Coverage Criterion in Sensor Networks Stable under Perturbation

Yasuaki Hiraoka
Genki Kusano

To the coverage problem of sensor networks, Vin de Silva and Robert Ghrist developed several approaches [4, 5] based on (persistent) homology theory. Their criterions for the coverage are formulated on the Rips complexes constructed by the sensors, in which their locations are supposed to be fixed. However, the sensors are in general affected by perturbations (e.g., natural phenomena), and hence the stability of the coverage criterions should be also discussed. In this paper, we present a coverage theorem stable under perturbation. This theorem is derived by extending the Rips interleaving theorem studied in the paper [3] into an appropriate relative version.

55N35, 93D09; 54E25

1 INTRODUCTION

A fundamental task of sensor networks is to extract information about a target domain by using sensors. Each sensor in the target domain obtains data around its location, as shown in Figure 1. Here, we are interested in a coverage problem, which appears in a variety of settings, such as military, environmental sensing, security, and so on [8, 9]. The problem is to determine whether the target domain is covered by the sensing region. Figure 2 shows a sensing region covering its target domain.

We define the target domain as a subset \( D \) of \( \mathbb{R}^d \) and the set of sensors as a finite subset \( X = \{ x_i \in D \} \). Each sensor \( x_i \) can monitor its surroundings within a cover radius \( r_c \). Let \( B_{r_c}(x_i) = \{ y \in \mathbb{R}^d \mid \| x_i - y \| \leq r_c \} \), where \( \| \cdot \| \) represents the Euclidean norm on \( \mathbb{R}^d \), be the sensing region of \( x_i \) and \( \mathcal{U}(X) = \bigcup_i B_{r_c}(x_i) \) be the whole sensing region of \( X \). Then the coverage problem is formulated as “Is the target domain \( D \) covered by the whole sensing region \( \mathcal{U}(X) \)?”.

From now on, we assume that each sensor does not have ability to gather absolute positional information, e.g., GPS. In other words, we cannot obtain the coordinates of the sensors. However, we assume that each sensor can communicate with other sensors.
if the distance between them is less than a certain communication radius. Here, we consider the coverage of a restricted target domain \( D' \subset D \), where \( D' \) is relatively large in \( D \). In this paper, our assumptions for sensor networks are the same as those used in [5].

A1. The cover radius is \( r_c \).

A2. We have two communication radii \( r_w \) and \( r_s \).

A3. The communication radii \( r_w \), \( r_s \) and the cover radius \( r_c \) satisfy

\[
 r_c \geq \frac{r_s}{\sqrt{2}}, \quad r_w \geq r_s \sqrt{10}.
\]

A4. The target domain \( D \) is a compact subset of \( \mathbb{R}^d \). In addition, the set

\[
 N_f(\partial D) := \{ x \in D \mid \|x - \partial D\| \leq r_f \}
\]

defines the set \( F \) of fence sensors as \( X \cap N_f(\partial D) \), where \( r_f \) is the fence detection radius.

A5. The restricted domain \( D - N_f(\partial D) \) is connected, where

\[
 N_f(\partial D) := \left\{ x \in D \mid \|x - \partial D\| \leq \hat{r} := r_f + \frac{r_s}{\sqrt{2}} \right\}.
\]

A6. The fence detection hypersurface \( \Sigma = \{ x \in D \mid \|x - \partial D\| = r_f \} \) has internal injectivity radius at least \( \frac{r_f}{\sqrt{2}} \) and external injectivity radius at least \( r_s \).

In these settings, Vin de Silva and Robert Ghrist proposed a criterion to solve the coverage problem from the communication data of sensors. Throughout, we call the criterion in [5, Theorem 3.4] as the dSG criterion for short.

Let \( \mathcal{R}(X; a) \) denote the Rips complex of \( X \) with parameter \( a \), which we define in Section 2. When \( F \) is a subset of \( X \), \( \mathcal{R}(F; a) \) is a subcomplex of \( \mathcal{R}(X; a) \), and \( H_k(\mathcal{R}(X, F; a)) \) means the relative homology group of this pair of the Rips complexes.

**dSG Criterion** (Theorem 3.4 of [5]) Let \( X \) be a set of sensors in a target domain \( D \subset \mathbb{R}^d \) satisfying assumptions A1-A6. If the homomorphism

\[
 \iota_* : H_d(\mathcal{R}(X, F; r_s)) \to H_d(\mathcal{R}(X, F; r_w))
\]

induced by the inclusion \( \iota : \mathcal{R}(F; r_s) \to \mathcal{R}(F; r_w) \) is nonzero, then the cover of the sensors \( U(X) \) contains the restricted domain \( D - N_f(\partial D) \).
The dSG criterion applies only for the fixed sensors. However, sensors are not stationary in general. For example, sensors can be affected by perturbations such as wind or earthquakes. They may move (Figure 3) and thus the coverage may change (Figure 4).

