Abstract. The correlation functions are calculated for the three – dimensional \( Z_2 \) electrodynamics for the particular values of the interaction energies and for the free boundary conditions.

1 Introduction

Onsager [1] invented the formula for the partition function of the two – dimensional Ising model. Another method for the calculation of the partition function was proposed by Kac and Ward [2]. They considered two formulae simultaneously: the determinant of the special matrix \( I + T \) (\( I \) is the identity matrix) is proportional to the partition function of the two – dimensional Ising model and it is proportional to the square of the partition function. For the proof of the first formula they used a topological statement. Sherman [3], [4] constructed a counter – example for this statement and gave some arguments for the equality

\[ Z^2 = C(\beta) \det(I + T) \] (1.1)

where \( Z \) is the partition function of the two – dimensional Ising model with the free boundary conditions and \( C(\beta) \) is the positive function of the inverse temperature \( \beta \). In the paper [5] the following formula

\[ Z^2 = C(\beta) \det(I - T) \] (1.2)

is proved for an arbitrary lattice lying on the plane. For the rectangular lattice the expression (1.2) is independent of the sign of the matrix \( T \). For an arbitrary lattice the formula (1.1) is wrong. In the paper [6] the free energy and the correlation functions of the two – dimensional Ising model with the free boundary conditions are calculated by using the formula (1.2).

In this paper we study the connection between the three – dimensional \( Z_2 \) electrodynamics and the two – dimensional Ising model. We consider a finite three – dimensional cubic lattice. The vertices of this lattice are given by the vectors \( p = (p_1, p_2, p_3) \) where the numbers \( p_i \),

*This work is supported in part by the Russian Foundation for Basic Research (Grant No. 00 – 01 – 00083)
\(i = 1, 2, 3\), are natural. The non – oriented edge of this lattice is given by the pair \(\{p, e\}\) where \(p\) is the edge initial vertex and the unit vector \(e\) is the edge direction. The unit vector \(e\) is one of six vectors: \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\). Since the edge is non – oriented, then \(\{p + e, -e\} = \{p, e\}\). The non – oriented face of this lattice is given by the triplet \(\{p, e_1, e_2\}\) where \(p\) is the face initial vertex and the unit vectors \(e_1, e_2\) are two orthogonal to each other from six unit vectors \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\). Since the face is non – oriented, then \(\{p, e_1, e_2\} = \{p, e_2, e_1\} = \{p + e_2, -e_1, e_2\} = \{p + e_1 + e_2, -e_1, -e_2\}\). By \(Z_2^{add}\) we denote the group of modulo 2 residuals. The modulo 2 residuals are multiplied by each other and the group \(Z_2^{add}\) is the field. We consider the function \(A(\{p, e\})\) on the non – oriented edges taking the values in the group \(Z_2^{add}\). The energy of \(Z_2\) electrodynamics has the following form

\[
H(A) = - \sum_{\{p, e_1, e_2\}} E(\{p, e_1, e_2\})(-1)^{A(\{p, e_1\}) + A(\{p + e_1, e_2\}) + A(\{p + e_2, e_1\}) + A(\{p, e_2\})} \tag{1.3}
\]

where the summing runs over all non – oriented faces of the lattice. The number \(E(\{p, e_1, e_2\})\) is called the interaction energy attached to the non – oriented face \(\{p, e_1, e_2\}\). For the infinite interaction energy the energy \((1.3)\) is also infinite. However, it is possible to consider the infinite values of the interaction energies \(E(\{p, e_1, e_2\})\) for the correlation functions and for the quantity

\[
\left( \prod_{\{p, e_1, e_2\}} \cosh \beta E(\{p, e_1, e_2\}) \right)^{-1} \sum_A \exp\{-\beta H(A)\}. \tag{1.4}
\]

(The product in \((1.4)\) runs over all non – oriented faces of the lattice. The sum in \((1.4)\) runs over all \(Z_2^{add}\) valued functions on the non – oriented edges of the lattice.) We will show that for the interaction energies \(E(\{p, (1, 0, 0), (0, 1, 0)\})\) equaled \(+\infty\) (or \(-\infty\)) the correlation functions and the quantity \((1.4)\) for \(Z_2\) electrodynamics are related to the correlation functions and the partition function of the two – dimensional Ising model with the free boundary conditions.

In the second section the definitions of the partition function and the correlation functions of the three – dimensional \(Z_2\) electrodynamics with the free boundary conditions are given. These quantities are calculated for the for the interaction energies \(E(\{p, (1, 0, 0), (0, 1, 0)\}) = 0\). The third section is devoted to the connection between the quantity \((1.4)\) for the interaction energies \(E(\{p, (1, 0, 0), (0, 1, 0)\}) = \pm \infty\) and the partition function of the two – dimensional Ising model with the free boundary conditions. In the fourth section we study the connection between the correlation functions of the three – dimensional \(Z_2\) electrodynamics with the interaction energies \(E(\{p, (1, 0, 0), (0, 1, 0)\}) = \pm \infty\) and the correlation functions of the two – dimensional Ising model with the free boundary conditions.

\section{\(Z_2\) Electrodynamics}

We consider a rectangular lattice formed by the points with the integral Cartesian coordinates \(x = k_1, y = k_2, z = k_3, M'_i \leq k_i \leq M_i, i = 1, 2, 3\), and the corresponding edges connecting the neighbour vertices. We denote this graph by \(G_3 = G(M'_1, M'_2, M'_3; M_1, M_2, M_3)\). The cell complex \(P(G_3)\) is called the set consisting of the cells (vertices, edges, faces, cubes). A vertex of \(P(G_3)\) is called a cell of dimension 0. It is denoted by \(s^0\). An edge of \(P(G_3)\) is
called a cell of dimension 1. It is denoted by \( s^1_i \). A face of \( P(G_3) \) is called a cell of dimension 2. It is denoted by \( s^2_i \). A cube of \( P(G_3) \) is called a cell of dimension 3. It is denoted by \( s^3_i \).

