Effective range function below threshold

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Abstract. We demonstrate that the kernel of the Lippmann-Schwinger equation, associated with interactions consisting of a sum of the Coulomb plus a short range nuclear potential, below threshold becomes degenerate. Taking advantage of this fact, we present a simple method of calculating the effective range function for negative energies. This may be useful in practice since the effective range expansion extrapolated to threshold allows to extract the low-energy scattering parameters: the Coulomb modified scattering length and the effective range.

PACS numbers: 03.65.Ge, 02.03.Rz, 21.10.Sf, 36.10.Gv

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1. Introduction

Effective range function plays central role in the analysis and interpretation of two-particle low-energy scattering data. It is well known from the simplest case of s-wave scattering by a short range potential that it is more advantageous to study the effective range function \( k \cot \delta \) than the phase shift \( \delta \). The former is an even function of the momentum \( k \) and close to threshold can be expanded in a power series of \( k^2 \), known as the effective range expansion. The first two coefficients in this expansion: the scattering length and the effective range, respectively, carrying important information about the underlying interaction, are regarded as fundamental parameters in low-energy scattering phenomenology. The inclusion of the long range Coulomb interaction is a non-trivial complication because the latter changes dramatically the singularity structure of the scattering matrix. The Coulomb corrections to the effective range parameters are not only model dependent but can be also quite sizable. Especially sensitive is the scattering length where, strictly speaking, there is even no upper limit to the size of the Coulomb correction, i.e. the scattering length that is finite in absence of Coulomb interaction, could become infinite when Coulomb interaction is turned on. Perhaps the best illustration that in real life Coulomb correction can be quite large is provided by the nucleon-nucleon scattering. With exact isospin invariance the nn and pp \( ^1S_0 \) scattering lengths in absence of Coulomb interaction should be equal but experimentally \( \hbar \)\( ^1S_0 \): \( a_{nn} = -18.6 \pm 0.5 \text{ fm} \) and \( a_{pp} = -7.828 \pm 0.008 \text{ fm} \). It has been confirmed by model calculations that most of the difference in these two values may be attributed to the Coulomb interaction.

The analytic continuation of the effective range function to negative energies presents considerable interest as it provides simple means to locate the near-threshold singularities of the scattering matrix, that might be associated with shallow bound states. For conducting a direct calculation of the effective range function below threshold, the standard Coulomb wave functions for negative energies are required, and, efficient algorithms for calculating them are currently available in the literature \( \hbar \)\( ^1S_0 \). Nevertheless, this approach leads to difficulties as the standard Coulomb wave functions are ill chosen for the purpose and for attractive Coulomb potential they are singular at the Coulomb bound state energies. Therefore, the usual procedure employs the effective range expansion for effecting the extrapolation to negative energies.

In this work we wish to present a direct method of calculating the effective range function below threshold based on the solution of the Lippmann-Schwinger equation. In this equation the Coulomb potential does not appear explicitly but has been accounted for exactly by introducing Coulomb-modified Green’s function and Coulomb distorted ingoing wave. By selecting a specific Coulomb Green’s function, intimately connected with the K-matrix, we have been able to devise a calculational scheme that is free from the Coulomb singularities. Another advantage of the proposed method is that it provides for a simple solution of the Lippmann-Schwinger equation. This is a consequence of the fact that for negative energies the Coulomb Green’s function has a separable representation of the form of a Sturmian expansion so that the kernel of the integral equation becomes degenerate. The task of solving the Lippmann-Schwinger equation is thereby reduced to that of solving a system of linear algebraic equations. The method can be applied either for calculating the scattering matrix, or the K-matrix below threshold. In particular, this approach can be used for locating the poles of the scattering matrix what is equivalent to solving the bound state problem. The K-matrix yields the effective range function and allows to extract the low-energy
scattering parameters.

The plan of the presentation is as follows. In the next section we outline
the Lippmann-Schwinger equation formalism necessary for calculating the scattering
matrix as well as the K-matrix. In section 3, we employ the Sturmian expansion of
the Coulomb Green’s function for converting the underlying integral equations into
a system of linear algebraic equations. Finally, in the last section we discuss possible
applications and examine numerically the performance of the presented method.

2. Effective range function

Consider a two-body, two-potential quantum mechanical scattering problem with
spherical symmetry where the interaction is a superposition of the Coulomb potential
plus a short ranged nuclear potential $V(r)$. It will be assumed in the following that
the nuclear potential for large separations $r$ has at least exponential fall-off
$\sim e^{-\lambda r}$ with $\lambda > 0$ being the inverse range parameter of the nuclear force. For small
$r$, we assume that the behaviour of $V(r)$ is not worse than $r^{-2}$. The Coulomb potential
corresponding to point-like charges is taken as $\alpha Z/r$ where $Z$ is dimensionless strength
parameter $Z < 0$ ($Z > 0$) for attraction (repulsion) and $\alpha$ is the fine structure constant.

