ANALYSIS OF DISCRETIZED PARABOLIC PROBLEMS MODELING ELECTROSTATIC MICRO-ELECTROMECHANICAL SYSTEMS

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Abstract. Our aim in this paper is to study discretized parabolic problems modeling electrostatic micro-electromechanical systems (MEMS). In particular, we prove, both for semi-implicit and implicit semi-discrete schemes, that, under proper assumptions, the solutions are monotonically and pointwise convergent to the minimal solution to the corresponding elliptic partial differential equation. We also study the fully discretized semi-implicit scheme in one space dimension. We finally give numerical simulations which illustrate the behavior of the solutions both in one and two space dimensions.

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1. Introduction. MEMS (micro-electromechanical systems) combine electronics and micro-size mechanical devices in order to decrease the scaling of electromechanical systems to micro-scale; this is similar to NEMS (nano-electromechanical systems) which go to nano-scale (see [27]). The idea of micro-machineries was presented by Feynman in his famous lecture (see [10]) at the end of the 1950’s. Several years later, the earliest micro-machinery, named resonant gate transistor and which served as a tuner for micro-electronic radios (see [24]), was created by Nathanson and his coworkers (see [25]) at Westinghouse research labs. From then on, MEMS devices have been extensively applied to many commercial systems, including inkjet printers, MEMS microphones in portable devices, accelerometers for airbag deployment and electronic stability control in modern cars, biosensors, silicon pressure sensors, such as disposable blood pressure sensors, and so on (see more examples in [6] and [27]).

From a mathematical point of view and with the fundamental works by Pelesko and Bernstein (see [27]), we consider an idealized MEMS device which is described in the following sketch (Fig. 1). The device mainly contains a thin and deformable elastic membrane with supported boundary and a parallel rigid ground electric plate.

![Figure 1. An idealized MEMS capacitor.](image)

The upper surface of the membrane, which is normally dielectric, is coated with a metallic conducting film and the thickness of the film is considered to be negligible. When applying a voltage to the conducting film, the elastic membrane deforms towards the ground plate. Considering both the dynamics and electrostatic processes (see [9] and [27] for details) and applying dimensionless analysis, we obtain the following idealized parabolic MEMS problem:

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(x)}{(1 - u)^2} \quad \text{in } \Omega,$$

$$u(t, x) = 0 \quad \text{on } \partial \Omega; \quad u(0, x) = 0 \quad \text{in } \Omega,$$

(1)

and the corresponding elliptic problem:

$$-\Delta u = \frac{\lambda f(x)}{(1 - u)^2} \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega; \quad 0 \leq u < 1 \quad \text{in } \Omega,$$

(2)

where $u = 1 - d$ and $d$ corresponds to the dimensionless distance between the membrane and the plate. Furthermore, $f$ describes the dielectric profile of the elastic membrane and $\lambda > 0$ characterizes the applied voltage.
There are several central problems in the study of problem (1) and its corresponding elliptic problem (2). For instance, when the applied voltage $\lambda$ increases to some threshold, the device cannot remain stable and a touchdown phenomenon appears, which means that $u$ goes to 1 in finite time. The threshold value is called the pull-in voltage, denoted by $\lambda^*$, and is defined by $$\lambda^*(\Omega, f) = \sup\{\lambda > 0 \mid \text{problem (2) possesses at least one solution}\}.$$ The questions of the evaluation of $\lambda^*$ and of how $\lambda$ as well as $\lambda^*$, influence the solutions to MEMS problems are among the central questions in the mathematical study of MEMS. Moreover, we list below the definitions of quenching time and quenching set (see [15], [16] and [21]) which are also important problems in the study of MEMS parabolic problem.

**Definition 1.1.** We call $T$ the quenching time of problem (1) if $T$ satisfies $$T = \sup\{t > 0 \mid \text{for } s \in [0, t], \sup_{\Omega} u(\cdot, s) < 1\}. \quad (3)$$

**Definition 1.2.** We call $\Sigma$ the quenching set of problem (1) if $\Sigma$ satisfies $$\Sigma = \{x \in \bar{\Omega} \mid \exists(x_n, t_n) \in \Omega \times (0, T), x_n \to x, t_n \to t, u(x_n, t_n) \to 1\}. \quad (4)$$

Besides, the diversification of materials for the membrane leads to different dielectric profiles $f$ and corresponding solutions. We refer the readers to [9], [11], [12], [17], [19], [20], [23], [27] and references therein for more details, including the evaluation of $\lambda^*$, discussions on the touchdown phenomenon and the existence and properties of the global solutions to both the elliptic and parabolic problems, as well as the study of the equations with varying dielectric properties from a theoretical perspective. Furthermore, we refer the readers to [22] for modified MEMS problems with a non-local term, to [29] for an advection term and to [8] and references therein for fourth-order problems.

We are interested in this paper in the numerical approximation of problem (1). In particular, we prove that, both for semi-implicit and implicit semi-discrete schemes, the solutions are monotonically and pointwise convergent to the minimal solution to the corresponding elliptic partial differential equation, under proper assumptions. We also study the fully discretized semi-implicit scheme in one space dimension. We finally give numerical simulations which illustrate the behavior of the solutions, as well as the touchdown phenomenon, with different schemes and different initial conditions.

2. **Setting of the problem.** We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(x)}{(1 - u)^2} \quad \text{in } \Omega,$$

$$u(t, x) = 0 \quad \text{on } \partial \Omega; \quad u(0, x) = u_0(x) \quad \text{in } \Omega,$$

where $f$ describes the permittivity profile of the elastic membrane and $\lambda > 0$ characterizes the applied voltage. We make the following assumptions:

- $\Omega$ is a bounded and regular domain of $\mathbb{R}^N$, $N = 1, 2$ or 3;
- $f \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1]$ and $f$ satisfies the condition $0 \leq f \leq 1$, but is not reduced to the null function.
- $u_0 \in L^2(\Omega)$ and $0 \leq u_0 < 1$ a.e.
In particular, when \( u_0 = 0 \), there exists \( \lambda^* > 0 \) such that, if \( 0 \leq \lambda \leq \lambda^* \), then (5) possesses a unique solution which globally converges as \( t \to +\infty \), monotonically and pointwise, to its unique minimal steady state. Furthermore, when \( \lambda > \lambda^* \), the unique solution reaches the singular value 1 in finite time. We refer the interested reader to [9] for more details.

3. **The semi-implicit scheme.** We set, for \( \tau > 0 \) given, \( t_n = n\tau \), \( n = 0, 1, \ldots \), and consider the semi-implicit semi-discrete scheme:

\[
\begin{cases}
    u_{n+1} - \tau \Delta u_{n+1} = u_n + \frac{\lambda \tau f(x)}{(1-u_n)^2} \quad \text{in } \Omega, \\
    u_{n+1} = 0 \quad \text{on } \partial\Omega,
\end{cases}
\]

where \( u_{n+1} \simeq u(t_{n+1}, x) \), \( u_n \simeq u(t_n, x) \) and \( u_0 \) is as in (5).

