GENERIC SIMPLICITY OF RESONANCES IN OBSTACLE SCATTERING

HAOREN XIONG

Abstract. We show that all resonances in Dirichlet obstacle scattering (in \( \mathbb{C} \) in odd dimensions and in the logarithmic cover of \( \mathbb{C} \setminus \{0\} \) in even dimensions) are generically simple in the class of obstacles with \( C^k \) (and \( C^\infty \)) boundaries, \( k \geq 2 \).

1. Introduction

The evolution of eigenvalues of second order elliptic operators under boundary perturbations have been studied through different perspectives since Hadamard [Ha08]. Uhlenbeck [Uh76] proved generic properties of eigenvalues and eigenfunctions of second order elliptic operators with respect to variation of the domain for general boundary conditions. Henry [He05] developed a general theory on perturbation of domains for second order elliptic operators. In this paper we prove that a generic boundary perturbation in obstacle scattering for the Dirichlet Laplacian splits the multiplicities of all resonances in both odd and even dimensions. We formulate the problem as follows:

Suppose that \( \mathcal{O} \subset \mathbb{R}^n \) is a bounded open set such that \( \partial \mathcal{O} \) is a \( C^k \) \( (k \geq 2) \) hypersurface in \( \mathbb{R}^n \). Let \( \Delta_\mathcal{O} \) be the self-adjoint Dirichlet Laplacian on \( \mathbb{R}^n \setminus \mathcal{O} \) with domain

\[
\mathcal{D}(\Delta_\mathcal{O}) := H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O}).
\] (1.1)

The resolvent of \(-\Delta_\mathcal{O}\),

\[
R_\mathcal{O}(\lambda) := (-\Delta_\mathcal{O} - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad \text{Im} \lambda > 0,
\]
continues meromorphically as an operator from \( L^2_{\text{comp}}(\mathbb{R}^n \setminus \mathcal{O}) \) to \( L^2_{\text{loc}}(\mathbb{R}^n \setminus \mathcal{O}) \) – see for instance Dyatlov–Zworski [DyZw19, §4.2] and a review in §2. When \( n \) is odd the continuation is to \( \lambda \in \mathbb{C} \) and when \( n \) is even to the logarithmic cover of \( \mathbb{C} \setminus \{0\} \):

\[
\Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\}).
\] (1.2)

We denote the set of poles of \( R_\mathcal{O}(\lambda) \) by \( \text{Res}(\mathcal{O}) \). The elements of \( \text{Res}(\mathcal{O}) \) are called scattering resonances for the obstacle \( \mathcal{O} \). We recall the following facts from [DyZw19, Theorem 4.19] (for \( n \) odd) and Christiansen [Ch17, §6] (for \( n \) even) that:

\[
0 \notin \text{Res}(\mathcal{O}), \text{ for } n \text{ odd}; \quad 0 \text{ is not a limit point of } \text{Res}(\mathcal{O}) \text{ for } n \text{ even.}
\] (1.3)
Thus for $\lambda \in \text{Res}(\mathcal{O})$, its multiplicity $m_\mathcal{O}(\lambda)$ satisfies
\[
m_\mathcal{O}(\lambda) := \text{rank} \int_\lambda R_\mathcal{O}(\zeta)d\zeta = \text{rank} \int_\lambda R_\mathcal{O}(\zeta)2\zeta d\zeta, \tag{1.4}
\]
where the integral is over a circle containing no other pole of $R_\mathcal{O}(\zeta)$ than $\lambda$, see [DyZw19, §4.2]. A resonance $\lambda \in \text{Res}(\mathcal{O})$ is called simple if $m_\mathcal{O}(\lambda) = 1$.

To describe the deformations of obstacles, we follow Pereira [Pe04] and introduce a set of $C^k$-smooth mappings ($k \geq 2$) which deforms the obstacle $\mathcal{O}$:
\[
\text{Diff}(\mathcal{O}) := \left\{ \Phi \in C^k(\mathbb{R}^n; \mathbb{R}^n) \text{ is a } C^k\text{-diffeomorphism : } \Phi(\partial \mathcal{O}) = \partial \Phi(\mathcal{O}), \text{ and } \Phi(x) = x, \forall |x| > R, \text{ for some } R > 0 \right\}. \tag{1.5}
\]
Let $X$ be the class of obstacles diffeomorphic to a fixed obstacle $\mathcal{O}_0$ (for example, $\mathcal{O}_0 = B_{\mathbb{R}^n}(0,1)$), that is,
\[
X = \{ \Phi(\mathcal{O}_0) : \Phi \in \text{Diff}(\mathcal{O}_0) \}. \tag{1.6}
\]
We introduce a topology in this set by defining a sub-basis of the neighborhoods of a given $\mathcal{O} \in X$ by
\[
\{ \Phi(\mathcal{O}) : \Phi \in \text{Diff}(\mathcal{O}), \|\Phi - \text{id}\|_{C^k(\mathbb{R}^n \setminus \mathcal{O})} < \varepsilon, \text{ with } \varepsilon \text{ sufficiently small.} \}
\]
Now we state the main result of this paper, by a generic set we mean an intersection of open dense sets:

**Theorem.** For any fixed obstacle $\mathcal{O}_0$ and the corresponding family $X$ given in (1.6), there exists a generic set $\mathcal{X} \subset X$ such that for every $\mathcal{O} \in \mathcal{X}$, all resonances $\lambda \in \text{Res}(\mathcal{O})$ are simple.

**Remark 1:** We should point out that an analogue of this result for Robin boundary condition (and in particular for the Neumann boundary condition) remains an open problem. The difficulty was overcome by Uhlenbeck [Uh72] in the case of Neumann eigenvalue problem in a bounded domain $\Omega$ by using Transversality Theorem in infinite dimensions and then deriving a contradiction from the equation $\nabla u \cdot \nabla v = \lambda uv$ on $\partial \Omega$ where $\lambda > 0$, $u,v \in C^2(\partial \Omega; \mathbb{R})$ and $uv \neq 0$ on an open dense subset of $\partial \Omega$, see also [He05, Example 6.4] for more details. In the case of obstacle scattering with Neumann boundary condition, this argument does not seem to apply for $\nabla u \cdot \nabla v = zuv$ when $u, v$ are complex-valued and $z$ is a complex resonance.

**Remark 2:** Klopp and Zworski [KlZw95] proved that a generic potential perturbation in black box scattering (for a definition see for instance [DyZw19, §4] and §2) splits the multiplicities of all resonances. This result was extended to scattering on asymptotically hyperbolic manifolds by Borthwick and Perry [BoPe02], in which the method of complex scaling used in [KlZw95] was replaced by Agmon’s perturbation theory of resonances [Ag98]. We will combine the strategies of [KlZw95] and [BoPe02]
in the proof of our theorem. However, the boundary perturbation produces additional difficulties.

The paper is organized as follows. In §2 we review the meromorphy of the resolvent $R_O(\lambda)$. More precisely, we show that $R_O(\lambda)$ admits a meromorphic continuation to $\lambda \in \mathbb{C}$ in odd dimensions, to $\lambda \in \Lambda$ in even dimensions, as an operator between some weighted Hilbert spaces (instead of $L^2_{\text{comp}} \to L^2_{\text{loc}}$) as a preparation for applying Agmon’s perturbation theory of resonances. In §3 we review Agmon’s perturbation theory of resonances [Ag98] in which the resonances are realized as eigenvalues of a non-self-adjoint operator on an abstractly constructed Banach space. We remark that the method of complex scaling is also capable of characterizing resonances in $\mathbb{C}$ in odd dimensions and resonances in $\Lambda$ in even dimensions with small argument, but Agmon’s method allows us to prove the generic simplicity of all resonances in the whole $\Lambda$ in even dimensions. In §4 we conjugate the Dirichelet Laplacian of the deformed obstacle $\Phi(O)$ by the pullback $\Phi^*$ to obtain an operator on the original domain $\mathbb{R}^n \setminus O$. As a result, the variation of the domain is transferred to the coefficients of the differential operator. In §5, we use Agmon’s theory to study the resonances of the deformed operators introduced in §4. The proof of the theorem is completed in §6 by adapting the strategy in [KlZw95] to the case of boundary perturbations.

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2. Meromorphic continuation

In this section we will follow [DyZw19, §4] to introduce a general class of compactly supported self-adjoint perturbations of the Laplacian in $\mathbb{R}^n$, $P$, which are called black box Hamiltonians.

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \quad (2.1)$$

where $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$ and $R_0$ is fixed. The corresponding orthogonal projections will be denoted by

$$u \mapsto u|_{B(0, R_0)} \text{ and } u \mapsto u|_{\mathbb{R}^n \setminus B(0, R_0)},$$

or simply by the characteristic function $1_L$ of the corresponding set $L$.