To every pair of the cells \( s^p_i, s^{p-1}_j \) there corresponds the number \( (s^p_i : s^{p-1}_j) \in \mathbb{Z}_2^{add} \) (incidence number). If the cell \( s^{p-1}_j \) is included into the boundary of the cell \( s^p_i \), then the incidence number \( (s^p_i : s^{p-1}_j) = 1 \). Otherwise the incidence number \( (s^p_i : s^{p-1}_j) = 0 \). For any pair of the cells \( s^{p+1}_i, s^p_j \) the incidence numbers satisfy the condition

\[
\sum_m (s^{p+1}_i : s^m)_n (s^{p+1}_m : s^p_j) = 0 \mod 2. \tag{2.1}
\]

Indeed, if the vertex \( s^0_j \) is not included into the boundary of the square \( s^2_i \), then the condition (2.1) is fulfilled. If the vertex \( s^0_j \) is included into the boundary of the square \( s^2_i \), then it is included into the boundaries of the six edges \( s^1_m \) two of which are included into the boundary of the square \( s^2_i \). The condition (2.1) is fulfilled again. If the edge \( s^1_j \) is not included into the boundary of the cube \( s^3_i \), then the condition (2.1) is fulfilled. If the edge \( s^1_j \) is included into the boundary of the cube \( s^3_i \), then it is included into the boundaries of the four squares \( s^2_m \) two of which are included into the boundary of the cube \( s^3_i \). The condition (2.1) is fulfilled again.

A cochain \( c^p \) of the complex \( P(G_3) \) with the coefficients in the group \( \mathbb{Z}^{add}_2 \) is a function on the \( p \) - dimensional cells taking values in the group \( \mathbb{Z}^{add}_2 \). Usually the cell orientation is considered and the cochains are antisymmetric functions: \( c^p(-s^p_i) = -c^p(s^p_i) \). However, \(-1 = 1 \mod 2\) and we can neglect the cell orientation for the coefficients in the group \( \mathbb{Z}^{add}_2 \).

The cochains form an Abelian group

\[
(c^p + c^p)(s^p_i) = c^p(s^p_i) + c^p(s^p_i) \mod 2. \tag{2.2}
\]

It is denoted by \( C^p(P(G_3), \mathbb{Z}^{add}_2) \). The mapping

\[
\partial c^p(s^{p-1}_i) = \sum_j (s^p_j : s^{p-1}_i) c^p(s^p_j) \mod 2 \tag{2.3}
\]

defines the homomorphism of the group \( C^p(P(G_3), \mathbb{Z}^{add}_2) \) into the group \( C^{p-1}(P(G_3), \mathbb{Z}^{add}_2) \). It is called the boundary operator. The mapping

\[
\partial^* c^p(s^{p+1}_i) = \sum_j (s^{p+1}_i : s^p_j) c^p(s^p_j) \mod 2 \tag{2.4}
\]

defines the homomorphism of the group \( C^p(P(G_3), \mathbb{Z}^{add}_2) \) into the group \( C^{p+1}(P(G_3), \mathbb{Z}^{add}_2) \). It is called the coboundary operator. The condition (2.1) implies \( \partial \partial = 0, \partial^* \partial^* = 0 \). The kernel \( Z_p(P(G_3), \mathbb{Z}^{add}_2) \) of the homomorphism (2.3) on the group \( C^p(P(G_3), \mathbb{Z}^{add}_2) \) is called the group of cycles of the complex \( P(G_3) \) with the coefficients in the group \( \mathbb{Z}^{add}_2 \). The image \( B_p(P(G_3), \mathbb{Z}^{add}_2) \) of the homomorphism (2.3) in the group \( C^p(P(G_3), \mathbb{Z}^{add}_2) \) is called the group of boundaries of the complex \( P(G_3) \) with the coefficients in the group \( \mathbb{Z}^{add}_2 \). Since \( \partial \partial = 0 \), the group \( B_p(P(G_3), \mathbb{Z}^{add}_2) \) is the subgroup of the group \( Z_p(P(G_3), \mathbb{Z}^{add}_2) \). Analogously, for the coboundary operator \( \partial^* \) the group of cocycles \( Z^p(P(G_3), \mathbb{Z}^{add}_2) \) and the group of coboundaries \( B^p(P(G_3), \mathbb{Z}^{add}_2) \) are defined.

It is possible to introduce the bilinear form on \( C^p(P(G_3), \mathbb{Z}^{add}_2) \):

\[
\langle f^p, g^p \rangle = \sum_i f^p(s^p_i) g^p(s^p_i) \mod 2. \tag{2.5}
\]
The definitions (2.3), (2.4) imply
\[ \langle f^p, \partial^* g^p \rangle = \langle \partial f^p, g^p \rangle \]
\[ \langle f^p, \partial g^{p+1} \rangle = \langle \partial^* f^p, g^{p+1} \rangle. \]
\hfill (2.6)

Let a cochain \( A^1 \in C^1(P(G_3), \mathbb{Z}^{add}_2) \). Let the energy be expressed in the form
\[ H'(\partial^* A^1) = \sum_{s_i^2 \in P(G_3)} h_i(\partial^* A^1(s_i^2)) \]
\hfill (2.7)
where \( h_i(\epsilon) \) is an arbitrary function on the group \( \mathbb{Z}^{add}_2 \):
\[ h_i(\epsilon) = D_i - E_i(-1)^\epsilon \]
\hfill (2.8)
and the constants
\[ D_i = \frac{1}{2}(h_i(1) + h_i(0)), \]
\[ E_i = \frac{1}{2}(h_i(1) - h_i(0)). \]

The substitution of the equality (2.8) into the equality (2.7) gives
\[ H'(\partial^* A^1) = \sum_{s_i^2 \in P(G_3)} D_i + H(\partial^* A^1) \]
\hfill (2.9)
where the function
\[ H(\partial^* A^1) = - \sum_{s_i^2 \in P(G_3)} E_i(-1)^{\partial^* A^1(s_i^2)} \]
\hfill (2.10)
is called the energy of \( \mathbb{Z}_2 \) electrodynamics. The number \( E_i = E(s_i^2) \) is the interaction energy attached to the face \( s_i^2 \).

