SIMPLE $\mathfrak{sl}(V)$-MODULES WHICH ARE FREE OVER AN ABELIAN SUBALGEBRA

JONATHAN NILSSON

Abstract. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{sl}(V)$ of maximal dimension and let $\mathfrak{n} \subset \mathfrak{p}$ be the corresponding nilradical. In this paper we classify the set of $\mathfrak{sl}(V)$-modules whose restriction to $U(\mathfrak{n})$ is free of rank 1. It turns out that isomorphism classes of such modules are parametrized by polynomials in $\dim V - 1$ variables. We determine the submodule structure for these modules and we show that they generically are simple.

1. Introduction

Lie algebras and their representations appear throughout multiple areas of mathematics, and the elemental objects of representation theory are simple modules. Unfortunately, a complete classification of simple modules for a Lie algebra $\mathfrak{g}$ is too broad a project, only for the Lie algebra $\mathfrak{sl}_2$ does a version of such a classification exist, see [B Maz1]. Nevertheless many classes of $\mathfrak{g}$-modules are well studied. For example, when $\mathfrak{g}$ is a simple finite-dimensional complex Lie algebra, all simple finite-dimensional modules were classified early, see [Ca Di]. More generally, simple highest weight modules (see [Di Hu BGG]) and simple weight modules with finite-dimensional weight spaces (see [BL Fu Fe Mat]) are also completely classified.

Several classes of non-weight modules have also been studied. These include Whittaker modules (see [Kos BM]), Gelfand-Zetlin modules (see [DFO]), and various others (see for example [FOS]).

Recently several authors have studied $\mathfrak{g}$-modules whose restriction to certain $\mathfrak{g}$-subalgebras are free. For example, when $\mathfrak{g}$ is a simple complex finite-dimensional Lie algebra, the set of modules which are free of rank 1 when restricted to the universal enveloping algebra of a Cartan subalgebra were classified in [N1 N2]. Corresponding and related results were also obtained for a multitude of other Lie algebras such as the Witt- and Virasoro-algebras, the Heisenberg-Virasoro algebra, Schrödinger algebras, and for basic Lie-super algebras, see [CC CG CLNZ CZ CTZ HCS LZ MP N3 TZ1 TZ2] and references therein. A common theme for many of the modules in the papers listed above are that they are free when restricted to some commutative $\mathfrak{g}$-subalgebra, often involving a Cartan subalgebra and central elements of $\mathfrak{g}$.

In the present paper we study $\mathfrak{sl}(V)$-modules which are free over another maximal commutative subalgebra: the nilradical of a parabolic subalgebra of maximal dimension. A concrete example of such a module is given in the following result which is a restatement of Theorem 16 in Section 4.2.

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and let $\mathfrak{n} = \text{span}(e_{1,n+1}, \ldots, e_{n,n+1}) \subset \mathfrak{g}$ (where as usual $e_{i,j}$ are the standard basis elements for $\mathfrak{g}_{i,j}$). Then $\mathfrak{n}$ is the nilradical of the parabolic
subalgebra corresponding to removing the simple root whose root space is spanned by \( e_{n+1,n} \).

**Theorem 1.** Fix a polynomial \( p(x) \in \mathbb{K}[x_1, \ldots, x_n] \) and for \( 1 \leq i, j \leq n \) define polynomials

\[
p_{ij} := x_i \frac{\partial p}{\partial x_j} + \delta_{ij}p(0)/n \quad \text{and} \quad q_i := -\frac{1}{x_i} \int_0^x \sum_{r=1}^n (p_{ri}^i p_{ri} + x_r p_{ri}^r + p_{ri}^r p_{rr}) dx_i,
\]

where upper indices indicate derivatives: \( p^k = \frac{\partial p}{\partial x_k} \).

Then the following action equips the space \( M(p) = \mathbb{K}[x_1, \ldots, x_n] \) with an \( \mathfrak{sl}_{n+1} \)-module structure:

\[
e_{i,j} \cdot f = x_i f, \\
h_i \cdot f = f p_{ii} + x_i \frac{\partial f}{\partial x_i}, \\
e_{i,j} \cdot f = f p_{ij} + x_j \frac{\partial f}{\partial x_j}, \\
e_{n+1,i} \cdot f = q_i f - \sum_r (p_{ri}^i \frac{\partial f}{\partial x_r} + p_{ri}^r \frac{\partial f}{\partial x_i} + x_r \frac{\partial^2 f}{\partial x_i \partial x_r}),
\]

for \( f \in \mathbb{K}[x_1, \ldots, x_n] \), \( 1 \leq i, j \leq n \), and \( h_i := e_{ii} - \frac{1}{x_i} I \).

Moreover, any \( \mathfrak{sl}_{n+1} \)-module \( M \) for which \( \text{Res}_{U(G)} U_{U(\mathfrak{n})} \) is free of rank 1 is isomorphic to \( M(p) \) for a unique polynomial \( p \in \mathbb{K}[x_1, \ldots, x_n] \).

The easiest case is when \( p \) is constant as in the following example:

**Example** Taking \( n = 2 \) and \( p = -\frac{1}{2} \lambda \in \mathbb{K} \) we obtain the following \( \mathfrak{sl}_2 \)-module structure on \( \mathbb{K}[x_1, x_2] \):

\[
e_{13} \cdot f = x_1 f, \quad e_{23} \cdot f = x_2 f, \\
h_1 \cdot f = x_1 \frac{\partial f}{\partial x_1}, \quad h_2 \cdot f = x_2 \frac{\partial f}{\partial x_2}, \\
e_{12} \cdot f = x_1 \frac{\partial f}{\partial x_2}, \quad e_{21} \cdot f = x_2 \frac{\partial f}{\partial x_1}, \\
e_{31} \cdot f = \lambda \frac{\partial f}{\partial x_1} - d(\frac{\partial f}{\partial x_1}), \\
e_{32} \cdot f = \lambda \frac{\partial f}{\partial x_2} - d(\frac{\partial f}{\partial x_2}).
\]

where we have written \( d \) for the degree operator \( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \). In this case our module \( M(-\frac{3\lambda}{2}) \) is actually parabolically induced: Let \( \mathfrak{p} = \text{span}(h_1, h_2, e_{12}, e_{21}, e_{31}, e_{32}) \), and let \( \mathbb{K}_\lambda \) be the 1-dimensional \( \mathfrak{p} \)-module where \( h_1 \) and \( h_2 \) both act by \( \lambda \), and the other basis elements of \( \mathfrak{p} \) act trivially. Then \( \text{Ind}_{U(\mathfrak{p})}^{U(\mathfrak{sl}_2)} \mathbb{K}_\lambda = U(\mathfrak{sl}_2) \otimes U(\mathfrak{p}) \mathbb{K}_\lambda \cong M(-\frac{3\lambda}{2}) \). However, when \( p \) is nonconstant \( M(p) \) is not parabolically induced.

The layout of this paper is as follows. In Section 2 we discuss parabolic subalgebras of \( \mathfrak{sl}(V) \) and their nilradicals, and we look at some general theory for modules free over subalgebras. In Section 3 we focus on \( \mathfrak{g} = \mathfrak{sl}_2 \), in this case we get somewhat nicer formulas for our module structure. We determine the Jordan-Häfner components of the modules we construct, and in Section 4 we give a Glebsch-Gordan style decomposition theorem for tensor products of \( U(\mathfrak{n}) \)-free modules and finite-dimensional modules. In Section 5 we generalize most of these results to \( \mathfrak{sl}_{n+1} \). In Section 6 we obtain the classification of \( U(\mathfrak{n}) \)-free modules of rank 1 for \( \mathfrak{sl}(V) \) and in Section 7 we prove that our modules are irreducible in general, and determine the submodule structure for the exceptional cases.
2. Preliminaries

Denote the nonnegative integers by \( \mathbb{N} \), and let \( \mathbb{K} \) be an algebraically closed field of characteristic zero.

2.1. Modules which are free over a subalgebra. We first discuss some general results relating the modules we study to previously known modules.

In this section let \( g \) be an arbitrary Lie algebra over \( \mathbb{K} \) and let \( a \subset g \) be a subalgebra.

First let us briefly recall that we have an adjunction between the functors \( \text{Res}^{U(g)} : U(a)\text{-Mod} \to U(g)\text{-Mod} \) and \( \text{Hom}_{U(a)}(U(g), -) : U(a)\text{-Mod} \to U(g)\text{-Mod} \).

In particular this means that for every \( g \)-module \( M \) and any \( a \)-module \( N \) we have a natural vector space isomorphism

\[
\text{Hom}_{U(a)}(\text{Res}^{U(g)}M, N) \cong \text{Hom}_{U(g)}(M, \text{Hom}_{U(a)}(U(g), N)).
\]

Here the \( g \)-action on \( \text{Hom}_{U(a)}(U(g), N) \) is given by \((x \cdot f)(y) := f(yx)\), and the correspondence above maps \( \varphi \in \text{Hom}_{U(a)}(\text{Res}^{U(g)}M, N) \) to \( \varphi \in \text{Hom}_{U(g)}(M, \text{Hom}_{U(a)}(U(g), N)) \), where \( \varphi(m) \in \text{Hom}_{U(a)}(U(g), N) \) is defined by \( \varphi(m)(x) := \varphi(x \cdot m) \).

