Derivatives of normal Jacobi operator on real hypersurfaces in the complex quadric

Hyunjin Lee, Juan de Dios Pérez and Young Jin Suh

Abstract

Suh (Math. Nachr. 290 (2017) 442–451) proved that there are no Hopf real hypersurfaces in the complex quadric that have parallel normal Jacobi operators. Motivated by this result, in this paper, we introduce the notions of $C$-parallel and Reeb parallel for normal Jacobi operators, which generalize the notion ‘parallel’. First we obtain a non-existence theorem of Hopf real hypersurfaces with $C$-parallel normal Jacobi operator in the complex quadric $Q^m$ for $m \geq 3$. We then prove that a Hopf real hypersurface has a Reeb parallel normal Jacobi operator if and only if it has an $\mathfrak{A}$-isotropic singular normal vector field.

1. Introduction

As an example of a Hermitian symmetric space of compact type, consider the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see [14–16, 19, 20]). The complex quadric can also be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [3]). Accordingly, the complex quadric $Q^m$ admits two important geometric structures, a complex conjugation structure $A$ and a Kähler structure $J$, which anti-commute with each other, that is, for which $AJ = -JA$. For $m \geq 3$, the triple $(Q^m, J, g)$ is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Kobayashi and Nomizu [7] and Reckziegel [13]).

In addition to the complex structure $J$, there is another distinguished geometric structure on $Q^m$, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, specifically, an $S^1$-bundle of complex conjugations $A$ on the tangent spaces of $Q^m$. This is denoted by $\mathfrak{A}_{[z]} = \{ A_\lambda \mid \lambda \in S^{1} \subset \mathbb{C} \}$, $[z] \in Q^m$. Now, $\mathfrak{A}_{[z]}$ is a parallel rank 2-subbundle of $\text{End} \ T_{[z]}Q^m$, $[z] \in Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^m$. Here the notion of a parallel vector bundle $\mathfrak{A}$ means that $1 \nabla_X A) Y = q(X) J_AY$ for any vector fields $X$ and $Y$ on $Q^m$, where $\nabla$ and $q$ denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$, respectively, (see [16]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for the complex hyperbolic quadric $Q^m$.

- If there exists a conjugation $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A) = \{ X \in T_{[z]}Q^m \mid AX = X \}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
• If there exist a conjugation $A \in \mathfrak{X}[z]$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{X}$-isotropic, where $V(A) = \{X \in T[z]Q^m \mid AX = X\}$ and $JV(A) = \{X \in T[z]Q^m \mid AX = -X\}$ are the $(+1)$-eigenspace and $(-1)$-eigenspace for the involution $A$ on $T[z]Q^m$, $[z] \in Q^m$.

Jacobi fields along geodesics of a given Riemannian manifold $(M, g)$ satisfy a well-known differential equation which naturally inspires the so-called Jacobi operator. That is, if $\bar{R}$ denotes the curvature operator of $M$, and $Z$ is a tangent vector field to $M$, then the Jacobi operator $\bar{R}_Z \in \mathcal{E}nd(T[z]M)$ with respect to $Z$ at $[z] \in M$, defined by $(\bar{R}_Z Y) ([z]) = (\bar{R}(Y, Z)Z) ([z])$ for any $Y \in T[z]M$, becomes a self-adjoint endomorphism of the tangent bundle $TM$ of $M$. Thus, each tangent vector field $Z$ to $M$ provides a Jacobi operator $\bar{R}_Z$ with respect to $Z$. In particular, let $M$ be a real hypersurface in $\tilde{M}$ and $N$ be a normal vector field of $M$ in $\tilde{M}$. Then, for the normal vector field $N$, we can define the Jacobi operator $\bar{R}_N \in \mathcal{E}nd T\tilde{M}$, which is said to be the normal Jacobi operator. Previously, several geometric properties — parallelism, invariancy, the commuting property — of the normal Jacobi operator $\bar{R}_N$ for a real hypersurface in a Kähler manifold have been studied by many geometers [4, 9–12].

The normal Jacobi operator $\bar{R}_N$ of $M$ is said to be parallel ($\mathcal{C}$-parallel or Reeb parallel, respectively) if $\bar{R}_N \in \mathcal{E}nd T\tilde{M}$ satisfies

$$\nabla_X \bar{R}_N = 0 \quad (\nabla_{\xi} \bar{R}_N = 0 \text{ or } \nabla_{\xi} \bar{R}_N = 0, \text{ respectively})$$

for any tangent vector field $X$ on $M$, where $\mathcal{C}$ denotes the orthogonal distribution to span{ξ} such that $\mathcal{C} = \{X \in T[z]M \mid X \perp \xi, [z] \in M\}$.