The effective range function is defined as the inverse of the K-matrix $K_\ell(k)$ and is
related to the scattering matrix by means of a generalized Heitler formula

$$\frac{1}{K_\ell(k)} = \frac{1}{\tau_\ell(k)} + g_\ell(k)$$

(1)

where $\ell$ denotes the orbital momentum and the independent complex variable $k$ on
the positive real axis becomes the physical c.m. momentum and the complex function
$g_\ell(k)$ can be interpreted as a generalized barrier penetration factor. This function has
been studied in several papers [6, 7, 8, 9, 10, 11], and, $g_\ell(k) = i k^{2\ell+1}/[(2\ell + 1)!]^2$ for
$Z = 0$, whereas for $Z \neq 0$ its explicit form is

$$g_\ell(k) = (2\mu\alpha Z)^{2\ell+1}/[(2\ell + 1)!]^2 \prod_{m=0}^{\ell} \left(1 + m^2/\eta^2\right)$$

$$\times \left\{ \frac{1}{2} \left[ \psi(1 + i\eta) + \psi(1 - i\eta) \right] - (\pi/2) \cot(i\eta \pi) - \log(i\mu|Z|/k) \right\},$$

(2)

where $\eta = \mu\alpha Z/k$, $\mu$ is the reduced mass of the system and $\psi$ denotes the logarithmic
derivative of the gamma function. We choose the units system where $\hbar = c = 1$. In
(2) the complex function $\tau_\ell(k)$ is the scattering matrix that will be obtained from the
solution of the Lippmann-Schwinger equation incorporating the appropriate boundary
condition

$$u_\ell(k,r) = \phi(k,r) + \int_0^\infty \langle r | G_\ell^+(k) | r' \rangle V(r') u_\ell(k,r') \, dr',$$

(3)

where $u_\ell(k,r)$ is the sought for wave function. In order to account for the Coulomb
interaction, in (3) the ingoing plane wave has been replaced by the regular Coulomb
wave function $\phi_\ell(k,r)$ and, respectively, the free resolvent by the Coulomb Green’s
function $\langle r | G_\ell^+(k) | r' \rangle$ satisfying the outgoing wave boundary condition. It is easily
checked that acting on both sides of (3) with the Coulomb Hamiltonian, we retrieve the
Schrödinger equation appropriate for the two-potential problem under consideration.

Given the solution of (3), the scattering matrix $\tau_\ell(k)$ is obtained from the formula

$$\tau_\ell(k) = -2\mu \int_0^\infty \phi_\ell(k,r) V(r) u_\ell(k,r) \, dr,$$

(4)
where the regular Coulomb solution \( \phi_\ell(k, r) \) satisfying the following boundary condition at \( r = 0 \)
\[
\lim_{r \to 0} r^{-\ell-1} \phi_\ell(k, r) = 1,
\]
is well known \([10]\) and can be expressed analytically as
\[
\phi_\ell(k, r) = r^{\ell+1} e^{\pm i kr} F_1(\ell + 1 \pm i\eta, 2\ell + 2; \mp 2i kr),
\]
where \( F_1(a; b; z) \) denotes the confluent hypergeometric function \([12]\). Since the boundary condition is independent of \( k \), it is evident that \( \phi_\ell(k, r) \) must be entire analytic function of \( k^2 \) for every non-negative value of \( r \) and is real for real \( k^2 \).

For physical momenta \( k \) the function \( \tau_\ell(k) \) is related to the Coulomb modified nuclear phase shift \( \delta_\ell \)
\[
1/\tau_\ell(k) = C_\ell(\eta)^2 k^{2\ell} (k \cot \delta_\ell - i k)
\]
where
\[
C_\ell(\eta)^2 k^{2\ell} = C_0(\eta)^2 \frac{(2\mu\alpha Z)^{2\ell+1}}{[(2\ell + 1)!]^2} \prod_{m=0}^{\ell} \left( 1 + \frac{m^2}{\eta^2} \right),
\]
and
\[
C_0(\eta)^2 = 2\pi\eta/\left[ \exp(2\pi\eta) - 1 \right].
\]
Although for physical momenta \( k \) the functions \( 1/\tau_\ell(k) \) and \( g_\ell(k) \) are both complex but their imaginary parts are equal in magnitude and have opposite signs. Indeed, in this case from \([3]\), one has
\[
g_\ell(k) = C_\ell(\eta)^2 k^{2\ell} \left\{ (2\mu\alpha Z)[\Re \psi(1 + i\eta) - \log(|\eta|)]/C_0(\eta)^2 + i k \right\}
\]
so that using \([3]\) and \([4]\) in \([4]\), the imaginary parts are cancelled and the resulting effective range function \([4]\) is real, as it should be. However, the effective range function is known to be real analytic function of \( k^2 \), so that it must be real also on the imaginary axis. For \( k \) located on the positive imaginary axis the Green’s function and \( \phi_\ell(k, r) \) are both real what implies that \( \tau_\ell(k) \) is necessarily real as well. Also the function \( g_\ell(k) \) is real on the imaginary axis
\[
g_\ell(k) = (2\mu\alpha Z)^{2\ell+1}/[(2\ell + 1)!]^2 \prod_{m=0}^{\ell} \left( 1 - m^2/\bar{\eta}^2 \right)
\]
\[
\times \left\{ \frac{1}{2} \left[ \psi(1 + \bar{\eta}) + \psi(1 - \bar{\eta}) \right] - (\pi/2) \cot(\bar{\eta}\pi) - \log(|\bar{\eta}|) \right\},
\]
with \( \bar{\eta} = \mu\alpha Z/|k| \) and this ensures that the ensuing effective range function is real. It should be noted that in \([11]\) the poles in \( \psi(1 - \bar{\eta}) \) are cancelled with those from the cotangent term and only the poles coming from \( \psi(1 + \bar{\eta}) \) will survive, they correspond to the Coulomb bound states.

