Object reconstruction from multiplexed quantum ghost images using reduction technique

D. A. Balakin¹  ·  A. V. Belinsky¹  ·  A. S. Chirkin¹,²

Received: 29 August 2018 / Accepted: 18 January 2019 / Published online: 4 February 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
We apply the measurement reduction technique to optimally reconstruct an object image from multiplexed ghost images (GI) while taking into account both GI correlations and object image sparsity. The measurement reduction technique is employed because it provides a unified framework for optimal processing of multiplexed and non-multiplexed images with different information about the measurement model, the research object and the research objective. We show that one can reconstruct an image in that way even if the object is illuminated by a small photon number. We consider frequency GI multiplexing using coupled parametric processes. We revealed that the imaging condition depends on the type of parametric process, namely, whether down- or up-conversion is used. Influence of information about sparsity in discrete cosine transform and Haar transform bases on reconstruction quality is studied. In addition, we compared ordinary images and GI when the detectors are additionally illuminated by noise photons in a computer experiment, which showed increased noise immunity of GI, especially with processing via the proposed technique.

Keywords Measurement reduction · Ghost images · Multiplexed ghost images · Entangled photons · Compressive sensing

The authors acknowledge the support by the Russian Foundation for Basic Research under the Project Number 18-01-00598-A.

D. A. Balakin
balakin_d_a@physics.msu.ru

A. V. Belinsky
belinsky@inbox.ru

A. S. Chirkin
aschirkin@physics.msu.ru

¹ Faculty of Physics, M. V. Lomonosov Moscow State University, Leninskie Gory, 1, bld 2, Moscow, Russia 119991
² The International Laser Center, M. V. Lomonosov Moscow State University, Leninskie Gory, 1, bld 62, Moscow, Russia 119991
1 Introduction

By now, to enhance human visual capability a vast high-tech base including highly sensitive, high-precision and high-speed cameras have been developed. Nevertheless, there still are objects whose direct optical observation is difficult. They are primarily halftone biological objects that are especially sensitive to illumination and thus have to be investigated very delicately. Ghost imaging is one way of solving this problem, as it allows to obtain object images without direct observation of its spatial structure. For ghost imaging, correlated light beams are necessary. Ghost imaging enables extraction of object information from spatial correlations between beams, one of which (in the object arm) interacts with the object, while the other one (in the reference arm) does not. In the object arm, a bucket detector is used, which provides only information about the total energy of the transmitted radiation. The other beam does not interact with the object, but is detected by a CCD matrix, which permits measuring the spatial correlation function of intensity between two arms. The information about transparency or reflectivity distribution of the research object is extracted from photocount correlations in the object and reference arms [1], see also [2–6].

In this paper, we study application of multicomponent entangled quantum light states that let us produce several GI simultaneously (to multiplex GI) [7–10] by using radiation with different frequencies in reference arms. Mutual correlations of multiplexed images are used as additional information to improve image processing in the presence of fluctuations. There are various ways of producing multi-frequency entangled light beams. The required states can be obtained, e.g., in consecutive coupled parametric interactions in nonlinear crystals located either outside [11,12] or inside an optical resonator, in nonlinear waveguide structures [14,15] where modes are coupled through evanescent modes, in a spatially modulated pump beam [16]. The considered GI multiplexing employs four-frequency entangled quantum states formed through parametric decay of pump photons into two photons with different frequencies that are mixed in the same crystal with pump photons, which produces photons with sum frequencies [17,18]. Quantum theory of this process has been systematically developed in recent years [19–22]. Note that in [4,23,24] GI were multiplexed via multi-frequency noncoherent radiation sources to simultaneously produce several GI that are superimposed afterward. Recently, polarization multiplexing has been used in works on ghost imaging to improve the reconstructed image quality [25] and to reveal the polarization structure of the image [26].

The ghost image processing methods considered in the literature usually rely on regularization. The regularizing functional is a characteristic of image sparsity in a given basis [27–30], and the minimized functional itself is the least squares one [28–30] or likelihood function [27]. Alternatively, a sparsity characteristic (e.g., the \(L^1\) norm in a given basis) is minimized [31] under the constraint that measuring the image reconstructed in that way would give the results actually obtained. Since such a functional is not connected to the error of the interpretation result, the obtained estimate is, generally speaking, not the optimal one. Unlike [27], measurement reduction method, including its proposed version, does not require only Poisson photocount distribution, and unlike [27–31], image sparsity in any basis is not required.
Note the main differences between this article and publications [8–10], in which multiplexed GI processing using measurement reduction technique, were employed as well. Firstly, in these works the situation was considered when the only information about transparency distribution available to the researcher was that its values belong to a unit interval. In this article, it is considered that the researcher also has information about transparency distribution sparsity in a given basis and wants to take advantage of it to improve estimation quality. Secondly, as this information enables reconstruction of acceptable quality even with a small number of photons illuminating the object, multiplexed ghost imaging with a small number of photons (∼ 1 ÷ 10 photons per pixel) and processing of acquired images is modeled (see Sect. 4). Thirdly, the presented version of measurement reduction technique differs from the one used in [8–10] in that projection (to take into account the information about the object) minimizes Mahalanobis distance instead of Euclidean distance, see Eq. (24). Finally, fourthly, in the studied multiplexed ghost imaging setup the object arm is lensless. This leads to imaging conditions depending on the type of parametric coupling of photon frequencies in the object arm and the reference arms.

The article structure is as follows. In Sect. 2, we discuss GI multiplexing setup with lensless object arm and with lenses in reference arms. In Sect. 3, the measurement reduction method is outlined. The information about the object that is available to the researcher and that is employed in reduction is summarized in Sect. 3.1. In Sect. 3.2, the algorithm of GI processing using reduction method that takes this information into account is described. Computer modeling results are given in Sect. 4. Main results of the article are summarized in the conclusion.

2 Frequency multiplexing of quantum ghost images

GI multiplexing setup is shown in Fig. 1. The illumination is provided by coupled parametric processes that produce four-frequency entangled light fields. Pump radiation incident into the nonlinear convertor (nonlinear photon crystal) has frequency $\omega_p$. In the crystal, pump photons decay into two photons with related frequencies $\omega_1$ and $\omega_2$: $\omega_p = \omega_1 + \omega_2$.

![Fig. 1 Multiplexed ghost imaging setup. NC is the nonlinear convertor; BS is the beam splitter; $\omega_p$ is pump frequency; $\omega_1, \ldots, \omega_4$ are frequencies of produced entangled photons; $O$ is the object; BD is the bucket detector in the object arm; $L_j$ are lenses with focal lengths $f_j$; CCD$^j$ are CCD in reference arms; C$^j$ are intensity correlators, $j = 2, 3, 4$](image-url)
Four-frequency fields appear as a result of further conversion of a part of photons with frequencies $\omega_1$ and $\omega_2$ to photons with frequencies $\omega_3$ and $\omega_4$ in frequency-mixing processes:

$$\omega_p + \omega_1 = \omega_3, \quad \omega_p + \omega_2 = \omega_4.$$  \hspace{1cm} (1)

Efficient energy exchange between interacting light waves in these processes can be achieved in aperiodically nonlinear photon crystals, e.g., in LiNbO$_3$, in the quasiphase matched regime, in which phase mismatch $\Delta k_j$ between interacting waves are compensated by the vectors of the inverse nonlinear lattice [17,18]. Note that the considered process was recently realized via a setup with two nonlinear photon crystals in the work [32], where the spectrum of a photon pair at frequency above pump frequency was studied.

