THE IRREDUCIBILITY OF THE MODULI SPACE OF STABLE VECTOR BUNDLES OF RANK 2 ON A QUINTIC IN $\mathbb{P}^3$

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Abstract. In this paper I consider a quintic surface in $\mathbb{P}^3$, general in the sense of Noether-Lefschetz theory. The vector bundles of rank 2 on this surface which are $\mu$-stable with respect to the hyperplane section and have $c_1 = K$, the canonical class of the surface and fixed $c_2$, are parametrized by a moduli space. This space is known to be irreducible for large $c_2$ (work of K.G. O’Grady). I give an explicit bound, namely $c_2 \geq 16$. 

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Recently there has been some progress in proving irreducibility of the moduli space of $\mu$-stable rank-2 vector bundles of large $c_2$, on a surface of general type. The first result in that direction was an ineffective statement due to D. Gieseker and J. Li [GL], the second one was a more effective result, giving a lower bound for $c_2$ to ascertain irreducibility. This was proved by K.G. O’Grady in [O2]. However, the bound that he obtained is only made effective for complete intersections of high degree, and it is a high one. So the question naturally arises whether one can do better in some simple cases.

In this paper I will consider quintic surfaces in $\mathbb{P}^3$ general with respect to the Noether-Lefschetz property, i.e. $\text{Pic}(S) = \mathbb{Z}$. For such a surface it appears that the moduli space $\mathcal{M}(K,c_2)$ of $\mu$-stable (with respect to the hyperplane section $H$) rank-2 vector bundles with Chern classes $c_1 = K$ and $c_2$, is irreducible for $c_2 \geq 16$. This is shown by establishing the appearance of non locally-free sheaves in the closure of every component of $\mathcal{M}(K,c_2)$.

One may ask what happens in the case of a quintic surface $S$ with $\text{Pic}(S) \neq \mathbb{Z}$. O’Grady’s proof ([O2, 4.15-4.17]) of the irreducibility of $\mathcal{M}(K,c_2)$ does not work here, since it uses the existence of boundary points in the non-general case, where we only have such a result for a general surface. Although it seems not that difficult to redo the existence theorem, allowing divisors that are not complete intersections (the only place where it is really used occurs in step 3 to prove the stability of the elementary modifications $F_y$), bounds will become worse and I do not think it is worth the effort. The main aim of this paper is to show that bounds on $c_2$ can be rather low in special cases.

One also can ask what the situation is in case of surfaces of higher degree in $\mathbb{P}^3$, if one looks at $\mathcal{M}(H,c_2)$. This also can be attacked in the same way as described in this paper, but again, bounds become worse, as could be expected, whereas the proofs become more messy.
§1 The boundary of the moduli space

First I need of course some notation. Let $S \subset \mathbb{P}^3$ be a surface of degree 5, general in the sense of Noether-Lefschetz, i.e. $\text{Pic}(S) = \mathbb{Z}$. Let $K$ be the canonical class. Here $K$ is actually the hyperplane class of $S$ and in view of this, it will sometimes be denoted by $H$. Denote by $\mathcal{M}(K, c_2)$ the moduli space of rank-2 vector bundles $E$ on $S$, with $c_1(E) = K$ and $c_2(E) = c_2$, μ-stable with respect to $H$. The existence and the construction of this space can be found in [M].

Furthermore let $\overline{\mathcal{M}}(K, c_2)$ be the closure of $\mathcal{M}(K, c_2)$ in the Gieseker-Maruyama moduli space (see [G]) of $G$-semistable torsion free sheaves. Define for a closed subset $X$ of the moduli space $\mathcal{M}(K, c_2)$ the boundary of $X$ as

$$\partial X = \{ [F] \in X \mid F \text{ is not locally free} \}.$$  

Notice that since $K$ is not divisible by 2, every μ-semistable sheaf is automatically μ-stable.

I will take a 'component' to be an irreducible component. When I am speaking about connected components I will always write 'connected components'.

**Proposition (1.1).** Let $V = V(c_2) = \{ [E] \in \mathcal{M}(K, c_2) \mid H^0(E) \neq 0 \}$. For $c_2 \geq 10$, $\dim V \leq 3c_2 - 11$.

