Estimation and Concentration of Missing Mass
of Functions of Discrete Probability
Distributions

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Abstract

Given a positive function $g$ from $[0,1]$ to the reals, the function’s missing mass in a sequence
of iid samples, defined as the sum of $g(\Pr(x))$ over the missing letters $x$, is introduced and studied.
The missing mass of a function generalizes the classical missing mass, and has several interesting
connections to other related estimation problems. Minimax estimation is studied for order-$\alpha$ missing
mass ($g(p) = p^\alpha$) for both integer and non-integer values of $\alpha$. Exact minimax convergence rates are
obtained for the integer case. Concentration is studied for a class of functions and specific results are
derived for order-$\alpha$ missing mass and missing Shannon entropy ($g(p) = -p\log p$). Sub-Gaussian tail
bounds with near-optimal worst-case variance factors are derived. Two new notions of concentration,
named strongly sub-Gamma and filtered sub-Gaussian concentration, are introduced and shown to result
in right tail bounds that are better than those obtained from sub-Gaussian concentration.

Index terms- Missing mass, Good-Turing estimator, missing mass of a function, entropy, Mean
squared error, minimax optimality, concentration, tail bounds, sub-Gaussian and sub-Gamma
tails.

I. INTRODUCTION

Let $P$ be an arbitrary discrete distribution on an alphabet $\mathcal{X}$. For $x \in \mathcal{X}$, let $p_x \triangleq P(x)$. Given
a positive function $g : [0,1] \to [0,\infty)$, we define a so-called additive function $G(P)$ over the
distribution $P$ as follows:

\[ G(P) \triangleq \sum_{x \in \mathcal{X}} g(p_x). \]
The problem of estimating such additive functions from \( n \) random samples \( X^n = (X_1, X_2, \ldots, X_n) \) with \( X_i \sim P \) iid has been studied before \([1]\). Estimation of Shannon entropy, which is an additive function with \( g(p) = -p \log p \), and Rényi entropy of order-\( \alpha \), which contains the additive function with \( g(p) = p^\alpha \), have been of particular interest \([2]\). While the applications of estimating Shannon entropy are numerous, estimation of Rényi entropy has applications, for instance, in the problem of guessing \([3], [4]\).

Another classical problem of interest in the area of estimation is that of estimating missing mass, which is the total probability of unseen letters in a sequence \([5]\). To define missing mass precisely, we need some notation. Let \( I(\cdot) \) and \( E[\cdot] \) denote indicator random variables and expectations, respectively. For \( x \in \mathcal{X} \), let \( F_x(X^n) = \sum_{i=1}^{n} I(X_i = x) \) denote the number of occurrences of \( x \) in \( X^n \). With this notation, missing mass, denoted \( M_0(X^n, P) \), is defined as

\[
M_0(X^n, P) \triangleq \sum_{x \in \mathcal{X}} p_x I(F_x(X^n) = 0).
\]  

We observe that missing mass is a random variable, and it is defined in an additive fashion over the letters of \( \mathcal{X} \). Estimation of missing mass \( M_0 \) has been of interest to statisticians and is widely used in domains like language modelling \([6], [7]\) and ecology \([8]\). The problem of estimation and concentration of missing mass and the properties of the classical Good-Turing estimator \([5]\) have been studied in \([9]–[16]\).

In this work, we combine the above two problems and consider estimation and concentration of the missing mass of functions of distributions. In other words, given independent samples from a discrete probability distribution, we consider estimating the contribution of the letters in the alphabet not seen in these samples to a positive function of the discrete distribution. More precisely, we define the missing mass of \( g(p) \) as

\[
G_0(X^n, P) \triangleq \sum_{x \in \mathcal{X}} g(p_x) I(F_x(X^n) = 0),
\]  

As seen above, \( G_0(X^n, P) \) generalizes the standard missing mass \( M_0(X^n, P) \) and includes interesting special cases such as the missing Shannon or Rényi entropy. Therefore, a study of estimation and concentration of \( G_0(X^n, P) \) are of theoretical interest. Defining the observed mass of \( g(p) \) as

\[
G_{1+}(X^n, P) \triangleq \sum_{x \in \mathcal{X}} g(p_x) I(F_x(X^n) \neq 0),
\]  

we obtain the identity

\[
G(P) = G_0(X^n, P) + G_{1+}(X^n, P).
\]
Understanding the estimation and concentration properties of \( G_0(X^n, P) \) can potentially shed light on the nature of the difficulty in estimating \( G(P) \) as well.

To the best of our knowledge, this is the first work to consider the above generalization of missing mass to functions of discrete distributions.

A. Minimax estimation

Our study of estimation will be in the worst case minimax sense. An estimator \( \hat{G}_0(X^n) \) for \( G_0(X^n, P) \) is a mapping from \( X^n \) to \( \mathbb{R} \). For a distribution \( P \), the \( L_2^2 \) or squared-error risk of the estimator \( \hat{G}_0(X^n) \) is
\[
R_{n,g}(\hat{G}_0, P) \triangleq E_{X^n \sim P}[(\hat{G}_0(X^n) - G_0(X^n, P))^2].
\]

(5)

The worst case risk of \( \hat{G}_0(X^n) \) is
\[
R_{n,g}(\hat{G}_0) \triangleq \max_P R_{n,g}(\hat{G}_0, P).
\]

(6)

The minimax risk of estimating \( G_0(X^n, P) \) is
\[
R^*_n \triangleq \min_{\hat{G}_0} R_{n,g}(\hat{G}_0).
\]

(7)

As is standard, the goal of our study is to characterize the rate of fall of \( R^*_n \) with \( n \) for a given function \( g(p) \). For the estimation part, in this article, we will primarily consider \( g(p) = p^\alpha \).

1) Missing Mass of Order-\( \alpha \): Missing mass of order-\( \alpha \) (\( \alpha > 0 \)), denoted \( M_{0,\alpha}(X^n, P) \), is obtained by choosing \( g(p) = p^\alpha \) in (2) so that
\[
M_{0,\alpha}(X^n, P) \triangleq \sum_{x \in X} p_x^n I(F_x(X^n) = 0).
\]

(8)

\( M_{0,\alpha}(X^n, P) \) is the unseen portion (in \( X^n \)) of the power sum \( S_\alpha(P) \) of order-\( \alpha \), defined as
\[
S_\alpha(P) \triangleq \sum_{x \in X} p_x^\alpha.
\]

(9)

Choosing \( \alpha = 1 \), we obtain the standard definition of missing mass \( M_0(X^n, P) \triangleq M_{0,1}(X^n, P) \). We will drop the arguments \( X^n \) and \( P \) whenever possible. A classical estimator for \( M_0 \) is the Good-Turing estimator [5], denoted \( M_0^{GT}(X^n) \), defined as
\[
M_0^{GT}(X^n) \triangleq \frac{\phi_1(X^n)}{n},
\]

(10)

where, for \( l \geq 0 \), \( \phi_l(X^n) \triangleq \sum_{x \in X} I(F_x(X^n) = l) \) denotes the number of letters that have occurred \( l \) times in \( X^n \). We will define and use generalized versions of the Good-Turing estimator for estimating missing mass of order-\( \alpha \).
B. Concentration of missing mass

The estimation of missing mass is considered to be complicated primarily because missing mass is a random variable dependant on the samples $X^n$ and the distribution $P$, while typical quantities for estimation are functions of $P$ alone. However, missing mass is known to concentrate about its expected value [9], [17], [18], and the concentration phenomenon plays an important role in the success of estimation of missing mass.

As a natural extension, we study the concentration of $G_0(X^n, P)$ (and, by implication, $G_{1+}(X^n, P)$ because of (4)) for a given function $g(p)$, and we provide specialised results for the case $g(p) = p^\alpha$ and $g(p) = -p \log p$. Our approach for proving concentration is different from previous ones, and, arguably, provides one of the simplest proofs for distribution-free concentration results of missing mass.

The rest of this article is organised as follows. We start by providing some relevant prior results in this area in Section II. We present a summary of our main results in Section III. In Section IV we provide the proofs for our results on the minimax estimation of $M_{0,\alpha}$ for $\alpha \geq 1$ under $L_2^2$ risk. In Section V we present the proofs for our concentration and tail bound results for $G_0$. Section VI presents a proof for a lemma outlining the choice of parameters in the concentration results. Section VII has derivations of specific, simplified tail bounds for generalized missing mass and missing Shannon entropy. We conclude in Section VIII with some remarks. Technical aspects of some proofs are collected in the appendix.

II. Prior work

As stated before, estimation and concentration of missing mass $M_0$ have been well-studied and we provide a few results that are important in the context of this article. Where applicable, we make remarks on possible extensions to missing mass of functions. For two non-negative sequences $a_n, b_n$, the notation $a_n =_n b_n$ denotes that $a_n = b_n \pm o(b_n)$. The notations $a_n \leq_n b_n$ and $a_n \geq_n b_n$ are similarly defined. The notation $a_n \preceq_n b_n$ indicates that there exists a universal constant $C$ such that $\sup_n \frac{a_n}{b_n} \leq C$.

A. Minimax results

Recall that $R^*_{n,p}$ denotes the minimax risk of estimating the missing mass $M_0 = \sum_x p_x I(F_x = 0)$ under $L_2^2$ loss. While $R^*_{n,p}$ was known to be $O(1/n)$ on the basis of concentration results [9],
bounds with small constants were found in [19], which showed that

\[
\frac{0.25}{n} \leq R_{n,p}^* \leq \frac{0.6179}{n}.
\]  

(11)

In [20], the bounds were further improved to

\[
\frac{0.570}{n} \leq R_{n,p}^* \leq \frac{0.608}{n}.
\]  

(12)

The upper bound in the above results come from the worst case risk of the Good-Turing estimator.

In the context of estimation of additive functions, minimax rate results for the power sum

\[ S_\alpha(P) = \sum_x p_x^\alpha \]  

are of relevance. Let \( R_{n}^*(S_\alpha) \) denote the minimax \( L_2 \) risk for estimating \( S_\alpha(P) \), and let \( K = |X| \) denote the support size of \( P \). As shown in [1],

\[
R_{n}^*(S_\alpha) = \begin{cases} 
\frac{K^2}{(n \log n)^{2\alpha}}, & (n \geq K^{1/\alpha} / \log K, \log n \leq \log K), \\
\frac{K^2}{(n \log n)^{2\alpha}}, & (n \geq K^{1/\alpha} / \log K), \\
\frac{1}{(n \log n)^2(\alpha - 1)}, & (K \leq n \log n), \\
\frac{1}{n}, & (1 < \alpha < 3/2) \quad \text{(Thm 2 in [4])}
\end{cases}
\]

\[
\alpha \geq 3/2. \quad \text{(13)}
\]

**Remarks:** For \( \alpha < 3/2 \), the minimax rates for \( S_\alpha(P) = \sum_x p_x^\alpha \) are conditional on the support size \( K \) relative to \( n \), while the minimax rate for the missing mass \( M_0 = \sum_x p_x I(F_x = 0) \) holds for any support size. For \( \alpha > 3/2 \), minimax rates for \( S_\alpha(P) \) are independent of both \( K \) and \( \alpha \). In this context, it is interesting to consider the minimax rate for order-\( \alpha \) missing mass \( M_{0,\alpha} = \sum_x p_x^\alpha I(F_x = 0) \), and observe its behaviour with respect to \( K \) and \( \alpha \).