In [4], Jeong, Kim and Suh investigated the parallel normal Jacobi operator of a Hopf real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ and they gave a non-existence theorem. Here a real hypersurface $M$ is said to be Hopf if the Reeb vector field $\xi$ of $M$ is principal for the shape operator $S$, that is, if $S\xi = g(S\xi, \xi)\xi = \alpha \xi$. In particular, if the Reeb curvature function $\alpha = g(S\xi, \xi)$ identically vanishes, we say that $M$ has a vanishing geodesic Reeb flow. Otherwise, $M$ has a non-vanishing geodesic Reeb flow. Moreover, for the complex quadric $Q^m$, Suh [21] proved the following:

**Theorem A.** There do not exist any real hypersurfaces in the complex quadric $Q^m$ for $m \geq 3$, with parallel normal Jacobi operator.

In addition, Lee and Suh [9] generalized this notion to the recurrent normal Jacobi operator, defined to satisfy $(\nabla_X \bar{R}_N)Y = \beta(X)\bar{R}_N Y$ for a certain 1-form $\beta$ and any vector fields $X, Y$ on $M$ in $Q^m$. They gave a non-existence theorem in this generality.

Motivated by these results, as a generalization of Theorem A, in this paper we want to give some classifications of Hopf real hypersurfaces in $Q^m$ with respect to $\mathcal{C}$-parallelism and Reeb parallelism for the normal Jacobi operator $\bar{R}_N$ which are, respectively, defined by

$$\nabla_{\xi} \bar{R}_N = 0$$

and

$$\nabla_{\xi} \bar{R}_N = 0,$$

where the orthogonal distribution $\mathcal{C}$ of the Reeb vector field $\xi$ is defined by

$$\mathcal{C} = \{X \in T[z]M \mid X \perp \xi, [z] \in M\}.$$
Berndt–Suh and Klein–Suh, respectively, gave new characterizations of model spaces of type $B$ in the complex quadric $Q^m$ by using the notion of contact (see [2, 6]). By virtue of these characterizations of contact real hypersurfaces in $Q^m$, in [10], Lee and Suh gave:

**Theorem B.** Let $M$ be a Hopf real hypersurface in the complex quadric $Q^m$ for some $m \geq 3$. The real hypersurface $M$ has an $\mathfrak{A}$-principal normal vector field in $Q^m$ if and only if $M$ is locally congruent to a model space of type $(T_B)$.

**Remark 1.** In Theorem B, a model space of type $(T_B)$ means a tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$ which is embedded in $Q^m$ as a real form of $Q^m$.

**Remark 2.** As a way to classify real hypersurfaces $M$ with $\mathfrak{A}$-isotropic singular normal vector field in $Q^m$, Berndt–Suh introduced the notion of *isometric Reeb flow* on $M$ in $Q^m$ for $m \geq 3$, and asserted that $M$ is locally congruent to a tube over a totally geodesic $CP^k$ in $Q^{2k}$, $m = 2k$. Also, it is known that the normal vector field of this kind of tube is $\mathfrak{A}$-isotropic (see [1, 17, 21, 22]).

Motivated by these results, we first prove the following:

**Theorem 1.** There does not exist a Hopf real hypersurface with $C$-parallel normal Jacobi operator in the complex quadric $Q^m$ for $m \geq 3$.

We then consider a Hopf real hypersurface with Reeb parallel normal Jacobi operator in $Q^m$. Using Theorem B mentioned above we will prove the following:

**Theorem 2.** Let $M$ be a Hopf real hypersurface in the complex quadric $Q^m$ for $m \geq 3$. The real hypersurface $M$ has an $\mathfrak{A}$-isotropic singular normal vector field if and only if $M$ has a Reeb parallel normal Jacobi operator.

### 2. The complex quadric

For more detailed background related to this section, we recommend [1, 2, 5, 7, 13, 17, 18, 20, 22]. The complex quadric $Q^m$ is the complex hypersurface in $CP^{m+1}$ which is defined by the equation $z^2_1 + \cdots + z^2_{m+2} = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $CP^{m+1}$. We equip $Q^m$ with the Riemannian metric which is induced from the Fubini Study metric on $CP^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $CP^{m+1}$ canonically induces a Kähler structure $(J, g)$ on the complex quadric. For a nonzero vector $z \in \mathbb{C}^{m+2}$, we denote by $[z]$ the complex span of $z$, that is, $[z] = \mathbb{C}z = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that, by definition, $[z]$ is a point in $CP^{m+1}$. For each $[z] \in Q^m \subset CP^{m+1}$, we identify $T_{[z]} CP^m$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see [7]). The tangent space $T_{[z]} Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus C \rho)$ of $\mathbb{C}z \oplus C \rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_{[z]} Q^m$ is a normal vector of $Q^m$ in $CP^{m+1}$ at the point $z$.