The function
\[ Z_{G_3} = \sum_{A^1 \in C^1(P(G_3), \mathbb{Z}^{add}_2)} \exp\{-\beta H(\partial^* A^1)\} \]
\hfill (2.11)
is called the partition function of \( \mathbb{Z}_2 \) electrodynamics.

Let the cochain \( \chi^1 \in C^1(P(G_3), \mathbb{Z}^{add}_2) \) take the value 1 at the edges \( s_1^1, ..., s_m^1 \) and be equal to 0 at all other edges of the graph \( G_3 \). The correlation function at the edges \( s_1^1, ..., s_m^1 \) of the lattice \( G_3 \) is the function
\[ W_{G_3}(\chi^1) = (Z_{G_3})^{-1} \sum_{A^1 \in C^1(P(G_3), \mathbb{Z}^{add}_2)} (-1)^{\langle \chi^1, A^1 \rangle} \exp\{-\beta H(\partial^* A^1)\}. \]
\hfill (2.12)

**Proposition 2.1.** The partition function of \( \mathbb{Z}_2 \) electrodynamics on the graph \( G_3 \)
\[ Z_{G_3} = 2^{#(E_{G_3})} \prod_{s_i^2 \in P(G_3)} \cosh \beta E(s_i^2) \]
\hfill (2.13)
where the reduced partition function of \( \mathbb{Z}_2 \) electrodynamics on the graph \( G_3 \)
\[ Z_{r,G_3} = \sum_{\xi^2 \in Z_2(P(G_3), \mathbb{Z}^{add}_2)} \prod_{s_i^2 \in P(G_3)} (\tanh \beta E(s_i^2))^{2(1-(-1)^{\xi^2(s_i^2)})} \]
\hfill (2.14)
Proof. The definition (2.10) implies
\[ W_{G_3}(\chi^1) = (Z_{G,3})^{-1} \sum_{\xi^2 \in C^2(P(G_3), Z_2^{add}), s_i^2 \in P(G_3)} \prod (\tanh \beta E(s_i^2))^{\frac{1}{2}(1 - (-1)^{\xi^2(s_i^2)})}. \] (2.15)

The relation (2.15) implies that the correlation function is not zero only for the cochain
\[ \chi^1 \in B_1(P(G_3), Z_2^{add}). \] In other words, the cochain \( \chi^1 \) is equal to 1 on the closed contours which are the boundaries of the two – dimensional domains.

The relation (2.15) implies that the correlation function is not zero only for the cochain
\[ \chi^1 \in B_1(P(G_3), Z_2^{add}). \] In other words, the cochain \( \chi^1 \) is equal to 1 on the closed contours which are the boundaries of the two – dimensional domains.

Let us consider a rectangular lattice formed by the points with the integral Cartesian coordinates \( x = k_1, y = k_2, M'_i \leq k_i \leq M_i, i = 1, 2, \) and the corresponding edges connecting neighbour vertices. We denote this graph by \( G_2 = G(M'_1, M'_2; M_1, M_2) \). The cell complex \( P(G_2) \) is defined analogously to the cell complex \( P(G_3) \) for the three – dimensional graph \( G_3 = G(M'_1, M'_2, M'_3; M_1, M_2, M_3) \). Let a cochain \( \sigma^0 \in C^0(P(G_2), Z_2^{add}) \). The Ising model energy is expressed in the following form similar to the form (2.10)
\[ H(\partial^* \sigma^0) = - \sum_{s_i^1 \in P(G_2)} E(s_i^1)(-1)^{\partial^* \sigma^0(s_i^1)}. \] (2.20)

The substitution of the equality (2.18) into the definition (2.14), the first relation (2.6) and the relation
\[ \sum_{\xi=0,1} (-1)^{\xi^2} = \begin{cases} 2, & \epsilon = 0, \\ 0, & \epsilon = 1, \end{cases} \] (2.19)
give the equalities (2.13), (2.14). The substitution of the equality (2.18) into the definition (2.14), the first relation (2.6) and the relation (2.19) give the equality (2.15). The proposition is proved.

Here we used the definitions and the methods of the paper [7].

The relation (2.15) implies that the correlation function is not zero only for the cochain
\[ \chi^1 \in B_1(P(G_3), Z_2^{add}). \] In other words, the cochain \( \chi^1 \) is equal to 1 on the closed contours which are the boundaries of the two – dimensional domains.

The partition function and the correlation function of the two – dimensional Ising model are defined by the relations similar to the relations (2.11), (2.12)
\[ Z_{G_2} = \sum_{\sigma^0 \in C^0(P(G_2), Z_2^{add})} \exp\{-\beta H(\partial^* \sigma^0)\}, \] (2.21)
\[ W_{G_2}(\chi^0) = (Z_{G_2})^{-1} \sum_{\sigma^0 \in C^0(P(G_2), Z_2^{\text{add}})} (-1)^{\langle \chi^0, \sigma^0 \rangle} \exp\{-\beta H(\partial^* \sigma^0)\}. \quad (2.22) \]

where the cochain \( \chi^0 \in C^0(P(G_2), Z_2^{\text{add}}) \).

It is possible to show \[8\] that similar to the relations (2.13) – (2.15)

\[ Z_{G_2} = 2^{\#(V G_2)} \left( \prod_{s_1^i \in P(G_2)} \cosh \beta E(s_1^i) \right) Z_{r,G_2}, \quad (2.23) \]

\[ Z_{r,G_2} = \sum_{\xi^1 \in C^1(P(G_2), Z_2^{\text{add}})} \prod_{s_1^i \in P(G_2)} (\tanh \beta E(s_1^i))^{\frac{1}{2}(1-(-1)^{\xi^1(s_1^i)})}, \quad (2.24) \]

\[ W_{G_2}(\chi^0) = (Z_{r,G_2})^{-1} \sum_{\xi^1 \in C^1(P(G_2), Z_2^{\text{add}})} \prod_{s_1^i \in P(G_2)} (\tanh \beta E(s_1^i))^{\frac{1}{2}(1-(-1)^{\xi^1(s_1^i)})}. \quad (2.25) \]

where \( \#(V G_2) \) is the total number of the vertices of the graph \( G_2 \).

