Multi Particle Semiclassical Process
in $\phi^4$ Theory\textsuperscript{1}

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Abstract

We have shown an example of semiclassical transition in $\phi^4$ model with positive coupling constant. This process describes a semiclassical transition between two coherent states with much smaller average number of particles in the initial state than in the final state. This transition is technically analogous to the one-instanton transition in the electroweak model. It is suppressed by the factor $\exp(-2S_0)$, where $S_0$ is Lipatov instanton action. It could be important to the problem of calculation of amplitudes for multiparticle production in $\phi^4$-type models.

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1. Recently, considerable efforts have been made to calculate amplitudes for multiparticle production in weakly coupled field theories. The study of this problem was initiated by the observation of the fact [1] that baryon-number violating processes in the electroweak theory, associated with multiparticle production, could become relevant at energy scale $E \sim 10 \text{ TeV}$. This problem gave impulse to study multiparticle amplitudes in the simpler case of $\phi^4$ model [2, 3], considered before in the context of large orders of perturbation theory [4].

The semiclassical methods for computing such amplitudes in the electroweak theory use Euclidean classical solutions of the equations of motion – instantons [5]. To apply the similar calculations to $\phi^4$ theory [3], we have to use Lipatov’s trick [4] and consider first the theory with negative coupling constant. Then, the theory with negative coupling constant allows a semiclassical instanton-like transitions.

In this paper we show that $\phi^4$ theory allows a semiclassical transition even for the case of positive coupling constant. This transition is described by a classical $O(4)$-invariant solution, considered on a contour in the complex time plane. The “type” of the transition is determined by the position of this contour with respect to the positions of the singularities of the classical solution.

The transition is technically analogous to the one-instanton transitions in the electroweak model. It is suppressed by the factor $\exp (-2S_0)$, where $S_0$ is equal to Lipatov instanton action – the action of the classical solution in the Euclidean theory with negative coupling constant [4].

This process describes a classically-forbidden transition between two coherent states with a much smaller number of particles in the initial state than in the final state – $n_{\text{final}} \sim n_{\text{initial}}^{5/7}/\lambda^{2/7}$ (where $\lambda$ is a small coupling constant). Therefore, it could be relevant to the calculation of amplitudes for multiparticle production in $\phi^4$-type models. We suppose that the contribution of such a process must be included into the corresponding multiparticle amplitude and, probably, can slow down the factorial growth of the perturbative amplitude [4].

2. Formally, we cannot calculate the transition probability for the process $\text{two} \rightarrow \text{many}$ particles in the semiclassical manner at all, because of the non-semiclassical nature of the initial two-particle state. Instead, as proposed in
Ref. [6], we can calculate the probability of transition between a semiclassical initial state with a “small” number of particles and a final semiclassical state with a “large” number of particles. The probability of such a transition can be considered as some approximation to the two particle cross section in one-instanton sector and gives us an upper bound for this cross section.

The starting point of this approach is the amplitude for a transition at fixed energy $E$ from the initial coherent state $| \{a_k\} \rangle$ (projected onto this energy) to the final coherent state $| \{b_k\} \rangle$.

$$A = \langle \{b_k\} | SP_E | \{a_k\} \rangle,$$

where the operator $P_E$ is a projector onto subspace of definite energy $E$; $S$ is the $S$-matrix.

When $E \sim 1/\lambda$ and $a_k, b_k \sim 1/\sqrt{\lambda}$ for small coupling constant $\lambda$, we can evaluate the transition amplitude (1) in the saddle-point approximation.

In the saddle-point approximation the functional integral (1) is dominated by the solution of the classical field equations with some specific boundary conditions [7], determined by the initial coherent state at early time $t \to -\infty$ and the final coherent states at late time $t \to +\infty$.

However, for the calculation of some classically-forbidden transition (for example tunneling) we cannot restrict ourselves to a pure Minkowski or Euclidean time, because we make deal simultaneously with the classically-allowed event (such as free evolution of the initial and final states) and...
classically-forbidden event. So we work with the contour in the complex 
time plane shown on Fig. (1) \[7\].

