Median Area for Broken Sticks

STEVEN FINCH

April 25, 2018

ABSTRACT. Breaking a line segment $L$ in two places at random, the three pieces can be configured as a triangle $T$ with probability $1/4$. We determine both the PDF and CDF for area($T$) in terms of elliptic integrals. In particular, if $L$ has length 1, then the median area $0.031458...$ can be calculated to arbitrary precision. We also mention the analog involving cyclic quadrilaterals – with corresponding probability $1/2$ – and ask some unanswered questions.

For simplicity’s sake, we start with a stick of length 2 (not 1). A triangle, formed by two independent uniform breaks in the stick, has sides $a+b+c=2$ satisfying

$$0 < a < b + c, \quad 0 < b < a + c, \quad 0 < c < a + b$$

hence

$$0 < a < 1, \quad 0 < b < 1, \quad 1 < a + b < 2.$$ 

The joint density function for $(a,b)$ is thus 2 (constant) over the shaded triangular region in Figure 1; the marginal density for $a$ is $2x$ if $0 < x < 1$; the cross-correlation between $a$ and $b$ is $-1/2$. By Heron’s formula, the mean area of the triangle is

$$E(\text{area}) = \int_0^1 \int_0^{1-x} 2\sqrt{(1-x)(1-y)(x+y-1)} \, dy \, dx = \frac{4\pi}{105}$$

and the mean square area is

$$E(\text{area}^2) = \int_0^1 \int_0^{1-x} 2(1-x)(1-y)(x+y-1) \, dy \, dx = \frac{4^2}{960} = \frac{1}{60}.$$ 

In contrast, a cyclic quadrilateral $[2\,3]$, formed by three independent uniform breaks in the stick, has sides $a+b+c+d=2$ in this order satisfying

$$0 < a < b + c + d, \quad 0 < b < a + c + d, \quad 0 < c < a + b + d, \quad 0 < d < a + b + c$$

hence

$$0 < a < 1, \quad 0 < b < 1, \quad 0 < c < 1, \quad 1 < a + b + c < 2.$$
Figure 1: Triangular support for bivariate side density

Figure 2: Hexahedral support for trivariate side density (two views)
The joint density for \((a, b, c)\) is thus \(3/2\) (constant) over the shaded hexahedral region in Figure 2; note the additional complexity of two missing corners, not just one. The marginal density for \(a\) is \(3/4 (1 + 2x - 2x^2)\) for \(0 < x < 1\); the cross-correlation between \(a\) and \(b\) is \(-1/3\). By Brahmagupta’s formula, the mean area of the cyclic quadrilateral is \(4\)

\[
E(\text{area}) = \int_0^1 \int_0^{1-x} \int_0^{1-y} \frac{3}{2} \sqrt{(1-x)(1-y)(1-z)(x+y+z-1)} \, dz \, dy \, dx
\]

\[
+ \int_0^1 \int_0^{1-2x-y} \int_0^{1-y} \frac{3}{2} \sqrt{(1-x)(1-y)(1-z)(x+y+z-1)} \, dz \, dy \, dx
\]

\[
= 4 \left( \frac{17\pi}{525} - \frac{\pi^2}{160} \right)
\]

and the mean square area is similarly \(4^2/560 = 1/35\).

As far as we know, no one has previously determined the exact density for the area of a triangle or a cyclic quadrilateral created via broken sticks. From such an expression would come a numerical estimate of the median area (50%-tile), obtained via a single integration. We succeed in finding the density for triangles, but unfortunately not for quadrilaterals. Even better would be an exact cumulative distribution function – allowing us to avoid the integration – and, surprisingly, this too is possible.