We can note that, if \( u_n \in H^2(\Omega) \cap H^1_0(\Omega) \), then \( u_{n+1} \in H^2(\Omega) \cap H^1_0(\Omega) \) and

\[ u_{n+1} = (I - \tau \Delta)^{-1}(u_n + \frac{\lambda \tau f}{(1-u_n)^2}), \]

as long as this makes sense, i.e., \( u_n \) does not reach the singular value 1. Actually, it follows from classical elliptic regularity results that, if \( f \in H^2(\Omega) \), then \( u_{n+1} \in H^4(\Omega) \). Thus, if \( u_0 \in H^2(\Omega) \) and \( f \in H^{2n}(\Omega) \), then \( u_n \in H^{2n+2}(\Omega) \) and, if \( u_0 \), \( f \in C^\infty(\Omega) \), then \( u_n \in C^\infty(\Omega) \) (as long as it exists).

We also know that (see [9]), for \( 0 \leq \lambda < \lambda^* \), where \( \lambda^* > 0 \) denotes the pull-in voltage and is the same as above, the elliptic problem

\[-\Delta u = \frac{\lambda f(x)}{(1-u)^2} \quad \text{in } \Omega, \quad 0 \leq u < 1 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \]

possesses at least one solution \( u = u_\lambda \). Furthermore, we assume that \( u_\lambda \) is the unique minimal solution (see [9]) to (7), i.e., for any other solution \( v \) to (7), there holds \( u_\lambda(x) \leq v(x) \) a.e. in \( \Omega \). Actually, there exists \( \bar{\nu} \in (0, 1) \) independent of \( \lambda \) such that

\[ \|u_\lambda\|_{L^\infty(\Omega)} \leq \bar{\nu}, \forall \lambda \in [0, \lambda^*). \]

We additionally assume that

\[ 0 \leq u_0(x) \leq u_\lambda(x), \text{ a.e. } x \in \Omega. \]

We first establish a number of propositions and corollaries as follows.

**Proposition 1.** There holds, for all \( n \in \mathbb{N} \cup \{0\} \),

\[ 0 \leq u_n(x) \leq u_\lambda(x), \text{ a.e. } x \in \Omega; \]

in particular, for all \( n \in \mathbb{N} \cup \{0\} \), \( u_n \) exists and satisfies

\[ 0 \leq u_n(x) < 1, \text{ a.e. } x \in \Omega. \]

**Proof.** We already assumed that (9) holds for \( n = 0 \). Let us assume that, for a given \( n \in \mathbb{N} \cup \{0\} \), there holds \( 0 \leq u_n(x) \leq u_\lambda(x), \text{ a.e. } x \in \Omega \). The function \( v_{n+1} = u_{n+1} - u_\lambda \) satisfies

\[ v_{n+1} - \tau \Delta v_{n+1} = u_n - u_\lambda + \frac{\lambda \tau f(x)}{(1-u_n)^2} - \frac{\lambda \tau f}{(1-u_\lambda)^2} \leq 0 \]

and

\[ v_{n+1} = 0 \quad \text{on } \partial\Omega. \]
Multiplying (11) by $v_{n+1}^+$, where $\cdot^+ = \max\{0, \cdot\}$, we obtain, integrating over $\Omega$ and by parts and owing to (12),
\[
\|v_{n+1}^+\|^2 \leq 0,
\]
where $\|\cdot\|$ denotes the $L^2$-norm. This means that $v_{n+1}^+ = 0$ and $u_{n+1} \leq u_\lambda$ a.e. $x \in \Omega$.

Alternatively, we can note that
\[
u_\lambda - \tau \Delta u_\lambda = u_\lambda + \frac{\lambda \tau f}{(1 - u_\lambda)^2} \geq u_n + \frac{\lambda \tau f}{(1 - u_n)^2},
\]
so that $u_\lambda$ is a supersolution to (6). Noticing that 0 is a subsolution to (6), it follows from the sub and supersolutions theorem (see, e.g., [7]) that
\[
0 \leq u_{n+1} \leq u_\lambda, \text{ a.e. } x \in \Omega,
\]
which completes the proof. \(\square\)

**Proposition 2.** There holds, for $n \in \mathbb{N} \cup \{0\}$,
\[
\|u_{n+1} - u_\lambda\| \leq \frac{1}{1 + c_0 \tau} \left(1 + \frac{2\lambda \tau}{(1 - \bar{u})^3}\right) \|u_n - u_\lambda\|, \tag{13}
\]
where $\bar{u}$ was given in (8), and $c_0 > 0$ is the optimal constant in the Poincaré inequality
\[
\|
\nabla v
\|_2 \geq c_0 \|v\|_2, \quad v \in H_0^1(\Omega). \tag{14}
\]

**Proof.** We have, setting $v_n = u_n - u_\lambda$ and $v_{n+1} = u_{n+1} - u_\lambda$,
\[
v_{n+1} - \tau \Delta v_{n+1} = v_n + \frac{\lambda \tau f}{(1 - u_n)^2} - \frac{\lambda \tau f}{(1 - u_\lambda)^2} \text{ in } \Omega,
v_{n+1} = 0 \text{ on } \partial \Omega.
\]
We thus deduce that
\[
v_{n+1} - \tau \Delta v_{n+1} = v_n + \frac{\lambda \tau f}{(1 - u_\lambda)^2} - \frac{\lambda \tau f}{(1 - u_n)^2} = v_n + \left[\frac{\lambda \tau f}{(1 - u_\lambda)^2} + \frac{\lambda \tau f}{(1 - u_n)^2}\right] v_n. \tag{15}
\]
Multiplying (15) by $v_{n+1}$ and integrating over $\Omega$ and by parts, we obtain
\[
\|v_{n+1}\|^2 + \tau \|\nabla v_{n+1}\|^2 = ((v_n, v_{n+1})) + ((\frac{\lambda \tau f}{(1 - u_\lambda)^2} + \frac{\lambda \tau f}{(1 - u_n)^2}) v_n, v_{n+1}),
\]
which yields, noting that $v_n$ and $v_{n+1}$ are nonnegative and recalling that $u_n \leq u_\lambda$ and the assumptions on $f$,
\[
\|v_{n+1}\|^2 + \tau \|\nabla v_{n+1}\|^2 \leq ((v_n, v_{n+1})) + ((\frac{2\lambda \tau}{(1 - \bar{u})^3} v_n, v_{n+1})) = ((\frac{2\lambda \tau}{(1 - \bar{u})^3}) v_n, v_{n+1})) \leq \left(1 + \frac{2\lambda \tau}{(1 - \bar{u})^3}\right) \|v_n\| \|v_{n+1}\|.
\]
It thus follows from (12) and the Poincaré inequality that
\[
(1 + c_0 \tau) \|v_{n+1}\|^2 \leq \left(1 + \frac{2\lambda \tau}{(1 - \bar{u})^3}\right) \|v_{n+1}\| \|v_n\|.
\]
This completes the proof. \hfill \Box

As a consequence of Proposition 1, we deduce the following corollary.

