The monotonicity properties for the rank of overpartitions

Huan Xiong\textsuperscript{1} and Wenston J.T. Zang\textsuperscript{2}

\textsuperscript{1}Institut de Recherche Mathématique Avancée, UMR 7501
Université de Strasbourg et CNRS, F-67000 Strasbourg, France

\textsuperscript{2}Institute of Advanced Study of Mathematics
Harbin Institute of Technology, Heilongjiang 150001, P.R. China

Email: \textsuperscript{1}xiong@math.unistra.fr, \textsuperscript{2}zang@hit.edu.cn

Abstract. The $D$-rank and $M_2$-rank of an overpartition were introduced by Lovejoy. Let $N(m, n)$ denote the number of overpartitions of $n$ with $D$-rank $m$, and let $N_2(m, n)$ denote the number of overpartitions of $n$ with $M_2$-rank $m$. In 2014, Chan and Mao proposed a conjecture on the monotonicity properties of $N(m, n)$ and $N_2(m, n)$. In this paper, we prove the Chan-Mao monotonicity conjecture. To be specific, we show that for any integer $m$ and nonnegative integer $n$, $N_2(m, n) \leq N_2(m, n + 1)$; and for $(m, n) \neq (0, 4)$ with $n \neq |m| + 2$, we have $N(m, n) \leq N(m, n + 1)$. Also, when $m$ increases, we prove that $N(m, n) \geq N(m + 2, n)$ and $N_2(m, n) \geq N_2(m + 2, n)$ for any $m, n \geq 0$, which is an analogue of Chan and Mao’s result for partitions.

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1 \ Introduction

The aim of this paper is to study the monotonicity properties of the $D$-rank and $M_2$-rank on overpartitions and therefore prove a conjecture of Chan and Mao \cite{ChMa14}.

Recall that a partition of a nonnegative integer $n$ is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\sum_{1 \leq i \leq \ell} \lambda_i = n$. Here $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are called parts of the partition $\lambda$ (see \cite{Andrews76}). The rank of a partition was defined by Dyson \cite{Dyson44} as the largest part of the partition minus the number of parts. Dyson first conjectured and then proved by Atkin and Swinnerton-Dyer \cite{AS63} that the rank can provide combinatorial interpretations to the following Ramanujan’s famous congruence for the partition function modulo 5 and 7, respectively:

\begin{align}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}.
\end{align}
where \( p(n) \) denotes the number of partitions of \( n \). Since then, various results on the rank of partitions have been obtained by many mathematicians (For example, see \([2,5,7,9,14,16,18,21,23,27,34,39]\)).

Let \( N(m, n) \) denote the number of partitions of \( n \) with rank \( m \). Chan and Mao \([16]\) established the following monotonicity properties for \( N(m, n) \).

**Theorem 1.1** (Chan and Mao \([16]\)). For \( n \geq 12, m \geq 0 \) and \( n \neq m + 2 \),

\[
N(m, n) \geq N(m, n - 1). \tag{1.3}
\]

**Theorem 1.2** (Chan and Mao \([16]\)). For \( n \geq 0 \) and \( m \geq 0 \),

\[
N(m, n) \geq N(m + 2, n). \tag{1.4}
\]

At the end of their paper, Chan and Mao \([16]\) proposed a conjecture on the monotonicity properties of the \( D \)-rank and \( M_2 \)-rank on an overpartition. Recall that an overpartition was defined by Corteel and Lovejoy \([19]\) as a partition of \( n \) in which the first occurrence of a part may be overlined. For example, there are 14 overpartitions of 4:

\[
(4), \quad (\overline{4}), \quad (3, 1), \quad (\overline{3}, 1), \quad (3, \overline{1}), \quad (\overline{3}, \overline{1}), \quad (2, 2),
\
(2, 2) \quad (2, 1, 1), \quad (2, 1, 1), \quad (\overline{2}, 1, 1), \quad (\overline{2}, \overline{1}, 1), \quad (1, 1, 1, 1), \quad (1, 1, 1, 1).
\]

Lovejoy \([35]\) defined the \( D \)-rank of an overpartition as the largest part in the partition minus the number of parts, which is an analogue of the rank on ordinary partitions. Let \( \overline{N}(m, n) \) denote the number of overpartitions of \( n \) with \( D \)-rank \( m \), Lovejoy \([35, Proposition 1.1] \) gave the following generating function of \( \overline{N}(m, n) \):

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) z^m q^n = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(zq; q)_k (q/z; q)_k}. \tag{1.5}
\]