Now let \( g \) be a finite-dimensional complex simple Lie algebra, and let \( a \subset g \) be a subalgebra. Let \( M \) be a \( g \)-module such that \( \text{Res}^{U(g)}M \) is a free module of rank 1. This just means that \( M \cong U(a) \) as an \( a \)-module. As vector spaces we then have

\[
\text{Hom}_{U(a)}(\text{Res}^{U(g)}M, N) \cong \text{Hom}_{U(a)}(U(a), N) \cong N,
\]

so the \( \text{Res} - \text{Hom} \) adjunction above gives

\[
\dim \text{Hom}_{U(a)}(M, \text{Hom}_{U(a)}(U(g), N)) = \dim N.
\]

For example, take \( a = \text{span}(z) \) for some \( z \in g \), and take and take \( N = \mathbb{K}_a \) to be the one-dimensional \( a \)-module where \( z \) acts by the scalar \( \alpha \). Then the space \( \text{Hom}_{U(a)}(U(a), \mathbb{K}_a) = \text{Hom}_{\mathbb{K}[z]}(\mathbb{K}[z], \mathbb{K}_a) \) is one-dimensional and spanned by the evaluation map \( \varphi_a \) where \( \varphi_a(f(z)) = f(\alpha) \). Corresponding to \( \varphi_a \) we get a \( g \)-submodule \( \text{Ker}(\varphi_a) \subset M \). We can describe this kernel explicitly: Let \( f(z) \in \mathbb{K}[z] = U(a) \). Then

\[
f \in \text{Ker}(\varphi_a) \iff \varphi_a(f(z)) = 0 \iff \varphi_a((x \cdot f(z))(x)) = 0 \forall x \in U(g)
\]

\[
\iff \varphi_a(x \cdot f(z)) = 0 \forall x \in U(g) \iff (x \cdot f)(\alpha) = 0 \forall x \in U(g).
\]

As an example we may take \( a \) to be a Cartan-subalgebra of \( g \). Then the situation becomes as in the papers \[N1\] \[N2\], where we studied simple module structures on \( U(\mathfrak{h}) \). In particular, if we take \( g = \mathfrak{sl}_2 \) and pick a basis \( \{x, y, h\} \) satisfying \( [h, x] = x, [h, y] = -y \) and \( [x, y] = 2h \), for any scalar \( b \) we have a module structure on \( \mathfrak{m}_b = U(\mathfrak{h}) \) in which

\[
h \cdot f(h) = h f(h), x \cdot f(h) = (h + b)f(h - 1), y \cdot f(h) = -(h - b)f(h + 1).
\]

As above, \( \text{Ker}(\varphi_a) \) is a submodule for each \( \alpha \), and the conditions from \[N1\] for \( \mathfrak{m}_b \) to be simple correspond precisely to our derived condition \( \text{Ker}(\varphi_a) = \mathfrak{m}_b \) as above.

If we stick with \( \mathfrak{sl}_2 \), another option is to instead take \( a = \text{span}(x) \) and study modules free over \( U(a) \). This is what we do in Section 3 below.

We can generalize these results to construct a new type of modules for \( \mathfrak{sl}_n \). Here instead of taking \( a \) as the Cartan subalgebra, we pick \( a \) as another abelian
subalgebra of dimension \( n - 1 \), namely the nilradical of a parabolic subalgebra of maximal dimension. This is discussed starting from Section 4.

2.2. Nilradicals of maximal parabolics. We first describe the set of parabolic subalgebras of \( \mathfrak{sl}(V) \) of maximal dimension. Given a proper nontrivial subspace \( \Delta \subset V \) we define subalgebras of \( \mathfrak{sl}(V) \) as follows:

\[
\mathfrak{p}_\Delta := \text{Stab}(\Delta) = \{ f \in \mathfrak{sl}(V) \mid f(\Delta) \subset \Delta \},
\]

\[
\mathfrak{n}_\Delta := \{ f \in \mathfrak{sl}(V) \mid f(V) \subset \Delta \}.
\]

We summarize some classical results on such subalgebras, see [Kob, Lemma 7.3.1] for details.

**Lemma 2.** We have

1. \( \mathfrak{p}_\Delta \) is a parabolic subalgebra of \( \mathfrak{sl}(V) \).
2. \( \mathfrak{p}_\Delta \) is maximal with respect to inclusion: it is not contained in any other parabolic subalgebra.
3. \( \mathfrak{n}_\Delta \) is the nilradical of \( \mathfrak{p}_\Delta \).
4. \( \mathfrak{n}_\Delta \) is an abelian subalgebra.
5. \( (\mathfrak{p}_\Delta)^\perp = \mathfrak{n}_\Delta \) with respect to the Killing form on \( \mathfrak{sl}(V) \).
6. \( \mathfrak{n}_\Delta \) is an ideal of \( \mathfrak{p}_\Delta \) and \( \mathfrak{p}_\Delta/\mathfrak{n}_\Delta \) is semi-simple.

**Lemma 3.** The following statements are equivalent:

1. \( \mathfrak{p}_\Delta \) is a parabolic subalgebra of maximal dimension.
2. \( \dim \mathfrak{n}_\Delta = \dim V - 1 \).
3. \( \dim \Delta = 1 \) or \( \text{codim} \Delta = 1 \).

For any subspace \( \Delta \subset V \), denote by \( \mathcal{C}_\Delta \) the full subcategory \( \mathfrak{sl}(V)\text{-Mod} \) consisting of modules which are free of rank 1 when restricted to \( U(\mathfrak{n}_\Delta) \).

When \( \Delta \subset V \) is a one-dimensional subspace, we may fix a basis \( v_1 \ldots v_n \) of \( V \) such that \( v_n \in \Delta \). We write \( \Delta^{\perp} \) for the subspace spanned by \( v_1, \ldots, v_{n-1} \). This choice of basis lets us identify \( \mathfrak{sl}(V) = \mathfrak{sl}_n \) which gives

\[
\mathfrak{p}_\Delta = \begin{bmatrix} * & * & \cdots & * & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}, \quad (\mathfrak{p}_\Delta)^\perp = \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}
\]

\[
\mathfrak{n}_\Delta = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 \end{bmatrix}, \quad (\mathfrak{n}_\Delta)^\perp = \begin{bmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\]

**Lemma 4.** There are equivalences of categories \( \mathcal{C}_\Delta \simeq \mathcal{C}_{\Delta'} \) for any pairs of subspaces \( \Delta \) and \( \Delta' \) of either dimension or codimension 1.

**Proof.** First assume that \( \dim \Delta = \dim \Delta' \). Then we may pick an invertible \( S \) that bijectively maps \( \Delta \) to \( \Delta' \). But then the automorphism \( \varphi : x \mapsto SxS^{-1} \) maps \( \mathfrak{n}_\Delta \) to \( \mathfrak{n}_{\Delta'} \). And therefore, if \( M \) is a module free over \( \mathfrak{n}_{\Delta'} \), then the twisted module \( \varphi M \) (in which the action is \( x \cdot m := \varphi(x) \cdot m \)) will be free over \( \mathfrak{n}_\Delta \). Similarly, we
note that if dim \( \Delta = 1 \) the category \( \mathcal{C}_\Delta \) is equivalent to \( \mathcal{C}_{\Delta^\perp} \) by twisting by the outer isomorphism \( x \mapsto -x^T \) (minus transpose). Finally, by combining the above statements we see that if \( \dim \Delta = 1 \) and \( \text{codim} \Delta' = 1 \) we have \( \mathcal{C}_\Delta \simeq \mathcal{C}_{\Delta^\perp} \simeq \mathcal{C}_{\Delta'} \). □

Thus we shall restrict our focus to modules which are free of rank 1 over the universal enveloping algebra of the fixed subalgebra \( n_{\Delta^\perp} \) of \( \mathfrak{sl}_n \) as described above. All other \( \mathfrak{sl}(V) \)-modules free over the nilradical of a maximal-dimensional parabolic can be obtained from these by twisting by automorphisms.

3. \( \mathfrak{sl}_2 \)-modules

We treat the case \( \dim V = 2 \) separately, because we obtain more extensive results and nicer formulas in this setting. We fix the standard basis \( \{ x, y, h \} \) for \( \mathfrak{sl}_2(\mathbb{K}) \). These elements satisfy \([h, x] = 2x, [h, y] = -2y, [x, y] = h\). When \( \dim V = 2 \) the only parabolic subalgebras of \( \mathfrak{sl}(V) \) are Borel-subalgebras \( \mathfrak{p} = h \oplus n \), where the corresponding Cartan and nilradical subalgebras both are 1-dimensional. By picking a basis \( (v_1, v_2) \) for \( V \) where \( v_1 \in h \) and \( v_2 \in n \), we obtain an identification \( \mathfrak{sl}(V) = \mathfrak{sl}_2, \mathfrak{p} = \text{span}(h, x), \text{and} \ n = \text{span}(x) \). Since \( U(n) = \mathbb{K}[x] \), our classification problem reduces to describing all possible \( \mathfrak{sl}_2 \)-module structures on \( \mathbb{K}[x] \) in which we have \( x \cdot f(x) = xf(x) \).

