WHEN POINTS ARE EQUIVALENT IN LONG TIME BEHAVIORS

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Abstract. We metric the distance between two points in a new way, and then show a new equivalent condition of ergodic measures. We also conclude a necessary and sufficient condition of uniquely ergodicity. What’s more, an equivalent condition of when the time average operator maps continuous functions to be continuous is shown.

1. Introduction

Throughout this paper, a topological dynamical system is a pair \((X, T)\), where \(X\) is a non-empty compact metric space with a metric \(d\) and \(T\) is a continuous map from \(X\) to itself.

When studying long time behaviors, people first focused on the equicontinuous systems, because they have simple dynamic behaviors \([1, 2]\). But just the cumulative effect of the points in the orbit can influence the long time behaviors, so we can ignore where the orbital is at some points if we study the long time behaviors. For this purpose, mean-L-stable systems were introduced\([3, 4, 5]\). We call a dynamic system \((X, T)\) mean-L-stable if for \(\forall \varepsilon > 0\), there is a \(\delta > 0\) such that \(d(x, y) < \delta\) implies \(d(T^n(x), T^n(y)) < \varepsilon\) for all \(n \in \mathbb{N}\) except a set of upper density less than \(\varepsilon\). Recently, Li, Tu and Ye\([12]\) introduced mean equicontinuous systems. A dynamic system is called mean equicontinuous if for \(\forall \varepsilon > 0\), there exists a \(\delta > 0\) such that whenever \(x, y \in X\) with \(d(x, y) < \delta\),

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^k(y)) < \varepsilon
\]

In their paper, they proved that a dynamic system is mean equicontinuous if and only if it is mean-L-stable. We refer to \([6, 7, 8, 9, 10, 11]\) for further study on mean equicontinuity and related subjects.

Sometimes we can also ignore the order of the points in the orbit when studying long time

2010 Mathematics Subject Classification. Primary: 54H20; Secondary: 37A20, 37B05, 37B45.

Key words and phrases. Generic point. Uniquely ergodicity. Time average. Weak mean equicontinuity.

The second author is supported by the NSF of China(No. 11671382), CAS Key Project of Frontier Sciences(No. QYZDJ-SSW-JSC003), the Key Lab. of Random Complex Structures and Data Sciences CAS and National Center for Mathematics and Interdisciplinary Sciences CAS.

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For this purpose, we introduce some new notions in our paper. For $\forall x, y \in X$, we define

$$\overline{F}(x, y) = \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^\sigma(x))$$

$$\overline{E}(x, y) = \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^\sigma(y))$$

$N(\overline{F}) = \{(x, y) \in X \times X | \overline{F}(x, y) = 0\}$

$N(\overline{E}) = \{(x, y) \in X \times X | \overline{E}(x, y) = 0\}$

where $S_n$ is the n-order permutation group. If $\overline{F}(x, y) = \overline{E}(x, y)$, we say $F(x, y)$ exists, and denote

$$F(x, y) = \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^\sigma(x))$$

$$N(F) = \{(x, y) \in X \times X | F(x, y) = 0\}$$

obviously,

$$N(\overline{F}) = N(F) \subset N(\overline{E})$$

In [3], Fomin proved that a minimal mean-L-stable system is uniquely ergodic, and then in [5], Oxtoby proved a more general result each transitive mean-L-stable system is uniquely ergodic. In this paper we shall show that each transitive mean-L-stable system is uniquely ergodic. In this paper we shall show ($X, T$) is uniquely ergodic if and only if $N(F) = X \times X$.

**Definition 1.1.** We say ($X, T$) is $\overline{F}$-continuous at $x$ if for $\forall \varepsilon > 0$, $\exists \delta > 0$, whenever $d(x, y) < \delta$, we have $\overline{F}(x, y) < \varepsilon$. Let $C(\overline{F})$ denote all $\overline{F}$-continuous points, if $C(\overline{F}) = X$, we say ($X, T$) is $\overline{F}$-continuous. Moreover, we say ($X, T$) is weak mean equicontinuous if it is $\overline{F}$-continuous.

**Definition 1.2.** We say ($X, T$) is $F$-continuous at $x$ if for $\forall \varepsilon > 0$, $\exists \delta > 0$, whenever $d(x, y) < \delta$, $F(x, y)$ exists and $F(x, y) < \varepsilon$. Let $C(F)$ denote all $F$-continuous points, if $C(F) = X$, we say ($X, T$) is $F$-continuous.

Obviously,

equicontinuity $\Rightarrow$ mean equicontinuity $\Rightarrow$ $F$-continuity $\Rightarrow$ $\overline{F}$-continuity

In this paper, we shall prove $F$-continuity is equivalent to $\overline{F}$-continuity. But in general, $F$-continuity doesn’t imply mean equicontinuity.

**Example**

$$T : S^1 \to S^1$$

$$T(x) = \begin{cases} 
1 - 2(x - \frac{1}{2})^2, & x \in [0, \frac{1}{2}] \\
\frac{3}{2} - 2(x - 1)^2, & x \in [\frac{1}{2}, 1]
\end{cases}$$

For $\forall x, y \in S^1$, $F(x, y) = 0$, so ($S^1, T$) is $F$-continuous, but 0 and $\frac{1}{2}$ are not the mean equicontinuous points.

Chaos shows a perturbation on the initial value can cause a large gap on orbits. Weak mean equicontinuous systems may be chaos, but in the view of measure, they are stable, for we can predict the distribution of the orbit even though there is a perturbation on the initial value. The electron cloud is a visual but loose example.

Birkhoff Ergodic Theorem shows the time average operator maps integrable functions to be integrable, one may ask in which case the time average operator maps continuous functions to be continuous? We shall answer this question clearly in our paper.

