Morris–Thorne wormholes in modified $f(R, T)$ gravity

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Abstract
Wormhole solutions obtained by Morris and Thorne (MT) in general relativity (GR) is investigated in a modified theory of gravity. In the gravitational action, we consider $f(R, T)$ which is a function of the Ricci scalar ($R$) and the trace of the energy-momentum tensor ($T$). In the framework of a modified gravity described by $f(R, T) = R + \alpha R^2 + \lambda T^\beta$, where $\alpha$, $\beta$ and $\lambda$ are coupling constants, MT wormhole (WH) solutions with normal matter are obtained for a relevant shape function. We have considered two different values of $\beta$ leading to two forms of $f(R, T)$-gravity. The energy conditions are probed at the throat and away from the throat of the WH. It is found that the coupling parameters, $\alpha$ and $\lambda$ in the gravitational action play an important role in deciding the matter composition in the wormholes. It is found that for a given $\lambda$, WH exists in the presence of exotic matter at the throat when $\alpha < 0$. It is demonstrated here that WH exists even without exotic matter for $\alpha > 0$ in the modified gravity. Two different shape functions are considered to obtain WH solutions that are permitted with or without exotic matter. It is noted that in a modified $f(R, T)$ gravity MT WH is permitted with normal matter which is not possible in GR. It is demonstrated that a class of WH solutions exist with anisotropic fluid for $\lambda \neq -8\pi$. However, for flat asymptotic regions with anisotropic fluids WH solutions cannot be realized when $\lambda = -8\pi$. All the energy conditions are found consistent with the hybrid shape function indicating existence of WH even with normal matter for $\lambda \to 0$.

Keywords Traversable Wormholes · Modified gravity · $f(R, T)$ gravity
1 Introduction

Einstein’s theory of General Relativity (GR) permits traversable wormholes which are considered to describe topological passage through a hypothetical tunnel like bridges connecting two distant regions of a universe or two different distant universes. The term wormhole was first introduced in 1957 by Wheeler and Misner [1,2]. Subsequently the interesting features of the wormhole led to a spurt in research activities in theoretical astrophysics [3,4]. The important feature of traversable wormhole is that it allows a shortcut passage between two distant regions of space-time. The observed features of the WH solutions are employed for constructing hypothetical time machines [5, 6]. Theoretically it is proposed that black holes and wormholes are inter-convertible structures and a stationary wormhole might be the final stage of an evaporating black hole [7]. In the literature [8–10], it is found that astrophysical accretion of ordinary matter could convert wormholes into black holes. It is known that WH exists in GR at the cost of violation of energy conditions of the matter. The energy-momentum tensor of the matter supporting such geometries violates the null energy condition (NEC) near the throat region of the WH [11–13] for its existence. A fundamental ingredient in wormhole physics is the flaring out condition of the throat, which can be obtained in GR with the violation of the NEC. Matter that violates the NEC is called exotic matter. In cosmology exotic matter described by dark energy or phantom fluid [14,15], is required to accommodate the late accelerating phase of the universe. The phantom fluid is considered [16] to realize wormholes in GR. The exotic matter at the throat of the wormhole signifies that an observer who moves through the throat with a radial velocity approaching the speed of light will observe presence of negative energy density leading to the violation of the energy conditions. However, it will be interesting if one can construct WH without exotic matter.

Recently, there is a growing interest to modify the gravitational sector of GR to accommodate the observed universe. One of the simplest modification of the Einstein–Hilbert action is the $f(R)$- theory of gravity, where the curvature scalar ($R$) in the gravitational action is replaced by a non-linear function, $f(R)$ which is a polynomial function of $R$. In $f(R) = R + \alpha R^2$ theory, Starobinsky [17] first obtained early inflationary universe solution long before the efficacy of the inflation was known. The
higher order gravity is important to look for other aspects in astrophysics. Recently, WH solutions are obtained in $f(R)$ theory of gravity [18,19]. Using specific shape functions and a constant redshift function, Lobo and Oliveira [20] explored the criteria that affect the wormhole structures. Thin shell like wormholes with charge is obtained in the framework of $f(R)$ gravity and their stability under perturbations was probed [21]. Sahoo et al. [22] obtained WH where NEC is violated in the framework of higher curvature and phantom field. Recently Godani and Samanta [23] studied wormhole solutions for different shape functions in $f(R)$ gravity and found that wormholes are filled with phantom fluid. However, in a ghost-free scalar tensor model of dark energy admitting phantom behaviour it is shown by Bronnikov and Starobinsky [24] that realistic WH are not permitted even in the presence of electric and magnetic fields. Both Brans–Dicke theory and $f(R)$ gravity are taken up to study WH, it is claimed [25] that no vacuum WH exists in Brans–Dicke theory but WH might exist in $f(R)$-gravity if it satisfies an extremum where the effective gravitational constant changes its sign. Bronnikov et al. [26] showed a no-go theorem in GR for obtaining wormhole solutions, according to that it excludes the existence of wormholes with flat and/or AdS asymptotic regions on both sides of the throat if the source matter is isotropic.

In the literature [27] the modified $f(R, T)$ -gravity which is an arbitrary function of the Ricci scalar ($R$) and of the trace of the energy momentum tensor ($T$) are considered widely for understanding the observed universe. In the $f(R, T)$ gravity framework the random requirement on $T$ with conceivable contributions from both non-minimal coupling and the unambiguous $T$ terms in the gravitational action may have rich structure which help in understanding the universe. Consequently $f(R, T)$ theory of gravity is important in understanding many different aspects considering different functional forms of the curvature scalar $R$ and trace of the energy-momentum tensor $T$ [28–30]. Azizi [31] obtained wormhole solutions with a shape function satisfying the null energy condition (NEC) in the framework of $f(R, T)$ gravity. Zubair et al. [32], however, considered three types of fluid in a static spherically symmetric wormhole under $f(R, T)$ gravity formalism and analysed the energy conditions. The modified $f(R, T)$- gravity permits WH in the presence of phantom fluid and it is found that for WHs, the radial NEC is violated, unlike the tangential NEC [33].

In $f(R,T)$ modified theory of gravity existence of spherically symmetric wormhole solutions are examined with non-commutative geometry in terms of Gaussian and Lorentzian distributions of string theory [34]. Stable wormhole solutions are found to exist which are used to estimate the deflection angle and found that it diverges at the wormhole throat. A hybrid form of shape function is considered [35] to examine the energy conditions in the framework of $f(R, T) = R + f(T)$ gravity. The radial null energy and weak energy conditions are found to satisfy with the shape function in the absence of exotic matter. WH solutions are obtained with both anisotropic and isotropic fluids in some cases. Godani and Samanta [36] considered another functional form of $f(R, T)$ -gravity relevant for dark energy model in cosmology to obtain wormhole solutions and found that wormhole solutions are permitted without the requirement of exotic matter for a given form of the shape function.

