AVERAGES AND HIGHER MOMENTS FOR THE \( \ell \)-TORSION IN
CLASS GROUPS

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Abstract. We prove upper bounds for the average size of the \( \ell \)-torsion \( \text{Cl}_K[\ell] \) of the
class group of \( K \), as \( K \) runs through certain natural families of number fields and \( \ell \) is a
positive integer. We refine a key argument, used in almost all results of this type, which
links upper bounds for \( \text{Cl}_K[\ell] \) to the existence of many primes splitting completely in
\( K \) that are small compared to the discriminant of \( K \). Our improvements are achieved
through the introduction of a new family of specialised invariants of number fields to
replace the discriminant in this argument, in conjunction with new counting results for
these invariants. This leads to significantly improved upper bounds for the average and
sometimes even higher moments of \( \text{Cl}_K[\ell] \) for many families of number fields
considered in the literature, for example, for the families of all degree-\( d \)-fields for \( d \in \{2, 3, 4, 5\} \) (and
non-\( D_4 \) if \( d = 4 \)). As an application of the case \( d = 2 \) we obtain the best upper bounds
for the number of \( D_p \)-fields of bounded discriminant, for primes \( p > 3 \).

1. Introduction

In this paper, we provide bounds for the average and higher moments of the size of the
\( \ell \)-torsion \( \text{Cl}_K[\ell] = \{[a] \in \text{Cl}_K; [a]^{\ell} = [O_K]\} \) of the ideal class groups of number fields \( K \)
in certain families, for arbitrary \( \ell \in \mathbb{N} = \{1, 2, 3, \ldots\} \). Throughout, we order number fields
\( K \) by the absolute value \( D_K \) of their discriminant. For real-valued maps \( f \) and \( g \) with
common domain we mean by \( f(t) \ll_a g(t) \) that there exists a positive constant \( C = C(a) \),
depending only on \( a \), such that \( |f(t)| \leq C|g(t)| \) for all \( t \) in the domain. Throughout this
article we assume \( X \geq 2 \). To give the reader a quick taste of the results in this paper, here
is our first theorem concerning quadratic fields.

Theorem 1.1. Let \( \varepsilon > 0 \) and \( k \geq 0 \) be real numbers and \( \ell \in \mathbb{N} \). As \( K \) ranges over all
quadratic number fields with \( D_K \leq X \) we have

\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{\ell, k, \varepsilon} X^{\frac{k}{2} + 1 - \min \{1, \frac{k}{1 + \varepsilon}\} + \varepsilon}.
\]

1.1. Background. Let us provide here some context for Theorem 1.1 and our further
results. Denote the degree of the number field \( K \) by \( d \). Landau (see, e.g., [Nar80, Theorem
4.4]) noticed that that the Minkowski bound implies the upper bound

\[
\# \text{Cl}_K \ll_{d, \varepsilon} D_K^{\frac{1}{2} + \varepsilon},
\]

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for arbitrarily small \( \varepsilon > 0 \). This bound is essentially sharp, and provides the “trivial” upper bound for the \( \ell \)-torsion
\[
\# \text{Cl}_K[\ell] \ll_{d,\varepsilon} D_K^{\frac{3}{2} + \varepsilon}.
\]
However, a standard conjecture asserts that
\[
\# \text{Cl}_K[\ell] \ll_{d,\ell,\varepsilon} D_K.
\]
For some references providing motivation and background for this conjecture, we refer to [PTBW17, Conjecture 1.2] and the discussion thereafter. The conjecture is known to hold for \( d = \ell = 2 \) by Gauß’ genus theory. Apart from that the only cases for which improvements over the trivial bound have been established are \( \ell = 3 \) for \( d \leq 4 \) by pioneering work of Pierce, Helfgott, Ellenberg and Venkatesh [Pie05, Pie06, HV06, EV07], and more recently the case \( \ell = 2 \) for arbitrary \( d \) by Bhargava et al. [BST+17].

Assuming the Riemann hypothesis for the Dedekind zeta function of the normal closure of \( K \), Ellenberg and Venkatesh [EV07] proved the bound
\[
\# \text{Cl}_K[\ell] \ll_{d,\ell,\varepsilon} D_K^{\frac{3}{2} - \frac{1}{2(\ell(d) - 1)} + \varepsilon}
\]
for all number fields \( K \). Taking up a key idea of Michel and Soundararajan and generalising it from imaginary quadratic to arbitrary number fields they show in [EV07, Lemma 2.3] that the presence of many small primes splitting completely in \( K \) leads to savings over (1.2.1). Together with the conditional effective version of Chebotarev’s density theorem, this leads directly to the bound (1.4). Small splitting primes were also used in [AD03] to lower bound the exponent of the class group of CM-fields.

Subsequently, several papers took the same approach using [EV07, Lemma 2.3], but tried to establish the existence of enough splitting primes unconditionally, at the cost of averaging or having to exclude a zero-density subset of fields in a given family. Number field counting techniques were used in combination with Erdős’ probabilistic method in [EPW17, FW18], the large sieve in [HBP17], and new effective versions of Chebotarev’s density theorem in [PTBW17, An18].

In this paper, we take a different direction by refining the core argument [EV07, Lemma 2.3] itself, see Proposition 2.1. A first step in this direction was taken by the second author in [Wid18], leading to improvements upon [EPW17] in some cases. Our new technique yields improvements on average in all cases of [EPW17] and [Wid18], as well as some results in [PTBW17, An18]. For example, the case \( k = 1 \) in Theorem 1.1 should be compared to the case \( d = 2 \) of [EPW17, Corollary 1.1.1], which gives an upper bound
\[
\sum_K \# \text{Cl}_K[\ell] \ll_{\ell,\varepsilon} X^{\frac{3}{2} - \frac{1}{2(\ell(d) - 1)} + \varepsilon},
\]
provided \( d \in \{2, 3, 4, 5\} \) and \( \ell \geq \ell(d) \), where \( \ell(2) = \ell(3) = 1, \ell(4) = 8 \) and \( \ell(5) = 25 \).

Note that control over averages is often enough for applications. For example, Theorem 1.1 leads to the best known bounds for the number of \( D_p \)-extensions of \( \mathbb{Q} \) of bounded discriminant, for fixed odd prime \( p \); see Corollary 1.6 later in the introduction. Moreover, having sufficiently good upper bounds for averages of arbitrarily large moments \( k \) would imply (1.3) as shown in [PTBW17, Proposition 8.1]. Here, sufficiently good means with an exponent on \( X \) independent of \( k \), and valid for arbitrarily large \( k \).
To our best knowledge, the only published results concerning higher moments are those of Heath-Brown and Pierce [HBP17] on imaginary quadratic fields. In particular, Theorem 1.1 provides the first non-trivial higher moment estimates over a full class of number fields of fixed degree.

Last but not least we should mention that there are very few but spectacular results for the averages of $\ell$-torsion in degree-$d$-fields that provide not only upper bounds but even asymptotics. The case $(d, \ell) = (2, 3)$ is due to Davenport-Heilbronn [DH71] (see also the recent improvements [BST13, TT13, Hou16]), and $(3, 2)$ due to Bhargava [Bha05]. Regarding 4-torsion in quadratic fields Fouvry and Klüners [FK07] have established the average value for $\# \text{Cl}_K[4]/\# \text{Cl}_K[2]$. Related results were obtained by Klys [Kly16] for 3-torsion in cyclic cubic fields, and by Milovich [Mil17] for the 16-rank in certain quadratic fields.

1.2. Further unconditional results. Let us next consider the other cases of [EPW17], concerning degree-$d$-fields for $d \in \{3, 4, 5\}$ (whose normal closure does not have Galois group $D_4$ in case $d = 4$). In this case, our result is as follows. Define $\delta_0(3) = 2/25$, $\delta_0(4) = 1/48$, and $\delta_0(5) = 1/200$.

**Theorem 1.2.** Suppose $d \in \{3, 4, 5\}$, and $\varepsilon > 0$. As $K$ ranges over number fields of degree $d$ with $D_K \leq X$ (and non-$D_4$ in the case $d = 4$), we have

$$\sum_K \# \text{Cl}_K[\ell] \ll_{\ell, \varepsilon} X^{\frac{2}{d} - \min \{\delta_0(d), \frac{1}{d-1}, \varepsilon\} + \varepsilon}.$$ 

This improves upon Ellenberg, Pierce, and Wood’s result mentioned in (1.5) (for large enough $\ell$), and moreover upon [Wid18, Corollary 1.5]. Our method also works for some families $S$ of number fields of fixed degree and Galois group, but it loses its power if the families are too thin, that is, their counting function satisfies $\# \{K \in S; D_K \leq X\} \ll X^\rho$ for $\rho < 1$ too small compared to the other parameters (see also the discussion after Theorem 1.7 later in the introduction). For cyclic extensions not covered by Theorem 1.1, we are able to improve upon [FW18, PTBW17] in the case $d = 3$ and, moreover, to cover higher moments.

