Spinor Parallel Propagator and Green’s Function in Maximally Symmetric Spaces

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December 8, 1999

Abstract

We introduce the spinor parallel propagator for maximally symmetric spaces in any dimension. Then, the Dirac spinor Green’s functions in the maximally symmetric spaces \(\mathbb{R}^n\), \(S^n\) and \(H^n\) are calculated in terms of intrinsic geometric objects. The results are covariant and coordinate-independent.

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1 Introduction

The study of field theory in Anti-de Sitter (AdS) spaces, which topologically are hyperbolic maximally symmetric spaces, has been revived over the past two years following the so-called Maldacena conjecture relating type IIB supergravity on AdS$_5 \times S^5$ with $\mathcal{N} = 4$, $U(N)$ Super Yang Mills theory in four dimensions.

More than a decade ago, the calculation of correlation functions in maximally symmetric spaces using only intrinsic geometric objects was presented in a series of papers starting with [2, 3]. In one of them [4], Green’s functions for two-component spinors in maximally symmetric four-spaces were considered using the $SL(2, R)$ formulation.

To our knowledge, this analysis has not been extended since to Dirac spinors in other space-time dimensions. However, it should be mentioned that spinor Green’s functions in AdS spaces have been considered and calculated by other means in the context of the AdS/CFT correspondence [5, 6].

In the present paper, we present an intrinsically geometric approach to spinor Green’s functions in maximally symmetric spaces. In section 2, we introduce the spinor parallel propagator for maximally symmetric spaces of dimension $n$ and find its covariant derivatives. Then, in section 3, we calculate the spinor Green’s functions for the spaces $\mathbb{R}^n$, $S^n$ and $H^n$. Finally, section 4 contains conclusions.

In the remainder of this section, we would like to review the elementary maximally bi-tensors, which have been discussed in detail by Allen and Jacobsen [3].

Consider a maximally symmetric space of dimension $n$ with constant scalar curvature $n(n-1)/R^2$. For the space $S^n$, the radius $R$ is real and positive, whereas for the hyperbolic space $H^n$, $R = il$ with $l$ positive, and in the flat case, $\mathbb{R}^n$, $R = \infty$.

Consider further two points $x$ and $x'$, which can be connected uniquely by a shortest geodesic. Let $\mu$ be the proper geodesic distance along this shortest geodesic between $x$ and $x'$. Then, the vectors

$$n_\nu (x, x') = D_\nu \mu (x, x') \quad \text{and} \quad n_{\nu'} (x, x') = D_{\nu'} \mu (x, x')$$

are tangent to the geodesic and have unit length. Furthermore, denote by $g_{\nu'} (x, x')$ the vector parallel propagator along the geodesic. Notice the relation $n^{\nu'} = -g_{\nu'} n^\mu$.

These elementary maximally bi-tensors $n^\mu$, $n_{\nu'}$ and $g_{\nu'}$ satisfy the following properties:

$$D_\mu n_\nu = A(g_{\mu\nu} - n_\mu n_\nu) \quad (2a)$$
$$D_{\mu'} n_\nu = C(g_{\mu'\nu} + n_{\mu'} n_\nu) \quad (2b)$$
$$D_\mu g_{\nu\lambda'} = -(A + C)(g_{\mu\nu} n_{\lambda'} + g_{\mu\lambda'} n_\nu) \quad (2c)$$

where $A$ and $C$ are functions of the geodesic distance $\mu$ and are given by

$$A = \frac{1}{R} \cot \frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R \sin(\mu/R)}. \quad (3)$$

Therefore, they satisfy the relations

$$dA/d\mu = -C^2, \quad dC/d\mu = -AC \quad \text{and} \quad C^2 - A^2 = 1/R^2. \quad (4)$$

Finally, our convention for covariant gamma matrices is $\{\Gamma^\mu, \Gamma^{\nu'}\} = 2g^{\mu\nu}$. 

