Complex formalism of Riemann – Silberstein – Majorana – Oppenheimer in Maxwell electrodynamics is extended to the case of arbitrary pseudo-Riemannian space – time in accordance with the tetrad recipe of Tetrode – Weyl – Fock – Ivanenko. In this approach, the Maxwell equations are solved exactly on the background of simplest static cosmological models, spaces of constant curvature of Riemann and Lobachevsky parameterized by spherical coordinates. Separation of variables is realized in the basis of Schrödinger – Pauli type, description of angular dependence in electromagnetic complex 3-vectors is given in terms of Wigner D-functions. In the case of compact Riemann model a discrete frequency spectrum for electromagnetic modes depending on the curvature radius of space and three discrete parameters is found. In the case of hyperbolic Lobachevsky model no discrete spectrum for frequencies of electromagnetic modes arises.

1 Introduction: matrix complex form of Maxwell theory

The task of the present paper is to obtain in explicit form spherical waves solutions to Maxwell equations in space of positive and negative curvature, spherical Riemann $S_3$ Lobachevsky $H_3$ models, when they are parameterized by extended spherical coordinates. This paper continues investigation of similar problems on searching solutions of the Maxwell equations in symmetrical space-time model [1, 2].

We will use the known complex form of Maxwell theory according to approach by Riemann, Silberstein, Oppenheimer, and Majorana [3], [4], [5], [6], (also see [8–32]), which is extended to the case of arbitrary curved space – time in accordance with general tetrad formalism by Tetrode – Weyl – Fock – Ivanenko [33-35]; also see [35]).

Let us start with Maxwell equations in vacuum at presence of sources: (with the use of notation $j^a = (\rho, J/c)$, $c^2 = 1/\varepsilon_0\mu_0$):

\[
\text{div } c\mathbf{B} = 0 , \quad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}, \\
\text{div } \mathbf{E} = \frac{\rho}{\varepsilon_0} , \quad \text{rot } c\mathbf{B} = \frac{j}{\varepsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}.
\]

In explicit form they are

\[
\partial_1 cB^1 + \partial_2 cB^2 + \partial_3 cB^3 = 0 , \quad \partial_2 E^3 - \partial_3 E^2 + \partial_0 cB^1 = 0 , \\
\partial_3 E^1 - \partial_1 E^3 + \partial_0 cB^2 = 0 , \quad \partial_1 E^2 - \partial_2 E^1 + \partial_0 cB^3 = 0 , \\
\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = j^0/\varepsilon_0 , \quad \partial_2 cB^3 - \partial_3 cB^2 - \partial_0 E^1 = j^1/\varepsilon_0 , \\
\partial_3 cB^1 - \partial_1 cB^3 - \partial_0 E^2 = j^2/\varepsilon_0 , \quad \partial_1 cB^2 - \partial_2 cB^1 - \partial_0 E^3 = j^3/\varepsilon_0 .
\]

Let us introduce a complex 3-component vector $\psi^k = E^k + icB^k$; eqs. (2) are easily combined
or in matrix form: (12 arbitrary parameters enter this matrix equation):

\[
\begin{align*}
\begin{pmatrix} 
0 & \psi^1 & \psi^2 & \psi^3 
\end{pmatrix} = 
\begin{pmatrix} 
a_0 & 0 & 0 & 0 
 a_1 & 1 & 0 & 0 
 a_2 & 0 & 1 & 0 
 a_3 & 0 & 0 & 1 
\end{pmatrix},
\end{align*}
\]

\[
(\alpha^0)^2 = 
\begin{pmatrix} 
 a_0a_0 & 0 & 0 & 0 
 a_1a_0 + a_1 & 1 & 0 & 0 
 a_2a_0 + a_2 & 0 & 1 & 0 
 a_3a_0 + a_3 & 0 & 0 & 1 
\end{pmatrix}.
\]

Taking into account

\[
(\alpha^0)^2 = +I, \quad a_0a_0 = 1, \quad a_1a_0 + a_1, \quad a_2a_0 + a_2, \quad a_3a_0 + a_3;
\]

the most simple solution is

\[
a_0 = \pm 1, \quad a_j = 0, \quad \alpha^0 = 
\begin{pmatrix} 
 \pm 1 & 0 & 0 & 0 
 0 & 1 & 0 & 0 
 0 & 0 & 1 & 0 
 0 & 0 & 0 & 1 
\end{pmatrix},
\]

\[
(\alpha^0)^2 = +I.
\]

