On $\tau$-Pseudo-$\nu$-Convex $\kappa$-Fold Symmetric Bi-Univalent Function Family

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Abstract: The object of this article is to explore a $\tau$-pseudo-$\nu$-convex $\kappa$-fold symmetric bi-univalent function family satisfying subordinations condition generalizing certain previously examined families. We originate the initial Taylor–Maclaurin coefficient estimates of functions in the defined family. The classical Fekete–Szegö inequalities for functions in the defined family of $\tau$-pseudo-$\nu$-convex family is also estimated. Furthermore, we present some of the special cases of the main results. Relevant connections with those in several earlier works are also pointed out. Our study in this paper is also motivated by the symmetry nature of $\kappa$-fold symmetric bi-univalent functions in the defined class.

Keywords: analytic functions; subordination; coefficient estimates; bi-univalent functions; $\kappa$-fold symmetric; Fekete–Szegö functional

MSC: 30C45; 30C50

1. Introduction

Let $\mathcal{D} = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ be the unit disc. The set of all analytic functions of the form

$$s(\zeta) = \zeta + \sum_{k=2}^{\infty} d_k \zeta^k.$$  \hspace{1cm} (1)

in $\mathcal{D}$ with normalization $s(0) = s'(0) - 1 = 0$ is denoted by $\mathcal{A}$. The subfamily of $\mathcal{A}$ which are univalent in $\mathcal{D}$ is symbolized by $\mathcal{S}$. For $\tau \geq 1$, we denote by $\mathcal{S}^\tau$ and $\mathcal{K}^\tau$, the family of $\tau$-pseudo-starlike functions and the family of $\tau$-pseudo-convex functions, respectively, where

$$\mathcal{S}^\tau = \left\{ s \in \mathcal{A} : \Re \left( \frac{\zeta s'(\zeta)}{s(\zeta)} \right)^\tau > 0, \quad \zeta \in \mathcal{D} \right\}$$

and

$$\mathcal{K}^\tau = \left\{ s \in \mathcal{A} : \Re \left( \frac{\left| s'(\zeta) \right|^\tau}{s(\zeta)} \right) > 0, \quad \zeta \in \mathcal{D} \right\}.$$ 

The family $\mathcal{S}^\tau$ was investigated by Babalola [1] and he proved that all $\tau$-pseudo-starlike are univalent in $\mathcal{D}$. The family $\mathcal{K}^\tau$ was defined by Guney and Murugusundaramoorthy [2]. In 1969, Mocanu [3] examined the family $\mathcal{M}(\nu)$ of $\nu$-convex functions $s \in \mathcal{A}$ satisfying

$$\Re \left( (1 - \nu) \frac{cs'(\zeta)}{s(\zeta)} + \nu \left| s'(\zeta) \right|^\nu \right) > 0, \quad (\nu \in \mathbb{R}, \zeta \in \mathcal{D}).$$

Clearly $\mathcal{M}(0) = \mathcal{S}^1 = \mathcal{S}$ and $\mathcal{M}(1) = \mathcal{K}^1 = \mathcal{K}$. It was shown in [4] that for all $\nu \in \mathbb{R}$, $\mathcal{M}(\nu) \subset \mathcal{S}^\nu \subset \mathcal{S}$.

In [5], the Koebe one-quarter theorem ensures that $s(\mathcal{D})$ contains a disc of radius $1/4$ for every function $s \in \mathcal{S}$. Hence, every function $s \in \mathcal{S}$ admits an inverse $g = s^{-1}$.
defined by \( s^{-1}(s(\zeta)) = \zeta, s(s^{-1}(\zeta)) = \zeta, |\zeta| < r_0(s), r_0(s) \geq 1/4 \), where \( \zeta, \zeta \in \mathbb{D} \) and a computation shows that \( g \) has the expansion of the form

\[
  g(\zeta) = s^{-1}(\zeta) = \zeta - d_2\zeta^2 + (2d_2^2 - d_3)\zeta^3 - (5d_2^3 - 5d_2d_3 + d_4)\zeta^4 + \cdots
\]  

A member \( s \) of \( A \) is called bi-univalent in \( \mathbb{D} \) if both \( s \) and \( s^{-1} \) are univalent in \( \mathbb{D} \). Let \( \sigma \) be the family of bi-univalent functions in \( \mathbb{D} \) given by (1). The systematic study of the family \( \sigma \) has its origin in a paper authored by Lewin [6], where coefficient-related investigations for elements of the family \( \sigma \) are examined. Lewin was the first to investigate the family \( \sigma \) and it was proved that \( |d_2| < 1.51 \) for members of the family \( \sigma \). Few years later, the estimation for \( |d_2| \) was further investigated by Brannan and Clunie [7] and they proved that \( |d_2| < \sqrt{2} \), if \( s \in \sigma \). In 1984, Tan [8] found initial coefficient estimates of functions in the family \( \sigma \). Brannan and Taha in [9], investigated bi-starlike and bi-convex functions, which are analogous to the concepts of starlike and convex functions. An investigation by Srivastava et al. [10] resurfaced the interest in the study of family \( \sigma \) and it opened the space for many thinkings in the topics of discussion of the paper. The trend in the last decade was to investigate coefficient-related non-sharp bounds for members of certain subfamilies of \( \sigma \) as it can be seen in papers [11–15].

An holomorphic function \( s \) in \( \mathbb{D} \) is said to be \( \kappa \)-fold symmetric if \( s(e^{2\pi i/\kappa}) = e^{2\pi i/\kappa}s(\zeta) \). For each \( \zeta \in \mathbb{S} \), the function \( s \) given by \( s(\zeta) = (f(\zeta))/\kappa \), \( \kappa \in \mathbb{N} \), is univalent and maps \( \mathbb{D} \) into a region with \( \kappa \)-fold symmetry. The class of \( \kappa \)-fold symmetric univalent functions in \( \mathbb{D} \) is symbolized by \( S_\kappa \). A function \( s \in S_\kappa \) has the form given by

\[
  s(\zeta) = \zeta + \sum_{k=1}^{\infty} d_k\zeta^{k+1} \quad (\kappa \in \mathbb{N}, \zeta \in \mathbb{D}).
\]  

Clearly \( S_1 = \mathbb{S} \).

