Generalized Fock Spaces and New Forms of Quantum Statistics

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Abstract

The recent discoveries of new forms of quantum statistics require a close look at the underlying Fock space structure. This exercise becomes all the more important in order to provide a general classification scheme for various forms of statistics, and establish interconnections among them whenever it is possible. We formulate a theory of generalized Fock spaces, which has a three tiered structure consisting of Fock space, statistics and algebra. This general formalism unifies various forms of statistics and algebras, which were earlier considered to describe different systems. Besides, the formalism allows us to construct many new kinds of quantum statistics and the associated algebras of creation and destruction operators. Some of these are: orthostatistics, null statistics or statistics of frozen order, quantum group based statistics and its many avatars, and ‘doubly-infinite’ statistics. The emergence of new forms of quantum statistics for particles interacting with singular potential is also highlighted.
1. Introduction

In recent years, many new kinds of quantum statistics have been postulated. In spite of large literature which now exists, a unified picture for various statistics and their associated algebra has not emerged. The aim of the present work is to provide such a general formalism. This is achieved by introducing the concept of generalized Fock spaces. Starting with this basic notion, it is possible to show that more than one statistics can be postulated in a given Fock space, and many different algebraic realizations can be constructed for any particular statistics.

We introduce new forms of quantum statistics, *viz.* null, orthofermi and Hubbard statistics, and doubly-infinite statistics in the next section. The null statistics corresponds to a situation wherein no permutation is allowed and particles are frozen in their initial order. Orthofermi and Hubbard statistics satisfy an exclusion principle which is more exclusive than the Pauli’s exclusion principle: an orbital state shall not contain more than one particle irrespective of their spin directions. Such a situation arises when the Coulomb repulsion $U$ between two electrons occupying a same orbital state becomes infinity. The $U$ infinity model has been extensively used in the context of strongly correlated electron systems [1]. A deformation of the orthofermi algebra subsequently leads to doubly-infinite statistics.

The theory of generalized Fock spaces is formulated in Sec.3. The key element is the notion of independence of the permutation ordered states. The largest linear vector space constructed in this way is the super Fock space. The subsequent specification of a subset of states in this space as null states leads to many reduced Fock spaces. All these spaces are collectively called as generalized Fock spaces. We construct creation ($c^\dagger$), annihilation ($c$) and number ($N$) operators in the generalized Fock spaces. The creation and annihilation operators, even for a particular Fock space, are not unique. Consequently, many statistics and algebras can exist in a given Fock space. On the other hand, a universal representation for the number operator valid for all forms of statistics and algebra exists.

In quantum mechanical calculations, algebraic relations involving only $c$ and $c^\dagger$ are required. One does not explicitely need a $c\ c$ relation. However, given a $c\ c^\dagger$ expression, corresponding $c\ c$ relation can be obtained in an elegant manner. This is demonstrated in Sec.4, and based on this approach, fractional statistics in one dimension is constructed. Sec.5 is devoted to summary and conclusions.
2. New Forms of Quantum Statistics

Newer forms of quantum statistics have been constructed by deforming the canonical commutation relations. For example, the deformation

\[ [c_j, c_k^\dagger] = \delta_{jk} \rightarrow [c_j, c_k^\dagger]_q = c_j c_k^\dagger + q c_k^\dagger c_j = \delta_{jk} ; \quad -1 < q < 1 \]

(1)
gives rise to infinite statistics [2,3]. Here no \( cc \) relation exists, and all permuted states are linearly independent.

As a counterpoint to infinite statistics, null statistics can also be constructed [4]. As mentioned earlier, no permutation is allowed in the null statistics. The defining algebra for this statistics is :

\[ c_k c_j^\dagger = 0 \text{ for } k \neq j ; \quad c_j c_j^\dagger = 1 - \sum_{k<j} c_k^\dagger c_k ; \quad c_i c_j = 0 \text{ for } i < j \]

(2)

Singular interparticle interactions also lead to new kinds of statistics. When in addition to Pauli’s exclusion principle, an infinite repulsion exists between two particles occupying the same orbital state (k or m) but having different spin directions (\( \alpha \) and \( \beta \)), we have

\[ c_{k\alpha} c_{k\beta} = 0 \]

(3)

