Asymptotic behavior of extremals for fractional Sobolev inequalities associated with singular problems

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Abstract

Let $\Omega$ be a smooth, bounded domain of $\mathbb{R}^N$, $\omega$ be a positive, $L^1$-normalized function, and $0 < s < 1 < p$. We study the asymptotic behavior, as $p \to \infty$, of the pair $(\sqrt[p]{\Lambda_p}, u_p)$, where $\Lambda_p$ is the best constant $C$ in the Sobolev type inequality

$$C \exp \left( \int_{\Omega} (\log |u|^p) \omega dx \right) \leq [u]_{s,p}^p \quad \forall \ u \in W_0^{s,p}(\Omega)$$

and $u_p$ is the positive, suitably normalized extremal function corresponding to $\Lambda_p$. We show that the limit pairs are closely related to the problem of minimizing the quotient $|u|_s / \exp \left( \int_{\Omega} (\log |u|) \omega dx \right)$, where $|u|_s$ denotes the $s$-Hölder seminorm of a function $u \in C_0^{0,s}(\Omega)$.

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1 Introduction

Let $\Omega$ be a smooth (at least Lipschitz) domain of $\mathbb{R}^N$ and consider the fractional Sobolev space

$$W_0^{s,p}(\Omega) := \left\{ u \in L^p(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } [u]_{s,p} < \infty \right\}, \quad 0 < s < 1 < p,$$
where

$$[u]_{s,p} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$ 

It is well-known that the Gagliardo seminorm $[\cdot]_{s,p}$ is a norm in $W^{s,p}_0(\Omega)$ and that this Banach space is uniformly convex. Actually,

$$W^{s,p}_0(\Omega) = C_c^\infty(\Omega)^{[\cdot]_{s,p}}.$$ 

Let $\omega$ be a nonnegative function in $L^1(\Omega)$ satisfying $\|\omega\|_{L^1(\Omega)} = 1$ and define

$$\mathcal{M}_p := \left\{ u \in W^{s,p}_0(\Omega) : \int_{\Omega} (\log |u|) \omega \, dx = 0 \right\}$$

and

$$\Lambda_p := \inf \left\{ [u]_{s,p}^p : u \in \mathcal{M}_p \right\}. \quad (1)$$

In the recent paper [9] is proved that $\Lambda_p > 0$ and that

$$\Lambda_p \exp \left( \int_{\Omega} (\log |u|^p) \omega \, dx \right) \leq [u]_{s,p}^p \quad \forall u \in W^{s,p}_0(\Omega), \quad (2)$$

provided that $\Lambda_p < \infty$. Moreover, the equality in this Sobolev type inequality holds if, and only if, $u$ is a scalar multiple of the function $u_p \in \mathcal{M}_p$ which is the only weak solution of the problem

$$\begin{cases}
(-\Delta_p)^s u = \Lambda_p u^{-1} \omega & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases} \quad (3)$$

Here, $(-\Delta_p)^s$ is the $s$-fractional $p$-Laplacian, formally defined by

$$(-\Delta_p)^s u(x) = -2 \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x)) (\varphi(x) - \varphi(y))}{|y - x|^{N+sp}} \, dy.$$ 

We recall that a weak solution of the equation in (3) is a function $u \in W^{s,p}_0(\Omega)$ satisfying

$$\langle (-\Delta_p)^s u, \varphi \rangle = \Lambda_p \int_{\Omega} u^{-1} \varphi \omega \, dx \quad \forall \varphi \in W^{s,p}_0(\Omega),$$

where

$$\langle (-\Delta_p)^s u, \varphi \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy$$

is the expression of $(-\Delta_p)^s$ as an operator from $W^{s,p}_0(\Omega)$ into its dual.
The purpose of this paper is to determine both the asymptotic behavior of the pair \((\sqrt[p]{\Lambda_p}, u_p)\), as \(p \to \infty\), and the corresponding limit problem of (3). In our study \(s \in (0, 1)\) is kept fixed.

After introducing, in Section 2, the notation used throughout the paper, we prove in Section 3 that \(\Lambda_p < \infty\) by constructing a function \(\xi \in C_0^1(\overline{\Omega}) \cap M_p\). In the simplest case \(\omega \equiv |\Omega|^{-1}\) this was made in [10] where the inequality (2) corresponding to the standard Sobolev Space \(W^{1,p}_0(\Omega)\) has been derived.

In Section 4, we show that the limit problem is closely related to the problem of minimizing the quotient

\[ Q_s(u) := \frac{|u|_s}{\exp(\int_\Omega (\log |u|) \omega dx)} \]

on the Banach space \(C^{0,s_0}(\overline{\Omega}), |\cdot|_s\) of the \(s\)-Hölder continuous functions in \(\overline{\Omega}\) that are zero on the boundary \(\partial \Omega\). Here, \(|u|_s\) denotes the \(s\)-Hölder seminorm of \(u\) (see (6)).

We prove that if \(p_n \to \infty\) then (up to a subsequence)

\[ u_{p_n} \to u_\infty \in C^{0,s_0}(\overline{\Omega}) \text{ uniformly in } \Omega, \quad \text{and} \quad \sqrt{\Lambda_{p_n}} \to |u_\infty|_s. \]

Moreover, the limit function \(u_\infty\) satisfies

\[ \int_\Omega (\log |u_\infty|) \omega dx \geq 0 \quad \text{and} \quad Q_s(u_\infty) \leq Q_s(u) \quad \forall u \in C^{0,s_0}(\overline{\Omega}) \setminus \{0\} \]

and the only minimizers of the quotient \(Q_s\) are the scalar multiples of \(u_\infty\).

One of the difficulties we face in Section 4 is that \(C_c(\Omega)\) is not dense in \(C^{0,s}(\overline{\Omega}), |\cdot|_s\). This makes it impossible to directly exploit the fact that \(u_p\) is a weak solution of (3). We overcome this issue by using a convenient technical result proved in [18, Lemma 3.2] and employed in [2] to deal with a similar approximation matter.

In Section 5, motivated by [3, 13, 17], we derive the limit problem of (3). Assuming that \(\omega\) is continuous and positive in \(\Omega\) we prove that \(u_\infty\) is a viscosity solution of

\[ \begin{cases} 
\mathcal{L}^- u + |u|_s = 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega 
\end{cases} \]

where

\[ (\mathcal{L}^- u)(x) := \inf_{y \in \mathbb{R}^N \setminus \{x\}} \left\{ \frac{u(y) - u(x)}{|y - x|^s} \right\}. \]

We also show \(u_\infty\) is a viscosity supersolution of

\[ \begin{cases} \mathcal{L}^- u = 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega 
\end{cases} \]

where

\[ \mathcal{L}_\infty := \mathcal{L}^- + \mathcal{L}^- \]

and

\[ (\mathcal{L}^+ u)(x) := \sup_{y \in \mathbb{R}^N \setminus \{x\}} \left\{ \frac{u(y) - u(x)}{|y - x|^s} \right\}. \]
This fact guarantees that \( u_\infty > 0 \) in \( \Omega \).

The existing literature on the asymptotic behavior (as \( p \to \infty \)) of solutions of problems involving the \( p \)-Laplacian is most focused on the local version of the operator, that is, on the problem

\[
\begin{cases}
-\Delta_p u = f(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]  

(4)

where \( \Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \) is the standard \( p \)-Laplacian. This kind of asymptotic behavior has been studied for at least three decades (see \([1,14,16]\)) and many new results, adding the dependence of \( p \) in the term \( f(x,u) \), are still being produced (see \([4–6,8]\)). The solutions of (4) are obtained in the natural Sobolev space \( W^{1,p}_0(\Omega) \) and an important property related to this space, crucial in the study of the asymptotic behavior of the corresponding family of solutions \( \{ u_p \} \), is the inclusion

\[
W^{1,p_2}_0(\Omega) \subset W^{1,p_1}_0(\Omega) \quad \text{whenever} \quad 1 < p_1 < p_2.
\]

It allows us to show that any uniform limit function \( u_\infty \) of the sequence \( \{ u_{p_n} \} \) (with \( p_n \to \infty \)) is admissible as a test function in the weak formulation of (4), so that \( u_\infty \) inherits certain properties of the functions of \( \{ u_{p_n} \} \).

