THUE EQUATIONS AND THE METHOD OF CHABAUTY-COLEMAN

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Abstract. Let $F(x, y) \in \mathbb{Z}[x, y]$ be a homogenous polynomial of degree $n$, assumed to have unit content. Let $h \in \mathbb{Z}$ be such that the polynomial $hz^n - F(x, y)$ is irreducible in $\mathbb{Q}[x, y]$. We denote by $X_{F,h}/\mathbb{Q}$ the nonsingular complete model of the projective plane curve $C_{F,h}/\mathbb{Q}$ defined by the equation $hz^n - F(x, y) = 0$. A classical Thue equation is an equation $F(x, y) = h$ where $F(x, 1)$ does not have repeated roots. Let $N(F, h)$ denote the cardinality of the set \{(x, y) \in \mathbb{Z}^2 | F(x, y) = h \text{ and } \gcd(x, y) = 1\}.

Let $r(X_{F,h})$ denote the Mordell-Weil rank over $\mathbb{Q}$ of the Jacobian of $X_{F,h}/\mathbb{Q}$. Our main theorem is:

Theorem. Assume that the genus $g$ of $X_{F,h}$ is at least 2. If $r(X_{F,h}) < g$, then $N(F, h) \leq 2n^3 - 2n - 3$.

To prove this theorem, we first refine the method of Chabauty-Coleman so that it can be applied to any regular model of a curve $X_{F,h}/\mathbb{Q}_p$ with $p > n$. (Coleman’s Theorem applies only to curves having good reduction at a prime $p > 2g$.) We then describe some regular open subsets of a normal model of the curve $X_{F,h}/\mathbb{Q}_p$ and prove our main theorem on primitive solutions of Thue equations using this model.

We also present some refinements of our main theorem in some special cases where we obtain a bound of the form $N(F, h) \leq O(n^2)$, and discuss examples of twists of Fermat curves.

Let $\mathcal{O}_K$ be a domain with field of fractions $K$. Let $F(x, y) \in \mathcal{O}_K[x, y]$ be a homogenous polynomial of degree $n$, assumed to have unit content (i.e., the coefficients of $F$ generate the unit ideal in $\mathcal{O}_K$). Assume that $\gcd(n, \text{char}(K)) = 1$. Let $h \in \mathcal{O}_K$ and assume that the polynomial $hz^n - F(x, y)$ is irreducible in $\overline{K}[x, y, z]$. We denote by $X_{F,h}/K$ the nonsingular complete model of the projective plane curve $C_{F,h}/K$ defined by the equation $hz^n - F(x, y) = 0$. Recall that $hz^n - F(x, y) = 0$ defines a nonsingular curve if $F(x, 1)$ has simple roots in $\overline{K}$, and that if $F(x, 1) = c \prod_{i=1}^{s} (x - \alpha_i)^{n_i}$ with $\prod_{i \neq j} (\alpha_i - \alpha_j) \neq 0$, then the genus $g$ of $X_{F,h}$ is given by the formula

$$2g - 2 = n(s - 2) - \sum_{i=1}^{s} \gcd(n, n_i).$$

We shall assume in this article that $g \geq 2$. When $K$ is a number field, Mordell’s Conjecture implies that $|X_{F,h}(K)| < \infty$. Caporaso, Harris, and Mazur ([CHM], 1.1) have shown that if Lang’s conjecture for varieties of general type is true, then for any number field $K$, the size $|X(K)|$ of the set of $K$-rational points of any curve

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Theorem (Bombieri-Schmidt, [B-S]). Assume that \( F(x, 1) \) has distinct roots in \( \overline{\mathbb{Q}} \). There exists a constant \( B_1 \), which can be taken to be 215 when \( n \) is sufficiently large, such that \( N(F, h) \leq B_1 n^{w(h)+1} \), where \( w(h) \) equals the number of prime factors of \( h \).

This bound depends on \( n \) and \( h \). Let \( r(X_{F,h}) \) denote the Mordell-Weil rank over \( \mathbb{Q} \) of the jacobian of \( X_{F,h}/\mathbb{Q} \).

Theorem (Silverman, [Si1]). Assume that \( F(x, 1) \) has distinct roots in \( \overline{\mathbb{Q}} \). There exists an ineffective constant \( h(F) \) such that for all \( n^{th} \) power-free \( h > h(F) \), then

\[
N(F, h) \leq n^{2n^2}(8n^3)^{r(X_{F,h})}.
\]

For a fixed \( F \), this bound depends only on \( n \) and \( r(X_{F,h}) \), but only works for \( h \) sufficiently large. Our main theorem is:

Theorem 2.25 If \( r(X_{F,h}) < g(X_{F,h}) \), then \( N(F, h) \leq 2n^3 - 2n - 3 \).

This bound only holds when \( r(X_{F,h}) \) is small, but when it holds, it depends only on \( n \). In section 3, we are able to refine our method in some special cases to obtain a bound of the form \( N(F, h) \leq O(n^2) \).

The theorems of [B-S] and [Si1] both hold when \( n = 3 \). In this case, \( X_{F,h} \) is an elliptic curve with potentially good reduction. When its rank is assumed to be zero, that is, under the hypothesis of Theorem 2.25, it is easy to find a bound for the order of the finite group \( X_{F,h}(\mathbb{Q}) \) by reducing modulo \( p \) for appropriate primes \( p \).

Both [B-S] and [Si1] make use of the Thue-Siegel-Roth theorem on approximations of algebraic numbers. Typically, this theorem yields a bound on the number of
solutions to an equation that depends on the size of the coefficients defining the equation. Bombieri and Schmidt transform the coefficients of Thue equations with elements of $\text{SL}_2(\mathbb{Z})$ to overcome this difficulty in the case $h = 1$; the general case of their theorem is proved via induction on the number of prime factors of $h$. Silverman works with a fixed $F$ and uses the Thue-Siegel-Roth theorem to dispose of solutions to $F(x, y) = h$ corresponding to the so-called large approximations of the roots of $F$. The number of these large approximations can only be bounded in terms of the size of the coefficients of $F$, but, in any case, there are only finitely many such approximations for a fixed $F$ and thus only finitely $h$ for which these approximations contribute to the number of solutions to $F(x, y) = h$.

The proof of Theorem 2.25, by contrast, does not involve diophantine approximation. As mentioned earlier, Theorem 2.25 is proved using the method of Chabauty-Coleman on the curve $X := X_{F,h}/\mathbb{Q}$. In order to use this method to bound $|X(\mathbb{Q})|$, one needs to pick a prime and compute enough of a regular model $\mathcal{X}/\mathbb{Z}_p$ of $X/\mathbb{Q}_p$ to be able to bound the number $N_1$ of components of multiplicity 1 in the special fiber $\mathcal{X}/\mathbb{F}_p$. The number $N_1$ is not, in general, bounded by a constant depending only on $g(X)$ (see 2.27). Hence, this method does not always enable us to bound $|X(\mathbb{Q})|$ in terms of $g(X)$ only. Surprisingly, however, it is possible to bound, in terms of $n$ only, the number of reduction classes in the special fiber of a regular model $\mathcal{X}/\mathbb{Z}_p$ of the integer solutions $(x, y)$ of $F(x, y) = h$ with $\gcd(x, y) = 1$. Note that there are only few families of curves $X_n/\mathbb{Q}$ with $\limsup g(X_n) = \infty$ for which a bound for $|X_n(\mathbb{Q})|$ is known and depends only on $g(X_n)$. For two such families, modular curves (work of Mazur) and Fermat curves (work of Wiles), the sets $X_n(\mathbb{Q})$ of rational points are in fact completely described.

The Mordell-Weil rank of a curve $X/K$ is in general very hard to compute. The case of superelliptic curves of the form $y^p = f(x)$ with $p \mid \deg(f)$ is treated in [P-S]. An effective algorithm in the case $p = 2$ and $\deg(f) = 6$ has been implemented with Magma [Sto], and this algorithm can be used to produce some explicit examples with $n = 6$ where the bound given in Theorem 2.25 holds. Indeed, as is recalled in [L8], the method of Chabauty-Coleman can be applied to the curve $X_{F,h}$ with $n = 6$ if its hyperelliptic quotient $h z^2 = F(x, 1)$ has Mordell-Weil rank at most 1.

We do not discuss here the problem of effectively determining all of the solutions to a Thue equation. For some recent work on this problem, we refer the reader to [B-H] and [TdW].

This paper is organized as follows. In the first section, we refine the method of Chabauty-Coleman so that it can be applied to any regular model of a curve $X_{F,h}/\mathbb{Q}_p$ with $p > n$. (In [Co2], the method applies only to curves having good reduction at a prime $p > 2g$.) In the second section, we first describe some regular models of the curves $X_{F,h}/\mathbb{Q}_p$ and we then prove our main theorem on primitive solutions of Thue equations using these models. In the last section, we present some refinements and discuss examples of twists of Fermat curves.

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1It may be possible to bound the number $N_1$ in terms of the size of $h$ and the coefficients of $F$. If such were the case, then the method of Chabauty-Coleman would provide a bound for $|X(\mathbb{Q})|$ in terms of $n$ and the size of $h$ and the coefficients of $F$. 
1. The method of Chabauty-Coleman

We begin by fixing some notation for this section. Let $K$ be any number field with a place $v$ over a prime $p$. Let $K_v$ denote the completion of $K$ at $v$, with uniformizer $\pi$ and residue field $\mathbb{F}_q$, where $q$ is a power of $p$. Let $X/K$ be any nonsingular, connected, complete, curve of genus $g$. Let $\mathcal{X}/\mathcal{O}_{K_v}$ be a proper model for $X/K_v$. Denote by $\overline{\mathcal{X}}$ the special fiber $\mathcal{X} \times_{\text{Spec}(\mathcal{O}_{K_v})} \text{Spec}(\mathcal{O}_{K_v}/(\pi))$. Let $J/K$ be the Jacobian of $X$. Since $\mathcal{X}$ is proper, we have a reduction map

$$r : \mathcal{X}(K_v) \longrightarrow \overline{\mathcal{X}}(\mathbb{F}_q),$$

which sends points in $\mathcal{X}(K_v)$ to points in $\overline{\mathcal{X}}(\mathbb{F}_q)$. If $Q \in \overline{\mathcal{X}}(\mathbb{F}_q)$, denote by $D_Q$ the set $r^{-1}(Q)$. When $P \in X(K_v)$, the set $D_{\nu(P)}$ is called the residue class of $P$.

Let $A/K$ be any abelian variety. As a $p$-adic Lie group, $A(K_v)$ is endowed with a logarithm map $\log : A(K_v) \to T_0(A)(K_v)$, where $T_0(A)(K_v)$ denote the tangent space to $A$ at 0. For an abelian variety $A$ over a number field $K$ with completion $K_v$, the Chabauty rank of $A$ at $v$ is the integer

$$\text{Chab}(A, K, v) := \dim_{K_v} (\log(A(K))) \otimes_{\mathbb{Z}} K_v.$$

Note that since $\log$ is a homomorphism, $\text{Chab}(A, K, v)$ is less than or equal to the Mordell-Weil rank of $A(K)$. Let us now recall briefly the main ideas of the method of Chabauty-Coleman. Consider again $A(K_v)$ as a $p$-adic Lie group. Given any global differential $\eta \in \Gamma(A, \Omega_{A/K_v})$, there is a unique $p$-adic analytic homomorphism

$$\lambda_{\eta} : A(K_v) \longrightarrow K_v$$

such that $d(\lambda_{\eta}) = \eta$ (see [Wet, Lemma 1.3.2]). In fact, it follows from standard facts on formal groups (see [Fre, Theorem 1]), for example) that one can take $\lambda_{\eta}$ to be

$$(\eta)_0 \circ \log : A(K_v) \longrightarrow K_v,$$

where $(\eta)_0$ is the stalk at 0 of $\eta$ considered as a dual element of the tangent bundle of $A$. The key to the method of Chabauty-Coleman is the remark that when $\text{Chab}(A, K, v) < \dim(A)$, then there exists a differential $\eta$ such that the homomorphism $\lambda_{\eta} : A(K_v) \to K_v$ vanishes on $A(K)$.

Suppose now that $A/K$ is the Jacobian $J/K$ of a curve $X/K$. Define the Chabauty rank of $X/K$ at $v$ to be $\text{Chab}(J, K, v)$, and denote it by $\text{Chab}(X, K, v)$. Assume that there is an embedding $j : X \longrightarrow J$ defined over $K$. This embedding induces an isomorphism $j^*$ from $\Gamma(A, \Omega_{A/K_v})$ to $\Gamma(X, \Omega_{X/K_v})$. Since every $\omega \in \Gamma(X, \Omega_{X/K_v})$ is $j^*(\eta)$ for some $\eta \in \Gamma(A, \Omega_{A/K_v})$, there is an analytic map

$$\lambda_{\omega} : X(K_v) \longrightarrow K_v$$

such that $d(\lambda_{\omega}) = \omega$. Furthermore, for any $P \in X(K_v)$, there is a power series that converges on the residue class of $P$ and is equal to $\lambda_{\omega}$ as a function on that residue class (see [Wet, Lemma 1.7.3]). When $\mathcal{X}/\mathcal{O}_{K_v}$ is a regular model for $X/K_v$ and $\omega$ is in $\Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_{K_v}})$, the power series has the form

$$\lambda_{\omega} = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{m} u^m,$$
where all of the $a_i$ are in $\mathcal{O}_{K_v}$, and $u : D_{v(P)} \rightarrow \pi \mathcal{O}_K$ is a local coordinate. This power series is obtained by formally integrating a power series expansion for $\omega$ with coefficients in $\mathcal{O}_{K_v}$. The method of Coleman-Chabauty allows one to bound $|X(K)|$ in terms of the number of zeros of the $p$-adic analytic function $\lambda_w$.

Coleman considers the case where the curve $X$ has good reduction at $v$, (that is, where $X/K_v$ has a smooth model $X$ over $\mathcal{O}_{K_v}$). In this case, some multiple of the differential $\omega$ reduces to a differential $\overline{\omega}$ on the special fiber $\overline{X}$ and the zeros of $\overline{\omega}$ give rise to information about the coefficients in the power series expansion of $\lambda_w$. Coleman uses this information, along with some simple Newton polygon arguments to obtain a variety of results on the size of $X(K)$ in [Co1] and [Co2]. For example, he shows in [Co2, 0.ii] that,

1.1 If $X$ is a curve of genus $g$ defined over a number field $K$ with completion $K_v$ unramified over $\mathbb{Q}_p$, and if $p > 2g$ and $X$ has good reduction over $\mathcal{O}_{K_v}$, then

$$|X(K)| \leq q - 1 + 2g(\sqrt{q} + 1),$$

whenever $\text{Chab}(J, K, v) < g$.

