GULBRANDSEN–HALLE–HULEK DEGENERATION AND HILBERT–CHOW MORPHISM

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ABSTRACT. For a semistable degeneration of surfaces without a triple point, we show that two models of degeneration of Hilbert scheme of points of the family, Gulbrandsen-Halle-Hulek degeneration given in [GHH16] and the one given by the author in [Nag16], are actually isomorphic.

INTRODUCTION

The Hilbert scheme of points on a surface appears as an interesting object in many branches of mathematics, such as holomorphic symplectic geometry, differential geometry, singularity theory, representation theory, and so on. If one wants to study a moduli behavior of Hilbert scheme of points on surfaces, it is natural to ask for a good model of degenerating family of Hilbert schemes.

For the first sight, one might regard this question a triviality; for a semistable degeneration $S \rightarrow C$ of quasi-projective surfaces, shouldn’t the relative Hilbert scheme of points $\text{Hilb}^n(S/C) \rightarrow C$ do the work? However, even though the family satisfies several good properties such as unipotency of monodromy operators on the cohomology groups, the singular fiber of $\text{Hilb}^n(S/C) \rightarrow C$ can be quite singular. In fact, it is not quite clear how to cut out the ‘main component’ of the relative Hilbert scheme. Moreover, even in the case of $n = 2$, the relative Hilbert scheme is not a minimal model in the sense of higher dimensional birational geometry [Nag08]. Therefore, a search for a minimal model that is very near to being semistable as a family over the base curve $C$ is a non-trivial problem.

At the time of writing, there are at least two approaches to the problem. One is an approach of Gulbrandsen-Halle-Hulek [GHH16] based on the notion of expanded degeneration due to Jun Li. They associate to the family $S \rightarrow C$ a family of expanded degeneration $S[n] \rightarrow \mathbb{A}^{n+1}$, consider the relative Hilbert scheme $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})$ of the expanded degeneration, and define $I^n_S/C$ to be a GIT quotient $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})_{ss} // G[n]$ for a natural action of $G[n] \cong (\mathbb{C}^*)^n$ with a certain linearization. We call the family $I^n_S/C \rightarrow C$ Gulbrandsen-Halle-Hulek degeneration (GHH degeneration as a shorthand). The other construction is in the previous work of the author [Nag16]; it works in a local situation that $S = \mathbb{A}^3 \rightarrow C = \mathbb{A}^1$ is given by $(x,y,z) \mapsto t = xy$, and analyzes the local structure of the singularities of the relative symmetric product $\text{Sym}^n(S/C)$. We construct a $\mathbb{Q}$-factorial terminalization
$Y^{(n)} \to \text{Sym}^n(S/X)$ explicitly; first we consider a small projective toric resolution $\tilde{Z}^{(n)}$ of the relative self-product $(S/C)^n$ and define $Z^{(n)} = \tilde{Z}^{(n)}/\mathbb{G}_n$. $Y^{(n)}$ is given as a crepant divisorial partial resolution of $Z^{(n)}$.

Each approach has its own merit; the construction of GHH degeneration is global in nature. Gulbrandsen et. al. clarified the necessary and sufficient condition that the GHH degeneration $I^n_{S/C}$ be projective over the base and analyzes the combinatorial properties of the degenerate fiber. On the other hand, the approach of [Nag16] clarifies the local singularities along the singular fiber in every step of the construction of the minimal model.

Now, another natural question is to ask the relationship between these two models. The main theorem of this article is the following:

**Main Theorem** (=Theorem 4.3.1). GHH degeneration $I^n_{S/C}$ and $Y^{(n)}$ are isomorphic to each other as a family over $C$.

The main device to prove the theorem is Hilbert-Chow morphism; we have a natural relative Hilbert-Chow morphism

$$\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss} \to \text{Sym}^n(S[n]/\mathbb{A}^{n+1}),$$

which is $G[n]$-equivariant. Taking GIT quotient by $G[n]$, we get a birational morphism

$$I^n_{S/C} = \text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss} // G[n] \to \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} // G[n].$$

The main technical claim is that the quotient stack $[\text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} // G[n]]$ is isomorphic to $[\tilde{Z}^{(n)}/\mathbb{G}_n]$ (Theorem 4.6.4). We prove this claim relying on toric geometry, in particular a description of torus quotient of a semi-projective toric variety via polyhedron. In the process, we will see that the choice of linearization that Gulbrandsen–Halle–Hulek made (we call it GHH linearization) is very natural also with a view toward toric–combinatorial aspect of the theory.

1. **Toric blowing-up and its GIT quotient**

1.1. **Toric variety via polyhedron.** First we review the description of a semi-projective toric varieties using lattice polyhedra. For details, we refer [CLS11], Chapter 7.

Let $T = (\mathbb{C}^*)^n$ be a torus, $M = \mathbb{Z}^n$ a character lattice of $T$, and $N = M^\vee$ a lattice of one-parameter subgroups of $T$. Let $\bar{P}$ be a lattice polyhedron on $M$ (op. cit. Definition 7.1.3). Then, $\bar{P}$ is a Minkowski sum of a lattice polytope $P$ and a strongly convex rational polyhedral cone $C$, the recession cone of $\bar{P}$. The normal fan $\Sigma_\bar{P}$ defines a toric variety $X_T(\bar{P}) = X(\bar{P})$. The dual cone $\sigma = C^\vee \subset N_R$ to the recession
cone may not be strongly convex, while \( \sigma \) is the union of the maximal cones in \( \Sigma \).

Let \( W = \mathbb{R}\text{-span}(\sigma \cap (-\sigma)) \) and define an affine toric variety \( U(\tilde{P}) = X_{N/W \cap N}(\sigma') \), where \( \sigma' \) is the image of \( \sigma \) in \( N_{\mathbb{R}}/W \). Then, the toric variety \( X(\tilde{P}) \) is equipped with a projective toric morphism

\[
\phi_P : X(\tilde{P}) \to U(\tilde{P}).
\]

A projective toric morphism over an affine toric variety can always be realized as \( \phi_P \) above (op. cit., Theorem 7.2.4 and Proposition 7.2.3). A projective birational toric modification is a special case in which the cone \( \sigma \) is strictly convex and \( U(\tilde{P}) = X(\sigma) \).

1.2. Toric blowing-up. Let \( \sigma \subset N_{\mathbb{R}} \) be a rational polyhedral cone and \( m_1, \ldots, m_r \in M \) a set of generators of the semigroup \( \sigma^\vee \cap M \). Then, the affine toric variety \( X(\sigma) \) is the closure of the image of a map

\[
T \to \mathbb{A}^r; \quad t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_r}(t))
\]

where \( \chi^m \) is a monomial with the exponent \( m \). Let us take another set of elements \( m'_1, \ldots, m'_s \in M \) and consider a polytope \( P \) that is the convex hull of \( \{m'_0 = 0, m'_1, \ldots, m'_s\} \) in \( M_{\mathbb{R}} \) and a polyhedron \( \tilde{P} \) given as the Minkowski sum \( P + \sigma^\vee \). Let us assume further the following.

**Assumption 1.2.1.** For every vertex \( v \in \tilde{P} \), the set

\[
\{m_1 - v, \ldots, m_r - v, -v, m'_1 - v, \ldots, m'_s - v\}
\]

generates the semigroup \( \sigma_v^\vee \cap M \) where \( \sigma_v \) is the normal cone to \( \tilde{P} \) at \( v \).

Note that this assumption immediately implies that the polyhedron \( \tilde{P} \) is very ample (op. cit., Definition 7.1.8). In this situation, the toric variety \( X(\tilde{P}) \) can be realized as the closure of the image of a monomial map

\[
T \to \mathbb{A}^r \times \mathbb{P}^s; \quad t \mapsto \left( (\chi^{m_1}(t), \ldots, \chi^{m_r}(t)), [1 : \chi^{m'_1}(t) : \cdots : \chi^{m'_s}(t)] \right).
\]

The canonical morphism \( \phi_P : X(\tilde{P}) \to X(\sigma) \), which we call a toric blowing-up, is nothing but the projection to the first factor \( \mathbb{A}^r \).

1.3. Fractional linearization and torus quotient. Now we consider an action of a sub-torus \( G \subset T \) on a toric blowing-up \( X(\tilde{P}) \) and discuss GIT quotients of \( X(\tilde{P}) \) by \( G \). The following argument is a slight generalization of [KSZ91], §3.

The sub-torus \( G \subset T \) acts in a trivial way on \( X(\tilde{P}) \) and \( X(\sigma) \) such that \( \phi_P \) is \( G \)-equivariant. Let \( L \) be a line bundle on \( X(\tilde{P}) \) that is the pull back of \( \mathcal{O}_{\mathbb{P}^s}(1) \). We call a linearization of \( L^\otimes k \) a fractional linearization of \( L \).
The cone $C(\bar{P})$ associated to $\bar{P}$ is a cone in $\tilde{M}_R = M_R \times \mathbb{R}$ such that

$$C(\bar{P}) \cap H_t = t\bar{P} = tP + \sigma^\vee$$

where $H_t = \{(m, t) \in M_R \times \mathbb{R} \mid m \in M_R\}$, the hyperplane of ‘height t’. The cone $C(\bar{P})$, in turn, determines a graded ring

$$S(\bar{P}) = \mathbb{C}[C(\bar{P}) \cap (M \times \mathbb{Z})],$$

and we know that $X(\bar{P}) \cong \text{Proj} \ S(\bar{P})$ (op. cit., Theorem 7.1.13). Note that $S(\bar{P})_0 = \mathbb{C}[\sigma^\vee \cap M]$ is the coordinate ring of the affine toric variety $X(\sigma)$ and $S(\bar{P})_k = H^0(X(\bar{P}), L^\otimes k) = \bigoplus_{m \in k\mathbb{P}} \mathbb{C}\chi^m$.

**Proposition 1.3.1.** We keep the notation above. Let $M_G$ be the character lattice of $G$ and $\alpha : M \to M_G$ the canonical projection corresponding to the embedding $G \subset T$.

1. The set of fractional linearizations of $L$ is naturally identified with $M_G \otimes \mathbb{Q}$.
2. Assume that $b \in M_G \otimes \mathbb{Q}$ is a fractional $G$-linearization of $L$. Then, the GIT quotient $X(\bar{P})^{ss}(L, b)/\!\!/G$ is a toric variety given by a polyhedron

$$\tilde{P}_b = \bar{P} \cap (\alpha \otimes \mathbb{R})^{-1}(-b),$$

which is naturally identified with a (fractional) lattice polyhedron on a sublattice $\text{Ker}(\alpha) \subset M$.

**Proof.** (1) After passing to sufficiently high Veronese embedding, namely passing $L$ to $L^{\otimes m}$ instead of $L$, if necessary, we may assume that $S(\bar{P})$ is generated by $S(\bar{P})_1 = \mathbb{C}[C(\bar{P}) \cap (M \times \{1\})]$, i.e., we assume that $\bar{P}$ is a normal polyhedron (op. cit., Definition 7.1.8). Then, to give a $G$-linearization of $L$ is the same as to give a dual $G$-action on the $S(\bar{P})_0$-module

$$S(\bar{P})_1 \to S(\bar{P})_1 \otimes \mathbb{C}[G]$$

that is compatible with the dual $G$-action on $S(\bar{P})_0$ (cf. [Muk03], Definition 6.23). The $G$-action on $S(\bar{P})_0 = \mathbb{C}[\sigma^\vee \cap M]$ is determined by the canonical projection $\alpha : M \to M_G$. The map (1.1) is determined by a map

$$l : \bar{P} \cap M \to M_G$$

satisfying $l(m' + m) = \alpha(m') + l(m)$. This immediately implies that the map $l$ is (a restriction of) an affine map $l : M \to M_G$ such that $l(m) = \alpha(m) + l(0)$ for all $m \in M$. Therefore, a fractional linearization of $L$ is in one to one correspondence with $b = l(0) \in M_G \otimes \mathbb{Q}$.

(2) A (integral) linearization $b \in M_G$ determines a diagonal action of $G$ on $S(\bar{P})_k = \bigoplus_{m \in k\mathbb{P}} \mathbb{C}\chi^m$, thus it determines the ring of invariants $S(\bar{P})_{(G, b)}$ with respect to this action. A monomial function $\chi^m$ is $G$-invariant if and only if $l_b(m) = 0$ for $l_b = l = \ldots$
\(\alpha + b\) as in \((1.2)\) corresponding to \(b\). The GIT quotient \(X^{ss}(L, b) \sslash G\) is defined to be the Proj of the graded ring \(S(\tilde{P})^{(G, b)}\). If we define an affine plane \(\tilde{M}_{\mathbb{R}, b} \subset \tilde{M}_{\mathbb{R}}\) by
\[
\tilde{M}_{\mathbb{R}, b} = \{(m, t) \in M_{\mathbb{R}} \times \mathbb{R} \mid (l_b \otimes \mathbb{R})(m) = 0\},
\]
the invariant ring is given by
\[
S(\tilde{P})^{(G, b)} = \mathbb{C}[C(\tilde{P}) \cap \tilde{M}_{\mathbb{R}, b} \cap (M \times \mathbb{Z})],
\]
which is nothing but \(S(\tilde{P}_b)\). Therefore, the GIT quotient \(X(\tilde{P})^{ss}(L, b) \sslash G\) is the toric variety associated with \(\tilde{P}_b\). The case of fractional linearization \(b \in M_G \otimes \mathbb{Q}\), we just pass to a sufficiently high truncation of the graded ring \(S(\tilde{P})\).