We start by defining some such modules.

**Proposition 5.** Fix a polynomial \( p(x) \in \mathbb{K}[x] \) and define a second polynomial

\[
q(x) := -\frac{1}{2x} \int_0^x \left( p(t)p'(t) + tp''(t) \right) dt.
\]

Then the following \( \mathfrak{sl}_2 \)-action defines an \( \mathfrak{sl}_2 \)-module structure on \( \mathbb{K}[x] \):

\[
\begin{align*}
x \cdot f(x) &= xf(x), \\
h \cdot f(x) &= p(x)f(x) + 2xf'(x), \\
y \cdot f(x) &= q(x)f(x) - p(x)f'(x) - xf''(x).
\end{align*}
\]

We denote this module by \( V(p) \).

**Proof.** We verify that the above action respects the \( \mathfrak{sl}_2 \) structure. We have

\[
\begin{align*}
x \cdot y \cdot f - y \cdot x \cdot f &= x \cdot (qf - pf' - xf'') - y \cdot (xf) \\
&= x(qf - pf' - xf'') - (qf(x) - p(f + xf')) - x(2f' + xf'') \\
&= pf + 2xf' = h \cdot f = [x, y] \cdot f,
\end{align*}
\]

and

\[
\begin{align*}
h \cdot x \cdot f - x \cdot h \cdot f &= h \cdot (xf) - x \cdot (pf + 2xf') \\
&= p(xf) + 2x(f + xf') - x(pf + 2xf') = 2xf = [h, x] \cdot f,
\end{align*}
\]
and
\[ h \cdot y \cdot f - y \cdot h \cdot f = h \cdot (q f - p f' - xf'') - y \cdot (p f + 2xf') \]
\[ = p(qf - pf' - xf'') + 2x(qf' + qf' - pf' - pf'' - f'' - xf''') \]
\[ - q(pf + 2xf') + p(p f' + 2f' + 2xf'') \]
\[ + x(pf'' + pf' + pf' + pf'' + 4f'' + 2xf'') \]
\[ = (2xq' + 2q + pp' + xp'')f - 2(qf - pf' - xf''') \]
\[ = (2 \frac{\partial}{\partial x}(xq) + pp' + xp'')f - 2(qf - pf' - xf''') \]
\[ = (pp' + xp'' - pp' - xp'')f - 2(qf - pf' - xf'') = -2y \cdot f = [h, y] \cdot f. \]

\[ \square \]

It turns out that the modules \( V(p) \) defined above are pairwise non-isomorphic and exhaust all modules whose restriction to \( \mathbb{K}[x] \) is free of rank 1.

**Proposition 6.** \( V(p) \simeq V(\overline{p}) \) if and only if \( p = \overline{p} \).

**Proof.** Let \( \varphi : V(p) \to V(\overline{p}) \) be an isomorphism. Then \( \varphi(f) = f \varphi(1) \) so \( \varphi(1) \) is a nonzero constant. The relation \( \varphi(h \cdot f) = h \cdot \varphi(f) \) is equivalent to the condition \( \varphi(1)(p - \overline{p}) = 0 \) so \( p = \overline{p} \).

**Proposition 7.** Any \( M \in \mathfrak{s}_2\text{-Mod} \) such that \( \text{Res}_{\mathfrak{g}[x]}^{U(\mathfrak{sl}_2)} M \) is free of rank 1 is isomorphic to \( V(p) \) for some polynomial \( p \).

**Proof.** Let \( M = \mathbb{K}[x] \) with a given \( \mathfrak{s}_2\)-module structure such that \( x \cdot f(x) = xf(x) \). Define \( p(x) := h \cdot 1 \). We claim that this implies that \( h \cdot x^k = (p(x) + 2k)x^k \). Indeed, it holds for \( k = 0 \), and by induction we have
\[ h \cdot x^{k+1} = h \cdot x \cdot x^k = x \cdot h \cdot x^k + [h, x] \cdot x^k = x(p(x) + 2k)x^k + 2x \cdot x^k \]
\[ = (p(x) + 2k)x^{k+1} + 2x^{k+1} = (p(x) + 2(k + 1))x^{k+1}. \]

Note that \( h \cdot x^k = (p(x) + 2k)x^k \) for all \( k \) can be written more compactly as \( h \cdot f = pf + 2xf' \) as in the definition of \( V(p) \) above.

Next, we define \( q(x) := y \cdot 1 \) and claim that this implies that \( y \cdot f = qf - pf' - xf'' \) as in the above definition. This equality is equivalent to \( y \cdot x^k = qx^k - kpx^{k-1} - k(k-1)x^{k-1} = (qx - kp - k(k-1))x^{k-1} \) for all \( k \). The latter statement can again be proved by induction: it holds trivially for \( k = 0 \) and we have
\[ y \cdot x^{k+1} = y \cdot x \cdot x^k = x \cdot y \cdot x^k + [y, x] \cdot x^k = x(y \cdot x^k) - h \cdot x^k \]
\[ = x((qx - kp - k(k-1))x^{k-1} - (p + 2k)x^k) = x((qx - k(k-1))x^{k-1} - (p + 2k)x^k) \]
\[ = (qx - kp - k(k-1) - p - 2k)x^k = (qx - (k+1)p - (k + 1)(k + 1) - 1)x^{(k+1)-1}. \]

This proves that the \( h \)- and \( y \)-action are completely determined by \( p \) and \( q \). It remains only to verify that \( q \) is uniquely determined by \( p \) as in the above definition.
For this we expand the equality \([h,y] \cdot f - (h \cdot y \cdot f - y \cdot h \cdot f) = 0\). Our previous considerations show that the left side expands as follows.

\[
0 = [h,y] \cdot f - (h \cdot y \cdot f - y \cdot h \cdot f) = -2y \cdot f - h \cdot (qf - pf' - xf'') + y \cdot (pf + 2xf')
\]

\[
= -2(qf - pf' - xf'') - p(qf - pf' - xf'') - 2x(qf + qf' - pf' - pf'' - f'' - xf^{(3)})
\]

\[
+ q(pf + 2xf') - p(pf'f + pf' + 2f' + 2xf'') - x(pf''f + 2f'pf' + pf''f + 4f''f + 2xf^{(3)})
\]

\[
= -2q + 2xf' + pp' + xp''f.
\]

This should hold for all \(f\), which implies that \(2q + 2xf' + pp' + xp''f = 0\). This can be rewritten \(\frac{\partial}{\partial x}(qf) = -\frac{1}{4}(pp' + xp'')\), which has the unique polynomial solution \(q = -\frac{1}{2} \int(pp' + xp'')dx\) (where the integration constant is forced to be zero). Thus \(q\) is determined by \(p\) and the module structure is just as in the above definition. \(\square\)

Next we investigate the simplicity of the modules \(V(p)\).

**Proposition 8.** The module \(V(p)\) is simple if and only if \(p(0) \not\in \mathbb{N}\). Otherwise \(V(p(x))\) has length 2 and we have a short exact sequence

\[
0 \to V(p(x) - 2p(0) + 2) \to V(p(x)) \to L(-p(0)) \to 0,
\]

where \(L(-p(0))\) is the simple highest weight module of highest weight \(-p(0) \in \mathbb{N}\).

**Proof.** Let \(S \subset V(p(x))\) be a proper nonzero submodule. We first claim that \(S\) is a homogeneous ideal of \(\mathbb{K}[x]\). This follows because \(d := \frac{1}{4}(h - p(x)) \in U(\mathfrak{sl}_2)\) is the degree operator, which acts by \(f \mapsto xf'(x)\), so by repeatedly acting by \((k - d)\) for different \(k \in \mathbb{N}\) we can reduce any element \(f\) to its lowest degree homogeneous component. Thus \(S = x^k \mathbb{K}[x]\) for some \(k > 0\). Then \(S \ni (q(x) - y) \cdot x^k = k(p(x) + (k - 1))x^{k-1}\), so \(x^k[k(p(x) + (k - 1))x^{k-1}\) and \((x)(p(x) + (k - 1))\), which in turn means that \(p(0) = 1 - k\). Thus if \(p(0) \not\in \mathbb{N}\), this is a contradiction so \(V(p(x))\) is simple. On the other hand, if \(p(0) = 1 - k \in \mathbb{N}\) then \(V(p(x))\) has a unique proper nontrivial submodule \(x^k \mathbb{K}[x]\), so \(V(p(x))\) has length 2. Finally we analyze the quotient \(V(p(x)) / (x^k)\) for \(p(0) = 1 - k \in \mathbb{N}\). This quotient is finite-dimensional and we have \(x \cdot x^{k-1} = 0\) and

\[
h \cdot x^{k-1} = (p(0) + 2(k - 1))x^{k-1} = (1 - k + 2(k - 1)x^{k-1} = (k - 1)x^{k-1}.
\]

so \(x^{k-1}\) is a highest weight vector, and the quotient is the simple highest weight module \(L(k - 1)\), and we recall that we had \(k - 1 = -p(0)\).

