Half-Quantum Vortices in Thin Film of Superfluid $^3$He

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Stability of a half-quantum vortex (HQV) in superfluid $^3$He has been discussed recently by Kawakami, Tsutsumi and Machida in Phys. Rev. B 79, 092506 (2009). We further extend this work here and consider the $A_2$ phase of superfluid $^3$He confined in thin slab geometry and analyze the HQV realized in this setting. Solutions of HQV and singly quantized singular vortex are evaluated numerically by solving the Ginzburg-Landau (GL) equation and respective first critical angular velocities are obtained by employing these solutions. We show that the HQV in the $A_2$ phase is stable near the boundary between the $A_2$ and $A_1$ phases. It is found that temperature and magnetic field must be fixed first in the stable region and subsequently the angular velocity of the system should be increased from zero to a sufficiently large value to create a HQV with sufficiently large probability. A HQV does not form if the system starts with a fixed angular velocity and subsequently the temperature is lowered down to the $A_1$ phase. It is estimated that the external magnetic field with strength on the order of 1 T is required to have a sufficiently large domain in the temperature-magnetic field phase diagram to have a stable HQV.

KEYWORDS: Half-quantum vortex, Singular vortex, First critical angular velocity

1. Introduction

A half-quantum vortex (HQV) in the A phase of superfluid $^3$He has been proposed first by Volovik and Mineev in 1976.$^1$ They fully utilized the peculiar structure of the order parameter manifold $M = S^2 × SO(3)/Z_2$ of the A phase, where $S^2$ is a manifold where the unit magnetic vector $\hat{d}$ resides, while $SO(3)$ is a group manifold representing the orbital degrees of freedom $\hat{m} + i\hat{n}$. They are intertwined by the $Z_2$ factor, showing the uniqueness of the magnetic part and orbital part are required only up to the sign flip; pairs $(\hat{d}, \hat{m} + i\hat{n})$ and $(-\hat{d}, -(\hat{m} + i\hat{n}))$ define the same ordered state.

Since then, there have been several theoretical$^{2-4}$ as well as experimental$^{5,6}$ works devoted to the stability of a HQV in a thin film of superfluid $^3$He-A. Recently, Kawakami, Tsutsumi and Machida published papers studying the Majorana modes trapped in a HQV$^{7,8}$ inspired by the NMR experiment conducted by Yamashita et al.$^9$ They show that a HQV is stabilized when the system is rotated and obtain the phase diagram in rotation velocity-system size space. To date, however, the existence of a HQV has not been demonstrated in superfluid $^3$He yet in spite of intensive challenges.

In the present paper, we further pursue their work and show that a stable HQV exists in the $A_2$ phase near the boundary between the $A_1$ and the $A_2$ phases. The stability domain of a HQV has the temperature range comparable to that of the $A_1$ phase if the external magnetic field has a strength on the order of 1 T.

The rest of the paper is organized as follows. In Sec. II, we introduce the Ginzburg-Landau free energy of superfluid $^3$He to establish our notation and convention. The order parameter describing a HQV is introduced in Sec. III. The critical angular velocity is defined in Sec. IV for a HQV and a singular vortex. Section V is devoted to numerical analysis. The order parameter profiles and comparison between the formation energy of a HQV and a singular vortex are given. Section VI concludes this paper.

2. Ginzburg-Landau Free Energy

Let $A_{a\mu}$ be the order parameter of a superfluid phase of $^3$He, where $\alpha$ is the spinor index while $i$ is the orbital index. Then the bulk free energy in the Ginzburg-Landau expansion takes the form

$$
F_B = -\alpha A_{a\mu} A_{a\mu} + \beta_1 A_{a\mu} A_{a\mu} A_{\beta j} A_{\beta j} + \beta_2 A_{a\mu} A_{a\mu} A_{\beta j} A_{\beta j} + \beta_3 A_{a\mu} A_{a\mu} A_{\beta j} A_{\beta j} + \beta_4 A_{a\mu} A_{a\mu} A_{\beta j} A_{\beta j} + \beta_5 A_{a\mu} A_{a\mu} A_{\beta j} A_{\beta j},
$$

