The problem of time in quantum mechanics

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Abstract
The problem of time in quantum mechanics concerns the fact that in the Schrödinger equation time is a parameter, not an operator. Pauli’s objection to a time-energy uncertainty relation analogue to the position-momentum one, conjectured by Heisenberg early on, seemed to exclude the existence of such an operator. However Dirac’s formulation of electron’s relativistic quantum mechanics (RQM) does allow the introduction of a dynamical time operator that is self-adjoint. Consequently, it can be considered as the generator of a unitary transformation of the system, as well as an additional system observable subject to uncertainty. In the present paper these aspects are examined within the standard framework of RQM.

Keywords: time operator; relativistic quantum mechanics; time-energy uncertainty relation

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1 Introduction

In the time dependent Schrödinger equation (TDSE) of quantum mechanics (QM) time appears as a parameter, not as an operator[1][2]. Pauli’s objection to the existence of a time operator, satisfying a commutation relation \([H, t] = -i\hbar\) as entertained by Heisenberg early on, is, to quote: "...from the C.R. written above it follows that \(H\) possesses continuously all eigenvalues from \(-\infty\) to \(+\infty\), whereas on the other hand, discrete eigenvalues of \(H\) can be present. We, therefore, conclude that the introduction of an operator \(t\) is basically forbidden and the time \(t\) must necessarily be considered as an ordinary number ("c" number) in Quantum Mechanics".

Pauli’s argument, sustained also by the fact that the system’s stability requires the energy to have a finite minimum, has given rise to a variety of alternative proposals for a time-energy uncertainty relation and an extensive discussion
of time in quantum mechanics throughout several decades. Within the time quantities considered one finds parametric (clock) time, tunneling times, decay times, dwell times, delay times, arrival times or jump times, i.e., both instantaneous values and intervals. To quote Ref.3: “In fact, the standard recipe to link the observables and the formalism does not seem to apply, at least in an obvious manner, to time observables”. The extensive experimental confirmation of the Schrödinger equation asserts that this parameter corresponds to the time coordinate of the laboratory frame of reference, in both QM and RQM (Relativistic Quantum Mechanics).

This is the problem of time in quantum mechanics. The questions to be answered are: 1) Can a time operator be found? 2) What is the status of a time-energy uncertainty relation? 3) How did the parameter \( t \) enter into the Schrödinger equation?

Recently it has been shown that Dirac’s formulation of relativistic quantum mechanics (RQM) does allow the introduction of a dynamical time operator that is self-adjoint. Consequently, it can be considered as the generator of a unitary transformation of the system, as well as an additional system observable subject to uncertainty. In the present paper these aspects are examined within the standard framework of RQM. The definition and main properties of the proposed time operator are recalled in Section 2. Section 3 analyses the effect of the corresponding unitary transformation. In Section 4 the ensuing time-energy uncertainty relation is derived and shown how it circumvents Pauli’s objection. It is also compared to the Mandelstam-Tamm formulation. Section 5 advances conclusions and possible developments.

Finally, to explain its presence in the TDSE has led to the consideration of time as an emergent property arising from the entanglement of a microscopic system with a classical environment in an overall closed time independent system, this property being apparent only to an internal observer.

The proof that the commutation relation \([\hat{x}, \hat{p}] = i\hbar\) necessarily implies that the corresponding spectra of \(\hat{x}\) and \(\hat{p}\) go from \(-\infty\) to \(+\infty\) continuously (as shown in Ref.14) is absent from most introductory textbooks on quantum mechanics. Also absent is the fact that the representations of the position and momentum operators in the configuration and momentum spaces follow from this commutation relation, as well as the fact that the representations of the state vector in configuration and momentum spaces are Fourier transform of each other. This is pedagogically unfortunate, as the student is induced to consider these as unrelated assumptions. In particular he will miss the connection of the infinite continuity of these spectra to unitary transformations and group properties, which is a cornerstone of the further development of quantum mechanics and of quantum field theory. Appendix A presents a unified description of the consequences of the commutation relation that could be incorporated in textbooks.
2 The dynamical time operator in RQM[11]

