Role of thermal noise in tripartite quantum steering

Meng Wang¹, Qihuang Gong¹,², Zbigniew Ficek³, and Qiongyi He¹,²,∗

¹State Key Laboratory of Mesoscopic Physics, School of Physics, Peking University, Beijing 100871, P. R. China
²Collaborative Innovation Center of Quantum Matter, Beijing 100871, P. R. China and
³National Centre for Applied Physics, KACST, P.O. Box 6086, Riyadh 11442, Saudi Arabia

The influence of thermal noise on bipartite and tripartite quantum steering induced by a short laser pulse in a hybrid three-mode optomechanical system is investigated. The calculation is carried out under the bad cavity limit, the adiabatic approximation of a slowly varying amplitude of the cavity mode, and with the assumption of driving the cavity mode with a blue detuned strong laser pulse. Under such conditions, explicit expressions of the bipartite and tripartite steering parameters are obtained, and the concept of collective tripartite quantum steering, recently introduced by He and Reid [Phys. Rev. Lett. 111, 250403 (2013)], is clearly explored. It is found that both bipartite and tripartite steering parameters are sensitive functions of the initial state of the modes and distinctly different steering behaviour could be observed depending on whether the modes were initially in a thermal state or not. For the modes initially in a vacuum state, the bipartite and tripartite steering occur simultaneously over the entire interaction time. This indicates that collective tripartite steering cannot be achieved. The collective steering can be achieved for the modes initially prepared in a thermal state. We find that the initial thermal noise is more effective in destroying the bipartite rather than the tripartite steering which, on the other hand, can persist even for a large thermal noise. For the initial vacuum state of a steered mode, the tripartite steering exists over the entire interaction time even if the steering modes are in very noisy thermal states. When the steered mode is initially in a thermal state, it can be collectively steered by the other modes. There are thresholds for the average number of the thermal photons above which the existing tripartite steering appears as the collective steering. Finally, we point out that the collective steering may provide a resource in a hybrid quantum network for quantum secret sharing protocol.

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I. INTRODUCTION

The main property of entanglement is that it is shared equally between the subsystems, that one cannot judge which of the subsystems is more or less responsible for the entanglement. Unlike entanglement, quantum steering distinguishes the role of each subsystem. In fact, it is a form of quantum nonlocality and gives a way to quantify how measurements by Alice on her local particle A can collapse the wavepacket of Bob’s particle B. The asymmetry reflects the asymmetric nature of the original Einstein–Podolsky–Rosen (EPR) paradox [14–16, 18]. In it, the reduced uncertainty levels of Alice’s predictions for Bob’s system that are relevant in establishing the paradox [14, 15, 18]. Recently, the concept of collective multipartite steering has been developed [19], which shows that special quantum states allow Einstein’s nonlocality to be shared among all observers involved.

An experimental challenge is to observe the quantum nonlocality predicted by EPR for macroscopic system. Optomechanical systems with nanomechanical oscillators provide a natural setting for testing quantum nonlocality of mesoscopic systems [20, 21]. The ability to cool optomechanical systems near their ground states resulted in the demonstration of a number of quantum effects such as quantum-state transfer [22, 27], mechanical entanglement [28, 30], mechanical squeezing [31, 32], and electromagnetically induced transparency [10, 42]. In addition, spatial entanglement between the macroscopic mirror and the microscopic cavity field in a pulsed two-mode optomechanical system has been demonstrated theoretically [23, 24] and experimentally observed [13]. Of particular interest is to observe an EPR paradox for the position and momentum of mesoscopic mechanical oscillators which could demonstrate the inconsistency of quantum mechanics with the local reality of a macroscopic object.

When an optomechanical system is composed of more than two modes, more complex correlations can be created. These correlations can significantly affect the two-mode entanglement and result in multimode entanglement. In searching for fully inseparable tripartite entanglement, first extension is to study the entanglement between any pair of the tripartite optomechanical systems [23, 44, 51]. Most of them produce a partially (at least one pair is entangled) or fully inseparable (any two pairs show entanglement) tripartite entanglement. It has been demonstrated that a genuine tripartite entanglement can be produced in a three-mode optomechanical system composed of an atomic ensemble located inside a single-mode cavity with a movable mirror [52].

In this paper, we examine the conditions for tripartite steering and a special form of the tripartite steering called collective tripartite quantum steering [19]. We study a hy-
brid optomechanical system composed of an ensemble of N identical two-level atoms located inside a single-mode cavity formed by two mirrors, a fixed semitransparent mirror and a movable fully reflective mirror. The cavity is driven by a short laser pulse and thus is free from the restriction of the stability requirements. We adopt the criteria for bipartite and tripartite steering determined by parameters which are simple functions of the variances and correlation functions of the quadrature components of the amplitudes of the output modes. We are particularly interested in the role of the thermal noise in the creation of collective tripartite steering between the modes. The treatment is restricted to the bad cavity limit under which the adiabatic approximation can be made of a slowly varying amplitude of the cavity mode. In addition, the laser pulses are assumed to be strong and blue detuned to the cavity and the atomic resonance frequencies. We show that the thermal noise presented in the input modes is more effective in destroying the bipartite than the tripartite steering which, on the other hand, can persist even for a large thermal noise. The threshold values for the average number of the thermal photons at which the bipartite steering disappears are easy calculated.

The paper is organized as follows. In Sec. II we introduce parameters that determine the conditions for bipartite and tripartite steering and briefly discuss the conditions required for the tripartite steering to be regarded as collective tripartite steering. In Sec. III the pulsed three-mode optomechanics is introduced. We define a set of normalized temporal modes and derive analytic expressions for the variances and correlation function of the quadrature components of the output fields. In Sec. IV we evaluate the parameters for the bipartite and tripartite steering and discuss in details the conditions for collective tripartite steering to occur in the system. We summarize our results in Sec. V. Finally, in the Appendix, we give general expressions for the bipartite and tripartite steering parameters in terms of the variances and correlation functions, and optimal weight factors that minimize the variances involved in the steering parameters.

II. DEFINITIONS AND IDENTIFICATION OF COLLECTIVE TRIPARTITE STEERING

For later convenience we start by introducing the definition of tripartite steering and explain in details how one could distinguish between the ordinary tripartite steering and collective tripartite steering. We introduce parameters that measure the degree of ordinary and collective tripartite steering.

