Foliation by free boundary constant mean curvature leaves

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Abstract

Let $M$ be a Riemannian manifold of dimension $n + 1$ with smooth boundary and $p \in \partial M$. We prove that there exists a smooth foliation around $p$ whose leaves are submanifolds of dimension $n$, constant mean curvature and its arrive perpendicular to the boundary of $M$, provided that $p$ is a nondegenerate critical point of the mean curvature function of $\partial M$.

1 Introduction

The strategy of the proof of this result was inspired by [2]. In this work, Rugang Ye considered the foliation by geodesic spheres around $p \in M$ of small radius and showed that this foliation can be perturbed into a foliation whose leaves are spheres of constant mean curvature, provided that $p$ is a nondegenerate critical point of the scalar curvature function of $M$. So we are going to consider a family of foliations whose leaves are submanifolds of $M$ with boundary contained in $\partial M$ and it’s arriving perpendicular to the boundary of $M$. The idea is then to perturb each leaves to obtain, via implicit function theorem, a foliation whose leaves are hemispheres of constant mean curvature and its arrive perpendicular to the boundary of $M$, provided that $p$ is a nondegenerate critical point of the mean curvature function of $\partial M$.

We refer to [1] for basic terminology in local Riemannian geometry. Let $(M, g)$ be an $(n+1)$-dimensional Riemannian manifold with smooth boundary $\partial M$, $n \geq 2$. We will denote by $\nabla$ and $\nabla$ the covariant derivatives and by $R$ and $\bar{R}$ the full Riemannian curvature tensor of $M$ and $\partial M$, respectively. The trace of second fundamental form of the boundary will be denoted by $h$. We will make use of the index notation for tensors, commas denoting covariant differentiation and we will adopt the summation convention.

**Definition 1.1** Let $T$ denote the inward unit normal vector field along $\partial M$. Let $\Sigma$ be a submanifold with boundary $\partial \Sigma$ contained in $\partial M$. The unit conormal of $\partial \Sigma$ that points outside $\Sigma$ will be denoted by $\nu$. $\Sigma$ is called free boundary when $\nu = -T$ on $\partial \Sigma$. 
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2 Fermi Coordinate System

Consider a point \( p \in \partial M \) and an orthonormal basis \( \{e_1, \ldots, e_n\} \) for \( T_p \partial M \). Let \( B_r = \{ x \in \mathbb{R}^n : |x| < r \} \) be the open ball in \( \mathbb{R}^n \). There are \( r_p > 0 \) and \( t_p > 0 \) for which we can define the Fermi coordinate system centered at \( p, \varphi^0 : B_{r_p} \times [0, t_p) \to M \), given by

\[
\varphi^0(x, t) = \exp^{\mathcal{M}}_{\varphi^0(x)}(t T(x))
\]

where \( \varphi^0(x) = \exp^{\partial M}_p(x^i e_i) \) is the normal coordinate system in \( \partial M \) centered at \( p \) and \( T(x) \) is the inward unit vector normal to the boundary at \( \varphi^0(x) \).

For each \( \tau = (\tau_1, \ldots, \tau_n) \in B_{r_p/2} \) we will consider the Fermi coordinate system centered at

\[
c(\tau) = \exp^{\partial M}_p(\tau^i e_i),
\]

which we denote by \( \varphi^\tau : B_{r_p/2} \times [0, t_p) \to M \), and it defined by

\[
\varphi^\tau(x, t) = \exp^{\mathcal{M}}_{\varphi^\tau(x)}(t T(x))
\]

where

\[
\phi^\tau(x) = \exp_{c(\tau)}(x^i e_i^\tau),
\]

\( e_i^\tau \) are the parallel transport of \( e_i \) to \( c(\tau) \) along the geodesic \( c(s) \|_{0 \leq s \leq 1} \) in \( \partial M \), and \( T(x) \) is the inward unit vector normal to the boundary at \( \varphi^\tau(x) \in \partial M \).

We will denote the metric tensor of \( M \) by \( ds^2 \), the coefficients of \( ds^2 \) in the coordinates system \( \varphi^\tau \) by \( g^{\tau}_{ij}(x, t) \), and \( g_{ij}^{-1} = (g^{\tau}_{ij})^{-1} = (g^{\tau}_{ij}) \). The expansion of \( g^{\tau}_{ij} \) (up to fourth order) in Fermi coordinates can be found in [1] p.1604.

