A NOTE ON THE ZAKHAROV-SHABAT TOPOLOGICAL MODEL

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ABSTRACT

In this note I discuss some features of the topological theory obtained from the Zakharov-Shabat (or general $sl(2, \mathbb{C})$) hierarchy, and comment on some possible physical and/or mathematical interpretations of it.

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In some recent papers (see for example refs. [1,2,3,4]) it has been shown that the double scaling limit of hermitian and anti-hermitian matrices in the one- and two-arc sector, leads to two dimensional quantum gravity models based on the following $sl(2,\mathbb{C})$ hierarchies: Korteweg-de Vries (KdV), modified KdV (mKdV), non-linear Schrödinger (NLS) and Zakharov-Shabat (ZS). Of particular interest are the two dimensional quantum gravity theories constructed from the ZS hierarchy. Indeed, the ZS hierarchy contains both the KdV and mKdV hierarchies as reductions, and its first (odd) critical point gives rise to a new “topological theory” [3], different from the one obtained from the KdV hierarchy [5,6,7]. (Notice that only the even critical points of the ZS hierarchy transform, under reduction, into those of the KdV hierarchy, whereas the odd critical points don’t.)

The purpose of this note is to discuss some features of the topological theory, which will be called “ZS topological model”, corresponding to the first critical point of the ZS hierarchy [3], and to make some comments on its possible physical and/or mathematical interpretations.

The Zakharov-Shabat hierarchy is simply defined by introducing two real, independent functions, $\psi$ and $\overline{\psi}$, of the parameters $(t_{-1}, x, t_1, \ldots)$ \footnote{I have adopted the conventions of ref. [4].} and by defining the flows as

$$
\frac{\partial \psi}{\partial t_k} = \frac{1}{2}(F_{k+1} - G_{k+1}) \quad \frac{\partial \overline{\psi}}{\partial t_k} = \frac{1}{2}(F_{k+1} + G_{k+1}) \quad (k \geq -1)
$$

(1)

where the polynomials $F_k$ and $G_k$ are given by

$$
F_{k+1} = G'_k + (\overline{\psi} - \psi)H_k \quad G_{k+1} = F'_k + (\overline{\psi} + \psi)H_k
$$

(2)

$$
H'_k = \overline{\psi}(G_k - F_k) - \psi(G_k + F_k)
$$

(3)

with

$$
F_0 = \overline{\psi} - \psi \quad G_0 = \psi + \overline{\psi} \quad H_0 = 0
$$

(4)

The connexion with the anti-hermitian 1-matrix models is given by the following formulæ:

$$
\langle PP \rangle = -\psi \overline{\psi}, \quad \langle O_i PP \rangle = \frac{\partial}{\partial t_i} \langle PP \rangle \quad O_i P = \frac{1}{2}H_{i+1}
$$

(5)
\[ \langle PP \rangle = \partial_x^2 \log Z = -F'', \quad t_0 = x. \] The string equations are

\[ \sum_{k=0}^{\infty} (k+1)t_k F_k = 0 \quad \sum_{k=0}^{\infty} (k+1)t_k G_k = 0. \tag{6} \]

The string equations and the hierarchy equations can be written as Virasoro constraints acting on the partition function of the matrix model (or on the tau-function of the ZS hierarchy since \( Z = \tau \)). These are the Virasoro constraints of an untwisted scalar field

\[ L_n(\alpha) Z = 0, \quad n \geq -1 \tag{7} \]

where \( \alpha \) is an a-priori arbitrary integration constant (see refs. [3,4] for a discussion of the Virasoro constraints in the ZS hierarchy).

The topological point of the ZS hierarchy is given by

\[ t_1 = \beta \quad t_n = 0 \quad (n \neq 1) \tag{8} \]

where \( \beta \) is a number to be determined. For convenience, it is better to shift \( t_1 \) by \(-\beta\) in such a way that the topological point is given by \( t_n = 0 \) for any \( n \).