In similar manner, taking

\[
(\alpha^1)^2 = 
\begin{pmatrix} 
 b_0^2 + b_1 & b_0 & 0 & 0 
 b_1b_0 & b_1 & 0 & 0 
 b_2b_0 - b_3 & b_2 & -1 & 0 
 b_3b_0 - b_2 & b_3 & 0 & -1 
\end{pmatrix}.
\]

for equation \((\alpha^1)^2 = -I\) let us use the most simple solution

\[
b_0 = 0, \quad b_1 = -1, \quad b_2 = 0, \quad b_3 = 0, \quad \alpha^1 = 
\begin{pmatrix} 
 0 & 1 & 0 & 0 
 -1 & 0 & 0 & 0 
 0 & 0 & 0 & -1 
 0 & 0 & 1 & 0 
\end{pmatrix}.
\]
Again,

\[
(a^2)^2 = \begin{vmatrix}
 c_0c_0 + c_2 & 0 & c_0 & 0 \\
 c_1c_0 + c_3 & -1 & c_1 & 0 \\
 c_2c_0 & 0 & c_2 & 0 \\
 c_3c_0 - c_1 & 0 & c_3 & -1 \\
\end{vmatrix} = -I ,
\]

that is

\[
c_0 = 0 , c_1 = 0 , c_2 = -1 , c_3 = 0 , \quad (a^2)^2 = \begin{vmatrix}
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 -1 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 \\
\end{vmatrix} , \quad (a^2)^2 = -I . \quad (7)
\]

And finally, for \(a^3\) we have

\[
(a^3)^2 = \begin{vmatrix}
 d_0d_0 + d_3 & 0 & 0 & d_0 \\
 d_1d_0 - d_2 & -1 & 0 & 0 \\
 d_2d_0 + d_1 & 0 & -1 & d_2 \\
 d_3d_0 & 0 & 0 & d_3 \\
\end{vmatrix} = -I ,
\]

so that

\[
d_0 = 0 , d_1 = 0 , d_2 = 0 , d_3 = -1, \quad (a^3)^2 = \begin{vmatrix}
 0 & 0 & 0 & 1 \\
 0 & 0 & -1 & 0 \\
 0 & 1 & 0 & 0 \\
 -1 & 0 & 0 & 0 \\
\end{vmatrix} , \quad (a^3)^2 = -I . \quad (8)
\]

Simple rules for their products hold:

\[
a^1a^2 = +a^3 , \quad a^2a^1 = -a^3 , \quad a^2a^3 = a^1 , \quad a^3a^2 = -a^1 , \quad a^3a^3 = +a^2 , \quad a^1a^3 = -a^2 ; \quad (9)
\]

Also we get

\[
k = \pm 1, \quad a^0a^1 = \begin{vmatrix}
 0 & k & 0 & 0 \\
 -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 \\
 0 & 0 & 1 & 0 \\
\end{vmatrix} , \quad a^1a^0 = \begin{vmatrix}
 0 & 1 & 0 & 0 \\
 -k & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 \\
 0 & 0 & 1 & 0 \\
\end{vmatrix} .
\]

only when \(k = +1\) we get the very simple commutation rule

\[
a^0 = I , \quad a^1a^0 = a^0a^1 = a^1 . \quad (10)
\]

Thus, Maxwell equations can be presented in the following simple matrix form

\[
(-i\partial_0 + a^i\partial_j)\Psi = J , \quad \Psi = \begin{vmatrix}
 0 \\
 \psi^1 \\
 \psi^2 \\
 \psi^3 \\
\end{vmatrix} , \quad J = \frac{1}{\epsilon_0} \begin{vmatrix}
 j^0 \\
 i \psi^1 \\
 i \psi^2 \\
 i \psi^3 \\
\end{vmatrix} ,
\]

\[
a^1 = \begin{vmatrix}
 0 & 1 & 0 & 0 \\
 -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 \\
 0 & 0 & 1 & 0 \\
\end{vmatrix} , \quad (a^1)^2 = -I , \quad (a^1)^2 = -I , \quad (a^1)^2 = -I ,
\]

\[
a^1a^2 = -a^2a^1 = a^3 , \quad a^2a^3 = -a^3a^2 = a^1 , \quad a^3a^1 = -a^1a^3 = a^2 . \quad (11)
\]
2 Matrix Maxwell equation in Riemannian space

Matrix Maxwell equation can be extended to the case of arbitrary Riemannian space – time in accordance with the tetrad approach of Tetrode – Weyl – Fock – Ivanenko:

\[
\alpha^\rho(x) \left[ \partial_{\rho} + A_\rho(x) \right] \Psi(x) = J(x) ,
\]

\[
\alpha^\rho(x) = \alpha^c e^\rho_{(c)}(x) , \quad A_\rho(x) = \frac{1}{2} J^{ab} e^\beta_{(a)} \nabla_\rho e_{(n)\beta} .
\]

(12)

where \( e^\rho_{(c)}(x) \) stands for the tetrad, \( J^{ab} \) stands for – generators of the complex vector representation of complex orthogonal group \( SO(3,C) \). Eq. (12) can be rewritten in terms of rotational Ricci coefficients

\[
\alpha^c \left( e^\rho_{(c)} \partial_{\rho} + \frac{1}{2} J^{ab} \gamma_{abc} \right) \Psi = J(x) ,
\]