Lemma 2.1 In Fermi coordinates \( (x_1, \ldots, x_n, t) \) centered at \( c(\tau) \in \partial M \) we have \( g^{\tau}_{ij}(x, t) = 1 \), \( g^{\tau}_{ir}(x, t) = 0 \), and

\[
g^{\tau}_{ij}(x, t) = \delta_{ij} + 2h_{ij} t + \cdots
\]

\[
+ \frac{1}{15} R_{ijkl} t x_k x_l + 2h_{ij, k} t x_k + (R_{titj} + 3h_{ik} h_{kj}) t^2 + \cdots
\]

\[
+ \frac{1}{15} R_{ijkl, m} t x_k x_l x_m + \left( \frac{2}{3} \text{Sym}_{ij}(R_{km} h_{mj}) + h_{ij, kl} \right) t x_k x_l + \cdots
\]

\[
+ (R_{titj, k} + 6 \text{Sym}_{ij} h_{il} h_{kj}) t^2 x_k + \cdots
\]

\[
+ \left( \frac{1}{15} R_{ijkl, mp} + \frac{1}{15} R_{ikql, m} R_{jmqp} \right) x_k x_l x_m x_p + \cdots
\]

\[
+ \left( \frac{2}{3} \text{Sym}_{ij}(R_{ilm} h_{pj}) + \frac{2}{3} \text{Sym}_{ij}(R_{ikl} h_{pj, m}) + \frac{1}{3} h_{ij, kl} \right) t x_k x_l x_m + \cdots
\]
We will work with the following set of functions

\[ C_T^{2,0}(S_+^n) = \left\{ \varphi \in C^{2,0}(S_+^n); \frac{\partial \varphi}{\partial e_4} = 0 \text{ in } \partial S_+^n \right\} \]  \tag{4}

3 Perturbation by free boundary submanifolds

We will work with the following set of functions

\[ C_T^{2,0}(S_+^n) = \left\{ \varphi \in C^{2,0}(S_+^n); \frac{\partial \varphi}{\partial e_4} = 0 \text{ in } \partial S_+^n \right\} \]  \tag{4}
where $S^n_+ = \{(x, t) \in \mathbb{R}^{n+1}; \ t^2 + |x|^2 = 1, \ t \geq 0\}$.

For $\varphi \in C^{2,\alpha}_{T}(S^n_+)$ we define

$$S^+_{\varphi} = \{(1 + \varphi(x, t))(x, t); \ (x, t) \in S^n_+\}$$

and

$$S_{r, \tau, \varphi} = \varphi^*(\alpha_r(S^+_{\varphi})) \quad (5)$$

where $\alpha_r$ is the dilation $(x, t) \mapsto (rx, rt)$ for $0 < r < r_0$ and $r_0$ sufficiently small such that $\alpha_{r_0}(\mathbb{B}_2 \times [0, 2)) \subset \mathbb{B}_{r_0/2} \times [0, t_p)$.

There are numbers $\delta_0 > 0$ and $r_0 > 0$ such that $S_{r, \tau, \varphi}$ is an embedded $C^2$ hypersurface in $M^{n+1}$ for any $\|\varphi\|_{C^1} \leq \delta_0$ and $0 < r < r_0$. In addition $S_{r, \tau, \varphi}$ is a free boundary submanifold of $M$, this is, $\partial S_{r, \tau, \varphi} \subset \partial M$ and its arrive perpendicular to the boundary of M, because $\partial \varphi / \partial e_t = 0$. We denote the inward mean curvature function of $S_{r, \tau, \varphi}$ by $h(r, \tau, \varphi)$.

For $(x, t) \in S^n_+$ we have $H(r, \tau, \varphi)(x, t) = r h(r, \tau, \varphi)(\varphi^*(r(1 + \varphi(x, t))(x, t)))$. \quad (6)

