A PRODUCT FOR PERMUTATION GROUPS AND TOPOLOGICAL GROUPS

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Abstract. In this paper we introduce a new non-associative product for permutation groups, which has a number of striking properties. The product takes as input two nontrivial permutation groups, $M$ and $N$, which may be finite or infinite, and produces an infinite group $M \boxtimes N$ which carries many permutational properties of $M$. When both $M$ and $N$ are generated by point stabilisers and at least one of them is transitive, $M \boxtimes N$ is simple.

One way in which we use this product is to answer an open problem from topological group theory by obtaining the first construction of uncountably many pairwise non-isomorphic simple topological groups which are totally disconnected, locally compact, compactly generated and non-discrete.

To construct the product, we build a group $U(M, N)$ which acts on an edge-transitive biregular tree $T$. The action of $U(M, N)$ on $T$ is locally-$(M, N)$, in that the stabiliser of any vertex $v$ induces either $M$ or $N$ on the neighbours of $v$. When $M$ and $N$ are transitive, the group $U(M, N)$ is universal, in that it contains an isomorphic copy of every other locally-$(M, N)$ group. This construction is analogous to the iconic universal group construction of Burger and Mozes for locally finite regular trees, and will be of independent interest.

The product $M \boxtimes N$ is the permutation group induced by $U(M, N)$ on one part of the bipartition of $T$. As a permutational product, its most significant property is the following: $M \boxtimes N$ is primitive if and only if $M$ is primitive but not regular, and $N$ is transitive. Despite this remarkable similarity with the wreath product in product action, $M \boxtimes N$ and $M \wr N$ are dissimilar in all other respects, and in some sense distort the actions of $M$ and $N$ in opposite ways.

1. Introduction

We introduce a new product for permutation groups, and detail some of its striking properties. By considering permutation groups to be topological groups under the permutation topology, this product can be used to answer a question of P.-E. Caprace and T. De Medts [4]: using the product one can easily construct uncountably many pairwise non-isomorphic simple topological groups which are totally disconnected, locally compact, compactly generated and non-discrete. To prove the existence of the product, we extend the influential universal group of Burger and Mozes ([2] Section 3.2) from locally finite regular trees to biregular trees which are not necessarily locally finite.

The product is non-associative. It takes two nontrivial permutation groups, $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, which may be finite or infinite, and yields a new
permutation group $M \Join N \leq \text{Sym}(V_T)$ which is always infinite, where $V_T$ denotes one part of the bipartition of the $(|X|,|Y|)$-biregular tree $T$. We shall define $M \Join N$ in Section 2 here we summarise three of its most important attributes.

(i) The product shares many properties with the unrestricted wreath product in product action, a product which is fundamental to the study of permutation groups. For example, the similarity between the following statements is remarkable:

- $M \text{Wr} N$ is primitive in its product action on $XY$ if and only if $M$ is primitive and not regular on $X$, and $Y$ is finite and $N$ is transitive;
- $M \Join N$ is primitive on $V_Y$ if and only if $M$ is primitive and not regular on $X$, and $N$ is transitive (see Theorem 19).

Like the wreath product, when $N$ is finite $M \Join N$ inherits many permutational properties from $M$; but unlike the wreath product it does not inherit finiteness.

(ii) The product $M \Join N$ is simple whenever $M$ and $N$ are transitive and generated by point stabilisers (see Theorem 16).

(iii) If $M, N$ are transitive, and $M, N$ and $M \Join N$ are thought of as topological groups under their respective permutation topologies, when $N$ is compact $M \Join N$ inherits many topological properties from $M$, but it does not inherit discreteness. For example, if every point stabiliser in $M$ is compact and $N$ is compact, then every point stabiliser in $M \Join N$ is compact (Theorem 21); moreover, if in addition $M$ is compactly generated, then $M \Join N$ is compactly generated (Theorem 24).

2. Preliminaries

2.1. Permutation groups. Let $V$ be a non-empty set, and suppose $G$ acts on $V$. If $x \in V$ and $g \in G$, we denote the image of $x$ under $g$ by $gx$, thus following the convention that our permutations act from the left. This notation extends naturally to sets, so if $\Phi \subseteq V$ then $g\Phi$ is the image $\{gx : x \in \Phi\}$. The setwise stabiliser of $\Phi$ is the group $G|\Phi| := \{g \in G : g\Phi = \Phi\}$, and the pointwise stabiliser is the group $G_{\{x\}} := \{g \in G : gx = x, \forall x \in \Phi\}$. If $\Lambda$ consists of a single element $x$, then the setwise and pointwise stabilisers of $\Lambda$ coincide; this group is called the stabiliser of $x$ and is denoted by $G_x$. The set $Gx := \{gx : g \in G\}$ denotes the orbit of $x$ (under $G$), and $G$ is said to be transitive if $V$ consists of a single orbit. If $\Phi$ is an orbit of $G$, then $G$ induces a subgroup of $\text{Sym}(\Phi)$ which we denote by $G|\Phi$. The orbits of point stabilisers in $G$ are called suborbits of $G$.

The action of $G$ gives rise to a homomorphism from $G$ to the group $\text{Sym}(V)$ of all permutations of $V$. If this homomorphism is injective, then $G$ is said to be acting faithfully; when this occurs we will often consider $G$ to be a subgroup of $\text{Sym}(V)$, identifying $G$ with its image in $\text{Sym}(V)$. Two permutation groups $G \leq \text{Sym}(V)$ and $H \leq \text{Sym}(\Delta)$ are permutation isomorphic if there exists a bijection $\phi : V \to \Delta$ such that the map $g \mapsto \phi g \phi^{-1}$ is an isomorphism from $G$ to $H$.

A transitive group $G \leq \text{Sym}(V)$ is primitive on $V$ if the only $G$-invariant equivalence relations on $V$ are the trivial relation (each element in $V$ is related only to itself) and the universal relation (each element in $V$ is related to every element in $V$). If $G$ is transitive, then it is primitive if and only if every point-stabiliser $G_x$ is a maximal subgroup of $G$.

2.2. Graphs. In this paper a graph $\Gamma$ consists of a set $VT$ and a set $ET$ of two-element subsets of $VT$. The elements in $VT$ are called the vertices of $\Gamma$, and the elements of $ET$ the edges. The graph is non-trivial if $ET$ is non-empty. If two distinct vertices $v$ and $w$ belong to the same edge, they are said to be adjacent. An arc
in $\Gamma$ is an ordered pair of adjacent vertices, and the set of all arcs is denoted by $\text{Aut} \Gamma$. Thus, our graphs contain no loops or multiple edges, and between any two adjacent edges there are two arcs, one in each direction. We denote the automorphism group of $\Gamma$ by $\text{Aut} \Gamma$.

If $a \in \text{Aut} \Gamma$, we denote by $o(a)$ and $t(a)$ the vertices such that $a = (o(a), t(a))$, and by $\pi$ the arc $(t(a), o(a))$. If $v$ is a vertex in $\Gamma$, then $A(v) := \{a \in \text{Aut} \Gamma : o(a) = v\}$ denotes those arcs originating from $v$, while $\overline{A}(v) := \{a \in \text{Aut} \Gamma : t(a) = v\}$ denotes those arcs which terminate at $v$. We denote the set of vertices adjacent to $v$ by $B(v)$. These notational conventions extend to sets of vertices, so for example if $W \subseteq \mathcal{V}$, then $A(W) := \{a \in \text{Aut} \Gamma : o(a) \in W\}$.

The valency of $v$ is the cardinal $|B(v)|$; if all valencies are finite the graph is locally finite. A path is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_i \in B(v_{i-1})$ for all integers $i$ satisfying $1 \leq i \leq n$; the length of this path is $n$. A path which consists of a single vertex is called trivial or sometimes empty. Two vertices are connected if there is a path between them, and the distance between two connected vertices $v, w$, which we denote by $d(v, w)$, is the length of the shortest path between them; if two vertices are not connected then their distance is infinite. A graph is connected if the distance between any two vertices is finite. A cycle is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $n > 1$ and $v_0, v_1, \ldots, v_{n-1}$ and $v_1, v_2, \ldots, v_n$ are nontrivial paths. A tree is a connected graph which contains no cycles. In a tree $T$, there is a unique path between any two vertices $v, w \in \mathcal{V}$ which we denote by $[v, w]_T$. If we wish to exclude $w$, we write $[v, \beta]_T := [v, \beta]_T \setminus \{\beta\}$.

If $W$ is any set of vertices in a graph $\Gamma$, we denote by $\Gamma \setminus W$ the subgraph of $\Gamma$ induced on $\mathcal{V} \setminus W$. If $\Gamma$ is connected, and $\Gamma \setminus \{v\}$ is disconnected for some $v \in \mathcal{V}$, then $\Gamma$ is said to have connectivity one and $v$ is called a cut vertex. The connected subgraphs of $\Gamma$ which are maximal subject to the condition that they do not have connectivity one are called lobes. If $\Gamma$ is vertex transitive and has connectivity one, then every vertex is a cut vertex, and there is a tree $T_1$, called the block-cut-vertex tree of $\Gamma$. Let $L$ be a set in bijective correspondence with the set of lobes of $\Gamma$. Since $\text{Aut} \Gamma$ acts on the set of lobes of $\Gamma$, there is an induced action of $\text{Aut} \Gamma$ on $L$. The vertex set of $T_1$ is the union $L \cup \mathcal{V}$, and $v \in \mathcal{V}T_1$ and $\ell \in L$ are adjacent in $T_1$ if and only if $v$ lies in the lobe associated to $\ell$. The action of $\text{Aut} \Gamma$ on $L \cup \mathcal{V}$ preserves this relation, and so $\text{Aut} \Gamma$ acts on $T_1$. This action is easily seen to be faithful, and we will frequently consider $\text{Aut} \Gamma$ to be a subgroup of $\text{Aut} T_1$. Figure 2 shows a graph with connectivity one and its block-cut-vertex tree.

