Unrestricted Douglas-Rachford algorithms for solving convex feasibility problems in Hilbert space

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ABSTRACT

In this work we focus on the convex feasibility problem (CFP) in Hilbert space. A specific method in this area that has gained a lot of interest in recent years is the Douglas-Rachford (DR) algorithm. This algorithm was originally introduced in 1956 for solving stationary and non-stationary heat equations. Then in 1979, Lions and Mercier adjusted and extended the algorithm with the aim of solving CFPs and even more general problems, such as finding zeros of the sum of two maximally monotone operators. Many developments which implement various concepts concerning this algorithm have occurred during the last decade. We introduce an unrestricted DR algorithm, which provides a general framework for such concepts. Using unrestricted products of a finite number of strongly nonexpansive operators, we apply this framework to provide new iterative methods, where, inter alia, such operators may be interlaced between the operators used in the scheme of our unrestricted DR algorithm.

1. Preliminaries

Suppose that $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$. We denote by $\text{Id} : H \to H$ the identity operator on $H$, that is, for all $x \in H$, $\text{Id}x = x$. For an operator $T : H \to H$, the fixed points set of $T$ is defined as $\text{Fix}(T) := \{ x \in H \mid Tx = x \}$. For each nonempty and convex set $C \subseteq H$, we denote by $P_C$ the (unique) metric projection onto $C$, the existence of which is guaranteed if $C$ is, in addition, closed. The expressions $x_n \rightharpoonup x$ and $x_n \to x$ denote, respectively, the weak and strong convergence to $x$ of a sequence $\{x_n\}_{n=0}^\infty$ in $(H, \| \cdot \|)$. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers starting from 0, and for any two integers $m$ and $n$, with $m \leq n$, we denote by $\{m, \ldots, n\}$ the set of all integers between $m$ and $n$ (including $m$ and $n$).

Now we recall the following types of algorithmic operators. For more information on such operators, see, for example, [9].

Definition 1.1: Let $T : H \to H$ be an operator and let $\lambda \in [0, 2]$. The operator $T_\lambda : H \to H$ defined by $T_\lambda := (1 - \lambda)\text{Id} + \lambda T$ is called a $\lambda$-relaxation of the operator $T$. The operator $T_2$ is called the reflection of the operator $T$. 
Remark 1.1: Clearly, Fix(T) = Fix(T_λ) for every operator T : H → H and every λ ∈ (0, 2].

Definition 1.2: An operator T : H → H is said to be nonexpansive if
\[ \| Tx - Ty \| \leq \| x - y \| \quad \forall x, y \in H. \]

For λ ∈ [0, 2], an operator T : H → H is said to be λ-relaxed nonexpansive if T is a λ-relaxation of a nonexpansive operator U, that is, T = U_λ.

Remark 1.2: Clearly, a composition of nonexpansive operators is nonexpansive.

Definition 1.3: An operator T : H → H is said to be quasi-nonexpansive if Fix(T) ≠ ∅ and
\[ \| Tx - z \| \leq \| z - x \| \quad \forall (x, z) \in H \times \text{Fix}(T). \]

Remark 1.3: It is clear that a nonexpansive operator with a fixed point is, in particular, quasi-nonexpansive.

Definition 1.4: An operator T : H → H is called firmly nonexpansive if
\[ \langle Tx - Ty, x - y \rangle \geq \| Tx - Ty \|^2 \quad \forall x, y \in H. \]

For λ ∈ [0, 2], an operator T : H → H is called λ-relaxed firmly nonexpansive if T is a λ-relaxation of a firmly nonexpansive operator U, that is, T = U_λ.

Theorem 1.5: Let T : H → H be an operator. The following conditions are equivalent.
(i) T is firmly nonexpansive.
(ii) T_λ is nonexpansive for each λ ∈ [0, 2].
(iii) There exists a nonexpansive operator N : H → H such that T = 2^{-1}(Id + N).

Proof: See Theorem 2.2.10 in [9].

Definition 1.6: For λ ∈ (0, 1), we say that an operator T : H → H is λ-averaged if it is a λ-relaxed nonexpansive operator. We say that an operator T : H → H is averaged if it is λ-averaged for some λ ∈ (0, 1).

Corollary 1.7: Let λ ∈ (0, 2). An operator T : H → H is λ-relaxed firmly nonexpansive if and only if it is (2^{-1}λ)-averaged.

Proof: See Corollary 2.2.17 in [9].

Theorem 1.8: Let n be a positive integer. For each i = 1, ..., n, let λ_i ∈ (0, 2) and let T_i : H → H be λ_i-relaxed firmly nonexpansive. Then T := T_nT_{n-1}⋯T_1 is a λ-relaxed firmly nonexpansive operator for some λ ∈ (0, 2). In addition, if \( \cap_{i=1}^{n} \text{Fix}(T_i) \neq \emptyset \), then Fix(T) = \( \cap_{i=1}^{n} \text{Fix}(T_i) \).
**Proof:** See Theorem 2.2.42 and Theorem 2.1.26 in [9].

The following corollary is an immediate consequence of Theorem 1.8 and Corollary 1.7.

**Corollary 1.9:** Let \( n \) be a positive integer. For each \( i = 1, \ldots, n \), let \( \lambda_i \in (0, 1) \) and let \( T_i : \mathcal{H} \to \mathcal{H} \) be \( \lambda_i \)-averaged. Then \( T := T_nT_{n-1}\cdots T_1 \) is a \( \lambda \)-averaged operator for some \( \lambda \in (0, 1) \). In addition, if \( \bigcap_{i=1}^n \text{Fix}(T_i) \neq \emptyset \), then \( \text{Fix}(T) = \bigcap_{i=1}^n \text{Fix}(T_i) \).

