \text{Imm}(S^3, \mathbb{R}^5) \) to \( \mathbb{Z} \). If \( f : S^3 \hookrightarrow \mathbb{R}^5 \) is an embedding, then \( L_1(f) = L_2(f) = 0 \), hence \( L_1 \pm L_2 \) is 0 on the 24-index subgroup \( \text{Emb}(S^3, \mathbb{R}^5) \) of \( \text{Imm}(S^3, \mathbb{R}^5) \cong \mathbb{Z} \). It follows that \( L_1 \pm L_2 \) is 0 for any stable immersion. \( \square \)

**Remark 2.2.7.** \( L \) can be defined also for nonstable immersions which do not have triple values. Any immersion \( f \) admits a small perturbation by regular homotopy to a stable immersion \( \tilde{f} \), and if \( f \) does not have triple values, then any two stable perturbations can be joined with a regular homotopy without stepping through triple point. Thus \( L(f) \) can be defined as \( L(\tilde{f}) \) of any small stable perturbation \( \tilde{f} \) of \( f \).

**Remark 2.2.8.** Ekholm also defined the invariant \( \text{St}(f) = (\Omega(f) + L(f))/3 \) for stable immersions \( f : S^3 \hookrightarrow \mathbb{R}^5 \), which is the analogue of Arnold’s ‘strangeness’ for plane curve immersions [7 2.2., 6.5.]. \( \text{St} \) changes by \( \pm 1 \) under triple point moves, it is additive with respect to connected sum, and changes sign under precomposing an immersion with a reflection in the source. For immersions associated with holomorphic germs we express \( L \) and \( \text{St} \) in terms of \( C \) and \( T \), see Subsection 3.1.5.

**2.2.3. Ekholm–Szűcs formulas.** Ekholm and Szűcs generalized the Hughes–Melvin formula 2.2.2 for immersions via generic singular Seifert surfaces, in two different ways: mapped either in \( \mathbb{R}^5 \) or in \( \mathbb{R}^6 \) [9], see also [10, 60].

If \( M^4 \) is a compact oriented 4-manifold and \( g : M^4 \to \mathbb{R}^5 \) is a stable \( \mathcal{C}^\infty \) map, then \( g \) has isolated \( \Sigma^{1,1} \)-points (cusps). These are the singular points of \( g \) restricted to its singular
locus $\Sigma^1 \subset M^4$. Each cusp point is endowed with a well-defined sign. Let $\# \Sigma^{1,1}(g)$ be their algebraic number (cf. [9]).

**Theorem 2.2.9** (Ekholm, Szűcs [9]). Let $f : S^3 \hookrightarrow \mathbb{R}^5$ be an immersion and $M^4$ be a compact oriented 4-manifold with boundary $S^3$. Let $\tilde{f} : M^4 \to \mathbb{R}^5$ be a generic map such that $\tilde{f}|_{\partial M^4}$ is regular homotopic to $f$ and $\tilde{f}$ has no singular points near the boundary. Then

$$\Omega(f) = \pm \frac{1}{2}(3\sigma(M^4) + \# \Sigma^{1,1}(\tilde{f})) .$$

The second formula uses stable $C^\infty$ maps $g : M^4 \to \mathbb{R}^6$ defined on compact oriented 4-manifolds $M^4$. It involves two topological invariants associated with such a map. Next we review their definitions. They will be computed for two concrete holomorphic maps in order to identify the missing sign in Section 3.6.

If $g$ is as above, then it has isolated triple values (three local sheets of $M^4$ intersecting in general position). Such a point is endowed with a well-defined sign ([9], 2.3]).

**Definition 2.2.10** ([9]). $t(g)$ denotes the algebraic number of the triple values of $g$.

Next, assume that $\partial M^4 = S^3$ and $g : (M^4, \partial M^4) \to (\mathbb{R}^6_+, \partial \mathbb{R}^6_+)$ is generic, it is nonsingular near the boundary, and $\tilde{g}^{-1}(\partial \mathbb{R}^6_+) = \partial M^4$. Here $\mathbb{R}^6_+$ is the closed half-space of $\mathbb{R}^6$. The set of double values of $g$ is an immersed oriented 2-manifold, denoted by $D(g)$. Its oriented boundary consists of two parts, the intersection of $D(g) \cap \partial \mathbb{R}^6_+$, and the other, disjoint with $\partial \mathbb{R}^6_+$, is the set of singular values $\Sigma(g)$ of $g$. Let $\Sigma'(g)$ be a copy of $\Sigma(g)$ shifted slightly along the outward normal vector field of $\Sigma(g)$ in $D(g)$. Then $\Sigma'(g) \cap g(M^4) = \emptyset$.

**Definition 2.2.11** ([9]). $l(g)$ denotes the linking number of $g(M^4)$ and $\Sigma'(g)$ in $(\mathbb{R}^6_+, \partial \mathbb{R}^6_+)$.

**Theorem 2.2.12** (Ekholm, Szűcs [9]). Let $f : S^3 \hookrightarrow \mathbb{R}^5$ be an immersion and $M^4$ be a compact oriented 4-manifold with boundary $\partial M^4 = S^3$. Let $\tilde{f} : (M^4, \partial M^4) \to (\mathbb{R}^6_+, \partial \mathbb{R}^6_+)$ be a generic map nonsingular near the boundary, such that $\tilde{f}^{-1}(\partial \mathbb{R}^6_+) = \partial M^4$ and $\tilde{f}|_{\partial M^4}$ is regular homotopic to $f$. Then

$$\Omega(f) = \pm \frac{1}{2}(3\sigma(M^4) + 3t(\tilde{f}) - 3l(\tilde{f}) + L(\tilde{f}|_{\partial M^4})).$$

### 2.3. Outline of other related results

#### 2.3.1. Immersions of $S^3$ to $\mathbb{R}^4$

Here we review a formula for the Smale invariant for immersions of $S^3$ to $\mathbb{R}^4$ [18, 11]. For some connections with our cases see Subsection 2.3.2 and Example 3.5.2.
The Smale invariant of an immersion \( f : S^3 \looparrowright \mathbb{R}^4 \) is an element of \( \pi_3(V_3(\mathbb{R}^4)) = \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z} \). The description of the generators of \( \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z} \) (usually denoted by \( \sigma \) and \( \rho \)) can be found in [66] or [11], and also in Section 3.2 of this thesis (where the first generator is denoted by \( L \) instead of \( \sigma \)).

The standard embedding \( i : \mathbb{R}^4 \hookrightarrow \mathbb{R}^5 \) induces an embedding \( \iota : SO(4) \hookrightarrow SO(5) \), and so a homomorphism \( \pi_3(\iota) : \pi_3(SO(4)) \rightarrow \pi_3(SO(5)) \). This homomorphism is \( \pi_3(\iota) : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \), \( (a, b) \mapsto a + 2b \) (cf. [66], [11] and Section 3.2). Thus for an immersion \( f : S^3 \looparrowright \mathbb{R}^4 \) with \( \Omega(f) = (a, b) \), the Smale invariant of the composition \( i \circ f : S^3 \looparrowright \mathbb{R}^5 \) is \( \Omega(i \circ f) = a + 2b \).

Hughes [18] expressed the Smale invariant in terms of an immersed Seifert surface as follows.

**Theorem 2.3.1 ([18, Theorem 3.1.]).** Let \( f : S^3 \looparrowright \mathbb{R}^4 \) be an immersion which admits an immersed Seifert surface, i.e. there is a compact oriented 4-manifold \( M^4 \) with boundary \( \partial M^4 \cong S^3 \) and an immersion \( F : M^4 \looparrowright \mathbb{R}^4 \) such that \( F|_{\partial M^4} = f \). Then

\[
\Omega(f) = \left( \chi(M^4) - 1, \frac{3\sigma(M^4) - 2(\chi(M^4) - 1)}{4} \right) \in \mathbb{Z} \oplus \mathbb{Z},
\]

where \( \chi(M^4) \) is the Euler characteristic, and \( \sigma(M^4) \) is the signature of \( M^4 \).

An immersion \( f : S^3 \looparrowright \mathbb{R}^4 \) induces a stable framing of \( S^3 \) via the bundle isomorphism \( E^1 \oplus TS^3 \cong f^*T\mathbb{R}^4 \), where \( E^1 \) denotes the trivial line bundle. Indeed, the Smale invariant measures the difference of this stable framing and the standard one (induced by the standard embedding). By [25] the homotopy class of the stable framing is characterized by two integers, the normal degree (also called the degree) \( D(f) \) of \( f \) and the Hirzebruch defect (also called signature defect) \( H(f) \). See for definitions [25, 5]. \( D(f) \) is described in detail in Example 2.1.14. Here we review some properties of \( H(f) \) which can be used as an alternative definition in our cases. When the stable framing extends to a framing of a compact oriented 4-manifold \( M^4 \) with boundary \( \partial M^4 \cong S^3 \), then \( D(f) = \chi(M^4) \) and \( H(f) = -3\sigma(M^4) \), thus (2.3.1) can be written in the form

\[
\Omega(f) = \left( D(f) - 1, \frac{-H(f) - 2(D(f) - 1)}{4} \right) \in \mathbb{Z} \oplus \mathbb{Z}.
\]

As it is proved in [11], the Smale invariant formula (2.3.2) holds for all immersions \( f : S^3 \looparrowright \mathbb{R}^4 \).

Note that the formulas (2.3.1) and (2.3.2) are compatible with (2.2.1). Indeed, by (2.3.1) and (2.3.2) the Smale invariant of \( i \circ f : S^3 \looparrowright \mathbb{R}^5 \) is \( \Omega(i \circ f) = -\frac{1}{2}H(f) = \frac{3}{2}\sigma(M^4) \), where \( M^4 \) is an immersed Seifert surface. See [11, 67].
For an arbitrary immersion \( f : S^3 \hookrightarrow \mathbb{R}^4 \) the Hirzebruch defect \( H(f) \) can be expressed in terms of a singular Seifert surface, i.e., a stable map \( F : M^4 \to \mathbb{R}^4 \) from a compact oriented 4-manifold with boundary \( \partial M^4 \cong S^3 \), such that \( F|_{\partial M^4} \) is regular homotopic with \( f \) and \( F \) is regular near the boundary. Such a map \( F \) has isolated \( \Sigma^{2,0} \)-points, also called umbilic points, i.e. the points where the rank of \( dF \) is 2, each of them can be equipped with a sign. Then
\[
H(f) = -3\sigma(M^4) - \sharp\Sigma^{2,0}(F),
\]
where \( \sharp\Sigma^{2,0}(F) \) is the algebraic number of the \( \Sigma^{2,0} \)-points. See [68, 11]. In [11, Theorem 2.7] the formula (2.3.3) is generalized for non-stable maps with isolated umbilic points. In this case each umbilic point can be equipped with an index, and if \( \sharp\Sigma^{2,0}(F) \) denotes the sum of the indices, (2.3.3) holds in the same form.

2.3.2. Immersions associated with plumbing graphs. Ekholm and Takase [11] and Kinjo [24] defined immersions associated with plumbing graphs, and calculated their Smale invariant by using the formulae (2.3.2) and (2.3.3). Here we summarize their results.

Let \( \Gamma \) be a plumbing graph with all genera 0 and even Euler numbers, see Subsection 4.2.2 or [56]. We denote by \( M^3(\Gamma) \) (resp. \( M^4(\Gamma) \)) the 3-manifold (resp. the 4-manifold with boundary) associated with \( \Gamma \). Note that \( \partial M^4(\Gamma) = M^3(\Gamma) \). Then \( \Gamma \) determines an immersion \( f_\Gamma : M^4(\Gamma) \hookrightarrow \mathbb{R}^4 \), and also \( f_\Gamma|_{M^3(\Gamma)} : M^3(\Gamma) \hookrightarrow \mathbb{R}^4 \) in the following way [11, 24].

If the algebraic number of the double values of a stable immersion \( S^2 \hookrightarrow \mathbb{R}^4 \) is \( n \), then the Euler number of its normal bundle is \((-2n\)). Therefore, for a vertex \( v \in \Gamma \) with genus \( g_v = 0 \) and Euler number \( e_v = -2n \) one associates a stable immersion \( S^2 \hookrightarrow \mathbb{R}^4 \) with \( n \) double values, and its tubular neighbourhood is the image of an immersion from the disc bundle over \( S^2 \) with Euler number \((-2n\)). Plumbing these immersions for each vertex according to \( \Gamma \) provides the immersion \( f_\Gamma \).

If \( M^3(\Gamma) \) is the quotient of \( S^3 \) by an action of a finite group \( G \), then the composition of the factorization \( S^3 \to S^3/G = M^3(\Gamma) \) and the immersion \( f_\Gamma|_{M^3(\Gamma)} : M^3(\Gamma) \hookrightarrow \mathbb{R}^4 \) defines an immersion \( g_\Gamma : S^3 \hookrightarrow \mathbb{R}^4 \).

In [11] \( \Gamma \) is the graph with one vertex and Euler number \( 2n \), and so \( M^3(\Gamma) \) is a lens space. The Smale invariant of the corresponding immersion \( g_n := g_\Gamma : S^3 \hookrightarrow \mathbb{R}^4 \) is
\[
(2.3.4) \quad \Omega(g_n) = (4n - 1, (n - 1)^2) \in \mathbb{Z} \oplus \mathbb{Z}.
\]

In [24] \( \Gamma \) is the weighted Dynkin diagram \( A_{n-1} \) (resp. \( D_{n+2} \)) with Euler number 2 of each vertex. So \( M^3(\Gamma) = L(n, 1) \) is a lens space (resp. \( M^3(\Gamma) = S^3/D_n \), where \( D_n \) is the binary dihedral group or also called dicyclic group). The Smale invariant of the
2.3. OUTLINE OF OTHER RELATED RESULTS

The corresponding immersion \( a_n := g_{A_{n-1}} : S^3 \looparrowright \mathbb{R}^4 \) (resp. \( d_n := g_{D_{n+2}} : S^3 \looparrowright \mathbb{R}^4 \)) is

\[
\Omega(a_n) = (n^2 - 1, 0) \in \mathbb{Z} \oplus \mathbb{Z}, \text{ resp.} \tag{2.3.5}
\]

\[
\Omega(d_n) = (4n^2 + 12n - 1, 0) \in \mathbb{Z} \oplus \mathbb{Z}. \tag{2.3.6}
\]

For any plumbing graph \( \Gamma \), the composition \( i \circ f_\Gamma|_{M^3(\Gamma)} : M^3(\Gamma) \looparrowright \mathbb{R}^5 \) (where \( i : \mathbb{R}^4 \looparrowright \mathbb{R}^5 \) is the standard embedding) can be deformed to an embedding by a regular homotopy in \( \mathbb{R}^5 \). Indeed, the composition \( S^2 \looparrowright \mathbb{R}^4 \looparrowright \mathbb{R}^5 \) can be deformed to an embedding by a regular homotopy, and this induces a regular homotopy of the tubular neighbourhood. Thus the immersions \( i \circ a_n \) and \( i \circ d_n \) have the same structure as the immersions associated with the covering maps \( (\mathbb{C}^2, 0) \to (X, 0) \subset (\mathbb{C}^3, 0) \) of the \( A_{n-1} \) and \( D_{n+2} \) singularities, namely \( S^3 \to S^3/G \looparrowright \mathbb{R}^5 \).

The immersions associated with the covering maps of the \( A_{n-1} \) and \( D_{n+2} \) quotient singularities have Smale invariant \(- (n^2 - 1)\), resp. \(- (4n^2 + 12n - 1)\), see Example 3.5.2. Therefore they are regular homotopic either with the immersion \( i \circ a_n \) (resp. \( i \circ d_n \)) or \( i \circ a_n \circ \tau \) (resp. \( i \circ d_n \circ \tau \)) from \( S^3 \) to \( \mathbb{R}^5 \), where \( i : \mathbb{R}^4 \looparrowright \mathbb{R}^5 \) is the standard embedding, and \( \tau : \mathbb{R}^4 \to \mathbb{R}^4 \) is the reflection \( \tau(x, y, z, w) = (x, y, z, -w) \). Note that precomposing with \( \tau \) changes the sign of the Smale invariant. But there are other differences between the two types of immersions as well, which may change the sign. In our construction \( S^3/G \) is the lens space \( L(n, n-1) \), which is diffeomorphic to \( L(n, 1) \) with reversed orientation. Indeed, the plumbing graph of \( S^3/G \) in our case is \( A_{n-1} \) (resp. \( D_{n+1} \)) with Euler numbers \(-2\).

Moreover the author of this thesis does not know the relation of the sign conventions for the Smale invariant used in [24] and here, see Section 3.2. It would be interesting to see a direct relation between these immersions, which applied to the \( E_6, E_7, E_8 \) graphs may provide the Smale invariant of the immersions associated with these graphs.

The immersion \( i \circ g_1 = i \circ a_2 : S^3 \to \mathbb{R}P^3 \looparrowright \mathbb{R}^5 \) was studied in several papers, e.g. [41, 7], and it has many connections with our work. Ekholm’s invariant \( L(i \circ g_1) \) is well-defined by Remark 2.2.7 as \( L \) of a small stable perturbation of \( i \circ g_1 \) with regular homotopy. A regular homotopy deforming \( i \circ g_1 \) to a stable immersion is given in [11] Section 4], and by the calculation from that article \( L(i \circ g_1) = 0 \) follows. Another stabilizing regular homotopy of \( i \circ g_1 \) (or \( i \circ g_1 \circ \tau \)) is given in Subsection 3.7.2 via the stabilization of the double covering map of the \( A_1 \) singularity, and \( L(i \circ g_1) = 0 \) follows too. Note that the number of the double curve components (Ekholm’s \( J \) invariant) is different in the two cases: the stable immersion given in [11] has one double curve component, while the one given in Subsection 3.7.2 has 3. The immersion associated with the corank–2 germ of Example 1.4.12 can also be considered as a stable version of \( i \circ g_1(\circ \tau) \) with 5 double curve
components, see Subsection 5.5.8. Indeed, \( C(\Phi) = 3 \) and \( T(\Phi) = 1 \) implies \( \Omega(\Phi|_{S^3}) = -3 \) and \( L(\Phi|_{S^3}) = 0 \), cf. 3.1.5 3.1.2.

2.3.3. Immersions of 3-manifolds. This subsection contains a summary of the immersions of arbitrary 3-manifolds. We mention here some results about their characterization up to regular homotopy according to 73, 60, and up to cobordism according to 18, 68.

Since every closed oriented 3-manifold is parallelizable, a choice of a trivialization \( \tau = (v_1, v_2, v_3) \) of the tangent bundle \( TM^3 \) determines a bijection between \( \text{Imm}(M^3, \mathbb{R}^q) \) and \( [M^3, V_3(\mathbb{R}^q)] \), which associates with an immersion \( f : M^3 \to \mathbb{R}^q \) the map \( p \mapsto df_p(\tau) \), cf. Theorem 2.1.2 If \( q = 4 \),

\[ [M^3, V_3(\mathbb{R}^4)] \cong [M^3, SO(4)] \cong [M^3, S^3 \times SO(3)] \cong [M^3, S^3] \times [M^3, SO(3)] \]

holds. Note that the degree (resp. the degree and the induced homomorphism between the fundamental groups) completely characterises the maps from \( M^3 \) to \( S^3 \) (resp. the maps from \( M^3 \) to \( SO(3) \)) up to homotopy.

Consider an immersion \( f : M^3 \to \mathbb{R}^4 \). By 73 the corresponding invariant in \( \mathbb{Z} \times \mathbb{Z} \times H^1(M^3, \mathbb{Z}_2) \) can be identified as follows.

The orientation determines a normal framing \( u \) of the immersion \( f \). The first integer is the normal degree \( D(f) \) of \( f \), i.e. the degree of the Gauss map \( M^3 \to S^3, p \mapsto u(p) \), cf. Example 2.1.14 and (2.3.1).

In \( \mathbb{R}^4 \cong \mathbb{H} \) any vector \( v \) determines a basis \( (v, iv, jv, kv) \), where \( (i, j, k) \) is the canonical basis of the pure imaginary quaternions. \( df_p(\tau) \) and \( (iu(p), ju(p), ku(p)) \) are two bases of \( u^+ \subset \mathbb{R}^4 \), and the transition matrix between them defines a map \( M^3 \to GL^+(3, \mathbb{R}) \), which composed with the Gram-Schmidt process \( GL^+(3, \mathbb{R}) \to SO(3) \) provides the map \( \beta_f : M^3 \to SO(3) \). The degree \( b_f \) of \( \beta_f \) is the second integer. The third invariant is the cohomology class \( h_f \) induced by the homomorphism \( \pi_1(\beta_f) : \pi_1(M^3) \to \pi_1(SO(3)) \cong \mathbb{Z}_2 \).

**Theorem 2.3.2 (73 Theorem 1).** The map \( \text{Imm}(M^3, \mathbb{R}^4) \to \mathbb{Z} \times \mathbb{Z} \times H^1(M^3, \mathbb{Z}_2) \) induced by the correspondence \( f \mapsto (D(f), b_f, h_f) \) is a bijection.

In the \( q = 5 \) case let us fix a generator \( g \in H^2(V_3(\mathbb{R}^5), \mathbb{Z}) \) such that \( 2g \) is the Euler-class of the \( S^1 \)-bundle \( SO(5) \to V_3(\mathbb{R}^5) \). Then the Wu-invariant of an immersion \( f : M^3 \to \mathbb{R}^5 \) with respect to the trivialization \( \tau = (v_1, v_2, v_3) \) of \( TM^3 \) is the element \( c(f) := (df(\tau))^*(g) \in H^2(M^3, \mathbb{Z}) \), cf. 60, Definition 3.3.]. The normal Euler class of \( f \) is equal to \( 2c(f) \) by 73, Theorem 2, see also 60, Theorem 3.1., Remark 3.2.

Now we restrict the discussion to the regular homotopy classes of immersions with trivial normal bundle, denoted by \( \text{Imm}(M^3, \mathbb{R}^5)_0 \). The normal Euler class of such an
immersion $f$ is 0, hence $c(f)$ is an order–2 element in $H^2(M^3, \mathbb{Z})$. Let $\Gamma_2(M^3) \subset H^2(M^3, \mathbb{Z})$ denote the set of the order–2 elements. Let $M^3_0 = M^3 \setminus B^3$ denote the non-closed manifold obtained from $M^3$ by removing a 3-ball.

**Theorem 2.3.3.** (a) [60] pg. 5] $f \mapsto c(f)$ induces a bijection between $\text{Imm}(M^3_0, \mathbb{R}^5)_0$ and $\Gamma_2(M^3)$.

(b) [60] pg. 5, 9] There is an integer valued regular homotopy invariant $i(f)$ such that the correspondence $f \mapsto (c(f), i(f))$ induces a bijection between $\text{Imm}(M^3, \mathbb{R}^5)_0$ and $\Gamma_2(M^3) \times \mathbb{Z}$.

(c) [60, Theorem 3.8.] For every element $C \in \Gamma_2(M^3)$ there exists an embedding $g : M^3 \hookrightarrow \mathbb{R}^5$ with $c(g) = C$.

The invariant $i(f)$ can be defined via a singular Seifert surface, it is the analogue of the Smale invariant via formulae (2.2.7) and (2.2.8).

Introduce the notation $\alpha(M^3) := \dim_{\mathbb{Z}}(\tau H_1(M^3, \mathbb{Z}) \otimes \mathbb{Z}_2)$ where $\tau H_1(M^3, \mathbb{Z})$ is the torsion subgroup of $H_1(M^3, \mathbb{Z})$. Any immersion $f : M^3 \hookrightarrow \mathbb{R}^5$ admits a normal framing $\nu$, however it is not unique up to homotopy. Define $L_\nu(g)$ as (2.2.5) for a stable immersion $g : M^3 \hookrightarrow \mathbb{R}^5$ with a fixed normal framing $\nu$.

Let $V^4$ be a compact oriented 4-manifold with boundary $M^3$. Let $\tilde{f} : V^4 \to \mathbb{R}^5$ be a generic map such that $\tilde{f}\mid_{\partial V^4}$ is regular homotopic to $f : M^3 \hookrightarrow \mathbb{R}^5$ and $\tilde{f}$ has no singular points near the boundary. Define

$$i_a(f) = \frac{3}{2}(\sigma(V^4) - \alpha(M^3)) + \frac{1}{2}\#\Sigma^{1,1}(\tilde{f})$$

(2.3.7)

Let $\tilde{f} : (V^4, \partial V^4) \to (\mathbb{R}^6_+, \partial \mathbb{R}^6_+)$ be a generic map nonsingular near the boundary, such that $\tilde{f}^{-1}(\partial \mathbb{R}^6_+) = \partial V^4$ and $\tilde{f}\mid_{\partial V^4}$ is regular homotopic to $f$. Define

$$i_b(f) = \frac{3}{2}(\sigma(V^4) - \alpha(M^3)) + \frac{1}{2}(3t(\tilde{f}) - 3l(\tilde{f}) + L_\nu(\tilde{f}\mid_{\partial V^4}))$$

(2.3.8)

**Theorem 2.3.4** (Lemmas 5.5., 5.7., theorems 5.6., 5.8. in [60]). (a) $i_a(f)$ is an integer, and it does not depend on the choice of $V^4$ and $\tilde{f}$. It depends only on the regular homotopy class of $f$.

(b) $i_b(f)$ is an integer, and it does not depend on the choice of $V^4$, $\tilde{f}$ and $\nu$. It depends only on the regular homotopy class of $f$.

(c) $i_a(f) = i_b(f)$.

(d) Two immersions $f, g : M^3 \hookrightarrow \mathbb{R}^5$ with $c(f) = c(g)$ are regular homotopic if and only if $i(f) = i(g)$, where $i(f) := i_a(f) = i_b(f)$.

Cobordism of immersions is an equivalence relation between immersions of different manifolds. Two immersions $f : M^n \hookrightarrow \mathbb{R}^q$ and $g : N^n \hookrightarrow \mathbb{R}^q$ of closed (resp. closed,
oriented) manifolds \( M^n \) and \( N^n \) are cobordant (resp. oriented cobordant) if there is a compact manifold \( V^{n+1} \) with boundary \( \partial V^{n+1} \simeq M^n \sqcup N^n \) (resp. a compact oriented manifold \( V^{n+1} \) with oriented boundary \( \partial V^{n+1} \simeq M^n \sqcup -N^n \), where \( -N^n \) denotes \( N^n \) with opposite orientation) and an immersion \( F : V^{n+1} \hookrightarrow \mathbb{R}^q \times [0,1] \) such that \( F|_{M^n} = f \) and \( F|_{N^n} = g \). Let \( \text{Imm}(n,k) \) (resp. \( \text{Imm}^{SO}(n,k) \)) denotes the cobordism classes of the immersions of \( n \)-manifolds to \( \mathbb{R}^{n+k} \) (resp. the oriented cobordism classes of the immersions of oriented \( n \)-manifolds to \( \mathbb{R}^{n+k} \)). \( \text{Imm}(n,k) \) and \( \text{Imm}^{SO}(n,k) \) form groups with respect to the disjoint union, which agrees with the connected sum up to (oriented) cobordism.

\[ \text{Imm}^{SO}(3,1) \cong \pi^s(3) \cong \mathbb{Z}_{24}, \] where \( \pi^s(3) \) denotes the third stable homotopy group of spheres [67, Lemma 1.4], [68]. Each class in \( \text{Imm}^{SO}(3,1) \) can be represented by an immersion of \( S^3 \) via the following diagram, cf. [67, Lemma 1.8], [68, pg. 41, 43].

\[
\begin{align*}
\text{Imm}(S^3, \mathbb{R}^4) & \rightarrow \text{Imm}(S^3, \mathbb{R}^5) \xrightarrow{J} \text{Imm}^{SO}(3,1) \\
\pi_3(SO(4)) & \rightarrow \pi_3(SO(5)) \xrightarrow{J} \pi^s(3) \\
\mathbb{Z} \oplus \mathbb{Z} & \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_{24} \\
(a,b) & \mapsto a + 2b \pmod{24}
\end{align*}
\]

(2.3.9)

The homomorphism on the right of the diagram is called \( J \)-homomorphism, its kernel is \( \text{Emb}(S^3, \mathbb{R}^5) \), cf. Theorem 2.2.2 [17, 67]. There are also geometric formulae to determine the oriented cobordism class of an immersion \( M^3 \hookrightarrow \mathbb{R}^4 \), see [68, Theorem 3.1., Remark 3.6.].

By [18, Theorem 4.1.] \( \text{Imm}(3,2) \cong \text{Imm}^{SO}(3,2) \cong \mathbb{Z}_2 \), and the cobordism class of a stable immersion of \( M^3 \) to \( \mathbb{R}^5 \) is completely determined by its total twist, which is the parity of the number of non-trivially covered double point curve components. In particular, the total twist of a stable immersion \( S^3 \hookrightarrow \mathbb{R}^5 \) is a regular homotopy invariant, it defines a homomorphism \( \text{Imm}(S^3, \mathbb{R}^5) \rightarrow \mathbb{Z}_2 \). From this fact we conclude a correspondence between analytic invariants in Subsection 5.1.3.

