Semi-analytic single-channel and cross-channel nonlinear interference spectra in highly-dispersed WDM coherent optical links with rectangular signal spectra

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Abstract

We provide new single-integral formulas of the power spectral density of single-channel and cross-channel nonlinear interference in highly-dispersed coherent optical links for which the Gaussian Noise model \cite{1}, \cite{2} applies.

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I. INTRODUCTION

The Gaussian Noise (GN) model has recently been shown to effectively predict the system performance of highly-dispersed wavelength division multiplexed (WDM) coherent optical transmission systems, such as high baud-rate dispersion-uncompensated (DU) systems \cite{1}, \cite{2}. In such a model, the GN reference formula (GNRF) provides a formally elegant and compact expression of the power spectral density (PSD) of the received nonlinear interference (NLI). However, the GNRF involves a double frequency integral which poses non-trivial numerical problems for multi-span wavelength division multiplexed (WDM) systems. Many of the numerical integration issues have been already addressed in \cite{2}. Given the practical importance of developing an accurate GNRF numerical evaluator, however, for debugging purposes it proves quite useful to have exact expressions of the NLI PSD in special realistic cases. The case of rectangular per-channel input spectra has already served in \cite{2} as a basic example to clarify the integration regions, and in \cite{3} to obtain novel explicit expressions of both NLI PSD and total received NLI power in the single-channel case, or equivalently in the Nyquist WDM case where the whole WDM spectrum is rectangular.

In this paper, we derive exact single-integral semi-analytic expressions of the NLI PSD in the GNRF for both Nyquist and non-Nyquist WDM systems with input rectangular per-channel spectra. We provide explicit PSD formulas for both the single-channel interference (SCI) and the cross-channel interference (XCI) \cite{2}. We formulate the GNRF in a generalized form that applies to any link configuration, be it with concentrated or distributed amplification, with or without in-line compensation, and with possibly different spans: the whole link complexity is summarized within the \textit{kernel} frequency function \cite{4}–\cite{6}.
II. THE GN REFERENCE FORMULA

In dual-polarization transmission, assuming uncorrelated signals with identical spectra on the two polarizations, the GN reference formula (GNRF) yields the power spectral density (PSD) of the nonlinear interference (NLI) as [1, 2, 5, 6]:

\[ G_{NLI}(f) = \frac{16}{9\pi} I(f) \]
\[ I(f) := \int_{-\infty}^{\infty} |\mathcal{K}(f_1 f_2)|^2 G(f + f_1)G(f + f_2)G(f + f_1 + f_2)df_1 df_2 \]  

where \( G(f) \) is the input PSD (i.e., that of the propagated channel in single-channel transmission, or the whole wavelength division multiplexed (WDM) spectrum in multi-channel transmission), and the scalar frequency-kernel when higher-order dispersion is neglected is [5, 6]:

\[ \mathcal{K}(v) := \int_0^{L} \gamma(s)G(s)e^{-j(2\pi)^2 C(s)v}ds \]

where \( L \) is total system length, \( \gamma(z) \) is the fiber nonlinear coefficient, \( G(s) \) is the power gain from 0 to \( s \), and \( C(s) \equiv C_0 - \int_0^s \beta_2(s')ds' \) is the cumulated dispersion from 0 to \( s \). \( C_0 \) is the (possibly present) pre-compensation, and \( C \) has here the sign of the dispersion coefficient. Note that the system function \( \mathcal{K} \) depends only on the product \( v \equiv f_1 f_2 \). A generalization including third-order dispersion is provided in [5].

Whenever the input PSD \( G(f) \) is symmetric in \( f \), then also \( G_{NLI}(f) \) is symmetric. In fact, for \( f \geq 0 \) we have:

\[ I(f) = \int_{-\infty}^{\infty} |\mathcal{K}(f_1 f_2)|^2 G(-f + f_1)G(-f + f_2)G(-f + f_1 + f_2)df_1 df_2 \]

\[ = \int_{-\infty}^{\infty} |\mathcal{K}(f_1 f_2)|^2 G(f - f_1)G(f - f_2)G(f - f_1 - f_2)df_1 df_2 \]

because of the symmetry of \( G(.) \). By substituting \( f_1 \) by \( -f_1 \) and \( f_2 \) by \( -f_2 \) we get \( I(f) \) again. Hence with symmetric input PSDs the \( G_{NLI}(f) \) needs to be calculated only at positive frequencies.

The trouble with the analytic formula (1) is that it involves a double frequency integration where the squared kernel \( |\mathcal{K}(f_1 f_2)|^2 \) is oscillating in frequency faster and faster as the number of spans increases and poses non-trivial integral convergence problems [2]. A first step towards easing the double integration comes from the GN reference formula (GNRF) which yields the power spectral density (PSD) of the nonlinear interference (NLI) as [1, 2, 5, 6]:

\[ I(f) = \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u}G(f + u)G(f + \frac{v}{u})G(f + u + \frac{v}{u})\frac{du}{u} \right] \]

\[ + \int_0^\infty \frac{1}{u}G(f - u)G(f + \frac{v}{u})G(f - u + \frac{v}{u})\frac{du}{u} \]

\[ + \int_0^\infty \frac{1}{u}G(f - u)G(f - \frac{v}{u})G(f - u - \frac{v}{u})\frac{du}{u} \]

\[ + \int_0^\infty \frac{1}{u}G(f + u)G(f - \frac{v}{u})G(f + u - \frac{v}{u})\frac{du}{u} \]  

where \( \mathcal{K}(v) \) is given in (2), and the four lines correspond to integration over the four quadrants of the \((f_1, f_2)\) plane. The pole at \( u = 0 \) in the inner integral does not pose convergence problems for any finite-power spectrum, since \( \lim_{f \to \pm\infty} G(f) = 0 \) and thus all triple products \( G(.)G(.)G(.) \) in the integrand go to zero sufficiently fast as \( u \to 0 \).

When the input WDM signals have rectangular spectra, the inner integral (4)-(7) can be solved exactly, and in the next sections we will present numerically stable single-integral formulas of the NLI PSD in such a case. The usefulness of these single-integral formulas is that they provide a case against which numerical double-integration routines of (4) can be checked for debugging.

III. SINGLE-CHANNEL / NYQUIST-WDM SYSTEMS

We tackle here the rectangular-spectrum single-channel case, or equivalently the WDM case where no bandwidth gaps are present between neighboring channels, known as the Nyquist-WDM case. The total power is \( P \) and the input PSD \( G(f) = \frac{P}{2\delta} \text{rect}_{2\delta}(f) \) is a rectangular gate centered at \( f = 0 \) with total two-sided bandwidth \( 2\delta \). The integrand in (4) is non-zero only over the shaded domains in quadrants I through IV shown
in Fig. 1 at several values of \( f \). For instance, since \( |\mathcal{K}(v)|^2 = |\mathcal{K}(-v)|^2 \) for any kernel (2), then the squared kernel is the same over the 4 quadrants, hence the integral (1) over quadrants II and IV has always the same value. Also, it is easy to see that the integrand support disappears if \( f > 3\delta \).

We can now state our main result on the PSD of the single-channel interference (SCI):

**SCI Theorem** If the input channel has a rectangular PSD \( G(f) = \frac{P}{2\delta} \text{rect}_{2\delta}(f + \delta) \) with bandwidth \( 2\delta \) and power \( P \), then the PSD of the SCI is given by (1). The normalized double integral \( \mathcal{I}(f) := I(f)/(P/(2\delta))^2 \) can be exactly derived from the \( I(f) \) expression in (4)-(7) as follows:

If \( |f| < \delta \):

\[
\mathcal{I}(f) = \int_0^{\sqrt{\delta^2 - f^2}} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta + f + \sqrt{(\delta + f)^2 - v}}{\delta - f + \sqrt{(\delta - f)^2 - v}} \right) dv + 2 \int_0^{\delta^2 - f^2} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta^2 - f^2}{v} \right) dv
\]

else if \( \delta \leq |f| < 3\delta \):

\[
\mathcal{I}(f) = \int_{(|f| - \delta)^2}^{2\delta^2} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta^2}{|f| - \delta} \right) dv + \int_{2\delta^2}^{(\delta + |f|)^2} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta^2 + \sqrt{(\delta^2 + |f|^2)^2 - v}}{\delta^2 - \sqrt{(\delta^2 + |f|^2)^2 - v}} \right) dv
\]

otherwise \( \mathcal{I}(f) = 0 \).

For \( f > 0 \), the first integral in (8) corresponds to integration over domain I in Fig. 1 the second term to integration over domains II+IV, and the last term over domain III. When \( 0 < \delta \leq f < 3\delta \) only integration over domain III is nonzero.

The proof is provided in Appendix A.

A. Value at \( f=0 \)

From (8), the value at \( f = 0 \) is found as:

\[
\mathcal{I}(0) = 2 \int_0^{\sqrt{\frac{\delta^2}{2}}} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta + \sqrt{(\delta)^2 - v}}{\delta - \sqrt{(\delta)^2 - v}} \right) dv + 2 \int_0^{\delta^2} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta^2}{v} \right) dv.
\]

Referring to [2] Fig. 1 or Fig. 1(a), the first term in the above sum corresponds to integration of \( |\mathcal{K}(f_1f_2)|^2 \) over the triangular domains in the I+III quadrants of the \((f_1, f_2)\) plane, while the second term to the square domains in quadrants II+IV.
B. Examples and Cross-Checks

1) A theoretical cross check: A simple theoretical example may be constructed by assuming the quadratic kernel function to have a constant value \( K(0) \equiv 1 \) at all \( f \). This physically corresponds to the zero dispersion case. In this case

\[
I(f) := \int_{-\infty}^{\infty} G(f + f_1)G(f + f_2)G(f + f_1 + f_2)f_1f_2 \ dx_1 \ dx_2
\]

and the value of \( I(f) := \frac{(\partial f)^3}{\partial x_1^3} \) corresponds exactly to the areas of the integration domains sketched in Fig. [I]. It can be readily seen from simple geometrical considerations on Fig. [I] that

\[
I(f) = \begin{cases} 
\frac{(\delta - |f|)^2}{2} + 2 \cdot (\delta^2 - |f|^2) + \frac{(\delta + |f|)^2}{2} & \text{if } |f| \leq \delta \\
\frac{(\delta^2 - |f|^2)}{2} & \text{if } \delta < |f| \leq 3\delta \\
0 & \text{if } |f| > 3\delta.
\end{cases}
\]

(12)

Let’s verify that the above expression indeed coincides with (8)-\( \delta \). Let’s start with the following general result valid for \( \delta > 0 \):

\[
\int_0^a \ln \left( \frac{a + \sqrt{a^2 - v}}{a - \sqrt{a^2 - v}} \right) \ dv = 2a^2 - 2a\sqrt{a^2 - v} + \ln \left( \frac{a + \sqrt{a^2 - v}}{a - \sqrt{a^2 - v}} \right) \bigg|_0^a = 2a^2 + \ln (1) a^2 - 2a^2 + 2a\sqrt{a^2} = 2a^2.
\]

(13)

By setting \( a = \left( \frac{\delta - f}{2} \right)^2 \), we get \( \frac{(\delta - f)^2}{2} \) for the first integral in equation (8). Furthermore, for \( a > 0 \) we have

\[
\int_0^a \ln \left( \frac{a}{\sqrt{v}} \right) \ dv = \ln \left( \frac{a}{\sqrt{v}} \right) \cdot v \bigg|_0^a = a.
\]

(14)

By setting \( a = \delta^2 - f^2 \), we get \( (\delta^2 - |f|^2) \) for the second integral in equation (8). Finally by setting \( a = \frac{\delta + f}{2} \), the result in (13) leads to the value \( \frac{(\delta + |f|)^2}{2} \) for the third integral. All together, these lead to formula (12) for the case \( |f| < \delta \).

For the case \( \delta < |f| \leq 3\delta \) we may again use the integration result (13). For the first partial integral in (9) we have

\[
\int_b^a \ln \left( \frac{a + \sqrt{a^2 - v}}{a - \sqrt{a^2 - v}} \right) \ dv = 2a^2 - 2a\sqrt{a^2 - v} + \ln \left( \frac{a + \sqrt{a^2 - v}}{a - \sqrt{a^2 - v}} \right) \bigg|_b^a = 2a^2 + \ln (1) a^2 - 2a^2 + 2a\sqrt{a^2 - b} - \ln \left( \frac{a + \sqrt{a^2 - b}}{a - \sqrt{a^2 - b}} \right) \cdot b
\]

(15)

Now

\[
a^2 - b = \left( \frac{\delta + |f|}{2} \right)^2 - 2\delta(|f| - \delta) = \frac{1}{4} (\delta + 2\delta|f| + |f|^2 - 8\delta|f| + 8\delta^2) = \frac{1}{4} (|f|^2 - 6\delta|f| + 9\delta^2)
\]

(16)

and we thus derive from (15):

\[
\int_{2\delta(|f| - \delta)}^{\frac{\delta + |f|}{2}} \ln \left( \frac{\delta + |f|}{2} + \sqrt{\frac{\delta + |f|}{2} - \nu} \right) \ dv
\]

\[
= 2\frac{\delta + |f|}{2} \left( 3\delta - |f| \right) - \ln \left( \frac{\delta + |f|}{2} + \frac{1}{2} (3\delta - |f|) \right) \cdot 2\delta(|f| - \delta)
\]

\[
= \frac{1}{2} (\delta + |f|)(3\delta - |f|) - 2\ln \left( \frac{2\delta}{|f| - \delta} \right) \delta(|f| - \delta).
\]

(17)
For the second partial integral in (2) we get by setting $a = (|f| - \delta)^2$ and $b = 2\delta(|f| - \delta)$:

$$
\int_a^b \ln \left( \frac{v}{a} \right) dv = \ln \left( \frac{v}{a} \right) \cdot v - v \bigg|_a^b = \ln \left( \frac{b}{a} \right) \cdot b - b - \ln \left( \frac{a}{a} \right) \cdot a + a \\
= \ln \left( \frac{b}{a} \right) \cdot b - b + a = 2\ln \left( \frac{2\delta}{|f| - \delta} \right) \delta(|f| - \delta) - 2\delta(|f| - \delta) + (|f| - \delta)^2.
$$

(18)

The sum of (17) and (18) gives:

$$
\frac{1}{2} (\delta + |f|)(3\delta - |f|) - 2\delta(|f| - \delta) + (|f| - \delta)^2 = \frac{1}{2} (3\delta^2 + 2\delta|f| - |f|^2) + 3\delta^2 - 4\delta |f| + |f|^2
$$

(19)

$$
= \frac{1}{2} (9\delta^2 - 6\delta |f| + |f|^2) = \frac{(3\delta - |f|)^2}{2}.
$$

This yields the formula (12) for the case $\delta < |f| < 3\delta$.

