ON TRANSFORMATIONS OF $A$-HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We propose a systematic study of transformations of $A$-hypergeometric functions. Our approach is to apply changes of variables corresponding to automorphisms of toric rings, to Euler-type integral representations of $A$-hypergeometric functions. We show that all linear $A$-hypergeometric transformations arise from symmetries of the corresponding polytope. As an application of the techniques developed here, we show that the Appell function $F_4$ does not admit a certain kind of Euler-type integral representation.

1. INTRODUCTION

Hypergeometric functions are among the most extensively studied mathematical functions. The archetypal hypergeometric function is the Gauss hypergeometric series:

$$2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1) \cdot b(b+1)\cdots(b+n-1)}{n! \cdot c(c+1)\cdots(c+n-1)} x^n, \quad |x| < 1,$$

where $a, b, c \in \mathbb{C}$ are considered parameters, and $c \notin \mathbb{Z}_{\leq 0}$.

Transformations are one of the characteristic phenomena exhibited by hypergeometric functions. Among the earliest noted ones are Pfaff’s transformations:

$$2F_1(a, b; c; x) = (1-x)^{-b} 2F_1(b, c-a; c; x),$$

which are valid when $|x| < 1/2$, and which can be used to explicitly analytically continue the Gauss hypergeometric series. We point out that Pfaff’s transformations can be proved by a change of variables in Euler’s integral

$$B(b, c-b)2F_1(a, b; c; x) = \int_0^1 z^{b-1} (1-z)^{c-b-1} (1-xz)^{-a} \, dz$$

valid for $\text{Re}(c) > \text{Re}(b) > 0$ and $|x| < 1$, and where $B$ denotes the beta function.

On the one hand, transformation formulas are abundant in the hypergeometric literature, see, e.g., reference works such as [AAR99, EMOT53, NIST]. Of particular note is the work of Vidūnas, culminating in [Vid09], which classifies algebraic transformations for the Gauss hypergeometric function. On the other hand, the known hypergeometric transformations mostly involve only a few of the most classical families of hypergeometric functions, essentially those named after Gauss, Appell, and Lauricella.

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In the late 1980s Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89] introduced a way of studying multivariate hypergeometric functions, which at once unified and vastly generalized them. The goal of this article is to study transformations in the more general context of the $A$-hypergeometric functions introduced by Gelfand, Graev, Kapranov, and Zelevinsky. (See Section 2.1 for details, and in particular, Example 2.2 for the relationship between the classical and $A$-hypergeometric functions.)

Let \( \{F_k(\beta; x)\}_{k \in I} \), where \( I \subset \mathbb{Z} \), be a family of hypergeometric functions in the variables \( x \) depending on parameters \( \beta \). Loosely speaking, a transformation of hypergeometric functions is an identity involving the functions \( F_k(\beta T_k; \varphi_k(x)) \), \( k \in I \) where \( T_k \) is an affine function and \( \varphi_k \) is an algebraic function for all \( k \in I \). Such an identity is not necessarily given by a closed formula, and is allowed to involve elementary functions of \( x \) and \( \beta \) as coefficients. Needless to say, the task of classifying all hypergeometric transformations, even for the most classical hypergeometric functions in one or two variables, is probably out of reach.

The advantage of working in the context of $A$-hypergeometric functions, where tools from combinatorics, toric geometry, $D$-module theory and combinatorial commutative algebra are available, is that a systematical, unified study may be undertaken. This way, we may attempt to understand which kinds of transformations are valid in the most general contexts, and which are valid only for specific families of hypergeometric functions satisfying additional (combinatorial) properties.

Just as is the case for Pfaff’s transformations, integral expressions for hypergeometric functions provide many of the proofs of the classical transformation formulas. Here we use Euler-type integrals [GKZ90, BFP14] to provide transformations of $A$-hypergeometric functions. To aid our purposes, we introduce in (2.4) a more symmetric version of these integrals, which has not appeared before.

We point out that in order to apply changes of variables to Euler-type integrals so as to produce transformations, both the integrand and the cycle of integration need to be carefully controlled. A challenge in using the integrals of [GKZ90] is that the (compact) cycles used there are not explicitly constructed. Even when using the explicit cycle from [Beu10], it is difficult to determine which changes of variables preserve it. On the other hand, the cycles used in [BFP14] are essentially orthants, and thus easier to control, with the drawback, however, of requiring stronger assumptions on the integrand in order to achieve convergence than in the case of compact cycles.

As can be seen from the previous paragraph, we consider individual Euler-type integrals, and study their transformations. On the other hand, $A$-hypergeometric functions are defined as solutions of $A$-hypergeometric systems of differential equations. If the parameters are sufficiently generic, it is known [GKZ90, Beu11, SW12] that such systems have irreducible monodromy representation. As such, if a single $A$-hypergeometric function satisfies a given transformation, by monodromy transformations, a basis of such solutions may be obtained that satisfy the transformation as well, with the proviso that some coefficients may appear in order to account for choices of branches. This provides transformations for any $A$-hypergeometric function (not necessarily the original transformation due to the aforementioned coefficients).

Regarding classical hypergeometric transformations, we observe that not all of these are proved by straight change of variables in an integral representation of the corresponding function. See, e.g., the quadratic and higher order transformations for the Gauss hypergeometric function. While
these transformations are valid for their \( A \)-hypergeometric counterparts, we have not been able to derive them using the techniques developed here.

