DIFFERENTIAL INVARIANTS OF GENERIC HYPERBOLIC MONGE–AMPÈRE EQUATIONS

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Abstract. In this paper basic differential invariants of generic hyperbolic Monge–Ampère equations with respect to contact transformations are constructed and the equivalence problem for these equations is solved.

1. Introduction

With this paper we start a systematic study of differential invariants of Monge–Ampère equations, with the objective of the classification problem, methods of integration and other applications. Complete proofs of the results announced in [14] are presented. We are interested in the classical case of two independent variables. The Monge–Ampère equations equations merit a special attention due to a large spectrum of various applications, first of all, in differential geometry and mathematical physics. Moreover, they form a natural testing area for new methods emerging in the modern theory of nonlinear PDE’s.

In spite of more than 200 years of history of Monge–Ampère equations and numerous publications devoted to them it would be an exaggeration to say that their nature is well understood. An important success was establishing the existence and uniqueness theorems by Lewy and others (see [10, 3] for local aspects and [20] for global ones). The classical Monge integration method was modernized by Matsuda [15, 16] and Morimoto [17], etc. Our interest in differential invariants is motivated not only by the classification problem but, no less, by hopes that they could illuminate many aspects of the theory of Monge–Ampère equations.

According to [22] (see also [1]) scalar differential invariants provide a key to solving the classification problem for any kind of geometrical structures. In fact, geometrical structures of a given type are classified by solutions of a naturally associated classifying (differential) equation, which describes “family ties” connecting the corresponding scalar differential invariants. More exactly, scalar differential invariants are smooth functions on the classifying diffeity, which is the infinite prolongation of the classifying equation. This diffeity has, generally, singularities and its singular strata classify those geometrical structures that possess nontrivial symmetries. Each of these strata
is also an infinitely prolonged differential equation in a lesser number of independent variables. For instance, homogeneous structures correspond to the zero-dimensional case. So, the classification problem consists of a complete description of all strata composing the classifying diffeety and, therefore, involves a complete symmetry analysis of the geometric structures under consideration. The interested reader will find an illustration of the above said in [23] where plane 3-webs, a rather simple geometrical structure, is considered.

The classification problem for Monge–Ampère equations dates back to Sophus Lie. For modern proofs of Lie’s theorems, classification problems for various strata of Monge-Ampère equations see, e.g., [13, 6, 7, 8, 9, 18] and references therein. In this paper we interpret a hyperbolic Monge–Ampère equation as a pair of 2-dimensional, skew-orthogonal non-lagrangian subdistributions of the contact distribution on 5-dimensional contact manifolds. Another approach to these equations was developed by V.V. Lychagin in [11, 12]. We look for differential invariants of Monge–Ampère equations, not only scalar, with respect to the group of contact transformations. Here we limit ourself to the case of generic hyperbolic equations. This is motivated by two reasons. First, the study of singular strata benefits much from the knowledge of the generic one. Second, for the hyperbolic equations differential invariants are easier visible due to the existence of bicharacteristics.

Differential invariants found in this paper give a solution of the classification problem for generic hyperbolic equations. This solution requires a substantial computer support in analysis of concrete cases and a further work is necessary to improve its efficiency.

Differential invariants for elliptic and parabolic Monge–Ampère equations can be obtained more or less straightforwardly by following the approach developed in this paper. This and a the study of singular strata will be the subject of subsequent publications.

2. Preliminaries

Below, all manifolds and maps are supposed to be smooth. By $[f]^k_p$, $k = 0, 1, 2, \ldots, \infty$, we denote the $k$-jet of a map $f$ at a point $p$. $\mathbb{R}$ stands for the field of real numbers, and $\mathbb{R}^n$ for the $n$-dimensional arithmetic space.

2.1. Jet bundles. Here we recall necessary definitions and facts about jet bundles, see [4, 5].

Let $M$ be an $n$-dimensional manifold, $E$ an $n + m$-dimensional manifold and

$$\pi : E \rightarrow M,$$

a fiber bundle. By

$$\pi_k : J^k \pi \rightarrow M, \quad \pi_k : [S]^k_p \mapsto p, \quad k = 0, 1, 2, \ldots$$

we denote the bundle of all $k$-jets of sections of $\pi$. For any $l > m \geq 0$, the natural projection is defined as

$$\pi_{l,m} : J^l \pi \rightarrow J^m \pi, \quad \pi_{l,m} : [S]^l_p \mapsto [S]^m_p.$$

Any section $S$ of $\pi$ generates the section $j_k S$ of the bundle $\pi_k$ by the formula

$$j_k S : p \mapsto [S]^k_p.$$
Put 

\[ L^k_S = \text{Im} j_k S. \]

Let \( \theta_{k+1} \) be an arbitrary point of \( J^{k+1} \pi \), \( \theta_k = \pi_{k+1,k}(\theta_{k+1}) \), and \( T_{\theta_k}(J^k \pi) \) the tangent space to \( J^k \pi \) at the point \( \theta_k \). Then \( \theta_{k+1} \) defines the subspace \( K_{\theta_{k+1}} \subset T_{\theta_k}(J^k \pi) \) by the formula

\[ K_{\theta_{k+1}} = T_{\theta_k}(L^k_S). \]

Clearly, \( \theta_{k+1} \) is identified with \( K_{\theta_{k+1}} \). It is easy to prove that

\[ T_{\theta_k}(J^k \pi) = K_{\theta_{k+1}} \oplus T_{\theta_k}(\pi^{-1}_k(p)). \]

Consider all submanifolds of the form \( L^k_S \) containing \( \theta_k \). Subspace spanned by their tangent spaces \( T_{\theta_k}(L^k_S) \) is denoted by \( \mathcal{C}(\theta_k) \) and it is called the **Cartan plane at** \( \theta_k \). The distribution

\[ \mathcal{C}_k : \theta_k \mapsto \mathcal{C}(\theta_k) \]

is called the **Cartan distribution on** \( J^k \pi \). The distribution \( \mathcal{C}_k, k \geq 1 \), can be defined as the kernel of the **Cartan form**

\[ U_k = \text{pr}_2 \circ (\pi_{k,k-1})_* , \]

where \( \text{pr}_2 : T_{\theta_{k-1}}(J^{k-1} \pi) \rightarrow T_{\theta_{k-1}}(\pi^{-1}_{k-1}(p)) \) is the projection generated by direct sum decomposition (1).

2.2. The contact structure. Consider the trivial bundle

\[ \tau : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2, \quad \tau : (x,y,z) \mapsto (x,y). \]

By \( x, y, z, p = z_x, q = z_y, r = z_xz, s = z_yz, t = z_yy \) we denote the standard coordinates in \( J^2 \tau \).

The Cartan distribution \( C_1 \) on \( J^1 \tau \) is identical to the contact structure on \( J^1 \tau \). The corresponding contact 1-form \( U_1 \) has the canonical form

\[ U_1 = dz - p dx - q dy. \]

in the standard coordinates.

A diffeomorphism \( \varphi : J^1 \tau \rightarrow J^1 \tau \) is called a contact transformation if it preserves the Cartan distribution. Obviously, a diffeomorphism \( \varphi \) is a contact transformation iff there exist a nowhere vanishing function \( \lambda \) such that

\[ \varphi^*(U_1) = \lambda U_1. \]

Any contact transformation \( \varphi \) can be lifted to the diffeomorphism

\[ \varphi^{(1)}_\tau : J^2 \tau \rightarrow J^2 \tau \]

by the formula

\[ \varphi^{(1)}_\tau : \theta_2 \equiv K_{\theta_2} \mapsto \varphi_*(K_{\theta_2}) \equiv \tilde{\theta}_2 = \varphi^{(1)}_\tau(\theta_2). \]

If \( \varphi \) is defined on an open set \( V \subset J^1 \tau \), then \( \varphi^{(1)}_\tau \) is defined on an open, everywhere dense subset of \( \tau_{2,1}^{-1}(V) \).

A vector field \( Z \) in \( J^1 \tau \) is a contact vector field if its flow \( \varphi_t \) consists of contact transformations. Clearly, \( Z \) is a contact vector field iff there exist a function \( \lambda \) such that

\[ L_Z(U_1) = \lambda U_1, \]

where \( L_Z \) is the Lie derivative with respect to \( Z \).
There exists a natural one-to-one correspondence between the set of all contact vector fields in \(J^1\tau\) and the set of all functions in \(J^1\tau\). It is defined by the formula
\[
Z \mapsto f = Z \mathcal{J}U_1.
\]
The function \(f = Z \mathcal{J}U_1\) is called the generating function of the contact vector field \(Z\). The contact vector field \(Z\) corresponding to \(f\) is denoted by \(Z_f\). In standard coordinates, the field \(Z_f\) is given by the formula
\[
Z_f = -f_p \frac{\partial}{\partial x} - f_q \frac{\partial}{\partial y} + \left( f - pf_p - qf_q \right) \frac{\partial}{\partial z} + \left( f_x + pf_z \right) \frac{\partial}{\partial p} + \left( f_y + qf_z \right) \frac{\partial}{\partial q}.
\]

2.3. Operations over vector-valued forms. Let \(M\) be a smooth \(n\)-dimensional manifold, \(\Lambda^i(M)\) the \(C^\infty(M)\)-module of \(i\)-forms on \(M\) and \(D(M)\) the \(C^\infty(M)\)-module of vector fields on \(M\). Let \(\alpha \in \Lambda^k(M)\), \(\beta \in \Lambda^r(M)\), and \(X, Y \in D(M)\). Then the Frölicher–Nijenhuis bracket \([\cdot, \cdot]\) of the vector-valued forms \(\alpha \otimes X\) and \(\beta \otimes Y\) is defined by the formula
\[
\begin{align*}
[\alpha \otimes X, \beta \otimes Y] &= \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge X(\beta) \otimes Y - Y(\alpha) \wedge \beta \otimes X \\
&\quad + (-1)^k d\alpha \wedge (X \mathcal{J} \beta) \otimes Y - (-1)^k (Y \mathcal{J} \alpha) \wedge d\beta \otimes X,
\end{align*}
\]
see [2].

The contraction \(\mathcal{J}\) of forms \(\alpha \otimes X\) and \(\beta \otimes Y\) is defined by the formula
\[
(\alpha \otimes X) \mathcal{J}(\beta \otimes Y) = \alpha \wedge (X \mathcal{J} \beta) \otimes Y.
\]

2.4. Projectors and their curvatures. The following simple construction allows one to associate a vector valued 2-form with a projector. Namely, let \(P, Q \in D(M)\) be endomorphisms of the \(C^\infty(M)\)-module \(D(M)\) such that \(QP = 0\). Then
\[
\Omega_{Q,P}(X,Y) = Q[P(X), P(Y)], \quad X, Y \in D(M),
\]
otherwise, is skew-symmetric and \(C^\infty(M)\)-bilinear, i.e., a vector valued form. More precisely, it takes values in \(\text{Im } Q \subset D(M)\). If \(P : D(M) \to D(M)\) is a projector, i.e., \(P^2 = P\), then the associated curvature form of \(P\) is defined to be
\[
\mathcal{R}_P = \Omega_{I - P, P}
\]
with \(I = \text{id}_{D(M)}\).

