Center vortices at $N > 4$ colors

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We discuss two issues related to the physics of center vortices in pure SU($N$) lattice gauge theory at large $N$: 1. Center vortices are stable classical solutions of the Wilson action, as well as of a wide class of improved lattice actions, for any $N > 4$. 2. The natural scaling of $k$-string tensions at large $N$, in the vortex picture of confinement, is $\sigma(k) = k\sigma(1)$. This is the common large $N$ limit of Casimir and Sine Law scaling. The crucial feature for explaining this behavior is the existence of center monopoles.

1. CENTER VORTICES AS CLASSICALLY STABLE OBJECTS FOR $N > 4$ COLORS

Lattice simulations provide abundant evidence for the important role of center vortices in the mechanism of color confinement [1]. However, it is generally believed that unlike instantons, which are known to be stable local minima of the pure SU($N$) gauge theory action, center vortices exist as stable classical solutions only in the case of gauge fields coupled to a set of adjoint Higgs fields, when the symmetry is broken from SU($N$) to Z$_N$.

Surprisingly, there is no need to break the SU($N$) symmetry to obtain stable vortex solutions on the lattice. The following remarkable result, by now almost forgotten, was obtained by Bachas and Dashen in 1982 [2]: Thin center vortices are stable local minima of the Wilson lattice action for any SU($N$) pure gauge theory with $N > 4$. We will briefly recall the proof of this statement, report on its extension to a wide class of two-parameter improved actions, and comment on its importance in the context of the renormalization group.

1.1. Wilson action

The proof for the Wilson action is rather trivial: Any thin center vortex in SU($N$) lattice gauge theory is gauge equivalent to

$$U_\mu(x) = Z_\mu(x) I_N$$

with

$$Z_\mu(x) = \exp\left(\frac{2\pi i n_\mu(x)}{N}\right), \quad n_\mu(x) = 1, 2, \ldots, N - 1.$$  

Now write a small deformation of this configuration as

$$U_\mu(x) = Z_\mu(x) V_\mu(x), \quad V_\mu(x) = e^{i A_\mu(x)}$$

with

$$A_\mu(x) = \sum_a A^a_\mu(x) L_a, \quad |A^a_\mu(x)| \ll 1.$$  

Substituting (2) into the Wilson action

$$S = \frac{\beta}{2N} \sum_P \left(2N - \text{Tr}[U_P] - \text{Tr}[U_P^\dagger]\right)$$

we obtain

$$S = \frac{\beta}{2N} \sum_P \left(2N - Z_P \text{Tr}[V_P] - Z^*_P \text{Tr}[V^\dagger_P]\right)$$

Writing the product of $V$-link variables around a plaquette in terms of a field strength

$$V_P = e^{i F_P} = I_N + i F_P - \frac{1}{2} F_P^2 + O(F_P^3),$$

and substituting for $Z_P$

$$Z_P = \exp\left(\frac{2\pi i n_P}{N}\right)$$
This vortex stability condition cannot be satisfied for

The action has a local minimum at \( \text{Tr}[F_p^2] = 0 \) providing that at each plaquette with \( n_p > 0 \)

\[
\cos \left( \frac{2 \pi n_p}{N} \right) > 0,
\]

i.e.

\[
\frac{n_p}{N} < \frac{1}{4} \quad \text{or} \quad \frac{N - n_p}{N} < \frac{1}{4}.
\]

This vortex stability condition cannot be satisfied for \( N \leq 4 \), however, beginning with \( n_p = 1, 4 \) at \( N = 5 \), vortex stability is obtained from the Wilson action already at the classical level.

### 1.2. Improved actions

The above result can be extended to more complicated lattice actions, and therein lies its physical relevance. Thin vortices, stable or not, are suppressed at weak couplings by a factor of order \( \exp(-\text{Vortex Area}/g^2) \) and do not percolate. The configurations of physical interest are center vortices having some finite thickness in physical units. To investigate the stable classical configurations with a certain length scale \( d \), starting from a lattice action at spacing \( a \) and including quantum fluctuations up to the scale \( d \), we can follow the RG approach and apply successively blocking transformations

\[
e^{-S'[\mathcal{U}]} = \int DU \delta[\mathcal{U} - F(U)] e^{-S[U]},
\]

where \( \mathcal{U} \) are links on the blocked lattice, and \( F(U) \) is the blocking function.

In this kind of approach, one usually assumes that only a few contours (plaquettes, 6- and 8-link loops, etc.) are important in the effective action. We have studied, in particular, the class of two-parameter improved lattice actions which consist of plaquette and \( 1 \times 2 \) rectangle terms:

\[
S_I = c_0 \sum_P (\text{N - ReTr}[U(P)])
+ c_1 \sum_{1 \times 2} (\text{N - ReTr}[U(R)]).
\]

This simple extension beyond the Wilson action includes many lattice actions discussed in the literature (tadpole-improved, Iwasaki, and DBW2 actions, two-parameter approximations to the Symanzik action), and has been applied to MCRG studies of the renormalization trajectory \([6]\).