This article is organized as follows. In Section 2, we introduce \( A \)-hypergeometric functions and their Euler-type integral representations. In Section 3, we discuss how certain changes of variables induce transformations of \( A \)-hypergeometric functions. In Section 4, we study changes of variables and transformations induced by symmetries of the polytope underlying a given \( A \)-hypergeometric systems, and show that all linear transformations of \( A \)-hypergeometric functions arise from such symmetries. In Section 5, we briefly consider transformations induced by automorphisms of the monoid ring which defines an \( A \)-hypergeometric system. In Section 6, we use the techniques developed in this article to show that the Appell function \( F_4 \) does not admit an Euler-type integral representation where the cycle of integration is a rotation of the positive orthant.

Throughout this article \( \mathbb{N} \) denotes the set \( \{0, 1, 2, \ldots\} \).

2. THE \( A \)-HYPERGEOMETRIC SYSTEM AND EULER TYPE INTEGRALS

2.1. The \( A \)-hypergeometric system. We set
\[
A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \in \mathbb{Z}^{d \times n} \tag{2.1}
\]
where \( a_1, \ldots, a_n \in \mathbb{Z}^d \). We assume, firstly, that the columns of \( A \) span \( \mathbb{Z}^d \) as a lattice and, secondly, that the all ones vector lies in the rowspan of \( A \). Thus, \( A \) has rank \( d \), and there exists a vector \( \xi \in \mathbb{N}^d \) such that
\[
\xi A = (1, \ldots, 1). \tag{2.2}
\]

We denote by \( z \) elements in the torus \((\mathbb{C}^*)^d\) or indeterminates in the coordinate ring of this torus.

Definition 2.1. Let \( A \) be as in (2.1), and let \( \beta \in \mathbb{C}^d \). The \( A \)-hypergeometric system with parameter \( \beta \), denoted \( H_A(\beta) \), is the following system of linear partial differential equations:
\[
\partial^u F - \partial^v F = 0 \quad \text{for all } u, v \in \mathbb{N}^n \text{ such that } Au = Av;
\]
\[
\sum_{j=1}^{n} a_{ij} x_j \partial_j F = \beta_i F \quad \text{for } i = 1, \ldots, d.
\]

The matrix \( A \) defines a projective toric variety, namely the closure in \( \mathbb{P}^{n-1} \) of the image of the map
\[
(\mathbb{C}^*)^d \to (\mathbb{C}^*)^n \quad \text{given by } z \mapsto (z^{a_1}, \ldots, z^{a_n}).
\]
We refer to this as the toric variety underlying the hypergeometric system \( H_A(\beta) \). The coordinate ring of this variety is the semigroup ring \( \mathbb{C}[\mathbb{N}A] \), where \( \mathbb{N}A \) is the semigroup (actually, monoid) of nonnegative integer combinations of the columns of \( A \).

We identify the space \( \mathbb{C}^A = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{C}\} \) with the family of polynomials
\[
\left\{ f(z) = \sum_{j=1}^{n} x_j z^{a_j} \bigg| (x_1, \ldots, x_n) \in \mathbb{C}^n \right\} \tag{2.3}
\]
where \( z = (z_1, \ldots, z_d) \). Note that we deviate from the standard approach: the Newton polytope (i.e. the convex hull of the exponent vectors of the monomials) of a polynomial \( f \in \mathbb{C}^A \) is at most \((d - 1)\)-dimensional. Indeed, the linear form \( \xi \) encodes the quasi-homogeneity
\[
f(\lambda^\xi z) = \lambda f(z),
\]
where $\lambda^\xi z = (\lambda^\xi_1 z_1, \ldots, \lambda^\xi_d z_d)$. It is common in the literature to reduce the number of variables by removing this homogeneity, and in effect consider $f$ as a $(d - 1)$-variate polynomial.

**Example 2.2.** Classical hypergeometric functions must be suitably homogenized to fit in the $A$-hypergeometric setting. Let us give some information on how to realize some of the more important classical hypergeometric functions in an $A$-hypergeometric way. Since only special matrices $A$ are involved, we note that the $A$-hypergeometric setting is indeed more general than the classical setting.

It can be shown that, when $A$ and $\beta$ are given by

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix} c - 1 \\ -a \\ -b \end{bmatrix},$$

then a function $F(x)$ is a solution of the Gauss hypergeometric equation if and only if the function $x_1^{c-1}x_2^{-a}x_3^{-b}F(x_1x_4/x_2x_3)$ is a solution of $H_A(\beta)$.

In a similar way, the hypergeometric functions $pF_{p-1}$ can be obtained using a $(2p - 1) \times 2p$ matrix $A$ consisting of a $(2p - 1) \times (2p - 1)$ identity matrix to which we adjoin an additional column whose entries are $\pm 1$ in such a way that the kernel of $A$ is spanned by $(1, \ldots, 1, -1, \ldots, -1)$ with $p$ ones and $p$ negative ones.

For the Appell functions, each of $F_1$ and $F_4$ arises from a single $4 \times 6$ matrix $A$ (see the proof of Theorem 6.1 for more details on $F_4$), while each of $F_2$ and $F_3$ correspond to a $5 \times 7$ matrix $A$.

More generally, the $m$-variate Lauricella function $F_A$ corresponds to a matrix $A$ of size $(2m + 1) \times (3m + 1)$, the function $F_B$ corresponds to a matrix $A$ of size $(2m + 1) \times (3m + 1)$, the function $F_C$ corresponds to a matrix $A$ of size $(m + 2) \times (2m + 2)$ (given explicitly in Example 4.3) and, finally, the function $F_D$ corresponds to a matrix $A$ of size $(m + 2) \times (2m + 2)$.

