A Candidate for Exact Continuum Dual Theory for Scalar QED$_3$.

A. Kovner* and P. Kurzepa **
Theory Division, T-8, Los Alamos National Laboratory, MS B-285
Los Alamos, NM 87545
and
B. Rosenstein†
Institute of Physics, Academia Sinica
Taipei 11529., Taiwan, R.O.C.

Abstract
We discuss a possible exact equivalence of the Abelian Higgs model and a scalar theory of a magnetic vortex field in 2+1 dimensions. The vortex model has a current - current interaction and can be viewed as a strong coupling limit of a massive vector theory. The fixed point structure of the theory is discussed and mapped into fixed points of the Higgs model.

*KOVNER@PION.LANL.GOV
**KURZEPA@PION.LANL.GOV
† BARUCH@PHYS.SINICA.EDU.TW
1 Introduction

Quantum electrodynamics in 2+1 dimensions has recently attracted much attention as a model theory for different physical phenomena. The theory with massless fermions is interesting as a model of some aspects of QCD since it exhibits spontaneous chiral symmetry breaking \[1\]. When endowed with Chern-Simons term the model exhibits the statistics transmutation phenomenon and is used as an effective theory for description of the Quantum Hall Effect \[2\]. Different variants of QED$_3$ have been also used to describe the high temperature superconductors \[3\].

Apart from this the scalar QED$_3$ - the Abelian Higgs model- has always been interesting as the Landau-Ginzburg theory for ordinary superconductors. For example the order of the Higgs-Coulomb phase transition for a long time has been a matter of controversy \[4,5\] and the model was studied by a variety of numerical \[6,7\] and analytical \[4,5,8\] methods.

Recently we have studied the Abelian Higgs model in 2+1 dimensions using continuum duality methods \[9\]. The results of this analysis can be summarized as follows. The Higgs-Coulomb phase transition can be understood as due to condensation of magnetic vortices. The phase transition can be described by a local order parameter - the vortex creation operator \(V(x)\). It has a nonvanishing expectation value in the Coulomb phase and zero VEV in the Higgs phase. The field \(V(x)\) is a scalar complex field. Its phase rotations are generated by the operator of the magnetic flux \(\Phi = \int d^2 x B(x)\).

\[
[V(x), \Phi] = \frac{2\pi}{e} V(x)
\]

This \(U(1)\) transformation is a symmetry of the theory since the magnetic flux is conserved. The corresponding conserved current is the dual field strength \(\tilde{F}_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda\) and its conservation equation is just the homogeneous Maxwell equation of the theory. The nonvanishing VEV of \(V(x)\) in the Coulomb vacuum therefore signals spontaneous breaking of the magnetic flux symmetry in the Coulomb phase. The SSB of a global continuous symmetry is followed by the appearance in the spectrum of a massless Goldstone mode. Here it is the massless photon. The electrically charged states of QED are topological solitons of the vortex field \(V(x)\) and asymptotically have the form \(V(x) \to \theta(x) \exp i\theta\) where \(\theta(x)\) is a polar angle.

In the Higgs phase the flux symmetry is restored. The massless excitation
disappears from the spectrum and the operator $V(x)$ itself interpolates a massive particle - the magnetic vortex.

This simple picture suggests that the phase transition itself and the low energy properties of the theory near the phase transition can be described by a Landau theory written in terms of the order parameter field $V(x)$.

$$L_{\text{Landau}} = \partial_\mu V \partial_\mu V^* - U(V^*V)$$

where the potential $U(V^*V)$ has a single minimum at $V = 0$ in the Higgs phase and degenerate minima $|V| = \text{const}$ in the Coulomb phase. Indeed it was shown in [9] and [10] that the Lagrangian eq.(2) describes correctly the low energy properties of the theory including the Aharonov - Bohm effect. The standard universality argument then suggests that the Coulomb - Higgs phase transition is in the same universality class as the 3D XY model. The elementary ”spin” of this XY model is the vortex field $V$. Moreover the same argument suggests that if the theory has a tricritical fixed point it must be the theory of a free vortex field $V$.

The interesting possibility is that the notion of dual Lagrangian may be more fundamental than just the effective low energy theory in the sense of Landau. It may be possible to construct a local Lagrangian in terms of the vortex field $V(x)$ only which is exactly equivalent to the scalar $QED_3$ at all scales and not just at low energies. Should such a Lagrangian exist, it would be a completely local description of $QED_3$ in terms of local gauge invariant variables. In the following we will refer to this model as Quantum Vortex Dynamics (QVD).