Let $X'/\mathcal{O}_{K_v}$ be any regular model of $X/K_v$. Let $\overline{X}_{ns}(\mathbb{F}_q)$ denote the nonsingular locus of $\overline{X}(\mathbb{F}_q)$. In this section, we extend Coleman’s result and prove:

**Theorem 1.2.** Let $X/K$ be a curve of genus $g$ defined over a number field $K$ with completion $K_v$ unramified over $\mathbb{Q}_p$. If $\text{Chab}(J, K, v) < g$ and $p^2 > 2g + 1$, then for any subset $U \subset \overline{X}_{ns}(\mathbb{F}_q)$ of the special fiber $\overline{X}$ of a regular model $X'/\mathcal{O}_{K_v}$ of $X/K_v$, we have

$$|r^{-1}(U) \cap X(K)| \leq |U| + \left(\frac{p - 1}{p - 2}\right)(2g - 2).$$

Thus, this theorem removes from the original method of Chabauty-Coleman the hypothesis that $X/K$ has good reduction at the place $v$. McCallum pointed out at the Arizona Winter School meeting of 1999 that such a generalization of the method should be possible. In fact, the early sections of [McC] can probably be used to obtain results similar to 1.1 in the case where the curve has a regular model with a reduced special fiber. Before presenting the proof of Theorem 1.2, let us prove a simple lemma that allow us to bound the number of zeros of $\lambda_w$ in terms of information about local power series expansions. Similar arguments can be found in [Co1], [Co2], [McC], and [Wet]. For simplicity, let us assume that $K_v/\mathbb{Q}_p$ is unramified. Let $\lambda$ be a $p$-adic analytic function

$$\lambda : X(K_v) \longrightarrow K_v.$$ 

Let $P \in X(K_v)$ with reduction $r(P) = Q$ in $X'$ and let $u : D_Q \rightarrow p\mathcal{O}_{K_v}$ be a local coordinate at $P$. Suppose that $\lambda$ has a power series expansion of the form

$$\lambda = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{m} u^m,$$

where $a_m \in \mathcal{O}_{K_v}$, and $v(a_m) = 0$ for some $m$. We can thus consider $\lambda$ as a power series $\lambda(u)$ in the variable $u$, converging on the disk $|u| \leq |p|$. The $p$-adic Weierstrass preparation theorem ([Kol, Thm. 14]) allows us to bound the number of zeros of
\(\lambda\) in \(D_Q\). As this result is most easily stated on the disc \(\mathcal{O}_{K_v}\), we will make the substitution \(z := u/p\). This gives us a power series expansion for \(\lambda\) in \(z\) as

\[
\lambda(z) = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{m} p^m z^m,
\]

converging for all \(z \in \mathcal{O}_{K_v}\). Let us define

\[
I(\lambda, D_Q) := \min\{m \mid v(a_m) = 0\}
\]

and

\[
J(\lambda, D_Q) := \min\{m \mid v(a_\ell p^\ell / \ell) > v(a_m p^m / m) \text{ for all } \ell > m\}.
\]

The Weierstrass preparation theorem then implies that the number of \(z \in \mathcal{O}_{K_v}\) for which \(\lambda(z) = 0\) is at most \(J(\lambda, D_Q)\). It also follows from this theorem that when \(I(\lambda, D_Q) > 0\), the number of \(z \in \mathcal{O}_{K_v}\) for which \(\lambda'(z) = 0\) is at most \(I(\lambda, D_Q) - 1\) (where \(\lambda'(z)\) denotes the formal derivative of \(\lambda(z)\)). It will be convenient in the lemma below to use the function

\[
\rho(x) := x - \log_p x.
\]

It is clear that \(\rho(m)\) is a lower bound for \(v(a_m p^m / m)\) when \(a_m \in \mathcal{O}_{K_v}\), since

\[
v(a_\ell p^\ell / \ell) = m + v(a_m) - v(m) \geq m - \log_p m.
\]

The derivative of \(\rho(x)\) is \(\rho'(x) = 1 - 1/x \ln p\). Thus, when \(p > 2\), \(\rho'(x) > 0\) for \(x \geq 1\), and \(\rho\) is therefore an increasing function for \(x \geq 1\).

**Lemma 1.3.** Let \(p > 2\). Assume that \(K_v/\mathbb{Q}_p\) is unramified and that \(I(\lambda, D_Q) < p^2 - 2\).

a) Suppose that \(p \mid (I(\lambda, D_Q) + 1)\). Then \(J(\lambda, D_Q) \leq I(\lambda, D_Q) + 1\).

b) Suppose that \(p \nmid (I(\lambda, D_Q) + 1)\). Then \(J(\lambda, D_Q) \leq I(\lambda, D_Q)\).

**Proof:** To simplify notation, we will denote \(I(\lambda, D_Q)\) as \(I\). Note that

\[
\rho(I + 2) = I + 2 - \log_p (I + 2) > I,
\]

since \(I + 2 < p^2\). Let us now prove a). Then \(I > 0\) and \(v(I) = 0\). Since \(v(a_I) = 0\), we see that \(v(\frac{a_I}{I} p^I) = I\). Hence,

\[
\rho(I + 2) > v(\frac{a_I}{I} p^I).
\]

Since \(\rho(x)\) is increasing for \(x > 1\), it follows that for all \(j \geq I + 2\), we have \(v(a_j p^j / j) \geq \rho(j) > I\). If \(v(\frac{a_I+1}{I+1} p^{I+1}) > I\), then \(J(\lambda, D_Q) = I(\lambda, D_Q)\), and if \(v(\frac{a_I+1}{I+1} p^{I+1}) = I\), then \(J(\lambda, D_Q) = I(\lambda, D_Q) + 1\).

Part b) is clear when \(I = 0\). To prove b) when \(I > 0\), it is easy to see that we need only show that \(v(\frac{a_I}{I} p^I) > I\) for all \(j > I\), since \(v(\frac{a_I}{I} p^I) \leq I\) for \(I > 0\). Now, since \(p \nmid (I(\lambda, D_Q) + 1)\), we find that

\[
v(\frac{a_{I+1}}{I+1} p^{I+1}) \geq I + 1 > I.
\]

Recall that \(\rho(I + 2) > I\). Using the fact that \(\rho(x)\) is increasing for \(x \geq 1\), we see that \(\rho(j) > I\) for all \(j > I\), and we are done.
Remark 1.4 Coleman uses arguments similar to the lemma above combined with
information about the terms $I(\lambda, D_Q)$ to obtain [11]. In [Co1] and [Co2], he controls
the terms $I(\lambda, D_Q)$ by counting the zeros of the pull-back $\omega$ of $\omega \in \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ to
the special fiber $\mathcal{X}$. Indeed, when the curve has good reduction, the differential
$\omega$ must have exactly $2g(X) - 2$ zeros, counted with multiplicity, since it is a nonzero
differential on a smooth irreducible curve. On a general regular model, however, $\omega$
may vanish on entire components of the special fiber, which makes it difficult to
count the zeros of $\omega$ in a sensible manner. Furthermore, when $\mathcal{X}$ is not reduced, its
dualizing sheaf may not even be a line bundle. Hence, we will not work with the
reduction of $\omega$, but will proceed as follows.

Let $\mathcal{X}/\mathcal{O}_K$ be any regular model of $X/K$. Let $Q \in \mathcal{X}_{ns}(\mathbb{F}_q)$. Denote by $\mathcal{O}_Q$
the local ring $\mathcal{O}_{\mathcal{X},Q}$. Let $P_0 \in X(K)$ be a point reducing to $Q$. The closure of
$P_0$ in $\mathcal{X}$ corresponds to a prime ideal of height 1 in $\mathcal{O}_Q$. Since $\mathcal{O}_Q$ is regular and,
thus, a UFD, we can write this prime ideal as $(u)$ for some prime $u \in \mathcal{O}_Q$. It
follows that the maximal ideal of $\mathcal{O}_Q$ is generated by $\pi$ and $u$. Let $\hat{\mathcal{O}}_Q$ denote
the completion of the ring $\mathcal{O}_Q$ at the prime $(u)$. One easily shows that the natural map
from the ring $\mathcal{O}_K[[u]]$ of formal power series to the ring $\hat{\mathcal{O}}_Q$ (which sends $u$ to $u$) is
an isomorphism. It is also easy to check that the $\hat{\mathcal{O}}_Q$-module of relative differentials
$\mathcal{O}_\mathcal{X}/\mathcal{O}_K$ is generated by $du$. Any differential $\omega \in \Omega_{\mathcal{O}_Q/\mathcal{O}_K}$ can thus be written as
a power series $\omega = \sum_{m=0}^{\infty} a_m u^m du$ with $a_m \in \mathcal{O}_K$ for all $m \in \mathbb{Z}_{\geq 0}$.

Let us fix a generator $\pi$ for the maximal ideal of $\mathcal{O}_K$. Since $Q$ is a nonsingular
point of $\mathcal{X}$, the local ring $\mathcal{O}_{\mathcal{X},Q}$ is a discrete valuation ring, and we denote by $v_Q$
its valuation. For any $P \in X_{K}$, we denote by $v_P$ the valuation of the local ring
$\mathcal{O}_{X_{K},P}$. A differential $\omega \in \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ pulls back to a differential $i^*\omega$ via the
natural map $i : X_{K} \rightarrow \mathcal{X}$ from the generic fiber $X_{K}$ of $\mathcal{X}$ to $\mathcal{X}$. We denote by
$(i^*\omega)_0$ the divisor of zeros of $i^*\omega$, and we shall write $(i^*\omega)_0 = \sum_P v_P(i^*\omega)P$.
We have the following proposition relating the power series expansion of some multiple
of $\omega$ to the zeros of $i^*\omega$ in $r^{-1}(Q)$.

**Proposition 1.5.** Keep the notation introduced above. Let $\omega \in \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ and
let $Q \in \mathcal{X}_{ns}(\mathbb{F}_q)$. Then there exists an element $t \in K$ such that $t\omega \in \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$
and has a local power series expansion

\begin{equation}
(2) \quad t\omega = \sum_{m=0}^{\infty} a_m u^m du,
\end{equation}

with $a_m \in \mathcal{O}_K$ for all $m \in \mathbb{Z}_{\geq 0}$, such that

\begin{equation}
(3) \quad \min\{m \mid v(a_{m+1}) = 0\} = \sum_{r(P) = Q} [K_{v}(P) : K_v] v_P(i^*\omega),
\end{equation}

where the sum is taken over all points $P$ of the scheme $X_{K}$ such that the intersection
of the closure of $P$ in $\mathcal{X}$ with $\mathcal{X}$ is $Q$.

**Proof:** Since $\Omega_{\mathcal{X}/\mathcal{O}_K}$ is locally free of rank 1, let $f$ denote a generator of the stalk
$\Omega_{\mathcal{O}_Q/\mathcal{O}_K}$ of $\Omega_{\mathcal{X}/\mathcal{O}_K}$ at $Q$. We can write the stalk of $\omega$ at $Q$ as $sf$, where $s \in \mathcal{O}_Q$.
Factor $s$ as

\[ s = \ell_1^{\ell_1} \cdots \ell_m^{\ell_m} \pi'^{e} \]
where the $\gamma_j$ are generators for primes corresponding to points $P_j$ on the generic fiber of $X$. It is not hard to see that $v_{P_j}(i^*\omega) = \ell_j$. Indeed, one obtains the local ring $\mathcal{O}_{X_{K_v}, P_j}$ localizing $\mathcal{O}_Q$ at the prime ideal generated by $\gamma_j$, so we see that the ideal generated by $s$ in $\mathcal{O}_{X_{K_v}, P_j}$ is just $\mathcal{M}_{P_j}^{\ell_j}$, where $\mathcal{M}_{P_j}$ is the maximal ideal in $\mathcal{O}_{X_{K_v}, P_j}$; since $f$ pulls back to a generator for the stalk of $\Omega_{X_{K_v}/K_v}$, $sf$ must pull back to a differential with order of vanishing equal to $v_{P_j}(s) = \ell_j$ for all $j$.

After dividing $s$ by $\pi^\ell$ we obtain an element $s_1$ that is not in $\pi \mathcal{O}_Q$. Now complete $\mathcal{O}_Q$ at $(u)$. We obtain a power series expansion for $s_1f$:

$$s_1f = \sum_{m=0}^{\infty} a_{m+1}u^m du.$$  

Let us denote by $s_2$ the element $\sum_{m=0}^{\infty} a_{m+1}u^m$ of $\hat{\mathcal{O}}_Q$. The elements $s_1$ and $s_2$ differ by a unit in $\hat{\mathcal{O}}_Q$. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_Q & \longrightarrow & \hat{\mathcal{O}}_Q \\
\downarrow & & \downarrow \\
\mathcal{O}_Q/\pi & \longrightarrow & \hat{\mathcal{O}}_Q/\pi.
\end{array}
\]

It is easy to check that $\hat{\mathcal{O}}_Q/\pi$ is the completion of $\mathcal{O}_Q/\pi$ at the maximal ideal $(u)$. Thus the valuation $v_Q$ of $\mathcal{O}_Q/\pi$ extends to a valuation on $\hat{\mathcal{O}}_Q/\pi$, again denoted by $v_Q$. Denoting by $\phi_\pi$ the map taking $\hat{\mathcal{O}}_Q$ to $\hat{\mathcal{O}}_Q/\pi$, it is clear that

$$v_Q(\phi_\pi(s_2)) = \sum_{j=1}^{m} \ell_j v_Q(\phi_\pi(\gamma_j)).$$

Since

$$v_Q(\phi_\pi(\gamma_j)) = [K_v(P_j) : K_v],$$

it will suffice to show that

$$v_Q(\phi_\pi(\gamma_j)) = [K_v(P_j) : K_v].$$

This follows from the fact that

$$v_Q(\phi_\pi(\gamma_j)) = \dim_{\mathbb{Q}_q} (((\mathcal{O}_Q/\pi \mathcal{O}_Q)/\phi_\pi(\gamma_j))$$

(4)  

\[= \text{rank}_{\mathcal{O}_{K_v}}(\mathcal{O}_Q/\gamma_j \mathcal{O}_Q) \]

$$= [K_v(P_j) : K_v],$$

since $(\mathcal{O}_Q/\gamma_j \mathcal{O}_Q)$ is a free $\mathcal{O}_{K_v}$-module (which follows from the fact that $\mathcal{O}_{K_v}$ is of course a principal ideal domain). This concludes the proof of (4).

Let us now apply Lemma 4.3 and Proposition 4.5 to the sort of $p$-adic analytic function that arises in the Chabauty-Coleman method.

**Proposition 1.6.** Let $X/K$ be a curve of genus $g$ defined over a number field $K$ with completion $K_v$ unramified over $\mathbb{Q}_p$. Let $\mathcal{X}/\mathcal{O}_{K_v}$ be any regular model for $X/K_v$, and let $\mathcal{U} \subset \Gamma_{ns}(\mathbb{P}_q)$. If $p^2 > 2g + 1$ and $\lambda_{\omega}$ is as in (1), then

$$|r^{-1}(\mathcal{U}) \cap \lambda_{\omega}^{-1}(0)| \leq |\mathcal{U}| + \left(\frac{p-1}{p-2}\right)(2g-2).$$
Proof: Choose $Q \in \mathcal{U}$. For any nonzero element $t \in K_v$, multiplying $\lambda_\omega$ by $t$ will not change the zeros of $\lambda_\omega$, and, furthermore, $\lambda_{tw} = t\lambda_\omega$, so

$$|r^{-1}(Q) \cap \lambda_\omega^{-1}(0)| = |r^{-1}(Q) \cap \lambda_{tw}^{-1}(0)|.$$  

Thus, we may choose $t \in K_v$ and apply Proposition 1.3 to obtain a power series expansion of the form (2) for which equation (3) holds. We denote by $\omega$ appearing in the statement of Proposition 1.5. It is clear that since $d(\lambda_{tw}) = tw$, we find that

$$Z(\omega, Q) = I(\lambda_{tw}, D_Q) - 1.$$  

Since

$$\sum_{Q \in \mathcal{U}} Z(\omega, Q) \leq \sum_{P \in X_{K_v}} [K_v(P) : K_v] v_P(i^*\omega) = 2g - 2 < p^2 - 3,$$

we find that $I(\lambda_{tw}, D_Q) < p^2 - 2$, and we can apply Lemma 1.3. We obtain

$$|r^{-1}(\mathcal{U}) \cap \lambda_\omega^{-1}(0)| \leq \sum_{Q \in \mathcal{U}} J(\lambda_{tw}, D_Q)$$  

(5)

$$\leq \sum_{p \mid (Z(\omega, Q) + 2)} (Z(\omega, Q) + 2) + \sum_{p \mid (Z(\omega, Q) + 2)} (Z(\omega, Q) + 1).$$  

If $p \mid (Z(\omega, Q) + 2)$, then $Z(\omega, Q) \geq p - 2$. Since $\sum_{Q \in \mathcal{U}} Z(\omega, Q) \leq 2g - 2$, there are at most $(2g - 2)/(p - 2)$ points $Q \in X_{ns}(\mathbb{F}_q)$ for which $p \mid (Z(\omega, Q) + 2)$. Hence, (5) becomes

$$|r^{-1}(\mathcal{U}) \cap \lambda_\omega^{-1}(0)| \leq \sum_{Q \in \mathcal{U}} Z(\omega, Q) + |\mathcal{U}| + \frac{2g - 2}{p - 2}$$  

$$\leq |\mathcal{U}| + \left(1 + \frac{1}{p - 2}\right)(2g - 2).$$  

We are now ready prove Theorem 1.2.