Since the inclusion \( W^{s,p_2}_0(\Omega) \subset W^{s,p_1}_0(\Omega) \) does not hold when \( 0 < s < 1 < p_1 < p_2 \) (see \([19]\)) the asymptotic behavior, as \( p \to \infty \), of the solutions of the problem

\[
\begin{cases}
(-\Delta)^s u = f(x,u) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]  

(5)

is more difficult to be determined. For example, in the case considered in the present paper \( f(x,u) = \omega(x)/u \) we cannot ensure that the property

\[
\int_\Omega (\log |u_{p_n}|) \omega \, dx = 0
\]

is inherited by the limit function \( u_\infty \) (see Remark 12). Actually, we are able to prove only that

\[
\int_\Omega (\log u_\infty) \omega \, dx \geq 0.
\]

As a consequence, the limit functions of the family \( \{ u_p \} \) might not be unique.

The study of the asymptotic behavior, as \( p \to \infty \), of the solutions of (5) is quite recent and restricted to few works. In \([17]\) the authors considered \( f(x,u) = \lambda_p |u|^{p-2} u \) where \( \lambda_p \) is the first eigenvalue of the \( s \)-fractional \( p \)-Laplacian. Among other results, they proved that

\[
\lim_{p \to \infty} \frac{\sqrt{\lambda_p}}{p} = R^{-s},
\]

where \( R \) is the radius of the largest ball inscribed in \( \Omega \), and that limit function \( u_\infty \) of the family \( \{ u_p \} \) is a positive viscosity solution of

\[
\max \{ \mathcal{L}_\infty u , \mathcal{L}_-^{s} u + R^{-s} u \} = 0.
\]
The equation in (5) with \( f = 0 \) and under the nonhomogeneous boundary condition \( u = g \) in \( \mathbb{R}^N \setminus \Omega \) was first studied in [3]. It is shown that the limit function is an optimal \( s \)-Hölder extension of \( g \in C^{0,s}(\partial\Omega) \) and also a viscosity solution of the equation

\[
\mathcal{L}_\infty u = 0 \quad \text{in } \partial\Omega.
\]

Moreover, some tools for studying the behavior as \( p \to \infty \) of the solutions of (5) are developed there.

In [13], also under the boundary condition \( u = g \) in \( \mathbb{R}^N \setminus \Omega \), the cases \( f = f(x) \) and \( f = f(u) = |u|^\theta(p)-2 u \) with \( \Theta := \lim_{p \to \infty} \theta(p)/p < 1 \) are studied. In the first case, different limit equations involving the operators \( \mathcal{L}_\infty, \mathcal{L}_\infty^+ \) and \( \mathcal{L}_\infty^- \) are derived according to the sign of the function \( f(x) \), what resembles the known results obtained in [1], where the standard \( p \)-Laplacian is considered. For example, the limit function \( u_\infty \) is a viscosity solution of

\[
-\mathcal{L}_\infty^- u = 1 \quad \text{in } \{ f > 0 \}.
\]

As for the second case, the limit equation is

\[
\min \{ -\mathcal{L}_\infty^- u - u^\Theta, -\mathcal{L}_\infty^- u \} = 0
\]

which is consistent with the limit equation obtained in [4] for the standard \( p \)-Laplacian and \( f(u) = |u|^\theta(p)-2 u \) satisfying \( \Theta := \lim_{p \to \infty} \theta(p)/p < 1 \).

2 Notation

The ball centered at \( x \in \mathbb{R}^N \) with radius \( \rho \) is denoted by \( B(x, \rho) \) and \( \delta \) stands for the distance function to the boundary \( \partial\Omega \), defined by

\[
\delta(x) := \min_{y \in \partial\Omega} |x - y|, \quad x \in \overline{\Omega}.
\]

We recall that \( \delta \in C^{0,1}_0(\overline{\Omega}) \) and satisfies \( |\nabla \delta| = 1 \) a.e. in \( \Omega \). Here,

\[
C^{0,\beta}_0(\overline{\Omega}) := \{ u \in C^{0,\beta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \}, \quad 0 < \beta \leq 1,
\]

where \( C^{0,\beta}(\overline{\Omega}) \) is the well-known \( \beta \)-Hölder space endowed with the norm

\[
\|u\|_{0,\beta} = \|u\|_\infty + |u|_\beta
\]

with \( \|u\|_\infty \) denoting the sup norm of \( u \) and \( |u|_\beta \) denoting the \( \beta \)-Hölder seminorm, that is,

\[
|u|_\beta := \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|_\beta}.
\]

(6)
We recall that \( \left( C^0_0(\Omega), |\cdot|_\beta \right) \) is a Banach space. The fact that the \( \beta \)-Hölder seminorm \( |\cdot|_\beta \) is a norm in \( C^0_0(\Omega) \) equivalent to \( \|u\|_{0, \beta} \) is a consequence of the estimate

\[
\|u\|_{\infty} \leq |u|_\beta \|\delta\|_{\infty} \quad \forall u \in C^0_0(\Omega),
\]

which in turn follows from the following

\[
|u(x)| = |u(x) - u(y_x)| \leq |u|_\beta |x - y_x|^\beta = |u|_\beta \delta(x)^\beta \quad \forall x \in \Omega,
\]

where \( y_x \in \partial\Omega \) is such that \( \delta(x) = |x - y_x| \).

We also define \( C^\infty_c(\Omega) := \{ u \in C^\infty(\Omega) : \text{supp}(f) \subset \subset \Omega \} \)

where

\[
\text{supp}(u) := \{ x \in \Omega : u(x) \neq 0 \}
\]

is the support of \( u \) and \( X \subset \subset Y \) means that \( \overline{X} \) is a compact subset of \( Y \). Analogously, we define \( E_c \) if \( E \) is a space of functions (e.g. \( C_c(\mathbb{R}^N) \), \( C_c(\mathbb{R}^N; \mathbb{R}^N) \), \( C^0_0,\beta_c(\Omega) \)).

3 Finiteness of \( \Lambda_p \)

Let us recall the Federer’s co-area formula (see [12])

\[
\int_\Omega g(x) |\nabla f(x)| \, dx = \int_{-\infty}^\infty \left( \int_{f^{-1}(t)} g(x) \, d\mathcal{H}_{N-1} \right) \, dt,
\]

which holds whenever \( g \in L^1(\Omega) \) and \( f \in C^{0,1}(\Omega) \). (In this formula \( \mathcal{H}_{N-1} \) stands for the \( (N - 1) \)-dimensional Hausdorff measure).

In the particular case \( f = \delta \) the above formula becomes

\[
\int_\Omega g(x) \, dx = \int_0^{\|\delta\|_{\infty}} \left( \int_{\delta^{-1}(t)} g(x) \, d\mathcal{H}_{N-1} \right) \, dt. \tag{8}
\]

Proposition 1 Let \( \omega \in L^1(\Omega) \) such that

\[
\int_\Omega \omega \, dx = 1 \quad \text{and} \quad \omega \geq 0 \quad \text{a.e. in } \Omega. \tag{9}
\]

There exists a nonnegative function \( \xi \in C(\overline{\Omega}) \) that vanishes on the boundary \( \partial\Omega \) and satisfies

\[
\int_\Omega (\log|\xi|) \omega \, dx = 0.
\]

If, in addition,

\[
K_\epsilon := \text{ess} \sup_{0 \leq t \leq \epsilon} \int_{\delta^{-1}(t)} \omega \, d\mathcal{H}_{N-1} < \infty \tag{10}
\]

for some \( \epsilon > 0 \), then \( \xi \in C^{0,1}_0(\overline{\Omega}) \).
Proof. Let $\sigma : [0, \|\delta\|_{\infty}] \to [0, 1]$ be the $\omega$-distribution associated with $\delta$, that is,

$$\sigma(t) := \int_{\Omega_t} \omega dx, \quad t \in [0, \|\delta\|_{\infty}]$$

where

$$\Omega_t := \{ x \in \Omega : \delta(x) > t \}$$

is the $t$-superlevel set of $\delta$.