**Corollary 1.** We assume that

\[
\frac{1}{1 + c_0 \tau} \left( 1 + \frac{2\lambda \tau}{(1 - \pi)^3} \right) < 1.
\]  

(16)

Then, \( u_n \) converges to \( u_\lambda \) in \( L^2(\Omega) \) as \( n \to +\infty \).

**Remark 1.** We can note that (16) holds for

\[ \lambda < \frac{1}{2} c_0 (1 - \pi)^3. \]

In particular, this estimate does not depend on the choice of \( \tau > 0 \).

**Remark 2.** Let us take \( u_0 = 0 \). It is clear that \( u_1 \geq u_0 = 0 \). Let us then assume that, for a given \( n \in \mathbb{N} \), \( u_n \geq u_{n-1} \). We have, setting \( v_{n+1} = u_{n+1} - u_n \),

\[
v_{n+1} - \tau \Delta v_{n+1} = u_n - u_{n-1} + \frac{\lambda \tau f}{(1 - u_n)^2} - \frac{\lambda \tau f}{(1 - u_{n-1})^2} \geq 0 \text{ in } \Omega \]

(17)

and

\[ v_{n+1} = 0 \text{ on } \partial \Omega. \]

Multiplying (17) by \(-v_{n+1}^-\), where \(-\max \{0, -\cdot\}\), we deduce that \( \|v_{n+1}^-\| = 0 \), whence \( u_{n+1} \geq u_n \). Therefore, for every \( x \in \Omega \), the sequence \( \{u_n(x)\} \) is monotone increasing and bounded from above by \( u_\lambda \). Hence it converges monotonically as \( n \to +\infty \).

Moreover, it is verified numerically (see [9] for details) that, if \( \lambda < \lambda^* \) and is close to \( \lambda^* \), with given \( f \) and \( N \), there may exist another stable solution to the elliptic problem (7) which is denoted by \( u_\lambda^+ \) and satisfies \( u_\lambda < u_\lambda^+ < 1 \). We now assume that the stationary problem possesses at least 2 solutions \( u_\lambda < u_\lambda^+ < 1 \) and that the initial condition satisfies \( u_\lambda \leq u_0 < u_\lambda^+ < 1 \). Then, we additionally have the following proposition.

**Proposition 3.** We consider the semi-implicit scheme (6), with \( 0 < \lambda < \lambda^* \) and \( f \) satisfying the assumptions mentioned in Section 2. Then, if the initial condition satisfies \( u_\lambda \leq u_0 < u_\lambda^+ < 1 \), there holds, for all \( n \in \mathbb{N} \cup \{0\} \), \( u_\lambda \leq u_n < u_\lambda^+ < 1 \), a.e. \( x \in \Omega \).

**Proof.** We proceed as above. On the one hand, for \( n = 0 \), then \( u_\lambda \leq u_0 < 1 \). We further assume that, for a given \( n \in \mathbb{N} \cup \{0\} \), there holds \( u_\lambda \leq u_n < 1 \), a.e. \( x \in \Omega \). Then, the function \( v_{n+1} = u_\lambda - u_{n+1} \) satisfies

\[
v_{n+1} - \tau \Delta v_{n+1} = v_n + \frac{\lambda \tau f(x)}{(1 - u_n)^2} - \frac{\lambda \tau f(x)}{(1 - u_{n+1})^2} \leq 0
\]

(18)

and

\[ v_{n+1} = 0 \text{ on } \partial \Omega. \]

(19)

Multiplying (18) by \( v_{n+1}^+ \) and integrating over \( \Omega \) and by parts, we obtain, owing to (19),

\[
\|v_{n+1}^+\|^2 + \tau \|
abla v_{n+1}^+\|^2 \leq 0.
\]

(20)

which implies \( v_{n+1}^+ \leq 0 \), i.e., \( u_\lambda \leq u_{n+1} \), a.e. \( x \in \Omega \). On the other hand, we already assumed that \( u_\lambda \leq u_0 < u_\lambda^+ < 1 \) and we further assume that, for a given \( n \in \mathbb{N} \cup \{0\} \),
there holds $u_\lambda \leq u_n < u_\lambda^+ < 1$, a.e. $x \in \Omega$. Then, the function $w_{n+1} = u_{n+1} - u_\lambda^+$ satisfies

$$w_{n+1} - \tau \Delta w_{n+1} = w_n + \frac{\lambda \tau f(x)}{(1-u_n)^2} - \frac{\lambda \tau f(x)}{(1-u_\lambda^+)^2} < 0$$  \hspace{1cm} (21)

and

$$w_{n+1} = 0 \quad \text{on} \quad \partial \Omega. \hspace{1cm} (22)$$

Multiplying (21) by $w_{n+1}^+$ and integrating over $\Omega$ and by parts, we obtain, owing to (22),

$$\|w_{n+1}^+\|^2 + \tau \|\nabla w_{n+1}^+\|^2 < 0,$$ \hspace{1cm} (23)

which implies $w_{n+1} < 0$, i.e., $u_{n+1} < u_\lambda^+$, a.e. $x \in \Omega$. ☐

**Remark 3.** Let us take $u_0$ such that $u_\lambda \leq u_0 < u_\lambda^+ < 1$. It was numerically verified that $u_\lambda \leq u_1 \leq u_0 < u_\lambda^+$. Assuming that, for a given $n \in \mathbb{N} \cup \{0\}$, $u_\lambda \leq u_n \leq u_{n-1} < u_\lambda^+$ and setting $v_{n+1} = u_{n+1} - u_n$, we deduce that

$$v_{n+1} - \tau \Delta v_{n+1} = v_n + \frac{\lambda \tau f(x)}{(1-u_n)^2} - \frac{\lambda \tau f(x)}{(1-u_{n-1})^2} \leq 0$$  \hspace{1cm} (24)

and

$$v_{n+1} = 0 \quad \text{on} \quad \partial \Omega. \hspace{1cm} (25)$$

Multiplying (24) by $v_{n+1}^+$ and integrating over $\Omega$ and by parts, we obtain, owing to (25),

$$\|v_{n+1}^+\|^2 + \tau \|\nabla v_{n+1}^+\|^2 \leq 0,$$ \hspace{1cm} (26)

whence $u_\lambda \leq u_{n+1} \leq u_n < u_\lambda^+$. Therefore, for $x \in \Omega$, the sequence $\{u_n(x)\}$ is monotonously decreasing and bounded from below by $u_\lambda$, i.e., it converges as $n \to +\infty$.

