Wythoff’s Game with a Pass

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Abstract

The author introduces the Wythoff’s game with a pass, a variant of the classical Wythoff game. The latter game is played with two piles of stones, and two players take turns removing stones from one or both piles; when removing stones from both piles, the numbers of stones removed from each pile must be equal. The player who removes the last stone or stones is the winner.

An equivalent description of the game is that a single chess queen is placed somewhere on a large grid of squares, and each player can move the queen towards the upper left corner of the grid: leftward, upward, or to the upper left, any number of steps. The winner is the player who moves the queen into the upper left corner.

In Wythoff’s game with a pass, we modify the standard rules of the game so as to allow for a one-time pass, i.e., a pass move which may be used at most once in a game, and not from a terminal position. Once the pass has been used by either player, it is no longer available.

The author discovered that, when a pass move is available, the set of $P$-positions (previsous player’s position) of the Wythoff’s game with a pass is equal to $T_0 \cup T_1 \cup B - A$, where $T_0 = \{(x, y) : G(x, y) = 0\}$, $T_0 = \{(x, y) : G(x, y) = 1\}$, $B$ is a finite subset of $\{(x, y) : G(x, y) = 4\}$, $A$ is a finite subset of $T_0$ and $G$ is the Grundy number of the classical Wythoff game. Here the size of $B$ is 6, and the size of $A$ is 7.

According to the research of U.Blass and A.S.Fraenkel, for a position $(x, y) \in T_1$ there is always a position $(x', y') \in T_0$ such that $|x - x'| \leq 2$ and $|y - y'| \leq 4$, and hence the graph of the set of $P$-positions of the Wythoff’s game with a pass is very similar to the graph of the set of $P$-positions of the classical Wythoff’s game.

1 Wythoff’s game

Let $\mathbb{Z}_{\geq 0}$ and $N$ be the sets of non-negative number and natural numbers. Wythoff’s game is a played with two piles of stones. Two players take turns removing stones from one or both piles; when removing stones from both piles, the numbers of stones removed from each pile must be equal. The player who removes the last stone or stones is the winner.

An equivalent description of the game is that a single chess queen is placed somewhere on a large grid of squares, and each player can move the queen towards the upper left corner of the grid: leftward, upward, or to the upper left, any number of steps. The winner is the player who moves the queen into the corner.

Figure [1] is the grid of squares, and Figure [1] shows the move of a queen.
Definition 1.1. We define move\((x, y)\) of Wythoff’ s game for \(x, y \in \mathbb{Z}_{\geq 0}\) with \(1 \leq x + y\) by
\[
\text{move}(x, y) = M_1(x, y) \cup M_2(x, y) \cup M_3(x, y),
\]
where
\[
M_1(x, y) = \{(u, y) : u < x\},
\]
\[
M_2(x, y) = \{(x, v) : v < y\}
\]
with \(u, v \in \mathbb{Z}_{\geq 0}\)
and
\[
M_3(x, y) = \{(x - t, y - t) : 1 \leq t \leq \min(x, y)\}
\]
with \(x, y \geq 1\).

Remark 1.1. \(M_1(x, y), M_2(x, y), M_3(x, y)\) are the sets made by the leftward, the upward and the upperleft move. \(M_3(x, y)\) is an empty set if \(x = 0\) or \(y = 0\).

When we study the Wythoff’ s games, there are two important positions of the queen.

Definition 1.2. (a) \(N\)-positions, from which the next player can force a win, as long as he plays correctly at every stage.
(b) \(P\)-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.

Definition 1.3. (i) The minimum excluded value (\(\text{mex}\)) of a set, \(S\), of non-negative integers is the least non-negative integer that is not in \(S\).
(ii) Each position \(p = (x, y)\) of an impartial game \(G\) has an associated Grundy number, and we denote it by \(G(p)\).
The Grundy number is calculated recursively: \(G(p) = \text{mex}\{G(h) : h \in \text{move}(p)\}\).

Theorem 1.1. Let \(G\) be the Grundy number. Then, \(h\) is a \(P\)-position if and only if \(G(h) = 0\)

For the proof of this theorem, see [4].
Definition 1.4. Let $T_i = \{(x, y) : G(x, y) = i\}$ for any natural number $i$.

