ON THE MINIMAL MODEL OF SEMI-ISOGENOUS MIXED SURFACES

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ABSTRACT. The aim of this paper is to determine minimal models of the semi-isogenous mixed surfaces with $\chi = 1$ and $K^2 > 0$ constructed in [CF18]. In order to do this, we further develop the idea of orbit divisors introduced in [FL19], to construct effective divisors on surfaces isogenous to a product of mixed type, extending it to the semi-isogenous mixed surfaces.

INTRODUCTION

When we construct a “new” surface of general type, one of the first natural questions is: “Is it minimal?”. In other words, “does it contains exceptional curves of the first kind? If yes, what is its minimal model?”. These questions are natural ones, being the first step to addressing classification and geographical problems. Although we know that a minimal model exists, to determine it, also to identify the $(-1)$-curves on our model, is in general not an easy task.

In recent years there has been growing interest in those surfaces birational to the quotient of the product of two curves by the action of a finite group and several new surfaces of general type with $p_g = q$ have been constructed in this way: see [BC04, BCG08, BCGP12, BP12, BP16, Fra13, FP15] for $p_g = 0$, [CP09, Pol08, Pol09, MP10, FP15] for $p_g = 1$, [Pen11, Zuc03] for $p_g = 2$.

In these articles the authors work under the assumption that the group action is free outside of a finite set of points, i.e. the induced map onto the quotient is quasi-étale in the sense of [Cat07].

Recently the research moved to the non-quasi-étale case. In [CF18] the following situation was considered: let $C$ be a smooth projective curve of genus $g(C)$ and $G$ a finite subgroup of $\text{Aut}(C)^2 \rtimes \mathbb{Z}_2$ whose action is mixed, i.e. there are elements in $G$ exchanging the two natural isotrivial fibrations of $C \times C$. Let $G^0 \triangleleft G$ be the index two subgroup $G \cap \text{Aut}(C)^2$, i.e. the subgroup consisting of those elements that do not exchange the factors. Assuming that $G^0$ acts freely on $C \times C$ then $X := (C \times C)/G$ is smooth and we call it a semi-isogenous mixed surface (see Section 1). The basic example of such a surface is the symmetric product $C(2)$ of a curve $C$, e.g. $\mathbb{P}^1(2) \cong \mathbb{P}^2$.

Dropping any assumption on the $G$-action on $C \times C$ the mixed quotient $X := (C \times C)/G$ may be singular and the minimal resolution $S \to X$ of its singularities is called mixed surface.

In the above mentioned papers, the minimality of the new surfaces, respectively the construction of their minimal model, has been proved using ad hoc arguments (e.g. see [MP10, BP12, FP15]). In particular, in [FP15, Theorem 4.5] we proved that any irregular mixed quasi-étale surface of general type is minimal, and by [Pig20, Theorem 3] any mixed surface with irregularity $q \geq 3$ is minimal.

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By [Bea82, CCML98, HP02, Pir02] we have a complete classification of minimal surfaces of general type with \( p_g = q \geq 3 \): among them there exists a unique family of semi-isogenous mixed surfaces: 
\[ S = C_{\frac{1}{3}}^{(2)} \], where \( C_3 \) is a genus 3 curve \( (p_g(S) = q(S) = 3, K_S^2 = 6) \).

In [CF18] we classified semi-isogenous mixed surfaces having \( K^2 > 0 \) and \( p_g = q \leq 2 \). Besides \( \mathbb{P}^2 \) there are 15 other families with \( p_g = q = 0, K^2 \in \{2, 6, 8\} \); 35 families with \( p_g = q = 1, K^2 \in \{2, 4, 6, 7, 8\} \); 9 families with \( p_g = q = 2, K^2 \in \{2, 4, 6, 7, 8\} \) (see Theorem 1.12). All surfaces in these 59 families are of general type.

We then proved that if \( X \) is a semi-isogenous mixed surface of general type with \( \chi(X) = 1 \) and \( K_X^2 \geq 6 \), then \( X \) is minimal (cf. Proposition 1.13). In most of the remaining cases we were able to use other classification results to conclude that the surfaces are not minimal.

The aim of this paper is to determine the minimal model in the open cases, giving an explicit description of the \((-1\))-curves (if any). We will prove the following.

**Theorem** (=Theorem 1.1). Let \( X := (C \times C)/G \) be a semi-isogenous mixed surface of general type with \( p_g(X) = q(X), \vert G^0 \vert \leq 2000, \neq 1024 \) and let \( X_{\min} \) be its minimal model.

- If \( K_X^2 \geq 6 \), then \( X \) is minimal.
- If \( K_X^2 = 4 \), then \( X \) is not minimal and \( K_{X_{\min}}^2 = 5 \).
- If \( K_X^2 = 2 \), then \( X \) is not minimal and \( K_{X_{\min}}^2 = 4 \).

To search for exceptional curves of the first kind on semi-isogenous mixed surfaces, we generalize the technique of “orbit divisors” (see Section 2) introduced in [FL19], to construct effective divisors on surfaces isogenous to a product of mixed type.

Let \( X = (C \times C)/G \) be a semi-isogenous mixed surface. Studying divisors on \( X \) is equivalent to studying \( G \)-invariant divisors on \( C \times C \). Therefore, we want to construct several \( G \)-invariant divisors on \( C \times C \) in order to increase our chances of finding \((-1\))-curves. Let \( H \) be a subgroup of \( \text{Aut}(C) \) containing \( G^0 \) and for each \( h \in H \), let \( \Delta_h = \{(x, hx) \mid x \in C\} \) denote its graph. The \( G \)-action on \( C \times C \) induces an action on the set \( \{\Delta_h \mid h \in H\} \), and from the \( G \)-orbits of this set of graphs, we obtain several \( G \)-invariant divisors on \( C \times C \), which descend to effective divisors on \( X \). We call these divisors orbit divisors induced by \( H \). Since we can compute the intersection numbers of these \( G \)-invariant divisors on \( C \times C \), we can compute self-intersection and intersection with \( K_X \) of the orbit divisors, and possibly identify \((-1\)-curves.

We remark that under certain assumptions (Proposition 2.4), this approach is the unique way to produce rational curves on a mixed quotient, and we raise the question whether this holds in general.

A natural candidate for \( H \) is the group \( G^0 \), but this does not always produce enough curves on \( X \). For example, if the curve \( C \) is a covering of an elliptic curve \( E \) branched over 2 points of multiplicity 2, or an étale covering of a curve \( C' \) of genus 2, we need to consider a larger group of automorphisms.

To do this we show that the involution of \( E \) swapping the 2 branch points (resp. the hyperelliptic involution of \( C' \)) lifts to an automorphism of \( C \) compatible with the \( G^0 \) action, whence \( \text{Aut}(C) \) contains the subgroup \( H \) generated by \( G^0 \) and a lift of the involution (see Section 3).

