Ising model on directed small-world Voronoi Delaunay random lattices

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Abstract. We investigate the critical properties of the Ising model in two dimensions on directed small-world lattice with quenched connectivity disorder. The disordered system is simulated by applying the Monte Carlo update heat bath algorithm. We calculate the critical temperature, as well as the critical exponents $\gamma/\nu$, $\beta/\nu$, and $1/\nu$ for several values of the rewiring probability $p$. We find that this disorder system does not belong to the same universality class as the regular two-dimensional ferromagnetic model. The Ising model on directed small-world lattices presents in fact a second-order phase transition with new critical exponents which do not depend on $p$ ($0 < p < 1$), but are identical to the exponents of the Ising model and the spin-1 Blume-Capel model on directed small-world network.

1 Introduction

Experimental studies of real magnetic materials show that their critical behavior can suffer the influence of either impurities or inhomogeneities [1]. A theoretical understanding of such impurities can be realized, with a very good approximation, in the case of quenched disorder. In this case, the criterion due to Harris [2] is an important theoretical tool to interpret the importance of the effect of quenched random disorder on the critical behavior of a physical system. The effect of randomness can be classified solely by the specific heat exponent of the pure system, $\alpha_{\text{pure}}$. This criterion asserts that for $\alpha_{\text{pure}} > 0$ the quenched random disorder is a relevant perturbation, leading to a different critical behavior than in the pure case (as for the three-dimensional Ising model). In particular, one expects [3] in the disordered system that $\nu \geq 2/d$, where $\nu$ is the correlation length exponent and $d$ is the dimension of the system. Assuming hyper-scaling to be valid, this implies $\alpha = 2 - d\nu \leq 0$. On the other hand, for $\alpha_{\text{pure}} < 0$ disorder is irrelevant (as, for instance, in the three-dimensional Heisenberg model) and, in the marginal case $\alpha_{\text{pure}} = 0$ (like the $d=2$ Ising model), no prediction can be made. For the case of (non-critical) first-order phase transitions it is known that the influence of quenched random disorder can lead to a softening of the transition [4]. Recently, the predicted softening effect at first-order phase transitions has been confirmed for $d = 3$ $q$-state Potts models with $q \geq 3$ using Monte Carlo [5–7] and high-temperature series expansion [8] techniques. The overall picture is even better in two dimensions ($d=2$) where several models with $\alpha_{\text{pure}} > 0$ [9–12] and the marginal ($\alpha_{\text{pure}} = 0$) [13–17] have been investigated.

In this paper we study a type of different quenched disorder, namely the effect of directed bond case with rewiring probability $p$ [18]. Specifically, we consider $d = 2$ small-world Voronoi Delaunay random lattices (SWVD) type, and performed an extensive computer simulations of the Ising model. We concentrated on the close vicinity of the transition point and applied finite-size scaling (FSS) techniques to extract the exponents and the “renormalized charges” $U^*_4$. Monte Carlo simulations of the disorder system was realized using the spin-flip heat bath algorithm to update the spins. Previous studies of connectivity disorder on $d = 2$ lattices have been realized by Monte Carlo simulations of $q$-state Potts models on quenched random lattices of Voronoi Delaunay type for $q = 2$ [19–21], $q = 3$ [22] and $q = 8$ [23,24]. In particular, it has been shown that for $q = 2$ [19–21] and $q = 3$ [22] the critical exponents are the same as those for the model on a regular $d = 2$ lattice. This is indeed a surprising result since the relevant criterion of the Delaunay triangulations reduces to the well-known Harris criterion such that disorder of this type should be relevant for any model with positive specific heat exponent [25]. This means that for $q = 3$ on $d = 2$ lattices, where $\alpha_{\text{pure}} > 0$, one would expect a universality class different from the non-random case. For the spin-1 Ising model, where
configuration of spins. We ran 4 × 10\(^5\) simulations for every outgoing link, we are left with a network with a density \(p = p_c\) nearest neighbors by both outgoing and incoming links. Then, with probability \(p\), we replace nearest-neighbor outgoing links by outgoing links to different sites chosen at random. After repeating this process for every outgoing link, we are left with a network with a density \(p\) of SWVD directed links. Therefore, with this procedure every site will have the old number of outgoing links and a varying (random) number of incoming links.