### 1. PDF for Triangle Area

We work with \(z = \text{area}^2\) for now, returning to \(\sqrt{z} = \text{area}\) at the conclusion. The system of equations

\[
z = (1-x)(1-y)(x+y-1), \quad w = y
\]

has two solutions:

\[
x = 1 - \frac{w}{2} \pm \frac{\sqrt{(1-w)w^2 - 4z}}{2\sqrt{1-w}}, \quad y = w
\]

and the map \((x, y) \mapsto (z, w)\) has absolute Jacobian determinant

\[
\left| \begin{array}{cc}
-(1-y)(2x+y-2) & -(1-x)(x+2y-2) \\
0 & 1
\end{array} \right| = \sqrt{1-w} \sqrt{(1-w)w^2 - 4z}.
\]

Since the joint density for \((x, y)\) is 2 and the map to \((z, w)\) is two-to-one, the joint density for \((z, w)\) is \[5\]

\[
\frac{4}{\sqrt{1-w} \sqrt{(1-w)w^2 - 4z}} = \frac{4}{\sqrt{(w-c)(w-a)(b-w)(1-w)}}
\]
where \( c(z) < 0 < a(z) < w < b(z) < 1 \) are the three zeroes of the cubic polynomial \((1 - w)w^2 - 4z\):

\[
a(z) = \frac{1}{3} + \frac{1 - i\sqrt{3}}{6} \theta(z)^{-1/3} + \frac{1 + i\sqrt{3}}{6} \theta(z)^{1/3},
\]

\[
b(z) = \frac{1}{3} + \frac{1 + i\sqrt{3}}{6} \theta(z)^{-1/3} + \frac{1 - i\sqrt{3}}{6} \theta(z)^{1/3},
\]

\[
c(z) = \frac{1}{3} - \frac{1}{3} \theta(z)^{-1/3} - \frac{1}{3} \theta(z)^{1/3}
\]

and

\[
\theta(z) = -1 + 54z + 6\sqrt{3}\sqrt{-z + 27z^2}.
\]

It follows that the marginal density for \( z \) is [6]

\[
g(z) = \int_a^b \frac{4dw}{\sqrt{1 - w}\sqrt{(1 - w)w^2 - 4z}} = \frac{8}{\sqrt{(1 - a)(b - c)}} K \left[ \frac{(b - a)(1 - c)}{(1 - a)(b - c)} \right]
\]

where \( K \) is the complete elliptic integral of the first kind:

\[
K[m] = \int_0^1 \frac{d\tau}{\sqrt{1 - \tau^2}\sqrt{1 - m\tau^2}}
\]

(consistent with Mathematica). The marginal density for \( \sqrt{z} \) is therefore

\[
f(\zeta) = \frac{d}{d\zeta} \mathbb{P} \{ \sqrt{z} < \zeta \} = \frac{d}{d\zeta} \mathbb{P} \{ z < \zeta^2 \} = 2\zeta \cdot g(\zeta^2)
\]

where \( 0 < \zeta < 1/(3\sqrt{3}) \); see Figure 3. Numerically solving the equation

\[
\int_0^\mu f(\zeta) \, d\zeta = \frac{1}{2}
\]

gives the median

\[
\mu = 0.125833843136510592028005... = (4)(0.0314584607846627648007001...)
\]

to precision limited only by the accuracy of the integration routine. A symbolic antiderivative of \( f(\zeta) \) would seem infeasible, at least at first glance.
Again, we denote sides by $x, y$ and area $z$ by $z$. Let $u = 2 - x - y$ and $v = y - x$, so that $0 < u < 1$, $-1 < v < 1$ and
\[(u+v)(u-v)(1-u) = (2-2x)(2-2y)[1 -(2-x-y)] = 4(1-x)(1-y)(x+y-1) = 4z\]

hence
\[u^2 - v^2 = \frac{4z}{1-u}\]

hence
\[|v| = \sqrt{u^2 - \frac{4z}{1-u}} = \sqrt{\frac{(1-u)u^2 - 4z}{1-u}} = q(z,u).\]