A dynamical self-adjoint "time operator"
\[ \hat{T} = \alpha \hat{\mathbf{r}}/c + \beta \tau_0 \]  
(1)

has been proposed in analogy to the Dirac free particle Hamiltonian \( \hat{H}_D = c\alpha \hat{\mathbf{p}} + \beta m_0 c^2 \), where \( \alpha_i (i = 1, 2, 3) \) and \( \beta \) are the \( 4 \times 4 \) Dirac matrices, satisfying the anticommutation relations:
\[ \alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij} \quad \alpha_i \beta + \beta \alpha_i = 0 \quad \beta^2 = 1 \]  
(2)

The parameter \( \tau_0 \) represents in principle an internal property of the system, to be determined. In the Heisenberg picture, the time evolution of the time operator is given by:
\[ \hat{T}(t) = \alpha(t) \hat{\mathbf{r}}(t)/c + \beta(t) \tau_0 = \hat{T}(0) + (c\hat{\mathbf{p}}/\hat{H}_D)^2 t + \text{oscillating terms} \]  
(3)

where use has been made of the following relations[14, 15, 16]:
\[ \alpha(t) = \alpha(0) + \{ \alpha(0) - c\hat{\mathbf{p}}/\hat{H}_D \} \{ \exp(-2i\hat{H}_D t/\hbar) - 1 \} \]  
(4)

\[ \beta(t) = \beta(0) + \{ \beta(0) - m_0 c^2/\hat{H}_D \} \{ \exp(-2i\hat{H}_D t/\hbar) - 1 \} \]  
(5)

\[ \hat{\mathbf{r}}(t) = \hat{\mathbf{r}}(0) + (c^2 \hat{\mathbf{p}}/\hat{H}_D) t + i(c\hbar/2\hat{H}_D) \{ \exp(-2i\hat{H}_D t/\hbar) - 1 \} \]  
(6)

\[ cc\alpha(0).\{c\hat{\mathbf{p}}/\hat{H}_D\} = \left[ \frac{d\hat{\mathbf{r}}}{dt} \right]_{t=0}.\{c\hat{\mathbf{p}}/\hat{H}_D\} = (c\hat{\mathbf{p}}/\hat{H}_D)^2 + \text{oscillating terms} \]  
(7)

Thus \( \hat{T}(t) \) exhibits a linear dependence on \( t \) with a superimposed oscillation (Zitterbewegung), as occurs with the time development of the position operator \( \hat{\mathbf{r}}(t) \).

Following step by step the algebra developed for the Dirac Hamiltonian[14, 15, 16], the eigenvalue equation
\[ \hat{T}|\tau> = \tau|\tau> \]  
(8)

yields:
\[ \tau = \pm \tau_r = \pm \sqrt{\left( r/c \right)^2 + \tau_0^2} \]  
(9)

as \( \hat{T}^2 = (\hat{\mathbf{r}}/c)^2 + \tau_0^2 \). The eigenvalue spectrum has two continuous branches, a positive and a negative one separated by a \( 2\tau_0 \) gap. As \( r \) goes from \( -\infty \) to \( +\infty \) the positive branch drops from \( +\infty \) to \( \tau_0 \) when \( r = 0 \), and then rises to \( +\infty \) again. The negative branch follows the opposite behaviour.

Each of these eigenvalues is doubly degenerate with respect to the component \( \sigma \cdot \hat{\mathbf{r}}/2r \) of the spin along the \( r \) direction, which commutes with \( \hat{T} \). Thus one
can find simultaneous eigenfunctions of $\sigma \cdot \hat{r} / 2r$ and $\hat{T}$, giving rise to altogether four eigenvalue pairs $|\tau, \sigma\rangle$. The "time eigenvectors" are:

$$|\pm \tau, \pm 1/2\rangle = u_i^r |r\rangle, \quad i = 1, 2, 3, 4 \quad (10)$$

where $|r\rangle$ is the eigenvector of the position operator $\hat{r}$ and $u_i^r$ is a four-component spinor independent of the linear momentum $p$. The four orthonormal spinors $u_i^r$ are listed in Ref.11.