**4. The implicit scheme.** We consider in this section the implicit semi-discrete scheme: for $n \in \mathbb{N} \cup \{0\}$,

$$\begin{cases} u_{n+1} - \tau \Delta u_{n+1} = u_n + \frac{\lambda \tau f(x)}{(1-u_{n+1})^2} \quad \text{in} \quad \Omega, \\
 u_{n+1} = 0 \quad \text{on} \quad \partial \Omega, \end{cases} \hspace{1cm} (27)$$

where, again, $u_{n+1} \simeq u(t_{n+1}, x)$, $u_n \simeq u(t_n, x)$. We suppose that the assumptions on $\Omega$, $f$, and (9)-(10) still hold.

**Proposition 4.** For $0 \leq u_n \leq u_\lambda < 1$, a.e. $x \in \Omega$, problem (27) possesses at least one solution such that

$$0 \leq u_{n+1} \leq u_\lambda < 1, \quad \text{a.e.} \quad x \in \Omega, \hspace{1cm} (28)$$

where $u_\lambda$ is the minimal solution to (7).

**Proof.** It is obvious that 0 is a subsolution to problem (27). Furthermore, we note that

$$u_\lambda - \tau \Delta u_\lambda = u_\lambda + \frac{\lambda \tau f(x)}{(1-u_\lambda)^2} \geq u_n + \frac{\lambda \tau f(x)}{(1-u_\lambda)^2},$$

so that $u_\lambda$ is a supersolution to (27). The proof is completed. ☐
Proceeding as in the proof of Proposition 2 and setting \( v_n = u_n - u \lambda \), \( v_{n+1} = u_{n+1} - u \lambda \), we have

\[
v_{n+1} - \tau \Delta v_{n+1} = v_n + \left[ \frac{\lambda \tau f}{1 - u \lambda} \right] v_{n+1} \quad \text{in} \quad \Omega, \tag{29}
\]

\[
v_{n+1} = 0 \quad \text{on} \quad \partial \Omega. \tag{30}
\]

Multiplying (29) by \( v_{n+1} \) and applying the Poincaré inequality, we obtain

\[
\| v_{n+1} \|^2 + c_0 \tau \| v_{n+1} \|^2 \leq \langle (v_n, v_{n+1}) \rangle + \frac{2 \lambda \tau}{(1 - \overline{u})^3} \| v_{n+1} \|^2
\]

\[
\leq \| v_n \| \| v_{n+1} \| + \frac{2 \lambda \tau}{(1 - \overline{u})^3} \| v_{n+1} \|^2,
\]

whence

\[
\left( 1 + c_0 \tau - \frac{2 \lambda \tau}{(1 - \overline{u})^3} \right) \| v_{n+1} \| \leq \| v_n \|. \tag{31}
\]

This yields the following result.

**Proposition 5.** We assume that,

\[
\lambda < \frac{1}{2} (1 - \overline{u})^3 c_0. \tag{32}
\]

Then, \( u_n \) converges to \( u \) in \( L^2(\Omega) \) as \( n \to +\infty \).

**Remark 4.** We again take \( u_0 = 0 \). Then, \( u_1 \geq u_0 \). Let us assume that, for a given \( n \in \mathbb{N} \), \( u_n \geq u_{n-1} \). Then, the function \( v_{n+1} = u_{n+1} - u_n \) satisfies

\[
v_{n+1} - \tau \Delta v_{n+1} = u_n - u_{n-1} + \frac{\lambda \tau f}{1 - u_{n+1}} - \frac{\lambda \tau f}{1 - u_n} \\
\geq \left( \frac{\lambda \tau f}{1 - u_{n+1}}(1 - u_n) + \frac{\lambda \tau f}{1 - u_n}(1 - u_{n+1}) \right) v_{n+1} \quad \text{in} \quad \Omega
\]

and

\[
v_{n+1} = 0 \quad \text{on} \quad \partial \Omega. \tag{33}
\]

Multiplying (33) by \(-v_{n+1}^-\) and integrating over \( \Omega \) and by parts, we find

\[
\| v_{n+1}^- \|^2 + \tau \| \nabla v_{n+1}^- \|^2 \leq \left( \frac{\lambda \tau f}{1 - u_{n+1}(1 - u_n)} + \frac{\lambda \tau f}{1 - u_n(1 - u_{n+1})} \right) \| v_{n+1}^- \|^2
\]

\[
\leq \frac{2 \lambda \tau}{(1 - \overline{u})^3} \| v_{n+1}^- \|^2,
\]

whence

\[
\left( 1 + c_0 \tau - \frac{2 \lambda \tau}{(1 - \overline{u})^3} \right) \| v_{n+1}^- \|^2 \leq 0. \tag{34}
\]

Therefore, for a given \( \lambda \), if \( \tau \leq \tau_0 \), where \( \tau_0 \) is small enough so that

\[
1 + c_0 \tau_0 - \frac{2 \lambda \tau_0}{(1 - \overline{u})^3} > 0, \tag{35}
\]

then \( v_{n+1}^- = 0 \) and thus \( u_{n+1} \geq u_n \). It thus follows from Proposition 4 that \( u_n \) converges monotonically and pointwise.
Remark 5. Multiplying (29) by $-\Delta v_{n+1}$ and integrating over $\Omega$ and by parts, we have
\[
\|\nabla v_{n+1}\|^2 + \|\Delta v_{n+1}\|^2 \leq \|\nabla v_n\|\|\nabla v_{n+1}\| + \frac{2\lambda \tau}{(1-\pi)^3}\|v_{n+1}\|\|\Delta v_{n+1}\|. \tag{36}
\]
Then, since
\[
\|\Delta v\|^2 \geq c_0\|\nabla v\|^2, \quad v \in H^2(\Omega) \cap H^1_0(\Omega),
\]
applying the Cauchy-Schwarz inequality, we obtain
\[
\|\nabla v_{n+1}\|^2 + c_1^{1/2}c_0^{1/2}\tau\|\nabla v_{n+1}\|\|\Delta v_{n+1}\| \leq \|\nabla v_n\|\|\nabla v_{n+1}\| + 2\lambda \tau c_1^{1/2}c_0^{1/2}(1-u)^{3/2}\|\nabla v_{n+1}\|\|\Delta v_{n+1}\|. \tag{37}
\]
whence
\[
\|\nabla v_{n+1}\| + \left(\frac{1}{c_0^{1/2}} - \frac{2\lambda \tau}{c_0^{1/2}(1-\pi)^3}\right)\|\Delta v_{n+1}\| \leq \|\nabla v_n\|.
\]
Therefore,
\[
(1 + c_0\tau - \frac{2\lambda \tau}{(1-\pi)^3})\|\nabla v_{n+1}\| \leq \|\nabla v_n\|. \tag{38}
\]
Finally, if
\[
\lambda < \frac{1}{2}(1-\pi)^3c_0,
\]
or equivalently
\[
\left(1 + c_0\tau - \frac{2\lambda \tau}{(1-\pi)^3}\right)^{-1} < 1,
\]
then $u_n$ converges to $u_\lambda$ in $H^1_0(\Omega)$ as $n \to +\infty$. In particular, in one space dimension, $u_n$ converges to $u_\lambda$ in $C(\Omega)$ as $n \to +\infty$.