**B. Concentration results**

The missing mass \( M_0(X^n, P) \) has been shown to concentrate around its mean in a manner independent of the underlying distribution \( P \) [9], [17], [18], [22], [23]. We quickly recall the general setting for such concentration results. For a random variable \( Z \), let

\[
L_Z(\lambda) \triangleq \log(\mathbb{E}[e^{\lambda(Z - \mathbb{E}[Z])}])
\]  

(14)

denote the log Moment Generating Function (MGF). A random variable \( Z \) is said to be sub-Gaussian with variance factor \( v \), denoted \( Z \sim \text{sub-Gaussian}(v) \), on the right tail if \( L_Z(\lambda) \leq \lambda^2 v/2, \lambda > 0 \).

If \( Z \) is sub-Gaussian(\( v \)) on the right tail, by the standard Chernoff method, we have the right tail bound

\[
\Pr(Z \geq \mathbb{E}[Z] + \epsilon) \leq e^{-\epsilon^2/2v}, \epsilon \geq 0.
\]  

(15)
For sub-Gaussian left tail bounds, we will need $-Z \sim \text{sub-Gaussian}(v)$ on the right tail, or equivalently $L_Z(\lambda) \leq \lambda^2 v / 2$, $\lambda < 0$.

One of the first concentration results for missing mass was shown in [9], where a high probability bound on a letter not being seen in $X^n$ is used in McDiarmid’s inequality [24] to show the right tail bound $\Pr(M_0 > E[M_0] + \epsilon) \leq e^{-\frac{\epsilon^2}{3}}$, $\epsilon \geq 0$. In [22], the above bound was improved to

$$\Pr(M_0 \geq E[M_0] + \epsilon) \leq e^{-\epsilon^2 / 2}, \epsilon \geq 0,$$

by showing that $M_0(X^n, P) \sim \text{sub-Gaussian}(0.5/n)$ on the right tail using negative association and the Kearns-Saul inequality. Further, [22] showed the left tail bound $\Pr(M_0 < E[M_0] - \epsilon) \leq e^{-1.35^{92} n \epsilon^2}$, $\epsilon \geq 0$, using negative association and connections between Chernoff entropy and Gibbs variance. In [17], the left tail bound was further improved to

$$\Pr(M_0 < E[M_0] - \epsilon) \leq e^{-1.92 n \epsilon^2}, \epsilon \geq 0,$$

by showing that $-M_0(X^n, P) \sim \text{sub-Gaussian}(\gamma/n)$ on the right tail, where $\gamma$ is defined as

$$\gamma = \max_{t > 0} t e^{-t} (1 - e^{-t}) = 0.2603 \ldots, \quad (1/2 \gamma) = 1.92 \ldots$$

Remarks: As we show in Section [VII-A], $\text{Var}(M_0) \leq \gamma / n$, and there exist uniform distributions that nearly achieve this variance upper bound [19]. Now, if $Z \sim \text{sub-Gaussian}(v)$, it is known that $v \geq \text{var}(Z)$ [24]. So, the constant 1.92 in the left tail bound $e^{-1.92 n \epsilon^2}$ is nearly optimal.

In our recent work [23], we provided evidence that the constant 1 is likely to be tight for the right tail bound $e^{-n \epsilon^2}$ in the sub-Gaussian regime. More precisely, we showed that, in the Poisson sampling model ($N \sim \text{Poisson}(n)$ samples), there exists $P$ such that $M_0(X^N, P)$ is not sub-Gaussian($c/n$) for $c < 0.5$.

Therefore, the best asymptotic left and right tail bounds for $M_0$ using sub-Gaussianity appear to be $e^{-1.92 n \epsilon^2}$ and $e^{-n \epsilon^2}$, respectively. However, it is interesting to consider tail bounds using other approaches such as sub-Poisson and sub-Gamma property [18], as these are known to improve upon sub-Gaussian tail bounds in some cases [24].

Extending concentration results from missing mass $M_0(X^n, P)$ to missing mass of functions $G_0(X^n, P)$ is challenging because the methods used for proving concentration results for $M_0$ are specific to the missing mass $M_0$ and to sub-Gaussian concentration results. For applications, obtaining the best possible concentration results by going beyond sub-Gaussianity, if necessary, is interesting and important.
III. SUMMARY OF RESULTS

In this section, we provide details of the problems addressed in this work, summarize the main results and discuss them.

A. Minimax estimation results for $M_{0,\alpha}$

Let $R_{n,\alpha}^* \triangleq R_{n,p,\alpha}^*$ denote the minimax risk of estimating the order-$\alpha$ missing mass $M_{0,\alpha}$ under $L_2^2$ loss as defined in (7). Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of positive integers.

**Theorem 1.**

1) For a positive integer $\alpha \in \mathbb{N}$ and $n > 2\alpha$,

$$\frac{c_l}{n^{2\alpha-1}} \leq R_{n,\alpha}^* \leq \frac{c_u}{n^{2\alpha-1}},$$

where $c_l, c_u$ are positive constants.

2) For a non-integer $\alpha \in (1, \infty) \setminus \mathbb{N}$,

$$\frac{c_l}{n^{2\alpha-1}} \leq R_{n,\alpha}^* \leq \frac{c_u}{n^{2(\alpha-1)}}.$$

Paraphrasing the first part of the theorem, the minimax risk of the order-$\alpha$ missing mass for a positive integer $\alpha$ falls as $1/n^{2\alpha-1}$. In a proof presented in Section IV we show the lower bound in (19) using Dirichlet priors and the upper bound using a generalized Good-Turing estimator for $M_{0,\alpha}(X^n, P)$, denoted $M_{0,\alpha}^\text{GT}(X^n)$, defined as

$$M_{0,\alpha}^\text{GT}(X^n) = \frac{\phi_\alpha}{\binom{n}{\alpha}}, \quad \alpha \in \mathbb{N}. \quad (21)$$

To understand why the above estimator works well, let us consider its bias, which can be simplified as follows:

$$E[M_{0,\alpha}^\text{GT}(X^n) - M_{0,\alpha}(X^n, P)] = \sum_{x \in \mathcal{X}} \frac{1}{\binom{n}{\alpha}} \Pr(F_x = \alpha) - p_x^\alpha \Pr(F_x = 0)$$

$$= \sum_{x \in \mathcal{X}} p_x^\alpha (1 - p_x)^{n-\alpha} [1 - (1 - p_x)^\alpha] \quad (\geq 0)$$

$$|E[M_{0,\alpha}^\text{GT}(X^n) - M_{0,\alpha}(X^n, P)]| \leq \frac{\alpha}{\binom{n}{\alpha}} \sum_{x \in \mathcal{X}} p_x \left[ \binom{n}{\alpha} p_x^\alpha (1 - p_x)^{n-\alpha} \right] \quad (22)$$

$$\leq \frac{\alpha^{\alpha+1}}{n^{\alpha}}, \quad (23)$$
where the change to absolute value in the LHS of (a) follows because the previous expression is clearly non-negative, and the RHS of (a) follows from the inequality $1 - (1 - p_x)^\alpha \leq \alpha p_x$ ($\alpha \geq 1$). Finally, (b) follows because $\binom{n}{\alpha} p_x^\alpha (1 - p_x)^{n-\alpha} \leq 1$ and $\binom{n}{\alpha} \geq (n/\alpha)^\alpha$. Hence, the bias falls as $1/n^\alpha$. The squared-error risk computation is more involved, and, as shown in the proof in Section [IV] it falls as $1/n^{2\alpha - 1}$ matching the lower bound.

At this juncture, we draw attention to (22), where the $p_x$ term inside the summation results in an averaging of a bounded function from $[0,1] \rightarrow \mathbb{R}$. The average is upper bounded by the maximum of the function, and this bound holds for all distributions with no assumptions needed on the alphabet size. This trick is a recurring one in the analysis of estimation of missing mass and its variants. Notice that the division by $\binom{n}{\alpha}$ in the definition of the estimator $M_{0,\alpha}^{GT}$ in (21) is instrumental in the additional $p_x$ term appearing inside the summation in the bias expression.

Remark on generalization: The bias upper bound of $O(1/n^\alpha)$ above can be generalized readily to the missing mass of $g(p)$ that satisfies

\[
\frac{g(p)(1-p)^\alpha}{p^\alpha} \geq a - O(p h(p)),
\]  

(24)

$\alpha \in \mathbb{N}$, $a$ is a constant and $\binom{n}{\alpha} p^\alpha (1 - p)^{n-\alpha} h(p) \leq O(1)$ by using the Good-Turing estimator $a M_{0,\alpha}^{GT}$. Linear combinations of a constant (possibly, $o(n)$) number of such $g(p)$’s can also be estimated with diminishing bias by using corresponding linear combinations of the Good-Turing estimators. Examples include $g(p) = p^a (1 - p)^b$ ($a \in \mathbb{N}$, $b > 1$), $p^a e^{-bp}$ ($a \in \mathbb{N}$) and so on. We do not explicitly study such classes of functions further in this article, and proceed to consider opposite situations where the Good-Turing or any other linear estimator may not be able to do better than a plugin estimator, which estimates anything missing as 0.

The second part of the theorem considers positive non-integer values of $\alpha > 1$. Let $\lfloor \alpha \rfloor$ be the integer part (largest integer $\leq \alpha$) and let $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ be the fractional part of $\alpha$. We see that the lower bound $1/n^{2\alpha - 1}$ is the same as in the integer case, and the upper bound has a rate of fall of $1/n^{2(\alpha - 1)}$, which is larger by a multiplicative factor of $n$. In the proof, we derive the upper bound as the worst case risk of the plugin estimator, which simply estimates the missing mass as zero.

A relevant question is whether an estimator such as a generalized Good-Turing estimator will do better than the plugin estimator and improve the convergence rate of $1/n^{2(\alpha - 1)}$ for a non-integer $\alpha$. At this point in our investigation, it looks unlikely that a linear estimator will provide improvement beyond logarithmic factors. Since the lower bound is away by a factor of $1/n$, it
appears some form of non-linear interpolation may be required, and this is a topic for future study.

The range \(0 < \alpha < 1\) is significantly different from \(\alpha > 1\). Consider the distribution \(P_\epsilon\) with \(\mathcal{X} = \mathbb{N}\) and \(p_k = A/k^{1+\epsilon}\) for \(\epsilon > 0\), \(k \in \mathbb{N}\), \(A^{-1} = \sum_{k=1}^{\infty} 1/k^{1+\epsilon}\). The probability that \(k \in \mathbb{N}\) does not appear in \(X^n\) is \(\Pr(F_k = 0) = (1 - A/k^{1+\epsilon})^n\). For \(\alpha \in (0, 1)\), the expected order-\(\alpha\) missing mass

\[
E[M_{0,\alpha}(X^n, P_\epsilon)] = A^\alpha \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+\epsilon}}(1 - A/k^{1+\epsilon})^n
\]

diverges if \(\alpha + \alpha \epsilon < 1\) or \(\epsilon \in (0, \frac{1}{\alpha} - 1)\). Because of the impossibility of distinguishing between \(P_\epsilon\) for \(\epsilon \in (0, \frac{1}{\alpha} - 1)\), consistent estimation of \(M_{0,\alpha}(X^n, P_\epsilon)\) is not possible unless the alphabet size \(|\mathcal{X}|\) is upper bounded. This is similar to the case of estimation of the power sum \(S_\alpha(P)\) for \(\alpha \in (0, 1)\). Because of such complications, we postpone the study of estimation of order-\(\alpha\) missing mass for the case \(\alpha \in (0, 1)\) to future work.

