HOMOGENIZATION OF THE FIRST INITIAL BOUNDARY-VALUE PROBLEM FOR PARABOLIC SYSTEMS: OPERATOR ERROR ESTIMATES

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Abstract. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O}; \mathbb{C}^n)$, we consider a self-adjoint matrix second order elliptic differential operator $B_{D,\varepsilon}$, $0 < \varepsilon \leq 1$, with the Dirichlet boundary condition. The principal part of the operator is given in a factorized form. The operator involves first and zero order terms. The operator $B_{D,\varepsilon}$ is positive definite; its coefficients are periodic and depend on $x/\varepsilon$. We study the behavior of the operator exponential $e^{-B_{D,\varepsilon}t}$, $t > 0$, as $\varepsilon \to 0$. We obtain approximations for the exponential $e^{-B_{D,\varepsilon}t}$ in the operator norm on $L_2(\mathcal{O}; \mathbb{C}^n)$ and in the norm of operators acting from $L_2(\mathcal{O}; \mathbb{C}^n)$ to the Sobolev space $H^1(\mathcal{O}; \mathbb{C}^n)$. The results are applied to homogenization of solutions of the first initial boundary-value problem for parabolic systems.

Introduction

The paper concerns homogenization theory of periodic differential operators (DO’s). We mention the books on homogenization [BaPa, BeLPap, ZhKO, Sa].

0.1. Statement of the problem. Let $\Gamma \subset \mathbb{R}^d$ be a lattice and let $\Omega$ be the elementary cell of the lattice $\Gamma$. For a $\Gamma$-periodic function $\psi$, we denote $\psi(x) := \psi(x/\varepsilon)$, where $\varepsilon > 0$, and $\overline{\psi} := |\Omega|^{-1} \int_{\Omega} \psi(x)\,dx$.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O}; \mathbb{C}^n)$, we study a selfadjoint matrix strongly elliptic second order DO $B_{D,\varepsilon}$, $0 < \varepsilon \leq 1$, with the Dirichlet boundary condition. The principal part of the operator $B_{D,\varepsilon}$ is given in a factorized form $A_{\varepsilon} = b(D)^*g^\varepsilon(x)b(D)$, where $b(D)$ is a matrix homogeneous first order DO, and $g(x)$ is a $\Gamma$-periodic bounded and positive definite matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(D)$ and $g(x)$ are given below in Subsection 1.3.) The operator $B_{D,\varepsilon}$ is given by the differential expression

$$B_{\varepsilon} = b(D)^*g^\varepsilon(x)b(D) + \sum_{j=1}^d (a_j^\varepsilon(x)D_j + D_ja_j^\varepsilon(x)^*) + Q^\varepsilon(x) + \lambda Q_0^\varepsilon(x)$$

with the Dirichlet condition on $\partial \mathcal{O}$. Here $a_j(x)$, $j = 1, \ldots, d$, and $Q(x)$ are $\Gamma$-periodic matrix-valued functions, in general, unbounded; a $\Gamma$-periodic matrix-valued function $Q_0(x)$ is such that $Q_0(x) > 0$ and $Q_0,Q_0^{-1} \in L_\infty$. The constant $\lambda$ is chosen so that the operator $B_{D,\varepsilon}$ is positive definite. (The precise assumptions on the coefficients are given below in Subsection 1.4.)

The coefficients of the operator $B_{\varepsilon}$ oscillate rapidly for small $\varepsilon$. Let $u_\varepsilon(x,t)$ be the solution of the first initial boundary-value problem:

$$
\begin{cases}
Q_0^\varepsilon(x)\partial_t u_\varepsilon(x,t) = -B_\varepsilon u_\varepsilon(x,t), & x \in \mathcal{O}, \ t > 0; \\
u_\varepsilon(x,t) = 0, & x \in \partial \mathcal{O}, \ t > 0; \ Q_0^\varepsilon(x)u_\varepsilon(x,0) = \varphi(x), & x \in \mathcal{O},
\end{cases}
$$

where $\varphi \in L_2(\mathcal{O}; \mathbb{C}^n)$. We are interested in the behavior of the solution in the small period limit.
0.2. Main results. It turns out that, as \( \varepsilon \to 0 \), the solution \( u_\varepsilon(\cdot,t) \) converges in \( L_2(O;C^n) \) to the solution \( u_0(\cdot,t) \) of the effective problem with constant coefficients:

\[
\begin{aligned}
Q_0 \partial_t u_\varepsilon(x,t) &= -B^0_\varepsilon u_\varepsilon(x,t), \quad x \in O, \ t > 0; \\
\partial_t u_\varepsilon(x,t) &= 0, \quad x \in \partial O, \ t > 0; \quad Q_0 u_0(x,0) = \varphi(x), \quad x \in O.
\end{aligned}
\] (0.3)

Here \( B^0_\varepsilon \) is the differential operator of the effective operator \( B^0_{\varepsilon_D} \). Our first main result is the estimate

\[
\|u_\varepsilon(\cdot,t) - u_0(\cdot,t)\|_{L_2(O)} \leq C \varepsilon(t + \varepsilon^2)^{-1/2} e^{-c t} \|\varphi\|_{L_2(O)}, \quad t \geq 0,
\] (0.4)

for sufficiently small \( \varepsilon \). For fixed time \( t > 0 \), this estimate is of sharp order \( O(\varepsilon) \). Our second main result is approximation of the solution \( u_\varepsilon(\cdot,t) \) in the energy norm:

\[
\|u_\varepsilon(\cdot,t) - v_\varepsilon(\cdot,t)\|_{H^1(O)} \leq C (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c t} \|\varphi\|_{L_2(O)}, \quad t > 0.
\] (0.5)

Here \( v_\varepsilon(\cdot,t) = u_0(\cdot,t) + \varepsilon K_D(t;\varepsilon) \varphi(\cdot) \) is the first order approximation of the solution \( u_\varepsilon(\cdot,t) \). The operator \( K_D(t;\varepsilon) \) is a corrector. It involves rapidly oscillating factors, and so depends on \( \varepsilon \). We have \( \|\varepsilon K_D(t;\varepsilon)\|_{L_2 \to H^1} = O(1) \). For fixed \( t \), estimate (0.5) is of order \( O(\varepsilon^{1/2}) \) due to the influence of the boundary layer. The presence of the boundary layer is confirmed by the fact that, in a strictly interior subdomain \( O' \subset O \), the order of the \( H^1 \)-estimate can be improved:

\[
\|u_\varepsilon(\cdot,t) - v_\varepsilon(\cdot,t)\|_{H^1(O')} \leq C \varepsilon(t^{-1/2} \delta^{-1} + t^{-1}) e^{-c t} \|\varphi\|_{L_2(O)}, \quad t > 0.
\]

Here \( \delta = \text{dist} \{O'; \partial O\} \).

In the general case, the corrector involves a smoothing operator. We distinguish conditions under which it is possible to use a simpler corrector which does not include the smoothing operator. Along with estimate (0.5), we obtain approximation of the flux \( \tilde{g}^\varepsilon b(D)u_\varepsilon(\cdot,t) \) in the \( L_2 \)-norm.

The constants in estimates (0.4) and (0.5) are controlled in terms of the problem data; they do not depend on \( \varphi \). Therefore, estimates (0.4) and (0.5) can be rewritten in the uniform operator topology. In a simpler case where \( Q_0(x) = 1_n \), we have

\[
\|e^{-B_{\varepsilon_D} t} - e^{-B_0 t}\|_{L_2(O) \to L_2(O)} \leq C \varepsilon(t + \varepsilon^2)^{-1/2} e^{-c t}, \quad t \geq 0,
\]

\[
\|e^{-B_{\varepsilon_D} t} - e^{-B_0 t} - \varepsilon K(t;\varepsilon)\|_{L_2(O) \to H^1(O)} \leq C (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c t}, \quad t > 0.
\]

The results of such type are called \textit{operator error estimates} in homogenization theory.

0.3. Operator error estimates. Survey. Currently, the study of operator error estimates is an actively developing area of homogenization theory. The interest in this subject arose in connection with the papers [BSn1, BSn2] by M. Sh. Birman and T. A. Suslina, where the operator \( A_\varepsilon \) of the form \( b(D)^*g^\varepsilon(x)b(D) \) acting in \( L_2(\mathbb{R}^d;C^n) \) was studied. By the \textit{spectral approach}, it was proved that

\[
\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon.
\] (0.6)

Here \( A^0 = b(D)^*g^0b(D) \) is an effective operator and \( g^0 \) is a constant effective matrix. Approximation for the operator \( (A_\varepsilon + I)^{-1} \) in the \( (L_2 \to H^1) \)-norm was obtained in [BSn3]:

\[
\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C \varepsilon.
\] (0.7)

Later T. A. Suslina carried over estimates (0.6) and (0.7) to more general operator \( B_\varepsilon \) of the form (0.1) acting in \( L_2(\mathbb{R}^d;C^n) \). We also mention the paper [Bo] by D. I. Borisov, where the expression for the effective operator \( B^0_\varepsilon \) was found and approximations (0.6), (0.7) for the resolvent were obtained. In [Bo], it was assumed that the coefficients of the operator depend not only on the rapid variable, but also on the slow variable; however, the coefficients of \( B_\varepsilon \) were assumed to be sufficiently smooth.

To parabolic systems, the spectral approach was applied in the papers [Su1, Su2] by T. A. Suslina, where the principal term of approximation was found, and in [Su3], where estimate with the corrector was proved:

\[
\|e^{-A_\varepsilon t} - e^{-A^0 t}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon(t + \varepsilon^2)^{-1/2}, \quad t \geq 0,
\] (0.8)

\[
\|e^{-A_\varepsilon t} - e^{-A^0 t} - \varepsilon K(t;\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C \varepsilon(t^{-1/2} + t^{-1}), \quad t \geq \varepsilon^2.
\] (0.9)
In these estimates, the exponentially decreasing function of $t$ is absent, because the bottom of the spectra of $A_\varepsilon$ and $A^\varepsilon$ is zero. The exponential of the operator $B_\varepsilon$ of the form (0.11) was studied in the paper [M] by Yu. M. Meshkova, where analogs of inequalities (0.8) and (0.9) were obtained.

A different approach to operator error estimates in homogenization theory was suggested by V. V. Zhikov [Zh2]. In [Zh2] and [ZhPas1], estimates of the form (0.6) and (0.7) for the acoustics and elasticity operators were obtained. The “modified method of the first order approximation” or the “shift method”, in the terminology of the authors, was based on analysis of the first order approximation to the solution and introduction of the additional parameter. Along with problems in $\mathbb{R}^d$, in [Zh2] and [ZhPas1], homogenization problems in a bounded domain $O \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were studied. To parabolic equations, the shift method was applied in [ZhPas2], where analogs of estimates (0.8) and (0.9) were proved. Further results of V. V. Zhikov, S. E. Pastukhova, and their students are discussed in the recent survey [ZhPas3].

Operator error estimates for the Dirichlet and Neumann problems for second order elliptic equations in a bounded domain were studied by many authors. Apparently, the first result is due to Sh. Moskow and M. Vogelius who proved an estimate

$$\|A_{D,\varepsilon}^{-1} - (A_D^0)^{-1}\|_{L_2(O) \to L_2(O)} \leq C\varepsilon;$$

(0.10) see [MoV, Corollary 2.2]. Here the operator $A_{D,\varepsilon}$ acts in $L_2(O)$, where $O \subset \mathbb{R}^2$, and is given by $-\nabla g^2(\mathbf{x})\nabla$ with the Dirichlet condition on $\partial O$. The matrix-valued function $g(\mathbf{x})$ is assumed to be infinitely smooth.

For arbitrary dimension, homogenization problems in a bounded domain were studied in [Zh2] and [ZhPas1]. The acoustics and elasticity operators with the Dirichlet or Neumann boundary conditions and without any smoothness assumptions on coefficients were considered. The authors obtained approximation with corrector for the inverse operator in the $(L_2 \to H^1)$-norm with error estimate of order $O(\sqrt{\varepsilon})$. The order deteriorates as compared with a similar result in $\mathbb{R}^d$; this is explained by the boundary influence. As a rough consequence, approximation of the form (0.10) with error estimate of order $O(\sqrt{\varepsilon})$ was deduced. Similar results for the operator $-\nabla g^2(\mathbf{x})\nabla$ in a bounded domain $O \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were obtained by G. Griso [Gr1, Gr2] with the help of the “unfolding” method. In [Gr2], for the same operator a sharp-order estimate (0.10) was proved. For elliptic systems similar results were independently obtained in [KeLiS] and in [PSu, Su5]. Further results and a detailed survey can be found in [Sn6, Sn7].

For the matrix operator of the form (0.11) with the Dirichlet condition, a homogenization problem was studied by Q. Xu [Xu1, Xu3]. The case of the Neumann boundary condition was studied in [Xn2]. However, in the papers by Q. Xu, the operator is subject to a rather restrictive condition of uniform ellipticity. Approximations of the generalizated resolvent of the operator (0.11) with two-parametric error estimates were obtained in the recent paper [MSu3] by the authors (see also a brief communication [MSu4]). We focus on these results in more detail, since they are basic for us. For $\zeta \in C \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and sufficiently small $\varepsilon$, we have

$$\|(B_{D,\varepsilon} - \zeta Q_0^0)^{-1} - (B_D^0 - \zeta Q_0^0)^{-1}\|_{L_2(O) \to L_2(O)} \leq C(\varepsilon)|\zeta|^{-1/2},$$

(0.11)

$$\|(B_{D,\varepsilon} - \zeta Q_0^0)^{-1} - (B_D^0 - \zeta Q_0^0)^{-1} - \varepsilon K_D(\varepsilon, \zeta)\|_{L_2(O) \to H^1(O)} \leq C(\varepsilon)(\varepsilon^{1/2}|\zeta|^{-1/4} + \varepsilon).$$

(0.12)

Note that the values $C(\phi)$ are controlled explicitly in terms of the problem data and the angle $\phi = \arg \zeta$. Estimates (0.11) and (0.12) are uniform with respect to $\phi$ in any domain of the form $\{\zeta \in C : |\zeta| \geq 1, \phi_0 \leq \phi \leq 2\pi - \phi_0\}$ with arbitrarily small $\phi_0 > 0$. Moreover, in [MSu3], analogs of estimates (0.11) and (0.12) in a wider domain of spectral parameter $\zeta$ were proved.

We proceed to discussion of the parabolic problems in a bounded domain. In the two-dimensional case, some estimates of operator type for elliptic and parabolic equations were obtained in [ChKonLe]. However, in [ChKonLe], the matrix $g$ was assumed to be $C^\infty$-smooth, and the initial data for a parabolic equation belonged to $H^2(O)$. In the case of arbitrary dimension and without smoothness assumptions on coefficients, approximation for the exponential of the operator $b(D)^\gamma g^2(\mathbf{x})b(D)$ (with the Dirichlet or Neumann conditions) was found in the
paper [MSu1] by the authors:
\[
\| e^{-AD,\varepsilon t} - e^{-A_0^D t} \|_{L^2(\Omega) \to L^2(\Omega)} \leq C\varepsilon(t + \varepsilon^2)^{-1/2}e^{-ct}, \quad t \geq 0,
\]
\[
\| e^{-AD,\varepsilon t} - e^{-A_0^D t} - \varepsilon K_D(t; \varepsilon) \|_{L^2(\Omega) \to H^1(\Omega)} \leq C\varepsilon^{1/2}t^{-3/4}e^{-ct}, \quad t \geq \varepsilon^2.
\]
The method of [MSu1] was based on employing the identity
\[
e^{-AD,\varepsilon t} = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t}(A_{D,\varepsilon} - \zeta I)^{-1} d\zeta,
\]
where \( \gamma \subset \mathbb{C} \) is a contour enclosing the spectrum of \( A_{D,\varepsilon} \) in positive direction. This identity allows us to deduce approximations for the operator exponential \( e^{-AD,\varepsilon t} \) from the corresponding approximations of the resolvent \( (A_{D,\varepsilon} - \zeta I)^{-1} \) with two-parametric error estimates (with respect to \( \varepsilon \) and \( \zeta \)). The required approximations for the resolvent were found in [Su7].

The operator with coefficients periodic in the space and time variables was studied by J. Geng and Z. Shen [GeS]. In [GeS], operator error estimates for the equation

\[
\begin{align*}
\Delta u_{\varepsilon}(x, t) &= -\varepsilon g(x, \varepsilon t) - \varepsilon^2 \nabla \cdot (a(x, t) \nabla u_{\varepsilon}(x, t)) \quad \text{in } \Omega \times \mathbb{R}, \\
\frac{1}{\varepsilon} \partial_t u_{\varepsilon}(x, t) &= 0 \quad \text{on } \partial \Omega \times \mathbb{R}, \\
u_{\varepsilon}(x, 0) &= u_0(x),
\end{align*}
\]

in a bounded domain of class \( C^{1,1} \) were obtained. The results of [GeS] were generalized to the case of Lipschitz domains by Q. Xu and Sh. Zhou [XuZ].

0.4. Method. We develop the method of the paper [MSu1]. It is based upon the following representation for the solution \( u_{\varepsilon} \) of the first initial boundary-value problem (0.2): \( u_{\varepsilon} (\cdot, t) = -\frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t}(B_{D,\varepsilon} - \zeta Q_0^{-1})\varphi d\zeta \), where \( \gamma \subset \mathbb{C} \) is a suitable contour. The solution of the effective problem (0.3) admits a similar representation. Hence,

\[
\begin{align*}
u_{\varepsilon}(\cdot, t) - u_0(\cdot, t) &= -\frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t}((B_{D,\varepsilon} - \zeta Q_0^{-1}) - (B_0^D - \zeta Q_0^{-1}))\varphi d\zeta.
\end{align*}
\]

Using the results of [MSu3] (estimate (0.11)), we obtain approximation of the resolvent for \( \zeta \in \gamma \) and employ representation (0.13). This leads to (0.4). Note that the dependence of the right-hand side of (0.11) on \( |\zeta| \) for large \( |\zeta| \) is important for us. Approximation with the corrector taken into account is obtained in a similar way.

0.5. Plan of the paper. The paper consists of five sections and Appendix (§§3–8). In §§1, 2 we describe the class of operators \( B_{D,\varepsilon} \), introduce the effective operator \( B_0^D \), and formulate the needed results about approximation of the operator \( (B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1} \). The main results of the paper are obtained in §§3, 4. In §§3, 4 these results are applied to homogenization of the solutions of the first initial boundary-value problem for nonhomogeneous parabolic equation. §§3, 4 are devoted to applications of the general results. In §§5, 6 we study an operator with a singular potential of order \( O(\varepsilon^{-1}) \) is considered. In §§5, 6 we prove some statements concerning removal of the smoothing operator in the corrector. The case of additional smoothness of the boundary is considered in §7 of a strictly interior subdomain is discussed in §§8. The needed properties of the oscillating factors in the corrector are obtained in §§6.

0.6. Notation. Let \( \mathcal{H} \) and \( \mathcal{H}_s \) be complex separable Hilbert spaces. The symbols \( (\cdot, \cdot)_D \) and \( \| \cdot \|_D \) stand for the inner product and the norm in \( \mathcal{H} \); the symbol \( \| \cdot \|_{\mathcal{H} \to \mathcal{H}_s} \) denotes the norm of a linear continuous operator acting from \( \mathcal{H} \) to \( \mathcal{H}_s \).

The set of natural numbers and the set of nonnegative integers are denoted by \( \mathbb{N} \) and \( \mathbb{Z}_+ \), respectively. We denote \( \mathbb{R}_+ := [0, \infty) \). The symbols \( (\cdot, \cdot) \) and \( |\cdot| \) denote the inner product and the norm in \( \mathbb{C}^n \); \( \mathbf{1}_n \) is the identity \((n \times n)\)-matrix. If \( a \) is an \((m \times n)\)-matrix, then the symbol \( |a| \) denotes the norm of a viewed as operator from \( \mathbb{C}^n \) to \( \mathbb{C}^m \). If \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_+ \) is a multiindex, \( |\alpha| \) denotes its length: \( |\alpha| = \sum_{j=1}^d \alpha_j \). For \( z \in \mathbb{C} \), the complex conjugate number is denoted by \( z^* \). (We use such nonstandard notation, because the upper line denotes the mean value of a periodic function over the periodicity cell.) We denote \( \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( iD_j = \partial_j = \partial / \partial x_j, \ j = 1, \ldots, d, \ D = -i\nabla = (D_1, \ldots, D_d) \). The \( L_p \)-classes of \( \mathbb{C}^n \)-valued functions in a domain \( \Omega \subset \mathbb{R}^d \) are denoted by \( L_p(\Omega; \mathbb{C}^n) \). The Sobolev classes of \( \mathbb{C}^n \)-valued functions in a domain \( \Omega \subset \mathbb{R}^d \) are denoted by \( H^s(\Omega; \mathbb{C}^n) \). By \( H^s_0(\Omega; \mathbb{C}^n) \) we denote
the closure of $C_0^\infty(\mathcal{O}; \mathbb{C}^n)$ in $H^1(\mathcal{O}; \mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes, if this does not lead to confusion, we use such simple notation for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0,T); \mathfrak{H})$, $1 \leq p \leq \infty$, denotes the $L_p$-space of $\mathfrak{H}$-valued functions on the interval $(0,T)$.

Various constants in estimates are denoted by $c, C, C, C, \mathcal{C}$ (probably, with indices and marks).

The main results of the present paper were announced in [MSu4].

1. THE RESULTS ON HOMOGENIZATION OF THE DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS

1.1. Lattices in $\mathbb{R}^d$. Let $\Gamma \subset \mathbb{R}^d$ be a lattice generated by the basis $a_1, \ldots, a_d \in \mathbb{R}^d$:

$$\Gamma = \{ a \in \mathbb{R}^d : a = \sum_{j=1}^d \nu_j a_j, \nu_j \in \mathbb{Z} \},$$

and let $\Omega$ be the elementary cell of the lattice $\Gamma$:

$$\Omega = \{ x \in \mathbb{R}^d : x = \sum_{j=1}^d \tau_j a_j, -\frac{1}{2} < \tau_j < \frac{1}{2} \}.$$ 

By $|\Omega|$ we denote the Lebesgue measure of the cell $\Omega$: $|\Omega| = \text{meas}\, \Omega$. We put $2r_1 := \text{diam}\, \Omega$.

Let $\tilde{H}^1(\Omega)$ denote the subspace of functions in $H^1(\Omega)$, whose $\Gamma$-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\text{loc}}(\mathbb{R}^d)$. If $\Phi(x)$ is a $\Gamma$-periodic matrix-valued function in $\mathbb{R}^d$, we put $\Phi^\varepsilon(x) := (\Phi(x)/\varepsilon, \varepsilon > 0); \Phi := [\Omega]^{-1} \int_\Omega \Phi(x) dx, \Phi := (|\Omega|^{-1} \int_\Omega \Phi(x)^{-1} dx)^{-1}$. Here, in the definition of $\Phi$ it is assumed that $\Phi \in L_{\text{loc}}(\mathbb{R}^d)$; in the definition of $\Phi^\varepsilon$ it is assumed that the matrix $\Phi$ is square and nondegenerate, and $\Phi^{-1} \in L_{\text{loc}}(\mathbb{R}^d)$. By $[\Phi^\varepsilon]$ we denote the operator of multiplication by the matrix-valued function $\Phi^\varepsilon(x)$.

1.2. The Steklov smoothing. The Steklov smoothing operator $S_\varepsilon^{(k)}$, $\varepsilon > 0$, acts in $L_2(\mathbb{R}^d; \mathbb{C}^k)$ (where $k \in \mathbb{N}$) and is given by

$$S_\varepsilon^{(k)}(u)(x) = |\Omega|^{-1} \int_\Omega u(x - \varepsilon z) dz, \quad u \in L_2(\mathbb{R}^d; \mathbb{C}^k).$$ (1.1)

We shall omit the index $k$ in the notation and write simply $S_\varepsilon$. Obviously, $S_\varepsilon D^\alpha u = D^\alpha S_\varepsilon u$ for $u \in H^\alpha(\mathbb{R}^d; \mathbb{C}^k)$ and any multiindex $\alpha$ such that $|\alpha| \leq \sigma$. Note that

$$\|S_\varepsilon\|_{H^\sigma(\mathbb{R}^d) \to H^\sigma(\mathbb{R}^d)} \leq 1, \quad \sigma \geq 0.$$ (1.2)

We need the following properties of the operator $S_\varepsilon$ (see [ZhPas1] Lemmas 1.1 and 1.2 or [PSu, Propositions 3.1 and 3.2]).

Proposition 1.1. For any function $u \in H^1(\mathbb{R}^d; \mathbb{C}^k)$, we have

$$\|S_\varepsilon u - u\|_{L_2(\mathbb{R}^d)} \leq \varepsilon r_1 \|Du\|_{L_2(\mathbb{R}^d)},$$

where $2r_1 = \text{diam}\, \Omega$.