1.7 Proof of 1.2: Each differential $\eta \in \Gamma(J, \Omega_{J/K_v})$ gives rise to a homomorphism $\lambda_\eta : J(K_v) \rightarrow K_v$. Since Chab($J, K, v$) $< \dim H^0(J, \Omega_{J/K_v})$, there must be a nonzero $\eta$ for which $\lambda_\eta(J(K)) = 0$. We may assume that $X(K)$ contains a point $Q$, as otherwise our assertion is trivial. Hence, we may embed $X(K_v)$ into $J$ via the mapping $j : X \rightarrow J$, which sends $P \in X(K_v)$ to the class of $P - Q$. Now, $\eta$ pulls back to a differential $\omega$ on $X$, and $\lambda_\eta$ restricts to a function $\lambda_\omega$ that vanishes on $X(K)$ (because $j$ sends points in $X(K)$ to points in $J(K)$). Applying Proposition 1.6 then gives the desired result.

1.8 Note that if an abelian variety $A/K$ is $K$-isogenous to a product $\prod A_i$, then Chab($A, K, v$) $= \sum$ Chab($A_i, K, v$). Thus, the method of Chabauty-Coleman can be applied to $A$ if and only if Chab($A_i, K, v$) $< \dim(A_i)$ for some $i$.

When the Chabauty rank of $X/K$ is zero, we can strengthen Theorem 1.2 as follows.
Let \( X/K \) be a curve of genus \( g \) defined over a number field \( K \) with completion \( K_v/\mathbb{Q}_p \) unramified over \( \mathbb{Q}_p \). Let \( \mathcal{X}/\mathcal{O}_{K_v} \) be any regular model for \( X/K_v \). If \( \text{Chab}(J,K,v) = 0 \), then \( |X(K)| \leq |\overline{\mathcal{X}}_{ns}(\mathbb{F}_q)| \).

**Proof:** We claim that for each \( Q \in \overline{\mathcal{X}}_{ns}(\mathbb{F}_p) \), the set \( r^{-1}(Q) \) contains at most one \( K \)-rational point of \( X \). We begin by noting that since \( \text{Chab}(J,K,v) = 0 \), every differential \( \eta \in \Gamma(J,\Omega_{J/K_v}) \) has the property that its formal \( v \)-adic integral \( \lambda_\eta \) is identically 0 on \( J(K) \). It follows that for every differential \( \omega \in \Gamma(X,\Omega_{X/K_v}) \), the formal integral \( \lambda_\omega \) is identically 0 on \( X(K) \). To prove our claim, we need only to show that for some \( w \in \Omega_{X/\mathcal{O}_{K_v}} \), the sum

\[
\sum_{r(P)=Q} [K_v(P) : K_v]v_P(i^* \omega)
\]

appearing in the statement of 1.3 is equal to 0. Indeed, the proof of Proposition 1.6 shows that this sum is equal to \( I(\lambda_w, D_Q) - 1 \), and that

\[
|D_Q \cap \lambda_w^{-1}(0)| \leq J(\lambda_w, D_Q) \leq I(\lambda_w, D_Q).
\]

Now, since the sheaf \( \Omega_{X/\mathcal{O}_{K_v}} \) is generated by global sections, there must be a global section \( \omega \in \Gamma(X,\Omega_{X/\mathcal{O}_{K_v}}) \) whose stalk at \( Q \) generates the stalk \( \Omega_{X/\mathcal{O}_{K_v}}(Q) \) of \( \Omega_{X/\mathcal{O}_{K_v}} \) at \( Q \) as an \( \mathcal{O}_{Q,X} \)-module. Otherwise, the stalks of all the global sections would be in \( \mathcal{M}_Q \Omega_{X/\mathcal{O}_{K_v}} \) and hence would not together generate \( \Omega_{X/\mathcal{O}_{K_v}}(Q) \) as an \( \mathcal{O}_{Q,v} \)-module. Since \( \omega \) has to generate the stalks of the differential sheaf at all of the \( P \) with \( r(P) = Q \), we find that the sum \( \sum_{r(P)=Q} [K_v(P) : K_v]v_P(i^* \omega) \) appearing in 1.3 must be 0. This concludes the proof of 1.9.

**Remark 1.10** When the Mordell-Weil rank of \( X/K \) is zero, Proposition 1.9 can be strengthened as follows. Let \( X/K \) be a curve of genus \( g \) defined over a number field \( K \) with completion \( K_v/\mathbb{Q}_p \) such that \( v(p) < p - 1 \). Let \( \mathcal{X}/\mathcal{O}_{K_v} \) be any regular model for \( X/K_v \). If the Mordell-Weil rank of \( X/K \) is zero, then \( |X(K)| \leq |\overline{\mathcal{X}}_{ns}(\mathbb{F}_q)| \).

To prove this statement, we proceed as follows. We claim that for each \( Q \in \overline{\mathcal{X}}_{ns}(\mathbb{F}_q) \), the set \( r^{-1}(Q) \) contains at most one \( K \)-rational point of \( X \). Indeed, suppose that \( P \) and \( P' \) belong to \( r^{-1}(Q) \cap X(K) \). Consider the embedding of \( X \) in \( J \) using the map \( D \mapsto D - \text{deg}(D)P \). The map \( X \to J \) extends to a map from the smooth part of \( \mathcal{X} \) to the Néron model of \( J \). By construction, \( P' - P \) reduces to the origin. In other words, \( P' - P \) belongs to the kernel of the reduction, which does not contain any torsion point (see for instance \[\text{Ser}], \text{LG 4.25-4.26}). Thus, \( P' - P \) has infinite order, contradicting the assumption that the Mordell-Weil rank over \( K \) is zero.

The following generalization of 1.1 will not be used in this paper.

**Corollary 1.11.** Let \( X/K \) be a curve of genus \( g \) defined over a number field \( K \) with completion \( K_v/\mathbb{Q}_p \) unramified over \( \mathbb{Q}_p \). Let \( \mathcal{X}/\mathcal{O}_{K_v} \) be any regular model for \( X/K_v \). If \( p > 2g \) and \( \text{Chab}(J,K,v) < g \), then \( |X(K)| \leq |\overline{\mathcal{X}}_{ns}(\mathbb{F}_q)| + 2g - 2 \).

**Proof:** We proceed exactly as in 1.2, with \( \mathcal{U} = \overline{\mathcal{X}}_{ns}(\mathbb{F}_q) \). Since \( p > 2g \), we find that \((2g - 2)/(p - 2) < 1 \), and \( |X(K)| \) is of course a whole number. So \( |X(K)| \leq |\overline{\mathcal{X}}_{ns}(\mathbb{F}_q)| + (2g - 2) \), as desired.
In view of [1,9] and [1,11], it is natural to wonder, under the hypotheses of [1,11], whether the bound for $|X(K)|$ can be made to depend on the precise value of $\text{Chab}(J, K, v)$, such as a bound of the form $|X(K)| \leq |\mathcal{X}_{ns}(\mathbb{F}_q)| + 2\text{Chab}(J, K, v)$.

2. Constructing regular models of curves

Let $K$ be a field with a discrete valuation $v_K$. Let $\mathcal{O}_K$ denote the ring of integers of $K$, with maximal ideal $(\pi_K)$ and residue field $k$. Let $p := \text{char}(k)$. When no confusion may ensue, we will write $\pi$ and $v$ instead of $\pi_K$ and $v_K$. Let $X/K$ be the nonsingular proper model of the plane curve $C/K$ given by a homogeneous equation $f(x, y, z) \in \mathcal{O}_K[x, y, z]$. Assume that the ideal of $\mathcal{O}_K$ generated by the coefficients of $f$ is $\mathcal{O}_K$. A regular model $\mathcal{X}/\mathcal{O}_K$ of $X/K$ can be theoretically obtained as follows. Consider first the model $\mathcal{D}/\mathcal{O}_K$ given by

$$\mathcal{D} := \text{Proj}(\mathcal{O}_K[x, y, z]/(f)).$$

and let $\mathcal{X}$ be the minimal desingularization of the normalization $\mathcal{D}^{\text{nor}}$ of $\mathcal{D}$. Let $\rho : \mathcal{X} \to \mathcal{D}^{\text{nor}}$ denote the desingularization map and let $\nu : \mathcal{D}^{\text{nor}} \to \mathcal{D}$ be the normalization map. Both $\rho$ and $\nu$ are usually quite difficult to describe explicitly, even in the case of a rather simple equation $f$. Obviously, we have the option of changing the defining equation $f$, but this does not always lead to more easily described maps $\rho$ and $\nu$.

Another way of constructing a regular model $\mathcal{X}/\mathcal{O}_K$ of $X/K$ was introduced by Viehweg in [Vie]. It may happen that over a Galois extension $L/K$, a normal model $\mathcal{V}/\mathcal{O}_L$ of $X_L/L$ can be described. If the Galois group $\text{Gal}(L/K)$ acts on $\mathcal{V}$, lifting its action on $\text{Spec}(\mathcal{O}_L)$, then we may consider the quotient $\mathcal{V}/\text{Gal}(L/K)$ as a scheme over $\text{Spec}(\mathcal{O}_K)$. The scheme $\mathcal{V}/\text{Gal}(L/K)$ is a normal model of $X/K$ and, thus, a desingularization $\rho : \mathcal{X} \to \mathcal{V}/\text{Gal}(L/K)$ leads to a regular model $\mathcal{X}/\mathcal{O}_K$ of $X/K$. A key feature of this method is the fact that the singularities of $\mathcal{V}/\text{Gal}(L/K)$ are quotient singularities and that when $L/K$ is tame, such singularities are well-understood.

In this section, we first use this method to construct regular models of curves having potentially good reduction after a tame extension $L/K$, such as the superelliptic curves $X := X_{F,h}$ with $\pi \nmid d^*(F)$ and $p > n$ (see [2,1] below). The model $\mathcal{V}/\mathcal{O}_L$ that we use in this case is simply the smooth minimal model of $X_L/L$, where $L/K$ is large enough to ensure that $X_L/L$ has good reduction.

We shall also use the above method to consider the more difficult case where $\pi \mid d^*(F)$ and $\pi \mid h$. In this case, we construct just enough of a regular model for $X/K$ to be able to bound the number of residue classes of primitive integral solutions to the Thue equation $F(x, y) = h$. Here the field $L/K$ will be the splitting field over $K$ of the polynomial $F(x, 1)$, and the model $\mathcal{V}/\mathcal{O}_L$ will be the normalization of the model

$$\mathcal{C} := \text{Proj} \mathcal{O}_L[x, y, z]/(hz^n - F(x, y)).$$

Some smooth open affine subsets of the model $\mathcal{V}$ are described below in [2,3]. Let $F(x, 1) = c \prod_{i=1}^{s} (x - \alpha_i)^{n_i}$ in $\overline{K}[x]$. Let

$$d^*(F) := c \prod_{i \neq j} (\alpha_i - \alpha_j) \in \overline{K}.$$
We shall say that \( \pi_K \nmid d^*(F) \) if \( \pi_K \not\in (d^*(F)) \) in \( \mathcal{O}_K \).

**Lemma 2.1.** Assume that \( \text{char}(k) \nmid n \). Let \( X := X_{F,h}/K \).

a) If \( \pi_K \nmid d^*(F) \) and \( \pi_K \nmid h \), then \( X/K \) has good reduction.

b) If \( \pi_K \nmid d^*(F) \) and \( \pi_K \mid h \), then \( X/K \) achieves good reduction over \( L := K(\sqrt[n]{h}) \).

**Proof:** Consider the model \( \mathcal{C}/\mathcal{O}_K \) given by \( \text{Proj}(\mathcal{O}_K[x,y,z]/(hz^n - F(x,y))) \) and its normalization \( \mathcal{C}^{\text{nor}}/\mathcal{O}_K \). The generic fiber of \( \mathcal{C}^{\text{nor}} \) has genus equal to

\[
2g(X) - 2 = n(s - 2) - \sum_{i=1}^s \gcd(n, n_i).
\]

If \( g(X) = 0 \), then \( X/K \) has obviously good reduction over \( \mathcal{O}_K \). Assume then that \( g(X) > 0 \). Since \( \pi_K \nmid h \), the reduction modulo \( \pi \) of \( hz^n - F(x,y) \) is irreducible. When \( \pi_K \nmid h \) and \( \pi_K \mid d^*(F) \), we find that the geometric genus of \( \mathcal{C}^{\text{nor}}_K \) is equal to \( g(X) \). Thus, \( \mathcal{C}^{\text{nor}}_K \) is non-singular since its arithmetical genus is equal to the genus of \( X \). It follows that \( \mathcal{C}^{\text{nor}}/\mathcal{O}_K \) is the (minimal) regular model of \( X/K \).

If \( \pi_K \mid h \), consider the change of variables \( z' = \sqrt[n]{h}z \), \( x' = x \) and \( y' = y \). Then \( \text{Proj}(\mathcal{O}_L[x',y',z']/(z'^n - F(x',y'))) \) is a model for \( X_L/L \). Hence, we may apply a) to find that \( X_L/L \) has good reduction. This concludes the proof of \( 2.1 \).

For most of the applications that we have in mind, the residue field \( \mathcal{O}_K/(\pi_K) \) will be \( \mathbb{F}_p \), and we will assume that \( p > n \). The following lemma shows that we may assume, under these hypotheses, that \( F(x,1) \) is monic.

**Lemma 2.2.** Assume that \( |\mathcal{O}_K/(\pi_K)| > n \). Then, up to a change of variable, we may assume that \( F(x,1) \) is monic in \( \mathcal{O}_K[x] \).

**Proof:** Write \( F(x,y) = \prod_{i=1}^n (\beta_i x - \rho_i y) \), with \( \beta_i, \rho_i \in \mathcal{O}_K \). By substituting \( y' = y + ux \), we can write

\[
F(x,y') = \prod_{i=1}^n ((\beta_i + \rho_i u)x - \rho_i y).
\]

Since the coefficients of \( F \) have no common factor, we must have \( \min(v(\beta_i), v(\rho_i)) = 0 \) for each \( i \), so we can choose \( u \in \mathcal{O}_K \) such that \( \rho_i u + \beta_i \in \mathcal{O}_K^* \) for all \( i \) by simply avoiding the residue classes of the \( -(\beta_i/\rho_i) \) for which \( v(\rho_i) = 0 \). There are at most \( n \) residue classes to be avoided and there are at least \( n + 1 \) residue classes in \( \mathcal{O}_K/(\pi_K) \) by hypothesis, so we can do this. This allows us to rewrite our original equation \( F(x,y) = h \) as

\[
\prod_{i=1}^n (x - \alpha_i y) = \mu \pi^w,
\]

where \( \mu \) is unit in \( \mathcal{O}_K \) and \( w = v(h) \).