(13)

where \( \gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta} \alpha e^\beta_{(a)} \alpha^c e^\rho_{(c)} \) and

\[
\begin{align*}
  & j^{23} = s_1 , \quad j^{01} = i s_1 , \quad j^{31} = s_2 , \quad j^{02} = i s_2 , \quad j^{12} = s_3 , \quad j^{03} = i s_3 , \\
  & s_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad s_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \quad s_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tau_3 \end{bmatrix} .
\end{align*}
\]

3 Spherical coordinates and tetrad in the Riemann space \( S_3 \)

In spherical coordinates in the Riemann space \( S_3 \) (see [38])

\[
dS^2 = c^2 dt^2 - d\chi^2 - \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) ,
\]

\[
x^\alpha = (ct, \chi, \theta, \phi) , \quad \gamma_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sin^2 \chi & 0 \\ 0 & 0 & -\sin^2 \chi \sin^2 \theta & 0 \end{bmatrix} .
\]

(14)

let us use the following tetrad

\[
e^\alpha_{(0)} = (1, 0, 0, 0) , \quad e^\alpha_{(3)} = (0, 1, 0, 0) , \quad e^\alpha_{(1)} = (0, 0, \frac{1}{\sin \chi}, 0) , \quad e^\alpha_{(2)} = (1, 0, 0, \frac{1}{\sin \chi \sin \theta}) .
\]

(15)

Christoffel symbols are given by

\[
\begin{align*}
  \Gamma^\chi_{\phi\phi} &= -\sin \chi \cos \chi \sin^2 \theta , \quad \Gamma^\chi_{\theta\theta} = -\sin \chi \cos \chi , \\
  \Gamma^\theta_{\phi\phi} &= -\sin \theta \cos \theta , \quad \Gamma^\phi_{\phi\theta} = \cos \chi \sin \chi , \quad \Gamma^\phi_{\phi\phi} = \frac{\cos \chi}{\sin \chi} .
\end{align*}
\]

(16)

The Ricci coefficients are \( \gamma_{ab0} = 0 \), \( \gamma_{ab3} = 0 \) and

\[
\begin{align*}
  \gamma_{ab1} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tan \chi} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\tan \chi} & 0 & 0 \end{bmatrix} , \quad \gamma_{ab2} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tan \theta \sin \chi} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\tan \theta \sin \chi} & 0 & 0 \end{bmatrix} .
\end{align*}
\]

(17)
Correspondingly, for \( \alpha^\alpha(x) \) and \( A_\alpha(x) \) we get

\[
\alpha^\alpha(x) = (\alpha^0, \alpha^3, \frac{\alpha^1}{\sin \chi}, \frac{\alpha^2}{\sin \chi \sin \theta}), \quad A_0(x) = 0,
\]

\[
A_\chi(x) = 0, \quad A_\theta(x) = j^{31}, \quad A_\phi(x) = \sin \theta j^{32} + \cos \theta j^{12}.
\] (18)

Therefore, eq. (13) takes the form

\[
\begin{bmatrix}
- i \partial_0 + \alpha^3 \partial_r + \frac{\alpha^1 j^{31} + \alpha^2 j^{32}}{\tan \chi} + \frac{1}{\sin \chi} \Sigma_{\theta, \phi} \\
\end{bmatrix} \Psi(x) = 0,
\]

\[
\Sigma_{\theta, \phi} = \alpha^1 \partial_\theta + \alpha^2 \frac{\partial_\phi + \cos \theta j^{12}}{\sin \theta}.
\] (19)

It is more convenient to have the matrix \( j^{12} \) as diagonal one. To this end, one needs to use a cyclic basis

\[
\Psi' = U_4 \Psi, \quad U_4 = \begin{bmatrix}
1 & 0 \\
0 & U
\end{bmatrix},
\]

(20)

where

\[
U = \begin{bmatrix}
-1/\sqrt{2} & i/\sqrt{2} & 0 \\
0 & 0 & 1 \\
1/\sqrt{2} & i/\sqrt{2} & 0
\end{bmatrix}, \quad U^{-1} = U_3^+ = \begin{bmatrix}
-1/\sqrt{2} & 0 & 1/\sqrt{2} \\
-i/\sqrt{2} & 0 & -i/\sqrt{2} \\
0 & 1 & 0
\end{bmatrix}.
\]