B. Concentration results and new tail bounds

To obtain concentration results and tail bounds for \(G_0(X^n, P)\) that are distribution-free, a standard approach is to upper bound the log Moment Generating Function (log MGF) \(L_{G_0}(\lambda) = \log E[e^{\lambda(G_0 - E[G_0])}]\), by a function of \(\lambda\), and use the Chernoff method. Now, \(G_0 = \sum_x g(p_x)I(F_x = 0)\), is a sum of random variables \(g(p_x)I(F_x = 0)\), \(x \in \mathcal{X}\), which are (1) dependent because \(F_x\) are dependent, and (2) highly heterogeneous because of the range of values of \(p_x\). To address dependency, we will follow [22] and use the idea of negative association. The log MGF of \(G_0\)

\[
L_{G_0}(\lambda) = \log E \left[ \prod_{x \in \mathcal{X}} \exp\{\lambda g(p_x)[I(F_x = 0) - \Pr(F_x = 0)]\} \right]
\]

\[
\leq \sum_{x \in \mathcal{X}} \log E \left[ \exp\{\lambda g(p_x)[I(F_x = 0) - \Pr(F_x = 0)]\}\right], \quad (25)
\]

where \((a)\) follows because of the following argument. The random variables \(F_x, x \in \mathcal{X}\), are negatively associated [9], [25] and

\[
T(F_x) \triangleq \exp\{\lambda g(p_x)[I(F_x = 0) - \Pr(F_x = 0)]\}
\]

is a monotonically non-increasing function of \(F_x\). So, \(T(F_x), x \in \mathcal{X}\), are negatively associated [25] resulting in \(E[T(F_x)T(F_{x'})] \leq E[T(F_x)]E[T(F_{x'})]\).

Following the above simplification, to address the heterogeneity and decouple \(\lambda\) and \(p_x\), [9], [17] use the Kearns-Saul inequality to upper bound \(E[T(F_x)]\). An alternative method, introduced
in [18], starts with Bennett’s inequality [24]. We will employ a stronger version of Bennett’s inequality for a zero-mean random variable $Z$ with $|Z| < 1$, which is proved as follows:

$$E[\exp(tZ)]^{(E[Z]=0)} \leq 1 + \sum_{k=2}^{\infty} \frac{t^k E[Z^k]}{k!}^{(|Z|<1)} \leq 1 + \sum_{k=2}^{\infty} \frac{t^k E[Z]}{k!} \leq 1 + \text{Var}(Z)(e^t - t - 1). \tag{26}$$

By using $1 + x \leq e^x$ on the RHS of (26) with $x = \text{Var}(Z)(e^t - t - 1)$ and taking logarithms, we obtain the usual Bennett’s inequality, given below:

$$\log E[\exp(tZ)] \leq \text{Var}(Z)(e^t - t - 1), \text{ all } t. \tag{27}$$

Setting $t = \lambda g(p_x)$ and $Z = I(F_x = 0) - \Pr(F_x = 0)$ with $\text{Var}(Z) = (1 - p_x)^n(1 - (1 - p_x)^n)$ in (26) and (27), and substituting the upper bounds in (25), we get the following two bounds on the log MGF:

$$L_{G_0}(\lambda) \leq \sum_{x \in \mathcal{X}} \log \left(1 + (1 - p_x)^n(1 - (1 - p_x)^n)(e^{\lambda g(p_x)} - \lambda g(p_x) - 1)\right), \tag{28}$$

$$L_{G_0}(\lambda) \leq \sum_{x \in \mathcal{X}} (1 - p_x)^n(1 - (1 - p_x)^n)(e^{\lambda g(p_x)} - \lambda g(p_x) - 1). \tag{29}$$

To obtain distribution-free concentration results and tail bounds, the next step is to upper bound the RHS above by a function of $\lambda$ (free of the distribution $p_x$) for $\lambda > 0$ to obtain right tail bounds and $\lambda < 0$ to obtain left tail bounds. Table I shows the different forms of the upper-bounding function of $\lambda$ along with the range of $\lambda$ and associated tail bounds obtained by the Chernoff method. The first row describes the general Chernoff bounding method.

Sub-Gaussianity is a standard notion, and has been extensively studied for the case of missing mass where $g(p) = p$. While sub-Gamma concentration is described in [24], we propose the notion of strongly sub-Gamma concentration, where the log-MGF is upper-bounded by $(v/c^2) \log \frac{e^{-\lambda c}}{1 - e^\lambda}$, which is the log-MGF of a Gamma random variable with variance parameter $v$ and scale parameter $c$. Since the function $\frac{\lambda^2 v/2}{1 - e^\lambda}$, introduced for sub-Gamma concentration in [24], does not correspond to the log-MGF of a Gamma random variable, the notion of strongly sub-Gamma concentration is, perhaps, a useful addition. It is easy to show the following implications:

$$Z \sim \text{sub-Gaussian}(v), \lambda > 0 \rightarrow Z \sim \text{Strongly sub-Gamma}(v, c), c > 0 \rightarrow Z \sim \text{sub-Gamma}(v, c). \tag{30}$$
So, while sub-Gaussianity is the strongest notion, the optimality of the variance factor is critical for obtaining the best possible tail bounds. If the variance factor of sub-Gaussian concentration does not coincide with the variance of \( Z \), the seemingly weaker notions of concentration, such as strongly sub-Gamma or sub-Gamma concentration, result in better tail bounds on \( \Pr(Z > E[Z] + \epsilon) \) for a restricted range of \( \epsilon \), as illustrated for missing mass later on.

Next, we propose the notion of filtered sub-Gaussian concentration, where the log-MGF is upper-bounded by \( f(\lambda) = \lambda^2 \sigma^2 / 2 + h(\lambda) \). In terms of distributions, this is equivalent to convolving a Gaussian distribution with another distribution with log-MGF \( h(\lambda) \) (assuming it is a valid log-MGF corresponding to some distribution). Hence, we use the terminology of “filtering” and call \( h(\lambda) \) as a filter. The convolution seemingly weakens concentration when compared to sub-Gaussian concentration. However, when the variance factor of sub-Gaussian concentration is not optimal, instead of bounding the tail with that of a Gaussian random variable of higher variance, one can possibly obtain better bounds by considering convolutions of a Gaussian
random variable (with lower variance) and other random variables.

The last row of the table describes the specific type of filtering that is useful for missing mass, namely, filtering a Gamma log-MGF by a polynomial. When the degree parameter $R = 1$, this type of concentration reduces to strongly sub-Gamma concentration. When $R = 2$, the log-MGF upper bound $\lambda^2 a_2/2 + (v/c^2) \log \frac{e^{-\lambda c}}{1-\lambda c}$ corresponds to a random variable whose distribution is the convolution of a Gaussian random variable with variance $a_2$ and a Gamma random variable with variance parameter $v$ and scale parameter $c$. As we illustrate later, such convolutions improve the right tail bound for missing mass. For $R > 2$, the polynomial part may result in the filter not being a valid log-MGF, but the tail bounds are still valid, and can be computed numerically.

If $Z \sim \text{sub-Gamma}(v, c)$ on the right tail, we get the following tail bound using the standard Chernoff method:

$$
\Pr(Z \geq E[Z] + \epsilon) \leq \exp\left\{ -\left( 1 + \frac{\epsilon c}{v} - \sqrt{1 + \frac{2\epsilon c}{v}} \right) \frac{v}{c^2} \right\}.
$$

(31)

Similarly, if $Z \sim \text{Strongly sub-Gamma}(v, c)$, we get

$$
\Pr(Z \geq E[Z] + \epsilon) \leq \exp\left\{ -\frac{1}{c} \left( \epsilon - \frac{v}{c} \log \left( 1 + \frac{\epsilon c}{v} \right) \right) \right\}.
$$

(32)

If $Z \sim \text{Poly-sub-Gamma filtered}(R, a, v, c)$, the minimization inside the exponent in Chernoff bound computation is easy to evaluate numerically for any $R$ because $f'(\lambda)$ is rational in $\lambda$ with a linear denominator. For $R = 2$, the tail bound can be expressed analytically as follows.

$$
\Pr(G_0 - E[G_0] \geq \epsilon) \leq \exp\left\{ -\frac{1}{c} \left( \frac{1}{2} - \frac{d_2}{d_1} \right) \epsilon + \frac{v}{c} \ln \left( 1 - \frac{2\epsilon c}{v} \right) \right\},
$$

(33)

where $d_1 = (a_2 + v + \epsilon) + \sqrt{(a_2 + v + \epsilon)^2 - 4a_2\epsilon c}$ and $d_2 = a_2 - (v + \epsilon c)$.

Using the definitions and notation in Table I, we now describe our main concentration result for the missing mass of the function $g(p)$, $G_0(X^n, P)$, in Theorem 2. Let

$$
u^*_r(n, g) \triangleq \max_{0<p<1} g(p)^r (1-p)^n \frac{1-(1-p)^n}{p}. \quad (34)
$$

Where necessary, we will drop the arguments $n, g$ in $\nu^*_r(n, g)$ to reduce clutter.

**Theorem 2.**

1) $G_0(X^n, P) \sim \text{sub-Gaussian} \left( \frac{0.519 \max_p (g(p)/p)^2}{n} \right)$, all $\lambda$.

(35)

Additionally, on the left tail,

$$
G_0(X^n, P) \sim \text{sub-Gaussian}(\nu^*_2(n, g)), \ \lambda < 0.
$$

(36)
2) Let $R \geq 1$ be a positive integer. Let $c > 0$ be such that
\[
\frac{u^*_r(n, g)}{(r - 1)!} \leq c \frac{u^*_{r-1}(n, g)}{(r - 2)!}, \quad r \geq 3.
\] (37)

Let $v = u^*_{R+1}(n, g)/(c^{R-1} R!)$, and
\[
a_r = \frac{u^*_r(n, g)}{(r - 1)!} - c^{r-2} v, \quad 2 \leq r \leq R.
\]

Then, for $a = [a_2 \ a_3 \ \cdots \ a_R]$, on the right tail,
\[
G_0(X^n, P) \sim \text{Poly-sub-Gamma-filtered}(R, a, v, c).
\] (38)

The first part of Theorem 2, proven in Section V-A, provides a sub-Gaussian right-tail and left-tail bound as shown in Table I. For the case of missing mass, i.e. $g(p) = p$, we have sub-Gaussianity with a variance of $0.519/n$, which is close to the best-known $0.5/n$. Additionally, since $u^*_2(n, p) \leq \gamma/n$ (proved in Section VII-A, recall that $\gamma = \max_{t > 0} t e^{-t}(1 - e^{-t}) = 0.2603\ldots$), this recovers the best-known sub-Gaussian left-tail bound $e^{-1.92n^2}$. Our proof of these tail bounds for $M_0$ are arguably simpler compared to the Gibbs-variance method used in [22] or the Kearns-Saul inequality method used in [17]. For all other $g(p)$, such as $g(p) = p^\alpha$ for power sum or $g(p) = p \log(1/p)$ for Shannon entropy, the sub-Gaussian concentration left tail results have been shown for the first time.

For the case of $g(p) = p^\alpha$ and $g(p) = p \log_2(1/p)$, we simplify the left tail bounds and present them in the following corollary.

**Corollary 3.** 1) The order-$\alpha$ missing mass $M_{0,\alpha}$ for $\alpha \geq 1$, satisfies
\[
\Pr(M_{0,\alpha} - E[M_{0,\alpha}] \leq -\epsilon) \leq \exp\left\{-\frac{n^{2\alpha-1}\epsilon^2}{2\gamma}\right\}, \quad n \geq (2\alpha - 1) \ln 2/(2\alpha - 1 - \ln 2),
\] (39)

where $\gamma = \max_{t > 0} t^{2\alpha-1} e^{-t}(1 - e^{-t})$.

