Actions of groups of homeomorphisms on one-manifolds

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Abstract

In this article, we describe all the group morphisms from the group of compactly-supported homeomorphisms isotopic to the identity of a manifold to the group of homeomorphisms of the real line or of the circle.

MSC: 37C85.

1 Introduction

Fix a connected manifold \( M \) (without boundary). For an integer \( r \geq 0 \), we denote by \( \text{Diff}^r(M) \) the group of \( C^r \)-diffeomorphisms of \( M \). When \( r = 0 \), this group will also be denoted by \( \text{Homeo}(M) \). For a homeomorphism \( f \) of \( M \), the support of \( f \) is the closure of the set:

\[
\{ x \in M, \ f(x) \neq x \}.
\]

We say that a homeomorphism \( f \) in \( \text{Diff}^r(M) \) with compact support is compactly isotopic to the identity if there exists a \( C^r \) map \( F : M \times [0, 1] \to M \) such that

1. For any \( t \in [0, 1] \), \( F(., t) \) belongs to \( \text{Diff}^r(M) \).
2. There exists a compact subset \( K \subset M \) such that, for any \( t \in [0, 1] \), the support of the diffeomorphism \( F(., t) \) is contained in \( K \).
3. \( F(., 0) = \text{Id}_M \) and \( F(., 1) = f \).

We denote by \( \text{Diff}^r_0(M) \) (\( \text{Homeo}_0(M) \) if \( r = 0 \)) the group of compactly supported \( C^r \)-diffeomorphisms of \( M \) which are compactly isotopic to the identity. The main reason why we are considering these groups is the following difficult theorem by Fisher, Mather and Thurston (see [1], [2], [5], [10], [11]).

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**Theorem** (Fisher, Mather, Thurston). *Let $M$ be a connected manifold. If $r \neq \dim(M) + 1$, the group $\Diff^r_0(M)$ is simple.*

This theorem will be used throughout the article. It implies for instance that any group morphism from a group of the form $\Diff^r_0(M)$ to another group is either one-to-one or trivial: the kernel of such a morphism is a normal subgroup of $\Diff^r_0(M)$ and hence is either trivial or the whole group. As an application of this theorem, let us prove that any group morphism $\Homeo_0(T^2) \to \Homeo(\mathbb{R})$ is trivial. Notice that the group $\Homeo_0(T^2)$ contains finite order elements (the rational translations) whereas the group of homeomorphisms of the real line does not. Hence, a finite order element has to be sent to the identity under such a morphism which is not one-to-one. Therefore, it is trivial.

In [7], Étienne Ghys asked whether the following statement was true: if $M$ and $N$ are two closed manifolds and if there exists a non-trivial morphism $\Diff^\infty_0(M) \to \Diff^\infty_0(N)$, then $\dim(M) \leq \dim(N)$. In [9], Kathryn Mann proved the following theorem. Take a connected manifold $M$ of dimension greater than 1 and a one-dimensional connected manifold $N$. Then any morphism $\Diff^\infty_0(M) \to \Diff^\infty_0(N)$ is trivial: she answers Ghys’s question in the case where the manifold $N$ is one-dimensional. Mann also describes all the group morphisms $\Diff^r_0(M) \to \Diff^r_0(N)$ for $r \geq 3$ when $M$ as well as $N$ are one-dimensional. The techniques involved in the proofs of these theorems are Kopell’s lemma (see [16] Theorem 4.1.1) and Szekeres’s theorem (see [16] Theorem 4.1.11). These theorems are valid only for a regularity at least $C^2$. In this article, we prove similar results in the case of a $C^0$ regularity. The techniques used are different.

**Theorem 1.1.** *Let $M$ be a connected manifold of dimension greater than 2 and let $N$ be a connected one-manifold. Then any group morphism $\Homeo_0(M) \to \Homeo(N)$ is trivial.*

The case where the manifold $M$ is one-dimensional is also well-understood.

Using bounded cohomology techniques, Matsumoto proved the following theorem (see [13] Theorem 5.3) which is also a key point in the proof of our theorems.

**Theorem (Matsumoto).** *Every group morphism $\Homeo_0(S^1) \to \Homeo_0(S^1)$ is a conjugation by a homeomorphism of the circle.*

Notice that any group morphism $\Homeo_0(S^1) \to \Homeo(\mathbb{R})$ is trivial. Recall that, as the group $\Homeo_0(S^1)$ is simple, such a group morphism is either one-to-one or trivial. However, the group $\Homeo_0(S^1)$ contains torsion elements whereas the group $\Homeo(\mathbb{R})$ does not: such a morphism cannot be one-to-one.

It remains to study the case of a morphism defined on $\Homeo_0(\mathbb{R})$.

**Theorem 1.2.** *Let $N$ be a connected one-manifold. For any group morphism $\varphi : \Homeo_0(\mathbb{R}) \to \Homeo(N)$, there exists a closed set $K \subset N$ such that:*
1. The set $K$ is pointwise fixed under any homeomorphism in $\varphi(\text{Homeo}_0(\mathbb{R}))$.

2. For any connected component $I$ of $N - K$, there exists a homeomorphism $h_I : \mathbb{R} \to I$ such that:
   \[ \forall f \in \text{Homeo}_0(\mathbb{R}), \quad \varphi(f)|_I = h_Ifh_I^{-1}. \]

Notice that the set $K$ has to be the set of points which are fixed under every element in $\varphi(\text{Homeo}_0(\mathbb{R}))$.

The following remark will be used repeatedly in the proof of Theorems 1.1 and 1.2. Consider a nontrivial morphism $\varphi$ from a group $G$ to the group $\text{Homeo}_+^+(\mathbb{R})$. Denote by $F$ the closed set of points of the real line which are fixed under every element in $\varphi(G)$. Take a connected component $I$ of $\mathbb{R} - F$. Any homeomorphism in $\varphi(G)$ preserves this connected component $I$. Consider then the morphism $G \to \text{Homeo}_+^+(I)$
\[ g \mapsto \varphi(g)|_I. \]
Notice that the image of this morphism has no global fixed point and that the interval $I$ is homeomorphic to the real line. We have just seen that any morphism $G \to \text{Homeo}_+^+(\mathbb{R})$ induces by restriction a morphism without global fixed point. Hence, to prove that any morphism $G \to \text{Homeo}_+^+(\mathbb{R})$ is trivial, it suffices to prove that any such morphism has a fixed point.

