Preparing tunable Bell-diagonal states on a quantum computer

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The class of two-qubit Bell-diagonal states has been the workhorse for developing understanding about the geometry, dynamics, and applications of quantum resources. In this article, we present a quantum circuit for preparing Bell-diagonal states on a quantum computer in a tunable way. We implement this quantum circuit using the IBM Q 5 Yorktown quantum computer and, as an experimental example, we measure the non-local, steering, entanglement, and discord quantum correlations and non-local quantum coherence of Werner states.

Keywords: Bell-diagonal states; Werner states; quantum computer; IBM Q 5 Yorktown; quantum resources

Quantum properties such as coherence [1], nonlocality [2], steering [3], entanglement [4], and discord [5] have been identified as resources enabling the implementation of diverse quantum computation and communication protocols [6–10]. The functions defined to quantify these quantum features based on the resource-theory framework [1, 11–14] are frequently hard to compute analytically for general quantum states [15, 16]. Motivated by that observation, a subset of two-qubit states, the so-called Bell-diagonal states (BDS), have been used extensively for better understanding some of these resources [17–37].

So, due to its central place within the study of quantum resources, the experimental preparation of BDS is of apparent need. Recently Liu et al. showed how to prepare tunable Werner states in a linear optical system via the implementation of a depolarizing channel applied to a Bell state [38]. Here we devise a simple quantum circuit that can be used to create tunable BDS on a quantum computer with the use of two auxiliary qubits. To exemplify the use of our protocol, we measure experimentally, using the IBM Q 5 Yorktown quantum computer [39], the quantum nonlocality, steering, entanglement, discord, and non-local coherence of Werner states, which are a one-parameter subset of the BDS.

Our protocol can find application in verifying experimentally several theoretical results from the literature. For instance, one can apply our circuit to verify the relation between the sudden change phenomenon of quantum discord and the worst case fidelity in quantum teleportation, discovered in [20]. The necessity of quantum entanglement, instead of quantum non-locality, for better than classical fidelity of quantum teleportation exemplified using the thermal state associated with the magnetic dipolar interaction Hamiltonian [27] can also be simulated using our protocol. This procedure can also be applied to verify the direct-dynamical entanglement-discord relations reported in [40]. Besides these three examples, one can easily find several other applications for our protocol, as e.g. in the experimental verification of the theoretical results reported in Refs. [18, 22, 31, 41, 42].

The remainder of this article is organized as follows. We begin describing the class of BDS and presenting the quantum circuit we propose for its preparation on a quantum computer (QC). In the sequence we outline the implementation of this circuit on the IBM Q 5 Yorktown QC, hereafter referred to as ibmqx2. Then we present the experimental results we obtained for the quantum correlations and non-local coherence of Werner states. Finally, we report on a simple model that we have introduced to explain the noise influence on the experimental data and give our conclusions.

As the name indicates, two-qubit Bell-diagonal states read

\[ \rho_{bd}^{\text{bd}} = \sum_{j,k=0}^{1} p_{jk} |\beta_{jk}\rangle \langle \beta_{jk}|, \]

where \(|\beta_{jk}\rangle = 2^{-1/2} (|0\rangle \otimes |k\rangle + (-1)^j |1\rangle \otimes |k \oplus 1\rangle)\) are the Bell’s base states [43], with \(\oplus\) being the modulo 2 sum, \(\{j\}_{j=0}^{1}\) is the computational basis, and \(p_{jk}\) is a probability distribution. This class of two-qubit states has the following four-qubit purification

\[ |\tau\rangle_{abcd} = \sum_{j,k=0}^{1} \sqrt{p_{jk}} |j\rangle_a \otimes |k\rangle_b \otimes |\beta_{jk}\rangle_{cd}. \]