**Proof.** If $E$ has a section, this gives rise to an exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{I}_Z(1) \to 0,$$

(see e.g. [L, §3] for the well-known facts) where $Z \in \text{Hilb}^{c_2}(S)$. Notice that I use that $E(-1)$ has no sections for stability reasons, therefore the section of $E$ can not vanish along a divisor. Now let $\mathcal{N}(V)$ be the space

$$\mathcal{N}(V) = \{ (E, s) \mid E \in V, s \in \mathbb{P}H^0(E) \}$$

and consider the following diagram

$$\begin{array}{ccc}
\mathcal{N}(V) & \xrightarrow{p_2} & \text{Hilb}^{c_2}(S) \\
p_1 \downarrow & & \\
V & & \\
\end{array}$$

Certainly, $p_1$ is surjective. Hence $\dim V \leq \dim \mathcal{N}(V)$. Furthermore

$$\dim p_2^{-1}(Z) = \dim \mathbb{P}\text{Ext}^1(\mathcal{I}_Z(1), \mathcal{O}) = h^1(\mathcal{I}_Z(2)) - 1.$$

If $c_2 \geq 10$ a general $Z \in \text{Hilb}^{c_2}(S)$ has $h^0(\mathcal{I}_Z(2)) = 0$, so $h^1(\mathcal{I}_Z(2)) = c_2 - 10$. $\text{Hilb}^{c_2}(S)$ has dimension $2c_2$, so the expected dimension of $\mathcal{N}(V)$ is $\leq 3c_2 - 11$ (also valid for $c_2 = 10$).

However we should be careful with the subsets

$$\Delta_i = \{ Z \in \text{Hilb}^{c_2}(S) \mid h^0(\mathcal{I}_Z(2)) \geq i \}.$$

In fact, their codimensions in $\text{Hilb}^{c_2}(S)$ should have to be $\geq i$. If we take inside

$$\mathbb{P}H^0(\mathcal{O}(2)) \times \text{Hilb}^{c_2}(S).$$
the incidence variety $I = \{(C, Z) \mid Z \subset C\}$ and denote by $\pi_1, \pi_2$ the projections, then the dimension of $\pi_1^{-1}(C)$ is at most $c_2$, so $9 + c_2 \geq \dim I \geq \dim \pi_2^{-1}(\Delta_i) \geq \dim \Delta_i + i - 1$. This implies that $\dim \Delta_i$ is bounded by $c_2 + 10 - i \leq 2c_2 - i$ and hence the codimension of $\Delta_i$ in $\text{Hilb}^{c_2}(S)$ is $\geq i$. □

**Theorem (1.2).** Let $t$ be a nonnegative integer. Let $X \subset \overline{\mathcal{M}}(K, c_2)$ be closed of dimension $\geq 4c_2 - 20 + t$. If $c_2 \geq \max(10, 16 - t)$ then $\partial X \neq \emptyset$.

**Proof.** We will follow O'Grady's approach [O2] very closely. So let's suppose that $\mathcal{E}$ is locally free for all $[\mathcal{E}] \in X$.

**Step 1.** For every smooth curve $C \in |H|$ there is a vector bundle $[\mathcal{E}] \in X$ such that $\mathcal{E} |_C$ is not $\mu$-stable, given that $c_2 \geq 9 - \frac{1}{4} t$.

Let $\mathcal{M}(C, H)$ be the moduli space of rank-2 semistable bundles on $C$ with determinant $H$. If $\mathcal{E} |_C$ were $\mu$-stable for all $\mathcal{E}$ then we would have a restriction map

$$\rho : X \to \mathcal{M}(C, H).$$

It is well known that the dimension of $\mathcal{M}(C, H)$ is $3(g_C - 1) = 15$. But for $c_2 \geq 9 - \frac{1}{4} t$ the dimension of $X$ is at least 16. Hence, setting $\Theta$ the theta divisor on $\mathcal{M}(C, H)$ (see [O2, prop.1.18], [DN]),

$$(\rho^* \Theta)^{\dim X} = 0,$$

contradicting the aforesaid proposition 1.18 of [O2].

