Towards canonical quantum gravity for $G_1$ geometries in 2+1 dimensions with a $\Lambda$-term

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Abstract
The canonical analysis and subsequent quantization of the (2+1)-dimensional action of pure gravity plus a cosmological constant term is considered, under the assumption of the existence of one spacelike Killing vector field. The proper imposition of the quantum analogues of two linear (momentum) constraints reduces an initial collection of state vectors, consisting of all smooth functionals of the components (and/or their derivatives) of the spatial metric, to particular scalar smooth functionals. The demand that the midi-superspace metric (inferred from the kinetic part of the quadratic (Hamiltonian) constraint) must define on the space of these states an induced metric whose components are given in terms of the same states, which is made possible through an appropriate re-normalization assumption, severely reduces the possible state vectors to three unique (up to general coordinate transformations) smooth scalar functionals. The quantum analogue of the Hamiltonian constraint produces a Wheeler–DeWitt equation based on this reduced manifold of states, which is completely integrated.

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1. Introduction

Dirac’s seminal work on his formalism for a self-contained treatment of systems with constraints [1–4] has paved the way for a systematic treatment of constrained dynamics. Some of the landmarks in the study of constrained systems have been the connection between constraints and invariances [5], the extension of the formalism to describe fields with half-integer spin through the algebra of Grassmann variables [6] and the introduction of the BRST formalism [7]. All the classical results obtained so far have made up an armory prerequisite for the quantization of gauge theories, and there are several excellent reviews studying constrained systems with a finite number of degrees of freedom [8] or constrained field theories [9], as well as more general presentations [10–15]. In particular, the conventional canonical analysis approach of quantum gravity has been initiated by B S DeWitt [16] based on earlier work of P G Bergmann [17].

In the absence of a full theory of quantum gravity, it is reasonably important to address the quantization of (classes of) simplified geometries. An elegant way to achieve a degree of simplification is to impose some symmetry. For example, the assumption of a $G_3$ symmetry group acting simply transitively on the surfaces of simultaneity, i.e. the existence of three independent spacelike Killing vector fields, leads to classical and subsequently quantum homogeneous cosmology (see, e.g., [18, 19]). The imposition of lesser symmetry, e.g. fewer Killing vector fields, results in the various inhomogeneous cosmological models [20]. The canonical analysis under the assumption of spherical symmetry, which is a $G_5$ group acting multiply transitively on two-dimensional spacelike subsurfaces of the three slices, has been first considered in [21, 22]. Quantum black holes have also been treated, for instance, in [23, 24] while in [25] a lattice regularization has been employed to deal with the infinities arising due to the ill-defined nature of the quantum operator constraints.

Another way to arrive at simplified models is to consider lower dimensions. For example, there is a vast literature on (2+1)-dimensional gravity (see, e.g., [26–28] and references therein). The role of non-commutative geometry in (2+1)-dimensional quantum gravity has been recently investigated in [29]. In this work, we consider the canonical quantization of all 2+1 geometries admitting one spacelike Killing vector field. In section 2, we give the reduced metrics, the space of classical solutions and the Hamiltonian formulation of the reduced Einstein–Hilbert action principle, resulting in one (quadratic) Hamiltonian and two (linear) momentum constraints, all being first class. In section 3, we consider the quantization of this constraint system following Dirac’s proposal of implementing the quantum operator constraints as conditions annihilating the wavefunction [4]. Our guideline is a conceptual generalization of the quantization scheme developed in [30, 31] for the case of constraint systems with finite degrees of freedom, to the present case. Even though after the symmetry reduction the system still represents a field theory (all remaining metric components depend on time and the radial coordinate), we manage to extract and subsequently completely solve a Wheeler–DeWitt equation in terms of three unique smooth scalar functionals of the appropriate components of the reduced spatial metric. This is achieved through an appropriate re-normalization assumption we adopt. Finally, some concluding remarks are included in the discussion.

2. Possible metrics and Hamiltonian formulation

Our starting point is the action principle:

$$I = \int d^3x \sqrt{-g} (R - 2\Lambda). \quad (2.1)$$
The equations of motion arising upon variation of this action are

\[ R_{IJ} - \frac{1}{2} g_{IJ} R + \Lambda g_{IJ} = 0, \]  

(2.2)

where \( I, J = 0, 1, 2 \). Of course, since in three dimensions the Riemann curvature tensor is expressible in terms of both the Ricci tensor and scalar, the space of solutions to (2.2) consists simply of all maximally symmetric 3D metrics (AdS3). If topological considerations are taken into account, the above space might be ‘enriched’ containing, for example, the stationary BTZ ‘black’ hole [32, 33]:

\[
d s^2 = -(M - \Lambda r^2) \, dt^2 - J \, dt \, d\phi + \left( M - \Lambda r^2 + \frac{J^2}{4r^2} \right)^{-1} \, dr^2 + r^2 \, d\phi^2 \]  

(2.3)

or the ‘cosmological’ solutions [34, 35]:

\[
d s^2 = -\frac{1}{4r^2 \Lambda} \, dr^2 + \frac{1}{2t \sqrt{\Lambda}} \, (dr^2 + d\phi^2), \]  

(4.4)

\[
d s^2 = -\left( \frac{4}{16r^2 - \Lambda} \right)^2 \, dr^2 + \frac{4}{16r^2 - \Lambda} \, dr^2 + \frac{4 e^{-4r}}{16r^2 - \Lambda} \, d \phi^2. \]  

(2.5)

Note that all these three line elements are locally AdS3, and therefore admit six local Killing fields. Their differences consist in the topological identifications. At this point, we deem it pertinent to explain our view concerning the issue of the bearing of topology on a local theory. The Hamiltonian formulation is by itself implying a spacetime topology \( R \times \Sigma^2 \). Consequently, what we are concerned with is the topology of the 2-slices. Since the theory is local, it is implicitly assumed that the entire analysis holds in a coordinate patch. Different topologies can only affect the number of patches needed to cover the space and, therefore, can only impose restrictions on the range of the coordinates and/or the range of validity of local fields, such as the symmetry generators admitted by these metrics. The paradigm of the cylinder may help clarify our point. The integral curves of rotations in the plane are circles, but if one tries to draw a circle of radius \( R \geq 2\pi L \) on the cylinder (\( L \) being the cylinder’s radius), crossings (or a pinch in the case of equality) will occur, indicating that the corresponding generator is ill defined. In such a situation one can, as many do, drop rotations altogether; this is the case in [32, 33], where four of the six Killing fields are considered as non-valid symmetries. On the other hand, one can accept integral curves (circles) of radius \( R < 2\pi L \) (by suitably restricting the range of validity of the Killing field), which would simply result in the need of two patches to cover the cylinder with these lines. We adopt the latter point of view, as it seems to us much more reasonable. We shall thus not specify any ranges for our coordinates \((t, r, \phi)\) precisely to allow for different topological options, which are not otherwise affecting our results.

In this spirit, we can say that the above metrics admit a \( G_6 \) symmetry group. In what follows, we consider a generalization consisting in the imposition of a \( G_1 \) symmetry only, i.e. we impose one Killing vector field, say \( \xi = \frac{\partial}{\partial \phi} \). Subsequently, all components of the metric become functions of both time and the radial coordinate only. The canonical decomposition of such a metric is given in terms of the spatial metric \( g_{ij}(t, r) \), the lapse function \( N^0(t, r) \) and the shift ‘vector’ \( N_i(t, r) \) [10]:

\[
d s^2 = (-N^0)^2 + g^{ij} N_i N_j \, dt^2 + 2N_i \, dt \, dx^i + g_{ij} \, dx^i \, dx^j, \]  

(2.5)

where

\[
g_{ij} = \left( \begin{array}{cc} \rho^2 + \sigma^2 \chi^2 & \sigma \chi \\ \sigma \chi & \sigma \end{array} \right), \quad g^{ij} = \left( \begin{array}{cc} \sigma & -\sigma \chi \\ -\sigma \chi & \rho^2 \end{array} \right), \]  

(2.7)
with \( i, j = 1, 2 \), and \( x^i = (r, \phi) \). The particular parametrization of \( g_{ij} \) above has been chosen in such a way that to simplify the second linear constraint (see below), and consequently the resulting algebra.