3. Hyperbolic Monge–Ampère equations

3.1. Monge–Ampère equations. The Monge–Ampère equation is a partial differential equation of the form
\[
N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0,
\]
where \(x, y\) are independent variables, \(z\) is a dependent variable, \(z_{xx} = \partial^2 z/\partial x^2\), \(z_{xy} = \partial^2 z/\partial x \partial y\), \(z_{yy} = \partial^2 z/\partial y^2\), and coefficients \(N, A, B, C, D\) are functions of \(x, y, z, z_x = \partial z/\partial x\) and \(z_y = \partial z/\partial y\).

We identify equation (5) with the submanifold \(E\) of the jet bundle \(J^2\tau\) determined by the equation
\[
N(rt - s^2) + Ar + Bs + Ct + D = 0.
\]
Obviously,
\[ \tau_{2,1}(\mathcal{E}) = J^1 \tau. \]

Let \( \theta_2 \in \mathcal{E}, \tau_{2,1}(\theta_2) = \theta_1, \) and \( F_{\theta_1} \) be the fiber of the projection \( \tau_{2,1} \) over the point \( \theta_1 \in J^1 \tau. \) Then the subspace
\[ \operatorname{Smlb}_{\theta_2} \mathcal{E} = T_{\theta_2} \mathcal{E} \cap T_{\theta_2} F_{\theta_1}, \]
where \( T_{\theta_2} \mathcal{E} \) is the tangent space to \( \mathcal{E} \) at the point \( \theta_2 \) is called the symbol of the equation \( \mathcal{E} \) at the point \( \theta_2 \in \mathcal{E}. \) In terms of standard coordinates, \( \operatorname{Smlb}_{\theta_2} \mathcal{E} \) is described by the linear equation
\[ N(t\tilde{r} + r\tilde{s} - 2s\tilde{t}) + A\tilde{r} + B\tilde{s} + C\tilde{t} = 0, \]
where \( \tilde{r}, \tilde{s}, \tilde{t} \) are the standard coordinates in \( T_{\theta_2} F_{\theta_1} \) generated by the standard coordinates on \( J^2 \tau. \)

Taking into account (9), we observe that all 1-ray subspaces form the cone \( \tau \mathcal{E} \) at the point \( \theta_2 \in \mathcal{E}. \) In terms of standard coordinates, \( \tau \mathcal{E} \) can be elliptic, parabolic, or hyperbolic. To introduce these notions, let us consider a one-dimensional subspace \( P \subset \mathcal{E}(\theta_1) \) such that \( (\tau_{1})_* P \neq 0. \) By definition, put
\[ l(P) = \{ \theta_2 \in F_{\theta_1} \mid P \subset K_{\theta_2} \}. \]

The submanifold \( l(P) \) is called a 1-ray. In terms of standard coordinates, let \( \theta_1 = (x,y,z,p,q), \) \( P = \langle v \rangle \) and
\[ v = \zeta_1 \frac{\partial}{\partial x} + \zeta_2 \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial z} + \eta_1 \frac{\partial}{\partial p} + \eta_2 \frac{\partial}{\partial q}. \]

Then \( (\tau_{1})_* P \neq 0 \) means that
\[ (\zeta_1, \zeta_2) \neq (0,0), \]
\[ v \in \mathcal{E}(\theta_1) \] means that
\[ \mu = \zeta_1 p + \zeta_2 q, \]
and \( P \subset K_{\theta_2} \) means that
\[ \begin{cases} \eta_1 = \zeta_1 r + \zeta_2 s, \\ \eta_2 = \zeta_1 s + \zeta_2 t, \end{cases} \]
where \( r, s, t \) are the standard coordinates of \( \theta_2 \) in the fiber \( F_{\theta_1}. \) From system (11), we see that \( l(P) \) is an affine straight line in \( F_{\theta_1}. \) By \( \ell_{\theta_2}(P) \) we denote the tangent space \( T_{\theta_2} l(P) \) to \( l(P) \) at the point \( \theta_2 \in l(P). \) We call it a 1-ray subspace. In terms of the standard coordinates \( \tilde{r}, \tilde{s}, \tilde{t} \) in \( T_{\theta_2} F_{\theta_1} \), vectors of \( \ell_{\theta_2}(P) \) satisfy
\[ \begin{cases} \zeta_1 \tilde{r} + \zeta_2 \tilde{s} = 0, \\ \tilde{1} \tilde{s} + \zeta_2 \tilde{t} = 0, \end{cases} \]
Obviously, \( \ell_{\theta_2}(P) \) is spanned by the vector
\[ (\tilde{r}, \tilde{s}, \tilde{t}) = (\zeta_2^2, -\zeta_1 \zeta_2, \zeta_1^2). \]
Taking into account (9), we observe that all 1-ray subspaces form the cone
\[ \mathcal{V}_{\theta_2} = \{ \tilde{r} \tilde{t} - \tilde{s}^2 = 0 \} \]
in the tangent space \( T_{\theta_2} F_{\theta_1}. \) This cone is called the cone of singular square forms. Obviously, the intersection \( \operatorname{Smlb}_{\theta_2} \mathcal{E} \cap \mathcal{V}_{\theta_2} \) is either zero, or a single 1-ray subspace, or two 1-ray subspaces. Correspondingly, the point \( \theta_2 \in \mathcal{E} \) is then called elliptic, parabolic or hyperbolic. It is not difficult to prove that a contact transformation takes an elliptic, parabolic, or hyperbolic point to
an elliptic, parabolic, or hyperbolic point, respectively. The equation $E$ is called elliptic, parabolic or hyperbolic if all its points are elliptic, parabolic or hyperbolic, respectively. In this work, we consider hyperbolic Monge–Ampère equations only. It is easy to see that $E$ is hyperbolic iff its coefficients satisfy the condition

$$\Delta = B^2 - 4AC + 4ND > 0.$$  \hspace{1cm} (14)

3.2. Skew-orthogonal distributions. Following [21], we show that a hyperbolic Monge–Ampère equation is equivalent to a pair of skew-orthogonal two-dimensional distributions in the Cartan distribution on $J^1\tau$.

Let $\theta_1$ be an arbitrary point of $J^1\tau$. By $Q_{\theta_1}$ we denote the union of all one-dimensional subspaces $P$ of $\mathcal{C}(\theta_1)$ such that $\tau_*P \neq 0$ and the 1-ray $l(P)$ is tangent to $E$ at least at one point.

**Proposition 3.1.** Let $E$ be a hyperbolic Monge–Ampère equation. Then $Q_{\theta_1}$ is the union of two-dimensional subspaces $D_1^E(\theta_1)$ and $D_2^E(\theta_1)$ of the Cartan plane $\mathcal{C}(\theta_1)$, so that

1. $\mathcal{C}(\theta_1) = D_1^E(\theta_1) \oplus D_2^E(\theta_1)$,
2. $D_1^E(\theta_1)$ and $D_2^E(\theta_1)$ are skew-orthogonal with respect to the symplectic form $dU_1 = dx \wedge dp + dy \wedge dq$ on $\mathcal{C}$.

**Proof.** We prove this proposition for Monge–Ampère equations such that $N \neq 0$. The proof for $N = 0$ follows from the fact that every Monge–Ampère equation can be transformed to one with $N \neq 0$ by an appropriate contact transformation.

Let $v \in Q_{\theta_1}$ and $P = \langle v \rangle$. The condition for $l(P)$ to be tangent to $E$ can be written in the following way. We can assume that $v$ is of the form (8). Then the vector of fiber coordinates $(\zeta_2^2, -\zeta_1\zeta_2, \zeta_1^2)$ is tangent to $l(P)$. Now using (7) we deduce that $l(P)$ is tangent to $E$ iff

$$N(r\zeta_1^4 + 2s\zeta_1\zeta_2 + t\zeta_2^2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0.$$  \hspace{1cm} (15)

Taking into account that the coordinates $\zeta_i$ and $\eta_i$ of $v$ are connected by equations (11), we reduce this equation to the form

$$N(\zeta_1\eta_1 + \zeta_2\eta_2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0.$$  \hspace{1cm} (15)

Then in view of (9) we assume that $\zeta_1 \neq 0$ (the case $\zeta_2 \neq 0$ is analogous). Then from (11) we get

$$r = \frac{1}{\zeta_1^2}(\eta_1\zeta_1 - \eta_2\zeta_2 + \zeta_2^2), \quad s = \frac{1}{\zeta_1^2}(\eta_2 - \zeta_2 t).$$

Substituting these expressions for $r$ and $s$ in equation (6) and taking into account equation (15), we obtain the equation

$$Nh_2^2 + (A\zeta_2 - B\zeta_1)\eta_2 - A\zeta_1\eta_1 - D\zeta_1^2 = 0.$$  \hspace{1cm} (16)

Solving the system of equations (15) and (16) with respect to $\eta_1$ and $\eta_2$, we find

$$\eta_1 = \frac{(B \pm \sqrt{A})\zeta_2 - 2C\zeta_1}{2N}, \quad \eta_2 = \frac{(B \pm \sqrt{A})\zeta_1 - 2A\zeta_2}{2N}.$$
Finally, in view of (10), we see that

\[
v = \zeta_1 \left( \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} - \frac{C}{N} \frac{\partial}{\partial p} + \frac{B \pm \sqrt{\Delta}}{2N} \frac{\partial}{\partial q} \right) + \zeta_2 \left( \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} + \frac{B \mp \sqrt{\Delta}}{2N} \frac{\partial}{\partial p} - \frac{A}{N} \frac{\partial}{\partial q} \right), \tag{17}
\]

This proves that \( \Omega_{\theta_1} = \langle X_1, X_2 \rangle \cup \langle X_3, X_4 \rangle \) with

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} - \frac{C}{N} \frac{\partial}{\partial p} + \frac{B - \sqrt{\Delta}}{2N} \frac{\partial}{\partial q}, \\
X_2 &= \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} + \frac{B + \sqrt{\Delta}}{2N} \frac{\partial}{\partial p} - \frac{A}{N} \frac{\partial}{\partial q}, \\
X_3 &= \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} - \frac{C}{N} \frac{\partial}{\partial p} + \frac{B + \sqrt{\Delta}}{2N} \frac{\partial}{\partial q}, \\
X_4 &= \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} + \frac{B - \sqrt{\Delta}}{2N} \frac{\partial}{\partial p} - \frac{A}{N} \frac{\partial}{\partial q}.
\end{align*}
\tag{18}
\]

Put

\[
\mathcal{D}_E^1(\theta_1) = \langle X_1, X_2 \rangle, \quad \mathcal{D}_E^2(\theta_1) = \langle X_3, X_4 \rangle.
\]

Now it is straightforward to verify that subspaces \( \mathcal{D}_E^1(\theta_1) \) and \( \mathcal{D}_E^2(\theta_1) \) are skew-orthogonal and \( \mathcal{D}_E^1(\theta_1) \cap \mathcal{D}_E^2(\theta_1) = \{0\} \). This completes the proof. \( \square \)

From (18) we see that for a Monge–Ampère equation such that \( N \neq 0 \), the map \( \tau_1 \ast \) projects \( \mathcal{D}_E^1(\theta_1) \) and \( \mathcal{D}_E^2(\theta_1) \) onto the tangent space to the base of the bundle \( \tau \) without degeneration.