To complete the above scheme for calculating the effective range function we need the explicit formula for the Coulomb Green’s function. The latter involves a product of the regular \( F_\ell(\eta, \rho) \) and the irregular \( G_\ell(\eta, \rho) \) Coulomb wave functions (with \( \rho = kr \)) defined in \([12]\), and takes the form
\[
\langle r' | G^+_\ell(k) | r \rangle = -(2\rho/k) [G_\ell(\eta, \rho_<) + i F_\ell(\eta, \rho_<)] F_\ell(\eta, \rho_<),
\]
where \( F_\ell(\eta, \rho) = C_\ell(\eta) k^{\ell+1} \phi_\ell(k, r) \).

Actually, the K-matrix can be calculated directly, i.e. without reference to the scattering matrix \( \tau_\ell(k) \). To this end, a different Green’s function that is real analytic
function of $k^2$ will be introduced. Such Green’s function, denoted hereafter as $G^K_\ell$, to emphasize its connection with the K-matrix, can be easily constructed and reads
\[ \langle r | G^K_\ell(k) | r' \rangle = \langle r | G^r_\ell(k) | r' \rangle + 2\mu g\ell(k) \phi\ell(k, r) \phi\ell(k, r'). \] (12)

Indeed, the above Green’s function has all what it takes. For physical momenta the imaginary parts in the two terms on the right hand side of (12) cancel each other and $\langle r | G^K_\ell(k) | r' \rangle$ is real. For $k$ values located on the positive imaginary axis both terms are real and so is $\langle r | G^K_\ell(k) | r' \rangle$. Using spectral representation of $\langle r | G^K_\ell(k) | r' \rangle$, it is easy to verify that the Coulomb bound state poles, present in $\langle r | G^K_\ell(k) | r' \rangle$ for attractive Coulomb potential, will be exactly cancelled with the corresponding poles that occur in $g\ell(k)$ (cf. (10)). As a result, the Green’s function $\langle r | G^K_\ell(k) | r' \rangle$ given by (12) is free from the Coulomb singularities and is a real analytic function of $k^2$ in any finite part of the complex $k$-plane for non-negative $r$ and $r'$. The K-matrix will be obtained from the solution of a Lippmann-Schwinger equation that parallels equation (3)

\[ w\ell(k, r) = \phi(k, r) + \int_0^\infty \langle r | G^K_\ell(k) | r' \rangle V(r') w\ell(k, r') \, dr', \] (13)

where $w\ell(k, r)$ is a new wave function that differs from $u\ell(k, r)$ by a constant factor. Given the solution of (13), the K-matrix is calculated from the formula

\[ K_\ell(k) = -2\mu \int_0^\infty \phi\ell(k, r) V(r) w\ell(k, r) \, dr, \] (14)

and the above function is meromorphic in any finite part of the $k$-plane. It is easy to check that setting $w\ell(k, r) = [1 - g\ell(k) K_\ell(k)] u\ell(k, r)$ in (13) and using (3) – (5), we immediately retrieve the Heitler formula (1). This shows that the Green’s function (12) is unique for a given choice of $g\ell(k)$. It should be recalled here that $g\ell(k)$ is not free from ambiguities and can be determined only up to a polynomial in $k^2$ (cf. (1, 8)) but the latter is normally set equal to zero. This has been also our choice in the present work.