5. The fully discretized semi-implicit scheme. We only consider the one-dimensional problem in this section. We believe that similar results hold in two-dimensional space and will address this elsewhere.

Let $M > 0$ be an integer, $h = (b - a)/(M + 1)$ denote the spatial mesh size, and $x_i = a + ih$, $i = 0, \cdots, M + 1$. We consider the fully discretized semi-implicit scheme as follows: for $n \in \mathbb{N} \cup \{0\}$,
\[
\begin{align*}
\frac{u^{n+1}_i - u^n_i}{\tau} - \frac{u^{n+1}_{i+1} - 2u^{n+1}_i + u^{n+1}_{i-1}}{h^2} & = \frac{\lambda f(x_i)}{1 - u^n_i} \cdot (1 - u^n_i)^2, \quad i = 1, \cdots, M; \\
u^{n+1}_0 = u^{n+1}_{M+1} = 0,
\end{align*}
\tag{39}
\]
where $u^n_i \approx u(t^n, x_i)$. Problem (39) can be rewritten equivalently as
\[
(1 + 2\frac{\tau}{h^2})u^{n+1}_i - \frac{\tau}{h^2}u^{n+1}_{i+1} + \frac{\tau}{h^2}u^{n+1}_{i-1} - \frac{\tau}{h^2}u^{n+1}_{i+1} = u^n_i + \frac{\lambda \tau f(x_i)}{1 - u^n_i} \cdot (1 - u^n_i)^2, \tag{40}
\]
We then rewrite (40) in vector form as
\[
AU^{n+1} = F(U^n), \quad U^0 = 0, \tag{41}
\]

where \( U^n = (u^n_1, u^n_2, \ldots, u^n_M)^t \) and
\[
A = \begin{pmatrix}
1 + 2\frac{\tau}{h^2} & -\frac{\tau}{h^2} & & -\frac{\tau}{h^2} \\
-\frac{\tau}{h^2} & 1 + 2\frac{\tau}{h^2} & -\frac{\tau}{h^2} & & \\
& \ddots & \ddots & \ddots & -\frac{\tau}{h^2} \\
& & -\frac{\tau}{h^2} & 1 + 2\frac{\tau}{h^2} & -\frac{\tau}{h^2} \\
& & & 1 + 2\frac{\tau}{h^2} & \tau
\end{pmatrix},
\]
\[
F(U^n) = \begin{pmatrix}
\frac{f(x_1)}{(1-u^n_1)^2} \\
\frac{f(x_2)}{(1-u^n_2)^2} \\
\vdots \\
\frac{f(x_M)}{(1-u^n_M)^2}
\end{pmatrix}.
\]
We can rewrite (41) equivalently as
\[
AU^{n+1} = U^n + G(U^n),
\]
where
\[
G(U^n) = \lambda \tau \begin{pmatrix}
\frac{f(x_1)}{(1-u^n_1)^2} \\
\frac{f(x_2)}{(1-u^n_2)^2} \\
\vdots \\
\frac{f(x_M)}{(1-u^n_M)^2}
\end{pmatrix}.
\]
We note that
\[
A = I + \frac{\tau}{h^2} B,
\]
where
\[
B = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}.
\]
It is well known that \( A \) is positive definite, and thus invertible. Furthermore,
\[
A = \left(1 + 2\frac{\tau}{h^2}\right) \left(I - \frac{\tau}{h^2} C\right),
\]
where
\[
C = \begin{pmatrix}
0 & 1 & & & \\
1 & 0 & 1 & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & 0 & 1 \\
& & & 1 & 0
\end{pmatrix},
\]
so that
\[
A^{-1} = \left(1 + 2\frac{\tau}{h^2}\right)^{-1} \left(I - \frac{\tau}{h^2} C\right)^{-1}
= \left(1 + 2\frac{\tau}{h^2}\right)^{-1} \sum_{j=0}^{+\infty} \left(\frac{\tau}{h^2} C\right)^j.
\]
This yields that \( A^{-1} \geq 0 \).
We now consider the equation
\[
AU^* = U^* + G(U^*),
\]
which is the centered difference scheme of the steady state problem (7).
Equation (44) can be rewritten as

\[ BU^* = \lambda h^2 \begin{pmatrix} \frac{f(x_1)}{(1-u^*_1)^2} \\ \frac{f(x_2)}{(1-u^*_2)^2} \\ \vdots \\ \frac{f(x_M)}{(1-u^*_M)^2} \end{pmatrix}. \] (45)

The solvability of the above problem is given in the following theorem.

**Theorem 5.1.** We assume that \( 0 \leq \lambda \leq \frac{8(1-\delta)^2}{(b-a)^2} \), for some \( \delta \in (0, 1) \). Then (45) possesses a solution \( U^* \) such that \( 0 \leq U^* < E \), where \( E = (1, 1, \ldots, 1)^t_{1 \times M} \).

**Proof.** First note that \( B \) is invertible and (as above) \( B^{-1} \geq 0 \). We rewrite (45) in the form

\[ U^* = \mathcal{H}(U^*), \] (46)

where

\[ \mathcal{H}(U^*) = \lambda h^2 B^{-1} \begin{pmatrix} \frac{f(x_1)}{(1-u^*_1)^2} \\ \frac{f(x_2)}{(1-u^*_2)^2} \\ \vdots \\ \frac{f(x_M)}{(1-u^*_M)^2} \end{pmatrix}. \]

We then consider the sequence

\[ W^{k+1} = \mathcal{H}(W^k), \quad W^0 = 0. \] (47)

Note that \( W^1 \) exists and, since \( B^{-1} \geq 0 \), \( W^1 \geq 0 \). Actually, as long as it exists,

\[ W^k \geq 0. \]

Furthermore,

\[ W^{k+2} - W^{k+1} = \lambda h^2 B^{-1} \begin{pmatrix} \frac{f(x_1)}{(1-u^*_{1+1})^2} - \frac{f(x_1)}{(1-u^*_1)^2} \\ \frac{f(x_2)}{(1-u^*_{2+1})^2} - \frac{f(x_2)}{(1-u^*_2)^2} \\ \vdots \\ \frac{f(x_M)}{(1-u^*_{M+1})^2} - \frac{f(x_M)}{(1-u^*_M)^2} \end{pmatrix}. \]

Therefore, since \( W^1 \geq W^0 \), we deduce that

\[ W^{k+1} \geq W^k, \] (48)

as long as this makes sense.