Lemma 1.1. $T_0 = \{(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n) : n \in \mathbb{Z}_{\geq 0}\} \cup \{\lfloor n\phi \rfloor + n, \lfloor n\phi \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where $\phi = \frac{1+\sqrt{5}}{2}$.

This is a well known result. See [3].

Lemma 1.2. (a) If $(x, y) \in T_0$, then move$(x, y) \cap T_0 = \emptyset$.
(b) If $(x, y) \in T_1$, then move$(x, y) \cap T_1 = \emptyset$.
(c) If $(x, y) \in T_i$ with $i \neq 0$, then move$(x, y) \cap T_0 = \emptyset$.
(d) If $(x, y) \in T_i$ with $i \neq 0, 1$, then move$(x, y) \cap T_1 = \emptyset$.

Proof. This is direct from the definition of Grundy number.

Lemma 1.3. Let $(a_n, b_n) = T_1$, where $a_n$ is increasing. Then $|b_n - (\lfloor n\phi \rfloor + n)| \leq 4$ and $|\lfloor n\phi \rfloor - 1 \leq a_n \leq \lfloor n\phi \rfloor + 2\nu$.

Proof. This result is proved in pp.330 in [2] as Corollary 5.14.

Remark 1.2. By Lemma 1.1 and Lemma 1.2, for each position $(x, y) \in T_1$ there exists a $(x', y') \in T_0$ such that $|x - x'| \leq 2$ and $|y - y'| \leq 4$.

Definition 1.5. (a) For any $n \in \mathbb{Z}_{\geq 0}$, let $V(n, 0) = \{(n, y) : y \in \mathbb{Z}_{\geq 0}\}$. We call this a vertical path.
(b) For any $n \in \mathbb{Z}_{\geq 0}$, let $H(0, n) = \{(x, n) : x \in \mathbb{Z}_{\geq 0}\}$ We call this a horizontal path.
(c) For any $n \in \mathbb{Z}_{\geq 0}$, let $D(n, 0) = \{(n + y, y) : y \in \mathbb{Z}_{\geq 0}\}$ and $D(0, n) = \{(x + n, y) : x \in \mathbb{Z}_{\geq 0}\}$. We call these diagonal paths.

Definition 1.6. Let $A = \{(0, 1), (1, 0), (2, 2), (3, 6), (6, 3), (5, 7), (7, 5)\}$ and $B = \{(0, 0), (1, 3), (3, 1), (2, 5), (5, 2), (6, 7), (7, 6)\}$.

Lemma 1.4. (i) For any $(x, y) \in T_1 \cup B - A$, move$(x, y) \cap (T_1 \cup B - A) = \emptyset$.
(ii) For any $(x, y) \notin T_0 \cup (T_1 \cup B - A)$, move$(x, y) \cap (T_1 \cup B - A) \neq \emptyset$.

Proof. We use a table of Grundy number. Figure 1 is a table of Grundy numbers of Wythoff’s game. This is mathematically the same as “The Grundy function of Wythoff’s game” of Table 1 in [1]. In Figure 1 the positions in $(T_1 \cup B - A)$ are in blue squares (or in gray square when printed in black and white.), and the positions in $A$ are in purple squares (or in light gray square when printed in black and white.)
(i) Let $(x, y) \in T_1 \cup B - A$.
(a) Suppose that $(x, y) \in T_1 - A$.
(a.1) By (b) of Lemma 1.2, move$(x, y) \cap T_1 = \emptyset$.
(a.2) Next, we prove that move$(x, y) \cap B = \emptyset$.
(a.2.1) We prove that we cannot reach from $(x, y) \in T_1 - A$ to any of positon of $B$ by moving leftward or upward. $A = \{(0, 1), (1, 0), (2, 2), (3, 6), (6, 3), (5, 7), (7, 5)\}$ \subset T_1$ and $(0, 1) \in V(0, 0) \cap A = \emptyset$, $(1, 0) \in V(1, 0) \cap A = \emptyset$, $(2, 2) \in V(2, 0) \cap A = \emptyset$, $(3, 6) \in V(3, 0) \cap A \neq \emptyset$, $(5, 7) \in V(5, 0) \cap A \neq \emptyset$, $(6, 3) \in V(6, 0) \cap A \neq \emptyset$ and $(7, 5) \in V(7, 0) \cap A \neq \emptyset$, where the vertical path is defined in Definition 1.5. We
Table 1: Table of Grundy numbers of the classical Wythoff’s game.

| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
| 1 1 2 0 4 5 3 7 8 6 10 11 9 13 14 12 16 |
| 2 2 0 1 5 3 4 8 6 7 10 11 9 12 14 17 13 |
| 3 3 2 1 2 0 4 5 3 4 8 6 7 10 11 12 16 15 |
| 4 4 2 1 2 0 5 4 3 7 8 6 10 11 9 14 12 15 |
| 5 5 3 4 0 6 8 10 13 2 16 17 13 17 10 19 12 |
| 6 6 4 5 3 2 7 6 9 10 3 3 4 5 13 0 11 12 16 |
| 7 7 6 8 1 2 0 1 5 6 9 13 12 11 16 15 10 19 |
| 8 8 7 6 9 0 1 2 3 4 5 3 3 4 5 13 14 15 17 12 |
| 9 9 8 7 10 1 2 3 4 5 3 4 5 13 14 15 17 12 16 |

Also have \((1, 0) \in H(0, 0) \cap A \neq \emptyset, (0, 1) \in H(0, 1) \cap A \neq \emptyset, (2, 2) \in H(0, 2) \cap A \neq \emptyset, (6, 3) \in H(0, 3) \cap A \neq \emptyset, (7, 5) \in H(0, 5) \cap A \neq \emptyset, (3, 6) \in H(0, 6) \cap A \neq \emptyset\) and \((5, 7) \in H(0, 7) \cap A \neq \emptyset\), where the horizontal path is defined in Definition 1.5. By (b) of Lemma 1.2 and Definition 1.5, there is only one position of \(T_1\) in each vertical and horizontal path, and hence we have \(V(0, 0) \cap T_1 = A = \emptyset, V(1, 0) \cap T_1 = A = \emptyset, V(2, 0) \cap T_1 = A = \emptyset, V(3, 0) \cap T_1 = A = \emptyset, V(5, 0) \cap T_1 = A = \emptyset, V(6, 0) \cap T_1 = A = \emptyset, V(7, 0) \cap T_1 = A = \emptyset, H(0, 0) \cap T_1 = A = \emptyset, H(0, 1) \cap T_1 = A = \emptyset, H(0, 2) \cap T_1 = A = \emptyset, H(0, 3) \cap T_1 = A = \emptyset, H(0, 5) \cap T_1 = A = \emptyset, H(0, 6) \cap T_1 = A = \emptyset, H(0, 7) \cap T_1 = A = \emptyset\). Since \(B \subset V(0, 0) \cup V(1, 0) \cup V(2, 0) \cup V(3, 0) \cup V(5, 0) \cup V(6, 0) \cup V(7, 0)\) and \(B \subset H(0, 0) \cup H(0, 1) \cup H(0, 2) \cup H(0, 3) \cup H(0, 5) \cup H(0, 6) \cup H(0, 7)\), there does not exist \((x, y) \in T_1\) from which we can reach the set \(B\) by the leftward move or the upward move.

(a.2.2) Next, we prove that we cannot reach from \((x, y) \in T_1\) to any position of \(B\) by moving to the upper left.

(a.2.2.1) The Diagonal path \(D(0, 0)\) contains the position \((0, 1) \in T_1 \cap A\). Since \(D(0, 0)\) contains only one position of \(T_1\), and hence \((T_1-A) \cap D(0, 0) = \emptyset\). Therefore there is no position of \(T_1-A\) from which we can reach the position \((0, 1) \in D(0, 1)\) by diagonal move. We prove almost the same thing for other positions in \(B\).