A ray (also called a half-line) in $\Gamma$ is a sequence of distinct vertices $R = \{v_i\}_{i \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ we have that $v_1v_2\ldots v_n$ is a path in $\Gamma$; thus a ray is a one-way infinite path. The vertex $v_1$ is often called root of $R$. The ends of $\Gamma$ are equivalence classes of rays: two rays $R_1$ and $R_2$ lie in the same end if there is a third ray $R_3$ which contains infinitely many vertices from both $R_1$ and $R_2$. In the special case of an infinite tree $T$, the ends of $T$ are particular easy to picture: for a fixed vertex $a \in \mathcal{V}$, each end $e$ contains precisely one ray whose root is $a$, and we denote this ray by $[a, e]_T$.

Let $m, n$ be non-zero cardinal numbers. A tree $T$ is $m$-regular, or sometimes regular, if the valency of every vertex equals $m$. There is a natural bipartition of any tree $T$, in which any pair of vertices whose distance is even lie in the same part of the partition. If all vertices in one part the this bipartition have valency $m$, and all vertices in the other part have valency $n$, we say that $T$ is $(m, n)$-biregular. If $a \in AT$ is an arc in $T$ and both connected components of $T \setminus \{a, \pi\}$ are infinite, then each connected component is called a half-tree of $T$. We denote the half-tree of $T \setminus \{a, \pi\}$ containing $o(a)$ by $T_\pi$, so the other connected component of $T \setminus \{a, \pi\}$ is $T_\tau$. 
2.3. Permutation groups as topological groups. For a thorough introduction see [12] and [17]. If \( V \) is any non-empty set, then \( \text{Sym}(V) \) can be given a natural topology, that of pointwise convergence, under which \( \text{Sym}(V) \) is a Hausdorff topological group. A basis of neighbourhoods of the identity is given by point stabilisers of finite subsets of \( V \), and so an open set in \( \text{Sym}(V) \) is a union of cosets of point stabilizers of finite subsets of \( V \). Under this topology, if \( G \leq \text{Sym}(V) \), then a subgroup of \( G \) is open if and only if it contains the pointwise stabiliser in \( G \) of some finite subset of \( V \).

For any finite subset \( \Phi \), the pointwise stabiliser \( \langle \text{Sym}(V) \rangle_\Phi \) is both open and closed in \( \text{Sym}(V) \), so with this topology \( \text{Sym}(V) \) is totally disconnected. Convergence in \( \text{Sym}(V) \) is natural: a set \( W \subseteq \text{Sym}(V) \) has a limit point \( h \in \text{Sym}(V) \) if and only if, for all finite subsets \( \Phi \subseteq V \), there exists \( g \in W \) distinct from \( h \) such that \( gh^{-1} \in \text{Sym}(V)_\Phi \).

Let \( G \) be a subgroup of \( \text{Sym}(V) \). The group \( G \) is closed if and only if some point stabiliser \( G_v \) is closed, and is compact if and only if all \( G \)-orbits on \( V \) are finite. The topology on \( G \) is locally compact if and only if \( G_\Phi \) is compact for some finite \( \Phi \subseteq V \) (see [3] Lemma 3.1 for example). In particular, if \( \Gamma \) is a connected locally finite graph, then any closed subgroup \( G \) of \( \text{Aut}(\Gamma) \) will be totally disconnected and locally compact.

A topological space in which every subset is open is called discrete, and so the permutation topology is discrete on \( G \) if and only if there is a finite subset \( \Phi \subseteq V \) such that \( G_\Phi \) is trivial.

3. Groups acting on trees without inversion: a universal group

Throughout, \( X \) and \( Y \) will be disjoint sets, each containing at least two elements, and \( M \leq \text{Sym}(X) \) and \( N \leq \text{Sym}(Y) \) will be nontrivial permutation groups. Let \( T \) denote the \((|X|,|Y|)\)-biregular tree. Let \( V_X \) be the set of vertices of \( T \) with valency \(|X|\), and \( V_Y \) the set of vertices with valency \(|Y|\), so \( V_X \) and \( V_Y \) are the two parts of the natural bipartition of \( T \). A function \( \mathcal{L} : AT \to X \cup Y \) is called a legal colouring of \( X \) and \( Y \) if it satisfies:

(i) for all \( v \in V_X \), the restriction \( \mathcal{L}|_{A(v)} : A(v) \to X \) is a bijection;

(ii) for all \( v \in V_Y \), the restriction \( \mathcal{L}|_{A(v)} : A(v) \to Y \) is a bijection; and

(iii) for all \( v \in V_T \), the image of \( \mathcal{L}|_{A(v)} \) contains precisely one element.

One may easily verify that it is always possible to construct a legal colouring of \( X \) and \( Y \). Figure 3 shows a legal colouring of the \((5,3)\)-biregular tree.

If \( M \leq \text{Sym}(X) \) and \( N \leq \text{Sym}(Y) \), and \( \mathcal{L} \) is a legal colouring of \( X \) and \( Y \), define

\[
\mathcal{U}_\mathcal{L}(M, N) := \left\{ g \in (\text{Aut } T)|_{V_X} : \mathcal{L}|_{A(gv)}|_{A(v)} \mathcal{L}^{-1}|_{A(v)} \in \begin{cases} M & \text{for all } v \in V_X \\ N & \text{for all } v \in V_Y \end{cases} \right\}.
\]

Verifying that \( \mathcal{U}_\mathcal{L}(M, N) \) is a subgroup of \( \text{Aut } T \) is tedious but not difficult. The group \( \mathcal{U}_\mathcal{L}(M, N) \) is similar to Burger and Mozes’ universal group \( U(F) \), which acts on a locally finite regular tree. Of course, a legal colouring in our setting is not the same as a legal colouring in [2] Section 3.2. But, in a sense, the product \( \mathcal{U}_\mathcal{L}(M, N) \) can be considered to be an extension of Burger and Mozes’ group, generalising from locally finite regular trees to biregular trees that need not be locally finite.

The first construction of the group \( \mathcal{U}_\mathcal{L}(M, N) \) was inspired by [15] Section 2.2, and can \( \mathcal{U}_\mathcal{L}(M, N) \) in fact be constructed using refinements of the arguments in [15] (which use relational structures), so long as \( M \) and \( N \) are closed (in their respective permutation topologies).
Since $U_\mathcal{L}(M, N)$ preserves the parts $V_X$ and $V_Y$, it induces a subgroup of $\text{Sym}(V_Y)$, which we denote by $M \boxtimes N$. As we shall see in Proposition 4 for two legal colourings $\mathcal{L}, \mathcal{L}'$ the groups $M \boxtimes N$ and $M \boxtimes N'$ permuationally isomorphic, and so we write $M \boxtimes N$ instead of $M \boxtimes N$ when there is no chance of ambiguity.

Let us say that a group $H \leq \text{Aut } T$ is locally-$(M, N)$ if $H$ fixes setwise the parts $V_X$ and $V_Y$ setwise, and for all vertices $v$ of $T$ the group $H_v \mid_{B(v)} \leq \text{Sym}(B(v))$ induced by the vertex stabiliser $H_v$ is permutation isomorphic to $M$ if $v \in V_X$ and $N$ if $v \in V_Y$. As a subgroup of $\text{Aut } T$, the group $U_\mathcal{L}(M, N)$ has the following properties.

**Theorem 1.** Suppose $X, Y$ are disjoint sets of cardinality at least two, and $T$ is the $(|X|, |Y|)$-biregular tree. Given permutation groups $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$, and a legal colouring $\mathcal{L}$ of $T$,

(i) $\Delta \subseteq X \cup Y$ is an orbit of $M$ or $N$ if and only if $t(\mathcal{L}^{-1}\Delta)$ is an orbit of $U_\mathcal{L}(M, N);$  
(ii) given legal colourings $\mathcal{L}, \mathcal{L}'$, the groups $U_\mathcal{L}(M, N)$ and $U_\mathcal{L}-(M, N)$ are conjugate in $\text{Aut } T;$  
(iii) if $M$ and $N$ are transitive, and $H \leq \text{Aut } T$ is locally-$(M, N)$, then for some suitable legal colouring $\mathcal{L}$ we have $H \leq U_\mathcal{L}(M, N).$

(iv) $U_\mathcal{L}(M, N)$ is locally-$(M, N);$  
(v) if $M$ and $N$ are closed, then $U_\mathcal{L}(M, N)$ is a closed subgroup of $\text{Aut } T;$

Thus, when $M$ and $N$ are transitive, $U_\mathcal{L}(M, N)$ is the universal locally-$(M, N)$ group, in that it contains a permutationally isomorphic copy of every locally-$(M, N)$ group acting on the biregular tree $T$. Theorem 1 follows from Lemma 2 Propositions 3 and Lemmas 6.

Our arguments in this section rely on manipulating sequences of bijections and restrictions of functions. Although these manipulations can look rather daunting, they are in fact very simple and involve only the following basic steps. If $g, h \in (\text{Aut } T)_{\{V_X\}}$, then for all $w \in VT$,

(i) $g|_{A(w)}^{-1} = g^{-1}|_{A(gw)}$;  
(ii) $g|_{A(hw)}h|_{A(w)}g^{-1}|_{A(gw)} = (ghg^{-1})|_{A(gw)}$;  
(iii) if $\mathcal{L} = \mathcal{L}'g$, then $\mathcal{L}|_{A(w)} = \mathcal{L}'|_{A(gw)}g|_{A(w)}$ and $\mathcal{L}|_{A(w)}^{-1} = g^{-1}|_{A(gw)}\mathcal{L}'|_{A(gw)}^{-1}$.

In what follows, $\mathcal{L}$ and $\mathcal{L}'$ are arbitrary legal colourings of $X$ and $Y$. To simplify notation, we associate each constant function $\mathcal{L}|_{A(w)}$ with its image in $X \cup Y$, so for all $w \in VT$ we have $\mathcal{L}|_{A(w)} \in X \cup Y$.