(1)

where the coefficient $\alpha$ of the second order term has a temperature dependence $\alpha = \alpha' t$ with a constant $\alpha'$ and $t = 1 - T/T_c$. We take account of the effect of the strong coupling through the paramagnon parameter $\delta$ in the fourth order terms $\beta_i$ as

$$
\beta_1 = -(1 + 0.1\delta)\beta_0, \quad \beta_2 = (2 + 0.2\delta)\beta_0, \quad \beta_3 = (2 - 0.05\delta)\beta_0, \quad \beta_4 = (2 - 0.55\delta)\beta_0, \quad \beta_5 = (2 + 0.76\delta)\beta_0.
$$

(2)

The gradient free energy is given by

$$
F_G = K_1 \partial_a A_{a\mu} \partial_a A_{a\mu} + K_2 \partial_a A_{a\mu} \partial_j A_{a\mu} + K_3 \delta \partial_a A_{a\mu} \partial_i A_{a\mu}.
$$

(3)

The coefficients $K_i$ satisfy

$$
K_1 = K_2 = K_3 = K
$$

(4)

in the weak coupling limit. We employ the relation (4) in the rest of this paper for simplicity. As a result, the coherence
length is uniquely defined as
\[ \xi(t) = \sqrt{\frac{K}{\alpha}} = \sqrt{\frac{K}{\alpha'} \frac{1}{\sqrt{t}}} \]  
(5)

In the following, we take superfluid $^3$He confined between two parallel plates, the distance of which is less than the dipole coherence length. The superfluid is rotated around the $z$-axis, which is perpendicular to the plates, and subject to a strong magnetic field along the $z$-axis so that the superfluid is in the $\Lambda_2$ phase. The parallel plates introduces the boundary condition such that the $\hat{I}$ vector is perpendicular to the plates at the boundary. As a result, the orbital states of the order parameter are restricted to $l_z = 1$ or $l_z = -1$ at the boundary. A strong magnetic field along the $z$-axis aligns the $\hat{d}$ vector in the $xy$-plane.

It turns out to be convenient in this setting to change the basis of the order parameter from $(x, y, z)$ to $(1, 0, -1)$, characterizing the $z$-component of the angular momentum. These two sets of basis vectors are related as
\[ \hat{e}_z = \frac{z}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y), \quad \hat{e}_0 = \hat{e}_z. \]  
(6)

From now on, we change the notation of the order parameter from $A_{\alpha i}$ with respect to the $\hat{e}_z = (\hat{e}_x, \hat{e}_y, \hat{e}_z)$ basis to $A_{\mu \nu}$ with respect to $\hat{e}_{\nu} = (\hat{e}_x, \hat{e}_y, \hat{e}_z)$ basis.

The magnetic field coupled to the superfluid changes the second order term in $F_B$ to
\[ - \sum_{\nu = \pm} [(\alpha + \eta H)A^*_\nu A_{\nu +} + (\alpha - \eta H)A^*_\nu A_{\nu -}] \]
\[ = - \sum_{\nu = \pm} \alpha' t[(1 + \hat{h})A^*_\nu A_{\nu +} + (1 - \hat{h})A^*_\nu A_{\nu -}], \]  
(7)

where $\hat{h} = h/t$. The variable $h = \eta H/\alpha'$ is a dimensionless parameter corresponding to the magnetic field strength. The parameter $\eta$ is a constant yielding coupling between $H$ and the condensate.

Let us analyze a uniform superfluid in the $\Lambda_2$ phase with $l_z = 1$ by employing these free energies. Since $A_{\nu -} = A_{\nu -} = 0$ for this state, the bulk free energy reduces to
\[ F_B = - \alpha' t(1 + \hat{h})|A_{\nu +}|^2 + \beta_{24}|A_{\nu +}|^4 
- \alpha' t(1 - \hat{h})|A_{\nu -}|^2 + \beta_{24}|A_{\nu -}|^4 
+ 2(\beta_{24} + 2\beta_5)|A_{\nu +}|^2|A_{\nu -}|^2. \]  
(8)\n
The bulk order parameter is fixed by minimizing Eq. (10) with respect to $|A_{\nu +}|^2$ and $|A_{\nu -}|^2$ as
\[ |A_{\nu +}|^2 = \frac{(1 + \hat{h})\beta_{24} - (1 - \hat{h})(\beta_{24} + 2\beta_5)}{-8\beta_5(\beta_{24} + \beta_5)} \]
\[ |A_{\nu -}|^2 = \frac{(1 - \hat{h})\beta_{24} - (1 + \hat{h})(\beta_{24} + 2\beta_5)}{-8\beta_5(\beta_{24} + \beta_5)}. \]  
(11)