It should be noted that if \( N = 0 \) (that is, if \( E \) is a quasilinear second order PDE), then the projections \( \tau_1 \ast (\mathcal{D}_E^1(\theta_1)) \) and \( \tau_1 \ast (\mathcal{D}_E^2(\theta_1)) \) are one-dimensional.

Thus an arbitrary hyperbolic Monge–Ampère equation generates two 2-dimensional skew-orthogonal subdistributions of the Cartan distribution \( \mathcal{C}_1 \) in \( J^1\tau \).

**Proposition 3.2.** Let \( E \) be a hyperbolic Monge–Ampère equation. Then \( \theta_2 \in \mathcal{E} \) if and only if one of the following equivalent conditions holds:

1. \( K_{\theta_2} \cap \mathcal{D}_E^1(\theta_1) \) is 1-dimensional,
2. \( K_{\theta_2} \cap \mathcal{D}_E^2(\theta_1) \) is 1-dimensional.

**Proof.** As in the proof of Proposition 3.1 one can assume that \( N \neq 0 \).

Let \( \theta_2 \in \mathcal{E} \). Then Smbl\( \theta_2 \) \( \mathcal{C} \cap V_{\theta_2} = \ell_{\theta_2}(\langle v \rangle) \cup \ell_{\theta_2}(\langle \tilde{v} \rangle) \), where \( \ell_{\theta_2}(\langle v \rangle) \) and \( \ell_{\theta_2}(\langle \tilde{v} \rangle) \) are different straight lines and, so, vectors \( v \) and \( \tilde{v} \) are independent. They are skew-orthogonal, since \( K_{\theta_2} \) is a Lagrangian plane in \( \mathcal{C}(\theta_1) \) and, by definition of \( \Omega_{\theta_1}, v, \tilde{v} \in \Omega_{\theta_1} \). This means that \( K_{\theta_2} \) intersects planes \( \mathcal{D}_E^1(\theta_1) \) and \( \mathcal{D}_E^2(\theta_1) \) along \( \langle v \rangle \) and \( \langle \tilde{v} \rangle \), respectively.

Let \( \theta_2 \) be a point of \( J^2\tau \) such that \( K_{\theta_0} \) intersects the plane \( \mathcal{D}_E^1(\theta_1) \) along a straight line, that is, \( K_{\theta_2} \cap \mathcal{D}_E^1(\theta_1) = \langle v \rangle \). By substituting coordinates \( \eta_1, \eta_2 \)
of the vector \( v \) given by formula (17) into eq. (11), we obtain
\[
\left( r + \frac{C}{N} \right) \zeta_1 + \left( s - \frac{B - \sqrt{\Delta}}{2N} \right) \zeta_2 = 0, \\
\left( s - \frac{B + \sqrt{\Delta}}{2N} \right) \zeta_1 + \left( r + \frac{A}{N} \right) \zeta_2 = 0.
\]
By hypothesis this system is of rank 1 (cf. (9)) and hence its determinant is zero. Now it remains to note that this is exactly equation (6) and, so, \( \theta_2 \in \mathcal{E} \). The case of \( \mathcal{D}_2^2(\theta_1) \) differs only by the sign at \( \sqrt{\Delta} \).

□

An important consequence of this proposition is that a hyperbolic Monge–Ampère equation \( \mathcal{E} \) is completely determined by one of the associated distributions \( \mathcal{D}_i \mathcal{E} \), \( i = 1, 2 \).

Thus, every hyperbolic Monge–Ampère equation \( \mathcal{E} \) is naturally equivalent to a pair of 2-dimensional, skew-orthogonal non-lagrangian subdistributions \( \mathcal{D}_1 \mathcal{E}, \mathcal{D}_2 \mathcal{E} \) of the Cartan distribution \( \mathcal{C}_1 \) in \( J^1 \tau \). In particular, the equivalence problem for hyperbolic Monge–Ampère equations with respect to contact transformations may be interpreted as the equivalence problem for pairs of 2-dimensional, skew-orthogonal non-lagrangian subdistributions of \( \mathcal{C}_1 \) with respect to contact transformations.

3.3. **Bundles of Monge–Ampère equations.** From now on we put \( M = J^1 \tau \).

3.3.1. **Bundles of hyperbolic Monge–Ampère equations.** Let \( \mathcal{E} \) be a Monge–Ampère equation (5). It is identified with the section
\[
S_\mathcal{E} : x \mapsto [N(x) : A(x) : B(x) : C(x) : D(x)]
\]
of the trivial bundle
\[
\rho : \mathbb{RP}^4 \times M \rightarrow M, \quad ([p^0 : p^1 : p^2 : p^3 : p^4], x) \mapsto x,
\]
where \( \mathbb{RP}^4 \) is the 4-dimensional projective space. Obviously, this identification is a bijection of the set of all Monge–Ampère equations onto the set of all sections of \( \rho \).

Consider the open subset \( E \) of the total space of \( \rho \) defined by the condition (14), i.e.,
\[
(p^2)^2 - 4p^1p^3 + 4p^4p^0 > 0.
\]
Clearly, the section \( S_\mathcal{E} \) corresponding to a hyperbolic Monge–Ampère equation \( \mathcal{E} \) takes values in \( E \). Thus we can define the bundle of hyperbolic Monge–Ampère equations by the formula
\[
\pi = \rho|_E : E \rightarrow M, \quad ([p^0 : p^1 : p^2 : p^3 : p^4], x) \mapsto x. \quad (19)
\]

We use local coordinates \( x, y, z, p, q, u^1, \ldots, u^4 \) in the total space \( E \) of \( \pi \), where \( x, y, z, p, q \) are the standard coordinates on \( M \), while the coordinates \( u^1, \ldots, u^4 \) on the fibres of \( \pi \) are defined as follows. Consider the affine hyperplane in \( \mathbb{R}^5 \) defined by the equation \( p^5 = 1 \). It generates the local chart in \( E \)
\[
[1 : p^1 : p^2 : p^3 : p^4] \mapsto (p^1, p^2, p^3, p^4).
\]
Following formulas (18), we introduce the local coordinates $u^1, \ldots, u^4$ along the fibres of $\pi$ by

$$u^1 = -p^3, \quad u^2 = \frac{p^2 - \sqrt{\Delta}}{2}, \quad u^3 = \frac{p^2 + \sqrt{\Delta}}{2}, \quad u^4 = -p^1,$$

(20)

where $\Delta = (p^2)^2 - 4p^1p^3 + 4p^4$.

These coordinates extend to the standard coordinates $x, y, z, p, q, u^1, u^2, u^i, u^j, u^k, u^l, \ldots, u^r, \ldots$, on $J^k\pi$, used in this paper until we replace them with a more convenient set in Sect. 4.3.

3.3.2. The lifting of contact transformations. Let $\varphi$ be a contact transformation defined in $M$. Then $\varphi$ transforms any Monge–Ampère equation $E$ to another Monge–Ampère equation $\tilde{E}$. In other words, $\varphi$ induces a transformation of the corresponding sections $S_E \mapsto S_{\tilde{E}}$ and, consequently, a diffeomorphism $\varphi^{(0)}$ of the total space of $\pi$ such that the diagram

$$
\begin{array}{c}
E \xrightarrow{\varphi^{(0)}} E \\
\pi \downarrow \quad \downarrow \pi \\
M \xrightarrow{\varphi} M
\end{array}
$$

is commutative (in the domain of $\varphi^{(0)}$). The diffeomorphism $\varphi^{(0)}$ is called the lifting of $\varphi$ to the bundle $\pi$.

The diffeomorphism $\varphi^{(0)}$, in its turn, can be lifted to a diffeomorphism $\varphi^{(k)}$ of $J^k\pi$ by the formula

$$
\varphi^{(k)}\left([S]_{x}^{k}\right) = \left[[\varphi^{(0)} \circ S \circ \varphi^{-1}]^{k}\right]_{\varphi(x)}.
$$

Obviously, for any $l > m$, the diagram

$$
\begin{array}{c}
J^l\pi \xrightarrow{\varphi^{(l)}} J^l\pi \\
\pi_{l,m} \downarrow \quad \downarrow \pi_{l,m} \\
J^m\pi \xrightarrow{\varphi^{(m)}} J^m\pi
\end{array}
$$

is commutative (in the domains of $\varphi^{(l)}$). The diffeomorphism $\varphi^{(k)}$ is called the lifting of $\varphi$ to the jet bundle $J^k\pi$.

3.3.3. The lifting of contact vector fields. Let $Z$ be a contact vector field in $M$ and let $\varphi_t$ be its flow. Then $\varphi_{t(k)}$ defines a vector field $Z^{(k)}$ in $J^k\pi$. This field is called the lifting of $Z$ to $J^k\pi$. Obviously,

$$
(\pi_{l,m} \circ Z^{(l)}) = Z^{(m)}, \quad \infty \geq l > m \geq -1,
$$

where $Z^{(-1)} = Z$.

It is not difficult to see that the map

$$
Z \mapsto Z^{(k)}
$$

is a homomorphism of the Lie algebra of all contact vector fields onto the Lie algebra generated by all vector fields of the form $Z^{(k)}$.

The local expression of $Z^{(k)}$ can be found as follows. First, change the notation by putting $x^1 = x, x^2 = y, x^3 = z, x^4 = p, x^5 = q$. Recall that
the operator $D_j$ of total derivative with respect to $x^j$ in $J_\infty$ is given by the formula

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|\sigma| \geq 0} \sum_{i=1}^4 u^i_{\sigma j} \frac{\partial}{\partial u^i_{\sigma}} , \quad j = 1, 2, \ldots, 5,$$

The operator of evolution differentiation corresponding to a generating function $\psi(Z) = (\psi^1(Z), \ldots, \psi^4(Z))^t$ is defined by the formula

$$\mathcal{E}_{\psi(Z)} = \sum_{|\sigma| \geq 0} \sum_{i=1}^4 D_\sigma (\psi^i(Z)) \frac{\partial}{\partial u^i_{\sigma}},$$

where $\sigma = \{j_1 \ldots j_r\}$, $D_\sigma = D_{j_1} \circ \ldots \circ D_{j_r}$, and $\psi(Z)$ is defined as follows.

