Long-time tails in the random transverse Ising chain

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Abstract

Taking one-dimensional random transverse Ising model (RTIM) with the double-Gaussian disorder for example, we investigated the spin autocorrelation function (SAF) and associated spectral density at high temperature by the recursion method. Based on the first twelve recurrants obtained analytically, we have found strong numerical evidence for the long-time tail in the SAF of a single spin. Numerical results indicate that when the standard deviation $\sigma_{JS}$ (or $\sigma_{BS}$) of the exchange couplings $J_i$ (or the random transverse fields $B_i$) is small, no long-time tail appears in the SAF. The spin system undergoes a crossover from a central-peak behavior to a collective-mode behavior, which is the dynamical characteristics of RTIM with the bimodal disorder. However, when the standard deviation is large enough, the system exhibits similar dynamics behaviors to those of the RTIM with the Gaussian disorder, i.e., the system exhibits an enhanced central-peak behavior for large $\sigma_{JS}$ or a disordered behavior for large $\sigma_{BS}$. In this instance, the long-time tails in the SAFs appear, i.e., $C(t) \sim t^{-2}$. Similar properties are obtained when the random variables ($J_i$ or $B_i$) satisfy other distributions such as the double-exponential distribution and the double-uniform distribution.

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The time-dependent behavior of quantum spin systems has been the subject of theoretical studies for quite some time. However, most literature deals with the dynamic correlation functions of the pure quantum spin systems. It is found that the spin autocorrelation functions (SAFs) of these pure systems are associated with the temperature. The transverse correlation functions show a power-law behavior at \( T = 0 \), an exponential behavior at \( 0 < T < \infty \), and a Gaussian behavior at \( T = \infty \), respectively. As to the longitudinal correlation functions, Niemeijer found that SAF of the XY spin chain is a damped oscillation function which may be expressed by the square of the Bessel function at \( T = \infty \), but it has a more complex expression for other temperatures. For the spin-1/2 van der Waals model, Dekeyser and Lee pointed out that the power-law long-time tails are found in the transverse component only at high temperature. Only recently, since the disorder effects besides the effects of the temperature has been shown to affect the dynamics behavior of spin systems in a drastic way providing a very rich area of investigation, more and more attention has been paid on the random quantum spin systems, especially on the low-dimensional systems, such as the random transverse Ising model (RTIM), XY chain, XX and XXZ chain, etc. The known results indicate that the disorder effects may drastically slow down the behavior of the disordered averaged spin-spin correlation function, including the SAFs. For example, as the exchange couplings vary, the Gaussian behavior of \( xx \)-relaxation of the single impurity in XY chain slows down to the stretched exponential-like relaxation at \( T = \infty \). However, as far as we know, although the short-time behaviors of the SAFs have been reported, the long-time behaviors of them are still not understood. An important issue in this work is to explore whether it is possible to realize a situation in which the exchange couplings and the transverse fields are independent random variables, one can obtain the long-time behavior of SAF of the random transverse Ising chain.

One-dimensional (1-D) RTIM is one of simplest but nontrivial example of the random quantum spin chains. It is defined by the following Hamiltonian:

\[
H = -1/2 \sum_i J_i \sigma_i^x \sigma_{i+1}^x - 1/2 \sum_i B_i \sigma_i^z, \tag{1}
\]

where 1/2 is the constant number introduced for the convenience of the derivation. \( \sigma_i^x \), \( \sigma_i^z \) denote Pauli matrices at site \( i \). The exchange couplings \( J_i \) (or the transverse fields \( B_i \)) are independent random variables obeying the distribution \( \rho(J_i) \) (or \( \rho(B_i) \)). This model has a
well-known physical interpretation in connection not only with some quasi-one-dimensional hydrogen-bonded ferroelectric crystals like Cs(H$_{1-x}$D$_x$)$_2$PO$_4$, PbH$_{1-x}$D$_x$PO$_4$ etc.\cite{20, 21}, but also with Ising spin glasses LiHo$_{0.167}$Y$_{0.833}$F$_4$\cite{22}. Besides, this model may capture vital features of the recent neutron scattering experiment in LiHoF$_4$\cite{14, 23}.