For usual fermions, the above relation is valid only when \( \alpha = \beta \). Consistent with the above exclusion principle, two different statistics, \( \text{viz.} \) orthofermi statistics

\[ c_{ka} c_{m\beta}^\dagger + \delta_{\alpha\beta} \sum_{\gamma} c_{m\gamma}^\dagger c_{k\gamma} = \delta_{km} \delta_{\alpha\beta} \quad ; \quad c_{ka} c_{m\beta} + c_{ma} c_{k\beta} = 0 \]

(4)

and Hubbard statistics

\[ \left\{ \begin{array}{l}
  c_{ka} c_{m\beta}^\dagger + (1 - \delta_{km}) c_{m\beta}^\dagger c_{ka} = \delta_{km} \delta_{\alpha\beta} \left( 1 - \sum_{\gamma} c_{k\gamma}^\dagger c_{k\gamma} \right) \\
  c_{ka} c_{m\beta} + (1 - \delta_{km}) c_{m\beta} c_{ka} = 0
\end{array} \right. \]  

(5)
can be postulated [5]. The orthofermi statistics is formulated in a representation invariant manner. The Hubbard statistics is not invariant under unitary transformation, and it depends on the representation.

The algebra for the orthobose statistics is obtained by replacing the positive signs by negative signs in Eq.(4).

Usually states of a system are characterized by a set of individual indices describing position, spin, internal degrees of freedom etc.. These are then mapped to a set of single
indices. The symmetry properties are then postulated with respect to these composite indices. As a result, symmetries with respect to the exchange of individual indices get correlated. For the ortho and Hubbard statistics, spatial and spin indices can not be mapped to a single composite index. But for Hubbard statistics, exchange between $k\alpha$ and $m\beta$ is still permissible. In orthostatistics, exchanges between $k$, $m$ and between $\alpha$, $\beta$ are uncorrelated. The former exchange leads to fermi or bose statistics, where as the later satisfies infinite statistics.

A deformation of $c c^\dagger$ algebra for the orthostatistics, that is,

$$c_{k\alpha}c_{m\beta}^\dagger + q\delta_{\alpha\beta} \sum_{\gamma} c_{m\gamma}^\dagger c_{k\gamma} = \delta_{km}\delta_{\alpha\beta} = -1 < q < 1 \quad (6)$$

gives rise to doubly-infinite statistics; one with respect to the pair of indices $(k, m)$ and the other for the pair $(\alpha, \beta)$.

3. Generalized Fock Spaces

Given a set of quantum numbers $g, h, i...$ with the respective occupancy being $n_g, n_h, n_i...$, all possible multiparticle state vectors are

$$|n_g, n_h \ldots n_m; \mu\rangle, \mu = 1, 2 \ldots s \quad (7)$$

where $s$ is the total number of distinct permutations and $\mu$ labels each of these permuted states. we assume the existence of a unique vacuum state

$$|0\rangle \equiv |0, 0, 0 \ldots 0\rangle \quad (8)$$

All these states are linearly independent, but need not be orthogonal or normalized

$$\langle n'_{g}, n'_{h} \ldots n'_{m}; \mu'|n_g, n_h \ldots n_m; \nu\rangle = \delta_{n'_{g}n_{g}} \delta_{n'_{h}n_{h}} \ldots \delta_{n'_{m}n_{m}} M_{\mu\nu} \quad (9)$$

with $M$ being a $s \times s$ hermitian matrix. We choose it to be positive definite.

From the set of linearly independent state vectors, an orthonormal set of vectors $\{|| n_g \ldots n_m; \mu \gg\}$ can be obtained

$$|| n_g \ldots n_m; \mu \gg = \sum_{\nu} X_{\nu\mu} |n_g \ldots n_m; \nu\rangle \quad (10)$$
Alternatively, starting with the orthonormal vectors \( \{ \| n_g \ldots n_m ; \mu \gg \} \), the vectors \( \{|n_g, n_h \ldots n_m; \mu \} \) can be constructed by taking the inverse of relation (10). \( X \) is a non-singular matrix. Although \( X \) is not unique and depends on the particular orthogonalization procedure, we have

\[
M^{-1} = XX^\dagger
\]  

Choosing a non-singular matrix \( X \) and determining the inner product matrix \( M \) as above will ensure the positivity of the matrix \( M \).