Quantum steering is normally identified by criteria which are a natural generalization of those for entanglement. They involve inequalities the variances must satisfy which, in fact, are stronger than those for entanglement [10]. Therefore, steering always certifies entanglement. Let us briefly discuss the criteria for bipartite and tripartite steering. The criteria are based on an accuracy of inference defined as the root mean square of the variances $\Delta_{\text{inf},j}^{2}X_{\text{out}}^{i}$ and $\Delta_{\text{out}}^{2}P_{\text{out}}^{i}$ of the conditional distributions $P(X_{\text{in}}^{out}|O_{\text{out}}^{j})$ and $P(P_{\text{in}}^{out}|O_{\text{out}}^{j})$ ($O_{\text{out}}^{j}, O_{\text{out}}^{j} = X_{\text{out}}^{i}, P_{\text{out}}^{i}$), for a result of measurement of the quadratures $X_{\text{out}}^{i}, P_{\text{out}}^{i}$ at $i$, based on the results $O_{\text{out}}^{j}, O_{\text{out}}^{j}$ of the measurement at $j$. Here, $O_{\text{out}}^{j}, O_{\text{out}}^{j}$ are arbitrary observables (quadratures) for system $j$ selected such that they minimize the variance product $\Delta_{\text{inf},j}^{2}X_{\text{out}}^{i}$. A useful strategy is to use a linear estimate $u_{j}O_{\text{out}}^{j}$, where $u_{j}$ is a constant chosen such that it minimizes the variance product [15, 16]. The inferred uncertainty $\Delta_{\text{inf},j}X_{\text{out}}^{i}$ can be written as

$$\Delta_{\text{inf},j}X_{\text{out}}^{i} = \Delta(X_{\text{out}}^{i} + u_{j}O_{\text{out}}^{j}),$$

where the quadrature $O_{\text{out}}^{j}$ is selected either $O_{\text{out}}^{j} = X_{\text{out}}^{i}$ or $O_{\text{out}}^{j} = P_{\text{out}}^{i}$, depending on the type of the correlations between the modes $i$ and $j$ [15, 16]. The best choice of $u_{j}$ can be calculated by linear regression and it is not difficult to show that the uncertainty is minimized for $u_{j} = -\Delta(P_{\text{out}}^{i}, O_{\text{out}}^{j})/\Delta^{2}O_{\text{out}}^{j}$. We say that the mode $i$ is steered by the mode $j$ if the product of the inferred variances satisfies the inequality ($\hbar = 1$) [12, 16]

$$E_{ij} = \Delta_{\text{inf},j}X_{\text{out}}^{i}\Delta_{\text{inf},j}P_{\text{out}}^{i} < \frac{1}{2},$$

The condition for tripartite steering is described in terms of the inferred variances of a linear combination of the quadrature components

$$\Delta_{\text{inf},jk}X_{\text{out}}^{i} = \Delta [X_{\text{out}}^{i} + (u_{j}O_{\text{out}}^{j} + u_{k}O_{\text{out}}^{k})],$$

where, depending on the type of correlations between the modes, the quadrature $O_{\text{out}}^{j(k)}$ can be selected either $X_{\text{out}}^{j(k)}$ or $P_{\text{out}}^{j(k)}$, and the weight factors $u_{j}, u_{k}$ are estimated to minimize the variance. $\Delta_{\text{inf},jk}P_{\text{out}}^{i}$ is defined similarly. Then, we say that the mode $i$ is steered by the group of modes $\{jk\}$ if

$$E_{i|jk} = \Delta_{\text{inf},jk}X_{\text{out}}^{i}\Delta_{\text{inf},jk}P_{\text{out}}^{i} < \frac{1}{2}.$$  

It involves a superposition of the modes $j$ and $k$ which can be treated a single “collective” mode $W_{\text{out}}^{i} = O_{\text{out}}^{j} + u_{jk}O_{\text{out}}^{k}$. Thus, the collective mode $W_{\text{out}}^{i}$ can be treated as a single mode that can steer the mode $i$.

The inequality is the sufficient condition for tripartite steering without any requirements about the bipartite steering between modes $i$ and $j$, and between $i$ and $k$. This means that in general for a tripartite steering we can have two distinct possibilities. Namely, we could have $E_{i|jk} < 1/2$ with both or either $E_{i|j}$ or $E_{i|k}$ smaller than 1/2. In this case, the tripartite steering is accompanied by a bipartite steering, and is referred to as
ordinary tripartite steering. The other case corresponds to
the inequality \( E_{ij} < 1/2 \) with both \( E_{ij} \geq 1/2 \) and \( E_{ij} \geq 1/2 \). In this case, the tripartite steering is not
accompanied by the bipartite steering. The mode \( i \) is
steered solely by the collective mode and therefore it is
referred to as collective steering \[19\]. In other words, to
demonstrate the existence of collective steering in a tri-
partite system we must show that whenever the condition
\( E_{ij} < 1/2 \) holds, the bipartite steering parameters are
\( E_{ij} \geq 1/2 \) and \( E_{ij} \geq 1/2 \). The collective steering is thus
a generalization of the ordinary tripartite steering to the
case when a given mode is steered only by the collective
mode of a linear superposition of the remaining modes.

III. HYBRID PULSED CAVITY
OPTOMECHANICAL SYSTEM

We now illustrate how the ordinary and collective tri-
partite steering may be created in a three mode sys-
tem. We choose a three-mode hybrid pulsed optome-
chanical system and investigate under which conditions
the ordinary tripartite steering can be created and under
what circumstances it is not accompanied by the bipar-
tite steering. The three-mode hybrid pulsed optome-
chanical system is known to exhibit bipartite steering \[52\].

\[
S^+ \approx \sqrt{N}c_a, \quad S^- \approx \sqrt{N}c_a, \quad S_z \approx \langle S_z \rangle \approx -N. \quad (6)
\]

If the atomic ensemble is weakly coupled to the cavity
mode the mean number of atoms transferred to the up-
per state \( |2_j \rangle \) is expected to be much smaller than the
total number of atoms, i.e., \( \langle c_a^\dagger c_a \rangle \ll N \). By expanding
the square root in Eq. (6) and neglecting terms of the
order of \( O(1/N) \), the collective atomic operators can be
approximated as \[54\]

\[
S^+ \approx \sqrt{N}c_a, \quad S^- \approx \sqrt{N}c_a, \quad S_z \approx \langle S_z \rangle \approx -N. \quad (6)
\]

It is easily verified that the operators \( c_a \) and \( c_a^\dagger \) satisfy
the fundamental commutation relation for boson operators,
\( \{c_a, c_a^\dagger\} = 1 \).

The Hamiltonian of the system, in a frame rotating
with the laser frequency \( \omega_L \), is given by \[52\]

\[
H = \hbar \Delta_c a_c^\dagger a_c + \hbar \omega_m a_m^\dagger a_m + \hbar \Delta_a c_a^\dagger c_a
\]
\[
+ \hbar g_0 c_a^\dagger a_m (a_m^\dagger + a_m) + \hbar g_a (c_a^\dagger a_c + a_c c_a^\dagger)
\]
\[
+ \hbar \left[ E(t) a_c^\dagger - E^*(t) a_c \right]. \quad (7)
\]

The first three terms represent the energy of the modes.
Here, \( \Delta_c = \omega_c - \omega_L \) and \( \Delta_a = \omega_a - \omega_L \) are
the detunings of the laser frequency \( \omega_L \) from the cavity and
the atomic transition frequencies, respectively. The fourth term
describes the interaction of the cavity mode with the movable
mirror. This is a nonlinear type interaction with the
strength determined by single-photon coupling constant
\( g_0 \). As we shall see this interaction results in a parametric
coupling between the modes when the cavity is driven by
a blue detuned laser \[53\]. The fifth term describes the
interaction between cavity mode and atomic excitation
mode with coupling constant \( g_a \). This is a beamsplitter-
like interaction. The last term in Eq. (7) describes the
interaction of the cavity mode with the coherent laser
field of the amplitude \( E(t) \). The laser field is injected
into the cavity mode through the fixed mirror. Note that
the atomic mode is not directly coupled to the mecha-
nical mode.