Now, recalling the assumptions on \( f \), we have

\[ \|\mathcal{H}(\chi)\|_{\infty} \leq \lambda h^2 \|B^{-1}\|_{\infty} \max_j \left( \frac{1}{(1-\chi_j)^2} \right). \] (49)

We denote by \( D_M \) the determinant of \( B \). It is easy to see that

\[ D_M = 2D_{M-1} - D_{M-2}. \]

Therefore, since \( D_2 = 3 \) and \( D_3 = 4 \), it follows that \( D_M = M + 1 \). It thus follows from [3] (see also [14]) that \( B^{-1} \) is the factorizable matrix \( \{M_{ij}\} \), \( M_{ij} = a_i b_j \), \( i \leq j \), \( M_{ij} = M_{ji} \), where

\[ a_i = i, \quad b_j = 1 - \frac{j}{M+1}. \]
We deduce from the structure of $B^{-1}$ that
\[
\|B^{-1}\|_{\infty} = \max \left( \sum_{i=1}^{M} b_j, \left( \sum_{i=1}^{M} a_i \right) b_M, \max_{i=2, \ldots, M-1} \left( b_i \sum_{j=1}^{i-1} a_j + a_i \sum_{j=i}^{M} b_j \right) \right),
\]
that is to say
\[
\|B^{-1}\|_{\infty} = \max \left( \frac{M}{2}, \max_{i=2, \ldots, M-1} \left( \frac{iM + 1 - i}{2} \right) \right).
\]
Studying the variations of the function $x \mapsto x\frac{M + 1 - x}{2}$, it is easy to see that
\[
\max_{i=2, \ldots, M-1} \left( \frac{iM + 1 - i}{2} \right) \leq \left( \frac{M+1}{2} \right)^2,
\]
so that (taking $M$ large enough)
\[
\|B^{-1}\|_{\infty} \leq \frac{(M+1)^2}{8} = \frac{(b-a)^2}{8h^2},
\]
whence
\[
\|H(\chi)\|_{\infty} \leq \frac{(b-a)^2}{8} \lambda \max_{i} \left( \frac{1}{(1-\chi_i)^2} \right). \tag{50}
\]
Let us now assume that
\[
0 \leq \chi_i \leq 1 - \delta, \ i = 1, \ldots, M, \ \delta \in (0,1).
\]
Then, we have
\[
\|H(\chi)\|_{\infty} \leq \frac{(b-a)^2}{8\delta^2} \lambda.
\]
Therefore, if
\[
\lambda \leq \frac{8(1-\delta)\delta^2}{(b-a)^2},
\]
then
\[
\|H(\chi)\|_{\infty} \leq 1 - \delta.
\]
This yields that $\{W^k\}$ exists for all $k \in \mathbb{N} \cup \{0\}$ and
\[
0 \leq W^k \leq \begin{pmatrix} 1 - \delta \\ 1 - \delta \\ \vdots \\ 1 - \delta \end{pmatrix}. \tag{51}
\]
Hence, each component of $\{W^k\}$ is bounded and monotone increasing, and thus converges. This finishes the proof. \(\square\)

**Remark 6.** Note that, when $\delta$ goes to $0$, i.e., the $\chi_i$’s approach $1$, then $\lambda$ goes to $0$.

We assume from now on that the assumptions of Theorem 5.1 hold, so that $0 \leq U^1 \leq U^*$. Let us assume that $0 \leq U^n \leq U^*$. Then, setting $V^n = U^n - U^*$, there holds
\[
V^{n+1} = A^{-1} \left( V^n + G(U^n) - G(U^*) \right),
\]
so that (since $A^{-1} \geq 0$)
\[
U^{n+1} \leq U^*.
\]
Furthermore, since \( A^{-1} \geq 0 \), \( U^{n+1} \geq 0 \). It thus follows that \( \{U^n\} \) is well-defined and
\[
0 \leq U^n \leq U^*, \quad n \in \mathbb{N} \cup \{0\}.
\]
We can also prove, proceeding as above, that
\[
0 \leq U^n \leq U^{n+1}, \quad n \in \mathbb{N} \cup \{0\},
\]
whence \( \{U^n\} \) converges.

**Remark 7.** We have, for all \( \{V^{n+1}\} \) with \( v_0^{n+1} = 0 \),
\[
((AV^{n+1}, V^{n+1}))
\]
\[
= (1 + \frac{2\tau}{h^2}) \sum_{j=1}^{M} (v_j^{n+1})^2 - \frac{2\tau}{h^2} \sum_{j=1}^{M-1} v_j^{n+1} v_{j+1}^{n+1}
\]
\[
= (1 + \frac{\tau}{h^2}) ((v_1^{n+1})^2 + (v_M^{n+1})^2) + \sum_{j=2}^{M-1} (v_j^{n+1})^2 + \frac{\tau}{h^2} \sum_{j=1}^{M-1} |v_j^{n+1} - v_{j+1}^{n+1}|^2
\]
\[
\geq \sum_{j=1}^{M} (v_j^{n+1})^2 + \frac{\tau}{h^2} \sum_{j=0}^{M-1} |v_j^{n+1} - v_{j+1}^{n+1}|^2,
\]
where \((\cdot, \cdot)\) denotes the usual Euclidean scalar product, with associated norm \( \| \cdot \| \).
It follows from the discrete Poincaré inequality that
\[
\sum_{j=0}^{M-1} |v_j^{n+1} - v_{j+1}^{n+1}|^2 \geq \frac{2h^2}{(b-a)^2} \sum_{j=1}^{M} |v_j^{n+1}|^2,
\]
which yields
\[
((AV^{n+1}, V^{n+1})) \geq \left( 1 + \frac{2\tau}{(b-a)^2} \right) \|V^{n+1}\|^2. \tag{52}
\]

We now have
\[
AV^{n+1} = V^n + G(U^n) - G(U^*),
\]
so that, taking the scalar product with \( V^{n+1} \),
\[
\left( 1 + \frac{2\tau}{(b-a)^2} \right) \|V^{n+1}\|^2 \leq \|V^n + G(U^n) - G(U^*)\| \|V^{n+1}\|.
\]
Therefore,
\[
\|V^{n+1}\| \leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \|G(U^n) - G(U^*)\| \right)
\]
\[
\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \tau \lambda \left( \sum_{j=1}^{M} \left( \frac{V_j^n (2 - u_j^n - u_j^*)}{(1 - u_j^n)^2(1 - u_j^*)^2} \right)^2 \right)^{\frac{1}{2}} \right)
\]
\[
\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + 2\lambda \tau \frac{1}{(1 - \|U^*\|)\|V^n\|} \|V^n\| \right)
\]
\[
\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( 1 + 2\lambda \tau \frac{1}{(1 - \|U^*\|)\|V^n\|} \|V^n\| \right).
\]
Therefore, if
\[
\left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( 1 + 2\lambda \tau \frac{1}{(1 - \|U^*\|)\|V^n\|} \right) < 1,
\]
i.e. \( \lambda < (1 - \|U^*\|_\infty)^3/(b - a)^2 \), then \( \{U^n\} \) converges to \( U^* \).