The Green’s function (12) can be also cast into an alternative form which is on a par with (1)

\[ \langle r | G^K_\ell(k) | r' \rangle = -2\mu \phi\ell(k, r_<) \theta\ell(k, r_>, \) (15)

where the imposed asymptotic boundary condition has been made explicit. The irregular Coulomb wave function $\theta\ell(k, r)$ present in (15) has been introduced in (1, 14). The latter is a real analytic function of $k^2$ for non-negative $r$ and can be expressed as a superposition of the standard Coulomb functions

\[ \theta\ell(k, r) = C\ell(\eta) k^\ell G\ell(\eta, \rho) + Rg\ell(k) \phi\ell(k, r). \]

Admittedly, the asymptotic behaviour ($r \to \infty$) of this function is rather complicated but it is worth to pay such a price in view of the fact that the Green’s function given by (13) is manifestly a real analytic function of $k^2$, and, as such, can be systematically expanded in a Taylor series in powers of $k^2$. The form (15) is particularly useful for calculating the K-matrix at $k = 0$. Following (10), one has

\[ \phi\ell(0, r) = \begin{cases} ((2\ell + 1)!/\beta^{\ell+1}) \sqrt{\beta r} I_{2\ell+1}(2\sqrt{\beta r}) & \text{for } Z > 0 \\ r^{\ell+1} & \text{for } Z = 0 \\ ((2\ell + 1)!/\beta^{\ell+1}) \sqrt{\beta r} J_{2\ell+1}(2\sqrt{\beta r}) & \text{for } Z < 0 \end{cases} \] (16)
where $\beta = 2 \mu \alpha |Z|$, and $(J_{2\ell+1}, I_{2\ell+1})$ are Bessel and modified Bessel functions, respectively [13]. The function $\theta_{\ell}(0, r)$ is given in terms of the Neuman $Y_{2\ell+1}$ and modified Neuman function $K_{2\ell+1}$, respectively [12].

$$\theta_{\ell}(0, r) = \begin{cases} \frac{[\beta^2/(2\ell + 1)]!}{\beta r} K_{2\ell+1}(2\sqrt{\beta r}) & \text{for } Z > 0 \\ r^{-\ell}/(2\ell + 1) & \text{for } Z = 0 \\ -\pi/[\beta^2/(2\ell + 1)]! \sqrt{\beta r} Y_{2\ell+1}(2\sqrt{\beta r}) & \text{for } Z < 0 \end{cases}$$

(17)

Since the K-matrix is a real analytic function of $k^2$ the effective range function can be expanded in a Taylor series in powers of $k^2$ around threshold

$$\frac{1}{K_{\ell}(k)} = \pm \frac{1}{a_{\ell}} + \frac{1}{2} r_{\ell} k^2 + \cdots ,$$

(18)

where the expansion coefficients for $\ell = 0$ have their standard meaning, i.e. $a_{0}$ is the scattering length and $r_{0}$ is the effective range and the higher order terms coefficients, that have been suppressed in (18), are known as shape parameters. Finally, the sign ambiguity in (18), reflecting the coexistence in the literature of two different definitions of the scattering length, has been resolved by adhering to the generally accepted convention that $(-)$ sign applies for baryon–baryon scattering and $(+)$ sign in all remaining cases.

When the scattering matrix has a bound state pole at $k = k_B$, then from (1) we obtain an equation for the binding energy $B = k_B^2/2\mu$ involving the effective range function continued below threshold

$$\frac{1}{K_{\ell}(k_B)} = g_{\ell}(k_B).$$

(19)

For shallow bound states the effective range function can be continued below threshold by employing the expansion (18). Formula (19) correlates then experimentally measurable quantities: the low-energy scattering parameters and the binding energy in a model independent way.

### 3. Sturmian expansion method

Following the scheme devised in the preceding section, the scattering matrix and the K-matrix will be obtained by solving a dedicated Lippmann-Schwinger equation. Actually, the solution of the latter below threshold presents a quite simple task thanks to the Sturmian representation of the Coulomb Green’s function.

The Sturmians, or Sturm-Liouville functions [13, 14, 15] $S_{\ell\nu}(b, r)$, where $b$ is a free parameter $(\text{Re } b > 0)$ are enumerated by the nodal quantum number $\nu = 0, 1, 2, \ldots$ and result as solution of the differential equation

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} + \frac{2b(\nu + \ell + 1)}{r} - b^2\right] S_{\ell\nu}(b, r) = 0,$$

(20)

with the following boundary conditions imposed, respectively, at zero, and at infinity

$$\lim_{r \to 0} S_{\ell\nu}(b, r) \sim r^{\ell+1};$$

$$\lim_{r \to \infty} S_{\ell\nu}(b, r) \sim e^{-br}.$$

(21) (22)

The eigenfunctions are given in a closed form

$$S_{\ell\nu}(b, r) = [\nu!/((2\ell + 1 + \nu))!]^{1/2} (2br)^{\ell+1} e^{-br} P_{\ell+1}^{2\ell+1}(2br),$$

(23)
where $L_{\nu}^{2\ell+1}$ denotes Laguerre polynomial. The functions (23) are orthogonal and form a complete set with respect to the weight function $1/r$.