For the Hamiltonian formulation of the system (2.6) (see, e.g., chapter 9 of [10]), we first define the vectors

\[
\eta^I = \frac{1}{N^0} (1, -N^i), \quad N^I \equiv g^{jk} N_k,
\]

\[
P^I = \eta^I \eta^J - \eta^J \eta^I.
\]

where \( I, J \) are spacetime indices, and ‘;’ stands for covariant differentiation with respect to (2.6). Then, utilizing the Gauss–Codazzi equation (see, e.g., [36]), we eliminate all second time derivatives from the Einstein–Hilbert action and arrive at an action quadratic in the velocities,

\[
H = \int (N^o \mathcal{H}_o + N^i \mathcal{H}_i) \, dr,
\]

where \( \mathcal{H}_o, \mathcal{H}_i \) are given by

\[
\mathcal{H}_o = \frac{1}{2} G^{\alpha \beta} \pi_\alpha \pi_\beta + V \\
\mathcal{H}_1 = \sigma' \pi_\sigma - \rho \pi_\rho - \chi \pi_\chi \\
\mathcal{H}_2 = -\pi'_\chi,
\]

the indices \((\alpha, \beta)\) take the values \((\rho, \sigma, \chi)\) and \( ' \equiv \partial / \partial r \). The Wheeler–DeWitt midi-superspace metric \( G^{\alpha \beta} \) reads

\[
G^{\alpha \beta} = \begin{pmatrix}
-\rho & -\sigma & 0 \\
-\sigma & 0 & 0 \\
0 & 0 & \rho/\sigma^2
\end{pmatrix},
\]

while the potential \( V \) is

\[
V = 2\Lambda \rho + \left( \frac{\sigma'}{\rho} \right)'.
\]

The requirement for preservation, in time, of the primary constraints leads to the secondary constraints

\[
\mathcal{H}_o \approx 0, \quad \mathcal{H}_1 \approx 0, \quad \mathcal{H}_2 \approx 0.
\]

At this stage, a tedious but straightforward calculation produces the following ‘open’ Poisson bracket algebra of these constraints:

\[
[H_o(r), H_o(\bar{r})] = [g^{11}(r)H_1(r) + g^{11}(\bar{r})H_1(\bar{r})] \delta'(r, \bar{r}) \\
[H_1(r), H_o(\bar{r})] = H_o(r) \delta'(r, \bar{r}) \\
[H_2(r), H_o(\bar{r})] = 0 \\
[H_1(r), H_1(\bar{r})] = H_1(r) \delta'(r, \bar{r}) - H_1(\bar{r}) \delta(r, \bar{r})' \\
[H_1(r), H_2(\bar{r})] = H_2(r) \delta'(r, \bar{r}) \\
[H_2(r), H_2(\bar{r})] = 0
\]

indicating that they are first class and also signaling the termination of the algorithm. Thus, our system is described by (2.12); the ‘dynamical’ Hamilton–Jacobi equations
\[
\frac{d\pi_{\rho}}{dt} = \{\pi_{\rho}, H\}, \quad \frac{d\pi_{\sigma}}{dt} = \{\pi_{\sigma}, H\}, \quad \frac{d\pi_{\chi}}{dt} = \{\pi_{\chi}, H\}
\]
are satisfied by virtue of the time derivatives of (2.12). One can readily check (as one must always do with reduced action principles) that these three equations, when expressed in the velocity phase space with the help of the definitions \(\frac{d}{dt} = \{\rho, H\}, \frac{d\sigma}{dt} = \{\sigma, H\}, \frac{d\chi}{dt} = \{\chi, H\}\), are completely equivalent to the three independent Einstein’s field equations satisfied by (2.6).

We end up this section by noting a few facts concerning the transformation properties of \(\rho(t, r), \sigma(t, r), \chi(t, r)\), and their spatial derivatives under changes of the radial variable \(r\) of the form \(r \rightarrow \tilde{r} = h(r)\). As it can easily be inferred from (2.6) and (2.7),

\[
\begin{align*}
\tilde{\rho}(\tilde{r}) &= \rho(r) \frac{dr}{d\tilde{r}}, \\
\tilde{\sigma}(\tilde{r}) &= \sigma(r), \\
\tilde{\chi}(\tilde{r}) &= \chi(r) \frac{dr}{d\tilde{r}},
\end{align*}
\]

(2.14)

where the \(t\)-dependence has been omitted for the sake of brevity. Thus, under the above coordinate transformations, \(\sigma, \chi\) are scalars, while \(\rho, \chi\) and the derivatives of \(\sigma, \chi\) are covariant rank 1 tensors (one forms), or, equivalently in one dimension, scalar densities of weight \(-1\). Therefore, the scalar derivative is not \(\frac{d}{dr}\) but rather \(\frac{d}{d\tilde{r}}\) or \(\frac{d}{d\tilde{r}} \equiv \tilde{\xi} \frac{d}{dr}\). Finally, if we consider an infinitesimal transformation \(r \rightarrow \tilde{r} = r - \eta(r)\), it is easily seen that the corresponding changes induced on the basic fields are

\[
\begin{align*}
\delta \rho(r) &= (\rho(r)\eta(r))', \\
\delta \sigma(r) &= \sigma'(r)\eta(r), \\
\delta \chi(r) &= (\chi(r)\eta(r))',
\end{align*}
\]

(2.15)
i.e., nothing but the one-dimensional analogue of the appropriate Lie derivatives.

With the use of (2.15), we can reveal the nature of the action of \(\mathcal{H}_1\) on the basic configuration space variables as that of the generator of spatial diffeomorphisms:

\[
\begin{align*}
\{\rho(r), \int d\tilde{r} \, \eta(\tilde{r}) \mathcal{H}_1(\tilde{r})\} &= (\rho(r)\eta(r))', \\
\{\sigma(r), \int d\tilde{r} \, \eta(\tilde{r}) \mathcal{H}_1(\tilde{r})\} &= \sigma'(r)\eta(r), \\
\{\chi(r), \int d\tilde{r} \, \eta(\tilde{r}) \mathcal{H}_1(\tilde{r})\} &= (\chi(r)\eta(r))'.
\end{align*}
\]

(2.16)

Thus, we are justified to consider \(\mathcal{H}_1\) as the representative, in phase space, of an arbitrary infinitesimal reparametrization of the radial coordinate. As far as \(\mathcal{H}_2\) is concerned, the situation is a little more complicated: the imposition of the symmetry generated by the Killing vector field \(\xi = \partial/\partial \phi\) has left all configuration variables without any \(\phi\) dependence; subsequently, we cannot expect \(\mathcal{H}_2\) to generate arbitrary infinitesimal reparametrization of \(\phi\). Nevertheless, we can identify a property of \(\mathcal{H}_2\) which links its existence to the existence of \(\xi\). This property is described by the relation \(\{\mathcal{H}_2(r), \{\mathcal{H}_2(r), \{\mathcal{H}_2(r), g_{ij}(\tilde{r})\}\}\} = 0 \iff \mathcal{L}_\xi g_{ij} = 0\).

3. Quantization

We are now interested in attempting to quantize this Hamiltonian system following Dirac’s general spirit of realizing the classical first class constraints (2.12) as quantum operator constraint conditions annihilating the wavefunctional. The main motivation behind such an approach is the justified desire to construct a quantum theory manifestly invariant under the ‘gauge’ generated by the constraints. To begin with, let us first note that, despite the simplification brought by the imposition of the symmetry \(\xi = \partial/\partial \phi\) \(\Rightarrow \mathcal{L}_\xi g_{ij} = 0\), the system is still a field theory in the sense that all configuration variables and canonical
conjugate momenta depend not only on time (as is the case in homogeneous cosmology), but also on the radial coordinate $r$. Thus, to canonically quantize the system in the Schrödinger representation, we first realize the classical momenta as functional derivatives with respect to their corresponding conjugate fields:

$$\hat{\pi}_\rho(r) = -i \frac{\delta}{\delta \rho(r)}, \quad \hat{\pi}_\sigma(r) = -i \frac{\delta}{\delta \sigma(r)}, \quad \hat{\pi}_\chi(r) = -i \frac{\delta}{\delta \chi(r)}.$$ 

Next, we have to decide on the initial space of state vectors. To elucidate our choice, let us consider the action of a momentum operator on some function of the configuration field variables, say

$$\hat{\pi}_\rho(r) \rho(\tilde{r})^2 = -2i \rho(\tilde{r}) \delta(\tilde{r}, r).$$