In the paper we consider a modified $f(R, T) = f(R) + f(T)$ gravity, where $f(T) = \lambda T^\beta$, $\beta$ being a constant, and $f(R) = R + \alpha R^2$ to study existence of wormholes. It may be pointed out that $T = 0$ was employed in black hole physics,
matter density perturbations [18,37–39] after its introduction in cosmology [17]. Compact astrophysical objects have been analysed in Starobinsky model [40–42]. In cosmology such $R^2$ models are also employed [43–47] to analyze the observed universe. Recently Sahoo et al. [48]) studied WH solutions using a modified gravity follow from $f(R, T) = f(R) + \lambda T$ with a shape function which was considered in Ref. ( [49]) to obtain WH in GR. The negative values of $\lambda$ was considered in $f(R, T) = R + \alpha R^2 + \lambda T$-gravity to obtain wormhole. In the literature WH geometry has been studied considering exponential form of $f(R, T)$ gravity [50], traceless $f(R, T)$ gravity [51], considering linear and quadratic terms on the trace of the energy-momentum tensor in the gravitational action [52]. We recommend following Refs. [53–55] for some important articles regarding this subject. The coupling factor of the geometry and matter in the modified gravity is taken up in recent times to investigate various issues in cosmology and astrophysics. The successes of the $f(R)$ -theory in astrophysics motivated us to probe wormhole solutions as in the limit $\lambda \rightarrow 0$ the $f(R, T)$-theory reduces to $f(R)$ gravity. Thus it is important to investigate the role of $\lambda$ in obtaining wormhole solutions with different forms of shape functions in the presence of matter. Although the reliability of the isotropy in the fluid description has been experimentally verified in many contexts, there are many situations for which anisotropies may originate at high and low energy densities which was reported in the context of compact objects [56]. The anisotropic distribution of fluids was used to study magnetized accretion disks around Kerr black holes [57] (see Ref. [26] for further studies on wormholes from anisotropic fluids). Shaikh [58] constructed a wide class of wormholes in Eddington-inspired Born-Infeld gravity with a stress energy which does not violate the weak or null energy condition. The existence of anisotropic solutions usually involves strong gravity effects, and in some cases they are in the realm where these effects arises from quantum theory of gravity [59]. The wormholes will be probed here in the presence of anisotropic fluid in the modified gravity.

The paper is organised as follows: In Sect. 2, the fundamentals of $f(R, T)$ gravity are presented. The general form of the field equation is obtained by varying the gravitational action with respect to the metric. In Sect. 3, we have considered the static spherically symmetric metric which describes the WH geometry and the necessary conditions for wormholes making use of two different shape functions. We consider two different functional forms of $f(R, T)$ gravity and the field equations are determined. In both the cases we determine the energy conditions needed for wormholes and analysed them for a given shape function. The physical analysis are carried out for both the models. The field equations are highly non-linear, the validity of energy conditions are thus studied numerically by plotting relevant graphs. The results are summarised in Sect. 4 followed by a brief discussion.

2 The Gravitational action in $f(R, T)$ modified theory and the field equations

The modified gravitational action is given by [27],
\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R, T) + \int d^4x \sqrt{-g} L_m \] (1)

where \( f(R, T) \) is a function of Ricci scalar \( R \) and the trace of the energy-momentum tensor \( T = g^{\mu\nu} T_{\mu\nu} \), \( g \) is the determinant of the metric and \( L_m \) is the matter Lagrangian density. Varying the action \( S \) with respect to the metric \( g_{\mu\nu} \) the field equation is obtained, which is given by \([27]\)

\[ R_{\mu\nu} f_R(R, T) - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R(R, T) = 8\pi T_{\mu\nu} - f_T(R, T) \Theta_{\mu\nu} - f_T(R, T) T_{\mu\nu} \] (2)

where we denote \( f_R(R, T) \equiv \frac{\partial f(R, T)}{\partial R} \), \( f_T(R, T) \equiv \frac{\partial f(R, T)}{\partial T} \), \( T_{\mu\nu} \) is the energy-momentum tensor and \( \Theta_{\mu\nu} \) is defined as

\[ \Theta_{\mu\nu} = g^{\alpha\beta} \delta T_{\alpha\beta}^{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu} L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}} \] (3)

in the above we consider natural units, \( i.e. \), speed of light, \( c = 1 \) and gravitational constant, \( G = 1 \). The energy momentum tensor for anisotropic fluid \([4]\) is given by

\[ T_{\mu\nu} = (\rho + p_r) u_\mu u_\nu - p_r g_{\mu\nu} + (p_r - p_t) X_\mu X_\nu, \] (4)

where \( \rho, p_r \) and \( p_t \) represent the energy density, radial pressure and tangential pressure respectively. In the above, \( u_\mu \) and \( X_\mu \) denotes the four-velocity vector and the radial unit four vector respectively, and satisfy the relations \( u_\mu u_\nu = 1 \) and \( X_\mu X_\nu = -1 \).

Kovisto \([60,61]\) demonstrated the vanishing of the covariant divergence of the energy-momentum tensor in modified theories of gravity. A functional form of the curvature scalar multiplying a matter Lagrangian was also considered. Using the Palatini variational principle it is found that the covariant conservation of energy-momentum tensor is satisfied in the field equations. However, in the conformal Einstein frame bimetricity emerges into the structure of the theories and it can be generalized to gravitational action beyond \( f(R) \). In \( f(R, T) \)-gravity similar approaches can be followed and the field equations are obtained. It may be mentioned here that the trace of the energy momentum tensor \( T \) may have the same role as a scalar field as is shown in case of hybrid metric-Palatini gravity \([62]\). Here we consider the matter Lagrangian density as follows: \( L_m = -P \), consequently one can rewrite eq. (3) as

\[ \Theta_{\mu\nu} = -2T_{\mu\nu} - Pg_{\mu\nu} \] (5)

where \( P = \frac{p_r + 2p_t}{3} \), and the trace of the energy-momentum tensor is \( T = \rho - 3P \). It may be mentioned here that following choices for the matter Lagrangian (i) \( L_m = \rho \), (ii) \( L_m = -P \) and (iii) \( L_m = T \) are found in the literature. In geometry-matter coupling gravity theories namely, \( f(R, T) \) theory, an extra force acts along the orthogonal direction to the direction of four velocities for a (non-)geodesic motion. The extra force depends on the choice of the matter Lagrangian which however found
to vanish when \( L_m = -P \). The covariant derivative of the stress-energy tensor is given by,

\[
∇μT_{μν} = \frac{f_T[(T_{μν} + Θ_{μν})∇μlnf_T + ∇μΘ_{μν} - \frac{1}{2} g_{μν}∇μT]}{8π - f_T}
\]

where \( f_T = \frac{∂f(R, T)}{∂T} \). It may be pointed out here that the covariant divergence of the stress-energy tensor \( T_{μν} \) in \( f(R, T) \) theory of gravity is not conserved unlike GR or \( f(R) \) theories of gravity. We consider here \( f(R, T) = R + αR^2 + λT^β \), where \( β \) is a constant. For \( β = 1 \) and \( β = \frac{1}{2} \) the gravitational Lagrangians are \( f(R, T) = f(R) + λT \) and \( f(R, T) = f(R) + λT^{\frac{1}{2}} \) respectively. It is important to note that a specific form of the \( f(R, T) \) Lagrangian guarantees conservation of the energy momentum tensor [63] corresponding to the EoS parameter \( ω \). We are motivated to choose the second form of \( f(T) \) in \( f(R, T) \)- gravity in the paper following Ref. [63].