Here and throughout the rest of this paper, we adopt the common \( q \)-series notation \([1]\):

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n) \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.
\]

The \( M_2 \)-rank on overpartitions was also introduced by Lovejoy \([36]\). For an overpartition \( \lambda \), let \( \lambda_1 \) denote the largest part of \( \lambda \), \( \ell(\lambda) \) denote the number of parts of \( \lambda \), and \( \lambda_o \) denote the partition consisting of the non-overlined odd parts of \( \lambda \). Then define

\[
M_2 \text{-rank}(\lambda) = \left\lfloor \frac{\lambda_1}{2} \right\rfloor - \ell(\lambda) + \ell(\lambda_o) - \chi(\lambda), \tag{1.6}
\]

where \( \chi(\lambda) = 1 \) if the largest part of \( \lambda \) is odd and non-overlined, and otherwise \( \chi(\lambda) = 0 \).
For instance, let \( \lambda = (7, 5, 4, 2, 1, 1, 1, 1, 1) \). Then \( \lambda_1 = 7 \), \( \ell(\lambda) = 8 \), \( \lambda_o = (5, 1, 1) \), \( \ell(\lambda_o) = 3 \) and \( \chi(\lambda) = 0 \). Therefore,

\[
M_2\text{-rank}(\lambda) = 3 - 8 + 3 = -2.
\]

Let \( \overline{N}_2(m, n) \) denote the number of overpartitions of \( n \) with \( M_2 \)-rank \( m \). Lovejoy \[36\] found the generating function of \( \overline{N}_2(m, n) \) as follows:

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}_2(m, n) z^n q^m = \sum_{k=0}^{\infty} \frac{(-1; q)_{2k} q^k}{(zq^2; q^2)_k (q^2/z; q^2)_k}.
\]

(1.7)

Various results on the \( D \)-rank and \( M_2 \)-rank of overpartitions can be found in \[3, 4, 15, 22, 24–26, 35–38\]. In 2014, Chan and Mao \[16\] proposed the following monotonicity conjecture on \( \overline{N}(m, n) \) and \( \overline{N}_2(m, n) \):

**Conjecture 1.3** (Chan and Mao \[16\]). For all integer \( m \in \mathbb{Z} \) and \( n \geq 0 \),

\[
\overline{N}_2(m, n) \geq \overline{N}_2(m, n - 1).
\]

(1.8)

For \( (m, n) \neq (0, 4) \) with \( n \neq |m| + 2 \), we have

\[
\overline{N}(m, n) \geq \overline{N}(m, n - 1).
\]

(1.9)

The main purpose of this paper is to give analogues of Theorems 1.1 and 1.2. To be specific, we obtain the following results:

**Theorem 1.4.** For \( m, n \geq 0 \) with \( n \neq m + 2 \) and \( (m, n) \neq (0, 4) \),

\[
\overline{N}(m, n) \geq \overline{N}(m, n - 1).
\]

(1.10)

For \( m, n \geq 0 \), we have

\[
\overline{N}_2(m, n) \geq \overline{N}_2(m, n - 1).
\]

(1.11)

**Theorem 1.5.** For \( m, n \geq 0 \), we have

\[
\overline{N}(m, n) \geq \overline{N}(m + 2, n),
\]

(1.12)

and

\[
\overline{N}_2(m, n) \geq \overline{N}_2(m + 2, n),
\]

(1.13)

By the generating functions (1.5) and (1.7), it is easy to see that \( \overline{N}(-m, n) = \overline{N}(m, n) \) and \( \overline{N}_2(-m, n) = \overline{N}_2(m, n) \). Therefore Theorem 1.4 verifies Conjecture 1.3.

This paper is organized as follows. Some preliminary results are given in Section 2. Then in Section 3 we establish a nonnegativity result Lemma 3.1 and use it to give a proof of Theorem 1.4. Section 4 is devoted to prove Theorem 1.5.
2 Preliminary

In order to prove Theorems 1.4 and Theorem 1.5, we need to recall the definition of $f_{m,k}(q)$, which was given by Chan and Mao [16].

**Definition 2.1** (Chan and Mao). Define $f_{m,k}(q)$ as follows:

$$
\sum_{m=-\infty}^{\infty} z^m f_{m,k}(q) := \frac{1 - q}{(zq;q)_k(q/z;q)_k}. \tag{2.1}
$$

When $k = 0$, by definition we see that $f_{0,0}(q) = 1 - q$ and $f_{m,0}(q) = 0$ for all $m \neq 0$. Chan and Mao [16, Lemma 9] gave the following expressions for $f_{m,1}(q)$ and $f_{m,2}(q)$.

