Nonnegatively curved 5-manifolds with non-abelian symmetry

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Known examples of manifolds which admit metrics of positive (sectional) curvature are rare when compared with nonnegatively curved examples. In fact, besides rank one symmetric spaces, compact manifolds with positive curvature are known to exist only in dimensions below 25, while to generate new nonnegatively curved manifolds from known ones it is enough, for example, to take products, quotients or biquotients (see [32] for a survey).

By the Soul Theorem any complete non-compact nonnegatively curved manifold is diffeomorphic to a vector bundle over a compact manifold with nonnegative curvature. For compact manifolds of positive curvature Bonnet-Myers implies that the fundamental group is finite and in nonnegative curvature a finite cover is diffeomorphic to the product of a torus with a compact simply-connected manifold of nonnegative curvature (see [5]). We will hence only consider compact simply-connected manifolds.

Recently, positively and nonnegatively curved manifolds were studied under the additional assumption of having a “large” isometry group (see the surveys [12] and [30]). The beginning of this subject was the result by Hsiang and Kleiner [17] that a compact simply-connected 4-dimensional Riemannian manifold with positive curvature and $S^1$-symmetry must be either $S^4$ or $CP^2$. The possible isometric circle-actions were classified in [10] and [15].

The classification of isometric circle actions on positively curved 5-manifolds is a very difficult problem and at the moment seems out of reach. In 2002, Rong [26] showed that a positively curved compact simply-connected 5-dimensional manifold with a 2-torus acting by isometries is diffeomorphic to a 5-sphere. In 2009 Galaz-Garcia and Searle [11], only assuming nonnegative curvature, showed that a simply-connected 5-manifold which admits an isometric action of a 2-torus is diffeomorphic to either $S^5$, $S^3 \times S^2$, the nontrivial $S^3$-bundle over $S^2$, denoted by $\tilde{S}^3 \times S^2$, or the Wu-manifold $W = SU(3)/SO(3)$. The description of the actions is not yet solved in any of these cases.

In this context, a question that naturally arises is which 5-manifolds admit a metric of non-negative (or positive) curvature with symmetry containing a connected non-abelian group $G$. In this paper we will classify such manifolds with nonnegative curvature and obtain a partial classification in positive curvature. For this purpose, we first classify all five-dimensional compact simply-connected manifolds which admit an action of a connected non-abelian Lie group without any geometric assumptions. They are either $S^5$, $S^3 \times S^2$, $S^3 \tilde{\times} S^2$, connected sums of $S^3 \times S^2$, or connected sums $kW \# lB$ of copies of the Wu-manifold $W$ and the Brieskorn variety $B$ of type $(2,3,3,3)$. Since any non-abelian connected Lie group contains $SO(3)$ or $SU(2)$ as a subgroup, it is natural to classify in addition the actions by these two groups up to equivariant diffeomorphisms. This is the content of Theorems C and D below.

To describe the actions we introduce the following key construction.

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Main example. Let \( m \leq n \) and \( l \) be nonnegative integers and consider the \( \mathbb{S}^1 \)-action on \( SU(2) \times \mathbb{S}^3 = \mathbb{S}^3 \times \mathbb{S}^3 \) given by

\[
x \cdot (p, (z, w)) = (p a^l, (x^m z, x^n w)),
\]

where we regard \( SU(2) \) as the group of unit quaternions, \( p \in SU(2) \), \( x \in \mathbb{S}^1 = \{ e^{i \theta} \in SU(2) \} \) and \((z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2 \). This action is free whenever \( \gcd(l, m) = \gcd(l, n) = 1 \). Notice that \( l = 1 \) if \( m = 0 \).

As we will see, the quotient \( \mathcal{N}^l_{m,n} := (SU(2) \times \mathbb{S}^3) / \mathbb{S}^1 \) is diffeomorphic to \( \mathbb{S}^3 \times \mathbb{S}^2 \) if \( m + n \) is even and diffeomorphic to \( \mathbb{S}^3 \times \mathbb{S}^2 \) otherwise. Consider \( \mathcal{N}^l_{m,n} \) as an \( SU(2) \)-manifold by defining

\[
g \cdot [(p, (z, w))] = [(gp, (z, w))],
\]

for \( g \in SU(2) \). This action has isotropy groups isomorphic to \( \mathbb{Z}_m, \mathbb{Z}_n \) and \( \mathbb{Z}_{\gcd(m,n)} \) if \( m \) and \( n \) are both positive, \( \mathbb{Z}_n \) and \( SO(2) \) if \( n > m = 0 \) and only one isotropy type \( (SO(2)) \) if \( m = n = 0 \). If \( \gcd(m, n) \) is even, the action has ineffective kernel \( \mathbb{Z}_2 \) and hence it is an effective action by \( SO(3) \).

Throughout this work, unless otherwise stated, \( G \) will denote \( SO(3) \) or \( SU(2) \) and \( M \) a simply-connected compact \( G \)-manifold of dimension 5. Our first result is a complete classification of all such nonnegatively curved \( G \)-manifolds.

**Theorem A.** If the \( G \)-manifold \( M \) admits an invariant metric with nonnegative curvature, then it is equivariantly diffeomorphic to either \( \mathbb{S}^5 \), \( \mathbb{S}^3 \times \mathbb{S}^2 \), or \( W = SU(3)/SO(3) \) with their natural linear \( G \)-actions, or \( \mathcal{N}^l_{m,n} \).

Notice that the natural metrics on these manifolds are invariant by the \( G \)-action and have nonnegative curvature and that the manifolds can all be written as biquotients.

For positive curvature we have the following partial classification.

**Theorem B.** If the \( G \)-manifold \( M \) admits an invariant metric with positive curvature, then it is either equivariantly diffeomorphic to \( \mathbb{S}^5 \) with a linear action, or possibly \( W \) with the linear \( SU(2) \)-action, or \( \mathcal{N}^l_{m,n} \) with trivial principal isotropy group, i.e., \( \gcd(m, n) = 1 \) or 2.

It is natural to conjecture that in the context of Theorem B only the linear actions on \( \mathbb{S}^5 \) admit invariant metrics of positive curvature. In any case a complete classification in Theorem B would answer the generalized Hopf conjecture for \( \mathbb{S}^3 \times \mathbb{S}^2 \) with non-abelian symmetry.

Theorems A and B will be a consequence of a general equivariant classification of \( SO(3) \) and \( SU(2) \)-actions in dimension five, a result that is of interest in its own right. We begin with the case without singular orbits.

**Theorem C.** If the \( G \)-manifold \( M \) does not have singular orbits, then it is equivariantly diffeomorphic to either \( \mathcal{N}^l_{0,0} \) or \( \mathcal{N}^l_{m,n} \) for some choice of positive integers \( m, n \) and \( l \). The \( SO(3) \)-manifolds correspond to \( \mathcal{N}^l_{m,n} \) with \( m \) and \( n \) even.
We will see that these actions are pairwise non-equivalent for different choices of the parameter \( l \) when \( \gcd(m,n) \geq 3 \) by showing that the fundamental group of the fixed point set of the principal isotropy group is isomorphic to \( \mathbb{Z}_l \). For \( \gcd(m,n) = 1 \) or 2, the actions on \( \mathcal{N}^l_{m,n} \) and \( \mathcal{N}'^l_{m,n} \) are equivalent precisely when \( l = l' \) modulo \( mn/\gcd(m,n) \).

Theorem C easily implies the following.

**Corollary 1.** A \( G \)-action without singular orbits on \( M \) extends to \( U(2) \times S^1 \) if \( G = SU(2) \) and to \( SO(3) \times T^2 \) if \( G = SO(3) \). In particular, \( M \) admits an effective \( T^3 \)-action.

Finally, for actions with singular orbits we have

**Theorem D.** If the \( G \)-manifold \( M \) has singular orbits, then \( M \) is equivariantly diffeomorphic to \( S^5 \) with a linear action or:

(a) \( G = SO(3) \) and \( M \) is equivalent to either \( \mathcal{N}^l_{0,2m} = S^3 \times S^2 \), connected sums of copies of the Wu-manifold and copies of the Brieskorn variety of type \((2,3,3,3)\), or connected sums of copies of \( S^3 \times S^2 \) with the linear action by \( SO(3) \subset SO(4) \) on the first factor;

(b) \( G = SU(2) \) and \( M \) is equivalent to either \( \mathcal{N}^l_{0,2m+1} = S^3 \times S^2 \), or the Wu-manifold with the left action by \( SU(2) \subset SU(3) \).

The case of \( G = SO(3) \) was studied in [18], although the author missed the equivariant connected sums of \( S^3 \times S^2 \) with the above \( SO(3) \)-action, and did not describe some of the actions explicitly. For partial results about differentiable classifications, see [22], [23], [24] and [25]. In [20] the \( SO(3) \) and \( SU(2) \)-actions on compact simply-connected 4-manifolds are classified.

This paper is organized as follows. In Section 1 we discuss preliminaries and describe the basic examples. In Section 2 we introduce the \( SU(2) \)-manifolds \( \mathcal{N}^l_{m,n} \), prove Corollary 1 assuming Theorem C and prove some results needed for the proof of Theorem C. Sections 3 and 4 are devoted to the proofs of Theorems C and D. In Section 5 we prove Theorems A and B.

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1 Preliminaries

In this section we fix our notation and define some of the objects we will work with. Here \( G \) is any compact Lie group not necessarily \( SO(3) \) or \( SU(2) \).

If \( G \) acts on \( M \) we denote by \( G_p = \{ g \in G : gp = p \} \) the isotropy group of the point \( p \in M \) and by \( (K) \) the conjugacy class of an isotropy group \( K \) and call it type of \( K \). The Principal Orbit Theorem guarantees that there exists one of smallest type, the principal isotropy group \( K \) which will be denoted by \( H \). By the Slice Theorem, a neighborhood of an orbit \( Gp \) can be describe up to equivariant diffeomorphism by \( G \times K \) \( D \) where \( K = G_p \) and \( D \) is a disk in the normal space to \( T_p(Gp) \). The linear action of \( K \) on \( D \) is called the slice representation at the point \( p \).

If \( H \) is a principal isotropy group then, \( \dim H \leq \dim K \) for any isotropy subgroup \( K \) of the action. If \( \dim H < \dim K \) then we call \( K \) a singular isotropy group. If \( \dim H = \dim K \) and \( K \) has more connected components than \( H \), then the group \( K \) is called an exceptional isotropy group.
The dimension of the orbit space $M/G$ is called the cohomogeneity of the action. About the orbit space we will need the following basic result (see [4] p. 91, 190, 207 and 211).

**Proposition 2.** If $M$ is simply-connected and $G$ is connected:

(a) The orbit space $M/G$ is simply-connected.

(b) If the action has cohomogeneity 2, then the orbit space is a topological surface with (or without) boundary. The boundary, if not empty, consists of the singular orbits and, in this case, there are no exceptional orbits.