In this formulation, $\tau_0$ plays the role of an invariant quantity in the $(r, \tau)$ space, i.e., $\tau_0^2 = \tau^2 - (r/c)^2$, as $m_0 c^2$ plays in the $(p, E)$ space, namely $(m_0 c^2)^2 = E^2 - (cp)^2$. To maintain the fundamental indeterminacy modulo $n^2 \pi$ $(n$ an integer) in the phase of the complex eigenfunctions one has to set, for $n = 1$:

$$\tau_0 = 2\pi \hbar / <\beta> \varepsilon = h/m_0 c^2 \quad (11)$$

This is the de Broglie period [18, 19]. Together with the Compton wave length, it sets a unified spacetime Compton scale that limits the wave packets width in space and time before negative energy and negative time components (particle and antiparticle) occur significantly. Moreover, it supports the existence of an internal property, the "de Broglie clock" with a period $\tau_0 = h/m_0 c^2 \quad (20)$ [21, 22].

3 The time operator as generator of a unitary transformation

The proposed operator is clearly self-adjoint and therefor can be the generator of a unitary transformation (Stone’s theorem [17]):

$$\hat{U}_T(\varepsilon) = e^{-ie\hat{T}/\hbar} = e^{-ie(\zeta / c + \beta \tau_0) / \hbar} \quad (12)$$

where $\varepsilon$ is real and has the dimensions of energy.

For infinitesimal transformations ($\delta \varepsilon << 1$), one can write:

$$\hat{U}_T(\varepsilon) \simeq e^{-i(\delta \varepsilon)(\alpha \cdot \hat{r}/c)/\hbar} e^{-i(\delta \varepsilon)\beta \tau_0 / \hbar} = e^{-i(\delta \varepsilon)\beta \tau_0 / \hbar} e^{-i(\delta \varepsilon)(\alpha \cdot \hat{r}/c) / \hbar} \quad (13)$$

as $[i(\delta \varepsilon)(\alpha \cdot \hat{r}/c), e^{-i(\delta \varepsilon)\beta \tau_0 / \hbar}] \propto (\delta \varepsilon)^2 \approx 0$ (Glauber proof[16, p.442]). Then the transformed Hamiltonian can be approximated as:

$$\hat{H}_D = \hat{U} \hat{H}_D \hat{U}^\dagger \simeq e^{i(\delta \varepsilon)\beta m_0 c^2 / \hbar} e^{i(\delta \varepsilon)\alpha \cdot \hat{r}/c} \hat{H}_D e^{-i(\delta \varepsilon)\alpha \cdot \hat{r}/c} \hat{H}_D e^{-i(\delta \varepsilon)\beta m_0 c^2 / \hbar} \quad (14)$$

Consider first:

$$\hat{h}_D \simeq e^{i(\delta \varepsilon)\alpha \cdot \hat{r}/c} \hat{H}_D e^{-i(\delta \varepsilon)\alpha \cdot \hat{r}/c} \simeq$$

$$\simeq \{I + i(\delta \varepsilon)\alpha \cdot \hat{r}/c + ..\} \hat{H}_D \{I - i(\delta \varepsilon)\alpha \cdot \hat{r}/c + ..\}$$

$$\simeq \hat{H}_D + i\{((\delta \varepsilon)/c)\alpha \cdot \hat{r}, \hat{H}_D\} + .. \quad (15)$$
Then using [14]:

\[
[\alpha \hat{r}, \hat{H}_D] = 3i\hbar I + 2\hat{H}_D\{\alpha - c\hat{p}/\hat{H}_D\}.\hat{r}
\]  

(16)

and substituting \(3I = \alpha.\alpha\), one obtains:

\[
\hat{H}_D \simeq \hat{H}_D[\hat{p}] + (\delta \varepsilon)\alpha.\alpha + i\{(\delta \varepsilon)/\hbar\}^2\{\hat{H}_D[\hat{p}]\alpha.\hat{r} - c\hat{p}.\hat{r}\} =
\]

\[
= c\alpha.\{\hat{p} + (\delta \varepsilon)\alpha/c\} + \beta m_0 c^2 + i2\{(\delta \varepsilon)/\hbar\}\{\hat{H}_D[\hat{p}]\alpha.\hat{r} - c\hat{p}.\hat{r}\} =
\]

\[
= \hat{H}_D[\hat{p}] + (\delta \varepsilon)\alpha/c + i2\{(\delta \varepsilon)/\hbar\}\{\hat{H}_D[\hat{p}]\alpha.\hat{r} - c\hat{p}.\hat{r}\}
\]  

(17)

Thus, the unitary transformation induces a shift in momentum by the amount:

\[
\delta \hat{p} = \{(\delta \varepsilon)/c\} \alpha = \{(\delta \varepsilon)/c^2\} c\alpha
\]  

(18)

as well as a Zitterbewegung behavior in the corresponding propagator \(U(t) = e^{-i\hat{H}_D t/\hbar}\).