**Lemma 2.** If the distance between $v, v' \in VT$ is even, then there exists $\sigma \in \text{Sym}(X \cup Y)_{\{X\}}$ satisfying $\sigma \mathcal{L}|_{A(v')} = \mathcal{L}|_{A(v)}$, and for all such $\sigma$ there exists a unique automorphism $g \in (\text{Aut } T)_{\{V_X\}}$ such that $\sigma v = v'$ and $\mathcal{L} = \sigma \mathcal{L} g$.

**Proof.** Since $\mathcal{L}$ and $\mathcal{L}'$ are legal colourings, we can always find $\sigma \in \text{Sym}(X \cup Y)_{\{X\}}$ satisfying $\sigma \mathcal{L}|_{A(v')} = \mathcal{L}'|_{A(v)}$. For each integer $n \geq 0$, let $B_n$ (resp. $B'_n$) be the subtree of $T$ induced by those vertices whose distance in $T$ from $v$ (resp. $v'$) is at most $n$. Because $d(v, v')$ is even, $v$ and $v'$ belong to the same part of the bipartition of $T$.

Let $g_0 : B_0 \to B'_0$ be the map taking $v$ to $v'$. Our proof that an appropriate element $g \in \text{Aut } T$ exists relies on an easy induction argument with the following induction hypothesis: for all integers $n \geq 1$, there exists a graph isomorphism
Proposition 3. Vertices \( v, v' \in V_X \) (resp. \( V_Y \)) lie in the same orbit of \( \mathcal{U}_\mathcal{L}(M, N) \) if and only if \( \mathcal{L}|_{\pi(v)} = \mathcal{L}|_{\pi(v')} \) lie in the same orbit of \( \mathcal{U}_\mathcal{L}(M, N) \) (resp. \( M \)).

Proof. Suppose \( v, v' \in V_X \) and there exists \( g \in \mathcal{U}_\mathcal{L}(M, N) \) such that \( gv = v' \). Fix \( (w, v) \in \mathcal{X}(v) \) and write \( \sigma := \mathcal{L}|_{\mathcal{A}(gw)} g|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} \in N \). Then \( \mathcal{L}(w, v) = \mathcal{L}(gw, v') \). By the definition of \( \mathcal{L} \), we have that \( \mathcal{L}|_{\pi(v)} = \mathcal{L}(w, v) \) and \( \mathcal{L}|_{\pi(v')} = \mathcal{L}(gw, v') \). Hence \( \mathcal{L}|_{\pi(v)} = \mathcal{L}|_{\pi(v')} \).

Conversely, suppose \( \mathcal{L}|_{\pi(v)} = \mathcal{L}|_{\pi(v')} \) lie in the same orbit of \( N \). Then there exists \( \sigma \in N \) such that \( \mathcal{L}|_{\pi(v)} = \mathcal{L}|_{\pi(v')} = \sigma \mathcal{L} \). Let \( \hat{\sigma} \) be the element in \( \mathcal{Y}(X \cup Y) \) that is \( \sigma \) on \( Y \) and trivial on \( X \). By Lemma 2, there exists a unique \( g \in \langle \mathcal{A}(T) \rangle_{V_Y} \) such that \( \hat{\sigma} \mathcal{L} g = \mathcal{L}(w, v) \). If \( w \in V_X \), then \( \mathcal{L}|_{\mathcal{A}(gw)} g|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} = (\mathcal{L}g)|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} = \hat{\sigma} \mathcal{L} \mathcal{L}^{-1}|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} = \hat{\sigma}^{-1} \mathcal{L}|_{\mathcal{A}(u)} \) in \( M \). Similarly, if \( w \in V_Y \) then \( \mathcal{L}|_{\mathcal{A}(gw)} g|_{\mathcal{A}(u)} \mathcal{L}^{-1}|_{\mathcal{A}(u)} = \hat{\sigma}^{-1} \mathcal{L}|_{\mathcal{A}(u)} \) in \( N \). Hence \( g \in \mathcal{U}_\mathcal{L}(M, N) \), and so \( v, v' \) lie in the same orbit of \( \mathcal{U}_\mathcal{L}(M, N) \).

A symmetric argument proves that \( v, v' \in V_Y \) lie in the same orbits of \( \mathcal{U}_\mathcal{L}(M, N) \) if and only if \( \mathcal{L}|_{\pi(v)} = \mathcal{L}|_{\pi(v')} \) lie in the same orbit of \( M \).

Any two legal colourings give rise to essentially the same permutation group.
Proposition 4. The groups $U_L(M,N)$ and $U_L'(M,N)$ are conjugate in $\text{Aut} T$.

Proof. Choose vertices $v,v' \in VT$ such that $L_{\pi(v)}$ and $L'_{\pi(v')}$ are equal. By Lemma [2] there is a unique element $g \in (\text{Aut} T)_{\{V_X\}}$ such that $gv = v'$ and $L = Lg$. Choose $h \in (\text{Aut} T)_{\{V_X\}}$, and set $h' := ghg^{-1}$. If $w' \in VT$, then $w := g^{-1}w'$ and $w'$ must both lie in $V_X$, or both lie in $V_Y$, and

$$L'_{A(h'w')}h'_{A(w')}L'_{A(w')} = L'_{A(ghw)}g_{A(hw)}h_{A(w)}g^{-1}_{A(gw)}L'_{A(gw)}^{-1} = L_{A(hw)}h_{A(w)}L_{A(w)}^{-1}.$$ 

Hence $h' \in U_L(M,N)$ if and only if $h \in U_L(M,N)$. □

If $M$ and $N$ are transitive, then $U_L(M,N)$ contains an isomorphic copy of every locally-$(M,N)$ group.

Proposition 5. If $M$ and $N$ are transitive, and $H \leq \text{Aut} T$ is locally-$(M,N)$, then $H \leq U_L(M,N)$ for some legal colouring $L$ of $T$.

Proof. Suppose $M$ and $N$ are transitive, and $H \leq \text{Aut} T$ is locally-$(M,N)$, so $H$ has precisely two orbits on $VT$, acting transitively on $V_X$ and $V_Y$. Fix adjacent vertices $p \in V_X$ and $q \in V_Y$. Since $H_p|_{B(p)}$ is permutation isomorphic to $M$ there
exists a bijection \( \phi : B(p) \to X \) such that \( \phi H_p|_{B(p)} \phi^{-1} = M \). Similarly, there exists a bijection \( \psi : B(q) \to Y \) such that \( \psi H_q|_{B(q)} \psi^{-1} = N \). For each integer \( n \geq 0 \), let \( S_n \) be the set of vertices in \( T \) whose distance from \( p \) is precisely \( n \). For \( v \in \bigcup_{i=0}^{n} S_i \), there is a unique path \([p,v]_T \) in \( T \) between \( p \) and \( v \); denote the unique vertex in \( B(v) \cap [p,v]_T \) by \( v_1 \) and the unique vertex in \( B(v_1) \cap [p,v_1]_T \) by \( v_2 \).

We inductively construct a subset of \( H \) which we will use to define a legal colouring \( L \) such that \( H \leq U_G(M, N) \). Let \( h_p \) be the identity, and for each \( v \in S_1 \), choose \( h_v \in H_p \) such that \( h_v q = v \). Notice that for all \( v \in \bigcup_{i=0}^{n} S_i \), \( h_v p = v \) if \( v \in V_X \) and \( h_v q = v \) if \( v \in V_Y \). Let \( n \in \mathbb{N} \), and suppose, for each \( v \in \bigcup_{i=0}^{n} S_i \), we have chosen an element \( h_v \in H \) such that \( h_v p = v \) if \( v \in V_X \) and \( h_v q = v \) if \( v \in V_Y \). For \( v \in S_{n+1} \), choose \( h \in H_{v_1} \) such that \( h v_2 = v \) (which we can do because \( H_{v_1} \) is transitive on \( B(v_1) \)) and set \( h_v := h h_{v_2} \). If \( v \in V_X \), then \( v_2 \in V_X \) and \( h_v p = v \), and similarly if \( v \in V_Y \) then \( h_v q = v \). Note that for all \( v \in \bigcup_{i=2}^{n} S_i \) we have \( h_v h_v^{-1} \in H_{v_1} \), and for all \( v, w \in S_1 \) we have \( h_w^{-1} h_v \in H_p \).

We now describe a colouring \( L \) and show that it is a legal colouring. Define \( L : AT : \to X \cup Y \) as follows: for \( v \in V_X \) let \( L|_{A(v)} : A(v) \to X \) be \( a \mapsto \phi h^{-1}_v t(a) \), and for \( v \in V_Y \) let \( L|_{A(v)} : A(v) \to Y \) be \( a \mapsto \psi h^{-1}_v t(a) \). We claim that \( L \) is a legal colouring. It is clear that for all \( v \in VT \) the map \( L|_{A(v)} \) is a bijection, so it remains to show that condition 1 of the definition holds; that is, the image of \( L|_{A(v)} \) has cardinality one for all \( v \in VT \). We show this first for \( p \), then for all \( v \in S_1 \), and finally for all \( v \in \bigcup_{i=2}^{n} S_i \).

If \((w,p)\) is any arc in \( \overline{A}(p) \), then \( w \in V_Y \) and \( L|_{\overline{A}(p)}(w,p) = L|_{A(w)}(w,p) = \psi h^{-1}_w p = \psi p \) (because \( w \in S_1 \) so \( h_w \in H_p \)). Therefore the cardinality of the image of \( L|_{\overline{A}(p)} \) is one.

