On $\alpha$-Square-Stable Graphs

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Abstract

The stability number of a graph $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set, and $\mu(G)$ is the cardinality of a maximum matching in $G$. If $\alpha(G) + \mu(G)$ equals its order, then $G$ is a König-Egerváry graph. We call $G$ an $\alpha$-square-stable graph if $\alpha(G) = \alpha(G^2)$, where $G^2$ denotes the second power of $G$. These graphs were first investigated by Randerath and Wolkmann, [18]. In this paper we obtain several new characterizations of $\alpha$-square-stable graphs. We also show that $G$ is an $\alpha$-square-stable König-Egerváry graph if and only if it has a perfect matching consisting of pendant edges. Moreover, we find that well-covered trees are exactly $\alpha$-square-stable trees. To verify this result we give a new proof of one Ravindra’s theorem describing well-covered trees, [19].

1 Introduction

All the graphs considered in this paper are simple, i.e., are finite, undirected, loopless and without multiple edges. For such a graph $G = (V, E)$ we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. By $G - F$ we denote the partial subgraph of $G$ obtained by deleting the edges of $F$, for $F \subset E(G)$, and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then $(A, B)$ stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. A set of pairwise non-adjacent vertices is a stable set of $G$. The stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. The stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. The stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. Let $\Omega(G)$ denotes $\{S : S$ is a maximum stable set of $G\}$. $\theta(G)$ is the clique covering number of $G$, i.e., the minimum number of cliques whose union covers $V(G)$. Recall also that $i(G) = \min\{|S| : S$ is a maximal stable set in $G\}$, and $\gamma(G) = \min\{|D| : D$ is a minimal domination set in $G\}$. A matching is a set of non-incident edges of $G$. A matching of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is a matching covering all the vertices of $G$. $M$ is an induced matching, if no edge of $G$ connects two edges of $M$ (some recent results on induced matchings can be found in [3], [4]). If $A, B$ are disjoint subsets of $V(G)$, we say that $A$ is uniquely matched into $B$ if there is a unique
matching $M \subseteq (A, B)$ that saturates all the vertices in $A$. $G$ is a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$, \cite{10,13,20}. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \{N(v) : v \in A\}$, for $A \subseteq V$. If $G[N(v)]$ is a complete subgraph in $G$, then $v$ is a simplicial vertex of $G$. A maximal clique in $G$ is called a *simplex* if it contains at least one simplicial vertex of $G$, \cite{10}. $G$ is said to be *simplicial* if every vertex of $G$ is simplicial or is adjacent to a simplicial vertex of $G$, \cite{10}. If $|N(v)| = |\{w\}| = 1$, then $v$ is a pendant vertex and $vw$ is a pendant edge of $G$. By $C_n$, $K_n$, $P_n$ we denote the chordless cycle on $n \geq 4$ vertices, the complete graph on $n \geq 1$ vertices, and respectively the chordless path on $n \geq 3$ vertices. A graph $G$ is $\alpha^-$-*stable* if $\alpha(G-e) = \alpha(G)$, for any $e \in E(G)$, and $\alpha^+$-*stable* if $\alpha(G+e) = \alpha(G)$, for any edge $e \in E(G)$, where $\overline{G}$ is the complement of $G$, \cite{14}. $G$ is well-covered if it has no isolated vertices and if every maximal stable set of $G$ is also a maximum stable set, i.e., it is in $\Omega(G)$, \cite{18}. $G$ is called *very well-covered*, \cite{14}. If provided $G$ is well-covered and $|V(G)| = 2\alpha(G)$.

The distance between two vertices $v, w \in V(G)$ is denoted by $\text{dist}_G(v, w)$, or $\text{dist}(v, w)$ if no ambiguity. $G^2$ denotes the second power of graph $G$, i.e., the graph with the same vertex set $V$ and an edge is joining distinct vertices $v, w \in V$ whenever $\text{dist}_G(v, w) \leq 2$. Clearly, any stable set of $G^2$ is stable in $G$, as well, while the converse is not generally true. Therefore, we may assert that $1 \leq \alpha(G^2) \leq \alpha(G)$. Let notice that the both bounds are tight. For instance, if $G$ is not a complete graph and $\text{dist}(a, b) \leq 2$ holds for any $a, b \in V(G)$, then $\alpha(G) \geq 2 > 1 = \alpha(G^2)$, e.g., for the $n$-star graph $G = K_{1,n}$, with $n \geq 2$, we have $\alpha(G) = n > \alpha(G^2) = 1$. On the other hand, if $G = P_4$, then $\alpha(G) = \alpha(G^2) = 2$.