We remark that $\sigma$ is continuous at each point $t \in [0, \|\delta\|_{\infty}]$ since the $t$-level set $\delta^{-1}\{t\}$ has Lebesgue measure zero. This follows, for example, from the Lebesgue density theorem (see [11], where the distance function to a general closed set in $\mathbb{R}^N$ is considered).

Thus, there exists a nonincreasing sequence $\{t_n\} \subset [0, \|\delta\|_{\infty}]$ such that

$$\sigma(t_n) = 1 - \frac{1}{2^n}.$$ 

Now, choose a nondecreasing, piecewise linear function $\varphi \in C([0, \|\delta\|_{\infty}])$ satisfying

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(t_n) = \frac{1}{2^n},$$

and take the function

$$\xi_1 := \varphi \circ \delta \in C_0(\overline{\Omega}).$$

Taking into account that

$$t_{n+1} \leq \delta(x) \leq t_n \quad \text{a.e. } x \in \Omega_{t_{n+1}} \setminus \Omega_{t_n}$$

one has

$$\frac{1}{2^{n+1}} = \varphi(t_{n+1}) \leq \xi_1(x) \leq \varphi(t_n) = \frac{1}{2^n} \quad \text{a.e. } x \in \Omega_{t_{n+1}} \setminus \Omega_{t_n}.$$ 

Consequently,

$$\int_{\Omega} |\xi_1|^{\epsilon} \omega dx \geq \int_{\Omega_{t_1}} |\xi_1|^{\epsilon} \omega dx + \sum_{k=1}^{n-1} \int_{\Omega_{t_{k+1}} \setminus \Omega_k} |\xi_1|^{\epsilon} \omega dx$$

$$\geq \frac{1}{2^\epsilon} \int_{\Omega_{t_1}} \omega dx + \sum_{k=1}^{n-1} \frac{1}{2^{\epsilon(k+1)}} \int_{\Omega_{t_{k+1}} \setminus \Omega_k} \omega dx$$

$$= \frac{1}{2^\epsilon} \sigma(t_1) + \sum_{k=1}^{n} \frac{1}{2^{\epsilon(k+1)}} (\sigma(t_{k+1}) - \sigma(t_k))$$

$$= \frac{1}{2^\epsilon} \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2^{\epsilon(k+1)}} \frac{1}{2^{k+1}} = \sum_{k=1}^{n+1} \left(\frac{(1/2)^{\epsilon+1}}{2^k}\right)^k.$$
It follows that
\[
\lim_{\epsilon \to 0} \left( \frac{1}{2} \right)^{1/\epsilon} \left( \int_{\Omega} |\xi_1|^\epsilon \omega dx \right)^{1/\epsilon} \geq \lim_{\epsilon \to 0} \left( \frac{1}{2^{(1/2)^{\epsilon+1}}} \right)^{1/\epsilon} = \lim_{\epsilon \to 0} \left( \frac{(1/2)^{\epsilon+1}}{1 - (1/2)^{\epsilon+1}} \right)^{1/\epsilon} = \frac{1}{4}.
\]

Taking \(\xi := k \xi_1\) with
\[
k = \lim_{\epsilon \to 0} \left( \int_{\Omega} |\xi_1|^\epsilon \omega dx \right)^{-1/\epsilon}
\]
we obtain, by L'Hôpital's rule,
\[
1 = \lim_{\epsilon \to 0^+} \left( \int_{\Omega} |\xi|^\epsilon \omega dx \right)^{1/\epsilon} = \exp \left( \int_{\Omega} (\log |\xi|) \omega dx \right).
\]

Hence,
\[
\int_{\Omega} (\log |\xi|) \omega dx = 0.
\]

We now prove that \(\xi_1 \in C^{0,1} (\overline{\Omega})\) under the additional hypothesis (10). Since the nondecreasing function \(\varphi\) can be chosen such that \(\varphi'\) is bounded in any closed interval contained in \((0, \|\delta\|_{\infty})\), we can assume that \(\nabla \xi_1 \in L^\infty_{\text{loc}} (\Omega)\) (note that \(|\nabla \xi_1| = |\varphi' (\delta) \nabla \delta| = |\varphi' (\delta)|\) a.e. in \(\Omega\).

Thus, it suffices to show that the quotient
\[
Q(x, y) := \frac{|\xi_1(x) - \xi_1(y)|}{|x - y|}
\]
is bounded uniformly with respect to \(y \in \partial \Omega\) and \(x \in \Omega_{\epsilon}^c := \{ x \in \overline{\Omega} : \delta(x) \leq \epsilon \}\), where \(\epsilon\) is given by (10).

Let \(x \in \Omega_{\epsilon}^c\) and \(y \in \partial \Omega\) be fixed and choose \(n \in \mathbb{N}\) sufficiently large such that
\[
t_{n+1} < \delta(x) \leq t_n \leq \epsilon.
\]

Since \(\xi_1(y) = 0\) and \(\varphi\) is nondecreasing one has
\[
|\xi_1(x) - \xi_1(y)| = \xi_1(x) \leq \varphi(t_n) = \frac{1}{2^n}.
\]

Moreover, \(t_{n+1} < \delta(x) \leq |x - y|\).

Hence,
\[
Q(x, y) \leq \frac{1}{2^n t_{n+1}} \quad \text{whenever} \quad y \in \partial \Omega \quad \text{and} \quad x \in \Omega_{\epsilon}^c.
\]

Applying the co-area formula (8) with \(g = \omega\) and \(\Omega = \Omega_{t_{n+1}}^c\) we find
\[
\frac{1}{2^{n+1}} = \int_{\Omega_{\epsilon}^{t_{n+1}}} \omega dx = \int_0^{t_{n+1}} \left( \int_{\delta^{-1}(t)} \omega d\mathcal{H}_{N-1} \right) dt \leq K_\epsilon t_{n+1}.
\]
It follows that
\[ Q(x, y) \leq \frac{1}{2^{n+1}} \leq K \frac{2^{n+1}}{2^n} = 2K \quad \text{whenever } y \in \partial \Omega \text{ and } x \in \Omega_c, \]
concluding thus the proof that \( \xi_1 \in C^{0,1}(\bar{\Omega}) \). \( \blacksquare \)

**Remark 2** The estimate (11) can also be obtained from the Weyl’s Formula (see [15]) provided that \( \omega \) is bounded on an \( \epsilon \)-tubular neighborhood of \( \partial \Omega \).

In the remaining of this section \( \xi \) denotes the function obtained in Proposition 1 extended as zero outside \( \Omega \). So,
\[ \xi \in C^{0,1}(\bar{\Omega}) \quad \text{and} \quad \int_{\Omega} (\log |\xi|) \omega \, dx = 0. \]
Since \( C^{0,1}(\bar{\Omega}) \subseteq W^{1,p}_0(\Omega) \subseteq W^{s,p}_0(\Omega) \) we have \( \xi \in M_p \) (for a proof of the second inclusion see [11]). Therefore,
\[ \Lambda_p \leq [\xi]_{s,p}^{p} \quad \forall \, p > 1. \quad (12) \]

Combining (12) with the results proved in [9, Section 4] (which requires \( \omega \in L^r(\Omega) \), for some \( r > 1 \)) we have the following theorem.