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Figure 2. Plot of $I(f)$ eq. (1) [mW/GHz] versus frequency [GHz] for a constant unit quadratic kernel and rectangular input signal with $P = 1$ mW and support $[-10, 10]$ GHz (left) and support $[-20, 20]$ GHz (right). Label “new formula”: $I(f) \cdot \left( \frac{P}{3\pi} \right)^3$, with $I(f)$ as in (12). Label “exact Gxpf”: direct numerical evaluation of frequency double integral (1).

Fig. 2 illustrates this result for a rectangular spectrum with support $[-10, 10]$ GHz (left figure) and support $[-20, 20]$ GHz (right figure). The theoretical result (8)-(9) (labeled “new formula”) was cross-checked in these figures with an ad-hoc numerical double-integration routine that we separately developed (labeled “exact Gxpf” in the figures). The numerical routine greatly benefited from the explicit formulas (8)-(9) for debugging purposes.

2) **Numerical cross-checks**: The formulas (8)-(9) have been cross-checked also against numerical double-integration for realistic kernel functions.

We used a single-channel transmission over a 5-span dispersion-uncompensated (DU) terrestrial link with 100 km fiber spans with dispersion 17 ps/nm/km (standard single mode (SMF) fiber) and attenuation 0.2 dB/km. The power was $P = 1$ mW.

Fig. 3 and 4 show the SCI PSD $G_{NLI}(f)/\sqrt{27} = I(f)$ [mW/GHz] for a unit-power rectangular input spectrum with various bandwidths. Again, theory using (8)-(9) (label “semianalytical”) was checked against direct numerical double-integration (label “numerical”).

The examples show perfect coincidence between the numerical results and the theory. Note that in all examples the numerical evaluation of $G_{NLI}(f)$ was done at 39 equidistant frequencies and took between 230 and 280 seconds. The evaluation of $G_{NLI}(f)$ with the new semi-analytic formulas (8)-(9) however took only between 0.3 and 0.8 seconds.

3) **Check of Nyquist-WDM**: In [2], Poggiolini presents an example of $G_{NLI}(f)$ in the Nyquist-WDM case over standard single-mode fiber (denoted as NY-SMF in [2]) with 20 spans, an overall optical bandwidth $B_{WDM} = 544$ GHz, equivalent to 17 Nyquist-WDM channels at 32 Gbaud. He used an WDM-input signal with an all flat, i.e. rectangular shaped, PSD in the frequency band of $[-272, 272]$ GHz. Clearly, even if formulas (8)-(9), as stated above, are conceived for a single channel, they can be applied to that particular case as well, because the 17 Nyquist-WDM channels may be identified with one single channel in the frequency band $[-272, 272]$ GHz.
Figure 3. $I(f)$ [mW/GHz] vs frequency [GHz] for a rectangular input spectrum with $P = 1$ mW and support $[-10, 10]$ GHz (left) and support $[-20, 20]$ GHz (right) over a 5x100km SMF DU link.

Figure 4. $I(f)$ [mW/GHz] vs frequency [GHz] for a rectangular input spectrum with $P = 1$ mW and support $[-5, 5]$ GHz (left) and support $[-15, 15]$ GHz (right) over a 5x100km SMF DU link.

Figure 5. NY-SMF system, with 20 spans. Green line: PSD of the transmitted signal $G_{WDM}(f)$, equivalent to 17 Nyquist-WDM channels at 32 Gbaud. Blue line: PSD of NLI noise $G_{NLI}(f)$. Spectra arbitrarily rescaled as in [2].

The NLI PSD $G_{NLI}(f)$ has been calculated with the new semi-analytic formula and the result is depicted in Fig. 5. This result coincides exactly with that in [2, Fig. 5]. Once more, the result confirms the correctness of formulas (8)-(9).

IV. NON-NYQUIST WDM SYSTEMS

We assume here a WDM system with a reference central channel, $N_c$ channels to its left and $N_c$ channels to its right on the frequency axis, with uniform frequency spacing $\Delta$. The WDM comb has input PSD

$$G(f) = \sum_{k=-N_c}^{N_c} G_k(f) := \sum_{k=-N_c}^{N_c} S(f - k\Delta). \quad (20)$$
where each lowpass equivalent channel envelope has power $P$ and a rectangular PSD with bandwidth $2\delta$, namely $S(f) = \frac{P}{2} \text{rect}_2 \delta(f + \delta)$. The Nyquist-WDM case has $2\delta = \Delta$. When channels do not spectrally overlap and have guard-bands, we have the traditional Non-Nyquist WDM system, for which $2\delta < \Delta$.

Substitution of (20) in (1) yields:

$$I(f) = \int_{-\infty}^{\infty} |K(f_1, f_2)|^2 G(f + f_1)G(f + f_2)G(f + f_1 + f_2) df_1 df_2$$

$$= \int_{-\infty}^{\infty} |K(f_1, f_2)|^2 \sum_{k=-N_c}^{N_c} \sum_{l=-N_c}^{N_c} \sum_{m=-N_c}^{N_c} G_k(f + f_1)G_l(f + f_2)G_m(f + f_1 + f_2) df_1 df_2$$

$$= \sum_{k,l,m=-N_c}^{N_c} \int_{-\infty}^{\infty} |K(f_1, f_2)|^2 G_k(f + f_1)G_l(f + f_2)G_m(f + f_1 + f_2) df_1 df_2. \quad (21)$$

In practice we have broken up the global integral into the sum of partial integrals over special integration domains or “islands”. Fig. 6 shows such domains, where the integrand $G(\cdot)G(\cdot)G(\cdot) > 0$ for rectangular channel spectra, a channel spacing $\Delta = 50$ GHz, a per-channel bandwidth $2\delta$ of 40 GHz, a frequency $f = \frac{2\delta}{3}$ GHz and $N_c = 1$ adjacent channel, i.e., a 3-channel WDM system. The set of integration ‘islands’ for rectangular spectra is also presented in the special case $f = 0$ in [2, Fig. 3]. Since integration is additive over the islands, the NLI PSD may be decomposed as the sum of single-channel interference (SCI), cross-channel interference (XCI) and multi-channel interference (MCI, also known as four-wave mixing (FWM)) [2]:

$$G_{NLI}(f) = G_{SCI}(f) + G_{XCI}(f) + G_{MCI}(f). \quad (22)$$

Integration over the central red island in [2, Fig. 3] corresponds to the SCI and can directly be obtained from (8), (9).

A. Cross-Channel-Interference (XCI)

Consider now only the case $k = 0$ and its symmetric case $l = 0$. For this portion of the NLI we get:

$$I(f) = \sum_{l,m=-N_c}^{N_c} \int_{-\infty}^{\infty} |K(f_1, f_2)|^2 G_0(f + f_1)G_1(f + f_2)G_m(f + f_1 + f_2) df_1 df_2. \quad (23)$$
Note that if $m \neq l$ the support of $G_m(f + f_1 + f_2)$ never intersects the support of the other two terms, and thus the contribution is zero. So we may simplify (23) to:

$$I(f) = 2 \sum_{m=-N_c}^{N_c} \int_{-\infty}^{\infty} |K(f_1 f_2)|^2 G_0(f + f_1)G_m(f + f_2)G_m(f + f_1 + f_2)df_1 df_2. \quad (24)$$

If we also exclude the term for $m = 0$ (which represents the SCI), then we get the cross-channel interference (XCI [2]) contribution to $I(f)$. XCI encompasses both scalar cross-phase modulation and cross-polarization modulation [7]. In summary, the XCI PSD is given by

$$G_{XCI}(f) = \frac{16}{27} I_{XCI}(f)$$

$$I_{XCI}(f) := \frac{2}{2} \sum_{m=1}^{N_c} (I_m(f) + I_{-m}(f)) \quad (25)$$

where $I_m$ is defined as

$$I_m(f) := \int_{-\infty}^{\infty} |K(f_1 f_2)|^2 G_0(f + f_1)G_m(f + f_2)G_m(f + f_1 + f_2)df_1 df_2. \quad (26)$$

After the usual change of variable, such an integral can be written as

$$I_m(f) = \int_{0}^{\infty} |K(v)|^2 \left[ \int_{0}^{\infty} \frac{1}{u} G_0(f + u)G_m(f + u + v)G_m(f + u + v)du \right. \quad (27)$$

$$+ \int_{0}^{\infty} \frac{1}{u} G_0(f - u)G_m(f - u + v)G_m(f - u + v)du \quad (28)$$

$$+ \int_{0}^{\infty} \frac{1}{u} G_0(f - u)G_m(f - u - v)G_m(f - u + v)du \quad (29)$$

$$+ \int_{0}^{\infty} \frac{1}{u} G_0(f + u)G_m(f - u - v)G_m(f + u + v)du \right] dv. \quad (30)$$

We can now state our main result on the XCI spectrum.

**XCI Theorem** If the input WDM system has a symmetric PSD $G(f) = \sum_{k=-N_c}^{N_c} S(f - k\Delta)$ with channel spacing $\Delta$, a rectangular per-channel spectrum $S(f) = \frac{P}{2\delta} \text{rect}_\delta(f + \delta)$ with bandwidth $2\delta$ and per-channel power $P$, then for any integer $m > 0$ the normalized double integral $I_m(f) \triangleq (I_m(f) + I_{-m}(f))/(P/2\delta)^3$ can be written as follows. Define

$$\eta := \delta - |f| \quad \text{and} \quad \epsilon := \delta + |f|$$

$$\eta^+ := m\Delta + \eta \quad \text{and} \quad \epsilon^+ := m\Delta + \epsilon$$

$$\eta^- := m\Delta - \eta \quad \text{and} \quad \epsilon^- := m\Delta - \epsilon. \quad (31)$$

Then, if $|f| < \delta$:

$$I_m(f) = \int_{0}^{\eta^-} |K(v)|^2 \ln \left( \frac{\eta}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 - v} \right) dv + \int_{\eta^-}^{\eta^+} |K(v)|^2 \ln \left( \frac{\eta}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 - v} \right) dv$$

$$+ \int_{\eta^+}^{\epsilon^+} |K(v)|^2 \ln \left( \frac{\eta}{\epsilon^+} \sqrt{(\frac{\Delta}{2})^2 + v} \right) dv + \int_{\epsilon^+}^{\epsilon^-} \int_{\eta^+}^{\eta^-} |K(v)|^2 \ln \left( \frac{\eta}{\epsilon^+} \sqrt{(\frac{\Delta}{2})^2 + v} \right) dv$$

$$+ \int_{\epsilon^-}^{\eta^-} \int_{\eta^-}^{\eta^+} |K(v)|^2 \ln \left( \frac{\eta}{\epsilon^-} \sqrt{(\frac{\Delta}{2})^2 - v} \right) dv$$

(32)

else if $\delta \leq |f| < 3\delta$:

$$I_m(f) = \int_{\eta^-}^{\eta^-} |K(v)|^2 \ln \left( \frac{\eta^-}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 + v} \right) dv + \int_{\eta^-}^{\eta^+} \int_{\eta^-}^{\eta^-} |K(v)|^2 \ln \left( \frac{\eta^-}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 + v} \right) dv$$

$$+ \int_{\eta^-}^{\eta^+} |K(v)|^2 \ln \left( \frac{\eta^-}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 - v} \right) dv + \int_{\eta^-}^{2\delta\eta^+} \int_{\eta^-}^{\eta^-} |K(v)|^2 \ln \left( \frac{\eta^-}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 + v} \right) dv$$

$$+ \int_{\eta^-}^{\epsilon^+} \int_{\eta^-}^{\epsilon^+} |K(v)|^2 \ln \left( \frac{\eta^-}{\eta^-} \sqrt{(\frac{\Delta}{2})^2 - v} \right) dv$$

(33)

otherwise $I_m(f) = 0$.

The details of the proof can be found in Appendix B.
Figure 7. The four domains corresponding to the four terms in (34) in the order they appear. For instance, \( \int_{D_1} |\mathcal{K}(f_1,f_2)|^2 df_1 df_2 = \int_{\delta m \Delta} ^{\delta \Delta_m} |\mathcal{K}(v)|^2 2 \ln \left( \frac{\delta \Delta_m}{v} \right) dv \). Dashed curves are hyperbolas \( f_1 f_2 = \text{const} \).