We take, without loss of generality, the following sign convention
\[ A_{\nu +} = A_{\nu +}^{(0)} = - \sqrt{\frac{(1 + \hat{h})\beta_{24} - (1 - \hat{h})(\beta_{24} + 2\beta_5)}{-8\beta_5(\beta_{24} + \beta_5)}} \]
\[ A_{\nu -} = A_{\nu -}^{(0)} = - \sqrt{\frac{(1 - \hat{h})\beta_{24} - (1 + \hat{h})(\beta_{24} + 2\beta_5)}{-8\beta_5(\beta_{24} + \beta_5)}}. \]  
(12)

This choice gives the $\hat{d}$-vector parallel to the $x$-axis in the $\Lambda$ phase resulting in the limit $\hat{h} \to 0$.

The coherence lengths of $A_{\nu +}$ and $A_{\nu -}$ in the presence of $\hat{h} \neq 0$ are $\xi_{\nu +} = 1/\sqrt{(1 + \hat{h})}$ and $\xi_{\nu -} = 1/\sqrt{(1 - \hat{h})}$ and they satisfy inequalities
\[ \xi_{\nu +} < \xi < \xi_{\nu -}. \]  
(13)

We now look at the gradient energy $F_G$. Consider a vortex along the $z$-axis and assume the order parameter is translationally invariant along this axis. Let us introduce the cylindrical coordinates $(r, \varphi, z)$ assuming $\partial/\partial z$ is a null operator. Let $n_{\mu \nu} \in \mathbb{Z}$ be the quantum number of the component $A_{\mu \nu}$ and write it as
\[ A_{\mu \nu} = C_{\mu \nu}(r)e^{in_{\mu \nu} \varphi}. \]  
(14)

The gradient term introduces the coupling between the orbital components $+$ and $-$, namely the coupling between $A_{\mu +}$ and $A_{\mu -}$. The quantum numbers $n_{\mu \pm}$ must satisfy the condition
\[ n_{\mu \pm} = n_{\mu \pm} + 2 \]  
(15)

for the vortex to be cylindrically symmetric around the $z$-axis. If this is the case, the gradient energy takes the form
\[ F_G = \sum_{\mu, \nu, \pm} \left[ \left( \frac{\partial}{\partial r} C_{\mu \nu} - \frac{n_{\mu \nu}}{r} C_{\mu \nu} \right)^2 + \frac{n_{\mu \nu}^2}{r^2} \left( C_{\mu \nu} \right)^2 \right], \]  
(16)

where the rescalings of $r \to \xi r$ and $F_G \to \left( (\alpha')^2/\beta_0 \right) F_G$ have been made as before.

3. Half-Quantum Vortex

The order parameter of a HQV proposed by Volokov and Mineev$^1$ takes the form
\[ A_{\alpha i} = \Delta_A d_\alpha (\hat{m} + i\hat{n}); \]
\[ = \Delta_A e^{\varphi/2} \left( \cos \frac{\varphi}{2} \hat{e}_x + \sin \frac{\varphi}{2} \hat{e}_y \right) \left( \hat{e}_z + i\hat{e}_x \right). \]  
(17)
in the A-phase with vanishing magnetic field $H = 0$, where it is assumed that the $\hat{L}$-vector is directed along the $z$-axis, while the $\hat{d}$-vector points in the $xy$-plane. Equation (17) is rewritten as

$$\Delta A \left( \hat{e}_+ - e^{i\varphi} \hat{e}_- \right) \hat{e}_{\pm i}$$

(18)

This shows that the order parameter of the HQV represented in the $(1, 0, -1)$ basis has a non-vanishing winding number only in the component $\hat{e}_-$. Similarly, there is an order parameter of a HQV, in which only the component $\hat{e}_+$ has a non-vanishing winding number.