### 2.3 Some regular affine subsets of the normalization of \( \mathcal{C} \).

In what follows, we assume that \( F(x,1) \) is monic, that \( \pi_K \mid h \) and that \( \pi_K \mid d^*(F) \). Assume also that \( K \) is complete, so that for any finite extension \( L/K \), the integral closure \( \mathcal{O}_L \) of \( \mathcal{O}_K \) in \( L \) is a local ring.

Let \( L/K \) be the splitting field over \( K \) of the polynomial \( F(x,1) \). We denote by \( v \) the valuation of \( \mathcal{O}_L \), and let \( \pi \) be a uniformizer of \( \mathcal{O}_L \). Let us say that \( P = (a,b) \)
is a primitive integral solution to \( F(x, y) = h \) if \( a, b \in \mathcal{O}_K \) and \( \gcd(a, b) = 1 \). We describe below an affine regular scheme \( \mathcal{U}/\mathcal{O}_L \) such that \( \mathcal{U} \times_{\Spec(\mathcal{O}_L)} \Spec(L) \) is open in \( X_{F,h,L} \) and \( P \in \mathcal{U}(L) \) has a non-trivial reduction modulo \( (\pi) \). In other words, the closure of \( P \) in \( \mathcal{U} \) includes a point on the special fiber of \( \mathcal{U} \).

Consider any root \( \alpha_0 \) of \( F(x, 1) \) such that \( v(a - \alpha_0 b) = \max_j (v(a - \alpha_j b)) \). Let \( t := v(a - \alpha_0 b) \). Change variables from \( x \) to \( z_0 := x - \alpha_i y \), so that \( F(z_0, y) = z_0 \prod_{j=1}^{s-1} (z_0 - \gamma_j y) \), where \( \gamma_j := \alpha_j - \alpha_i \) for \( j < i \) and \( \gamma_i := \alpha_{i+1} - \alpha_i \) for \( j \geq i \). Define \( s_0 := 0 \), and then recursively define

\[
s_k := \min \{ v(\gamma_j) \mid t \geq v(\gamma_j) > s_{k-1} \},
\]

for \( k \geq 1 \). We obtain in this way a finite increasing sequence of integers. If \( t \) is not the largest integer of this sequence, add \( t \) to the sequence. Denote the elements of the new sequence by

\[
s_0 < s_1 < \cdots < s_m = t.
\]

Define, for \( k < m \),

\[
S_k := \{ \gamma_j \mid v(\gamma_j) = s_k \}.
\]

The set \( S_m \) is defined to be \( \{ \gamma_j \mid v(\gamma_j) \geq s_m \} \). If \( \gamma \) is a root of \( F(z_0, y) \), let \( n(\gamma) \) denote its multiplicity. Then, for \( k \leq m \), define \( z_k \) to be \( z_0/\pi^{s_k} \), and let \( F_k \) be the polynomial

\[
F_k(z_k, y) := \prod_{j=0}^{k} \prod_{\gamma \in S_j} (\pi^{s_k-s_j} z_k - \gamma \pi^{-s_j} y)^{n(\gamma)} \prod_{\gamma \notin \cup_{j=0}^{k} S_j} (z_k - \gamma \pi^{-s_k} y)^{n(\gamma)}.
\]

Set

\[
u_k := \sum_{j=0}^{k} (\sum_{\gamma \in S_j} n(\gamma)) s_j + \sum_{j=k+1}^{m} (\sum_{\gamma \in S_j} n(\gamma)) s_k.
\]

Then \( F_k(z_k, y) = F_0(z_0, y) \pi^{-u_k} \). Finally, let

\[
A_k := \mathcal{O}_L[z_k, y]/(F_k(z_k, y) - \mu \pi^{w-u_k}),
\]

for \( k \leq m \) (recall that \( h = \mu \pi^w \)). Note now that when \( (a, b) \) is primitive and \( \pi \mid h \), then \( v(b) = 0 \). Indeed, if \( v(b) > 0 \) and \( (a, b) \) is primitive, then \( v(a) = 0 \). Thus, \( v(a - \alpha_j b) = 0 \) for all \( j \), contradicting the fact that \( v(F(a, b)) = v(h) > 0 \). Hence, \( v(b) = 0 \). If follows that for \( j \neq i \), the inequality

\[
v(a - \alpha b) \geq \min(v(a - \alpha_i b), v(\alpha b - \alpha_j b))
\]

implies that either \( v(a - \alpha b) = t \) and \( v(\alpha_i - \alpha_j) \geq t \), or \( v(a - \alpha b) < t \) and \( v(a - \alpha b) = v(\alpha_i - \alpha_j) \). In particular, we find that when \( (a, b) \) is primitive, \( w = u_m \).

**Lemma 2.4.** The ring \( A_m \) is regular and \( \Spec(A_m)/\Spec(\mathcal{O}_L) \) is smooth.

**Proof:** The generic fiber of \( A_m \) is easily checked to be smooth. Hence, we need only check points on the special fiber of \( A_m \). We note that modulo \( \pi \), the equation \( F_m(z_m, y) = \mu \) is equivalent to the equation

\[
\left( \prod_{\gamma \in S_m} \left( z_m - \frac{\gamma}{\pi^t} y \right)^{n(\gamma)} \right) \left( \prod_{j=0}^{m-1} \prod_{\gamma \in S_j} \left( -\frac{\gamma}{\pi^{s_j}} y \right)^{n(\gamma)} \right) - \mu = 0,
\]
Because \( \pi^{w-u_m} = 1 \) as noted earlier. Since \( \overline{\pi} \neq 0 \), this equation defines a nonsingular affine curve in \( \mathbb{A}^2 \). Thus, the special fiber of \( A_m \) is nonsingular; therefore all the points on the special fiber of \( A_m \) are regular and \( A_m \) is a regular ring.

Let \( \mathcal{Y}/\mathcal{O}_L \) denote the normalization of the scheme
\[
\mathcal{C} := \text{Proj} \mathcal{O}_L[x, y, z]/(hz^n - F(x, y)).
\]

**Lemma 2.5.** The affine scheme \( \text{Spec} A_m \) is an open subset of \( \mathcal{Y} \).

**Proof:** There is a natural ring homomorphism \( A_{k-1} \rightarrow A_k \) that sends \( z_{k-1} \) to \( \pi^{s_k-s_{k-1}} z_k \). Define
\[
G_k(z_k, y) := \prod_{j=0}^{k} \prod_{\gamma \in S_j} (\pi^{s_k-s_j} z_k - \gamma \pi^{-s_j} y)^{n(\gamma)}.
\]

Let \( S_k \) denote the multiplicative subset of \( A_k \) generated by \( G_k(z_k, y) \). We claim that \( A_k \) is integral over \( S_{k-1}^{-1}(A_{k-1}) \). Indeed, it suffices to show that \( z_k \) is integral over \( S_{k-1}^{-1}(A_{k-1}) \). Recall that in \( A_k \),
\[
F_k(z_k, y) - \mu \pi^{w-u_k} = G_{k-1}(z_{k-1}, y) \prod_{j=k}^{m} \prod_{\gamma \in S_j} (z_k - \gamma \pi^{-s_k} y)^{n(\gamma)} - \mu \pi^{w-u_k} = 0.
\]

Thus, the image of \( z_k \) in \( A_k \) is the root of a monic polynomial over \( S_{k-1}^{-1}(A_{k-1}) \) (since \( G_{k-1} \) is of course a unit in this ring). Hence, it follows that the map \( \text{Spec}(A_k) \rightarrow \text{Spec}(A_{k-1}) \) is quasi-finite for any \( k \geq 1 \). Since \( A_m \) is normal because it is regular, we have a natural map \( j : \text{Spec}(A_m) \rightarrow \mathcal{Y} \), and our discussion above shows that this map is quasi-finite. Since \( \mathcal{Y} \) is normal and \( j \) is generically an isomorphism, we can apply Zariski’s Main Theorem to find that \( j \) is an open immersion.

2.6 Let \( \mathcal{U}(\alpha_i) := \text{Spec}(A_m) \). The primitive point \( P = (a, b) \) in \( X_{F, h}(L) \) corresponds to the point \( (\pi^{-t}(a - \alpha_i b), b) \) in \( \mathcal{U}(\alpha_i)(L) \). Since this point is integral, it has a non-trivial reduction in the special fiber of \( \mathcal{U}(\alpha_i) \). Denote by \( \mathcal{P} \) the set of integral primitive solutions, so that
\[
\mathcal{P} := \{(x, y) \in (\mathcal{O}_K)^2 \mid F(x, y) = h \text{ and } \gcd(x, y) = 1\}.
\]

Our next lemma shows that the closure of \( \mathcal{P} \) in the normalization \( \mathcal{Y}/\mathcal{O}_L \) of \( \mathcal{C}/\mathcal{O}_L \) is contained in at most \( n \) regular affine open sets; namely, this closure is contained in the union of the sets \( \mathcal{U}(\alpha_i) \), where \( \alpha_i \) runs through all the roots of \( F(x, 1) \) such that there exists a primitive point \( (a, b) \) with \( v(a - \alpha_i b) = \max_j (v(a - \alpha_j b)) \).

**Lemma 2.7.** Let \( \pi \mid h \). Let \( (a, b) \) and \( (a', b') \) be elements of \( \mathcal{P} \). Suppose that \( v(a - \alpha_i b) = \max_j (v(a - \alpha_j b)) \) and \( v(a' - \alpha_i b') = \max_j (v(a' - \alpha_j b')) \). Then \( v(a - \alpha_i b) = v(a' - \alpha_i b') \).

**Proof:** Recall that \( v(b) = 0 \) when \( \pi \mid h \) and \( (a, b) \) is primitive. Suppose that \( v(a - \alpha_i b) \) and \( v(a' - \alpha_i b') \) are not equal. We may assume without loss of generality that \( v(a/b - \alpha_i) > v(a'/b' - \alpha_i) \). We claim that this inequality implies that \( v(a/b - \alpha_j) \geq v(a'/b' - \alpha_j) \) for all \( j \). Indeed, \( v(a/b - \alpha_j) \geq v(a'/b' - \alpha_j) \) is clear if \( v(a'/b' - \alpha_j) \leq v(a/b - \alpha_j) \). Thus we may assume that \( v(a'/b' - \alpha_i) > v(a/b - \alpha_j) \). From \( v(a/b - \alpha_i) > v(a'/b' - \alpha_i) \) we find that \( v(a/b - a'/b') = v(a'/b' - \alpha_i) \). It follows
from \(v(a/b - a'/b') > v(a/b - \alpha_j)\) that \(v(a'/b' - \alpha_j) = v(a/b - \alpha_j)\), and our claim is proved. This claim contradicts the fact that \(v(F(a, b)) = v(F(a', b')) = v(h)\), and the lemma follows.

**Example 2.8** The above lemma is not correct if the hypothesis that both \((a, b)\) and \((a', b')\) be primitive solutions is dropped. Indeed, consider \(F(x, y) = (x - y)(x - p^2y)(x - (p^2 - p + 1)y)^4 \in \mathbb{Z}[x, y]\) with \((a, b) = (p^2 + 1, 1)\) and \((a', b') = (p, 0)\).

Slightly more can be said about the closure of \(\mathcal{P}\) in \(\mathcal{Y}/\mathcal{O}_L\). Consider the following two schemes, \(\mathcal{U}(\alpha_i)\) attached to a primitive point \((a, b)\) with associated valuation \(t\), and \(\mathcal{U}(\alpha_j)\) attached to a primitive point \((a', b')\) with associated valuation \(t'\). We claim that if \(v(\alpha_i - \alpha_j) \geq \text{min}(t, t')\), then the images of \(\mathcal{U}(\alpha_i)\) and \(\mathcal{U}(\alpha_j)\) in \(\mathcal{Y}\) are equal. Assume \(t' \leq t\). It follows that \(v(a' - \alpha_i b') \geq t'\). Thus, \(v(a' - \alpha_i b') = t'\), and Lemma 2.7 shows that \(t = t'\). We may then define an isomorphism from \(\mathcal{U}(\alpha_i)\) to \(\mathcal{U}(\alpha_j)\) on the level of rings

\[
\mathcal{O}_L[u', y]/(F_{m'}(u', y) - \mu) \longrightarrow \mathcal{O}_L[u, y]/(F_{m}(u, y) - \mu)
\]

by setting \(u' \mapsto u + \pi^{-t}(\alpha_i - \alpha_j)y\) and \(y \mapsto y\).

We have thus shown that there exist at most \(n\) disjoint disks in \(\mathcal{O}_L\), each centered at a root of \(F(x, 1)\), such that if \(\alpha_i\) and \(\alpha_j\) belong to the same disk (and have primitive solutions attached to them), then the images of \(\mathcal{U}(\alpha_i)\) and \(\mathcal{U}(\alpha_j)\) in \(\mathcal{Y}\) are equal. Note now that if a disk contains a single root of \(F(x, 1)\), say \(\alpha_i\), then by construction the special fiber of \(\mathcal{U}(\alpha_i)\) is a rational curve, given by an equation \(uy^{n-1} = \pi z^n\). In general, if a disk contains \(r\) distinct roots and \(\alpha_i\) is one of them, then the special fiber of \(\mathcal{U}(\alpha_i)\) is given by an equation of the form \(f_r(u, y)y^{n-r} = \pi\), where \(f_r\) is a homogeneous polynomial of degree \(r\), coprime to \(y\).

**2.9 The quotient construction.**

Let \(X/K\) be a smooth proper geometrically connected curve of genus \(g\). Let \(L/K\) be a cyclic Galois extension with Galois group \(\text{Gal}(L/K) = \langle \sigma \rangle\). Let \(\mathcal{Y}/\mathcal{O}_L\) be a normal model of \(X_L/L\) such that \(\text{Gal}(L/K)\) acts on \(\mathcal{Y}\), lifting its natural action on \(\text{Spec}(\mathcal{O}_L)\). An example of such a model \(\mathcal{Y}\) is the normalization in \(L(X)\) of a normal model over \(\mathcal{O}_K\) of \(X/K\). Another example is the minimal regular model \(\mathcal{Y}/\mathcal{O}_L\) of \(X_L/L\). Indeed, the following is well-known.

**2.10** Let \(\mathcal{Y}/\mathcal{O}_L\) be the minimal regular model of \(X_L/L\). The map \(\sigma\) induces a canonical morphism \(X_L \rightarrow X_L\) over the map \(\sigma: \text{Spec}(L) \rightarrow \text{Spec}(L)\). Since \(X_L\) is the generic fiber of \(\mathcal{Y}\), the map \(\sigma\) induces a birational proper map \(\mathcal{Y} \longrightarrow \mathcal{Y} \times_{\text{Spec}(\mathcal{O}_L)} \text{Spec}(\mathcal{O}_L)\) over \(\text{Spec}(\mathcal{O}_L)\). By the universal property of a minimal model ([C-S, page 310]), this map extends to a morphism from \(\mathcal{Y}\) to \(\mathcal{Y} \times_{\text{Spec}(\mathcal{O}_L)} \text{Spec}(\mathcal{O}_L)\) over \(\text{Spec}(\mathcal{O}_L)\). Since \(\mathcal{Y}\) is reduced and separated, this extension is unique. Hence, there exists then a unique automorphism \(\tau\) of \(\mathcal{Y}\) making the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{Y} & \overset{\tau}{\longrightarrow} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}_L) & \overset{\sigma}{\longrightarrow} & \text{Spec}(\mathcal{O}_L)
\end{array}
\]

**2.11** Let \(G = \langle \tau \rangle\). The following fact is standard: Since \(\mathcal{Y}/\mathcal{O}_L\) is projective, the quotient \(\mathcal{Z} = \mathcal{Y}/G\) can be constructed in the usual way by gluing together the rings of invariants of \(G\)-invariant affine open sets of \(\mathcal{Y}\). The scheme \(\mathcal{Z}/\mathcal{O}_K\) is normal and,
hence, its singular points are closed points of its special fiber. We let \( f : \mathcal{Y} \rightarrow \mathcal{Z} \) denote the quotient map.