The main theorem is then proved (Section 4) case by case, constructing many orbit divisors and checking their intersection numbers. Some of the computations are performed using the MAGMA [BCP97] script available at [http://www.staff.uni-bayreuth.de/~bt301744/].

**Notation.** We work over \( \mathbb{C} \) and use the standard notation in surface theory: for a smooth projective surface \( X \) we denote by \( p_g(X) := h^2(X, \mathcal{O}_X) \) its geometric genus, by \( q(X) := h^1(X, \mathcal{O}_X) \) its irregularity,
by \( \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) \) its holomorphic Euler-Poincaré characteristic, and by \( K_X^2 \) the self-intersection of its canonical divisor. For a smooth compact curve (Riemann surface) \( C \) we denote by \( g(C) \) its genus.

We shall also borrow a standard notation from group theory: we denote by \( \mathbb{Z}_n \) the cyclic group of order \( n \), by \( S_n \) the symmetric group on \( n \) letters, by \( Q \) the group of quaternions, by \( D_n \) the dihedral group of order \( 2n \), and by \( D_{p,q,r} \) the group \( \langle x, y | x^p = y^q = 1, xyx^{-1} = y^r \rangle \).

1. Semi-isogenous Mixed Surfaces

In this section we recall the definition of a semi-isogenous mixed surface and a few general results on them; we refer to [CF18, Pig20] for further details.

Let \( C \) be a smooth projective curve of genus \( g(C) \), let \( \sigma \) be the involution on \( C \times C \) swapping the factors: for \( x, y \in C \), \( \sigma(x, y) = (y, x) \), and let \( \text{Aut}(C)^2 < \text{Aut}(C \times C) \) be the subgroup of automorphisms of \( C \times C \) preserving the factors. We denote by \( \text{Aut}(C)(2) \) the smallest subgroup of \( \text{Aut}(C \times C) \) containing \( \text{Aut}(C)^2 \) and \( \sigma \): \( \text{Aut}(C)(2) := \text{Aut}(C)^2 \times \alpha \mathbb{Z}_2, \alpha : 1 \mapsto \sigma \). Note that if \( g(C) \geq 2 \) then \( \text{Aut}(C)(2) = \text{Aut}(C \times C) \) (see [Cat00, Corollary 3.9]) and it is a finite group.

**Definition 1.1.** Let \( G \) be a finite subgroup of \( \text{Aut}(C)(2) \). We say that its action on \( C \times C \) is **mixed** if \( G \) is not contained in \( \text{Aut}(C)^2 \), i.e. there are elements in \( G \) exchanging the two isotrivial fibrations of \( C \times C \).

We denote by \( G^0 \triangleleft G \) the index two subgroup \( G \cap \text{Aut}(C)^2 \), i.e. the subgroup consisting of those elements that preserve the factors.

The action is **minimal** if the group \( G^0 \) acts faithfully on both factors.

**Remark 1.2.** By [Cat00, Proposition 3.13] we may assume (and from now on we do) that the action is minimal. In this case we identify \( G^0 < \text{Aut}(C) \times \text{Aut}(C) \) with its projection onto the first factor.

We have the following description of minimal mixed actions:

**Theorem 1.3** (cf. [Cat00, Proposition 3.16]). Let \( G \) be a finite subgroup of \( \text{Aut}(C)(2) \) whose action is minimal and mixed. Fix \( \tau^i \in G \setminus G^0 \); let \( \tau := \tau^2 \in G^0 \) and let \( \varphi \in \text{Aut}(G^0) \) be defined by \( \varphi(h) := \tau^i h \tau^{i-1} \). Then, up to a coordinate change, \( G \) acts as follows:

\[
\begin{align*}
g(x, y) &= (gx, \varphi(g)y) \\
\tau^i g(x, y) &= (\varphi(g)y, \tau gx)
\end{align*}
\]

for \( g \in G^0 \).

Conversely, for every finite subgroup \( G^0 < \text{Aut}(C) \) and \( G \) extension of degree 2 of \( G^0 \) fixed \( \tau^i \in G \setminus G^0 \) and defined \( \tau \) and \( \varphi \) as above, (1.1) defines a minimal mixed action on \( C \times C \).

**Definition 1.4.** Let \( C \) be a smooth projective curve and let \( G < \text{Aut}(C)(2) \) be a finite group whose action is mixed. The quotient surface \( X := (C \times C)/G \) is a **mixed quotient**, and its minimal resolution of the singularities \( S \to X \) is a **mixed surface**.

Assume that \( G^0 \triangleleft G \) acts freely on \( C \times C \), then the quotient surface \( X := (C \times C)/G \) is called a **semi-isogenous mixed surface**.

If the whole group \( G \) acts freely on \( C \times C \), then \( X \) is a **surface isogenous to a product** of mixed type.

**Proposition 1.5** ([CF18, Corollary 2.11]). Every semi-isogenous mixed surface is smooth.
Let $X := (C \times C)/G$ be a semi-isogenous mixed surface. Since $G^0$ acts freely, any non-trivial element of $G$ having fixed points on $C \times C$ lies in $G \setminus G^0$ and must have order 2.

**Definition 1.6.** Let $O_2 := \{g \in G \setminus G^0 : g^2 = 1\}$, and for each $g \in O_2$ let $R_g$ denote its fixed locus: $R_g := \text{Fix}(g)$, i.e. $R_g$ is the graph of the automorphism $\tau^g \in G^0$: $R_g := \{(x, (\tau^g \cdot x) : x \in C\}$.

We denote by $\text{Cl}(g)$ the conjugacy class of $g \in G$ and we define $\text{Cl}(O_2) := \{\text{Cl}(g) : g \in O_2\}$.

**Proposition 1.7 (cf. [CF18 Section 3]).** Let $X := (C \times C)/G$ be a semi-isogenous mixed surface, and let $\eta: C \times C \to X$ be the quotient map. Then:

i) the ramification locus $\mathcal{R}$ of $\eta$ is the disjoint union of the curves $R_g$, for $g \in O_2$; the stabilizer of $R_g$ is $\langle g \rangle \cong \mathbb{Z}_2$.

ii) the branch locus $\mathcal{B}$ of $\eta$ is the disjoint union $\mathcal{B} = B_{g_1} \cup \cdots \cup B_{g_t}$, where $t := |\text{Cl}(O_2)|$, $\{g_1, \ldots, g_t\}$ is a set of representative of the conjugacy classes in $\text{Cl}(O_2)$ and $B_{g_i} := \eta(R_{g_i})$ is an irreducible smooth curve of genus $g(B_{g_i}) = \frac{2g(C) - 1}{|\text{Cl}(g)|} + 1$.

iii) $X$ has invariants

$$q(X) = g(C/G^0), \quad \chi(O_X) = \frac{(g(C) - 1)^2}{|G|} - \frac{p_a(B) - 1}{2}, \quad K_X^2 = 8\chi(O_X) - (p_a(B) - 1).$$

**Remark 1.8** (see [Pig20 Proposition 1.7]). The formula for the irregularity holds for any mixed surface $S$ resolution of a mixed quotient $S \to (C \times C)/G$, namely $q(S) = g(C/G^0)$.

