A DENSITY-INDEPENDENT FORMULATION OF SMOOTHED PARTICLE HYDRODYNAMICS

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ABSTRACT

The standard formulation of the smoothed particle hydrodynamics (SPH) assumes that the local density distribution is differentiable. This assumption is used to derive the spatial derivatives of other quantities. However, this assumption breaks down at the contact discontinuity. At the contact discontinuity, the density of the low-density side is overestimated while that of the high-density side is underestimated. As a result, the pressure of the low-density (high-density) side is overestimated (underestimated). Thus, unphysical repulsive force appears at the contact discontinuity, resulting in the effective surface tension. This tension suppresses fluid instabilities. In this paper, we present a new formulation of SPH, which does not require the differentiability of density. Instead of the mass density, we adopt the internal energy density (pressure) and its arbitrary function, which are smoothed quantities at the contact discontinuity, as the volume element used for the kernel integration. We call this new formulation density-independent SPH (DISPH). It handles the contact discontinuity without numerical problems. The results of standard tests such as the shock tube, Kelvin–Helmholtz and Rayleigh–Taylor instabilities, point-like explosion, and blob tests are all very favorable to DISPH. We conclude that DISPH solved most of the known difficulties associated with the shock, while the artificial conductivity adds physically non-existent dissipation. One needs to fine-tune the conductivity coefficient to prevent unnecessary smoothing. This means that the conductivity must be nonlinear. Second, if there is a jump in the chemical composition, then thermal conductivity is not enough. However, whether the use of artificial chemical diffusion is justified or not is an open question. He also showed that when the Ritchie & Thomas formulation (Ritchie & Thomas 2001) was used, the Kelvin–Helmholtz instability grew but the growth of the instability was insufficient. Read et al. (2010) suggested that the Kelvin–Helmholtz instability took place when a higher order kernel with a large enough number of neighbor particles and the momentum equation of Ritchie & Thomas (2001) were used. Abel (2011) used the relative pressure, which was first proposed by Morris (1996), instead of the absolute values of pressures in the equation of motion. This formulation improves the treatment of the Kelvin–Helmholtz instability, but breaks Newton’s third law. Garcia-Senz et al. (2012) considered the use of the integral form of the first derivative, which also improved the treatment of hydrodynamical instabilities.

In this paper, we describe a new formulation of SPH which does not use the smoothed mass density for the volume element. Instead, we use an arbitrary function of the internal energy density for the volume element to obtain other quantities and their spatial derivatives. The reason why we adopt the energy density instead of the mass density is that it is the fundamental quantity of hydrodynamics. We call this new formulation density-independent SPH (DISPH). In DISPH, the pressure is calculated without using the mass density. Thus, unphysical jumps in pressure at the contact discontinuity disappear. The special case where the volume element is defined by the internal energy density or pressure leads to the equation of motion proposed by Ritchie & Thomas (2001). Our formulation...
can be used to derive the SPH equation for an arbitrary quantity, while how we can apply the Ritchie & Thomas formulation to equations other than energy equation and equation of motion is not clear. The results of various tests indicate that our formulation is highly advantageous.

The structure of this paper is as follows. In Section 2, we analyze the problem of standard SPH at discontinuities. The derivation of DISPH is described in Section 3. We then generalize DISPH by adopting an arbitrary function of pressure in Section 4. The comparison of the results of test calculations with DISPH and the standard formulation of SPH are shown in Section 5. Summary and discussion are presented in Section 6.

2. STANDARD SPH AND ITS DIFFICULTY AROUND DISCONTINUITIES

In SPH, the fluid is expressed by discrete particles and physical quantities are approximated by kernel interpolation. In the standard formulation of SPH, the local density is first calculated and then the rest of the necessary physical quantities, such as the pressure gradient and the time derivative of the internal energy, are calculated. Thus, the accuracy of the solution depends on the accuracy of the density estimate. In this section, we reexamine the derivation of the equation of motion in SPH to understand its problem.

A physical quantity \( f \) at position \( r \) can be expressed as follows:

\[
f(r) = \int f(r') \delta(|r - r'|) dr'.
\]

A smoothed value of \( f \) at position \( r \), \( \langle f \rangle(r) \), is given by the convolution of \( f \) and a kernel function \( W(|r - r'|, h) \):

\[
\langle f \rangle(r) = \int f(r') W(|r - r'|, h) dr',
\]

where \( h \) is the size of the kernel function and corresponds to the spatial resolution. This smoothing is the base of SPH. Here, the kernel function must satisfy the following three conditions: (1) it becomes the delta function in the limit of \( h \to 0 \), (2) it is normalized as unity, and (3) it is a function with compact support. A cubic spline function is most widely used as the kernel function:

\[
W(|r - r'|, h) = \frac{\sigma}{h^D} \begin{cases}
\left( 1 - \frac{3}{2} s^2 + \frac{3}{4} s^3 \right) & 0 \leq s < 1, \\
\frac{1}{4} (2 - s)^3 & 1 \leq s < 2, \\
0 & 2 \leq s,
\end{cases}
\]

where \( s = |r - r'|/h \), \( D \) is the dimension, and the normalized factors \( \sigma \) in one, two, and three dimensions are 2/3, 10/7\( \pi \), and 1/\( \pi \), respectively. We first derive the equations of motion and energy with the constant kernel size, and then we generalized them to the individual kernel size.

The first derivative of the smoothed \( f \) is given by

\[
\langle \nabla f \rangle(r) = \int \nabla f(r') W(|r - r'|, h) dr'.
\]

By making use of the partial integral and the fact that the kernel function has compact support, Equation (4) becomes

\[
\langle \nabla f \rangle(r) = \int f(r') \nabla W(|r - r'|, h) dr'.
\]

We need to discretize Equation (2) to evaluate the physical quantities at positions of particles. To convert the integral into summation, a volume element \( dr' \) is replaced by the discrete volume element \( \Delta V_j = m_j/\rho_j \), where \( m_j \) and \( \rho_j \) are the mass and density of the particle \( j \). In addition, the positions of particles \( i \) and \( j \) are expressed by \( r_i \) and \( r_j \), and \( f(r') \) is replaced by \( f_j \). Thus, the value of \( f \) at the position of particle \( i \) is

\[
f_i = \sum_j m_j \frac{f_j}{\rho_j} W_{ij}(h),
\]

where \( f_i \equiv \langle f \rangle(r_i) \) and \( W_{ij} = W(|r_i - r_j|, h) \). Hereafter, we refer to the SPH formulation with this type of discretization as the standard SPH. At this point, we do not know \( \rho_j \). By substituting \( \rho \) into \( f \), we obtain

\[
\rho_i = \sum_j m_j W_{ij}(h),
\]

where \( \rho_i \equiv \langle \rho \rangle(r_i) \) is the smoothed density at the position of particle \( i \). Note that the right-hand side of Equation (7) includes no unknown quantities. Thus, densities should first be calculated in the standard SPH.

The equation of motion is

\[
d^2 r_i \over dt^2 = - \nabla P_i / \rho_i,
\]

where \( t \) is time and \( P \) is pressure. The SPH approximation of Equation (8) is given by

\[
\frac{d^2 r_i}{dt^2} = - \sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla W_{ij}(h).
\]

This form satisfies Newton’s third law. We used the following relation to obtain Equation (9):

\[
\frac{\nabla P_i}{\rho_i} = \nabla \left( \frac{P_i}{\rho_i} \right) - \frac{P_i}{\rho_i^2} \nabla \rho_i.
\]

In order for Equation (9) to be meaningful, \( \rho \) must be differentiable, since its derivative is used in Equation (10).

