On the synectic metric in the tangent bundle of a Riemannian manifold

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Abstract

The purpose of this paper is to investigate applications the covariant derivatives of the covector fields and killing vector fields with respect to the synectic lift $Sg = Cg + V a$ in a the Riemannian manifold to its tangent bundle $T(M_n)$, where $Cg$-complete lift of the Riemannian metric, $V a$-vertical lift of the symmetric tensor field of type $(0, 2)$ in $M_n$.

Keywords: Tensor bundle; Metric connection; Covector field; Levi-Civita connections; Killing vector field

1. Introduction

Suppose that there is given the following Riemannian metric

$$Sg_{CB} dx^C dx^B = a_{ji} dx^j dx^i + 2 g_{ji} dx^j \delta y^i$$

in tangent bundle in $T(M_n)$ over a Riemannian manifold $M_n$ with metric $g$, where $a_{ji}$ are components of a symmetric tensor field of type $(0, 2)$ in $M_n$ and $\delta y^h = dy^h + \Gamma^h_i dx^i$, $\Gamma^h_i = y^i \Gamma^{h}_{ij}$ with respect to the induced coordinates $(x^h, y^h)$ in $\pi^{-1}(U) \subset T(M_n)$. We call this metric the synectic metric. The synectic metric $Sg = Cg + V a$ has respectively components

$$Sg = (Sg_{CB}) = \left( \begin{array}{cc} a_{ji} + \partial g_{ji} & g_{ji} \\ g_{ji} & 0 \end{array} \right), \quad Sg^{CB} = \left( \begin{array}{ccc} 0 & x^s \partial_s g^{ji} - a'^{ji} \\ 0 & x^s \partial_s g^{si} - a'^{si} \end{array} \right)$$

where $\partial g_{ji} = x^s \partial_s g_{ji}$ and $a'^{si} = g^{hji} a_{js} g^{si}$.

Components of the Riemannian connection determined by the synectic metric $Sg$ are

$$\left\{ \begin{array}{ll} S\Gamma^k_{ji} = \Gamma^k_{ji}, & S\Gamma^{k}_{ji} = \Gamma^{k}_{ji}, \\ S\Gamma^{k}_{ji} = \Gamma^{k}_{ji}, & S\Gamma^{k}_{ji} = \Gamma^{k}_{ji}, \\ S\Gamma^{k}_{ji} = 0, & S\Gamma^{k}_{ji} = 0, \\ S\Gamma^{k}_{ji} = x^s \partial_s \Gamma^{k}_{ji} + H^k_{ji} \end{array} \right.$$
with respect to the induced coordinates in $T(M_n)$, $\Gamma^k_{ji}$ being Christoffel symbols constructed with $g_{ji}$. Where $H^k_{ji} = \frac{1}{2} g^{ks} (\nabla_j a_{si} + \nabla_i a_{js} - \nabla_s a_{ji})$ is a tensor of type $(1,2)$ and $\nabla_s a_{ji} = \partial_s a_{ji} - \Gamma^l_{kj} a_{li} - \Gamma^l_{ki} a_{jl}$.

The metric connection $\nabla$ of the synectic metric satisfies $\nabla_C g_{BA} = 0$ and has non-trivial torsion tensor $\Gamma^A_{CB}$, which is skew-symmetric in the indices $C$ and $B$. Then the metric connection $\nabla$ of the synectic metric has components $^\nabla$

$$\begin{cases} \Gamma^h_{ji} = \Gamma^h_{ji} = \Gamma^h_{ji}, \\
\Gamma^h_{ji} = \Gamma^h_{ji} = \Gamma^h_{ji}, \\
\Gamma^h_{ji} = x^a \partial_a \Gamma^h_{ji} + H^h_{ji} - y^k R_{kjih} 
\end{cases}$$

with respect to the induced coordinates, $\Gamma^h_{ji}$ being Christoffel symbols formed with $g_{ji}$, where $H^h_{ji} = \frac{1}{2} g^{ks} (\nabla_j a_{si} + \nabla_i a_{js} - \nabla_s a_{ji})$.