**Example 1.10:** Let \( C \subset \mathcal{H} \) be a nonempty, closed and convex set, and let \( P_C : \mathcal{H} \to \mathcal{H} \) be the metric projection onto \( C \). Then the operator \( P_C \) is firmly nonexpansive and hence averaged. For the proof, see Theorem 2.2.21 in [9].

**Definition 1.11:** We write that an operator \( T : \mathcal{H} \to \mathcal{H} \) satisfies Condition (S) if for each pair of sequences \( \{x_n\}_{n=0}^\infty \) and \( \{y_n\}_{n=0}^\infty \) in \( \mathcal{H} \),

\[
\begin{cases}
\{x_n - y_n\}_{n=0}^\infty & \text{is bounded} \\
\|x_n - y_n\| - \|Tx_n - Ty_n\| & \to 0
\end{cases}
\implies x_n - y_n - (Tx_n - Ty_n) \to 0.
\]

The operator \( T \) is called **strongly nonexpansive** if it is nonexpansive and satisfies Condition (S).

**Theorem 1.12:** Let \( T : \mathcal{H} \to \mathcal{H} \) be a firmly nonexpansive operator and let \( \lambda \in (0, 2) \). Then the \( \lambda \)-relaxation \( T_\lambda \) of \( T \) is strongly nonexpansive.

**Proof:** See Theorem 2.3.4 in [9].

The following corollary is an immediate consequence of Theorem 1.12 and Corollary 1.7.

**Corollary 1.13:** Let \( T : \mathcal{H} \to \mathcal{H} \) be an averaged operator. Then \( T \) is strongly nonexpansive.

**Definition 1.14:** An operator \( T : \mathcal{H} \to \mathcal{H} \) is **asymptotically regular** if for each \( x \in \mathcal{H} \), we have

\[
\|T^{n+1}x - T^nx\| \to 0.
\]

**Lemma 1.15:** Let \( T : \mathcal{H} \to \mathcal{H} \) be a strongly nonexpansive operator such that \( \text{Fix}(T) \neq \emptyset \). Then \( T \) is asymptotically regular.

**Proof:** See Lemma 3.4.9 in [9].

**Definition 1.16:** Let \( C \) be nonempty, closed and convex subset of \( \mathcal{H} \). An operator \( T : C \to \mathcal{H} \) is **weakly regular** (satisfies Opial’s demi-closedness principle, \( T-I \) is demiclosed at 0) if for any sequence \( \{x_n\}_{n=0}^\infty \subset \mathcal{H} \) and any \( x \in \mathcal{H} \), the following implication holds:

\[
\begin{cases}
x_n \rightharpoonup x \\
Tx_n - x_n \to 0
\end{cases}
\implies x \in \text{Fix}(T).
\]
Lemma 1.17: Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator. Then $T$ is weakly regular.

Proof: See Lemma 3.2.5 in [9]. ■

Next, we recall the following generalization of Opial’s Theorem, which we will use in the sequel.

Theorem 1.18: Let $C \subset \mathcal{H}$ be a nonempty, closed and convex set, and let $S : C \to \mathcal{H}$ be a weakly regular operator such that $\text{Fix}(S) \neq \emptyset$. Assume that $x_0 \in C$ is arbitrary and $\{T_n\}_{n=0}^\infty$ is a sequence of quasi-nonexpansive operators $T_n : C \to C$ such that $\text{Fix}(S) \subset \bigcap_{n=0}^\infty \text{Fix}(T_n)$. If the sequence $\{x_n\}_{n=0}^\infty$ generated by the recurrence $x_n = T_{n-1}(x_{n-1})$ satisfies

$$
\|Sx_n - x_n\| \to 0,
$$

then it converges weakly to a point $x_* \in \text{Fix}(S)$.

Proof: See Theorem 3.6.2 along with Remark 3.6.4 in [9]. ■

The following theorem is an immediate consequence of Theorem 1.18. It is a well-known theorem established by Opial in [16].

Theorem 1.19: Let $C \subset \mathcal{H}$ be a nonempty, closed and convex set. If $T : C \to C$ is a nonexpansive and asymptotically regular operator with $\text{Fix}(T) \neq \emptyset$, then for any $x_0 \in C$, the sequence $\{x_n\}_{n=0}^\infty$ generated by the recurrence $x_n = T(x_{n-1})$ converges weakly to a point $x_* \in \text{Fix}(T)$.