In the present spin-1/2 Ising model on directed SWVD lattice we will show that the critical behavior is quite similar to that observed by Fernandes et al. in the spin-1 case [26]. However, now one has only a second-order phase transition for all \(p\) values studied. The critical exponents do not belong to the same universality class as the regular two-dimensional ferromagnetic model, but they agree with the critical exponents of Blume-Capel model on a Voronoi-Delaunay lattice for \(p < p_c\). In the next section we present the model and the simulation. The results and conclusions are discussed in the last section.

2 Model and simulation

We consider the ferromagnetic spin-1/2 Ising model, on directed SWVD random lattice by a set of spins variables \(S_i\) taking the values ±1 situated on every site of a directed SWVD random lattice with \(N = L \times L\) sites, were \(L\) is the side of square cluster. In this random lattice, similar to Sánchez et al. [18], we start from a two-dimensional Voronoi-Delaunay random lattice consisting of sites linked to their \(c\) (where \(3 < c < 20\) and different for each site of network) nearest neighbors by both outgoing and incoming links. Then, with probability \(p\), we replace nearest-neighbor outgoing links by outgoing links to different sites chosen at random. After repeating this process for every outgoing link, we are left with a network with a density \(p\) of SWVD directed links. Therefore, with this procedure every site will have the old number of outgoing links and a varying (random) number of incoming links.

The evolution in time of these systems is given by a single spin-flip like heat bath dynamics with a probability \(P_i\) given by

\[
P_i = 1/[1 + \exp(2E_i/k_BT)],
\]

where \(T\) is the temperature, \(k_B\) is the Boltzmann constant, and \(E_i\) is the energy of the configuration obtained from the Hamiltonian

\[
H = -J \sum_{\langle i,j \rangle} S_i S_j,
\]

where the sum runs over all neighbor pairs of sites (including the nearest-neighbor and the long-ranged ones determined by the probability \(p\)) and the spin-1/2 variables \(S_i\) assume values ±1. In the above equation \(J\) is the exchange coupling. The spin-1/2 case on square lattices is well known in the literature [27–29].

The simulations have been performed on different SWVD random lattice sizes comprising a number \(N = 5000, 10000, 20000, 40000, 60000\) and \(80000\) of sites. For simplicity, the length of the system is defined here in terms of the size of a regular lattice \(L = N^{1/2}\). For each system size quenched averages over the connectivity disorder are approximated by averaging over \(R = 20\) independent realizations. For each simulation we have started with a uniform configuration of spins. We ran \(4 \times 10^5\) Monte Carlo steps (MCS) per spin with \(2 \times 10^5\) configurations discarded for thermalization using the “perfect” random-number generator [30]. We do not see any significant change by increasing the number of \(R\) and MCS. So, for the sake of saving computer time, the present values seem to give reasonable results for our simulation.

In both cases we have employed the heat bath algorithm and for every MCS, the energy per spin, \(e = E/N\), and the magnetization per spin, \(m = \sum_i S_i/N\), were measured. From the energy measurements we can compute the average energy, specific heat and energetic fourth-order parameter, given, respectively, by

\[
u(T) = \langle E\rangle_{av}/N, \quad C(T) = T^2 2(N[\langle e^2 \rangle - \langle e \rangle^2]_{av}, \quad B(T) = 1 - \left[ \frac{\langle e^4 \rangle}{3\langle e^2 \rangle^2} \right]_{av}.
\]