One option to this situation is to check the dSG criterion again to determine the coverage of the sensors after the perturbation. Instead, we improve the dSG criterion to be stable under perturbation. Namely, once we examine the original induced map, we can automatically know the coverage of the sensors even after the perturbation.

Main theorem  Let $X$ be a set of sensors which satisfies assumptions A1-A6 and let $f : X \to D$ satisfy $\sup_{x \in X} \| x - f(x) \| \leq \frac{\varepsilon}{2}$, $f(F) \subset N_f(\partial D)$, and $f(X - F) \subset D - N_f(\partial D)$. If the homomorphism

$$t_* : H_d(\mathcal{R}(X, F; r_s - \varepsilon)) \to H_d(\mathcal{R}(X, F; r_w + \varepsilon))$$


induced by the inclusion \( \iota : \mathcal{R}(X; r_s - \varepsilon) \rightarrow \mathcal{R}(X; r_w + \varepsilon) \) is nonzero, then the cover of the sensors after the perturbation \( U(f(X)) \) contains the restricted domain \( D - N_{\iota}(\partial D) \).

Here, we view \( f \) as a perturbation; that is, each sensor moves within \( \frac{\varepsilon}{2} \) from \( x \in X \) via \( f \). We emphasize that this criterion does not require the further communication data of the sensors after the perturbation. In other words, for \( X \) and \( f \) satisfying the conditions of the main theorem, we can automatically conclude that the map

\[
t_* : H_d(\mathcal{R}(f(X), f(F); r_s)) \rightarrow H_d(\mathcal{R}(f(X), f(F); r_w))
\]

is nonzero, which guarantees the coverage criterion for the perturbed sensors \( f(X) \).

A key mathematical concept for studying stability of the coverage is the interleaving of persistence modules [2, 3]. This characterizes the similarity of two persistence modules, and hence some features in one persistence module can be studied by another. In this paper, we study the stability property of the coverages between the original and perturbed sensors by interleavings. In the derivation of the main theorem, our mathematical contribution is to extend the Rips interleaving theorem studied in [3] into an appropriate relative version.

This paper is organized as follows. In Section 2, the basic concepts of correspondences, the Gromov-Hausdorff distance, persistence modules, Rips complexes, and Rips interleaving are introduced. In Section 3, we extend some properties of persistence modules to the relative version and prove relative Rips interleaving. In Section 4, we prove the main theorem using relative Rips interleaving.

## 2 PRELIMINARIES

In this section, we review the basic concepts of correspondences and persistence modules. We refer the reader to [1] for correspondences, [7, 10] for homology groups and [2, 6] for persistence modules. We basically follow the exposition in [3] with appropriate modifications. Let \( K \) be a field. Throughout this paper, we assume that all vector spaces are defined over \( K \) and that the coefficient group of the homology groups is \( K \).

### 2.1 Correspondence

Let \( X \) and \( Y \) be two sets. A **correspondence** \( C \) from \( X \) to \( Y \), denoted by \( C : X \rightrightarrows Y \), is a subset of \( X \times Y \) satisfying the following conditions: for every \( x \in X \) there exists
at least one \( y \in Y \) such that \((x, y) \in C\), and for every \( y \in Y \) there exists at least one \( x \in X \) such that \((x, y) \in C\). Since \( C = X \times Y \) is a correspondence, there exists at least one correspondence between any two sets.

**Example 2.1** For any map \( f : X \to Y \), its graph \( G(f) = \{(x, f(x)) \in X \times Y \mid x \in X\} \subset X \times f(X) \) is a correspondence from \( X \) to \( f(X) \).

The composition of two correspondences \( C : X \Rightarrow Y \) and \( D : Y \Rightarrow Z \) is the correspondence \( D \circ C : X \Rightarrow Z \) defined by

\[
D \circ C = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } (x, y) \in C, \ (y, z) \in D\}.
\]

Let \( C \) be a subset of \( X \times Y \). The **transpose** of \( C \), denoted by \( C^T \), is defined by

\[
C^T = \{(y, x) \in Y \times X \mid (x, y) \in C\}.
\]

A subset \( C \subset X \times Y \) is a correspondence from \( X \) to \( Y \) if and only if its transpose \( C^T \) is a correspondence from \( Y \) to \( X \). For a subset \( C \) of \( X \times Y \) and a subset \( \sigma \) of \( X \), its **image** \( C(\sigma) \) is the set \( \{y \in Y \mid (x, y) \in C, \ x \in \sigma\} \).