(c) If the action has cohomogeneity 3, then $M/G$ is a simply-connected topological manifold possibly with boundary.

Every nontrivial subgroup of SO(3) is isomorphic to either the cyclic group $\mathbb{Z}_k$, the dihedral group $D_m$, the tetrahedral group $T$, the octahedral group $O$, the icosahedral group $I$, the circle $\mathbb{S}^1$ or the normalizer of $\mathbb{S}^1$ in SO(3), which is $O(2)$ (see [31] Section 7.1).

The special unitary group $SU(2)$ can be seen as $S^3 \subset \mathbb{C}^2$, or the unit quaternions, $S^3 \subset \mathbb{H}$. For $\alpha$ and $\beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ we have the following correspondence between the three expressions of an element in $SU(2)$:

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathbb{C}^{2\times2} \sim (\alpha, \beta) \in \mathbb{C}^2 \sim \alpha + \beta j \in \mathbb{H}.$$  

The quaternion notation will be generally used for the group $SU(2)$, while the $S^3 \subset \mathbb{C}^2$ notation when considering $SU(2)$ just as a manifold.

Denoting by $\phi : SU(2) \to SO(3)$ the 2-fold cover. The subgroups of $SU(2)$ are isomorphic to $\mathbb{Z}_{2k+1}$ or the pre-images by $\phi$ of the subgroups of SO(3) (see [1] §2). Any closed nontrivial subgroup of $SU(2)$ is then isomorphic to $\mathbb{Z}_k$, the dicyclic group $Dic_n$, the binary tetrahedral group $T^*$, the binary octahedral group $O^*$, the binary icosahedral group $I^*$, the circle or the $Pin(2)$ group.

We introduce here some of the $G$-actions with $G = SO(3)$ or $SU(2)$ that will appear in our classification.

**Example 1.** The linear $G$-actions on $S^5$ are given by:

(a) $A \in SO(3) \mapsto \text{diag}(A,1,1,1) \in SO(6)$ with isotropy groups SO(2) and SO(3).

(b) $A \in SO(3) \mapsto \text{diag}(A,A) \in SO(6)$ with isotropy groups $\{\text{Id}\}$ and SO(2).

(c) $B \in SU(2) \mapsto \text{diag}(B,1) \in SU(3)$ with isotropy $\{\text{Id}\}$ and a circle of fixed points.

(d) The representation of SO(3) on $\mathbb{R}^6$ is defined by $A \cdot X = AXA^{-1}$ where $X \in \mathbb{R}^6$ is a 3 by 3 symmetric matrix. The action is by isometries on $\mathbb{R}^6$ with the standard inner product. The isotropy groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$, $O(2)$ and SO(3) and the orbit space is a sector of angle $\pi/3$ in $\mathbb{R}^2$.

Thus, it induces an SO(3)-action in $S^5$ with the same isotropy groups and quotient a topological 2-disk with angle $\pi/3$ in each of the two fixed points.

**Example 2.** The linear $G$-actions on $S^3 \times S^2$.

They are given by the nonequivalent embeddings of $G$ in SO(4) × SO(3).
We start with the fixed point set is the union of two copies of 2-spheres, any other point has isotropy SO(2) and the orbit space is diffeomorphic to $S^2 \times [-1, 1]$.

(b) SO(3) in the second factor. The unique isotropy type is (SO(2)). It is equivalent to $\mathcal{N}_0^{1,1}$.

(c) The diagonal inclusion of SO(3), that is $A \in SO(3) \mapsto (\text{diag}(A, 1), A) \in SO(4) \times SO(3)$. The isotropy groups are $\{\text{Id}\}$ and SO(2). The action is equivalent to $\mathcal{N}_0^{1,2}$.

(d) SU(2) in the first factor. The action of SU(2) on $S^3$ is equivalent to the left multiplication of Lie groups, thus it is free with quotient $S^2$. This action is equivalent to $\mathcal{N}_0^{0,1}$. Note that this is equivalent to the linear action given by the diagonal inclusion of SU(2) in SO(4) $\times$ SO(3), by the classification of fiber bundles over spheres (see, Corollary 18.6 in [28]).

Example 3. The SU(2)-action on the Wu-manifold $W := SU(3)/SO(3)$.

Given $B \in SU(2)$ and $[C] \in W$, the action is $B \cdot [C] = [(\text{diag}(B, 1)C)]$. The isotropy types are $\{\text{Id}\}$ and (SO(2)) and the quotient is a 2-disk.

Example 4. The SO(3)-action on $W$.

The group SO(3) acts on $W$ as $A \cdot [B] = [AB]$. The isotropy types are $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, (O(2)) and SO(3). If the action is by isometries, the quotient is a flat triangle with vertices the fixed points and each edge corresponds to one of the three distinct embeddings of O(2) in SO(3). It was proved in [18] that this is the unique SO(3)-action on the Wu-manifold up to conjugacy.

Example 5. The SO(3)-action on the Brieskorn variety, $B$ of type $(2, 3, 3, 3)$.

The Brieskorn variety of type $(2, 3, 3, 3)$ can be defined as

$$B = \{(z_o, z_1, z_2, z_3) \in \mathbb{C}^4; \quad z_o^2 + z_1^3 + z_2^3 + z_3^3 = 0 \text{ and } |z_o|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}.$$ 

In [18] this action is constructed as an example of an SO(3)-manifolds with isotropy types $(\mathbb{Z}_2 \times \mathbb{Z}_2)$, (O(2)) and four fixed points by a process of gluing four open sets and the manifold is identified computing topological invariants. As far as we know an explicit description of the action is not known.

Given two $n$-dimensional $G$-manifolds with fixed points, choose Riemannian metrics invariant under the $G$-actions and consider a small ball of radius $r$ around a fixed point in each manifold. If the isotropy actions of $G$ on the slices of those fixed points are the same, then the actions on the boundaries of the balls are equivalent and we can form a connected sum of the two $G$-manifolds gluing along those spheres to, possibly, obtain a new $G$-manifold. Particular examples of this are:

Example 6. The SO(3)-action on the connected sum of $k$ copies of $S^3 \times S^2$.

We start with the SO(3)-action on $S^3 \times S^2$ from Example 5. At a fixed point the isotropy representation is given by SO(3)-action on $\mathbb{R}^3 \times \mathbb{R}^2$ that is standard in the first factor and trivial on the second. We can now take the connected sum of 2 copies of $S^3 \times S^2$ at the fixed points. This provides a connected sum of two fixed 2-spheres, so if we do this for $k$ copies of $S^3 \times S^2$, we obtain $k + 1$ fixed 2-spheres. The orbit space of the action is diffeomorphic to a 3-sphere with $k + 1$ three-disks removed.

The SO(3)-manifolds in Example 5 were overlooked in the classification in [18].

Example 7. The SO(3)-actions on $kW \# lB$.

The isotropy representation around an isolated fixed point of an SO(3)-manifold of dimension 5 must
be the unique irreducible one (see Example 1 (d)). The SO(3)-action on $S^4$ where the connected sum takes place has quotient an interval, so there are exactly two ways to connect the manifolds. In [18], it is shown that depending on the way that $W$ and $B$ are connected we get distinct SO(3)-manifolds.

2 The main example

In this section we present the main class of examples of our classifications and prove some of its properties that will be used to obtain Theorem C. The following construction is crucial since it generates all 5-dimensional $G$-manifolds without singular orbits and most actions with singular orbits for $G = SO(3)$ or SU(2).

Example 8 (Main example). Let $m$, $n$ and $l$ be nonnegative integers, we assume that $m \leq n$ and set $l = 1$ whenever $m = 0$. Consider the $S^1$-action on $SU(2) \times S^3$ given by

$$x \ast (p, (z, w)) = (px^l, (x^mz, x^nw)),$$

where $p \in SU(2)$ and $(z, w) \in S^3 \subset \mathbb{C}^2$. We regard the SU(2) factor as the group of unit quaternions. Take $S^1 = \{e^{i\theta}\}$ to the inclusion of the circle in both $\mathbb{C}$ and $\mathbb{H}$. The quotient

$$\mathcal{N}_{m,n}^l = SU(2) \times_{S^1} S^3$$

is a manifold whenever $\gcd(l, m) = \gcd(l, n) = 1$. It is clear from the sequence of homotopies that the manifold $\mathcal{N}_{m,n}^l$ is simply-connected.

Proposition 3. The manifolds $\mathcal{N}_{m,n}^l$ are diffeomorphic to $S^3 \times S^2$ if $m + n$ is even, otherwise they are diffeomorphic to the nontrivial $S^3$-bundle over $S^2$, denoted by $S^3 \tilde{\times} S^2$.

Proof. The sequence of homotopies of the principal bundle $S^1 \to S^3 \times S^3 \to \mathcal{N}_{m,n}^l$ and Hurewicz isomorphism guarantee that $H_2(\mathcal{N}_{m,n}^l) \simeq \pi_2(\mathcal{N}_{m,n}^l) \simeq \mathbb{Z}$. By [8] p. 77 the second Stiefel-Whitney class $w_2 = 0$ if $m + n$ is even and $w_2 = 1$ otherwise. So, the result follows from Barden-Smale Theorems, c.f. [2] and [27].

Now we define the SU(2)-action on $\mathcal{N}_{m,n}^l$ by

$$g \cdot [(p, (z, w))] = [(gp, (z, w))].$$

We will also denote this SU(2)-manifold by $\mathcal{N}_{m,n}^l$. This action has the same isotropy structure as the $S^1$-action on the second factor $S^3$. If the integers $l$, $m$ and $n$ are nonzero, the isotropy group of the point $[(p, (z, w))]$ is respectively $\mathbb{Z}_m$, $\mathbb{Z}_n$ or $\mathbb{Z}_{\gcd(m,n)}$ according $w = 0$, $z = 0$ or both $z$ and $w$ are nonzero. For convenience we set $\gcd(0,0) = 1$.

In general, for SU(2)-actions, if the principal isotropy group contains $\mathbb{Z}_2$ as a subgroup, the action is ineffective since $\mathbb{Z}_2$ is normal in SU(2). Therefore, $\mathcal{N}_{2m,2n}^l$ becomes an effective SO(3)-manifold with isotropy groups $\mathbb{Z}_m$, $\mathbb{Z}_n$ and $\mathbb{Z}_{\gcd(m,n)}$. Observe that in this case the underlying manifold is diffeomorphic to $S^3 \times S^2$ and $l$ has to be odd.