B. Value at \( f=0 \)

As a corollary, the value at \( f=0 \) is found from (32) as follows. Define \( \Delta_m^+ := m \Delta + \delta \) and \( \Delta_m^- := m \Delta - \delta \). Then,

\[
\mathcal{I}_m(0) = 2 \left\{ \int_0 ^{\delta \Delta_m} |\mathcal{K}(v)|^2 2 \ln \left( \frac{\delta \Delta_m}{v} \right) dv + \int_{\delta \Delta_m} ^{\delta \Delta_m + \sqrt{(\frac{\delta \Delta_m}{v})^2 + \delta \Delta_m}} |\mathcal{K}(v)|^2 2 \ln \left( \frac{\delta \Delta_m}{v} \right) dv \right\}. \tag{34}
\]

Fig. 7 shows a geometric interpretation of the 4 terms in the curly bracket in (34) as the integral of \( |\mathcal{K}(f_1,f_2)|^2 \) over the shown domains \( D_1 \) through \( D_4 \) in the \( (f_1,f_2) \) plane (in the order they appear in eq. (34)).

C. Examples and Cross Checks

1) A theoretical cross check: When the quadratic kernel has a constant value 1, then the double integral is proportional to the area of the integration islands. As seen in Fig. 6 such islands all have the same area. Hence \( G_{XCI}(f) \) in this case is simply \( 4 N_c \)-times the value \( G_{SCI}(f) = \left( \frac{1}{N_c} \right)^3 \mathcal{I}(f) \), with \( \mathcal{I}(f) \) as given in (12), since there are \( 2 N_c \) XCI islands on every axis. Fig. 8 shows the calculation of the theoretical \( G_{SPM}(f) \) and \( G_{XPM}(f) \) with a unity squared kernel and \( N_c = 5 \) adjacent channels. Note that the scale on the y-axis of the XPM-figure is \( 20 = 4 N_c \)-times larger than that of the SPM-figure.

Figure 8. (left) Function \( G_{SCI}(f) \) and (right) function \( G_{XCI}(f) \) for \( N_c = 5 \) adjacent channels.
2) Numerical cross-checks: The formulas \((32)-(33)\) have been cross-checked also against numerical double-integration for realistic kernel functions.

We used an 11-channel \((N_c = 5)\) WDM non-Nyquist transmission with spacing \(\Delta = 50\text{GHz}\) over a 5-span dispersion-uncompensated terrestrial link with 100 km fiber spans with dispersion \(17 \text{ps/nm/km}\) and attenuation \(0.2 \text{dB/km}\). The power per channel was \(P = 1 \text{mW}\). Figure 9 shows the XCI PSD \(G_{XCI}(f)/16 = I_{XCI}(f) \text{[mW/GHz]}\) for rectangular per-channel input spectra with various bandwidths. Theory using \((32)-(33)\) (label “semianalytic formula”) was checked against direct numerical double-integration (label “XPM simulated”). Some discrepancies between theory and numerical double integration are visible in the figures. We later found that the double integration routine had mis-convergence problems, that were finally fixed to perfectly match with theory.

![Figure 9](image-url)

Figure 9. XCI PSD on central channel vs frequency for rectangular signal spectra, with support \([-5,5]\) GHz (left) and support \([-20,20]\) GHz (right) over a 5x100km SMF DU link. Spacing \(\Delta = 50\text{GHz}\), 11 channels \((N_c = 5)\).

V. Conclusions

We have presented new semi-analytical power spectral density formulas of the received nonlinear interference, both for single-channel and cross-channel interference. The great value of these formulas is twofold:

1) they represent a benchmark against which more general GNRF solvers can be tested;
2) it is now possible to easily analyze the separate behavior of SCI and XCI in order to quickly find out the dominant nonlinear effect \([7]\) in highly-dispersed nonlinear coherent transmissions. This second aspect will be developed in a future publication.

Acknowledgments

The present paper is a synthesis due to the first author of two reports \([8],[9]\) due to the second author, both written at the end of his 6-month sabbatical leave at the Department of Information Engineering of Parma University, Italy. The authors gladly acknowledge discussions on the developments of this work with Dr. Paolo Serena and Dr. Nicolaos Mantzoukis.

Appendix A: Proof of SCI Integral \(I(f)\)

In this Appendix we prove the expressions of the SCI integral \(I(f)\) given in \([8],[9]\). By the symmetry \((3)\) we only need calculations at \(f \geq 0\).
Calculation of partial integral (4), quadrant I:

Regarding the integrand of the inner integral (4) we deduce:

\[ G(f + u) \neq 0 \iff f + u \leq \delta \iff u \leq \delta - f := \tilde{\delta}. \] (35)

Note that this implies for the following analysis of (4) that \( \tilde{\delta} := \delta - f > 0 \) since by definition \( u \geq 0 \). Otherwise the factor \( G(f + u) \) is zero and the integral (4) disappears. This implies also that integral (4) disappears for \( f > \delta \) (Cfr Fig. 1c). For the second factor we have:

\[ G\left(f + \frac{u}{u}\right) \neq 0 \iff f + \frac{v}{u} \leq \delta \iff u \leq \frac{v}{\delta - f} = \frac{v}{\delta}. \] (36)

Note that we used \( \tilde{\delta} := \delta - f > 0 \) in the last transformation of the inequality. For the third factor we have (note \( u > 0 \)):

\[ G\left(f + u + \frac{v}{u}\right) \neq 0 \iff u + \frac{v}{u} \leq \tilde{\delta} \iff u^2 - \tilde{\delta} u + v \leq 0. \] (37)

Since

\[ u^2 - \tilde{\delta} u + v \leq 0 \iff \left(u - \frac{\tilde{\delta}}{2}\right)^2 - \left(\frac{\delta}{2}\right)^2 + v \leq 0 \]
\[ \iff \left(u - \frac{\tilde{\delta}}{2}\right)^2 \leq \left(\frac{\delta}{2}\right)^2 - v \] (38)

the factor \( G\left(f + u + \frac{v}{u}\right) \) is always 0 if \( \left(\frac{\tilde{\delta}}{2}\right)^2 < v \). If \( \left(\frac{\delta}{2}\right)^2 \geq v \) then (38) has solutions and

\[ \left(u - \frac{\tilde{\delta}}{2}\right)^2 \leq \left(\frac{\delta}{2}\right)^2 - v \]
\[ \iff u \leq \sqrt{\left(\frac{\delta}{2}\right)^2 - v} + \frac{\tilde{\delta}}{2} := u^{(1)} \text{ and } u \geq -\sqrt{\left(\frac{\delta}{2}\right)^2 - v} + \frac{\tilde{\delta}}{2} := u^{(0)}. \] (39)

Thus using (35), (36) and (39) the partial integral (4) reads for \( f < \delta \):

\[
\int_0^\infty |K(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G(f + u) G\left(f + \frac{v}{u}\right) u \cdot G\left(f + u + \frac{v}{u}\right) du \right] dv
\]
\[ = \int_0^\infty |K(v)|^2 \left[ \min\{u^{(1)}, \tilde{\delta}\} \right. \int_{\max\{u^{(0)}, \frac{\delta}{2}\}}^\infty \frac{1}{u} \cdot G(f + u) G\left(f + \frac{v}{u}\right) G\left(f + u + \frac{v}{u}\right) du \left. \right] dv. \] (40)

Note once more that for \( f \geq \delta \) the partial integral (4) is zero. Since

\[ u^{(1)} = \sqrt{\left(\frac{\delta}{2}\right)^2 - v + \frac{\delta}{2}} \leq \frac{\delta}{2} + \frac{\tilde{\delta}}{2} = \tilde{\delta} \] (41)

and

\[ u^{(0)} = -\sqrt{\left(\frac{\delta}{2}\right)^2 - v + \frac{\delta}{2}} \geq \frac{v}{\delta} \iff \left(\frac{\delta}{2} - \frac{v}{\delta}\right)^2 \geq \left(\frac{\delta}{2}\right)^2 - v \]
\[ \iff -v + \left(\frac{v}{\delta}\right)^2 \geq -v \] (42)
is always true, then the integral limits for the first inner integral are \( u^{(0)} \) and \( u^{(1)} \). We thus get for \( f < \delta \):\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G(f + u) G\left(f + \frac{v}{u}\right) G\left(f + u + \frac{v}{u}\right) \, du \right] \, dv \\
= \left( \frac{P}{2\delta} \right)^3 \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^{u^{(1)}} \frac{1}{u} \, du \right] \, dv = \left( \frac{P}{2\delta} \right)^3 \int_0^\infty |\mathcal{K}(v)|^2 \ln \left( \frac{u^{(1)}}{u^{(0)}} \right) \, dv
\] (43)

Calculation of partial integral (7), quadrant IV:

Regarding the integrand of the inner integral (7) we deduce according to (55):\[
G(f + u) \neq 0 \iff u \leq \tilde{\delta}.
\] (44)

Again this implies for the following analysis of (7) that \( \tilde{\delta} > 0 \) and that integral (7) disappears for \( f > \delta \). For the second factor we have:

\[
G(f - \frac{v}{u}) \neq 0 \iff f - \frac{v}{u} \leq \delta \quad \text{and} \quad f - \frac{v}{u} \geq -\delta
\]
\[
\iff \frac{-v}{u} \leq \tilde{\delta} \quad \text{and} \quad \frac{-v}{u} \geq -\tilde{\delta} - f = -(\delta + f) := -\tilde{\delta}
\] (45)

\[
\iff \frac{v}{u} \geq -\tilde{\delta} \quad \text{and} \quad \frac{v}{u} \leq \tilde{\delta}
\]

Since \( \tilde{\delta} > 0 \) and \( u, v \geq 0 \) the first inequality doesn’t represent a constraint. So we have:

\[
G(f - \frac{v}{u}) \neq 0 \iff \frac{v}{u} \leq \tilde{\delta} \iff u \geq \frac{v}{\tilde{\delta}}.
\] (46)

Note that we may exclude the special case \( \delta = 0 \) since the whole double integral will be zero in this case. The quotient \( \frac{\tilde{\delta}}{\delta} \) is therefore well defined. For the third factor we have:

\[
G(f + u - \frac{v}{u}) \neq 0 \iff u - \frac{v}{u} \leq \tilde{\delta} \quad \text{and} \quad u - \frac{v}{u} \geq -\tilde{\delta}.
\] (47)

For the first inequality we deduce:

\[
u - \frac{v}{u} \leq \tilde{\delta} \iff u^2 - \tilde{\delta} u - v \leq 0
\]
\[
\iff \left( u - \frac{\tilde{\delta}}{2} \right)^2 - \left( \frac{\tilde{\delta}}{2} \right)^2 - v \leq 0 \iff \left( u - \frac{\tilde{\delta}}{2} \right)^2 \leq \left( \frac{\tilde{\delta}}{2} \right)^2 + v.
\] (48)

This implies

\[
u \leq \frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v} \quad \text{and} \quad -u \leq -\frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v}.
\] (49)

Since \( u, v \geq 0 \) the last inequality is always fulfilled and doesn’t represent a constraint. So finally the first inequality implies

\[
u \leq u^{(3)} := \frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v}.
\] (50)

For the second inequality in (47) we deduce:

\[
u - \frac{v}{u} \geq -\tilde{\delta} \iff u^2 + \tilde{\delta} u - v \geq 0
\]
\[
\iff \left( u + \frac{\tilde{\delta}}{2} \right)^2 - \left( \frac{\tilde{\delta}}{2} \right)^2 - v \geq 0 \iff \left( u + \frac{\tilde{\delta}}{2} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 + v.
\] (51)

This implies
Thus using (44), (46), (50) and (52) the partial integral (7) reads for $f < \delta$:

$$
\int_0^\infty |K(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G(f + u) G \left( f - \frac{v}{u} \right) G \left( f + u - \frac{v}{u} \right) \, du \right] \, dv
$$

$$
= \int_0^\infty |K(v)|^2 \left[ \frac{\min \left\{ u^{(3)}, \delta \right\}}{\max \left\{ u^{(2)}, \delta \right\}} \right] \, dv.
$$

(53)

Since

$$
u^{(3)} = \frac{\delta}{2} + \sqrt{\left( \frac{\delta}{2} \right)^2} + v \geq \frac{\delta}{2} + \sqrt{\left( \frac{\delta}{2} \right)^2} = 2 \frac{\delta}{2} = \delta
$$

we have

$$
\min \left\{ u^{(3)}, \delta \right\} = \delta.
$$

(54)

Additionally since

$$
\frac{v}{\delta} \geq u^{(2)} = -\frac{\delta}{2} + \sqrt{\left( \frac{\delta}{2} \right)^2} + v \iff \frac{v}{\delta} + \frac{\delta}{2} \geq + \sqrt{\left( \frac{\delta}{2} \right)^2} + v
$$

$$
\iff \left( \frac{v}{\delta} \right)^2 + v + \left( \frac{\delta}{2} \right)^2 \geq \left( \frac{\delta}{2} \right)^2 + v
$$

we have

$$
\max \left\{ u^{(2)}, \frac{v}{\delta} \right\} = \frac{v}{\delta}.
$$

(55)

Thus

$$
\frac{v}{\delta} \leq u \leq \delta
$$

(58)

otherwise the partial integral (7) disappears. This however imposes a restriction on $v$, because it implies $v \leq \delta \cdot \delta$! Thus finally the partial integral (7) reads for $f < \delta$:

$$
\int_0^\infty |K(v)|^2 \left[ \int_0^{\frac{\delta}{2}} \frac{1}{u} \cdot G(f + u) G \left( f - \frac{v}{u} \right) G \left( f + u - \frac{v}{u} \right) \, du \right] \, dv
$$

$$
= \left( \frac{P}{2\delta} \right)^3 \cdot \int_0^{\frac{\delta}{2}} |\tilde{\eta}(v)|^2 \left[ \int_0^{\frac{\delta}{2}} \frac{1}{u} \, du \right] \, dv = \left( \frac{P}{2\delta} \right)^3 \cdot \int_0^{\frac{\delta}{2}} |K(v)|^2 \ln \left( \frac{\delta}{v} \right) \, dv.
$$

(59)

Calculation of partial integral (5), quadrant II:

Regarding the integrand of the inner integral (5) we deduce:

$$
G(f - u) \neq 0 \iff -u \leq \delta \quad \text{and} \quad -u \geq -\delta
\iff u \leq \delta \quad \text{and} \quad u \geq -\delta.
$$

(60)

For the second factor we have like in (56):

$$
G \left( f + \frac{v}{u} \right) \neq 0 \iff \frac{v}{u} \leq \delta \quad \iff \quad u \geq \frac{v}{\delta}.
$$

(61)

Note that we may suppose $\delta > 0$ since otherwise because of $\frac{v}{u} \leq \delta$ and the fact that $u, v \geq 0$ the factor $G \left( f + \frac{v}{u} \right)$ and consequently the whole integral would be zero. Hence once more the partial integral (5) disappears if $f > \delta$! Since $\delta > 0$ the second inequality in (60) doesn’t represent a constraint. So we have

$$
G(f - u) \neq 0 \quad \iff \quad u \leq \delta.
$$

(62)
$G \left( f + \frac{u}{v} \right) \neq 0 \iff u \geq \frac{v}{\bar{\delta}}$.