The normal scheme \( \mathcal{Z} \) has quotient singularities. A desingularization \( \nu : \mathcal{X} \rightarrow \mathcal{Z} \) leads to a regular model \( \mathcal{X}/\mathcal{O}_K \) of \( X/K \). Let \( K^{nr} \) denote the maximal unramified extension of \( K \), and assume now that \( K = K^{nr} \). When \( L/K \) is a tamely ramified field extension, the quotient singularities of \( \mathcal{Z} \) are well-understood. We recall their properties below, closely following Viehweg’s article \cite{Vie}. We refer the reader to his work for more details. Though he states at the beginning of his paper that he considers only the equicharacteristic case, his proofs of the facts listed below are also correct in the mixed characteristic case. If \( \mathcal{X}/\mathcal{O}_K \) is any scheme, let us denote by \( \overline{\mathcal{X}} \) the special fiber \( \mathcal{X} \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_K/(\pi)) \).

2.12 (\cite{Vie}, page 303) Let \( \overline{\mathcal{Y}} \rightarrow \mathcal{Y} \) and \( \overline{\mathcal{Y}}^{red} : \mathcal{Y}^{red} \rightarrow \overline{\mathcal{Y}}^{red} \) be the natural morphisms induced by \( \tau \). Then the natural map
\[
\overline{\mathcal{Y}}^{red}/<\tau^{red}> \rightarrow \overline{\mathcal{Z}}^{red} = \overline{\mathcal{Y}}/<\tau>^{red}
\]
is an isomorphism of schemes over the residue field.

For any irreducible component \( Y_i \subset \mathcal{Y} \), let
\[
D(Y_i) := \{ \mu \in G \mid \mu(Y_i) = Y_i \} \quad \text{and} \quad I(Y_i) := \{ \mu \in G \mid \mu|_{Y_i} = \text{id} \}.
\]

2.13 (\cite{Vie}, page 303) Let \( m_i \) be the multiplicity if \( Y_i \) in \( \mathcal{Y} \) and let \( Z_j := f(Y_i) \). The multiplicity of \( Z_j \) in \( \overline{\mathcal{Z}} \) is equal to \( m_i \cdot [L : K]/|I_i| \).

Recall the following terminology. Let \((C \cdot D)\) denote the intersection number on a regular model \( \mathcal{X} \) of two divisors \( C \) and \( D \). Let us call chain of rational curves on \( \mathcal{X} \) a divisor \( D \) such that

1. \( D = \bigcup_{i=1}^{q} E_i \), \( E_i \) smooth and rational curve for \( i = 1, \ldots, q \).
2. \((E_i \cdot E_{i+1}) = 1 \) for all \( i = 1, \ldots, q - 1 \) and \((E_i \cdot E_j) = 0 \) for all \( j \neq i + 1 \).

Moreover, \((E_i \cdot E_i) \leq -2 \) for all \( i \). Let us call \( E_1 \) and \( E_q \) the end-components of the chain.

Consider again a normal model \( \mathcal{Y}/\mathcal{O}_L \) with an action of \( \text{Gal}(L/K) \) lifting the action on \( \text{Spec}(\mathcal{O}_L) \). Assume that \( \mathcal{U}/\mathcal{O}_L \) is a smooth open subset of \( \mathcal{Y}/\mathcal{O}_L \) such that \( \mathcal{U} \) is invariant under the action of \( G \). Let \( \mathcal{Z} := \mathcal{U}/G \).

2.14 (\cite{Vie}, section 6) There exists a regular scheme \( \mathcal{X}/\mathcal{O}_K \) and a proper birational morphism \( \nu : \mathcal{X} \rightarrow \mathcal{Z} \) such that \( \nu \) induces an isomorphism between \( \mathcal{X} - \{ \nu^{-1}(\mathcal{Z}_{\text{sing}}) \} \) and \( \mathcal{Z} - \{ \mathcal{Z}_{\text{sing}} \} \) and such that, for any \( z \in \mathcal{Z}_{\text{sing}} \), \( \nu^{-1}(z) \) is a connected chain of rational curves. The point \( z \) belongs to an end-component of the chain. Since \( \mathcal{U} \) is smooth, we find that if \( z \) is a singular point of \( \mathcal{Z} \), then \( \nu^{-1}(z) \) intersects the rest of the special fiber \( \overline{\mathcal{X}} \) with normal crossings in exactly one point, say on \( E_1 \). (Viehweg states in 8.1.d) on page 306 of \cite{Vie} that the model \( \mathcal{X} \), obtained by taking the quotient of \( \mathcal{U} \) and then resolving the singularities, has normal crossings.) Let us call the component \( E_q \) the terminal component of the chain \( \nu^{-1}(z) \). The other end-component of the chain \( \nu^{-1}(z) \) is attached to an irreducible component of \( \overline{\mathcal{X}} \setminus \nu^{-1}(z) \).

2.15 (\cite{Vie}, section 6) Let \( f : \mathcal{U} \rightarrow \mathcal{Z} \) denote the quotient map. Let \( z_1, \ldots, z_d \) be the closed points of \( \mathcal{Z} \) that are ramification points of the morphism \( \overline{f} : \overline{\mathcal{Y}} \rightarrow \overline{\mathcal{Z}}^{red} \). Then \( \{ z_1, \ldots, z_d \} \) is the set of singular points of \( \mathcal{Z} \). Moreover, if \( \nu : \mathcal{X} \rightarrow \mathcal{Z} \) is
the desingularization of $Z$ described in \ref{2.14}, then the multiplicity of the terminal component on the chain $\nu^{-1}(z_i)$ is equal to the number of closed points in the fiber $\overline{f}^{-1}(z_i)$.

We now apply the quotient construction to the case where the model $\mathcal{Y}/\mathcal{O}_L$ is smooth. The scheme $\overline{Z} = \mathcal{Y}/<\tau>$ has an irreducible special fiber. The reduced special fiber $\overline{Z}^{\text{red}}$ is obtained as the quotient of $\overline{\mathcal{Y}}$ by the action of $<\tau>$ and is then a smooth and proper curve. The multiplicity of $\overline{Z}$ in $Z$ equals $[L : K]/I(\overline{\mathcal{Y}})$. The regular model $X/\mathcal{O}_K$ obtained as the minimal desingularization of $Z$ is thus very simple.

### 2.16 Applications of the method of Chabauty-Coleman

We may now apply the method of Chabauty-Coleman to the case of Thue equations. Let $g := g(X_{F,h})$.

**Proposition 2.17.** Let $X_{F,h}/\mathbb{Q}$ be such that for some prime $p > n$, $p \nmid h$ and $p \nmid d^*(F)$. Let $K$ be any number field having an unramified prime $\mathfrak{P}$ of norm $p$. Assume that $\text{Chab}(X_{F,h}, K, \mathfrak{P}) < g$. Then

$$|X_{F,h}(K)| \leq (2g - 2)\frac{p - 1}{p - 2} + |X_{F,h}(\mathbb{F}_p)|.$$  

**Proof:** As noted in \ref{2.1}, $X_{F,h}/\mathbb{Q}_p$ has good reduction. Thus, we can apply \ref{1.2}.

**Proposition 2.18.** Let $X_{F,h}/\mathbb{Q}$ be such that for some prime $p > n$, $p \nmid h$ and $p \nmid d^*(F)$. Let $s$ denote the number of distinct roots of $F(x, 1)$ in $\overline{\mathbb{Q}}$. Let $K$ be any number field having an unramified prime $\mathfrak{P}$ of norm $p$. Assume that $\text{Chab}(X_{F,h}, K, \mathfrak{P}) < g$. Then $|X_{F,h}(K)| \leq (2g - 2)\frac{p - 1}{p - 2} + sp$.

**Proof:** Let $X := X_{F,h}$. As noted in \ref{2.1}, $X/\mathbb{Q}_p$ has good reduction after a tame extension of $\mathbb{Q}_p$. Thus, we may apply the quotient construction to describe a regular model of $X/\mathbb{Q}_p^{nr}$ over $\mathbb{Z}_p^{nr}$. Let $L = \mathbb{Q}_p^{nr}(\sqrt[p]{h})$. The extension $L/\mathbb{Q}_p^{nr}$ is Galois with cyclic Galois group. Let $\xi_n$ be a primitive $n$-th root of unity, and denote by $\sigma : L \to L$, with $\sigma(\sqrt[p]{h}) = \xi_n \sqrt[p]{h}$, a generator of $\text{Gal}(L/K)$. The morphism $\sigma$ lifts to a morphism $\sigma : X_L \to X_L$ by setting

$$L[u, v, w]/(F(u, v) - hw^n) \xrightarrow{\sigma} L[u, v, w](F(u, v) - hw^n)$$

with $\sigma(u) = u$, $\sigma(v) = v$ and $\sigma(w) = w$. Let $\mathcal{Y}$ denote the normalization of $\text{Proj}(\mathcal{O}_L[x, y, z]/(F(x, y) - z^n))$. Then $\mathcal{Y}/\mathcal{O}_L$ is the smooth minimal model of $X_L/L$ (see \ref{2.1}). The morphism $\sigma : X_L \to X_L$ extends to a morphism $\sigma : \mathcal{Y} \to \mathcal{Y}$ by setting

$$\mathcal{O}_L[x, y, z]/(F(x, y) - z^n) \xrightarrow{\sigma} \mathcal{O}_L[x, y, z]/(F(x, y) - z^n)$$

with $\sigma(x) = x$, $\sigma(y) = y$, and $\sigma(z) = \xi_n z$. When restricted to the special fiber $\mathcal{Y}$ of $\mathcal{Y}$, the morphism $\sigma$ becomes an automorphism $\sigma$ over $\mathbb{F}_p$ of $\mathcal{Y}$ which lifts the
standard automorphism of \( \text{Proj}(k[x, y, z]/(F - z^n)) \). The quotient map \( \overline{\mathcal{Y}} \to \overline{\mathcal{Y}}/\langle \sigma \rangle \) is ramified over at most \( s \) points. It follows from \[2.15\] that the desingularization \( \mathcal{X} \) of \( \mathcal{Y}/\langle \sigma \rangle \) has a special fiber containing at most \( s \) (smooth rational) components of multiplicity one.

Consider now the minimal regular model \( \mathcal{X}_0/\mathbb{Z}_p \) of \( X/\mathbb{Q}_p \). A point in \( X(\mathbb{Q}_p) \) specializes in the special fiber \( \overline{\mathcal{X}_0}/\mathbb{F}_p \) to a smooth point, belonging to a geometrically integral irreducible component \( C/\mathbb{F}_p \) of multiplicity one. Let \( \overline{X}_0 := \mathcal{X}_0 \times_{\text{Spec}(\mathbb{Z}_p)} \text{Spec}(\mathbb{Z}^{nr}_p) \). Since the self-intersection of \( C \) in \( \mathcal{X}_0 \) equals the self-intersection of \( C \) in \( \overline{X}_0 \) (see, e.g., \[3-1\], 1.4), we find that \( C \) cannot be contracted in \( \overline{X}_0 \) and, thus, corresponds to a component in the minimal regular model \( \overline{\mathcal{X}}_{00} \) of \( X/\mathbb{Q}_p^{nr} \). Since there is a natural morphism \( \mathcal{X} \to \overline{\mathcal{X}}_{00} \), our description above of the special fiber of \( \mathcal{X} \) implies that there are at most \( s \) components of \( \overline{\mathcal{X}}_0 \) that can contain the reduction of a \( \mathbb{Q}_p \)-point, and that each such component is a smooth rational curve. Moreover, each such component \( C \) meets the divisor \( \overline{\mathcal{X}}_0 - C \) in exactly one \( \mathbb{F}_p \)-point. Hence, the number of points in \( \overline{\mathcal{X}}_0 \) that can be reductions of \( \mathbb{Q}_p \)-rational points is at most \( sp \).

**Remark 2.19** Let \( K \) be any field with a discrete valuation. If \( \pi_K \not| d^*(F) \) and \( \pi_K \mid h \), then \( X/K \) achieves good reduction over \( L := K(\sqrt[n]{h}) \). The quotient construction used in the above proof can be used to show that if \( M \) does not contain \( L \), then \( X_M/M \) does not have good reduction at any maximal ideal of \( \mathcal{O}_M \).

Let \( K \) be any number field, and let \( \mathfrak{p} \) be a maximal ideal of \( \mathcal{O}_K \). Let \( N(F, h, K, \mathfrak{p}) \) denote the number of solutions \( (x, y) \in (\mathcal{O}_K)^2_{\mathfrak{p}} \) of \( F(x, y) = h \) with \( \gcd(x, y) = 1 \).

**Proposition 2.20.** Let \( X_{F,h}/\mathbb{Q} \) be such that for some prime \( p > n \), \( p \not| h \) but \( p \mid d^*(F) \). Let \( K \) be any number field having an unramified prime \( \mathfrak{p} \) of norm \( p \). Assume that \( \text{Chab}(X_{F,h}, K, \mathfrak{p}) < g \). Let \( a(p) \) denote the number of \( \mathbb{F}_p \)-rational points of the affine curve \( F(x, y) - h = 0 \mod p \). Then

\[
N(F, h, K, \mathfrak{p}) \leq (2g - 2) \frac{p - 1}{p - 2} + a(p).
\]

**Proof:** Consider the model \( \mathcal{C}/\mathbb{Z}_p \) given by

\[
\mathcal{C} = \text{Proj}(\mathbb{Z}_p[x, y, z]/(F - h z^n)).
\]

The special fiber \( \overline{\mathcal{C}}/\mathbb{F}_p \) is a plane projective curve with possible singularities only at points \( (x : y : z) \) with \( z = 0 \). None of the singular points of \( \overline{\mathcal{C}} \) can be the reduction of a primitive integral solution of \( X_{F,h}(\mathbb{Q}) \). Resolve the singularities of \( \mathcal{C} \) to obtain a regular model \( \mathcal{X}/\mathbb{Z}_p \) of \( X_{F,h}/\mathbb{Q} \). The only points in \( \overline{\mathcal{X}}/\mathbb{F}_p \) that can be reduction of primitive points in \( X_{F,h}(\mathbb{Q}) \) are the points in \( \overline{\mathcal{X}}(\mathbb{F}_p) \) that correspond to \( \mathbb{F}_p \)-rational points of \( \overline{\mathcal{C}} \) with \( z \neq 0 \). Apply \[1.2\].