If \( v \in S_1 \) and \((w,v) \in \overline{A}(v) \), then either \( w = p \), or \( w \in S_2 \subseteq V_X \). If \( w = p \) then \( L|_{\overline{A}(v)}(w,v) = \psi \phi h^{-1}_w v \). On the other hand, if \( w \in S_2 \) then \( w_1 = v \) and \( w_2 = p \) and so \( L|_{\overline{A}(v)}(w,v) = L|_{A(v)}(w,v) = \phi h^{-1}_w v \). Now \( w \in S_2 \), so \( h_w \in H_{w_1} h_{w_2} = H_v h_p = H_v \). Hence \( L|_{\overline{A}(v)}(w,v) = \phi v \). Therefore, if \( v \in S_1 \) and \((w,v) \in \overline{A}(v) \) then the cardinality of the image of \( L|_{\overline{A}(v)} \) is always one.

Suppose that \( v \in S_n \), for some \( n \geq 2 \). Choose \((w,v) \in \overline{A}(v) \) such that \( w \neq v_1 \); then \( v = w_1 \) and \( v_1 = w_2 \). If \( v \in V_X \), then \( w \in V_Y \) and \( L|_{\overline{A}(v)}(w,v) = L|_{A(v)}(w,v) = \psi h^{-1}_w v = \psi h^{-1}_w w_1 v \). Since \( h^{-1}_w h^{-1}_w H_w = H_{w_1} \), we have \( L|_{\overline{A}(v)}(w,v) = \psi h^{-1}_w w_1 \psi h^{-1}_w v = L|_{\overline{A}(v)}(v_1,v) \). Hence, the cardinality of the image of \( L|_{\overline{A}(v)} \) is one. A symmetric argument shows that the same is true if \( v \) is instead chosen from \( V_Y \). We have thus demonstrated that \( L \) is a legal colouring.

Finally, we show \( H \leq U_G(M, N) \). Choose \( g \in H \) and \( v \in VT \). If \( v \in V_X \) then \( g v \in V_X \) and \( h^{-1}_v g h_v \in H_p \). Hence \( L|_{A(g v)} g|_{A(v)} L|_{A(v)}^{-1} \in \phi H_p|_{B(p)} \phi^{-1} = M \). Similarly, if \( v \in V_Y \) then \( g v \in V_Y \) and \( h^{-1}_v g h_v \in H_q \), so \( L|_{A(g v)} g|_{A(v)} L|_{A(v)}^{-1} \in \psi H_q|_{B(q)} \psi^{-1} = N \). Therefore \( g \in U_G(M, N) \).

In the proposition above, the requirement that \( M \) and \( N \) be transitive is essential. For example, if \( G \) is the automorphism group of the graph \( \Gamma \) pictured in Figure 2 then \( G \) is transitive on the vertices of \( \Gamma \) and induces a faithful action on its block-cut-vertex tree \( T_r \). In its action on \( T_r \), it is easily seen that the group \( G \) has two orbits: one orbit consists of the \( T_r \)-vertices corresponding to the vertices of \( \Gamma \), and the other consists of those \( T_r \)-vertices which correspond to the lobes of \( \Gamma \). Choose any pair \( v \) and \( w \) of adjacent vertices in \( T_r \), and let \( M \) denote \( G_v|_{B(v)} \) and \( N \)
Lemma 6. $g_{w|_{B(w)}}$ and $\sigma$ denote guarantees that there exists a unique element $g$ such that it is surjective. Fix $\sigma \in M$, and let $\delta \in \text{Sym}(X \cup Y)$ be the permutation which equals $\sigma$ on $X$ and is the identity on $Y$. Now $\delta \mathcal{L}$ is a legal colouring, and Lemma 2 guarantees that there exists a unique element $g \in (A \cup T)^{(v_x)}$ such that $g w = v$ and $L g = \delta \mathcal{L}$. Hence, for all $w \in VT$ we have $L_{A(w)} g_{A(w)} L_{A(w)}^{-1} = \delta_{A(w)}$, and from this it follows that $g \in (\mathcal{U}_g(M, N))_v$ and $L_{A(v)} g_{A(v)} L_{A(v)}^{-1} = \sigma$.

The above argument can now be repeated, with $X$ and $M$ interchanged with $Y$ and $N$; this gives a proof that, for $v \in V_Y$, the map $g_{A(v)} \mapsto L_{A(v)} g_{A(v)} L_{A(v)}^{-1}$ is an isomorphism from $(\mathcal{U}_g(M, N))_v$ to $N$. The lemma then follows from the observation that the permutation groups induced by $(\mathcal{U}_g(M, N))_v$ on $A(v)$ and $B(v)$ are permutation isomorphic. 

Proposition 7. Suppose $v_1, v_2 \in V_X$ and $w_1, w_2 \in V_Y$. Edges $\{v_1, w_1\}$ and $\{v_2, w_2\}$ in $T$ lie in the same orbit of $\mathcal{U}_g(M, N)$ if and only if $v_1, v_2$ lie in the same orbit of $\mathcal{U}_g(M, N)$ and $w_1, w_2$ lie in the same orbit of $\mathcal{U}_g(M, N)$.

Proof. Let $G$ denote $\mathcal{U}_g(M, N)$, and choose $v_2 \in G v_1$ and $w_2 \in G w_1$. We wish to show that edges $\{v_2, w_2\}$ and $\{v_1, w_1\}$ lie in the same $G$-orbit and so we may, without loss of generality, suppose that $w_1 = w_2$. By Proposition 2 there exists $\sigma \in N$ such that $\sigma \mathcal{L}_{|_A(v_1)} = L_{|_A(v_1)}$. By Lemma 2 $G_{w_1} A_{(w_1)}$ is permutation isomorphic to $N$ via the map $g_{A(w_1)} \mapsto L_{A(w_1)} g_{A(w_1)} L_{A(w_1)}^{-1}$ for $g \in G_{w_1}$. Choose some $g \in G_{w_1}$ whose image under this permutation isomorphism is $\sigma$. Then $g(w_1, v_1) = L_{A(w_1)}^{-1} \sigma L_{A(v_1)} (w_1, v_1) = L_{A(w_1)}^{-1} L_{A(v_1)} (w_1, v_2) = L_{A(w_1)}^{-1} \sigma L_{A(v_1)} (w_1, v_2)$. Thus
we have established the proposition in one direction; the converse is obvious and requires no proof.

If we are given $\mu \in M$, let $\tilde{\mu} \in \text{Sym}(X \cup Y)$ be such that $\tilde{\mu}|_X = \mu$ and $\tilde{\mu}|_Y = 1_Y$. By Lemma 2 for each vertex $v \in V_X$ there is a unique automorphism $g_{\mu,v} \in (\text{Aut}T)_V$ such that $g_{\mu,v}v = v$ and $\mathcal{L} = \tilde{\mu}Lg_{\mu,v}$. One may quickly verify that $g_{\mu,v} \in \mathcal{U}_C(M,N)$. For $v \in V_Y$ and $\tau \in N$ we define $g_{\tau,v}$ similarly.

**Proposition 8.** If $v \in V_X$ and $w \in V_Y$, then $M(v) := \{g_{\mu,v} : \mu \in M\}$ is a subgroup of $\mathcal{U}_C(M,N)_v$ and $\tilde{N}(w) := \{g_{\tau,w} : \tau \in N\}$ is a subgroup of $\mathcal{U}_C(M,N)_w$.

**Proof.** Suppose $v \in V_X$. If $\mu, \tau \in M$, then $g_{\mu,v}g_{\tau,v} = v = g_{\tau,\mu,v}$ and $\tilde{\tau}\tilde{\mu}\mathcal{L}g_{\tau,\mu,v} = \mathcal{L} = \tilde{\mu}\tilde{\tau}\mathcal{L}g_{\mu,v}g_{\tau,v}$. Since $\tilde{\tau}$ fixes $Y$ pointwise, we have $\tilde{\tau}\tilde{\mu}\mathcal{L}g_{\tau,\mu,v} = \mathcal{L}|_{\mathcal{L}(v)}$. It follows immediately from Lemma 2 that $g_{\tau\mu,v} = g_{\mu,v}g_{\tau,v}$. Similarly, it follows from Lemma 2 that if $e$ denotes the identity in $M$, then $g_{e,v} = 1 \in \mathcal{U}_C(M,N)$. Finally, we have $g_{\mu,v}g_{\mu,v}^{-1} = g_{\mu^{-1},v}g_{\mu,v} = g_{e,v} = 1 \in \mathcal{U}_C(M,N)$, so $g_{\mu,v}^{-1} = g_{\mu,v}^{-1}$. Hence $M(v) \leq \mathcal{U}_C(M,N)_v$. A similar argument holds for $w \in V_Y$ and $\tilde{N}(w)$.

**Corollary 9.** Suppose $v \in V_X$ and $w \in V_Y$. Then $\mathcal{U}_C(M,N)_v$ contains a subgroup which is isomorphic to $M$, and $\mathcal{U}_C(M,N)_w$ contains a subgroup which is isomorphic to $N$.

**Proof.** We claim that for $\mu, \tau \in M$ we have $g_{\mu,v} = g_{\tau,v}$ if and only if $\mu = \tau$. Indeed, if $\mu = \tau$, then $\tilde{\tau}\tilde{\mu}\mathcal{L}g_{\tau,\mu,v} = \mathcal{L} = \tilde{\mu}\tilde{\tau}\mathcal{L}g_{\mu,v}$, and so $g_{\mu,v} = g_{\tau,v}$ by Lemma 2. On the other hand, if $g_{\tau,v} = g_{\mu,v}$ then $1 = g_{\mu,v}g_{\tau,v}^{-1} = g_{\tau^{-1}\mu,v}$. Hence $\mathcal{L} = \tilde{\tau}^{-1}\mu\mathcal{L}g_{\tau^{-1}\mu,v} = \tilde{\tau}^{-1}\mu\mathcal{L}$, so $\tau$ and $\mu$ must be equal.