Some work has been done in this direction in the late seventies and early eighties but the form of the exact dual Lagrangian is not known. There are two approaches to the derivation of the QVD Lagrangian. One is to attempt to perform an exact dual transformation in the lattice regularized theory [8],[5]. In this way one can quite easily dualize the Villain version of the fixed length Abelian Higgs model. However this Villain model and the original theory are supposed to be equivalent only near the infrared fixed point and therefore this excercise does not tell one much about the ultraviolet form of the dual theory. Moreover it would probably be too ambitious to try to find an exact equivalent on the lattice. The lattice dual theory could in principle contain a host of irrelevant terms (some even nonlocal) which would all disappear in the continuum limit and therefore would be totally
irrelevant for the continuum duality. Another approach is therefore to perform a dual transformation directly in the continuum \([11]\). However a formal dual transformation in continuum has the same drawback. It generates a complicated nonpolynomial Lagrangian totally unappealing from both, aesthetic and computational points of view. One expects again that most of its terms are irrelevant and therefore can be dropped with the only effect of renormalizing the coefficients of the remaining terms. It is however not clear which are the remaining relevant terms.

In this paper therefore we will take a different approach to determining the exact dual theory. In Sec.2 we will explain several conditions which the dual theory must satisfy given that the dynamics of the Abelian Higgs model in the ultraviolet regime can be probed with the help of perturbation theory. We also describe the known fixed point structure of the RG flow in the model. In Sec.3 we present a model of the vortex field \(V(x)\) which satisfies all these conditions. We discuss the fixed point structure of this model and show that it is consistent with the known (and expected) fixed points of scalar QED\(_3\) and the assumption of duality. Sec. 4 is devoted to a short discussion.

2 Necessary conditions and the fixed points.

In this section we summarize the conditions that the dual theory must satisfy which can be inferred with a large measure of certainty from the standard perturbative calculations in the framework of QED. The Lagrangian of QED\(_3\) is

\[
L_{\text{Higgs}} = -\frac{1}{4g^2} F_{\mu\nu}^2 + \left| (\partial_\mu + iA_\mu)\phi \right|^2 + \mu^2 \phi^* \phi - \kappa (\phi^* \phi)^2
\]  

(3)

1. The most obvious condition is that QVD must have a well defined continuum limit or in the language of continuum theory must be renormalizable. This renormalizability does not have to be perturbative, in a sense that some couplings of the dual theory may be not small but rather take values in the vicinity of some finite fixed point. In fact it is pretty obvious that this must be the case. If all the couplings of the vortex model were small in the ultraviolet (asymptotically free), the UV fixed point would be a free theory of the vortex field. However the Abelian Higgs model in the UV region is a theory of free photons and free charges rather than free vortices.
2. Consistent with the universality assumption in the low energy region the QVD Lagrangian must reduce to the Landau theory eq.(2). Formally this means that in addition to the usual potential energy the interaction Lagrangian can have higher derivative terms which are important in the UV (consistent with the previous point) but are very small in the IR.

3. In the framework of the Abelian Higgs model the vortex operator $V(x)$ is a local operator for any nonzero value of the finite structure constant $e^2$. The charged field $\phi(x)$ is on the other hand nonlocal in any physical gauge. In the limit $e^2 \rightarrow 0$ however the charged field decouples from the photon fields and becomes itself gauge invariant and local. In the same limit magnetic vortices (which are nonlocal configurations of $\phi$) become nonlocal objects and the vortex operator $V(x)$ ceases to be a local field. Another manifestation of the singular nature of the limit $e^2 \rightarrow 0$ is that the quantum numbers of the vortex operator are changed. For any nonzero $e^2$ the vortex configuration that carries a nonzero vorticity $\int d^2 x e \epsilon_{ij} \partial_i (\phi^* \partial_j \phi)$, carries also nonzero magnetic flux. For $e = 0$ however the photon field is decoupled and the vortices therefore do not carry magnetic flux. In the dual theory as discussed in [10] the charged field creates topological solitons of $V(x)$. The dual theory should therefore have a well defined limit in which these solitons become local and the field $V$ looses locality.

4. Another hint on the form of the QVD interaction can be gathered from the spectrum of QED$_3$ in the Higgs phase. Perturbatively the spectrum contains a massive vector particle (massive photon) and a massive scalar Higgs. In terms of the vortex model those should be evidently interpreted as vortex - antivortex bound states in the disordered phase. However a scalar theory with just potential interaction of the type $(V^*V)^2$ can not have bound states. The interaction of this type must be repulsive, since otherwise the theory is unstable. Moreover in any theory of this type the vortex-vortex and vortex-antivortex interactions are the same which is not the case in QED$_3$. One simple possibility to induce bound states is to have an interaction of the vector-vector type $(V^* \overleftrightarrow{\partial}_\mu V)^2$. In that case one can have attraction in the vortex-antivortex channel and repulsion in the vortex-vortex channel allowing for appearance of bound states without destabilizing the theory. Also, the massive photon and the Higgs particle in perturbation theory are light whereas the vortices themselves are nonperturbatively heavy. This means that the vector type interaction, if present must be very strong
for small electromagnetic coupling $e^2$ to produce such a large binding energy.  