Proposition 1.2. Let $\Phi$ be a $\Gamma$-periodic function in $\mathbb{R}^d$ such that $\Phi \in L_2(\Omega)$. Then the operator $[\Phi^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d)$ and

$$\|\Phi^\varepsilon S_\varepsilon\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} \|\Phi\|_{L_2(\Omega)}.$$ 1.3. The operator $A_{D,\varepsilon}$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O}; \mathbb{C}^n)$, we consider the operator $A_{D,\varepsilon}$ given formally by the differential expression $A_\varepsilon = b(D) + g^\varepsilon(x)b(D)$ with the Dirichlet condition on $\partial\mathcal{O}$. Here $g(x)$ is a $\Gamma$-periodic Hermitian $(m \times m)$-matrix-valued function (in general, with complex entries). It is assumed that $g(x) > 0$ and $g, g^{-1} \in L_\infty(\mathbb{R}^d)$. The differential operator $b(D)$ is given by $b(D) = \sum_{j=1}^d b_j D_j$, where $b_j, j = 1, \ldots, d$, are constant matrices of size $m \times n$ (in general, with complex entries). Assume that $m \geq n$ and that the symbol $b(\xi) = \sum_{j=1}^d b_j \xi_j$ of the operator $b(D)$ has maximal rank: $\text{rank}\, b(\xi) = n$ for $0 \neq \xi \in \mathbb{R}^d$. This condition is equivalent to the estimates

$$\alpha_1 1_n \leq b(\theta)^* b(\theta) \leq \alpha_1 1_n, \quad \theta \in S^{d-1}; \quad 0 < \alpha_0 \leq \alpha_1 < \infty,$$ (1.3)
with some positive constants \( \alpha_0 \) and \( \alpha_1 \). From (1.3) it follows that
\[
|b_j| \leq \alpha_1^{1/2}, \quad j = 1, \ldots, d.
\]

The precise definition of the operator \( A_{D,\varepsilon} \) is given in terms of the quadratic form
\[
a_{D,\varepsilon}[u, u] = \int g(x)b(D)u, b(D)u\,dx, \quad u \in H^1_0(O; \mathbb{C}^n).
\]

Extending \( u \in H^1_0(O; \mathbb{C}^n) \) by zero to \( \mathbb{R}^d \setminus O \) and taking (1.3) into account, we find
\[
\alpha_0\|g^{-1}L^{-1}Q u\|^2_{L_2(O)} \leq a_{D,\varepsilon}[u, u] \leq \alpha_1\|g\|_{L_\infty}\|Du\|^2_{L_2(O)}, \quad u \in H^1_0(O; \mathbb{C}^n).
\]

1.4. Lower order terms. The operator \( B_{D,\varepsilon} \). We study the selfadjoint operator \( B_{D,\varepsilon} \) whose principal part coincides with \( A_{\varepsilon} \). To define the lower order terms, we introduce \( \Gamma \)-periodic \((n \times n)\)-matrix-valued functions (in general, with complex entries) \( a_j \), \( j = 1, \ldots, d \), such that
\[
a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2, \quad j = 1, \ldots, d.
\]

Next, let \( Q \) and \( Q_0 \) be \( \Gamma \)-periodic Hermitian \((n \times n)\)-matrix-valued functions (with complex entries) such that
\[
Q \in L_s(\Omega), \quad s \leq 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2;
\]
\[
Q_0(x) > 0; \quad Q_0, Q_0^{-1} \in L_\infty(\mathbb{R}^d).
\]

For convenience of further references, the following set of variables is called the “problem data”:
\[
d, m, n, \rho, s; \quad \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L_\rho(\Omega)}, \quad j = 1, \ldots, d;
\]
\[
\|Q\|_{L_s(\Omega)}; \quad \|Q_0\|_{L_\infty}, \|Q_0^{-1}\|_{L_\infty}; \quad \text{the parameters of the lattice } \Gamma; \quad \text{the domain } O.
\]

In \( L_2(O; \mathbb{C}^n) \), we consider the operator \( B_{D,\varepsilon} \), \( 0 < \varepsilon \leq 1 \), formally given by the differential expression
\[
B_\varepsilon = b(D)^*g(x)b(D) + \sum_{j=1}^d (a_j^*(x)D_j + D_ja_j^*(x))^* + Q^*(x) + \lambda Q_0^*(x)
\]
with the Dirichlet boundary condition. Here the constant \( \lambda \) is chosen so that the operator \( B_{D,\varepsilon} \) is positive definite (see (1.6) below). The precise definition of the operator \( B_{D,\varepsilon} \) is given in terms of the quadratic form
\[
b_{D,\varepsilon}[u, u] = (g^*b(D)u, b(D)u)_{L_2(O)} + 2\text{Re} \sum_{j=1}^d (a_j^*D_ju, u)_{L_2(O)}
\]
\[
+ (Q^*u, u)_{L_2(O)} + \lambda(Q_0^*u, u)_{L_2(O)}, \quad u \in H^1_0(O; \mathbb{C}^n).
\]

Let us check that the form \( b_{D,\varepsilon} \) is closed. By the Hölder inequality and the Sobolev embedding theorem, it can be shown that for any \( \nu > 0 \) there exist constants \( C_j(\nu) > 0 \) such that
\[
\|a_j^*u\|^2_{L^2(\mathbb{R}^d)} \leq \nu\|Du\|^2_{L^2(\mathbb{R}^d)} + C_j(\nu)\|u\|^2_{L^2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n),
\]
\[
j = 1, \ldots, d; \text{ see } [\text{Su4}], (5.11)-(5.14)]. \quad \text{By the change of variables } y := \varepsilon^{-1}x \text{ and } u(x) = : v(y), \text{ we deduce}
\]
\[
\|(a_j^*)^*u\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |a_j(\varepsilon^{-1}x)^*u(x)|^2 \,dx = \varepsilon^d \int_{\mathbb{R}^d} |a_j(y)^*v(y)|^2 \,dy
\]
\[
\leq \varepsilon^d\nu \int_{\mathbb{R}^d} |D_yv(y)|^2 \,dy + \varepsilon^dC_j(\nu)\int_{\mathbb{R}^d} |v(y)|^2 \,dy
\]
\[
\leq \nu\|Du\|^2_{L^2(\mathbb{R}^d)} + C_j(\nu)\|u\|^2_{L^2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
Then, by (1.5), for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that
\[
\sum_{j=1}^{d} \|(a_j^e)^* u\|_{L^2(\mathbb{R}^d)}^2 \leq \nu \|(g^e)^{1/2} b(D) u\|_{L^2(\mathbb{R}^d)}^2 + C(\nu) \|u\|_{L^2(\mathbb{R}^d)}^2,
\] (1.12)
where $u \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, $0 < \varepsilon \leq 1$.

If $\nu$ is fixed, then $C(\nu)$ depends only on $d$, $\rho$, $\alpha_0$, the norms $\|g^{-1}\|_{L^\infty}$, $\|a_j\|_{L^\infty(\Omega)}$, $j = 1, \ldots, d$, and the parameters of the lattice $\Gamma$.

By (1.3), for $u \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ we have
\[
\|D u\|_{L^2(\mathbb{R}^d)}^2 \leq c_1^2 \|(g^e)^{1/2} b(D) u\|_{L^2(\mathbb{R}^d)}^2,
\] (1.13)
where $c_1 := \alpha_0^{-1/2} \|g^{-1}\|_{L^\infty}^{1/2}$. Combining this with (1.12), we obtain
\[
\left| 2 \text{Re} \sum_{j=1}^{d} (D_j u, (a_j^e)^* u)_{L^2(\mathbb{R}^d)} \right| \leq \frac{1}{4} \|(g^e)^{1/2} b(D) u\|_{L^2(\mathbb{R}^d)}^2 + c_2 \|u\|_{L^2(\mathbb{R}^d)}^2,
\] (1.14)
where $c_2 := 8c_1^2 C(\nu_0)$ with $\nu_0 := 2^{-6} \alpha_0 \|g^{-1}\|_{L^\infty}^{-1}$.

Next, by condition (1.8) on $Q$, for any $\nu > 0$ there exists a constant $C_Q(\nu) > 0$ such that
\[
\|(Q^\varepsilon u, u)_{L^2(\mathbb{R}^d)}\| \leq \nu \|D u\|_{L^2(\mathbb{R}^d)}^2 + C_Q(\nu) \|u\|_{L^2(\mathbb{R}^d)}^2,
\] (1.15)
where $u \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, $0 < \varepsilon \leq 1$.

For fixed $\nu$, the constant $C_Q(\nu)$ is controlled in terms of $d$, $s$, $\|Q\|_{L^s(\Omega)}$, and the parameters of the lattice $\Gamma$.

We fix a constant $\lambda$ in (1.10) as in [MS02] Subsection 2.8:
\[
\lambda := (C_Q(\nu_s) + c_2) \|Q_{0\varepsilon}\|_{L^\infty} \quad \text{for } \nu_s := 2^{-1} \alpha_0 \|g^{-1}\|_{L^\infty}^{-1}.
\] (1.16)

We return to the form (1.11). Extending the function $u \in H^1_0(\mathbb{O}; \mathbb{C}^n)$ by zero to $\mathbb{R}^d \setminus \mathbb{O}$ and using (1.5), (1.13), (1.14), and (1.15) with $\nu = \nu_s$, we obtain the lower estimate for the form (1.11):
\[
\begin{align*}
\tilde{b}_{D,\varepsilon}(u, u) & \geq \frac{1}{4} a_{D,\varepsilon}(u, u) \geq c_3 \|D u\|_{L^2(\mathbb{O})}^2, \\
& \quad u \in H^1(\mathbb{O}; \mathbb{C}^n); \\
c_3 & := \frac{1}{4} \alpha_0 \|g^{-1}\|_{L^\infty}^{-1}.
\end{align*}
\] (1.17)

Next, by (1.6), (1.14), and (1.15) with $\nu = 1$, we have
\[
\tilde{b}_{D,\varepsilon}(u, u) \leq C_* \|u\|_{H^1(\mathbb{R}^d)}^2, \\
& \quad u \in H^1(\mathbb{O}; \mathbb{C}^n),
\] where $C_* := \max\{\frac{2}{3} \alpha_1 \|g^{-1}\|_{L^\infty} + 1 : C_Q(1) + \lambda \|Q\|_{L^\infty} + c_2\}$. Thus, the form $\tilde{b}_{D,\varepsilon}$ is closed. The corresponding selfadjoint operator in $L^2(\mathbb{O}; \mathbb{C}^n)$ is denoted by $B_{D,\varepsilon}$.

By the Friedrichs inequality, (1.17) implies that
\[
\tilde{b}_{D,\varepsilon}(u, u) \geq c_* (\text{diam } \mathbb{O})^{-2} \|u\|_{L^2(\mathbb{O})}^2, \\
& \quad u \in H^1_0(\mathbb{O}; \mathbb{C}^n); \\
d_3 & := \frac{1}{4} \alpha_0 \|g^{-1}\|_{L^\infty}^{-1}.
\] (1.19)

Hence, the operator $B_{D,\varepsilon}$ is positive definite. By (1.17) and (1.19),
\[
\|u\|_{H^1(\mathbb{O})} \leq c_3 \|B_{D,\varepsilon}^{1/2} u\|_{L^2(\mathbb{O})}, \\
& \quad u \in H^0_0(\mathbb{O}; \mathbb{C}^n); \\
c_3 & := c_*^{-1/2} (1 + (\text{diam } \mathbb{O})^2)^{1/2}.
\] (1.20)

We also need an auxiliary operator $\tilde{B}_{D,\varepsilon}$. We factorize the matrix $Q_0(x)$: there exists a $\Gamma$-periodic matrix-valued function $f(x)$ such that $f, f^{-1} \in L^\infty(\mathbb{R}^d)$ and
\[
Q_0(x) = (f(x))^* f(x)^{-1}.
\] (1.22)
(For instance, one can choose $f(x) = Q_0(x)^{1/2}$.) Let $\tilde{B}_{D,\varepsilon}$ be a selfadjoint operator in $L^2(\mathbb{O}; \mathbb{C}^n)$ generated by the quadratic form
\[
\tilde{b}_{D,\varepsilon}(u, u) := b_{D,\varepsilon}(f^* u, f^* u)
\] (1.23)
on the domain $\text{Dom} \, \tilde{b}_{D,\varepsilon} := \{ u \in L_2(O; \mathbb{C}^n) : f^* u \in H^1_0(O; \mathbb{C}^n) \}$. In other words, $\tilde{B}_{D,\varepsilon} = (f^*)^* B_{D,\varepsilon} f^*$. Let $\tilde{B}_{\varepsilon}$ denote the differential expression $(f^*)^* B_{\varepsilon} f^*$. Note that

$$
(B_{D,\varepsilon} - \zeta Q_0)\xi = f^* (\tilde{B}_{D,\varepsilon} - \zeta I)^{-1} (f^*)^*.
$$

(1.24)

1.5. The effective matrix and its properties. The effective operator for $A_{D,\varepsilon}$ is given by the differential expression $A^0 = b(D)^* g^0 b(D)$ with the Dirichlet condition on $\partial O$. Here $g^0$ is a constant effective matrix of size $m \times m$. The matrix $g^0$ is expressed in terms of the solution of an auxiliary problem on the cell. Let an $(n \times m)$-matrix-valued function $\Lambda(x)$ be the (weak) $\Gamma$-periodic solution of the problem

$$
b(D)^* g(x) / (b(D) \Lambda(x) + 1_m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0.
$$

(1.25)

Then the effective matrix is given by

$$
g^0 := |\Omega|^{-1} \int_\Omega \tilde{g}(x) \, dx,
$$

(1.26)

$$
\tilde{g}(x) := g(x) / (b(D) \Lambda(x) + 1_m).
$$

(1.27)

It can be checked that the matrix $g^0$ is positive definite.

According to [BSn3] (6.28) and Subsection 7.3, the solution of problem (1.25) satisfies

$$
\|\Lambda\|_{H^1(\Omega)} \leq M.
$$

(1.28)

Here the constant $M$ depends only on $m$, $\alpha_0$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, and the parameters of the lattice $\Gamma$.

The effective matrix satisfies the estimates known as the Voigt–Reuss bracketing (see, e. g., [BSn2] Chapter 3, Theorem 1.5).

Proposition 1.3. Let $g^0$ be the effective matrix (1.26). Then

$$
\underline{g} \leq g^0 \leq \overline{g}.
$$

(1.29)

If $m = n$, then $g^0 = g$.

From (1.29) it follows that

$$
|g^0| \leq \|g\|_{L^\infty}, \quad |(g^0)^{-1}| \leq \|g^{-1}\|_{L^\infty}.
$$

(1.30)

Now we distinguish the cases where one of the inequalities in (1.29) becomes an identity, see [BSn2] Chapter 3, Propositions 1.6 and 1.7).

Proposition 1.4. The identity $g^0 = \overline{g}$ is equivalent to the relations

$$
b(D)^* g_k(x) = 0, \quad k = 1, \ldots, m,
$$

(1.31)

where $g_k(x)$, $k = 1, \ldots, m$, are the columns of the matrix $g(x)$.

Proposition 1.5. The identity $g^0 = \underline{g}$ is equivalent to the representations

$$
l_k(x) = l^0_k + b(D) w_k, \quad l^0_k \in \mathbb{C}^m, \quad w_k \in \tilde{H}^1(\Omega; \mathbb{C}^m), \quad k = 1, \ldots, m,
$$

(1.32)

where $l_k(x)$, $k = 1, \ldots, m$, are the columns of the matrix $g(x)^{-1}$.

1.6. The effective operator. To describe how the lower order terms of the operator $B_{D,\varepsilon}$ are homogenized, we consider a $\Gamma$-periodic $(n \times n)$-matrix-valued function $\tilde{\Lambda}(x)$ which is the (weak) solution of the problem

$$
b(D)^* g(x) / (b(D) \tilde{\Lambda}(x)) + \sum_{j=1}^d D_j a_j(x)^* = 0, \quad \int_\Omega \tilde{\Lambda}(x) \, dx = 0.
$$

(1.33)

According to [Su4] (7.51) and (7.52), we have

$$
\|\tilde{\Lambda}\|_{H^1(\Omega)} \leq \tilde{M},
$$

(1.34)

where the constant $\tilde{M}$ depends only on $n$, $\rho$, $\alpha_0$, $\|g^{-1}\|_{L^\infty}$, $\|a_j\|_{L^\rho(\Omega)}$, $j = 1, \ldots, d$, and the parameters of the lattice $\Gamma$. 

Next, we define constant matrices $V$ and $W$ as follows:

$$V := |\Omega|^{-1} \int_{\Omega} (b(D)r(x))^* g(x)(b(D)r(x)) \, dx,$$

(1.35)

$$W := |\Omega|^{-1} \int_{\Omega} (b(D)r(x))^* g(x)(b(D)r(x)) \, dx.$$  

(1.36)

In $L_2(\mathcal{O}; \mathbb{C}^n)$, consider the quadratic form

$$b_D^0[u, u] = (g^0 b(D)u, b(D)u)_{L_2(\mathcal{O})} + 2\text{Re} \sum_{j=1}^d (\sigma_j D_j u, u)_{L_2(\mathcal{O})}$$

$$- 2\text{Re}(V u, b(D)u)_{L_2(\mathcal{O})} - (W u, u)_{L_2(\mathcal{O})} + \lambda(\nabla u, u)_{L_2(\mathcal{O})}, \ u \in H_0^1(\mathcal{O}; \mathbb{C}^n).$$

The following estimates were proved in [MSu3] (2.22) and (2.23):

$$c_s \|Du\|^2_{L_2(\mathcal{O})} \leq b_D^0[u, u] \leq c_1 \|u\|_{H^1(\mathcal{O})}^2, \quad u \in H_0^1(\mathcal{O}; \mathbb{C}^n),$$  

(1.37)

$$b_D^0[u, u] \geq c_s (\text{diam } \mathcal{O})^{-2} \|u\|_{L_2(\mathcal{O})}^2, \quad u \in H_0^1(\mathcal{O}; \mathbb{C}^n).$$  

(1.38)

Here the constant $c_s$ depends only on the problem data (1.9). A selfadjoint operator in $L_2(\mathcal{O}; \mathbb{C}^n)$ corresponding to the form $b_D^0$ is denoted by $B_D^0$. By (1.37) and (1.38),

$$\|u\|_{H^1(\mathcal{O})} \leq c_3 \|(B_D^0)^{1/2} u\|_{L_2(\mathcal{O})}, \quad u \in H_0^1(\mathcal{O}; \mathbb{C}^n),$$  

(1.39)

where $c_3$ is given by (1.21).

Due to condition $\partial \mathcal{O} \in C^{1,1}$, the operator $B_D^0$ is given by

$$B_D^0 = b(D)^* g^0 b(D) - b(D)^* V - V^* b(D) + \sum_{j=1}^d (a_j + a_j^*) D_j - W + \nabla + \lambda Q_0$$  

(1.40)

on the domain $H^2(\mathcal{O}; \mathbb{C}^n) \cap H_0^1(\mathcal{O}; \mathbb{C}^n)$, and we have

$$\|(B_D^0)^{-1}\|_{L_2(\mathcal{O}) \to H^2(\mathcal{O})} \leq \hat{c}.$$  

(1.41)

Here the constant $\hat{c}$ depends only on the problem data (1.9). To justify this fact, we refer to the theorems about regularity of solutions of the strongly elliptic systems (see [McL, Chapter 4]).

**Remark 1.6.** Instead of condition $\partial \mathcal{O} \in C^{1,1}$, one could impose the following implicit condition: a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^d$ is such that estimate (1.41) holds. For such domain the results of the paper remain true. In the case of scalar elliptic operators, wide conditions on $\partial \mathcal{O}$ ensuring estimate (1.41) can be found in [KoE] and [MaSh, Chapter 7] (in particular, it suffices to assume that $\partial \mathcal{O} \in C^{\alpha}$, $\alpha > 3/2$).

Denote

$$f_0 := (Q_0)^{-1/2}.$$  

(1.42)

By (1.22),

$$|f_0| \leq \|f\|_{L_\infty} = \|Q_0^{-1}\|_{L_\infty}^{1/2}, \quad |f_0^{-1}| \leq \|f^{-1}\|_{L_\infty} = \|Q_0\|_{L_\infty}^{1/2}.$$  

(1.43)

In what follows, we will need the operator $\tilde{B}_D^0 := f_0 B_D^0 f_0$ corresponding to the quadratic form

$$\tilde{b}_D^0[u, u] := b_D^0[f_0 u, f_0 u], \quad u \in H_0^1(\mathcal{O}; \mathbb{C}^n).$$  

(1.44)

Note that $(B_D^0 - \lambda Q_0)^{-1} = f_0 (\tilde{B}_D^0 - \lambda I)^{-1} f_0.$
1.7. Approximation of the generalized resolvent \((B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1}\). Now we formulate the results of the paper [MSu3], where the behavior of the generalized resolvent \((B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1}\) was studied. Suppose that \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+\) and \(|\zeta| \geq 1\). The principal term of approximation of the generalized resolvent \((B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1}\) was found in [MSu3] Theorem 2.5; approximation of this resolvent in the \((L_2 \rightarrow H^1)\)-norm with the corrector taken into account was found in [MSu3] Theorem 2.6; an approximate approximation of the operator \(g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1}\) (corresponding to the “flux”) was obtained in [MSu3] Proposition 10.7.

We choose the numbers \(\varepsilon_0, \varepsilon_1 \in (0, 1]\) according to the following condition.

**Condition 1.7.** Let \(O \subset \mathbb{R}^d\) be a bounded domain. Denote

\[
(\partial O)_\varepsilon := \left\{ x \in \mathbb{R}^d : \text{dist}(x, \partial O) < \varepsilon \right\}.
\]

Suppose there exists a number \(\varepsilon_0 \in (0, 1]\) such that the strip \((\partial O)_\varepsilon\) can be covered by a finite number of open sets admitting diffeomorphisms of class \(C^{0,1}\) rectifying the boundary \(\partial O\). Denote \(\varepsilon_1 := \varepsilon_0(1 + r_1)^{-1}\), where \(2r_1 = \text{diam} \Omega\).

Obviously, the number \(\varepsilon_1\) depends only on the domain \(O\) and the lattice \(\Gamma\). Note that Condition 1.7 is ensured only by the assumption that \(\partial O\) is Lipschitz; we imposed a more restrictive condition \(\partial O \in C^{1,1}\) in order to ensure estimate (1.41).

**Theorem 1.8** ([MSu3]). Let \(O \subset \mathbb{R}^d\) be a bounded domain of class \(C^{1,1}\). Suppose that the assumptions of Subsections 1.3 and 1.6 are satisfied. Let \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+\) and \(|\zeta| \geq 1\).

Denote

\[
c(\phi) := \begin{cases} |\sin \phi|^{-1}, & \phi \in (0, \pi/2) \cup (3\pi/2, 2\pi), \\ 1, & \phi \in [\pi/2, 3\pi/2]. \end{cases}
\]

Suppose that \(\varepsilon_1\) is subject to Condition 1.7. Then for \(0 < \varepsilon \leq \varepsilon_1\) we have

\[
|||B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D^0 - \zeta Q_0^{-1})|||_{L_2(O) \rightarrow L_2(O)} \leq C_1 c(\phi)^{1/2}|\zeta|^{-1/2}.
\]

The constant \(C_1\) depends only on the problem data (1.9).

We fix a linear continuous extension operator

\[
P_O : H^\sigma(O; \mathbb{C}^n) \rightarrow H^\sigma(\mathbb{R}^d; \mathbb{C}^n), \quad \sigma \geq 0.
\]

Such a “universal” extension operator exists for any Lipschitz bounded domain (see [R]). We have

\[
|||P_O|||_{H^\sigma(\mathbb{R}^d) \rightarrow H^\sigma(\mathbb{R}^d)} \leq C_0(\sigma), \quad \sigma \geq 0,
\]

(1.46)

where the constant \(C_0(\sigma)\) depends only on \(\sigma\) and the domain \(O\). By \(R_O\) we denote the operator of restriction of functions in \(\mathbb{R}^d\) to the domain \(O\). We put

\[
K_D(\varepsilon; \zeta) := R_O ([\Lambda^\varepsilon b(D)] + [\widetilde{\Lambda}^\varepsilon]) S_P O (B_D^0 - \zeta Q_0)^{-1}.
\]

(1.47)

The corrector \(1.47\) is a continuous mapping of \(L_2(O; \mathbb{C}^n)\) to \(H^1(O; \mathbb{C}^n)\). This can be easily checked with the help of Proposition 1.2 and relations \(\Lambda, \widetilde{\Lambda} \in H^1(\Omega)\). Note that \(|||\varepsilon K_D(\varepsilon; \zeta)|||_{L_2(\Omega) \rightarrow H^1(\Omega)} = O(1)\) for small \(\varepsilon\) and fixed \(\zeta\).