The Dirac delta function renders the outcome of this action a distribution rather than a function. Also, if the momentum operator were to act at the point at which the function is evaluated, i.e., if $\tilde{r} = r$, then its action would produce a $\delta(0)$ and would therefore be ill defined. Both of these unwanted features are rectified, as far as expressions linear in momentum operators are concerned, if we choose as our initial collection of states all smooth functionals (i.e., integrals over $r$) of the configuration variables $\rho(r), \sigma(r), \chi(r)$ and their derivatives of any order. Indeed, as we infer from the previous example,

$$\hat{\pi}_\rho(r) \int d\tilde{r} \rho(\tilde{r})^2 = -2i \int d\tilde{r} \rho(\tilde{r}) \delta(\tilde{r}, r) = -2i \rho(r);$$

thus, the action of the momentum operators on all such states will be well defined (no $\delta(0)$’s) and will also produce only local functions and not distributions. However, even so, $\delta(0)$’s will appear as soon as local expressions quadratic in momenta are considered, e.g.,

$$\hat{\pi}_\rho(r) \hat{\pi}_\rho(r) \int d\tilde{r} \rho(\tilde{r})^2 = \hat{\pi}_\rho(r) \left( -2i \int d\tilde{r} \rho(\tilde{r}) \delta(\tilde{r}, r) \right) = \hat{\pi}_\rho(r) (-2i \rho(r)) = -2i \delta(r, r).$$

Another problem of equal, if not greater, importance has to do with the number of derivatives (with respect to $r$) considered. A momentum operator acting on a smooth functional of degree $n$ in derivatives of $\rho(r), \sigma(r), \chi(r)$ will, in general, produce a function of degree $2n$, e.g.,

$$\hat{\pi}_\rho(r) \int d\tilde{r} \rho''(\tilde{r})^2 = -2i \int d\tilde{r} \rho''(\tilde{r}) \delta''(\tilde{r}, r) = -2i \rho^{(4)}(r).$$

Thus, clearly, more and more derivatives must be included if we desire the action of momentum operators to keep us inside the space of integrands corresponding to the initial collection of smooth functionals; eventually, we have to consider $n \rightarrow \infty$. This, in a sense, can be considered as the reflection to the canonical approach, of the non-re-normalizability results existing in the so-called covariant approach. The way to deal with these problems is, loosely speaking, to regularize (i.e., render finite) the infinite distribution limits, and re-normalize the theory by, somehow, enforcing $n$ to terminate at some finite value.

In the following, we are going to present a quantization scheme of our system which (a) avoids the occurrence of $\delta(0)$’s, (b) reveals the value $n = 1$, as the only possibility of obtaining a closed space of state vectors, and (c) extracts a finite-dimensional Wheeler–DeWitt equation governing the quantum dynamics. The scheme closely parallels, conceptually, the quantization developed in [30, 31] for finite systems with one quadratic and a number of linear first class constraints. Therefore, we deem it appropriate and instructive to present a brief account of the essentials of this construction.

To this end, let us consider a system described by a Hamiltonian of the form

$$H \equiv \mu \chi_i + \mu^i \chi_i$$

$$= \mu \left( \frac{1}{2} G^{AB}(\bar{Q}^i) P_A P_B + U^A(\bar{Q}^i) P_A + V(\bar{Q}^i) \right) + \mu^i \phi^i(\bar{Q}^i) P_A.$$  

(3.1)
where $A, B, \Gamma \ldots = 1, 2, \ldots, M$ count the configuration space variables and $i = 1, 2, \ldots, N < (M - 1)$ numbers the super-momenta constraints $\chi_i \approx 0$, which along with the super-Hamiltonian constraint $X \approx 0$ are assumed to be first class:

$$\{X, X\} = 0, \quad \{X, \chi_i\} = XC_i + C^j_i \chi_j, \quad \{\chi_i, \chi_j\} = C^k_{ij} \chi_k.$$  \hspace{1cm} (3.2)

where the first (trivial) Poisson bracket has been included only to emphasize the difference from the first of (2.13).

The physical state of the system is unaffected by the ‘gauge’ transformations generated by $(X, \chi_i)$, but also under the following three changes:

1. Mixing of the super momenta with a non-singular matrix

$$\tilde{\chi}_i = \lambda^j_i (Q/\Gamma) \chi_j.$$

2. Gauging of the super Hamiltonian with the super momenta

$$\tilde{X} = X + \kappa (A_i (Q/\Gamma) \phi B_j (Q/\Gamma) P_A P_B + \sigma^i (Q/\Gamma) \phi^A_i (Q/\Gamma) P_A,$$

3. Scaling of the super Hamiltonian

$$\tilde{X} = \tau^2 (Q/\Gamma) X.$$

Therefore, the geometrical structures on the configuration space that can be inferred from the super Hamiltonian are really equivalence classes under actions (I), (II) and (III); for example, (II) and (III) imply that the super-metric $G_{AB}$ is known only up to conformal scalings and additions of the super-momenta coefficients $\tilde{G}_{AB} = \tau^2 (G_{AB} + \kappa (A_i \phi^B_j))$. It is thus mandatory that, when we Dirac-quantize the system, we realize the quantum operator constraint conditions on the wavefunction in such a way as to secure that the whole scheme is independent of actions (I), (II) and (III). This is achieved by the following steps:

1. Realize the linear operator constraint conditions with the momentum operators to the right:

$$\hat{\chi}_i \Psi = 0 \leftrightarrow \phi^A_i (Q/\Gamma) \frac{\partial \Psi(Q/\Gamma)}{\partial Q^A} = 0,$$

which maintains the geometrical meaning of the linear constraints and produces the $M - N$ independent solutions to the above equations $q^\alpha (Q/\Gamma), \alpha = 1, 2, \ldots, M - N$ called physical variables, since they are invariant under the transformations generated by $\hat{\chi}_i$.

2. In order to make the final states physical with respect to the ‘gauge’ generated by the quadratic constraint $\tilde{X}$ aswell.

Define the induced structure $g^{a\beta} = G_{AB} \frac{\partial q^a}{\partial Q^A} \frac{\partial q^\beta}{\partial Q^B}$ and realize the quadratic in momenta part of $X$ as the conformal Laplace–Beltrami operator based on $g_{ab}$. Note that in order for this construction to be self consistent, all components of $g_{ab}$ must be functions of the physical coordinates $q^a$. This can be proven to be so by virtue of the classical algebra the constraints satisfy (for specific quantum cosmology examples see [19]).

We are now ready to proceed with the quantization of our system, in close analogy to the scheme outlined above. In order to realize the equivalent to step 1, we first define the quantum analogue of $\hat{H}_1(r) \approx 0$ as

$$\hat{\mathcal{H}}_1 (r) \Phi = 0 \leftrightarrow - \rho (r) \left( \frac{\delta \Phi}{\delta \rho (r)} \right) + \sigma (r) \frac{\delta \Phi}{\delta \sigma (r)} - \chi (r) \left( \frac{\delta \Phi}{\delta \chi (r)} \right) = 0.$$  \hspace{1cm} (3.3)
As explained in the beginning of the section, the action of $\hat{\mathcal{H}}_1^\prime(r)$ on all smooth functionals is well defined, i.e., produces no $\delta(0)$’s. It can be proven that, in order for such a functional to be annihilated by this linear quantum operator, it must be scalar, i.e. have the form

$$
\Phi = \int \rho(\tilde{r}) f(\Sigma^{(0)}, \Sigma^{(1)}, \ldots, \Sigma^{(n)}, X^{(0)}, X^{(1)}, \ldots, X^{(n)}) \, d\tilde{r},
$$

(3.4a)

$$
\Sigma^{(0)} \equiv \sigma(\tilde{r}), \quad \Sigma^{(1)} \equiv \sigma'(\tilde{r}), \ldots, \quad \Sigma^{(n)} \equiv \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left( \cdots \sigma(\tilde{r}) \right),
$$

(3.4b)

$$
X^{(0)} \equiv \frac{\chi(\tilde{r})}{\rho(\tilde{r})}, \quad X^{(1)} \equiv \frac{1}{\rho(\tilde{r})} \left( \frac{\chi(\tilde{r})}{\rho(\tilde{r})} \right)', \ldots, \quad X^{(n)} \equiv \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left( \cdots \frac{\chi(\tilde{r})}{\rho(\tilde{r})} \right),
$$

(3.4c)

where $f$ is any function of its arguments. We note that, as is discussed at the end of the previous section, $\sigma'$ is the only scalar first derivative of $\sigma$, and likewise for the higher derivatives. The proof of this statement is analogous to the proof of the corresponding result concerning full gravity [37]: consider an infinitesimal $r$-reparametrization $\tilde{r} = r - \eta(r)$. Under such a change, the left-hand side of (3.4), being a number, must remain unaltered. If we calculate the change induced on the right-hand side we arrive at

$$
0 = \int \left[ f \delta \rho + \rho \frac{\delta f}{\delta \sigma} + \rho \frac{\delta f}{\delta (\chi/\rho)} \delta \frac{\chi}{\rho} \right] \, dr = \int [\rho \hat{\mathcal{H}}_1^\prime(f)] \eta(r) \, dr,
$$

(3.5)

where use of (2.15) and a partial integration has been made. Since this must hold for any $\eta(r)$, the result sought for is obtained.

