Nonequilibrium Self-Interacting Quantum Fields in Cosmology: The Liouville-Neumann Approach*

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We present the so-called Liouville-Neumann (LN) approach to nonequilibrium quantum fields. The LN approach unifies the functional Schrödinger equation and the LN equation for time-independent or time-dependent quantum systems and for equilibrium or nonequilibrium quantum systems. The LN approach is nonperturbative in that at the lowest order of coupling constant it gives the same results as those of the Gaussian effective potential at the zero and finite temperature in a Minkowski spacetime. We study a self-interacting quantum field in an expanding Friedmann-Robertson-Walker Universe. By studying a toy model of anharmonic oscillator and finding the underlying algebraic structure we propose a scheme to go beyond the Gaussian approximation.

I. INTRODUCTION

Recently there have been much interests in the cosmological scenarios. These models have been developed to describe the evolution of the Universe from the early stage to the present epoch. Quantum effects or high temperature thermal effects of matter fields have dominated in the early Universe. But in the cosmological scenarios the Universe has been evolving and expanding. Quantum fields in such an expanding background spacetime could not be described properly as thermal equilibrium. They are characterized by nonequilibrium quantum processes. Quantum nature of the matter fields predominated as it dated back into the early stage of the Universe. Several field theoretical methods have been introduced to describe properly the nonequilibrium quantum process. Typical and frequently used methods are introduced time path and thermal field dynamics. Closed time paths integrate over contours of imaginary (or complex) time and real time. Thermal properties of quantum field are exhibited through the contours of imaginary time. Thermal field dynamics doubles the degrees of freedom of the system and thermal states are condensation of particles of the system and its fictitious counterpart. Despite wide applications closed time paths are difficult to do integrals for self-interacting quantum fields and to incorporate various initial quantum conditions. It would be more than just an interest to have another method, particularly a canonical method, for nonequilibrium quantum fields.

In this paper we present another field theoretical method, the so-called Liouville-Neumann approach, to describe the nonequilibrium quantum processes in cosmology. It is a nonperturbative canonical method that unifies the Liouville-Neumann (LN) equation and the functional Schrödinger equation. It is nonperturbative in that at the lowest order of coupling constant the wave functionals take into account nonperturbative quantum effects and are equivalent to an infinite summation of daisy and superdaisy diagrams or the Gaussian effective potential. In particular, it is the LN equation that describes the nonequilibrium quantum process. The underlying assumption of the LN approach is that all nonequilibrium processes are consequence of microscopic processes which are well described by quantum theory. An introduction of canonical method will enlarge our understanding of the nonequilibrium quantum processes that have mostly been dealt with the closed time path integral methods but are still not completely understood. Furthermore, the LN approach has several advantageous points compared with the path integral methods. First of all, it is truly canonical. Many useful concepts and ideas from quantum mechanics can straightforwardly be applied to quantum field theory by treating a quantum field as a system with infinite degrees of freedom. Secondly, the Hilbert space of the exact or approximate wave functionals is constructed explicitly. Therefore, it is relatively easy to incorporate various initial quantum conditions and to evolve them. Moreover, the LN approach provides a scheme to go beyond the Gaussian approximation. Thirdly, the LN approach can readily be unified with thermal field dynamics due to the canonical nature. Besides these there are also some other useful points.

The organization of this paper is as follows. In Sec. II we introduce the LN equation to solve time-dependent quantum systems and unify it with the functional Schrödinger equation to describe quantum fields. In Sec. III the LN approach is applied to a massive scalar field in an expanding Friedmann-Robertson-Walker (FRW) Universe and the quantum states or Wightmann functions are found for various initial conditions. In Sec. IV we apply the LN approach to a self-interacting quantum field in the FRW Universe and find various quantum states or wave functionals. A toy model of anharmonic oscillator is studied to go beyond the Gaussian approximation. Comparison of the LN approach is made with the Gaussian effective potential. In Sec. V the LN approach is further exploited beyond the Gaussian approximation and reveals the algebraic struc-
tecture of the anharmonic oscillator.

II. LIOUVILLE-NEUMANN APPROACH

As mentioned in the Introduction there are several motivations to study the LN approach to quantum field theory. The LN approach can be used in a unified way to both time-independent and time-dependent quantum systems, and to both equilibrium and nonequilibrium systems. It is a canonical method that deals with the nonperturbative quantum effects. It is canonical because it provides a method to solve the functional Schrödinger equation. As a field theoretical method we wish to unify the LN equation and the functional Schrödinger-picture to make the so-called LN approach.

A. Liouville-Neumann equation

The LN approach to quantum mechanics has been introduced by Lewis and Riesenfeld [3] to solve time-dependent quantum systems, especially time-dependent harmonic oscillators. It is based on the observation that the Schrödinger equation ($\hbar = 1$)

$$i \frac{\partial}{\partial t} \Psi(t) = \hat{H}(t)\Psi(t)$$

(1)

for time-dependent as well as time-independent systems can be solved in terms of the following operator

$$i \frac{\partial}{\partial t} \hat{O} + [\hat{O}, \hat{H}] - i \frac{\delta}{\delta\phi}(\frac{\delta}{\delta\pi}) = 0,$$

$$\hat{O}|\eta_n(t)\rangle = \eta_n(t)|\eta_n(t)\rangle.$$

(2)

The exact quantum states are indeed given by

$$\Psi(t) = \sum_n c_n \exp[i \int dt \langle \eta_n(t)|i \frac{\partial}{\partial t} - \hat{H}|\eta_n(t)\rangle] |\eta_n(t)\rangle.$$

(3)

In particular, when $\hat{O}$ satisfies the LN equation

$$i \frac{\partial}{\partial t} \hat{O} + [\hat{O}, \hat{H}] = 0,$$

(4)

its eigenvalues are time-independent, $\dot{\eta}_n = 0$. Such an operator is called the Lewis-Riesenfeld invariant operator or generalized invariant operator or LN operator.