In this rapid communication, we investigate the dynamics of RTIM with the double-Gaussian disorder, i.e., $J_i$ or $B_i$ respectively satisfy the double-Gaussian distribution

$$\rho(\beta_i) = p\rho_1(\beta_i) + (1 - p)\rho_2(\beta_i),$$  \hspace{1cm} (2)

where

$$\rho_1(\beta_i) = \frac{1}{\sqrt{2\pi\sigma_{\beta}}} \exp \left[-\frac{(\beta_i - \beta_{1m})^2}{2\sigma_{\beta}^2}\right]$$ \hspace{1cm} (3)

and

$$\rho_2(\beta_i) = \frac{1}{\sqrt{2\pi\sigma_{\beta}}} \exp \left[-\frac{(\beta_i - \beta_{2m})^2}{2\sigma_{\beta}^2}\right]$$ \hspace{1cm} (4)

are the standard Gaussian distributions, wherein $\beta_{1m}$ ($J_{1m}$ or $B_{1m}$) and $\beta_{2m}$ ($J_{2m}$ or $B_{2m}$) denote the corresponding mean values of the random variables $\beta_i$ ($J_i$ or $B_i$), $\sigma_{\beta}$ ($\sigma_J$ or $\sigma_B$) is the standard deviation. The physical sense of Eq. (2) is that the random variable $\beta_i$ satisfies $\rho_1(\beta_i)$ with probability $p$ and $\rho_2(\beta_i)$ with probability $(1 - p)$, respectively. The mean value and the standard deviation of the random variables $\beta_i$ obeying distribution $\rho(\beta_i)$ are respectively $p\beta_{1m} + (1 - p)\beta_{2m}$ and $\sigma_{\beta S} = \sqrt{p(1 - p)(\beta_{1m} - \beta_{2m})^2 + \sigma_{\beta}^2}$. Since $\sigma_{\beta S}$ is an increasing function of $\sigma_{\beta}$, for convenience, $\sigma_{\beta}$ is used to denote $\sigma_{\beta S}$ in the following unless specified otherwise.

Obviously, the cases discussed in references \cite{10} and \cite{11} are two special cases of the RTIM with the double-Gaussian disorder. When $\sigma_{\beta} \to 0$, Eq. (2) becomes the bimodal distribution

$$\rho(\beta_i) = p \delta(\beta_i - \beta_{1m}) + (1 - p) \delta(\beta_i - \beta_{2m}).$$ \hspace{1cm} (5)

When $p = 1$ (or $p = 0$), Eq. (2) becomes a standard Gaussian distribution. Of course, $\rho_1(\beta_i)$ and $\rho_2(\beta_i)$ may be other distributions, such as an exponential distribution or a uniform distribution. In this way, $\rho(\beta_i)$ becomes the double-exponential distribution or the double-uniform distribution.

The basic tool employing in this work is the recursion method\cite{24}, which is very powerful in the study on many-body dynamics\cite{3, 10, 11, 15, 17, 25, 29}. It may help us obtain the
SAF and associated spectral density. They are respectively defined as

\[ C(t) = \langle \sigma_j^x(t) \sigma_j^x(0) \rangle, \tag{6} \]

and

\[ \Phi(\omega) = \int_0^{+\infty} dt C(t) e^{-i\omega t}, \tag{7} \]

where \( \langle \cdots \rangle \) denotes an average over the random variables is performed after the statistical average. At \( T \rightarrow \infty \), the SAF may be expanded into a power series

\[ C(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mu_{2k} t^{2k}, \tag{8} \]

where \( \mu_{2k} \) is the \( 2k \)th moment of the SAF. And the spectral density may be determined via the relation

\[ \Phi(\omega) = \lim_{\varepsilon \to 0} Re a_0(z) \big|_{z=\varepsilon + i\omega}, \tag{9} \]

where

\[ a_0(z) = 1/(z + \Delta_1/\{z + \Delta_2/\{z + \Delta_3/ (z + \ldots) \}\}), \tag{10} \]

which is the Laplace transform of the SAF \( (\mathbf{1}) \). The continued-fraction coefficients (recurrants) \( \Delta_\nu (\nu = 1, 2, \cdots) \) of the SAF may be derived by the recurrence relation I given in reference \( [24] \).

Although only a finite number of recurrants can be deduced analytically, they play vital roles in the whole study process. On the one hand, since the first \( 2k \) moments may be expressed by the first \( k \) recurrants \( [24] \), SAF can be determined by the Padé approximant constructed out of the known moments. On the other hand, one can determine the spectral density of the SAF by employing the known recurrants and so-called Gaussian terminator \( [10, 24, 28, 30] \). Due to the first 12 recurrants have been deduced analytically, we extract more relevant information from them in this paper than we did in our previous work \( [11] \) in which only the first 9 recurrants have been obtained analytically.