In the above equations \(\langle \ldots \rangle\) stands for thermodynamic averages and \([\ldots]_{av}\) for averages over different realizations. Similarly, we can derive from the magnetization measurements the average magnetization, the susceptibility, and the fourth-order magnetic cumulant,

\[
m(T) = \langle |m| \rangle_{av}, \quad \chi(T) = TN \left[ \langle m^2 \rangle - \langle |m| \rangle^2 \right]_{av}, \quad U_4(T) = 1 - \left[ \frac{\langle m^4 \rangle}{3\langle |m| \rangle^2} \right]_{av}.
\]
In order to calculate the exponents of this model, we apply finite-size scaling (FSS) theory. We then expect, for large system sizes, an asymptotic FSS behavior of the form

$$C = C_{\text{reg}} + L^{\alpha/\nu} f_C(x)[1 + \ldots],$$

$$\langle|m|\rangle_{\text{av}} = L^{-\beta/\nu} f_m(x)[1 + \ldots],$$

$$\chi = L^{\gamma/\nu} f_\chi(x)[1 + \ldots],$$

where $C_{\text{reg}}$ is a regular background term, $\nu$, $\alpha$, $\beta$, and $\gamma$ are the usual critical exponents, and $f_i(x)$ are FSS functions with $x = (T - T_c)L^{1/\nu}$ being the scaling variable. The dots in the brackets [$1 + \ldots$] indicate corrections-to-scaling terms. We calculated the error bars from the fluctuations among the different realizations. Note that these errors contain both, the average thermodynamic error for a given realization and the theoretical variance for infinitely accurate thermodynamic averages which are caused by the variation of the quenched, random geometry of the lattices.

### 3 Results and conclusion

By applying the standard heat bath algorithm to each of the $R$ energy data we determine the temperature dependence of $C_i(T)$, $\chi_i(T)$, $\ldots$, $i = 1, \ldots, R$. Once the temperature dependence is known for each realization, we can easily compute the disorder average, e.g., $C(T) = \sum_{i=1}^{T} C_i(T)/R$, and then determine the maxima of the averaged quantities, e.g., $C_{\text{max}}(T_{\text{max}}) = \max_T C(T)$. The variable $R (= 20)$ represents the number of replicas in our simulations.

In fig. 1 we show the behavior of the magnetization versus temperature for several different lattice sizes and rewiring probability $p = 0.5$. Figure 2 displays the behavior of the susceptibility versus temperature for the same parameters used in fig. 1. From here on we set $J$ and $k_B$ to unity. One can see a typical behavior of a second-order phase transition. In order to estimate the critical temperature we calculate the fourth-order Binder cumulant given by eq. (8). It is well known that these quantities are independent of the system size and should intercept at the critical temperature [31]. In fig. 3 the fourth-order Binder cumulant is shown as a function of $T$ for several lattice sizes for the rewiring probability $p = 0.5$. Taking the largest lattices we have $T_c = 5.118(4)$. To estimate $U^{*}_4(p)$ we note that it varies little at $T_c$, so we have $U^{*}_4 = 0.283(4)$. One can see that $U^{*}_4$ is different from the universal value $U^{*}_4 \sim 0.61$ for the Ising model on the regular $d = 2$ lattice, and also for the Ising model on Voronoi-Delaunay random lattice in two dimensions [19–21]. By following this same procedure one can get the corresponding results for other values of $p$.

The correlation length exponent can be estimated from $T_c(L) = T_c + bL^{-1/\nu}$, where $T_c(L)$ is the pseudo-critical temperature for the lattice size $L$, $T_c$ is the critical temperature in the thermodynamic limit, and $b$ is a non-universal constant. In fig. 4 it is shown a plot of $\ln[T_c(L) - T_c]$ as a function of $\ln L$ for several values of $p$. One can clearly see that the exponent is, within the errors, independent of $p$, in agreement with universality ideas. The actual values of $1/\nu$ are displayed in table 1.
Fig. 2. (Color online) The same as fig. 1 for susceptibility versus temperature $T$.

Fig. 3. (Color online) The same as fig. 1 for the fourth-order Binder cumulant as a function of $T$.