Following the concept of \( S_\kappa, \kappa \in \mathbb{N} \), Srivastava et al. [16] examined the family \( \sigma_\kappa \) of \( \kappa \)-fold symmetric bi-univalent functions. They found some interesting results, such as the series for \( s^{-1} = g, s \in \sigma_\kappa \), which is as follows:

\[
  s^{-1}(\zeta) = g(\zeta) = \zeta - d_{k+1}\zeta^{k+1} + [(\kappa + 1)d_{k+1}^2 - d_{2k+1}]\zeta^{2k+1}
\]  

We obtain (2) from (4) on taking \( \kappa = 1 \). Note that \( \sigma_1 = \sigma \). Some examples in the class \( \sigma_\kappa \) are

\[
  \left[ \frac{1}{2} \log \left( \frac{1 + \zeta^\kappa}{1 - \zeta^\kappa} \right) \right]^{1/\kappa} , \left( \frac{\zeta^\kappa}{1 - \zeta^\kappa} \right)^{1/\kappa} , \left[ - \log (1 - \zeta^\kappa) \right]^{1/\kappa} , \ldots.
\]  

Inverse functions of them are as follows:

\[
  \left( \frac{e^{2\pi x} - 1}{e^{2\pi x} - 1} \right)^{1/\kappa} , \left( \frac{x^\kappa}{1 + x^\kappa} \right)^{1/\kappa} , \left( \frac{e^{2\pi x} - 1}{e^{2\pi x} - 1} \right)^{1/\kappa} , \ldots.
\]  

The momentum on investigations of functions in certain subfamilies of \( \sigma_\kappa \) was gained in recent years due to the paper [16] and it has led to a large number of papers on subfamilies of \( \sigma_\kappa \) [17–20]. Inspired by these works, many researchers have investigated several interesting subfamilies of \( \sigma_\kappa \) and found non-sharp estimates on initial coefficients and the Fekete–Szegő functional problem \(|d_{2k+1} - \delta d_{k+1}^2|, \delta \in \mathbb{R} \) [21] of functions belonging to these subfamilies (see, for example [22–25]) and this continued to appear in [26,27], showing the developments in the subject area of this paper.

Motivated by the works of [3,4,20,28], we define a \( \tau \)-pseudo-\( \nu \)-convex \( \kappa \)-fold symmetric bi-univalent function family \( \mathfrak{M}_\kappa^\tau(\eta, \nu, \varphi) \), in Section 2. We derive estimations for \( |d_{k+1}| \), \( |d_{2k+1}| \) and \( |d_{2k+1} - \delta d_{k+1}^2|, \delta \in \mathbb{R} \), for functions \( \mathfrak{M}_\kappa^\tau(\eta, \nu, \varphi) \). In Section 3, we investigate bounds on \( |d_{k+1}|, |d_{2k+1}| \) and \( |d_{2k+1} - \delta d_{k+1}^2|, \delta \in \mathbb{R} \), for functions \( \mathfrak{M}_\kappa^\tau(\eta, \nu, \varphi) = \)}
In Section 4, we initiate bounds on $|d_{k+1}|$, $|d_{2k+1}|$ and $|d_{2k+1} - d_{2k+1}^2| \ (\delta \in \mathbb{R})$, for functions $f \in X^C_{\sigma}(\eta, \nu, \xi) = M^C_{\psi}(\eta, \nu, \xi)$, $\phi \leq 1$. The results obtained are not sharp. We discuss several related families and indicate connections to earlier defined classes.

2. The Function Family $M^C_{\psi}(\eta, \nu, \phi)$

Throughout our investigations in the present paper, it is assumed that $\eta \in \mathbb{C}^* = \mathbb{C} - \{0\}, g(\varphi) = s^{-1}(\varphi)$ as stated in (4) and $\varphi(\xi)$ is an analytic function with $\Re(\varphi(\xi)) > 0 \ (\xi \in \mathbb{D})$ such that $\varphi(\mathbb{D})$ is symmetric with respect to the real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. The function $\varphi(\xi)$ has an expansion given by

$$\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \cdots \ (B_1 > 0).$$

We symbolize by $P$ the family of analytic functions having the series form $p(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots$, satisfying $\Re(p(\xi)) > 0 \ (\xi \in \mathbb{D})$. In view of Pommerenke [29], the $\kappa$-fold symmetric function $p \in P$ is of the form $p(\xi) = 1 + p_\kappa\xi + p_{2\kappa}\xi^{2\kappa} + p_{3\kappa}\xi^{3\kappa} + \cdots$.

Let $h(\xi)$ and $p(\varphi)$ be analytic in $\mathbb{D}$ with $h(0) = 0 = p(0)$ and $\max\{|h(\xi)|; |p(\xi)|\} < 1$. We suppose that $h(\xi) = h_\kappa\xi^\kappa + h_{2\kappa}\xi^{2\kappa} + h_{3\kappa}\xi^{3\kappa} + \cdots$ and $p(\varphi) = p_\kappa\varphi^\kappa + p_{2\kappa}\varphi^{2\kappa} + p_{3\kappa}\varphi^{3\kappa} + \cdots$. Additionally, we know that

$$|h_k| < 1; |h_{2k}| \leq 1 - |h_k|^2 ; |p_k| < 1; |p_{2k}| \leq 1 - |p_k|^2.$$

After simple calculations using (5), we get

$$\varphi(h(\xi)) = 1 + B_1\xi + B_2\xi^2 + \cdots \ (|\xi| < 1) \quad (7)$$

and

$$\varphi(p(\varphi)) = 1 + B_1p_\kappa \varphi^\kappa + B_2p_{2\kappa} \varphi^{2\kappa} + \cdots \ (|\varphi| < 1). \quad (8)$$

**Definition 1.** A function $s \in \sigma_{s}(\xi, \nu, \phi)$, $\nu > 0$, $\tau > 1$, if it fulfills the following subordination conditions:

$$\left[ 1 + \frac{1}{\eta} (1 - \nu) \left( \frac{c(s(\xi))}{s(\xi)} + v \left[ \frac{s'(s(\xi))}{s(\xi)} - 1 \right] \right) \right] \prec \varphi(\xi)$$

and

$$\left[ 1 + \frac{1}{\eta} (1 - \nu) \left( \frac{c(\varphi(\xi))}{\varphi(\xi)} + v \left[ \frac{\varphi'(\varphi(\xi))}{\varphi(\xi)} - 1 \right] \right) \right] \prec \varphi(\xi),$$

where $\xi, \varphi \in \mathbb{D}$.