### 2.2. Euler type integrals.

In this article, a central role is played by solutions of $H_A(\beta)$ representable by an integral

$$F_\sigma(\beta; x) = \int_\sigma z^{-\beta} f(z)^{[\xi, \beta]} \, d\eta, \quad (2.4)$$

where $f$ is as in (2.3) and $\xi$ is as in (2.2). The relationship between $A$-hypergeometric functions and toric geometry is here manifest through the Haar measure

$$d\eta = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}$$

of the complex torus $(\mathbb{C}^*)^d$. We discuss the multivaluedness of (2.4) in § 2.3. In order for the integral (2.4) to be well defined, we need the $d$-cycle $\sigma$ to avoid the singular locus of the integrand. That is, we require that $\sigma \subset (\mathbb{C}^*)^d \setminus V(f)$, where $V(f)$ denotes the zero locus of the polynomial $f$. The inclusion $\sigma \subset (\mathbb{C}^*)^d \setminus V(f)$ is also valid if we perturb the coefficients of the polynomial $f$. Hence, the integral (2.4) defines a germ of a meromorphic function in $x$, under the assumption that it converges.

As we do not follow the standard approach, we provide a proof of the following theorem.

**Theorem 2.3 (almost [GKZ90, Theorem 2.7]).** Provided sufficient convergence properties, the integral (2.4) defines a germ of an $A$-hypergeometric function in the variables $x$. 

Let us comment on the sufficient convergence properties required in Theorem 2.3. In order to perform the steps in the following proof, we need to interchange the order of the integration in (2.4) and partial derivatives with respect to parameters $\beta$. This interchange is allowed if, e.g., the cycle $\sigma$ is compact as in [GKZ90] or, in case of a noncompact cycle $\sigma$, provided the integrand has sufficiently rapid decay along its boundary as in [BFP14]. In the latter case one might be forced to impose restrictions on the parameter $\beta$; however, these restrictions can be handled by considering a meromorphic extension in the sense of Riesz and Hadamard. As the explicit cycles considered in this work are of the forms appearing in [GKZ90] and [BFP14], we refer the reader to those articles for further details on these aspects.

Proof of Theorem 2.3. We need to show that $F_{\sigma}(\beta; x)$ solves the differential equations set forward in Definition 2.1. We consider them in order.

Firstly, let $u \in \mathbb{N}^n$. Then,

$$\partial^u F_{\sigma}(\beta; x) = (\langle \xi, \beta \rangle)_{|u|} \int_{\sigma} z^{-\beta} f(z)^{(\xi, \beta) - |u|} z^{Au} \, d\eta,$$

where $|u| = u_1 + \cdots + u_n$ and $(\langle \xi, \beta \rangle)_{|u|}$ denotes the descending factorial $(\alpha)_m = \prod_{j=0}^{m-1} (\alpha - j)$. Since $u$ has only nonnegative components, we have that $|u| = \xi Au$, and hence the right hand side of (2.5) depends, in terms of $u$, only on the vector $Au$. It follows that $F_{\sigma}(\beta; x)$ solves the first set of equations in Definition 2.1.

Secondly, we have that

$$\sum_{j=1}^{n} a_{ij} x_j \partial_j F_{\sigma}(\beta; x) = \int_{\sigma} z^{-\beta} z_i f_i'(z) \langle \xi, \beta \rangle f(z)^{(\xi, \beta) - 1} \, d\eta = \beta_i F_{\sigma}(\beta; x), \quad i = 1, \ldots, d,$$

where the last equality is obtained through integration by parts with respect to $z_i$. This is precisely the second set of equations in Definition 2.1.

Remark 2.4. The integral (2.4) is a homogeneous version of the hypergeometric integrals appearing in [GKZ90] and [BFP14]. In those papers, the matrix $A$ was given in the form

$$A = \begin{bmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
a_{11} & \cdots & a_{1n_1} & a_{21} & \cdots & a_{2n_2} & a_{m1} & \cdots & a_{mn_m}
\end{bmatrix},$$

where $n = n_1 + \cdots + n_m$ and $a_{ij} \in \mathbb{Z}^r$. Thus, $d = m + r$. The existence of $\xi$ ensures that $A$ can always be written in this form with $m \geq 1$. Notice that, with $A$ as in (2.6),

$$\xi = (1, \ldots, 1, 0, \ldots, 0),$$

where $m$ times and $r$ times.