2) Let $H_0(X^n, P) \triangleq \sum_{x \in X} p_x \log_2(1/p_x) I(F_x(X^n) = 0)$ be the missing Shannon entropy in $X^n$ of a distribution $P$ with $p_x \geq 1/k$ for all $x \in X$. $H_0(X^n, P)$ satisfies
\[
\Pr(H_0 - E[H_0] \leq -\epsilon) \leq \exp\left\{-\frac{n\epsilon^2}{2\gamma(\log_2 k)^2}\right\}, \quad n \geq 3,
\] (40)

where $\gamma = \max_{t > 0} t e^{-t}(1 - e^{-t}) = 0.2603\ldots$.

A proof of Corollary 3 is given in Section VII.
The second part of Theorem \[2\] proven in Section \[V-B\] provides right tail bounds using the Poly-sub-Gamma-filtered approach with a Gamma MGF and a polynomial filter as described in Table \[1\]. The choice of the scale parameter \( c \) needs elaboration. From (37), it is clear that \( c \) quantifies the rate of fall of \( u^*_r(n, g)/(r - 1)! \), and it will depend on the function \( g(p) \). An important requirement is that \( c \) should fall with \( n \) so that the confidence intervals from the tail bounds shrink with increasing \( n \). The following rate of fall lemma describes the choice of \( c \) as a decreasing function of \( n \) for two types of functions \( g(p) \).

**Lemma 4.** Let \( g(p) \) be differentiable for \( p \in (0, 1) \), and let \( g'(p) \) denote its derivative. Let \( u^*_r(n, g) \) be as defined in (34). There exist \( b_i, 0 \leq i \leq 5 \), independent of \( r \), such that

\[
c = \max(b_0, g(b_1/(n + b_2)), b_3, g(b_4/(n + b_5))) \text{satisfies } \frac{u^*_r(n, g)}{(r - 1)!} \leq c \frac{u^*_{r-1}(n, g)}{(r - 2)!}, \quad r \geq 3,
\]

if \( g(p) \) belongs to one of the following two types:

1) **Type A:** For some \( \mu > 0 \),

\[
0 < g'(p) \leq \mu g(p)/p, \quad p \in (0, 1).
\]

2) **Type B:** For some \( p^* \in (0, 1) \),

\[
0 < g'(p) \leq (1/p - 1/p^*) \frac{g(p)}{1 - p}, \quad p \in (0, p^*),
\]

\[
g'(p) < 0, \quad p \in (p^*, 1).
\]

A proof of Lemma 4 is given in Section \[VI\]. In the proof, the values of \( b_i \) are explicitly specified for a \( g(p) \) of Type A or B. Interesting examples such as \( g(p) = p^\alpha, \alpha > 0 \), belong to Type A with \( \mu = \alpha \) and \( g(p) = p \log_2(1/p) \) belongs to Type B with \( p^* = 1/e \). For these cases, we further simplify the right tail bound for \( R = 1 \) (strongly sub-gamma concentration) in the following corollary.

**Corollary 5.**

1) The missing mass \( M_0 \) satisfies

\[
\Pr(M_0 - E[M_0] \geq \epsilon) \leq \exp \left\{ -\frac{2(n + 2)}{3} \left( \epsilon - \frac{2\gamma_n}{3} \log \left( 1 + \frac{3\epsilon}{2\gamma_n} \right) \right) \right\}, \quad n \geq 3,
\]

where \( \gamma_n = \gamma(1 + 2/n), \gamma = \max_{t>0} te^{-t}(1 - e^{-t}) = 0.2603 \ldots \)

2) Consider the order-\( \alpha \) missing mass \( M_{0,\alpha} \), \( \alpha > 1 \). For \( n > 1 + 4\alpha^2/(1 - \alpha)^2 \),

\[
\Pr(M_{0,\alpha} - E[M_{0,\alpha}] \geq \epsilon) \leq \exp \left\{ -\frac{n - b}{a} \epsilon + \frac{\gamma_\alpha}{a^2} (n - b)^{3 - 2\alpha} \log \left( 1 + \frac{an^{2\alpha-1}}{\gamma_\alpha(n - b) \epsilon} \right) \right\},
\]

(43)
where \( b = 1 + \frac{2\alpha}{\alpha - 1} \), \( a = (b - 1) \left( \frac{2(\alpha - 1)}{\alpha + 1} \right)^\alpha \) and \( \gamma_\alpha = \max_{t > 0} t^{2\alpha - 1} e^{-t(1 - e^{-t})} \).

3) For the missing Shannon entropy in \( X^n, H_0(X^n, P) \), of a distribution \( P \) with \( p_x \geq 1/k \) for all \( x \in X \),

\[
\Pr(H_0 - E[H_0] \geq \epsilon) \leq \exp\left\{ -2n_0 \frac{\gamma_k}{\log_2 n_0} \left( \epsilon - \frac{\gamma_k}{\log_2(n_0)} \log_2 \left( 1 + \epsilon \frac{\log_2(n_0)}{\gamma_k} \right) \right) \right\}, \quad n \geq 3,
\]

where \( n_0 = (1/3)(n - 1) + e \) and \( \gamma_k = 2\gamma(1/3 + (e - 1/3)/n)(\log_2 k)^2 \).

A proof of Corollary 5 is given in Section VII.

For \( M_0 \) (i.e. \( g(p) = p \)), Fig. 1 shows a plot comparing the different right tail bounds for \( n = 20, 100, 1000 \). The new bounds, for \( R = 2 \) and \( R = 5 \), provide a noticeable improvement over the sub-Gaussian bound \( e^{-n\epsilon^2} \) for all three values of \( n \) over a significant range of \( \epsilon \).

The rest of this article contains proofs of the results in this section.

**IV. PROOF OF THEOREM**

We start by proving the upper bound of Part 1 of the theorem.
A. Upper bound on $R_{n,\alpha}^*$ for $\alpha \in \mathbb{N}$

Consider the generalized Good-Turing estimator $M_{0,\alpha}^{\text{GT}}(X^n) = \frac{\phi_{\alpha}(X^n)}{\binom{n}{\alpha}}$, $\alpha \in \mathbb{N}$. By the definition of minimax risk, we have

$$R_{n,\alpha}^* = \min_{\hat{M}_{0,\alpha}} \max_P E[(\hat{M}_{0,\alpha}(X^n) - M_{0,\alpha}(X^n, P))^2]$$

$$\leq \max_P E[(M_{0,\alpha}^{\text{GT}}(X^n) - M_{0,\alpha}(X^n, P))^2] \triangleq R_{n,\alpha}(M_{0,\alpha}^{\text{GT}}).$$

(45)

The rest of the proof is to upper bound the worst-case squared-error risk of the generalized Good-Turing estimator $R_{n,\alpha}(M_{0,\alpha}^{\text{GT}})$, and this is provided in the next lemma.

**Lemma 6.**

$$R_{n,\alpha}(M_{0,\alpha}^{\text{GT}}) \leq O\left(1/n^{2\alpha-1}\right) + o\left(1/n^{2\alpha-1}\right).$$

(46)

**Proof.**

$$M_{0,\alpha}^{\text{GT}} - M_{0,\alpha} = \left(\sum_{x \in \mathcal{X}} \frac{1}{\binom{n}{\alpha}} I(F_x = \alpha) - p_x^0 I(F_x = 0)\right) \left(\sum_{y \in \mathcal{X}} \frac{1}{\binom{n}{\alpha}} I(F_y = \alpha) - p_y^0 I(F_y = 0)\right)$$

$$= \left[ \sum_{x \in \mathcal{X}} \left(\frac{1}{\binom{n}{\alpha}} I(F_x = \alpha) + p_x^2 I(F_x = 0)\right)\right]$$

$$+ \left[ \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, y \neq x} \frac{1}{\binom{n}{\alpha}^2} I(F_x = F_y = \alpha) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, y \neq x} p_x^0 I(F_x = 0, F_y = \alpha)\right]$$

$$- \left[ \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, y \neq x} p_y^0 I(F_y = 0, F_x = \alpha) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, y \neq x} (p_x p_y)^0 I(F_x = F_y = 0)\right].$$

(47)

Taking expectations, we consider each of the terms inside square brackets above and bound them by $O(1/n^{2\alpha-1})$.

The following bounds are used repeatedly in the simplifications. Let $f_1 : [0, 1] \to \mathbb{R}^+$ and $f_2 : [0, 1]^2 \to \mathbb{R}^+$ be non-negative functions, and let $a$, $b$, $c$ be positive numbers.

$$\sum_x p_x f_1(p_x) \leq \max_{p \in [0,1]} f_1(p)$$

(48)

$$\sum_x \sum_y p_x p_y f_2(p_x, p_y) \leq \max_{p,q \in [0,1]} f_2(p, q)$$

(49)
\[
\max_{p \in [0,1]} p^a(1-p)^b = \frac{a^b b^a}{(a + b)^{a + b}}
\] (50)

\[
\max_{p,q \in [0,1]} p^a q^b (1-p-q)^c = \frac{a^b b^c c}{(a + b + c)^{a + b + c}}
\] (51)

Let us first consider the first term in square brackets in (47).

\[
\sum_{x \in X} \left( \frac{1}{\binom{n}{a}^2} I(F_x = \alpha) + p_x^{2\alpha} I(F_x = 0) \right) = \sum_{x \in X} \frac{1}{\binom{n}{a}^2} p_x^a (1-p_x)^{n-a} + \sum_{x \in X} p_x^{2\alpha} (1-p_x)^n \leq \frac{(\alpha - 1)^{a-1}(n - \alpha)^{n-a}}{(n-1)^{n-1}} + \frac{(2\alpha - 1)^{2\alpha-1}(n)^n}{(n + 2\alpha - 1)^{n+2\alpha-1}} \leq n O\left( \frac{1}{n^{2\alpha-1}} \right),
\] (52)

where (a) uses (48), (50) and the notation \( \leq_n \) is as defined in Section III. Next, we consider the second term in square brackets in (47). We will need the notation \( \binom{n}{a,b} = \frac{n!}{a!b!(n-a-b)!} \). Further, we will assume \( n > 2\alpha \).

\[
\sum_{x \in X} \sum_{y \in X, y \neq x} E\left( \frac{1}{\binom{n}{a}^2} I(F_x = F_y = \alpha) - \frac{p_x^a}{\binom{n}{a}} I(F_x = 0, F_y = \alpha) \right) = \sum_{x \in X} \sum_{y \in X, y \neq x} (p_x p_y)^\alpha \left( \frac{\binom{n}{a}}{\binom{n}{a}^2} (1-p_x - p_y)^{n-2\alpha} - (1-p_x - p_y)^{n-\alpha} \right)
\]

\leq \sum_{x \in X} \sum_{y \in X, y \neq x} (p_x p_y)^\alpha (1-p_x - p_y)^{n-2\alpha} (1-(1-p_x - p_y)^\alpha)

\leq \sum_{x \in X} \sum_{y \in X, y \neq x} (p_x p_y)^\alpha (p_x + p_y) \left( \sum_{i=0}^{\alpha-1} (1-p_x - p_y)^{n-2\alpha+i} \right)

= 2 \sum_{i=0}^{\alpha-1} \sum_{x \in X} \sum_{y \in X, y \neq x} p_x p_y \left( p_x^\alpha p_y^{\alpha-1} (1-p_x - p_y)^{n-2\alpha+i} \right)

\leq 2 \sum_{i=0}^{\alpha-1} \alpha^\alpha (\alpha - 1)^{\alpha-1}(n - 2\alpha + i)^{n-2\alpha+i} \leq n O\left( \frac{1}{n^{2\alpha-1}} \right),
\] (53)

where (a) follows because \( \binom{n}{a,b} \leq \binom{n}{a}^2 \), (b) follows using the identity \( 1 - (1-a)^m = a \sum_{i=0}^{m-1} (1-a)^i \), (c) follows because of the symmetry between \( x \) and \( y \) in the summation, and (d) uses (49), (51).
Finally, we consider the third term in square brackets in (47).

\[
\sum_{x \in X} \sum_{y \in X, y \neq x} E\left( \frac{p_y}{(n/\alpha)} I(F_x = \alpha, F_y = 0) - (p_x p_y)^\alpha I(F_x = F_y = 0) \right)
\]

\[= \sum_{x \in X} \sum_{y \in X, y \neq x} (p_x p_y)^\alpha (1 - p_x - p_y)^{n-\alpha} (1 - (1 - p_x - p_y)\alpha) \]

\[\leq n \cdot O\left( \frac{1}{n^{2\alpha-1}} \right), \tag{54}\]

where the final inequality follows using steps similar to those used to obtain (53). Using (52), (53) and (54), the proof is complete. \qed

B. Upper bound on \( R_{n,\alpha}^* \) for \( \alpha \in (1, \infty) \setminus \mathbb{N} \)

Using the plug-in estimator or \( \hat{M}_{0,\alpha} = 0 \) in (45), we get \( R_{n,\alpha}^* \leq \max_P E[(M_{0,\alpha}(X^n, P))^2] \). The rest of the proof is to upper bound the worst case expectation of \( M_{0,\alpha}^2 \) and this is provided in the next lemma.