The aim of this paper is to develop this LN approach to quantum mechanical systems as a field theoretical method in the functional Schrödinger-picture.

B. Functional Schrödinger-Picture

The functional Schrödinger-picture approach to quantum field theory is a canonical method in which the wave functionals of a field as quantum states are determined by the functional Schrödinger equation [6]

$$i \frac{\partial}{\partial t} \Psi(\phi, t) = \hat{H}(\phi, -i \frac{\delta}{\delta\phi})\Psi(\phi, t),$$

(5)

where $\phi$ represents the quantum field. The Hamiltonian describes an infinite dimensional system since the field has infinite degrees of freedom. The wave functionals $\Psi(\phi, t)$ are specified by assigning c-numbers to the field $\phi(x)$ at a fixed time. The wave functionals constitute the Hilbert space equipped with the inner product

$$\langle \Psi_1|\Psi_2 \rangle = \int D[\phi]\Psi_1^\dagger(\phi)\Psi_2(\phi).$$

(6)

Operators act on the wave functional of the Hilbert space

$$\hat{O}(\phi, \pi)|\Psi(\phi, \pi)\rangle \rightarrow \hat{O}(\phi, -i \frac{\delta}{\delta\phi})|\Psi(\phi, \pi)\rangle.$$

(7)

The Gaussian wave functionals are a generalization of the Gaussian states of a harmonic oscillator:

$$\Psi_0(\phi) = N \exp\left[-\int_{x,y} (\phi(x) - \bar{\phi}(x, t))^2 \right] \times \left(\frac{1}{4G(x, y, t)} - i \Sigma(x, y, t)\right)(\phi(y) - \bar{\phi}(y, t))$$

$$+ i \int_x \bar{\pi}(x, t)(\phi(x) - \bar{\phi}(x, t)) \right],$$

(8)

where

$$\langle \Psi_0|\bar{\phi}(x)|\Psi_0 \rangle = \bar{\phi}(x, t),$$

$$\langle \Psi_0|\bar{\pi}(x)|\Psi_0 \rangle = \bar{\pi}(x, t),$$

$$\langle \Psi_0|\bar{\phi}(x)\phi(y)|\Psi_0 \rangle = \bar{\phi}(x, t)\bar{\phi}(y, t) + G(x, y, t),$$

$$\langle \Psi_0|\bar{\pi}(x)\pi(y)|\Psi_0 \rangle = \bar{\pi}(x, t)\bar{\pi}(y, t) + \Sigma(x, y, t).$$

(9)

III. FREE SCALAR FIELD IN THE FRW UNIVERSE

As an application of the LN approach, we consider a free massive scalar field in a spatially flat FRW Universe with the metric

$$ds^2 = -dt^2 + R^2(t)dx^2.$$  

(10)

The scalar field has the Lagrangian

$$L = \int d^3x \frac{R^3}{2} \left(\dot{\phi}^2 - \frac{(\nabla\phi)^2}{R^2} - (m^2 + \xi^{(4)}R)\phi^2 \right),$$

(11)
where \( \xi = 0, \frac{1}{2} \) for the minimal and conformal couplings, respectively, and \( ^{(4)}R \) is the scalar curvature. The Hamiltonian has the form

\[
H = \int d^3x \left[ \frac{\pi^2}{2R^3} + \frac{R}{2} (\nabla \phi)^2 + \frac{R^3}{2} (m^2 + \xi ^{(4)}R)^2 \right].
\]

We quantize the scalar field according to the functional Schrödinger-picture

\[
i \frac{\partial}{\partial t} \Psi(\phi) = \hat{H}(\phi, -i \frac{\delta}{\delta \phi}) \Psi(\phi).
\]

Knowing the Fock space of exact quantum states for a time-dependent harmonic oscillator, we decompose the Hamiltonian into a sum of harmonic oscillators. For this purpose, we decompose the field into Fourier-modes

\[
\phi(x, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \hat{\phi}_k(t) e^{ik \cdot x},
\]

and refine the field modes as

\[
\phi_k^{(+)} = \frac{1}{2} (\hat{\phi}_k + \hat{\phi}_{-k}),
\]

\[
\phi_k^{(-)} = \frac{1}{2} (\hat{\phi}_k - \hat{\phi}_{-k}).
\]

Space integrations can easily be done for the field and momentum:

\[
\int d^3x \hat{\phi}_k^{(+)}(x, t) = \int d^3k \hat{\phi}_k(\phi, \hat{\phi}_{-k}) \equiv \sum_{\alpha} \hat{\phi}_{\alpha},
\]

\[
\int d^3x \hat{\phi}_k^{(-)}(x, t) = \int d^3k \hat{\pi}_k(\hat{\pi}_{-k}) \equiv \sum_{\alpha} \hat{\pi}_{\alpha},
\]

where \( \alpha \) denotes \((\pm, k)\). Then the Hamiltonian takes the form

\[
\hat{H} = \sum_{\alpha} \left[ \frac{\hat{\pi}_{\alpha}^2}{2R^3} + \frac{R^3 \omega_{\alpha}^2}{2} \hat{\phi}_{\alpha} \right] \equiv \sum_{\alpha} \hat{H}_\alpha,
\]

where

\[
\omega_{\alpha}^2(t) = m^2 + \frac{k^2}{R^2} + \xi ^{(4)}R.
\]

That is, the Hamiltonian is a countably infinite sum of time-dependent harmonic oscillators. The functional Schrödinger equation (13) becomes the ordinary Schrödinger equation for the infinite system

\[
i \frac{\partial}{\partial t} \Psi(\phi) = \sum_{\alpha} \hat{H}_\alpha(\phi, \hat{\phi}_{\alpha}) \Psi(\phi).
\]

The wave functional is a product of each wave function

\[
\Psi(\phi) = \prod_{\alpha} \Psi(\phi_{\alpha}),
\]

In the end the right hand side will be expressed in terms of the field \( \phi \) by taking an appropriate inverse Fourier-transform.