The pair $(u, v)$ is uniform on the domain $\{(u,v) : 0 < u < 1$ and $|v| < u\}$, a triangle of unit area; thus the probability that area $z$ exceeds $z$ is
\[
\frac{b(z)}{a(z)} \int_{a(z)-q(z,u)}^{b(z)} q(z,u) \, dv \, du = 2 \int_{a(z)}^{b(z)} q(z,u) \, du
\]
where the zeros \( c(z) < a(z) < u < b(z) < 1 \) are exactly as before. If \( t = \sqrt{1-u} \), then \( u = 1 - t^2 \) and \( du = -2t \, dt \); it follows that

\[
q(z, u) \, du = \sqrt{\frac{t^2 (1-t^2)^2 - 4z}{t^2}} (-2t \, dt) = -2\sqrt{(1-t^2)^2 t^2 - 4z} \, dt
\]

\[
= -2\sqrt{(1-t^2) t - 2\sqrt{z} \sqrt{(1-t^2) t + 2\sqrt{z}}} \, dt
\]

\[
= -2\sqrt{(t^2 - \alpha^2)(\beta^2 - t^2)(\gamma^2 - t^2)} \, dt
\]

where \( \beta = \sqrt{1-a} \), \( \alpha = \sqrt{1-b} \) and \( \gamma = \sqrt{1-c} \). From \( 1-c > 1-a > 1-u > 1-b > 0 \), we have \( 0 < \alpha(z) < t < \beta(z) < \gamma(z) \). The preceding argument leading to the formula

\[
P\{\text{area} > \zeta\} = 4 \int_{\alpha(\zeta)}^{\beta(\zeta)} \sqrt{(1-t^2)^2 t^2 - 4\zeta^2} \, dt = 4J(\zeta)
\]

is due to an anonymous student [7]. Our only contribution is to link this with Dieckmann’s [8] integral evaluation:

\[
8J = \beta \sqrt{\gamma^2 - \alpha^2} (\alpha^2 + \beta^2 + \gamma^2) E\left[\frac{(\beta^2 - \alpha^2)^2}{(\gamma^2 - \alpha^2) \beta^2}\right] + \frac{\alpha^2 \beta}{\sqrt{\gamma^2 - \alpha^2}} (\alpha^2 + \beta^2 - 5\gamma^2) K\left[\frac{(\beta^2 - \alpha^2)^2}{(\gamma^2 - \alpha^2) \beta^2}\right] - \frac{\alpha^2}{\beta \sqrt{\gamma^2 - \alpha^2}} (\alpha + \beta - \gamma) (\alpha - \beta - \gamma) (\alpha - \beta + \gamma) (\alpha + \beta + \gamma) \Pi\left[\frac{(\beta^2 - \alpha^2)^2}{\beta^2}, \frac{(\beta^2 - \alpha^2)^2}{(\gamma^2 - \alpha^2) \beta^2}\right]
\]

where \( E \) and \( \Pi \) are complete elliptic integrals of the second and third kind:

\[
E[m] = \int_0^1 \frac{\sqrt{1 - m\tau^2}}{\sqrt{1 - \tau^2}} \, d\tau, \quad \Pi[n, m] = \int_0^1 \frac{d\tau}{(1 - n\tau^2) \sqrt{1 - \tau^2} \sqrt{1 - m\tau^2}}.
\]

Solving numerically the equation \( 8J(\mu) = 1 \) gives the median to essentially infinite precision.

3. Cyclic Quadrilaterals

On the one hand, arbitrary angles \( \alpha \) and \( \beta \) in a cyclic quadrilateral are distributed according to what we call a bivariate tent density:

\[
\begin{cases}
\varphi(\pi - y, x) & \text{if } \pi - y < x < y < \pi, \\
\varphi(x, y) & \text{if } x < y < \pi - x \text{ and } 0 < x < \pi/2, \\
\varphi(\pi - x, y) & \text{if } \pi - x < y < x \text{ and } \pi/2 < x < \pi, \\
\varphi(y, x) & \text{if } y < x < \pi - y \text{ and } 0 < y < \pi/2.
\end{cases}
\]
Figure 4: Bivariate tent density on $[0, \pi] \times [0, \pi]$

where

$$\varphi(x, y) = \frac{[4 \cos(x) - 3 \cos(2y) - 1] \tan (x/2)^2}{2 [\sin(x) + \sin(y)]^2 \sin(y)^2}.$$ 