For $\eta = 1$, a function in the class $M^C_{\psi}(1, \nu, \phi)$ is called $\tau$-pseudo-bi-$\nu$-convex $\kappa$-fold symmetric function of Ma-Minda type. For $\tau = 1$, a function in the class $M^C_{\psi}(1, \nu, \phi)$ is called $\tau$-pseudo-bi-Mocanu-convex function of Ma-Minda type of complex order $\eta$.

In [30], Mishra and Soren have illustrated that $s(\xi) = \xi + d_2\xi^2$ is univalent starlike function of order $\eta$, $0 \leq \eta < 1$, if $|d_2| \leq \frac{1 - \xi^2}{2(1 - \xi^2)}$. They have further shown that $s^{-1}(\omega) = \frac{1 + \sqrt{1 + 4d_2\omega}}{2d_2}$ is a univalent starlike function of order $\xi$. Therefore $g(\xi)$ is bi-starlike function of order $\eta$. On similar lines of [30], one can show that $s(\xi) = \xi + d_2\xi^2$ is bi-convex function of order $\xi$. Hence, the function $s \in M^C_{\psi}(1, \nu, \phi)$.

**Remark 1.** The family $H^C_{\psi}(\eta, \phi) \equiv M^C_{\psi}(\eta, 0, \phi), \tau > 1$, was examined by Aldawish et al. [24].

We observe that certain choices of $\tau$ and $\nu$ in $M^C_{\psi}(1, \nu, \phi)$ lead to the subfamilies $\mathcal{A}_{\psi}(\eta, \nu, \phi)$ and $\mathcal{N}_{\psi}(\eta, \nu, \phi)$, as given below:
(i). If we set $\tau = 1$ in the class $\mathcal{M}_{\kappa}^{r}(\eta, \nu, \varphi)$, then we get the subclass $\mathcal{A}_{\kappa}^{r}(\eta, \nu, \varphi) \equiv \mathcal{M}_{\kappa}^{1}(\eta, \nu, \varphi)$ of functions $s \in \sigma_{k}$ satisfying

$$
\left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{s'(\varsigma)}{s(\varsigma)} + \nu \frac{(s'(\varsigma))'}{s'(\varsigma)} - 1 \right) \right] < \varphi(\varsigma)
$$

and

$$
\left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{g'(\varsigma)}{g(\varsigma)} + \nu \frac{(g'(\varsigma))'}{g'(\varsigma)} - 1 \right) \right] < \varphi(\varsigma),
$$

where $\varsigma, \kappa \in \mathcal{D}$ and $\nu \geq 0$.

(ii). If we allow $\nu = 1$ in the class $\mathcal{M}_{\kappa}^{r}(\eta, \nu, \varphi)$, then we get the subclass $\mathcal{A}_{\kappa}^{r}(\eta, \varphi) \equiv \mathcal{M}_{\kappa}^{1}(\eta, 1, \varphi)$ of functions $s \in \sigma_{k}$ satisfying

$$
\left[ 1 + \frac{1}{\eta} \left( \left( \frac{(s'(\varsigma))'}{s'(\varsigma)} \right)^{1^{r}} - 1 \right) \right] < \varphi(\varsigma)\quad \text{and} \quad \left[ 1 + \frac{1}{\eta} \left( \left( \frac{(g'(\varsigma))'}{g'(\varsigma)} \right)^{1^{r}} - 1 \right) \right] < \varphi(\varsigma),
$$

where $\varsigma, \kappa \in \mathcal{D}$ and $\tau \geq 1$.

**Remark 2.** The family $\mathcal{M}_{\kappa}(v, \varphi) \equiv \mathcal{A}_{\kappa}(1, v, \varphi)$ was investigated by Tang et al. [20]. A function in the class $\mathcal{M}_{\kappa}(v, \varphi)$ is called bi-Mocanu-convex function of Ma-Minda type studied by Ali et al. [31].

**Theorem 1.** Let $\tau \geq 1, \nu \geq 0$ and $\delta \in \mathbb{R}$. If a function $s$ in $\mathcal{A}$ belongs to $\mathcal{M}_{\kappa}^{r}(\eta, \nu, \varphi)$, then

$$
|d_{\kappa+1}| \leq \frac{|\eta| B_{1} \sqrt{2B_{1}^{2}}}{\sqrt{\{M(\kappa + 1) + N(2 + \tau(\tau - 1)(\kappa + 1)^{2} - 2\tau(\kappa + 1))\} \eta B_{1}^{2} - 2L^{2}B_{2} + 2L^{2}B_{1}},
$$

$$
|d_{2\kappa+1}| \leq \frac{|\eta| B_{1} M}{\sqrt{\{M(\kappa + 1) + N(2 + \tau(\tau - 1)(\kappa + 1)^{2} - 2\tau(\kappa + 1))\} \eta B_{1}^{2} - 2L^{2}B_{2} + 2L^{2}B_{1}}},
$$

$$
|d_{2\kappa+1} - \delta d_{\kappa+1}^{2}| \leq \frac{|\eta| B_{1}^{2} |m+1-2\delta|}{\sqrt{\{M(\kappa + 1) + N(2 + \tau(\tau - 1)(\kappa + 1)^{2} - 2\tau(\kappa + 1))\} \eta B_{1}^{2} - 2L^{2}B_{2} + 2L^{2}B_{1}}},
$$

$$
J = \frac{\{M(\kappa + 1) + N(2 + \tau(\tau - 1)(\kappa + 1)^{2} - 2\tau(\kappa + 1))\} \eta B_{1}^{2} - 2L^{2}B_{2}}{\eta MB_{1}^{2}}.
$$

$$
L = (\kappa \nu + 1)(\tau(\kappa + 1) - 1),
$$

$$
M = (2\kappa v + 1)(2\kappa + 1 - 1)
$$

and

$$
N = \kappa(\kappa + 2)v + 1.
$$

**Proof.** Let the function $s$ in $\mathcal{A}$ belongs to $\mathcal{M}_{\kappa}^{r}(\eta, \nu, \varphi)$. Then there are holomorphic functions $b, p : \mathcal{D} \rightarrow \mathcal{D}$ with $b(0) = 0 = p(0)$ satisfying

$$
1 + \frac{1}{\eta} \left( (1 - \nu) \frac{s'(\varsigma)}{s(\varsigma)} + \nu \frac{(s'(\varsigma))'}{s'(\varsigma)} - 1 \right) = \varphi(b(\varsigma)),
$$

and

$$
1 + \frac{1}{\eta} \left( (1 - \nu) \frac{g'(\varsigma)}{g(\varsigma)} + \nu \frac{(g'(\varsigma))'}{g'(\varsigma)} - 1 \right) = \varphi(p(\varsigma)),
$$

where $\varsigma, \kappa \in \mathcal{D}$. 