**Main theorem**

- $N(F) = X \times X$ if and only if ($X, T$) is uniquely ergodic.
- ($X, T$) is $\overline{F}$-continuous if and only if ($X, T$) is $F$-continuous.
• $(X, T)$ is weak mean equicontinuous if and only if for $\forall f \in C(X)$, the time average $f^*$ is well defined and continuous.

2. Preliminaries

In this section we recall some notions of topological dynamical system.

2.1. Density. Denote by $\mathbb{N}$ the set of all non-negative integers. Let $F \subset \mathbb{N}$, we define the upper density $D(F)$ of $F$ by

$$
D(F) = \lim_{n \to +\infty} \frac{\#(F \cap [0, n-1])}{n}
$$

where $\#(\cdot)$ is the number of elements of a set. Similarly, the lower density $\underline{D}(F)$ of $F$ is defined by

$$
\underline{D}(F) = \lim_{n \to +\infty} \frac{\#(F \cap [0, n-1])}{n}
$$

One may say $F$ has density $D(F)$ if $D(F) = \underline{D}(F)$, in which case $D(F)$ is equal to this common value.

2.2. Invariant measures. $(X, T)$ is a compact topological dynamic system, let $M(X)$ denote all regular Borel probability measures, $M(X, T)$ denote all $T$-invariant regular Borel probability measures. It is well known that $M(X)$ and $M(X, T)$ are nonempty, moreover, $M(X)$ and $M(X, T)$ are compact spaces. An invariant measure is ergodic if and only if it is an extreme point of $M(X, T)$.

We say $(X, T)$ is uniquely ergodic if $M(X, T)$ consists of a single measure, what’s more, if the only invariant measure supports on $X$, we call $(X, T)$ is strictly ergodic.

Define

$$
\delta_x(A) = \begin{cases} 
1, & x \in A \\
0, & x \notin A
\end{cases}
$$

then for $\forall x \in X$, $\left\{\frac{1}{n} \sum_{k=1}^{n} \delta_{T^k(x)}\right\}_{n=1}^{\infty} \subset M(X)$, let $M_x$ denote all limit points of $\left\{\frac{1}{n} \sum_{k=1}^{n} \delta_{T^k(x)}\right\}_{n=1}^{\infty}$, obviously, $M_x \neq \emptyset$ and $M_x \subset M(X, T)$. Define

$$
C_x = \bigcup_{\mu \in M_x} \Lambda_{\mu}
$$

$$
M(T) = \bigcup_{x \in X} C_x
$$

where $\Lambda_{\mu}$ is the support set of $\mu$. We call $C_x$ the minimal center of attraction of $x$ and $M(T)$ the minimal center of attraction of $T$.

Let $x \in X$, if for $\forall f \in C(X)$, the time average

$$
f^*(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(x))
$$

exists, then $x$ is called a generic point.

**Proposition 2.1.** $x \in X$ is a generic point iff $M_x$ consists of a single measure.

**Proposition 2.2.** $(X, T)$ is a compact topological dynamic system, $\mu$ is ergodic on $(X, T)$, then there exists a generic point $x \in X$ and $M_x = \{\mu\}$. 

If \( x \) is a generic point, we say \( \mu_x \in M_x \) is generated by \( x \). Denote
\[
Q = \{ x \in X | x \text{ is a generic point} \}
\]
\[
Q_T = \{ x \in Q | \mu_x \text{ is an ergodic measure} \}
\]

**Definition 2.1.** A Borel subset \( E \subset X \) is invariant measure one if \( \mu(E) = 1 \) for all \( \mu \in M(X, T) \).

**Proposition 2.3.** (\([5]\)) \( Q \) and \( Q_T \) are invariant measure one.

For any \( A \subset X \), we define
\[
\chi_A(x) = \begin{cases} 
1, & x \in A \\
0, & x \notin A
\end{cases}
\]

**Proposition 2.4.** \( x \in X \) is a generic point and \( \mu \) is generated by \( x \), then for any open set \( U \subset X \) and any closed set \( V \subset X \), we have
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(U(T^k(x))) \geq \mu(U)
\]
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(V(T^k(x))) \leq \mu(V)
\]

### 2.3. Recurrence.

**Definition 2.2.** \( x \in X \) is a almost period point, if for \( \forall \varepsilon > 0 \), there is a \( N_\varepsilon > 0 \) such that
\[
\#(\{ r | T^r(x) \in O(x, \varepsilon), n < r \leq n + N_\varepsilon \}) \geq 1, \forall n > 0
\]

**Definition 2.3.** \( x \in X \) is a weak almost period point, if for \( \forall \varepsilon > 0 \), there is a \( N_\varepsilon > 0 \) such that
\[
\#(\{ r | T^r(x) \in O(x, \varepsilon), 0 < r < nN_\varepsilon \}) \geq n, \forall n > 0
\]

**Definition 2.4.** \( x \in X \) is a quasi-weak almost period point, if for \( \forall \varepsilon > 0 \), there are \( N_\varepsilon > 0 \) and \( \{ n_i \}_{i=1}^{\infty} \) such that
\[
\#(\{ r | T^r(x) \in O(x, \varepsilon), 0 < r < n_iN_\varepsilon \}) \geq n_i, \forall i > 0
\]

Let \( A(T) \) denote all almost period points, \( W(T) \) denote all weak almost period points, and \( QW(T) \) denote all quasi-weak almost period points, obviously,
\[
A(T) \subset W(T) \subset QW(T)
\]

Moreover, we have the follow proposition,

**Proposition 2.5.**
\[
\overline{A(T)} \subset \overline{W(T)} = \overline{QW(T)} = M(T)
\]

We refer readers to the textbook\([13]\) for more information.