Remark 4. The $S^1$-action $[(p, (z, w))]$ is a restriction of the SU(2) $\times T^3$-action on $(SU(2) \times S^3)$ given by

$$(g, (r, s, t)) \cdot (p, (z, w)) = (gpr, (sz, tw))$$
where \( r, s, t \in S^1 \). In our example \( \{1\} \times \Delta S^1 = \{(1, (x^1, x^m, x^n))\} \subset SU(2) \times T^3 \) is the \( S^1 \) group that acts on \( SU(2) \times S^3 \). Hence, the quotient \( N^l_{m,n} \) admits an action by \( SU(2) \times (T^3/\Delta S^1) \) with kernel generated by \((-1, [(-1,1,1)])\). Notice that if \( m \) and \( n \) are even then \((1,(-1,1,1)) \in \Delta S^1 \), so the kernel is \( \mathbb{Z}_2 \subset SU(2) \) and \( SO(3) \times T^2 \) acts effectively on \( N^l_{m,n} \). Otherwise, the ineffective kernel is the diagonal \( \mathbb{Z}_2 \) in \( SU(2) \times S^1 \subset SU(2) \times T^2 \), thus \( U(2) \times S^1 \) acts effectively on \( N^l_{m,n} \). This proves Corollary [1] assuming Theorem [C].

As we pointed out, if \( lmn \neq 0 \), the isotropy types of \( N^l_{m,n} \) are \((\mathbb{Z}_m), (\mathbb{Z}_n)\) and \((\mathbb{Z}_{gcd(m,n)})\). Hence, if \( m = n \), the action has only one isotropy type \((\mathbb{Z}_m)\). In particular, we get free actions on \( S^3 \times S^2 \) when \( m \) and \( n \) are both equal to either 1 or 2. The \( SU(2) \)-manifolds \( N^l_{1,1} \) are all equivalent since there is only one isomorphism class of \( SU(2) \)-principal bundles over \( S^2 \) (c.f. Corollary 18.6 in [28]).

The same result shows that there are two non-equivalent \( SO(3) \)-principal bundles over \( S^2 \), but just one of them is simply-connected, \( N^l_{2,2} \), with the free \( SO(3) \)-action. Notice that the free \( SU(2) \)-manifold \( N^l_{0,1} \) corresponds to the left multiplication on the first factor of \( SU(2) \times S^2 \).

If \( m = n = 0 \) then \( l = 1 \) and the \( S^1 \)-action on the first factor reduces to the Hopf action with quotient diffeomorphic to \( S^2 \cong SU(2)/SO(2) \). Thus the \( SU(2) \)-action is the natural product on the cosets and has unique isotropy type equal to \((SO(2))\). On the other hand, if \( m = 0 < n \), then \( l = 1 \) again and the isotropy types are \((\mathbb{Z}_n)\) and \((SO(2))\).

Remark 5. We do not obtain new \( G \)-manifolds by taking negative integer parameters. In fact, if we regard the \( S^1 \)-action on the first factor of \( SU(2) \times S^3 \) considering \( SU(2) \subset \mathbb{C}^2 \) rather than the unit quaternions, for \( x \in S^1 \) and \((u,v) \in SU(2)\) we obtain the action \( x \cdot (u,v) = (ux^j,vx^l) \). Therefore, the \( SU(2) \)-manifolds \( N^l_{m,n} \) and \( N^{-l}_{m,n} \) are equivalent by switching \((u,v)\) to \((v,u)\). In the same way we can consider the \( SU(2) \)-equivariant diffeomorphism \( f : N^l_{m,n} \rightarrow N^{-l}_{m,n} \) which takes \([p,(z,w)]\) to \([p,(\bar{z},w)]\). The equivalence for \( n \) negative is analogous.

Hereafter in this section \( N^l_{m,n} \) will be denoted by \( N^l_{n_1,n_2} \) and its elements are now represented by \([p,(z_1,z_2)]\). Now we will prove some results that will be crucial to obtain Theorem [C]. Proposition [6] and Lemma [7] describe respectively the slice representations in the neighborhoods of the exceptional orbits of \( N^l_{n_1,n_2} \) and the clutching function of the decomposition into two equivariant neighborhoods of the exceptional orbits. The slice representations and the homotopy class of the clutching function arise as invariants to be used in the proof of Theorem [C].

**Proposition 6.** The \( SU(2) \)-manifold \( N^l_{n_1,n_2} \) with \( n_1 \) and \( n_2 \) positive, when written as the union of the slice representations has the form

\[
N^l_{n_1,n_2} = SU(2) \times_{\mathbb{Z}_{n_1}} D^2 \bigcup_{\varphi} SU(2) \times_{\mathbb{Z}_{n_2}} D^2,
\]

where \( \mathbb{Z}_{n_j} \) acts on \( SU(2) \times D^2 \) by \( \xi \cdot (p,z) = (p\xi, \xi^{n_j-1}z) \) and \( \varphi^{-1} \) is the inverse of \( \varphi \) in \( \mathbb{Z}_{n_j} \) for \( 1 \leq i \neq j \leq 2 \).

**Proof.** First consider \( S^3 = B_1 \cup B_2 \) where \( B_j = \{(z_1,z_2) \in S^3 : |z_j| \geq 1/\sqrt{2}\} \) for \( j = 1 \) or \( 2 \), and the identification of the boundaries is the trivial one (the identity map). Note that

\[
N^l_{n_1,n_2} = SU(2) \times_{S^1} B_1 \bigcup_{Id} SU(2) \times_{S^1} B_2.
\]

Now we describe an equivalence between \( SU(2) \times_{S^1} B_j \) and a certain quotient of \( SU(2) \times D^2 \) by \( \mathbb{Z}_{n_j} \). Assume \( j = 2 \), the other case being analogous. For \([p,(z_1,z_2)]\) \( \in SU(2) \times_{S^1} B_2 \) we have \( z_2 \neq 0 \)
therefore we can write \([p, (z_1, z_2)] = [(p, (z_1, \sqrt{z_2/|z_2|})].\) Take \(x = \zeta \eta_2 \in S^1\) with \(\eta_2^{n_2} = 1\) and \(\zeta^{n_2} = \sqrt{z_2/|z_2|}\) with \(\arg(\zeta) < 2\pi/n_2\) in order to obtain \(x^{n_2} = \sqrt{z_2/|z_2|}\). Then

\[
[(p, (z_1, z_2))] = [(p \zeta^{n_2}, (\eta_2^{n_1} \zeta^{n_1} z_1, \sqrt{1 - |\zeta_1|^2})],
\]

where \(\eta_2 \in \mathbb{Z}_{n_2} \subset S^1 \subset \mathbb{C}, \) we called \(\hat{p} = p \zeta\) for each \(p \in SU(2)\) and \(\hat{z}_1 = \zeta^{n_1} z_1.\) So, for some equivariant diffeomorphism \(\varphi: SU(2) \times_{\mathbb{Z}_{n_1}} S^1 \to SU(2) \times_{\mathbb{Z}_{n_2}} S^1\) we have

\[
\mathcal{N}_{n, n_2}^d = SU(2) \times_{\mathbb{Z}_{n_1}} D^2 \bigcup SU(2) \times_{\mathbb{Z}_{n_2}} D^2,
\]

with the actions \(\mathbb{Z}_{n_j} \circ SU(2) \times D^2\) given by \(\eta_j \cdot (p, z) = (pn_j, \eta_j^{n_j} z).\) The result follows from considering the generator \(\xi_j = \eta_j^{n_j}\) of \(\mathbb{Z}_{n_j}.\)

Denote \(\gcd(n_1, n_2) = d\) and \(n_j = dq_j.\) Notice that the principal isotropy group \(\mathbb{Z}_d \subset \mathbb{Z}_{n_j}\) acts trivially on the slice \(D^2,\) so \(SU(2) \times_{\mathbb{Z}_{n_j}} D^2\) is equivalent to \(SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_j}} D^2\) and the equivalence is given by \([(p, z)] \sim [(p\mathbb{Z}_d, z)].\) Therefore the clutching function \(\varphi\) is an equivariant map defined from \(SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} S^1\) to \(SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} S^1.\) It is thus sufficient to compute \(\varphi\) along the path \(t \mapsto [(\mathbb{Z}_d, \mu(t)^{n_2})]\) where

\[
\mu(t) = e^{2\pi i t/dq_1 q_2} \in \mathbb{C} \subset \mathbb{H}.
\]

Whenever necessary to make arguments more clear, we will use \([\cdot, \cdot]\) for the classes in \(G / \mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} D^2.\)

**Lemma 7.** The clutching function \(\varphi: SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} S^1 \to SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} S^1\) is

\[
\varphi([(\mathbb{Z}_d, \mu(t)^{n_2})]) = [(\mu(t)^d \mathbb{Z}_d, \mu(t)^{n_1})].
\]

**Proof.** Consider the notation introduced in the proof of Proposition 6. First, identify the elements \([(\mathbb{Z}_d, \mu(t)^{n_2})] \in SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_1}} S^1\) with \([(1, (1 / \sqrt{2}, \mu(t)^{n_2} / \sqrt{2})]\) that lie in the boundary of \(SU(2) \times_{S^1} B_1.\) Let \(x(t) = \mu(t) \eta_2 \in S^1,\) recall that \(\eta_2^{n_2} = 1,\) and use the equivalence by \(S^1\) to get \([(1, (1 / \sqrt{2}, \mu(t)^{n_2} / \sqrt{2})] = [(\mu(t)^d \eta_2, (\mu(t)^d \eta_2^{n_1} / \sqrt{2}, 1 / \sqrt{2})].\) If we see this last element in \(SU(2) \times_{S^1} B_2,\) then the previous identification yields \([(\mu(t)^d \mathbb{Z}_d, \mu(t)^{n_1})] \in SU(2) / \mathbb{Z}_d \times_{\mathbb{Z}_{q_2}} D^2.\)

Proposition 6 implies that if \(\mathcal{N}_{n_1, n_2}^d\) and \(\mathcal{N}_{n_1, n_2}^{d'}\) are equivalent, then \(d' \equiv \pm l\) mod \(q_1 q_2.\) Indeed, we will see in Theorem 22 that it can be improved to \(d' \equiv \pm l\) mod \(2q_1 q_2\) if \(d = 2\) and for \(d \geq 3,\) the next result guarantees that the parameter \(l \geq 0\) itself is an invariant of the action.