Let $S$ be a section of $\pi$ defined in the domain of $Z$, $\theta_1 = [S]^1_\pi$, and $x = \pi_1(\theta_1)$; then

$$\psi(Z)(\theta_1) = \frac{d}{dt} (\varphi^0_t \circ S \circ \varphi^{-1}_t)|_{t=0}(x).$$

If

$$Z = \sum_{i=1}^5 Z^i \frac{\partial}{\partial x^i},$$

then the lifting $Z^{(\infty)}$ is defined by the formula (see [4, 5])

$$Z^{(\infty)} = \sum_{j=1}^5 Z^j D_j + \mathcal{E}_{\psi(Z)}.$$

It follows from this formula that

$$Z^{(k)} = \sum_{j=1}^5 Z^j D_j^k + \mathcal{E}_{\psi(Z)}^k,$$

where

$$D_j^k = \frac{\partial}{\partial x^j} + \sum_{0 \leq |\sigma| \leq k} \sum_{i=1}^4 u^i_{\sigma j} \frac{\partial}{\partial u^i_{\sigma}}, \quad \mathcal{E}_{\psi(Z)}^k = \sum_{0 \leq |\sigma| \leq k} \sum_{i=1}^4 D_\sigma (\psi^i(Z)) \frac{\partial}{\partial u^i_{\sigma}}.$$

Let $f$ be the generating function of the contact vector field $Z$ (see formula (2)) and $\theta_1 = (x, y, z, p, q, u^i, u^i_x, u^i_y, u^i_z, u^i_p, u^i_q)$. Then the vector $\psi(Z_f)(\theta_1)$
particular, that\( k \)th-order scalar invariants are first integrals of all contact transformations of \( M \). Its action is lifted to \( J^k \pi \), \( k \geq 0 \), as it was explained above.

A function (vector field, differential form, or any other natural geometric object on \( J^k \pi \)) is a \( k \)th-order differential invariant of \( \Gamma \) if for any \( \varphi \in \Gamma \) the lifted transformation \( \varphi^{(k)} \) preserves this object. In this work these differential invariants are called also differential invariants (of order \( k \)) of Monge–Ampère equations or simply differential invariants (of order \( k \)).

Let \( \mathcal{E} \) be a Monge–Ampère equation, \( S_\mathcal{E} \) the section of \( \pi \) identified with \( \mathcal{E} \), and \( I \) a differential invariant of order \( k \). Then the value of \( I \) on \( \mathcal{E} \) is defined as \( (j_k S_\mathcal{E})^*(I) \) and denoted by \( I_\mathcal{E} \). If a contact transformation \( f \) transforms \( \mathcal{E} \) to \( \tilde{\mathcal{E}} \), then, obviously, \( f^{(k)} \) transforms \( I_\mathcal{E} \) to \( I_{\tilde{\mathcal{E}}} \), for any \( k \)th order invariant \( I \).

Differential invariants that are functions are also called scalar differential invariants. By \( A_k \) we denote the \( \mathbb{R} \)-algebra of all scalar differential invariants of order \( \leq k \). By identifying \( A_k \) with \( \pi^{l,k}_{l,k}(A_k) \subset A_l \), \( \forall k \leq l \), one gets a sequence of inclusions

\[
A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} \subset \ldots
\]

The \( \mathbb{R} \)-algebra \( A = \bigcup_{k=0}^{\infty} A_k \) is called the algebra of scalar differential invariants of Monge–Ampère equations.

**Remark 3.3.** It is worth noticing that a scalar differential invariant \( I \) is completely determined by its values \( I_\mathcal{E} \) on concrete equations \( \mathcal{E} \). This observation will be used below.

Let \( Z \) be a contact vector field in \( M \) and \( I \) a differential invariant of order \( k \). Then \( L_{Z(k)}(I) = 0 \), where \( L \) stands for the Lie derivative. This means, in particular, that \( k \)th order scalar invariants are first integrals of all contact
vector fields lifted to $J^k\pi$. Obviously, a scalar differential invariant of order $k$ is constant on any orbit of the action of $\Gamma$ on $J^k\pi$. Such an orbit consists, generally, of two components, since contact transformations need not be orientation preserving (e.g., the famous Legendre transformation $x' = p$, $y' = q$, $z' = xp + yq - z$, $p' = x$, $q' = y$ is not). In other words, the above-mentioned first integrals of $Z^{(k)}$ are, generally, invariant only with respect to the unit component of $\Gamma$ and will be called almost invariant. Anyway, generic orbits of contact transformations and of contact vector fields have the same dimension:

**Proposition 3.4.**

1. $J^k\pi$ is an orbit of the action of $\Gamma$ iff $k = 0, 1$.
2. Codimension of a generic orbit of $J^2\pi$ is equal to 2.
3. Codimension of a generic orbit of $J^3\pi$ is equal to 29.

**Proof.** Let $\theta_k$ be a generic point of $J^k\pi$ and $\text{Orb}_{\theta_k}$ the orbit of the action of $\Gamma$ on $J^k\pi$ passing through $\theta_k$. Then $\text{codim Orb}_{\theta_k} = \dim J^k\pi - \dim \text{Orb}_{\theta_k}$. The dimension of $\text{Orb}_{\theta_k}$ is the dimension of the subspace spanned by all vectors $X^{(k)}(\theta_k)$ which can be calculated with the help of computer algebra using formulas (22) and (23). □

An immediate consequence of the above proposition is

**Corollary 3.5.**

1. The algebra of scalar differential invariants $A_2$ is generated by 2 functionally independent invariants.
2. The algebra of scalar differential invariants $A_3$ is generated by 29 functionally independent invariants.

Differential invariants constructed below come mainly from natural geometric constructions without saying that these are invariant with respect to the full pseudo-group $\Gamma$. Although not impossible, it is quite challenging task to obtain first integrals of $Z^{(k)}$ analytically even for small $k$.

4. Differential invariants on $J^2\pi$

The next step to be done is explicit construction of differential invariants that generate $A_2$ as a $C^\infty$-closed algebra.

4.1. Base projectors. Let $D$ be a distribution on $M$. Denote by $D^{(1)}$ the distribution generated by all vector fields $X$ and $[X,Y]$, $\forall X, Y \in D$. Setting $D^{(0)} = D$, we define $D^{(r+1)}$, $r = 0, 1, \ldots$, inductively by the formula $D^{(r+1)} = (D^{(r)})^{(1)}$.

**Lemma 4.1.** For a hyperbolic Monge–Ampère equation $\mathcal{E}$

$$\dim(D^1_{\mathcal{E}})^{(1)} = \dim(D^2_{\mathcal{E}})^{(1)} = 3.$$ 

**Proof.** Let $\omega \in \Lambda^1(M)$ and $X, Y \in D(M)$ be such that $\omega(X) = \omega(Y) = 0$. Then, by applying formula $d\omega(X, Y) = L_X(Y \mathcal{J} \omega) - L_Y(X \mathcal{J} \omega) = [X,Y] \mathcal{J} \omega$, one easily finds that

$$\omega([X,Y]) = -d\omega(X,Y).$$

If now $\omega = U_1$ and vector fields $X, Y \in D^i_{\mathcal{E}}, i = 1, 2$, are independent, then $dU_1(X,Y) \neq 0$ due to hyperbolicity of $\mathcal{E}$. So, the above formula shows that $U_1([X,Y]) \neq 0$, i.e., that $[X,Y]$ does not belong to the Cartan distribution on $M$. So, $[X,Y]$ is independent on $X$ and $Y$. □
Restricting ourselves to the generic case only, we assume from now on that
\[ \dim(D^1_\mathcal{E}(2)) = \dim(D^2_\mathcal{E}(2)) = 5. \] (24)

Suppose that vector fields \( X_1, X_2 \) generate the distribution \( D^1_\mathcal{E} \) and vector fields \( X_3, X_4 \) generate the distribution \( D^2_\mathcal{E} \). The 3-dimensional generic distributions \( \langle X_1, X_2, [X_1, X_2] \rangle \) and \( \langle X_3, X_4, [X_3, X_4] \rangle \) intersect along a one-dimensional subdistribution \( D^3_\mathcal{E} = \langle X_1, X_2, [X_1, X_2] \rangle \cap \langle X_3, X_4, [X_3, X_4] \rangle \). Hence, equation \( \mathcal{E} \) generates a direct sum decomposition
\[ T(M) = D^1_\mathcal{E} \oplus D^2_\mathcal{E} \oplus D^3_\mathcal{E}. \] (25)

This decomposition generates six projections
\[
\begin{align*}
P_i : T(M) &\to D^i_\mathcal{E}, \quad i = 1, 2, 3, \\
P_j^{(1)} : T(M) &\to D^1_\mathcal{E} \oplus D^3_\mathcal{E}, \quad j = 1, 2, \\
P_c : T(M) &\to \mathcal{C} = D^1_\mathcal{E} \oplus D^2_\mathcal{E}.
\end{align*}
\]

These projections may be viewed as vector-valued 1-forms. Namely, let \( X_5 \) be a vector field generating \( D^3_\mathcal{E} \). Consider the co-frame \( \{\omega^1, \ldots, \omega^5\} \) on \( M \) dual to the frame \( \{X_1, \ldots, X_5\} \), i.e., \( \omega_i(X_j) = \delta_{ij} \). Then
\[
\begin{align*}
P_1 &= \omega^1 \otimes X_1 + \omega^2 \otimes X_2, \\
P_2 &= \omega^3 \otimes X_3 + \omega^4 \otimes X_4, \\
P_3 &= \omega^5 \otimes X_5, \\
P_j^{(1)} &= P_j + P_3, \quad j = 1, 2, \\
P_c &= P_1 + P_2.
\end{align*}
\]

These vector-valued differential 1-forms are, obviously, differential invariants of \( \mathcal{E} \) with respect to contact transformations. Moreover, according to proposition 3.2, the original equation \( \mathcal{E} \) is completely determined by each of the projectors \( P_1, P_2 \).