The equalities (2.23), (2.24) were proved for the first time in the paper \[8\].

Let us consider the relations (2.13) – (2.15) for the particular case of the interaction energy \( E(s_1^2) = 0 \) for any face \( s_1^2 \) orthogonal to the coordinate axis \( z \). For the graph \( G_3 = G(M_1, M_2, M_3; M_1, M_2, M_3) \) the group \( Z_2(P(G_3), Z_2^{\text{add}}) \) coincides with the group \( B_3(P(G_3), Z_2^{\text{add}}) \). (The homology group is trivial). A boundary of any set of the cubes contains a face \( s_1^2 \) orthogonal to the coordinate axis \( z \). Hence the equality \( E(s_1^2) = 0 \) implies that the sum (2.14) consists of the only term corresponding to the cycle which is equal to zero on any face. Thus the relations (2.13), (2.14) imply

\[ Z_{G_3} = 2^{\#(E G_3)} \prod_{s_1^2 \in P(G_3)} \cosh \beta E(s_1^2). \quad (2.26) \]

Let us consider the sum (2.15). It is possible that the equation \( \partial \xi^2 = \chi^1 \) has no solution \( \xi^2 \in C^2(P(G_3), Z_2^{\text{add}}) \) such that the cochain \( \xi^2(s_1^2) = 0 \) for any face \( s_1^2 \) orthogonal to the coordinate axis \( z \). In this case the relation (2.15) implies

\[ W_{G_3}(\chi^1) = 0. \quad (2.27) \]

If the equation \( \partial \xi^2 = \chi^1 \) has a solution \( \xi^2 \in C^2(P(G_3), Z_2^{\text{add}}) \) such that the cochain \( \xi^2(s_1^2) = 0 \) for any face \( s_1^2 \) orthogonal to the coordinate axis \( z \), then this solution is unique since the equality \( \xi^2(s_1^2) = 0 \) for any face \( s_1^2 \) orthogonal to the coordinate axis \( z \) implies that the cycle \( \xi^2 \in Z_2(P(G_3), Z_2^{\text{add}}) \) is equal to zero on any face. It follows from the relations (2.13) – (2.15), (2.26) that in this case

\[ W_{G_3}(\chi^1) = \prod_{s_1^2 \in P(G_3), \partial \xi^2 = \chi^1} (\tanh \beta E(s_1^2))^{\frac{1}{2}(1-(-1)^{\xi^2(s_1^2)})}. \quad (2.28) \]

It is possible to set the infinite interaction energies \( E(s_1^2) = \pm \infty \) into the reduced partition function (2.14) and the correlation functions (2.13).
3 Partition Function

In this section we study the connection between the reduced partition function \(Z_{r,G_2}\) of the three-dimensional \(\mathbb{Z}_2\) electrodynamics and the reduced partition function \(Z_{r,G_2}\) of the two-dimensional Ising model.

We consider the finite three-dimensional lattice \(G(M_1', M_2', M_3'; M_1, M_2, M_3)\). The non-oriented edge of this lattice is given by the pair \(\{p, e\}\) where \(p\) is the edge initial vertex and the unit vector \(e\) is the edge direction. The unit vector \(e\) is one of six vectors: \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\). Since the edge is non-oriented, then \(\{p + e, -e\} = \{p, e\}\). The initial vertex \(p\) and the final vertex \(p + e\) are the vertices of the graph \(G(M_1', M_2', M_3; M_1, M_2, M_3)\). Hence the components \(p_i, i = 1, 2, 3\), are the natural numbers and \(M_i' \leq p_i \leq M_i, M_i' \leq p_i + e_i \leq M_i, i = 1, 2, 3\). For the edge \(\{p, e\}\) the incidence numbers \((\{p, e\} : p) = (\{p, e\} : p + e) = 1\). All other incidence numbers are equal to zero.

The non-oriented face of the lattice \(G(M_1', M_2', M_3'; M_1, M_2, M_3)\) is given by the triplet \(\{p, e_1, e_2\}\) where \(p\) is the face initial vertex and the unit vectors \(e_1, e_2\) are two orthogonal to each other vectors from six unit vectors \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\). Since the face is non-oriented, then \(\{p, e_1, e_2\} = \{p + e_1, -e_1, e_2\} = \{p + e_2, e_1, -e_2\} = \{p + e_1 + e_2, -e_1 - e_2\}\). Any of four vertices \(p, p + e_1, p + e_2, p + e_1 + e_2\) of the face \(\{p, e_1, e_2\}\) is the vertex of the graph \(G(M_1', M_2', M_3'; M_1, M_2, M_3)\). Hence the components \(p_i, i = 1, 2, 3\), are the natural numbers and \(M_i' \leq p_i \leq M_i, M_i' \leq p_i + (e_1)_i \leq M_i, M_i' \leq p_i + (e_2)_i \leq M_i, i = 1, 2, 3\). For the non-oriented face \(\{p, e_1, e_2\}\) the incidence numbers \((\{p, e_1, e_2\} : \{p, e_1\}) = (\{p, e_1, e_2\} : \{p, e_2\}) = (\{p, e_1, e_2\} : \{p + e_1, e_2\}) = 1\). All other incidence numbers are equal to zero.