It remains only to verify that the submodule \(x^k \mathbb{K}[x]\) is isomorphic to \(V(p(x) - 2p(0) + 2)\). The submodule \(x^k \mathbb{K}[x]\) is free of rank 1 over \(\mathbb{K}[x]\) so by Proposition 7 we have \(x^k \mathbb{K}[x] \simeq V(\overline{p}(x))\) for some polynomial \(\overline{p}\). Any isomorphism \(\varphi : x^k \mathbb{K}[x] \to V(\overline{p}(x))\) must be a multiple of \(\varphi : V(\overline{p}(x)) \to V(\overline{p}(x))\) defined by \(\varphi(f) = x^k f\) since both modules are free over \(\mathbb{K}[x]\) and \(\varphi\) needs to be bijective map between \(V(\overline{p}(x))\) and the submodule \(x^k \mathbb{K}[x] \subset V(p(x))\). Since \(\varphi\) is an isomorphism we have

\[
x^k pf + 2kx^{k-1} + 2x^{k+1} f' = h \cdot \varphi(f) = \varphi(h \cdot f) = x^k \overline{p} f + 2x^{k+1},
\]

from which it follows that \(\overline{p}(x) = p(x) + 2k = p(x) + 2(1 - p(0))\). \(\square\)

Actually, the family of modules \(V(p(x))\) includes the lowest weight Verma-modules as seen below. Take \(p = \lambda \in \mathbb{K}\). Then \(q = 0\) and the action on the
basis \{x^k\} of the module \(V(\lambda)\) is given by
\[
x \cdot x^k = x^{k+1}, \\
h \cdot x^k = (\lambda + 2k)x^k, \\
y \cdot x^k = -k(\lambda + (k - 1))x^{k-1}.
\]
Thus each \(x^k\) is a weight vector of weight \(\lambda + 2k\), and \(V(\lambda)\) is a weight module of lowest weight \(\lambda\). As in the proposition, \(V(\lambda)\) is reducible precisely when \(p(0) = \lambda \in -\mathbb{N}\). The quotient is the unique simple highest weight module of highest weight \(-\lambda\).

3.1. Tensor product decomposition. In this section we shall give a formula for decomposing \(V(p(x)) \otimes E\) when \(E\) is a finite-dimensional and \(V(p(x))\) is simple.
Let \(L(k)\) be the unique simple \(\mathfrak{sl}_2\) module of dimension \(k + 1\). For natural numbers \(k \geq m\) we then have
\[
L(k) \otimes L(m) \simeq L(k + m) \oplus L(k + m - 2) \oplus \cdots \oplus L(k - m),
\]
which is known as the Clebsch-Gordan formula (see for example [Maz1]).
Recall that \(L(1)\) is isomorphic to the natural module; it has a basis \(\{e_1, e_2\}\) on which \(\mathfrak{sl}_2\) acts by \(e_{ij} \cdot e_k = \delta_{jk}e_i\). Then
\[
V(p) \otimes L(1) = \{(f, g) := f \otimes e_1 + g \otimes e_2 \mid f, g \in \mathbb{K}[x]\},
\]
and using Proposition 5 we see that the \(\mathfrak{sl}_2\) action on the tensor product is given by
\[
x \cdot (f, g) = (xf + g, xg), \\
h \cdot (f, g) = ((p + 1)f + 2xf', (p - 1)g + 2xg'), \\
y \cdot (f, g) = (qf - pf' - xf'', qg - pg' - xg'' + f).
\]

Lemma 9. For any polynomial \(p\) we have
\[
V(p) \otimes L(1) \simeq V(p - 1) \oplus V(p + 1).
\]

Proof. The action of \(\mathbb{K}[x]\) on the tensor product can be written \(r(x) \cdot (f, g) = (rf + r'g, rg)\), so any submodule isomorphic to \(\mathbb{K}[x]\) can be generated by a single element \((f, g)\). By taking \((f, g) := (\frac{1}{2x}(p - p(0)), 1)\) we get a \(\mathbb{K}[x]\)-submodule as follows: Define \(\varphi : \mathbb{K}[x] \to V(p) \otimes L(1)\) by
\[
\varphi(f) := \left(\frac{1}{2x}(p - p(0))f + f', f\right).
\]
Then
\[
\nabla := \operatorname{Im} \varphi = \{\varphi(f) = (\frac{1}{2x}(p - p(0))f + f', f) \in V(p) \otimes L(1) \mid f \in \mathbb{K}[x]\}
\]
is a \(\mathbb{K}[X]\)-submodule in which \(x \cdot \varphi(f) = \varphi(xf)\). We claim that \(\nabla\) in fact is an \(\mathfrak{sl}_2\)-submodule. Since \(\varphi(f) = f \cdot \varphi(1)\) it suffices to verify that \(h \cdot \varphi(1) \in \nabla\) and \(h \cdot \varphi(1) \in \nabla\). We have
\[
h \cdot \varphi(1) = h \cdot (\frac{1}{2x}(p - p(0)), 1) = ((p + 1)\frac{1}{2x}(p - p(0)) + 2x\frac{1}{2x}(p - p(0))'), p - 1) \\
= ((p + 1)\frac{1}{2x}(p - p(0)) - \frac{1}{x}(p - p(0)) + p', p - 1) \\
= ((p - 1)\frac{1}{2x}(p - p(0)) + p', p - 1) = \varphi(p - 1).
\]
Next we claim that $y \cdot \varphi(1) = \varphi(q + \frac{1}{2x}(p - p(0)))$. To see this first note that 

$$-2xq' = (-2xq)' + 2q = \frac{\partial}{\partial x} \left( \int_0^x (pp' + xp'')dt \right) + 2q = pp' + xp'' - p(0)p'(0) + 2q.$$ 

We now calculate 

$$(y \cdot \varphi(1)) - \varphi(q + \frac{1}{2x}(p - p(0))) =$$ 

$$= \left( (p - p(0)) \left( \frac{q}{2} + \frac{p'}{2} - \frac{1}{2x} \right) + \frac{p'}{2} (2 - p) - \frac{q}{2x} q + \frac{1}{2x}(p - p(0)) \right)$$ 

$$- \left( \frac{1}{2x} (p - p(0)) (q + \frac{1}{2x}(p - p(0))) + q' - \frac{1}{2x} (p - p(0)) + \frac{p'}{2}, q + \frac{1}{2x}(p - p(0)) \right).$$ 

In this difference the second component is indeed zero and in the first component we obtain the following after multiplying by $2x$: 

$$-2xq' + (p - p(0)) \left( \frac{q}{2} - \frac{1}{2x} \right) - pp' + p' - xp''.$$ 

and we need to show that this is zero too. Inserting our expression above for $-2xq'$ and multiplying again by $x$ we get 

$$2xq - xp(0)p'(0) + (p - p(0)) \left( \frac{\frac{q}{2}+p(0)}{2x} - 1 \right) + xp',$$ 

which is zero when $x = 0$. The derivative is 

$$-(pp' + xp'' - p(0)p'(x)) - (p(0)p'(0)) + p' \left( \frac{p(0)}{2x} - 1 \right) + (p - p(0)) \frac{p'}{x} + xp'' + p' = 0.$$ 

Hence we have shown that $y \cdot \varphi(1) = \varphi(q + \frac{1}{2x}(p - p(0))) \in \overline{V}$, and it follows that $

\overline{V}$ is an $\mathfrak{sl}_2$-submodule.

Next we define $\psi : \mathbb{K}[x] \rightarrow V(p) \otimes L(1)$ by 

$$\psi(f) = \left( \frac{1}{x} (p + p(0)) f + xf', xf \right)$$ 

so that 

$$\hat{V} := \text{Im} \psi = \{ \psi(f) = \left( \frac{1}{x} (p + p(0)) f + xf', xf \right) \in V(p) \otimes L(1) \mid f \in \mathbb{K}[x] \}.$$ 

We claim that $\hat{V}$ is a $\mathfrak{sl}_2$ submodule complementary to $\overline{V}$. Verification of this is analogous to the calculations above and we omit it here. We note however that $x \cdot \psi(f) = \psi(xf)$, $h \cdot \psi(1) = \psi(p + 1)$ and $y \cdot \psi(1) = \psi(q - \frac{1}{2x}(p - p(0)))$.