## 2 Proofs of Theorems 1.1 and 1.2

Fix integers $d \geq k \geq 0$. We will call embedded $k$-dimensional ball of $\mathbb{R}^d$ the image of the closed unit ball of $\mathbb{R}^k = \mathbb{R}^k \times \{0\}^{d-k} \subset \mathbb{R}^d$ under a homeomorphism of $\mathbb{R}^d$. Take an embedded $k$-dimensional ball $D \subset \mathbb{R}^d$ (which is a single point if $k = 0$). We denote by $G^d_D$ the group of homeomorphisms of $\mathbb{R}^d - D$ with compact support which are compactly isotopic to the identity. As any homeomorphism in the group $G^d_D$ is equal to the identity near the embedded ball $D$, it can be continuously extended by the identity on the ball $D$. Hence the group $G^d_D$ can be seen as a subgroup of $\text{Homeo}_0(\mathbb{R}^d)$.

Finally, if $G$ denotes a subgroup of $\text{Homeo}(\mathbb{R}^d)$, a point $p \in \mathbb{R}^d$ is said to be fixed under the group $G$ if it is fixed under all the elements of this group. We denote by $\text{Fix}(G)$ the (closed) set of fixed points of $G$.

The theorems will be deduced from the following propositions. The two first propositions will be proved respectively in Sections 3 and 4.

**Proposition 2.1.** Let $d > 0$ and let $\varphi : \text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(\mathbb{R})$ be a group morphism. Suppose that no point of the real line is fixed under the group $\varphi(\text{Homeo}_0(\mathbb{R}^d))$. Then, for any embedded $(d - 1)$-dimensional ball $D \subset \mathbb{R}^d$, the group $\varphi(G^d_D)$ admits at most one fixed point.
Proposition 2.2. Let $d > 0$ and $\varphi : \text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(\mathbb{R})$ be a group morphism. Then, for any point $p$ in $\mathbb{R}^d$, the group $\varphi(G_p^d)$ admits at least one fixed point.

Proposition 2.3. Let $d > 0$. For any group morphism $\psi : \text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(S^1)$, the group $\psi(\text{Homeo}_0(\mathbb{R}^d))$ has a fixed point.

Proof of Proposition 2.3. Recall that the group $\text{Homeo}_0(\mathbb{R}^d)$ is infinite and simple and that the group $\text{Homeo}(S^1)/\text{Homeo}_0(S^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence any morphism $\text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(S^1)/\text{Homeo}_0(S^1)$ is trivial. Therefore, the image of a morphism $\text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(S^1)$ is contained in $\text{Homeo}_0(S^1)$.

The rest of the proof of this proposition uses a result by Ghys. Ghys associates to any morphism from a group $G$ to the group $\text{Homeo}_0(S^1)$ an element of the second bounded cohomology group $H_b^2(G, \mathbb{Z})$ of the discrete group $G$, which we call the bounded Euler class of this action of $G$. This class vanishes if and only if the action has a global fixed point on the circle. For some more background about the bounded cohomology of groups and the bounded Euler class of a group acting on a circle, see Section 6 in [6].

By a theorem by Matsumoto and Morita (see Theorem 3.1 in [14]):

$$H_b^2(\text{Homeo}_0(\mathbb{R}^d), \mathbb{Z}) = \{0\}.$$ 

Therefore, the bounded Euler class of a morphism $\text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}_0(S^1)$ vanishes: this action has a fixed point.

Proof of Theorem 1.1. Let $d = \dim(M)$. The theorem will be deduced from the following lemma.

Lemma 2.4. For any $d > 1$, any group morphism $\text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(\mathbb{R})$ is trivial.

Using Proposition 2.3, we obtain the following immediate corollary.

Corollary 2.5. For any $d > 1$, any group morphism $\text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(S^1)$ is trivial.

Let us see why this lemma and this corollary implies the theorem. Consider a morphism $\text{Homeo}_0(M) \to \text{Homeo}_0(N)$. Take an open set $U \subset M$ homeomorphic to $\mathbb{R}^d$ and let us denote by $\text{Homeo}_0(U)$ the subgroup of $\text{Homeo}_0(M)$ consisting of homeomorphisms supported in $U$. By Lemma 2.4 and Corollary 2.5, the restriction of this morphism to the subgroup $\text{Homeo}_0(U)$ is trivial. Moreover, as the group $\text{Homeo}_0(M)$ is simple, such a group morphism is either one-to-one or trivial: it is necessarily trivial in this case.
Proof of Lemma 2.4. Take a group morphism $\varphi : \text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(\mathbb{R})$. Suppose by contradiction that this morphism is nontrivial. Replacing if necessary $\mathbb{R}$ with a connected component of the complement of the closed set $\text{Fix}(\varphi(\text{Homeo}_0(\mathbb{R}^d)))$, we can suppose that the group $\varphi(\text{Homeo}_0(\mathbb{R}^d))$ has no fixed points.

Claim 2.6. For any points $p_1 \neq p_2$ in $\mathbb{R}^d$:

$$\text{Fix}(\varphi(G^d_{p_1})) \cap \text{Fix}(\varphi(G^d_{p_2})) = \emptyset.$$ 

Proof. The proof of this claim requires the following lemma which will be proved afterwards.

Lemma 2.7. Let $d \geq 1$ and $d \geq k \geq 0$ be integers. Let $D_1$ and $D_2$ be two disjoint embedded $k$-dimensional balls of $\mathbb{R}^d$. Then, for any homeomorphism $f$ in $\text{Homeo}_0(\mathbb{R}^d)$, there exist homeomorphisms $f_1$, $f_3$ in $G^d_{D_1}$ and $f_2$ in $G^d_{D_2}$ such that:

$$f = f_1 f_2 f_3.$$ 

Take two points $p_1$ and $p_2$ in $\mathbb{R}^d$. Suppose by contradiction that $\text{Fix}(\varphi(G^d_{p_1})) \cap \text{Fix}(\varphi(G^d_{p_2})) \neq \emptyset$. By Lemma 2.7 applied to the 0-dimensional balls $\{p_1\}$ and $\{p_2\}$, a point in this set is a fixed point of the group $\varphi(\text{Homeo}_0(\mathbb{R}^d))$, a contradiction. 