Set $X_C = \{[\mathcal{E}] \in X \mid \mathcal{E} |_C \text{ is not stable}\}$. This set is not empty as is shown in step 1. Recall that $V = \{[\mathcal{E}] \in \mathcal{M}(K, c_2) \mid H^0(\mathcal{E}) \neq 0\}$ as in proposition (1.1)

**Step 2.** $X_C \setminus V \neq \emptyset$ provided that $c_2 \geq \max(10, 16 - t)$.

Let $\mathcal{E}$ be a vector bundle in $X_C$. If $B$ is a neighbourhood of $[\mathcal{E}]$ in $X$, the subset $B^{\text{ns}} \subset B$ of vector bundles which are nonstable when restricted to $C$ has codimension at most $g_C = 6$ by [O2, prop.5.47]. This means that $\dim B^{\text{ns}} \geq 4c_2 - 26 + t$. But $\dim V \leq 3c_2 - 11$. So the claim should hold for dimension reasons.

**Step 3.** The construction of a family of stable sheaves.

Let $\mathcal{F}$ be a vector bundle whose existence is guaranteed by step 2 and let

$$0 \to \mathcal{L}_0 \to \mathcal{F} |_C \to \mathcal{Q}_0 \to 0$$

be a fixed destabilizing sequence. In particular $c_1(\mathcal{L}_0) > c_1(\mathcal{Q}_0)$ since $c_1(\mathcal{F} |_C)$ is not divisible by 2. To this sequence elementary modifications of $\mathcal{F}$ are associated, namely, first consider the elementary transformation ([L, ex.3.17])

$$0 \to \mathcal{E} \to \mathcal{F} \to \iota_* \mathcal{Q}_0 \to 0,$$

where $\iota : C \hookrightarrow S$ is the natural injection. Notice that we have $c_1(\mathcal{E}) = c_1(\mathcal{F}) - [C] = 0$ and $H^0(\mathcal{E}) \hookrightarrow H^0(\mathcal{F}) = 0$, which implies that $\mathcal{E}$ is $\mu$-stable. Restricting this sequence to $C$ one gets
0 \rightarrow \mathcal{Q}_0(-C) \rightarrow \mathcal{E} \mid_C \rightarrow \mathcal{L}_0 \rightarrow 0.

Now let
\[ Y_{\mathcal{F}} = \text{Quot}(\mathcal{E} \mid_C, \mathcal{L}_0) \]
be the Grothendieck Quot-scheme parametrizing quotients of \( \mathcal{E} \mid_C \), that have the same Hilbert polynomial as \( \mathcal{L}_0 \) (see [O2, p.8]). This set parametrizes a family \( \{ \mathcal{F}_y \}_{y \in Y_{\mathcal{F}}} \) of modifications of \( \mathcal{F} \), where \( \mathcal{F}_y = \mathcal{G}_y(C) \) for a subsheaf \( \mathcal{G}_y \) of \( \mathcal{E} \), which is defined as the kernel in the exact sequence
\[ 0 \rightarrow \mathcal{G}_y \rightarrow \mathcal{E} \rightarrow \iota_* \mathcal{L}_y \rightarrow 0. \]

Then it is easily shown that \( \mathcal{F}_y \) is \( \mu \)-stable for all \( y \in Y_{\mathcal{F}} \), namely notice that this is equivalent to the stability of \( \mathcal{G}_y \). Then take a subsheaf \( A = \mathcal{O}(a) \hookrightarrow \mathcal{G}_y \hookrightarrow \mathcal{E} \). Since \( \mathcal{G}_y = \mathcal{F}_y(-C) \), \( \mu(\mathcal{G}_y) = \mu(\mathcal{F}_y) - 5 = -\frac{5}{2} \). So we have to prove that \( \mu(A) < -\frac{5}{2} \) i.e. \( a < 0 \). But this follows directly from the fact that \( \mathcal{E} \) has no sections. Notice that this is the place where I make a crucial use of the generality of \( S \).

Thus we obtain a map
\[ \varphi : Y_{\mathcal{F}} \rightarrow \overline{\mathcal{M}}(K, c_2) \]
and therefore a subset \( \varphi^{-1}(X) \subset Y_{\mathcal{F}} \). The dimension of this subset is bounded by the following elementary lemma

**Lemma (1.3).** \( \dim \varphi^{-1}(X) \geq \dim X + \dim Y_{\mathcal{F}} - \dim T_{[\mathcal{F}]} \mathcal{M}(K, c_2). \)