The complex projective space $CP^{m+1}$ is a Hermitian symmetric space of the special unitary group $SU_{m+2}$, namely $CP^{m+1} = SU_{m+2} / S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in CP^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $CP^{m+1}$ with cohomogeneity one. The orbit containing $o$ is a totally geodesic real projective space $RP^{m+1} \subset CP^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2} / SO_{m} \cdot SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric $Q^m$ as the Grassmann manifold $G^+_2(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of $Q^m$ as a Hermitian symmetric space.
of rank 2. The complex quadric $Q^1$ is isometric to a sphere $S^2$ with constant curvature, and $Q^2$ is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason, we will assume $m \geq 3$ from now on.

For a unit normal vector $\rho$ of $Q^m$ at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of $Q^m$ in $\mathbb{CP}^{m+1}$ with respect to $\rho$. The shape operator is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace of $A_\rho$ and $JV(A_\rho)$ is the $(-1)$-eigenspace of $A_\rho$. Geometrically, this means that the shape operator $A_\rho$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, it is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of $Q^m$ in $\mathbb{CP}^{m+1}$ is 2, this induces an $S^1$-subbundle $\mathcal{A}$ of the endomorphism bundle $\text{End}(TQ^m)$ consisting of complex conjugations. There is a geometric interpretation of these conjugations. Indeed, the complex quadric $Q^m$ can be viewed as the complexification of the $m$-dimensional sphere $S^m$. Through each point $[z] \in Q^m$, there exists a one-parameter family of real forms of $Q^m$ which are isometric to the sphere $S^m$. These real forms are congruent to each other under action of the center $SO_2$ of the isotropy subgroup of $SO_{m+2}$ at $[z]$. The isometric reflection of $Q^m$ in such a real form $S^m$ is an isometry, and the differential of $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way, the family $\mathcal{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms $S^m$ of $Q^m$ containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms $S^m$ of $Q^m$.

The Gauss equation for $Q^m \subset \mathbb{CP}^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and the complex conjugations $A \in \mathcal{A}$:

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ + g(AY,Z)AX - g(AX,Z)AY - g(JAX,Z)JAY$$

for any vector fields $X, Y$ and $Z$ in $T_{[z]}Q^m$, $[z] \in Q^m$.

It is well known that for every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathcal{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$W = \cos(t)Z_1 + \sin(t)JZ_2$$

for some $t \in [0, \pi/4]$ (see [13]). The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$, then the unique maximal flat containing $W$ is $\mathbb{R}Z_1 \oplus \mathbb{R}JZ_2$.

3. Real hypersurfaces in $Q^m$

Let $M$ be a real hypersurface in $Q^m$ and denote by $(\phi, \xi, \eta, g)$ the induced almost contact metric structure. By the Gauss and Weingarten formulas, the left-hand side of (2.1) becomes

$$\bar{R}(X,Y)Z = R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY + \{g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z)\}N,$$

where $R$ and $S$ denote the Riemannian curvature tensor and the shape operator of $M$ in $Q^m$, respectively.

Note that $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where $\phi X$ is the tangential component of $JX$ and $N$ is a (local) unit normal vector field of $M$. The tangent bundle $TM$ of $M$ splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta = \{X \in TM \mid g(X, \xi) = \eta(X) = 0\}$ is the maximal complex subbundle of $TM$. The structure tensor field $\phi$ restricted to $\mathcal{C}$ coincides with the complex structure $J$ restricted to $\mathcal{C}$, and $\phi \xi = 0$. Moreover, since the complex quadric $Q^m$...
has also a real structure $A$, we can decompose $AX$ into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$:

$$AX = BX + \rho(X)N,$$

where $BX$ is the tangential component of $AX$ and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

From these notations, taking the tangential and normal components of (2.1), we obtain

$$R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY = g(Y,Z)X - g(X,Z)Y + g(JY,Z)\phi X - g(JX,Z)\phi Y - 2g(JX,Y)\phi Z + g(AY,Z)BX - g(AX,Z)BY + g(JAY,Z)\phi BX - g(JAX,Z)\rho(Y)\xi$$

and

$$(\nabla_X S)Y - (\nabla_Y S)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi - g(X,\phi A\xi)BY + g(\phi A\xi,Y)BX + g(AX,Y)\phi BY + g(A\xi,Y)g(\phi A\xi,Y)\xi - g(A\xi,Y)\phi BX - g(A\xi,Y)\phi A\xi - g(\phi A\xi,Y)\xi,$$

which are called the equations of Gauss and Codazzi, respectively.