By Proposition 2.2, the sets $\text{Fix}(\varphi(G^d_p))$, for $p \in \mathbb{R}^d$ are nonempty. We just saw that they are pairwise disjoint. Recall that, for any embedded $(d-1)$-dimensional ball $D$, the set $\text{Fix}(\varphi(G^d_D))$ contains the union of the sets $\text{Fix}(\varphi(G^d_p))$ over the points $p$ in the closed set $D$. Hence, this set has infinitely many points as $d \geq 2$, a contradiction with Proposition 2.1.

Proof of Lemma 2.7. To prove this lemma, we use the following theorem by Brown and Gluck (see Theorem 7.1 in [3]), which is also a consequence of the annulus theorem by Kirby and Quinn (see [8] and [17]).

Theorem (Brown-Gluck). Let $d \geq 1$ and let $B_1$ and $B_2$ be two $d$-dimensional balls of $\mathbb{R}^d$ such that the ball $B_1$ is contained in the interior of $B_2$. Let $h$ be any homeomorphism in $\text{Homeo}_0(\mathbb{R}^d)$ such that the ball $h(B_1)$ is also contained in the interior of $B_2$. Then there exists a homeomorphism $\tilde{h}$ in $\text{Homeo}_0(\mathbb{R}^d)$ with the following properties.

1. The homeomorphism $\tilde{h}$ is supported in $B_2$.
2. $\tilde{h}|_{B_1} = h|_{B_1}$.

Take a homeomorphism $f$ in $\text{Homeo}_0(\mathbb{R}^d)$.

Claim 2.8. There exists a homeomorphism $f_1 G^d_{D_1}$ such that $f_1^{-1}$ sends the $k$-dimensional embedded ball $f(D_1)$ to a $k$-dimensional embedded ball which lies in the same connected component of $\mathbb{R}^d - D_2$ as the embedded ball $D_1$. 

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Notice that, if \( d \neq 1 \), the set \( \mathbb{R}^d - D_2 \) is connected. In the case \( d \neq 1 \), this lemma amounts to finding a homeomorphism which sends the ball \( f(D_1) \) to a ball which is disjoint from \( D_2 \).

**Proof.** First suppose that \( d = 1 \). If \( \sup(D_1) < \inf(D_2) \), take as \( f_1^{-1} \) any homeomorphism in \( \text{Homeo}_0(\mathbb{R}) \) supported in \((\sup(D_1), +\infty)\) which sends the point \( \sup(h(D_1)) \) to a point \( x < \inf(D_2) \). If \( \sup(D_2) < \inf(D_1) \), take as \( f_1^{-1} \) any homeomorphism in \( \text{Homeo}_0(\mathbb{R}) \) supported in \((-\infty, \inf(D_1))\) which sends the point \( \inf(h(D_1)) \) to a point \( x > \inf(D_2) \).

Now suppose that \( d \geq 2 \). It is not difficult to find a \( d \)-dimensional embedded ball \( B \) which contains the \( k \)-dimensional ball \( D_2 \) and a point \( p \) outside \( f(D_1) \) in its interior: using the definition of an embedded ball, find first a \( d \)-dimensional \( B_0 \) which contains \( D_2 \) in its interior. If this ball is not contained in \( f(D_1) \) take \( B = B_0 \). Otherwise take any point \( p \) which does not belong to \( f(D_1) \) and consider a tubular neighbourhood \( T \) of a path in \( \mathbb{R}^d - D_1 \) which joins the ball \( B_0 \) and the point \( p \) to construct the ball \( B \) out of \( T \) and \( B \).

Changing coordinates if necessary, we can suppose that \( p = 0 \in \mathbb{R}^d \) and that the ball \( B \) is the unit ball. Consider any vector field \( X \) of \( \mathbb{R}^d \) which is supported in \( B \) and which is equal to \( x \mapsto x \) on a ball centered at 0 containing \( D_2 \). Let \( V \) be a neighbourhood of the point 0 which is disjoint from the embedded ball \( f(D_1) \). Denote by \( \varphi_X^t \) the time \( t \) of the flow of the vector field \( X \). Observe that there exists \( T > 0 \) such that \( \varphi_X^t(B - V) \cap D_2 = \emptyset \). Hence \( \varphi_X^t(f(D_1)) \cap D_2 = \emptyset \). It suffices to take \( f_1^{-1} = \varphi_X^T \).

Take a \( d \)-dimensional ball \( B_2 \) with the following properties:

1. It contains \( D_1 \) and \( f_1^{-1}f(D_1) \).
2. It is disjoint from the embedded ball \( D_2 \).

Consider a \( d \)-dimensional ball \( B_1 \) contained in the interior of the embedded ball \( B_2 \) such that \( f_1^{-1}f(B_1) \) is contained in the interior of \( B_2 \). Apply the theorem by Brown and Gluck above with the balls \( B_1, B_2 \) and the homeomorphism \( h = f_1^{-1}f \); there exists a homeomorphism \( f_2 \) in \( G_{D_2}^d \) which is equal to \( f_1^{-1}f \) in a neighbourhood of the \( k \)-dimensional embedded ball \( D_1 \).

Notice that the homeomorphism \( f_2^{-1}f_1^{-1}f \) pointwise fixes a neighbourhood of the embedded ball \( D_1 \). However, its restriction to \( \mathbb{R}^d - D_1 \) might not be compactly isotopic to the identity. Nevertheless, this homeomorphism of \( \mathbb{R}^d - D_1 \) is compactly isotopic to a homeomorphism \( \eta \) whose support is contained in a small neighbourhood of the embedded ball \( D_1 \) and is disjoint from the embedded ball \( D_2 \); in order to see it, conjugate the homeomorphism \( f_2^{-1}f_1^{-1}f \) with the flow at a sufficiently large time of a vector field for which a small neighbourhood of the embedded ball \( D_1 \) is an attractor.