We now turn to the second linear constraint and try to see what are the restrictions it brings into our space of state vectors. We define

$$
\hat{\mathcal{H}}_2(r) \Phi = 0 \iff \left( \frac{\delta \Phi}{\delta X(r)} \right)' = 0 \iff \frac{\delta \Phi}{\delta X(r)} = k,
$$

(3.6)

where $k$ is any constant (with respect to $r$) independent of the basic fields and their derivatives, and $\Phi$ is given by (3.4a)-(3.4c). As we argued before, the functional derivative $\frac{\delta}{\delta X(r)}$ acting on $X^{(n)}$ will produce, upon partial integration of the $n$th derivative of the Dirac delta function, a term proportional to $X^{(2n)}$. Since the arguments of $f$ in (3.4a) reach only up to $X^{(n)}$, it is evident that $f$ must be such that the coefficient of $X^{(2n)}$ vanishes; more precisely:

$$
\frac{\delta \Phi}{\delta X(r)} = k \iff \ldots + \int \rho(\tilde{r}) \frac{\partial f}{\partial X^{(n)}(\tilde{r})} \frac{\delta X^{(0)}(\tilde{r})}{\delta X(r)} \, d\tilde{r} = k \iff
$$

$$
\ldots + \int \rho(\tilde{r}) \frac{\partial f}{\partial X^{(n)}(\tilde{r})} \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} \left( \cdots \frac{\delta (r, \tilde{r})}{\rho(\tilde{r})} \right) \, d\tilde{r} = k \iff
$$

$$
\ldots + (-1)^n \int \frac{\partial^2 f}{\partial (X^{(n)}(\tilde{r}))^2} X^{(2n)}(\tilde{r}) \delta (r, \tilde{r}) \, d\tilde{r} = k \iff
$$

$$
\ldots + (-1)^n \frac{\partial^2 f}{\partial (X^{(n)})^2} X^{(2n)} = k.
$$

Thus, since all the terms hidden in $\ldots$ do not involve $X^{(2n)}$, and (3.6) must be satisfied identically for all $X^{(n)}$’s $k = 0, 1, \ldots, 2n$, we conclude that $\frac{\partial^2 f}{\partial (X^{(2n)})^2} = 0$ in order for this equation to have the possibility of being satisfied. Subsequently,

$$
f = f_1(\Sigma^{(0)}, \ldots, \Sigma^{(n)}, X^{(0)}, \ldots, X^{(n-1)}) X^{(n)} + f_2(\Sigma^{(0)}, \ldots, \Sigma^{(n)}, X^{(0)}, \ldots, X^{(n-1)}).
$$
Now, the term in $\Phi$ corresponding to $f_1$ is, up to a surface term, equivalent to a general term depending on $X(0), \ldots, X(n-1)$ only; indeed,

$$\Phi_1 = \int \rho(\tilde{r}) f_1 \frac{1}{\rho(\tilde{r})} \frac{d}{d\tilde{r}} X^{(n-1)} d\tilde{r},$$

which upon subtraction of the surface term

$$A = \int d\tilde{r} \frac{d}{d\tilde{r}} \left( \int dX^{(n-1)} f_1 \right)$$

produces a smooth functional with arguments up to $X^{(n-1)}$ only. Since a surface term in $\Phi$ does not affect the outcome of the variational derivative $\frac{\delta \Phi}{\delta \chi(r)}$, we conclude that only $f_2$ is important for the local part of $\Phi_1$. The entire argument can be repeated successively for $n-1, n-2, \ldots, 1$; therefore, all $X^{(n)}$'s are suppressed from $f$ except when $n = 0$. Thus, finally, upon inserting into (3.6) the resulting functional

$$\Phi = \int \rho(\tilde{r}) h(\Sigma(0), \ldots, \Sigma^{(0)}, X(0)) d\tilde{r},$$

we obtain

$$\frac{\delta \Phi}{\delta \chi(r)} = k \leftrightarrow \int \rho(\tilde{r}) \frac{\partial h}{\partial X^{(0)}} \delta(r, \tilde{r}) \frac{d\tilde{r}}{\rho(\tilde{r})} = k \leftrightarrow \frac{\partial h}{\partial X^{(0)}} = k \leftrightarrow \frac{h}{\rho(r)} = k \leftrightarrow h = k \chi(r) + L(\Sigma(0), \ldots, \Sigma^{(n)}).$$

We have thus reached the conclusion that the imposition of both linear quantum operators $\hat{H}_1$ and $\hat{H}_2$ dictates the form of the smooth functional to be

$$\Phi = k \int d\tilde{r} \chi(\tilde{r}) + \int d\tilde{r} \rho(\tilde{r}) L(\Sigma^{(0)}, \ldots, \Sigma^{(n)}). \quad (3.7)$$

We now try to realize step 2 of the programme previously outlined. We have to define the equivalent of Kuchař’s induced metric on the so far space of ‘physical’ states $\Phi$ described by (3.7) which are the analogues, in our case, of Kuchař’s physical variables $q^a$. Let us start our investigation by considering one initial candidate of the above form. Then, generalizing the partial to functional derivatives, the induced metric will be given by

$$g^{\Phi\Phi} = G^{a\beta} \frac{\delta \Phi}{\delta x^a} \frac{\delta \Phi}{\delta x^\beta}, \quad (3.8)$$

where $(x^1, x^2, x^3) = (\rho, \sigma, \chi)$, and $G^{a\beta}$ is given by (2.10). Note that this metric is well defined since it contains only first functional derivatives of the state vectors, as opposed to any second-order functional derivative operator that might have been considered as a quantum analogue of the kinetic part of $\hat{H}_0$. Nevertheless, $g^{\Phi\Phi}$ is a local function and not a smooth functional. It is thus clear that, if we want the induced metric $g^{\Phi\Phi}$ to be composed out of the ‘physical’ states annihilated by $\hat{H}_1, \hat{H}_2$, we must establish a correspondence between local functions and smooth functionals. A way to achieve this is to adopt the following ansatz.

**Assumption.** We assume that, as part of the re-normalization procedure, we are permitted to map local functions to their corresponding smeared expressions, e.g., $\chi(r) \leftrightarrow \int d\tilde{r} \chi(\tilde{r})$.

Let us be more specific, concerning the meaning of the above assumption. Let $\mathcal{F}$ be the space which contains all local functions, and define the equivalence relations

$$\sim: \{ f_1(r) \sim f_2(\tilde{r}), \tilde{r} = g(r) \}, \quad \approx: \{ h_1(r) \approx h_2(\tilde{r}), \frac{d\tilde{r}}{dr}, \tilde{r} = g(r) \} \quad (3.9)$$

for scalars and densities, respectively.
Now, let \( F_o = \{ f \in F, \mod(\sim, \approx) \} \) and \( F_I \) the space of the smeared functionals. We define the one-to-one maps \( G, G^{-1} : \)

\[
\begin{align*}
G : F_o &\rightarrow F_I : \chi(r) \mapsto \int \chi(r) \, dr, \\
G^{-1} : F_I &\rightarrow F_o : \int \chi(r) \, dr \mapsto \chi(r).
\end{align*}
\]

The necessity to define the maps \( G, G^{-1} \) on the equivalence classes and not on the individual functions stems out of the fact that we are trying to develop a quantum theory of the geometries \((2.6), (2.7)\) and not of their coordinate representations. If we had tried to define the map \( G \) from the original space \( F \) to \( F_I \) we would end up with states which would not be invariant under spatial coordinate transformations \( (r \text{-reparametrizations}) \).

Indeed, one can make a correspondence between local functions and smeared expressions, but smeared expressions must contain another arbitrary smearing function, say \( s(r) \). Then the map between functions and smeared expressions is one to one (as is also the above map) and is given by multiplying with \( s(r) \) and integrating over \( r \); while the inverse map is given by varying w.r.t. \( s(r) \).

However, this would be in the opposite direction from that which led us to the states \((3.7)\) by the imposition of the linear operator constraints. As an example, consider the action of these operators on two particular cases of the states \((3.7)\), containing the structure \( \Sigma^{(n)} \):

\[
\hat{H}_1(r) \int s(\tilde{r}) \rho(\tilde{r}) \sigma(\tilde{r}) \, d\tilde{r} = -s'(r) \rho(r) \sigma(r) \neq 0 \quad \text{for arbitrary } s(r),
\]

\[
\hat{H}_2(r) \int s(\tilde{r}) \chi(\tilde{r}) \, d\tilde{r} = s'(r) \neq 0 \quad \text{for arbitrary } s(r).
\]

Thus, every foreign to the geometry structure \( s(r) \) is not allowed to enter the physical states.