**Theorem 1.3.** Let $\varepsilon > 0$ and $k \geq 0$ be real numbers, and $\ell \in \mathbb{N}$. As $K$ ranges over cubic $A_3$-extensions of $\mathbb{Q}$ with $D_K \leq X$, we have

$$\sum_K \# \text{Cl}_K[\ell]^k \ll_{\ell, k, \varepsilon} X^{\frac{4k}{k+1} - \min \{\frac{k}{4}, \frac{1}{4}, \varepsilon\} + \varepsilon}.$$ 

For comparison, summing up the $k$-th power of the pointwise bound for almost all $A_3$-fields from [PTBW17, Theorem 1.19] would lead to the bound $\sum_K \# \text{Cl}_K[\ell]^k \ll_{\ell, k, \varepsilon} X^{\frac{4k}{4k+1} - \frac{k}{4} + \varepsilon}$.

We can also get improvements in the case of quintic fields whose normal closure has Galois group $D_5$, the dihedral group of order 10. Note that no asymptotics for the counting function of these fields are known, see [1.3] for more details. Moreover, we need to impose the same ramification restrictions as in [PTBW17], since we rely on results from that paper to count small splitting primes. If the rational prime $p$ ramifies tamely in a number field $K$ whose normal closure $\tilde{K}$ has Galois group $G$ then the inertia group $I(\mathfrak{B}) \subset G$ is cyclic.
Theorem 1.5. classified in [An18, Condition 1.3].

For distinct and squarefree such that \( S \) contains the biquadratic field
\[
\mathbb{Q}(\sqrt{a}, \sqrt{b}).
\]

Note that, by [PTBW17, Proposition 2.5], any \( \rho \) in Corollary 1.6 that
\[
\text{a reflection is the set of all reflections. Keeping this in mind we say that the ramification type of a tamely ramified prime } p \text{ is generated by a reflection if each } I(\mathfrak{B}) \text{ is generated by a reflection.}
\]

Theorem 1.4. Let \( \varepsilon > 0 \) and \( k \geq 0 \) be real numbers, and \( \ell \in \mathbb{N} \). Let \( S \) be the family of all quintic \( D_5 \)-extensions of \( \mathbb{Q} \) for which the ramification type of \( p \) is generated by a reflection in \( D_5 \) for every tamely ramified rational prime \( p \). Suppose moreover that \( \rho, c_1 > 0 \) are such that
\[
\#\{K \in S; D_K \leq X\} \leq c_1 X^\rho
\]
holds for all \( X \geq 2 \). Then, as \( K \) ranges over all \( K \in S \) with \( D_K \leq X \), we have
\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{\rho, c_1, \ell, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{12k^2}{372 + 24} + \varepsilon} + X^{\frac{k}{2} + \frac{1}{4} + \varepsilon}.
\]

For comparison, summing up \( k \)-th powers of the pointwise bound (with \( \ll_{\ell, \varepsilon} X^{1/4 + \varepsilon} \) exceptions) from [PTBW17 Theorem 1.19] and (1.2) for the exceptions would yield a bound
\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{\rho, c_1, \ell, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{1}{8} + \varepsilon} + X^{\frac{k}{2} + \frac{1}{4} + \varepsilon}.
\]

Note that, by [PTBW17 Proposition 2.5], any \( \rho \) with (1.6) must satisfy \( \rho \geq 1/2 \). We prove in Corollary [1.6] that \( \rho = 19/28 + \varepsilon \) is a valid choice.

Finally, we can get improvements for certain families of quartic \( D_4 \)-fields studied in very recent work of An [An18]. For distinct and squarefree \( a, b \in \mathbb{Z} \setminus \{0, 1\} \), we denote by \( S_4(a, b) \) the family of quartic number fields whose normal closure has Galois group \( D_4 \) and contains the biquadratic field \( \mathbb{Q}(\sqrt{a}, \sqrt{b}) \). It is shown in [An18] that the normal closure of every \( D_4 \)-field contains a unique biquadratic field, and the pairs \((a, b)\) with \( S_4(a, b) \neq \emptyset \) are classified in [An18, Condition 1.3].

Theorem 1.5. Let \( \varepsilon > 0 \) and \( k \geq 0 \) be real numbers, and \( \ell \in \mathbb{N} \). Let \( a, b \in \mathbb{Z} \setminus \{0, 1\} \) be distinct and squarefree such that \( S_4(a, b) \neq \emptyset \). Suppose moreover that \( \rho, c_1 > 0 \) are such that
\[
\#\{K \in S_4(a, b); D_K \leq X\} \leq c_1 X^\rho
\]
holds for all \( X \geq 2 \). Then, as \( K \) ranges over the fields in \( S_4(a, b) \) with \( D_K \leq X \), we have
\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{a, b, \rho, c_1, \ell, k, \varepsilon} X^{\frac{k}{2} + \rho - \min\{\frac{3k}{136}, \frac{k}{4}\} + \varepsilon}.
\]

One can take the exponent \( \rho = 1 \) by [DO02]. By [An18 Theorem 1.2], any \( \rho \) with (1.8) must satisfy \( \rho \geq 1/2 \), and one might expect \( \rho = 1/2 \) to be the correct order of magnitude. Theorem 1.5 yields an improvement over summing the pointwise bound (with \( \ll_{\ell, \varepsilon} X^{\varepsilon} \) exceptions) from [An18 Theorem 1.1] whenever \( \rho > 1/2 - 1/9 + 1/(3\ell) \).
1.3. Application. We now discuss an application of Theorem 1.1. For a transitive permutation group \(G\) of degree \(d\) and \(X > 0\), let \(N(d, G, X)\) be the number of field extensions \(K/\mathbb{Q}\) of degree \(d\) within a fixed algebraic closure \(\overline{\mathbb{Q}}\) with \(D_K \leq X\) and whose normal closure has Galois group isomorphic to \(G\) as a permutation group. Malle’s conjecture \([\text{Mal02}, \text{Mal04}]\) predicts an asymptotic formula for \(N(d, G, X)\) as \(X \to \infty\). Let \(p\) be an odd prime and \(D_p, D_p(2p)\) the Dihedral group of order \(2p\) and its regular permutation representation. In these cases, Malle’s conjecture predicts the formulas

\[
N(p, D_p, X) \sim c_p X^{\frac{3}{p-1}} \quad \text{and} \quad N(2p, D_p(2p), X) \sim c_{2p} X^{\frac{1}{2}}
\]

with positive constants \(c_p, c_{2p}\) (see [Klii06, Example after Conjecture 1.1]). Currently the best upper bounds for \(p > 3\) are

\[
N(p, D_p, X) \ll_{p, \epsilon} X^{\frac{3}{p-1} - \frac{1}{p(p-1)} + \epsilon} \quad \text{and} \quad N(2p, D_p(2p), X) \ll_{p, \epsilon} X^{\frac{1}{2} + \epsilon},
\]

the first due to Cohen and Thorne [CT17, Theorem 1.1], the second due to Klünners [Klii06, Theorem 2.7]. As an immediate consequence of Klünners’ method and the case \(k = 1\) in Theorem 1.1 we can improve both bounds for all primes \(p > 3\).

Corollary 1.6. Let \(p\) be an odd prime and \(\epsilon > 0\). Then we have

\[
N(p, D_p, X) \ll_{p, \epsilon} X^{\frac{3}{p-1} - \frac{2}{p(p+1)} + \epsilon} \quad \text{and} \quad N(2p, D_p(2p), X) \ll_{p, \epsilon} X^{\frac{1}{2} - \frac{1}{p(p+1)} + \epsilon}.
\]

The special case \(p = 5\) was also considered by Larsen and Rolen [LR12]. They suggest to improve Klünners’ bound \(X^{0.75 + \epsilon}\) [Klii06, Theorem 2.7] by counting integral points on a variety defined by a norm equation. While counting these points seems a difficult matter, their numerical experiments provide evidence that the number of these points is \(\ll X^{0.698}\), which, if true, would provide the same bound for \(N(5, D_5, X)\). The exponent \(0.7 + \epsilon\) of Cohen and Thorne is just slightly above the latter. Our bound is \(X^{0.678-\epsilon}\), and hence is slightly better than the bound suggested by the numerical experiments in [LR12].

1.4. Conditional results. Our techniques can also provide improved average and higher moment bounds for some conditional results.

Theorem 1.7. Let \(\epsilon > 0\), let \(S\) be any family of number fields of degree \(d\), and assume that

(i) the Dedekind zeta function of the normal closure of each field in \(S\) satisfies the Riemann hypothesis,

(ii) the numbers \(\rho, c_1 > 0\) are such that \(\#\{K \in S; D_K \leq X\} \leq c_1 X^\rho\) for all \(X \geq 2\).