Let us check that the homeomorphism \( \eta|_{\mathbb{R}^d - D_2} \) is compactly isotopic to the
identity. To prove it, it suffices to conjugate this homeomorphism by a continuous family of homeomorphisms $(h_t)_{t \in [0, +\infty)}$ supported in $\mathbb{R}^d - D_2$ such that:

1. $h_0 = Id$.
2. the family of compact sets $(h_t(\text{supp}(\eta)))_{t \geq 0}$ converges to a point for the Hausdorff topology as $t \to +\infty$.

Hence the continuous family of homeomorphisms $h_t \eta h_t^{-1}$ converges to the identity as $t \to +\infty$ (this the well-known Alexander trick).

To finish the proof of the lemma, it suffices to take $f_2 = \hat{f} \eta$ and $f_3 = f_2^{-1} f_1^{-1} f$.

Proof of Theorem 1.2. Let $\varphi : \text{Homeo}(\mathbb{R}) \to \text{Homeo}(N)$ be a nontrivial group morphism. By Proposition 2.3, we can suppose that the manifold $N$ is the real line $\mathbb{R}$. Replacing $\mathbb{R}$ with a connected component of the complement of the closed set $\text{Fix}(\varphi(\text{Homeo}(\mathbb{R})))$ if necessary, we can suppose that the group $\varphi(\text{Homeo}(\mathbb{R}))$ has no fixed point (see the remark at the end of the introduction). Recall that the group $\text{Homeo}(\mathbb{R})$ is simple. Hence any morphism $\text{Homeo}(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ is trivial. Thus, any element of the group $\varphi(\text{Homeo}(\mathbb{R}))$ preserves the orientation of $\mathbb{R}$.

By Propositions 2.1 and 2.2, for any real number $x$, the group $\varphi(G^1_x)$ has a unique fixed point $h(x)$. Take a homeomorphism $f$ in $\text{Homeo}(\mathbb{R})$ which sends a point $x$ in $\mathbb{R}$ to a point $y$ in $\mathbb{R}$. Then $f G^1_x f^{-1} = G^1_y$ and, taking the image under $\varphi$, $\varphi(f) \varphi(G^1_x) \varphi(f)^{-1} = \varphi(G^1_y)$. Hence $\varphi(f)(\text{Fix}(\varphi(G^1_x))) = \text{Fix}(\varphi(G^1_y))$. Therefore, for any homeomorphism $f$ in $\text{Homeo}(\mathbb{R})$, $\varphi(f)h = hf$.

Let us prove that the map $h$ is one-to-one. Suppose by contradiction that there exist real numbers $x \neq y$ such that $h(x) = h(y)$. The point $h(x)$ is fixed under the groups $\varphi(G^1_x)$ and $\varphi(G^1_y)$. However, the groups $G^1_x$ and $G^1_y$ generate the group $\text{Homeo}(\mathbb{R})$ by Lemma 2.7. Therefore, the point $h(x)$ is fixed under the group $\varphi(\text{Homeo}(\mathbb{R}))$, a contradiction.

Now we prove that the map $h$ is either strictly increasing or strictly decreasing. Fix two points $x_0 < y_0$ of the real line. For any two points $x < y$ of the real line, let us consider a homeomorphism $f_{x,y}$ in $\text{Homeo}(\mathbb{R})$ such that $f_{x,y}(x_0) = x$ and $f_{x,y}(y_0) = y$. As $\varphi(f_{x,y})h = hf_{x,y}$, the homeomorphism $\varphi(f_{x,y})$ sends the ordered pair $(h(x_0), h(y_0))$ to the ordered pair $(h(x), h(y))$. As the homeomorphism $\varphi(f_{x,y})$ is strictly increasing:

$$h(x) < h(y) \iff h(x_0) < h(y_0)$$

and

$$h(x) > h(y) \iff h(x_0) > h(y_0).$$

Hence the map $h$ is either strictly increasing or strictly decreasing.

Now, it remains to prove that the map $h$ is onto to complete the proof. Suppose by contradiction that the map $h$ is not onto. Notice that the set $h(\mathbb{R})$ is preserved under the group $\varphi(\text{Homeo}(\mathbb{R}))$. If this set had a lower bound or an upper bound,
then the supremum of this set or the infimum of this set would provide a fixed point for the group $\varphi(\text{Homeo}_0(\mathbb{R}^d))$, a contradiction. This set has neither upper bound nor lower bound. Let $C$ be a connected component of the complement of the set $h(\mathbb{R})$. Replacing if necessary $h$ by $-h$ and the morphism $\varphi$ by its conjugate under $-Id$, we can suppose that the map $h$ is increasing. Let us denote by $x_0$ the supremum of the set of points $x$ such that the real number $h(x)$ is lower than any point in the interval $C$. Then the point $h(x_0)$ is necessarily in the closure of $C$: otherwise, there would exist an interval in the complement of $h(\mathbb{R})$ which strictly contains the interval $C$. Hence the point $h(x_0)$ is either the infimum or the supremum of the interval $C$. As the proof is analogous in these two cases, we suppose from now on that the point $h(x_0)$ is the supremum of the interval $C$.

Choose, for each couple $(z_1, z_2)$ of real numbers, a homeomorphism $g_{z_1, z_2}$ in $\text{Homeo}_0(\mathbb{R})$ which sends the point $z_1$ to the point $z_2$. Then the sets $g_{z_0, x}(C)$, for $x$ in $\mathbb{R}$, are pairwise disjoint: they are pairwise distinct as their suprema are pairwise distinct (the supremum of the set $g_{z_0, x}(C)$ is the point $h(x)$). Moreover, those sets do not contain any point of $h(\mathbb{R})$ and the infima of those sets are accumulated by points in $h(\mathbb{R})$. Hence, these sets are pairwise disjoint. Then the set $C$ has necessarily an empty interior as the topological space $\mathbb{R}$ is second-countable. Therefore $C = \{h(x_0)\}$, which is not possible. \qed

3 Proof of Proposition 2.1

Fix $d > 0$ and a group morphism $\varphi : \text{Homeo}_0(\mathbb{R}^d) \to \text{Homeo}(\mathbb{R})$. We want to prove that, for any $(d-1)$-dimensional embedded ball $D$, the group $\varphi(G^d_D)$ has at most one global fixed point.