For repeated infinitesimal applications one obtains a momentum displacement \(\Delta \hat{p}\) whose expectation value is

\[
< \Delta \hat{p} > = (\varepsilon/c^2)\nu_{gp} = \gamma m_0 \nu_{gp}
\]  

(19)

where \(\gamma = \{1 - (\nu_{gp}/c)^2\}^{-1/2}\) is the Lorentz factor and \(\nu_{gp}\) the group velocity.

It also induces a phase shift. Indeed:

\[
\langle \Psi | \hat{H}_D[\hat{p}] | \Psi \rangle = \langle \Psi | e^{i(\delta \varepsilon)\beta \tau_0/\hbar} \hat{H}_D e^{-i(\delta \varepsilon)\beta \tau_0/\hbar} | \Psi \rangle = \langle \Phi | \hat{H}_D[\hat{p}] + \alpha \delta \varepsilon/c | \Phi \rangle + ...
\]  

(20)

where

\[
| \Phi \rangle = e^{-i(\delta \varepsilon)\beta \tau_0/\hbar} | \Psi \rangle
\]  

(21)

The phase shift is \(\delta \varphi = (\delta \varepsilon)\beta \tau_0/\hbar\). For a finite transformation, its expectation value is

\[
< \Delta \varphi > = \{(\delta \varepsilon)\beta \tau_0/\hbar\} < \beta > \sim (1/\gamma)m_0 c^2 \tau_0/\hbar
\]  

(22)

as \(< \beta > = m_0 c^2/\hbar < H_D > = \pm m_0 c^2/\varepsilon = \pm 1/\gamma\), for a positive (negative) energy wave packet that contains both positive and negative energy free particle solutions [11]. Thus the sign of \(< \beta >\) distinguishes the positive or negative energy branch where the momentum displacement takes place. To maintain the fundamental indeterminacy modulo \(n\pi\) (\(n\) an integer) in the phase of the complex eigenfunctions one has to set, for \(n = 1\):

\[
\tau_0 = 2\pi \hbar/ < \beta > \varepsilon = h/m_0 c^2
\]  

(23)

This is the de Broglie period. One has then:

\[
h/p = h/\gamma m_0 \nu_{gp} = hc^2/\gamma m_0 c^2 \nu_{gp} = (h/\varepsilon)(c^2/\nu_{gp}) = (1/\nu) v_{ph}
\]  

(24)

which is precisely the de Broglie wavelength, that is, the product of the phase velocity by the period derived from the Planck relation \(\varepsilon = h\nu\), as originally assumed by de Broglie [15].

In conclusion, the dynamical time operator \(T = \alpha.\hat{r}/c + \beta(h/m_0 c^2)\) (where the parameter \(\tau_0\) is equated to de Broglie period \(h/m_0 c^2\)), generates the Lorentz
boost that gives rise to the de Broglie wave. Indeed the fact that a rest frame oscillation gives rise to a travelling wave has been shown to be a simple consequence of special relativity applied to the complex-phase oscillation of stationary states through the Lorentz transformation of the time dependence [18, 19, 20]. To quote Baylis: "What in the rest frame is a synchronized oscillation in time is seen in the laboratory to be a wave in space".

What about Pauli’s objection? The continuous displacement in momentum implies also a continuous displacement in energy only within the positive and the negative branches. No crossing of the energy gap is involved. Thus Pauli’s objection is circumvented.

Finally, it is also interesting to note the following. In the same way as above, in the case of an infinitesimal time lapse (δt << 1) the unitary operator

\[ U_{HD}(\delta t) = e^{i(\delta t)(\alpha \hat{p} + \beta m_0 c^2)/\hbar} \]

can be approximated as:

\[ U(\delta t) \simeq e^{i(\delta t)(\alpha \hat{p}/\hbar)} e^{i(\delta t)(\beta m_0 c^2/\hbar)} \quad (25) \]

In configuration space this yields a displacement

\[ \delta \hat{r} = < \hat{r} + c\alpha(\delta t) + v_{gp}(\delta t) > + \hat{r} \]

and a phase shift \( \phi = (\delta t) < \beta > m_0 c^2/\hbar \). For repeated infinitesimal time displacements to reach \( \Delta t = \gamma \tau_0 = \gamma h/m_0 c^2 \) (the boosted de Broglie period), the phase shift is

\[ \Delta \phi = (\gamma h/m_0 c^2)(m_0 c^2/ < H >) m_0 c^2/\hbar = h/h = 2\pi \quad (26) \]

These results are in agreement with the fact that the Hamiltonian is actually the generator of the time development of a system described by a wave packet. The approximate treatment provides only the displacement, neglecting the dispersion of the wave.