### 3. Invariant measures

In this section, we will study the invariant measures. We obtain a sufficient condition of \( M_x = M_\mu \), and in the case that \( x \) is a generic point, the condition can be a necessary and sufficient condition. Moreover, we find out several equivalent relations in ergodic theroy. At first, we show a property of invariant measures.
Lemma 3.1. Let $\mu$ be an invariant measure, then for $\forall \varepsilon > 0$, there are finite mutually disjoint closed sets $\{\Lambda_k\}_{k=1}^{k_0}$ such that
\[
\text{diam}(\Lambda_k) \leq \varepsilon \quad \forall k = 1, 2, \cdots, k_0
\]
\[
\mu\left(\bigcup_{k=1}^{k_0} \Lambda_k\right) \geq 1 - \varepsilon
\]

Proof. Obviously, $\{O(x, \frac{\varepsilon}{2})\}_{x \in X}$ cover $X$, then there are $\{O(x_k, \frac{\varepsilon}{2})\}_{k=1}^{k_0}$ cover $X$ for $X$ is compact. $\forall k = 1, 2, \cdots, k_0$, there exists closed subset $V_k \subset X$ such that
\[
V_k \subset O(x_k, \frac{\varepsilon}{2})
\]
\[
\mu(O(x_k, \frac{\varepsilon}{2}) \setminus V_k) \leq \frac{\varepsilon}{k_0}
\]

Let $W_k = X \setminus O(x_k, \frac{\varepsilon}{2})$, then $W_k$ is closed, and $W_k \cap V_k = \emptyset$. We construct $\{\Lambda_k\}_{k=1}^{k_0}$ as follow,
\[
\Lambda_1 = V_1
\]
\[
\Lambda_2 = V_2 \cap W_1
\]
\[
\vdots
\]
\[
\Lambda_{k_0} = V_{k_0} \cap \bigcap_{k=1}^{k_0-1} W_k
\]

It’s easy to get that $\{\Lambda_k\}_{k=1}^{k_0}$ are mutually disjoint closed sets. Next, we shall prove for $\forall k = 1, 2, \cdots, k_0$,
\[
\sum_{i=1}^{k} \mu(\Lambda_i) \geq \mu\left(\bigcup_{i=1}^{k} O(x_i, \frac{\varepsilon}{2})\right) - \frac{k \varepsilon}{k_0}
\]
when $k = 1$, with the definition of $\Lambda_1$, we can easily obtain
\[
\mu(\Lambda_1) \geq \mu(O(x_1, \frac{\varepsilon}{2})) - \frac{\varepsilon}{k_0}
\]
if $k = n, n \leq k_0 - 1$, the following inequation is true,
\[
\sum_{i=1}^{n} \mu(\Lambda_i) \geq \mu\left(\bigcup_{i=1}^{n} O(x_i, \frac{\varepsilon}{2})\right) - \frac{n \varepsilon}{k_0}
\]
when $k = n + 1$, we deduce
\[
\mu(V_k) = \mu(V_{n+1}) \geq \mu(O(x_{n+1}, \frac{\varepsilon}{2})) - \frac{\varepsilon}{k_0}
\]
then
\[
\mu(\Lambda_k) = \mu(V_{n+1} \cap \bigcap_{i=1}^{n} W_i) \geq \mu(O(x_{n+1}, \frac{\varepsilon}{2}) \cap \bigcap_{i=1}^{n} W_i) - \frac{\varepsilon}{k_0}
\]
\[
\sum_{i=1}^{n+1} \mu(\Lambda_i) \geq \mu(O(x_{n+1}, \frac{\varepsilon}{2}) \cap \bigcap_{i=1}^{n} W_i) + \mu\left(\bigcup_{i=1}^{n} O(x_i, \frac{\varepsilon}{2})\right) - \frac{n + 1}{k_0} \varepsilon
\]
\[
\geq \mu(O(x_{n+1}, \frac{\varepsilon}{2}) \cap \bigcap_{i=1}^{n} W_i) \bigcup \bigcup_{i=1}^{n} O(x_i, \frac{\varepsilon}{2}) - \frac{n + 1}{k_0} \varepsilon
\]
since
\[(\bigcap_{i=1}^{n} W_i)^c = \bigcup_{i=1}^{n} W_i^c = \bigcup_{i=1}^{n} O(x_i, \varepsilon/2)\]
we have
\[O(x_{n+1}, \varepsilon/2) \bigcap (\bigcap_{i=1}^{n} W_i)^c \subset \bigcup_{i=1}^{n} O(x_i, \varepsilon/2)\]
then
\[(O(x_{n+1}, \varepsilon/2) \bigcap \bigcap_{i=1}^{n} W_i) \bigcup (\bigcup_{i=1}^{n} O(x_i, \varepsilon/2))\]
\[= O(x_{n+1}, \varepsilon/2) \bigcup \bigcup_{i=1}^{n} O(x_i, \varepsilon/2)\]
Thus,
\[\sum_{i=1}^{n+1} \mu(A_i) \geq \mu\left(\bigcup_{i=1}^{n+1} O(x_i, \varepsilon/2)\right) - \frac{n+1}{k_0} \varepsilon\]
As a result, we can deduce
\[\mu\left(\bigcup_{k=1}^{k_0} \Lambda_k\right) = \sum_{k=1}^{k_0} \mu(\Lambda_k)\]
\[\geq \mu\left(\bigcup_{k=1}^{k_0} O(x_k, \varepsilon/2)\right) - \varepsilon\]
\[= 1 - \varepsilon\]

Next, we shall study in which case two points can generate the same measure set.