**Theorem 3** Let \( \omega \) be a function in \( L^r(\Omega) \), for some \( r > 1 \), satisfying (9)-(10). For each \( p > 1 \), the infimum \( \Lambda_p \) in (1) is attained by a function \( u_p \in M_p \) which is the only positive weak solution of
\[ (-\Delta_p)^s u = \Lambda_p u^{-1} \omega, \quad u \in W^{s,p}_0(\Omega). \]

Summarizing,
\[ [u_p]_{s,p}^p = \Lambda_p := \min \left\{ [u]_{s,p}^p : u \in M_p \right\} \leq [\xi]_{s,p}^p \quad \forall \, p > 1, \quad (13) \]
and \( u_p \) is the unique function in \( W^{1,p}_0(\Omega) \) satisfying
\[ u_p > 0 \quad \text{in } \Omega \quad \text{and} \quad \langle (-\Delta_p)^s u_p, \phi \rangle = \Lambda_p \int_{\Omega} \omega(u_p)^{-1} \phi \, dx \quad \forall \, \phi \in W^{s,p}_0(\Omega). \]

We also have
\[ 0 < \sqrt[p]{\Lambda_p} \leq \frac{[u]_{s,p}}{\exp \left( \int_{\Omega} (\log |u|) \omega \, dx \right)} \quad \forall \, u \in W^{s,p}_0(\Omega), \]
since the quotient is homogeneous.

**Remark 4** It is worth pointing out that
\[ \int_{\Omega} (\log |u|) \omega \, dx = -\infty \quad (14) \]
for any function \( u \in L^\infty(\Omega) \) whose \( \text{supp } u \) is a proper subset of \( \text{supp } \omega \). Indeed, in this case we have
\[ 0 \leq \exp \left( \int_{\Omega} (\log |u|) \omega \, dx \right) = \lim_{t \to 0^+} \left( \int_{\Omega} |u|^t \omega \, dx \right)^{\frac{1}{t}} \leq \| u \|_{L^\infty} \lim_{t \to 0^+} \left( \int_{\text{supp } u} \omega \, dx \right)^{\frac{1}{t}} = 0. \]

Thus, if \( \omega > 0 \) almost everywhere in \( \Omega \) then (14) holds for every \( u \in C^\infty_c(\Omega) \setminus \{0\} \).
The asymptotic behavior as $p \to \infty$

In this section we assume that the weight $\omega$ satisfies the hypothesis of Theorem 3. Our goal is to relate the asymptotic behavior (as $p \to \infty$) of the pair $(p, u^p_p, u_p^p)$ with the problem of minimizing the homogeneous quotient $Q_s : C^{0,s}_0(\Omega) \setminus \{0\} \to (0, \infty)$ defined by

$$Q_s(u) := \frac{|u|^s}{k(u)} \quad \text{where} \quad k(u) := \exp \left( \int_\Omega (\log |u|) \omega \, dx \right).$$

Note that $k(u) = 0$ if, and only if, $u$ satisfies (14). In particular, according to Remark 4,

$$\omega > 0 \text{ a.e. in } \Omega \implies Q_s(u) = \infty \quad \forall \ u \in C^\infty_c(\Omega) \setminus \{0\}.$$

We also observe that

$$0 \leq k(u) \leq \int_\Omega |u| \omega \, dx < \infty \quad \forall \ u \in C^{0,s}_0(\Omega) \setminus \{0\}, \quad (15)$$

where the second inequality is consequence of the Jensen’s inequality (since the logarithm is concave):

$$\int_\Omega (\log |u|) \omega \, dx \leq \log \left( \int_\Omega |u| \omega \, dx \right). \quad (16)$$

Now, let us define

$$\mu_s := \inf_{u \in C^{0,s}_0(\Omega) \setminus \{0\}} Q_s(u).$$

Thanks to the homogeneity of $Q_s$ we have

$$\mu_s = \inf_{u \in M_s} |u|^s,$$

where

$$M_s := \{ u \in C^{0,s}_0(\Omega) : k(u) = 1 \}.$$

Combining (15) and (17) we obtain

$$1 \leq \int_\Omega |u| \omega \, dx \leq |u|^s \int_\Omega \delta^s \omega \, dx \quad \forall \ u \in M_s,$$

what yields the following positive lower bound to $\mu_s$

$$\left( \int_\Omega \delta^s \omega \, dx \right)^{-1} \leq \mu_s.$$

In the sequel we show that $\mu_s$ is in fact a minimum, attained at a unique nonnegative function. Before this, let us make an important remark.
Remark 5 If \( v \) minimizes \(|\cdot|_s\) in \( \mathcal{M}_s \) the same holds for \(|v|\), since the function \( w = |v| \) belongs to \( \mathcal{M}_s \) and satisfies \(|w|_s \leq |v|_s\).

Proposition 6 There exists a unique nonnegative function \( v \in \mathcal{M}_s \) such that
\[ \mu_s = |v|_s. \]

Proof. Let \( \{v_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_s \) be such that
\[ \lim_{n \to \infty} |v_n|_s = \mu_s. \] (17)
Since the function \( w_n = |v_n| \) belongs to \( \mathcal{M}_s \) and satisfies \(|w_n|_s \leq |v_n|_s\) we can assume that \( v_n \geq 0 \) in \( \Omega \).

It follows from (17) that \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( C^{0,s}_{0,0}(\Omega) \). Hence, the compactness of the embedding \( C^{0,s}_{0,0}(\Omega) \hookrightarrow C_0(\Omega) \) allows us to assume (by renaming a subsequence) that \( \{v_n\}_{n \in \mathbb{N}} \) converges uniformly to a function \( v \in C_0(\Omega) \). Of course, \( v \geq 0 \) in \( \Omega \).

Letting \( n \to \infty \) in the inequality
\[ |v_n(x) - v_n(y)| \leq |v_n|_s |x - y|^s \quad \forall x, y \in \overline{\Omega} \]
and taking (17) into account we obtain
\[ |v(x) - v(y)| \leq \mu_s |x - y|^s \quad \forall x, y \in \overline{\Omega}. \]
This implies that \( v \in C^{0,s}_{0,0}(\Omega) \) and
\[ |v|_s \leq \mu_s. \] (18)

Thus, to prove that \( \mu_s = |v|_s \) it suffices to verify that \( v \in \mathcal{M}_s \). Since
\[ 1 = k(v_n) = \lim_{\epsilon \to 0^+} \left( \int_{\Omega} |v_n|^\epsilon \omega dx \right)^\frac{1}{\epsilon} \leq \left( \int_{\Omega} |v_n|^t \omega dx \right)^\frac{1}{t} \quad \forall t > 0 \]
the uniform convergence \( v_n \to v \) yields
\[ 1 \leq \left( \int_{\Omega} |v|^t \omega dx \right)^\frac{1}{t} \quad \forall t > 0. \]
Hence,
\[ 1 \leq \lim_{t \to 0^+} \left( \int_{\Omega} |v|^t \omega dx \right)^\frac{1}{t} = k(v). \]

Thus, noticing that \( (k(v))^{-1}v \in \mathcal{M}_s \) and taking (18) into account we obtain
\[ \mu_s \leq |(k(v))^{-1}v|_s = (k(v))^{-1} |v|_s \leq |v|_s \leq \mu_s. \]
Therefore, \( k(v) = 1, v \in \mathcal{M}_s \) and \(|v|_s = \mu_s\).
Now, let \( u \in \mathcal{M}_s \) be a nonnegative minimizer of \(|.|_s\) and consider the convex combination

\[
w := \theta u + (1 - \theta)v \quad \text{with} \quad 0 < \theta < 1.
\]

Since the logarithm is a concave function, we have

\[
\int_{\Omega} (\log w) \omega dx \geq \int_{\Omega} (\theta \log(u) + (1 - \theta) \log(v)) \omega dx
\]

\[
= \theta \int_{\Omega} (\log u) \omega dx + (1 - \theta) \int_{\Omega} (\log v) \omega dx = 0.
\]

This implies that \( c^{-1}w \in \mathcal{M}_s \) where \( c := k(w) \geq 1 \).