2. Introduction

The Convex Feasibility Problem (CFP) in $\mathcal{H}$ is the problem of finding a point $x_* \in \bigcap_{\alpha \in I} C_{\alpha}$, where $\{C_{\alpha}\}_{\alpha \in I}$ is a family of nonempty, closed and convex subsets of $\mathcal{H}$. There are numerous iterative methods for solving this problem in the literature (see, for example, [4,5,9] and references therein). The Douglas-Rachford (DR) algorithm for a finite family of sets, originally introduced by Douglas and Rachford in [12] for solving the heat equation, is one such method. It has attracted considerable interest in the last few years. Many investigations of this algorithm have recently been undertaken in diverse directions, as one can observe, for example, in [1,6,8,11,14] and references therein. Borwein and Tam presented in [7] a cyclic-DR algorithm for CFPs in which the number of sets is allowed to be greater than 2. In this DR algorithm the original one is applied over subsequent pairs of sets. Artacho et al. provided in [2] a more general framework for the cyclic DR algorithm, where the proper employment of certain sets and operators in this algorithm is studied. In the present paper we offer a further extension of these ideas by introducing an unrestricted DR algorithm, where the sets are chosen (almost) unrestrictedly, that is, our DR-algorithm is (almost) unrestricted in the sense of the choice of sets which are used to obtain its outcome. We begin by recalling the notion of an $r$-set Douglas-Rachford ($r$-set DR) operator, which was defined in [11] in the following way.
Definition 2.1: Given a finite family \( \{C_i\}_{i=1}^r \) of nonempty, closed and convex subsets of \( \mathcal{H} \), the composite reflection operator \( \mathcal{V}_{\{C_i\}_{i=1}^r} : \mathcal{H} \to \mathcal{H} \) is defined by

\[
\mathcal{V}_{\{C_i\}_{i=1}^r} := \mathcal{R}_{C_r} \ldots \mathcal{R}_{C_1},
\]

where \( \mathcal{R}_{C_i} = 2P_{C_i} - \text{Id} \) is the reflection with respect to the corresponding \( C_i \) for each \( i = 1, \ldots, r \). The \( r \)-set DR operator \( \mathcal{T}_{\{C_i\}_{i=1}^r} : \mathcal{H} \to \mathcal{H} \) is defined by

\[
\mathcal{T}_{\{C_i\}_{i=1}^r} := 2^{-1} \left( \text{Id} + \mathcal{V}_{\{C_i\}_{i=1}^r} \right).
\]

We recall the following property of the \( r \)-set DR operator established in [11].

**Lemma 2.2:** Let \( \{C_i\}_{i=1}^r \) be a finite family of nonempty, closed and convex subsets of \( \mathcal{H} \) such that \( \text{int} \bigcap_{i=1}^r C_i \neq \emptyset \). Then

\[
\text{Fix} \left( \mathcal{T}_{\{C_i\}_{i=1}^r} \right) = \bigcap_{i=1}^r C_i.
\]

**Proof:** See Corollary 23 in [11].

Given a finite family \( \{C_i\}_{i=1}^m \) of nonempty, closed and convex subsets of \( \mathcal{H} \), a mapping \( f \) from \( \mathbb{N} \) onto \( \{1, \ldots, m\} \) and an integer \( r > 1 \), set \( \{C_{m,r}(n)(j)\}_{j=1}^r \) to be the finite family of subsets of \( \mathcal{H} \) defined by

\[
C_{m,r}(n)(j) := C_f((r-1)n+j-1)
\]

for each \( j = 1, \ldots, r \) and each natural number \( n \). Define a sequence of unrestricted \( r \)-set DR operators \( \{S_n\}_{n=0}^\infty \) by

\[
S_n := \mathcal{T}_{\{C_{m,r}(n)(j)\}_{j=1}^r}
\]

for each \( n \in \mathbb{N} \). For an arbitrary point \( x_0 \in \mathcal{H} \), we intend to analyze the convergence of the sequence \( \{x_n\}_{n=0}^\infty \) generated by the following recurrence:

\[
x_n := S_{n-1} \left( x_{n-1} \right).
\]

The sequence \( \{x_n\}_{n=0}^\infty \) is the outcome of our unrestricted DR algorithm.

In the setting above, let \( j_f \) be a natural number such that

\[
f \left( \{1, \ldots, j_f\} \right) = \{1, \ldots, m\}.
\]

We define a composite unrestricted DR operator \( Q : \mathcal{H} \to \mathcal{H} \) by

\[
Q := S_{j_f} \ldots S_0
\]

and study the convergence properties of the sequence \( \{y_n\}_{n=0}^\infty \) generated by the following recurrence, for an arbitrary point \( y_0 \in \mathcal{H} \):

\[
y_n := Q \left( y_{n-1} \right)
\]

This sequence is the outcome of an alternative unrestricted DR algorithm.
For a finite (possibly empty) family of operators \( \{T_i\}_{i=1}^q \) defined on \( \mathcal{H} \), where \( p \) and \( q \) are integers, recall that for each integer \( q' \leq q \), the product (composition) of \( \{T_i\}_{i=p}^q \), which we denote by \( T_q \cdots T_p \) (it is sometimes denoted in the literature also by \( \prod_{i=p}^q T_i \)) is defined by the following recurrence:

\[
T_q \cdots T_p := \begin{cases} 
\text{Id} & \text{if } q' < p \\
T_{q'} \circ (T_{q'-1} \cdots T_p) & \text{if } q' \geq p.
\end{cases}
\]

Recall also that the unrestricted product of a family \( \{T_\alpha\}_{\alpha \in I} \) of operators determined by a mapping \( h : \mathbb{N} \to I \) is the sequence of operators \( \{P_n\}_{n=0}^\infty \) defined by \( P_n := T_{h(n)} \cdots T_{h(0)} \) for each \( n \in \mathbb{N} \). For more information and applications concerning unrestricted products, we refer the reader to, for example, [10,13].

The following theorem concerning the convergence of an unrestricted product of two strongly nonexpansive operators was established in [13]. It provides a powerful tool for finding a common fixed point of two such operators.