Let ghost images be obtained by means of the optical system setup shown in Fig. 1, where a detector integrating radiation over the entire aperture is used in the object arm. It is assumed that the length of nonlinear photon crystal is chosen so that the transversal wave number amplification band of the nonlinear convertor substantially exceeds the width of wave spectrum of the object image. Details of four-frequency entangled state formation were considered in [7–10].

In the setup of Fig. 1, the object is illuminated by radiation with frequency $\omega_1$, which is detected by the bucket detector (BD) over the entire beam aperture, and therefore lacks spatial resolution. Radiation with other frequencies $\omega_2, \omega_3, \omega_4$ after their spatial separation enters reference arms with lenses in them. Focal lengths $f_j$ of lenses and their positions between the beam splitter (BS) and CCD cameras are chosen according to imaging conditions. These conditions depend on the type of relation to the frequency of object illumination. For frequencies $\omega_2, \omega_4$, the imaging condition has the form (cf. the analogous condition in [1] for equal frequencies)

$$\frac{1}{f_j} = \frac{1}{l_{j2}} + \frac{1}{l_{j1} + (\lambda_1/\lambda_j)l_{11}}, \quad j = 2, 4,$$  \hspace{1cm} (2)

while for frequency $\omega_3$ it reads (cf. the analogous condition in [33] for equal frequencies)

$$\frac{1}{f_3} = \frac{1}{l_{32}} + \frac{1}{l_{31} - (\lambda_1/\lambda_3)l_{11}}.$$  \hspace{1cm} (3)

In expressions (2), (3) the length $l_{11}$ is the distance from BS to the object, $l_{j1}$ is the distance from BS to the lens, $l_{j2}$ is the distance from the lens to the detector CCD$ _j$, $\lambda_j$ is the length of the wave of corresponding frequency. The derivation of Eqs. (2), (3) is described. They are a generalization of known ones to the case of different frequencies of radiation illuminating the object and radiation in reference arms.

The theory of formation of entangled quantum four-beam states in processes (1) was developed in [19–21]. We present some of its results. The positive-frequency part of the interacting wave fields is written as
\[ \hat{E}_j^\dagger(\mathbf{r}, z, t) = \hat{A}_j(\mathbf{r}, z, t) \exp(-i(\omega_j t - k_j z)), \]  

(4)

where \( \hat{A}_j(\mathbf{r}, z) \) is the slowly varying amplitude operator in Heisenberg representation, \( k_j \) is wave number, \( z \) is the direction of propagation of interacting waves in the nonlinear crystal, the vector \( \mathbf{r} \) lies in the plane perpendicular to the \( z \) axis.

In a given field of intense classical monochromatic pumping, spatial evolution of amplitude operators \( \hat{A}_j(\rho; z) \) of interacting waves is described in the quasioptical approximation by the coupled equations:

\[
\begin{aligned}
\left( \frac{\partial}{\partial z} - \frac{i}{2k_1} \Delta_\perp \right) \hat{A}_1(\mathbf{r}, z) &= i \beta \hat{A}_2^\dagger(\mathbf{r}, z) + i \gamma_1^* \hat{A}_3(\mathbf{r}, z), \\
\left( \frac{\partial}{\partial z} - \frac{i}{2k_2} \Delta_\perp \right) \hat{A}_2(\mathbf{r}, z) &= i \beta \hat{A}_1^\dagger(\mathbf{r}, z) + i \gamma_2^* \hat{A}_4(\mathbf{r}, z), \\
\left( \frac{\partial}{\partial z} - \frac{i}{2k_3} \Delta_\perp \right) \hat{A}_3(\mathbf{r}, z) &= i \gamma_1 \hat{A}_1(\mathbf{r}, z), \\
\left( \frac{\partial}{\partial z} - \frac{i}{2k_4} \Delta_\perp \right) \hat{A}_4(\mathbf{r}, z) &= i \gamma_2 \hat{A}_2(\mathbf{r}, z),
\end{aligned}
\]

5

Here \( \Delta_\perp = \Delta_\perp(x, y) \) is the transverse Laplacian. The nonlinearity coefficient \( \beta \) is responsible for parametric frequency down-conversion, while the coefficients \( \gamma_1 \) and \( \gamma_2 \) are responsible for the sum-frequency generation.

The solution of Eq. (5) is found using the Fourier transform

\[ \hat{a}_j(\mathbf{q}, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{A}_j(\rho, z) \exp(-i\mathbf{q}\cdot\rho) d\rho, \]

6

The operator \( \hat{a}_j(\mathbf{q}, z) \) is the annihilation operator of plane mode photons with frequency \( \omega_j \) and transversal wave vector \( \mathbf{q} \).

Bose operators of annihilation \( \hat{a}_j(\mathbf{q}, z) \) and creation \( \hat{a}_j(\mathbf{q}, z)^\dagger \) of photons at nonlinear crystal output are represented in the following matrix form:

\[ \hat{a}(\mathbf{q}, l) = Q(\mathbf{q}, l)\hat{v}(\mathbf{q}), \]

7

Here \( \hat{a} \) and \( \hat{v} \) are columns of Bose operators at crystal output and input, respectively. They are of the form \( \hat{a} \equiv (\hat{a}_1, \hat{a}_2^\dagger, \hat{a}_3, \hat{a}_4^\dagger)^T \), where \( T \) denotes transposition, \( \hat{a}_1 = \hat{\alpha}_1(\mathbf{q}, l), \hat{a}_2^\dagger = \hat{\alpha}_2^\dagger(-\mathbf{q}, l), \hat{a}_3 = \hat{\alpha}_3(\mathbf{q}, l), \hat{a}_4^\dagger = \hat{\alpha}_4^\dagger(-\mathbf{q}, l), l \) is the length of nonlinear crystal. Operators in the column \( \hat{v} \equiv (\hat{v}_1, \hat{v}_2^\dagger, \hat{v}_3, \hat{v}_4^\dagger)^T \) refer to vacuum field state.

\( Q \) is a \( 4 \times 4 \) matrix whose elements \( Q_{mn} \) describe field conversion from frequency \( \omega_n \) to frequency \( \omega_m \). The form of the matrix \( Q \) and its properties in the quasioptical approximation are given in [19]. The elements of \( Q \) depend on crystal length, pump intensity and transversal wave number \( \mathbf{q} \).
After BS, their amplitude operators are defined by the following relations

\[ \hat{B}_j(r_j) = \int H_j(r_j, \rho_j) \hat{A}_j(\rho_j, l) d\rho_j, \]  

(8)

in the detector plane, integration being over the light beam aperture. \( H_j(r_j, \rho) \) is the medium response function for radiation propagation from the crystal to the detector in \( j \)-th arm. We assume for simplicity that beam splitting takes place directly at nonlinear crystal output. In other words, BS is considered to be thin.