But, by the Lemma 2.2 in [1, p.1604],

$$ds^2_{\tau, r}(x, t)(v, w) = \langle v, w \rangle_{\mathbb{R}^{n+1}} + O(r)$$

with $O(r) \to 0$ when $r = |(x, t)| \to 0$. One readily checks $ds^2_{\tau, r}$ extends smoothly to the euclidean metric when $r$ goes to zero. Hence $H(r, \tau, \varphi)$ also extends to $r = 0$. Then by a straightforward computation the inward mean curvature function of $S^+_{\varphi}$ at $(\bar{x}, \bar{t}) = (1 + s\varphi(x, t))(x, t)$ with respect to the metric $ds^2_{\tau, r}$ on $\mathbb{B}^{2+}_2$, can be written as
Corollary 3.2
The following holds true

\[ H(r, \tau, s\varphi)(\bar{x}, \bar{t}) = \frac{1}{\Psi_s} \left( \Delta \rho - s \Delta \varphi - \frac{s^2}{2} \Delta \varphi^2 \right) \]

\[ -1 + s\varphi \left[ \frac{\partial \Psi_s}{\partial t} \left( t - s \frac{\partial \varphi}{\partial t} \right) + \sum_{i,j} g^{ij} \left( x_j - s \frac{\partial \varphi}{\partial x_j} \right) \right] \]

where \( \rho(x, t) = \frac{(t^2 + |x|^2)^2}{2} \), \( g^{ij} = g^{ij}(\bar{x}, \bar{t}) \),

\[ \bar{\varphi}(x, t) = \varphi \left( \frac{x}{\sqrt{t^2 + |x|^2}}, \frac{\bar{t}}{\sqrt{t^2 + |x|^2}} \right), \]

\[ \Psi_s = \Psi_s(r, x, t) = \left[ \left( t - s(1+s\varphi) \frac{\partial \varphi}{\partial t} \right)^2 + \sum_{i,j} g^{ij}(tx, rt) \left( x_i - s(1+s\varphi) \frac{\partial \varphi}{\partial x_i} \right) \left( x_j - s(1+s\varphi) \frac{\partial \varphi}{\partial x_j} \right) \right]^{\frac{1}{2}} \]

and \( \Delta \) is the standard Laplace operator on \( B_2^+ \) relative to the metric \( ds^2_{r,t} \).

**Lemma 3.1** We have

\[ H(r, \tau, 0)(x, t) = n + [h^i_t t - (n + 3)h^i_j tx_i x_j] r + \left[ \frac{3n+2}{2} h^i_j h^k_i h^r_j t^2 x_i x_j x_k x_i \right. \]

\[ - (n + 4) h^i_j h^k_l tx_i x_j x_k + \left( - \frac{n+4}{2} R_{ijkl} - \frac{3n+20}{2} h^i_k h^j_l - h^i_j h^k_l \right) t^2 x_i x_j \]

\[ + \frac{1}{4} \bar{R}_{kikl} x_i x_j + 2 h^i_j x_i x_j + 2(h^i_j)^2 t^2 \right] r^2 + \left[ \int_0^1 \frac{(1 - \eta)^2}{2} H_{rrr}(\eta r, \tau, 0) d\eta \right] r^3 \]

where every coefficient is computed at \( c(\tau) \).

**Corollary 3.2** The following holds true

\[ H(0, \tau, 0) = \lim_{r \to 0} H(r, \tau, 0) = n. \]

Now we consider \( H(r, \tau, \cdot) \) as a mapping from \( C^2(\bar{S}_\varphi^+ \right) \) into \( C^0(\bar{S}_\varphi^+) \) and let \( H_\varphi \) denote the differential of \( H \) with respect to \( \varphi \). In order to calculate \( H_\varphi \), we consider the variation of \( S_\varphi^+ \) by smooth maps \( f : S_\varphi^+ \times (-\epsilon, \epsilon) \to \mathbb{B}_2^+ \) given by \( f(x, t, s) = (1 + s \varphi(x, t))(x, t) \). For each \( s \in (-\epsilon, \epsilon) \) we denote \( f^s(x, t) = f(x, t, s) \). Note that \( f^s(\bar{S}_\varphi^+) = S_\varphi^+ \) is an embedded \( C^2 \) in \( \mathbb{B}_2^+ \) with \( \partial S_\varphi^+ \subset \partial \mathbb{B}_2^+ \). We will denote by \( N_s(r, \tau, \varphi) \) a unit vector field normal to \( S_\varphi^+ \) and \( \bar{H}(r, \tau, s\varphi) \) the mean curvature of \( S_\varphi^+ \). We decompose the variational vector field

\[ \partial_s = \varphi(x, t)(x, t) = \partial^T_s + v^s N_s \]

where \( v^s \) is the function on \( S_\varphi^+ \) defined by \( v^s = ds^2_{r,t}(\partial_s, N_s) \).