In the LN approach, one looks for the following first order operators

\[
\hat{a}_{\alpha} = i(\varphi_{\alpha}^{*}(t)\hat{\pi}_{\alpha} - \varphi_{\alpha}(t)\hat{\phi}_{\alpha}),
\]

\[
\hat{a}_{\alpha}^{\dagger} = -i(\varphi_{\alpha}^{*}(t)\hat{\pi}_{\alpha} - \varphi_{\alpha}(t)\hat{\phi}_{\alpha}).
\]

And then one imposes the LN equations

\[
i \frac{\partial}{\partial t} \hat{a}_{\alpha} + [\hat{a}_{\alpha}, \hat{H}_\alpha] = 0,
\]

\[
i \frac{\partial}{\partial t} \hat{a}_{\alpha}^{\dagger} + [\hat{a}_{\alpha}^{\dagger}, \hat{H}_\alpha] = 0.
\]

Equations (22) are satisfied when each mode satisfies the classical equation of motion

\[
\ddot{\phi}_{\alpha}(t) + \frac{3R}{R^3} \dot{\phi}_{\alpha}(t) + \omega_{\alpha}^2 \phi_{\alpha}(t) = 0.
\]

One further requires \( \hat{a}_{\alpha} \) and \( \hat{a}_{\alpha}^{\dagger} \) to form the basis for the Fock space:

\[
[\hat{a}_{\alpha}, \hat{a}_{\beta}^{\dagger}] = \delta_{\alpha, \beta}.
\]

These commutation relations lead to the boundary conditions on \( \varphi_{\alpha} \)

\[
R^3(\dot{\varphi}_{\alpha}^{*} \varphi_{\alpha} - \dot{\varphi}_{\alpha} \varphi_{\alpha}^{*}) = i.
\]

Then the Fock space of the number states of each mode is constructed by

\[
\hat{a}_{\alpha}|0_{\alpha}, t\rangle = 0,
\]

\[
|n_{\alpha}, t\rangle = \frac{1}{\sqrt{n_{\alpha}}}(\hat{a}_{\alpha})^{n_{\alpha}}|0_{\alpha}, t\rangle.
\]

The vacuum state of the field is the product of vacuum states for each mode

\[
|0, t\rangle = \prod_{\alpha}|0_{\alpha}, t\rangle.
\]

A few comments are in order. The vacuum state has all the symmetries of the Lagrangian. Knowing the Fock space of exact quantum states, it is easy to incorporate various initial quantum conditions.

Usually initial conditions are prepared at a particular time \( t_0 \). The vacuum and the number states are denoted by \( |0_{\alpha}, t_0\rangle \) and \( |n_{\alpha}, t_0\rangle \). First, with respect to the initial vacuum state, one finds the Wightmann function

\[
\langle \hat{\phi}(\mathbf{x}, t) \hat{\phi}(\mathbf{x}', t') \rangle_{\text{vac}} = \frac{1}{(2\pi)^3} \sum_{\alpha_1, \alpha_2} e^{i(k_1 \cdot x - k_2 \cdot x')} \langle \hat{\phi}_{\alpha_1}(t) \hat{\phi}_{\alpha_2}(t') \rangle_{\text{vac}}.
\]

\[
= \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x - x')} \varphi_k(t) \varphi_k(t').
\]

(28)
Second, for an initial thermal state
\[ \hat{\rho}_\alpha(t_0) = \frac{1}{\text{Tr}e^{-\beta H_\alpha(t_0)}} e^{-\beta H_\alpha(t_0)}, \] (29)
the Wightmann function is found
\[ \langle \hat{\phi}(\mathbf{x}, t) \hat{\phi}(\mathbf{x}', t') \rangle_{\text{therm.}} = \langle \hat{\phi}(\mathbf{x}, t) \hat{\phi}(\mathbf{x}', t') \rangle_{\text{vac.}} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\omega_k(t_0)} - 1} \times \left( \varphi_k(t) \varphi_k(t') + \varphi_k(t) \varphi_k^*(t') \right). \] (30)

Finally, let us consider a coherent state described by the density operator
\[ \hat{\rho}_{\alpha,\text{coh}}(t_0) = \frac{1}{\text{Tr}\hat{\rho}_{\alpha,\text{coh}}(t_0)} \exp \left[ -\beta(\omega_\alpha(t_0) \hat{a}_\alpha^\dagger \hat{a}_\alpha + \gamma_\alpha \hat{a}_\alpha^\dagger + \gamma_\alpha^* \hat{a}_\alpha + \epsilon_\alpha) \right], \] (31)
where \( \epsilon_\alpha = \omega_\alpha^2 + \omega_\alpha \). Equation (31) indeed describes a thermal state displaced by
\[ \hat{D}_\alpha = e^{-\frac{\omega_\alpha}{2} \hat{a}_\alpha^\dagger \hat{a}_\alpha + \frac{\omega_\alpha}{2} \hat{a}_\alpha}. \] (32)

The Wightmann function is found
\[ \langle \hat{\phi}(\mathbf{x}, t) \hat{\phi}(\mathbf{x}', t') \rangle_{\text{coh.}} = \langle \hat{\phi}(\mathbf{x}, t) \hat{\phi}(\mathbf{x}', t') \rangle_{\text{therm.}} + \int \frac{d^3k}{(2\pi)^3} \varphi_{k,c}(t) \varphi_{k,c}(t'), \] (33)
where
\[ \varphi_{k,c} = \frac{1}{\omega_k}(\gamma_k \varphi_k^* + \gamma_k^* \varphi_k), \] (34)
is a classical field.