**Proposition 8.** If the principal isotropy group \(H\) of \(\mathcal{N}_{n_1, n_2}^d\) is isomorphic to \(\mathbb{Z}_d\) for \(d \geq 3,\) then the fixed point set \((\mathcal{N}_{n_1, n_2}^d)^H\) is a disjoint union of two copies of lens spaces \(S^3 / \mathbb{Z}_d.\)

**Proof.** Let \(H \simeq \mathbb{Z}_d\) be the subgroup of \(SU(2)\) generated by \(e^{2\pi i / d}\) and notice that \(N(H) = N(S^1),\) where \(H \subset S^1\) and \(N(K)\) is the normalizer of the subgroup \(K \subset G\) in \(G.\) This easily implies that an element \([(p, (z_1, z_2))]\) belongs to \(M^H\) if, and only if, \(p \in N(H).\) Thus \((\mathcal{N}_{n_1, n_2}^d)^H = N(H) \times_{S^1} S^3.\) Therefore

\[
(\mathcal{N}_{n_1, n_2}^d)^H = (S^1 \times_{S^1} S^3) \bigcup (S^1 \times_{S^1} S^3),
\]

since \(N(\mathbb{Z}_d) \simeq \text{Pin}(2).\)
Notice that every $[(y, (z_1, z_2))] \in S^1 \times S^1 S^3$ has a representative with $y = 1$. In fact, $[(y, (z_1, z_2))] = [(1, (\zeta^{n_1} \zeta^{m_1} z_1, \zeta^{n_2} \zeta^{m_2} z_2))]$ where $\zeta^l = y$ with $\arg(\zeta) < 2\pi/l$ and $\zeta^l = 1$. Now define $\hat{z}_1 = \zeta^{n_1} z_1$ and $\hat{z}_2 = \zeta^{m_2} z_2$, as a new parametrization for the $S^3$-sphere. Thus $S^1 \times S^1 S^3$ is diffeomorphic to the quotient of $S^3$ by the $S^1$-action $\xi \cdot (z_1, z_2) = (\zeta^{n_1} z_1, \zeta^{m_2} z_2)$. Hence $S^1 \times S^1 S^3$ is a Lens space $S^3/\mathbb{Z}_l$. \hfill \Box

Remark 9. Clearly Proposition 8 only has an assumption on the principal isotropy group, so the fixed point set $M^H$ when $M$ is either $N^2_{0,0}$ or $N^2_{0,n}$ is the disjoint union of two copies of a 3-sphere.

3 Actions without singular orbits and the proof of Theorem C

For the proof of Theorem C we first verify that the quotient is homeomorphic to a two or three-dimensional sphere. Then we show that, if the action has only one isotropy type, the circle and cyclic groups are the only possible isotropy subgroups of the action and classify the actions with exactly one isotropy type.

For the proof of Theorem C we first verify that the quotient is homeomorphic to a two or three-dimensional sphere. Then we show that, if the action has only one isotropy type, the circle and cyclic groups are the only possible isotropy subgroups of the action and classify the actions with exactly one isotropy type.

3.1 Actions with only one (noncyclic) isotropy type

In this section we show that a simply-connected $G$-manifold with only one orbit type has isotropy either $\mathbb{Z}_m$ or $SO(2)$, classifying the actions in the second case. The classification of the actions with only one isotropy type ($\mathbb{Z}_m$) will be done later together with the classification of $G$-manifolds with exceptional orbits.

For general $G$-actions with only one isotropy type we have the following:
Proposition 11. There are as many equivalence classes of $G$-manifolds with only one isotropy type $(H)$ and orbit space $M/G \simeq \mathbb{S}^n$ as elements in $\pi_{n-1}(\Gamma_H)$.

Proof. It is known (c.f. A. Borel [2]) that given a closed subgroup $H \subset G$, there is a bijective correspondence between the set of isomorphism classes of principal bundles $\Gamma_H \rightarrow P \rightarrow B$, where $\Gamma_H = N(H)/H$, and the set of equivalence classes of $G$-manifolds $M$ with unique orbit type $(G/H)$. The set of isomorphism classes of $F$-bundles over $\mathbb{S}^n$ is in bijection with $\pi_{n-1}(F)$ whenever $F$ is arcwise connected (c.f. [28], Corollary 18.6). Now, the proposition follows from the fact that for a Lie group $K$, the principal $K$-bundle is determined by the $K_o$-fiber bundle over the same basis, where $K_o$ is the identity component of $K$. \hfill \Box

By Proposition [11] the number of $G$-manifolds with only one orbit type equal to $(G/H)$ and quotient an $n$-sphere is the order of the $(n-1)$-th homotopy group of $N(H)/H$, where $N(H)$ is the normalizer of $H$ in $G$. These homotopy groups are presented in Table 1 (see [13] for SO(3) and [11] for SU(2)), for $G = SO(3)$ or SU(2).

$$G = SO(3)$$

| $H$ | \{1\} | $\mathbb{Z}_2$ | $\mathbb{Z}_m$ | $D_2$ | $D_m$ | T | I | O | SO(2) | O(2) |
| N(H) | SO(3) | O(2) | O(2) | O | $D_{2m}$ | O | I | O | O(2) | O(2) |
| N(H)/H | SO(3) | SO(2) | O(2) | $D_3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | \{1\} | \{1\} | $\mathbb{Z}_2$ | \{1\} |
| $\pi_{n-1}(N(H)/H)$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} |

$$G = SU(2)$$

| $H$ | \{1\} | $\mathbb{Z}_2$ | $\mathbb{Z}_m$ | $\text{Dic}_2$ | $\text{Dic}_m$ | $T^*$ | $I^*$ | O$^*$ | SO(2) | Pin(2) |
| N(H) | SU(2) | SU(2) | Pin(2) | O$^*$ | $\text{Dic}_{2m}$ | O$^*$ | $I^*$ | O$^*$ | Pin(2) | Pin(2) |
| N(H)/H | SU(2) | SO(3) | Pin(2) | $D_3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | \{1\} | \{1\} | $\mathbb{Z}_2$ | \{1\} |
| $\pi_{n-1}(N(H)/H)$ | \{1\} | $\mathbb{Z}_2$ | $\mathbb{Z}$ | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} | \{1\} |

Table 1: Here $m \geq 3$ and $n$ is the cohomogeneity of the action, i.e. $n = 2 + \dim H$

The following are the simplest examples.

Example 9. Let $H$ be a Lie subgroup of $G$ and $X$ be a manifold. Define a $G$-action on $(G/H) \times X$ by $g \cdot (kH, x) = (gkH, x)$. This action has unique isotropy type $(H)$ and its orbit space is $X$.

From Table 1 we see that there are two distinct free $G$-manifolds for $G = SO(3)$ but only one if $G = SU(2)$. One of the two free SO(3)-manifolds is SO(3) $\times \mathbb{S}^2$ which is not simply-connected, the other is $N^3_{2,2}$, hence simply-connected. The free SU(2)-action is $N^3_{1,1}$, see Example 2 [4].

The SO(3)-action with unique isotropy type (SO(2)) is $N^3_{0,0}$. It is the linear SO(3)-action in Example 2 [6]. Notice that SU(2) is a rank one Lie group whose center is $\mathbb{Z}_2$, thus all the circle subgroups of SU(2) contain $\mathbb{Z}_2$, hence the SU(2)-action with unique isotropy (SO(2)) also corresponds to $N^3_{0,0}$. Finally, all the G-manifolds with unique isotropy $H$ equal to $D_m$ with $m \geq 2$, $T$, $I$, $O$ or $O(2)$, if $G = SO(3)$ and for $H$ isomorphic to $\text{Dic}_m$ with $m \geq 2$, $T^*$, $I^*$, $O^*$ or Pin(2) if $G = SU(2)$ are described by Example 9 [6]. But none of them is simply-connected since in all these cases the fundamental group of $G/H$ is nontrivial.

For each $m \geq 3$ there are infinitely many examples of G-manifolds with unique isotropy type ($\mathbb{Z}_m$). The same holds for $m = 2$ and $G = SO(3)$, but for $G = SU(2)$ there are exactly
two such actions, they are ineffective and coincide with the free $\text{SO}(3)$-manifolds. We will see in the next section that $\mathcal{N}_{m,m}^l$ are precisely the examples with isotropy $\mathbb{Z}_m$.

### 3.2 Actions with exceptional orbits or unique cyclic isotropy type

In this section we conclude the proof of Theorem C. We will classify the $G$-manifolds with exceptional orbits and actions with only one isotropy type ($\mathbb{Z}_m$). The latter can be seen as a particular case of the former when the principal and exceptional isotropy groups coincide. The condition of simply-connectedness imposes strong restrictions on the isotropies (c.f., Proposition 12) and limits the number of exceptional orbits to two (c.f., Lemma 14). In this situation we can construct $M$ as a union of the neighborhoods of the exceptional orbits using the Slice Theorem.

**Proposition 12.** If the $G$-manifold $M$ has exceptional orbits, then the pair of principal and exceptional isotropy types, $(H, K)$, is either $(\mathbb{Z}_d, \mathbb{Z}_m)$, $(D_2, T)$ or $(\text{Dic}_2, T^*)$. Moreover, exceptional orbits are isolated and $H$ is the kernel of the slice representation of $K$.

In order to prove Proposition 12 the following will be essential.

**Lemma 13.** Let $K \subset G$ be a finite subgroup with a normal subgroup $N \triangleleft K$ such that $K/N \simeq \mathbb{Z}_n$ for some $n \geq 3$. Then the pair $(N, K)$ is either $(\mathbb{Z}_d, \mathbb{Z}_m)$, $(D_2, T)$ or $(\text{Dic}_2, T^*)$.

The proof of the lemma is a case-by-case analysis on the finite subgroups of $\text{SO}(3)$ and $\text{SU}(2)$ and will be omitted. We refer to [2] and [31] Section 7.1 for details about these groups.

*Proof of Proposition 12.* As observed in Lemma 10, the isotropy groups are finite. Let $K \subset G$ be an exceptional isotropy group and consider the slice representation $\rho : K \to O(2)$ of $K$ with $N = \ker(\rho)$. The quotient $\mathbb{R}^2/(K/N) \simeq \mathbb{R}^2$ since $M/G$ is homeomorphic to $S^2$, therefore $K/N \simeq \mathbb{Z}_n \subset \text{SO}(2)$ with $n \geq 3$. So, the pairs $(N, K)$ are determined by Lemma 13.

We claim that $H = N$. This is clear if $K$ is cyclic. If it is not cyclic then $N$ is an index three subgroup of $K$ and since $N \subset H \subset K$, we get $H = N$. The slice action of $K$ on $\mathbb{R}^2$, represented by $\rho$, only fixes the origin, thus the exceptional orbits are isolated. \hfill \Box

The action of the tetrahedral group and the binary tetrahedral group on the linear slice $\mathbb{R}^2$ have kernel $D_2$ and $\text{Dic}_4$ respectively, since in our cases the principal isotropy groups are normal subgroups of $K$. So, the effective actions to be considered on $\mathbb{R}^2$ in these cases are by $\mathbb{Z}_3 \simeq T / D_2 = T^* / \text{Dic}_2$.