That is to say,

\[ \rho_{bd} = p_{cd} = \text{Tr}_{ab} (|\tau\rangle \langle \tau|_{abcd}), \]
Figure 1: Quantum circuit we propose to generate tunable Bell-diagonal states on a quantum computer. The $R$ gates generate one-qubit superposition states. The CNOTs are used to copy the states of the qubits $a$ and $b$ to the qubits $c$ and $d$, respectively. By its turn, the Hadamard and the last CNOT gate are used for changing from the computational to the Bell basis. At the output, the joint state of qubits $c$ and $d$ is equivalent to $\rho_{bd}$ of Eq. (1) with $p_{jk}$ given as in Eq. (7).

with $\text{Tr}_{ab}$ being the partial trace function \[44\]. Here we report that the quantum circuit shown in Fig. 1 generates the 4-qubit state $|\tau\rangle_{abcd}$, and therefore that it can be used to prepare any BDS.

In the circuit shown in Fig. 1, we used the rotation

$$R(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix},$$

the controlled-not gate

$$\text{CNOT}_{s\rightarrow s'} = |0\rangle\langle 0|_s \otimes \sigma_0^{s'} + |1\rangle\langle 1|_s \otimes \sigma_1^{s'},$$

and the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$  

Above $\sigma_0^{s} \equiv \sigma_0$ is the 2x2 identity matrix and $\{\sigma_j^{s}\}_{j=1}^3$ are the Pauli matrices acting on qubit $s$ [43].

For the circuit in Fig. 1, the relations among the probabilities in the BDS and rotation angles are seen to be

$$p_{jk} = \left( \cos^2 \frac{\theta}{2} \right)^{1-j} \left( \sin^2 \frac{\theta}{2} \right)^{j} \left( \cos^2 \frac{\alpha}{2} \right)^{1-k} \left( \sin^2 \frac{\alpha}{2} \right)^{k}.$$  

(7)

For the calculation of quantum correlations, one usually start studying a maximally-mixed marginals state, $\rho_{3m} = 2^{-2}(\sigma_0 \otimes \sigma_0 + \sum_{j,k=1}^3 c_{jk}^j \sigma_j \otimes \sigma_k)$, put to the normal form \[45\],

$$\rho_n = 2^{-2} \left( \sigma_0 \otimes \sigma_0 + \sum_{j=1}^3 c_{jj} \sigma_j \otimes \sigma_j \right),$$  

(8)

via local unitaries. The states $\rho_n$ are diagonal in the Bell basis, having the following eigenvalue–eigenvector pairs

$$\left( p_{jk} = \frac{1}{4} \left( 1 + (-1)^j c_1 + (-1)^{j+k-1} c_2 + (-1)^k c_3 \right), |\beta_{jk}\rangle \right),$$  

(9)

where we used $c_j \equiv c_{jj}$.

Hence, from Eqs. (7) and (9) we see that given $\rho_n$, we can prepare any BDS in a tunable way by using as input to the quantum computer rotations $R(\theta/2)$ and $R(\alpha/2)$ the angles:

$$\theta = 2 \arccos \sqrt{p_{00} + p_{01}},$$

$$\alpha = 2 \arccos \sqrt{p_{00} + p_{10}}.$$  

(10)

(11)

For the implementation of the quantum circuit of Fig. 1 on the ibmqx2, we use $R(x/2) = U_3(x, 0, 0)$ with

$$U_3(\theta, \phi, \lambda) = \begin{bmatrix} \cos \frac{\theta}{2} & e^{i\lambda} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & e^{i(\lambda+\phi)} \cos \frac{\theta}{2} \end{bmatrix}.$$  

(12)
### Table I: Averages of the calibration data of the IBM Q 5 Yorktown quantum computer with which the experiments were performed. The temperature was $T = 0.0159$ K.