**IV. SELF-INTERACTING SCALAR FIELDS**

In this section we apply the LN approach to a self-interacting quantum field. There have been introduced different methods to investigate the nonperturbative quantum effects. The most typical nonperturbative method is the Gaussian effective potential (variational Gaussian approximation) approach \( \tilde{\mathcal{G}} \). But this method has been applied to time-independent quantum fields. The Gaussian wave functionals have also been considered in Ref. \( \mathcal{G} \). The problem of solving the covariant kernel is not completed. Compared with these methods, the LN approach provides us with the explicit wave functionals in terms of the field equations satisfied by each mode. The LN approach has recently been applied to quantum fields in curved spacetime \( \tilde{\mathcal{G}} \).

**A. (0 + 1)-Dimensional Toy Model**

To illustrate how the LN approach works for a self-interacting quantum field, first we consider a time-dependent anharmonic oscillator. More than just a quantum mechanical system, the anharmonic oscillator has been used as an important toy model to test various field theoretical methods.

Let us consider the anharmonic oscillator
\[ \hat{H} = \frac{\hat{p}^2}{2} + \omega_0^2(t) \hat{x}^2 + \lambda(t) \frac{\hat{x}^4}{4}. \] (35)
The mass has been rescaled for convenience. As in the case of harmonic oscillators in Sec. III, at the lowest order of the coupling constant one optimizes the Hilbert space by a Fock space of the annihilation and creation operators
\[ \hat{a}(t) = i(y^*(t) \hat{p} - y(t) \hat{q}), \]
\[ \hat{a}^\dagger(t) = -i(y(t) \hat{p} - y^*(t) \hat{q}). \] (36)

We require these operators to minimize the energy or the Dirac action in the time-independent case or time-dependent case, respectively. In the time-dependent case we can not use the minimization of the energy, because the energy is not conserved. In the time-independent case, these two principles of minimization give the identical result. The resulting quantum state is the Gaussian wave function. In the time-dependent case, it has been found that the LN approach gives the same result as that of the Dirac action principle at the lowest order of the coupling constant. However, the LN approach has several advantageous points. First, it is an algebraic method for the anharmonic oscillator problem. The underlying group structure proves very convenient and useful. The underlying structure of the anharmonic oscillator will be studied in detail in the next section. Second, the quantum states can be found nonperturbatively to any desired order of the coupling constant.

Then the anharmonic oscillator has the oscillator representation
\[ \hat{H} = \hat{H}_G + \lambda \hat{H}', \] (37)
where
\[ \hat{H}_G = \left[ \hat{y}^* \hat{y} + \omega_0^2 y^2 y^* y + 3 \lambda(y^* y) \right] \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \frac{3 \lambda}{4} (y^* y)^2 \\
+ \frac{1}{2} \left[ \hat{y}^* \hat{y} + \omega_0^2 y^2 y^* y + 3 \lambda(y^* y) y^2 \right] \hat{a}^\dagger \hat{a} \\
+ \frac{1}{2} \left[ \hat{y}^2 + \omega_0^2 y^2 + 3 \lambda(y^* y) y^2 \right] \hat{a}^2, \] (38)
\[ \hat{H}' = -\frac{1}{4} \sum_{k=0}^{4} \left( \begin{array}{c} 4 \\ k \end{array} \right) y_k^* \hat{y}^k \hat{a}^\dagger \hat{a}^k. \] (39)

At the lowest order of the coupling constant, we use the approximate LN equations
The recursive relations can be solved iteratively
\[ \hat{B}(1) = i \int [\hat{a}, \hat{H}'], \]
\[ \vdots \]
\[ \hat{B}(n) = i^n \int \cdots \int \left[ \left[ \cdots \left[ \hat{a}, \hat{H}' \right], \hat{H}' \right], \cdots, \hat{H}' \right]. \] (49)

For instance the first order corrections are found
\[ \hat{B}(1) = 4 \left( i \int y^4 \right) \hat{a}^3 + 3 \left( i \int y^3 y \right) \hat{a}^2 \hat{a} \]
\[ + 2 \left( i \int y^2 y^2 \right) \hat{a}^2 \hat{a}^2 + \left( i \int y^2 y^2 \right) \hat{a}^3. \] (50)

An improved ground state is obtained from
\[ \hat{A}(1)[0][1] = 0, \] (51)
where
\[ \hat{A}(1) = \hat{a} + \lambda \hat{B}(1). \] (52)

This procedure can be repeated to solve Eq. (46). The quantum states at any order of the coupling constant are obtained therefrom.

A comment is in order. In the language of field theory, the wave functional determined by \( \hat{a} \) and \( \hat{a}^\dagger \) corresponds to a free propagator with a renormalized mass and that determined by \( \hat{A} \) and \( \hat{A}^\dagger \) in Eq. (46) corresponds to a summation of all loop diagrams. In this sense Eq. (49) can be interpreted as the Feynman rules on the closed time path for the time-dependent system.