(63)

For the third factor we have:

$G \left( f - u + \frac{v}{u} \right) \neq 0 \iff -u + \frac{v}{u} \leq \tilde{\delta}$ and $-u + \frac{v}{u} \geq -\tilde{\theta}$.

(64)

For the first inequality we deduce:

$-u + \frac{v}{u} \leq \tilde{\delta} \iff -u^2 - \tilde{\delta}u + v \leq 0 \iff u^2 + \tilde{\delta}u - v \geq 0$

$\iff \left( u + \frac{\tilde{\delta}}{2} \right)^2 - \left( \frac{\tilde{\delta}}{2} \right)^2 - v \geq 0 \iff \left( u + \frac{\tilde{\delta}}{2} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 + v.$

(65)

This implies

$u \geq u^{(4)} := -\frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v}.$

(66)

For the second inequality we have:

$-u + \frac{v}{u} \geq -\bar{\delta} \iff -u^2 + \bar{\delta}u + v \geq 0$

$\iff \left( u - \frac{\bar{\delta}}{2} \right)^2 - \left( \frac{\bar{\delta}}{2} \right)^2 - v \leq 0 \iff \left( u - \frac{\bar{\delta}}{2} \right)^2 \leq \left( \frac{\bar{\delta}}{2} \right)^2 + v.$

(67)

This implies

$u \leq u^{(5)} := \frac{\bar{\delta}}{2} + \sqrt{\left( \frac{\bar{\delta}}{2} \right)^2 + v}.$

(68)

Using (62), (63), (66) and (68) the partial integral (5) reads for $f < \bar{\delta}$:

$$
\int_{0}^{\infty} |K(v)|^2 \left[ \int_{0}^{\infty} \frac{1}{u} \cdot G \left( f + u \right) G \left( f - \frac{v}{u} \right) G \left( f + u - \frac{v}{u} \right) \, du \right] \, dv = \int_{0}^{\infty} |K(v)|^2 \left[ \int_{\max \left\{ u^{(4)} , \frac{\bar{\delta}}{2} \right\}}^{\min \left\{ u^{(5)} , \bar{\delta} \right\}} \frac{1}{u} \cdot G \left( f + u \right) G \left( f - \frac{v}{u} \right) G \left( f + u - \frac{v}{u} \right) \, du \right] \, dv.
$$

(69)

Since

$$u^{(5)} = \frac{\bar{\delta}}{2} + \sqrt{\left( \frac{\bar{\delta}}{2} \right)^2 + v} \geq \frac{\bar{\delta}}{2} + \sqrt{\left( \frac{\bar{\delta}}{2} \right)^2} = 2 \frac{\bar{\delta}}{2} = \bar{\delta},$$

we have

$$\min \left\{ u^{(5)} , \bar{\delta} \right\} = \bar{\delta}.$$  

(70)

Additionally since

$$\frac{v}{\delta} \geq u^{(4)} = -\frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v} \iff \frac{v}{\delta} + \frac{\tilde{\delta}}{2} \geq \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 + v}$$

$\iff \left( \frac{v}{\delta} \right)^2 + v + \left( \frac{\tilde{\delta}}{2} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 + v.$

(72)

we have

$$\max \left\{ u^{(4)} , \frac{v}{\delta} \right\} = \frac{v}{\delta}.$$  

(73)
Thus
\[ \frac{v}{\delta} \leq u \leq \delta \]  
otherwise the partial integral \((5)\) disappears. This however imposes a restriction on \(v\), because it implies again \(v \leq \delta \cdot \tilde{\delta}\). Thus finally the partial integral \((5)\) reads for \(f < \delta\):

\[
\int_{0}^{\infty} |K(v)|^2 \left[ \int_{0}^{\infty} \frac{1}{u} \cdot G(f + u) G\left(f - \frac{v}{u}\right) du \right] dv = \left( \frac{P}{2\delta} \right)^3 \cdot \int_{0}^{\delta \cdot \tilde{\delta}} |K(v)|^2 \ln \left( \frac{\delta \cdot \tilde{\delta}}{v} \right) dv.
\]  

\((75)\)

Calculation of partial integral \((6)\), quadrant III

The forth integral is the only one for which \(f < \delta\) doesn’t follow necessarily as a condition for not being zero.

So we have to make a distinction between the two cases \(f < \delta\) and \(f \geq \delta\).

The partial integral \((6)\) for \(f < \delta\): Regarding the integrand of the inner integral \((6)\) we have according to \((60)\):

\[ G\left(f - u\right) \neq 0 \iff -u \leq \tilde{\delta} \quad \text{and} \quad u \geq -\tilde{\delta} \]

\[ \iff u \leq \tilde{\delta} \quad \text{and} \quad u \geq -\tilde{\delta}. \]  

\((76)\)

Since by assumption \(\tilde{\delta} = \delta - f > 0\), then the condition \(u \geq -\tilde{\delta}\) is always fulfilled and the only remaining restriction is:

\[ u \leq \tilde{\delta}. \]  

\((77)\)

For the second factor we have according to \((46)\):

\[ G\left(f - \frac{u}{v}\right) \neq 0 \iff f - \frac{u}{v} \leq \delta \quad \text{and} \quad f - \frac{u}{v} \geq -\delta \]

\[ \iff \frac{v}{u} \geq -\tilde{\delta} \quad \text{and} \quad \frac{v}{u} \leq \delta. \]  

\((78)\)

Again since \(\tilde{\delta} > 0\) the first inequality is always fulfilled and we have (note that \(\tilde{\delta} = f + \delta > 0\) by definition):

\[ u \geq \frac{v}{\tilde{\delta}}. \]  

\((79)\)

For the third factor we have:

\[ G\left(f - u - \frac{u}{v}\right) \neq 0 \iff -u - \frac{v}{u} \leq \tilde{\delta} \quad \text{and} \quad -u - \frac{v}{u} \geq -\tilde{\delta} \]

\[ \iff u + \frac{v}{u} \geq -\tilde{\delta} \quad \text{and} \quad u + \frac{v}{u} \leq \tilde{\delta}. \]  

\((80)\)

Again since \(u, v, \tilde{\delta} > 0\) the first inequality doesn’t deliver a restriction and we get for the second one:

\[ u^2 - \tilde{\delta}u + v \leq 0 \iff \left(u - \frac{\tilde{\delta}}{2}\right)^2 - \left(\frac{\tilde{\delta}}{2}\right)^2 + v \leq 0 \]

\[ \iff \left(u - \frac{\tilde{\delta}}{2}\right)^2 \leq \left(\frac{\tilde{\delta}}{2}\right)^2 - v. \]  

\((81)\)

The partial integral \((6)\) therefore disappears if \(v > \left(\frac{\tilde{\delta}}{2}\right)^2\). For \(v < \left(\frac{\tilde{\delta}}{2}\right)^2\) we have

\[ u \leq u^{(7)} := \frac{\tilde{\delta}}{2} + \sqrt{\left(\frac{\tilde{\delta}}{2}\right)^2 - v} \]

and

\[ \frac{\tilde{\delta}}{2} - u \leq \sqrt{\left(\frac{\tilde{\delta}}{2}\right)^2 - v} \iff u \geq u^{(6)} := \frac{\tilde{\delta}}{2} - \sqrt{\left(\frac{\tilde{\delta}}{2}\right)^2 - v}. \]  

\((82)\)

\((83)\)
Using (77), (79), (82) and (83) the partial integral (6) reads for \( f < \delta \):

\[
\int_{0}^{\infty} |\mathcal{K}(v)|^2 \left[ \int_{0}^{\infty} \frac{1}{u} \cdot G(f-u) G \left( f - \frac{v}{u} \right) G \left( f - u - \frac{v}{u} \right) \, du \right] \, dv.
\]

\[
= \left( \frac{\pi}{2} \right)^2 \int_{0}^{\infty} |\mathcal{K}(v)|^2 \left[ \min \left\{ u^{(7)}, \tilde{\delta} \right\} \right] \, dv.
\]

It is easy to see that

\[
\min \left\{ u^{(7)}, \tilde{\delta} \right\} = u^{(7)} = \frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v}.
\]

Since \( v < \left( \frac{\tilde{\delta}}{2} \right)^2 = \frac{1}{4} \tilde{\delta}^2 \) we deduce:

\[
2v \leq \delta^2 \iff v \leq \delta \iff v - \frac{\tilde{\delta}}{2} \leq 0 \iff \frac{\tilde{\delta}}{2} - \frac{v}{\delta} \geq 0.
\]

Then

\[
u^{(6)} \geq \frac{v}{\delta} \iff \frac{\tilde{\delta}}{2} - \frac{v}{\delta} \geq \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v} \iff \left( \frac{\tilde{\delta}}{2} - \frac{v}{\delta} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 - v \iff \left( \frac{v}{\delta} \right)^2 \geq 0.
\]

Thus

\[
\max \left\{ u^{(6)}, \frac{v}{\delta} \right\} = u^{(6)} = \frac{\tilde{\delta}}{2} - \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v}.
\]

Finally in the case \( f < \delta \) for the partial integral (6) follows:

\[
\int_{0}^{\infty} |\mathcal{K}(v)|^2 \left[ \int_{0}^{\infty} \frac{1}{u} \cdot G(f-u) G \left( f - \frac{v}{u} \right) G \left( f - u - \frac{v}{u} \right) \, du \right] \, dv
\]

\[
= \left( \frac{P}{2\delta} \right)^3 \cdot \int_{0}^{\infty} |\mathcal{K}(v)|^2 \left[ \frac{u^{(7)}}{u^{(6)}} \right] \, dv = \left( \frac{P}{2\delta} \right)^3 \cdot \int_{0}^{\infty} |\mathcal{K}(v)|^2 \ln \left( \frac{u^{(7)}}{u^{(6)}} \right) \, dv
\]

Together with (43), (59) and (75) this proves equation (8).

The partial integral (6) for \( f > \delta \): Again we have according to (60):

\[
G(f-u) \neq 0 \iff -u \leq \delta \quad \text{and} \quad -u \geq -\delta
\]

\[
\iff u \leq \frac{\tilde{\delta}}{2} \quad \text{and} \quad u \geq \frac{\tilde{\delta}}{2}.
\]

This time \( u \geq -\tilde{\delta} \) is a genuine restriction because \( -\tilde{\delta} = f - \delta > 0 \) by assumption. For the second factor we have like in (78):

\[
G \left( f - \frac{v}{u} \right) \neq 0 \iff \frac{v}{u} \leq \delta \quad \text{and} \quad \frac{v}{u} \geq -\delta
\]

\[
\iff \frac{v}{u} \geq -\tilde{\delta} \quad \text{and} \quad \frac{v}{u} \leq \tilde{\delta}.
\]

Since \( \tilde{\delta} > 0, -\tilde{\delta} > 0 \) this leads to:

\[
u \geq \frac{v}{\delta} \quad \text{and} \quad u \leq -\frac{v}{\delta} = \frac{v}{-\delta}.
\]
Especially (90) and (92) imply the following restrictions on \( v \):
\[
\frac{v}{\delta} \leq \tilde{\delta} \quad \text{and} \quad -\tilde{\delta} \leq \frac{v}{-\delta}.
\] (93)

Thus
\[
\left( -\delta \right)^2 = \tilde{\delta}^2 \leq v \leq \tilde{\delta}^2.
\] (94)

So the partial integral (6) reads for \( f > \delta \):
\[
\int_0^\infty |K(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G(f - u) G\left( f - \frac{u}{u} \right) \right. du \right] dv
\]
\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \int_{\frac{\delta^2}{\tilde{\delta}^2}} |K(v)|^2 \left[ \min\left\{ \frac{-\tilde{\delta}^2 + \sqrt{\left( \frac{\delta}{2} \right)^2 - v}}{-\tilde{\delta}} \right\} \right. du \right] dv
\]
\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \int_{\frac{\delta^2}{\tilde{\delta}^2}} |K(v)|^2 \left[ \int \frac{1}{u} \cdot G\left( f - u - \frac{v}{u} \right) du \right] dv.
\] (95)