For $\text{Im} \ k > 0$ the Coulomb Green’s function $G^0_\nu(k)$ turns out to be diagonal in a very specific Sturmian basis, namely when the parameter $b$ has been set equal to $-ik$. Therefore, in coordinate representation, following [14], we have

$$
| r \rangle \langle G^\nu_\ell(k) | r' \rangle = \frac{\mu}{ik} \sum_{\nu=0}^{\infty} S_{\ell\nu}(-ik, r) S_{\ell\nu}(-ik, r') \frac{1}{\nu + \ell + 1 + i\eta} \tag{24}
$$

but it should be clearly stated that for real $k$ the above series diverges so that (24) is invalid for physical momenta. Formula (24) is quite remarkable in that the Green’s function is given as a sum of the poles at the Coulomb bound states yet the same expression holds for Coulomb repulsion as well as in the Coulomb-free case. Unlike the function $g_\nu(k)$, the Green’s function is analytic function of the Coulomb strength parameter $Z$. Most importantly, however, the residues in (24) factorize what results in a separable representation very useful in practical applications. In particular, this implies that the kernel of the Lippmann-Schwinger equation either (3), or (13) becomes degenerate. With such a simplification the solution of equation (13) is straightforward.

Setting $k = ip$ with real $p > 0$ and inserting (24) together with (12) in (13), the solution of (13) is

$$
w_\ell(k, r) = [1 - g_\ell(k) K_\ell(k)] \phi_\ell(k, r) - \mu \sum_{\nu=0}^{\infty} \frac{S_{\ell\nu}(p, r) X_{\ell\nu}(p)}{\ell + 1 + \nu + i\eta}, \tag{25}
$$

where

$$X_{\ell\nu}(p) = \int_{0}^{\infty} S_{\ell\nu}(p, r) V(r) w_\ell(k, r) dr. \tag{26}
$$

In order to determine the unknown coefficients $X_{\ell\nu}(p)$, as well as the K-matrix, the expression (27) for $w_\ell(k, r)$ is inserted in (26) and in (14). This gives an infinite system of linear algebraic equations where the unknowns are the coefficients $X_{\ell\nu}$ together with $K_\ell(k)$

$$
\sum_{\nu'=0}^{\infty} \left[ \delta_{\ell\nu'} + \frac{\mu}{p} \frac{\langle \ell\nu' | V | \ell\nu' \rangle}{\ell + 1 + \nu' + i\eta} \right] X_{\ell\nu'}(p) + g_\ell(k) A_{\ell\nu}(p) K_\ell(k) = A_{\ell\nu}(p), \tag{27}
$$

where $\nu = 0, 1, 2, \ldots$, and this infinite set is supplemented by the equation

$$
\frac{\mu}{p} \sum_{\nu'=0}^{\infty} A_{\ell\nu'}(p) X_{\ell\nu'}(p) \frac{1}{\ell + 1 + \nu' + i\eta} + [g_\ell(k) \langle \phi_\ell | V | \phi_\ell \rangle - (1/2\mu)] K_\ell(k) = \langle \phi_\ell | V | \phi_\ell \rangle. \tag{28}
$$

In (27) and (28) we have introduced the following abbreviations

$$A_{\ell\nu}(p) = \int_{0}^{\infty} S_{\ell\nu}(p, r) V(r) \phi_\ell(k, r) dr, \tag{29}
$$

$$\langle \ell\nu | V | \ell\nu' \rangle$$

denotes the expectation value of the potential in the Sturmian basis

$$\langle \ell\nu | V | \ell\nu' \rangle = \int_{0}^{\infty} S_{\ell\nu}(p, r) V(r) S_{\ell\nu'}(p, r) dr, \tag{30}
$$

and $\langle \phi_\ell | V | \phi_\ell \rangle$ is the Born term

$$\langle \phi_\ell | V | \phi_\ell \rangle = \int_{0}^{\infty} \phi_\ell(k, r) V(r) \phi_\ell(k, r) dr. \tag{31}$$
Effective range function below threshold

The integrals \((29) - (31)\), carrying important information about the potential, should be regarded as input and to proceed further it is sufficient to truncate the infinite Sturmian basis approximating it by a finite dimensional one. When the resulting finite set of linear algebraic equations has been solved by any of the standard methods, one of the solutions contains the K-matrix.

