Finite action solutions of $SO(2, 1)$ Hitchin’s equations

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Abstract

We present a 1-parameter family of finite action solutions to the $SO(2, 1)$ Hitchin’s equations and explore some of its basic properties. For a fixed value of the parameter, the solution is smooth. We conclude by showing a multi-particle generalization of our basic solutions.
1 Introduction

This brief paper is dedicated to find a special class of explicit solutions of the self-dual Yang-Mills equations. These are originally defined on euclidean 4-space; the physically relevant solutions are the ones with finite action and are called instantons. The same equations may be dimensionally reduced to euclidean 3-space by imposing invariance under translations in one direction. Such equations are also physically relevant, for its finite action solutions are interpreted as magnetic monopoles. If we take one step further and consider the solutions which are invariant under translations along two directions we obtain a set of equations in the plane with no clear physical meaning.

Indeed, it is conjectured that there are no finite action solutions whatsoever to these equations, with gauge group $SU(2)$ (see [H]). For instance, if one tries to apply the so-called 't Hooft ansatz [JNR], which produces the basic solutions to the higher dimensional instanton and monopole equations, one will soon run into trouble. Recall that the 't Hooft ansatz reduces the self-dual Yang-Mills equation to find a nowhere vanishing solution to the Laplace's equation. In dimensions 4 and 3, the fundamental solution of the laplacian are proportional to $\frac{1}{r^2}$ and $\frac{1}{r}$, respectively; but in dimension 2, such solution is logarithmic, what makes the ansatz useless.

Nonetheless, these equations were extensively studied by Hitchin [H] for gauge group $SU(2)$, leading to the discovery of very interesting mathematical structures; similar results were generalized to gauge groups $SU(n)$ by other authors.

We now take a different path, and show the existence of finite action solutions to the so-called Hitchin’s equations with gauge group $SO(2,1)$. Although there is no clear physical meaning attached to these solutions, we expect that the solutions here presented might inspire the study of gauge theories with non-compact gauge groups, as well as the search for less obvious physical interpretations.

Hitchin’s equations. We begin with a brief review of Hitchin’s equations. Let $\mathbb{R}^4$ be parametrised by coordinates $(x, y; u, v)$. Consider a $SO(2,1)$ bundle $E \longrightarrow \mathbb{R}^4$ with a connection, whose entries are $\mathfrak{so}(2,1)$ matrices depend-
ing only on the two first coordinates:

\[ A = A_1(x, y)dx + A_2(x, y)dy + \phi_1(x, y)du + \phi_2(x, y)dv \]  

(1.1)

Now let \( \Phi = \phi_1 + i\phi_2 \) and \( dz = du - idv \), hence:

\[ A = A_1(x, y)dx + A_2(x, y)dy + \Phi(x, y)dz + \Phi^*(x, y)d\overline{z} \]  

(1.2)

Let \( \tilde{A} = A_1dx + A_2dy \); the self-duality equations are then given by:

\[
\begin{cases}
F_{\tilde{A}} + [\Phi, \Phi^*] = 0 \\
\partial_{\tilde{A}} \Phi = 0
\end{cases}
\]  

(1.3)

which under this form are known as Hitchin’s equations; these are now regarded as equations on a two-dimensional plane. Here, \( F_{\tilde{A}} \) denotes the curvature component along the \( (x, y) \) plane. Conformal invariance of the self-duality equations implies that such equations also make sense over any Riemann surface.

2 \( SO(2, 1) \) solutions

First, let us recall some of the elementary properties of the group \( SO(2, 1) \) and its Lie algebra \( \text{so}(2, 1) \). Consider the usual Pauli matrices:

\[
\sigma_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]  

(2.1)

which represent the generators of \( \text{su}(2) \) in \( \text{End}(\mathbb{C}^2) \) and satisfy the following commutation relations:

\[
\begin{cases}
[\sigma_1, \sigma_2] = \sigma_3 \\
[\sigma_1, \sigma_3] = -\sigma_2 \\
[\sigma_2, \sigma_3] = \sigma_1
\end{cases}
\]  

(2.2)

The Lie algebra \( \text{so}(2, 1) \) is defined by slightly different relations:

\[
\begin{cases}
[\tau_1, \tau_2] = \tau_3 \\
[\tau_1, \tau_3] = -\tau_2 \\
[\tau_2, \tau_3] = -\tau_1
\end{cases}
\]  

(2.3)
and can be represented in $\text{End}(\mathbb{C}^2)$ as:

\[
\tau_1 = \sigma_1 \quad \tau_2 = i\sigma_2 \quad \tau_3 = i\sigma_3
\]  

(2.4)

Note that choosing this representation is equivalent to regard the bundle $E \rightarrow \mathbb{R}^4$ as a rank 2 complex bundle.