I will also introduce the following notation for the correlation functions: \( \langle O_{n_1} \ldots O_{n_p} \rangle_t \) will denote a correlation function computed not at the ZS topological point (which means that it is a function of the \( t \)'s), \( \langle O_{n_1} \ldots O_{n_p} \rangle \) will, instead, denote a correlation function computed at the ZS topological point (i.e. at \( t_n = 0 \)).

The Virasoro constraints, eq. (7), can then be written in the following way

\[ \langle P \rangle_t = \frac{1}{2\beta} \left[ \sum_{k \geq 1} (k+1)t_k \langle O_{k-1} \rangle_t + \frac{\alpha t_0}{2} \right] \]

\[ \langle O_1 \rangle_t = \frac{1}{2\beta} \left[ \sum_{k \geq 1} (k+1)t_k \langle O_k \rangle_t + \frac{\alpha^2}{4} \right] \tag{9} \]

\[ \langle O_{n+1} \rangle_t = \frac{1}{2\beta} \left[ \sum_{k \geq 1} (k+1)t_k \langle O_{k+n} \rangle_t + \alpha \langle O_{n-1} \rangle_t + \right. \]

\[ + \sum_{k=1}^{n-2} \{ \langle O_k \rangle_t \langle O_{n-k-2} \rangle_t + \langle O_k O_{n-k-2} \rangle_t \} \] .

It is now necessary to consider the meaning of \( \alpha \). If \( \alpha \) is a parameter in the theory, from eq. (9) it follows immediately that \( \alpha \) cannot assume the value zero, otherwise at
the ZS topological point all the correlation functions are zero. Moreover, to construct a
topological theory one needs to assign ghost charges to the operators. It is not difficult to
verify that a ghost charge assignment is possible only if $\alpha = 0$, leading to a trivial theory.

The other possibility, which I will assume from now on, is that $\alpha$ is a coupling constant,
\[\alpha \neq 0\]
i.e. there is an operator $Q$ associated to $\frac{\partial}{\partial \alpha}$ in the sense that
\[
\left. \langle Q \prod_{i} O_{m_i} \rangle \right|_{t_n=0, \alpha=0} = 0.
\]
(10)
Notice that if $\alpha$ is a coupling constant, then the ZS topological point is given by $t_n = 0$
and $\alpha = 0$. As it is obvious from its definition, the $Q$ operator has a different origin from
all the other operators and understanding its meaning could be a key to understand the
ZS topological theory.

The Virasoro constraints can be recasted in a way to show explicitly the topological
recursion relations. Consider the correlation function
\[
\left. \langle \prod_{i} O_{n_i} \rangle \right|_{t=0} = \frac{\partial}{\partial t_n} \langle \prod_{i} O_{n_i} \rangle |_{t=0}.
\]
(11)
One can eliminate $\frac{\partial}{\partial t_n}$ using the Virasoro constraints (eqs. (7) or (9)). One has:
\[
\left. \langle \prod_{i \in S} O_{m_i} Q^p \rangle \right|_{t=0} = \frac{1}{2^\beta} \left[ \sum_{i \in S} (m_i + 1) \langle O_{m_i+n-1} \prod_{j \neq i} O_{m_j} Q^p \rangle + \right.
\]
\[
+ \sum_{m=0}^{n-3} \langle O_m O_{n-3-m} \prod_{i \in S} O_{m_i} Q^p \rangle + \right.
\]
\[
+ \sum_{m=0}^{n-3} \sum_{X \subseteq Y=S} \sum_{a+b=p} \langle O_m \prod_{j \in X} O_{m_j} Q^a \rangle \langle O_{n-3-m} \prod_{k \in Y} O_{m_k} Q^b \rangle + \right.
\]
\[
+ p \langle O_{n-2} \prod_{i \in S} O_{m_i} Q^{p-1} \rangle \left. \right] \right.
\]
and the exceptions
\[
\langle PPQ \rangle = \frac{1}{4^\beta} = \langle O_1 QQ \rangle
\]
(13)
with the convention that $O_{-1} \equiv 0$. 3
Now one has to assign ghost charges to the operators $O_n$, $P$ and $Q$. As usual, since $t_1$ has been shifted, one gives ghost number zero to $O_1$. A good assignment of ghost charges is:

$$ q(O_n) = (n-1) \quad q(P) = -1 \quad q(Q) = -2 . \quad (14) $$

It is convenient to introduce the notation

$$ \left\langle \prod_{i \in S} O_{m_i} Q^p \right\rangle_h , \quad h = \sum_{i \in S} q(O_{m_i}) - 2p \quad (15) $$

i.e. $h$ is the total ghost charge of the operators inside the v.e.v. . One can prove the following results:

1) $\langle \cdots \rangle_h = 0$ if $h < -4$ ;

2) all correlation functions without insertions of $Q$ are zero;

3) all correlation functions with $h \neq 4(g-1)$ and $g = 0, 1, 2, \ldots$, are zero, with the exception of $\langle Q \rangle_{h=-2}$.

Thus, there is the following ghost charge conservation law:

$$ \sum_i q_i = 4(g-1) \quad (16) $$

and I’ll set $\langle Q \rangle = 0$. I will also label the correlation functions by the genus $g$ instead that by ghost charge $h$.

It is useful to change normalization of the operators $O_m$ and $Q$. Define $\tilde{O}_m$ and $\tilde{Q}$ by:

$$ O_m = \frac{(m+1)!}{(2\beta)^m} \tilde{O}_m \quad m \geq 0 , \quad Q = \beta \tilde{Q} . \quad (17) $$

Then eqs. (12) and (13) become

$$ \left\langle \tilde{O}_n \prod_{i \in S} \tilde{O}_{m_i} \tilde{Q}^p \right\rangle_g = $$

$$ \sum_{i \in S} \frac{(m_i+1)(m_i+n)!}{(n+1)!(m_i+1)!} \left\langle \tilde{O}_{m_i+n-1} \prod_{j \neq i} \tilde{O}_{m_j} \tilde{Q}^p \right\rangle_g + $$

$$ + (2\beta)^2 \sum_{m=0}^{n-3} \frac{(m+1)!(n-2-m)!}{(n+1)!} \left\langle \tilde{O}_m \tilde{O}_{n-3-m} \prod_{i \in S} \tilde{O}_{m_i} \tilde{Q}^p \right\rangle_{g-1} + $$

$$ + (2\beta)^2 \sum_{m=0}^{n-3} \sum_{X \cup Y = S} \sum_{a+b=p \atop g_1+g_2=g} \frac{(m+1)!(n-2-m)!}{(n+1)!} \left\langle \tilde{O}_m \prod_{j \in X} \tilde{O}_{m_j} \tilde{Q}^a \right\rangle_{g_1} . $$

4
\[
\cdot \left\langle \tilde{\mathcal{O}}_{n-3-m} \prod_{k \in Y} \tilde{\mathcal{O}}_{m_k} \tilde{Q}^b \rightangle_{g_2} + 2p \frac{(n-1)!}{(n+1)!} \left\langle \tilde{\mathcal{O}}_{n-2} \prod_{i \in S} \tilde{\mathcal{O}}_{m_i} \tilde{Q}^{p-1} \right\rangle_g
\]
and
\[
\left\langle PP\tilde{Q} \right\rangle_{g=0} = \frac{1}{4\beta^2} = \left\langle \tilde{\mathcal{O}}_1 \tilde{Q} \tilde{Q} \right\rangle_{g=0}.
\]  
(19)

In particular
\[
\left\langle P \prod_{i \in S} \tilde{\mathcal{O}}_{m_i} \tilde{Q}^p \right\rangle_g = \sum_{i \in S} \left\langle \tilde{\mathcal{O}}_{m_i-1} \prod_{j \neq i} \tilde{\mathcal{O}}_{m_j} \tilde{Q}^p \right\rangle_g  
\]  
(20)
\[
\left\langle \tilde{\mathcal{O}}_1 \prod_{i=1}^{s-p} \tilde{\mathcal{O}}_{m_i} \tilde{Q}^p \right\rangle_g = (2g-2+s) \left\langle \prod_{i=1}^{s-p} \tilde{\mathcal{O}}_{m_i} \tilde{Q}^p \right\rangle_g  
\]  
(21)
where, in the last formula, the ghost charge conservation has been used.