Given a vector field $\tilde{X}$ in $T(M_n)$, the $1$-form $\tilde{\omega}$ defined by $\tilde{\omega}(\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y})$, $\tilde{Y}$ being an arbitrary element of $T_0^1(M_n)$, is called the covector field associated with $\tilde{X}$ and denoted by $\tilde{X}^\ast$. If $\tilde{X}$ has local components $\tilde{X}^A$, then the associated covector field $\tilde{X}^\ast$ of $\tilde{X}$ has local components $\tilde{X}_C = \tilde{g}_{CA} \tilde{X}^A$.

Let $\omega$ be a $1$-form in $M_n$ with components $\omega_i$. Then the vertical, complete and horizontal lifts of $\omega$ to $T(M_n)$ have respectively components $^\ast$

$$(^V \omega_B) = (\omega_i, 0), \quad (^C \omega_B) = (\partial_i \omega_j, \omega_j), \quad (^H \omega_B) = (-\Gamma^i_{h} \omega_i, \omega_i)$$

with respect to the induced coordinates in $T(M_n)$.

The associated covector fields of the vertical, complete and horizontal lifts to $T(M_n)$, with the synectic metric, of a vector field $X$ with components $X^h$ in $M_n$ are respectively

$$(X_i, 0), \quad (y^a \partial_a X_i + a_{ij} X^j, X_i), \quad (\Gamma^h_{ji} X_h + a_{ij} X^j, X_i)$$

with respect to the induced coordinates, where $X_j = g_{ji} X^i$ are components of the covector field $X^\ast$ associated with $X$.

A vector field $X \in \mathfrak{X}(M_n)$ is said to be a Killing vector field of a Riemannian manifold with metric $g$, if $L_X g = 0$ $^3$. In terms of components $g_{ji}$ of $g$, $X$ is a Killing vector field if and only if

$$L_X g = X^\alpha \nabla_\alpha g_{ji} + g_{al} \nabla_j X^\alpha + g_{ja} \nabla_i X^\alpha = \nabla_j X_i + \nabla_i X_j = 0,$$

$X^\alpha$ being components of $X$, where $\nabla$ is the Riemannian connection of the metric $g$.

2. Main Results

We now take a vector field $X$ in $M_n$ with components $X^h$. Then, since the associated covector fields of the lifts of $X$ have respectively components give by (5), we have by (3) and (5)
with respect to the induced coordinates, where

\[
\begin{align*}
S \nabla^V_B X_A &= \begin{pmatrix} \nabla_j X_i & 0 \\ 0 & 0 \end{pmatrix}, \\
S \nabla^C_B X_A &= \begin{pmatrix} \nabla_j X_i + \nabla_i X_j - H^m_{ji} X_m & \nabla_j X_i \\ \nabla_j X_i & 0 \end{pmatrix}
\end{align*}
\]

(6)

and consequently

\[
\begin{align*}
S \nabla^V_B X_A + S \nabla^V_A X_B &= \begin{pmatrix} \nabla_j X_i + \nabla_i X_j & 0 \\ 0 & 0 \end{pmatrix}, \\
S \nabla^C_B X_A + S \nabla^C_A X_B &= \begin{pmatrix} S \nabla^C_B X_i + S \nabla^C_A X_j & S \nabla^C_B X_i + S \nabla^C_A X_j \\ S \nabla^C_B X_i + S \nabla^C_A X_j & S \nabla^C_B X_i + S \nabla^C_A X_j \end{pmatrix}
\end{align*}
\]

(7)

with respect to the induced coordinates, where \( X_j = g_{jk} X^k \). From (7) we have

**Theorem 1** Necessary and sufficient conditions in order that (a) the vertical, (b) complete lifts to \( T(M_n) \), with the symplectic metric, of a vector field \( X \) in \( M_n \) be a Killing vector field in \( T(M_n) \) are that, respectively, (a) \( X \) is a Killing vector field in \( M_n \) and (b) \( X \) is Killing vector field with vanishing covariant derivative in \( M_n \) and the covariant derivative of symmetric tensor field \( a \) of type (0,2) vanishes.