As mentioned in Section 2, since the normal vector field $N$ belongs to $T_{[z]}Q^m$, $[z] \in M$, we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$N = N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [13]). Note that $t$ is a function on $M$. If $t = 0$, then $N = Z_1 \in V(A)$, therefore we see that $N$ becomes an $\mathfrak{A}$-principal singular tangent vector field. On the other hand, if $t = \frac{\pi}{4}$, then $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$. That is, $N$ is an $\mathfrak{A}$-isotropic singular tangent vector field. In addition, since $\xi = -JN$, we have

$$\begin{cases}
\xi = \sin(t)Z_2 - \cos(t)JZ_1, \\
AN = \cos(t)Z_1 - \sin(t)JZ_2, \\
A\xi = \sin(t)Z_2 + \cos(t)JZ_1.
\end{cases}$$

This implies $g(\xi, AN) = 0$ and $g(AX, \xi) = -g(AN, N) = -\cos(2t)$ on $M$. At each point $[z] \in M$ we define the maximal $\mathfrak{A}$-invariant subspace of $T_{[z]}M$, $[z] \in M$, as follows:

$$Q_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}.$$  

It is known that if $N_{[z]}$ is $\mathfrak{A}$-principal, then $Q_{[z]} = C_{[z]}$. But if $N_{[z]}$ is not $\mathfrak{A}$-principal, then $C_{[z]} = Q_{[z]} \oplus \span\{AN, A\xi\}$ (see [17, 20]).

We now assume that $M$ is a Hopf real hypersurface in the complex quadric $Q^m$. Then, the shape operator $S$ of $M$ in $Q^m$ satisfies $S\xi = \alpha \xi$ with the Reeb function $\alpha = g(S\xi, \xi)$ on $M$. In particular, if $\alpha$ identically vanishes (otherwise, respectively), then it is said that $M$ has a vanishing (non-vanishing, respectively) geodesic Reeb flow. By virtue of the Codazzi equation (3.3), we obtain the following lemma.
**Lemma 3.1** [9]. Let $M$ be a Hopf real hypersurface in $Q^m$ for $m \geq 3$. We have

$$X\alpha = (\xi\alpha)\eta(X) + 2g(A\xi, \xi)g(X, AN)$$

(3.5)

and

$$2S\phi SX = \alpha\phi SX + \alpha S\phi X + 2\phi X + 2g(X, \phi A\xi)A\xi$$

$$-2g(X, A\xi)\phi A\xi - 2g(\xi, A\xi)g(\phi A\xi, X)\xi + 2\eta(X)g(\xi, A\xi)\phi A\xi$$

(3.6)

for any tangent vector field $X$ on $M$.

**Remark 3.2.** From (3.5), we know that if $M$ has either vanishing geodesic Reeb flow or constant Reeb curvature, then the normal vector $N$ is singular. In fact, for such real hypersurfaces (3.5) becomes $g(A\xi, \xi)g(X, AN) = 0$ for any tangent vector field $X$ on $M$. Since $g(A\xi, \xi) = -\cos(2t)$, $0 \leq t \leq \frac{\pi}{4}$, the case of $g(A\xi, \xi) = 0$ implies that the normal vector field $N$ is $\mathfrak{A}$-isotropic. On the other hand, if $g(A\xi, \xi) \neq 0$, that is, $g(AN, X) = 0$ for all $X \in TM$, then the vector field $AN \in TQ^m$ is given

$$AN = \sum_{i=1}^{2m} g(AN, e_i)e_i + g(AN, N)N = g(AN, N)N$$

for any basis $\{e_1, e_2, \ldots, e_{2m-1}, e_{2m} = N | e_i \in TM, i = 1, 2, \ldots, 2m - 1\}$ of $TQ^m$. As $A^2 = I$, we then get

$$N = A^2 N = g(AN, N)AN.$$

Taking the inner product with $N$ of this equation, we get $g(AN, N) = \pm 1$. Since $g(AN, N) = \cos(2t)$ where $t \in [0, \frac{\pi}{4}]$, we assert that $g(AN, N) = 1$. That is, $AN = N$. Hence $N$ is $\mathfrak{A}$-principal.