**Remark 8.** The above results, as well as those in the previous sections, can be improved by taking an assumption on \( f \) of the form

\[
0 \leq f \leq f_0 \leq 1.
\]

In particular, in Theorem 5.1, we then have the more general assumption

\[
0 \leq \lambda \bar{f} \leq 8(1 - \delta)\delta^2/(b - a)^2,
\]

which allows for a larger \( \lambda \) when \( \bar{f} \) is small.

6. **Numerical simulations.** In this section, we give several numerical simulations which show the behavior of the solutions \( u \) corresponding to different schemes, different \( \lambda \)'s and different initial conditions (see Proposition 3 and Remark 3). In particular, it is verified that, when \( 0 < \lambda \leq \lambda^* \), the solutions to problem (5) tend to a stable solution \( u_\lambda \) as time grows. Furthermore, if \( \lambda > \lambda^* \), one can observe the so-called touchdown phenomenon. The numerical simulations are performed with MATLAB.

6.1. **The elliptic problem.** In this subsection, we study the elliptic problem (7). Theoretically, it has been shown (see, e.g., [9]) that there exists \( \lambda^* \) depending on the domain \( \Omega \) as well as the permittivity profile \( f \) such that when \( \lambda \leq \lambda^* \), problem (7) possesses a unique minimal solution.

6.1.1. **The 1D elliptic problem.** Employing the continuation method described in Appendix A and considering the one-dimensional elliptic problem, we draw the branch of solutions \( u(0) \) as a function of \( \lambda \). Setting \( \Omega = (-0.5, 0.5), M = 199 \) and \( f(x) = |2x| \) or \( f(x) \equiv 1 \), we obtain Fig. 2 in which we observe the existence of \( \lambda^* \) (when \( f(x) = |2x|, \lambda^* \approx 4.388 \), see Fig. 2(a), and, when \( f \equiv 1, \lambda^* \approx 1.4 \), see Fig. 2(b)) and that, when \( \lambda < \lambda^* \) but is close to \( \lambda^* \), the branches display two solutions \( u_\lambda(0) \) and \( u_\lambda^*(0) \). The results are consistent with [9].

![Figure 2](image-url)
6.1.2. The 2D elliptic problem. As far as the 2D elliptic problem is concerned, applying the 5-point centered difference to approximate the Laplace operator, we have the discrete scheme:

$$\begin{cases} -4u_{i,j} + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} + \lambda h^2 f(x_i, y_j) = 0, & i, j = 1, \ldots, M, \\ u_{0,j} = u_{M+1,j} = u_{i,0} = u_{i,M+1} = 0, & i, j = 0, \ldots, M+1, \end{cases}$$

where $h = 1/(M+1)$ denotes the spatial mesh size of the computational domain $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ and $u_{i,j} \approx u(x_i, y_j)$, with $(x_i, y_j) = (-0.5 + ih, -0.5 + jh)$, $i, j = 0, \ldots, M+1$. Similarly, we display the branch of solutions $u(0, 0)$ as a function of $\lambda$ in Fig. 3(a) for $f(x, y) = \sqrt{x^2 + y^2}$ and in Fig. 4(a) for $f(x, y) \equiv 1$; here $M = 29$. We observe in Fig. 3(a) and Fig. 4(a) that there exists a maximal value of $\lambda$ such that, if $0 < \lambda \leq \lambda^*$, problem (7) possesses at least one solution, and, if $\lambda > \lambda^*$, there does not exist a solution to the elliptic problem. Simultaneously, we observe that, when $\lambda$ is less than but is close to $\lambda^*$, there are two values of $u(0, 0)$ which are denoted by $u_\lambda(0, 0)$ and $u_\lambda^+(0, 0)$, as illustrated in Fig. 3(a) for $\lambda = 10$ and in Fig. 4(a) for $\lambda = 2.5$. In Fig. 3(b) (resp. Fig. 4(b)), we display the two corresponding solutions $u_\lambda$, $u_\lambda^+$ with $\lambda = 10$ (resp. $\lambda = 2.5$) and $u_\lambda^*$. Furthermore, Fig. 4(c) shows four solutions to problem (7) with $\lambda = 1.6$ which become sharper and sharper as the computation goes on. In the two dimensional simulations, here and below, otherwise specified, the solutions correspond to the section $y = 0$.

The branches and solutions for the one- and two-dimensional elliptic problems are helpful in view of the study of the parabolic problems.

6.2. The parabolic problem. We set $t^n = n\tau$, $n = 0, \ldots, K$, and write the time steps as superscripts and the spatial nodes as subscripts.

6.2.1. The 1D problem. Setting $\Omega = (-0.5, 0.5)$, we display in Fig. 5 the one-dimensional results for problem (5) applying both the semi-implicit and implicit schemes when $\lambda = 4.0$, $f(x) = |2x|$, $M = 199$ and $u_{\text{init}} = 0$; these are consistent with the theoretical results.
Figure 4. The branch of solutions $u(0,0)$ as a function of $\lambda$: (a) $f(x,y) = 1$; (b) $\lambda = 2.5$, $u_\lambda$, $u_\lambda^+$ and $u_\lambda^{++}$; (c) $\lambda = 1.6$, four corresponding solutions.

Moreover, the results for the semi-implicit scheme for the initial condition

$$u_{\text{ini}} \equiv u_{\text{ini},\text{pw}} = \frac{9}{10} u_\lambda^+ + \frac{1}{10} u_\lambda$$

(53)

which is larger than $u_\lambda$ and less than but close to $u_\lambda^+$ show that the solution decreases and converges to $u_\lambda$ (see Fig. 6(a)). In Fig. 6(b), the solution for the nonsymmetric initial condition

$$u_{\text{ini}} \equiv u_{\text{ini},\text{nonsym}} = 4(x + 0.33)^2(x + 0.5)^2|x - 0.5|, \quad x \in [-0.5, 0.5],$$

(54)

also converges to $u_\lambda$. However, for $\lambda = 4.45 > \lambda^*$ and a smaller time step $\tau = 0.001$, we observe in Fig. 6(c) that, after 2654 steps, there does not exist a stable solution to the parabolic problem. We have similar results when applying the implicit scheme; these are not displayed here.

6.2.2. The 2D problem. Applying the centered scheme for the spatial discretization, we have the fully discrete semi-implicit scheme

$$(h^2 - 4\tau) u_{i,j}^{n+1} - \tau u_{i-1,j}^{n+1} - \tau u_{i+1,j}^{n+1} - \tau u_{i,j-1}^{n+1} - \tau u_{i,j+1}^{n+1} = h^2 u_{i,j}^n + \lambda \tau h^2 f(x_i, y_j)$$

(55)

with $i, j = 1, \ldots, M$, $u_{0,j}^n = u_{M+1,j}^n = u_{i,0}^n = u_{i,M+1}^n = 0$ ($i, j = 0, \ldots, M + 1$), and $n = 0, \ldots, K$. Then, one can easily get the two-dimensional fully-discrete implicit scheme, keeping the same notation and boundary conditions. Here, again, $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$. 