The evolution of the system is studied using the
Heisenberg equations of motion \[24, 52\]. The equa-
tions form a set of coupled nonlinear differential equa-
tions, which we solve in the limit of a strong pulse,
\( |E(t)| \gg g_0, g_a \). In this case, we can make the semi-
classical approximation in which we write the operators
of modes as composed of a large classical amplitude and
a small fluctuation operator, i.e. \( a_i \rightarrow a_i + \delta a_i \)
and \( c_a \rightarrow \tilde{c}_a + \delta c_a \). The fluctuation operators, in a frame
rotating with \( \omega_m \) and under the rotating-wave approxi-
mation in which we ignore all terms oscillating with \( 2\omega_m \),
satisfy the following linearized quantum Langevin equa-
tions

\[
\delta \dot{a}_m = -\gamma_m \delta a_m - ig \delta a_c^\dagger - \sqrt{2\gamma_m} \xi_m^a,
\]
\[
\delta \dot{a}_c = -\kappa \delta a_c - ig \delta a_m - \sqrt{2\kappa} \xi_m^a,
\]
\[
\delta \dot{c}_a = -\gamma_a \delta c_a - ig \delta a_c - \sqrt{2\gamma_a} \xi_m^c,
\]

where \( g = g_0 |a_c| \) is the effective optomechanical
coupling constant and we have assumed that the atomic

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Figure 1. (Color online) Schematic diagram of a driven hy-
brid optomechanical system. The cavity mode, atoms, and
movable mirror constitute a three-mode system and are
represented by annihilation operators \( a_c, c_a \) and \( a_m \), respectively. Here, \( a_{in} \) and \( a_{out} \) denote input and output cavity fields.

We consider an optomechanical system composed of a
single-mode cavity with a movable fully reflective mirror,
containing an atomic ensemble and driven by light pulses
of duration \( \tau \), as shown in Fig. 1. The cavity mode has
a frequency \( \omega_c \) and a decay rate \( \kappa \). The atomic ensemble
contains \( N \) identical two-level atoms each composed of a
ground state \( |1_j \rangle \) and an excited state \( |2_j \rangle \) \( (j = 1, \cdots N) \),
separated by the transition frequency \( \omega_a \). We represent
the atomic ensemble in terms of the collective dipole low-
ering \( S^- = \sum_j |1_j \rangle \langle 1_j | \), raising \( S^+ = \sum_j |2_j \rangle \langle 2_j | \) and
population inversion \( S_z = \sum_j (|2_j \rangle \langle 2_j | - |1_j \rangle \langle 1_j |) \) op-

tators. We assume that the atomic ensemble is composed of
a large number of atoms \( (N \gg 1) \) which allows us to make
use of the Holstein-Primakoff representation \[52\] that
transforms the collective atomic operators into bosonic
annihilation and creation operators \( c_a \) and \( c_a^\dagger \):

\[
S^+ = (S^-)^\dagger = c_a^\dagger \sqrt{N - c_a^\dagger c_a}, \quad S_z = c_a^\dagger c_a - N. \quad (5)
\]
detuning $\Delta_\delta = -\omega_m$ and the effective cavity detuning $\Delta_\epsilon + g_0(\alpha_m + \alpha_m^*) = -\omega_m$. The choice we have made for the detunings corresponds to the laser pulse driving on the blue sideband of the cavity and the atomic resonances.

In particular, when we make an adiabatic approximation, we assume that the environments are in thermal vacuum states characterized by the correlation functions $(\xi_{\text{in}}(t)^*\xi_{\text{in}}(t') + \xi_{\text{in}}(t')\xi_{\text{in}}(t)) = (2n_i + 1)\delta(t - t')$, where $n_i$ is the average number of thermal photons in the environment coupled to the mode.

A. Normalized temporal modes

In the bad cavity limit $\kappa \gg g_a, g$ and for short evolution times $t \sim 1/\kappa$, we may neglect the relaxations of the atoms and the mechanical mirror ($\gamma_a = \gamma_m = 0$). In this case, Eqs. (8) are simple enough to be solved analytically. In particular, when we make an adiabatic approximation, $\delta a_c \approx 0$, which is justified for $\kappa \gg g_a, g$, we arrive at the following equations

$$a_c(t) \approx -i \frac{g_a}{\kappa} a_c(t) - i \frac{g_a^*}{\kappa} a_m(t) - \sqrt{\frac{2}{\kappa}} a_{\text{in}}(t),$$

$$a_m(t) = a_m(0)e^{Gt} + \sqrt{G}G_a e^{Gt} \int_0^t dt' c^{\dagger}_a(t') e^{-Gt'},$$

$$c^{\dagger}_a(t) = c^{\dagger}_a(0)e^{-G^*t} - \sqrt{G}G_a e^{-G^*t} \int_0^t dt' a_m(t') e^{G^*t'},$$

where $G = g^2/\kappa$ and $G_a = g_a^2/\kappa$. Note that for simplicity of the notation, we have dropped $\delta$.

The structure of the solutions given by Eqs. (9)-(11) suggests the introduction of normalized temporal modes of the input and output cavity fields, which in the case of $G > G_a$ are defined by

$$A_{\text{in}} = \sqrt{\frac{2(G - G_a)}{1 - e^{-2(G - G_a)\tau}}} \int_0^\tau dt \ a_{\text{in}}(t)e^{-(G - G_a)t},$$

$$A_{\text{out}} = \sqrt{\frac{2(G - G_a)}{e^{2(G - G_a)\tau} - 1}} \int_0^\tau dt \ a_{\text{out}}^c(t)e^{(G - G_a)t},$$

where $a_{\text{out}}^c(t)$ is the annihilation operator of the output cavity field, given by the standard cavity input-output relation, $a_{\text{out}}^c(t) = a_{\text{in}}(t) + \sqrt{2\kappa a_c(t)} [13]$, and $\tau$ is the duration of the laser pulse. We may also define the normalized input and output operators of the atomic and the mirror modes, $B_{\text{in}} = a_m(0), \ B_{\text{out}} = a_m(\tau), \ C_{\text{in}} = c_a(0), \ C_{\text{out}} = c_a(\tau)$, and then find using Eqs. (9)-(11) the solution for the quadrature components $X_i$ and $P_i$ of the output fields can be expressed only in terms of the quadrature components of the input fields. Assuming that the input modes are in thermal states characterized by the variances

$$\Delta^2 X_m = \Delta^2 P_m = \left(n_0 + \frac{1}{2}\right),$$

$$\Delta^2 X_a = \Delta^2 P_a = \left(n_0 + \frac{1}{2}\right),$$

$$\Delta^2 X_c = \Delta^2 P_c = \left(n_0 + \frac{1}{2}\right),$$

in which $n_0$ is the average numbers of thermal photons in the mirror mode $m$ and $n_1$ is the average number of thermal photons in the cavity mode $a$ and atomic mode $c$, we arrive to the following expressions for the variances of the output fields

$$\Delta^2 X_{m} = \Delta^2 P_{m} = \left(n_0 + \frac{1}{2}\right),$$

$$\Delta^2 X_{a} = \Delta^2 P_{a} = \left(n_0 + \frac{1}{2}\right),$$

$$\Delta^2 X_{c} = \Delta^2 P_{c} = \left(n_0 + \frac{1}{2}\right),$$

$$\langle X_{m}, P_{a} \rangle = \langle P_{m}, X_{a} \rangle = -\left(n_0 + n_1 + 1\right)\alpha \sqrt{e^{2\tau} - 1} \left(\alpha^2 e^{\tau} - \beta^2\right),$$