### 2.3.4. Regular homotopy class of the embedded link.

We mention here two results about a very similar topic to the material discussed in Chapter 3. The articles [10] and [23] study links of isolated hypersurface singularities (cf. Chapter 4) up to regular homotopy.

Although [10] contains a generalization of the Hughes-Melvin theorem [2.2.1] and of singular Seifert surface formulae [2.2.7] and [2.2.8] for immersions and embeddings of homotopy \((4k-1)\)-spheres, here we only concentrate to [10, Section 5.1.]. Consider the Brieskorn equations \( f_k(z) = z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 \), which determine isolated singularities...
(X_k, 0) = f_k^{-1}(0) \subset (\mathbb{C}^5, 0)$. Their links $K_k = X \cap S^9_\epsilon \simeq \Sigma_k^7$ are homotopy spheres (also called exotic spheres): $\Sigma_k^7$ is homeomorphic, but not diffeomorphic to $S^7$ for all $k \neq 28n$. $\Sigma_{28}^7$ is diffeomorphic to $S^7$, and $\Sigma_k^7$ is diffeomorphic to $\Sigma_l^7$ if and only if $k \equiv l \pmod{28}$.

By [10], the embeddings of $\Sigma_{28n+k}^7$ into $S^9$ are not regular homotopic for different values of $n$, and they represent all the regular homotopy classes of the immersions which contain embeddings, i.e. $\text{Emb}(\Sigma_k^7, S^9) \subset \text{Imm}(\Sigma_k^7, S^9)$. In particular, $k = 28n$ provide embeddings $S^7 \hookrightarrow S^9$ which are not regular homotopic to the standard embedding, each appears as the link of a complex hypersurface singularity. In contrast with this fact, nonstandard embeddings $S^3 \hookrightarrow \mathbb{R}^5$ cannot be realized as links of surface singularities in $\mathbb{C}^3$, cf. Theorem [3.1.3]

A similar question is studied in [23] about the link of $(X_k, 0) = f_k^{-1}(0) \subset (\mathbb{C}^4, 0)$, where $f_k(x, y, z, w) = x^2 + y^2 + z^2 + w^k$. Its link depends on the parity of $k$, that is

$$K_k = X_k \cap S^7_\epsilon \simeq \begin{cases} S^5 & \text{if } k = 2d + 1, \\ S^3 \times S^2 & \text{if } k = 2d. \end{cases}$$

The question is that in the case of links with different equations but with the same diffeomorphism type do they represent different regular homotopy classes?

For $k$ odd, $\text{Imm}(S^5, S^7) = \pi_5(SO(7)) = 0$ (see [2.2.3]) by Bott periodicity [2]. Thus the question is irrelevant in this case.

In the other case $k = 2d$ it turns out that the question is not well-posed: precomposing an immersion $f : S^3 \times S^2 \hookrightarrow S^7$ with a self-diffeomorphism of $S^3 \times S^2$ can change the regular homotopy type of $f$. In other words the regular homotopy class of the inclusion $K_{2d} \subset S^7$ depends on the choice of the identification of $K_{2d}$ with $S^3 \times S^2$.

A well-defined notion is the image regular homotopy type: the immersions $f$ and $g$ $S^3 \times S^2 \hookrightarrow S^7$ are image regular homotopic if there is a self diffeomorphism $\phi$ of $S^3 \times S^2$ such that $g$ is regular homotopic with $f \circ \phi$. By [23] there are two image regular homotopy classes of immersions $S^3 \times S^2 \hookrightarrow S^7$, and the image regular homotopy class of the inclusion $K_{2d} \subset S^7$ depends on the parity of $d$. 

By
CHAPTER 3

Immersions associated with holomorphic germs

3.1. Summary of the results

3.1.1. This chapter includes the results of the paper [53]. Our main goal is to analyse the complex analytic realizations of the elements of the groups Imm(S³, S⁵) and Emb(S³, S⁵). Recall that Imm(S³, S⁵) ∼= π₃(SO(5)) ∼= Z via the Smale invariant. We use (2.2.3) for its definition throughout Chapter 3.

Let Φ : (C², 0) → (C³, 0) be a holomorphic germ. We assume that Φ is singular only at the origin, that is \{z : \text{rk}(dΦ_z) < 2\} ⊂ \{0\} in a small representative of (C², 0). Such a germ, at the level of links of the spacegerms (C², 0) and (C³, 0), provides an immersion f : S³ ↪ S⁵, cf. Definition 1.1.3 and Theorem 1.4.6. If an element of Imm(S³, S⁵), or Emb(S³, S⁵) respectively, can be realized (up to regular homotopy) by such an immersion, we call it holomorphic. The corresponding subsets will be denoted by Imm_{hol}(S³, S⁵) and Emb_{hol}(S³, S⁵) respectively.

As we will see, Imm_{hol}(S³, S⁵) is not symmetric with respect to a sign change of Z, hence, in order to identify the subset Imm_{hol}(S³, S⁵) without any sign-ambiguity, we will fix a ‘canonical’ generator of π₃(SO(5)). This will be done via the isomorphisms π₃(U(3)) → π₃(SO(6)) → π₃(SO(5)) and by fixing a canonical generator in π₃(U(3)) (see 3.2.3). Sometimes, to emphasize that we work with the Smale invariant with this fixed sign convention, we refer to it as the sign–refined Smale invariant. Our second goal is to determine the correct signs (compatibly with the above choice of generators) in the Smale invariant formulas (2.2.2) and (2.2.7), (2.2.8), which were stated only up to a sign–ambiguity.

3.1.2. The set Imm_{hol}(S³, S⁵). One expects that the analytic geometry of holomorphic realization imposes some rigidity restrictions, and also provides some further connections with the properties of complex analytic spaces. Mumford already in 1961 in his seminal article [50] asked for the characterization of the Smale invariant of a holomorphic (algebraic) immersion in terms of the analytic/algebraic geometry. This chapter provides a complete answer to his question. A more precise formulation of our guiding questions is:
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**Question 3.1.1.** (a) Which are the regular homotopy classes \( \text{Imm}_{\text{hol}}(S^3, S^5) \) and \( \text{Emb}_{\text{hol}}(S^3, S^5) \) represented by holomorphic germs?

(b) How can a certain regular homotopy class be identified via complex singularity theory, that is, via algebraic or analytic invariants of the involved analytic spaces? Furthermore, if some \( \Phi \) realizes some Smale invariant (e.g., if its Smale invariant is zero), then what kind of specific analytic properties \( \Phi \) must have?

The main results of this chapter provide the following answer in the case of immersions.

**Theorem 3.1.2.** (a) \( \text{Imm}_{\text{hol}}(S^3, S^5) \) is identified via the sign–refined Smale invariant \( \Omega(f) \) by the set of non–positive integers.

(b) If the immersion \( f \) is induced by the holomorphic germ \( \Phi \), then \( \Omega(f) = -C(\Phi) \), where \( C(\Phi) \) is the number of complex Whitney umbrella points of a stabilization of \( \Phi \).

\( C(\Phi) \) can be calculated in an algebraic way, as the codimension of the ideal generated by the determinants of the \( 2 \times 2 \)-minors of the Jacobian matrix of \( \Phi \). Cf. Subsection 1.4.2.

The main tool of the proof of Theorem 3.1.2 is the concept of complex Smale invariant of the germ \( \Phi \). We introduce it in Section 3.3 and then we prove that it agrees with \( C(\Phi) \). Next, in Section 3.4 we identify the complex Smale invariant of a germ \( \Phi \) with the (classical) Smale invariant of the link of \( \Phi \). The proof of the part (b) of Theorem 3.1.2 is then ready up to sign. In 3.2.3 we fix explicit generators of the groups \( \pi_3(U) \) and \( \pi_3(SO) \) and calculate the homomorphism between them. With this convention the complex Smale invariant of \( \Phi \) is equal to \( C(\Phi) \) and is opposite to the sign–refined Smale invariant.

Part (b) of Theorem 3.1.2 implies that the sign–refined Smale invariant of a complex analytic realization is always non–positive. The proof of part (a) is then completed by Example 3.5.1 which provides analytic representatives for all non–positive \( \Omega(f) \).

Note that in the present literature the known (\( C^\infty \)) realizations of certain Smale invariants \( \Omega(f) \) are rather involved (similarly, as the computation of \( \Omega(f) \) for any concrete \( f \)), see e.g. [18, 7]. Here we provide very simple polynomial maps realizing all non–positive Smale invariants. Furthermore, the computation of \( C(\Phi) \) for any \( \Phi \) is extremely simple.

Moreover, precomposing the above complex realizations with the \( C^\infty \) reflection \( (s, t) \mapsto (s, \bar{t}) \), we get explicit representatives for all positive Smale numbers as well, compare [7, Lemma 3.4.2]. In this way Theorem 3.1.2 together with Example 3.5.1 provides an answer to Smale’s question 2.1.8.

**3.1.3. The set** \( \text{Emb}_{\text{hol}}(S^3, S^5) \). Recall that \( \text{Emb}_{\text{hol}}(S^3, S^5) \) consists of regular homotopy classes (that is, sign–refined Smale invariants in \( \mathbb{Z} \)) represented by holomorphic germs \( \Phi \) whose induced immersions \( S^3 \hookrightarrow S^5 \) might not be embeddings, but are regular homotopic with embeddings.
3.1. SUMMARY OF THE RESULTS

A more restrictive subset consists of those regular homotopy classes (Smale invariants), which can be represented by holomorphic gems, whose restrictions off origin are embeddings. Cf. Hughes–Melvin theorem 2.2.1.

Theorem 3.1.3. (a) \( \text{Emb}_{hol}(S^3, S^5) = (24 \cdot \mathbb{Z}) \cap \mathbb{Z}_{\leq 0} \).

(b) Assume that the immersion \( f \) is the restriction at links level of a holomorphic germ \( \Phi \) as above, \( f = \Phi |_{S^3} \). Then the following facts are equivalent:

1. \( \text{rk}(d\Phi_0) = 2 \) (hence \( \Phi \) is not singular),
2. \( \Omega(f) = 0 \),
3. \( f : S^3 \hookrightarrow S^5 \) is an embedding,
4. \( f : S^3 \rightarrow S^5 \) is the trivial embedding.

Again, we wish to emphasize that the previous construction of the generator of \( 24 \cdot \mathbb{Z} = \text{Emb}(S^3, S^5) \) (that is, of a smooth embedding with \( \Omega(f) = \pm 24 \)) is complicated, it is more existential than constructive [17]. On the other hand, by our complex realizations, for any given \( \Omega(f) \in 24 \cdot \mathbb{Z} \) we provide several easily defined germs, which are immersions, and are regular homotopic with embeddings. Moreover, part (b) says that it is impossible to find holomorphic representatives \( \Phi \) such that \( \Phi |_{S^3} \) is already embedding (except for \( \Omega(f) = 0 \)).

The essential parts of Theorem 3.1.3(b) are the implications (2) \( \Rightarrow \) (1) and (3) \( \Rightarrow \) (1), which conclude an analytic statement from topological ones. The proof (2) \( \Rightarrow \) (1) is based on Theorem 3.1.2, which recovers the vanishing of the analytic invariant \( C(\Phi) \) from the ‘topological vanishing’ \( \Omega(f) = 0 \).

A possible proof of \( (\Phi |_{S^3} \text{ embedding}) \Rightarrow (\text{rk}(d\Phi_0) = 2) \) is based on a deep theorem of Mumford, which says that if the link of a complex normal surface singularity is \( S^3 \) then the germ should be non–singular [50]. We will provide two other possible proofs too: one of them is based on Mond’s Theorem 1.4.3, the other on a theorem of Ekholm–Szűcs 2.2.12.

3.1.4. The literature of singular analytic germs \( \Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0) \) is huge with several deep and interesting results and invariants, see e.g. the articles of D. Mond and V. Goryunov [13, 14, 15, 44, 45, 46] and the references therein, or Chapter 1 of this thesis. In singularity classifications finitely determined or finite codimensional germs are central (with respect to some equivalence relation). For germs \( \Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0) \) Mond proved that the finite \( A \)-determinacy is equivalent with the finiteness of three invariants \( C(\Phi), \ T(\Phi) \) and \( N(\Phi) \), see [45] or Subsection 1.4.2. This is more restrictive than our assumption \( \{ z : \text{rk}(d\Phi_z) < 2 \} \subset \{ 0 \} \), which requires the finiteness of \( C(\Phi) \) only, cf. Theorem 1.4.6.

However, it is advantageous to consider this larger class, since there are many key families of germs with infinite right-left codimension, but with finite \( C(\Phi) \), and they produce interesting connections with other areas as well (see e.g. the next example).
Example 3.1.4. Consider a simple hypersurface singularity \((X, 0) \subset (\mathbb{C}^3, 0)\) (that is, of type A–D–E). They are quotient singularities, that is \((X, 0) \simeq (\mathbb{C}^2, 0)/G\) for certain finite subgroup \(G \subset GL(2, \mathbb{C})\), cf. examples 1.3.5 1.4.14 1.4.15 1.4.17. Let \(K\) be the link of \((X, 0)\) (e.g., it is a lens space for A-type), and consider the regular \(G\)-covering \(S^3 \to K\). This composed with the inclusion \(K \hookrightarrow S^5\) provides an immersion \(S^3 \↬ S^5\). Hence, the universal cover of each A–D–E singularity automatically provides an element of \(\text{Imm}_{\text{hol}}(S^3, S^5)\), which usually have infinite right–left codimension. The corresponding Smale invariants are given in Section 3.5. E.g., \(-\Omega(A_{n-1}) = n^2 - 1\), hence \(A_4\) represents (up to regular homotopy) a generator of \(24 \cdot \mathbb{Z} = \text{Emb}(S^3, S^5)\).

Recently Kinjo, using the plumbing graphs of the links of A–D singularities and \(C^\infty\)-techniques, constructed immersions with the same Smale invariants as our \(-C(\Phi)\) up to sign. See [24] or the discussion in Subsection 2.3.2. Hence, the natural complex analytic maps \((\mathbb{C}^2, 0) \to (X, 0) \subset (\mathbb{C}^3, 0)\) provide analytic realizations of the \(C^\infty\) constructions of [24], and emphasize their distinguished nature.

3.1.5. Smale invariants and the geometry of Seifert surfaces. In Section 2.2 we reviewed three major topological theorems, which recover the classical Smale invariant in terms of the geometry of their (singular) Seifert surfaces, namely the Hughes–Melvin Theorem 2.2.2, and two theorems of Ekholm–Szűcs [9], (2.2.7) and (2.2.8). All of them carry the sign ambiguity of the Smale invariant (which sometimes is also caused by the nature of their proofs).

Section 3.6 has two goals. First, we will indicate the correct sign in all these formulae, whenever the Smale invariant is replaced by the sign–refined Smale invariant. Moreover, we also determine the Seifert type invariants in terms of \(C(\Phi)\) and \(T(\Phi)\), whenever the immersion is induced by a holomorphic germ \(\Phi\).

When \(f\) is a stable immersion, the invariant \(L(f)\) of stable immersions introduced by Ekholm (cf. Subsection 2.2.2) is also expressed in terms of \(C(\Phi)\) and \(T(\Phi)\), namely \(L(f) = C(\Phi) - 3T(\Phi)\). In other words, Ekholm’s ‘strangeness’ invariant \(\text{St}(f)\) is equal to \(-T(\Phi)\), see Remark 2.2.8.

From this point of view our results can be considered as complex singular Seifert surface formulae expressing the invariants of an immersion. The disentanglement (see 1.4.4) plays the role the singular Seifert surface.

3.1.6. \(C^\infty\)-characterisation of \(C(\Phi)\) and \(T(\Phi)\). The formulae connecting the holomorphic invariants \(C(\Phi)\) and \(T(\Phi)\) with \(C^\infty\)-invariants \(\Omega(f)\) and \(L(f)\) have the following consequence.
Theorem 3.1.5. Assume that the analytic germs $\Phi$ and $\Phi'$ : $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ are $C^\infty$ $\mathcal{A}$-equivalent (that is, $\Phi' = \Lambda \circ \Phi \circ \psi$ holds for some germs of orientation preserving diffeomorphisms $\psi : (\mathbb{R}^4,0) \to (\mathbb{R}^4,0)$ and $\Lambda : (\mathbb{R}^6,0) \to (\mathbb{R}^6,0)$). Then

(a) $C(\Phi) = C(\Phi')$.

(b) If additionally $\Phi$ is finitely $\mathcal{A}$-determined as a holomorphic germ, then $T(\Phi) < \infty$ and $T(\Phi) = T(\Phi')$.

Part (a) of Theorem 3.1.5 is proved as Corollary 3.4.2, while part (b) in Remark 3.6.4.

In fact, one might even ask for the topological analogue of Theorem 3.1.5: is it true that if $\Phi' = \Lambda \circ \Phi \circ \psi$ holds for some germs of orientation preserving homeomorphisms $\psi : (\mathbb{R}^4,0) \to (\mathbb{R}^4,0)$ and $\Lambda : (\mathbb{R}^6,0) \to (\mathbb{R}^6,0)$, then $C(\Phi) = C(\Phi')$ and $T(\Phi) = T(\Phi')$? In Remark 3.6.4 we show the following.

Corollary 3.1.6. In the presence of a topological left–right equivalence as above, if $\Phi$ and $\Phi'$ are finitely $\mathcal{A}$-determined, then $L(f) = L(f')$, hence $C(\Phi) - 3T(\Phi) = C(\Phi') - 3T(\Phi')$.

The full extension of Theorem 3.1.5 from $C^\infty$ to topological category is obstructed by the following facts. Though we identify the analytic invariant $C(\Phi)$ with the smooth Smale invariant $\Omega(f)$, it is not yet known if $\Omega(f)$ is stable with respect to topological left–right equivalence. A possible way to prove this requires the extension of the formulae from Section 2.2.3 from smooth to more general Seifert surfaces, which exceeds the aims of the present thesis. (Also, we do not know how the topological left–right equivalence behaves with respect to analytic deformations used in Section 3.6.)

3.2. Distinguished generators and sign conventions

3.2.1. Smale invariant and orientation. Recall that we use (2.2.3) for definition of the Smale invariant throughout Chapter 3. We wish to emphasize the following facts regarding orientations of $S^3$ and $\mathbb{R}^5$ and their effects on definition (2.2.3). (This might serve also as a small guide for the next sections.)

Let us think about $S^3$ as the subset of $\mathbb{R}^4$, the boundary of the 4–ball $B^4$ in $\mathbb{R}^4$, or via embedding $\mathbb{R}^4 \subset \mathbb{R}^5$, as a subset of $\mathbb{R}^5$. We do not wish to fix any orientation on it as the orientation of $\partial B^4$ (that would depend on the convention how one defines the orientation of the boundary of an oriented manifold — called, say, ‘boundary convention’).

Note that in the definition (2.2.3) of the Smale invariant, not the orientation of $S^3$ is used, but the orientation of the tubular neighbourhood $U \subset \mathbb{R}^5$ and the orientation of the target $\mathbb{R}^5$. Moreover, $\Omega(f)$ is unsensitive to the orientation change simultaneously in both $\mathbb{R}^5$. In this way we get an element $\Omega(f) \in [S^3, SO(5)]$, which is independent of the
orientation of \( \mathbb{R}^5 \) and does not use any orientation of \( S^3 \). Furthermore, if we define (this will done in 3.2.3) a generator \([L]\) in \([S^3, SO(5)]\), using again only the embedding \( S^3 \subset \mathbb{R}^5 \) (and no other orientation data), then \( \Omega(f) \) identifies with an element of \( \mathbb{Z} \), such that its definition is independent of any orientations of \( S^3 \) and \( \mathbb{R}^5 \), hence also of the ‘boundary convention’.

All our discussions are in this spirit (except sections 2.2.3 and 3.6 where oriented Seifert surfaces are treated): we run orientation and ‘boundary convention’ free definitions and statements (associated with \( S^3 \), regarded as a subset of \( \mathbb{R}^5 \), and immersions \( S^3 \hookrightarrow \mathbb{R}^5 \)).

However, if we fix a ‘boundary convention’, then \( S^3 \) (in \( \mathbb{R}^5 \)) will get an orientation (as \( \partial B^4 \)). Then, for any oriented abstract \( S^3 \), let us denote it by \( S^3 \), and immersion \( S^3 \hookrightarrow \mathbb{R}^5 \), we can define the Smale invariant \( \Omega^{\text{a}}(f) \in \mathbb{Z} \) (here ‘a’ refers to the ‘abstract’ \( S^3 \)) by identifying \( S^3 \) with the embedded \( S^3 \subset \mathbb{R}^5 \) by an orientation preserving diffeomorphism and taking \( \Omega(S^3 \to S^3 \hookrightarrow \mathbb{R}^5) \). This \( \Omega^{\text{a}}(f) \) depends on the ‘boundary convention’, since the identification \( S^3 \to S^3 \) depends on it: changing the convention we change \( \Omega^{\text{a}}(f) \) by a sign.

This point of view should be adapted when \( S^2 \) will be the (oriented) boundary of an oriented Seifert surface. But till Section 3.6 we will focus on the first version, \( \Omega(f) \).

Next, in the definition of \( \Omega(f) \), one can replace \( \mathbb{R}^5 \) by \( S^5 \), where \( S^5 \) is the boundary of the ball in \( \mathbb{R}^6 \), and \( S^3 \) is embedded naturally in \( S^5 \), cf. Example 2.1.14. By taking a generic point \( P \in S^5 \) we identify \( S^5 \setminus \{P\} \) with \( \mathbb{R}^5 \), and \( U \) will be replaced by a tubular neighbourhood of \( S^3 \) in \( S^5 \). Then the definition (2.2.3) of \( \Omega(f) \) can be repeated for any immersion \( S^3 \hookrightarrow S^5 \) (where \( S^3 \subset S^5 \)) providing an element \([S^3, SO(5)]\), which becomes an integer once a generator \([L]\) is constructed from the embedding \( S^3 \subset S^5 \). Again, this Smale invariant \( \Omega(f) \) will be independent of the orientations of \( S^3 \) and \( S^5 \), hence of the ‘boundary convention’ as well.

For immersions defined in Subsection 1.1.2 \( \mathcal{G}^3 \) evidently sits naturally in \( \mathbb{C}^2 = \mathbb{R}^4 \) (hence also in a certain \( \mathcal{G}^5 = S^5 \subset \mathbb{C}^3 = \mathbb{R}^6 \), cf. 3.4.1). This together with Definition (2.2.3) provide \( \Omega(f) \) (which becomes an integer once \([L]\) will be constructed in 3.2.3).

### 3.2.2. Switch complex to real.

There is a natural map \( \tau \) from \( \text{Hom}(\mathbb{C}^3, \mathbb{C}^3) \) to \( \text{Hom}(\mathbb{R}^6, \mathbb{R}^6) \), which replaces any entry \( M_{ij} = a + bi \) of a matrix \( M \in \text{Hom}(\mathbb{C}^3, \mathbb{C}^3) \) by the real \( 2 \times 2 \)-matrix \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \). \( \tau(U(3)) \subset SO(6) \) holds. A map \( F : \mathbb{C}^3 \to \mathbb{C}^3 \) can be regarded as a map \( \tilde{F} : \mathbb{R}^6 \to \mathbb{R}^6 \): if we denote by \( z_j = x_j + iy_j \) \((j = 1, 2, 3)\) the coordinates of \( \mathbb{C}^3 \), then for the components of \( F \) and \( \tilde{F} \) one has

\[
F_j(z_1, z_2, z_3) = \tilde{F}_{2j-1}(x_1, y_1, x_2, y_2, x_3, y_3) + i\tilde{F}_{2j}(x_1, y_1, x_2, y_2, x_3, y_3).
\]
Then \( \tau(d_C F) = d_\mathbb{R} \tilde{F} \) holds for the complex Jacobian of \( F \) and the real Jacobian of \( \tilde{F} \).

Let \( j : SO(5) \hookrightarrow SO(6) \) denote the inclusion. It is well-known (see e.g. [19]) that

(3.2.1) \[ \pi_3(j) : \pi_3(SO(5)) \to \pi_3(SO(6)) \] is an isomorphism.

**Proposition 3.2.1.** The homomorphism \( \pi_3(\tau) : \pi_3(U(3)) \to \pi_3(SO(6)) \) is an isomorphism too.

**Proof.** First, we provide a more conceptual proof, which does not identify distinguished generators. Both sides are in the stable range (see [19]), hence we can switch to the homomorphism \( \pi_3(U) \to \pi_3(O) \) induced by the embedding \( \tau : U \hookrightarrow O \). By (a proof of) Bott periodicity, the quotient \( O/U \) is homotopically equivalent to the loopspace \( \Omega O \) of \( O \), cf. [2]. Hence \( \pi_i(O/U) = \pi_i(\Omega O) = \pi_{i+1}(O) = 0 \) for \( i = 3 \) and \( 4 \). Then the isomorphism follows from the homotopy exact sequence of the fibration \( O \to O/U \) with fibre \( U \). \( \Box \)

In 3.2.3 we will give another, more computational proof, where we will be able to fix distinguished generators for \( \pi_3(U(3)) \) and \( \pi_3(SO(6)) \), and via these generators we identify \( \pi_3(\tau) \) with multiplication by \(-1\).

**3.2.3. Conventions, identifications.** First, we identify \( \mathbb{H} \) and \( \mathbb{R}^4 \) and \( \mathbb{C}^2 \) in the obvious way: we identify the quaternion \( q = a + bi + cj + dk = z + wj \in \mathbb{H} \) with \((a, b, c, d) \in \mathbb{R}^4 \) and with the complex pair \((z, w) \in \mathbb{C}^2 \), where \( z = a + bi \) and \( w = c + di \). Also, we identify \( S^3 \) with the quaternions of unit length: \( S^3 = \{ q = a + bi + cj + dk \in \mathbb{H} | a^2 + b^2 + c^2 + d^2 = 1 \} \).

We define the following maps. Set

\[ u : S^3 \to U(2) , \ u_q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} , \]

where \( q = z + wj \). \( u_q \) is the (complex) matrix of the right (quaternionic) multiplication with \( q \), that is, of the map \( \mathbb{H} \to \mathbb{H} , p \mapsto pq \). Note that the left multiplication by \( q \) is not a complex unitary transformation, in general. Next, set

\[ L : S^3 \to SO(4) , \ L_q = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} , \]

where \( q = a + bi + cj + dk \). \( L_q \) is the (real) matrix of the left multiplication with \( q \) (i.e., of the map \( \mathbb{H} \to \mathbb{H} , p \mapsto qp \)).

Let \( R : S^3 \to SO(4) \) be the map which assigns for a \( q \in S^3 \) the (real) matrix \( R_q \) of the right multiplication with \( q \) (i.e., of the map \( \mathbb{H} \to \mathbb{H} , p \mapsto pq \)).
Let $\rho : S^3 \to SO(4)$ be the map which assigns for a $q \in S^3$ the (real) matrix $\rho q$ of the conjugation with $q$ (i.e., of the map $\mathbb{H} \to \mathbb{H}$, $p \mapsto qpq^{-1}$).

We use the same notation for the compositions of these maps with the inclusions $SO(4) \hookrightarrow SO$ and $U(2) \hookrightarrow U$. Note that these inclusions commute with $\tau$.

**Proposition 3.2.2.** [19, Section 7, Subsection 12]

- $(a)$ $\pi_3(U(2)) = \pi_3(U) = \mathbb{Z} \langle [u] \rangle$.
- $(b)$ $\pi_3(SO(4)) = \mathbb{Z} \langle [L] \rangle \oplus \mathbb{Z} \langle [\rho] \rangle$.
- $(c)$ $\pi_3(SO) = \mathbb{Z} \langle [L] \rangle$ and $[\rho] = 2[L]$ in $\pi_3(SO)$.

In the sequel, using these base choices $[u]$ and $[L]$ we identify the groups $\pi_3(U)$ and $\pi_3(SO)$ with $\mathbb{Z}$. Now we can state the explicit version of Proposition 3.2.1.

**Proposition 3.2.3.** $\pi_3(\tau)([u]) = -[L] \in \pi_3(SO)$ holds for $[u] \in \pi_3(U)$.

**Proof.** From definitions $\tau \circ u = R$ and $\rho R = L$, thus $\pi_3(\tau)([u]) = [R] = [L] - [\rho] = -[L]$ by part $(c)$ of Proposition 3.2.2. □

**Remark 3.2.4.** Let $p : U(2) \to S^3$ be the projection (choosing the first or the second column of the matrix). Then $[u] \in \pi_3(U(2))$ is the unique generator for which $\deg(u \circ p) = 1$.