Q.E.D.

We note that if we set \(\bar{\sigma}^\vee = \sigma^\vee \cap \text{Ker}(\alpha) \otimes \mathbb{R}\) and \(P_b = P \cap (\alpha \otimes \mathbb{R})^{-1}(-b)\), we have \(\tilde{P}_b = P_b + \bar{\sigma}^\vee\), that is, the GIT quotient \(X(\tilde{P})^{ss}(L, b) \sslash G\) is the toric blow-up of the affine quotient \(X(\sigma) \sslash G\) determined by the polytope \(P_b\).

2. Toric description of a family of expanded degeneration

2.1. Family of expanded degenerations \(X[n]\). Now we study the local model of expanded degeneration using toric geometry. For details, we refer [GHH16]. We also follow the notation in op. cit.

Let \(X = \mathbb{A}^2\) and \(C = \mathbb{A}^1\) with coordinates \((x, y)\) and \(t\), respectively, and consider the morphism \(X \to C = \mathbb{A}^1\) defined by \(t = xy\). The base change \(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}\) by
\[
\mathbb{A}^{n+1} \to \mathbb{A}^1; \quad (t_1, \ldots, t_{n+1}) \mapsto t_1 \ldots t_{n+1}
\]
is an affine variety defined by \(xy - t_1 \ldots t_{n+1} = 0\) in \(\mathbb{A}^{n+3}\). The \(n\)-th family of expanded degeneration \(X[n] \to \mathbb{A}^{n+1}\) is a successive blowing-up \(X[n]\) of \(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}\) by the strict transform of a subvariety defined by the ideal \((t_i, x)\) for \(i = 1, \ldots, n\) in this order, equipped with the natural projection to \(\mathbb{A}^{n+1}\).

One can easily see that \(X[n]\) is a toric variety by the construction. Let us describe \(X[n]\) via a polyhedron. It is easy to see that \(X[n]\) is the closure of the image of a map
\[
\Phi'[n]: T[n] = \left(\mathbb{C}^n\right)^{n+2} \to \left(\mathbb{A}^2 \times \mathbb{A}^{n+1}\right) \times (\mathbb{P}^1)^n
\]
defined by
\[
(s, t_1, \ldots, t_{n+1}) \mapsto \left(\frac{t_1 \ldots t_{n+1}}{s}, s, t_1, \ldots, t_{n+1}\right), \quad \left[1: \frac{t_2 \ldots t_{n+1}}{s}\right], \quad \left[1: \frac{t_3 \ldots t_{n+1}}{s}\right], \ldots, \quad \left[1: \frac{t_{n+1}}{s}\right].
\]
Here we note that we have the relations
\[
x = \frac{t_1 \ldots t_{n+1}}{s} \quad \text{and} \quad y = s.
\]
Composing with the Segre embedding \( (\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1} \), we get a monomial map
\[
\Phi[n] : T[n] \to \mathbb{A}^{n+3} \times \mathbb{P}^{2^n-1}.
\]
By the description in §1.1, we have a polyhedron \( \tilde{P}[n] \) on the character lattice \( M[n] = \mathbb{Z}^{n+3} \) of \( T[n] \) such that
\[
\phi_{\tilde{P}[n]} : X[n] = X(\tilde{P}[n]) \to X \times \mathbb{A}^{n+1} = X(\sigma[n])
\]
is the composite of blowing-ups described above. Here \( \sigma[n] \) is the dual cone of the recession cone of the polyhedron \( \tilde{P}[n] \), namely we have a Minkowski sum decomposition
\[
\tilde{P}[n] = P[n] + \sigma[n]^\vee
\]
with \( P[n] \) a lattice polytope on \( M[n] \).

The cone \( \sigma[n]^\vee \) is easy to describe: taking a basis of \( M[n] \) corresponding to the coordinate \((s, t_1, \ldots, t_{n+1})\), \( \sigma[n]^\vee \subset M[n]_{\mathbb{R}} \) is the cone generated by the column vectors of the matrix
\[
\sigma[n]^\vee = \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 1
\end{pmatrix},
\]
corresponding to the monomials \( x, y, t_1, \ldots, t_{n+1} \) in this order.

Let \( \Box_n \) be a hypercube in \( \mathbb{R}^n \) whose \( 2^n \) vertices are the vectors whose entries are 0 or 1. Then we define \( P[n] \) to be the image of \( \Box_n \) under the linear map \( \mathbb{R}^n \to \mathbb{R}^{n+2} = M_{\mathbb{R}} \) defined by the left multiplication of a matrix
\[
\begin{pmatrix}
-1 & -1 & \cdots & -1 & -1 \\
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0 \\
1 & 1 & \cdots & 1 & 1
\end{pmatrix}.
\]

It is straightforward to see that \( P[n] \) is a lattice polytope whose vertices are generated by the vectors corresponding to the monomials that appear as the entries of the monomial map \( \text{pr}_2 \circ \Phi[n] : T \to \mathbb{P}^{2^n-1} \). It is also easy to check that the polyhedron \( \tilde{P}[n] \) is very ample, thus it satisfies Assumption 1.2.1.

2.2. Self-product \( W[n] \). For later use, we calculate the polyhedron \( \tilde{W}[n] \) corresponding to the \( n \)-fold self-product of \( X[n] \) over the base \( \mathbb{A}^{n+1} \),
\[
W[n] = (X[n] / \mathbb{A}^{n+1})^n = X[n] \times \mathbb{A}^{n+1} \cdots \times \mathbb{A}^{n+1} X[n].
\]
It is easy to see that \( W[n] \) is the closure of the image of a map
\[
\Phi_W[n] : T_W[n] := (\mathbb{C}^*)^{2n+1} \rightarrow (\mathbb{A}^2)^n \times \mathbb{A}^{n+1} \times (\mathbb{P}^1)^n
\]
defined by
\[
\begin{align*}
\Phi_W[n] & : T_W[n] \rightarrow (\mathbb{P}^1)^n, \\
\Phi_W[n] & \mapsto \left( \frac{t_1 \cdots t_{n+1}}{s_1}, \ldots, \frac{t_1 \cdots t_{n+1}}{s_n}, (t_1, \ldots, t_{n+1}) \right), \\
& \mapsto \left( \frac{t_2 \cdots t_{n+1}}{s_1}, \ldots, \frac{t_2 \cdots t_{n+1}}{s_n}, \frac{t_{n+1}}{s_1}, \ldots, \frac{t_{n+1}}{s_n} \right).
\end{align*}
\]

Let \( M_W[n] \) be the character lattice of \( T_W[n] \). The recession cone \( \sigma_W[n] \) of \( \tilde{P}_W[n] \) is a rational polyhedral cone on \( M_W[n] \) generated by the column vectors of a \((2n+1, 3n+1)\) matrix
\[
\sigma_W[n] = \begin{pmatrix}
-I_n & I_n & O \\
1 & \cdots & 1 \\
\vdots & \vdots & \vdots \\
1 & 1 & 1
\end{pmatrix},
\]
while the polytopal part \( P_W[n] \) is the image of the hypercube \( \square_{n^2} \subset \mathbb{R}^{n^2} \) by a linear map defined by the matrix
\[
L[n] = \begin{pmatrix}
-I_n & -I_n & \cdots & -I_n \\
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix}
\]
of size \((2n+1, n^2)\). As \( \tilde{P}[n] \) satisfies Assumption \([\text{1.2.1}]\), \( \tilde{P}_W[n] \) also satisfies the assumption.

3. Relative Symmetric Product of an Expanded Degeneration and Its Quotient

3.1. Small resolution \( Z^{(n)'} \) of \((X/C)^n\). Next we review the construction of a small crepant resolution \( \tilde{Z}^{(n)'} \) of the relative \( n \)-fold self-product \((X/C)^n\) of the family \( X \to C \) in \([\text{Nag16}]\). For details, we refer \textit{op. cit.}, §§1 and 2.
Let $\tilde{X}^{(n)} = (X/C)^n = X \times_C \cdots \times_C X$ be the $n$-fold self-product of $X$ over $C$. It is an affine toric variety defined by the equations

$$z_{11}z_{12} = z_{21}z_{22} = \cdots = z_{n1}z_{n2}$$

in $\mathbb{A}^{2n}$ with coordinates $(z_{11}, z_{12}, \ldots, z_{n1}, z_{n2})$. The symmetric group $\mathfrak{S}_n$ acts on $X^{(n)'}$ by the permutation of the first index $i$ of $z_{ij}$.

Let $N = \mathbb{Z}^{n-1}$ and $e_i \in N$ a vector whose $i$-th entry is one and all the other entries are 0. For a nonempty subset $I \subset \{1, 2, \ldots, n\}$, we define $e_I = \sum_{i \in I} e_i$ and call such vectors primitive positive weight vectors. We define primitive negative weight vectors as the negation of positive vectors. The positive vectors and negative vectors span a full dimensional smooth fan $\tilde{\Delta}^n$ in $N_\mathbb{R}$, which is isomorphic to the Coxeter complex of $A_{n-1}$-root system. The simple reflections acts on $N$ by

$$\begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & I_{n-k-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

One can easily check that the cone $\tilde{\delta}^{(n)}$ generated by the column vectors of

$$\tilde{\delta}^{(n)} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and its $\mathfrak{S}_n$-translates are exactly the maximal cones of the fan $\tilde{\Delta}^{(n)}$. We have the corresponding projective toric variety $X(\tilde{\Delta}^{(n)})$. We denote the variety by $X(A_{n-1})$ for simplicity.

Let $N = \mathbb{Z} \oplus \overline{N} \oplus \mathbb{Z} = \mathbb{Z}^{n+1}$ and consider an $\mathfrak{S}_n$-action defined by

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & I_{n-k-1} \end{pmatrix} \quad \text{for } k = 1, \ldots, n-2 \text{ and}$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix} \quad (3.1)$$

and

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$


Then, the projection $N \to \overline{N}$ is $\mathbb{S}_n$-equivariant. Let $\delta^{(n)}$ be the maximal cone in $N_\mathbb{R}$ spanned by the column vectors of

$$
\delta^{(n)} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 0 \\
0 & 1 & 1 & \ldots & 1 & 0 \\
0 & 0 & 1 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
$$

and $\Delta^{(n)}$ be the fan consisting of faces of maximal cones $s\delta^n$ ($s \in \mathbb{S}_n$). Then, it is easy to see that the toric variety $\tilde{X}^{(n)r} = X(\Delta^{(n)})$ is the total space of $\mathbb{C}^2$-bundle $\mathcal{O}_{X(\mathbb{A}^{n-1})}(-D_{pos}) \oplus \mathcal{O}_{X(\mathbb{A}^{n-1})}(-D_{neg})$ over $X(\mathbb{A}^{n-1})$, where $D_{pos}$ is the sum of torus invariant divisors corresponding to positive vectors and $D_{neg}$ is defined similarly;

$$
D_{pos} = \sum \delta_{e_1}, \quad D_{neg} = \sum \delta_{-e_1}.
$$

Let $\sigma^{(n)}$ be the union of all the maximal cones in $\Delta^{(n)}$. It is easy to see that no ray in $\Delta^{(n)}$ is in the relative interior of $\sigma^{(n)}$, and that $\sigma^{(n)}$ is generated by the column vectors of

$$(3.2) \quad \sigma^{(n)} = \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0
\end{pmatrix}
$$

This implies that the associated projective birational toric morphism $X(\Delta^{(n)}) \to X(\sigma^{(n)})$ is small i.e., its exceptional set has no divisorial component.

**Proposition 3.1.1** ([Nag16], Proposition 1.4, Proposition 2.5). The affine toric variety $X(\sigma^{(n)})$ is the relative n-fold product $\tilde{X}^{(n)r}$ of $X$ over $\mathbb{C}$. Therefore, $\tilde{X}^{(n)r} = X(\Delta^{(n)})$ is an $\mathbb{S}_n$-equivariant small projective resolution of $\tilde{X}^{(n)r}$.

We can also describe the toric variety $X(\Delta^{(n)})$ in terms of polyhedron. It is well-known that the Coxeter complex $\widetilde{\Delta}^{(n)}$ of $\mathbb{A}^{n-1}$-root system is a normal fan to the n-th permutahedron $P^{(n)}$. One of a realization of $P^{(n)}$ is as follows; define $P^{(n)}$ as the convex hull of the vertex set

$$
\left\{ v_s = \begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(n-1) \end{pmatrix} \right| s \in \mathbb{S}_n \right\} \subset \mathbb{R}^{n-1}.
$$

This is clearly a lattice polytope on $\overline{M} = \mathbb{Z}^{n-1}$, the dual of $\overline{N}$. The normal Fan to $P^{(n)}$ agrees with our $\widetilde{\Delta}^{(n)}$. Actually, the vertices adjacent to $v_e = (0, \ldots, 0)^T$ is given
by $v_s$ for all simple transpositions $s = (1\ 2), (2\ 3), \ldots, (n-1\ n)$, namely the column vectors of

$$B_e = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 \\
-1 & 1 & \cdots & \cdots & 0 & 0 \\
0 & -1 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \ddots & 1 & 0 \\
0 & 0 & \cdots & \cdots & -1 & 1
\end{pmatrix},$$

(3.3)

The dual cone to the cone $B_e$ spanned by the column vectors of the matrix is the normal cone to $P(n)$ at $v_e$, which agrees with the positive Weyl chamber $\delta(n)$. Moreover, one can easily check that the normal cone at $v_{s-1}$ ($s \in S_n$) is $s\delta(n)$.