On the other hand, from the fact that $g(A\xi, N) = 0$ we assert that $A\xi$ is a unit tangent vector field on $M$ in $Q^m$. Hence by Gauss formula, $\nabla_X Y = \nabla_X Y + \sigma(X, Y)$, and $\nabla_X \xi = \phi SX$ for $X, Y \in TM$, it induces

$$\nabla_X (A\xi) = \nabla_X (A\xi) - \sigma(X, A\xi)$$

$$= q(X)JA\xi + A(\nabla_X \xi) + g(SX, \xi)AN - g(SX, A\xi)N$$

$$= q(X)JA\xi + A\phi SX + g(SX, \xi)AN - g(SX, A\xi)N.$$  

From $AN = AJ\xi = -JA\xi$ and $JA\xi = \phi A\xi + \eta(A\xi)N$, we get

$$\begin{cases} 
\text{Tangential Part} : \quad \nabla_X (A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi, \\
\text{Normal Part} : \quad q(X)g(A\xi, \xi) = -g(AN, \phi SX) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi). 
\end{cases}$$

(3.7)

In particular, if $M$ is Hopf, then the second equation in (3.7) becomes

$$q(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi).$$

(3.8)

Now, we want to introduce the following proposition as a typical characterization of real hypersurfaces in $Q^m$ with $\mathfrak{A}$-principal normal vector field due to Berndt and Suh in [2].

**Proposition B.** Let $(T_B)$ be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$ in $Q^m$. Then the following hold.

(i) $(T_B)$ is a Hopf real hypersurface.

(ii) The normal bundle of $(T_B)$ consists of $\mathfrak{A}$-principal singular vector fields.

(iii) $(T_B)$ has three distinct constant principal curvatures.
\( \alpha = -\sqrt{2} \cot(\sqrt{2}r) \)
\( \lambda = \sqrt{2} \tan(\sqrt{2}r) \)
\( \mu = 0 \)

| principal curvature | eigenspace | multiplicity |
|---------------------|------------|--------------|
| \( T_\alpha = \text{Span}\{\xi\} \) | \( \{X \in \mathcal{C} \mid AX = X\} \) | 1 |
| \( T_\lambda = \{X \in \mathcal{C} \mid AX = -X\} \) | \( \{X \in \mathcal{C} \mid AX = X\} \) | \( m - 1 \) |
| \( T_\mu = \{X \in \mathcal{C} \mid AX = -X\} \) | \( \{X \in \mathcal{C} \mid AX = X\} \) | \( m - 1 \) |

(iv) \( S\phi + \phi S = 2\delta\phi, \delta = -\frac{1}{\alpha} \neq 0 \) (contact hypersurface).

### 4. \( \mathcal{C} \)-parallel normal Jacobi operator

In this section, we assume that \( M \) is a Hopf real hypersurface in the complex quadric \( Q^m \) for \( m \geq 3 \), with \( \mathcal{C} \)-parallel normal Jacobi operator, meaning

\[
(\nabla_X \bar{R}_N)Y = 0
\]

for \( X \in \mathcal{C} \) and \( Y \in TM \). Recall that the distribution \( \mathcal{C} \) is given by \( \mathcal{C} = \{X \in TM \mid X \perp \xi\} \).

It is mentioned in [9, Section 5] that the normal Jacobi operator \( \bar{R}_N \in \text{End}(TM) \) and its covariant derivative are given for any \( Y, Z \in TM \), respectively, by

\[
\bar{R}_NZ = Z + 3\eta(Z)\xi - g(A\xi,\xi)BZ - g(\phi A\xi, Z)\phi A\xi - g(A\xi, Z)A\xi, \tag{4.1}
\]

and

\[
(\nabla_Y \bar{R}_N)Z = 3g(Z, \phi SY)\xi + 3\eta(Z)\phi SY - 2g(A\xi, \phi SY)BZ - g(Y)g(A\xi, \xi)\phi BZ - g(Y)g(\phi A\xi, \xi) - g(BZ, SY)\phi A\xi - g(\phi A\xi, Z)BSY - g(BZ, \phi SY)A\xi - g(\phi A\xi, Z)B\phi SY + g(\phi A\xi, Z)g(SY, \xi)A\xi + g(A\xi, Z)g(SY, \phi A\xi), \tag{4.2}
\]

where we have used:

\[
g(AN, N) = -g(A\xi, \xi),
\]

\[
AN = AJ\xi = -JA\xi = -\phi A\xi - \eta(A\xi)N,
\]

and

\[
JAZ = J(BZ + g(AZ, N)N) = \phi BZ + \eta(BZ)N - g(AZ, N)\xi = \phi BZ + g(Z, \phi A\xi)\xi + \eta(BZ)N.
\]

Assuming that the Jacobi operator is a \( \mathcal{C} \)-parallel normal Jacobi operator, we now prove that the unit normal vector field \( N \) of \( M \) in \( Q^m \) is singular. In order to do this, we consider two cases: either the Reeb curvature function \( \alpha = g(S\xi, \xi) \) vanishes, or it does not vanish. When the Reeb curvature function \( \alpha \) vanishes, by Remark 3.2 we know that \( N \) is singular. Now, for the case that the Reeb curvature function \( \alpha \) does not vanish we prove the following lemma.