$$\langle X_{m}, X_{c} \rangle = -\langle P_{m}, P_{c} \rangle = -\left(n_0 + n_1 + 1\right)\alpha \beta \left(e^{\tau} - 1\right)\left(\alpha^2 e^{\tau} - \beta^2\right),$$

$$\langle P_{a}, X_{c} \rangle = -\langle X_{a}, P_{c} \rangle = \left(n_0 + n_1 + 1\right)\alpha^2 \beta \sqrt{e^{2\tau} - 1} \left(e^{\tau} - 1\right),$$

$$\langle P_{m}, X_{c} \rangle = \langle X_{c}, P_{m} \rangle = \langle P_{c}, X_{m} \rangle = \langle X_{m}, P_{c} \rangle = 0.$$ (14)

where $r_\alpha = (G - G_a)\tau = G\tau/\alpha^2 = r/\alpha^2$ is the normalized interaction time parameter, $\alpha = \sqrt{G/(G - G_a)}$, and $\beta = \sqrt{G_a/(G - G_a)}$. Note that the parameter $r_\alpha$ has the physical meaning of the squeezing parameter [24].

The solutions for the variances of the output fields are used in the following section to analyze the criteria for tripartite steering and collective tripartite steering. We address the question of the role of the thermal noise in the creation of ordinary tripartite steering and collective tripartite steering.

IV. TRIPARTITE STEERING AND COLLECTIVE STEERING

We now proceed to evaluate the parameters $E_{iijk}$ for different combinations of the three modes of the optome-
mechanical system to determine the role of the inter-modal interactions and thermal noise in the creation of tripartite steering. Our objective is to find simple analytic forms for the steering parameters. We examine separately several cases of different initial (input) states of the modes, the vacuum state with \( n_0 = n_1 = 0 \), thermal states with equal \( (n_0 = n_1) \) and different \( (n_0 \neq n_1) \) average numbers of thermal photons. Next, we turn to the problem of tripartite collective steering which requires the absence of bipartite steering between the modes. In trying to produce such conditions, we note the constructive role of the thermal noise in the collective steering of the modes.

**A. Tripartite steering**

For the three-mode system considered here, each mode can be steered by the remaining two modes. In this case we have three combinations for the steering parameter. In the first combination, \( E_{m|ac} \) describes steering of the mirror mode \( m \) by the cavity and atomic modes, \( E_{c|am} \) describes steering of the atomic mode by the cavity and mirror modes, and \( E_{a|mc} \) describes steering of the cavity mode by the mirror and atomic modes.

To evaluate the tripartite steering parameters \( E_{ij|k} \), we take suitable linear combinations of the quadrature components and find

\[
E_{m|ac} = \Delta [X_m + (u_a P_a + u_c X_c)] \Delta [P_m + (u_a X_a - u_c P_c)]
\]

\[
= \Delta^2 X_m + u_c^2 \Delta^2 P_c + u_c \alpha^2 \Delta^2 X_c + 2 u_a \langle X_m, P_a \rangle
\]

\[+ 2 u_c \langle X_m, X_c \rangle + 2 u_a u_c \langle P_a, X_c \rangle, \quad (15)
\]

\[
E_{c|am} = \Delta [X_c + (u_a P_a + u_m X_m)] \Delta [P_c - (u_a X_a + u_m P_m)]
\]

\[
= \Delta^2 X_c + u_c^2 \Delta^2 P_a + u_c \alpha^2 \Delta^2 X_m + 2 u_a \langle X_c, P_a \rangle
\]

\[+ 2 u_m \langle X_c, X_m \rangle + 2 u_a u_m \langle P_a, X_m \rangle, \quad (16)
\]

\[
E_{a|mc} = \Delta [X_a + (u_m P_m + u_c X_c)] \Delta [P_a - (u_m X_m - u_c P_c)]
\]

\[
= \Delta^2 X_a + u_c^2 \Delta^2 P_m + u_c \alpha^2 \Delta^2 P_c + 2 u_m \langle X_a, P_m \rangle
\]

\[+ 2 u_c \langle X_a, P_c \rangle + 2 u_m u_c \langle P_m, P_c \rangle. \quad (17)
\]

This shows that, apart from the variances of the quadratures involved, the parameters depend on correlations between the modes. Since the variances \( \Delta^2 X_i \geq 1/2 \) and \( \Delta^2 P_i \geq 1/2 \), we see that the mechanism for steering is in the correlations between the modes. Steering will occur if the correlations are sufficiently large and negative to enforce the inequality \( E_{ij|k} < 1/2 \). The minimum requirement for this to be possible is that there are negative correlations at least between two modes. If the modes are uncorrelated, \( \langle X_i, P_j \rangle = \langle X_i, X_j \rangle = \langle P_i, X_j \rangle = 0 \), and then all the steering parameters are greater than 1/2. Therefore, correlations between the modes are necessary to produce quantum steering. Note that the requirement that the modes should be correlated is necessary but not sufficient for steering. The correlations may be negative but not large enough to reduce the steering parameter below the threshold for steering.

The steering parameters \( E_{ij|k} \) involve correlations between the steering modes \( j \) and \( k \). It is well known that negative correlations is one mechanism for entanglement between modes. Therefore, we would expect an enhancement of steering when \( j \) and \( k \) are entangled. However, we will see that a better tripartite steering is obtained when the correlations between the steering modes are positive rather than negative.

To see if tripartite steering exists in the system and especially what is the role of the input thermal noise, we evaluate the steering parameters given by the expressions \[^{15}^{17}\]. The general solutions, Eq. \(^{14}\), for the variances and the correlation functions involved in these expressions are simple enough to obtain the analytical expressions for the steering parameters. After straightforward but somewhat tedious manipulation of terms, we arrive at the following general solutions

\[
E_{m|ac} = \left( n_0 + \frac{1}{2} \right) \times \left\{ 1 - \frac{(2\bar{n} + 1)(\alpha^2 \epsilon^\alpha - \beta^2)^2}{(n_0 + \frac{1}{2}) + (2\bar{n} + 1) \left( \alpha^2 \epsilon^\alpha - \beta^2 \right)^2} \right\},
\]

\[
E_{c|am} = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 \beta^2 (2\bar{n} + 1)(\epsilon^\beta - 1)^2}{(n_0 + \frac{1}{2}) + \alpha^2 \beta^2 (2\bar{n} + 1)(\epsilon^\beta - 1)^2} \right],
\]

\[
E_{a|mc} = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 (2\bar{n} + 1)(\epsilon^{2\alpha} - 1)^2}{(n_0 + \frac{1}{2}) + \alpha^2 (2\bar{n} + 1)(\epsilon^{2\alpha} - 1)^2} \right], \quad (18)
\]

in which \( \bar{n} = (n_0 + n_1) \).