5. Finally, another strong constraint comes from the known ultraviolet behaviour of QED$_3$. In the ultraviolet the theory is asymptotically free. One can therefore calculate the UV behaviour of the vortex operator. This was done in [9] with the following result

$$< TV(x)V^*(y) > \rightarrow |x-y| \rightarrow 0 \frac{\pi}{e^2|x-y|}$$

This behaviour is somewhat unusual, since it is telling us that in the UV region the field $V$ cannot be assigned a scaling dimension. Its correlators decay exponentially and it is therefore exponentially irrelevant near the UV fixed point. On the other hand the correlators of the bilinears of the type $V^*V$ or $V^* \partial_\mu V$ scale with powers of distance. QVD must accommodate this capricious UV behaviour.

Let us now discuss the fixed point structure of scalar QED. This is important since if the dual theory is to be exactly equivalent it must have the same fixed points in its parameter space.

The theory has three couplings: the electromagnetic coupling $e^2$, the scalar self coupling $\kappa$ and the scalar mass $\mu^2$. In the following discussion we only consider a physically massless theory, that is at a given value of $e^2$ and $\kappa$ the mass $\mu^2$ is tuned in such a way that the theory is critical (the physical mass vanishes). The remaining two couplings at every point on a critical surface define two energy scales $M_\kappa$ and $M_{e^2}$. Physically at energies much higher than $M_{e^2}$ the electromagnetic interaction is negligibly small, while far above $M_\kappa$ the scalar coupling becomes negligible. Naively by taking each one of these scales to infinity (or rather to the UV cutoff $\Lambda$ which is infinite from the point of view of the continuum) or to zero one obtains a theory without a scale. It is usually the case that such a scale invariant theory is also conformally invariant and corresponds to a fixed point of the renomalization group flow. In the present model one therefore expects four fixed points: A. $M_{e^2} = M_\kappa = 0$, B. $M_{e^2} \rightarrow \infty(\propto \Lambda)$, $M_\kappa \rightarrow \infty(\propto \Lambda)$; C. $M_{e^2} = 0$, $M_\kappa \rightarrow \infty(\propto \Lambda)$; D. $M_{e^2} \rightarrow \infty(\propto \Lambda)$, $M_\kappa = 0$. With some modifications these are indeed the fixed points of the Abelian Higgs model. A. $M_\kappa = M_{e^2} = 0$. A theory of noninteracting charges and free massless photon. Small changes in both couplings change the low energy physics and therefore the perturbations associated with both couplings are relevant. On the two dimensional critical surface both directions are relevant and this is
the UV fixed point. This of course follows directly from the perturbation theory. In terms of the bare couplings $e^2 = \kappa = 0$. It is customary to characterize the fixed points not in terms of bare dimensionful couplings but rather in terms of dimensionless physical couplings. Those are defined as dimensionless Green’s functions at a given scale. For example

$$\kappa_R(\mu) \equiv \mu^7 G^4(p_1\ldots p_4)|_{p_ip_j=1/2\mu^2\delta_{ij}}, \quad e^2_R(\mu) \equiv \mu^7 G^4_{\mu\nu}(p_1\ldots p_4)|_{p_ip_j=1/2\mu^2\delta_{ij}}$$  \hspace{1cm} (5)

where $G^4$ and $G^4_{\mu\nu}$ are the connected Green’s functions of four scalar fields and two scalar and two vector fields respectively. At a fixed point the dependence on the scale $\mu$ of course disappears. The UV fixed point then is $e^2_R = \kappa_R = 0$.

B. $M_\varepsilon \to \infty(\propto \Lambda), M_\kappa \to \infty(\propto \Lambda)$. On dimensional grounds it is clear that this fixed point is achieved when both bare couplings are of the order of the UV cutoff $\Lambda$. In terms of dimensionless couplings $e^2_R = e^2_C, \kappa_R = \kappa_C$. This is the IR fixed point of the theory. The reason is that once all the couplings of the theory are of the order of the UV cutoff, the low energy physics should be insensitive to small changes in any of the couplings (provided of course, the mass is suitably changed so that the theory remains massless). All the directions on the critical surface are thus irrelevant and the fixed point is IR stable. The existence of this IR fixed point has been a matter of some controversy. It is now however well established that the theory does indeed have a second order phase transition, and therefore an IR fixed point exists. The evidence comes from lattice simulations [6], analytic calculations on the lattice [7], 1/N expansion in the continuum strongly coupled model [13], and perturbative loop expansion [14]. Formally taking the limit $\kappa \to \infty$ and then $e^2 \to \infty$ one arrives at a $N \to 1$ limit of a $CP^{N-1}$ theory. We will refer to this fixed point therefore as the $CP^0$ model[1].