Using (3) in (16) and (17), we obtain
\[
1 + \frac{1}{\eta} \left\{ Ld_{k+1} + \left[ M(d_{k+1}^2 - d_{2k+1}) + \left( 1 + \frac{\tau (\tau - 1)}{2} \right) (\kappa + 1)^2 - \tau (\kappa + 1) \right] N d_{k+1}^2 \right\} x^{2k + \cdots}
\] (18)
and
\[
1 + \frac{1}{\eta} \left\{ -Ld_{k+1} x^k + \left[ M((\kappa + 1)d_{k+1}^2 - d_{2k+1}) + \left( 1 + \frac{\tau (\tau - 1)}{2} \right) (\kappa + 1)^2 - \tau (\kappa + 1) \right] N d_{k+1}^2 \right\} x^{2k + \cdots}
\] (19)
where L, M and N are given by (13), (14) and (15), respectively.

Comparing (7) and (18), we get
\[
Ld_{k+1} = \eta B_1 h_k
\] (20)

\[
Md_{2k+1} + N \left( 1 + \frac{\tau (\tau - 1)}{2} (\kappa + 1)^2 - \tau (\kappa + 1) \right) d_{k+1}^2 = \eta [B_1 h_{2k} + B_2 p_{2k}],
\] (21)

Comparing (8) and (19), we obtain
\[
-Ld_{k+1} = \eta B_1 p_k
\] (22)

and
\[
M((\kappa + 1)d_{k+1}^2 - d_{2k+1}) + N \left( 1 + \frac{\tau (\tau - 1)}{2} (\kappa + 1)^2 - \tau (\kappa + 1) \right) d_{k+1}^2 = \eta [B_1 p_{2k} + B_2 p_{2k}^2].
\] (23)

From (20) and (22), we get
\[
h_k = -p_k
\] (24)
and
\[
2L^2 d_{k+1}^2 = \eta^2 B_1^2 (h_k^2 + p_k^2).
\] (25)

For finding the bound on \(|d_{k+1}|\), we add (21) and (23) and then we use (25) to obtain
\[
\left| \left\{ M((\kappa + 1) + N(2 + \tau (\tau - 1)(\kappa + 1)^2 - 2\tau (\kappa + 1))) \right\} \eta B_1^2 - 2L^2 B_2 \right| d_{k+1}^2 = \eta^2 B_1^3 (h_{2k} + p_{2k})
\] (26)

By using (6), (20) and (24) in (26) for the coefficients \(h_{2k}\) and \(p_{2k}\), we obtain
\[
\left| \left\{ M((\kappa + 1) + N(2 + \tau (\tau - 1)(\kappa + 1)^2 - 2\tau (\kappa + 1))) \right\} \eta B_1^2 - 2L^2 B_2 \right| + 2L^2 B_1 |d_{k+1}| \leq 2\eta^2 B_1^3.
\] (27)

Inequality (27) implies the assertion (9).

To obtain bound on \(|d_{2k+1}|\), we subtract (23) from (21):
\[
d_{2k+1} = \frac{\eta B_1 (h_{2k} - p_{2k})}{2M} + \left( \frac{\kappa + 1}{2} \right) d_{k+1}^2.
\] (28)

In view of (20), (24), (28) and applying (6), it follows that
\[
|d_{2k+1}| \leq \frac{|\eta B_1|}{M} + \left( \frac{\kappa + 1}{2} \right) \frac{L^2}{|\eta B_1 M|}
\]
\[
\left| \left\{ M((\kappa + 1) + N(2 + \tau (\tau - 1)(\kappa + 1)^2 - 2\tau (\kappa + 1))) \right\} \eta B_1^2 - 2L^2 B_2 \right| + 2L^2 B_1.
\] (29)

Inequality (29) gets the desired estimate (10).

From (26) and (28), for \(\delta \in \mathbb{R}\), we get
\[
d_{2k+1} - \delta d_{k+1}^2 = \frac{\eta B_1}{2} \left[ \left( T(\delta) + \frac{1}{M} \right) h_{2k} \left( T(\delta) - \frac{1}{M} \right) p_{2k} \right],
\]
where

\[ T(\delta) = \frac{\eta B_1^2 (\kappa + 1 - 2\delta)}{\{M(\kappa + 1)M + N(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(\kappa + 1))\} \eta B_1^2 - 2L^2 B_2^2}. \]

In view of (6), we conclude that

\[ |d_{2\alpha + 1} - \delta d_{\alpha + 1}| \leq \begin{cases} \frac{|\eta|B_1}{M} & ; 0 \leq |T(\delta)| < \frac{1}{M} \\ |\eta|B_1 |T(\delta)| & ; |T(\delta)| \geq \frac{1}{M} \end{cases}, \]

from which we obtain the desired assertion (11) with \( f \) as in (12). So the proof is completed. \( \square \)

**Remark 3.** (i). If we take \( \nu = 1 \) in Theorem 1, then we obtain Corollary 2 of Aldawish et al. [24]. (ii). If we set \( \nu = 1, \tau = 1 \) and \( \eta = 1 \) in Theorem 1, we get Corollaries 2.2 and 2.11 of [32]. Further, we get Corollaries 2.6 and 2.13 of [32], when \( \kappa = 1 \).

We obtain the results stated below, if we set \( \tau = 1 \) in Theorem 1.