Finally, we derive the energy equation in the standard SPH. The energy equation is

\[
\frac{du_i}{dt} = - \frac{P_i}{\rho_i} \nabla \cdot v_i,
\]

where \( u \) is the internal energy and \( v \) is the velocity. To obtain the SPH formulation of the energy equation, we need the SPH expression of \( \nabla \cdot v \). We use

\[
\nabla (\rho v) = \nabla \rho v + \rho \nabla \cdot v.
\]

The SPH formulation of \( \nabla \cdot v \) is given by

\[
\rho_i \nabla \cdot v_i = \sum_j m_j v_{ij} \cdot \nabla W_{ij}(h) - v_i \cdot \sum_j m_j \nabla W_{ij}(h) = - \sum_j m_j v_{ij} \cdot \nabla W_{ij}(h),
\]

where \( v_{ij} = v_i - v_j \). Therefore, the energy equation in the standard SPH is

\[
\frac{du_i}{dt} = \sum_j m_j \frac{P_i}{\rho_i} v_{ij} \cdot \nabla W_{ij}(h).
\]
Equations (7), (9), and (14) close with the equation of state (EOS),

\[ P = (\gamma - 1)\rho u, \]

where \( \gamma \) is the specific heat ratio. There is no need to solve the continuity equation in SPH since it is satisfied automatically.

When we use the variable and individual kernel size, the above equations should be modified slightly. Here, we adopt a simple and traditional approach. First, the density evaluation equation is rewritten as

\[ \rho_i = \sum_j m_j W_{ij}(h_i). \]

This is the same as the so-called gather interpretation of the summation (Hernquist & Katz 1989). When a variable kernel size is employed, an iterative approach is used to determine both \( \rho \) and \( h \) imposing a condition, for instance, that the number of neighbor particles is kept in a fixed range. In equations of motion and energy, the gather-and-scatter interpretation is used (Hernquist & Katz 1989). Thus, Equations (9) and (14) become

\[ \frac{d^2 \rho_i}{dt^2} = - \sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla \tilde{W}_{ij}, \]

and

\[ \frac{d\mu_i}{dt} = \sum_j m_j \frac{P_i}{\rho_i} v_{ij} \cdot \nabla \tilde{W}_{ij}, \]

where \( \nabla W_{ij}(h_i) \) is replaced by \( \nabla \tilde{W}_{ij} = 0.5[\nabla W_{ij}(h_i) + \nabla W_{ij}(h_j)] \) so that the equation of motion can satisfy Newton’s third law. It is also possible to use \( \tilde{W}_{ij} = W_{ij}[0.5(h_i + h_j)] \). We adopt the first form throughout this paper.

The Lagrangian formulation (Springel & Hernquist 2002) provides the derivative of the kernel size. We will show the comparison results in Section 5.6.

In the derivation of the standard SPH discretization, the differentiability of \( \rho \) is used for both the equation of motion and the energy equation. However, \( \rho \) is discontinuous at the contact discontinuity. In the following, we illustrate the consequence of the discontinuity of the density.

In Figure 1, we show the values of density and pressure around a contact discontinuity evaluated by the standard formulation of SPH. Equation (16) is used and \( \rho = (\gamma - 1)\rho u \). To set up this contact discontinuity, we place particles on a regular grid in three dimensions and set \( \rho = 1 \) for \( x < 0.5 \) and \( \rho = 0.125 \) for \( x > 0.5 \). We used equal-mass particles in the first two configurations. In these two setups, the positions of particles in the less dense region are determined by taking the center of mass of the eight particles in the cube of the particle separation. In the last configuration, we adopted an equal separation for both regions, which means that the mass of particles in the less dense region is 1/8 of that of particles in the dense region. The internal energy was set to 1.5 (\( x < 0.5 \)) and 12 (\( x > 0.5 \)), and the specific heat ratio was 5/3. Velocities of particles were set to zero. The kernel size is determined to keep the neighbor number, \( N_{nb} \), to the range 32±2 in the first and last tests. In the second test, a constant \( h \) fixed to twice the particle separation in the less dense region is used.

The top panels show the distribution of particles. The panels in the second row show the SPH density. Though the initial setup has the discontinuity at \( x = 0.5 \), it is smoothed by the kernel. As a result, the SPH density of particles next to the discontinuity has very large errors, as shown in the panels in the third row. This large error in the density causes a similarly large error in the pressure (fourth row). The pressure of particles at the end of the low-density region is grossly overestimated, while that at the end of the high-density region is underestimated only modestly. This non-symmetric error in the pressure is the origin of the repulsive force at the contact discontinuity, as has been pointed out in previous studies (e.g., Ritchie & Thomas 2001; Okamoto et al. 2003; Agertz et al. 2007). This large error in the pressure also exists in both of the constant kernel size case (the middle column) and the equal-separation case (the right column).

Consider the following density and pressure distribution:

\[ \rho = \begin{cases} 
\rho_1 & x \geq 0, \\
\rho_2 & x < 0, 
\end{cases} \]

and

\[ P = P_0. \]

Obviously, we have

\[ \langle \rho \rangle(x) \rightarrow \frac{\rho_1 + \rho_2}{2}, \quad \text{for } x \rightarrow 0, \]

and therefore,

\[ \lim_{x \rightarrow 0} \langle P \rangle(x) = \frac{\rho_1 + \rho_2}{2n} P_0, \]

\[ \lim_{x \rightarrow 0} \langle P \rangle(x) = \frac{\rho_1 + \rho_2}{2n} P_0. \]

Thus, if \( \rho_1 \ll \rho_2 \), then the error of the pressure can be arbitrarily large. Note that the existence of this error does not imply an inconsistency of SPH. In this limit of \( h \rightarrow 0 \), the volume of the regime affected by this error approaches zero, which means that the original differential equation is restored almost everywhere. In other words, SPH satisfies the weak form of the original equation. However, it means that the convergence is slow and of first order.

One might think that this error is caused by an inadequate initial thermal energy (or density) distribution. However, that is not the case. If we initialize the internal energy (and/or density) of particles near the contact discontinuity so that the pressure is smooth initially, then there is no problem. However, as the particles move, the change of the density can sharpen, resulting in the numerical problem described above. We thus need continuous adjustment to suppress the pressure error throughout the time integration. Price’s artificial conductivity (Price 2008) provides such a continuous adjustment. Though the artificial conductivity works beautifully in test calculations for the Kelvin–Helmholtz instability, whether or not its use in actual astrophysical simulation is justified is a bit questionable. First, in the case of the discontinuity of chemical composition, not only the jump in the internal energy but also that in the chemical composition should be smoothed, but that is clearly not adequate. Second, the artificial heat conduction can significantly enhance the thermal relaxation of the system, which is again unwanted.

3. A DENSITY-INDEPENDENT FORMULATION OF SPH

In Section 2, we have seen that the standard SPH breaks down at the contact discontinuity because the continuity and differentiability of the density are necessary to guarantee the
3.1. Concept

Here, we propose an alternative formulation of SPH in which we discretize Equation (2) using the EOS of fluid, not the mass density. The new volume element of $j$th particle is

$$d\mathbf{r}' = \Delta V_j = \frac{(\gamma - 1)m_j u_j}{P_j} = \frac{U_j}{q_j}, \quad (24)$$

where $q_j \equiv \rho_j u_j$ and $U_j = m_j u_j$ are the energy density and the internal energy of particle $j$, respectively. Substituting Equation (24) into Equation (2) and using the gathered summation, we obtain a new SPH approximation of smoothed $f$: convergence of the SPH approximation. The basic reason for this problem is the use of $m_j/\rho_j$ for the volume element. Thus, if we use something else as the volume element, then we might be able to avoid this difficulty altogether. As shown in Section 4, the formulation we show in this section is the special case of the density-independent formulation of SPH.