Now, after the correspondence has been established, we can come to the basic property the induced metric must have. In the case of finite degrees of freedom, the induced metric depends, up to a conformal scaling, on the physical coordinates \( q^\alpha \) by virtue of \((3.2)\). In our case, due to the dependence of the configuration variables on the radial coordinate \( r \), the above property is not automatically satisfied; e.g. the functional derivative \( \frac{\delta}{\delta \sigma(r)} \) acting on \( \Sigma^{(n)} \) will produce, upon partial integration of the \( n \)th derivative of the Dirac delta function, a term proportional to \( \Sigma^{(2n)} \).

Therefore, since \( L \) in \((3.7)\) contains derivatives of \( \sigma(r) \) up to \( \Sigma^{(n)} \), the above-mentioned property must be enforced. The need for this can also be traced to the substantially different first Poisson bracket in \((2.13)\), which signals a non-trivial mixing between the dynamical evolution generator \( \mathcal{H}_o \) and the linear generators \( \mathcal{H}_i \).

Thus, according to the above reasoning, in order to proceed with the generalization of the Kuchař's method, we have to demand that

**Requirement.** \( L(\Sigma^{(0)}, \ldots, \Sigma^{(n)}) \) must be such that \( g^{\Phi \Phi} \) becomes a general function, say \( F(k \chi(r) + \rho(r) L(\Sigma^{(0)}, \ldots, \Sigma^{(n)})) \) of the integrand of \( \Phi \), so that it can be considered as a function of this state: \( g^{\Phi \Phi} \leftrightarrow F(k \int \chi(\tilde{r}) \, d\tilde{r} + \int \rho(\tilde{r}) L(\Sigma^{(0)}, \ldots, \Sigma^{(n)}) \, d\tilde{r}) = F(\Phi) \).

At this point, we must emphasize that the application of the requirement in the subsequent development of our quantum theory will result in very severe restrictions on the form of \((3.7)\). Essentially, \( \chi(r) \) as well as all higher derivatives of \( \sigma(r) \) (i.e. \( \Sigma^{(2)} \ldots \Sigma^{(n)} \)) are eliminated from \( \Phi \) (see \((3.14), (3.23)\)). This might, at first sight, strike as odd; indeed, the common belief is that all the derivatives of the configuration variables should enter the physical states. However, before the imposition of both the linear and the quadratic constraints there are no
truly physical states. Thus, no physical states are lost by the imposition of the requirement; ultimately, the only true physical states are the solutions to (3.24).

Having clarified the way in which we view the assumption and requirement above, we now proceed to the restrictions implied by their use.

A first consequence of the requirement that 
\[ g_{\Phi}\Phi = F(k\chi(r) + \rho(r)L(\Sigma^{(0)}, \ldots, \Sigma^{(n)})) \]
is the vanishing of \( k \). This follows from (a) the property that 
\[ g_{\Phi}\Phi \] is homogenous in the functional derivative \( \frac{\delta}{\delta \chi(r)} \), (b) that \( G_{\alpha\beta} \) in (2.10) does not contain any \( \chi(r) \); namely
\[ g_{\Phi}\Phi = \cdots + 2G_{12} \frac{\delta}{\delta \rho(r)} \frac{\delta}{\delta \sigma(r)} , \]
where the functional derivatives are
\[ \frac{\delta}{\delta \sigma} = \cdots + \int \frac{\rho}{\sigma^{(n)}} \frac{\partial L}{\partial \Sigma^{(n)}} \frac{\partial}{\partial \Sigma^{(n)}} \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \]
\[ = \cdots - \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \right) \]
\[ = \cdots + (-1)^n \int \frac{\rho(\tilde{r})}{\sigma^{(n)}} \frac{\partial^2 L}{\partial \Sigma^{(n)} \partial \rho(r)} \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \]
\[ = \cdots + (-1)^n \rho \frac{\partial^2 L}{\partial \Sigma^{(n)} \partial \rho(r)} \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \]

and
\[ \frac{\delta}{\delta \rho} = \cdots + \int \frac{\rho}{\sigma^{(n)}} \frac{\partial L}{\partial \rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} = \cdots + \int \frac{\rho}{\sigma^{(n)} \rho} \frac{\partial L}{\partial \rho} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]
\[ = \cdots + \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]
\[ = \cdots - \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]

Since . . . are terms not involving \( \chi(r) \), the final identification is possible iff \( k = 0 \). Thus, \( \Phi \) is reduced to
\[ \Phi = \int d\tilde{r} \rho(\tilde{r})L(\Sigma^{(0)}, \ldots, \Sigma^{(n)}) . \quad (3.13) \]

We now turn to the degree of derivatives \( (n) \) of \( \sigma(r) \). The situation is similar to the corresponding case with \( X(n) \) considered before; again the functional derivative \( \frac{\delta}{\delta \sigma(r)} \) acting on \( \Phi \) will bring a maximum term \( \Sigma^{(2n)} \) while \( \frac{\delta}{\delta \rho(r)} \), a corresponding term \( \Sigma^{(2n-1)} \). More precisely,
\[ g_{\Phi}\Phi = \cdots + 2G_{12} \frac{\delta}{\delta \rho(r)} \frac{\delta}{\delta \sigma(r)} \]

where the functional derivatives are
\[ \frac{\delta}{\delta \sigma} = \cdots + \int \frac{\rho}{\sigma^{(n)}} \frac{\partial L}{\partial \Sigma^{(n)}} \frac{\partial}{\partial \Sigma^{(n)}} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \right) \]
\[ = \cdots - \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \right) \]
\[ = \cdots + (-1)^n \int \rho(\tilde{r}) \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \frac{\partial}{\partial (\Sigma^{(n)})^2} \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \]
\[ = \cdots + (-1)^n \rho \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \sigma'(\tilde{r}) \frac{\partial}{\partial \rho(r)} \frac{\partial}{\partial \sigma(r)} d\tilde{r} \]

and
\[ \frac{\delta}{\delta \rho} = \cdots + \int \frac{\rho}{\sigma^{(n)}} \frac{\partial L}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} = \cdots + \int \frac{\rho}{\sigma^{(n)} \rho} \frac{\partial L}{\partial \rho} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]
\[ = \cdots + \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]
\[ = \cdots - \int \frac{d}{d\tilde{r}} \left( \frac{\partial L}{\partial \Sigma^{(n)}} \right) \frac{1}{\rho} \frac{d}{d\tilde{r}} \left( \cdots \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} d\tilde{r} \right) \]
\[ g^{\Phi\Phi} = \cdots + (-1)^{n-1} \int \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n-1)} \Sigma^{(1)} \delta(r, \tilde{r}) \, d\tilde{r} \]
\[ = \cdots + (-1)^{n-1} \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(2n-1)} \Sigma^{(1)}. \]

Therefore,
\[ g^{\Phi\Phi} = \cdots - 2\rho \sigma (-1)^{2n-1} \left( \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} \Sigma^{(1)} \Sigma^{(2n-1)} \Sigma^{(2n)} \right), \]

where the \( \cdots \) stands for all other terms, not involving \( \Sigma^{(2n)} \). Now, according to the aforementioned requirement we need this to be a general function, say \( F(\rho L) \), and for this to happen the coefficient of \( \Sigma^{(2n)} \) must vanish, i.e.
\[ \frac{\partial^2 L}{\partial (\Sigma^{(n)})^2} = 0 \Leftrightarrow L = L_1(\Sigma^{(0)}, \ldots, \Sigma^{(n-1)}) + L_2(\Sigma^{(0)}, \ldots, \Sigma^{(n-1)}). \]

Again the term of \( \Phi_1 \) corresponding to \( L_1 \) is, up to a total derivative, equivalent to a local smooth functional containing \( \Sigma^{(0)}, \ldots, \Sigma^{(n-1)} \). The case \( n = 1 \) needs separate consideration since, upon elimination of the linear in the \( \Sigma^{(2)} \) term we are left with a local function of \( \Sigma^{(1)} \), and thus the possibility arises to meet the requirement by solving a differential equation for \( L \). In more detail, if
\[ \Phi = \int \rho(\tilde{r}) L(\sigma, \Sigma^{(1)}) \, d\tilde{r}, \quad (3.14) \]
g\( \Phi \Phi \) reads
\[ g^{\Phi\Phi} = -\rho \left( L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} \right) \left[ L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} + 2\alpha \left( \frac{\partial L}{\partial \sigma} - \Sigma^{(1)} \frac{\partial^2 L}{\partial \sigma \partial \Sigma^{(1)}} \right) \right] \]
\[ + 2\rho \sigma \left( L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} \right) \frac{\partial^2 L}{\partial (\Sigma^{(1)})^2} \Sigma^{(2)}. \quad (3.15) \]