**Theorem 1.9** ([MSu3]). Suppose that the assumptions of Theorem 1.8 are satisfied. Let \(K_D(\varepsilon; \zeta)\) be given by (1.47). Then for \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, \ |\zeta| \geq 1, \) and \(0 < \varepsilon \leq \varepsilon_1\) we have

\[
|||B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D^0 - \zeta Q_0^{-1}) - \varepsilon K_D(\varepsilon; \zeta)|||_{L_2(O) \rightarrow H^1(O)} \leq C_2 c(\phi)^{1/2}|\zeta|^{-1/4} + C_3 c(\phi)^{1/2} \varepsilon.
\]

(1.48)

Let \(g(\mathbf{x})\) be the matrix-valued function \(1.27\). We put

\[
G_D(\varepsilon; \zeta) := g^\varepsilon S b(D) P_O (B_D^0 - \zeta Q_0)^{-1} + g^\varepsilon (b(D) \widetilde{\Lambda})^\varepsilon S_P O (B_D^0 - \zeta Q_0)^{-1}.
\]

(1.49)

Then for \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, \ |\zeta| \geq 1, \) and \(0 < \varepsilon \leq \varepsilon_1\) the operator \(g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1}\) corresponding to the “flux” satisfies

\[
|||g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1} - G_D(\varepsilon; \zeta)|||_{L_2(O) \rightarrow L_2(O)} \leq C_2 c(\phi)^{5/2} \varepsilon^{1/2} |\zeta|^{-1/4}.
\]

(1.50)
The constants $C_2$, $C_3$, and $\tilde{C}_2$ depend only on the problem data \[1.19\].

In [MSn3, Theorem 9.2], estimates in a wider domain of the spectral parameter were obtained. It was assumed that $\zeta \in \mathbb{C} \setminus [c_0, \infty)$, where $c_0$ is a common lower bound of the operators $B_{D,\varepsilon}$ and $\tilde{B}_D$. We put

$$
c_{2} := 4^{-1} \alpha_0 \|g^{-1}\|_{L^\infty_{\lambda}}^{-1} \|Q_0\|_{L^\infty_{\lambda}}^{-1} (\text{diam } O)^{-2},
$$

using relations (1.18), (1.19), (1.22), (1.23), (1.38), (1.33), and (1.44).

**Theorem 1.10** (MSn3). Let $O \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. Suppose that the assumptions of Subsections 1.3.1.6 are satisfied. Let $K_D(\varepsilon; \zeta)$ be the corrector \[1.46\] and let $G_D(\varepsilon; \zeta)$ be the operator \[1.39\]. Suppose that $\zeta \in \mathbb{C} \setminus [c_0, \infty)$, where $c_0$ is given by \[1.51\]. Denote

$$
\phi(\varepsilon) := \begin{cases} 
c(\psi)^2 |\zeta - c_0|^2, & |\zeta - c_0| < 1, \\
c(\psi)^2, & |\zeta - c_0| \geq 1. 
\end{cases}
$$

Suppose that the number $\varepsilon_1$ is subject to Condition \[1.7\]. For $0 < \varepsilon \leq \varepsilon_1$ we have

$$
\| (B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta \tilde{Q}_0)^{-1} \|_{L_2(O) \rightarrow L_2(O)} \leq C_4 \varepsilon \phi(\zeta),
$$

$$
\| (B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta \tilde{Q}_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) \|_{L_2(O) \rightarrow H^1(O)} \leq C_5 \varepsilon^{1/2} \phi(\zeta)^{1/2} + \varepsilon |1 + |\zeta|^{1/2} \phi(\zeta)|,
$$

$$
\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(O) \rightarrow L_2(O)} \leq \tilde{C}_2 \varepsilon^{1/2} \phi(\zeta)^{1/2} + \varepsilon |1 + |\zeta|^{1/2} \phi(\zeta)|.
$$

The constants $C_4$, $C_5$, and $\tilde{C}_2$ depend only on the problem data \[1.19\].

**Remark 1.11.** 1) In (1.52), expression $c(\psi)^2|\zeta - c_0|^2$ is inverse to the square of the distance from $\zeta$ to $[c_0, \infty)$. 2) The number (1.51) in Theorem 1.10 can be replaced by any common lower bound of the operators $B_{D,\varepsilon}$ and $\tilde{B}_D$. Let $\kappa > 0$ be an arbitrarily small number. According to (1.53) (with $\zeta = 0$), $B_{D,\varepsilon}$ converges to $B_D^0$ in the norm-resolvent sense. Therefore, for sufficiently small $\varepsilon$ one can take $c_0 = \lambda_0^0 \|Q_0\|_{L^\infty_{\lambda}}^{-1} - \kappa$, where $\lambda_0^0$ is the first eigenvalue of the operator $B_D^0$. Under such choice of $c_0$, the constants in estimates become dependent on $\kappa$. 3) It makes sense to use estimates (1.52), (1.54) for bounded values of $|\zeta|$ and small $\varepsilon \phi(\zeta)$. In this case, the value $\varepsilon^{1/2} \phi(\zeta)^{1/2} + \varepsilon |1 + |\zeta|^{1/2} \phi(\zeta)|$ is controlled in terms of $C \varepsilon^{1/2} \phi(\zeta)^{1/2}$. For large $|\zeta|$ and for $\phi$ separated from the points 0 and $2\pi$, it is preferable to use Theorems 1.3 and 1.9.

1.8. **Removal of the smoothing operator in the corrector.** It turns out that the smoothing operator in the corrector can be removed under some additional assumptions on the matrix-valued functions $\Lambda(x)$ and $\tilde{\Lambda}(x)$.

**Condition 1.12.** Suppose that the $\Gamma$-periodic solution $\Lambda(x)$ of problem \[1.23\] is bounded, i. e., $\Lambda \in L_{\infty}(\mathbb{R}^d)$.

Some cases ensuring that Condition 1.12 is satisfied were distinguished in [BSu4, Lemma 8.7].

**Proposition 1.13** (BSu4). Suppose that at least one of the following assumptions is satisfied:

1°) $d \leq 2$;

2°) dimension $d \geq 1$ is arbitrary, and the differential expression $A_\varepsilon$ is given by $A_\varepsilon = D^*g^0(x)D$, where $g(x)$ is a symmetric matrix with real entries;

3°) dimension $d$ is arbitrary, and $g^0 = g$, i. e., relations (1.32) are satisfied.

Then Condition 1.12 holds.

In order to remove $S_\varepsilon$ in the term of the corrector involving $\tilde{\Lambda}^\varepsilon$, it suffices to impose the following condition.

**Condition 1.14.** Suppose that the $\Gamma$-periodic solution $\tilde{\Lambda}(x)$ of problem \[1.33\] is such that

$$
\tilde{\Lambda} \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \geq 3.
$$

The following result was checked in [Su4, Proposition 8.11].
Proposition 1.15 (Su3). Suppose that at least one of the following assumptions is satisfied:

1°) $d \leq 4$;
2°) dimension $d$ is arbitrary, and $A_\varepsilon$ is given by $A_\varepsilon = D^*g^\varepsilon(x)D$, where $g(x)$ is a symmetric matrix with real entries.

Then Condition 1.14 is satisfied.

Remark 1.16. If $A_\varepsilon = D^*g^\varepsilon(x)D$, where $g(x)$ is a symmetric matrix with real entries, then from [LaU] Chapter III, Theorem 13.1 it follows that $\Lambda, \tilde{\Lambda} \in L_\infty$ and the norm $\|\Lambda\|_{L_\infty}$ does not exceed a constant depending on $d$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and $\Omega$, while the norm $\|\tilde{\Lambda}\|_{L_\infty}$ is controlled in terms of $d, \rho$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_p(\Omega)}$, $j = 1, \ldots, d, \Omega$. In this case, Conditions 1.12 and 1.14 hold.

In [MSu3] Theorem 7.6 the following result was obtained.

Theorem 1.17 (MSu3). Suppose that the assumptions of Theorem 1.9 are satisfied. Suppose that $\Lambda(x)$ is subject to Condition 1.12, and $\Lambda(x)$ satisfies Condition 1.14. We put

$$K_0^0(\varepsilon; \zeta) := (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon)(B_D^0 - \zeta Q_0)^{-1} - g^\varepsilon b(\mathbf{D})(B_D^0 - \zeta Q_0)^{-1} + g^\varepsilon(b(\mathbf{D})\Lambda^\varepsilon)(B_D^0 - \zeta Q_0)^{-1}. \quad (1.56)$$

Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon \leq \varepsilon_1$ we have

$$\|\varepsilon K_0^0(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_2(\phi)^2\varepsilon |\zeta|^{-1/4} + C_6\phi^4\varepsilon,$$

$$\|g^\varepsilon b(\mathbf{D})(B_D^0 - \zeta Q_0)^{-1} - G_D^0(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_2(\phi)^2\varepsilon |\zeta|^{-1/4} + C_6\phi^4\varepsilon.$$

Here the constants $C_2, \tilde{C}_2$ are as in (1.38) and (1.50). The constants $C_6$ and $\tilde{C}_6$ depend only on the problem data (1.30), $p$, and the norms $\|\Lambda\|_{L_\infty}, \|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Approximations in a wider domain of the spectral parameter were found in [MSu3] Theorem 9.8.

Theorem 1.18 (MSu3). Suppose that the assumptions of Theorem 1.10 and Conditions 1.12 and 1.14 are satisfied. Let $K^0_D(\varepsilon; \zeta)$ be the corrector (1.56). Let $G_D^0(\varepsilon; \zeta)$ be given by (1.57). Then for $0 < \varepsilon \leq \varepsilon_1$ and $\zeta \in \mathbb{C} \setminus \overline{[0, \infty)}$ we have

$$\|\varepsilon K_D^0(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_7(\varepsilon \theta_0(\zeta)^{1/2} + \varepsilon|1 + \zeta|^{1/2} \theta_0(\zeta)),$$

$$\|g^\varepsilon b(\mathbf{D})(B_D^0 - \zeta Q_0)^{-1} - G_D^0(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_7(\varepsilon \theta_0(\zeta)^{1/2} + \varepsilon|1 + \zeta|^{1/2} \theta_0(\zeta)),$$

Here the constants $C_7$ and $\tilde{C}_7$ depend only on the problem data (1.30), $p$, and the norms $\|\Lambda\|_{L_\infty}, \|\tilde{\Lambda}\|_{L_p(\Omega)}$.

According to [MSu3] Remarks 7.9 and 9.9, we observe the following.

Remark 1.19. If only Condition 1.12 (respectively, Condition 1.14) is satisfied, then the smoothing operator $S_\varepsilon$ can be removed in the term of the corrector involving $\Lambda^\varepsilon$ (respectively, in the term containing $\tilde{\Lambda}^\varepsilon$).

1.9. The case where the corrector is equal to zero. Suppose that $g^0 = \tilde{g}$, i.e., relations (1.31) hold. Then the $\Gamma$-periodic solution of problem (1.25) is equal to zero: $\Lambda(x) = 0$. Suppose in addition that

$$\sum_{j=1}^d D_j a_j(x)^* = 0. \quad (1.58)$$

Then the $\Gamma$-periodic solution of problem (1.33) is also equal to zero: $\tilde{\Lambda}(x) = 0$. According to [MSu3] Propositions 7.10 and 9.12, in this case the $(L_2 \rightarrow H^1)$-estimate of sharp order $O(\varepsilon)$ holds.

Proposition 1.20 (MSu3). Suppose that relations (1.31) and (1.58) are satisfied.

1°. Under the assumptions of Theorem 1.8, for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon \leq 1$ we have

$$\|(B_D^0 - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_8\varepsilon. \quad (1.59)$$
2°. Under the assumptions of Theorem 1.10 for \( \zeta \in \mathbb{C} \setminus [c_0, \infty) \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^{0} - \zeta Q_0^0)^{-1} \|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq (C_0 + C_{10})|1 + \zeta|^{1/2} \varepsilon \theta_0(\zeta). \tag{1.60}
\]

The constants \( C_8, C_9, \) and \( C_{10} \) depend only on the problem data (1.9).

1.10. Estimates in a strictly interior subdomain. It is possible to improve the \( H^1 \)-estimates in a strictly interior subdomain \( \Omega' \) of the domain \( \Omega \). In Theorems 8.1 and 9.14 of [MSu3], the following result was obtained.

**Theorem 1.21** ([MSu3]). Let \( \Omega' \) be a strictly interior subdomain of the domain \( \Omega \). Denote
\[
\delta := \min \{ 1; \text{dist} \{ \Omega'; \partial \Omega \} \}. \tag{1.61}
\]

1°. Under the assumptions of Theorem 1.9 for \( \zeta \in \mathbb{C} \setminus R_+, |\zeta| \geq 1, \) and \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^{0} - \zeta Q_0^0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq c(\phi)^6(\varepsilon C_{11}^1 |\zeta|^{-1/2} \delta^{-1} + C''_{11}),
\]
\[
\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq c(\phi)^6(\varepsilon \tilde{C}_{11}' |\zeta|^{-1/2} \delta^{-1} + \tilde{C}_{11}'').
\]

The constants \( C_{11}', \tilde{C}_{11}' \), and \( \tilde{C}_{11}'' \) depend only on the problem data (1.9).

2°. Under the assumptions of Theorem 1.10 for \( \zeta \in \mathbb{C} \setminus [c_0, \infty) \) and \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^{0} - \zeta Q_0^0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq \varepsilon(c_{12}^2 \delta^{-1} \theta_0(\zeta)|\zeta|^{1/2} + C_{12}''|1 + \zeta|^{1/2} \theta_0(\zeta)), \tag{1.62}
\]
\[
\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq \varepsilon(\tilde{C}_{12}' \delta^{-1} \theta_0(\zeta)|\zeta|^{1/2} + \tilde{C}_{12}''|1 + \zeta|^{1/2} \theta_0(\zeta)). \tag{1.63}
\]

The constants \( C_{12}', \tilde{C}_{12}' \), and \( \tilde{C}_{12}'' \) depend only on the problem data (1.9).

If the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) satisfy some additional assumptions, this result remains true with a simpler corrector. Approximations for \( \zeta \in \mathbb{C} \setminus R_+, |\zeta| \geq 1 \), were found in [MSu3] Theorem 8.2.

**Theorem 1.22** ([MSu3]). Suppose that the assumptions of Theorem 1.21 (1°) are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) satisfy Conditions 1.12 and 1.14 respectively. Let \( K_D^0(\varepsilon; \zeta) \) and \( G_D^0(\varepsilon; \zeta) \) be the operators defined by (1.56) and (1.57). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \zeta \in \mathbb{C} \setminus [c_0, \infty) \), we have
\[
\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^{0} - \zeta Q_0^0)^{-1} - \varepsilon K_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq c(\phi)^6(\varepsilon C_{11}' |\zeta|^{-1/2} \delta^{-1} + C_{13}),
\]
\[
\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq c(\phi)^6(\varepsilon \tilde{C}_{11}' |\zeta|^{-1/2} \delta^{-1} + \tilde{C}_{13}).
\]

The constants \( C_{13}', \tilde{C}_{13}' \) are as in Theorem 1.21. The constants \( C_{13} \) and \( \tilde{C}_{13} \) depend on the problem data (1.9), \( p \), and the norms \( \| \Lambda \|_{L_\infty}, \| \tilde{\Lambda} \|_{L_p(\Omega)} \).

Approximations in a wider domain of the parameter \( \zeta \) are obtained in [MSu3] Theorem 9.15.

**Theorem 1.23** ([MSu3]). Suppose that the assumptions of Theorem 1.21 (2°) are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) are subject to Conditions 1.12 and 1.14 respectively. Let \( K_D^0(\varepsilon; \zeta) \) be the corrector (1.56), and let \( G_D^0(\varepsilon; \zeta) \) be the operator (1.57). Then for \( \zeta \in \mathbb{C} \setminus [c_0, \infty) \) and \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^{0} - \zeta Q_0^0)^{-1} - \varepsilon K_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq \varepsilon(c_{12}' \delta^{-1} \theta_0(\zeta)|\zeta|^{1/2} + C_{14}|1 + \zeta|^{1/2} \theta_0(\zeta)),
\]
\[
\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq \varepsilon(\tilde{C}_{12}' \delta^{-1} \theta_0(\zeta)|\zeta|^{1/2} + \tilde{C}_{14}|1 + \zeta|^{1/2} \theta_0(\zeta)).
\]

Here the constants \( C_{12}', \tilde{C}_{12}' \) are as in (1.62) and (1.63). The constants \( C_{14} \) and \( \tilde{C}_{14} \) depend on the problem data (1.9), \( p \), and the norms \( \| \Lambda \|_{L_\infty}, \| \tilde{\Lambda} \|_{L_p(\Omega)} \).
2. Statement of the problem. Main results

2.1. Statement of the problem. We study the behavior of the solution of the first initial boundary-value problem

\[
\begin{aligned}
Q_0'(x) \frac{\partial u_\varepsilon}{\partial t}(x, t) &= -B_\varepsilon u_\varepsilon(x, t), \quad x \in \mathcal{O}, \quad t > 0; \\
u_\varepsilon(\cdot, t)|_{\partial \mathcal{O}} &= 0, \quad t > 0; \\
Q_0(x)u_\varepsilon(x, 0) &= \varphi(x), \quad x \in \mathcal{O}.
\end{aligned}
\]  

(2.1)

Here \( \varphi \in L_2(\mathcal{O}; \mathbb{C}^n) \). (The solution is understood in the weak sense.) Let us find relation between \( u_\varepsilon(\cdot, t) \) and \( \varphi \). According to (2.22), the function \( s_\varepsilon(x, t) := (f^\varepsilon(x))^{-1}u_\varepsilon(x, t) \) is the solution of the problem

\[
\begin{aligned}
\frac{\partial s_\varepsilon}{\partial t}(x, t) &= -\tilde{B}_\varepsilon s_\varepsilon(x, t), \quad x \in \mathcal{O}, \quad t > 0; \\
s_\varepsilon(\cdot, t)|_{\partial \mathcal{O}} &= 0, \quad t > 0; \\
s_\varepsilon(x, 0) &= (f^\varepsilon(x))\varphi(x), \quad x \in \mathcal{O}.
\end{aligned}
\]

Then \( s_\varepsilon(\cdot, t) = e^{-\tilde{B}_\varepsilon t}(f^\varepsilon)^*\varphi \) and \( u_\varepsilon(\cdot, t) = f^\varepsilon s_\varepsilon(\cdot, t) = f^\varepsilon e^{-\tilde{B}_\varepsilon t}(f^\varepsilon)^*\varphi \).

Our goal is to study the behavior of the generalized solution \( u_\varepsilon \) of the first initial boundary-value problem (2.1) in the small period limit. In other words, we are interested in approximations of the sandwiched operator exponential \( f^\varepsilon e^{-\tilde{B}_\varepsilon t}(f^\varepsilon)^* \) for small \( \varepsilon \).

The corresponding effective problem is given by

\[
\begin{aligned}
\frac{\partial u_0}{\partial t}(x, t) &= -B_0^0 u_0(x, t), \quad x \in \mathcal{O}, \quad t > 0; \\
u_0(\cdot, t)|_{\partial \mathcal{O}} &= 0, \quad t > 0; \\
Q_0 u_0(x, 0) &= \varphi(x), \quad x \in \mathcal{O}.
\end{aligned}
\]  

(2.2)

By (1.42), the solution of the effective problem is given by

\[
u_0(\cdot, t) = f_0 e^{-\tilde{B}_0^0 t} f_0 \varphi(\cdot).
\]  

(2.3)

2.2. The properties of the operator exponential. We prove the following simple statement about estimates for the operator exponentials \( e^{-\tilde{B}_\varepsilon t} \) and \( e^{-B_0^0 t} \).

Lemma 2.1. For \( 0 < \varepsilon \leq 1 \) we have

\[
\begin{aligned}
\|e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} &\leq e^{-c_0 t}, \quad t \geq 0, \\
\|f^\varepsilon e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow H^1(\mathcal{O})} &\leq c_3 t^{-1/2} e^{-c_3 t/2}, \quad t > 0, \\
\|e^{-B_0^0 t}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} &\leq e^{-c_0 t}, \quad t \geq 0, \\
\|f_0 e^{-B_0^0 t}\|_{L_2(\mathcal{O}) \rightarrow H^1(\mathcal{O})} &\leq c_3 t^{-1/2} e^{-c_3 t/2}, \quad t > 0, \\
\|f_0 e^{-B_0^0 t}\|_{L_2(\mathcal{O}) \rightarrow H^1(\mathcal{O})} &\leq c_3 t^{-1/2} e^{-c_3 t/2}, \quad t > 0.
\end{aligned}
\]  

(2.4)

(2.5)

(2.6)

(2.7)

(2.8)

Here the constants \( c_3 \) and \( c_0 \) are given by (1.21) and (1.51). The constant \( c \) depends only on the problem data (1.9).

Proof. Since the number \( c_0 \) defined by (1.51) is a common lower bound of the operators \( \tilde{B}_D,\varepsilon \) and \( B_0^0 \), estimates (2.4) and (2.6) are obvious.

By (1.20) and (1.23),

\[
\|f^\varepsilon e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow H^1(\mathcal{O})} \leq c_3 \|B_\varepsilon^{1/2} f^\varepsilon e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} = c_3 \|\tilde{B}_\varepsilon^{1/2} e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})}.
\]  

(2.9)

Since \( \tilde{B}_\varepsilon \geq c_0 I \), then

\[
\|\tilde{B}_\varepsilon^{1/2} e^{-\tilde{B}_\varepsilon t}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} \leq \sup_{x \geq c_0} x^{1/2} e^{-xt} \leq e^{-c_0 t/2} \sup_{x \geq c_0} x^{1/2} e^{-xt/2} \leq t^{-1/2} e^{-c_0 t/2}.
\]  

(2.10)

Combining this with (2.9), we obtain inequality (2.5). Similarly, (1.39) and (1.44) imply estimate (2.7).
From (1.11), (1.13), and the identity \( \tilde{B}_D^0 = f_0B_D^0f_0 \) it follows that
\[
\|f_0e^{-\tilde{B}_D^0t}\|_{L_2(\gamma) \to H^2(\gamma)} \leq \tilde{c}\|B_D^0f_0e^{-\tilde{B}_D^0t}\|_{L_2(\gamma) \to L_2(\gamma)} \leq \tilde{c}\|f^{-1}\|_{L_\infty} \|\tilde{B}_D^0e^{-\tilde{B}_D^0t}\|_{L_2(\gamma) \to L_2(\gamma)}.
\]
Hence,
\[
\|f_0e^{-\tilde{B}_D^0t}\|_{L_2(\gamma) \to H^2(\gamma)} \leq \tilde{c}\|f^{-1}\|_{L_\infty} \sup_{x \geq \gamma} x e^{-\gamma t} \leq \tilde{c}\|f^{-1}\|_{L_\infty} t^{-1} e^{-c_0t/2}.
\]
This proves estimate (2.8) with the constant \( \tilde{c} = \tilde{c}\|f^{-1}\|_{L_\infty}. \) \( \square \)

2.3. Approximation of the solution in \( L_2(\mathbb{O}; \mathbb{C}^n) \).

**Theorem 2.2.** Let \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain of class \( C^{1,1} \). Suppose that the assumptions of Subsections 1.3–1.6 are satisfied. Let \( B_{D,\varepsilon} \) be the operator in \( L_2(\mathcal{O}; \mathbb{C}^n) \) corresponding to the quadratic form \( 1.3–1.6 \). Let \( B_D^0 \) be the operator in \( L_2(\mathcal{O}; \mathbb{C}^n) \) given by expression (1.40) on \( H^2(\mathcal{O}; \mathbb{C}^n) \cap H_0^1(\mathcal{O}; \mathbb{C}^n) \). We put \( \tilde{B}_{D,\varepsilon} = (f^\varepsilon)^*B_{D,\varepsilon}f^\varepsilon \) and \( \tilde{B}_D^0 = f_0B_D^0f_0 \), where the matrix-valued function \( f \) is defined by (1.22), and the matrix \( f_0 \) is given by (1.42). Let \( u_\varepsilon \) be the solution of problem (2.11), and let \( u_0 \) be the solution of the corresponding effective problem (2.2). Suppose that the number \( \varepsilon_1 \) is subject to Condition (1.7). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\|u_\varepsilon(\cdot, t) - u_0(\cdot, t)\|_{L_2(\gamma)} \leq C_{15}(t + \varepsilon)^{-1/2}e^{-c_0t/2}\|\varphi\|_{L_2(\gamma)}, \quad t \geq 0.
\]
In the operator terms,
\[
\|f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0e^{-\tilde{B}_D^0t}f_0\|_{L_2(\gamma) \to L_2(\gamma)} \leq C_{15}(t + \varepsilon)^{-1/2}e^{-c_0t/2}, \quad t \geq 0. \tag{2.11}
\]
Here the constant \( c_0 \) is given by (1.51). The constant \( C_{15} \) depends only on the problem data (1.9).