For the object arm

\[ H_1(r_1, \rho_1) = \int_{-\infty}^{+\infty} H_1(r_1 - \rho'_1; l_{12}) T(\rho'_1) H_j(\rho'_1 - \rho_1; l_{11}) d\rho'_1. \]  

(9)

Here \( T(\rho'_1) \) is the object transmission coefficient, \( H_1(r - \rho; l_{1j}) \) is the Green’s function

\[ \int H_1(r - \rho; l_{1j}) = -i \frac{k_1}{2\pi l_{1j}} \exp \left( i \frac{k_1(r - \rho)^2}{2l_{1j}} \right). \]  

(10)

As noted above, \( l_{11} \) is the distance between BS and the object and \( l_{12} \) is the distance between the object and the bucket detector.

Response functions of reference arms containing thin lenses with focal length \( f_j \) can be represented as (see [34,35])

\[ H_j(r_j, \rho_j) = -i \frac{k_j}{2\pi L_j} \exp \left( i \frac{k_j(r_j - \rho_j)^2}{2L_j} \right) \left[ (r_j - \rho_j)^2 - \left( l_{11}r_j^2 + l_{12}\rho_j^2 \right) / f_j \right], \]  

(11)

where

\[ L_j = l_{11} + l_{12} - l_{11}l_{12} / f_j. \]

Intensity operators of the obtained beams are \( \hat{I}_j(r_j) = \hat{B}_j^\dagger(r_j) \hat{B}_j(r_j) \). Mutual intensity correlation functions of the object arm and reference arms, taking into account Gaussian field statistics, are determined by the following formulas: for radiation with frequency \( \omega_3 \) the correlation function is

\[ G_{13}(r_1, r_3) = \langle \hat{I}_1(r_1) \hat{I}_3(r_3) \rangle - \langle \hat{I}_1(r_1) \rangle \langle \hat{I}_3(r_3) \rangle = |\langle \hat{B}_1(r_1) \hat{B}_3^\dagger(r_3) \rangle|^2, \]  

(12)

while for radiation with frequencies \( \omega_2 \) or \( \omega_4 \) the correlation function is

\[ G_{1j}(r_1, r_j) = \langle \hat{I}_1(r_1) \hat{I}_j(r_j) \rangle - \langle \hat{I}_1(r_1) \rangle \langle \hat{I}_j(r_j) \rangle = |\langle \hat{B}_1(r_1) \hat{B}_j(r_j) \rangle|^2, \quad j = 2, 4. \]  

(13)
The difference in the definitions of the correlation functions under consideration is due to the type of parametric conversion and the initial vacuum fluctuations. As a consequence, only the vacuum operators in antinormal ordering contribute to correlations.

Under imaging conditions (2), (3), the expressions can be transformed to

\[
G_{13}(\mathbf{r}_1, \mathbf{r}_3) = |\Gamma_3|^2 \left| l_{31}(\lambda_1/\lambda_3) l_{11} \overline{l_{12}l_{32}} T(-\alpha_3 \mathbf{r}_3) \right|^2, \quad \alpha_3 \equiv l_{31} - (\lambda_1/\lambda_3) l_{11} l_{32}, \quad (14)
\]

\[
G_{1j}(\mathbf{r}_1, \mathbf{r}_j) = |\Gamma_j|^2 \left| l_{j1}(\lambda_1/\lambda_j) l_{11} \overline{l_{12}l_{j2}} T(-\alpha_j \mathbf{r}_j) \right|^2, \quad \alpha_j \equiv l_{j1} + (\lambda_1/\lambda_j) l_{11} l_{j2}. \quad (15)
\]

These formulas are derived under the assumption that at the nonlinear crystal output, mutual correlation functions of the radiation

\[
\Gamma_{1j}(\rho_1 - \rho_j) = \langle \hat{A}_1(\rho_1, l) \hat{A}_j(\rho_j, l) \rangle = (2\pi)^{-1} \int Q_{11j}(\mathbf{q}) \exp(i\mathbf{q}(\rho_1 - \rho_j))d\mathbf{q}.
\]

\[j = 2, 4,
\]

\[
\Gamma_{13}(\rho_1 - \rho_3) = \langle \hat{A}_1(\rho_1, l) \hat{A}_3^+(\rho_3, l) \rangle = (2\pi)^{-1} \int Q_{113}(\mathbf{q}) \exp(-i\mathbf{q}(\rho_1 - \rho_3))d\mathbf{q}.
\]

where \( Q_{11n}(\mathbf{q}) \) and \( Q_{13n}(\mathbf{q}) \) are substituted by \( \delta \)-functions

\[
\Gamma_{1n}(\rho_1 - \rho_n) = \Gamma_n \delta(\rho_1 - \rho_n), \quad \Gamma_n = \int \Gamma_{1n}(\rho)d\rho.
\]

These substitutions are valid if the radiation correlation radius is much smaller than a characteristic spatial scale of object image change. Thus, the correlation functions (14) and (15) contain information about the image. In the case of \( \gamma_1 = \gamma_2 = 0 \), we arrive at the conventional, non-multiplexed quantum ghost image at frequency \( \omega_2 \) [2–6] (see also the discussion of Fig. 4 on page 14).

The expressions (14), (15) coincide, up to a factor before the image transmission coefficient, with the expression obtained for another experimental setup [9,10]. In [9, 10], ghost image correlations determined by fourth-order intensity correlations (eighth-order field ones) were studied as well. Obviously, in the setup under consideration they will be the same as in [9,10]. After integration over the area \( s \) of the beam in the object arm (over \( d\mathbf{r}_1 \) intensity correlation functions of the second order, in accordance with (14), (15), become

\[
G_j(\mathbf{r}_j) \sim s \left| T(-\alpha_j \mathbf{r}_j) \right|^2, \quad (16)
\]

while the GI correlation function determined by eighth-order field correlation function becomes

\[
K_{ij}^{GI}(\mathbf{r}_i, \mathbf{r}_j) \sim s^2 \left| T(-\alpha_i \mathbf{r}_i) \right|^2 \left| T(-\alpha_j \mathbf{r}_j) \right|^2. \quad (17)
\]
Naturally, the coefficients $\alpha_2, \alpha_3, \alpha_4$ can be made equal by choice of setup parameters. In addition, in the following formulas (19) and (20) the factors dependent on the measurement unit choice will be omitted for brevity.

As mentioned above, the correlation functions derived above provide the information about the measuring (image acquisition) process that is used in the measurement reduction technique along with the information about the object. The following section focuses on the measurement reduction technique itself and the information about the object. Not all of the information mentioned above is equal in importance: only the correlation function (16) and finiteness of $L^2$ norm of the correlation function (17) for all $i, j = 2, 3, 4$ are strictly necessary for image reconstruction. Nevertheless, additional information about both the measuring process (in our case, the form of correlation function (17)) and the object (sparsity of its transparency distribution) can vastly improve reconstruction quality, as it will be shown below.