**Remark 3.2.5.** The conventions we use are not universal. For example, Kirby and Melvin in [25] have chosen the same generators of $\pi_3(U)$ and $\pi_3(SO)$ (these are $[u]$ and $[L]$ with our notations), but they identified $\mathbb{R}^4$ and $\mathbb{C}^2$ differently than us. Namely, they identified the quaternion $q = a + bi + cj + dk = z + jw \in \mathbb{H}$ with $(a, b, c, d) \in \mathbb{R}^4$ and the complex pair $(z, w) \in \mathbb{C}^2$, where $z = a + bi$ and $w = c - di$. With that identification $u_q$ becomes the (complex) matrix of the quaternionic left multiplication with $q$. In that identification $\pi_3(\tau)[u]$ would be equal to $-[R]$, since that is the homotopy class of the map $S^3 \to SO(4)$ given by the composition $\tau \circ u \circ \kappa$, where $\kappa$ is the reflection $\kappa(z, w) = (z, \bar{w})$.

### 3.3. The complex Smale invariant

**3.3.1.** In this section we define the complex Smale invariant $\Omega_C(\Phi)$ for a holomorphic germ $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, singular only at 0. It will be the bridge between $C(\Phi)$ and $\Omega(f)$.

**Definition 3.3.1.** Consider the map (with target the complex Stiefel variety $V_2(\mathbb{C}^3)$):

$$d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3)$$
defined via the natural trivialization of the complex tangent bundles \( T\mathbb{C}^2 \) and \( T\mathbb{C}^3 \). By definition, the complex Smale invariant of \( \Phi \) is the homotopy class:

\[
\Omega_C(\Phi) = [d\Phi|_{S^3}] \in \pi_3(V_2(\mathbb{C}^3)).
\]

By the connectivity of the group of local coordinate transformations, \( \Omega_C(\Phi) \) is independent of the choice of local coordinates in \((\mathbb{C}^2, 0)\) and \((\mathbb{C}^3, 0)\).

**Remark 3.3.2.** The projection \( U(3) \to V_2(\mathbb{C}^3) \) induces an isomorphism between \( \pi_3(V_2(\mathbb{C}^3)) \) and \( \pi_3(U(3)) = \pi_3(U) \) (see e.g. [19]). Hence, if we choose a complex normal vector field \( N_\Phi \) of \( \Phi \), then the map

\[
(d\Phi, N_\Phi)|_{S^3} : S^3 \to GL(3, \mathbb{C})
\]

represents \( \Omega_C(\Phi) \) in \( \pi_3(GL(3, \mathbb{C})) = \pi_3(U(3)) = \pi_3(U) \). A canonical choice of \( N_\Phi \) could be the complex conjugate of the cross product of the partial derivatives of \( \Phi \):

\[
N_\Phi(s, t) = \partial_s \Phi(s, t) \times \partial_t \Phi(s, t).
\]

**Remark 3.3.3.** \( \pi_3(U) \cong \mathbb{Z} \) and in 3.2.3 we identify them through a fixed isomorphism. In this way \( \Omega_C(\Phi) \) becomes a well-defined integer without any sign-ambiguity.

**Proposition 3.3.4.** Let \( \Phi : \mathbb{C}^2 \to \mathbb{C}^3 \) be the cross cap, i.e. \( \Phi(s, t) = (s^2, st, t) \). Then \( \Omega_C(\Phi) = [u] \).

**Proof.** The map \( d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3) \) represents \( \Omega_C(\Phi) \). We should compose this with \( V_2(\mathbb{C}^3) \to U(3) \), then with the inverse of the inclusion \( U(2) \to U(3) \), and finally with the projection \( U(2) \to S^3 \); and then calculate the degree of the resulting map \( S^3 \to S^3 \). In fact, along these compositions we will use (the homotopically equivalent groups) \( GL(2, \mathbb{C}) \) and \( GL(3, \mathbb{C}) \) instead of \( U(2) \) and \( U(3) \). Therefore, we will arrive in \( \mathbb{C}^2 \setminus \{0\} \) instead of \( S^3 \).

\[
d\Phi|_{S^3} : S^3 \to V_2(\mathbb{C}^3) , (s, t) \mapsto \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 1 \end{pmatrix}.
\]

The first composition gives the map

\[
S^3 \to GL(3, \mathbb{C}) , (s, t) \mapsto \begin{pmatrix} 2s & 0 & N_1 \\ t & s & N_2 \\ 0 & 1 & N_3 \end{pmatrix},
\]
where \( N_1 = \bar{t} \), \( N_2 = -2\bar{s} \) and \( N_3 = 2\bar{s}^2 \) are the coordinates of the normal vector \( N_\Phi \) (see Remark 3.3.2). This is modified by the homotopy

\[
S^3 \times [0, 1] \to GL(3, \mathbb{C}) , \ (s, t, h) \mapsto \begin{pmatrix} 2s & 0 & N_1 \\ t & hs & N_2 \\ 0 & 1 & hN_3 \end{pmatrix},
\]

which maps \((s, t, 0)\) into \( GL(2, \mathbb{C}) \subset GL(3, \mathbb{C}) \). [Note that the determinant is \( |t|^2 + 4|s|^2 + 4h^2|s|^4 \neq 0 \), thus the image is indeed in \( GL(3, \mathbb{C}) \).] Hence, we obtain the map

\[
S^3 \to GL(2, \mathbb{C}) , \ (s, t) \mapsto \begin{pmatrix} 2s & N_1 \\ t & N_2 \end{pmatrix},
\]

which composes with the projection (first column) provides \( S^3 \to \mathbb{C}^2 \setminus \{0\} \), \((s, t) \mapsto (2s, t)\). After a normalisation, the degree of the resulting map is 1. \( \square \)

The proof of Proposition 3.3.4 works for all germs of the form

\[
\Phi(s, t) = (g_1(s, t), g_2(s, t), t)
\]

and implies that

\[
\Omega_\mathbb{C}(\Phi) = \deg \left( S^3 \to S^3 , \ (s, t) \mapsto \frac{\left( \partial_s g_1(s, t), \partial_s g_2(s, t) \right)}{|(\partial_s g_1(s, t), \partial_s g_2(s, t))|} \right).
\]

This degree agrees with the intersection multiplicity of \( \partial_s g_1 \) and \( \partial_s g_2 \) in \((\mathbb{C}^2, 0)\), hence we proved the following, cf. Example 1.4.10.

**Proposition 3.3.5.**

\[
\Omega_\mathbb{C}(\Phi) = \dim_\mathbb{C} \frac{O_{(\mathbb{C}^2, 0)}}{(\partial_s g_1, \partial_s g_2)} = C(\Phi)
\]

holds for corank–1 germs \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) singular only at the origin.

This identity \( \Omega_\mathbb{C}(\Phi) = C(\Phi) \) will be proved in the general case in Section 3.3.2.

**Remark 3.3.6.** Conversely, the identity \( \Omega_\mathbb{C}(\Phi) = C(\Phi) \) is proved in Section 3.3.2 independently of Theorem 1.4.3, therefore (3.3.2) together with Theorem 3.3.7 give a new proof for Theorem 1.4.3 in the case of germs which satisfy \( \text{rk}(d\Phi_0) = 1 \).

**3.3.2. The identity** \( \Omega_\mathbb{C}(\Phi) = C(\Phi) \). Next we identify the complex Smale invariant with the number of cross caps.

**Theorem 3.3.7.** \( \Omega_\mathbb{C}(\Phi) = C(\Phi) \).
3.4. The proof of Theorems 3.1.2 and 3.1.3

Proof. Consider the following diagram:

\[ d\Phi : \mathbb{C}^2 \rightarrow \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \]

where \( \mathcal{D} = \{ M \in \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \mid \text{rk}(M) < 2 \} \). \( \mathcal{D} \) is an irreducible algebraic variety of complex codimension 2, its Zariski open set \( \mathcal{D}^1 = \{ M \in \mathcal{D} \mid \text{rk}(M) = 1 \} \) is smooth.

First we prove that \( \Omega_C(\Phi) \) is equal to the linking number of \( d\Phi|_{S^3} \) and \( \mathcal{D} \) in \( \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \).

This is defined as follows. If \( g : S^3 \rightarrow \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \backslash \mathcal{D} \) is a smooth map, and \( \tilde{g} \) is a smooth extension defined on the ball such that \( \tilde{g}|_{S^3} = g \) and \( \tilde{g} \) intersects \( \mathcal{D} \) transversally along \( \mathcal{D}^1 \), then the linking number of \( g \) and \( \mathcal{D} \) is the algebraic number of the intersection points of \( \tilde{g} \) and \( \mathcal{D} \). By standard argument it is a homotopy invariant of maps \( S^3 \rightarrow \text{Hom}(\mathbb{C}^2, \mathbb{C}^3) \backslash \mathcal{D} \).

The linking number gives a group homomorphism \( \text{lk} : \pi_3(V_2(\mathbb{C}^3)) \rightarrow \mathbb{Z} \). Next lemma shows that this homomorphism is surjective.

Lemma 3.3.8. Let \( \Phi(s,t) = (s^2, st, t) \) be the cross cap. If \( g = d\Phi|_{S^3} \) and \( \tilde{g} = d\Phi|_{B^4} \), then \( \tilde{g}(0) \in \mathcal{D} \) is the only intersection point and the intersection is transverse at that point.

Proof. This is a straightforward local computation left to the reader. The transversality follows also from the conceptual fact that the cross cap is a stable map.

The sign of the intersection multiplicity at the intersection point of two complex submanifolds is always positive. For \( g \) described in 3.3.8 the linking number of \( g(S^3) \) and \( \mathcal{D} \) is 1. This shows not only that the homomorphism given by the linking number is surjective (hence an isomorphism too), but also that this isomorphism agrees with the chosen one in 3.2.3. This follows from the fact that the complex Smale invariant of the cross cap is exactly the chosen generator, see Proposition 3.3.4. Hence, the homomorphisms \( \Omega_C \) and \( \text{lk} \) coincide.

Next, we show that \( \text{lk}_{\text{Hom}(\mathbb{C}^2, \mathbb{C}^3)}(d\Phi|_{S^3}(S^3), \mathcal{D}) = C(\Phi) \). Take a generic perturbation \( \Phi_\epsilon \) of \( \Phi \). \( d\Phi_\epsilon|_{S^3} \) is homotopic to \( d\Phi|_{S^3} \), hence their linking numbers are the same. \( \Phi_\epsilon \) has only cross cap singularities, their number is \( C(\Phi) \). This means that \( d\Phi_\epsilon|_{B^4} \) intersects transversally \( \mathcal{D} \) in \( C(\Phi) \) points. Intersection of complex manifolds provides positive signs.

Corollary 3.3.9. \( \Omega_C(\Phi) \geq 0 \).

3.4. The proof of Theorems 3.1.2 and 3.1.3

3.4.1. Theorem 3.1.2 follows from Theorem 3.3.7, Proposition 3.2.3 and the next identity.
Proposition 3.4.1. $\pi_3(\tau)(\Omega_C(\Phi)) = \pi_3(j)(\Omega(f))$.

Proof. By the definition of the Smale invariant, one has to extend $f$ to a neighbourhood of the standard embedding of $S^5$ in an $\mathbb{R}^5$ (cf. [2.2.3]). On the other hand $\Phi$ extends $f$ in the $\mathbb{C}^2$ direction. We will compare these two extensions using a common extension $F : W \to \mathbb{C}^3$, where $W$ is a suitable neighbourhood of $S^5$ in $\mathbb{C}^3$.

Let us consider a fixed $\epsilon$ which satisfies the properties of Corollary [1.1.2]. We also write $B^6_\epsilon = \{ z : |z| \leq \epsilon \} \subset \mathbb{C}^3$, $S^5_\epsilon = \partial B^6_\epsilon$, $\mathcal{B}^4_\epsilon := \Phi^{-1}(B^6_\epsilon)$ for the $C^\infty$ ball in $\mathbb{C}^2$, and $\mathcal{G}^3_\epsilon := \Phi^{-1}(S^5_\epsilon)$ for its boundary. (Late we will drop some of the $\epsilon$'s.) For positive $\epsilon_1$, $\epsilon_2$ sufficiently closed to $\epsilon$, $\epsilon_1 < \epsilon < \epsilon_2$, and for $0 < \rho \ll \epsilon$ one defines $F(s,t,r) = \Phi(s,t) + r \cdot N_\Phi(s,t)$, where $(s,t,r) \in W := \Phi^{-1}(z : \epsilon_1 < |z| < \epsilon_2) \times D^2_\rho$, $D^2_\rho$ is the $\rho$-disc in $\mathbb{C}$, and $N_\Phi$ is the complex normal vector of $\Phi$, see Remark [3.3.2]. Since the normal bundle of $f$ in $S^5_\epsilon$ is trivial (and since the transversality is an open property), we get that $F^{-1}(S^5_\epsilon)$ is diffeomorphic to $\mathcal{G}^3_\epsilon \times D^2_\rho$. In fact, if $p : \mathbb{C}^2 \times D^2_\rho \to D^2_\rho$ is the natural projection, then for any $r \in D^2_\rho$ we can define $\mathcal{G}^3_{\epsilon,r} := F^{-1}(S^5_\epsilon) \cap p^{-1}(r)$. Then each $\mathcal{G}^3_{\epsilon,r}$ is a $C^\infty$ 3-sphere, being the boundary of the $C^\infty$ 4-ball $\mathcal{B}^4_{\epsilon,r} \subset \mathbb{C}^2 \times \mathbb{C}$ is a thickened tubular neighbourhood of $\mathcal{B}^4_{\epsilon,r}$. Moreover, $\mathcal{B}^6 := \cup_{r \in D^2_\rho} \mathcal{B}^4_{\epsilon,r} \subset \mathbb{C}^2 \times \mathbb{C}$ is a thickened tubular neighbourhood of $\mathcal{B}^4_{\epsilon,r}$ (diffeomorphic to $B^4 \times S^1$).

In a point $(s,t,0) \in \mathcal{G}^3_\epsilon \times \{0\}$ the differential of $F$ is

$$dF(s,t,0) = (\partial_s F(s,t,0), \partial_t F(s,t,0), \partial_r F(s,t,0)) = (\partial_s \Phi(s,t), \partial_t \Phi(s,t), N_\Phi(s,t)) .$$

Thus, the homotopy class of $dF|_{\mathcal{G}^3_\epsilon}$ equals $\Omega_C(\Phi)$ (cf. [3.3.2]). Therefore, taking the real function $\tilde{F} : W \to \mathbb{R}^6$ (cf. [3.2.2]), its real Jacobian satisfies $[d\tilde{F}|_{\mathcal{G}^3_\epsilon}] = \pi_3(\tau)(\Omega_C(\Phi))$.

On the other hand we show that $[d\tilde{F}|_{\mathcal{G}^3_\epsilon}] = \pi_3(j)(\Omega(f))$. In order to recover the Smale invariant $\Omega(f)$ of $f = \Phi|_{\mathcal{G}^3} : \mathcal{G}^3 \to S^5$, first we need to fix a global coordinate system in a contractible neighbourhood of the source $\mathcal{G}^3$ in $\mathcal{G}^5$ and also in $\mathbb{R}^5 \simeq S^5_\epsilon \setminus \{\text{a point}\}$ containing im$(f)$. Let us introduce the ‘outward normal at the end’ convention to orient compatibly a manifold and its boundary. In this way we fix an orientation of $\mathcal{G}^5 = \partial \mathcal{B}^6$ and $S^5 = \partial B^6$. (According to [3.2.3] the output of the proof is independent of the convention choice.)

In the first case we introduce a coordinate system in $\mathcal{G}^5 \setminus \{Q\} \simeq \mathbb{R}^5$ compatibly with the orientation, where $Q \in \mathcal{G}^5 \setminus \mathcal{G}^3$ is an arbitrary point (e.g. $(0,0,\rho)$). Let $\nu'$ denote the framing of $T(\mathcal{G}^5 \setminus \{Q\}) \simeq \partial \mathbb{R}^5$ induced by this coordinate system. We can extend the outward normal frame $\nu_0$ of $\mathcal{G}^3$ in $\mathbb{C}^2$ to the rest of $\mathcal{G}^5 \setminus \{Q\}$ (as the outward normal vector of $\mathcal{G}^5$). This framing can be extended to a neighbourhood $V$ of $\mathcal{G}^5 \setminus \{Q\}$ in $\mathbb{C}^3$. 


Let \( \nu : V \to GL^+(6, \mathbb{R}) \) denote this framing (or more precisely, \( \nu \) is the transition function from the standard framing inherited from \( \mathbb{R}^6 \) to the one just constructed).

The target is the standard \( S^5 \subset \mathbb{R}^6 \). We can choose a point \( P \in S^5 \setminus f(\mathbb{S}^3) \) and a coordinate system on \( S^5 \setminus \{P\} \) compatibly with the orientation. The coordinate system induces a framing \( \eta' \) of the tangent bundle \( T(S^5 \setminus \{P\}) \) of \( S^5 \setminus \{P\} \). In the points of the target of \( \tilde{F} \) the vectors of \( \eta' \) and \( d\tilde{F}(\nu_6) \) are linearly independent, that is, \( d\tilde{F}(\nu_6) \) behaves like a normal framing (this follows from the transversality property of 1.1.2). We can extend it to a normal framing \( \eta_6 \) of \( S^5 \setminus \{P\} \) in \( \mathbb{R}^6 \). In this way we get a framing of the tangent bundle of a neighbourhood \( V' \) of \( S^5 \setminus \{P\} \) in \( \mathbb{R}^6 \). Let \( \eta : V' \to GL^+(6, \mathbb{R}) \) denote the transition from the framing on \( V' \) inherited from \( \mathbb{R}^6 \) to the framing just defined.

The Smale invariant \( \Omega(f) \) is constructed in the following way, cf. (2.2.3). Take

\[ J_{(\nu', \eta')}(\tilde{F}|_\mathcal{I}), \]

the Jacobian of \( \tilde{F} \) restricted to \( \mathcal{I} = F^{-1}(S^5) \) prescribed in the framings \( \nu' \) and \( \eta' \). The homotopy class of this matrix restricted to \( \mathbb{S}^3 \) (as a map \( \mathbb{S}^3 \to GL^+(6, \mathbb{R}) \)) is equal to \( \Omega(f) \). (Since \( \tilde{F} \) preserves the orientation, \( \tilde{F}|_\mathcal{I} \) does as well.)

\[ J_{(\nu, \eta)}(\tilde{F})|_{\mathbb{S}^3} = j(J_{(\nu', \eta')}(\tilde{F}|_\mathcal{I})) \]

because \( d\tilde{F}(\nu_6) = \eta_6 \), thus the homotopy class of \( J_{(\nu, \eta)}(\tilde{F})|_{\mathbb{S}^3} \) equals \( \pi_3(j)(\Omega(f)) \).

On the other hand \( J_{(\nu, \eta)}(\tilde{F}) = (\eta^{-1} \circ \tilde{F}) \cdot d\tilde{F} \cdot \nu \). As maps \( \mathbb{S}^3 \to GL^+(6, \mathbb{R}) \), \( (\eta^{-1} \circ \tilde{F})|_{\mathbb{S}^3} \) and \( \nu|_{\mathbb{S}^3} \) are nullhomotopic because the vector fields are defined on the contractible spaces \( \mathbb{S}^5 \setminus \{Q\} \) and \( S^5 \setminus \{P\} \). Therefore (cf. Remark 2.1.4)

\[ [J_{(\nu, \eta)}(\tilde{F})|_{\mathbb{S}^3}] = [(\eta^{-1} \circ \tilde{F})|_{\mathbb{S}^3}] + [d\tilde{F}|_{\mathbb{S}^3}] + [\nu|_{\mathbb{S}^3}] = [d\tilde{F}|_{\mathbb{S}^3}] . \]

The left hand side of this identity is \( \pi_3(j)(\Omega(f)) \), while the right hand side \( \pi_3(\tau)(\Omega_C(\Phi)) \).

\[ \square \]

**Corollary 3.4.2.** Assume that the analytic germs \( \Phi \) and \( \Phi' : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) are \( C^\infty \) left-right equivalent, that is, \( \Phi' = \Lambda \circ \Phi \circ \psi \) holds for some germs of orientation preserving diffeomorphisms \( \psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0) \) and \( \Lambda : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0) \). Then \( C(\Phi) = C(\Phi') \).

**Proof.** For a sufficiently small \( \epsilon \) take \( \mathbb{S}^3 = (\Phi')^{-1}(S^5) \simeq S^3 \) and let \( f' : \mathbb{S}^3 \hookrightarrow S^5 \) be the immersion associated with the germ \( \Phi' \) (cf. 1.1.2).

As in the proof of Proposition 3.4.1 let \( \tilde{F} : W \to \mathbb{R}^6 \) be the extension of \( \Phi \) viewed as a real function. Then, by that proof, \( [d\tilde{F}|_{\mathbb{S}^3}] = \pi_3(j)(\Omega(f)) \in \pi_3(GL^+(6, \mathbb{R})) \), which is \( -C(\Phi) \) under the above identification.
Let us define $\Psi : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0)$ by
\[
\Psi(x_1, x_2, x_3, x_4, x_5, x_6) = (\psi(x_1, x_2, x_3, x_4), x_5, x_6).
\]
$\Psi$ is a germ of orientation preserving diffeomorphism extending $\psi$.

Then $\Lambda \circ \Phi \circ \Psi$ restricted on some neighbourhood $W'$ of $S'$ plays the role of a real extension of $\Phi'|_{S^5} = f'$. One can verify that the proof of Proposition 3.4.1 works for this extension as well, since it sends the normal vector of $S^5$ into a non-tangent vector of $S^5$.

Therefore,
\[
[d(\Lambda \circ \Phi \circ \Psi)|_{S^5}] = \pi_3(j)(\Omega(f')).
\]

Finally, note that $[d(\Lambda \circ \Phi \circ \Psi)|_{S^5}] = [d\tilde{F}|_{S^5}]$. This follows from the fact that the functions $d\Lambda \circ \Phi \circ \Psi$ and $d\Psi$ (with images in $(GL^+(6, \mathbb{R}))$ extend to the ball $B^6$.

3.4.2. Proof of Theorem 3.1.3. Part (a) follows from Theorem 3.1.2 and [17].

In part (b), the implications $(1) \Rightarrow (2, 3, 4)$, and $(4) \Rightarrow (3)$ are clear.

The proof of $(2) \Rightarrow (1)$: $(2)$ implies $C(\Phi) = 0$ by Theorem 3.1.2, while this vanishing implies $(1)$ via Mond’s Theorem 1.4.3. For $(3) \Rightarrow (1)$ we provide three proofs, each of them emphasize a different geometrical/topological aspect.

(A) (Based on Mumford’s Theorem.) If $f$ is an embedding then the image $(X, 0)$ of $\Phi$ is an isolated hypersurface singularity in $(\mathbb{C}^3, 0)$. Moreover, its link is $S^3$, hence by Mumford’s theorem [50] $(X, 0)$ is smooth. Hence its normalization $\Phi$ is an isomorphism.

(B) (Based on Mond’s Theorem.) Let us take the generic deformation $\Phi_\lambda$, and consider the closure $D$ of the preimage of the the set of double values. It is a 1–dimensional closed complex analytic subspace of the disc in $\mathbb{C}^2$. The preimages of cross cap and triple points are interior points of the closure of $D$, while its boundary is $D \cap S^3$ is the preimage of the double points of the immersion of $f : S^3 \hookrightarrow S^5$. If $f$ is an embedding then $\partial D = \emptyset$, hence $D$ is a compact analytic curve in $(\text{the disc of})$ $\mathbb{C}^2$, hence it should be empty. This shows that $\Phi_\lambda$ has no cross cap and triple points either. Hence $C(\Phi) = 0$, which implies $(1)$ by 1.4.3 as before.

(C) (Based on Ekholm–Szűcs Theorem.) As above, we get that $\Phi_\lambda$ is an embedding. Since $\Phi|_{S^3}$ is an embedding, this embedding is regular homotopic to $\Phi_\lambda|_{S^3}$, hence they have the same Smale invariant. In the second case it can be determined by an Ekholm–Szűcs formula [9] (recalled as Theorem 2.2.12): since $\text{im}(\Phi_\lambda)$ is an embedded Seifert surface with signature zero we get $\Omega(f) = 0$. This basically proves $(3) \Rightarrow (2)$. Then we continue with the already shown $(2) \Rightarrow (1)$.

In fact, the main point of this last proof is already coded in Hughes–Melvin Theorem [17] (2.2.2 here), but in that statement the Seifert surface is in $\mathbb{R}^5$ and not in $\mathbb{R}^6_+$ (or
3.5. Examples

Example 3.5.1 (\(S_{k-1}\) from Mond’s list \([45]\)). \(\Phi_{-k}(s,t) = (s, t^2, t^3 + s^k t) \ (k \in \mathbb{Z}_{\geq 0})\).

The ramification ideal \(J(\Phi_{-k})\) is generated by \((2t, 3t^2 + s^k, -2kt^2s^{k-1}) = (t, s)\), cf. Subsection 1.4.2 and Example 1.4.10. Hence \(\Omega(f_{-k}) = -C(\Phi_{-k}) = -k\).

This family gives representatives for every regular homotopy class with non-positive sign–refined Smale invariant. Furthermore, we can represent any regular homotopy class with Smale invariant \(k\) in the form \(\Phi_{-k} \circ \tau\), where \(\tau\) is the reflection \(\tau(z, w) = (z, \bar{w})\) (c.f. [7] Lemma 3.4.2].)

Example 3.5.2 (Quotient singularities). The covering map germs of the quotient singularities of type \(A, D, E\) are discussed in examples 1.3.5, 1.4.14, 1.4.15 and 1.4.17, where the \(C\) invariant of them is calculated. Their associated immersions decompose in the form \(S^3 \to S^3/G \hookrightarrow S^5\), cf. Example 3.1.4. The list of their Smale invariants \(\Omega(f) = -C(\Phi)\) is:

\[
\begin{align*}
A_{n-1} : \quad & \Omega(f) = -(n^2 - 1), \\
D_{n+2} : \quad & \Omega(f) = -(4n^2 + 12n - 1), \\
E_6 : \quad & \Omega(f) = -167, \\
E_7 : \quad & \Omega(f) = -383, \\
E_8 : \quad & \Omega(f) = -1079.
\end{align*}
\]

The Smale invariant of the \(A\) and \(D\) types agrees up to sign with the Smale invariant of the immersions constructed by Kinjo \([24]\), its discussion is in Subsection 2.3.2. Finding a direct relation between our immersions and Kinjo’s constructions would enable us to determine the Smale invariant of the immersions associated with the graphs \(E_6, E_7, E_8\).

3.6. Ekholm-Szűcs formulae for holomorphic germs \(\Phi\)

3.6.1. In this section, from a stabilization of \(\Phi\) we construct a singular Seifert surface, and we express the topological summands of (2.2.8) in terms of holomorphic invariants.

The topological formulae (2.2.1), (2.2.7) and (2.2.8) targeting the Smale invariant in terms of the geometry of oriented Seifert surfaces are stated and proved only up to a sign ambiguity. In this next section we will show that the sign–refined Smale invariant appears in all these expressions with a unique well–defined sign, and we determine it simultaneously for all formulae. The discussion has an extra output as well: the topological ingredients in the formulae below will get reinterpretations in terms of complex analytic invariants, provided that the immersion is induced by a holomorphic germ \(\Phi\).

First of all we have to reinterpret the formulae (2.2.1), (2.2.7) and (2.2.8) in the spirit of the discussion of Subsection 3.2.1. We replace \(S^3\) (which is the unit sphere in \(\mathbb{R}^4\)) with
S\(^3\), the ‘oriented abstract S\(^3\)’, and we also need to fix a ‘boundary convention’, in order to have the notion of oriented \(\partial M^4\). Then \(\partial M^4 = S^3\) has a well-defined meaning, and \(\partial B^4 = S^3\) inherits an orientation, hence the boundary convention determines whether a diffeomorphism \(S^3 \to S^3\) is orientation preserving. \(\Omega(f)\) denotes the Smale invariant (given by any of its definitions, still having its sign-ambiguity). Nevertheless, the sign-corrected formulae will be ‘boundary convention’ free, cf. Theorem 3.6.5.

3.6.2. Singular Seifert surface associated with a stabilization. Let \(\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\) be a holomorphic germ singular only at the origin and let \(f : S^3 \leftrightarrow S^5\) be the immersion associated with \(\Phi\). We take an \(\epsilon\) as in Corollary 1.1.2, that is, we fix in the target a ball \(B^6_{\epsilon}\). We also consider a holomorphic stabilization \(\Phi_\lambda\) of \(\Phi_0 = \Phi\) (cf. Section 1.4.1), and we fix \(\lambda\) sufficiently small, \(0 < |\lambda| \ll \epsilon\), such that the cross caps and (if \(T(\Phi) < \infty\)) the triple points of \(\Phi_\lambda\) sit in \(B^6_{\epsilon}\). We set \(\mathcal{B}^4_{\epsilon, \lambda} := \Phi^{-1}_\lambda(B^6_{\epsilon})\), it is a \(C^\infty\) non-metric ball in \(\mathbb{C}^2\). Its boundary is \(S^3_{\epsilon, \lambda} := \Phi^{-1}_\lambda(S^5_{\epsilon})\), it is canonically diffeomorphic to \(S^3\).