We now consider an unbounded self-adjoint operator

$$P : \mathcal{H} \to \mathcal{H} \text{ with domain } \mathcal{D}(P). \quad (2.2)$$

Assume that

$$\mathcal{D}(P)|_{\mathbb{R}^n \setminus B(0, R_0)} \subset H^2(\mathbb{R}^n \setminus B(0, R_0)), \quad (2.3)$$
and conversely, \( u \in \mathcal{D}(P) \) if \( u \in H^2(\mathbb{R}^n \setminus B(0,R_0)) \) and \( u \) vanishes near \( B(0,R_0) \):

\[
1_{B(0,R_0)}(P + i)^{-1} \text{ is compact.} \tag{2.4}
\]

We also assume that,

\[
1_{\mathbb{R}^n \setminus B(0,R_0)}Pu = -\Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \quad \text{for all } u \in \mathcal{D}(P). \tag{2.5}
\]

It is well known that the resolvent \( (P - \lambda^2)^{-1}, \operatorname{Im} \lambda > 0, \lambda^2 \notin \operatorname{Spec}_{\text{point}}(P), \) has a meromorphic continuation to \( \mathbb{C} \) when \( n \) is odd; to \( \Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\}) \) when \( n \) is even: as an operator \( (P - \lambda^2)^{-1} : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}, \) see [SjZw91, Theorem 1.1] and [DyZw19, Theorem 4.4]. However, we need to meromorphically continue \( (P - \lambda^2)^{-1} \) as an operator between some Banach spaces to apply Agmon’s method [Ag98] and prove our theorem. For that we define a weighted subspace of \( \mathcal{H} \) for any large constant \( A > 0, \)

\[
\mathcal{H}_0^A := \mathcal{H}_{R_0} \oplus e^{-A|x|}L^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{2.6}
\]

and a larger space containing \( \mathcal{H} \):

\[
\mathcal{H}_1^A := \mathcal{H}_{R_0} \oplus e^{A|x|}L^2(\mathbb{R}^n \setminus B(0,R_0)). \tag{2.7}
\]

The space \( \mathcal{D}_1^A(P) \) is defined using (2.7),

\[
\mathcal{D}_1^A(P) := \{u \in \mathcal{H}_1^A : \chi \in C^\infty_c(\mathbb{R}^n), \chi|_{B(0,R_0)} \equiv 1 \Rightarrow \chi u \in \mathcal{D}(P), \Delta((1 - \chi)u) \in \mathcal{H}_1^A\}. \tag{2.8}
\]

We also denote the strips in \( \mathbb{C} \) by

\[
T_A := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > -A\}, \tag{2.9}
\]

and a family of subsets of \( \Lambda \) by

\[
S_m := \{\lambda \in \Lambda : -m\pi < \arg \lambda < m\pi\}, \quad m \in \mathbb{N}_+,
\]

\[
\Lambda_A := \{\lambda \in \Lambda : 0 < \arg \lambda < \pi\} \cup \{\lambda \in S_{\lfloor A \rfloor} : |\lambda| < A\}. \tag{2.10}
\]

We are now ready to state the main result of this section:

**Proposition 2.1.** Suppose that \( P \) is a black box Hamiltonian. Then

\[
R(\lambda) := (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}(P) \text{ is meromorphic for } \operatorname{Im} \lambda > 0. \tag{2.11}
\]

Moreover, when \( n \) is odd, the resolvent extends to a meromorphic family

\[
R(\lambda) : \mathcal{H}_0^A \to \mathcal{D}_1^A(P), \quad \lambda \in T_A. \tag{2.12}
\]

When \( n \) is even (2.12) holds with \( T_A \) replaced by \( \Lambda_A. \)
are meromorphic in $H$ asymptotics of $L$ operator between written in terms of the Hankel functions of the first kind:

$$J \equiv x$$

The proof is the same as the one of [DyZw19, Theorem 4.4]. The only difference is that unlike the free resolvent $R_0(\lambda) := (-\Delta - \lambda^2)^{-1}$ meromorphically continued as an operator between $L^2_{\text{comp}}(\mathbb{R}^n)$ and $H^2_{\text{loc}}(\mathbb{R}^n)$ there, we have to show that

$$\lambda \mapsto R_0(\lambda) : e^{-A|x|}L^2(\mathbb{R}^n) \to e^{-A|x|}L^2(\mathbb{R}^n),$$

$$\lambda \mapsto [\Delta, \chi]R_0(\lambda) : e^{-A|x|}L^2(\mathbb{R}^n) \to L^2_{\text{comp}}(\mathbb{R}^n), \quad \forall \chi \in C_c^\infty(\mathbb{R}^n),$$

(2.13)

are meromorphic in $\lambda \in T_A$ when $n$ is odd, $\lambda \in \Lambda_A$ when $n$ is even.

Denote by $R_0(\lambda, x, y)$ the Schwartz kernel of the free resolvent $R_0(\lambda)$, which can be written in terms of the Hankel functions of the first kind:

$$R_0(\lambda, x, y) = c_n \lambda^{n-2}(\lambda|x-y|)^{-\frac{\alpha}{2}}H^{(1)}_{\frac{\alpha}{2}-1}(\lambda|x-y|).$$

(2.14)

We recall some well known facts about $R_0(\lambda, x, y)$ as follows, see for instance [DyZw19, §3.1] for a detailed account. When $n$ is odd, (2.14) admits a finite expansion:

$$R_0(\lambda, x, y) = \lambda^{n-2}e^{i\lambda|x-y|} \sum_{j=\frac{n-1}{2}}^{n-2} \frac{C_{n,j}}{(\lambda|x-y|)^j}. \quad (2.15)$$

For $x \neq y$ this form extends meromorphically to $\lambda \in \mathbb{C}$. When $n$ is even, using the relation:

$$R_0(e^{i\pi} \lambda, x, y) - R_0(\lambda, x, y) = c_n \ell(-1)^{\frac{\alpha}{2}(\ell+1)} \lambda^{\frac{\alpha}{2}}|x-y|^{-\frac{\alpha}{2}}J_{\frac{\alpha}{2}}(\lambda|x-y|),$$

(2.16)

where $J_{\ell}(z)$ is the Bessel function, we see that $R_0(\lambda, x, y), x \neq y$ extends to $\lambda \in \Lambda$. In view of (2.15), when $n$ is odd we have the upper bounds for $\lambda \in \mathbb{C}$:

$$|R_0(\lambda, x, y)| \leq \begin{cases} C(\lambda)|x-y|^{2-n}, & |x-y| \leq |\lambda|^{-1}; \\ C(\lambda)e^{-\text{Im} \lambda|x-y|} |\lambda|^{\frac{\alpha}{2}} |x-y|^{\frac{1-n}{2}}, & |x-y| \geq |\lambda|^{-1}. \end{cases}$$

(2.17)

For $n$ even, $n \neq 2$, the bounds (2.17) hold for $-\pi < \text{arg} \lambda < 2\pi$. This follows from the asymptotics of $H^{(1)}_{\ell}(z)$, see also Galkowski–Smith [GaSm14] for more details. Using (2.16) and the following formulas about $J_{\ell}(z)$:

$$J_{\ell}(z) \sim \frac{1}{\Gamma(d+1)} \left(\frac{z}{2}\right)^d, \quad \text{as } z \to 0, \text{ when } d \in \mathbb{Z},$$

$$J_{\ell}(z) = \sqrt{\frac{2}{\pi z}} \left(\cos \left(z - \frac{d\pi}{2} - \frac{\pi}{4}\right) + e^{\text{Im} z}O(|z|^{-1})\right), \quad \text{as } |z| \to \infty, |\text{arg } z| < \pi.$$

we can extend (2.17) to any $\lambda \in \Lambda$, arg $\lambda \leq -\pi$ or arg $\lambda \geq 2\pi$:

$$|R_0(\lambda, x, y)| \leq \begin{cases} C(\lambda)|x-y|^{2-n}, & |x-y| \leq |\lambda|^{-1}; \\ C(\lambda)e^{\text{Im} \lambda|x-y|} |\lambda|^{\frac{\alpha}{2}} |x-y|^{\frac{1-n}{2}}, & |x-y| \geq |\lambda|^{-1}. \end{cases}$$

(2.18)
In the case that \( n = 2 \), \(|x-y|^{2-n}\) in (2.18) is replaced by \(-\ln |x-y|\) when \(|x-y| \leq |\lambda|^{-1}\).

Now we can conclude from (2.17) and (2.18) that for any (except possible poles) \( \lambda \in T_A \) when \( n \) is odd, \( \lambda \in \Lambda_A \) when \( n \) is even,

\[
\sup_{x} \int_{\mathbb{R}^n} e^{-A|x-y|} |R_0(\lambda, x, y)| dy, \quad \sup_{y} \int_{\mathbb{R}^n} e^{-A|x-y|} |R_0(\lambda, x, y)| dx < \infty.
\]

Using the formula about derivatives of the Hankel functions

\[
\frac{d}{dz} H^{(1)}_m(z) = H^{(1)}_m(z) - \frac{m}{z} H^{(1)}_m(z),
\]

we can also conclude from the bounds (2.17) that

\[
\sup_{x} \int_{\mathbb{R}^n} |[\Delta_x, \chi] R_0(\lambda, x, y)| e^{-A|y|} dy, \quad \sup_{y} \int_{\mathbb{R}^n} |[\Delta_x, \chi] R_0(\lambda, x, y)| e^{-A|y|} dx < \infty.
\]

Hence (2.13) follows by the Schur test.

### 3. Agmon’s Perturbation Theory of Resonances

In this section we adapt Agmon’s perturbation theory of resonances [Ag98] to study resonances in obstacle scattering, in which resonances are realized as eigenvalues of a non-self-adjoint operator on an abstractly constructed Banach space.

Let \( \mathcal{O} \subset \mathbb{R}^n \) be an obstacle and \( \Delta_\mathcal{O} \) be the corresponding self-adjoint Dirichlet Laplacian on \( \mathbb{R}^n \setminus \mathcal{O} \), whose resolvent admits a meromorphic continuation by Proposition 2.1. More precisely, for any fixed obstacle \( \mathcal{O} \) and constant \( A > 0 \) let

\[
B_0 := e^{-A|x|} L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad B_1 := e^{A|x|} L^2(\mathbb{R}^n \setminus \mathcal{O}),
\]

the resolvent of \(-\Delta_\mathcal{O}\) extends to a meromorphic family

\[
(-\Delta_\mathcal{O} - \lambda^2)^{-1} : B_0 \to \mathcal{D}_1(\mathcal{O}) \subset B_1, \quad \lambda \in T_A \text{ when } n \text{ odd, } \lambda \in \Lambda_A \text{ when } n \text{ even},
\]

where \( \mathcal{D}_1(\mathcal{O}) \) is the same as (2.8) except that \( B_1 \) replaces \( \mathcal{H}_1 \) there:

\[
\mathcal{D}_1(\mathcal{O}) = \left\{ u \in B_1 \cap H^2_{\text{loc}}(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial \mathcal{O}} = 0, \Delta u \in B_1 \right\}, \quad (3.1)
\]

and \( T_A, \Lambda_A \) are given by (2.9), (2.10). The poles in this meromorphic continuation are called scattering resonances for the obstacle \( \mathcal{O} \).