### 3 Processing of acquired images

The output of $i$-th correlator, denoted as $\xi^{(i)}(\mathbf{r})$, can be considered as the impact of a measuring transducer (MT) on the input signal $f(\mathbf{r}) \sim |T(-\mathbf{r})|^2$. Here and below, unlike the previous section, $f$ denotes the vector describing the object transparency distribution instead of focal length. We assume for simplicity that $\alpha_2 = \alpha_3 = \alpha_4 = 1$.

We will consider piecewise constant images, i.e., transparency is constant within each pixel. Areas of constant transparency and constant brightness corresponding to pixels are considered to be ordered in an arbitrary but fixed way. Due to that it is sufficient for us to consider a finite number of values of $r$. Thus, $f$ as the vector of transparencies is an element of finite-dimensional Euclidean space $\mathcal{F}$.

An image processing algorithm ought to provide the most accurate estimate of the feature of the original image $f$ that is of interest to the researcher based on obtained data $\xi$, which consists of acquired ghost images $\xi^{(i)}(\mathbf{r}), i = 2, 3, 4$. Measurement reduction method allows to obtain such an estimate. Let us formulate the measurement model as

$$\xi = \mathbf{A}f + \nu,$$

(18)

where $f$ is an a priori unknown vector that describes the transparency distribution of the object, $\nu$ is measurement error with zero expectation, $\mathbb{E}\nu = 0$, which means absence of systematic measurement error, and covariance matrix $\Sigma_\nu = \mathbb{E}\nu\nu^*$. The matrix $\mathbf{A}$ describes ghost imaging and GI acquisition: the matrix element $A_{ij}$ is equal to the mean output of $i$-th detector for unit transparency of $j$-th element of the object and zero transparency of other object elements (i.e., whose indices differ from $j$). The dimension of vector $f$ is the number of pixels in the object image, while the dimension of $\xi$ is the number of pixels in all CCD. The condition of systematic measurement error absence $\mathbb{E}\nu = 0$ means, in particular, that the expectation of the component of measurement results caused by detector dark noises is subtracted from the measurement results, similar to Shi et al. [36].
Matrices $\mathbf{A}$ and $\mathbf{\Sigma}_\nu$ are related to the correlation functions considered above. The measuring setup employs correlators that measure correlations between the object arm and other arms. Therefore, the matrix $\mathbf{A}$, which models the impact of MT on the image, is a block matrix and consists of three blocks describing correlator outputs, i.e., correlations between the object arm and reference arms:

$$
\mathbf{A} = \begin{pmatrix}
\mathbf{B}_2 \mathbf{C}_2 \\
\mathbf{B}_3 \mathbf{C}_3 \\
\mathbf{B}_4 \mathbf{C}_4
\end{pmatrix}.
$$

(19)

Under the conditions used to derive the intensity correlation functions (16) and (17), the matrices $\mathbf{C}_2–\mathbf{C}_4$ are identity ones multiplied by pixel size and the factor before $|T(r_i)|^2$ in expression (16) for the correlation function $G_j$. The matrices $\mathbf{B}_2–\mathbf{B}_4$ model the detectors. Specifically, the matrix element $(\mathbf{B}_i)_{pk}$ is equal to the output of the detector in $i$-th arm at $p$-th position for unit brightness of $k$-th pixel and zero brightness of other pixels.

Noise covariance matrix has block form as well:

$$
\mathbf{\Sigma}_\nu = \begin{pmatrix}
\mathbf{B}_2 \mathbf{\Sigma}_{22}(f) \mathbf{B}_2^* \\
\mathbf{B}_3 \mathbf{\Sigma}_{32}(f) \mathbf{B}_3^* \\
\mathbf{B}_4 \mathbf{\Sigma}_{42}(f) \mathbf{B}_2^*
\end{pmatrix} \\
\begin{pmatrix}
\mathbf{B}_2 \mathbf{\Sigma}_{23}(f) \mathbf{B}_3^* \\
\mathbf{B}_3 \mathbf{\Sigma}_{33}(f) \mathbf{B}_3^* \\
\mathbf{B}_4 \mathbf{\Sigma}_{43}(f) \mathbf{B}_3^*
\end{pmatrix} \\
\begin{pmatrix}
\mathbf{B}_2 \mathbf{\Sigma}_{24}(f) \mathbf{B}_4^* \\
\mathbf{B}_3 \mathbf{\Sigma}_{34}(f) \mathbf{B}_4^* \\
\mathbf{B}_4 \mathbf{\Sigma}_{44}(f) \mathbf{B}_4^*
\end{pmatrix} + \mathbf{\Sigma}_{\nu'}.
$$

(20)

Here the element with indices $k, k'$ of the block $\mathbf{\Sigma}_{ij}$ is equal to the integral of $K_{GI}^{ij}(17)$ over the values of $r_i$ belonging to $k$-th pixel and over the values of $r_j$ belonging to $k'$-th pixel, for the same pixel ordering as in the matrix $\mathbf{A}$. Hence, the dependence of (20) on $f$ is caused by the dependence on $|T(\cdot)|^2$ of the correlation function $K_{Gj}^{ij}(17)$. The term $\mathbf{\Sigma}_{\nu'}$ is the covariance matrix of the noise component $\nu'$ that is unrelated to ghost imaging, e.g., thermal noise in circuits and digitization error. Most of the noise arising before the correlators is suppressed by them if noise in object and reference arms is independent, but this does not apply to noise arising after the correlators. Besides, due to finite coincidence circuit match time some of noise photons contribute to the noise as well, see discussion of Fig. 4.

It should be noted that the algorithm proposed below can be applied for an image multiplexing method that differs from the one considered in Sect. 2 if the measurement model has the form (18). Specifically the expectation of measurement result has to be the product of a matrix $\mathbf{A}$ and the transparency distribution vector of the measured object, and the error has to be able to be considered additive. For that, the fourth-order intensity correlation function (an analog of (16)) has to linearly depend on the transparency distribution, and the eighth-order intensity correlation function (an analog of (17)) has to “sufficiently weakly” depend on the transparency distribution so that an unknown covariance matrix could be estimated using measurement results. If, in addition to that, photon detections in reference arms are conditionally independent under fixed output of the bucket detector in the object arm (output of a detector in a reference arm does not affect output of detectors in other reference arms), then what was said about the form of matrices $\mathbf{A}$ and $\mathbf{\Sigma}_\nu$ remains valid.
The estimation problem consists of reconstruction of the most accurate estimate of the signal \( Uf \) from the measurement result \( \xi \), where the matrix \( U \) describes a measuring device that is ideal (for the researcher). We consider the case when the researcher is interested in reconstruction of the object image itself, and imaging does not distort the object, therefore, \( U = I \).