Then

\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, \ell, k, \epsilon} X^{\frac{k}{2} + k} + \epsilon.
\]

Theorem 1.7 improves upon the bound

\[
\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, \ell, k, \epsilon} X^{\frac{k}{2} + k} - \frac{k}{2(d-1)} + \epsilon,
\]

which one would get from summing up \(k\)-th powers of the GRH-bound (1.4) from [EV07], as soon as \(\rho > \frac{1}{2} + \frac{1}{d-1}\), thus giving an impression of the density of \(S\) that is required for our method to yield improvements. In [PTBW17], the assumption of GRH was replaced for
certain families of number fields by other assumptions, at the price of introducing certain ramification conditions and allowing a small exceptional set. We can also improve some of these conditional results on average.

**Theorem 1.8.** Let $\varepsilon > 0$ and $k \geq 0$ be real numbers, and $\ell \in \mathbb{N}$. Let $d \geq 3$ and $S$ be the family of all number fields of degree $d$ with squarefree discriminant, whose normal closure has full Galois group $S_d$ over $\mathbb{Q}$. Suppose that

(i) the strong Artin conjecture holds for all irreducible Galois representations over $\mathbb{Q}$ with image $S_d$,

(ii) the numbers $\tau < 1/2 + 1/d$ and $c_2$ are such that for every integer $D$, there are at most $c_2 D^\tau$ fields $K \in S$ with $D_K = D$,

(iii) the numbers $\rho, c_1 > 0$ are such that $\# \{ K \in S; D_K \leq X \} \leq c_1 X^\rho$ for all $X \geq 2$.

Then, as $K$ ranges over all elements of $S$ with $D_K \leq X$, we have

$$\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, c_2, \ell, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{1}{2(d-1)} + \varepsilon} + X^{\frac{k}{2} + \tau + \varepsilon}.$$

The assumptions (i) and (ii) of Theorem 1.8 are the same as in [PTBW17, Theorem 13] for $d \geq 6$. For a precise formulation of the strong Artin conjecture, see [PTBW17, Conjecture L]. For $d \in \{3, 4, 5\}$, our assumptions can be weakened as in [PTBW17]. If $d = 3, 4$, the result is unconditional if one takes $\rho = 1$ (using [DH71] and [Bha05]) and $\tau = 1/3$ or $\tau = 1/2$, respectively (see Theorem 5.3). If $d = 5$, one still needs (i), but one can take $\rho = 1$ and the upper bound for $\tau$ in (ii) can be replaced by 1 (see Theorem 5.3).

For comparison, summing the $k$-th power of the pointwise bound from [PTBW17, Theorem 1.19] and (1.2) for the exceptions would yield

$$\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, c_2, \ell, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{1}{2(d-1)} + \varepsilon} + X^{\frac{k}{2} + \tau + \varepsilon}.$$

Note that Bhargava, Shankar and Wang [BSW16] have shown that $\rho \geq 1/2 + 1/d$, and Bhargava [Bha14] conjectured that (iii) is sharp with $\rho = 1$. On the other hand, it is conjectured that (ii) holds with $\tau = 0$ (see [EV05]). Finally, we can also improve the conditional result of [PTBW17] on $A_d$-extensions for all $d \geq 5$.

**Theorem 1.9.** Let $\varepsilon > 0$ and $k \geq 0$ be real numbers. Let $d \geq 5$ and $S$ be the family of all number fields of degree $d$, whose normal closure has Galois group $A_d$ over $\mathbb{Q}$. Suppose that

(i) the strong Artin conjecture holds for all irreducible Galois representations over $\mathbb{Q}$ with image $A_d$,

(ii) the numbers $\rho, c_1 > 0$ are such that $\# \{ K \in S; D_K \leq X \} \leq c_1 X^\rho$ for all $X \geq 2$.

Then, as $K$ ranges over all fields in $S$ with $D_K \leq X$, we have

$$\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{1}{2(d-1)} + \varepsilon}.$$

For comparison, the pointwise bound (with a few exceptions) from [PTBW17, Theorem 1.19] would lead to the average bound

$$\sum_K \# \text{Cl}_K[\ell]^k \ll_{d, \rho, c_1, k, \varepsilon} X^{\frac{k}{2} + \rho - \frac{1}{2(d-1)} + \varepsilon},$$
which is worse than our bound (in case $\ell > (k-2)/(d-3/2)$) whenever
\[ \rho > \frac{1}{2} - \frac{1}{4(d-1)} + \frac{1}{\ell(d-1)}. \]
Note that Malle’s conjecture predicts the optimal exponent $\rho = 1/2$, in which case our bound provides an improvement as soon as $\ell > 4$.

1.5. Plan of the paper. In \[2\] we introduce invariants $\eta_\ell(K)$ of number fields $K$ and use them to refine the key lemma [EV07, Lemma 2.3] of Ellenberg and Venkatesh. In \[3\] we prove two general results that use the refined key lemma to deduce average and moment bounds for $\ell$-torsion from certain asymptotic counting results. In \[4\] we provide such counting results for fields $K$ of bounded $\eta_\ell(K)$. In \[5\] we recall results from the literature that guarantee the existence of enough small split primes. In \[6\] we deduce all of our theorems, and in \[7\] we prove Corollary \[1.6\].

2. A refined key lemma

Let
\[ H_K(\alpha) = \prod_{v \in M_K} \max\{1, |\alpha|_v\}^{d_v} \]
be the multiplicative Weil height of $\alpha \in K$ relative to $K$. Here $M_K$ denotes the set of places of $K$, and for each place $v$ we choose the unique representative $|\cdot|_v$ that either extends the usual Archimedean absolute value on $\mathbb{Q}$ or a usual $p$-adic absolute value on $\mathbb{Q}$, and $d_v = [K_v : \mathbb{Q}_v]$ denotes the local degree at $v$.

For every prime ideal $p$ of $K$ lying above a rational prime $p$, we write $e(p) = e(p/p)$ for the ramification index and $f(p) = f(p/p)$ for the inertia degree of $p$ over $p$. For each $\ell \in \mathbb{N}$ we introduce a new invariant of number fields $K$,
\[ \eta_\ell(K) = \inf \left\{ H_K(\alpha) : \alpha \in K, \alpha \mathcal{O}_K = (p_1 p_2^{-1})^\ell, \text{ where } p_1 \neq p_2 \text{ are prime ideals of } \mathcal{O}_K \text{ with } e(p_i) = f(p_i) = 1 \text{ for } i = 1, 2 \right\}. \]

We will show in Lemma \[1.1\] that an element $\alpha$ of this special form necessarily generates $K$, and moreover its minimal polynomial has a restricted shape. This will allow us to deduce upper bounds for the number of fields $K$ of bounded $\eta_\ell(K)$ which lead to the improved bounds in our theorems. The following proposition is a refinement of [EV07, Lemma 2.3] and central to all our improvements.

**Proposition 2.1.** Let $K$ be a number field of degree $d$, $\delta < 1/\ell$, and $\varepsilon > 0$. Moreover, suppose that there are $M$ prime ideals $p$ of $\mathcal{O}_K$ with norm $N(p) \leq \eta_\ell(K)^{\delta}$ that satisfy $e(p) = f(p) = 1$. If $M > 0$, we have
\[ \# \text{Cl}_K[\ell] \ll_{d, \delta, \varepsilon} D_K^{1/2+\varepsilon} M^{-1}. \]

**Proof.** We may assume that $\eta_\ell(K) \geq 2$. Set $G := \text{Cl}_K / \text{Cl}_K[\ell]$ so that $\# \text{Cl}_K[\ell] \cdot \#G \cdot R_K = \# \text{Cl}_K R_K \ll_{d, \varepsilon} D_K^{1/2+\varepsilon}$. Hence, we need to show that $\#G \gg_{d, \varepsilon} M/R_K$. Fix a constant $c > 0$ and write $R := \lceil cR_K \rceil$. Our goal is to show that $\#G \geq M/R$, if $c$ was chosen sufficiently large in terms of only $d$ and $\delta$. Since $R_K \gg_d 1$, we may assume that $R \geq 2$. 

Suppose \( \#G < M/R \). Then, by the pigeon hole principle, the classes \([p]\) of at least \( R + 1 \) out of our \( M \) prime ideals \( p \) must lie in the same coset in \( G \). We call these prime ideals \( p_1, \ldots, p_{R+1} \) to obtain \([p_{R+1}] Cl_K[\ell] = [p_i] Cl_K[\ell] \) for all \( 1 \leq i \leq R \), and thus find \( \alpha_i \in K \) with

\[
\alpha_i \mathcal{O}_K = (p_i p_{R+1})^e.
\]

First suppose that \( K \) is imaginary quadratic. We choose distinct \( i \) and \( j \) between 1 and \( R \) and conclude

\[
H_K(\alpha_i/\alpha_j) \leq \max\{N(p_i), N(p_j)\}^e < \eta_\ell(K),
\]

which contradicts the minimality assumption in the definition of \( \eta_\ell(K) \).