Finally, recall that both $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ generate the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$; $\mathfrak{su}(2)$ is its unique compact real sub-algebra and $\mathfrak{so}(2, 1)$ is one of its non-compact real form. Hence, one can think of $\mathfrak{so}(2, 1)$ as a 3-dimensional plane sitting diagonally in the 6-dimensional space $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2) = \mathfrak{so}(2, 1) \oplus i\mathfrak{so}(2, 1)$.

As a group, $SO(2, 1)$ is clearly not compact; its maximal compact subgroup is the 1-parameter subgroup generated by $\tau_1$. $SO(2, 1)$ is therefore homotopically equivalent to $S^1$, and has classifying space $BSO(2, 1) = \mathbb{C}P^\infty$.

**Non-compactness issues.** One of the problems of working with a non-compact group is that the Killing form of its algebra is not negative definite. In fact, using the commutation relations (2.3) one quickly verifies that the Killing form of $SO(2, 1)$ is given by:

\[
< \tau_i, \tau_j > = \eta_{ij} = \frac{1}{2} \text{diag}(+ + -)
\]  

(2.5)

In particular, this implies that the action:

\[
\mathcal{S}(A) = -\int \langle F_A \wedge *F_A \rangle
\]  

(2.6)

is also indefinite, but it is gauge-invariant. Nonetheless, the solutions here presented will be shown to have strictly positive action density, when computed with the natural pairing above.

An alternative approach would be to note that although $SO(2, 1)$ is non-compact, its algebra has a negative definite bilinear form, and we might use such form to compute the action (2.6). The pairing we have in mind is:

\[
< \tau_i, \tau_j > = \text{Tr}(\tau_i \overline{\tau_j}) = -\frac{1}{2} \delta_{ij}
\]  

(2.7)

where by $\overline{\tau_j}$ we mean the complex conjugate to $\tau_j$. Now, the expression (2.6) is always strictly positive. The problem is that this pairing has no invariant
meaning and depends on a choice of basis on $\mathbb{C}^2$. Hence, it might happen that a gauge transformation of a connection with finite action result in a connection with divergent action.

In the next paragraph, we present our 1-parameter family of solutions to (1.3) with gauge group $SO(2, 1)$ and then we proceed to compute its action with respect the Killing form (2.3).

**The ansatz.** Our starting point is the following ansatz; reparametrize $(x, y)$, the first two coordinates of $\mathbb{R}^4$, by polar coordinates $(r, \theta)$ and consider the connection:

$$A = f(r).\tau_1 d\theta + g(r).\tau_2 du + h(r).\tau_3 dv$$

(2.8)

Note that such ansatz is more general than it seems, for most connections can be put in this form after gauge transformations. First, gauge away the $dr$ component; then, if the three remaining components are linearly independent, apply constant gauge transformations so that they lie along the generators of the algebra.

In terms of the fields involved on Hitchin’s formulation, we have that the ansatz:

$$\tilde{A} = -yf(x, y).\tau_1 dx + xf(x, y).\tau_1 dy \quad \Phi = g(x, y).\tau_2 + ih(x, y).\tau_3$$

(2.9)

The equations (1.3) are then given by:

$$\begin{cases}
\frac{1}{r} \frac{df}{dr} - gh = 0 \\
\frac{dg}{dr} + \frac{1}{r} fh = 0 \\
\frac{dh}{dr} + \frac{1}{r} fg = 0
\end{cases}$$

(2.10)