### 1.1. The Classification of semi-isogenous mixed surface with $\chi = 1$

By Theorem 1.3, a semi-isogenous mixed surface is equivalent to the following data:

1. A curve $C$ together with a faithful $G^0$-action;
2. A degree 2 extension $G$ of $G^0$, such that the induced $G^0$-action on $C \times C$ is free.

The translation into algebraic terms of semi-isogenous mixed surface is then accomplished through the theory of Galois coverings between projective curves (cf. [Mir95 Section III.3, III.4]).

**Definition 1.9.** Given integers $g' \geq 0$, $m_1, \ldots, m_r > 1$ and a finite group $H$, a generating vector for $H$ of type $[g'; m_1, \ldots, m_r]$ is a $(2g' + r)$-tuple of elements of $H$: $V := (d_1, e_1, \ldots, d_{g'}, e_{g'}; h_1, \ldots, h_r)$ such that $V$ generates $H$, $\prod_{i=1}^{g'} [d_i, e_i] \cdot h_1 \cdot h_2 \cdots h_r = 1$ and $\text{ord}(h_i) = m_i$.

**Theorem 1.10** (Riemann Existence Theorem). A finite group $H$ acts as a group of automorphisms of some compact Riemann surface $C$ of genus $g(C)$, if and only if there exists a generating vector $V := (d_1, e_1, \ldots, d_{g'}, e_{g'}; h_1, \ldots, h_r)$ for $H$ of type $[g'; m_1, \ldots, m_r]$, such that the Hurwitz formula holds:

$$2g(C) - 2 = |H| \left(2g' - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}\right).$$

In this case $g'$ is the genus of the quotient Riemann surface $C' := C/H$ and the $H$-covering $c : C \to C'$ is branched over $r$ points $\{x_1, \ldots, x_r\}$ with branching indices $m_1, \ldots, m_r$, respectively.

**Remark 1.11.** The points of the fiber $c^{-1}(x_i)$ are in 1-1 correspondence with the cosets $\{gK_i\}_{g \in H}$ of the cyclic group $K_i := \langle h_i \rangle$ and the stabilizer of the point $y \in c^{-1}(x_i)$ corresponding to $gK_i$ is $gK_i g^{-1}$. 

The stabilizer set $\Sigma_V$ of $V$ is the subset of $H$ consisting of the automorphisms of $C$ having fixed points:

$$\Sigma_V := \bigcup_{g \in H} \bigcup_{j \in \mathbb{Z}} \bigcup_{i=1}^{r} \{g \cdot h_i^j \cdot g^{-1}\}.$$ 

To give a semi-isogenous mixed surface is then equivalent to give the following:

1. A finite group $G^0$, a curve $C'$ (which will be $C/G^0$), $r$ points on $C'$ and a generating vector $V$ for $G^0$ of type $[g(C')]; m_1, \ldots, m_r$ (these give the Galois-covering $C \to C/G^0$).
2. A degree 2 extension $G$ of $G^0$ (this defines the $G$-action on $C \times C$), such that the following holds: let $\varphi \in \text{Aut}(G^0)$ as in Theorem [1.3] then the stabilizer sets $\Sigma_V$ and $\Sigma_{\varphi(V)} (= \varphi(\Sigma_V))$ are disjoint, i.e. $\Sigma_V \cap \Sigma_{\varphi(V)} = \{1\}$ (this ensures that $G^0$ acts freely on $C \times C$).

Hence to classify semi-isogenous mixed surface is equivalent to classify groups with the right properties. This algebraic description has been used in [CF18] to obtain the following classification (cf. also [BCG08, Fra13, CP09, Pen11] for the isogenous case: $K^2 = 8$).

**Theorem 1.12** (see [CF18] Theorems A, B, C).

- Let $X:=(C \times C)/G$ be a semi-isogenous mixed surface with $p_g(X) = q(X) = 0$ and $K_X^2 > 0$, such that $|G^0| \leq 2000, \neq 1024$. Then either $X$ is $\mathbb{P}^2$, or $X$ is of general type.
- The latter form 15 families: 5 with $K^2 = 8$, 8 with $K^2 = 6$ and 2 with $K^2 = 2$.
- The semi-isogenous mixed surfaces with $p_g = q = 1$ and $K^2 > 0$, are of general type and form 35 families: 3 with $K^2 = 8$, 2 with $K^2 = 7$, 12 with $K^2 = 6$, 2 with $K^2 = 4$ and 16 with $K^2 = 2$.
- The semi-isogenous mixed surfaces with $p_g = q = 2$ and $K_X^2 > 0$, are of general type and form 9 families: 1 with $K^2 = 8$, 1 with $K^2 = 7$, 3 with $K^2 = 6$, 1 with $K^2 = 4$, and 3 with $K^2 = 2$.

The condition $|G^0| \leq 2000, \neq 1024$ in Theorem [1.12] is a computational assumption, since its proof relies on the MAGMA [BCP97] database of small groups, which has some technical limitations (cf. [CF18] Remark 6.5)) .

1.2. **On the minimality.** In [CF18] we could only partially answer the question of minimality for the surfaces we constructed. More precisely, using the Hodge Index Theorem we proved:

**Proposition 1.13** ([CF18] Proposition 7.6). Let $X$ be a semi-isogenous mixed surface of general type with invariants $p_g(X) = q(X)$ and let $\rho: X \to X_{\text{min}}$ be the projection to its minimal model. Then

- for $K_X^2 \in \{6, 7, 8\}$, $\rho$ is the identity map: $X = X_{\text{min}}$;
- for $K_X^2 \in \{4, 5\}$, $\rho$ is the contraction of at most one $(-1)$-curve;
- for $K_X^2 \in \{2, 3\}$, $\rho$ is the contraction of at most two $(-1)$-curves;
- for $K_X^2 = 1$, $\rho$ is the contraction of at most three $(-1)$-curves.

In particular if $X$ is a semi-isogenous mixed surface of general type with $\chi(X) = 1$ and $K_X^2 \geq 6$, then $X$ is minimal. In most of the remaining cases we used other classification results to conclude that the surfaces are not minimal, but we did not determine their minimal model, except for 4 families (see [CF18] Section 7)):

- if $p_g(X) = q(X) = 2$ and $K_X^2 = 4$, then $X$ is not minimal and $K_{X_{\text{min}}}^2 = 5$.
- if $p_g(X) = q(X) = 2$ and $K_X^2 = 2$, then $X$ is not minimal and $K_{X_{\text{min}}}^2 = 4$. 

The open cases are collected in Table 1 and 2 and aim of this paper is to explicitly construct \((-1)\)-curves in these cases, in order to determine their minimal model.

