Conformally-coupled dark spinor and FRW universe

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We study conformal coupling of dark spinor fields to gravity and calculate the energy density and the pressure of the spinor in FRW spacetime. We consider the renormalizable potential of the spinor field. In the cases where the field is proportional to some power of the cosmic scale factor \(a(t)\), we determine the Hubble parameter as a function of the scale factor and find analytic solutions for \(a(t)\) when the spinor field matter dilutes as the universe expands. We discuss the possibility that both matter- and dark energy-dominated eras of our universe can be described by the dark spinor.

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I. INTRODUCTION

Recently various proposals have appeared in the literature to study the properties of spinor fields in cosmology. Relatively earlier attempts are the standard spinor fields with a non-linear self interaction term [1]. Although introducing a non-linear fermionic potential term may cause a problem at the quantum level, it was treated classically and a number of cosmological issues were investigated. For instance, the problem of initial singularity, the issue of isotropization, and their possible role in the late time acceleration have been studied in this context.

More recently a proposal was put forward to use the so-called ELKO spinors [2] as a dark matter candidate. One of the most interesting properties associated with these non-standard spinors is the fact that their dominant interaction is via the gravitational field, which is an essential property for any dark matter candidate. Another interesting feature is that these dark spinors have a canonical mass dimension one instead of 3/2 for the standard spinors. This feature makes wider range of perturbatively renormalizable self interactions possible. Consequently, many studies on the cosmological aspects of the ELKO spinor model have followed [3].

In Ref. [4], however, it was pointed out that the construction of ELKO spinors itself implicitly violates Lorentz invariance. They proposed a non-local but Lorentz invariant version of the dark spinors. They also pointed out that some crucial errors were made in the calculation of stress energy tensors in the previous works on ELKO model. Based upon a careful recalculation they give a correct expression for the ELKO spinor field and apply the result to show the existence of de Sitter like solutions. Dynamical analyses of the ELKO model have followed this work using the correct form of the stress energy momentum tensor and it was shown that scaling attractor solutions do not exist in this model [3].

In this paper we study the cosmology with the (ELKO) dark spinor field treating it as the main part of the matter. For self interaction we choose the most general renormalizable form of the potential. Most importantly we include the conformal coupling of the spinor field with the gravitational field. Although this possibility was briefly mentioned in Ref. [4], it was not pursued further. We will focus on the existence of the cosmological solution where the spinor field is proportional to the simple power of the scale factor. Such solutions would be possible only when the potential terms are chosen appropriately. However even if we restrict it to the renormalizable type of potential, it turns out that many different types of solutions are possible.

We fix the notation following Ref. [4] with the metric sign convention changed and introduce the conformal coupling of dark spinor field to gravity in Sec. II. Considering the renormalizable potential of the field, in the cases where the field is proportional to some power of the cosmic scale factor we study the Friedmann-Robertson-Walker (FRW) cosmology in Sec. III, and we will discuss on their physical implications in Sec. IV.
II. CONFORMAL COUPLING OF DARK SPINOR TO GRAVITY

We study the conformal coupling of dark spinor field to gravity. Let us consider the action of the form

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} \frac{\beta}{2} \bar{\psi} \psi R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - V(\bar{\psi} \psi) \right] + S_m, \tag{1} \]

where \( \psi \) is the dark spinor field \( \bar{2}, \bar{3} \), \( V \) is its potential, and \( S_m \) is the action for other matter. The covariant derivative on a spinor and its dual \( \bar{\psi} \) are given by

\[ \nabla_\mu \psi \equiv \partial_\mu \psi - \Gamma_\mu \psi \tag{2} \]

and

\[ \nabla_\mu \bar{\psi} \equiv \partial_\mu \bar{\psi} + \bar{\psi} \Gamma_\mu, \tag{3} \]

where \( \Gamma_\mu = \frac{i}{4} \omega_\mu^{ab} f_{ab} \) with the spin connection \( \omega_\mu^{ab} = e^\alpha_{\mu} (\partial_\mu e^\alpha + \Gamma^\nu_{\mu\rho} e^\rho^\alpha), \) \( f_{ab} = -\frac{1}{2} [\gamma_a, \gamma_b], \) and \( \gamma_a \)-matrices satisfy the Clifford algebra, \( \{\gamma_a, \gamma_b\} = -2\eta_{ab}, \) in a locally flat inertial coordinate with \( \eta_{ab} = \text{diag}(-1, 1, 1, 1). \)