**Theorem 2.2** (Chan and Mao [16]). For all integer $m$,

$$f_{m,1}(q) = \sum_{n=|m|}^{\infty} (-1)^{m+n} q^n = \frac{q^{|m|}}{1 + q}. \tag{2.2}$$

For $m = 0$,

$$f_{0,2}(q) = -q + \frac{1}{1 - q^3} + \frac{q^2}{1 - q^4} + \frac{q^8}{(1 - q^3)(1 - q^4)} \tag{2.3}$$

and for $m \neq 0$,

$$f_{m,2}(q) = q^{|m|} \left( \frac{1 - q^{m+1}}{(1 - q^2)(1 - q^3)} + \frac{q^{m+3}}{(1 - q^3)(1 - q^4)} \right). \tag{2.4}$$

Chan and Mao [16, Lemma 11] also found the following nonnegative property for $f_{m,k}(q)$ when $k \geq 2$. For the remainder part of this paper, let $\{b_n\}_{n=0}^\infty$ be any sequence of nonnegative integers but not necessarily the same in different equations.

**Theorem 2.3** (Chan and Mao [16]). When $k \geq 2$,

$$f_{0,k}(q) = -q + q^2 + \sum_{n=0}^{\infty} b_n q^n; \tag{2.5}$$

$$f_{1,k}(q) = q^{k+2} + \sum_{n=0}^{\infty} b_n q^n; \tag{2.6}$$

$$f_{m,k}(q) = \sum_{n=0}^{\infty} b_n q^n, \text{ for } m \geq 2. \tag{2.7}$$

By definition, it is easy to check that the constant term of $f_{0,k}(q)$ is equal to 1. Hence (2.5) yields the following corollary:
Corollary 2.4. When $k \geq 2$,
\[
f_{0,k}(q) = 1 - q + q^2 + \sum_{n=0}^{\infty} b_n q^n. \tag{2.8}
\]

We also need the following two lemmas in [16].

Lemma 2.5 (See Lemma 8 of Chan and Mao [16]). When $k \geq 0$, we have
\[
f_{m,k+1}(q) = \sum_{n=-\infty}^{\infty} f_{n,k}(q) q^{(k+1)|m-n|} \frac{1}{1 - q^{2k+2}}.
\]

Lemma 2.6 (See Lemma 10 of Chan and Mao [16]). For any positive integer $m$,
\[
\frac{1 - q^{m+1}}{(1 - q^2)(1 - q^4)}
\]
has nonnegative power series coefficients.

3 The proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To this end, we need the following lemma.

Lemma 3.1. For any nonnegative integer $a$, $b$ and $c$, the coefficient of $q^n$ in
\[
\frac{q^a}{1 + q^c} + \frac{q^b}{(1 - q^3)(1 - q^4)}
\]
is nonnegative for $n \geq b + 6$.

Proof. It is clear that
\[
\frac{q^b}{(1 - q^3)(1 - q^4)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{b+3i+4j}.
\]
Note that for any $n \geq 6$, there exists $i, j \geq 0$ such that $3i + 4j = n$. To be specific,
\[
(i, j) = \begin{cases} (k, 0) & \text{if } n = 3k; \\ (k - 1, 1) & \text{if } n = 3k + 1; \\ (k - 2, 2) & \text{if } n = 3k + 2. \end{cases} \tag{3.1}
\]
Hence we see that, the coefficient of $q^n$ in
\[
\frac{q^b}{(1 - q^3)(1 - q^4)} \tag{3.2}
\]
is at least 1. On the other hand,
\[
\frac{q^a}{1 + q^c} = \sum_{m=0}^{\infty} (-1)^m q^{cm+a}. \tag{3.3}
\]
Evidently, for any nonnegative integer \(n\), the coefficient of \(q^n\) in \(\sum_{m=0}^{\infty} (-1)^m q^{cm+a}\) is either \(-1, 0\) or \(1\). Thus when \(n \geq b + 6\), the coefficient of \(q^n\) in
\[
\frac{q^a}{1 + q^c} + \frac{q^b}{(1 - q^3)(1 - q^4)}
\]
is nonnegative. This yields the desired result.

We are now in a position to prove Theorem 1.4.