In this paper we characterize the graphs $G$ for which the upper bound of the above inequality is achieved, i.e., $\alpha(G) = \alpha(G^2)$. These graphs we call $\alpha$-square-stable, or shortly square-stable. We show that any square-stable graph is $\alpha^+$-stable and that none of them is $\alpha^-$-stable. We give a complete description of square-stable König-Egerváry graphs extending the investigation of well-covered trees, started in \cite{19}.

Randerath and Volkmann, \cite{18}, prove that:

**Theorem 1.1** \cite{18} For a graph $G$ the following statements are equivalent:

(i) every vertex of $G$ belongs to exactly one simplex of $G$;

(ii) $G$ satisfies $\alpha(G) = \alpha(G^2)$;

(iii) $G$ satisfies $\theta(G) = \theta(G^2)$;

(iv) $G$ satisfies $\alpha(G^2) = \theta(G^2) = \gamma(G) = i(G) = \alpha(G) = \theta(G)$.

**Remark 1.1** In general, it can be shown (e.g., see \cite{18}) that the graph invariants appearing in the above theorem are related by the following inequalities:

$$\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G).$$

The graph $C_{12}$ indicates that no other non-trivial equality (except $\alpha(G) = \alpha(G^2)$ and $\theta(G) = \theta(G^2)$) of a pair of the above invariants ensures that all of them are equal, namely, $\alpha(C_{12}^2) = i(C_{12}) = 4$, while $\alpha(C_{12}) = 6$. 

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The following characterization of maximum stable sets in a graph, due to Berge, we shall use in the sequel.

**Proposition 1.2** \[1\] \( S \in \Omega(G) \) if and only if every stable set \( A \) of \( G \), disjoint from \( S \), can be matched into \( S \).

Other useful results are:

**Proposition 1.3** \[13\] A graph \( G \) is very well-covered if and only if it is a well-covered König-Egerváry graph.

**Proposition 1.4** \[13\] A König-Egerváry graph is well-covered if and only if it is very well-covered.

**Proposition 1.5** \[13\] A graph \( G \) is:

(i) \( \alpha^+ \)-stable if and only if \(|\cap\{S : S \in \Omega(G)\}| \leq 1\);  
(ii) \( \alpha^- \)-stable if and only if \(|N(v) \cap S| \geq 2\) is true for every \( S \in \Omega(G) \) and any \( v \in V(G) - S \).

By Proposition 1.5, an \( \alpha^+ \)-stable graph may have either \(|\cap\{S : S \in \Omega(G)\}| = 0\) or \(|\cap\{S : S \in \Omega(G)\}| = 1\). This motivates the following definition.

**Definition 1.6** \[13\] A graph \( G \) is called:

(i) \( \alpha^+_0 \)-stable whenever \(|\cap\{S : S \in \Omega(G)\}| = 0\);  
(ii) \( \alpha^+_1 \)-stable provided \(|\cap\{S : S \in \Omega(G)\}| = 1\).

For instance, the graph in Figure 1 is an \( \alpha^+_1 \)-stable graph.

![Figure 1: Graph \( K_3 + e \).](image)

In \[13\] it was shown that an \( \alpha^+ \)-stable tree can be only \( \alpha^+_0 \)-stable, and this is exactly the case of trees possessing a perfect matching. This result was generalized to bipartite graphs in \[13\]. Nevertheless, there exist both \( \alpha^+_1 \)-stable König-Egerváry graphs (e.g., the graph in Figure 1), and \( \alpha^+_0 \)-stable König-Egerváry graphs (e.g., all \( \alpha^+ \)-stable bipartite graphs).

## 2 Square-stable graphs

Clearly, any complete graph is square-stable. Moreover, since \( K^2_n = K_n \), we get that  
\[
\Omega(K_n) = \Omega(K^2_n) = \{\{v\} : v \in V(K_n)\}.
\]
**Proposition 2.1** Graph $G$ is square-stable if and only if $\Omega(G^2) \subseteq \Omega(G)$.

**Proof.** Clearly, any stable set $A$ of $G^2$ is stable in $G$, too. Consequently, if $G$ is square-stable, then any maximum stable set of $G^2$ is a maximum stable set of $G$, as well, i.e., $\Omega(G^2) \subseteq \Omega(G)$.