### 3 Wormholes in \( f(R, T) \) gravity

In this section static spherically symmetric metric is considered for describing wormholes which is given by [4,48,64],

\[
ds^2 = e^{2φ(r)}dt^2 - e^{2γ(r)}dr^2 - r^2(dθ^2 + sin^2θdφ^2)
\]

where \( φ(r) \) denote the redshift function, and for the wormhole geometry we consider \( e^{2γ(r)} = \frac{1}{1 - \frac{b(r)}{r}} \), where \( b(r) \) is the shape function. The radial coordinate \( r \) in this case increases from a minimum value \( r_0 \) to \( ∞ \), \( r_0 \) being the WH throat radius. The WH metric must satisfy the following conditions as mentioned below:

- The range of radial coordinate \( r \) is, \( r_0 ≤ r ≤ ∞ \), with \( r_0 \) being the throat radius.
- The shape function \( b(r) \) satisfies the condition \( b(r_0) = r_0 \), at the throat and away from the throat i.e. for \( r > r_0 \) it must satisfy the constraint condition
  \[
  1 - \frac{b(r)}{r} > 0.
  \]

- For a physical flaring out condition, the shape function \( b(r) \) at the throat of a WH must satisfy, \( b(r_0) = r_0 \) and \( b'(r_0) < 1 \) with \( r_0 \) being the throat radius.
- For an asymptotic flatness of the spacetime geometry one obtains
  \[
  \frac{b(r)}{r} \to 0 \quad as \quad |r| → ∞
  \]

- At the throat \( r_0 \), the redshift function \( φ(r) \) must be a finite non-vanishing function.

For simplicity we assume here that the redshift function is a constant \( (φ \to 0) \) accommodating the asymptotic de Sitter or anti-de Sitter solution of the WH metric [4].
particular choice which is also known as the tidal force solution is acceptable as the redshift function $\phi(r)$ stays finite for all values of $r$ to evade an event horizon $[4]$. We consider two different models of $f(R, T)$ gravity to analyze WH solutions as follows.

3.1 Model-I: $f(R, T) = R + \alpha R^2 + \lambda T$

In $f(R, T) = R + \alpha R^2 + \lambda T$ gravity the covariant derivative of the energy momentum tensor in non-zero. It is usual to write the field equation (2) as an effective Einstein field equation as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{\text{eff}}$$  \hspace{1cm} (10)

where the effective stress-energy tensor is

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{f_R(R, T)} \left[ (8\pi + f_T(R, T)) T_{\mu\nu} + P g_{\mu\nu} f_T(R, T) \right]$$

$$+ \frac{1}{f_R(R, T)} \left[ \frac{1}{2} [f(R, T) - R f_R(R, T)] g_{\mu\nu} \right]$$

$$- \frac{1}{f_R(R, T)} \left[ (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R(R, T) \right].$$  \hspace{1cm} (11)

The effective stress energy tensor ($T_{\mu\nu}^{\text{eff}}$) is determined by the matter stress-energy tensor $T_{\mu\nu}$ and the curvature quantities originating from the $f(R, T)$ modified theory of gravity.

In $f(R, T) = R + \alpha R^2 + \lambda T$ theory of gravity, the covariant derivative of effective energy momentum tensor $T_{\mu\nu}^{\text{eff}}$ is zero, i.e.,

$$\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0.$$  \hspace{1cm} (12)

The geometry matter coupling in $f(R, T)$ theory leads to a non-vanishing four divergence of the energy momentum tensor hence it is the effective energy momentum tensor that represents the conservation equation. Thermodynamical interpretation of this geometry matter coupling has been studied by Harko [65] and it is pointed out that the non-conservation of the matter energy momentum tensor is related with irreversible matter creation process. It is also shown that the model parameters decides the creation pressure and the rate of particle production. It is pointed out that the types of particles created can not be predicted with certainty. Using quantum analogy, it was predicted that most of the particles created in such geometry matter coupling may be due to some scalar particles or bosons which in the cosmological scale may contribute to the dark matter content in the universe.
The effective field equations for the modified gravitational theory \( f(R, T) = R + \alpha R^2 + \lambda T \) are obtained for the wormhole metric given by eq. (10) as,

\[
\frac{b'}{r^2} = \frac{\left[ (8\pi + \frac{3\lambda}{2})\rho - \frac{\lambda(p_r + 2p_t)}{6} - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1},
\]

\[
\frac{b}{r^3} = \frac{\left[ -(8\pi + \frac{7\lambda}{6})p_r + \frac{\lambda}{2}(\rho - \frac{2p_t}{3}) - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1},
\]

\[
\frac{b'r - b}{2r^3} = \frac{\left[ -(8\pi + \frac{4\lambda}{3})p_t + \frac{\lambda}{2}(\rho - \frac{p_r}{3}) - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1},
\]

The Ricci scalar for the WH metric is given by,

\[
R = \frac{2b'}{r^2}.
\]

The components of the effective stress energy tensor \( T_{\mu\nu}^{\text{eff}} \) are determined as,

\[
\rho^{\text{eff}} = \frac{\left[ (8\pi + \frac{3\lambda}{2})\rho - \frac{\lambda(p_r + 2p_t)}{6} - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1},
\]

\[
p_r^{\text{eff}} = \frac{\left[ -(8\pi + \frac{7\lambda}{6})p_r + \frac{\lambda}{2}(\rho - \frac{2p_t}{3}) - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1},
\]

\[
p_t^{\text{eff}} = \frac{\left[ -(8\pi + \frac{4\lambda}{3})p_t + \frac{\lambda}{2}(\rho - \frac{p_r}{3}) - \frac{2ab^2}{r^4} \right]}{2\alpha R + 1}.
\]

In GR, a fundamental point in wormhole physics is the energy condition violations, thus it needs to be investigated in modified theories of gravity as the gravitational field equations differ from the classical relativistic Einstein equations. In the framework of modified gravity we consider the generalized energy conditions [66–70]. Thus the generalised NEC in this case is \( T_{\mu\nu}^{\text{eff}} K^\mu K^\nu < 0 \), where \( K^\mu \) is a null vector. The generalised NEC obtained here reduces to that in GR for \( \lambda = 0 \) and \( \alpha = 0 \). In the \( f(R, T) \)- theory, one obtains a different picture where in principle the matter stress energy tensor satisfies the standard NEC, i.e., \( T_{\mu\nu} K^\mu K^\nu \geq 0 \), while the generalized NEC is violated in order to ensure the flaring out condition. Using the field equations the generalized NEC can be written as,

\[
\rho^{\text{eff}} + p_r^{\text{eff}} = \frac{b + b'r}{r^3} < 0.
\]

The effective anisotropy is given by

\[
p_t^{\text{eff}} - p_r^{\text{eff}} = \frac{8\pi + \lambda}{1 + 2\alpha R} \Delta
\]
where $\Delta = p_r - p_t$ is the measure of anisotropy. It is evident that the effective anisotropic pressure vanishes if (i) $p_r = p_t$ or (ii) $\lambda = -8\pi$. For static spherically symmetric space-times in $f(R, T)$-gravity, the no go theorem of general relativity which excludes the existence of wormholes with flat and/or AdS asymptotic regions on both sides of the throat with isotropic source of matter [26] for $\phi + \gamma = 0$ in eq. (7) is also true. In addition to that $f(R, T)$-theory adds one more criteria that no-go theorem is also valid with anisotropic fluid source when $\lambda = -8\pi$. In this paper we obtain wormhole solutions for $\lambda \neq -8\pi$ with anisotropic fluid.