4.2. Coordinate-wise description of base projectors. In order to find local expressions for the above projectors, consider vector fields \( X_1, \ldots, X_4 \) given by (18) and use the notation (20), i.e.,
\[
\begin{align*}
X_1 &= \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} + u^1 \frac{\partial}{\partial p} + u^2 \frac{\partial}{\partial q}, \\
X_2 &= \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} + u^3 \frac{\partial}{\partial p} + u^4 \frac{\partial}{\partial q}, \\
X_3 &= \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} + u^1 \frac{\partial}{\partial p} + u^3 \frac{\partial}{\partial q}, \\
X_4 &= \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} + u^3 \frac{\partial}{\partial p} + u^4 \frac{\partial}{\partial q}.
\end{align*}
\] (27)

The remaining field \( X_5 \) is defined by the relation
\[ X_5 = \lambda^1 X_1 + \lambda^2 X_2 + \kappa [X_1, X_2] = \lambda^3 X_3 + \lambda^4 X_4 + \chi [X_3, X_4]. \] (28)

A simple computation shows that
\[ \lambda^3 = \lambda^1, \quad \lambda^4 = \lambda^2, \quad \chi = -\kappa \neq 0, \]
with
\[
\lambda^1 = \frac{1}{u^2 - u^3} \left( (u^2 + u^3) y + q(u^2 + u^3) z + u^4(u^2 + u^3) q - 2(u^4 + pu^2 + u_1 u_2) - (u^2 + u^3) u_q^4 + u^3 u_p^2 + u^2 u_q^3) \right),
\]
\[
\lambda^2 = \frac{1}{u^2 - u^3} \left( (u^2 + u^3) x + p(u^2 + u^3) y + u^1(u^2 + u^3) p - 2(u^1 + qu^2 + u_4 u_1) - (u^2 + u^3) u_p^1 + u^2 u_q^3 + u^3 u_q^2) \right),
\]
provided that \( X_5 \) is normalized by the requirement \( \kappa = 1 \).

Brackets of vector fields \( X_1, \ldots, X_5 \) are described by means of the coefficients \( b_{jk}^i \):
\[
[X_j, X_k] = \sum_{i=1}^{5} b_{jk}^i X_i.
\]
Obviously, \( b_{jk}^i = -b_{kj}^i \).

4.3. Convenient coordinates on \( J^k \pi \). Vector fields \( X_i, i = 1, \ldots, 5 \) induce vector fields \( \mathcal{X}_i \) on the bundle \( J^k \pi \), uniquely defined by the condition \( j^k(S_c) \mathcal{X}_i = X_i \) for all sections \( S_c \). Thus, \( \mathcal{X}_1 = D_1 + pD_3 + u^1 D_4 + u^2 D_5 \), etc., where \( D_i \) denote the total derivatives, see Sect. 3.3.3.

Differential invariants of hyperbolic Monge–Ampère equations constructed below are described in terms of the quantities \( \mathcal{X}_{i_1} \ldots \mathcal{X}_{i_h} b_{ij}^k \). So, we need to know all algebraic relations connecting them, at least for \( h = 0, 1 \). To find these efficiently it is convenient to use a non-standard local chart in \( J^k \pi \).

**Lemma 4.2.** Functions
\[
\tilde{u}_{i_1 \ldots i_h}^j = \mathcal{X}_{i_1} \ldots \mathcal{X}_{i_h} u^j, \quad i_1 \leq \ldots \leq i_h, \ h \leq k,
\]

together with functions \( x^i, w^j \) constitute a local chart on \( J^k \pi \). Moreover, the standard jet coordinates on \( J^k \pi \) are rational functions of these coordinates.

**Proof.** For \( k = 2 \) the assertion is verified directly. For \( k > 1 \) one can express the standard jet coordinates \( u_{i_1 \ldots i_h}^j = D_{i_1 \ldots i_h} u^j \) in terms of coordinates (30) by making use of the following obvious facts. First, fields \( D_i \) are linear combinations of fields \( \mathcal{X}_i \) with coefficients in \( C^\infty(J^2 \pi) \). Second, the coefficients \( b_{i_1 i_2}^j \) are functions on \( J^2 \pi \). Third, \( \mathcal{X}_{i_2} \mathcal{X}_{i_1} f = -b_{i_1 i_2}^j \mathcal{X}_j f + \mathcal{X}_{i_1} \mathcal{X}_{i_2} f \) for every function \( f \in C^\infty(J^k \pi), \ k \geq 2 \). \( \square \)

A complete system of relations connecting functions \( b_{ij}^k \) can be found by routine computations and taking into consideration geometric properties of fields \( X_1, \ldots, X_5 \). For instance, \( b_{12}^1 = b_{12}^1 = 0 \), since \( [X_1, X_2] \) belongs to the distribution \( (D_c^1)^{(1)} \) generated by \( X_1, X_2 \) and \( X_5 \), etc. The final result is as
follows:

\[ b_{34}^1 = 0, \quad b_{34}^2 = 0, \]
\[ b_{12}^1 = 0, \quad b_{12}^1 = 0, \quad b_{12}^5 = 1, \]
\[ b_{13}^1 = -b_{13}^1, \quad b_{13}^4 = -b_{13}^1, \quad b_{13}^5 = 0, \]
\[ b_{23}^1 = -b_{23}^1, \quad b_{23}^4 = -b_{23}^2, \quad b_{23}^5 = 0, \]
\[ b_{14}^1 = -b_{14}^1, \quad b_{14}^4 = -b_{14}^2, \quad b_{14}^5 = 0, \]
\[ b_{24}^1 = -b_{24}^1, \quad b_{24}^4 = -b_{24}^2, \quad b_{24}^5 = 0, \]
\[ b_{34}^1 = -b_{12}^1, \quad b_{34}^4 = -b_{12}^2, \quad b_{34}^5 = -1, \]
\[ b_{45}^1 = -b_{14}^1 - b_{14}^2, \quad b_{45}^5 = -b_{24}^2 - b_{23}^2, \]
\[ b_{35}^1 = -b_{13}^1 - b_{13}^2, \quad b_{35}^5 = -b_{24}^2 - b_{14}^2, \]
\[ b_{15}^1 = -b_{35}^3 + b_{25}^3 + b_{15}^1, \]

Henceforth we shall simplify the notation by using \( X_i \) for \( X_i \).

4.4. Curvatures. Using formulas (3), (4) and the direct sum decomposition (25), it is easy to compute the curvature forms of projectors \( P_1, P_2, P_1^{(1)} \), \( P_2^{(1)} \), \( P_3 \), which are

\[ R_1 = \omega^1 \wedge \omega^2 \otimes X_5, \]
\[ R_2 = -\omega^3 \wedge \omega^4 \otimes X_5, \]
\[ R_1^1 = -(b_{15}^1 \omega^1 + b_{25}^3 \omega^2) \wedge \omega^5 \otimes X_3 - (b_{15}^4 \omega^1 + b_{25}^2 \omega^2) \wedge \omega^5 \otimes X_4, \]
\[ R_2^1 = -(b_{35}^3 \omega^3 + b_{45}^4 \omega^4) \wedge \omega^5 \otimes X_1 - (b_{35}^2 \omega^3 + b_{45}^2 \omega^4) \wedge \omega^5 \otimes X_2, \]
\[ R = R_1 + R_2, \tag{32} \]

respectively. It is clear that these curvature forms are differential invariants of \( \mathcal{E} \).

Frölicher–Nijenhuis brackets of base projectors give new invariant vector-valued forms. These, however, turn out to be linear combinations of curvature forms. More exactly, a direct computation, which is omitted, shows that

\[ [P_1, P_2] = \frac{1}{2}(-[P_1, P_1] - [P_2, P_2] + [P_3, P_3]), \]
\[ [P_1, P_3] = \frac{1}{2}(-[P_1, P_1] + [P_2, P_2] - [P_3, P_3]), \]
\[ [P_2, P_3] = \frac{1}{2}([P_1, P_1] - [P_2, P_2] - [P_3, P_3]) \]

and

\[ [P_1, P_1] = -2(R_2^1 + R_1), \quad [P_2, P_2] = -2(R_1^1 + R_2), \]
\[ [P_3, P_3] = -2(R_1 + R_2). \]
4.5. **Scalar invariants on** $J^2 \pi$. The following three invariant 5-forms with values in $\mathcal{D}_\xi^5 = (X_5)$:

$$
\frac{1}{2} (\mathcal{R}_2 \mathcal{J} \mathcal{R}_1) \mathcal{J} (\mathcal{R}_2 \mathcal{J} \mathcal{R}_1) = \Lambda_1 \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5,
$$

$$
\frac{1}{2} (\mathcal{R}_2 \mathcal{J} \mathcal{R}_2) \mathcal{J} (\mathcal{R}_2 \mathcal{J} \mathcal{R}_2) = \Lambda_2 \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5,
$$

$$
(\mathcal{R}_2 \mathcal{J} \mathcal{R}_1) \mathcal{J} (\mathcal{R}_2 \mathcal{J} \mathcal{R}_2) = \Lambda_{12} \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5,
$$

with

$$
\Lambda_1 = b_{35}^2 b_{45}^1 - b_{35}^1 b_{45}^2, \quad \Lambda_2 = b_{15}^4 b_{25}^3 - b_{15}^3 b_{25}^4,
$$

$$
\Lambda_{12} = b_{15}^3 b_{35}^1 + b_{15}^4 b_{45}^1 + b_{25}^3 b_{35}^2 + b_{25}^4 b_{45}^2.
$$

are proportional. Therefore, the corresponding proportionality factors are scalar differential invariants. In particular, such are

$$
I^1 = \Lambda_{12}/\Lambda_1,
$$

$$
I^2 = \Lambda_{12}/\Lambda_2.
$$

Below it will be shown that $\Lambda_1, \Lambda_2$ are nowhere zero.

**Theorem 4.3.** The algebra of scalar differential invariants on $J^2 \pi$ is generated by the invariants $I^1$ and $I^2$.

**Proof.** In view of Corollary 3.5, it is sufficient to show that $I^1$ and $I^2$ are functionally independent (on $J^2 \pi$). But this is straightforward from (31).