**Table 1.** \(p_g = q = 0\)

| \(K^2_X\) | \(G\) | \(Id(G)\) | \(G^0\) | \(Id(G^0)\) | \(g(C)\) | Type | Branch Locus \(B\) | \(H_1(X, \mathbb{Z})\) |
|------------|-------|------------|---------|-------------|---------|------|----------------|----------------|
| 2 \((\mathbb{Z}^2 \times D_4) \times \mathbb{Z}^2\) | \(256, 47930\) | \(\mathbb{Z}^2 \times D_4\) | 128, 1135 | 33 | [0; 2\(^5\)] | (3, −8)\(^3\) | \(\mathbb{Z}^2 \times \mathbb{Z}_4\) |
| 2 \((\mathbb{Z}^2 \times \mathbb{Z}^2_2) \times \mathbb{Z}^2_2\) | \(256, 45303\) | \(\mathbb{Z}^4 \times D_4\) | 128, 1135 | 33 | [0; 2\(^5\)] | (3, −8)\(^2\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_4\) |

**Table 2.** \(p_g = q = 1\)

| \(K^2_X\) | \(G\) | \(Id(G)\) | \(G^0\) | \(Id(G^0)\) | \(g(C)\) | Type | Branch Locus \(B\) | \(H_1(X, \mathbb{Z})\) |
|------------|-------|------------|---------|-------------|---------|------|----------------|----------------|
| 4 \(S_3 \times D_4\) | 48,38 | \(\mathbb{Z}^2_2 \times S_3\) | 24,14 | 13 | [1; 2\(^2\)] | (2, −4), (4, −12) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 4 \(D_{12} \times \mathbb{Z}_2\) | 48,37 | \(\mathbb{Z}_4 \times S_3\) | 24,5 | 13 | [1; 2\(^2\)] | (2, −4), (4, −12) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2 \times \mathbb{Z}_2^3) \times \mathbb{Z}_2\) | 64,153 | \(D_{2,8,5} \times \mathbb{Z}_2\) | 32,7 | 17 | [1; 2\(^2\)] | (3, −8), (5, −16) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \(\mathbb{Z}_4 \times D_4\) | 64,150 | \(D_4 \times \mathbb{Z}_4\) | 32,9 | 17 | [1; 2\(^2\)] | (3, −8), (5, −16) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \(\mathbb{Z}_2^3 \times D_8\) | 64,147 | \(D_4 \times \mathbb{Z}_4\) | 32,9 | 17 | [1; 2\(^2\)] | (2, −4)\(^2\), (5, −16) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2 \times D_8) \times \mathbb{Z}_2\) | 64,128 | \(\mathbb{Z}_2 \times D_8\) | 32,39 | 17 | [1; 2\(^2\)] | (2, −4)\(^2\), (3, −8)\(^2\) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \(Q \times D_4\) | 64,123 | \(\mathbb{Z}_2 \times D_{2,8,3}\) | 32,40 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \(D_4 \times D_4\) | 64,134 | \(\mathbb{Z}_4 \times \mathbb{Z}_2^2\) | 32,43 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2 \times D_4) \times \mathbb{Z}_2^3\) | 64,227 | \(\mathbb{Z}_2 \times \mathbb{Z}_4\) | 32,22 | 17 | [1; 2\(^2\)] | (3, −8)\(^2\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2 \times D_4) \times \mathbb{Z}_2^2\) | 64,227 | \(\mathbb{Z}_2 \times \mathbb{Z}_4\) | 32,22 | 17 | [1; 2\(^2\)] | (3, −8)\(^2\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_4 \times D_4) \times \mathbb{Z}_2\) | 64,228 | \((\mathbb{Z}_4 \times \mathbb{Z}_4) \times \mathbb{Z}_2\) | 32,23 | 17 | [1; 2\(^2\)] | (3, −8)\(^2\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_4 \times D_4) \times \mathbb{Z}_2\) | 64,234 | \((\mathbb{Z}_4 \times \mathbb{Z}_4) \times \mathbb{Z}_2\) | 32,23 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_4 \times D_4) \times \mathbb{Z}_2\) | 64,234 | \(\mathbb{Z}_4 \times \mathbb{Z}_2\) | 32,24 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_4 \times Q) \times \mathbb{Z}_2\) | 64,236 | \(\mathbb{Z}_4 \times \mathbb{Z}_2\) | 32,24 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \(\mathbb{Z}_2^3 \times \mathbb{Z}_2\) | 64,219 | \(\mathbb{Z}_4 \times D_4\) | 32,25 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \(\mathbb{Z}_2^3 \times \mathbb{Z}_2\) | 64,221 | \(\mathbb{Z}_4 \times D_4\) | 32,25 | 17 | [1; 2\(^2\)] | (3, −8)\(^3\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2^3 \times D_4) \times \mathbb{Z}_2\) | 64,213 | \(\mathbb{Z}_4 \times D_4\) | 32,25 | 17 | [1; 2\(^2\)] | (3, −8)\(^2\), (2, −4)\(^2\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |
| 2 \((\mathbb{Z}_2^3 \times \mathbb{Z}_2) \times \mathbb{Z}_2\) | 64,206 | \(\mathbb{Z}_4 \times D_4\) | 32,25 | 17 | [1; 2\(^2\)] | (3, −8), (2, −4)\(^4\) | \(\mathbb{Z}^2_2 \times \mathbb{Z}_2\) |

In Table 1 and 2 every row corresponds to a family and we use the following notation: columns \(Id(G)\) and \(Id(G^0)\) report the MAGMA identifier of the groups \(G\) and \(G^0\): the pair \((a, b)\) denotes the \(b^{th}\) group of order \(a\) in the database of Small Groups.

The column Type gives the type of the generating vector for \(G^0\) in a short form, e.g. \([0; 2^5]\) stands for \((0; 2, 2, 2, 2, 2)\). The Branch Locus \(B\) of \(\eta: C \times C \to X\) is also given in a short form, e.g. \((3, −8)^2, (2, −4)^2\) means that \(B\) consists of 4 curves, two of genus 3 and self-intersection −8 and two of genus 2 and self-intersection −4.

2. **Orbit divisors on Semi-isogenous Mixed Surfaces**

In this section we explain a method to construct effective divisors on a semi-isogenous mixed surface, or more generally on a mixed quotient.
Let $C$ be a smooth curve. For $f \in \text{Aut}(C)$ we denote by $\Delta_f$ its graph:
\[\Delta_f := \{(x, fx) \in C \times C \mid x \in C\}.
\]
Let $G$ be a finite subgroup of $\text{Aut}(C)(2)$ whose action on $C \times C$ is mixed and let $H$ be a subgroup of $\text{Aut}(C)$ containing $G^0$: $G^0 < H < \text{Aut}(C)$. The $G$-action on $C \times C$ induces a $G$-action on the set $\{\Delta_f \mid f \in H\}$:

**Lemma 2.1** ([FL19 Lemma 3.1]). Let $h$ be in $G^0$, then $h(\Delta_f) = \Delta_{\varphi(h)f h^{-1}}$ and $\tau h(\Delta_f) = \Delta_{\tau h f^{-1} \varphi(h^{-1})}$.