Using eqs. (13)–(16), energy density, radial pressure and tangential pressure can be rewritten as,

$$\rho = \frac{b'[\lambda(2r^2 + 11\alpha b') + 12\pi (r^2 + 6\alpha b')]}{3(\lambda + 4\pi)(\lambda + 8\pi)r^4} \quad (21)$$

$$p_r = \frac{X}{3(\lambda + 4\pi)(\lambda + 8\pi)r^3} \quad (22)$$

$$p_t = -\frac{Y}{6(\lambda + 4\pi)(\lambda + 8\pi)r^5} \quad (23)$$

where, $X = -3b(r^2 + 4b'\alpha)(4\pi + \lambda) + b'(r^2\lambda + b'\alpha(-24\pi + \lambda))$ and $Y = 3b(r^2 + 4b'\alpha)(4\pi + \lambda) - b' r(12\pi (r^2 + 8b'\alpha) + (r^2 + 10b'\alpha)\lambda)$. The EoS for tangential pressure is

$$p_t = \omega \rho \quad (24)$$

where, the equation of state parameter $\omega$ is a function of $r$. We consider here anisotropic fluid distribution where the radial and tangential pressures are different and the corresponding EoS is,

$$p_r = \omega_1 p_t \quad (25)$$

where $\omega_1$ is also a function of $r$. It may be mentioned here that wormhole solutions are obtained considering a hyperbolic function for $\omega_1$ in Ref. [36]. But it is not necessary to assume a functional form of the EoS parameter. As the field equations are highly non-linear, the EoS state parameters vary with $r$ in addition to its dependence on the coupling parameters $\alpha$ and $\lambda$ of the gravitational action.

There are three equations and four unknowns namely, $\rho$, $p_r$, $p_t$ and $b(r)$, therefore, we choose the shape functions for wormholes to obtain solutions. For a given shape function $b(r)$, we determine the matter necessary for wormhole solution.

The effective equation of state parameters represented in eqs. (24) and 25 are given by,

$$\omega = \frac{3b(r^2 + 4b'\alpha)(4\pi + \lambda) - b'r[12\pi (r^2 + 8b'\alpha) + (r^2 + 10b'\alpha)\lambda]}{2b' r[12\pi (r^2 + 6b'\alpha) + (2r^2 + 11b'\alpha)\lambda]} , \quad (26)$$
\[ \omega_1 = -\frac{2[3b(r^2 + 4b'\alpha)(4\pi + \lambda) + b'r[b'\alpha(24\pi - \lambda) - r^2\lambda]]}{3b(r^2 + 4b'\alpha)(4\pi + \lambda) - b'r[12\pi(r^2 + 8b'\alpha) + (r^2 + 10b'\alpha)\lambda]} \]  

(27)

For a given shape function the variation of the EoS parameters are determined with the coupling constants \( \alpha \) and \( \lambda \) of the action.

We consider two different shape functions: [35, 71] (I). \( b(r) = r_0 e^{\frac{1}{r} - \frac{r_0}{r}} \) and (II). \( b(r) = r_0^2 \frac{e^{r_0 - r}}{r} \), to study the WH energy conditions. The wormhole models considered here are asymptotically flat as for \( r \to \infty \), \( \left( 1 - \frac{b(r)}{r} \right) \to 1 \).

**Case I: Shape Function** : \( b(r) = r_0 e^{1 - \frac{r_0}{r}} \).

The shape function [71] satisfies all the criterion for accommodating wormholes as mentioned in Sect. 3. The throat radius of the WH is at \( r_0 = 0.5 \). Using eqs. (21)-(23) and (26), (27), we get \( \rho, p_r, \) and \( p_t \) as follows,

\[ \rho = \frac{e^{1 - \frac{r}{r_0}}(-2e^{r_0}r^2(6\pi + \lambda) + e\alpha(72\pi + 11\lambda))}{3r^4(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(28)

\[ p_r = -\frac{e^{1 - \frac{r}{r_0}}(e^{r_0}r^2(12\pi r_0 + (r + 3r_0)\lambda) + e\alpha(24\pi(r - 2r_0) - (r + 12r_0)\lambda))}{3r^3(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(29)

\[ p_t = \frac{e^{1 - \frac{r}{r_0}}(e^{r_0}r^2(12\pi(r + r_0) + (r + 3r_0)\lambda) - 2e\alpha(24\pi(2r + r_0) + (5r + 6r_0)\lambda))}{6r^3(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(30)

The energy-density, radial pressure and tangential pressure are functions of the coupling parameters \( \lambda \) and \( \alpha \) of the gravitational action. The energy conditions can be studied for a wide range of values of the parameters \( \lambda \) and \( \alpha \).

**Case II: Shape Function II** \( b(r) = r_0^2 \frac{e^{r_0 - r}}{r} \).

In this case the WH solutions are obtained for the shape function \( b(r) = r_0^2 \frac{e^{r_0 - r}}{r} \) [35]. The throat radius is found to be \( r_0 = 1 \). The energy density (\( \rho \)), radial pressure and tangential pressure are given by,

\[ \rho = \frac{e^{-2r + r_0}(1 + r)r_0^2(-2e^r r^4(6\pi + \lambda) + e^0(1 + r)r_0^2\alpha(72\pi + 11\lambda))}{3r^8(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(31)

\[ p_r = -\frac{e^{-2r + r_0}r_0^2(e^r r^4(12\pi + (4 + r)\lambda) + e^0(1 + r)r_0^2\alpha(24\pi(-1 + r) - (13 + r)\lambda))}{3r^8(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(32)

\[ p_t = -\frac{e^{-2r + r_0}r_0^2(-e^r r^4(12\pi(2 + r) + (4 + r)\lambda) + 2e^0(1 + r)r_0^2\alpha(24\pi(3 + 2r) + (11 + 5r)\lambda))}{6e^r(32\pi^2 + 12\pi\lambda + \lambda^2)} \]  

(33)
3.1.1 Physical analysis for model I

The radial variation of energy conditions namely, dominant energy condition (DEC), weak energy condition (WEC), null energy condition (NEC) and the strong energy condition (SEC) are plotted to examine the validity in both the cases. We probe the admissibility of matter from the validity of the following energy conditions in the entire geometry for wormholes:

1. DEC: \( \rho \geq |p_i| \),
2. WEC : \( \rho \geq 0; \rho + p_i > 0; \)
3. NEC: \( \rho + p_i \geq 0; \)
4. SEC: \( \rho + p_r + 2p_t \geq 0; \)

where \( i = r, t \).

In modified \( f(R, T) \) gravity the generalised energy conditions are also important which are to be tested for a stable and traversable WH solution. It has been shown in the context of modified theories of gravity that the matter threading the wormhole may satisfy the NEC, however the generalised NEC which involves the effective stress energy tensor may not be satisfied [66,72]. The generalised NEC is also studied in \( f(R) \) gravity [20], curvature matter coupling [73,74], conformal Weyl geometry [75], braneworld scenario [76] for wormhole existences. The generalised NEC is given by \( \rho_{eff} + p_{i(eff)} \geq 0 \) where \( i = r, t \). We study the energy conditions considering two different shape functions in the following.

**Case I**

The energy density (\( \rho \)), the radial pressure (\( p_r \)) and the tangential pressure (\( p_t \)) for the shape function \( b(r) = r_0 e^{1- \frac{|\alpha|}{\lambda}} \) given in Case I, are given in eqs. (28)–(30). Using these equations the Energy conditions WEC, NEC, DEC and SEC are analyzed.