The proof of the proposition is similar to the proofs of Lemmas 3.6 and 3.7 in [15]. For an embedded $(d-1)$-dimensional ball $D$, let $F_D = \text{Fix}(\varphi(G^d_D))$. Let us prove that these sets are pairwise homeomorphic. Take two embedded $(d-1)$-dimensional balls $D$ and $D'$ and take a homeomorphism $h$ in $\text{Homeo}_0(\mathbb{R}^d)$ which sends the set $D$ onto $D'$. Observe that $hG^d_Dh^{-1} = H^d_{D'}$ and that $\varphi(h)\varphi(G^d_D)\varphi(h)^{-1} = \varphi(H^d_{D'})$. Therefore: $\varphi(h)(F_D) = F_{D'}$.

In the case where these sets are all empty, there is nothing to prove. We suppose in what follows that they are not empty.

Given two disjoint embedded $(d-1)$-dimensional balls $D$ and $D'$, Lemma 2.7 implies, as in the proof of Lemma 2.4:

$$F_D \cap F_{D'} = \emptyset.$$ 

**Lemma 3.1.** Fix an embedded $(d-1)$-dimensional ball $D_0$ of $\mathbb{R}^d$. For any connected component $C$ of the complement of $F_{D_0}$, there exists an embedded $(d-1)$-dimensional ball $D$ disjoint from $D_0$ such that the set $F_D$ meets $C$. 

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Proof. Let \((a_1, a_2)\) be a connected component of the complement of the set \(F_{D_0}\). It is possible that either \(a_1 = -\infty\) or \(a_2 = +\infty\). Consider a homeomorphism \(e : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d\) such that \(e(B^{d-1} \times \{0\}) = D_0\), where \(B^{d-1}\) denotes the unit closed ball in \(\mathbb{R}^{d-1}\). For any real number \(x\), let \(D_x = e(B^{d-1} \times \{x\})\). Given two real \(x \neq y\), take a homeomorphism \(\eta_{x,y}\) in \(\text{Homeo}_0(\mathbb{R})\) which sends the point \(x\) to the point \(y\). Consider a homeomorphism \(h_{x,y}\) such that the following property is satisfied. The restriction of \(e h_{x,y} e^{-1}\) to \(B^{d-1} \times \mathbb{R}\) is equal to the map:

\[
B^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^{d-1} \times \mathbb{R} \\
(p, z) \mapsto (p, \eta_{x,y}(z))
\]

Notice that, for any real numbers \(x\) and \(y\), \(h_{x,y}(D_x) = D_y\).

Let us prove by contradiction that there exists a real number \(x \neq 0\) such that \(F_{D_x} \cap (a_1, a_2) \neq \emptyset\). Suppose that, for any such embedded ball \(D_x\), \(F_{D_x} \cap (a_1, a_2) = \emptyset\). We claim that the open sets \(\varphi(h_{0,x})(\{(a_1, a_2)\})\) are pairwise disjoint. It is not possible as there would be uncountably many pairwise disjoint open intervals in \(\mathbb{R}\).

Indeed, suppose by contradiction that there exists real numbers \(x \neq y\) such that \(\varphi(h_{0,x})(\{(a_1, a_2)\}) \cap \varphi(h_{0,y})(\{(a_1, a_2)\}) \neq \emptyset\). Notice that the homeomorphisms \(h_{0,x}^{-1} h_{0,y}\) and \(h_{0,y}^{-1} h_{0,x}\) send respectively the set \(D_0\) to sets of the form \(D_z\) and \(D_{z'}\), where \(z, z' \in \mathbb{R}\). Hence, for \(i = 1, 2\), the homeomorphisms \(\varphi(h_{0,x}^{-1} h_{0,y})\) (respectively \(\varphi(h_{0,y}^{-1} h_{0,x})\)) sends the point \(x_i \in F_{D_0}\) to a point in \(F_{D_z}\) (respectively in \(F_{D_{z'}}\)). By hypothesis, these points do not belong to \((a_1, a_2)\). Therefore

\[
\varphi(h_{0,x}^{-1} h_{0,y})(a_1, a_2) = (a_1, a_2)
\]

or

\[
\varphi(h_{0,y}^{-1} h_{0,x})(a_1, a_2) = \varphi(h_{0,y})(a_1, a_2).
\]

But this last equality cannot hold as the real endpoints of the interval on the left-hand side belong to \(F_{D_z}\) and the real endpoints point of the interval on the right-hand side belongs to \(F_{D_{z'}}\). Moreover, we saw that these two closed sets were disjoint, a contradiction. \(\square\)

Lemma 3.2. Each set \(F_D\) contains only one point.

Proof. Suppose that there exists an embedded \((d-1)\)-dimensional ball \(D\) such that the set \(F_D\) contains two points \(p_1 < p_2\). By Lemma 3.1, there exists an embedded \((d-1)\)-dimensional ball \(D'\) disjoint from \(D\) such that the set \(F_{D'}\) has a common point with the open interval \((p_1, p_2)\). Take a real number \(r < p_1\). Then, for any homeomorphisms \(g_1\) in \(G_D\), \(g_2\) in \(G_{D'}\) and \(g_3\) in \(G_{D'}\),

\[
\varphi(g_1) \circ \varphi(g_2) \circ \varphi(g_3)(r) < p_2.
\]

By Lemma 2.7, this implies that the following inclusion holds:

\[
\{\varphi(g)(r), g \in \text{Homeo}_0(\mathbb{R}^d)\} \subset (-\infty, p_2].
\]

9
The supremum of the left-hand set provides a fixed point for the action $\varphi$, a contradiction.

4 Proof of Proposition 2.2

This proof uses the following lemmas. For a subgroup $G$ of $\text{Homeo}_0(\mathbb{R}^d)$, we define the support $\text{Supp}(G)$ of $G$ as the closure of the set:

$$\{x \in \mathbb{R}^d, \exists g \in G, gx \neq x\}.$$ 

Let $\text{Homeo}_Z(\mathbb{R}) = \{f \in \text{Homeo}(\mathbb{R}), \forall x \in \mathbb{R}, f(x + 1) = f(x) + 1\}$.

To prove Proposition 2.2, we need the following lemmas.

**Lemma 4.1.** Let $G$ and $G'$ be subgroups of the group $\text{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms of the real line. Suppose that the following conditions are satisfied:

1. The groups $G$ and $G'$ are isomorphic to the group $\text{Homeo}_Z(\mathbb{R})$.
2. The subgroups $G$ and $G'$ of $\text{Homeo}_+(\mathbb{R})$ commute: $\forall g \in G, g' \in G', gg' = g'g$.