**Proof.** The proof is based on the results of Theorems 1.8 and 1.10 and representations for the exponentials of the operators \( \tilde{B}_{D,\varepsilon}, \tilde{B}_D^0 \) in terms of the contour integrals of the corresponding resolvents.

We have (see, e.g., [Ka, Chapter IX, Section 1.6])
\[
e^{-\tilde{B}_{D,\varepsilon}t} = -\frac{1}{2\pi i} \int e^{-\zeta t}(\tilde{B}_{D,\varepsilon} - \zeta I)^{-1}d\zeta, \quad t > 0. \tag{2.12}
\]
Here \( \gamma \subset \mathbb{C} \) is a contour enclosing the spectrum of the operator \( \tilde{B}_{D,\varepsilon} \) in positive direction. The exponential of the operator \( \tilde{B}_D^0 \) satisfies a similar representation. Since the constant (1.51) is a common lower bound of the operators \( \tilde{B}_{D,\varepsilon} \) and \( \tilde{B}_D^0 \), it is convenient to choose the contour of integration as follows:
\[
\gamma = \{ \zeta \in \mathbb{C} : \text{Im} \zeta \geq 0, \text{Re} \zeta = \text{Im} \zeta + c_0/2 \} \cup \{ \zeta \in \mathbb{C} : \text{Im} \zeta \leq 0, \text{Re} \zeta = -\text{Im} \zeta + c_0/2 \}.
\]

Multiplying (2.12) by \( f^\varepsilon \) from the left and by \((f^\varepsilon)^*\) from the right and using identity (1.24), we obtain
\[
f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* = -\frac{1}{2\pi i} \int e^{-\zeta t}(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}d\zeta, \quad t > 0.
\]
Similarly,
\[
f_0e^{-\tilde{B}_D^0t}f_0 = -\frac{1}{2\pi i} \int e^{-\zeta t}(B_0^0 - \zeta Q_0)^{-1}d\zeta, \quad t > 0.
\]
Hence,
\[
f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0e^{-\tilde{B}_D^0t}f_0 = -\frac{1}{2\pi i} \int e^{-\zeta t}((B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_0^0 - \zeta Q_0)^{-1})d\zeta. \tag{2.13}
\]

By Theorems 1.8 and 1.10 we estimate the difference of the generalized resolvents for \( \zeta \in \gamma \) uniformly in \( \text{arg} \zeta \). Recall the notation \( \psi = \text{arg} (\zeta - c_0) \). Note that for \( \zeta \in \gamma \) and \( \psi = \pi/2 \) or \( \psi = 3\pi/2 \) we have \( |\zeta| = \sqrt{5c_0}/2 \). We apply Theorem 1.10 for \( \zeta \in \gamma \) such that \( |\zeta| \leq \hat{\epsilon} \), where
\[
\hat{\epsilon} := \max\{1; \sqrt{5c_0}/2\}. \tag{2.14}
\]
Obviously, \( \psi \in (\pi/4, 7\pi/4) \) on the contour \( \gamma \) and
\[
\rho_\delta(\zeta) \leq 2 \max\{1; 8\delta^{-2}\} =: \mathcal{C}, \quad \zeta \in \gamma.
\] (2.15)

Therefore, (1.53) implies that
\[
\|(B_{D, \varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_{D}^0 - \zeta Q_0^0)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4 \varepsilon \leq C_4' |\zeta|^{-1/2} \varepsilon,
\]
\[
\zeta \in \gamma, \quad |\zeta| \leq \delta, \quad 0 < \varepsilon \leq \varepsilon_1; \quad C_4' := C_4 \delta^{1/2}.
\] (2.16)

For other \( \zeta \in \gamma \), we have
\[
|\sin \phi| \geq 5^{-1/2}, \quad \zeta \in \gamma, \quad |\zeta| > \delta.
\] (2.17)

and, by Theorem 1.8,
\[
\|(B_{D, \varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_{D}^0 - \zeta Q_0^0)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4'' |\zeta|^{-1/2} \varepsilon,
\]
\[
\zeta \in \gamma, \quad |\zeta| > \delta, \quad 0 < \varepsilon \leq \varepsilon_1,
\] (2.18)

where \( C_4'' := 5^{5/2} C_1 \). As a result, combining (2.16) and (2.18), for \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\|(B_{D, \varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_{D}^0 - \zeta Q_0^0)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leq \tilde{C}_1 |\zeta|^{-1/2} \varepsilon, \quad \zeta \in \gamma,
\] (2.19)

where \( \tilde{C}_1 := \max\{C_4'; C_4''\} \).

From (2.13) and (2.19) it follows that
\[
\|f^t e^{-B_{D, \varepsilon}^{t} \cdot} (f^t)^* - f_0 e^{-B_{D}^{0} t} f_0\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2 \pi^{-1} \tilde{C}_1 e^{t} \Gamma(1/2) e^{-c_1 t/2}.
\]

Taking into account that \( \Gamma(1/2) = \pi^{1/2} \), we find
\[
\|f^t e^{-B_{D, \varepsilon}^{t} \cdot} (f^t)^* - f_0 e^{-B_{D}^{0} t} f_0\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2 \pi^{-1/2} \tilde{C}_1 |\varepsilon (t + \varepsilon^2)^{-1/2} e^{-c_1 t/2}, \quad t \geq \varepsilon^2,
\] (2.20)

where \( \tilde{C}_1 := 2 \sqrt{2} \pi^{-1/2} \tilde{C}_1 \). For \( t \leq \varepsilon^2 \) we use a rough estimate
\[
\|f^t e^{-B_{D, \varepsilon}^{t} \cdot} (f^t)^* - f_0 e^{-B_{D}^{0} t} f_0\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2 \|f\|_{L_\infty} e^{-c_1 t}
\]
\[
\leq 2 \sqrt{2} \|f\|_{L_\infty} e(t + \varepsilon^2)^{-1/2} e^{-c_1 t /2}, \quad t \leq \varepsilon^2.
\] (2.21)

Relations (2.20) and (2.21) imply the required inequality (2.11) with the constant \( C_1 := \max\{C_{15}; 2 \sqrt{2} \|f\|_{L_\infty}\} \).

### 2.4. Approximation of the solution in \( H^1(\Omega; \mathbb{C}^n) \)

We introduce a corrector
\[
K_D(t; \varepsilon) := R_0 \left( \left[ \Lambda^\varepsilon \right] S_{t\varepsilon} b(D) + \left[ \Lambda^\varepsilon \right] S_{t\varepsilon} \right) \mathcal{P}_0 f_0 e^{-B_{D}^{0} t} f_0.
\] (2.22)

For \( t > 0 \) the operator (2.22) is a continuous mapping of \( L_2(\Omega; \mathbb{C}^n) \) to \( H^1(\Omega; \mathbb{C}^n) \). Indeed, according to (2.5), for \( t > 0 \) the operator \( f_0 e^{-B_{D}^{0} t} f_0 \) is continuous from \( L_2(\Omega; \mathbb{C}^n) \) to \( H^2(\Omega; \mathbb{C}^n) \). Hence, the operator \( b(D) \mathcal{P}_0 f_0 e^{-B_{D}^{0} t} f_0 \) is continuous from \( L_2(\Omega; \mathbb{C}^n) \) to \( H^1(\mathbb{D}; \mathbb{C}^n) \). Obviously, the operator \( \mathcal{P}_0 f_0 e^{-B_{D}^{0} t} f_0 \) is also continuous from \( L_2(\Omega; \mathbb{C}^n) \) to \( H^1(\mathbb{D}; \mathbb{C}^n) \). It remains to use the continuity of the operators \( \left[ \Lambda^\varepsilon \right] S_{t\varepsilon} : H^1(\mathbb{D}; \mathbb{C}^n) \to H^1(\mathbb{D}; \mathbb{C}^n) \) and \( \left[ \Lambda^\varepsilon \right] S_{t\varepsilon} : H^1(\mathbb{D}; \mathbb{C}^n) \to H^1(\mathbb{D}; \mathbb{C}^n) \) which follows from Proposition (1.2) and relations \( \Lambda, \tilde{\Lambda} \in H^1(\Omega) \).

We put \( \tilde{u}_0(\cdot, t) := \mathcal{P}_0 u_0(\cdot, t) \). By \( \varepsilon \) we denote the first order approximation of the solution \( u_\varepsilon \) of problem (2.1):
\[
\tilde{v}_\varepsilon(\cdot, t) = \tilde{u}_0(\cdot, t) + \varepsilon \Lambda^\varepsilon S_{t\varepsilon} b(D) \tilde{u}_0(\cdot, t) + \varepsilon \tilde{\Lambda}^\varepsilon S_{t\varepsilon} \tilde{u}_0(\cdot, t),
\]
\[
\varepsilon(\cdot, t) := \tilde{v}_\varepsilon(\cdot, t) |\Omega.
\] (2.23)

So, \( \varepsilon(\cdot, t) = f_0 e^{-B_{D}^{0} t} f_0 \varphi(\cdot) + \varepsilon K_D(t; \varepsilon) \varphi(\cdot) \).

**Theorem 2.3.** Suppose that the assumptions of Theorem 2.2 are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) are \( \Gamma \)-periodic solutions of the problems (1.25) and (1.33),
respectively. Let $S_\ell$ be the Steklov smoothing operator \([1.45]\), and let $P_\ell$ be the extension operator \([1.45]\). We put $u_0(\cdot, t) = P_\ell u_0(\cdot, t)$. Suppose that $v_\varepsilon$ is defined by \([2.23]\). Then for $0 < \varepsilon \leq \varepsilon_1$ and $t > 0$ we have

$$
\|u_\varepsilon(\cdot, t) - v_\varepsilon(\cdot, t)\|_{H^1(\Omega)} \leq C_16(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_\varepsilon t/2}\|\varphi\|_{L_2(\Omega)}.
$$

In the operator terms,

$$
\|f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0 e^{-\tilde{B}_{D}t}f_0 - \varepsilon\mathcal{K}_D(t; \varepsilon)\|_{L_2(\Omega) \Rightarrow H^1(\Omega)} \leq C_16(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_\varepsilon t/2},
$$

(2.24)

where $\mathcal{K}_D(t; \varepsilon)$ is the corrector \([2.22]\). Suppose that the matrix-valued function $\tilde{g}(x)$ is defined by \([1.27]\). For $0 < \varepsilon < \varepsilon_1$ and $t > 0$ the flux $p_\varepsilon := g^\varepsilon b(D)u_\varepsilon$ satisfies

$$
\|p_\varepsilon(\cdot, t) - \tilde{g}^\varepsilon S_\varepsilon b(D)\tilde{u}_0(\cdot, t) - g^\varepsilon (b(D)\tilde{\Lambda})^{1/2}S_\varepsilon \tilde{u}_0(\cdot, t)\|_{L_2(\Omega)} \leq \tilde{C}_16\varepsilon^{1/2}t^{-3/4}e^{-c_\varepsilon t/2}\|\varphi\|_{L_2(\Omega)}.
$$

In the operator terms,

$$
\|g^\varepsilon b(D)f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - \mathcal{G}_D(t; \varepsilon)\|_{L_2(\Omega) \Rightarrow L_2(\Omega)} \leq \tilde{C}_16\varepsilon^{1/2}t^{-3/4}e^{-c_\varepsilon t/2}.
$$

(2.25)

Here

$$
\mathcal{G}_D(t; \varepsilon) := \tilde{g}^\varepsilon S_\varepsilon b(D)P_\ell f_0 e^{-\tilde{B}_{D,\varepsilon}t}f_0 + g^\varepsilon (b(D)\tilde{\Lambda})^{1/2}S_\varepsilon P_\ell f_0 e^{-\tilde{B}_{D}t}f_0.
$$

The constants $C_16$ and $\tilde{C}_16$ depend only on the problem data \([1.37]\).

**Proof.** As in the proof of Theorem \([2.2]\) we use representations for the sandwiched operator exponentials in terms of the contour integrals of the corresponding generalized resolvents. We have

$$
f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0 e^{-\tilde{B}_{D}t}f_0 - \varepsilon\mathcal{K}_D(t; \varepsilon) = -\frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} \left( (B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D - \zeta Q_0)^{-1} - \varepsilon\mathcal{K}_D(\varepsilon; \zeta) \right) d\zeta.
$$

(2.26)

Here $\mathcal{K}_D(\varepsilon; \zeta)$ is the operator \([1.47]\).

Similarly to \([2.16] - [2.19]\), by Theorems \([1.9]\) and \([1.10]\)

$$
\| (B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D - \zeta Q_0)^{-1} - \varepsilon\mathcal{K}_D(\varepsilon; \zeta) \|_{L_2(\Omega) \Rightarrow H^1(\Omega)} \leq C_16 \left( \varepsilon^{1/2} |\zeta|^{-1/4} + \varepsilon \right), \quad \zeta \in \gamma, \quad 0 < \varepsilon \leq \varepsilon_1,
$$

(2.27)

with the constant $\tilde{C}_16 := \max\{C'_{16}; C''_{16}\}$, where $C'_{16} := (1 + \varepsilon)^{1/2}C_3\mathcal{C}$ and $C''_{16} := \max\{5C_2; 25C_3\}$. Relations \([2.20]\) and \([2.27]\) imply the required estimate \([2.21]\) with the constant

$C_{16} := 2\pi^{-1}\Gamma(3/4)\tilde{C}_{16}$.

Similarly, the identity

$$
g^\varepsilon b(D)f^\varepsilon e^{-\tilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - \mathcal{G}_D(t; \varepsilon) = -\frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} (g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta)) d\zeta
$$

(2.28)

and estimates \([1.50], [1.55]\) yield the inequality \([2.25]\) with the constant

$$
\tilde{C}_{16} := 2\pi^{-1}\Gamma(3/4) \max \left\{ 5^{5/4}\tilde{C}_{2}; 2\varepsilon^{1/4}(1 + \varepsilon)^{1/2}\tilde{C}_{3}\mathcal{C} \right\}.
$$

\[ \square \]

By Remark \([1.11]\)\(2\), we observe the following.

**Remark 2.4.** Let $\lambda_1^\varepsilon$ be the first eigenvalue of the operator $B_D^0$, and let $\kappa > 0$ be an arbitrarily small number. Due to the norm-resolvent convergence, for sufficiently small $\varepsilon_0$, the number $\lambda_1^\varepsilon\|Q_0\|_{L_\infty}^{-1} - \kappa/2$ is a common lower bound of the operators $B_{D,\varepsilon}$ for all $0 < \varepsilon \leq \varepsilon_0$. Therefore, we can shift the integration contour so that it will intersect the real axis at the point $\zeta := \lambda_1^\varepsilon\|Q_0\|_{L_\infty}^{-1} - \kappa$ instead of $c_\varepsilon/2$. By this way, we obtain estimates \([2.11], [2.24]\), and \([2.25]\) with $e^{-c_\varepsilon t/2}$ replaced by $e^{-\zeta t}$ in the right-hand sides. The constants in estimates become dependent on $\kappa$. 

2.5. Estimates for small time. Note that for $0 < t < \varepsilon^2$ it makes no sense to apply estimates (2.24) and (2.25), since it is better to use the following simple statement (which is valid, however, for all $t > 0$).

Proposition 2.5. Suppose that the assumptions of Theorem 2.2 are satisfied. Then for $t > 0$ and $0 < \varepsilon \leq 1$ we have

$$
\|f^*e^{-\tilde{B}_{D,t}}(f^*)^* - f_0e^{-\tilde{B}_{D,t}^0}f_0\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq C_{17}t^{-1/2}e^{-c_0t/2},
$$

(2.29)

$$
\|g^b(D)f^*e^{-\tilde{B}_{D,t}}(f^*)^*\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \tilde{C}_{17}t^{-1/2}e^{-c_0t/2},
$$

(2.30)

$$
\|g^b(D)f_0e^{-\tilde{B}_{D,t}^0}f_0\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \tilde{C}_{17}t^{-1/2}e^{-c_0t/2},
$$

(2.31)

where the constants $C_{17} := 2c_3\|f\|_{L_{\infty}}$ and $\tilde{C}_{17} := \|g\|_{L_{\infty}}\|f\|_{L_{\infty}}$ depend only on the problem data (1.9).

Proof. Inequality (2.29) follows from (1.43), (2.5), and (2.7).

Next, by (1.23),

$$
\|g^b(D)f^*e^{-\tilde{B}_{D,t}}(f^*)^*\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \|g\|_{L_{\infty}}\|f\|_{L_{\infty}}\|\tilde{B}_{D,t}^{1/2}e^{-\tilde{B}_{D,t}}\|_{L^2(\Omega) \rightarrow L^2(\Omega)}.
$$

Together with (2.10), this yields (2.30). By (1.33) and (1.44), estimate (2.31) is checked similarly. \qed

2.6. Removal of the smoothing operator $S_\varepsilon$ in the corrector. It is possible to remove the smoothing operator in the corrector if the matrix-valued functions $\Lambda(x)$ and $\tilde{\Lambda}(x)$ satisfy Conditions 1.12 and 1.14 respectively. The following result is checked similarly to Theorem 2.3 by using Theorems 1.17 and 1.18.

Theorem 2.6. Suppose that the assumptions of Theorem 2.3 are satisfied. Suppose that the matrix-valued functions $\Lambda(x)$ and $\tilde{\Lambda}(x)$ satisfy Conditions 1.12 and 1.14 respectively. We put

$$
K_D^0(t; \varepsilon) := (\Lambda^*b(D) + \tilde{\Lambda}^*)f_0e^{-\tilde{B}_{D,t}^0}f_0,
$$

(2.32)

$$
g_D^0(t; \varepsilon) := \tilde{g}^\varepsilon b(D)f_0e^{-\tilde{B}_{D,t}^0}f_0 + g^\varepsilon(b(D)\tilde{\Lambda})e^{-\tilde{B}_{D,t}^0}f_0.
$$

(2.33)

Then for $t > 0$ and $0 < \varepsilon \leq \varepsilon_1$ we have

$$
\|f^*e^{-\tilde{B}_{D,t}}(f^*)^* - f_0e^{-\tilde{B}_{D,t}^0}f_0 - \varepsilon K_D^0(t; \varepsilon)\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq C_{18}\left(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1}\right)e^{-c_0t/2},
$$

$$
\|g^b(D)f^*e^{-\tilde{B}_{D,t}}(f^*)^* - g_D^0(t; \varepsilon)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \tilde{C}_{18}\left(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1}\right)e^{-c_0t/2}.
$$

The constants $C_{18}$ and $\tilde{C}_{18}$ depend on the problem data (1.9), $p$, and the norms $\|\Lambda\|_{L_{\infty}}$ and $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

By Remark 1.19 we observe the following.

Remark 2.7. If only Condition 1.12 (Condition 1.14 respectively) is satisfied, then the smoothing operator $S_\varepsilon$ can be removed in the term of the corrector containing $\Lambda^\varepsilon$ ($\tilde{\Lambda}^\varepsilon$, respectively).

2.7. The case of smooth boundary. It is also possible to remove the smoothing operator $S_\varepsilon$ in the corrector by increasing smoothness of the boundary. In this subsection, we consider the case where $d \geq 3$, because for $d \leq 2$ we can apply Theorem 2.6 (see Propositions 1.13 and 1.15).

Lemma 2.8. Let $k \geq 2$ be an integer. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the boundary $\partial \Omega$ of class $C^{k-1,1}$. Then for $t > 0$ the operator $e^{-\tilde{B}_{D,t}^0}$ is a continuous mapping of $L^2(\Omega; C^k)$ to $H^q(\Omega; C^k)$, $0 \leq q \leq k$, and

$$
\|e^{-\tilde{B}_{D,t}^0}\|_{L^2(\Omega) \rightarrow H^q(\Omega)} \leq \tilde{C}_q t^{-q/2}e^{-c_0t/2}, \quad t > 0.
$$

(2.34)

The constant $\tilde{C}_q$ depends only on $q$ and the problem data (1.9).
Proof. It suffices to check estimate (2.34) for integer $q \in [0, k]$; then the result for non-integer $q$ follows by interpolation. For $q = 0, 1, 2$ estimate (2.34) has already been proved (see Lemma 2.1).

So, let $q$ be an integer such that $2 \leq q \leq k$. By theorems about regularity of solutions of strongly elliptic systems (see, e. g., [McL, Chapter 4]), the operator $(\bar{B}_D^0)^{-1}$ is continuous from $H^q(\Omega; \mathbb{C}^n)$ to $H^{q+2}(\Omega; \mathbb{C}^n)$ under the assumption $\partial \Omega \in C^{\sigma+1,1}$, where $\sigma \in \mathbb{Z}$. We also take into account that the operator $(\bar{B}_D^0)^{-1/2}$ is continuous from $L_2(\Omega; \mathbb{C}^n)$ to $H^1(\Omega; \mathbb{C}^n)$. It follows that, under the assumptions of Lemma, for integer $q \in [2, k]$ the operator $(\bar{B}_D^0)^{-q/2}$ is a continuous mapping of $L_2(\Omega; \mathbb{C}^n)$ to $H^q(\Omega; \mathbb{C}^n)$. We have

$$
\|(\bar{B}_D^0)^{-q/2}\|_{L_2(\Omega) \rightarrow H^q(\Omega)} \leq C_q,
$$

where the constant $C_q$ depends on $q$ and the problem data (1.9). From (2.35) it follows that

$$
\|e^{-\bar{B}_D^0 t}\|_{L_2(\Omega) \rightarrow H^q(\Omega)} \leq C_q \|e^{-\bar{B}_D^0 t} - \bar{B}_D^0\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \hat{C}_q \sup_{x \geq t} x^{q/2} e^{-\epsilon t}
$$

$$
\leq \hat{C}_q t^{-q/2} e^{-\epsilon t/2} \sup_{x \geq 0} x^{q/2} e^{-x/2} \leq \hat{C}_q t^{-q/2} e^{-\epsilon t/2},
$$

where $\hat{C}_q := C_q (q/\epsilon)^{q/2}$.

Using Lemma 2.8, the properties of the matrix-valued functions $\Lambda(x)$ and $\bar{\Lambda}(x)$, and the properties of the operator $S_\epsilon$, we can estimate the difference of the correctors (2.22) and (2.32).

**Lemma 2.9.** Let $d \geq 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^{d+1/2}$ if $d$ is even and of class $C^{(d+1)/2}$ if $d$ is odd. Let $\mathcal{K}_D(t; \epsilon)$ be the operator (2.22), and let $\mathcal{K}_D^0(t; \epsilon)$ be the operator (2.32). Then for $0 < \epsilon \leq 1$ and $t > 0$ we have

$$
\|\mathcal{K}_D(t; \epsilon) - \mathcal{K}_D^0(t; \epsilon)\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq \hat{C}_d (t^{-1} + t^{-d/4-1/2}) e^{-\epsilon t/2}.
$$

The constant $\hat{C}_d$ depends only on the problem data (1.9).

**Lemma 2.9** and **Theorem 2.10** imply the following result.

**Theorem 2.10.** Suppose that the assumptions of Theorem 2.2 are satisfied, and $d \geq 3$. Suppose that the domain $\Omega$ satisfies the assumptions of Lemma 2.9. Let $\mathcal{K}_D^0(t; \epsilon)$ be the corrector (2.32). Let $\mathcal{G}_D^0(t; \epsilon)$ be the operator (2.33). Then for $t > 0$ and $0 < \epsilon \leq \epsilon_1$ we have

$$
\|f^\epsilon e^{-\bar{B}_D, t}(f^\epsilon) - f_0 e^{-\bar{B}_D, t} f_0 - \epsilon \mathcal{K}_D^0(t; \epsilon)\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq \hat{C}_d (\epsilon^{1/2} t^{-3/4} + \epsilon t^{-d/4-1/2}) e^{-\epsilon t/2},
$$

$$
\|g^\epsilon b(D) f^\epsilon e^{-\bar{B}_D, t}(f^\epsilon) - \mathcal{G}_D^0(t; \epsilon)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \hat{C}_d (\epsilon^{1/2} t^{-3/4} + \epsilon t^{-d/4-1/2}) e^{-\epsilon t/2}.
$$

The constants $\hat{C}_d$ and $\hat{C}_d$ depend only on the problem data (1.9).

The proofs of **Lemma 2.9** and **Theorem 2.10** are presented in Appendix (see §7) in order not to clutter the main presentation. Clearly, it is convenient to apply **Theorem 2.10** if $t$ is separated from zero. For small $t$ the order of the factor $(\epsilon^{1/2} t^{-3/4} + \epsilon t^{-d/4-1/2})$ grows with dimension. This is a “charge” for the removal of the smoothing operator.