Now suppose that \( K \) is not imaginary quadratic. Let \( l : K^* \to \mathbb{R}^{q+1} \) be the classical logarithmic embedding, where \( q + 1 \) is the number of Archimedean places of \( K \). After multiplying \( \alpha_i \) by a unit we can assume that \( l(\alpha_i) = (d_v | \alpha_i|_v)_{v|\infty} \in F + (d_v)_{v|\infty} (-\infty, \infty) \), where \( F \) is a fundamental cell of the unit lattice \( l(\mathcal{O}^*) \subset \mathbb{R}^{q+1} \). We take \( F = [0,1]u_1 + \cdots + [0,1]u_q \) where \( u_1, \ldots, u_q \) is a Minkowski reduced basis of the unit lattice. Write \( l(\alpha_i) = v_i + \gamma_i(d_v)_{v|\infty} \), where \( v_i \in F \) and \( \gamma_i \in (-\infty, \infty) \). We note that the Euclidean length \( |u_i| \gg d \) 1, which follows easily from Northcott’s Theorem (see, e.g., [Wid10] below (8.2)). Since \( F \) comes from a Minkowski reduced basis we can partition \( F \) into at most \( R - 1 \) subcells of diameter \( \ll_d (R_K/R)^{1/q} \leq c^{-1/q} \leq c^{-1/d} \). Again by the pigeon hole principle, we find distinct \( i \) and \( j \) such that \( v_i \) and \( v_j \) lie in the same subcell and hence \( |(v_i - v_j)|_v \leq c^{-1/d} \) for all \( v|\infty \). Without loss of generality, we may assume that \( \gamma_i \leq \gamma_j \). Since \( |\alpha_i/\alpha_j|_v = e^{(1/d_v)(v_i-v_j)+(|\gamma_i-\gamma_j|)} \), we conclude that

\[
|\alpha_i/\alpha_j|_v = e^{O_d(c^{-1/d}+|\gamma_i-\gamma_j|)} \leq e^{O_d(c^{-1/d})} \text{ holds for all } v|\infty.
\]

Since \( \alpha \mathcal{O}_K = (p_i p_j)^e \), this shows that

\[
H_K(\alpha_i/\alpha_j) \leq e^{O_d(c^{-1/d})} N(p_j)^e \leq e^{O_d(c^{-1/d})} \eta_\ell(K)^\delta.
\]

Since \( \ell \delta < 1 \) and \( \eta_\ell(K) \geq 2 \), we can choose \( c \) large enough in terms of \( d, \delta \) to ensure that \( H_K(\alpha_i/\alpha_j) < \eta_\ell(K) \), contradicting the definition of \( \eta_\ell(K) \). Thus, with this choice of \( c \) we get \( \#G \geq M/R \gg_d M/R_K \).

\[\square\]

3. Framework

Let \( d > 1 \) be an integer. We set

\[(3.1) S_{\mathbb{Q},d} = \{K \subset \mathbb{Q} \; ; \; [K : \mathbb{Q}] = d\}\]

for the collection of all number fields of degree \( d \). For a subset \( S \subset S_{\mathbb{Q},d} \) we set

\[
S_X := \{K \in S \; ; \; X \leq D_K < 2X\},
\]

\[
\mathcal{B}_S(X; Y, M) := \{K \in S_X \; ; \; \text{ at most } M \text{ primes } p \leq Y \text{ split completely in } K\},
\]

\[
N_{\eta_\ell}(S, X) := \#\{K \in S \; ; \; \eta_\ell(K) < X\},
\]

\[
N_D(S, X) := \#S_X.
\]
Throughout this section we assume that $\theta, \rho, c_1, c_3 > 0$ are such that for all $X \geq 2$
\begin{align}
N_D(S, X) &\leq c_1 X^\rho, \label{eq:bd1} \\
N_{\eta}(S, X) &\leq c_3 X^\theta. \label{eq:bd2}
\end{align}

We can now formulate our two main propositions. They differ in their assumption on $\#\mathcal{B}_S(X; X^\delta, cX^x / \log X))$. In the first case we have an upper bound that gets worse when $\delta$ gets smaller. This situation happens in the work [EPW17] based on Erdös’ probabilistic method. For $d = 2$ one can establish a good upper bound uniform in $\delta$ by using the large sieve, as shown in [HBPT17]. It is a new innovation of the recent work [PTBW17] that such uniform bounds are also available for a much larger class of families $S$.

**Proposition 3.1.** Suppose $S \subset S_{Q,d}$, $\delta > 0$, and that \eqref{eq:bd1}, \eqref{eq:bd2} hold for $\theta, \rho, c_1, c_3 > 0$. Moreover, suppose for every $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, 1)$ there are positive $c_4(\delta, \varepsilon)$ and $c_5(\delta, \varepsilon)$ such that
\[ \#\mathcal{B}_S(X; X^\delta, c_4(\delta, \varepsilon) X^\delta / \log X)) \leq c_5(\delta, \varepsilon) X^{\rho - \delta + \varepsilon} \]
holds for all $X \geq 2$. Then we have, for all $\varepsilon \in (0, 1)$,
\[ \sum_{K \in S_X} \#\text{Cl}_K[\ell] \ll_{d, \varepsilon, \rho, c_1, c_3, \delta_0, c_4(\cdot), c_5(\cdot)} X^{1 + \frac{\rho - \min\{\delta_0, \frac{\varepsilon}{\theta + 1}\}}{2} + \varepsilon}. \]

**Proof.** Let $\varepsilon \in (0, 1)$. For sake of clarity, we suppress the dependence of implicit constants in our notation and write $\ll$ instead of $\ll_{d, \varepsilon, \rho, c_1, c_3, \delta_0, c_4(\cdot), c_5(\cdot)}$ throughout the proof. We define
\[ \gamma_0 := \frac{\rho \ell}{\ell \theta + 1}. \]
Hence we have
\[ \gamma_0 \theta = \rho - \frac{\gamma_0}{\ell}. \]
First let us assume that $\ell \leq \frac{1}{\theta}(\frac{\rho}{\delta_0} - 1)$, and thus
\[ \gamma_0 \geq \delta_0 \ell. \]
We decompose $S_X$ into the three subsets
\begin{align*}
M_0 &= \{ K \in S_X ; \eta_\ell(K) \leq D_K^{\delta_0 \ell} \}, \\
M_1 &= \{ K \in S_X ; \eta_\ell(K) > D_K^{\delta_0 \ell} \} \setminus \mathcal{B}_S(X; X^{(1-\varepsilon)\delta_0}, cX^{(1-\varepsilon)\delta_0} / \log X), \\
M_1'' &= \{ K \in S_X ; \eta_\ell(K) > D_K^{\delta_0 \ell} \} \cap \mathcal{B}_S(X; X^{(1-\varepsilon)\delta_0}, cX^{(1-\varepsilon)\delta_0} / \log X),
\end{align*}
where $c = c_4((1-\varepsilon)\delta_0, \varepsilon)$ comes from the assumptions of the proposition. Using \eqref{eq:bd2}, we get
\[ \sum_{K \in M_0} \#\text{Cl}_K[\ell] \ll \sum_{K \in M_0} D_K^{1+\varepsilon} \leq \#M_0 \cdot (2X)^{\frac{1}{2}+\varepsilon}. \]
Since $\#M_0 \leq N_{\eta}(S, (2X)^{\delta_0 \ell}) \leq X^{\delta_0 \ell}$ and $\delta_0 \ell \leq \gamma_0$ we conclude
\[ \sum_{K \in M_0} \#\text{Cl}_K[\ell] \ll X^{\frac{1}{2}+\gamma_0+\varepsilon} = X^{\frac{1}{2}+\rho-\delta_0+\varepsilon}. \]
Since by assumption \( \#M''_1 \ll X^{\rho-(1-\varepsilon)\delta_0+\varepsilon} \), we find similarly
\[
\sum_{K \in M''_1} \# \text{Cl}_K[\ell] \ll X^{\frac{1}{2}+\rho-\delta_0+(2+\delta_0)\varepsilon}.
\]

For the sum over \( M'_1 \) we use Proposition 3.2 with the valid choice \( M = cX^{(1-\varepsilon)\delta_0}/\log X \), and then bound \( \#M'_1 \) by (3.2) to conclude that
\[
\sum_{K \in M'_1} \# \text{Cl}_K[\ell] \ll \sum_{K \in M'_1} D_K^{\frac{1}{2}-(1-\varepsilon)\delta_0+2\varepsilon} \leq \#M'_1 \cdot (2X)^{\frac{1}{2}-(1-\varepsilon)\delta_0+2\varepsilon} \ll X^{\frac{1}{2}+\rho-\delta_0+(2+\delta_0)\varepsilon}.
\]

This proves the proposition when \( \ell \leq \frac{1}{\delta_0}(\frac{\rho}{\delta_0} - 1) \). Now let us assume that \( \ell > \frac{1}{\delta_0}(\frac{\rho}{\delta_0} - 1) \), and thus
\[
\gamma_0 < \delta_0 \ell.
\]