The converse is obvious. ■

It is quite evident that $G$ and $G^2$ are simultaneously connected or disconnected. In addition, if $H_i$, $1 \leq i \leq k$ are the connected components of graph $G$, then $S \in \Omega(G)$ if and only if $S \cap V(H_i) \in \Omega(H_i), 1 \leq i \leq k$. Henceforth, using Proposition 2.1 we infer that:

**Proposition 2.2** A disconnected graph is square-stable if and only if any of its connected components is square-stable.

Therefore, in the rest of the paper all the graphs are connected, unless otherwise stated.

**Lemma 2.3** Every $S \in \Omega(G^2)$ has the property that

$$\text{dist}_G(a, b) \geq 3$$

holds for any distinct $a, b \in S$.

**Proof.** If $S \in \Omega(G^2)$ and $a, b \in S$, then $\text{dist}_G(a, b) \geq 3$, since otherwise $ab \in E(G^2)$, contradicting the stability of $S$ in $G^2$. ■

**Proposition 2.4** A graph $G$ is square-stable if and only if there is some $S \in \Omega(G)$ such that $\text{dist}_G(a, b) \geq 3$ holds for any distinct $a, b \in S$.

**Proof.** If $G$ is square-stable, then Proposition 2.1 ensures that $\Omega(G^2) \subseteq \Omega(G)$, and by above Lemma 2.3, $\text{dist}(a, b) \geq 3$ is valid for every $S \in \Omega(G^2)$ and any $a, b \in S$.

Conversely, if $S \in \Omega(G)$ and $\text{dist}(a, b) \geq 3$ holds for any $a, b \in S$, then $S$ is stable in $G^2$, and therefore, $|S| \leq \alpha(G^2) \leq \alpha(G) = |S|$ implies that $\alpha(G^2) = \alpha(G)$, i.e., $G$ is square-stable. ■

**Lemma 2.5** If $G \neq K_{|V|}$ is square-stable, then for every $S \in \Omega(G^2)$ and any $a \in S$, there is $b \in S$ with $\text{dist}_G(a, b) = 3$.

**Proof.** Suppose, on the contrary, that there are $S \in \Omega(G^2)$ and some $a \in S$, such that $\text{dist}_G(a, b) \geq 4$ holds for any $b \in S$. Let $v \in V$ be with $\text{dist}_G(a, v) = 2$; hence $\text{dist}_G(v, w) \geq 2$ is valid for any $w \in S$, and consequently, $S \cup \{v\}$ is stable in $G$, a contradiction, because $S$ is a maximum stable set in $G$. ■

**Lemma 2.6** If $G$ is square-stable, then $\Omega(G^2) = \Omega(G)$ if and only if $G$ is a complete graph.

**Proof.** Suppose, on the contrary, that $\Omega(G^2) = \Omega(G)$ holds for a non-complete square-stable graph $G$. Let $S \in \Omega(G^2)$ and $a \in S$. According to Lemma 2.1, there is $b \in S$ with $\text{dist}_G(a, b) = 3$. Now, if $c \in N(a)$ and $\text{dist}_G(c, b) = 2$, Proposition 2.4 implies that $S \cup \{c\} \setminus \{a\} \in \Omega(G) - \Omega(G^2)$, contradicting the relation $\Omega(G^2) = \Omega(G)$.

The converse is clear. ■

Combining Proposition 2.1 and Lemma 2.6 we obtain the following assertion:
Theorem 2.7 $\Omega(G^2) = \Omega(G)$ if and only if $G$ is a complete graph.

Let $A \triangle B$ denotes the symmetric difference of the sets $A, B$, i.e., the set

$$A \triangle B = (A - B) \cup (B - A).$$

Theorem 2.8 For a graph $G$ the following assertions are equivalent:

(i) $G$ is square-stable;

(ii) there exists $S_0 \in \Omega(G)$ that satisfies the property

P1: any stable set $A$ of $G$ disjoint from $S_0$ can be uniquely matched into $S_0$;

(iii) any $S \in \Omega(G^2)$ has property P1;

(iv) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has a unique perfect matching;

(v) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has a perfect matching;

(vi) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has an induced perfect matching.