**Theorem 3.2.** \(\forall x, y \in X, \text{ if } F(x, y) = 0, \text{ then } M_x = M_y.\)

**Proof.** \(\forall \mu \in M_x, \exists \{n_r\}_{r=1}^\infty\) such that for \(\forall f \in C(X)\)
\[\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) = \int_X f \, d\mu\]
\(\forall \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that whenever } x_1, x_2 \in X \text{ with } d(x_1, x_2) < \delta, \text{ we have}\)
\[|f(x_1) - f(x_2)| < \varepsilon\]
With the assumption, we obtain
\[\lim_{r \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma\,k}(y)) = 0\]
let $M = \max_{z \in X} \{|f(z)|\}$, for $\forall \sigma \in S_{n_r}$, we have
\[
\frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma(k)}(y)) \geq \delta \times \frac{\# \{|k|d(T^k(x), T^{\sigma(k)}(y)) \geq \delta, k = 1, 2, \cdots, n_r\}}{n_r}
\]
\[
\frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^{\sigma(k)}(y)) \leq \frac{1}{n_r} \sum_{k=1}^{n_r} |f(T^k(x)) - f(T^{\sigma(k)}(y))|
\]
\[
\leq \varepsilon + 2M \times \frac{\# \{|k|d(T^k(x), T^{\sigma(k)}(y)) \geq \delta, k = 1, 2, \cdots, n_r\}}{n_r}
\]
\[
\leq \varepsilon + \frac{2M}{\delta} \times \frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma(k)}(y))
\]
so
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^{\sigma(k)}(y)) \leq \varepsilon
\]
let $\varepsilon \to 0$, we get
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^{\sigma(k)}(y)) = 0
\]
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) = \lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^{\sigma(k)}(y))
\]
hence, for $\forall f \in C(X)$,
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y)) = \int_X f \, d\mu
\]
which implies $\mu \in M_y$, thus, $M_x \subset M_y$. Similarly, we can conclude $M_y \subset M_x$, so $M_x = M_y$. $\square$

If $x$ is a generic point, the condition can be a necessary and sufficient condition.

Theorem 3.3. $\forall x, y \in X$, and $x$ is a generic point, then $F(x, y) = 0$ iff $M_x = M_y$.

Proof. We need only to prove $x$ is a generic point and $M_x = M_y$ can infer $F(x, y) = 0$. Let $\mu$ be the measure generated by $x$, then for $\forall f \in C(X)$, we have
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(x)) = \int_X f \, d\mu
\]
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(y)) = \int_X f \, d\mu
\]

If $F(x, y) \neq 0$, then there is an $\varepsilon > 0$ such that $\exists (x, y) = \varepsilon$, we can suppose
\[
\lim_{r \to +\infty} \inf_{\sigma \in S_{n_r}} \frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma(k)}(y)) = \varepsilon
\]

Let $\eta = \min \{\frac{\varepsilon}{m}, \frac{\varepsilon}{2m}\}$, where $m = \text{diam}(X)$, by Lemma 3.1, there are mutually disjoint closed sets $\{\Lambda_s\}_{s=1}^{s_0}$ such that
\[
\text{diam}(\Lambda_s) \leq \eta \quad \forall s = 1, 2, \cdots, s_0
\]
\[ \mu\left(\bigcup_{s=1}^{s_0} \Lambda_s\right) \geq 1 - \eta \]

then there exist mutually disjoint open sets \( \{U_s\}_{s=1}^{s_0} \) such that

\[ \Lambda_s \subset U_s \quad \forall s = 1, 2, \cdots, s_0 \]

\[ \text{diam}(U_s) \leq \frac{4}{3} \eta \quad \forall s = 1, 2, \cdots, s_0 \]

For \( \forall s = 1, 2, \cdots, s_0 \), we have

\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{U_s}(T^k(x)) \geq \mu(U_s) \]

\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{U_s}(T^k(y)) \geq \mu(U_s) \]

then for \( \forall \delta > 0 \), there is a \( r_0 > 0 \) such that for \( \forall r > r_0, \forall s = 1, 2, \cdots, s_0 \)

\[ \frac{1}{n r} \sum_{k=1}^{n r} \chi_{U_s}(T^k(x)) \geq \mu(U_s) - \delta \geq \mu(\Lambda_s) - \delta \]

\[ \frac{1}{n r} \sum_{k=1}^{n r} \chi_{U_s}(T^k(y)) \geq \mu(U_s) - \delta \geq \mu(\Lambda_s) - \delta \]

and we obtain

\[ \inf_{\sigma \in S_{n r}} \frac{1}{n r} \sum_{k=1}^{n r} d(T^k(x), T^{\sigma(k)}(y)) \leq \frac{4}{3} \eta \times \sum_{s=1}^{s_0} (\mu(\Lambda_s) - \delta) + m \times (1 - \sum_{s=1}^{s_0} (\mu(\Lambda_s) - \delta)) \]

\[ \leq \frac{4}{3} \eta + m \eta + m \delta s_0 \]

\[ \lim_{r \to +\infty} \inf_{\sigma \in S_{n r}} \frac{1}{n r} \sum_{k=1}^{n r} d(T^k(x), T^{\sigma(k)}(y)) \leq \frac{4}{3} \eta + m \eta + m \delta s_0 \]

let \( \delta \to 0 \), we get

\[ \lim_{r \to +\infty} \inf_{\sigma \in S_{n r}} \frac{1}{n r} \sum_{k=1}^{n r} d(T^k(x), T^{\sigma(k)}(y)) \leq \frac{4}{3} \eta + m \eta = \frac{7}{12} \varepsilon \]

this is a contradiction, so \( F(x, y) = 0 \). \( \Box \)

By **Theorem 3.3**, we can get the following results.