For the third factor we have like in (80):
\[
G\left( f - u - \frac{v}{u} \right) \neq 0 \iff -u - \frac{v}{u} \leq \tilde{\delta} \quad \text{and} \quad -u - \frac{v}{u} \geq -\tilde{\delta}
\]
\[
\iff u + \frac{v}{u} \geq -\tilde{\delta} \quad \text{and} \quad u + \frac{v}{u} \leq \tilde{\delta}.
\] (96)

For the first inequality we get equivalently:
\[
\left( u + \frac{\tilde{\delta}}{2} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 - v.
\] (97)

For the second inequality we have:
\[
\left( u - \frac{\tilde{\delta}}{2} \right)^2 \leq \left( \frac{\tilde{\delta}}{2} \right)^2 - v.
\] (98)

The last inequality implies that the whole integral is zero if \( v \) exceeds \( \left( \frac{\tilde{\delta}}{2} \right)^2 \). So the upper limit of the first integral of (95) is \( \left( \frac{\tilde{\delta}}{2} \right)^2 \) instead of \( \tilde{\delta}^2 \). Since by (94) \( v \geq \tilde{\delta}^2 \), inequality (97) is no constraint. Inequality (98) is obviously fulfilled iff
\[
\frac{\tilde{\delta}}{2} - \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v} \leq u \leq \frac{\tilde{\delta}}{2} + \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v}.
\] (99)

The partial integral (6) for \( f > \delta \) is consequently:
\[
\int_0^\infty |K(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G\left( f - u - \frac{v}{u} \right) G\left( f - \frac{u}{u} \right) du \right] dv
\]
\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \int_{\frac{\delta^2}{\tilde{\delta}^2}} |K(v)|^2 \left[ \int \frac{1}{u} \cdot G\left( f - u - \frac{v}{u} \right) du \right] dv
\]
\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \int_{\frac{\delta^2}{\tilde{\delta}^2}} |K(v)|^2 \left[ \int \frac{1}{u} \cdot G\left( f - u - \frac{v}{u} \right) du \right] dv.
\] (100)

Further we deduce
\[
\frac{v}{\delta} \leq \frac{\tilde{\delta}}{2} - \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v} \iff \frac{\tilde{\delta}}{2} - \frac{v}{\delta} \geq \sqrt{\left( \frac{\tilde{\delta}}{2} \right)^2 - v}
\]
\[
\iff \frac{v^2}{\delta^2} - v \geq \left( \frac{\tilde{\delta}}{2} \right)^2 - v
\]
\[
\iff \frac{v^2}{\delta^2} - v + \left( \frac{\tilde{\delta}}{2} \right)^2 \geq \left( \frac{\tilde{\delta}}{2} \right)^2 - v
\]
\[
\iff \frac{v^2}{\delta^2} \geq 0.
\] (101)
Since this condition is always fulfilled we get:

\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G(f-u) G\left(f-u-v\right) \, du \right] \, dv
\]

\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \frac{(\frac{\pi}{2})^2}{4} \cdot \left[ \int_{\delta}^\infty |\mathcal{K}(v)|^2 \left[ \min\left\{ \frac{v}{\pi \delta^2 + \sqrt{\left( \frac{v}{\pi \delta} \right)^2 - v}} \right\} \right. \right.
\]

\[
\left. \int_{\max\left\{ -\delta, \frac{v}{\pi \delta} - \sqrt{\left( \frac{v}{\pi \delta} \right)^2 - v} \right\}}^{\min\left\{ \frac{\pi}{\delta^2} \right\}} \frac{1}{u} \cdot G\left(f-u-v\right) \, du \right] \, dv.
\]

(102)

We further have:

\[
\frac{v}{\delta} \leq \frac{\delta}{2} + \sqrt{\left( \frac{\delta}{2} \right)^2 - v} \quad \iff \quad \left( \frac{v}{\delta} + \frac{\delta}{2} \right)^2 \leq \left( \frac{\delta}{2} \right)^2 - v
\]

\[
\iff \quad \frac{v^2}{\delta^2} + \frac{\delta v}{\delta^2} + \frac{\delta}{2} \leq \left( \frac{\delta}{2} \right)^2 - v
\]

\[
\iff \quad v \leq -\delta^2 - \delta \delta = -\left( \delta - f \right)^2 - \left( \delta^2 - f^2 \right)
\]

\[
\iff \quad v \leq -\delta^2 + 2\delta^2 f - f^2 - \delta^2 + f^2 = -2\delta^2 + 2\delta^2 f = 2\delta(f - \delta).
\]

So if \( v \leq 2\delta(f - \delta) \) then \( \frac{\pi}{\delta} \) is the upper limit of the inner integral of (100) else \( \frac{\pi}{\delta} + \sqrt{\left( \frac{\pi}{\delta} \right)^2 - v} \) is the upper limit. Note that

\[
2\delta(f - \delta) \leq \left( \frac{\delta}{2} \right)^2 \quad \iff \quad 8\delta f - 8\delta^2 \leq f^2 + 2\delta f + \delta^2
\]

\[
\iff \quad 0 \leq 9\delta^2 - 6\delta f + f^2 = (3\delta - f)^2
\]

\[
\iff \quad f \leq 3\delta.
\]

(104)

This condition is always fulfilled since we may restrict the analysis to that case, knowing that the Nonlinearity Double Integral is always 0 for \( f > 3\delta \). We also have

\[
-\delta \leq \frac{\delta}{2} - \sqrt{\left( \frac{\delta}{2} \right)^2 - v} \quad \iff \quad \left( \pm \left( \delta - \frac{\delta}{2} \right) \right)^2 \leq \left( \frac{\delta}{2} \right)^2 - v
\]

\[
\iff \quad (-\delta)^2 + \delta \delta + \left( \frac{\delta}{2} \right)^2 \leq \left( \frac{\delta}{2} \right)^2 - v
\]

\[
\iff \quad v \leq -\delta^2 - \delta \delta = 2\delta(f - \delta).
\]

(105)

So if \( v \leq 2\delta(f - \delta) \) then \( -\delta \) is the lower limit of the inner integral of (100) else \( \frac{\pi}{\delta} - \sqrt{\left( \frac{\pi}{\delta} \right)^2 - v} \) is the lower limit. This leads to

\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G(f-u) G\left(f-u-v\right) \, du \right] \, dv
\]

\[
= \left( \frac{P}{2\delta} \right)^2 \cdot \frac{(\frac{\pi}{2})^2}{4} \cdot \left[ \int_{-\delta}^{\frac{2\delta(f-\delta)}{-\delta}} |\mathcal{K}(v)|^2 \left[ \int_0^1 \frac{1}{u} \, du \right] \, dv
\]

\[
+ \left( \frac{P}{2\delta} \right)^2 \cdot \frac{(\frac{\pi}{2})^2}{4} \cdot \left[ \int_{\frac{2\delta(f-\delta)}{-\delta}}^{\frac{\pi}{\delta}} |\mathcal{K}(v)|^2 \left[ \int_0^1 \frac{1}{u} \, du \right] \, dv
\]

(106)

and proves together with the remark following equation (104) the equation (9). The result for negative \( f < -\delta \) follows from the symmetry property (3).
The partial integral (6) for $f = \delta$: The value of the partial integral (6) for $f = \delta$ is simply deduced by letting $|f|$ tend to $\delta$ in (8) or (9). It can be easily seen that in both cases the limit value is:

$$\mathcal{I}(f) = \int_{0}^{\delta^2} |\mathcal{K}(v)|^2 \ln \left( \frac{\delta + \sqrt{\delta^2 - v}}{\delta - \sqrt{\delta^2 - v}} \right) dv.$$ 

APPENDIX B: PROOF OF THE XCI INTEGRALS $\mathcal{I}_m(f)$

In this Appendix we prove the expressions of the XCI integrals $\mathcal{I}_m(f)$ given in (32)-(33). By the symmetry (3) we only need calculations at $f \geq 0$.

A. Proof for $f < \delta$ (resp. $|f| < \delta$)

**Calculation of partial integral (27), quadrant I:**

Regarding the integrand of the inner integral (27) we deduce:

$$G_0 (f + u) \neq 0 \iff f + u \leq \delta \iff u \leq \delta - f := \eta. \quad (107)$$

Note that this implies for the following analysis of (27) that $\eta > 0$ since by definition $u \geq 0$. Otherwise the factor $G_0 (f + u)$ is zero and the integral (27) vanishes. This implies also that integral (27) vanishes for $f > \delta$.

For the second factor we have:

$$G_m \left( f + \frac{v}{u} \right) \neq 0 \iff f + \frac{v}{u} - m\Delta \leq \delta \quad \text{and} \quad f + \frac{v}{u} - m\Delta \geq -\delta$$

$$\iff \frac{v}{u} \leq \eta + m\Delta \quad \text{and} \quad \frac{v}{u} \geq m\Delta - (\delta + f)$$

$$\iff \frac{v}{u} \leq \eta_+^m \quad \text{and} \quad \frac{v}{u} \geq \varepsilon_m^-$$

where we defined

$$\eta_+^m := m\Delta + \eta \quad \text{and} \quad \varepsilon_m^- := m\Delta - (\delta + f). \quad (109)$$

Since $0 < \eta \leq \delta < \Delta$ we see that the first inequality of (108) is never fulfilled for $m < 0$. For $m > 0$ we get (since all terms are positive)

$$G_m \left( f + \frac{v}{u} \right) \neq 0 \iff \frac{u}{v} \geq \frac{1}{\eta_+^m} \quad \text{and} \quad \frac{u}{v} \leq \frac{1}{\varepsilon_m^-}$$

$$\iff \frac{u}{\eta_+^m} \geq \frac{v}{\varepsilon_m^-} \quad \text{and} \quad \frac{u}{\varepsilon_m^-} \leq \frac{v}{\eta_+^m}. \quad (110)$$

Putting (107) and (110) together this leads to the restrictions:

$$\frac{v}{\eta_+^m} \leq u \leq \min \left\{ \eta, \frac{v}{\varepsilon_m^-} \right\}. \quad (111)$$

Note that this implies:

$$\min \left\{ \eta, \frac{v}{\varepsilon_m^-} \right\} = \eta \iff v \geq \varepsilon_m^- \eta$$

and

$$\min \left\{ \eta, \frac{v}{\varepsilon_m^-} \right\} = \frac{v}{\varepsilon_m^-} \iff v < \varepsilon_m^- \eta. \quad (112)$$

For the third factor we have (note $u > 0$):

$$G_m \left( f + u + \frac{v}{u} \right) \neq 0 \iff u + \frac{v}{u} \leq \eta_+^m \quad \text{and} \quad u + \frac{v}{u} \geq \varepsilon_m^-$$

$$\iff u^2 - \eta_+^m u + v \leq 0 \quad \text{and} \quad u^2 - \varepsilon_m^- u + v \geq 0.$$

Note that for $m > 0$ the second inequality is a genuine restriction because $\varepsilon_m^- > 0$. Since

$$u^2 - \eta_+^m u + v \leq 0 \iff \left( u - \frac{\eta_+^m}{2} \right)^2 \leq \left( \frac{\eta_+^m}{2} \right)^2 - v \quad (114)$$
the factor $G_m \left( f + u + \frac{u^2}{2} \right)$ is always 0 if $\left( \frac{\eta_m^+}{\eta_m^-} \right)^2 < v$. If $\left( \frac{\eta_m^+}{\eta_m^-} \right)^2 \geq v$ then (114) has solutions and

$$
\left( u - \frac{\eta_m^+}{2} \right)^2 \leq \left( \frac{\eta_m^+}{2} \right)^2 - v
\quad \iff \quad u \leq \frac{\eta_m^+}{2} + \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 - v} := u^{(1)}
\quad \text{or} \quad u \geq \frac{\eta_m^+}{2} - \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 - v} := u^{(0)}.
$$

(115)

Since $\frac{\eta_m^+}{\eta_m^-} > m \Delta \geq \eta$ taking into account (111) the only remaining restriction is $u^{(0)}$. For the second inequality we get:

$$
u^2 - \varepsilon_m^- u + v \geq 0 \quad \iff \quad \left( u - \frac{\varepsilon_m^-}{2} \right)^2 \geq \left( \frac{\varepsilon_m^-}{2} \right)^2 - v.
$$

(116)

This is no restriction if $v > \left( \frac{\varepsilon_m^-}{2} \right)^2$. If $v \leq \left( \frac{\varepsilon_m^-}{2} \right)^2$ then the condition is equivalent to:

$$
u \leq u^{(k)} \triangleq \frac{\varepsilon_m^-}{2} - \sqrt{\left( \frac{\varepsilon_m^-}{2} \right)^2 - v} \quad \text{or} \quad u \geq u^{(n)} \triangleq \frac{\varepsilon_m^-}{2} + \sqrt{\left( \frac{\varepsilon_m^-}{2} \right)^2 - v}.
$$

(117)

Since $\frac{\varepsilon_m^-}{\varepsilon_m^n} > m \Delta \geq \eta$ taking again into account (111) the only remaining restriction is $u^{(k)}$. Note however that in general

$$
a - \sqrt{a^2 - x} \leq b - \sqrt{b^2 - x}.
$$

(118)

if $a \geq b$. Since $\frac{\varepsilon_m^-}{\varepsilon_m^n} \geq \eta$ this leads to

$$
\nu^{(k)} \leq \nu^{(0)}
$$

(119)

and consequently $\nu^{(0)}$ is the lower limit for $u$. Thus using all this the terms of the partial integral (27) read for $f < \delta$ and $m > 0$:

$$
\int_0^\infty |K(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G_0 \left( f + u \right) G_m \left( f + u + \frac{v}{u} \right) du \right] dv
= \varepsilon_m^n |K(v)|^2 \left[ \int_0^{\frac{\varepsilon_m^n}{\varepsilon_m^+}} \frac{1}{u} du \right] dv + \int_0^\infty \frac{1}{u} |K(v)|^2 \left[ \int_0^{\frac{1}{u}} \frac{1}{u} du \right] dv.
$$

