Abstract—In this paper, we concentrate on the peak age of information (PAoI) in a discrete-time slotted ALOHA network comprised of $M$ buffer-less nodes, capable of keeping just one packet at each slot. In this network, a collision happens whenever at least two nodes transmit their packets simultaneously. Thus, there is some interaction among queues, and the transmission delay of a packet might prolong more than one slot. The packets are generated at each node stochastically and independently. The nodes follow the preemptive queueing policy. In this network, we propose a trellis-based model to analyze packet system time and derive the exact average PAoI of symmetric and asymmetric slotted ALOHA networks. We confirm our analysis by simulation results in different conditions. By exploiting numerical results, we optimize the average PAoI of the symmetric slotted ALOHA network, find the PAoI-constrained regions, and compare the PAoI of slotted ALOHA networks with and without retransmission of the failed packets.

Index Terms—Age-of-information (AoI), peak AoI (PAoI), PAoI-constrained region, preemptive policy, slotted ALOHA.

I. INTRODUCTION

Status updating is a crucial task to be done in modern communication networks, e.g., vehicular ad hoc networks (VANETs), sensor networks, etc. In fact, in these networks, in order to decide correctly, the decision-maker (destination) needs updated status information. Age-of-information (AoI) is a performance metric introduced in [1], which considers not only the end-to-end packet delay but also the time duration that the destination remains outdated. At any time $t$, the AoI function $\Delta(t)$ is defined as the difference between $t$ and the generation moment of the freshest packet delivered at the destination before $t$.

In [2], the peak AoI (PAoI) is introduced as another related metric that only concentrates on local peaks of $\Delta(t)$. The merit of the PAoI metric is that it shows the maximum time interval that the destination is outdated. Obviously, regarding different effective factors, PAoI is a random variable, and we focus on its average in our work. If the destination is a decision-making node that decides continually based on the available information, PAoI determines the worst-case situation because it corresponds to the maximal outdated information used for decision making.

From a historical point of view, primary works on AoI considered a communication link comprised of a transmitter-receiver pair and studied AoI for different queueing policies. The first-come-first-serve (FCFS) policy was analyzed in [3] for M/M/1, M/D/1, and D/M/1 queues and seemed to be inefficient when the packet arrival rate is relatively high. Preemptive and non-preemptive last-come-first-serve (LCFS) policies were investigated in [4] and [5] with exponentially and Gamma distributed service times, respectively. The authors in [6] showed that in multi-server queues, the preemptive policy is optimal when service times are distributed exponentially. The authors in [7] showed that by using infinite incremental redundancy and finite redundancy hybrid ARQ (HARQ), the blocking policy is beneficial over the preemptive policy, while the authors in [8] showed that in time-slotted scenarios equipped with HARQ, each policy could be superior to the other one based on network conditions.

Recently, analysis of AoI in multi-source or multi-hop networks has got attention. AoI in multi-hop networks was investigated in some works (e.g., [9], [10]). Scheduled transmissions in broadcast networks were investigated in several works (e.g., [11], [12], [13], [14]), and in [12], it was shown that Whittle’s index and max-weight policies are able to achieve near optimal AoI among other scheduling policies.

Different papers studied AoI in some well-known multiple access protocols. The authors in [15] and [16] compared AoI in TDMA and FDMA, while the authors in [17] compared AoI in non-orthogonal multiple access (NOMA) and orthogonal multiple access (OMA). In [18], the authors investigated AoI of an independent queue with and without packet management in a time-slotted scenario, and then, based on simulation results, they studied AoI in round-robin, work-conserving scheduled access and random access protocols.

There are several works that analyzed AoI in random access protocols. These works can be divided into two main categories. The first category includes the papers assuming that packet generation is at will, i.e., a fresh version of information is available at any desired time. Within the first category, by proposing AoI-dependent transmission schemes where the channel access probability of each node depends on the amount of the corresponding AoI at the destination, [9], [20], [21], [22] have redesigned the original slotted ALOHA protocol to achieve a smaller AoI in massive random access networks. The authors in [23] followed a game-theoretical approach in a carrier sense multiple
access (CSMA) network with both AoI-interested and throughput-interested nodes, which try to maximize their own utilities. The authors in [24] compared the slotted ALOHA protocol with scheduled transmissions by considering that each node replaces the available packet with a new one at each time slot. In [25], the authors studied AoI under randomized scheduling policies in a wireless communication network. In [26], AoI of the irregular repetition slotted ALOHA network has been investigated. It is important to note that in many practical networks, the generation-at-will assumption seems to be impossible. For example, in VANETs, some sensors generate an information packet when a special event or an abnormality happens, e.g., when an unexpected velocity change is detected. As these special events happen stochastically, a fresh version of information is not available at any desired time.

The second category of works studying AoI in random access protocols includes papers with stochastic packet generation processes. In this category, we will encounter the challenge of interactive queueing systems, i.e., the transmission time (service time) of a packet at a typical node depends on the state of others. For example, if transmissions of two nodes have collided in a time slot, both nodes would definitely have a packet to transmit in the next slot if retransmission is allowed, while they do not always have a packet to transmit. Thus, the analysis of each node could not be isolated from the analysis of other nodes. Moreover, if nodes adopt the preemptive policy and consider retransmitting the packets with failed transmission (i.e., failed packets), due to stochastic packet arrivals, there is a possibility that a failed packet is preempted by the arrival of a new packet. The interaction among nodes and probable packet preemptions make the AoI challenging. Actually, packet preemption can be interpreted as a time dependency between the packets, as every new arrival readjusts the age of the next packet to be received at the destination (i.e., its end-to-end delay). It is worth noting that in the first category, if each node transmits a fresh packet with a constant probability at each slot, neither the challenge of interactive queueing nor the challenge of probable packet preemption does exist.

The challenge of interactive queueing has led to some approximate analyses of random access protocols in the aforementioned last category, and most works could not find the exact AoI or PAoI in their studied scenarios. In [27], the authors found an upper bound on AoI in the CSMA environment as they considered that all nodes always transmit a packet, no matter have a real one or not. Similarly, in [28], the authors studied the case that except for one node, others are in a saturated situation (i.e., always ready to transmit) to obtain the average AoI in the worst case. Authors in [29] have provided asymptotic results for optimizing AoI in random access channels when the number of sources approaches infinity. In [30] and [31], the authors have studied the slotted ALOHA network by the approximation that each node sees the traffic load of other nodes on average. In [32] and [33], the authors have analyzed a CSMA environment with symmetric nodes while they have discarded the dependency between the states of the nodes. There are only a few works that found the exact AoI or PAoI in interactive systems with stochastic packet arrivals. In this respect, the authors in [34] studied a multiple access network consisting of two nodes where packets are arrived based on a Bernoulli process in one of them, and in the other one, packet generation is at will. Moreover, the authors in [35] investigated PAoI in the frameless slotted ALOHA network in the case that retransmission of the failed packets is not allowed, so the interaction is not challenging.