It holds that $\mathbb{C}^A \simeq \prod_{i=1}^{m} \mathbb{C}^{A_i}$, where $i = 1, \ldots, m$, and where $\mathbb{C}^{A_i}$ denotes the family of $r$-variate polynomials

$$\left\{ f_i(w) = \sum_{j=1}^{n_i} x_{ij} w^{a_{ij}} \left| (x_{i1}, \ldots, x_{in_i}) \in \mathbb{C}^{n_i} \right. \right\},$$
where \( w = (w_1, \ldots, w_r) \). The Euler-type hypergeometric integral considered in, e.g., [GKZ90] and [BFP14] takes the form

\[
M_\varepsilon(\beta; x) = \int_\varepsilon w_1^{-\beta_1} \cdots w_r^{-\beta_r} \prod_{i=1}^m f_i(w)^{\beta_i} d\eta.
\]

where \( \varepsilon \) is some \( r \)-cycle. Let us consider instead the integral from Definition 2.4, and the change of variables \( z \mapsto \varphi(z) \) defined by

\[
\begin{align*}
    z_i &\mapsto z_i/f_i(w), & \text{for } i = 1, \ldots, m, \text{ and } \\
    z_i &\mapsto w_{i-m}, & \text{for } i = m + 1, \ldots, d.
\end{align*}
\]

If \( \sigma = \varphi^{-1}(\tau) \times \varepsilon \), where \( \tau \) is some \( m \)-cycle, then

\[
F_\sigma(\beta; x) = K(\beta; \tau) \cdot M_\varepsilon(\beta; x)
\]

where

\[
K(\beta; \tau) = \int_\tau z_1^{-\beta_1} \cdots z_m^{-\beta_m} (z_1 + \cdots + z_m)^{\beta_1 + \cdots + \beta_m} d\eta
\]

is constant with respect to \( x \). We say that \( F_\sigma(\beta; x) \) is the homogenized version of \( M_\varepsilon(\beta; x) \) or, conversely, that \( M_\varepsilon(\beta; x) \) is a dehomogenized version of \( F_\sigma(\beta; x) \). In this nomenclature, the dependence on the cycle \( \tau \) is suppressed; typically one considers \( \tau \) as simple as possible to ensure convergence and nontriviality of \( K(\beta; \tau) \), e.g., as the skeleton of a polydisc centered at the origin. Throughout this text, we consider dehomogenized integrals in the examples, to emphasize the relationship with integral representations of classical hypergeometric functions.

**Remark 2.5.** A parameter \( \beta \) is called **nonresonant** (with respect to \( A \)) if it does not belong to any integer translate of a supporting hyperplane of a facet of the real cone spanned by the columns of the matrix \( A \). Thus, nonresonant parameters are **very generic** in the sense that they avoid a locally finite (but infinite) collection of algebraic varieties. It follows from Remark 2.4 and [GKZ90] that if \( \beta \) is nonresonant, one can always find a basis of the solution space of \( H_A(\beta) \) by considering integrals (2.4) taken over \( \text{vol}(A) \)-many distinct cycles \( \sigma \).

### 2.3. Multivaluedness

An additional difficulty to overcome is the multivaluedness of the integrand in (2.4). Indeed, we need to choose branches of the exponential functions \( f(z)^{\langle \gamma, \beta \rangle} \) and \( z^{-\beta} \). A change of branch alters the value of the integral by a factor of \( e^{\langle \gamma, \beta \rangle} \), where \( \gamma \in \Gamma \) for some subgroup \( \Gamma = \Gamma(\sigma) \subset \mathbb{C}^d \) depending on the cycle \( \sigma \). (Note that the one-dimensional subspace of the solution space generated by (2.4) is well-defined.) In particular, the validity of any identity involving Euler-type integrals depends on an appropriate choice of branches. We mention this, as it has implications relevant for any study comparing transformations with computations of monodromy; the coefficient \( e^{\langle \gamma, \beta \rangle} \) can appear through the action of analytic continuation.

**Example 2.6.** Consider the \( A \)-hypergeometric system defined by the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}.
\]

Let us consider a neighbourhood of a point \((x_1, x_2, x_3) \in \mathbb{R}^3 \) such that \( x_2^2 < 4x_1x_3 \), and assume that \( \beta = (\beta_1, \beta_2) \) is sufficiently generic (nonresonant suffices). Then, the solution space of the
system $H_A(\beta)$ is spanned by the two dehomogenized Euler-type integrals (as in \cite{BFP14})

$$F_1(\beta_1, \beta_2; x_1, x_2, x_3) = \int_{\mathbb{R}_+} (x_1 + x_2 z + x_3 z^2)^{\beta_1} z^{-\beta_2} d\eta$$

and

$$F_2(\beta_1, \beta_2; x_1, x_2, x_3) = \int_{\mathbb{R}_-} (x_1 + x_2 z + x_3 z^2)^{\beta_1} z^{-\beta_2} d\eta.$$  

Here $\mathbb{R}_+$ and $\mathbb{R}_-$ refer to the nonnegative and nonpositive real half-lines, respectively. By the multivaluedness of the exponential functions, the integrals $F_1$ and $F_2$ are well-defined only up to a multiple of $e^{2\pi i(k, \beta)}$ for $k \in \mathbb{Z}^2$. Applying the change of variables $z \mapsto 1/z$ for each of the above integrals we obtain the transformation

$$F_j(\beta_1, \beta_2; x_1, x_2, x_3) = F_j(\beta_1, 2\beta_1 - \beta_2; x_3, x_2, x_1), \quad \text{for } j = 1, 2.$$  

On the other hand the change of variables $z \mapsto e^{i\pi z}$ gives

$$F_1(\beta_1, \beta_2; x_1, x_2, x_3) = e^{-i\pi \beta_2} F_2(\beta_1, \beta_2; x_1, -x_2, x_3).$$

Thus, applying the second identity twice, and the transformation once, we obtain,

$$F_1(\beta_1, \beta_2; x_1, x_2, x_3) = e^{-2i\pi (\beta_1 + \beta_2)} F_1(\beta_1, 2\beta_1 - \beta_2; x_3, x_2, x_1).$$

This seeming contradiction is caused by inconsistencies in the choices of branches. The transformation is valid – but it does not commute with meromorphic continuations.