Figure 1. Density and pressure fields evaluated with the standard SPH and our SPH around the contact discontinuity with the density ratio of 1:8. Equal-mass particles are used for the first and second configurations (the left and middle columns). The positions of the less dense region are determined by taking the center of mass of the eight particles in the cube of the particle separation. The equal separation is used in the last configuration (the right column). In this configuration, the mass of particles in the less dense region is $1/8$ of that of particles in the dense region. For the left and right columns, the constant neighbor number, $32 \pm 2$, is used. In the middle column, a constant kernel size of 0.03125 is used. The top row shows the distribution of particles projected on the $x$–$y$ plane. The second row shows the density of each SPH particle evaluated with Equation (16). The third row shows the density contrast between the evaluated density and the true one. The fourth row shows the corresponding pressure. In the bottom row, the pressure of each particle calculated with DISPH is shown.
\[ f_i = \sum_j (\gamma - 1) \frac{m_j u_j f_i}{P_j} W_j(h_i), \]  
\[ = \sum_j (\gamma - 1) \frac{U_j f_i}{P_j} W_j(h_i). \]  
By substituting \( f \) with the energy density, \( q \), we have
\[ q_i = \sum_j U_j W_j(h_i), \]  
where we used \( q_i \equiv \langle q \rangle(r) \). By replacing \( dr_j \) in Equation (5) with Equation (24), we have the gradient of \( \langle \tilde{f} \rangle \):
\[ \langle \nabla f \rangle(r_i) = \sum_j U_j \frac{f_i}{q_j} \nabla W_j(h_i). \]  
We adopt Equations (26) and (27) as the basis of DISPH. We derive the equations of motion and energy from this basis. We note that DISPH is also Galilei invariant.

One might think that the use of \( U \) for the calculation of the volume element would cause some inconsistency, since \( U \) is not a conserved quantity. The mass of a particle is constant, and thus looks safer. In the following, we show that we can construct a consistent set of equations using \( U \), that it has many advantages over the standard SPH, and that it retains important characteristics such as force symmetry and energy conservation. We first derive the energy equation and then the equation of motion. We then discuss the formulation for the estimate of the density and the implementation of the artificial viscosity.

### 3.2. Energy Equation

We need an expression of \( \nabla \cdot \mathbf{v} \) to derive the energy equation. We start with
\[ \nabla(q \mathbf{v}) = \nabla q \mathbf{v} + q \nabla \cdot \mathbf{v}, \]  
which is obtained by replacing \( \rho \) in Equation (12) with \( q \). By applying Equation (28) to Equation (29), we obtain an analogy of Equation (13):
\[ q_i \nabla \cdot \mathbf{v}_i = \sum_j U_j \mathbf{v}_j \cdot \nabla W_j(h_i) - \mathbf{v}_i \cdot \sum_j U_j \nabla W_j(h_i) \]
\[ = - \sum_j U_j \mathbf{v}_i \cdot \nabla W_j(h_i). \]  
The energy equation is then given by
\[ \frac{dU_i}{dt} = \sum_j \frac{P_j}{\rho_j q_i} \mathbf{v}_{ij} \cdot \nabla \tilde{W}_{ij}. \]  
Equation (31) contains \( \rho_i \) since \( u \) is the energy per unit mass. The equation for \( U_i \) is obtained by multiplying Equation (31) by \( m_i \):
\[ \frac{dU_i}{dt} = m_i \frac{1}{\rho_i} \sum_j \frac{P_j}{q_i} \mathbf{v}_{ij} \cdot \nabla \tilde{W}_{ij}. \]  
Here, \( m_i / \rho_i \) is the volume associated with particle \( i \) which can be replaced by \( U_i / q_i = (\gamma - 1) U_i / P_i \). Thus, we have
\[ \frac{dU_i}{dt} = (\gamma - 1) \sum_j \frac{U_i U_j}{q_i} \mathbf{v}_{ij} \cdot \nabla \tilde{W}_{ij}. \]  

### 3.3. Equation of Motion

From the energy equation, Equation (33), we derive the equation of motion. The change in the internal energy of particles \( i \) and \( j \) due to their relative motion is
\[ \frac{dU_i}{dt} + \frac{dU_j}{dt} = (\gamma - 1) U_i U_j \left( \frac{1}{q_i} + \frac{1}{q_j} \right) \mathbf{v}_{ij} \cdot \nabla \tilde{W}_{ij}. \]  
This change is the same as the change of the kinetic energy of particles with an opposite sign. Thus, we have
\[ \frac{m_i m_j}{m_i + m_j} \mathbf{v}_{ij} \cdot \left( \frac{dU_i}{dt} - \frac{dU_j}{dt} \right) = - \left( \frac{dU_i}{dt} + \frac{dU_j}{dt} \right). \]  
Substituting Equation (34) into Equation (35), we obtain
\[ \left( \frac{dU_i}{dt} - \frac{dU_j}{dt} \right) = - (\gamma - 1) \frac{m_i m_j}{m_i + m_j} U_i U_j \left( \frac{1}{q_i} + \frac{1}{q_j} \right) \nabla \tilde{W}_{ij}. \]  
Since the motion of the center of mass of two particles is unchanged by the interaction of two particles, we have
\[ \frac{d}{dt}(m_i \mathbf{v}_i + m_j \mathbf{v}_j) = 0. \]  
Thus, we have
\[ m_i \frac{d\mathbf{v}_i}{dt} = - (\gamma - 1) U_i U_j \left( \frac{1}{q_i} + \frac{1}{q_j} \right) \nabla \tilde{W}_{ij} \]  
as the contribution of particle \( j \) to the equation of motion of particle \( i \).

The equation of motion for particle \( i \) is obtained by taking the summation over neighbor particles:
\[ m_i \frac{d\mathbf{v}_i}{dt} = - (\gamma - 1) \sum_j U_i U_j \left( \frac{1}{q_i} + \frac{1}{q_j} \right) \nabla \tilde{W}_{ij}. \]  
The right-hand side of Equation (39) contains only the energy \( U \) and energy density \( q \). Thus, as far as \( q \) is smooth, Equation (39) is likely to be well-behaved. The equation of motion of the standard SPH (Equation (9)) requires that both \( P \) and \( \rho \) are smooth. Thus, in our formulation, there is nothing special about the contact discontinuity. We can therefore expect that the treatment of the contact discontinuity is improved. We will see this in Section 3.6.

Note that Equation (39) is mathematically equivalent to the equation of motion obtained by Ritchie & Thomas (2001), while the derivation is completely different. Ritchie & Thomas (2001) started from Equation (27) and the density estimate \( \rho = mq / U \), but still tried to use the standard SPH estimate of Equation (6). In order to eliminate \( \rho \) from the equation of motion, they used the following formal relationship:
\[ \nabla \frac{P}{\rho} = \nabla P + \frac{P}{\rho} \nabla 1 \]  
and the formal identity
\[ 1 = \sum_j m_j \frac{1}{\rho_j} \nabla W_j(h) \simeq 0. \]  
Thus, their derivation was a heuristic modification of the standard SPH and they did not employ the volume element \( (\gamma - 1)m_j u_j / P_j \) explicitly. We have shown that by choosing \( (\gamma - 1)m_j u_j / P_j \) as the volume element, we can derive a consistent set of SPH equations naturally.
3.4. Artificial Viscosity

To deal with shocks, the standard SPH needs an artificial viscosity term. DISPH also needs an artificial viscosity term. We utilize artificial viscosity terms which are widely used in simulations with the standard SPH.