**Theorem 3.1.** Let the reduced partition function \(Z_{r,G_3}\) of the three-dimensional \(\mathbb{Z}_2\) electrodynamics be given by the relation \((2.14)\). Let the reduced partition function \(Z_{r,G_2}\) of the two-dimensional Ising model be given by the relation \((2.24)\). Then

\[
\begin{align*}
Z_{r,G(M_1', M_2', M_3'; M_1, M_2, M_3)} | E((p, (1,0,0),(0,1,0))) &= \infty = \\
Z_{r,G(M_1', M_2', M_3'; M_1, M_2, M_3)} | E((p,(0,1,0),(0,0,1))) &= -\infty = \\
\prod_{i=M_3-1}^{M_3-1} Z_{r,G(M_i', M_2', M_3'; M_1, M_2, M_3)} & (3.1)
\end{align*}
\]

where \(\{p, (1,0,0),(0,1,0)\}\) is any face orthogonal to the coordinate axis \(z\). The reduced partition function \(Z_{r,G(M_i', M_2', M_3'; M_1, M_2, M_3)}\) is defined by the relation \((2.24)\) with the interaction energies \(E(\{(p_1, p_2), e\}) = E(\{(p_1, p_2, i), e, (0,0,1)\})\).

**Proof.** Let us consider a cycle \(\xi^2 \in Z_2(P(G_3), \mathbb{Z}_2^{odd})\) for the graph \(G_3 = G(M_1', M_2', M_3'; M_1, M_2, M_3)\). By using this cycle we define \(M_3 - M_3'\) cochains from the group \(C^3(P(G_2), \mathbb{Z}_2^{odd})\) for the graph \(G_2 = G(M_1', M_2', M_1, M_2)\)

\[
\xi^i(\{(p_1, p_2), e\}) = \xi^2(\{(p_1, p_2, i), e, (0,0,1)\}) (3.2)
\]

where \(i = M_3', ..., M_3 - 1\) and the unit vector \(e\) is orthogonal to the unit vector \((0,0,1)\). The incidence numbers satisfy the following relations

\[
\{(p_1, p_2), e\} : (q_1, q_2) = (\{(p_1, p_2, i), e, (0,0,1)\} : (q_1, q_2, i), (0,0,1)\} (3.3)
\]

where \(i = M_3', ..., M_3 - 1\). Hence the definition \((2.3)\) implies

\[
\partial \xi^i((p_1, p_2)) = \partial \xi^2((p_1, p_2, i), (0,0,1)). (3.4)
\]

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Since $\xi^2 \in Z_2(P(G_3), Z^{add}_2)$ the equality (3.4) implies $\xi^1_i \in Z_1(P(G_2), Z^{add}_2)$.

By using a cycle $\xi^2 \in Z_2(P(G_3), Z^{add}_2)$ we define $M_3 - M'_3 + 1$ cochains from the group $C^2(P(G_2), Z^{add}_2)$ for the graph $G_2 = G(M'_1, M'_2, M_1, M_2)$

$$\xi^2_i \{(p_1, p_2), e_1, e_2\} = \xi^2((p_1, p_2, i), e_1, e_2)$$

(3.5)

where $i = M'_3, ..., M_3$ and the unit vectors $e_1, e_2$ are orthogonal to the unit vector $(0,0,1)$. Since $\xi^2 \in Z_2(P(G_3), Z^{add}_2)$ the definition (2.3) and the relations (3.2), (3.3) imply

$$\partial \xi^2_i \{(p_1, p_2, e)\} = \xi^1_{i-1} \{(p_1, p_2, e)\} + \xi^1_i \{(p_1, p_2, e)\}$$

(3.6)

$$i = M'_3 + 1, ..., M_3 - 1,$$

$$\partial \xi^2_{M'_3} \{(p_1, p_2, e)\} = \xi^1_{M'_3} \{(p_1, p_2, e)\}$$

(3.7)

$$\partial \xi^2_{M_3} \{(p_1, p_2, e)\} = \xi^1_{M_3 - 1} \{(p_1, p_2, e)\}$$

(3.8)

where the unit vector $e$ is orthogonal to the unit vector $(0,0,1)$.

For the graph $G_2 = G(M'_1, M'_2, M_1, M_2)$ the group $Z_1(P(G_2), Z^{add}_2)$ coincides with the group $B_1(P(G_2), Z^{add}_2)$ and the group $Z_2(P(G_2), Z^{add}_2)$ consists of the only cochain which is equal to zero on any face. (The homology groups are trivial). Let the arbitrary cycles $\xi^1_i \in Z_1(P(G_2), Z^{add}_2), i = M'_3, ..., M_3 - 1,$ be given. Then the equations (3.10)–(3.8) define the cochains $\xi^2_i, i = M'_3, ..., M_3$, uniquely. Now the relations (3.2), (3.5) define uniquely the cycle $\xi^2 \in Z_2(P(G_3), Z^{add}_2)$. Hence the cycle $\xi^2 \in Z_2(P(G_3), Z^{add}_2)$ is given uniquely by the cycles $\xi^1_i \in Z_1(P(G_2), Z^{add}_2), i = M'_3, ..., M_3 - 1$. The relations (3.2), (3.5) imply

$$\prod_{\{p, e_1, e_2\} \in P(G_3)} \frac{1}{\prod_{i = M'_3}^{M_3} \prod_{\{p_1, p_2, i, e_1, e_2\} \in P(G_2)} (\tanh \beta E(\{(p_1, p_2, i), e_1, e_2\}))^{1/2(1-(-1)^2((p_1, p_2, i, e_1, e_2)))}}$$

(3.9)

$$\left(\prod_{\{p, e_1, e_2\} \in P(G_3)} (\tanh \beta E(\{(p_1, p_2, i), e_1, e_2\}))^{1/2(1-(-1)^2((p_1, p_2, i, e_1, e_2)))}\right) \prod_{\{p_1, p_2, i\} \in P(G_2)} (\tanh \beta E(\{(p_1, p_2, i), e, (0,0,1)\}))^{1/2(1-(-1)^2((p_1, p_2, i, e)))}$$

(3.10)

Since $\beta > 0$, then

$$(\tanh \beta E(\{p, e_1, e_2\}))|_{E(\{p, e_1, e_2\}) = \pm \infty} = \pm 1.$$  

If $\epsilon = \pm 1$, then for any natural number $m$ we have

$$\epsilon^{1(1-(-1)m)} = \epsilon^m.$$  

(3.11)