3. Changes of Variables, Automorphisms, and Transformations

We saw in Example \ref{2} that a change of variables in the integral \ref{2} can induce a transformation of $A$-hypergeometric functions. We now consider the general situation. In order to deduce an identity, we need to perform a change of variables under which the cycle $\sigma$ is invariant up to homotopy. To simplify notation, we introduce the following assumptions.

Let $\varphi : \sigma \to \sigma$ be a diffeomorphism such that the toric Jacobian $J_\varphi(z)$ is nonvanishing.

Let $g$ denote the pullback $\varphi^*(f)$. Then, performing the change of variables $z \mapsto \varphi(z)$ in \ref{2} we obtain

$$\int_{\sigma} z^{-\beta} f(z)^{\langle \xi, \beta \rangle} d\eta = \int_{\sigma} \varphi(z)^{-\beta} g(z)^{\langle \xi, \beta \rangle} J_\varphi(z) d\eta. \quad \text{(3.1)}$$

We now wish to interpret the right hand side of \ref{3} as an $A$-hypergeometric function. We present here only a sufficient result in this direction. There is a flexibility in considering, for example, special values of the parameters $\beta$, which could greatly simplify the right hand side of \ref{3}.

One natural requirement to enable the right hand side of \ref{3} to represent an $A$-hypergeometric function is that the pullback $\varphi^*$ defines a map

$$\varphi^* : \mathbb{C}^A \to \mathbb{C}^A. \quad \text{(3.2)}$$

The form $\xi$ from \ref{2} induces a grading on the semigroup ring $\mathbb{C}[NA]$. We say that an automorphism of $\mathbb{C}[NA]$ is homogeneous if it preserves homogeneous elements under this grading. Then $\varphi$ is induced by a homogeneous automorphism of the semigroup ring $\mathbb{C}[NA]$ if and only if \ref{3} holds and $\varphi^*$ is a $\mathbb{C}$-linear map. Conversely, we have the main theorem of this section.

$$\varphi^* : \mathbb{C}^A \to \mathbb{C}^A.$$
**Theorem 3.1.** Assume that \( \varphi \) is induced by a homogeneous automorphism of \( \mathbb{C}[\mathbb{N}A] \). Then, provided sufficient convergence, the identity (3.1) encodes a transformation of \( A \)-hypergeometric functions.

The requirement of sufficient convergence depends heavily on the automorphism and cycle in question. We refer to reader to [4, 5] for details. In this section we provide only the formal computations.

Before we prove Theorem 3.1 we need a few auxiliary results. We remark already at this point that by abuse of notation we identify \( \varphi \in \text{Aut}(\mathbb{C}[\mathbb{N}A]) \) with its induced rational map \( \varphi: \mathbb{C}^d \rightarrow \mathbb{C}^d \), whose existence follows from Lemma 3.3.

**Lemma 3.2.** Let \( \varphi \) be a homogeneous automorphism of \( \mathbb{C}[\mathbb{N}A] \). Then, there exist linear homogeneous polynomials \( p_j \in \mathbb{C}[w_1, \ldots, w_n] \), for \( j = 1, \ldots, n \), such that \( \varphi(z^{a_j}) = p_j(z^{a_1}, \ldots, z^{a_n}) \).

**Proof.** Since \( \varphi \) is an automorphism, it is injective, and hence \( \varphi(z^{a_j}) \) is nonconstant. Then, since \( \varphi \) is a homogeneous automorphism, \( p_j \) must be a polynomial with vanishing constant term for each \( j = 1, \ldots, n \).

Let \( q_m \), for \( m = 1, \ldots, n \), denote the corresponding polynomials associated to the inverse \( \varphi^{-1} \) of \( \varphi \). Let \( h(z) = p_j(z^{a_1}, \ldots, z^{a_n}) \). Then, since \( \langle \xi, a_k \rangle = 1 \) for all \( k = 1, \ldots, n \), the degree of \( p_j \) is equal to the degree of \( h(\lambda^\xi z) \) in \( \lambda \). We have that

\[
 z^{a_j} = \varphi^{-1}(\varphi(z^{a_j})) = \varphi^{-1}(p_j(z^{a_1}, \ldots, z^{a_n})).
\]

After the substitution \( z \mapsto \lambda^\xi z \) we obtain in the left hand side a polynomial of degree one in \( \lambda \).

Since the polynomials \( q_m \) all have positive degree, we obtain in the right hand side a polynomial in \( \lambda \) of degree at least the degree of \( p_j \). Hence, the degree of \( p_j \) is at most one, which implies that it is equal to one. \( \square \)

**Lemma 3.3.** Every automorphism of \( \mathbb{C}[\mathbb{N}A] \) is induced from an automorphism of the field of rational functions \( \mathbb{C}(z) \).