In this paper, we consider a slotted ALOHA network with erasure channels, comprised of $M$ buffer-less nodes within the coverage of each other and with Bernoulli packet arrival process while the preemptive policy is considered at each node and retransmission is allowed. We evaluate the exact average PAoI in this scenario. In most practical situations, the number of nodes within the transmission range of each node (i.e., the number of interactive queues) is finite. Hence, evaluating the exact average PAoI in finite interactive queueing systems is practically valuable. The most important challenge in this network is the interaction among queues. As mentioned before, most analytic works on AoI with random access MAC protocols do not completely face the challenge of interactive queues, either by considering some assumptions which lead to the same successful transmission probability for each node in different time slots (like [27], [28]), or by considering some approximations like that each node sees the traffic load of the other nodes in average. Although these approximations could be acceptable in some special cases, for example, when the number of nodes is relatively high, the difference between PAoIs in the approximated and real scenarios is not negligible in general.

The main challenge in our analysis, i.e., interaction among queues, has been considered before in the analytical evaluation of the stability region, i.e., the achievable throughput region, in the slotted ALOHA network in a few cases, e.g., the case of two nodes [36] and three nodes [37]. In [38], an approximate approach for finding the delay in random access networks has been presented but does not completely model the interaction among the queues. However, due to the preemption of packets in our scenario, delay analysis for evaluating AoI and PAoI could become even harder than the classical queueing analysis for random access protocols. Thus, our main goal is to design an analytical framework that not only deals with interaction among queues but also considers packet preemption and failed packet retransmission in the case of queues with limited-size buffers and stochastic arrivals. Our main contributions are as in the following:

- We propose a trellis-based model for the analysis of system time under interaction among queues, suitable for interactive scenarios, esp., those in which finite buffer nodes adopt the preemptive policy as well as stochastic packet arrival.
- By exploiting the proposed trellis-based model, we derive the average PAoI in the asymmetric slotted ALOHA network analytically and then analyze the symmetric one (where all nodes have the same parameters) with much lower complexity.
- We find the PAoI-constrained regions (i.e., regions over arrival rates and transmission probabilities where the average PAoI of different nodes is constrained to be less
than a specific threshold) numerically. By observing the results, we see that the regions are convex.

- By numerical analysis, we minimize the average PAoI in the symmetric network and study the effect of the number of nodes on this metric. By numerical results, we show that in order to minimize the average PAoI of the symmetric slotted ALOHA network, \((1,1/M)\) is the optimal choice for the values of arrival rate and transmission probability, where \(M\) is the number of nodes. However, these values do not necessarily lead to the maximum throughput.

- We compare the optimal PAoI in slotted ALOHA networks with and without the assumption of retransmitting the packets with a failed transmission (i.e., failed packets). In fact, not retransmitting these packets lowers the collision probability, leading to less transmission delay, but it may degrade throughput because any failed packet has been dropped. So there is a challenge whether the retransmission of failed packets leads to less PAoI or not. We will observe that when the packet arrival rate is low, the retransmission of failed packets has a significant effect in reducing PAoI.

- We compare the achievable throughput region of the buffer-less slotted ALOHA network with the unlimited buffer one for similar channel access probabilities. We observe that the achievable throughput region in the former includes the one in the latter. This shows that not only are buffer-less nodes preferable from the AoI point of view, but they might also be superior for achieving a higher amount of throughput.

The rest of the paper is organized as follows. In Section II, we introduce the network scenario and the main problem we dealt with. In Section III, we find the average PAoI of the asymmetric ALOHA network, and in Section IV, we find the average PAoI of the symmetric ALOHA network with much lower complexity. In Section V, we present some numerical results to understand some characteristics of PAoI in the slotted ALOHA network in different scenarios. Section VI concludes the paper.

II. NETWORK SCENARIO AND PROBLEM DESCRIPTION

The network considered in this paper is a slotted ALOHA network consisting of a destination and \(M\) source nodes, as shown in Fig. 1. The time is slotted, and at node \(i\), status updates (i.e., new packets) are independently arrived (generated) at the end of each slot with probability \(\lambda_i\) (i.e., Bernoulli process) and could be transmitted in the next time slots. When packets are generated independently (e.g., when some abnormalities occur in the system), Bernoulli arrival process is a reasonable assumption. At each slot, all nodes follow the slotted ALOHA protocol, i.e., at the beginning of each time slot, if node \(i\) has a packet, it will transmit its packet with probability \(p_i\), and each packet transmission, if being successful, takes one slot. Simultaneous packet transmission and packet arrival at a typical time slot are possible (i.e., the nodes are full-duplex); however, the former occurs earlier in this case. Thus, an arrived packet cannot be transmitted at the same slot. In the rest of the paper, we will interchangeably use arrival rate (AR) and transmission rate (TR) for \(\lambda_i\) and \(p_i\), respectively.

Transmission of a packet could be unsuccessful for two reasons. First, a collision happens whenever some nodes (at least two nodes) transmit their packets at the same slot. Thus, these packets need to be retransmitted. Second, if at a time slot, only node \(i\) transmits its packet and the other nodes do not transmit any packet, though the transmission would not face a collision, due to physical channel imperfections and independent of previous packet erasures, it may be erased at the destination with probability \(q_i\) (see Fig. 1). This can model the scenario where the nodes follow the simple ARQ protocol and the channel condition is independent of the previous time slots (i.e., the length of the slots is relatively larger than the channel coherence time). It has been assumed that after each packet transmission, the destination sends an acknowledgment (ACK)/negative ACK (NACK) at the end of the same slot in an error-free channel. Although our analytic approach could be deployed for solving more complex scenarios, e.g., multiple packet reception, in order to emphasize how we deal with interaction among preemptive queues in deriving the average PAoI, we constrain our work to the case of simple nodes and no requirement to channel state information exchange.

If during a typical time slot, the transmission of node \(i\) has been successful (the transmitter becomes aware by receiving ACK from the destination), the transmitted packet departs from the corresponding node (server). Suppose a packet has not been delivered yet (due to possible collisions, packet erasures, or randomized access of nodes). In this case, we consider that the packet will be transmitted in the next time slots with probability \(p_i\) unless a new packet arrives at node \(i\) and preempts the old packet (i.e., the new packet will take the place of the older one and the older packet will be dropped). In technical words, the preemption in service policy (as in [39]) has been assumed, so the server keeps only the last generated packet, and each node could be considered as a buffer-less queue with a single server. This packet management policy is considered since transmitting the freshest available packet instead of older ones is obviously age-beneficial if the residual transmission time of an old packet has the same distribution as the transmission time of a new packet, as with this assumption, the delivery instance of the new packet is identical to the older one but the former is fresher (i.e., the age of this packet at the delivery instance is lower), which leads to a more decrease in the AoI function. Since the packet erasures in different time slots are assumed to be independent, this condition is satisfied, so we adopt the preemptive policy.
III. PAoI IN ASYMMETRIC SLOTTED ALOHA NETWORK

In this section, we find the average PAoI of the asymmetric slotted ALOHA network. As mentioned before, we only evaluate the PAoI of the first node as the average PAoI of other nodes could be evaluated similarly. To this end, after modeling the slotted ALOHA network with a discrete time Markov chain (DTMC), we find \( \mathbb{E}(H_i) \) in (1) based on the DTMC model. Then, we find \( \mathbb{E}(Y_i) \) in (1) based on a trellis-based model with some matrix algebra. For ease of reading, we have summarized the most important notations that we introduce in this section in Table I.