Proof. (i) $\Rightarrow$ (ii) By Proposition 2.1 we get that $\Omega(G^2) \subseteq \Omega(G)$ holds for $G$ square-stable. Now, if $S \in \Omega(G^2)$, and $A$ is a stable set in $G$ disjoint from $S$, Proposition 2.3 implies that $A$ can be matched into $S$. If there exists another matching of $A$ into $S$, then at least one vertex $a \in A$ has two neighbors in $S$, say $b, c$. Hence, $bc \in E(G^2)$ and this contradicts the stability of $S$. Therefore, any $S \in \Omega(G^2) \subseteq \Omega(G)$ has property P1.

(ii) $\Rightarrow$ (i) Suppose, on the contrary, that $G$ is not square-stable. It follows that $S_0 \notin \Omega(G^2)$, i.e., there are $v, w \in S_0$ with $vw \in E(G^2)$. Henceforth, there exists $u \in V \setminus \{v, w\}$, such that $vw, uw \in E(G)$. Consequently, there are two matchings of $A = \{u\}$ into $S_0$, contradicting the fact that $S_0$ has property P1.

(iii) $\Rightarrow$ (iv) Let $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$. Then $|S_2| \leq |S_1|$, and since $S_1 - S_2$ is stable in $G$ and disjoint from $S_2$, we infer that $S_1 - S_2$ can be uniquely matched into $S_2$, precisely into $S_2 - S_1$, and because $|S_2 - S_1| \leq |S_1 - S_2|$, this matching is perfect. In conclusion, $G[S_1 \triangle S_2]$ has a unique perfect matching.

(iv) $\Rightarrow$ (v) It is clear.

(v) $\Rightarrow$ (i) If $G[S_1 \triangle S_2]$ has a perfect matching, for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, it follows that $|S_1 - S_2| = |S_2 - S_1|$, and this implies $|S_1| = |S_2|$, i.e., $\alpha(G) = \alpha(G^2)$ is valid.

(i) $\Rightarrow$ (vi) According to (iv), $G[S_1 \triangle S_2]$ has a unique perfect matching $M$, for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$. By (ii), $|N(v) \cap S_2| = 1$ holds for any $v \in S_1 - S_2$. Therefore, $M$ must be induced.

(vi) $\Rightarrow$ (iv) It is evident.

Corollary 2.9 There are no $\alpha^-$-stable square-stable graphs.

Proof. According to Proposition 2.3, $G$ is $\alpha^-$-stable provided $|N(v) \cap S| \geq 2$ holds for every $S \in \Omega(G)$ and any $v \in V(G) - S$. If $G$ is also square-stable, then there exists some $S_0 \in \Omega(G)$ satisfying property P1, which implies that $|N(v) \cap S_0| = 1$ holds for any $v \in V(G) - S_0$. This incompatibility concerning $S_0$ proves that $G$ cannot be simultaneously square-stable and $\alpha^-$-stable.
In Figure 2 are shown two non-square-stable graphs: $C_6$, which is both $\alpha^-$-stable and $\alpha^+$-stable, and the diamond, which is only $\alpha^-$-stable.

**Corollary 2.10** Any square-stable graph is $\alpha^+$-stable.

**Proof.** Suppose that $G$ is a non-$\alpha^+$-stable square-stable graph. Hence, according to Proposition 1.5, there are $a, b \in \cap\{S : S \in \Omega(G)\}$, and since $G$ is square-stable, we infer that $a, b \in \cap\{S : S \in \Omega(G^2)\}$, as well. Let $S_0 \in \Omega(G^2)$ and $c \in N(a)$ in $G$. Clearly, $a, b \in S_0$, and by Lemma 2.3, $\text{dist}_G(a, v) \geq 3$ holds for any $v \in S_0 - \{a\}$. Consequently, $\text{dist}_G(c, v) \geq 2$ holds for any $v \in S_0 - \{a\}$. It follows that $S_1 = S_0 \cup \{c\} - \{a\} \in \Omega(G)$, but this contradicts the assumption on $a$, namely that $a \in \cap\{S : S \in \Omega(G)\}$.

Moreover, we can strengthen Corollary 2.10 to the following:

**Corollary 2.11** Any square-stable graph is well-covered.

**Proof.** Assume, on the contrary, that $G$ is not well-covered, i.e., there is some maximal stable set $A$ that is not maximum. According to Theorem 2.8, for any $S \in \Omega(G^2)$, there is a unique matching of $B = A - S \cap A$ into $S$, in fact, into $S - A$. Consequently, $S \cup B - N(B) \cap S$ is a maximum stable set of $G$ that includes $A$, in contradiction with the fact that $A$ is a maximal stable set.