**Remark 2.11.** Instead of the smoothness assumption on $\partial \Omega$ from **Lemma 2.9** we could impose the following implicit condition: a bounded domain $\Omega$ with Lipschitz boundary is such that estimate (2.34) holds for $q = d/2 + 1$. In such domain the statements of **Lemma 2.9** and **Theorem 2.10** remain valid.

**2.8. The case of zero corrector.** Suppose that $g^0 = \overline{g}$, i. e., relations (1.31) are satisfied. Suppose also that condition (1.58) is satisfied. Then the $\Gamma$-periodic solutions of problems (1.23) and (1.33) are equal to zero: $\Lambda(x) = 0$ and $\bar{\Lambda}(x) = 0$. Using Proposition 1.20, we obtain the following result.

**Proposition 2.12.** Suppose that relations (1.31) and (1.58) are satisfied. Then, under the assumptions of **Theorem 2.2** for $0 < \epsilon \leq 1$ we have

$$
\|f^\epsilon e^{-\bar{B}_D, t}(f^\epsilon) - f_0 e^{-\bar{B}_D, t} f_0\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_{19} \epsilon t^{-1} e^{-\epsilon t/2}, \quad t > 0,
$$

where $C_{19}$ is a constant.
where the constant $C_{19}$ depends only on the problem data (1.9).

**Proof.** We rely on identity (2.13). For $|\zeta| \leq \hat{\zeta}$, where $\hat{\zeta}$ is the constant (2.14), we use (1.60) and (2.13). For $|\zeta| > \hat{\zeta}$ we apply (1.59) and (2.17). As a result, we see that for $0 < \varepsilon \leq 1$

$$
\|(B_D,\varepsilon - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq \tilde{C}_{19} \varepsilon, \quad \zeta \in \gamma;
$$

$$
\tilde{C}_{19} := \max\{ (C_9 + C_{10}(1 + \hat{\zeta})^{1/2}) \varepsilon, 25C_8 \}.
$$

Together with (2.13), this yields (2.39) with the constant $C_{19} := 2\pi^{-1} \tilde{C}_{19}$.

\[ \square \]

2.9. **Special case.** Now, we assume that $g^0 = g$, i.e., relations (1.32) are satisfied. Then, by Proposition 1.13 (3°), Condition 1.12 is satisfied. By [BSn, Remark 3.5], the matrix-valued function (1.27) is constant and coincides with $g^0$, i.e., $\tilde{g}(x) = g^0 = g$. Thus, $\tilde{g} \cdot b(D) f_0 e^{-B_D t} f_0 = g^0 b(D) f_0 e^{-B_D t} f_0$.

Suppose in addition that relation (1.58) is satisfied. Then $\tilde{\Lambda}(x) = 0$. The following result can be deduced from Theorem 2.9 and Proposition 1.1.

**Proposition 2.13.** Suppose that the relations (1.32) and (1.58) are satisfied. Then, under the assumptions of Theorem 2.2 for $0 < \varepsilon \leq \varepsilon_1$ and $t > 0$ we have

$$
\| g^0 b(D) f^e e^{-B_D t}(f^e)^* - g^0 b(D) f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_{16}^{\prime} t^{-3/4} e^{-c_t t/2}. \tag{2.40}
$$

The constant $\tilde{C}_{16}^{\prime}$ depends only on the problem data (1.9).

**Proof.** From Theorem 2.3 it follows that

$$
\| g^0 b(D) f^e e^{-B_D t}(f^e)^* - g^0 b(D) P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_{16}^{\prime} t^{-3/4} e^{-c_t t/2}. \tag{2.41}
$$

On the one hand, Proposition 1.1 and relations (1.3), (1.30), (1.43), (1.46), (2.8) imply that

$$
\| g^0 (S_\varepsilon - I) b(D) P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow L_2(\mathbb{R}^d)} \leq \varepsilon \| g \|_{L_{\infty}} r_1 \alpha_1^{1/2} \| P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow H^2(\mathbb{R}^d)} \leq \varepsilon \| g \|_{L_{\infty}} \| f \|_{L_{\infty}} r_1 \alpha_1^{1/2} C_{\Omega}^{(2)} c_3 t^{-1/2} e^{-c_t t/2}. \tag{2.42}
$$

On the other hand, from (1.2), (1.3), (1.30), (1.43), (1.46), and (2.7) it follows that

$$
\| g^0 (S_\varepsilon - I) b(D) P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow L_2(\mathbb{R}^d)} \leq 2 \| g \|_{L_{\infty}} \alpha_1^{1/2} \| P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow H^2(\mathbb{R}^d)} \leq 2 \| g \|_{L_{\infty}} \| f \|_{L_{\infty}} \alpha_1^{1/2} C_{\Omega}^{(2)} c_3 t^{-1/2} e^{-c_t t/2}. \tag{2.43}
$$

By (2.42) and (2.43),

$$
\| g^0 (S_\varepsilon - I) b(D) P_0 f_0 e^{-B_D t} f_0 \|_{L_2(\Omega) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_{16}^{\prime} t^{-3/4} e^{-c_t t/2},
$$

where $\tilde{C}_{16} := \| g \|_{L_{\infty}} \| f \|_{L_{\infty}} \alpha_1^{1/2} (2 r_1 C_{\Omega}^{(1)} C_{\Omega}^{(2)} c_3)^{1/2}$. Combining this with (2.41), we obtain estimate (2.40) with the constant $\tilde{C}_{16}^{\prime} := \tilde{C}_{16} + \tilde{C}_{16}^{\prime}$.

\[ \square \]

2.10. **Estimates in a strictly interior subdomain.** Using Theorem 1.21 we improve error estimates in a strictly interior subdomain.

**Theorem 2.14.** Suppose that the assumptions of Theorem 2.3 are satisfied. Let $\Omega'$ be a strictly interior subdomain of the domain $\Omega$, and let $\delta$ be defined by (1.61). Then for $0 < \varepsilon \leq \varepsilon_1$ and $t > 0$ we have

$$
\| f^e e^{-B_D t}(f^e)^* - f_0 e^{-B_D t} f_0 - \varepsilon K_D(t; \varepsilon) \|_{L_2(\Omega') \rightarrow H^1(\Omega')} \leq \varepsilon (C_{20} t^{-1/2} \delta^{-1} + C_{21} t^{-1/2} e^{-c_t t/2}), \tag{2.44}
$$

$$
\| g^0 b(D) f^e e^{-B_D t}(f^e)^* - G_D(t; \varepsilon) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq \varepsilon (\tilde{C}_{20} t^{-1/2} \delta^{-1} + \tilde{C}_{21} t^{-1/2} e^{-c_t t/2}).
$$

The constants $C_{20}, C_{21}, \tilde{C}_{20}$, and $\tilde{C}_{21}$ depend only on the problem data (1.9).

**Proof.** The proof is based on application of Theorem 1.21 and relations (2.20), (2.28). Also, estimates (2.15) and (2.17) are used. We omit the details.

\[ \square \]
The following result is checked similarly with the help of Theorems 2.14 and 2.15.

**Theorem 2.15.** Suppose that the assumptions of Theorem 2.14 are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) satisfy Conditions 1.11 and 1.14, respectively. Let \( K_0(t; \varepsilon) \) be the corrector (2.32), and let \( \tilde{K}_0(t; \varepsilon) \) be the operator (2.33). Then for \( t > 0 \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\| f^s e^{-\tilde{B}_D,t}(f^s) - f_0 e^{-\tilde{B}_D,t} f_0 - \varepsilon K_0(t; \varepsilon) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq \varepsilon (C_{20} t^{-1/2} \delta^{-1} + C_{22} t^{-1}) e^{-c_0 t/2},
\]

\[
\| g^b(D) f^s e^{-\tilde{B}_D,t}(f^s) - \tilde{K}_0(t; \varepsilon) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq \varepsilon (\tilde{C}_{20} t^{-1/2} \delta^{-1} + \tilde{C}_{22} t^{-1}) e^{-c_0 t/2}.
\]

The constants \( C_{20} \) and \( \tilde{C}_{20} \) are the same as in Theorem 2.14. The constants \( C_{22} \) and \( \tilde{C}_{22} \) depend on the problem data (1.19), \( p \), and the norms \( \| \Lambda \|_{L_{\infty}}, \| \Lambda_n \|_{L_p(\Omega)} \).

Note that it is possible to remove the smoothing operator \( S_\varepsilon \) in the corrector in estimates of Theorem 2.14 without any additional assumptions on the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \). For this, the additional smoothness of the boundary is not required. We consider the case where \( d \geq 3 \) (otherwise, by Propositions 1.11 and 1.13, we can apply Theorem 2.15). We know that for \( t > 0 \) the operator \( e^{-\tilde{B}_D,t} \) is continuous from \( L_2(\Omega; C^n) \) to \( H^2(\Omega; C^n) \) and estimate (2.33) holds. Moreover, the following property of “regularity improvement” inside the domain is valid: for \( t > 0 \) the operator \( e^{-\tilde{B}_D,t} \) is continuous from \( L_2(\Omega; C^n) \) to \( H^\sigma(\Omega'; C^n) \) for any integer \( \sigma \geq 3 \).

We have

\[
\| e^{-\tilde{B}_D,t} \|_{L_2(\Omega) \rightarrow H^\sigma(\Omega')} \leq C_d t^{-1/2} (e^{-2 t} + O(t^{-1}))(\sigma-1)/2 e^{-c_0 t/2},
\]

\[
t > 0, \quad \sigma \in \mathbb{N}, \quad \sigma \geq 3.
\]

The constant \( C_d \) depends on \( \sigma \) and the problem data (1.19). For the scalar parabolic equations, the property of “regularity improvement” inside the domain was obtained in [LaSoU, Chapter 3, § 12]. In a similar way, it can be checked for the operator \( \tilde{B}_D \). It is easy to deduce the qualified estimates (2.33), noticing that the derivatives \( \tilde{D}^\sigma u_0 \) (where \( u_0 \) is the function (2.3) with \( \sigma \geq 1 \) are solutions of a parabolic equation \( \tilde{Q}\partial_t \tilde{D}^\sigma u_0 = -\tilde{B}D^\sigma u_0 \). We multiply this equation by \( \chi^2 \tilde{D}^\sigma u_0 \) and integrate over the cylinder \( \Omega \times (0, t) \). Here \( \chi \) is a smooth cut-off function equal to zero near the lateral surface and the bottom of the cylinder. The standard analysis of the corresponding integral identity together with the already known inequalities of Lemma 2.1 leads to estimates (2.45).

Using the properties of the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \), and also the properties of the operator \( S_\varepsilon \), we can deduce the following statement from relation (2.45).

**Lemma 2.16.** Suppose that the assumptions of Theorem 2.14 are satisfied and that \( d \geq 3 \). Let \( K_0(t; \varepsilon) \) be the operator (2.32). Denote

\[
h_d(\varepsilon; t) := t^{-1} + t^{-1/2} (\delta^{-2} + t^{-1} \gamma^{d/4}).
\]

Let \( 2r_1 = \text{diam } \Omega \). Then for \( 0 < \varepsilon \leq (4r_1)^{-1} \delta \) and \( t > 0 \) we have

\[
\| K_0(t; \varepsilon) - K_0(t; \varepsilon) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq C_d h_d(\varepsilon; t)e^{-c_0 t/2},
\]

(2.47)

The constant \( C_d \) depends only on the problem data (1.19).

From Lemma 2.16 and Theorem 2.14 we deduce the following result.

**Theorem 2.17.** Suppose that the assumptions of Theorem 2.14 are satisfied, and \( d \geq 3 \). Let \( K_0(t; \varepsilon) \) be the corrector (2.32), and let \( \tilde{K}_0(t; \varepsilon) \) be the operator (2.33). Let \( 2r_1 = \text{diam } \Omega \). Then for \( 0 < \varepsilon \leq \min\{\varepsilon_1; (4r_1)^{-1} \delta\} \) and \( t > 0 \) we have

\[
\| f^s e^{-\tilde{B}_D,t}(f^s) - f_0 e^{-\tilde{B}_D,t} f_0 - \varepsilon \tilde{K}_0(t; \varepsilon) \|_{L_2(\Omega) \rightarrow H^1(\Omega')} \leq \varepsilon C_d h_d(\varepsilon; t)e^{-c_0 t/2},
\]

\[
\| g^b(D) f^s e^{-\tilde{B}_D,t}(f^s) - \tilde{K}_0(t; \varepsilon) \|_{L_2(\Omega) \rightarrow L_2(\Omega')} \leq \varepsilon \tilde{C}_d h_d(\varepsilon; t)e^{-c_0 t/2}.
\]

Here \( h_d(\varepsilon; t) \) is given by (2.46), the constants \( C_d \) and \( \tilde{C}_d \) depend only on the problem data (1.19).

The proofs of Lemma 2.16 and Theorem 2.17 are presented in Appendix (see § 2) in order not to clutter the main presentation. Clearly, it is convenient to apply Theorem 2.17 if \( t \) is separated from zero. For small \( t \) the order of the factor \( h_d(\varepsilon; t) \) grows with dimension. This is a “charge” for removal of the smoothing operator.
3. Homogenization of the First Initial Boundary Value Problem for Nonhomogeneous Equation

3.1. The principal term of approximation. In this section, we study the behavior of the solution of the first initial boundary value problem for a nonhomogeneous parabolic equation:

\[
\begin{align*}
Q_0^r(x)\frac{\partial u_r}{\partial t}(x,t) &= -B(x)u_r(x,t) + F(x,t), \quad x \in \mathcal{O}, \quad t > 0; \\
u_r(\cdot, 0)|_{\partial \mathcal{O}} &= 0, \quad t > 0; \\
Q_0^r(x)u_r(x,0) &= \varphi(x), \quad x \in \mathcal{O}.
\end{align*}
\]  
(3.1)

Here \(F \in \mathcal{H}_r(T) := L_r((0,T); L_2(\mathcal{O}; \mathbb{C}^n)), 0 < T \leq \infty, \) with some \(1 \leq r \leq \infty. \) Then

\[
u_r(\cdot, t) = f^\varepsilon e^{-B_d(t)}\varphi(\cdot) + \int_0^t f^\varepsilon e^{-B_d(t-s)}(f^\varepsilon)\varphi(\cdot) + (3.5)
\]  
(3.2)

The corresponding effective problem takes the form

\[
\begin{align*}
Q_0^r(x)\frac{\partial u_0}{\partial t}(x,t) &= -B^0u_0(x,t) + F(x,t), \quad x \in \mathcal{O}, \quad t > 0; \\
u_0(\cdot, 0)|_{\partial \mathcal{O}} &= 0, \quad t > 0; \\
Q_0^r u_0(x,0) &= \varphi(x), \quad x \in \mathcal{O}.
\end{align*}
\]  
(3.3)

The solution of this problem is given by

\[
u_0(\cdot, t) = f_0^\varepsilon e^{-B^0t}f_0\varphi(\cdot) + \int_0^t f_0^\varepsilon e^{-B^0(t-s)}f_0^\varepsilon F(\cdot, t) \, dt.
\]  
(3.4)

Subtracting (3.4) from (3.2) and using Theorem 2.2 (see (2.11)), we conclude that for \(0 < \varepsilon \leq \varepsilon_1 \) and \(t > 0\)

\[
\|\nu_r(\cdot, t) - \nu_0(\cdot, t)\|_{L_2(\mathcal{O})} \leq C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_3t/2}\|\varphi\|_{L_2(\mathcal{O})} + C_{15}\varepsilon\mathcal{L}(\varepsilon; t; F),
\]

where

\[
\mathcal{L}(\varepsilon; t; F) := \int_0^t e^{-c_3(t-s)}(s^2 + t - s)^{-1/2}\|F(\cdot, s)\|_{L_2(\mathcal{O})} \, ds.
\]

Estimating the term \(\mathcal{L}(\varepsilon; t; F),\) for the case \(1 < r \leq \infty\) we obtain the following result. Its proof is completely analogous to the proof of Theorem 5.1 from [MSu1].

**Theorem 3.1.** Suppose that \(\mathcal{O} \subset \mathbb{R}^d\) is a bounded domain of class \(C^{1,1}.\) Suppose that the assumptions of Subsections 1.3, 1.6 are satisfied. Let \(\nu_r\) be the solution of problem (5.1), and let \(\nu_0\) be the solution of the effective problem (5.3) with \(\varphi \in L_2(\mathcal{O}; \mathbb{C}^n)\) and \(F \in \mathcal{H}_r(T), 0 < T \leq \infty,\) with some \(1 \leq r \leq \infty.\) Then for \(0 < \varepsilon \leq \varepsilon_1\) and \(0 < t < T\) we have

\[
\|\nu_r(\cdot, t) - \nu_0(\cdot, t)\|_{L_2(\mathcal{O})} \leq C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_3t/2}\|\varphi\|_{L_2(\mathcal{O})} + c_3\varepsilon(\varepsilon; r)\|F\|_{\mathcal{H}_r(T)}.
\]

Here \(\theta(\varepsilon, r)\) is given by

\[
\theta(\varepsilon, r) = \begin{cases} 
\varepsilon^{2-2/r}, & 1 < r < 2, \\
\varepsilon(\ln \varepsilon + 1)^{-1/2}, & r = 2, \\
\varepsilon, & 2 < r \leq \infty.
\end{cases}
\]  
(3.5)

The constant \(c_3\) depends only on \(r\) and the problem data (1.9).

By analogy with the proof of Theorem 5.2 from [MSu1], we can deduce approximation of the solution of problem (3.1) in \(\mathcal{H}_r(T)\) from Theorem 2.2.

**Theorem 3.2.** Suppose that the assumptions of Theorem 3.1 are satisfied. Let \(\nu_r\) and \(\nu_0\) be the solutions of problems (3.1) and (3.3), respectively, with \(\varphi \in L_2(\mathcal{O}; \mathbb{C}^n)\) and \(F \in \mathcal{H}_r(T), 0 < T \leq \infty,\) for some \(1 \leq r < \infty.\) Then for \(0 < \varepsilon \leq \varepsilon_1\) we have

\[
\|\nu_r - \nu_0\|_{\mathcal{H}_r(T)} \leq c_3(\theta(\varepsilon, r'))\|\varphi\|_{L_2(\mathcal{O})} + C_{23}\varepsilon\|F\|_{\mathcal{H}_r(T)}.
\]

Here \(\theta(\varepsilon, \cdot)\) is given by (3.5), \(r^{-1} + (r')^{-1} = 1. \) The constant \(C_{23}\) depends only on the problem data (1.9), the constant \(c_{3r}\) depends on the same parameters and \(r.\)
Remark 3.3. For the case where \( \varphi = 0 \) and \( F \in \mathcal{H}_\infty(T) \), Theorem 3.1 implies that
\[
\|u_\epsilon - u_0\|_{\mathcal{H}_\infty(T)} \leq c_\infty \|F\|_{\mathcal{H}_\infty(T)}, \quad 0 < \varepsilon \leq \varepsilon_1.
\]

3.2. Approximation of the solution in \( H^1(\Omega; \mathbb{C}^n) \). Now, we obtain approximation of the solution of problem (3.1) in the \( H^1(\Omega; \mathbb{C}^n) \)-norm with the help of Theorem 2.3. The difficulties arise in consideration of the integral term in (3.2), because estimate (2.21) “deteriorates” for small \( t \). Assuming that \( t \geq \varepsilon^2 \), we divide the integration interval in (3.2) into two parts: \((0, t - \varepsilon^2)\) and \((t - \varepsilon^2, t)\). On the interval \((0, t - \varepsilon^2)\) we apply (2.24), and on \((t - \varepsilon^2, t)\) we use (2.29).

Denote
\[
w_\epsilon(\cdot, t) := f_0 e^{-\overline{B}_D t^2} f_0^{-1} u_0(\cdot, t - \varepsilon^2),
\]
where \( u_0 \) is the solution of problem (3.3). By (3.4),
\[
w_\epsilon(\cdot, t) = f_0 e^{-\overline{B}_D t} f_0 \varphi(\cdot) + \int_0^{t-\varepsilon^2} f_0 e^{-\overline{B}_D (t-\tilde{t})} f_0 F(\cdot, \tilde{t}) \, d\tilde{t}.
\]

The following statement can be checked similarly to Theorem 5.4 from [MSm1].

Theorem 3.4. Suppose that the assumptions of Theorem 3.1 are satisfied. Suppose that \( u_\epsilon \) and \( u_0 \) are the solutions of problems (3.1) and (3.3), respectively, with \( \varphi \in L_2(\Omega; \mathbb{C}^n) \) and \( F \in \mathcal{H}_\infty(T) \), \( 0 < T \leq \infty \), for some \( 2 < r \leq \infty \). Let \( w_\epsilon(\cdot, t) \) be given by (3.6). Let \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) be the \( \Gamma \)-periodic matrix solutions of problems (1.25) and (1.33), respectively. Suppose that \( P_\Omega \) is a linear continuous extension operator (1.15). Let \( S_\epsilon \) be the Steklov smoothing operator (1.1). We put \( w_\epsilon(\cdot, t) := P_\Omega w_\epsilon(\cdot, t) \) and denote
\[
v_\epsilon(\cdot, t) := u_0(\cdot, t) + \varepsilon^4 S_\epsilon b(D) \tilde{w}_\epsilon(\cdot, t) + \varepsilon^4 S_\epsilon \tilde{w}_\epsilon(\cdot, t).
\]
Let \( p_\epsilon(\cdot, t) := g^r b(D) u_\epsilon(\cdot, t) \), and let \( \tilde{g}(x) \) be the matrix-valued function (1.27). We put
\[
q_\epsilon(\cdot, t) := \tilde{g}^r S_\epsilon b(D) \tilde{w}_\epsilon(\cdot, t) + g^r (b(D) \tilde{\Lambda})^r S_\epsilon \tilde{w}_\epsilon(\cdot, t).
\]
Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \varepsilon^2 \leq t < T \) we have
\[
\|u_\epsilon(\cdot, t) - v_\epsilon(\cdot, t)\|_{H^1(\Omega)} \leq 2C_1 \varepsilon^{1/2} \left( t^{3/4} e^{-c_\varepsilon t/2} \|\varphi\|_{L_2(\Omega)} + c_\varepsilon \omega(\varepsilon, r) \|F\|_{\mathcal{H}_\infty(T)} \right),
\]
\[
\|p_\epsilon(\cdot, t) - q_\epsilon(\cdot, t)\|_{L_2(\Omega)} \leq \tilde{C}_1 \varepsilon^{1/2} \left( t^{3/4} e^{-c_\varepsilon t/2} \|\varphi\|_{L_2(\Omega)} + c_\varepsilon \omega(\varepsilon, r) \|F\|_{\mathcal{H}_\infty(T)} \right).
\]

Here constants \( \tilde{c}_\varepsilon \) and \( \tilde{c}_\varepsilon \) depend only on the problem data (1.9) and \( r \), and
\[
\omega(\varepsilon, r) := \begin{cases} \varepsilon^{1-2/r}, & 2 < r < 4, \\ \varepsilon^{1/2} (| \ln \varepsilon | + 1)^{3/4}, & r = 4, \\ \varepsilon^{1/2}, & 4 < r \leq \infty. \end{cases}
\]

Since the right-hand side of estimate (2.21) grows slowly than the right-hand side in estimate (2.24), as \( t \to 0 \), for \( r > 4 \) we can approximate the flux \( p_\epsilon \) in terms of
\[
h_\epsilon(\cdot, t) := \tilde{g}^r S_\epsilon b(D) \tilde{u}_0(\cdot, t) + g^r (b(D) \tilde{\Lambda})^r S_\epsilon \tilde{u}_0(\cdot, t).
\]

Proposition 3.5. Suppose that the assumptions of Theorem 3.1 are satisfied. Suppose that \( u_\epsilon \) and \( u_0 \) are the solutions of problems (3.1) and (3.3), respectively, with \( \varphi \in L_2(\Omega; \mathbb{C}^n) \) and \( F \in \mathcal{H}_\infty(T) \), \( 0 < T \leq \infty \), for some \( 4 < r \leq \infty \). Let \( p_\epsilon(\cdot, t) = g^r b(D) u_\epsilon(\cdot, t) \) and let \( h_\epsilon(\cdot, t) \) be given by (3.8). Then for \( 0 < t < T \) and \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\|p_\epsilon(\cdot, t) - h_\epsilon(\cdot, t)\|_{L_2(\Omega)} \leq \tilde{C}_1 \varepsilon^{1/2} t^{3/4} e^{-c_\varepsilon t/2} \|\varphi\|_{L_2(\Omega)} + C_{24}^{(r)} \varepsilon^{1/2} \|F\|_{\mathcal{H}_\infty(T)}.
\]
The constant \( C_{24}^{(r)} \) depends only on the problem data (1.9) and \( r \).