| Parameter                  | Q0  | Q1  | Q2  | Q3  | Q4  |
|----------------------------|-----|-----|-----|-----|-----|
| Frequency (GHz)            | 5.29| 5.23| 5.02| 5.29| 5.08|
| $T_1$ ($\mu$s)             | 50.81| 59.80| 64.93| 56.37| 56.81|
| $T_2$ ($\mu$s)             | 45.89| 39.70| 63.14| 31.60| 32.32|
| Gate error ($10^{-3}$)     | 2.82 | 1.83 | 4.65 | 4.36 | 2.54 |
| Readout error ($10^{-2}$)  | 4.16 | 1.89 | 1.93 | 2.87 | 4.61 |
| MultiQubit gate error      | $CX_{0\_1}$ | 4.15 | $CX_{1\_2}$ | 3.81 | $CX_{3\_2}$ | 7.09 | $CX_{4\_2}$ | 3.84 |
|                           | $CX_{0\_2}$ | 4.42 | $CX_{3\_4}$ | 5.28 |           |     |           |     |

being one of the ibmqx2 quantum gates \cite{ref39}. The other gates we need are themselves directly included in the ibmqx2 set of ready-to-use quantum gates.

The experiments were carried out with the calibration parameters for the ibmqx2 shown in Table I. We have chosen the following encoding (see Table I) for implementation of the circuit in Fig. 1:

\[ a \to Q1, \ b \to Q3, \ c \to Q2, \ d \to Q4. \]  

(13)

With these settings, we prepared Werner states \cite{ref4},

\[ \rho_w = (1 - w)\frac{\sigma_0 \otimes \sigma_0}{4} + w|\beta_{11}\rangle\langle\beta_{11}|, \]  

(14)

for eleven values of $w \in [0, 1]$. We observe that $\rho_w$ is equivalent to $\rho_n$ if $c_1 = c_2 = c_3 = -w$.

In order to experimentally reconstruct the prepared states, we consider general two-qubit states written in the form

\[ \rho = \frac{1}{4} \sum_{j,k=0}^3 c_{jk} \sigma_j \otimes \sigma_k, \]  

(15)

with $c_{jk} = \langle \sigma_j \otimes \sigma_k \rangle_{\rho}$. All of these averages can be obtained from the joint probability distributions of the local measurements of $\sigma_j$ and $\sigma_k$. Let

\[ p_{j\pm,k\pm} := \text{Prob}(\sigma_j = \pm 1, \sigma_k = \pm 1). \]  

(16)

Then, for $j, k = 1, 2, 3$:

\[ c_{jk} = p_{j+,k+} + p_{j-,k-} - p_{j+,k-} - p_{j-,k+}. \]  

(17)

Using the marginal probability distributions

\[ p_{j\pm} = p_{j\pm,k+} + p_{j\pm,k-} \quad \text{and} \quad p_{k\pm} = p_{j+,k\pm} + p_{j-,k\pm} \]  

(18)

we calculate

\[ c_{j0} = p_{j+} - p_{j-} \quad \text{and} \quad c_{0k} = p_{k+} - p_{k-} \]  

(19)

for $j = 1, 2, 3$ and for $k = 1, 2, 3$. Finally, because $\text{Tr}(\rho) = 1$, we have $c_{00} = 1$. Measurements of $\sigma_3$ are part of the ready-to-use operations of ibmqx2. To measure $\sigma_1$, we first applied the Hadamard gate $H$ and then measured $\sigma_3$. For measuring $\sigma_2$, we applied

\[ S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, \]  

(20)

then applied $H$, and finally measured $\sigma_3$. Above $^\dagger$ denotes the transpose conjugate. With these measurement procedures, the probability distributions $p_{j\pm,k\pm}$ were estimated with 8192 runs of the given quantum circuit and
Figure 2: (color online) Computational basis representation of the real and imaginary parts of the theoretical-target and experimentally prepared density matrices corresponding to Werner states with three different values of the weight $w$. 
corresponding measurements. The computational basis representation of the reconstructed Werner states density matrices is shown in Fig. 2 for three values of \( w \).

In the sequence, we shall describe the quantumness measures we consider in this article. We begin by the \( l_1 \)-norm coherence, with the standard basis used as the reference basis \([1]\):

\[
C_{l_1}(\rho) = \sum_{j \neq k} |\langle j|\rho|k \rangle| \tag{21}
\]

for \( j, k = 1, \cdots, d \) with \( d \) being the dimension of the regarded state space. A natural candidate for quantifying the non-local extent of the quantum coherence of a bipartite system is \([46, 47]\):

\[
C(\rho) = C_{l_1}(\rho) - [C_{l_1}(\rho_a) + C_{l_1}(\rho_b)], \tag{22}
\]

with the reduced states given by \([44]\): \( \rho_a = \text{Tr}_b(\rho) \) and \( \rho_b = \text{Tr}_a(\rho) \).