For $f \in H$, the sum (taken with the reduced structure) of the divisors in the orbit $\{\gamma(\Delta_f)\}_{\gamma \in G}$ gives the effective divisor
\[\hat{D} := \left(\sum_{\gamma \in G} \gamma(\Delta_f)\right)_{\text{red}} = \Delta_{f_1} + \ldots + \Delta_{f_n},\]
where $n$ is the index of the subgroup $\{\gamma \mid \gamma(\Delta_f) = \Delta_f\} \subset G$.

Let $\eta: C \times C \to X := (C \times C)/G$ be the quotient map. The divisor $\hat{D}$ is $G$-invariant, so it yields an effective divisor $D := \eta_*(\hat{D})_{\text{red}}$ on $X$. Note that $D$ is irreducible, as the image of an irreducible divisor: $D = \eta(\Delta_f)$.

Each $G$-orbit yields a distinct effective divisor on $X$, so we call $D$ the **orbit divisor** induced by $f$, e.g. by Proposition 1.7 the branch curve $B_g$ is the orbit divisor induced by $\tau g \in G^0$.

**Lemma 2.2.** Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and let $R := \sum_{g \in O_2} \Delta_{\tau' g}$ be the ramification divisor of $\eta: C \times C \to X$.

Let $D := \eta_*(\Delta_{f_1} + \ldots + \Delta_{f_m})_{\text{red}}$ and $D' := \eta_*(\Delta_{f'_1} + \ldots + \Delta_{f'_m})_{\text{red}}$ be two distinct orbit divisors on $X$ induced by $f$ and $f'$ respectively. Then
\[D.D' = \frac{\alpha}{|G|} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_{f_i} \cdot \Delta_{f'_j},\]
where $\alpha := 0$ if $D$ and $D'$ are both branch curves, $\alpha := 2$ if only one of them is a branch curve, $\alpha := 1$ if neither of them is a branch curve.

If $D$ is not a branch curve of $\eta$, then
\[
D^2 = \frac{-2(g(C) - 1)n}{|G|} + \frac{2}{|G|} \sum_{1 \leq i < j \leq n} \Delta_{f_i} \cdot \Delta_{f_j},
\]
\[
K_X.D = \frac{4(g(C) - 1)n}{|G|} - \frac{1}{|G|} \sum_{j=1}^{n} \sum_{g \in O_2} \Delta_{\tau' g} \cdot \Delta_{f_j}.
\]
If $D$ is a branch curve of $\eta$, then
\[
D^2 = \frac{-8(g(C) - 1)n}{|G|} + \frac{8}{|G|} \sum_{1 \leq i < j \leq n} \Delta_{f_i} \cdot \Delta_{f_j},
\]
\[
K_X.D = \frac{12(g(C) - 1)n}{|G|}.
\]

**Proof.** If $D$ and $D'$ are branch curves, then $D.D' = 0$ since they are disjoint by Proposition 1.7. Otherwise let $\tilde{D} := \Delta_{f_1} + \ldots + \Delta_{f_n}$ and set $m := 2$ if $D$ is a branch curve for $\eta$, and $m := 1$ if not. By
definition we have $\eta^*D = m\tilde{D}$ and the same goes for $D'$. By the projection formula the first formula follows and we obtain
\[
D^2 = \frac{1}{|G|} (\eta^*D)^2 = \frac{1}{|G|} \left( m \cdot \sum_{i=1}^n \Delta_{f_i} \right)^2 = \frac{m^2}{|G|} \left( \sum_{i=1}^n \Delta_{f_i}^2 + 2 \sum_{1 \leq i < j \leq n} \Delta_{f_i}\Delta_{f_j} \right).
\]

The canonical divisor $K_{C \times C}$ of $C \times C$ is numerically equivalent to $2(g(C) - 1)(F_1 + F_2)$, where $F_1, F_2$ are general fibers of the projection of $C \times C$ respectively onto the first and the second coordinate, whence the values $\Delta_{f_i}^2$ are easily computed using the adjunction formula $K_{C \times C}.\Delta_{f_i} + \Delta_{f_i}^2 = 2(g(C) - 1)$:
\[
K_{C \times C}.\Delta_{f_i} = 2(g(C) - 1)(F_1 + F_2).\Delta_{f_i} = 4(g(C) - 1), \text{ and so } \Delta_{f_i}^2 = -2(g(C) - 1).
\]
Both formulas for $D^2$ follow.

Since $\eta$ is branched along some curves, each with ramification index 2, the ramification formula implies $\eta^*K_X = K_{C \times C} - R$ and the projection formula yields
\[
K_{X}.D = \frac{1}{|G|} \eta^*K_X.\eta^*D = \frac{m}{|G|} K_{C \times C}.\tilde{D} - \frac{m}{|G|} R.\tilde{D}.
\]

The first summand can be easily computed as above: $K_{C \times C}.\tilde{D} = 4(g(C) - 1)n$, so we are left with $R.\tilde{D}$. We distinguish 2 cases.

If $D$ is not a branch curve, then none of the $\Delta_{f_i}$ is a component of $R$, whence
\[
R.\tilde{D} = \sum_{i=1}^n R.\Delta_{f_i} = \sum_{i=1}^n \sum_{g \in O_2} \Delta_{x^g}\Delta_{f_i}.
\]

If $D$ is a branch curve, then the curves $\Delta_{f_i}$ are irreducible components of $R$, and so $\tilde{R} := R - \tilde{D}$ is an effective divisor. By Lemma 1.7 the ramification curves are pairwise disjoint, whence
\[
R.\tilde{D} = (\tilde{D} + \tilde{R}).\tilde{D} = \tilde{D}^2 = \sum_{i=1}^n \Delta_{f_i}^2 = -2(g(C) - 1)n.
\]

\[
\square
\]

Remark 2.3. By Lemma 3.3, if $f_1, f_2$ are different elements of $\text{Aut}(C)$, then their graphs intersect transversally, whence to determine $\Delta_{f_1}, \Delta_{f_2} = |\Delta_{f_1} \cap \Delta_{f_2}| = |\{x \in C \mid f_1(x) = f_2(x)\}|$, it is enough to count the number of points on $C$ fixed by the non-trivial automorphism $f := f_1^{-1}f_2 \in H$.