It is seen from Eq. \(^{18}\) that in the absence of the thermal noise \( n_0 = n_1 = 0 \) all the parameters are smaller than 1/2 indicating that a tripartite steering of each mode occurs immediately when the laser pulse is turned on, \( r_\alpha > 0 \). In the presence of the thermal noise the parameters are enhanced and thermal barriers appear that the tripartite steering of a given mode occurs at a finite \( r_\alpha \). It is interesting that the thermal barriers are determined by the thermal noise present at the steered mode only. For example, the parameter \( E_{m|ac} \) is enhanced by the factor \((n_0 + 1/2)\), the thermal noise at the steered mode \( m \). Similarly, the parameters \( E_{c|am} \) and \( E_{a|mc} \) are enhanced by the factor \((n_1 + 1/2)\), the thermal noise at the steered modes \( c \) and \( a \).

The steering parameters increase with the thermal noise but remain smaller than 1/2 at least for some maximum (threshold) values of \( n_0 \) and \( n_1 \) determined by \( r_\alpha \) and \( \alpha \). For example, in the case when the modes are equally affected by the thermal noise, \( n_0 = n_1 = n \), the parameters can be rewritten as

\[
E_{m|ac} = \frac{1}{2} + \frac{n - \alpha^2 (\epsilon^\alpha - 1)(\alpha^2 \epsilon^\alpha - \beta^2 + 1)}{2 [\alpha^2 (\epsilon^\alpha - 1) + 1]^2 - 1},
\]

\[
E_{c|am} = \frac{1}{2} + \frac{n - \alpha^2 \beta^2 (\epsilon^\beta - 1)^2}{2 \alpha^2 \beta^2 (\epsilon^\beta - 1)^2 + 1},
\]

\[
E_{a|mc} = \frac{1}{2} + \frac{n - \alpha^2 (\epsilon^{2\alpha} - 1)}{2 \alpha^2 (\epsilon^{2\alpha} - 1) + 1}. \quad (19)
\]
The threshold values of $n$ at which the tripartite steering of the modes disappears are given by

$$n_{th} = \alpha^2 (e^{r_n} - 1) (\alpha^2 e^{r_n} - \beta^2 + 1), \quad \text{for } E_{m|ac},$$

$$n_{th} = \alpha^2 \beta^2 (e^{r_n} - 1)^2, \quad \text{for } E_{c|am}$$

$$n_{th} = \alpha^2 (e^{2r_n} - 1), \quad \text{for } E_{a|mc}. \quad (20)$$

Note that the threshold values increase exponentially with $r_n$, thus tripartite steering can be preserved even in the presence of a large thermal noise.

Viewed as a function of $r_n$, the tripartite steering of the modes appears at a finite $r_n$. For example, for the case of the mode $m$, steering occurs at

$$r_n = \ln \left( 1 + \frac{\sqrt{n+1} - 1}{\alpha^2} \right), \quad (21)$$

which is different from zero when $n \neq 0$. The threshold value of $r_n$ increases with $n$. This shows that a larger thermal noise requires a larger squeezing to produce a tripartite steering.

The above considerations are illustrated in Fig. 2. In the absence of thermal noise ($n = 0$) the tripartite steering is present over the entire range of $r$ and perfect steering, $E_{m|ac} = 0$, is achieved for $r \to \infty$. In the presence of thermal noise ($n \neq 0$), there is a threshold for $r$ above which the steering takes place.

Although the steering parameters (19) go up with an increasing $n$, it does not prevent us from achieving perfect steering of the modes. It is easily verified from Eq. (19) that in the limit of large squeezing, $r \gg 1$, the steering parameters reduce to simple expressions

$$E_{m|ac} \approx \frac{n + 1/2}{2\alpha^4 e^{2r_n}},$$

$$E_{c|am} \approx \frac{n + 1/2}{2\alpha^2 \beta^2 e^{2r_n}}.$$ 

$$E_{a|mc} \approx \frac{n + 1/2}{2\alpha^2 e^{2r_n}}. \quad (22)$$

We see here that, even when the thermal noise is large, the steering parameters can be made negligibly small by increasing the squeezing parameter $r_n$. This shows that the effect of the thermal noise on the tripartite steering is not dramatic and perfect steering can be observed even for large $n$.

Figure 3 shows the variation of the parameter $E_{m|ac}$ with $\alpha^2$ and $n$ for $r = 15$. Note that $\alpha^2$ depends on the relative strength of the coupling constants $G$ and $G_a$ between the modes. For small $\alpha^2$, corresponding to the parametric type interaction dominating over the beamsplitter type interaction, we see that perfect steering can be observed over the entire range of $n$. This tendency continues until at $\alpha^2 \approx 2$ the degree of steering starts to decrease and becomes independent of $\alpha^2$ for $\alpha^2 \gg 1$. A considerable tripartite steering still is present even for large $n$. In order to see it explicitly, we take the limit of $\alpha^2 \gg r$ and expand the exponents appearing in the expressions (19) into Taylor series and obtain

$$E_{m|ac} \approx \frac{n + 1/2}{2(r + 1)^2 - 1},$$

$$E_{c|am} \approx \frac{n + 1/2}{2\beta^2 r^2/\alpha^2 + 1},$$

$$E_{a|mc} \approx \frac{n + 1/2}{4r + 1}. \quad (23)$$

We see that the steering parameter $E_{m|ac}$ are independent of $\alpha^2$. What this means is that the tripartite steering can be present over a large range of $n$ even if the beamsplitter type interaction, determined by $G_a$, is comparable to the parametric type interaction, determined by $G$.

Let us now comment about the dependence of the steering parameters on the sign of the correlations between the steering modes. In steering of the mode $m$ by the pair $\{ac\}$, Eq. (19), the correlation between the steering modes is described by the correlation function $\langle P_m, X_c \rangle$. In steering of the mode $c$, Eq. (19), the correlation between the steering modes is described by
\( \langle P_a, X_m \rangle \). According to Eq. (13), \( \langle P_a, X_c \rangle \) is positive whereas \( \langle P_c, X_m \rangle \) is negative. This suggests that the involvement of the negative correlation should result in a better steering of the mode \( c \) by the pair \( \{am\} \) than the mode \( m \) by the pair \( \{ac\} \). However, this is not the case, a negative correlation between the steering modes not necessarily leads to a better steering. To demonstrate this feature we take the ratio \( E_{c|am}/E_{m|ac} \) and find

\[
\frac{E_{c|am}}{E_{m|ac}} = 1 + \frac{2\alpha^2 (e^{2r_m} - 1)}{2\alpha^2 (e^{r_m} - 1)^2 + 1}.
\] (24)

Obviously the ratio is always greater than 1, so \( E_{c|am} > E_{m|ac} \). This implies that a negative rather than a positive correlation between steering modes reduces the ability of the modes for steering. It should be mentioned that the negative correlation between two modes may result in an entanglement between these modes. Thus, one can conclude that entanglement between the steering modes leads to a reduction of the ability of these modes to steer the other mode.

**B. Collective tripartite steering**

We now turn to an interesting problem of collective tripartite steering which occurs when a given mode \( i \) is steered by the remaining modes collectively, \( E_{i|jk} < 1/2 \), and simultaneously is not steered by each of the modes alone, \( E_{i|j} \geq 1/2 \) and \( E_{i|k} \geq 1/2 \).