By the Proposition 16 in [3, p.14] we have

\[ H_\varphi(r, \tau, 0) \varphi = (\partial_s H(r, \tau, s\varphi)) \big|_{s=0} = dH(r, \tau, 0)(\partial^T_0) - L_{r,\tau} v_0 \]
and
\[ \partial s ds^2\tau,\tau(N_s, e_t)|_{s=0} = -\partial v^0 \partial e_t + ds^2\tau,\tau(N_0, \nabla c_t) v^0 \]
where \( L_{\tau,\tau} = \Delta (s^2, ds^2\tau,\tau) + Ric_{\tau,\tau}(N_0, N_0) + \|B_{r,\tau}\| \) is the Jacobi operator.

In particular
\[ H_{\varphi,\varphi}(0, \tau, 0) = L_{\varphi} := -(\Delta_{S^n} + n)\varphi, \quad (11) \]
where \( \Delta_{S^n} \) is the standard Laplace operator in \( S^n \).

**Lemma 3.3** We have
\[ H_{\varphi\varphi}(0, \tau, 0) = 2n \varphi^2 - (n-2) \left( \frac{\partial \varphi}{\partial x_i} \right)^2 - (n-2) \sum_i \left( \frac{\partial \varphi}{\partial x_i} \right)^2 \quad (13) \]
where \( \varphi \) was defined in (8).

The Jacob operator
\[ L : C^{2,\alpha}_T(S^n_+) \to C^{0,\alpha}(S^n_+) \]
\[ L = \Delta_{S^n} + n \] has an \( n \)-dimensional kernel \( K \) consisting of first order spherical harmonic functions \( x^i = x^i|_{S^n_+} \), \( i = 1, \ldots, n \), which satisfy
\[ \partial \partial e_t x^i|_{S^n_+} = 0 \text{ in } \partial S^n_+. \]

In addition we have the \( L_2 \)-decompositions of spaces \( C^{2,\alpha}_N(S^n_+) = K \oplus K^\perp \) and \( C^{0,\alpha}(S^n_+) = K \oplus L(K^\perp) \). Let \( P \) denote the orthogonal projection from
$C^{0,\alpha}(S^n_+)$ onto $K$, and $T: K \rightarrow \mathbb{R}^n$ be the isomorphism sending $x^i|_{S^n_+}$ to $e_i$, the $i$th coordinate basis. Define $\tilde{P} = T \circ P$, that is,

$$\tilde{P}(f) = \frac{2}{w_{n+1}} \left( \int_{S^n_+} f x^i \right) e_i$$

because

$$\int_{S^n_+} x^i x^j = \frac{w_{n+1}}{2} \delta_{ij},$$

where $w_{n+1} = \text{Vol}(B_1)$.

**Lemma 3.4** We have

$$\tilde{P}(H(r, \tau, 0)) = -\frac{2 w_n r^2}{(n+2)w_{n+1}} h_{jj,k} e_i + O(r^3). \quad (14)$$

**Proof.** From Lemma 3.1 and the fact that $\tilde{P}(x^{n+1}) = \tilde{P}(x^i x^j x^{n+1}) = 0$ we have

$$\tilde{P}(H(r, \tau, 0)) = \left[ -(n+4)h_{ij,k}^+ \tilde{P}(x^{n+1} x^i x^j x^k) + 2 h_{ji,2}^+ \tilde{P}(x^{n+1} x^i) \right] r^2 + O(r^3)$$

where

$$\tilde{P}(x^{n+1} x^i x^j x^k) = \frac{2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) w_n}{(n+2)(n+4)w_{n+1}} e_i$$

and

$$\tilde{P}(x^{n+1} x^i) = \frac{w_n \delta_{ij}}{w_{n+1}(n+2)} e_j.$$

Hence (14) follows.

**Lemma 3.5** If $\varphi^\tau: S^n_+ \rightarrow \mathbb{R}$ is the solution of the Neumann problem

$$\begin{cases}
-(\Delta_{S^n_+} \varphi + n \varphi) = h_{ii}^+ x^{n+1} - (n+3) h_{ij}^+ x^i x^j x^{n+1} \\
\frac{\partial \varphi}{\partial e_{n+1}} = 0 \text{ on } \partial(S^n_+) \quad (15)
\end{cases}$$

then

$$\tilde{P}(H_{\varphi^\tau}(0, \tau, 0) \varphi^\tau) = 0 \quad (16)$$

and

$$\tilde{P}(H_{\varphi^\tau}(0, \tau, 0) \varphi^\tau \varphi^\tau) = 0. \quad (17)$$

for $||\tau|| < r_p/2$.