C. $M_\varepsilon = 0, M_\kappa \to \infty(\propto \Lambda)$. Since the scalar interaction by itself does not induce vector coupling this fixed point appears at $e^2 = 0, \kappa \propto \Lambda$ (or $e^2_R = 0, \kappa_R = \kappa_1$). The photon is decoupled from the matter fields. The dynamics of the matter fields is that of strongly interacting $U(1)$ invariant $\phi^4$. This just means that the scalar theory is defined at its IR fixed point, which is the 3D XY model. Clearly, the direction of the coupling $\kappa$ near this

\[\text{Naively it seems that the } CP^0 \text{ model does not contain any degrees of freedom. However it really means only that the charged degrees of freedom are irrelevant at the IR fixed point. This is in full agreement with the universality argument which in the present case suggests that the IR fixed point should be the XY model of magnetic vortices.}\]
Figure 1: Schematic phase diagram of the Abelian Higgs model. The arrows show the direction of the renormalization group flow. The point A is UV stable and has two relevant directions. The point B is IR stable and has no relevant directions. The point C corresponds to decoupled photon. The point D is the tricritical fixed point and the direction along the tricritical line AD is irrelevant.

point is irrelevant, whereas $e^2$ is relevant. So in the space of parameters of QED this fixed point is stable in one direction which is the $e^2 = 0$ axis.

D. Naively one expects to find a fixed point at $M_\kappa^2 \to \infty (\propto \Lambda)$, $M_\kappa = 0$. However in the present model it is impossible to lower the scalar coupling scale $M_\kappa$ (or equivalently $\kappa_R$) all the way to zero at nonzero $M_\kappa^2$ (or $e^2_R$). The reason is that for $\kappa_R < \kappa_T$ the vector coupling induces interactions that change the order of the phase transition from second to first. The critical surface in the parameter space of QED$_3$ is therefore not infinite but ends on a line of tricritical points. This tricritical line naturally also ends at a fixed point - the tricritical fixed point $e^2_R = e^2_T$, $\kappa_R = \kappa_T$. The existence of this point has been shown analytically [5] and numerically its measured critical exponents are consistent with gaussian critical exponents, which is what one expects in 3D [7]. This fixed point then must have one irrelevant direction - the direction along the tricritical line on the critical surface. The direction corresponding to the flow from the tricritical point $(e^2_T, \kappa_T)$ to the IR fixed point $(e^2_C, \kappa_C)$ is relevant.

The phase diagram and the directions of the RG flow near different fixed points is schematically depicted on Fig.1.

3 A Candidate Theory.

Consider the following Lagrangian

$$L_{\text{dual}} = -\frac{1}{4g^2}G_{\mu\nu}^2 + |(\partial_\mu + iB_\mu)V|^2 - \frac{m^2}{2}B_\mu^2 - M^2V^*V - \lambda(V^*V)^2 \quad (6)$$

where $V(x)$ is a scalar complex field, $B_\mu$ a real vector field and $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The field $V(x)$ should be thought of as a vortex operator of QED$_3$ while $B_\mu$ is closely related to the dual field strength $\tilde{F}_\mu$. We shall argue in this
section that in a certain strong coupling limit this theory is equivalent to scalar QED$_3$.

The first thing to note about the Lagrangian eq.(6) is that the mass term of the vector field breaks explicitly the $U(1)$ "gauge" symmetry. Therefore the field $V$ is a local physical field and the $U(1)$ symmetry indeed does have a local order parameter. This is what one expects if this symmetry is to be identified with the flux symmetry of QED.