Coefficients $\Lambda_\sigma$, $\sigma = 1, 2, 12$, introduced in (34) have a geometrical meaning explained below. Fix a generator $W = fX_5$ in $\mathcal{D}_\xi^5$ and consider maps

$$
\square^W_1: \mathcal{D}^2 \to \mathcal{D}^1, \quad \square^W_2: \mathcal{D}^1 \to \mathcal{D}^2,
$$

defined by formulas

$$
\square^W_1(Z_2) = \mathcal{P}_1([Z_2, W]), \quad \square^W_2(Z_1) = \mathcal{P}_2([Z_1, W]),
$$

with $Z_1 \in \mathcal{D}^1$, $Z_2 \in \mathcal{D}^2$. Since $\mathcal{P}_1(\mathcal{D}_\xi^2) = \mathcal{P}_2(\mathcal{D}_\xi^1) = 0$ both $\square^W_1$ and $\square^W_2$ are $C^\infty(M)$-linear. This is seen as well from their local expressions

$$
\square^W_1 = f b_{j_3}^i \omega^j \otimes X_i, \quad i = 1, 2, \quad j = 3, 4,
$$

$$
\square^W_2 = f b_{j_3}^i \omega^j \otimes X_i, \quad i = 3, 4, \quad j = 1, 2.
$$

Consider also 2-forms $\rho^W_i: \mathcal{D}_i \times \mathcal{D}_i \to \mathbb{R}$, $i = 1, 2$, defined by

$$
\rho^W_i(U_i, V_i)W = \mathcal{R}_i(U_i, V_i), \quad U_i, V_i \in \mathcal{D}_\xi^i.
$$

Then, obviously, $\rho^W_1 = (1/f)\omega^1 \wedge \omega^2$, $\rho^W_2 = -(1/f)\omega^3 \wedge \omega^4$, so that both are volume forms of $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively. Moreover, we have

$$
(\square^W_1)^*(\rho^W_1) = f^2 \Lambda_2 \rho^W_2,
$$

$$
(\square^W_2)^*(\rho^W_2) = f^2 \Lambda_1 \rho^W_1,
$$

$$
\text{tr}(\square^W_1 \circ \square^W_2) = \text{tr}(\square^W_1 \circ \square^W_1) = f^2 \Lambda_{12}.
$$

**Proposition 4.4.** If $\mathcal{E}$ is generic, then functions $\Lambda_1, \Lambda_2$ are nowhere zero.

**Proof.** By genericity condition (24), $\square^W_1$ and $\square^W_2$ are surjective, hence $\Lambda_1, \Lambda_2$ are nonzero.
4.5.1. Now consider operators $\nabla_1^W = \Box_1^W \circ \Box_2^W$ and $\nabla_2^W = \Box_2^W \circ \Box_1^W$ acting on $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively. It follows from (37) that

$$\lambda^2 - f^2 \Lambda_{12} \lambda + f^4 \Lambda_1 \Lambda_2$$

(38)

is the characteristic polynomial for each of them. Another peculiarity of the situation is that $\Box_1^W$ send eigenvectors of $\nabla_2^W$ to that of $\nabla_1^W$ and similarly for $\Box_2^W$.

The discriminant of polynomial (38) is

$$f^4 \Lambda_1 \Lambda_2 (I_1 I_2 - 4).$$

Its sign coincides, obviously, with the sign of $I_1 I_2 (I_1 I_2 - 4)$.

This proves that generic hyperbolic Monge–Ampère equations are subdivided into three subclasses as follows:

1. subclass “h”: the operator $\nabla_1^W$ has two different real eigenfunctions $\iff I_1 I_2 (I_1 I_2 - 4) > 0$,

2. subclass “p”: the operator $\nabla_i$ has a unique real eigenfunction $\iff I_1 I_2 (I_1 I_2 - 4) = 0$,

3. subclass “e”: the operator $\nabla_i$ has no real eigenfunctions $\iff I_1 I_2 (I_1 I_2 - 4) < 0$.

4.5.2. Some almost invariants. The previous considerations lead to an almost invariant choice of generator $W = f X_5$ in $\mathcal{D}^3_\xi$. Namely, define functions $\Lambda_i^W$, $i = 1, 2$, by relations

$$(\Box_i^W)^* (\rho_1^W) = \Lambda_2^W \rho_2^W, \quad (\Box_i^W)^* (\rho_2^W) = \Lambda_1^W \rho_1^W.$$  

Obviously, $\Lambda_i^W = f^2 \Lambda_i$. This shows that, up to sign, vector fields

$$W_i = \frac{1}{\sqrt{|\Lambda_i^W|}} W, \quad i = 1, 2,$$

do not depend on the choice of $W$. In particular, $\Lambda_i^{X_5} = \Lambda_i$, so that

$$W_i = \frac{1}{\sqrt{|\Lambda_i|}} X_5, \quad i = 1, 2.$$

By duality, 1-forms

$$\vartheta_i = \sqrt{|\Lambda_i|} \omega_5, \quad i = 1, 2,$$

are almost invariant as well.

It is not difficult to construct further almost invariant forms. For instance, the forms

$$\vartheta_{ij} = \mathcal{R}_{ij} \vartheta_j, \quad i = 1, 2,$$

are manifestly almost invariant and have the following local expressions:

$$\vartheta_{1j} = \sqrt{|\Lambda_j|} \omega^1 \wedge \omega^2 \quad \vartheta_{2j} = \sqrt{|\Lambda_j|} \omega^3 \wedge \omega^4.$$

The products

$$\rho_j = (-\text{sign} \Lambda_j) \vartheta_{1j} \wedge \vartheta_{2j} = \Lambda_j \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4, \quad j = 1, 2,$$

(39)
which are volume forms on the Cartan distribution $\mathcal{D}^1_\mathcal{E}$, $\mathcal{D}^2_\mathcal{E}$, are, obviously, fully invariant. This is a very simple example on how an invariant can be constructed from almost invariants. Forms $\rho_j$ can be described in a manifestly invariant way as follows:

$$\rho_1 = \frac{1}{2} \langle (\mathcal{R}^1_2 \mathcal{J} \mathcal{R}_1) \mathcal{J} (\mathcal{R}^1_2 \mathcal{J} \mathcal{R}_1) \rangle, \quad \rho_2 = \frac{1}{2} \langle (\mathcal{R}^1_3 \mathcal{J} \mathcal{R}_2) \mathcal{J} (\mathcal{R}^1_3 \mathcal{J} \mathcal{R}_2) \rangle$$

where $\langle \cdot, \cdot \rangle$ stands for convolution. Note that the form

$$\rho^{12} = I^i \rho_j = \Lambda_{12} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$$

is invariant too.

Similarly, one can construct many other invariant forms. Some of them are:

$$\langle \mathcal{R}^1_3 \mathcal{J} \mathcal{R}_1 \rangle = -(b_{35}^1 \omega^3 + b_{15}^1 \omega^4) \wedge \omega^2 + (b_{35}^2 \omega^3 + b_{15}^2 \omega^4) \wedge \omega^1$$

$$\langle \mathcal{R}^1_3 \mathcal{J} \mathcal{R}_2 \rangle = (b_{35}^3 \omega^1 + b_{25}^3 \omega^2) \wedge \omega^4 - (b_{15}^4 \omega^1 + b_{25}^4 \omega^2) \wedge \omega^3$$

$$\mathcal{R}^1_3 \mathcal{J} (\mathcal{R}^1_2 \mathcal{J} \mathcal{R}_1) = 2 \Lambda_2 \omega^1 \wedge \omega^2 \wedge \omega^5$$

Now it is easy to construct almost invariant volume forms:

$$\vartheta_j \wedge \rho_j = |\Lambda_j|^{3/2} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5, \quad j = 1, 2.$$  \hfill (42)

5. Differential invariants on $J^3\pi$

Since $\omega^k(X_i) = \text{const}$, namely, $\delta_{ki}$, we have

$$d\omega_k(X_i, X_j) = -\omega^k([X_i, X_j]) .$$

(see the proof of lemma 4.1). This implies the useful formula

$$d\omega^k = -\sum_{i<j} b_{ij}^k \omega^i \wedge \omega^j .$$  \hfill (43)

5.1. The complete parallelism. First, note that invariant differential 1-forms $dI^1$ and $dI^2$ live on $J^3\pi$. This leads us immediately to another set of invariant differential 1-forms on $J^3\pi$:

$$\Omega^1 = \mathcal{P}_1 \mathcal{J} dI^1 = X_1(I^1) \omega^1 + X_2(I^1) \omega^2,$$

$$\Omega^2 = \mathcal{P}_1 \mathcal{J} dI^2 = X_1(I^2) \omega^1 + X_2(I^2) \omega^2,$$

$$\Omega^3 = \mathcal{P}_2 \mathcal{J} dI^1 = X_3(I^1) \omega^3 + X_4(I^1) \omega^4,$$

$$\Omega^4 = \mathcal{P}_2 \mathcal{J} dI^2 = X_3(I^2) \omega^3 + X_4(I^2) \omega^4,$$

$$\Omega^5_1 = \mathcal{P}_3 \mathcal{J} dI^1 = X_5(I^1) \omega^5, \quad \Omega^5_2 = \mathcal{P}_3 \mathcal{J} dI^2 = X_5(I^2) \omega^5 .$$

Supposing that $\mathcal{E}$ is a generic equation, we henceforth assume that

$$X_5(I^1) \neq 0, \quad X_5(I^2) \neq 0,$$  \hfill (45)

and

$$\Delta_1 = \begin{vmatrix} X_1(I^1) & X_2(I^1) \\ X_1(I^2) & X_2(I^2) \end{vmatrix} \neq 0, \quad \Delta_2 = \begin{vmatrix} X_3(I^1) & X_4(I^1) \\ X_3(I^2) & X_4(I^2) \end{vmatrix} \neq 0 .$$  \hfill (46)

This means that two sets of forms $\{ \Omega^1, \ldots, \Omega^4, \Omega^5_1 \}$ and $\{ \Omega^1, \ldots, \Omega^4, \Omega^5_2 \}$ are invariant coframes on $M$ (we omit the subscript $\mathcal{E}$ according to Remark
Each of these coframes determines an invariant complete parallelism on $M$.

The frames \( \{Y_1, \ldots, Y_4, Y_5^1\} \) and \( \{Y_1, \ldots, Y_4, Y_5^2\} \), dual to the above constructed coframes, are, obviously, invariant. An explicit description of them is:

\[
\begin{align*}
Y_1 &= \frac{1}{\Delta_1} (X_2(I^2)X_1 - X_1(I^2)X_2), \\
Y_2 &= \frac{1}{\Delta_1} (-X_2(I^1)X_1 + X_1(I^1)X_2), \\
Y_3 &= \frac{1}{\Delta_2} (X_4(I^2)X_3 - X_3(I^2)X_4), \\
Y_4 &= \frac{1}{\Delta_2} (-X_4(I^1)X_3 + X_3(I^1)X_4), \\
Y_5^1 &= \frac{1}{X_5(I^1)} X_5, \quad Y_5^2 = \frac{1}{X_5(I^2)} X_5.
\end{align*}
\]

5.2. More scalar invariants on $J^3\pi$. Among numerous invariants constructed previously there are functions, (vector-valued) differential forms, and vector fields. Further invariants can be obtained just by applying various operations of tensor algebra, Frölicher–Nijenhuis brackets, etc, to these objects. Moreover, components of an invariant object with respect to an invariant basis are scalar differential invariants as well as its proper differential invariants. These simple general tricks are rather efficient and were already used in constructing differential invariants on $J^2\pi$. As for $J^3\pi$ we shall proceed along these lines as well.

The invariant 1-forms $\Omega_1^5$ and $\Omega_2^5$ are proportional. So, the proportionality factor

\[
I^3 = \frac{X_5(I^1)}{X_5(I^2)}
\]

is a scalar differential invariant on $J^3\pi$.