Now we consider $M = \mathbb{Z} \oplus \mathbb{M} \oplus \mathbb{Z}$, the dual of $N = \mathbb{Z} \oplus \mathbb{N} \oplus \mathbb{Z}$ and let $tP(n)$ be the image of $P(n)$ under the natural injection $\iota: \mathbb{M} \to M$. We define a polyhedron

$$\tilde{B}(n) = \sigma(n)^\vee + tP(n).$$

**Proposition 3.1.2.** The toric variety $X(\tilde{B}(n))$ associated to the polyhedron $\tilde{B}(n)$ is isomorphic to $X(\Delta(n))$.

**Proof.** Note that the set of vertices of $\tilde{B}(n)$ is the same as the set of vertices of $tP(n)$,

$$\{ \tilde{v}_s = \begin{pmatrix} 0 \\ v_s \\ 0 \end{pmatrix} \in \mathbb{R} \oplus \mathbb{M} \oplus \mathbb{R} \mid s \in S_n \}. $$

The cone $\sigma(n)^\vee$ is generated by the column vectors of

$$\sigma(n)^\vee = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1
\end{pmatrix},$$

by *op. cit.*, §1.6 (note that we are working on a basis modified by $Q$ in *op. cit.*, Proof of Proposition 2.5). One can easily see that a cone $\sigma(n)^\vee + tB_e$ is generated by the column vectors of

$$\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
-1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\
0 & -1 & \ddots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \ddots & 1 & 0 & 0 \\
0 & 0 & \cdots & \cdots & -1 & 1 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & 1
\end{pmatrix},$$

(3.4)
Therefore, the normal cone to $\tilde{P}(n)$ at $\tilde{v}_e$, which is the dual cone to $\sigma^{(n)}/ + tB_e$, agrees with $\check{\delta}^{(n)}$. At the vertex $\tilde{v}_{s^{-1}} (s \in G_n)$, the normal cone is the dual cone to $\sigma^{(n)}/ + t(s^{-1}B_e)$ as $\sigma^{(n)}/ is invariant under the action of $G_n$. However, as we know that the dual cone of $s^{-1}B_e$ is nothing but $s\check{\delta}^{(n)}$, the normal cone to $\tilde{P}(n)$ at $\tilde{v}_{s^{-1}}$ must be the same as $s\check{\delta}^{(n)}$. This implies that the normal fan of $\tilde{P}(n)$ is exactly the fan $\Delta^{(n)}$.

Q.E.D.

3.2. GHH linearization. Let us go back to the self-product $W[n]$ of the expanded degeneration $X[n]$. We keep the notation in [2.2] Gulbrandsen, Halle, and Hulek introduced in [GHH16] a specific fractional linearization on the expanded degeneration $X[n]$ with respect to the embedding

$$\Phi[n] : X[n] \to (\mathbb{A}^2 \times \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n \times \mathbb{A}^{n+1} \times \mathbb{P}^{2n-1}.$$

Here we describe the fractional linearization in the framework of [1.3].

Let us consider the torus $(\mathbb{C}^*)^{n+1}$ of the base space of the $n$-th expanded degeneration $X[n] \to \mathbb{A}^{n+1}$ with coordinate $(t_1, \ldots, t_{n+1})$. We define

$$G[n] := \{(t_1, \ldots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid t_1 \ldots t_{n+1} = 1\}.$$

We note that we can naturally regard $G[n]$ as a sub-torus of $T[n]$ or $T_W[n]$ by our consistent choice of coordinate. We also note that $G[n]$ has a natural action of $G[n]$ on $(\mathbb{P}^1)^n$ through the map $\Phi'[n]$. Following [GHH16], we introduce another coordinate $(\tau_1, \ldots, \tau_n)$ of $G[n]$ by

$$\tau_i = \prod_{j=1}^{i} t_j \quad (i = 1, \ldots, n).$$

In other words, we have $t_1 = \tau_1$ and $t_i = \tau_i/\tau_{i-1}$ for $i = 2, \ldots, n+1$. Now we let $G[n]$ act on $(\mathbb{A}^2)^n$ by

$$(\ldots, (u_i, v_i), \ldots) \mapsto (\ldots, (\tau_i^{l_{i+1}}u_i, \tau_i^{l_i+1}v_i), \ldots).$$

Putting

$$[u_i : v_i] = \left[ 1 : \frac{l_{i+1} \cdots l_{n+1}}{s} \right],$$

we see that this is a lifting of the $G[n]$ action on $(\mathbb{P}^1)^n$ induced by $\Phi'[n]$. This determines a linearization on $L^{\otimes n+1}$ where $L$ is a pull-back to $X[n]$ of $\mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ under $\Phi[n]$, hence we get a fractional linearization on $L$, which we call GHH fractional linearization. As we saw in [1.3], we have a corresponding affine map

$$l_{GHH} : M[n] \to M_G[n],$$

where we will always take a dual basis on $M_G[n]$ to the coordinate $(\tau_1, \ldots, \tau_n)$. The origin of $M[n]$ is a vertex of the polytope $P[n]$ that corresponds to the monomial
$u_1 \ldots u_n$. As $\tau_i$ acts on the monomial via a character $\tau_i \mapsto \tau_i^{\frac{n}{n+1}}$, we know that

$$b_{GHH} = l_{GHH}(0) = \left( \begin{array}{c} \frac{n}{n+1} \\ \vdots \\ \frac{n}{n+1} \end{array} \right).$$

The GHH fractional linearization induces a fractional linearization of the self-product $W[n]$ with respect to the embedding

$$\Phi_W[n] : W[n] \ni \phi_W[n] \mapsto ((\mathbb{A}^2)^n \times \mathbb{A}^{n+1}) \times \mathbb{P}^{n+1} \ni \sigma_{GHH} \mapsto ((\mathbb{A}^2)^n \times \mathbb{A}^{n+1}) \times \mathbb{P}^{2n-1}.$$

The origin of $M_W[n]$ is a vertex of the polytope $P_W[n]$ corresponding to a monomial $(u_1 \ldots u_n)^n$, and therefore the induced fractional linearization is given by

$$b_{GHH}^{W[n]} = \left( \begin{array}{c} \frac{2n}{n+1} \\ \frac{n}{n+1} \\ \vdots \\ \frac{n}{n+1} \end{array} \right).$$

For later use, we note that the linear part $\alpha_W[n] : M_W[n] \to M_G[n]$ of the GHH linearization

$$l_{GHH}^W[n] = \alpha_W[n] + b_{GHH}^{W[n]} : M_W[n]_\mathbb{Q} \to M_G[n]_\mathbb{Q}$$

is given by the $(n, 2n + 1)$ matrix

$$\alpha_W[n] = \left( \begin{array}{cccccc} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{array} \right)_{(n,n)}$$

where $O_n$ is the zero matrix of size $(n,n)$.

### 3.3. GIT quotient of $W[n]$.

Now we are prepared to prove the following

**Theorem 3.3.1.** Notation as above. The GIT quotient $W[n]^{ss} // G[n]$ with respect to the GHH fractional linearization is isomorphic to $\tilde{Z}^{(n)}$.

According to Proposition 1.3.1, the polyhedron

$$P_b[n] := P_W[n] \cap (\alpha_W[n] \otimes \mathbb{R})^{-1}(-b)$$

with $b = b_{GHH}^W[n]$ determines the quotient $W[n]^{ss} // G[n]$. First we calculate the recession cone $\sigma'[n] \vee$ of $\tilde{P}_b[n]$. We have a short exact sequence of lattices

$$0 \to M'[n] \to M_W[n] \to M_G[n] \to 0.$$

By dualizing the sequence, we get a surjective map $\pi : N_W[n] \to N'[n] \cong \mathbb{Z}^{n+1}$. Then by [Hu02], Lemma 10.1, the dual to the recession cone $\sigma'[n]$ is just the image
\( \pi(\sigma_W[n]) \). By making an appropriate choice of basis for \( N'[n] \), \( \pi \) is given by the matrix

\[
\pi = \begin{pmatrix}
0 & 0 & \cdots & 0 & -1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 0 & -1 & 0 & 1 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & \cdots & 0 & 1 & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}_{O_{n,n+1}}.
\]

Lemma 3.3.2. (1) The dual cone \( \sigma_W[n] \) of the recession cone of \( \tilde{P}_W[n] \) is a rational polyhedral cone generated by the vectors

\[
v = \begin{pmatrix}
a_1 \\
\vdots \\
a_n \\
\hline
b_1 \\
\vdots \\
b_{n+1}
\end{pmatrix}
\]

such that (i) \( a_i \) is either 0 or 1 for \( i = 1, \ldots, n \), and (ii) exactly one among \( b_j \)'s is 1 and others are all 0.

(2) The image \( \sigma'[n] = \pi(\sigma_W[n]) \) coincides with \( \sigma^{(n)} \).

Proof. We have a list (2.2) of generators for \( \sigma_W[n]^\vee \). The two blocks on the right implies that if \( v \in \sigma_W[n] \), all \( a_i \) and \( b_j \) are non-negative. The condition from leftmost block is

\[
a_i \leq \sum_{j=1}^{n+1} b_j \quad (i = 1, \ldots, n).
\]

Thus, \( \sigma_W[n] \) is a family of hypercubes in \( (a_1, \ldots, a_n) \) of size \( \sum b_j \) over the positive orthant in \( (b_1, \ldots, b_{n+1}) \). This proves (1). Let \( v \in \sigma_W[n] \) be one of the generators listed in (1). Then, we have

\[
\pi(v) = \begin{pmatrix}
-a_n + 1 \\
1 & -a_n \\
\vdots \\
\vdots \\
(-a_n - 1) & -a_n \\
a_n
\end{pmatrix}
\]

If \( a_n = 0 \), \( \pi(v) \) is of the form

\[
\begin{pmatrix}
1 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
u \\
0
\end{pmatrix}
\]
where \( u \) is either a positive primitive weight vector \( e_l \) or zero. If \( a_n = 1 \),
\[
\pi(v) = \begin{pmatrix}
0 \\
\vdots \\
u \\
-1
\end{pmatrix}
\]
where \( u \) is either a negative primitive weight vector \(-e_l \) or zero. Comparing with (3.2), we immediately conclude that \( \pi(\sigma_W[n]) = \sigma^{(n)} \). Q.E.D.

Next, let us calculate the polytopal part \( P_b[n] \) of \( \tilde{P}_b[n] \), which is given by
\[
P_b[n] = P_W[n] \cap (\alpha_W[n] \otimes \mathbb{R})^{-1}(-b),
\]
for \( b = b_{GHH}^W = (\frac{n}{n+1}, \frac{2n}{n+1}, \ldots, \frac{n^2}{n+1}) \). Recalling that \( P_W[n] \) is the image of the hypercube \( \square_{n^2} \) under the linear map \( L[n] \), first we look at
\[
\square_{n^2} \cap (\alpha_W[n]L[n])^{-1}(-b).
\]
We represent a vector in \( \mathbb{R}^{n^2} \) by a transpose of
\[
\begin{pmatrix}
c_{11} & \cdots & c_{1n} \\
c_{21} & \cdots & c_{2n} \\
\vdots & \ddots & \vdots \\
c_{n1} & \cdots & c_{nn}
\end{pmatrix}
\]
As one can easily check that the composition \( \alpha_W[n]L[n] \) is given by \( (n,n^2) \)-matrix
\[
\alpha_W[n]L[n] = \begin{pmatrix}
-1 & \cdots & -1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -1 & \cdots & -1 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & \cdots & -1
\end{pmatrix},
\]
the subspace \( (\alpha_W[n]L[n])^{-1}(-b) \) is cut out by \( n \) hyperplanes
\[
H_i : c_{i1} + c_{i2} + \cdots + c_{in} = \frac{in}{n+1} (i = 1, \ldots, n).
\]
If we define a polytope \( R_i[n] \) in the subspace with coordinates \( (c_{i1}, c_{i2}, \ldots, c_{in}) \) as the intersection \( \square_{n} \cap H_i \), we get a decomposition
\[
\square_{n^2} \cap (\alpha_W[n]L[n])^{-1}(-b) = R_1[n] \times \cdots \times R_{n}[n].
\]
As \( R_i[n] \) is a hyperplane cut of a hypercube \( \square_{n} \), a vertex of \( R_i[n] \) is on an edge of \( \square_{n} \), that is, a vertex is of the form \( (c_{i1}, \ldots, c_{in}) \) with \( c_{ij} = 0 \) or 1 for all \( j \) but a unique \( k \) and \( 0 \leq c_{ik} \leq 1 \). Combining with the equation for \( H_i \), one sees that the vertex set of \( R_i[n] \) is given by
\[
w_i = t(1, \ldots, 1, \underbrace{n-i+1}_{(i-1)}, 0, \ldots, 0)
\]
and its permutation of components \( s w_i (s \in \mathfrak{S}_n) \). This implies that the polytope \( P_b[n] \), the image of \( \square_{n^2} \cap (\alpha_W[n]L[n])^{-1}(-b) \) under \( L[n] \), is the convex hull of vectors
\[
\left\{ L[n] \begin{pmatrix} s_1 w_1 \\ \vdots \\ s_n w_n \end{pmatrix} \middle| s_1, \ldots, s_n \in \mathfrak{S}_n \right\}.
\]
As we have

\[
L[n] = \begin{pmatrix}
    c_{11} \\
    \vdots \\
    c_{1n} \\
    \vdots \\
    c_{n1} \\
    \vdots \\
    c_{nn}
\end{pmatrix} = \begin{pmatrix}
    -\sum_{i=1}^{n} c_{i1} \\
    \vdots \\
    -\sum_{i=1}^{n} c_{in} \\
    0 \\
    \sum_{j=1}^{n} c_{1j} \\
    \sum_{j=1}^{n} c_{2j} \\
    \vdots \\
    \sum_{j=1}^{n} c_{nj}
\end{pmatrix},
\]

the last \((n + 1)\) entries of \(L[n](s_{1}w_{1}, \ldots, s_{n}w_{n})\) is always \(t\left(0, \frac{n}{n+1}, \frac{3n}{n+1}, \ldots, \frac{n^2(n+1)}{n+1}\right)\) independent of permutations \(s_{1}, \ldots, s_{n} \in S_{n}\). As the kernel of \(\alpha_{W}[n]\) is a free \(\mathbb{Z}\)-module with a basis

\[
\begin{pmatrix}
    1 \\
    \vdots \\
    0 \\
    \vdots \\
    0 \\
    \vdots \\
    0
\end{pmatrix}, \quad \begin{pmatrix}
    0 \\
    \vdots \\
    1 \\
    \vdots \\
    0 \\
    \vdots \\
    1
\end{pmatrix}, \quad \begin{pmatrix}
    0 \\
    \vdots \\
    0 \\
    \vdots \\
    0 \\
    \vdots \\
    1
\end{pmatrix},
\]

it is sufficient to look at the image of \(P_{b}[n]\) under a projection to first \(n\) components. The first \(n\) components of \(L[n](s_{1}w_{1}, \ldots, s_{n}w_{n})\) is given by

\[
u = t\left(-\frac{n^2+n-1}{n+1}, -\frac{n^2-3}{n+1}, \ldots, -\frac{n+3}{n+1}, -\frac{1}{n+1}\right).
\]

Here we note that the difference of every two consecutive numbers is \(1 + \frac{1}{n+1}\).