In view of (1.3), resonances lie in \( T_A \setminus i[0, \infty) \) when \( n \) is odd. We consider the map:

\[
T_A \setminus i[0, \infty) \ni \lambda = re^{i\theta} \mapsto z = r^2e^{2i\theta} = \lambda^2 \in \Lambda,
\]

which is invertible. Throughout this section, we will replace parameter \( \lambda \) by \( z \) with \( z = \lambda^2 \) defined above. We introduce the image of \( T_A \setminus i[0, \infty) \) or \( \Lambda_A \) under this map:

\[
D_+ := \begin{cases} \{ \lambda^2 \in \Lambda : \lambda \in T_A \setminus \{0\}, \ -\frac{3\pi}{2} < \arg \lambda < \frac{\pi}{2} \} & \text{when } n \text{ is odd;} \\ \{ z : 0 < \arg z < 2\pi \} \cup \{ z \in S_{2|A|} : |z| < A^2 \} & \text{when } n \text{ is even}. \end{cases} \quad (3.2)
\]
We write the resolvent of $\Delta_O$ as follows:

$$R(z) := (-\Delta_O - z)^{-1} : B_0 \to B_1, \quad z \in D_+,$$

which is a meromorphic family by Proposition 2.1. We denote by $\text{Res}(O)$, the poles of $R(z)$, $z \in D_+$, which is also the image of the resonances under the map $\lambda \mapsto z = \lambda^2$.

We note that $-\Delta_O$ as an operator acting on $B_1$ is closable. Denote by $P_1$ the closure of $-\Delta_O$ in $B_1$, by (3.1) we have

$$P_1 = -\Delta : B_1 \to B_1 \quad \text{with domain} \quad D_1(O).$$

Let us take $B = L^2(\mathbb{R}^n \setminus O)$, $D = \{z \in \mathbb{C} : \text{Im}z > 0\}$. Then one can check that $P := -\Delta_O$ satisfies the abstract hypotheses of Agmon’s theory:

**Hypothesis 3.1.**

(i) $P$ is a closed, densely defined operator acting in some Banach space $B$;

(ii) The resolvent $(P - z)^{-1}$ is a meromorphic family of operators in $\mathcal{L}(B)$ for $z \in D$;

(iii) There are two reflexive Banach spaces $B_0$ and $B_1$ such that $B_0 \subset B \subset B_1$;

(iv) $P$ as an operator on $B_1$ is closable, and denoting the closure of $P$ in $B_1$ by $P_1$, the resolvent $(P_1 - z)^{-1}$ exists for some $z \in D$ as an operator in $\mathcal{L}(B_1)$;

(v) The resolvent $(P - z)^{-1}$ admits a meromorphic continuation from $D$ to $D_+$ as an operator in $\mathcal{L}(B_0, B_1)$.

(iv) can be seen from the following calculation

$$e^{-A|x|}(-\Delta - z)e^{A|x|} = -\Delta - \frac{2Ax}{|x|} \cdot \nabla - \frac{(n-1)A}{|x|} - A^2 - z,$$

which is invertible as an operator from $\mathcal{D}(\Delta_O)$ to $L^2(\mathbb{R}^n \setminus O)$ for $z \in D$, $\text{Im}z \gg A^2$.

Now we fix a resonance $z_0 \in \text{Res}(O) \subset D_+$, $z_0 \neq 0$ then choose $\Sigma$, a bounded domain containing $z_0$, with a $C^1$ boundary $\Gamma$, satisfying

(i) $\Sigma \subset D_+$;  
(ii) $\Gamma \cap \text{Res}(O) = \emptyset$;  
(iii) $\Sigma \cap D \neq \emptyset$.  

(3.4)

Having chosen $\Sigma$ we denote by $B_\Gamma$, the subspace of $B_1$ consisting of elements $f$, admitting a representation of the form:

$$f = g + \int_{\Gamma} R(\zeta) \varphi(\zeta) d\zeta, \quad g \in B_0, \ \varphi \in C(\Gamma; B_0),$$

(3.5)

We recall [Ag98] that $B_\Gamma$ is a Banach space with the norm

$$\|f\|_{B_\Gamma} := \inf_{g,\varphi}(\|g\|_{B_0} + \|\varphi\|_{C(\Gamma; B_0)}),$$

(3.6)

where the infimum is taken over all $g \in B_0$ and $\varphi \in C(\Gamma; B_0)$ which verify (3.5). Then $B_0 \subset B_\Gamma \subset B_1$ are continuous inclusions. Agmon [Ag98] also introduced a linear...
operator \(R_\Gamma(z)\) on \(B_\Gamma\) associated to any \(z \in \Sigma \setminus \text{Res}(\mathcal{O})\),
\[
R_\Gamma(z)f := \mathcal{R}(z)g + \int_\Gamma (\zeta - z)^{-1}(\mathcal{R}(\zeta) - \mathcal{R}(z))\varphi(\zeta)d\zeta,
\] (3.7)
where \(f \in B_\Gamma\) is given by (3.5). Under Hypothesis 3.1, Agmon [Ag98] showed that \(R_\Gamma(z)\) is a well-defined operator in \(\mathcal{L}(B_\Gamma)\), which is actually the resolvent of an operator \(P_\Gamma\) acting on \(B_\Gamma\):
\[
R_\Gamma(z) = (P_\Gamma - z)^{-1} \quad \text{for} \quad z \in \Sigma \setminus \text{Res}(\mathcal{O}),
\] (3.8)
where \(P_\Gamma\) is closed linear operator in \(B_\Gamma\) defined as follows:
\[
\mathcal{D}(P_\Gamma) = \text{Ran} \ R_\Gamma(w_0), \quad P_\Gamma u = w_0u + f
\] (3.9)
for \(u = R_\Gamma(w_0)f \in \mathcal{D}(P_\Gamma)\), \(f \in B_\Gamma\). Here \(w_0\) is a fixed point in \(\Sigma \cap D\). Moreover, \(P_1\) extends \(P_\Gamma\) in the sense that
\[
\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O}), \quad P_\Gamma u = P_1u \quad \text{for} \quad u \in \mathcal{D}(P_\Gamma),
\] (3.10)
where \(\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O})\) is continuous if they are equipped with the graph norms:
\[
\|u\|_{\mathcal{D}(P_\Gamma)} := \|u\|_{B_\Gamma} + \|P_\Gamma u\|_{B_\Gamma}; \quad \|u\|_{\mathcal{D}_1(\mathcal{O})} := \|u\|_{B_1} + \|\Delta u\|_{B_1}.
\]
We recall from [Ag98] the following properties that relate \(P_\Gamma\) to \(-\Delta_\mathcal{O}\):

**Proposition 3.2.** \(P_\Gamma\) has a discrete spectrum in \(\Sigma\), given by \(\text{Res}(\mathcal{O}) \cap \Sigma\). Furthermore, let \(z_0 \in \text{Res}(\mathcal{O}) \cap \Sigma\) be an eigenvalue of \(P_\Gamma\), \(E_\Gamma(z_0)\) denote the generalized eigenspace of \(P_\Gamma\) at \(z_0\), then
\[
E_\Gamma(z_0) := \left( \int_{z_0} (P_\Gamma - \zeta)^{-1}d\zeta \right)(B_\Gamma) = \left( \int_{z_0} \mathcal{R}(\zeta)d\zeta \right)(B_0),
\] (3.11)
where the integral is over a circle containing no other resonance than \(z_0\). In particular, the multiplicity of \(z_0 \in \text{Spec}(P_\Gamma)\) satisfies
\[
m_\Gamma(z_0) := \dim E_\Gamma(z_0) = m_\mathcal{O}(\lambda_0), \quad \text{with} \quad z_0 = \lambda_0^2.
\] (3.12)

Let us now turn to the perturbation theory for resonances. Let \(\Omega\) be an open neighborhood of the origin in \(\mathbb{C}\). We assume the following:

**Hypothesis 3.3.** There exists a family of linear operators \(V(t) : \mathcal{D}_1(\mathcal{O}) \to B_0, t \in \Omega,\) with \(V(0) = 0\), such that
\[
\begin{align*}
(1) \quad & \|V(t)u\|_{B_0} = \mathcal{O}(t)\|u\|_{\mathcal{D}_1(\mathcal{O})}, \quad \forall u \in \mathcal{D}_1(\mathcal{O}) \quad \text{as} \quad \Omega \ni t \to 0; \\
(2) \quad & \Omega \ni t \mapsto V(t)u \text{ is an analytic } B_0\text{-valued function, for any } u \in \mathcal{D}_1(\mathcal{O}).
\end{align*}
\]

Then we consider a family of operators on \(B_\Gamma\):
\[
P_\Gamma(t) = P_\Gamma + V(t), \quad \text{with domain } \mathcal{D}(P_\Gamma(t)) := \mathcal{D}(P_\Gamma).
\] (3.13)
Since \(P_\Gamma\) is closed, it follows from the bound in Hypothesis 3.3 and a well-known result by Kato [Ka13] that \(P_\Gamma(t)\) is also closed in \(B_\Gamma\) provided \(|t|\) sufficiently small. Shrinking
Ω if necessary, we assume from now on that \( P_t(t) \) is a closed operator for all \( t \in \Omega \), then we can apply analytic perturbation theory to the eigenvalues of \( P_t(t) \) – see [Ka13, Chapter VII, §1] for a full treatment.

Fixing a resonance \( z_0 \in \text{Res}(\mathcal{O}) \cap \Sigma \), we choose \( \Sigma' \in \Sigma \) with \( z_0 \in \Sigma' \), \( z_0 \) is also an eigenvalue of \( P_t \) by Proposition 3.2. We recall the following perturbation result from [Ag98, Theorem 7.4]:

**Proposition 3.4.** There exists an open neighborhood of \( 0 \in \mathbb{C} \), \( \Omega_0 \subset \Omega \), such that

(i) for each \( t \in \Omega_0 \), \( P_t(t) \) has a discrete spectrum in \( \Sigma' \);

(ii) the spectrum of \( P_t(t) \) depends analytically on \( t \) in the following sense: for each \( t \in \Omega_0 \), there exist a polynomial \( q^t(z) \), of degree independent of \( t \), with coefficients analytic in \( t \), such that the zeros of \( q^t(z) \) in \( \Sigma' \) coincide with the eigenvalues of \( P_t(t) \) in \( \Sigma' \), with agreement of multiplicities.