In this work we analyze two types of RTIM: a random-bonds model, and a random-fields model. In the former, the exchange couplings \( J_i \) are chosen independently from the double-Gaussian distributions defined by Eq. \( (2) \), while the transverse fields \( B_i \) remain unaltered. To compare our results with the previous results presented in reference \( [10] \), we set the same parameters as those in it: \( J_{1m} = 1.0, J_{2m} = 0.4 \) and \( B_i = B = 1 \) which fixes the energy scale, and \( \sigma_J \) varies from 0.01 to 2. \( C(t) \) and \( \Phi(\omega) \) are plotted in Fig. 1 and Fig. 2, respectively.
Each inset in Fig. 2 gives the recurrants $\Delta_\nu (\nu = 1, \cdots, 12)$ for the same parameters in each figure, and the lines in it are just a guide to the eye.

As $\sigma_J \to 0$, the double-Gaussian distribution turns into the bimodal distribution. Taking $\sigma_J = 0.01$ for example, as shown in Figs. 1 (a) and 2 (a), the dynamics of the system undergoes a crossover from a central-peak behavior to a collective-mode behavior as $p$ decreases from 1 to 0. Although we cannot obtain $C(t)$ for large time, comparison shows that Fig.1 (a) offers a better improvement in SAFs for short time than those given by Florencio, et al. [10]. Comparing all the figures in Fig. 1 and Fig. 2, one finds that as $\sigma_J$ increases from 0.01 to 2, the damped oscillations of the SAFs diminish in magnitude, meanwhile the recurrants for the same parameters tend to fall into a straight line. In this process, the spectral weights at $\omega = 0$ are enhanced, and the peaks’ locations of the spectral lines move towards $\omega = 0$ gradually (e.g., line $p = 0$). The solid lines with $p = 1$ in Fig. 2 illustrate how an almost pure Gaussian profile of $\Phi(\omega)$ evolves into a curve with some additional structure which consists of a central peak of increasing height and decreasing width. Above analysis indicates that the collective-mode behavior becomes weaker and weaker, while the dynamics becomes increasingly dominated by a central-peak behavior as $\sigma_J$ increases. When $\sigma_J$ are large enough (see Figs. 1(d) and 2(d)), the crossover vanishes, and the system exhibits an enhanced central-peak behavior. These results are in agreement with those in our previous work[11].

The infrared (i.e., low-frequency) singularity in the spectral density signals the SAF should manifest itself a characteristic quantum-diffusion long-time tail[31]. We find, as expected, long-time tails exist in the SAFs when $\sigma_J$ are large enough. Fig. 3 gives the log-log plot of the absolute values of $C(t)$ versus time. The solid lines with $p = 1$ in Figs. 1 and 3 show how a Gaussian behavior of $C(t)$, due to the disorder effects, slows down to a power-law relaxation. Calculations indicate that $C(t) \sim t^{-2}$ for $t > 20$. This result is identical with the pure XY interaction case ($R = 0$) of the spin-1/2 van der Waals model[7]. In fact, the XY model and the transverse Ising model are dynamically equivalent to the spin van der Waals model[4]. These systems are indicated by the same geometry of their respective dynamics Hilbert spaces. Finally, we should like to emphasize that success in obtaining the long-time tails of the SAFs is owing to the fact that the true long-time behavior is only nebulously encoded in the first few continued-fraction coefficients[31].

For the random-fields model, the random transverse fields $B_i$ are independent and satisfy
the double-Gaussian distributions given by Eq. (2), while the exchange couplings are uniform (i.e., \( J_i = J = 1 \)). For convenience to compare our results with the known results\[10\], we set \( B_{1m} = 0.6, B_{2m} = 1.4 \), and \( \sigma_B \) varies from 0.01 to 2. The SAFs and the corresponding spectral densities are shown in Fig. 4 and Fig. 5, respectively. The inserts in them respectively give the double-logarithmic plots of the absolute values of \( C(t) \) versus time and the associated recurrants \( \Delta_\nu (\nu = 1, \cdots, 12) \) for the same parameters in each figure.