Through the definition
\[ H \equiv L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}}, \quad (3.16) \]
we obtain
\[ \frac{\partial H}{\partial \sigma} = \frac{\partial L}{\partial \sigma} - \Sigma^{(1)} \frac{\partial^2 L}{\partial \sigma \partial \Sigma^{(1)}}, \quad \frac{\partial H}{\partial \Sigma^{(1)}} = -\Sigma^{(1)} \frac{\partial^2 L}{\partial (\Sigma^{(1)})^2}. \]

Thus, (3.15) assumes the form
\[ g^{\Phi\Phi} = -\rho \left( H^2 + 2\alpha H \frac{\partial H}{\partial \sigma} + \frac{2\alpha}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} \Sigma^{(2)} \right), \]
which upon addition, by virtue of the assumption, of the surface term
\[ A = \frac{d}{dr} \left( \int \frac{2\alpha}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} \, d\Sigma^{(1)} \right) \]
gives
\[ g^{\Phi\Phi} = -\rho \left( H^2 + 2\alpha H \frac{\partial H}{\partial \sigma} - \Sigma^{(1)} \frac{\partial}{\partial \sigma} \int \frac{2\alpha}{\Sigma^{(1)}} H \frac{\partial H}{\partial \Sigma^{(1)}} \, d\Sigma^{(1)} \right). \quad (3.17) \]
Since in the last expression we have only a multiplicative \( \rho(\sigma) \), it is obvious that the requirement
\[ g^{\Phi\Phi} = F(\rho L) \]

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can be satisfied only by
\[ g^{\Phi \Phi} = -\kappa \rho L, \tag{3.18} \]
with \( g^{\Phi \Phi} \) given by (3.17). Upon differentiation of this equation with respect to \( \Sigma^{(1)} \) we get
\[- \frac{\partial}{\partial \sigma} \int 2\sigma \frac{\partial H}{\partial \Sigma^{(1)}} \frac{\partial H}{\partial \Sigma^{(1)}} = \kappa \frac{\partial L}{\partial \Sigma^{(1)}}. \]
Multiplying the last expression by \( \Sigma^{(1)} \) and subtracting it from (3.18) we end up with the autonomous necessary condition for \( H(\sigma, \Sigma^{(1)}) \):
\[ H \left( H + 2\sigma \frac{\partial H}{\partial \sigma} - \kappa \right) = 0, \]
where (3.16) was also used. The above equation can be readily integrated giving
\[ H = 0, \]
\[ H = \kappa + \frac{a(\Sigma^{(1)})}{\sqrt{\sigma}}, \]
where \( a(\Sigma^{(1)}) \) is an arbitrary function of its argument. The first possibility gives according to (3.16) \( L = \lambda \Sigma^{(1)} \) which, however, contributes to \( \Phi \), a surface term, and can thus be ignored. Inserting the second solution into (3.16) we construct a partial differential equation for \( L \), namely
\[ L - \Sigma^{(1)} \frac{\partial L}{\partial \Sigma^{(1)}} = \kappa + \frac{a(\Sigma^{(1)})}{\sqrt{\sigma}}, \]
which upon integration gives
\[ L = \kappa - \Sigma^{(1)} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)} \sqrt{\sigma}} d\Sigma^{(1)} + c_1(\sigma) \Sigma^{(1)}. \]
Since this form of \( L \) emerged as a necessary condition, it must be inserted (along with \( H \)) in (3.18). The result is that \( c_1(\sigma) = 0 \). Thus \( L \) reads
\[ L = \kappa - \Sigma^{(1)} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)} \sqrt{\sigma}} d\Sigma^{(1)}, \tag{3.19} \]
By assuming that the \( \Sigma^{(1)} \)-dependent part of \( L \) equals \( b(\Sigma^{(1)}) \), i.e.
\[- \Sigma^{(1)} \int \frac{a(\Sigma^{(1)})}{\Sigma^{(1)} \sqrt{\sigma}} d\Sigma^{(1)} = b(\Sigma^{(1)}), \]
we get, upon a double differentiation with respect to \( \Sigma^{(1)} \), the ordinary differential equation
\[- \frac{a'(\Sigma^{(1)})}{\Sigma^{(1)}} = b''(\Sigma^{(1)}), \]
with the solution
\[ a(\Sigma^{(1)}) = b(\Sigma^{(1)}) + \kappa_1 - \Sigma^{(1)} b'(\Sigma^{(1)}), \]
where \( \kappa_1 \) is a constant. Substituting this equation into (3.19) and performing a partial integration we end up with
\[ L = \kappa + \frac{\kappa_1}{\sqrt{\sigma}} + \frac{b(\Sigma^{(1)})}{\sqrt{\sigma}}, \tag{3.20} \]
$\kappa$, $\kappa_1$ and $b(\Sigma^{(1)})$ being completely arbitrary and to our disposal; the two simpler choices $\kappa_1 = 0$, $b(\Sigma^{(1)}) = 0$ and $\kappa = 0$, $b(\Sigma^{(1)}) = 0$ lead respectively to the following two basic local smooth functionals:

$$q^1 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}, \quad q^2 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}. \quad (3.21)$$

The next simplest choice $\kappa = 0$, $\kappa_1 = 0$ and $b(\Sigma^{(1)})$ arbitrary leads to a generic $q^3 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}$. However, it can be proven that, for any choice of $b(\Sigma^{(1)})$, the corresponding renormalized induced metric

$$g^{AB} = G^\alpha_\beta \frac{\delta q^A}{\delta x^\alpha} \frac{\delta q^B}{\delta x^\beta}$$

is singular. The calculation of $g^{AB}$ gives

- $g^{11} = G^\alpha_\beta \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^1}{\delta x^\beta} = -\rho \quad \text{Assumption} \quad \Rightarrow \quad s^{11}_{\text{ren}} = -q^1$, 
- $g^{12} = G^\alpha_\beta \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = -\frac{\rho}{2\sqrt{\sigma}} \quad \text{Assumption} \quad \Rightarrow \quad s^{12}_{\text{ren}} = -\frac{q^2}{2}$, 
- $g^{22} = G^\alpha_\beta \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^2}{\delta x^\beta} = 0 \quad \text{Assumption} \quad \Rightarrow \quad s^{22}_{\text{ren}} = 0$, 
- $g^{13} = G^\alpha_\beta \frac{\delta q^1}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \rho \left( -\frac{b}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} b' + \sqrt{\sigma} \Sigma^{(2)} b'' \right) \quad \text{Assumption} \quad \Rightarrow \quad s^{13}_{\text{ren}} = \int d\tilde{r} \rho \left( -\frac{b}{2\sqrt{\sigma}} + \frac{\Sigma^{(1)}}{2\sqrt{\sigma}} b' + \sqrt{\sigma} \Sigma^{(2)} b'' \right)$
  \[ = -\int d\tilde{r} \rho \left( -\frac{b}{2\sqrt{\sigma}} \right) - \frac{q^3}{2}, \]
- $g^{23} = G^\alpha_\beta \frac{\delta q^2}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = \rho \left( \Sigma^{(2)} b'' = \frac{d}{dr} b' \right) \quad \text{Assumption} \quad \Rightarrow \quad s^{23}_{\text{ren}} = 0$, 
- $g^{33} = G^\alpha_\beta \frac{\delta q^3}{\delta x^\alpha} \frac{\delta q^3}{\delta x^\beta} = 2\rho(b - \Sigma^{(1)} b') \Sigma^{(2)} b'' \quad \text{Assumption} \quad \Rightarrow \quad s^{33}_{\text{ren}} = 2 \int d\tilde{r} \rho(b - \Sigma^{(1)} b') \Sigma^{(2)} b'' - 2 \int d\tilde{r} \frac{d}{dr} \left( \int d\Sigma^{(1)} (b - \Sigma^{(1)} b') b'' \right) = 0,$

where by $'$ we denote differentiation with respect to $\Sigma^{(1)}$. Thus, the renormalized induced metric reads

$$s^{AB}_{\text{ren}} = -\frac{1}{2} \begin{pmatrix} 2q^1 & q^2 & q^3 \\ q^2 & 0 & 0 \\ q^3 & 0 & 0 \end{pmatrix}. $$

Effecting the transformation $(\tilde{q}^1, \tilde{q}^2, \tilde{q}^3) = (q^1, q^2, \ln q^3)$ we bring $s^{AB}_{\text{ren}}$ into a manifestly degenerate form:

$$s^{AB}_{\text{ren}} = -\frac{1}{2} \begin{pmatrix} 2q^1 & q^2 & 0 \\ q^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