**Lemma 7.**

\[ \max_P E[(M_{0,\alpha}(X^n, P))^2] \leq n \cdot O\left( \frac{1}{n^{2(\alpha-1)}} \right), \tag{55}\]

where the notation \( \leq_n \) is as defined in Section \( \text{II} \).

**Proof.**

\[ M_{0,\alpha}^2 = \left( \sum_{x \in X} p_x^\alpha I(F_x = 0) \right) \left( \sum_{y \in X} p_y^\alpha I(F_y = 0) \right) \]

\[= \sum_{x \in X} p_x^{2\alpha} I(F_x = 0) + \sum_{x \in X} \sum_{y \in X, y \neq x} (p_x p_y)^\alpha I(F_x = F_y = 0). \tag{56}\]

Taking expectation, we bound each of the above two terms. For the first term, we proceed in exactly the same way as in the proof of Lemma \( \text{6} \) and get

\[ E\left[ \sum_{x \in X} p_x^{2\alpha} I(F_x = 0) \right] = \sum_{x \in X} p_x^{2\alpha}(1 - p_x)^n \leq \frac{(2\alpha - 1)^{2\alpha-1}(n)^n}{(n + 2\alpha - 1)^{n+2\alpha-1}} = n \cdot O\left( \frac{1}{n^{2\alpha-1}} \right). \tag{57}\]
Next, we bound the expectation of the second term in (56) in the following way:

\[
\sum_{x \in X} \sum_{y \in X, y \neq x} E\left[ (p_x p_y)^\alpha I(F_x = F_y = 0) \right] = \sum_{x \in X} \sum_{y \in X, y \neq x} p_x^\alpha p_y^\alpha (1 - p_x - p_y)^n
\]

\[\leq (a) \sum_{x \in X} \sum_{y \in X, y \neq x} p_x^\alpha p_y^\alpha (1 - p_x)^n (1 - p_y)^n
\]

\[= \sum_{x \in X} p_x^\alpha (1 - p_x)^n \left( \sum_{y \in X, y \neq x} p_y^\alpha (1 - p_y)^n \right)
\]

\[\leq (b) \left( \sum_{x \in X} p_x^\alpha (1 - p_x)^n \right) \left( \sum_{y \in X} p_y^\alpha (1 - p_y)^n \right)
\]

\[\leq (c) O\left( \frac{1}{n^{2(\alpha - 1)}} \right), \quad (58)
\]

where (a) follows because \((1 - a_1 - a_2) \leq (1 - a_1)(1 - a_2)\) when both \(a_1\) and \(a_2\) have the same sign, (b) follows by adding \(p_x^\alpha (1 - p_x)^n\) to \(\sum_{y \in X, y \neq x} p_y^\alpha (1 - p_y)^n\) and (c) uses (48) and (50). Using (57) and (58), the proof is complete.

\(\square\)

C. Lower bound on minimax risk \(R_{n,\alpha}^*\)

The lower bound of \(O\left(1/n^{2\alpha-1}\right)\) for all \(\alpha \geq 1\) is obtained by following the Dirichlet prior approach. This approach was used in [19] for the case of missing mass \(M_0\) with \(\alpha = 1\). The same approach is generalized here for \(\alpha \geq 1\).

Let \(\Delta_k\) be the set of all probability distributions on the alphabet \(X = \{1, 2, \ldots, k\}\). Let \(P\) be a random variable on \(\Delta_k\), generated according to a Dirichlet distribution [26] with parameters \(\beta = (\beta_1, \beta_2, \ldots, \beta_k)\). From the family of distributions \(\Delta_k\), \(P\) is chosen according to Dirichlet(\(\beta\)) and then \(X^n \sim P\) is sampled iid. By the standard prior method, the minimax rate \(R_{n,\alpha}^*\) is lower bounded as

\[
R_{n,\alpha}^* \geq E_{X^n} \left[ \text{Var} [M_{0,\alpha}(X^n, P)|X^n] \right], \quad (59)
\]

where \(\text{Var} [M_{0,\alpha}(X^n, P)|X^n]\) is the conditional variance of \(M_{0,\alpha}(X^n, P)\) conditioned on \(X^n\).

**Lemma 8.** Let \(P \sim \text{Dirichlet}(\beta_1, \beta_2, \ldots, \beta_k)\) with \(k = cn^2, c > 0, \text{ and } \beta_i = 1/n\). Let \(X^n \sim P\), iid. Then,

\[
E_{X^n} \left[ \text{Var} [M_{0,\alpha}(X^n, P)|X^n] \right] \geq_n O\left(1/n^{2\alpha-1}\right), \quad (60)
\]

where the notation \(\geq_n\) is as defined in Section [I].
Proof. A proof is given in Appendix A. □

Combining Lemma 8 and (59), the proof of the lower bound is complete.

V. PROOF OF THEOREM 2

A. Proof of Part 1 of Theorem 2

Starting with (28), we simplify as follows:

\[
L_{G_0}(\lambda) \leq \sum_x (1 - p_x^n (1 - (1 - p_x)^n) (e^{\lambda g(p_x)} - \lambda g(p_x) - 1))
\]

\[
= \frac{\lambda^2}{n} \sum_x p_x \left[ \frac{n}{\lambda^2 p_x} \log \left( 1 + e^{-np_x (e^{\lambda g(p_x)} - \lambda g(p_x) - 1)} \right) \right]
\]

\[
\leq \frac{\lambda^2}{n} \max_p \left[ \frac{n}{\lambda^2 p} \log \left( 1 + e^{-np (e^{\lambda g(p)} - \lambda g(p) - 1)} \right) \right]
\]

\[
= \frac{\lambda^2}{n} \max_p \left[ \left( \frac{g(p)}{p} \right)^2 \frac{np}{(\lambda g(p))^2} \log \left( 1 + e^{-np (e^{\lambda g(p)} - \lambda g(p) - 1)} \right) \right]
\]

\[
\leq \frac{\lambda^2}{n} \max_p \left( \frac{g(p)}{p} \right)^2 \max_{u,v} \frac{n}{u^2} \log \left( 1 + e^{-n (e^v - v - 1)} \right)
\]

\[
\leq \frac{0.2595 \lambda^2 \max_p (g(p)/p)^2}{n}, \quad (61)
\]

where (a) follows by using \((1 - p_x^n (1 - (1 - p_x)^n) \leq (1 - p_x)^n \leq e^{-np_x}\), (b) follows because average is lesser than maximum, and (c) follows because \(\max_{u,v} \frac{n}{u^2} \log \left( 1 + e^{-n (e^v - v - 1)} \right) = 0.2595\) using some calculus and computations. From (61), we get the sub-Gaussianity result in (35).

Recall, from (29), the upper bound

\[
L_{G_0}(\lambda) \leq \sum_{x \in \mathcal{X}} (1 - p_x^n (1 - (1 - p_x)^n) (e^{\lambda g(p_x)} - \lambda g(p_x) - 1)).
\]

Since we are concerned with the left tail bound, we consider \(\lambda < 0\). Using \(e^t - t - 1 \leq t^2/2, t \leq 0\), and setting \(t = \lambda g(p_x)\), we get

\[
L_{G_0}(\lambda) \leq \sum_{x \in \mathcal{X}} (1 - p_x^n (1 - (1 - p_x)^n) \frac{(\lambda g(p_x))^2}{2}
\]

\[
\leq \frac{\lambda^2 u_2^*(n,g)}{2}, \quad \lambda \leq 0,
\quad (62)
\]

where \(u_2^*(n,g)\) is as defined in (34). This completes the proof of Part 1 of Theorem 2.
B. Proof of Part 2 of Theorem 2

For right tail bounds, we consider $\lambda > 0$. The following lemma provides a power series upper bound on $L_{G_0}(\lambda)$ and is critical in the proof of Part 2 of Theorem 2.

**Lemma 9** (Power series upper bound).

$$L_{G_0}(\lambda) \leq \sum_{r=2}^{\infty} \frac{\lambda^r}{r!} u^*_r(n, g), \quad \lambda > 0,$$

where $u^*_r(n, g)$ is as defined in (34).

**Proof.** Using $e^t - t - 1 = \sum_{r=2}^{\infty} \frac{t^r}{r!}$, $t \geq 0$, with $t = \lambda g(p_x)$ in (29), we get

$$L_{G_0}(\lambda) \leq \sum_{x \in \mathcal{X}} (1 - p_x)^n (1 - (1 - p_x)^n) \left( \sum_{r=2}^{\infty} \frac{(\lambda g(p_x))^r}{r!} \right)$$

$$= \sum_{r=2}^{\infty} \frac{\lambda^r}{r!} \left( \sum_{x \in \mathcal{X}} g(p_x)^r (1 - p_x)^n (1 - (1 - p_x)^n) \right)$$

$$= \sum_{r=2}^{\infty} \frac{\lambda^r}{r!} \left( \sum_{x \in \mathcal{X}} p_x g(p_x)^r (1 - p_x)^n \frac{(1 - (1 - p_x)^n)}{p_x} \right)$$

$$\leq \sum_{r=2}^{R} \frac{\lambda^r}{r!} u^*_r(n, g), \quad \lambda \geq 0,$$

where (a) follows by using (48) for $\lambda \geq 0$.