Proof. To check estimate (3.8), we use inequality (2.25) and identities (3.2), (3.4). If \( r = \infty \), we deduce (3.3) with \( C_{24}^{(\infty)} := (2/c_\varepsilon)^1/4 \tilde{T}(1/4) \tilde{C}_{16} \). If \( 4 < r < \infty \), we apply the Hölder inequality:
\[
\|p_\epsilon(\cdot, t) - h_\epsilon(\cdot, t)\|_{L_2(\Omega)} \leq \tilde{C}_1 \varepsilon^{1/2} t^{3/4} e^{-c_\varepsilon t/2} \|\varphi\|_{L_2(\Omega)} + \tilde{C}_{16} \varepsilon^{1/2} \|F\|_{\mathcal{H}_\infty(T)} \|J_\varepsilon(\varepsilon, t)^{1/r'} \|^{-1} (r')^{-1} = 1.
\]
Here
\[ \mathcal{J}_r(\varepsilon, t) := \int_0^t T^{-3r'/4} e^{-c_r t^2/2} d\tau \leq (c_r T'/2)^{3r'/4-1} \Gamma(1 - 3r'/4). \]
This implies (3.9) with the constant \( C_{24}^{(r)} := (c_r T'/2)^{3/4-1/r'} \Gamma(1 - 3r'/4)^{1/r'} \tilde{C}_{16}. \)

Combining Proposition 2.10 and Theorem 2.6 we deduce the following result.

**Theorem 3.6.** Suppose that the assumptions of Theorem 3.3 are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) satisfy Conditions 1.12 and 1.14 respectively. Denote
\[ \tilde{v}_e(\cdot, t) := u_0(\cdot, t) + \varepsilon \Lambda^\varepsilon b(D)w_e(\cdot, t), \quad \tilde{q}_e(\cdot, t) := \tilde{g}^\varepsilon b(D)w_e(\cdot, t), \]
(3.10) (3.11)

Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \varepsilon^2 \leq t < T \) we have
\[ \|\tilde{u}_e(\cdot, t) - \tilde{v}_e(\cdot, t)\|_{H^1(O)} \leq 2C_{18} \varepsilon^{1/2 t^{-3/4} e^{-c_t t/2}} \|\phi\|_{L_2(O)} + c_{18}^\varepsilon e^{-c_t t/2} \|\tilde{\phi}\|_{H^1(O)}), \]
\[ \|\tilde{p}_e(\cdot, t) - \tilde{q}_e(\cdot, t)\|_{L_2(O)} \leq 2\tilde{C}_{18} \varepsilon^{1/2 t^{-3/4} e^{-c_t t/2}} \|\phi\|_{L_2(O)} + c_{18}^\varepsilon e^{-c_t t/2} \|\tilde{\phi}\|_{H^1(O)}), \]
(3.10) (3.11)
The constants \( c_{18}^\varepsilon \) and \( \tilde{c}_{18}^\varepsilon \) depend only on the initial data \( 1.9, r, p, \) and the norms \( \|\Lambda\|_{L^\infty}, \|\tilde{\Lambda}\|_{L^\infty}, \)
For the case of sufficiently smooth boundary, we could apply Theorem 2.10. However, because of the strong growth of the right-hand side in estimates (2.37), (2.38) for small \( t \), we obtain a meaningful result only in the three-dimensional case and only for \( r > 4 \).

**Proposition 3.7.** Suppose that the assumptions of Theorem 3.3 are satisfied with \( d = 3 \) and \( r > 4 \). Suppose that \( \partial O \in C^{2,1} \). Let \( \tilde{v}_e \) and \( \tilde{q}_e \) be given by (3.10) and (3.11). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \varepsilon^2 \leq t < T \) we have
\[ \|\tilde{u}_e(\cdot, t) - \tilde{v}_e(\cdot, t)\|_{H^1(O)} \leq \tilde{C}_{3} \varepsilon^{1/2 t^{-3/4} + \varepsilon t^{-5/4}} e^{-c_t t/2} \|\phi\|_{L_2(O)} + c_{4}\varepsilon^{1/2 - 2/r} \|\tilde{\phi}\|_{H^1(O)}), \]
\[ \|\tilde{p}_e(\cdot, t) - \tilde{q}_e(\cdot, t)\|_{L_2(O)} \leq \tilde{C}_{3} \varepsilon^{1/2 t^{-3/4} + \varepsilon t^{-5/4}} e^{-c_t t/2} \|\phi\|_{L_2(O)} + c_{4} \varepsilon^{1/2 - 2/r} \|\tilde{\phi}\|_{H^1(O)}), \]
(3.10) (3.11)
The constants \( \tilde{c}_{4}\) and \( \tilde{c}_4 \) depend only on the problem data \( 1.9 \) and \( r \).

3.3. Approximation of the solution in a strictly interior subdomain. From Theorem 2.14 and Proposition 2.3 we deduce the following result.

**Theorem 3.8.** Suppose that the assumptions of Theorem 3.3 are satisfied. Let \( O' \) be a strictly interior subdomain of the domain \( O \). Let \( \delta \) be given by (1.61). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \varepsilon^2 \leq t < T \) we have
\[ \|\tilde{u}_e(\cdot, t) - \tilde{v}_e(\cdot, t)\|_{H^1(O')} \leq \varepsilon (C_{20} t^{-1/2} e^{-c_t t/2} + C_{21} t^{-1}) e^{-c_t t/2} \|\phi\|_{L_2(O)} + k_r \vartheta(\varepsilon, \delta, r) \|\tilde{\phi}\|_{H^1(O')}, \]
\[ \|\tilde{p}_e(\cdot, t) - \tilde{q}_e(\cdot, t)\|_{L_2(O')} \leq \varepsilon \tilde{C}_{20} t^{-1/2} e^{-c_t t/2} \|\phi\|_{L_2(O)} + \tilde{k}_r \vartheta(\varepsilon, \delta, r) \|\tilde{\phi}\|_{H^1(O')}. \]
(3.10) (3.11)
Here the constants \( k_r \) and \( \tilde{k}_r \) depend only on the problem data \( 1.9 \) and \( r \), and
\[ \vartheta(\varepsilon, \delta, r) := \begin{cases} \varepsilon^{-1} + \varepsilon^{-1/2} r, & 2 < r < \infty, \\ \varepsilon^{-1} + \varepsilon(|\ln \varepsilon| + 1), & r = \infty. \end{cases} \]
Finally, under Conditions 1.12 and 1.14 Theorem 2.15 implies the following result.

**Theorem 3.9.** Suppose that the assumptions of Theorem 3.8 are satisfied. Suppose that the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \) satisfy Conditions 1.12 and 1.14 respectively. Suppose that \( \tilde{v}_e \) and \( \tilde{q}_e \) are given by (3.10) and (3.11). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \varepsilon^2 \leq t < T \) we have
\[ \|\tilde{u}_e(\cdot, t) - \tilde{v}_e(\cdot, t)\|_{H^1(O')} \leq \varepsilon (C_{20} t^{-1/2} e^{-c_t t/2} + C_{21} t^{-1}) e^{-c_t t/2} \|\phi\|_{L_2(O)} + k_r \vartheta(\varepsilon, \delta, r) \|\tilde{\phi}\|_{H^1(O')}, \]
\[ \|\tilde{p}_e(\cdot, t) - \tilde{q}_e(\cdot, t)\|_{L_2(O')} \leq \varepsilon \tilde{C}_{20} t^{-1/2} e^{-c_t t/2} \|\phi\|_{L_2(O)} + \tilde{k}_r \vartheta(\varepsilon, \delta, r) \|\tilde{\phi}\|_{H^1(O')}. \]
(3.10) (3.11)
The constants \( \tilde{k}_r \) and \( \tilde{k}_r \) depend only on the problem data \( 1.9 \), \( r \), \( p \), and the norms \( \|\Lambda\|_{L^\infty}, \|\tilde{\Lambda}\|_{L^\infty}. \)
Applications

For elliptic systems in the whole space $\mathbb{R}^d$, the examples considered below were studied in [Su4, MSu2]. For elliptic systems in a bounded domain, these examples were considered in [MSu3].

4. Scalar elliptic operator with a singular potential

4.1. Description of the operator. We consider the case where $n = 1, m = d$, $b(D) = D$, and $g(x)$ is a $\Gamma$-periodic symmetric $(d \times d)$-matrix-valued function with real entries such that $g, g^{-1} \in L_\infty$ and $g(x) > 0$. Obviously (see (1.3)), $\alpha_0 = \alpha_1 = 1$ and $b(D)^*g^\varepsilon(x)b(D) = -\text{div} g^\varepsilon(x)\nabla$.

Next, let $A(x) = \text{col}\{A_1(x), \ldots, A_d(x)\}$, where $A_j(x), j = 1, \ldots, d$, are $\Gamma$-periodic real-valued functions such that

$$A_j \in L_\rho(O), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2; \quad j = 1, \ldots, d.$$  (4.1)

Let $v(x)$ and $V(x)$ be real-valued $\Gamma$-periodic functions such that

$$v, V \in L_s(O), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2; \quad \int_O v(x) \, dx = 0.$$  (4.2)

In $L_2(O)$, we consider the operator $B_{D, \varepsilon}$ given formally by the differential expression

$$B_{\varepsilon} = (D - A^\varepsilon(x))^*g^\varepsilon(x)(D - A^\varepsilon(x)) + \varepsilon^{-1}\varepsilon^\varepsilon(x) + V^\varepsilon(x)$$  (4.3)

with the Dirichlet condition on $\partial O$. The precise definition of the operator $B_{D, \varepsilon}$ is given in terms of the quadratic form

$$b_{D, \varepsilon}[u, u] = \int_O \left( (g^\varepsilon(D - A^\varepsilon(x))^*g^\varepsilon(x)D - A^\varepsilon(x)) + \varepsilon^{-1}\varepsilon^\varepsilon(x) + V^\varepsilon(x) \right) |u|^2 \, dx, \quad u \in H_0^1(O).$$

It is easily seen (cf. [Su4, Subsection 13.1]) that expression (4.3) can be written as

$$B_{\varepsilon} = D^*g^\varepsilon(x)D + \sum_{j=1}^d (a_j^\varepsilon(x)D_j + D_j(a_j^\varepsilon(x))^*) + Q^\varepsilon(x).$$  (4.4)

Here $Q(x)$ is a real-valued function defined by

$$Q(x) = V(x) + (g(x)A(x), A(x)).$$  (4.5)

The complex-valued functions $a_j(x)$ are given by

$$a_j(x) = -\eta_j(x) + i\xi_j(x), \quad j = 1, \ldots, d.$$  (4.6)

Here $\eta_j(x)$ and $\xi_j(x)$ are the components of the vector-valued function $\eta(x) = g(x)A(x)$, and the functions $\xi_j(x)$ are defined by $\xi_j(x) = -\partial_j\Phi(x)$, where $\Phi(x)$ is the $\Gamma$-periodic solution of the problem $\Delta \Phi(x) = v(x), \int_O \Phi(x) \, dx = 0$. We have

$$v(x) = -\sum_{j=1}^d \partial_j\xi_j(x).$$  (4.7)

It is easy to check that the functions (4.6) satisfy condition (1.7) with a suitable $\rho'$ depending on $\rho$ and $s$, and the norms $\|a_j\|_{L_\rho(O)}$ are controlled in terms of $\|g\|_{L_\infty}, \|A\|_{L_\rho(O)}, \|v\|_{L_\rho(O)}$, and the parameters of the lattice $\Gamma$. (See [Su4, Subsection 13.1].) The function (4.5) satisfies condition (1.8) with a suitable $s' = \min\{s; \rho/2\}$.

Let $Q_0(x)$ be a positive definite and bounded $\Gamma$-periodic function. According to (1.10), we introduce a positive definite operator $B_{D, \varepsilon} := B_{D, \varepsilon} + \lambda^\varepsilon Q_0$. Here the constant $\lambda$ is chosen according to condition (1.10) for the operator $B_{D, \varepsilon}$ with the coefficients $g, a_j, j = 1, \ldots, d$, $Q$, and $Q_0$ defined above. The operator $B_{D, \varepsilon}$ is given by

$$B_{\varepsilon} = (D - A^\varepsilon(x))^*g^\varepsilon(x)(D - A^\varepsilon(x)) + \varepsilon^{-1}\varepsilon^\varepsilon(x) + V^\varepsilon(x) + \lambda Q_0(x).$$  (4.8)

We are interested in the behavior of the exponential of the operator $B_{D, \varepsilon} := f^\varepsilon B_{D, \varepsilon} f^\varepsilon$, where $f(x) := Q_0(x)^{-1/2}$. 
For the scalar elliptic operator \( \mathcal{L} \), the problem data \( \mathcal{B} \) are reduced to the following set of parameters:

\[
d, \rho, s; \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|A\|_{L_\infty(\Omega)}, \|\psi\|_{L_\infty(\Omega)}, \|V\|_{L_\infty(\Omega)},
\]

\[
\|Q_0\|_{L_\infty}, \|Q_0^{-1}\|_{L_\infty};
\]

the parameters of the lattice \( \Gamma \); the domain \( \mathcal{O} \).

### 4.2. The effective operator

Let us write down the effective operator. In the case under consideration, the \( \Gamma \)-periodic solution of problem (1.25) is a row: \( \Lambda(x) = i\Psi(x), \Psi(x) = (\psi_1(x), \ldots, \psi_d(x)) \), where \( \psi_j \in H^1(\Omega) \) is the solution of the problem

\[
div g(x)(\nabla \psi_j(x) + e_j) = 0, \quad \int_\Omega \psi_j(x) \, dx = 0.
\]

Here \( e_j, j = 1, \ldots, d \), is the standard orthonormal basis in \( \mathbb{R}^d \). Clearly, the functions \( \psi_j(x) \) are real-valued, and the entries of \( \Lambda(x) \) are purely imaginary. By (1.27), the columns of the \((d \times d)\)-matrix-valued function \( \tilde{g}(x) \) are the vector-valued functions \( g(x)(\nabla \psi_j(x) + e_j), j = 1, \ldots, d \). The effective matrix is defined according to (1.26): \( g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(x) \, dx \). Clearly, \( \tilde{g}(x) \) and \( g^0 \) have real entries.

According to (4.6) and (4.7), the periodic solution of problem (1.33) is represented as \( \tilde{\Lambda}(x) = \tilde{\Lambda}_1(x) + i\tilde{\Lambda}_2(x) \), where the real-valued \( \Gamma \)-periodic functions \( \tilde{\Lambda}_1(x) \) and \( \tilde{\Lambda}_2(x) \) are the solutions of the problems

\[
- \text{div} \; g(x)(\nabla \tilde{\Lambda}_1(x) + v(\tilde{\Lambda}_1(x))) = 0, \quad \int_\Omega \tilde{\Lambda}_1(x) \, dx = 0;
\]

\[
- \text{div} \; g(x)(\nabla \tilde{\Lambda}_2(x) + \text{div} \; g(x)A(x)) = 0, \quad \int_\Omega \tilde{\Lambda}_2(x) \, dx = 0.
\]

The column \( V \) (see (1.35)) has the form \( V = V_1 + iV_2 \), where \( V_1, V_2 \) are the columns with real entries defined by

\[
V_1 = |\Omega|^{-1} \int_\Omega (\nabla \tilde{\Psi}(x))^t g(x) \nabla \tilde{\Lambda}_2(x) \, dx, \quad V_2 = -|\Omega|^{-1} \int_\Omega (\nabla \tilde{\Psi}(x))^t g(x) \nabla \tilde{\Lambda}_1(x) \, dx.
\]

According to (1.36), the constant \( W \) is given by

\[
W = |\Omega|^{-1} \int_\Omega \left( (g(x)\nabla \tilde{\Lambda}_1(x), \nabla \tilde{\Lambda}_1(x)) + (g(x)\nabla \tilde{\Lambda}_2(x), \nabla \tilde{\Lambda}_2(x)) \right) \, dx.
\]

The effective operator for \( \mathcal{B}_{D,\varepsilon} \) acts as follows

\[
\mathcal{B}^0_{D,\varepsilon} u = -\text{div} \; g^0 \nabla u + 2i(\nabla u, V_1 + \overline{\eta}) + (-W + \overline{Q} + \lambda\overline{Q_0})u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega).
\]

The corresponding differential expression can be written as

\[
\mathcal{B}^0 = (D - A^0)^* g^0(D - A^0) + V^0 + \lambda\overline{Q_0},
\]

where

\[
A^0 = (g^0)^{-1}(V_1 + g\overline{A}), \quad V^0 = \overline{V} + (g\overline{A}, \overline{A}) - (g^0 \overline{A}, \overline{A}) - W.
\]

Let \( f_0 := (Q_0)^{-1/2} \). Denote \( \tilde{\mathcal{B}}^0_{D,\varepsilon} := f_0 \mathcal{B}^0_{D,\varepsilon} f_0 \).

### 4.3. Approximation of the sandwiched operator exponential

According to Remark 1.16 in the case under consideration, Conditions 1.12 and 1.14 are satisfied, and the norms \( \|\Lambda\|_{L_\infty} \) and \( \|\tilde{\Lambda}\|_{L_\infty} \) are estimated in terms of the problem data (4.9). Therefore, we can use the corrector which does not involve the smoothing operator:

\[
\mathcal{K}^0_D(t; \varepsilon) := \left( [\varepsilon A^0] D + [\varepsilon \tilde{A}^0] \right) f_0 e^{-\tilde{\mathcal{B}}^0_{D,\varepsilon} t} f_0 = \left( [\varepsilon A^0] \nabla + [\varepsilon \tilde{A}^0] \right) f_0 e^{-\tilde{\mathcal{B}}^0_{D,\varepsilon} t} f_0.
\]

The operator (2.33) takes the form \( \mathcal{G}^0_D(t; \varepsilon) = -i\mathcal{G}^0_D(t; \varepsilon) \), where

\[
\mathcal{G}^0_D(t; \varepsilon) = \tilde{g}^0 \nabla f_0 e^{-\tilde{\mathcal{B}}^0_{D,\varepsilon} t} f_0 + g^0 (\nabla \tilde{A}^0) f_0 e^{-\tilde{\mathcal{B}}^0_{D,\varepsilon} t} f_0.
\]

Theorems 2.2 and 2.6 imply the following result.
Proposition 4.1. Suppose that the assumptions of Subsections 4.1 and 4.2 are satisfied. Suppose that the operators $K^0_D(t; \varepsilon)$ and $\Theta^0_D(t; \varepsilon)$ are given by (4.11) and (4.12), respectively. Suppose that the number $\varepsilon_1$ is subject to Condition 1.7. Then for $0 < \varepsilon \leq \varepsilon_1$ we have
\[
\|f^\varepsilon e^{-\tilde{B}_D, t} f^\varepsilon - f_0 e^{-\tilde{B}_D, t} f_0 \|_{L^2(\Omega) \to L^2(\Omega)} \leq C_{15} (t + \varepsilon^2)^{-1/2} e^{-c_{0}t/2} , \quad t \geq 0 ;
\]
\[
\|f^\varepsilon e^{-\tilde{B}_D, t} f^\varepsilon - f_0 e^{-\tilde{B}_D, t} f_0 - \varepsilon K^0_D(t; \varepsilon) \|_{L^2(\Omega) \to H^1(\Omega)} \leq C_{18} (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c_{0}t/2} , \quad t > 0 ;
\]
\[
\|g^\varepsilon \nabla f^\varepsilon e^{-\tilde{B}_D, t} f^\varepsilon - \Theta^0_D(t; \varepsilon) \|_{L^2(\Omega) \to L^2(\Omega)} \leq \tilde{C}_{18} (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c_{0}t/2} , \quad t > 0 .
\]

The constants $C_{15}, C_{18},$ and $\tilde{C}_{18}$ depend only on the problem data (4.9).

4.4. Homogenization of the first initial boundary-value problem for parabolic equation with a singular potential. Consider the first initial boundary-value problem for nonhomogeneous parabolic equation with a singular potential:
\[
\begin{align*}
Q_0^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial t} (x, t) &= -(D - A^\varepsilon(x))^* f^\varepsilon(x)(D - A^\varepsilon(x)) u^\varepsilon(x, t) \\
&\quad - (\varepsilon^{-1} r^\varepsilon(x) + \nu^\varepsilon(x) + \lambda Q_0^\varepsilon(x)) u^\varepsilon(x, t) + F(x, t), \quad x \in \Omega, \quad t > 0; \\
u^\varepsilon(\cdot, t)|_{\partial \Omega} &= 0, \quad t > 0; \\
Q_0^\varepsilon(x) u^\varepsilon(x, 0) &= \varphi(x), \quad x \in \Omega.
\end{align*}
\]

Here $\varphi \in L^2(\Omega)$ and $F \in \mathcal{S}_r(T) := L^2((0, T); L^2(\Omega))$, $0 < T \leq \infty$, for some $1 \leq r \leq \infty$.

According to (3.3) and (4.10), the effective problem takes the form
\[
\begin{align*}
Q_0^\varepsilon(x) \frac{\partial u_0}{\partial t} (x, t) &= -(D - A^0)^* g^0(D - A^0) u_0(x, t) - (\nu^0 + \lambda Q_0) u_0(x, t) \\
&\quad + F(x, t), \quad x \in \Omega, \quad t > 0; \\
u_0(\cdot, t)|_{\partial \Omega} &= 0, \quad t > 0; \\
Q_0^\varepsilon u_0(x, 0) &= \varphi(x), \quad x \in \Omega.
\end{align*}
\]

Applying Theorems 3.1 and 3.6 we obtain the following result.

Proposition 4.2. Suppose that the number $\varepsilon_1$ is subject to Condition 1.7. Suppose that the assumptions of Subsection 4.1 are satisfied, and $1 < r \leq \infty$. Then for $0 < \varepsilon \leq \varepsilon_1$ and $0 < t < T$ we have
\[
\|u^\varepsilon(\cdot, t) - u_0(\cdot, t)\|_{L^2(\Omega)} \leq C_{15} (t + \varepsilon^2)^{-1/2} e^{-c_{0}t/2} \|\varphi\|_{L^2(\Omega)} + c_r \theta(\varepsilon, r) \|F\|_{\mathcal{S}_r(T)}.
\]

Here $\theta(\varepsilon, r)$ is given by (3.5).

Assuming that $t \geq \varepsilon^2$, we put $w^\varepsilon(\cdot, t) := f_0 e^{-\tilde{B}_D, \varepsilon} f_0^{-1} u_0(\cdot, t - \varepsilon^2)$. Denote $\tilde{u}^\varepsilon(\cdot, t) := u_0(\cdot, t) + \varepsilon \tilde{\Psi}^\varepsilon \nabla w^\varepsilon(\cdot, t) + \varepsilon \tilde{\Lambda}^\varepsilon w^\varepsilon(\cdot, t)$ and $\tilde{q}^\varepsilon(\cdot, t) := \tilde{g}^\varepsilon \nabla w^\varepsilon(\cdot, t) + g^\varepsilon (\tilde{\nabla}^\varepsilon)^2 w^\varepsilon(\cdot, t)$. In addition, assume that $2 < r \leq \infty$. Then for $0 < \varepsilon \leq \varepsilon_1$ and $\varepsilon^2 \leq t < T$ we have
\[
\|\tilde{u}^\varepsilon(\cdot, t) - \tilde{w}^\varepsilon(\cdot, t)\|_{H^1(\Omega)} \leq 2C_{18} \varepsilon^{1/2} t^{-3/4} e^{-c_{0}t/2} \|\varphi\|_{L^2(\Omega)} + c_r \omega(\varepsilon, r) \|F\|_{\mathcal{S}_r(T)} ,
\]
\[
\|\tilde{g}^\varepsilon \nabla w^\varepsilon(\cdot, t) - \tilde{q}^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq 2 \tilde{C}_{18} \varepsilon^{1/2} t^{-3/4} e^{-c_{0}t/2} \|\varphi\|_{L^2(\Omega)} + c_r \omega(\varepsilon, r) \|F\|_{\mathcal{S}_r(T)} .
\]

Here $\omega(\varepsilon, r)$ is given by (3.7). The constants $C_{15}, C_{18},$ and $\tilde{C}_{18}$ depend only on the problem data (4.9). The constants $c_r$, $c_r'$, and $c_r''$ depend on the same parameters and on $r$.