Figure 5. 1D, $\tau=0.01$. (a) Semi-implicit scheme; (b) Implicit scheme; (c) Convergence: error $= \|u^n - u_\lambda\|$. 

Figure 6. (a) $\lambda = 1.6$; (b) $\lambda^* = 2.684$, $u_\lambda^*(0,0)$; (c) $\lambda = 2.5$, $u_\lambda(0,0)$. The branch of solutions $u(0,0)$ as a function of $\lambda$.
Figure 6. 1D, $M = 199$, $f(x) = |2x|$. (a) $\tau = 0.01$, $\lambda = 4.0$, $u_{ini} = \text{pointwise}_\text{ini};$ (b) $\tau = 0.01$, $\lambda = 4.0$, $u_{ini}$ is nonsymmetric; (c) touchdown phenomenon: $\lambda = 4.45$, $\tau = 0.001$.

In Fig. 7, we illustrate the two-dimensional results for $\lambda = 10.0$, $f(x,y) = \sqrt{x^2 + y^2}$ and $u_{ini} = 0$, using both the semi-implicit and implicit schemes, as well as the corresponding convergence. We observe that, with each scheme, the solution increases and pointwise converges to $u_\lambda$.

Figure 7. 2D, $\lambda = 10.0$, $f(x,y) = \sqrt{x^2 + y^2}$, $\tau = 0.01$, $M = 29$. (a) The global solution when $t = 100\tau$; (b) Semi-implicit scheme; (c) Implicit scheme; (d) Convergence.
As shown in Fig. 8, when the initial value $u_{ini}$ is less than $u_\lambda$ (more precisely, we apply the cubic Lagrange polynomial on five points, (-0.5, -0.5, 0), (-0.5, 0.5, 0), (0, 0, 0.1), (0.5, -0.5, 0) and (0.5, 0.5, 0), to interpolate the initial value $u_{ini}$), the solutions corresponding to the semi-implicit and implicit schemes both increase and converge to $u_\lambda$.

Finally, setting $\tau = 0.001$, $\lambda = 11.5$ (which is larger than $\lambda^*$) and $M = 35$, the touchdown phenomenon can be observed (see Fig. 9).

**Appendix A. The continuation method.** In this appendix, we explain how to compute the upper bound on $\lambda$ (namely, the pull-in voltage $\lambda^*$) by applying a continuation method (see [1]) which we describe for the two-dimensional problem. The elliptic system (6.1.2) can be rewritten as

$$H(w) = 0,$$

where $w = (U \lambda)^t$ and $U = (u_{1,1}, \cdots, u_{M,1}, \cdots, u_{1,M}, \cdots, u_{M,M})$ is the reordered row vector of the solution to the elliptic problem (6.1.2). Let $H'(u)$ denote the Jacobian of $H(w)$ and $t(A)$ denote the tangent vector induced by $A$, which is defined by

$$H'(u)^t = \begin{bmatrix}
\frac{\partial H(u)}{\partial u_{1,1}} & \cdots & \frac{\partial H(u)}{\partial u_{1,M}} \\
\vdots & \ddots & \vdots \\
\frac{\partial H(u)}{\partial u_{M,1}} & \cdots & \frac{\partial H(u)}{\partial u_{M,M}}
\end{bmatrix},$$

$$t(A) = \begin{bmatrix}
\frac{\partial t(A)}{\partial u_{1,1}} & \cdots & \frac{\partial t(A)}{\partial u_{1,M}} \\
\vdots & \ddots & \vdots \\
\frac{\partial t(A)}{\partial u_{M,1}} & \cdots & \frac{\partial t(A)}{\partial u_{M,M}}
\end{bmatrix},$$

and

$$t(A)^t = \begin{bmatrix}
\frac{\partial t(A)}{\partial u_{1,1}} & \cdots & \frac{\partial t(A)}{\partial u_{1,M}} \\
\vdots & \ddots & \vdots \\
\frac{\partial t(A)}{\partial u_{M,1}} & \cdots & \frac{\partial t(A)}{\partial u_{M,M}}
\end{bmatrix}.$$
Definition A.1. Let $A$ be an $M \times (M + 1)$ matrix with maximal rank. The unique vector $t(A) \in \mathbb{R}^{M+1}$ satisfying the conditions
\[ At = 0, \|t\| = 1, \, \det \left( \begin{bmatrix} A \\ t^* \end{bmatrix} \right) > 0, \]
(57)
is called the tangent vector induced by $A$, where $(\cdot)^*$ denotes the Hermitian transpose.

Here and below, $M = M^2$.

We then use an approximate Euler predictor and Newton-type iterations as corrector steps, described in the following algorithm, see [1].

\begin{algorithm}{Continuation Method}
\begin{algorithmic}[1]
\Require $N_{pas}$, $\epsilon$ and $w$ such that $H(w) = 0$
\For { $n = 1, N_{pas}$ }
\State Estimate $A = H'(w)$
\State Compute $A^+$ and the tangent vector $t(A)$
\State $w := w + \epsilon t(A)$ \Comment{prediction}
\While { error larger than tolerance and Newton’s iteration step is bounded }
\State $w := w - A^+ H(w)$ \Comment{corrections}
\EndWhile
\EndFor
\end{algorithmic}
\end{algorithm}

Figure 10. The main idea of the continuation method.

In this algorithm, $A^+$ denotes the Moore–Penrose inverse of $A$ and $\epsilon$ stands for the current step size. We refer the readers to [5] and [26] for Newton’s method based on the Moore–Penrose inverse and [2] for a convergence result which ensures that the above algorithm is applicable and effective under reasonable assumptions. To compute $A^+$ and $t(A)$, we use a QR factorization (see [1] for details), that is to say, $A$ being a maximal rank $M \times (M + 1)$ matrix and $A^+$ representing its conjugate transpose, we have
\[ A^+ = A^* (A A^*)^{-1} = Q \begin{bmatrix} (R^*)^{-1} \\ 0^* \end{bmatrix} = Q \begin{bmatrix} R \\ 0^* \end{bmatrix}, \]
(58)
with the QR factorization, $A^* = Q \begin{bmatrix} R \\ 0^* \end{bmatrix}$, where $Q$ is an $(M+1) \times (M+1)$ orthogonal matrix and $R$ is a nonsingular $M \times M$ upper triangular matrix. Then, we compute
\( t(A) = \sigma q \), where \( q \) denotes the last column of \( Q \) and, in order to satisfy the orientation which has been defined in (1)(2.5), we can choose \( \sigma = \text{sign}(\det Q \det R) \).

The main idea of the continuation method is displayed in Fig. 10, using a rough prediction and several steps of corrections.

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