The positivity of the energy density (\( \rho \)) at the throat gives rise to the following constraints between \( \alpha \) and \( \lambda \): (i) for \( \alpha > \left( \frac{2|\lambda|+6\pi}{(11|\lambda|+72\pi)} \right) \), (ii) \( \alpha > \frac{1}{6} \) with \( \lambda = 0 \), (iii) Both \( \alpha \) and \( \lambda \) are negative satisfying the inequalities \( |\alpha| < \left( \frac{2(||\lambda|+6\pi|)}{(72\pi-11|\lambda|)} \right) \) with \( 6\pi < |\lambda| < \frac{72\pi}{11}, \) (iv) For a positive \( \alpha \), the following limiting values are found \( \alpha < \left( \frac{2(||\lambda|+6\pi|)}{(11|\lambda|+72\pi)} \right) \) and \( \frac{72\pi}{11} < |\lambda| < 6\pi \).

The radial variation of the NECs are plotted in Fig. 1 for \( \alpha = 1 \) with different \( \lambda \). The plot shows that the tangential NEC, \( \rho + p_i \geq 0 \) is violated near the throat in all cases. It is evident that away from the throat NEC is valid for a short length and then again violated, but at infinite radial distance it obeys again. The radial variation of \( \rho - |p_r| \) is plotted in Fig. 2 for different \( \lambda \) and \( \alpha = 1 \). It is evident that at the throat it violates thereafter it obeys. For negative \( \lambda \) the transition is rapid for \( \lambda = -10 \) than \( \lambda = 15 \), for positive \( \lambda \), the transition is sharp but the duration of NEC violation region is shorter for smaller \( \lambda \) values. The radial variation of NECs for different \( \alpha \) is plotted in Fig. 3, it is found that NEC is violated at the throat but away from the throat it obeys. The violation of NEC implies presence of exotic matter at the throat of the WH described by shape function-I in \( f(R, T) \) gravity framework.

In Fig. 4 the radial variation of energy density is plotted for three different cases of \( \alpha \), it is found that \( \rho > 0 \) only for positive \( \alpha \). The generalised energy conditions
Fig. 1 $\rho + p_t$ with $\alpha = 1$ and different $\lambda$ for shape function I

Fig. 2 $\rho - |p_r|$ with $\alpha = 1$ and different $\lambda$ for shape function I

are plotted in Fig. 5. It is interesting to note that the generalised energy conditions depend on the choice of the shape function which are observed to be violated near the throat of the wormhole [66,72].

Case II
We consider here hybrid shape function given by $b(r) = r_0^2 e^{\frac{0-r}{r}}$ for wormhole solutions. The energy density ($\rho$) at the throat is positive when (i) for positive $\alpha > \left( \frac{2(\lambda + 6\pi)}{11\lambda + 72\pi} \right)$, (ii) $\alpha > \frac{1}{6}$ with $\lambda = 0$, (iii) Both $\alpha$ and $\lambda$ are negative satisfying the inequalities $|\alpha| < \left( \frac{2(|\lambda| - 6\pi)}{72\pi - 11|\lambda|} \right)$ with $6\pi < |\lambda| < \frac{72\pi}{11}$, (iv) $\alpha$ positive satisfying the inequalities $\alpha > \left( \frac{2(6\pi - |\lambda|)}{72\pi - 11|\lambda|} \right)$ and $\frac{72\pi}{11} < |\lambda| < 6\pi$.

We note that for a range of $\lambda$ values the energy density, $\rho$ remains positive. For $\alpha = 1$, $\rho$ is positive for $\lambda > -10$. There is a small range of negative $\lambda$ values for
Morris–Thorne wormholes in modified $f(R, T)$ gravity

**Fig. 3** Radial variation of $\rho + p_r$ (Solid line) and $\rho + p_t$ (Dot Dashed lines) with $\lambda = 1$ and different $\alpha$ for shape function I

**Fig. 4** Radial variation of Energy Density ($\rho$) with $\lambda = 1$ with different $\alpha$ for shape function I

which it is positive thereafter it becomes negative again. For $\lambda = 1$, energy density is always positive when $\alpha > 0$ but it becomes negative for $\alpha \leq 0$. Thus wormhole models will be studied in this case with positive $\alpha$ for different values of $\lambda$. We note that wormholes solutions exist with normal matter when $\alpha > 0$ and $\lambda = 0$, this result is different from that obtained in Ref. [77].

We consider $\alpha = 1$ and $\lambda = 5$ to plot all the energy conditions in Fig. 6. All the energy conditions except the SEC are found to satisfy. However, for a large value say, $\lambda = 80$, we plot all the energy conditions in Fig. 7, it is clear from the plot that all the energy conditions are satisfied accommodating wormholes. This is interesting as traversable wormhole are permitted with normal matter in the modified gravity for this specific shape function. We note that SEC is violated at small values of $\lambda$ but it is valid for $\lambda > 75$ with $\alpha = 1$. 
The radial variation of the generalised energy conditions are plotted in Fig. 8 for the hybrid shape function. The generalised ECs are violated near the throat of the wormhole as found in the previous case.

Thus it comes out that existence of wormholes in the modified gravity depends on the coupling parameters (viz. $\alpha$ and $\lambda$) and we note two different cases: (1) NEC is satisfied, SEC violated and (2) both NEC and SEC are satisfied simultaneously signalling wormhole solutions with normal matter.
3.2 Model-II: $f(R, T) = R + \alpha R^2 + \lambda T^2$

In this section, we consider the functional form of $f(R, T)$ to be $f(R, T) = R + \alpha R^2 + \lambda T^2$. Alvarenga et al. [63] using scalar perturbation in the metric formalism have shown that some specific models of $f(R, T)$ gravity can satisfy the energy conservation equation which is in general violated. Shabani and Farhoudi [78] have investigated cosmological solutions in the $f(R, T)$ gravity framework for perfect fluid in spatially FLRW metric through phase space analysis considering such a non-linear
form of $f(T)$. We consider such a non-linear function of $T$ to study the WH solutions for the two shape functions.

The field equations in this case are given by,

$$\frac{b'}{r^2} = \frac{(8\pi + \frac{\lambda}{2\sqrt{T}} + \frac{\lambda}{2})\rho + (\frac{\lambda}{6\sqrt{T}} - \frac{\lambda}{2})(p_r + 2p_t) - \frac{2ab'^2}{r^4}}{2\alpha R + 1}, \quad (34)$$

$$\frac{b}{r^3} = \frac{- (8\pi + \frac{\lambda}{2} + \frac{\lambda}{2\sqrt{T}} - \frac{\lambda}{6\sqrt{T}})pr + \lambda(p + \frac{p_r}{2} - pr) - \frac{2ab'^2}{r^4}}{2\alpha R + 1}, \quad (35)$$

$$\frac{b'r - b}{2r^3} = \frac{- (8\pi + \frac{\lambda}{2\sqrt{T}} - \frac{\lambda}{3\sqrt{T}} + \lambda)p_t + \frac{\lambda}{2}(\rho + \frac{\lambda}{3\sqrt{T}} p_r - pr) - \frac{2ab'^2}{r^4}}{2\alpha R + 1}. \quad (36)$$

where, $T = \rho - p_r - 2p_t$, is the trace of the energy momentum tensor. The field equations are highly non-linear because of the presence of the $\sqrt{T}$ term, hence one cannot obtain exact analytical solutions for the energy density and pressures like the first model. We have used numerical techniques to find the solutions of the field equations considering two wormhole shape functions. The results are summarised in the following.