Therefore, to examine the occurrence of collective tripartite steering we must look at the properties of the bipartite steering parameters \( E_{i|jk} \) defined in Eq. (2). They are readily evaluated using the solutions for the variances and correlation functions of the output fields, Eq. (14). We then obtain analytical expressions for the bipartite steering parameters. For the case of steering the cavity mode \( a \), we have

\[
E_{a|m} = \Delta(X_a + u_m P_m) \Delta(P_a + u_m X_m) = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 (2n_1 + 1) (e^{2r_m} - 1)}{\Delta^2 P_m} \right],
\] (25)

and

\[
E_{a|c} = \Delta(X_a + u_c P_c) \Delta(P_a - u_c X_c) = \left( n_1 + \frac{1}{2} \right) \left[ 1 + \frac{\alpha^2 (2n_1 + 1) (e^{2r_m} - 1)}{\Delta^2 P_c} \right].
\] (26)

For steering of the atomic mode \( c \) by cavity mode \( a \), we get

\[
E_{c|a} = \Delta(X_c + u_a P_a) \Delta(P_a - u_a X_a) = \left( n_1 + \frac{1}{2} \right) \left[ 1 + \frac{\alpha^2 \beta^2 (2n_1 + 1) (e^{2r_a} - 1)^2}{\Delta^2 X_a} \right],
\] (27)

and by mirror mode \( m \)

\[
E_{c|m} = \Delta(X_c + u_m P_m) \Delta(P_m - u_m X_m) = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 \beta^2 (2n_1 + 1) (e^{2r_a} - 1)^2}{\Delta^2 X_m} \right].
\] (28)

Finally, for steering of the mirror mode \( m \) by cavity mode \( a \), we find

\[
E_{m|a} = \Delta(X_m + u_a P_a) \Delta(P_m + u_a X_a) = \left( n_0 + \frac{1}{2} \right) \left[ \frac{1 - \left( 1 - (2n_1 + 1) \frac{e^{r_a} - 1}{2} (2n_0 + 1) \frac{e^{r_a} + 1}{2} \right)^2}{\alpha^2 (2n_1 + 1) (e^{2r_a} - 1)} \right],
\] (29)

and by atomic mode \( c \)

\[
E_{m|c} = \Delta(X_m + u_c P_c) \Delta(P_m - u_c X_m) = \left( n_0 + \frac{1}{2} \right) \left[ \frac{1 + \left( 2n_1 + 1 \right)}{\alpha^2 (2n_1 + 1) (e^{2r_a} - 1)} \right].
\] (30)

First, we note that the bipartite parameters (25)-(30), in the limit \( r_a \to \infty \), satisfy the inequality

\[
E_{c|ij} E_{i|jk} = \left( n_1 + \frac{1}{2} \right)^2 \geq \frac{1}{4},
\] (31)

which shows that the bipartite steering properties are in accordance with the monogamy relation that mode \( i \) cannot be simultaneously steered by modes \( j \) and \( k \) [57].

From Eqs. (25-30), we find that the parameters \( E_{a|c} \) and \( E_{c|a} \) are always greater than 1/2. Therefore, we need only to consider the other four parameters in order to search for conditions to remove the bipartite steering. It should be noted that the behavior of the parameters \( E_{m|a} \) and \( E_{m|c} \), describing steering properties of the mode \( m \), is quite different that either \( E_{m|a} \) or \( E_{m|c} \) can be smaller than 1/2. It depends on whether \( \beta^2 < (2n_1 + 1)/(2n_0 + 1) \) or \( \beta^2 > 1 \). For \( \beta^2 < 1 \), corresponding to \( G_a < G/2 \), the parameter \( E_{m|c} \) is always greater than 1/2, while \( E_{m|a} \) can be reduced below 1/2. On the other hand, for \( \beta^2 > 1 \) this relationship is reversed and \( E_{m|c} \) is the parameter which can be reduced below 1/2.

Let us examine in details the dependence of these parameters on \( n_0 \) and \( n_1 \). In the absence of the thermal noise, \( n_0 = n_1 = 0 \), and then \( E_{a|m} \) and \( E_{c|m} \) are always smaller than 1/2. Moreover, depending on whether \( \beta < 1 \) or \( \beta > 1 \), either \( E_{m|a} \) or \( E_{m|c} \) can be always smaller than 1/2. Hence, in the absence of the thermal noise, bipartite steering always occurs.

This is illustrated in Fig. [3]. We see that the tripartite steering is accompanied by the bipartite steering over the entire range of \( r \). Thus, the collective steering does not
Figure 4. (Color online) Variation of the tripartite and the corresponding bipartite steering parameters with \( r \) for \( \alpha = 1.2 \) and \( n_0 = n_1 = 0 \).

occur. It is interesting to note that the bipartite steering occurs only between those modes which are coupled through the parametric interaction. For example, the modes \( a \) and \( m \) are directly coupled through the parametric interaction and it is apparent from the figure that the modes steer each other. There is no steering between the modes \( a \) and \( c \) since the modes are coupled through the beam-splitter type interaction. The mode \( m \) steers the mode \( c \) due to the indirect coupling through the parametric interaction. Note an asymmetry in the steering between the modes \( m \) and \( c \) that the mode \( c \) is steered by \( m \) (\( E_{cm} < 1/2 \) shown in Fig. 4(c)) but the mode \( m \) is not steered by \( c \) (\( E_{mc} > 1/2 \) shown in Fig. 4(a)).

In the presence of the thermal noise, the steering properties of the modes change dramatically. When all modes are affected by thermal noise with equal average numbers of photons at each mode, \( n_0 = n_1 = n \neq 0 \), there are minimum (threshold) values for \( n \) at which the bipartite steering parameters become greater than \( 1/2 \). It is easy to see from Eqs. (26) and (29), and Eqs. (30) and (28) that the minimum (threshold) values for \( n \) are

\[
\begin{align*}
n_{th} &= \frac{\alpha^2 (e^{2r_{\alpha}} - 1)}{1 + 2\alpha^2 \beta^2 (e^{r_{\alpha}} - 1)^2}, \quad \text{for } E_{am}, \\
n_{th} &= \frac{\alpha^2 (e^{r_{\alpha}} - 1) \left[ \alpha^2 + (1 - \beta^2) e^{r_{\alpha}} \right]}{1 + 2\alpha^2 \beta^2 (e^{r_{\alpha}} - 1)^2}, \quad \text{for } E_{ma}, \\
n_{th} &= \frac{\alpha^2 (e^{r_{\alpha}} - 1) \left[ (\beta^2 - 1) e^{r_{\alpha}} - \alpha^2 \right]}{1 + 2\alpha^2 (e^{2r_{\alpha}} - 1)}, \quad \text{for } E_{mc}, \\
n_{th} &= \frac{\alpha^2 (e^{r_{\alpha}} - 1)^2}{1 + 2\alpha^2 (e^{2r_{\alpha}} - 1)}, \quad \text{for } E_{cm},
\end{align*}
\]

Note that, in contrast with the corresponding threshold values for tripartite steering, Eq. (20), the above threshold values for bipartite steering do not increase exponentially with \( r \). In other words, the threshold values cannot be made arbitrarily large, they rather saturate at finite values as \( r \to \infty \). This makes it possible to wipe out the bipartite steering even at small \( n \). This is demonstrated in Fig. 5 which shows the variation of the threshold value of \( n \) with the squeezing parameter \( r \) for \( \alpha = 1.2 \), corresponding to \( \beta < 1 \), and \( \alpha = 2 \), corresponding to \( \beta > 1 \).