Consider now invariant 2-forms on $J^3\pi$:

\[
\begin{align*}
\mathcal{R}_1 \mathcal{J} \, dI^1 &= I^6\Omega^1 \wedge \Omega^2, \\
\mathcal{R}_1 \mathcal{J} \, dI^2 &= I^7\Omega^1 \wedge \Omega^2, \\
\mathcal{R}_2 \mathcal{J} \, dI^1 &= I^8\Omega^3 \wedge \Omega^4, \\
\mathcal{R}_2 \mathcal{J} \, dI^2 &= I^9\Omega^3 \wedge \Omega^4, \\
\mathcal{R}_1^1 \mathcal{J} \, dI^1 &= I^{10}\Omega^1 \wedge \Omega_1^5 + I^{11}\Omega^2 \wedge \Omega_1^5, \\
\mathcal{R}_1^1 \mathcal{J} \, dI^2 &= I^{12}\Omega^1 \wedge \Omega_1^5 + I^{13}\Omega^2 \wedge \Omega_1^5, \\
\mathcal{R}_2^1 \mathcal{J} \, dI^1 &= I^{14}\Omega^3 \wedge \Omega_1^5 + I^{15}\Omega^4 \wedge \Omega_1^5, \\
\mathcal{R}_2^1 \mathcal{J} \, dI^2 &= I^{16}\Omega^3 \wedge \Omega_1^5 + I^{17}\Omega^4 \wedge \Omega_1^5.
\end{align*}
\]

Their components $I^6, \ldots, I^{17}$ with respect to the base $\Omega^1, \ldots, \Omega^5$ are further scalar differential invariants on $J^3\pi$. The simplest among them are $I^6 = \Delta_1/X_5(I^1)$ and $I^8 = \Delta_2/X_5(I^1)$.

In the same manner one easily finds numerous non-scalar differential invariants on $J^3\pi$. For instance, such are 3-forms $[\mathcal{P}_i, \mathcal{R}_j]$ or $[\mathcal{P}_i, \mathcal{R}_j^1]$, 4-forms $[\mathcal{P}_i, (\mathcal{R}_j \mathcal{J}[\mathcal{P}_k, \mathcal{R}_l]^1)]$, 5-forms $[\mathcal{P}_i, \mathcal{R}_j^1] \mathcal{J}[\mathcal{P}_k, \mathcal{R}_l]^1$, etc.
5.3. Better manageable invariants. From the above said one can see that there are sufficient resources for constructing differential invariants and the main problem becomes to select functionally independent ones in the simplest possible way. From technical point of view this forces us to look for manageable invariants, for instance, those that have local expression as simple as possible. In the considered context a help comes from almost invariant objects as it is illustrated below.

In view of (39), (40) and (43), for \( \sigma = 1, 2, 12 \) we have the invariant 5-forms

\[
d\rho_\sigma = d(\Lambda_\sigma \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4) = (X_5(\Lambda_\sigma) + \Lambda_\sigma B) \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5,
\]

where \( B = b_{15}^1 + b_{25}^2 + b_{35}^3 + b_{45}^4 = 2(b_{15}^1 + b_{25}^2) = 2(b_{35}^3 + b_{45}^4) \) according to (31).

By comparing these 5-forms with (42) we obtain almost scalar invariants

\[
I_j \sigma = X_5(\Lambda_\sigma) + \Lambda_\sigma B|_{\Lambda_j}^{3/2}, \quad \sigma = 1, 2, 12, \quad j = 1, 2.
\]

on \( J^3(\pi) \) which are better manageable in comparison to those constructed in the previous subsection. The squares \((I_j^\sigma)^2\) are, obviously, full scalar invariants. Apart from the obvious relation \((I_1^1/I_2^2)^3 = (I_1^2/I_2^1)^3\) they are functionally independent. Some of the earlier constructed invariants can be expressed in terms of \(I_j^\sigma\)'s, e.g.,

\[
\frac{X_5(I_1^1)}{X_5(I_2^2)} = \frac{(I_{12}^3 - I_{12}^1 I_1^3)I_1^3}{(I_{12}^3 - I_{12}^1 I_2^3)I_2^3}, \quad j = 1, 2.
\]

6. The equivalence problem

So far we obtained two independent second-order scalar invariants \(I^1, I^2\) (see (35)) and a number of third-order invariants. Put (see (51))

\[
I^3 = (I_1^1)^2, \quad I^4 = (I_2^2)^2, \quad I^5 = (I_{12}^1)^2,
\]

The following statement can be checked by a direct computer-supported calculation in coordinates (30):

**Theorem 6.1.** Invariants \(I^1, I^2, I^3, I^4, \) and \(I^5\) are functionally independent on \(J^3(\pi)\).

Of course, this choice of basic scalar invariants is not unique. For instance, invariants \(I^1, I^2, I^3, I^6, I^8\) (see (48), (49)) are functionally independent as well. However, this and other reasonable choices are "less manageable" with respect to those made in the above theorem. Unfortunately, this fact is not clearly seen from the above exposition, since we were forced to skip technical details of computations.

According to "the principle of \(n\) invariants" [22], any quintuple of functionally independent scalar invariants gives a solution of the equivalence problem for generic hyperbolic Monge–Ampère equations. Theorem 6.1 guarantees existence of a such one, namely, \(I^1, \ldots, I^5\).
More exactly, let $\mathcal{E}$ be a generic hyperbolic Monge–Ampère equation considered as a section of the bundle $\pi$. Since invariants $I^1, \ldots, I^5$ are functionally independent their values $I^1_z, \ldots, I^5_z$ on $\mathcal{E}$ form a (local) chart in $M$. In terms of these coordinates, the 1-forms $\Omega_1, \ldots, \Omega_5$, defining an absolute parallelism on $M$, are described in terms of functions $\Omega^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$ coming from the decomposition

$$
\Omega_i = \sum_{j=1}^5 \Omega^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})dI^j_\mathcal{E}, \quad i = 1, \ldots, 5.
$$

**Theorem 6.2.** The (local) equivalence class of a generic equation $\mathcal{E}$ with respect to contact transformations is uniquely determined by the family of functions $\Omega^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$, $i = 1, \ldots, 5$.

**Proof.** Let $\tilde{\mathcal{E}}$ be another generic Monge–Ampère equation such that there exists a contact transformation transforming it to $\mathcal{E}$. Then, obviously, the functions $\Omega^j(I^1_\tilde{\mathcal{E}}, \ldots, I^5_\tilde{\mathcal{E}})$ and $\tilde{\Omega}^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$ coincide for all $i$ and $j$.

Let $\mathcal{E}, \tilde{\mathcal{E}}$ be Monge–Ampère equations such that for all $i$ and $j$ the functions $\Omega^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$ and $\tilde{\Omega}^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$ coincide. Let $I^1_\mathcal{E} = (I^1, \ldots, I^5)$ and $I^1_\tilde{\mathcal{E}} = (I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})$ be invariant coordinate systems in $M$ for $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. Then $I^1_\mathcal{E} \circ I^1_\tilde{\mathcal{E}}$ is a locally defined diffeomorphism $M \to M$. This diffeomorphism is a contact transformation because it transforms $\Omega^i = \sum_{j=1}^5 \Omega^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})dI^j_\mathcal{E}$ to $\sum_{j=1}^5 \tilde{\Omega}^j(I^1_\mathcal{E}, \ldots, I^5_\mathcal{E})dI^j_\tilde{\mathcal{E}} = \tilde{\Omega}^5$, $i = 1, \ldots, 5$, and, in particular, the contact form $\Omega_5$ to the contact form $\tilde{\Omega}_5$. By obvious reasons it also transforms the pair of distributions $\mathcal{D}^1_\mathcal{E}, \mathcal{D}^2_\mathcal{E}$ to the pair $\mathcal{D}^1_\tilde{\mathcal{E}}, \mathcal{D}^2_\tilde{\mathcal{E}}$ and hence $\mathcal{E}$ to $\tilde{\mathcal{E}}$. \hfill \Box

7. Examples

Examples discussed in this section aim to illustrate the character and complexity of problems related with actual computations and use of differential invariants. Henceforth invariants $I^i$ are denoted by $I_i$.

**Example 7.1.** Consider the equation

$$
\frac{1}{2}(z_{xx}z_{yy} - z_{xy}^2) + y^2z_{xx} - 2xyz_{xy} + x^2z_{yy} + x^2y^2z^2 = 0.
$$

The first two invariants are $I_1 = zn_+/d$, $I_2 = zn_-/d$, where

$$
n_{\pm} = 2(z + 3y^4 + 2)x^2z_{x}^2 - (z + 12x^2y^2)xyz_{z} + 2(z + 3x^4 + 2)y^2z_{y}^2
$$
$$
+ (z^2 + 8x^2y^2z + 4y^4z + 16x^4 + 4y^2 + 16y^4 - 12)x_{x}z_{x}
$$
$$
+ (z^2 + 4x^4 + 8z^2y^2 + 4y^4 + 16x^4 + 16y^2 - 12)y_{y}z_{y}
$$
$$
+ 2x^3 + 36x^4y^4z^3 + 6y^4z^2 - 4x^2y^2z^2 + 6x^4z^2
$$
$$
- 8z + 16x^4z + 16y^4z + 8x^4 + 16x^2y^2 + 8y^4,
$$
$$
d = 4(z^2 + 3y^4z + 4)x^2z_{x}^2 - 2(z^2 + 12x^2y^2z - 16)xyz_{z}
$$
$$
+ 4(z^2 + 3x^4z + 4)y^2z_{y}^2 + 2(z^2 + 8x^2y^2z + y^2 + 20)x_{x}z_{x}
$$
$$
+ 2(z^2 + 4x^4z + 8x^2y^2z + 20)y_{y}z_{y} + 4(18x^4y^4z^3 + z^3
$$
$$
+ 3x^4z^2 - 2x^2y^2z^2 + 3y^4z^2 + 12z + 4x^4 + 8x^2y^2 + 4y^4).z.
$$
The invariants \( I_3, I_4, I_5 \) are large fractions whose non-reducible numerators are polynomials of order three in \( z_x, z_y \), five in \( z \), and six in \( x, y \). Invariants \( I_s, s > 5 \), are even more cumbersome.

Computation shows that the jacobian \( \partial(I_1, I_2, I_3, I_4, I_5)/\partial(x, y, z, z_x, z_y) \) is nonzero, hence the first five invariants are functionally independent and can be chosen to be local coordinates on \( J^1(\tau) \). Although an explicit inversion is rather hopeless, one can still find algorithmically the relations connecting principal invariants \( I_1, \ldots, I_5 \) and higher \( I_k \) at least in principle. This kind of procedure is outlined in Example 7.2 below.