**Corollary 1.** Let \( 0 \leq \nu < 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( \mathcal{A} \) belongs to \( \mathcal{A}_n(\eta, \nu, \varphi) \), then

\[ |d_{\alpha + 1}| \leq \frac{|\eta|B_1 \sqrt{B_1}}{\kappa \sqrt{|(\nu + 1)\eta B_1^2 - (\nu + 1)^2 B_2 + (\nu + 1)^2 B_1|}}, \]

\[ |d_{2\alpha + 1}| \leq \begin{cases} \frac{|\eta|B_1}{2(\nu + 1)} & ; 0 < B_1 < \frac{\kappa(\nu + 1)^2}{|\eta[(\nu + 1)2(\nu + 1)]^2 B_1^2|} \\ \frac{|\eta|^2 B_1^2 (\kappa + 1 - 2\delta)}{(\nu + 1)^2 B_2^2 (\kappa + 1)^2 B_2^2) ; |\kappa + 1 - 2\delta| \geq \frac{\kappa(\nu + 1)^2}{|\eta[(\nu + 1)2(\nu + 1)]^2 B_1^2|} \end{cases}. \]

**Remark 4.** (i). We obtain the bound on \( |d_{\alpha + 1}| \) stated in Theorems 5 and 6 of Tang et al. [20] from Corollary 1, if we allow \( \eta = 1 \). (ii). We also obtain Corollary 19 of [20] from Corollary 1, when \( \eta = 1 \) and \( \kappa = 1 \). Further for \( \delta = 1 \), we get Corollary 20 of [20].

We obtain the results stated below, if we set \( \nu = 1 \) in Theorem 1.

**Corollary 2.** Let \( \tau \geq 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( \mathcal{A} \) belongs to \( \mathcal{A}_n(\eta, \varphi) \), then

\[ |d_{\alpha + 1}| \leq \begin{cases} \frac{|\eta|B_1}{M} & ; 0 < B_1 < \frac{2L^2 B_1^2}{|\eta|M(\kappa + 1)} \\ \frac{|\eta|^2 B_1^2 (\kappa + 1 - 2\delta)}{|\eta|B_1^2 M(\kappa + 1) ; |\kappa + 1 - 2\delta| \geq \frac{2L^2 B_1^2}{|\eta|M(\kappa + 1)} \end{cases}. \]
where

\[
J_1 = \left| \frac{M_1(\kappa + 1) + N_1(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(\kappa + 1))}{\eta M_1 B_1^2} \right|,
\]

\[
L_1 = (\kappa + 1)(\tau(\kappa + 1) - 1),
\]

\[
M_1 = (2\kappa + 1)(\tau(2\kappa + 1) - 1)
\]

and

\[
N_1 = (\kappa + 1)^2.
\]

If \( \kappa = 1 \) in the above corollary, then we have

**Corollary 3.** Let \( \tau \geq 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \in \mathcal{A} \in \mathcal{F}(\eta, \varphi) \), then

\[
|d_2| \leq \frac{\vert \eta \vert B_1 \sqrt{B_1}}{\sqrt{|(8\tau^2 - 7\tau + 1)\eta B_1^2 - 4(2\tau - 1)^2 B_2| + 4(2\tau - 1)^2 B_1}},
\]

\[
|d_3| \leq \left\{ \begin{array}{ll}
\frac{\vert \eta \vert B_1}{\delta(3\tau - 1)} & ; |1 - \delta| < \frac{3}{2} |(8\tau^2 - 7\tau + 1)\eta B_1^2 - 4(2\tau - 1)^2 B_2| + 4(2\tau - 1)^2 B_1; B_1 \geq \frac{(2\tau - 1)^2}{\delta(3\tau - 1)} \\
\frac{|\eta|^2 B_1^2}{|\eta|^2 B_1^2 - 4(2\tau - 1)^2 B_2} & ; |1 - \delta| \geq \frac{3}{2} |(8\tau^2 - 7\tau + 1)\eta B_1^2 - 4(2\tau - 1)^2 B_2| + 4(2\tau - 1)^2 B_1
\end{array} \right.
\]

and

\[
|d_3 - \delta^2| \leq \left\{ \begin{array}{ll}
\frac{\vert \eta \vert B_1}{\delta(3\tau - 1)} & ; |1 - \delta| < \frac{3}{2} |(8\tau^2 - 7\tau + 1)\eta B_1^2 - 4(2\tau - 1)^2 B_2| + 4(2\tau - 1)^2 B_1; B_1 \geq \frac{(2\tau - 1)^2}{\delta(3\tau - 1)} \\
\frac{|\eta|^2 B_1^2}{|\eta|^2 B_1^2 - 4(2\tau - 1)^2 B_2} & ; |1 - \delta| \geq \frac{3}{2} |(8\tau^2 - 7\tau + 1)\eta B_1^2 - 4(2\tau - 1)^2 B_2| + 4(2\tau - 1)^2 B_1
\end{array} \right.
\]

Taking \( \kappa = 1 \) and \( \nu = 0 \) in Definition 1, we get the subfamily \( \mathcal{G}_{\tau, \varphi} \equiv \mathcal{M}_{\tau, \varphi}(\eta, 0, \varphi) \) of functions \( s \in \mathcal{A} \) satisfying

\[
\left[ 1 + \frac{1}{\eta} \left( \frac{(\zeta g'(\zeta))^\tau}{s(\zeta)} - 1 \right) \right] \prec \varphi(\zeta) \text{ and } \left[ 1 + \frac{1}{\eta} \left( \frac{\varphi'(\varphi')}{\varphi(\varphi')} - 1 \right) \right] \prec \varphi(\varphi),
\]

where \( \zeta, \varphi \in \mathcal{D} \) and \( \tau \geq 1 \).

**Corollary 4.** Let \( \tau \geq 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \in \mathcal{A} \in \mathcal{G}_{\tau, \varphi} \), then

\[
|d_2| \leq \frac{\vert \eta \vert B_1 \sqrt{B_1}}{\sqrt{|(2\tau - 1)\eta B_1^2 - (2\tau - 1)^2 B_2| + (2\tau - 1)^2 B_1}},
\]

\[
|d_3| \leq \left\{ \begin{array}{ll}
\frac{\vert \eta \vert B_1}{\delta(2\tau - 1)} & ; |1 - \delta| < \frac{3}{2} |(2\tau - 1)\eta B_1^2 - (2\tau - 1)^2 B_2| + (2\tau - 1)^2 B_1; B_1 \geq \frac{(2\tau - 1)^2}{\delta(2\tau - 1)} \\
\frac{|\eta|^2 B_1}{|\eta|^2 B_1^2 - (2\tau - 1)^2 B_2} & ; |1 - \delta| \geq \frac{3}{2} |(2\tau - 1)\eta B_1^2 - (2\tau - 1)^2 B_2| + (2\tau - 1)^2 B_1
\end{array} \right.
\]

and

\[
|d_3 - \delta^2| \leq \left\{ \begin{array}{ll}
\frac{\vert \eta \vert B_1}{\delta(2\tau - 1)} & ; |1 - \delta| < \frac{3}{2} |(2\tau - 1)\eta B_1^2 - (2\tau - 1)^2 B_2| + (2\tau - 1)^2 B_1; B_1 \geq \frac{(2\tau - 1)^2}{\delta(2\tau - 1)} \\
\frac{|\eta|^2 B_1^2}{|\eta|^2 B_1^2 - (2\tau - 1)^2 B_2} & ; |1 - \delta| \geq \frac{3}{2} |(2\tau - 1)\eta B_1^2 - (2\tau - 1)^2 B_2| + (2\tau - 1)^2 B_1
\end{array} \right.
\]
3. The Function Family $\mathcal{W}_{c_{\nu}}(\eta, \nu, q)$