Shrinking \( \Omega_0 \) if necessary, we may assume by Hypothesis 3.3 that

\[
\mathcal{R}(z, t) := (-\Delta_{\mathcal{O}} + V(t) - z)^{-1} : B_0 \to \mathcal{D}_1(\mathcal{O})
\]

exists for \( \text{Im} z > c > 0 \) for all \( t \in \Omega_0 \). It was shown in [Ag98, Theorem 7.5] that for any \( t \in \Omega_0 \), \( \mathcal{R}(z, t) \) admits a meromorphic continuation with poles of finite rank to \( z \in \Sigma' \) given by

\[
\mathcal{R}(z, t)f = (P_t(t) - z)^{-1}f, \quad \forall f \in B_0, \quad z \in \Sigma' \setminus \text{Spec}(P_t(t)).
\]

The connection between the poles of \( z \mapsto \mathcal{R}(z, t) \) and the eigenvalues of \( P_t(t) \) was established in [Ag98, Theorem 7.7], which shows that these two discrete sets are identical, more precisely, the multiplicity of \( z_t \) as an eigenvalue of \( P_t(t) \) equals its rank as a pole of \( \mathcal{R}(z, t) \). This correspondence and Proposition 3.4 yield the following perturbation result for resonances – see [Ag98, Proposition 8.1]:

**Proposition 3.5.** Suppose that the multiplicity of resonance \( z_0 \) equals \( m \). Let \( K \subset \Sigma' \) be any disc centered at \( z_0 \) containing no other resonances. Then there exists a neighborhood of \( 0 \in \mathbb{C} \), \( \Omega'_0 \subset \Omega_0 \), such that for any \( t \in \Omega'_0 

(i) The total rank of the poles of \( \mathcal{R}(z, t) \) in \( K \) is equal to \( m \).

(ii) Denote by \( z_1(t), \ldots, z_m(t) \) the poles of \( \mathcal{R}(z, t) \) in \( K \), each repeated with respect to its rank. Then \( z_j(t) \to z_0 \) as \( t \to 0 \), \( j = 1, \ldots, m \).

(iii) The average \( \hat{z}(t) := m^{-1} \sum_{j=1}^{m} z_j(t) \) is an analytic function of \( t \) in \( \Omega'_0 \).

4. Deformation of obstacle

In this section we study the deformation of obstacle and the corresponding deformed Dirichlet Laplacian. Let \( \mathcal{O} \) be an obstacle as in §1, we use \( \text{Diff}(\mathcal{O}) \) defined in (1.5) to describe the deformations of \( \mathcal{O} \). For every \( \Phi \in \text{Diff}(\mathcal{O}) \), \( \Phi(\mathcal{O}) \) is a deformed obstacle.
satisfying all the requirements as \( \mathcal{O} \) does, thus we can define the Dirichlet Laplacian \( \Delta_{\Phi(\mathcal{O})} \). We conjugate \( \Delta_{\Phi(\mathcal{O})} \) by the pullback \( \Phi^* \). This will transform the deformed domain \( \mathbb{R}^n \setminus \Phi(\mathcal{O}) \) to the original one. As a result, the variation is transferred to the coefficients of the newly-defined differential operator. For \( \mathcal{O} \), an obstacle, and \( \Phi \in \text{Diff}(\mathcal{O}) \) given in (1.5), the pullback \( \Phi^* \) is a bounded operator from \( L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O})) \) to \( L^2(\mathbb{R}^n \setminus \mathcal{O}) \), which is invertible with the inverse \((\Phi^{-1})^*\). In view of (1.1), the restricted map \( \Phi^* : \mathcal{D}(\Delta_{\Phi(\mathcal{O})}) \to \mathcal{D}(\Delta_{\mathcal{O}}) \) is also invertible with the inverse \((\Phi^{-1})^*\), since \( \Phi^* \) preserves the Dirichlet boundary condition. Hence we can define the deformed operator \( \Delta^\Phi_{\mathcal{O}} \) of \( \Delta_{\mathcal{O}} \) associated to the deformation \( \Phi \):

\[
\Delta^\Phi_{\mathcal{O}} := \Phi^* \Delta_{\Phi(\mathcal{O})}(\Phi^{-1})^* : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \text{ with domain } \mathcal{D}(\Delta_{\mathcal{O}}). \tag{4.1}
\]

Let \( J^i_j(x) \) denote \([D\Phi(x)^{-1}]_{ij}\), by a direct calculation we have

\[
\Phi^* \Delta(\Phi^{-1})^* = \sum_{i,j} J^i_j \partial^2_{xixj} + \sum_{i,j} (\partial^2_{xixj} \Phi^m) J^i_j \Phi^m \Phi^* \partial_{xj},
\]

where \( \Phi^m(x) \) is the \( m \)-th component of \( \Phi(x) = (\Phi^1(x), \ldots, \Phi^n(x)) \). Now we define

\[
V := \Delta - \Phi^* \Delta(\Phi^{-1})^* = \sum_{i,j} a_{ij}(x) \partial^2_{xixj} + \sum_{j} b_j(x) \partial_{xj},
\]

where \( a_{ij} = \delta_{ij} - \sum_{\ell} J^\ell_i J^\ell_j \), \( b_j = - \sum_{i,\ell, q} (\partial^2_{xixj} \Phi^m) J^i_j \Phi^m \Phi^* \partial_{xj} \).

Then by (1.5) we obtain that for all \( 1 \leq i, j \leq n \),

\[
a_{ij} \in C^m_c(\mathbb{R}^n), \quad b_j \in C^{m-2}_c(\mathbb{R}^n), \quad ||a_{ij}||_{\infty}, ||b_j||_{\infty} \leq C ||\Phi - \text{id}||_{C^2}. \tag{4.3}
\]

We note that \(-\Delta_{\Phi(\mathcal{O})}\) is a self-adjoint black box Hamiltonian, whose resolvent admits a meromorphic continuation by Proposition 2.1. More precisely,

\[
(-\Delta_{\Phi(\mathcal{O})} - \lambda^2)^{-1} : e^{-A|x|}L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O})) \to \mathcal{D}_1(\Phi(\mathcal{O})),
\]

is a meromorphic family of operators for \( \lambda \in \mathbb{C} \) when \( n \) is odd, \( \lambda \in \Lambda \) when \( n \) is even. Here \( \mathcal{D}_1(\Phi(\mathcal{O})) \) is defined as in (3.1). Since \( \Phi^* \) gives isomorphisms between

\[
\mathcal{D}_1(\Phi(\mathcal{O})) \xrightarrow{\Phi^*} \mathcal{D}_1(\mathcal{O}), \quad e^{-A|x|}L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O})) \xrightarrow{\Phi^*} e^{-A|x|}L^2(\mathbb{R}^n \setminus \mathcal{O})
\]

respectively, it follows from (4.1) that the resolvent of \(-\Delta^\Phi_{\mathcal{O}}\) also has a meromorphic continuation given by

\[
(-\Delta^\Phi_{\mathcal{O}} - \lambda^2)^{-1} = \Phi^*(\Delta_{\Phi(\mathcal{O})} - \lambda^2)^{-1}(\Phi^{-1})^* = \Phi^* R_{\Phi(\mathcal{O})}(\lambda)(\Phi^{-1})^*, \tag{4.4}
\]

whose poles, denoted by \( \text{Res}(-\Delta^\Phi_{\mathcal{O}}) \), coincide, with agreement of multiplicities, with the resonances of \( \Phi(\mathcal{O}) \).
5. Agmon’s theory and boundary perturbation

In this section we consider Agmon’s theory for the deformed operators $-\Delta^\phi_\Omega$. It follows from (4.1) and (4.2) that $-\Delta^\phi_\Omega$ is also closable on $B_1$ with the closure

$$P^\phi_1 := \Phi^* (-\Delta)(\Phi^{-1})^* = -\Delta + V : B_1 \to B_1 \quad \text{with domain } D_1(\Omega).$$  \hspace{1cm} (5.1)

Thus $-\Delta^\phi_\Omega$ also satisfies the abstract hypotheses of Agmon’s theory reviewed in §3.

For a fixed domain $\Sigma$ with boundary $\Gamma$ satisfying (3.4), by Proposition 2.1 we can assume that

$$\sup_{\zeta \in \Gamma} \| \mathcal{R}(\zeta) \|_{\mathcal{L}(B_0, D_1(\Omega))} < C_\Gamma \quad \text{for some constant } C_\Gamma > 0,$$  \hspace{1cm} (5.2)

where $D_1(\Omega)$ is equipped with the graph norm:

$$\| u \|_{D_1(\Omega)} = \| u \|_{B_1} + \| \Delta u \|_{B_1}.$$  

Since $V$ defined in (4.2) is a second order differential operator with compactly supported coefficients, it can be viewed as an operator in $\mathcal{L}(D_1(\Omega), B_0)$ satisfying

$$\| V \|_{\mathcal{L}(D_1(\Omega), B_0)} \leq C \| \Phi - \text{id} \|_{C^2},$$  \hspace{1cm} (5.3)

there exists $\delta_\Gamma > 0$ sufficiently small such that for all $\zeta \in \Gamma$,

$$\| \Phi - \text{id} \|_{C^2} < \delta_\Gamma \implies \| V \mathcal{R}(\zeta) \|_{\mathcal{L}(B_0, B_0)} < 1/2,$$  \hspace{1cm} (5.4)

which guarantees that $I + V \mathcal{R}(\zeta) : B_0 \to B_0$ is invertible by a Neumann series argument. Thus we have

$$\mathcal{R}_\phi(\zeta) := (-\Delta^\phi_\Omega - \zeta)^{-1} = \mathcal{R}(\zeta)(I + V \mathcal{R}(\zeta))^{-1}, \quad \zeta \in \Gamma,$$  \hspace{1cm} (5.5)

which can be justified first for $\zeta$ near $\Gamma \cap \{ z : 0 < \text{Im} \, z < \pi \}$ and then by meromorphic continuation. In particular, $\Gamma \cap \text{Res}(-\Delta^\phi_\Omega) = \emptyset$. Hence for the same domain $\Sigma$ with boundary $\Gamma$, we can define $B_{\Gamma, \Phi}, R_{\Gamma, \Phi}$ and $P_{\Gamma, \Phi}$ for the deformed operator $-\Delta^\phi_\Omega$, as in (3.5), (3.7) and (3.9) with $\mathcal{R}(\zeta)$ replaced by $\mathcal{R}_\phi(\zeta)$.