Then $\text{Supp}(G) \subset \text{Fix}(G')$ and $\text{Supp}(G') \subset \text{Fix}(G)$.

**Lemma 4.2.** Let $d > 0$. Take any nonempty open subset $U$ of $\mathbb{R}^d$. Then there exists a subgroup of $\text{Homeo}_0(\mathbb{R}^d)$ isomorphic to $\text{Homeo}_Z(\mathbb{R})$ which is supported in $U$.

Lemma 4.1 will be proved in the next section. We now provide a proof of Lemma 4.2.

**Proof of Lemma 4.2.** Take a closed ball $B$ contained in $U$. Changing coordinates if necessary, we can suppose that $B$ is the unit closed ball in $\mathbb{R}^d$. Take an orientation-preserving homeomorphism $h : \mathbb{R} \rightarrow (-1, 1)$. For any orientation-preserving homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$, we define the homeomorphism $\lambda_h(f) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ in the following way.

1. The homeomorphism $\lambda_h(f)$ is equal to the identity outside the interior of the ball $B$.
2. For any $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \cap \text{int}(B)$:

$$\lambda_h(f)(x_1, x') = \left(\sqrt{1 - ||x'||^2} h \circ f \circ h^{-1}\left(\frac{x_1}{\sqrt{1 - ||x'||^2}}\right), x'\right).$$

The map $\lambda_h$ defines an embedding of the group $\text{Homeo}_+(\mathbb{R})$ into the group $\text{Homeo}_0(\mathbb{R}^d)$. The image under $\lambda_h$ of the group $\text{Homeo}_Z(\mathbb{R})$ is a subgroup of $\text{Homeo}_0(\mathbb{R}^d)$ which is supported in $U$. \qed
Let us complete now the proof of Proposition 2.2.

Proof of Proposition 2.2. Fix a point \( p \) in \( \mathbb{R}^d \). Take a closed ball \( B \subset \mathbb{R}^d \) which is centered at \( p \). Let us prove that \( \text{Fix}(\varphi(G^d_B)) \neq \emptyset \).

Take a subgroup \( G_1 \) of \( \text{Homeo}_0(\mathbb{R}^d) \) which is isomorphic to \( \text{Homeo}_Z(\mathbb{R}) \) and supported in \( B \). Such a subgroup exists by Lemma 4.2. This subgroup commutes with any subgroup \( G_2 \) of \( \text{Homeo}_0(\mathbb{R}^d) \) which is isomorphic to \( \text{Homeo}_Z(\mathbb{R}) \) and supported outside \( B \).

If the group \( \varphi(\text{Homeo}_0(\mathbb{R}^d)) \) admits a fixed point, there is nothing to prove. Suppose that this group has no fixed point. As the group \( \text{Homeo}_0(\mathbb{R}^d) \) is simple, the morphism \( \varphi \) is one-to-one. Moreover, any morphism \( \text{Homeo}_0(\mathbb{R}^d) \to \mathbb{Z}/2\mathbb{Z} \) is trivial: the morphism \( \varphi \) takes values in \( \text{Homeo}_+(\mathbb{R}) \). Hence the subgroups \( \varphi(G_1) \) and \( \varphi(G_2) \) of \( \text{Homeo}(\mathbb{R}) \) satisfy the hypothesis of Lemma 4.1. By this lemma:

\[
\emptyset \neq \text{Supp}(\varphi(G_1)) \subset \text{Fix}(\varphi(G_2)).
\]

Claim 4.3. The group \( G^d_B \) is generated by the union of its subgroups isomorphic to \( \text{Homeo}_Z(\mathbb{R}) \).

This claim implies that

\[
\emptyset \neq \text{Supp}(\varphi(G_1)) \subset \text{Fix}(\varphi(G^d_B)).
\]

Proof. For \( d \geq 2 \), observe that the open set \( \mathbb{R}^d - B \) is connected. Hence, as we recalled in the introduction, the group \( G^d_B \) is simple by a theorem by Fisher (see [5]). The claim is a direct consequence of the simplicity of this group. In the case where \( d = 1 \), the open set \( \mathbb{R} - B \) has two connected components. Denote by \([a,b] \) the compact interval \( B \). The inclusions of the groups \( \text{Homeo}_0((\infty, a)) \) and \( \text{Homeo}_0((b, +\infty)) \) induce an isomorphism

\[
\text{Homeo}_0((\infty, a)) \times \text{Homeo}_0((b, +\infty)) \to G^d_B.
\]

The simplicity of each factor of this decomposition implies the claim. \( \square \)

Claim 4.4. The set \( \text{Fix}(\varphi(G^d_B)) \) is compact.

Proof. Suppose by contradiction that there exists a sequence \((a_k)_{k \in \mathbb{N}} \) of real numbers in \( \text{Fix}(\varphi(G^d_B)) \) which tends to \( +\infty \) (if we suppose that it tends to \( -\infty \), we obtain of course an analogous contradiction). Let us choose a closed ball \( B' \subset \mathbb{R}^d \) which is disjoint from \( B \). Observe that the subgroups \( G^d_B \) and \( G^d_{B'} \) are conjugate in \( \text{Homeo}_0(\mathbb{R}^d) \) by a homeomorphism which sends the ball \( B \) to the ball \( B' \). Then the subgroups \( \varphi(G^d_B) \) and \( \varphi(G^d_{B'}) \) are conjugate in the group \( \text{Homeo}_+(\mathbb{R}) \). Hence the sets \( \text{Fix}(\varphi(G^d_B)) \) and \( \text{Fix}(\varphi(G^d_{B'})) \) are homeomorphic: there exists a sequence \((b_k)_{k \in \mathbb{N}} \) of elements in \( \text{Fix}(\varphi(G^d_{B'})) \) which tends to \( +\infty \). Take positive integers
$n_1, n_2$ and $n_3$ such that $a_{n_1} < b_{n_2} < a_{n_3}$. Fix $x_0 < a_{n_1}$. We notice then that for any homeomorphisms $g_1 \in G^d_B$, $g_2 \in G^d_B$, and $g_3 \in G^d_B$, the following inequality is satisfied:

$$\varphi(g_1)\varphi(g_2)\varphi(g_3)(x_0) < a_{n_3}.$$  

However, by Lemma 2.7, any element $g$ in Homeo$(\mathbb{R}^d)$ can be written as a product $g = g_1 g_2 g_3$, where $g_1$ and $g_3$ belong to $G^d_B$ and $g_2$ belongs to $G^d_B'$. The proof of this fact is similar to that of Lemma 2.7. Therefore:

$$\{\varphi(g)(x_0), \; g \in \text{Homeo}_0(\mathbb{R}^d)\} \subset (-\infty, a_{n_3}].$$

The greatest element of the left-hand set is a fixed point of the image of $\varphi$; this is not possible as this image was supposed to have no fixed point.