**Theorem 3.4.** \( \mu \) is a invariant measure of \((X, T)\), then \( \mu \) is ergodic iff \( \mu \times \mu(N(F)) = 1 \).

**Proof.** If \( \mu \) is ergodic, by **Theorem 3.3**, there exists \( \Lambda \subset X \) such that \( \mu(\Lambda) = 1 \) and \( \Lambda \times \Lambda \subset N(F) \), thus, \( \mu \times \mu(N(F)) = 1 \).

\( \forall x \in X \), we define

\[ N(F, x) = \{ y \in X | F(x, y) = 0 \} \]

obviuosly,

\[ \mu \times \mu(N(F)) = \int_X \mu(N(F, x))d\mu(x) \]

If for \( \forall x \in X \), \( \mu(N(F, x)) = 0 \), then \( \mu \times \mu(N(F)) = 0 \); if there is \( x_0 \in X \) such that \( \mu(N(F, x_0)) = a \) with \( a \in (0, 1) \), let \( b = \max\{a, 1 - a\} \), we can deduce \( \mu(N(F, x)) \leq b \) for \( \forall x \in X \), then \( \mu \times \mu(N(F)) \leq b < 1 \). Hence, there is a \( x \in X \) such that \( \mu(N(F, x)) = 1 \) when \( \mu \times \mu(N(F)) = 1 \), by
Proposition 2.3, there is a \( y \in Q_T \cap N(F, x) \), and by Theorem 3.3, we can get \( f^*(z) = \int_X f \, d\mu_y \) for \( \forall z \in N(F, x) \) and \( \forall f \in C(X) \). Therefore, we can deduce

\[
\int_X f \, d\mu = \lim_{n \to +\infty} \int_{N(F, x)} \frac{1}{n} \sum_{k=1}^{n} f(T^k(z)) \, d\mu(z)
\]

\[
= \int_{N(F, x)} \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(z)) \, d\mu(z)
\]

\[
= \int_X f \, d\mu_y
\]

this shows \( \mu = \mu_y \), thus, \( \mu \) is ergodic. \( \square \)

Theorem 3.5. \( N(F) = X \times X \) iff \((X, T)\) is uniquely ergodic

Proof. If \((X, T)\) is uniquely ergodic, then \( \forall x, y \in X \), we have \( M_x = M_y \). Moreover, it is easy to obtain \( x \) is a generic point, by Theorem 3.3, we can conclude \( F(x, y) = 0 \), which implies \( N(F) = X \times X \).

Let \( \mu_1 \) and \( \mu_2 \) are ergodic measures of \((X, T)\), then there exist \( x, y \in X \) such that \( M_x = \{\mu_1\}, M_y = \{\mu_2\} \). With the assumption, we obtain \( F(x, y) = 0 \), by Theorem 3.3, we can conclude \( M_x = M_y \), which implies \((X, T)\) is uniquely ergodic. \( \square \)

By Theorem 3.5, the following corollaries are obviously.

Corollary 3.6. ([3]) If a minimal system is mean-L-stable, then it is uniquely ergodic.

Corollary 3.7. ([5]) Each transitive mean-L-stable system is uniquely ergodic.

Next, we study the properties of \( F(x, y) \).

Theorem 3.8. \( \forall x, y \in X \), if \( F(x, y) = 0 \), then \( M_x \cap M_y \neq \emptyset \).

Proof. Since \( F(x, y) = 0 \), there exist \( \{n_r\}_{r=1}^{\infty} \) such that

\[
\lim_{r \to +\infty} \inf_{\sigma \in S_{n_r}} \frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma(k)}(y)) = 0
\]

without loss of generality, we can assume that there exists \( \mu \in M_x \) such that for \( \forall f \in C(X) \),

\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) = \int_X f \, d\mu
\]

\( \forall \varepsilon > 0 \), there is a \( \delta > 0 \) such that whenever \( x_1, x_2 \in X \) with \( d(x_1, x_2) < \delta \), we obtain

\[
|f(x_1) - f(x_2)| < \varepsilon
\]

Let \( M = \max_{z \in X} \{|f(z)|\} \), for \( \forall \sigma \in S_{n_r} \), we have

\[
\frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^{\sigma(k)}(y)) \geq \delta \times \frac{\#\{k|d(T^k(x), T^{\sigma(k)}(y)) \geq \delta, k = 1, 2, \ldots, n_r\}}{n_r}
\]

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\[
\left| \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y)) \right| \leq \frac{1}{n_r} \sum_{k=1}^{n_r} |f(T^k(x)) - f(T^\sigma(k)(y))| \\
\leq \varepsilon + 2M \times \frac{\#\{k | d(T^k(x), T^\sigma(k)(y)) \geq \delta, k = 1, 2, \ldots, n_r\}}{n_r} \\
\leq \varepsilon + 2M \times \frac{1}{\delta} \times \frac{1}{n_r} \sum_{k=1}^{n_r} d(T^k(x), T^\sigma(k)(y))
\]

so
\[
\lim_{r \to +\infty} \left| \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y)) \right| \leq \varepsilon
\]

let \( \varepsilon \to 0 \), we get
\[
\lim_{r \to +\infty} \left| \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) - \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y)) \right| = 0
\]
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(x)) = \lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y))
\]
hence, for \( \forall f \in C(X) \),
\[
\lim_{r \to +\infty} \frac{1}{n_r} \sum_{k=1}^{n_r} f(T^k(y)) = \int_X f \, d\mu
\]
which implies \( \mu \in M_y \), thus, \( M_x \cap M_y \neq \emptyset \).