**Remark 2.21** The integer \( a(p) \) can be bounded using the Weil bound for singular curves (where the genus is replaced by the arithmetic genus) and applying it to each irreducible components of \( \overline{\mathcal{C}} \). When \( p \) is not too large compared to \( n \) a better bound for \( a(p) \) can be obtained as follows. Project the curve \( \overline{\mathcal{C}}/\mathbb{F}_p \) onto an \( \mathbb{F}_p \)-rational projective line, using when possible one of the points at \( \infty \) of \( \overline{\mathcal{C}}(\mathbb{F}_p) \). Then the projection map has degree at most \( n \) (or \( n - 1 \) when \( \overline{\mathcal{C}}(\mathbb{F}_p) \neq \emptyset \)). It follows that \( a(p) \leq np \).
Proposition 2.22. Let $X_{F,h}/\mathbb{Q}$ be such that for some prime $p > n$, $p \mid d^*(F)$ and $p \mid h$. Let $K$ be any number field having an unramified prime $\mathfrak{p}$ of norm $p$. Assume that $\text{Chab}(X_{F,h}, K, \mathfrak{p}) < g$. Then

$$N(F, h, K, \mathfrak{p}) \leq (2g - 2)\frac{p-1}{p-2} + \text{snp}.$$  

Proof: Let $X := X_{F,h}$. Let $L/\mathbb{Q}_p^{nr}$ denote the splitting field of $F(x, 1)$ over $\mathbb{Q}_p^{nr}$. Since $p > n$, the extension $L/\mathbb{Q}_p^{nr}$ is tame and, thus, cyclic. Let $\mathcal{Y}/\mathcal{O}_L$ be the normalization of

$$\mathcal{C}/\mathcal{O}_L := \text{Proj}(\mathcal{O}_L[x, y, z]/(F(x, y, z) - h z^n)).$$

Let $\langle \sigma \rangle = \text{Gal}(L/K)$. The morphism $\sigma$ induces obvious automorphisms

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\sigma} & \mathcal{C} \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}_L) & \xrightarrow{\sigma} & \text{Spec}(\mathcal{O}_L).
\end{array}$$

We shall denote by $G := \langle \sigma \rangle$ the group of automorphisms of $\mathcal{Y}$, resp. $\mathcal{C}$, generated by $\sigma$. Fix a root $\alpha_i$ of $F(x, 1)$ such that there exists a primitive solution $P = (a, b)$ with $t := v_L(a - \alpha_i) = \max_j(v_L(a - \alpha_j))$. Let $U(\alpha_i)$ denote the regular open subset of $\mathcal{Y}/\mathcal{O}_L$ described in 2.6. Recall that $U(\alpha_i) = \text{Spec}(\mathcal{O}_L[u, y]/(F_m(u, y) - \mu))$. Thus its special fiber may not be irreducible. The following lemma, whose proof is omitted, describes the possible components of $U(\alpha_i)$.

Lemma 2.23. Let $k$ be any algebraically closed field. Let $n \in \mathbb{N}$ with $\text{char}(k) \nmid n$. Let $\xi_n$ denote a primitive $n$-th root of unity in $k$. Let $f(x, y)$ be homogeneous of degree $n$ in $k[x, y]$, and let $\mu \in k^*$. Then $f(x, y) - \mu z^n$ factors in $k[x, y, z]$ if and only if there exist $d \mid n$ and $g \in k[x, y]$ with $f = g^{n/d}$. Then $f - \mu z^n = \prod_{i=1}^{n/d}(g - \xi_n^{id}/\sqrt[n]{\mu}z^d)$.

We will also need the following lemma describing the action of $G$ on the components of $U(\alpha_i)$. Recall the definitions of $D(Y_\ell)$ and $I(Y_\ell)$ in 2.12.

Lemma 2.24. Let $Y_1, \ldots, Y_{n/d}$ denote the irreducible components of $\overline{\mathcal{Y}}$ whose generic points belong to $U(\alpha_i)$. Then $D(Y_\ell) = D(Y_j) = G$ and $I(Y_\ell) = I(Y_j)$ for all $\ell, j \in \{1, \ldots, n/d\}$.

Proof: Since $p \nmid n$, the group of $n$-th roots of unity is contained in $\mathbb{Q}_p^{nr}$, and acts on $\mathcal{C}/\mathcal{O}_L$ as follows. A generator $\xi_n$ induces an automorphism $\varphi : \mathcal{C} \to \mathcal{C}$ given by:

$$\mathcal{O}_L[x, y, z]/(F(x, y) - h z^n) \xrightarrow{\varphi^*} \mathcal{O}_L[x, y, z]/(F(x, y) - h z^n),$$

where $x \mapsto x$, $y \mapsto y$, and $z \mapsto \xi_n z$. The automorphism $\varphi$ induces an automorphism $\varphi : \mathcal{Y} \to \mathcal{Y}$. The generator $\xi_n$ also induces an automorphism $\varphi : U(\alpha_i) \to U(\alpha_i)$ given by

$$\mathcal{O}_L[u, y]/(F_m(u, y) - \mu) \xrightarrow{\varphi^*} \mathcal{O}_L[u, y]/(F_m(u, y) - \mu),$$
where $u \mapsto \xi_n^{-1}u$ and $y \mapsto \xi_n^{-1}y$. Let

$$\psi^* : \mathcal{O}_L[x, y]/(F(x, y) - h) \rightarrow \mathcal{O}_L[u, y]/(F_m(u, y) - \mu).$$

be given by $x \mapsto \pi_L^* u + \alpha y$, and $y \mapsto y$. The induced morphism $\psi : \mathcal{U}(\alpha_i) \rightarrow \mathcal{C}$ was shown to induce an open immersion $\psi : \mathcal{U}(\alpha_i) \rightarrow \mathcal{Y}$ in \cite{2}. The reader will easily verify that the diagram

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\varphi} & \mathcal{Y} \\
\uparrow{\psi} & & \uparrow{\psi} \\
\mathcal{U}(\alpha_i) & \xrightarrow{\varphi} & \mathcal{U}(\alpha_i)
\end{array}$$

is commutative. As above, let $\sigma$ also denote the automorphisms induced on $\mathcal{Y}$ and $\mathcal{C}$ by a generator $\sigma$ of $\text{Gal}(L/K)$. Since $\sigma$ and $\varphi$ commute, and since $\varphi$ acts transitively on $\{Y_1, \ldots, Y_{n/d}\}$, we find that $D(Y_i) = D(Y_j)$ and $I(Y_i) = I(Y_j)$ for all $\ell, j \in \{1, \ldots, n/d\}$. We let $D := D(Y_j)$ and $I := I(Y_j)$. Note now that $D = G$. Indeed, if $P = (a, b)$ reduces to $Y_j$ for some $j$, then $\sigma(P)$ reduces to $\sigma(Y_j)$. Since $(a, b) \in (\mathbb{Z}_p^{nr})^2$, we find that $P$ reduces to a point in $Y_j \cap \sigma(Y_j)$. Since $P$ reduces to a non-singular point of $\mathcal{U}(\alpha_i)$, we find that $Y_j = \sigma(Y_j)$. This concludes the proof of Lemma \cite{2}.

Consider the following $G$-invariant subset of $\mathcal{Y}$:

$$\mathcal{V}(\alpha_i) := \bigcap_{\tau \in G} \tau(\mathcal{U}(\alpha_i)).$$

Let $\tilde{P}$ denote the closure of $P \in X(L)$ in $\mathcal{Y}$. Then $\tilde{P} \in \mathcal{U}(\alpha_i)$. Since $P$ is fixed by $\tau$, $\tau(\mathcal{U}(\alpha_i))$ contains $\tilde{P}$ and, thus, $\tilde{P} \in \mathcal{V}(\alpha_i)$.

We need to understand the desingularization of $\mathcal{V}(\alpha_i)/G$. Consider first the case where $I = G$. (This case happens for instance if $\alpha_i \in \mathbb{Q}_{p}^{nr}$.) Then

$$\mathcal{V}(\alpha_i) \longrightarrow \mathcal{V}(\alpha_i)/G$$

is an isomorphism. Let

$$\begin{align*}
\mathcal{C} & := \text{Proj}(\mathcal{O}_L[x, y, z]/(F - h z^n)) \\
\mathcal{D} & := \text{Proj}(\mathbb{Z}_p^{nr}[x, y, z]/(F - h z^n)) \\
\mathcal{D}' & := \text{Proj}(\mathbb{Z}_p[x, y, z]/(F - h z^n)).
\end{align*}$$

Let $\mathcal{Z}'/\mathbb{Z}_p$ and $\mathcal{Z}/\mathbb{Z}_p^{nr}$ denote the normalization of $\mathcal{D}'$ and $\mathcal{D}$, respectively. Clearly $\mathcal{Z} = \mathcal{Y}/G$. We have the following commutative diagram:

$$\begin{array}{cccc}
Y_j & \subset & \mathcal{Y} & \xrightarrow{\rho} & \mathcal{C} & \longrightarrow & \text{Spec}(\mathcal{O}_L) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z_\ell & \subset & \mathcal{Z} & \xrightarrow{\varepsilon} & \mathcal{D} & \longrightarrow & \text{Spec}(\mathbb{Z}_p^{nr}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z'_\ell & \subset & \mathcal{Z}' & \xrightarrow{\varepsilon'} & \mathcal{D}' & \longrightarrow & \text{Spec}(\mathbb{Z}_p).
\end{array}$$
The map $\rho$ induces a morphism $\rho_j : Y_j \to \rho(Y_j)$, given in coordinates by the bottom horizontal map below:

\[
\begin{align*}
\mathcal{O}_L[x,y]/(F(x,y) - h) & \longrightarrow \mathcal{O}_L[u,y]/(F_m(u,y) - \mu) \\
\downarrow & \\
\mathcal{O}_L[x,y]/(\pi_L - \alpha_i y) & \longrightarrow \mathcal{O}_L[u,y]/(\pi_L F_m(u,y) - \mu).
\end{align*}
\]

This morphism is clearly of degree at most $n$. The morphism

\[\varepsilon'_i : Z'_i \to \varepsilon'_i(Z'_i)\]

induced by $\rho_j$ is also of degree at most $n$. The curve $\varepsilon'_i(Z'_i)/\mathbb{F}_p$ is a smooth projective line. The primitive integral point $(a, b)$ cannot reduce to the intersection point $Q$ of all the components of the special fiber of $\mathcal{D}'$. The morphism $\varepsilon'_i$ is defined over $\mathbb{F}_p$, and there are at most $np \mathbb{F}_p$-rational points in the preimage of $\varepsilon'_i(Z'_i) \setminus \{Q\}$. We conclude that at most $np$ points in the image of $\mathcal{V}(\alpha_i)$ in $Z'$ can be residue classes of primitive integral points.

Let us now consider the case where $I \subseteq D$. Then the image $Z$ of $Y_j$ in $Z = \mathcal{Y}/G$ has multiplicity $|D|/|I| > 1$, and [2.13] indicates that to count the components of multiplicity one (in a desingularization of $Z'$) which contain the reduction of primitive integral points, one first needs to count the number of totally ramified points in the branch locus of $Y_j \to Z$. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y} \\
\uparrow{\psi} & & \uparrow{\psi} \\
\mathcal{V}(\alpha_i) & \xrightarrow{\sigma} & \mathcal{V}(\alpha_i),
\end{array}
\]

where $\sigma : \mathcal{V}(\alpha_i) \to \mathcal{V}(\alpha_i)$ is defined so that the diagram commutes. Consider an open set $U$ of $\mathcal{V}(\alpha_i)$ that is dense in each fiber and is a special open set of $U(\alpha_i)$. We find that on the level of rings, $\sigma : U \to U(\alpha_i)$ induces the top horizontal map below

\[
\begin{align*}
\mathcal{O}_L[u,y]/(F_m(u,y) - \mu) & \xrightarrow{\sigma} S^{-1} \mathcal{O}_L[u,y]/(F_m(u,y) - \mu) \\
\uparrow & \\
\mathcal{O}_L[x,y]/(F(x,y) - h) & \xrightarrow{\sigma} \mathcal{O}_L[x,y]/(F(x,y) - h) \\
\uparrow & \\
\mathcal{O}_L & \xrightarrow{\sigma} \mathcal{O}_L,
\end{align*}
\]

with $\sigma(x) = x$ and $\sigma(y) = y$ so that $\sigma(\pi_L u + \alpha_i y) = \pi_L u + \alpha_i y$. Since $\sigma(\pi_L u + \alpha_i y) = \sigma(\pi_L)\sigma(u) + \sigma(\alpha_i) y$, we find that

\[
\sigma(u) = \frac{\pi_L u + \alpha_i - \sigma(\alpha_i)}{\sigma(\pi_L)} y.
\]

(Note that both $\pi_L / \sigma(\pi_L)$ and $(\alpha_i - \sigma(\alpha_i))/\sigma(\pi_L)$ belong to $\mathcal{O}_L$.) By hypothesis, $\sigma := \sigma|_{Y_j}$ does not act trivially on $Y_j$. The points where the morphism $Y_j \to Y_j/ < \sigma >$ is totally ramified is the set of fixed points of the map $\sigma$. On the plane curve $F_m(u,y) - \mu = 0$, the automorphism $\sigma$ is given by $u \mapsto cu + dy$ and $y \mapsto y$, for some $c, d \in k$. Thus the fixed points of $\sigma$ lie on the line $(c-1)u + dy = 0$, and we
find that there are at most $n$ such points. Let now $\nu : \mathcal{X} \to \mathcal{Z}$ denote the minimal desingularization of $\mathcal{Z}$. As we recalled in 2.13, the special fiber of $\mathcal{X}$ contains at most $n$ components of multiplicity one, each smooth and rational, and each meeting the rest of the special fiber in a single point.

Consider now the minimal regular model $\mathcal{X}_0/\mathbb{Z}_p$ of $X/\mathbb{Q}_p$. A point in $X(\mathbb{Q}_p)$ specializes in the special fiber $\overline{\mathcal{X}}_0/\mathbb{F}_p$ to a smooth point, belonging to a geometrically integral irreducible component $C/\mathbb{F}_p$ of multiplicity one. Let $\overline{\mathcal{X}}_0 := \mathcal{X}_0 \times_{\text{Spec}(\mathbb{Z}_p)} \text{Spec}(\mathbb{F}_p)$. Since the self-intersection of such a component $C$ in $\mathcal{X}_0$ equals the self-intersection of $C$ in $\overline{\mathcal{X}}_0$ (see, e.g., [B-L], 1.4), we find that $C$ cannot be contracted in $\overline{\mathcal{X}}_0$ and, thus, corresponds to a component in the minimal regular model $\overline{\mathcal{X}}_{00}$ of $X/\mathbb{Q}_p^{nr}$. Since there is a natural morphism $\mathcal{X} \to \overline{\mathcal{X}}_{00}$, our description above of the special fiber of $\mathcal{X}$ implies that there are at most $n$ components of $\overline{\mathcal{X}}_0$ that can contain the reduction of a $\mathbb{Q}_p$-point, and that each such component is a smooth rational curve. Moreover, each such component $C$ meets the divisor $\overline{\mathcal{X}}_0 - C$ in exactly one $\mathbb{F}_p$-point. Hence, the number of points in $\overline{\mathcal{X}}_0$ that can be reductions of $\mathbb{Q}_p$-rational points is at most $np$. Since the contribution of an open set of the form $\mathcal{V}(\alpha_i)$ to the number of reductions of primitive integral points in the special fiber of the model $\mathcal{X}_0$ is bounded by $np$, and since the primitive integral points are contained in at most $s$ such open sets (2.7), we find that the reduction of the primitive integral points in the special fiber of the model $\mathcal{X}_0$ consists in at most $snp$ points. This concludes the proof of Proposition 2.22.

Let us now state our main theorem. Let $N(F, h)$ denote the number of solutions $(x, y) \in \mathbb{Z}^2$ of $F(x, y) = h$ with $\gcd(x, y) = 1$.

**Theorem 2.25.** Let $p$ be a prime with $n < p < 2n$. Assume that the Chabauty rank with respect to $(p)$ of $X_{F,h}/\mathbb{Q}$ is less than $g := g(X_{F,h})$. Then

$$N(F, h) \leq 2n^3 - 2n - 3.$$ 

More precisely, choose a prime $p$ with $n < p < 2n$.

a) If $p \nmid h$ and $p \nmid d^*(F)$, then $|X_{F,h}(\mathbb{Q})| \leq 2g + s - 5 + 2n(n-1)$.

b) If $p \mid h$ and $p \nmid d^*(F)$, then $|X_{F,h}(\mathbb{Q})| \leq 2g - 5 + 2sn$.

c) If $p \nmid h$ and $p \mid d^*(F)$, then $N(F, h, \mathbb{Q}, p) \leq 2g + s - 5 + n(2n-1)$.

d) If $p \mid h$ and $p \mid d^*(F)$, then $N(F, h, \mathbb{Q}, p) \leq 2g + s - 5 + sn(2n-1)$.