To integrate this system, suppose that $g = h$, change coordinates to $r = e^{-t}$ and let $F = 1 - f$ and $G = e^{-t}g$. Then (2.10) are reduced to:

$$\begin{cases}
\frac{dF}{dt} - G^2 = 0 \\
\frac{dG}{dt} + FG = 0
\end{cases}$$

(2.11)

whose solutions are

$$F(t) = c \tanh(ct) \quad G(t) = c \text{sech}(ct)$$

(2.12)
for any constant $c > 0$. Changing these back to the $r$ coordinate, get:

$$f(r) = \frac{(1-c)^2 c}{1+r^{2c}} \quad g(r) = h(r) = 2c r^{c-1} \quad (2.13)$$

and substituting these in (2.8) we have:

$$A = \frac{(1-c) - (1+c)r^{2c}}{1+r^{2c}} \tau_1 d\theta + 2c \frac{r^{c-1}}{1+r^{2c}} (\tau_2 du + \tau_3 dv)$$

which is the promised 1-parameter family of solutions to the $SO(2,1)$ Hitchin’s equations.

Note that this is not the unique solution to the system (1.3). One could, for instance, set $g = -h$ after the same change of coordinates, but setting $F = f + 1$, obtain:

$$\begin{cases}
\frac{dF}{dt} + G^2 = 0 \\
\frac{dG}{dt} + FG = 0
\end{cases} \quad (2.15)$$

whose solutions are:

$$F(t) = c \coth(ct) \quad G(t) = c \csch(ct) \quad (2.16)$$

for any $c > 0$. In terms of the original coordinates, we get:

$$f(r) = \frac{(c-1)+(c+1)r^{2c}}{1+r^{2c}} \quad g(r) = h(r) = 2c r^{c-1} \quad (2.17)$$

but the singularity at $r = 1$ makes this solution useless for our purposes.

3 Some properties.

We analyze some of the properties of the solutions (2.13). First, we show that they have finite action. Then, we observe that for $c = 1$ the solution (2.13) is smooth but singular for any other value of $c$. Finally, we compute its holonomy around an arbitrarily large disc centered at the origin and show multi-particle generalizations of our smooth solution.
Computing the action. We use the Killing form (2.5) to compute (2.6) and show that although the bilinear form (2.5) is not positive definite, the action of (2.14) has positive density. Indeed, plugging (2.14) into (2.6), we have:

\[ S(A) = \int_{\mathbb{R}^2} \frac{1}{r^2} \left( \frac{df}{dr} \right)^2 dxdy = 2\pi c^4 \int_0^\infty \frac{r^{4c}}{r^3(1 + r^{2c})^4} dr \] (3.1)

which is clearly convergent for any \( \frac{1}{2} < c < \infty \). In particular, if \( c = 1 \), we have:

\[ S(A) = 2\pi \int_0^\infty \frac{r}{(1 + r^2)^4} dr = \frac{\pi}{3} \] (3.2)

As we pointed out before, the above quantity has invariant meaning and is independent of the choice of gauge. Note also that all connections of the form (2.8) have strictly positive action.

Smooth and singular solutions. In Euclidean coordinates, the solution (2.14) can be written as:

\[ A = \frac{(1 - c) - (1 + c)(x^2 + y^2)^c}{1 + (x^2 + y^2)^c} \cdot \frac{-y\tau_1 dx + x\tau_1 dy}{x^2 + y^2} + \]

\[ + \frac{c(x^2 + y^2)^{c-1}}{1 + (x^2 + y^2)^c} \cdot (\tau_2 du + \tau_3 dv) \] (3.3)

and one can see that \( A \) is smooth if and only if \( c = 1 \). Writing it explicitly, the smooth, finite energy solution of Hitchin’s equations that motivated the present paper:

\[ \tilde{A} = \frac{2}{1 + x^2 + y^2} (-ydx + xdy)\tau_1 \quad \Phi = \frac{1}{1 + x^2 + y^2} (\tau_2 + i\tau_3) \] (3.4)

For other values of the parameter, such that \( c > 1 \), then \( A \) has a singularity of codimension 2 at \( x = y = 0 \) of type \( \frac{1}{r} \). Such singular solutions were studied by several authors in dimensions two and four (see, for instance, [KM] and [FHP]), and the interested reader should refer to these works. This type of singular field configuration is also known as meron, for they carry fractional topological charge.
Limiting holonomy. As mentioned before, we want to compute the holonomy of (2.14) around an arbitrarily large circle centered at the origin. More precisely, we want to solve the initial value problem parametrized by the radial distance $r$ for $\gamma_r : S^1 \to SO(2,1)$:

$$\begin{cases}
\frac{d}{dr} \gamma_r + A_\theta \gamma_r = 0 \\
\gamma(0) = I
\end{cases} \quad (3.5)$$