It is also possible to see the above result stated implicitly in the proof of Theorem 1.1 from [8], but our proof is different.

The converse of Corollary 2.11 is not generally true; e.g., $C_5$ is well-covered, but is not square-stable. The square-stable graphs do not coincide with the very well-covered graphs. For instance, $P_4$ is both square-stable and very well-covered, $C_4$ is very well-covered and non-square-stable, but there are square-stable graphs that are not very well-covered; e.g., the graph in Figure 3.

![Figure 3: A square-stable graph $G$ and its $G^2$. $G$ is not very well-covered.](image)

**Corollary 2.12** Any square-stable graph is $\alpha_0^+$-stable.
Theorem 2.13 For a graph $G$ the following statements are equivalent:

(i) $G$ is square-stable;

(ii) there is $S_0 \in \Omega(G)$ that has the property $P_2$; for any stable set $A$ of $G$ disjoint from $S_0$, $A \cup S^* \in \Omega(G)$ holds for some $S^* \subset S_0$;

(iii) every $S \in \Omega(G^2)$ has property $P_2$.

Proof. (i) $\Rightarrow$ (ii), (iii) By Theorem 2.8, for every $S \in \Omega(G^2)$ and any stable set $A$ in $G$, disjoint from $S$, there is a unique matching of $A$ into $S$. Consequently, $S^* = S - N(A) \cap S$ has $|S^*| = |S| - |A|$ and $S^* \cup A \in \Omega(G)$.

(ii) $\Rightarrow$ (i) It suffices to show that $S_0 \in \Omega(G^2)$. If $S_0 \notin \Omega(G)$, there must exist $a, b \in S_0$ such that $ab \in E(G^2)$, and this is possible provided $a, b \in N(e) \cap S_0$ for some $c \in V - S_0$. Hence, $|S_0 \cup \{c\} - \{a, b\}| < |S_0|$ and this implies that $\{c\} \cup S^* \notin \Omega(G)$ holds for any $S^* \subset S$, contradicting the fact that $S_0$ has the property $P_2$. Therefore, we get that $S_0 \in \Omega(G^2)$, and this implies that $\alpha(G) = \alpha(G^2)$.

(iii) $\Rightarrow$ (i) Let $S \in \Omega(G^2)$, $b \in S$ and $a \in V - S$ be such that $ab$ is an edge in $G$. Since $\{a\}$ is stable and disjoint from $S$, and $S$ has property $P_2$, there exists $S^* \subset S$ so that $S^* \cup \{a\} \in \Omega(G)$. Hence, $|S^*| = \alpha(G) - 1$ and consequently, $|S| = |S^*| + 1 = \alpha(G)$, i.e., $S \in \Omega(G)$ holds for any $S \in \Omega(G^2)$. By Proposition 2.1, $G$ is square-stable.

Combining Theorem 1.1 and our results on square-stable graphs, we obtain:

Theorem 2.14 For a graph $G$ the following statements are equivalent:

(i) every vertex of $G$ belongs to exactly one simplex of $G$;

(ii) $G$ is square-stable;

(iii) $G$ satisfies $\theta(G) = \theta(G^2)$;

(iv) $G$ satisfies $\alpha(G^2) = \theta(G^2) = \gamma(G) = \omega(G) = \alpha(G) = \theta(G)$;

(v) $\Omega(G^2) \subseteq \Omega(G)$;

(vi) there is some $S \in \Omega(G)$ such that $\text{dist}(a, b) \geq 3$ holds for any distinct $a, b \in S$;

(vii) there exists $S_0 \in \Omega(G)$ that satisfies the property $P_1$:

P1: any stable set $A$ of $G$ disjoint from $S_0$ can be uniquely matched into $S_0$;

(viii) any $S \in \Omega(G^2)$ has property $P_1$;

(ix) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has a unique perfect matching;

(x) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has a perfect matching;

(xi) for any $S_1 \in \Omega(G)$ and $S_2 \in \Omega(G^2)$, $G[S_1 \triangle S_2]$ has an induced perfect matching;

(xii) there is $S_0 \in \Omega(G)$ that has the property $P_2$; for any stable set $A$ of $G$ disjoint from $S_0$, $A \cup S^* \in \Omega(G)$ holds for some $S^* \subset S_0$;

(xiii) any $S \in \Omega(G^2)$ has property $P_2$.