A sketch of the proof is given in Appendix I; a less complicated example appears in [9]. Clearly $\alpha$ and $\beta$ are uncorrelated yet dependent. The univariate density for $\alpha$ is

$$\psi_1(x) = 16 \psi_2(x) \ln(\sin(x)/2) + \psi_3(x) \ln(\tan(x/2))$$

where trigonometric polynomials $\psi_1$, $\psi_2$, $\psi_3$ are given by

$$\psi_1(x) = -25 \cos(x) + 7 \cos(3x) + 17 \cos(5x) + \cos(7x),$$

$$\psi_2(x) = 42 \cos(x) + 19 \cos(3x) + 3 \cos(5x),$$

$$\psi_3(x) = 378 + 489 \cos(2x) + 150 \cos(4x) + 7 \cos(6x)$$

(Figures 4 and 5).

It follows that $E(\alpha) = \pi/2$, but a closed-form expression for

$$E(\alpha^2) = 3.0252500344067143300547137...$$

remains open.

On the other hand, finding the density for area has eluded us – witness Appendix II – and computer simulation suggests that it is approximately linear (Figure 6).
Figure 5: Angle density for cyclic quadrilaterals

Figure 6: Area density for cyclic quadrilaterals
We nonrigorously estimate the median area to be 0.1696... via such experimentation. Can this be calculated to high precision?

Of all quadrilaterals with sides \(a, b, c, d\) in this order, there is a unique one with maximal area, the cyclic quadrilateral \([10, 11]\). The natural analog of this theorem to \(n\)-gons for \(n \geq 5\) is true \([12, 13]\).

When breaking a stick in \(n - 1\) places at random, the \(n\) pieces can be configured as an \(n\)-gon with probability \(1 - \frac{n}{2^{n-1}}\) \([14]\).

A concise formula for the area of a cyclic pentagon, generalizing those of Heron and Brahmagupta, apparently does not exist. It is known that \((4 \cdot \text{area})^2\) satisfies a \(7\)th degree polynomial equation with coefficients involving elementary symmetric functions \(\sigma_k\) of squares of sides \([15, 16, 17, 18, 19, 20, 21]\). One of two \(7\)th degree polynomials is satisfied for cyclic hexagons. For \(n \geq 7\), the equations become inconceivably lengthy, possessing degree 38 for cyclic heptagons and octagons, and almost a million terms when expanding with regard to \(\sigma_k\).

Unless a theoretical breakthrough occurs, broken sticks will never be fully understood for large \(n\). A numerical approach is perhaps mandatory. We wonder if even the mean area (let alone the median area) of a cyclic pentagon is too much for which to ask.

\[\text{Median Area for Broken Sticks}\]

The bivariate density for two angles of a triangle is easily obtained in \([22]\); the corresponding work for a cyclic quadrilateral is harder. Let adjacent angles \(\alpha_1, \alpha_2\) be opposite angles \(\alpha_3 = \pi - \alpha_1, \alpha_4 = \pi - \alpha_2\). Let sides \(s_1, s_2\) determine \(\alpha_3\) and sides \(s_3, s_4\) determine \(\alpha_1\) (see the picture in \([9]\)). By the Law of Cosines,

\[
\begin{align*}
    s_1^2 + s_2^2 - 2s_1s_2 \cos(\alpha_3) &= s_3^2 + s_4^2 - 2s_3s_4 \cos(\alpha_1), \\
    s_2^2 + s_3^2 - 2s_2s_3 \cos(\alpha_4) &= s_1^2 + s_4^2 - 2s_1s_4 \cos(\alpha_2)
\end{align*}
\]

hence

\[
\begin{align*}
    s_1^2 + s_2^2 - s_3^2 - s_4^2 &= -2(s_1s_2 + s_3s_4) \cos(\alpha_1), \\
    s_2^2 + s_3^2 - s_1^2 - s_4^2 &= -2(s_2s_3 + s_1s_4) \cos(\alpha_2)
\end{align*}
\]