2
2 Spinor Parallel Propagator

To start, consider a bi-spinor \( \Lambda(x', x)_{\alpha' \beta} \), which acts as parallel propagator for Dirac spinors in a maximally symmetric space-time, i.e. it performs the parallel transport
\[
\Psi'(x')^{\alpha'} = \Lambda(x', x)_{\alpha' \beta} \Psi(x)^{\beta}.
\]

The spinor parallel propagator \( \Lambda(x', x) \) can be uniquely defined for any space-time dimension by the following properties:
\[
\begin{align*}
\Lambda(x', x) &= [\Lambda(x, x')]^{-1}, \\
\Gamma^{\nu}(x') &= \Lambda(x', x) \Gamma^{\mu}(x) \Lambda(x, x') g^{\nu}_{\mu'}(x', x), \\
n^{\mu} D_{\mu} \Lambda(x, x') &= 0.
\end{align*}
\]

Eqn. (5a) implies that \( \Lambda(x, x)_{\alpha' \beta} = \delta_{\alpha' \beta} \), whereas eqn. (5b) conveniently formulates the parallel transport of the covariant gamma matrices. Finally, eqn. (5c) says that \( \Lambda(x, x') \) is covariantly constant along the geodesic of parallel transport.

We would like to evaluate now a particular property of \( \Lambda(x, x') \), namely its covariant derivative. Therefore, combine eqns. (5a) and (5b) to
\[
\Gamma^{\nu} \Lambda(x, x') = \Lambda(x, x') \Gamma^{\mu'} g^{\nu}_{\mu'}
\]
and differentiate covariantly with respect to \( x \) to obtain
\[
\Gamma^{\nu} D_{\lambda} \Lambda(x, x') = D_{\lambda} \Lambda(x, x') \Gamma^{\mu'} g^{\nu}_{\mu'} - (A + C) \Lambda(x, x') \Gamma^{\mu'} (\delta_{\lambda}^{\nu} n_{\mu'} + g_{\lambda \mu'} n^{\nu}),
\]
where we have used the property (2c) of the vector parallel propagator. Now, use eqn. (6) for the second term on the right hand side of eqn. (7) and multiply with \( \Gamma^{\lambda} \), which yields
\[
2 D^{\nu} \Lambda(x, x') - \Gamma^{\nu} \Psi \Lambda(x, x') = \Psi \Lambda(x, x') \Gamma^{\mu'} g^{\nu}_{\mu'} + (A + C) (\Gamma^{\nu} \Gamma^{\rho} n_{\rho} - n n^{\nu}) \Lambda(x, x').
\]
Thus, a multiplication with \( \Gamma^{\nu} \) leads to
\[
(2 - n) \Psi \Lambda(x, x') = \Gamma_{\nu} \Psi \Lambda(x, x') \Gamma^{\mu'} g^{\nu}_{\mu'},
\]
the solution of which is
\[
\Psi \Lambda(x, x') = B n_{\nu} \Gamma^{\mu} \Lambda(x, x'),
\]
where \( B \) is some function of the geodesic distance \( \mu \). Then, substituting eqn. (8) into eqn. (9) yields
\[
2 D^{\nu} \Lambda(x, x') = 2 B n^{\nu} \Lambda(x, x') + (A + C) (\Gamma^{\nu} \Gamma^{\rho} n_{\rho} - n n^{\nu}) \Lambda(x, x').
\]
Moreover, multiplying this with \( n_{\nu} \) and using eqn. (5c) one determines \( B \) to be
\[
B = \frac{1}{2} (n - 1) (A + C).
\]
Therefore, one finally obtains
\[
D_{\mu} \Lambda(x, x') = \frac{1}{2} (A + C) \left( \Gamma_{\mu} \Gamma^{\nu} n_{\nu} - n_{\mu'} \right) \Lambda(x, x').
\]