This can be done by applying the Riemann Existence Theorem (Theorem 1.10) to the covering $C \to C/H$. According to Remark 1.1 (we use here the same notation) the automorphism $f \in H$ fixes the point $gK_j(=: y)$ if and only if $f \in gK_jg^{-1}(=\text{Stab}(y))$. Since every coset has $|K_j| = m_j$ representatives, we obtain
\[
|\text{Fix}(f)| = \sum_{j=1}^r \frac{1}{m_j} \sum_{g \in H} 1_{gK_jg^{-1}}(f),
\]
where $1_{gK_jg^{-1}}$ denotes the indicator function of the set $gK_jg^{-1}$.

We conclude this section showing that, under certain assumption, the unique way to obtain rational curves (and so a $(-1)$-curve) on a mixed quotient $X$ with irregularity $q = 1, 2$ is by considering orbit divisors (cf. Proposition 1.8, Theorem 4.3 for the case $q \geq 3$).
Proposition 2.4. Let $S \to X := (C \times C)/G$ be a mixed surface with irregularity $q := q(S)$ and let $E$ be a rational curve on the mixed quotient $X$. Assume that one of the following holds:

- $q = 2$ and the hyperelliptic involution $\sigma$ of the genus 2 curve $C/G^0$ lifts to an automorphism of $C$;
- $q = 1$ and the involution $\sigma : x \mapsto p - x$ of the elliptic curve $C/G^0$ lifts to an automorphism of $C$.

Then $E$ is the orbit divisor induced by a lift $\hat{\sigma} \in \text{Aut}(C)$ of $\sigma$.

Proof. Let $C'$ be the genus $q$ curve $C/G^0$ and let $\text{Sym}^2(C')$ be its symmetric product.

By construction (cf. [Pig20, Remark 1.6]), there is a finite dominating morphism $\pi : X \to \text{Sym}^2(C')$, which fits into the commutative diagram (all morphism are finite):

$$
\begin{array}{ccc}
C \times C & \xrightarrow{Q} & C' \times C' \\
\eta \downarrow & & \downarrow s \\
X & \xrightarrow{\pi} & \text{Sym}^2(C')
\end{array}
$$

where the map $Q$ is the product map $Q := f \times f$, being $f : C \to C'$ the quotient map.

If $g(C') = 2$ we borrow an argument from [PPT17, Proposition 1.5]. In this case the Abel-Jacobi map $a : \text{Sym}^2(C') \to J(C')$ is the contraction of the rational curve $E'$ given by the divisors in the unique $g_2^1$, and $\text{Sym}^2(C')$ does not contain other rational curves, since $J(C')$ is a torus. This implies $\pi(E) = E'$.

On the other hand $E'$ is the image of $\Gamma := \{(x, i(x)) \mid x \in C'\}$, the graph of the hyperelliptic involution $\sigma$ of $C'$, via the quotient map $s : C' \times C' \to \text{Sym}^2(C')$.

Since every irreducible component of $\eta^{-1}(E)$ dominates $\Gamma$ and the involution $\sigma$ lifts to an automorphism $\hat{\sigma} \in \text{Aut}(C)$, we have that every irreducible component of $\eta^{-1}(E)$ has the form $\{(u, g\hat{\sigma}(u)) \mid u \in C\} = \Delta_{g\hat{\sigma}}$ for $g \in G^0$ and the first claim follows.

If $g(C') = 1$, then the rational curves on $\text{Sym}^2(C')$ are precisely the fibers of the Abel-Jacobi map $a : \text{Sym}^2(C') \to J(C'), \{v, w\} \mapsto v + w$, whence $a(\pi(E)) = \{p\}$, for $p \in C'$. It follows that $\pi(E)$ is the image of $\Gamma := \{(x, p - x) \mid x \in C'\}$, the graph of the involution $\sigma : x \mapsto p - x$ of $C'$, via the quotient map $s : C' \times C' \to \text{Sym}^2(C')$. Arguing as above we get the second claim. $\square$

We pose the question whether a similar description for the rational curves holds, when the involution does not lift to an automorphism of $C$.

3. Lifting Automorphisms

We give now sufficient conditions for an involution on $C/G^0$ to lifting to an automorphism of $C$.

Lemma 3.1. Let $C$ be a smooth projective curve, and let $H^0 < \text{Aut}(C)$ be a finite group acting faithfully on $C$, such that a generating vector $V := (a_1, b_1; c_1, e_2)$ for $H^0$ associated to the Galois covering $f_1 : C \to C/H^0 := E$ is of type $[1; m, m]$, i.e. $E$ is an elliptic curve and $f_1$ is branched in 2 points $p_1, p_2$ with index $m$. Then:

i) the involution $\iota : E \to E, x \mapsto (p_1 + p_2) - x$ lifts to an automorphism $i$ of $C$ compatible with $f_1$, i.e. $f_1 \circ i = \iota \circ f_1$.

ii) the group $H < \text{Aut}(C)$ generated by $H^0$ and $i$ is a degree 2 extension of $H^0$: $H \cong H^0 \times \mathbb{Z}_2$. 

iii) The Galois covering $f : C \rightarrow C/H \cong \mathbb{P}^1$ is branched in 5 points: the image (via the quotient map $f_2 : E \rightarrow E/(i) \cong \mathbb{P}^1$) of $p_1$ with multiplicity $m$ and of the four halves of $(p_1 + p_2)$ with multiplicity 2; i.e the covering has a generating vector of type $[0;2^4,m]$. Moreover, such a generating vector has the form $(-;h_1,h_2,h_3,h_4,c_1)$ where $h_j \in H \setminus H^0$.

Proof. The involution $\iota$ has four fixed points: $\mathbb{P}^1 \setminus E[2]$ the four halves of $p_1 + p_2 \in E$, hence the Galois covering $f_2 : E \rightarrow E/(i) \cong \mathbb{P}^1$ is branched in four points $q_1,q_2,q_3,q_4$, each with ramification index 2. These four points together with $p := f_2(p_1) = f_2(p_2)$ form the branch locus $B \subset \mathbb{P}^1$ of the covering $f := f_2 \circ f_1 : C \rightarrow \mathbb{P}^1$.

Claim. The map $f : C \rightarrow \mathbb{P}^1$ is a Galois covering, with Galois group $H$: a degree 2 extension of $H^0$.

Taking the Claim for granted, the statement follows immediately.

ii) We have $\deg(f) = 2 \cdot |H^0| = |H|$, and each $\eta_i$ has $|H^0|$ preimages, i.e. ramification index $\frac{|H|}{|H^0|} = 2$, while $p$ has $2 \cdot \frac{|H^0|}{m}$ preimages, i.e. ramification index $m$.