**Lemma 3.3.3.** The projection \(\overline{P}_{b}[n]\) of \(P_{b}[n]\) to the first \(n\) components is the convex hull of the set \(\{su \mid s \in S_{n}\}\). In particular, \(P_{b}[n]\) agrees with the permutahedron \(P^{(n)}\) up to a multiplication of rational scalar and a translation.

**Proof.** It is sufficient to show that \(L[n](s_{1}w_{1}, \ldots, s_{n}w_{n})\) is in the convex hull of \(\{su \mid s \in S_{n}\}\) for every \((s_{1}, \ldots, s_{n}) \in (S_{n})^{n}\). By symmetry under the diagonal action of \(S_{n}\), this is equivalent to say that \(L[n](s_{1}w_{1}, \ldots, s_{n}w_{n})\) is in a cone \(C\) spanned by \(\{(ii+1)u \mid i =
\)
1, \ldots, n-1} with the vertex \( u \). One can easily check that the cone \( C \) is defined by

\[
a_1 + \cdots + a_k \geq -\frac{k^2(n+2) - k(2n^2 + 3n)}{2(n+1)} \quad \text{for } k = 1, \ldots, n-1, \text{ and}
\]

\[
a_1 + \cdots + a_n = -\frac{n^2}{2},
\]

where \( a_1, \ldots, a_n \) are the first \( n \) coordinates of \( MW[n]_R \) as in Lemma 3.3.2. As \( w_i \) satisfies

\[
c_i1 \geq c_i2 \geq \cdots \geq c_in,
\]

for all \( i \), the sum \( a_1 + \cdots + a_k = -\sum_{i=1}^{n} \sum_{j=1}^{k} c_{ij} \) only increases under the action of

\[
(s_1, \ldots, s_n) \in (\mathcal{S}_n)^n \text{ on } \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.
\]

Q.E.D.

Now we finish the proof of Theorem 3.3.1. On one hand \( \tilde{Z}(n)' \) is the toric variety corresponding to a polyhedron \( \tilde{P}(n) = \sigma(n)'^{+} + \iota P(n) \) by Proposition 3.1.2. On the other hand, the quotient \( W[n]^{ss}/G[n] \) is also a toric variety determined by the polyhedron \( \tilde{P}_b[n] \). By Lemma 3.3.2 the conical part of \( \sigma'[n] \) coincides with \( \sigma(n)' \) under the basis of \( M'[n] \) consisting of column vectors of the transpose \( t\pi \) of \( \pi \) in (3.5). Let us denote by \( Q' \) the matrix of basis change such that

\[
\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = Q' \left( \begin{pmatrix} P_{b}[n] - u \end{pmatrix} \right).
\]

Then, we have

\[
\iota P(n) = \frac{n+1}{n+2} : Q'(P_{b}[n] - u).
\]

This implies that \( \tilde{P}_b[n] \) agrees with the polyhedron \( \tilde{P}(n) \) after a multiplication of rational scalar \( \frac{n+1}{n+2} \) and a translation. Therefore, after taking sufficiently high truncation of the graded rings, we get an isomorphism \( Z(n)' = X(\tilde{P}(n)) \cong X(\tilde{P}_b[n]) = W[n]^{ss}/G[n] \).

Remark 3.3.4. In the same way, one can easily verify that \( X[n]/G[n] \) is isomorphic to the original family \( X \). In this case, the polytopal part \( P_b \) of the quotient becomes just one point so that the polyhedron \( \tilde{P}_b \) is just a cone that corresponds to the invariant ring of \( X \times A^1 A_{n+1} \).
4. HILBERT-CHOW MORPHISM FOR GHH DEGENERATION

4.1. GHH degeneration of Hilbert schemes. Let us define $S = X \times \mathbb{A}^{m-1}$ and endow it with a morphism $S \to C = \mathbb{A}^1$ given by the composition
$$S \xrightarrow{pr_1} X \longrightarrow C = \mathbb{A}^1.$$ 
This is a local model for a simple degeneration in the sense of [GHH16], Definition 1.1. The expanded degeneration of the family $S \to C$ is just given by the composition
$$S[n] := X[n] \times \mathbb{A}^{m-1} \to X[n] \to \mathbb{A}^{n+1}.$$ 
Here we remark that the torus $G[n]$ acts trivially on the factor $\mathbb{A}^{m-1}$ in $S$ or $S[n]$. Gulbrandsen, Halle, and Hulek considered in op. cit. the relative Hilbert scheme
$$\text{Hilb}^n(S[n]/\mathbb{A}^{n+1}) \to \mathbb{A}^{n+1}$$ 
of the expanded degeneration $S[n] \to \mathbb{A}^{n+1}$. Since the Hilbert scheme admits a natural action of $G[n]$, they define
$$I^n_{S/C} = \text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss} // G[n],$$
where the GIT stability and the GIT quotient are considered under the GHH linearization as in [3.2]. $I^n_{S/C}$ has a natural morphism
$$I^n_{S/C} \to \mathbb{A}^{n+1} // G[n] \cong \mathbb{A}^1,$$
whose general fiber over $t \in \mathbb{A}^1$ is isomorphic to $\text{Hilb}^n(S_t)$, where $S_t$ is the fiber over $t$ of the original semistable family $S \to \mathbb{A}^1$. Let us call the family $I^n_{S/C} \to \mathbb{A}^1$ Gulbrandsen-Halle-Hulek degeneration, or GHH degeneration of Hilbert schemes associated with the family $S \to C$.

This construction is most interesting in the case where $m = 2$, namely in the case where $S \to C$ is a semistable family of surfaces whose singular fiber has no triple point. As $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1}) \to \text{Sym}^n(S[n]/\mathbb{A}^{n+1})$ is $G[n]$-equivariant, we get a projective birational morphism
$$\Psi : I^n_{S/C} \to \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} // G[n].$$
We call the morphism $\Psi$ Hilbert-Chow morphism of GHH degeneration.

4.2. A small partial resolution $Z^{(n)}$ of $\text{Sym}^n(S/C)$. To analyze a degeneration of Hilbert schemes, we have another approach, namely we can also start from the symmetric product $\text{Sym}^n(S/C)$ of the family $S \to C$. As $S = X \times \mathbb{A}^1$, it is just an $\mathcal{O}_n$-quotient of $(S/C)^n = \tilde{X}^{(n)'} \times \mathbb{A}^n$. The projective toric small resolution $\tilde{Z}^{(n)'} \to \tilde{X}^{(n)'}$ as in Proposition 3.1.1 immediately gives a toric small resolution
$$\tilde{Z}^{(n)} = \tilde{Z}^{(n)'} \times \mathbb{A}^n \to \tilde{X}^{(n)'} \times \mathbb{A}^n = (S/C)^n.$$
We note that the resolution is $\mathfrak{S}_n$-equivariant. By taking $\mathfrak{S}_n$-quotient of both sides, we get a projective small resolution

$$Z^{(n)} := \tilde{Z}^{(n)}/\mathfrak{S}_n \rightarrow \text{Sym}^n(S/C).$$

The self-product $(S[n]/\mathbb{A}^{n+1})^n$ of the expanded degeneration $S[n] = X[n] \times \mathbb{A}^1 \rightarrow \mathbb{A}^{n+1}$ is just

$$(S[n]/\mathbb{A}^{n+1})^n = W[n] \times \mathbb{A}^n \xrightarrow{\text{pr}} W[n] \rightarrow \mathbb{A}^{n+1}.$$ Again the torus $G[n]$ acts trivially on the factor $\mathbb{A}^n$, and Theorem 3.3.1 gives an isomorphism

$$\tilde{\varepsilon}^{(n)} : (W[n] \times \mathbb{A}^n)^{ss}/G[n] \sim \tilde{Z}^{(n)} = Z^{(n)\prime} \times \mathbb{A}^n.$$ Moreover the $G[n]$-action on $W[n] \times \mathbb{A}^n$ commutes with the natural $\mathfrak{S}^n$-action, $\tilde{\varepsilon}^{(n)}$ descends to an isomorphism

$$\varepsilon^{(n)} : \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n] \rightarrow Z^{(n)},$$ and thus we have a natural birational morphism

$$\psi^{(n),GHH} = \varepsilon^{(n)} \circ \Psi : I_{S/C}^n \rightarrow Z^{(n)}.$$ Here we note that there is a commutative diagram

$$\begin{array}{ccc}
(W[n] \times \mathbb{A}^n)^{ss} & \xrightarrow{\text{Sym}^n} & \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} \\
\downarrow & & \downarrow \\
\tilde{Z}^{(n)} & \xrightarrow{\sim} & Z^{(n)}
\end{array}$$

### 4.3. A crepant resolution of $Z^{(n)}$

By the construction, the general fiber of $Z^{(n)} \rightarrow C = \mathbb{A}^1$ is just the symmetric product of the general fiber $\text{Sym}^n(S_t)$. More precisely, as the restriction of the family $S \rightarrow C$ to $C^0 = C \setminus \{0\}$ is a trivial family of $\mathbb{C}^* \times \mathbb{A}^1$, the restriction

$$Z^{(n)\circ} := Z^{(n)} \times_C C^0 \rightarrow C^0$$

is a trivial family of $\text{Sym}^n(\mathbb{C}^* \times \mathbb{A}^1)$. Therefore, the ordinary Hilbert-Chow morphism gives a crepant divisorial resolution

$$\psi^{(n)\circ} : Y^{(n)\circ} \rightarrow Z^{(n)\circ}.$$ In [Nag16], Theorem 4.1, we constructed an extension of $\psi^{(n)\circ}$ to a projective crepant divisorial birational morphism

$$\psi^{(n)} : Y^{(n)} \rightarrow Z^{(n)}.$$ For $Z \in \text{Hilb}^n(S[n]/\mathbb{A}^{n+1})$, we define $t_i(Z)$ to be the $i$-th component of the image of $Z$ in $\mathbb{A}^{n+1}$. Since $Z \in \text{Hilb}^n(S[n]/\mathbb{A}^{n+1})$ has trivial stabilizer under the action of $G[n]$ if $t_i(Z) \neq 0$ for all $i$, we see that the restriction of $\psi^{(n),GHH}$ agrees with the
trivial family of Hilbert-Chow morphisms $\psi^{(n)}$. In the rest of the article, we prove the following

**Theorem 4.3.1.** The Hilbert-Chow morphism of GHH degeneration $\Psi$ (or $\psi^{(n)}_{GHH}$) is isomorphic to the projective crepant divisorial partial resolution $\psi^{(n)}$.

### 4.4. Orbifold structures.

The key to prove the theorem is natural orbifold structures on $I^n_S/C$ and $Y^{(n)}$. The orbifold structure on $I^n_S/C$ is explained in [GHH16], §3: by the numerical criterion of stability ([GHH16], Theorem 2.9), a semistable point in $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})$ is automatically stable under the GHH linearization, and therefore the stabilizer subgroup of $G[n]$ at a point is finite. Since the semistable locus $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss}$ is contained in the smooth locus of $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1}) \rightarrow \mathbb{A}^{n+1}$ ([GHH16], Lemma 3.6 and 3.7), Luna’s étale slice theorem implies that the quotient stack

$$\mathcal{I}^n_{S/C} = [\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n]]$$

is a smooth Deligne-Mumford stack. The canonical morphism to the coarse moduli scheme $\mathcal{I}^n_{S/C} \rightarrow I^n_{S/C}$ gives the orbifold structure on $I^n_{S/C}$. Similarly, the GIT quotient $\text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n]$ has a natural covering structure coming from the corresponding quotient stack $[\text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n]]$. Again by the numerical criterion of stability (loc. cit., see also the following remark), one sees that the stack is also (non-smooth) Deligne-Mumford stack.