For completeness, we also give the expression for \( D_{\mu'} \Lambda(x, x') \). It is easily obtained from eqn. (10) using eqn. (5a) and is given by
\[
D_{\mu'} \Lambda(x, x') = - \frac{1}{2} (A + C) \Lambda(x, x') \left( \Gamma_{\mu'} \Gamma^{\nu'} n_{\nu'} - n_{\mu'} \right).
\]
3 Spinor Green’s Function

Using the spinor parallel propagator \( \Lambda(x, x') \) calculated in section 2, we would now like to find the spinor Green’s function \( S(x, x') \) satisfying
\[
[(\not\!D - m)S(x, x')]^\alpha_{\beta'} = \frac{\delta(x - x')}{\sqrt{g(x)}} \delta^\alpha_{\beta'}.
\] (12)

Here, we have written the indices explicitly in order to emphasize that this is a bi-spinor equation. We shall henceforth omit the indices.

Now, we make the general ansatz
\[
S(x, x') = [\alpha(\mu) + \beta(\mu)n_\nu \Gamma^\nu] \Lambda(x, x'),
\] (13)
where \( \alpha \) and \( \beta \) are functions of the geodesic distance \( \mu \) still to be determined. We substitute the ansatz (13) into eqn. (12) and, after using eqn. (10), obtain the two coupled differential equations
\[
\beta' + \frac{1}{2} (n - 1)(A - C)\beta - m\alpha = \frac{\delta(x - x')}{\sqrt{g(x)}},
\] (14)
\[
\alpha' + \frac{1}{2} (n - 1)(A + C)\alpha - m\beta = 0,
\] (15)
where the prime denotes differentiation with respect to \( \mu \).

In order to proceed, multiply eqn. (14) with \( m \) and substitute \( m\beta \) from eqn. (15). One finds
\[
\alpha'' + \frac{n - 1}{\mu} \alpha' - m^2 \alpha = m \frac{\delta(x - x')}{\sqrt{g(x)}},
\] (16)
where eqn. (4) has been used. We shall solve equation (16) separately for the spaces \( \mathbb{R}^n, S^n \) and \( H^n \).

3.1 Green’s Function for \( \mathbb{R}^n \)

For \( \mathbb{R}^n \), we have \( A = -C = 1/\mu, R = \infty \) and \( \mu = |x - x'| \). Thus, eqn. (16) becomes
\[
\alpha'' + \frac{n - 1}{\mu} \alpha' - m^2 \alpha = m \delta(x - x').
\] (17)

The solution to eqn. (17) is
\[
\alpha(\mu) = -\left(\frac{m}{2\pi}\right)^{\frac{\mu}{2}} \mu^{1-\frac{\mu}{2}} K_{\frac{\mu}{2}-1}(m\mu),
\] (18)
where the functional form was obtained by solving eqn. (17) for \( \mu \neq 0 \), and the constant was found by matching the singularity. Furthermore, one finds from eqn. (15) \( m\beta = \alpha' \), i.e. \( n_\nu \beta = \partial_\nu \alpha/m \), so that the final result for the spinor Green’s function in \( \mathbb{R}^n \) is
\[
S(x, x') = -\frac{1}{m} \left(\frac{m}{2\pi}\right)^{\frac{\mu}{2}} (\not\!D + m) \mu^{1-\frac{\mu}{2}} K_{\frac{1}{2}}(m\mu).
\] (19)

Upon Fourier transforming it, one obtains the more familiar expression
\[
S(x, x') = -(\not\!D + m) \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(x-x')} \frac{1}{k^2 + m^2}.
\] (20)
3.2 Green’s Function for $S^n$

In order to solve eqn. (14), we consider first $x \neq x'$ and make the substitution

$$z = \cos^2 \frac{\mu}{2R}.$$  

(21)