### B. Self-Interacting Scalar Field

We now turn to a self-interacting quantum field. A quantum field is equivalent to a system of infinitely mode-decomposed harmonic or anharmonic oscillators. Thus the results from the anharmonic oscillator system are expected to be useful in studying the self-interaction quantum field. Though the formalism (LN approach) to be developed in this paper can be applied to any quantum field theory, either renormalizable or nonrenormalizable, we shall confine our attention to the \( \phi^4 \)-theory. The reason is that the results from our formalism can readily be compared with those from other methods.

The \( \phi^4 \)-theory in the FRW Universe has the Hamiltonian density
\[ H = \frac{\pi^2}{2R^3} + \frac{R}{2} (\nabla \phi)^2 + R^3 \left( \frac{m^2}{2} \phi^2 + \lambda \phi^4 \right). \] (53)

Before the Fourier-decomposition, the field is divided into a classical background field and quantum fluctuations
\[ \phi = \phi_c + \phi_f. \] (54)
The classical background field depends only on the co-moving time and equals to the space average of the field (zero-mode). Quantum mechanically it is a coherent state of the field
\[ \langle \hat{\phi} \rangle = \phi_c(t). \] (55)
We assume the fluctuations to have symmetric quantum states
\[ \langle \hat{\phi}_f \rangle = 0. \] (56)
Then the Hamiltonian density is the sum of classical background field part and quantum fluctuations part:
\[ H = \left[ \frac{\pi^2}{2R^3} + R^3 \left( \frac{m^2}{2} \phi_c^2 + \frac{\lambda}{4} \phi_c^4 \right) \right] + \frac{\pi f}{2R^3} + \frac{R}{2} \langle \nabla \phi_f \rangle^2 \\
+ R^3 \left( \frac{m^2}{2} + \frac{3\lambda}{2} \phi_c^2 \right) \phi_f^2 + \frac{\lambda R^3}{4} \phi_f^4 \right] \\
+ \left[ \frac{\pi f}{R^3} + R^3 \left( \frac{m^2}{2} + \lambda \phi_c^2 \right) \phi_f + \lambda R^3 \phi_c \phi_f^3 \right]. \] (57)
The expectation values of the terms in the last square bracket vanish with respect to the symmetric quantum states of fluctuations. So these terms will not considered any more.
First, we study the classical background field. Only those terms whose expectation values do not vanish with respect to the symmetric quantum states of fluctuations contribute to the Hamiltonian density for the classical background field
\[ H_c = \frac{\pi^2}{2R^3} + R^3 \left( \frac{m^2}{2} + \frac{3\lambda}{2} \langle \hat{\phi}_f^2 \rangle \right) \phi_c^2 + \frac{\lambda}{4} \phi_c^4. \] (58)
The classical background field is frequently treated as a classical system. Then the equation of motion is
\[ \ddot{\phi}_c + \frac{R}{R^3} \dot{\phi}_c + (m^2 + 3\lambda \langle \hat{\phi}_f^2 \rangle) \phi_c + \lambda \phi_c^3 = 0. \] (59)
The infinite contribution from the quantum fluctuations should be regulated in a suitable manner. But more proper treatment is to quantize even the classical background field. Therefore, the background field is nothing but an anharmonic oscillator with the mass shifted to \( m^2 + 3\lambda \langle \hat{\phi}_f^2 \rangle \). The classical background field plays the role of an inflaton in the inflationary scenarios. The importance of such a quantum background field has been shown in Ref. [10]. The annihilation and creation operators of the Fock space are
\[ \hat{a}_c = i(\phi_c^* \hat{\pi}_c - R^3 \hat{\phi}_c^* \phi_c) - i(\phi_c^* \pi_c - R^3 \phi_c^* \phi_c), \]
\[ \hat{a}_c^\dagger = -i(\phi_c(t) \hat{\pi}_c - R^3 \phi_c(t) \phi_c) + i(\phi_c(t) \pi_c - R^3 \phi_c(t) \phi_c). \] (60)
The \( \hat{a}_c \) and \( \hat{a}_c^\dagger \) are chosen so that the vacuum state gives the expectation values
\[ \langle \hat{\phi}_c \rangle = \phi_c, \]
\[ \langle \hat{\pi}_c \rangle = \pi_c = R^3 \phi_c. \] (61)
Requiring the LN equations for \( \hat{a}_c \) and \( \hat{a}_c^\dagger \), it is shown that \( \phi_c \) satisfies the equation
\[ \ddot{\phi}_c + \frac{3R}{R^3} \dot{\phi}_c + (m^2 + 3\lambda \langle \hat{\phi}_f^2 \rangle) \phi_c + \lambda (\phi_c^* \phi_c) \phi_c = 0. \] (62)
In the above procedure, \( \phi_c \) is required to satisfy the same equation (52) as \( \varphi_c \). So the terms proportional to the identity operator in Eq. (51) become the Wronskian of the equation of motion, and constant. This makes our procedure self-consistent. In the minimal uncertainty proposal for the inflation [10], a particular choice of coherent state was assumed
\[ \phi_c = \pi_c = 0, \] (63)
and the quantum states of the inflaton were again the symmetric Gaussian states.
Next, we consider the quantum fluctuations. The terms in the second square bracket of Eq. (57) give rise to the Hamiltonian density for quantum fluctuations
\[ H_f = \frac{\pi f}{2R^3} + \frac{R}{2} \langle \nabla \phi_f \rangle^2 + R^3 \left( \frac{m^2}{2} + \frac{3\lambda}{2} \phi_c^2 \right) \phi_f^2 + \frac{\lambda R^3}{4} \phi_f^4. \] (64)
Upon decomposing into Fourier-modes, we keep only those terms with nonvanishing contribution
\[ \int d^3x \phi_f^\dagger(x, t) = \frac{3}{(2\pi)^3} \sum_{\alpha \neq \beta} \phi_{\alpha}^\dagger \phi_{\beta}^\dagger + \frac{1}{(2\pi)^3} \sum_{\alpha} \phi_{\alpha}^\dagger. \] (65)
We express the fluctuations and their momenta in terms of the annihilation and creation operators
\[ \hat{\phi}_{\alpha} = \phi_{\alpha} \hat{a}_{\alpha} + \phi_{\alpha}^* \hat{a}_{\alpha}^\dagger, \]
\[ \hat{\pi}_{\alpha} = R^3 \left( \phi_{\alpha} \hat{\pi}_{\alpha} + \phi_{\alpha}^* \hat{\pi}_{\alpha}^\dagger \right). \] (66)
As for the anharmonic oscillator, we find the oscillator representation of the Hamiltonian for fluctuations
\[ \hat{H}_f = \hat{H}_{f,G} + \lambda \hat{H}_f', \] (67)
where \( \hat{H}_{f,G} \) and \( \hat{H}_f' \) are the quadratic and non-quadratic parts. At the lowest order of the coupling constant we are interested in \( \hat{H}_{f,G} \):
\[ \hat{H}_{f,G} = R^3 \sum_{\alpha} \left[ \phi_{\alpha}^\dagger \phi_{\alpha} + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \phi_{\alpha}^\dagger \phi_{\alpha} \right. \]
\[ + 3\lambda \left( \frac{1}{2\pi} \right)^3 \sum_{\beta} (\phi_{\beta} \phi_{\beta}^*) \phi_{\alpha}^\dagger \phi_{\alpha} \left( \hat{a}_{\alpha}^\dagger \hat{a}_{\alpha} + \frac{1}{2} \right) \]
\[ - \frac{3\lambda R^3}{4} \left( \frac{1}{(2\pi)^3} \sum_{\alpha} \phi_{\alpha}^\dagger \phi_{\alpha} \right)^2. \]
The equations of motion for the variables $\lambda \Phi$ are determined from the LN equations of its number states. The quantum state of the fluctuations is a product of such quantum states and can be relatively easily incorporated. This LN approach is expected to be a powerful and convenient method especially for studying the early Universe and the time-dependent process of critical phenomena.