(120)

Finally we should take into account that

$$
\frac{\eta_m^+}{2} - \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 - v} \leq \eta \quad \iff \quad \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 - v} \geq \frac{\eta_m^+}{2} - \eta
\quad \iff \quad \frac{\eta_m^+}{2} - v \geq \left( \frac{\eta_m^+}{2} \right)^2 - \eta_m^+ \eta^2
\quad \iff \quad v \leq \eta \left( \eta_m^+ - \eta \right) = m \Delta \eta
$$

(121)

which imposes an upper restriction on the admissible values of $v$. In the end we get for $f < \delta$ and $m > 0$:

$$
\int_0^\infty |K(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G_0 \left( f + u \right) G_m \left( f + u + \frac{v}{u} \right) du \right] dv
= \varepsilon_m^n |K(v)|^2 \left[ \int_0^{\frac{\varepsilon_m^n}{\varepsilon_m^+}} \frac{1}{u} du \right] dv + \int_0^\infty \frac{1}{u} |K(v)|^2 \left[ \int_0^{\frac{1}{u}} \frac{1}{u} du \right] dv
$$

(122)

$$
= \varepsilon_m^n |K(v)|^2 \ln \left( \frac{\eta_m^+}{\eta_m^-} - \sqrt{\left( \frac{\eta_m^+}{\eta_m^-} \right)^2 - v} \right) dv + \int_0^\infty \frac{1}{u} |K(v)|^2 \ln \left( \frac{\eta_m^+}{\eta_m^-} - \sqrt{\left( \frac{\eta_m^+}{\eta_m^-} \right)^2 - v} \right) dv.
$$
The first part (27) of $I_{XCI}(f)$ now reads:

$$2 \left( \frac{P}{2\delta} \right)^3 \sum_{m=1}^{N_c} \left[ \int_0^{\eta_m} \left| \mathcal{K}(v) \right|^2 \ln \left( \frac{v}{\eta_m} \right)^2 \ln \left( \frac{\eta_m}{\eta_m - \sqrt{\left( \frac{\eta_m}{2} \right)^2 - v}} \right) \, dv + \int_{\eta_m}^{m\Delta\eta} \left| \mathcal{K}(v) \right|^2 \ln \left( \frac{\eta}{\eta_m} \right)^2 \ln \left( \frac{\eta_m}{\eta_m - \sqrt{\left( \frac{\eta_m}{2} \right)^2 - v}} \right) \, dv \right].$$

(123)

Figure 10. Geometric interpretation for result (123)

There is a useful geometrical interpretation for this result. Fig. 10 depicts the corresponding situation in the $(f_1, f_2)$-plane. Note that the transformation to the $(u, v)$-plane is such that $u = f_1$ and $f_2 = \frac{v}{u}$. Integration with respect to $u$ geometrically means integration along the depicted equipotential lines $\frac{v}{u}$. For a fixed small $v \approx 0$ those lines intersect the lozenge-shaped domain of $G(\cdot)G(\cdot)G(\cdot)$ between the upper limit

$$m\Delta + (\delta - f) - f_1 = \eta_m^+ - u$$

(124)

and the lower limit

$$m\Delta - (\delta + f) = \eta_m^-$$

(125)

until the point $A$ is reached. At this point

$$\frac{v}{\delta - f} = \epsilon_m^- \iff v = \eta \varepsilon_m^-.$$  

(126)

As long as $0 < v \leq \eta \varepsilon_m^-$ for a given $v$ the equipotential line intersects first at the solution of

$$\eta_m^+ - u = \frac{v}{u}$$

(127)

which is

$$\frac{\eta_m^+}{2} = \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 - v}.$$  

(128)
and the solution of
\[ \varepsilon_m^- = \frac{v}{u} \]
which is
\[ \frac{v}{\varepsilon_m^-} \]
This explains the first integral in (122). If \( v \) increases and the equipotential line passes point \( A \), it intersects still at the solution (128) of \( \eta_m^+ - u = \frac{v}{\eta} \) and then at the right limit line. In this case at \( u = \eta \). This is true until point \( B \) is reached. At this point the equation
\[ \frac{v}{\eta} = m\Delta \quad \iff \quad v = m\Delta \eta \]
holds. All this explains second integral in (123).

**Calculation of partial integral (30), quadrant IV:**
For the integrand of the inner integral (30) we deduce according to (107):
\[ G_0 (f + u) \neq 0 \quad \iff \quad f + u \leq \delta \quad \iff \quad u \leq \delta - f := \eta. \]
Note that this again implies for the following analysis of (30) that \( \eta > 0 \) since by definition \( u \geq 0 \). Otherwise the factor \( G_0 (f + u) \) is zero and the integral (30) disappears. This implies also that integral (30) disappears for \( f > \delta \).

For the second factor we have:
\[ G_m (f - \frac{v}{u}) \neq 0 \quad \iff \quad f - \frac{v}{u} - m\Delta \leq \delta \quad \text{and} \quad f - \frac{v}{u} - m\Delta \geq -\delta \]
\[ \iff \quad -\frac{v}{u} \leq \eta_m^+ \quad \text{and} \quad -\frac{v}{u} \geq \varepsilon_m^- \]
\[ \iff \quad \frac{v}{u} \geq -\eta_m^+ \quad \text{and} \quad \frac{v}{u} \leq -\varepsilon_m^- \]
Since
\[ -\varepsilon_m^- = (\delta + f) - m\Delta \leq 2\delta - m\Delta \leq \Delta - m\Delta \]
we see that the second inequality of (133) is never fulfilled for \( m > 0 \). So the integral (30) is zero for \( m > 0 \).
We then consider only \( m < 0 \) in the following. For \( m < 0 \) we get (since all terms are positive)
\[ G_m (f - \frac{v}{u}) \neq 0 \quad \iff \quad \frac{u}{v} \leq \frac{1}{-\eta_m^+} \quad \text{and} \quad \frac{u}{v} \geq \frac{1}{-\varepsilon_m^-} \]
\[ \iff \quad u \leq \frac{v}{-\eta_m^+} \quad \text{and} \quad u \geq \frac{v}{-\varepsilon_m^-}. \]
Now (132) and (135) together give:
\[ \frac{v}{-\varepsilon_m^-} \leq u \leq \min \left\{ \eta, \frac{v}{-\eta_m^+} \right\}. \]
Note that this implies:
\[ \min \left\{ \eta, \frac{v}{-\eta_m^+} \right\} = \eta \quad \text{iff} \quad v \leq -\varepsilon_m^- \eta \]
and
\[ \min \left\{ \eta, \frac{v}{-\eta_m^+} \right\} = \frac{v}{-\eta_m^+} \quad \text{iff} \quad v > -\varepsilon_m^- \eta. \]
For the third factor we have:
\[ G_m (f + u - \frac{v}{u}) \neq 0 \quad \iff \quad u - \frac{v}{u} \leq \eta_m^+ \quad \text{and} \quad u - \frac{v}{u} \geq -\varepsilon_m^- \]
\[ \iff \quad u^2 - \eta_m^+ u - v \leq 0 \quad \text{and} \quad u^2 + \varepsilon_m^- u - v \geq 0. \]
For the first inequality we deduce:
\[ u^2 - \eta_m^+ u - v \leq 0 \quad \iff \quad \left( u - \frac{\eta_m^+}{2} \right)^2 - \left( \frac{\eta_m^+}{2} \right)^2 - v \leq 0 \]
\[ \iff \quad \left( u - \frac{\eta_m^+}{2} \right)^2 \leq \left( \frac{\eta_m^+}{2} \right)^2 + v. \]
(139)
This implies
\[
   u \leq \frac{\eta_m^+}{2} + \sqrt{\left(\frac{\eta_m^+}{2}\right)^2 + v} := u^{(1)} \quad \text{or} \quad u \geq \frac{\eta_m^+}{2} - \sqrt{\left(\frac{\eta_m^+}{2}\right)^2 + v} := u^{(0)}.
\]  

(140)

However, $\eta_m^+$ is negative since $m < 0$ and so the last condition doesn’t represent a restriction and
\[
   u \leq \frac{\eta_m^+}{2} + \sqrt{\left(\frac{\eta_m^+}{2}\right)^2 + v} = u^{(1)}
\]  

remains. Now (note that $\eta - \frac{\eta_m^+}{2}$ is positive since $m < 0$):

\[
   u^{(1)} = \frac{\eta_m^+}{2} + \sqrt{\left(\frac{\eta_m^+}{2}\right)^2 + v} \leq \eta \iff \sqrt{\left(\frac{\eta_m^+}{2}\right)^2 + v} \leq \eta - \frac{\eta_m^+}{2}
\]

\[
   \iff \left(\frac{\eta_m^+}{2}\right)^2 + v \leq \eta^2 - \eta \frac{\eta_m^+}{2} \left(\frac{\eta_m^+}{2}\right)^2
\]

\[
   \iff v \leq \eta \left(\eta - \eta_m^+\right) = -m \Delta \eta.
\]  

(142)

So for $v \leq -m \Delta \eta < -\varepsilon_m \eta$ the upper limit of the inner integral is $u^{(1)}$ if $v \leq -m \Delta \eta$ else the upper limit is $\eta$. For the second inequality in (138) we deduce:

\[
   u^2 + \varepsilon_m u - v \geq 0 \iff \left(\frac{u + \varepsilon_m}{2}\right)^2 - \left(\frac{\varepsilon_m}{2}\right)^2 - v \geq 0
\]

\[
   \iff \left(\frac{u + \varepsilon_m}{2}\right)^2 \geq \left(\frac{\varepsilon_m}{2}\right)^2 + v.
\]  

(143)

This implies

\[
   u \geq u^{(k)} \triangleq -\frac{\varepsilon_m}{2} + \sqrt{\left(\frac{\varepsilon_m}{2}\right)^2 + v} \quad \text{or} \quad u \leq u^{(n)} \triangleq -\frac{\varepsilon_m}{2} - \sqrt{\left(\frac{\varepsilon_m}{2}\right)^2 + v}.
\]  

(144)

Since $-\frac{\varepsilon_m}{2} > -m \Delta \eta \geq \eta$ and since $u^{(n)}$ is always negative, this doesn’t impose new restrictions. Consequently $\frac{\eta - \varepsilon_m}{2}$ is always the lower limit for $u$. Using all this, the terms of the partial integral (40) read for $f < \delta$ and $m < 0$:

\[
   \int_0^\infty |K(v)|^2 \left[ \int_0^\infty \frac{1}{u} G_0 (f + u) G_m \left( f - \frac{v}{u} \right) G_m \left( f + u - \frac{v}{u} \right) \, du \right] \, dv
\]

\[
   = -m \Delta \eta \left[ \int_0^{\frac{\eta - \varepsilon_m}{2}} \frac{1}{u} \, du \right] + \int_0^{\frac{\eta}{\varepsilon_m}} \left[ \int_0^{\frac{\eta}{\varepsilon_m}} \frac{1}{u} \, du \right] \, dv
\]

\[
   = -m \Delta \eta \int_0^{\frac{\eta - \varepsilon_m}{2}} |K(v)|^2 \ln \left( \frac{\frac{\eta - \varepsilon_m}{2}}{\frac{\eta}{\varepsilon_m}} \right) \, dv + \int_{-m \Delta \eta}^{\frac{\eta}{\varepsilon_m}} |K(v)|^2 \ln \left( \frac{\frac{\eta}{\varepsilon_m}}{\frac{\eta - \varepsilon_m}{2}} \right) \, dv.
\]  

(145)

The second part (40) of $I_{xcl}(f)$ now reads:

\[
   2 \left( \frac{P}{2 \delta} \right)^3 \sum_{m = -N_c}^{-1} \left[ \int_0^{\frac{\eta - \varepsilon_m}{2}} |K(v)|^2 \ln \left( \frac{\frac{\eta - \varepsilon_m}{2}}{\frac{\eta}{\varepsilon_m}} \right) \, dv + \int_{-m \Delta \eta}^{\frac{\eta}{\varepsilon_m}} |K(v)|^2 \ln \left( \frac{\frac{\eta}{\varepsilon_m}}{\frac{\eta - \varepsilon_m}{2}} \right) \, dv \right].
\]  

(146)
Calculation of partial integral (28), quadrant II:

From the inner integral (28) we deduce:
\[ G_0 (f-u) \neq 0 \iff -\delta \leq f-u \leq \delta \iff -\varepsilon := -(\delta + f) \leq -u \leq \eta (147) \]

Note that since we are supposing in this analysis that \( \eta := \delta - f > 0 \) the first inequality is no restriction. So we get in this case:

\[ G_0 (f-u) \neq 0 \iff u \leq \varepsilon. (148) \]

For the second factor we have according to (132):

\[ G_m \left( f + \frac{v}{u} \right) \neq 0 \iff \frac{v}{u} \leq \eta_m^+ \quad \text{and} \quad \frac{v}{u} \geq \frac{\varepsilon}{m} (149) \]

Reasoning analogously to (132) the first inequality of (149) is never fulfilled for \( m < 0 \). So the integral (28) is zero for \( m < 0 \). For \( m > 0 \) we get again (since all terms are positive)

\[ G_m \left( f + \frac{v}{u} \right) \neq 0 \iff u \geq \frac{v}{\eta_m} \quad \text{and} \quad u \leq \frac{v}{\varepsilon_m}. (150) \]