Observe that the group $\varphi(G^d_B)$ is the union of its subgroup of the form $\varphi(G^d_B')$, with $B'$ varying over the set $B_p$ of closed balls centered at the point $p$. By compactness, the set

$$\text{Fix}(\varphi(G^d_B)) = \bigcap_{B' \in B_p} \text{Fix}(G^d_B')$$

is nonempty. Proposition 2.2 is proved.

\section{Proof of Lemma 4.1}

We start this section by recalling some facts about the group Homeo$^\infty(\mathbb{R})$ of homeomorphisms of the real line which commute with integral translations. Observe that the center of the group Homeo$^\infty(\mathbb{R})$ is the subgroup generated by the translation $x \mapsto x + 1$. The quotient of this group by its center is the group Homeo$^\infty(S^1)$. The following lemma is classical.

\textbf{Lemma 5.1.} Any group morphism Homeo$^\infty(\mathbb{R}) \to \mathbb{Z}$ or Homeo$^\infty(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ is trivial.

\textit{Proof of Lemma 5.1.} Actually, any element in Homeo$^\infty(\mathbb{R})$ can be written as a product of commutators, i.e. elements of the form $aba^{-1}b^{-1}$, with $a, b \in \text{Homeo}^\infty(\mathbb{R})$. For an explicit construction of such a decomposition, see Section 2 in [4].

\textbf{Lemma 5.2.} Let $\psi : \text{Homeo}^\infty(\mathbb{R}) \to \text{Homeo}^+_\infty(\mathbb{R})$ be a group morphism. Denote by $F$ the closed set of fixed points of the group $\psi(\text{Homeo}^\infty(\mathbb{R}))$. Then, for any connected component $K$ of the complement of $F$, there exists a homeomorphism $h_K : \mathbb{R} \to K$ such that:

$$\forall f \in \text{Homeo}^\infty(\mathbb{R}), \forall x \in K, \; \psi(f)(x) = h_K f h_K^{-1}.$$
This lemma is similar to Matsumoto’s theorem about morphisms \( \text{Homeo}_0(S^1) \to \text{Homeo}_0(S^1) \) (see introduction) and the proof of this lemma relies heavily on Matsumoto’s theorem. Before proving this lemma, let us see how it implies Lemma 4.1.

**Proof of Lemma 4.1.** Recall that we are given two subgroups \( G \) and \( G' \) of \( \text{Homeo}^+_\mathbb{R} \) isomorphic to the group \( \text{Homeo}_2\mathbb{R} \).

Let \( \alpha \) (respectively \( \alpha' \)) be a generator of the center of \( G \) (respectively of \( G' \)). Let \( A_\alpha = \mathbb{R} - \text{Fix}(\alpha) \) and \( A_{\alpha'} = \mathbb{R} - \text{Fix}(\alpha') \).

As the homeomorphisms \( \alpha \) and \( \alpha' \) commute:

\[
\begin{align*}
\{ \alpha'(A_\alpha) &= A_\alpha \\
\alpha(A_{\alpha'}) &= A_{\alpha'} \}
\end{align*}
\]

**Claim 5.3.** Take any connected component \( I \) of \( A_\alpha \) and any connected component \( I' \) of \( A_{\alpha'} \). Then the interval \( I \) and \( I' \) are disjoint.

This claim is proved below. Let us complete now the proof of Lemma 4.1. By Lemma 5.2, \( A_\alpha = \text{Fix}(G) \) and \( A_{\alpha'} = \text{Fix}(G') \). Hence, we have proved that any connected component of the complement of \( \text{Fix}(G) \) is disjoint from the complement of \( \text{Fix}(G') \). Therefore \( \text{Supp}(G) \subset \text{Fix}(G') \). We have also proved that \( \text{Supp}(G') \subset \text{Fix}(G) \).

**Proof of claim 5.3.** This claim is a direct consequence of the three following claims.

**Claim 5.4.** Either \( I \) is contained in \( I' \), or \( I' \) is contained in \( I \), or \( I \) and \( I' \) are disjoint.

**Claim 5.5.** The interval \( I \) is not strictly contained in the interval \( I' \).

Of course, the case where the interval \( I' \) is strictly contained in \( I \) is symmetric and cannot occur.

**Claim 5.6.** The interval \( I \) and \( I' \) are distinct.

**Proof of Claim 5.4.** Suppose for a contradiction that the conclusion of this claim does not hold. Changing the roles of \( \alpha \) and \( \alpha' \) if necessary, we can suppose that the supremum \( b \) of \( I \) is contained in the open interval \( I' \) and the infimum \( a' \) of \( I' \) is contained in the open interval \( I \). Then either the sequence \((\alpha^k(b))_{k>0}\) converges to the point \( a' \) as \( k \to +\infty \) or the sequence \((\alpha'^{-k}(b))_{k>0}\) converges to the point \( a' \) as \( k \to +\infty \). In any case, a sequence of points in \( A_\alpha \) converge to the point \( a' \).
As the set $A_\alpha$ is closed, this means that the point $a'$ belongs to $A_\alpha$. This is not possible as $a'$ belongs to $I$ which is a connected component of the complement of $A_\alpha$.