Let $\varphi : M(v) \to M$ be given by $g_{\mu,v} \mapsto \mu^{-1}$. Our claim guarantees that this map is well-defined and injective, and it is obviously surjective. As in the proof of Proposition 8 for $\mu, \tau \in M$ we have $g_{\mu,v}g_{\tau,v} = g_{\tau\mu,v}$, so $\varphi$ is a homomorphism. Hence, by Proposition 8 $\mathcal{U}_C(M,N)_v$ contains a subgroup which is isomorphic to $M$. A symmetric argument shows that $\mathcal{U}_C(M,N)_w$ contains a subgroup which is isomorphic to $N$.

**Lemma 10.** If $M$ is a closed subgroup of $\text{Sym}(X)$ and $N$ is a closed subgroup of $\text{Sym}(Y)$, then $\mathcal{U}_C(M,N)$ is a closed subgroup of $\text{Aut}T$.

**Proof.** One may easily verify that $A := (\text{Aut}T)_{V_X}$ is a closed subgroup of $\text{Aut}T$. Let $G := \mathcal{U}_C(M,N)$. We show that $A \setminus G$ is open. Fix $v \in V_T$, and for $g \in A$ write $\overline{g} := \mathcal{L}|_{A(v)}^\dagger|_{A(v)}^{-1}$. Fix such an element $g$. We claim if $v \in V_X$ and $\overline{g} \notin M$, then there exists an open subset of $A \setminus G$ which contains $g$.

Suppose $v \in V_X$ and $\overline{g} \notin M$. Since $M$ is a closed subgroup of $\text{Sym}(X)$, and $\overline{g} \in \text{Sym}(X)$, there exists an open subset $U \subseteq \text{Sym}(X) \setminus M$ containing $\overline{g}$. Now $\overline{g}^{-1}U$ is a neighbourhood of the identity, and so it contains the pointwise stabiliser (in $\text{Sym}(X)$) of some finite subset $F \subseteq X$. Hence $\overline{g} \in \overline{g}\text{Sym}(X)_{(F)} \subseteq U \subseteq \text{Sym}(X) \setminus M$. Let $\Phi$ denote the finite set $\{v\} \cup \{w \in B(v) : \mathcal{L}|_{A(v)}^{-1}(v,w) \in F\}$.

To prove our claim, it is sufficient to show that the open set $gA(\Phi)$ is contained in $A \setminus G$. If $h \in gA(\Phi)$, then $hv = gv$. For all $x \in F$ there exists $w \in B(v)$ such that $x = \mathcal{L}|_{A(v)}^{-1}(v,w)$; since $hw = gw$, it follows that $\overline{h}x = \overline{g}x$. Therefore $\overline{h} \in \overline{g}\text{Sym}(X)_{(F)}$ and $\overline{h} \notin M$, and from this we may conclude that $h \notin G$. Thus $gA(\Phi) \subseteq A \setminus G$ and our claim is true.

If one interchanges $X$ and $M$ with $Y$ and $N$ in the above argument, then we obtain a proof that if $v \in V_Y$ and $g \in A$ satisfies $\overline{g} \notin N$, then there exists an open subset of $A \setminus G$ which contains $g$. Hence, every element in $A \setminus G$ lies in an open subset of $A$ which is disjoint from $G$, so $A \setminus G$ is open and $G$ is closed.
The following observation is obvious but useful. Recall that $T_a$ and $T_\pi$ are the two half-trees of $T \setminus \{a, \overline{a}\}$.

**Lemma 11.** If $a$ is any arc in $T$, then for all $h \in \mathcal{U}_\mathcal{L}(M, N)_a$ there exists $g \in \mathcal{U}_\mathcal{L}(M, N)$ such that $g$ fixes $T_\pi$ pointwise and $g\big|_{T_a} = h\big|_{T_a}$.

**Proof.** Suppose $a$ is any arc in $T$. Choose $h \in (\mathcal{U}_\mathcal{L}(M, N))_a$ and let $g \in \text{Sym}(VT)$ be defined by

$$gv = \begin{cases} hv & \text{if } v \in T_a; \\
v & \text{if } v \in T_\pi, \end{cases}$$

Notice that $A(o(a)) \setminus \{a\}$ is contained in the arc set of $T_a$, and $ga = a = ha$, so $h^{-1}g$ fixes the set $A(o(a))$ pointwise. Furthermore, if $v \in VT_a \setminus \{o(a)\}$, then $A(v)$ is contained in the arc set of $T_a$ and so $h^{-1}g$ fixes $A(v)$ pointwise.

Hence, if $v \in VT_a$, then $L|_{A(gv)}g|_{A(v)}L^{-1}_{A(v)} = L|_{A(hv)}h|_{A(v)}L^{-1}_{A(v)}$ which lies in $M$ if $v \in V_X$ and in $N$ if $v \in V_Y$. On the other hand, if $v \in VT_\pi$, then $L|_{A(gv)}g|_{A(v)}L^{-1}_{A(v)}$ is the identity, and so it lies in $M$ if $v \in V_X$ and in $N$ if $v \in V_Y$. \hfill \Box

4. Simplicity

In his influential paper \cite{Tits1972}, Jacques Tits introduced the following property $(P)$, which is sometimes known as Tits’ independence property. Let $G$ act on a (not necessarily locally-finite) tree $T$. If $P$ is a non-empty finite or infinite path in $T$, for each vertex $v$ in $T$ there is a unique vertex $\pi_P(v)$ in $P$ which is closest to $v$. This gives rise to a well-defined map on $VT$, in which each vertex $v$ is mapped to $\pi_P(v)$. For each vertex $q$ in $P$, the set $\pi_P^{-1}(q)$ of vertices in $T$ which are mapped to $q$ by $\pi_P$ is the vertex set of a subtree of $T$. The pointwise stabilizer $G_P(v)$ of $P$ leaves each of these subtrees invariant, and so we can define $G^q_P$ to be the subgroup of $\text{Sym}(\pi_P^{-1}(q))$ induced by $G_P(v)$. Thus, we have homomorphisms $\varphi_q : G_P(v) \to G^q_P$ for each $q \in VP$ from which we obtain the natural homomorphism

$$\varphi : G_P(v) \to \prod_{q \in VP} G^q_P.$$ \hfill (2)

The group $G$ is said to have property $(P)$ if the homomorphism $\varphi$ is an isomorphism for every possible choice of $P$. Intuitively, property $(P)$ means that $G_P(v)$ acts independently on each of the subtrees branching from $P$.

**Theorem 12** (\cite{Tits1972 Théorème 4.5]). Suppose $T$ is a tree. If $G \leq \text{Aut } T$ satisfies property $(P)$, no proper non-empty subtree of $T$ is invariant under $G$ and no end of $T$ is fixed by $G$, then the group $G^+ := \langle G(v, w) : \{v, w\} \in ET \rangle$ is simple (and possibly trivial). \hfill \Box

**Theorem 13.** Let $X$ and $Y$ be finite or infinite sets whose cardinality is at least two. Suppose $M \leq \text{Sym } (X)$ and $N \leq \text{Sym } (Y)$ are two permutation groups, and let $T$ be the $(|X|, |Y|)$-biregular tree. If $\mathcal{L}$ is a legal colouring of $X$ and $Y$, then $\mathcal{U}_\mathcal{L}(M, N)$ satisfies Tits’ independence property $(P)$. Furthermore, if $\mathcal{U}_\mathcal{L}(M, N)$ leaves invariant no proper non-empty subtree of $T$ and fixes no end of $T$, then $(\mathcal{U}_\mathcal{L}(M, N))^+$ is simple.

**Proof.** Let $P$ be any non-empty path in $T$ and let $G$ denote $\mathcal{U}_\mathcal{L}(M, N)$. Suppose we are given $g \in G_P(v)$ and $p \in VP$. Note that $B(p)$ contains either one or two vertices in $P$, so we may write $B(p) \cap VP = \{q, q'\}$, where $q, q'$ could be equal. By Lemma 11 there exists $g' \in G$ such that $g'$ fixes $T_{(q,p)}$ pointwise and agrees with $g$ on $T_{(p,q)}$. The element $g'$ fixes $P$ pointwise, and so in particular it lies in $G_{(p,q')}$,
and so we may apply the lemma again to deduce the existence of \( g_p \in G \) such that \( g_p \) fixes \( T(q',p) \) pointwise and agrees with \( g' \) on \( T(p,q') \). The element \( g_p \) therefore fixes \( VT \setminus \pi^{-1}(p) \) pointwise, agrees with \( g \) on \( \pi^{-1}(p) \), and satisfies \( g_p \big|_{\pi^{-1}(p)} \in G^p \).

Now \( g = \prod_{p \in V \setminus \{R\}} g_p \), and so the map given by \( g \mapsto \prod_{p \in V \setminus \{R\}} g_p \big|_{\pi^{-1}(p)} \) is a homomorphism from \( G(T) \) to \( \prod_{p \in V \setminus \{R\}} G^p \). It is easily verified that this map is an isomorphism, from which it follows that \( G \) has property (P).

There are many mild conditions under which \( U_L(M,N) \) leaves invariant no proper non-empty subtree of \( T \) and fixes non end of \( T \). For example: \( M \) and \( N \) contain no fixed points.

We will presently cite a result in J. P. Serre’s book [13], but we must do so with care since Serre’s definition of a graph differs from ours in that loops (arcs \( a \) which satisfy \( o(a) = t(a) \)) and multiple arcs (distinct arcs \( a, a' \) which satisfy \( o(a) = o(a') \) and \( t(a) = t(a') \)) are permitted. Trees in [13] cannot contain loops nor multiple arcs, and so we may use the term tree without ambiguity (see [13, pp. 13–17]).

Suppose \( T \) is a tree and \( G \) acts on \( T \) without inversion. Following [13] pp. 25], \( G/T \) consists of a set of vertices and a set of edges. The vertices (resp. edges) are the orbits of \( G \) on the vertices (resp. edges) of \( T \). Notions like adjacency and arcs extend naturally to \( G \setminus T \). One should note, however, that \( G \setminus T \) may not be a graph, since there may be more than one edge between two given vertices.