Hence,

\[
\mu_s \leq c^{-1} |w|_s \leq |w|_s \leq \theta |u|_s + (1 - \theta) |v|_s = \theta \mu_s + (1 - \theta) \mu_s = \mu_s.
\]

It follows that \( c = 1 \) and the convex combination \( w \) minimizes \(|.|_s\) in \( \mathcal{M}_s \). Consequently,

\[
0 = \int_{\Omega} [\log(\theta u + (1 - \theta)v)] \omega dx \geq \int_{\Omega} [\theta \log(u) + (1 - \theta) \log(v)] \omega dx = 0.
\]

Since the concavity of the logarithm is strict, one must have \( u = Cv \) for some positive constant \( C \). Taking account that \( 1 = k(u) = Ck(v) = C \), we have \( u = v \). \( \blacksquare \)

From now on, \( v_s \in \mathcal{M}_s \) denotes the only nonnegative minimizer of \(|.|_s\) on \( \mathcal{M}_s \), given by Proposition 6. The main result of this section, proved in the sequence, shows that if \( p_n \to \infty \) then a subsequence of \( \{u_{p_n}\}_{n \in \mathbb{N}} \) converges uniformly to a scalar multiple of \( v_s \), say \( u_\infty = k_\infty v_s \) where \( k_\infty \geq 1 \).

In the next section (see (37)) we show that \( u_\infty \) is strictly positive in \( \Omega \), implying thus that \( -v_s \) and \( v_s \) are the only minimizers of \(|.|_s\) on \( \mathcal{M}_s \). As consequence, the minimizers of \( Q_s \) on \( C^{0,s}_0(\Omega) \setminus \{0\} \) are precisely the scalar multiples of \( v_s \) (or, equivalently, the scalar multiples of \( u_\infty \)). Further, we derive an equation satisfied by \( v_s \) and \( \mu_s \) in the viscosity sense (see Corollary 16).

**Lemma 7** Let \( u \in C^{0,s}_0(\Omega) \) be extended as zero outside \( \Omega \). If \( u \in W^{s,q}(\Omega) \) for some \( q > 1 \), then \( u \in W^{0,s,p}_0(\Omega) \) for all \( p \geq q \) and

\[
\lim_{p \to \infty} [u]_{s,p} = |u|_s.
\]

**Proof.** First, note that the inequality

\[
|u(x) - u(y)| \leq |u|_s |x - y|^s
\]

is valid for all \( x, y \in \mathbb{R}^N \), not only for those \( x, y \in \partial \Omega \). In fact, this is obvious when \( x, y \in \mathbb{R}^N \setminus \overline{\Omega} \). Now, if \( x \in \Omega \) and \( y \in \mathbb{R}^N \setminus \overline{\Omega} \) then take \( y_1 \in \partial \Omega \) such that \( |x - y_1| \leq |x - y| \) (such \( y_1 \) can be taken on the straight line connecting \( x \) to \( y \)). Since \( u(y) = u(y_1) = 0 \), we have

\[
|u(x) - u(y)| = |u(x)| = |u(x) - u(y_1)| \leq |u|_s |x - y_1|^s \leq |u|_s |x - y|^s.
\]
For each \( p > q \) we have
\[
[u]_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-q}}{|x-y|^{s(p-q)}} \frac{|u(x) - u(y)|^q}{|x-y|^{N+sq}} \, dx \, dy \leq (|u|_s)^{(p-q)} [u]_{s,q}^q.
\]
Thus, \( u \in W^{s,p}_0(\Omega) \) and
\[
\limsup_{p \to \infty} [u]_{s,p} \leq \lim_{p \to \infty} |u|_s^{(p-q)/p} [u]_{s,q}^{q/p} = |u|_s.
\]

Now, noticing that (by Fatou’s lemma)
\[
\int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^q \, dx \, dy \leq \liminf_{p \to \infty} \int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^p \, dx \, dy
\]
and (by Hölder’s inequality)
\[
\int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^q \, dx \, dy \leq |\Omega|^{2(1-\frac{q}{p})} \left( \int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^p \, dx \, dy \right)^{\frac{q}{p}}
\]
we obtain
\[
\left( \int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^q \, dx \, dy \right)^{\frac{1}{q}} \leq |\Omega|^{2/q} \liminf_{p \to \infty} [u]_{s,p}.
\]
Hence, taking into account that
\[
|u|_s = \lim_{q \to \infty} \left( \int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^q \, dx \, dy \right)^{\frac{1}{q}}
\]
we arrive at
\[
|u|_s \leq \lim_{q \to \infty} |\Omega|^{2/q} \left( \liminf_{p \to \infty} [u]_{s,p} \right) = \liminf_{p \to \infty} [u]_{s,p}.
\]
This estimate combined with (20) leads us to (19). ■

It is known (see [7, Theorem 8.2]) that if \( p > \frac{N}{s} \) then there exists of a positive constant \( C \) such that
\[
\|u\|_{C^{0,\beta}([\Omega])} \leq C [u]_{s,p} \quad \forall u \in W^{s,p}_0(\Omega),
\]
where \( \beta := s - \frac{N}{p} \in (0,1) \). As pointed out in [13, Remark 2.2] the constant \( C \) in (21) can be chosen uniform with respect to \( p \).

We remark that the family of positive numbers \( \{ \sqrt[p]{\lambda_p} \}_{p>1} \) is bounded. Indeed, combining (12) with the previous lemma we obtain
\[
\limsup_{p \to \infty} \sqrt[p]{\lambda_p} \leq |\xi|_s.
\]

The next lemma, where \( \text{Id} \) stands for the identity function, is extracted of the proof of [18, Lemma 3.2]. It helps us to overcome the fact that \( C^{\infty}_c(\Omega) \) is not dense in \( C^{0,s}_0(\Omega) \).
Lemma 8 (see [18, Lemma 3.2]) Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain. There exist $\phi \in C^\infty_c(\mathbb{R}^N, \mathbb{R}^N)$ and $0 < \tau_0 < (|\phi |_1)^{-1}$ such that, for each $0 \leq \tau \leq \tau_0$, the map
\[
\Phi_\tau := \text{Id} + \tau \phi : \mathbb{R}^N \to \mathbb{R}^N
\]
is a diffeomorphism satisfying
\begin{enumerate}
\item $\Phi_\tau(\Omega) \subset \subset \Omega$,
\item $\Phi_\tau \to \text{Id}$ and $(\Phi_\tau)^{-1} \to \text{Id}$ as $\tau \to 0^+$ uniformly on $\mathbb{R}^N$,
\item $|(\Phi_\tau)^{-1}(x) - (\Phi_\tau)^{-1}(y)| \leq \frac{|x - y|}{1 - \tau |\phi|_1}$.
\end{enumerate}

Lemma 9 Let $u \in C^{0,s}_0(\overline{\Omega})$ be a nonnegative function extended as zero outside $\Omega$. There exists a sequence of nonnegative functions $\{u_k\}_{k \in \mathbb{N}} \subset C^{0,s}_0(\overline{\Omega}) \cap W^{s,p}_0(\Omega)$, for all $p > 1$, converging uniformly to $u$ in $\overline{\Omega}$ and such that
\[
\limsup_{k \to \infty} |u_k|_s \leq |u|_s.
\]

Proof. For each $k \in \mathbb{N}$ let $\Psi_k$ denote the inverse of $\Phi_{1/k}$, given by Lemma 8 and set $\Omega_k := \Phi_{1/k}(\Omega)$. Since $\Omega_k \subset \subset \Omega$ there exists $U_k$, a subdomain of $\Omega$, such that $\overline{\Omega_k} \subset U_k \subset \overline{U_k} \subset \Omega$.

Let $\eta \in C^\infty(\mathbb{R}^N)$ be a standard convolution kernel: $\eta(z) > 0$ if $|z| < 1$, $\eta(z) = 0$ if $|z| \geq 1$ and $\int_{|z| \leq 1} \phi(z)dz = 1$.

Define the function
\[
u_k = (u \circ \Psi_k) \ast \eta_k \in C^\infty(\mathbb{R}^N),
\]
where
\[
\eta_k(x) := (\epsilon_k)^{-N} \eta \left( \frac{x}{\epsilon_k} \right), \quad x \in \mathbb{R}^N
\]
and $\epsilon_k < \text{dist}(\Omega_k, \partial U_k)$. Note that $\epsilon_k \to 0$.