The part A of this contour is shifted upward and runs parallel to the real 
axis $t = t' + iT$. Evolution of the system with respect to $t'$ corresponds to 
initial state propagation, while the real part of the contour describes final 
state propagation.

The boundary value problem is conveniently formulated on this contour 
\[7\]. We assume below that the classical solution $\phi$ becomes free at large 
initial and final time, which means that its spatial Fourier transform can be 
written as a superposition of plane waves. Then, the negative frequency part 
of the classical field, considered on the part A of the contour at the limit 
$t' \to -\infty$, is determined by the initial state. The positive frequency part of 
the field is determined by the final state on Minkowski part of the contour 
at large positive time $t \to +\infty$.

Thus, to find the transition probability, we have to solve the field equa-
tions with fixed negative frequency part of the field at early time and positive 
frequency part of the field at late time. This is an extremely difficult prob-
lem for arbitrary initial and final states, even in the case of the $\phi^4$ theory. 
So we are forced to restrict ourselves to a less general problem, proposed in 
Ref. \[8\]: we find first some real Minkowski-time solution and then find the 
corresponding initial and final states as asymptotics of this solution.

We have to make some remarks about the choice of the “appropriate” 
solution.

First, we consider only real solutions because, as it has been shown in 
Ref.\[8\], the probability of the transition from the given initial state to all 
possible final states is saturated by a single final state which is real at real 
time. Therefore, the real saddle-point configuration corresponds to the tran-
sition from the given initial state to the most probable final state.

The second condition is that this solution should have an appropriate 
singularity structure in the complex time plane - we have to be able to choose 
the contour of Fig.(1) and avoid any singularities of the solution.

We will show below that $\phi^4$ theory possesses such solutions.

The semiclassical suppression in the transition probability is determined 
by the imaginary part of the classical action, calculated along the time con-
tour of Fig. (1) \[8\] (see also \[9\])

$$
\sigma = |A|^2 \sim \exp(-2\text{Im } S). 
$$

\(2\)
The probability of the transition does not depend on the choice of the contour and we can move the contour upward or downward until we reach a singularity of the classical solution. So the "type" of the transition is determined by the position of the contour with respect to the position of the singularities of the classical solution in the complex time plane.

3. Now we apply this formalism to $\phi^4$ model.

The action of the model (we consider a real scalar field), written in conformally invariant form [11], is

$$S = \int d^4x \left( -\frac{1}{2} \dot{\phi} \partial_\mu \partial^\mu \phi - \frac{\lambda}{4} \phi^4 \right),$$

where $\lambda > 0$ is the small coupling constant. The corresponding classical field equation is

$$\partial^2 \phi + \lambda \phi^3 = 0 \quad (3)$$

$O(4)$-invariant solutions of this equation [12] can be easily found using the invariance of the massless theory under the Minkowski conformal group. This invariance can be made explicit by projecting the theory onto the surface of the hypertorus [13]. Then, $O(4)$-invariant solutions can be found by solving a one-dimensional equation and they correspond to the oscillations with amplitude $a$ in the one-dimensional potential $V(x) = \frac{x^2}{2} + \lambda x^4/4$.

The $O(4)$-invariant solution can be expressed in terms of elliptic functions

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{\lambda}} \frac{2a}{\sqrt{(r^2 - (t - i)^2)(r^2 - (t + i)^2)}} \text{cn}(\sqrt{1 + a^2} \zeta - \zeta_0, k^2), \quad (4)$$

where $r = |\vec{x}|$, $k^2 = a^2/(2(1 + a^2))$ and

$$\zeta = \frac{1}{2t} \ln \frac{r^2 - (t - i)^2}{r^2 - (t + i)^2}.$$

Here $\text{cn}$ stands for the Jacobi elliptic cosine (see, for example, [14]) and $k$ is the modulus of this function. The arbitrary integration constants are $a$ and $\zeta_0$. We choose $\zeta_0 = K$ (where $K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$ is the complete elliptic integral), in which case $\phi = 0$ at $t = 0$. The constant $a$, as we will see below, is related to the energy.
According to the above described approach, we are going to calculate a transition corresponding to the saddle point configuration (4), considered on the time contour of Fig.(1) for some value of parameter $T = \text{Im} t$.

First, we investigate the analytic structure of the solution in the complex time plane.