A. DTMC Model and Evaluation of \( \mathbb{E}(H_i) \)

Before computing \( \mathbb{E}(H_i) \), we first model our network with a DTMC. As explained in the previous section, each node is considered to have a server and no buffer. Thus, each node could have one of the states of being empty (does not have a packet at its server) or being full (its server is occupied by a packet) at the beginning of a time slot. Note that for considering the probable collisions, knowing these states is sufficient. By knowing the current state of the network, i.e., knowing that each node is empty or full, the future of our slotted ALOHA network is independent of its past. Hence, it could be modeled as a DTMC with \( 2^M \) states. It is important to note that packet preemption does not affect the state of the network, as in our DTMC model, it is only important that a node is empty or full.

Let \( S \in \{0,1\}^M \) be a vector of \( M \) binary elements like \((S_1,S_2, \ldots, S_M)\), where \( S_i \) shows the current state of node \( i \). \( S_i = 0 \) denotes that node \( i \) is empty and \( S_i = 1 \) shows it is full. Let \( f_{x}(x) \) determine the binary representation of decimal number \( x \) with \( a \) digits. For example, \( f_{2}(0) = (0,0,0) \), or \( f_{3}(7) = (1,1,1) \). This notation helps to present the state

\[
\begin{align*}
&\textbf{TABLE I} \\
&\text{DESCRIPTION OF IMPORTANT NOTATIONS IN SECTION III} \\
&\begin{array}{|l|l|} 
\hline & \text{Description} \\
\hline \lambda_i & \text{Packet generation probability at node } i \\
p_i & \text{Packet transmission probability at node } i \text{ when it is full} \\
q_i & \text{Packet erasure probability at node } i \text{ due to channel imperfections} \\
P_{S_iS'} & \text{Transition probability form } S \text{ to } S' \text{ in the slotted ALOHA DTMC} \\
\pi(S) & \text{Steady-state probability of state } S \\
f_a(x) & \text{Binary representation of } x \text{ with } a \text{ digits} \\
S & \text{State of nodes } 2,3, \ldots, M \\
d & \text{Probability of successful transmission of node 1 when the state of the other nodes is } S \\
\varphi & \text{Vector comprised of } (S_2 \in \{0,1\}^{M-1}) \text{ based on (9)} \\
\pi(S) & \text{Vector comprised of } (S_2 \in \{0,1\}^{M-1}) \text{ based on (10)} \\
f_{S,S'} & \text{Probability that a node 1 packet does not leave the system in one time slot, while the state of the other nodes transits from } S \text{ to } S' \\
t & \text{Matrix where } t_{(s,s')} = f_{M-1}(i-1),f_{M-1}(j-1) \\
E_{n,S} & \text{Event that the system time of a typical node 1 packet exceeds } n \text{ slots while at the } (n+1)\text{-th slot of its system time, the state of the other nodes is } S \\
G_{n,S} & \text{Probability of } E_{n,S} \\
G_n & \text{Vector comprised of } G_{n,S} (S \in \{0,1\}^{M-1}) \text{ based on (16)} \\
e_n & \text{Probability that a node 1 packet is transmitted successfully at the } n\text{-th slot of its system time} \\
\hline 
\end{array}
\end{align*}
\]
Similarly, we define $A_{(S,S')} = \{i \mid 1 \leq i \leq M, S_i = 0, S'_i = 1\}$. (2)

Similarly, we define $B_{(S,S')}$ as

$$B_{(S,S')} = \{i \mid 1 \leq i \leq M, S_i = 1, S'_i = 0\},$$

and $C_{(S)}$ as

$$C_{(S)} = \{i \mid 1 \leq i \leq M, S_i = 1\}.$$ (4)

Let $P_{S,S'}$ denote the transition probability from state $S$ to state $S'$ in this DTMC. Since in the slotted ALOHA network, at a time slot at most one source could send its packet without collision, so in order to have a non-zero $P_{S,S'}$, it is required that $|B_{(S,S')}| = 0$ or $|B_{(S,S')}| = 1$, otherwise we have $P_{S,S'} = 0$. For the case $|B_{(S,S')}| = 1$, let $\eta$ be the only member of $B_{(S,S')}$. Thus,

$$P_{S,S'} = p_\eta (1 - q_\eta) (1 - \lambda_\eta) \prod_{i \in C_{(S)}} (1 - p_i) \prod_{j \in A_{(S,S')}} \lambda_j \prod_{l \notin A_{(S,S')} \cup C_{(S)}} (1 - \lambda_l).$$ (5)

When $|B_{(S,S')}| = 0$, all of the nodes that are currently full should still have a packet at the beginning of the next time slot. Thus, we have

$$P_{S,S'} = \prod_{j \in A_{(S,S')}} \lambda_j \prod_{l \notin A_{(S,S')} \cup C_{(S)}} (1 - \lambda_l) \left[1 - \sum_{k \in C_{(S)}} p_k (1 - q_k) (1 - \lambda_k) \prod_{i \in C_{(S)}, \ i \neq k} (1 - p_i)\right].$$ (6)

An example of this DTMC model is illustrated in Fig. 3 for the case that $M = 2$. Since the number of states is finite and the corresponding DTMC is irreducible and aperiodic, the steady-state probability of being in state $S$ at the beginning of a time slot, i.e., $\pi_{(S)}$, satisfies the following equation (i.e., the global balance equation (GBE)):

$$\pi_{(S)} \left( \sum_{S' \in \{0,1\}^M} P_{S,S'} \right) = \sum_{S' \in \{0,1\}^M} \pi_{(S')} P_{S',S}.$$ (7)

Let $d_S$ be the probability of successful transmission of node 1 in a time slot which at its beginning, the state of the network is $(1, S)$. We have

$$d_S = p_1 (1 - q_1) \prod_{i \in C_{(1, S)}, \ i \neq 1} (1 - p_i).$$ (8)

As an example, in case that $M = 2$, $d = [d_0, d_1]$ where $d_0$ is the probability of successful transmission of node 1 in a time slot where node 1 is full and node 2 is empty, which is equal to $p_1 (1 - q_1)$ and $d_1$ is the probability of successful transmission of node 1 in a time slot where node 1 is full and node 2 is full, which is equal to $p_1 (1 - q_1) (1 - p_2)$. Let $\pi_{(1)}$ be a vector comprised of $\pi_{(1),S}$ $(S \in \{0,1\}^{(M-1)})$ as

$$\pi_{(1)} \triangleq \left[\pi_{(1,f_{M-1}(0))}, \pi_{(1,f_{M-1}(1))}, \ldots, \pi_{(1,f_{M-1}(2^{M-1}-1))}\right].$$ (10)

As an example, in case that $M = 2$, $\pi_{(1)} = [\pi_{(1,0)}, \pi_{(1,1)}]$. The probability of successful transmission of node 1 at a typical time slot is $\pi_{(1)} d^T$ since with probability $\pi_{(1),S}$, the state of the network is equal to $(1, S)$ and in this state, node 1 could deliver its packet successfully during this time slot with probability $d_S$. Thus, if $N(j)$ denotes the number of delivered packets of node 1 before the $j$-th time slot, due to the ergodicity of the DTMC, $N(j)/j$ converges to $\pi_{(1)} d^T$ as $j \to \infty$. In other words, the throughput (i.e., the long-term average number of successfully transmitted packets per slot) of node 1 is equal to $\pi_{(1)} d^T$. On the other hand, as the average elapsed time between deliveries of two consecutive packets of node 1 is $E(H_1)$, and due to the ergodicity conditions, $N(j)/j$ converges to $1/E(H_1)$. Hence, we have

$$E(H_1) = \frac{1}{\pi_{(1)} d^T}. \quad (11)$$

In the next subsection, we will find $E(Y_1)$ to complete the evaluation of the average PAoI based on (1).
B. Evaluating $\mathbb{E}(Y_i)$