**Proof.** This is easily obtained by the following construction. Notice that, since \( \varphi \) is an injection, \( \dim \varphi^{-1}(X) = \dim X \cap Y_{\mathcal{F}} \). Since \([\mathcal{F}] \in X \cap Y_{\mathcal{F}}\), this space is not empty. Embed a sufficiently small neighbourhood of \([\mathcal{F}] \in \mathcal{M}(K, c_2)\) into a suitable \( \mathbb{C}^* \), namely with \( e = \text{emb.dim}_{[\mathcal{F}]} \mathcal{M}(K, c_2) \). By [GR, p.115] \( e = \dim T_{[\mathcal{F}]} \mathcal{M}(K, c_2) \). Then the lemma follows from the result on intersection dimensions in affine space. \( \square \)

The well known facts on \( \mathcal{M}(K, c_2) \) (see e.g. [OV] for a résumé) as well as the assumption that \( \dim X \geq \exp. \dim \mathcal{M}(K, c_2) \), imply that \( \dim T_{[\mathcal{F}]} \mathcal{M}(K, c_2) - \dim X \) is bounded from above by \( h^2(\mathcal{E} \text{nd}^0 \mathcal{F}) = h^0(\mathcal{E} \text{nd}^0 \mathcal{F}(1)) \). Here \( \mathcal{E} \text{nd}^0 \mathcal{F} \) denotes the traceless endomorphisms of \( \mathcal{F} \). If we can arrange that the dimension of \( \varphi^{-1}(X) > 1 \) then [O2, lemma 1.15] assures the existence of a boundary point in \( \partial X \). Therefore lemma (1.3) shows that we are done if we prove the following lemma

**Lemma (1.4).** \( \dim Y_{\mathcal{F}} > 1 + h^0(\mathcal{E} \text{nd}^0 \mathcal{F}(1)) \)

**Proof.** By definition of \( Y_{\mathcal{F}} \):

\[
\dim Y_{\mathcal{F}} = h^0(Q^*(C) \otimes \mathcal{L}) \\
= \chi(Q^*(C) \otimes \mathcal{L}) + h^1(Q^*(C) \otimes \mathcal{L}) \\
= 1 + h^0(L^* \otimes Q(1))
\]

(we skip the subscripts in \( Q_0 \) and \( L_0 \) here). So let’s assume that

\[ h^0(\mathcal{E} \text{nd}^0 \mathcal{F}(1)) \geq \dim \text{Hom}(\mathcal{L}, \mathcal{O}(1)) \]
In view of the diagram

\[
\begin{array}{c}
0 \longrightarrow \mathcal{L} \overset{\alpha}{\longrightarrow} \mathcal{F} \mid_C \longrightarrow \mathcal{Q} \longrightarrow 0 \\
\downarrow s \\
0 \longrightarrow \mathcal{L}(1) \longrightarrow \mathcal{F}(1) \mid_C \overset{\beta}{\longrightarrow} \mathcal{Q}(1) \longrightarrow 0
\end{array}
\]

this means that there is a section \( s \in H^0(\text{End}^0 \mathcal{F}(1)) \) such that \( \beta \circ s \circ \alpha \equiv 0 \), or equivalently that \( s \) restricts to a map \( a \) from \( \mathcal{L} \) to \( \mathcal{L}(1) \). Then \( a \) can be viewed as an element of \( H^0(\mathcal{O}_C(1)) \) and since \( C \) is a complete intersection, this comes from an element of \( H^0(\mathcal{O}_S(1)) \), which we will denote by \( a \) too.

Consider \( \tilde{s} = s - a : \mathcal{F} \to \mathcal{F}(1) \). Then \( \mathcal{L} \hookrightarrow \ker \tilde{s} \mid_C \) and we have the following commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{O}_D \\
\downarrow \\
0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \mid_C \longrightarrow \mathcal{Q} \longrightarrow 0 \\
\downarrow \\
\mathcal{O}_D \\
\downarrow \\
0 \longrightarrow \ker \tilde{s} \mid_C \longrightarrow \mathcal{F} \mid_C \longrightarrow \text{coker} \tilde{s} \mid_C \longrightarrow 0 \\
\downarrow \\
0
\end{array}
\]

where \( D \) is a divisor on \( C \). But since \( \mathcal{Q} \) is a destabilizing quotient of \( \mathcal{F} \) I can assume it to be torsion free. Hence \( D = 0 \), \( \mathcal{L} = \ker \tilde{s} \mid_C \) and the map \( \mathcal{F} \mid_C \to \mathcal{Q} \) is nothing else as \( \tilde{s} \mid_C \). If we now take the elementary transformation associated to the composition \( \mathcal{F} \to \mathcal{F} \mid_C \to \mathcal{Q} \) and denote it by \( \mathcal{E} \) then \( \mathcal{E} \) fits into the exact sequence

\[
0 \to \mathcal{E} \overset{j}{\longrightarrow} \mathcal{F} \overset{i}{\longrightarrow} \mathcal{Q} \to 0.
\]