In particular, if the Mordell-Weil rank of $X_{F,h}/\mathbb{Q}$ is less than $g$, then $N(F, h) \leq 2n^3 - 2n - 3$.

**Proof:** We apply our previous results using the estimate $p \leq 2n - 1$ and $s \leq n$. The term $(2g-2)(p-1)/(p-2)$ is bounded by $2g + s - 5$. To prove a), apply 2.17 and bound $|\overline{X}_{F,h}(\mathbb{F}_p)|$ using a projection from a smooth point of $\overline{X}_{F,h}(\mathbb{F}_p)$ to a $\mathbb{F}_p$-line; we find in this case that $|\overline{X}_{F,h}(\mathbb{F}_p)| \leq (n-1)(p+1)$. (Note that we may assume that $\overline{X}_{F,h}(\mathbb{F}_p)$ is not empty because otherwise $X_{F,h}(\mathbb{Q})$ is empty.) To prove c), apply 2.20, and bound $a(p)$ using a projection from a point of $\overline{X}_{F,h}(\mathbb{F}_p)$ to a line; we find again that $a(p) \leq (n-1)(p+1)$. To prove b) and d), use 2.18 and 2.22.
Note now that by Bertrand’s postulate, there exists a prime \( p \) with \( n < p < 2n \). If the Mordell-Weil rank of \( X_{F,h}/\mathbb{Q} \) is less than \( g \), then the Chabauty rank with respect to \( (p) \) of \( X_{F,h}/\mathbb{Q} \) is also less than \( g \), and we find that \( N(F,h) \leq 2n^3 - 2n - 3 \).

**Remark 2.26** Recall that for any \( a \in \mathbb{Z} \), the curve \( X_{F,h,a} \) is isomorphic over \( \mathbb{Q} \) to \( X_{F,h} \). It follows that the Chabauty rank of \( X_{F,h,a} \) is equal to the Chabauty rank of \( X_{F,h} \). Thus, when the method of Chabauty-Coleman can be applied to bound \( N(F,h,\mathbb{Q},p) \), it can also be used to bound the size of subsets of \( X_{F,h}(\mathbb{Q}) \) other than the subset of primitive \( p \)-integral solutions. For instance, consider the case where \( F(x,1) \) has distinct roots and let, for \( i \geq 0 \),

\[
S_i := \{ (x : y : z) \in X_{F,h}(\mathbb{Q}) \mid x, y, z \in \mathbb{Z}(p), p \nmid z, (x,y) = p^i\mathbb{Z}(p) \}.
\]

The set \( S_0 \) is the set of primitive \( p \)-integral points of \( X_{F,h}/\mathbb{Q} \). The set \( S_i \) is in bijection with the set of primitive \( p \)-integral points of \( X_{F,h,p^{-m}}/\mathbb{Q} \). The set

\[
T_i := \{ (x : y : z) \in X_{F,h}(\mathbb{Q}) \mid x, y, z \in \mathbb{Z}(p), (x,y,z) = \mathbb{Z}(p), v_p(z) = i \}
\]

is in bijection with the set of primitive \( p \)-integral points on \( X_{F,h,p^{-m}}/\mathbb{Q} \).

**Remark 2.27** The method of Chabauty-Coleman bounds the number of rational points in \( X_{F,h}(\mathbb{Q}) \) in terms of the number \( N \) of irreducible components (of the special fiber of a regular model) of multiplicity one that contain \( \mathbb{F}_p \)-points. Thus, the method provides a bound for \( |X_{F,h}(\mathbb{Q})| \) in terms of \( n \) if \( N \) can be bounded in terms of \( n \) only. As the following example with \( n = 6 \) shows, the integer \( N \) is not bounded in terms of \( n \) in general.

Consider a genus 2 curve \( Y/\mathbb{Q} \) with a regular model \( \mathcal{Y}/\mathbb{Z}_p \) whose special fiber consists of a chain of projective lines over \( \mathbb{F}_p \) with one elliptic curve attached to each end of the chain. All intersections in \( \overline{\mathcal{Y}} \) are transverse and all components have multiplicity 1. Thus \( \text{Jac}(Y)/\mathbb{Q} \) has good reduction modulo \( p \). An example of such a curve when \( p \geq 7 \) is the curve \([I_0 - I_0 - m]\) in \([N-U]\) given by

\[
y^2 = (x^3 + \alpha x + 1)(x^3 + \beta p^{4m}x + p^{6m}), \quad \text{with} \ \alpha, \beta \in \mathbb{Z}_p^*.
\]

The number of rational curves in the special fiber of the minimal model \( \mathcal{Y}/\mathbb{Z}_p \) is \( m \) (and not \( m + 1 \) as stated in \([N-U]\)). This can be shown using Liu’s algorithm (\([Liu]\), Thm. 1 and Prop. 2). Consider now the curve \( X/\mathbb{Q} \) given by

\[
y^6 = (x^3 + \alpha x + 1)(x^3 + \beta p^{4m}x + p^{6m}).
\]

We claim that the minimal model \( \mathcal{X}/\mathcal{O}_K \) has a special fiber \( \overline{\mathcal{X}} \) consisting of two curves of genus 4 linked by three chains of rational curves, as in the picture below on the left. The special fiber \( \overline{\mathcal{Y}} \) is represented below.

Let us only give a brief sketch of the proof of this claim. First, the map \( \varphi : E \to D \) is the natural map between the curve \( E \) given by \( y^6 = (x^3 + \alpha x + 1)x^3 \) and the curve \( D \) given by \( y^2 = (x^3 + \alpha x + 1)x^3 \). Making the change of variable \( u := x/t^{2m} \), the curve \( E' \) is given by \( y^6 = x^3 + \beta x + 1 \) and the curve \( D' \) by \( y^2 = x^3 + \beta x + 1 \). The map \( \varphi' : E' \to D' \) is the natural map. The maps \( \varphi \) and \( \varphi' \) are not ramified over the points
at \( \infty \) of \( D \) and \( D' \). Thus the graph associated with a regular model of \( X \) having normal crossings must contain at least two independent loops. Since \( g(X) = 10 \) and \( g(E) = g(E') = 4 \), we conclude that \( X \) has semistable reduction over \( O_K \). The automorphism \( \sigma : (x, y) \mapsto (x, \xi y) \) induces an action on the minimal regular model \( X'/O_K \) of \( X/K \). It can be shown that \( X'/\sigma \) is a semi-stable regular model for \( Y/K \). Since the minimal model \( Y/O_K \) has a special fiber with \( m \) projective lines, we conclude that \( X' \) contains \( 3m \) projective lines.

3. Some refinements

We use in this section the fact that any subfield of \( \mathbb{Q}(\xi_{p-1}) \) has an unramified prime of norm \( p \) to obtain in some cases bounds for \( N(F, h, \mathbb{Q}, p) \) of the form \( O(n^2) \).

**Theorem 3.1.** Let \( n \geq 5 \) be a prime so that \( p := an + 1 \) is also prime for some \( a > 1 \). Assume that the Mordell-Weil rank of \( \text{Jac}(X_{F,h})(\mathbb{Q}) \) is less than \((n - 3)/2\).

**Proof:** Let \( X := X_{F,h} \). Recall ([P-S], 13.4) that when \( n \) is prime, 

\[
\text{rank}_Z(\text{Jac}(X/\mathbb{Q}))(n-1) = \text{rank}_Z(\text{Jac}(X/\mathbb{Q}(\xi_n))).
\]

Thus, our hypothesis implies that the Mordell-Weil rank of \( \text{Jac}(X/\mathbb{Q}(\xi_n)) \) is less than \( g(X) \). We may then apply the results of the previous section. We bound \((2g - 2)(p - 1)/(p - 2)\) by \( n^2 - 2n - 3 \), and \( s \) by \( n \). Let \( u \) denote the number of points in \( X(\mathbb{Q}) \) with \( z = 0 \). Clearly, \( u \leq n \). Let \( v := |X(\mathbb{Q})| - u \). Then

\[
|X(\mathbb{Q}(\xi_n))| \geq u + nv.
\]

For part (a), we use the bound \([2.17]\):

\[
|X(\mathbb{Q}(\xi_n))| \leq n^2 - 2n - 3 + (n - 1)(p + 1).
\]

(To bound \( |\overline{X}(\mathbb{F}_p)| \), use a projection from a point in \( \overline{X}(\mathbb{F}_p). \)). It follows that

\[
u + nv \leq (a + 1)n^2 - an - 5.
\]

Hence,

\[
v \leq (a + 1)n - a - 5/n - u/n.
\]

Thus, \( v \leq (a + 1)(n - 1) \). We find that

\[
|X(\mathbb{Q})| = u + v \leq n + (a + 1)(n - 1).
\]

To prove part (b), we use \([2.18]\):

\[
|X(\mathbb{Q}(\xi_n))| \leq n^2 - 2n - 3 + np.
\]

Thus,

\[
 u + nv \leq (a + 1)n^2 - n - 3.
\]

Hence, \( v \leq (a + 1)n - 2 \), and \( |X(\mathbb{Q})| \leq (a + 2)n - 2 \). To prove part c), we use \([2.20]\):

\[
nN(F, h, \mathbb{Q}, p) \leq N(F, h, \mathbb{Q}(\xi_n), p) \leq n^2 - 2n - 3 + (n - 1)(p + 1).
\]
(To bound \(a(p)\), we use the fact that \(a(p) \leq |X(\mathbb{F}_p)|\), and project \(X(\mathbb{F}_p)\) to a line using one of its points). Finally, to prove part d), we use 2.22:

\[ nN(F, h, \mathbb{Q}, p) \leq N(F, h, \mathbb{Q}(\xi_n), p) \leq n^2 - 2n - 3 + n^2p. \]

**Theorem 3.2.** Let \(p \geq 5\) be prime and let \(n := p - 1\). Let \(X := X_{F,h}\). Assume that \(\text{Chab}(X, \mathbb{Q}(\xi_{p-1}), (p)) < g(X)\). This is the case, for instance, if the Mordell-Weil rank of \(X/\mathbb{Q}\) is less than \((s - 2)/2\). Then

a) If \(p \nmid d^*(F)\), then \(N(F, h, \mathbb{Q}, p) \leq 4n - 3\) and \(|X(\mathbb{Q})| \leq 5n - 3\).

b) If \(p \nmid h\) and \(p \mid d^*(F)\), then \(N(F, h, \mathbb{Q}, p) \leq 4n - 3\).

c) If \(p \mid h\) and \(p \mid d^*(F)\), then \(N(F, h, \mathbb{Q}, p) \leq 2n^2 + 4n - 5\).

**Proof:** We proceed as in the previous theorem, with \(|X(\mathbb{Q}(\xi_n))| \geq u + \nu/2\). For part (a), we use the bound 2.17:

\(|X(\mathbb{Q}(\xi_n))| \leq n^2 - 2n - 3 + (n - 1)(p + 1)|\)

(To bound \(|X(\mathbb{F}_p)|\), use a projection from a point in \(X(\mathbb{F}_p)\).) It follows that \(u + \nu/2 \leq 2n^2 - n - 5\). Hence, \(v \leq 4n - 2 - 10/n - 2u/n\). Thus, \(v \leq 4n - 3\). We find that

\(|X(\mathbb{Q})) = u + v \leq n + 4n - 3.\)

To prove part (b), we use 2.18:

\(|X(\mathbb{Q}(\xi_n))| \leq n^2 - 2n - 3 + np.\)

Thus, \(u + \nu/2 \leq 2n^2 - n - 3\). Hence, \(v \leq 4n - 3\), and \(|X(\mathbb{Q})| \leq 5n - 3\). To prove part (c), we use 2.20:

\[ nN(F, h, \mathbb{Q}, p)/2 \leq N(F, h, \mathbb{Q}(\xi_n), p) \leq n^2 - 2n - 3 + (n - 1)(p + 1). \]

(To bound \(a(p)\), we use the fact that \(a(p) \leq |X(\mathbb{F}_p)|\), and project \(X(\mathbb{F}_p)\) to a line using one of its points). Finally, to prove part d), we use 2.22:

\[ nN(F, h, \mathbb{Q}, p)/2 \leq N(F, h, \mathbb{Q}(\xi_n), p) \leq n^2 - 2n - 3 + n^2p. \]

The assertion that, if the Mordell-Weil rank of \(X/\mathbb{Q}\) is less than \((s - 2)/2\), then the Chabauty rank of \(\text{Jac}(X/\mathbb{Q}(\xi_{p-1}))\) is less than \(g(X)\), is a consequence of the following general fact proved in 3.3 below. Let \(X/\mathbb{Q}\) denote the smooth proper model of the affine curve given by an equation \(hy^n = f(x)\), with \(n \mid \deg(f)\). Write \(f(x) = \prod_{i=1}^s (x - a_i)^{n_i} \prod_{i \neq j}(a_i - a_j) \neq 0\), and assume that \(\gcd(n, \nu_i)\) for all \(i\). Lemma 13.4 in \[P-Σ\] states that when \(n\) is prime, then

\(\text{rank}_Z(\text{Jac}(X/\mathbb{Q}))(n - 1) = \text{rank}_Z(\text{Jac}(X/\mathbb{Q}(\xi_n)))).\)

Fix \(\xi_n\), a primitive \(n\)-th root of unity. Denote by \(σ\) the automorphism of \(\mathbb{F}_p\) induced by \((x, y) \mapsto (x, \xi_ny)\). The proof of 13.4 when \(n\) is prime relies strongly on the fact that the minimal polynomial of \(σ\) acting on \(\text{Jac}(X)(\mathbb{Q}(\xi_n))\) is the minimal polynomial of \(ξ_n\). When \(n\) is not prime, the minimal polynomial of \(σ\) on \(\text{Jac}(X)(\mathbb{Q}(\xi_n))\) divides \(t^{n-1} + t^{n-2} + \cdots + t + 1\) and may not be irreducible; the proof of 13.4 does not apply. We can nevertheless show that the following weaker statement holds.

**Proposition 3.3.** Assume that the Chabauty rank (with respect to any prime) of \(X/\mathbb{Q}(\xi_n)\) is equal to \(g(X)\). Then the Mordell-Weil rank of \(X/\mathbb{Q}\) is at least equal to \((s - 2)/2\).
Proof: Let \( \phi(t) = (t^n-1)/(t-1) \). Let \( \varphi_d(t) \) denote the \( d \)-th cyclotomic polynomial, so that \( \phi(t) = \prod_{d|n} \varphi_d(t) \). Let \( \phi_d(t) := \phi(t)/\varphi_d(t) \). Let \( d | n \), and consider the abelian variety

\[
A_d := \text{Im}(\phi_d(\sigma)) \subset \text{Jac}(X)/\mathbb{Q}(\xi_n).
\]

(Note that when \( d < \gcd(n,n_i) \) for some \( i \), it may happen that \( A_d \) is trivial.) It is clear that \( A_d \subset \text{Ker}(\varphi_d(\sigma)) \). If \( d \neq d' \), \( \varphi_d(t) \) and \( \varphi_{d'}(t) \) are coprime, and we conclude that \( A_d \cap A_{d'} = \{0\} \). Since the polynomials \( \{\phi_d(t)\}_{d|n} \) are coprime, we can find \( \{a_d \in \mathbb{Z}, d | n, d \neq 1\} \) such that \( \sum a_d \phi_d(t) = 1 \). Hence, given \( P \in \text{Jac}(X) \),

\[
P = \sum_{d \neq 1} \phi_d(\sigma)(a_dP) \in \langle A_d, d | n, d \neq 1 \rangle \subset \text{Jac}(X)
\]

and, thus, \( \text{Jac}(X) = \bigoplus_{d|n} A_d \).