The viscosity term for the equation of motion is

\[
m_i \frac{d^2 r_i}{dt^2} = -m_j \sum_j m_j \Pi_{ij} \nabla \hat{W}_{ij},
\]

and the corresponding form of it for the energy equation is

\[
dU_i \over dt = m_i \sum_j m_j \Pi_{ij} v_{ij} \cdot \nabla \hat{W}_{ij},
\]

where \( \Pi_{ij} \) is the function of the strength of the artificial viscosity.

There are two types of artificial viscosity term, \( \Pi_{ij} \), which are commonly used. The most commonly used one (Lattanzio et al. 1985) is

\[
\Pi_{ij} = \begin{cases} \frac{-\alpha c_{ij} \mu_{ij} + \beta \mu_{ij}^2}{\rho_{ij}} & v_{ij} \cdot r_{ij} < 0, \\ 0 & v_{ij} \cdot r_{ij} \geq 0, \end{cases}
\]

where \( \alpha \) and \( \beta \) are the control parameters for the strength of the artificial viscosity, \( c_{ij} \) is the arithmetic average of the sound speeds of particles \( i \) and \( j \), \( \rho_{ij} = 0.5(\rho_i + \rho_j) \), and

\[
\mu_{ij} = \frac{h_{ij} v_{ij} \cdot r_{ij}}{r_{ij}^2 + \epsilon h_{ij}^2}.
\]

The constant \( \epsilon \) is introduced to avoid divergence and its fiducial value is \( \sim 0.01 \).

The other one is proposed by Monaghan (1997) from the analogy of the Riemann solver:

\[
\Pi_{ij} = \begin{cases} \frac{\alpha v_{ij}^B w_{ij}}{2 \rho_{ij}} & v_{ij} \cdot r_{ij} < 0, \\ 0 & v_{ij} \cdot r_{ij} \geq 0, \end{cases}
\]

where \( v_{ij}^B = c_i + c_j - 3w_{ij} \) and \( w_{ij} = v_{ij} \cdot r_{ij} / r_{ij} \).

Since we have the density estimate \( \rho = q/u \), we have

\[
\rho_{ij} = \frac{1}{2} \left( \frac{q_i}{u_i} + \frac{q_j}{u_j} \right)
\]

for our formulation. However, this modification of \( \rho_{ij} \) leads to unstable behavior under strong shocks. It seems to be safe to use the smoothed mass densities of particles \( i \) and \( j \) that were evaluated using Equation (7). The viscosity relates to the inertial force. Therefore, the use of the matter distribution is reasonable. We use the smoothed mass density to evaluate the artificial viscosity term. In Section 5.6, we investigate the effect of the choice of the averaged density in the artificial viscosity term. It is also safer to use the smoothed mass density when one uses SPH for the simulation with a radiative cooling term.

We use the standard Balsara switch (Balsara 1995) to suppress the shear viscosity. It is given by

\[
r_{iBalsara} = \frac{|\nabla \cdot v_i|}{|\nabla \cdot v_i| + |\nabla \times v_i| + \epsilon_b c_i / h_i},
\]

and \( \Pi_{ijBalsara} = 0.5(F_{iBalsara} + F_{jBalsara}) \Pi_{ij} \). Here, \( \epsilon_b \) is a small value (typically \( 10^{-4} \)). The rotation of velocity in the standard SPH is found in the literature (e.g., Monaghan 1992). The rotation of velocity in our SPH is calculated as follows:

\[
\nabla \times v_i = \frac{1}{q_i} \sum_j U_j v_{ij} \times \nabla W_{ij}(h_i).
\]

3.5. Grad-h Term

To obtain a consistent formulation with the variable kernel size, we have to take into account not only the gradient of the kernel and physical quantities with respect to \( r \) but also with respect to \( h \). Here, we take the contribution of the variable kernel size into account. This was accomplished by using the Lagrangian formulation (Springel & Hernquist 2002; Rosswog 2009; Springel 2010; Hopkins 2013). Since the variation of \( h \) is the first-order term, the contribution of this term is rather limited. Indeed, we find that the contribution of this term is prominent only in extremely strong shock problems like the Sedov problem (see Section 5.6).

We start from the Lagrangian

\[
L(\mathbf{Q}, \mathbf{Q}) = \sum_i \frac{1}{2} m_i \dot{\mathbf{Q}}_i^2 - \sum_i U_i(\mathbf{Q}).
\]

where \( \mathbf{Q} \equiv (r_1, r_2, \ldots, r_N, h_1, h_2, \ldots, h_N) \). We adopt the following constraint:

\[
\phi_i = \frac{4\pi}{3} (2h_i)^3 \frac{q_i}{U_i} - N_{ngh} = 0.
\]

This constraint means that the spherical region with radius \( 2h_i \) covers a volume of \( N_{ngh} \Delta h_i \).

The Euler–Lagrange equation with a constraint is as follows:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{Q}}_i} - \frac{\partial L}{\partial \mathbf{Q}_i} = \sum_j \lambda_j \frac{\partial \phi_j}{\partial h_i}.
\]

First, we solve this equation regarding the kernel size. Since the Lagrangian and constraint do not have the first-order derivative of the kernel size, Equation (52) is

\[
\frac{\partial L}{\partial h_i} = \sum_j \lambda_j \frac{\partial \phi_j}{\partial h_i}.
\]

The left-hand-side of this equation is

\[
-\frac{\partial L}{\partial h_i} = \frac{\partial U_i}{\partial h_i},
\]

\[
= \frac{\partial U_i}{\partial \Delta V_i} \frac{\partial \Delta V_i}{\partial h_i},
\]

\[
= P_i U_i \frac{\partial q_i}{\partial h_i},
\]

where we used the first law of thermodynamics in an adiabatic state, \( dU = -PdV \).

The right-hand-side of Equation (53) is

\[
\sum_j \lambda_j \frac{\partial \phi_j}{\partial h_i} = \sum_j \lambda_j \frac{1}{h_j} \left[ \frac{4\pi}{3}(2h_j)^3 \frac{q_j}{U_j} - N_{ngh} \right],
\]

\[
= \lambda_i 32\pi h_i^2 q_i \frac{1}{U_i} \left( 1 + \frac{h_i}{3} \frac{\partial q_i}{\partial h_i} \right),
\]
Hence, we have
\[ \lambda_i = \frac{3P_i}{32\pi h_i^3} \frac{U^2}{q_i} \psi_i, \tag{56} \]
where
\[ \psi_i = \frac{h_i}{3q_i} \frac{\partial q_i}{\partial h_i} \left( 1 + \frac{h_i}{3q_i} \frac{\partial q_i}{\partial h_i} \right)^{-1}. \tag{57} \]

Next, let us solve the Euler–Lagrange equation regarding \( r \). According to Equation (52), the equation of motion is
\[ m_i \frac{d\psi_i}{dt} = -\nabla_i \sum_j U_j + \sum_j \lambda_j \nabla_i \phi_j \]
\[ = -\sum_j \frac{\partial U_j}{\partial \nabla V_j} \nabla_i \Delta V_j \]
\[ + \sum_j \frac{3P_j}{32\pi h_j^3} \frac{U_j^2}{q_j^2} \psi_j \nabla_j \left[ \frac{4\pi}{3} (2h_j)^3 \frac{q_j}{U_j} - N_{\text{ngb}} \right] \]
\[ = -\sum_j \frac{P_j U_j}{q_j^2} (1 - \psi_j) \nabla_i q_j \]
\[ = -\sum_j \frac{P_j U_j}{q_j^2} f_j^{\text{grad}} \nabla_i q_j, \tag{58} \]
where
\[ f_j^{\text{grad}} \equiv (1 - \psi_j) = \left( 1 + \frac{h_j}{3q_j} \frac{\partial q_j}{\partial h_j} \right)^{-1}. \tag{59} \]