The thin vortex configuration, eq. (1), is easily shown to be a stable minimum of the action \( S_I \) providing both \( c_0 \) and \( c_1 \) are positive; the proof is as simple as in the case of the Wilson action. But for most improved actions of this type one has \( c_0 > 0 \) and \( c_1 < 0 \), and the proof of stability is a little more involved. We will only report here our basic result, which will be derived in full in a subsequent publication \([4]\): Thin center vortex configurations are stable local minima of the two-parameter action if, first, the trivial vacuum \( U_0(x) = I_N \) is the global minimum of the action, which is satisfied iff

\[
c_0 + 8c_1 > 0.
\]

The second condition for vortex stability is the same as for the Wilson action, namely, that eq. (11) satisfied.

So it seems that the result first obtained by Bachas and Dashen \([2]\) is quite robust: Center vortices at \( N > 4 \) are stable minima of lattice actions in a large region of coupling constant space associated with improved actions. Assuming they remain as local minima all along the renormalization trajectory, their effects must become apparent at some scale. The reason is that the entropy factor increases with vortex surface area as \( \exp[\pm \text{const} \cdot (\text{Vortex Area})] \), while the Boltzmann suppression factor goes like \( \exp[-(\text{Vortex Area})/\kappa^2(d)] \). As \( d \) increases, so does \( \kappa^2(d) \). Eventually entropy wins over action, and vortices at that scale will percolate through the lattice.

At large distance scales, plaquette terms in the adjoint representation should also appear and become important in the lattice effective action; such terms can stabilize center vortices even in the \( N \leq 4 \) case. This mechanism has been demonstrated explicitly in the context of strong-coupling lattice gauge theory in ref. \([5]\).

### 2. \( k \)-STRING TENSIONS AT LARGE \( N \)

For \( SU(N) \) gauge theories with \( N > 3 \) there are a number of color representations in which color charge cannot be screened by gluons. These unscreenable representations correspond to the lowest dimensional
SU(N) representation with N-ality k. There are two predictions for the k dependence of string tensions between such color charges: Casimir scaling

$$\sigma(k) = \frac{k(N-k)}{N-1} \sigma(1), \quad (14)$$

based on dimensional-reduction arguments, and the Sine Law

$$\sigma(k) = \frac{\sin(\pi k/N)}{\sin(\pi/N)} \sigma(1), \quad (15)$$

motivated by MQCD.

In the large N limit, Casimir scaling becomes exact due to the factorization property. In this limit, the difference between the above formulas disappears. Both predict, for k-string tensions with $k \ll N$,

$$\sigma(k) = k \sigma(1) \quad (16)$$

which we refer to as “k-scaling”. Is this property a feature of the center vortex confinement mechanism?

The affirmative answer to this question is based on the following chain of arguments:

1. The distribution of center vortices in an SU(N) gauge theory is very likely controlled by an effective $Z_N$ gauge theory, namely, the P-vortex effective action. Calculation of center-projected observables in an SU(N) theory fixed to an adjoint (e.g. maximal center) gauge is equivalent to calculation of observables in the effective $Z_N$ gauge theory. These calculations have been carried out extensively in the $N=2$ case [3] (there are some results for SU(3) as well).

2. The effective $Z_N$ action is certainly non-local at the lattice scale, but should be local at the color-screening scale. Its excitations, at this scale, are thin center vortices and center monopoles (for $N \geq 3$). For large $N$, the $Z_N$ gauge group approximates a U(1) group, and center monopoles go over to the abelian monopoles of compact QED. A Wilson loop of N-ality k then becomes a Wilson loop of k units of abelian charge.

3. Finally, the charge dependence of string tensions in the U(1) theory can be calculated à la Polyakov [6], using the monopole Coulomb gas representation. This was done, in $D=3$ dimensions, in Ref. [7]: the result is precisely k-scaling.

Invoking center dominance, k-scaling in the effective $Z_N$ theory carries over to k-scaling of k-string tensions in the full SU(N) theory. This is how the k-scaling property at large N is obtained in the center vortex confinement picture.

We conclude with a comment about vortex densities at large $N$. If we assume that center flux is essentially uncorrelated among regions of area $\mathcal{A} > \mathcal{A}_{\text{min}}$ in a plane, then we can subdivide the plane into square regions of area $\mathcal{A}_{\text{min}}$ and ask for the number of these regions, per unit area, which are pierced by center flux of magnitude $2\pi l/N$. This number per unit area defines a vortex density $\rho(l,N)$. Given k-scaling, it is then straightforward to calculate $\rho(l,N)$, and we find that it falls like $1/N$ at large $N$. This is in contrast to a recent result in ref. [8], which argues that a vortex density consistent with k-scaling would have to increase linearly with $N$. We believe that the discrepancy can be traced to a dilute gas approximation used in ref. [8], which appears to be inconsistent with the existence of a lower bound $\mathcal{A}_{\text{min}}$ at large $N$ [4].

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