We can obtain the following theorem by Theorem 3.5 and Theorem 3.8.

**Theorem 3.9.** \( N(\mathcal{F}) = X \times X \) iff \( (X, T) \) is uniquely ergodic.

**Proof.** If \((X, T)\) is uniquely ergodic, by theorem 3.5, we conclude \( N(F) = X \times X \), since \( N(F) \subset N(\mathcal{F}) \), it is easy to obtain \( N(\mathcal{F}) = X \times X \).

Let \( \mu_1 \) and \( \mu_2 \) are ergodic measures of \((X, T)\), then \( \exists x, y \in X, s.t. \ M_x = \{\mu_1\}, M_y = \{\mu_2\} \). With the assumption, we obtain \( F(x, y) = 0 \), by Theorem 3.8, we can conclude \( M_x \cap M_y \neq \emptyset \), which implies \( \mu_1 = \mu_2 \), therefore, \((X, T)\) is uniquely ergodic.

By theorem 3.5 and theorem 3.9, we have the following corollary.

**Corollary 3.10.** \( N(\mathcal{F}) = X \times X \) iff \( N(F) = X \times X \).

4. The existence of \( F(x, y) \)

In this section, we study the functions \( \overline{F}(x, y) \) and \( F(x, y) \). We show \( \overline{F}(x, y) \) and \( F(x, y) \) satisfy triangle inequality, and obtain a sufficient condition of the existence of \( F(x, y) \).

**Proposition 4.1.** For \( \forall x, y, z \in X \), we have
\[
\overline{F}(x, y) \leq \overline{F}(x, z) + \overline{F}(y, z)
\]
if \( F(x, y) \), \( F(x, z) \) and \( F(y, z) \) exist, we also have
\[
F(x, y) \leq F(x, z) + F(y, z)
\]
Proof. \( \forall x, y, z \in X, \forall n \in \mathbb{N}^+ \), there are \( \sigma_1, \sigma_2 \in S_n \) such that

\[
\inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(z)) = \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma_1(k)}(z))
\]

\[
\inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma(k)}(y)) = \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma_2(k)}(y))
\]

Let \( \sigma_3 = \sigma_2 \sigma_1 \), we have

\[
\frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma_3(k)}(y)) \leq \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma_1(k)}(z)) + \frac{1}{n} \sum_{k=1}^{n} d(T^{\sigma_1(k)}(z), T^{\sigma_3(k)}(y))
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma_1(k)}(z)) + \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma_2(k)}(y))
\]

\[
= \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(z)) + \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma(k)}(y))
\]

and then we can conclude

\[
\inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y)) \leq \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(z)) + \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma(k)}(y))
\]

\[
\lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y))
\]

\[
\leq \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(z)) + \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(z), T^{\sigma(k)}(y))
\]

this shows

\[
\overline{F}(x, y) \leq \overline{F}(x, z) + \overline{F}(y, z)
\]

\( \square \)

**Theorem 4.1.** If \( x, y \) are generic points, then \( F(x, y) \) exists.

**Proof.** Let

\[
\alpha = \overline{F}(x, y) - \overline{F}(x, y)
\]

we assume \( F(x, y) \) does not exist, then \( \alpha > 0 \).

Denote \( \mu_x \) be the invariant measure generated by \( x \) and \( \mu_y \) be the invariant measure generated by \( y \). Let \( \varepsilon = \frac{\alpha}{\sum_{r=1}^{r_0} \mu_x(\Lambda_r)} \) where \( M = \text{diam}(X) \), then there exist mutually disjoint open sets \( \{\Lambda_r\}_{r=1}^{r_0} \) and mutually disjoint open sets \( \{V_s\}_{s=1}^{s_0} \) such that

\[
\text{diam}(\Lambda_r) < \varepsilon, \forall r = 1, 2, \cdots, r_0
\]

\[
\mu_x(\Lambda_r) > 0, \forall r = 1, 2, \cdots, r_0
\]

\[
\sum_{r=1}^{r_0} \mu_x(\Lambda_r) > 1 - \varepsilon
\]

\[
\text{diam}(V_s) < \varepsilon, \forall s = 1, 2, \cdots, s_0
\]

\[
\mu_y(V_s) > 0, \forall s = 1, 2, \cdots, s_0
\]