4 The time-energy uncertainty relation

The time operator and the Dirac Hamiltonian satisfy the commutation relation:

\[ [\hat{t}, \hat{H}_D] = i\hbar\{I + 2\beta K\} + 2\beta\{\tau_0 \hat{H}_D - m_0 c^2 \hat{T}\} \quad (27) \]

where \( K = \beta (2s.l/\hbar^2 + 1) \) is a constant of motion[14]. In the usual manner an uncertainty relation follows, namely:

\[ (\Delta T)(\Delta H_D) \geq (\hbar/2) |\{1 + 2 < \beta K >\}| = (\hbar/2) |\{3 + 4 < s.l/\hbar^2 >\}| \quad (28) \]

To be noted now is that, with the present definition of the time operator, its uncertainty is related to the uncertainty in position \( \Delta \hat{r} \), in the same way as the energy uncertainty is related to the momentum uncertainty \( \Delta \hat{p} \). Indeed:

\[ (\Delta T)^2 = \langle \hat{T}^2 \rangle - \langle \hat{T} \rangle^2 = \langle \hat{r}^2/c^2 + \tau_0^2 \rangle - \langle \hat{T} \rangle^2 = \]

\[ = \{(\Delta r)^2 + (\langle \hat{r} \rangle)^2\}/c^2 + \tau_0^2 - \langle \hat{T} \rangle^2 \]

\[ = \{(\Delta r)^2 + (\langle \hat{r} \rangle)^2\}/c^2 + \tau_0^2 - (\alpha \hat{r}/c + \beta \tau_0)^2 \geq (\Delta r)^2/c^2 \quad (29) \]
\[ (\Delta H_D)^2 = \langle \hat{H}_D^2 \rangle - \langle \hat{H}_D \rangle^2 = \]
\[ = c^2\{(\Delta p)^2 + (\hat{p})^2\} + \langle (m_0c^2)^2 - (c\alpha \hat{p} + \beta m_0c^2) \rangle^2 \geq c^2 (\Delta p)^2 \]

Then
\[ (\Delta T)(\Delta \hat{H}) \geq (\Delta r)(\Delta p) \geq \left(\frac{3\hbar}{2}\right) \]

This corresponds to Bohr’s interpretation: the width of a wave packet, complementary to its momentum dispersion, measures the uncertainty in the time of passage at a certain point, and is thereby complementary to its energy dispersion.

4.0.1 The Mandelstam-Tamm uncertainty relation[16, p.319]

As an observable, the time operator can be subject to the Mandelstam-Tamm (MT) formulation of a time-energy uncertainty relation within standard QM, to wit: any observable \( \hat{O} \) represented by a self-adjoint operator \( \hat{O} \) not explicitly dependent on time, satisfies the dynamical equation:
\[ (i\hbar) \frac{d}{dt} \left< \hat{O} \right> = \left< [\hat{O}, \hat{H}] \right> \]

From the commutator \([\hat{O}, \hat{H}]\) it follows that the uncertainties defined \( \Delta \hat{O} \) and \( \Delta \hat{H} \) satisfy the relation:
\[ (\Delta \hat{O})(\Delta \hat{H}) \geq (1/2) \left| \langle [\hat{O}, \hat{H}] \rangle \right| \]

Then, associated to any system observable \( \hat{O} \), a related time uncertainty is defined as:
\[ \Delta \hat{T}_{\hat{O}}^{MT} = \frac{\Delta \hat{O}}{\left| \frac{d}{dt} \left< \hat{O} \right> \right|} \]

From Eqs.29 and 30, it then follows that:
\[ (\Delta \hat{T}_{\hat{O}}^{MT})(\Delta \hat{H}) \geq (\hbar/2) \]

This is the Mandelstam-Tamm time-energy uncertainty relation. \( \Delta \hat{T}_{\hat{O}}^{MT} \) can be interpreted as "the time required for the center \( \left< \hat{O} \right> \) of this distribution to be displaced by an amount equal to its width \( \Delta \hat{O} \)."