So, it seems as though the relevant part of the renormalized metric is described by the upper $2 \times 2$ block. This fact is consistent with the form of the renormalized potential $V = 2\Lambda q^1$ which indeed does not contain any $\Sigma^{(1)}$ term.
However, this is not the end of our investigation for a suitable space of state vectors: the argument leading to \( q^3, q^2 \) depends upon the original choice of one initial candidate smooth scalar functional \((3, 14)\); to complete the search we must close the circle by starting with the two already secured smooth functionals \((q^1, q^2)\), and a third of the general form

\[
q^3 = \int dr \, \rho L(\Sigma^{(1)}),
\]

since the \( \sigma \)-dependence has already been fixed to either 1 or \( \frac{1}{\sqrt{\sigma}} \). The calculation of the, related to \( q^3 \), components of the induced metric \( g^{AB} \) gives

\[
\begin{align*}
\rho \left( -L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) \quad \text{Assumption} \\
\rho \left( -L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) \quad \text{Assumption}
\end{align*}
\]

\[
\begin{align*}
g^{13} &= \rho \left( -L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) \quad \text{Assumption} \\
g^{23} &= \rho \left( -L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) \quad \text{Assumption}
\end{align*}
\]

\[
\begin{align*}
g^{23} &= \int dr \left( -L + \Sigma^{(1)} L' + \sigma \Sigma^{(2)} L'' \right) - \int dr \left( \int d\Sigma^{(1)} \sigma L'' \right) = - \int dr \rho L \\
&= - q^3,
\end{align*}
\]

\[
\begin{align*}
g^{33} &= -\rho \left( L - \Sigma^{(1)} L' \right)^2 + 2 \rho \sigma \left( L - \Sigma^{(1)} L' \right) \Sigma^{(2)} L''.
\end{align*}
\]

By following the procedure presented between \((3.15)\) and \((3.17)\) we end up with the expression

\[
g^{33} = \rho \left( L - \Sigma^{(1)} L' \right)^2 - \Sigma^{(1)} \int d\Sigma^{(1)} \frac{\partial}{\partial \Sigma^{(1)}} \left( L - \Sigma^{(1)} L' \right)^2,
\]

the expression inside the square brackets being a generic function of \( \Sigma^{(1)} \) and therefore, also of \( L \): let this function be parametrized as \( L(\Sigma^{(1)})^2 = \frac{4F(L(\Sigma^{1}))^2}{3F[F(L(\Sigma^{1}))]} \): this 'peculiar' parametrization of the arbitrariness in \( L(\Sigma^{(1)}) \) has been chosen in order to facilitate the subsequent proof that this freedom is a pure general coordinate transformation (gct) of the induced re-normalized metric. Indeed, let us first take the simplest non-trivial choice \( L(\Sigma^{(1)}) = \Sigma^{(1)} \) which results in the re-normalized metric

\[
\begin{align*}
g^{AB}_{\text{ren}} &= \frac{1}{2} \begin{pmatrix}
2q^1 & q^2 & 2q^3 \\
q^2 & 0 & q^2 \frac{\partial^2}{\partial q^1} \\
2q^3 & q^2 \frac{\partial^2}{\partial q^1} & -2q^2 \frac{\partial^2}{\partial q^1}
\end{pmatrix}, \\
g_{\text{ren}} &= \frac{1}{2} \begin{pmatrix}
\frac{3}{2q^1} & -\frac{4}{q^1} & -\frac{3}{2q^1} \\
-\frac{4}{q^1} & \frac{8q^1}{q^1} & 0 \\
-\frac{3}{2q^1} & 0 & \frac{3q^1}{2q^1}
\end{pmatrix}.
\end{align*}
\]

Considering a generic \( L(\Sigma^{(1)}) \), i.e. \( x^3 = \int dr \rho L(\Sigma^{(1)}) \) (along with \((3.21)\)) we are led to

\[
\begin{align*}
g^{13} &= -x^3, \\
g^{33} &= -\frac{q^2 x^3}{2q^1},
\end{align*}
\]

and

\[
\begin{align*}
g^{33} &= -\rho \left[ L^2 - \frac{4F[L]^2}{3F[F[L]^2]} \right] = - \frac{(\rho L)^2}{\rho} + \frac{4\rho F[\frac{\rho L}{\rho}]}{3F[F[\frac{\rho L}{\rho}]]} \\
g^{33}_{\text{ren}} &= \frac{(x^3)^2}{q^1} + \frac{4q^1 F[\frac{x^3}{q^1}]}{3F[F[\frac{x^3}{q^1}]]^2}.
\end{align*}
\]
Remarkably enough, the new induced re-normalized metric can be put in gct equivalence with the metric (3.22) through the transformation
\[
(q^1, q^2, x^3) = \left( q^1, q^2, q^1 F^{-1} \left( \frac{q^1}{q^1} \right) \right),
\]
with \( F^{-1} \) denoting the function inverse to \( F \), i.e. \( F^{-1}(F(x)) = x \).

We can therefore consider, without loss of generality, the reduced re-normalized manifold to be parametrized by the following three smooth scalar functionals:
\[
q^1 = \int d\tilde{r} \rho(\tilde{r}), \quad q^2 = \int d\tilde{r} \frac{\rho(\tilde{r})}{\sqrt{\sigma(\tilde{r})}}, \quad q^3 = \int d\tilde{r} \frac{\sigma'(\tilde{r})^2}{\rho(\tilde{r})}.
(3.23)
\]

Any other functional, say \( q^4 = \int d\tilde{r} \rho(\tilde{r}) L(\sigma(\tilde{r}), \Sigma^{(1)}(\tilde{r})) \), can be considered as a function of \( q^1, q^2, q^3 \); indeed, since the scalar functions appearing in the integrands of \( q^2, q^3 \) form a base in the space of \( \sigma, \Sigma^{(1)} \), we can express the generic \( L \) in \( q^4 \) as \( F(\frac{\rho}{\sqrt{\sigma}}, \frac{\Sigma^{(1)}}{\rho}) \), which (through the assumption) gives \( q^4 = q^1 F(\frac{\Sigma^{(1)}}{\rho}, \frac{\rho}{\sqrt{\sigma}}) \).

The geometry of this space is described by the induced re-normalized metric (3.22). Any function \( \Psi(q^1, q^2, q^3) \) on this manifold is of course annihilated by the quantum linear constraint \( \hat{\mathcal{H}}_\sigma \). According to the above exposition, we postulate that the quantum gravity of the geometries given by (2.6) and (2.7) will be described by the following partial differential equation (in terms of the \( q^A \)’s):
\[
\hat{\mathcal{H}}_\sigma \Psi \equiv \left[ -\frac{1}{2} \Box + V_{\text{ren}} \right] \Psi(q^1, q^2, q^3) = 0,
(3.24)
\]
with
\[
\Box \equiv \Box + \frac{d - 2}{4(d - 1)} R
(3.25)
\]
being the conformal Laplacian based on \( g_{\text{ABren}} \), \( R \) being the Ricci scalar, and \( d \) being the dimensions of \( g_{\text{ABren}} \). Metric (3.22) is conformally flat with Ricci scalar \( R = \frac{1}{\sqrt{\sigma}} \), and its dimension is \( d = 3 \). The re-normalized form of the potential (2.11) offers us the possibility of introducing, in a dynamical way, topological effects into our wavefunctional. Indeed, under our assumption, the first term becomes \( 2\Lambda q^1 \) while the second, being a total derivative, becomes \( A_T \equiv \frac{\sigma}{\rho} \partial_{\rho} \) (if \( \alpha < r < \beta \)). In the spirit previously explained we should drop this term; however, one could keep it, thus arriving at \( V_{\text{ren}} = 2\Lambda q^1 + A_T \), and the Wheeler–DeWitt equation is finally given as
\[
2q^1 \Lambda \Psi(q^1, q^2, q^3) + A_T \Psi(q^1, q^2, q^3) - \frac{1}{32q^1} \Psi(q^1, q^2, q^3) + q^3 \frac{q^3}{12q^1} \frac{\partial^3 \Psi(q^1, q^2, q^3)}{\partial q^3} + q^2 \frac{q^2}{8q^1} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^2} + q^1 \frac{q^1}{2q^1} \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^1} + 3 \frac{\partial \Psi(q^1, q^2, q^3)}{\partial q^1} = 0.
\]
\[
+ q^2 \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^1 \partial q^3} + \frac{q^2}{2} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial q^1 \partial q^2} + \frac{q^1}{2} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial (q^1)^2} - \frac{(q^1)^2}{6q^1} \frac{\partial^2 \Psi(q^1, q^2, q^3)}{\partial (q^1)^2} = 0.
\]