Einstein’s field equations in the presence of the conformal coupling become:

\[ (1 - \beta \kappa \psi \bar{\psi})(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = \kappa (T^{(c)}_{\mu\nu} + T^{(m)}_{\mu\nu} + \beta T^{(c)}_{\mu\nu}), \tag{4} \]

where \( T^{(c)}_{\mu\nu} = \nabla_\mu \bar{\psi} \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} (\nabla^\rho \bar{\psi} \nabla_\rho \psi + 2V) + \frac{1}{4} \nabla_\rho J^{\rho}_{\mu\nu} \) is the energy-momentum tensor of dark spinor and this accurate expression containing contributions due to the spin connection has been obtained in Ref. \( \bar{4} \) with \( J^{\rho}_{\mu\nu} = -\frac{1}{2} [\bar{\psi} f_{\rho \sigma} \psi] \nabla_\sigma \nabla_\nu \psi + \bar{\psi} f^a_{\mu} \nabla^a \psi]. \) The second term (proportional to \( \beta \)) in the left hand side of Eq. \( \bar{4} \) and the third term in the right hand side, \( T^{(c)}_{\mu\nu} = \nabla_\mu \bar{\psi} \nabla_\nu \psi - g_{\mu\nu} \psi^{\rho\sigma} \nabla_\rho \bar{\psi} \nabla_\sigma \psi \), are contributions from the conformal coupling in the action \( \bar{1} \), \( -\frac{1}{2} \bar{\psi} \psi R \), and \( \sqrt{-g} T^{(m)}_{\mu\nu} = -2\delta S_m / \delta g^{\mu\nu} \) with \( \kappa = 8\pi G \).

III. DARK SPINOR IN THE FRW COSMOLOGY

To study effects of the dark spinor on cosmology in the flat FRW spacetime with a metric of the form

\[ g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \tag{5} \]

where \( a(t) \) is the scale factor of our universe, we assume

\[ \psi(a^\mu) = \varphi(t) \xi \tag{6} \]

with a homogeneous real scalar field \( \varphi(t) \) and a constant spinor \( \xi \) such that \( \xi \xi = 1 \) but \( \nabla_\mu \xi \neq 0 \) \( \bar{1} \).

We comment that the spinor nature of the dark spinor remains alive in the real scalar field \( \varphi(t) \) defined in Eq. \( \bar{1} \) and the ensuing equations contain the contributions, \( 3/8H^2 \varphi^2 + ... \) in Eqs. \( \bar{9} \) and \( \bar{10} \), from the spin connection which are absent in the case of a genuine scalar field. It was shown that these new contributions of the dark spinor to \( \rho_c \) and \( \rho_m \) supply fundamentally different aspects in cosmology \( \bar{4} \).

Applying Einstein’s equations \( \bar{4} \) in the previous section to the metric \( \bar{5} \), we have two differential equations with \( H = \dot{a}/a: \)

\[ 3(1 - \beta \kappa \varphi^2)H^2 = \kappa (\rho_c + \rho_m), \tag{7} \]

\[ -(1 - \beta \kappa \varphi^2)(2 \frac{\dot{a}}{a} + H^2) = \kappa (p_c + p_m), \tag{8} \]

where \( \rho_c \) is the energy density of dark spinor, \( p_c \) is its pressure, and the energy density and pressure of other matter, \( \rho_m \) and \( p_m \):

\[ \rho_c = \frac{\dot{\varphi}^2}{2} + V + \frac{3}{8} H^2 \varphi^2 + 6\beta H \varphi \dot{\varphi}, \tag{9} \]

\[ p_c = \frac{\dot{\varphi}^2}{2} - V - \frac{3}{8} H^2 \varphi^2 - \frac{1}{4} H \varphi^2 + \frac{1}{2} H \dot{\varphi}^2 - \beta (4H \varphi \dot{\varphi} + \dot{\varphi}^2 \varphi^2). \tag{10} \]