**Proof of Theorem 1.4.** We first prove (1.10) with the aid of Lemma 3.1, and then show (1.11).

From (1.5), it is clear to see that
\[
1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left( N(m, n) - N(m, n - 1) \right) z^m q^n = \sum_{k=0}^{\infty} \frac{(-1; q)_k}{(zq; q)_k(q/z; q)_k} q^{k(k+1)/2} (1 - q) \tag{3.4}
\]

By the definition of \(f_{m,k}(q)\) (see (2.1)), we derive that
\[
1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left( N(m, n) - N(m, n - 1) \right) z^m q^n = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} (-1; q)_k q^{k(k+1)/2} f_{m,k}(q). \tag{3.5}
\]
Hence for fixed integer \(m \neq 0\),
\[
\sum_{n=1}^{\infty} \left( N(m, n) - N(m, n - 1) \right) q^n = \sum_{k=0}^{\infty} (-1; q)_k q^{k(k+1)/2} f_{m,k}(q). \tag{3.6}
\]
When \(m = 0\), by (3.5), (3.6) and Theorem 2.2 we find that
\[
\sum_{n=1}^{\infty} \left( N(0, n) - N(0, n - 1) \right) q^n = -q + \frac{2q}{1 + q} + 2(1 + q)q^3 \left( -q + \frac{1}{1 - q^3} + \frac{q^2}{1 - q^4} + \frac{q^4}{(1 - q^3)(1 - q^4)} \right) + \sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} f_{0,k}(q). \tag{3.7}
\]
By Corollary 2.4, we derive that
\[
\sum_{n=1}^{\infty} \left( N(0, n) - N(0, n - 1) \right) q^n
\]
\[- q - 2q^4 - 2q^5 + \frac{2(1 + q)q^3}{1 - q^3} + \frac{2(1 + q)q^5}{1 - q^4} + \frac{2q^{12}}{(1 - q^3)(1 - q^4)} \]
\[+ \frac{2q}{1 + q} + \frac{2q^{11}}{(1 - q^3)(1 - q^4)} \]
\[+ \sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} \left( 1 - q + q^2 + \sum_{n=0}^{\infty} b_n q^n \right). \tag{3.8} \]

The last term in (3.8) can be transformed as follows:
\[
\sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} \left( 1 - q + q^2 + \sum_{n=0}^{\infty} b_n q^n \right) = \sum_{k=3}^{\infty} 2(1 + q)(-q^2; q)_k q^{k(k+1)/2} (1 - q + q^2) + \sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} \sum_{n=0}^{\infty} b_n q^n 
\]
\[= \sum_{k=3}^{\infty} 2(1 + q^3)(-q^2; q)_k q^{k(k+1)/2} + \sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} \sum_{n=0}^{\infty} b_n q^n, \tag{3.9} \]

which clearly has nonnegative coefficients. Moreover, by Lemma 3.1 the coefficient of \( q^n \) in
\[\frac{2q}{1 + q} + \frac{2q^{11}}{(1 - q^3)(1 - q^4)} \]
is nonnegative for \( n \geq 17 \). From the above analysis, we see that
\[\overline{N}(0, n) \geq \overline{N}(0, n - 1) \]
for \( n \geq 17 \). It is trivial to check that for \( 1 \leq n \leq 16 \),
\[\overline{N}(0, n) \geq \overline{N}(0, n - 1) \]
extcept for \( n = 2 \) or \( n = 4 \). Therefore Theorem 1.4 holds for \( m = 0 \).

We now assume that \( m \geq 1 \). Substituting (2.2) and (2.4) into (3.6), we have
\[
\sum_{n=1}^{\infty} \left( \overline{N}(m, n) - \overline{N}(m, n - 1) \right) q^n = 2q^{m+1} + \sum_{k=3}^{\infty} (-1; q)_k q^{k(k+1)/2} f_{m,k}(q) 
\]
\[+ 2(1 + q)q^{m+3} \left( \frac{1 - q^{m+1}}{(1 - q^2)(1 - q^3)} + \frac{q^{m+3}}{(1 - q^3)(1 - q^4)} \right). \tag{3.10} \]

From Theorem 2.3 we see that for \( k \geq 3 \), \( f_{m,k}(q) \) has nonnegative coefficients. We proceed to show the coefficients of \( q^n \) in
\[\frac{2q^{m+1}}{1 + q} + 2(1 + q)q^{m+3} \left( \frac{1 - q^{m+1}}{(1 - q^2)(1 - q^3)} + \frac{q^{m+3}}{(1 - q^3)(1 - q^4)} \right) \tag{3.11} \]
is nonnegative for all \( n \geq m + 3 \).