3.2.1 Physical analysis for model II

The radial variation of the energy conditions are plotted to examine the validity for both shape functions. Since the functional form of the energy density and pressure are not known, their dependence on the model parameters $\alpha$ and $\lambda$ are studied numerically.

Case I

We have plotted the energy conditions for $\lambda = 1$ and $\alpha = 1$ in Fig. 9. It is evident from the figure that the NEC corresponding to the tangential pressure is violated near the throat. Thus WH solutions cannot be obtained without exotic matters in this case. The SEC is also violated in this case. As the $\lambda$ value is increased keeping $\alpha$ fixed the energy density and pressures exhibit discontinuous behaviour. Another point to note here is that for $f(T) = \lambda T^{\frac{1}{2}}$ model numerical solutions cannot be obtained for $\alpha \leq 0$.

Case II

In Fig. 10 we have shown the radial variation of the energy conditions for the hybrid shape function considering $\lambda = 1$ and $\alpha = 1$. It is seen from the figure that the NEC for the radial as well as transverse pressure are satisfied throughout in this case. The SEC is also violated near the throat of the wormhole. We note that as $\lambda$ increased keeping $\alpha$ fixed the NEC and SEC are violated over a range of radial co-ordinates. In this case also numerical solutions cannot be obtained for negative $\alpha$ values which differs from the first $f(R, T)$ model.

Thus from this analysis we conclude that the choice of the shape function plays an important role in deciding the type of matter content threading the wormhole. In both $f(R, T)$ models with linear and non-linear form of $f(T)$, the NEC corresponding to
the transverse pressure is violated for shape function-I whereas it is satisfied for shape function-II. We have summarised the results in the next section.

4 Results and discussion

In the paper wormhole solutions are obtained in the modified $f(R, T) = R + \alpha R^2 + f(T)$ gravity with static spherically symmetric metric. We have considered two differ-
ent forms of $f(T)$ viz. $f(T) = \lambda T$ and $f(T) = \lambda T^{\frac{1}{2}}$. The field equations are highly non-linear and intractable in known forms, we have studied the energy density, radial and transverse pressures of the wormhole solutions for a given shape function. As the equation of state of matter for wormhole is not known, the energy conditions are studied in the theoretical framework with gravitational coupling parameters making use of known shape functions. The results are as follows:

4.1 Model-I: $f(R, T) = R + \alpha R^{\frac{1}{2}} + \lambda T$ gravity

For $b(r) = r_0 e^{1-\frac{r}{r_0}}$:

- NEC corresponding to transverse pressure is plotted in Fig. 1 which shows that NEC is violated near the throat for $\alpha = 1$ and different $\lambda$. DEC corresponding to the radial pressure is also seen to be violated near the throat for different $\lambda$ keeping $\alpha$ fixed. For a given $\lambda$, NEC is checked with different $\alpha$ in Fig. 3 which shows that NEC is satisfied for $\alpha > 0$. It is evident from Fig. 4 that the energy density is positive for $\alpha > 0$. The generalised energy conditions are plotted in Fig. 5. We note that the generalised ECs which include the effective energy density and pressures are violated in this case. In this case WH exists in the presence of exotic matter.

- The radial variation of the effective EoS parameter in Fig. 11 show that for $\alpha = 0$ and $-1$ with $\lambda = 1$ the EoS parameter for the shape function I, is always negative. The EoS parameter for positive values of $\alpha$ with $\lambda = 1$ show that the effective EoS parameter is always negative with a discontinuity at,

$$e^{\frac{r}{r_0}-1}r^2 = \alpha \left( \frac{72\pi + 11\lambda}{12\pi + 2\lambda} \right).$$ (37)

Far away from the throat EoS parameter decreases ultimately attaining a value $\omega \to -0.6$. The effective EoS parameter remains negative at late times indicating the presence of exotic matter accommodating an accelerating universe. A detail numerical analysis for the shape function in the present model is displayed in Tables 1 and 2.

For $b(r) = \frac{r_0 e^{0-\frac{r}{r_0}}}{r}$:

- Radial variation of energy density is positive for $\alpha > 0$ with $\lambda = 1$ as shown in Fig. 6.

- We note that in this case NEC is valid from a small range of negative $\lambda$ to large positive values, thus existence of wormhole satisfying NEC over a wide range of values of $\lambda$ is possible.

- The radial variation of energy conditions for $\alpha = 1$ and $\lambda = 5$ plotted in Fig. 6 shows that the energy conditions except SEC are valid. But for $\alpha = 1$ and large $\lambda$ ($\lambda = 80$) as shown in Fig. 7, it is evident that all the energy conditions are obeyed right from the wormhole throat. We note that for $\alpha < 0$ all the energy conditions are violated. Thus wormhole solution with normal matter at the throat is permitted.
for the hybrid shape function in \( f(R, T) = R + \alpha R^\frac{1}{2} + \lambda T \) gravity with positive \( \alpha \).

- The radial variation of the effective EoS parameter for different \( \alpha \) is plotted in Fig. 12. The effective EoS is found to be always positive with \( \lambda = 1 \). There is a minimum for \( \alpha < 0 \), then it increases, attains a maximum thereafter decreases slowly. For \( \alpha > 0 \) it is evident that \( \omega \) decreases from a positive to a negative value with a discontinuity at,

\[
\frac{2e^{r-r_0}r^4}{(1+r)r_0^2} = \alpha \left( \frac{72\pi + 11\lambda}{6\pi + \lambda} \right).
\]  

(38)
Table 1 Summary of results for $b(r) = r_0 e^{1 - \frac{r}{\lambda}}$ with $\alpha = 2$ and different $\lambda$.

| Terms | $\lambda = 1$ | $\lambda = -1$ | $\lambda = -30$ |
|-------|---------------|----------------|-----------------|
| $\rho$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_t$ | $< 0$ for $r \in (0, 0.5)$ | $< 0$ for $r \in (0, 0.5)$ | $< 0$ for $r \in (0, 1.28)$ |
| $\rho - |p_r|$ | $> 0$ for $r \in (0.5, 1.25)$ | $> 0$ for $r \in (0.5, 1.3)$ | $> 0$ for $r \in (1.28, \infty)$ |
| $\rho - |p_t|$ | $\leq 0$ for $r \in (1.25, \infty)$ | $\leq 0$ for $r \in (1.3, \infty)$ | $\leq 0$ for $r \in (1.3, \infty)$ |

Table 2 Summary of results for $b(r) = r_0 e^{1 - \frac{r}{\lambda}}$ with $\lambda = 1$ and different $\alpha$.

| Terms | $\alpha > 0$ | $\alpha < 0$ |
|-------|-------------|-------------|
| $\rho$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_t$ | $< 0$ for $r \in (0, 0.5)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho - |p_r|$ | $> 0$ for $r \in (0.5, 1.25)$ | $\leq 0$ for $r \in (0, 1.3)$ |
| $\rho - |p_t|$ | $\leq 0$ for $r \in (1.3, \infty)$ | $\leq 0$ for $r \in (1.3, \infty)$ |

A detail analysis of the models for obtaining wormholes for shape functions II is displayed in Tables (3) and (4).