So we obtain a map \( \mathcal{E} \overset{\tilde{s} \circ j}{\longrightarrow} \mathcal{E}(1) \) which is zero on \( C \). Hence either there is a non-trivial map \( \mathcal{E}(C) \to \mathcal{E}(1) \) or \( \tilde{s} \circ j \) is the zero map on \( \mathcal{E} \). In the first case one concludes that \( \tilde{s} \circ j \in H^0(\text{End} \mathcal{E}) \), which contains only multiples of the identity by the stability of \( \mathcal{E} \). In the second case \( \tilde{s} \) is the zero map on \( \mathcal{F} \), i.e. \( s = a \). In both cases there occurs a contradiction with \( \text{tr} s = 0 \).

Now it remains to exclude the case \( h^0(\text{End}^0 \mathcal{F}(1)) = \dim \text{Hom}(\mathcal{L}, \mathcal{Q}(1)) \). Diagram (1) shows that there is a map

\[
\psi : H^0(\text{End}^0 \mathcal{F}(1)) \to \text{Hom}(\mathcal{L}, \mathcal{Q}(1))
\]
Moreover, the argument used above shows that $\psi$ restricted to the subspace $H^0(\mathcal{E}nd^0\mathcal{F}(1))$ is injective. But if, as we have assumed, the dimensions of these spaces match, the restricted $\psi$ is surjective too. But that implies that the map $b : \mathcal{F} \to \mathcal{F}(1)$, where $b \in H^0(\mathcal{O}(1))$ has the same image under $\psi$ as a certain traceless map $s_b$, say. This means that

$$\beta \circ (s_b - b) \circ \alpha \equiv 0.$$  

Now one can copy the above argument mutatis mutandis to obtain the same contradiction. \qed

We close this section with the following proposition, showing that if there is a boundary, it is large.

**Proposition (1.5).** ([O2, prop.3.3]) Let $X \subset \overline{\mathcal{M}}(K,c_2)$ be a closed subspace that has a non-empty boundary. Then $\partial X$ is closed and $\text{codim} (\partial X, X) \leq 1$. \qed
§2 The goodness of the moduli space

We should start with saying what a good subset of the moduli space is

**Definition (2.1).** A closed subset \( X \subset \overline{\mathcal{M}}(K,c_2) \) is called good if in every irreducible component of \( X \) there is a sheaf \( E \) with \( H^2(\text{End}^0 E) = 0 \). Here \( \text{End}^0 E \) denotes the traceless endomorphisms of \( E \). Bad is not good.

To simplify notation in the sequel I will use the following definition.

**Definition (2.2).** Let \( X \) be a closed subset of \( \mathcal{M}(K,c_2) \). Define

\[
 h(X) = \min \{ h^2(\text{End}^0 F) \mid [F] \in X \}
\]

Notice that, if a component of the moduli space is good, it is reduced and of the expected dimension (\( \text{exp.dim } \mathcal{M}(K,c_2) = 4c_2 - 20 \)). It turns out to be useful to know how large the vector space \( H^2(\text{End}^0 F) \) can be, for then we also know the maximal dimension of a component of \( \mathcal{M}(K,c_2) \). By Serre duality the dimension of this space equals \( h^0(\text{End}^0 F(1)) \).

At a first glance one finds a bound from the sequence

\[
0 \to O(-1) \to \text{End}^0 F \to G \to 0
\]

for an appropriate \( G \) (that one can prove to be a vector bundle). Namely it implies that \( h^2(\text{End}^0 F) \leq h^2(O(-1)) = 10 \) for all \( F \).

However we can do much better when we have a vector bundle \( F \), whose restriction to a certain hyperplane section \( C \) is not \( \mu \)-stable. The existence of such an \( F \) for large enough \( c_2 \) is guaranteed by step 1 of theorem (1.2).

**Proposition (2.3).** For \( F \) a vector bundle, whose restriction to a certain hyperplane section is not \( \mu \)-stable, \( h^0(\text{End}^0 F(1)) \leq 1 \).