As usual a continuum model is defined in the critical region of the second order phase transition. In the present case it is the breaking of this $U(1)$ symmetry that is responsible for the existence of the continuum limit. The fact that the continuum limit does exist at least for some values of parameters can be demonstrated in perturbation theory. For small values of the couplings $g$ and $\lambda$ the theory is renormalizable perturbatively [13]. This can be easily shown as follows. Let us perform a canonical transformation that decouples the longitudinal part of the vector field $B_\mu$

$$V = \bar{V} e^{-\frac{i}{\xi} B_\mu}$$

(7)

The field $\bar{V}$ now interacts only with the transverse part of $B_\mu$ and therefore only the transverse part of the propagator of $B_\mu$ enters the Feynman diagrams of the theory. But the UV asymptotics of the transverse part of the massive vector propagator is the same as that of the photon propagator in QED$_3$ in the Landau gauge. Therefore as far as the UV asymptotics are concerned, the correlators of $\bar{V}$ and transverse $B_\mu$ are the same as in Abelian Higgs model and are renormalizable. To get back to the correlators of the original field $V(x)$ one just has to undo the canonical transformation eq.(7). Since $\partial_\mu B_\mu$ is a noninteracting field this is straightforward. From the Lagrangian we have

$$< \partial_\mu B_\mu(x)\partial_\nu B_\nu(y) > = -\frac{\Box}{m^2} \delta^3(x-y)$$

(8)

Therefore for any $U(1)$ invariant correlator of the field $V$ one has

$$< V(x_1)\ldots V(x_n) V^*(y_1)\ldots V^*(y_n) > = < \bar{V}(x_1)\ldots \bar{V}(x_n) \bar{V}^*(y_1)\ldots \bar{V}^*(y_n) >$$

$$\exp \left\{ -\frac{n\Lambda}{4\pi m^2} + \sum_{i,j} \frac{1}{4\pi m^2|x_i-y_j|} - \sum_{i\neq j} \left[ \frac{1}{4\pi m^2|x_i-x_j|} + \frac{1}{4\pi m^2|y_i-y_j|} \right] \right\}$$

(9)

where $\Lambda$ is the ultraviolet cutoff. Evidently multiplying the vortex operator $V$ by the wave function renormalization factor $Z = \exp\left\{ \frac{\Lambda}{8\pi m^2} \right\}$ makes all the correlators finite. The theory is therefore perturbatively renormalizable.
It is however not the weakly coupled limit of eq. (6) that we have in mind as a candidate for QVD. The reason is that for weak coupling the UV fixed point of the model is at zero \( g^2 \) and \( \lambda \) and is a free theory of vortices rather than a free theory of charges. However the discussion in a previous section suggests near which UV fixed point one has to consider eq. (6). First of all, obviously in the ultraviolet the mass term of a vector field is unimportant and therefore the UV fixed point must be among one of the fixed points of scalar QED discussed earlier. Now, from all the fixed points of QED_3 only the “tricritical” point D is a good candidate. By a universality argument the tricritical point of QED must be a free theory of magnetic vortices. This is also consistent with lattice simulations [3]. But after omission of the mass term, the Lagrangian eq. (3) describes a gauge interaction of vortices. So in the Lagrangian eq. (3) the roles of charges and vortices are interchanged relative to the original Lagrangian of scalar QED_3. Therefore the UV fixed point relevant to our discussion must be the point D with the obvious change of \( e^2 \) into \( g^2 \) and \( \kappa \) into \( \lambda \). It describes free electric charges. QVD should therefore be defined by eq. (3) with couplings near the fixed point \( g_R^2 = e_T^2, \lambda_R = \kappa_T \) (the renormalized couplings \( g_R^2 \) and \( \lambda_R \) are defined by eq. (5) interchanging the charged field into the vortex field and the vector potential into the vector field \( B_\mu \)). In terms of the bare couplings it is a formal limit \( g^2 \to \infty, \lambda \to \infty \) with the ratio \( g^2/\lambda \) finite and fixed.

The perturbative renormalizability tells us that the theory remains renormalizable also at strong bare couplings as long as both theories are connected by a critical line. The physical picture is simple. The renormalized theory with small couplings \( g \) and \( \lambda \) has a certain energy scale \( M_g \) associated with the vector coupling. Above this scale the asymptotic UV behaviour sets in and the correlators are perturbatively calculable. Below this scale the perturbation theory breaks down for small enough physical mass since the effective dimensionless expansion parameter is \( g^2/M \). However the renormalizability of the theory assures that all the correlators are finite also for these momenta. In perturbation theory this crossover scale is of order \( M_g \propto g^2 \). The limit that interests us involves raising this crossover scale to the scale of the UV cutoff and simultaneously fine tuning \( \lambda \) so that the effective quartic scalar self coupling at \( M_g \) be as small as possible. However since this is done already on the level of renormalized theory, the correlators remain finite in this limit. It is therefore reasonable to expect that going from the weakly coupled massive scalar QED to QVD will not introduce new UV divergencies and the theory
will remain UV finite.