We now define \( M_0, M'_1 \) and \( M''_1 \) exactly in the same way but with \( \delta_0 \) replaced by \( \gamma_0/\ell \). Arguing in exactly the same way as in the previous case we get
\[
\sum_{K \in S_X} \# \text{Cl}_K[\ell] \ll X^{\frac{1}{2}+\rho-\frac{\delta_0}{\ell}+(2+\delta_0)\varepsilon} \leq X^{\frac{1}{2}+\rho-\frac{\delta_0}{\ell}+(2+\rho)\varepsilon}.
\]

\( \Box \)

**Proposition 3.2.** Suppose \( S \subset S_{Q,d} \), \( \tau \geq 0 \), and that (3.2), (3.3) hold for \( \theta, \rho, c_1, c_3 > 0 \). Moreover, suppose for every \( \delta > 0 \) and \( \varepsilon \in (0, 1/\ell) \) there are positive \( c_4(\delta, \varepsilon) \) and \( c_5(\delta, \varepsilon) \) such that
\[
\# \mathcal{B}_S(X; X^\delta, c_4(\delta, \varepsilon)X^\delta/\log X) \leq c_5(\delta, \varepsilon)X^{\tau+\varepsilon}
\]
holds for all \( X \geq 2 \). Then we have, for all \( k \geq 0 \) and \( \varepsilon \in (0, 1/\ell) \),
\[
\sum_{K \in S_X} \# \text{Cl}_K[\ell]^k \ll_{d, \rho, \theta, c_1, c_3, c_4(\cdot, \cdot), c_5(\cdot, \cdot), \ell, k, \tau, \varepsilon} X^{\frac{1}{2}+\rho-\frac{\delta_0}{\ell}+\varepsilon} + X^{\frac{1}{2}+\tau+\varepsilon}.
\]

**Proof.** Let \( \varepsilon \in (1, 1/\ell) \). We decompose \( S_X \) into \( N+2 \) subsets \( M_i \), where \( N = N(\varepsilon) \) will be chosen later. Let \( 0 = \gamma_{-1} \leq \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_N \) and set
\[
M_i = \{ K \in S_X : D_K^{\gamma_{i-1}} \leq \eta_\ell(K) < D_K^{\gamma_i} \} \quad (0 \leq i \leq N),
M_{N+1} = \{ K \in S_X : D_K^{\gamma_N} \leq \eta_\ell(K) \}.
\]
Furthermore, for \( 1 \leq i \leq N+1 \) we decompose \( M_i \) into the two sets
\[
M'_i = M_i \setminus \mathcal{B}_S(X; X^{\gamma_{i-1}(1/\ell-\varepsilon)}, c'_iX^{\gamma_{i-1}(1/\ell-\varepsilon)}/\log X),
M''_i = M_i \cap \mathcal{B}_S(X; X^{\gamma_{i-1}(1/\ell-\varepsilon)}, c'_iX^{\gamma_{i-1}(1/\ell-\varepsilon)}/\log X),
\]
where \( c'_i = c_4(\gamma_{i-1}(1/\ell-\varepsilon), \varepsilon) \). Hence, we have partitioned \( S_X \) into the \( 1+2(N+1) \) subsets \( M_0, M'_1, M''_1 \) \((1 \leq i \leq N+1) \). Throughout this proof, we suppress the implicit constants in our notation and write \( \ll \) for \( \ll_{d, \theta, \rho, c_1, c_3, c_4(\cdot, \cdot), c_5(\cdot, \cdot), \ell, k, \tau, \varepsilon, \gamma_0, \ldots, \gamma_N} \). The values of \( \gamma_0, \ldots, \gamma_N \)
are fixed later in the proof depending only on the other parameters. Next we record the estimates
\[
\#M_0 \leq N_{\eta_0}(S, (2X)^{\gamma_0}) \ll X^{\gamma_0 \theta},
\]
\[
\#M_i \leq \#M_i \leq N_{\eta_i}(S, (2X)^{\gamma_i}) \ll X^{\gamma_i \theta} \quad (1 \leq i \leq N),
\]
\[
\#M'_{N+1} \leq \#M_{N+1} \leq N_D(S, X) \ll X^\rho,
\]
\[
\#M''_i \ll X^{\tau+\varepsilon} \quad (1 \leq i \leq N + 1).
\]
We use (1.2) to estimate the sums over \(M_0\) and \(M''_i\) (1 \(\leq i \leq N + 1),
\[
\sum_{K \in M_0} \# \text{Cl}_K[\ell]^k \ll \sum_{K \in M_0} D_K^{(\frac{1}{2} + \varepsilon)k} \leq \#M_0 \cdot (2X)^{\frac{1}{2} + k\varepsilon} \ll X^{\frac{1}{2} + \gamma_0 \theta + k\varepsilon},
\]
\[
\sum_{K \in M''_i} \# \text{Cl}_K[\ell]^k \ll \sum_{K \in M''_i} D_K^{(\frac{1}{2} + \varepsilon)k} \leq \#M''_i \cdot (2X)^{\frac{1}{2} + k\varepsilon} \ll X^{\frac{1}{2} + \tau + (k+1)\varepsilon}.
\]
From Proposition 2.1 with the eligible choice \(M = \ell X^{\gamma_i-1(1/\ell - \varepsilon)}/\log X\), we conclude for 1 \(\leq i \leq N\) that
\[
\sum_{K \in M'_i} \# \text{Cl}_K[\ell]^k \ll \sum_{K \in M'_i} D_K^{\left(\frac{1}{2} - \gamma_i - \frac{1}{\ell} - 2\varepsilon\right)k} \ll X^{\frac{1}{2} - \frac{1}{\ell} + \gamma_0 + k(2 + 2\gamma)\varepsilon}
\]
and similarly
\[
\sum_{K \in M'_{N+1}} \# \text{Cl}_K[\ell]^k \ll X^{\frac{1}{2} + \rho - \frac{\gamma_0 k}{\ell} + k(2 + 2\gamma)\varepsilon}.
\]
For 0 \(\leq i \leq N\), we define \(Q_i = \sum_{r=0}^i q^r\), where \(q = \frac{k}{\ell \theta}\). With these quantities in place, we proceed to choose our \(\gamma_i\) as follows,
\[
\gamma_0 = \gamma_0(N) = \frac{\rho \ell}{\ell \theta + kQ_N} \quad \text{and} \quad \gamma_i = \gamma_0 Q_i \leq \frac{\rho \ell}{k} \quad (1 \leq i \leq N).
\]
Then a quick computation shows that
\[
\frac{k}{2} + \gamma_0 \theta = \frac{k}{2} - \frac{\gamma_i - k}{\ell} + \gamma_i \theta = \frac{k}{2} + \rho - \frac{\gamma_N k}{\ell},
\]
which allows us to estimate
\[
\sum_{K \in S_X} \# \text{Cl}_K[\ell]^k \ll X^{\frac{1}{2} + \gamma_0 \theta + (2k + \rho \ell)\varepsilon} + X^{\frac{1}{2} + \tau + (k+1)\varepsilon}.
\]
The only task left is to choose \(N = N(\varepsilon)\). We observe that
\[
\tilde{\gamma}_0 := \lim_{N \to \infty} \gamma_0(N) = \begin{cases} \frac{\rho}{\ell} - \frac{\rho k}{\ell \theta^2} & \text{if } q < 1, \\ 0 & \text{if } q \geq 1. \end{cases}
\]
Hence, choosing \(N = N(\varepsilon)\) big enough to ensure \(\gamma_0 \theta \leq \tilde{\gamma}_0 \theta + \varepsilon\), we conclude that
\[
\sum_{K \in S_X} \# \text{Cl}_K[\ell]^k \ll X^{\frac{1}{2} + \tilde{\gamma}_0 \theta + (1 + 2k + \rho \ell)\varepsilon} + X^{\frac{1}{2} + \tau + (k+1)\varepsilon},
\]
which proves the proposition. \(\square\)
4. Counting fields of bounded $\eta_\ell(K)$

For $\alpha \in \overline{\mathbb{Q}}$ we write $D_\alpha \in \mathbb{Z}[x]$ for the minimal polynomial of $\alpha$ over $\mathbb{Z}$, i.e., the irreducible polynomial with positive leading coefficient that satisfies $D_\alpha(\alpha) = 0$. Our estimates for $N_{\eta_\ell}(S, X)$ hinge upon the following observation.

**Lemma 4.1.** Let $\alpha \in K$ be such that $\alpha \mathcal{O}_K = (p_1p_2^{-1})^\ell$, with distinct prime ideals $p_1, p_2$ of $\mathcal{O}_K$ that satisfy $e(p_i) = f(p_i) = 1$ for $i = 1, 2$. Then $K = \mathbb{Q}(\alpha)$ and the minimal polynomial $D_\alpha$ has the form

\begin{equation}
D_\alpha = p^e x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + q^f,
\end{equation}

where $a_1, \ldots, a_{d-1} \in \mathbb{Z}$ and $p, q$ are the primes below $p_2$ and $p_1$, respectively.