**Proof.** The proof of lemma (1.4) shows that \( h^0(\text{End}^0 F(1)) < h^0(\mathcal{L}^* \otimes Q(1)) \), where \( \mathcal{L} \) and \( Q \) are the line bundles from the destabilizing sequence on \( C \). But the degree on \( C \) of \( \mathcal{L}^* \otimes Q(1) \) is less or equal than 4. Then Clifford’s theorem [ACGH, p.107] implies that its space of sections has dimension \( \leq 3 \), with equality if and only if \( C \) is hyperelliptic. But \( C \) is a complete intersection, so it has ample canonical divisor and hence it is not hyperelliptic. So \( h^0(\mathcal{L}^* \otimes Q(1)) \leq 2 \) and \( h^0(\text{End}^0 F(1)) \leq 1 \). \( \square \)

**Corollary (2.4).** If \( X \) is a closed subset of \( \mathcal{M}(K,c_2) \) of dimension at least 16 then \( h(X) \leq 1 \).

**Proof.** Immediate from the proof of step 1 of theorem (1.2) \( \square \)

Let me say some words about the double-dual stratification of \( \partial X \), for a closed subset \( X \) of \( \mathcal{M}(K,c_2) \). For \( [F] \in \partial X \) we have a canonical exact sequence

\[
0 \to F \to F^{**} \to Q_F \to 0,
\]

where \( Q_F \) is a skyscraper sheaf of finite length \( l(Q_F) = h^0(Q_F) \). The \( \mu \)-stability of \( F \) implies \( \mu \)-stability of \( F^{**} \) (this is an easy lemma, see [Mo, lemma 2.2.1]). Notice that \( c_2(F^{**}) = c_2(F) - l(Q_F) \). Although in general it will not be possible to glue the bundles \( F^{**} \) for \( F \) in \( \partial X \) to a global family because their second Chern classes may jump, there is a stratification of \( \partial X \) by locally closed subsets, such that the duals of sheaves parametrized by points of the same stratum, locally do glue to
a neat family. This follows from lemma 3.5 of [O1] (note that one has locally a universal sheaf on \( \mathcal{M}(K, c_2) \)). We are interested in the open strata which have by proposition (1.5) codimension 1 in \( X \).

**Theorem (2.5).** Let \( t \) be a positive integer. For \( c_2 \geq \max(10, 16 - t) \), \( \mathcal{M}(K, c_2) \) has no closed subsets \( X \) of dimension \( \geq 4c_2 - 20 + t \) with \( h(X) \geq 1 \). In particular, \( \mathcal{M}(K, c_2) \) is good for \( c_2 \geq 16 \).

**Proof.** Let \( X_0 \) be a closed subset of \( \mathcal{M}(K, c_2) \) of dimension \( \geq 4c_2 - 20 + t \) and \( h(X_0) \geq 1 \). Since \( c_2 \geq 9 - \frac{1}{4}t \) by corollary (2.4) \( h(X_0) = 1 \) and hence \( t \leq 1 \) too. So we may assume that \( c_2 \geq 15 \). Notice that \( \partial X_0 \) is not empty (theorem.(1.2)), and proceed in the following way. Let \( Y_0 \) be an irreducible component of an open stratum of the double-dual stratification. Define \( Y_0^{**} := \{ F^{**} \mid [F] \in Y_0 \} \) and set \( X_1 = \overline{Y_0^{**}} \). Then \( X_1 \) lies in a certain moduli space \( \mathcal{M}(K, c'_2) \), with \( c'_2 < c_2 \). If still \( \dim X_1 \geq 16 \) then \( h(X_1) = 1 \) and by prop.3.8 together with cor.3.6 of [O2] one gets

\[
\dim X_1 \geq 4c'_2 + (c_2 - c'_2) - 20 + t.
\]

On the other hand \( \dim X_1 \leq 4c'_2 - 20 + 1 \). So \( (c_2 - c'_2) + t \leq 1 \), i.e. \( c_2 - c'_2 = 1 \) and \( t = 0 \). In particular \( c'_2 \geq 15 \) and \( \dim X_1 = 4c'_2 - 20 + 1 \), hence also \( \partial X_1 \neq \emptyset \) and we are able to construct \( X_2 \) in the given way. And again, if \( \dim X_2 \geq 16 \), then \( (c_2 - c'_2) + t \leq 1 \), which cannot occur.