By considering the condition (15), the order parameter (18) yields a vortex with quantum numbers

$$((n_{++}, n_{+-}), (n_{--}, n_{-+})) = ((0, 2), (1, 3)).$$

(19)

We call this vortex as a vortex of type $(0, 1)$ to distinguish it from other types of vortices introduced in the following. When the superfluid is rotated in the opposite sense, the resulting vortex has an order parameter in which $e^{i\varphi}$ is replaced by $e^{-i\varphi}$ in Eq. (18), which will be called a vortex of type $(0, -1)$ having quantum numbers

$$((n_{++}, n_{+-}), (n_{--}, n_{-+})) = ((0, 2), (-1, 1)).$$

(20)

It is important to realize that the structure of a vortex of type $(0, 1)$, obtained by rotating the superfluid in the positive sense with respect to the $\hat{L}$-vector, is different from that of a vortex of type $(0, -1)$ obtained by rotating the superfluid in the opposite direction. The condensate with orbital angular momentum spontaneously breaks the rotational invariance and hence the clockwise rotation and anticlockwise rotation are not mirror reflections of each other.

4. First Critical Angular Velocity

Let $R$ be the radius of a cylindrical container and $\Omega$ be the angular velocity with which the cylinder rotates. Now we obtain the condition under which a vortex stably exists at the center of the container. The gradient free energy in the rotating system is obtained by replacing the $\varphi$-derivative as

$$\frac{\partial \varphi}{\partial r} \rightarrow \frac{\partial \varphi}{\partial r} - \frac{2m}{h} (2\Omega \times \mathbf{r})_{\varphi},$$

(21)

where $m$ is the mass of a $^3$He atom. Let us first consider a HQV, in which the component $\hat{e}_-$ with lower creation energy has a non-vanishing quantum number $n_{--}$. There are two terms of the form

$$\left( \frac{1}{r} - \frac{2m}{h} \Omega r \right)^2 C_{\pm}(r)^2$$

(22)

in the gradient energy (16). The coefficient of a term linear in $\Omega$ is nothing but the angular momentum and the total angular momentum of the system is found to be

$$L^{(-)} = 2 \times 4\pi \frac{2m}{h} \int_0^R r dr C_{\pm}(r)^2 = 4\pi \frac{2m}{h} (A_{\pm}^{(0)})^2 R^2,$$

(23)

where we noted that the contribution of the vortex core to the total angular momentum is negligible.

The vortex formation energy measured with respect to the uniform bulk energy $F_0$ is evaluated as

$$F_{\text{vor}}^{(-)} = 2\pi \int_0^R r dr (F - F_0) = 4\pi (A_{\pm}^{(0)})^2 (\ln R + C_{\pm}).$$

(24)

The parameter $C_{\pm}$ will be evaluated numerically later. The first critical angular velocity for a formation of a vortex with $\hat{e}_-$ spin component, namely a vortex in the spin component $\downarrow\downarrow$ is obtained by solving

$$F_{\text{vor}} - \Omega L = 0$$

(25)

as

$$\Omega_{\text{c}}^{(-)} = \frac{h}{2m} \frac{F_{\text{vor}}^{(-)}}{C_{\pm}} = \frac{h}{2m R^2} (\ln R + C_{\pm}).$$

(26)

An angular velocity will be scaled by $\hbar/2mR^2$ from now on. As a result, the critical angular velocity is written as

$$\Omega_{\text{c}}^{(-)} = \ln R + C_{\pm}.$$  

(27)

A singular vortex (SV) with a winding number 1 is obtained by setting the quantum numbers of $A_{\mu r}$ to

$$((n_{++}, n_{+-}), (n_{--}, n_{-+})) = ((1, 3), (1, 3)).$$

(28)

The total angular momentum of a SV is

$$L^{(s)} = 2 \times 4\pi \frac{2m}{h} \int_0^R r dr \left[ C_{\pm}(r)^2 + C_{\mp}(r)^2 \right] = 4\pi \frac{2m}{h} \left[ (A_{\pm}^{(0)})^2 + (A_{\mp}^{(0)})^2 \right] R^2.$$  

(29)

The formation energy of a SV is

$$F_{\text{vor}}^{(s)} = 4\pi \left[ (A_{\pm}^{(0)})^2 + (A_{\mp}^{(0)})^2 \right] (\ln R + C_{\mp})$$

(30)

and the first critical angular velocity is

$$\Omega_{\text{c}}^{(s)} = \ln R + C_{\mp}.$$  

(31)

in the dimensionless form.