Regarding the second condition of the previous section, it is evidently satisfied by the Lagrangian eq. (3). The kinetic term of the vector field \( B_\mu \) is negligible at low energies and the field \( B_\mu \) for the purposes of low energy physics can be expressed in terms of the vortex field \( V(x) \)

\[
B_\mu = \frac{i}{-m^2 + 2V^* V} V^* \frac{\partial}{\partial \mu} V
\]  

(10)

Substituting it back in the Lagrangian one finds

\[
L_{\text{dual}} = \partial_\mu V^* \partial_\nu V + \frac{1}{-2m^2 + 4V^* V} (V^* \frac{\partial}{\partial \mu} V)^2 - M^2 V^* V - \lambda (V^* V)^2
\]  

(11)

In the infrared region the interaction terms which involve derivatives become less important than the potential terms and therefore the theory degenerates into the Landau theory eq. (3).

Now let us discuss the limit of QVD which reproduces main features of the \( e^2 \to 0 \) limit in QED\(_3\). This will also help us to establish roughly the correspondence between some of the couplings in eq. (3) with the QED couplings. To this end let us calculate the infrared divergent contribution to the energy of a hedgehog soliton in the phase where \(< V > = v \neq 0\).

Asymptotically the hedgehog configuration is

\[
V(x) = ve^{i\theta}, B_i = -\frac{2v^2}{-m^2 + 2v^2} \partial_i \theta(x)
\]  

(12)

where \( \theta(x) \) is a planar angle function. Assuming \( m^2 << v^2 \) (that is that we are far from the phase transition line, which is the region where the perturbation theory in QED is valid), we obtain

\[
E_{\text{soliton}} = \frac{\pi m^2}{2} \ln(LM) + \text{IR finite}
\]  

(13)

where \( L \) is the infrared cutoff. The hedgehog soliton in the dual theory represents the charged particle of QED\(_3\). But for weak coupling the energy of the charged scalar in QED\(_3\) also diverges in the IR as \( \frac{e^2}{8\pi} \ln L \). We find therefore that to lowest order in \( e^2 \) the following relation between the couplings should hold

\[
m^2 = \frac{e^2}{4\pi^2}
\]  

(14)
The $e^2 \to 0$ limit in QED$_3$ therefore should be likened to the $m^2 \to 0$ limit in QVD.

Indeed the $m^2 \to 0$ limit is singular. In this limit the $U(1)$ phase rotation in the dual theory eq.(6) becomes a local gauge symmetry. The field $V(x)$ becomes then nongauge invariant. Moreover, since we are interested in the limit $g^2 \propto \Lambda$, the gauge field $B_\mu$ does not have a kinetic term at any finite momentum and the global $U(1)$ transformations do not generate a global symmetry but rather a part of the gauge group and act trivially on all physical states[14]. The model still has however a conserved topological current $J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu B_\lambda$. A solitonic configuration that carries a unit of this topological charge eq.(12) at $m^2 = 0$ differs from vacuum asymptotically only by a gauge transformation. Therefore the soliton is now a local field configuration and the operator creating the soliton becomes a local operator. The soliton energy accordingly becomes IR finite. As for the elementary excitations of eq.(7), the vortices are not interpolated by $V$. To interpolate them one would have to construct some gauge invariant operator with the same global quantum numbers. However as discussed in [14] such an operator does not exist precisely due to the fact that the gauge field does not have a kinetic term. There still are excitations that carry the vorticity $\epsilon_{\mu\nu\lambda} \partial_\nu J_\lambda$[14]. There is a "transmutation" of quantum numbers. The elementary excitations instead of carrying the global $U(1)$ magnetic flux carry only vorticity. They are however nonlocal configurations of gauge invariant fields and are logarithmically confined. This picture is precisely the same as described in item 3 of the previous section.

The smallness of $m^2$ for small $e^2$ tells us also another thing. Let us concentrate on a phase in which $\langle V \rangle = 0$. In that case the coefficient of the vector-vector interaction in eq.(11) is proportional to $1/e^2$ and is very large. The vortex and antivortex therefore attract each other strongly. Therefore the formation of tightly bound vortex-antivortex bound states is favored by the dynamics. Since the $\langle V \rangle = 0$ phase in this model is dual to the Higgs phase of scalar QED$_3$ these bound states represent the massive photon and the massive Higgs particle.