**Theorem 14 ([13, Corollary 1 in Section I.5.4]).** Suppose \( T \) is a tree, and let \( G \) be a group acting on \( T \) without inversion. Let \( R \) be the group generated by all vertex stabilisers \( G_v \), \( v \in VT \). Then \( R \) is a normal subgroup of \( G \), and \( G/R \) is isomorphic to the fundamental group of \( G \setminus T \).

It is an immediate consequence of this theorem that \( G = R \) if and only if \( G \setminus T \) is a tree (see [13] Exercise 2 in Section I.5.4, for example).

As before, let \( X \) and \( Y \) be finite or infinite sets with cardinality at least two, \( M \leq \Sym (X) \) and \( N \leq \Sym (Y) \), denote the \( (|X|,|Y|) \)-biregular tree by \( T \) and choose some legal colouring \( L \) of \( T \). Our next lemma follows from Propositions 3 and 7. In particular, we have that \( U_L(M,N) \setminus T \) contains no loops or multiple edges.

**Lemma 15.** If \( m \) (resp. \( n \)) denotes the number of orbits of \( M \) (resp. \( N \)), then \( U_L(M,N) \setminus T \) is the complete bipartite graph \( K_{m,n} \).

If \( M \) and \( N \) are transitive, then Lemma 15 and [13] Section 4.1, Theorem 6] imply that \( G := U_L(M,N) \) has an amalgamated free product structure,

\[
G = G_u \ast_{G_{(u,v)}} G_v,
\]

where \( u \) and \( v \) are any two adjacent vertices in \( T \).

Recall that \( M \boxtimes N \) is the subgroup of \( \Sym (V_T) \) which is induced by \( U_L(M,N) \). Following [4] Section 4.1 and [10], we say a permutation group \( G \leq \Sym (V) \) generated by point stabilisers if \( G = \langle G_\alpha : \alpha \in V \rangle \).

**Theorem 16.** Suppose \( M \) and \( N \) are permutation groups of (not necessarily finite) degree at least two, both groups are generated by point stabilisers and at least one group is nontrivial. Then \( M \boxtimes N \) is simple if and only if \( M \) or \( N \) is transitive.

**Proof.** Suppose \( M \) is a subgroup of \( \Sym (X) \) and \( N \) is a subgroup of \( \Sym (Y) \); let \( T \) be the \( (|X|,|Y|) \)-biregular tree, and let \( L \) be a legal colouring of \( X \) and \( Y \). Write \( G := U_L(M,N) \). We claim that \( (G_v)^+ = G_v \) for all \( v \in VT \). Fix \( v \in VT \). By Lemma 3, the group \( G_v \big|_{B(v)} \) is generated by point stabilisers. Moreover, \( G_{(v,w)} \big|_{B(v)} \leq (G_v)^+ \big|_{B(v)} \) for all \( w \in B(v) \). Hence \( (G_v)^+ \big|_{B(v)} = G_v \big|_{B(v)} \). In
particular, \((G_v)^+\) and \(G_v\) both have the same orbits on \(B(v)\). Since \(G_{(v,w)} \leq (G_v)^+\) for all \(w \in B(v)\), our claim follows.

Hence \(R := \langle G_v : v \in VT \rangle \leq G^+\). Since \(M\) or \(N\) is nontrivial, \(R\) is nontrivial. Moreover, by Theorem \[14\] \(R\) is a normal subgroup of \(G\), and \(G = R\) if and only if \(G\)'T is a tree.

If \(M\) and \(N\) are intransitive, then \(G\)'T is not a tree by Lemma \[13\] and so \(R\) is a nontrivial proper normal subgroup of \(G\). Conversely, suppose \(M\) or \(N\) is transitive. Then \(G\)'T is a tree, and so \(G = R = G^+\). Let \(\epsilon\) be an end in \(T\), and suppose \(T'\) is some non-empty proper subtree of \(T\). Now \(G_v|_{B(v)}\) is a tree for all \(v\) in one part of the bipartition of \(T\), and so we may choose such a vertex \(v\) from \(VT \setminus VT'\). Clearly \(G_v\) does not fix \(\epsilon\), nor does \(G_v\) leave \(T'\) invariant. Thus, by Theorem \[13\] \(G = G^+\) is simple. Since \(M \boxtimes N \cong G\), the result follows.

\textbf{Corollary 17.} If \(M\) and \(N\) are non-regular primitive permutation groups, then \(M \boxtimes N\) is simple.

\textbf{Corollary 18.} Under the conditions of Theorem \[16\] the group \(U_G(M, N)\) is simple if and only if \(M\) or \(N\) is transitive.

5. Permutational Properties

\textbf{Theorem 19.} Given nontrivial permutation groups \(M \leq \text{Sym}(X)\) and \(N \leq \text{Sym}(Y)\), the permutation group \(M \boxtimes N \leq \text{Sym}(V_Y)\) satisfies:

(i) \(M \boxtimes N\) is transitive if and only if \(M\) is transitive; and

(ii) \(M \boxtimes N\) is primitive if and only if \(M\) is primitive but not regular, and \(N\) is transitive.

Compare these properties with those of the unrestricted wreath product \(M \mathrm{Wr}_Y N \leq \text{Sym}(X^Y)\) in its product action.

(i) \(M \mathrm{Wr}_Y N\) is transitive if and only if \(M\) is transitive;

(ii) \(M \mathrm{Wr}_Y N\) is primitive if and only if \(M\) is primitive but not regular, and \(N\) is transitive and finite;

Despite this striking similarity, \(M \mathrm{Wr}_Y N\) and \(M \boxtimes N\) distort the actions of \(M\) and \(N\) in opposite ways. This is most apparent when \(M\) is subdegree-finite and primitive but not regular and \(N\) is subdegree-finite and transitive. All nontrivial orbital graphs of \(M \boxtimes N\) are tree-like: they are locally finite, connected and have infinitely many ends; on the other hand nontrivial orbital graphs of \(M \mathrm{Wr}_Y N \leq \text{Sym}(X^Y)\) are locally finite and connected but with at most one-end (see \[13\] Theorem 2.4).

\textbf{Proof of Theorem 19.} Part \[4\] is Theorem \[1\]; \[4\] is Theorem \[1\]. Write \(G := U_G(M, N)\) for some legal colouring of \(X\) and \(Y\), let \(T\) be the \((|X|,|Y|)\)-biregular tree.

Suppose that \(N\) is not transitive on \(Y\). Fix \(v \in V_X\) and distinct vertices \(w, w' \in B(v) \subseteq V_Y\). Choose \(w' \in B(w)\) such that \(L(w, v)\) and \(L(w, v')\) lie in distinct orbits of \(N\). By Proposition \[3\] the vertex \(v'\) does not lie in the orbit \(Gv\). Let \(\Gamma\) be the orbital digraph whose vertex set is \(V_Y\) and arc set is \(G(w, w')\). Notice that if two vertices in \(\Gamma\) are adjacent, then their distance in \(T\) is two. Therefore, if \(x \in V_Y\) lies in the connected component of \(T \setminus \{w\}\) which contains \(v'\), then any path in \(\Gamma\) from \(x\) to \(w\) must contain an arc (in \(\Gamma\)) between \(w\) and some vertex \(w'' \in B(v')\). But this implies that \((w, w'')\) or \((w'', w)\) lie in \(G(w, w')\), and both possibilities require that \(v' \in Gv\), which is false. Hence \(G\) has a nontrivial orbital digraph on \(V_Y\) which is not connected, and so the action of \(G\) on \(V_Y\) is not primitive.

Suppose \(M\) is not primitive on \(X\). If \(M\) is not transitive then \(G\) is not transitive (and therefore not primitive) on \(V_Y\) by Proposition \[3\]. Suppose, then, that \(M\) is imprimitive. By D. G. Higman’s Theorem (\[9\] 1.12), there exists an orbital graph
of $M$ which is nontrivial and not connected. Choose a vertex $v \in V_X$. By Lemma 5, some nontrivial orbital graph $\Delta$ of $G_v |_{B(v)}$ is not connected; let $\{w, w'\}$ be an edge in $\Delta$. Now take $\Gamma$ to be the orbital graph whose vertex set is $V_Y$ and whose edge set is the orbital $G\{w, w'\}$. We claim that $\Gamma$ is not connected. Indeed, any pair of adjacent vertices in $\Gamma$ are at distance 2 in the tree $T$. Therefore any $\Gamma$-path between distinct vertices in $B(v)$ contains only vertices in $B(v) = V_\Delta$. It follows that $\Gamma$ is not connected, and thus that $G$ is not primitive on $V_Y$.

Suppose that $M$ is regular. Let $\Gamma'$ be a new graph on $V_Y$, in which vertices are adjacent if and only if their distance in $T$ is two. Thus $\Gamma'$ has connectivity one. If $|X| = 2$, then $\Gamma'$ is a regular tree and the bipartition of $\Gamma'$ is a system of imprimitivity for $G$ on $V_Y$ so the action of $G$ on $V_Y$ is not primitive. If $|X| \geq 3$, then the lobes of $\Gamma'$ contain at least three vertices, $G|_{V_Y} \leq \text{Aut } \Gamma'$ acts vertex transitively, and the block-cut-vertex tree of $\Gamma'$ is $T$. Because $M$ is regular, $G_{w,v} = G_{w',v}$. By [13] Theorem 2.5), the action of $G$ on $V_Y$ is not primitive.