**Theorem 2.3:** Let \( T_1 \) and \( T_2 \) be two strongly nonexpansive operators such that \( \text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \emptyset \). Assume \( h : \mathbb{N} \to \{1, 2\} \) is a mapping such that \( h^{-1}(i) \) is an infinite set for each \( i = 1, 2 \). Then for each \( x \in \mathcal{H} \), the sequence \( \{P_nx\}_{n=0}^\infty \), where \( \{P_n\}_{n=0}^\infty \) is the unrestricted product of \( \{T_i\}_{i=1}^2 \) determined by the mapping \( h \), converges weakly to a common fixed point of \( T_1 \) and \( T_2 \).

**Proof:** See Theorem 1 in [13].

It is still an open question, whether this result can be extended to more than two operators in the most general case. But from the optimization theory point of view, we can actually choose how often each operator belonging to a given family of operators should appear in the concrete method which we apply in order to find a common fixed point of the operators in this family. It turns out that in the case where each operator of a given finite family appears in the product sufficiently frequently, Theorem 2.3 can be extended to an arbitrary finite number of given operators, as is implied by Theorem 2.5 below. Before formulating it, we present the following notion of quasi-periodicity, which was used in [13]. In similar settings, it is sometimes called intermittency (see, for example, [3,15]).

**Definition 2.4:** Assume that \( M \) is a positive integer and that \( h \) is a mapping defined on \( \mathbb{N} \). We say that \( h \) is \( M \)-quasi-periodic if for each \( n \in \mathbb{N} \), we have

\[
h\left(\{n, \ldots, n + M - 1\}\right) = h\left(\mathbb{N}\right).
\]

Given a finite family \( \{T_i\}_{i=1}^m \) of strongly nonexpansive operators, a positive integer \( M \) and an \( M \)-quasi-periodic mapping \( h \) which is onto \( \{1, \ldots, m\} \), the following theorem, which was first proved in [13], establishes the convergence of the sequence \( \{P_nx\}_{n=0}^\infty \), where \( \{P_n\}_{n=0}^\infty \) is the unrestricted product of the family \( \{T_i\}_{i=1}^m \) determined by \( h \) and \( x \in \mathcal{H} \) is arbitrary. Due to the high importance of this theorem and for the convenience of the reader, we provide a detailed proof of it in Section 4.
Theorem 2.5: Assume that \( \{T_i\}_{i=1}^{m} \) is a finite family of strongly nonexpansive operators satisfying \( \cap_{i=1}^{m} \text{Fix}(T_i) \neq \emptyset \), \( M \) is a positive integer and \( h \) is an \( M \)-quasi-periodic mapping onto \( \{1, \ldots, m\} \). Then for each \( x \in \mathcal{H} \), the sequence \( \{P_n x\}_{n=0}^{\infty} \) converges weakly to a common fixed point of the operators \( \{T_i\}_{i=1}^{m} \), where \( \{P_n\}_{n=0}^{\infty} \) is the unrestricted product of \( \{T_i\}_{i=1}^{m} \) determined by \( h \).

In order to prove the above theorem we need the following two technical lemmata.

Lemma 2.6: Let \( T : \mathcal{H} \to \mathcal{H} \) be a nonexpansive mapping and \( \{v_n\}_{n=0}^{\infty} \) be a sequence in \( \mathcal{H} \). Suppose \( \{v_n\}_{n=0}^{\infty} \) and \( \{Tv_n\}_{n=0}^{\infty} \) both converge weakly to some \( v \in \mathcal{H} \) and that \( ||v_n|| - ||Tv_n|| \to 0 \). Then \( v \in \text{Fix}(T) \).

Proof: See Lemma 1 in [13].

Lemma 2.7: Let \( \{T_\alpha\}_{\alpha \in I} \) be a family of nonexpansive operators and \( h : \mathbb{N} \to I \) be a mapping. Assume that \( \{P_{n_k}\}_{k=0}^{\infty} \) and \( \{P_{n'_k}\}_{k=0}^{\infty} \) are two subsequences of the unrestricted product \( \{P_n\}_{n=0}^{\infty} \) of \( \{T_\alpha\}_{\alpha \in I} \) determined by \( h \) and let \( x \in \mathcal{H} \). If \( x_1 \) and \( x_2 \) are common fixed points of the family \( \{T_\alpha\}_{\alpha \in I} \) above such that \( P_{n_k} x \to x_1 \) and \( P_{n'_k} x \to x_2 \), then \( x_1 = x_2 \).

Proof: See Lemma 3 in [13].

The rest of the paper is organized as follows. In Section 3 we formulate our main results, as well as discuss a few remarks and examples. Section 4 is devoted to several auxiliary results. Finally, the proofs of our main results are presented in Section 5.

3. Main results

Below we state our three main theorems. We establish them in Section 5.

Theorem 3.1: Assume that \( \{C_i\}_{i=1}^{m} \) is a finite family of nonempty, closed and convex subsets of \( \mathcal{H} \), \( r > 1 \) is an integer and \( f \) is a mapping from \( \mathbb{N} \) onto \( \{1, \ldots, m\} \). Suppose that the sequences \( \{S_n\}_{n=0}^{\infty} \) and \( \{x_n\}_{n=0}^{\infty} \) are defined, respectively, by (4) and (5), \( j_f \) is defined by (6), \( Q \) is defined by (7) and \( \{y_n\}_{n=0}^{\infty} \) is defined by (8). Then the following assertions hold:

(i) If there exists \( S : \mathcal{H} \to \mathcal{H} \), which is a weakly regular operator with a fixed point such that \( \text{Fix}(S) \subset \cap_{i=1}^{m} C_i \), and the sequence \( \{x_n\}_{n=0}^{\infty} \) satisfies

\[ \|Sx_n - x_n\| \to 0, \]

then \( \{x_n\}_{n=0}^{\infty} \) converges weakly to a point \( x_s \in \cap_{i=1}^{m} C_i \).