We first assume that \( m \neq 1, 3 \). In this case, we transform (3.11) as follows:

\[
\frac{2q^{m+1}}{1+q} + 2(1+q)q^{m+3} \left( \frac{1-q^{m+1}}{(1-q^2)(1-q^3)} + \frac{q^{m+3}}{(1-q^3)(1-q^4)} \right) \\
= \frac{2q^{m+1}}{1+q} + 2q^{m+4} \frac{1-q^{m+1}}{(1-q^2)(1-q^3)} \\
+ 2q^{m+3} \frac{1-q^{m+1}}{(1-q^2)(1-q^3)} + 2(1+q) \frac{q^{2m+6}}{(1-q^3)(1-q^4)}. \tag{3.12}
\]

By Lemma 2.6, we find that

\[
\frac{2q^{m+3}}{(1-q^2)(1-q^3)}
\]

has nonnegative coefficients in \( q^n \) for all \( n \geq 1 \). Moreover,

\[
\frac{2q^{m+1}}{1+q} + 2q^{m+4} \frac{1-q^{m+1}}{(1-q^2)(1-q^3)} = \frac{2q^{m+1}}{1+q} + 2q^{m+4} \frac{1-q^3 + q^3 - q^{m+1}}{(1-q^2)(1-q^3)} \\
= \frac{2q^{m+1}}{1+q} + 2q^{m+4} \frac{1-q^{m+1}}{1-q^2} + 2q^{m+7} \frac{1-q^{m-2}}{(1-q^2)(1-q^3)} \\
= 2q^{m+1} \frac{1-q + q^3}{1-q^2} + 2q^{m+7} \frac{1-q^{m-2}}{(1-q^2)(1-q^3)} \\
= 2q^{m+1} - 2q^{m+2} + 2q^{m+7} \frac{1-q^{m-2}}{(1-q^2)(1-q^3)}.
\]

Notice that when \( m \neq 1, 3 \), by Lemma 2.6 we obtain

\[
2q^{m+7} \frac{1-q^{m-2}}{(1-q^2)(1-q^3)} = \sum_{n=0}^{\infty} b_n q^n.
\]

This yields that (3.12) has nonnegative coefficients in \( q^n \) for \( n \geq m + 3 \), as desired.

It remains to consider the case \( m = 1 \) or \( 3 \). For \( m = 1 \), it is trivial to calculate that

\[
\frac{2q^2}{1+q} + \frac{2q^4 + 2q^5}{(1-q^3)(1-q^4)}. \tag{3.13}
\]

From Lemma 3.1 we see that for \( n \geq 10 \), the coefficient of \( q^n \) in

\[
\frac{2q^2}{1+q} + \frac{2q^4}{(1-q^3)(1-q^4)}
\]

is nonnegative. Hence we derive that \( N(1, n) \geq N(1, n - 1) \) for \( n \geq 10 \). It is trivial to check that for \( 4 \leq n \leq 9 \), \( N(1, n) \geq N(1, n - 1) \) also holds. This yields the case for \( m = 1 \).
Finally, for $m = 3$, (3.11) is equal to:

$$
\frac{2q^4}{1 + q} + \frac{2q^{12}}{(1 - q^2)(1 - q^4)} + \frac{2q^{13}}{(1 - q^3)(1 - q^4)} + \frac{2(1 + q)(1 + q^2)q^6}{1 - q^2}.
$$

(3.14)

Using Lemma 3.1, we find that for $n \geq 18$, the coefficient of $q^n$ in

$$
\frac{2q^4}{1 + q} + \frac{2q^{12}}{(1 - q^3)(1 - q^4)}
$$

(3.15)

is nonnegative. This yields that $N(3, n) \geq N(3, n - 1)$ for $n \geq 18$. After checking $N(3, n) \geq N(3, n - 1)$ for $6 \leq n \leq 17$, we find that (1.10) is valid for $m = 3$.