Let $R \geq 1$. For $r \geq R + 2$, repeatedly applying $\frac{u^*_r(n, g)}{(r-1)!} \leq c \frac{u^*_{r-1}(n, g)}{(r-2)!}$, $r \geq 3$, we have

$$\frac{u^*_r(n, g)}{(r-1)!} \leq c^{r-(R+1)} \frac{u^*_{R+1}(n, g)}{R!}, \quad r \geq R + 1.$$

(64)

Omitting the arguments $n, g$ from $u^*_r(n, g)$ for brevity and using the above inequality in (63), we get

$$L_{G_0}(\lambda) \leq \sum_{r=2}^{R} \frac{\lambda^r}{r!} u^*_r + \sum_{r=R+1}^{\infty} \frac{\lambda^r}{r} c^{r-(R+1)} \frac{u^*_{R+1}}{(R)!}$$

$$= \sum_{r=2}^{R} \frac{\lambda^r}{r!} u^*_r + \frac{u^*_{R+1}}{c^{R+1}R!} \sum_{r=R+1}^{\infty} \frac{(c\lambda)^r}{r},$$

$$= \left( \sum_{r=2}^{R} \frac{\lambda^r}{r!} \left( u^*_r - c^{r-2} v (r-1)! \right) \right) - \frac{v}{c^2} (c\lambda) + \frac{v}{c^2} \left( \sum_{r=1}^{\infty} \frac{(c\lambda)^r}{r} \right),$$

(65)

where (a) follows by setting $v \triangleq \frac{u^*_{R+1}}{c^{R+1}R!}$ and adding/subtracting terms between the two summations. Using the Taylor expansion $-\log(1 - x) = \sum_{r=1}^{\infty} \frac{x^r}{r}$, $x < 1$, completes the proof.
VI. PROOF OF LEMMA 4

In this section, we present a proof for Lemma 4. Let

\[ u_r(p, n, g) \triangleq g(p)^r (1 - p)^n (1 - (1 - p)^n)/p \]

be the function that is maximized over \( p \in (0, 1) \) in (34), and let \( p_r^* = \arg \max_{p \in (0, 1)} u_r(p, n, g) \). So, we have \( u_r^*(n, g) = u_r(p_r^*, n, g) \). In the next lemma, we upper bound the rate of fall of \( u_r^*(n, g)/(r - 1)! \) using \( p_r^* \).

**Lemma 10.**

\[
\frac{u_r^*(n, g)}{(r - 1)!} \leq \frac{g(p_r^*)}{r - 1} \frac{u_{r-1}^*(n, g)}{(r - 2)!}, \quad r \geq 3. \tag{66}
\]

**Proof.** By the definition of \( u_r(p, n, g), u_r^*(n, g) \) and \( p_r^* \) we get

\[
\frac{u_r^*(n, g)}{(r - 1)!} = \frac{u_r(p_r^*, n, g)}{(r - 1)!} \overset{(a)}{=} \frac{g(p_r^*)}{r - 1} \frac{u_{r-1}(p_r^*, n, g)}{(r - 2)!} \overset{(b)}{\leq} \frac{g(p_r^*)}{r - 1} \frac{u_{r-1}^*(n, g)}{(r - 2)!},
\]

where (a) follows from \( u_r(p, n, g) = g(p)u_{r-1}(p, n, g) \) and (b) follows from \( u_{r-1}(p_r^*, n, g) \leq u_{r-1}^*(n, g) \).

In the rest of the proof, we find an upper bound on \( c(r) \triangleq \frac{g(p_r^*)}{r - 1} \) that is independent of \( r \) and decreasing with \( n \) for functions \( g(p) \) that are differentiable for \( p \in (0, 1) \) and fall under either Type A or Type B as described in Lemma 4. Recall that \( g'(p) \) denotes the derivative of \( g(p) \).

**Type A:** For this type, we have that for some \( \mu > 0 \),

\[ 0 < g'(p) \leq \mu g(p)/p, \quad p \in (0, 1). \]

Taking partial derivative of \( u_r(p, n, g) \) with respect to \( p \), letting \( q \triangleq 1 - p \) and simplifying, we get

\[
\frac{\partial u_r(p, n, g)}{\partial p} = \frac{q^{n-1}}{p^2} g(p)^r (1 - q^n) \left[ ((r - 1)q - np)g(p) + r pq \left( g'(p) - \frac{g(p)}{p} \right) \right]. \tag{67}
\]

Using \( g'(p) \leq \mu g(p)/p \) in the above equation, we get

\[
\frac{\partial u_r(p, n, g)}{\partial p} \leq \frac{q^{n-1}}{p^2} g(p)^r (npq^n + (1 - q^n)(r \mu - 1 - (n + r \mu - 1)p)) \overset{(a)}{<} 0, \quad \text{if } r \mu - 1 - (n + r \mu - 1)p < -1, \tag{68}
\]

\[
< 0, \quad \text{if } r \mu - 1 - (n + r \mu - 1)p < -1.
\]
where (a) follows because \( npq^n - (1 - q^n) < 0 \). So, \( u_r(p, n, g) \) decreases for \( p > r\mu/(n + r\mu - 1) \) implying that

\[
p^*_r \leq \frac{r\mu}{n + r\mu - 1}.
\]

Since \( g(p) \) is an increasing function \((g'(p) > 0)\),

\[
c(r) = \frac{g(p^*_r)}{r - 1} \leq \frac{1}{r - 1} g\left(\frac{r\mu}{n + r\mu - 1}\right), \quad r \geq 3. \tag{69}
\]

In the next lemma, we present upper bounds on \( \frac{1}{r - 1} g\left(\frac{r\mu}{n + r\mu - 1}\right) \) that are independent of \( r \) and are in the form of \( (41) \) for different ranges of \( \mu \) and \( n \).

**Lemma 11.**

1) If either \( 0 < \mu \leq 1 \), or \( \mu > 1 \) and \( n < 1 + \frac{4\mu^2}{(\mu - 1)^2} \),

\[
c(r) \leq 0.5 g\left(\frac{3\mu}{n + 3\mu - 1}\right).
\]

To get the form of \( (41) \), set \( b_0 = b_3 = 0.5 \), \( b_1 = b_4 = 3\mu \) and \( b_2 = b_5 = 3\mu - 1 \).

2) For \( \mu > 1 \) and \( n \geq 1 + \frac{4\mu^2}{(\mu - 1)^2} \),

\[
c(r) \leq \max\left(0.5 g\left(\frac{3\mu}{n + 3\mu - 1}\right), \frac{1}{r_2 - 1} g\left(\frac{r_2\mu}{n + r_2\mu - 1}\right)\right),
\]

where \( r_2 = 0.5(n - 1)(1 - \frac{1}{\mu}) \left(1 + \sqrt{1 - \frac{4\mu^2}{(n-1)(\mu-1)^2}}\right) \). The above bound is directly in the form of \( (41) \).

**Proof.** A proof is given in Appendix B. \( \square \)

**Type B:** For this type, we have that, for some \( p^* \in (0, 1) \),

\[
0 < g'(p) \leq \left(\frac{1/p - 1/p^*}{1 - p}\right) \frac{g(p)}{1 - p}, \quad p \in (0, p^*),
\]

\[
g'(p) < 0, \quad p \in (p^*, 1).
\]

We rewrite the partial derivative of \( u_r(p, n, g) \) in \( (67) \) as

\[
\frac{\partial u_r(p, n, g)}{\partial p} = \frac{q^{n-1}}{p^2} g(p)^{r-1} \left(np g(p)(2q^n - 1) + (1 - q^n)q \left[(r - 1)g(p) + rp \left(g'(p) - \frac{g(p)}{p}\right)\right]\right) \tag{70}
\]

We will consider the two ranges \( p \in (0, p^*] \) and \( p \in (p^*, 1) \) separately. For \( p \in (p^*, 1) \), using \( g'(p) < 0 \) in \( (70) \) and simplifying, we get

\[
\frac{\partial u_r(p, n, g)}{\partial p} \leq \frac{q^{n-1}}{p^2} g(p)\left(np(2q^n - 1) - (1 - q^n)q\right) < 0, \tag{71}
\]
where the last inequality can be readily verified. So, \( u_r(p, n, g) \) decreases for \( p > p^* \) implying that \( p_r^* \leq p^* \).

For \( p \in (0, p^*) \), using \( g'(p) \leq \frac{g(p)}{1-p} \left( \frac{1}{p} - \frac{1}{p^*} \right) \) in (70) and simplifying, we get

\[
\frac{\partial u_r(p, n, g)}{\partial p} \leq \frac{q^n-1}{p^2} g(p)^r \left( [r - 1 - p(n + (r/p^*) - 1)] (1 - q^n) + npq^n \right)
\]

\( \leq 0, \) if \( r - 1 - p(n + (r/p^*) - 1) < -1 \),

(72)

where (a) follows from the fact that \( npq^n - (1 - q^n) < 0 \) for \( p \in (0, 1) \). So, \( u_r(p, n, g) \) decreases for \( r/(n + (r/p^*) - 1) < p < p^* \) implying that

\[
p_r^* \leq \frac{r}{n + (r/p^*) - 1}.
\]

Since \( g(p) \) is an increasing function for \( p \in (0, p^*) \), we get

\[
c(r) = \frac{g(p_r^*)}{r - 1} \leq \frac{1}{r - 1} g \left( \frac{r}{n + (r/p^*) - 1} \right), r \geq 3.
\]

(73)

In the next lemma, we present upper bounds on \( \frac{1}{r - 1} g \left( \frac{r}{n + (r/p^*) - 1} \right) \) that are independent of \( r \) and are in the form of (41).

**Lemma 12.**

1) If either \( g(p)/p \) is non-increasing, or \( n < 1 + \frac{4}{(1-p^*)^2} \),

\[
c(r) \leq 0.5 \ g \left( \frac{3}{n + (3/p^*) - 1} \right).
\]

To get the form of (41), set \( b_0 = b_3 = 0.5, b_1 = b_4 = 3 \) and \( b_2 = b_5 = (3/p^*) - 1 \).

2) If \( n \geq 1 + \frac{4}{(1-p^*)^2} \),

\[
c(r) \leq \max \left( 0.5 \ g \left( \frac{3}{n + (3/p^*) - 1} \right), \frac{1}{r_4 - 1} g \left( \frac{r_4}{n + (r_4/p^*) - 1} \right) \right),
\]

where \( r_4 = 0.5 \ (n - 1) \ (1 - p^*) \left( 1 + \sqrt{1 - \frac{4}{(n-1)(1-p^*)^2}} \right) \). The above bound is directly in the form of (41).

**Proof.** A proof is given in Appendix C.

VII. PROOF OF COROLLARIES

We begin with deriving upper bounds for \( u_r^*(n, g(p)) \) for \( g(p) = p^\alpha \) and \( g(p) = p \log(1/p) \). These are used in the simplifications of the tail bound expression.
A. Upper bounds on $u^*_2(n, g(p))$

Recall that for $g(p) = p$, $u_2(n, p) = p(1-p)^n(1-(1-p)^n)$, $u^*_2(n, p) = \max_{p \in (0,1)} u_2(n, p)$ and

$$\gamma = \max_{np} np e^{-np}(1-e^{-np}), \ p \in (0,1).$$

Since $\arg\max_p p(1-p)^n = 1/(n + 1)$ and $1 - (1-p)^n$ is increasing in $p$, we have that $\arg\max_{0<p<1} u_2(n, p) > \frac{1}{n+1}$.

If $p \geq \ln 2/n$ or $e^{-np} \leq 1/2$, since $0 \leq 1 - p \leq e^{-p}$, we have

$$e^{-np} + (1-p)^n \leq 1 \text{ if } p \geq \ln 2/n.$$ 

Multiplying by the positive quantity $np(e^{-np} - (1-p)^n)$ and rearranging, we get

$$np(1-p)^n(1-(1-p)^n) \leq np e^{-np}(1-e^{-np}) \text{ if } p \geq \frac{\ln 2}{n}. \quad (74)$$

Since $\arg\max_{0<p<1} u_2(n, p) > \frac{1}{n+1} > \frac{\ln 2}{n}$ for $n \geq 3$, we have $u^*_2(n, p) \leq \gamma/n$, for $n \geq 3$.

The generalization for $\alpha \geq 1$ is very similar to the proof above. So, we omit the details and present only the final result. For $\alpha \geq 1$,

$$u^*_2(n, p^\alpha) = \max_{p \in (0,1)} p^{2\alpha-1}(1-p)^n(1-(1-p)^n).$$

For $n \geq (2\alpha - 1) \ln 2/(2\alpha - 1 - \ln 2)$, the following bound holds:

$$u^*_2(n, p^\alpha) \leq \gamma_\alpha/n^{2\alpha-1},$$

where $\gamma_\alpha = \max_{t>0} t^{2\alpha-1} e^{-t(1-e^{-t})}$.