The field equation for dark spinor can be written by means of the scalar field in Eq. \( \bar{6} \):

\[ \ddot{\varphi} + 3H \dot{\varphi} + V_{,\varphi} + (\beta R - \frac{3}{4} H^2) \varphi = 0, \tag{11} \]

where “\( V_{,\varphi} \)” denotes the derivative of \( V \) with respect to \( \varphi \) and

\[ R = 6\left( \frac{\dot{a}}{a} + H^2 \right) \tag{12} \]

is the scalar curvature.

We take a renormalizable potential of the dark spinor field with mass dimension one, in the form

\[ V = V_0 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4 \tag{13} \]

with constants \( V_0 = \Lambda / \kappa \), \( m \), and \( \lambda \). \( \Lambda \) stands for the (positive) cosmological constant \( \bar{6} \).

A. Ansatz for dark spinor field and consistency condition for renormalizable potential

To solve Eqs. \( \bar{11} \), \( \bar{8} \), and \( \bar{11} \), we assume the following simple form \( \bar{1} \) for the scalar field in Eq. \( \bar{6} \)

\[ \varphi(t) = \varphi_0 \ a^n(t) \tag{14} \]
with constants \( \varphi_0 \) and \( n \). When \( \rho_m = 0 \) and \( p_m = 0 \), we can recast the equations as

\[
H^2(1 - c_1 \kappa \varphi^2) = \kappa V/3 \tag{15}
\]

with \( c_1 = \left[ \frac{1}{8} + \frac{n^2}{6} + \beta(2n + 1) \right] \),

\[
\dot{H}(1 - c_2 \kappa \varphi^2) = \kappa V/2 - \frac{3}{2} H^2(1 - c_3 \kappa \varphi^2) \tag{16}
\]

with \( c_2 = \left[ \frac{1}{8} + \beta(n + 1) \right] \) and \( c_3 = \left[ 1/8 + n/6 - n^2/6 + \beta/(3 + 4n + 4n^2) \right] \), and

\[
V_{\varphi \varphi} = \varphi(-n + 6\beta)\dot{H} + c_4 H^2 \tag{17}
\]

with \( c_4 = 3/4 - 3n - n^2 - 12\beta \).

Eqs. (15)-(17) give us the consistency condition for the renormalizable potential \( V(\varphi) \) in Eq. (13):

\[
(1 - c_1 \kappa \varphi^2)[(n + 6\beta)\kappa V + 2(1 - c_2 \kappa \varphi^2)\frac{V_{\varphi \varphi}}{\varphi}] = \frac{\kappa V}{3}(h_1 - h_2 \kappa \varphi^2) \tag{18}
\]

with constants \( h_1 = 3/2 - 3n - 2n^2 - 6\beta \) and \( h_2 = (1 - 2n)(1 - 4\beta)/(3 + 4n^2 + 24\beta(1 + 2n))/16 \). From the above equation (18) we determine the parameters of the potential (13) as

\[
m^2 = \frac{\kappa \Lambda}{12n}(n + 2)(2n + 3)(1 - 2n) = -2\kappa \Lambda[c_1(n) + c_2(n)] \tag{19}
\]

and

\[
\lambda = 4\kappa^2 \Lambda c_1(n)c_2(n), \tag{20}
\]

with

\[
\begin{aligned}
c_1(n) &= \frac{1}{24n}(2n + 3)(2n - 1)(n + 1), \\
c_2(n) &= \frac{1}{24n}(2n + 3)(2n - 1), \\
c_1(n) + c_2(n) &= \frac{1}{24n}(2n + 3)(2n - 1)(n + 2) \tag{21}
\end{aligned}
\]

where

\[
\beta(n) = \frac{1}{6} - \frac{1}{8n} \tag{22}
\]

is used.