This yields the differential equation

$$\left[ z(1-z) \frac{d^2}{dz^2} + \frac{n}{2} (1-2z) \frac{d}{dz} - \frac{(n-1)^2}{4} - m^2 R^2 - \frac{n-1}{4z} \right] \alpha(z) = 0. \quad (22)$$

Then, writing $\alpha(z) = \sqrt{z} \gamma(z)$, one obtains a hypergeometric equation for $\gamma$,

$$H(a, b; c; z) \gamma(z) = 0,$$  

(23a)

where

$$H(a, b; c; z) = z(1-z) \frac{d^2}{dz^2} + [c - (a+b+1)z] \frac{d}{dz} - ab$$  

(23b)

is the hypergeometric operator, and its parameters are

$$a = \frac{n}{2} - i|m|R, \quad b = \frac{n}{2} + i|m|R, \quad c = \frac{n}{2} + 1.$$  

(23c)

The solution of eqn. (23) which is singular at $z = 1$ is

$$\gamma(z) = \lambda F(a, b; c; z) = \lambda F(n/2 - i|m|R, n/2 + i|m|R; n/2 + 1; z),$$  

(24)

where $\lambda$ is a proportionality constant. Therefore, $\alpha(z)$ is

$$\alpha(z) = \lambda \frac{\Gamma(n/2 + 1) \Gamma(n/2 - 1)}{\Gamma(n/2 - i|m|R) \Gamma(n/2 + i|m|R)} \left( \frac{\mu}{2R} \right)^{2-n}.$$  

We can now determine the constants $\lambda$ by matching the singularity in eqn. (15). This is equivalent to demanding the singularity of $\alpha$ at $\mu = 0$ to have the same strength as in the case of $\mathbb{R}^n$. One finds from eqn. (23)

$$\alpha \rightarrow \lambda \frac{\Gamma(n/2 + 1) \Gamma(n/2 + 1)}{\Gamma(n/2 - i|m|R) \Gamma(n/2 + i|m|R)} \left( \frac{\mu}{2R} \right)^{2-n},$$

whereas in $\mathbb{R}^n$ we have, from eqn. (18),

$$\alpha \rightarrow -\frac{m}{4} \Gamma(n/2 - 1) \pi^{-n/2} \mu^{2-n}. \quad (26)$$

Comparing these two expressions we find

$$\lambda = -\frac{m}{4} \frac{\Gamma(n/2 - i|m|R) \Gamma(n/2 + i|m|R)}{\Gamma(n/2 + 1) \pi^{n/2} 2^n} R^{2-n}. \quad (27)$$

Finally, one can calculate $\beta$ from eqn. (17), which yields

$$\beta(z) = -\frac{1}{m} \left[ \frac{1}{R} \sqrt{z(1-z)} \frac{d}{dz} + \frac{n-1}{2R} \sqrt{\frac{1-z}{z}} \right] \alpha(z)$$

$$= -\frac{\lambda}{mR} \sqrt{1-z} \left[ z F(n/2 + 1 - i|m|R, n/2 + 1 + i|m|R; n/2 + 2; z) \right.$$

$$+ \frac{n}{2} F(n/2 - i|m|R, n/2 + i|m|R; n/2 + 1; z) \right]. \quad (28)$$

It should be noticed that $\beta$ has a finite $m \to 0$ limit, whereas $\alpha$ vanishes.
3.3 Green’s Function for $H^n$

For $H^n$, we can start with eqn. (23) and set $R = il$, i.e. we have to solve

$$H(a, b; c; z)\gamma(z) = 0$$  \hspace{1cm} (29a)

with

$$a = \frac{n}{2} + |m|l, \quad b = \frac{n}{2} - |m|l, \quad c = \frac{n}{2} + 1.$$  \hspace{1cm} (29b)