(ii) If \( \cap_{i=1}^{m} C_i \) has an interior point, then the sequence \( \{y_n\}_{n=0}^{\infty} \) converges weakly to a point \( y_s \in \cap_{i=1}^{m} C_i \).

Note that part (ii) of Theorem 3.1 above generalizes Theorem 3.7 in [2] (see also Remark 3.1 (c) below). Since it is often difficult to verify the existence of an operator \( S \) as in part (i), we introduce a so-called \( M \)-quasi-periodic unrestricted DR algorithm in our next result, where we make a modest assumption concerning the frequency of the sets defined...
by (3). This algorithm converges weakly if \( \bigcap_{i=1}^{n} C_i \neq \emptyset \) and if, in addition, \( \bigcap_{i=1}^{n} C_i \) has an interior point, then its weak limit is a solution of the CFP defined by the family \( \{ C_i \}_{i=1}^{m} \).

**Theorem 3.2:** Assume that \( \{ C_i \}_{i=1}^{m} \) is a finite family of nonempty, closed and convex subsets of \( \mathcal{H} \), \( r > 1 \) and \( M \) are positive integers. Let \( f \) be a mapping from \( \mathbb{N} \) onto \( \{ 1, \ldots, m \} \). For each natural number \( n \), let \( \{ C_{n,r}(n)(j) \}_{j=1}^{r} \) be the finite family of subsets of \( \mathcal{H} \) defined by (3) so that the sequence of families \( \{ \{ C_{n,r}(n)(j) \}_{j=1}^{r} \}_{n=0}^{\infty} \) is \( M \)-quasi-periodic (as a mapping defined on \( \mathbb{N} \)). Suppose that the sequences \( \{ S_n \}_{n=0}^{\infty} \) and \( \{ x_n \}_{n=0}^{\infty} \) are defined, respectively, by (4) and (5). Then the following assertions hold:

(i) If \( \bigcap_{i=1}^{n} C_i \neq \emptyset \), then \( \{ x_n \}_{n=0}^{\infty} \) converges weakly to a common fixed point of the operators \( \{ S_n \}_{n=0}^{\infty} \).

(ii) If \( \text{int} \bigcap_{i=1}^{n} C_i \neq \emptyset \), then \( \{ x_n \}_{n=0}^{\infty} \) converges weakly to a point \( x_\ast \in \bigcap_{i=1}^{m} C_i \).

**Theorem 3.3:** Assume that \( \{ C_i \}_{i=1}^{m} \) is a finite family of nonempty, closed and convex subsets of \( \mathcal{H} \) such that \( \bigcap_{i=1}^{m} C_i \) has an interior point, and that \( r > 1 \) and \( M \) are positive integers. Suppose further that \( f \) is a mapping from \( \mathbb{N} \) onto \( \{ 1, \ldots, m \} \), the sequence \( \{ S_n \}_{n=0}^{\infty} \) is defined by (4), \( j_f \) is defined by (6) and \( Q \) is defined by (7). Let \( \{ T_j \}_{j=1}^{k} \) be a family of strongly nonexpansive operators with a common fixed point, let \( h \) be an \( M \)-quasi-periodic mapping onto \( \{ 1, \ldots, k \} \), let \( x \in \mathcal{H} \) and let \( \{ P_n \}_{n=0}^{\infty} \) be the unrestricted product of the family \( \{ T_j \}_{j=0}^{k} \) determined by \( h \). Then the following assertions hold:

(i) If for each \( n \in \{ 1, \ldots, j_f \} \), the operator \( S_n \) is an element of \( \{ T_j \}_{j=1}^{k} \), then \( \{ P_n x \}_{n=0}^{\infty} \) converges weakly to a point \( x_\ast \in \bigcap_{i=1}^{m} C_i \).

(ii) If \( Q \) is an element of \( \{ T_j \}_{j=1}^{k} \), then \( \{ P_n x \}_{n=0}^{\infty} \) converges weakly to a point \( x_\ast \in \bigcap_{i=1}^{m} C_i \).

**Remark 3.1:** (a) If \( \mathcal{H} \) is a finite dimensional space, then the convergence in our results is strong.

(b) In particular, due to Lemma 1.17, Theorem 3.1(i) can be applied to the case where the operator \( S \) is a nonexpansive operator.

(c) The cyclic DR algorithm, first introduced in [2], is obtained by choosing in Theorem 3.2 \( f : \mathbb{N} \to \{ 1, \ldots, m \} \) to be defined by \( f(n) := n \mod m + 1 \) for each \( n \in \mathbb{N} \) and \( M = m \). Here we obtain, in addition, that it always converges to a solution of the CFP defined by the family \( \{ C_i \}_{i=1}^{m} \) under the assumption that \( \bigcap_{i=1}^{m} C_i \) has an interior point.