The stabilizers of the points in $f^{-1}(p)$ are conjugate to each other in $H$, and by assumption one of them is generated by $c_1$. On the other hand, a fixed point in $f^{-1}(\eta_i)$ has stabilizer of order 2 generated by an element in $H \setminus H^0$, since $H^0$ fixes no points outside $f_1^{-1}(p_1)$ and $f_1^{-1}(p_2)$. Therefore, a generating vector associated to the covering $f$ has the form $(-;h_1,h_2,h_3,h_4,c_1)$, $h_i \in H \setminus H^0$.

i)-ii) Being a degree 2 extension of $H^0$, the group $H$ is generated by the subgroup $H^0$ and a single element of $H \setminus H^0$, which is then a lift of $\iota$ compatible with $f_1$. Moreover there exists a short exact sequence

$$1 \rightarrow H^0 \rightarrow H \rightarrow \mathbb{Z}_2 \rightarrow 1$$

which splits, because $h_i \in H \setminus H^0$ has order 2, whence $H \cong H^0 \rtimes \mathbb{Z}_2$.

Proof of the claim. Let $x' \in \mathbb{P}^1 \setminus B$ and consider the monodromy map

$$\mu : \pi_1(\mathbb{P}^1 \setminus B, x') \rightarrow \mathfrak{S}_F$$

of the covering $f : C \rightarrow \mathbb{P}^1$, where $F := f^{-1}(x')$. Recall that for a loop $\gamma$ in $\mathbb{P}^1 \setminus B$ with base point $x'$, the bijection $\mu(\gamma) \in \mathfrak{S}_F$ of the fiber $F$ is defined as follows: given a point $y \in F$, we map it to the end point $\tilde{\gamma}_y(1)$ of the unique lift $\tilde{\gamma}_y$ of $\gamma$ with starting point $y$.

This map depends on the base point, and to change it, amounts to a relabelling of the element of $F$, hence the image of $\mu$ as abstract group is well defined (see [Mir95, Section III.4]). The covering $f$ is then Galois if and only if $|\text{Im}(\mu)| = |F| = 2|H^0|$, in this case $\text{Im}(\mu)$ is the Galois group of the covering.

Let $x_1$, $x_2$ be the 2 preimages of the base point $x'$ via the map $f_2$, and fix a path $\tilde{\beta} : [0,1] \rightarrow E \setminus f_2^{-1}(B)$ with starting point $\tilde{\beta}(0) = x_1$ and end point $\tilde{\beta}(1) = x_2$.

This path descends to a loop $\beta := f_2 \circ (\tilde{\beta})$ in $\mathbb{P}^1 \setminus B$, and we set $b := \mu(\beta) \in \mathfrak{S}_F$. This permutation $b$ defines a bijection between the following subsets of $F$: $A_1 := f_1^{-1}(x_1)$ and $A_2 := f_1^{-1}(x_2)$, and it induces an isomorphism of the images of the monodromy maps

$$\nu_1 : \pi_1(E \setminus \{p_1,p_2\}, x_1) \rightarrow H^0 \leq \mathfrak{S}_{A_1} ,$$

$$\nu_2 : \pi_1(E \setminus \{p_1,p_2\}, x_2) \rightarrow H^0 \leq \mathfrak{S}_{A_2} .$$
There are 2 types of loops in \( \pi_1(\mathbb{P}^1 \setminus B, x') \): either they lift to loops on \( E \) with starting point \( x_i \) (\( i = 1, 2 \)), or they lift to paths joining \( x_1 \) and \( x_2 \).

Let \( \gamma \in \pi_1(\mathbb{P}^1 \setminus B, x') \) be a loop of the first type and let \( \gamma_1 \) be its lift on \( E \) with base point \( x_i \). By definition of the monodromy maps we have that

\[
\mu(\gamma) = (\nu_1(\gamma_1), \nu_2(\gamma_2)) = (g_{\gamma}, g'_{\gamma}) \in H^0 \times H^0 \leq \mathfrak{S}_A_1 \times \mathfrak{S}_A_2 \leq \mathfrak{S}_F
\]

hence we can identify the pair \((g_{\gamma}, g'_{\gamma})\) with its first entry and the loops of the first type generate a subgroup \( L \cong H^0 \) inside \( \mathfrak{S}_A_1 \times \mathfrak{S}_A_2 \leq \mathfrak{S}_F \).

Note that \( b \) swaps \( A_1 \) and \( A_2 \), so \( b = \mu(\beta) \notin L \). On the other hand the unique lift of \( b^2 \) with starting point \( x_1 \) (or \( x_2 \)) is a loop, whence \( b^2 = \mu(\beta^2) \in L \).

Let now \( \delta \in \pi_1(\mathbb{P}^1 \setminus B, x') \) be a loop, whose unique lift on \( E \setminus f_1^{-1}(B) \) with starting point \( x_i \) is not a loop. Then \( \beta^{-1} \ast \delta \) lifts to loops with base points \( x_i \) and so

\[
\mu(\delta) = \mu(\beta \ast \beta^{-1} \ast \delta) = \mu(\beta) \cdot \mu(\beta^{-1} \ast \delta) = b \cdot (g_{\beta^{-1} \ast \delta}, g'_{\beta^{-1} \ast \delta}) \in b \cdot L .
\]

Finally, we have \( b \cdot (g_{\gamma}, g'_{\gamma}) \cdot b^{-1} = \mu(\beta \ast \gamma \ast \beta^{-1}) = \mu(\gamma') = (g_{\gamma'}, g'_{\gamma'}) \), whence \( L \leq_2 (b, L) \leq \mathfrak{S}_F \).

Since the image \( \text{Im}(\mu) =: H \) of the monodromy map has order \( 2|H^0| = |f_1^{-1}(x')| \), and \( H^0 \leq_2 \text{Im}(\mu) \), the claim is proved. \( \square \)

Remark 3.2. An analogous argument also works in the following cases:

- the generating vector of \( H^0 \) is of type \([2; -]\). In this case we lift the hyperelliptic involution, getting a group \( H \) of order \( 2|H^0| \) and a covering \( C \to C/H \cong \mathbb{P}^1 \) branched in 6 points (images of the 6 Weierstrass points via the hyperelliptic map) with multiplicity 2, i.e. having generating vector of type \([0; 2^6]\).

- the generating vector of \( H^0 \) is of type \([0; m, m, m]\). In this case we lift the symmetric group \( S_3 \): the group of symmetries of \( \mathbb{P}^1 \) with marked points \( 1, \xi := \exp(\frac{2\pi i}{3}) \) and \( \xi^2 \) (cf. [FL19, Section 2.1]). In this way we get a group \( H \) of order \( 6|H^0| \) and a covering \( C \to C/H \cong \mathbb{P}^1 \) with generating vector of type \([0; 2, 3, 2m]\).

4. PROOF OF THE MAIN THEOREM

In this section we prove the following theorem.

**Theorem 4.1.** Let \( X := (C \times C)/G \) be a semi-isogenous mixed surfaces with \( p_g(X) = q(X) \) and \( |G^0| \leq 2000, \neq 1024 \).

- If \( K^2_X = 2 \), then there are two \((-1)\)-curves on \( X \), and \( K^2_{X_{\text{min}}} = 4 \).
- If \( K^2_X = 4 \), then there is one \((-1)\)-curve on \( X \), and \( K^2_{X_{\text{min}}} = 5 \).