**Proof.** Since the columns of \( A \) span \( \mathbb{Z}^d \) as a lattice, each of the standard basis vectors \( e_1, \ldots, e_d \) has a representation

\[
e_i = \sum_{j=1}^n m_{ij} a_j, \quad i = 1, \ldots, d,
\]

(3.3)

where \( m_{ij} \in \mathbb{Z} \) for all \( i = 1, \ldots, d \) and \( j = 1, \ldots, n \). Then, \( \varphi \) extends to a map \( \mathbb{C}(z) \rightarrow \mathbb{C}(z) \) by

\[
 \varphi(z_i) = \prod_{j=1}^n (p_j(z^{a_1}, \ldots, z^{a_n}))^{m_{ij}}.
\]

(3.4)

To see that this is an automorphism of \( \mathbb{C}(z) \), apply the same procedure to the inverse automorphism \( \varphi^{-1}: \mathbb{C}[\mathbb{N}A] \rightarrow \mathbb{C}[\mathbb{N}A] \). \( \square \)

We remark that the converse of Lemma 3.3 is not true, as most automorphisms of \( \mathbb{C}(z) \), or of \( \mathbb{C}[z^{\pm 1}] \), do not restrict to well defined maps \( \mathbb{C}[\mathbb{N}A] \rightarrow \mathbb{C}[\mathbb{N}A] \). For more information on automorphisms of \( \mathbb{C}[\mathbb{N}A] \) we refer to [BG99], where a classification of homogeneous automorphisms of \( \mathbb{C}[\mathbb{N}A] \) is provided under the assumption that \( \mathbb{C}[\mathbb{N}A] \) is normal.
Proof of Theorem 3.1. Since \( p_j \) for \( j = 1, \ldots, n \) are homogeneous linear forms the pullback \( \varphi^* \) defines a linear map \( \varphi^* : \mathbb{C}^A \to \mathbb{C}^A \). In particular, \( g = \varphi^*(f) \in \mathbb{C}^A \).

From (3.4) we can conclude that each \( \varphi(z_i) \) is a rational function of \( z \) for \( i = 1, \ldots, d \). It follows that also the toric Jacobian \( J_\varphi(z) \) is a rational function. By use of the generalized binomial theorem, under proper assumptions to ensure convergence, we can expand the factor \( \varphi(z)^{-\beta} J_\varphi(z) \) of the integrand as a generalized Laurent series. Each term of such a series is an \( A \)-hypergeometric function, which finishes the proof. \( \square \)

We have assumed that \( \varphi \) restricts to a diffeomorphism of the cycle \( \sigma \). Given an explicit automorphism, to deduce a valid transformation, we must describe explicitly the cycle \( \sigma \). What sparked our investigation of the subject matter was not the realization provided by Theorem 3.1, but the study of Euler-type \( A \)-hypergeometric integrals over explicit cycles in [NP13] and [BFP14].

4. LINEAR ALGEBRAIC TRANSFORMATIONS FROM POLYTOPE SYMMETRIES

Consider a matrix \( A \) as in (2.1), and a monomial homogeneous automorphism \( \varphi : \mathbb{C}[\mathbb{N}A] \to \mathbb{C}[\mathbb{N}A] \). That is, all polynomials \( p_j \), for \( j = 1, \ldots, n \), are monomials. It follows from these assumptions and Lemma 3.3 that there exist vectors \( t_i \in \mathbb{Z}^d \) such that

\[
\varphi(z_i) = z^{t_i}, \text{ for } i = 1, \ldots, d.
\]

Let us denote by \( T \) the matrix \([t_1 \ t_2 \ldots \ t_d]\).

The fact that \( \varphi \) is a monomial homogeneous automorphism, together with Lemma 3.2 implies that \( \varphi(z^{a_j}) = z^{\pi(j)} \) for each \( j = 1, \ldots, n \), where \( \pi(j) \in \{1, \ldots, n\} \). By injectivity of \( \varphi \), we conclude that \( \pi \in \mathfrak{S}_n \) is a permutation of the columns of \( A \). Let \( P = P(\pi) \) denote the corresponding permutation matrix. Then,

\[
TA = AP. \tag{4.1}
\]

The pair \((T, P)\) encodes a polytope symmetry of the Newton polytope \( \Delta_A = \text{conv}(A) \), the convex hull of the columns of the matrix \( A \). In general, however, a polytope symmetry of \( \Delta_A \) need not induce an automorphism of \( \mathbb{C}[\mathbb{N}A] \). If \( A \) is saturated, that is, if \( \mathbb{N}A = \mathbb{R}_+A \cap \mathbb{Z}^d \), or equivalently, if \( \mathbb{C}[\mathbb{N}A] \) is normal, then any polytope symmetry does induce an element of \( \text{Aut} (\mathbb{C}[\mathbb{N}A]) \).

Corollary 4.1. The monomial homogeneous automorphism \( \varphi \) induces the transformation

\[
F_\sigma(\beta; x) = |T| F_\sigma(T\beta; xP), \tag{4.2}
\]

where \( \sigma \simeq (\mathbb{R}_+)^{d-1} \times S^1 \) and \( xP \) and \( T\beta \) denotes the standard matrix multiplication.

Proof. Let us first remark on the choice of cycle. Applying a linear transformation we can write \( A \) in the form (2.6) with \( m = 1 \). In the notation of Remark 2.4 we set \( \tau \simeq S^1 \) to ensure convergence, and nonvanishing, of the constant (2.7). The dehomogenized integral is taken over the cycle \( \varepsilon = \mathbb{R}^{d-1} \). Note that \( \varphi \) restricts to monomial transformation in \( d-1 \) variables of the dehomogenized integral, which we also denote by \( \varphi \).

