A note on visible islands

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Abstract

Given a finite point set $P$ in the plane, a subset $S \subseteq P$ is called an island in $P$ if $\text{conv}(S) \cap P = S$. We say that $S \subseteq P$ is a visible island if the points in $S$ are pairwise visible and $S$ is an island in $P$. The famous Big-line Big-clique Conjecture states that for any $k \geq 3$ and $\ell \geq 4$, there is an integer $n = n(k, \ell)$, such that every finite set of at least $n$ points in the plane contains $\ell$ collinear points or $k$ pairwise visible points. In this paper, we show that this conjecture is false for visible islands, by replacing each point in a Horton set by a triple of collinear points. Hence, there are arbitrarily large finite point sets in the plane with no 4 collinear members and no visible island of size 13.

1 Introduction

Given a finite point set $P$ in the plane, two points $p, q \in P$ are visible in $P$ if no other point in $P$ lies in the interior of the segment $[pq]$. The famous Big-line Big-clique Conjecture, introduced by Kára, Pór, and Wood [3], states that for any $k \geq 3$ and $\ell \geq 3$, there is an integer $n = n(k, \ell)$, such that every finite set of at least $n$ points in the plane contains either $\ell$ collinear points or $k$ pairwise visible points. Clearly, the conjecture holds when $k = 3$ or $\ell = 3$. Kára, Pór, and Wood showed that the conjecture holds when $k = 4$ and $\ell \geq 4$, and Abel et al. [1] verified the conjecture for $k = 5$ and $\ell \geq 4$. The conjecture remains open for all $k \geq 6$ and $\ell \geq 4$. See [6, 4] for more related results.

A natural approach to the Big-line Big-clique Conjecture is to find holes in planar point sets. Given a finite point set $P$ in the plane, a $k$-subset $Q \subseteq P$ is called a hole in $P$ if $Q$ is in convex position and $\text{conv}(Q) \cap P = Q$. Clearly, if $Q$ is a hole in $P$, then $Q$ consists of $k$ pairwise visible points in $P$. Does every sufficiently large finite point set $P$ in the plane contain $\ell$ collinear points or a $k$-hole? In [1], Abel et al. proved this to be true when $k = 5$, and conjectured it to be true when $k = 6$. However, a famous construction due to Horton [2] shows that this is false for $k \geq 7$ (See also Chapter 3 in [5]). In this paper, we study a relaxed version of this question by replacing holes with visible islands.

Given a finite point set $P$ in the plane, a subset $S \subseteq P$ is called an island in $P$ if $\text{conv}(S) \cap P = S$. We say that $S \subseteq P$ is a visible island if the points in $S$ are pairwise visible and $S$ is an island in $P$.

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Problem 1. Given integers $k, \ell \geq 4$, is there an integer $n = n(k, \ell)$ such that every $n$-element planar point set $P$ contains either $\ell$ collinear points or a visible island of size $k$?

For $\ell \leq 3$, clearly we have $n(k, \ell) = k$. For $\ell \geq 4$ and $k \leq 5$, $n(k, \ell)$ exists by the result of Abel et al. [1] stated above. Our main result shows that by modifying Horton’s construction [2], $n(k, \ell)$ does not exist for $\ell = 4$ and $k \geq 13$.

Theorem 2. There exist arbitrarily large, finite point sets in the plane with no 4 collinear points and no visible island of size 13.

When $\ell \geq 4$ and $6 \leq k \leq 12$, Problem [1] remains open.

2 Proof of Theorem [2]

Let us begin by recalling the definition of Horton sets. Given finite point sets $P$ and $Q$ in the plane, we say that $P$ is high above $Q$ (or, equivalently, $Q$ is deep below $P$) if each line determined by two points of $P$ lies above all the points of $Q$, and each line determined by two points of $Q$ lies below all of the points of $P$. Finally, given a point $p$ in the plane, we denote $x(p)$ to be the $x$-coordinate of $p$.

Throughout the proof, we will only consider point sets whose members have distinct $x$-coordinates.