(a.2.2.2) The position \((1, 0) \in T_1 \cap D(1, 0) \cap A\), and hence \((T_1-A) \cap D(1, 0) = \emptyset\). Therefore there is no position of \(T_1-A\) from which we can reach the position \((7, 6) \in D(1, 0)\) by the move to the upper left.

(a.2.2.3) The position \((2, 2) \in T_1 \cap D(0, 0) \cap A\). Since the diagonal path \(D(1, 0)\) contains only one position of \(T_1\), \((T_1-A) \cap D(0, 0) = \emptyset\). Therefore there is no position of \(T_1-A\) from which we can reach the position \((0, 0) \in D(0, 0)\) by the move to the upper left.

(a.2.2.4) The position \((3, 6) \in T_1 \cap D(0, 3) \cap A\), and hence \((T_1-A) \cap D(0, 3) = \emptyset\). Therefore there is no position of \(T_1-A\) from which we can reach the position \((2, 5) \in D(0, 3)\) by the move to the upper left.
(a.2.2.5) The position \((5, 7) \in T_1 \cap D(0, 2) \cap A\), and hence \((T_1 - A) \cap D(0, 2) = \emptyset\). Therefore there is no position of \(T_1 - A\) from which we can reach the position \((1, 3) \in D(0, 2)\) by the move to the upper left.

(a.2.2.6) The position \((6, 3) \in T_1 \cap D(3, 0) \cap A\), and hence \((T_1 - A) \cap D(3, 0) = \emptyset\). Therefore there is no position of \(T_1 - A\) from which we can reach the position \((5, 2) \in D(3, 0)\) by the move to the upper left.

(a.2.2.7) The position \((7, 5) \in T_1 \cap D(2, 0) \cap A\), and hence \((T_1 - A) \cap D(2, 0) = \emptyset\). Therefore there is no position of \(T_1 - A\) from which we can reach the position \((3, 1) \in D(2, 0)\) by the move to the upper left.

(b) Suppose that \((x, y) \in B\). Since \(B \subseteq \{(u, v) : u \leq 7 \text{ and } v \leq 7\}\) and move\((x, y) \subseteq \{(u, v) : u \leq 7 \text{ and } v \leq 7\}\), by Figure 1 it is clear that move\((x, y) \cap (T_1 \cup B - A) \neq \emptyset\).

(ii) Let \((x, y) \notin T_0 \cup (T_1 \cup B - A)\). We prove that move\((x, y) \cap (T_1 \cup B - A) \neq \emptyset\).

(c) Suppose that \((x, y) \in \{(u, v) : u \leq 7 \text{ and } v \leq 7\}\). Then, move\((x, y) \subseteq \{(u, v) : u \leq 7 \text{ and } v \leq 7\}\). By Figure 1 we prove move\((x, y) \cap (T_1 \cup B - A) \neq \emptyset\).

(d) Next, we suppose that

\[
(x, y) \notin \{(u, v) : u \leq 7 \text{ and } v \leq 7\}.
\]  

(2)

Then, \(A \subseteq \{(u, v) : u \leq 7 \text{ and } v \leq 7\}\), and hence \((x, y) \notin T_0 \cup T_1\). This implies that \(G(x, y) \neq 0, 1\). Therefore, by (d) of Lemma 1.2 there exists \((s, t) \in T_1 \cap \text{move}(x, y)\).

(d.1) If \((s, t) \in T_1 - A\), then move\((x, y) \cap (T_1 \cup B - A) \neq \emptyset\).

(d.2) Suppose that \((s, t) \in A\). We prove that we can choose \((s', t') \in \text{move}(x, y) \cap A\).

(d.2.1) Suppose that \((s, t) = (5, 7)\).

(d.2.1.1) If we can reach \((5, 7)\) from \((x, y)\) by the upward move, then we can reach \((5, 2) \in B\) by the upward move.

(d.2.1.2) If we can reach \((5, 7)\) from \((x, y)\) by the leftward move, then \(y = 7 \leq 7\). Then by (2) \(x \geq 8\), and hence we can reach \((6, 7) \in B\) from \((x, y)\) by the leftward move.

(d.2.1.3) If we can reach \((5, 7)\) from \((x, y)\) by the move to the upper left, then we can reach \((1, 3) \in B\) by the upward move.