Now we explore the relationships between $B_{\Gamma, \Phi}, R_{\Gamma, \Phi}, P_{\Gamma, \Phi}$ and $B_{\Gamma}, R_{\Gamma}, P_{\Gamma}$. Assuming that $\| \Phi - \text{id} \|_{C^2} < \delta_\Gamma$, by (5.5) we have for any $f \in B_{\Gamma}$,

$$f = g + \int_{\Gamma} \mathcal{R}(\zeta) \varphi(\zeta) d\zeta = g + \int_{\Gamma} \mathcal{R}_\phi(\zeta)(I + V \mathcal{R}(\zeta)) \varphi(\zeta) d\zeta.$$  

Since $(I + V \mathcal{R}(\zeta)) \varphi(\zeta) \in C(\Gamma; B_0), f \in B_{\Gamma, \Phi}$ thus we have $B_{\Gamma} \subset B_{\Gamma, \Phi}$. Furthermore, (5.4) implies that

$$\| g \|_{B_0} + \| (I + V \mathcal{R}(\zeta)) \varphi(\zeta) \|_{C(\Gamma; B_0)} \leq \frac{3}{2} (\| g \|_{B_0} + \| \varphi \|_{C(\Gamma; B_0)}),$$  

by taking the infimum as in (3.6), we obtain that $\|f\|_{B_{\Gamma, \Phi}} \leq 3/2 \|f\|_{B_{\Gamma}}$. Similarly, for $f \in B_{\Gamma, \Phi}$ we have

$$f = g + \int_{\Gamma} \mathcal{R}_\Phi(\zeta) \varphi(\zeta) d\zeta = g + \int_{\Gamma} \mathcal{R}(\zeta)(I + V \mathcal{R}(\zeta))^{-1} \varphi(\zeta) d\zeta \in B_{\Gamma},$$

and again by (5.4) we can deduce that $\|f\|_{B_{\Gamma}} \leq 2 \|f\|_{B_{\Gamma, \Phi}}$. Therefore,

$$B_{\Gamma, \Phi} = B_{\Gamma}, \| \cdot \|_{B_{\Gamma, \Phi}} \text{ and } \| \cdot \|_{B_{\Gamma}} \text{ are equivalent, if } \|\Phi - \text{id}\|_{C^2} < \delta_{\Gamma}. \quad (5.6)$$

Henceforth, we identify $B_{\Gamma, \Phi}$ with $B_{\Gamma}$. Suppose that $f = g + \int_{\Gamma} \mathcal{R}_\Phi(\zeta) \varphi(\zeta) d\zeta \in B_{\Gamma}$, then for $w_0$ chosen in (3.9), in view of (3.7) and (5.5) we have

$$R_{\Gamma, \Phi}(w_0) f = \mathcal{R}_\Phi(w_0) g + \int_{\Gamma} (\zeta - w_0)^{-1} (\mathcal{R}_\Phi(\zeta) - \mathcal{R}_\Phi(w_0)) \varphi(\zeta) d\zeta$$

$$= \mathcal{R}(w_0) g_1 + \int_{\Gamma} (\zeta - w_0)^{-1} (\mathcal{R}(\zeta) - \mathcal{R}(w_0)) \varphi_1(\zeta) d\zeta,$$

where $\varphi_1(\zeta) := (I + V \mathcal{R}(\zeta))^{-1} \varphi(\zeta) \in C(\Gamma; B_0)$ and

$$g_1 := (I + V \mathcal{R}(w_0))^{-1} g + \int_{\Gamma} (I + V \mathcal{R}(\zeta))^{-1} - (I + V \mathcal{R}(w_0))^{-1} \frac{\varphi(\zeta) d\zeta}{\zeta - w_0} \in B_{\Gamma}.$$

Thus $R_{\Gamma, \Phi}(w_0) f = R_{\Gamma}(w_0) f_1$ for $f_1 := g_1 + \int_{\Gamma} \mathcal{R}(\zeta) \varphi_1(\zeta) d\zeta \in B_{\Gamma}$, which implies that $\text{Ran } R_{\Gamma, \Phi}(w_0) \subset \text{Ran } R_{\Gamma}(w_0)$. We can also derive that $\text{Ran } R_{\Gamma}(w_0) \subset \text{Ran } R_{\Gamma, \Phi}(w_0)$ by similar arguments. Therefore, recalling (3.9) we obtain that

$$D(P_{\Gamma, \Phi}) := \text{Ran } R_{\Gamma, \Phi}(w_0) = \text{Ran } R_{\Gamma}(w_0) = D(P_{\Gamma}).$$

We recall [Ag98] that $P_1^\Phi$ extends $P_{\Gamma, \Phi}$ as in (3.10), then for any $u \in D(P_{\Gamma})$, (5.1) and (3.10) imply that

$$P_{\Gamma, \Phi} u = P_1^\Phi u = P_1 u + V u = P_{\Gamma} u + V u \quad (5.7)$$

Hence $P_{\Gamma, \Phi}$ and $P_{\Gamma}$ are related as follows

$$P_{\Gamma, \Phi} = P_{\Gamma} + V : B_{\Gamma} \to B_{\Gamma} \quad \text{with domain } D(P_{\Gamma}). \quad (5.8)$$

Now we substitute $P_{\Gamma}$ by $P_{\Gamma, \Phi}$ in Proposition 3.2 and recall (4.4) to conclude:

**Proposition 5.1.** Let $\Sigma$ with boundary $\Gamma$ be chosen as in (3.4) and suppose that $\Phi \in \text{Diff}(\mathcal{O})$ satisfies $\|\Phi - \text{id}\|_{C^2} < \delta_{\Gamma}$ for some $\delta_{\Gamma} > 0$ in (5.4), then $P_{\Gamma, \Phi}$ has a discrete spectrum in $\Sigma$, given by $\text{Res}(\Phi(\mathcal{O})) \cap \Sigma$.

Furthermore, let $z \in \text{Res}(\Phi(\mathcal{O})) \cap \Sigma$ be an eigenvalue of $P_{\Gamma, \Phi}$, denote by $\mathcal{E}_{\Gamma, \Phi}(z)$ the generalized eigenspace of $P_{\Gamma, \Phi}$ at $z$, then

$$\mathcal{E}_{\Gamma, \Phi}(z) := \left( \int_{\delta} (P_{\Gamma, \Phi} - \zeta)^{-1} d\zeta \right) (B_{\Gamma}) = \left( \int_{\delta} \mathcal{R}_\Phi(\zeta) d\zeta \right) (B_0) \quad (5.9)$$

where the integral is over a circle containing no other resonance than $z$. In particular, the multiplicity of $z \in \text{Spec } P_{\Gamma, \Phi}$ satisfies

$$m_{\Gamma, \Phi}(z) := \text{dim } \mathcal{E}_{\Gamma, \Phi}(z) = m_{\Phi(\mathcal{O})}(\lambda), \quad \text{with } z = \lambda^2. \quad (5.10)$$
6. Generic simplicity of resonances in obstacle scattering

We will follow the strategy of [KlZw95] and [BoPe02] in the case of potential perturbations to prove our theorem. However we have to overcome the additional difficulties produced by boundary perturbations using the results obtained in §4 and §5. For simplicity we identify $\mathbb{C} \setminus i[0, \infty)$ with $\{ \lambda \in \Lambda : -3\pi/2 < \arg \lambda < \pi/2 \}$ when $n$ is odd. Let $X$ be the class of obstacles diffeomorphic to a fixed obstacle $O_0$ — see (1.6), that is for some $k \geq 2$,

$$X := \{ \Phi(O_0) : \Phi \in \text{Diff}(O_0) \},$$

with Diff($O_0$) defined by (1.5). We introduce a topology in this set by defining a sub-basis of the neighborhoods of any $O \in X$ by

$$V_{\varepsilon}(O) := \{ \Phi(O) : \Phi \in \text{Diff}(O), \| \Phi - \text{id} \|_{C^k} < \varepsilon \text{ with } \varepsilon \text{ sufficiently small} \}.$$

For any $\theta_1, \theta_2 \in \mathbb{R}$ and $r > 1$, we define

$$S^{\theta_1,\theta_2}_{\theta} := \{ \lambda \in \Lambda : \theta_1 < \arg \lambda \theta_2, \ 1/r < |\lambda| < r \},$$

$$E^{r,\theta}_{\theta} := \{ O \in X : m_O(\lambda) \leq 1, \ \forall \lambda \in S^{\theta_1,\theta_2}_{\theta} \}.$$  \hspace{1cm} (6.1)

To prove our theorem it suffices to show that for each $\theta_1, \theta_2$ and $r$, $E^{r,\theta}_{\theta_1,\theta_2}$ is open and dense in $X$, since we can then obtain the generic set $\mathcal{X}$ by taking

$$\mathcal{X} := \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} E^{N,\pi,\pi}_{-m,\pi,\pi} \text{ when } n \text{ is even; } \mathcal{X} := \bigcap_{N=1}^{\infty} E^{N,\pi,\pi}_{-\pi,\pi} \text{ when } n \text{ is odd.}$$

We proceed the proof of our theorem in steps:

**Proof.** Step 1. As in §3, for $O \in X$ we write Res($O$) for the image of resonances under the map $\lambda \mapsto z = \lambda^2$, and for any $z$ the multiplicity is given by $m_O(z) := m_O(\lambda)$ provided $z = \lambda^2$. Then

$$E^{r,\theta}_{\theta_1,\theta_2} = \{ O \in X : m_O(z) \leq 1, \ \forall z \in S^{2,2}_{2\theta_1,2\theta_2} \}.$$

In view of (3.2), we can choose $A > 0$ large enough, such that $S^{2,2}_{2\theta_1,2\theta_2} \subseteq D_+$ (when $n$ is odd, we only need to check for $\theta_1 = -3\pi/2, \theta_2 = \pi/2$). Suppose that there is exactly one resonance $z_0$ in $B(z_0, 2\delta) \subset S^{2,2}_{2\theta_1,2\theta_2}$, where $B(z_0, r)$ denotes the disc in $\mathbb{C}$ centered at $z_0$ with radius $r$. For $\Omega := B(z_0, \delta)$ we then define

$$\Pi_O(\Omega) := -\frac{1}{2\pi i} \int_{\partial \Omega} (-\Delta_O - \zeta)^{-1} d\zeta, \ m_O(\Omega) := \text{rank } \Pi_O(\Omega).$$  \hspace{1cm} (6.2)