hence

\[
\begin{align*}
    \alpha_1 &= \arccos \left( \frac{s_3^2 + (2 - s_1 - s_2 - s_3)^2 - s_4^2 - s_2^2}{s_3(2 - s_1 - s_2 - s_3) + s_1s_2} \right), \\
    \alpha_2 &= \arccos \left( \frac{s_1^2 + (2 - s_1 - s_2 - s_3)^2 - s_2^2 - s_3^2}{s_1(2 - s_1 - s_2 - s_3) + s_2s_3} \right)
\end{align*}
\]

because the stick has length 2. The map \((s_1, s_2, s_3) \mapsto (\alpha_1, \alpha_2, s_3)\) has absolute Jacobian determinant

\[
\frac{2(s_1 + s_3)(1 - s_3)}{|s_1s_2 + s_3(2 - s_1 - s_2 - s_3)| s_2s_3 + s_1(2 - s_1 - s_2 - s_3)|}.
\]
We first rewrite this in terms of $\alpha_1, \alpha_2, s_3$, remembering not only $s_1 > 0$, $s_2 > 0$ but also $s_1 + s_2 + s_3 < 2$. To do this, perform the substitutions

$$s_1 = 1 - (1 - s_3) \tan \left(\frac{\alpha_2}{2}\right),$$

$$s_2 = -s_3 \sin \left(\alpha_1 + \frac{\alpha_2}{2}\right) + (2 - s_3) \sin \left(\frac{\alpha_2}{2}\right) \cdot \frac{\sin(\alpha_1 - \frac{\alpha_2}{2}) + \sin \left(\frac{\alpha_2}{2}\right)}{\sin(\alpha_1 - \frac{\alpha_2}{2}) + \sin \left(\frac{\alpha_2}{2}\right)}.$$

The reciprocal of the determinant is then integrated with respect to $s_3$, with lower limit

$$\max \left\{1 - \tan \left(\frac{\alpha_1}{2}\right), 0\right\}$$

(from $s_1 = 0$) and upper limit

$$\min \left\{\frac{\sin \left(\frac{\alpha_2}{2}\right)}{\cos \left(\frac{\alpha_1}{2}\right) \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right)}, \frac{\cos \left(\frac{\alpha_1}{2}\right)}{\sin \left(\frac{\alpha_1 + \alpha_2}{2}\right)}\right\}$$

(from $s_2 = 0$ and $s_1 + s_2 + s_3 = 2$). As an example, $s_2 = 0$ when

$$s_3 = \frac{2 \sin \left(\frac{\alpha_2}{2}\right)}{\sin \left(\alpha_1 + \frac{\alpha_2}{2}\right) + \sin \left(\frac{\alpha_2}{2}\right)} = \frac{\sin \left(\frac{\alpha_2}{2}\right)}{\cos \left(\frac{\alpha_1}{2}\right) \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right)}$$

since, by the sum-to-product identity,

$$\sin \left(\alpha_1 + \frac{\alpha_2}{2}\right) + \sin \left(\frac{\alpha_2}{2}\right) = 2 \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right) \cos \left(\frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_2}{2}\right) = 2 \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right) \cos \left(\frac{\alpha_2}{2}\right).$$

Likewise, at the end of the derivation,

$$\frac{[4 \cos(\alpha_1) - 3 \cos(2\alpha_2) - 1] \tan \left(\frac{\alpha_1}{2}\right)^2}{32 \cos \left(\frac{\alpha_1 - \alpha_2}{2}\right)^2 \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right)^2 \cos \left(\frac{\alpha_2}{2}\right)^2} = \frac{[4 \cos(\alpha_1) - 3 \cos(2\alpha_2) - 1] \tan \left(\frac{\alpha_1}{2}\right)^2}{2 \left[\sin(\alpha_1) + \sin(\alpha_2)\right]^2 \sin(\alpha_2)^2}$$

by a double angle formula and since, by the product-to-sum identity,

$$2 \cos \left(\frac{\alpha_1 - \alpha_2}{2}\right) \sin \left(\frac{\alpha_1 + \alpha_2}{2}\right) = \sin \left(\frac{\alpha_1 - \alpha_2}{2} + \frac{\alpha_1 + \alpha_2}{2}\right) - \sin \left(\frac{\alpha_1 - \alpha_2}{2} - \frac{\alpha_1 + \alpha_2}{2}\right) = \sin(\alpha_1) + \sin(\alpha_2).$$