We can now characterize the square-stable graphs that are also simplicial or chordal, by extending two results from [17].

Proposition 2.15 For a graph $G$ the following assertions are equivalent:

(i) $G$ is square-stable;

(ii) $G$ is simplicial and well-covered;

(iii) every vertex belongs to exactly one simplex of $G$. 

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Proof. The equivalence (ii) ⇔ (iii) is proved in [17], and Theorem 2.14 ensures that (i) ⇔ (iii). 

Proposition 2.16 For a chordal graph $G$ the following assertions are equivalent:
(i) $G$ is square-stable;
(ii) $G$ is well-covered;
(iii) every vertex belongs to exactly one simplex of $G$.

Proof. The equivalence (ii) ⇔ (iii) is proved in [17], and Theorem 2.14 ensures that (i) ⇔ (iii). 

As another consequence of Theorem 2.14, we obtain that $\Omega(G)$ is the set of bases of a matroid on $V(G)$ provided $G$ is a complete graph.

Lemma 2.17 $\Omega(G)$ is the set of bases of a matroid on $V$ if and only if $\Omega(G^2) = \Omega(G)$.

Proof. If $\Omega(G)$ is the set of bases of a matroid on $V$, then any $S \in \Omega(G)$ must have the property P2. By Theorem 2.13, $G$ is square-stable and therefore $\Omega(G^2) \subseteq \Omega(G)$. Suppose that there exists $S_0 \in \Omega(G) - \Omega(G^2)$; it follows that there are $a, b \in S_0$ and $c \in N(a) \cap N(b)$. Hence, \{c\} is stable in $G$ and disjoint from $S_0$, but $S^* \cup \{c\} \notin \Omega(G)$ for any $S^* \subset S_0$, a contradiction, since $S_0$ has property P2. Consequently, the equality $\Omega(G^2) = \Omega(G)$ is true.

Conversely, according to Theorem 2.13, any $S \in \Omega(G^2) = \Omega(G)$ has the property P2. Therefore, $\Omega(G)$ is the set of bases of a matroid on $V$. 

Combining Theorem 2.7 and Lemma 2.17, we get the following:

Proposition 2.18 $\Omega(G)$ is the set of bases of a matroid on $V$ if and only if $G$ is a complete graph.

For graphs that are not necessarily connected, we may deduce the following:

Proposition 2.19 $\Omega(G)$ is the set of bases of a matroid on $V(G)$ if and only if $G$ is a disjoint union of cliques.

3 Unique pendant perfect matching graphs

In general, a graph having a unique perfect matching is not necessarily square-stable. For instance, $K_3 + e$ has a unique perfect matching, but is not square-stable. Further, we pay attention to graphs having a perfect matching consisting of pendant edges, which is obviously unique.

Proposition 3.1 If $G$ has a perfect matching consisting of pendant edges, then the following statements are valid:
(i) $\Omega(G^2) = \{S_0\}$, where $S_0 = \{v : v$ is a pendant vertex in $G\}$;
(ii) $G$ is square-stable.
According to Proposition 3.1, it follows that \( G = \{ a \in V : v \text{ is a pendant vertex in } G \} \) is stable in \( G \), and \( |S_0| = |V - S_0| \leq \alpha(G) \). Let \( S_1 \in \Omega(G) \) and suppose that \( |S_1| > |S_0| \). Hence, both \( S_1 \cap S_0 \) and \( S_1 \cap (V - S_0) \) are non-empty, and \( |S_1 \cap S_0| > |V - S_0 - (S_1 \cap (V - S_0))| \).

In addition, we have that \( |S_1 \cap S_0, S_1 \cap (V - S_0)| = \emptyset \), because \( S_1 \) is stable, and therefore \( S_1 \cap S_0 \) can not be matched into \( V - S_0 - (S_1 \cap (V - S_0)) \), contradicting the fact that \( G \) has a perfect matching. Consequently, \( S_0 \in \Omega(G) \), and because \( \text{dist}_{G}(a, b) \geq 3 \) holds for any \( a, b \in S_0 \), we get that \( S_0 \in \Omega(G^2) \), i.e., \( G \) is square-stable.