The quantum correlation (QC) named quantum discord is related to the minimal extent to which the correlations in a composite system are to be deprecated by local non-selective projective measurements. Here we use Ollivier-Zurek’s discord \([5]\):

\[
D(\rho) = I(\rho) - \text{max } I(\Pi_b(\rho)), \tag{23}
\]

with the quantum mutual information being \( I(x) = S(x_a) + S(x_b) - S(x) \), where \( x_a \) and \( x_b \) are reduced operators computed as mentioned above. By its turn, the measured state is defined as \( \Pi_b(\rho) = \sum_j \sigma_0 \otimes \Pi^b_j \rho \sigma_0 \otimes \Pi^b_j \) with \( \Pi^b_j \Pi^b_k = \delta_{jk} \Pi^b_j \) and \( \sum_j \Pi^b_j = \sigma_0 \). We observe that once there is no known analytical formula for \( D \) of general states (even for two qubits), the results we present in the sequence are obtained using numerical optimization.

Discord is known to be a weaker quantum correlation when compared to entanglement. This last type of quantum correlation, the non-separable correlations, are quantified here using the entanglement negativity \([48]\):

\[
E(\rho) = ||T_\rho(\rho)||_{tr} - 1, \tag{24}
\]

where \( ||X||_{tr} = \text{Tr}\sqrt{X^\dagger X} \) is the trace norm and \( T_\rho \) is the partial transposition operation \([49]\).

For the two strongest forms of quantum correlations known, steering and non-locality, which cannot be explained using a local hidden state and a local hidden variable model, respectively, we use the formulas reported in \([50]\). These authors considered measures for these quantities given by the maximum extend to which a given related inequality \([51–53]\) is violated. For deriving their analytical formulas, they used the standard form for two-qubit states \([45]\):

\[
4\rho = \sigma_0 \otimes \sigma_0 + \vec{a} \cdot \vec{\sigma} \otimes \sigma_0 + \sigma_0 \otimes \vec{b} \cdot \vec{\sigma} + \sum_{j=1}^3 c_j \sigma_j \otimes \sigma_j. \tag{25}
\]

This form can be obtained via local unitary transformations applied locally to a general two-qubit state, i.e., \( \rho = U_a \otimes U_b \rho_u U_a^\dagger \otimes U_b^\dagger \), for \( U_a U_a^\dagger = U_b U_b^\dagger = U_a^\dagger U_a = U_b^\dagger U_b = I \), with

\[
4\rho_b = \sigma_0 \otimes \sigma_0 + \vec{x} \cdot \vec{\sigma} \otimes \sigma_0 + \sigma_0 \otimes \vec{y} \cdot \vec{\sigma} + \sum_{j,k=1}^3 c_{jk} \sigma_j \otimes \sigma_k, \tag{26}
\]

where \( \vec{a} = O_{a} \vec{x} \), \( \vec{b} = O_{b} \vec{y} \), and \( \text{diag}(c_1, c_2, c_3) = O_{a} CO_{b}^T \), with \( C = (c_{jk}) \) being the correlation matrix and \( O_{a} \) and \( O_{b} \) are orthonormal matrices, i.e., \( O_{a} O_{a}^T = O_{b} O_{b}^T = O_{a}^T O_{b} = O_{b}^T O_{a} = I \) for \( x^T \) denoting the transpose of the matrix \( x \).