\[ + \frac{R^3}{2} \sum_{\alpha} \left[ \dot{\Phi}_{\alpha}^2 + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \Phi_{\alpha}^2 \right] \\
+ 3\lambda \left( \frac{1}{(2\pi)^3} \sum_{\beta} \Phi_{\beta}^2 \Phi_{\beta} \Phi_{\alpha} \right) \Phi_{\alpha}^2 \\
+ \frac{R^3}{2} \sum_{\alpha} \left[ \dot{\Phi}_{\alpha}^2 + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \Phi_{\alpha}^2 \right] \Phi_{\alpha}^2 \\
+ 3\lambda \left( \frac{1}{(2\pi)^3} \sum_{\beta} \Phi^2 \Phi_{\beta} \Phi_{\alpha} \right) \Phi_{\alpha}^2. \tag{68} \]

The equations of motion for the variables $\Phi_{\alpha}$

\[ \ddot{\Phi}_{\alpha} + \frac{R}{2} \dot{\Phi}_{\alpha} + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \Phi_{\alpha} \\
+ 3\lambda \left( \frac{1}{(2\pi)^3} \sum_{\beta} \Phi^2 \Phi_{\beta} \Phi_{\alpha} \right) \Phi_{\alpha} = 0, \tag{69} \]

are determined from the LN equations

\[ \frac{\partial}{\partial t} \dot{\Phi}_{\alpha} + [\dot{\Phi}_{\alpha}, \hat{H}_{f,G}] = 0, \tag{70} \]

\[ i \frac{\partial}{\partial t} \Phi_{\alpha} + [\Phi_{\alpha}, \hat{H}_{f,G}] = 0. \]

The state annihilated by all annihilation operators is an approximate ground state

\[ \hat{a}_{\alpha}[0,t]|0\rangle = 0. \tag{71} \]

The vacuum expectation value of the Hamiltonian for fluctuations is

\[ [0,0] \langle 0,0| \hat{H}_{f,G} |0,0\rangle = \frac{R^3}{2} \sum_{\alpha} \left[ \dot{\Phi}_{\alpha}^2 + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \Phi_{\alpha}^2 \right] \\
+ 3\lambda \left( \frac{1}{(2\pi)^3} \sum_{\beta} \Phi^2 \Phi_{\beta} \Phi_{\alpha} \right) \Phi_{\alpha}^2. \tag{72} \]

The number states are similarly constructed by

\[ |n_{\alpha},t\rangle = \frac{1}{\sqrt{n_{\alpha}}} (\hat{a}^{\dagger}_{\alpha})^n |0,0\rangle. \tag{73} \]

The quantum state of each mode is a linear superposition of its number states. The quantum state of the fluctuations is a product of such quantum states

\[ \Psi_{f,[0]}(\phi_f) = \prod_{\alpha} \Psi_{\alpha,[0]}(\Phi_{\alpha}). \tag{74} \]

As emphasized before, the LN approach constructs the Hilbert space for the classical background field and fluctuations on which various initial conditions can be relatively easily incorporated. This LN approach is expected to be a powerful and convenient method especially for studying the early Universe and the time-dependent process of critical phenomena.