By Propositions 8 and 7 it follows from the facts that $h \cdot \varphi(1) = \varphi(p - 1)$ and $h \cdot \psi(1) = \psi(p + 1)$ that $\varphi$ is an isomorphism $V(p - 1) \rightarrow \overline{V}$ and that $\psi$ is an isomorphism $V(p + 1) \rightarrow \hat{V}$.

Finally we verify that $V(p) = \overline{V} \oplus \hat{V}$. Since 

$$\varphi(xf) - \psi(f) = \left( \frac{1}{x} (p - p(0)) f + (xf)', xf \right) - \left( \frac{1}{x} (p + p(0)) f + xf', xf \right) = (1 - p(0)) f, 0),$$ 

and since $p(0) \neq 1$ we have $(\mathbb{K}[x], 0) \subset \overline{V} \oplus \hat{V}$ and then clearly also $(0, \mathbb{K}[x]) \subset \overline{V} \oplus \hat{V}$ since we may form $\varphi(g) - \left( \frac{1}{x} (p - p(0)) g + g', 0 \right) = (0, g)$. Thus $V(p) \otimes L(1) = \overline{V} \oplus \hat{V}$.

Next, assume that $\varphi(g) = \psi(f)$, then $g = xf$ and $\left( \frac{1}{x} (p - p(0)) f + (xf)', xf \right) = \left( \frac{1}{x} (p + p(0)) f + xf', xf \right)$ so $\frac{1}{x} (p - p(0)) f + xf' + f = \frac{1}{x} (p + p(0)) f + xf'$ implying $(1 - p(0)) f = 0$. Since $p(0) \neq 1$ we get $f = g = 0$ showing that $\overline{V} \cap \hat{V} = \{0\}$. Thus $V(p) \otimes L(1) = \overline{V} \oplus \hat{V}$.

Since each finite-dimensional module $E$ is a direct sum of modules $L(k)$, the following proposition determines the tensor product decomposition of $V(p) \otimes E$ completely.
Proposition 10. We have
\[ V(p(x)) \otimes L(k) = \bigoplus_{i=0}^{k} V(p(x) + k - 2i). \]

Proof. We proceed by induction. The statement holds trivially for \( k = 0 \), and also for \( k = 1 \) by Lemma [1]. Using the inductive assumption and Lemma [1] we find that
\[
(V(p) \otimes L(k)) \otimes L(1) = \bigoplus_{i=0}^{k} (V(p + k - 2i) \otimes L(1))
\]
\[
= \bigoplus_{i=0}^{k} (V(p + (k + 1) - 2i) \oplus V(p + (k - 1) - 2i))
\]
\[
= \bigoplus_{i=0}^{k} V(p + (k + 1) - 2i) \oplus V(p - k - 1) \bigoplus_{i=1}^{k-1} V(p + (k - 1) - 2i)
\]
\[
= \bigoplus_{i=0}^{k+1} V(p + (k + 1) - 2i) \bigoplus_{i=0}^{k-1} V(p + (k - 1) - 2i).
\]
But on the other hand we can use the Clebsch-Gordan formula to obtain
\[
(V(p) \otimes L(k)) \otimes L(1) = V(p) \otimes (L(k) \otimes L(1)) = V(p) \otimes (L(k + 1) \oplus L(k - 1))
\]
\[
= (V(p) \otimes L(k + 1)) \bigoplus_{i=0}^{k-1} V(p + (k - 1) - 2i).
\]
By cancelling the isomorphic summands \( \bigoplus_{i=0}^{k-1} V(p + (k - 1) - 2i) \) in the two above expressions we finally get
\[
V(p) \otimes L(k + 1) = \bigoplus_{i=0}^{k+1} V(p(x) + (k + 1) - 2i),
\]
and the statement of the proposition follows by induction. \( \square \)

4. \( \mathfrak{sl}_{n+1} \)-modules

In this section we generalize most of the results of the previous section from \( \mathfrak{sl}_2 \) to \( \mathfrak{sl}_{n+1} \).

4.1. Preliminaries. For \( 1 \leq i, j \leq n+1 \), let \( e_{ij} \) be the standard basis for \( \mathfrak{sl}_{n+1} \), and recall that these satisfy \( [e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{jk} \). Let \( \mathfrak{n} \subset \mathfrak{sl}_{n+1} \) be the subalgebra with basis \( \{ e_{i,n+1} \mid 1 \leq i \leq n \} \). Since \( \mathfrak{n} \) is abelian we have \( U(\mathfrak{n}) \simeq \mathbb{K}[x_1, \ldots, x_n] \).

For \( f \in \mathbb{K}[x_1, \ldots, x_n] \) write \( f^i := \frac{\partial}{\partial x_i} f \) for the partial derivative, and define degree operators
\[
d_i, d : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n] \text{ by } d_i(f) = x_i f^i \text{ and } d(f) = \sum_{i=1}^{n} x_i f^i.
\]
Note that \( d_i \) and \( d \) are \( \mathbb{K} \)-linear derivations of \( \mathbb{K}[x_1, \ldots, x_n] \). We also note that \( d \) is invertible on the space of polynomials with zero constant term. We define maps \( d', d'' : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n] \) by
\[
d'(x_1^{a_1} \cdots x_n^{a_n}) = \frac{1}{\sum a_i} x_1^{a_1} \cdots x_n^{a_n} \text{ when } \sum a_i > 0, \text{ and } d'|_\mathbb{K} := \text{id}.
\]
and
\[ d'_i(x_1^{a_1} \cdots x_n^{a_n}) = \frac{1}{a_i} x_1^{a_1} \cdots x_n^{a_n} \text{ when } a_i > 0, \text{ and } d'_i(x_1^{a_1} \cdots x_n^{a_n}) = x_1^{a_1} \cdots x_n^{a_n} \text{ when } a_i = 0. \]

Then the following lemma is easy to verify.

**Lemma 11.** For all \( 1 \leq i, j \leq n \) we have
\[
d \circ d'(f) = f - f(0, \ldots, 0) = d' \circ d(f),
\]
\[
d_i \circ d'_j(f) = f - f(x_1, \ldots, 0_i, \ldots, x_n) = d'_i \circ d_j(f),
\]
\[
d(x_i, \frac{\partial}{\partial x_j}) = x_i \frac{\partial}{\partial x_j},
\]
\[
[x_i, d] = \frac{\partial}{\partial x_i}.
\]

For \( 1 \leq i \leq n \) define \( x_i := e_{i,n+1} \). Let \( I := \sum_{i=1}^{n+1} e_{ii} \) and for \( 1 \leq i \leq n + 1 \) define \( h_i := e_{ii} - \frac{1}{n+1} I \) and \( \overline{h} = \sum_{i=1}^{n} h_i \). Note that \( h_{n+1} = -\sum_{i=1}^{n} h_i = -\overline{h} \).

We fix the basis \( \{ e_{ij} \mid 1 \leq i < j \leq n + 1; i \neq j \} \cup \{ h_1, \ldots, h_n \} \) for \( \mathfrak{sl}_{n+1} \) and note that for \( 1 \leq i, j \leq n \) and \( i \neq j \) we have
\[
[h_i, x_j] = \delta_{ij} x_j \text{ and } [x_i, e_{n+1,i}] = \overline{h} + h_i.
\]

The following lemma tells us how to commute elements of \( \mathfrak{sl}_{n+1} \) with the \( x_i \).

**Lemma 12.** The following relations hold in \( U(\mathfrak{sl}_{n+1}) \) for all \( 1 \leq i, j, k \leq n \) with \( i \neq j \) and for all \( m \in \mathbb{N} \).

1. \( x_i x_k^m = x_k^m x_i \)
2. \( h_i x_k^m = x_k^m h_i + \delta_{ik} mx^m \)
3. \( e_{ij} x_k^m = x_k^m e_{ij} + \delta_{kj} mx x_k^{m-1} \)
4. \( e_{n+1,i} x_j^m = x_j^m e_{n+1,i} - mx_j^{m-1} e_{ji} \)
5. \( e_{n+1,i} x_i^m = x_i^m e_{n+1,i} - mx_i^{m-1} (\overline{h} + h_i) - m(m - 1)x_i^{m-1} \)

**Proof.** These relations can be easily proved by induction on \( m \). We verify only Equation (5). It holds trivially for \( m = 0 \) and also for \( m = 1 \). Assuming it holds for a fixed \( m \), we have
\[
e_{n+1,i} x_i^{m+1} = (x_i^m e_{n+1,i} - mx_i^{m-1} (\overline{h} + h_i) - m(m - 1)x_i^{m-1}) x_i
= x_i^m (x_i e_{n+1,i} - (\overline{h} + h_i)) - mx_i^{m-1} (x_i (\overline{h} + h_i) + 2x_i) - m(m - 1)x_i^m
= x_i^{m+1} e_{n+1,i} - (m + 1)x_i^m (\overline{h} + h_i) - m(m + 1)x_i^m,
\]
where in the second equality we used Equations (2) and (3) for \( m = 1 \). Thus Equation (5) holds for all \( m \in \mathbb{N} \) by induction. \( \square \)
Corollary 13. Let \( f \in \mathbb{K}[x_1, \ldots, x_n] = U(n) \). The following relations hold in \( U(\mathfrak{sl}_{n+1}) \) for all \( 1 \leq i, j \leq n \) with \( i \neq j \).