By using
\[ \nabla_i q_j = \nabla_i \sum_k U_k W_{jk}(h_j) \]
\[ = U_i \nabla_i W_{ij}(h_j) + \delta_{ij} \sum_k U_k \nabla_i W_{ik}(h_i), \tag{60} \]
Equation (58) becomes
\[ m_i \frac{d\psi_i}{dt} = -\sum_j \frac{P_j U_j}{q_j^2} f_j^{\text{grad}} \]
\[ \left( U_i \nabla_i W_{ij}(h_j) + \delta_{ij} \sum_k U_k \nabla_i W_{ik}(h_i) \right). \]
\[ = -(\gamma - 1) \sum_j U_i U_j \]
\[ \left( \frac{1}{q_i} f_i^{\text{grad}} \nabla_i W_{ij}(h_i) + \frac{1}{q_j} f_j^{\text{grad}} \nabla_i W_{ij}(h_j) \right). \tag{61} \]

The energy equation can be obtained as follows:
\[ \frac{dU_i}{dt} = -P_i \frac{d\Delta V_i}{dt}, \]
\[ = P_i \frac{U_i d\psi_i}{dt}. \tag{62} \]
Taking a time derivative of \( q_i \), we have
\[ \frac{dq_i}{dt} = \frac{d}{dt} \sum_j U_j W_{ij}(h_i), \]
\[ = \sum_j U_j \frac{d\psi_i}{dt} \cdot \nabla_i W_{ij}(h_i) + \sum_j U_j \frac{dh_i}{dt} \frac{\partial W_{ij}(h_i)}{\partial h_i}, \]
\[ = \sum_j U_j v_{ij} \cdot \nabla_i W_{ij}(h_i) + \frac{dq_i}{dt} \frac{dh_i}{dt} \sum_j U_j \frac{\partial W_{ij}(h_i)}{\partial h_i}. \tag{63} \]

Then,
\[ \left( 1 - \frac{\partial h_i}{\partial q_i} \sum_j U_j \frac{\partial W_{ij}(h_i)}{\partial h_i} \right) \frac{dq_i}{dt} = \sum_j U_j v_{ij} \cdot \nabla_i W_{ij}(h_i). \tag{64} \]

Since \( h_i^3 q_i = \text{constant} \), we have
\[ \left( \frac{\partial h_i}{\partial q_i} \right)^{-1} = -3 \frac{q_i}{h_i}. \tag{65} \]

Therefore, Equation (64) becomes
\[ \left( 1 + \frac{h_i}{3q_i} \frac{\partial q_i}{\partial h_i} \right) \frac{dq_i}{dt} = \sum_j U_j v_{ij} \cdot \nabla_i W_{ij}(h_i), \tag{66} \]
and using Equation (59),
\[ \frac{dq_i}{dt} = f_i^{\text{grad}} \sum_j U_j v_{ij} \cdot \nabla_i W_{ij}(h_i). \tag{67} \]

By substituting Equation (67) into Equation (62), we finally obtain the energy equation,
\[ \frac{dU_i}{dt} = (\gamma - 1) \sum_j U_i U_j f_i^{\text{grad}} v_{ij} \cdot \nabla_i W_{ij}(h_i). \tag{68} \]
This energy equation and the equation of motion, Equation (61), are also density independent.

The Lagrangian formulation guarantees the conservation of energy and momentum. We note that the set of equations of energy and motion, Equations (33) and (39), also conserve energy and momentum, since these equations are derived by assuming the conservation of energy and momentum.

### 3.6. Pressure in Contact Discontinuities

The pressure around the contact discontinuity calculated with our SPH equation, Equation (27), is shown in the bottom panels of Figure 1. In the case of the equal-mass particle and the fixed neighbor number (the left panel), we can see that the jump of the pressure at the contact discontinuity in DISPH is much smaller than that in the standard SPH. In the case of the constant kernel size (the middle panel), the result of DISPH is almost flat, while that of the standard SPH has a large error.

In these two equal-mass cases, pressure still has small jumps at the contact discontinuity. The reason is that in both cases the distribution of particles is asymmetric. In the high-density region, the particle separation is smaller, resulting in a small integration error. As a result, a small error appears when the kernel contains the contribution from both low- and high-density regions. In the case of the equal separation of particles, there is no jump in the pressure distribution at the contact discontinuity, as shown in the rightmost panel.
4. A GENERALIZED FORM OF DISPH

In Section 3, we have derived equations of energy and motion which do not depend on the density in their right-hand side. To derive these formulations, we used $q$ instead of $\rho$ as the basis of the formulation. In an ideal fluid, $q$ is identical to $P$, except for the factor of $\gamma - 1$ and that the pressure is a continuous quantity across the contact discontinuities.

As we will show in the next section, this formulation works quite well in many tests, such as the shock tube test, and the Kelvin–Helmholtz and Rayleigh–Taylor instability tests. However, this formulation is not good for the extremely strong shock tests where the pressure jump at the shock is very large.

In this section, we generalize DISPH by using an arbitrary function of $\rho$ as the basis of form of the function that depends weakly on the pressure, such a formulation might work well even for an extremely strong shock. As we show in Section 5.6, we found that it indeed works well. In the following, we derive a generalized form of DISPH.

4.1. Equations of a Generalized DISPH

We start from the following relation:

$$ y_i = G(P_i). $$

(69)

where $G(P_i)$ is an arbitrary function of $P_i$. Formally, any kind of function is possible as $G$. When we introduce a physical quantity, $Z$, we can define a new volume element as

$$ \Delta V_i = \frac{Z_i}{y_i}. $$

(70)

By applying Equation (70) to Equation (2), we have

$$ f_i = \sum_j Z_j f_{ij} W_{ij}(h_i). $$

(71)

When we assume $f = y$, we have

$$ y_i = \sum_j Z_j W_{ij}(h_i). $$

(72)

The quantity $y$ given by Equation (72) is the basic quantity of this generalized DISPH. This is an implicit equation for $y$ and $Z$. We will show how to solve this equation in Section 4.3. Here, we continue the derivation of equations. The first derivative of the physical quantity $f$ is given by

$$ \langle \nabla f \rangle (r) = \sum_j Z_j f_{ij} \nabla W_{ij}(h_i). $$

(73)

The equations of energy and motion are

$$ \frac{dU_i}{dt} = \sum_j P_i Z_j y_i \sum_j Z_j \left( \frac{P_i}{y_i} + \frac{P_j}{y_j} \right) \nabla W_{ij}, $$

(74)

$$ m_i \frac{dv_i}{dt} = -\sum_j Z_j Z_j \left( \frac{P_i}{y_i} + \frac{P_j}{y_j} \right) \nabla W_{ij}, $$

(75)

where we use the following relation:

$$ \nabla (y v) = \nabla y v + y \nabla \cdot v, $$

(76)

which can be rewritten as

$$ \nabla \cdot v_i = -\frac{1}{y_i} \sum_j Z_j v_{ij} \cdot \nabla W_{ij}(h_i). $$

(77)

The rotation of velocity in this generalized DISPH is expressed as

$$ \nabla \times v_i = \frac{1}{y_i} \sum_j Z_j v_{ij} \times \nabla W_{ij}(h_i). $$

(78)

The time derivative of $Z$ is obtained from Equation (70). By taking the time derivative of Equation (70), we have

$$ \frac{dZ_i}{dt} = Z_i \left( \frac{1}{\Delta V_i} \frac{\partial \Delta V_i}{\partial t} + \frac{1}{y_i} \frac{\partial y_i}{\partial t} \right), $$

(79)

$$ \frac{dZ_i}{dt} = Z_i \left( \frac{1}{\Delta V_i} \frac{\partial \Delta V_i}{\partial t} + \frac{1}{y_i} \frac{\partial y_i}{\partial t} \frac{\partial \Delta V_i}{\partial \Delta V_i} \right), $$