- For $\lambda = 0$, the modified gravity reduces to the $f(R)$-gravity and existence of wormhole is studied considering $f(R) = R + \alpha R^n$ theory of gravity for a given shape function in Ref [77]. In the present paper we consider $n = 2$ and different energy conditions are examined to obtain wormhole solutions. It is found that wormhole solutions with exotic and non-exotic matter are permitted for different values of the parameters $\alpha$ and $\lambda$. The results obtained here are different from that derived in GR where energy conditions are violated indicating requirement of exotic matter. This is an interesting result which leads to a distinction between the GR and the modified theories of gravity.

- In the $f(R, T)$- gravity, wormhole solutions obtained with the shape function in Ref. [48] is different from that considered in this paper. We note that there exists a class of wormholes with normal matter in our model with a wide range of positive
Table 3  Summary of results for $b(r) = r_0^2 e^{\alpha - r} / r$ with $\alpha = 1$ and different $\lambda$

| Terms | $\lambda = 1$ | $\lambda = 76$ | $\lambda = -13$ |
|-------|--------------|----------------|-----------------|
| $\rho$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ |
| $\rho + p_t$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | Violated for a range of $r$ |
| $\rho - |p_r|$ | $\geq 0$ for $r \in (0, \infty)$ | Violated for small $r$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho - |p_t|$ | $\geq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r + 2p_t$ | $\leq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ |

Table 4  Summary of results for $b(r) = r_0^2 e^{\alpha - r} / r$ with $\lambda = 1$ and different $\alpha$

| Terms | $\alpha > 0$ | $\alpha < 0$ |
|-------|--------------|--------------|
| $\rho$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_t$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho - |p_r|$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho - |p_t|$ | $\geq 0$ for $r \in (0, \infty)$ | $\leq 0$ for $r \in (0, \infty)$ |
| $\rho + p_r + 2p_t$ | $\leq 0$ for $r \in (0, \infty)$ | $\geq 0$ for $r \in (0, \infty)$ |

Fig. 13  NEC and DEC, with $\alpha = 10$ and $\lambda = 0$ for shape function II. Here $\rho + p_r > 0$ (Blue), $\rho > |p_r|$ (Green) and $\rho > |p_t|$ (Purple)

as well as negative values of $\lambda$. Thus the choice of shape function is important as in Ref. [48], wormholes are obtained for negative values of $\lambda$ where NEC for radial pressure is obeyed but the other two energy conditions namely, NEC for transverse pressure and DEC for radial pressure are violated, for positive $\lambda$, all the energy conditions are violated, a different result from that is obtained in our model with shape function II. In Fig. 13 the plots of NEC and DEC with $\lambda = 0$ and $\alpha = 10$ are shown for shape function II. The energy conditions are satisfied in this case. Thus wormholes with normal matter can be obtained in $f(R)$ model also.
4.2 Model-II: \( f(R, T) = R + \alpha R^\frac{1}{2} + \lambda T^\frac{1}{2} \) gravity:

We have obtained WH solutions in the \( f(R, T) = R + \alpha R^\frac{1}{2} + \lambda T^\frac{1}{2} \) model considering two shape functions. The motivation behind the choice of such a non-linear material correction term lies in the fact that such specific forms of \( f(T) \) can lead to the conservation of energy-momentum which is otherwise violated [63,78]. Thus the effective energy-momentum tensor is not considered in this case. The field equations obtained in eqs. (34)–(36) are highly non-linear and exact analytical solutions cannot be obtained. So using numerical techniques we have studied the energy conditions satisfied by the matter stress energy tensor \( T_{\mu\nu} \). The results are as follows.

For \( b(r) = r_0 e^{\frac{1}{r_0} - r} \):

- From the radial variation of the energy conditions as seen in Fig. 9 it is evident that the NEC corresponding to the transverse pressure is violated in this case with \( \lambda = 1 \) and \( \alpha = 1 \). If we move away from the GR domain by increasing \( \lambda \) solutions exhibit discontinuous nature. Another thing to note here is that in this model the analysis fails for \( \alpha \leq 0 \) which is different from model-I. Thus for shape function-I, WHs cannot be constructed without exotic matter in this model also.

For \( b(r) = r^2_0 e^{\frac{r}{r_0} - r} \):

- In case of the hybrid shape function, the ECs are plotted in Fig. 10. All the ECs except the SEC are satisfied. So we conclude that in this case one can construct WHs without exotic matter as the NECs are satisfied throughout the WH. In this case also the field equations cannot be solved for negative \( \alpha \). Increasing \( \lambda \) keeping \( \alpha \) fixed leads to similar behaviour. The choice of shape function is crucial as in both the \( f(R, T) \) models we see that shape function-II satisfies the NEC which is violated in case of shape function-I.

The absence of observational signature of wormholes made the geometrical and the material properties of these objects unpredictable, therefore, wormholes are assigned with a shape function and EoS. Recently De Faciaco et. al. [79] developed a model-independent procedure to single out static and spherically symmetric wormhole solutions based on the general relativistic Poynting-Robertson effect and studied the extension of the ray-tracing formalism in generic static and spherically symmetric wormhole metrics. Simulating the flux emitted by the Poynting-Robertson critical hypersurface they have reconstructed wormhole solutions to the emission region, which are in agreement with the high-energy astrophysical observational data. This method can give valuable insight if wormhole evidences are detected. As the Poynting-Robertson critical hypersurfaces can be located in regions of strong gravitational field, they can behave as an astrophysical probe to observationally search for the existence of WHs [80]. However this method cannot be applied in case of wormholes with constant redshift function \( (\phi) \) which is considered in this paper. The Poynting-Robertson effect in case of modified \( f(R, T) \) gravity considering a varying redshift function will be the subject of future investigations.

It is generally believed that the present universe emerged out from an inflationary phase at an early era thereafter it enters into present accelerating phase followed by matter domination. Another interesting scenario called emergent universe [81,82] where the universe begins from a static Einstein Universe which transits to other
phases and encompasses finally the present observed universe may be realized in the framework of wormholes geometry. The wormhole throat may be considered as the seed of the early static Einstein phase and away from the throat the late accelerating phase may be realized which will be discussed elsewhere.