**Remark 4.4.1.** Although the numerical criterion of stability [GHH16], Theorem 2.9 is stated only for a point in the Hilbert scheme $\text{Hilb}^n(S[n]/\mathbb{A}^{n+1})$, one can easily verify that the proof works in exactly the same way for the case of the symmetric product $\text{Sym}^n(S[n]/\mathbb{A}^{n+1})$ and a self-fiber product $(S[n]/\mathbb{A}^{n+1})^n$. We will repeatedly use this fact.

On the other hand, $Y^{(n)}$ also has a natural orbifold structure. We recall the construction of $Y^{(n)}$ in some detail (we refer [Nag16], in particular §2.7, Lemma 2.9, Lemma 4.5, and §4.6, for full detail). Let us take a point $q \in Z^{(n)}$ and $\tilde{q} = (\tilde{q}_1, \tilde{q}_2) \in \tilde{Z}^{(n)} = \tilde{Z}^{(n)} \times \mathbb{A}^n$, a point above $q$. We have a sequence of $\mathbb{G}_n$-equivariant projections

$$Z^{(n)'} = X(\Delta^{(n)}) \xrightarrow{\text{\tilde{\Delta}}\text{-bundle}} X(A_{n-1}) \xrightarrow{\text{birational}} \mathbb{P}^{n-1}.$$  

If we denote the homogeneous coordinate on $\mathbb{P}^{n-1}$ by $[\xi_1 : \cdots : \xi_n]$ and take the coordinate $\left(\frac{\xi_1}{\xi_n}, \cdots, \frac{\xi_{n-1}}{\xi_n}\right)$ for the torus $(\mathbb{C}^*)^{n-1} \subset \mathbb{P}^{n-1}$, an toric affine open neighborhood $X(\Delta^{(n)})$ of $X(A_{n-1})$ is $\mathbb{A}^{n-1}$ with the coordinates

$$\left(\frac{\xi_1}{\xi_n}, \frac{\xi_2}{\xi_n}, \cdots, \frac{\xi_{n-1}}{\xi_n}\right).$$
by (3.3), and therefore the toric coordinate on $x(s \tilde{\delta}^{(n)})$; ($s \in \mathcal{S}_n$) is given
\[
\begin{pmatrix}
\xi_{s(1)} \\
\xi_{s(2)} \\
\vdots \\
\xi_{s(n)}
\end{pmatrix}
\]
for some $s \in \mathcal{S}_n$. Let us decompose $s \in \mathcal{S}_n$ into cycles as
\[
s = (i_1 \ldots i_1)(i_{l_1+1} \ldots i_{l_2}) \ldots (i_{l_{r-1}+1} \ldots i_{l_r}),
\]
If $\tilde{q}_1$ is fixed by $s$, we necessarily have
\[
\frac{\xi_{i_{l_k-1}+1}}{\xi_{i_{l_k-1}+2}} = \cdots = \frac{\xi_{i_{l_k-1}}}{\xi_{i_{l_k}}} = \alpha_k
\]
for $k = 1, \ldots, r$, where $l_0 = 0$ by convention and $\alpha_k$ is an $(l_k - l_{k-1})$-th root of unity. We say that $\tilde{q}$ is an $s$-fixed point of trivial angle type if
\[
\alpha_1 = \cdots = \alpha_r = 1.
\]
One can easily see that $\tilde{q}$ is an $s$-fixed point of trivial angle if and only if
\[
\frac{\xi_i}{\xi_{s(i)}} = 1 \quad \text{for all} \quad i \text{ with } s(i) \neq i
\]
and $\tilde{q}_2 \in \mathcal{A}^n$ is an $s$-fixed point with respect to the standard permutation action. From this characterization, one sees that
\[
(4.2) \quad \text{Stab}_{\mathcal{S}_n}^0(\tilde{q}) = \{ s \in \text{Stab}_{\mathcal{S}_n}(\tilde{q}) \mid \tilde{q} \text{ is an } s \text{-fixed point of trivial angle type} \}
\]
is a Young subgroup of $\mathcal{S}_n$ and is a normal subgroup of $\text{Stab}_{\mathcal{S}_n}(\tilde{q})$ ([Nag16], Lemma 4.5). The tangent space $T_{\tilde{q}}\tilde{Z}^{(n) \prime} \cong \mathbb{C}^{n+1}$ seen as a representation of $\text{Stab}_{\mathcal{S}_n}^0(\tilde{q})$ is a direct sum of the restriction of standard permutation representation and a one-dimensional trivial representation. Therefore, for a sufficiently small neighborhood $\tilde{U}_{\tilde{q}} \subset \tilde{Z}^{(n)}$ of $\tilde{q}$, the quotient $U_{\tilde{q}} = \tilde{U}_{\tilde{q}}/\text{Stab}_{\mathcal{S}_n}(\tilde{q})$ is isomorphic to an open neighborhood of $(\gamma, 0) \in \text{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1$, where $\gamma = \sum \mu_i p_i \in \text{Sym}^n(\mathbb{A}^2)$ if $\text{Stab}_{\mathcal{S}_n}^0(\tilde{q})$ is isomorphic to a Young subgroup $\mathcal{S}_\mu$ associated with a partition $\mu = (\mu_i)$ of $n$. Now restricting the Hilbert-Chow morphism $\text{Hilb}^n(\mathbb{A}^2) \times \mathbb{A}^1 \to \text{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1$ to $U_{\tilde{q}}$, we get a crepant divisorial resolution
\[
\tilde{U}_{\tilde{q}} \to U_{\tilde{q}} = \tilde{U}_{\tilde{q}}/\text{Stab}_{\mathcal{S}_n}^0(\tilde{q}).
\]
As $\text{Stab}_{\mathcal{S}_n}^0(\tilde{q})$ is a normal subgroup of $\text{Stab}_{\mathcal{S}_n}(\tilde{q})$, the quotient group
\[
(4.3) \quad G(\tilde{q}) = \text{Stab}_{\mathcal{S}_n}(\tilde{q})/\text{Stab}_{\mathcal{S}_n}^0(\tilde{q})
\]
acts on $U_{\tilde{q}}$ and moreover the action lifts to $\tilde{U}_{\tilde{q}}$. Nothing that $U_{\tilde{q}}/G(\tilde{q})$ is isomorphic to a neighborhood of the image $q \in Z^{(n)}$ of the point $\tilde{q} \in \tilde{Z}^{(n)}$, we see that the quotients with canonical morphism
\[
\hat{U}_{\tilde{q}}/G(\tilde{q}) \to U_{\tilde{q}}/G(\tilde{q}) \subset Z^{(n)}
\]
patch together along \( Z^{(n)} \) to give a crepant partial resolution
\[
\psi^{(n)} : Y^{(n)} \to Z^{(n)}.
\]
Therefore, the family of quotient stacks \( \{[\hat{U}_q/G(\hat{q})] \to Z^{(n)} \} \) defines a smooth Deligne-Mumford stack \( \mathcal{Y}^{(n)} \) whose coarse moduli space is \( Y^{(n)} \).

4.5. \textbf{Semistable locus} \( W[n]^{ss} \) \textbf{and the quotient map}. In this subsection, we describe explicitly the local behavior of the quotient map \( W[n]^{ss} \to W[n]/\tilde{G}[n] = \tilde{Z}^{(n)'} \).

The cone \( \sigma[n] \) corresponding to the base change \( X(\sigma[n]) = X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \) is generated by the column vectors of \((n+2, 2n)\)-matrix
\[
\sigma[n] = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1
\end{pmatrix}.
\]

We can see this as follows; let \( \sigma_X \) be a cone generated by \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) in \( N_X \mathbb{R} = \mathbb{R}^2 \), and \( \sigma_{\mathbb{A}^{n+1}} \) be the positive orthant in \( N_{\mathbb{A}^{n+1}} \mathbb{R} = \mathbb{R}^{n+1} \). Then, by [Nag16], Lemma 1.5, \( \sigma[n] \) is just a fiber product of cones \( \sigma_X \times_{\mathbb{R}} \sigma_{\mathbb{A}^{n+1}} \) under the maps
\[
(0 1) : N_X = \mathbb{Z}^2 \to \mathbb{Z}, \quad \text{and} \quad (1 1 \cdots 1) : N_{\mathbb{A}^{n+1}} = \mathbb{Z}^{n+1} \to \mathbb{Z}.
\]

Then the cone \( \sigma_W[n] \) corresponding to the self-product
\[
X(\sigma[n]) \times_{\mathbb{A}^{n+1}} \cdots \times_{\mathbb{A}^{n+1}} X(\sigma[n])
\]
is given by the fiber product \( \sigma_W[n] \) of \( n \)-copies of the cone \( \sigma[n] \) with respect to the projection to the last \( n \)-factors, which is generated by the vectors
\[
v_{I, j} = \begin{pmatrix} e_I \\ e_j \end{pmatrix}
\]
for \( e_I = \sum_{i \in I} e_i \in \mathbb{Z}^n \) with a (possibly empty) subset \( I \subset \{1, \ldots, n\} \) and \( e_i \in \mathbb{Z}^n \) \((i = 1, \ldots, n)\) the standard basis, and similarly for \( e_j \in \mathbb{Z}^{n+1} \) \((j = 1, \ldots, n+1)\). As \( W[n] \to (X(\Sigma[n]))/\mathbb{A}^{n+1})^n \) is a toric small birational morphism, the set of rays in the normal fan \( \Sigma_W \) to the polyhedron \( \tilde{P}_W[n] \) coincides with the set of rays generated by \( v_{I, j} \).

Therefore, the set of torus invariant divisors on \( W[n] \) is in one to one correspondence with the set \( \{v_{I, j}\} \). We denote by \( D_{I, j} \) the torus invariant divisor corresponding to \( v_{I, j} \). \( \tilde{P}_W[n] \) is cut out by halfspaces defined by
\[
v_{I, j} \geq d_{I, j}
\]
for some \( d_{I, j} \in \mathbb{Z} \), where \( v_{I, j} \) is seen as a linear functional on \( M_W[n] \). The ample divisor corresponding to the polyhedron \( \tilde{P}_W[n] \) is given by
\[
D_{\tilde{P}_W[n]} = \sum (-d_{I, j})D_{I, j}.
\]
Since the functional $v_{I,j}$ is non-negative on the conical part $\sigma_W[n]$, the constants $d_{I,j}$ is defined by

$$d_{I,j} = \min(\{\langle v_{I,j}, u \rangle \mid u \in P_W[n] \cup \{0\}\}).$$

As the polytopal part $P_W[n]$ is the image of hypercube $\Box_n$ under $L[n]$, we have

$$d_{I,j} = \min(\{\langle t L[n]v_{I,j}, \tilde{u} \rangle \mid \tilde{u} \in \Box_n \cup \{0\}\}).$$

On the other hand, the equality

$$t L[n]v_{I,j} = \begin{pmatrix} e_I^C \\ \vdots \\ -e_I \end{pmatrix}_{j-1}$$

infers that

$$d_{I,j} = -(n - j) \cdot \#(I).$$

The space of sections of $\mathcal{O}(kD_{P_W[n]})$ is isomorphic to the vector space spanned by the monomials $m \in kP_W[m]$. Therefore the complete linear system $|kD_{P_W[n]}|$ consists of divisors of the form

$$\sum(\langle v_{I,j}, m \rangle - kd_{I,j})D_{I,j}.$$

Therefore, the subsystem of $G[n]$-invariant divisors is given by

$$\Lambda_{b,k} = \left\{ \sum(\langle v_{I,j}, m \rangle - kd_{I,j})D_{I,j} \mid m \in kP_b[n] \right\}.$$

**Proposition 4.5.1.** The stable base locus $\bigcap_k \text{Bs}(\Lambda_{b,k})$, namely the locus of unstable points with respect to the GHH-linearization is

$$W[n] \backslash W[n]^{ss} = \bigcup_{\#(I) \neq j} D_{I,j}.$$

**Proof.** It is sufficient to prove

$$\langle v_{I,j}, m \rangle - d_{I,j} \geq 0$$

for every rational point $m \in P_b[n]$ and the equality is attained if and only if $j = \#(I)$. It is equivalent to say that the same condition holds for every vertex $m \in P_b[n]$. By
Lemma 3.3.3, a vertex of $P_b[n]$ is of the form

$$m_s = \begin{pmatrix} su \\ 0 \\ \frac{n}{n+1} \\ \frac{sn}{n+1} \\ \vdots \\ \frac{1/n^2(n+1)}{n+1} \end{pmatrix}$$

for $u = t\left(-\frac{n^2+n-1}{n+1}, -\frac{n^2-3}{n+1}, \ldots, -\frac{n+3}{n+1}, -\frac{1}{n+1}\right)$ and $s \in S_n$, we have

$$\langle v_{I,j}, m_s \rangle - d_{I,j} = \sum_{i \in I} (su)_i + \frac{j(j+1)}{2} \frac{n}{n+1} - \#(I) \cdot (n-j).$$

Its minimum is given by

$$\sum_{i=1}^{\#(I)} \left( \frac{n^2+n-1}{n+1} - \frac{(i-1)(n+2)}{n+1} + \frac{j(j+1)}{2} \frac{n}{n+1} - \#(I) \cdot (n-j) \right) = \frac{1}{2(n+1)} (j - \#(I))(j - \#(I))n + n - 2 \cdot \#(I)).$$

It is elementary exercise to show that this amount is always non-negative and equals to zero if and only if $j = \#(I)$. Q.E.D.