In is matter of simple calculation to find

\[
U \tau_1 U^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & -i & 0 \\
-i & 0 & -i \\
0 & -i & 0
\end{bmatrix} = \tau'_1, \quad j^{23} = s'_1 = \begin{bmatrix}
0 & 0 \\
0 & \tau'_1
\end{bmatrix},
\]

\[
U \tau_2 U^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} = \tau'_2, \quad j^{31} = s'_2 = \begin{bmatrix}
0 & 0 \\
0 & \tau'_2
\end{bmatrix},
\]

\[
U \tau_3 U^{-1} = -i \begin{bmatrix}
+1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} = \tau'_3, \quad j^{12} = s'_3 = \begin{bmatrix}
0 & 0 \\
0 & \tau'_3
\end{bmatrix}.
\]

\[
\alpha' = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & -i & 0 \\
0 & -i & 0 & -i
\end{bmatrix}, \quad \Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & -i & 0 & -i \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{bmatrix}, \quad \alpha'' = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & +i
\end{bmatrix}.
\]

In cyclic basis, eq. (19) reads

\[
\begin{bmatrix}
- i \frac{\partial}{\partial t} + \alpha'^1 \frac{\partial}{\partial r} + \frac{\alpha'^1 s'_1 - \alpha'^2 s'_2}{\tan \chi} + \frac{1}{\sin \chi} \Sigma'_{\theta, \phi} \\
\end{bmatrix} \Psi'(x) = 0,
\]

\[
\Sigma'_{\theta, \phi} = \alpha'^1 \partial_\theta + \alpha'^2 \frac{\partial_\phi + \cos \theta s'_3}{\sin \theta}.
\] (21)
4 Separation of variables and Wigner functions

Let us construct electromagnetic spherical waves, then the field function should be taken in the form

$$
\psi = e^{-i\omega t} \begin{vmatrix}
0 \\
f_1(r)D_{-1} \\
f_2(r)D_0 \\
f_3(r)D_{+1}
\end{vmatrix}
$$

(22)

where the Wigner $D$-function are used $D_{\sigma} = D^{2}_{-m,\sigma}(\phi, \theta, 0)$, $\sigma = -1, 0, +1$; $j, m$ determine $J^2$ and $J_3$ eigenvalues. When separating the variables we will need the recurrent relations for Wigner’s function [3]:

$$
\begin{align*}
\partial_\theta D_{-1} &= \frac{1}{2} (a D_{-2} - \nu D_0), \\
\partial_\theta D_0 &= \frac{1}{2} (\nu D_{-1} - \nu D_{+1}), \\
\partial_\theta D_{+1} &= \frac{1}{2} (\nu D_0 - a D_{+2}), \\
\nu &= \sqrt{j(j+1)}, \quad a = \sqrt{(j-1)(j+2)}.
\end{align*}
$$

(23)

Let us find the action of the angular operator (the factor $e^{-i\omega t}$ will be omitted for shortness)

$$
\sqrt{2} \Sigma'_{\theta\phi} \Psi' = \sqrt{2} \left[ \alpha'^1 \partial_\theta + \alpha'^2 \frac{\partial_\phi + \cos \theta s'_3}{\sin \theta} \right] \begin{vmatrix}
0 \\
f_1(r)D_{-1} \\
f_2(r)D_0 \\
f_3(r)D_{+1}
\end{vmatrix} =
$$

$$
\begin{align*}
\begin{vmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & -i & 0 \\
0 & -i & 0 & -i \\
-1 & 0 & -i & 0
\end{vmatrix} \partial_\theta 
&= \begin{vmatrix}
0 \\
f_1D_{-1} \\
f_2D_0 \\
f_3D_{+1}
\end{vmatrix} + \begin{vmatrix}
-1 & 0 & -i & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & -i & 0
\end{vmatrix} \\
&= \frac{1}{2} \begin{vmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & -i & 0 \\
0 & -i & 0 & -i \\
-1 & 0 & -i & 0
\end{vmatrix} \begin{vmatrix}
0 & 1 & 0 & 1 \\
+1 & 0 & -i & 0 \\
+1 & 0 & i & 0 \\
0 & i & 0 & 1
\end{vmatrix} \\
&= \begin{vmatrix}
0 & 1 & 0 & 1 \\
+1 & 0 & -i & 0 \\
+1 & 0 & i & 0 \\
0 & i & 0 & 1
\end{vmatrix}
\end{align*}
$$

from whence we arrive at

$$
\Sigma'_{\theta\phi} \Psi' = \frac{\nu}{\sqrt{2}} \begin{vmatrix}
(f_1 + f_3)D_0 \\
- i f_2D_{-1} \\
i (f_1 - f_3)D_0 \\
i f_2D_{+1}
\end{vmatrix}
$$