Since $B(x, \epsilon_k) \subset \mathbb{R}^N \setminus \Omega_k \quad \forall x \in \mathbb{R}^N \setminus U_k,$
we have
$\Psi_k(B(x, \epsilon_k)) \subset \mathbb{R}^N \setminus \Omega \quad \forall x \in \mathbb{R}^N \setminus U_k.$

Hence, observing that
\[

u_k(x) = \int_{\mathbb{R}^N} \eta_k(x - z)u(\Psi_k(z))dz = \int_{\mathbb{R}^N} \eta(z)u(\psi_k(x - \epsilon_k z))dz \quad \forall x \in \mathbb{R}^N
\]
and that
\[ |x - \epsilon_k z - x| \leq \epsilon_k \quad \forall z \in B(0, 1) \]
we conclude that
\[ u_k(x) = 0 \quad \forall x \in \mathbb{R}^N \setminus U_k. \]
Therefore, \( u_k \in C_c^\infty(\Omega) \subset W^{1,p}_0(\Omega) \) for all \( p > 1 \).

Now, let \( x, y \in \overline{\Omega} \) be fixed. According to item 3 of Lemma 8
\[ |u_k(x) - u_k(y)| \leq \int_{B(0,1)} \eta(z) |u(\Psi_k(x - \epsilon_k z)) - u(\Psi_k(y - \epsilon_k z))| \, dz \]
\[ \leq |u|_s \int_{B(0,1)} \eta(z) |\Psi_k(x - \epsilon_k z) - \Psi_k(y - \epsilon_k z)|^s \, dz \]
\[ \leq \frac{|u|_s}{(1 - (1/k)|\phi|_1)^s} \int_{B(0,1)} \eta(z) |x - y|^s \, dz \]
\[ = \frac{|u|_s}{(1 - (1/k)|\phi|_1)^s} |x - y|^s. \]

It follows that \( u_k \in C^{0,s}_0(\overline{\Omega}) \) and
\[ \limsup_{k \to \infty} |u_k|_s \leq \lim_{k \to \infty} \frac{|u|_s}{(1 - (1/k)|\phi|_1)^s} = |u|_s. \]
Consequently, up to a subsequence, \( u_k \to \tilde{u} \in C(\overline{\Omega}) \) uniformly in \( \overline{\Omega} \). Hence, \( \tilde{u} = u \) since item 2 of Lemma 8 implies that
\[ \lim_{k \to \infty} u_k(x) = \int_{B(0,1)} \eta(z) u(\lim_{k \to \infty} \Psi_k(x - \epsilon_k z)) \, dz = u(x) \int_{B(0,1)} \eta(z) \, dz = u(x). \]

**Theorem 10** Let \( p_n \to \infty \). Up to a subsequence, \( \{u_{p_n}\}_{n \in \mathbb{N}} \) converges uniformly to a nonnegative function \( u_\infty \in C^{0,s}_0(\overline{\Omega}) \) such that
\[ |u_\infty|_s = \lim_{n \to \infty} \sqrt[n]{\Lambda_{p_n}}. \]
Furthermore,
\[ v_s = (k_\infty)^{-1} u_\infty \quad (22) \]
where
\[ k_\infty := k(u_\infty) = \exp \left( \int_{\Omega} (\log |u_\infty|) \omega \, dx \right) \geq 1. \quad (23) \]
Proof. Let $p_0 > \frac{N}{s}$ be fixed and take $\beta_0 = s - \frac{N}{p_0}$. For each $(x, y) \in \Omega \times \Omega$, with $x \neq y$, we obtain from (21)

$$\frac{|u_p(x) - u_p(y)|}{|x - y|^{s - \frac{N}{p_0}}} = \frac{|u_p(x) - u_p(y)|}{|x - y|^{s - \frac{N}{p} - \frac{1}{p}}}
\leq C [u_p]_{s,p} \text{diam}(\Omega)^{s - \frac{N}{p}} = C [u_p]_{s,p} \text{diam}(\Omega)^{s - \frac{N}{p}}, \quad \forall p \geq p_0,$$

where $C$ is uniform with respect to $p$ and $\text{diam}(\Omega)$ is the diameter of $\Omega$. Hence, in view of (13) and (12) the family $\{u_p\}_{p \geq p_0}$ is bounded in $C^{0, \beta_0}(\overline{\Omega})$, implying that, up to a subsequence, $u_{p_n} \to u_\infty \in C(\overline{\Omega})$ uniformly in $\Omega$. Of course, the limit function $u_\infty$ is nonnegative in $\Omega$ and vanishes on $\partial \Omega$.

Letting $n \to \infty$ in the inequality (which follows from (21))

$$\frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{s - \frac{N}{p_n}}} \leq C [u_{p_n}]_{s,p_n} = C \text{r}_p \Lambda_{p_n}$$

and taking (12) into account we conclude that $u_\infty \in C^{0, s}(\overline{\Omega})$.

Up to another subsequence, we can assume that

$$r_p \Lambda_{p_n} \to L.$$

Let $q > \frac{N}{s}$ be fixed. By Fatou’s Lemma and Hölder’s inequality,

$$\int_\Omega \int_\Omega \left( \frac{|u_\infty(x) - u_\infty(y)|}{|x - y|^{s}} \right)^q \, dxdy \leq \liminf_{n \to \infty} \int_\Omega \int_\Omega \left( \frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{s - \frac{N}{p_n} + s}} \right)^q \, dxdy$$

$$\leq \liminf_{n \to \infty} |\Omega|^{2(1 - \frac{N}{p_n})} \left( \int_\Omega \int_\Omega \left( \frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{s - \frac{N}{p_n} + s}} \right)^p \, dxdy \right) \frac{2}{p_n}$$

$$\leq |\Omega|^2 \liminf_{n \to \infty} [u_{p_n}]_{s,p_n}^q = |\Omega|^2 \lim_{n \to \infty} \left( \frac{\text{r}_p \Lambda_{p_n}}{|\Omega|^\frac{2}{q}} \right)^q = |\Omega|^2 L^q.$$

Therefore,

$$|u_\infty|_s = \lim_{q \to \infty} \left( \int_\Omega \int_\Omega \left( \frac{|u_\infty(x) - u_\infty(y)|}{|x - y|^s} \right)^q \, dxdy \right)^{1/q} \leq \lim_{q \to \infty} |\Omega|^\frac{2}{q} L = L. \quad (24)$$

To prove that $k_\infty \geq 1$ we first note that

$$\lim_{t \to 0^+} \left( \int_\Omega |u_{p_n}|^t \, \omega dx \right)^{\frac{1}{t}} = \inf_{0 < t < 1} \left( \int_\Omega |u_{p_n}|^t \, \omega dx \right)^{\frac{1}{t}} \leq \left( \int_\Omega |u_{p_n}|^e \, \omega dx \right)^{\frac{1}{t}} \quad \forall \epsilon \in (0, 1).$$

Consequently,

$$1 = k(u_{p_n}) = \lim_{t \to 0^+} \left( \int_\Omega |u_{p_n}|^t \, \omega dx \right)^{\frac{1}{t}} \leq \left( \int_\Omega |u_{p_n}|^e \, \omega dx \right)^{\frac{1}{t}}.$$
The uniform convergence \( u_{p_n} \to u_\infty \) then yields

\[
1 \leq \lim_{n \to \infty} \left( \int_{\Omega} |u_{p_n}|^\epsilon \omega dx \right)^{\frac{1}{\epsilon}} = \left( \int_{\Omega} |u_\infty|^\epsilon \omega dx \right)^{\frac{1}{\epsilon}}.
\]

Therefore,

\[
k_\infty = k(u_\infty) = \lim_{\epsilon \to 0^+} \left( \int_{\Omega} |u_\infty|^\epsilon \omega dx \right)^{\frac{1}{\epsilon}} \geq 1.
\]

It follows that \( (k_\infty)^{-1}u_\infty \in \mathcal{M}_s \), so that

\[
\mu_s \leq |(k_\infty)^{-1}u_\infty|_s = (k_\infty)^{-1}|u_\infty|_s.
\]

In the next step we prove that

\[
\int_{\Omega} \frac{u}{u_\infty} \omega dx \leq \frac{|u|_s}{L} \quad \forall u \in C^{0,s}_0(\Omega).
\]