It is clear that \( \varphi \) maps \((\mathbb{R}_+)^{d-1}\) into itself. As \( \varphi^{-1} \) is also a monomial automorphism, we find that \( \varphi \) preserves the positive orthant. The toric Jacobian \( J_\varphi(z) \) is equal to the determinant \(|T|\), which
is nonvanishing since \( \varphi \) is surjective. Furthermore, the identity \( TA = AP \) implies that, with the notation of Theorem 3.1,

\[
g(z) = \sum_{j=1}^{n} (xP_j) z^a_j.
\]

Finally, that \( \varphi(z)^\beta = z^T \beta \) is immediate. Thus, the statement follows from Theorem 3.1. \( \square \)

**Example 4.2.** Consider the point configuration

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}.
\]

There is a group of eight transformations generated by the polytope symmetries encoded by the pairs

\[
T_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad P_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The corresponding identities for the hypergeometric functions reads as

\[
\begin{align*}
F_\sigma(\beta_1, \beta_2, \beta_3; x_1, x_2, x_3) & = F_\sigma(\beta_1, \beta_2, \beta_3; x_1, x_2, x_3) \\
F_\sigma(\beta_1, \beta_2, \beta_3; x_1, x_2, x_3) & = F_\sigma(\beta_1, \beta_2, \beta_3; x_1, x_2, x_3)
\end{align*}
\]

Using a classical integral representation of Gauss hypergeometric function \( _2F_1(a, b; c; x) \), with \( a = -\beta_2, b = -\beta_3, c = -\beta_1, \) and \( x = 1 - \frac{1 + x}{x_1 x_2 x_3} \), the first transformation translates to the in terms of series trivial identity

\[
_2F_1(a, b; c; x) = \frac{x}{x_1 x_2 x_3}.
\]

while the second transformation translates to the Pfaff transformation

\[
_2F_1(a, b; c; x) = (1 - x)^{-b} _2F_1(c - a, b; c; \frac{x}{x_1 x_2 x_3}).
\]

**Example 4.3.** Suitably homogenized, the Lauricella hypergeometric function \( F_C^{(m)} \) is a solution to the \( A \)-hypergeometric system defined by the \( (m + 2) \times (2m + 2) \)-matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & I_m & 0 & -I_m
\end{bmatrix},
\]

where \( I_m \) denotes the \( m \times m \) identity matrix, and bold numbers are to be interpreted as a vectors of appropriate dimensions. We can view \( \text{conv}(A) \) as the convex hull of two \( m \)-simplices in \( \mathbb{R}^{m+2} \), and find two families of transformations of \( A \). The first corresponds to a permutation of the vertices of the simplices: let \( P \) be a permutation matrix of size \( m \times m \), then

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & P
\end{bmatrix} A \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & P^{-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & P^{-1}
\end{bmatrix} = A.
\]

The second family of transformations corresponds to swapping two vertices between the simplices:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & I_{m-k} & 0 \\
0 & 0 & 0 & I_{m-1}
\end{bmatrix} A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & I_{m-1} & 0 & 0 & I_{m-1} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -I_{m-1}
\end{bmatrix},
\]
which corresponds to transposing columns \( k + 1 \) and \( m + k + 2 \) of \( A \). The two families of permutations commute, generating a group of \( 2^m m! \) linear transformations of \( F_C^{(m)} \).

Let us end this section with a partial converse of Corollary 4.1.

**Theorem 4.4.** Let \( F(\beta; x) \) be an \( A \)-hypergeometric function for which there exists a transformation valid for generic parameters \( \beta \);

\[
F(\beta; x) = K(\beta) F(T\beta; xP),
\]

where \( P \) is a permutation matrix and \( K(\beta) \) is a constant with respect to \( x \). Then, \( TA = AP \). That is, \( P \) encodes a polytope symmetry of \( A \).

**Proof.** We recall from \([\text{Mat09}]\) Theorem 2.7 and Corollary 2.8 the fact that if \( \beta \) is sufficiently generic, and \( F = F(x) \) is a solution of \( H_A(\beta) \), then any differential operator annihilating \( F \) must belong to \( H_A(\beta) \). Using this result, the equation (4.3) implies that \( H_A(\beta) = H_{AP}(T\beta) \).

Since the toric ideals underlying \( H_A(\beta) \) and \( H_{AP}(T\beta) \) coincide, we see that \( A \) and \( AP \) have the same integer kernel, and therefore (since \( P \) has full rank), the same rational rowspan. Now considering the second set of equations from Definition 2.1 for \( H_A(\beta) \) and \( H_{AP}(T\beta) \), we conclude that \( TA = AP \). □

5. Linear Algebraic Transformations from Elementary Automorphisms

It follows from \([\text{BG99}]\) that in the case when \( A \) is saturated all homogeneous automorphisms of \( \mathbb{N}A \) are given by the polytope symmetries considered in \( \S 4 \) and elementary (toric) automorphisms which we consider in this section. These are generated by mappings

\[
z_i \mapsto z_i + t \ z^a,
\]

where \( t \) is a scalar and \( a \in A \). These automorphisms also generate identities of hypergeometric functions. However, in contrast to the situation in \( \S 4 \) one must perform an expansion using the generalized binomial theorem, as in the proof of Theorem 3.1. This requires a specialization of \( \beta \) to a family of hyperplanes in the parameter space. To simplify the exposition, we deduce the corresponding transformations in examples only.