Let $\varphi(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\nu} = 1 + 2q\xi + 2q^2\xi^2 + \cdots$. Then we have from Definition 1 the subclass $\mathcal{W}_{c_{\nu}}(\eta, \nu, q) = \mathcal{W}_{c_{\nu}}(\eta, \nu, \left(\frac{1+\xi}{1-\xi}\right)^{\nu})$ of functions $s \in \sigma_m$ satisfying

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2}$$

and

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2},$$

where $\zeta, \varphi \in \mathcal{D}, 0 < q \leq 1$, and $\tau \geq 1$.

**Remark 5.** The family $\mathcal{W}_{c_{\nu}}(\eta, q) = \mathcal{W}_{c_{\nu}}(\eta, 0, q), 0 < q \leq 1, \tau \geq 1$, was investigated by Aldawish et al. [24].

We observe that the choice $\tau = 1$ and $\nu = 1$ in $\mathcal{W}_{c_{\nu}}(\eta, \nu, q)$ lead to the subfamilies $\mathcal{R}_{c_{\nu}}(\eta, \nu, q)$ and $\mathcal{C}_{c_{\nu}}(\eta, q)$, respectively, as given below:

(i) $\mathcal{R}_{c_{\nu}}(\eta, \nu, q) \equiv \mathcal{W}_{c_{\nu}}(\eta, 0, q)$, is the family of $s \in \sigma_v$ satisfying

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2}$$

and

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2},$$

where $\zeta, \varphi \in \mathcal{D}, 0 < q \leq 1$ and $\nu \geq 0$.

(ii) $\mathcal{C}_{c_{\nu}}(\eta, q) \equiv \mathcal{W}_{c_{\nu}}(\eta, 1, q)$ is the family of $s \in \sigma_m$ satisfying

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2}$$

and

$$\left|\arg\left[1 + \frac{1}{\eta} \left((1 - \nu)\frac{c_s(\xi)}{s(\xi)} + \nu \frac{(cs(\xi))^\prime}{s(\xi)} - 1\right)\right]\right| < \frac{\varphi\pi}{2},$$

where $\zeta, \varphi \in \mathcal{D}, 0 < q \leq 1$ and $\tau \geq 1$.

**Remark 6.** We note that $\mathcal{R}_{c_{\nu}}(\eta, 1, q) \equiv \mathcal{C}_{c_{\nu}}(\eta, q) \equiv \mathcal{K}_{c_{\nu}}(\eta, \omega)$. The function class $\mathcal{K}_{c_{\nu}}(\eta, \omega)$ was considered by Kumar et al. [33]. The function family $\mathcal{R}_{c_{\nu}}(1, 0, q)$ coincides with the family of strongly bi-starlike functions of order $\omega$, which was studied by Branan and Taha [9].

If $\varphi(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\nu}$, then Theorem 1 reduce to the corollary stated below:

**Corollary 5.** Let $\tau \geq 1, 0 \leq \nu \leq 1, 0 < q \leq 1$ and $\delta \in \Re$. If a function $s \in A$ belongs to $\mathcal{W}_{c_{\nu}}(\eta, \nu, q)$, then

$$|d_{k+1}| \leq \frac{2|\eta|}{\sqrt{\nu}}\frac{\xi}{[M(k+1) + N(2 + \tau(\tau - 1)(k+1)^2 - 2\tau(k+1))]\eta - L^2 + L^2},$$

$$|d_{2k+1}| \leq \left\{\begin{array}{ll}
\frac{2|\eta|}{M} & ; 0 < q < \frac{L^2}{|\eta|(k+1)M} \\
\frac{2|\eta|}{M} + \left(k + 1 - \frac{L^2}{|\eta| e M}\right) \frac{2\eta^2 \nu^2}{\nu([M(k+1) + N(2 + \tau(\tau - 1)(k+1)^2 - 2\tau(k+1))]\eta - L^2 + L^2)} & ; 0 < q \geq \frac{L^2}{|\eta|(k+1)M},
\end{array}\right.$$

and

$$|d_{2k+1} - \delta d_{2k+1}^2| \leq \left\{\begin{array}{ll}
\frac{2|\eta|}{M} & ; |k + 1 - 2\delta| < f_2 \\
\frac{2|\eta|}{M} e^{2|\eta|\delta} & ; |k + 1 - 2\delta| \geq f_2,
\end{array}\right.$$
where
\[
J_2 = \left| \frac{M_1(k+1) + N_1(2 + \tau(t - 1)(k+1)^2 - 2\tau(k+1))}{\eta M^2} \right|.
\]

L, M, and N are as in (13), (14) and (15), respectively.

**Remark 7.** (a) If we take \( \nu = 0 \) in Corollary 5, then we obtain Corollary 5 of Aldawish et al. [24].
(b) (i) The bound on \( |d_k+1| \) stated in Corollary 6 of [34] is got, if we set \( \nu = 0, \tau = 1 \) and \( \eta = 1 \) in Corollary 5. (ii) Our result on \( |d_{2k+1}| \) is better than the bound stated in Corollary 6 of [34], in terms of \( \delta \) ranges as well as the bounds, if \( \nu = 0, \tau = 1 \) and \( \eta = 1 \) in Corollary 5.

Setting \( \tau = 1 \), Corollary 5 reduce to the next corollary.