**Example 7.2.** Put \( \zeta = z_x + z_y + e \) and consider the family of equations

\[
(4z_x z_y + \zeta^2)(z_{xx} z_{yy} - z_{xy}^2) + 4\zeta^2(z_y z_{xx} + z_x z_{yy} + \zeta^2) = 0. \tag{52}
\]

depending on parameter \( e \). Assuming that \( e \neq 0 \), we have

\[
I_1 = 2\left(\frac{z_x + z_y}{5ez_x + ez_y + 4e^2} + 3e(z_x + z_y) + 4e^2\right),
\]

\[
I_2 = 2\left(\frac{z_x + z_y}{ez_x + 5ez_y + 4e^2} + 3e(z_x + z_y) + 4e^2\right),
\]

\[
I_3 = 2^{3/2}\left(\frac{7z_x^2 + 6z_x z_y - z_y^2 + 33ez_x + 5ez_y + 21e^2}{e^{1/2}(5z_x + z_y + 4e)^{3/2}}\right),
\]

\[
I_4 = 2^{3/2}\left(\frac{-z_x^2 + 6z_x z_y + 7z_y^2 + 5ez_x + 33ez_y + 21e^2}{e^{1/2}(5z_x + z_y + 4e)^{3/2}}\right),
\]

\[
I_5 = 2^{5/2}\left(\frac{(z_x + z_y)^3 + 7e(z_x + z_y)^2 + 17e^2(z_x + z_y) + 21e^3}{e^{3/2}(5z_x + z_y + 4e)^{3/2}}\right).
\]

All invariants are independent of \( x, y, z \), reflecting the fact that \( x \mapsto x + t_1, y \mapsto y + t_2, z \mapsto z + t_3 \) are symmetries of equation (52). One easily checks that \( I_1, I_2 \) are functionally independent, but it is still not straightforward to express \( z_x, z_y \) in terms of \( I_1, I_2 \) explicitly.

To establish the dependence of \( I_s, s > 3 \), on \( I_1, I_2 \), we observe that for every \( s \) there exists a polynomial \( P_s(z_x, z_y, I_s) \) such that \( I_s \) is a solution of the equation \( P_s = 0 \). Then what we need is eliminating \( z_x, z_y \) from the system

\[
I_1 - 2\left(\frac{z_x + z_y}{5ez_x + ez_y + 4e^2} + 3e(z_x + z_y) + 4e^2\right) = 0,
\]

\[
I_2 - 2\left(\frac{z_x + z_y}{ez_x + 5ez_y + 4e^2} + 3e(z_x + z_y) + 4e^2\right) = 0,
\]

\[
P_s(z_x, z_y, I_s) = 0.
\]

To this end, it suffices to compute the Gröbner basis of the last system with respect to an “elimination ordering” of monomials. With the help of the
Groebner package of Maple 10 the following quadratic equation for $I_3$,

$$0 = 4096 I_2^6 I_3^2 - I_2^4 (729 I_1^3 I_2^3 - 1971 I_1^2 I_2^2 + 20493 I_1^2 I_2^2 + 3563 I_1^4 I_2 - 51114 I_1^2 I_2^2 + 183915 I_1 I_2^2)$$

$$+ 3951 I_1^3 - 52723 I_1^2 I_2 + 45517 I_1 I_2^2 + 102191 I_2^3) I_3$$

$$+ (27 I_1^3 I_2^2 - 81 I_1^2 I_2^3 - 32 I_1^3 I_2 + 426 I_1^2 I_2^2 - 1206 I_1 I_2^3$$

$$- 44 I_1^3 + 270 I_1^2 I_2 - 800 I_1 I_2^2 - 1114 I_2^3)^2$$

can be found rather quickly as well as similar quadratic equations for $I_4, I_5$. The assumptions of Sect. 5.1 are satisfied as well. In particular, $\Delta_1, \Delta_2$ are nonzero since

$$\Delta_1 = \Delta_2 = -128 \frac{(z_x + z_y)(3z_x + 3z_y + 8e)(z_x + z_y + e)^2}{e^2(z_x + 5z_y + 4e)^2(5z_x + z_y + 4e)^2}$$

$$\times \frac{z_x^2 + 2z_x z_y + z_y^2 + 3ez_x + 3ez_y + 4e^2}{z_x^2 + 6ez_x + z_y^2 + 2ez_x + 2ez_y + e^2}.$$  

This enables us to compute the higher invariants. For instance, $I_6$ is solution of the quadratic equation

$$0 = -16I_1^6 (27I_1^3 I_2 - 27I_1^2 I_2^2 + 22I_1^2 - 56I_1 I_2 - 22I_2^2$$

$$+ 8I_1^2 I_2 - 42I_1 + 50I_1 I_2 + 28I_1^2 + 56I_1 I_2 + 28I_2^2) I_6$$

$$+ I_1(I_1 I_2 - I_1 - I_2)(9I_1 I_2 + 7I_1 + 7I_2)(3I_1^2 I_2 - 3I_1^2 I_2$$

$$- 26I_1^3 - 34I_1^2 I_2 - 8I_1 I_2 + 18I_1^2 + 36I_1 I_2 + 18I_2^2) I_6$$

$$+ (I_1 + I_2)(I_1 I_2 - I_1 - I_2)(9I_1 I_2 + 7I_1 + 7I_2)(I_1 I_2 - 2I_1 - 2I_2)^2.$$  

Although every invariant computed so far depends on $e$, its expression in terms of $I_1, I_2$ does not. This suggests the idea that the parameter $e$ is removable. And indeed, after substitution $z \mapsto ez$ equation (52) becomes equivalent to itself with $e = 1$. Thus, the family of equations (52) consists of a continuum of generic members with $e \neq 0$, which are all mutually equivalent, and a single non-generic member with $e = 0$ (in which case $\Delta_1 = \Delta_2 = 0$).

**Example 7.3.** Consider the family of equations

$$\frac{1}{4}(z_{xx}z_{yy} - z_{xy}^2) + y^2 z_{xx} - 2xy z_{xy} + x^2 z_{yy} + e x^2 y^2 = 0,$$

depending on a real parameter $e \neq 4$. Then the first five invariants are constants

$$I_1 = I_2 = 2\frac{e + 12}{e - 4},$$

$$I_3 = I_4 = \frac{800}{e - 4},$$

$$I_5 = 3200\frac{(e + 12)^2}{(e - 4)^3},$$

while the higher invariants $I_s$ are undefined.

The equation belongs to the subclass “h”, or “p”, or “e” (see 4.5.1) if $e > -4$ or $e = -4$ or $e < -4$, respectively.
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References

[1] D.V. Alekseevskiy, A.M. Vinogradov and V.V. Lychagin, Basic ideas and concepts of
differential geometry. in: Geometry, I Encyclopaedia Math. Sci. 28, Springer, Berlin,
1991, 1–264.

[2] A. Frölicher and A. Nijenhuis, Theory of vector valued differential forms. Part I:
Derivations in the graded ring of differential forms, Indag. Math. 18 (1956) 338–359.

[3] P. Hartman and A. Wintner, On hyperbolic partial differential equations, American
Journal of Mathematics 74 (1952) 834–864.

[4] I.S. Krasil’shchik, V.V. Lychagin and A.M. Vinogradov, Geometry of Jet Spaces and
Nonlinear Partial Differential Equations, Gordon and Breach, New York, 1986.

[5] I.S. Krasil’shchik and A.M. Vinogradov, Editors, Symmetries and Conservation Laws
for Differential Equations of Mathematical Physics, Translations of Mathematical
Monographs. Vol.182, Providence RI: American Mathematical Society, 1999.

[6] B.S. Kruglikov, Some classificational problems in four-dimensional geometry: distri-
butions, almost complex structures and the generalized Monge–Ampère equations,
Math. Sbornik 189 (1998) (11) 61-74 (in Russian); English translation in Sb. Math.
186 (1998) (11–12) 1643–1656; e-print: http://xxx.lanl.gov/abs/dg-ga/9611005.

[7] B.S. Kruglikov, Symplectic and contact Lie algebras with application to the Monge–
Ampère equations Trudy Mat. Inst. Steklova 221 (1998) 232—246 (in Russian);
English translation in Proc. Steklov Math. Inst. 221 (1998) (2) 221–235; e-print:
http://xxx.lanl.gov/abs/dg-ga/9709004

[8] B.S. Kruglikov, Classification of Monge–Ampère equations with two variables, in: Ge-
ometry and Topology of Caustics - CAUSTICS ’98 (Warsaw), Banach Center Publications
50 (Polish Acad. Sci., Warsaw, 1999) 179-194.

[9] A. Kushner, Monge–Ampère equations and $e$-structures, Dokl. Akad. Nauk 361 (1998)
(5) 595–596.

[10] H. Lewy, Über das Anfangswertproblem bei einer hyperbolischen nichtlinearen parti-
tellen Differentialgleichung zweiter Ordnung mit zwei unabhängigen Veränderlichen,
Math. Annalen 98 (1928) 179–191.

[11] V.V. Lychagin, Contact geometry and non-linear second order differential equations,
Russian Math. Surveys 34 (1979) 149–180.

[12] V.V. Lychagin, Lectures on Geometry of Differential Equations, Universita “La
Sapienza,” Roma, 1992, 133 p.

[13] V.V. Lychagin, V.N. Rubtsov and I.V. Chekalov, A classification of Monge-Ampere
equations, Ann. Sc. Ecole Norm. Sup. (4) 26 (1993), 281-308.

[14] M. Marvan, A.M. Vinogradov and V.A. Yumaguzhina, Differential invariants of generic
hyperbolic Monge–Ampère equations Russian Acad. Sci. Dokl. Math. 405 (2005) 299–
301 (in Russian) English translation in: Doklady Mathematics 72 (2005) 883–885.

[15] M. Matsuda, Two methods of integrating Monge–Ampère’s equations and $e$-structures, Dokl.
Acad. Nauk 361 (1998) (5) 595–596.

[16] M. Matsuda, Two methods of integrating Monge–Ampère’s equations. II, Trans.
Amer. Math. Soc. 150 (1970) 327–343.

[17] T. Morimoto, Monge–Ampère equations viewed from contact geometry. in: Symplectic
Singularities and Geometry of Gauge Fields (Warsaw, 1995), 105–121, Banach Center
Publ., 39, Polish Acad. Sci., Warsaw, 1997.

[18] O.P. Tchij, Contact geometry of hyperbolic Monge–Ampère equations, Lobachevskii
Journal of Mathematics 4 (1999) 109–162.

[19] D.V. Tunitskiy, Monge–Ampère equations and functors of characteristic connection,
Izv. RAN, Ser. Math. 65 (6) (2001) 173–222.

[20] D.V. Tunitskiy, On the global solvability of hyperbolic Monge–Ampère equations,
Izv. Ross. Akad. Nauk Ser. Mat. 61 (1997), No. 5, 177–224 (in Russian); translation
in Izv. Math 61 (1997), No. 5, 1069–1111.
[21] A.M. Vinogradov, Geometry of nonlinear differential equations, manuscript, Diffiety Institute, 1987.

[22] A.M. Vinogradov, Scalar differential invariants, diffieties and characteristic classes, in: Mechanics, Analysis and Geometry: 200 Years after Lagrange, ed. M. Francaviglia (North-Holland), pp.379–414, 1991.

[23] A.M. Vinogradov and V.A. Yumaguzhin, Differential invariants of webs on 2-dimensional manifolds, Mat. Zametki 48 (1990), No. 1, 46–68 (in Russian).

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