The map \(\Phi_\lambda\) is stable as a holomorphic map, but it is not stable as a \(C^\infty\) map, cf. Example 1.2.19. The \(C^\infty\) stability is obstructed by its cross cap points. We will modify \(\Phi_\lambda\) in the neighbourhood of these points according to the following local model.

Let us fix local holomorphic coordinate systems in the source and the target such that \(\Phi_\lambda\) in the neighbourhood of a cross cap has local equation \(\Phi_{\text{loc}}(s,t) = (s^2, st, t)\). We consider its real smooth deformation (with \(0 \leq \tau \ll |\lambda|\)):

\[
\Phi^\text{loc}_\tau(s,t) = (s^2 + 2\tau s, st + \tau s, t).
\]

Since the restriction of \(\Phi^\text{loc}_\tau\) near the boundary of the local 4-ball is stable, by a \(C^\infty\) bump function the local deformation can be glued to the trivial deformation of \(\Phi_\lambda\) outside of local neighbourhoods of the cross caps. This gives a \(C^\infty\) global deformation \(\Phi_{\lambda, \tau}\) of \(\Phi_\lambda\) and \(\Phi\). The map \(\tilde{f} = \Phi_{\lambda, \tau} : (\mathcal{B}^4_{\epsilon, \lambda}, S^3_{\epsilon, \lambda}) \to (B^6_{\epsilon}, S^5_{\epsilon})\) is the singular Seifert surface we will consider. Its restriction, \(f_\lambda = \Phi_{\lambda, \tau}|_{S^3_{\epsilon, \lambda}} = \Phi_{\lambda}|_{S^3_{\epsilon, \lambda}}\) is the immersion associated with \(\Phi_\lambda\).

**Proposition 3.6.1.**

(a) \(\tilde{f} : \mathcal{B}^4_{\epsilon, \lambda} \to \mathbb{C}^3\) is a stable smooth map, nonsingular near the boundary.

(b) \(f_\lambda\) is a stable immersion and it is regular homotopic to \(f\).

(c) If \(f\) is a stable immersion, then \(f_\lambda\) is regular homotopic to \(f\) through generic immersions. In this case \(L(f) = L(f_\lambda)\).

**Proof.** (a) First one checks that the local \(\Phi^\text{loc}_\tau\) is stable. This follows from the computation from Section 3.7.1. Its most complicated singularities are \(\Sigma^{1,0}\) (fold) points, the singular values constitute an \(S^1\), which – together with the double values of the image of the boundary of the local ball – bounds the 2-manifold of the double values. Cf. [9, 2.3].
In the complement of local balls $\Phi_{\lambda, \tau}$ agrees with $\Phi_{\lambda}$, hence it has only simple points, self-transverse double points and isolated triple points. All of them are stable. Hence $\tilde{f}$ has all the local properties (multigerms) of a stable map, thus it is stable by Remark 1.2.6.

(b) $\Phi_{\lambda|S^3_\epsilon, \lambda}$ is stable in real sense too: it has only simple points and self-transverse double points. $\Phi_{h\lambda|S^3_\epsilon, \lambda}$ is a regular homotopy between $f$ and $f_\lambda$ ($h \in [0, 1]$).

(c) Being a stable immersion is an open condition (cf. [9, 2.1.]). Furthermore, $L$ is constant along a regular homotopy through generic immersions, cf. [7]. □

Next, we return back to the formula (2.2.8), applied for $\tilde{f}$. Clearly, $C(\Phi) - 3T(\Phi)$.

Theorem 3.6.2. Let $\Phi_\lambda$ be a holomorphic stabilization of $\Phi$ with fixed $\lambda \neq 0$ and the corresponding maps $\tilde{f}$ and $f_\lambda$ as above. Then the following statements hold.

(a) $t(\tilde{f}) = T(\Phi_\lambda)$ (cf. 1.4.9).
(b) $l(\tilde{f}) = C(\Phi)$.
(c) $L(f_\lambda) = C(\Phi) - 3T(\Phi_\lambda)$.

For the proof see 3.6.3.

Note that $L(f_\lambda)$ is an analytic invariant of $\Phi$, since it is defined as a (topological) invariant of an analytic deformation. Recall from Proposition 1.4.9 that if $T(\Phi) < \infty$, then $T(\Phi_\lambda)$ is independent of the deformation $\Phi_\lambda$ and is equal to $T(\Phi)$.

Corollary 3.6.3. If $T(\Phi) < \infty$, then $t(\tilde{f}) = T(\Phi)$ and $L(f_\lambda) = C(\Phi) - 3T(\Phi)$ is also independent of the holomorphic stabilization $\Phi_\lambda$ of $\Phi$.

Remark 3.6.4. (a) Assume that $\Phi$ is finitely determined, that is, $\Phi|_{S^3_\epsilon} = f$ is a stable immersion, cf. Theorem 1.4.4. Then $T(\Phi)$ is finite, cf. Theorem 1.4.3, and for any holomorphic stabilization $\Phi_\lambda$ one has $L(f_\lambda) = L(f)$ (cf. Proposition 3.6.1), hence $C(\Phi) - 3T(\Phi) = L(f)$.

(b) Note that if $\Phi$ is finitely determined and the holomorphic germ $\Phi'$ is $C^\infty$ left-right equivalent with $\Phi$ (see Corollary 3.4.2 for precise definition), then the immersion $f'$ associated with $\Phi'$ is also stable immersion and $L(f) = L(f')$. Therefore by Corollaries 3.4.2 and 3.6.3 we have $T(\Phi) = T(\Phi')$ too.

(c) More generally, $\Phi$ and $\Phi'$ are topological left–right equivalent and both of them are finitely $\mathcal{A}$-determined (so $f$ and $f'$ are stable immersions), then $L(f) = L(f')$, hence Corollary 3.1.6 follows too.

(d) If $C(\Phi)$ and $T(\Phi)$ are finite, then $f$ is not necessarily stable immersion, but it does not have triple points. Then by Remark 2.2.7 $L(f)$ is well-defined as $L$ of any small stable perturbation of $f$ with a regular homotopy. This provides an other proof for Corollary 3.6.3. However it is not clear, how can $L(f)$ be determined from the topology
of $f$ itself, without stable perturbation. That obstructs the generalization of part (b) and (c) for germs with finite $C$ and $T$.

**Theorem 3.6.5.** With our sign-convention, if in the left hand side of the formulae \[2.2.1\], \[2.2.7\] and \[2.2.8\] we put the sign-refined Smale invariant $\Omega^a(f)$, then the formulae are valid if we put the positive sign on the right hand sides.

In particular, the validity of these sign-corrected formulae (e.g., $\Omega^a(f) = \frac{3}{2}\sigma(M^4)$) is independent of the ‘boundary convention’: changing the boundary convention changes the sign in both sides of the formulae simultaneously.

The proof will appear in 3.6.3.

**Remark 3.6.6.** The formula \[2.2.7\], involving the (algebraic) number of real cusps of maps $g : M^4 \to \mathbb{R}^5$ is the real analogue of our theorem $\Omega(f) = -C(\Phi)$, involving the number of cross caps of holomorphic deformations. This suggests that if we replace a holomorphic deformation by a smooth generic map, then we trade each cross cup by $-2$ real cusps.

**3.6.3. Proof.** We prove Theorems 3.6.2 and 3.6.5 simultaneously (see also the discussion from Subsection 3.2.1).

In the definitions of the invariants $t$, $l$ and $L$ one uses very specific sign/orientation conventions, based on the orientation of the involved subspaces in their definition.

For a triple value, the sign is determined in such a way that it is $+1$ whenever the triple value is obtained from a holomorphic triple point (hence the orientations agree with the complex orientations).

Since in the local deformation $\Phi^\text{loc}_\tau$ we do not create any new triple value, see e.g. the computation of Section 3.7 all the triple values of $\tilde{f}$ come from the complex triple points of the holomorphic $\Phi_\lambda$, hence (a) follows.

The proof of the remaining parts are based on computations of the invariants $C(\Phi)$, $T(\Phi)$, $l(\tilde{f})$ and $L(f)$ for two concrete cases. For the integers $l$ and $L$ the definitions (orientation conventions) are not immediate even in simple cases. Therefore, in our computation we determine them only up to a sign. The point is that computing ‘sufficiently many’ examples, the formula \[2.2.8\], even with its sign ambiguity in front of the right hand side, and even with the (new) sign ambiguities of the integers $l$ and $L$, determine uniquely all these signs. (This also shows that, in fact, there is a unique universal way to fix the orientation conventions and signs in the definitions of $l$ and $L$ such that \[2.2.8\] works universally.)
In Section 3.7 we will determine the following data:

(i) For cross cup: \( C(\Phi) = 1, \ T(\Phi) = 0, \ l = \pm 1, \ L = \pm 1 \).

(ii) For \( A_1 \): \( C(\Phi) = 3, \ T(\Phi) = 1, \ L = 0 \).

\[ (3.6.2) \]

The singular values of \( \tilde{f} \) are concentrated near the cross caps of \( \Phi_\lambda \). For \( \Phi_{\lambda \tau}^{loc} \) the value \( l \) is \( \pm 1 \), see (i). Since the sign is the same for all cross caps, \( l(\tilde{f}) = \pm C(\Phi) \).

We introduce the notation

\[ (3.6.3) \quad \Omega'(f_\lambda) := \frac{1}{2}(3t(\tilde{f}) - 3l(\tilde{f}) + L(f_\lambda)). \]

\( \Omega'(f_\lambda) \) agrees with \( \Omega(f) \) up to sign, thus \( \Omega'(f_\lambda) = \pm C(\Phi) \). Substituting this and the data (i) of the cross cap in (3.6.3) we conclude that \( l(\tilde{f}) = -\Omega'(f_\lambda) \) and \( L(f_\lambda) = \pm C(\Phi) - 3T(\Phi_\lambda) \).

Next, using the date (ii) for \( A_1 \), all the remaining sign ambiguities can be eliminated: \( L(f_\lambda) = C(\Phi) - 3T(\Phi_\lambda), \ l(\tilde{f}) = C(\Phi) \) and \( \Omega'(f_\lambda) = -C(\Phi) = \Omega(f) \).

The universal signs in formulae (2.2.1), (2.2.7) and (2.2.8) are related by common examples, hence one of them determines all of them.

3.7. Calculations. The proof of (3.6.2).

We show the main steps of the computations. Note that if the germ \( \Phi \) is weighted homogeneous, then \( \epsilon_0 = 1 \) can be chosen.

3.7.1. The case of cross cap. \( T(\Phi) = 0 \) and \( C(\Phi) = 1 \) is clear, cf. examples 1.4.10 and 1.4.11. Next we compute \( l \) and \( L \). Set \( \Phi(s, t) = (s^2, st, t) \) and the smooth perturbation \( \tilde{f}(s, t) = (s^2 + 2\epsilon s, st + \epsilon s, t) \). The singular locus is \( \tilde{\Sigma} = \{ (s, t) \mid s = t, \ |s| = |t| = \epsilon \} \simeq S^1 \).

\( \tilde{f}|_{\tilde{\Sigma}} \) has no singular point, hence \( \tilde{f} \) has no cusp points. The most complicated singularities of \( \tilde{f} \) are \( \Sigma^{1,0} \) (or fold) points. The closure of the set of the double points \( \tilde{D} \) of \( \tilde{f} \) is

\[ \text{cl}(\tilde{D}) = \{ (s, t) \in \mathbb{C}^2 \mid (s - t)t + \epsilon(\tilde{s} - \tilde{t}) = 0 \} \]

with the involution \( (s, t) \mapsto (s', t) = (2t - s, t) \). The fix point set of the involution is \( \{ s = t \} \). Thus the set of the double points is

\[ \tilde{D} = \{ (s, t) \in \mathbb{C}^2 \mid (s - t)t + \epsilon(\tilde{s} - \tilde{t}) = 0 \} \setminus \{ s = t \}. \]

Each double point has exactly one pair with the same value, hence \( \tilde{f} \) has no triple point.

A parametrization of \( \tilde{D} \) is \( (\rho, \alpha) \mapsto (-\epsilon e^{-2\alpha i} + \rho e^{i\alpha}, -\epsilon e^{-2\alpha i}) \), where \( \rho \in \mathbb{R}_+, \ \alpha \in [0, 2\pi] \).
The parametrization shows that the closure of $\tilde{D}$ is a Möbius band. For $\rho = 0$ we get $\tilde{\Sigma}$, which is the midline of the Möbius band. The set of double values is

$$D = \tilde{f}(\tilde{D}) = \{(s^2 + 2\epsilon \bar{s}, st + \epsilon \bar{s}, t) \mid (s, t) \in \tilde{D}\} = \{(\rho^2 e^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2 (e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i}) \mid \rho \in \mathbb{R}_+, \alpha \in [0, 2\pi]\}.$$ 

Writing $\rho = 0$ we get the singular values of $\tilde{f}$,

$$\Sigma = \tilde{f}(\tilde{\Sigma}) = \{(\epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2 (e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i}) \}.$$ 

The inward normal field of $\Sigma$ in $D$ is the derivative of the curve

$$\gamma(t) = (te^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2 (e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i})$$

at $t = 0$, that is $\gamma'(t)|_{t=0} = (\epsilon^{2i\alpha}, 0, 0)$. The pushing out of $\Sigma$ (cf. Definition 2.2.11) is

$$\Sigma' = \Sigma - \delta \cdot \gamma'(t)|_{t=0} = \{(-\delta e^{2i\alpha} + \epsilon^2 e^{2i\alpha}(e^{-6i\alpha} - 2), \epsilon^2 (e^{-4i\alpha} - e^{2i\alpha}), -\epsilon e^{-2\alpha i})\},$$

where $0 < \delta \ll \epsilon$. By Definition 2.2.11 we need the linking number of $\tilde{f}(\mathbb{R}^4)$ and $\Sigma'$ in $\mathbb{R}^6$. To calculate it we fill in $\Sigma' \simeq S^1$ with a ‘membrane’, which here will be the disc

$$H = \{(-\delta w + \epsilon^2 (\bar{w}^2 - 2w), \epsilon^2 (\bar{w}^2 - w), -\epsilon \bar{w}) \mid w \in \mathbb{C}, |w| \leq 1\}.$$ 

$l(\tilde{f})$ is the algebraic number of the intersection points of $H$ and $\tilde{f}(\mathbb{R}^4)$. The only solution is $w = 0$, $(s, t) = (0, 0)$, and the intersection at this point is transverse. Hence, for the smooth perturbation $\tilde{f}$ of the cross cap $l(\tilde{f}) = \pm 1$.

Next we compute $L$. The set of the double points of $\Phi$ is $\tilde{D} = \{(s, 0) \mid s \neq 0\} \subset \mathbb{C}^2$.

The set of the double values is $D = \Phi(\tilde{D}) = \{(s^2, 0, 0) \mid s \neq 0\} \subset \mathbb{C}^3$, and the set of the double values of $f$ is $D_f = D \cap S^5 = \{(s^2, 0, 0) \mid |s| = 1\} \subset S^5$.

The sum of the normal vectors at $(s^2, 0, 0)$ is $(0, 0, \bar{s}^2)$. Hence the shifted copy of $D$ along $N$ is $D' = D_f + \delta N = \{(s^2, 0, \delta \bar{s}^2) \mid |s| = 1\}$.

Since $D'$ does not intersect $\Phi(\mathbb{C}^2)$ for $\delta \in (0, 1]$, we can choose $\delta = 1$. An injective parametrization of $D_f + \delta N$ is $D' = \{(z, 0, \bar{z}) \mid |z| = 1\}$, where $z = s^2$. To calculate the linking number of $\Phi(\mathbb{C}^2)$ and $D'$ in $\mathbb{R}^6$, we need a membrane which fills in $D$. We take

$$H = \{(z, \sqrt{1 - |z|^2}, \bar{z}) \mid |z| \leq 1\} \simeq D^2.$$ 

$L(f)$ is the algebraic number of the intersection points of $\Phi(\mathbb{C}^2)$ and $H$. But there is only one such point, namely $P := \Phi(\sqrt{\xi}, 0) = (\xi, 0, \xi \sqrt{\xi}, 0, \xi)$, where $\xi$ is the real root of $g(z) := z^3 + z^2 - 1 = 0$. Moreover, this intersection is transverse.
3.7.2. The $A_1$ singularity. By \[1.3.5\] \[1.4.14\] its covering is given by $\Phi_0(s,t) = (s^2, t^2, st)$. The immersion $f_0$ associated with $\Phi_0$ is not generic, $f_0$ is the 2-fold covering of the projective space composed with the inclusion. Thus all points of $S^3$ are double points of the immersion $f_0$. Compare $f_0$ with the immersion $i \circ g_1$ described in the end of Subsection \[2.3.2\] see also [\[1\]. Section 4]. $i \circ g_1$ is regular homotopic to $f_0$ (up to pre-composing with a reflection), and has the same structure $S^3 \to \mathbb{RP}^3 \hookrightarrow \mathbb{R}^5$ and vanishing $L$-invariant.

On the other hand, by \[1.4.14\] $C(\Phi_0) = 3$, and the codimension of the second fitting ideal shows that $T(\Phi_0) = 1$, see \[1.3.5\]. The finiteness of these invariants shows that the number of cross caps and triple points of a generic deformation of $\Phi_0$ are independent of the chosen deformation. Below we give a concrete deformation $\Phi_\epsilon$ of $\Phi_0$ and we calculate the invariant $L$ of the generic immersion $f_\epsilon$ associated with $\Phi_\epsilon$.

The deformation is $\Phi_\epsilon(s,t) = ((s - \epsilon)s, (t - \epsilon)t, st)$, see Figure \[1.3\] The vector field

$$\tilde{N}(s,t) = \partial_s \Phi_\epsilon(s,t) \times \partial_t \Phi_\epsilon(s,t) = \begin{pmatrix} \tilde{t}(2\tilde{t} - \epsilon) \\ -\tilde{s}(2\tilde{s} - \epsilon) \\ (2\tilde{s} - \epsilon)(2\tilde{t} - \epsilon) \end{pmatrix}$$

is 0 at the points $(0, \epsilon/2), (\epsilon/2, 0)$ and $(\epsilon/2, \epsilon/2)$. These are the cross caps.

The defining equation $\Phi_\epsilon(s,t) = \Phi_\epsilon(s', t')$ (where $(s,t) \neq (s', t')$) of the double points leads to the system of equations

$$(s - s')(s + s' - \epsilon) = 0, \quad (t - t')(t + t' - \epsilon) = 0, \quad st = s't'.$$

Thus the double locus $\tilde{D}$ has three parts and these parts correspond to the three cross caps. The first part comes from the solution $s' = s$ and $t' = t - \epsilon$, which implies $s = 0$, hence $\tilde{D}_1 = \{(0,t) \mid t \neq \epsilon/2\}$ with $\Phi_\epsilon(0,t) = \Phi_\epsilon(0,\epsilon - t)$. This provide the double value set

$$D_1 = \Phi_\epsilon(\tilde{D}_1) = \{(0,t(t - \epsilon),0) \mid t \neq \epsilon/2\}.$$  

The second part comes from the solution $s' = \epsilon - s$ and $t' = t$, which implies $t = 0$, and $\tilde{D}_2 = \{(s,0) \mid s \neq \epsilon/2\}$ with $\Phi_\epsilon(s,0) = \Phi_\epsilon(\epsilon - s, 0)$. The set of double values is

$$D_2 = \Phi_\epsilon(\tilde{D}_2) = \{(s(s - \epsilon),0,0) \mid s \neq \epsilon/2\}.$$  

The third part comes from the solution $s' = \epsilon - s$ and $t' = \epsilon - t$, which implies $s + t = \epsilon$, and $\tilde{D}_3 = \{(s,\epsilon - s) \mid s \neq \epsilon/2\}$ with $\Phi_\epsilon(s,\epsilon - s) = \Phi_\epsilon(\epsilon - s, s)$. The set of double values is

$$D_3 = \Phi_\epsilon(\tilde{D}_3) = \{(s(s - \epsilon),s(s - \epsilon),-s(s - \epsilon)) \mid s \neq \epsilon/2\}.$$
Let \( D_i(f) = D_i \cap S^5 \) \((i = 1, 2, 3)\) denote the disjoint components of the set of the double values of \( f \). Clearly \( L(f) = L_1(f) + L_2(f) + L_3(f) \), where \( L_i(f) \) is the linking number corresponding to the component \( D_i(f) \). But \( L_1(f) = L_2(f) = L_3(f) \). Indeed, \( D_1 \) and \( D_2 \) is interchanged via the transformations \( \phi(s, t) = (t, s) \) (of \( \mathbb{C}^2 \)) and \( \psi(X, Y, Z) = (Y, X, Z) \) (of \( \mathbb{C}^3 \)), and \( D_3 \) and \( D_2 \) via \( \phi(s, t) = (\epsilon - s - t, t) \) and \( \psi(X, Y, Z) = (X + Y + 2Z, Y, -Y - Z) \). Thus, it is enough to calculate \( L_1(f) \). The needed vector field along \( D_1 \) is

\[
N(0, t(t - \epsilon), 0) = \tilde{N}(0, t) + \tilde{N}(0, \epsilon - t) = ((2\ell - \epsilon)^2, 0, 0).
\]

The set of the double values of \( f \) corresponding to \( D_1 \) is

\[
D_1(f) = D_1 \cap S^5 = \{(0, t(t - \epsilon), 0) \mid |t(t - \epsilon)| = 1 \}.
\]

The shifted \( D_1(f) \) along \( N \) is

\[
D'_1 = D_1(f) + \delta N = \{(\delta(2\ell - \epsilon)^2, t(t - \epsilon), 0) \mid |t(t - \epsilon)| = 1 \},
\]

where \( \delta \) is small enough. Nevertheless, we can choose \( \delta = 1 \), because \( D'_1 \cap \Phi(\mathbb{C}^2) = \emptyset \) for any \( \delta \in (0, 1] \). With the notation \( z = t(t - \epsilon) \) we give an injective parametrization \( D'_1 = \{(4\bar{z} + \epsilon^2, z, 0) \mid |z| = 1 \} \). We fill it with the membrane

\[
H = \{(4\bar{z} + \epsilon^2, z, i\sqrt{1 - |z|^2}) \mid |z| \leq 1 \}.
\]

Computing the intersection points of \( H \) and \( \Phi(\mathbb{C}^2) \) leads to the equations

\[
4\bar{z} + \epsilon^2 = a(a - \epsilon), \quad z = b(b - \epsilon), \quad i\sqrt{1 - |z|^2} = ab,
\]

with \( |z| \leq 1 \) and \( \epsilon \) small. The first two equations imply that \(|a| < 5 \) and \(|b| < 2 \). Multiplying the first two equations one gets

\[
z(4\bar{z} + \epsilon^2) = a^2b^2 - a^2b\epsilon - ab^2\epsilon + ab\epsilon^2.
\]

From the third equation follows \( a^2b^2 = |z|^2 - 1 \), hence

\[
3|z|^2 = -1 - z\epsilon^2 - a^2b\epsilon - ab^2\epsilon + ab\epsilon^2,
\]

and the right hand side is negative if \( \epsilon \) is small enough. Hence \( H \cap \Phi(\mathbb{C}^2) = \emptyset \), and \( L(f) = 0 \).

3.8. A note on the real case

3.8.1. Immersions associated with real germs. There is a real version of part (b) of Theorem 3.1.2 which follows directly from the result of Whitney and Smale.
Let $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{2n+1}, 0)$ be a real analytic germ singular only at 0. With the same method as in the complex case we can associate an immersion $f : S^n \hookrightarrow S^{2n}$ with $\Phi$ (see 1.1.2). The set of regular homotopy classes is

$$\text{Imm}(S^n, S^{2n}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ \mathbb{Z}/2 & \text{if } n \text{ is odd,} \end{cases}$$

and the invariant equals to the algebraic number of self-intersection points of a stable immersion regular homotopic to $f$ (mod 2 if $n$ is odd). Cf. examples 2.1.11, 2.1.13 and 2.1.14.

A stabilization $\Phi'$ of $\Phi$ has only cross cap type singularities, i.e. locally right–left equivalent with germs of the form $(s,t) \mapsto (s^2, st, t)$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}^n$, cf. Example 1.2.18 and Remark 1.4.8. These cross caps are isolated, and if $n$ is even, we can associate a sign for each of them as a boundary component of the corresponding oriented double point curve. $\Phi'$ restricted to the boundary is a stable immersion $f' : S^n \hookrightarrow S^{2n}$. $f'$ and $f$ are regular homotopic, and $f'$ has two kinds of double values:

(a) double values related to a cross cap (that is, they are connected by a segment consisting of double values of $\Phi'$),

(b) double values not related to a cross cap.

When $n$ is even, the sign associated to a cross cap agrees with the sign associated with the self intersection point of $f'$ related to the cross cap. Thus the algebraic number of such cross caps is equal to the algebraic number of double values of type (a) (mod 2 if $n$ is odd). The double points of type (b) are pairwise joined up by segments of the double values of $\Phi'$, thus the algebraic number of them is 0. Moreover, it can happen that two cross caps are joined by a segment consisting of double values of $\Phi'$, but then they will have different algebraic sign, hence they will not contribute in the sum. Hence, we proved:

**Proposition 3.8.1.** The Smale invariant of $f$ agrees with the algebraic number of the cross cap points appearing in a stabilization of $\Phi$ (mod 2 if $n$ is odd).

**Remark 3.8.2.** In [33] a complete topological invariant of finitely determined real germs $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is introduced and calculated in many cases: the Gauss word of the associated immersion $\Phi|_{S^1} : S^1 \hookrightarrow S^2$. 
Singular holomorphic spacegerms

4.1. The Milnor fibre

4.1.1. Isolated, complete intersection and normal singularities. This chapter serves as an introduction and background to Chapter 5, where we study the Milnor fibre of non-isolated hypersurfaces.

The aim of this subsection is to give a short introduction of the basic types of singularities, and clarify the position of our mostly interested type of surface singularities in this classification, see Remark 4.1.5. For the definitions and basic properties we refer to [29, 20, 51].

A complex analytic spacegerm \( (X, 0) \subset (\mathbb{C}^{n+k}, 0) \) is the zero set of a holomorphic germ \( f = (f_1, f_2, \ldots, f_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0) \) [20, Definition 3.4.2]. We assume that the ideal \( I := (f_1, f_2, \ldots, f_k) \subset O(\mathbb{C}^{n+k}, 0) \) is a radical ideal (i.e. \( \sqrt{I} = I \)), in this case \( I \) provides reduced analytic structure of \( (X, 0) \). (In general, by local Nullstellensatz [20, Theorem 3.4.4.], \( \sqrt{I} \) is equal to the set of germs vanishing on \( (X, 0) \)).

If \( k = 1 \), then \( (X, 0) \) is called hypersurface in \( (\mathbb{C}^{n+1}, 0) \), see [29, (1.1)]. If the rank of \( df_0 \) is equal to \( k \), then \( (X, 0) \) is regular (or smooth), i.e. it is the germ of a complex manifold. The dimension of \( (X, 0) \) is defined as \( (X, 0) = n + k - r \), where \( r \) is the rank of the Jacobian \( df \) at a generic point \( p \in X \). \( (X, 0) \) is called complete intersection if \( \dim_{\mathbb{C}}(X, 0) = n \) [29, (1.5)]. Any hypersurface singularity is a complete intersection by [29, (1.1)]. If there is a small enough representative of \( (X, 0) \) such that \( \text{rk}(df_p) = r \) for all \( p \in X \setminus \{0\} \) and \( \text{rk}(df_0) < r \), then \( (X, 0) \) has an isolated singularity at \( 0 \) (we also say that \( (X, 0) \) is isolated).

\( (X, 0) \) is irreducible if for any decomposition \( (X, 0) = (X_1, 0) \cup (X_2, 0) \), where \( (X_1, 0) \) and \( (X_2, 0) \) are complex analytic spacegerms, \( (X_1, 0) = (X, 0) \) or \( (X_2, 0) = (X, 0) \) holds [20, Definition 3.4.17.]. \( (X, 0) \) is irreducible if and only if \( I \subset O(\mathbb{C}^{n+k}, 0) \) is a prime ideal [20, Corollary 3.4.18.]. For a hypersurface singularity \( (X, 0) \subset (\mathbb{C}^{n+1}, 0) \) that means \( f \) is irreducible in \( O(\mathbb{C}^{n+1}, 0) \).