**Proof.** We prove the statement case by case, considering a suitable automorphism group \( H : G^0 < H < \text{Aut}(C) \), and constructing the induced orbit divisors. In each case we find the maximal possible number of \((-1)\)-curves predicted by Proposition 1.13.

According to Tables 1 and 2, we have to consider 3 cases.

**Case 1,** \( p_g(X) = q(X) = 0 \). Looking at Table 1, we have to consider 2 families, both with \( K^2_X = 2 \), and determined by a group \( G \) of order 256, a subgroup \( G^0 < G \) of order 128 and a generating vector of type \([0; 2^5]\) for \( G^0 \). As automorphism group \( H \) we consider the group \( G^0 \).
The MAGMA script in Appendix shows that in both cases the 128 curves $\Delta_f$, $f \in G^0$, induce 18 orbit divisors on $X$, and among them there are two $(-1)$-curves.

By Proposition 1.13 there are no further $(-1)$-curves, whence contracting them we get the minimal model $X_{\text{min}}$ of $X$: $K^2_{X_{\text{min}}}$ = 4.

**Case 2**, $p_g(X) = q(X) = 1$, $K^2_X = 4$. Looking at Table 1 we have to consider 2 families, both determined by a group $G$ of order 48, a subgroup $G^0 < G$ of order 24 and a generating vector of type $[1; 2, 2]$ for $G^0$, i.e. the covering $C \to C/G^0$ is branched in 2 points and $C/G^0$ has genus 1.

By Lemma 3.1 there exists a degree 2 extension $H$ of $G^0$ acting as a group of automorphism of $C$.

The supporting MAGMA script available at [http://www.staff.uni-bayreuth.de/~bt301744/] shows that in both cases among the orbit divisors induced by the 48 curves $\Delta_f$, $f \in H$ there is a $(-1)$-curve.

By Proposition 1.13 there are no further $(-1)$-curves, whence contracting it we get the minimal model $X_{\text{min}}$ of $X$: $K^2_{X_{\text{min}}}$ = 4.

**Case 3**, $p_g(X) = q(X) = 1$, $K^2_X = 2$. This case is analogous to Case 2: all families are determined by a group $G$ of order 64, a subgroup $G^0 < G$ of order 32 and a generating vector of type $[1; 2, 2]$ for $G^0$, and there exists a degree 2 extension $H$ of $G^0$ acting as a group of automorphism of $C$.

The MAGMA script shows that in each case among the orbit divisors induced by the 64 curves $\Delta_f$, $f \in H$ there are two $(-1)$-curves.

By Proposition 1.13 there are no further $(-1)$-curves, whence contracting them we get the minimal model $X_{\text{min}}$ of $X$: $K^2_{X_{\text{min}}}$ = 4.

**Remark 4.2.** It is worth mentioning that the technique of orbit-divisors used in the previous proof also works in the case $p_g(X) = q(X) = 2$.

In [CF18, Section 7.1] we explicitly construct a $(-1)$-curve on each semi-isogenous mixed surface with $p_g(X) = q(X) = 2$, $K^2_X = 4$. It turns out that the idea used there is a rough application of the technique of the orbit-divisors.

If $K^2_X = 2$ the existence of two $(-1)$-curves is predicted by Debarre’s inequality (see [Deb82]). Nevertheless we can explicitly construct them. How?

Looking at [CF18, Table 3], all families with $q(X) = 2$ are determined via a generating vector of type $[2; -]$, i.e. the covering $c_1 : C \to C/G^0 = C'$ is unbranched and $g(C') = 2$, hence $C'$ is hyperelliptic. Following Proposition 2.4 we check if the hyperelliptic involution $\sigma : C' \to C'$ lifts to an automorphism of $C$, and it does lift by Remark 3.2. We obtain a Galois covering $C \to \mathbb{P}^1$ branched in 6 points (the images of the 6 Weierstraß points) with ramification index 2 and Galois group $H$: the subgroup of Aut($C$) generated by $G^0$ and a lift of $\iota$.

Considering such a group $H$ as automorphism group of $C$ we find in each case the maximal numbers of exceptional curves of the first kind.

APPENDIX A. MAGMA SCRIPT

```magma
// This function counts the number of fixed points of f in H < Aut(C)
CountingIntersections := function(f, seq, H)
int := 0;
for j in [1..#seq] do c := 0; K := sub<H|seq[j]>
```
for g in H do
    if f in \{g*k*g^{-1} : k in K\} then c:=c+1;
    end if; end for;
    int:=int*(c/#K);
end for; return int;
end function;

// The algebraic data defining the family are
G:=SmallGroup(256, 47930);
gv:=[G.2 * G.5 * G.6 * G.7 * G.8, G.1 * G.3 * G.7, G.2 * G.3 * G.4 * G.5 * G.8, G.1 * G.2 * G.5 * G.6 * G.8, G.2 * G.4 * G.5 * G.6 * G.6 ];

// For the other family replace with:
G:=SmallGroup(256, 45303);
gv:=[G.1 * G.2 * G.3 * G.5 * G.6 * G.7 * G.8, G.3 * G.5 * G.6, G.3 * G.6, G.2 * G.4 * G.5 * G.6 * G.7, G.1 * G.3 * G.4 * G.5 ];*/

// g(C)-1

G0:=sub<G|gv>; t:=Rep({x: x in G | xnotin G0}); // \tau'
O2:={x: x in G| xnotin G0 and Order(x) eq 2};
K:=(t*x: x in O2); // the ramification curves represented by elements in G0

// We construct now the orbit divisors, distinguishing between branch and non-branch curves
Curves:={}; P:=Set(G0);
P:=P diff K; // We discard the branch curves, since we know they are not rational
while not IsEmpty(P) do f:=Rep(P); //it represents the curve Delta_f, graph of the automorphism f of C
    Gamma:=(t*h*t^{-1}*f*h^{-1} : h in G0) join (t^{-2}*h*f^{-1}*t*h^{-1}*t^{-1} : h in G0); // it is the G-orbit of Delta_f
    Append(~Curves, Gamma); P:=P diff Gamma;
end while;

//For each orbit divisor D we compute D^2 and K_X.D, and we save in Curves_1 the (-1)-curves
Curves_1:={};
for a in [1..#Curves] do Gamma:=Curves[a]; n:=#Gamma; self:=-2*n* genus_1; GK:=4*n*genus_1;
    for g1 in Gamma do
        for g2 in Gamma diff \{g1\} do
            int:=CountingIntersections(g1^{-1}*g2,gv,G0); self:=self+int;
            end for;
        end for;
    end for;
    for r in RamCurves do
        intK:=CountingIntersections(g1^{-1}*r,gv,G0); GK:=GK-intK;
        end for;
    end for;
    if self/#G eq -1 and GK/#G eq -1 then Include(~Curves_1, a); end if;
end for;

#Curves_1; // it returns the number of (-1)-curves.

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