Using expression (2.13), it is easy to see that:

$$\lim_{r \to \infty} \gamma_r(\theta) = \begin{pmatrix}
\exp \left( \frac{1}{2} i (1 + c) \theta \right) & 0 \\
0 & \exp \left( -\frac{1}{2} i (1 + c) \theta \right)
\end{pmatrix} \quad (3.6)$$

Such procedure also allows us to characterize multi-particle solutions, just like the usual multi-instanton solutions on $\mathbb{R}^4$ are characterized through the degree of a mapping from the 3-sphere at infinity to $SU(2) \equiv S^3$. The difference in the present case is that $\gamma_\infty(\theta)$ represents a map from the circle at infinity of the plane to $S^1 \subset SO(2,1)$, regarded as the maximal compact subgroup of $SO(2,1)$, which, as we have mentioned before, classifies $SO(2,1)$-bundles topologically.

Multi-instanton solutions. Again, fix $c = 1$. The smooth solution (3.4) might be generalized as follows:

$$A = \sum_{k=1}^{N} \frac{2}{1 + (x - x_k)^2 + (y - y_k)^2} \cdot \frac{-(y - y_k)\tau_1 dx + (x - x_k)\tau_1 dy}{(x - x_k)^2 + (y - y_k)^2} + \frac{2}{1 + (x - x_k)^2 + (y - y_k)^2} \left(\tau_2 du + \tau_3 dv\right) \quad (3.7)$$

The $\gamma_\infty(\theta)$ map (3.6) associated to this solution is then given by:

$$\begin{pmatrix}
\exp(iN\theta) & 0 \\
0 & \exp(-iN\theta)
\end{pmatrix} \quad (3.8)$$

whose degree is $N$. Following the above analogy, we can interpret solutions of the form (3.7) as a multi-particle solution. Each point $(x_k, y_k) \in \mathbb{R}^2$ corresponds to the position of a particle.
It hence easy to conclude that the space of solutions of the \( SO(2,1) \) Hitchin’s equations (1.3), up to gauge equivalence, is at least a \( 2N \)-dimensional manifold, parametrized by the coordinates \((x_k, y_k)\).

A still more general non-smooth, multi-particle solution can be obtained by superposing instantons with different values of the parameter \( c \):

\[
A = \sum_{k=1}^{N} \frac{(1 - c_k) - (1 + c_k)((x - x_k)^2 + (y - y_k)^2)^{c_k}}{1 + ((x - x_k)^2 + (y - y_k)^2)^{c_k}} \cdot
\]
\[
- (y - y_k)\tau_1 dx + (x - x_k)\tau_1 dy +
\]
\[
+ 2c_k \frac{((x - x_k)^2 + (y - y_k)^2)^{c_k-1}}{1 + ((x - x_k)^2 + (y - y_k)^2)^{c_k}} (\tau_2 du + \tau_3 dv) \quad (3.9)
\]

The interpretation of the \( \gamma_\infty(\theta) \) map (3.6) now is less obvious, but it may be understood as counting fractionally charged particles.

**Conclusion.** We have shown that although there are no known finite action \( SU(2) \) solutions of Hitchin’s equations, it is possible to write down finite action solutions of the \( SO(2,1) \) version of these equations. It seems likely that the non-compactness of the structural group has a deeper role. Such issue is certainly worth of further investigation. For instance, are there finite action \( SU(n) \) solutions, for \( n \geq 2 \)? What about the non-compact forms \( SO(p, n - p) \)?

Another point would be to determine the correct framework which gives a natural interpretations to the equations and solutions presented above. One might also expect to find some relation with the well-known phenomenon of fractional statistics on \( \mathbb{R}^2 \).

On the mathematical side, it is interesting to ask how the Chern-Weil theory adapts to non-compact gauge groups, what would fit in the bigger programme of understanding the gauge theory of non-compact Lie groups. The \( SO(2,1) \) seems a good choice for acquiring some intuition on such an unexplored subject.

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