\[
\sum_{s=1}^{s_0} \mu_y(V_s) > 1 - \varepsilon
\]
Let \( \{x_r\}_{r=1}^{r_0} \) and \( \{y_s\}_{s=1}^{s_0} \) are sequences such that
\[
x_r \in \Lambda_r, \forall r = 1, 2, \cdots, r_0 \\
y_s \in V_s, \forall s = 1, 2, \cdots, s_0
\]
we denote \( a_1 = \min\{\mu_x(\Lambda_r), r = 1, 2, \cdots, r_0\} \), \( a_2 = \min\{\mu_y(V_s), s = 1, 2, \cdots, s_0\} \), \( a = \min\{a_1, a_2, \frac{\varepsilon}{r_0 + s_0}\} \), then for \( \forall r = 1, 2, \cdots, r_0 \), and \( \forall s = 1, 2, \cdots, s_0 \), there are \( n_r, m_s \in \mathbb{N}^+ \) such that
\[
an_r \leq \mu_x(\Lambda_r) < a(n_r + 1) \\
am_s \leq \mu_y(V_s) < a(m_s + 1)
\]
Let \( K = \min\{\sum_{i=1}^{r_0} n_r, \sum_{s=1}^{s_0} m_s\} \), we creat the sequences \( \{\overline{x}_i\}_{i=1}^{K} \) and \( \{\overline{y}_i\}_{i=1}^{K} \) as follow,
\[
\overline{x}_i = \begin{cases} x_1, & i \leq n_1 \\ x_{r+1}, & \sum_{j=1}^{r} n_j < i \leq \sum_{j=1}^{r+1} n_j \end{cases} \\
\overline{y}_i = \begin{cases} y_1, & i \leq m_1 \\ y_{s+1}, & \sum_{j=1}^{s} m_j < i \leq \sum_{j=1}^{s+1} m_j \end{cases}
\]
then for \( \forall \beta > 0, \exists N_1 > 0 \) such that for \( \forall n > N_1 \), we can get
\[
\inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y)) \leq \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} (a - \beta)(d(\overline{x}_i, \overline{y}_{\sigma(i)}) + 2\varepsilon) + M(1 - K(a - \beta)) \\
= (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) + 2K\varepsilon(a - \beta) + M(1 - Ka) + MK\beta \\
\leq (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) + 2\varepsilon + M\varepsilon + MK\beta
\]
\[
\inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y)) \geq \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} (a - \beta)(d(\overline{x}_i, \overline{y}_{\sigma(i)}) - 2\varepsilon) - M(1 - K(a - \beta)) \\
= (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) - 2K\varepsilon(a - \beta) - M(1 - Ka) - MK\beta \\
\geq (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) - 2\varepsilon - M\varepsilon - MK\beta
\]
thus,
\[
\overline{F}(x, y) \leq (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) + 2\varepsilon + M\varepsilon + MK\beta \\
\overline{F}(x, y) \geq (a - \beta) \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) - 2\varepsilon - M\varepsilon - MK\beta
\]
let \( \beta \to 0 \), we have
\[
\overline{F}(x, y) \leq a \inf_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d(\overline{x}_i, \overline{y}_{\sigma(i)}) + 2\varepsilon + M\varepsilon
\]
\[
\bar{F}(x, y) \geq a \inf_{\sigma \in S_K} \sum_{i=1}^{K} d(\bar{\tau}_i, \bar{y}_{\sigma(i)}) - 2\varepsilon - M\varepsilon
\]

and then we can obtain
\[
\bar{F}(x, y) - \bar{F}(x, y) \leq 4\varepsilon + 2M\varepsilon = \frac{1}{2} \alpha
\]

this is a contradiction. \qed

5. Weak mean equicontinuity

In this section, we study \( \bar{F} \)-continuity and \( F \)-continuity. Several equivalent conditions of \( \bar{F} \)-continuous systems and \( F \)-continuous systems are shown. As a result, we deduce \( \bar{F} \)-continuity is equivalent to \( F \)-continuity. Moreover, we show almost period points is dense in \( M(T) \) if \( (X, T) \) is weak mean equicontinuous.

**Proposition 5.1.** \((X, T)\) is \( \bar{F} \)-continuous iff \( \forall \varepsilon > 0 \), there is a \( \delta > 0 \) such that whenever \( x, y \in X \) with \( d(x, y) < \delta \), we have \( \bar{F}(x, y) < \varepsilon \).

**Proposition 5.2.** \((X, T)\) is \( F \)-continuous iff \( \forall \varepsilon > 0 \), there is a \( \delta > 0 \) such that whenever \( x, y \in X \) with \( d(x, y) < \delta \), we have \( F(x, y) < \varepsilon \).

**Theorem 5.1.** If \((X, T)\) is \( \bar{F} \)-continuous, then for \( \forall f \in C(X) \), \( f^*(x) \) is well defined.

**Proof.** \( \forall x \in X \), we define
\[
N(\bar{F}, x) = \{ y \in X | \bar{F}(x, y) = 0 \}
\]
\( \forall y \in N(\bar{F}, x) \), there are \( \{y_n\}_{n=1}^{\infty} \subset N(\bar{F}, x) \) such that \( \lim_{n \to \infty} y_n = y \). for \( \forall y_n \), we have
\[
\bar{F}(x, y) \leq \bar{F}(x, y_n) + \bar{F}(y_n, y) = \bar{F}(y_n, y)
\]

this implies that \( \bar{F}(x, y) = 0 \), thus, \( N(\bar{F}, x) \) is closed. By **Theorem 3.5**, we conclude \( (N(\bar{F}, x), T) \) is uniquely ergodic, so all the points in \( N(\bar{F}, x) \) are generic, in particular \( x \) is a generic point. \( \square \)

**Corollary 5.2.** If \((X, T)\) is \( F \)-continuous, then for \( \forall f \in C(X) \), \( f^*(x) \) is well defined.

**Corollary 5.3.** If \( M(T) \subset C(\bar{F}) \), then for \( \forall f \in C(X) \), \( f^*(x) \) is well defined.

**Proof.** \( M(T), T \) is \( \bar{F} \)-continuous, then by **Theorem 5.1**, all the points in \( M(T) \) are generic. \( \forall x \in X \), there is \( y \in \omega(x) \cap M(T) \), it’s easy to show \( F(x, y) = 0 \), by **Theorem 3.3**, we can deduce \( x \) is a generic point. \( \square \)

By **Theorem 4.1** and **Theorem 5.1**, we have the following theorem.