Putting (148) and (150) together this leads to the restrictions:

\[ \frac{v}{\eta_m} \leq u \leq \min \left\{ \varepsilon, \frac{v}{\varepsilon_m} \right\}. (151) \]

Note that this immediately implies:

\[ \min \left\{ \varepsilon, \frac{v}{\varepsilon_m} \right\} = \varepsilon \iff v \geq \varepsilon_m \varepsilon \quad \text{and} \quad \min \left\{ \varepsilon, \frac{v}{\varepsilon_m} \right\} = \frac{v}{\varepsilon_m} \iff v < \varepsilon_m \varepsilon. (152) \]

For the third factor we have (note \( u > 0 \)):

\[ G_m \left( f - u + \frac{v}{u} \right) \neq 0 \iff -u + \frac{v}{u} \leq \eta_m^+ \quad \text{and} \quad -u + \frac{v}{u} \geq \frac{\varepsilon}{m} (153) \]

\[ \iff -u^2 + \eta_m^+ u - v \leq 0 \quad \text{and} \quad -u^2 + \frac{\varepsilon}{m} u + v \geq 0 \]

We get

\[ u^2 + \eta_m^+ u - v \geq 0 \iff \left( u + \frac{\eta_m^+}{2} \right)^2 \geq \left( \frac{\eta_m^+}{2} \right)^2 + v (154) \]

\[ \text{and} \quad u^2 + \varepsilon_m u - v \leq 0 \iff \left( u + \frac{\varepsilon_m}{2} \right)^2 \leq \left( \frac{\varepsilon_m}{2} \right)^2 + v. \]

This leads to the conditions:

\[ u \geq u^{(0)} := -\frac{\eta_m^+}{2} + \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 + v} \quad \text{or} \quad u \leq u^{(1)} := -\frac{\eta_m^+}{2} - \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 + v} (155) \]

and

\[ u \geq u^{(k)} := -\frac{\varepsilon_m}{2} - \sqrt{\left( \frac{\varepsilon_m}{2} \right)^2 + v} \quad \text{or} \quad u \leq u^{(n)} := -\frac{\varepsilon_m}{2} + \sqrt{\left( \frac{\varepsilon_m}{2} \right)^2 + v}. (156) \]

Since the expression \(-\frac{\eta_m^+}{2} < 0\) and \(-\frac{\varepsilon_m}{2} < 0\) for \( m > 0 \) the first inequality of (154) is equivalent to

\[ u \geq u^{(0)} := -\frac{\eta_m^+}{2} + \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 + v} (157) \]

and the remaining restriction in the second is:

\[ u \leq u^{(n)} := -\frac{\varepsilon_m}{2} + \sqrt{\left( \frac{\varepsilon_m}{2} \right)^2 + v}. (158) \]
Consequently the terms of the partial integral \(28\) read for \(f < \delta\) and \(m > 0\):

\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G_0 \left( f - u \right) G_m \left( f + \frac{v}{u} \right) G_m \left( f - u + \frac{v}{u} \right) \, du \right] \, dv \\
= \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv + \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv.
\]

(159)

Now

\[
u^{(0)} \leq \frac{\nu}{\eta_m} \quad \Leftrightarrow \quad -\frac{\eta_m^+}{2} + \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 + \nu} \leq \frac{\nu}{\eta_m}
\]

\[
\Leftrightarrow \sqrt{\left( \frac{\eta_m^+}{2} \right)^2 + \nu} \leq \frac{\eta_m^+}{2} + \frac{\nu}{\eta_m}
\]

\[
\Leftrightarrow \left( \frac{\eta_m^+}{2} \right)^2 + \nu \leq \left( \frac{\eta_m^+}{2} \right)^2 + \nu + \left( \frac{\nu}{\eta_m} \right)^2 \quad \Leftrightarrow \quad 0 \leq \left( \frac{\nu}{\eta_m} \right)^2
\]

which is always true. So \(\nu\) is always the lower limit of the inner integrals. Since

\[
u^{(n)} \leq \frac{\nu}{\epsilon_m} \quad \Leftrightarrow \quad -\frac{\epsilon_m}{2} + \sqrt{\left( \frac{\epsilon_m}{2} \right)^2 + \nu} \leq \frac{\nu}{\epsilon_m}
\]

\[
\Leftrightarrow \sqrt{\left( \frac{\epsilon_m}{2} \right)^2 + \nu} \leq \frac{\epsilon_m}{2} + \frac{\nu}{\epsilon_m}
\]

\[
\Leftrightarrow \left( \frac{\epsilon_m}{2} \right)^2 + \nu \leq \left( \frac{\epsilon_m}{2} \right)^2 + \nu + \left( \frac{\nu}{\epsilon_m} \right)^2 \quad \Leftrightarrow \quad 0 \leq \left( \frac{\nu}{\epsilon_m} \right)^2
\]

is also always true, \(\nu^{(n)}\) is the upper limit of the partial integral of the inner integral. Additionally

\[
\epsilon \leq u^{(n)} \quad \Leftrightarrow \quad \epsilon \leq -\frac{\epsilon_m}{2} + \sqrt{\left( \frac{\epsilon_m}{2} \right)^2 + \nu}
\]

\[
\Leftrightarrow \frac{\epsilon_m}{2} + \epsilon \leq \sqrt{\left( \frac{\epsilon_m}{2} \right)^2 + \nu}
\]

\[
\Leftrightarrow \left( \frac{\epsilon_m}{2} \right)^2 + \epsilon \frac{\epsilon_m}{2} + \epsilon^2 \leq \left( \frac{\epsilon_m}{2} \right)^2 + \nu
\]

\[
\Leftrightarrow \epsilon \left( \epsilon_m + \epsilon \right) = m \Delta \epsilon \leq \nu.
\]

So for \(f < \delta\) and \(m > 0\) we get:

\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G_0 \left( f - u \right) G_m \left( f + \frac{\nu}{u} \right) G_m \left( f - u + \frac{\nu}{u} \right) \, du \right] \, dv \\
= \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv + \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv.
\]

(163)

Finally we note that

\[
\frac{\nu}{\eta_m} \leq \epsilon \quad \Leftrightarrow \quad \nu \leq \eta_m \epsilon
\]

(164)

which leads to a corresponding restriction on \(\nu\) for the first integral. Consequently we arrive at:

\[
\int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \cdot G_0 \left( f - u \right) G_m \left( f + \frac{\nu}{u} \right) G_m \left( f - u + \frac{\nu}{u} \right) \, du \right] \, dv \\
= \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv + \int_0^\infty |\mathcal{K}(v)|^2 \left[ \int_0^\infty \frac{1}{u} \, du \right] \, dv
\]

(165)
The third part \(28\) of \(I_{XC1}(f)\) now reads:

\[
2 \left( \frac{P}{2\delta} \right)^3 \cdot N \left. \sum_{m=1}^{2m\Delta \varepsilon} \int_0^{\eta_m} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon}{2} + \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + v}}{\eta_m} \right) \, dv + \int_{m\Delta \varepsilon}^{2m\Delta \varepsilon} |K(v)|^2 \ln \left( \frac{\eta_m^+ \varepsilon}{v} \right) \, dv \right].
\] (166)

**Calculation of partial integral \(29\), quadrant III:**

For the integrand of the inner integral \(29\) we derive according to \(148\) (note that we we suppose \(f < \delta\)):

\[
G_0 (f - u) \neq 0 \iff u \leq \varepsilon.
\] (167)

For the second factor we have, following \(133\):

\[
G_m \left( f - \frac{v}{u} \right) \neq 0 \iff \frac{v}{u} \geq -\eta_m^+ \quad \text{and} \quad \frac{v}{u} \leq -\varepsilon_m^-
\] (168)

and again we note that the second inequality of \(168\) is never fulfilled for \(m > 0\) and therefore the integral \(29\) is zero for \(m > 0\). We thus consider only \(m < 0\) in the following. For \(m < 0\) we get (since we suppose \(f < \delta\) and all terms are positive)

\[
G_m \left( f - \frac{v}{u} \right) \neq 0 \iff \frac{v}{u} \leq -\eta_m^+ \quad \text{and} \quad \frac{v}{u} \geq -\varepsilon_m^-.
\] (169)

Now \(167\) and \(169\) together give:

\[
\frac{v}{u} \leq \eta_m^{-1} \leq \varepsilon \leq \min \left\{ \varepsilon, \frac{v}{-\eta_m^-} \right\}.
\] (170)

Note that this implies:

\[
\min \left\{ \varepsilon, \frac{v}{-\eta_m^-} \right\} = \varepsilon \quad \text{iff} \quad \frac{v}{u} \leq -\eta_m^+ \varepsilon
\] (171)

and

\[
\min \left\{ \varepsilon, \frac{v}{-\eta_m^-} \right\} = \frac{v}{u} \quad \text{iff} \quad \frac{v}{u} \geq -\varepsilon_m^- \varepsilon.
\]

For the third factor we have:

\[
G_m \left( f - u - \frac{v}{u} \right) \neq 0 \iff -u - \frac{v}{u} \leq \eta_m^- \quad \text{and} \quad -u - \frac{v}{u} \geq \varepsilon_m^-.
\]

\[
\iff -u^2 - \eta_m^+ u - v \leq 0 \quad \text{and} \quad -u^2 - \varepsilon_m^- u - v \geq 0
\]

\[
\iff u^2 + \eta_m^- u + v \geq 0 \quad \text{and} \quad u^2 + \varepsilon_m^- u + v \leq 0.
\] (172)

The first inequality is not a new restriction since by \(168\)

\[
\frac{v}{u} \leq -\eta_m^+
\] (173)

and \(u \geq 0\). We thus get the condition

\[
u^2 + \varepsilon_m^- u + v \leq 0 \iff \left( u + \frac{\varepsilon_m^-}{2} \right)^2 \leq \left( \frac{\varepsilon_m^-}{2} \right)^2 - v.
\] (174)

The inequality shows that the factor \(G_m \left( f - u - \frac{v}{u} \right)\) is always 0 if \(\left( \frac{\varepsilon_m^-}{2} \right)^2 < v\). If \(\left( \frac{\varepsilon_m^-}{2} \right)^2 \geq v\) then \(174\) has solutions and

\[
\left( u + \frac{\varepsilon_m^-}{2} \right)^2 \leq \left( \frac{\varepsilon_m^-}{2} \right)^2 - v
\]

\[
\iff u \leq u^{(1)} := -\frac{\varepsilon_m^-}{2} + \sqrt{\left( \frac{\varepsilon_m^-}{2} \right)^2 - v} \quad \text{or} \quad u \geq u^{(0)} := -\frac{\varepsilon_m^-}{2} - \sqrt{\left( \frac{\varepsilon_m^-}{2} \right)^2 - v}.
\] (175)

Since (note that \(m < 0\)) \(u^{(1)} \geq -\frac{\varepsilon_m^-}{2} \geq \varepsilon\), the first condition is no further restriction if we take \(167\) into account. Thus the only remaining restriction is

\[
u \geq u^{(0)} := -\frac{\varepsilon_m^-}{2} - \sqrt{\left( \frac{\varepsilon_m^-}{2} \right)^2 - v}.
\] (176)
Since by (170) \( u^{(0)} \) should be less than \( \varepsilon \) we get

\[
\begin{aligned}
    u^{(0)} \leq \varepsilon & \iff \sqrt{\frac{\varepsilon m}{2}} - v \geq \frac{\varepsilon m}{2} - \varepsilon \\
    & \iff -v \geq \varepsilon \left( \frac{\varepsilon m}{2} + \varepsilon \right) \\
    & \iff v \leq -m \Delta \varepsilon.
\end{aligned}
\]  

(177)

So we get a new upper limit for the admissible \( v \). Since for \( v \leq -\eta_m^+ \varepsilon \)

\[
    u^{(0)} \leq \frac{v}{-\eta_m^+}
\]  

(178)

we get for the terms of the partial integral (29) (note \( f < \delta \) and \( m < 0 \)):

\[
\int_0^\infty |K(v)|^2 \left[ \int_0^{u^{(0)}} \frac{1}{u} \cdot G_0 \left( f - u \right) G_m \left( f - u - \frac{v}{u} \right) du \right] dv
\]

\[=
\int_0^{-\eta_m^+ \varepsilon} |K(v)|^2 \left[ \int_{u^{(0)}}^\infty \frac{1}{u} du \right] dv + \int_{-\eta_m^+ \varepsilon}^{\varepsilon} |K(v)|^2 \int_{u^{(0)}}^\infty \frac{1}{u} du dv
\]

\[=
\int_0^{-\eta_m^+ \varepsilon} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon u}{\eta_m^+} - \frac{\varepsilon m}{\eta_m^+} - \frac{\varepsilon}{\eta_m^+} \frac{\varepsilon m}{\eta_m^+}}{-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} \right) dv + \int_{-\eta_m^+ \varepsilon}^{\varepsilon} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} {-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} \right) dv.
\]  

(179)

The fourth part (29) of \( I_{XCl}(f) \) now reads:

\[
2 \left( \frac{P}{2\delta} \right)^3 \sum_{m=-N_\varepsilon}^{-1} \int_0^{-\eta_m^+ \varepsilon} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon u}{\eta_m^+} - \frac{\varepsilon m}{\eta_m^+} - \frac{\varepsilon}{\eta_m^+} \frac{\varepsilon m}{\eta_m^+}}{-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} \right) dv + \int_{-\eta_m^+ \varepsilon}^{\varepsilon} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} {-\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} \right) dv.
\]  