**Proof.** The function $\varphi^\tau = \frac{1}{2} h_{ij}^+ x^i x^j x^{n+1}$ is the solution of the problem (15).

Now we can use (12), (13) and the fact

$$\tilde{P}(x^{i_1} \ldots x^{i_k} x^{n+1}) = 0,$$

for all integer $k \geq 0$, to prove (16) and (17).
4 Main Theorem

Definition 4.1 Consider \( p \in \partial M \) and let \( U \) be a neighborhood of \( p \) on \( M \cup \partial M \). A smooth codimension 1 foliation \( \mathcal{F} \) of \( \cup \{ p \} \) for a neighborhood \( U \) of \( p \) is called a free boundary foliation centered at \( p \), provided that its leaves are all closed and free boundary.

Theorem 4.2 If \( p \in \partial M \) is a nondegenerate critical point of the mean curvature function of \( \partial M \), then there exist \( \delta > 0 \) and smooth functions \( \tau = \tau(r) \) and \( \phi = \phi(r) \) such that
\[
H(r, \tau(r), r\phi(r)) \equiv n \quad \text{for all } 0 \leq r < \epsilon.
\]
Hence the family \( \mathcal{F} = \{ S_r := S_{r, \tau(r), r\phi(r)} : 0 \leq r < \epsilon \} \) is a smooth family of constant mean curvature spheres with \( S_r \) having mean curvature \( n/r \). Furthermore \( \mathcal{F} \) is a free boundary foliation centered at \( p \).

Proof. We will use the Taylor’s formula with integral remainder
\[
H(r, \tau, r\phi) = n + [H_\tau(0, \tau, 0)\varphi + H_r(0, \tau, 0)] r
\]
\[
+ \left[\frac{1}{2} H_{\varphi\varphi}(0, \tau, 0)\varphi \varphi + H_{\varphi r}(0, \tau, 0)\varphi + \frac{1}{2} H_{rr}(0, \tau, 0)\right] r^2 + R(r, \tau, \varphi) r^3
\]
where
\[
R(r, \tau, \varphi) = \int_0^1 (1 - \eta) H_{\varphi r r}(\eta r, \tau, 0)\varphi d\eta + \frac{1}{2} \int_0^1 H_{\varphi r \varphi}(\eta r, \tau, 0)\varphi d\eta
\]
\[
+ \frac{1}{2} \int_0^1 (1 - \eta)^2 H_{\varphi \varphi \varphi}(r, \tau, \eta \varphi)\varphi \varphi d\eta.
\]
We are interested in solving the equation \( H(r, \tau, r\varphi) = n \), but first we are going to treat the equation
\[
P^\perp(H(r, \tau, r\varphi) - n) = 0,
\]
where \( P^\perp \) denotes the \( L^2 \) orthogonal projection from \( C_0^{\text{\alpha}} \) onto \( K^\perp \).

By \( \{11\} \) and the fact \( PL = PH_r(0, \tau, 0) = 0 \) we can write the equation in \( \{18\} \) as follows (after division by \( r \))
\[
L\varphi + H_r(0, \tau, 0) + \bar{R}(r, \tau, \varphi) r = 0,
\]
where
\[
\bar{R}(r, \tau, \varphi) = \frac{1}{2} P^\perp(H_{\varphi\varphi}(0, \tau, 0)\varphi \varphi) + P^\perp(H_{\varphi r}(0, \tau, 0)\varphi) + P^\perp\left(\frac{1}{2} H_{rr}(0, \tau, 0)\right)
\]
\[
+ P^\perp(R(r, \tau, \varphi)) r.
\]
Consider the mapping \( G : [0, r_p/8] \times \mathbb{B}_{r_p/4} \times \mathbb{B}_{\delta_0} \to K^\perp \) given by
\[
G(r, \tau, \varphi) = L\varphi + H_r(0, \tau, 0) + \bar{R}(r, \tau, \varphi) r,
\]
where \( \mathbb{B}_{\delta_0} = \{ \varphi \in K^\perp : \|\varphi\|_{C^{2,\alpha}} < \delta_0 \} \).
For \( \tau = 0 \), let \( \varphi_0 \in C^2_\mathcal{N}(\mathbb{R}^n) \) be a solution of the equation
\[
L\varphi_0 + H_r(0, 0, 0) = 0.
\]
One sees that \( G(0, 0, \varphi_0) = 0 \) and \( G\varphi_0(0, 0, \varphi_0) = (-\Delta - n) : K^\perp \to L(K^\perp) \) is a bounded invertible linear transformation. By the implicit function theorem we can solve \( P^\perp(H(r, \tau, r\varphi(r, \tau)) - n) = 0 \) for a function \( \varphi : [0, \delta) \times \mathbb{B}_\delta \to K^\perp \), for some \( 0 < \delta \leq r_p/8 \), with \( \varphi(0, 0) = \varphi_0 \). Furthermore
\[
\varphi_r(0, 0) = -G\varphi(0, 0, \varphi_0)^{-1}G_r(0, 0, \varphi_0)
\]