Assume that there is \( S_2 \in \Omega(G^2) \), \( S_0 \neq S_2 \). Then \( S_2 \in \Omega(G) \) and \( \text{dist}_{G}(a, b) \geq 3 \) holds for any \( a, b \in S_2 \). Let denote \( S_0 = \{ v_i : 1 \leq i \leq \alpha(G) \} \) and \( N(v_i) = \{ w_i \} \), for \( 1 \leq i \leq \alpha(G) \). Since \( S_0 \neq S_2 \), we may assume that, for instance, \( w_i \in S_2 \), and because \( w_1 \) is not pendant, it follows that \( |N(w_1)| \geq 2 \). Without loss of generality, we may suppose that \( w_2 \in N(w_1) \). Hence, \( v_1, v_2, w_2 \notin S_2 \), and this implies that \( |S_2| < |S_0| \), because for any \( i \geq 3 \), \( S_2 \) contains either \( v_i \) or \( w_i \), but never both of them. So, we may conclude that \( \Omega(G^2) = \{ S_0 \} \).

Let us notice that there are square-stable graphs with more than one maximum stable set, and having no perfect matching; e.g., the graph in Figure 3.

**Proposition 3.2** For a König-Egerváry graph \( G \) of order \( n \geq 2 \) the following assertions are equivalent:

(i) \( G \) square-stable;

(ii) \( G \) has a perfect matching consisting of pendant edges;

(iii) \( G \) is very well-covered with exactly \( \alpha(G) \) pendant vertices.

**Proof.** (i) \( \Rightarrow \) (ii) By Proposition 2.11, \( G \) is well-covered, and according to Proposition 1.3 it is also very well-covered. Hence, we get that \( \alpha(G) = \mu(G) = n/2 \), and \( G \) has a perfect matching \( M \). Let \( S_0 = \{ a_i : 1 \leq i \leq \alpha(G) \} \in \Omega(G^2) \) and \( b_i \in V(G) - S_0 \) be such that \( a_i b_i \in M \) for \( 1 \leq i \leq \alpha(G) \). By Proposition 2.4, \( \text{dist}_{G}(v, w) \geq 3 \) holds for any \( v, w \in S_0 \). We claim that every \( a_i \in S_0 \) is pendant, i.e., \( N(a_i) = \{ b_i \} \), since otherwise, if \( b_j \in N(a_i) \) for some \( i \neq j \), it follows that \( \text{dist}_{G}(a_i, a_j) = 2 \), in contradiction with \( \text{dist}_{G}(a_i, a_j) \geq 3 \). Therefore, \( M \) consists only of pendant edges.

(ii) \( \Rightarrow \) (iii) Let \( M = \{ v_i w_i : 1 \leq i \leq n/2 \} \) be the perfect matching of \( G \), consisting only of pendant edges, and suppose that all vertices in \( S_0 = \{ v_i : 1 \leq i \leq n/2 \} \) are pendant. By Proposition 3.1, we get that \( S_0 \in \Omega(G) \), i.e., \( \alpha(G) = \mu(G) = n/2 \).

Assume that \( G \) is not well-covered, that is there exists some maximal stable set \( A \) in \( G \) such that \( A \notin \Omega(G) \). Since \( S_0 \) contains all pendant vertices of \( G \), it follows that \( A \cup \{ v_i : v_i \in S_0, N(v_i) \cap A = \emptyset \} \) is stable and larger than \( A \), in contradiction with the maximality of \( A \). In conclusion, \( G \) is very well-covered.

(iii) \( \Rightarrow \) (i) Since \( G \) is very well-covered with exactly \( \alpha(G) \) pendant vertices, we infer that \( S_0 = \{ v : v \text{ is a pendant vertex} \} \in \Omega(G) \) and also that the matching \( M = \{ vw : vw \in E(G), v \in S_0 \} \) is perfect and consists of only pendant edges. According to Proposition 3.1, it follows that \( G \) is square-stable.

**Remark 3.1** Well covered König-Egerváry graphs do not have to be square-stable, for instance, the graph \( C_4 \).
Remark 3.2 A König-Egerváry graph with a unique perfect matching is not always square-stable, e.g., the graphs $P_6$ (by the way, it is also a tree) and $K_3 + e$ (i.e., the graph in Figure 4).

Remark 3.3 A non-König-Egerváry graph with a unique perfect matching $M$ may be square-stable, even if $M$ does not consist of only pendant edges (for instance, see the graph in Figure 4).