(d.2.2) Suppose that \((s, t) = (7, 5)\). Then we can use the method that is used in (d.2.1).

(d.2.3) Suppose that \((s, t) = (3, 6)\).

(d.2.3.1) If we can reach \((3, 6)\) from \((x, y)\) by the upward move, then we can reach \((3, 1) \in B\) by the upward move.

(d.2.3.2) If we can reach \((3, 6)\) from \((x, y)\) by the leftward move, then we can reach \((7, 6) \in B\) by the leftward move. Note that \(x \geq 8\) by (2).

(d.2.3.3) If we can reach \((3, 6)\) from \((x, y)\) by the move to the upper left, then we can reach \((2, 5) \in B\) by the move to the upper left.

(d.2.4) Suppose that \((s, t) = (6, 3)\). Then we can use the method that is used in (d.2.3).

(d.2.5) Suppose that \((s, t) = (2, 2)\).

(d.2.5.1) If we can reach \((2, 2)\) from \((x, y)\) by the upward move, then we can reach \((2, 5) \in B\) by the upward move. Note that \(y \geq 8\) by (2).

(d.2.5.2) If we can reach \((2, 2)\) from \((x, y)\) by the leftward move, then we can reach
(5, 2) ∈ B by the leftwar move. Note that x ≥ 8 by (2).
(d.2.5.3) If we can reach (2, 2) from (x, y) by the move to the upper left, then we can reach (0, 0) ∈ B by the move to the upper left.
(d.2.6) Suppose that (s, t) = (0, 1).
(d.2.6.1) If we can reach (0, 1) from (x, y) by the upward move, then we can reach (0, 0) ∈ B by the upward move.
(d.2.6.2) If we can reach (0, 1) from (x, y) by the leftwar move, then we can reach (3, 1) ∈ B by the leftwar move. Note that x ≥ 8 by (2).
(d.2.6.3) If we can reach (0, 1) from (x, y) by the move to the upper left, then y = x + 1. Then, by (2) x ≥ 7 and y ≥ 8. Therefore, we can reach (6, 7) ∈ B by the move to the upper left.
(d.2.7) Suppose that (s, t) = (1, 0). Then we can use the method that is used in (d.2.6).

□

2 Wythoff Game with a Pass

Throughout the remainder of this paper, we modify the standard rules of Wythoff’s game so as to allow for a one-time pass, i.e., a pass move which may be used at most once in a game, and not from a terminal position. Once the pass has been used by either player, it is no longer available. We call the Wythoff’s game without a pass “the classical Wythoff’s game”.

In this section, we denote the position of the queen with three coordinates \{x, y, p\}, where x, y define the position of the queen on the chess board, and p = 1 if the pass is still available and p = 0 if not.

When p = 0, then the game is mathematically the same as the classical Wythoff’s game.

we define the move of a queen in the Wythoff’s game with a pass in Definition 2.1.

Definition 2.1.
Let MP_1(x, y, p) = \{(s, t, p) : (s, t) ∈ move(x, y)\} for x, y ∈ Z_≥0 and p = 1, 2 and
MP_2(x, y, p) = {(x, y, 0)} for x, y ∈ Z_≥0 with 1 ≤ x + y and p = 1, (3)
where move(x, y) is defined in Definition 1.1.

We define move_pass(x, y, p) = MP_1(x, y, p) ∪ MP_2(x, y, p).

Remark 2.1. MP_2(x, y, p) is the set of the pass move, and it is an empty set if x = 0 or y = 0 or p = 0.

Definition 2.2. Each position (x, y, p) of the the Wythoff’s game has an associated Grundy number, and we denote it by G_{pass}(x, y, p). The Grundy number is calculated recursively: G_{pass}((x, y, p) = mex{G_{pass}(h) : h ∈ move_pass(x, y, p)}.
Definition 2.3. Let $\mathcal{T}_0 = \{(x, y, 0) : (x, y) \in T_0\}$ and $\mathcal{T}_1 = \{(x, y, 0) : (x, y) \in T_1\}$.