Now we choose a bounded domain $\Sigma$ containing $B(z_0, 2\delta)$ with boundary $\Gamma$ satisfying (3.4). We also assume that $\Sigma \subseteq S^{2,2}_{2\theta_1,2\theta_2}$. By Proposition 3.2, elements in Res($O$) coincide with the eigenvalues of $P_\Gamma$ in $\Sigma$. In view of (3.12), we have the relationship:

$$\Pi_\Gamma(\Omega) := -\frac{1}{2\pi i} \int_{\partial \Omega} (P_\Gamma - \zeta)^{-1} d\zeta, \text{ then } m_\Gamma(\Omega) := \text{rank } \Pi_\Gamma(\Omega) = m_O(\Omega).$$  \hspace{1cm} (6.3)
Let $\mathcal{U}_\varepsilon(\mathcal{O})$ be a set of deformations defined for small $\varepsilon > 0$, 

$$\mathcal{U}_\varepsilon(\mathcal{O}) := \{ \Phi \in \text{Diff}(\mathcal{O}) : \| \Phi - \text{id} \|_{C^k} < \varepsilon \}.$$ 

Assuming that $\varepsilon < \delta_\Gamma$ for constant $\delta_\Gamma$ given in (5.4), then for every $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$ Proposition 5.1 implies that

$$\Pi_{\Gamma, \Phi}(\Omega) := -\frac{1}{2\pi i} \int_{\partial\Omega} (P_{\Gamma, \Phi} - \zeta)^{-1} d\zeta, \quad m_{\Gamma, \Phi}(\Omega) := \text{rank} \Pi_{\Gamma, \Phi}(\Omega) = m_{\Phi(\mathcal{O})}(\Omega). \quad (6.4)$$

We recall (5.8) that $P_{\Gamma, \Phi} = P_{\Gamma} + V$ with $V$ defined in (4.2), then by (5.3) we obtain that if $\varepsilon$ is sufficiently small, then for $\zeta \in \partial\Omega$ and $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$,

$$(P_{\Gamma, \Phi} - \zeta)^{-1} = (P_{\Gamma} - \zeta)^{-1} (I + V(P_{\Gamma} - \zeta)^{-1})^{-1}$$

and $\sup_{\zeta \in \partial\Omega} \|(P_{\Gamma, \Phi} - \zeta)^{-1} - (P_{\Gamma} - \zeta)^{-1}\|_{B_{\Omega} \rightarrow B_{\Omega}} < C(\Omega)\varepsilon$. Then we can derive that $\Pi_{\Gamma}(\Omega)$ and $\Pi_{\Gamma, \Phi}(\Omega)$ have the same rank for any $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$ if $\varepsilon$ is sufficiently small. We restate this as follows:

$$m_{\Phi(\mathcal{O})}(\Omega) \text{ is constant for } \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ if } \varepsilon \text{ is sufficiently small.} \quad (6.5)$$

Hence $\mathcal{O} \in E^r_{\theta}$ implies that $\{ \Phi(\mathcal{O}) : \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \} \subset E^r_{\theta}$ for some $\varepsilon$ sufficiently small, in other words, $E^r_{\theta}$ is an open subset of $X$.

Step 2. It remains to show that $E^r_{\theta}$ is dense in $X$, which is equivalent to:

$$\forall \mathcal{O} \in X \text{ and } \varepsilon > 0, \quad \exists \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ such that } \Phi(\mathcal{O}) \in E^r_{\theta}. \quad (6.6)$$

Since the number of resonances for the obstacle $\mathcal{O}$ in $S^r_{\theta_1, \theta_2}$ is finite, it is enough to prove a local statement as it can be applied successively to obtain (6.6) (once a resonance is simple it stays simple under small deformations due to (6.5)). We will define $\Omega$ for any given $\mathcal{O}$ and $z_0 \in \text{Res}(\mathcal{O})$ as in Step 1, thus to obtain (6.6) it suffices to prove that for

$$\forall \mathcal{O} \in X, \quad z_0 \in \text{Res}(\mathcal{O}) \text{ and } \varepsilon > 0, \quad \exists \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ s.t. } m_{\Phi(\mathcal{O})}(z) \leq 1, \forall z \in \Omega. \quad (6.7)$$

To establish (6.7) we proceed by induction. We note that for each $\mathcal{O} \in X, z_0 \in \text{Res}(\mathcal{O})$, one of the following cases has to occur:

$$\forall \varepsilon > 0, \quad \exists \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}) \text{ s.t. } 1 \leq m_{\Phi(\mathcal{O})}(z) < m_{\Phi(\mathcal{O})}(\Omega), \quad \forall z \in \Omega, \quad (6.8)$$

or

$$\exists \varepsilon > 0, \text{ s.t. } \forall \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}), \exists z = z(\Phi) \in \Omega, \quad m_{\Phi(\mathcal{O})}(z) = m_{\Phi(\mathcal{O})}(\Omega) > 1. \quad (6.9)$$

The first possibility means that by applying an arbitrarily small deformation $\Phi$ to $\mathcal{O}$ we can obtain at least two distinct resonances for $\Phi(\mathcal{O})$ in $\Omega$. The second possibility means that under any small deformations the maximal multiplicity persists.

Step 3. Assuming (6.8) we can prove (6.7) by induction on $m_{\mathcal{O}}(z_0)$. If $m_{\mathcal{O}}(z_0) = 1$ there is nothing to prove. Assuming that we proved (6.7) in the case $m_{\mathcal{O}}(z_0) < M$, we now
assume that \( m_\mathcal{O}(z_0) = M \). We note that for any \( \Phi_1 \in \text{Diff}(\mathcal{O}) \) and \( \Phi_2 \in \text{Diff}(\Phi_1(\mathcal{O})) \), there exists \( C = C(k, n) \) such that

\[
\| \Phi_2 \circ \Phi_1 - \text{id} \|_{C^k} \leq C\left( \| \Phi_2 - \text{id} \|_{C^k} + \| \Phi_1 - \text{id} \|_{C^k} \right).
\]

In view of (6.8) we can find \( \Phi_0 \in \text{Diff}(\mathcal{O}) \) with \( \| \Phi_0 - \text{id} \|_{C^k} < \varepsilon/(2C)^M \) such that \( m_{\mathcal{O}(\mathcal{O})}(\Omega) = m_\mathcal{O}(\Omega) \) (using (6.5)) and that all resonances in \( \Omega \), denoted by \( z_1, \ldots, z_\ell \), satisfy \( m_{\Phi_0(\mathcal{O})}(z_j) < M \). We now find \( r_j \) such that

\[
B(z_j, 2r_j) \subset \Omega, \quad \{ z_j \} = B(z_j, 2r_j) \cap \text{Res}(\Phi_0(\mathcal{O})), \quad B(z_j, 2r_j) \cap B(z_i, 2r_i) = \emptyset.
\]

We put \( \Omega_j := B(z_j, r_j) \) and apply (6.7) successively to \( \Phi_{j-1} \circ \cdots \circ \Phi_0(\mathcal{O}) \), \( j = 1, \ldots, \ell \) with \( \| \Phi_j - \text{id} \|_{C^k} < \varepsilon/(2C)^{\ell+1-j} \) (by (6.5) we can assume that \( \Phi_j \) is sufficiently close to the identity map such that resonances in \( \Omega_0, \ldots, \Omega_{j-1} \) that are already simple stay simple while total multiplicities in \( \Omega_{j+1}, \ldots, \Omega_\ell \) are invariant). Then we obtain the desired \( \Phi = \Phi_\ell \circ \cdots \circ \Phi_0 \in U_\varepsilon(\mathcal{O}) \) since (note that \( \ell < M \))

\[
\| \Phi_\ell \circ \cdots \circ \Phi_0 - \text{id} \|_{C^k} < \sum_{j=1}^\ell C^{\ell+1-j} \frac{\varepsilon}{(2C)^{\ell+1-j}} + C^{\ell} \frac{\varepsilon}{(2C)^M} \leq \varepsilon.
\]

Step 4. It remains to show (6.9) is impossible. For that, we shall argue by contradiction. Suppose that there exist an obstacle \( \mathcal{O} \in X \) and a resonance \( z_0 \in \Omega \) with some disc \( \Omega = B(z_0, r) \) containing no other resonances, such that (6.9) holds. In fact we may assume further that \( \mathcal{O} \) has \( C^\infty \)-boundary since we can deform \( \mathcal{O} \) to a smooth obstacle \( \tilde{\mathcal{O}} \) through some \( \tilde{\Phi} \in \text{Diff}(\mathcal{O}) \) with \( \| \tilde{\Phi} - \text{id} \|_{C^k} \ll \varepsilon \), decreasing \( \varepsilon \) if necessary, then (6.9) still holds with \( \tilde{\mathcal{O}} \) and \( \tilde{z}_0 = z(\tilde{\Phi}) \) replacing \( \mathcal{O} \) and \( z_0 \). Hence we assume in the following that \( \mathcal{O} \) is a smooth obstacle.

Let \( M = m_\mathcal{O}(\Omega) \). Suppose that \( \Sigma \) and \( \Gamma \) are chosen as in Step 1. Using (6.3) and (6.4) we obtain an equivalent statement to (6.9):

\[
\exists \varepsilon > 0, \ \text{s.t.} \ \forall \Phi \in U_\varepsilon(\mathcal{O}), \ \exists z = z(\Phi) \in \Omega, \ m_{\Gamma, \Phi}(z) = m_{\Gamma, \Phi}(\Omega) > 1. \quad (6.10)
\]

For \( \Phi \in U_\varepsilon(\mathcal{O}) \), we define

\[
q(\Phi) := \min\{ q \in \mathbb{N} : (P_{\Gamma, \Phi} - z(\Phi))^q \Pi_{\Gamma, \Phi}(\Omega) = 0 \},
\]

then \( 1 \leq q(\Phi) \leq M \). It follows from (5.8) and (4.2) that if \( \| \Phi_j - \Phi \|_{C^2M} \to 0 \) and \( (P_{\Gamma, \Phi_j} - z(\Phi_j))^q \Pi_{\Gamma, \Phi_j}(\Omega) = 0 \), then \( (P_{\Gamma, \Phi} - z(\Phi))^q \Pi_{\Gamma, \Phi}(\Omega) = 0 \). We now define

\[
q_0 := \max\{ q(\Phi) : \Phi \in U_{\varepsilon/2}(\mathcal{O}) \},
\]

and assume that the maximum is attained at \( \Phi_0 \) i.e. \( q(\Phi_0) = q_0 \), then there exists \( \varepsilon' > 0 \) such that

\[
\| \Phi - \Phi_0 \|_{C^2M} < \varepsilon' \implies q(\Phi) = q_0.
\]
Therefore, we can choose a $\tilde{\Phi}_0 \in \text{Diff}(O)$ that is in $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\tilde{\Phi}_0 - \Phi_0\| \ll \varepsilon$.