Denote the integral by $I(\alpha_1, \alpha_2)$. The tent-like appearance of the surface plot of $I$ suggests necessary simplifications leading to our formula for the joint density. Finally, the details of further integrating out $\alpha_2$ are elaborate and thus omitted.
5. Appendix II

We work with $r_1 = \text{area}^2$. The system of equations

$$
r_1 = (1 - s_1)(1 - s_2)(1 - s_3)(s_1 + s_2 + s_3 - 1), \quad r_2 = s_2, \quad r_3 = s_3
$$

has two solutions:

$$
s_1 = 1 - \frac{r_2 + r_3}{2} \pm \frac{\sqrt{(1 - r_2)(1 - r_3)(r_2 + r_3)^2 - 4r_1}}{2\sqrt{1 - r_2}\sqrt{1 - r_3}}, \quad s_2 = r_2, \quad s_3 = r_3
$$

and the map $(s_1, s_2, s_3) \mapsto (r_1, r_2, r_3)$ has absolute Jacobian determinant

$$
\sqrt{1 - r_2}\sqrt{1 - r_3}\sqrt{(1 - r_2)(1 - r_3)(r_2 + r_3)^2 - 4r_1}.
$$

Since the joint density for $(s_1, s_2, s_3)$ is $3/2$ and the map to $(r_1, r_2, r_3)$ is two-to-one, the joint density for $(r_1, r_2, r_3)$ is

$$
\frac{3}{\sqrt{1 - r_2}\sqrt{1 - r_3}\sqrt{(1 - r_2)(1 - r_3)(r_2 + r_3)^2 - 4r_1}}\frac{3}{\sqrt{(r_3 - c)(r_3 - a)(b - r_3)(1 - r_3)(1 - r_2)}
$$

where $c(r_1, r_2) < a(r_1, r_2) < r_3 < b(r_1, r_2) < 1$ are the three zeroes of the cubic polynomial $(1 - r_2)(1 - r_3)(r_2 + r_3)^2 - 4r_1$ (regarded as a function of $r_3$ only). What troubles us is that, given sufficiently small $r_1 > 0$, there is a nonempty interval $\Omega \subseteq [0, 1]$ for which $r_2 \in \Omega$ implies $a(r_1, r_2) < 0$. (As an example, if $r_1 = 0.03$, then $\Omega = [0.4807..., 0.8227...].$) This implies that an integral with respect to $r_3$ must possess lower limit $\max\{0, a(r_1, r_2)\}$. While this should not present an obstacle numerically, it does create havoc symbolically. To find exactly the endpoints of $\Omega$, that is, to solve the equation $a(r_1, r_2) = 0$ for two values $0 < r_2' < r_2'' < 1$ via computer algebra, introduces a complexity roadblock in our stochastic analysis. Such difficulties did not arise in Section 1 because $a(z)$ was always positive. Our hope is that someone else will see a workaround.

6. Acknowledgements

I am grateful to Andreas Dieckmann [5], who promptly evaluated the integral containing three quadratic factors at my request. It is impressive to see the recent work of students in [4, 7, 23, 24] on variations of the broken stick; I appreciate efforts of their teachers in keeping the flame of original research alive.
REFERENCES

[1] D. E. Dobbs, The average area of a triangle, Mathematics and Computer Education 21 (1987) 178–181.

[2] R. A. Johnson, Advanced Euclidean Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle, Dover Publications, 1960, pp. 81–85; MR0120538 (22 #11289).

[3] H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited, Math. Assoc. of Amer., 1967, pp. 56–60; MR315365.