According to Lemma 9 there exists a sequence of nonnegative functions \( \{u_k\}_{k \in \mathbb{N}} \subset C^{0,s}_0(\Omega) \cap W^{s,p}_0(\Omega) \), for all \( p > 1 \), converging uniformly to \( u \) in \( C(\Omega) \) and such that

\[
\limsup_{k \to \infty} |u_k|_s \leq |u|_s.
\]

Since \( u_p \) is the weak solution of (3) and \( \Lambda_p = [u_p]_{s,p}^p \) we use Hölder’s inequality to get

\[
\Lambda_p \int_{\Omega} \frac{u_k}{u_p} \omega dx = \langle (-\Delta_p)^s u_p, u_k \rangle \leq [u_p]_{s,p}^{p-1} |u_k|_{s,p} = (\Lambda_p)^{p-1} [u_k]_{s,p}.
\]

It follows that

\[
\sqrt[p]{\Lambda_p} \int_{\Omega} \frac{u_k}{u_p} \omega dx \leq |u_k|_{s,p}.
\]

Combining Fatou’s lemma with the uniform convergence \( u_{p_n} \to u_\infty \) and the Lemma 7 we obtain

\[
L \int_{\Omega} \frac{u_k}{u_\infty} \omega dx \leq L \liminf_{n \to \infty} \int_{\Omega} \frac{u_k}{u_{p_n}} \omega dx \leq \liminf_{n \to \infty} [u_k]_{s,p} = |u_k|_s,
\]

that is,

\[
L \int_{\Omega} \frac{u_k}{u_\infty} \omega dx \leq |u_k|_s.
\]

Letting \( k \to \infty \) and applying Fatou’s lemma again we arrive at (26):

\[
L \int_{\Omega} \frac{u}{u_\infty} \omega dx \leq L \liminf_{k \to \infty} \int_{\Omega} \frac{u_k}{u_\infty} \omega dx \leq \liminf_{k \to \infty} |u_k|_s \leq |u|_s.
\]

Taking \( u = u_\infty \) in (26) we obtain

\[
L \leq |u_\infty|_s.
\]
and combining this with (24) we conclude that

\[ L = |u_\infty|_s. \]  

(27)

Now, let \( 0 \leq u \in M_s \) be fixed. Then (16) yields

\[- \int_\Omega (\log u_\infty) \omega dx = \int_\Omega (\log u) \omega dx - \int_\Omega (\log u_\infty) \omega dx = \int_\Omega (\log \frac{u}{u_\infty}) \omega dx \leq \log \left( \int_\Omega \frac{u}{u_\infty} \omega dx \right).\]

Hence, (26) and (27) imply that

\[ (k_\infty)^{-1} \leq \int_\Omega \frac{u}{u_\infty} \omega dx \leq \frac{|u|_s}{|u_\infty|_s} \quad \text{whenever} \quad 0 \leq u \in M_s. \]  

(28)

Combining these estimates at \( u = v_s \) with (25) we obtain

\[ (k_\infty)^{-1} \leq \int_\Omega \frac{v_s}{u_\infty} \omega dx \leq \frac{|v_s|_s}{|u_\infty|_s} = \frac{\mu_s}{|u_\infty|_s} \leq (k_\infty)^{-1}, \]

which leads us to conclude that

\[ \mu_s = \left| (k_\infty)^{-1} u_\infty \right|_s \quad \text{and} \quad (k_\infty)^{-1} = \int_\Omega \frac{v_s}{u_\infty} \omega dx. \]

Corollary 11 The following inequalities hold

\[ k(u) \leq \int_\Omega \frac{|u|}{v_s} \omega dx \leq \frac{|u|_s}{\mu_s} \quad \forall u \in C_0^{0,s}(\Omega). \]  

(29)

Proof. Since we already know that \( L = |u_\infty|_s \) and \( u_\infty = k_\infty v_s \) the second inequality in (29) follows from (26), with \( u \) replaced with \( w = |u| \) (note that \( |w|_s \leq |u|_s \)). The first inequality in (29) is obvious when \( k(u) = 0 \) and, when \( k(u) > 0 \), it follows from the first inequality in (25), with \( w = (k(u))^{-1} |u| \in M_s. \)

Remark 12 In contrast with what happens in similar problems driven by the standard \( p \)-Laplacian, we are not able to prove that \( u_\infty \in W_0^{s,q}(\Omega) \) for some \( q > 1 \). Such a property would guarantee that \( u_\infty = v_s \) and, consequently,

\[ \lim_{p \to \infty} u_p = v_s \]

(that is, \( v_s \) would be the only limit point of the family \( \{u_p\}_{p>1} \), as \( p \to \infty \)). Indeed, if \( u_\infty \in W_0^{s,q}(\Omega) \) for some \( q > 1 \) then, according to Lemma 7, \( u_\infty \in W_0^{s,p_n}(\Omega) \) for all \( n \) sufficiently large (such that \( p_n \geq q \)) and

\[ \lim_{n \to \infty} [u_\infty]_{s,p_n} = |u_\infty|_s. \]
Hence, proceeding as in the proof of Theorem 10, we would arrive at

\[ 1 \leq k_\infty \leq \int_\Omega \frac{u_\infty}{u_{pn}} \omega dx \leq \frac{[u_\infty]_{s, pn}}{\sqrt[p]{\Lambda_{pn}}}. \]

Since \( \lim_{n \to \infty} [u_\infty]_{s, pn} = \lim_{n \to \infty} p\sqrt[2p]{\Lambda_{pn}} = |u_\infty|_s \) we would conclude that \( k_\infty = 1 \) and \( u_\infty = v_s \).

5 The limit problem

For a matter of compatibility with the viscosity approach we add the hypotheses of continuity and strict positiveness to the weight \( \omega \). So, we assume in this section that

\[ \omega \in C(\Omega) \cap L^r(\Omega), \ r > 1, \ \omega > 0 \ in \ \Omega, \ and \ \int_\Omega \omega dx = 1. \]

Note that such \( \omega \) satisfies the hypotheses of Theorem 3.

For \( 1 < p < \infty \) we write the \( s \)-fractional \( p \)-Laplacian, in its integral version, as \((-\Delta_p)^s = -\mathcal{L}_p\) where

\[
(\mathcal{L}_pu)(x) := 2\int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|y-x|^{N+sp}} dy. \tag{30}
\]

Corresponding to the case \( p = \infty \) we define operator \( \mathcal{L}_\infty \) by

\[
\mathcal{L}_\infty := \mathcal{L}_\infty^+ + \mathcal{L}_\infty^- \tag{31}
\]

where

\[
(\mathcal{L}_\infty^+ u)(x) := \sup_{y \in \mathbb{R}^N \setminus \{x\}} \frac{u(y) - u(x)}{|y-x|^s} \quad \text{and} \quad (\mathcal{L}_\infty^- u)(x) := \inf_{y \in \mathbb{R}^N \setminus \{x\}} \frac{u(y) - u(x)}{|y-x|^s}. \tag{32}
\]

In the sequel we consider, in the viscosity sense, the problem

\[
\begin{cases}
\mathcal{L}u = 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{33}
\]

where either \( \mathcal{L}u = \mathcal{L}_pu + \Lambda_p u^{-1}\omega \), with \( 1 < p < \infty \), or

\[
\mathcal{L}u = \mathcal{L}_\infty u \quad \text{or} \quad \mathcal{L}u = \mathcal{L}_\infty^- u + |u_\infty|_s.
\]

We recall some definitions related to the viscosity approach for the problem (33).

**Definition 13** Let \( u \in C(\mathbb{R}^N) \) such that \( u > 0 \) in \( \Omega \) and \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \). We say that \( u \) is a viscosity supersolution of the equation (33) if

\[
(\mathcal{L}\varphi)(x_0) \leq 0
\]
for all pair \((x_0, \varphi) \in \Omega \times C^1_0(\mathbb{R}^N)\) satisfying
\[
\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \leq u(x) \quad \forall x \in \mathbb{R}^N.
\]

Analogously, we say that \(u\) is a viscosity subsolution of (33) if
\[
(L\varphi)(x_0) \geq 0
\]
for all pair \((x_0, \varphi) \in \Omega \times C^1_0(\mathbb{R}^N)\) satisfying
\[
\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \geq u(x) \quad \forall x \in \mathbb{R}^N.
\]

We say that \(u\) is a viscosity solution of (33) if it is simultaneously a subsolution and a supersolution of (33).