Replacing $O$ in (6.10) by $\tilde{\Phi}_0(O)$ and decreasing $\varepsilon$ such that $\varepsilon \ll \varepsilon'$, we assume in the following that

$$\forall \Phi \in \text{Diff}(O), \|\Phi - \text{id}\|_{C^{2M}} < \varepsilon, \exists z(\Phi) \text{ and } 1 \le q_0 \le M \text{ such that}$$

$$m_{\Gamma, \Phi}(z(\Phi)) = \text{rank } \Pi_{\Gamma, \Phi}(\Omega) = M > 1,$$

$$(P_{\Gamma, \Phi} - z(\Phi))^{q_0} \Pi_{\Gamma, \Phi}(\Omega) = 0, \quad (P_{\Gamma, \Phi} - z(\Phi))^{q_0 - 1} \Pi_{\Gamma, \Phi}(\Omega) \neq 0.$$  \hspace{1cm} (6.11)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Deformation $\varphi^h_i$ in Diff($O$) acting near a fixed point on $\partial O$, which is used in Step 5 of the proof.}
\end{figure}

Step 5. Before proving (6.11) is impossible we introduce a family of deformations in Diff($O$) acting near a point on $\partial O$. For any fixed $x_0 \in \partial O$, we consider the normal coordinates near $x_0$, that is there is some $U = B_{\mathbb{R}^n}(x_0, 2r_0)$ such that for each $x \in U$ there exist unique $(x', x_n) \in \partial O \times \mathbb{R}$ with $x = x' + x_n \nu(x')$, where $\nu(x')$ is the normal vector at $x'$ pointing to the interior of $O$. Let $\rho \in C^\infty_c(\mathbb{R}; [0,1])$ be a bump function such that $\rho(0) = 1$ and $\text{supp } \rho \subset (-r_0, r_0)$. Fixing $h_0 > 0$ small, we choose a family of functions $\chi_h \in C^\infty(\partial O; [0, \infty))$ depending continuously in $h \in (0, h_0]$ such that

$$\int_{\partial O} \chi_h(x')dS(x') = 1, \quad \text{supp } \chi_h \subset B_{\partial O}(x_0, h) \subset U, \quad \forall h \in (0, h_0],$$

where $B_{\partial O}(x_0, h)$ is a geodesic ball on $\partial O$ with center $x_0$ and radius $h$. For each $h \in (0, h_0]$, we construct a smooth vector field $V_h \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as follows

$$V_h(x) = \chi_h(x')\rho(x_n)\nu(x'), \quad \text{for } x = x' + x_n\nu(x') \in U,$$

and $V_h(x) = 0$ for all $x \in \mathbb{R}^n \setminus U$. \hspace{1cm} (6.13)

Then we introduce a family of smooth deformations produced by $V_h$:

$$\varphi^h_i \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad \varphi^h_i(x) := x + tV_h(x).$$

It follows from (6.13) that for every $h \in (0, h_0]$ there is $t_0 = t_0(h) \ll 1$ such that

$$\forall t \in (-t_0, t_0), \quad \varphi^h_i \in \text{Diff}(O), \quad ||\varphi^h_i - \text{id}||_{C^{2M}} < \varepsilon.$$
Step 6. To show that (6.11) is impossible we first assume the case \( q_0 > 1 \). We recall (3.11) that \( \Pi_\Gamma(\Omega)(B_\Gamma) = \Pi_\Gamma(\Omega)(B_0) \), let
\[
C^\infty_c(\mathbb{R}^n \setminus \partial \Omega) := \{ f \in C^\infty_c(\mathbb{R}^n) : \text{supp } f \subset \mathbb{R}^n \setminus \partial \Omega \}
\]
then \( \text{Ran} \Pi_\Gamma(\Omega) = \Pi_\Gamma(\Omega)(C^\infty_c(\mathbb{R}^n \setminus \partial \Omega)) \) since \( \Pi_\Gamma(\Omega) \) is finite rank and \( C^\infty_c(\mathbb{R}^n \setminus \partial \Omega) \) is dense in \( B_0 \). Thus by (6.11) we can find \( w \in C^\infty_c(\mathbb{R}^n \setminus \partial \Omega) \) such that
\[
u := (P_\Gamma - z_0)^{q_0-1} \Pi_\Gamma(\Omega) w \neq 0, \quad \text{here } z_0 = z(\text{id}).
\]
Fixing \( x_0 \in \partial \Omega \) and \( h \in (0, h_0] \), we define \( \varphi^t_h \) as in Step 5 and write \( \Phi_t := \varphi^t_h, t \in (-t_0, t_0) \). If we set
\[
u(t) := (\Phi_t^{-1})^* \nu(t), \quad \nu(t) := (P_{\Gamma, \Phi_t} - z(t))^{q_0-1} \Pi_{\Gamma, \Phi_t}(\Omega) w, \quad z(t) := z(\Phi_t).
\]
Then by (6.11) we have for any
\[
\forall t \in (-t_0, t_0), \quad m_{\Gamma, \Phi_t}(z(t)) = \text{rank } \Pi_{\Gamma, \Phi_t}(\Omega) = M, \quad (P_{\Gamma, \Phi_t} - z(t)) \nu(t) = 0.
\]
Recalling (5.1) and (5.7), we obtain the equation for \( \nu(t) \):
\[
(\Delta - z(t)) \nu(t) = 0 \quad \text{on } \mathbb{R}^n \setminus \Phi_t(\Omega),
\]
in the sense of \( L^2_{\text{loc}} \) functions.

We next aim to show that \( z(t) \) is differentiable at 0. For that we extend (6.14) to \( \Phi_t \in C^\infty(\mathbb{R}^n, \mathbb{C}^n), t \in \mathbb{C} \):
\[
\Phi_t(x) := x + t V_h(x), \quad t \in \mathbb{C}
\]
We set \( t_1 = t_1(h) \) sufficiently small such that for all \( |t| < t_1 \) and \( x \in \mathbb{R}^n \), \( D \Phi_t(x) = I + t D V_h(x) \) is invertible. Denoting by \( J_{ij}^{\ell}(x) = [D \Phi_t(x)^{-1}]_{ij} \), \( J_{ij}^{\ell}(x) \) the \( m \)-th component of \( \Phi_t(x) \), we replace \( \Phi \) by \( \Phi_t \) in (4.2) to define
\[
\nu(t) := \sum_{i,j} a_{ij}(t, x) \partial_{x_i,x_j}^2 + \sum_j b_j(t, x) \partial_{x_j},
\]
where
\[
a_{ij} = \delta_{ij} - \sum_{\ell} J_{ij}^{\ell} J_{ij}^{\ell}, \quad b_j = - \sum_{\ell,m,q} \partial_{x_i,x_{j}}^2 \Phi_{\ell}^m J_{ij}^{\ell} J_{ij}^{m} J_{ij}^{q}. \]
Repeating the calculation that yields (4.3), we also have for some \( C = C(h) > 0 \),
\[
a_{ij}(t, \cdot), \quad b_j(t, \cdot) \in C^\infty_c(\mathbb{R}^n), \quad \|a_{ij}(t, \cdot)\|_\infty, \quad \|b_j(t, \cdot)\|_\infty < C|t|.
\]
It follows that \( \nu(t), |t| < t_1 \) satisfies Hypothesis 3.3. For \( t \in \mathbb{C}, |t| < t_1 \), we follow (3.13) to define
\[
\Pi_\Gamma(t) = \Pi_\Gamma + \nu(t), \quad \text{with domain } \mathcal{D}(\Pi_\Gamma(t)) := \mathcal{D}(\Pi_\Gamma).
\]
Recalling Propositions 3.4 and 3.5, decreasing \( t_0 \) if necessary, for any \( t \in \mathbb{C}, |t| < t_0 \), \( \Pi_\Gamma(t) \) has a discrete spectrum in some neighborhood \( K \) of \( z_0 \) and the total multiplicity of the eigenvalues of \( \Pi_\Gamma(t) \) in \( K \) equals \( M \). Moreover, if we denote by \( z_1(t), \ldots, z_M(t) \) the eigenvalues of \( \Pi_\Gamma(t) \) in \( K \), repeated with multiplicity, then \( \hat{z}(t) = M^{-1} \sum_{j=1}^M z_j(t) \)
is an analytic function in $t \in \mathbb{C}$, $|t| < t_0$. On the other hand, if we consider real $t$, $t \in (-t_0, t_0)$, then (4.2) and (5.8) imply that

$$P_t(t) = P_t + V(t) = P_{t, \Phi_t}, \quad -t_0 < t < t_0.$$ 

It follows from (6.17) that for $t \in (-t_0, t_0)$ the eigenvalues of $P_t(t)$ near $z_0$ don’t split, i.e. $z_j(t) = z(t), j = 1, \ldots, M$. Thus $z(t) = \dot{z}(t)$ when $t$ is real, $t \in (-t_0, t_0)$. The analyticity of $\dot{z}(t)$ gives the smoothness of $z(t)$ on $(-t_0, t_0)$. As a consequence, $u(t)$ and $v(t)$ defined in (6.16) also depend smoothly on $t \in (-t_0, t_0)$.