**Lemma 14.** There are at most two exceptional orbits.

*Proof.* The exceptional orbits in the quotient $M/G \simeq S^2$ (topologically) represent orbifold singularities, in fact, a neighborhood of an exceptional orbit is parametrized by $\mathbb{R}^2/\mathbb{Z}_m$ with $m \geq 3$. It is known that a 2-dimensional orbifold with underlying space $S^2$ with more than two singularities has nontrivial orbifold fundamental group (c.f. Thurston [29], p. 304). On the other hand, in our situation, there is an onto map from $\pi_1(M)$ to the orbifold fundamental group of the quotient $M/G$ (c.f. Molino [21], p. 273 and 274). Therefore, there are at most two exceptional orbits since $M$ is simply-connected. \hfill \Box

A neighborhood of an exceptional orbit $G/K$ is given by $A = G \times_K D^2$, where $D^2 \subset \mathbb{C}$ is the 2-disk and the linear slice action of $K$ on $D^2$ has kernel equal to the principal isotropy group $H \subset K$ of the $G$-action. The action of $G$ on $A$ is given by $g \cdot [(p, z)] = [(gp, z)]$. The slice action
only fixes the origin in $D^2$, since the exceptional orbits are isolated. The manifold $M$ with at most two exceptional orbits can be written as

$$M = A_1 \bigcup_{\varphi} A_2, \quad A_j = G \times K_j D^2,$$

where $\varphi : G \times K_1 S^1 \to G \times K_2 S^1$ is a $G$-equivariant diffeomorphism. Since $H$ is only acting on the first factor of the product $G \times D^2$, we can write $G \times K_j D^2 = (G/H) \times K_j/H D^2$. Recall that in all our cases $K_j/H$ is a cyclic group, say $\mathbb{Z}_{q_j}$. Since $\varphi$ is $G$-equivariant, it is a bundle map between the fiber bundles

$$G/H \to (G/H) \times K_j/H S^1 \to S^1/\mathbb{Z}_{q_j},$$

for $j = 1, 2$. Also here, $\varphi$ is completely determined by the image of the path $t \mapsto [(H, e^{2\pi it/q_j})]$ in $\partial A_1$ for $0 \leq t \leq 1$. Notice that if we take $t \in [0, 1]$ fixed, the map $\varphi$ becomes a $G$-equivariant diffeomorphism of $G/H$ on itself, so it is identified with an element $\kappa(t)$ of $N(H)/H$, where we can assume that $\kappa(0) = H$. Therefore,

$$\varphi[(H, e^{2\pi it/q_j})] = [(\kappa(t), e^{2\pi it/q_j})].$$

Before applying this construction to the cyclic and noncyclic isotropy types we compute the fundamental groups of $A_j$ and $A_1 \cap A_2 \simeq \partial(A_1)$. Each component $A_j$ deformation retracts to $G/K_j$, so the fundamental group of $A_j$ is isomorphic to $K_j$. Observe that $\mathbb{Z}_{q_j} \to G/H \times S^1 \to G/H \times \mathbb{Z}_{q_j} S^1$ is a principal bundle. So, considering the sequence of homotopies of this bundle and of the bundle in (5) we see that $H$ and $q_j \mathbb{Z}$ are normal subgroups of the fundamental group of $A_1 \cap A_2$ and that $\pi_1(A_1 \cap A_2) \simeq H \rtimes \mathbb{Z}$.

We claim that the elements of $H$ and $\mathbb{Z}$ in $\pi_1(A_1 \cap A_2)$ commute if $\text{gcd}(q_1, q_2) = 1$. In fact, in this case the whole subgroup $\mathbb{Z}$ is normal in the fundamental group, therefore, $\pi_1(A_1 \cap A_2) \simeq H \times \mathbb{Z}$, if $H = \mathbb{Z}_d$.

We show next that the isotropy groups are cyclic.

**Lemma 15.** The 5-dimensional compact simply-connected $G$-manifolds with exceptional orbits only have cyclic isotropy groups.

**Proof.** By Proposition 12 the pair $(H, K)$ is either $(D_2, T)$, $(\text{Dic}_2, T^*)$ or both are cyclic. We have $M = G/H \times_{\mathbb{Z}_3} D^2 \cup_{\varphi} G/H \times_{\mathbb{Z}_3} D^2$ where $\mathbb{Z}_3$ is the quotient $K/H$. We assume $G = \text{SU}(2)$ since $\text{SO}(3)/D_2 = \text{SU}(2)/\text{Dic}_2$. Note that there are exactly two non-equivalent $\mathbb{Z}_3$-actions on $G/H \times D^2$, namely $\xi \cdot (pH, z) = (p\xi H, \xi^{c_j} z)$ for $c_j = 1$ or 2 and $\xi = e^{2\pi i/3}$. Recall that $A_j = G/H \times_{(K_j/N)} D^2$. Moreover, the clutching function $\varphi$ is trivial since the normalizer $N(H)/H$ is discrete (see Table 1).

It is known that $T^* \simeq \text{Dic}_2 \rtimes \mathbb{Z}_3$ where the $\mathbb{Z}_3$ is generated by $w = -1/2(1 + i + j + k)$ and the $\mathbb{Z}_3$-action on $\text{Dic}_2$ is the automorphism that cyclically rotates $i$, $j$ and $k$ (see [7] p. 76). The isomorphism between $\text{Dic}_2 \rtimes \mathbb{Z}_3$ and $T^*$ takes $(x, w)$ to $xw \in T^*$. So, the action of $T^*$ on $\text{SU}(2) \times D^2$, which has quotient $G/H \times_{\mathbb{Z}_3} D^2$ is defined by $(x, w) \cdot (p, z) = (pw^{-1}x^{-1}, w^{c_j} z)$.

Also, the action of $\text{Dic}_2 \rtimes \mathbb{Z}$ on $\text{SU}(2) \times \mathbb{R}$, which has quotient $G/H \times_{\mathbb{Z}_3} S^1$ is given by $(x, a) \cdot (g, s) = (gw^{-a}x^{-1}, s + 2\pi c_j a/3)$. Since $\varphi$ is trivial, the induced maps $i_j^* : \text{Dic}_2 \rtimes \mathbb{Z} \to T^*$ take $(x, a)$ to $(x, w^{ac_j})$ and it is clear that $\pi_1(M) \simeq (T^* \rtimes T^*)/T^*$ is nontrivial when the quotient is provided by the amalgamation property $i_1(x, a) = i_2(x, a)$ for all $(x, a) \in \text{Dic}_2 \rtimes \mathbb{Z}$ and any choice of $c_1$ and $c_2$. \qed
By Lemmas 14 and 15 we only need to consider SU(2)-actions with isotropy types (\(\mathbb{Z}_{n_1}\), \(\mathbb{Z}_{n_2}\)) and (\(\mathbb{Z}_d\)) where \(d \mid \gcd(n_1, n_2)\). To avoid ambiguity assume \(n_1 \leq n_2\). The SO(3)-manifolds will be just the SU(2)-actions with ineffective kernel \(\mathbb{Z}_2\).

Consider \(\mathbb{Z}_{n_1}\) and \(\mathbb{Z}_{n_2}\) as subgroups of the same circle parametrized by \(t \mapsto e^{2\pi it} \in \text{SU}(2)\), using quaternion notation. Let \(n_j = dq_j\) and \(\xi_j = e^{2\pi i/d_j}\) be a generator of \(\mathbb{Z}_{n_j}\). For the isotropy action in a neighborhood of an exceptional orbit it is easy to see the following.

**Lemma 16.** The \(\mathbb{Z}_{n_j}\)-actions on \(\text{SU}(2) \times D^2\) are given by

\[
\xi_j \cdot (p,z) = (p \xi_j^a, \xi_j^d z),
\]

for some \(a_j\) with \(\gcd(a_j, q_j) = 1\) and \(0 \leq a_j < q_j\) for \(j = 1, 2\).

**Remark 17.** The isotropy representation at a point with isotropy \(\mathbb{Z}_{n_j}\) is determined by the number \(a_j\). It is well known that two different representations, \(\rho\) and \(\rho' : \mathbb{Z}_m \to \text{O}(2)\), are equivalent if and only if \(\rho' = \rho\). Notice that they rotate the 2-plane in opposite directions, so \(A_j\) is unchanged if we consider \(q_j - a_j\) instead of \(a_j\) in the isotropy representation. However, the orientation of the slice in \(A_j\) is reversed under this change.

A simply-connected SU(2)-manifold \(M\) with exceptional orbits, or with only one cyclic isotropy type, is determined by the isotropies, the parameters \(a_1\) and \(a_2\), and the clutching function \(\varphi\). The elements \(a_j \in \mathbb{Z}_{q_j}\) have inverse, say \(b_j = a_j^{-1}\), so we write \(M = M(b_1, b_2, \varphi)\).

For \(d \geq 3\) if we consider an orientation on the manifold and on \(G\), it naturally defines an orientation on \(G/\mathbb{Z}_d\) and on the slices through the exceptional orbits. Therefore, by Remark 17 the \(G\)-manifolds \(M(b_1, b_2, \varphi)\) and \(M(q_1 - b_1, b_2, \varphi)\) cannot be equivariantly diffeomorphic, although they have equivalent slice representations.

The proof of the next result is inspired by Theorem 5.1 of [4].

**Proposition 18.** The SU(2)-manifolds \(M(b_1, b_2, \varphi_o)\) and \(M(b_1, b_2, \varphi_1)\) are equivalent if and only if the clutching functions \(\varphi_o\) and \(\varphi_1\) are homotopic.

**Proof.** Call \(M_o = M(b_1, b_2, \varphi_o)\) and \(M_1 = M(b_1, b_2, \varphi_1)\). Let \(H\) be a homotopy between \(\varphi_o\) and \(\varphi_1\). Define \(F : \partial A_1 \times I \to \partial A_2 \times I\) by \(F(p, t) = (H(p, t), t)\) and the G-manifold

\[
N = A_1 \times I \bigcup_F A_2 \times I,
\]

with trivial action on the intervals. This makes \(N\) a cylinder with \(M_o\) on the bottom and \(M_1\) on the top. Observe that \(N/G\) is homeomorphic to \(S^2 \times I\) and that if \(\pi\) is the projection of \(N\) in \(S^2 \times I\), then \(\pi^{-1}(S^2 \times \{0\}) = M_o\), thus the Covering Homotopy Theorem (see, e.g. [4] p. 93) asserts that \(N\) is equivalent to \(M_o \times I\) (the product of \(G\)-manifolds with trivial action on the interval) and, therefore, \(M_o\) and \(M_1\) are equivalent.