#### C. Comparison with the Gaussian Effective Potential

In order to compare with the results from other methods, let us consider the Minkowski spacetime

\[ R(t) = 1. \tag{75} \]

The solutions to Eq. (69) are easily found

\[ \phi_{\alpha} = \frac{1}{\sqrt{2\Omega_{\alpha}}} e^{-i\Omega_{\alpha} t}. \tag{66} \]

The $\Omega_{\alpha}$ satisfies the gap equation

\[ \Omega_{\alpha}^2 = k^2 + \mu^2(\phi_c), \tag{77} \]

where

\[ \mu^2(\phi_c) = m^2 + 3\lambda \phi_c^2 + 3\lambda \frac{1}{(2\pi)^3} \sum_{\beta} \frac{1}{2\Omega_{\beta}}, \tag{78} \]

is the renormalized mass. From the equations of the motion, one is left with the quadratic Hamiltonian

\[ \hat{H}_{f,G} = R^3 \sum_{\alpha} \left[ \dot{\Phi}_{\alpha}^* \dot{\Phi}_{\alpha} + \left( m^2 + \frac{k^2}{R^2} + 3\lambda \phi_c^2 \right) \Phi_{\alpha}^* \Phi_{\alpha} \right] \\
+ 3\lambda \left( \frac{1}{(2\pi)^3} \sum_{\alpha} \Phi^2 \Phi_{\beta} \Phi_{\alpha} \right) \Phi_{\alpha}^* \Phi_{\alpha} \\
- \frac{3\lambda R^3}{4} \left( \sum_{\alpha} \Phi_{\alpha} \Phi_{\alpha}^* \right)^2. \tag{79} \]

First, we compute the effective potential at the zero temperature. The effective potential is obtained by taking the expectation of the Hamiltonian (77) except the term $\hat{a}_{\alpha}^*: \Psi_{e_{\alpha}} = m^2 \phi_c^2 + \frac{\lambda}{4} \phi_c^4 + I_0(\mu) - \frac{3\lambda}{2} I_0^2(\mu), \tag{80} \]

where

\[ I_0(\mu) = \frac{1}{(2\pi)^3} \sum_{\alpha} \frac{1}{\Omega_{\alpha}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\Omega_{\alpha}} \tag{81} \]

\[ I_1(\mu) = \frac{1}{(2\pi)^3} \sum_{\alpha} \frac{1}{2} \Omega_{\alpha} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \Omega_{\alpha}. \]

We have thus shown that the effective potential is equal to the Gaussian effective potential.

Next, we compute the effective potential at finite temperature. The effective potential equals to the Helmholtz free energy

\[ F = -\frac{1}{\beta} \ln(Z) = V_{\text{eff}}, \tag{82} \]

where $Z$ is the partition function.
sider again the anharmonic oscillator model for a self-interacting quantum field, we shall con-

-54x-1108o to quantum field theory. As a (0 + 1)-dimensional toy

V. BEYOND THE GAUSSIAN APPROXIMATION

-54x-655the Gaussian effective potential to any desired order

V_{ef}.(\phi_c, \mu) = \frac{m^2}{2} \phi_c^2 + \frac{\lambda}{4} \phi_c^4 + (I_1 + I_1^I) - \frac{3\lambda}{2} (I_0 + I_0^2)^2,

\( V_{ef}.(\phi_c, \mu) = \frac{m^2}{2} \phi_c^2 + \frac{\lambda}{4} \phi_c^4 + (I_1 + I_1^I) - \frac{3\lambda}{2} (I_0 + I_0^2)^2, \)

(85)

where

\[ I_0^\beta(\mu) = \frac{1}{(2\pi)^3} \sum_\alpha \frac{1}{\Omega_\alpha} e^{\beta \Omega_\alpha} \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\Omega(k)} e^{\beta \Omega(k)} - 1, \]

\[ I_1^\beta(\mu) = \frac{1}{(2\pi)^3} \sum_\alpha \ln \left( 1 - e^{-\beta \Omega_\alpha} \right) \]

\[ = \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - e^{-\beta \Omega(k)} \right). \]

(86)

The finite temperature effective potential can also be rewritten as

\[ V_{ef}.(\phi_c, \mu) = V_{ef}.(\phi_c, \mu) + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - e^{-\beta \Omega(k)} \right), \]

(87)

in terms of the renormalized mass

\[ M^2 = m^2 + 3\lambda (I_0 + I_0^2 + \phi_c^2). \]

(88)

Hence the effective potential at finite temperature is equal to the Gaussian effective potential at finite temperature. We have shown that the LN approach to quantum field theory at the lowest order of the coupling constant gives the same results as those of the Gaussian effective potential approach. The real vantage of the LN approach is that one can compute readily the effective potential beyond the Gaussian effective potential to any desired order as will be sketched in the next section.

V. BEYOND THE GAUSSIAN APPROXIMATION

We introduce an algebraic, nonperturbative method to quantum field theory. As a (0 + 1)-dimensional toy model for a self-interacting quantum field, we shall consider again the anharmonic oscillator

\[ \hat{H} = \frac{1}{2} \hat{p}^2 + \frac{m^2}{2} \hat{q}^2 + \frac{\lambda}{4} \hat{q}^4. \]

(89)

One of results of the NL approach is that the anharmonic oscillator has an algebraic structure which has not been much noticed in the literature. In the Fock space we can represent the Hamiltonian in the form

\[ \hat{H} = \Omega_G \hat{A} \hat{A}^I + E_{vac}. \]

(90)

The operators \( \hat{A} \) and \( \hat{A}^I \) do not satisfy the standard commutation relation. They rather satisfy a generalized deformed algebra of the form

\[ \hat{A} \hat{A}^I = F(\hat{A} \hat{A}^I). \]

(91)

The standard commutation relation is given by \( F(x) = x + 1 \), and the \( q \)-deformed algebra by \( F(x) = Q^2 x + 1 \). In the Fock space representation \( F \) is in general a polynomial of \( x \).