This solution is real on the real time axis, so, as has been shown in Ref. [8], it corresponds to a transition from the given initial state to the most probable final state.

The solution has essential “light-cone” singularities at $t = \pm x \pm i$. Hence, we have to choose $T < 1$ for the contour of Fig.(1) not to cross the “light-cone” singularity.

In addition, there are singularities (poles) at the “points” where

$$\sqrt{1 + a^2} \, \zeta - K = 2mK + (2n + 1)i \, K'.$$

Here $K' (k^2) = K(1 - k^2)$, $m, n = 0, \pm 1, \pm 2, \ldots$. These “points” are poles of the elliptic cosine [14]. Because $\zeta$ is a function of radial coordinate and complex time, the solutions of this equation determine the singularity curves in the coordinate axes $r, \text{Re} t, \text{Im} t$.

We will consider below only the case $a << 1$, which, as will be shown below, corresponds to the case of a “small” number of final-state particles ($n_{\text{final}} << 1/\lambda$). In this limit $K \approx \pi/2$ and only $m = -1$ and $n \geq 0$ case corresponds to the singularities in the region $\text{Im} t \geq 0, \text{Re} t \leq 0$. The singularities curves (numeralated by integer number $n$) $t = t_n(r)$ run asymptotically “parallel” to the “light-cone” and have $(\text{Im} t)$ coordinate close to 1

$$t_n = i \left( 1 - \left( \frac{a^2}{16} \right)^{2n+1} \right) - r$$

at $r \to +\infty$ and $n = 0, 1, 2, \ldots$. We have shown in Fig. (2) two curves in the region $\text{Im} t \geq 0, \text{Re} t \leq 0$. The structure of singularities of the classical solution is similar to the structure of singularities of particular classical solutions in two-dimensional $\sigma$-model and four dimensional Yang-Mills theory, investigated in this context in Ref. [8] and [10].

We can see that the structure of the singularities of this solution is “appropriate” – we are able to choose the contour of Fig.(1) and not to cross any singularities. We choose the contour with exactly one singularity curve under it (i.e. with $1 - a^2/16 < T < 1 - (a^2/16)^3$). It will be shown below that
this choice corresponds to a classically-forbidden (exponentially suppressed) transition.

\[ S = \frac{\lambda}{2} \int d^3x \int_C dt \phi^A(\vec{x}, t), \]

where we have used the equation of motion. For every \( x \) the time integral along the contour of Fig. (1) is equal to the sum of the integral along the real time axis (which is real) and contribution of the pole \( t_0 \), corresponding to the singularity (5) at \( n = 0 \). The pole contribution can be calculated using
the expression for the \( cn \) near the singularity \( -2\tilde{K} + iK' \):

\[
\text{cn} \left( -2\tilde{K} + iK' + u \right) = -\frac{1}{iku} - \frac{1}{6iku} (1 - 2k^2) u + O(u^2)
\]

and expanding \( \zeta \) in Taylor series up to the fourth order.

After lengthy calculations we obtain for the imaginary part of the action

\[
\text{Im} S = \frac{8\pi^2}{3\lambda}.
\]

It is exactly equal to Lipatov instanton action: the Euclidean action of the classical solution in \( \phi^4 \) theory with negative coupling constant \( \lambda \) (our normalization of \( \lambda \) differs from the normalization of \( \lambda \) in \cite{4} by factor 6). Thus, the choice of the contour between the first and the second singularity line corresponds to the classically forbidden transition suppressed by the factor

\[
\sigma \sim \exp(-2S_0),
\]

where \( S_0 \) is equal to Lipatov instanton action. So this process is analogous to the one-instanton transition in the electroweak model or to the “instanton-like” transition in \( \phi^4 \) theory with negative coupling constant \( \lambda \).

Our calculation of the transition probability has some resemblance to Landau approach to the calculation of the classically-forbidden reflection from a potential barrier in one-dimensional quantum mechanics \cite{15} (for energy larger then the height of barrier). In Landau approach the reflection probability is also determined by the imaginary part of the classical action on a trajectory in the complex coordinate plane, “wound” around a singular “turning” point.