It follows from the relations (3.9)–(3.11)

$$\left(\prod_{\{p, e_1, e_2\} \in P(G_3)} (\tanh \beta E(\{(p, e_1, e_2)\}))^{1/2(1-(-1)^2((p, e_1, e_2)))}\right)_{E(\{p, (1,0,0), (0,1,0)\}) = \pm \infty} =$$

$$\left(\prod_{\{p_1, p_2, i\} \in P(G_2)} \sum_{\{p_1, p_2, e_1, e_2\} \in P(G_2)} \xi^1_i \{(p_1, p_2, e_1, e_2)\}\right) \prod_{\{p_1, p_2, i\} \in P(G_2)} (\tanh \beta E(\{(p_1, p_2, i), e_1, e_2\}))^{1/2(1-(-1)^2((p_1, p_2, i, e_1, e_2)))}$$

(3.12)
where $E(\{p, (1, 0, 0), (0, 1, 0)\})$ is an interaction energy attached to a face
\{p, (1, 0, 0), (0, 1, 0)\} orthogonal to the coordinate axis $z$ and $\epsilon = \pm 1$ in correspondence with the relation (3.10).

By summing up the relations (3.6) – (3.8) we have
\[
\partial \left( \sum_{i=M_1^2}^{M_3} \xi_i^2 \right) = 0. \tag{3.13}
\]

The group $Z_2(P(G_2), Z^2_2)$ consists of the only cochain which is equal to zero on any face. (The homology group is trivial). Hence the relations (2.14), (2.24), (3.12) and (3.13) imply the equality (3.13). The theorem is proved.

Let us consider the particular case when the interaction energy depends only on the face orientation: $E(\{p, (1, 0, 0), (0, 0, 1)\}) = E_1$, $E(\{p, (0, 1, 0), (0, 0, 1)\}) = E_2$. Then the relation (3.1) implies
\[
Z_{r,G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)}|E((p, (1, 0, 0), (0, 1, 0))) = \infty = Z_{r,G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)}|E((p, (1, 0, 0), (0, 0, 1))) = \infty = (Z_{r,G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)})^{M_3-M_4} \tag{3.14}
\]

where the reduced partition function $Z_{r,G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)}$ is defined by the relation (2.24) for the interaction energies $E((p_1, p_2), (1, 0)) = E_1$, $E((p_1, p_2), (0, 1)) = E_2$.

The total number $(EG_3)$ of the non–oriented edges of the graph $G_3 = G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)$ is equal to

$$(M_1 - M_1^2 + 1)(M_2 - M_2^2 + 1)(M_3 - M_3^2) + (M_1 - M_1^2 + 1)(M_2 - M_2^2)(M_3 - M_3^2 + 1) + (M_1 - M_1^2)(M_2 - M_2^2 + 1)(M_3 - M_3^2 + 1).$$

Hence the relation (3.14) and Theorem 4.2 from the paper [3] imply
\[
\lim_{G_3 \to X^3, G_3^{-1}(\#(EG_3))} \frac{1}{3} \sum_{i=1,2} \lim_{M_i \to \infty, M_i^2 \to -\infty} (M_1 - M_1^2 + 1)^{-1}(M_2 - M_2^2 + 1)^{-1} \ln Z_{r,G(M_1^2, M_2^2, M_3^2, M_1, M_2, M_3)} = \frac{1}{6}(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2] \tag{3.15}
\]

where the variables $z_i = \tanh \beta E_i$, $i = 1, 2$, satisfy the estimate $|z_i| < 1/3$.

It is easy to show that the integral (3.15) has the singularity when
\[
|z_1||z_2| + |z_1| + |z_2| = 1. \tag{3.16}
\]

### 4 Correlation Functions

In this section we study the connection between the correlation functions of the three–dimensional $\mathbb{Z}_2$ electrodynamics and the correlation functions of the two–dimensional Ising model.
For the fixed cochain $\chi^0 \in C^0(P(G_2(\chi^0)), \mathbb{Z}^{add}_2)$ where the graph $G_2(\chi^0) = G(M_1(\chi^0), M_2(\chi^0); M_1(\chi^0), M_2(\chi^0))$ we define

$$W_{Z^{\times 2}}(\chi^0) = \lim_{G_2(\chi^0) \to \mathbb{Z}^{add}_2} W_{G_2}(\chi^0). \quad (4.1)$$

Due to [1] we describe the correlation function (4.1). The oriented edge of the graph $Z^{\times 2}$ is given by the pair $(p, e)$ where $p$ is the edge initial vertex and the unit vector $e$ is the edge direction. The unit vector $e$ is one of four vectors: $(\pm 1, 0), (0, \pm 1)$. A closed path is a sequence of the oriented edges $C = ((p_1, e_1), \ldots, (p_k, e_k))$ such that

$$p_{i+1} = p_i + e_i, \quad i = 1, \ldots, k-1, \quad p_1 = p_k + e_k.$$ \quad (4.2)

The number $|C| = k$ is the length of the closed path $C = ((p_1, e_1), \ldots, (p_k, e_k))$. If $C$ is a closed path, the support $||C||$ is the set of all non-oriented edges $\{p, e\}$ such that the oriented edge $(p, e)$ is included into the path $C$ or the oriented edge $(p + e, -e)$ is included into the path $C$. The closed path $((p_1, e_1), \ldots, (p_k, e_k))$ is called reduced if it satisfies the following condition

$$p_{i+1} = p_i + e_i, \quad i = 1, \ldots, k-1, \quad p_1 = p_k + e_k, \quad p_1 + e_1 \neq p_k.$$ \quad (4.3)

The set of all reduced closed paths on the graph $Z^{\times 2}$ is denoted by $RC(Z^{\times 2})$.