(3.26)

The change to new coordinates \((x^1, x^2, x^3)\) described by

\[(q^1, q^2, q^3) = (e^{2\xi}, e^{2i(x^1\xi^2)}, e^{2i\xi^3})\]

transforms the metric into the manifestly conformally flat form \(\text{diag}(e^{2\xi}, -e^{2\xi}, -e^{2\xi})\) and brings (3.26) into the form

\[2 e^{2\xi} \Lambda \Psi(x^1, x^2, x^3) + A_T e^{2\xi} \Psi(x^1, x^2, x^3) - \frac{1}{32} \Psi(x^1, x^2, x^3) + \frac{1}{4} \frac{\partial \Psi(x^1, x^2, x^3)}{\partial x^2} = 0.\]

(3.27)

This equation is readily solved by the method of separation of variables: assuming \(\Psi(x^1, x^2, x^3) = X^1(x^1)X^2(x^2)X^3(x^3)\) and dividing (3.27) by \(\Psi\) we get the three ordinary differential equations:

\[
\frac{1}{2X^1(x^1)} \frac{d^2 X^1(x^1)}{dx^1} + \frac{1}{32} = m + n,
\]

(3.28a)

\[
\frac{1}{2X^2(x^2)} \frac{d^2 X^2(x^2)}{dx^2} + \frac{1}{4X^2(x^2)} \frac{dX^2(x^2)}{dx^2} = 2 e^{2\xi} \Lambda + A_T e^{2\xi} = n,
\]

(3.28b)

\[
\frac{1}{2X^3(x^3)} \frac{d^2 X^3(x^3)}{dx^3} = m,
\]

(3.28c)

where \(m\) and \(n\) are separation constants. Their solutions for \(A_T = 0\) are

\[
X^1(x^1) = c_1 e^{i\sqrt{2m+2\xi}\sqrt{\Lambda} - x^1} + c_2 e^{-i\sqrt{2m+2\xi}\sqrt{\Lambda} - x^1},
\]

(3.29a)

\[
X^2(x^2) = c_3 e^{-i\xi/4} J_{-\sqrt{2m+2\xi}\sqrt{\Lambda}}(2 e^{2\xi} \sqrt{\Lambda}) + c_4 e^{-i\xi/4} J_{\sqrt{2m+2\xi}\sqrt{\Lambda}}(2 e^{2\xi} \sqrt{\Lambda})
\]

(3.29b)

\[
X^3(x^3) = c_5 e^{i\sqrt{2m}\xi^3} + c_6 e^{-i\sqrt{2m}\xi^3}.
\]

(3.29c)

where \(J_{\pm \sqrt{2m\xi^{\sqrt{\Lambda}}}\sqrt{\Lambda}}(2 e^{2\xi} \sqrt{\Lambda})\) are Bessel functions of the first kind and of non-integer order.

4. Discussion

We have considered the canonical analysis and subsequent quantization of the (2+1)-dimensional action of pure gravity plus a cosmological constant term, under the assumption of the existence of one Killing vector field. The implementation of the Dirac algorithm for this action results, at the classical level, in two linear (momentum) and one quadratic (Hamiltonian) first class constraints. The first linear constraint (2.9b) is shown to correspond to arbitrary changes of the radial coordinate. The second linear constraint (2.9c) owes its existence to the \(G_1\) symmetry imposed, a fact that is by itself worth mentioning. The quadratic constraint (2.9a) is, as usual, the generator of the time evolution (using the classical equation of motion, see p 21 of [26]). To avoid an ill-defined action of the quantum analogues of the linear constraints, we adopt as our initial collection of state vectors all smooth (integrals over the radial coordinate \(r\)) functionals. The first quantum linear constraint entails a reduction of this collection to...
all smooth scalar functionals (3.4). The subsequent imposition of the second quantum linear constraint further reduces these states to (3.7). At this stage, we need to somehow obtain, through the midi-superspace metric (2.10), an induced metric (3.8) whose components are given in terms of the same states, since this is a basic property of the corresponding structure in the case of finite degrees of freedom (see item 2 on p 7). Thus, the need for adopting the assumption emerges as a prerequisite; \( g^{\Phi \Phi} \), being a local function, cannot be directly compared to the state(s) to assess if it is composed out of this state(s). It is only after adopting the assumption that this comparison is made possible. Even so, however, unlike the case of finite degrees of freedom, the consideration of the derivatives w.r.t. \( r \) implies that \( g^{\Phi \Phi} \) cannot be automatically considered as a function of \( \Phi \); if we want to have this property, we must enforce it by adopting the requirement. This enforcement leads (see pp 11–17) to the three unique functionals (3.23) which are ‘physical’ under the ‘gauge’ generated by the linear operator constraints (invariant under spatial diffeomorphisms), and define an induced re-normalized metric (3.22) composed out of these states. Note that, as we make clear in the text, the map defined by the assumption must not be one to one between the spaces of the relevant functions and functionals, but only between the relevant spaces of equivalence classes (under \( r \)-reparametrizations) of functions and functionals (see p 10). The kinetic part of (2.9a) is then realized as the conformal Laplace–Beltrami operator based on the induced re-normalized metric (3.22), resulting in the Wheeler–DeWitt equation (3.26). It is in this way that the states become physical, i.e. invariant under spacetime diffeomorphisms as well. Effecting an appropriate change of variables the equation is made separable and, subsequently, completely integrated. We thus arrive at the only truly physical states, i.e. the solutions (3.29a).

We now come to two issues that we deem worthwhile discussing.

The first has to do with the apparent absence of the quantum analogue of the classical Poisson algebra (2.13). It seems to us that the primary purpose of searching for a (self-adjoint) representation of this algebra on a Hilbert space is to secure, through Frobenius’ theorem, the consistency of the quantum theory emanating from the chosen operator constraints (3.3), (3.6) and (3.24). But this aim is superseded by the finding of the common kernel, i.e. the solutions (3.29a). Furthermore, if after the issue of the measure is resolved, the Hilbert space is to be composed out of these states, the algebra of the operator constraints will be reduced to an Abelian one.

The second concerns our choice of following Dirac’s proposal to implement the first class constraints (2.9b), (2.9c) and (2.9a) as operator conditions annihilating the wavefunctional, rather than ‘imposing’ them on the classical level, as is the case for the vast majority of relevant works in 2 + 1 gravity. Within Dirac’s theory for constrained systems the only correct way that we are aware of to impose the first class constraints at the classical level is to choose a ‘gauge’, i.e. to select a phase-space function for each first class constraint, so that constraints plus ‘gauge’ fixing conditions become second class: then and only then is one allowed to solve them all, at the very important expense of being obliged there afterwards to use Dirac rather than Poisson brackets. Since the construction of these brackets makes use of the matrix formed by the second class constraints, it is obvious that one will, in general, be carrying to the subsequent quantization procedure properties of the choice made. In such a situation, one is never certain of how and/or to what extent the ‘gauge’ fixing chosen will infiltrate and affect the emanating quantum theory, especially if the ‘gauge’ involved is so immense and complicated as the group of spacetime coordinate transformations. This constitutes our primary motivation for following Dirac’s proposal which we interpret as an elimination of the ‘gauge’ freedom at the quantum level. The fact that in 2+1 dimensions it seems more easy to classically separate the ‘gauge’ from the ‘true’ degrees of freedom does not at all diminish
the strength of this motivation, much more in view of the fact that our method is meant to be applicable to spherically symmetric 3+1 geometries as well.

Generally (and somewhat loosely) speaking, the point of the exercise as we see it is, at a first stage, to assign a unique number between 0 and 1 to each and every geometry (2.6)–(2.7), in a way that is independent of the coordinate system used to represent the metric. Of course, at the present status of things we cannot do this, since the following two problems remain to be solved: (i) render finite the three smooth functionals (3.23) and (ii) select an appropriate inner product.

The first will need a final regularization of $q^1$, $q^2$, $q^3$, but most probably, the detailed way to do this will depend upon the particular geometry under consideration. For example, it is obvious that for the metric (2.3) three segments of the range $(0, \infty)$ of the radial coordinate have to be separately considered, while for the metrics (2.4) and (2.5) one segment (the entire range) is enough.

For the second, a natural choice would be the determinant of the induced re-normalized metric, although the problem with the positive definiteness may dictate another choice.

An analogous treatment of the (3+1)-dimensional spherically symmetric configurations can be carried through a task that we have already under active consideration.

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