The scattering matrix \(\tau_{\ell}(k)\) can be calculated in a similar way. By inserting \((24)\) in \((3)\), we get

\[
u_{\ell}(k, r) = \phi_{\ell}(k, r) - \mu \sum_{\nu=0}^{\infty} \frac{S_{\ell\nu}(p, r) Y_{\ell\nu}(p)}{\ell + 1 + \nu + \bar{\eta}}.
\]

(32)

where the new expansion coefficients defined as

\[
Y_{\ell\nu}(p) = \int_{0}^{\infty} S_{\ell\nu}(p, r) V(r) \nu_{\ell}(k, r) \, dr.
\]

(33)

are obtained by solving the following system of linear algebraic equations

\[
\sum_{\nu'=0}^{\infty} \left[ \delta_{\nu\nu'} + \frac{\mu}{p} \frac{\langle \ell\nu \vert V \vert \ell\nu' \rangle}{\ell + 1 + \nu' + \bar{\eta}} \right] Y_{\ell\nu'}(p) = A_{\ell\nu}(p).
\]

(34)

Using \((32)\) in \((3)\), we arrive at the ultimate expression for the scattering matrix

\[
\tau_{\ell}(k) = -2\mu \left[ \langle \phi_{\ell} \vert V \vert \phi_{\ell} \rangle - \mu \sum_{\nu'=0}^{\infty} A_{\ell\nu'}(p) Y_{\ell\nu'}(p) \right].
\]

(35)

Similarly as before, for practical purposes the infinite Sturmian basis must be approximated by a finite dimensional one. The values of \(k\) for which the determinant of the system \((34)\) vanishes correspond to the bound states. For attractive Coulomb potential \((Z < 0)\) they are infinite in number, all shifted from the pure Coulomb position owing to the nuclear interaction. For \(Z \geq 0\) the bound states may, or may not exist, depending upon the nature of the nuclear potential. Since the case of bound states has been discussed elsewhere \([16]\), we assume in the following that for the considered \(k\) value the system of equations \((34)\) is non-singular. Once \(\tau_{\ell}(k)\) has been determined, the effective range function may be obtained by making use of Heitler formula but this approach is equivalent to solving the system of equations \((27)\) and \((28)\). The determinant of that system may also vanish at some values of \(k\) and they correspond to poles of the K-matrix. Clearly, the poles of the scattering matrix are distinct from the poles of the K-matrix.

The proposed scheme has, of course, its limitations and breaks down not only for large \(p\) but also for very small \(p\). The large \(p\) limit is rather obvious and stems from the fact that for large \(r\) the regular Coulomb wave function \(\phi_{\ell}(k, r)\) increases like \(\sim e^{pr}\) and therefore the Born term integral \((31)\) diverges for \(p > \lambda/2\). Consequently, the largest admitted \(p\) value is determined by the range of the nuclear potential. There is also a lower bound for \(p\), as in the limit of small \(p\) the convergence rate of the Sturmian expansion gradually deteriorates eventually rendering the whole scheme impractical. The lower limit for \(p\), according to our experience, depends upon the characteristics of the nuclear potential. In this situation, the effective range function must be extrapolated to zero. If the extrapolation is not satisfactory the K-matrix at \(k = 0\) can be obtained by solving the Lippmann-Schwinger equation with the Green’s function given by \((15)\). Although the kernel is not degenerate in that case but the resulting integral equation can be easily solved by standard methods. Another alternative is a numerical integration of the appropriate Schrödinger equation. A stepwise integration
of the latter from \( r = 0 \) up to some large \( r = R \), well beyond the range of the nuclear potential, yields the regular solution \( u_\ell(0, R) \) and the derivative \( u'_\ell(0, R) \). With these two functions in hand, the effective range function is

\[
\frac{1}{K_\ell(0)} = -\frac{W[u_\ell(0, R), \theta_\ell(0, R)]}{W[u_\ell(0, R), \phi_\ell(0, R)]},
\]

where \( W[f, g] = f g' - g f' \) denotes the Wronskian determinant and the Coulomb wave functions are given by (16) and (17), respectively.

4. Applications

In order to examine the performance of the calculational scheme presented in the preceding section, the Sturmian expansion method will be subjected to a rather demanding test. To achieve this goal, for the nuclear potential \( V(r) \) we have selected Reid nucleon-nucleon \(^1S_0\) potential \([17]\). This potential is given as a superposition of three Yukawa potentials, has a one-pion-exchange tail and a short range very strong repulsive term. Since the shape of this potential shows a good deal of structure comprising of attraction and repulsion, the rank \( N \) of the Sturmian approximation must be large enough to account for the rapid variation of the potential. In addition to that, the other unfavourable circumstance is that the long one-pion-exchange tail greatly reduces the range of the admissible momenta which must not exceed 65 MeV. In the actual computations, we have been able to determine the effective range function in the interval \( 20 \text{ MeV} < \rho < 65 \text{ MeV} \). The rank of the Sturmian approximation \( N \) at which convergence has been attained varied from about hundred to several hundred at the low-energy end. It is perhaps worth noting that when \( V(r) \) has a Yukawa shape then all the necessary integrals are given in an analytic form. To be more explicit, \( A_{\ell\nu} \) is obtained from

\[
\int_0^\infty S_{\ell\nu}(b, r) \frac{e^{-\lambda r}}{r} \phi_\ell(k, r) \, dr = \sqrt{\frac{(2\ell + 1 + \nu)}{(\nu)!}} \left[ \frac{2b}{(\lambda + b)^2 + k^2} \right]^{\nu+1} \left( \frac{\lambda - b + ik}{\lambda + b + ik} \right)^\nu 
\times \left( \frac{\lambda + b - ik}{\lambda + b + ik} \right)^{in} \sqrt{\frac{2F_1(-\ell, \ell + 1 - in; 2\ell + 2; 1 - z)}{\lambda + b + ik}}
\]