Now let \( \hat{O} \) be the dynamical time operator in RQM \( \hat{T} = (\alpha \hat{r})/c + \beta \tau_0 \). Then, from Eq. 32:
\[ \Delta T_{\hat{T}}^{MT} \approx \frac{\Delta \hat{T}}{|\langle I + 2\beta K \rangle|} \]
It follows that:

\[
\frac{\Delta \hat{T}}{\langle I + 2\beta K \rangle} (\Delta \hat{H}_D) \geq \langle h/2 \rangle
\]  

(37)

or

\[
(\Delta \hat{T})(\Delta \hat{H}_D) \geq \langle h/2 \rangle |\langle I + 2\beta K \rangle|
\]  

(38)

In the non relativistic limit \( \langle \hat{H}_D \rangle \approx m_0c^2 \), neglecting the oscillating terms,

\[
\hat{T}(t) \simeq \tau_0 + (cp/m_0c^2)^2 t + ...
\]  

(39)

Thus:

\[
\frac{d \langle \hat{T} \rangle}{dt} = \langle (cp/m_0c^2)^2 \rangle = (v_{gp}/c)^2
\]  

(40)

and

\[
\Delta T^{MT}_{\hat{T}} \simeq \frac{\Delta \hat{T}}{(v_{gp}/c)^2} \gg \Delta \hat{T}
\]  

(41)

as \( v_{gp} \ll c \). The uncertainty of the Mandelstam-Tamm time operator associated with the observable \( \hat{T} \) overestimates largely the usual uncertainty.

In the ultra relativistic limit \( \langle \hat{H}_D \rangle \approx cp \):

\[
\hat{T}(t) \simeq t + (m_0c^2/cp)^2 \tau_0 + ... \approx t
\]  

(42)

and

\[
\Delta T^{MT}_{\hat{T}} \simeq \Delta \hat{T}
\]  

(43)

5 Conclusion

A self-adjoint internal "time operator" can be defined within Dirac’s formulation of relativistic quantum mechanics (RQM). As with all observables, it is accorded in the Heisenberg picture a dependence on the external laboratory time parameter \( t \) in the TDSE, which is attributed to the entanglement of the microscopic system with a classical environment. Also as an observable, it is in general subject to an uncertainty shown to be proportional to the position uncertainty; and consequently to a time-energy uncertainty relation where the energy uncertainty is related to the momentum one, supporting Bohr’s original interpretation as the uncertainty in the instant of passage at a point of the trajectory. It nevertheless circumvents Pauli’s objection, due to the fact that, as generator of a unitary transformation, it actually produces displacements in the continuous momentum spectrum, and thus only indirectly in energy. This resolves the problem of time in quantum mechanics.

Based on the position observable, the time operator is expected to exhibit a Zitterbewegung behaviour about its linear dependence on \( t \). As occurs with the position Zitterbewegung, its observation is be beyond current
technical possibilities. However it may be observable in systems that simulate Dirac’s Hamiltonian, where position Zitterbewegung has already been exhibited experimentally [23, 24, 25]. A corresponding time operator can be constructed in each case and its properties examined.

Finally, general relativity accords a dynamical behaviour to space-time, firmly confirmed recently by the detection of gravitational waves. As a dynamical time is definitively incompatible with a time parameter, this becomes from the start a fundamental ‘problem of time’ in quantum gravity [26, 27, 28]. Whether the time operator here introduced has a relevance in this subject, is a venue to be considered [29].

6 Appendix A. The lore of \([\hat{x}, \hat{p}] = i\hbar\)

For pedagogical purposes this Appendix collects all the properties derivable from the commutation relation, that are usually dispersed in quantum mechanical books.