The upper bound on $u^*_2(n, p)$ can be extended to an upper bound on $u^*_2(n, g(p))$ in cases where $g(p)/p$ is bounded by using the following:

$$u^*_2(n, g(p)) = \max_{0 \leq p \leq 1} (g(p)^2/p)(1-p)^n(1-(1-p)^n)$$

$$\leq \max_{0 \leq p \leq 1} (g(p)/p)^2 \max_{0 \leq p \leq 1} p(1-p)^n(1-(1-p)^n)$$

$$\leq \frac{\gamma}{n} \max_{0 \leq p \leq 1} (g(p)/p)^2. \quad (n > 3) \quad (75)$$

For example, consider the missing Shannon entropy with $g(p) = p \log_2(1/p)$ for distributions $P$ satisfying $p_x \geq 1/k$ for all $x \in X$. In this case, we get

$$u^*_2(n, p \log_2(1/p)) \leq \frac{\gamma}{n} \max_{0 \leq p \leq 1} (\log_2(1/p))^2 = (\log_2 k)^2 \gamma/n, \quad (n \geq 3).$$
B. **Proof of Corollary 3**

Using Part 1 of Theorem 2 with \( u_2^*(n, p^\alpha) \leq \gamma_{\alpha}/n^{2\alpha-1} \) and \( u_2^*(n, p\log_2(1/p))) \leq (\log_2 k)^2\gamma/n \) (as shown above) in the Chernoff method results in the left tail bounds presented in Corollary 3 for \( M_{0,\alpha}, \alpha \geq 1 \) and \( H_0 \), respectively.

C. **Proof of Corollary 5**

The proof for each part of Corollary 5 is given below. For every \( g(p) \), we identify the scale parameter \( c \) either from Lemma 11 if \( g(p) \) is of Type A, or from Lemma 12 if \( g(p) \) is of Type B. We bound the scale parameter, if necessary. Then, the variance parameter is found by bounding \( u_2^*(n, g) \) using one of the methods described in Section VII-A. Finally, \( c \) and \( v \) are used in the strongly sub-Gamma right tail bound (32).

1) For \( g(p) = p \), we get \( c = 3/(2(n + 2)) \) from Lemma 11 Part 1 Using \( c = 3/(2(n + 2)) \) and \( v = u_2^*(n, p) \leq \gamma/n \) (as shown above) in (32) gives (42).

2) For \( g(p) = p^\alpha \), \( \alpha > 1 \) and \( n \) sufficiently large, using Lemma 11 Part 2 with \( \mu = \alpha \), it can be shown that

\[
c = \frac{1}{r_2 - 1}(\alpha r_2/(n - 1 + \alpha r_2))^\alpha \leq \frac{2\alpha(2(\alpha - 1)/(\alpha + 1))^\alpha}{(n - 1)(\alpha - 1) - 2\alpha},
\]

(76)

where \( r_2 \) is as defined in Lemma 11 and the inequality is obtained by using \( 0 \leq \sqrt{1 - 4\alpha^2/((n - 1)(\alpha - 1)^2) \leq 1 \) appropriately. Using the \( c \) above in (32) along with \( v = u_2^*(n, p^\alpha) \leq \gamma_{\alpha}/n^{2\alpha-1} \) (as shown above) gives (43).

3) For \( g(p) = p\log_2(1/p) \), from Lemma 12 Part 1 we get \( c = 0.5 \ (\log_2 n_0)/n_0 \), where \( n_0 = (1/3)(n - 1) + e \). Using this value of \( c \) in (32) along with \( v = u_2^*(n, p\log_2(1/p))) \leq (\log_2 k)^2\gamma/n \) (as shown above) gives (44).

VIII. **Conclusion and Future Directions**

We have generalized the notion of missing mass to missing mass of functions. Estimation and concentration of missing mass of functions was studied and several new results were shown. In particular, by generalizing Good-Turing estimators, we showed that estimation better than that of a plugin estimator is possible for several interesting classes of functions. However, there are important classes of functions where linear estimators perform only as good as the plugin estimator. Non-linear estimators are an interesting possibility to explore in future work.
As far as concentration results are concerned, we introduced two new notions of concentration, named strongly sub-Gamma and filtered sub-Gaussian, that result in tail bounds better than that of sub-Gaussian concentration for missing mass. In other situations where sub-Gaussian concentration has a sub-optimal variance factor, these new notions, particularly the idea of filtering a Gaussian distribution, is worth exploring.

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A. Proof of Lemma 8

Recall that \( P = (p_1, p_2, \ldots, p_k) \) on \( \Delta_k \) has a Dirichlet distribution with parameters \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \). Let \( \beta_0 \triangleq \sum_{i=1}^{k} \beta_i \). Let \( \Gamma(u) \triangleq \int_{0}^{\infty} x^{u-1} e^{-x} dx \) denote the Gamma function, and let \( \tau(u, v) \triangleq \Gamma(u + v)/\Gamma(u) \). Recall the notation \( F_x(X^n) = \sum_{i=1}^{n} I(X_i = x) \) denoting the number of occurrences of \( x \) in \( X^n \). The following properties of the Dirichlet distribution are useful:

\[
E[p_i^a] = \tau(\beta_i, a)/\tau(\beta_0, a), \quad a > 0 \tag{77}
\]
\[
E[p_i^a p_j^b] = \tau(\beta_i, a) \tau(\beta_j, b)/\tau(\beta_0, a + b), \quad a, b > 0
\tag{78}
\]
\[
P|X^n \sim \text{Dirichlet}(\beta_1 + F_1(X^n), \beta_2 + F_2(X^n), \ldots, \beta_k + F_k(X^n)). \tag{79}
\]

Since \( M_{0, \alpha} = \sum_{x \in X} p_x^\alpha I(F_x(X^n) = 0) \) and \( F_x \) is a function of only \( X^n \), we get

\[
E[M_{0, \alpha}(X^n, P)|X^n] = \sum_{x \in X} E[p_x^\alpha|X^n] I(F_x = 0).
\]

The conditional variance of \( M_{0, \alpha}(X^n, P) \) simplifies as follows:

\[
\text{Var}[M_{0, \alpha}(X^n, P)|X^n] = E \left[ \left( \sum_{x \in X} I(F_x = 0)(p_x^\alpha - E[p_x^\alpha|X^n]) \right)^2 | X^n \right]
\]

\[
= \sum_{x \in X} I(F_x = F_y = 0) (E[p_x^\alpha p_y^\alpha|X^n] - E[p_x^\alpha|X^n]E[p_y^\alpha|X^n])
\]

\[
+ \sum_{x \in X} I(F_x = 0) (E[p_x^{2\alpha}|X^n] - E[p_x^\alpha|X^n]^2)
\]

\[
(a) \quad \sum_{x, y \in X; x \neq y} I(F_x = F_y = 0) \frac{\tau(\beta_x + F_x, \alpha) \tau(\beta_y + F_y, \alpha)}{\tau(\beta_0 + n, 2\alpha)} \left( \frac{1}{\tau(\beta_0 + n, 2\alpha)} - \frac{1}{\tau^2(\beta_0 + n, \alpha)} \right)
\]

\[
+ \sum_{x \in X} I(F_x = 0) \left( \frac{\tau(\beta_x + F_x, 2\alpha)}{\tau(\beta_0 + n, 2\alpha)} - \frac{\tau^2(\beta_x + F_x, \alpha)}{\tau^2(\beta_0 + n, \alpha)} \right)
\]

\[
(b) \quad \sum_{x, y \in X; x \neq y} I(F_x = F_y = 0) \frac{\tau(\beta_x, \alpha) \tau(\beta_y, \alpha)}{\tau(\beta_0 + n, 2\alpha)} \left( \frac{1}{\tau(\beta_0 + n, 2\alpha)} - \frac{1}{\tau^2(\beta_0 + n, \alpha)} \right)
\]

\[
+ \sum_{x \in X} I(F_x = 0) \left( \frac{\tau(\beta_x, 2\alpha)}{\tau(\beta_0 + n, 2\alpha)} - \frac{\tau^2(\beta_x, \alpha)}{\tau^2(\beta_0 + n, \alpha)} \right), \tag{80}
\]
where (a) follows by using (79) and (77), (78), and (b) follows because of the presence of the indicators $I(F_x = F_y = 0)$ and $I(F_x = 0)$. Taking expectation over $X^n$ on both sides of (80),

$$E_{X^n} \left[ \text{Var} \left[ M_{0,\alpha}(X^n, P) | X^n \right] \right]$$

$$= \sum_{x,y \in X : x \neq y} \Pr(F_x = F_y = 0) \tau(\beta_x, \alpha) \tau(\beta_y, \alpha) \left( \frac{1}{\tau(\beta_0 + n, 2\alpha)} - \frac{1}{\tau^2(\beta_0 + n, \alpha)} \right)$$

$$+ \sum_{x \in X} \Pr(F_x = 0) \left( \frac{\tau(\beta_x, 2\alpha)}{\tau(\beta_0 + n, 2\alpha)} - \frac{\tau^2(\beta_x, \alpha)}{\tau^2(\beta_0 + n, \alpha)} \right)$$

$$= \sum_{x \in X} \frac{\tau(\beta_0 - \beta_x, n)}{\tau(\beta_0, n)} \left( \frac{\tau(\beta_x, 2\alpha)}{\tau(\beta_0 + n, 2\alpha)} - \frac{\tau^2(\beta_x, \alpha)}{\tau^2(\beta_0 + n, \alpha)} \right)$$

$$+ \sum_{x,y \in X : x \neq y} \frac{\tau(\beta_0 - \beta_x - \beta_y, n)}{\tau(\beta_0, n)} \tau(\beta_x, \alpha) \tau(\beta_y, \alpha) \left( \frac{1}{\tau(\beta_0 + n, 2\alpha)} - \frac{1}{\tau^2(\beta_0 + n, \alpha)} \right),$$

where (a) follows because

$$\Pr(F_x(X^n) = 0) = E[(1 - p_x)^n] = \tau(\beta_0 - \beta_x, n) / \tau(\beta_0, n),$$

$$\Pr(F_x(X^n) = F_y(X^n) = 0) = E[(1 - p_x - p_y)^n] = \tau(\beta_0 - \beta_x - \beta_y, n) / \tau(\beta_0, n).$$

Let $D_{n,c}$ denote the Dirichlet distribution with $\beta_i = 1/n$, $i = 1, 2, \ldots, k$ and $k = cn^2$ with $c > 0$. Let

$$T(D_{n,c}) \triangleq E_{X^n} \left[ \text{Var} \left[ M_{0,\alpha}(X^n, D_{n,c}) | X^n \right] \right].$$

Setting $P = D_{n,c}$ in (81), we get

$$T(D_{n,c}) = cn^2 \tau(cn) \frac{\tau(cn - 1/n, n)}{\tau(cn, n)} \left( \frac{\tau(1/n, 2\alpha)}{\tau((c + 1)n, 2\alpha)} - \frac{\tau^2(1/n, \alpha)}{\tau^2((c + 1)n, \alpha)} \right)$$

$$+ cn^2(2n - 1) \frac{\tau(cn - 2/n, n)}{\tau(cn, n)} \tau^2(1/n, \alpha) \left( \frac{1}{\tau((c + 1)n, 2\alpha)} - \frac{1}{\tau^2((c + 1)n, \alpha)} \right),$$

$$\triangleq cn^2 A_1(A_2 - A_3) + cn^2(2n - 1)A_4A_5(A_6 - A_7),$$

where $A_i$, $1 \leq i \leq 7$, denote the corresponding terms in the previous equation. We consider two different cases.