Let $t = (t_1, \ldots, t_{n+1}) \in \mathbb{A}^{n+1}$ and $X[n]_t$ the fiber of the expanded degeneration of $X = \mathbb{A}^2 \rightarrow \mathbb{A}^1$, $(x,y) \mapsto t = xy$. We recall the description of the fiber $X[n]_t$ (see [GHH16], Proposition 1.11 for detail). Of course, if all the $t_i$’s are non-zero, then the fiber is just $\mathbb{C}^*$. The case $t = 0 = (0, \ldots, 0)$ is the ‘most degenerate’ case; $X[n]_0$ consists of $(n+1)$ curves $\Delta^0, \ldots, \Delta^{n+1}$ that form a straight tree.

Here, $\Delta^0$ and $\Delta^{n+1}$ are $\mathbb{A}^1$ and all the other $\Delta^i$’s are $\mathbb{P}^1$. The intersection $\Delta^{i-1} \cap \Delta^i$ is defined by $t_i = 0$. The intermediate case is a partial smoothing of the degeneration. Let us define

$$I_t = \{i \mid t_i = 0\}.$$

If $I_t = \{i_1 < i_2 < \cdots < i_r\}$, the fiber $X_t$ consists of $\Delta^0, \Delta^{i_1}, \ldots, \Delta^{i_r}$ and is a result of smoothing along the coordinate $t_i$ for $i \notin I_t$.
Again $\Delta^0$ and $\Delta^i$ are $\mathbb{A}^1$ and all the other components are $\mathbb{P}^1$. Let $\Delta^{i,0} = \Delta^i \setminus \text{Sing}(X_t)$. We note that $\Delta^{i,0} \cong \mathbb{C}^*.$

Let us take a cycle $\gamma = \sum m_i (p_i, p'_i) \in \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}$, where $(p_i, p'_i)$ is a point of $S[n] = X[n] \times \mathbb{A}^1$. We denote the image of $\gamma$ under the projection by

$$t(\gamma) = (t_1(\gamma), \ldots, t_{n+1}(\gamma)) \in \mathbb{A}^{n+1}.$$  

Then, $\gamma$ is supported on the fiber $X[n]_{t(\gamma)} \times \mathbb{A}^1$. The numerical criterion of stability (op. cit., Theorem 2.9, see also §4, (19)) imposes strong constraints on the distribution of points; all $p_i$'s are in the smooth locus of $X[n]_{t(\gamma)}$ and the degree of $\gamma|_{\Delta^i \times \mathbb{A}^1}$ is $i_{i+1} - i_i$ (here we set $i_0 = 1$ and $i_{r+1} = n+1$).

Let us take a point

$$\tilde{\gamma}_1 = (p_1, \ldots, p_n) \in W[n] = X[n] \times \mathbb{A}^{n+1} \times \cdots \times \mathbb{A}^{n+1} X[n],$$

namely $n$-tuple of points $p_j \in X[n]$ such that

$$t(p_1) = \cdots = t(p_n) \in \mathbb{A}^{n+1},$$

where $t(p_j) = (t_1(p_j), \ldots, t_{n+1}(p_j))$ stands for the image of $p_j$ in $\mathbb{A}^{n+1}$, as before. Combining with a point $\tilde{\gamma}_2 = (p'_1, \ldots, p'_n) \in \mathbb{A}^n$, we specify a point

$$\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in W[n] \times \mathbb{A}^n.$$  

We recall that we have toric charts $W_k \subset X[n]$ $(k = 1, \ldots, n+1)$ defined by

$$u_i \neq 0 \text{ for } i < k \text{ and } v_i \neq 0 \text{ for } i \geq k,$$

(see [GHHT16], Remark 1.6). $W_k$ is isomorphic to $\mathbb{A}^{n+2}$ with toric coordinates

$$(x, \frac{u_1}{v_1}, t_2, \ldots, t_{n+1}), \quad (k = 1)$$

$$(t_1, \ldots, t_{k-1}, \frac{v_{k-1}}{u_{k-1}}, \frac{u_k}{v_k}, t_{k+1}, \ldots, t_{n+1}), \quad (1 < k < n + 1)$$

$$(t_1, \ldots, t_n, \frac{v_n}{u_n}, y), \quad (k = n+1)$$

We also note that we have the relations

$$(4.4) \quad x \cdot \frac{u_1}{v_1} = t_1, \quad \frac{v_{k-1}}{u_{k-1}} \cdot \frac{u_k}{v_k} = t_k, \quad \text{and} \quad \frac{v_n}{u_n} \cdot y = t_{n+1}.$$  

If the image

$$\gamma = \sum (p_i, p'_i) \in \text{Sym}^n(S[n]/\mathbb{A}^{n+1})$$
of $\tilde{\gamma}$ is semistable, by the stability criterion (op. cit., Theorem 2.9), we may assume $p_i \in W_i$ after renumbering, and hence we may assume

$$\tilde{\gamma}_1 = (p_1, p_2, \ldots, p_n) \in W_1 \times_{\mathbb{A}^{n+1}} W_2 \times_{\mathbb{A}^{n+1}} \cdots \times_{\mathbb{A}^{n+1}} W_n.$$  

We write the coordinate of $p_k$ as

$$p_k = \left( t_1, \ldots, t_{k-1}, \frac{v_{k,k-1}}{v_{k,k}}, t_{k+1}, \ldots, t_{n+1} \right). \quad (1 < k \leq n)$$

Then, $W_1 \times_{\mathbb{A}^{n+1}} \cdots \times_{\mathbb{A}^{n+1}} W_n$ is an affine space with coordinate

$$(w_1, w_2, w_3, \ldots, w_n; w_{n+1}, \ldots, w_{2n}; w_{2n+1}) = \left( \frac{u_{11}}{v_{11}}, \ldots, \frac{u_{nn}}{v_{nn}}, \frac{v_{11}}{u_{11}}, \ldots, \frac{v_{n,n-1}}{u_{n,n-1}}, t_{n+1} \right).$$

As the right hand side of (2.1) equals to

$$((x_1, y_1, \ldots, x_n, y_n), (t_1, \ldots, t_{n+1}))$$

$$\left( [u_{11} : v_{11}], \ldots, [u_{nn} : v_{nn}] \right), \ldots, \left( [u_{n1} : v_{n1}], \ldots, [u_{nn} : v_{nn}] \right),$$

in our notation, where the image of $p_k$ under the map $X[n] \to X = \mathbb{A}^2$ is $(x_k, y_k)$, we have

$$(w_1, w_2, w_3, \ldots, w_n; w_{n+1}, \ldots, w_{2n}; w_{2n+1})$$

$$= \left( \frac{s_1}{t_1 \cdots t_{n+1}}, \frac{s_2}{t_2 \cdots t_{n+1}}, \ldots, \frac{s_n}{t_n}, \frac{t_1 \cdots t_{n+1}}{s_1}, \frac{t_2 \cdots t_{n+1}}{s_2}, \ldots, \frac{t_{n+1}}{s_n} \right).$$

The corresponding cone of monomials $\sigma_{1\ldots n}^\vee$ on $M_W[n]_{\mathbb{R}}$ is generated by the column vectors of

$$\sigma_{1\ldots n}^\vee = \begin{pmatrix}
1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 \\
-1 & -1 & \cdots & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & 1 & 1 & \cdots & 1 & 0 \\
-1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}.$$
One sees that its dual cone $\sigma_{1...n}$ is generated by the columns of

\[
\sigma_{1...n} = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
& & & & & \ddots & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 0 \\
& & & & & \ddots & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

and $X(\sigma_{1...n}) = W_1 \times_{\mathbb{A}^{n+1}} \cdots \times_{\mathbb{A}^{n+1}} W_n \subset W[n]$. The affine quotient $X(\sigma_{1...n})//G[n]$ is an affine toric variety $X(\pi \sigma_{1...n})$ for $\pi$ defined in (3.5). A direct calculation immediately shows that

\[
\pi \sigma_{1...n} = \delta^{(n)}
\]

and its dual cone generated by the column vectors of

\[
(\pi \sigma_{1...n})^\vee = \delta^{(n)\vee} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & \ddots & 0 \\
& & & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]

The columns correspond to the invariant monomial functions $f_0, \ldots, f_n$ that generates the coordinate ring of the quotient $X(\sigma_{1...n})//G[n]$. As we have

\[
^t \pi (\pi \sigma_{1...n})^\vee = \begin{pmatrix}
-1 & 1 & \cdots & 0 & 0 \\
0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

we get the relations

\[
\begin{align*}
f_0 &= w_{n+1} = x_1, \\
f_k &= w_k w_{n+k+1} = \frac{u_{k,k} v_{k+1,k}}{v_{k,k} u_{k+1,k}}, \quad (0 < k < n) \\
f_n &= w_n w_{n+1} = \frac{u_{n,n}}{v_{n,n}} \cdot t_{n+1}.
\end{align*}
\]
Among the generators of $\sigma_{1...n}$, the one that corresponds to an irreducible component of the locus of unstable points is of the form

$$e_1 + \cdots + e_{j+1}$$

(j = 0, 1, ..., n − 1)

by Proposition 4.5.1 and the corresponding divisor $D_{\{1,...,j+1\},j}$ is defined by $w_{j+1} = 0$. Therefore, the locus of semistable points $X(\sigma_{1...n})^{ss}$ is given by $w_1 \ldots w_n \neq 0$. Summarizing everything up, we get the following

**Proposition 4.5.2.** Notation as above. Let $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2) \in X(\sigma_{1...n})^{ss} \times \mathbb{A}^n$, and denote the value of the function $w_j$ at $\bar{\gamma}_1$ by $w_j(\bar{\gamma}_1)$. Then the affine subspace of $X(\sigma_{1...n}) \times \mathbb{A}^n$ defined by

$$w_j = w_j(\bar{\gamma}_1) \quad (j = 1, \ldots, n)$$

gives a $\text{Stab}_{\sigma_n}(\bar{\gamma})$-invariant slice $\bar{V}_\gamma$ at $\bar{\gamma}$ to the quotient map

$$W[n]^{ss} \times \mathbb{A}^n \rightarrow \tilde{Z}^{(n)}$$

$$X(\sigma_{1...n})^{ss} \times \mathbb{A}^n \rightarrow \tilde{Z}^{(n)}$$

by $w$ \text{ } (\text{see } 4.2). Then the affine subspace of $X(\sigma_{1...n}) \times \mathbb{A}^n$ defined by

$$w_j = w_j(\bar{\gamma}_1) \quad (j = 1, \ldots, n)$$

gives a $\text{Stab}_{\sigma_n}(\bar{\gamma})$-invariant slice $\bar{V}_\gamma$ at $\bar{\gamma}$ to the quotient map

$$W[n]^{ss} \times \mathbb{A}^n \rightarrow \tilde{Z}^{(n)}$$

$$X(\sigma_{1...n})^{ss} \times \mathbb{A}^n \rightarrow \tilde{Z}^{(n)}$$

namely, the quotient map restricted to $\bar{V}_\gamma$ gives an isomorphism $\bar{V}_\gamma \cong X(\delta^{(n)}) \times \mathbb{A}^n$.

**4.6. Comparison of stabilizer subgroups.** To prove Theorem 4.3.1, we compare the Deligne-Mumford stacks $\mathfrak{q}_{S_{1/C}}$ and $\mathfrak{q}^{(n)}$.

**Lemma 4.6.1.** Let $\bar{\gamma} = ((p_1, p_1^1), \ldots, (p_n, p_n^1)) \in X(\sigma_{1...n})^{ss} \times \mathbb{A}^n$ and $\bar{\delta} \in X(\delta^{(n)}) \times \mathbb{A}^n \subset \tilde{Z}^{(n)}$ be its image. Recall that we defined the sequence $I_\gamma$ as

$$I_\gamma = \{i \mid i \in I(\bar{\gamma}) = \{i \mid t_0(\bar{\gamma}) = 0\} = \{i_1 < \cdots < i_r\}.$$

and $i_0 = 1, i_{r+1} = n + 1$ by convention. Then,

1. If $s \in \text{Stab}_{\sigma_n}(\bar{\delta})$ and $i_j \leq j < i_{j+1}$, we have $i_j \leq s(j) < i_{j+1}$.
2. $\text{Stab}_{\sigma_n}(\bar{\gamma}) = \text{Stab}_{\sigma_n}^{(0)}(\bar{\delta})$ (see (4.2)).

**Proof.** (1) We may assume $j < s(j)$. $s \in \text{Stab}_{\sigma_n}(\bar{\delta})$ implies that there exists a root of unity $\alpha$ such that

$$\frac{\xi_j}{\xi_{s(j)}} = \alpha.$$

As we have

$$\left(\frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \ldots, \frac{\xi_{n-1}}{\xi_n}\right) = (f_1, \ldots, f_{n-1})$$
in the coordinate ring of $X(\delta^{(n)})$ by (3.4) and (4.6), we know
\begin{align}
\alpha &= \frac{\xi_j \xi_{j+1} \cdots \xi_{s(j)-1}}{\xi_{j+1} \xi_{j+2} \cdots \xi_{s(j)}} \\
&= f_j f_{j+1} \cdots f_{s(j)-1} \\
&= u_{j,i} v_{j+1,i} \cdots u_{j,i+1, j} v_{j+2,j+1} \cdots u_{s(j)-1, s(j)-1} v_{s(j), s(j)-1}.
\end{align}
(4.8)
As we have $0 = t_{i+1} = \frac{v_{i+1,j+1} - u_{i+1,j+1}}{v_{i+1,j+1} - u_{i+1,j+1}}$ as in (4.4), if $s(j) \geq i+1$, the product (4.8) must be zero, which is a contradiction.