For $n \geq 0$, a Horton set $H_n$ is a set of $2^n$ points in the plane with no three collinear members, defined recursively as follows. Set $H_0$ to be a single point in the plane. Having constructed $H_{n-1} = \{p_1, p_2, \ldots, p_{2^n-1}\}$, whose elements are ordered by increasing $x$-coordinate, set

$$H_{n-1}^{(1)} = \{p_1, p_2, \ldots, p_{2^n-1}\},$$

$$H_{n-1}^{(2)} = H_{n-1}^{(1)} + (\varepsilon, K),$$

where $K$ is a sufficiently large number such that $H_{n-1}^{(1)}$ lies deep below $H_{n-1}^{(2)}$. Likewise, we set $\varepsilon > 0$ to be sufficiently small such that for each $i$,

$$x(p_i) < x(p_i) + \varepsilon < x(p_{i+1}).$$

Then we set $H_n = H_{n-1}^{(1)} \cup H_{n-1}^{(2)}$. It is known that $H_n$ has the following property (see Chapter 3 in [5]).

Lemma 3 [2, 5]. If $S \subset H_n$ such that $|S| = 7$, then the interior of $\text{conv}(S)$ contains a point from $H_n$.

For each $p_i \in H_n$, replace $p_i$ with three collinear points $q_i, u_i, v_i$ that are very close together, while avoiding the creation of four collinear points. The points $q_i, u_i, v_i$ are called triplets of each other and $p_i$ is the parent of them. Let $\hat{H}_n$ be the resulting set. Then $\hat{H}_n$ contains no visible island on 13 points. Indeed, for sake of contradiction, suppose $\hat{S} \subset \hat{H}_n$ is a visible island in $\hat{H}_n$ and $|\hat{S}| = 13$. Since at most two members of a triplet can belong to $\hat{S}$, by the pigeonhole principle, there are 7 points $p_{i_1}, \ldots, p_{i_7} \in H_n$ that are parents of points in $\hat{S}$. By setting $S = \{p_{i_1}, \ldots, p_{i_7}\}$, Lemma 3 implies that there is a point $p_j \in H_n$ that lies in the interior of $\text{conv}(S)$. However, this implies that the triplet $q_j, u_j, v_j \in \text{conv}(\hat{S})$, a contradiction. \(\square\)
3 Concluding remarks

Our initial goal was to find arbitrarily large visible islands in point sets with no four collinear members. Unfortunately, Theorem 2 shows that this is not possible. However, the following conjecture remains open, which would imply the Big-line Big-clique Conjecture for $\ell = 4$ by applying an induction on $k$ and setting $n$ sufficiently large. Given a finite point set $P$, the neighborhood of $p \in P$ is the set of points in $P$ that are visible to $p$.

**Conjecture 4.** Every $n$-element planar point set $P$ with no four collinear members, contains a point $p \in P$ such that its neighborhood contains an island of size $f(n)$, where $f(n)$ tends to infinity as $n$ tends to infinity.

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**References**

[1] A. Abel, B. Ballinger, P. Bose, S. Collette, V. Dujmović, F. Hurtado, S.D. Kominers, S. Langerman, A. Pór, D. Wood, Every large point set contains many collinear points or an empty pentagon, *Graphs and Combinatorics* 27 (2011), 47–60.

[2] J. D. Horton, Sets with no empty convex 7-gons, *Canad. Math. Bull.* 26 (1983), 482–484.

[3] J. Kára, A. Pór, D. Wood, On the chromatic number of the visibility graph of a set of points in the plane, *Discrete Comput. Geom.* 34 (2005), 497–506.

[4] J. Matoušek, Blocking Visibility for Points in General Position, *Discrete Comput Geom* 42 (2009), 219–223.

[5] J. Matoušek, *Lectures on Discrete Geometry*, Springer–Verlag, New York, 2002.

[6] A. Pór, D. Wood, On visibility and blockers, *J. Comput. Geom.* 1 (2010), 29–40.