From eq. (20) it follows that \( P \) could play a rôle similar to that of the puncture operator in the standard (KdV) topological gravity [5,6,7], with the \( \tilde{\mathcal{O}}_m \) \((m \geq 1)\) acting as “descendants” and \( \tilde{Q} \) as a kind of second “primary” operator. Moreover, eq. (21) tells us the \( \tilde{\mathcal{O}}_1 \) behaves exactly as the dilaton operator in the standard (KdV) topological gravity, with the only exception of the correlation function \( \left\langle \tilde{\mathcal{O}}_1 \tilde{Q} \tilde{Q} \right\rangle_{g=0} \). Anyway, it is clear from the recursion relations eq. (18) that the ZS topological gravity has very different characteristics from the usual KdV topological gravity due to the distinctive rôle played by the operator \( \tilde{Q} \).

Let first try to make a comparison between the ZS topological gravity and the well known KdV topological gravity [6]. Behind the \( \tilde{Q} \) operator, also \( \mathcal{O}_1 \) shows a different behaviour. Indeed, in KdV topological gravity \( \mathcal{O}_1 \) is a descendant operator, which means that every correlation function in which it appears can be turned, using the recursion relations, into a correlation function involving only primary operators. This is not what happens in eq. (19). Thus, it is not clear if the standard definition of primary and descendant operators can be applied to the ZS topological model.

Indeed, one could think of \( Q, P \) and \( \mathcal{R} \stackrel{\text{def}}{=} \mathcal{O}_1 \) as “primary” operators. In this case, on the Small-Phase-Space at genus zero, one has
\[
\log Z = -F = \frac{t_Q(t_P)^2}{8(\beta-t_R)} + \frac{(t_q)^2}{8\beta} \log \left(1 - \frac{t_R}{\beta} \right) + \frac{1}{2} \delta(t_Q)^2  
\]  
(22)
where $\delta$ is an arbitrary integration constant which must be set to zero if $\langle QQ \rangle = 0$, and the constitutive equations:

\[
\begin{align*}
\langle PP \rangle_t &= \frac{tQ}{4(\beta - tR)} \\
\langle PQ \rangle_t &= \frac{tP}{4(\beta - tR)} \\
\langle QQ \rangle_t &= -\frac{1}{4} \log \left( 1 - \frac{tR}{\beta} \right) + \delta \\
\langle RR \rangle_t &= 16 \langle PP \rangle_t \langle PQ \rangle_t^2 + 2 \langle PP \rangle_t^2 \\
\langle PR \rangle_t &= 4 \langle PP \rangle_t \langle PQ \rangle_t \\
\langle RQ \rangle_t &= 2 \langle PQ \rangle_t^2 + \langle PP \rangle_t .
\end{align*}
\] (23)

If $\langle PP \rangle_t$ would had been exchanged with $\langle RQ \rangle_t$, these constitutive equations would had resembled those for a three-primary KdV topological model.

To understand the behaviour of the $Q$ and $O_1$ operators, another possibility [8] is to consider the $O_m$ ($m \geq 0$) operators as linear combinations of operators coming from the topological gravitational sector and a topological matter sector, for example a topological $U(1)$-gauge theory. Although this interpretation is very promising and the behaviour of $O_1$ can be easily understood, I haven’t yet found a consistent way of applying it.

Another possible way of understanding the ZS topological model is that of looking for a mathematical description in terms of an intersection theory on Riemann surfaces [9,10,11]. An intersection theory on Riemann surfaces which could be a good candidate for being the algebro-geometrical description of the ZS topological model can be constructed as follows [8].