Figure 4: $G$ is square-stable and has a unique perfect matching containing not only pendant edges.

Proposition 3.2 is true for bipartite graphs as well, since any bipartite graph is also a König-Egerváry graph. It is worth recalling here that for a bipartite graph (see [12], and for trees see [14]) to have a perfect matching is equivalent to be $\alpha^+$-stable. In general, we have shown in [14] that any $\alpha^+$-stable König-Egerváry graph has a perfect matching, while the converse is not true (see, for instance, the diamond, Figure 2).

Proposition 3.3 [15] Any well-covered tree $T$ non-isomorphic to $K_1, K_2$, contains at least one edge $e$ connecting two non-pendant vertices, such that $T - e = T' \cup K_2$ and $T'$ is a well-covered tree.

For trees, Propositions 3.2 and 3.3 lead to the following extension of the characterization that Ravindra gave to well-covered trees in [19]:

Corollary 3.4 If $T$ is a tree of order $n \geq 2$, then the following statements are equivalent:

(i) $T$ is well-covered;
(ii) $T$ is very well-covered;
(iii) $T$ has a perfect matching consisting of pendant edges;
(iv) $T$ is square-stable.

Proof. Let us notice that for general graphs: $(iv) \Rightarrow (i)$ is true according to Corollary 2.11 and the implication $(iii) \Rightarrow (ii)$ is clear. Further, for König-Egerváry graphs, the assertions $(iii)$, $(iv)$ are equivalent according to Proposition 3.2, and $(i)$, $(ii)$ are equivalent by Proposition 1.4. Thus, to complete the proof of the corollary, it is sufficient to show that for trees $(i)$ implies $(iii)$. Since $(i)$ and $(ii)$ are equivalent, the order $n$ of $T$ must be even. We use induction on $n$. The assertion is true for $n = 2$. If $T$ has $n > 2$ vertices, then according to Proposition 3.3, $T$ contains at least one edge $e$ connecting two non-pendant vertices, such that $T - e = T' \cup K_2$ and $T'$ is a well-covered tree. By the induction hypothesis, $T'$ has a perfect matching $M$ consisting of pendant edges. Hence, $M \cup \{e\}$ is a perfect matching of $T$ consisting of...
Let us notice that the equivalences appearing in Corollary 3.4 fail for bipartite graphs. For instance, the graph in Figure 5 is very well-covered, but is not square-stable.

Combining Proposition 1.3 and Proposition 3.2, we obtain:

**Corollary 3.5** \( G \) is square-stable and very well-covered if and only if \( G \) is a König-Egerváry graph with exactly \( \alpha(G) \) pendant vertices.

**Corollary 3.6** If \( G \) is a square-stable König-Egerváry graph, then \( G^2 \) is also a König-Egerváry graph.

**Remark 3.4** Figure 5 brings an example of a König-Egerváry graph whose square is not a König-Egerváry graph.

Another consequence of Proposition 3.2 is the following extension of the characterization that Finbow, Hartnell and Nowakowski give in [7] for graphs having the girth \( \geq 6 \).

**Proposition 3.7** Let \( G \) be a graph of girth \( \geq 6 \), which is isomorphic to neither \( C_7 \) nor \( K_1 \). Then the following assertions are equivalent:
1. \( G \) is well-covered;
2. \( G \) has a perfect matching consisting of pendant edges;
3. \( G \) is very well-covered;
4. \( G \) is a König-Egerváry graph with exactly \( \alpha(G) \) pendant vertices;
5. \( G \) is a König-Egerváry square-stable graph.

**Proof.** The equivalences (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) are done in [7]. In [13] it has been proved that (iii) \( \Leftrightarrow \) (iv). Finally, (ii) \( \Leftrightarrow \) (v) is true by Propositions 3.1 and 3.2. ■

**Remark 3.5** \( C_7 \) is not a König-Egerváry graph.
4 Conclusions

In this paper we continue the investigations, started by Randerath and Volkmann [15], on the class of square-stable graphs. We think that the characterization of Koenig-Egervary square-stable graphs obtained here may be extended to some new classes of square-stable graphs. It is also important to mention that square-stable trees have a very specific recursive structure (see [14]).

It also seems interesting to study graphs satisfying some equalities between the invariants appearing in the following series of inequalities: $\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G)$, for instance $\alpha(G^2) = i(G)$.

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