It has been found that up to the first order of the coupling constant the anharmonic oscillator has the supersymmetric form

\[ \hat{H} = \frac{\Omega_{[1]}}{2} \left( \hat{A}_{[1]} \hat{A}_{[1]} + \hat{A}_{[1]} \hat{A}^I_{[1]} \right), \]

(92)

where the frequency is determined from the gap equation

\[ \Omega_{[1]} = \Omega_G - \frac{3\lambda}{8\Omega_G^3}. \]

(93)

The explicit form of \( \hat{A} \) and \( \hat{A}^I \) is found from the LN equations. Remarkably these operators satisfy the \( q \)-deformed algebra

\[ \hat{A}_{[1]} \hat{A}^I_{[1]} - Q^2 \hat{A}^I_{[1]} \hat{A}_{[1]} = 1, \]

(94)

where

\[ Q^2 = 1 + \frac{3\lambda}{4\Omega_G^3}. \]

(95)

The first improved vacuum state beyond the Gaussian approximation is defined by

\[ \hat{A}_{[1]} |0\rangle_{[1]} = 0. \]

(96)

The \( q \)-deformed algebra enables one to find the excited states

\[ |n\rangle_{[1]} = \frac{1}{\sqrt{|n|!}} \left( \hat{A}^I_{[1]} \right)^n |0\rangle_{[1]}, \]

(97)

where

\[ |n| = \frac{Q^{2n} - 1}{Q^2 - 1}. \]

(98)

It is straightforward to obtain the energy eigenvalues

\[ E_{[1], n} = \frac{\Omega_{[1]}}{2} \left( |n| + |n + 1| \right). \]

(99)

To any order of the coupling constant, we are able to find the Hamiltonian in the form \( \hat{H} \). For instance,
to the third order of the coupling constant, we find the Hamiltonian
\[
\hat{H} = \hbar \Omega_G \hat{A}_3^\dagger \hat{A}_3 + E_{3,\text{vac}},
\]
where
\[
E_{3,\text{vac}} = \hbar \Omega_G \left[ \frac{1}{2} - \frac{3}{16} \left( \frac{\hbar \lambda}{\Omega_G} \right) + \frac{3}{128} \left( \frac{\hbar \lambda}{\Omega_G} \right)^2 + \frac{25}{1024} \left( \frac{\hbar \lambda}{\Omega_G} \right)^3 \right].
\]

\( E_{3,\text{vac}} \) is the vacuum energy up to the same order of the coupling constant. The generalized deformed algebra \([3]\) is found to be
\[
\hat{A}_3^\dagger \hat{A}_3^\dagger = 1 + \hat{A}_3 \hat{A}_3 + \frac{3}{4} \frac{\hbar \lambda}{\Omega_G^2} \hat{A}_3 + \frac{9}{32} \frac{\hbar \lambda}{\Omega_G^2} \hat{A}_3^\dagger \hat{A}_3 + \frac{3}{16} \frac{\hbar \lambda}{\Omega_G^2} (\hat{A}_3^\dagger \hat{A}_3)^2 + \frac{69}{64} (\hat{A}_3^\dagger \hat{A}_3)^3.
\]

Therefore, one obtains the better improved vacuum state beyond the Gaussian approximation from
\[
\hat{A}_3 |0_3\rangle = 0.
\]
As in the harmonic oscillator case, the excited states are obtained by applying \( \hat{A}_3^\dagger \) to the vacuum state:
\[
|k_3\rangle = \frac{1}{\sqrt{\prod_{k=0}^{k-1} N_k}} (\hat{A}_3)^k |0_3\rangle,
\]
where
\[
\hat{A}_3^\dagger \hat{A}_3 |k_3\rangle = N_k |k_3\rangle.
\]
The eigenstates are constructed to be orthonormal
\[
\langle 0_3 | k_3 \rangle = \delta_{kl}.
\]
It is straightforward to see that the eigenvalues satisfy the following recursive relations
\[
N_k = F(N_{k-1}), \quad N_0 = 0.
\]
Therefore, the anharmonic oscillator has the energy
\[
E_{3,k} = \hbar \Omega_G N_k + E_{3,\text{vac}}.
\]

Though not shown explicitly, the algebraic structure of the quartic anharmonic oscillator with \( q \)-deformed algebra at the first order of the coupling constant or with the generalized deformed algebra at the higher order of coupling constant is expected to play a pivotal role in nonperturbative quantum field theory. For instance the \( q \)-deformed algebra results in the partition function of the anharmonic oscillator valid even for the strong coupling limit. The extension of the \( q \)-deformed algebra to the self-interacting quantum field in Sec. IV will give rise to much more correctly the effective potential at the zero or finite temperature than the Gaussian effective potential.

\textbf{VI. SUMMARY AND OUTLOOK}

In this paper we have introduced the so-called Liouville-Neumann (LN) approach to nonequilibrium quantum fields. It is a canonical method that unifies the functional Schrödinger equation with the LN equation. The LN approach is nonperturbative in that at the lowest order of the coupling constant it gives the same results as the Gaussian effective potential. The LN approach is universal since it is equally applicable to time-independent and time-dependent quantum systems, and to equilibrium and nonequilibrium systems. Though not shown in this paper, the LN approach can be unified with thermal field dynamics by doubling the degrees of freedom. This is achieved by introducing a fictitious Hamiltonian \( \hat{H} \) and enlarging the Hilbert space constructed in this paper. The LN approach combined with thermal field dynamics is expected to give more physical intuition to our understanding of nonequilibrium quantum fields. The renormalization problem and the partition function up to higher orders of the coupling constant are not dealt with. Interestingly enough, fermion systems and gauge theories are challenging problems to the LN approach.

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