The set of state vectors considered here constitute super Fock space. Infinite statistics resides in this Fock space. Using the projection operator

\[
P(n_g \ldots n_k \ldots n_m) = \sum_{\lambda, \nu} |n_g \ldots n_m; \nu\rangle (M^{-1})_{\nu \lambda} \langle n_g \ldots n_m; \lambda|
\]

the number operator can be written as

\[
N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k P(n_g \ldots n_k \ldots n_m)
\]

which satisfies the following properties

\[
N_k |n_g \ldots n_k \ldots n_m; \mu\rangle = n_k |n_g \ldots n_k \ldots n_m; \mu\rangle ; \quad [N_k, N_j]_\pm = 0
\]

The creation operator is defined as

\[
c_j^\dagger = \sum_{n_g \ldots n_j \ldots n_m} \sum_{\mu \nu} A_{\mu \nu} |1_j n_g \ldots n_j \ldots n_m; \mu\rangle \langle n_g \ldots n_j \ldots n_m; \nu|
\]

and \( c_j \) as the hermitian conjugate of \( c_j^\dagger \). \( A_{\mu \nu} \) are a set of arbitrary (complex) numbers. Even at this stage, it is possible to verify that

\[
[c_j^\dagger, N_k]_\pm = -c_j^\dagger \delta_{jk}
\]

The ordered state vectors can be constructed using a string of \( c_j^\dagger \) acting on the vacuum state. Consequently we also have

\[
\sum_{\nu} A_{\mu \nu} M_{\nu \lambda} = \delta_{\mu \lambda} ; \quad A = M^{-1}
\]  

We have provided here a unique representation of the number operator. But many different representations of creation and annihilation operators are possible through different choices of matrices \( A, X \) and \( M \). The number operator \( N \) can be expressed in terms of \( c_j^\dagger \)
and $c$ by solving Eqs.(12,13) and (15). Since $c^\dagger$ and $c$ are not uniquely defined many different expressions for $N$ in terms of $c^\dagger$, $c$ can be obtained.

Next we consider reduced Fock spaces. All known forms of statistics other than the infinite statistics reside in reduced Fock spaces, which are obtained by postulating relations like

$$\sum_\mu B_\mu^n |n_g, n_h \ldots ; \mu > 0 ; p = 1, 2, \ldots r$$

where $r < s$ and $B_\mu^n$ are constants. The vector space dimension in the sector $\{n_g, n_h \ldots\}$ is now reduced to $d = s - r$. The formalism developed for the super Fock space is also valid for reduced Fock spaces. But $\mu$ and $\nu$ now range from 1...$d$, and $X$, $M$ and $A$ are $d \times d$ matrices.

Arbitrariness in the matrix $A$ appearing in the creation operator expression (15) can be exploited to generate many relations involving $c^\dagger c^\dagger$. Therefore, many different forms of statistics specified by different $c^\dagger c^\dagger$ relations can be constructed in a given Fock space. All these statistics are interconnected. No connection exists between the statistics and the associated algebras residing in different Fock spaces. It may also be mentioned here that multiplicity of statistics are not possible in the super Fock space and in the Fock space of frozen order. Only infinite and null statistics respectively reside in these two Fock spaces. But even here, many algebras involving $c$ and $c^\dagger$ are possible. For example, depending on the different choice of the the inner product matrix $M$ in the super Fock space, new $c c^\dagger$ relations in addition to the one given in Eq.(1) are possible. Some of these are ($p > 1$ or $p < -1$):

$$c_i c_j^\dagger - q \delta_{ij} \sum_k c_k^\dagger c_k = \delta_{ij} ; \quad -1 < q < \infty$$

$$c_i c_j^\dagger - c_j^\dagger c_i = \delta_{ij} p^2 \sum_{k<i} N_k p^{N_i}$$

$$c_i c_j^\dagger - p^{-1} c_j^\dagger c_i = 0 \quad \text{for} \quad i \neq j ; \quad c_i c_i^\dagger - c_i^\dagger c_i = p^{N_i}$$

The $q$ in Eq.(1) can be made a complex number, provided the indices are ordered.

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j$$

For completeness this relation has to be supplemented with ($p$ real)

$$c_i c_i^\dagger - p c_i^\dagger c_i = 1$$

$p = |q|$ corresponds to infinite statistics when $|q| < 1$. $p = -1$ and $|q| < 1$ leads to infinite statistics with an exclusion principle.
Similarly, many algebraic relations can be obtained in the bosonic Fock space \((d = 1)\). Taking \(q\) and \(p\) to be complex numbers and \(\phi\) as any arbitrary function of number operator, a general \(c_i c_i^\dagger\) relation in this space is written as

\[
c_j c_j^\dagger - p c_j^\dagger c_j = |q|^2 \sum_{i<j} N_i f(N_j) ; \quad f(N_j) = \left| \frac{\phi(N_j)}{\phi(N_j + 1)} \right|^2 - p \left| \frac{\phi(N_j - 1)}{\phi(N_j)} \right|^2
\]  