Consider a Riemann surface $\Sigma$ with $g$ handles and $n = s + t$ marked points, and its Deligne-Mumford compactified moduli space $\overline{M}_{g,n}$. By the Riemann-Roch theorem this is a space of complex dimension $\dim \overline{M}_{g,n} = 3g - 3 + n$. Consider the line bundle

\[
L \overset{\text{def}}{=} \bigotimes_{i=1}^{s} O(q_i)^{-1} \bigotimes_{j=1}^{t} O(p_j)^{n_j}, \quad n_j \geq 0
\] (24)

over $\overline{M}_{g,n}$. That is, its sections are functions that can have poles of order $n_j$ at the point $p_j$ or zero of first order at the point $q_i$. The degree of $L$ is $\deg(L) = \sum n_j - s$. Consider

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2 I thank E. Witten for his patience in explaining to me many mathematical details on the algebro-geometrical formulation of topological gravity.
now a $D$-dimensional vector bundle $V$ over the moduli space, whose fiber $V_{\Sigma}$ at a point $\Sigma \in \mathcal{M}_{g,n}$ is defined as the space of holomorphic sections of $L$:

$$V_{\Sigma} \overset{\text{def}}{=} H^0(\Sigma, L) \quad \text{if} \quad H^1(\Sigma, L) = 0. \quad (25)$$

(This definition is correct only if $H^1(\Sigma, L) = 0$. A more general definition that takes into account that $H^1(\Sigma, L)$ may not always vanish, can be done following ref. [10].) This vector bundle has Chern classes and in particular the Top Chern class $c_{\text{top}}(V) = c_D(V)$. Notice that the Riemann-Roch theorem tells us that the fiber of $V$ has complex dimension $D = 1 - g + \sum n_j - s$. Now, the possible connexion with the ZS topological model is given by the following formula:

$$\left\langle \prod_{j=1}^{t} \mathcal{O}_{n_j} \cdot Q^s \right\rangle_g \overset{?}{=} \left\langle c_D(V), \mathcal{M}_{g,n} \right\rangle. \quad (26)$$

To investigate if this conjecture is correct, one must study the properties of the intersection numbers $\left\langle c_D(V), \mathcal{M}_{g,n} \right\rangle$ and compare them to those of the ZS topological correlation functions. These intersection numbers vanish unless a dimensional condition is obeyed, namely $\sum_i q_i = 4(g - 1)$. This is exactly the ghost selection rule (16). A few examples of intersection numbers permitted by this rule are:

- $D = 0$ \hspace{1cm} $\langle \mathcal{O}_1 Q^2 \rangle_0 = 1 = \langle P^2 Q \rangle_0 \quad (27)$
- $D = 1$ \hspace{1cm} $\langle \mathcal{O}_3 Q^3 \rangle_0$, $\langle \mathcal{O}_2 Q^2 \rangle_0$, $\langle P \mathcal{O}_2 Q^2 \rangle_0$
- $\langle P^2 \mathcal{O}_1 Q \rangle_0$, $\langle P^4 \rangle_0$, $\langle \mathcal{O}_1 \rangle_1$

... where I have actually written the correlation functions corresponding to the intersection numbers (i.e. $\langle \mathcal{O}_1 Q^2 \rangle_0$ stays for $\langle \mathcal{O}_0(V), \mathcal{M}_{0,3} \rangle$ where $L = \mathcal{O}(q_1)^{-1} \otimes \mathcal{O}(q_2)^{-1} \otimes \mathcal{O}(p_1)^{+1}$, etc.).

Although the situation up to now looks promising, many thing must be proven in the mathematical theory even before proposing this intersection theory as the algebrog-geometrical description of the ZS topological model. For example, one should prove that all the intersection numbers without insertions of $Q$ (i.e. with $s = 0$) are zero and that at least eqs. (20) and (21) hold. To do that, one has to understand the meaning of “descendant” operator from the mathematical point of view, and this requires the full mathematical apparatus of ref. [10] (especially section 2).
In conclusion, in this note I have discussed some properties and possible interpretations of the ZS topological theory. This theory, although similar in many aspects to the (pure) KdV topological gravity, has various interesting new features and its physical and/or mathematical interpretation is still unclear. Obviously, it will be very interesting to get a better understanding of this new two dimensional topological theory.

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