To represent observables the operators \(\hat{x}\) and \(\hat{p}\) are self-adjoint \((\hat{x} = \hat{x}^\dagger, \hat{p} = \hat{p}^\dagger)\), which insures real eigenvalues. Then:

1) Spectrum

Consider the eigenvalue equation:

\[
|\hat{x}|x\rangle = x|\rangle (A.1)
\]

By Stone–von Neumann’s theorem [17] [10] the operator \(U(\alpha) = \exp(-i\alpha\hat{p}/\hbar)\) with \(\alpha\) real is unitary. Then:

\[
\hat{x}\{U(\alpha)|x\rangle\}
= \hat{x}\{1 + (1/1!)(-i\alpha\hat{p}/\hbar) + (1/2!)(-i\alpha\hat{p}/\hbar)^2 + (1/3!)(-i\alpha\hat{p}/\hbar)^3 + \ldots\}|x\rangle
= (\hat{x} + (-i\alpha/\hbar)\hat{p} + (1/2)(-i\alpha/\hbar)^2\hat{p}^2 + \ldots)|x\rangle
= (\hat{x} + (-i\alpha/\hbar)|\hat{p}\rangle + [\hat{x}, \hat{p}] + (1/2)(-i\alpha/\hbar)^2[\hat{p}^2\hat{x} + [\hat{x}, \hat{p}^2]] + \ldots)|x\rangle
= (\hat{x} + (-i\alpha/\hbar)|\hat{p}\rangle + [\hat{x}, \hat{p}] + (1/2)(-i\alpha/\hbar)^2\hat{p}^2\hat{x} + 2i\alpha\hat{p} + \ldots)|x\rangle
= (\hat{x} + \alpha)|\rangle (1 + (-i\alpha\hat{p}/\hbar) + (1/2)(-i\alpha\hat{p}/\hbar)^2 + \ldots)|\rangle = (x + \alpha)|\rangle (U(\alpha)|x\rangle
\]

One concludes that:

\[
\{U(\alpha)|x\rangle\} = |x + \alpha\rangle (A.2)
\]

As \(\alpha\) is arbitrary, it follows that the eigenvalues of \(\hat{x}\) are continuous from \(-\infty\) \(a + \infty\), and that the eigenvectors satisfy:

\[
\langle x' | x \rangle = \delta(x' - x) \int dx \ |x\rangle \langle x| = I \quad (A.3)
\]

where \(\delta(x' - x)\) is the Dirac delta function and \(I\) is the identity operator.
In the same way one can prove that the eigenvalues in $\hat{p} |p\rangle = p |p\rangle$ span a continuum from $-\infty$ to $+\infty$ and that the eigenvectors satisfy:

$$\langle p' | p \rangle = \delta(p' - p) \quad \int dp |p\rangle \langle p| = I \quad (A.4)$$

2) Representations

In configuration space one has:

$$\Phi(x) = \langle x | \Phi \rangle = \langle x | \hat{x} | \Psi \rangle = x \langle x | \Psi \rangle = x \Psi(x) \quad (A.5)$$

and:

$$\langle x' | [\hat{x}, \hat{p}] | x \rangle = i\hbar \delta(x' - x) = \langle x' | \hat{x} \hat{p} - \hat{p} \hat{x} | x \rangle$$

$$= \int dx'' \langle x' | \hat{x} | x'' \rangle \langle x'' | \hat{p} | x \rangle - \int dx'' \langle x' | \hat{p} | x'' \rangle \langle x'' | \hat{x} | x \rangle$$

$$= \int dx'' \delta(x' - x'') \langle x'' | \hat{p} | x \rangle - \int dx'' \langle x' | \hat{p} | x'' \rangle x \delta(x'' - x)$$

$$= x' \langle x' | \hat{p} | x \rangle - x \langle x' | \hat{p} | x \rangle = (x' - x) \langle x' | \hat{p} | x \rangle$$

$$\langle x' | \hat{p} | x \rangle = \frac{i\hbar \delta(x' - x)}{x' - x} \quad \Rightarrow \quad x' \rightarrow x \quad \Rightarrow \quad \frac{d}{dx'} \delta(x' - x) \quad (A.6)$$

Then:

$$\Phi(x) = \langle x | \Phi \rangle = \langle x | \hat{p} | \Psi \rangle =$$

$$= \int dx' \langle x | \hat{p} | x' \rangle \langle x' | \Psi \rangle = i\hbar \int dx' \frac{d}{dx'} \delta(x' - x) \langle x' | \Psi \rangle \quad (A.7)$$

$$= [\delta(x' - x) \Psi(x')]_{-\infty}^{+\infty} - i\hbar \int dx' \delta(x' - x) \frac{d}{dx'} \Psi(x') = -i\hbar \frac{d}{dx} \Psi(x)$$

i.e., the representation in configuration space of the vector $|\Phi\rangle = \hat{p} |\Psi\rangle$ is obtained by taking the derivative of the representation of the vector $|\Psi\rangle$, while the representation in configuration space of the vector $|\Theta\rangle = \hat{x} |\Psi\rangle$ is obtained multiplying by $x$ the representation of $|\Psi\rangle$.