**Proof of Claim 5.5.** Suppose for a contradiction that the interval $I$ is strictly contained in the interval $I'$. Let $\sim$ be the equivalence relation defined on $I'$ by

$$x \sim y \iff (\exists k \in \mathbb{Z}, x = \alpha^k(y)).$$

The topological space $I'/\sim$ is homeomorphic to a circle. By Lemma 5.2, the group $G'$ preserves the interval $I'$. Notice that the group $G'/\langle \alpha' \rangle \cong \text{Homeo}_0(S^1)$ acts on the circle $I'/\sim$. As the group $G'$ commutes with the homeomorphism $\alpha$, this action preserves the nonempty set $(A_\alpha \cap I')/\sim$. As $\alpha'(A_\alpha) = A_\alpha$, the points of the interval $I$ are sent to points in the complement of $A_\alpha$ under the iterates of the homeomorphism $\alpha'$. Hence the set $(A_\alpha \cap I')/\sim$ is not equal to the whole circle $I'/\sim$. However, by Theorem 5.3 in [13] (see the remark below Theorem 1.2), a non-trivial action of the group $\text{Homeo}_0(S^1)$ on a circle has no non-empty proper invariant subset. Hence, the group $G'/\langle \alpha' \rangle$ acts trivially on the circle $I'/\sim$: for any element $\beta'$ of $G'$, and any point $x \in I'$, there exists an integer $k(x, \beta') \in \mathbb{Z}$ such that $\beta'(x) = \alpha^{k(x, \beta')}(x)$. Fixing such a point $x$, we see that the map

$$G' \to \mathbb{Z}$$

$$\beta' \mapsto k(x, \beta')$$

is a group morphism. Such a group morphism is trivial by Lemma 5.1. Therefore, the group $G'$ acts trivially on the interval $I'$, a contradiction.

**Proof of Claim 5.6.** Suppose that $I = I'$. Take any element $\beta'$ in $G'$. As the homeomorphism $\beta'$ commutes with $\alpha$, by Lemma 5.2, the homeomorphism $\beta'$ is equal to some element of $G$ on $I$. As the homeomorphism $\beta'$ commute with any element of $G$, there exists a unique integer $k(\beta')$ such that $\beta'|_I = \alpha^{k(\beta')}_I$. The map $k : \mathbb{Z} \to G$ is a nontrivial group morphism. But such a map cannot exist by Lemma 5.1.

It remains to prove Lemma 5.2.

**Proof of Lemma 5.2.** Denote by $t$ a generator of the center of the group $\text{Homeo}_2(\mathbb{R})$.

**Claim 5.7.** The connected components of the complement of $\text{Fix}(\psi(t))$ are each preserved by the group $\psi(\text{Homeo}_2(\mathbb{R}))$. Moreover

$$\text{Fix}(\psi(\text{Homeo}_2(\mathbb{R}))) = \text{Fix}(\psi(t)).$$

**Claim 5.8.** Any action of the group $\text{Homeo}_2(\mathbb{R})$ on $\mathbb{R}$ without fixed points is conjugate to the standard action.
It is clear that these two claims imply Lemma 5.2.

Proof of Claim 5.7. The set $\text{Fix}(\psi(t))$ is preserved under any element in $\psi(\text{Homeo}_Z(\mathbb{R}))$, because any element of this group commutes with the homeomorphism $\psi(t)$. Moreover, any element in $\psi(\text{Homeo}_Z(\mathbb{R}))$ preserves the orientation by Lemma 5.1. Hence the action $\psi$ induces an action of the group $\text{Homeo}_Z(\mathbb{R})/\langle t \rangle$, which is isomorphic to $\text{Homeo}_0(\mathbb{S}^1)$, on the set $F = \text{Fix}(\psi(t))$ by increasing homeomorphisms. As the group $\text{Homeo}_0(\mathbb{S}^1)$ is simple, the induced morphism from the group $\text{Homeo}_0(\mathbb{S}^1)$ to the group $\text{Homeo}_0(F)$ of increasing homeomorphisms of $F$ is either one-to-one or trivial. However, the group $\text{Homeo}_0(\mathbb{S}^1)$ contains some non-trivial finite order elements whereas the group $\text{Homeo}_0(F)$ does not: such a morphism is trivial. Hence any element of the group $\psi(\text{Homeo}_Z(\mathbb{R}))$ fixes any point in $\text{Fix}(\psi(t))$: any element of this group preserves each connected component of the complement of $\text{Fix}(\psi(t))$.

Proof of Claim 5.8. We denote by $\varphi : \text{Homeo}_Z(\mathbb{R}) \to \text{Homeo}(\mathbb{R})$ a morphism such that the group $\varphi(\text{Homeo}_Z(\mathbb{R}))$ of homeomorphisms of $\mathbb{R}$ has no fixed point.

By Claim 5.7, the homeomorphism $\varphi(t)$ has no fixed point. Changing coordinates if necessary, we can suppose that the homeomorphism $\varphi(t)$ is the translation $x \mapsto x + 1$. The morphism $\varphi$ induces an action $\hat{\varphi}$ of the group $\text{Homeo}_Z(\mathbb{R})/\langle t \rangle \cong \text{Homeo}_0(\mathbb{S}^1)$ on the circle $\mathbb{R}/\mathbb{Z}$. This action is nontrivial: otherwise, there would exist a nontrivial group morphism $\text{Homeo}_0(\mathbb{S}^1) \to \mathbb{Z}$. By the theorem by Matsumoto that we recalled earlier (see the introduction of this article), there exists a homeomorphism $h$ of the circle $\mathbb{R}/\mathbb{Z}$ such that, for any homeomorphism $f$ in $\text{Homeo}_Z(\mathbb{R})/\langle t \rangle$ (which can be canonically identified with $\text{Homeo}_0(\mathbb{R}/\mathbb{Z})$):

$$\hat{\varphi}(f) = hfh^{-1}.$$ 

Take a lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ of $h$. For any integer $n$, denote by $T_n : \mathbb{R} \to \mathbb{R}$ the translation $x \mapsto x + n$. For any homeomorphism $f$ in $\text{Homeo}_Z(\mathbb{R})$, there exists an integer $n(f)$ such that

$$\varphi(f) = T_{n(f)} \tilde{h} \tilde{f} \tilde{h}^{-1}.$$ 

However, the map $n : \text{Homeo}_Z(\mathbb{R}) \to \mathbb{Z}$ is a group morphism: it is trivial by Lemma 5.1. This completes the proof of Claim 5.8.

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