**Lemma 4.1.** Let \( M \) be a Hopf real hypersurface in the complex quadric \( Q^m \) for \( m \geq 3 \), with non-vanishing geodesic Reeb flow. If the normal Jacobi operator \( \bar{R}_N \) is \( \mathcal{C} \)-parallel, then the normal vector field \( N \) is singular.

**Proof.** From (4.2), the condition of \( \mathcal{C} \)-parallel normal Jacobi operator gives us

\[
(\nabla_X \bar{R}_N)\xi = 0
\]

\[
\iff 3\phi SX - 2g(A\xi, \phi SX)A\xi - g(X)g(A\xi, \xi)\phi A\xi - g(A\xi, SX)\phi A\xi - g(A\xi, \phi SX)A\xi - g(A\xi, \xi)B\phi SX = 0 \tag{4.3}
\]
if \( Z = \xi \) and \( Y = X \in C \). On the other hand, from the property that \( JA = -AJ \), we obtain that

\[
\phi BZ + g(\phi A\xi, Z)\xi = -B\phi Z + \eta(Z)\phi A\xi, \quad Z \in TM.
\]

By using this formula, (4.3) can be rewritten as

\[
3\phi SX - 2g(A\xi, \phi SX)A\xi - q(X)g(A\xi, \xi)\phi A\xi - g(A\xi, SX)\phi A\xi
\]

\[-g(\alpha, \phi SX)A\xi + g(A\xi, \xi)\phi BSX + g(A\xi, \xi)g(\phi A\xi, SX)\xi = 0.
\]

Moreover, from (3.7) we obtain \( q(X)g(A\xi, \xi) = 2g(SA\xi, X), \ X \in C \). So, the above equation becomes

\[
3\phi SX + 3g(S\phi A\xi, X)A\xi - 3g(SA\xi, X)\phi A\xi
\]

\[+g(A\xi, \xi)\phi BSX + g(A\xi, \xi)g(S\phi A\xi, X)\xi = 0. \tag{4.4}
\]

Taking the inner product of (4.4) with \( \xi \), we get

\[g(A\xi, \xi)g(S\phi A\xi, X) = 0 \]

for all \( X \in C \).

When \( g(A\xi, \xi) = 0 \), the normal vector field \( N \) should be \( \mathfrak{A} \)-isotropic. Hence from now on we consider \( g(A\xi, \xi) \neq 0 \). This implies \( g(S\phi A\xi, X) = 0 \), which leads to

\[S\phi A\xi = g(S\phi A\xi, \xi)\xi = 0. \]

From this, (4.4) becomes

\[
3\phi SX - 3g(SA\xi, X)\phi A\xi + g(A\xi, \xi)\phi BSX = 0, \tag{4.5}
\]

which yields

\[
-3\alpha SX + 3g(SA\xi, X)A\xi - 3g(SA\xi, X)g(A\xi, \xi)\xi
\]

\[-g(\alpha, \xi)BSX + g(\alpha, \xi)g(ASX, \xi)\xi = 0,
\]

if we apply the structure tensor \( \phi \) to (4.5) and use \( \phi^2 Z = -Z + \eta(Z)\xi, \ A\xi = B\xi. \) Since \( \alpha = g(S\xi, \xi) \neq 0 \), we consequently have

\[
-3\alpha SX + 3g(SA\xi, X)A\xi - 3g(SA\xi, X)g(A\xi, \xi)\xi
\]

\[-\alpha g(\alpha, \xi)BSX + \alpha g(\alpha, \xi)g(X, SA\xi)\xi = 0. \tag{4.6}
\]

From (3.6) and \( S\phi A\xi = 0 \), we have \( \alpha SA\xi = \beta(\alpha^2 + 2\beta^2)\xi - 2\beta^2 A\xi \), where \( \beta = g(A\xi, \xi) \). Thus it follows, for any \( X \in C \)

\[\alpha g(SA\xi, X) = -2\beta^2 g(A\xi, X). \tag{4.7}
\]

So, equation (4.6) yields

\[-3\alpha SX - 6\beta^2 g(A\xi, X)A\xi + 6\beta^3 g(A\xi, X)\xi - \alpha\beta BSX - 2\beta^3 g(A\xi, X)\xi = 0.
\]

Taking the inner product with \( A\xi \) of this equation, we get

\[-3\alpha g(SX, A\xi) - 6\beta^2 g(A\xi, X) + 4\beta^4 g(A\xi, X) - \alpha\beta g(BSX, A\xi) = 0.
\]