Whether a HQV forms or a SV forms as the angular velocity is raised from zero when $\Omega \rightarrow 0$ and $\hbar > 0$ to begin with. It follows from the inequality $\xi_{\pm} < \xi_{\mp}$ that a vortex in $A_{\pm}$ is energetically favorable than that in $A_{\mp}$ and it is expected that $\Omega_{\text{c}}^{(-)} < \Omega_{\text{c}}^{(s)}$ is satisfied. In case $\delta > 0$ and $h \equiv 0$, the coupling between $A_{\pm}$ and $A_{\mp}$ is attractive and a SV is expected to be favorable compared to a HQV. This is because low magnitude $A_{\pm}$ and $A_{\mp}$ overlap at the common vortex core in a SV while they do not in a HQV, thus gaining more negative energy for the former. Then an inequality $\Omega_{\text{c}}^{(-)} > \Omega_{\text{c}}^{(s)}$ is expected to be satisfied.

It is expected from the above arguments that a SV is formed first as the angular velocity is raised from zero when $\delta > 0$ and $\hbar$ is small. When the external magnetic field is strong enough, in contrast, there is a region in the temperature-angular velocity domain in which a HQV is formed first. These statements will be verified numerically in the next section.

5. Numerical Analysis

We have solved the Ginzburg-Landau equation with respect to $(C_{\pm}(r), C_{\mp}(r), C_{\pm}(r), C_{\mp}(r))$ numerically. Four choices of the quantum numbers are considered:

(a) A HQV $(0, 1)$ with $((n_{++}, n_{+-}), (n_{--}, n_{-+})) = ((0, 2), (1, 3))$, 

(b) A SV $(1)$ with $((n_{++}, n_{+-}), (n_{--}, n_{-+})) = ((1, 3), (1, 3))$, both with $\Omega > 0$ and
(c) A HQV \((0, -1)\) with \((n_{++}, n_{+-}, n_{--}, n_{-+}) = ((0, 2), (-1, 1))\),
(d) A SV \((-1)\) with \((n_{++}, n_{+-}, n_{--}, n_{-+}) = ((-1, 1), (-1, 1))\),
both with \(\Omega < 0\). The parameters \(\hat{h}\) and \(\delta\) are changed from 0.0 to 0.5 with a step 0.1. The boundary condition at \(r = R\) does not affect the formation energy since \(R > 1\) is assumed.

We take the boundary condition
\[
(A_{++}(R), A_{+-}(R), A_{--}(R), A_{-+}(R)) = (A_{++}^{(0)}, A_{+-}^{(0)}, 0),
\]
which corresponds to a vortex embedded in a uniform \(\hat{t}\) texture with \(l = +1\).

The order parameter profiles for HQV \((0, 1)\), SV \((1)\), HQV \((0, -1)\) and SV \((-1)\) with \(\delta = 0.2, \hat{h} = 0.2\), and \(R = 30\) are shown in Fig. 1.

Next the first critical angular velocities are obtained by evaluating the free energies of HQV’s and SV’s with our numerical solutions and then employing Eqs. (27) and (31). Furthermore, we repeat the same calculation with a function \(\Omega = \hat{A} \ln R + C, \hat{A}\) and \(C\) being constants. The result shows that \(\hat{A}\) is in fact 1 with a good precision, as expected, and we have determined the \(\hat{h}\) and \(\delta\)-dependences of \(C_{-}\) and \(C_{+}\), the C value of a HQV \((0, 1)\) and a SV \((1)\), respectively. Figure 2 shows \(C_{-} = \Omega_{c}^{(0)} - \ln R\) and \(C_{+} = \Omega_{c}^{(1)} - \ln R\) for cases (a) and (b) as functions of \(\hat{h}\) for \(\delta = 0.2\). The dimensionless magnetic field \(\hat{h}\) also takes values \(\hat{h} = 0.0, 0.1, \ldots, 0.5\). The result shows that, for \(\delta \neq 0\), there exists \(\hat{h}_{c}\) at which the inequality \(\Omega_{c}^{(+)} > \Omega_{c}^{(0)}\) flips to \(\Omega_{c}^{(+)} < \Omega_{c}^{(0)}\) as \(\hat{h}\) is increased. The critical magnetic field \(\hat{h}_{c}\) vanishes for \(\delta = 0\), showing there is a range of \(\Omega\) in which a HQV is stable for any \(\hat{h} > 0\). The first critical angular velocity \(\hat{h}_{c}\) has been estimated by finding the intersection of numerically interpolated curves \(\Omega_{c}^{(-)}\) and \(\Omega_{c}^{(0)}\) as functions of \(\hat{h}\). Figure 3 depicts the \(\delta\)-dependence of \(\hat{h}_{c}\) thus obtained. In case \(\Omega_{c}^{(-)} < \Omega_{c}^{(0)}\), we reach a region in the \(\Omega\) axis in which a SV is stabilized if the angular velocity is further increased beyond \(\Omega_{c}^{(-)}\). The boundary between two stability regions along the \(\Omega\)-axis is found from
\[
F_{\text{vor}}^{(-)} - \Omega L^{(-)} = F_{\text{vor}}^{(0)} - \Omega L^{(0)}.
\]
Figure 2 also shows \(C_{+}(\rightarrow) = \Omega_{c}^{(-)}(\rightarrow) - \ln R\). The region in which a HQV is stable is bounded by two curves \(\Omega_{c}^{(-)}(\rightarrow)\) and \(\Omega_{c}^{(0)}\) in Fig. 2.