5. The scalar operator with a strongly singular potential of order $\varepsilon^{-2}$

Homogenization of the first initial boundary-value problem for parabolic equation with a strongly singular potential was studied in [AICPSiVa]. Some motivations can be found in [AICPSiVa §1]. However, the results of [AICPSiVa] cannot be formulated in the uniform operator topology.
5.1. Description of the operator. Let \( \tilde{g}(x) \) be a \( \Gamma \)-periodic symmetric \((d \times d)\)-matrix-valued function in \( \mathbb{R}^d \) with real entries such that \( \tilde{g}, \tilde{g}^{-1} \in L_\infty \) and \( \tilde{g}(x) > 0 \). Let \( \tilde{v}(x) \) be a real-valued \( \Gamma \)-periodic function such that
\[
\tilde{v} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2.
\]
By \( \hat{A} \) we denote the operator in \( L_2(\mathbb{R}^d) \) corresponding to the quadratic form
\[
\int_{\mathbb{R}^d} ((\tilde{g}(x)Du, Du) + \tilde{v}(x)|u|^2) \, dx, \quad u \in H^1(\mathbb{R}^d).
\]
Adding a constant to the potential \( \tilde{v}(x) \), we assume that the bottom of the spectrum of \( \hat{A} \) is the point zero. Then the operator \( \hat{A} \) admits a factorization with the help of the eigenfunction of the operator \( D^* \tilde{g}(x)D + \tilde{v}(x) \) on the cell \( \Omega \) (with periodic boundary conditions) corresponding to the eigenvalue \( \lambda = 0 \) (see [BSu2, Chapter 6, Subsection 1.1]). Apparently, such factorization trick was first used in homogenization problems in [Z1, K].

In \( L_2(\mathcal{O}) \), we consider the operator \( \tilde{A}_D \) given by the expression \( D^* \tilde{g}(x)D + \tilde{v}(x) \) with the Dirichlet condition on \( \partial\mathcal{O} \). The precise definition of the operator \( \tilde{A}_D \) is given in terms of the quadratic form
\[
\tilde{a}_D[u, u] = \int_{\mathcal{O}} ((\tilde{g}(x)D_\gamma u, D_\gamma u) + \tilde{v}(x)|u|^2) \, dx, \quad u \in H^1_0(\mathcal{O}). \tag{5.1}
\]
The operator \( \tilde{A}_D \) inherits factorization of the operator \( \hat{A} \). To describe this factorization, we consider the equation
\[
D^* \tilde{g}(x)D\omega(x) + \tilde{v}(x)\omega(x) = 0. \tag{5.2}
\]
There exists a \( \Gamma \)-periodic solution \( \omega \in \widetilde{H}^1(\Omega) \) of this equation defined up to a constant factor. We can fix this factor so that \( \omega(x) > 0 \) and
\[
\int_{\Omega} \omega^2(x) \, dx = |\Omega|. \tag{5.3}
\]
Moreover, the solution is positive definite and bounded: \( 0 < \omega_0 \leq \omega(x) \leq \omega_1 < \infty \). The norms \( \|\omega\|_{L_\infty} \) and \( \|\omega^{-1}\|_{L_\infty} \) are controlled in terms of \( \|\tilde{g}\|_{L_\infty}, \|\tilde{g}^{-1}\|_{L_\infty}, \) and \( \|\tilde{v}\|_{L_s(\Omega)} \). Note that \( \omega \) and \( \omega^{-1} \) are multipliers in \( H^1_0(\mathcal{O}) \).

Substituting \( u = \omega z \) and taking (5.2) into account, we represent the form (5.1) as
\[
\tilde{a}_D[u, u] = \int_{\mathcal{O}} \omega(x)^2 (\tilde{g}(x)D_\gamma z, D_\gamma z) \, dx, \quad u = \omega z, \quad z \in H^1_0(\mathcal{O}).
\]
Hence, the differential expression for the operator \( \tilde{A}_D \) admits a factorization
\[
\tilde{A} = \omega^{-1} D^* \tilde{g} D \omega^{-1}, \quad g = \omega^2 \tilde{g}. \tag{5.4}
\]
Now, we consider the operator \( \tilde{A}_{D,\varepsilon} \) with rapidly oscillating coefficients acting in \( L_2(\mathcal{O}) \) and given by
\[
\tilde{A}_\varepsilon = (\omega^\varepsilon)^{-1} D^* \tilde{g}^\varepsilon D (\omega^\varepsilon)^{-1}, \quad g = \omega^2 \tilde{g}^\varepsilon. \tag{5.5}
\]
with the Dirichlet boundary condition. In the initial terms, expression (5.5) takes the form
\[
\tilde{A}_\varepsilon = D^* \tilde{g}^\varepsilon D + \varepsilon^{-2} \tilde{v}^\varepsilon. \tag{5.6}
\]
Next, let \( A(x) = \text{col} \{ A_1(x), \ldots, A_d(x) \} \), where \( A_j(x) \) are \( \Gamma \)-periodic real-valued functions satisfying (4.1). Let \( \tilde{v}(x) \) and \( \tilde{V}(x) \) be \( \Gamma \)-periodic real-valued functions such that
\[
\tilde{v}, \tilde{V} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2; \quad \int_{\Omega} \tilde{v}(x)\omega^2(x) \, dx = 0. \tag{5.7}
\]
In \( L_2(\mathcal{O}) \), we consider the operator \( \tilde{B}_{D,\varepsilon} \) given formally by the differential expression
\[
\tilde{B}_\varepsilon = (D - A^\varepsilon)^* \tilde{g}^\varepsilon (D - A^\varepsilon) + \varepsilon^{-2} \tilde{v}^\varepsilon + \varepsilon^{-1} \tilde{v}^\varepsilon + \tilde{V}^\varepsilon
\]
with the Dirichlet condition on \( \partial\mathcal{O} \). The precise definition is given in terms of the quadratic form.
We put
\[ v(x) := \tilde{w}(x)\omega^2(x), \quad \nu(x) := \tilde{\nu}(x)\omega^2(x). \]  
(5.8)

By (5.3) and (5.6), we have \( \tilde{B}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}B_{D,\varepsilon}(\omega^\varepsilon)^{-1} \), where the operator \( B_{D,\varepsilon} \) is given by the expression (4.3) with the Dirichlet condition on \( \partial\Omega \); \( g \) is defined by (5.3), and \( \nu, \tilde{\nu} \) are given by (5.5). By (5.7) and the properties of \( \omega \), the coefficients \( v \) and \( \nu \) satisfy (4.2). Then the operator \( B_{D,\varepsilon} \) can be represented in the form (4.1), where \( a_j, j = 1, \ldots, d \), and \( Q \) are constructed in terms of \( g, A, v, \) and \( \nu \) according to (4.5), (4.6).

The constant \( \lambda \) is chosen according to condition (1.16) for the operator with the same coefficients \( g, a_j, j = 1, \ldots, d \), and \( Q \), as the coefficients of \( B_{D,\varepsilon} \), and the coefficient \( Q_0(x) := \omega^2(x) \). Then the operators \( B_{D,\varepsilon} := B_{D,\varepsilon} + \lambda \nu \) and \( B_{D,\varepsilon} := B_{D,\varepsilon} + \lambda Q_0^\varepsilon \) are related by \( \tilde{B}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}B_{D,\varepsilon}(\omega^\varepsilon)^{-1} \).

The following set of parameters is called the “problem data”:
\[
\begin{align*}
   d, \rho, s: \quad & \|\tilde{g}\|_{L_\infty}, \|\tilde{g}^{-1}\|_{L_\infty}, \|A\|_{L^\rho(\Omega)}, \|\tilde{\nu}\|_{L^\rho(\Omega)}, \|\tilde{\nu}\|_{L^s(\Omega)}, \|\tilde{\nu}\|_{L^s(\Omega)}; \quad (5.9) \\
   & \text{the parameters of the lattice } \Gamma; \quad \text{the domain } \Omega.
\end{align*}
\]

5.2. Homogenization of the first initial boundary-value problem for the parabolic equation with strongly singular potential. We apply Proposition 4.1 to the operator \( B_{D,\varepsilon} \) described in Subsection 5.1. We have \( f(x) = \omega(x)^{-1} \), whence, by (5.3), \( f_0 = 1 \) and \( B_{D,\varepsilon} = B_0^\varepsilon \). The coefficients \( g_0^\varepsilon, A_0^\varepsilon, \) and \( \nu_0^\varepsilon \) of the effective operator are constructed in terms of \( g, A, v, \) and \( \nu \) (see (5.5) and (5.8)), as described in Subsection 4.2. We apply the results to homogenization of the solution of the first initial boundary-value problem
\[
\begin{align*}
   & \left\{ \begin{array}{ll}
      \frac{\partial u_\varepsilon}{\partial t}(x,t) &= - (D - A^\varepsilon(x))^\ast g^\varepsilon(x)(D - A^\varepsilon(x))u_\varepsilon(x,t) \\
      & \quad - (\varepsilon^{-2}\tilde{\nu}^\varepsilon + \varepsilon^{-1}\tilde{\nu}^\varepsilon)^\ast(\tilde{\nu}^\varepsilon(x) + \lambda \nu_\varepsilon(x,t))u_\varepsilon(x,t), \\
      u_\varepsilon(\cdot,0) &= \omega^\varepsilon(x)^{-1}\varphi(x), & x \in \Omega, & t > 0;
   \end{array} \right.
\end{align*}
\]
Here \( \varphi \in L_2(\Omega) \). (For simplicity, we consider a homogeneous equation.) Then \( u_\varepsilon(\cdot, t) = e^{-\tilde{B}_{D,\varepsilon}t}(\omega^\varepsilon)^{-1}\varphi \).

Let \( u_0 \) be the solution of the homogenized problem
\[
\begin{align*}
   & \left\{ \begin{array}{ll}
      \frac{\partial u_0}{\partial t}(x,t) &= -(D - A^0)^\ast g^0(D - A^0)u_0(x,t) - (\nu^0 + \lambda)u_0(x,t), \\
      u_0(\cdot,0) &= 0, & x \in \Omega, & t > 0; \\
      u_0(\cdot,0) |_{\partial\Omega} &= 0, & t > 0; \\
      u_0(x,0) &= \varphi(x), & x \in \Omega.
   \end{array} \right.
\end{align*}
\]

Proposition 4.1 implies the following result.

Proposition 5.1. Suppose that the assumptions of Subsection 5.2 are satisfied. Denote
\[
\begin{align*}
   \tilde{v}_\varepsilon(\cdot, t) &= u_0(\cdot, t) + \varepsilon \Psi^\varepsilon \nabla u_0(\cdot, t) + \varepsilon\tilde{\Lambda}^\varepsilon u_0(\cdot, t), \\
   \tilde{\varphi}_\varepsilon(\cdot, t) &= \tilde{g}^\varepsilon \nabla u_0(\cdot, t) + g^\varepsilon(\nabla \tilde{\Lambda})^\varepsilon u_0(\cdot, t).
\end{align*}
\]
Then for \( 0 < \varepsilon \leq \varepsilon_1 \) we have
\[
\begin{align*}
   & \|\omega^\varepsilon\|^{-1}u_\varepsilon(\cdot, t) - u_0(\cdot, t)\|_{L_2(\Omega)} \leq C_{15}\varepsilon(t + \varepsilon^2)^{-1/2}e^{-c_0t/2}\|\varphi\|_{L_2(\Omega)}, & t \geq 0; \\
   & \|\omega^\varepsilon\|^{-1}u_\varepsilon(\cdot, t) - \tilde{v}_\varepsilon(\cdot, t)\|_{H^1(\Omega)} \leq C_{18}(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_0t/2}\|\varphi\|_{L_2(\Omega)}, \\
   & \|g^\varepsilon\nabla(\omega^\varepsilon)^{-1}u_\varepsilon(\cdot, t) - \tilde{\varphi}_\varepsilon(\cdot, t)\|_{L_2(\Omega)} \leq \tilde{C}_{18}(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_0t/2}\|\varphi\|_{L_2(\Omega)}, \\
   & t > 0. \quad \text{The constants } C_{15}, C_{18}, \text{ and } \tilde{C}_{18} \text{ depend on the problem data (5.9).}
\end{align*}
\]

Note that, in the presence of a strongly singular potential in the equation, not the solution \( u_\varepsilon \), but the product \( (\omega^\varepsilon)^{-1}u_\varepsilon \) admits a “good approximation.” This shows that the nature of the results of (3) and (4) is different.
APPENDIX

In Appendix, we consider the case where \( d \geq 3 \) and prove the statements about removal of the smoothing operator \( S_\varepsilon \) in the case of sufficiently smooth boundary (Lemma 2.9 and Theorem 2.10) and in the case of a strictly interior subdomain (Lemma 2.10 and Theorem 2.17).

6. The properties of the matrix-valued functions \( \Lambda \) and \( \widetilde{\Lambda} \)

We need the following results; see [PSu, Lemma 2.3] and [MSu2, Lemma 3.4].

**Lemma 6.1.** Let \( \Lambda \) be the \( \Gamma \)-periodic solution of problem (1.25). Then for any function \( u \in C_0^\infty(\mathbb{R}^d) \) and \( \varepsilon > 0 \) we have

\[
\int_{\mathbb{R}^d} |(D\Lambda)^\varepsilon(x)|^2|u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \int_{\mathbb{R}^d} |\Lambda^\varepsilon(x)|^2 |Du(x)|^2 \, dx.
\]

The constants \( \beta_1 \) and \( \beta_2 \) depend on \( m, d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \) and \( \|g^{-1}\|_{L_\infty} \).

**Lemma 6.2.** Let \( \widetilde{\Lambda} \) be the \( \Gamma \)-periodic solution of problem (1.33). Then for any function \( u \in C_0^\infty(\mathbb{R}^d) \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\int_{\mathbb{R}^d} |(D\widetilde{\Lambda})^\varepsilon(x)|^2|u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{H^1(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \int_{\mathbb{R}^d} |\widetilde{\Lambda}^\varepsilon(x)|^2 |Du(x)|^2 \, dx.
\]

The constants \( \beta_1 \) and \( \beta_2 \) depend only on \( n, d, \alpha_0, \alpha_1, \rho, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty} \), the norms \( \|\alpha_j\|_{L_\rho(\Omega)} \), \( j = 1, \ldots, d \), and the parameters of the lattice \( \Gamma \).

Below in \( \S 7 \) we will need the following multiplier properties of the matrix-valued functions \( \Lambda(x) \) and \( \widetilde{\Lambda}(x) \).

**Lemma 6.3.** Suppose that a matrix-valued function \( \Lambda(x) \) is the \( \Gamma \)-periodic solution of problem (1.25). Let \( d \geq 3 \) and \( l = d/2 \).

1. For \( 0 < \varepsilon \leq 1 \) and \( u \in H^{l-1}(\mathbb{R}^d; \mathbb{C}^m) \) we have \( \Lambda^\varepsilon u \in L_2(\mathbb{R}^d; \mathbb{C}^m) \) and

\[
\|\Lambda^\varepsilon u\|^2_{L_2(\mathbb{R}^d)} \leq C(0) \|u\|^2_{H^{l-1}(\mathbb{R}^d)}.
\]

2. For \( 0 < \varepsilon \leq 1 \) and \( u \in H^l(\mathbb{R}^d; \mathbb{C}^m) \) we have \( \Lambda^\varepsilon u \in H^l(\mathbb{R}^d; \mathbb{C}^m) \) and

\[
\|\Lambda^\varepsilon u\|^2_{H^l(\mathbb{R}^d)} \leq C(1) \varepsilon^{-1} \|u\|^2_{L_2(\mathbb{R}^d)} + C(2) \|u\|^2_{H^l(\mathbb{R}^d)}.
\]

The constants \( C(0), C(1), \) and \( C(2) \) depend on \( m, d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty} \), and the parameters of the lattice \( \Gamma \).

**Proof.** It suffices to check (6.1) and (6.2) for \( \varepsilon = 0, \varepsilon = \varepsilon_0 \). Substituting \( x = \varepsilon y, \varepsilon_0 = 1/d, \) we obtain

\[
\|\Lambda^\varepsilon u\|^2_{L_2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |\varepsilon(x)|^2|u(x)|^2 \, dx = \int_{\mathbb{R}^d} |\Lambda(y)|^2|U(y)|^2 \, dy
\]

\[
= \sum_{a \in \Gamma} \int_{\Omega + a} |\Lambda(y)|^2|U(y)|^2 \, dy \leq \sum_{a \in \Gamma} \|\Lambda\|^2_{L_2(\Omega)} \|U\|^2_{L_2(\Omega + a)},
\]

where \( \nu^{-1} + (\nu')^{-1} = 1 \). We choose \( \nu \) so that the embedding \( H^l(\Omega) \hookrightarrow L_{2\nu}(\Omega) \) is continuous, i. e., \( \nu = d(d - 2)^{-1} \). Then

\[
\|\varepsilon\|^2_{L_2(\Omega)} \leq c_\Omega \|\Lambda\|^2_{H^l(\Omega)},
\]

where the constant \( c_\Omega \) depends only on the dimension \( d \) and the lattice \( \Gamma \). We have \( 2\nu = d \). Since the embedding \( H^{l-1}(\Omega) \hookrightarrow L_{2\nu}(\Omega) \) is continuous, we have

\[
\|U\|^2_{L_2(\Omega + a)} \leq c_\Omega' \|U\|^2_{H^{l-1}(\Omega + a)},
\]

where the constant \( c_\Omega' \) depends only on the dimension \( d \) and the lattice \( \Gamma \). Now, from (6.3)–(6.5) it follows that

\[
\int_{\mathbb{R}^d} |\Lambda^\varepsilon(x)|^2|u(x)|^2 \, dx \leq c_\Omega c_\Omega' \|\Lambda\|^2_{H^{l}(\Omega)} \|U\|^2_{H^{l-1}(\Omega + a)}.
\]
Obviously, for \(0 < \varepsilon \leq 1\) we have \(\|U\|_{H^{l-1}(\mathbb{R}^d)} \leq \|u\|_{H^{l-1}(\mathbb{R}^d)}\). Combining this with (6.3) and (1.28), we see that

\[
\int_\mathbb{R}^d |\Lambda^\varepsilon(x)|^2 |u(x)|^2 \, dx \leq c \Omega c' M^2 \|u\|^2_{H^{l-1}(\mathbb{R}^d)}, \quad u \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m),
\]

which proves estimate (6.1) with the constant \(C(0) := (c \Omega c')^{1/2} M\).

Next, by Lemma 6.1,

\[
\|D(\Lambda^\varepsilon u)\|^2_{L^2(\mathbb{R}^d)} \leq 2\varepsilon^{-2} \int_\mathbb{R}^d |(\nabla \Lambda^\varepsilon) u(x)|^2 \, dx + 2 \int_\mathbb{R}^d |\Lambda^\varepsilon(x)|^2 |Du(x)|^2 \, dx
\]

\[
\leq 2\beta_1 \varepsilon^{-2} \int_\mathbb{R}^d |u(x)|^2 \, dx + 2(1 + \beta_2) \int_\mathbb{R}^d |\Lambda^\varepsilon(x)|^2 |Du(x)|^2 \, dx.
\]

From (6.7) (with \(u\) replaced by the derivatives \(\partial_j u\)) it follows that

\[
\int_\mathbb{R}^d |\Lambda^\varepsilon(x)|^2 |Du(x)|^2 \, dx \leq c \Omega c' M^2 \|u\|^2_{H^{l-1}(\mathbb{R}^d)}, \quad u \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m).
\]

As a result, relations (6.7), (6.8) imply inequality (6.2) with the constants \(C(1) := (2\beta_1)^{1/2}\) and \(C(2) := M(3 + 2\beta_2)^{1/2}(c \Omega c')^{1/2}\). \(\square\)

Using the extension operator \(P_\Omega\) satisfying estimates (1.40), we deduce the following statement from Lemma 6.3 (1°).

**Corollary 6.4.** Suppose that the assumptions of Lemma 6.3 are satisfied. Then the operator \(\Lambda^\varepsilon\) is continuous from \(H^{l-1}(\mathcal{O}; \mathbb{C}^m)\) to \(L^2(\mathcal{O}; \mathbb{C}^n)\) and

\[
\|\Lambda^\varepsilon\|_{H^{l-1}(\mathcal{O}) \to L^2(\mathcal{O})} \leq C(0) C(1)^{l-1}.\]

The following statement can be checked similarly to Lemma 6.3 by using Lemma 6.2 and estimate (1.34).

**Lemma 6.5.** Suppose that a matrix-valued function \(\tilde{\Lambda}(x)\) is the \(\Gamma\)-periodic solution of problem (1.33). Let \(d \geq 3\) and \(l = d/2\).

1°. For \(0 < \varepsilon \leq 1\) and \(u \in H^{l-1}(\mathbb{R}^d; \mathbb{C}^m)\) we have \(\tilde{\Lambda}^\varepsilon u \in L^2(\mathbb{R}^d; \mathbb{C}^n)\) and

\[
\|\tilde{\Lambda}^\varepsilon u\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}(0) \|u\|_{H^{l-1}(\mathbb{R}^d)}.
\]

2°. For \(0 < \varepsilon \leq 1\) and \(u \in H^l(\mathbb{R}^d; \mathbb{C}^n)\) we have \(\tilde{\Lambda}^\varepsilon u \in H^l(\mathbb{R}^d; \mathbb{C}^n)\) and

\[
\|\tilde{\Lambda}^\varepsilon u\|_{H^l(\mathbb{R}^d)} \leq \tilde{C}(1) \varepsilon^{-1} \|u\|_{H^l(\mathbb{R}^d)} + \tilde{C}(2) \|u\|_{H^l(\mathbb{R}^d)}.
\]

The constants

\[
\tilde{C}(0) := (c \Omega c')^{1/2} \tilde{M}, \quad \tilde{C}(1) := (2\beta_1)^{1/2}, \quad \tilde{C}(2) := \sqrt{2}(\tilde{\beta}_2 + 1)^{1/2}(c \Omega c')^{1/2} \tilde{M}
\]

depend only on the problem data (1.9).

Using the extension operator \(P_\Omega\), we deduce the following corollary from Lemma 6.5 (1°).