### B. Cosmological solutions

We restrict our interest to special cases where the cosmic scale factor \( a(t) \) increases but the scalar field \( \varphi(t) \propto a^n(t) \) with \( n < 0 \) decreases with time, and we find useful solutions to the following equation:

\[
a^2 + V_{\text{eff}}(a) = 0 \tag{23}
\]

with

\[
V_{\text{eff}}(a) = \frac{\Lambda}{3}\left[c_2 \kappa \varphi^2 a^{2(n+1)} - a^2\right] \tag{24}
\]

which is derived from Eq. (15) and the renormalizable potential \( V(\varphi) = V_0(1 - c_1 \kappa \varphi^2)(1 - c_2 \kappa \varphi^2) \) in Eq. (13) recast with the help of Eqs. (19) and (20).

Depending on the sign of \( c_2 \), two types of solutions are given as follows.

1. \( c_2 > 0 \) case

When \( -3/2 < n < 0 \), \( c_2 > 0 \) and solving Eq. (23) yields

\[
a(t) \propto \cosh\frac{\sqrt{n}}{\Lambda}\left[-n\sqrt{\frac{\Lambda}{3}}(t - t_c)\right] \tag{25}
\]

with a constant \( t_c \). For \( -1 < n < 0 \) the renormalizable potential \( V(\varphi) \) has stable ground states, while for \( -3/2 < n < -1 \) the potential \( V(\varphi) \) has no stable ground state. In this case, \( a \propto \text{const.} + (t - t_c)^2 \) for \( t_c \lesssim t \), and the universe becomes dark energy-dominated for \( t_c << t \).

2. \( c_2 < 0 \) case

When \( n < -3/2 \), \( c_2 < 0 \) and solving Eq. (23) yields

\[
a(t) \propto \sinh\frac{\sqrt{n}}{\Lambda}\left[-n\sqrt{\frac{\Lambda}{3}}t\right]. \tag{26}
\]

For \( -2 < n < -3/2 \) the renormalizable potential \( V(\varphi) \) has a locally stable ground state, while for \( n \leq -2 \) potential \( V(\varphi) \) has no stable ground state. Note that \( n = -3 \) case looks similar to Eq. (33) of Ref. [7], where stiff fluid- and dark energy-dominated era of the universe could be described by a special realization of nonlinear sigma model.

We note that the above equation (26) leads to

\[
a(t) \sim t^{-\frac{1}{\beta}} \tag{27}
\]

for small \( t \) and that

\[
a \sim e^{\sqrt{\frac{t}{t_c}}} \tag{28}
\]

for large \( t \). When we focus on the case where \( -2 < n < -3/2 \) and thus \( V(\varphi) \) has a metastable ground state, we might say that at early times the field \( \varphi \) acts as dark matter with energy density and pressure \( p_e = \omega \rho_e \) \( (0 < \omega < 1/3) \) and that at sufficiently late times the field dilutes \( (\varphi \propto a^n \rightarrow 0 \text{ with } n < 0) \) and the cosmological constant \( \Lambda = \kappa V_0 \) becomes dominant, leading to an acceleration in the universe’s expansion \[6\], for \( V_{\text{eff}}(a) \) in Eq. (24) approaches \(-\Lambda a^2/3\).
When we calculate the equation of state \( \omega = p_e/\rho_e \) for the dark spinor field, it is given by

\[
\omega = -1 + \frac{n}{6} \left( \frac{(2n-1)(2n+3)\kappa_\varphi^2}{(6n - 4)(2n - 1)(2n+3)\kappa_\varphi^2} \right).
\] (29)

We notice that \( \omega > -1 \) for large \( t \) in the case \( n < -3/2 \) and but \( \omega < -1 \) in the case \( n > -3/2 \). (\( \omega = -1 \) when \( n = -3/2 \)) The late-time value of the equation-of-state parameter \( \omega \) for the dark spinor field can reach the parameter value, \(-1.06_{-0.43}^{+0.43}\), reported by using WMAP data [8], even without any source field carrying negative kinetic energy.