The local ring of analytic functions defined on \( (X, 0) \) is \( O_{(X, 0)} = O(\mathbb{C}^{n+k}, 0) / \sqrt{I} \), cf. [20, Definition 3.4.19., Lemma 3.4.20.]. An irreducible singularity \( (X, 0) \) is called normal if \( O_{(X, 0)} \) is an integrally closed ring in its field of fractions, or equivalently: if any bounded
holomorphic function \( g : X \setminus \{0\} \to \mathbb{C} \) can be extended to an analytic function defined on \( X \), see [51, Definition 1.3.], and also [20, Theorem 4.4.15.].

The normalization of an irreducible germ \((X, 0)\) is a normal germ \((\bar{X}, 0)\) together with a finite, generically 1 to 1 map germ \( g : (\bar{X}, 0) \to (X, 0) \), cf. [20, Definition 4.4.5.]. Every irreducible germ admits a unique normalization. If \( g : (\bar{X}, 0) \to (X, 0) \) is the normalization of \((X, 0)\), then \( \mathcal{O}(\bar{X}, 0) \) is isomorphic with the integral closure of \( \mathcal{O}(X, 0) \) [51, 1.5.]. See [20, Remark 4.4.6., Theorem 4.4.8.].

**Example 4.1.1 (Curves).** Let \((X, 0)\) be an irreducible curve singularity, i.e. \( \dim_\mathbb{C} X = 1 \). \((X, 0)\) is normal if and only if it is smooth. Hence the normalization of a curve is a parametrization \( p : (\mathbb{C}, 0) \to (X, 0) \), see [20, Theorem 4.4.9., 4.4.10.].

**Example 4.1.2 (Surfaces).** A normal 2-dimensional singularity is always isolated (or regular), and a complete intersection surface germ is normal if and only if it has (at most) an isolated singularity [51, 1.7 (b), 1.8.]. For instance, the quotient singularities in Example 3.1.4 are normal hypersurface singularities, hence their covering germs \( \Phi : (\mathbb{C}^2, 0) \to (X, 0) \subset (\mathbb{C}^3, 0) \) are not normalizations.

**Example 4.1.3 (Finitely determined germs).** A finitely determined germ \( \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \) is the normalization of its image \((X, 0) = f^{-1}(0)\), where \( f \) is the generator of the 0-th Fitting ideal \( \mathcal{F}_0(\Phi_\ast \mathcal{O}(\mathbb{C}^n, 0)) \), cf. Section 1.3. It can be illustrated by the Whitney umbrella \((X, 0) \subset (\mathbb{C}^3, 0)\) parametrized by \( \Phi(s, t) = (s, t^2, st) \) or given as the zero set of \( f(x, y, z) = x^2y - z^2 \). Since \( t^2 - y = 0, t = z/x \) is an element of the integral closure of \( \mathcal{O}(X, 0) \) in the fraction field of \( \mathcal{O}(X, 0) \). The extension of \( \mathcal{O}(X, 0) \) with \( t \) is the integral closure of \( \mathcal{O}(X, 0) \), and it is isomorphic with \( \mathcal{O}(\mathbb{C}^2, 0) \). See [20, 4.4.7. (5)] for details.

**Example 4.1.4 (Cuspidal edge).** The germ \( \Phi(s, t) = (s, t^2, t^3) \) in Example 1.4.13 is also the normalization of its image.

**Remark 4.1.5.** None of the sets \( \{ \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \mid \Phi \text{ is singular only at } 0 \} \) and \( \{ \Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0) \mid \Phi \text{ is the normalization of its image} \} \) includes the other one, as it is shown by the examples 1.1.2, 1.1.3, 1.1.4. Indeed the second class contains exactly the finite, generically 1 to 1 map germs. However the intersection of the two sets includes very important classes of germs, for instance the finitely \( A \)-determined germs.

**4.1.2. Milnor fibration.** In this subsection we introduce the notion of the Milnor number and Milnor fibration [42, 52, 29].

Throughout this subsection let \((X, 0) = f^{-1}(0) \subset (\mathbb{C}^{n+1}, 0)\) be a hypersurface singularity of dimension \( n \), where \( f \in \mathcal{O}(\mathbb{C}^{n+1}, 0) \) is square free, i.e. it is a reduced equation of
(X, 0). Let \( J = (\partial_i f)_{i=1,\ldots,n+1} \subset \mathcal{O}(\mathbb{C}^{n+1}) \) be the Jacobian ideal of \( f \) generated by the partial derivatives of \( f \). The Milnor number of \((X, 0)\) is \( \mu(X, 0) := \dim_{\mathbb{C}} \mathcal{O}(\mathbb{C}^{n+1})/J \) [29 (1.4)].

### Theorem 4.1.6

(a) [29 Proposition (1.2)] \( \mu(X, 0) \) is finite if and only if \((X, 0)\) has (at most) an isolated singularity.

(b) [52 E.1.6.] If \((X, 0)\) is isolated, then any stabilization of \( f \) has \( \mu(X, 0) \) Morse-points.

Since the defining germ \( f \) of an isolated singularity \((X, 0)\) is finitely determined, and \( f \) is stable if and only if it is a Morse function, cf. Example [1.2.15] Theorem 4.1.6 is analogue with theorems [1.4.2 and 1.4.3].

By [42 Corollary 2.9., Remark 2.11., 52 Theorem 1.10.] there exists an \( 0 < \epsilon_0 \) such that for all \( 0 < \epsilon \leq \epsilon_0 \) the oriented homeomorphism type of the intersection \( K_\epsilon \) of the sphere \( S_{\epsilon}^{n+1} \subset \mathbb{C}^{n+1} \) with radius \( \epsilon \) and \( X \) does not depend on \( \epsilon \), that is, the pair \( (S_{\epsilon}^{n+1}, K_\epsilon) \) has the same homeomorphism type for all \( 0 < \epsilon \leq \epsilon_0 \). Let us fix an \( 0 < \epsilon \leq \epsilon_0 \). \( K := X \cap S_{\epsilon}^{n+1} \) is the link of \( X \). The embedded link determines the embedded topological type of \((X, 0) \subset (\mathbb{C}^{n+1}, 0)\), namely the pair \((B_{\epsilon}^{2n+2}, X \cap B_{\epsilon}^{2n+2})\) is homeomorphic to the cone over \((S_{\epsilon}^{2n+1}, K)\). (In the non-isolated case this fact is proved in [4]. Note that the link can also be defined when \((X, 0) \subset (\mathbb{C}^{n+k}, 0)\) is not hypersurface, as the intersection of \( X \) with the sphere in \( \mathbb{C}^{n+k} \) of small enough radius, or as the \( \epsilon \)-level set of any real analytic germ \( \rho : (X, 0) \to ([0, \infty), 0), \rho^{-1}(0) = 0 \) for small enough \( \epsilon \) [29 (2.4), (2.5)].)

The Milnor fibre of \((X, 0)\) can be defined via two different fibre bundles, which happen to be isomorphic. The first is the local fibration of \( f \), see [52 Theorem 2.5.], [29 Theorem 2.8].

### Theorem 4.1.7

(a) For \( 0 < \delta \ll \epsilon \)

\[
(4.1.1) \quad f : f^{-1}(D_{\delta}^2 \setminus \{0\}) \cap B_{\epsilon}^{2n+2} \to D_{\delta}^2 \setminus \{0\}
\]

is a \( C^\infty \) locally trivial fibration, such that its restriction \( f : f^{-1}(D_{\delta}^2 \setminus \{0\}) \cap S_{\epsilon}^{2n+1} \to D_{\delta}^2 \setminus \{0\} \) is also a \( C^\infty \) locally trivial fibration.

(b) If, additionally, \((X, 0)\) is isolated, then the fibration \( f : f^{-1}(D_{\delta}^2 \setminus \{0\}) \cap S_{\epsilon}^{2n+1} \to D_{\delta}^2 \setminus \{0\} \) extends to a \( C^\infty \) locally trivial fibration \( f : f^{-1}(D_{\delta}^2) \cap S_{\epsilon}^{2n+1} \to D_{\delta}^2 \), which is in fact trivial fibration, since \( D_{\delta}^2 \) is contractible.

Next, we present the second fibration, the so-called Milnor fibration, see [42 Theorem 4.8.], [52 Theorem 3.5.]. Let \( U \) be an open tubular neighborhood of \( K \subset S_{\epsilon}^{2n+1} \), and define \( \phi(z) = f(z)/|f(z)| \) for \( f(z) \neq 0 \).
Theorem 4.1.8. (a) 
(4.1.2) \( \phi : S^{2n+1} \setminus U \to S^1 \)

is a \( C^\infty \) locally trivial fibration.

(b) The local fibration (4.1.1) and the Milnor fibration (4.1.2) are bundle isomorphic.

The Milnor fibre \( F \) is the fibre of either of the bundles (4.1.1) and (4.1.2), that is, \( F \simeq f^{-1}(\delta \theta) \cap B^{2n+2}_\epsilon \simeq \phi^{-1}(\theta) \) with any \( 0 < \delta \ll \epsilon, |\theta| = 1 \). The Milnor fibre is well defined up to orientation preserving diffeomorphism, that is, it is independent of the different choices.

Here we collect some important properties of \( K \) and \( F \).

Theorem 4.1.9. (a) [42, Theorem 5.1.] \( F \) is an oriented, compact, smooth, parallelizable \( 2n \)-manifold with boundary (in fact, \( F \) is complex manifold), and has the homotopy type of a finite CW-complex of dimension \( n \).

(b) [42, Theorem 5.2.] \( K \) is \((n-2)\)-connected.

Isolated singularities have more restrictive properties for \( K \) and \( F \).

Theorem 4.1.10. Let \((X,0)\) be an isolated hypersurface singularity in \((\mathbb{C}^{n+1},0)\). Then

(a) [42, Corollary 2.9.] \( K \) is a smooth, oriented, closed \((2n-1)\)-submanifold of \( S^{2n+1} \).

(b) (Corollary of Theorem 4.1.7, part (b).) The boundary \( \partial F \) of \( F \) is diffeomorphic to \( K \), actually, they are isotopic submanifolds of \( S^{2n+1} \).

(c) [42, Theorem 6.5.] \( F \) is homotopically equivalent with a bouquet of \( n \)-spheres.

(d) [42, Theorem 7.2.] The number of spheres in the bouquet is equal to the Milnor number \( \mu(X,0) \), i.e. \( \text{rk}(H_n(F,\mathbb{Z})) = \mu(X,0) \).

4.2. Resolution and plumbing

4.2.1. Resolution graph. The topological information of a resolution of the normal surface singularity \((X,0)\) can be encoded in the resolution graph, which also serves as a plumbing graph for the link. We refer to [51, 55, 56].

Let \((X,0)\) be normal singularity of dimension \( 2 \), recall that \((X,0)\) is isolated. A resolution of \((X,0)\) is a complex analytic map \( p : (\tilde{X},E) \to (X,0) \), where \( \tilde{X} \) is a smooth complex manifold of dimension \( 2 \) with boundary, \( p^{-1}(0) = E \subset \tilde{X} \) is a complex analytic curve and \( p|_{\tilde{X}\setminus E} : \tilde{X} \setminus E \to X \setminus \{0\} \) is a biholomorphism [51, Definition 1.19.(a)]. Let \( E = \bigcup_{v \in \mathcal{V}} E_v \) be the irreducible decomposition of \( E \), where \( \mathcal{V} \) is a finite set. The components \( E_v \) of \( E \) are called exceptional divisors. The resolution is called good if each \( E_v \) is smooth curve, they intersect each other transversally, in particular the intersection of any distinct three of them is empty. A good resolution always exists and it is not unique.
\[ \partial \tilde{X} \text{ is diffeomorphic with the link } K \text{ of } (X, 0). \] Hence the resolution \( \tilde{X} \) and the Milnor fibre \( F \) are two complex fillings of \( K \), which fact allows to compare their invariants, see for instance the formulas of Laufer \([27]\) or Durfee \([5]\) and their generalizations, e.g. \([70]\).

The tubular neighbourhood \( N(E_v) \) of \( E_v \subset \tilde{X} \) is diffeomorphic with the total space of (the disc bundle of) the normal bundle of \( E_v \), which is a \( C^\infty D^2\)-bundle over the Riemannian surface \( E_v \). Thus the genus \( g_v \) of \( E_v \) and the Euler number \( e_v \) of the normal bundle of \( E_v \subset \tilde{X} \) determines \( N(E_v) \) with its \( D^2\)-bundle structure. Furthermore \( \tilde{X} \) is diffeomorphic to \( \bigcup_{v \in V} N(E_v) \). The topology of \( \tilde{X} \), hence the topology of \( K \) too, can be encoded in a decorated graph \( G \), called the resolution graph, whose vertex set is \( V \), and the vertexes \( v \) and \( w \) are joined with \( k \) edges if \( |E_v \cap E_w| = k \). Each vertex \( v \) is decorated by two weights, the genus \( g_v \) and the Euler number \( e_v \) \([51]\) Definition 1.19.(b)]. We omit the genus if it is 0.

Define \( E_v \cdot E_w = |E_v \cap E_w| \) for \( v \neq w \), and \( E_v \cdot E_v = e_v \). \( H_2(\tilde{X}, \mathbb{Z}) \cong H_2(E, \mathbb{Z}) \cong \mathbb{Z}^{|V|} \) is generated by the classes of \( E_v \), and \( E_v \cdot E_w \) is the intersection number of the corresponding cycles. These numbers form the intersection matrix, which contains the same information as the resolution graph, whenever \( g_v = 0 \) for all \( v \in V \). A decorated graph \( G \) occurs as a resolution graph of a normal surface singularity if and only if the corresponding intersection matrix is negative definite \([51]\) 1.23.(a)]

\( \tilde{X} \) and \( K \) can be recovered from \( G \) by plumbing, see below. On the other hand, \( K \), or equivalently, the homeomorphism type of \( (X, 0) \) determines the graph \( G \), hence the diffeomorphism type of \( \tilde{X} \) (up to plumbing operations) by \([56]\) Theorem 2.

### 4.2.2. Plumbing.

In Chapter \([5]\) we use the plumbing construction in a more general context than it occurs for the resolution of normal singularities. Let \( G \) be a connected graph which can have multiple edges and loops, and each vertex \( v \) is decorated by the weights \(([g_v], e_v), g_v \geq 0 \). Furthermore each edge is decorated with a sign \( \oplus \) or \( \ominus \). Then the plumbing construction associates two manifolds to \( G \), a compact oriented smooth 4-manifold \( M^4(G) \), and its boundary, the closed oriented smooth 3-manifold \( M^3(G) \), as follows. For each vertex \( v \) fix a smooth, oriented \( D^2 \)-bundle \( p_v : N_v \rightarrow E_v \) with Euler number \( e_v \), whose base space \( E_v \) is a closed oriented surface with genus \( g_v \). If \( v \) is the endpoint of \( s_v \) edges, then fix \( s_v \) basepoints in \( E_v \), fix disc-neighborhoods \( \{D^2_{v,i}\}_{i=1,...,s_v} \) of these points, as well as orientation preserving local trivializations \( p_v^{-1}(D^2_{v,i}) \cong D^2_{v,i} \times D^2 \). Any edge between \( v \) and \( w \) determines a pair of multidisks \( D^2_{v,i} \times D^2 \) and \( D^2_{w,j} \times D^2 \) in \( N_v \) and \( N_w \) respectively. Identify them by \( (x, y) \sim (y, x) \), if the edge has \( \oplus \) sign, and by \( (x, y) \sim (\bar{y}, \bar{x}) \), if the edge has \( \ominus \) sign (where \( \bar{x} \) is the complex conjugate of \( x \)). After these
gluings we obtain a 4-manifold with corners, but these corners can be smoothed out. Cf. [51] 1.25, [56], [55] 4.1.

The resolution graph $G$ corresponding to a good resolution $\tilde{X}$ of a normal surface singularity $(X,0)$ does not have loops and each edge decoration is $\oplus$. The associated 4-manifold $M^4(G)$ is diffeomorphic to $\tilde{X}$, whose boundary $M^3(G)$ is diffeomorphic with the link $K$.

The 3-manifolds in the form $M^3(G)$ associated with a plumbing graph $G$ are called plumbed 3-manifolds.

The plumbing graph of a plumbed 3-manifold is not unique. The possible operations of a plumbing graph which do not change the oriented diffeomorphism type of $M(G)$ are listed in [56], [55] 4.2. We mention here some of them, which will be used in the examples of Chapter 5. We follow the notations of [56].

The graph operation [R0.(a)] is reversing the signs on all edges other than loops adjacent to any fixed vertex. By [R0.(a)] the sign of the edges can be reversed preserving the only relevant information, the parity of the number of $\ominus$–edges along the cycles of the graph.

[R1] corresponds to the (inverse of the) blowing up of a point $q \in E_v \subset M^4(G)$. This operation is the key tool in the construction of embedded resolution of plane curve singularities, see Subsection 4.2.3. $q$ can be a generic point, an intersection point of $E_v$ and $E_w$, or the self intersection point of $E_v$. The three cases of [R1] are shown in the picture, where $\epsilon = \pm 1$ and the edge signs $\epsilon_0, \epsilon_1, \epsilon_2$ are related by $\epsilon_0 = -\epsilon_1 \epsilon_2$.

We also use the 0-chain absorption [R3] and the oriented handle absorption [R5]. The edge signs $\epsilon'_i$ ($i = 1, ..., s$) are related by $\epsilon'_i = -\epsilon \epsilon_i$ provided that the edge sign in question is not on a loop, and $\epsilon'_i = \epsilon_i$, if it is on a loop.
We also use plumbing graphs with arrowhead vertexes. An arrowhead vertex based on the vertex \( v \) denotes a generic (oriented) fibre \( D^2 \) of \( N_v \subset M^4(G) \). Its boundary is an oriented knot \( S^1 \subset M^3 \). Additionally, the \( S^1 \)-bundle structure of \( \partial N_v \) determines a trivialization \( S^1 \times D^2 \) of the tubular neighbourhood of the knot.

In [56] there is a third decoration, the non-negative integer \( h_v \), which denotes the number of boundary components of the genus–\( g_v \) surface. This concept is closely related to the arrowhead vertexes. In fact, the resulted 3-manifold has \( h_v \) boundary components for each vertex \( v \), and it can be obtained from the closed 3-manifold by removing the open tubular neighbourhoods of \( h_v \) knots correspond to \( h_v \) arrowhead vertexes based on \( v \).

### 4.2.3. Embedded resolution of plane curves.

Let \( (D, 0) \subset (\mathbb{C}^2, 0) \) be a plane curve singularity defined as the zero set of the reduced germ \( d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \). Let \( d = \Pi d_i \) be the irreducible decomposition of \( d \in \mathcal{O}(\mathbb{C}^2, 0) \), and let \( (D, 0) = \bigcup_{i=1}^l(D_i, 0) \) be the irreducible decomposition of \( (D, 0) \), where \( (D_i, 0) = d_i^{-1}(0) \).

A **good embedded resolution** of \( (D, 0) \) is a good resolution \( \pi : (\widetilde{\mathbb{C}}^2, E) \to (\mathbb{C}^2, 0) \) of \( (\mathbb{C}^2, 0) \) with the additional property, that \( (d \circ \pi)^{-1}(0) \) is a normal crossing divisor in \( \widetilde{\mathbb{C}}^2 \), that is, its irreducible components are smooth and intersects each other transversally [51, 1.28.]. Note that each component \( E_v \) of the exceptional divisor \( E \) of any good resolution of \( (\mathbb{C}^2, 0) \) is diffeomorphic to \( \mathbb{C}P^1 \approx S^2 \).

The irreducible components of \( (d \circ \pi)^{-1}(0) \) are the exceptional divisors \( E_v \) and the strict transforms corresponding to the components of \( (D, 0) \). That is, the components \( \tilde{D}_i \) of the strict transform \( \tilde{D} = (d \circ \pi)^{-1}(0) \setminus E \subset \widetilde{\mathbb{C}}^2 \) of \( D \). Each \( \tilde{D}_i \) intersects (transversally) only one exceptional divisor, say \( E_{v(i)} \).

The **total transform** of \( D_i \) is the divisor \( \text{div}(d_i \circ \pi) = \Sigma_{v \in V} m_i(v) \cdot E_v + \tilde{D}_i \), where the multiplicity \( m_i(v) \in \mathbb{Z}_{>0} \) is the vanishing order of \( d_i \circ \pi \) along \( E_v \).

Let \( \Gamma \) be the embedded resolution graph of \( (D, 0) \subset (\mathbb{C}^2, 0) \) associated with the resolution \( \pi \), i.e. the resolution graph of \( \pi \), where the arrowhead vertexes \( \{a_i\}_{i=1}^l \) codify the strict transforms \( \{\tilde{D}_i\}_{i=1}^l \). Note that \( M^3(\Gamma) \simeq S^3 \) since it is the link of \( (\mathbb{C}^2, 0) \), and the link \( \bigcup_{i=1}^l S^1 \subset S^3 \) determined by the arrowhead vertexes is the link of \( (D, 0) \).
The multiplicities $m_i(v) \ (v \in \mathcal{V}, \ 1 \leq i \leq l)$ are determined by $\Gamma$ via the identities (see e.g. [26, 51])

\[(4.2.1) \quad \sum_{v \in \mathcal{V}} m_i(v)(E_v \cdot E_w) + (\xi_i \cdot E_w) = 0 \quad \text{for all } w \in \mathcal{V}.\]

A good embedded resolution of a plane curve can be obtained by a sequence of blowings-ups, whose local model is the blowing-up of the origin in $\mathbb{C}^2$. That is the complex manifold $\mathcal{B} = \{(p, l) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid p \in l\}$ with its natural projection $\pi : \mathcal{B} \rightarrow \mathbb{C}^2$. The exceptional divisor $\pi^{-1}(0)$ is $\mathbb{C}P^1$, and $\text{pr}_2 : \mathcal{B} \rightarrow \mathbb{C}P^1$ defines a complex line bundle over it with Euler number $(-1)$.

$\mathcal{B}$ admits a natural atlas with two charts whose domains are $U_x = \pi^{-1}(\{x \neq 0\})$ and $U_y = \pi^{-1}(\{y \neq 0\})$, where $x$ and $y$ denote the coordinates in $\mathbb{C}^2$. The coordinates $(s, t)$ of $U_x$ are chosen such that $\pi(s, t) = (s, st)$ holds, similarly, $\pi(u, v) = (uv, v)$ holds in the coordinates $(u, v)$ of $U_y$. The transition map on $U_x \cap U_y$ is given by $t = 1/u$ and $s = uv$.

4.2.4. Examples.

**Example 4.2.1** (The quotient singularities $A-D-E$). The link of the normal hypersurface singularity described in Example 3.1.4 is the quotient of $S^3 \subset \mathbb{C}^2$ with the group action of $G$. For these singularities the Milnor fibre $F$ is diffeomorphic to the resolution $\tilde{X}$, however they are not biholomorphic, since $\tilde{X}$ contains closed analytic curves (namely, the exceptional divisors $E_i$), while $F \subset \mathbb{C}^3$ does not contain (in fact, $F$ is always a Stein manifold). Their resolution graphs are the $A-D-E$ graphs with $g_v = 0$ and $e_v = -2$ for each vertex, see [51] 1.20.(b)-(f).

**Example 4.2.2** (The plane curve $A_1$). Let $(X, 0) \subset (\mathbb{C}^2, 0)$ be the zero set of $f = xy$. Its link is $S^1 \cup S^1 \subset S^3$, the two components correspond to the irreducible components of $f$. Each $S^1$ is unknotted in $S^3$ and their linking number is 1, they form a Hopf-link. Since $\mu(X, 0) = 1$, the Milnor fibre is diffeomorphic to $S^1 \times I$. In fact, the Milnor fibre $F = f^{-1}(\delta) \cap B^4_{\epsilon}$ ($0 < \delta \ll \epsilon$) can be parametrized with $S^1 \times [r_0, r_1]$ as $x = \sqrt{\delta}re^{ai}$, $y = \sqrt{\delta}e^{-ai}$, $-\pi \leq \alpha \leq \pi$, $r_0 \leq r \leq r_1$, where $r_0r_1 = 1$ and $\delta(r_0^2 + r_1^2) = \epsilon^2$. We use this parametrization in Chapter 5 where $A_1$ appears as the transverse type of the non-isolated surfaces we study.

The embedded resolution of $(X, 0)$ can be obtained by blowing up $(\mathbb{C}^2, 0)$ once. It is easier to show this blowing up with the germ $x^2 + y^2 = (x + iy)(x - iy)$, which is equivalent with $f$, since we need only one chart for it. The total transform on the $U_x$-chart is $(s + ist)(s - ist) = s^2(1 + it)(1 - it)$, where $\{s = 0\}$ is the exceptional divisor, $\{t = \pm i\}$ are the strict transforms of the components of $(X, 0)$. The resolution graph is
with the multiplicities $m_1(v) = m_2(v) = 1$.

**Example 4.2.3.** The curve $(D, 0) = \{d(x, y) = xy^2 + x^k = 0\} \subset (\mathbb{C}^2, 0)$ (where $k \geq 1$) appears as the double point locus of the family $C_k$ of finitely determined germs, see Subsection [5.5. We present here the calculation of its embedded resolution graph, it serves as a model also for the other examples.

The irreducible decomposition of $d$ depends on the parity of $k$: it is $d(x, y) = x(y^2 + x^{k-1})$ for $k = 2n$ and $d(x, y) = x(iy + x^n)(-iy + x^n)$ for $k = 2n + 1$. After a blowing up the total transform of $d$ is $(f \circ \pi)(s, t) = s((st)^2 + s^{k-1}) = s^3(t^2 + s^{-3})$ on the $U_s$-chart, and $(f \circ \pi)(u, v) = uv(v^2 + (uv)^{k-1}) = v^3u(1 + uk^{-1}v^{-3})$ on $U_y$. The exceptional divisor $E_1$ is defined by $\{s = 0\}$ and $\{v = 0\}$ on the charts, the Euler number of its normal bundle is $(1)$ The strict transform of $D_1 = \{x = 0\}$ is $\tilde{D}_1 = \{u = 0\}$ on $U_y$, and it is not visible on $U_x$, although one can write it as $\tilde{D}_1 = \{t = \infty\}$. $\tilde{D}_1$ is smooth and intersects the exceptional divisor $E_1$ transversally.

The strict transform of the other component(s) $\{y^2 + x^{k-1} = 0\}$ is $\{t^2 + s^{k-3} = 0\}$ on $U_x$ (and is not visible on $U_y$). It is not smooth, if $k > 4$, and for $k = 4$ it is smooth, but its intersection with $E_1$ is not transverse. We continue with the blow up of the point $s = t = 0$. Since it is on $E_1$, the Euler number of $E_1$ changes by $(1)$, hence it will be $(2)$ after the second blow up.

The strict transform of $\{y^2 + x^{k-1} = 0\}$ after $m$ blowing-ups is $\{t^2_m + s^{k-1-2m} = 0\}$, where $(s_m, t_m)$ denotes the coordinates of the first chart of the $m$-th blow up. Moreover the strict transform intersects only the $m$-th exceptional divisor $E_m$, if $m < n$. The sequence of $m$ blow-ups results a line of $(-2)$-vertexes (a ‘bamboo’) with a $(1)$-vertex at the end $(m < n)$.

**Case 1: $k = 2n + 1$.**

For $k = 2n + 1$ the process reaches to the strict transform $\{t_n^2 + 1 = 0\} = \{t_n^2 = \pm 1\}$ after $n$ steps, which is smooth, the components are disjoint and intersect $E_n$ transversally. Hence the resolution graph is

\[
\begin{array}{c}
-2 \quad -2 \quad \ldots \quad -2 \quad -1
\end{array}
\]

where the number of $(-2)$-vertices in the middle is $n - 1$. By (4.2.1) the multiplicities are $m_1(E_i) = 1$ for all $i = 1 \ldots n$ and $m_2(E_i) = m_3(E_i) = i$. 
Case 2: \( k = 2n \).

For \( k = 2n \) the process reaches to the strict transform \( \{ t_{n-1}^2 + s_{n-1} = 0 \} \) after \( n - 1 \) steps, which is smooth, but its intersection with \( E_{n-1} \) is not transverse. The \( n \)-th blowing up results the strict transform \( \{ u_n + v_n = 0 \} \) on the second chart, which is smooth and intersects \( E_n = \{ v_n = 0 \} \) transversally, but it intersects also the strict transform \( \{ u_n = 0 \} \) of \( E_{n-1} = \{ s_{n-1} = 0 \} \). One more blow-up is needed at \( u_n = v_n = 0 \), which is the intersection point of \( E_n \) and the strict transform of \( E_{n-1} \). The new exceptional divisor \( E_{n+1} \) intersects (the strict transforms of) \( E_n \) and \( E_{n-1} \), and their Euler number changes by \((-1)\), hence \( e_{n-1} = -3, e_n = -2, e_{n+1} = -1 \). The resolution graph is

```
  -2   -2   -2   -3   -1
```

where the number of \((-2)\)-vertices in the middle is \( n - 2 \). By (4.2.1) the multiplicities are \( m_1(E_i) = 1 \) for \( i \neq n + 1 \) and \( m_1(E_{n+1}) = 2 \), \( m_2(E_i) = 2i \) for \( i < n \) and \( m_2(E_n) = 2n - 1 \), \( m_2(E_{n+1}) = 4n - 2 \).