Since \( BA\xi = A^2 \xi = \xi \), together with (4.7) and \( \beta \neq 0 \), it gives us \( g(A\xi, X) = 0 \) for all \( X \in C \). Hence, it follows

\[A\xi = g(A\xi, \xi)\xi = \beta\xi. \tag{4.8}
\]

As \( A \) is a real structure, we have \( A^2 = I \). So, (4.8) becomes \( \xi = A^2 \xi = \beta A\xi \). By using (4.8) again, it leads to \( \beta^2 = 1 \). As stated in Section 3, we see that \( \beta = g(A\xi, \xi) = -\cos(2t) \) where \( t \in [0, \frac{\pi}{2}] \), since \( \beta \neq 0 \). Hence \( \beta^2 = \cos^2(2t) = 1 \) implies \( t = 0 \), which means that the normal
vector field $N$ is $\mathfrak{A}$-principal. In fact, if $t = 0$, then $N$ can be expressed as $N = V_1$ for some $V_1 \in V(A)$.

This completes the proof of the lemma. \hfill \square

By Remark 3.2 and Lemma 4.1, if we consider a Hopf real hypersurface $M$ with $C$-parallel normal Jacobi operator in $Q^m$ for $m \geq 3$, then the normal vector field $N$ is singular, that is, it is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. We thus consider the case that $M$ has an $\mathfrak{A}$-principal normal vector field $N$ in $Q^m$. In this case, (4.4) becomes $3\phi SX - \phi BSX = 0$ for all $X \in \mathcal{C}$. Applying the structure tensor $\phi$ to this equation, we get

$$-3SX + BSX = 0$$

(4.9)

for $X \in \mathcal{C}$.

On the other hand, when a Hopf real hypersurface $M$ in $Q^m$ has an $\mathfrak{A}$-principal normal vector field $N$, it follows that $A\xi = -\xi$ and $AN = N$. In particular, from $AN = N$, we obtain:

$$BSZ = ASZ - g(ASZ, N)N = ASZ = SZ - 2\alpha \eta(Z)\xi,$$

where we have used the Weingarten formula, $\nabla_Z N = -SZ$ for any $Z \in TM$, in the third equality. Thus (4.9) gives us $SX = 0$ for all $X \in \mathcal{C}$. That is, the diagonal components of the shape operator $S$ of $M$ are given by

$$S = \text{diag}(\alpha, 0, 0, \ldots, 0),$$

which implies that the shape operator $S$ of $M$ should be anti-commuting: $S\phi + \phi S = 0$. Using the main theorem in [10], we then get that there does not exist a Hopf real hypersurface with $C$-parallel normal Jacobi operator and $\mathfrak{A}$-principal normal vector field $N$ of $M$ in $Q^m$ for $m \geq 3$.

Next, let us assume that $N$ is $\mathfrak{A}$-isotropic. This implies $SA\xi = SAN = -S\phi A\xi = 0$. So, (4.4) becomes $\phi SX = 0$ for any $X \in \mathcal{C}$. Applying the structure tensor $\phi$ to this equation, we get $SX = 0$ for any $X \in \mathcal{C}$. This yields that $S$ satisfies the anti-commuting property, $S\phi + \phi S = 0$. So, we obtain that there does not exist a Hopf real hypersurface with $C$-parallel normal Jacobi operator and $\mathfrak{A}$-isotropic tangent vector field $N$ in $Q^m$ for $m \geq 3$, and this completes the proof of our Theorem 1.

5. **Reeb parallel normal Jacobi operator**

In this section we assume that $M$ is a Hopf real hypersurface in the complex quadric $Q^m$ for $m \geq 3$, with Reeb parallel normal Jacobi operator, that is,

$$(\nabla_\xi \bar{R}_N)Y = 0$$

for all tangent vector fields $Y$ of $M$. Under this assumption, equation (4.2) together with (3.8) yield

$$(\nabla_\xi \bar{R}_N)Y = 0
\iff q(\xi)g(A\xi, \xi)\{\phi BY + g(\phi A\xi, Y)\xi\} = 0
\iff 2q(\xi)\{\phi BY + g(\phi A\xi, Y)\xi\} = 0.$$

(5.1)

By virtue of Remark 3.2, if the Reeb function $\alpha = g(S\xi, \xi)$ vanishes identically, then the normal vector field $N$ of $M$ in $Q^m$ is singular. Hence, now, let us consider the case of $\alpha \neq 0$. From (5.1), we can divide the study into the following two cases.