(2) It is sufficient to prove that $s \in \text{Stab}_{\mathfrak{S}_n}(\bar{\gamma})$ if and only if $(p_s, p'_s) = (p_t, p'_t)$ for every $t$. As $\mathfrak{S}_n$ acts on $W[n] \times \mathbb{A}^n$ by simultaneous permutations, $\text{Stab}_{\mathfrak{S}_n}(\bar{\gamma})$ is a Young subgroup. As we know that $\text{Stab}_{\mathfrak{S}_n}(\bar{\gamma})$ is also a Young subgroup, we may assume that $s$ is a transposition $(i j)$ for $i < j$. Then, $s \in \text{Stab}_{\mathfrak{S}_n}(\bar{\gamma})$ is equivalent to $x_i = x_j = 1$ and $\bar{\gamma}_2 = \bar{\gamma}_2$ is $s$-invariant. The first condition can be rewritten as
\[
f_i \cdot f_{i+1} \cdots f_{j-1} = \frac{u_{i,i} v_{i+1,i} \cdots u_{i,i+1,j} v_{i+2,j+1} \cdots u_{j-1,j-1} v_{j-1,j-1} v_{j,j-1}}{v_{i,i} u_{i+1,i} \cdots u_{i,i+1,j} v_{i+2,j+1} \cdots u_{j-1,j-1} u_{j,j-1}} = 1.
\]
Using the relations (4.4), one can further rephrase the condition as
\[
1 = \frac{u_{i,i} v_{i+1,i} \cdots u_{i,i+1,j} v_{i+2,j+1} \cdots u_{j-1,j-1} v_{j,j-1}}{v_{i,i} u_{i+1,i} \cdots u_{i,i+1,j} v_{i+2,j+1} \cdots u_{j-1,j-1} u_{j,j-1}} = \frac{u_{i,j} v_{i+1,i+1} \cdots u_{i,j-1} v_{i+2,j-1}}{v_{i,i} u_{i+1,i} \cdots u_{i,i+1,j} v_{i+2,j+1} \cdots u_{j-1,j-1} u_{j,j-1}},
\]
which is clearly equivalent to $p_t = p_j$. Q.E.D.

Let us write $\text{Stab}_{\mathfrak{S}_n}(\bar{\gamma}) = \mathfrak{S}_M(\bar{\gamma})$, where $\mathfrak{S}_M(\bar{\gamma})$ is a Young subgroup associated with a partition $M(\bar{\gamma}) = \{M(\bar{\gamma})_k\}$,

\[
\{1, \ldots, n\} = \bigsqcup_k M(\bar{\gamma})_k.
\]

Lemma 4.6.1 (1) implies that by further renumbering of $\bar{\gamma}$ staying inside $X(\sigma_1 \ldots n) = W_1 \times \mathbb{A}^{n+1} \cdots \times \mathbb{A}^{n+1} W_n$, we may assume that
\[
M(\bar{\gamma})_k = \{m_k, m_k + 1, \ldots, m_k + 1\}.
\]
(4.9)
for a sequence $1 = m_1 < m_2 < \cdots < m_V < m_{V+1} = n$ and the partition $M(\bar{\gamma})$ is a sub-partition of the partition determined by $I(\bar{\gamma})$. More precisely, for each $l$, we have a partition
\[
i_l = m_{k_l} < m_{k_l+1} < \cdots < m_{k_l+\beta_l-1} < m_{k_l+\beta_l} = m_{k_l+1} = i_{l+1}.
\]
We prepare the following notation:

\[ \begin{align*}
\mu_k &= m_{k+1} - m_k = \#(M(\bar{\gamma})_k), \\
K_l &= \{k_l, k_l + 1, \ldots, k_l + \beta_l - 1\}, \\
K^d &= \{k \mid \mu_k = d\}, \\
K^d_l &= K_l \cap K^d, \\
K^d_l &= \{K^d_l\}_d \text{ a partition of } K_l, \\
M_l &= \{M(\bar{\gamma})_k \mid k \in K_l\}, \\
M^d_l &= \{M(\bar{\gamma})_k \mid k \in K^d\}, \\
M^d_l &= M_l \cap M^d.
\end{align*} \]

(4.10)

**Lemma 4.6.2.** Notation as above.

1. \( s \in \text{Stab}_{S_n}(\bar{\gamma}) \) induces a permutation of the set \( M^d_l \) for each \( d \).
2. There is an injective homomorphism

\[ \rho : G(\bar{\gamma}) = \text{Stab}_{S_n}(\bar{\gamma})/\text{Stab}_{S_n}^0(\bar{\gamma}) \to \prod_l \mathfrak{S}_{K^*_l} \]

where \( \mathfrak{S}_{K^*_l} \subset \mathfrak{S}_{K_l} \) is the Young subgroup associated with the partition \( K^*_l \) of \( K_l \).

**Proof.** (1) Take a cyclic permutation \( c_k = (m_k \cdots m_{k+1} - 1) \in \mathfrak{S}_M(\bar{\gamma})_k \) in accordance with (4.9) for each \( k \). Since an element \( s \in \text{Stab}_{S_n}(\bar{\gamma})/\text{Stab}_{S_n}^0(\bar{\gamma}) \) normalizes \( \text{Stab}_{S_n}^0(\bar{\gamma}) = \mathfrak{S}_M(\bar{\gamma}), s c_k s^{-1} = (s(m_k) \cdots s(m_{k+1} - 1)) \in \mathfrak{S}_M(\bar{\gamma}) \). Therefore, there is \( k' \) such that \( \mu_{k'} \geq \mu_k \) and \( s(m_k), \ldots, s(m_{k+1} - 1) \in M(\bar{\gamma})_{k'} \). By a similar argument for \( s^{-1}c_{k'}s \), we conclude that \( \mu_k = \mu_{k'} \). This implies that \( s \) induces a permutation of the set \( M^d_l \) for each \( d \). Moreover, Lemma 4.6.1 (1) asserts that this permutation leave \( M^d_l \) invariant.

(2) (1) implies that an element of \( \text{Stab}_{S_n}(\bar{\gamma}) \) induces a permutation of the set \( K^d_l \). Therefore, we have a natural map \( \bar{\rho} : \text{Stab}_{S_n}(\bar{\gamma}) \to \prod_l \mathfrak{S}_{K^*_l} \). Taking (1) into account, it is straightforward to check that for \( s, s' \in \text{Stab}_{S_n}(\bar{\gamma}), \bar{\rho}(s) = \bar{\rho}(s') \) if and only if \( s^{-1}s' \in \mathfrak{S}_M(\bar{\gamma}) = \text{Stab}_{S_n}^0(\bar{\gamma}) \), and hence \( \bar{\gamma} \) descends to an injective homomorphism \( \rho : G(\bar{\gamma}) \to \prod_l \mathfrak{S}_{K^*_l} \).

Q.E.D.

**Lemma 4.6.3.** Let \( \gamma \in \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} \) and assume \( I_{\iota(\gamma)} = \{i_1 < \cdots < i_r\} \). Let us denote the restriction of \( \gamma \) to \( \Delta^{i_l, \iota} \times \mathbb{A}^1 \) by \( \gamma_l \) \((l = 0, \ldots, r)\). Then, the stabilizer subgroup of \( \gamma \) under the action of \( G[n] \) is given by

\[ \text{Stab}_{G[n]}(\gamma) = \prod_{1 \leq l \leq r - 1} \text{Stab}(\gamma_l), \]

where

\[ \text{Stab}(\gamma_l) = \{ \tau \in \mathbb{C}^* \mid \tau \cdot \gamma_l = \gamma \} \]

is the stabilizer subgroup where \( \mathbb{C}^* \) acts on \( \Delta^{i_l, \iota} \times \mathbb{A}^1 \) by multiplication on the first factor and trivially on the second factor.
Proof. Let us take \((\lambda_1, \ldots, \lambda_{n+1}) \in G[n]\) with \(\lambda_1 \cdots \lambda_{n+1} = 1\). If \(t_i(\gamma) \neq 0\), then \(\lambda_i\) acts freely in the orbit \(G[n] \cdot \gamma\). Therefore, the stabilizer subgroup \(\text{Stab}_{G[n]}(\gamma)\) is actually a subgroup of \((\mathbb{C}^*)^{r-1}\) with coordinate \((\lambda_{i_1}, \ldots, \lambda_{i_r})\) up to the relation \(\lambda_{i_1} \cdots \lambda_{i_r} = 1\). Since \(u_{ij}/v_{ij}\) gives a coordinate of \(\Delta^{ij,\circ}\), if we introduce \(\tau\)-coordinate

\[\tau_{ij} = \lambda_{i_1} \cdots \lambda_{i_r}\]

as in \([3.2]\) \(\tau_{ij}\) acts on \(\Delta^{ij,\circ}\) by multiplication and trivially on the other components \(\Delta^{ij,\circ}\), from which the lemma immediately follows. Q.E.D.

Let us take a sufficiently small neighborhood \(\tilde{U}_{\tilde{q}}\) of \(\tilde{q} \in \tilde{Z}^{(n)}\) and replace \(\tilde{V}_\gamma\) by its inverse image so that the quotient map \(W[n]^{ss} \times \mathbb{A}^n \to \tilde{Z}^{(n)}\) restricts to an isomorphism \(\tilde{V}_\gamma \tilde{\to} \tilde{U}_{\tilde{q}}\). Lemma \[4.6.1\] (2) asserts that we have an induced isomorphism

\[V_\gamma = V_{\tilde{\gamma}}/\text{Stab}_{\mathfrak{c}_n}(\tilde{\gamma}) \tilde{\to} \tilde{U}_{\tilde{q}}/\text{Stab}_{\mathfrak{c}_n}^0(\tilde{q}) = U_{\tilde{q}}.\]

Here we note that \(V_\gamma\) is identified with a slice at the image \(\gamma \in \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}\) of \(\tilde{\gamma}\) with respect to the quotient map \(\text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} \to Z^{(n)}\).

**Theorem 4.6.4.** The quotient map \(\text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss} \to Z^{(n)}\) induces an isomorphism

\[\text{Stab}_{G[n]}(\gamma) \tilde{\to} G(\tilde{q}) = \text{Stab}_{\mathfrak{c}_n}(\tilde{q})/\text{Stab}_{\mathfrak{c}_n}^0(\tilde{q})\]

and the isomorphism \(V_\gamma \tilde{\to} U_{\tilde{q}}\) is equivariant.

Before giving a proof of this theorem, let us look at a handy case.

**Example 4.6.5.** 1) Let \(n = 9\) and consider \(\gamma = \sum_{i=1}^9 p_i p_i' \in \text{Sym}^9(S[9]/\mathbb{A}^{10})^{ss}\). Let us assume \(I_{t(\gamma)} = \{1 < 7 < 10\}\). Then, the fiber \(X[9]_{t(\gamma)}\) consists of 4 components. By stablility, six points are on \(\Delta^{1,\circ} \times \mathbb{A}^1\) and three points are on \(\Delta^{7,\circ} \times \mathbb{A}^1\).

Let \(\zeta_1 = \frac{\omega}{\omega_{11}}\) and \(\zeta_4 = \frac{\omega}{\omega_{44}}\), the coordinate of \(p_1\) and \(p_4\), respectively, and \(\zeta_7 = \frac{\omega}{\omega_{77}}\), the coordinate of \(p_7\). Let \(\omega\) be a primitive third root of unity, and assume

\[p_2 = \omega p_1,\quad p_3 = \omega^2 p_1,\quad p_1' = p_3',\quad p_5 = \omega p_4,\quad p_6 = \omega^2 p_4,\quad p_4' = p_5',\quad p_8 = \omega p_7,\quad p_9 = \omega^2 p_7,\quad p_7' = p_8',\quad p_9' = p_7'.\]

Furthermore, we assume that \((p_1, p_1')\) and \((p_4, p_4')\) are in general position, namely \(p_1 \neq p_4\) or the ratio \(\zeta_1/\zeta_4\) is not a sixth root of unity. Then by Lemma \[4.6.3\]...
\[ \text{Stab}_{G[n]}(\gamma) \cong (\mathbb{Z}/3\mathbb{Z})^2, \] which is generated by multiplications of \( \omega \) on \( \Delta^1 \) and \( \Delta^7 \). Now we calculate the value of the invariant functions \( f_k \). Since \( t_1 = t_{10} = 0 \), we have \( f_0 = f_9 = 0 \), while \( f_k = \frac{u_{kk} v_{k,k+1}}{v_{kk} u_{k,k+1}} = t_j(\gamma) \), and \( t_7 = 0 \) imply that
\[ f_1 = f_2 = f_4 = f_5 = f_7 = f_8 = \omega^{-1} \quad \text{and} \quad f_6 = 0. \]

In terms of the toric coordinate of \( X(\delta^{(n)}) \), the image point \( \tilde{q}_1 \) has coordinate
\[ \left( \frac{f_0}{\xi_2}, \frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_4}, \frac{\xi_3}{\xi_5}, \frac{\xi_4}{\xi_6}, \frac{\xi_5}{\xi_7}, \frac{\xi_6}{\xi_8}, \frac{\xi_7}{\xi_9}, f_0 \right) = (0, \omega^{-1}, \omega^{-1}, \omega^{-1}, \omega^{-1}, 0, \omega^{-1}, 0). \]

A permutation of \( \{\xi_1, \xi_2, \xi_3\} \) that preserves the ratio \( \xi_i / \xi_{i+1} \) is either of cyclic permutations \((1 \ 2 \ 3) \) or \((1 \ 3 \ 2) \). We have the same thing for \( \{\xi_4, \xi_5, \xi_6\} \) and \( \{\xi_7, \xi_8, \xi_9\} \).