(d) By Theorem 1.12, Corollary 1.13 and Example 1.10, Theorem 3.3 can be applied to the family of strongly nonexpansive operators \( \{ T_i \}_{i=1}^{k} \), which contains, in particular, \( \lambda \)-relaxed firmly nonexpansive operators with the relaxation parameter lying in the interval \( (0, 2) \), averaged operators and metric projections.

4. Auxiliary results

In this section, we present several technical lemmata, which will be used in the proofs of our main results.
Lemma 4.1: For each family \( \{C_i\}_{i=1}^r \) of nonempty, closed and convex subsets of \( \mathcal{H} \), the unrestricted \( r \)-sets-DR operator \( T_{\{C_i\}} \) is averaged and hence strongly nonexpansive. In particular, if the family \( \{C_{m,r}(n)(j)\}_{j=1}^r \) is defined by (3), where \( \{C_i\}_{i=1}^m \) is a given family of nonempty, closed and convex subsets of \( \mathcal{H} \), \( r > 1 \) is an integer and \( f \) is a mapping from \( N \) onto \( \{1, \ldots, m\} \), then for each \( n \in \mathbb{N} \), the operator \( S_n \) defined by (4) is averaged and hence strongly nonexpansive. As a result, the composite unrestricted DR operator \( \mathcal{Q} \) defined by (7) is averaged and hence strongly nonexpansive.

Proof: Let \( \{C_i\}_{i=1}^r \) be a family of nonempty, closed and convex subsets of \( \mathcal{H} \). For each \( i = 1, \ldots, r \), the reflection \( R_{C_i} = 2P_{C_i} - Id \) is nonexpansive by Example 1.10 and Theorem 1.5(i) and (ii). Hence the composite reflection operator \( V_{\{C_i\}} \) is nonexpansive by Remark 1.2. By (2) and Theorem 1.5 (iii) and (i), \( T_{\{C_i\}} \) is firmly nonexpansive and by Corollary 1.7, it is averaged and hence strongly nonexpansive by Corollary 1.13. The rest of the statement of the lemma follows from (4), (7), Corollary 1.9, and Corollary 1.13. ■

Lemma 4.2: Assume that \( \{C_i\}_{i=1}^m \) is a family of nonempty, closed and convex subsets of \( \mathcal{H} \), \( r > 1 \) is an integer and \( f \) is a mapping from \( N \) onto \( \{1, \ldots, m\} \). Then

\[
\bigcap_{i=1}^m C_i \subset \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \subset \bigcap_{n=0}^{j_f(n)} \text{Fix}(S_n),
\]

where \( \{S_n\}_{n=0}^{\infty} \) and \( j_f \) are defined by, respectively, (4) and (6). If, in addition, \( \bigcap_{i=1}^m C_i \neq \emptyset \), then

\[
\bigcap_{i=1}^m C_i = \bigcap_{n=0}^{\infty} \text{Fix}(S_n) = \bigcap_{n=0}^{j_f(n)} \text{Fix}(S_n).
\]

Proof: Clearly, by (4), Remark 1.1 and (1), for each \( n \in \mathbb{N} \),

\[
\text{Fix}(S_n) = \text{Fix} \left( T_{\{C_{m,r}(n)(j)\}_{j=1}^r} \right) = \text{Fix} \left( V_{\{C_{m,r}(n)(j)\}_{j=1}^r} \right) \supset \bigcap_{i=1}^m C_i.
\]

Hence

\[
\bigcap_{i=1}^m C_i \subset \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \subset \bigcap_{n=0}^{j_f(n)} \text{Fix}(S_n).
\]

Assume that \( \bigcap_{i=1}^m C_i \neq \emptyset \). Let \( z \in \bigcap_{n=0}^{j_f(n)} \text{Fix}(S_n) \) and \( p \in \{1, \ldots, m\} \) be arbitrary. There exist \( j \in \mathbb{N} \) and unique \( N, k \in \mathbb{N} \) such that \( f(j) = p, j \leq j_f \), \( k < r-1 \) and \( j = (r - 1)N + k \). Since \( N \leq j \leq j_f \) and \( k + 1 \leq r - 1 \), by (4), Lemma 2.2 and (3), we have

\[
z \in \text{Fix}(S_N) = \text{Fix} \left( T_{\{C_{m,r}(N)(j)\}_{j=1}^r} \right) = \bigcap_{j=1}^r C_f((r-1)N+j-1) \subset C_f((r-1)N+k) = C_p.
\]

Hence

\[
\bigcap_{n=0}^{\infty} \text{Fix}(S_n) \subset \bigcap_{n=0}^{j_f(n)} \text{Fix}(S_n) \subset \bigcap_{i=1}^m C_i
\]

and the result follows. ■

The rest of this section is devoted to the proof of Theorem 2.5.

Lemma 4.3: Assume that \( \{T_i\}_{i=1}^m \) is a finite family of operators, \( M \) is a positive integer and \( h \) is a \( M \)-quasi-periodic mapping onto \( \{1, \ldots, m\} \). Then for each strictly increasing sequence
of natural numbers such that \( n_k \geq n_{k-1} + M \) for each positive integer \( k \), there exist a strictly increasing sequence \( \{n'_k\}_{k=0}^\infty \) of natural numbers and a finite sequence \( \{l_i\}_{i=1}^M \) so that the set of values of \( \{l_i\}_{i=1}^M \) is \( \{1, \ldots, m\} \) and \( h(j) = l_{j-n'_{k}} \) for each \( k \in \mathbb{N} \) and each \( j \in \{n'_{k} + 1, \ldots, n'_{k} + M\} \). As a result,

\[
P_{n_{n_{k+1}}^k} = T_{h(n'_{k+1})} \cdots T_{h(n'_{k}+M+1)} T_{l_{M}} \cdots T_{l_{1}} P_{n'_{k}},
\]

where \( \{P_n\}_{n=0}^\infty \) is the unrestricted product of \( \{T_i\}_{i=1}^m \) determined by \( h \).