\begin{align*}
(6) & \quad x_if = f x_i \\
(7) & \quad h_if = f h_i + x_if^i \\
(8) & \quad e_{ij}f = fe_{ij} + x_if^j \\
(9) & \quad e_{n+1,i}f = fe_{n+1,i} - \sum_{k \neq i} f^k e_{ki} - f^i(\bar{h} + h_i) - d(f^i)
\end{align*}

Proof.} Since the above formulas are linear in \( f \), it suffices to prove them when \( f \) is a monomial. We do by using Lemma \([12]\) repeatedly. Equations \((6)\)–\((7)\) follows easily from Equations \((1)\)–\((2)\). For \((8)\) we take \( f = \prod x_k^{a_k} \) and compute

\[
e_{ij}f = \sum_{k \neq j} x_j^a x_k^{a_k} = \sum_{k \neq j} x_j^a x_k^{a_k} \cdot (x_j^a e_{ij} + a_j x_i x_j^{a_j-1}) \prod_{k \neq j} x_k^{a_k} \\
= x_j^a \prod_{k \neq j} x_k^{a_k} e_{ij} + a_j x_i x_j^{a_j-1} \prod_{k \neq j} x_k^{a_k} = fe_{ij} + x_if^j
\]

where we used that \( e_{ij} \) commutes with \( x_k \) for \( k \neq j \). For \((9)\) we instead proceed by induction. The equation holds for \( f = 1 \) and assuming that the equation holds for a fixed monomial \( f \) it suffices to prove that it holds when we replace \( f \) by \( x_j f \). We divide this into cases: for \( j \neq i \) we have

\[
(x_j f)e_{n+1,i} - \sum_{k \neq i} (x_j f)^k e_{ki} - (x_j f)^i(\bar{h} + h_i) - d((x_j f)^i)
= x_j f e_{n+1,i} - (x_j \sum_{k \neq i} f^k e_{ki} + f e_{ji}) - x_j f^i(\bar{h} + h_i) - x_j (d(f^i) + f^i)
\]

And similarly, if we instead take \( j = i \) we have

\[
(x_i f)e_{n+1,i} - \sum_{k \neq i} (x_i f)^k e_{ki} - (x_i f)^i(\bar{h} + h_i) - d((x_i f)^i)
= x_i f e_{n+1,i} - x_i \sum_{k \neq i} f^k e_{ki} - (x_i f^i + f)(\bar{h} + h_i) - x_i (d(f^i) + f^i) - d(f)
\]

Therefore Equation \((9)\) holds by induction. \( \square \)

4.2. Classification. In this section we shall study all possible \( \mathfrak{sl}_{n+1} \)-modules \( M \) such that \( \text{Res}_{\mathfrak{U}(n)} U(\mathfrak{sl}_{n+1}) \) is free of rank 1.

Let \( M \) be a given \( \mathfrak{sl}_{n+1} \)-modules \( M \) such that \( \text{Res}_{\mathfrak{U}(n)} U(\mathfrak{sl}_{n+1}) \) is free of rank 1. Without loss of generality we may assume that \( M = \mathbb{K}[x_1, \ldots, x_n] \) as a vector space and that \( e_{i,n+1} f = x_i f \) for \( f \in M \). For \( 1 \leq i, j \leq n \) and \( i \neq j \), define \( p_{ij} := e_{ij} \cdot 1, p_{ii} := h_i \cdot 1 \) and \( q_i := e_{n+1,i} \cdot 1 \). Also let \( p := \sum_{i=1}^n p_{ii} \) and \( p := d'(\mathfrak{p}) \).
Lemma 14. The elements $p_{ij}, q_i \in \mathbb{K}[x_1, \ldots, x_n]$ uniquely determines the module structure on $M$. Explicitly, for $i \neq j$ we have

$$
x_i \cdot f = x_i f
$$

$$
h_i \cdot f = p_{ii} f + x_i f^i
$$

$$
e_{ij} \cdot f = p_{ij} f + x_i f^j
$$

$$
e_{n+1,i} \cdot f = q_i f - \sum_{k=1}^n p_{ki} f^k - \mathcal{P} f^i - d(f^i)
$$

$$
= q_i f - \sum_r (p_{ri} f^r + p_{rr} f^i + x_r f^{ir})
$$

where $\mathcal{P} = \sum_{i=1}^n p_{ii}$.

Proof. This immediately follows by acting on $1 \in \mathbb{K}[x_1, \ldots, x_n]$ in both sides of each equation of Corollary 13.

Next we determine what relations are required between polynomials $p_{ij}$ and $q_i$ in order that the action in Lemma 14 should define an $\mathfrak{sl}_{n+1}$-module structure on $\mathbb{K}[x_1, \ldots, x_n]$.

First, let $\mathfrak{p} \subset \mathfrak{sl}_{n+1}$ be the parabolic subalgebra spanned by $\{h_1, \ldots, h_n\}$ and all $e_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq n + 1$ where $i \neq j$.

First we note that

$$
h_i \cdot h_j \cdot f - h_j \cdot h_i \cdot f = h_i(p_{jj} f + x_j f^j) - h_j(p_{ii} f + x_i f^i)
$$

$$
= p_{ii}(p_{jj} f + x_j f^j) + x_i(p_{ij} f + p_{jj} f^i + x_j f^{ji})
$$

$$
- p_{jj}(p_{ii} f + x_i f^i) - x_j(p_{ij} f + p_{ii} f^j + x_i f^{ij})
$$

$$
= (x_i p_{ij} - x_j p_{ii}) f
$$

so the conditions that $h_i \cdot h_j \cdot f - h_j \cdot h_i \cdot f = [h_i, h_j] \cdot f = 0$ for all $f$ reduces to the conditions

$$
x_i p_{ij} = x_j p_{ii}
$$

(10)

Summing over $j$ we obtain $x_i \mathcal{P} = d(p_{ii})$ and applying $d'$ we have

$$
d'(x_i \mathcal{P}) = d'(d(p_{ii})) = p_{ii} - p_{ii}(0, \ldots, 0)
$$

Thus each $p_{ii}$ is determined up to addition of a constant by $\mathcal{P}$. We conclude that

$$
p_{ii} = d'(x_i \mathcal{P}) + c_i
$$

for some constants $c_i$.

Next, for $1 \leq i, j, k \leq n$ with $i \neq j$ we have

$$
h_k \cdot e_{ij} \cdot f - e_{ij} \cdot h_k \cdot f
$$

$$
= h_k(p_{ij} f + x_i f^j) - e_{ij}(p_{kk} f + x_k f^k)
$$

$$
= p_k(p_{ij} f + x_i f^j) + x_k(p_{ij} f + p_{ij} f^k + x_i f^{jk} + \delta_k f^j)
$$

$$
- p_{ij}(p_{kk} f + x_k f^k) - x_i(p_{kk} f + p_{kk} f^j + x_k f^{kj} + \delta_k f^k)
$$

$$
= (x_k p_{ij} - x_i p_{kk}) f + x_k \delta_k f^j - x_i \delta_k f^k .
$$

On the other hand,

$$
[h_k, e_{ij}] \cdot f = (\delta_k e_{kj} - \delta_k e_{ik}) \cdot f = (\delta_{ki} - \delta_{kj})e_{ij} \cdot f = (\delta_{ki} - \delta_{kj})(p_{ij} f + x_i f^j)
$$

.$$
So the conditions \([h_k, e_{ij}] \cdot f = h_k \cdot e_{ij} \cdot f - e_{ij} \cdot h_k \cdot f\) translates to the conditions

\[x_k p^k_{ij} - x_i p^i_{kk} = (\delta_{ki} - \delta_{kj})p_{ij} \quad 1 \leq i, j, k \leq n.\]

Note that by Equation (10), the above equality also holds when \(i = j\).