**Proof.** First, suppose $\mathbb{Q}(\alpha) = F \subsetneq K$. Let $q_1$ be the prime ideal of $\mathcal{O}_F$ below $p_1$. Then $e(p_1/q_1) = f(p_1/q_1) = 1$. Hence, as $[K : F] > 1$, there must be another prime ideal $p'_1$ of $\mathcal{O}_K$ above $q_1$. For the corresponding discrete valuations, we get $v_{p'_1}(\alpha) = e(p'_1/q_1)v_{q_1}(\alpha) = e(p'_1/q_1)v_{p_1}(\alpha) = e(p'_1/q_1)\ell > 0$. But there is no other prime ideal of $\mathcal{O}_K$ at which $\alpha$ has positive valuation. Hence, $\mathbb{Q}(\alpha) = K$. The second assertion follows immediately from the well-known formula

\[ a_0 = \prod_{v \mid \infty} \max\{1, |\alpha|_v\}^{d_v}, \]

where $a_0$ is the leading coefficient of $D_\alpha$ and the product runs over all non-Archimedean places of $\mathbb{Q}(\alpha)$. The latter formula in turn is essentially a consequence of Gauß' Lemma applied to $D_\alpha$ and each non-Archimedean place of the splitting field of $D_\alpha$. $\square$

**Lemma 4.2.** Suppose $S \subset S_{Q, d}$, and $\theta = d - 1 + 2/\ell$. Then

\[ N_{\eta_\ell}(S, X) \ll_d X^\theta. \]

**Proof.** Let $P_S$ be the set of all $\alpha \in \overline{\mathbb{Q}}$ such that $\mathbb{Q}(\alpha) \in S$ and $\alpha \mathcal{O}_{\mathbb{Q}(\alpha)} = (p_1p_2^{-1})^\ell$, for prime ideals $p_1 \neq p_2$ of $\mathcal{O}_{\mathbb{Q}(\alpha)}$ with $e(p_i) = f(p_i) = 1$ for $i = 1, 2$. Moreover, let

\[ N_H(P_S, X) := \#\{\alpha \in P_S ; H_{\mathbb{Q}(\alpha)}(\alpha) \leq X\}. \]

Using Lemma 4.1, we observe that the image of the map $\alpha \to \mathbb{Q}(\alpha)$ with domain

\[ \{\alpha \in P_S ; H_{\mathbb{Q}(\alpha)}(\alpha) \leq X\} \]

covers the set

\[ \{K \in S ; \eta_\ell(K) \leq X\}. \]

Hence, we get

\[ N_{\eta_\ell}(S, X) \leq N_H(P_S, X). \]

Now if $\alpha \in P_S$ then, as noted in (4.1), the first and last coefficient of its minimal polynomial $D_\alpha$ are, up to sign, $\ell$-th prime powers. For $\alpha$ to be counted in $N_H(P_S, X)$, we also require $H_{\mathbb{Q}(\alpha)}(\alpha) \leq X$. Now the maximum norm of the coefficient vector of $D_\alpha$ is bounded from above by $2^d H_{\mathbb{Q}(\alpha)}(\alpha)$, and hence by $2^d X$. Thus, we have at most \(\ll_d X^{d-1+2/\ell}\) possibilities for these minimal polynomials and thus for $\alpha$. $\square$
The following proofs were inspired by [Die12] and use (sometimes slightly refined or modified) results and techniques from that paper. Hence, we keep our notation similar to that of [Die12]. In particular, we will write \( n \) instead of \( d \) for the degree of certain polynomials. For any field \( K \) of characteristic 0 and \( n \in \mathbb{N} \), we consider polynomials

\[
f = x^n + a_1x^{n-1} + \cdots + a_n \in K[x]
\]

with distinct roots \( \alpha_1, \ldots, \alpha_n \) in an algebraic closure of \( K \). Let \( G \subset S_n \) be a subgroup, then the Galois resolvent from [Die12, Lemma 5] is defined as

\[
(4.2) \quad \phi(z; a_1, \ldots, a_n) = \prod_{\sigma \in S_n/G} \left( z - \sum_{\tau \in G} \alpha_{\sigma(\tau(1))}^2 \alpha_{\sigma(\tau(2))} \cdots \alpha_{\sigma(\tau(n))}^n \right).
\]

It is a polynomial in \( z, a_1, \ldots, a_n \) with integer coefficients that do not depend on \( K \). It is monic in \( z \) of degree \( \#(S_n/G) \). It has a root \( z \in K \) whenever the Galois group of \( f \), as a subgroup of \( S_n \) acting on \( a_1, \ldots, a_n \), is contained in \( G \). In case \( K = \mathbb{Q} \) and \( a_1, \ldots, a_n \in \mathbb{Z} \), this root must clearly lie in \( \mathbb{Z} \). Moreover, we denote by \( \Delta_\phi(a_1, \ldots, a_n) \in K \) the discriminant of \( \phi(z; a_1, \ldots, a_n) \in K[z] \). Again, this discriminant is a polynomial in \( a_1, \ldots, a_n \) with integer coefficients independent of \( K \).

**Lemma 4.3.** Fix \( a_n \in \mathbb{Q}, a_n \neq 0 \). Then \( \Delta_\phi(a_1, \ldots, a_{n-1}, a_n) \) is not identically zero as a polynomial in \( a_1, \ldots, a_{n-1} \).

**Proof.** This is a refinement of [Die12, Lemma 7]. Fix \( a_n \neq 0 \). Then it is enough to find \( a_1, \ldots, a_{n-1} \in \mathbb{C} \) such that \( \Delta_\phi(a_1, \ldots, a_n) \neq 0 \). For any choice of \( a_1, \ldots, a_{n-1} \), it is clear from (4.2) that the roots of \( \phi(z; a_1, \ldots, a_n) \) are the complex numbers

\[
(4.3) \quad \sum_{\tau \in G} \alpha_{\sigma(\tau(1))}^2 \alpha_{\sigma(\tau(2))} \cdots \alpha_{\sigma(\tau(n))}^n,
\]

where \( \sigma \) ranges over a set of representatives for the cosets in \( S_n/G \). All \( \#(S_n/G) \) of these expressions are distinct homogeneous polynomials of degree \( n(n+1)/2 \) in \( \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \). Hence, there is a non-empty Zariski-open subset of points \( (\alpha_1 : \cdots : \alpha_n) \in \mathbb{P}^{n-1} \) for which all the expressions in (4.3) are distinct. In particular, we find such \( (\alpha_1 : \cdots : \alpha_n) \) whose homogeneous coordinates \( \alpha_i \in \mathbb{C} \) are all distinct and non-zero. Picking a correctly scaled representative of such a point, we get \( a_1, \ldots, a_n \in \mathbb{C} \) that satisfy all of the previous conditions and moreover that \( (-1)^n a_1 \cdots a_n = a_n \). Let \( a_1, \ldots, a_{n-1} \in \mathbb{C} \) be the other coefficients of the polynomial \( \prod_{i=1}^n (x - \alpha_i) \). Then, by our choice of \( a_1, \ldots, a_n \), all zeros of \( \phi(z; a_1, \ldots, a_n) \) are distinct, and hence its discriminant satisfies \( \Delta_\phi(a_1, \ldots, a_n) \neq 0 \). \( \Box \)

**Lemma 4.4.** Let \( n \geq 3 \) and \( a_2, \ldots, a_{n-2}, a_n \in \mathbb{Z} \) such that \( a_n \neq 0 \). Then the polynomial

\[
x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + tx + a_n \in \mathbb{Q}(t)[x]
\]

has, for all but \( \ll_n 1 \) values of \( a_1 \in \mathbb{Z} \), the full symmetric group \( S_n \) as Galois group acting on its roots in an algebraic closure of the rational function field \( \mathbb{Q}(t) \).

**Proof.** This is similar to [Die12, Lemma 2]. By [Her70 Satz 1], the Galois group is \( S_n \) for all but finitely many values of \( a_1 \in \mathbb{Z} \). As described in [Die12, Lemma 2] and the introduction of [Her72], the proof of [Her70 Satz 1] provides the upper bound \( n^2 \) for the number of excluded values of \( a_1 \). \( \Box \)
Lemma 4.5. Let \( n \geq 2 \) and \( a_1, \ldots, a_{n-2}, a_n \in \mathbb{Z} \) such that the polynomial
\[
x^n + a_1 x^{n-1} + \cdots + a_{n-2} x^2 + tx + a_n \in \mathbb{Q}(t)[x]
\]
has Galois group \( S_n \) over the rational function field \( \mathbb{Q}(t) \). Moreover, suppose that
\[
\Delta_\phi(a_1, \ldots, a_{n-2}, t, a_n) \neq 0 \text{ in } \mathbb{Q}(t).
\]
Then the polynomial \( \phi(z; t) = \phi(z; a_1, \ldots, a_{n-2}, t, a_n) \in \mathbb{Q}[z, t] \) is irreducible over \( \mathbb{Q} \).