For \( n \) beyond \( n = 2 \), which is above the thresholds defined by Eq. (32), all bipartite steering parameters then go above \( 1/2 \) that the bipartite steering becomes impos-
Figure 6. (Color online) Variation of the bipartite and tripartite steering parameters with \( r = G\tau \) for \( \alpha = 1.2 \) and the thermal noise present at three modes with equal average number of photons, \( n_0 = n_1 = 2 \). The shaded region marks the range of the squeezing parameter \( r \) over which a given mode can be steered collectively by the remaining two modes.

Figure 7. (Color online) Variation of the bipartite and tripartite steering parameters with \( r = G\tau \) for \( \alpha = 1.2 \) and the thermal noise present only at the mirror mode, \( n_0 = 4 \) and \( n_1 = 0 \).

It is clear from the figure that unlike the tripartite steering, the bipartite steering can be removed by the thermal noise.

One can notice from Fig. 6 that in the case of thermal noise affecting all three modes \( (n_0 = n_1) \), the tripartite and collective tripartite steering occur over the same range of \( r \). The situation differs if initially only one or two modes of the system were in the thermal state, either \( n_0 \neq 0, n_1 = 0 \) or \( n_0 = 0, n_1 \neq 0 \). A close look at the parameters (25)-(30) reveals that similar to the tripartite steering, the bipartite steering parameters are mostly affected by the thermal noise present initially in the steered mode. Namely, the steering of the mode \( m \) is limited by the factor \( (n_0 + 1/2) \), whereas the steering of the modes \( a \) and \( c \) is limited by the factor \( (n_1 + 1/2) \). This clearly shows that the bipartite steering can be wiped out only by the thermal noise present in the steered mode. Equa-
tions \(24), (30)\) also show that the thermal noise at the steering mode has a marginal effect on the bipartite steering. It is particularly well seen in the case of \(n_0 \gg 1\), in which the steering parameters \(E_{a|m}\), Eq. (25), and \(E_{c|m}\), Eq. (28), can be simplified to

\[
E_{a|m} = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 e^{2\alpha - 1}}{(\alpha^2 e^{2\alpha - 2})^2} \right], \tag{33}
\]

and

\[
E_{c|m} = \left( n_1 + \frac{1}{2} \right) \left[ 1 - \frac{\alpha^2 \beta^2 (e^{2\alpha - 1})^2}{(\alpha^2 e^{2\alpha - 2})^2} \right]. \tag{34}
\]

Evidently, the parameters are independent of \(n_0\), and could be larger than 1/2 only if \(n_1 \neq 0\).

The variation of the bipartite and tripartite steering parameters with \(r = G\tau\) for \(\alpha = 4\) and \(\alpha = 1.2\) and the thermal noise present at two modes, the cavity and atomic modes, \(n_0 = 0\) and \(n_1 = 4\). The shaded region marks the range of the squeezing parameter \(r\) over which a given mode can be steered collectively by the remaining two modes. The narrower green region (before the blue point) in (a) marks the range of \(r\) where the tripartite steering is present but the collective steering can never occur.

Figure 8. (Color online) Variation of the bipartite and tripartite steering parameters with \(r = G\tau\) for \(\alpha = 4\) and the thermal noise initially present only in the mirror mode, \(n_0 = 4\) and \(n_1 = 0\). The shaded region marks the range of the squeezing parameter \(r\) over which the mode \(m\) can be steered collectively by the modes \(a\) and \(c\).

Figure 9. (Color online) Variation of the bipartite and tripartite steering parameters with \(r = G\tau\) for \(\alpha = 1.2\) and the thermal noise present at two modes, the cavity and atomic modes, \(n_0 = 0\) and \(n_1 = 4\). The shaded region marks the range of the squeezing parameter \(r\) over which a given mode can be steered collectively by the remaining two modes. The narrower green region (before the blue point) in (a) marks the range of \(r\) where the tripartite steering is present but the collective steering can never occur.

Figure 9. (Color online) Variation of the bipartite and tripartite steering parameters with \(r = G\tau\) for \(\alpha = 1.2\) and the thermal noise present at two modes, the cavity and atomic modes, \(n_0 = 0\) and \(n_1 = 4\). The shaded region marks the range of the squeezing parameter \(r\) over which a given mode can be steered collectively by the remaining two modes. The narrower green region (before the blue point) in (a) marks the range of \(r\) where the tripartite steering is present but the collective steering can never occur.
the tripartite steering cannot be always regarded as the collective steering. In the opposite situation, shown in Fig. 9(b) where initially the steered mode and steering mode \( c \), and in Fig. 9(c) where initially the steered mode \( c \) and steering mode \( a \) were in the thermal state, both tripartite and collective steering occur in the same range of \( r \). Thus, if the steered and one of the steering modes were initially in the thermal state, the tripartite steering could coincide with the collective steering with appropriate values of \( \alpha \) and \( n_1 \).

We may summarize that the results presented in Figs. 6-9 clearly show that tripartite steering always occurs in the system regardless of the initial state of the modes, but collective tripartite steering can occur only if the modes are initially in a thermal state. The questions whether the tripartite steering coincides with the collective steering depends strongly on the redistribution of the thermal noise between the modes.

In closing this section, we briefly comment on the role of collective tripartite steering in security of hybrid quantum networks as a resource for quantum secret sharing. Suppose Alice wishes to transmit a secret message to two parties, Bob and Charlie. Before transmitting the message, Alice may send a quantum encryption key separately to Bob and Charlie or she can distribute the key among them, so in the later case Bob and Charlie must collaborate to decipher the message. The important feature is that steering ability, unlike ordinary entanglement, is constrained by the violation of inferred Heisenberg relation, \( \Delta_{m,j} X_i \Delta_{n,f,j} P_i \geq \frac{1}{2} \), which cannot be performed by classical means. When receiving the message from Alice (the result of Alice’s measurement), Bob and Charlie have to deduce the value of the amplitude of Alice’s system by demonstrating the violation of inferred Heisenberg uncertainty relation by collective measurements on their systems.

V. CONCLUSIONS

We have studied the steering properties of the bosonic modes of a hybrid pulsed cavity optomechanical system composed of a single-mode cavity with a movable fully reflective mirror and containing an ensemble of two-level atoms. The cavity mode was driven by light pulses and the variances and correlation functions of the amplitudes of the output modes were evaluated. The treatment was restricted to the bad cavity limit under which the adiabatic approximation was made of a slowly varying amplitude of the cavity mode. The laser pulses were assumed to be strong and blue detuned to the cavity and the atomic resonance frequencies. The solutions were then used to obtain analytic expressions for the steering parameters. We were particularly interested in the dependence of the bipartite and tripartite steering on the initial state of the modes. We have found that the initial thermal noise presented in the modes is more effective in destroying the bipartite rather than the tripartite steering. In fact, the tripartite steering can persist even for a large thermal noise. When the bipartite steering is destroyed then the existing tripartite steering can be regarded as collective tripartite steering. A detailed analysis has shown that the occurrence of the collective tripartite steering is highly sensitive to number of modes being initially in thermal states and to whether the noise affected mode appears as a steered or steering mode. In the case where initially only a steered mode is in a thermal state, the bipartite steering could be completely destroyed and then the existing tripartite steering corresponds to the collective steering. On the other hand, when initially only the steering modes were in thermal states, the collective steering of the remaining mode may occur but it is present in a very restricted range of the interaction time. In particular, the collective steering is absent when only one of the steering modes was initially in a thermal state. When initially both steering modes were in a thermal state, a collective steering may occur in a less restricted range of the interaction time. When all modes are initially in thermal states, the bipartite steering can be destroyed completely leaving the collective steering as the only steering present in the system. We note that the collective steering has potential application for quantum secret sharing in a hybrid quantum network.