(24)
From Maxwell equation, taking into account (24) and identities

\[
\Psi = e^{-i\omega t} \begin{vmatrix}
\frac{\alpha'_1 s_2' - \alpha'_2 s_1'}{\tan \chi} & 0 & 0 & 0 & 0 & 0 \\
-\omega f_1 - i f_1' - \frac{i}{\tan \chi} f_1 - \frac{i \nu}{\sin \chi \sqrt{2}} f_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix},
\]

we get the radial equations

\[
\begin{align*}
\frac{d f_2}{d\chi} & + \frac{1}{\tan \chi} f_2 & + \frac{1}{\sin \chi} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0, \\
-\omega f_1 - i f_1' - \frac{i}{\tan \chi} f_1 & - \frac{i \nu}{\sin \chi \sqrt{2}} f_2 & = 0, \\
-\omega f_2 & + \frac{i}{\sin \chi \sqrt{2}} (f_1 - f_3) = 0, \\
-\omega f_3 & + i f_3' + \frac{i}{\tan \chi} f_3 + \frac{i \nu}{\sin \chi \sqrt{2}} f_2 & = 0.
\end{align*}
\]  

(26)

With the use of substitutions

\[
f_1 = \frac{1}{\sin \chi} F_1, \quad f_2 = \frac{1}{\sin \chi} F_2, \quad f_3 = \frac{1}{\sin \chi} F_3,
\]

the systems reads simpler

\[
\begin{align*}
(1) \quad & \left( \frac{d}{d\chi} + \frac{1}{\tan \chi} \right) \omega F_2 + \frac{\omega \nu}{\sqrt{2} \sin \chi} (F_1 + F_3) = 0, \\
(2) \quad & -\omega^2 F_1 - i \omega F_1' - \frac{i \nu}{\sqrt{2} \sin \chi} \omega F_2 = 0, \\
(3) \quad & \omega F_2 = \frac{i \nu}{\sqrt{2} \sin \chi} (F_1 - F_3), \\
(4) \quad & -\omega^2 F_3 + i \omega F_3' + \frac{i \nu}{\sqrt{2} \sin \chi} \omega F_2 = 0.
\end{align*}
\]  

(27)

Combining eqs. (2) and (4) we get

\[
(2) + (4),
\]

\[
-\omega (F_1 + F_3) - i (F_1' - F_3') = 0 
\]  

(28)

\[
-(2) + (4),
\]

\[
\omega^2 (F_1 - F_3) + i \omega (F_1' + F_3') + \frac{2i \nu}{\sqrt{2} \sin \chi} \frac{i \nu}{\sqrt{2} \sin \chi} (F_1 - F_3) = 0
\]  

(29)

Allowing for eq. (3) and (28), from eq. (3) in (29) we arrive at an identity

\[
\left( \frac{d}{d\chi} + \frac{1}{\tan \chi} \right) \frac{i \nu}{\sqrt{2} \sin \chi} (F_1 - F_3) + \frac{\omega \nu}{\sqrt{2} \sin \chi} (-i)(F_1' - F_3') = 0.
\]  

(30)
Therefore, the Maxwell equations reduce to only three independent equations:

\[
\omega F_2 = \frac{i\nu}{\sqrt{2}\sin \chi}(F_1 - F_3), \quad -\omega(F_1 + F_3) - i(F'_1 - F'_3) = 0,
\]

\[
\omega^2(F_1 - F_3) + i\omega(F'_1 + F'_3) + \frac{2i\nu}{\sqrt{2}\sin \chi} \frac{i\nu}{\sqrt{2}\sin \chi}(F_1 - F_3) = 0.
\]  

(31)

Let us introduce new variables:

\[
F = \frac{F_1 + F_3}{\sqrt{2}}, \quad G = \frac{F_1 - F_3}{\sqrt{2}},
\]

then (31) will read

\[
F_2 = \frac{i\nu}{\omega \sin \chi} G, \quad F = -\frac{i}{\omega} \frac{d}{d\chi} G, \quad \frac{d^2}{d\chi^2} G + \omega^2 G - \frac{\nu^2}{\sin^2 \chi} G = 0.
\]  

(32)

5 Solution of the radial equation in $S_3$

Let us solve eq. (32). To this end one need to introduce a new variable

\[
z = 1 - e^{-2i\chi}, \quad z = 2 \sin \chi e^{i(-\chi + \pi/2)} ;
\]  

(33)

\[z \text{ runs along closed path in the complex plane:}
\]

\[
\text{Fig. 1 (The variable } z)\]

\[
\chi = \pi/4
\]

\[
\chi = 2\pi/4
\]

\[
\chi = 3\pi/4
\]

Allowing for identities

\[
\frac{d}{d\chi} = 2i(1 - z) \frac{d}{dz}, \quad \frac{\cos \chi}{\sin \chi} = \frac{2 - z}{z}, \quad \frac{1}{\sin^2 \chi} = \frac{4(1 - z)}{z^2},
\]

eq. (32) reduces to

\[
4(1 - z)^2 \frac{d^2 G}{dz^2} - 4(1 - z) \frac{dG}{dz} - \omega^2 G - \frac{4(1 - z)\nu^2}{z^2} G = 0.
\]  

(34)