**Corollary 6.6.** Under the assumptions of Lemma 6.5, the operator \(\tilde{\Lambda}^\varepsilon\) is continuous from \(H^{l-1}(\mathcal{O}; \mathbb{C}^m)\) to \(L^2(\mathcal{O}; \mathbb{C}^n)\) and

\[
\|\tilde{\Lambda}^\varepsilon\|_{H^{l-1}(\mathcal{O}) \to L^2(\mathcal{O})} \leq \tilde{C}(0) C(1)^{l-1}.
\]
7. Removal of the smoothing operator in the corrector 
in the case of sufficiently smooth boundary

7.1. Proof of Lemma 2.9 Suppose that the assumptions of Lemma 2.9 are satisfied. Let \( u_0 \) be given by (2.3), where \( \varphi \in L_2(O; \mathbb{C}^n) \). We put

\[
\tilde{u}_0(\cdot, t) = P_O u_0(\cdot, t).
\]

According to (2.22) and (2.32), we have

\[
\begin{align*}
K_D(t; \varepsilon) & = (\Lambda^\varepsilon S_\varepsilon b(D) + \tilde{\Lambda}^\varepsilon S_\varepsilon) \tilde{u}_0(\cdot, t), \\
K_0^D(t; \varepsilon) & = (\Lambda^\varepsilon b(D) + \tilde{\Lambda}^\varepsilon) u_0(\cdot, t).
\end{align*}
\]

We need to estimate the following value

\[
\|K_D(t; \varepsilon) \varphi - K_0^D(t; \varepsilon) \varphi\|_{H^{1}(O)} \leq \|\Lambda^\varepsilon((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)}
\]

\[
+ \|\tilde{\Lambda}^\varepsilon((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)}.
\]

Under the above assumptions, by Lemma 2.8, we have \( u_0 \in H^{l+1}(O; \mathbb{C}^n) \), whence \( \tilde{u}_0 \in H^{l+1}(\mathbb{R}^d; \mathbb{C}^n) \). This gives us possibility to apply Lemma 6.3(2') to estimate the first summand in the right-hand side of (7.3):

\[
\|\Lambda^\varepsilon((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq C^{(1)} \varepsilon^{-1}\|((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{L_2(\mathbb{R}^d)}
\]

\[
+ C^{(2)} \|((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^0(\mathbb{R}^d)}.
\]

where \( l = d/2 \). The first term in the right-hand side of (7.4) is estimated with the help of Proposition 1.1 and relations (1.3), (1.43), (1.46), (2.3), and (2.8):

\[
\varepsilon^{-1}\|((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{L_2(\mathbb{R}^d)} \leq r_1 \|D^b(D)\tilde{u}_0(\cdot, t)\|_{L_2(\mathbb{R}^d)}
\]

\[
\leq r_1 \alpha_1^{1/2} C^{(3)} \|u_0(\cdot, t)\|_{H^2(\mathbb{R}^d)} \leq C^{(3)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)}.
\]

where \( C^{(3)} := r_1 \alpha_1^{1/2} C^{(3)} \varepsilon^{-1/2} \|f\|_{L_\infty} \). To estimate the second term in the right-hand side of (7.4), we apply (1.2) and (1.3):

\[
\|((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^0(\mathbb{R}^d)} \leq 2 \alpha_1^{1/2} \|\tilde{u}_0(\cdot, t)\|_{H^{l+1}(\mathbb{R}^d)}.
\]

By (1.3), (1.46), (2.3), and Lemma 2.8,

\[
\|\tilde{u}_0(\cdot, t)\|_{H^{l+1}(\mathbb{R}^d)} \leq C^{(l+1)} \tilde{C}_{l+1} \|f\|_{L_\infty} \leq C^{(l+1)} \tilde{C}_{l+1} \|f\|_{L_\infty}.
\]

From (1.3) and (1.4) it follows that

\[
\|((S_\varepsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^0(\mathbb{R}^d)} \leq C^{(4)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)}.
\]

where \( C^{(4)} := 2 \alpha_1^{1/2} C^{(l+1)} \tilde{C}_{l+1} \|f\|_{L_\infty} \). Now we estimate the second term in the right-hand side of (7.3). By Lemma 6.3(2'),

\[
\|\tilde{\Lambda}^\varepsilon((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq C^{(1)} \varepsilon^{-1}\|((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)}
\]

\[
+ C^{(2)} \|(S_\varepsilon - I)\tilde{u}_0(\cdot, t)\|_{H^0(\mathbb{R}^d)} \leq C^{(5)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)}; \quad C^{(5)} := r_1 C^{(3)} \|f\|_{L_\infty}.
\]

The first summand in the right-hand side of (7.9) is estimated by using Proposition 1.1 and relations (1.3), (1.46), (2.3), (2.8):

\[
\varepsilon^{-1}\|((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq r_1 C^{(2)} \|u_0(\cdot, t)\|_{H^2(O)} \leq C^{(5)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)};
\]

\[
C^{(5)} := r_1 C^{(3)} \|f\|_{L_\infty}.
\]

The second summand in (7.9) is estimated with the help of (1.2) and (7.7):

\[
\|((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq 2 \|\tilde{u}_0(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq 2 \|\tilde{u}_0(\cdot, t)\|_{H^{l-1}(\mathbb{R}^d)}
\]

\[
\leq C^{(6)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)}; \quad C^{(6)} := 2 C^{(l+1)} \tilde{C}_{l+1} \|f\|_{L_\infty}.
\]

As a result, relations (7.8)–(7.10) and (7.3)–(7.11) imply that

\[
\|K_D(t; \varepsilon) \varphi - K_0^D(t; \varepsilon) \varphi\|_{H^{1}(O)} \leq (C^{(7)} \varepsilon^{-1} + C^{(8)} \varepsilon^{-1/2} \|\varphi\|_{L_2(O)}.
\]
where \( l = d/2 \), \( C^{(7)} := C^{(1)}C^{(3)} + \tilde{C}(1)C^{(5)}, \) and \( C^{(8)} := C^{(2)}C^{(4)} + \tilde{C}(2)C^{(6)}. \) This proves estimate (2.38) with the constant \( \tilde{C}_d := \max\{C^{(7)}; C^{(8)}\}. \)

7.2. Proof of Theorem 2.10. Inequality (2.37) directly follows from (2.24) and (2.33). Here-with, \( C_d := 2(\tilde{C}_d + C_{16}). \) Above, we took into account that for \( t > 1 \) the term \( ct^{-1} \) does not exceed \( \epsilon^{1/2}t^{-3/4} \), and for \( t \leq 1 \) it does not exceed \( \epsilon t^{-d/4-1/2} \) since \( d \geq 3. \)

Let us consider (2.38). By (2.37) and (1.4),

\[
\left\| g^\varepsilon b(D) \left( f^\varepsilon e^{-\tilde{B}_0^\varepsilon t} f^\varepsilon(\varepsilon)^{\varepsilon} - \tilde{B}_0^\varepsilon f^\varepsilon \right) - \varepsilon \left( \Lambda^\varepsilon b(D) + \tilde{B}_0^\varepsilon \right) f^\varepsilon \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) \right\|_{L^2 \to L^2} \leq \|g\|_{L_\infty}(d\alpha_1)^{1/2} C_d(\epsilon^{1/2}t^{-3/4} + \epsilon t^{-d/4-1/2})e^{-\epsilon t/2}.
\]

We have

\[
\varepsilon g^\varepsilon b(D) \left( \Lambda^\varepsilon b(D) + \tilde{B}_0^\varepsilon \right) f^\varepsilon \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) f^\varepsilon = g^\varepsilon \left( (b(D)\Lambda)^\varepsilon + (b(D)\tilde{B}_0^\varepsilon)^\varepsilon \right) f^\varepsilon \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) f^\varepsilon + \varepsilon \sum_{k,j=1}^d g^\varepsilon b_k^\varepsilon b_j^\varepsilon D_k^\varepsilon D_j^\varepsilon f^{\varepsilon} \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) f^{\varepsilon}.
\]

The norm of the second summand in the right-hand side of (7.13) is estimated with the help of (1.4), (1.43), Lemma 2.8, and Corollary 6.4

\[
\varepsilon \left\| \sum_{k,j=1}^d g^\varepsilon b_k^\varepsilon b_j^\varepsilon D_k^\varepsilon D_j^\varepsilon f^\varepsilon \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) f^\varepsilon \right\|_{L^2(O) \to L^2(O)} \leq C^{(9)} \varepsilon t^{-((l+1)/2)}e^{-\epsilon t/2},
\]

where \( l = d/2 \), \( C^{(9)} := c_1 d C^{(6)} C^{(l+1)} \tilde{C}_l \|g\|_{L_\infty} \|f\|_{L_\infty}^2 \). The third summand in the right-hand side of (7.13) is estimated by using (1.4), (1.43), Lemma 2.8, and Corollary 6.4

\[
\varepsilon \left\| \sum_{j=1}^d g^\varepsilon b_j^\varepsilon D_j^\varepsilon f^{\varepsilon} \left( \tilde{B}_0^\varepsilon f^\varepsilon \right) f^\varepsilon \right\|_{L^2(O) \to L^2(O)} \leq C^{(10)} \varepsilon t^{-((l+1)/2)}e^{-\epsilon t/2},
\]

where \( l = d/2 \) and

\[
C^{(10)} := (d\alpha_1)^{1/2} \tilde{C}_d C^{(l+1)} \tilde{C}_l \|g\|_{L_\infty} \|f\|_{L_\infty}^2.
\]

As a result, relations (7.12)-(7.15) imply inequality (2.38) with the constant \( \tilde{C}_d := \|g\|_{L_\infty}(d\alpha_1)^{1/2}C_d + C^{(9)} + C^{(10)}. \)

8. Removal of the smoothing operator in the reformulation of a strictly interior subdomain

8.1. One property of the operator \( S_\varepsilon \). Now we proceed to estimates in a strictly interior subdomain. We start with one simple property of the operator \( S_\varepsilon \).

Let \( \mathcal{O}' \) be a strictly interior subdomain of the domain \( \mathcal{O} \), and let \( \delta \) be given by (1.61). Denote \( \mathcal{O}'' := \{x \in \mathcal{O} : \text{dist} \{x; \partial \mathcal{O}\} > \delta/2\}, \quad \mathcal{O}''' := \{x \in \mathcal{O} : \text{dist} \{x; \partial \mathcal{O}\} > \delta/4\} \).

Lemma 8.1. Let \( S_\varepsilon \) be the operator (1.1). Let \( \mathcal{O}_r := \text{diam} \Omega \). Suppose that \( \mathbf{v} \in L^2(\mathbb{R}^d; \mathbb{C}^m) \) and \( \mathbf{v} \in H^s(\mathcal{O}''); \mathbb{C}^m \) with some \( s \in \mathbb{Z}_+ \). Then for \( 0 < \varepsilon \leq (4\mathcal{O}_r)^{-1}s \) we have \( S_\varepsilon \mathbf{v} \in H^s(\mathcal{O}'''; \mathbb{C}^m) \)

\[
\|S_\varepsilon \mathbf{v}\|_{H^s(\mathcal{O}''')} \leq \|\mathbf{v}\|_{H^s(\mathcal{O}''')}.
\]

Proof. According to (1.1),

\[
\|S_\varepsilon \mathbf{v}\|_{H^s(\mathcal{O}''')} = |\Omega|^{-2} \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha \mathbf{v}(x - \varepsilon z)|^2 dz \leq |\Omega|^{-1} \sum_{|\alpha| \leq s} \int_{\Omega} \int_{\Omega} |D^\alpha \mathbf{v}(x - \varepsilon z)|^2 dz.
\]

Since \( 0 < \varepsilon \leq (\delta/4) \), for \( \mathbf{x} \in \mathcal{O}'' \) and \( \mathbf{z} \in \Omega \) we have \( \mathbf{x} - \varepsilon \mathbf{z} \in \mathcal{O}''' \). Hence, changing the order of integration in (8.1), we obtain the required estimate.
8.2. A cut-off function $\chi(x)$. We fix a smooth cut-off function $\chi(x)$ such that

$$
\chi \in C_0^\infty(\mathbb{R}^d), \quad 0 \leq \chi(x) \leq 1; \quad \chi(x) = 1, \ x \in \mathcal{O}';
$$

$$
supp \chi \subset \mathcal{O}''; \quad |D^\alpha \chi(x)| \leq \kappa_\sigma \delta^{-\sigma}, \quad |\alpha| = \sigma, \quad \sigma \in \mathbb{N}. \quad (8.2)
$$

The constants $\kappa_\sigma$ depend only on $d$, $\sigma$, and the domain $\mathcal{O}$.

**Lemma 8.2.** Suppose that $\chi(x)$ is a cut-off function satisfying conditions (8.2). Let $k \in \mathbb{Z}_+$. 1°. For any function $v \in H^k(\mathbb{R}^d; \mathbb{C}^m)$ we have

$$
\|\chi v\|_{H^k(\mathbb{R}^d)} \leq C_k^{(11)} \sum_{j=0}^k \delta^{-(k-j)} \|v\|_{H^j(\mathcal{O}')}.
$$

(8.3) 2°. For any function $v \in H^{k+1}(\mathbb{R}^d; \mathbb{C}^m)$ we have

$$
\|\chi v\|_{H^{k+1}(\mathbb{R}^d)} \leq C_{k+1/2}^{(11)} \left( \sum_{j=0}^{k+1} \delta^{-(k+1-j)} \|v\|_{H^j(\mathcal{O}')} \right)^{1/2} \left( \sum_{i=0}^k \delta^{-(k-i)} \|v\|_{H^i(\mathcal{O}')} \right)^{1/2}.
$$

(8.4)

The constants $C_k^{(11)}$ and $C_{k+1/2}^{(11)}$ depend on $d$, $k$, and the domain $\mathcal{O}$.

**Proof.** Inequality (8.3) follows from the Leibniz formula for the derivatives of the product $\chi v$ and from the estimates for the derivatives of $\chi$ (see (8.2)). To check (8.3), we should also take into account that

$$
\|w\|_{H^{k+1/2}(\mathbb{R}^d)}^2 \leq \|w\|_{H^{k+1}(\mathbb{R}^d)} \|w\|_{H^k(\mathbb{R}^d)}, \quad w \in H^{k+1}(\mathbb{R}^d; \mathbb{C}^m). 
$$

\[\square\]

8.3. **Proof of Lemma 2.16** Suppose that the assumptions of Lemma 2.16 are satisfied. Let $u_0$ be given by (2.3) with $\varphi \in L_2(\mathcal{O}; \mathbb{C}^m)$. According to (1.43) and (2.7), (2.8), we have

$$
\|D u_0(\cdot, t)\|_{L_2(\mathcal{O})} \leq \|u_0(\cdot, t)\|_{H^1(\mathcal{O})} \leq C_{l_2} \|\int L_{\infty}^t e^{-c_0 t/2} \|\varphi\|_{L_2(\mathcal{O})},
$$

(8.5)

$$
\|D u_0(\cdot, t)\|_{H^1(\mathcal{O})} \leq \|u_0(\cdot, t)\|_{H^2(\mathcal{O})} \leq C_{l_2} \|\int L_{\infty}^t e^{-c_0 t/2} \|\varphi\|_{L_2(\mathcal{O})}. \quad (8.6)
$$

Let $\tilde{u}_0 = P_{\mathcal{O}} u_0$. Relations (7.1) and (7.2) remain valid. We need to estimate the following value:

$$
\|\mathcal{K}_D(t; \varepsilon) \varphi - \mathcal{K}_D^0(t; \varepsilon) \|_{H^1(\mathcal{O}')}, \quad \|\mathcal{L}_\varepsilon^\delta \left( (S_{\varepsilon} - I) b(D) \tilde{u}_0 \right) (\cdot, t)\|_{H^1(\mathbb{R}^d)}
$$

(8.7)

Recall (cf. discussion in Subsection 2.10) that $u_0(\cdot, t) \in H^\sigma(\mathcal{O}''; \mathbb{C}^m)$ for any $\sigma \in \mathbb{Z}_+$. Then the function $\tilde{u}_0(\cdot, t)$ satisfies the assumptions of Lemma 8.1 for any $\sigma \in \mathbb{Z}_+$. Hence, $(S_{\varepsilon} \tilde{u}_0)(\cdot, t) \in H^\sigma(\mathcal{O}'''; \mathbb{C}^m)$ for $0 < \varepsilon \leq (4\pi)^{-1} \delta$. Then we can apply Lemma 6.3(2°) to estimate the first summand in the right-hand side of (8.7):

$$
\|\mathcal{L}_\varepsilon^\delta \left( (S_{\varepsilon} - I) b(D) \tilde{u}_0 \right) (\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq C^{(1)} \varepsilon^{-1} \|\chi (S_{\varepsilon} - I) b(D) \tilde{u}_0 (\cdot, t)\|_{L_2(\mathbb{R}^d)} \quad (8.8)
$$

$l = d/2$. The first term in the right-hand side of (8.8) is estimated by using inequality (7.5) (which holds without additional smoothness assumption on $\partial \mathcal{O}$):
If $l = d/2 = k + 1/2$, then
\[
\|\chi b(D)\tilde{u}_0(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C_l^{(11)}(d\alpha_1)^{1/2} \left( \sum_{j=0}^{k+1} \delta^{-(k+1-j)} \|D_u0(\cdot, t)\|_{H^j(\mathcal{O}')} \right)^{1/2} \times \left( \sum_{\sigma=0}^{k} \delta^{-(k-\sigma)} \|D_u0(\cdot, t)\|_{H^{\sigma}(\mathcal{O}')} \right)^{1/2}. \tag{8.12}
\]

The norms of $D_u0(\cdot, t)$ in $L_2(\mathcal{O}; \mathbb{C}^n)$ and in $H^1(\mathcal{O}; \mathbb{C}^n)$ are estimated in (8.5) and (8.6). By (1.43), (2.2), and (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}'$),
\[
\|D_u0(\cdot, t)\|_{H^\sigma(\mathcal{O}'')} \leq C_{\sigma+1}' \|f\|_{L_\infty} 2^\sigma t^{-1/2}(\delta^{-2} + t^{-1})^{\sigma/2} e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})}, \tag{8.13}
\]

$\sigma \geq 2$. Using (8.5), (8.6), and (8.11)–(8.13), we arrive at the inequality
\[
\|\chi b(D)\tilde{u}_0(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C(12) t^{-1/2}(\delta^{-2} + t^{-1})^{d/4} e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})}. \tag{8.14}
\]

The constant $C(12)$ depends only on the problem data (1.9). To estimate the first term in the right-hand side of (8.10), we apply Lemmas 8.1 and 8.2. Assume that $0 < \epsilon \leq (4\tau_1)^{-1} \delta$. By (1.4), in the case of integer $l$, we have
\[
\|\chi(S_\epsilon b(D)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C_l^{(11)}(d\alpha_1)^{1/2} \sum_{\sigma=0}^{l} \delta^{-(l-\sigma)} \|D_u0(\cdot, t)\|_{H^{\sigma}(\mathcal{O}')} \tag{8.15}
\]

The norms of $D_u0(\cdot, t)$ in $L_2(\mathcal{O}; \mathbb{C}^n)$ and in $H^1(\mathcal{O}; \mathbb{C}^n)$ are estimated in (8.5) and (8.6). By (1.43), (2.2) and (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}'$)
\[
\|D_u0(\cdot, t)\|_{H^\sigma(\mathcal{O}')} \leq C_{\sigma+1}' \|f\|_{L_\infty} 4^\sigma t^{-1/2}(\delta^{-2} + t^{-1})^{\sigma/2} e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})} \tag{8.16}
\]

$\sigma \geq 2$. From (8.5), (8.6), (8.15), and (8.16), it follows that
\[
\|\chi(S_\epsilon b(D)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C(13) t^{-1/2}(\delta^{-2} + t^{-1})^{d/4} e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})} \tag{8.17}
\]

The constant $C(13)$ depends only on the problem data (1.9). Estimate (8.17) in the case of half-integer $l$ is checked similarly. Combining (8.8), (8.10), (8.14), and (8.17), we estimate the first summand in the right-hand side of (8.7):
\[
\|A^{\varphi} \chi((S_\epsilon - I)b(D)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C(14) \left( t^{-1} + t^{-1/2}(\delta^{-2} + t^{-1})^{d/4} \right) e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})} \tag{8.18}
\]

Here $C(14) := \max\{C(1)C(3); C(2)(C(12) + C(13))\}$. The second summand in the right-hand side of (8.7) is estimated by Lemma 6.5 (2): \[
\|A^{\varphi} \chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C(15) \epsilon^{-1} \|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} + \tilde{C}(2)\|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)}, \tag{8.19}
\]

where $l = d/2$. To estimate the first summand in the right-hand side of (8.19), we use (8.2) and inequality (1.10) (which holds without extra smoothness assumption on the boundary):
\[
\epsilon^{-1} \|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq \epsilon^{-1} \|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} + \epsilon^{-1} \|D\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{L_2(\mathbb{R}^d)} \leq C(5) t^{-1} e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})} + \epsilon^{-1} \kappa_1 \delta^{-1} \|(S_\epsilon - I)\tilde{u}_0(\cdot, t)\|_{L_2(\mathbb{R}^d)}. \]

Combining this with Proposition 1.11 and relations (1.43), (1.40), (2.2), and (2.7), we obtain
\[
\|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq C(15) \delta t^{-1/2} + t^{-1}) e^{-\epsilon t/2} \|\varphi\|_{L_2(\mathcal{O})}, \tag{8.20}
\]

where $C(15) := \max\{C(5); \kappa_1 \tau_1 C(1)\}$.

If $l = d/2$ is integer, the second summand in the right-hand side of (8.19) is estimated by analogy with (8.15):
\[
\|\chi((S_\epsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^l(\mathbb{R}^d)} \leq 2C_l^{(11)} \sum_{\sigma=0}^{l} \delta^{-(l-\sigma)} \|u_0(\cdot, t)\|_{H^{\sigma}(\mathcal{O}')} \tag{8.21}
\]
$0 < \varepsilon \leq (4r_1)^{-1}\delta$. The norms of $u_0$ in $L_2(\mathcal{O}; C^0)$, $H^1(\mathcal{O}; C^0)$, and $H^2(\mathcal{O}; C^0)$ are estimated by Lemma 2.1 and relations (1.43), (2.3). For $\sigma \geq 3$ the norm $\|u_0(\cdot, t)\|_{H^\sigma(\mathcal{O}''')}^\prime$ is estimated by using (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}''$):

$$
\|u_0(\cdot, t)\|_{H^\sigma(\mathcal{O}''')} \leq C\sigma_{\sigma+1}\|f\|_{L^\infty}^4 \varepsilon t^{-1/2}(\delta^{-2} + t^{-1})^{\sigma/2}e^{-c_1t/2}\|\varphi\|_{L_2(\mathcal{O})},
$$

Combining these arguments with (8.21), we deduce that

$$
\|\chi((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq C(16)t^{-1/2}(\delta^{-2} + t^{-1})^{d/4}e^{-c_1t/2}\|\varphi\|_{L_2(\mathcal{O})},
$$

with the constant $C(16)$ depending only on the problem data (1.9). For the case of half-integer $l$, estimate (8.22) is checked similarly. As a result, relations (8.19), (8.20), and (8.22) imply the following estimate for the second summand in the right-hand side of (7.7):

$$
\|\tilde{A}^x\chi((S_\varepsilon - I)\tilde{u}_0)(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leq \bar{C}(1)(\delta^{-1}t^{-1/2} + t^{-1})e^{-c_1t/2}\|\varphi\|_{L_2(\mathcal{O})} + \bar{C}(2)C(16)l^{-1/2}(\delta^{-2} + t^{-1})^{d/4}e^{-c_1t/2}\|\varphi\|_{L_2(\mathcal{O})}.
$$

Together with (8.7) and (8.18), this implies inequality (2.47) with the constant $C_d'' := C(14) + \bar{C}(1)C(15) + \bar{C}(2)C(16)$. We have taken into account that the term $\delta^{-1}t^{-1/2}$ does not exceed $t^{-1/2}(\delta^{-2} + t^{-1})^{d/4}$.

8.4. Proof of Theorem 2.17. Inequality (2.48) follows directly from (2.44) and (2.47). Herewith, $C_d := \max\{C_{20}; C_{21}\} + C_d''$. Let us check (2.49). From (2.48), (1.4), and (2.52) it follows that

$$
\|g^\varepsilon b(D)(f^\varepsilon e^{-\tilde{B}_dB_t}f^\varepsilon)^* - (I + \varepsilon A^x b(D) + \varepsilon \tilde{A}^x) f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0)\|_{L_2(\mathcal{O})\rightarrow L_2(\mathcal{O}')} \leq \|g\|_{L_\infty(\mathbb{R}^d)}(\alpha_1)^{1/2}C_\varepsilon h_d(\delta; t)e^{-c_1t/2}.
$$

Let us apply identity (7.13). The norm of the second summand in the right-hand side of (7.13) is estimated with the help of (1.4), (8.2), and Lemma 6.3 (13):

$$
\varepsilon \|\sum_{k,j=1}^d g^\varepsilon b_k A^x b_j D_k D_j f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0\|_{L_2(\mathcal{O})\rightarrow L_2(\mathcal{O}')} \leq \varepsilon \alpha_1 \|g\|_{L_\infty(\mathbb{R}^d)}\sum_{k,j=1}^d \|\chi D_k D_j f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0\|_{L_2(\mathcal{O})\rightarrow H^{l-1}(\mathbb{R}^d)}, \quad l = d/2.
$$

Next, we apply Lemma 8.2. If $l$ is integer, (1.43) yields

$$
\sum_{k,j=1}^d \|\chi D_k D_j f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0\|_{L_2(\mathcal{O})\rightarrow H^{l-1}(\mathbb{R}^d)} \leq \sum_{k,j=1}^d \|\chi D_k D_j f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0\|_{L_2(\mathcal{O})\rightarrow H^{l-1}(\mathbb{R}^d)}, \quad l = d/2.
$$

The norm $\|f_0e^{-\tilde{B}_D^{\tilde{t}}t}\|_{L_2(\mathcal{O})\rightarrow H^2(\mathbb{R}^d)}$ satisfies (2.8). If $i \geq 1$, relations (1.43) and (2.45) (with $\mathcal{O}$ replaced by $\mathcal{O}''$) imply that

$$
\|f_0e^{-\tilde{B}_D^{\tilde{t}}t}\|_{L_2(\mathcal{O})\rightarrow H^{i+2}(\mathbb{R}^d)} \leq C_{i+2}\|f\|_{L_\infty}2^{i+1}t^{-1/2}(\delta^{-2} + t^{-1})^{(i+1)/2}e^{-c_1t/2}.
$$

Combining this with (2.8), (8.24), and (8.25), we obtain

$$
\varepsilon \|\sum_{k,j=1}^d g^\varepsilon b_k A^x b_j D_k D_j f_0e^{-\tilde{B}_D^{\tilde{t}}t}f_0\|_{L_2(\mathcal{O})\rightarrow L_2(\mathcal{O}')} \leq C(17)\varepsilon t^{-1/2}(\delta^{-2} + t^{-1})^{d/4}e^{-c_1t/2},
$$

where the constant $C(17)$ depends only on the problem data (1.9). If $l$ is half-integer, inequality (8.26) is checked by using Lemma 8.2 (20).
The third summand in the right-hand side of (8.23) is estimated similarly by using (1.4), (8.2), Lemma 6.5(1°), and Lemma 8.2. As a result, we obtain
\[ \sum_{j=1}^{d} \|g\|_{L_{\infty}(d\alpha_{1})}^{1/2} \|_{\mathcal{D}} f_{0} e^{-\mathcal{R}_{0}t} f_{0} \|_{L_{2}(\Omega) \rightarrow L_{2}(\Omega^{*})} \leq C(18) \varepsilon t^{-1/2} (\delta^{-2} + t^{-1})^{d/4} e^{-c_{5}t/2}. \]  

Here the constant $C(18)$ depends only on the problem data (1.9).

Finally, relations (1.27), (7.13), (8.23), (8.26), and (8.27) imply inequality (2.49) with the constant $C(17) + C(18)$. \qed

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