Conversely, let us now suppose that $M$ is primitive and not regular, and $N$ is transitive. Let $\sim$ be a nontrivial $G$-invariant equivalence relation on $V_Y$. Choose distinct $w, w' \in V_Y$ with $w \sim w'$. Let $v$ denote the vertex adjacent to $w'$ in the path $[w, w']_T$, and let $w''$ denote the vertex adjacent to $v$ in $[w, w']_T$. Since $G_v |_{B(v)}$ is permutation isomorphic to $M$, it is primitive and not regular on $B(v)$, and so there is an element $h \in G_{v,w''}$ which does not fix $w'$. By Lemma 11 there exists $g \in G$ such that $g$ fixes the half-tree $T_{(w'', v)}$ pointwise (so it fixes $w$) and $g|_{T_{(v, w'')}} = h|_{T_{(v, w'')}}$. Since $w'$ lies in $T_{(v, w'')}$, this element $g$ does not fix $w'$. Hence $w' \sim w = gw \sim gw'$, and $d_T(w', gw') = 2$. However, $G_v |_{B(v)}$ is primitive, so $\sim$ must be a universal relation on $B(v)$. Since $N$ is transitive, it follows that $\sim$ is universal on $V_Y$. Hence there are no proper non-trivial $G$-invariant equivalence relations on $V_Y$, so the action of $G$ on $V_Y$ is primitive.

Recall that a permutation group is subdegree-finite if any point stabiliser has only orbits of finite length.

**Proposition 20.** Suppose $M$ and $N$ are nontrivial permutation groups. Then $M \boxtimes N$ is subdegree-finite if and only if $M$ is subdegree-finite and all orbits of $N$ are finite.

**Proof.** Let $G := U_2(M, N)$. Suppose $M$ is subdegree-finite and all orbits of $N$ are finite. Choose distinct $w, w' \in V_Y$ and let $w_0w_1 \cdots w_n$ denote the path $[w, w']_T$. Since $G_{w,w'} = G_{w_0, \ldots, w_n}$ we have

$$|G_{w,w'}| = |G_{w_0} : G_{w_0, \ldots, w_n}| \leq |G_{w_0}w_1| \prod_{i=1}^{n-1} |G_{w_{i-1}, w_i}w_{i+1}|.$$  

Now $|G_{w_0}w_1|$ is finite because $N$ has only finite orbits, and each $|G_{w_{i-1}, w_i}w_{i+1}|$ is finite because both $M$ and $N$ are subdegree-finite. Hence the action of $G$ on $V_Y$ is subdegree-finite.

On the other hand, if $M$ is not subdegree-finite, then for all $v \in V_X$ there exist $w, w' \in B(v) \subseteq V_Y$ such that $G_{w,v}w'$ is infinite. Hence $G_{w,v}w'$ is not finite. If $N$ has an infinite orbit, then for all $w \in V_Y$ there exists $v \in B(w) \subseteq V_X$ such that $G_{w,v}$ is infinite. Hence for any $w' \in B(v) \setminus \{w\}$, the orbit $G_{w,v}$ is not finite.

6. **Topological properties of the product**

If we bestow $\text{Sym}(X)$ and $\text{Sym}(Y)$ with their respective permutation topologies, then $M \boxtimes N$ under the permutation topology of $\text{Sym}(V_Y)$ preserves some topological properties of $M$ and $N$ but does not preserve discreteness.
Theorem 21. Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are nontrivial and closed. Then $M \Join N$ is closed in $\text{Sym}(V_T)$ and the following are equivalent:

(i) every point stabiliser in $M \Join N$ is compact in $\text{Sym}(V_T)$;

(ii) $N$ is compact and every point stabiliser in $M$ is compact.

This theorem is an immediate consequence of Lemma 10 and the following two lemmas.

Lemma 22. Let $T$ be a tree with no vertex of valence one. Suppose $G \leq \text{Aut} T$ is closed and fixes setwise the two parts $V_1, V_2$ of the bipartition of $T$. Then $G|_{V_i}$ is closed in $\text{Sym}(V_i)$ for $i = 1, 2$.

Proof. We prove that $G|_{V_1}$ is closed in $\text{Sym}(V_1)$; a symmetric argument will show that $G|_{V_2}$ is closed in $\text{Sym}(V_2)$. Let $A := (\text{Aut} T)|_{V_1}$, and note that $A$ is closed in $\text{Sym}(V_T)$. Let $\varphi : A \to A|_{V_1}$ be the map taking $g \in A$ to $g|_{V_1} \in A|_{V_1}$. Since the tree $T$ contains no vertices of valence one, the kernel of this map is trivial. One may then easily verify it is an isomorphism. We must show that $\varphi(G)$ is closed in $\text{Sym}(V_1)$.

Let us first prove that $\varphi(A)$ is closed in $\text{Sym}(V_1)$. Let $A$ be the graph with vertex set $V_1$, in which two vertices $w, w' \in V_1$ are adjacent if and only if $d_T(w, w') = 2$. We claim that $\varphi(A) = \text{Aut} A$. Since $G$ is closed in $\text{Sym}(V_1)$, it will follow from our claim that $\varphi(A)$ is closed. Since $\varphi(A)$ preserves the distance metric on $T$, we have that $\varphi(A) \leq \text{Aut} A$. Suppose then that $g \in \text{Aut} A$. We will find $\hat{g} \in G$ such that $g = \varphi(\hat{g}) \in \varphi(A)$. If $w, w' \in V_1$ lie at distance two in $T$, then there is a unique vertex between them, which we denote by $b(w, w') \in V_2$. Since $T$ has no vertices of valence one, for any $v \in V_2$ there exist $w, w' \in V_1$ such that $v = b(w, w')$. Notice that for all $w, w', y, y' \in V_1$ we have $b(w, w') = b(y, y')$ if and only if $b(\varphi(w), \varphi(w')) = b(\varphi(y), \varphi(y'))$. Therefore, we can extend the action of $g$ on $V_1$ to a well-defined action of $g$ on $V_T$ by specifying $\varphi(b(w, w')) := b(\varphi(w), \varphi(w'))$ for all $b(w, w') \in V_2$. Moreover, one may easily verify that $e \in ET$ if and only if $ge \in ET$, so this action of $g$ on $VT$ induces an automorphism $\hat{g}$ of $T$ which preserves $V_1$ setwise. Since $\varphi(\hat{g})$ and $g$ agree on $V_1$, it follows that $\varphi(\hat{g}) = g \in \text{Aut} A$. Hence $\text{Aut} A = \varphi(A)$, and $\varphi(A)$ is closed in $\text{Sym}(V_1)$.

Now show that $\varphi$ maps open sets in $A$ to open sets in $\varphi(A)$. Let $\Phi \subseteq VT$ be finite. If $v \in V_2$, then there exist $w, w' \in V_1$ such that $v = b(w, w')$ and so $A_{\Phi_{\Phi'}} \subseteq A_{\Phi}$. Hence there exists a finite set $\Phi' \subseteq V_1$ such that $A_{\Phi'} \subseteq A_{\Phi}$. Now $\varphi(A_{\Phi'}) = (\varphi(A))(\Phi')$, and the latter is open in $\varphi(A)$. Hence, given any $a \in A_{\Phi}$, we have that $a \in aA_{\Phi'}$, so $\varphi(a)$ lies in the open set $\varphi(a)(\varphi(A))(\Phi') \subseteq \varphi(A_{\Phi'})$. Hence $\varphi(A_{\Phi'})$ is open in $\varphi(A)$. Since any open set in $A$ is a union of cosets of finite subsets of $VT$, it follows that the image under $\varphi$ of any open set in $A$ is open in $\varphi(A)$.

By assumption $G \setminus A$ is open in $A$, so $\varphi(G) \setminus \varphi(A) = \varphi(G) \setminus A$ is open in $\varphi(A)$ and hence $\varphi(G)$ is closed in $\varphi(A)$. Since $\varphi(A)$ is closed in $\text{Sym}(V_1)$, the result follows.

Lemma 23. Let $T$ be a tree with no vertex of valence one. Suppose $G \leq \text{Aut} T$ is closed and fixes setwise the two parts $V_1, V_2$ of the bipartition of $T$, and for all $v \in VT$ the group $G_v|_{B(v)}$ is closed. Then the following are equivalent:

(i) for all $v \in V_1$ and $w \in V_2$, all point stabilisers in $G_v|_{B(v)}$ are compact and $G_{\pi_v}|_{B(w)}$ is compact;

(ii) all point stabilisers in $G|_{V_2}$ are compact.
Proof. Write $H := G|_{V_2}$. Then $H$ is closed by the preceding lemma. Now $\Box$ is true if and only if all orbits of $G_w|_{B(w)}$ are finite and all suborbits of $G_v|_{B(v)}$ are finite. On the other hand $\Box$ is true if and only if $H$ is subdegree-finite. An argument similar to that used in the proof of Proposition 20 shows that $H$ is subdegree-finite if and only if, for all $v \in V_1$ and $w \in V_2$, all orbits of $G_w|_{B(w)}$ are finite and $G_v|_{B(v)}$ is subdegree-finite. Hence $\Box$ and $\Box$ are equivalent.

**Theorem 24.** Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$ are transitive. If

(i) $M$ is compactly generated and every point stabiliser in $M$ is compact; and

(ii) $N$ is compact,

then $M \bowtie N$ is compactly generated.

**Proof.** Suppose $\Box$ and $\Box$ are true. Let $L$ be a legal colouring of the $([X], [Y])$-biregular tree $T$ and write $G := U_L(M, N)$. Fix $v \in V_X$ and a neighbour $w \in B(v) \subseteq V_Y$. Let $S$ be a compact generating set for $H := G_w|_{B(v)}$. Since $H_w$ is open, $\bigcup_{s \in S} sH_w$ is an open cover of $S$ and so we may choose some finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that $S \subseteq \bigcup_{i=1}^n s_iH_w$. Hence $H = \langle H_w, s_1, \ldots, s_n \rangle$. For each integer $i$ satisfying $1 \leq i \leq n$, choose $g_i \in G_v$ such that $g_i|_{B(v)} = s_i$. Write $S' := \{g_1, \ldots, g_n\}$. If $K := \langle G_w \cup S' \rangle$, then $K_w$ is transitive on $B(w)$ and $K_v$ is transitive on $B(v)$, so $K$ has two orbits, $V_X$ and $V_Y$, on $V_T$. Thus $G_w \leq K \leq G$ and $K_w = V_Y = Gw$, so $K$ and $G$ are equal.