**Proof:** Let \( \{n'_k\}_{k=0}^\infty \) be a strictly increasing sequence of natural numbers such that \( n'_k \geq n'_{k-1} + M \) for each positive integer \( k \). Let \( k \in \mathbb{N} \). Since \( h \) is an \( M \)-quasi-periodic mapping onto \( \{1, \ldots, m\} \), we have \( \{1, \ldots, m\} = h(\{n'_k + 1, \ldots, n'_k + M\}) \) for each natural number \( k \).

Since the number of mappings \( s : \{1, \ldots, M\} \to \{1, \ldots, m\} \) is finite, it follows that there exists a strictly increasing sequence \( \{n'_k\}_{k=0}^\infty \) of natural numbers and a finite sequence \( \{l_i\}_{i=1}^M \) with all the asserted properties.

**Lemma 4.4:** Assume that \( \{v_n\}_{n=0}^\infty \) is a bounded sequence in \( \mathcal{H} \), \( \{T_i\}_{i=1}^m \) is a finite family of strongly nonexpansive operators such that the origin is their common fixed point and \( v \in \mathcal{H} \) is such that

\[
v_n \to v \quad \text{and} \quad \|T_{i-1} \cdots T_1 v_n\| - \|T_1 \cdots T_1 v_n\| \to 0 \tag{9}
\]

for each \( i = 1, \ldots, m \). Then the sequence \( \{T_m \cdots T_1 v_n\}_{n=0}^\infty \) converges weakly to \( v \), which is also a common fixed point of the family \( \{T_i\}_{i=1}^m \).

**Proof:** The proof is by induction on \( m \). For \( m = 0 \) the statement is clear. Assume that \( m > 0 \). By the induction hypothesis \( \{T_{m-1} \cdots T_1 v_n\}_{n=0}^\infty \) converges weakly to \( v \), which is a common fixed point of the family \( \{T_i\}_{i=1}^{m-1} \). Since each \( T_i \) is nonexpansive, since the origin is a common fixed point of the family \( \{T_i\}_{i=1}^{m-1} \) and since the sequence \( \{v_n\}_{n=0}^\infty \) is bounded, we see that the sequence \( \{T_{m-1} \cdots T_1 v_n\}_{n=0}^\infty \) is also bounded. Define a sequence \( \{w_n\}_{n=1}^\infty \subset \mathcal{H} \) by \( w_n := 0 \) for all \( n \in \mathbb{N} \). Since \( T_m \) satisfies Condition (S) and

\[
\|T_{m-1} \cdots T_1 v_n - w_n\| - \|T_m T_{m-1} \cdots T_1 v_n - T_m w_n\| \to 0,
\]

it follows that

\[
T_{m-1} \cdots T_1 v_n - T_m \cdots T_1 v_n \to 0.
\]

As a result, \( T_m \cdots T_1 v_n \) converges weakly to \( v \). Combining this with the induction hypothesis, (9) and Lemma 2.6 applied to the sequence \( \{T_{m-1} \cdots T_1 v_n\}_{n=0}^\infty \) and the operator \( T_m \), we obtain the desired result.

**Lemma 4.5:** Assume that \( \{T_i\}_{i=1}^m \) is a finite family of strongly nonexpansive operators such that the origin is their common fixed point, \( M \) is a positive integer and \( h \) is an \( M \)-quasi-periodic mapping onto \( \{1, \ldots, m\} \). Assume that \( x \in \mathcal{H} \) and that \( v \) is a weak cluster point of the sequence \( \{P_n x\}_{n=0}^\infty \), where \( \{P_n\}_{n=0}^\infty \) is the unrestricted product of \( \{T_i\}_{i=1}^m \) determined by \( h \). Then \( v \) is also a common fixed point of the family \( \{T_i\}_{i=1}^m \).
Proof: Clearly, there exists a strictly increasing sequence \( \{n_k\}_{k=0}^{\infty} \) such that \( P_{n_k}x \to v \). Without any loss of generality, we may assume that \( n_k \geq n_{k+1} + M \) for each positive integer \( k \). By Lemma 4.3, we may also assume that there exists a finite sequence \( \{l_i\}_{i=1}^{M} \) so that the set of values of \( \{l_i\}_{i=1}^{M} = \{1, \ldots, m\} \) and \( h(j) = l_{j-n_k} \) for each \( k \in \mathbb{N} \) and each \( j \in \{n_k + 1, \ldots, n_k + M\} \). As a result,

\[
P_{n_k+1} = T_{h(n_k+1)} \cdots T_{h(n_k+1+M+1)} T_{M} \cdots T_{1} P_{n_k}. \tag{10}
\]

Since the origin is a common fixed point of the family \( \{T_i\}_{i=1}^{m} \) and since each \( T_i \) is nonexpansive, it follows that \( \lim_{n \to \infty} \|P_{n_k}x\| \) exists. Hence the sequence \( \{P_{n_k}x\}_{n=0}^{\infty} \) is bounded and by (10)

\[
\|T_{i_{n_k}} \cdots T_{i_1} P_{n_k}x\| = \|T_{i_1} \cdots T_{i_{n_k}} P_{n_k}x\| \to 0
\]

for each \( i = 1, \ldots, M \) as \( k \to \infty \). Now we apply Lemma 4.4 to the sequence \( \{P_{n_k}x\}_{k=0}^{\infty} \) and the family \( \{T_i\}_{i=1}^{M} \) to obtain that the sequence \( \{T_{M} \cdots T_{1} P_{n_k}x\}_{k=0}^{\infty} \) converges weakly to \( v \), which is a common fixed point of the family \( \{T_i\}_{i=1}^{m} \) and hence is a common fixed point of the family \( \{T_i\}_{i=1}^{m} \), because the set of values of \( \{l_i\}_{i=1}^{M} = \{1, \ldots, m\} \).