Now, we find $\mathbb{E}(Y_i)$ as another effective term in (1). This could be challenging due to interactions among the nodes, packet preemptions, and retransmission of the failed packets. Consider that $T''$ show the beginning of the first time slot within interval $H_i$ that node 1 is not empty. As shown in Fig. 3, some packets of node 1 may be preempted during interval $H_i$, according to the preemptive policy. Moreover, the state of nodes $2, 3, \ldots, M$ may change during the interval $[T''_i, T_i]$. Therefore, the probability of successful transmission of node 1 would change between the elements of set $\{d_S | S \in \{0,1\}^{M-1}\}$ during this interval, as the state of nodes $2, 3, \ldots, M$ may change. Thus, it is necessary to follow the changes in the state of nodes $2, 3, \ldots, M$ during the system time of a node 1 packet.

Let $t_{\bar{S}, \bar{S}'} (\bar{S}, \bar{S}' \in \{0,1\}^{M-1})$ be the probability that at a typical time slot which the state of the network is $(1, \bar{S}')$ at its beginning, the packet of node 1 does not leave the system (i.e., neither be transmitted successfully nor be preempted) and state of the other nodes transits from $\bar{S}'$ to $\bar{S}$ during this time slot. This would happen if and only if all of the following three events happen together:

1) Node 1 does not transmit its packet successfully during this time slot due to collision, randomized access of node 1 to the channel (i.e., not transmitting in that time slot), or channel erasure.

2) The current packet of node 1 is not preempted at the end of this time slot. Thus, no new packet has arrived at node 1 during this time slot.

3) The state of nodes $2, 3, \ldots, M$ transits from $\bar{S}'$ to $\bar{S}$ during this time slot.

We have

$$t_{\bar{S}, \bar{S}'} = (1 - \lambda_1) \left[ P_{(1, \bar{S}')} | (1, \bar{S}) - \frac{\lambda_1 P_{(1, \bar{S}')} | (0, \bar{S})}{1 - \lambda_1} \right]. \tag{12}$$

In fact, the packet of node 1 is not preempted with probability $(1 - \lambda_1)$. In addition, the state of the network should transit from $(1, \bar{S}')$ to $(1, \bar{S})$ while node 1 cannot transmit its packet successfully. In some transitions from state $(1, \bar{S}')$ to $(1, \bar{S})$, it is probable that node 1 successfully transmits its packet and then a new packet arrives at node 1 at the beginning of the next time slot. The probability of such a transition is $\left(\lambda_1 P_{(1, \bar{S}')} | (0, \bar{S})\right)/(1 - \lambda_1)$, because this is the probability of transition from state $(1, \bar{S}')$ to $(0, \bar{S})$ such that instead of not having an arrival at node 1 at this slot (with probability $(1 - \lambda_1)$), we have an arrival (with probability $\lambda_1$). Hence, $P_{(1, \bar{S}')} | (1, \bar{S}) - (\lambda_1 P_{(1, \bar{S}')} | (0, \bar{S}))/(1 - \lambda_1)$ is the probability of transition from state $(1, \bar{S}')$ to $(1, \bar{S})$, node 1 does not transmit successfully. Thus, multiplication of $(1 - \lambda_1)$ and $\left[P_{(1, \bar{S}')} | (1, \bar{S}) - (\lambda_1 P_{(1, \bar{S}')} | (0, \bar{S}))/ (1 - \lambda_1)\right]$ is the probability that at a typical time slot, node 1 packet does not leave the system while the state of the other nodes transits from $\bar{S}'$ to $\bar{S}$ during this slot, which proves (12).

As an example, in case where $M = 2$, $t_{0,0}$ is the probability that at a time slot which node 1 is full and node 2 is empty at its beginning, node 1 packet does not leave the system, and node 2 stays empty at the beginning of the next time slot. To this end, node 1 would not have a successful transmission with probability $1 - d_0 = 1 - p_1(1 - q_1)$, packet of node 1 should not be preempted with probability $(1 - \lambda_1)$ and no packet should arrive at node 2 at the beginning of the next time slot with probability $(1 - \lambda_2)$. Hence, $t_{0,0} = (1 - \lambda_1)(1 - p_1(1 - q_1))(1 - \lambda_2)$ and the reader could validate that this is compatible with (12). We define a $2^{M-1} \times 2^{M-1}$ matrix $t$ where for $t_{[i,j]}$, i.e., the entry of the $i$-th row and the $j$-th column, we have

$$t_{[i,j]} \equiv t_{f_{M-1}(i-1), f_{M-1}(j-1)}. \tag{13}$$

As an example, in case where $M = 2$, we have

$$t = \begin{bmatrix} t_{0,0} & t_{0,1} \\ t_{1,0} & t_{1,1} \end{bmatrix}. \tag{14}$$

To evaluate $\mathbb{E}(Y_i)$, we first focus on the system time of a typical packet. Note that the system time of a packet is at least one time slot. Let $E_{n, \bar{S}}$ be the event that the system time of a node 1 packet exceeds $n$ slots (i.e., it has not been transmitted or preempted in the first $n$ slots) while at the beginning of the $(n+1)$-th slot of its system time, the state of nodes $2, 3, \ldots, M$ is equal to $\bar{S}$ and $G_{n, \bar{S}} (n \geq 0)$ be the probability of this event. We have

$$G_{n, \bar{S}} = \sum_{S \in \{0,1\}^{M-1}} G_{n-1, S} \cdot t_{\bar{S}, S}. \tag{15}$$

For the proof, note that the probability of $E_{n, \bar{S}}$ subject to event $E_{n-1, \bar{S}'}$ is $t_{\bar{S} \bar{S}'}$, since with this probability, the packet of node 1 does not leave the system during the $n$-th slot of its system time and the state of the other nodes transits from $\bar{S}'$ to $\bar{S}$. As $G_{n-1, \bar{S}'}$ is the probability of event $E_{n-1, \bar{S}'}$, (15) is justified according to the total probability theorem. Let $G_n$ be a vector comprised of $G_{n, \bar{S}} (\bar{S} \in \{0,1\}^{M-1})$ as

$$G_n \equiv [G_{n, f_{M-1}(0)}, G_{n, f_{M-1}(1)}, \ldots, G_{n, f_{M-1}(2^{M-1}-1)}]. \tag{16}$$

It is worth noting that based of the definition of $E_{n, \bar{S}}$, the probability that system time of a typical packet exceeds $n$ slots is

$$\sum_{\bar{S} \in \{0,1\}^{M-1}} G_{n, \bar{S}} = G_n \mathbf{1}_{2^{M-1}-1}. \tag{17}$$

Also, from (15) we have

$$G_n = G_{n-1} t = G_{0} t^n. \tag{18}$$

Hence, it is enough to find $G_0$ in order to obtain $G_n$ for $n \geq 1$. Actually, $G_0, \bar{S}$ is the probability that at the beginning of a slot which a node 1 packet arrives, the state of the other nodes is equal to $\bar{S}$. As the arrival process at node 1 is a Bernoulli process, according to BASTA (Bernoulli arrivals see time average) theorem [40], $G_0, \bar{S}$ is equal to the steady-state probability that the state of nodes $2, 3, \ldots, M$ is equal to $\bar{S}$ at the beginning of a time slot. In other words, for all $\bar{S} \in \{0,1\}^{M-1}$, we have

$$G_{0, \bar{S}} = \pi(0, \bar{S}) + \pi(1, \bar{S}). \tag{19}$$

Thus, after finding the steady-state probabilities of DTMC, $G_0$ could be evaluated based on (19).
like in Fig. 4, for the case $E = 2$.