We also have by (2) and (6)

\[
\begin{align*}
S \nabla^V_B X_A - S \nabla^V_A X_B &= \begin{pmatrix} \nabla_j X_i - \nabla_i X_j & 0 \\ 0 & 0 \end{pmatrix}, \\
S \nabla^C_B X_A - S \nabla^C_A X_B &= \begin{pmatrix} S \nabla^C_B X_i - S \nabla^C_A X_j & S \nabla^C_B X_i - S \nabla^C_A X_j \\ S \nabla^C_B X_i - S \nabla^C_A X_j & S \nabla^C_B X_i - S \nabla^C_A X_j \end{pmatrix}
\end{align*}
\]

(8)

with respect to the induced coordinates, where \( S g^{BA} S \nabla^V_B X_A = 0 \) and \( S g^{BA} S \nabla^C_B X_A = 2g^{ij} \nabla_j X_i \). Thus we have, from (8), respectively,
**Theorem 2** The vertical lift of a vector field in $M_n$ to $T(M_n)$ with the synectic metric $S g$ is harmonic if and only if the vector field in $M_n$ is closed.

**Theorem 3** The complete lift of a vector field in $M_n$ to $T(M_n)$ with the synectic metric $S g$ is harmonic if and only if the vector field in $M_n$ is harmonic and the covariant derivative of symmetric tensor field $a$ of type $(0,2)$ vanishes.

We consider a vector field $X \in \mathfrak{X}_1(M_n)$. Then its vertical, complete and horizontal lifts have components of the form

$$
V X = \left( \begin{array}{c} 0 \\ X^h \end{array} \right), \quad C X = \left( \begin{array}{c} X^h \\ \frac{\partial X^h}{\partial X^h} \end{array} \right), \quad H X = \left( \begin{array}{c} X^h \\ -\Gamma^h_i X^i \end{array} \right)
$$

with respect to the induced coordinates in $T(M_n)$, where $\Gamma^h_i X^i = \gamma^h_s \Gamma^h_s X^i$.

Let $X$ be a vector field in $M_n$ with local components $X^k$. Then, from (9) and (4), we see that, the covariant derivatives of the vertical, complete and horizontal lifts of $X \in \mathfrak{X}_1(M_n)$ with the metric connection $\tilde{\nabla}$ have respectively components

$$
\tilde{\nabla}_B V^X = \left( \begin{array}{c} 0 \\ \nabla_j X^h \end{array} \right),
\tilde{\nabla}_B C X = \left( \begin{array}{c} \partial (\nabla_j X^h) + H^h_{jm} X^m - \gamma^k \Gamma^h_{ij} X^m \nabla_j X^h \\ 0 \\ \nabla_j X^h \end{array} \right),
\tilde{\nabla}_B H X = \left( \begin{array}{c} -\Gamma^h_i (\nabla_j X^i) + H^h_{jm} X^m \\ 0 \end{array} \right)
$$

with respect to the induced coordinates $T(M_n)$.

**Remark 4** $\tilde{\nabla} = \nabla + V H$, where $\nabla$ is the metric connection with the metric $C g$.

**Remark 5** The metric connection $\nabla$ coincides with the horizontal lift $H \nabla$ of Levi Civita connection $\nabla$ of $g$ in $M_n$. Thus we have

**Proposition 6** Necessery and sufficient conditions in order that (a) the vertical, (b) complete and horizontal lifts of a vector field in $M_n$ to $T(M_n)$ with the metric connection $\tilde{\nabla}$ be parallel in $T(M_n)$ are that, respectively, (a) the vector field given in $M_n$ is parallel (b) the vector field given in $M_n$ is parallel and the covariant derivative of symmetric tensor field $a$ of type $(0,2)$ vanishes.