Since measurement results linearly depend on \( f \), to solve the estimation problem we can use the model \([A, \Sigma_v, U]\) described in [37], see also [38–41]. If the estimation process is described by a linear operator \( R \) (\( R\xi \) is the result of processing the measurement \( \xi \)), the corresponding mean squared error (MSE) in the worst case of \( f \),

\[
h(R, U) = \sup_{f \in \mathcal{F}} \mathbb{E}\|R\xi - Uf\|^2
\]

as shown in [37], is minimal for \( R \) that is equal to the linear unbiased reduction operator

\[
R_* \equiv U(A^* \Sigma_v^{-1} A)^{-1} A^* \Sigma_v^{-1}, \tag{21}
\]

where \(-\) denotes pseudoinverse. \( h(R_*, U) = \text{tr} U(A^* \Sigma_v^{-1} A)^{-1} U^* \), and the covariance matrix of the linear reduction estimate \( R_* \xi \) is

\[
\Sigma_{R_* \xi} = U(A^* \Sigma_v^{-1} A)^{-1} U^*. \tag{22}
\]

Estimation is possible (MSE is finite) if the condition \( U(I - A A^*) = 0 \) holds, where, as noted above, \( A \) characterizes the real measuring device, while \( U \) characterizes an ideal one with the point spread function required by the researcher, and, therefore, any desired resolution, if this condition is fulfilled. Note that, unlike fluorescence-based superresolution techniques, see e.g., [42], the proposed technique does not require attaching fluorescent molecules to the object. However, as a rule, the better the desired resolution of the ideal measuring device compared to the resolution of the real one, the larger MSE of the obtained estimate. By choosing \( U \), one can select an acceptable (to him) compromise between obtained resolution and noise magnitude. In the case under consideration, as seen from (19), diagonal elements of each block (which, up to a nonzero factor, are equal to the factor before \( |T(\mathbf{r}_j)|^2 \) in the expression for correlation function \( G_j \)) are nonzero. Therefore, each block \( A_j \) is non-degenerate, so for non-degenerate \( B_j \) the reduction error takes only finite values. For a different multiplexing method and thus, different form of the matrix \( A \) this is generally not so.

The measurement reduction technique for the case when it is known that the value \( u \) of the feature of interest is an arbitrary element not of the entire \( \mathcal{U} \) but of its convex closed subset \( \mathcal{U}_{\text{pr}} \) was considered in [41,43]. The estimate refinement which takes advantage of this information is determined by solving the equation

\[
\hat{u} = \Pi \Sigma_{R_* \xi} \left( \tilde{R}_{R_* \xi} \left( \xi^T, \hat{u}^T \right)^T \right) \tag{23}
\]

for \( \hat{u} \), where \( \tilde{R}_{R_* \xi} \) is the measurement reduction operator for a MT \( (A^T, U^T)^T \) and noise with covariance matrix

\[
\begin{pmatrix}
\Sigma_v & 0 \\
0 & \Sigma_{R_* \xi}
\end{pmatrix},
\]

and the operator

\[ \Box \text{ Springer} \]
\[ \Pi_{R_{\xi}}(u) \overset{\text{def}}{=} \arg\min_{v \in U_{pr}} (v - u, \Sigma_{R_{\xi}}^{-1} (v - u)) \] (24)

describes projection onto \( U_{pr} \) by minimizing Mahalanobis distance \( \| \Sigma_{R_{\xi}}^{-1/2} \cdot \| \) that is related to covariance matrix \( \Sigma_{R_{\xi}} \) (22) of error of the linear reduction estimate \( R_{\xi} \). Note that the version of reduction technique proposed in [10,41] for similar information used minimization of the “ordinary” Euclidean distance instead of Mahalanobis distance. In [43], the advantages of minimizing Mahalanobis distance instead of Euclidean distance during projection are shown. In that case, the covariance matrix (22) of linear reduction estimate error is an upper bound on the covariance matrix of the obtained estimate.

3.1 Representation of the object information that is available to the researcher

It is obvious that a priori \( |T(-r_j)|^2 \in [0, 1] \), hence \( f \in [0, 1]^{\dim F}, U f \in [0, 1]^{\dim F} \).

It is assumed that the transparency distribution of the object is not “entirely” arbitrary: transparencies of neighboring pixels usually do not differ much, so the image is sparse (many of its components are zero) in a given (a priori known) basis, similarly to compressed sensing ghost imaging [27,30,36].

The researcher also knows the matrix \( A(19) \) that describes image acquisition conditions and, up to the vector \( f \), the matrix \( \Sigma_{\nu}(20) \) that describes measurement errors. Note that the worst case of \( f \) is realized if all pixels are equally transparent. For a different multiplexing method, one considers the worst case in the sense of reduction MSE of the object in step 1 of the algorithm below.

3.2 Reduction algorithm

The proposed algorithm of multiplexed GI processing using measurement reduction technique that is based on the indicated prior information has the following form.

1. Calculation of linear unbiased reduction estimate \( R_{\xi}(21) \) based on the acquired GI, assuming for calculation of covariance matrix (20) that all pixels have the same brightness.

2. Refinement of the estimate \( R_{\xi} \) using the information \( U_{pr} = [0, 1]^{\dim F} \) by the method (23) by fixed-point iteration, i.e., by consecutive application of the mapping (23) with \( \Pi_{R_{\xi}} \) as the initial approximation. We denote the obtained estimate by \( \hat{u} \).

3. Application of the sparsity-inducing transformation \( T \) to \( \hat{u} \). “Sparsity-inducing” means that the transformation is chosen by the researcher so that, in his opinion, the transform of the true transparency distribution of the object is sparse.

4. Calculation of maximal (in the worst case of \( f \)) variances \( \sigma^2_{T \hat{u}} = (\sigma^2_{(T \hat{u})_1}, \ldots, \sigma^2_{(T \hat{u})_\dim F}) \) of the components of \( T \hat{u} \), i.e., the diagonal matrix elements of \( T \Sigma_{R_{\xi}} T^* \), and calculation of \( T_{\text{thr}}: (T_{\text{thr}})^{\text{def}} = 0 \) if \( |(T \hat{u})_i| < \lambda \sigma((T \hat{u})_i) \), otherwise \( (T_{\text{thr}})^{\text{def}} = (T \hat{u})_i \).
5. Inverse transformation $T^{-1}$ of $\hat{T}_{\text{thr}}$ (if $T$ is a unitary transformation, then $T^{-1} = T^*$), i.e., calculation of $\hat{u}_{\text{thr}} \overset{\text{def}}{=} T^{-1}T\hat{u}_{\text{thr}}$.

5. Calculation of the projection $\Pi_{\Sigma_{e,\epsilon}}(\hat{u}_{\text{thr}})$ that is considered to be the result of processing obtained ghost images.

The value of $\lambda \geq 0$ is a parameter of the algorithm. It reflects a compromise between noise suppression (the larger the value of $\lambda$, the greater the noise suppression) and distortion of images whose components are close to 0. Step 4 can be considered as testing a statistical hypothesis, according to which $(TUf)_i = 0$ (for the alternative hypothesis that $(TUf)_i \neq 0$) for all $i$. In this paper to do that we employ in step 4 a simple criterion based on Chebyshev’s inequality: if $(TUf)_i = 0$, then $\Pr\left(|(TUf)_i| \geq \lambda \sigma(T\hat{u}_i)| \leq \lambda^{-2}$.