Now, we are able to have a better graphical view of the system time of packets in our scenario. Consider that events $E_n, S (n = 0, 1, \ldots)$ are shown on a trellis like in Fig. 4, for the case $M = 2$. The $j$-th level of the trellis is comprised of $E_j, S (S \in \{0, 1\}^{M-1})$. Each packet starts its journey in the system in one of the phases of the 0-th level (i.e., $E_0, S$), based on distribution $G_0$. In the phases of the $n$-th level (i.e., when the packet did not leave the system in the previous $n$ slots), this packet might be transmitted successfully based on transition vector $d$ or might be preempted based on transition vector $C = \lambda(1_{1 \times 2^{M-1}} - d)$ or the state of trellis might transit to the next level, based on the transition matrix $t$.

In this trellis, each packet leaves the system whenever it enters one of the being preempted or being successfully transmitted states. Due to possible packet preemptions and randomized channel access, system time is a random variable. Hence, after evaluating $G_0$, $d$ and $t$, we are able to analyze the packet system time, but to find the end-to-end delay of the tagged packets, we should only concentrate on the ones which have entered the being successfully transmitted state.

Let $e_n (n \geq 1)$ be the probability that a typical packet is transmitted successfully at the $n$-th slot after its arrival. We have

$$e_n = \sum_{S \in \{0, 1\}^{M-1}} G_n, S \ d_S = G_0 \ t^{n-1} d^T. \quad (20)$$

This could be proved by conditioning on the state of nodes $2, 3, \ldots, M$ at the $n$-th time slot of the system time of a typical packet at node 1. With probability $G_n, S, E_n, S$ happens, and in this case, the probability that this packet is successfully transmitted in the $n$-th time slot is equal to $d_S$. So, the first equality of (20) has been proved. As $\sum_{S \in \{0, 1\}^{M-1}} G_n, S \ d_S = G_n \ d^T$ and based on (18), the second equality could be proved.

Note that $Y_i$ is the transmission time of the $i$-th tagged packet. Thus, $P(Y_i = n)$ is the probability that the system time of a node 1 packet equals $n$, provided that this packet has not been preempted by any other packet. Then according to Bayes’ rule, we have

$$P(Y_i = n) = \frac{e_n}{\sum_{j=1} e_j}. \quad (21)$$

Hence, by replacing $e_n$ based on (20), we have

$$E(Y_i) = \frac{\sum_{n=1}^{\infty} n e_n}{\sum_{n=1}^{\infty} e_n} = G_0 (\sum_{n=1}^{\infty} n t^{n-1}) d^T$$

$$= G_0 \left(\sum_{n=1}^{\infty} t^{n-1}\right) \left(\sum_{n=1}^{\infty} n t^{n-1}\right) d^T$$

$$= G_0 (I_{2^{M-1} \times 2^{M-1}} - t)^{-1} t^{-2} d^T$$

$$= G_0 (I_{2^{M-1} \times 2^{M-1}} - t)^{-1} t^{-1} d^T. \quad (22)$$

where $I_{2^{M-1} \times 2^{M-1}}$ is the identity matrix with dimension of $2^{M-1}$. Note that $t$ is a sub-stochastic matrix, as $\sum_{j=1}^{2^{M-1}} t_{i,j}$ is equal to the probability that a node 1 packet is not transmitted successfully or preempted at a time slot which the state of the network is equal to $(1, f_{i-1}(i-1))$ at the beginning of this time slot. Therefore, $\left(\sum_{n=1}^{\infty} t^{n-1}\right)$ is not divergent and could be replaced by $(I_{2^{M-1} \times 2^{M-1}} - t)^{-1}$. From (1), (11) and (22), the average PAoI could be derived as

$$E(A_i) = \frac{1}{\pi(1)} d^T + G_0 (I_{2^{M-1} \times 2^{M-1}} - t)^{-1} t^{-1} d^T. \quad (23)$$

Hence, after finding the steady-state probabilities of DTMC by solving GBEs, $G_0$ and $\pi(1)$ could be evaluated based on (19) and (10), respectively. Then, the average PAoI could be found according to (23).

It is worth noting that we can evaluate the average PAoI in similar scenarios with interaction among the queues by a similar analysis. It is necessary to model the network by a DTMC with a sufficient set of states (like $S = \{S_1, S_2, \ldots, S_M\}$ in our scenario), evaluate $E(H_i)$ according to the steady-state probabilities of the corresponding DTMC and the probability of successful transmission in each state, and find $E(Y_i)$ by a similar trellis-based approach where the changes in the state of the network during the system time of node 1 (as a typical node) packets are followed. For example, for phase-type inter-arrival and transmission processes, we can use the same approach where $S_i$ shows the status of the $i$-th node (i.e., whether it is empty or full) as well as the phase of the transmission and inter-arrival processes at this node. However, BASTA property no longer works for non-Bernoulli arrival processes. Thus, $G_0$ (i.e., the initial distribution of the trellis model) should be found by a different approach.

For the asymmetric slotted ALOHA network, the complexity of computing the average PAoI grows exponentially with the number of nodes since the number of states in the DTMC is $2^M$. In the next section, we focus on a symmetric ALOHA network where the exact average PAoI could be evaluated with polynomial complexity.