4.3. Non-isolated case

For a non-isolated singularity \((X, 0)\) the link \( K \), the boundary \( \partial F \) of the Milnor fibre and the boundary \( \partial \tilde{X} \) of a resolution are three different spaces. Indeed, \( \partial F \) and \( \partial \tilde{X} \) are smooth manifolds, while \( K \) is not smooth. Moreover, any resolution \( \tilde{X} \) of \((X, 0)\) is the resolution of the normalization of \( X \) as well, thus it contains only limited information about \((X, 0)\).

Let \((X, 0) = f^{-1}(0) \subset (\mathbb{C}^3, 0)\) be a surface singularity, whose singular set \((\Sigma, 0) = (df)^{-1}(0) \cap (X, 0)\) has dimension 1. The Némethi–Szilárd book [55] provides a general algorithm to determine \( \partial F \) as a plumbed 3-manifold. Here we just sketch the decomposition of \( \partial F \) as the union of two pieces. For details see [55 2.3., 3.4.], [62] and Section 5.3 in our special case.

Let \( F = f^{-1}(\delta) \cap B_\epsilon^6 (0 < |\delta| \ll \epsilon) \) be the Milnor fibre of \((X, 0)\), its boundary is \( \partial F = f^{-1}(\delta) \cap S_\epsilon^5 \). Let \( N(\Sigma) \) be a closed tubular neighbourhood of \( \Sigma \cap S_\epsilon^5 \) in \( S_\epsilon^5 \) and let \( N^0(\Sigma) \) be its interior. Then the two parts mentioned above are \( \partial F \setminus N^0(\Sigma) \) and \( \partial F \cap N(\Sigma) \). \( \partial F \) can be obtained from these pieces by gluing them along their common boundary, which is a union of tori.

\( \partial F \setminus N^0(\Sigma) \) is diffeomorphic with \( K \setminus N^0(\Sigma) \), where \( K = X \cap S_\epsilon^5 \) is the link of \( X \). The normalization map \( n : (X_{norm}, 0) \rightarrow (X, 0) \) induces a diffeomorphism between \( n^{-1}(K \setminus \partial F) \).
$N^o(\Sigma) \subset K_{\text{norm}}$ and $K \setminus N^o(\Sigma)$, where $K_{\text{norm}}$ is the link of $(X_{\text{norm}}, 0)$. Hence this piece can be inherited from the resolution graph of the normalization of $(X, 0)$.

$\partial F \cap N(\Sigma)$ is the disjoint union of the pieces $\{\partial F \cap N(\Sigma_j)\}_j$, where $\Sigma = \bigcup_j \Sigma_j$ is the irreducible decomposition of $\Sigma$. The projection of $\partial F \cap N(\Sigma_j)$ to $\Sigma_j \cap S^5_\epsilon \simeq S^1$ induces a bundle structure on $\partial F \cap N(\Sigma_j)$ over $S^1$. Its fibre is the Milnor fibre of the transverse curve singularity of $\Sigma_j$. This is the plane curve singularity defined as the zero set of $f|_{(S,q)} : (S, q) \to (\mathbb{C}, 0)$, where $(S, q) \subset (\mathbb{C}^3, q)$ biholomorphic to $(\mathbb{C}^2, 0)$ is a $\Sigma_j$–transverse slice at an arbitrary point $q \in \Sigma_j \setminus \{0\}$. 
CHAPTER 5

Boundary of the Milnor fibre

5.1. Preliminaries

5.1.1. This chapter contains the results of the article [54]. We assume that \((X, 0) = (f^{-1}(0), 0)\) is the image of a finitely determined complex analytic germ \(\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\). This means that \(\Phi\) is a stable immersion off the origin, cf. Theorem 1.4.4 or [71, 45]. From an other approach \((X, 0)\) is a non-isolated hypersurface singularity in \((\mathbb{C}^3, 0)\) such that the transverse curve of the singular set has type \(A_1\) and the normalization of \((X, 0)\) is smooth. Then its normalization map \(\Phi : (\mathbb{C}^2, 0) \to (X, 0) \subset (\mathbb{C}^3, 0)\) is a finitely determined germ.

The main result of this chapter provides the plumbing graph of the boundary \(\partial F\) of the Milnor fibre \(F\) as a surgery, starting from the embedded resolution graph of the double points \((D, 0) \subset (\mathbb{C}^2, 0)\). The needed additional pieces, which will be glued to this primary object correspond to certain fibre bundles over \(S^1\) with fibres the local Milnor fibre of the transverse singularity type (and monodromy the corresponding vertical monodromy). The surgery itself is characterized by some homologically determined integers combined with the newly defined ‘vertical index’ associated with the irreducible components of the singular locus of \(f^{-1}(0)\).

This chapter is organized as follows. In this section we introduce the notations, in Section 5.2 we describe by several characterizations the surgery pieces associated with the components of the singular locus. In Section 5.3 and in Section 5.4 we describe the gluing and its invariants, while the last section contains several concrete examples.

The basic notions and properties of the Milnor fibre, resolution and plumbing construction are introduced in Chapter 4. For the properties of a finitely determined germ \(\Phi\) we refer to Chapter 1.

5.1.2. Double point curves. Let \(\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\) be a complex analytic germ singular only at the origin. We assume that \(\Phi\) is finitely \(e\)-determined. This happens exactly when \(\Phi\) is a stable immersion off the origin, that is, off the origin it has only single and double values and at each double value the intersection of the two smooth branches is transverse, cf. Example 1.2.17.
Write \((X, 0) := (\text{im}(\Phi), 0)\) and let \(f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)\) be the reduced equation of \((X, 0)\). \(f\) can be determined by Fitting ideal method, as a generator of \(\mathcal{F}_0(\Phi^* \mathcal{O}_{(\mathbb{C}^n, 0)})\), cf. Section 1.3. Note that \((X, 0)\) is a non-isolated hypersurface singularity, except when \(\Phi\) is a regular map, see Theorem 3.1.3 We denote by \((\Sigma, 0) = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)^{-1}(0) \subset (\mathbb{C}^3, 0)\) the reduced singular locus of \((X, 0)\) (which equals the closure of the set of double values of \(\Phi\)), and by \((D, 0)\) the reduced double point curve \(\Phi^{-1}(\Sigma) \subset (\mathbb{C}^2, 0)\). (In fact, the finite determinacy of the germ \(\Phi\) is equivalent with the fact that the double point curve is reduced; see Theorem 1.5.1.)

Let \(B^6_\epsilon\) be the \(\epsilon\)-ball in \(\mathbb{C}^3\) centred at the origin, and \(S^5_\epsilon\) its boundary \((\epsilon \in \mathbb{R}_{>0})\). Then for \(\epsilon\) sufficiently small \((B^6_\epsilon, 0)\) is a Milnor ball for the pair \((\Sigma, 0) \subset (X, 0)\), and, furthermore, \(\mathcal{B}_\epsilon := \Phi^{-1}(B^6_\epsilon)\) is a (non–metric) \(C^\infty\) ball in \((\mathbb{C}^2, 0)\), which might serve as a Milnor ball for \((D, 0)\), cf. [4.1.2]. We set \(\mathcal{G}^3 = \Phi^{-1}(S^5_\epsilon) = \partial \mathcal{B}_\epsilon, \) diffeomorphic to \(S^3\), and we treat it as the usual Milnor–ball boundary 3–sphere. Recall that the immersion associated with \(\Phi\) at the level of local neighbourhood boundaries is \(\Phi|_{\mathcal{G}^3} : \mathcal{G}^3 \to S^5\), cf. Definition 1.1.3 and Theorem 1.4.6.

5.1.3. Components and links of \(\Sigma\) and \(D\). Let \(\Upsilon \subset S^5_\epsilon\) be the link of \(\Sigma\). It is exactly the set of double values of \(\Phi|_{\mathcal{G}^3}\). Let \(L = \Phi^{-1}(\Upsilon) \subset \mathcal{G}^3\) denote the set of double points of \(\Phi|_{\mathcal{G}^3}\), that is, \(L \subset \mathcal{G}^3\) is the link of \(D\). Assume that the reduced equation of \(D\) is \(d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\), whose irreducible decomposition is \(d = \prod^l_{i=1} d_i\). The irreducible components of \(D\) are denoted by \((D_i, 0) = d_i^{-1}(0)\) and their link components in \(L\) by \(L_i\), \(1 \leq i \leq l\). \(D\) is equipped with an involution \(\iota : D \to D\) which pairs the double points. \(\iota|_L\) induces a permutation (pairing) \(\sigma\) of \(\{1, 2, \ldots, l\}\), such that \(\iota(L_i) = L_{\sigma(i)}\). Moreover, \(\Phi|_L : L \to \Upsilon\) is a double covering with \(\Phi(L_i) = \Phi(L_{\sigma(i)})\). If \(i = \sigma(i)\) for some \(i\), then \(\Phi|_{L_i}\) is a nontrivial double covering of its image, while above the other components the covering is trivial. Let \(J\) be the set of pairs \(\{i, \sigma(i)\}\) \((1 \leq i \leq l)\), it is the index set of the components of \(\Upsilon\); they will be denoted by \(\{\Upsilon_j\}_{j \in J}\).

All link components are considered with their natural orientations.

For the good embedded resolution of \((D, 0)\) and the corresponding notations see Subsection 4.2.3. Note that \(2|J| - l\) is the number of non-trivially covered double point curve components, its parity is the total twist of \(\Phi|_{\mathcal{G}^3}\), cf. the end of Subsection 2.3.3. Since \(2|J| - l \mod 2\) is an additive regular homotopy invariant and its value for the complex Whitney umbrella is 1, by Theorem 3.1.2 it follows that

**Corollary 5.1.1.** \(C(\Phi) \equiv 2|J| - l \mod 2\).
5.2. The manifold $Y$

In the construction of the boundary of the Milnor fibre we will need a special 3–
manifold with torus boundary. Its several realizations and properties will be discussed in
this section.

In the sequel $S^1$ (as the boundary of the unit disc of $\mathbb{C}$) and the real interval $I := [-1, 1]$ are considered with their natural orientations; $\overline{\cdot}$ denotes the complex conjugation of $\mathbb{C}$.

5.2.1. The definition of $Y$. Consider the $\mathbb{Z}_2$–action on $S^1 \times S^1 \times I$ defined by the
involution $(x, y, z) \mapsto (-x, \overline{y}, -z)$, and define $Y$ as the quotient

\begin{equation}
Y = \frac{S^1 \times S^1 \times I}{(x, y, z) \sim (-x, \overline{y}, -z)}.
\end{equation}

$Y$ is a 3–manifold with a boundary diffeomorphic with $S^1 \times S^1$. The projections to
different components provide different ‘realizations’ of $Y$.

1) The projection to the first coordinate $x$ gives a fibration

\begin{equation}
S^1 \times I \to Y \quad \downarrow \quad S^1
\end{equation}

where the base space $S^1 = S^1/\{x \sim -x\}$ is parametrized by $x^2$, and the monodromy
diffeomorphism $S^1 \times I \to S^1 \times I$ over the base space is $(y, z) \mapsto (\overline{y}, -z)$.

2) The projection to the first two coordinates $(x, y)$ realizes $Y$ as the total space of a
fibration

\begin{equation}
I \to Y \quad \downarrow \quad \mathcal{K}
\end{equation}

with fibre $I$ and base space $\mathcal{K} := (S^1 \times S^1)/\{(x, y) \sim (-x, \overline{y})\}$, the Klein bottle. The
factorization $S^1 \times S^1 \to \mathcal{K}$ is the orientation double cover of $\mathcal{K}$. In particular, the fibration\n(5.2.3) is the segment bundle of the orientation line bundle of $\mathcal{K}$, hence the orientation
double cover of $\mathcal{K}$ is realized also by the restriction of the bundle map to the boundary
$\partial Y \simeq S^1 \times S^1$.

3) The projection to the $(x, z)$ coordinates realizes $Y$ as the total space of a fibration

\begin{equation}
S^1 \to Y \quad \downarrow \quad \mathcal{M}
\end{equation}
over the base space \( \mathcal{M} := (S^1 \times I)/\{(x, z) \sim (-x, -z)\} \), the Möbius band. In this way, \( Y \) appears as the tangent circle bundle of \( \mathcal{M} \), i.e. as the sub–bundle of the tangent bundle \( T\mathcal{M} \) consisting of unit tangent vectors. This follows from the fact that both circle bundles have the same monodromy map along the midline of \( \mathcal{M} \), namely \( S^1 \to S^1, y \mapsto \bar{y} \).

(4) The projection to the \((y, z)\) coordinates realizes \( Y \) as the total space of a projection

\[
\begin{array}{ccc}
S^1 & \to & Y \\
\downarrow & & \downarrow \\
D^2
\end{array}
\]

to the base space \((S^1 \times I)/\{(y, z) \sim (\bar{y}, -z)\}\), the 2–disc \( D^2 \). Although the involution \((y, z) \mapsto (\bar{y}, -z)\) has two fix points \((-1, 0)\) and \((1, 0)\), the quotient \( D^2 \) can be smoothed. However, the projection \( Y \to D^2 \) is not a locally trivial fibration: it is a Seifert fibration with two exceptional fibres sitting above \((-1, 0)\) and \((1, 0)\). We will refer to this \( S^1 \)–fibration as the canonical Seifert fibration of \( Y \).

The Seifert invariants of the exceptional fibres can be calculated as in p. 307 of [56]. The two exceptional fibres of the canonical Seifert fibration (5.2.5) can be seen in the projection (3) as well: they correspond to the tangent vectors of the midline. On the other hand, a generic orbit consists of those unit tangent vectors of \( \mathcal{M} \), which form non–zero angle \( \pm \alpha \) (with \( \alpha \) fixed) with the midline. Thus, a generic fibre in a neighbourhood of an exceptional fibre goes around twice and both Seifert invariants are \((2, 1)\). Hence, cf. [56], a plumbing graph of \( Y \) is:

\[
\begin{array}{ccc}
[0, 1] & \bullet & 2 \\
\bullet & & \bullet & 2 \\
\end{array}
\]

Here \([0, 1]\) denotes a genus 0 core-space with one disc removed. The Euler number of the \( S^1 \)-bundle corresponding to the middle vertex is irrelevant, the resulted 3-manifolds with boundary are diffeomorphic with each other (hence with \( Y \) too). However, the restriction of the canonical Seifert fibration to \( \partial Y \simeq S^1 \times S^1 \) determines an \( S^1 \)–fibration of the boundary. Moreover, \( Y \) admits a unique closed Seifert 3–manifold \( \bar{Y} \), from which \( Y \) can be obtained by omitting a tubular neighbourhood of a generic fibre (that is, \( \bar{Y} \) is obtained by extending the fibration of \( \partial Y \) to an \( S^1 \)–fibration without any new special Seifert fibres. In this way there is a canonical choice for the Euler number of the ‘middle’ vertex, which is the ‘middle’ Euler number of the plumbing graph of \( \bar{Y} \). We will calculate it below. Using this Euler number, the graph also determines a parametrization (framing) of \( \partial Y \simeq S^1 \times S^1 \).
5.2.2. Homotopical properties of \( Y \). In order to understand better the structure of \( Y \) we consider its fundamental domain (in coordinates \((s, t, z)\)):

\[
\begin{align*}
(0, t, z) &\sim (\pi, -t, -z) \quad \text{and} \quad (s, -\pi, z) \sim (s, \pi, z).
\end{align*}
\]

The boundary is

\[
\partial Y = \frac{[0, \pi] \times [-\pi, \pi] \times \{-1, 1\}}{(0, t, \pm 1) \sim (\pi, -t, \mp 1), \ (s, -\pi, \pm 1) \sim (s, \pi, \pm 1)} \cong S^1 \times S^1.
\]

The \( S^1 \)-action determining the canonical Seifert fibration \( \{5.2.5\} \) is induced by the translation along the \( s \)-axis. The exceptional fibres are

\[
\lambda = \{(s, -\pi, 0) \mid s \in [0, \pi]\} = \{(s, \pi, 0) \mid s \in [0, \pi]\} \quad \text{and} \quad \bar{\lambda} = \{(s, 0, 0) \mid s \in [0, \pi]\}.
\]

Any fixed \((t, z) \notin \{(0, 0), (\pm \pi, 0)\}\) determines a generic fibre in the form

\[
\{(s, t, z) \mid s \in [0, \pi]\} \cup \{(s, -t, -z) \mid s \in [0, \pi]\},
\]

where \((0, \pm t, \pm z)\) are glued together with \((\pi, \mp t, \mp z)\). For example, \(c = c_1 \cup c_2 \subset \partial Y\) is a generic fibre.

The base space \( D^2 \) of the canonical Seifert-fibration can be represented as

\[
\frac{\{0\} \times [-\pi, \pi] \times [-1, 0]}{(0, -\pi, z) \sim (0, \pi, z), \ (0, t, 0) \sim (0, -t, 0)}.
\]

Its boundary is the class of \( m \), a circle.

Next, we describe the fundamental group and the homology of \( Y \). \( Y \) is homotopically equivalent with the Klein bottle \( \mathcal{K} \). Let us choose the base point \( P_0 = (0, -\pi, 0) \). All four vertexes of the rectangle representing \( \mathcal{K} \) represent \( P_0 \). Thus the fundamental group of \( Y \)
can be presented as

\[ \pi_1(Y) = \langle \mu, \lambda \mid \mu \cdot \lambda \cdot \mu = \lambda \rangle, \]

where \( \mu \) and \( \lambda \) denote also the class of \( \mu \) and \( \lambda \) in \( \pi_1(Y) = \pi_1(K) \); cf. with the description (2) from 5.2.1). A more precise description can be given via the next diagrams, provided by the \( \{ z = 0 \} \) subspace of \( Y \) (which can be identified with \( K \)).

The first diagram shows homological cycles. In order to rewrite the fundamental group, let \( \bar{\lambda} \) be the closed path shown in the second diagram by the dashed line. Then \( \bar{\lambda} = \mu \cdot \lambda \) in \( \pi_1(Y) \). Note that \( \lambda^2 = \mu \cdot \lambda \cdot \mu \cdot \lambda = \lambda \cdot \mu \cdot \lambda \cdot \mu \), thus \( \lambda^2 = \bar{\lambda}^2 = \mu^{-1} \cdot \lambda^2 \cdot \mu \) and this element commutes with \( \mu \). The fundamental group can be also presented as

\[ \pi_1(Y) = \langle \lambda, \bar{\lambda} \mid \bar{\lambda}^2 = \lambda^2 \rangle, \]

according to the third picture above.

On the other hand, the fundamental group of the boundary is

\[ \pi_1(\partial Y) = H_1(\partial Y, \mathbb{Z}) \cong \mathbb{Z} \langle m \rangle \oplus \mathbb{Z} \langle c \rangle. \]

The \( \partial Y \cong S^1 \times S^1 \hookrightarrow Y \) embedding (which is homotopically the same as the orientation covering \( S^1 \times S^1 \to K \)) induces a monomorphism \( \pi_1(\partial Y) \to \pi_1(Y) \). It is determined by the images of the generators, which are

\[ m \mapsto \mu \text{ and } c \mapsto \lambda^2 = \bar{\lambda}^2. \]

A direct computation shows that \( [\pi_1(Y), \pi_1(Y)] = \mathbb{Z} \langle \mu^2 \rangle \) and

\[ H_1(Y, \mathbb{Z}) \cong \mathbb{Z} \langle \lambda \rangle \oplus \mathbb{Z}_2 \langle \mu \rangle. \]

Note that \( m = m' \) in \( H_1(\partial Y, \mathbb{Z}) \), and analysing \( \{ s = 0 \} \subset Y \) one obtains that \( m = -m' \) in \( H_1(Y, \mathbb{Z}) \). Hence the class of \( m \) in \( H_1(Y, \mathbb{Z}) \) has order 2, it is exactly \( \mu \).

The next Lemma shows that the classes \( m \) and \( c \) in \( H_1(\partial Y, \mathbb{Z}) \) have certain universal properties with respect to the inclusion \( \partial Y \subset Y \).

**Lemma 5.2.1.** (a) \( \pm m \) are the unique primitive elements of \( H_1(\partial Y, \mathbb{Z}) \) with the property that their doubles vanish in \( H_1(Y, \mathbb{Z}) \).
(b) $\pm c$ are the unique primitive elements of $H_1(\partial Y, \mathbb{Z}) = \pi_1(\partial Y)$ whose images in $\pi_1(Y)$ are in the center of $\pi_1(Y)$.

**Proof.** (a) is clear. For (b) first note that any element of $\pi_1(Y)$ can be written in the form $\lambda^k \mu^l$ for some $k, l \in \mathbb{Z}$, and then using this one verifies that the center of $\pi_1(Y)$ is $\langle \lambda^2 \rangle$. $\square$

### 5.2.3. Homological properties of $\tilde{Y}$

The closed Seifert 3–manifold $\tilde{Y}$ considered in 5.2.1(4) is constructed as follows. First we consider a new disc $D^2_{\text{new}}$ and the trivial fibration $D^2_{\text{new}} \times S^1$. Then we paste $m$ with the boundary of $D^2_{\text{new}}$ and we extend the canonical Seifert–fibration of $Y$ above this disc (as base space) of the trivial fibration $D^2_{\text{new}} \times S^1$. This leads to the Seifert fibred closed manifold

\[ \tilde{Y} = \frac{Y \cup (D^2_{\text{new}} \times S^1)}{\partial Y \sim \partial D^2_{\text{new}} \times S^1, m \sim \partial D^2_{\text{new}} \times \ast, c \sim \ast \times S^1}. \]

$H_1(\tilde{Y}, \mathbb{Z})$ can be determined by the Mayer-Vietoris sequence of the decomposition (5.2.10):\[
\begin{align*}
H_1(\partial Y, \mathbb{Z}) &\to H_1(Y, \mathbb{Z}) \oplus H_1(D^2_{\text{new}} \times S^1, \mathbb{Z}) \to H_1(\tilde{Y}, \mathbb{Z}) \to 0 \\
\mathbb{Z}\langle m \rangle \oplus \mathbb{Z}\langle c \rangle &\to \mathbb{Z}\langle \lambda \rangle \oplus \mathbb{Z}_2\langle \mu \rangle \oplus \mathbb{Z}\langle c' \rangle \to H_1(\tilde{Y}, \mathbb{Z}) \to 0
\end{align*}
\]
where $m \mapsto \mu$ and $c \mapsto 2\lambda + c'$. Thus

\[ H_1(\tilde{Y}, \mathbb{Z}) \cong \mathbb{Z}\langle \lambda \rangle. \]

### 5.2.4. A plumbing graphs of $Y$ and $\tilde{Y}$

By [57], $\tilde{Y}$ has a plumbing graph $G$ of the form

```
      -2
     / \\
-2 - e \-2
```

(see also the discussion from 5.2.1(4)) and the Euler number $e$ should be chosen such that $H_1(M^3(G), \mathbb{Z}) \cong \mathbb{Z}$ (cf. (5.2.11)). Here $M^3(G)$ denotes the plumbed 3–manifold associated with the graph $G$, it is the boundary of $M^4(G)$, the plumbed 4–manifold associated with $G$. This (via the long cohomological exact sequence of the pair $(M^4(G), M^3(G))$) imposes the degeneracy of the intersection matrix of the plumbing (cf. e.g. [55] 15.1.3]). Hence $e = -1$. In particular, $\tilde{Y}$ (without its Seifert fibration structure) is diffeomorphic to $S^1 \times S^2$ (which also shows that $\tilde{Y}$ admits an orientation reversing diffeomorphism).

Furthermore, consider the graph
The arrow denotes a knot \( K \simeq S^1 \subset \bar{Y} \), which is a generic \( S^1 \)-fibre associated with the middle vertex by the plumbing construction of \( \bar{Y} \). Let \( N(K)^\circ \) be an open tubular neighbourhood of \( K \) in \( \bar{Y} \). Then \( Y \simeq \bar{Y} \setminus N(K)^\circ \), and the induced (singular/Seifert) \( S^1 \)-fibration associated with the middle vertex (by the plumbing construction) on \( Y \) agrees with the canonical Seifert fibration of \( Y \). The Euler number \(-1\) of the middle vertex determines a parametrization (framing) of \( \partial Y \simeq S^1 \times S^1 \).

We can present \( \pi_1(Y) \) also from the plumbing graph using the description of [50]. Let \( \lambda, \bar{\lambda}, c \) be oriented \( S^1 \)-fibres associated with the three vertices provided by the plumbing construction (and extended by convenient connecting paths to a base point as in [50]). Next, let \( m \) be the meridian of \( K \) corresponding to the arrowhead (and extended by a convenient path to the base point).

Then, by [50], there is a choice of the connecting paths such that \( \pi_1(Y) \) is generated by \( \lambda, \bar{\lambda}, c \) and \( m \), and they satisfies the relations \( \lambda^2 = \bar{\lambda}^2 = c, \) and \( c = m\lambda\bar{\lambda} \). This is compatible with the description from Subsection 5.2.2, cf. (5.2.6) and (5.2.9).

Note that by plumbing calculus (cf. [56]) one has the equivalence of plumbed manifolds (where \( \bar{\Gamma} \) is any graph):

Then, \( Y \) or \(-Y\) spliced along \( K \) to any 3–manifold give rise to diffeomorphic manifolds.

### 5.3. The boundary of the Milnor fibre

#### 5.3.1. Let \( F = f^{-1}(\delta) \cap B^6_\epsilon \) be the Milnor fibre of \( f \), where \( \delta \in \mathbb{C}^* \), \( |\delta| \ll \epsilon \). We wish to construct the 3–manifold \( \partial F = f^{-1}(\delta) \cap S^5_\epsilon \) as a surgery of \( S^3 \) along the link \( L \).
Let $N_j$ be a sufficiently small tubular neighbourhood of $L_i$ in $S^3$. For each $j = \{i, \sigma(i)\}$ we define $X_j$ as

$$X_j = \begin{cases} S^1 \times S^1 \times I & \text{if } i \neq \sigma(i), \\ Y & \text{if } i = \sigma(i), \end{cases}$$

where $Y$ is the 3–manifold described in Section 5.2. Recall that $\partial Y \simeq S^1 \times S^1$.

**Proposition 5.3.1.** One has an orientation preserving diffeomorphism

$$\partial F \simeq \left( S^3 \setminus \bigcup_{i=1}^l \text{int}(N_i) \right) \cup \phi \left( \bigcup_{j \in J} X_j \right),$$

where $\phi : \partial(S^3 \setminus \bigcup_i \text{int}(N_i)) \to -\partial(\bigcup_{j \in J} X_j)$ is a collection $(\phi_j)_{j \in J}$ of diffeomorphisms

$$\phi_{i,\sigma(i)} : \begin{cases} -\partial N_i \cup -\partial N_{\sigma(i)} \to -\partial(S^1 \times S^1 \times I) & \text{if } i \neq \sigma(i), \\ -\partial N_i \to -\partial Y & \text{if } i = \sigma(i). \end{cases}$$

**Proof.** The decomposition follows from the general decomposition proved in [62], see also [55, 2.3]. For the convenience of the reader we sketch the construction. Recall that $\Upsilon_j \subset S^5_\epsilon$ is the link of the component $\Sigma_j$ of $\Sigma$, $j = \{i, \sigma(i)\} \in J$. Consider a sufficiently small tubular neighbourhood $N(\Upsilon_j)$ of it in $S^5_\epsilon$. We can assume that $\Phi^{-1}(N(\Upsilon_j)) = N_i \cup N_{\sigma(i)}$. Furthermore, for $\epsilon$ small, the intersection of $\partial N(\Upsilon_j)$ with $K := X \cap S^5_\epsilon$ is transverse. Therefore, for $0 < |\delta| \ll \epsilon$, the intersection of $\partial N(\Upsilon_j)$ with $\partial F$ is still transverse in $S^5_\epsilon$, and, in fact, $K \setminus \bigcup_j N(\Upsilon_j)$ is diffeomorphic with $\partial F \setminus \bigcup_j N(\Upsilon_j)$. But, the former space can be identified via $\Phi$ by $S^3 \setminus \bigcup_i (N_i \cup N_{\sigma(i)})$. This is the space in the first parenthesis of (5.3.2).