Proof: Without any loss of generality, we may assume that the origin is a common fixed point of the family \( \{T_i\}_{i=1}^{m} \). Assume that the point \( x \in \mathcal{H} \). Since the sequence \( \{P_{n_k}x\}_{n=0}^{\infty} \) is bounded, it has a weak cluster point \( v \). Let \( v_1, v_2 \in \mathcal{H} \) be two weak cluster points of \( \{P_{n_k}x\}_{n=0}^{\infty} \). By Lemma 4.5, they are common fixed points of the family \( \{T_i\}_{i=1}^{M} \). By Lemma 2.7, we have \( v_1 = v_2 \). As a result, the sequence \( \{P_{n_k}x\}_{n=0}^{\infty} \) converges weakly to \( v \), which is a common fixed point of the family \( \{T_i\}_{i=1}^{m} \).

5. Proofs of the main results

Proof: (i) By Lemma 4.2, we have

\[
\bigcap_{i=1}^{m} C_i \subset \bigcap_{n=0}^{\infty} \text{Fix}(S_n). \tag{11}
\]

Let \( n \in \mathbb{N} \). By Lemma 4.1, \( S_n \) is strongly nonexpansive and hence nonexpansive. Since \( \text{Fix}(S) \subset \bigcap_{i=1}^{m} C_i \), it follows that \( \bigcap_{i=1}^{m} C_i \neq \emptyset \). Relation (11) and Remark 1.3 imply that \( S_n \) is quasi-nonexpansive. Applying Theorem 1.18, we arrive at the result.

(ii) By Lemmas 4.2, 4.1 and Corollary 1.9,

\[
\text{Fix}(Q) = \bigcap_{i=1}^{m} C_i \neq \emptyset. \tag{12}
\]

By Lemma 4.1, \( Q \) is strongly nonexpansive and therefore nonexpansive. By Lemma 1.15, it is asymptotically regular. The desired result now follows from (12) and Theorem 1.19.

Remark 5.1: In light of Lemma 4.2, part (ii) of Theorem 3.1 can also be proved by using Theorem 2.5 (without using Theorem 1.19): define a quasi-periodic mapping \( h \) from \( \mathbb{N} \) onto \( \{1, \ldots, j_f + 1\} \) by \( h(n) := \text{n mod (j_f + 1)} + 1 \) for each \( n \in \mathbb{N} \) and a sequence of operators \( \{T_i\}_{i=1}^{j_f+1} \) by \( T_i := S_{i-1} \) for each \( i \in \{1, \ldots, j_f + 1\} \).

Proof: Let \( F := \{S_n \mid n \in \{0, \ldots, M - 1\}\} \). Since the sequence \( \{(C_{m,r}(n)(j)_{j=1}^{j_f+1})_{n=0}^{\infty} \) is \( M \)-quasi-periodic, it is clear that \( F \) contains all possible operators \( S_n \) (for all \( n \in \mathbb{N} \)). Set \( T :
\( \{1, \ldots, p\} \to F \) to be a bijection, where \( p \) is the cardinality of the set \( F \). Define \( h : \mathbb{N} \to \{1, \ldots, p\} \) by \( h(n) = T^{-1}_n(S_n) \). Clearly, \( h \) is a quasi-periodic mapping onto \( \{1, \ldots, p\} \). Then \( S_n = T_{h(n)} \) for each natural number \( n \). Now (i) follows from Lemma 4.1 and Theorem 2.5, while (ii) follows from (i) and Lemma 4.2.

**Proof:** Since \( \cap_{j=1}^k \text{Fix}(T_j) \neq \emptyset \), by Theorem 2.5, the sequence \( \{P_n x\}_{n=0}^\infty \) converges weakly to a point \( x_* \in \cap_{j=1}^k \text{Fix}(T_j) \). Since \( \cap_{i=1}^m C_i \) has an interior point, by Lemma 4.2, \( \cap_{n=1}^J \text{Fix}(S_n) = \cap_{i=1}^m C_i \neq \emptyset \). By Lemma 4.1 and Corollary 1.9, \( \text{Fix}(Q) = \cap_{n=1}^J \text{Fix}(S_n) \). As a result, in both cases (i) and (ii)

\[
x_* \in \cap_{j=1}^k \text{Fix}(T_j) \subset \text{Fix}(Q) = \cap_{n=1}^J \text{Fix}(S_n) = \cap_{i=1}^m C_i.
\]

**Acknowledgments**

The authors would like to thank the editor and the anonymous referees for their comments on the manuscript which helped them improve an earlier version of this paper.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

Simeon Reich was partially supported by the Israel Science Foundation (Grant 820/17), the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

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