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References

1. Wheeler, J.A.: Phys. Rev. 97, 511 (1959)
2. Misner, C.W., Wheeler, J.A.: Ann. Phys. 2, 525 (1957)
3. Morris, M.S., Thorne, K.S., Yurtsewe, U.: Phys. Rev. Lett. 61, 1446 (1988)
4. Morris, M.S., Thorne, K.S.: Am. J. Phys. 56, 395 (1988)
5. Visser, M.: Phys. Rev. D 47, 554 (1993)
6. Bertolami, O., Ferreira, R.Z.: Phys. Rev. D 85, 104050 (2012)
7. Hayward, S.A.: Int. J. Mod. Phys. D 8, 373 (1999)
8. Kardashev, N.S., Novikov, I.D., Shatskiy, A.A.: Int. J. Mod. Phys. D 16, 909 (2007)
9. Kuhfittig, P.K.F.: Schol. Res. Exch 16, 296158 (2008)
10. Sushkov, S.V., Zaslavskii, O.B.: Phys. Rev. D 79, 067502 (2009)
11. Hochberg, D., Visser, M.: Phys. Rev. Lett 81, 746 (1998)
12. D. Hochberg, M. Visser, arXiv:gr-qc/9901020
13. Hochberg, D., Visser, M.: Phys. Rev. D 58, 044021 (1998)
14. Josset, T., Perez, A., Sudarsky, D.: Phys. Rev. Lett. 118, 021102 (2017)
15. Sushkov, S.V.: Phys. Rev. D 71, 043520 (2005)
16. Wang, D., Meng, X.H.: Eur. Phys. J. C 76, 171 (2016)
17. Starobinsky, A.A.: Phys. Lett. B 91, 99 (1980)
18. Pavlovic, P., Sossich, M.: Eur. Phys. J. C 75, 117 (2015)
19. Mazharimousavi, S.H., Halilsoy, M.: Mod. Phys. Lett. A 31, 1650192 (2016)
20. Lobo, F.S.N., Oliveira, M.A.: Phys. Rev. D 80, 104012 (2009)
21. Eiroa, E.F., Aguirre, G.F.: Eur. Phys. J. C 76, 132 (2016)
22. Sahoo, P.K., Moraes, P.H.R.S., Sahoo, P., Ribeiro, G.: Int. J. Mod. Phys. D 27, 1950004 (2019)
23. Godani, N., Samanta, G.C.: Int. J. Mod. Phys. D 28, 1950039 (2018)
24. Bronnikov, K.A., Starobinsky, A.A.: JETP Lett. 85, 1 (2007)
25. Bronnikov, K.A., Skvortsova, M.V., Starobinsky, A.A.: Grav. Cosmol. 16, 216 (2010)
26. Bronnikov, K.A., Balevskikh, K.A., Skvortsova, M.V.: Phys. Rev. D 96, 124039 (2017)
27. Hanko, T., et al.: Phys. Rev. D 84, 024020 (2011)
28. Houndjo, M.J.S., et al.: Int. J. Mod. Phys. D 21, 1250003 (2012)
29. Shabani, H., Farhoudi, M.: Phys. Rev. D 90, 044031 (2014)
30. Shabani, H., Ziae, A.H.: Eur. Phys. J. C 77, 31 (2017)
31. Azizi, T.: Int. J. Theo. Phys. 52, 3486 (2013)
32. Zubair, M., Waheed, S., Ahmad, Y.: Eur. Phys. J. C 76, 444 (2016)
33. Sahoo, P., Kirschner, A., Sahoo, P.K.: Mod. Phys. Lett. A 34, 1950303 (2019)
34. Zubair, M., Mustafa, G., Waheed, S., Abbas, G.: Eur. Phys. J. C 77, 680 (2017)
35. Mandal, S., Sahoo, P., Sahoo, P.K.: New Astron. 80, 101421 (2020)
36. Godani, N., Samanta, G.C.: Chin. J. Phys. 62, 161 (2019)
37. Fu, X., et al.: Eur. Phys. J. C 68, 271 (2010)
38. Hendi, S.H., Momeni, D.: Eur. Phys. J. C 71, 1823 (2011)
39. Kaneda, S., Ketov, S.V.: Eur. Phys. J. C 76, 26 (2016)
40. Noureen, I., Zubair, M.: Eur. Phys. J. C 75, 62 (2015)
41. Noureen, I., et al.: Eur. Phys. J. C 75, 323 (2015)
42. Zubair, M., Noureen, I.: Eur. Phys. J. C 75, 265 (2015)
43. Starobinsky, A.A.: JETP Lett. 86, 157 (2007)
44. Eliis, J., et al.: Phys. Lett. B 732, 380 (2014)
45. Eliis, J., et al.: Phys. Rev. Lett 111, 111301 (2013)
46. Eliis, J., et al.: JCAP 10, 009 (2013)
47. Koshelev, A.S., et al.: JCAP 11, 067 (2016)
48. Sahoo, P.K., Moraes, P.H.R.S., Sahoo, P.: Eur. Phys. J. C 78, 46 (2018)
49. Heydarzade, Y., Riazi, N., Moradpour, H.: Can. J. Phys. 93, 1523 (2015)
50. Moraes, P.H.R.S., Sahoo, P.K.: Eur. Phys. J. C 79, 677 (2019)
51. Sahoo P., Moraes P.H.R.S, Marcelo M. Lapola, Sahoo P.K.: arXiv:2012.00258 (2020)
52. Moraes, P.H.R.S., Sahoo, P.K.: Phys. Rev. D 97, 024007 (2018)
53. Moraes, P.H.R.S., et al.: Int. Jour. Mod. Phys. D 28(8), 1950098 (2019)
54. Moraes, P.H.R.S., Sahoo, P.K.: Phys. Rev. D 96, 044038 (2017)
55. Moraes, P.H.R.S., et al.: JCAP 07, 029 (2017)
56. Herrera, L., Santos, N.O.: Phys. Rep. 286, 53 (1997)
57. Gimeno-Soler, S., Font, J.A.: arXiv: 1707.03867
58. Shaikh, R.: Phys. Rev. D 92, 024015 (2015)
59. DeBenedictis, A.: Phys. Rev. D 84, 104030 (2011)
60. Kovisto, T.: Class. Quant. Grav. 23, 4289 (2006)
61. Kovisto, T.: It Phys. Rev. D 83, 101501 (2011)
62. Capozziello, S., et al.: Phys. Rev. D 86, 127504 (2012)
63. Alvarenga, F.G., et al.: Phys. Rev. D 87, 129905 (2013)
64. Visser, M.: Lorentzian wormholes: From Einstein to Hawking. AIP Press, New York (1995)
65. Harko, T.: Phys. Rev. D 90, 044067 (2014)
66. Harko, T., et al.: Phys. Rev. D 87, 067504 (2013)
67. Capozziello, S., et al.: Phys. Rev. D 91, 124019 (2015)
68. Capozziello, S., et al.: Phys. Lett. B 730, 280–283 (2014)
69. Alvarenga, F.G., et al.: J. Mod. Phys. 4, 1 (2013)
70. Rajabi, F., Nozari, K.: Eur. Phys. J. C 81, 247 (2021)
71. Moraes, P.H.R.S., Sahoo, P.K., Kulkarni, S.S., Agarwal, S.: Chin. Phys. Lett 36, 120401 (2019)
72. Lobo, F.S.N.: AIP Conf. Proc. 1458, 447 (2012)
73. Garcia, N.M., Lobo, F.S.N.: Phys. Rev. D 82, 104018 (2010)
74. Garcia, N.M., Lobo, F.S.N.: Class. Quantum Grav. 28, 085018 (2011)
75. Lobo, F.S.N.: Class. Quant. Grav. 25, 175006 (2008)
76. Lobo, F.S.N.: Phys. Rev. D 75, 064027 (2007)
77. Godani, N., Samanta, G.C.: Eur. Phys. J. C 80, 30 (2020)
78. Shabani, H., Farhoudi, M.: Phys. Rev. D 79, 044048 (2013)
79. De Falco, V., et al.: Phys. Rev. D 103, 044007 (2021)
80. De Falco, V., et al.: Phys. Rev. D 101, 104037 (2020)
81. Mukherjee, S., Paul, B.C., Dadhich, N.K., Maharaj, S.D., Beesham, A.: Class. Q. Grav. 23, 6927 (2006)
82. Paul, B.C., Majumdar, A.S.: Class. Quantum Grav. 35, 065001 (2018)

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