To conclude, in configuration space one has:

$$\hat{x} \Longrightarrow x \quad \hat{p} \Longrightarrow -i\hbar \frac{d}{dx} \quad (A.8)$$

and in the same way in momentum space:

$$\hat{x} \Longrightarrow i\hbar \frac{d}{dp} \quad \hat{p} \Longrightarrow p \quad (A.9)$$

3) Transformation between representations
Consider: \[ \langle x | [\hat{x}, \hat{p}] | p \rangle = i\hbar \langle x | p \rangle \] (A.10)

Developing:
\[
\langle x | [\hat{x}, \hat{p}] | p \rangle = \langle x | \hat{x}\hat{p} | p \rangle - \langle x | \hat{p}\hat{x} | p \rangle
\]
\[
= xp \langle x | p \rangle - i\hbar \int dx' \frac{d}{dx'} \delta(x' - x) \langle x' | x' \rangle \langle x' | p \rangle
\]
\[
= xp \langle x | p \rangle - i\hbar \delta(x' - x) \langle x' | x' \rangle \langle x' | p \rangle
\]
\[
= xp \langle x | p \rangle + i\hbar [\langle x | p \rangle + i\hbar \frac{d}{dx} \langle x | p \rangle]
\]
Substituting in Eq.(A.10) one obtains:
\[
xp \langle x | p \rangle + i\hbar [\langle x | p \rangle + i\hbar \frac{d}{dx} \langle x | p \rangle] = i\hbar \langle x | p \rangle
\]
which is satisfied if:
\[
\langle x | p \rangle = Ce^{ipx/\hbar} \quad \langle p | x \rangle = C^* e^{-ipx/\hbar}
\] (A.12)

Finally:
\[
\Phi(p) = \langle p | \Psi \rangle = \int dx \langle p | x \rangle \langle x | \Psi \rangle = C^* \int dx \ e^{-ipx/\hbar} \Psi(x)
\] (A.13)
and:
\[
\Psi(x) = \langle x | \Psi \rangle = \int dp \langle x | p \rangle \langle p | \Psi \rangle = C \int dx \ e^{ipx/\hbar} \Phi(p)
\] (A.14)
i.e., the representations of the state vector in the configuration and momentum spaces are Fourier transforms of each other. To preserve normalization one requires \( C = C^* = 1/\sqrt{2\pi\hbar} \).

4) Uncertainty relation
Consider the state vectors
\[
|\Phi\rangle = (\hat{x} - \langle x \rangle) |\Psi\rangle \quad \text{and} \quad |\Xi\rangle = (\hat{p} - \langle p \rangle) |\Psi\rangle
\] (A.15)
Then
\[
\langle \Phi | \Phi \rangle = \langle \Psi| \hat{x}^2 |\Psi \rangle - \langle \Psi| \hat{x} |\Psi \rangle^2 = \langle \Delta x \rangle^2 \Omega
\] (A.16)
and
\[ \langle \Xi | \Xi \rangle = \langle \Psi | \hat{p}^2 | \Psi \rangle - \langle \Psi | \hat{p} | \Psi \rangle^2 = (\Delta \hat{p})^2 \]  \hspace{1cm} (A.17)

By Schawrz inequality one has
\[ (\Phi | \Phi) \langle \Xi | \Xi \rangle \geq |\langle \Phi | \Xi \rangle|^2 = \left| \langle \Psi | \frac{1}{2}[\hat{x}, \hat{p}] + \frac{1}{2}\{\hat{x}, \hat{p}\} - \langle x \rangle \langle p \rangle | \Psi \rangle \right|^2 \geq \left| \langle \Psi | \frac{1}{2}[\hat{x}, \hat{p}] | \Psi \rangle \right|^2 = (\hbar/2)^2 \]  \hspace{1cm} (A.18)

Finally
\[ (\Delta x)_\Psi (\Delta p)_\Psi \geq \hbar/2 \]  \hspace{1cm} (A.19)

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