We next prove (1.11). From (1.7), we see that

$$
1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} (N^2(m, n) - N^2(m, n - 1))z^m q^n = 1 - q + 2 \sum_{k=0}^{\infty} \frac{(-1; q)_{2k}q^k}{(zq^2; q^2)_k(q^2/z; q^2)_k}.
$$

(3.16)

Hence

$$
1 + \sum_{n=1}^{\infty} (N^2(0, n) - N^2(0, n - 1))q^n = 1 - q + 2 \sum_{k=1}^{\infty} (-q^2; q)_{2k-2}q^k f_{0,k}(q^2),
$$

(3.17)

and for $m \geq 1$,

$$
\sum_{n=1}^{\infty} (N^2(m, n) - N^2(m, n - 1))q^n = 2 \sum_{k=1}^{\infty} (-q^2; q)_{2k-2}q^k f_{m,k}(q^2).
$$

(3.18)

Similar as the proof of (1.10), we first assume that $m = 0$. From Theorem 2.2 and Corollary 2.4, we deduce that

$$
1 + \sum_{n=1}^{\infty} (N(0, n) - N(0, n - 1))q^n = 1 - q + 2 \sum_{k=1}^{\infty} (-q^2; q)_{2k-2}q^k f_{0,k}(q^2).
$$

(3.10)

and for $m \geq 1$,

$$
2 \sum_{k=1}^{\infty} (-q^2; q)_{2k-2}q^k f_{m,k}(q^2).
$$

(3.18)
This allows us to transform
implies that
By Theorem 2.3 the coefficient of
Thus the coefficient of
Setting
We proceed to show that (1.11) holds for

\[
\sum_{k=3}^\infty (-q^2; q)_{2k-2} q^k \left( 1 - q^2 + q^4 + \sum_{n=0}^\infty b_n q^{2n} \right)
\]
\[
= 1 - q - 2q^4 - 2q^6 - 2q^7 - 2q^9 + \frac{2q}{1 + q^2} + \frac{2q^{18} + 2q^{20} + 2q^{21} + 2q^{23}}{(1 - q^6)(1 - q^8)}
\]
\[
+ 2(1 + q^2)(1 + q^3)q^2 \left( \frac{1}{1 - q^6} + \frac{q^4}{1 - q^8} \right)
\]
\[
+ 2 \sum_{k=3}^\infty (-q^2; q)_{2k-2} q^k \sum_{n=0}^\infty b_n q^{2n} + 2 \sum_{k=3}^\infty (-q^3; q)_{2k-3} q^k (1 + q^6).
\]

(3.19)

Setting \( a = 0, b = 10 \) and \( q = q^2 \) in Lemma 3.1 we find that for \( n \geq 33 \), the coefficient of \( q^n \) in

\[
\frac{2q}{1 + q^2} + \frac{2q^{21}}{(1 - q^6)(1 - q^8)}
\]

is nonnegative. Thus the coefficient of \( q^n \) in (3.19) is nonnegative for \( n \geq 33 \), which implies that \( \overline{N_2}(0, n) \geq \overline{N_2}(0, n - 1) \) for \( n \geq 33 \). It is trivial to check that for \( 1 \leq n \leq 32 \), \( \overline{N_2}(0, n) \geq \overline{N_2}(0, n - 1) \) also holds. This yields (1.11) for \( m = 0 \).

We proceed to show that (1.11) holds for \( m \geq 1 \). From Theorem 2.2 and (3.18), we have

\[
\sum_{n=1}^\infty (\overline{N_2}(m, n) - \overline{N_2}(m, n - 1)) q^n
\]
\[
= 2q f_{m,1}(q^2) + 2(-q^2; q)_4 q^3 f_{m,3}(q^2) + 2 \sum_{k=2 \hspace{1cm} k \neq 3}^\infty (-q^2; q)_{2k-2} q^k f_{m,k}(q^2)
\]
\[
= \frac{2q^{2m+1}}{1 + q^2} + 2(-q^2; q)_4 q^3 f_{m,3}(q^2) + 2 \sum_{k=2 \hspace{1cm} k \neq 3}^\infty (-q^2; q)_{2k-2} q^k f_{m,k}(q^2).
\]

(3.20)

From Lemma 2.3, we see that

\[
f_{m,3}(q) = \sum_{n=-\infty}^\infty f_{n,2}(q) \frac{q^{3|m-n|}}{1 - q^6} = f_{m,2}(q) + f_{m,2}(q) \frac{q^6}{1 - q^6} + \sum_{n=-\infty}^\infty f_{n,2}(q) \frac{q^{3|m-n|}}{1 - q^6}.
\]

(3.21)

By Theorem 2.3 the coefficient of \( q^n \) in \( f_{m,2}(q) \) is nonnegative for all integer \( m \) and \( n \geq 0 \). This allows us to transform \( f_{m,3}(q) \) as follows:

\[
f_{m,3}(q) = f_{m,2}(q) + \sum_{n=0}^\infty b_n q^n
\]
\[
= q^m \left( \frac{1 - q^{m+1}}{(1 - q^2)(1 - q^4)} + \frac{q^{m+3}}{(1 - q^3)(1 - q^4)} \right) + \sum_{n=0}^\infty b_n q^n.
\]