**Theorem 5.4.** \((X, T)\) is \( \bar{F} \)-continuous iff \((X, T)\) is \( F \)-continuous.

**Proof.** If \((X, T)\) is \( F \)-continuous, it is easy to obtain \((X, T)\) is \( \bar{F} \)-continuous. We need only to prove \( \bar{F} \)-continuity can imply \( F \)-continuity. By **Theorem 5.1**, we obtain all the points in \( X \) are generic points if \((X, T)\) is \( \bar{F} \)-continuous, then by **Theorem 4.1**, we have \( F(x, y) \) exists for \( \forall x, y \in X \), so \((X, T)\) is \( F \)-continuous. \( \square \)

**Theorem 5.5.** \((X, T)\) is weak mean equicontinuous iff \( \forall f \in C(X) \), \( f^*(x) \) is well defined and continuous.
Proof. Part 1: We shall prove if \((X, T)\) is weak mean equicontinuous, then for \(\forall f \in C(X), f^*(x)\) is well defined and continuous.

By Theorem 5.1, we know for \(\forall f \in C(X), f^*\) is well defined. Let \(M = \max\{\|f(x)\|, \text{then for } \forall \nu > 0, \text{there is an } \varepsilon > 0 \text{ such that whenever } d(x, y) < \varepsilon, |f(x) - f(y)| < \frac{\varepsilon}{2M+1}, \text{and there exists a } \delta > 0 \text{ such that whenever } d(x, y) < \delta, F(x, y) < \frac{\varepsilon\nu}{2(2M+1)}, \forall \sigma \in S_n, \text{we have}\)

\[
\frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y)) \geq \varepsilon \times \frac{\#\{k \mid d(T^k(x), T^{\sigma(k)}(y)) \geq \varepsilon, k = 1, 2, \ldots, n\}}{n}
\]

\[
\left| \frac{1}{n} \sum_{k=1}^{n} f(T^k(x)) - \frac{1}{n} \sum_{k=1}^{n} f(T^{\sigma(k)}(y)) \right| \leq \frac{1}{n} \sum_{k=1}^{n} |f(T^k(x)) - f(T^{\sigma(k)}(y))| \leq \frac{\nu}{2(2M+1)} + 2M \times \frac{\#\{k \mid d(T^k(x), T^{\sigma(k)}(y)) \geq \varepsilon, k = 1, 2, \ldots, n\}}{n}
\]

then for \(\forall d(x, y) < \delta, \) we conclude

\[
\lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^{\sigma(k)}(y))) \geq \lim_{n \to +\infty} \inf_{\sigma \in S_n} \frac{\#\{k \mid d(T^k(x), T^{\sigma(k)}(y)) \geq \varepsilon, k = 1, 2, \ldots, n\}}{n} \times \varepsilon
\]

\[
\lim_{n \to +\infty} \left| \frac{1}{n} \sum_{k=1}^{n} f(T^k(x)) - \frac{1}{n} \sum_{k=1}^{n} f(T^{\sigma(k)}(y)) \right| \leq \frac{\nu}{2(2M+1)} + 2M \times \frac{\nu}{2(2M+1)} = \frac{\nu}{2}
\]

thus, \(|f^*(x) - f^*(y)| < \nu, \) which implies \(f^*(x) \in C(X).\)

Part 2: We shall prove if for \(\forall f \in C(X), f^*(x)\) is well defined and continuous, then \((X, T)\) is weak mean equicontinuous.

If \((X, T)\) is not weak mean equicontinuous, then \(\exists x \in X, \exists \varepsilon > 0 \) and \(\exists \{x_m\}_{m=1}^{\infty} \subseteq X\) such that \(\lim_{m \to +\infty} x_m = x\) but \(F(x, x_m) \geq \varepsilon.\) Let \(\mu = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \delta(T^k(x)), \) \(N = diam(X), \eta = \frac{\varepsilon}{2(N+1)}, \) then there exist mutually disjoint closed sets \(\{\Lambda_s\}_{s=1}^{s_0}\) such that

\[
diam(\Lambda_s) < \eta, \forall s = 1, 2, \cdots, s_0
\]

\[
\mu(\bigcup_{s=1}^{s_0} \Lambda_s) > 1 - \eta
\]

We take \(\delta = \min \{d(\Lambda_{s_1}, \Lambda_{s_2})\}, \) \(r = \min\{\frac{\delta}{4}, \eta\} \) and \(\alpha = \frac{\varepsilon}{4s_0} \). For \(\forall s = 1, 2, \cdots, s_0, \) we define \(U_s = O(\Lambda_s, r), V_s = O(\Lambda_s, 2r), \) then for \(\forall x_m \in X, \exists s_m \in \{1, 2, \cdots, s_0\}, \)

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{U_{s_m}}(T^k(x)) > \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{V_{s_m}}(T^k(x_m)) + \alpha
\]

if not, for \(\forall s \in 1, 2, \cdots, s_0, \)

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{U_s}(T^k(x)) \leq \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{V_s}(T^k(x_m)) + \alpha
\]
we conclude

Theorem 5.8. If

(13) Lemma 5.7.

A

Thus

this is a contradiction.

In general, we can only obtain $\overline{A(T)} \subset M(T)$, but if $(X, T)$ is weak mean equicontinuous, then $\overline{A(T)} = M(T)$.

Lemma 5.6. ([13]) $(X, T)$ is a compact topological dynamic system, $x \in A(T)$ iff $x \in \omega(x)$ and $(\omega(x), T)$ is minimal.

Lemma 5.7. ([13]) If $(X, T)$ is strictly ergodic, then $(X, T)$ is minimal.

Theorem 5.8. If $(X, T)$ is weak mean equicontinuous, then $A(T)$ is dense in $M(T)$. 

1
Proof. \( \forall x \in X, (C_x, T) \) is strictly ergodic, by Lemma 5.7, we obtain \( (C_x, T) \) is minimal, then by Lemma 5.6, we conclude \( C_x \subset A(T) \), this implies \( M(T) \subset A(T) \). On the other hand, we know \( A(T) \subset M(T) \) from Proposition 2.5, thus \( A(T) = M(T) \). \( \square \)

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