So it remains to exclude the cases where \( \dim X_i < 16 \) occurs for \( i = 1, 2 \). Since by cor.3.6 of [O2]

\[
\dim X_i \geq 4c_2^{(i)} + (c_2 - c_2^{(i)}) - 21 + t \\
\geq 3c_2^{(i)} - 5,
\]

this corresponds to \( c_2^{(i)} < 7 \). But in that case \( \dim X_i \leq \dim V \leq 3c_2^{(i)} - 11 \) by proposition (1.1). □
§3 THE irreducibility of the moduli space

Let $X$ and $Y$ be two irreducible components of $\mathcal{M}(K, c_2)$, meeting in a subset $Z$. Assume that both $X$ and $Y$ are good and have boundaries. If their boundaries do not meet each other they are coming from components $X_1$ and $Y_1$ of $\mathcal{M}(K, c_2 - 1)$, whose closures are disjoint. Thus $\overline{\mathcal{M}}(K, c_2 - 1)$ will be disconnected. If on the contrary, both boundaries do meet, $Z$ has a boundary, which gives rise to a bad component of $\mathcal{M}(K, c_2 - 1)$. This shows that to prove irreducibility it will suffice to prove that $\overline{\mathcal{M}}(K, c_2 - 1)$ is connected. Therefore it is useful to have the following lemma

**Lemma (3.1).** $\overline{V}(c_2) := \{[\mathcal{F}] \in \overline{\mathcal{M}}(K, c_2) | h^0(\mathcal{F}) \neq 0\}$ is connected for $c_2 \geq 10$.

**Proof.** If $\mathcal{F}$ has a section, the cokernel of that section is torsion free since $\mu(\mathcal{O}) = 0$ is maximal in the set $\{\mu(\mathcal{A}) | \mathcal{A} \leftrightarrow \mathcal{F}\}$. So exactly as in the case of a vector bundle one obtains the exact sequence

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{I}_Z(1) \to 0$$

and we can define the same sets and maps as we did in proposition (1.1). Notice that this uses the generality in the sense of Noether-Lefschetz of $S$, however we will see at the end of this section that this dependence is not essential.

For $c_2 \geq 11$, a general point in $\text{Hilb}^{c_2}(S)$ has the Cayley-Bacharach property with respect to quadrics, so $p_2(\mathcal{N}(\mathcal{V})) \subset \text{Hilb}^{c_2}(S)$ is dense and hence connected. Furthermore, the fibres of $p_2$ are the projective spaces $\mathbb{P}^\text{Ext}_1(\mathcal{I}_Z(1), \mathcal{O})$.

For $c_2 = 10$, $\mathcal{N}(\mathcal{V})$ is mapped by $p_2$ onto the set $\{Z | h^0(\mathcal{I}_Z(2)) \neq 0\} \subset \text{Hilb}^{10}(S)$, with connected fibres. But this set is certainly connected. $\square$

**Proposition (3.2).** For $c_2 \geq 10$, $\overline{\mathcal{M}}(K, c_2)$ is connected.

**Proof.** Notice that the proof of theorem (1.2) shows that an irreducible component $X$ of $\mathcal{M}(K, c_2)$, for $c_2 \geq 10$, has a boundary unless $V(c_2) \cap X = \emptyset$. So the moduli space $\overline{\mathcal{M}}(K, c_2)$ has a connected component $C(V)$ containing $\overline{V}$ and all other connected components have boundaries. The latter are coming from a moduli space $\overline{\mathcal{M}}(K, c'_2)$, with $c'_2 = c_2 - 1$. But for $c_2 \geq 11$ $C(V)$ has a non-empty boundary too, since for $Z$ being $c_2 - 1$ points on a quadric plus a point not on that quadric, certainly $h^1(\mathcal{I}_Z(2)) \neq 0$, but $Z$ does not have the Cayley-Bacharach property with respect to quadrics, so the torsion free sheaf defined by $Z$ is not locally free. So the number of connected components of $\overline{\mathcal{M}}(K, c_2)$ is a decreasing function of $c_2$ for $c_2 \geq 10$.