With the use of the substitution

\[
G = z^a(1 - z)^b g(z),
\]

\[
G' = az^{a-1}(1 - z)^b g(z) - bz^a(1 - z)^{b-1}g(z) + z^a(1 - z)^b \frac{dg(z)}{dz},
\]

\[
G'' = a(a - 1)z^{a-2}(1 - z)^b g(z) - abz^{a-1}(1 - z)^{b-1}g(z) + az^{a-1}(1 - z)^b \frac{dg(z)}{dz} -
\]

\[-abz^{a-1}(1 - z)^{b-1}g(z) + b(b - 1)z^a(1 - z)^{b-2}g(z) - bz^a(1 - z)^{b-1} \frac{dg(z)}{dz} +
\]

\[+az^{a-1}(1 - z)^b \frac{dg(z)}{dz} - bz^a(1 - z)^b \frac{dg(z)}{dz} + z^a(1 - z)^b \frac{d^2 g(z)}{dz^2}.
\]  

(35)
from (34) we arrive at
\[ z(1 - z) \frac{d^2 g}{dz^2} + [2a - (2a + 2b + 1)z] \frac{dg}{dz} + \\
\left[ \frac{\omega^2}{4} - (a + b)^2 + (a(a - 1) - \nu^2) \frac{1}{z} + (b^2 - \frac{\omega^2}{4}) \frac{1}{1 - z} \right] g = 0. \]

Requiring
\[ a(a - 1) - \nu^2 = 0, \quad b^2 - \frac{\omega^2}{4} = 0 \quad \Rightarrow \quad a = j + 1, -j, \quad b = \pm \frac{\omega}{2}, \quad (36) \]
for \( g \) we obtain a simpler equation
\[ z(1 - z) \frac{d^2 g}{dz^2} + [2a - (2a + 2b + 1)z] \frac{dg}{dz} - \left[ (a + b)^2 - \frac{\omega^2}{4} \right] g = 0, \quad (37) \]
which is of hypergeometric type
\[ z(1 - z) F'' + [\gamma - (\alpha + \beta + 1)z] F' - \alpha \beta F = 0 \]
with parameters
\[ \gamma = 2a, \quad \alpha + \beta = 2a + 2b, \quad \alpha \beta = (a + b)^2 - \frac{\omega^2}{4}, \]
or
\[ \alpha = a + b - \frac{\omega}{2}, \quad \beta = a + b + \frac{\omega}{2}. \quad (38) \]
The function \( G \) is given by
\[ G = z^a(1 - z)^b g(z) = \left[ 2i \sin \chi e^{-i\chi} \right]^a \left[ 1 - 2i \sin \chi e^{-i\chi} \right]^b g(z); \quad (39) \]
it is finite at the points \( \chi = 0 \) and \( \chi = \pi \) only when \( a \) is positive (see (35)):
\[ a = j + 1. \quad (40) \]
also we must require \( b = -\omega/2 \) when hypergeometric series can be reduced to a polynomial (we take \( \omega > 0 \)):
\[ a = j + 1 - \omega = -n = \{0, -1, -2, ...\} \quad \Rightarrow \quad \omega = n + 1 + j; \quad (41) \]
Thus, physical solutions of the Maxwell equations in the Riemann space \( S_3 \) are given by relations:
\[ G = z^a(1 - z)^b g(z) = \left[ 2i \sin \chi e^{-i\chi} \right]^a \left[ 1 - 2i \sin \chi e^{-i\chi} \right]^b g(z), \]
\[ g(z) = F(-n, j + 1, 2j + 2; z) = F(-n, j + 1, 2j + 2; 2i \sin \chi e^{-i\chi}) \quad (42) \]
where
\[ \omega = n + 1 + j, \quad j = 0, 1, 2, ..., \quad n = 0, 1, 2, ...; \quad (43) \]
or in usual units
\[ \omega = \frac{c}{\rho} (n + 1 + j), \quad (44) \]
\( \rho \) stands for the curvature radius of the space, \( c \) is velocity of the light.
6 Spherical coordinates and tetrad in Lobachevsky space $H_3$

Let us consider Maxwell equation in spherical coordinates of the Lobachevsky space model $H_3$

$$dS^2 = c^2 dt^2 - d\chi^2 - \text{sh}^2 \chi (d\theta^2 + \text{sin}^2 \theta d\phi^2) ,$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\text{sh}^2 \chi & 0 \\ 0 & 0 & 0 & -\text{sh}^2 \chi \text{sin}^2 \theta \end{pmatrix}.$$ (45)

in the following tetrad

$$e_\alpha^{(0)} = (1, 0, 0, 0), \quad e_\alpha^{(3)} = (0, 1, 0, 0), \quad e_\alpha^{(1)} = (0, 0, \frac{1}{\text{sh} \chi}, 0), \quad e_\alpha^{(2)} = (1, 0, 0, \frac{1}{\text{sh} \chi \text{sin} \theta}).$$ (46)