Due to that one can suppose that image distortion is insignificant for, at least, $\lambda \leq 1$, as such distortion would be indistinguishable from the noise. Step 4 can be also interpreted as replacement of the original matrix $U$ with one whose kernel contains the estimate components after the specified transform that are affected by the noise the most.

In [10], the matrix $U$ was chosen to suppress the noise more, even at the cost of potential image distortion (e.g., worse resolution), by discarding the most noisy components of the image. Unlike [8–10], here we consider components of the image in a basis specified by the researcher instead of the eigenbasis [37, ch. 8] of the measurement interpretation model, i.e., a basis determined by error properties. In this article, the basis is defined by the transformation whose result for the true transparency distribution is sparse, but the discarded components are determined, as in [10], by the measurement error. Thus, to improve estimation quality not only information about the noise is used, but also information about the object, namely, the properties of the transparency distribution (in the opinion of the researcher) and its features of interest.

### 4 Computer modeling results

The results of processing of obtained GI as described above are shown in Figs. 2 and 3. The detectors in reference arms are identical ones that are three times as large as an element of the object image. Therefore, image processing via measurement reduction increases resolution in addition to noise suppression. Modeling was carried out for the same parameters of the optical setup as in [10]: beam wave numbers $k_1 = 6 \times 10^4 \text{ cm}^{-1}$, $k_3 = 1.7 \times 10^5 \text{ cm}^{-1}$, ratio of the nonlinear coefficients $\xi = \gamma_1 / \beta = 0.4$ and $\gamma_2 = \gamma_1$.

It should be noted that we carried out the calculations for achievable experimental conditions. When registering 10 photons per pixel, for a $64 \times 64$ pixel CCD the total number of photons is about $4.1 \times 10^4$ photons. If the detection efficiency is 30%, the CCD should be illuminated by approximately $1.4 \times 10^5$ photons. Estimates showed that when pumping a 10-mm long aperiodically poled lithium niobate crystal with continuous radiation of a wavelength of $1.064 \ \mu\text{m}$ and with a power of 200 mW, the photon generation rate can be $1.8 \times 10^6$ photons/s (see also [44]). Such bright sources of correlated photons have been experimentally implemented [45]. Thus, in the case under consideration the acquisition time should be about 80 $\mu$s.
Fig. 2 GI processing by the developed algorithm: (a) is the object, 64 × 64 pixels, that is illuminated by 1 photon per pixel on average, (b) are its acquired GI, and (c–h) are image reduction results: (c) is the result of reduction without sparsity information, (d–h) are results of reduction using information about sparsity in (d, e) discrete cosine transform (DCT), (f–h) Haar transform bases. The parameter $\lambda \geq 0$ of the image processing algorithm reflects a compromise between noise suppression (the larger the value of $\lambda$, the greater the noise suppression) and distortion of images whose components are close to 0.

One can see that additional information about sparsity allows to suppress noise more but its impact on obtained resolution is weak. As expected, for $0 \leq \lambda \leq 1$ the distortion is undistinguishable from the noise. Further increase of $\lambda$ leads to better noise suppression (cf. Fig. 2c, d), but also leads to more severe distortions caused by discarding “significant” image components as well (cf., e.g., Figs. 2e, 3d). For large $\lambda$, their influence outweighs the improvement in image quality due to noise suppression, as small-scale image details are suppressed as well. Therefore, the optimal value of $\lambda$ depends on one’s intentions: one should choose the maximal value of $\lambda$ that preserves the details of interest. To do that, one can model acquisition of a test image that contains the required details and choose the largest value of $\lambda$ that preserves them, or specify the value of $\lambda$ after comparing reduction results for different $\lambda$. In the case of an object with sharp transparency changes (Fig. 2), the additional information allowed to suppress false signal where the object is opaque, but only for Haar transform (discrete cosine transform (DCT) causes increased false signal in that region).

The transform whose result for the transparency distribution of the object is sparse that is usually employed in ghost image processing by the means of compressed...
Fig. 3 GI processing by the developed algorithm: a is the object, 64 × 64 pixels, that is illuminated by 10 photons per pixel on average, b are its acquired GI, and c–f are results of reduction using information about sparsity in c, d discrete cosine transform (DCT), e, f Haar transform bases.

sensing is DCT [27,30,36]. In [46], several transforms (identity transform, discrete wavelet transform and DCT) were reviewed and the advantages of DCT were shown. However, it seems that Haar transform may be preferable in the case of a transparency distribution that contains areas of weakly changing transparency with sharp borders if these areas are large compared to the resolution of the ideal measuring transducer and the location of the borders is important to the researcher. This assumption is verified by Fig. 2f, g, where one can see that Haar transform in this case, as opposed to Fig. 3, allows larger λ values without causing significant distortions, cf., e.g., Fig. 2e, g, where the usage of DCT causes blurring of transversal slit borders for the same value of λ.

In Fig. 4 GI are compared with ordinary images if noise photons, which do not carry information about the object, but increase noise, are present. Unlike the previous Figs. 2 and 3, the detectors in reference arms are five times as large as an element of the object image. Due to employing correlations to acquire GI, noise photons usually do not affect detected images, as this requires simultaneous detection of a noise photon by one detector and another photon by a different detector. Nevertheless, due to finite coincidence windows and finite widths of the light filters before the detectors the noise photons do increase the measurement errors. One can see that due to suppression of most noise photons the quality of the reconstructed image is better than the quality of the image reconstructed using the ordinary image for the same number of noise photons. Moreover, when taking advantage of sparsity information ghost imaging allows to exploit larger λ values and thus to suppress the noise more (cf., e.g., Fig. 4b, d). In this case, multiplexing provides the means for further noise suppression if noise photons in different arms are detected independently. In the case of non-multiplexed ghost images (Fig. 4e) obtained by “turning off” the frequency-mixing processes (1) by choosing NC parameters γ_1 = γ_2 = 0 (and, therefore, ξ = γ_1 / β = 0), the difference is less pronounced, but still present: the result of reduction in the non-multiplexed ghost image (Fig. 4f) for the same value of the algorithm parameter λ is more blurred.
Fig. 4 Ordinary and ghost image processing by the developed algorithm. a is the ordinary image of the object from Fig. 3a obtained by illuminating it by 10 photons per pixel on average and 10 noise photons, and b is the result of its reduction with information about sparsity in DCT base. c are GI impacted by 5 noise photons, respectively. d is the result of their reduction with information about sparsity in DCT base. e is a single, non-multiplexed ghost image and f is the result of its reduction with information about sparsity in DCT base.