The last point concerns the UV behavior of different correlators. As mentioned in the previous section, the correlators of vortex operators in QED have exponential rather than power UV asymptotics, see eq.(4). But as shown in the discussion of the renormalization in the dual theory this is precisely the UV asymptotic behavior of the correlators of $V(x)$ in eq.(3).
In fact even the coefficient in the exponential in eq.(11) is the same as in eq.(10) if the correspondence between $e^2$ and $m^2$ of eq.(14) is used. Moreover, the composite operators of the type $V^*V$ are invariant under the canonical transformation, eq.(11) and therefore scale in the same way as $\bar{V}^*\bar{V}$, that is have a power law behavior. The operator $V^*\partial_\mu V$ under the canonical transformation eq.(12) transforms into $\bar{V}^*\bar{\partial}_\mu \bar{V} - i\bar{V}^*\bar{V}\frac{\partial_\lambda \partial_\mu}{\partial_\lambda} B_\mu$. However both, polynomials of $V^*V$ and of $B_\mu$ scale as powers therefore this operator also has a power law behavior in the UV. Evidently this argument holds for any operator which is invariant under global $U(1)$ transformation, since under the canonical transformation, eq.(11), it will transform into a finite polynomial of $\bar{V}$, $B_\mu$ and their derivatives. We then conclude that any local operator invariant under global $U(1)$ transformation in the dual model, eq.(11), scales in the UV region as a power of distance, but any eigenoperator of $U(1)$ with nonzero eigenvalue scales as an exponential times a power. This is precisely the behavior expected from the perturbation theory in QED$_3$.

The dual theory we discussed is strongly coupled and therefore is not easily accessible to a quantitative analysis. One can give however several qualitative arguments which elucidate further the connection between this model and the Higgs model and support their conjectured equivalence. Based on very general assumptions we discuss now the fixed point structure of QVD, eq.(11), and argue that there is a one to one correspondence between its fixed points and the fixed points of QED$_3$ and that this correspondence is totally consistent with duality.

i. The UV fixed point. As we have discussed earlier QVD is defined at the UV fixed point $g_R^2 = e_T^2$, $\lambda_R = \kappa_T$. Being dual to the tricritical point of QED$_3$ the theory must therefore describe free charges. However if this is to be equivalent to UV point of QED$_3$ it must also contain a noninteracting massless pseudoscalar particle - the QED photon. In fact it is easy to see that this is correct. QED$_3$ is defined initially with the zero mass of the vector field and the longitudinal part of the photon (which at zero mass is completely decoupled) is thrown out of the Hilbert space. The UV fixed point of QVD on the other hand is the limit $m^2 \to 0$ of eq.(11). At any finite $m$ the longitudinal part of the vector field $B_\mu$ is in fact in the Hilbert space of the theory. In the limit $m^2 \to 0$ it is decoupled but still exists. Since $B_\mu$ represents the dual field strength of QED, it is a pseudovector field. Its longitudinal part for $m = 0$ describes therefore a decoupled massless pseudoscalar particle which
evidently can be thought of as a photon.

ii. The IR fixed point. In the IR limit all dimensional couplings are of the order of the UV cutoff. Therefore we have \( g^2 \propto \Lambda, \lambda \propto \Lambda, m^2 \propto \Lambda \). The mass of a vector particle behaves as \( m_V^2 \propto g^2 m^2 \). The renormalized vector coupling then scales as \( g_R^2(\mu) \propto g^2 \mu^3 / m_V^4 = \mu^3 / g^2 m^4 \to 0 \). The vector field therefore completely decouples from the scalar and also becomes infinitely heavy. The theory then becomes just the strongly interacting \((V*V)^2\) model - the XY model for the vortex field. As we have discussed in the previous section this is precisely what one expects as the IR fixed point of scalar QED on the grounds of universality.

iii. Taking the limit \( m^2 \to 0 \) and also \( \lambda_R = \kappa_C, g_R^2 = e_C^2 \), QVD moves to a fixed point dual to the point C of QED. Now it is the \( CP^0 \) model of the vortex field. Additionally this fixed point contains a massless pseudoscalar particle - the longitudinal component of \( B_\mu \). If indeed the \( CP^0 \) model of charges is equivalent to the XY model of vortices, then by duality this fixed point must be equivalent to the XY model of charges plus a decoupled photon.

iv. Finally one can take \( m^2 \to \infty \) at a fixed \( g^2 \). This again leads to \( g_R^2 = 0 \). Then the vector field disappears from the spectrum and decouples from the scalar. It is then possible to take the limit \( \lambda \to 0 \), which gives a theory of free noninteracting massless vortices. This is the tricritical point and should be equivalent to the point D in the figure.

The correspondence between the fixed points of QED and QVD is summarized in Table 1.

4 Discussion.

In this note we have presented a continuum field theory which has many features similar to scalar QED. Those include the phase structure of the theory, the infrared asymptotics, the leading ultraviolet behavior of correlators and the fixed point structure and the renormalization group flow on the critical surface. On the basis of these similarities we conjecture that this theory which we call QVD is equivalent to scalar QED on all energy scales. It should be noted that the QVD Lagrangian is written explicitly in terms of local gauge invariant fields. The gauge dynamics in 2+1 dimensions is thereby described without the use of nongauge invariant vector potentials.