There are two solutions to eqn. (29) which behave asymptotically like a power of $z$ for $z \to \infty$. These are

$$\gamma_{\pm}(z) = \lambda_{\pm}z^{-\left(\frac{n}{2} \pm |m|l\right)} F\left(\frac{n}{2} \pm |m|l, \pm|m|l; 1 \pm 2|m|l; \frac{1}{z}\right),$$  \hspace{1cm} (30)

where $\lambda_{\pm}$ are constants. The choice of the minus sign is not always possible. In fact, for $1 - 2|m|l = 0, -1, -2, \ldots$, the hypergeometric series is indeterminate. Thus, we shall include the solution with the minus sign only, if $|m|l < 1/2$. Hence, we have two solutions for $\alpha$,

$$\alpha_{\pm}(z) = \lambda_{\pm}z^{-\left(\frac{n}{2} \pm |m|l\right)} F\left(\frac{n}{2} \pm |m|l, \pm|m|l; 1 \pm 2|m|l; \frac{1}{z}\right),$$  \hspace{1cm} (31)

and we can now proceed to determine the constants $\lambda_{\pm}$ in a similar fashion as in the $S^n$ case. From eqn. (31) we find for $\mu \to 0$

$$\alpha \to \lambda_{\pm} \left(\frac{\mu}{2l}\right)^{2-n} \frac{\Gamma(1 \pm 2|m|l)\Gamma(n/2 - 1)}{\Gamma(n/2 \pm |m|l)\Gamma(\pm|m|l)}.$$  \hspace{1cm} (32)

Comparing this expression to the $\mathbb{R}^n$ case, eqn. (24), we find

$$\lambda_{\pm} = \mp \text{sgn} m \frac{2^{-(n\pm2|m|l)}(n/2 \pm |m|l)}{\pi^{(n-1)/2}} \frac{\Gamma(n/2 \pm |m|l)}{\Gamma(\pm|m|l)},$$  \hspace{1cm} (33)

where the doubling formula for Gamma functions has been used.

Finally, let us calculate $\beta$ from eqn. (13). Using a recursion formula for hypergeometric functions we find

$$\beta_{\pm}(z) = \frac{1}{m} \left[\frac{1}{l} \sqrt{z(z-1)} \frac{d}{dz} \frac{n-1}{2l} \sqrt{\frac{z-1}{z}}\right] \alpha_{\pm}(z)$$

$$= \mp \text{sgn} m \lambda_{\pm} \sqrt{z-1} z^{-\left(\frac{n}{2} \pm |m|l\right)} F\left(\frac{n}{2} \pm |m|l, 1 \pm |m|l; 1 \pm 2|m|l; \frac{1}{z}\right).$$  \hspace{1cm} (34)

It is interesting to note that in the limit $m \to 0$ the functions $\beta_+$ and $\beta_-$ become identical, whereas $\alpha_+$ and $\alpha_-$ do not, but differ in their signs. The reason is, of course, that, for $m = 0$, eqns. (14) and (15) decouple, and $\alpha$ can be a solution of eqn. (15) with arbitrary proportionality constant. Moreover, for $m = 0$, the common value of $\beta_{\pm}$ is a rational function of $z$,

$$\beta_{\pm}(z) = \frac{\Gamma(n/2)}{(2\pi)^n} l^{1-n} (z-1)^{-(n-1)/2}.$$  \hspace{1cm} (35)
4 Conclusions

We have introduced the spinor parallel propagator for maximally symmetric spaces in any dimension. This enabled us to find expressions for the Dirac spinor Green’s functions in the maximally symmetric spaces $\mathbb{R}^n$, $S^n$ and $H^n$ in terms of intrinsic geometric objects. Although there are obstructions to the quantization of spinors in odd dimensional manifolds with boundary \cite{8}, our results should be applicable to the AdS/CFT correspondence, because of the classical dynamics in AdS space.

Acknowledgments

I would like to thank K. S. Viswanathan and R. C. Rashkov for stimulating discussions. Moreover, financial support from Simon Fraser University is gratefully acknowledged.

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