The next lemma can be proved by following, step by step, the proof of Proposition 11 of [17].

**Lemma 14** Let \(u \in W^{s,p}_0(\Omega) \cap C(\Omega)\) be a positive weak solution of (3). Then \(u\) is a viscosity solution of
\[
\begin{aligned}
\{ & L_p u + \Lambda_p u^{-1} \omega = 0 \quad \text{in} \quad \Omega \\
& u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]

Our main result in this section is the following, where \(u_\infty \in C^0_0(\Omega)\) is the function given by Theorem 10.

**Theorem 15** The function \(u_\infty \in C^0_0(\Omega)\), extended as zero outside \(\Omega\), is both a viscosity supersolution of the problem
\[
\begin{aligned}
\{ & L_\infty u = 0 \quad \text{in} \quad \Omega \\
& u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega
\end{aligned}
\]
and a viscosity solution of the problem
\[
\begin{aligned}
\{ & L^-_\infty u + |u_\infty|_s = 0 \quad \text{in} \quad \Omega \\
& u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]

Moreover, \(u_\infty\) is strictly positive in \(\Omega\) and the only minimizers of \(|\cdot|_s\) on \(\mathcal{M}_s\) are
\[-v_s \quad \text{and} \quad v_s.\]

**Proof**. We begin by proving that \(u_\infty\) is a viscosity supersolution of (36). For this, let us fix \((x_0, \varphi) \in \Omega \times C^1_0(\mathbb{R}^N)\) satisfying
\[
\varphi(x_0) = u_\infty(x_0) \quad \text{and} \quad \varphi(x) \leq u_\infty(x) \quad \forall x \in \mathbb{R}^N.
\]

Without loss of generality we can assume that
\[
\varphi(x) < u_\infty(x) \quad \forall x \in \mathbb{R}^N,
\]
what allows us to assure that \( u_{pn} - \varphi \) assumes its minimum value at a point \( x_n \), with \( x_n \to x_0 \).

Let \( c_n := u_{pn}(x_n) - \varphi(x_n) \). Of course, \( c_n \to 0 \) (due to the uniform convergence \( u_{pn} \to u_\infty \)). By construction,

\[
\varphi(x_n) + c_n = u_{pn}(x_n) \quad \text{and} \quad \varphi(x) + c_n \leq u_{pn}(x) \quad \forall x \in \mathbb{R}^N.
\]

According to the previous lemma, \( u_p \) is a viscosity supersolution of (34) since it is a viscosity solution of the same problem. Therefore,

\[
(L_{pn} \varphi)(x_n) + \Lambda_{pn} \frac{\omega(x_n)}{u_{pn}(x_n)} = (L_{pn}(\varphi + c_n))(x_n) + \Lambda_{pn} \frac{\omega(x_n)}{\varphi(x_n) + c_n} \leq 0,
\]

an inequality that can be rewritten as

\[
A_{pn}^{-1} + C_{pn}^{-1} \leq B_{pn}^{-1}
\]

where

\[
A_{pn}^{-1} = 2 \int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(x_n)|^{p-2} (\varphi(y) - \varphi(x_n))^+}{|y - x|^{N + sp_{n}}} \, dy \geq 0,
\]

\[
B_{pn}^{-1} = 2 \int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(x_n)|^{p-2} (\varphi(y) - \varphi(x_n))^-}{|y - x|^{N + sp_{n}}} \, dy \geq 0,
\]

and

\[
C_{pn}^{-1} = \Lambda_{pn} \frac{\omega(x_n)}{u_{pn}(x_n)} > 0.
\]

(Here, \( a^+ := \max\{a, 0\} \) and \( a^- := \max\{-a, 0\} \), so that \( a = a^+ - a^- \).)

According to Lemma 6.1 of [13], which was adapted from Lemma 6.5 of [3], we have

\[
\lim_{n \to \infty} A_n = (L_\infty^+ \varphi)(x_0) \quad \text{and} \quad \lim_{n \to \infty} B_n = - (L_\infty^- \varphi)(x_0).
\]

Hence, noticing that

\[
A_{pn}^{-1} \leq A_{pn}^{-1} + C_{pn}^{-1} \leq B_{pn}^{-1}
\]

we conclude that

\[
(L_\infty \varphi)(x_0) = (L_\infty^+ \varphi)(x_0) + (L_\infty^- \varphi)(x_0) \leq 0
\]

since

\[
(L_\infty^+ \varphi)(x_0) = \lim_{n \to \infty} A_n \leq \lim_{n \to \infty} B_n = - (L_\infty^- \varphi)(x_0).
\]

We have proved that \( u_\infty \) is a supersolution of (35). Therefore, by directly applying Lemma 22 of [17] we conclude \( u_\infty > 0 \) in \( \Omega \).

The strict positiveness of \( u_\infty \) in \( \Omega \) and the uniqueness of the nonnegative minimizers of \(|\cdot|_s \) on \( \mathcal{M}_s \) imply that if \( w \in \mathcal{M}_s \) is such that

\[
|w|_s = \min_{u \in \mathcal{M}_s} |u|_s
\]
then $|w| = v_s = (k\infty)^{-1}u_\infty > 0$ in $\Omega$ (recall that $|w|$ is also a minimizer). The continuity of $w$ then implies that either $w > 0$ in $\Omega$ or $w < 0$ in $\Omega$. Consequently, $w = v_s$ or $w = -v_s$.

Now, recalling that
\[
\lim_{n \to \infty} (\Lambda_{p_n})^{\frac{1}{p_n} - 1} = |u_\infty|_s
\]
and using that $\omega(x_0) > 0$ and $u_\infty(x_0) > 0$ we have
\[
\lim_{n \to \infty} C_n = |u_\infty|_s
\]

Hence, since
\[
C_n^{p_n - 1} \leq A_n^{p_n - 1} + C_n^{p_n - 1} \leq B_n^{p_n - 1},
\]
we obtain
\[
|u_\infty|_s = \lim_{n \to \infty} C_n \leq \lim_{n \to \infty} B_n = - (\mathcal{L}_{\infty}^- \varphi)(x_0).
\]
It follows that $u_\infty$ is a viscosity supersolution of (36).

Now, let us take a pair $(x_0, \varphi) \in \Omega \times C^1_0(\mathbb{R}^N)$ satisfying
\[\varphi(x_0) = u_\infty(x_0) \quad \text{and} \quad \varphi(x) \geq u_\infty(x) \quad \forall x \in \mathbb{R}^N. \quad (39)\]

Since
\[
-|u_\infty|_s \leq \frac{u_\infty(x) - u_\infty(x_0)}{|x - x_0|^s} \leq \frac{\varphi(x) - \varphi(x_0)}{|x - x_0|^s} \quad \forall x \in \mathbb{R}^N \setminus \{x_0\},
\]
we have
\[
-|u_\infty|_s \leq \inf_{x \in \mathbb{R}^N \setminus \{x_0\}} \frac{\varphi(x) - \varphi(x_0)}{|x - x_0|^s} = (\mathcal{L}_{\infty}^- \varphi)(x_0).
\]
Therefore, $u_\infty$ is a viscosity subsolution of (36). \[\blacksquare\]

Since $v_s = (k\infty)^{-1}u_\infty$ is the only positive minimizer of $|\cdot|_s$ on $C^0_0(\overline{\Omega}) \setminus \{0\}$ and $\mathcal{L}_{\infty}^- (ku) = k\mathcal{L}_{\infty}^- u$ for any positive constant $k$, the following corollary is immediate.

**Corollary 16** The minimizer $v_s$ is a viscosity solution of the problem
\[
\begin{cases}
\mathcal{L}_{\infty}^- u + \mu_s = 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

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