The authors of \([50]\) obtained analytically the steering for three measurements per qubit,

\[
S(\rho) = \max \left( 0, \frac{||\vec{c}|| - 1}{\sqrt{3} - 1} \right), \tag{27}
\]

and the quantum non-locality for two measurements per qubit,

\[
N(\rho) = \max \left( 0, \frac{\sqrt{||\vec{c}||^2 - c_{\text{min}}^2} - 1}{\sqrt{2} - 1} \right), \tag{28}
\]

with \( c_{\text{min}} \) being the minimum value among the components of the correlation vector \( \vec{c} = (c_1, c_2, c_3) \). Here we use as the correlation vector the singular values of the correlation matrix \( C = (c_{jk}) \), for \( j, k = 1, 2, 3 \). We emphasize that
Figure 3: (color online) The $x$-like black points show the preparation fidelities, $F(\rho_w, \rho_{\text{exp}}^w) = Tr\sqrt{\rho_w \rho_{\text{exp}}^w \sqrt{\rho_w}}$. The fidelity and the functions indicated in the legend by the subscript $e$ refer to averages, and the associated standard deviations, computed for the experimentally prepared states using seven rounds of experiments. $C$ stands for non-local coherence, shown in gray. In magenta is plotted the Ollivier-Zurek discord $D$. The entanglement negativity $E$ is shown in blue. The steering $S$ was given the color red and the non-locality $N$ is shown in cyan.

the standard form is obtained via local unitary transformations, which do not affect the non-locality and steering functions above. Besides, we utilize the original state (reconstructed or theoretical) for the calculation of non-local coherence, discord, and entanglement.

The results for the state preparation fidelity and for all these quantum non-local coherence and correlation measures are presented in Fig. 3. The code we used to compute these functions is freely available at https://github.com/jonasmaziero/libPyQ.

Even though the preparation fidelities shown in Fig. 3 have, in general, values quite close to the maximum value 1, we see in this figure that the environmental noise and the quantum computer imperfections have significant detrimental effects on the quantum properties of the prepared states. This fact indicates that state preparation fidelity is not a reliable figure of merit if one’s main purpose is the production and utilization of quantum correlations. For a related discussion, see Ref. [54].

It is interesting noticing that not only are the different quantum resources affected unevenly by those external influences, but the stronger the quantum correlation is, the more it is impacted. This fact can be qualitatively well explained in a simplified manner through the application of the composition of the amplitude damping and phase damping channels [43, 55] to one of the qubits of the theoretical-target Werner states:

$$\rho^d_w(a,p) = \sum_{j=0}^2 K_j(a,p) \otimes \sigma_0 \rho_w K_j^+(a,p) \otimes \sigma_0,$$

with the Kraus’ operators given by:

$$K_0(a,p) = \sqrt{p(1-a)} |1\rangle \langle 1|,$$

$$K_1(a,p) = \sqrt{a} |0\rangle \langle 1|,$$

$$K_2(a,p) = \sqrt{1-a} |0\rangle \langle 0|.$$
The quantum non-local coherence and correlations of $\rho_{\text{d}}(0.25, 0.25)$ and of $\rho_{\text{e}}(0.15, 0.15)$ are shown in Fig. 4.

The results in Fig. 4 show that our simplified model of Eq. (29) describes well the main qualitative features of the experimental data. Besides, we see in the inset of Fig. 4 that by lowering the values of the amplitude and phase noise rates, one could significantly increase the values of the quantumness functions of the generated states.

In conclusion, in this article we gave a quantum circuit that can be used to prepare tunable Bell-diagonal states with a quantum computer. We implemented this quantum circuit using the IBM Q 5 Yorktown quantum computer and measured the non-local quantum coherence and discord, entanglement, steering, and non-local quantum correlations of experimentally reconstructed Werner states. Even though the noise and imperfections of the hardware utilized had a quite strong detrimental effect on the measured quantum correlations, we succeeded in verifying a hierarchy relation for quantum resources (see e.g. Ref. [50]) of the produced states: $N \Rightarrow S \Rightarrow E \Rightarrow D$. The simple zero-temperature composite noise model we made to explain the obtained experimental results indicates that access to quantum computers with lower noise rates will allow for the application of our quantum circuit to produce even the strongest kinds of quantum correlation and also to test several interesting theoretical results that have been reported in the recent quantum information science literature using Bell-diagonal states.

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