Fig. 4. (Color online) $\ln[T_c(L) - T_c]$ as a function of $L$ for several values of $p$. The solid lines are the best linear fits.
Table 1. The critical exponents, for spin-1/2 on directed SWVD random lattice with probability $p$. $\gamma/\nu^{\text{max}}$ are the results from the maximum of the magnetic susceptibility. Error bars are statistical only.

| $p$  | $1/\nu$      | $\beta/\nu$ | $\gamma/\nu$ | $\gamma/\nu^\text{max}$ |
|------|--------------|--------------|---------------|-------------------------|
| 0.1  | 1.036(49)    | 0.489(8)     | 1.003(11)     | 1.001(13)               |
| 0.2  | 1.098(82)    | 0.538(68)    | 1.016(11)     | 1.016(5)                |
| 0.3  | 1.009(49)    | 0.463(4)     | 0.924(98)     | 1.012(3)                |
| 0.4  | 0.886(8)     | 0.491(9)     | 1.017(14)     | 1.012(8)                |
| 0.5  | 0.987(64)    | 0.494(10)    | 0.998(18)     | 1.005(66)               |
| 0.6  | 0.927(92)    | 0.486(10)    | 1.042(13)     | 1.004(7)                |
| 0.7  | 1.107(60)    | 0.486(10)    | 1.016(13)     | 1.003(10)               |
| 0.8  | 0.972(57)    | 0.493(16)    | 1.018(23)     | 1.021(7)                |
| 0.9  | 1.032(66)    | 0.471(12)    | 1.038(16)     | 0.991(69)               |

Fig. 5. (Color online) Plot of the logarithm of the modulus of the magnetization at the inflection point as a function of the logarithm of $L$. The solid lines are the best linear fits.

In order to go further in the present analysis we have also computed the modulus of the magnetization at the inflection point and the magnetic susceptibility at $T_c$. The logarithm of these quantities as a function of the logarithm of $L$ are presented in figs. 5 and 6, respectively. A linear fit of these data gives $\beta/\nu$ from the magnetization and $\gamma/\nu$ from the susceptibility. In addition, we plotted in fig. 7 the logarithm of the maximum value of the susceptibility $\chi^\text{max}$ as a function of $\ln L$ for several values of $p$. One can also see that the exponents $\beta/\nu$ and $\gamma/\nu$ are also independent of $p$, as expected. They are different from $\beta/\nu = 0.125$ and $\gamma/\nu = 1.75$ obtained for a regular $d = 2$ lattice, but obey hyper-scaling relation (within the error bars)

$$2\frac{\beta}{\nu} + \frac{\gamma}{\nu} = d,$$

where $d = 2$. The numerical values of the ratio $\beta/\nu$ and $\gamma/\nu$ are also shown in table 1.

In figs. 8 and 9 we display the data collapse for the magnetisation and the susceptibility for $p = 0.5$. In these cases, we see that the estimates of the critical exponent ratios $\beta/\nu$ and $\gamma/\nu$ are in good agreement for all lattice sizes. The same qualitative results are obtained for other values of $p$. 


4.2 4.4 4.6 4.8 5.0 5.2 5.4 5.6
Ln L

1.5 2.0 2.5 3.0 3.5 4.0

Fig. 6. (Color online) Log-log plot of the susceptibility $\chi$ at $T_c$ as a function of the logarithm of $L$. The solid lines are the best linear fits.

4.2 4.4 4.6 4.8 5.0 5.2 5.4 5.6
Ln L

1.5 2.0 2.5 3.0 3.5 4.0

Fig. 7. (Color online) Log-log plot of the susceptibility maxima $\chi_{\text{max}}$ as a function of the logarithm of $L$. The solid lines are the best linear fits.

-2 -1 0 1 2
$(T - T_c)L^{1/v}$

Table: $L = \{70.71, 100.00, 141.42, 200.00, 244.94, 282.84\}$

Fig. 8. (Color online) Data collapse of magnetisation for various values of $L$ and $p = 0.5$. 
In summary, from the above results, there is a strong indication that the spin-1/2 Ising model on a directed SWVD random lattice is in a different universality class than the model on a regular two-dimensional lattice. The exponents here obtained are independent of $p$ ($0 < p < 1$) and different from the Ising model on regular $d = 2$ lattice, but they agree with the exponents of the Ising model and the spin-1 Blume-Capel model on directed small-world network [32]. One possible explanation for this change in universality can be ascribed to the influence of long-range interactions that occur with the presence of $p$ directed bonds. However, our results agree with the Harris criterion for directed SWVD random lattice.

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