(3.22)
Hence
\[ 2(-q^2; q)q^3 f_{m,3}(q^2) = 2(-q^2; q)q^{2m+3} \left( \frac{1 - q^{2m+2}}{(1 - q^2)(1 - q^3)} + \frac{q^{2m+6}}{(1 - q^2)(1 - q^3)} + \sum_{n=0}^{\infty} b_n q^n \right) = 2(-q^4; q)q^{2m+3} \left( \frac{1 - q^{2m+2}}{(1 - q^2)(1 - q^3)} + \frac{2q^{4m+9}}{(1 - q^2)(1 - q^4)} + \sum_{n=0}^{\infty} b_n q^n \right) = 2(-q^4; 2q^{2m+3} \left( \frac{1 - q^{2m+2}}{(1 - q^2)(1 - q^3)} + \frac{2q^{4m+9}}{(1 - q^2)(1 - q^4)} + \sum_{n=0}^{\infty} b_n q^n \right) \]

Moreover, from Theorem 2.3 we see that
\[ \sum_{n=0}^{\infty} (-q^2; q)_{2k-2} q^k f_{m,k}(q^2) = \sum_{n=0}^{\infty} b_n q^n. \] (3.25)

Next we show that $N2(m, n) \geq N2(m, n - 1)$ for $m \geq 2$. Substituting (3.24) and (3.25) into (3.20), we derive that
\[ \sum_{n=1}^{\infty} (N2(m, n) - N2(m, n - 1)) q^n = \frac{2q^{2m+1}}{1 + q^2} + \frac{2q^{2m+3}}{(1 - q^2)(1 - q^3)} + \frac{1 - q^{2m+2}}{(1 - q^2)(1 - q^3)} + \frac{2q^{4m+9}}{(1 - q^2)(1 - q^3)} + \sum_{n=0}^{\infty} b_n q^n \]
By Lemma 2.6, we see that when \( m \geq 2 \),
\[
q^{2m+6} \frac{1 - q^{2m-1}}{(1 - q^2)(1 - q^3)} = \sum_{n=0}^{\infty} b_n q^n.
\]
This gives \( \overline{N}_2(m, n) \geq \overline{N}_2(m, n - 1) \), as desired.

Finally, we consider the case \( m = 1 \). In this case, by (3.24),
\[
2(-q^2; q)q^3 f_{1,3}(q^2) = 2q^3 \frac{1 + q^2}{1 - q^3} + \frac{2q^{13}}{(1 - q^3)(1 - q^4)} + \sum_{n=0}^{\infty} b_n q^n. \tag{3.27}
\]
Substituting (3.25) and (3.27) into (3.20), we see that
\[
\sum_{n=1}^{\infty} (\overline{N}_2(1, n) - \overline{N}_2(1, n-1)) q^n = \frac{2q^3}{1 + q^2} + 2q^5 \frac{1 + q^2}{1 - q^3} + \frac{2q^{13}}{(1 - q^3)(1 - q^4)} + \sum_{n=0}^{\infty} b_n q^n. \tag{3.28}
\]
From Lemma 3.1, we find that for \( n \geq 19 \), the coefficient of \( q^n \) in
\[
\frac{2q^3}{1 + q^2} + \frac{2q^{13}}{(1 - q^3)(1 - q^4)}
\]
is nonnegative. This gives \( \overline{N}_2(1, n) \geq \overline{N}_2(1, n - 1) \) for \( n \geq 19 \). It can be checked that for \( 1 \leq n \leq 18 \), \( \overline{N}_2(1, n) \geq \overline{N}_2(1, n - 1) \) still holds. This completes the entire proof. \( \blacksquare \)

4 The proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. To this end, we need the following lemma.

Lemma 4.1. For integer \( k \geq 0 \), let
\[
\frac{1}{(qz; q)_k(q/z; q)_k} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{k,m}(n) z^m q^n.
\]
Then for \( m \geq 0 \), \( a_{k,m}(n) \geq a_{k,m+2}(n) \). Equivalently, for \( m \geq 0 \), the coefficient of \( z^m q^n \) in
\[
\frac{1 - z^{-2}}{(qz; q)_k(q/z; q)_k}
\]
is nonnegative.