So it remains to show that $\overline{\mathcal{M}}(K, 10)$ is connected. As is said already, we certainly have the connected component $C(V)$. Assume that there is another connected component $X$. Then $X$ has a boundary and we can choose an element $[\mathcal{F}] \in \partial X$. $\mathcal{F}$ gives rise to $\mathcal{F}^{**}$, which fits into

$$0 \to \mathcal{O} \to \mathcal{F}^{**} \to \mathcal{I}_Z(1) \to 0,$$

since by Riemann-Roch $h^0(\mathcal{F}^{**}) \geq 10 - c_2(\mathcal{F}^{**}) > 0$. Now choose a set $W$ of $l(Q_F)$ different points, disjoint from $Z$. Let $\mathcal{G}$ be the torsion free sheaf corresponding to the image of the extension class of $\mathcal{F}^{**}$ under the map

$$\text{Ext}_1^1(\mathcal{I}_Z(1), \mathcal{O}) \to \text{Ext}_1^1(\mathcal{I}_{Z'}(1), \mathcal{O})$$
This means that \( G \hookrightarrow F^{**} \) is defined as the kernel of the composite map
\[
F^{**} \to I_Z(1) \to \mathcal{O}_W.
\]

Thus we obtain the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \mathcal{O} & \longrightarrow & G & \longrightarrow & I_{Z+W}(1) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O} & \longrightarrow & F^{**} & \longrightarrow & I_Z(1) & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{O}_W & \longrightarrow & \mathcal{O}_W \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0
\end{array}
\]

which shows that \( G \) sits in the same connected component of the boundary \( \partial X \) as \( F \), since \( W \) can be obtained from a continuous disturbance of the points in the support of \( Q_F \). Moreover it shows that \( H^0(G) \neq 0 \). This contradicts our assumption that \( X \) is a connected component different from \( C(V) \). \( \square \)

But by what is said above we can make an improvement of theorem (2.6).

**Theorem (3.3).** \( \mathcal{M}(K,c_2) \) is good for \( c_2 \geq 13 \)

**Proof.** Let \( X \) be a bad component and assume that there is another component \( Y \). Since the moduli space is connected, they intersect. But \( h(X \cap Y) \) will be greater than \( h(X) = 1 \), so by corollary (2.4) \( \dim X \cap Y \leq 15 \), or equivalently, its codimension in \( Y \) is \( \geq 4c_2 - 35 \). But this codimension may not exceed 10, because \( h^0(\text{End}^0 F(1)) \) is bounded by 10. So \( c_2 \leq 11 \), a contradiction.

This shows that \( X \) is the only component of \( \mathcal{M}(K,c_2) \). But then it certainly has a boundary, coming from a bad component \( X_1 \) of \( \mathcal{M}(K,c_2 - 1) \) and by the same argument \( X_1 \) is coming from an even worse component \( X_2 \), which does not exist (see the argument of theorem (2.5)). \( \square \)

The final and main theorem is

**Theorem (3.4).** \( \mathcal{M}(K,c_2) \) is irreducible for \( c_2 \geq 16 \).

**Proof.** This is easy now: it is just the remark at the beginning of this section, using theorem (3.3). \( \square \)

**Remark (3.5).** The situation that can a priori occur for \( c_2 = 15 \) is that there are many good components without boundary.

Now the question arises what we can do in case of a surface \( S \) with \( \text{Pic } S \neq \mathbb{Z} \). We have used that \( \text{Pic } S = \mathbb{Z} \) in lemma (3.1), where I indicated that proposition (3.2) can be proven without this assumption. Indeed, lemma (3.1) shows that in any case the closure of \( V(c_2) \) is connected. So when \( \text{Pic } S \neq \mathbb{Z} \), it remains possible...
that there are boundaries, disjoint from $C(V)$, whose corresponding sheaves have sections. But the diagram at the end of the proof of proposition (3.2) shows that after some small disturbance the cokernel of the section is torsion free. So it has to meet $C(V)$ and proposition (3.2) remains valid even in the non-general case.

As already mentioned in the introduction the only place where the assumption $\text{Pic}S = \mathbb{Z}$ further is used, is in step 3 of theorem (1.2), where it is indicated. So it is used to find boundary points in a component. However the proof of theorem (3.3) shows that you do not need to prove the existence of boundary points in case you want to prove that the moduli space is good, since in bad components you obtain your boundary points for free. So theorem (3.3) holds for an arbitrary quintic surface. However this is not enough to prove the irreducibility. I really need boundary points there (see [O2, 4.15-4.17]).

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