3. Massless case (\( c_2 = 0 \))

In the previous two cases 1 and 2, we exclude one specific case with \( n = -3/2 \), where \( c_2 = 0 \), \( c_1 = 0 \), \( V(\varphi) = V_0 \) as seen in Eqs. [19]-[21], and thus \( a \propto e^{\sqrt{t} \varphi} \). If the dark spinor had a mass in this case, it could play a role of dark matter with the scale factor, \( a \propto t^{2/3} \) for small \( t \), corresponding to matter-dominated era of our universe.

The other massless case is the \( n = -2 \) where \( H^2 \propto \text{const.} + 1/a^4 \), which can describe both radiation- and dark energy-dominated era of the universe, even though the potential \( V(\varphi) \) has no stable ground state.

IV. SUMMARY AND DISCUSSIONS

Considering dark spinor fields which are non-minimally coupled to scalar curvature, we have calculated their contribution to the energy density and pressure in FRW spacetime in Eqs. [9]-[10] of Sec. III. We have used the renormalizable potential with the spinor field decomposed into a homogeneous real scalar field and a constant spinor field. In the case where the scalar field is proportional to some power of the cosmic scale factor, we have found analytic solutions for the scale factor when the scalar field matter dilutes as the universe expands. For \(-3/2 < n < 0 \) we have got the scale factor expressed as hyperbolic cosine functions of the cosmic time in Eq. [25], even though only for \(-1 < n < 0 \) the potential \( V(\varphi) \) has stable ground states. For \( n < -3/2 \) we have obtained the scale factor expressed as hyperbolic sine functions of the comic time in Eq. [26], even though only for \(-2 < n < -3/2 \) the potential has a locally stable ground state. In the latter case, as seen Sec. III B 2, we have discussed the possibility that both matter- and dark energy-dominated era of our universe can be described by the dark spinor.

It is interesting that two specific cases, \( n = -3/2 \) and \( n = -2 \), are not included among the aforementioned cases with \(-2 < n < -3/2 \). The dark spinor has no self interaction in the case of \( n = -3/2 \), and the scale factor is determined only by the cosmological constant \( \Lambda \). In the case of \( n = -2 \) the potential \( V(\varphi) \) has no stable ground state. When we see the early time behavior shown in Eq. (27) in the case of \(-2 < n < -3/2 \) where \( 11/48 < \beta < 1/4 \), we might think that it corresponds to the intermediate one between radiation- and matter-dominated era \((0 < \omega < 1/3, \omega = p_e/\rho_e)\). If we add the ordinary matter contribution in our analysis, then it shall be complicated but the qualitative features of its results will not be changed.

Even in the case \(-1 < n < 0 \) of Sec. III B 1, we have a non-zero conformal coupling to gravity, \( 7/24 < \beta \), and we thus notice the importance of the conformal coupling [4] for the late-time universe to be described well through the dark spinor. The late-time value of the parameter \( \omega \) in Eq. (20) for the dark spinor converges to a parameter value within the bound produced by using WMAP data [8], which has been become possible in this article without the need for any phantom-like matter [9], as discussed in Ref [10].

Our solutions and derived results crucially depend on the assumptions in Eqs. [13] and [14] of Sec. III. It would be interesting to check whether relaxing these conditions, on the renormalizability of \( V(\varphi) \) or on the configuration of the scalar field \( \varphi(t) \), could lead to some more general solutions.

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