Quadrupole effects on the motion of extended bodies in Schwarzschild spacetime

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Abstract

The motion of an extended body up to the quadrupolar structure is studied in the Schwarzschild background following Dixon’s model and within certain restrictions (constant frame components for the spin and the quadrupole tensor, center of mass moving along a circular orbit, etc). We find a number of interesting situations in which deviations from the geodesic motion, due to the internal structure of the particle, can originate measurable effects. However, the standard clock effect for a pair of co-counter-rotating bodies spinning up/down is not modified by the quadrupolar structure of the particle.

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1. Introduction

The equations of motion for an extended body in a given gravitational background were deduced by Dixon [1–5] (hereafter the ‘relativistic extended body model’ or simply ‘Dixon’s model’) in multipole approximation to any order. In the quadrupole approximation they read

\begin{align}
\frac{DP^\mu}{d\tau_U} &= -\frac{1}{2} R^\mu_{\nu\alpha\beta} U^\nu S^\alpha\beta - \frac{1}{6} J^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}^{\quad\mu} \equiv F^{(\text{spin})\mu} + F^{(\text{quad})\mu} \\
\frac{DS^\mu}{d\tau_U} &= 2P^{[\mu U^\nu]} - \frac{4}{3} J_{\alpha\beta\gamma}^{\quad\mu} R_{\alpha\beta\gamma}^{\quad\nu},
\end{align}

where $P^\mu = mU^\mu_p$ (with $U_p \cdot U_p = -1$) is the total 4-momentum of the particle and $S^\mu\nu$ is a (antisymmetric) spin tensor; $U$ is the timelike unit tangent vector of the ‘center of mass line’
C\text{U} used to make the multipole reduction, parametrized by the proper time $\tau_{U}$. The tensor $J^{\alpha\beta\gamma\delta}$ is the quadrupole moment of the stress–energy tensor of the body and has the same algebraic symmetries as the Riemann tensor. Using standard spacetime splitting techniques it can be reduced to the following form:

$$J^{\alpha\beta\gamma\delta} = \Pi^{\alpha\beta\gamma\delta} - \bar{u}^{[\alpha} \pi^{\beta]\gamma\delta} - \bar{u}^{[\gamma} \pi^{\delta]\alpha\beta} - 3\bar{u}^{[\alpha} Q^{\beta]\gamma\delta],$$  \hfill (1.3)

where $Q^{\alpha\beta} = Q^{(\alpha\beta)}$ represents the quadrupole moment of the mass distribution as measured by an observer with 4-velocity $\bar{u}$. Similarly $\pi^{\alpha\beta\gamma\delta} = \pi^{(\alpha\beta\gamma\delta)}$ (with the additional property $\pi^{(\alpha \beta \gamma \delta)} = 0$) and $\Pi^{\alpha\beta\gamma\delta} = \Pi^{(\alpha\beta\gamma\delta)}$ are essentially the body’s momentum and stress quadrupole. Moreover, the various fields $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma\delta}$ and $\Pi^{\alpha\beta\gamma\delta}$ are all spatial (i.e. give zero after any contraction by $\bar{u}$). The number of independent components of $J^{\alpha\beta\gamma\delta}$ is 20: 6 in $Q^{\alpha\beta}$, 6 in $\Pi^{\alpha\beta\gamma\delta}$ and 8 in $\pi^{\alpha\beta\gamma\delta}$. When the observer $\bar{u} = U_p$, i.e. in the frame associated with the momentum of the particle, the tensors $Q^{\alpha\beta}$, $\pi^{\alpha\beta\gamma\delta}$ and $\Pi^{\alpha\beta\gamma\delta}$ have an intrinsic meaning.

There are no evolution equations for the quadrupole as well as higher multipoles as a consequence of the Dixon’s construction, so their evolution is completely free, depending only on the considered body. Therefore the system of equations is not self-consistent, and one must assume that all unspecified quantities are known as intrinsic properties of the matter under consideration.

In order the model to be mathematically correct the following additional condition should be imposed on the spin tensor:

$$S^{\mu\nu} U_{\mu\nu} = 0.$$  \hfill (1.4)

Such supplementary conditions (or Tulczyjew–Dixon conditions [1, 6]) are necessary to ensure the correct definition of the various multipolar terms.

Dixon’s model for structured particles originated to complete and give a rigorous mathematical support to the previously introduced Mathisson–Papapetrou model [7–10], i.e. a multipole approximation to any order which includes evolution equations along the ‘center line’ for all the various structural quantities. The models are then different and a comparison between the two is possible at the dipolar order but not once the involved order is the quadrupole.

In this paper, we limit our considerations to Dixon’s model under the further simplifying assumption [11, 12] that the only contribution to the complete quadrupole moment $J^{\alpha\beta\gamma\delta}$ stems from the mass quadrupole moment $Q^{\alpha\beta}$, so that $\pi^{\alpha\beta\gamma\delta} = 0 = \Pi^{\alpha\beta\gamma\delta}$ and

$$J^{\alpha\beta\gamma\delta} = - 3U_{\rho} Q^{(\alpha\beta\gamma\delta)}, \quad Q^{\alpha\beta} U_{\rho\beta} = 0.$$  \hfill (1.5)

Let us introduce the spin vector by spatial (with respect to $U_{\rho}$) duality

$$S^{\beta} = \frac{1}{2} \eta^{\beta\gamma\delta} U_{\rho} S_{\rho\gamma\delta},$$  \hfill (1.6)

where $\eta^{\beta\gamma\delta} = \sqrt{-g} \epsilon^{\beta\gamma\delta}$ is the unit volume 4-form and $\epsilon_{0123} = 1$ is the Levi-Civita alternating symbol, as well as the scalar invariant

$$s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}.$$  \hfill (1.7)

In general $s$ is not constant along the trajectory of a spinning particle.

The assumption that the particle under consideration is a test particle means that its mass, its spin as well as its quadrupole moments must all be small enough not to contribute significantly to the background metric. Otherwise, backreaction must be taken into account.
2. Motion of extended bodies in Schwarzschild spacetime

Let us consider the case of the Schwarzschild spacetime, with the metric written in standard Schwarzschild coordinates,
\[
ds^2 = -\left(1 - \frac{2M}{r}\right) dr^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]
and introduce an orthonormal frame adapted to the static observers
\[
e_j = (1 - 2M/r)^{-1/2} \partial_j, \quad e_\phi = (1 - 2M/r)^{1/2} \partial_\phi,
\]
with dual
\[
\omega^j = (1 - 2M/r)^{1/2} dr, \quad \omega^\phi = (1 - 2M/r)^{-1/2} dr,
\]
Let us assume that \(U\) is tangent to a (timelike) spatially circular orbit, hereafter denoted as the \(U\)-orbit, with
\[
U = \Gamma [\partial_\theta + \xi \partial_\phi] = \gamma [e_j + v e_\phi], \quad \gamma = (1 - v^2)^{-1/2},
\]
where \(\xi\) is the angular velocity with respect to infinity and \(\Gamma\) is a normalization factor
\[
\Gamma = (-g_{tt} - \xi^2 g_{\phi\phi})^{-1/2}
\]
which assures that \(U \cdot U = -1\); here dot means scalar product with respect to the metric (2.1). The angular velocity \(\xi\) is related to the local proper linear velocity \(v\) measured in the frame (2.2) by
\[
\xi = \sqrt{-\frac{g_{tt}}{g_{\phi\phi}}} v.
\]
Here \(\xi\) and therefore also \(v\) are assumed to be constant along the \(U\)-orbit. We limit our analysis to the equatorial plane (\(\theta = \pi/2\)) of the Schwarzschild solution; as a convention, the physical (orthonormal) component along \(-\partial_\phi\), perpendicular to the equatorial plane, will be referred to as along the positive \(z\)-axis and will be indicated by \(\xi\), when necessary.

Among the circular orbits particular attention is devoted to the co-rotating (\(\xi_+\)) and counter-rotating (\(\xi_-\)) timelike circular geodesics, having respectively \(\xi_\pm = \pm \xi_K = \pm (M/r^3)^{1/2}\), so that
\[
U_\pm = \gamma_K [e_j \pm v_K e_\phi], \quad v_K = \left[\frac{M}{r - 2M}\right]^{1/2}, \quad \gamma_K = \left[\frac{r - 2M}{r - 3M}\right]^{1/2},
\]
with the timelike condition \(v_K < 1\) satisfied if \(r > 3M\). It is convenient to also introduce the Lie relative curvature [13, 14] of each orbit
\[
k_{(\text{lie})} = -\partial_t \ln \sqrt{g_{\phi\phi}} = -\frac{1}{r} \left(1 - \frac{2M}{r}\right)^{1/2} = -\frac{\xi_K}{v_K}.
\]
Let \(P = mU_p\) such that
\[
U_p = \gamma_p [e_j + v_p e_\phi]\quad \gamma_p = (1 - v_p^2)^{-1/2},
\]
i.e. let us assume that \(U_p\) also is tangent to a circular orbit and set up an orthonormal frame adapted to \(U_p\) given by
\[
e_0 = U_p, \quad e_1 = e_j, \quad e_2 = \gamma_p [v_p e_j + e_\phi], \quad e_3 = e_\phi;
\]
hereafter all frame components of the various fields are meant to be referred to such a frame. Note that the assumption of having both $U$ and $U_{\nu}$ aligned with a circular orbit certainly represents a restriction to our analysis, but leads to great simplifications to Dixon’s equations. On the other hand, dealing with the model in its complete generality, i.e. with $U_{\nu}$ having no relation a priori with $U$, is a hard task to pursue analytically due to its mathematical complexity, so that a numerical analysis would be needed.

The spin vector is orthogonal to $U_{\nu}$, so that

$$S = S^1 e_1 + S^2 e_2 + S^3 e_3.$$  \hspace{1cm} (2.11)

Furthermore, we have

$$Q_{00} = Q_{01} = Q_{02} = Q_{03} = 0.$$ \hspace{1cm} (2.12)

We also assume that $S^1 = 0 = S^2$ and that the remaining component $S^3 = S^3_x = s$ (mimic of a particle moving on a circular orbit and spinning around the $z$-axis) as well as the surviving components of the mass quadrupole moment are all constant along the path. The latter assumption corresponds to the definition of ‘quasirigid motion’ (or ‘quasirigid bodies’) due to Ehlers and Rudolph [12]. Clearly, in a more realistic situation the latter hypothesis should be released.

Consider first the evolution equations (1.2) for the spin tensor. They imply that

$$Q_{12} = Q_{13} = Q_{23} = 0,$$ \hspace{1cm} (2.13)

and introducing the following ‘structure functions’ of the extended body

$$Q_{11} = Q_{33} + f, \quad Q_{22} = Q_{33} + f',$$ \hspace{1cm} (2.14)

they also give

$$0 = (v v_p - v_k^2) s + m \frac{v_k}{\xi_k} (v - v_p) + 3 v_k \xi_k \frac{y_p v_p f}{\gamma}.$$ \hspace{1cm} (2.15)

Consider then the equations of motion (1.1). They imply that

$$0 = v_k \xi_k (2 v_p + v) s - \frac{3}{2} \frac{\xi_k^2}{\gamma f} \left[ -f' + (2 + v_p^2) f \right] + m (v v_p - v_k^2).$$ \hspace{1cm} (2.16)

After eliminating $s$ through equation (2.15), the previous equation becomes

$$0 = (v v_p - v_k^2) (-f' + 2 v_p^2 f) + y_p^2 \left[ 2 v v_p v_k^2 + v_p^2 (v v_p + v_k^2) + 2 v_k^2 v_p f \right]$$

$$- \frac{2 m v_p y_p}{3 \xi_k} \left( v^2 v_p^2 + 2 v_k^2 v_p^2 - 3 v v_p v_k^2 - v^2 v_k^2 + v_k^4 \right),$$ \hspace{1cm} (2.17)

where second- and fourth-order polynomial expressions in the velocities have been collected. Solving the last two equations for $v$ and $v_p$ in terms of $s$ and $f$, $f'$ completely determines the motion.

The quadrupole moment tensor of a mass distribution of density $\mu$ is defined classically by

$$Q^a_{\text{class}} = \int_V \mu (3 x^a x^b - r^2 \delta^{ab}) \, d^3x \quad a = 1, 2, 3$$ \hspace{1cm} (2.18)

with $r^2 = \delta_{ab} x^a x^b$ and it is tracefree. It is therefore natural to assume the same property holding for the relativistic quadrupole moment tensor, obtaining

$$0 = Q_{11} + Q_{22} + Q_{33} = Q_{33} + f + Q_{33} + f' + Q_{33} = 3 Q_{33} + f + f',$$ \hspace{1cm} (2.19)

so that the components $Q_{ab}$ in this case are completely determined by the two ‘structure functions’ $f$ and $f'$:

$$Q_{11} = \frac{2}{3} f - \frac{1}{3} f', \quad Q_{22} = -\frac{1}{3} f + \frac{2}{3} f', \quad Q_{33} = -\frac{1}{3} (f + f').$$ \hspace{1cm} (2.20)
If the body is axially symmetric about the z-axis, then \( f = f' \) and the frame components of \( Q \) reduce to

\[
Q_{ab} = \text{diag}[f/3, f/3, -2f/3]. \tag{2.21}
\]

Without entering the problems of a relativistic definition of the quadrupole moment tensor \( Q_{ab} \) generalizing equation (2.18) in terms of integrals over the volume of the body itself (the correct procedure is outlined in Dixon’s works [1–5]), we note that \( Q_{ab} \) (only defined all along the worldline with tangent vector \( U \)) can be interpreted as the mass quadrupole moment of the extended body as measured by the observer \( U_p \).

It is quite natural to introduce the following rescaled dimensionless angular and quadrupolar momentum quantities,

\[
\sigma = \frac{s}{m} \zeta_K, \quad F = \frac{f}{m} \zeta_K^2, \quad F' = \frac{f'}{m} \zeta_K^2, \tag{2.22}
\]
due to the fact that along a circular orbit \( r = \text{const} \). The quantities \( \sigma, F \) and \( F' \) are necessarily small. Although the quadrupolar terms \( f \) and \( f' \) are small only for a quasi-spherical body, the further rescaling by \( \zeta_K = (M/r^3)^{1/2} \) makes indeed them small in any case. In fact, the radius of the orbit is assumed to be large enough in comparison with certain natural length scales such as \( |s|/m \) (also known as the Møller radius [15] of the body), \( (|f|/m)^{1/2}, (|f'|/m)^{1/2} \) associated with the body itself in order to avoid backreaction effects. Furthermore, we also require that the characteristic length associated with the quadrupole is small if compared to the Møller radius of the body, in order that the multipolar expansion of the body’s stress–energy tensor underlying the Dixon’s model be consistent.

Equations (2.15) and (2.17) then become

\[
0 = (\nu v_p - \nu_K^2)\sigma + \nu_K (v - v_p) + 3 \nu_K \frac{\gamma_p v_p}{\gamma} F \tag{2.23}
\]

and

\[
0 = (\nu v_p - \nu_K^2)(-F' + 2\gamma_p^2 F + \gamma_p^2 [2\nu v_p v_K^2 + \nu^2 (\nu v_p + v_K^2) + 2v_K^2 v_p^2] F
- \frac{2}{\gamma} \gamma_p (v^2 v_p^2 + 2v_K^2 v_p^2 - 3 v_p v_K^2 - v^2 v_K^2 + v_K^4)). \tag{2.24}
\]

The above relations define the kinematical conditions allowing circular motion of the extended body taking into account its spinning and quadrupolar structures. The contribution due to the spinning structure disappears (i.e. remains arbitrary) when \( \nu v_p = \nu_K^2 \), i.e. if \( \nu_K \) is taken to be the geometrical mean of \( \nu \) and \( v_p \). In fact, in this case, we have

\[
F = -\frac{\gamma (v - v_p)}{3 \gamma_p v_p} \tag{2.25}
\]

and

\[
0 = (\nu_K^2 + 2\nu_p^2)(v - v_p) + (2\nu_p^2 - v^2 - \nu_K^2) v_p. \tag{2.26}
\]

After some manipulation using the condition \( \nu v_p = \nu_K^2 \), the latter equation turns out to be identically satisfied, whereas equation (2.25) becomes

\[
F = -\frac{\gamma}{3 \nu_K^2} \sqrt{1 - \frac{v_K^2}{v^2}(v^2 - \nu_K^2)}, \tag{2.27}
\]

with \( F' \) arbitrary.

Equations (2.23) and (2.24) can be examined in other special cases; for example,
(i) $\sigma = 0$, $F \neq 0$, $F' \neq 0$.

$$0 = v - v_p + 3\gamma_p v_p, \quad 0 = (v v_p - v_k^2)\left(F' + 2\gamma_p^2 F + \gamma_p^2 \left[2 v v_p v_k^2 + v_k^2 (vv_p + v_k^2) + 2 v_k^2 v_p^2\right]F\right)$$

$$- \frac{2}{3} \gamma_p \left(v^2 v_p^2 + 2 v_p^2 v_k^2 - 3 v v_p v_k^2 - v^2 v_k^2 + v_k^2\right). \quad (2.28)$$

We note that if $F = 0$ these equations imply $v = v_p$ and

$$0 = (v_p^2 - v_k^2)\left(F' + \frac{3}{2} \gamma_p^2 (v_p^2 - v_k^2)\right), \quad (2.29)$$

that is $v_p = \pm v_k$, or if $v_p^2 \neq v_k^2$

$$F' = - \frac{3}{2} \gamma_p^2 (v_p^2 - v_k^2), \quad (2.30)$$

allowing nongeodesic motion only due to the quadrupole moment tensor.

(ii) $\sigma \neq 0$, $F = 0$, $F' \neq 0$.

$$0 = (v v_p - v_k^2)\sigma + v_k^2 (v - v_p), \quad 0 = (v v_p - v_k^2)F' + \frac{3}{2} \gamma_p (v^2 v_p^2 + 2 v_p^2 v_k^2 - 3 v v_p v_k^2 - v^2 v_k^2 + v_k^2). \quad (2.31)$$

(iii) $\sigma \neq 0$, $F \neq 0$, $F' = 0$.

$$0 = (v v_p - v_k^2)\sigma + v_k^2 (v - v_p) + 3 v_k^2 \gamma_p v_p F, \quad 0 = \gamma_p^2 \left[2 v v_p (1 + v_k^2) + v_k^2 (vv_p + 3 v_k^2) - 2 v_k^2\right]F$$

$$- \frac{3}{2} \gamma_p (v^2 v_p^2 + 2 v_p^2 v_k^2 - 3 v v_p v_k^2 + v^2 v_k^2 + v_k^2). \quad (2.32)$$

The case of a spinning particle with vanishing quadrupole moment tensor, i.e. $F = 0 = F'$, has been already studied in [16].

Let us turn to equations (2.23)–(2.24) and investigate the case of extended bodies with internal structure (dipolar and quadrupolar) compatible with a nearly geodesic motion. In the case of vanishing quadrupole moments, equations (2.23)–(2.24) reduce to [16]

$$0 = (v v_p - v_k^2)\sigma + v_k^2 (v - v_p), \quad 0 = v^2 v_p^2 + 2 v_p^2 v_k^2 - 3 v v_p v_k^2 - v^2 v_k^2 + v_k^4. \quad (2.33)$$

In the limit of small spin $\sigma$ we find

$$v = \pm v_k - \frac{3}{2} v_k \sigma + O(\sigma^2), \quad v_p = v + O(\sigma^2). \quad (2.34)$$

If the contribution of quadrupolar terms can be considered negligible with respect to the dipolar ones and comparable with second-order terms in the spin itself, one can consider corrections to equation (2.34) as given by

$$v_{\pm} \simeq \pm v_k - \frac{3}{2} v_k \sigma \pm \frac{3}{8} (2 F + \sigma^2) v_k, \quad v_{\pm}^{(\pm)} \simeq v_{\pm} \pm 3 (F - \sigma^2) v_k, \quad (2.35)$$

where the signs $\pm$ correspond to co/counter rotating orbits. The corresponding angular velocity $\xi_{\pm} = (\xi_k / v_k) v_{\pm}$ and its reciprocal are

$$\xi_{\pm} \simeq \pm \xi_k \left[1 \mp \frac{3}{2} \sigma \pm \frac{3}{8} (2 F + \sigma^2)\right], \quad (2.36)$$

$$\frac{1}{\xi_{\pm}} \simeq \pm \frac{1}{\xi_k} \pm \frac{3}{2 \xi_k} \sigma \pm \frac{3}{8 \xi_k} (5 \sigma^2 - 2 F).$$
Furthermore, the period of revolution around the central source will consist of three different terms

\[ T = \frac{2\pi}{|\xi|} = T_K |1 \pm \lambda_d + \lambda_q|, \]  

(2.37)

where

\[ T_K = \frac{2\pi}{\xi_K}, \quad \lambda_d = \frac{3}{2} \sigma, \quad \lambda_q = \frac{3}{8} (5\sigma^2 - 2F). \]  

(2.38)

A direct measurement of \( T \) will then allow us to estimate the quantity \( F \) determining the quadrupolar structure of the body, if its spin is known. Note that the fraction \( \lambda_d \) due to the spin is different depending on whether the body is spinning up or down, whereas the term \( \lambda_q \) due to the quadrupole has a definite sign once the shape of the body is known (\( F \) cannot change its sign).

In the case of the Earth the nondimensional quantities (2.22) turn out to be given by \( \sigma \approx 2.1 \times 10^{-15} \) and \( F = F' \approx -1.8 \times 10^{-20} \), since \( s/m_\oplus \approx 3.4 \times 10^2 \) cm and \( f = f' = -J_2 m_\oplus \sigma^2 \), with \( J_2 \approx 10^{-3} \), and the distance between the Earth and the Sun is \( r \approx 1.5 \times 10^{13} \) cm. The correction to the geodesic value \( T_K \approx 9.425 \times 10^{17} \) cm due to the spin is \( \lambda_d \approx 3.4 \times 10^{-15} \), whereas the correction due to the quadrupole turns out to be \( \lambda_q \approx 1.3 \times 10^{-20} \) as from equation (2.38).

An interesting opportunity to test the quadrupole effect of extended body would arise from the motion of a binary pulsar system around Sgr A*, the supermassive (\( M \approx 10^6 M_\odot \)) black hole located at the galactic center [17, 18]. To illustrate the order of magnitude of the effect, we may consider the binary pulsar system PSR J0737-3039 as orbiting Sgr A* at a distance of \( r \approx 10^9 \) km. The PSR J0737-3039 system consists of two close neutron stars (their separation is only \( d_{AB} \approx 8 \times 10^5 \) km) of comparable masses \( m_A \approx 1.4 M_\odot, m_B \approx 1.2 M_\odot \), but very different intrinsic spin period (23 ms of pulsar A versus 2.8 s of pulsar B) [19]. Its orbital period is about 2.4 h, the smallest yet known for such an object. Since the intrinsic rotations are negligible with respect to the orbital period, we can treat the binary system as a single object with reduced mass \( \mu_{AB} \approx 0.7 M_\odot \) and intrinsic rotation equal to the orbital period. The spin parameter thus turns out to be equal to \( \sigma \approx 6 \times 10^{-8} \), whereas the quadrupolar parameters are \( F = F' \approx 9.6 \times 10^{-10} \), since we have taken \( f = f' = \mu_{AB} d_{AB}^2 \) as a rough estimate. The correction to the geodesic value \( T_K \approx 1.6 \times 10^{16} \) cm due to the spin is \( \lambda_d \approx 9 \times 10^{-8} \), whereas the correction due to the quadrupole turns out to be \( \lambda_q \approx -7.2 \times 10^{-10} \) as from equation (2.38).

The black hole at the galactic center is actually expected to be rotating [20]. Therefore, a more detailed analysis would take into account the effect of rotation of Sgr A* on the estimate of the various contributions to the period of revolution of the PSR J0737-3039 binary system. We will discuss such an extension of the present treatment in a forthcoming paper.

3. Quadrupolar corrections to the clock effect for spinning test particles

Bini, de Felice and Geralico have investigated in [16] the gravitomagnetic clock effect appearing for oppositely orbiting both spin-up or spin-down particles in the Schwarzschild spacetime. They found that spinning test particles move on circular orbits which, to first order in the spin parameter \( \sigma \), are close to a geodesic, with

\[ \frac{1}{\xi_{\pm, \pm}} = \pm \frac{1}{\xi_K} \pm \frac{3}{2\xi_K} |\sigma|, \]  

(3.1)

where the signed magnitude \( \sigma = \pm |\sigma| \) of the spin parameter has been introduced. The signs in front of \( 1/\xi_K \) correspond to co/couter-rotating orbits while the signs in front of \( |\sigma| \)
refer to a positive or negative spin direction along the $z$-axis; for instance, the quantity $\zeta_{(+,-)}$ denotes the angular velocity of $U$ corresponding to a co-rotating orbit (+) with spin-down (−) alignment, etc. Therefore, one can measure the difference in the arrival times after one complete revolution with respect to a static observer. The coordinate time difference is given by

$$
\Delta t_{(+,+;-,+)} = 2\pi \left( \frac{1}{\zeta_{(+,+)}} + \frac{1}{\zeta_{(-,+)}} \right) = \frac{6\pi}{\xi K} |\sigma|,
$$

(3.2)

and analogously for $\Delta t_{(+,-;-,+)}$.

Let us now analyze the introduction of quadrupolar terms. Equation (2.36) implies that equation (3.1) becomes

$$
\frac{1}{\zeta_{(k,b)}} = \pm \frac{1}{\zeta K} \pm \frac{3}{2\xi K} |\sigma| \pm \frac{3}{8\xi K} (5\sigma^2 - 2F),
$$

(3.3)

so that the last term does not contribute to the clock effect, since the ± signs in front of it correspond to co/counter-rotating orbits (like those in front of $1/\xi K$) and thus cancels anyway. Therefore, no modifications are induced by quadrupolar terms in equation (3.2).

4. Concluding remarks

We have studied the motion of quadrupolar particles on a Schwarzschild background following Dixon’s model. In the simplified situation of constant frame components (with respect to a natural orthonormal frame) of both the spin and the quadrupole tensor of the particle we have found the kinematical conditions to be imposed on the particle’s structure in order that the orbit of the particle itself be circular and confined on the equatorial plane. Co-rotating and counter-rotating particles result in a non-symmetric speed in spite of the spherical symmetry of the background, due to their internal structure. This fact has been anticipated when studying spinning particles only, i.e. with vanishing quadrupole moments [16].

We have then discussed the modifications due to the quadrupole which could be eventually observed in experiments. Such experiments, however, cannot concern standard clock effects, because we have shown that there are no contributions arising from the quadrupolar structure of the body in this case. In contrast, the effect of the quadrupole terms could be important when considering the period of revolution of an extended body around the central source. Measuring the period will provide an estimate of the quantities $F, F'$ determining the quadrupolar structure of the body, if its spin is known. On the other hand, the complete knowledge of the internal structure of the body will allow the estimate of the period of revolution. In the latter case, the comparison between the measured period (known from observations) with the value predicted by the Dixon’s model could provide a test for the model itself.

It would be of great interest to extend this analysis to systems with varying quadrupolar structure and emitting gravitational waves without perturbing significantly the background spacetime. Usually, variable quadrupole moment is generated in a test astronomical body of mass $m$ because of tides produced by the central source of mass $M \gg m$. Now the net gravitational radiation due to motion of $m$ is due to its orbit around $M$, the time varying tides and the interference between these two [21, 22]. We deserve such an investigation in future works.

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