Since $\nabla_j X^h = t \delta^h_j$ with constant $t$ implies $R^h_{kjm} X^m = 0$, we have also

**Proposition 7** The complete lift of a vector in $M_n$ to $T(M_n)$ with the metric connection $\nabla$ is concurrent if and only if the vector field given in $M_n$ is concurrent and the covariant derivative of symmetric tensor field $a$ of type $(0,2)$ vanishes.
A vector field $X \in \mathfrak{X}_{0}^{1}(M_{n})$ is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric $g$, if $L_{X}g = 0$. In terms of components $g_{ij}$ of $g$, $X$ is an infinitesimal isometry if and only if

$$X^{\gamma}\partial_{\gamma}g_{\alpha\beta} + g_{\gamma\beta}\partial_{\alpha}X^{\gamma} + g_{\gamma\alpha}\partial_{\beta}X^{\gamma} = 0, \quad (11)$$

$X^{\alpha}$ being components of $X$, where the indices $\alpha, \beta$ and $\gamma$ run over the range $\{1, 2, ..., m\}$.

Let there be given in $M_{n}$ a Riemannian metric $g$ with components $g_{ij}$. Let $\tilde{X}$ be a vector field with components $\left(\tilde{X}^{k}\right)$ with respect to the induced coordinates in $T(M_{n})$. Then, taking account of (2), we see by virtue of (11) that $\tilde{X}$ is an infinitesimal isometry in $T(M_{n})$ with metric $g$ if and only if

$$\left\{ \begin{array}{l}
\tilde{X}^{h}\partial_{h}g_{ji} + \tilde{X}^{h}\partial_{h}g_{ji} + \left( \partial_{g_{jh}}\partial_{h}X^{h} + g_{jh}\partial_{i}X^{h} \right) \\
+ \left( \partial_{g_{hi}}\partial_{j}X^{h} + g_{hi}\partial_{j}X^{h} \right) + \left( \tilde{X}^{h}\partial_{h}a_{ji} + a_{jh}\partial_{i}X^{h} + a_{hi}\partial_{j}X^{h} \right) = 0 \quad (12) \\
\tilde{X}^{h}\partial_{h}g_{ji} + g_{jh}\partial_{i}X^{h} + \left( \partial_{g_{hi}}\partial_{j}X^{h} + g_{hi}\partial_{j}X^{h} \right) + a_{hi}\partial_{j}X^{h} = 0 \quad (13) \\
\tilde{X}^{h}\partial_{h}g_{ji} + g_{hi}\partial_{j}X^{h} + \left( \partial_{g_{jh}}\partial_{i}X^{h} + g_{jh}\partial_{i}X^{h} \right) + a_{jh}\partial_{i}X^{h} = 0 \quad (14) \\
g_{jh}\partial_{i}X^{h} + g_{hi}\partial_{j}X^{h} = 0 \quad (15)
\end{array} \right.$$

We shall now prove

**Lemma 8** Let $C$ be an element of $\mathfrak{X}_{0}^{1}(M_{n})$. Then $L_{\iota C}g = 0$ holds if and only if $C = 0$.

**Proof.** Denote by $C^{k}_{i}$ the local components of $C$. Then $\iota C$ has components

$$\begin{pmatrix}
0 \\
y^{i}C^{k}_{i}
\end{pmatrix}$$

with respect to the induced coordinates in $T(M_{n})$. Thus, substituting $\tilde{X}^{k} = 0$ and $\tilde{X}^{h} = y^{i}C^{k}_{i}$ in (13), we have $g_{ki}C^{k}_{j} = 0$, which implies $C^{k}_{j} = 0$, i.e., $C = 0$. $\blacksquare$

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