So the evolution along the imaginary part of the contour of Fig. (1) can be interpreted as a classically-forbidden reflection in the \( \phi^4 \) potential. Part A of the contour describes the free propagation of an incoming spherically-symmetric shell (4) at early time. The Minkowski part of the contour corresponds to an outgoing wave at late time. Therefore, we can call the semiclassical process in the \( \phi^4 \) model, described by the nontrivial “trajectory” (lying between the singular lines) in the complex time plane, as a “transmission after classically-forbidden reflection”.

4. As has been mentioned before, the final state is determined via Eq.(6) by the asymptotics of the classical solution on the Minkowski part of the contour.
of Fig.(1) in the limit $t \to +\infty$, while the initial state corresponds to the asymptotics of the solution on part A of the contour in the limit $t' \to -\infty$, where $t = iT + t'$. The average numbers of particles are determined by the negative frequency Fourier components $g_k$ of the classical solution (4), considered on the corresponding parts of the contour

$$n = \int dk \, g_k^* g_k.$$

We analyze only case $a << 1$ which, we will see below, corresponds to the case $n_{final} << 1/\lambda$.

In this limit we obtain for the average number of the final particle

$$n_{final} = \frac{\pi^2 a^2}{\lambda}.$$

The number of the initial particles is given by

$$n_{initial} = \left(\frac{3}{4^4}\right)^{1/5} \frac{\pi^2 a^{14/5}}{2 \lambda} = \frac{1}{2} \left(\frac{3}{4^4}\right)^{1/5} a^{4/5} n_{final}.$$

This result implies

$$n_{final} \sim n_{initial}^{5/7} \lambda^{2/7}.$$

We can also define the average momentum \[8\]

$$k_{average} \approx E/n$$

(energy in terms of amplitude $a$ is expressed by $E = \frac{\pi^2 a^2}{\lambda}$). Then, the initial average momentum is related with the final momentum by

$$k_{initial_{average}} \sim \frac{k_{final_{average}}}{a^{4/5}}.$$

We can see that for small coupling constant the number of the final “soft” particles is much larger then the number of the initial “hard” particles.

Thus, the classical solution, considered on the contour of Fig.(1) in the complex time plane above the singularity line, corresponds to the transition between two coherent states with a “strong” violation of particle number, $n_{final} >> n_{initial}$. 

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5. To conclude, we have studied the semiclassical process in $\phi^4$ theory with positive coupling constant, which describes transition between two coherent states. This transition is suppressed by the factor $\exp(-2S_0)$, where $S_0$ is equal to the Lipatov instanton action – the Euclidean action of the classical solution in the theory with negative coupling constant.

The initial and final states, corresponding to this transition, have different numbers of particles ($n_{\text{final}} \gg n_{\text{initial}}$) and different average momenta ($k_{\text{final}} \ll k_{\text{initial}}$), so this transition approximates some multiparticle scattering process with a large number of “soft” final particles.

The process is technically analogous to the one-instanton transition in electroweak model and could serve as a good model for studying the instanton effects. It seems that we can also describe some “multi-instanton” processes using the solution (4) and choosing the contour of Fig.(1) above several singularity lines.

We believe that we have to include the contributions of these instanton-like processes into the corresponding “total” amplitude for multiparticle production. Such contributions might slow down the factorial growth of the perturbative amplitude and unitarize the high energy cross section.

The energy dependence of the transition probability of this process is a very interesting problem. The growth of the transition probability is related to the presence of the external particles. An accurate estimation of the contribution of the external particles requires, however, including mass term effects into consideration.

We have to add the mass term for the following reason. Calculation of the transition probability requires summing the contributions from different “sizes” of the classical field. In this paper we consider only the contribution of the solution with a “unit” size (field configuration (4)). This integration is divergent at the large “sizes” and should be regularized by introducing a mass term into the action in the manner of the “constrained instanton” approach [16]. We do not consider the effects of the mass term in this paper, so this problem requires a more detailed investigation.

The important point is that the framework of the formalism allows one to analyze, in principle, the most interesting case $n_{\text{final}} \geq 1/\lambda$. This case is analogous to multi-particle scattering at the sphaleron energy in the standard model, where the behavior of multi-particle cross section is still far from being understood.
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