IV. PAoI IN SYMMETRIC SLOTTED ALOHA NETWORK

In this section, we investigate the average PAoI of the symmetric slotted ALOHA network where $(\lambda_i, p_i, q_i) = (\lambda, p, q)$ for all $i$. We use a similar analysis as in the previous section to find the average PAoI of the symmetric slotted ALOHA network with lower complexity than the general asymmetric case. The main point is that in the symmetric network, instead of knowing which nodes are full and which are empty, it suffices to know how many are full and how many are empty.
We can model this scenario by a DTMC with state space \{0, 1, \ldots, M\} where state \(k\) denotes that there are \(k\) full nodes among all \(M\) nodes in the symmetric network. Similar to the previous section, packet preemption does not affect the state of the network in our DTMC, as only one packet could be transmitted in a time slot, we have \(\hat{P}_{k,k'} = 0\) for \(k' \leq k - 2\). For the case where \(k' = k - 1\), we have
\[
\hat{P}_{k,k-1} = k_z(k - \lambda)^{M-k+1}, \quad k \geq 1, \tag{24}
\]
where \(z_k = p(1-q)(1-p)^{k-1}\). Actually, \(k_z\) shows the probability that one node could transmit its packet during a time slot which \(k\) nodes are full at its beginning. Moreover, \((1-\lambda)^{M-k+1}\) is the probability that no packet arrives at \(M-k\) empty nodes and the one which transmits successfully. We also have
\[
\hat{P}_{k,k'} = k_z\left\{\begin{array}{ll}
\binom{M-k+1}{k'-k+1}\lambda^{k'-k+1}(1-\lambda)^{M-k'} & \text{if } k' \leq k - 1 \\
(1-k_z)\binom{M-k}{k'-k}\lambda^{k'-k}(1-\lambda)^{M-k'} & \text{if } k' > k - 1
\end{array}\right. \tag{25}
\]
Eq. (25) could be proved by considering whether any of the nodes has transmitted its packet successfully or not. With probability \(k_z\), one node has transmitted its packet successfully, so \(k' = k - 1\) nodes out of \(M-k+1\) nodes (\(M-k\) empty nodes and the one that has a successful transmission) must have a packet arrival to transit to state \(k'\). With probability \((1-k_z)\), none of the full nodes could transmit its packet successfully, and in this case, \(k' = k\) nodes out of \(M-k\) nodes (i.e., empty nodes) must have a packet arrival. Similar to the previous section, the steady-state probability of being in state \(k\), i.e., \(\hat{\pi}_k\), satisfies the following GBE:
\[
\hat{\pi}_k(\sum_{k'=0}^M \hat{P}_{k,k'} = \sum_{k'=0}^M \hat{\pi}_{k'}\hat{P}_{k',k}. \tag{26}
\]
An example of this DTMC has been illustrated in Fig. 5 for the case \(M = 2\).

Let \((a,b)\) denote the situation where node 1 has \(a\) packets (\(a \in \{0,1\}\)) and \(b\) nodes are full among nodes \(2,3,\ldots,M\). Let \(\hat{\pi}_{(a,b)}\) denote the steady-state probability of this situation. Based on the symmetry among the nodes, we have
\[
\hat{\pi}_{(0,b)} = \frac{(b+1)\hat{\pi}_{(b+1)}}{M}, \tag{27a}
\]
\[
\hat{\pi}_{(1,b)} = \frac{(b+1)\hat{\pi}_{(b+1)}}{M}, \tag{27b}
\]
since when there are \(b\) full nodes among the nodes, due to the symmetry of the network, node 1 is full with probability \(b/M\) and empty with probability \((M-b)/M\).

Let \(\hat{d}_b\) be the probability that node 1 could transmit its packet during the time slot which \(b\) nodes are full among nodes \(2,3,\ldots,M\) at the beginning of this slot. We have
\[
\hat{d}_b = z_{b+1} = p(1-q)(1-p)^b. \tag{28}
\]
Similar to the previous section, we define two vectors as
\[
\hat{a} = [\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_{M-1}], \tag{29}
\]
\[
\hat{\pi}_1 = [\hat{\pi}_{(1,0)}, \hat{\pi}_{(1,1)}, \ldots, \hat{\pi}_{(1,M-1)}]. \tag{30}
\]
With the same argument as in the previous section, we have
\[
\mathbb{E}(H_1) = \frac{1}{\hat{\pi}_1 \hat{a}^T}. \tag{31}
\]
Let \(\hat{t}_{(b',b)}\) be the probability that during a time slot which the state of the network is \((1,b')\) at its beginning, the packet of node 1 does not leave the system (i.e., neither being preempted nor being successfully transmitted) and the number of full nodes among nodes \(2,3,\ldots,M\) changes from \(b\) to \(b'\). We have \(\hat{t}_{(b',b)} = 0\) for \(b \leq b' - 2\). For the case that \(b = b' - 1\), we have
\[
\hat{t}_{(b',b)} = z_{b+1}(1-\lambda)^{M-b'+1}; \quad b' \geq 1. \tag{32}
\]
since in this case, one of the \(b'\) full nodes among nodes \(2,3,\ldots,M\) must transmit its packet successfully, and no packet should arrive at the empty nodes, the node that has a successful transmission, and node 1. For the other cases, we have
\[
\hat{t}_{(b',b)} = (1-(b'+1)z_{b'+1})(1-\lambda)^{M-b}; \quad b \geq b'. \tag{33}
\]
Proof of (33) is similar to (25). If none of the current full nodes \((b'+1)\) has a successful transmission (which happens with probability \(1-(b'+1)z_{b'+1}\)), then no packet should arrive at node 1 and \(M-b-1\) nodes out of the empty nodes \((M-b')\) must have a packet arrival. If one of the full nodes among nodes \(2,3,\ldots,M\) \((b')\) has a successful transmission (which happens with probability \(b'z_{b'+1}\)), then no packet should arrive at node 1 and \(M-b-1\) nodes out of \(M-b'\) nodes (empty nodes and the node which has a successful transmission).

Let \(\hat{\mathbf{t}}\) be an \(M \times M\) matrix where
\[
\hat{t}_{(i,j)} = \hat{t}_{(i-1),(j-1)} \tag{34}
\]
As an example, in case that \(M = 2\), we have
\[
\hat{\mathbf{t}} = \begin{bmatrix}
\hat{t}_{0,0} & \hat{t}_{0,1} \\
\hat{t}_{1,0} & \hat{t}_{1,1}
\end{bmatrix}. \tag{35}
\]
Let \(\hat{E}_{n,b}\) be the event that the system time of a typical packet at node 1 exceeds \(n\) slots while at the beginning of the \((n+1)\)-th slot of system time, \(b\) nodes are full among the
nodes $2, 3, \ldots, M$. Also, let $\hat{G}_{n,b}$ be the probability of $\hat{E}_{n,b}$ and

$$\hat{G}_n = [\hat{G}_{n,0}, \hat{G}_{n,1}, \ldots, \hat{G}_{n,M-1}].$$

With the same argument as in the previous section we have

$$\hat{G}_n = \hat{G}_0 \hat{\tau}^n,$$

where

$$\hat{G}_0 = [\hat{\pi}_0(0,0) + \hat{\pi}_1(0,1), \ldots, \hat{\pi}_0(0,M-1) + \hat{\pi}_1(1,M-1)].$$

Similar to the previous section, we have

$$\hat{e}_n = \hat{G}_0 \hat{\tau}^{n-1} \hat{d}^T,$$

where $\hat{e}_n$ is the probability that a packet of node 1 in the symmetric network is transmitted successfully at the $n$-th slot after its arrival. Thus, based on an argument similar to the one in the previous section, we have

$$E(Y_n) = \frac{\hat{G}_0 (I_{M \times M} - \hat{\tau})^{-2} \hat{d}^T}{\hat{G}_0 (I_{M \times M} - \hat{\tau})^{-1} \hat{d}^T}. \quad (40)$$

Hence, from (1), (31), and (40), we have

$$E(A) = \frac{1}{\hat{\pi}_1(1)\hat{d}^T} + \frac{\hat{G}_0 (I_{M \times M} - \hat{\tau})^{-2} \hat{d}^T}{\hat{G}_0 (I_{M \times M} - \hat{\tau})^{-1} \hat{d}^T}. \quad (41)$$

Now, we find the complexity of evaluating the average PAoI in the symmetric slotted ALOHA network. Based on (24) and (25), the order of operations for evaluating all transition probabilities in the DTMC model is at most $O(M^3)$ (the order of operations for evaluating each transition probability is at most $O(M)$). Similarly, $\hat{d}$ and $\hat{\tau}$ can be found with at most $O(M^2)$ and $O(M^3)$ operations, respectively. The steady-state probabilities of the DTMC can be found by solving the system of $M + 1$ linear equations obtained based on (26) (i.e., the GBEs) with $O(M^3)$ operations [41]. After finding the steady-state probabilities, the complexity of computing $\hat{G}_0$ and $\hat{\pi}_1$ are at most $O(M)$. $(I_{M \times M} - \hat{\tau})^{-1}$ can be found by $O(M^3)$ operations [41] and the vector and matrix multiplications in (41) can be done with at most $O(M^3)$ operations. Hence, the complexity of evaluating the exact average PAoI in the symmetric networks has a polynomial growth with the number of nodes (i.e., $O(M^3)$).