Proof. Note that (4.4) states that the roots of \( \phi(z; t) \) in an algebraic closure of \( \mathbb{Q}(t) \) are all distinct. Hence, we are precisely in the situation of \[\text{[Die12, Lemma 6]}. \]
except that we use the variable \( t \) for the linear coefficient, whereas Dietmann uses \( t \) for the constant coefficient. The proof of \[\text{[Die12, Lemma 6]}. \] is agnostic of this difference and works verbatim in our case. \( \square \)

The following result is \[\text{[Die12, Lemma 8]}. \] which follows from \[\text{[BHB05, Theorem 1]}. \]

Lemma 4.6. Let \( F \in \mathbb{Z}[x_1, x_2] \) be of degree \( d \) and irreducible over \( \mathbb{Q} \). Let \( P_1, P_2 \geq 1 \), and
\[
T = \max \{ P_1^{e_1} P_2^{e_2} \},
\]
where \((e_1, e_2)\) runs through all pairs for which the monomial \( x_1^{e_1} x_2^{e_2} \) appears in \( F \) with non-zero coefficient. Then, for \( \varepsilon > 0 \),
\[
\# \{ x \in \mathbb{Z}^2 ; F(x) = 0 \text{ and } |x_i| \leq P_i \text{ for } i = 1, 2 \} \ll_{d, \varepsilon} \max \{ P_1, P_2 \}^\varepsilon \exp \left( \frac{\log P_1 \log P_2}{\log T} \right).
\]

Note that the implicit constant depends only on the degree, but not on the values of the coefficients of \( F \). This is crucial for our application.

Proposition 4.7. Let \( n \geq 2 \), \( G \) a transitive subgroup of \( S_n \) and \( \ell \in \mathbb{N} \). For \( B \geq 2 \), let \( N_{n,G}(B) \) be the number of polynomials \( f = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \) such that
\[
(1) \ a_0, \ldots, a_n \in \mathbb{Z} \cap [-B, B],
(2) \ a_0, a_n \text{ are } \ell\text{-th powers in } \mathbb{Z} \setminus \{0\},
(3) \ f \text{ is irreducible over } \mathbb{Q},
(4) \text{ the Galois group of } f \text{ acts on the roots of } f \text{ (enumerated in a fixed order) as } G.
\]

Then, for \( \varepsilon > 0 \), we have the upper bound
\[
N_{n,G}(B) \ll_{n, \varepsilon} B^{n-2+2/(t+\#(S_n/G)-1)}+\varepsilon.
\]

Proof. The result follows from Lemma \[\text{[12]}. \] in case \( n = 2 \), so we assume from now on that \( n \geq 3 \). Conditions (3) and (4) are invariant under replacing \( f \) by
\[
a_0^{n-1} f(x/a_0) = x^n + a_1 x^{n-1} + \cdots + a_0^{n-3} a_{n-2} x^2 + a_0^{n-2} a_{n-1} x + a_0^{n-1} a_n,
\]
so we have to bound the number of \( a_0, \ldots, a_n \) subject to (1) and (2), for which the polynomial in (4.5) satisfies (3) and (4). Lemma \[\text{[12]}. \] shows that, for every choice of \( a_0, a_2, \ldots, a_n \), there are \( \ll_n 1 \) choices of \( a_1 \) for which the polynomial
\[
g(x; t) = x^n + a_1 x^{n-1} + \cdots + a_0^{n-3} a_{n-2} x^2 + tx + a_0^{n-1} a_n \in \mathbb{Q}(t)[x]
\]
does not have full Galois group $S_n$ over the rational function field $\mathbb{Q}(t)$. The total number of $a_0, \ldots, a_n$ for which this holds is thus $\ll_n B^{n-2+2/\ell}$. In view of the desired bound (4.5), we may thus restrict our attention to those $a_0, \ldots, a_n$ for which

$$g(x; t) \text{ has full Galois group } S_n \text{ over } \mathbb{Q}(t).$$

For these polynomials, we consider the corresponding Galois resolvents

$$\phi(z; t) = \phi(z; a_1, \ldots, a_0 a_{n-2}, t, a_0^{-1} a_n) \in \mathbb{Z}[z],$$

defined in (4.2), and their discriminants $\Delta_{\phi}(t) = \Delta_{\phi}(a_1, \ldots, a_0 a_{n-2}, t, a_0^{-1} a_n) \in \mathbb{Z}[t]$.

Lemma 4.3 shows that, for any fixed permitted choice of $a_0, a_n$, the discriminant $\Delta_{\phi}(t)$ does not vanish identically as a polynomial in $a_1, \ldots, a_{n-2}, t$. Hence, there are at most $\ll_n B^{n-3}$ choices of $a_1, \ldots, a_{n-2}$ with (1), for which $\Delta_{\phi}(t) = 0$ in $\mathbb{Q}(t)$. Summing this over all possible choices of $a_0, a_{n-1}, a_n$ with (1) and (2), we obtain a contribution $\ll_n B^{n-2+2/\ell}$ in total, which is negligible when compared to (4.5). Hence, we may assume from now on that $\Delta_{\phi}(t) \neq 0$ for all our tuples $a_0, \ldots, a_n$ under consideration. In this case, together with our previous assumption (4.7), we see from Lemma 4.5 that $\phi(z; t)$ is irreducible over $\mathbb{Q}$ for all choices of $a_0, \ldots, a_{n-2}, a_n$. Fixing such a choice, suppose that the polynomial $g(x; a_0^{a_{n-2}} a_{n-1})$ from (4.6) satisfies (3) and (4) for some $a_{n-1}$ subject to (1).

Then all complex roots of $g(x; a_0^{a_{n-2}} a_{n-1})$ are distinct and moreover the Galois resolvent $\phi(z; a_0 a_{n-2} a_{n-1})$ has a root $z \in \mathbb{Z}$. Since the roots of a complex polynomial are bounded polynomially in terms of its coefficients (see, e.g., [Die12, Lemma 1]), this root satisfies $|z| \leq B^\alpha$, for some $\alpha > 0$ that depends at most on $n$. Since the polynomial $\phi(z; t)$, and thus also $\phi(z; a_0 a_{n-2} t)$, is irreducible over $\mathbb{Q}$, we can apply Lemma 4.6 to bound the number of $(z, a_{n-1}) \in \mathbb{Z}^2$ with $|z| \leq P_1 := B^\alpha$ and $|a_{n-1}| \leq P_2 := B$ for which $\phi(z; a_0^{-2} a_{n-1}) = 0$. Since the monomial $z^{\#(S_n/G)}$ appears in $\phi(z; t)$, we get $T \geq B^{\alpha \#(S_n/G)}$, and thus the number of such pairs $(z, a_{n-1})$ is

$$\ll_{n, \varepsilon} B^\varepsilon \exp\left(\frac{\alpha (\log B)^2}{\alpha \#(S_n/G) \log B}\right) = B^{\#(S_n/G)-1+\varepsilon}.$$ 

Summing this over all viable choices of $a_0, \ldots, a_{n-2}, a_n$ yields the bound (4.5). \hfill \Box

**Corollary 4.8.** Suppose $S \subset S_{Q,d}$ consists of all $A_d$-extensions and $\theta > d - 3/2 + 2/\ell$. Then

$$N_{n_{\ell}}(S, X) \ll_{d, \theta} X^\theta.$$ 

**Proof.** The set $P_S$ from the proof of Lemma 4.2 now has the additional property that $\mathbb{Q}(\alpha)$ is an $A_d$-extension of $\mathbb{Q}$, and hence the minimal polynomial $D_\alpha$ is counted by $N_{d,A_d}(2^d X)$. With Proposition 4.7 we see that

$$N_{n_{\ell}}(S, X) \ll d N_{d,A_d}(2^d X) \ll_{d, \theta} X^\theta.$$ 

\hfill \Box

**Corollary 4.9.** Suppose $S \subset S_{Q,5}$ consists of all $D_5$-extensions and $\theta > 3 + 1/12 + 2/\ell$. Then

$$N_{n_{\ell}}(S, X) \ll_{\theta} X^\theta.$$ 

**Proof.** The proof is analogous to Corollary 4.8. Note that $\#(S_5/D_5) = 12$. \hfill \Box
Corollary 4.10. Suppose $S \subset S_{\mathbb{Q},d}$ consists of all $D_4$-extensions and $\theta > 2 + 1/3 + 2/\ell$. Then

$$N_{\mathfrak{m}}(S, X) \ll_{\theta} X^\theta.$$

Proof. Again, the proof is analogous to Corollary 4.8. Note that $\#(S_d/D_d) = 3$. \hfill $\Box$

5. Bounding the number of bad fields

Recall that $d > 1$ is an integer, $S_{\mathbb{Q},d} = \{K \subset \overline{\mathbb{Q}} ; [K : \mathbb{Q}] = d\}$, and for $S \subset S_{\mathbb{Q},d}$ we defined $\mathcal{B}_S(X; Y; M)$ as the set

$$\{K \in S ; X \leq D_K < 2X, \text{ at most } M \text{ primes } p \leq Y \text{ split completely in } K\}.$$