The second one is a union of spaces of type $X_j := \partial F \setminus N(\Upsilon_j)$, which fibres over $\Upsilon_j \simeq S^1$. The fibre of the fibration is the Milnor fibre of the corresponding transverse plane curve singularity (of $\Sigma_j$). Since the transverse type is $A_1$, this fibre is $F_j := S^1 \times I$. The monodromy of the fibration is the so-called geometric vertical monodromy of the transverse type, it is orientation preserving self-diffeomorphism of $S^1 \times I$. If it does not permute the two components of $\partial F_j$ then it preserves the orientation of $I$, hence of $S^1$ too, hence up to isotopy it is the identity. If it permutes the components of $\partial F_j$ then up to isotopy it is $(\alpha, t) \to (\alpha, -t)$, where $(\alpha, t) \in S^1 \times I$. The two types of vertical monodromies provide the two choices of $X_j$ in formula (5.3.1), cf. description (1) of $Y$ in Subsection 5.2.1.

**5.3.2. Preliminary discussion regarding the gluing.** Our next aim is to describe the gluing functions $\phi_j$. In both cases two tori must be glued: if $i \neq \sigma(i)$ then basically one should identify $\partial N_i$ and $-\partial N_{\sigma(i)}$, otherwise $\partial Y$ and $\partial N_i$. Up to diffeotopy an orientation
reversing diffeomorphism between tori is given by an invertible $2 \times 2$ matrix over $\mathbb{Z}$ with determinant $-1$. It turns out that in our cases all these gluing matrices have the form

$$f(\cdot) = \begin{pmatrix} -1 & n_j \\ 0 & 1 \end{pmatrix}$$

hence its only relevant entry is the off-diagonal one.

This integer will be determined by a newly introduced invariant, the \textit{vertical index}, associated with each $j \in J$. This is done using a special germ $H : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$, which will have a double role. First, it provides some kind of framing along $\Sigma \setminus \{0\}$, and also helps to identify generators from the boundaries of $K \setminus \partial N_j(N_i \cup N_{\sigma(i)})$ respectively (constructed in two different levels: in the target and in the source of $\Phi$).

We will use three parametrizations of $\partial N_i \simeq S^1 \times S^1$ with the same meridian but different longitudes. The topological longitude is the usual knot-theoretical Seifert–framing of $L_i \subset S^3$. The \textit{resolution longitude} is determined via a good embedded resolution of $(D, 0) \subset (\mathbb{C}^2, 0)$, it creates the bridge with the decorations and the combinatorics of the resolution graph. Finally, the \textit{sectional longitude} depends on $H$ and it will be used for describing the gluing. In fact, the sectional longitude allows us to compare the source and the target of $\Phi$: being defined by the geometry of $H$, the function $H$ and its pull-back $\Phi^*H$ plays the role of transportation of the invariants from $\mathbb{C}^3$ level to $\mathbb{C}^2$ level.

\textbf{5.3.3. The local form of $f$ along $\Sigma$.} In the sequel we will use the notation $\Sigma^* = \Sigma \setminus \{0\}$ and $\Sigma_j^* = \Sigma_j \setminus \{0\}$. Recall that in a small neighbourhood (in $\mathbb{C}^3$) of any point $p \in \Sigma^*$ the space $(X, 0)$ has two local components, both smooth and intersecting each other transversally.

A more precise local description along $\Sigma^*$ is the following. Let us fix a point $p_0 \in \Sigma_j^*$ and let $U_0$ be a small neighbourhood of $p_0$ in $\mathbb{C}^3$. In $U_0$ the function $f$ is a product $f_1 \cdot f_2$, where both $f_n$ are holomorphic, $\{f_n = 0\}$ are smooth and intersect each other transversally. (The intersection is $\Sigma^* \cap U_0$; later the fact that at $p_0$ the local parametrization of $\Sigma^* \cap U_0$ together with $f_1$ and $f_2$ might serve as local coordinates will be exploited further.)

$f_1$ and $f_2$ are well-defined up to a multiplication by an invertible holomorphic function $\iota$ of $U_0$; that is, $(f_1, f_2)$ can be replaced by $(\iota f_1, \iota^{-1} f_2)$. At any point $p \in \Sigma_j^* \cap U_0$ the linear term of $f_n$, $n \in \{1, 2\}$, (say, in the Taylor expansion) is $T_1(f_n) = \sum_{k=1}^3 u_{nk}(p)(x_k - p_k)$, where $(x_1, x_2, x_3)$ are the fixed coordinates of $(\mathbb{C}^3, 0)$. Let us code this in the non-zero vectors $u_n(p) := (u_{n1}(p), u_{n2}(p), u_{n3}(p))$. Hence, at any $p \in \Sigma_j^* \cap U_0$ we have two vectors $u_1(p)$ and $u_2(p)$ well-defined up to multiplication by $\iota|_{\Sigma_j^* \cap U_0}$ (in the sense described above). Their classes $[u_n(p)] \in \mathbb{CP}^2$ for $p \in \Sigma_j^* \cap U_0$ are independent of the $\iota$-ambiguity, hence are well-defined elements. In particular, they determine a global pair of elements $[u_1(p)]$
and \([u_2(p)] \in \mathbb{CP}^2\) for \(p \in \Sigma^*_j\), well-defined whenever \(i \neq \sigma(i)\), and well-defined up to permutation whenever \(i = \sigma(i)\).

In fact, we can do even more: there exists a splitting of \(f\) into product \(f_1 \cdot f_2\) along \(\Sigma^*_j\) without any invertible element ambiguity (but preserving the permutation ambiguity whenever \(i = \sigma(i)\)).

Indeed, assume that we are in the trivial covering \((i \neq \sigma(i))\) case, and let us cover \(\Sigma^*_j\) by small discs \(\{U_\alpha\}_\alpha\) such that on each \(U_\alpha\) we can fix a splitting \(f(p) = f_{1,\alpha}(p) \cdot f_{2,\alpha}(p), \ p \in U_\alpha\). For any intersection \(U_{\alpha\beta} = U_\alpha \cap U_\beta\) the two splittings can be compared: we define \(\iota_{\alpha\beta} \in O(U_{\alpha\beta})^*\) by \(f_{1,\alpha}|_{U_{\alpha\beta}} = \iota_{\alpha\beta} \cdot f_{1,\beta}|_{U_{\alpha\beta}}\). From this definition follows that \(\{\iota_{\alpha\beta}\}_{\alpha,\beta}\) form a Čech 1–cocycle.

**Lemma 5.3.2.** \(H^1(\Sigma^*_j, O_{\Sigma^*_j}) = 0\).

**Proof.** From the exponential exact sequence \(0 \to \mathbb{Z} \to O \to O^* \to 0\) over \(\Sigma^*_j\), we get that it is enough to prove the vanishing \(H^1(\Sigma^*_j, O_{\Sigma^*_j}) = 0\), a fact which follows from Cartan’s Theorem, since \(\Sigma^*_j\) is Stein. 

Since \(H^1(\Sigma^*_j, O_{\Sigma^*_j}) = 0\), the cocycle \(\{\iota_{\alpha\beta}\}_{\alpha,\beta}\) is a coboundary. This means that we can find invertible functions \(\iota_\alpha\) on each \(U_\alpha\) such that on \(U_{\alpha\beta}\) one has \(\iota_\alpha|_{U_{\alpha\beta}} = \iota_{\alpha\beta} \cdot \iota_\beta|_{U_{\alpha\beta}}\). This means that the local functions \(\tilde{f}_{1,\alpha} := f_{1,\alpha} \cdot \iota_\alpha^{-1}, \tilde{f}_{2,\alpha} := f_{2,\alpha} \cdot \iota_\alpha\) on \(U_\alpha\) provide a splitting (that is, \(f(p) = \tilde{f}_{1,\alpha}(p) \cdot \tilde{f}_{2,\alpha}(p), \ p \in U_\alpha\)), but in this new situation the local splittings glue globally: \(\tilde{f}_{1,\iota}|_{U_{\alpha\beta}} = \tilde{f}_{1,\iota}|_{U_{\alpha\beta}}\). If \(i = \sigma(i)\) then we repeat the proof on \(D_i\).

**5.3.4. The special germ \(H\).** Next, we treat the ‘aid’–germ \(H\).

**Definition 5.3.3.** Let us fix \(\Phi, f, \Sigma\) as above. A germ \(H : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)\) is called transverse section along \(\Sigma\) if \(\Sigma \subset H^{-1}(0), H^{-1}(0)\) at any point of \(\Sigma^*\) is smooth and intersects both local components of \((X, 0)\) transversally.

We claim that transverse sections always exist.

**Proposition 5.3.4.** There exist complex numbers \(a_1, a_2, a_3\) such that \(H = a_1\partial_{x_1}f + a_2\partial_{x_2}f + a_3\partial_{x_3}f\) is a transverse section.

**Proof.** For such \(H\) one has \(T_1(H) = \sum_{k=1}^3 a_k u_{1k} \cdot T_1(f_2) + \sum_{k=1}^3 a_k u_{2k} \cdot T_1(f_1)\). In particular, we have to show that for certain coefficients \(\{a_k\}_k\) the expressions \(\beta_n(p) = \sum_{k=1}^3 a_k u_{nk}(p)\) (for \(n \in \{1, 2\}\)) have no zeros for \(p \in \Sigma^*\).

If \(\gamma : X^N \to X\) is the Nash transform of \(X\), then \(\gamma^{-1}(0)\) is the set of limits of tangent spaces of \(X \setminus \{0\}\), it is an algebraic set of \(\mathbb{CP}^2\) of dimension \(\leq 1\) (for details see e.g. [28] and references therein). The existence of Nash transform guarantees that
\[ [\ell_n] = \lim_{p \to 0} [u_n(p)] \subset \mathbb{CP}^2 \] exist. Indeed, the set \( \{[l_1], [l_2]\} \) is the intersection of \( \gamma^{-1}(0) \) with the strict transform of \( \Sigma_j \). Then let \( [a_1 : a_2 : a_3] \) be generic such that \( \sum_k a_k \ell_{nk} \neq 0 \) for \( n \in \{1, 2\} \). With this choice \( \sum_{k=1}^3 a_k u_{nk}(p) \neq 0 \) for \( p \neq 0 \) and in a small representative of \( \Sigma_j \).

Fix again \( j = \{i, \sigma(i)\} \in J \), and let \( p : (\mathbb{C}, 0) \to (\Sigma_j, 0) \subset (\mathbb{C}^3, 0), \tau \mapsto p(\tau) \), be a parametrization (normalization) of \( \Sigma_j \). For any point \( p_0 = p(\tau_0) \) and neighbourhood \( \Sigma^*_j \cap U_0 \ni p(\tau) \) the discussion from the second paragraph of 5.3.3 can be repeated, in particular we have the holomorphic vectors \( u_n(p(\tau)) \) (\( n \in \{1, 2\} \)) (with the choice ambiguities described there). Additionally, choose some \( H \) as in Definition 5.3.3. The assumption regarding \( H \) guarantees that \( T_1(H)(p(\tau)) = \beta_1(\tau) T_1(f_1)(p(\tau)) + \beta_2(\tau) T_1(f_2)(p(\tau)) \) for some holomorphic functions \( \beta_1 \) and \( \beta_2 \) on \( p^{-1}(\Sigma^*_j \cap U_0) \). If we replace \( (u_1, u_2) \) by \( (\iota u_1, \iota^{-1} u_2) \) then \( (\beta_1, \beta_2) \) will be replaced by \( (\iota^{-1} \beta_1, \iota \beta_2) \), hence the product \( \beta_1 \beta_2 \) is independent of all the \( \iota \) and permutation ambiguities. It is a holomorphic function on \( p^{-1}(\Sigma^*_j \cap U_0) \) depending only on the equations \( f \) and \( H \). This uniqueness also guarantees that taking different points of \( \Sigma^*_j \) and repeating the construction, the output glues to a unique holomorphic function \( b_j(\tau) \) on a small punctures disc of \( (\mathbb{C}, 0) \). Usually \( b_j(\tau) \) has no analytic extension to the origin, however one has the following.

**Lemma 5.3.5.** \( b_j(\tau) \) is a Laurent series on \( (\mathbb{C}, 0) \).

**Proof.** The finiteness of the poles follows e.g. from the homological identities from Theorem 5.3.9 or from Corollary 5.3.17 combined with (5.3.9). \( \square \)

We wish to emphasize that \( b_j(\tau) \) and its pole order usually depends on the choice of \( H \).

**Definition 5.3.6.** Let \( a\tau^{v_j} \) (for some \( a \in \mathbb{C}^* \)) be the non-zero monomial with smallest power of \( \tau \) in the Laurent series of \( b_j(\tau) \). The integer \( v_j \) is called the vertical index of \( f \) along \( \Sigma_j \) with respect to \( H \) (or, the \( H \)-vertical index).

### 5.3.5. Computation of the gluing functions \( \phi_j \)

We fix a transverse section \( H \) (cf. 5.3.3). Then the divisor \( \Phi^*(H) \) is \( H \circ \Phi = d \cdot d_z \) for some (not necessarily reduced) germ \( d_z : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) (such that \( d \) and \( d_z \) have no common components). Let \( (D_z, 0) \) be the (non-reduced) divisor associated with \( d_z \), and \( N(L_z) \) be a small tubular neighbourhood of the reduced link \( L_z := \text{red}(D_z) \cap \mathbb{S}^3 \) in \( \mathbb{S}^3 \).

Let the Milnor fibre of \( \Phi^*(H) \) (in \( \mathbb{S}^3 \)) be

\[
F_{\Phi^*(H)} = \{(s, t) \in \partial \mathbb{B}_\epsilon = \mathbb{S}^3 \mid H(\Phi(s, t)) > 0\}.
\]
Let $\Lambda_i$ and $\Lambda_\sigma$ denote the components of the oriented intersection $F_{\Phi^*(H)} \cap \partial N_i$ and $F_{\Phi^*(H)} \cap \partial N(L_z)$ with the tubular neighborhood boundaries of $L_i$ and $L_z$ respectively. ($\Lambda_2$ might have several components, in this notation we collect all of them.)

Furthermore, let $M_i \subset \partial N_i$ be an oriented meridian of $\partial N_i$ such that $\text{lk}(M_i, L_i) = 1$ and fix also (the oriented Seifert framing of $D_i$) $L'_i \subset \partial N_i$ with $\text{lk}(L'_i, L_i) = 0$. (Here the linking numbers are considered in oriented 3–sphere $\mathbb{S}^3$.)

**Definition 5.3.7.** We call $L'_i$ the topological longitude of the torus $\partial N_i$, while $\Lambda_i$ the sectional longitude of $\partial N_i$ associated with the transverse section $H$.

Clearly we have the following facts (where $[\cdot]$ denotes the corresponding homology class)

(a) $H_1(N_i, \mathbb{Z}) \cong \mathbb{Z}([\Lambda_i]) \cong \mathbb{Z}([L'_i])$, 

(b) $H_1(\partial N_i, \mathbb{Z}) \cong \mathbb{Z}([\Lambda_i]) \oplus \mathbb{Z}([M_i]) \cong \mathbb{Z}([L'_i]) \oplus \mathbb{Z}([M_i])$.

We want to express $[\Lambda_i]$ in terms of $[M_i]$ and $[L'_i]$.

**Lemma 5.3.8.** Define $\lambda_i = -\sum_{k \neq i} D_k \cdot D_i - D_2 \cdot D_i$, where $C_1 \cdot C_2$ denotes the intersection multiplicity of $(C_1, 0)$ and $(C_2, 0)$ at $0 \in \mathbb{C}^2$. Then $[\Lambda_i] = [L'_i] + \lambda_i \cdot [M_i]$ in $H_1(\partial N_i, \mathbb{Z})$.

**Proof.** First note that $[\Lambda_i] = [L'_i] + \text{lk}(\Lambda_i, L_i) \cdot [M_i]$. Write $F'_{\Phi^*(H)} := F_{\Phi^*(H)} \setminus (\text{int}(\bigcup_i N_i \cup N(L_z)))$. Then $0 = \text{lk}(\partial F'_{\Phi^*(H)}, L_i) = \sum_k \text{lk}(\Lambda_k, L_i) + \text{lk}(\Lambda_2, L_i) = \text{lk}(\Lambda_i, L_i) + \sum_{k \neq i} D_k \cdot D_i + D_2 \cdot D_i = \text{lk}(\Lambda_i, L_i) - \lambda_i$. \hfill $\square$

**Theorem 5.3.9.** For any $j = \{i, \sigma(i)\} \in J$, the gluing functions $\phi_j$ from Proposition [5.3.1] is characterized up to homotopy by the following identities

**Case 1:** $i \neq \sigma(i)$. Identify the homology groups $H_1(S^1 \times S^1 \times \{1\}, \mathbb{Z})$, $H_1(S^1 \times S^1 \times \{-1\}, \mathbb{Z})$ and $H_1(S^1 \times S^1, \mathbb{Z})$ via the natural homotopies

$S^1 \times S^1 \times \{-1\} \xrightarrow{h} S^1 \times S^1 \times [-1, 1] \xrightarrow{h} S^1 \times S^1 \times \{1\}$.

Then in this homology group one has

$\phi_j([M_i]) = -\phi_j([M_{\sigma(i)}])$ and $\phi_j([\Lambda_i]) = \phi_j([\Lambda_{\sigma(i)}] + v_j \cdot [M_{\sigma(i)}])$.

**Case 2:** $i = \sigma(i)$.

$\phi_j([M_i]) = -m$ and $\phi_j([\Lambda_i]) = c + v_j \cdot m$,

where $m$ and $c$ are the two generators of $H_1(\partial Y, \mathbb{Z})$, see [5.2.2] (especially (5.2.8) and (5.2.9)).
5. BOUNDARY OF THE MILNOR FIBRE

PROOF. Recall that $\tau \mapsto p(\tau)$ is the normalization of $\Sigma_j$. At any point $p(\tau_0)$ of $\Sigma_j^*$ one can consider $\tau$ as a local complex coordinate, which can be completed with two other local complex coordinates $(x, y)$ (local coordinates in a transverse slice of $\Sigma_j$ at $p(\tau_0)$) such that $(\tau, x, y)$ form a local coordinate system of $(\mathbb{C}^3, p(\tau))$, and locally $f(x, y) = xy$. These two local coordinates correspond to a splitting $f = f_1 \cdot f_2$ of $f$. According to the discussion from 5.3.3, the splitting of $f$ can be done globally along the whole $\Sigma_j^*$, hence these coordinated $(x, y)$ (corresponding to the components $f_1$ and $f_2$) can also be chosen globally along $\Sigma_j^*$ (with the permutation ambiguity whenever $i = \sigma(i)$, a fact which will be handled below).

Furthermore, in these coordinates, $T_1 H = \beta_1(\tau)x + \beta_2(\tau)y$. Since $\beta_1(\tau)$ have no zeros and poles in the small representative of $\Sigma_j^*$, $(\tau, \beta_1 x, \beta_1^{-1} y)$ are also local coordinates, and in these coordinates the equations transform into $f(x, y) = xy$ and $T_1 H(x, y) = x + b_j(\tau)y$.

If we concentrate on the points of $\Upsilon_j = \Sigma_j \cap S^5_\epsilon$, and its neighbourhood in $S^5_\epsilon$, then similarly as above, we have the real coordinate $\tau \in \Upsilon_j$, and the two complex (transverse) local coordinates $(x, y)$, with equations $f(x, y) = xy$ and $T_1 H(x, y) = x + b_j(\tau)y$ as before.

**Case 1.** The above local description globalises as follows (compare also with the first part of the proof of Proposition 5.3.1). The space $\partial F \cap N(\Upsilon_j)$ has a product decomposition $\Upsilon_j \times F_j = S^1 \times S^1 \times I$, where $\Upsilon_j = S^1$ is the parameter space of $\tau$, and $F_j$ is the local Milnor fibre $F_j = \{xy = \delta\} \cap B^4_\epsilon$, $0 < \delta < \epsilon \ll \epsilon$, diffeomorphic to $S^1 \times I$. In other words, $\partial F \cap N(\Upsilon_j)$ is the space $\{(\tau, x, y) \in S^1 \times B^4_\epsilon \mid xy = \delta\}, 0 < \delta < \epsilon \ll \epsilon$ (here we will use the same $\tau$ notation for the parameter of $S^1$). Note that for $\delta \ll \epsilon$, the boundary of $F_j$ is ‘very close’ to the two circles $\{|x| = \epsilon, y = 0\}$ and $\{x = 0, |y| = \epsilon\}$ of $\partial B^4_\epsilon$. Using isotopy in the neighbourhoods of these two circles, $\partial F_j$ can be identified with these two circles (similarly as we identify via Ehresmann’s fibration theorem the boundary of the Milnor fibre of an isolated singularity with the link), and in order to simplify the presentation, we will make this identification. Hence, the boundary components of $\partial F \cap N(\Upsilon_j)$ can be identified (by isotopy in $\partial N(\Upsilon_j)$) with $\partial_i := \{(\tau, x, y) \mid \tau \in S^1, |x| = \epsilon, y = 0\}$ and $\partial_{\sigma(i)} := \{(\tau, x, y) \mid \tau \in S^1, x = 0, |y| = \epsilon\}$. (The choice of indices $i$ and $\sigma(i)$ is arbitrary and symmetric.) These two tori are identified homologically since they are boundary components of $S^1 \times F_j$. In $\partial_i$ we have the meridian $\tilde{M}_i = \{(\tau, x, y) \mid \tau = 1, |x| = \epsilon, y = 0\}$, while in $\partial_{\sigma(i)}$ we have the meridian $\tilde{M}_{\sigma(i)} = \{(\tau, x, y) \mid \tau = 1, x = 0, |y| = \epsilon\}$ (both naturally oriented as the complex unit circle). Since $\partial(\{(\tau = 1) \times F_j\}) = \tilde{M}_i \cup -\tilde{M}_{\sigma(i)}$, the first wished identity follows.

Now, we would like to study the intersection curve of $\partial_i$ and the Milnor fibre of $H$ associated with a positive argument, that is $\tilde{\Lambda}_i := \partial_i \cap \{H > 0\}$. This curve is homotopic
with \( \partial_i \cap \{ T_1 H = x + b_j(\tau)y > 0 \} \) and also with \( \partial_i \cap \{ x + \tau^0 y > 0 \} \) in \( \partial_i \), cf. the Definition 5.3.6. Similarly, \( \tilde{\Lambda}_i(\sigma(i)) := \partial_{\sigma(i)} \cap \{ H > 0 \} \) is homotopic with \( \partial_{\sigma(i)} \cap \{ x + \tau^0 y > 0 \} \) in \( \partial_{\sigma(i)} \).

Thus \( \tilde{\Lambda}_i \) is homotopic with \( \{ (\tau, x, y) : \tau \in S^1, |x| = \varepsilon, y = 0 \} \cap \{ x + \tau^0 y > 0 \} \) = \( \{ (\tau, x, y) : \tau \in S^1, x = \varepsilon, y = 0 \} \) and \( \tilde{\Lambda}_{\sigma(i)} \) with \( \{ (\tau, x, y) : \tau \in S^1, x = 0, y = \varepsilon \tau^{-v} \} \). Hence homologically \( \tilde{\Lambda}_{\sigma(i)} + \nu_j \tilde{M}_{\sigma(i)} \) is represented by the circle \( \{ (\tau, x, y) : \tau \in S^1, x = 0, y = \varepsilon \} \), which is homologous in \( S^1 \times F_j \) with \( \tilde{\Lambda}_i \). This is the second identity. Obviously, these identities can be transferred from the boundary \( \partial_i \cup \partial_{\sigma(i)} \) of \( S^1 \times F_j \) into similar identities in \( \partial \tilde{N}_i \cup \partial \tilde{N}_{\sigma(i)} \), via the diagram, where all the maps are orientation preserving diffeomorphisms (cf. the proof of 5.3.1):

\[
\begin{align*}
\Phi : \ G^3 \cup \tilde{N}_i & \rightarrow K \cup j.N(\Upsilon_j) \\
\partial \tilde{N}_i \cup \partial \tilde{N}_{\sigma(i)} & \rightarrow \partial_i \cup \partial_{\sigma(i)} \\
\Lambda_i, \Lambda_{\sigma(i)} & \rightarrow \tilde{\Lambda}_i, \tilde{\Lambda}_{\sigma(i)} \\
\tilde{M}_i, \tilde{M}_{\sigma(i)} & \rightarrow \tilde{M}_i, \tilde{M}_{\sigma(i)}.
\end{align*}
\]

Case 2. We use similar notations and conventions as in Case 1. Let us parametrize \( \Upsilon_j \) as \( \tau = e^{2\pi i}, s \in [0, \pi] \). Then \( \partial F \cap N(\Upsilon_j) \) is \( ([0, \pi] \times F_j) / \sim \), where by \( \sim \) we identify \( (0, x, y) \sim (\pi, y, x) \) for all \( (x, y) \in F_j \). Let us parametrize \( F_j \) as \( x = \sqrt{\delta}r e^{it} \) and \( y = \sqrt{\delta}r^{-1}e^{-it} \), where \( t \in [-\pi, \pi] \) and \( r \in [r_0, r_1] \) such that \( r_0 r_1 = 1 \) and \( \delta(r_0^2 + r_1^2) = \varepsilon^2 \). Denote \( z = \log_{r_1} r \). Then we can parametrize \( \partial F \cap N(\Upsilon_j) \) by \( (s, t, z) \), thus \( \partial F \cap N(\Upsilon_j) \) is just \( ([0, \pi] \times [-\pi, \pi] \times [-1, 1]) / \sim \), where by \( \sim \) we identify \( (0, t, z) \sim (\pi, -t, -z) \) and \( (s, -\pi, z) \sim (s, \pi, z) \). We regard this as parametrization of \( \partial F \cap N(\Upsilon_j) \) by \( Y \), cf. 5.2.2.

Set \( \tilde{M}_{j_1} = \{ (\tau = 1, x = \sqrt{\delta}r_1 e^{it}, y = \sqrt{\delta}r_0 e^{-it}) \} \) and \( \tilde{M}_{j_2} = \{ (\tau = 1, x = \sqrt{\delta}r_0 e^{-it}, y = \sqrt{\delta}r_1 e^{it}) \} \).

The are the two oriented meridians of \( \partial F \cap N(\Upsilon_j) \), parametrized by \( t \in [-\pi, \pi] \). In terms of \( (s, t, z) \) they are

\[
\tilde{M}_{j_1} = \{ (s = 0, t, z = 1) \} = \{ (s = \pi, -t, z = -1) \} \subset \partial Y
\]

\[
\tilde{M}_{j_2} = \{ (s = 0, -t, z = -1) \} = \{ (s = \pi, t, z = 1) \} \subset \partial Y,
\]

thus with the notations of 5.2.2 \( \tilde{M}_{j_1} = -m' \) and \( \tilde{M}_{j_2} = -m \).

Similarly as in Case 1, by an isotopy in \( \partial N(\Upsilon_j) \) the boundary \( \partial_j \) of \( \partial F \cap N(\Upsilon_j) \) can be identified with \( \partial_{j_1} \cup \partial_{j_2} \), where

\[
\partial_{j_1} := \{ (s, x, y) : s \in [0, \pi], |x| = \varepsilon, y = 0 \}
\]

\[
\partial_{j_2} := \{ (s, x, y) : s \in [0, \pi], x = 0, |y| = \varepsilon \}.
\]
The two parts of $\partial_j$ are glued together along the image of the oriented meridians
\[ \tilde{M}_{j1} = \{(s = 0, x = \varepsilon e^{it}, y = 0)\} = \{(s = \pi, x = 0, y = \varepsilon e^{it})\} \quad \text{and} \quad \tilde{M}_{j2} = \{(s = 0, x = 0, y = \varepsilon e^{it})\} = \{(s = \pi, x = \varepsilon e^{it}, y = 0)\}. \]

(See also the description in [5.2.2]) Since the oriented boundary $\partial(\{\tau = 1\} \times F_j)$ is $\tilde{M}_{j1} \sqcup \tilde{M}_{j2}$, $M_{j1}$ is homologous with $-\tilde{M}_{j2}$ in $F_j \subset Y$. On the other hand $M_{j1}$ is homologous with $\tilde{M}_{j2}$ in $\partial Y \subset Y$, thus $[\tilde{M}_{j1}] = [\tilde{M}_{j2}]$ is an order–2 element in $H_1(Y, \mathbb{Z})$.

Consider the closed curve $C$ obtained as union of $C_1$ and $C_2$, where
\[ C_1 = \{(s, x = \varepsilon, y = 0) : s \in [0, \pi]\} = \{(s, t = 0, z = 1) : s \in [0, \pi]\}, \quad \text{and} \quad C_2 = \{(s, x = 0, y = \varepsilon) : s \in [0, \pi]\} = \{(s, t = 0, z = -1) : s \in [0, \pi]\}. \]

Note that $C_1$ connects the points $A_1 = (s = 0, x = \varepsilon, y = 0) = (s = 0, t = 0, z = 1)$ with $B_1 = (s = \pi, x = \varepsilon, y = 0) = (s = \pi, t = 0, z = 1)$, while $C_2$ connects the points $A_2 = (s = 0, x = 0, y = \varepsilon) = (s = 0, t = 0, z = -1)$ with $B_2 = (s = \pi, x = 0, y = \varepsilon) = (s = \pi, t = 0, z = -1)$. Since $A_1 \sim B_2$ and $A_2 \sim B_1$, they form a closed curve. Note that $[C] = [c]$ in $H_1(\partial Y, \mathbb{Z})$, see [5.2.2].