(36)

where \( z = [\lambda^2 - (b + ik)^2]/[\lambda^2 - (b - ik)^2] \). The expectation value of the potential in the Sturmian basis is obtained from

\[
\int_0^\infty S_{\ell\nu}(b, r) \frac{e^{-\lambda r}}{r} S_{\ell\nu'}(b, r) \, dr = \frac{1}{(2\ell + 1)!} \sqrt{\frac{(2\ell + 1 + \nu)}{(\nu)!}} \left( \frac{2(\ell + 1 + \nu)}{(\ell + 1 + \nu')!} \right) \times x^{2\ell+2} (x + 1)^{-2\ell-2-\nu-\nu'} \sqrt{\frac{2F_1(-\nu, \nu; 2\ell + 2; x^2)}{\lambda + b + ik}}.
\]

(37)

where \( x = 2b/\lambda \). Finally, the Born term, takes the form

\[
\int_0^\infty \phi_\ell(k, r) \frac{e^{-\lambda r}}{r} \phi_\ell(k, r) \, dr = \frac{(2\ell + 1)!}{(\lambda + 2ik)^{\nu+1}} \left( \frac{\lambda - 2ik}{\lambda + 2ik} \right)^{in} \sqrt{\frac{2F_1(\ell + 1 - in, \ell + 1 + in; 2\ell + 2; y)}{\lambda + 2ik}},
\]

(38)

where \( y = 4k^2/(\lambda^2 + 4k^2) \). In the above expressions we assumed that \( \lambda^2 > 4k^2 \) and \( 2F_1(a, b; c; z) \) denotes the hypergeometric function.

The effective range functions resulting from our computations are presented in Figure 1 as functions of the c.m. energy \( E = k^2/2\mu \). The calculations have been carried out for three values of the Coulomb strength parameter \( Z \): \( Z = \pm 1 \) and
Effective range function below threshold

$Z = 0$. Strictly speaking, attractive Coulomb potential would be appropriate for the proton-antiproton pair. However, in that case the Reid potential, devised to fit the nucleon-nucleon scattering data, is not applicable and the inferred effective range parameters have no relevance to proton-antiproton scattering. Nevertheless, this case is of theoretical interest and has been included here for testing purposes. In Figure 1 the continuous curves above threshold result from a numerical integration of the appropriate Schrödinger equation. The entries below threshold, depicted by asterisk symbols on the plot, have been obtained by using the Sturmian expansion method. As expected, close to $E = 0$ the values of the effective range function calculated below and above threshold appear to be lying on the same straight line. In order to qualify that statement, the values calculated below threshold have been subsequently used to extrapolate each of the three functions to zero. This gave the intersection points at zero together with the slopes of the curves. The scattering lengths and effective ranges, extracted that way with the aid of (18), are presented in Table I (referred to as "below"). For comparison, we give also their counterparts calculated from the above threshold solutions of the Schrödinger equation (referred to as "above" in Table I). The agreement between the below threshold and the above threshold computations is in all three cases very good. The error on $r_0$ is slightly bigger than that on $a_0$, as the computation of the former involves the differences of the function values what results in some loss of accuracy. The effective range seems to be little affected by the Coulomb interaction what is also reflected in Figure 1 in that the three curves have almost the same slope near zero. By contrast, the Coulomb corrections to the scattering length are quite large especially for Coulomb attraction when $a_0$ has been pushed far away to positive values. The K-matrix, unlike the effective range function, shows rapid variation as function of energy and in all three cases exhibits a pole not far from the origin. At the pole the effective range function goes to zero, so that the actual positions of the poles of the K-matrix are apparent from Figure 1. It is interesting to note that for attractive Coulomb potential the pole of the K-matrix lies in the physical region, very close to threshold. This example is rather instructive, because it shows that any phenomenological attempt to approximate here the K-matrix by a polynomial in $k^2$, with coefficients to be fitted to the low-energy data, would have a disastrous effect. The problem is immediately alleviated when instead of the K-matrix the effective range function is employed.