We have that $M \bowtie N = G|_{V_Y} = \langle G_w|_{V_Y} \cup S'|_{V_Y} \rangle$. Now $N$ is a compact subgroup of the Hausdorff group $\text{Sym}(Y)$, so $N$ is closed. Similarly, every point stabiliser in $M$ is closed, so $M$ is closed. Therefore, by Lemma 23 every point stabiliser in $G|_{V_Y}$ is compact. Hence $G_w|_{V_Y} \cup S'|_{V_Y}$ is a compact set and $G|_{V_Y}$ is compactly generated. $\Box$

Recall that a permutation group is **semi-regular** if every point stabiliser is trivial. A regular permutation group is transitive and semi-regular.

**Theorem 25.** Suppose $M \leq \text{Sym}(X)$ and $N \leq \text{Sym}(Y)$. Then $M \bowtie N$ is discrete if and only if $M$ and $N$ are semi-regular.

**Proof.** Let $L$ be a legal colouring of the $([X], [Y])$-biregular tree $T$ and write $G := U_L(M, N)$. Suppose $M$ and $N$ are both semi-regular. If $w, w' \in V_Y$ are at distance two in $T$, there is a unique element $v \in V_X$ which is adjacent to both vertices, and $G_{w, w'} = G_{w, v}G_{v, w'}$. By Lemma 6, $G_{w, v}$ fixes $B(w)$ and $B(v)$ pointwise, and, since $T$ is connected, $G_{w, v}$ must therefore fix $VT$ pointwise; it is thus trivial. Hence $(M \bowtie N)|_{w, w'}$ is trivial and $M \bowtie N$ is discrete in $\text{Sym}(V_Y)$.

Let $\Phi \subseteq V_Y$ be any finite set. Suppose $M$ is not semi-regular, and choose $x \in X$ such that $M_x$ is non-trivial. Choose any vertex $w' \in V_Y$ such that $\Phi \subseteq w'$. Since $L|_{A(w')} \in B(w')$. Choose a vertex $v'' \in V_X$ which does not lie in $\mathcal{C} \cup B(v')$. Since $L|_{A(w')} \in B(w')$. Choose $\sigma \in M_x \setminus \{1\}$. By Lemma 6, there exists $h \in G_v$ such that $L|_{A(v)}h|_{A(v)}L|_{A(v)}^{-1} = \sigma$. Hence $h \in G_{v, w}$ and $h \notin G_{B(v)}$. By Lemma 11, there exists $g \in G_{v, w}$ such that $g|_{T(w, v)}$ is trivial and $g|_{T(v, w)} = h|_{T(v, w)}$. Hence $g \in G_{v, N}$ is non-trivial. A similar argument shows that if $N$ is not semi-regular, then $G_{v, N}$ is non-trivial. It follows that if $M$ or $N$ is not semi-regular, then $M \bowtie N$ is not discrete. $\Box$
7. An Uncountable Set of Compactly Generated Simple Groups

In [1], \( \mathcal{S} \) is defined to be the set of non-discrete, compactly generated, topologically simple, totally disconnected, locally compact groups. P.-E. Caprace and T. De Medts remark that they do not know whether it is sufficient; let us now fix such a prime \( p \).

By Corollary 9, then \( G = H/N \) is infinite and torsion so it contains a non-trivial normal subgroup of \( G \).

Theorem 27 ([7, Theorem 28.7]). For every sufficiently large prime number \( p \), there is a continuum of pairwise non-isomorphic infinite groups of exponent \( p \) all of whose proper non-trivial subgroups have order \( p \).

In [7, pp. 304] Ol’shanskiı notes that taking \( p > 10^{75} \) in the above theorem is sufficient; let us now fix such a prime \( p \). The groups whose existence is guaranteed by Theorem 27 we shall call Taski-Ol’shanskiı Monsters.

Recall from [13, pp. 58] the definition of Serre’s property (FA). Suppose \( G \) is a group acting on a tree \( T \). It acts without inversion if there is no edge \( \{v, w\} \in T \) which satisfies \( \{v, w\} \in G(v, w) \).

By Theorem 27 we shall call \( \mathcal{S} \) the set of non-discrete, compactly generated, totally disconnected, locally compact.

In [5] the known examples of groups in \( \mathcal{S} \) are grouped into five broad classes. The examples, of which there are countably many, that are structurally similar to the groups we describe here are due to M. Burger and S. Mozes ([2]), R. G. Möller and J. Vonk ([10]), and C. Banks M. Elder and G. A. Willis ([11]).

Recall from [13, Example 6.3.1], a group \( G \) has property (FA) if it acts without inversion.

For denumerable groups, having property (FA) is equivalent to being finitely generated but not an amalgam, with no non-trivial quotient isomorphic to \( Z \) (see [13, Theorem 15]). A particularly rich class of groups (for our purposes) which all have property (FA) is the class of finitely generated torsion groups.

Proposition 26 ([13, Example 6.3.1]). A finitely generated torsion group has property (FA).

Lemma 28. Let \( M_1, M_2, N_1, N_2 \) be non-trivial permutation groups. Suppose \( M_1 \) has property (FA), and no non-trivial quotient of \( M_1 \) is isomorphic to any subgroup of \( M_2 \) or \( N_2 \). Then \( M_1 \otimes N_1 \) and \( M_2 \otimes N_2 \) are not (abstractly) isomorphic.

Proof. For \( i = 1, 2 \), suppose \( M_i \leq \text{Sym}(X_i) \) and \( N_i \leq \text{Sym}(Y_i) \) and let \( T_i \) denote the \(|X_i|, |Y_i|\)-biregular tree. Let \( L_i \) be a legal colouring of \( X_i \) and \( Y_i \), and write \( G := \mathcal{U}_{L_1}(M_1, N_1) \) and \( H := \mathcal{U}_{L_2}(M_2, N_2) \). It suffices to show that \( G \) is not isomorphic to \( H \). Let us suppose, for a contradiction, that \( G \) and \( H \) are isomorphic. By Corollary 9, \( G \) contains a subgroup which is isomorphic to \( M_1 \), and so \( M_1 \) is isomorphic to subgroup \( K \) of \( H \).

On one hand, \( K \) cannot fix any vertex in \( T_2 \). Indeed, if \( K \) fixes some vertex \( v \in T_2 \), then \( K/K(B(v)) \cong K|_{B(v)} \leq H|_{B(v)} \). Since \( H|_{B(v)} \) is isomorphic to either \( M_2 \) or \( N_2 \), this is only possible if the quotient \( K/K(B(v)) \) is trivial; that is, if \( K \) fixes \( B(v) \) pointwise. But if this is so, then we can repeat this argument, since \( K \) now fixes \( w \in B(v) \). Since \( T_2 \) is connected, it follows that \( K \) must be trivial which is absurd.
On the other hand, $K$ must fix a vertex in $T_2$, since $K$ acts on $T_2$ without inversion and has property (FA).

**Theorem 29.** There are uncountably many non-isomorphic groups which are totally disconnected, locally compact, compactly generated, simple and not discrete.

**Proof.** Let $T$ be a Taski-Ol’Shanskiı Monster, and let $S_3$ denote the symmetric group on 3 elements. Fix a non-trivial proper subgroup $H \leq T$. The group $T$ acts faithfully and transitively on the coset space $X := (T : H)$, and so we think of $T$ as being a subgroup of $\text{Sym}(X)$, and bestow upon it the permutation topology. Point stabilisers in $T$ are finite, so $T$ is totally disconnected and locally compact. Moreover, $T$ is finitely generated, so it is compactly generated.

The group $T \boxtimes S_3$ is a subgroup of $\text{Sym}(V_Y)$. Under the permutation topology, it is totally disconnected because its action is faithful. It is locally compact by Theorem 21, it is compactly generated by Theorem 24, and it is non-discrete by Theorem 25.

Any point stabiliser in $T$ is a maximal subgroup, so $T$ is generated by point stabilisers; so too is $S_3$. Hence, by Theorem 16, the group $T \boxtimes S_3$ is simple.

If $R$ is another Taski-Ol’Shanskiı Monster which is not isomorphic to $T$, then $T \boxtimes S_3$ and $R \boxtimes S_3$ are not isomorphic by Lemma 28. The theorem now follows immediately from Theorem 27. □

Using a result by R. G. Möller, we now have a pleasant method for constructing examples of non-discrete simple groups which are totally disconnected, locally compact, and compactly generated.

(i) Take any locally finite, vertex-transitive, connected graph $\Gamma$ which contains at least two vertices, and choose a closed vertex-transitive group $M \leq \text{Aut} \Gamma$ which is generated by vertex-stabilisers. In the permutation topology of $\text{Sym}(V\Gamma)$, such a group will be totally disconnected and locally compact, and all vertex stabilisers will be compact and open. By Möller’s result ([11, Corollary 1]) the group $M$ is compactly generated.

For example, one could take $M$ to be a closed subdegree-finite non-regular primitive permutation group of degree at least three.

(ii) Choose any finite transitive permutation group $N \leq \text{Sym}(Y)$ of degree at least two, which is generated by point stabilisers. For example, one could take $N$ be any finite primitive non-regular permutation group of degree at least three.

(iii) Choose any legal colouring $L$ of the $(|X|, |Y|)$-biregular tree $T$, and let $V_Y$ be the part of the bipartition of $T$ whose elements have valence $|Y|$.

(iv) The group $M \boxtimes N \leq \text{Sym}(V_Y)$ is simple (by Theorem 16), and with the permutation topology of $\text{Sym}(V_Y)$ it is totally disconnected (because it is faithful), locally compact (by Theorem 21), compactly generated (by Theorem 24) and non-discrete (by Theorem 25). Point stabilisers in $M \boxtimes N$ are compact.

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