(80)

$$ \frac{dZ_i}{dt} = \frac{Z_i}{\Delta V_i} \frac{\partial \Delta V_i}{\partial t} \left( 1 + \frac{\Delta V_i}{y_i} \frac{\partial y_i}{\partial \Delta V_i} \right). $$

(81)

By adopting the relations

$$ \frac{1}{\Delta V_i} \frac{\partial \Delta V_i}{\partial t} = \nabla \cdot v_i, $$

(82)

$$ \frac{d \log P_i}{d \log \Delta V_i} = -\gamma, $$

(83)

$$ \frac{d \log y_i}{d \log P_i} = \frac{d \log G(P_i)}{d \log P_i} = \xi(P_i), $$

(84)

we finally obtain

$$ \frac{dZ_i}{dt} = Z_i (\xi(P_i)\gamma - 1) \frac{1}{\Delta V_i} \frac{\partial \Delta V_i}{\partial t}, $$

(85)

$$ \frac{dZ_i}{dt} = (\xi(P_i)\gamma - 1) \sum_j \frac{Z_j}{y_i} v_{ij} \cdot \nabla W_{ij}(h_i). $$

(86)

Note that these equations are reduced to those of DISPH shown in Section 3 if we choose $q_i$ as $y_i$. Thus, the equations of DISPH obtained in Section 3 are a special case of the generalized DISPH. In this special case, the time derivative of $Z$ is identical to the energy equation.

4.2. A Generalized DISPH with the Grad-h Term

We can obtain the grad-h term by using the new volume element, Equation (70). We begin the derivation by the Lagrangian, Equation (30), and a constraint with the new volume element of the generalized DISPH,

$$ \phi_i = \frac{4\pi}{3}(2h_i)^3 \frac{y_i}{Z_i} - N_{nhb} = 0. $$

(87)

By solving the Euler–Lagrange equation, we finally obtain the following set of equations:

$$ m_i \frac{dv_i}{dt} = -\sum_j Z_j Z_j \times \left( \frac{P_i}{y_i} f_{ij}^{\text{grad}} \nabla W_{ij}(h_i) + \frac{P_i}{y_j} f_{ij}^{\text{grad}} \nabla W_{ij}(h_j) \right). $$

(88)
Here, we consider the following power-law formulation is to reduce errors induced by huge pressure jumps, since our purpose in introducing such a function for the integral scheme. Thus, we should “correct” the value of \( \frac{\partial \zeta}{\partial y} \), \( \frac{\partial \zeta}{\partial h} \), and the error in \( \zeta \) is amplified by the direct substitution when \( \zeta(P) \) is reduced. However, when we choose \( \zeta = 0.1 \), the pressure contrast with 10 orders of magnitude is reduced to that with an order of magnitude. It is worth noting that \( \zeta = 1 \) is the special case in which our purpose is beyond the scope of this paper. We will study this formulation elsewhere.

5. NUMERICAL EXPERIMENTS

In this section, we show the results of several standard tests for fluid schemes, for both the standard SPH and our new SPH. In Section 5.1, we describe our numerical code briefly. In Section 5.2, we show the results of the shock tube tests. Then, we show the evolution of the system which is initially in hydrostatic equilibrium in Section 5.3. In Sections 5.4 and 5.5, tests for two important fluid instabilities are carried out. Point-like explosion tests are shown in Section 5.6. In Section 5.7, we show the results of the blob tests which were first proposed by Agertz et al. (2007). As an extra test, we describe the mixing of a two-phase fluid with a solid-body-like spoon which is found in Springel (2010a). In all tests, our new SPH shows a much better result compared to that of the standard SPH.

5.1. Numerical Method

We used ASURA, a parallel N-body/SPH code, as the framework for the current numerical experiments. ASURA adopts the leap-frog method for time integration. For simplicity, we used the shared steps with variable time steps. The time step is given by

\[
dt = \min_i dt_i, \tag{96}
\]

where

\[
dt_i = \alpha_{\text{CFL}} \frac{2h_i}{\max_j \frac{\nu_j}{\text{sig}}}, \tag{97}
\]

and \( \alpha_{\text{CFL}} = 0.3 \).

For the standard SPH, we first evaluated the densities and kernel sizes of particles using Equation (16) and iteration. Then, we calculated the pressure gradient and the time derivative of the internal energy using Equations (17) and (18). In DISPH, we computed \( q \) using Equation (27) and kernel sizes first, and then we calculated the pressure gradient and the time derivative of the internal energy using Equations (39) and (33). For most of the tests, we used a special DISPH which adopted \( q \) as a fundamental value. We investigated the advantages of the generalized DISPH and the \( h \)-term in the strong shock test in Section 5.6. We used Equation (46) as the artificial viscosity term in both cases and we adopted \( \alpha = 1 \) as a fiducial value. In
the loop where we compute $q$, we also compute smoothed mass densities since they are used in the calculation of the artificial viscosity term. The Balsara switch was also applied. To avoid the pairing instability, we used a first derivative of the kernel which has a cuspy core (Thomas & Couchman 1992). Note that this modification leads to an inconsistent sound speed and other quantities within $s = |r - r'|/h < 2/3$ (see footnote 8 in Price 2012). The essential solution is to use kernels that do not show the pairing instability, for instance, the kernel proposed by Read et al. (2010) and the Wendland kernels (Dehnen & Aly 2012).

The kernel size of each particle is determined to keep the number of neighbor particles within the range of $32 \pm 2$. As an exception, in the one-dimensional tests shown in Section 5.2, the kernel size is evaluated by

$$h = \eta \left( \frac{m}{\rho} \right),$$

where $\eta = 1.2$ for the Sod’s shock tube tests and $\eta = 2.4$ for the strong shock tube tests. For DISPH, we used the smoothed density for the evaluation of the kernel size.

5.2. Shock Tube Tests

The Sod shock tube (Sod 1978) is the most basic test for numerical schemes for compressible fluid. This test shows the shock-capturing ability of schemes. In SPH, not only the profile of the shock front but also the behavior of the contact discontinuity is important. Here, we show the results of one- and three-dimensional shock tube tests.

The setup is as follows. We prepared the periodic domain of $-1 \leq x < 1$ for the one-dimensional tests and $-1 \leq x < 1$, $-1/16 \leq y < 1/16$, and $-1/16 \leq z < 1/16$ for the three-dimensional tests. The initial condition is given by

$$\begin{cases}
\rho = 1, & P = 1, \quad v = 0, \quad x < 0, \\
\rho = 0.25, & P = 0.1795, \quad v = 0, \quad x \geq 0.
\end{cases}$$

To express this initial condition, we use equal-mass particles and regularly place 800 and 200 particles in the left and right domains, respectively, in the one-dimensional tests. In three-dimensional tests, we place 40,000 and 10,000 particles in the left and right domains, respectively, and a glass-like particle distribution was used. We set $\gamma = 1.4$ and gave the internal energy to each particle to ensure the given $P$.

In addition to the Sod shock tube, we performed a one-dimensional strong shock test. The initial condition for this test is given by

$$\begin{cases}
\rho = 1, & P = 1000, \quad v = 0, \quad x < 0, \\
\rho = 1, & P = 0.01, \quad v = 0, \quad x \geq 0.
\end{cases}$$

We use 1000 equal-mass particles in the computational domain of $-1 \leq x < 1$ with equal separation.

Figure 2 shows the results of the one-dimensional shock tube tests with the standard SPH and DISPH. The density (upper row) and pressure (bottom row) of each particle are plotted by circles. The red curves represent the analytic solutions. The pressure shows
large variations near the contact discontinuity, though it should be constant. Since Equation (9) of the standard SPH contains a large error near the contact discontinuity, in order to achieve zero acceleration, the pressures of particles must have large variations. This result is the same as the results of previous works with the standard SPH (e.g., Springel 2005; Price 2008).