The Christoffel symbols are

$$\begin{align} 
\Gamma^\chi_{\phi \phi} &= -\text{sh} \chi \text{ch} \chi \text{sin}^2 \theta, \\
\Gamma^\chi_{\theta \theta} &= -\text{sh} \chi \text{ch} \chi, \\
\Gamma^\theta_{\phi \phi} &= -\sin \theta \cos \theta, \\
\Gamma^\phi_{\theta \theta} &= \frac{\text{ch} \chi}{\text{sh} \chi}, \\
\Gamma^\phi_{\chi \phi} &= \text{ctg} \theta, \\
\Gamma^\chi_{\phi \phi} &= \frac{\text{ch} \chi}{\text{sh} \chi}. 
\end{align}$$ (47)

and the Ricci rotation coefficients are $\gamma_{ab0} = 0$, $\gamma_{ab3} = 0$ and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\text{th} \chi} \\ 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{\text{th} \chi} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\frac{1}{\text{tg} \theta \text{sh} \chi} \\ 0 & -\frac{1}{\text{tg} \theta \text{sh} \chi} & 0 & -\frac{1}{\text{th} \chi} \\ 0 & 0 & +\frac{1}{\text{th} \chi} & 0 \end{pmatrix}.$$ (48)

For $\alpha^\alpha(x)$ and $A_\alpha(x)$ we get

$$\alpha^\alpha(x) = (\alpha^0, \alpha^3, \frac{\alpha^1}{\text{sh} \chi}, \frac{\alpha^2}{\text{sh} \chi \text{sin} \theta}), \quad A_0(x) = 0, \quad A_\chi(x) = 0, \quad A_\theta(x) = j^{31}, \quad A_\phi(x) = \text{sin} \theta j^{32} + \text{cos} \theta j^{12}.$$ (49)

Therefore, Maxwell equation reads (the cyclic will be used)

$$\left[-i \frac{\partial}{\partial t} + \alpha^3 \frac{\partial}{\partial r} + \frac{\alpha^1 s^3 - \alpha^2 s^1}{\text{th} \chi} + \frac{1}{\text{sh} \chi} \Sigma'_{\theta, \phi}\right] \Psi'(x) = 0,$$

$$\Sigma'_{\theta, \phi} = \alpha^1 \partial_\theta + \alpha^2 \frac{\partial_\phi + \text{cos} \theta s^3}{\text{sin} \theta}.$$ (50)

7 Separation of variables

We start with spherical substitution

$$\psi = e^{-i\omega t} \begin{pmatrix} 0 \\ f_1(r)D_{-1} \\ f_2(r)D_0 \\ f_3(r)D_{+1} \end{pmatrix}.$$ (51)
Further calculations are completely the same that were used in previous case, so that we can go just to the final result:

\[ f_2' + \frac{2}{\text{th} \chi} f_2 + \frac{1}{\text{sh} \chi} \frac{\nu}{\sqrt{2}} (f_1 + f_3) = 0, \]

\[ -\omega f_1 - i f_1' - \frac{i}{\text{th} \chi} f_1 - \frac{i}{\text{sh} \chi} \frac{\nu}{\sqrt{2}} f_2 = 0, \]

\[ -\omega f_2 + \frac{i}{\text{sh} \chi} \frac{\nu}{\sqrt{2}} (f_1 - f_3) = 0, \]

\[ -\omega f_3 + i f_3' + \frac{i}{\text{th} \chi} f_3 + \frac{i}{\text{sh} \chi} \frac{\nu}{\sqrt{2}} f_2 = 0. \]  

(52)

The system becomes simpler with the substitution

\[ f_1 = \frac{1}{\text{sh} \chi} F_1, \quad f_2 = \frac{1}{\text{sh} \chi} F_2, \quad f_3 = \frac{1}{\text{sh} \chi} F_3, \]

so we get to

\[ (1) \quad \left( \frac{d}{d\chi} + \frac{1}{\text{th} \chi} \right) \omega F_2 + \frac{\omega \nu}{\sqrt{2 \text{sh} \chi}} (F_1 + F_3) = 0, \]

\[ (2) \quad -\omega^2 F_1 - i\omega F_1' - \frac{i \nu}{\sqrt{2 \text{sh} \chi}} \omega F_2 = 0, \]

\[ (3) \quad \omega F_2 = \frac{i \nu}{\sqrt{2 \text{sh} \chi}} (F_1 - F_3), \]

\[ (4) \quad -\omega^2 F_3 + i\omega F_3' + \frac{i \nu}{\sqrt{2 \text{sh} \chi}} \omega F_2 = 0. \]  

(53)

Combining (2) (4) we get

\[ (2) + (4), \]

\[ -\omega (F_1 + F_3) - i(F_1' - F_3') = 0 \]  