The permutations \((1 \ 2 \ 3) \) alone does not fix the point \( \tilde{q} \) because
\[ (1 \ 2 \ 3) \cdot \frac{\xi_3}{\xi_4} = \frac{\xi_3}{\xi_4} = \omega^{-3} \frac{\xi_3}{\xi_4} = \omega f_3 \]
and \( f_3 \) is non-zero as \( t_4 \neq 0 \). More generally, as we have
\[ (4.11) \quad (1 \ 2 \ 3)^a(4 \ 5 \ 6)^b \cdot \frac{\xi_3}{\xi_4} = \omega^{a-b} \frac{\xi_3}{\xi_4} \]
the permutation is in \( \text{Stab}_{G_9}(\tilde{q}) \) only if \( a - b = 0 \). On the other hand, \((7 \ 8 \ 9) \) fixes the point \( \tilde{q} \) as \( \frac{\xi_6}{\xi_7} = 0 \). Therefore, we know that
\[ G(\tilde{q}) = \text{Stab}_{G_9}(\tilde{q}) = \langle (1 \ 2 \ 3)(4 \ 5 \ 6), (7 \ 8 \ 9) \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2 \cong \text{Stab}_{G[n]}(\gamma) \]
as stated in the theorem (note that \( \text{Stab}_{G_9}^0(\tilde{q}) = \{\text{id}\} \) in this case).

(2) Next we consider the case \( n = 6 \) and \( \gamma = \sum_{i=1}^6 (p_i, p'_i) \in \text{Sym}^6(S[6]/\mathbb{A}^7)^{ss} \) with \( I_{t(\gamma)} = \{1 < 7\} \). We assume further
\[ (p_1, p'_1) = (p_2, p'_2), (p_3, p'_3) = (p_4, p'_4), (p_5, p'_5) = (p_6, p'_6) \]
so that \( \gamma = 2(p_1, p'_1) + 2(p_3, p'_3) + 2(p_5, p'_5) \), and for a primitive third root of unity \( \omega \)
\[ (p_3, p'_3) = \omega (p_1, p'_1), \quad (p_5, p'_5) = \omega^2 (p_1, p'_1). \]

In this case, the coordinate of \( \tilde{q}_1 \) is given by
\[ \left( f_0, \frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \frac{\xi_3}{\xi_4}, \frac{\xi_4}{\xi_5}, \frac{\xi_5}{\xi_6}, f_6 \right) = (0, 1, \omega^{-1}, 1, \omega^{-1}, 1, 0). \]
and
\[ \text{Stab}_{G_6}^0(\tilde{q}) = \mathbb{S}\{1, 2\} \times \mathbb{S}\{3, 4\} \times \mathbb{S}\{5, 6\}. \]

Therefore a cyclic permutation \((1 \ 3 \ 5 \ 2 \ 4 \ 6) \) gives an element of \( \text{Stab}_{G_6}(\tilde{q}) \) and its residue class generates \( G(\tilde{q}) \). Thus we know \( \text{Stab}_{G[n]}(\gamma) \cong G(\tilde{q}) \cong \mathbb{Z}/3\mathbb{Z}. \)
Proof of Theorem 4.6.4.} We keep all the assumptions and the notation above. The equality \( \text{Stab}_{\mathcal{E}_n}(\gamma) = \mathcal{G}_{M(\gamma)} \) for the partition \( \{M(\gamma)_k\} \) in Lemma 4.6.1 implies that
\[
(p_{m_k}, p'_{m_k}) = (p_{m_{k+1}}, p'_{m_{k+1}}) = \cdots = (p_{m_{k+1-1}}, p'_{m_{k+1-1}}) \neq (p_{m_{k+1}}, p'_{m_{k+1}})
\]
and the corresponding cycle \( \gamma \in \text{Sym}^n(S[n]/\mathbb{A}^{n+1}) \) is of the form
\[
\gamma = \sum_k \mu_k(p_{m_k}, p'_{m_k}),
\]
where \( \mu_k = m_{k+1} - m_k \). The restriction \( \gamma_l \) of \( \gamma \) to \( \Delta_l \times \mathbb{A}^1 \) is
\[
\gamma_l = \sum_{j=0}^{\beta_l-1} \mu_{k_l+j}(p_{m_{k_l+j}}, p'_{m_{k_l+j}}).
\]
Now we assume
\[
\mu_{k_l} \geq \mu_{k_l+1} \geq \cdots \geq \mu_{k_l+\beta_l-1}
\]
by a further renumbering. Then there is a partition of \( \beta_l \)
\[
0 = \beta_{l,0} < \beta_{l,1} < \cdots < \beta_{l,e_l-1} < \beta_l
\]
and
\[
d_{l,0} > d_{l,1} > \cdots > d_{l,e_l-1} > 0
\]
such that
\[
d_{l,i} = \mu_{k_l+i} = \cdots = \mu_{k_l+i+1-1}.
\]
Being a finite subgroup of \( \mathbb{C}^* \), \( \text{Stab}(\gamma_l) \) is a cyclic group of finite order consisting of roots of unity. Let \( \tau_l \in \mathbb{C}^* \) be a generator and \( r_l \) the order of \( \tau_l \). The action of \( \tau_l \) induces a cyclic permutation among the set of points
\[
\{(p_{m_{k_l+j}}, p'_{m_{k_l+j}}) | \beta_{l,i} \leq j < \beta_{l,i+1}\}
\]
and decomposes the set into a disjoint union of orbits each of which consists of \( r_l \) points. In particular, we have \( \beta_{l,i+1} - \beta_{l,i} = r_l \cdot \beta'_{l,i} \) for some positive integer \( \beta'_{l,i} \). We may assume for \( 0 \leq \kappa < \beta_l \) and \( 0 \leq j < r_l \),
\[
(p_{m_{k_l+i+\kappa r_l+j}}, p'_{m_{k_l+i+\kappa r_l+j}}) = \tau_l^j(p_{m_{k_l+i+\kappa r_l}}, p'_{m_{k_l+i+\kappa r_l}}).
\]
Summerizing, we configured the index set \( K_l^{d_{l,i}} \) as the following:

\[
\begin{array}{ccccccc}
\tau_l & m_{k_l+\beta_{l,i}+\kappa r_l} & m_{k_l+\beta_{l,i}+\kappa r_l+1} & \cdots & m_{k_l+\beta_{l,i}+\kappa r_l+d_{l,i}-1} \\
\tau_l & m_{k_l+\beta_{l,i}+\kappa r_l+1} & m_{k_l+\beta_{l,i}+\kappa r_l+1+1} & \cdots & m_{k_l+\beta_{l,i}+\kappa r_l+1+d_{l,i}-1} \\
\vdots & \vdots & \vdots & & \vdots \\
\tau_l & m_{k_l+\beta_{l,i}+(\kappa+1)r_l-1} & m_{k_l+\beta_{l,i}+(\kappa+1)r_l-1+1} & \cdots & m_{k_l+\beta_{l,i}+(\kappa+1)r_l-1+d_{l,i}-1}
\end{array}
\]
As the invariant function $\frac{\xi_k}{\xi_{k+1}} = f_k$ is the ratio of the $u_k/v_k$-coordinates of $p_k$ and $p_{k+1}$ by (4.7), we get

$$
\begin{pmatrix}
    f_{m_{k_1} + \beta_{i_1} + \kappa_{i_1}} & \cdots & f_{m_{k_2} + \beta_{i_2} + \kappa_{i_2} + d_{i_2} - 1} \\
    f_{m_{k_3} + \beta_{i_3} + \kappa_{i_3} + 1} & \cdots & f_{m_{k_2} + \beta_{i_2} + \kappa_{i_2} + 1 + d_{i_2} - 1} \\
    \vdots & & \vdots \\
    f_{m_{k_2} + \beta_{i_2} + (\kappa_1 + 1) r_{i_2} - 1} & \cdots & f_{m_{k_2} + \beta_{i_2} + (\kappa_1 + 1) r_{i_2} - 1 + d_{i_2} - 1}
\end{pmatrix} = \begin{pmatrix}
    1 & 1 & \cdots & 1 & \tau_i^{-1}
    1 & 1 & \cdots & 1 & \tau_i^{-1}
    \vdots & & \vdots & & \vdots
    1 & 1 & \cdots & 1 & \tau_i^{-1}
\end{pmatrix}
$$

Therefore, a cyclic permutation $c_{m_{k_1}, i, \kappa}$ of length $r_i \cdot d_{i,1}$ “along the column”

$$
\begin{pmatrix}
m_{k_1} + \beta_{i_1} + \kappa_{i_1} & \cdots & m_{k_2} + \beta_{i_2} + \kappa_{i_2} + 1 & \cdots & m_{k_2} + \beta_{i_2} + \kappa_{i_2} + d_{i_2} - 1 \\
m_{k_3} + \beta_{i_3} + \kappa_{i_3} + 1 & \cdots & m_{k_2} + \beta_{i_2} + \kappa_{i_2} + 1 + d_{i_2} - 1 \\
\vdots & & \vdots & & \vdots \\
m_{k_2} + \beta_{i_2} + (\kappa_1 + 1) r_{i_2} - 1 & \cdots & m_{k_2} + \beta_{i_2} + (\kappa_1 + 1) r_{i_2} - 1 + d_{i_2} - 1
\end{pmatrix}
$$

fixes this block of coordinates. Therefore, the correspondence

$$
\prod_l \tau_l \mapsto \left[ \prod_{l, i, \kappa} c_{m_{k_l}, i, \kappa} \right]
$$

gives a group homomorphism $\text{Stab}_{G[n]}(\gamma) \to \prod_l \mathcal{S}_{K_l^i}$ whose image is contained in $\rho(G(\bar{\gamma}))$. On the other hand, as the invariant functions $f_k$ determines the relative position of all $p_k$’s on a component $\Delta^{i, \circ}$, a permutation in $\mathcal{S}_{K_l^i}$ that fixes the point $\bar{\gamma}$ is necessarily a power of $[\prod_{l, i, \kappa} c_{m_{k_l}, i, \kappa}]$, as we saw in the previous example, therefore we get an isomorphisms $\text{Stab}_{G[n]}(\gamma) \cong G(\bar{\gamma})$. Q.E.D.

**Conclusion of the proof of Theorem 4.3.1** Let us keep our assumptions on

$$
\gamma = \sum (p_i, p_i') = \sum_k \mu_k(p_{m_k}, p_{m_k}') \in \text{Sym}^n(S[n]/\mathbb{A}^n+1)^{ss},
$$

and its lifting $\bar{\gamma} \in W[n]^{ss} \times \mathbb{A}^n$. Let us recall that the slice $\bar{V}_\gamma$ is cut out by the equations $w_j = w_j(\bar{\gamma})$. More precisely, if we define a non-zero constant $c_j$ by $c_j = w_j(\bar{\gamma})$, we have

$$
t_j = \frac{v_{j, j-1} u_{jj}}{u_{j, j-1} v_{jj}} = w_{n+j} \cdot c_j.
$$

This means that on $\bar{V}_\gamma$, a variation of a point $(p_j, p_j')$ is parametrized by the coordinate $t_j$ and the $j$-th coordinate of the factor $\mathbb{A}^n$ of $W[n]^{ss} \times \mathbb{A}^n$. Therefore, we know that $V_\gamma = \bar{V}_\gamma / \text{Stab}_{\mathbb{A}_n}(\bar{\gamma})$ is locally isomorphic to

$$
\left( \prod_k \text{Sym}^{\mu_k}(\mathbb{A}^2) \right) \times \mathbb{A}^1 \subset \text{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1.
$$

where the last factor $\mathbb{A}^1$ is the line with coordinate $t_{n+1}$. Moreover the fiber product

$$
\hat{V}_\gamma = \text{Hilb}^n(S[n]/\mathbb{A}^{n+1}) \times_{\text{Sym}^n(S[n]/\mathbb{A}^{n+1})} V_\gamma \to V_\gamma
$$
is isomorphic to a restriction of the Hilbert-Chow morphism \( \text{Hilb}^n(\mathbb{A}^2) \times \mathbb{A}^1 \to \text{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1 \) to \( V_\gamma \). By the universality of Hilbert scheme, the \( G(\tilde{q}) \)-equivariant isomorphism \( V_\gamma \to U_{\tilde{q}} \) lifts to an equivariant isomorphism

\[
\hat{V}_\gamma \sim \to \hat{U}_{\tilde{q}}.
\]

As \( \hat{V}_\gamma \) gives a slice to a quotient \( \text{Hilb}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n] \), this implies that we have an isomorphism of smooth Deligne-Mumford stacks

\[
\gamma_n^{(n)}_{S/C} \sim \to \gamma(n),
\]

over \( Z^{(n)} \equiv \text{Sym}^n(S[n]/\mathbb{A}^{n+1})^{ss}/G[n] \), which completes the proof of Theorem \([4.3.1]\) Q.E.D.

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