Similarly as in Case 1, the function $H$ can be replaced by $x + \tau^\nu y$, hence its level set associated with a positive value determines the curve $\tilde{\Lambda}_j = \{x + \tau^\nu y > 0\} \cap \partial_j$. This consists of two parts, $\tilde{\Lambda}_{j1}$ and $\tilde{\Lambda}_{j2}$, where
\[ \tilde{\Lambda}_{j1} = \{(s, x = \varepsilon, y = 0) : s \in [0, \pi]\} = \{(s, t = 0, z = 1) : s \in [0, \pi]\} \subset \partial_{j1}, \]
while $\tilde{\Lambda}_{j2} = \{(s, x = 0, y = \varepsilon e^{-2i\pi^\nu}) : s \in [0, \pi]\}$ equals
\[ \{(s, t = 2\nu s \mod [-\pi, \pi], z = -1) : s \in [0, \pi]\} \subset \partial_{j2}. \]

$\tilde{\Lambda}_{jn}$ has the same end–points as $C_n$, hence $\tilde{\Lambda}_{j1}$ and $\tilde{\Lambda}_{j2}$ form together a closed curve, as we expect. Furthermore, $\tilde{\Lambda}_j + v_j \tilde{M}_{j2}$ is homologous in $\partial Y$ with $C$.

The source and the target are connected by the restriction of $\Phi$, which gives the orientation preserving diffeomorphisms:
\[
\Phi : \mathbb{S}^3 \setminus \bigcup_i N_i \to K \setminus \bigcup_j N(\Upsilon_j)
\]
\[
\begin{align*}
\partial N_i &\to \partial_j \\
\Lambda_i &\to \tilde{\Lambda}_j \\
M_i &\to \tilde{M}_{j1} \text{ homologous with } \tilde{M}_{j2}
\end{align*}
\]
(5.3.6)

Since $C$ identifies with $c$ and $\tilde{M}_{j1}$ and $\tilde{M}_{j2}$ with $-m$, $\tilde{\Lambda}_i = c + v_j \cdot m$ follows. \hfill \Box

5.3.6. The $H$–independent description of the gluing. The ‘vertical index’.
Recall that the sectional longitudes $\Lambda_i$ and the corresponding $H$–vertical indexes $v_j$ depend on the choice of $H$. The goal of this paragraph is to replace $\left( M_i, \Lambda_i \right)$ by the $H$–independent $\left( M_i, L'_i \right)$ and $v_j$ by an $H$–independent number.

**Definition 5.3.10.** For any $j = \{i, \sigma(i)\}$ define $v_{ij}$ by

\begin{align}
(v_{ij}) := \begin{cases}
\lambda_i + \lambda_{\sigma(i)} + v_j & \text{if } i \neq \sigma(i), \\
\lambda_i + v_j & \text{if } i = \sigma(i).
\end{cases}
\end{align}

We call $v_{ij}$ the vertical index of $\Sigma_j$.

The next statement follows from 5.3.8 and 5.3.9 (see also 5.3.2).

**Corollary 5.3.11.** For $i \neq \sigma(i)$

$$
\phi_{j*}([M_i]) = -\phi_{j*}([M_{\sigma(i)}]) \quad \text{and} \\
\phi_{j*}([L'_i]) = \phi_{j*}([L'_{\sigma(i)}]) + v_{ij} \cdot [M_{\sigma(i)}],
$$

and for $i = \sigma(i)$

$$
\phi_{j*}([M_i]) = -m \quad \text{and} \\
\phi_{j*}([L'_i]) = c + v_{ij} \cdot m
$$

hold in the sense described in 5.3.9.

**Corollary 5.3.12.** The integer $v_{ij}$ does not depend on the choice of $H$, thus it is an invariant of $f$ and $\Sigma_j$.

5.3.7. The plumbing graph of $\partial F$. We construct a plumbing graph for $\partial F$ by modifying a good embedded resolution graph of $(D, 0) \subset (\mathbb{C}^2, 0)$. (For notations see Subsection 4.2.3.) The gluing of the plumbing construction uses different set of longitudes (cf. [56]). First we define them and then we rewrite the above established identities regarding $\phi_{j*}$ in this language.

We choose small tubular neighbourhoods $N(E_v)$ of $E_v \subset \tilde{\mathbb{C}}^2$ such that

$$
\pi^{-1}(\mathbb{S}^3) \simeq \partial \left( \bigcup_{v \in V} N(E_v) \right) \quad \text{(diffeomorphic with $\mathbb{S}^3$ via $\pi$)}
$$

after smoothing the corners of $\bigcup_{v \in V} N(E_v)$. The tubular neighbourhood $N(E_v)$ of $E_v \subset \tilde{\mathbb{C}}^2$ is a $D^2$ (real 2–disk) bundle over $E_v$ with Euler number $e_v$. We can choose this bundle structure in such a way that $\tilde{D}_i$ is one of the fibres of $N(E_{v(i)})$ for each $i = 1, \ldots, l$.

We choose another generic fibre $F_i \simeq D^2$ of the bundle $N(E_{v(i)})$ near $\tilde{D}_i$, and we set $L_i = \pi(\partial F_i) \subset \mathbb{S}^3$. By the choice of the bundle structure of $N(E_{v(i)})$ and by the choice of the fibre $F_i$ we can assume that $L_i \subset \partial N_i$. 

5. BOUNDARY OF THE MILNOR FIBRE

**Definition 5.3.13.** \( L_i^\pi \subset \partial N_i \subset \mathbb{S}^3 \) is called the resolution longitude of \( L_i \subset \mathbb{S}^3 \) associated with the resolution \( \pi \).

We fix a resolution longitude \( L_i^\pi \) for each \( L_i \). Clearly,

\[
H_1(N_i, \mathbb{Z}) = \mathbb{Z}[\langle L_i^\pi \rangle] \quad \text{and} \quad H_1(\partial N_i, \mathbb{Z}) = \mathbb{Z}[\langle L_i^\pi \rangle] \oplus \mathbb{Z}[\langle M_i \rangle].
\]

The following facts are well–known (cf. (5.3.9)).

**Proposition 5.3.14.** (a) \( lk(L_i^\pi, L_i) = m_i(v(i)) \).

(b) \( [L_i^\pi] = [L_i^\pi'] + m_i(v(i)) \cdot [M_i] \) holds in \( H_1(\partial N_i, \mathbb{Z}) \).

**Corollary 5.3.15.** \( [L_i^\pi] = [\Lambda_i] + (m_i(v(i)) - \lambda_i) \cdot [M_i] \) holds in \( H_1(\partial N_i, \mathbb{Z}) \).

**Definition 5.3.16.** For any \( j = \{ i, \sigma(i) \} \) define \( \alpha_j \) by:

\[
\alpha_j = \left\{ \begin{array}{ll}
-m_i(v(i)) + \lambda_i - m_{\sigma(i)}(v(\sigma(i))) + \lambda_{\sigma(i)} + v_j & \text{if} \quad i \neq \sigma(i), \\
\lambda_i - m_i(v(i)) + v_j & \text{if} \quad i = \sigma(i).
\end{array} \right.
\]

Then Theorem [5.3.9] and Corollary [5.3.15] give the following.

**Corollary 5.3.17.** **Case 1:** For \( i \neq \sigma(i) \) in \( H_1(S^1 \times S^1, \mathbb{Z}) \) the following identities hold:

\[
\phi_{j^\ast}([M_i]) = -\phi_{j^\ast}([M_{\sigma(i)}]) \quad \text{and} \quad \phi_{j^\ast}([L_i^\pi]) = \phi_{j^\ast}([L_{\sigma(i)}^\pi] + \alpha_j \cdot [M_{\sigma(i)}]).
\]

**Case 2:** For \( i = \sigma(i) \) in \( H_1(\partial Y, \mathbb{Z}) \) the following identities hold:

\[
\phi_{j^\ast}([M_i]) = -m \quad \text{and} \quad \phi_{j^\ast}([L_i^\pi]) = c + \alpha_j \cdot m.
\]

**5.3.8. The construction of the plumbing graph.** From the embedded resolution graph \( \Gamma \) of \( (D, 0) \subset (\mathbb{C}^2, 0) \) associated with the resolution \( \pi \) we construct a plumbing graph \( \tilde{\Gamma} \).

Recall that \( \Gamma \) has \( l \) arrowhead vertices representing the strict transforms of \( \{D_i\}_{i=1}^l \), and the \( i \)-th arrowhead is supported (via a unique edge) by the vertex \( v(i) \in V \). We obtain the plumbing graph \( \tilde{\Gamma} \) from \( \Gamma \) as follows.

1. Fix \( j = \{ i, \sigma(i) \} \) and assume that \( i \neq \sigma(i) \). Then we identify the two arrowheads and we replace it by a single new vertex of \( \tilde{\Gamma} \). We define the Euler number of the new vertex by \( \alpha_j \). Both two edges (which in \( \Gamma \) supported the arrowheads) will survive as edges of this new vertex (connecting it with \( v(i) \) and \( v(\sigma(i)) \)) respectively, however one of them will have a negative sign, the other one a positive sign. By plumbing calculus, cf. [56] Prop. 2.1. \( \text{R0}(\alpha) \), the choice of the edge which has negative sign – denoted by \( \ominus \) – is irrelevant. (The edges without any decorations, by convention, are edges with positive sign.)
(2) Fix \( j = \{ i, \sigma(i) \} \) such that \( i = \sigma(i) \). Then the arrowhead associated with \( i = \sigma(i) \) will be replaced by a new vertex and it will be decorated by Euler number \( \alpha_j \). Furthermore, to this new vertex we attach the plumbing graph of \( Y \) (cf. 5.2.4) as indicated below. The edge connecting the new vertex and the graph of \( Y \) will have a negative sign \( \ominus \).

(3) We do all these modification for all \( j \in J \), otherwise we keep the shape and decorations of \( \Gamma \).

More precisely, if the schematic picture of \( \Gamma \) is the following,

\[
\begin{array}{ccc}
\Gamma & \rightarrow & (\tilde{D}_i) \\
\vdots & \vdots & \vdots \\
& (\tilde{D}_v) & (j = \{ i, \sigma(i) \}, i \neq \sigma(i))
\end{array}
\]

then the schematic picture of \( \hat{\Gamma} \) is

\[
\begin{array}{ccc}
\hat{\Gamma} : & \rightarrow & \alpha_j \\
\vdots & \vdots & \vdots \\
& \Theta & -2
\end{array}
\]

In fact, by plumbing calculus (cf. Subsection 4.2.2 or [56, Prop. 2.1. R0(a)]), whenever \( i = \sigma(i) \) the edge sign from the newly created ‘branch’ (subtree) can be omitted, however at this point we put it since this is the graph provided by the proof (which reflects properly the corresponding base changes).

Note also that usually the graph \( \hat{\Gamma} \) (that is, the associated intersection form) is not negative definite (or, it is not even equivalent via plumbing calculus by a negative definite graph).

Above (when \( i \neq \sigma(i) \)) the coincidence \( v(i) = v(\sigma(i)) \) might happen, in fact, in all the cases we analysed this coincidence (on minimal graph) does happen.

**Theorem 5.3.18.** The plumbing 3-manifold associated with the plumbing graph \( \hat{\Gamma} \) is orientation preserving diffeomorphic with \( \partial F \).

**Proof.** This follows from Corollary 5.3.17 and the relationships between this base–change and the plumbing construction as it is described in [56], pages 318–319. In the case \( i = \sigma(i) \) the plumbing graph of \( Y \) from 5.2.4 should be also used. Note that in both cases Corollary 5.3.17 provides a base change matrix from the left hand side of (5.3.10),
this decomposes as a product as in the right hand side.

\[
\begin{pmatrix} -1 & \alpha_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\alpha_j & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

This, according to [56] pages 318–320 is interpreted as a ‘gluing’ by a string (with length one), this is the new vertex (for each \(j\)) given by the construction of \(\Gamma\).

\[\square\]

5.4. The vertical index for \(\Sigma^{1,0}\) type germs

5.4.1. \(\Sigma^{1,0}\) type germs. Assume that \(\Phi(s, t) = (s, t^2, td(s, t))\), where \(d(s, t) = g(s, t^2)\) for some germ \(g\), such that \(d(s, t)\) is not divisible by \(t\). In this case the equation of the image is \(f = yg^2(x, y) - z^2 = 0\). These germs (more precisely, their \(\mathcal{A}\)-equivalence classes) are labelled by the Boardman symbol \(\Sigma^{1,0}\), and they are exactly the finitely determined corank–1 map germs with no triple points in their stabilization, cf. Example 1.4.11, see [43, 33] for details.

The set of double points is \(D = \{(s, t) \in \mathbb{C}^2 \mid d(s, t) = 0\}\). It is equipped with the involution \(\iota : D \to D\), \(\iota(s, t) = (s, -t)\). The set of double values is \(\Sigma(f) = \{(x, y, z) \in \mathbb{C}^3 \mid z = 0 \text{ and } g(x, y) = 0\}\).

There are several options for the choice of the transverse section. E.g., Proposition 5.3.4 works with \(a_1 = a_2 = a_3 = 1\). Furthermore, one can also take \(H_2(x, y, z) = z\) or \(H_3(x, y, z) = g(x, y)\).

If we take \(H(x, y, z) = z\) then the \(H\)–vertical indexes \(v_j\) are zero. This fact follows directly from the product decomposition of \(f\). (However, for different other choices it can be nonzero as well.)

**Proposition 5.4.1.** For \(\Sigma^{1,0}\)–type germs \(\Sigma_j v_i = -\Sigma_{i \neq k} D_i \cdot D_k - C(\Phi)\), where \(C(\Phi)\) is the number of crosscaps of a stabilization of \(\Phi\), cf. Subsection 1.4.3 or [53, 45].

**Proof.** With the choice of \(H = z\), \(\Phi^*(H) = td(s, t)\) and \(v_j = 0\). If \(i \neq \sigma(i)\), then

\[v_i j = -\Sigma_{k \neq \sigma(i)} \{D_i + D_{\sigma(i)}\} \cdot D_k - \{D_i + D_{\sigma(i)}\} \cdot \{t = 0\},\]

while for \(i = \sigma(i)\) (see 5.3.6) one has

\[v_i j = -\Sigma_{k \neq i} D_i \cdot D_k - D_i \cdot \{t = 0\}.\]

Taking the sum for all \(j\) we get the identity, once we verify

\[\Sigma_{i=1}^l D_i \cdot \{t = 0\} = D \cdot \{t = 0\} = \dim \mathcal{O}(\mathbb{C}^2, 0) \cdot \frac{\mathcal{O}(\mathbb{C}^2, 0)}{(t, d(s, t))}.\]
5.5. Examples

In the next paragraphs we provide some concrete examples. The families are taken from D. Mond’s list of simple germs [45, Table 1]. In the sequel we provide the resolution graph of $D$ and the plumbing graph of $\partial F$. The computations are left to the reader. Example 4.2.3 serves as a model to determine the resolution graph of $D$. By Section 5.4 the equation of $D$ and all the gluing invariants can be determined for $\Sigma^{1,0}$ type germs.

The last two examples are more special than the previous ones: they are not of type $\Sigma^{1,0}$. In the first family ($H_k$ from the list [45, Table 1]) the calculation has some nontrivial steps, hence we provide more details. The last example 5.5.8 is a corank 2 map germ from [31], cf. Example 1.4.12.

5.5.1. Whitney umbrella, or cross-cap. $\Phi(s,t) = (s, t^2, ts)$ and $f(x,y,z) = x^2y - z^2 = 0$. The graph of $D$ is on the left, while our algorithm provides the graph from the right for $\partial F$

\[
\begin{array}{c}
-1 \\
\end{array} \quad \begin{array}{c}
-1 \\
\end{array} -2 -1 -2
\]

which after plumbing calculus (blowing down twice followed by a 0-chain absorption, see Subsection 4.2.2) transforms into $-4$ (Compare also with Example 10.4.2 from [55].)

5.5.2. $S_1$. Set $\Phi(s,t) = (s, t^2, t^3 + s^2t)$, hence $f(x,y,z) = y(y + x^2)^2 - z^2$. The graph of $D$ is the first diagram, while the other two equivalent graphs represent $\partial F$

\[
\begin{array}{c}
-1 \\
\end{array} \quad \begin{array}{c}
-1 \\
\end{array} -6 \quad \Theta \\
\end{array} \quad \begin{array}{c}
-4 \\
\end{array} \quad \begin{array}{c}
-4 \\
\end{array}
\]

or

5.5.3. The family $S_{k-1}$, $k \geq 2$. One has $\Phi(s,t) = (s, t^2, t^3 + s^k t)$ and $f(x,y,z) = y(y + x^k)^2 - z^2 = 0$. Cf. Examples 1.4.10, 3.5.1.

Case 1: $k = 2n$. 

\[
\begin{array}{c}
D: \\
\end{array} \quad \begin{array}{c}
-2 \\
\end{array} -2 \ldots -2 -1
\]

Here, the last identity follows from the algebraic definition of $C(\Phi)$, as the codimension of the Jacobian ideal of $\Phi$, see Subsection 1.4.2 and Example 1.4.10. In our case this ideal is generated by $t$ and $d(s,t)$. \qed
Here the number of $(-2)$-vertices is $n - 1$.

**Case 2:** $k = 2n + 1$.

The graph of $D$ is

where the number of $(-2)$-vertices is $k - 1$.

Note that the double point curve $D$ does not depend on the parity of $k$, however $\Sigma$ has one component, when $k$ is odd, and two components, when $k$ is even. Thus the pairing $\sigma$ changes the components of $D$ in the odd case, and $\sigma$ is the identity in the even case.

**Case 1:** $k = 2n + 1$.

where the number of $(-2)$-vertices is $k - 1$.

**Case 2:** $k = 2n$. 

5.5.4. The family $B_k$ ($k \geq 1$). $\Phi(s, t) = (s, t^2, s^2t + t^{2k+1})$ and $f = y(x^2 + y^k) - z^2 = 0$.

The graph of $D$ is
where the number of $(-2)$-vertexes on the left is again $k-1$.

**5.5.5. The family $C_k$ ($k \geq 1$).** $\Phi(s, t) = (s, t^2, st^3 + s^ht)$ and $f = y(x+y^k)^2 - z^2 = 0$. The resolution of $D$ is calculated in Example 4.2.3.

*Case 1: $k = 2n+1.*

$$
 D : \quad -2 -2 \ldots -2 -1
$$

where the number of $(-2)$-vertices in the middle is $n - 1$.

*Case 2: $k = 2n.*

$$
 D : \quad -2 -2 \ldots -2 -3 -1
$$

and the graph of $\partial F$ is

$$
 \partial F : \quad -2 -1 -4 -2 -2 \ldots -2 -1 -3k + 1
$$

where the number of $(-2)$-vertices in the middle is $n - 2$.

**5.5.6. $F_4$.** $\Phi(s, t) = (s, t^2, s^3t + t^5)$ and $f = y(x^3 + y^2)^2 - z^2 = 0.$
5.5.7. The family $H_k$, $k \geq 1$. In this case $\Phi(s, t) = (s, st + t^{3k-1}, t^3)$ and the equation of the image is calculated as the $0^{th}$ fitting ideal of $\Phi_*(\mathcal{O}_{(\mathbb{C}^2, 0)})$ in Example 1.3.4 $f(x, y, z) = z^{3k-1} - y^3 + x^3z + 3xyz^k = 0$.

When $k > 1$, the local form of $T^2 f$ along $p(\tau) = (\tau^{3k-2}, -\tau^{3k-1}, \tau^3) \in \Sigma^*$ is

$$\frac{z^{3k-1}}{12k^2 - 12k + 4} \left[ - \left( (3k-3) - (3k-1)\sqrt{3}i \right) \tau x' + \left( 3k + (3k-2)\sqrt{3}i \right) y' + (6k^2 - 6k + 2)\tau^{3k-4}z' \right].$$

$$\cdot \left[ - (3k-3) + (3k-1)\sqrt{3}i \right] \tau x' + \left( 3k - (3k-2)\sqrt{3}i \right) y' + (6k^2 - 6k + 2)\tau^{3k-4}z',$$

where $x' = x - \tau^{3k-2}$, $y' = y + \tau^{3k-1}$, $z' = z - \tau^3$. With the choice $H(x, y, z) = \partial_x f(x, y, z) + \partial_y f(x, y, z) + \partial_z f(x, y, z)$, one has $\nu = 3k - 1$.

For $k = 1$ one gets

$$T^2(f) = \left[ z' + \left( \frac{3}{2} \right) \tau y' + \sqrt{3}i\tau^2x' \right] \cdot \left[ z' + \left( \frac{3}{2} - \frac{\sqrt{3}}{2}i \right) \tau y' - \sqrt{3}i\tau^2x' \right],$$

where $x' = x - \tau$, $y' = y + \tau^2$, $z' = z - \tau^3$. In this case $\nu = 0$.

In all cases $\lambda_1 + \lambda_2 + \nu = -3k - 1$.

5.5.8. A corank 2 map germ. In this case $\Phi(s, t) = (s^2, t^2, s^3 + t^3 + st)$ and $f(x, y, z) = x^6 - 2x^4y + x^2y^2 - 2x^3y^3 - 2xy^4 + y^6 - 8x^2y^2z - 2x^3z^2 - 2xyz^2 - 2y^3z^2 + z^4 = 0$.

In the special case $k = 1$ the germs $H_1$ and $S_1$ are analytic $\mathscr{A}$-equivalent.
\partial F:

\begin{align*}
-2 & \quad -1 & \quad -5 & \quad -1 & \quad -5 & \quad -1 & \quad -2 \\
-2 & \quad -1 & \quad -5 & \quad -5 & \quad -2 \\
-2 & \quad -2 & \quad -2 & \quad -2 \\
-2 & \quad -2 & \quad -2 & \quad -2
\end{align*}
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Summary

In the thesis we study holomorphic germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, which are singular only at
the origin, with a special emphasis on the distinguished class of finitely determined germs. The results are published in two articles, joint with András Némethi, they are contained in Chapter 3 and 5.

Stability and finite determinacy of germs are discussed in Chapter 1. Here we review several theorems of Mather and Gaffney following the articles of Wall, Mond, Marar and Nuño-Ballesteros. Furthermore, we review the analytic invariants of germs introduced by Mond, namely the number of Whitney umbrellas and triple values of a stabilization.

In Chapter 2 we introduce the Smale invariant, which classifies the immersions of spheres up to regular homotopy. We review formulas of Hughes–Melvin and Ekholm–Szűcs, which express the Smale invariant of an immersion from the 3-sphere to the 5-sphere using singular Seifert surfaces. We introduce Ekholm’s invariants for stable immersions. Here we mainly refer to the articles of Hughes, Szűcs, Ekholm and Takase.

In Chapter 3 we identify the Smale invariant of immersions associated with holomorphic germs with an analytic invariant of the germs, namely with the number of Whitney umbrellas of a stabilization, answering a question of Mumford from 1961. We give representatives for all regular homotopy classes of immersions from the 3-sphere to the 5-sphere using the simple germs from Mond’s list, answering Smale’s question in this dimension. We give a new proof for Mond’s theorem for corank–1 germs about the number of Whitney umbrellas via the newly defined complex Smale invariant. We identify the correct sign in the Hughes–Melvin and Ekholm–Szűcs formulas. We express the Ekholm invariant of stable immersions associated with finitely determined germs in terms of the analytic invariants of Mond. We conclude that these analytic invariants are $\mathcal{C}^\infty$ invariants, and their combination expressing the Ekholm invariant is a topological invariant of the finitely determined germs.

The image of a finitely determined germ provides a non-isolated hypersurface singularity in $\mathbb{C}^3$. In Chapter 5 we determine the boundary of the Milnor fibre of these singularities. We summarize in Chapter 4 the main properties of the Milnor fibre and the resolution of isolated singularities, in which case the boundary of the Milnor fibre is diffeomorphic to the link and to the boundary of any resolution of the singularity as well. Here we use the de Jong–Pfister book and the articles of Milnor, Looijenga and Némethi. In the non-isolated case the Némethi–Szilárd book provides a general algorithm to determine the Milnor fibre boundary, however for concrete examples it is preferable to find a more direct procedure.

In Chapter 5 the Milnor fibre boundary is constructed as a surgery of the 3-sphere along the double point curves of the immersion associated with the holomorphic germ. The gluing of the pieces along their torus boundaries is described by the newly defined vertical indices and homological invariants. For a certain class of germs the sum of these invariants is expressed with the number of Whitney umbrellas. In practice our algorithm provides a plumbing graph of the Milnor fibre boundary by modifying a good embedded resolution graph of the double point locus of the finitely determined germ. The examples we discuss are the simple germs from Mond’s list and a corank–2 germ from an article of Marar and Nuño-Ballesteros.
Összefoglalás (Summary in Hungarian)

Az értekezésben olyan, \( C^2 \)-ből \( C^3 \)-be képező holomorf leképezésesírákkal foglalkozunk, melyek csak az origóban szingulárisak, illetve ezeknek egy speciális osztályával, a végesen determinált csírákkal. Az eredményeket két, Némethi Andrással közös cikkben publikáltuk, ezeket a 3. és 5. fejezet tartalmazza.

Az 1. fejezetben csírák stabilitását és véges determináltságát tárgyaljuk Mather és Gaffney eredményei, Wall, Mond, Marar és Nuño-Ballesteros cikkei alapján. Ismertetjük Mond analitikus invariánsait, a stabilizálás során megjelenő Whitney-esernyők ill. háromszoros pontok számát.

A 2. fejezetben bevezetjük a gömbök imerzióit reguláris homotópia erejéig teljesen osztályozó Smale-invariánsat. Kimondjuk a 3 dimenziós gömb 5 dimenziós térbe menő imerzióiról szóló Hughes–Melvin és Ekholm–Szűcs formulákat, melyek szinguláris Seifert-felület segítségével fejezik ki a Smale-invariánsat. Ismertetjük a stabil imerziók Ekholm–féle invariánsait. Fő forrásaink Hughes, Takase, Ekholm és Szűcs cikkei.

A 3. fejezetben holomorf csírához tartozó imerzió Smale-invariánsát azonosítjuk a csíra egyik analitikus invariánsával, a Whitney-esernyők számával, válaszolva ezzel Mumford 1961-es kérdésére. Mond listájáról származó csírákhoz tartozó imerziókkal reprezentáljuk a 3 dimenziós gömbből 5 dimenziós gömbbe képező imerziók minden reguláris homotópiaosztályát, válaszolva Smale kérdésére ebben a dimenzióban. Új bizonyítást adunk Mond Whitney-esernyők számáról szóló tételére 1–korangú csírák esetén az általunk bevezetett komplex Smale-invariáns segítségével. Azonosítjuk a Hughes–Melvin és az Ekholm–Szűcs formulák előjelét. Végesen determinált csírákhoz tartozó stabil imerziókra az Ekholm-invariáns kifejezése a Mond-féle analitikus invariánsokkal. Megmutatjuk, hogy az említett analitikus invariánsok \( C^\infty \) invariánsok is, és az Ekholm-invariáns kifejező kombinációnjuk a végesen determinált csírák topológikus invariánsa.

Végesen determinált csírák képei speciális nem-izolált felületszingularitásokat határoznak meg \( C^3 \)-ban. Az 5. fejezetben ezek Milnor-fibrumának a peremét határozzuk meg. Ehhez a 4. fejezetben összefoglaljuk az izolált felületszingularitások Milnor-fibrumával és rezolúciójával kapcsolatos eredményeket a de Jong–Pfister könyv, Milnor és Looijenga cikkei és Némethi összefoglaló cikkei alapján. Izolált felületszingularitás Milnor-fibrumának a pereme differenciálja a szingularitás linkjével, mely egyben a szingularitás rezolúciójának a pereme is. Nem-izolált esetben a Némethi–Szilárd könyvben található algoritmus a Milnor-fibrum peremének meghatározására, konkret családokra azonban érdemes közvetlenebb eljárásokat keresni.

Az 5. fejezetben a 3-dimenziós gömbből kiindulva, a holomorf csírához tartozó imerzió kettőspont-görbét mentén végzett műtettel szerkesztjük meg a Milnor-fibrum peremét. A tóruszok menti ragasztásokat az általunk bevezetett vertikális indexekkel és homológius invariánsokkal írjuk le, melyek összegét kifejezzük a Whitney-esernyők számával a csírák egy osztályára. Technikailag a végesen determinált csíra kettőspont-görbénként beágyazott rezolúciós gráfját módosítva adjuk meg a csíra által meghatározott nem-izolált felületszingularitás Milnor-fibrumának peremét plumbing gráf formájában. A példák Mond listájáról és egy Marar–Nuño-Ballesteros cikkből származnak.