Case 1. $g(A\xi, \xi) = 0$

From the definition of $\mathfrak{A}$-isotropic singular vector field, the normal vector field $N$ is $\mathfrak{A}$-isotropic.
Case 2. \( g(A\xi, \xi) \neq 0 \)

It implies that \( \phi BY + g(\phi A\xi, Y)\xi = 0 \) for any tangent vector field \( Y \in TM \). Applying the structure tensor \( \phi \), we have

\[
BY = \eta(BY)\xi = g(A\xi, Y)\xi.
\]

Since \( BY = AY - g(AY, N)N \), it follows

\[
AY - g(AY, N)N = g(A\xi, Y)\xi. \tag{5.2}
\]

Applying the real structure \( A \) to (5.2), we obtain

\[
Y - g(AY, N)AN = g(A\xi, Y)A\xi. \tag{5.3}
\]

Since \( AN = -\phi A\xi - g(A\xi, \xi)N \), it leads to

\[
Y + g(AY, N)A\xi + g(AY, N)g(A\xi, \xi)N = g(A\xi, Y)A\xi.
\]

From this, we get

\[
\begin{cases}
\text{Tangential Part : } Y + g(AY, N)\phi A\xi = g(A\xi, Y)A\xi \\
\text{Normal Part : } g(AY, N)g(A\xi, \xi) = 0
\end{cases}
\]

Since \( g(A\xi, \xi) \neq 0 \), the normal part gives \( g(AN, Y) = 0 \) for all \( Y \in TM \). This implies that the vector field \( AN \in TQ^n \) can be expressed by \( AN = g(AN, N)N \). It follows that \( N = g(AN, N)AN = (g(AN, N))^2N \), which implies \( g(AN, N) = \pm 1 \). On the other hand, as mentioned in Section 3, \( g(AN, N) = -\cos(2t), t \in [0, \frac{\pi}{2}] \). So, we get \( g(AN, N) = 1 \), when \( t = 0 \). It implies that \( N \) should be \( \mathfrak{A} \)-principal.

Putting these observations together, we get the following:

**Lemma 5.1.** Let \( M \) be a Hopf real hypersurface in the complex quadric \( Q^m \) for \( m \geq 3 \), with Reeb parallel normal Jacobi operator. Then \( M \) has a singular normal vector field, that is, \( N \) is either \( \mathfrak{A} \)-principal or \( \mathfrak{A} \)-isotropic.

Now, we assume that \( M \) has an \( \mathfrak{A} \)-principal normal vector field in \( Q^m \). By virtue of Theorem B given in Introduction, we see that \( M \) is locally congruent to a model space of type \((T_B)\).

But a real hypersurface of type \((T_B)\) in \( Q^m \) does not satisfy the property of Reeb parallel normal Jacobi operator. To see this, let us assume that the normal Jacobi operator \( \hat{R}_N \) of type \((T_B)\) real hypersurface is Reeb parallel. This implies

\[
\alpha \phi BY = 0 \tag{5.4}
\]

from (5.1).

On the other hand, if \( N \) is \( \mathfrak{A} \)-principal, we obtain that \( AY \in TM \) for any \( Y \in TM \). So, (5.4) becomes \( \alpha \phi AY = 0 \) for all \( Y \in TM \). On \((T_B)\), the principal curvature \( \alpha = -\sqrt{2} \cot(\sqrt{2}r) \) is a non-zero constant function for \( r \in (0, \frac{\pi}{\sqrt{2}}) \). So, we consequently have \( \phi AY = 0 \), which implies \( AY = \eta(AY)\xi = -\eta(Y)\xi \), together with \( A\xi = -\xi \). From the fact that \( A^2 = I \), we get

\[
Y = -\eta(Y)A\xi = \eta(Y)\xi
\]

for all \( Y \in T(T_B) \), where \( T(T_B) \) denotes the tangent space of \((T_B)\). This yields \( \dim T(T_B) = 1 \), which gives us a contradiction. In fact, according to the Proposition B, we see that the dimension of \( T(T_B) \) is \( 2m - 1 \). Therefore \( m = 1 \), which contradicts \( m \geq 3 \). Hence it is shown that, if \( M \) has an \( \mathfrak{A} \)-principal normal vector field, then it does not have Reeb parallel normal Jacobi operator.

This together with Lemma 5.1 assures that, if \( M \) has a Reeb parallel normal Jacobi operator, then it has an \( \mathfrak{A} \)-isotropic normal vector field.
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Hyunjin Lee
The Research Institute of Real and Complex Manifolds (RIRCM)
Kyungpook National University
Daegu 41566
Republic of Korea
lhjibis@hanmail.net

Juan de Dios Pérez
Departamento de Geometria y Topologia & IEMATH
Universidad de Granada
Campus de Fuentenueva s/n
Granada 18071
Spain
jdperez@ugr.es

Young Jin Suh
Department of Mathematics & RIRCM
Kyungpook National University
Daegu 41566
Republic of Korea
yjsuh@knu.ac.kr