**Example 5.1.** Consider the matrix

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},
\]

which we consider to be in the form (2.6) with \( m = 1 \). Consider the automorphism of \( \mathbb{C}[\mathbb{N}A] \) defined by \( z_2 z_1 \mapsto z_2 z_1 + z_1 \), which for the dehomogenized integral induces the change of variables \( z \mapsto z + 1 \). A cycle preserved under this transformation is \( \sigma = \mathbb{R} \), and under the restriction that \( f(z) \) is nonvanishing on \( x \), the integral (2.4) converges for \( \beta \) in some open domain \([\text{BFPT14}]\). We find that

\[
\int_{\mathbb{R}} (x_1 + x_2 z + x_3 z^2)^{\beta_1} z^{-\beta_2} \, d\eta = \int_{\mathbb{R}} ((x_1 + x_2 + x_3) + (x_2 + 2x_3)z + x_3 z^2)^{\beta_1} (z + 1)^{-\beta_2} \, d\eta.
\]

In order to obtain an identity of \( A \)-hypergeometric functions with shifted parameters, we need to expand the binomial \( (z + 1)^{-\beta_2} \). Since the cycle is \( \sigma = \mathbb{R} \), this imposes the requirement that \( \beta_2 \) is
integer. For \( \beta_2 = -N \), we obtain the identity

\[
F_R(x_1, x_2, x_3, \beta_1, -N) = \sum_{K=0}^{N-1} \binom{N-1}{K} F_R(x_1 + x_2 + x_3, x_2 + 2x_3, x_3, \beta_1, 1 - K).
\]

**Example 5.2.** Consider the matrix from Example 4.2. Then, all automorphisms of \( \mathbb{C}[NA] \) are generated by the monomial automorphism of that example, and the toric automorphism induced by \( z_1 \mapsto z_1 + 1 \) in the dehomogenized integral. The latter gives the identity

\[
\int_{\mathbb{R}} (x_1 + x_2 z_1 + x_3 z_2 + x_4 z_1 z_2)^{\beta_1} z_1^{-\beta_2} z_2^{-\beta_3} d\eta
\]

\[
= \int_{\mathbb{R}} ((x_1 + x_2) + x_2 z_1 + (x_3 + x_4) z_2 + x_4 z_1 z_2)^{\beta_1} (z_1 + 1)^{-\beta_2} z_2^{-\beta_3} d\eta,
\]

which yields an identity of \( A \)-hypergeometric integrals in the case when \( \beta_2 \) is a negative integer using the same reasoning as in the previous example.

## 6. On the absence of integral representations of Apell’s \( F_4 \).

The standard form of an integral representation of classical hypergeometric functions is as a dehomogenized Euler type integral (2.4) over the positive orthant \( \mathbb{R}^{d-1}_+ \), with a coefficient which is a quotient of gamma functions in the parameters \( \beta \). Such expressions are known, e.g., for Gauss’ hypergeometric function and Lauricella \( F_D \). However, such an expression is not known in the case of Apell’s hypergeometric function \( F_4 \), a special case of Lauricella \( F_C \). In this section we prove the following theorem, as an application of the results of [4]

**Theorem 6.1.** The Apell hypergeometric function \( F_4 \) does not admit any integral representation in the form of a dehomogenized Euler type integral taken over a cycle \( \sigma \) which is a rotation of the positive orthant.

**Proof.** Apell’s hypergeometric function \( F_4 \) can be realized as an \( A \)-hypergeometric function using the setup from Example 4.3. More precisely, let \( A \) be as in the case \( m = 2 \) of that example. In its dehomogenized version, Apell’s \( F_4 \) admits the series representation

\[
F_4(a, b; c, a'; y_1, y_2) = \sum_{r,s=0}^{\infty} \frac{(a)_r (b)_s}{(c)_r (a')_s} r! s! y_1^r y_2^s
\]

Then, for \( \kappa \) the transpose of \([-a, c - 1, a' - 1, -b, 0, 0]\), the function

\[
\frac{x_2^{c-1} x_3^{a'-1}}{x_1^a x_4^b} F_4\left(a, b; c, c'; x_2 x_3, x_4, x_1, x_4\right)
\]

is \( A \) hypergeometric of parameter \( \beta = A \kappa \).

Let us note that Apell’s \( F_4 \) satisfies the transformation

\[
F_4(a, b; c, a'; y_1, y_2) = K_1(\beta)(-y_2)^{-a} F_4\left(a, a - c' + 1; c, a - b + 1; \frac{y_1}{y_2}, \frac{1}{y_2}\right)
\]

\[
+ K_2(\beta)(-y_2)^{b} F_4\left(b - c' + 1, b; c, b - a + 1; \frac{y_1}{y_2}, \frac{1}{y_2}\right)
\]
where $K_1$ and $K_2$ are certain quotients of $\Gamma$ functions in the parameters $\beta = A\kappa$, see [EMOT53, §5.11]. We consider the polytope symmetry of $A$ encoded by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & -1 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. $$

Assume that $F_4$ admits an integral representation as a dehomogenized version of the Euler type integral (2.4) taken over some cycle $\sigma$ which is a rotation of the positive orthant. It then follows from Corollary 4.1, using $T$ and $P$ as above, that $F_4$ satisfies, for generic parameters, an identity of the form

$$F_4(a, b; c, c'; y_1, y_2) = K_3(\beta)(-y_2)^b F_4\left(b-c'+1, b; c, b-a+1; \frac{y_1}{y_2}, \frac{1}{y_2}\right).$$

However, this contradicts the validity of the transformation (6.1). Indeed, composing the two transformations, we find that the two functions appearing in the right hand side of (6.1) are linearly dependent for generic parameters $\beta$; this is absurd. □

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