In DISPH, unlike the case of the standard SPH, the pressure around the contact discontinuity does not show a large jump. The reason is simply that the energy density is used instead of the mass density. The energy density is constant at the contact discontinuity. The reason why there is a small change in the pressure is that the particle separation changes at the contact discontinuity. As we showed in Figure 1, our new SPH still has small error in the pressure due to the finite number of particles in the kernel. This error caused the change in the pressure in Figure 2.

The results of the three-dimensional shock tube tests for the standard SPH and our SPH are shown in Figure 3. In this figure, the circles represent the average values of particles in bins with the width of the mean particle separation at the high-density part. Again, we can see a variation in the pressure around the contact discontinuity in the case of the standard SPH. In our SPH, there is no such variation.

Figure 4 shows the results of the strong shock tube tests for the standard SPH and DISPH. The shock front and the contact discontinuity in the density distribution are well reproduced in both of the cases. In this extreme test, both runs show jumps in the pressure distribution around the contact discontinuity. The absolute value of the pressure jump in DISPH is much smaller than that in the standard SPH. The jump found in the pressure in our SPH is caused by the asymmetry in the particle distribution (see Section 3.6). Overall, DISPH can handle such a strong shock problem, even when a very large pressure jump exists initially. This result is quite reassuring. In DISPH, it is assumed that the pressure is smooth, which is not a valid assumption at the shock front. Thus, it could fail to capture very strong shocks. The result shown in Figure 4 shows that it is not the case and new SPH can handle very strong shocks.

5.3. Hydrostatic Equilibrium Tests

As is shown in Section 2, in the standard SPH, particles feel an unphysical repulsive force at the interface of the contact discontinuity. Therefore, in order to establish the hydrostatic equilibrium, the distance between particles at the different sides of the contact discontinuity must become larger than the “true” value. What is the consequence of this repulsive force? Here, we show the result of a simple test which helps us to understand the problem of the unphysical repulsive force. A similar test has been used in Heß & Springel (2010).

We follow the evolution of two fluids with different density values, but with the same pressure. We performed two-dimensional tests. The computational domain is a square of the unit size, $0 \leq x < 1$ and $0 \leq y < 1$, with a periodic boundary condition. The initial conditions are

$$\rho = \begin{cases} 4 & 0.25 \leq x \leq 0.75 \text{ and } 0.25 \leq y \leq 0.75, \\ 1 & \text{otherwise}, \end{cases}$$

(101)

$$P = 2.5,$$

(102)

$$\gamma = 5/3.$$  

(103)
We tried two different realizations. In the first one, the particle mass is the same for the entire computational region. Thus, the inter-particle distance is smaller in the high-density region. In the second one, particles in the high-density region are four times more massive than particles in the low-density region. In both cases, particles are initially in a regular grid. For the equal-mass case, the number of particles in the dense region is 4096 and that in the ambient is 3072. For the equal-separation cases, those are 1024 and 3072, respectively. The initial velocities of the particles were set to zero. Since the system is initially in the hydrostatic equilibrium, particles should not move, except for small local adjustments.

Figure 5 shows the time evolution up to $t = 8$. There is a clear difference between the result of the standard SPH and that of DISPH. With the standard SPH, the high-density region, which initially has a square shape, quickly becomes rounder and almost completely circular by $t = 8$. We can understand this unphysical rounding as follows. As we stated in Sections 2 and 3.6, an unphysical repulsive force operates at the contact discontinuity. We can see the effect of this force in the development of the gap of the distribution of particles near the boundary of two fluids. Because of this gap, the bulk of the system is slightly compressed. The system seeks to achieve the energy minimum by minimizing the surface area of the contact discontinuity. Thus, the high-density region evolves to a circular shape, which minimizes the length of the boundary. In other words, the repulsive force effectively adds the "surface tension."

DISPH gives a far better solution, as we see in the lower two rows of Figure 5. The overall square shape remains there till the end of the simulation in the equal-mass case. The result of the unequal-mass case is even better. The equation of motion of DISPH eliminates the unphysical surface tension completely.

Figure 6 shows the final state of the two-fluid system with the density contrast of 64. Our SPH handles the system without any problem (right panel). On the other hand, in the calculation with the standard SPH, a wide and empty ring structure is formed between two fluids.

5.4. Kelvin–Helmholtz Instability Tests

After the work by Agertz et al. (2007) which clearly demonstrated that the standard SPH cannot deal with the Kelvin–Helmholtz instability correctly, many researchers proposed modifications of SPH to solve the problem (see Section 1). In this section, we investigate how DISPH handles the Kelvin–Helmholtz instability.

We prepared a two-dimensional computational domain, $0 \leq x < 1$ and $0 \leq y < 1$. The periodic boundary condition was used. The initial density is

$$
\rho = \begin{cases} 
1(\equiv \rho_l) & 0 \leq y < 0.25, 0.75 \leq y < 1, \\
2(\equiv \rho_h) & 0.25 \leq y < 0.75.
\end{cases}
$$

We used equal-mass particles. The numbers of particles in the high- and low-density regions are 131,072 and 65,522, respectively. We set $P = 2.5$ and $\gamma = 5/3$. The high- and low-density regions had initial velocities of $v_{x,h} = 0.5$ and $v_{x,l} = -0.5$ in the $x$-direction, respectively.

We have used $N_{nb} = 32 \pm 2$ as the neighbor number. This value might seem a bit large, but we found that it guarantees a good sampling of the particles in the low-density
Figure 5. Snapshots of a two-fluid system at $t = 0.1, 0.3, 0.5, 1,$ and $8$. The red and blue points indicate the positions of particles with $\rho = 4$ and $\rho = 1$, respectively. The upper two rows are the results of the standard SPH, while the lower two rows are those of DISPH. The first and third rows show the results of the equal-mass cases, whereas the second and fourth rows show those of the equal-separation and unequal-mass cases.

(A color version of this figure is available in the online journal.)

Figure 6. Final state ($t = 8$) of a two-fluid system with the density contrast of 64. The red and blue points are the positions of particles with $\rho = 64$ and $\rho = 1$, respectively. The particle separation is constant and the particle mass difference is 1:64.

(A color version of this figure is available in the online journal.)

region at the interface. When we used $N_{nb} = 16 \pm 2$, the variation of the pressure at the interface became too large. For the artificial viscosity, we used $\alpha = 1$ with the Balsara switch.

We added a small velocity perturbation to the particles near the interfaces, following Price (2008). The velocity perturbation in the $y$-direction is as follows:

$$
\Delta v_y = \begin{cases} 
A \sin[-2\pi(x + 0.5)/\lambda], & |y - 0.25| < 0.025 \\
A \sin[2\pi(x + 0.5)/\lambda], & |y - 0.75| < 0.025,
\end{cases}
$$

(105)

where $\lambda = 1/6$ and $A = 0.025$. 

Figure 7. Density maps from the two-dimensional shear flow test at \( t = 1, 2, 4, \) and \( 8 \tau_{\text{kh}} \). The upper panels show the results of the standard SPH, while the bottom panels show those of DISPH. The color code of the density is given at the bottom. (A color version of this figure is available in the online journal.)

Figure 8. Pressure of each particle along the \( y \)-direction at \( t = 0.4 \tau_{\text{kh}} \). The left panel shows the result of the standard SPH, whereas the right panel shows that of DISPH. Particles initially in the high-density (low-density) region are expressed with red (blue) points. (A color version of this figure is available in the online journal.)