Let with any pair $(p_1, e_1), (p_2, e_2)$ of the oriented edges of the graph $Z^{\times 2}$ such that $p_2 = p_1 + e_1, \ p_2 + e_2 \neq p_1$ there correspond the number

$$\rho((p_1, e_1); (p_2, e_2)) = \exp\{\frac{i}{2}(e_1, e_2)\} \quad (4.4)$$

where $(e_1, e_2)$ is the radian measure of the angle between the direction of the vector $e_1$ and the direction of the vector $e_2$. With any reduced closed path $C = ((p_1, e_1), \ldots, (p_k, e_k))$ on the graph $Z^{\times 2}$ there corresponds the number

$$\rho(C) = \rho((p_1, e_1); (p_2, e_2))\rho((p_2, e_2); (p_3, e_3)) \cdots \rho((p_{k-1}, e_{k-1}); (p_k, e_k)) \times \rho((p_k, e_k); (p_1, e_1)). \quad (4.5)$$

The number $\rho(C) = \exp\{\frac{i}{2}\phi(C)\}$ where $\phi(C)$ is the total angle through which the tangent vector of the path $C$ turns along the path $C$.

Let a cochain $\xi^1 \in C^1(P(Z^{\times 2}), \mathbb{Z}^{add}_2)$. The support $||\xi^1||$ is the set of all non-oriented edges of the graph $Z^{\times 2}$ on which a cochain $\xi^1$ takes the value 1. Let a cochain $\chi^0 \in C^0(P(Z^{\times 2}), \mathbb{Z}^{add}_2)$. The support $||\xi^1||$ is called $\chi^0$-connected if any connected component of the support $||\xi^1||$ contains the non-oriented edges incident to the vertices on which a cochain $\chi^0$ equals 1. Let $i(||\xi^1||)$ be the set of all non-oriented edges incident to the vertices incident to the edges of the support $||\xi^1||$.

Due to [3] the correlation function (4.1) has the following form

$$W_{Z^{\times 2}}(\chi^0) = \sum_{\xi^1 \in C^1(P(Z^{\times 2}), \mathbb{Z}^{add}_2), \phi^1 = \chi^0, \chi^0 \text{- connected } ||\xi^1||} \exp\frac{1}{2} \sum_{C \in RC(Z^{\times 2}), ||C|| \neq 0, ||\xi^1|| \neq 0} |C|^{-1} \rho(C) \prod_{(p, e) \in C} \tanh \beta E(\{p, e\}) \times \prod_{(p, e) \in P(Z^{\times 2})} \tanh \beta E(\{p, e\}) \frac{1}{2^i(1-(-1)^i(\xi^1(p, e)))} \quad (4.6)$$
The series (4.6) is absolutely convergent when the following estimate is valid

\[ |\tanh \beta E(\{p, e\})| < 1/3 \]  

and the values of the interaction energies \( E(\{p, e\}) \) have the same sign for all collinear vectors \( e \).

**Theorem 4.1.** Let for the graph \( G_3(\chi^1) = G(M'_1(\chi^1), M'_2(\chi^1), M'_3(\chi^1); M_1(\chi^1), M_2(\chi^1), M_3(\chi^1)) \) the cochain \( \chi^1 \in B_1(P(G_3(\chi^1)), \mathbb{Z}_{2}^{odd}) \) be given. For the graph \( G'_2(\chi^1) = G(M'_1(\chi^1), M'_2(\chi^1); M_1(\chi^1), M_2(\chi^1)) \) we define \( M_3(\chi^1) - M'_3(\chi^1) \) cochains from the group \( C^0(P(G'_2(\chi^1)), \mathbb{Z}_{2}^{odd}) \)

\[ \chi^0_i((p_1, p_2, i)) = \chi^1((p_1, p_2, i), (0, 0, 1)) \]  

(4.8)

\[ i = M'_3(\chi^1), \ldots, M_3(\chi^1) - 1, \text{ and } M_3(\chi^1) - M'_3(\chi^1) + 1 \text{ cochains from the group } C^1(P(G'_2(\chi^1)), \mathbb{Z}_{2}^{odd}) \]

\[ \chi^1_i((p_1, p_2, e)) = \chi^1((p_1, p_2, i, e)) \]  

(4.9)

where \( i = M'_3(\chi^1), \ldots, M_3(\chi^1) \) and the unit vector \( e \) is orthogonal to the unit vector \((0, 0, 1)\).

If the interaction energies of the three-dimensional \( \mathbb{Z}_2 \) electrodynamics satisfy the estimate

\[ |\tanh \beta E(\{p, e, (0, 0, 1)\})| < 1/3 \]  

(4.10)

and the values of the interaction energies \( E(\{p, e, (0, 0, 1)\}) \) have the same sign for all collinear vectors \( e \), then the correlation functions of the three-dimensional \( \mathbb{Z}_2 \) electrodynamics

\[ \lim_{G_3 \to \mathbb{Z} \times 3, \ G_3(\chi^1) \subseteq G_3} W_{G_3}(\chi^1)|_{E(\{(p, (1,0,0),(0,1,0)\})} = \infty = \sum_{i=M'_3(\chi^1)}^{M_3(\chi^1)-1} W_{\mathbb{Z} \times 2,i} (\chi^0_i) \]  

(4.11)

\[ \lim_{G_3 \to \mathbb{Z} \times 3, \ G_3(\chi^1) \subseteq G_3} W_{G_3}(\chi^1)|_{E(\{(p, (1,0,0),(0,1,0)\})} = -\infty = (-1)^{\sum_{i=M'_3(\chi^1)}^{M_3(\chi^1)-1} \chi^1_i} \prod_{i=M'_3(\chi^1)}^{M_3(\chi^1)-1} W_{\mathbb{Z} \times 2,i} (\chi^0_i) \]  

(4.12)

where \( \{(p, (1,0,0),(0,1,0)\}) \) is a face orthogonal to the coordinate axis \( z \). The correlation function \( W_{\mathbb{Z} \times 2,i} (\chi^0_i) \) is given by the relation (4.3) for the interaction energies \( E(\{(p_1, p_2, e) = E(\{(p_1, p_2, i, e), (0, 0, 1)\}) \). For any cochain \( \chi^1 \in B_1(P(G_3(\chi^1)), \mathbb{Z}_{2}^{odd}) \) the cochain

\[ \sum_{i=M'_3(\chi^1)}^{M_3(\chi^1)-1} \chi^1_i = \partial \zeta^2 \]  

where the cochain \( \zeta^2 \in C^2(P(G_2(\chi^1)), \mathbb{Z}_{2}^{odd}) \) is defined uniquely. Then

\[ (-1)^{\sum_{i=M'_3(\chi^1)}^{M_3(\chi^1)-1} \chi^1_i} = (-1)^{\sum_{\{(p_1, p_2, e_1, e_2) \in P(G_2(\chi^1))} \zeta^2(\{(p_1, p_2, e_1, e_2)) \} \]  