We claim now that \( A_d \) is an abelian variety defined over \( \mathbb{Q} \). Indeed, let \( P = (a, b) \), where \( a, b \in \overline{\mathbb{Q}} \), be a solution of \( y^n = f(x) \). Let \( \mu \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Then \( \xi_n = \xi_n^c \) for some \( c \in \mathbb{Z} \) with \( (c, n) = 1 \). It follows that on \( X(\overline{\mathbb{Q}}) \), we have

\[
\sigma^c \circ \mu = \mu \circ \sigma
\]

for some \( c \in \mathbb{Z} \). If an element \( Z \) of \( \text{Jac}(X)(\overline{\mathbb{Q}}) \) is of the form

\[
Z = \phi_d(\sigma)(\sum_{i=1}^{s} a_i P_i), \quad \text{with} \quad P_i \in X(\overline{\mathbb{Q}}),
\]

then \( \mu(\phi_d(\sigma)(\sum_{i=1}^{s} a_i P_i)) = \phi_d(\sigma^c)(\sum_{i=1}^{s} a_i \mu(P_i)) \). Since \( (c, n) = 1 \) and \( d | n \), we find that \( \phi_d(t) \) divides \( \phi_d(t^c) \). Hence, \( \mu(Z) \in A_d(\overline{\mathbb{Q}}) \), for all \( \mu \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). It follows that \( A_d \) is defined over \( \mathbb{Q} \). We determine the dimension of \( A_d \) below.

**Lemma 3.4.** Suppose that \( d | n \) and that \( d > \gcd(n,n_i) \) for all \( i = 1, \ldots, s \). Let \( \varphi(d) \) denote the Euler \( \varphi \)-function. Then \( \dim(A_d) = \varphi(d)(s-2)/2 \).

**Proof:** By construction, \( \sigma|_{A_d} \) is such that \( \varphi_d(\sigma|_{A_d}) = 0 \). The characteristic polynomial \( \text{char}(t) \) of \( \sigma \) acting on \( H_1(X(\mathbb{C}), \mathbb{C}) \) is computed in [Lor, 4.1]:

\[
\text{char}(\sigma)(t) = \phi(t)^{s-2} \prod_{i=1}^{s} \left( \frac{t^{\gcd(n,n_i)} - 1}{t - 1} \right)^{-1}.
\]

Hence, \( \text{rank}_\mathbb{Z}(\text{Ker}(\varphi_d(\sigma)|_{H_1(X(\mathbb{C}), \mathbb{C})})) = (s-2)\varphi(d) \). Using the duality between \( H_1(X(\mathbb{C}), \mathbb{C}) \) and \( H^1(X(\mathbb{C}), \mathbb{C}) \) as well as the fact that

\[
0 \to H^0(X(\mathbb{C}), \Omega_X) \to H^1(X(\mathbb{C}), \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0
\]

is exact, with \( H^0(X(\mathbb{C}), \Omega_X) \) and \( H^1(X, \mathcal{O}_X) \) related by Serre duality, we find that \( \dim A_d = \varphi(d)(s-2)/2 \). This concludes the proof of Lemma 3.4.

Now, the reader will easily check that the proof of Lemma 13.4 in [P-S] can be used, *mutatis mutandis*, to show that, for any \( d | n \),

\[
\text{rank}_\mathbb{Z}(A_d(\mathbb{Q})) \varphi(d) = \text{rank}_\mathbb{Z}(A_d(\mathbb{Q}(\xi_d))).
\]

In particular, if the Chabauty rank of \( \text{Jac}(X)/\mathbb{Q} \) equals \( g(X) \), then

\[
\text{rank}_\mathbb{Z}(A_n(\mathbb{Q}(\xi_n))) \geq \dim(A_n),
\]

so that \( \text{rank}_\mathbb{Z}(A_n(\mathbb{Q})) \geq (s - 2)/2 \). This concludes the proof of 3.3.
Remark 3.5 The bound obtained in Theorem 2.23 is probably not optimal. The interest in Theorem 3.1 and 3.2 is that they exhibit cases where the bound for $N(F, h)$ is $O(n)$. There are no known examples in the literature of a family of Thue equations $(F_n(x, y) = h_n)_{n=1}^\infty$ with the degree of $F_n$ going to infinity for which $N(F_n, h_n) > O(\text{deg } F_n)$. The following simple examples of Thue equations with $|N(F, h)| \geq n + 1$ are well-known. Take

$$F(x, y) = \prod_{i=1}^n (x - a_i y) + hy^n, \quad \text{with } \prod_{i \neq j} (a_i - a_j) \neq 0, a_i \in \mathbb{Z}.$$ 

Then $\{(a_i, 1), i = 1, \ldots, n\}$ are primitive solutions (note that $(\ell, 0)$ is also a solution if $\ell^n = h$). If $n$ is even, $\{(-a_i, -1), i = 1, \ldots, n\}$ are also solutions. Now pick $q \in \mathbb{Z}$ and set $a_1 := q^{n-1}$. Take $h := \prod_{i=2}^n (1 - a_i q)$. Then $(1, q)$ is an additional primitive solution. The determination of all solutions of such a Thue equation can sometimes be done (see [Heu] and its bibliography list). The current record on the number of integral points on plane curves can be found in [R-A] and its review in the Math Reviews.

Example 3.6 Let $A, B, C \in \mathbb{Z}$ with $\gcd(A, B, C) = 1$. Consider the generalized Fermat equation

$$Ax^n + By^n = Cz^n.$$ 

Let $F_{A,B,C,n}/\mathbb{Q}$ denote the projective curve defined by this equation. The reader will easily check that if $F_{A,B,C,n}(\mathbb{Q})$ is non-empty, then there exist $A', B'$, and $C'$, such that $F_{A,B,C,n}$ and $F_{A', B', C', n}$ are isomorphic over $\mathbb{Q}$ and at most one of the coefficients $A', B'$, and $C'$ is divisible by $p$. We will thus restrict our attention to the case where $p \nmid AB$. Let $\mu_n$ denote the group of $n$-th roots of unity in $\mathbb{Q}$. The curve $F_{A,B,C,n}$ has at least $n^2$ automorphisms:

$$(x, y) \mapsto (\zeta_n^a x, \zeta_n^b y), \quad 1 \leq a, b \leq n.$$ 

In other words, the group $\mu_n \times \mu_n$ acts on $F_{A,B,C,n}(\mathbb{Q}(\zeta_n))$. Thus, given a single point $P = (x : y : z)$ in $F_{A,B,C,n}(\mathbb{Q}(\zeta_n))$ with $xyz \neq 0$, we obtain $n^2$ distinct points in $F_{A,B,C,n}(\mathbb{Q}(\zeta_n))$ by considering the orbit of $P$ under $\mu_n \times \mu_n$. When $A = B$ and $P = (x : y : z)$ is such that $x \neq y$, then we obtain $2n^2$ points in $F_{A,B,C,n}(\mathbb{Q}(\zeta_n))$ using the extra automorphism $(x : y : z) \mapsto (y : x : z)$.

Proposition 3.7. Let $n = p - 1$ and $p \nmid AB$. If the Mordell-Weil rank over $\mathbb{Q}$ of $F_{A,B,C,p-1}/\mathbb{Q}$ is smaller than $(p - 3)/2$, then there exists at least one triple $(x^{p-1}, y^{p-1}, z^{p-1})$ with $x, y, z \in \mathbb{Z}$, $xyz \neq 0$, and such that $Ax^{p-1} + By^{p-1} = Cz^{p-1}$. Moreover, if $A = B$, then no such triple can exist with $x \neq y$.

Proof: Using 3.3, we find that our hypothesis implies that the Chabauty rank of $F_{A,B,C,p-1}$ over $\mathbb{Q}(\zeta_{p-1})$ is less than the genus of $F_{A,B,C,p-1}$. Thus we may apply 2.18 to show that $|F_{A,B,C,p-1}(\mathbb{Q}(\zeta_{p-1}))| < 2(p - 1)^2$. Proposition 3.7 follows from the fact that a solution $P = (x : y : z)$ with $xyz \neq 0$ produces $(p - 1)^2$ distinct points in $F_{A,B,C,p-1}(\mathbb{Q}(\zeta_{p-1}))$.

Let us say that two solutions $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ of the generalized Fermat equation are non-equivalent if $(x_1^n, y_1^n, z_1^n) \neq \lambda(x_2^n, y_2^n, z_2^n)$. We shall say that a solution $(x, y, z)$ is non-trivial if $xyz \neq 0$. 


Recall that the jacobian of Fermat curve $F_{A,B,C,n}/\mathbb{Q}$ has many quotients. In particular, let $a,b \in \mathbb{N}$, and $d := \gcd(n,a,b)$. Consider the curve $Y/Q$ given by $B^{b/d}y^{n/d} = x^{a/d}(C - Ax)^{b/d}$. We have a map $F_{A,B,C,n} \to Y$, given in coordinates by $(x : y : 1) \mapsto (xa^b, x^a)$. 

**Corollary 3.8.** Let $p$ be an odd prime. Assume that the curve $Ax^{p-1} + By^{p-1} = Cz^{p-1}$ has two non-trivial non-equivalent solutions $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in $\mathbb{Q}^3$ and that $p \nmid AB$. Let $a, b \in \mathbb{N}$. Let $\ell$ be an odd prime with $\ell \mid p - 1$ and $\ell \nmid ab$. Let $Y/Q$ be given by the equation $B^{b/d}y^\ell = x^a(C - Ax)^b$. If $Y$ has positive genus, then its Mordell-Weil rank over $Q$ is positive.

**Proof:** Let $F := F_{A,B,C,p-1}/\mathbb{Q}$. Our hypotheses imply that Chab($F, \mathbb{Q}(\xi_{p-1}),(p)) = g(F)$. Thus, any quotient of Jac($F)/\mathbb{Q}$ defined over $\mathbb{Q}$ has Chabauty rank over $\mathbb{Q}(\xi_{p-1})$ equal to its dimension. Since $\ell$ divides $p - 1$, the curve $F_{A,B,C,\ell}$ is a surjective image of $F$, the curve $Y/Q$ is a surjective image of $F_{A,B,C,\ell}$, and the Jacobian of $Y/Q$ is a quotient of Jac($F)/\mathbb{Q}$). Hence, the Chabauty rank of the Jacobian of $Y/Q$ is equal to its dimension and we may apply $3.3$ with $s = 3$. Indeed, let $\ell f$ denote the smallest positive multiple of $\ell$ that is greater than or equal to $a + b$. Change $x$ to $x' + m$ for some appropriate $m$ so that $x' = 0$ is not in the branch locus of the natural map $Y \to \mathbb{P}^1$. Then change coordinates to $u := 1/x'$ and $v = y/x'^\ell$ to get an equation for $Y$ of the form $v^\ell = u^{f\ell - a - b}(u - \alpha_1)^a(u - \alpha_2)^b$. Thus, when $g(Y) > 0$, $3.3$ with $s = 3$ shows that the Mordell-Weil rank of $Y$ over $\mathbb{Q}$ is greater than or equal to $(s - 2)/2 > 0$, which completes our proof.

Let $Y'/Q$ be given by the equation $B^{b/d}y^\ell = x^{a/d}(C - Ax)^{b/d}$. Since $\ell$ divides $(p - 1)/d$, the curve $Z$ given by $B^{b/d}y^{n/d} = x^{a/d}(C - Ax)^{b/d}$ is a quotient of $F$, and $Y'$ is a quotient of $Z$. An isomorphism between the curves $Y'$ and $Y$ is given by $(x,y) \mapsto (x,y')$.

**Remark 3.9** It is easy to produce triples $(A,B,C)$ with $ABC \neq 0$ such that $F_{A,B,C,n}(\mathbb{Q})$ has 2 non-trivial non-equivalent solutions and $p$ divides at most one of $A, B, C$. Namely, pick appropriate triples of pairwise coprime integers $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ with $p \nmid x_1y_1z_1x_2y_2z_2$ and $(x_1^n, y_1^n, z_1^n) \neq (x_2^n, y_2^n, z_2^n)$. Then solve for $(A, B, C)$ in the simultaneous equations

$$\begin{align*}
Ax_1^n + By_1^n &= Cz_1^n \\
Ax_2^n + By_2^n &= Cz_2^n.
\end{align*}$$

It follows from $3.7$ that for such a choice of $(A, B, C)$ with $n = p - 1$, the Mordell-Weil rank of Jac($F_{A,B,C,p-1}/\mathbb{Q}$) is greater than or equal to $(p - 3)/2$. On the other hand, it does seem to follow from $3.7$ that for such a choice of $(A, B, C)$ with two solutions $P$ and $Q$, the $\mathbb{Q}$-rational point $P - Q$ has infinite order in the jacobian of $F_{A,B,C,p-1}$.

It would be interesting to determine, for any $n$, whether, when $F_{A,B,C,n}(\mathbb{Q})$ contains two non-trivial non-equivalent points $P$ and $Q$, then $P - Q$ has infinite order in the jacobain of $F_{A,B,C,n}$. We can construct examples where $P - Q$ has infinite order as follows. Choose a prime $q > 2$ such that $q \nmid x_1y_1z_1(x_1 - y_1)(x_1 - z_1)(y_1 - z_1)$ and let $(x_2, y_2, z_2) := (x_1, y_1, z_1) + (q, q, q)$. With these choices, we find that $q \nmid ABC$, where $(A, B, C)$ is a solution to the above system of equations with $\gcd(A, B, C) = 1$. It
follows that $F_{A,B,C,n}$ has good reduction at $q$. Moreover, by construction, $P - Q$ is in the kernel of the reduction mod $q$. Since $q > 2$, the kernel of the reduction does not contain any point of finite order and, thus, $P - Q$ has infinite order.

When $F_{A,B,C,n}$ has an elliptic quotient $E/\mathbb{Q}$, $P - Q$ has infinite order in the jacobian of $F_{A,B,C,n}$ if the image $P' - Q'$ of $P - Q$ in $E(\mathbb{Q})$ has infinite order. For instance, when $6 \mid n$, we find that $F_{A,B,C,n}$ has $Ax^3 + B = Cz^2$ as quotient, which can be rewritten as $u^2 = u^3 + C^3A^2B$. When $C^3A^2B$ is neither a square, nor a cube, nor equals $-1$, $E(\mathbb{Q})_{\text{tors}}$ is trivial. It is easy to check in this case that $P' - Q'$ has infinite order.

**Remark 3.10** Gross and Rohrlich have shown in [G-R], 2.1, that if $\ell > 7$ is prime, all but three of the $\ell - 2$ Fermat quotients $y^\ell = x^a(1-x)^b$ have positive Mordell-Weil ranks over $\mathbb{Q}$.

Suppose that $A = B$ and $(x : x : z) \in F_{A,A,C,n}(\mathbb{Q})$. Then the curve $F_{A,A,C}$ is isomorphic over $\mathbb{Q}$ to the curve $F_{1,1,2}$, with the point $(x : x : z)$ corresponding to the point $(1 : 1 : 1)$. The rational points of the curve $x^n + y^n = 2z^n$ are determined in [D-M]. Other cases are treated in [Sit]. For conjectures regarding the solutions of generalized Fermat equations, see [Gr].

The curve $F_{A,B,C,n}$ is a twist of the curve $F_{1,1,1,n}$. It is shown in [Si2], Thm. 1, that the number of points over a number field $K$ of a twist $C_x/K$ of a curve $C/K$ can be bounded in terms of a constant $\gamma(C/K)$ and the Mordell-Weil rank of $C_x/K$.

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