Since $\Phi_t(O) \subset O$ for $t \geq 0$, we can restrict (6.18) to the region $\mathbb{R}^n \setminus O$ then differentiate the equation in $t$, by taking $t = 0$, we obtain that

$$(-\Delta - z_0)\partial_t u(0, x) = z'(0)u(x) \quad \text{on} \quad \mathbb{R}^n \setminus O. \tag{6.19}$$

We recall (6.16) that $u(t, x) = v(t, \Phi_t^{-1}(x))$, using $u(0, x) = v(0, x) = u(x)$ and (6.14) we can calculate the derivative in $t$:

$$\partial_t u(0, x) = \partial_t v(t, \Phi_t^{-1}(x))|_{t=0} = \partial_t v(0, x) - \partial_x u \cdot V_h(x).$$

In view of (3.11) and (6.15), $u \in \mathcal{E}(z_0)$ is a resonant state of $-\Delta_O$ at $z_0$, thus we recall [DyZw19, Theorem 4.7] that $u \in C^\infty(\mathbb{R}^n \setminus O)$. Then by (6.13) we conclude that

$$(-\Delta - z_0)(\partial_t v(0, x) - f) = z'(0)u(x) \quad \text{on} \quad \mathbb{R}^n \setminus O,$$

$$f := \partial_x u \cdot V_h(x) \in C^\infty_c(\mathbb{R}^n \setminus O), \quad f|_{\partial O} = \chi_h \partial_v u. \tag{6.20}$$

It follows from $v(t, x) \in \mathcal{D}(P_t)$, $t \in (-t_0, t_0)$ that $\partial_t v(0, x) \in \mathcal{D}(P_t)$, thus the first equation in (6.20) reduces to

$$(P_t - z_0)\partial_t v(0, x) = (-\Delta - z_0)f + z'(0)u \quad \text{on} \quad \mathbb{R}^n \setminus O. \tag{6.21}$$

We introduce the bilinear form on $B_0 \times B_1$ (no complex conjugation),

$$\langle u, v \rangle := \int_{\mathbb{R}^n \setminus O} uv \, dx, \quad u \in B_1, \ v \in B_0.$$ 

We now apply the projection $\Pi_t$ (omitting $\Omega$) to both sides of (6.21), pair with $(P_t - z_0)^{q_0-1}w \in B_0$ (since $w \in C^\infty_c(\mathbb{R}^n \setminus O)$), use the fact that $(P_t - z_0)\Pi_t g = \Pi_t (P_t - z_0)g$, $\forall \ g \in \mathcal{D}(P_t)$ to obtain that

$$\langle (P_t - z_0)\Pi_t \partial_t v(0, x), (P_t - z_0)^{q_0-1}w \rangle$$

$$= \langle \Pi_t (-\Delta - z_0) f, (P_t - z_0)^{q_0-1}w \rangle + z'(0)\langle u, (P_t - z_0)^{q_0-1}w \rangle.$$ 

By Green’s formula, $\langle P_t g_1, g_2 \rangle = \langle g_1, P_t g_2 \rangle$ for any $g_1 \in \mathcal{D}(P_t), \ g_2 \in C^\infty_c(\mathbb{R}^n \setminus O)$. It then follows from (6.11) and (6.15) that

$$\langle (P_t - z_0)\Pi_t \partial_t v(0, x), (P_t - z_0)^{q_0-1}w \rangle = \langle (P_t - z_0)^{q_0} \Pi_t \partial_t v(0, x), w \rangle = 0,$$

and that

$$\langle u, (P_t - z_0)^{q_0-1}w \rangle = \langle (P_t - z_0)u, (P_t - z_0)^{q_0-2}w \rangle = 0.$$
Since $\langle \Pi_{\Gamma} f_1, f_2 \rangle = \langle \Pi_{\Gamma} f_2, f_1 \rangle$ for any $f_1, f_2 \in B_0$, we conclude that
\[
0 = \langle \Pi_{\Gamma} (-\Delta - z_0) f, (P_{\Gamma} - z_0)^{q_0-1} w \rangle = \langle (-\Delta - z_0) f, (P_{\Gamma} - z_0)^{q_0-1} \Pi_{\Gamma} w \rangle = \langle (-\Delta - z_0) f, u \rangle.
\]
Now we apply Green’s formula and recall (6.20) and $u_{\partial \Omega} = 0$ to obtain
\[
0 = \langle (-\Delta - z_0) f, u \rangle = \int_{\partial \Omega} f \, \partial_{\nu} u \, dS = \int_{\partial \Omega} \chi_h(x') (\partial_{\nu} u(x'))^2 dS(x').
\]
Since the above equation holds for any $h \in (0, h_0]$, $(u$ is independent of $x_0$ and $h)$ sending $h$ to $0^+$, by (6.12) we can derive that $\partial_{\nu} u(x_0) = 0$. We note that $x_0 \in \partial \Omega$ can be chosen arbitrarily, thus $\partial_{\nu} u|_{\partial \Omega} = 0$. However, it follows from (6.11) and (6.15) that $u \in D_1(\Omega)$ satisfying $(-\Delta - z_0) u = 0$ on $\mathbb{R}^n \setminus \Omega$. Extending $u$ into $\Omega$ by $u|_{\partial \Omega} = 0$, it then follows from (6.18) and the boundary values $u|_{\partial \Omega} = 0$, $\partial_{\nu} u|_{\partial \Omega} = 0$ that $u \in H^1_{\text{loc}}(\mathbb{R}^n)$ is a weak solution of $(-\Delta - z_0) u = 0$ on $\mathbb{R}^n$. The unique continuation property of second order elliptic differential equations shows that $u \equiv 0$, which contradicts (6.15).

Step 7. It remains to consider the case $q_0 = 1$ in (6.11). Let $\{w_j\}_{j=1}^M$ be a set of vectors in $C^\infty(\mathbb{R}^n \setminus \overline{\Omega})$ such that $\{\Pi_{\Gamma} w_j\}_{j=1}^M$ is a basis for $\text{Ran} \, \Pi_{\Gamma}$. Since $\Pi_{\Gamma}$ is symmetric with respect to the bilinear form $\langle \cdot, \cdot \rangle$ on $B_0 \times B_0$, the matrix $A$, $A_{ij} := \langle \Pi_{\Gamma} w_i, w_j \rangle$ is a complex symmetric matrix. To see $A$ is nondegenerate, we suppose that
\[
\exists x \in C^M, \quad \langle \Pi_{\Gamma} w_i, \sum_j x_j w_j \rangle = 0, \quad i = 1, \ldots, M.
\]
Since $\{\Pi_{\Gamma} w_i\}_{i=1}^M$ spans $\text{Ran} \, \Pi_{\Gamma}$, we have $\langle \Pi_{\Gamma} w, \sum_j x_j w_j \rangle = 0$ for all $w \in B_0$, which implies that $\langle \sum_j x_j \Pi_{\Gamma} w_j, w \rangle = 0$, $\forall w \in B_0$. Hence $\sum_j x_j \Pi_{\Gamma} w_j = 0 \Rightarrow x = 0$. We apply the Takagi factorization to the matrix $A$ to obtain that
\[
A = U^T \text{Diag}(r_1, \ldots, r_M) U, \quad \text{where } U \text{ is unitary, } r_j^2 \text{ are the eigenvalues of } AA^*.
\]
We remark that $U^T$ is the real transpose. Then we can write $A = B^T B$, $B$ nondegenerate due to the nondegeneracy of $A$. Transforming $\{w_j\}_{j=1}^M$ by the matrix $B$ and putting $u_j := \Pi_{\Gamma} w_j$, we may assume now that
\[
\text{Ran} \, \Pi_{\Gamma} = \text{span}\{u_j\}_{j=1}^M, \quad \langle u_j, w_i \rangle = \delta_{ij}.
\]
For any fixed $x_0 \in \partial \Omega$ and $h \in (0, h_0]$, we define the evolution of each $u_j$ as in (6.16):
\[
\begin{align*}
u_j(t) := (\Phi_t^{-1})^* v_j(t), & \quad v_j(t) := \Pi_{\Gamma, \Phi_t}(\Omega) w_j, \quad z(t) := z(\Phi_t). \quad (6.22)
\end{align*}
\]
We note that (6.21) still holds with $\partial_{\nu} v(0, x), u, f$ replaced by $\partial_{\nu} v_j(0, x), u_j$ and $f_j$ defined as in (6.20). The same arguments as in Step 6 show that
\[
\langle (P_{\Gamma} - z_0) \Pi_{\Gamma} v_j(0), w_i \rangle = \langle \Pi_{\Gamma} (-\Delta - z_0) f_j, w_i \rangle + z'(0) \langle u_j, w_i \rangle.
\]
Since $(P_{\Gamma} - z_0) \Pi_{\Gamma} = 0$ by (6.11) with $q_0 = 1$, it then follows that
\[
\langle (-\Delta - z_0) f_j, u_i \rangle = -z'(0) \delta_{ij}.
\]
We apply Green’s formula with boundary value of \( f_j \) like (6.20) to obtain that
\[
-\z'(0)\delta_{ij} = \langle (-\Delta - z_0)u_i, f_j \rangle + \int_{\partial\Omega} f_j \partial_\nu u_i \, dS = \int_{\partial\Omega} \chi_h(\partial_\nu u_i)(\partial_\nu u_j) \, dS.
\]
Since \( M \geq 2 \), for any \( x_0 \in \partial\Omega \) and \( h \in (0, h_0] \) we have
\[
\int_{\partial\Omega} \chi_h(\partial_\nu u_1)^2 \, dS = \int_{\partial\Omega} \chi_h(\partial_\nu u_2)^2 \, dS; \quad \int_{\partial\Omega} \chi_h(\partial_\nu u_1(\partial_\nu u_2) \, dS = 0.
\]
Sending \( h \to 0^+ \), it follows from (6.12) that
\[
(\partial_\nu u_1(x_0))^2 = (\partial_\nu u_2(x_0))^2, \quad \partial_\nu u_1(x_0) = \partial_\nu u_2(x_0) = 0.
\]
and thus \( \partial_\nu u_1 = \partial_\nu u_2 = 0 \). Since \( x_0 \in \partial\Omega \) is arbitrary, \( \partial_\nu u_1 \equiv 0 \). Hence the same arguments as in the end of Step 6 show that \( u_1 \equiv 0 \), which gives a contradiction. \( \Box \)

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Email address: xiong@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA