Spontaneous generation of dark-bright and dark-antidark solitons upon quenching a particle-imbalanced bosonic mixture

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We unveil the dynamical formation of multiple localized structures in the form of dark-bright and dark-antidark solitary waves that emerge upon quenching a one-dimensional particle-imbalanced Bose-Bose mixture. Interspecies interaction quenches drive the system out-of-equilibrium while the so-called miscible/immiscible threshold is crossed in a two directional manner. Dark-bright entities are spontaneously generated for quenches towards the phase separated regime and dark-antidark states are formed in the reverse process. The distinct mechanisms of creation of the aforementioned states are discussed in detail and their controlled generation is showcased. In both processes, it is found that the number of solitary waves generated is larger for larger particle imbalances, a result that is enhanced for stronger postquench interspecies interactions. Additionally the confining geometry highly affects the production of both types of states with a decaying solitary wave formation occurring for tighter traps. Finally, and also in both of the aforementioned transitions, the breathing frequencies measured for the species differ significantly for highly imbalanced mixtures.

I. INTRODUCTION

Over the past two decades, nonlinear wave phenomena in Bose-Einstein condensates (BECs) have attracted considerable interest. Early experiments on single component BECs triggered a vast amount of studies devoted in investigating the properties of non-linear excitations that arise in them, such as dark and bright solitons. These structures exist as stable configurations in highly elongated quasi one-dimensional (1D) BECs featuring repulsive and attractive interatomic interactions respectively. Besides single component BECs, the experimental realization of two-component mixtures enabled the investigation of more complex compounds, the so-called vector solitons. These include multiple dark and/or bright states such as dark-dark and dark-bright (DB) solitons (see also the review [35]). Moreover, considerable attention has been paid in understanding their properties and interactions not only in the so-called integrable Manakov limit [52, 53], but also upon breaking integrability. The latter breaking can be achieved e.g. by the presence of external trapping [51, 55], or by considering different interactions between the same or different atomic species [27, 54, 57].

The generation of DB states is a topic of ongoing research activity for a multitude of reasons. Remarkably these states appear in binary BECs with repulsive interparticle interactions, an environment into which bright solitons, being localized density peaks, would disperse if dark solitons were absent. This symbiotic nature of DB solitons leads to their remarkably prolonged lifetimes compared to their single component analogues. Thus DB states constitute ideal candidates for applications as e.g. coherent storage as well as processing of optical fields, multi-channel signals and their switching [24, 58–60].

DB solitons exist both in weakly miscible and immiscible BEC mixtures. The miscibility condition within mean-field theory is determined by \( a_{AB}^2 < a_{AA} a_{BB} \) [61, 64] where \( a_{ij} \) is the intra- and \( a_{ij} \) denotes the interspecies scattering lengths. In turn these scattering lengths can be experimentally tuned via Feshbach resonances allowing the realization of a transition between the miscible and immiscible regime of interactions [65, 70]. The tunability of the scattering lengths opened an new path for studying, in a controllable manner, the properties of binary BECs and the solitonic structures that form in them. In this way, vector solitons have been identified upon considering e.g. temporal [71–73] and spatial [74, 75] variations of the scattering lengths. In the same context control over the system’s parameters has been examined by performing temporal modulations of the confinement frequencies [71, 73] and also by adding appropriate time-dependent gain and loss terms [24, 60]. Moreover, control over the system’s dynamical evolution and pattern formation has been reported in [77], upon quenching the binary BEC towards miscibility. Also very recently the spontaneous generation of vector solitons upon quenching the interspecies interactions crossing the miscibility/immiscibility threshold both within mean-field but also considering the many-body treatment of this problem has been reported [41].

Here, robustly propagating DAD states were shown to dynamically emerge. The latter observation rises a natural question on how different variations of the systems’
parameters can lead to the controllable creation of these states at least within mean-field theory.

Motivated by the aforementioned studies, in the present work we take a closer look on the multi-soliton formation upon considering the interspecies interaction-quenched dynamics of a particle imbalanced BEC mixture while crossing the miscibility/immiscibility threshold in both directions. In this way we aim at deepening our understanding regarding vector soliton generation in an integrability broken scenario. In this context dynamical instabilities lead to peculiar evolution of the resulting states including mass redistribution between the solitons \[54, 57\], and thus shape changing collisions \[60, 78\]. In particular, we investigate how the number of vector solitons formed can be controlled under different variations of the binary systems’ parameters as e.g. the trapping frequency and the associated with each species particle number.

In the miscible-to-immiscible quench scenario, the unstable dynamics leads to the filamentation of the density of both species in line with our earlier findings \[61\]. This filamentation process entails the formation of DB solitary waves. The formation of the latter is verified upon fitting the waveforms corresponding to a dark and a bright soliton stemming from the exact (in the Manakov limit) single DB state \[26, 27, 57\]. Focusing on the initial stages of the nonequilibrium dynamics, an increase of the number of the DB states generated is found upon varying the particle number of the minority species until the particle balanced limit is reached. A result that is more enhanced for transitions that enter deeper in the immiscible phase \[41\]. Lastly, a decaying formation of DB states is observed as the trapping frequency is increased. In all cases both clouds undergo a collective breathing motion. Moreover, the generated DB states are seen to oscillate within the parabolic trap, featuring multiple collisions with one another. However, due to the non-integrable nature of the system, these collisions are also accompanied by a transfer of mass between the soliton constituents \[55, 57\]. The latter effect reduces, during evolution, the number of states generated with significantly fewer DB solitary waves remaining and robustly propagating at large evolution times. Finally, inspecting the corresponding breathing frequency of each bosonic cloud we demonstrate that significant differences occur for highly particle imbalanced mixtures \[79\].

Following the reverse process, namely an immiscible-to-miscible transition, robust DAD solitons are formed \[41\]. The binary system, is the 1D analogue of the so-called “ball and shell” configuration \[80, 81\], with one species occupying the center of the trap and the other species residing on the two sides of the harmonic potential. After the quench the outside species is allowed to “fall” towards the central region inducing a counterflow. The latter leads in turn to the appearance of interference fringes \[82, 84\] and dark solitons emerge in the regions of destructive interference \[51\]. It is the effective potential created by the emergent dark states that causes the other species to split and fill the newly formed density dips, leading in turn to the formation of DAD structures. An almost linear increase of the number of DAD states formed occurs either as the particle number of the inner species is increased or entering deeper in the miscible phase. Also, tighter trapping results to fewer DAD solitons occurring during the nonequilibrium dynamics for larger postquench interspecies interactions. Finally, different breathing frequencies are measured for the two species independently of the postquench interaction strength and the particle imbalance.

Our presentation is structured as follows. In section \[II\] we provide the theoretical framework of our mean-field approach in the form of a set of two coupled Gross-Pitaevskii equations (GPEs). We also briefly comment on the numerical methods utilized herein. In section \[III\] we present our results for miscible-to-immiscible quenches while section \[IV\] contains our results for the reverse process. Finally section \[V\] summarizes our findings and also provides future perspectives.

### II. THEORETICAL FRAMEWORK

#### A. Mean-Field ansatz for a binary Bose-mixture

We consider a binary mixture of repulsively interacting BECs composed of two different hyperfine states, namely \(|F = 1, m_F = -1\rangle\) and \(|F = 2, m_F = 1\rangle\), of \(87\)Rb \[85\] being confined in a one-dimensional (1D) harmonic oscillator potential. Such a cigar-shaped geometry can be realized experimentally \[9, 22, 36\] upon considering a highly anisotropic trap with the longitudinal and transverse trapping frequencies obeying \(\omega_2 \ll \omega_\perp\). Within mean-field theory the dynamics of this binary mixture can be described by the following system of coupled GPEs \[11, 2\]:

\[
\begin{align*}
    i\hbar \partial_t \tilde{\Psi}_\sigma &= \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \tilde{V}(\tilde{x}) - \mu\sigma + \sum_{k=A,B} \tilde{g}_{\sigma k} |\tilde{\Psi}_k|^2 \right) \tilde{\Psi}_\sigma.
\end{align*}
\]

In the above expression \(\tilde{\Psi}_\sigma(\tilde{x}, \tilde{t}) (\sigma = A, B)\) denotes the wavefunction for the A- and B-species respectively. Each \(\tilde{\Psi}_\sigma(\tilde{x}, \tilde{t})\) is normalized to the corresponding number of atoms i.e. \(N_\sigma = \int_{-\infty}^{+\infty} |\tilde{\Psi}_\sigma|^2 d\tilde{x}\). Also \(\tilde{V}(\tilde{x})\) represents the external trapping potential, while \(m_A = m_B = m\) and \(\mu_\sigma\) refer to the atomic mass and chemical potential for each of the species respectively. The effective 1D coupling constants are given by \(\tilde{g}_{\sigma k} = 2\hbar \omega_\perp a_{\sigma k}\), where \(a_{\sigma k}\) are the three \(s\)-wave scattering lengths (with \(a_{AB} = a_{BA}\)) accounting for collisions between atoms that belong to the same (\(a_{\sigma\sigma}\)) or different (\(a_{\sigma k}, \sigma \neq k\)) species. We note that both the intra and interspecies scattering lengths can be manipulated experimentally by means of Feshbach \[88, 89\] or confinement induced resonances \[88, 89\].
For the numerical findings to be presented below, we choose to express the system of Eqs. (1) in the following dimensionless form:

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial \Psi_A}{\partial t} &= \left[ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V + g_{AA}|\Psi_A|^2 + g_{AB}|\Psi_B|^2 - \mu_A \right] \Psi_A, \\
\frac{i}{\hbar} \frac{\partial \Psi_B}{\partial t} &= \left[ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V + g_{BB}|\Psi_B|^2 + g_{AB}|\Psi_A|^2 - \mu_B \right] \Psi_B.
\end{align*}
\]

(2)

(3)

In the above Eqs. (2)-(3), the interaction coefficients are normalized to the intraspecies scattering length, \(a_{AA}\), of the \(A\)-species, namely \(g_{AB} = \bar{g}_{AB}/\bar{g}_{AA}\), and \(g_{BB} = \bar{g}_{BB}/\bar{g}_{AA}\). Moreover, densities \(|\Psi_\sigma|^2\), length, energy and time are measured in units of \(2a_{AA}, \sqrt{\hbar/m_\sigma}, \hbar\omega_\perp\) and \(\omega_\perp^{-1}\) respectively. Finally, \(V(x) = \frac{1}{2} \omega x^2\) is the dimensionless trapping potential with \(\omega_A = \omega_B = \omega = \omega_\perp/\omega_\perp\).

In the present work we aim at revealing the out-of-equilibrium dynamics of the aforementioned binary system [see Eqs. (2)-(3)] upon performing an interspecies interaction quench from the miscible to the immiscible regime of interactions and vice versa. To this end we fix the intraspecies coefficients to the experimentally relevant values for a \(^{87}\text{Rb}\) mixture, i.e. \(g_{AA} = 1.0, g_{BB} = 0.95\) [85]. To induce the dynamics an interspecies interaction quench is performed with the system being initially relaxed to its ground state lying within the miscible regime of interactions. The latter is characterized by an initial (prequench) interaction \(g^{i}_{AB}\) subject to the miscibility condition \(g^{i}_{AB} < \sqrt{g_{AA}g_{BB}}\) [89]. Consecutively the system is quenched towards the immiscible (alias phase separated) regime having the desired postquench amplitude \(g^{f}_{AB}\) [see Sec.III]. The same procedure is followed for the inverse process, namely considering a relaxed state for immiscible components \((g^{f}_{AB} > \sqrt{g_{AA}g_{BB}})\) and quenching the system towards the miscible domain [see Sec.IV]. In both quench scenarios under investigation in order to obtain the ground state configuration a fixed point iteration scheme is employed [90]. To simulate the nonequilibrium dynamics of the binary mixture governed by Eqs. (2)-(3), a fourth-order Runge-Kutta integrator is used with spatial and time discretization \(dx = 0.1\) and \(dt = 0.005\) respectively. Moreover our numerical computations are restricted to a finite region by employing hard-wall boundary conditions. The latter are chosen wide enough to avoid boundary effects. In particular, for the majority of our simulations, they are located at \(x_\pm = \pm 80\) and we do not observe any appreciable density for \(|x| > 60\).

III. QUENCH DYNAMICS FROM THE MISCIBLE TO THE IMMISCIBLE REGIME

In the following we discuss the vector soliton formation considering quenches from the miscible to the immiscible regime, namely for \(g^{f}_{AB} > \sqrt{g_{AA}g_{BB}}\). Before proceeding, it is worth mentioning at this point that the latter relation is valid for the homogeneous case \((V=0)\). Nonetheless, corrections due to the presence of the trap are only significant for strong trapping frequencies \([91]\) that are not considered in the present work. In the following the trapping frequency is fixed to \(\omega = 0.05\) unless it is stated otherwise. Additionally, since the miscibility depends on the interaction strength between the species, for a \(^{87}\text{Rb}\) mixture considered herein the transition between the two regimes occurs at \(g^{f}_{AB} = \sqrt{0.95} \approx 0.974\). Thus for quenches with \(g^{f}_{AB} < g^{h}_{AB}\) it is expected that the two species prefer overlapping thereby residing around the trap center. However, for postquench interspecies interactions having \(g^{f}_{AB} > g^{h}_{AB}\) component separation takes place.

\[\text{FIG. 1. Spatio-temporal evolution of the one-body density } |\Psi_\sigma|^2 \text{ of species } \sigma = A \text{ (left panels) and species } \sigma = B \text{ (right panels). An interspecies interaction quench from } g^{AB} = 0.2 \text{ to different postquench values (a), (b) } g^{AB} = 0.8, \text{ (c), (d) } g^{AB} = 1.5 \text{ and (e), (f) } g^{AB} = 2.5 \text{ is performed. In all cases the particle number of species is } N_A = 65 \text{ and } N_B = 25 \text{ respectively, while both species are trapped in a harmonic oscillator of frequency } \omega = 0.05. \text{ The system is prepared in its ground state with } g^{f}_{AB} = 1.0, g^{f}_{BB} = 0.95 \text{ and } g^{f}_{AB} = 0.2. \text{ The dashed vertical lines mark time instants before and after specific collision events indicated by the circles.}\]

A. Density evolution within the miscible domain and breathing mode

Figures [1](a)-(f) illustrate the quenched density evolution upon increasing \(g^{f}_{AB}\). In all cases the system is initially prepared in the miscible regime with \(g^{i}_{AB} = 0.2\), while \(N_A = 65\) and \(N_B = 25\). As per our discussion above when the interaction is quenched e.g to \(g^{f}_{AB} = 0.8\) depicted in Figs. [1](a)-(b) for \(A\)-species and \(B\)-species respectively, component separation is absent. Instead
the two species are overlapping while residing around the trap center for all evolution times. Notice also that since $g_{BB} < g_{AA}$, $B$-species is localized closer to the trap center when compared to the $A$-species. Moreover, a collective breathing motion consisting of a periodic expansion and contraction of each bosonic cloud is observed during evolution. Let us now investigate the effect that different variations of the binary systems’ parameters have on the breathing frequency. In all cases illustrated in Figs. 2 (a)-(c) in order to obtain the breathing frequency, $\omega_{br}^f$, we start from the non-interacting limit ($g_{AB}^f = 0$) and we measure the Fourier transform of $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi_\sigma|^2 dx$ for each species $\sigma = A, B$. In this way we deduce $\omega_{br}^f$ upon varying either $g_{AB}^f$ [see Figs. 2 (a) and (b)] or the particle number, $N_B$, of the $B$-species [see Fig. 2(c)]. It is found that for particle number imbalances $N_B/N_A = 0.5$, $\omega_{br}^f$ is independent of the $g_{AB}^f$ variation acquiring the constant value of $\omega_{br} \approx \sqrt{3} \omega$ [79], see Fig. 2 (a). However, for larger particle imbalances as e.g. $N_B/N_A = 0.1$ depicted in Fig. 2 (b) a completely different behavior is observed, with the breathing frequency of the minority $B$-species being drastically affected by the postquench variation. In particular as $g_{AB}^f$ increases towards the miscibility/immiscibility threshold, a monotonic decrease of $\omega_{br}^f$ is observed [79] e.g. with $\omega_{br} = 0.5$ for $g_{AB}^f = 0.9$. This decrease of $\omega_{br}^f$ suggests that instead of the harmonic trapping a modified frequency comes into play. Since the $A$-species is the majority component, we assume that it creates an effective potential [92, 93] of the form $V_{eff}(x) = V(x) + g_{AB}^f |\Psi_A(x,0)|^2$ [see Eqs. (2)-(3)] into which the $B$-species is trapped and that $|\Psi_A(x,0)|^2$ has roughly the form of a Thomas-Fermi (TF) profile. As a consequence we obtain the following effective trapping frequency $\omega_{eff} = \sqrt{1 - \frac{2N_B}{\rho_{AA}}}$. Thus the corresponding effective breathing frequency is $\omega_{br}^{B,eff} = \sqrt{3}\omega_{eff}$. A remarkable agreement between $\omega_{br}^{B,eff}$ and $\omega_{br}^f$ is also shown in Fig 2(b) (see dashed black line). Finally, in line with the above discussion, deep in the miscible regime $\omega_{br}^f$ shows an increasing tendency for larger $N_B$ and acquires the constant value of $\omega_{br} \approx \sqrt{3} \omega$ for imbalances $N_B/N_A \gtrsim 0.3$ [see Fig. 2(c)].

**B. Density evolution towards the immiscible regime and identification of DB states**

In contrast to the above-observed dynamics within the miscible domain, for values of $g_{AB}^f$ that are above a certain threshold the miscible-to-immiscible phase transition takes place. As it is evident in Figs. 1 (c)-(f) the dynamics in this regime is unstable and the initially localized, for each species, configuration breaks into several filaments [41]. The filaments refer to the individual density branches appearing in $|\Psi_\sigma|^2$ during evolution for each of the species. For further details regarding this dynamical instability we refer the reader e.g. to Ref. [44] and references therein where the homogeneous case is investigated or the recent work of Ref. [41] that includes the presence of the trap. By comparing the filamentation of each density for two different postquench interactions, we observe that for values of $g_{AB}^f$ that enter deeper into the immiscible phase more filaments are formed [compare Figs. 3 (c) and (d) to Figs. 1 (e) and (f) respectively]. Yet another important observation is that in all cases the bosonic mixture is harmonically trapped with $\omega = 0.05$ and it is initialized in its ground state characterized by $g_{AB}^f = 0$. At $t = 0$ a quench is performed to $g_{AB}^f$.}

**FIG. 2.** (a), (b) Breathing mode frequency, $\omega_{br}^f$, of each component ($\sigma = A, B$) of the binary bosonic mixture for increasing the postquench interspecies interaction strength $g_{AB}^f$. In (a) the system consists of $N_A = 200$ and $N_B = 100$ particles, while in (b) $N_A = 200$ and $N_B = 20$ (see legends). (c) $\omega_{br}^f$ for varying particle number of the $B$-species. The remaining system parameters are fixed (see legend). In all cases the bosonic mixture is harmonically trapped with $\omega = 0.05$ and it is initialized in its ground state characterized by $g_{AB}^f = 0$. At $t = 0$ a quench is performed to $g_{AB}^f$.
the density dips of the majority one being filled by density peaks of the minority species. Since such a filling mechanism resembles the formation of DB solitons in defocusing media below we attempt an identification of the structures formed to these symbiotic states. As a first step towards confirmation we invoke for the integrable Manakov limit the exact single DB soliton solution \[26, 27, 57\]. The latter reads \( \Psi_A(x, t) = A(x, t) = A(\cos \phi \tanh [D(x - x_0(t))] + i \sin \phi) \) and \( \Psi_B(x, t) = B(x, t) = B \sech [D(x - x_0(t))] e^{i(kx + \theta(t)) + i(\mu_B - \mu_A)t} \). Here, \( \phi \) denotes the soliton’s phase angle, while \( A \) and \( B \) refer to the amplitude of the dark and the bright soliton respectively. Furthermore, \( D \) and \( x_0(t) \) correspond to the common inverse width and the soliton’s center respectively. Finally, \( k = D \tan \phi \) is the constant wave-number of the bright soliton and \( \theta(t) \) its phase. The above expressions are used for the density profile fits shown with dashed lines in Figs. 3(a) and (b). Evidently a remarkably good agreement between our numerical simulations and the fitted profiles is seen (see also Table I). The latter verifies that indeed the filamentation process leads to structures that possess a DB solitary wave character. We note that these emergent dark states are characterized by a phase jump being a multiple of \( \pi \) (results not shown here for brevity). The same overall phenomenology is observed for even higher values of \( g_{AB} \) such as the one illustrated in the profile snapshots of Figs. 3(c) and (d) for \( g_{AB} = 2.5 \). Here, the number of DB states is significantly increased compared to \( g_{AB} = 1.5 \) scenario but again an excellent agreement between the numerical data and the fitted DB waveforms can be inferred (see also Table I). Notice also that since in this case the number of solitary waves generated is large, already from these early stages of the dynamics collisions between the DB states are clearly captured by this snapshot. Indeed, the collisions leave their fingerprints in the deformed densities, see the dashed circles which mark the two most inner pairs symmetrically placed around \( x = 0 \) depicted in Figs. 3(c) and (d).

After the initial stages of their formation the DB solitary waves dynamically evolve within the parabolic trap [see Figs. 1(c)-(d) for \( t > 50 \)]. A closer inspection of the evolution of the density of each species reveals a collective breathing motion with four contraction or focus points where the DB states are closest. It is this breathing motion that leads in turn to the collision events that occur during evolution. Several collision events take place both around the center of the trap as well as at the edges of each bosonic cloud. These events are much more pronounced for smaller \( g_{AB} \) [compare Figs. 1(c)-(d) with Figs. 1(e)-(f)] and the DB states that are less mobile, i.e. those closer to the trap center, featuring more dramatic collisions. Recalling now that we operate in the nonintegrable limit these collisions can be rather asymmetric \[57\] leading to a transfer of mass between the solitary wave constituents and thus to shape changing collisions \[57, 90, 78\]. It is this mass redistribution that results in a decreasing number of DB solitons but renders them wider and robustly propagating for large evolution times.

In order to monitor such events and shed light on the observed dynamics we consider several collision points at different times instants during evolution [see the circles in Figs. 1(c), (d)]. For these selected events we perform fittings prior and after its collision vertex. In doing so we immediately have access to the relevant amplitudes and widths of the states and we can further compare the number of particles, \( N_b \), contained in each bright solitary wave prior and after a collision. In order to perform the latter calculation we make use of the analytical expression for the single DB state \[26, 27, 57\], i.e. \( N_b = \int dx |\Psi_B|^2 = 2B^2/D \), that is exact at the integrable limit but approximately holds also in our setup \[93\]. The corresponding outcomes of such a process are summarized in Table II for e.g. postquench amplitude \( g_{AB}' = 1.5 \). In particular, in this Table we present the fitted solitary wave parameters when monitoring the two most outer DB states labeled as “1” and “2” in Figs. 1(c)-(d). These DB waves participate in two distinct collision events that take place during evolution [see the light blue and green circles in Figs. 1(c)-(d)] one at around \( t \approx 140 \) and a second one around \( t \approx 210 \). By measuring the number of particles, \( N_b \), contained in each bright solitary wave prior and after the first collision namely at \( t = 106 \) and \( t = 175 \) respectively (see Table II) it is found that indeed a mass redistribution between the solitary...
TABLE I. Fitted DB soliton characteristics referring to the different collisional events that take place during the spatio-temporal evolution of both species shown in Figs. 1 (c) and (d). Recall that $A \cos \phi$ [B] refers to the amplitude of the dark [bright] solitons generated in the $A \ [B]$ species. The selected time instances, i.e., $t = 35$, $t = 106$, $t = 175$, and $t = 250$, for which we have performed the fits in the density profiles are indicated in Figs. 1 (c) and (d) by vertical dashed lines, and the collision events during propagation are marked with coloured circles. The number of particles of each bright solitary wave, $N_b$, prior and after each collision point is depicted in boldface. Note also that in all cases the accuracy of the fitted values is of about 0.98.

| DB soliton 1 | $t = 35$ | $t = 106$ | $t = 175$ | $t = 250$ |
|--------------|----------|----------|----------|----------|
| $A \cos \phi$ | 1.1683 | 1.1344 | 1.2908 | 1.2301 |
| $B$ | 0.9087 | 1.0299 | 0.7435 | 1.1371 |
| $D$ | 0.5684 | 0.5372 | 0.4484 | 0.4195 |
| $x_{DB}^1$ | 27.3425 | 27.3687 | 30.8764 | 22.6316 |
| $x_{DB}^2$ | 27.3715 | 27.3613 | 31.1916 | 22.4180 |
| $N_b$ | 2.9053 | 3.9491 | 2.4656 | 6.1651 |

| DB soliton 2 | $t = 35$ | $t = 106$ | $t = 175$ | $t = 250$ |
|--------------|----------|----------|----------|----------|
| $A \cos \phi$ | 1.1023 | 1.2351 | 1.2136 | 1.2906 |
| $B$ | 0.9788 | 1.0675 | 1.1102 | 1.1444 |
| $D$ | 0.5604 | 0.7378 | 0.5444 | 0.4788 |
| $x_{DB}^1$ | 17.8409 | 20.4372 | 19.7581 | 19.7446 |
| $x_{DB}^2$ | 17.8281 | 20.5576 | 19.7446 | 19.7446 |
| $N_b$ | 3.4119 | 3.0893 | 4.5278 | 4.5278 |

In particular, Fig. 4 (a) illustrates $N_{DB}$ upon increasing the particle number, $N_A$, of the $A$-species for $N_B = 50$. As it can be seen, solitary wave formation occurs already from small particle numbers ($N_A > 10$) in the $A$-species. As $N_A$ increases $N_{DB}$ increases in a square root fashion with more DB states being generated for larger postquench interactions $g_{AB}^f$. This behaviour can be intuitively understood by considering the size of the two BECs. To estimate the system’s size, we start from Eqs. (2) (3) and by making use of the TF approximation we derive, within the miscible regime, the following expressions for the corresponding radii of the two clouds.

$$R_{\sigma}^f = \omega^{-2/3} \left( \frac{3}{2} (g_{\sigma\sigma} N_{\sigma} + g_{\sigma\bar{\sigma}} N_{\bar{\sigma}}) \right)^{1/3}, \quad (4)$$

$$R_{\bar{\sigma}}^f = \omega^{-2/3} \left( \frac{3}{2} g_{\sigma\bar{\sigma}} N_{\sigma} - g_{\sigma\sigma} N_{\bar{\sigma}} \right)^{1/3}. \quad (5)$$

In the above expressions, $\sigma = A, B \ (\bar{\sigma} = B, A)$ denotes the majority (minority) component. It is now important to note that DB solitary waves will only form in regions where both species are present. The latter sets the scale for solitary wave formation as the region defined by the smaller of the two aforementioned radii. In the inset of Fig. 3 (a) $R_{A}^f$, and $R_{B}^f$ are shown as a function of $N_A$ for $g_{AB}^f = 1.5$. For $N_A < N_B$ the TF radius of the $A$-species is the smaller one and as such is provided by Eq. (4). Indeed, $R_{A}^f$ increases until the particle balanced limit is reached ($N_A = N_B = 50$) where $R_{A}^f = R_{B}^f$. It is in this region, i.e. $N_A < N_B$, that the growth rate of $N_{DB}$ is significant. On the other hand when $N_A > N_B$, $R_{B}^f$ becomes the smaller radius. In this case, unlike $R_{A}^f$, $R_{B}^f$ remains almost constant for increasing $N_A$. Thus, since $R_{B}^f$ remains constant as $N_A$ is increased, a plateau of almost constant DB soliton production appears.

The same qualitative picture is expected to hold also upon varying the number of particles, $N_B$, of the second species, since fixing $N_A$ and varying $N_B$ simply interchanges the role of the two components. Indeed as illustrated in Fig. 3 (b), solely focusing on the case where $N_B < N_A$, again a square-root-like increase of $N_{DB}$ is observed leading to more solitary waves being generated as we enter deeper into the immiscible regime of interactions. Notice that for $N_B > N_A$ again an almost constant production of DB states is observed. However since in this case $N_A$ is fixed to a larger value compared to the previous variation, $N_{DB}$ acquires larger values before the particle balanced limit is reached.

The influence of the postquench interspecies interaction $g_{AB}$ on the DB formation is shown in Fig. 4 (c). Here and as per our discussion above, for $g_{AB}^f < g_{AB}^{th} = 0.974$ (i.e. within the miscible domain) $N_{DB} = 0$. Solitary wave formation takes place only for values of $g_{AB}^f > g_{AB}^{th}$, with $N_{DB}$ increasing in an almost square root manner for increasing postquench interactions. Additionally here,
pared in its ground state having a fixed $N_{DB}$ as a function of (c) the final postquench interaction $g_{AB}^f$ and (d) the trapping frequency $\omega$. In all cases the different parameters used are indicated in the respective legends. Additionally, $g_{AA} = 1.0$, $g_{BB} = 0.95$ and the system is initialized in its ground state characterized by the corresponding $g_{AB}^i$.

$N_{DB}$ is larger even for slightly larger particle number imbalances. Recall that the observed dynamics shown in Figs. 4(c)-(f) dictates also an increase of $N_{DB}$ being generated faster for larger $g_{AB}^f$. The latter is consistent with earlier predictions \cite{[44]} regarding particle balanced mixtures. Furthermore, in order to examine the effect of the trapping geometry on the solitary wave formation, we next calculate $N_{DB}$ upon varying the trapping frequency $\omega$. The outcome of this parametric variation is depicted in Fig. 5(d). $N_{DB}$ decreases rapidly as $\omega$ increases but acquires larger values for larger $g_{AB}^f$. Note also that as we approach the homogeneous limit, $\omega \to 0$, $N_{DB}$ tends to infinity a result that is explained by the increasing size of the condensates. Indeed the radii of the condensates [see Eqs. (4)-(5)] depend on $\omega^{-2/3}$ which is in accordance to what we observe.

IV. QUENCH DYNAMICS FROM THE IMMICIBLE TO THE IMMICIBLE REGIME

Up to now we only considered quenches from the miscible to the immiscible regime of interactions. Next, immiscible-to-miscible transitions will be investigated. For this reverse process, the binary system is initially prepared in its ground state having a fixed $g_{AB}^i = 1.5$. Then, the $A$-species resides at the edges of the harmonic trap since it possesses a larger intraspecies coefficient ($g_{AA}$) while the $B$-species occupies the central region. Such an initial state configuration represents the 1D analogue of the so-called "ball and shell" structure found in higher dimensional mixtures \cite{[30],[31]}. After this initial state preparation the system is abruptly quenched to a lower $g_{AB}^i$ value lying within the miscible regime of interactions.

FIG. 4. Number of DB solitons, $N_{DB}$, formed when considering a quench from the miscible to the immiscible phase for varying system parameters. (a) $N_{DB}$ upon increasing the particle number, $N_A$, of the $A$-species. Inset illustrates the TF radii, $R_{TF}^A$ with $\sigma = A,B$ and $\sigma = B,A$, for the same variation. (b) The same as (a) but varying the particle number, $N_B$, of the $B$-species. $N_{DB}$ as a function of (c) the final postquench interaction $g_{AB}^f$ and (d) the trapping frequency $\omega$. In all cases the different parameters used are indicated in the respective legends. Additionally, $g_{AA} = 1.0$, $g_{BB} = 0.95$ and the system is initialized in its ground state characterized by the corresponding $g_{AB}^i$.

FIG. 5. One-body density evolution $|\Psi_\sigma|^2$ of $\sigma = A$ (left panels) and $\sigma = B$-species (right panels) respectively, performing an interspecies interaction quench from $g_{AB}^i = 1.5$ to $g_{AB}^f = 0.75$. The system consists of (a), (b) $N_A = 300$, $N_B = 100$ and (c), (d) $N_A = 100$, $N_B = 300$ bosons. In all cases both species are trapped in a harmonic oscillator of frequency $\omega = 0.05$. The system is initialized in its ground state with $g_{AA} = 1.0$, $g_{BB} = 0.95$ and $g_{AB} = 1.5$. 
A. Effective counterflow dynamics and emergence of DAD states

Case examples of the spatio-temporal evolution of the densities of both species for two different particle number imbalances, namely for $N_B/N_A = 1/3$ and $N_B/N_A = 3$, but for the same postquench interspecies repulsion are shown in Figs. 5 (a), (b) and (c), (d) respectively. In particular, Figs. 5 (a) and (c) depict the evolution of the density of the $A$-species and Figs. 5 (b) and (d) the corresponding evolution of the $B$-species. In both cases pattern formation takes place from the early stages of the nonequilibrium dynamics but with a dramatically larger number of states formed when the inner $B$-species is the majority component of the bosonic mixture. The latter outcome can be inferred by inspecting the corresponding profile snapshots illustrated in Figs. 6 (a)-(d). Here, Figs. 6 (a), (b) and (c), (d) show the density profiles of both species at the initial stages of formation, i.e. at $t = 45$ and $t = 60$ respectively, for $N_B/N_A = 1/3$ and $N_B/N_A = 3$. Evidently, at least twelve localized entities appear in the former snapshots, namely when the outer $A$-species is the majority component, while at least thirty occur when the inner $B$-species is the majority component. In this latter case the localized structures closely resemble interference fringes. Notice that in both cases, since component mixing is favored in this regime, the $B$-species tend to fill the quench induced minima of the $A$-species. However, when the $B$-species is the majority component an overfill of the corresponding $A$-species minima occurs [see the relevant profiles in Figs. 6 (b), (d)] and thus no robust localized configuration can be realized. Before delving in the associated dynamics, we should emphasize that in contrast to the previous quench scenario, the structures formed in this case are not only different in nature when compared to the DB solitary waves previously generated, but appear also to be more robust during evolution [compare Figs. 6 (a)-(d) to Figs. 4 (c)-(f)]. Thus the breaking of integrability within the miscible regime does not seem to affect the evolution of the states generated, a result that is in line with earlier predictions [11]. In particular, a closer inspection of the spatio-temporal evolution of the densities [see Figs. 5 (a)-(d) and Figs. 6 (a)-(d)] it can be deduced that dark soliton entities emerge during evolution in the $A$-species while antidark solitons [27, 29, 41], namely states consisting of a density peak on top of the BEC background, appear in the corresponding $B$-species.

Thus in contrast to the DB solitary waves formed in the previous quench scenario, here multiple DAD solitons are spontaneously generated. The formation of these DAD states has been reported also in the particle balanced case [41]. Below, and in contrast to earlier findings we will explore in detail the mechanism of the formation of these DAD states and we will also attempt to control their generation upon considering different variations of the binary systems’ parameters.

To unravel the underlying mechanism of formation of the aforementioned DAD states we must first recall that the initial state configuration consists of two spatially separated components. As the system is dynamically quenched towards the species miscible regime, the interspecies interaction energy, $\Delta E_{\text{int}} = 2 |\Psi_A|^2 |\Psi_B|^2 (g_{AB}^f - g_{AB}^p)$, decreases the more the smaller the $g_{AB}^f$ becomes. Then, the $B$-species is no longer deemed to reside at the central region of the parabolic potential but instead it can expand. More importantly the $A$-species residing previously symmetrically around the edges of the trap is now allowed to “fall” towards the center. It is this counterflow of the $A$-species that causes the formation of the dark solitons right at the trap center where the two parts of the gas destructively interfere [22]. Note also that in our initial state preparation we start with a constant zero phase in the $A$-species and thus we expect and indeed observe an even number of dark solitons to emerge [27] [see also here the profiles shown in Figs. 6 (a), (c)]. This dark soliton formation leads in turn, due to the effective potential that these states create, to the breaking of the $B$-species into several density peaks each of which filling the newly formed density dips. Essentially it is energetically beneficial for the $B$-species to avoid regions where the $A$-species has a high density, since the interaction between the species is repulsive ($g_{AB}^f > 0$). Thus the $B$-species tends to fill the $A$-species minima. This effect is more severe for weaker repulsions that enter deeper in the miscible regime, with the number of the emergent DAD states found to increase for smaller $g_{AB}^f$ (see our discussion below). However since the binary system is in the miscible regime where a mixed state is favorable the generated density peaks are formed on top of their BEC background. The above-discussed mechanism of formation of the observed DAD states is fundamentally different in nature, and is to be contrasted with the unstable dynamics that leads to the formation of the DB struc-
tures identified in the miscible-to-immiscible transition.

To testify the formation of dark solitons in our setup, we next rely on predictions stemming from counterflow experiments [10]. It is known that for single component BECs a critical distance between the two initially separated BECs exists below which soliton formation takes place [83]. This critical distance in the dimensionless units adopted herein reads \( D_c = \pi \left( \frac{6 \pi N_A g_{A \lambda}}{\omega_{br}^2} \right)^{1/3} \).

Here, \( \omega_{eff}^A = \omega \sqrt{1 - \frac{g_{AB}^2}{g_{br}^2}} \) is the modified trapping frequency that incorporates the postquench cross interaction term \( g_{AB}^f \). This effective potential picture [92, 93], \( V_{eff} = V(x) + g_{AB}^f \psi_B(x, 0)^2 \), is analogous to the one introduced in section III. We remark here that in contrast to the previous quench scenario [section III] the \( A \)-species performs the counterflow dynamics with the \( B \)-species residing at the trap center. Thus, we consider the dynamics of the \( A \)-species into the effective potential formed by the \( B \)-species and the external trap. For the case example of \( N_B/N_A = 1/3 \) presented in Fig. 3 (a) \( D_c = 197 \) while \( D = 20.8 \), verifying that the observed structures building upon the majority \( A \)-species are indeed dark solitons.

**B. Breathing mode**

Besides the formation of the DAD structures the two species also undergo a collective breathing motion as it can be readily seen in Fig. 7. Results for the corresponding normalized breathing frequency, \( \omega_{br}^A/\omega^* \), are shown in Figs. 7(a), (b), upon considering the effect of different particle number imbalances and postquench interspecies interaction strengths respectively. Referring to a fixed postquench interaction strength \( \omega_{br}^A \) and \( \omega_{br}^B \) are significantly different especially in the region where \( N_A > N_B \), see Fig. 7(a). This deviation is much more pronounced for smaller \( g_{AB}^f \). Within the region \( N_A < N_B \), the frequencies \( \omega_{br}^A \) and \( \omega_{br}^B \) tend to approach each other but they never coincide. For fixed particle number imbalances, i.e. \( N_B/N_A = 1/3 \), an increase of \( \omega_{br}^A/\omega^* \) is observed as the system is quenched to lower values of \( g_{AB}^f \) that enter deeper in the miscible regime of interactions [see Fig. 7(b)]. More specifically, both species appear to have almost the same breathing frequency for quenches near the miscibility/immiscibility threshold but deviate significantly for values of \( g_{AB}^f \) within the interval \([0.6, 0.1] \). In this later case, \( \omega_{br}^B/\omega^* \) appears to be higher for all values of the postquench interaction \( g_{AB}^f \) until the noninteracting limit is reached, with both species acquiring the value \( \omega_{br}^A/\omega^* \approx \sqrt{3} \). Moreover, the effective breathing frequency, \( \omega_{br}^{A,eff} \) (see also section III), seems to underestimate \( \omega_{br}^A/\omega^* \) more, the closest we are to the miscibility/immiscibility threshold. The observed deviations are mainly attributed to the fact that the effective model assumes a TF profile for the minority \( B \)-species which differs considerably from the actual density of this species due to the dynamical generation of antidark solitons.

**C. Controlled creation of DADs**

Having discussed the mechanism of formation of DAD states, we next investigate whether their number can be tuned via parametric variations. Figures 8(a)-(d) summarize our findings. We find that the number of DAD states, \( N_{DAD} \), strongly depends on variations of the particle number either of the \( A \)- or the \( B \)-species [see Figs. 8(a), (b)]. In particular, in both cases illustrated in Fig. 8(a) a linear increase of \( N_{DAD} \) for increasing \( N_A \) is observed. This increase is followed by a plateau of constant soliton count until the particle balanced limit is reached, with \( N_{DAD} \) being larger for larger \( N_B \). Further increase of \( N_A \), namely for \( N_B/N_A < 1 \) leads to a gradual decrease of \( N_{DAD} \) with less entities formed for smaller ratios. This behavior can be intuitively explained by recalling that increasing \( N_A \) also increases the size of the \( A \)-species. The latter initially reside at the edges of the trap squeezing more and more the central \( B \)-species and thus reducing the initial separation of the outer BECs. Smaller initial spatial separation leads in turn to less kinetic energy for...
Here, \( n \) and \( \omega \) varying the particle number, \( N \), of the \( A \)-species. The above-mentioned tendency can be understood carefully the case of \( \omega = 0 \), and thus resulting in a larger \( N_DAD \). However, it is important to note that as the non-interacting limit is approached the DAD states become gradually less robust. Focusing on the antidarks formed in the \( B \)-species it is found that as \( f_{AB}^{f} \) decreases the amplitude of these states also decreases. To appreciate this behaviour we introduce the visibility

\[
\nu = \frac{n_{\max} - n_{\min}}{n_{\max} + n_{\min}} \tag{6}
\]

Here, \( n_{\max} \) is the density at the peak of an antidark and \( n_{\min} \) is the corresponding background density in the minimum. In the case that the minimum densities are notably different on the left/right side of the peak, we define \( n_{\min} \) as the average of the two minima. As an example the maximum visibility of the central antidark soliton, is \( \nu = 80.5\% \) in Fig. \( 6 \)(a) and grows to \( \nu = 82.8\% \) in Fig. \( 6 \)(c), while is found to be \( \nu = 20.4\% \) and \( \nu = 25.7\% \) in Figs. \( 6 \)(b), (d) respectively. Moreover, the maximum visibility measured for the corresponding central antidark state for \( f_{AB}^{f} = 0.1 \) is \( \nu = 10\% \), while it approaches zero for \( f_{AB}^{f} \to 0 \). Thus, in the following, we define a visibility threshold for DAD formation. In particular, we consider as DAD states the ones whose central antidark structure exhibits a visibility \( \nu \geq 10\% \). For instance in the case of \( N_B/N_A = 1/3 \), according to our simulations this condition is fulfilled only for quenches with \( f_{AB}^{f} > 0.1 \) [see also Fig. 5\( (c) \)].

Finally, the role of the trapping geometry is examined in Fig. 8\( (d) \). Here, a hyperbolic-like increase of \( N_DAD \) can be seen for a decreasing trapping frequency \( \omega \), leading to more DAD structures being spontaneously generated for larger \( N_B/N_A \) ratios. The observed increase in \( N_DAD \) can be easily explained by the overall size of the system which for lower \( \omega \)-s increases providing in this way more space for solitons to form.

V. CONCLUSIONS

In the present work the interaction quench dynamics of a 1D harmonically confined particle imbalanced BEC mixture has been investigated within a mean-field theoretical approach. In particular upon considering quenches crossing the miscibility/immiscibility threshold
in both directions the dynamical formation of multiple solitary waves in this non-integrable model has been unravelled. For miscible-to-immiscible transitions the unstable dynamics leads to the filamentation of the density of both species in line with our recent findings regarding particle balanced mixtures \cite{41}. However for the particle imbalanced scenario studied herein we were able to identify that via the filamentation process DB solitary waves emerge. The identification of these states is based on fitting the waveforms corresponding to the exact (approximate) at the integrable (non-integrable) limit single DB soliton solution to our numerically obtained states. Excellent agreement is found between our data and the analytical expressions, not only at the initial stages of the dynamics where these states emerge but also for larger evolution times. However, due to the non-integrable nature of the system highly asymmetric collisions with mass redistribution \cite{57} between the solitary wave constituents occur during evolution that result to fewer DB solitary waves remaining and robustly propagating. Focusing on the initial stages of the out-of-equilibrium dynamics we further aimed at the control of the generation of these states. To achieve the latter different variations of the binary system’s parameters have been investigated. In particular, an increase of the DB solitary wave production is observed upon varying the corresponding particle number of each species until the case of a particle balanced mixture is reached. A result that is enhanced as we enter deeper into the immiscible regime of interactions. Furthermore we find that the number of DB states generated increases in a square root manner for increasing postquench interspecies repulsions. A decaying number of the DB structures is observed upon considering tighter traps resulting in turn to fewer states generated for smaller postquench interactions. Finally, and since both clouds undergo a collective breathing motion, we further evaluate the relevant breathing frequency for each species. We find that the two breathing frequencies are substantially different for strong particle imbalances. A deviation that is more dramatic for larger postquench repulsions. This is in line with earlier predictions in binary BEC mixtures \cite{79}.

On the other hand, for immiscible-to-miscible transitions the spontaneous creation of multiple DAD solitons is observed linking to previous studies \cite{41}. Here, we extend our investigations shedding light on the underlying mechanism that leads to the formation of these states. It is showcased that these states emerge due to a counterflow process that takes place as the system is quenched towards miscibility. Additionally, the generated DAD solitons are found to be robust throughout the evolution. This result is to be contrasted to the observed decrease of the DB states formed in the previous transition, indicating that the breaking of integrability in this regime does not seem to affect the dynamics. Measuring the breathing frequency of each bosonic cloud for such a transition, we find that it differs from one component to the other independently of the postquench interaction strength or the particle number imbalance. The above behavior of the breathing frequency for this immiscible-to-miscible transition, is to the best of our knowledge, reported herein for the first time. Also, the controlled creation of multiple DAD states is revealed. As in the previous quench scenario an increase of the number of DAD structures is observed upon varying the particle number of each species. This result is more pronounced for postquench interactions that enter deeper into the miscible domain. Moreover, an almost linear increase of the DAD states formed is evidenced for decreasing the postquench interaction. Finally, a decay of the number of DAD solitons occurs for tighter confinement. Here, the generated DAD states are fewer for postquench repulsions that are closer to the miscibility/immiscibility threshold.

There are several possible extensions of the present work that can be considered for future endeavours. Of particular interest would be to examine the quench induced dynamics of 1D spinor BECs \cite{99}. In such settings dark-dark-bright and dark-bright-bright soliton complexes are known to form \cite{99,100}. Therefore, attempting to control the generation of multiple such states upon quenching would be a natural generalization of our current effort. Also for these spinorial BEC environment one could further go beyond the mean-field approximation \cite{37,101,102}, addressing the fate of multiple solitonic entities in the presence of quantum fluctuations. Finally, a generalization of the quench-induced dynamics of binary BEC mixtures to higher dimensions represents an intriguing perspective. It is well-known that e.g. in two-dimensions there exist vortex-bright soliton states \cite{103,104} and it would be interesting to examine if such structures can be dynamically produced under the influence of an interspecies interaction quench.

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