Proof. By definition, we see that
\[
a_{k,m}(n) = a_{k,-m}(n). \tag{4.1}
\]
Moreover, it is clear that
\[
\sum_{n=0}^{\infty} a_{k+1,m}(n)z^m q^n = \frac{1}{(qz;q)_{k+1}(q/z;q)_{k+1}} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{k,m}(n)z^m q^n
\]
\[
= \sum_{r=0}^{\infty} \sum_{i=0}^{r} z^{r+i} q^{r(k+1)} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{k,m}(n)z^m q^n.
\] (4.2)

Thus we have
\[
a_{k+1,m}(n) = \sum_{r=0}^{\infty} \sum_{i=0}^{r} a_{k,m-r+2i}(n-r(k+1))
\] (4.3)

We prove this lemma by induction on \(k\). For \(k = 1\), it is trivial to check that
\[
a_{1,m}(n) = \begin{cases} 1 & \text{if } m \equiv n \pmod{2} \text{ and } n \geq |m|; \\ 0 & \text{otherwise}. \end{cases}
\]

This gives our desired result.

Set \(b_{k,m}(n) = a_{k,m}(n) - a_{k,m+2}(n)\) and assume that \(b_{k,m}(n) \geq 0\) for \(m \geq 0\). From (4.3), we derive that
\[
b_{k+1,m}(n) = \sum_{r=0}^{\infty} \sum_{i=0}^{r} b_{k,m-r+2i}(n-r(k+1)).
\] (4.4)

Moreover, by (4.4), we see that
\[
b_{k,m}(n) = -b_{k,-m-2}(n)
\] (4.5)

and therefore
\[
\sum_{r=m+1}^{\infty} \sum_{i=0}^{r-m-1} b_{k,m-r+2i}(n-r(k+1)) = 0.
\] (4.6)

Thus by (4.4) and (4.6), we derive that for \(m \geq 0\),
\[
b_{k+1,m}(n) = \sum_{r=0}^{m} \sum_{i=0}^{r} b_{k,m-r+2i}(n-r(k+1)) + \sum_{r=m+1}^{\infty} \sum_{i=0}^{r} b_{k,m-r+2i}(n-r(k+1))
\]
\[
= \sum_{r=0}^{m} \sum_{i=0}^{r} b_{k,m-r+2i}(n-r(k+1)) + \sum_{r=m+1}^{\infty} \sum_{i=r-m}^{\infty} b_{k,m-r+2i}(n-r(k+1)).
\] (4.7)

From induction hypothesis, we find that each term in the above summation is nonnegative. Thus \(b_{k+1,m}(n) \geq 0\). This completes the proof.
We now give a proof of Theorem 1.5.

Proof of Theorem 1.5. By (1.5), for \( m \geq 0 \), \( \overline{N}(m, n) \geq \overline{N}(m+2, n) \) is equivalent to the coefficient of \( z^m \) in

\[
\sum_{k=0}^{\infty} \frac{(-1; q)_k q^{k(k+1)/2}(1 - z^{-2})}{(zq; q)_k (q/z; q)_k} \]

is nonnegative. But by Lemma 4.1,

\[
[z^m] \sum_{k=0}^{\infty} \frac{(-1; q)_k q^{k(k+1)/2}(1 - z^{-2})}{(zq; q)_k (q/z; q)_k} = \sum_{k=0}^{\infty} \frac{(-1; q)_k q^{k(k+1)/2}[z^m]}{(zq; q)_k (q/z; q)_k} \frac{1 - z^{-2}}{(zq^2; q^2)_k (q^2/z; q^2)_k},
\]

which is clearly has nonnegative coefficients, where \([z^m]f(z)\) denotes the coefficient of \( z^m \) in \( f(z) \). This yields (1.12).

Similarly, by (1.7), for \( m \geq 0 \), \( \overline{N}_2(m, n) \geq \overline{N}_2(m+2, n) \) is equivalent to the coefficient of \( z^m \) in

\[
\sum_{k=0}^{\infty} \frac{(-1; q)_{2k} q^k(1 - z^{-2})}{(zq^2; q^2)_k (q^2/z; q^2)_k}
\]

is nonnegative. Again using Lemma 4.1 we see that

\[
[z^m] \sum_{k=0}^{\infty} \frac{(-1; q)_{2k} q^k(1 - z^{-2})}{(zq^2; q^2)_k (q^2/z; q^2)_k} = \sum_{k=0}^{\infty} (-1; q)_{2k} [z^m] \frac{1 - z^{-2}}{(zq^2; q^2)_k (q^2/z; q^2)_k},
\]

which has nonnegative coefficients. This completes the proof.

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