Next, the \(\hat{h}\)-dependences of \(C_{-}\) and \(C_{+}\) for cases (c) and (d), respectively, are depicted in Fig. 4. They are different from those of cases (a) and (c), reflecting upon the difference in the vortex structures for \(\Omega > 0\) and \(\Omega < 0\). Figure 4 also shows \(\Omega_{c}^{(-)}(\rightarrow)\), similarly to Fig. 2. A HQV is stable in the region bounded by two curves \(\Omega_{c}^{(-)}(\rightarrow)\) and \(\Omega_{c}^{(0)}\). Let us evaluate the critical value \(\hat{h}_{c}\), at which the HQV-stable region appears. The critical value will turn out to be almost the same as that with \(\Omega > 0\), in contrast with \(C_{-}\) and \(C_{+}\). The region in the \(\hat{h} - \hat{\rho}\)-plane (the temperature-pressure plane) where a HQV is stable is obtained from the \(\delta\)-dependence of \(\hat{h}_{c}\). The phase diagram for a given \(h = \eta H / a'\) is shown in Fig. 5, where \(t = h / h_{s}\) has been used. The \(A_{1}\) phase is also shown in Fig. 5 for comparison. Although we have ignored the \(\delta\)-dependences of \(\eta\) and the superfluid transition temperature \(T_{c}\) at \(H = 0\), the comparison between the width of the \(A_{1}\) phase and that of the HQV-stable region is meaningful for a fixed \(\delta\). Figure 5 shows that the width of the HQV-stable region along the \(t / h\)-axis is comparable to that of the \(A_{1}\) phase. Moreover, the former increases compared with the latter for small \(\delta\) (low pressure) region.
6. Conclusion and Discussion

We have obtained conditions with which a half-quantum vortex stably exists and have shown that the stability region of a HQV has a comparable range to that of the $A_1$ phase along the $t/h$-axis.

To obtain the HQV, the temperature and the magnetic field must be fixed in the HQV-stable region with no rotation first and subsequently the angular velocity must be increased beyond the critical angular velocity. The opposite scenario, in which the system is rotated beyond the critical angular velocity first and then the temperature is lowered to form the $A_2$ phase through the $A_1$ phase, does not lead to a HQV formation. This is because a singular vortex forms in the $A_2$ phase, does not lead to a HQV formation. It is desirable to observe the direct NMR signal from a HQV for its detection. Nonetheless, direct observation can be rather challenging in the presence of a strong magnetic field. Note, however, that, when HQV formation takes place, there are two transitions associated with vortex formation in the vicinities of $\Omega > 0$. For comparison, the $A_1$ phase, given $h = \eta H/\alpha'$, is also shown here. The origin of the horizontal axis corresponds to $T = T_c$, at $H = 0$. The critical temperature $T_c$ of the $A_1$-phase is 1 in the present scaling. The $\delta$ dependence of the phase transition takes place as $\delta > 0$, while it exists for any $h$ when $\delta = 0$. The intersecting point of the solid, the dashed and the dotted lines gives $\delta_{cp}(\delta)$.

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