Fig. 4 Ordinary and ghost image processing by the developed algorithm. a is the ordinary image of the object from Fig. 3a obtained by illuminating it by 10 photons per pixel on average and 10 noise photons, and b is the result of its reduction with information about sparsity in DCT base. c are GI impacted by 5 noise photons, respectively. d is the result of their reduction with information about sparsity in DCT base. e is a single, non-multiplexed ghost image and f is the result of its reduction with information about sparsity in DCT base.

Therefore, formalization of the researcher’s information about sparsity of the object transparency distribution by Haar transform is preferable if it has areas of weakly changing transparency with sharp borders that are large compared to the resolution of the ideal measuring transducer and the location of the borders is important to the researcher. DCT is preferable if the transparency distribution has small transparency changes that have to be present in the estimate, e.g., biological objects without high-contrast borders. The values of $\lambda \sim 1 \div 1.5$ are optimal if small-scale details are present and are of interest. Otherwise, larger $\lambda$ values are advisable.

5 Conclusion

In photocounting mode, the problem of increasing noise immunity is exacerbated due to higher information content of each photon or its absence. Noise immunity can be
improved by ghost imaging, as its usage of correlators in image acquisition means that most noise photons (that is, the photons that have not interacted with the object and, therefore, do not transmit any information about it) do not affect the acquired images. Multiplexing ghost images provides additional opportunities to extract information about the illuminated object, as illustrated by the comparison with ordinary images and non-multiplexed ghost images. Further improvement in noise immunity and, thus, of the quality of the estimated object image can be obtained by using the information about the object or about the measurement model that is available to the researcher. In order to take the available information into account, the measurement reduction technique is employed, as it provides a unified framework for optimal processing of multiplexed and, for comparison, non-multiplexed images with different information about the measurement model and the research object. As we consider multiplexed ghost images, the additional information about the measuring process in this work is the correlation functions of multiplexed ghost images. The information about the object is the information that the object transparency distribution is not arbitrary, namely, transparencies of neighboring pixels, as a rule, differ only slightly. This information is formalized as sparsity of the result of a given transform (e.g., DCT) of the transparency distribution, similar to compressed sensing.

In compressed sensing, as a rule, the measurement error is modeled as an arbitrary vector with bounded norm. Instead, in the proposed method it is modeled as a random vector, and selection of the estimate components which are considered to be zero is based on the statistical properties of the estimate components, namely, their variances. The use of covariances of the estimate components in addition to their variances is a subject of further research.

We consider that computer modeling based on the developed algorithm showed high efficiency of the developed reduction technique of ghost image processing in the sense of improvement in both their quality and their noise immunity. It is of interest to apply this technique in the field of quantum image processing for parametric amplification of images and frequency conversion.

Acknowledgements The authors are grateful for help to T. Yu. Lisovskaya.

References

1. Belinskii, A.V., Klyshko, D.N.: Two-photon optics: diffraction, holography, and transformation of two-dimensional signals. J. Exp. Theor. Phys. 78(3), 259–262 (1994)
2. Gatti, A., Brambilla, E., Bache, M., Lugiato, L.A.: Correlated imaging, quantum and classical. Phys. Rev. A 70(1), 013802 (2004). https://doi.org/10.1103/PhysRevA.70.013802
3. Gatti, A., Brambilla, E., Bache, M., Lugiato, L.A.: Ghost imaging. In: Kolobov, M.I. (ed.) Quantum Imaging, pp. 79–111. Springer, New York (2007)
4. Chan, K.W.C., O’Sullivan, M.N., Boyd, R.W.: High-order thermal ghost imaging. Opt. Lett. 34(21), 3343–3345 (2009). https://doi.org/10.1364/ol.34.003343
5. Erkman, B.I., Shapiro, J.H.: Ghost imaging: from quantum to classical to computational. Adv. Opt. Photonics 2(4), 405–450 (2010). https://doi.org/10.1364/aop.2.000405
6. Shapiro, J.H., Boyd, R.W.: The physics of ghost imaging. Quantum Inf. Process. 11(4), 949–993 (2012). https://doi.org/10.1007/s11128-011-0356-5
7. Chirkin, A.S.: Multiplication of a ghost image by means of multimode entangled quantum states. JETP Lett. 102(6), 404–407 (2015). https://doi.org/10.1134/S0021364015180046
8. Balakin, D.A., Belinsky, A.V., Chirkin, A.S., Yakovlev, V.S.: Multiplicated ghost images reconstruction. In: ICONOLAT 2016 Technical Digest, ICONO-03 Quantum and Atom Optics (2016)

9. Balakin, D.A., Belinsky, A.V., Chirkin, A.S.: Correlations of multiplexed quantum ghost images and improvement of the quality of restored image. J. Russ. Laser Res. 38(2), 164–172 (2017). https://doi.org/10.1007/s10496-017-9630-z

10. Balakin, D.A., Belinsky, A.V., Chirkin, A.S.: Improvement of the optical image reconstruction based on multiplexed quantum ghost images. J. Exp. Theor. Phys. 125(2), 210–222 (2017). https://doi.org/10.1134/S1063776117070147

11. Rodionov, A.V., Chirkin, A.S.: Entangled photon states in consecutive nonlinear optical interactions. JETP Lett. 79(6), 253–256 (2004). https://doi.org/10.1134/1.1759404

12. Ferraro, A., Paris, M.G.A., Bondani, M., Allevi, A., Puddu, E., Andreoni, A.: Three-mode entanglement by interlinked nonlinear interactions in optical $\chi^{(2)}$ media. JOSA B 21(6), 1241–1249 (2004). https://doi.org/10.1364/JOSAB.21.001241

13. Olsen, M.K., Drummond, P.D.: Entanglement and the Einstein–Podolsky–Rosen paradox with coupled intracavity optical down-converters. Phys. Rev. A 71(5), 053803 (2005). https://doi.org/10.1103/PhysRevA.71.053803

14. Solntsev, A.S., Sukhorukov, A.A., Neshev, D.N., Kivshar, Y.S.: Spontaneous parametric down-conversion and quantum walks in arrays of quadratic nonlinear waveguides. Phys. Rev. Lett. 108(2), 023601 (2012). https://doi.org/10.1103/physrevlett.108.023601

15. Kruse, R., Katzschmann, F., Christ, A., Schreiber, A., Wilhelm, S., Laiho, K., Gábris, A., Hamilton, C.S., Jex, I., Silberhorn, C.: Spatio-spectral characteristics of parametric down-conversion in waveguide arrays. New J. Phys. 15(8), 083046 (2013). https://doi.org/10.1088/1367-2630/15/8/083046

16. Daems, D., Bernard, F., Cerf, N.J., Kolobov, M.I.: Tripartite entanglement in parametric down-conversion with spatially structured pump. JOSA B 27(3), 447–451 (2010). https://doi.org/10.1364/josab.27.000447

17. Chirkin, A.S., Shutov, I.V.: On the possibility of the nondegenerate parametric amplification of optical waves at low-frequency pumping. JETP Lett. 86(11), 693–697 (2008). https://doi.org/10.1134/S0021364008070014