When \((X, d_X)\) and \((Y, d_Y)\) are metric spaces, the **distortion** of a correspondence \( C : X \Rightarrow Y \) is defined as follows:

\[
dis C = \sup \left\{ |d_X(x, x') - d_Y(y, y')| \mid (x, y), (x', y') \in C \right\}.
\]

The distortion of \( C : X \Rightarrow Y \) is related to the distance between \( X \) and \( Y \) as follows.

**Proposition 2.2** (Theorem 7.2.5 of [1]) For any two metric spaces \((X, d_X)\) and \((Y, d_Y)\),

\[
d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis } C \mid C : X \Rightarrow Y \},
\]

where \( d_{GH} \) is the Gromov-Hausdorff distance.

We here recall the definitions of the Hausdorff distance and the Gromov-Hausdorff distance. A **semi-metric** (psuedometric) is a function \( d : X \times X \to \mathbb{R} \) satisfying for all \( x, y, z \in X \)

1. \( d(x, y) \geq 0 \),
2. \( d(x, x) = 0 \),
3. \( d(x, y) = d(y, x) \),
4. \( d(x, y) + d(y, z) \geq d(x, z) \).
When $X$ and $Y$ are subspaces of some metric space $(Z, d)$,

$$d_H(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}$$

is the **Hausdorff distance** between $X$ and $Y$. When $(X, d_X)$ and $(Y, d_Y)$ are metric spaces with possibly different metrics, it is possible to make a semi-metric $d$ on $X \sqcup Y$ satisfying $d|_{X \times X} = d_X$ and $d|_{Y \times Y} = d_Y$ (see [1, Remark 7.3.12]). Regarding $X$ and $Y$ as subspaces of $(X \sqcup Y, d)$, the Hausdorff distance between $X$ and $Y$ is thus defined.

We also denote the Hausdorff distance by $d_H(X, Y; d)$ to emphasize the dependency on the semi-metric. Then the **Gromov-Hausdorff distance** is defined by

$$d_{GH}(X, Y) := \inf_d d_H(X, Y; d),$$

where the infimum is taken over all semi-metrics on $X \sqcup Y$ satisfying $d|_{X \times X} = d_X$ and $d|_{Y \times Y} = d_Y$. Hence, Proposition 2.2 provides a characterization of $d_{GH}(X, Y)$ by means of the distortions.

### 2.2 Persistence modules

A **persistence module** $V = (V_a, v^b_a)$ over the real numbers $\mathbb{R}$ is an indexed family of vector spaces $\{V_a | a \in \mathbb{R}\}$ and a doubly-indexed family of linear maps

$$\{v^b_a : V_a \to V_b | a \leq b\}$$

satisfying for all $a \leq b \leq c$

$$v^c_a = v^c_b \circ v^b_a, \quad v^a_a = 1_{V_a}.$$  

Let $U = (U_a, u^b_a)$ and $V = (V_a, v^b_a)$ be persistence modules over $\mathbb{R}$. Given $\varepsilon > 0$, a **homomorphism of degree** $\varepsilon$ is a family of linear maps $\Phi = \{\phi_a : U_a \to V_{a+\varepsilon} | a \in \mathbb{R}\}$ such that $v^b_{a+\varepsilon} \circ \phi_a = \phi_b \circ u^b_a$ whenever $a \leq b$. We denote the set of homomorphisms of degree $\varepsilon$ from $U$ to $V$ by $\text{Hom}^\varepsilon(U, V)$. The **composition** of two homomorphisms $\Phi = \{\phi_a\} \in \text{Hom}^\varepsilon(U, V)$ and $\Psi = \{\psi_a\} \in \text{Hom}^\delta(V, W)$ is the homomorphism of degree $\varepsilon + \delta$ defined by:

$$\Psi \circ \Phi = \{\psi_{a+\varepsilon} \circ \phi_a : U_a \to W_{a+\varepsilon+\delta}\} \in \text{Hom}^{\varepsilon+\delta}(U, W).$$

For any persistence module $V = (V_a, v^b_a)$,

$$1^\varepsilon_V := \{v^a_{a+\varepsilon} : V_a \to V_{a+\varepsilon} | a \in \mathbb{R}\}$$

is a homomorphism of degree $\varepsilon$ from $V$ to $V$. 


A filtered simplicial complex $\mathbb{S} = \{S_a \mid a \in \mathbb{R}\}$ is a family of simplicial complexes such that $S_a$ is a subcomplex of $S_b$ whenever $a \leq b$. When the vertex set is unchanged in the filtered simplicial complex $\mathbb{S}$ (say $X$), $X$ is called the vertex set of $\mathbb{S}$.

Example 2.3 Let $\mathbb{S} = \{S_a\}$ be a filtered simplicial complex and $u_a^b : H_k(S_a) \rightarrow H_k(S_b)$ be the linear map induced by the inclusion $S_a \hookrightarrow S_b$. Then $H_k(\mathbb{S}) := (H_k(S_a), u_a^b)$ forms a persistence module.

Let $\mathbb{S} = \{S_a\}, \mathbb{T} = \{T_a\}$ be two filtered simplicial complexes with vertex sets $X, Y$, respectively. A correspondence $C : X \rightleftharpoons Y$ is $\varepsilon$-simplicial from $\mathbb{S}$ to $\mathbb{T}$ if, for any $a \in \mathbb{R}$ and any simplex $\sigma \in S_a$ (recall that $\sigma$ is a subset of $X$), every finite subset of $C(\sigma)$ is a simplex of $T_{a+\varepsilon}$.

Two persistence modules $\mathbb{U}$ and $\mathbb{V}$ are said to be $\varepsilon$-interleaved if there exist two homomorphisms

$$\Phi \in \text{Hom}^\varepsilon(\mathbb{U}, \mathbb{V}), \quad \Psi \in \text{Hom}^\varepsilon(\mathbb{V}, \mathbb{U})$$

such that $\Psi \circ \Phi = 1_{\mathbb{U}}^{2\varepsilon}$ and $\Phi \circ \Psi = 1_{\mathbb{V}}^{2\varepsilon}$; in other words, if the following four diagrams commute whenever $a \leq b$:

Interleaving is useful to study some features of $\mathbb{V}$ by $\mathbb{U}$ (and vice versa). For instance, when $u_{a+\varepsilon}^{b+2\varepsilon}$ is nonzero, we can deduce that the vector space $V_{a+\varepsilon}$ is nonzero (see the upper right diagram).

2.3 Rips complex

Let $(X, d_X)$ be a metric space. For $a \geq 0$, we define a $k$-simplex $[x_{i_0} \cdots x_{i_k}]$ as a subset $\{x_{i_0}, \ldots, x_{i_k}\}$ of $X$ which satisfies $x_{i_p} \neq x_{i_q}$ ($p \neq q$) and $d_X(x_{i_p}, x_{i_q}) \leq a$ for all $p, q = 0, \ldots, k$. The set of these simplices forms a simplicial complex, called the Rips complex.
complex of $X$ with parameter $a$, denoted by $\mathcal{R}(X; a)$. For $a < 0$, we define $\mathcal{R}(X; a)$ as a simplicial complex only consisting of the vertex set $X$ for convenience. Since there is a natural inclusion $\mathcal{R}(X; a) \hookrightarrow \mathcal{R}(X; b)$ whenever $a < b$, $\mathcal{R}(X) = \{\mathcal{R}(X; a) \mid a \in \mathbb{R}\}$ is a filtered simplicial complex on the vertex set $X$. Figure 5 shows that the communication graph whose edges $[x_i, x_j]$ are defined by $d(x_i, x_j) \leq a$ and Figure 6 shows its Rips complex.

![Figure 5: A communication graph.](image1)

![Figure 6: The Rips complex of Figure 5.](image2)

**Proposition 2.4** (Lemma 4.3 of [3]) Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. If $\delta_{GH}(X, Y) < \frac{\varepsilon}{2}$, then the persistence modules $H_k(\mathcal{R}(X))$ and $H_k(\mathcal{R}(Y))$ are $\varepsilon$-interleaved.

Proposition 2.4 is called the Rips interleaving. The Rips interleaving is useful to study homological features of $\mathcal{R}(Y)$ by $\mathcal{R}(X)$, without directly computing the persistence module of $\mathcal{R}(Y)$.

Using Proposition 2.2, we rewrite Proposition 2.4 as follows.

**Proposition 2.5** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. If there exists a correspondence $C : X \cong Y$ satisfying $\text{dis } C \leq \varepsilon$, then the persistence modules $H_k(\mathcal{R}(X))$ and $H_k(\mathcal{R}(Y))$ are $\varepsilon$-interleaved.

We extend Proposition 2.5 to a relative homology version in the next section.

## 3 RELATIVE INTERLEAVING

Let $A$ be a subspace of a metric space $X$. Then $\mathcal{R}(A; a)$ is a subcomplex of $\mathcal{R}(X; a)$ for all $a \in \mathbb{R}$. We consider a pair $\mathcal{R}(X, A) = (\mathcal{R}(X), \mathcal{R}(A))$ of filtered Rips complexes and its relative homology group $H_k(\mathcal{R}(X, A)) = H_k(\mathcal{R}(X; a), \mathcal{R}(A; a))$. In this section, we extend some results in [3] to relative versions, especially relative Rips interleaving.
3.1 Relative correspondence

Let $X, Y$ be two sets and let $A, B$ be subsets of $X, Y$, respectively. A correspondence $C$ from $(X, A)$ to $(Y, B)$, denoted by $C : (X, A) \mapsto (Y, B)$, is a correspondence $C : X \Rightarrow Y$ satisfying $C(A) \subset B$ and $C^T(B) \subset A$. If $C : (X, A) \mapsto (Y, B)$ is a correspondence, then the transpose $C^T$ is also a correspondence from $(Y, B)$ to $(X, A)$.

Given a map $f : X \rightarrow Y$ satisfying $f(A) \subset B$, we write $f : (X, A) \rightarrow (Y, B)$. For $f : (X, A) \rightarrow (Y, B)$, its graph $G(f) = \{(x, f(x)) \in X \times Y \mid x \in X\} \subset X \times f(X)$ is not always a correspondence from $(X, A)$ to $(f(X), f(A))$ unlike Example 2.1. We can easily make counterexamples showing $G(f)^T(f(A)) \not\subset A$, because $G(f)^T(f(A)) = f^{-1}(f(A))$. The following examples give some conditions on $f$ so that $f$ will satisfy $G(f)^T(f(A)) = f^{-1}(f(A)) \subset A$.

**Example 3.1** If $f$ is injective, then $G(f)$ is a correspondence from $(X, A)$ to $(f(X), f(A))$.

**Example 3.2** If $f(A) \subset B$ and $f(X - A) \subset Y - B$, then $G(f)$ is a correspondence from $(X, A)$ to $(f(X), f(A))$.

For a map $f : (X, A) \rightarrow (Y, B)$, if $G(f)$ is a subset of $C : (X, A) \mapsto (Y, B)$, then the map $f : (X, A) \rightarrow (Y, B)$ is said to be subordinate to $C$, denoted by $f : (X, A) \overset{C}{\rightarrow} (Y, B)$.

Let $A$ be a subset of $X$ and $S = \{S_a\}$ is a filtered simplicial complex with vertex set $X$. The restriction $S^A = \{S^A_a \mid a \in \mathbb{R}\}$ of $S$ to $A$ is a filtered simplicial complex such that $S^A_a$ is the maximal subcomplex of $S_a$ whose vertex set is $A$. Each inclusion map $S_a \rightarrow S_b$ induces a homomorphism $u^b_a : H_k(S_a, S^A_a) \rightarrow H_k(S_b, S^A_b)$ whenever $a \leq b$, and $H_k(S, S^A) = (H_k(S_a, S^A_a), u^b_a)$ forms a persistence module.

Let $S = \{S_a\}, T = \{T_a\}$ be two filtered simplicial complexes with vertex sets $X, Y$, and let $A, B$ be subsets of $X, Y$, respectively. A correspondence $C : (X, A) \mapsto (Y, B)$ is $\varepsilon$-simplicial from $(S, S^A)$ to $(T, T^B)$ if $C : X \mapsto Y$ is $\varepsilon$-simplicial, and for any $a \in \mathbb{R}$ and any simplex $\sigma \in S^A_a$, every finite subset of $C(\sigma)$ is a simplex of $T^B_{a+\varepsilon}$.

**Proposition 3.3** Let $S = \{S_a\}, T = \{T_a\}$ be two filtered simplicial complexes with vertex sets $X, Y$, and let $A, B$ be subsets of $X, Y$, respectively. Let $C : (X, A) \mapsto (Y, B)$ be $\varepsilon$-simplicial from $(S, S^A)$ to $(T, T^B)$. Then any subordinate map $f : (X, A) \overset{C}{\rightarrow} (Y, B)$ induces a canonical map $H_k(f) \in \text{Hom}^\varepsilon(H_k(S, S^A), H_k(T, T^B))$. Moreover any maps induced by subordinate maps to $C$ are equal.
Proof Let \( f \) be a subordinate map \( f : (X, A) \xrightarrow{C} (Y, B) \) and \( \sigma \) be a simplex in \( S^A_\alpha \). Then \( f(\sigma) \) is a finite subset of \( C(\sigma) \). This means \( f(\sigma) \) is a simplex of \( T^B_{a+\varepsilon} \). By applying the same argument to \( X \) and \( Y \), \( f \) induces a simplicial map \( (S_a, S^A_\alpha) \rightarrow (T^B_{a+\varepsilon}, T^B_{a+\varepsilon}) \) for all \( a \in \mathbb{R} \). Moreover the following diagram

\[
\begin{array}{ccc}
(S_a, S^A_\alpha) & \rightarrow & (S_b, S^A_b) \\
\downarrow & & \downarrow \\
(T^B_{a+\varepsilon}, T^B_{a+\varepsilon}) & \rightarrow & (T^B_{b+\varepsilon}, T^B_{b+\varepsilon})
\end{array}
\]

commutes whenever \( a \leq b \), where the horizontal maps are inclusions. Therefore \( f \) induces \( H_k(f) \in \text{Hom}^i(H_k(\mathbb{S}, \mathbb{S}^A), H_k(\mathbb{T}, \mathbb{T}^B)) \).

Any two subordinate maps \( f_1, f_2 : (X, A) \xrightarrow{C} (Y, B) \) induce simplicial maps from \( (S_a, S^A_\alpha) \) to \( (T^B_{a+\varepsilon}, T^B_{a+\varepsilon}) \) that are contiguous (see [10] for the definition and properties of contiguous maps). In fact, for any \( \sigma \in S^A_\alpha \), the vertices of two simplices \( f_1(\sigma) \) and \( f_2(\sigma) \) span a simplex of \( T^B_{a+\varepsilon} \), since these vertices comprise a finite subset of \( C(\sigma) \). Because any two contiguous maps are homotopic, we conclude that \( H_k(f_1) = H_k(f_2) \). \( \Box \)

Therefore the map \( H_k(C) : H_k(\mathbb{S}, \mathbb{S}^A) \rightarrow H_k(\mathbb{T}, \mathbb{T}^B) \) is well-defined as \( H_k(f) \) by a subordinate map \( f : (X, A) \xrightarrow{C} (Y, B) \).

Proposition 3.4 Let \( \mathbb{S}, \mathbb{T} \) be two filtered simplicial complexes with vertex sets \( X, Y \), and let \( A, B \) be subsets of \( X, Y \), respectively. If \( C : (X, A) \Rightarrow (Y, B) \) is a correspondence such that \( C \) and \( C^T \) are both \( \varepsilon \)-simplicial, then \( H_k(\mathbb{S}, \mathbb{S}^A) \) and \( H_k(\mathbb{T}, \mathbb{T}^B) \) are \( \varepsilon \)-interleaved.

Proof The diagonal set \( \mathbb{I}_X := \{(x, x) \mid x \in X \} \) is a correspondence from \( (X, A) \) to \( (X, A) \) and satisfies \( \mathbb{I}_X \subset C^T \circ C \). Moreover, \( \mathbb{I}_X \) is \( 2\varepsilon \)-simplicial from \( (\mathbb{S}, \mathbb{S}^A) \) to \( (\mathbb{S}, \mathbb{S}^A) \). The identity map from \( (X, A) \) to \( (X, A) \) is subordinate to \( \mathbb{I}_X \) and it is also subordinate to \( C^T \circ C \). Thus by using Proposition 3.3

\[
\mathbb{I}_{H_k(\mathbb{S}, \mathbb{S}^A)}^{2\varepsilon} = H_k(\mathbb{I}_X) = H_k(C^T \circ C) = H_k(C^T) \circ H_k(C).
\]

Similarly, \( \mathbb{I}_{H_k(\mathbb{T}, \mathbb{T}^B)}^{2\varepsilon} = H_k(C) \circ H_k(C^T) \), and the proof is complete. \( \Box \)

3.2 Relative Rips interleaving

If we find a correspondence between two filtered simplicial complexes satisfying the assumptions of Proposition 3.4, then there persistence modules are interleaved. Here, we construct such a correspondence for filtered Rips complexes.
Theorem 3.5 (Relative Rips interleaving) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \(A, B\) be subspaces of \(X, Y\), respectively. If there exists a correspondence \(C : (X, A) \to (Y, B)\) satisfying \(\text{dis} \ C \leq \varepsilon\), then the persistence modules \(H_k(\mathcal{R}(X, A))\) and \(H_k(\mathcal{R}(Y, B))\) are \(\varepsilon\)-interleaved.

Proof Let \(\sigma\) be a finite subset of \(A\) and \(\tau\) be any finite subset of \(C(\sigma)\). For any \(y, y' \in \tau\), there exist vertices \(x, x'\) of \(\sigma\) such that \((x, y), (x', y') \in C\). If \(\sigma\) is a simplex of \(\mathcal{R}(A; a)\), then \(d_X(x, x') \leq a\) for any two vertices \(x, x'\) of \(\sigma\). It follows from \(\text{dis} \ C \leq \varepsilon\) that
\[
d_Y(y, y') \leq d_X(x, x') + \varepsilon \leq a + \varepsilon.
\]
Thus \(\tau\) is a simplex of \(\mathcal{R}(B; a + \varepsilon)\). By substituting \(X\) and \(Y\) for \(A\) and \(B\) in the above argument, it can be seen that \(C\) is \(\varepsilon\)-simplicial from \(\mathcal{R}(X, A)\) to \(\mathcal{R}(Y, B)\). Symmetrically, \(C^T\) is also \(\varepsilon\)-simplicial from \(\mathcal{R}(Y, B)\) to \(\mathcal{R}(X, A)\). Then the conclusion follows from Proposition 3.4.

The following proposition is a trivial extension of Proposition 2.2.

Proposition 3.6 If a correspondence \(C : (X, A) \to (Y, B)\) satisfies \(\text{dis} \ C \leq \varepsilon\), then
\[
d_{GH}(X, Y) \leq \frac{\varepsilon}{2} \quad \text{and} \quad d_{GH}(A, B) \leq \frac{\varepsilon}{2}.
\]

Conversely, with some additional assumptions, we show that \(d_H(X, Y) \leq \frac{\varepsilon}{2}\) and \(d_H(A, B) \leq \frac{\varepsilon}{2}\) imply the existence of \(C : (X, A) \to (Y, B)\) satisfying \(\text{dis} \ C \leq \varepsilon\).

Proposition 3.7 Let \(X_1, X_2, Y_1,\) and \(Y_2\) be subspaces of a metric space \((Z, d)\). If \(d_H(X_i, Y_i) < \frac{\varepsilon}{2}\) for \(i = 1, 2\) and \(X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset\), then there exists a correspondence \(C : (X_1 \cup X_2, X_i) \to (Y_1 \cup Y_2, Y_j)\) satisfying \(\text{dis} \ C \leq \varepsilon\) for \(j = 1, 2\).

Proof If \(d_H(X_i, Y_i) \leq \frac{\varepsilon}{2}\), then the set
\[
C_i = \left\{(x, y) \in X_i \times Y_i \mid d(x, y) \leq \frac{\varepsilon}{2}\right\}
\]
forms a correspondence \(C_i : X_i \to Y_i\).

Furthermore, we have \(\text{dis} \ C_i \leq \varepsilon\). Indeed, for any \(x, x', y, y' \in Z\), the triangle inequalities
\[
d(x, x') \leq d(x, y) + d(y, x'), \quad d(x', y) \leq d(x', y') + d(y, y')
\]
imply
\[
d(x, x') - d(y, y') \leq d(x, y) + d(x', y').
\]
Corollary 3.9

Let \( N \subseteq X \). Since

\[ \text{Proof} \]

In this section, we prove the main theorem.

4 PROOF OF THE MAIN THEOREM

We write \( X_{12} = X_1 \sqcup X_2 \) and \( Y_{12} = Y_1 \sqcup Y_2 \). Then the set

\[ C = \{(x, y) \in X_{12} \times Y_{12} \mid (x, y) \in C_1 \text{ or } (x, y) \in C_2 \} \]

is a correspondence from \( X_{12} \) to \( Y_{12} \) satisfying \( \text{dis} \ C \leq \varepsilon \).

Moreover, from \( X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset \), we have

\[ C(X_j) = \{ y \in Y_{12} \mid (x, y) \in C, x \in X_j \} = \{ y \in Y_j \mid (x, y) \in C_j \} \subseteq Y_j. \]

Similarly we obtain \( C^T(Y_j) \subseteq X_j \), and hence \( C \) is a correspondence from \( (X_{12}, X_j) \) to \( (Y_{12}, Y_j) \) satisfying \( \text{dis} \ C \leq \varepsilon \).

As a corollary of Theorem 3.5 and Proposition 3.7, we have the following.

**Corollary 3.8** Let \( X_1, X_2, Y_1, \) and \( Y_2 \) be subspaces of a metric space \( (Z, d) \). If \( d_H(X_i, Y_i) < \frac{\varepsilon}{2} \) for \( i = 1, 2 \) and \( X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset \), then \( H_k(\mathbb{R}(X_1 \sqcup X_2, X_j)) \) and \( H_k(\mathbb{R}(Y_1 \sqcup Y_2, Y_j)) \) are \( \varepsilon \)-interleaved for \( j = 1, 2 \).

For \( A \subseteq X \subseteq D \subseteq \mathbb{R}^d \), if \( f : X \to D \) satisfies \( \sup_{x \in X} \| x - f(x) \| \leq \frac{\varepsilon}{2} \), then \( d_H(A, f(A)) \leq \frac{\varepsilon}{2} \) and \( d_H(X - A, f(X - A)) \leq \frac{\varepsilon}{2} \). Let us substitute \( A, X - A, f(A), \) and \( f(X - A) \) for \( X_1, X_2, Y_1, \) and \( Y_2 \), respectively. If \( f(A) \cap f(X - A) = \emptyset \), then we get the following.

**Corollary 3.9** Let \( A \subseteq X \subseteq D \subseteq \mathbb{R}^d \) and \( f : X \to D \) satisfy \( \sup_{x \in X} \| x - f(x) \| \leq \frac{\varepsilon}{2} \). If \( f(A) \cap f(X - A) = \emptyset \), then \( H_k(\mathbb{R}(X, A)) \) and \( H_k(\mathbb{R}(f(X), f(A))) \) are \( \varepsilon \)-interleaved.

4 PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem.

**Proof** Since \( f(F) \subseteq N_f(\partial D) \) and \( f(X - F) \subseteq D - N_f(\partial D) \), we have \( f(F) = f(X) \cap N_f(\partial D) \). Thus \( f(F) \) is a set of fence sensors of \( f(X) \). Moreover

\[ f(F) \cap f(X - F) \subseteq N_f(\partial D) \cap (D - N_f(\partial D)) = \emptyset, \]
and it follows from Corollary 3.9 that \( H_d(\mathcal{R}(X, F)) \) and \( H_d(\mathcal{R}(f(X), f(F))) \) are \( \varepsilon \)-interleaved. Then the diagram

\[
\begin{array}{ccc}
H_d(\mathcal{R}(X, F; r_s - \varepsilon)) & \xrightarrow{i_*} & H_d(\mathcal{R}(X, F; r_w - \varepsilon)) \\
H_d(G(f)) & \xrightarrow{H_d(G(f))} & H_d(G(f^T)) \\
H_d(\mathcal{R}(f(X), f(F); r_s)) & \xrightarrow{j_*} & H_d(\mathcal{R}(f(X), f(F); r_w))
\end{array}
\]

commutes, where all horizontal maps are linear maps induced by inclusions. Because \( \iota_* \) is nonzero and

\[
\iota_* = j_* \circ i_* = H_d(G(f^T)) \circ H_d(G(f)) \circ i_* = H_d(G(f^T)) \circ j_* \circ H_d(G(f)),
\]

\( j_* \) is nonzero. Therefore it follows from the dSG criterion that \( \mathcal{U}(f(X)) \) contains the restricted domain \( D - N_f(\partial D) \).

\section{5 CONCLUSION}

In this paper, we have presented a new coverage criterion for the coverage problem stable under perturbation. The main mathematical contribution of this paper is the extension of Rips interleaving to the relative version. Relative Rips interleaving enables us to study the coverage of slightly moved sensors. With our criterion, even if we only have the communication data of the sensors before perturbation, we can determine the coverage of the perturbed sensors.

\section{ACKNOWLEDGEMENT}

The authors wish to express their sincere gratitude to Emerson Escolar for valuable discussions on this paper. This work is partially supported by JSPS 24684007.

\section{References}

[1] D. Burago, Y. Burago, and S. Ivanov. A Course in Metric Geometry. volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI. 2001.
[2] F. Chazal, D. Cohen-Steiner, M. Glisse, L.J. Guibas, and S.Y. Oudot. Proximity of persistence modules and their diagrams. In SCG. 237-246. 2009.

[3] F. Chazal, V. de Silva, and S. Oudot. Persistence stability for geometric complexes. Geometriae Dedicata. 2013.

[4] V. de Silva and R. Ghrist. Coordinate-free coverage in sensor networks with controlled boundaries. Int. J. Robotics Research 25 1205-1222. 2006.

[5] V. de Silva and R. Ghrist. Coverage in sensor networks via persistent homology. Algebraic and Geometric Topology 7 339-358. 2007.

[6] H. Edelsbrunner and J. Harer. Computational Topology: an Introduction. American Mathematical Society, Providence, RI. 2010.

[7] A. Hatcher. Algebraic Topology. Cambridge Univ Press. 2001.

[8] R. Mulligan and H. M. Ammari. Coverage in Wireless Sensor Networks: A Survey. Network Protocols and Algorithms vol. 2, 27-53. 2010.

[9] S. Meguerdichian, F. Koushanfar, M. Potkonjak, and M. Srivastava. Coverage problems in wireless ad-hoc sensor network. IEEE INFOCOM vol.3, 1380-1387. 2001

[10] J. R. Munkres. Elements of Algebraic Topology. Westview Press. 1984.

Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan
Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

hiraoka@imi.kyushu-u.ac.jp, ma214017@math.kyushu-u.ac.jp