Lemma 5.1. Let $d \geq 2$, and let $S \subset S_{\mathbb{Q},d}$ be a family of degree-$d$-fields. Suppose that the Riemann hypothesis holds for the Dedekind zeta function of the normal closure of each field in $S$. Then for every $\delta > 0$ there exists $c = c(d, \delta) > 0$ such that

$$\#\mathcal{B}_S(X; X^\delta, cX^\delta / \log X) \ll_{d, \delta} 1.$$

Proof. This is an immediate consequence of the conditional effective version of Chebotarev’s density theorem due to Lagarias and Odlyzko [LO77]. \hfill $\square$

Theorem 5.2. ([EPW17, Theorem 2.1]) Let $d \in \{3, 4, 5\}$, let $S = S_{\mathbb{Q},d}$ if $d \neq 4$ and $S = S_{\mathbb{Q},4}^*$ the family of all quartic non-$D_4$ fields, if $d = 4$, and let $\epsilon > 0$. Recall the definition of $\delta_0(d)$ (just before Theorem 1.2), and put

$$\delta_0 = \delta_0(d).$$

Then for every $0 < \delta \leq \delta_0$ there exists $c = c(\delta) > 0$ such that

$$\#\mathcal{B}_S(X; X^\delta, cX^\delta / \log X) \ll_{\delta, \epsilon} X^{1-\delta+\epsilon}.$$

Consider families $S = S(G, \mathcal{I}) \subset S_{\mathbb{Q},d}$ of fields $K$ whose normal closure $\overline{K}$ has Galois group $G$, and such that for each rational prime $p$ that is tamely ramified in $K$, its ramification is of type $\mathcal{I}$, where $\mathcal{I}$ specifies one or more conjugacy classes in $G$. By this we mean the inertia group $I(\mathfrak{B}) \subset G$ of any prime ideal $\mathfrak{B} \subset \mathcal{O}_K$ above $p$ (which is cyclic if $p$ is tamely ramified in $K$) is generated by an element in the conjugacy class (or classes) specified by $\mathcal{I}$ (see [PTBW17, §2.3]). The following result collects some special cases of [PTBW17, Corollary 1.17.1].

Theorem 5.3 (Pierce, Turnage-Butterbaugh, Wood). Let $\epsilon > 0$, let $S = S(G, \mathcal{I}) \subset S_{\mathbb{Q},d}$ be from one of the following five families, and let $\tau = \tau_S$ as below. Then for every $\delta > 0$ there exists $c = c(S, \delta) > 0$ such that

$$\#\mathcal{B}_S(X; X^\delta, cX^\delta / \log X) \ll_{S, \delta, c, \tau, \epsilon} X^{\tau+\epsilon}.$$

1. $G$ is a cyclic group of order $d \geq 2$ with $\mathcal{I}$ comprised of all generators of $G$ (equivalently, every rational prime that is tamely ramified in $K$ is totally ramified), and $\tau = 0$.
2. $d$ is an odd prime, and $G = D_d$ the Dihedral group of symmetries of a regular $d$-gon, with $\mathcal{I}$ being the conjugacy class of reflections and $\tau = 1/(p - 1)$. 


3. \( d \geq 5 \), \( G = A_d \) and \( \mathcal{I} = G \) (so no restriction on inertia type), and \( \tau = 0 \). Moreover, assume that the strong Artin conjecture holds for all irreducible Galois representations over \( \mathbb{Q} \) with image \( A_d \).

4. \( d \in \{3, 4\} \), \( G = S_d \), with \( \mathcal{I} \) being the conjugacy class of transpositions, and \( \tau = 1/3 \) if \( d = 3 \) and \( \tau = 1/2 \) if \( d = 4 \).

5. \( d \geq 5 \), \( G = S_d \), with \( \mathcal{I} \) being the conjugacy class of transpositions, and the following two conditions hold:
   (i) the strong Artin conjecture holds for all irreducible Galois representations over \( \mathbb{Q} \) with image \( S_d \),
   (ii) \( \tau \) and \( c_2 \) are numbers such that \( \tau < 1 \) if \( d = 5 \) and \( \tau < 1/2 + 1/d \) if \( d \geq 6 \), and for every fixed integer \( D \) there are at most \( c_2 D^\tau \) fields \( K \in S \) with \( D_K = D \).

For the families \( S_4(a, b) \) in Theorem 1.5, we have the following bounds, which follow from \[\text{An18}] \) Theorem 1.6 and Proposition 6.1.

**Theorem 5.4** (An). Let \( \varepsilon > 0 \), and let \( a, b \in \mathbb{Z} \setminus \{0, 1\} \) be distinct squarefree numbers. Then for every \( \delta > 0 \) there exists \( c = c(a, b, \delta) > 0 \) such that
\[
\# \mathcal{B}_{S_4(a, b)}(X; X^\delta, cX^\delta / \log X) \ll a, b, \delta, \varepsilon X^{\varepsilon}.
\]

6. **Proofs of theorems**

Each of our Theorems follows immediately from one of the Propositions [3.1] or [3.2] with suitable parameters, combined with a simple application of dyadic summation.

6.1. **Proof of Theorem 1.1.** Apply Proposition 3.2 with \( \theta = 1 + 2/\ell \) (by Lemma 4.2), \( \rho = 1 \), and \( \tau = 0 \) (by Theorem 5.3).

6.2. **Proof of Theorem 1.2.** Apply Proposition 3.1 with \( \theta = d - 1 + 2/\ell \) (by Lemma 4.2), \( \rho = 1 \) (by \[\text{DH71}] \ [Bha05] \ [Bha10] \), and \( \delta_0 = \delta_0(d) \) (by Theorem 5.2).

6.3. **Proof of Theorem 1.3.** For sufficiently small \( \varepsilon' > 0 \), we apply Proposition 3.2 with \( \theta = 3/2 + 2/\ell + \varepsilon' \) (by Corollary 4.8), \( \rho = 1/2 \) (by \[\text{Wri89}] \), and \( \tau = 0 \) (by Theorem 5.3).

6.4. **Proof of Theorem 1.4.** For sufficiently small \( \varepsilon' > 0 \), we apply Proposition 3.2 with \( \theta = 3 + 1/12 + 2/\ell + \varepsilon' \) (by Corollary 4.9) and \( \tau = 1/4 \) (by Theorem 5.3).

6.5. **Proof of Theorem 1.5.** For sufficiently small \( \varepsilon' > 0 \), we apply Proposition 3.2 with \( \theta = 2 + 1/3 + 2/\ell + \varepsilon' \) (by Corollary 4.10) and \( \tau = 0 \) (by Theorem 5.4).

6.6. **Proof of Theorem 1.6.** Apply Proposition 3.2 with \( \theta = d - 1 + 2/\ell \) (by Lemma 4.2) and \( \tau = 0 \) (by Lemma 5.1).

6.7. **Proof of Theorem 1.8.** First we note (cf. \[\text{PTBW17}] \ Lemma 3.1]) that for each \( S_d \)-extension of degree \( d \) with squarefree discriminant, the ramification type of each ramified prime \( p \) that is tamely ramified is the conjugacy class of transpositions. Now apply Proposition 3.2 with \( \theta = d - 1 + 2/\ell \) (by Lemma 4.2) and \( \tau \) as in the statement of the theorem (by Theorem 5.3).

6.8. **Proof of Theorem 1.9.** For sufficiently small \( \varepsilon' > 0 \), we apply Proposition 3.2 with \( \theta = d - 3/2 + 2/\ell + \varepsilon' \) (by Corollary 4.8) and \( \tau = 0 \) (by Theorem 5.3).
7. Upper bounds for Dihedral extensions

The aim of this section is to prove Corollary 1.6. In the proof of [Klu06, Theorem 2.5], Klüners has shown the estimates

\[ N(p, D^p, X) \leq \sum_{D^p b^{p-1} \leq X} \frac{p^{\omega(b)+r_K} - 1}{p - 1}, \]

\[ N(2p, D^p(2p), X) \leq \sum_{D^p b^{2(p-1)} \leq X} \frac{p^{\omega(b)+r_K} - 1}{p - 1}, \]

where both sums are taken over positive integers \( b \) and quadratic fields \( K \) with \( D_K \) in the indicated range, \( \omega(b) \) denotes the number of distinct prime divisors of \( b \), and \( r_K \) is the \( p \)-rank of \( \text{Cl}_K \), so that \( p^{r_K} = \#\text{Cl}_K[p] \). For the first sum we find

\[ N(p, D^p, X) \leq \sum_{D^p b^{p-1} \leq X} \frac{p^{\omega(b)+r_K} - 1}{p - 1} \leq \sum_{b^{p-1} \leq X} \frac{p^{\omega(b)} - 1}{b^2} \#\text{Cl}_K[p]. \]

Plugging in the bound from Theorem 1.1 in case \( k = 1 \) proves the claim for \( N(p, D^p, X) \). The second sum is handled similarly.

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