Lemma 2.1. $\mathcal{T}_0 = \{(x, y, 0) : \mathcal{G}(x, y, 0) = 0\}$.

Proof. This is a direct result of Definition 2.1, Definition 2.2 and Definition 2.2.

Therefore, the set $T_0$ that is the set of $\mathcal{P}$-positions in the classical Wythoff’s game is the set of $\mathcal{P}$-positions of the Wythoff’s game with a pass when $p = 0$.

Definition 2.4. Let $\mathcal{A} = \{(s, t, 1) : (s, t) \in A\} = \{(0, 1, 1), (1, 0, 1), (2, 2, 1), (3, 6, 1), (6, 3, 1), (5, 7, 1), (7, 5, 1)\}$ and $\mathcal{B} = \{(s, t, 1) : (s, t) \in B\} = \{(0, 0, 1), (1, 3, 1), (3, 1, 1), (2, 5, 1), (5, 2, 1), (6, 7, 1), (7, 6, 1)\}$, where the sets $A, B$ are defined in Definition 1.6.

Lemma 2.2. (i) For any $(x, y, 1) \in T_1 \cup \mathcal{B} - \mathcal{A}$, $\text{move}_{\text{pass}}(x, y, 1) \cap (T_1 \cup \mathcal{B} - \mathcal{A}) = \emptyset$.

Proof. This lemma is a direct from Lemma 1.4 and Definition 2.1.

Theorem 2.1. The set of $\mathcal{P}$-positions of Wythoff’s game with a pass is $\mathcal{P} = T_0 \cup T_1 \cup B - A$.

Proof. (a) Let $(x, y, p) \in \mathcal{P}$. We prove that $\text{move}_{\text{pass}}(x, y, p) \cap \mathcal{P} = \emptyset$.

(a, 1) If $p = 0$, then $(x, y, 0) \in \mathcal{T}_0$. Since $\text{move}_{\text{pass}}(x, y, 0) \subset \{(s, t, u) : s = 0\}$, by (a) of Lemma 1.2 and Definition 2.1, $\text{move}_{\text{pass}}(x, y, 0) \cap \mathcal{P} = \emptyset$.

(a, 2) Suppose that $p = 1$. If $(x, y, p) \in T_1 \cup \mathcal{B} - \mathcal{A}$, by Lemma 2.2 and Definition 2.3, $\text{move}_{\text{pass}}(x, y, 1) \cap (T_1 \cup \mathcal{B} - \mathcal{A}) = \emptyset$.

(b) Let $(x, y, p) \notin \mathcal{P}$.

(b, 1) Suppose that $p = 0$, then $(x, y, 0) \notin T_0$. Then $(x, y) \notin T_0$, and by (c) of Lemma 1.2, $\text{move}(x, y) \cap T_0 \neq \emptyset$. Therefore $\text{move}_{\text{pass}}(x, y, 0) \cap T_0 \neq \emptyset$.

(b, 2) Suppose that $p = 1$, then $(x, y, 1) \notin T_1 \cup \mathcal{B} - \mathcal{A}$.

(b, 2.1) If $(x, y, 1) \notin \{(x, y, 1) : (x, y) \in T_0\}$, then by (ii) of Lemma 2.2, $\text{move}_{\text{pass}}(x, y, 1) \cap (T_1 \cup \mathcal{B} - \mathcal{A}) \neq \emptyset$.

(b, 2.2) If $(x, y, 1) \in \{(x, y, 1) : (x, y) \in T_0\}$, then $(x, y, 0) \in T_0$.

Proof. This a direct result of Definition 2.1, Definition 2.2 and Definition 2.2.

By Lemma 1.3 and Remark 1.2, the positions in $T_1$ are very near to the positions in $T_0$, and hence by Theorem 2.1, the graph of the set of $\mathcal{P}$-positions of the classical Wythoff’s game with a pass is very similar to the graph of the set of $\mathcal{P}$-positions of the classical Wythoff’s game. See Graphs in Figures 3 and 4.
Figure 3: The graph of the set of $P$-positions of the classical Wythoff’s game

Figure 4: The set of $P$-positions of the Wythoff’s game with a pass

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