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Appendix A

1. Bipartite steering parameters and optimal gain factors

The parameter \( E_{m|a} \) determining bipartite steering of the mirror mode \( m \) by the cavity mode \( a \) is determined by

\[
E_{m|a} = \Delta [X_m + u_a P_a] \Delta [P_m + u_a X_a] = \Delta^2 X_m + u_a^2 \Delta^2 P_a + 2 u_a \langle X_m, P_a \rangle ,
\]

(A1)
where $\langle X_m, P_a \rangle = \frac{1}{2} \langle X_m P_a + P_a X_m \rangle - \langle X_m \rangle \langle P_a \rangle$. This can be minimized with the optimal weight factor

$$u_a = \frac{\langle X_m, P_a \rangle}{\Delta^2 P_a}. \tag{A2}$$

The steering of the mirror mode $m$ by the atomic mode $c$ is given by

$$E_{m|c} = \Delta [X_m + u_c X_c] \Delta [P_m - u_c P_c]$$
$$= \Delta^2 X_m + u_c^2 \Delta^2 X_c + 2 u_c \langle X_m, X_c \rangle, \tag{A3}$$

which can be minimized by the optimal weight factor

$$u_c = -\frac{\langle X_m, X_c \rangle}{\Delta^2 X_c}. \tag{A4}$$

The steering of the cavity mode $a$ by the mirror mode $m$ is given by

$$E_{a|m} = \Delta [X_a + u_m X_m] \Delta [P_a + u_m P_m]$$
$$= \Delta^2 X_a + u_m^2 \Delta^2 P_m + 2 u_m \langle X_a, P_m \rangle, \tag{A5}$$

which can be minimized by the optimal weight factor

$$u_m = -\frac{\langle X_a, P_m \rangle}{\Delta^2 P_m}. \tag{A6}$$

The steering of the cavity mode $a$ by the atomic mode $c$ is given by

$$E_{a|c} = \Delta [X_a + u_c P_c] \Delta [P_a - u_c X_c]$$
$$= \Delta^2 X_a + u_c^2 \Delta^2 P_c + 2 u_c \langle X_a, P_c \rangle, \tag{A7}$$

which can be minimized by the optimal weight factor

$$u_c = -\frac{\langle X_a, P_c \rangle}{\Delta^2 P_c}. \tag{A8}$$

The steering of atomic mode $c$ by the cavity mode $a$ is given by

$$E_{c|a} = \Delta [X_c + u_a P_a] \Delta [P_c - u_a X_a]$$
$$= \Delta^2 X_c + u_a^2 \Delta^2 P_a + 2 u_a \langle X_c, P_a \rangle, \tag{A9}$$

which can be minimized by the optimal weight factor

$$u_a = -\frac{\langle X_c, P_a \rangle}{\Delta^2 P_a}. \tag{A10}$$

The steering of atomic mode $c$ by the mirror mode $m$ is given by

$$E_{c|m} = \Delta [X_c + u_m X_m] \Delta [P_c - u_m P_m]$$
$$= \Delta^2 X_c + u_m^2 \Delta^2 X_m + 2 u_m \langle X_c, X_m \rangle, \tag{A11}$$

which can be minimized by the optimal weight factor

$$u_m = -\frac{\langle X_c, X_m \rangle}{\Delta^2 X_m}. \tag{A12}$$

2. Tripartite steering parameters and optimal gain factors

The steering of the mirror mode $m$ by the group $\{ac\}$ is determined by the parameter

$$E_{m|ac} = \Delta [X_m + (u_a P_a + u_c X_c)] \Delta [P_m + (u_a X_m - u_c P_c)]$$
$$= \Delta^2 X_m + u_a^2 \Delta^2 P_a + u_c^2 \Delta^2 X_c + 2 u_a \langle X_m, P_a \rangle + 2 u_c \langle X_m, X_c \rangle + 2 u_a u_c \langle P_a, X_c \rangle, \tag{A13}$$

which can be minimized by the optimal weight factors

$$u_a = \frac{\langle X_c, X_m \rangle}{\Delta^2 X_m}, \tag{A14}$$

$$u_c = \frac{\langle X_m, P_a \rangle - \langle X_m, X_c \rangle \langle P_a, X_c \rangle}{\langle P_a, X_c \rangle^2 - \Delta^2 P_m \Delta^2 X_c}. \tag{A15}$$

The parameter $E_{a|mc}$ determining tripartite steering of the cavity mode $a$ by the group $\{mc\}$ is given by

$$E_{a|mc} = \Delta [X_a + (u_m P_m + u_c P_c)] \Delta [P_a + (u_m X_m - u_c X_c)]$$
$$= \Delta^2 X_a + u_m^2 \Delta^2 P_m + u_c^2 \Delta^2 P_c + 2 u_m \langle X_a, P_m \rangle + 2 u_c \langle X_a, P_c \rangle + 2 u_m u_c \langle P_m, P_c \rangle, \tag{A16}$$

which can be minimized by the optimal weight factors

$$u_m = \frac{\Delta^2 P_m \langle X_a, P_m \rangle - \langle X_a, P_c \rangle \langle P_m, P_c \rangle}{\langle P_m, P_c \rangle^2 - \Delta^2 P_m \Delta^2 X_c}, \tag{A17}$$

$$u_c = \frac{\Delta^2 P_m \langle X_a, P_c \rangle - \langle X_m, P_a \rangle \langle P_m, P_c \rangle}{\langle P_m, P_c \rangle^2 - \Delta^2 P_m \Delta^2 X_c}. \tag{A18}$$

Finally, the parameter $E_{c|am}$ determining tripartite steering of the atomic mode $c$ by the group $\{am\}$ is given by

$$E_{c|am} = \Delta [X_c + (u_a P_a + u_m X_m)] \Delta [P_c - (u_a X_a + u_m P_m)]$$
$$= \Delta^2 X_c + u_a^2 \Delta^2 P_a + u_m^2 \Delta^2 X_m + 2 u_a \langle X_c, P_a \rangle + 2 u_m \langle X_c, X_m \rangle + 2 u_a u_m \langle P_a, X_m \rangle, \tag{A19}$$

which can be minimized by the optimal weight factors

$$u_a = \frac{\Delta^2 X_c \langle X_c, P_a \rangle - \langle X_c, X_m \rangle \langle P_a, X_m \rangle}{\langle P_a, X_m \rangle^2 - \Delta^2 P_a \Delta^2 X_m}, \tag{A20}$$

$$u_m = \frac{\Delta^2 P_a \langle X_c, X_m \rangle - \langle X_c, P_a \rangle \langle P_a, X_m \rangle}{\langle P_a, X_m \rangle^2 - \Delta^2 P_a \Delta^2 X_m}. \tag{A21}$$

[1] R. F. Werner, Phys. Rev. A 40, 4277 (1989).

[2] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys.