\[
= -(-\Delta - n)^{-1} \frac{\partial}{\partial r} [\bar{R}(r, \tau, \varphi) r]|_{r=0, \varphi=\varphi_0} = -(-\Delta - n)^{-1} \bar{R}(0, 0, \varphi_0)
\]

where
\[
\bar{R}(0, 0, \varphi_0) = \frac{1}{2} H_{\varphi\varphi}(0, 0, 0)\varphi_0\varphi_0 + H_{\varphi r}(0, 0, 0)\varphi_0.
\]

Since
\[
L\varphi(r, \tau) + H_r(0, \tau, 0) + O(r) = 0,
\]
we have, for \( r = 0 \),
\[
L\varphi(0, \tau) + H_r(0, \tau, 0) = 0.
\]

Then, by Lemma \[3.5\]
\[
\tilde{P}(H_{\varphi r}(0, \tau, 0)\varphi(0, \tau)) = \tilde{P}(H_{\varphi\varphi}(0, \tau, 0)\varphi_0\varphi(0, \tau)) = 0.
\]

On the other hand,
\[
\varphi(r, \tau) = \varphi(0, \tau) + r \int_0^1 \varphi_r(\eta r, \tau) \, d\eta,
\]

so that
\[
\tilde{P}(H_{\varphi r}(0, \tau, 0)\varphi(r, \tau)) = r \tilde{P} \left( H_{\varphi r}(0, \tau, 0) \left( \int_0^1 \varphi_r(\eta r, \tau) \, d\eta \right) \right) \quad (19)
\]

and
\[
\tilde{P}(H_{\varphi\varphi}(0, \tau, 0)\varphi(r, \tau)) = r \tilde{P} \left( H_{\varphi\varphi}(0, \tau, 0)\varphi(r, \tau) \left( \int_0^1 \varphi_r(\eta r, \tau) \, d\eta \right) \right) + O(r^2).
\]

Now we consider the mapping \((r, \tau) \mapsto H(r, \tau, r\varphi(r, \tau)) - n\) whose values lie in \( K \) by the construction of \( \varphi(r, \tau) \). Let us solve the equation
\[
H(r, \tau, r\varphi(r, \tau)) - n = 0,
\]
which is equivalent to equation \( \tilde{P}(H(r, \tau, r\varphi(r, \tau)) - n) = 0 \) and, after division by \( r^2 \), it is equivalent to
\[
\frac{1}{2} \tilde{P}(H_{\varphi\varphi}(0, \tau, 0)\varphi) + \tilde{P}(H_{\varphi r}(0, \tau, 0)\varphi) + \frac{1}{2} \tilde{P}(H_{rr}(0, \tau, 0)) + \tilde{P}(R(r, \tau, \varphi)) r = 0,
\]

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where \( \varphi = \varphi(r, \tau) \). By [19] and Lemma 0.4 the above equation may be written as follows

\[-\frac{2w_n}{(n+2)w_{n+1}} h_{jj,i}^r e_i + R_1(r, \tau, \varphi) r = 0,\]

with

\[R_1(r, \tau, \varphi) = \hat{P}(R(r, \tau, \varphi)) + r \hat{P}\left(H_{\varphi r}(0, \tau, 0) \left(\int_0^1 \varphi_{r}(\eta r, \tau) d\eta\right)\right)\]

\[+ r \hat{P}\left(H_{\varphi \tau}(0, \tau, 0) \varphi_{r}(r, \tau) \left(\int_0^1 \varphi_{r}(\eta r, \tau) d\eta\right)\right) + O(r^2).\]