Since the number of relevant directions at all fixed points of QVD and
Table 1: Fixed points of Abelian Higgs model in terms of QED$_3$ and QVD$_3$ couplings

QED$_3$ is the same there should be a one to one mapping between the parameters of QVD and QED which maps one theory exactly into the other. It would be very interesting to find this mapping explicitly. Unfortunately in the region of parameter space in which QED is weakly interacting, the interactions of the vortex field in QVD is strong and vice versa. Finding this exact mapping is therefore a very involved matter and we do not have adequate analytic tools to do so. Several qualitative features can however be determined. We have shown for example that for weak gauge coupling the vector mass term in QVD is proportional to the electromagnetic coupling $m^2 = e^2/4\pi^2$. One can also calculate the mass of the magnetic vortex in the framework of QED semiclassically in the Higgs phase far from the phase transition \[17\]. One then gets the following relation between the scalar mass coefficient of QVD and the QED couplings:

$$M = \frac{\pi \mu^2}{4\kappa} \ln(e^2/\kappa)$$  \hspace{1cm} (15)$$

Reversing the argument and performing the analogous calculation deep in the Coulomb phase in the limit $m^2 \to 0$ in QVD one obtains the mass of the charged particle as

$$\mu = \frac{\pi M^2}{4\lambda} \ln(g^2/\lambda)$$ \hspace{1cm} (16)$$
Unfortunately since the semiclassical approximation is only valid in a small region of parameter space and since we are dealing with the strongly interacting theories the bulk of our discussion has been qualitative. It would be very interesting to check numerically the equivalence picture presented here. Although of course the numerical proof of equivalence is impossible, it is nowadays possible to measure scaling dimensions of various operators in computer simulations with great accuracy. It would be interesting therefore to verify the main features of the RG flow and scaling laws discussed here in a large scale computer simulation.

Acknowledgements We thank H. Kleinert, A. Krasnitz and A.M.J. Schakel for interesting discussions.

References

[1] T.W. Appelquist, M. Bowick, D. Karabali and L.C.R. Wijewardhana, *Phys. Rev.* **D33**, 3704 (1986); T.W. Appelquist, D. Nash and L.C.R. Wijewardhana, *Phys. Rev. Lett.* **60**, 2575 (1988); R.D. Pisarski, *Phys. Rev. D44*, 1866 (1991); E. Dagotto, A. Kocic and J. Kogut, *Phys. Rev. Lett.* **62**, 1083 (1989);

[2] Su Cheng Zhang, *Int. J. Mod. Phys.* **B6**, 25 (1992);

[3] A. Kovner and B. Rosenstein, *Phys. Rev. B42*, 4748 (1990); N. Dorey and N. Mavromatos, *Nucl. Phys. B386*, 614 (1992);

[4] B. Halperin, T.C. Lubensky and S.K. Ma, *Phys. Rev. Lett.* **32**, 292 (1974);

[5] H. Kleinert, *Lett. Nuovo Cimento* **35**, 405 (1982);

[6] C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981);

[7] J. Bartholomew, *Phys. Rev. B28*, 5378 (1983);

[8] M. Peskin, *Ann. of Phys.* **113**, 122 (1978); P.O. Thomas and M. Stone, *Nucl. Phys. B144*, 513 (1978);

[9] A. Kovner, B. Rosenstein and D. Eliezer, *Mod. Phys. Lett.* **A5**, 2661 (1990); *Nucl. Phys. B350*, 325 (1991);
[10] A. Kovner and B. Rosenstein, *Phys. Rev. Lett.* **67**, 1490 (1991);
[11] K. Bardakci and S. Samuel, *Phys. Rev.* **D18**, 2849 (1978);
[12] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Clarendon press, Oxford, 1989);
[13] J.C. Collins, *Renormalization*, (Cambridge University Press, Cambridge, UK, 1984);
[14] B. Rosenstein and A. Kovner, *Nucl. Phys.* **B346**, 576 (1990);
[15] I. Ya. Arefyeva and S.I. Azakov *Nucl. Phys.* **B162** (1980), 298; A. Kovner and B. Rosenstein, *Phys. Lett.* **B261**, 104 (1991);
[16] A.M.J. Schakel and H. Kleinert, *One loop critical exponents for Ginzburg-Landau theory with Chern-Simons term*. Free University of Berlin preprint 1992,
[17] J.Preskill, *Ann. Rev. Nuc. Part. Science* **34** (1984);