Now for \(1 \leq i, j, k, l \leq n\) where \(i \neq j\) and \(k \neq l\) we have

\[e_{ij} \cdot e_{kl} \cdot f - e_{kl} \cdot e_{ij} \cdot f = e_{ij} \cdot (p_{kl}f + x_k f^l) - e_{kl} \cdot (p_{ij}f + x_i f^j) \]

\[= p_{ij}(p_{kl}f + x_k f^l) + x_i(p_{lj}f + p_{kl}f^j + \delta_{kj}f^l + x_k f^{lj}) \]

\[- p_{kl}(p_{ij}f + x_i f^j) - x_k(p_{lj}f + p_{ij}f^l + \delta_{il}f^j + x_i f^{il}) \]

\[= (x_k p^k_{lj} - x_k p^k_{ij})f + \delta_{kj}x_i f^l - \delta_{il}x_k f^j \]

while

\[[e_{ij}, e_{kl}] \cdot f = \delta_{jk}e_{il} \cdot f - \delta_{il}e_{kj} \cdot f = \delta_{jk}(p_{il}f + x_i f^l) - \delta_{il}(p_{kj} + x_k f^j),\]

So the condition \([e_{ij}, e_{kl}] \cdot f = e_{ij} \cdot e_{kl} \cdot f - e_{kl} \cdot e_{ij} \cdot f\) for all \(f\) is equivalent to

\[(12) \quad x_k p^k_{lj} - x_k p^k_{ij} = \delta_{kj}p_{il} - \delta_{il}p_{kj} \]

Lemma 15. We have

\[q_i = -\frac{1}{x_i} \int \left(\sum_{r=1}^{n} (p_{ir} p_{rt} + x_i p^i_{tr} + p^i_{rt} p_{tr})\right) dx_i = -\frac{1}{x_i} d_i' \left(\int \sum_{r=1}^{n} (p^i_{ir} p_{ri} + x_i p^i_{ri} + p^i_{ri} p_{rr})\right) dx_i\]
Proof. Since \([h_k, e_{n+1,i}] \cdot f = h_k \cdot e_{n+1,i} \cdot f - e_{n+1,i} \cdot h_k \cdot f\) for all \(f\), we have

\[
0 = h_k \cdot (q_i f - \sum_r (p_{ri} f^r + p_{rr} f^i + x_r f^{ir}))
- e_{n+1,i} \cdot (p_{kk} f + x_k f^k) + \delta_k e_{n+1,i} \cdot f
= p_{kk} (q_i f - \sum_r (p_{ri} f^r + p_{rr} f^i + x_r f^{ir}))
+ x_k (q_i^k f + q_i f^k - \sum_r (p_{ki} f^{kr} + p_{ki} f^i + p_{rr} f^{ik} + \delta_k f^{ir} + x_k f^{ir}))
- q_i (p_{kk} f + x_k f^k) + \sum_r p_{ri} (p_{ki} f + p_{kk} f^r + \delta_k f^k + x_k f^{ki})
+ \sum_r p_{rr} (p_{ki} f + p_{kk} f^i + \delta_k f^i + x_k f^{ki})
+ \sum_r x_r (p_{ki} f^{ir} + p_{ki} f^i + p_{kk} f^{ir} + \delta_k f^{ki} + x_k f^{ir})
+ \delta_k (q_i f - \sum_r (p_{ri} f^r + p_{rr} f^i + x_r f^{ir}))
\]

\[
= \left( x_k q_i^k + \delta_k q_i + \sum_{r=1}^n (p_{ki} p_{ri} + x_r p_{kk}^i + p_{ki} p_{rr}) \right) f
+ \left( \sum_r (x_r p_{ki} - x_k p_{rr}) \right) f^i + \sum_r \left( x_r p_{ki} - x_k p_{ri} - \delta_k p_{ri} \right) f^r + p_{ki} f^k
= \left( x_k q_i^k + \delta_k q_i + \sum_{r=1}^n (p_{ki} p_{ri} + x_r p_{kk}^i + p_{ki} p_{rr}) \right) f
\]

where the last equality followed by using equations (10) and (11). Since the above equality holds for all \(f\) we have

\[
(13) \quad x_k q_i^k + \delta_k q_i + \sum_{r=1}^n (p_{ki} p_{ri} + x_r p_{kk}^i + p_{ki} p_{rr}) = 0
\]

Taking \(k = i\) we obtain

\[
x_i q_i^i + q_i + \sum_{r=1}^n (p_{ii} p_{ri} + x_r p_{ii}^r + p_{ii} p_{rr}) = 0
\]

\[
\Leftrightarrow \frac{\partial}{\partial x_i} (x_i q_i) = -\sum_{r=1}^n (p_{ir} p_{ri} + x_r p_{ii}^r + p_{ir} p_{rr})
\]

\[
\Leftrightarrow q_i = -\frac{1}{x_i} \int \sum_{r=1}^n (p_{ir} p_{ri} + x_r p_{ii}^r + p_{ir} p_{rr}) dx_i
= -\frac{1}{x_i} \frac{d}{dx_i} \left( x_i \sum_{r=1}^n (p_{ir} p_{ri} + x_r p_{ii}^r + p_{ir} p_{rr}) \right).
\]

Note that since \(q_i\) is a polynomial the integral above is well-defined since it needs to be divisible by \(x_i\). \(\square\)
Theorem 16. Fix a polynomial $p \in \mathbb{K}[x_1, \ldots, x_n]$ and for $1 \leq i, j \leq n$ define polynomials $p_{ij} := x_i \frac{\partial p}{\partial x_j} + \delta_{ij} p(0) / n$ and $q_i := -\frac{1}{x_i} \int \sum_{r=1}^n (p_i^r p_{ri} + x_r p_i^r r + p^r_{i r r}) dx_i$.

Then the following action defines an $\mathfrak{sl}_{n+1}$-module structure on the space $M(p) = \mathbb{K}[x_1, \ldots, x_n]$:

$$e_{i+n+1} \cdot f = x_i f$$
$$h_i \cdot f = \frac{1}{n+1} \sum_{j=1}^{n+1} e_{ij} \cdot f$$
$$e_{ij} \cdot f = f p_{ij} + x_i f^j$$
$$e_{n+1,i} \cdot f = q_i f - \sum_r (p_i^r f^r + p_{rr} f^i + x_r f^{ir})$$

where $h_i = e_{ii} - \frac{1}{n+1} \sum_{j=1}^{n+1} e_{jj}$.

Moreover, any $\mathfrak{sl}_{n+1}$-module $M$ for which $\text{Res}_U^{U(\mathfrak{sl}_{n+1})} M$ is free of rank 1 is isomorphic to $M(p)$ for a unique $p \in \mathbb{K}[x_1, \ldots, x_n]$.

Proof. We first verify that the definition in the theorem indeed gives an $\mathfrak{sl}_{n+1}$-module structure. Lemma 13 guarantees that for any $y \in \mathfrak{sl}_{n+1}$ we have $[y, x_k] \cdot f = y \cdot x_k \cdot f - x_k \cdot y \cdot f$.

By equations (10), (11), (12), and (15) we see that the relations

$$[h_i, h_j] \cdot f = h_i \cdot h_j \cdot f - h_j \cdot h_i \cdot f$$
$$[e_{ij}, h_k] \cdot f = e_{ij} \cdot h_k \cdot f - h_k \cdot e_{ij} \cdot f$$
$$[e_{ij}, e_{kl}] \cdot f = e_{ij} \cdot e_{kl} \cdot f - e_{kl} \cdot e_{ij} \cdot f$$

hold for all $f$ if and only if

$$x_i p_{kl}^l - x_k p_{ij}^l = \delta_{kj} p_{il} - \delta_{il} p_{kj}$$

for all $1 \leq i, j, k, l \leq n$. We verify these relations for $p_{ij} = x_i p^j + \delta_{ij} c$:

$$x_i p_{kl}^l - x_k p_{ij}^l = x_i (x_k p^l)^l - x_k (x_i p^l)^l$$

holds for all $f$ if and only if

$$x_i p_{kl}^l - x_k p_{ij}^l = \delta_{kj} p_{il} - \delta_{il} p_{kj}$$

for all $1 \leq i, j, k, l \leq n$. We verify these relations for $p_{ij} = x_i p^j + \delta_{ij} c$:

$$x_i p_{kl}^l - x_k p_{ij}^l = x_i (x_k p^l)^l - x_k (x_i p^l)^l$$

holds for all $f$ if and only if

$$x_i p_{kl}^l - x_k p_{ij}^l = \delta_{kj} p_{il} - \delta_{il} p_{kj}.$$
For $k = i$, this equation holds by the definition of $q_i$, so assume that $k \neq i$. Then we have

$$x_k q_i^k + \delta_{k,i} q_i + \sum_{r=1}^{n} (p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i) = 0$$

$$\Leftrightarrow d_k(d_i(-x_i q_i)) = d_i(x_i \sum_{r=1}^{n} (p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i))$$

$$\Leftrightarrow x_i \sum_{r=1}^{n} d_k(p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i)) = d_i(x_i \sum_{r=1}^{n} (p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i))$$

$$\Leftrightarrow \sum_{r=1}^{n} x_r \frac{\partial}{\partial x_k} (p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i)) = \frac{\partial}{\partial x_i} (x_i \sum_{r=1}^{n} (p_{ rr}^i + x_r p_{ kk}^i + p_{ kk}^i p_{ rr}^i))$$

Substituting $p_{ij} = x_i p_{ij} + \delta_{ij} p(0)/n$, the verification of the above equality reduces to a simple but long calculation which we omit here.