The two-potential problem considered in this work can be easily generalized to comprise complex nuclear potentials. Such potentials appear in various problems of nuclear physics and, in particular, they have been widely used for a phenomenological description of the hadronic atoms phenomena. The introduction of a complex nuclear potential into the description of a hadronic atom has two observable effects: a shift $\epsilon$ in the energy of the otherwise hydrogen-like level position, and a broadening $\Gamma$ of the atomic level arising from the fact that the system can now decay through the absorptive strong interaction. When the orbital momentum is large enough the level shift in a hadronic atom is usually quite small in comparison with the Coulomb energy. This observation has prompted several authors (cf. [18, 19, 20, 21, 22]) to introduce small shift approximation (SSA) based on the idea that if atomic bound state pole has been shifted from the Coulomb position $k_n = -i \mu_0 Z/n$, where $n$ is the principal quantum number, by $\delta k$ then $|\delta k/k_n|$ can be regarded as a small expansion parameter. One of the implications of SSA is the shift formula expressing the energy shift in a hadronic atom in terms of hadron-nucleus low-energy scattering parameters. This formula relating only experimentally measurable quantities without reference to the
underlying nuclear potential has very important phenomenological applications.

In order to derive the shift formula we are going to obtain an approximate solution of the bound state equation \( (19) \). As mentioned before, the function \( g_\ell(k) \) that enters \( (19) \) is singular at \( k = k_n \) and for \( k \) such that \( |k - k_n| \) is small in comparison with the separation of the unperturbed levels \( |k_n - k_{n+1}| \), we can approximate \( g_\ell(k) \) by a single pole term

\[
g_\ell(k) \approx \frac{k_n R_{\ell n}}{k - k_n}. \tag{39}
\]

The residue \( R_{\ell n} \) can be worked out from the definition of \( g_\ell(k) \) and reads

\[
R_{\ell n} = \left\{ 2|k_n|^2 \int_0^\infty \phi_\ell(k_n, r)^2 \, dr \right\}^{-1} \tag{40}
\]

where the normalization integral, is

\[
\int_0^\infty \phi_\ell(k_n, r)^2 \, dr = \frac{n(n - \ell - 1)![(2\ell + 1)!\ell!]}{4|k_n|^{2\ell+3}(\ell + n)!}. \tag{41}
\]

In order to obtain an approximate solution of equation \( (19) \) it is assumed that the left hand side is a slowly varying function of \( k \) so that \( 1/K_\ell(k) \) may be replaced by \( 1/K_\ell(k_n) \). Using the single pole approximation \( (39) \), the solution of \( (19) \) is

\[
k \approx k_n[1 + K_\ell(k_n) R_{\ell n}], \tag{42}
\]

and this yields the level shift formula in its most general form

\[
\delta E = -\epsilon - \frac{i\hbar^2}{2\mu} \Gamma = E - E_n \approx -K_\ell(k_n) \left\{ 2\mu \int_0^\infty \phi_\ell(k_n, r)^2 \, dr \right\}^{-1} \tag{43}
\]

where \( E_n = k_n^2/2\mu \) and it is understood that for complex \( V(r) \) the K-matrix is necessarily also complex. Approximating in \( (19) \) the K-matrix by the scattering length one obtains the celebrated Deser–Trueman formula \([18, 19]\). However, the latter approximation is hardly necessary as the K-matrix below threshold can be calculated with the same amount of labour by using the Sturmian expansion method described in the preceding section. The other alternative, that does not require the SSA ansatz, is to locate the poles of the scattering matrix. Using Sturmian expansion, the problem reduces to finding the values of \( k \) at which the determinant of the system \( (34) \) vanishes (cf. \([16]\)).

In conclusion, we have presented a simple and efficient method for calculating the effective range function below threshold and the same calculational scheme can be used for solving the bound state problem. The main advantage of the proposed method lies in the exact treatment of the Coulomb interaction what is accomplished by using the Lippmann-Schwinger equation whose kernel involves the Coulomb Green’s function. Since this kernel below threshold assumes degenerate form, the solution of the integral equation is straightforward. The essential point is that the Coulomb Green’s function and the regular Coulomb wave function, constituting input to the Lippmann-Schwinger equation, are real analytic functions of \( k^2 \), free from Coulomb singularities. Both are given in analytic form suitable for numerical computations and the irregular Coulomb wave function is never needed.
Effective range function below threshold

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### Table 1. Coulomb-modified low-energy scattering parameters for Reid $^1S_0$ NN potential calculated below and above threshold. All entries are in fm.

| method | $Z = 1$ | $Z = 0$ | $Z = -1$ |
|--------|--------|--------|---------|
|        | $a_0$  | $r_0$  | $a_0$  | $r_0$  | $a_0$  | $r_0$  |
| below  | -7.77  | 2.75   | -17.1  | 2.83   | 148.   | 2.91   |
| above  | -7.77  | 2.69   | -17.1  | 2.78   | 147.   | 2.88   |
Figure captions

Figure 1. Coulomb-modified effective range function vs. c.m. energy for Reid $^1S_0$ NN potential. The different curves correspond to Coulomb attraction ($Z = -1$), Coulomb repulsion ($Z = 1$) and Coulomb-free case ($Z = 0$).