(180)

Now define

\[
\eta_m := m \Delta - \eta \quad \text{and} \quad \varepsilon_m^+ = m \Delta + (\delta + f).
\]  

(181)

Then for \( m > 0 \) we get the following correspondences:

\[
\eta_m^- := -\varepsilon_m^- = -(-m \Delta + \eta) \quad \text{and} \quad \varepsilon_m^- = -\varepsilon_m^- = -(-m \Delta + (\delta + f)).
\]  

(182)

This allows to express (145) and (179) in a unified form as a \( \sum_{m=1}^{N_\varepsilon} \) instead of a \( \sum_{m=-N_\varepsilon}^{-1} \). The second part (29) of \( I_{XCl}(f) \) can now be expressed as:

\[
2 \left( \frac{P}{2\delta} \right)^3 \sum_{m=1}^{N_\varepsilon} \int_0^{m \Delta \eta} |K(v)|^2 \ln \left( \frac{-\frac{\varepsilon_m^+}{\eta_m^+} + \sqrt{\left( \frac{\varepsilon_m^+}{\eta_m^+} \right)^2} + v}{\frac{\varepsilon_m^+}{\eta_m^+} + \sqrt{\left( \frac{\varepsilon_m^+}{\eta_m^+} \right)^2} + v} \right) dv + \int_{m \Delta \eta}^{\varepsilon \eta_m^+ \varepsilon} |K(v)|^2 \ln \left( \frac{\varepsilon_m^- \eta}{\eta_m^+ \varepsilon} \right) dv.
\]  

(183)

The forth part (29) of \( I_{XCl}(f) \) now reads:

\[
2 \left( \frac{P}{2\delta} \right)^3 \sum_{m=1}^{N_\varepsilon} \int_0^{\eta_m^- \varepsilon} |K(v)|^2 \ln \left( \frac{v}{\eta_m^+} \right) dv + \int_{\eta_m^- \varepsilon}^{m \Delta \eta} |K(v)|^2 \ln \left( \frac{\varepsilon}{\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} {\frac{\varepsilon}{\eta_m^+} - \sqrt{\left( \frac{\varepsilon}{\eta_m^+} \right)^2} - v} \right) dv.
\]  

(184)
For symmetry reasons, the results may be generalized to all $-\delta < f < \delta$ if we define

$$\eta := \delta - |f| \quad \text{and} \quad \varepsilon := \delta + |f|.$$

(185)

We are now ready to summarize the formula for expressing $I_{XCI}(f) \triangleq I_{XCI}(f)/\left(\frac{P}{2\pi}\right)^3$ in the case of rectangular shaped input signals for all $-\delta < f < \delta$ and thus finally prove the theorem:

$$I_{XCI}(f) = 2 \sum_{m=1}^{N} \int_{0}^{\eta_{m}} |K(v)|^2 \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{\eta_{m} - \frac{v}{m \Delta}}\right) dv + \int_{\eta_{m}}^{m \Delta \eta} |K(v)|^2 \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{v}\right) dv$$

$$+ \sum_{m=1}^{N} \sum_{m \Delta \eta}^{m \Delta \eta} \frac{m \Delta \eta}{|K(v)|^2} \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{v}\right) dv + \int_{m \Delta \eta}^{m \Delta \eta} |K(v)|^2 \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{v}\right) dv$$

$$+ \sum_{m=1}^{N} \sum_{m \Delta \eta}^{m \Delta \eta} \frac{m \Delta \eta}{|K(v)|^2} \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{v}\right) dv + \int_{m \Delta \eta}^{m \Delta \eta} |K(v)|^2 \ln \left(\frac{\eta_{m} + \frac{v}{m \Delta}}{v}\right) dv$$

(186)

$B$. Proof for $f > \delta$ (resp. $|f| > \delta$)

First note that we may restrict the analysis to the case $\delta < f < 3\delta$ since the Nonlinearity Double Integral is generally 0 for $f \geq 3\delta$. In section $\text{V-A}$ we showed that the partial integrals (27) and (30) are zero if $f$ exceeds $\delta$. So the only contribution to $I_{XCI}(f)$ in the case $\delta < f < 3\delta$ is due to partial integrals (28) and (29). The proof follows the guidelines of that of section $\text{V-A}$. In both cases the condition $G_{0}(f - u)$ leads to

$$-\eta \leq u \leq \varepsilon.$$  

(187)

The condition $G_{m}(f + \frac{v}{u})$ for the partial integral (28) leads to:

$$\frac{v}{u} \leq \eta_{m} = m \Delta + \eta \quad \text{and} \quad \frac{v}{u} \geq \varepsilon_{m}.$$  

(188)

Since $0 \geq \eta \geq -2\delta$ the first inequality of (188) is never fulfilled for $m < 0$. So the integral (28) is zero for $m < 0$. For $m > 0$ we get since $\eta_{m} = m \Delta + \eta > \Delta - 2\delta > 0$:

$$\max \left\{ \frac{v}{\eta_{m}}, -\eta \right\} \leq u.$$  

(189)

Now:

$$\max \left\{ \frac{v}{\eta_{m}}, -\eta \right\} = -\eta \quad \text{iff} \quad v \leq -\eta_{m}^{+}$$

and

$$\max \left\{ \frac{v}{\eta_{m}}, -\eta \right\} = \frac{v}{\eta_{m}} \quad \text{iff} \quad v \geq -\eta_{m}^{+}.$$  

(190)

Taking into account $G_{m}(f - u + \frac{v}{u})$ we deduce the restrictions

$$\frac{v}{u} \leq \eta_{m}^{+} + u \quad \text{and} \quad \frac{v}{u} \geq \varepsilon_{m}^{+} + u.$$  

(191)

This implies together with (187) and (188) the following restrictions:

$$v \geq (\varepsilon_{m}^{+} + u)u \geq (\varepsilon_{m}^{+} - \eta)(-\eta) = \eta(u - \varepsilon_{m}^{+})$$

and

$$\eta_{m}^{+} \geq \varepsilon_{m}^{+} + u \iff u \leq \eta_{m}^{+} - \varepsilon_{m}^{+} = \delta - f + (\delta + f) = 2\delta.$$  

(192)

For this maximum $u$ we have using (188) again a further restriction for $v$:

$$\frac{v}{u} = \frac{v}{2\delta} \leq \eta_{m}^{+} \iff v \leq 2\delta \eta_{m}^{+}.$$  

(193)
Consequently we have:

$$
\int_0^\infty \left| K(v) \right|^2 \left[ \int_0^\infty \frac{1}{u} G_0 \left( f - u \right) G_m \left( f - u + \frac{v}{u} \right) \, du \right] \, dv
= \int_{-\eta^- m}^{-\eta^+ m} \left| K(v) \right|^2 \left[ \int_0^1 \frac{1}{u} \, du \right] \, dv + \int_{-\eta^- m}^{-\eta^+ m} \left| K(v) \right|^2 \left[ \int_1^\infty \frac{1}{u} \, du \right] \, dv
$$

and we have to fill-in the correct integration limits of the inner integration. In the interval \([\eta(\eta - \varepsilon_m), -\eta^+ m]\) we derived

$$
u \geq -\eta.
$$

In the interval \([-\eta^+ m, 2\delta \eta^- m]\) we derived

$$
u \geq \frac{\eta}{\varepsilon_m + u}.
$$

Further we always have:

$$
u \leq \frac{\eta}{\varepsilon_m + u}.
$$

Since (note that one solution of the quadratic equation doesn’t give a restriction):

$$
u \leq \frac{\eta}{\varepsilon_m + u} \iff \left( u + \frac{\varepsilon_m}{2} \right)^2 \leq \left( \frac{\varepsilon_m}{2} \right)^2 + \eta
\iff \nu \leq \left( \frac{\varepsilon_m}{2} \right)^2 + \sqrt{\left( \frac{\varepsilon_m}{2} \right)^2 + \eta}
$$

Hence we got the integration limits in the inner integral:

$$
\int_0^\infty \left| K(v) \right|^2 \left[ \int_0^\infty \frac{1}{u} G_0 \left( f - u \right) G_m \left( f - u + \frac{v}{u} \right) \, du \right] \, dv
= \int_{-\eta^- m}^{-\eta^+ m} \left| K(v) \right|^2 \left[ \int_{-\eta}^{-\eta^- m} \frac{1}{u} \, du \right] \, dv + \int_{-\eta^- m}^{-\eta^+ m} \left| K(v) \right|^2 \left[ \int_{-\eta^+ m}^{-\eta^- m} \frac{1}{u} \, du \right] \, dv
$$

The condition \(G_m \left( f - \frac{v}{u} \right)\) for the partial integral (29) leads to:

$$
-\frac{v}{u} \leq \eta^+ m \quad \text{and} \quad -\frac{v}{u} \geq \varepsilon^- m
\iff \frac{v}{u} \geq -\eta^+ m \quad \text{and} \quad \frac{v}{u} \leq -\varepsilon^- m.
$$

Since \(0 \geq \eta \geq -2\delta\) the second inequality of (203) is never fulfilled for \(m > 0\). So the integral (29) is zero for \(m > 0\). Taking into account \(G_m \left( f - u - \frac{v}{u} \right)\) we deduce the restrictions

$$
-\frac{v}{u} \leq \eta^+ m + u \quad \text{and} \quad -\frac{v}{u} \geq \varepsilon^- m + u
$$

we see that the that we get instead of (203)

$$
-\frac{v}{u} \leq \eta^+ m \quad \text{and} \quad -\frac{v}{u} \geq \varepsilon^- m + u
$$

since \(-\eta^+ m = |m| \Delta - \eta \geq \Delta - 2\delta > 0\) and \(-\varepsilon^- m = |m| \Delta + \varepsilon > 0\). Then

$$
u \leq \frac{v}{-\eta^+ m} \quad \text{and} \quad \nu \geq \frac{v}{-\varepsilon^- m - u}
$$

and consequently using (187):

$$
\max \left\{ \frac{v}{-\varepsilon^- m - u} - \eta \right\} \leq u.
$$
Now it follows that:

\[
\max \left\{ \frac{v}{\varepsilon_m}, -\eta \right\} = -\eta \quad \text{iff} \quad v \leq \eta(\varepsilon_m - \eta)
\]

and

\[
\max \left\{ \frac{v}{\varepsilon_m}, -\eta \right\} = \frac{v}{-\varepsilon_m} \quad \text{iff} \quad v \geq \eta(\varepsilon_m - \eta).
\]  

(205)

Equations (204) and (202) imply

\[
\eta_m^+ \geq -\frac{v}{u} \geq \varepsilon_m + u
\]

(206)

which implies

\[
u \leq 2\delta
\]

(207)

and

\[
v \leq -2\delta \eta_m^+.
\]

(208)

On the other hand

\[
v \geq -\eta_m^+ u \quad \text{and} \quad u \geq -\eta
\]

(209)

imply

\[
v \geq \eta_m^+ \eta.
\]

(210)

Thus we get:

\[
\int_0^\infty |K(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G_0(f - u) G_m(f - \frac{v}{u}) G_m(f - u - \frac{v}{u}) \, du \right] \, dv
\]

(211)

\[
= \int_{\eta_m^+ \eta}^{\eta(\varepsilon_m - \eta)} |K(v)|^2 \left[ \int_{-\eta}^{\frac{v}{-\varepsilon_m}} \frac{1}{u} \, du \right] \, dv + \int_{\eta(\varepsilon_m - \eta)}^{2\delta \eta_m^+} |K(v)|^2 \left[ \int_{\frac{v}{\varepsilon_m + \eta}}^{\frac{v}{\varepsilon_m + \eta}} \frac{1}{u} \, du \right] \, dv.
\]

In the interval \([\eta(\varepsilon_m - \eta), -2\delta \eta_m^+]\) we derive from the second inequality (204) like in that (212) that

\[
u \geq -\varepsilon_m \geq \sqrt{\left(\frac{\varepsilon_m}{2}\right)^2 - v}.
\]

(212)

We therefore have

\[
\int_0^\infty |K(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G_0(f - u) G_m(f - \frac{v}{u}) G_m(f - u - \frac{v}{u}) \, du \right] \, dv
\]

(213)

\[
= \int_{\eta_m^+ \eta}^{\eta(\varepsilon_m - \eta)} |K(v)|^2 \left[ \int_{-\eta}^{\frac{v}{-\varepsilon_m}} \frac{1}{u} \, du \right] \, dv + \int_{\eta(\varepsilon_m - \eta)}^{2\delta \eta_m^+} |K(v)|^2 \left[ \int_{\frac{v}{\varepsilon_m + \eta}}^{\frac{v}{\varepsilon_m + \eta}} \frac{1}{u} \, du \right] \, dv.
\]

Using the correspondences (182) we finally arrive at

\[
\int_0^\infty |K(v)|^2 \left[ \int_0^1 \frac{1}{u} \cdot G_0(f - u) G_m(f - \frac{v}{u}) G_m(f - u - \frac{v}{u}) \, du \right] \, dv
\]

(214)

\[
= \int_{-\eta_m^+ \eta}^{-\eta(\varepsilon_m + \eta)} |K(v)|^2 \ln \left( -\frac{v}{\eta \eta_m^+} \right) \, dv + \int_{-\eta(\varepsilon_m + \eta)}^{2\delta \eta_m^+} |K(v)|^2 \ln \left( \frac{v}{\eta \eta_m^+} - (\frac{v}{\varepsilon_m + \eta} - \left(\frac{v}{\varepsilon_m + \eta}\right)^2 - v) \right) \, dv.
\]

which completes the proof. □