(4.13)
Proof. For the graph $G_3 = G(M_1', M_2', M_3'; M_1, M_2, M_3), G_3(\chi^1) \subset G_3$, we consider a cochain $\xi^2 \in C^2(P(G_3), \mathbb{Z}_2^{add})$ satisfying the equation $\partial \xi^2 = \chi^1$. The relations (4.2) define the cochains $\xi^1_i \in C^1(P(G_2), \mathbb{Z}_2^{add}), i = M_3', ..., M_3 - 1$, for the graph $G_2 = G(M_1', M_2'; M_1, M_2)$. The relations (3.4), (1.8) imply

$$\partial \xi^1_i = \chi^0_i$$

where $\chi^0_i = 0$ for $i < M_3'(\chi^1)$ and $i \geq M_3(\chi^1)$.

For the same cochain $\xi^2 \in C^2(P(G_3), \mathbb{Z}_2^{add})$ the relations (3.3) define the cochains $\xi^1_i \in C^2(P(G_2), \mathbb{Z}_2^{add}), i = M_3', ..., M_3$. Since $\partial \xi^2 = \chi^1$, the definition (2.3) and the relations (3.2), (3.3), (1.3) imply

$$\partial \xi^2_M((p_1, p_2), e)) = \chi^1_M((p_1, p_2), e)) + \xi^1_{M_3}(\xi^1_{M_3}(p_1, p_2, e)) + \xi^1_M((p_1, p_2, e)),$$

$$\partial \xi^2_M((p_1, p_2), e)) = \chi^1_M((p_1, p_2, e)) + \xi^1_{M_3}(\xi^1_{M_3}(p_1, p_2, e)) + \xi^1_M((p_1, p_2, e))$$

where $\xi^1_{M_3}(p_1, p_2, e)$ is a non-oriented face of the graph $G_2$ and the cochains $\chi^1_i = 0$ for $i < M_3'(\chi^1)$ and $i > M_3(\chi^1)$.

Let the arbitrary cochains $\xi^1_i \in C^1(P(G_2), \mathbb{Z}_2^{add}), i = M_3', ..., M_3 - 1$, satisfying the equations (4.14) be given. Since $\chi^1 \in B_1(P(G_3), \mathbb{Z}_2^{add}) = Z_1(P(G_3), \mathbb{Z}_2^{add})$, the equations (4.14) imply that the right hand sides of the equations (4.13) - (4.17) are the cycles, i.e. they belong to the group $Z_1(P(G_2), \mathbb{Z}_2^{add})$. For the graph $G_2 = G(M_1', M_2'; M_1, M_2)$ the group $Z_1(P(G_2), \mathbb{Z}_2^{add})$ coincides with the group $B_1(P(G_2), \mathbb{Z}_2^{add})$ and the group $Z_2(P(G_2), \mathbb{Z}_2^{add})$ consists of the only cochain which is equal to zero on any face. (The homology groups are trivial). Therefore the cochains $\xi^2_i \in C^2(P(G_2), \mathbb{Z}_2^{add})$ are defined uniquely by the equations (4.14). Then the relations (3.2), (3.3) define uniquely the cochain $\xi^2 \in C^2(P(G_3), \mathbb{Z}_2^{add})$ satisfying the equation $\partial \xi^2 = \chi^1$. Hence the cochain $\xi^2 \in C^2(P(G_2), \mathbb{Z}_2^{add})$ satisfying the equation $\partial \xi^2 = \chi^1$ is given uniquely by cochains $\xi^1_i \in C^1(P(G_2), \mathbb{Z}_2^{add}), i = M_3', ..., M_3 - 1$, satisfying the equations (4.14).

The relations (3.2), (3.3) imply

$$\left(\prod_{\{p, e, 1, e\}_i \in P(G_2)} (\tanh \beta E(\{p, e, 1, e\}_i)) \right)^{\frac{1}{2}} \frac{(1 - (-1)^{\xi^2((p, e, 1, e))})}{(1 - (-1)^{\xi^1((p, e, 1, e))})}$$

Now the relations (2.13), (2.23), (3.4), (4.14) and Theorem 4.3 from the paper [9] imply the relation (4.11).

The relations (3.2), (3.5), (3.10), (3.11) imply

$$\left(\prod_{\{p, e, 1, e\}_i \in P(G_3)} (\tanh \beta E(\{p, e, 1, e\}_i)) \right)^{\frac{1}{2}} \frac{(1 - (-1)^{\xi^2((p, e, 1, e))})}{(1 - (-1)^{\xi^1((p, e, 1, e))})}$$

$$\times \left(\prod_{i = M'_3 \{p, e, 1, e\}_i \in P(G_2)} (\tanh \beta E((p, e, 1, e)) \right)^{\frac{1}{2}} \frac{(1 - (-1)^{\xi^1((p, e, 1, e))})}{(1 - (-1)^{\xi^1((p, e, 1, e))})}.$$
Summing up the relations (4.15) – (4.17) we obtain
\[ \partial \left( \sum_{i=M_3'} M_3 \sum_{i=M_3'} \xi_i^2 \right) = \sum_{i=M_3'} M_3 \chi_i^1. \] (4.20)

The group \( Z_2(P(G_2, Z_2^{add})) \) consists of the only cochain which is equal to zero on any face. (The homology group is trivial). Hence the equation (4.20) defines the cochain \( \sum_{i=M_3'} M_3 \sum_{i=M_3'} \xi_i^2 \) uniquely. Now the relations (2.15), (2.25), (3.1), (4.1), (4.19), (4.20) and Theorem 4.3 from the paper [6] imply the relations (4.12), (4.13). The theorem is proved.

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