(54)

\[ -(2) + (4), \]

\[ \omega^2 (F_1 - F_3) + i\omega (F_1' + F_3') + \frac{2i \nu}{\sqrt{2 \text{sh} \chi}} \frac{i \nu}{\sqrt{2 \text{sh} \chi}} (F_1 - F_3) = 0 \]  

(55)

eq. (1) in (53) reduce to identity 0 \equiv 0 when allowing for (3) and (54), 0 \equiv 0. So we have only three independent equations:

\[ \omega F_2 = \frac{i \nu}{\sqrt{2 \text{sh} \chi}} (F_1 - F_3), \quad -\omega (F_1 + F_3) - i(F_1' - F_3') = 0, \]

\[ \omega^2 (F_1 - F_3) + i\omega (F_1' + F_3') + \frac{2i \nu}{\sqrt{2 \text{sh} \chi}} \frac{i \nu}{\sqrt{2 \text{sh} \chi}} (F_1 - F_3) = 0. \]  

(56)

In new field variables

\[ F = \frac{F_1 + F_3}{\sqrt{2}}, \quad G = \frac{F_1 - F_3}{\sqrt{2}}, \]

eqs. (56) read

\[ F_2 = \frac{i \nu}{\omega \text{sh} \chi} G, \quad F = -\frac{i}{\omega} \frac{d}{d\chi} G, \quad \frac{d^2}{d\chi^2} G + \omega^2 G - \frac{\nu^2}{\text{sh}^2 \chi} G = 0. \]  

(57)
8 Solution of the radial equations in $H_3$

In eq. (57) one needs to introduce new variable
\[ z = 1 - e^{-2\chi}, \quad z = 2\text{sh} \chi e^{-\chi}. \] (58)

Allowing for identities
\[ \frac{d}{d\chi} = 2(1 - z) \frac{d}{dz}, \quad \text{ch} \chi = \frac{2 - z}{z}, \quad \frac{1}{\text{sh}^2 \chi} = \frac{4(1 - z)}{z^2}, \]
eq (57) reduces to
\[ 4(1 - z)^2 \frac{d^2 G}{dz^2} - 4(1 - z) \frac{dG}{dz} + \omega^2 G - \frac{4(1 - z)\nu^2}{z^2} G = 0. \] (59)

With the use of the substitution
\[ G = z^a (1 - z)^b g(z), \]
we arrive at
\[ z(1 - z) \frac{d^2 g}{dz^2} + [2a - (2a + 2b + 1)z] \frac{dg}{dz} + \]
\[ + \left[ -\frac{\omega^2}{4} - (a + b)^2 + (a(a - 1) - \nu^2)\frac{1}{z} + (b^2 + \frac{\omega^2}{4})\frac{1}{1 - z} \right] g = 0. \]

With additional restrictions
\[ a(a - 1) - \nu^2 = 0, \quad b^2 + \frac{\omega^2}{4} = 0 \quad \implies \quad a = j + 1, -j, \quad b = \pm \frac{i\omega}{2}. \] (60)

for $g$ we obtain equation
\[ z(1 - z) \frac{d^2 g}{dz^2} + [2a - (2a + 2b + 1)z] \frac{dg}{dz} - \left[ (a + b)^2 + \frac{\omega^2}{4} \right] g = 0, \] (61)
of hypergeometric type with parameters
\[ \gamma = 2a, \quad \alpha + \beta = 2a + 2b, \quad \alpha\beta = (a + b)^2 + \frac{\omega^2}{4}, \]
or
\[ \alpha = a + b - \frac{i\omega}{2}, \quad \beta = a + b + \frac{i\omega}{2}. \] (62)

Therefore, the function $G$ is given by
\[ G = z^a (1 - z)^b g(z) = \left[ 2\text{sh} \chi e^{-\chi} \right]^a \left[ 1 - 2\text{sh} \chi e^{-\chi} \right]^b g(z). \] (63)
9 Conclusions

Complex formalism of Riemann – Silberstein – Majorana – Oppenheimer in Maxwell electrodynamics is extended to the case of arbitrary pseudo-Riemannian space-time in accordance with the tetrad recipe of Tetrode – Weyl – Fock – Ivanenko. In this approach, the Maxwell equations are solved exactly on the background of simplest static cosmological models, spaces of constant curvature of Riemann and Lobachevsky parameterized by spherical coordinates. Separation of variables is realized in the basis of Schrödinger – Pauli type, description of angular dependence in electromagnetic complex 3-vectors is given in terms of Wigner D-functions. In the case of compact Riemann model a discrete frequency spectrum for electromagnetic modes depending on the curvature radius of space and three discrete parameters is found. In the case of hyperbolic Lobachevsky model no discrete spectrum for frequencies of electromagnetic modes arises.

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