Equation (24)

Most interestingly, the corresponding \(c_i c_i^\dagger\) relation for \(i < j\) is still given by Eq.(22). Thus replacing Eq.(23) by Eq.(24) takes us from super Fock space and infinite statistics to bosonic Fock space and deformed bose statistics satisfying the symmetry relation

\[
c_i^\dagger c_j^\dagger - q c_j c_i^\dagger = 0 \quad \text{for} \quad i > j
\]  

Equation (25)

Various limiting cases, like \(\phi\) being a constant, or \(p = 0\), or \(f = 1\) and \(p\) real are possible for Eq.(24). A particular interesting case is \(p = |q|^2\). The complete algebra (Eqs.(22,24,25)) now becomes covariant under \(SU_q(n)\) or quantum group transformation [6-8]. This shows that the algebra covariant under quantum group is a particular case of the more general algebra that can be derived from the formalism of generalized Fock spaces.

The canonical Bose statistics as well as the q-bose statistics given by relation (25) reside in the bosonic Fock space and are interconnected [8]. The underlying configuration space for the q-statistics is non-commutative. It has been shown that a complete Fock space realization of the differential calculus on a non-commutative space leads to a new concept of simultaneous transmutation between quanta satisfying different quantum statistics [9].

Restricting state occupancy to zero and one in the bosonic Fock space leads to the fermionic Fock space. Note that no restriction is placed on the symmetry properties of the state vectors. Consequently, it is possible to construct anticommuting bosons in the bosonic Fock space and commuting fermions in the fermionic Fock space. In addition, many distinct algebras can be obtained in the fermionic Fock space too.

A detailed list of statistics and algebras in various Fock spaces corresponding to single, and two-indexed systems (e.g., orthofermi, and particles obeying ‘doubly-infinite’ statistics) are provided in reference [8].
4. \textit{cc} relations from \textit{cc$^\dagger$} algebra

A general form of \textit{cc$^\dagger$} algebra which allows us to to calculate vacuum matrix element of any polynomial in \textit{c} and \textit{c$^\dagger$} is given as

\[
c_i c_j^{\dagger} = A_{ij} + \sum_{k,m} B_{ijkm} c_k^{\dagger} c_m
\]  

(26)

where \(A_{ij}\) and \(B_{ijkm}\) are constants or functions of number operators.

Symmetry of particles under exchange is obtained by making the operator \(Q_{ij} \equiv c_i c_j - q c_j c_i\) a null operator (\(Q_{ij} = 0\)). This can be achieved if it can be shown that

\[
Q_{ij} c_k^{\dagger} = \sum_{i'j'k'} F_{ijk;i'j'k'} c_k^{\dagger} Q_{ij'}
\]

(27)

for all \(i, j\) and \(k\) where \(F_{ijk;i'j'k'}\) may be a \(c\)-number or operator. The successive applications of the above equation over any string of creation operators, and then allowing both side of the resulting expression to act on the vacuum state \(|0\rangle\) finally leads to the operator identity \(Q_{ij} = c_i c_j - q c_j c_i = 0\), which is the \textit{cc} relation sought after.

Employing the above methodology with the \textit{cc$^\dagger$} relations given in Eqs.(22,23), we can show that fractional statistics with (without) exclusion principle occurs in one dimension when \(q = e^{i\theta}\) and \(p = -1\) (\(\neq -1\)) [10]. For other values of \(q\), no \textit{cc} relation exists. This provides an analytical method to prove the absence of a \textit{cc} relation for infinite statistics.

5. Summary and Conclusions

By decoupling the notion of the underlying Fock space from \textit{c} and \textit{c$^\dagger$}, we are able to define different forms of statistics in a representation independent manner. Subsequently, one can construct creation, annihilation operators and their algebra in any desired representation.

The general formalism not only unifies and classifies various forms of quantum statistics, but also enables us to construct many new kinds of statistics and algebras for single and two-indexed systems in a systematic manner. Some of these are: (i) null statistics, (ii) orthostatistics, (iii) doubly-infinite statistics, (iv) complex \(q\) or fractional statistics in one dimension. Many \textit{cc$^\dagger$} algebras representing these statistics are also constructed. Besides, the notion of generalized Fock space leads to the concept of statistical transmutation in a quantum plane.
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