With a similar analysis, we can find the average PAoI in the group-symmetric networks where the nodes are partitioned into some groups (e.g., $F$) of size $M$, and if the $j$-th node belongs to the $j$-th group, $(\lambda_i, p_i, q_i) = (\lambda^{(j)}, p^{(j)}, q^{(j)})$. In the group-symmetric networks, it suffices to know how many nodes in each group are full. Hence, the network can be modeled with a DTMC with $(M + 1)^F$ states. Similar to the analysis in this section, the system time of the packets of a specific group can be modeled with a trellis with $M(M + 1)^{F-1}$ states at each level. Therefore, with a similar complexity analysis, in this network, the complexity of evaluating the average PAoI of a specific group has a polynomial growth with $M$ but an exponential growth with $F$.

V. NUMERICAL RESULTS

In this section, we provide some numerical results based on the analysis in the previous sections. At first, in order to confirm our analysis, in Table II, we consider an asymmetric ALOHA network with three nodes where $(\lambda, p, q)$ for the nodes are $(0.5, 0.3, 0.1), (0.6, 0.4, 0.2)$ and $(0.7, 0.5, 0.3)$, respectively, and compare the average PAoI obtained based on the analytical results in Section III with simulation results. The simulation results are obtained by simulating an Aol sample function over 100 million time slots and evaluating the average of the local peaks in this sample function. In order to confirm the analysis of the symmetric ALOHA network, we compare the analytically evaluated average PAoI with simulation results for two cases in Table III. In case (A), the arrival rate of nodes decreases with the number of nodes while their transmission probability is constant. In case (B), the transmission probability of nodes decreases with the number of nodes while their arrival rate is constant. In case (B), we can observe an approximately linear increase in the average PAoI when $M$ increases. While in case (A), we can observe a super-linear increase. In both cases, for large values of $M$, the network is in a near-saturated condition (i.e., most nodes have a packet to transmit with a high probability). When all nodes have a packet, the probability of successful transmission for each node in case (A) is $0.5^M$, which decreases exponentially with $M$. However, in case (B), this value is $(1 - \frac{1}{2M})^{M-1}/(2M)$, which for large values of $M$ is approximately equal to $1/(2M/\sqrt{\pi} \epsilon)$ ($\epsilon$ is the Euler’s number). This justifies how the average PAoI increases in these cases. As the analytic results have been confirmed, the rest of the numerical results in this section are obtained by the analytical formulas in Sections III and IV, which is much less time-consuming than finding the average PAoI based on the simulation results. Thus, after finding the steady-state probabilities by solving the GBEs, we evaluate the average PAoI by using (23) and (41).

Now, we concentrate on the asymmetric slotted ALOHA network comprised of two nodes. For the sake of simplicity, we consider that the erasure probability for both nodes is equal to $0.1$ ($q_1 = q_2 = 0.1$). Let $\Delta_1$ and $\Delta_2$ denote the average

| $M$ | Analytic PAoI (case A) | Simulation PAoI (case A) | Analytic PAoI (case B) | Simulation PAoI (case B) |
|-----|------------------------|--------------------------|------------------------|--------------------------|
| 2   | 8.0525                 | 8.0513                   | 8.3558                 | 8.3419                   |
| 4   | 15.4919                | 15.5167                  | 15.7167                | 15.6973                  |
| 6   | 30.1984                | 30.2220                  | 23.0649                | 23.0738                  |
| 8   | 141.1337               | 140.9916                 | 30.4032                | 30.4276                  |
| 10  | 970.5689               | 971.648                  | 37.7374                | 37.7559                  |
PAoI of nodes 1 and 2, respectively. In this network, it is interesting to find the regions over the ARs, i.e., $\lambda_1$ and $\lambda_2$, and TRs, i.e., $p_1$ and $p_2$ such that $\Delta = (\Delta_1 + \Delta_2)/2$ does not exceed a specific threshold. In this work, these regions are called PAoI-constrained regions. We have shown $\Delta$ contours in Fig. 6 for different sets of parameters using the analytic relation obtained in Section III. In Figs. 6(a) and 6(b), the contours specify the boundary of the regions over ARs where $\Delta$ does not exceed some specific thresholds. Some of these contours do not seem to be closed, which means $\lambda_1 = 1$ or $\lambda_2 = 1$, i.e., the maximum possible rates of arrivals, are the other boundaries of these regions. For more clearance, the $\Delta$-constrained regions for $\Delta \leq 6$ have been hatched in Fig. 6. Similar contours in Fig. 6(c) specify the boundary of the regions over TRs. We observe that the constrained regions are convex. Also, by increasing the threshold, a wider range of ARs and TRs are included in the PAoI-constrained regions. Moreover, Fig. 6(b) shows that if TR of node 1 is lower than TR of node 2, the $\Delta$-constrained regions mostly cover the points where AR of node 1 is relatively higher than AR of node 2. This makes a balance between the nodes with different TRs, as the average PAoI of both nodes is equally important in $\Delta$. In so many words, both throughput and transmission delay affect PAoI. Roughly speaking, AR directly affects the former, and TR is effective in the latter. The same result could be concluded for $\Delta$-constrained regions over TRs, as observed in Fig. 6(c).

Next, in the symmetric slotted ALOHA network, for a fixed value of $q$, we investigate the optimal throughput (i.e., the maximum possible throughput at each node by changing the value of $(\lambda, p)$) and the optimal PAoI (i.e., the minimum possible average PAoI at each node by changing the value of $(\lambda, p)$). Let $(\lambda_*, p_*)$ and $(\lambda^*_A, p^*_A)$ be the optimal solutions that maximize the throughput and minimize the average PAoI of the symmetric slotted ALOHA network, respectively. For different values of $(\lambda, p)$, varied by the step size 0.01, we have computed the average PAoI based on (41) and the throughput of each node, i.e., $\hat{\pi}(1)\hat{d}^T$ (as explained in Section III-A), and find $(\lambda^*_T, p^*_T)$ and $(\lambda^*_A, p^*_A)$, as reported in Table IV for different values of $M$. For a better comparison, in Table IV, we report the network throughput, i.e., $M\hat{\pi}(1)\hat{d}^T$, instead of the throughput of a single node. It is observed that to minimize the average PAoI of a symmetric network, $(1, 1/M)$ is the optimal choice for $(\lambda, p)$. Actually, it is easy to show that for $\lambda = 1$, $p = M/1$ is PAoI-optimal. When $\lambda = 1$, we have $Y_i = 1$, as a new packet arrives at each time slot at each node. Hence, PAoI minimization is aligned with throughput maximization (i.e., minimization of $\mathbb{E}(H_i)$). When $\lambda = 1$, at each time slot, each node has a successful transmission with probability $p(1 - p)^{M-1}$, which is maximized at $p = 1/M$. Thus, $p = 1/M$ minimizes the average PAoI when $\lambda = 1$. However, $p = 1/M$ does not necessarily minimize the average PAoI when $\lambda < 1$.