18. Chirkin, A.S., Shutov, I.V.: Parametric amplification of light waves at low-frequency pumping in aperiodic nonlinear photonic crystals. J. Exp. Theor. Phys. 109(4), 547–556 (2009). https://doi.org/10.1134/S1063776109100021

19. Saygin, M.Y., Chirkin, A.S.: Simultaneous parametric generation and up-conversion of entangled optical images. J. Exp. Theor. Phys. 111(1), 11–21 (2010). https://doi.org/10.1134/S1063776110070022

20. Saygin, M.Y., Chirkin, A.S.: Quantum properties of optical images in coupled nondegenerate parametric processes. Opt. Spectrosc. 110(1), 97–104 (2011). https://doi.org/10.1134/S0030400X111010152

21. Saygin, M.Y., Chirkin, A.S., Kolobov, M.I.: Quantum holographic teleportation of entangled two-color optical images. JOSA B 29(8), 2090–2098 (2012). https://doi.org/10.1364/josab.29.002090

22. Tlyachev, T.V., Chebotarev, A.M., Chirkin, A.S.: A new approach to quantum theory of multimode coupled parametric processes. Phys. Scr. T153, 014060 (2013). https://doi.org/10.1088/0031-8998/2013/T153/014060

23. Duan, D., Du, S., Xia, Y.: Multiwavelength ghost imaging. Phys. Rev. A 88(5), 053842 (2013). https://doi.org/10.1103/physreva.88.053842

24. Zhang, D.J., Li, H.G., Zhao, Q.L., Wang, S., Wang, H.B., Xiong, J., Wang, K.: Wavelength-multiplexing ghost imaging. Phys. Rev. A 92(1), 013823 (2015). https://doi.org/10.1103/physreva.92.013823

25. Shi, D., Zhang, J., Huang, J., Wang, K., Yuan, K., Cao, K., Xie, C., Liu, D., Zhu, W.: Polarization-multiplexing ghost imaging. Opt. Lasers Eng. 102, 100–105 (2018). https://doi.org/10.1016/j.optlaseng.2017.10.022

26. Chirkin, A.S., Gostev, P.P., Agapov, D.P., Magnitskiy, S.A.: Ghost polarimetry: ghost imaging of polarization-sensitive objects. Laser Phys. Lett. 15(11), 115404 (2018). https://doi.org/10.1088/1612-201x/aa4e46

27. Morris, P.A., Aspden, R.S., Bell, J.E.C., Boyd, R.W., Padgett, M.J.: Imaging with a small number of photons. Nat. Commun. 6, 5913 (2015). https://doi.org/10.1038/ncomms6913

28. Zerom, P., Chan, K.W.C., Howell, J.C., Boyd, R.W.: Entangled-photon compressive ghost imaging. Phys. Rev. A 84(6), 061804 (2011). https://doi.org/10.1103/physreva.84.061804

29. Gong, W., Han, S.: Experimental investigation of the quality of lensless super-resolution ghost imaging via sparsity constraints. Phys. Lett. A 376(17), 1519–1522 (2012). https://doi.org/10.1016/j.physleta.2012.03.027
30. Gong, W., Han, S.: High-resolution far-field ghost imaging via sparsity constraint. Sci. Rep. 5(1), 9280 (2015). https://doi.org/10.1038/srep09280
31. Katz, O., Bromberg, Y., Silberberg, Y.: Compressive ghost imaging. Appl. Phys. Lett. 95(13), 131110 (2009). https://doi.org/10.1063/1.3238296
32. Suchowski, H., Bruner, B.D., Israel, Y., Ganany-Padowicz, A., Arie, A., Silberberg, Y.: Broadband photon pair generation at \( 3\omega/2 \). Appl. Phys. B 122(2), 25 (2016). https://doi.org/10.1007/s00340-015-6304-9
33. Vyunishev, A.M., Arkhipkin, V.G., Chirkin, A.S.: Theory of second-harmonic generation in a chirped 2d nonlinear optical superlattice under nonlinear raman-nath diffraction. JOSA B 32(12), 2411–2416 (2015). https://doi.org/10.1364/josab.32.002411
34. Akhmanov, S.A., D’yakov, Y.E., Chirkin, A.S.: Introduction to Statistical Radiophysics and Optics. Nauka, Moscow (1981). [in Russian]
35. Goodman, J.W.: Introduction to Fourier Optics, 3rd edn. Roberts & Company Publishers, Englewood (2004)
36. Shi, X., Huang, X., Nan, S., Li, H., Bai, Y., Fu, X.: Image quality enhancement in low-light-level ghost imaging using modified compressive sensing method. Laser Phys. Lett. 15(4), 045204 (2018). https://doi.org/10.1088/1612-202x/aaa56
37. Pyt’ev, Y.P.: Methods of Mathematical Modeling of Measuring-Computing Systems, 3rd edn. Fizmatlit, Moscow (2012). [in Russian]
38. Pyt’ev, Y.P., Chulichkov, A.I.: Foundations for a theory of computer assisted superhigh resolution measurement systems. Meas. Tech. 41(2), 111–121 (1998). https://doi.org/10.1007/BF02524537
39. Pyt’ev, Y.P.: On the problem of superresolution of blurred images. Pattern Recognit. Image Anal. 14(1), 50–59 (2004)
40. Pyt’ev, Y.P.: Measurement-computation converter as a measurement facility. Autom. Remote Control 71(2), 303–319 (2010). https://doi.org/10.1134/s0005117910020116
41. Balakin, D.A., Pyt’ev, Y.P.: A comparative analysis of reduction quality for probabilistic and possibilistic measurement models. Moscow Univ. Phys. Bull. 72(2), 101–112 (2017). https://doi.org/10.3103/S0027134917020047
42. Solomon, O., Mutzafi, M., Segev, M., Eldar, Y.C.: Sparsity-based super-resolution microscopy from correlation information. Opt. Express 26(14), 18238–18269 (2018). https://doi.org/10.1364/oe.26.018238
43. Balakin, D.A., Pyt’ev, Y.P.: Improvement of measurement reduction in the case when the feature of interest to the researcher belongs to an a priori known convex closed set [in russian]. In: Lomonosov readings—2018. Proceedings of Physics section., pp. 155–158. M. V. Lomonosov Moscow State University. Faculty of Physics, Moscow (2018)
44. Peřina, J., Svozílk, J.: Randomly poled nonlinear crystals as a source of photon pairs. Phys. Rev. A 83(3), 033808 (2011). https://doi.org/10.1103/physreva.83.033808
45. Pelton, M., Marsden, P., Ljunggren, D., Tengner, M., Karlsson, A., Fragemann, A., Canalias, C., Laurell, F.: Bright, single-spatial-mode source of frequency non-degenerate, polarization-entangled photon pairs using periodically poled KTP. Opt. Express 12(15), 3573–3580 (2004). https://doi.org/10.1364/opex.12.003573
46. Du, J., Gong, W., Han, S.: The influence of sparsity property of images on ghost imaging with thermal light. Opt. Lett. 37(6), 1067–1069 (2012). https://doi.org/10.1364/ol.37.001067

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.