In order to solve the above equation, consider \( F : [0, \delta) \times B(0, \delta) \to \mathbb{R}^n \) defined by

\[F(r, \tau) = -\frac{2w_n}{(n+2)w_{n+1}} h_{jj,i}^r e_i + R_1(r, \tau, \varphi) r.\]  \hspace{1cm} (20)

By the assumption \( h_{jj,i}^r |_{\tau=0} = 0 \), we have \( F(0, 0) = 0 \) and the Hessian matrix

\[\frac{\partial F}{\partial \tau}(0, 0) = \left(\frac{\partial}{\partial \tau_j} h_{jj,i}^r \right)_{i,j} \bigg|_{\tau=0}\]

is nonsingular. Applying the implicit function theorem we obtain a solution \( \tau = \tau(r) \) of the equation \( H(r, \tau, r \varphi(r, \tau)) = n \) around \( (r, \tau) = (0, 0) \), \( 0 \leq r < \epsilon \), for some \( 0 < \epsilon \leq \delta \).

### 4.1 The Foliation

It is clear that \( \mathcal{F} = \{ S_r = S_{r, \tau(r), r \varphi(r)} : 0 < r \leq \epsilon \} \) is a smooth family of embedded constant mean curvature hemisphere with \( S_r \) having mean curvature \( n/r \). We need to show that this family constitutes a foliation.

In order to prove this we consider the application \( X : (0, \epsilon) \times B_1 \to M \) given by

\[X(r, x) = \varphi^r(r(1 + r \varphi(r)(x, t(x))))(x, t(x))\]

where \( t(x) = \sqrt{1 - |x|^2} \) and \( \varphi^r \) is the Fermi coordinate system defined in [2]. Observe that \( X(B_1, r) = S_r \). Thus, it is sufficient to prove that \( X \) is a parametrization of \( M \), for small \( \epsilon > 0 \). It is enough to prove that

\[Y(r, x) = \left(\varphi^0\right)^{-1}(X(r, x))\]

is an immersion, where \( \varphi^0 \) is a Fermi coordinate system centred at \( p \) defined in [1].

**Claim:** The function \( \tau = \tau(r) \) satisfies \( \tau(r) = O(r^2) \).

We have that \( \tau(r) \) is a solution of the equation \( F(r, \tau(r)) = 0 \), where \( F \) is defined in [20]. By the implicit function theorem

\[\frac{\partial \tau}{\partial r}(0) = -\left(\frac{\partial f_1}{\partial \tau_j}(0)\right)^{-1} \left(\frac{\partial F_1}{\partial r}(0), \ldots, \frac{\partial F_n}{\partial r}(0)\right)\]

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and
\[ \frac{\partial F_i}{\partial r}(0,0) = \langle R_1(0,0,\varphi_0), e_i \rangle = \langle \tilde{P}(R(0,0,\varphi_0)), e_i \rangle = 0. \]

Then,
\[ \frac{\partial \tau}{\partial r}(0) = 0. \]  \hspace{1cm} (21)

Now
\[ \frac{\partial \Upsilon}{\partial r}(r,x) = (d_x \Upsilon)((1 + r\varphi(r)(x,t))(x,t) + r((1 + r\varphi(r)(x,t))_x(x,t)) + \left( \frac{\partial \Upsilon}{\partial r} \right) (r(1 + r\varphi(r)(x,t))(x,t)) \]

and
\[ \frac{\partial \Upsilon}{\partial r}(0,0) = \left. \frac{\partial}{\partial \tau} \left( (\varphi^0)^{-1} \circ \varphi^\tau(r) \right) \right|_{\tau=0} \frac{\partial \tau^i}{\partial r}(0) = 0. \]

Using (21) we conclude that
\[ (\partial \Upsilon / \partial r)(0,x) = (x,t(x)), \]
i.e.,
\[ \Upsilon(r,x) = r(x,t(x)) + O(r^2). \]

Consequently
\[ (\partial \Upsilon / \partial r)(r,x) = (x,t(x)) + O(r), \]
\[ (\partial \Upsilon / \partial x_i)(r,x) = re_i - r(x_i/t(x))e_i + O(r^2) \]

and
\[ \det \left( \frac{\partial \Upsilon}{\partial r} \frac{\partial \Upsilon}{\partial x_1} \ldots \frac{\partial \Upsilon}{\partial x_n} \right)(r,x) = r^2 \left( \frac{1}{t(x)} + O(r) \right) > 0, \]
for all \( 0 < r < \epsilon' \leq \epsilon \) and some \( \epsilon' \) enough small.

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