**Case I:** $\alpha$ is an integer

In this case, we use

$$\tau(u, v) = \Gamma(u + v) / \Gamma(u) = \prod_{l=0}^{v-1} (u + l), \quad v \in \mathbb{N},$$
to simplify $A_i$ and find their dominant terms as follows.

$$A_1 = \prod_{l=0}^{n-1} \left(1 - \frac{1}{n(cn+l)}\right) = n^{-1}, \quad A_2 = \prod_{l=0}^{2\alpha-1} \frac{1/n + l}{(c+1)n + l} = n \frac{(2\alpha-1)!}{(c+1)^{2\alpha}n^{2\alpha+1}},$$

$$A_3 = \prod_{l=0}^{\alpha-1} \left(\frac{(1/n + l)^2}{((c+1)n + l)^2}\right) = n O(1/n^{2\alpha+2}), \quad A_4 = \prod_{l=0}^{n-1} \left(1 - \frac{2}{n(cn+l)}\right) = n^{-1},$$

$$A_5 = \prod_{l=0}^{\alpha-1} \left(\frac{(1/n + l)^2}{((c+1)n + l)^2}\right) = n \frac{((\alpha-1)!)^2}{n^2},$$

$$A_6 = \prod_{l=0}^{2\alpha-1} \frac{1}{((c+1)n + l)^2} = n \frac{1}{((c+1)n)^{2\alpha} + (\alpha-1)(2\alpha-1)((c+1)n)^{2\alpha-1}},$$

$$A_7 = \prod_{l=0}^{\alpha-1} \frac{1}{((c+1)n + l)^2} = n \frac{1}{((c+1)n)^{2\alpha} + (\alpha-1)(\alpha-2)((c+1)n)^{2\alpha-1}}.$$ 

For $A_6$ and $A_7$, the first two terms are retained because they are being subtracted and the leading terms are identical. Using the above in (82) we get

$$T(D_{n,c}) = cn^2 \left(\frac{(2\alpha-1)!}{(c+1)^{2\alpha}n^{2\alpha+1}} - O(1/n^{2\alpha+2})\right) + cn^2(cn^2 - 1) \frac{((\alpha-1)!)^2}{n^2} \frac{1}{((c+1)n)^{2\alpha} + (\alpha-1)(2\alpha-1)((c+1)n)^{2\alpha-1}} \frac{1}{((c+1)n)^{2\alpha} + (\alpha-1)(\alpha-2)((c+1)n)^{2\alpha-1}}$$

$$\geq n \frac{c(2\alpha-1)!}{(c+1)^{2\alpha}n^{2\alpha+1}} - cn^2(cn^2 - 1) \frac{((\alpha-1)!)^2}{n^2} \frac{\alpha^2}{((c+1)n)^{2\alpha+1}}$$

$$= n \frac{c/(c+1)}{(c+1)n)^{2\alpha-1}} \left(2\alpha - 1\right) \left(1 - \frac{c(\alpha)\alpha^2}{c+1}\right). \quad (83)$$

**Case II**: $\alpha$ is not an integer.

Recall that $\alpha = \lfloor \alpha \rfloor + \{\alpha\}$, where $\lfloor \alpha \rfloor$ is the largest integer lesser than $\alpha$ and $\{\alpha\}$ is the fractional part of $\alpha$. Also, $\lceil \alpha \rceil$ is the smallest integer greater than $\alpha$. Gautshi’s inequality presented in the following lemma is a critical result for the non-integer case.

**Lemma 13** (Gautshi’s Inequality [27]). For $z \in (0, \infty)$ and $s \in (0, 1)$,

$$z^{1-s} \leq \frac{\Gamma(z+1)}{\Gamma(z+s)} \leq (z+1)^{1-s}. \quad (84)$$
We first consider the case when \( \{\alpha\} < 0.5 \). As before, we simplify \( A_i \) to bound or obtain dominant terms as shown below.

\[
A_1 = n, A_5 \leq n \frac{1/n^2}{[\alpha]^{2(1-\alpha)}} ([\alpha]!^2, \quad A_6 - A_7 \geq_n \frac{[\alpha](1 - 2\{\alpha\} - [\alpha]) + 2(\{\alpha\} - 1)}{(c+1)n^{2\alpha+1}}
\]

\[
A_4 = n, A_2 \geq_n \frac{([2\alpha] - 1)!/[2\alpha]^{1-2\alpha}}{(c+1)^{2\alpha} n^{2\alpha+1}}, \quad A_3 \leq_n \frac{([\alpha] - 1)!/[2\alpha]^{1-2\alpha}}{(c+1)^{2\alpha} n^{2\alpha+2}}, \quad (85)
\]

In the above, \( A_3 \) is upper bounded because it appears with a negative sign. Also, \( A_5 \) is upper bounded because the dominant term of \( A_6 - A_7 \) is negative.

We will show details of the simplifications for lower bounding \( A_2 \). Using the definition \( \tau(u, v) = \frac{\Gamma(u+v)}{\Gamma(u)} \), we get

\[
A_2 = \frac{\Gamma(2\alpha + 1/n)}{\Gamma(1/n)} \frac{\Gamma((c+1)n)}{\Gamma((c+1)n + 2\alpha)}
\]

\[
= \frac{\Gamma(2\alpha + 1/n)}{\Gamma([2\alpha] + 1/n)} \frac{\Gamma((c+1)n + [2\alpha])}{\Gamma((c+1)n + 2\alpha)} \frac{\Gamma((c+1)n)}{\Gamma((c+1)n + [2\alpha])}
\]

The first and third term above are lower-bounded using Gautschi’s inequality because the difference of the Gamma function arguments is \([2\alpha] - 2\alpha \in (0, 1)\). The second and fourth term are expanded out as a product because the difference is the integer \([2\alpha] \). So, we get

\[
A_2 \geq_n \left( \frac{(c+1)n + [2\alpha]}{2\alpha + 1/n} \right)^{1-2\alpha} \left( \frac{(c+1)n}{c+1} \right)^{1-2\alpha} \left( \frac{([2\alpha] - 1)!}{(c+1)[2\alpha] n^{2\alpha+1}} \right)
\]

The bounding of the other \( A_i \) is similar, and we skip the details.

So, for \( \{\alpha\} < 0.5 \), combining the dominant term bounds for \( A_i \), \( 1 \leq i \leq 7 \), we get the following lower bound on \( T(D_{n,c}) \).

\[
T(D_{n,c}) \geq_n \frac{c/(c+1)}{(c+1)n^{2\alpha-1}} \left( \frac{([2\alpha] - 1)!}{[2\alpha]^{1-2\alpha}} - \frac{c}{c+1} \frac{([\alpha]!)^2}{[\alpha]^2} \right)
\]

\[
(86)
\]

For the case \( \{\alpha\} > 0.5 \), noting that \( 2\alpha = 2\{\alpha\} - 1 \) and following a procedure similar to the case \( \{\alpha\} < 0.5 \), we get

\[
T(D_{n,c}) \geq_n \frac{c/(c+1)}{(c+1)n^{2\alpha-1}} \left( \frac{([2\alpha] - 1)!}{[2\alpha]^{1-2\alpha}} - \frac{c}{c+1} \frac{([\alpha]!)^2}{[\alpha]^2} \right)
\]

\[
(87)
\]
Finally, for the case \( \alpha = 0.5 \), repeating a similar procedure, we get

\[
T(D_{n,c}) \geq n \cdot \frac{c/(c + 1)}{((c + 1)n)^{2\alpha - 1}} \left( (2\alpha - 1)! - \frac{c}{c + 1} \frac{([\alpha][\alpha] + 1)([\alpha])^2}{[\alpha]} \right). \tag{88}
\]

Combining the above, the proof is complete.

B. Proof of Lemma 11

Let \( p(r) \triangleq r\mu/(n + r\mu - 1) \) and \( h(p) \triangleq g(p)/p \). The general idea of the proof is to start with (69), which states

\[
c(r) \leq c_{\max}(r) \triangleq g(p(r))/(r - 1),
\]

and try to maximize \( c_{\max}(r) \) over \( r \geq 3 \).

1) Case 1(a) \((0 < \mu < 1)\): Since \( g'(p) = ph'(p) + h(p) \leq \mu h(p) \) or \( h'(p) \leq (\mu - 1)h(p)/p < 0 \), we see that \( h(p) \) decreases with \( p \). Rewriting \( c_{\max}(r) \) in terms of \( h(p(r)) \), we have

\[
c(r) \leq \frac{p(r)}{r - 1} h(p(r)) \leq \frac{p(3)}{2} h(p(r)) \leq \frac{p(3)}{2} h(p(3)) = 0.5g(p(3)),
\]

where \((a)\) follows because \( p(r)/(r - 1) \) decreases with \( r \) and \( r \geq 3 \), and \((b)\) follows because \( h(p(r)) \) decreases with \( r \) \((p(r) \) increases with \( r \), \( h(p) \) decreases with \( p \)) and \( r \geq 3 \).

This completes the proof for Case 1(a).

2) Case 1(b) \((\mu > 1 \text{ and } n < 1 + 4(\mu/(\mu - 1))^2)\): The derivative of \( c_{\max}(r) \) can be simplified as follows.

\[
c_{\max}'(r) = \frac{1}{r - 1} \left( \frac{(n - 1)\mu}{(n + r\mu - 1)^2} g'(p(r)) - \frac{1}{(r - 1)} g(p(r)) \right) \tag{89}
\]

\[
\leq g(p(r)) \frac{1}{r - 1} \left( \frac{(n - 1)\mu}{r(n + r\mu - 1)} - \frac{1}{r - 1} \right) \\
= -\mu g(p(r)) \frac{r^2 - (n - 1)(1 - 1/\mu)r + (n - 1)}{r(r - 1)^2(n + r\mu - 1)} \\
= -\mu g(p(r)) \frac{[r - (n - 1)(1 - 1/\mu)/2]^2 + (n - 1)[1 - (n - 1)(1 - 1/\mu)^2]/4}{r(r - 1)^2(n + r\mu - 1)},
\]

where \((a)\) uses \( g'(p(r)) \leq \mu g(p(r))/p(r) \).

If \( 1 - (n - 1)(1 - 1/\mu)^2/4 > 0 \) or \( n < 1 + 4\mu^2/(\mu - 1)^2 \), we see that \( c_{\max}' < 0 \) and \( c_{\max}(r) \) decreases with \( r \). So, for \( r \geq 3 \),

\[
c(r) \leq c_{\max}(r) \leq c_{\max}(3) = 0.5g(p(3)).
\]

This completes the proof for Case 1(b).
3) Case 2 (\(\mu > 1\) and \(n \geq 1 + 4(\mu / (\mu - 1))^2\)): From (89), \(c'_{\text{max}}(r)\) is negative for \(r > r_2\), where

\[ r_2 \triangleq 0.5(n - 1)(1 - 1/\mu)(1 + \sqrt{1 - 4\mu^2/((n - 1)(\mu - 1)^2)}) \]

is the largest of the roots of the quadratic polynomial in the numerator of (89). So, we have

\[ c(r) \leq c_{\text{max}}(r) \leq \max\left(c_{\text{max}}(3), c_{\text{max}}(r_2)\right) = \max\left(g(p(3))/2, g(p(r_2))/(r_2 - 1)\right). \]

This completes the proof for Case 2.

C. Proof of Lemma [12]

The proof mirrors the above proof of Lemma [11] with some minor changes to the functional forms, and we skip the details.