**Corollary 6.** Let \( 0 \leq \nu < 1, 0 < \delta \leq 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( A \) belongs to \( \mathcal{C}_\nu^\ell(\eta, \nu, \delta) \), then
\[
|d_{k+1}| \leq \frac{2|\eta|\delta}{\kappa \sqrt{\eta(2(\nu+1)\eta - (\nu+1)^2)^2 + (\nu+1)^2}},
\]
and
\[
|d_{k+1} - \delta d_{k+1}^2| \leq \begin{cases} \frac{|\eta|\delta}{\kappa(2\nu+1)} & ; |\kappa + 1 - 2\delta| < \kappa \frac{2(\nu+1)\eta - (\nu+1)^2}{2|\eta|\delta(2\nu+1)} \\ \frac{|\eta|\delta^2}{\kappa(2\nu+1)} & ; |\kappa + 1 - 2\delta| \geq \kappa \frac{2(\nu+1)\eta - (\nu+1)^2}{2|\eta|\delta(2\nu+1)} \end{cases}.
\]

We obtain the results stated below, if we take \( \nu = 1 \) in Corollary 5.

**Corollary 7.** Let \( \tau \geq 1, 0 < \delta \leq 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( A \) belongs to \( \mathcal{C}_\nu^\ell(\eta, \delta) \), then
\[
|d_{k+1}| \leq \frac{2|\eta|\delta}{\sqrt{\eta} |M_1(k+1) + N_1(2 + \tau(t - 1)(k+1)^2 - 2\tau(k+1))| \eta - L_1^2 + L_1^2},
\]
and
\[
|d_{k+1} - \delta d_{k+1}^2| \leq \begin{cases} \frac{2|\eta|\delta}{M_1^2} & ; |\kappa + 1 - 2\delta| < J_3 \\ \frac{2|\eta|\delta^2}{M_1(2\nu+1)} & ; |\kappa + 1 - 2\delta| \geq J_3 \end{cases},
\]
where
\[
J_3 = \left| \frac{M_1(k+1) + N_1(2 + \tau(t - 1)(k+1)^2 - 2\tau(k+1))}{\eta M_1} \right|.
\]

Results analogous to Corollary 3 and Corollary 4 can be obtained, by taking \( \kappa = 1 \) in Corollary 7 and \( \kappa = 1, \nu = 0 \) in Corollary 5, respectively.
4. The Function Family $\mathcal{X}_{\nu, \xi}^\tau (\eta, \nu, \xi)$

Let $\varphi(\xi) = \frac{1 + (1 - 2\xi)^{2\eta}}{2\eta}, \ 0 \leq \xi < 1.$ then from Definition 1, we obtain the subclass $\mathcal{X}_{\nu, \xi}^\tau (\eta, \nu, \xi) = \mathfrak{M}_{\nu, \xi}^\tau (\eta, \nu, \left(\frac{1 + (1 - 2\xi)^{2\eta}}{2\eta}\right))$ of functions $s \in \sigma_{\nu, \xi}$ satisfying

$$\Re \left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{\xi s'(\xi)}{s(\xi)} + \nu \left( \frac{[\xi s'(\xi)]'}{s'(\xi)} - 1 \right) \right) \right] > \xi$$

and

$$\Re \left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{\xi g'(\xi)}{g(\xi)} + \nu \left( \frac{[\xi g'(\xi)]'}{g'(\xi)} - 1 \right) \right) \right] > \xi,$$

where $\xi, \nu \in \mathfrak{D}.$

**Remark 9.** The family $\mathfrak{M}_{\nu, \xi}^\tau (\eta, \nu, \xi) \equiv \mathcal{X}_{\nu, \xi}^\tau (\eta, \nu, \xi)$ was investigated by Aldawish et al. [24].

We remark that certain values of $\tau$ and $\nu$ in $\mathcal{X}_{\nu, \xi}^\tau (\eta, \nu, \xi)$ lead to the subfamilies as mentioned below:

(i). $\mathcal{Y}_{\nu, \xi}^\tau (\eta, \nu, \xi) \equiv \mathcal{X}_{\nu, \xi}^\tau (\eta, 1, \xi)$ is the family of $s \in \sigma_{\nu, \xi}$ satisfying

$$\Re \left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{\xi s'(\xi)}{s(\xi)} + \nu \left( \frac{[\xi s'(\xi)]'}{s'(\xi)} - 1 \right) \right) \right] > \xi$$

and

$$\Re \left[ 1 + \frac{1}{\eta} \left( (1 - \nu) \frac{\xi g'(\xi)}{g(\xi)} + \nu \left( \frac{[\xi g'(\xi)]'}{g'(\xi)} - 1 \right) \right) \right] > \xi,$$

where $0 < \nu \leq 1, \ \nu \geq 0$ and $\xi, \nu \in \mathfrak{D}.$

(ii). $\mathfrak{B}_{\nu, \xi}^\tau (\eta, \xi) \equiv \mathcal{X}_{\nu, \xi}^\tau (\eta, 1, \xi)$ is the family of $s \in \sigma_{\nu, \xi}$ satisfying

$$\Re \left[ 1 + \frac{1}{\eta} \left( \frac{[\xi s'(\xi)]'}{s'(\xi)} - 1 \right) \right] > \xi$$

and

$$\Re \left[ 1 + \frac{1}{\eta} \left( \frac{[\xi g'(\xi)]'}{g'(\xi)} - 1 \right) \right] > \xi,$$

where $0 \leq \xi < 1, \ \tau \geq 1$ and $\xi, \nu \in \mathfrak{D}.$

**Remark 9.** We note that $\mathcal{Y}_{\nu, \xi}^\tau (\eta, 1, \xi) \equiv \mathfrak{B}_{\nu, \xi}^\tau (\eta, \xi) \equiv \mathfrak{C}_{\nu, \xi}^\tau (\eta, \xi).$ The function class $\mathfrak{C}_{\nu, \xi}^\tau (\eta, \xi)$ was considered by Kumar et al. [33]. The function family $\mathcal{Y}_{\nu, \xi}^\tau (1, 0, \xi)$ coincides with the family of bi-starlike functions of order $\xi,$ which was studied by Branan and Taha [9].