When \( \sigma_B \) is small (e.g., \( \sigma_B \leq 0.10 \)), the system undergoes a crossover from a central-peak behavior to a collective-mode behavior as \( p \) decreases from 1 to 0, as shown in Figs. 4 (a) and 5 (a). This result is consistent with that obtained by Florencio, et al.\[10\]. However, as \( \sigma_B \) increases, this crossover disappears gradually, see Fig. 5 (b), (c) and (d). Meanwhile, the SAFs exhibit an interesting phenomenon. From Fig. 4 (b) we find that the decays of long-time tails in lines \( p = 0 \) and \( p = 1 \) are proportional to \(-t^{-2}\), and those in other lines are proportional to \( t^{-2} \). When \( \sigma_B \) is large enough (e.g., \( \sigma_B = 1.00 \)), SAFs fall into three successive phases. In the initial phase \((0 < t < 1)\), the decay of \( C(t) \) is Gaussian regime. In the third phase \((t > 20)\), the long-time tails of SAFs obey the power-law decay \( t^{-2} \). There is a cross-over time between the Gaussian regime and the power-law regime. In this cross-over, the oscillation of \( C(t) \) implies that the \( x \) component of the spin may be coherently coupled to some mode with small fluctuation\[7\]. Thus, slow decay may emerge. The strong peaks at \( \omega = 0 \) also signal the existence of the long-time tails in the SAFs. The cross-over time becomes shorter as \( \sigma_B \) is increased further. While more and more high-frequency modes are involved in the corresponding spectral densities. The system is in a disordered state when \( \sigma_B \) is large enough.

Above we have investigated the dynamics of the RTIM with the double-Gaussian disorder at high temperature by the recursion method. This generalized model may recover the dynamics of both the RTIM with the bimodal disorder and that with the Gaussian disorder. One may wonder if these spin dynamics features are universal or not. Therefore, we have studied the dynamics of the RTIM with other types of disorder, in which the variables independently satisfy the double-Exponential distribution, and the double-Uniform distribution. Numerical results indicate that these systems show a similar dynamical behavior to those of the RTIM with the double-Gaussian disorder. Generally speaking, when \( \sigma_{JS} \) (or \( \sigma_{BS} \)) is small, the dynamics of the system undergoes a crossover from a central-peak behavior to a collective-mode one as \( p \) (or the mean value of \( J_i \) or \( B_i \)) varies. No long-time tail in SAF
is obtained. As $\sigma_{JS}$ (or $\sigma_{BS}$) increases, the crossover vanishes gradually, and the power-law decay appears in the SAFs when the standard deviation is large enough. For the case of large $\sigma_{JS}$ the system exhibits a central-peak behavior, but it shows a disordered behavior for the case of large $\sigma_{BS}$. As expected, we found that, independently of the type of disorder, the SAFs in 1-D RTIM decay as $t^{-2}$ for large time due to quantum spin diffusion.

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Figure Captions

Fig. 1. (Color online) Spin autocorrelation functions \( C(t) \) of RTIM with the double-Gaussian random bonds. Better improvement in SAFs is made than those given in references [10] and [11].

Fig. 2. (Color online) The associated spectral densities for the same parameters as in Fig. 1. The insets show the recurrants \( \Delta_\nu (\nu = 1, \cdots, 12) \). The lines in the insets are just a guide to the eye. For small \( \sigma_J \) (e.g., \( \sigma_J \leq 0.01 \)), (a) recovers \( \Phi(\omega) \) of RTIM with the bimodal distribution. As \( \sigma_J \) increases, the crossover from a central-peak behavior to a collective-mode behavior vanishes gradually. The system only exhibits an enhanced central-peak behavior when \( \sigma_J \) is large enough.

Fig. 3. (Color online) Log-log plot of \( |C(t)| \) versus \( t \) for different values of \( p \) and \( \sigma_J \). Black dashed line in each figure represents a power-law decay \( t^{-2} \).

Fig. 4. (Color online) SAFs \( C(t) \) of RTIM with the double-Gaussian random fields. When \( \sigma_B \) is small (e.g., \( \sigma_B = 0.01 \)), (a) approximately recovers the results presented in Ref. [10]. The inserts present log-log plot of \( |C(t)| \) versus \( t \). The black dashed line in each insert represents a power-law decay \( t^{-2} \).

Fig. 5. (Color online) The corresponding spectral densities for the same parameters as in Fig. 4. For small \( \sigma_B \) (e.g., \( \sigma_B = 0.01 \)), the system moves from a collective mode dynamics to a central peak type of dynamics as \( p \) varies from 0 to 1. However, the system is in a disordered state when \( \sigma_B \) is large enough (e.g., \( \sigma_B = 2.00 \)).