Conversely, let \(f : M_o \to M_1\) be a \(G\)-equivariant diffeomorphism. Assume that the manifolds are both written as above using the Slice Theorem and that \(f\) restricted to \(A_1\) is the identity map. Considering \(\xi_j(t) = e^{2\pi it/n_j}\), the clutching functions are

\[
\varphi_i([\mathbb{Z}_d, \xi_1(t)^d]) = [(\kappa_i(t), \xi_2(t)^d)],
\]

for \(i = 0\) or \(1\), as in (6). So, \(f|_{\text{SU}(2) \times \mathbb{Z}_{n_2} S^1}\) is an SU(2)-equivariant diffeomorphism of \(\text{SU}(2) \times \mathbb{Z}_{n_2} S^1\) given by

\[
[(\mathbb{Z}_d, \xi_2(t)^d)] \mapsto [(\kappa_o(t)^{-1}, \kappa_1(t), \xi_2(t)^d)];
\]

(7)
that extends equivariantly to $SU(2) \times_{\mathbb{Z}_{n_2}} D^2$. Since the slice representation is a parametrization of the orbit space this extension must take the slice $[(Z_d, s\xi_2(t)^d)]$ to $[(h_\ast(t), s\xi_2(t)^d)]$, for some $h_\ast(t) \in SU(2)/Z_d$ where $s \in [0, 1]$, $h_1(t) = \kappa_0(t)^{-1}\kappa_1(t)$ and observe that $h_0(t)$ does not depend on $t$ since $[(h_\ast(t), 0)] = f([\mathbb{Z}_d, 0])$. So, the path $\kappa_0, \kappa_1$ on $N(\mathbb{Z}_d)/\mathbb{Z}_d$ is homotopically trivial, and therefore the clutching functions $\varphi_0$ and $\varphi_1$ are homotopic.

For $d = 1$ the clutching function $\varphi$ has only one homotopy class, since $N(H) = SU(2)$, so $M$ can be represented by $M(b_1, b_2)$. We have seen in Remark 17 that the $SU(2)$-manifolds $M(b_1, b_2)$ and $M(b_1', b_2')$ are equivalent if and only if $b_j' = b_j$ or $b_j' = n_j - b_j$ for both $j = 1$ and $2$ simultaneously. The manifold $M(b_1, b_2)$ is equivalent to $\mathcal{N}_{n_1, n_2}$, when $l = b_1n_2 + b_2n_1$ since the isotropy representations coincide as one can see from Proposition 6. Thus Theorem C is proved if $\gcd(n_1, n_2) = 1$.

For $d \geq 2$, we analyze the clutching function $\varphi$ in more detail. The path $\kappa$ defined in (9) must satisfy $\kappa(1) = \xi_1^{b_1} \xi_2^{b_2} \kappa(0)$, since $\varphi$ is $SU(2)$-equivariant. This follows from

\[
[(\xi_1^{b_1} \kappa(0), 1)] = \varphi([\xi_1^{b_1} Z_d, 1]) = \varphi([\xi_2^{b_2}, \kappa(1), 1]) = [(\kappa(1), \xi_2^{b_2})] = [(\xi_2^{b_2} \kappa(1), 1)].
\]

Recall our notation $\xi_j(t) = e^{2\pi i t/n_j}$. By Proposition 18 we can assume that the path $\kappa$ is given by $\kappa(t) = \xi_1(t)^{b_1} \xi_2(t)^{b_2 + kqZ_d}$ with $k \in \mathbb{Z}$. Therefore

\[
\varphi([\xi_1^{b_1} Z_d, \xi_1(t)^d)] = [(\xi_2(b_2 + kqZ_d, \xi_2(t)^d)]
\]

Thus the homotopy class of $\varphi$ is precisely represented by $k$.

Notice that for $\mu(t) = e^{2\pi i t/dq_1q_2} \in SU(2)$ and $l = b_1q_2 + b_2q_1 + kq_1q_2$ the clutching function has the form

\[
\varphi([Z_d, \mu(t)^{n_2}]) = [(\mu(t)^{n_2} Z_d, \mu(t)^{n_1})],
\]

which is the same expression as in Proposition 6 by changing $l$ by $-l$. The sign does not matter for our purposes since $\mathcal{N}_{n_1, n_2} = \mathcal{N}_{n_1, n_2}^{-1}$ as observed in Remark 1.

The map $\varphi$ has two homotopy classes if $d = 2$ and depends on the number $k \in \mathbb{Z}$ if $d \geq 3$. Thus $M$ can be represented by $M(b_1, b_2, \epsilon)$ or $M(b_1, b_2, k)$ respectively, for $\epsilon \in \{0, 1\}$ and $k \in \mathbb{Z}$.

\textbf{Proposition 19.} The fundamental group of $M(b_1, b_2, k)$ is a cyclic group of order $\gcd(n_1, n_2, l)$.

\textit{Proof.} To compute the fundamental group of $M$, we describe the action of $\pi_1(SU(2) \times_{\mathbb{Z}_{n_1}} S^1)$ on the universal covering $SU(2) \times \mathbb{R}$, such that the quotient is $A_1 \cap A_2 = SU(2) \times_{\mathbb{Z}_{n_1}} S^1$. Take curves $\alpha$ and $\beta$ in $A_1 \cap A_2$ that are generators of the fundamental group. For each $j = 1, 2$, we include $A_1 \cap A_2$ in the component $A_j$ by the inclusion $i_j$ and use the $\pi_1(A_j)$-action on the universal covering $SU(2) \times D^2$ of $A_j$ to regard the loops $\alpha$ and $\beta$ as elements of $\pi_1(A_j)$. Van Kampen’s Theorem asserts that the fundamental group of $M$ is the free product $\pi_1(A_1) * \pi_1(A_2)$ with relations $i_{1*}[\alpha] = i_{2*}[\alpha]$ and $i_{1*}[\beta] = i_{2*}[\beta]$, where the maps $i_{1*} : \pi_1(A_1 \cap A_2) \to \pi_1(A_j)$, for $j = 1, 2$ are induced by $i_j$ on the fundamental groups.

The action of the fundamental group $\pi_1(A_1 \cap A_2) \simeq \mathbb{Z}_d \times \mathbb{Z}$ on $SU(2) \times \mathbb{R}$ with quotient $SU(2) \times_{\mathbb{Z}_{n_1}} S^1$ is given by

\[
(\bar{a}, k) \cdot (p, s) = (p\xi_1^{-\frac{aq_1}{n_1} + k}, s + 2\pi k/q_1),
\]

where $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$ and $a_1b_1 + u_1q_1 = 1$. Observe that the quotient $(SU(2) \times \mathbb{R})/(\mathbb{Z}_d \times \mathbb{Z})$ is exactly the orbit space $(SU(2) \times S^1)/\mathbb{Z}_{n_1}$. Indeed, since $\gcd(b_1, q_1) = 1$, for any $0 \leq r < n_1$ there
are integers \( u \) and \( k \) with \( 0 \leq u < d \) such that \( r = uq_1 + kb_1 \). So, the \( \mathbb{Z}_{n_1} \)-action can be written as

\[
\xi_1^u \cdot (p, e^{is}) = (p\xi_1^{-(uq_1 + kb_1)}, \exp(s + 2\pi u a_1 + 2\pi k a_1 b_1/q_1)) = (p\xi_1^{-(uq_1 + kb_1)}, \exp(s + 2\pi k/q_1)),
\]

that clearly defines the same quotient as \((\text{SU}(2) \times \mathbb{R})/(\mathbb{Z}_d \times \mathbb{Z})\).

It is convenient to define the loops in \( \text{SU}(2)/\mathbb{Z}_d \times \mathbb{Z} \) that generate its fundamental group, \( \mathbb{Z}_d \times \mathbb{Z} \). The loop \( \alpha : [0, 1] \to A_1 \cap A_2 \) defined by \( \alpha(t) = [(\xi_1(t) - nT, 1)] \) corresponds to \((\overline{1}, 0)\) in \( \mathbb{Z}_d \times \mathbb{Z} \). In fact, let \( \tilde{\alpha}(t) = (\xi_1(t)^{-q_1}, 0) \) be a lifting of \( \alpha \) by \((1, 0) \in \text{SU}(2) \times \mathbb{R} \). So, \( \tilde{\alpha}(1) = (\xi_1^{-q_1}, 0) = (\overline{1}, 0)\mathbb{Z}_d \times \mathbb{Z} \cdot \tilde{\alpha}(0) \).

On the other hand, the loop \( \beta : [0, 1] \to A_1 \cap A_2 \) defined by \( \beta(t) = [(\xi_1(t)^{-b_1}, \mathbb{Z}_d, \xi_1(t)^{-d})] \) corresponds to \((\overline{1}, 1) \in \mathbb{Z}_d \times \mathbb{Z} \). In fact, consider \( \tilde{\beta}(t) = (\xi_1(t)^{-b_1}, 2\pi t/q_1) \), a lifting of \( \beta \) to \( \text{SU}(2) \times \mathbb{R} \) by \( \tilde{\beta}(0) = (1, 0) \). Then \( \tilde{\beta}(1) = (\xi_1^{-b_1}, 2\pi/q_1) = (\overline{1}, 1)\mathbb{Z}_d \times \mathbb{Z} \cdot \beta(0) \). So, \( \beta \) corresponds to \((\overline{1}, 1) \in \mathbb{Z}_d \times \mathbb{Z} \).

If \( \varepsilon_j \) is a generator of \( \mathbb{Z}_n \) in the free product \( \mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} \), the induced loop \((i_1 \circ \alpha)(t)\) corresponds to \( e_1^{i_1} \). The loop \((i_1 \circ \beta) \) in \( A_1 \) corresponds to \( e_1^{i_1} \in \mathbb{Z}_{n_1} \). In fact, the lifting of \( i_1 \circ \beta \) by the point \((1, 1) \) of \( \text{SU}(2) \times \mathbb{D} \) is the curve \( t \mapsto (\xi_1(t)^{-b_1}, \xi_1(t)^{-d}) \).

That is, \( i_{1*} : \mathbb{Z}_d \times \mathbb{Z} \to \mathbb{Z}_{n_1} \) takes \((\alpha^u, \beta^v)\) to \( e_1^{uq_1 + nb^v} \).

We need to include the loops \( \alpha \) and \( \beta \) in \( A_2 \). To do this we simply use the composition, \( i_2 \), of the clutching function \( \varphi \) with the inclusion of \( \text{SU}(2)/\mathbb{Z}_d \times \mathbb{Z} \mathbb{Z} \times \mathbb{S}^1 \) in \( A_2 \). The induced loop \((i_2 \circ \alpha)(t) = [[\xi_2(t)^{-q_2} \mathbb{Z}_d, 1]] \) has lifting by \((1, 1) \in \text{SU}(2) \times \mathbb{D} \), with end point \( (\xi_1^{-q_2}, 1) = \xi_2^{q_2} \cdot (1, 1) \). So, \( (i_2 \circ \alpha) \) corresponds to \( e_2^{q_2} \in \mathbb{Z}_{n_2} \). The loop \((i_2 \circ \beta)(t) = [[\xi_2(t)^{-b_2 + kq_2} \mathbb{Z}_d, \xi_2(t)^{-d}]] \) has lifting \( t \mapsto (\xi_2(t)^{-b_2 + kq_2}, \xi_2(t)^{-d}) \) that starts at \((1, 1) \in \text{SU}(2) \times \mathbb{D} \) and that ends at \( (\xi_2^{-b_2 - kq_2}, 1) \), so the loop corresponds to \( e_2^{-(b_2 + kq_2)} \in \mathbb{Z}_{n_2} \). Therefore the map \( i_{2*} \) takes \((a^u, \beta^v)\) to \( e_2^{u-kq_2 - nb_2} \).