Finally we note that the conditions

$$[e_{n+1,i}, e_{n+1,j}] \cdot f = e_{n+1,i} \cdot e_{n+1,j} \cdot f - e_{n+1,j} \cdot e_{n+1,i} \cdot f$$

$$[e_{n+1,i}, e_{jk}] \cdot f = e_{n+1,i} \cdot e_{jk} \cdot f - e_{jk} \cdot e_{n+1,i} \cdot f$$

reduce to the two conditions

$$\sum_r (p_{rr} q_i^k + p_{ rr} q_i^i + x_r p_{ rr}^i) = 0$$

$$x_j q_i^k + \sum_r (p_{rr} p_{jk}^j + p_{ rr} p_{jk}^j + x_r p_{ rr}^j) = 0$$

which can be verified similarly.

Next we address the uniqueness claim of the theorem. Suppose that we are given an $\mathfrak{sl}_{n+1}$-module structure on $M = \mathbb{K}[x_1, \ldots, x_n]$ such that $e_{i,n+1} \cdot f = x_i f$ for $1 \leq i \leq n$. Define $p_{ij} := e_{ij} \cdot 1$ and $q_i := e_{n+1,i} \cdot 1$ as before. By Lemma 14 $M$ is determined by these polynomials up to isomorphism. We need to show that $M \simeq M(p)$ for some $p \in \mathbb{K}[x_1, \ldots, x_n]$.

Equation (10) says that $x_i p_{ij}^j = x_j p_{ij}^i$ for all $1 \leq i, j \leq n$. Summing over $j$ we obtain $x_i \overrightarrow{p} = d(p_{ii})$ and applying $d'$ we have

$$d'(x_i \overrightarrow{p}) = d' \circ d(p_{ii}) = p_{ii} - p_{ii}(0, \ldots, 0)$$

Thus each $p_{ii}$ is determined up to addition of a constant by $\overrightarrow{p}$. We conclude that

$$p_{ii} = d'(x_i \overrightarrow{p}) + c_i$$

for some constants $c_i$.

Next recall that by Equation 12 we had $x_i p_{kl}^j - k p_{ij}^l = \delta_{kj} p_{il} - \delta_{il} p_{kj}$. Taking $k = i$ and $j = l$ here we obtain

$$x_i p_{ij}^i - x_j p_{ij}^j = p_{ii} - p_{jj}.$$ (15)

Substituting $p_{ii} = d'(x_i \overrightarrow{p}) + c_i$ in (15) and considering the constant term it follows that $c_i = c_j$ for all $i, j$ and

$$p_{ii} = d'(x_i \overrightarrow{p}) + c = x_i \frac{\partial}{\partial x_i} d' (\overrightarrow{p}) + c.$$
Define \( p := d'(\overline{p}) \). Then \( p_{ii} = x_i p^j + c \), and \( \overline{p}(0) = \sum_{i=1}^n p_{ii}(0) = nc \), so

\[
c = \frac{\overline{p}(0)}{n} = \frac{d'(\overline{p})(0)}{n} = \frac{p(0)}{n}.
\]

Finally, recall that by (11) we had \( x_k p^k_{ij} - x_i p^k_{kk} = (\delta_{ki} - \delta_{kj})p_{ij} \). Summing over \( 1 \leq k \leq n \) in we get

\[
d(p_{ij}) - x_i \overline{p}^j = \left( \sum_k \delta_{ki} - \sum_k \delta_{kj} \right) p_{ij} = (1 - 1)p_{ij} = 0.
\]

Thus \( p_{ij} - p_{ij}(0) = d' \circ d(p_{ij}) = d'(x_i \overline{p}^j) \). Taking \( k = i \) in (11) we get \( x_i p^i_{ij} - x_i p^i_{ii} = p_{ij} \) which shows that \( p_{ij}(0) = 0 \), and we conclude that

\[ p_{ij} = d'(x_i \overline{p}^j) = x_i p^j. \]

Thus we have shown that the \( p_{ij} \) defined above are the same as those in \( M(p) \). Finally, by Lemma 15 each \( q_i \) is uniquely determined by the \( p_{ij} \). We have therefore proved that \( M \cong M(p) \), which shows that the \( sl_{n+1} \) modules of form \( M(p) \) exhaust the set of modules which are free when restricted to \( U(n) \).

4.3. Submodule structure.

**Proposition 17.** We have \( M(p) \simeq M(\tilde{p}) \) if and only if \( p = \tilde{p} \).

**Proof.** Let \( \varphi : M(p) \to M(\tilde{p}) \) be an isomorphism. Since \( \varphi(f) = \varphi(f \cdot 1) = f \cdot \varphi(1) \), \( \varphi(1) \) is a nonzero constant. For \( i \neq j \) we have \( \varphi(e_{ij} \cdot 1) = e_{ij} \cdot \varphi(1) \), which implies \( x_i p^j = x_i \tilde{p}^j \) and \( p^j = \tilde{p}^j \). Since this holds for each \( j \), \( p = \tilde{p} \) is a constant. Finally, \( \varphi(h_j \cdot 1) = h_j \cdot \varphi(1) \) implies \( x_j p^j + p(0)/n = x_j \tilde{p}^j + \tilde{p}(0)/n \) and in turn \( p(0) = \tilde{p}(0) \). Thus \( p = \tilde{p} \).

**Lemma 18.** Every submodule \( N \) of \( M(p) \) is a homogeneous ideal of form

\[ N = \{ f \in M(p) | \deg(f) \geq k \} \]

for some \( k \in \mathbb{N} \).

**Proof.** Let \( N \) be a submodule of \( M(p) \). Freeness of \( M(p) \) over \( \mathbb{K}[x_1, \ldots, x_n] \) implies that any submodule is an ideal. Note that \( D_i := (h_i - x_i p^i - p(0)/n) \in U(sl_{n+1}) \) acts as the \( i \)-degree operator on \( M(p) \), and \( n - D_i \) acts on \( f \) by killing all terms of \( f \) which has \( i \)-degree \( n \). Thus any \( f \in N \) can be reduced to each of its monomial terms. Next, let \( f \in N \). Then for \( i \neq j \), \( (e_{ij} - x_i p^j) \cdot f = x_i f^j \) which shows that \( f \) can be mapped to any polynomial of the same degree by a product of such elements in \( U(sl_{n+1}) \). Thus \( N \supset \{ g | d(g) = d(f) \} \), and since \( \mathbb{K}[x] \) acts freely on \( N \), we have \( N \supset \{ g | d(g) \geq d(f) \} \). Therefore if \( f \in N \) is taken to have minimal degree we see that \( N \) has the form stated in the lemma.

**Theorem 19.**

\( M(p) \) is a simple \( sl_{n+1} \)-module if and only if \( k := -\frac{n+1}{n} p(0, \ldots, 0) \notin \mathbb{N}_+ \). Otherwise, if \( k \in \mathbb{N}_+ \), the module \( M(p) \) has length 2 and the top of \( M \) is a simple highest weight module.

**Proof.** Let \( N \subset M(p) \) be a nonzero proper submodule, and \( f \in N \) be a monomial with \( \deg f \) minimal. Then

\[
(q_i - c_{n+1,i}) \cdot f = \sum_r (x_r p^i + \delta_{ri} p(0)/n)f^r + (x_r p^r + p(0)/n)f^i + x_r f^{ir}.
\]
Subtracting terms of degree $\geq \deg(f)$ we obtain

$$(p(0)/n)f^i + n(p(0)/n)f^i + d(f^i) = (\frac{n+1}{n}p(0) + \deg(f^i))f^i \in N.$$  

But by the minimality of $\deg(f)$, the coefficient of $f^i$ must be zero and we have $\deg(f^i) = -\frac{n+1}{n}p(0)$. This is a contradiction if $k \notin \mathbb{N}_+$, so $M(p)$ is simple in this case. Conversely, assume $k \in \mathbb{N}_+$ and let

$$W_k := \text{span}\{x_1^{a_1} \cdots x_n^{a_n} \in M(p) | \sum a_i \geq k\}.$$  

Then $W_k$ is invariant under the action of each $e_{n+1,i}$ by the above calculation. $W_k$ is also invariant under the remaining basis elements of $\mathfrak{sl}_{n+1}$ because $\deg(x_if^j) = \deg(f)$ for all $1 \leq i, j \leq n$. Thus $W_k$ is the unique submodule of $M(p)$. This means that $M(p)/W_k$ is simple finite-dimensional module, and therefore a weight module. We note that the image of $x_1^k$ in the quotient is a highest weight vector since it is annihilated by all $e_{ij}$ for $i < j$. Since

$$h_i \cdot x_1^k = \delta_{ik} (k + p(0)/n)x_1^k = \delta_{ik} n+2p(0)x_1^k,$$

we have $M(p)/W_k \cong L(\lambda)$ where $\lambda \in \mathfrak{h}^*$ is given by $\lambda(h_i) = \delta_{ik} \frac{n+2}{n}p(0)$. We note that $\dim M(p)/W_k = (k+n-2)/(k-1)$, the dimension of the space of polynomials of degree less than $k$.

\[\square\]

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Mathematical Sciences, Chalmers University of Technology, Sweden
E-mail address: jonathn@chalmers.se