[4] P. A. CrowdMath, The Broken Stick Problem: (ii), MIT PRIMES/AoPS, 2017, http://artofproblemsolving.com/polymath/mitprimes2017b/p.

[5] A. Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 1965, pp. 187–206; MR0176501 (31 #773).

[6] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th ed., Elsevier/Academic Press, 2007, p. 275, s. 3.147; MR2360010 (2008g:00005).

[7] P. A. CrowdMath, The Broken Stick Problem: (iii), MIT PRIMES/AoPS, 2017, http://artofproblemsolving.com/polymath/mitprimes2017b/p.

[8] A. Dieckmann, Table of Indefinite Integrals, Universität Bonn, http://www-elsa.physik.uni-bonn.de/~dieckman/IntegralsIndefinite/IndefInt.html.

[9] S. R. Finch, Random cyclic quadrilaterals, arXiv:1610.00510.

[10] I. Niven, Maxima and Minima Without Calculus, Math. Assoc. Amer., 1981, pp. 47–55, 253–256; MR0654149 (83i:52011).

[11] T. Peter, Maximizing the area of a quadrilateral, College Math. J. 34 (2003) 315-316.

[12] V. Janković, On the impossibility of one ruler-and-compass construction, Mat. Vesnik 48 (1996) 73–75; MR1454529 (98c:51025).

[13] D. King, Maximum Polygon Area, http://www.drking.org.uk/hexagons/misc/polymax.html.

[14] C. D’Andrea and E. Gómez, The broken spaghetti noodle, Amer. Math. Monthly 113 (2006) 555–557; MR2231141.

[15] D. P. Robbins, Areas of polygons inscribed in a circle, Discrete Comput. Geom. 12 (1994) 223–236; MR1283889 (95g:51027).
[16] D. P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly* 102 (1995) 523–530; MR1336638 (96k:51024).

[17] F. Miller Maley, D. P. Robbins and J. Roskies, On the areas of cyclic and semicyclic polygons, *Adv. in Appl. Math.* 34 (2005) 669–689; MR2128992 (2006b:51016).

[18] I. Pak, The area of cyclic polygons: recent progress on Robbins’ conjectures, *Adv. in Appl. Math.* 34 (2005) 690–696; MR2128993 (2006b:51017).

[19] D. Svrtan, D. Veljan and V. Volenc, Geometry of pentagons: from Gauss to Robbins, [arXiv:math/0403503](https://arxiv.org/abs/math/0403503).

[20] P. Pech, Computations of the area and radius of cyclic polygons given by the lengths of sides, *Automated Deduction in Geometry*, Lect. Notes in Comput. Sci. 3763, Springer-Verlag, 2006, pp. 44–58; MR2259087 (2008g:51021).

[21] P. Pech, Computation with pentagons, *J. Geom. Graph.* 12 (2008) 151–160; MR2519392 (2010f:51022).

[22] S. R. Finch, Uniform triangles with equality constraints, [arXiv:1411.5216](https://arxiv.org/abs/1411.5216).

[23] L. Kong, L. Lkhamsuren, A. Turner, A. Uppal and A. J. Hildebrand, Random Points, Broken Sticks, and Triangles, UIUC, 2013, [https://faculty.math.illinois.edu/~hildebr/ugresearch/brokenstick-spring2013report.pdf](https://faculty.math.illinois.edu/~hildebr/ugresearch/brokenstick-spring2013report.pdf).

[24] A. Page, Y. Semibratova, Y. Xuan, E. R. Zhang, M. T. Phaovibul and A. J. Hildebrand, The Broken Stick Problem in Higher Dimensions, UIUC, 2015, [https://faculty.math.illinois.edu/~hildebr/ugresearch/Hildebrand-Calculus-Spring2015-report.pdf](https://faculty.math.illinois.edu/~hildebr/ugresearch/Hildebrand-Calculus-Spring2015-report.pdf).

Steven Finch  
MIT Sloan School of Management  
Cambridge, MA, USA  
steven_finch@harvard.edu

©Copyright © 2018 by Steven R. Finch. All rights reserved.