We conclude that the fundamental group of \( M \) is generated by \( e_1 \) and \( e_2 \) with the relations \( e_1^{q_1} = e_2^{q_2} = e_1^{2} = e_2^2 \) and \( e_1^{b_1} = e_2^{-b_2 - kq_2} \). From these three identities and \( \gcd(b_j, q_j) = 1 \) we see that \( \pi_1(M) \simeq \mathbb{Z}_{\gcd(n_1, n_2, b_1 q_2 + b_2 q_1 + k q_2 q_1)} \).

Remark 20. As a consequence of Proposition 19 \( d = \gcd(n_1, n_2) \) when the \( \text{SU}(2) \)-manifold is simply-connected. In fact, \( \gcd(n_1, n_2, l) = 1 \) and \( \gcd(b_j, q_j) = 1 \) imply that \( \gcd(q_j, l) = 1 \), therefore \( \gcd(q_1, q_2) = 1 \). So, the \( \text{SU}(2) \)-action depends on a triple of integer parameters that belongs to \( \mathcal{P} = \{(b_1, b_2, k) \in \mathbb{Z}^3 : 0 \leq b_j < q_j, (b_j, q_j) = 1, j = 1, 2\} \).

The proof of the following result is a simple computation, then it will be omitted.

**Proposition 21.** The map \( l(b_1, b_2, k) = b_1 q_2 + b_2 q_1 + k q_1 q_2 \) is a bijection between the sets \( \mathcal{P} = \{(b_1, b_2, k) \in \mathbb{Z}^3 : 0 \leq b_j < q_j, (b_j, q_j) = 1, j = 1, 2\} \) and \( Q = \{l \in \mathbb{Z} : (l, q_j) = 1, j = 1, 2\} \).

We will show now that the \( G \)-manifold \( M(b_1, b_2, k) \) is equivariantly diffeomorphic to \( N_{n_1, n_2}^l \) where \( l = b_1 q_2 + b_2 q_1 + k q_1 q_2 \). For the next result recall that we have defined \( \gcd(0, 0) = 1 \).

**Theorem 22.** Let \( n_1 \leq n_2 \) be positive integers with \( d = \gcd(n_1, n_2) \geq 2 \), set \( q_j = n_j/d \) and take \( b_j \in \mathbb{Z} \) coprime with \( q_j \) satisfying \( 0 \leq b_j < q_j \), for \( j = 1, 2 \). Let \( k \) be an integer for \( d \geq 3 \) or \( k \in \mathbb{Z}_2 \) for \( d = 2 \). Then, the \( \text{SU}(2) \)-manifold \( M(b_1, b_2, k) \) is equivariantly diffeomorphic to \( N_{n_1, n_2}^l \), where \( l = b_1 q_2 + b_2 q_1 + k q_1 q_2 \). Moreover, these \( \text{SU}(2) \)-manifolds are pairwise nonequivalent except for \( M(b_1, b_2, k) = M(q_1 - b_1, q_2 - b_2, -k - 2) \).
Proof. It is clear that $M(b_1, b_2, k)$ is determined by the isotropy representations around the exceptional orbits and the homotopy class of the clutching function $\varphi$. So, $M(b_1, b_2, k) = \mathcal{N}_{n_1, n_2}^l$ with $l = b_1 q_2 + b_2 q_1 + k q_1 q_2$ since by Proposition 6 they coincide in both representations and also have the same clutching function, up to homotopy. Moreover, Propositions 19 and 21 show that each $l$ is reached exactly once by that formula.

For $d = 2$ we know that $k \in \{0, 1\}$ since the homotopy class of the clutching function is defined modulo 2. We use the identity $M(b_1, b_2, k) = \mathcal{N}_{n_1, n_2}^l$ and $\mathcal{N}_{n_1, n_2}^{-l} = \mathcal{N}_{n_1, n_2}^l$ (see, Remark 5) to conclude that $M(b_1, b_2, k) = M(q_1 - b_1, q_2 - b_2, -k - 2) = M(q_1 - b_1, q_2 - b_2, k)$ for $k = 0$ or 1. Remark 17 shows that otherwise these SU(2)-manifolds are pairwise distinct.

For $d \geq 3$ two $G$-manifolds $M(b_1, b_2, k)$ are equivalent if and only if they have the same number $l = b_1 q_2 + b_2 q_1 + k q_1 q_2$, see Proposition 8. This and Remark 17 imply that $M(b_1, b_2, k)$ is equivalent to $M(b'_1, b'_2, k')$ if and only if the parameters are exactly the same, or $b'_j = q_j - b_j$ and $k' = -k - 2$. This corresponds to replace $l$ by $-l$ in $\mathcal{N}_{n_1, n_2}^l$.

Observe that it is a consequence of the discussion in the proof of Theorem 22 that the manifolds $\mathcal{N}_{n_1, n_2}^l$ and $\mathcal{N}_{n_1, n_2}^l$ are equivalent if, and only if, $l \equiv l' \mod 2q_1 q_2$ for $d = 2$. This concludes the proof of Theorem 22.

4 Actions with singular orbits and the proof of Theorem D

For simply-connected compact $G$-manifolds of dimension 5 with singular orbits the number of orbit types cannot exceed 3, c.f., Lemma 23 below. In Section 4.1 we use a classical result by Bredon [1], Hsiang and Hsiang [16] and Janich [19] to classify the actions with exactly two orbit types. In Section 4.2 we make a few comments about Hudson’s work [18] on SO(3)-manifolds with three orbit types.

The classification of SO(3)-actions on simply-connected 5-manifolds with singular orbits and cohomogeneity 2 was carried out by Hudson in [18]. The actions with cohomogeneity 3 were discussed in the same paper, but the SO(3)-manifolds in Example 6 were overlooked. For the sake of completeness we classify again the SO(3)-manifolds without fixed points since it can be obtained together with the SU(2) case.

The following lemma gives strong restrictions to the possible chains of isotropy subgroups. It is inspired by Lemma 1A in [18].

Lemma 23. Let $M$ be a 5-dimensional simply-connected $G$-manifold with singular orbits.

(a) If the action has exactly two orbit types, say $(H) \leq (K)$, then the pair of principal and singular isotropy groups $(H, K)$ is $(\mathbb{Z}_m, \text{SO}(2))$, $(\mathbb{D}_m, \text{O}(2))$ or $(\text{SO}(2), \text{SO}(3))$ if $G = \text{SO}(3)$ and $(\mathbb{Z}_m, \text{SO}(2))$ or $(\{1\}, \text{SU}(2))$, if $G = \text{SU}(2)$, for $m \geq 1$;

(b) If the action has three orbit types then the isotropy types are $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \text{O}(2) \subset \text{SO}(3)$. There is no SU(2)-action with three isotropy types;

(c) Neither SO(3) nor SU(2) acts on $M$ with more than three orbit types.

Sketch of the proof. In general, for a singular point $p \in M^n$, if $K = G_p$ and $k$ is the codimension of $G(p)$, it is known that the slice action $K \odot \mathbb{R}^k$, of the isotropy group $K$ on the tangent space of a slice at $p$, is a linear action which has the same isotropy structure as the action $G \odot M$ in a
neighborhood of \( p \). So the chain of isotropy types \((H) \leq \cdots \leq (K)\) is possible for a \( G \)-action on \( M \) only if there is a representation \( \rho : K \to O(k) \) such that the action \( \rho(K) \cap \mathbb{R}^k \) has the same chain of isotropy types. Now, the proof of [a] and [b] follow of a case-by-case analysis in the subgroups of \( \text{SO}(3) \) and \( \text{SU}(2) \) and its representations in \( \text{O}(3) \).

The subgroups of \( G \) have dimensions either zero, one or three. If there are three isotropy types and \((G)\) is not one of them, two of them have the same dimension. Since there are no exceptional orbits, there are two one-dimensional isotropy types, say \((K)\) and \((K')\). Let \( M_{(K)} \) and \( M_{(K')} \) be the set of points in \( M \) whose isotropy groups belong to \((K)\) and \((K')\), respectively. Both sets \( M_{(K)} \) and \( M_{(K')} \) are submanifolds of \( M \) and are projected to the boundary of the disk in the quotient. So there is a point \( p \in M \) such that in any neighborhood of \( p \) there are orbits of type \((G/K)\) and \((G/K')\), but there is no representation neither of \( O(2) \) nor \( \text{Pin}(2) \) on \( O(3) \) with isotropy \( \text{SO}(2) \).

This shows that if the action has more than two distinct orbit types, then there is a fixed point. It also shows that there is no 5-dimensional \( G \)-action with more than three orbit types and the lemma follows.

### 4.1 Actions with singular orbits and two orbit types

The Second Classification Theorem in [4, p. 257 and p. 331] shows, in particular, that given a contractible topological manifold \( X \) with non-empty boundary and two subgroups \( H \subset K \) of a group \( G \) with \( \dim H < \dim K \), the set of classes of \( G \)-manifolds with orbit space \( X \) and isotropy types \((H)\) and \((K)\) is in one-to-one correspondence with the quotient

\[
\pi_{n-1} \left( N(H)/(N(H) \cap N(K)) \right) / \pi_o (N(H)/H). \tag{10}
\]

For our purposes an explicit expression for the \( \pi_o(N(H)/H) \)-action above is not needed since the groups involved in (10) are quite simple, see Table 2.

| H  | K  | G  | \( \pi_{n-1} \left( N(H)/(N(H) \cap N(K)) \right) \) | \( \pi_o (N(H)/H) \) | \( l \) |
|----|----|----|---------------------------------|-------------------|-----|
| (a) | \{1\} | SO(2) | SO(3) | \( \pi_1(\mathbb{RP}^2) \) | 1 | 2 |
| (b) | \{1\} | SO(2) | SU(2) | \( \pi_1(\mathbb{RP}^2) \) | 1 | 2 |
| (c) | \{1\} | SU(2) | SU(2) | \{0\} | 1 | 1 |
| (d) | \mathbb{Z}_2 | SO(2) | SO(3) | \{0\} | 1 | 1 |
| (e) | \mathbb{Z}_2 | SO(2) | SU(2) | \{0\} | 1 | \mathbb{Z} |
| (f) | \mathbb{Z}_m | SO(2) | SO(3) | \{0\} | 1 | 1 |
| (g) | \mathbb{Z}_m | SO(2) | SU(2) | \{0\} | 1 | \mathbb{Z} |
| (h) | \mathbb{Z}_2 | O(2) | SO(3) | \( \pi_1(\mathbb{S}^1) \) | \mathbb{Z}_2 | 1 |
| (i) | D_m | O(2) | SO(3) | \{0\} | \mathbb{Z}_2 | 1 |
| (j) | SO(2) | SO(3) | SO(3) | \{0\} | 1 | 1 |

Table 2: \( l \) is an upper bound for the number of actions with 2 isotropy types \( H \subset K \)

Due to Lemma 23 and the Second Classification Theorem we obtain an upper bound \( l \) (see Table 2) for the number of \( G \)-manifolds with exactly two orbit types and quotient a 2 or 3-disk, hence the classification of \( G \)-actions with contractible orbit space is complete by showing that we have as many examples as possible. The examples below represent the corresponding enumeration in Table 2.
Examples.