Based on the results of Table IV, we can observe that $(\lambda^*_A, p^*_A)$ does not maximize the total network throughput. However, the difference between the optimal throughput (i.e., the throughput at $(\lambda^*_T, p^*_T)$) and the throughput at $(\lambda^*_A, p^*_A)$ is negligible, while the difference between the average PAoI at $(\lambda^*_T, p^*_T)$ and $(\lambda^*_A, p^*_A)$ is significant. Also, it could be observed that as $M$ increases, the optimal throughput converges to $(1 - q)/e$. In Fig. 7, the average PAoI and throughput of a symmetric ALOHA network with $M = 4, q = 0.1$ has been illustrated where $(\lambda^*_A, p^*_A)$ and $(\lambda^*_T, p^*_T)$ are determined in this figure.

In Fig. 8, we compare the performance of the slotted ALOHA network as a decentralized access protocol with the randomized scheduling introduced in [13] as a centralized protocol. In this respect, we consider a symmetric case with $M$ nodes. In the decentralized case, each node transmits based on the same protocol studied in this paper with transmission rate $p$, where the value of $p$ has been optimized numerically (i.e., by changing $p$ with the step size 0.01 and finding
the best value). While in the centralized case, a scheduler determines the transmitting node at each slot randomly. Thus, at each slot, each node might be scheduled to transmit (even if it does not have a packet to transmit) with probability $1/M$ (as explained in [13] and the symmetric network we considered). The results show that when the arrival rate is low, slotted ALOHA is preferable, while when the arrival rate or the number of nodes increases, randomized scheduling as a centralized protocol is preferable. The reason behind this is that the randomized scheduling protocol introduced in [13] is a non-work-conservative policy since even a node with an empty server may be scheduled to transmit at a time slot. In contrast, the slotted ALOHA protocol is a work-conserving protocol because an empty node does not attempt to transmit. However, the slotted ALOHA protocol suffers from probable collisions when at least two nodes transmit simultaneously. Hence, when the arrival rates are relatively low, the slotted ALOHA protocol is preferable since it is work-conserving and the collision probability is low. On the other hand, when the arrival rates increase, the network becomes more crowded. Thus, the centralized policy is preferable as the result of removing collisions.

Next, we compare the optimal average PAoI of the symmetric buffer-less preemptive slotted ALOHA networks with and without the assumption of retransmitting the previously failed packets. We have analyzed the former scenario in Section IV, and with the same logic as in that section, we have analyzed the latter scenario in which each packet will be dropped from the system if it has a failed transmission (due to possible collisions or channel erasures). In Fig. 9, we compare the optimal average PAoI of these networks for different values of $\lambda$ when the value of $(M, q)$ is fixed. The optimal PAoI of these scenarios have been found numerically by changing $p$ with the step size 0.01 and using (41) in Section IV and the similar relation we have found for the average PAoI in the case that retransmission of failed packets is not permitted. Our observations show that when the arrival rate of nodes is relatively low, retransmitting a previously failed packet, if it has not been preempted yet, is preferable, like in Fig. 9 that when $M = 3, q = 0.5$, and $\lambda = 0.05$, retransmitting the failed packets will result in a $37\%$ lower average PAoI. However, when the arrival rate of nodes increase, due to the increase in the network communication load, it is better not to retransmit failed packets. Actually, retransmitting these packets could help to decrease PAoI by increasing the throughput and decreasing $E(H_i)$, but could increase the average PAoI as it increases the communication traffic of the network, which leads to an increase in $E(Y_i)$. When the arrival rate is very high, both schemes are similar because, with a high probability, at each node, a new packet arrives at each time slot and preempts the existing packet (if any exists). Fig. 9 shows that the positive effect of retransmitting the previously failed packets degrades with the increase of $M$ and decrease of $q$.

Next, we consider the asymmetric network with $M = 2$ and compare the throughput of our scenario, for fixed values of TRs, with the traditional slotted ALOHA network where the buffer size of each node is infinite, and packets of each node...
Node 1 is buffer-less, as explained in Section III-A, the throughput of the region, which has been evaluated in [36]. When the nodes are served distinctly. For the sake of simplicity, we consider that \( q_1 = q_2 = 0 \). The achievable throughput region of the traditional slotted ALOHA network is the same as its stability region, which has been evaluated in [36]. When the nodes are buffer-less, as explained in Section III-A, the throughput of node 1 is \( \pi_1 \mathbf{1}^T \) (similarly the throughput of node 2 can be found). It is important to note that in the buffer-less slotted ALOHA network, due to packet preemptions, the throughput is different from the arrival rate. Fig. 10 shows the achievable throughput region of both networks for fixed values of TRs. We observed that although the total achievable throughput region of both nodes are equally important in the symmetric slotted ALOHA network and observed that the optimal PAoI is obtained when the arrival rate equals one and the throughput point of view. But it is also superior from the achievable throughput point of view.

At the end, similar to [42], which has found AoI-limited capacity of massive MIMO systems, in Fig. 11, for fixed values of TRs, we have investigated the achievable throughput region of the slotted ALOHA network with two nodes when the sum of node 1 and node 2 average PAoIs (i.e., \( \Delta_1 + \Delta_2 \)) is constrained to be less than a specific threshold. Interestingly, this figure shows that by decreasing the threshold, the minimal throughput for each node increases, leading to a smaller achievable throughput region. Moreover, as the average PAoI is affected by the amount of throughput and the average PAoI of both nodes are equally important in \( \Delta_1 + \Delta_2 \), decreasing the threshold of \( \Delta_1 + \Delta_2 \) leads to regions with the lower difference between the throughputs of nodes.

VI. CONCLUSION

In this paper, we studied PAoI in a discrete-time slotted ALOHA network consisting of buffer-less preemptive nodes. We proposed a trellis model to analyze the packet system time and analytically evaluated the exact average PAoI in both asymmetric and symmetric slotted ALOHA networks with packet retransmission. Based on our analytical results, we found PAoI-constrained regions for different values of ARs and TRs. We also numerically optimized the the average PAoI in the symmetric slotted ALOHA network and observed that the optimal PAoI is obtained when the arrival rate equals one and transmission probability equals \( 1/M \) (\( M \) is the number of nodes). Furthermore, we compared the performance of the slotted ALOHA protocol with a non-work-conserving randomized scheduling (centralized) access protocol and concluded that when the ARs are relatively low, slotted ALOHA is superior. By comparing slotted ALOHA networks with and without the assumption of retransmitting the failed packets, we observed that at low arrival rates, retransmission of the failed packets significantly improves PAoI, and at medium arrival rates, it degrades PAoI smoothly. We also compared the achievable throughput region of the slotted ALOHA networks with unlimited buffer and buffer-less nodes for fixed values of TRs. We observed that the achievable throughput region of the buffer-less case includes the achievable throughput region of the unlimited buffer case.

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