If we take $\varphi(\xi) = \frac{1 + (1 - 2\xi)^{2\eta}}{2\eta}$ in Theorem 1, we get

**Corollary 8.** Let $\tau \geq 1, \ \nu \geq 0, 0 \leq \xi < 1$ and $\delta \in \Re.$ If a function $s$ in $A$ belongs to $\mathcal{X}_{\nu, \xi}^\tau (\eta, \nu, \xi),$ then

$$|d_{k+1}| < \frac{2|\eta|(1 - \xi)}{\sqrt{\left\{|M(k + 1) + N(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(k + 1))\eta(1 - \xi) - L^2\right| + L^2}},$$

$$|d_{2k+1}| \leq \frac{2|\eta|(1 - \xi)}{M \left( \frac{\kappa + 1}{|\eta|(1 - \xi) M} \right)^2 + \left( \kappa + 1 - \frac{L^2}{|\eta|(1 - \xi) M} \right)^2 \left( \frac{2\eta^2 \phi^2}{|\eta|(k + 1) M} \right) \left| \frac{1}{|\eta|(k + 1) M} \right| < \xi < 1, \ 0 \leq \xi \leq 1 - \frac{L^2}{|\eta|(k + 1) M}.$$
and

\[ |d_{2k+1} - \delta d_{k+1}^2| \leq \begin{cases} \frac{2|\eta(1-\xi)|}{M} ; |\kappa + 1 - 2\delta| < J_4 \\ \frac{2|\eta|^2(1-\xi)(\kappa + 1 - 2\delta)}{|(M(k+1) + N(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(\kappa + 1)))\eta(1-\xi) - L_2^2|} ; |\kappa + 1 - 2\delta| \geq J_4, \end{cases} \]

where

\[ J_4 = \frac{|(M(k+1) + N(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(\kappa + 1)))\eta(1-\xi) - L_2^2|}{\eta M(1-\xi)}, \]

and

\[ |d_{2k+1}| \leq \begin{cases} \frac{|\eta(1-\xi)|}{\kappa \sqrt{|(\kappa \nu + 1)(\kappa + 1)| - (\kappa \nu + 1)^2}} ; |\kappa + 1 - 2\delta| < \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)} < \xi < 1 \\ \frac{2\eta^2\rho^2}{\kappa^2|\eta(1-\xi)|^2(2\kappa + 1)^2} \left( 1 - \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)} \right) ; 0 \leq \xi \leq 1 - \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)}, \end{cases} \]

and

\[ |d_{2k+1} - \delta d_{k+1}^2| \leq \begin{cases} \frac{|\eta(1-\xi)|}{\kappa^2|\eta(1-\xi)|^2(2\kappa + 1)^2} ; |\kappa + 1 - 2\delta| < \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)} \\ \frac{2\eta^2\rho^2}{\kappa^2|\eta(1-\xi)|^2(2\kappa + 1)^2} \left( 1 - \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)} \right) ; |\kappa + 1 - 2\delta| \geq \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)}, \end{cases} \]

Remark 10. (a) If we allow \( \nu = 0 \) in Corollary 8, then we have Corollary 8 of Aldawish et al. [24].
(b) (i) The bound on \( |d_{k+1}| \) stated in Corollary 7 of [34] is obtained, if \( \nu = 0 \), \( \tau = 1 \) and \( \eta = 1 \) in Corollary 8. (ii) Our result on \( |d_{2k+1}| \) in Corollary 8 is better than the bound stated in Corollary 7 of [34] in terms of \( \xi \) ranges as well as the bounds, when \( \nu = 0 \), \( \tau = 1 \) and \( \eta = 1 \).

We obtain the results stated below, if we set \( \tau = 1 \) in Corollary 8.

Corollary 9. Let \( 0 \leq \xi < 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( \mathcal{A} \) belongs to \( \mathcal{B}_{\alpha\nu}(\eta, \nu, \xi) \), then

\[ |d_{2k+1}| \leq \frac{2|\eta(1-\xi)|}{\kappa \sqrt{|(\kappa \nu + 1)(\kappa + 1)| - (\kappa \nu + 1)^2}} \]

and

\[ |d_{2k+1} - \delta d_{k+1}^2| \leq \frac{2|\eta(1-\xi)|}{\kappa \sqrt{|(\kappa \nu + 1)(\kappa + 1)| - (\kappa \nu + 1)^2}} ; |\kappa + 1 - 2\delta| < \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)} \]

We obtain the results stated below, if we set \( \nu = 1 \) in Corollary 8.

Corollary 10. Let \( \tau \geq 1, 0 \leq \xi < 1 \) and \( \delta \in \mathbb{R} \). If a function \( s \) in \( \mathcal{A} \) belongs to \( \mathcal{B}_{\alpha\nu}(\eta, \nu, \xi) \), then

\[ |d_{2k+1}| \leq \frac{2|\eta(1-\xi)|}{\kappa \sqrt{|(\kappa \nu + 1)(\kappa + 1)| - (\kappa \nu + 1)^2}} \]

and

\[ |d_{2k+1} - \delta d_{k+1}^2| \leq \frac{2|\eta(1-\xi)|}{\kappa \sqrt{|(\kappa \nu + 1)(\kappa + 1)| - (\kappa \nu + 1)^2}} ; |\kappa + 1 - 2\delta| < \frac{\kappa(\kappa + 1)^2}{2\eta(1-\xi)(2\kappa + 1)} \]

where

\[ J_5 = \frac{|(M(k+1) + N(2 + \tau(\tau - 1)(\kappa + 1)^2 - 2\tau(\kappa + 1)))\eta(1-\xi) - L_2^2|}{\eta M(1-\xi)}, \]

\[ L_1, M_1 \text{ and } N_1 \text{ are as in (30), (31) and (32), respectively.} \]
5. Conclusions

In the current study, a \( \kappa \)-fold bi-univalent function family \( \mathcal{M}_k^\nu (v, \eta, \varphi) \) is introduced and the original results about the upper bounds of \( |d_{k+1}| \) and \( |d_{2k+1}| \) are estimated for functions belonging to this family. Furthermore, the estimate of Fekete–Szegő problem \( |d_{2k+1} - \delta^2 d_{k+1}|, \delta \in \mathbb{R} \), for functions in \( \mathcal{M}_k^\nu (v, \eta, \varphi) \) is also examined. Various subfamilies of \( \mathcal{M}_k^\nu (v, \eta, \varphi) \) are also discussed. The problem to determine bound on \( |d_{k+1}|, (k \in \mathbb{N} - \{1,2\} \) for the classes that have been examined in this paper remain open. Since the only investigation on the defined family was related to coefficient bounds, it could inspire many researchers for further investigations related to different other aspects associated with (i) \( q \)-derivative operator [35], (ii) integrodifferential operator [36], (iii) Hohlov operator linked with legendary polynomials [37] and so on.

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