(a) The two $\text{SO}(3)$-actions with isotropies $\{1\}$ and $\text{SO}(2)$. They are linear actions on $\mathbb{S}^5$ and $\mathbb{S}^3 \times \mathbb{S}^2$ described in Examples 1(b) and 2(c), the last one is $\mathcal{N}_{1,0}^1$.

(b) The two $\text{SU}(2)$-actions with isotropies $\{1\}$ and $\text{SO}(2)$. They are $\mathcal{N}_{1,0}^1 = \mathbb{S}^3 \times \mathbb{S}^2$ and the action on $\mathcal{W} = \text{SU}(3)/\text{SO}(3)$ given by $B \cdot [C] = [\text{diag}(B,1)C]$, described in Example 3.

(c) The linear $\text{SU}(2)$-action on $\mathbb{S}^5$ with isotropies $\{1\}$ and $\text{SU}(2)$. See Example 1(c).

(d), (e), (f) and (g) The $G$-action with isotropies $\mathbb{Z}_m$ and $\text{SO}(2)$ with $m \geq 2$. They are $\mathcal{N}_{1,0}^1$. Recall that $\text{SU}(2)$-actions with $H = \mathbb{Z}_{2m}$ are effective $\text{SO}(3)$-actions with principal isotropy $\mathbb{Z}_m$ (see Section 2).

(h) and (i) The $\text{SO}(3)$-actions with isotropy types $\mathbb{Z}_2$ and $\text{O}(2)$ or $\text{D}_m$ and $\text{O}(2)$. Remark 3C in [18] asserts that these examples are not simply-connected for any $m$.

(j) The linear $\text{SO}(3)$-action on $\mathbb{S}^5$ with isotropy types $\text{SO}(2)$ and $\text{SO}(3)$. See Example 1(a).

If the orbit space is three dimensional then the quotient is a topological 3-manifold with boundary, c.f. Proposition 2(c). In this case, the isotropy groups are $\text{SO}(2)$ and $\text{SO}(3)$ (see Lemma 23). In a neighborhood of a fixed point, the $\text{SO}(3)$ slice action on $\mathbb{R}^5$ is the one with two fixed directions, thus $M/G$ is smooth and has non-empty boundary. When $K = G$, the Second Classification Theorem guarantees that even for a non-contractible topological manifold $X$, there is only one class of $G$-manifolds with isotropy types $(H)$ and $(G)$, and orbit space $X$. Finally, the next proposition completes the possible orbit spaces for a cohomogeneity 3 action.

**Proposition 24.** A simply-connected compact 3-dimensional manifold with boundary is diffeomorphic to a 3-sphere with $k$ open 3-disks removed.

**Proof.** Let $X$ be a simply-connected compact 3-manifold with boundary. We claim that the boundary of $X$ is a disjoint union of 2-spheres. In fact, Poincaré duality to the pair $(X, \partial X)$ guarantees that $H_2(X, \partial X) \simeq H^1(X)$, so from the exactness of the relative homology sequence,

$$\cdots \to H_2(X, \partial X) \to H_1(\partial X) \to H_1(X) \to \cdots$$

we conclude that $H_1(\partial X) = 0$. So each connected component of $\partial X$ is homeomorphic to $\mathbb{S}^2$ by the classification of compact surfaces.

The manifold obtained from $X$ by covering each connected component with a 3-disk is simply-connected compact without boundary, so it is a 3-sphere and the proposition is proved.

Summarizing the discussion above we have the following.

**Proposition 25.** There is precisely one $\text{SO}(3)$-action (up to equivalence) with isotropy types $(\text{SO}(2))$ and $\text{SO}(3)$ with quotient a 3-sphere with $k$ three-disks removed, $k > 0$.

These actions are precisely those described in Example 6. This concludes the classification of actions with singular orbits and two orbit types. The remaining part of the proof of Theorem D follows from [18] since only $\text{SO}(3)$-manifolds have three isotropy types.
4.2 Actions with three orbit types

By Lemma 23, the \( G \)-actions with three orbit types have isotropy groups \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( O(2) \) and \( SO(3) \), thus \( G = SO(3) \). So, all these actions have fixed points and the isotropy representation around a fixed point is the irreducible \( SO(3) \)-action on \( \mathbb{R}^5 \). From this representation it is easy to conclude that the fixed points are isolated and there are at least two fixed points. Hudson [18] showed that if there are exactly two fixed points, it is the linear action on \( S^5 \) described in Example 1 (d). If it has three fixed points then the action is equivalent to left multiplication on the cosets of the Wu-manifold \( W = SU(3)/SO(3) \) (see Example 4). Moreover, there are two \( SO(3) \)-manifolds with exactly four fixed points: the Brieskorn variety, \( B \) of type \( (2, 3, 3, 3) \), and the connected sum of two \( W \)-manifolds with the action above. All other examples of \( SO(3) \)-manifolds with three orbit types have more than four fixed points. In [18] it is proved that the \( SO(3) \)-manifolds with more than two isolated fixed points are connected sums of copies of \( B \) and \( W \) (see Example 7).

5 Five-manifolds with nonnegative curvature

In this section we prove Theorems A and B. Theorem B is a consequence of Theorems C and D, by using Frankel’s Lemma [9] and the classification of the \( G \)-manifolds \( M \) with fixed point set with codimension one or two in \( M/G \) (c.f., [13] and [14]). In our context, the following lemma provides a more elementary proof.

**Lemma 26.** Let \( M \) be a simply-connected compact \( SO(3) \)-manifold of dimension 5. If \( M \) admits an invariant metric of nonnegative (resp. positive) curvature, then the number of isolated fixed points cannot exceed 3 (resp. 2).

**Proof.** If the action has isolated fixed points, then the quotient is a topological 2-disk since the isotropy action in a neighborhood of an isolated fixed point is equivalent to the irreducible \( SO(3) \)-representation in \( SO(5) \), and thus, the isotropy types are \( (\mathbb{Z}_2 \times \mathbb{Z}_2) \), \( (O(2)) \) and \( SO(3) \). The orbit strata of a group action is well known to be totally geodesic (see e.g. [12]). Hence the boundary of \( M/\text{SO}(3) \) consists of a geodesic polygon with \( n \) edges and \( n \) vertices. The edges correspond to the singular isotropies \( O(2) \) and vertices are the fixed points. From the isotropy representation it follows that the angles between the edges are all equal to \( \pi/3 \) (see Example 1 (d)).

By O’Neill’s formula the (interior of the) quotient inherits a metric of nonnegative (resp. positive) curvature if \( M \) has an invariant metric of nonnegative (resp. positive) curvature. Thus, by the Gauss-Bonnet Theorem, the sum of inner angles of the \( n \)-polygon \( M/\text{SO}(3) \) is equal to, or bigger (resp. strictly bigger) than \( \pi (n - 2) \). So, \( n = 2 \) or 3 if the curvature is nonnegative and \( n = 2 \) if the curvature is positive. \( \square \)

**Proof of Theorem A** By Theorem D and Section 4.2 the \( G \)-manifolds with more than 3 isolated fixed points are \( kW \# lB \) with \( (k, l) \neq (1, 0) \) (since \( SO(3) \circ \mathcal{W} \) has exactly 3 fixed points). So, by Lemma 26 they do not admit invariant metrics with nonnegative curvature.

The connected sum \( M \) of \( k \) copies of \( S^3 \times S^2 \) (see Example 4) has quotient \( X \) diffeomorphic to \( S^3 \) with \( k + 2 \) three-disks removed. If \( M \) admits a metric of nonnegative curvature, then \( X \) with the induced metric also has nonnegative curvature. As follows from the proof of the Soul Theorem [6], a compact nonnegatively curved manifold \( X \) with non-empty convex boundary contains a totally geodesic compact submanifold \( \Sigma \) without boundary and \( \Sigma \) is a deformation retract of \( X \). In our case \( \dim \Sigma \neq 0 \) since \( X \) is not a disk. Also, \( \dim \Sigma \neq 1 \) since \( X \) is simply-connected. Thus \( \Sigma \) is
a simply-connected surface and a neighborhood of \( \Sigma \) is diffeomorphic to \( S^2 \times (-1,1) \). Using the flow of the gradient like vector field in the proof of the Soul Theorem it follows that \( \partial X \) has two connected components. Therefore, \( k = 0 \) and only \( S^3 \times S^2 \) with the linear \( SO(3) \)-action on the first factor admits an invariant metric of nonnegative curvature.

On the other hand, all the other actions in Theorems \([\text{C}]\) and \([\text{D}]\) i.e., the linear actions on \( S^5 \) and \( S^3 \times S^2 \), the \( SO(3) \) or \( SU(2) \) left multiplication on the cosets in \( W \), and \( N_{m,n}^l \) clearly admit an invariant metric of nonnegative curvature.

**Proof of Theorem \([\text{B}]\)**. We restrict ourselves to the actions in Theorem \([\text{A}]\). By Theorems \([\text{C}]\) and \([\text{D}]\) the \( G \)-manifolds diffeomorphic to \( S^3 \times S^2 \) are \( N_{m,n}^l \) and the \( SO(3) \)-manifold in Example \([2]\). This last one has quotient \( X \) diffeomorphic to a 3-sphere with two 3-disks removed, thus its soul is homeomorphic to a 2-sphere and by the Soul Theorem \( X \) cannot be positively curved. So neither \( S^3 \times S^2 \) admits an invariant metric of positive curvature. Also, the \( SO(3) \)-action on \( W \) has three fixed points (see Example \([4]\), and therefore does not admit an invariant metric of positive curvature by Lemma \([26]\).

We finally observe that \( N_{m,n}^l \) with \( \gcd(m,n) \geq 3 \) does not admit a metric of positive curvature. In fact, by Proposition \([3]\) the fixed point set of the principal isotropy group \( \mathbb{Z}_{\gcd(m,n)} \) has two connected components of dimension three and by Frankel’s Lemma \([9]\) in a positively curved manifold \( M^n \) the sum of the dimensions of two totally geodesic submanifolds cannot exceed \( n \). It is also clear that the fixed point set of the \( SO(3) \)-manifold \( N_{1,0}^l \) is the disjoint union of two 3-spheres, thus also cannot admit metric of positive curvature.

**Remark 27.** Using standard arguments of equivariancy it can be shown that for fixed \( m \) and \( n \) with \( \gcd(m,n) = 1 \) or 2, only 3 of the manifolds \( N_{m,n}^l \) are candidates to admit positive curvature.

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