The timescale of the growth of the Kelvin–Helmholtz instability is

\[
\tau_{\text{kh}} = \frac{\lambda (\rho_h + \rho_l)}{\sqrt[4]{\rho_h \rho_l} |v_{x,h} - v_{x,l}|}. \tag{106}
\]

For our test setup, \( \tau_{\text{kh}} = 0.35 \). We followed the evolution up to \( t = 8 \tau_{\text{kh}} \).

The results are shown in Figure 7. The difference between the two results is clear. In the run with the standard SPH, perturbations grow until \( t = \tau_{\text{kh}} \), but the unphysical surface tension inhibited the growth of roll-like structures. The stretched high-density fluids break apart (\( t = 4 \tau_{\text{kh}} \)) and form blobs (\( t = 8 \tau_{\text{kh}} \)). This evolution is completely different from those obtained by Euler codes (e.g., Agertz et al. 2007). On the other hand, DISPH shows a very good result which is comparable to those obtained with the Euler codes and with the SPH codes adopting the Ritchie & Thomas (2001) equation of motion or the artificial conductivity (see Price 2008). Price (2008) reported that the instability grew but did not develop prominently when the Ritchie & Thomas formulation was used. He argued that this failure was because of noise when the Ritchie & Thomas formulation was used and the low resolution (see Figure 5 in his paper). In our test, we observed extended Kelvin–Helmholtz instabilities with the same resolution as that used in Price (2008). When we turned off the Balsara switch, instabilities did not develop, and we obtained a result similar to that of Price. The use of the Balsara switch is important for such a simulation of shear flows.

Figure 8 shows the cross section of the pressure distribution along the \( y \)-axis. We can see that a very large pressure jump exists around the contact interfaces in the case of the standard SPH. The surface tension at the interface of the two fluids prevents the normal growth of the Kelvin–Helmholtz instability. On the other hand, there is no such jump in the case of DISPH.
The solid and dotted curves indicate density and entropy, respectively. We show the result with our SPH as well as that following the development of the Rayleigh–Taylor instability since the pressure and particle distribution is well-behaved at the interface, the growth of the Kelvin–Helmholtz instability is not suppressed.

5.5. Rayleigh–Taylor Instability Tests

Abel (2011) demonstrated that the standard SPH cannot follow the development of the Rayleigh–Taylor instability correctly. We show the result with our SPH as well as that with the standard SPH.

The initial setup is as follows. We prepared a two-dimensional computational domain of $0 \leq x < 1$ and $0 \leq y < 1$. We placed two fluids separated at $y = 0.5$. The density just above (below) the interface was set to $\rho_h \equiv 2$ ($\rho_l \equiv 1$). These two fluids were initially in the hydrostatic equilibrium. Furthermore, we assumed that each fluid was initially isentropic. The density distributions of these fluids in the vertical direction are given by

$$\rho = \begin{cases} 
\rho_l \left[ 1 + \gamma - 1 \frac{\rho_l g(y - 0.5)}{P_0} \right] \gamma^{-\gamma} & y < 0.5, \\
\rho_h \left[ 1 + \gamma - 1 \frac{\rho_h g(y - 0.5)}{P_0} \right] \gamma^{-\gamma} & y \geq 0.5,
\end{cases} \quad (107)$$

where $g = -0.5$ is the gravitational constant, $P_0 = 10/7$ is the value of pressure at the interface, and $\gamma = 1.4$. The initial density and entropy profiles are shown in Figure 9. To ensure the initial density distribution given by Equation (107), we first placed equal-mass particles on the regular grid with the separation of $1/512$. Then, we adjusted the vertical separation of each particle set having the same $y$ to reproduce the density distribution. The particle mass was set to $5.7 \times 10^{-6}$ and the total number of particles was 247,296. The periodic boundary condition was imposed on the $x$-direction. Particles with $y < 0.1$ and $y > 0.9$ were fixed at the initial positions and were not allowed to change their internal energy.

The velocity perturbation in the vertical direction was added as the seed of the instabilities. We carried out runs with two kinds of seeds. For the first test, we added the velocity perturbation to particles in the range of $0.3 < y < 0.7$, and the form of the perturbation is

$$\Delta v_y(x, y) = \delta_{y_0} [1 + \cos(4\pi x)] [1 + \cos(5\pi(y - 0.5))]. \quad (108)$$

We set $\delta_{y_0} = 0.025$. For the second test, we added a velocity perturbation of the form

$$\Delta v_y(x, y) = \sum_{j=20}^{40} a_j n_j \cos(k_j x) \exp(-0.05k_j |y - 0.5|), \quad (109)$$

and

$$n_j^2 = k_j |g| \left( \frac{\rho_h - \rho_l}{\rho_h + \rho_l} \right), \quad (110)$$

where $n_j$ is the linear growth rate of the Rayleigh–Taylor instability and $k_j = j\pi/L (\equiv 1)$ is the wave number of the perturbation. The amplitude of each mode, $a_j$, was drawn from a Gaussian distribution with the variance of unity at random. This initial velocity perturbation is based on Youngs (1984) with slight modifications. Velocities of the particles outside the perturbed region were set to zero. We call these two tests single-mode and multi-mode tests, respectively.

In Figure 10, the growth of the Rayleigh–Taylor instability in the case of the single-mode test is shown. The Rayleigh–Taylor instability develops in calculations with both the standard SPH and our SPH, but their structures are quite different. The unphysical surface tension of the standard SPH again prevents the development of the fine structures on the surface of the Rayleigh–Taylor fingers. Thus, the result looks quite different from those obtained with the Euler schemes. On the other hand, in the calculation with our SPH, the overall evolution of the Rayleigh–Taylor instability in our SPH shows excellent agreement with those with Euler schemes and the moving mesh scheme (see Springel 2010a).

Figure 11 shows the growth of the Rayleigh–Taylor instability with the multi-mode perturbations with the standard SPH and DiSPH. The global-phase mixing of fluids can be seen in the result with our SPH. On the other hand, due to the unphysical surface tension, the mixing is significantly suppressed in that of the standard SPH. The distribution of the two fluids looks like a mixture of oil and water.

5.6. Point-like Explosion Tests

In this section, we describe the results of the test calculations for a point-like explosion.

We prepared a three-dimensional computational domain of $0 \leq x, y, z < 1$ with a periodic boundary condition. Then, we placed $64^3$ equal-mass particles in that domain and made a glass-like distribution. The particle mass is $1/64^3$. Thus, the initial density was unity. The explosion energy was injected into the center of the domain. The total thermal energy of unity was distributed following the shape of the SPH kernel with the kernel size in which $N_k$ particles were included. Here, we tested $N_k = 32$ and $N_k = 256$. After that, we added the thermal energy of $10^{-6}$ of the particles with the maximum energy to all particles. This energy difference between the hot region and the ambient corresponds to a supernova explosion in a cold cloud. In this test, we adopted the viscosity parameter $\alpha = 3$.

We compared results of three schemes, i.e., the standard SPH and our SPH with $q$ and $y = P^5$. For DiSPH with $y = P^5$, we adopted $\zeta = 0.1$ and the three iterations to determine $Z$ in
Figure 10. Density maps of the two-dimensional Rayleigh–Taylor instability tests at \( t = 1, 3, 6, \) and 10. The upper panels show the results of the standard SPH, while the bottom panels show those of DISPH. The color code of the density is given at the bottom.

(A color version of this figure is available in the online journal.)

Figure 11. Same as Figure 10, but for the multi-mode perturbations.

(A color version of this figure is available in the online journal.)

each time step. We investigated the effects of the grad-\( h \) term, the adopted value of \( N_s \), and the use of the density derived from the EOS for the artificial viscosity term. The equations of motion and energy with the grad-\( h \) term for the standard SPH are Equations (25) and (22) in Springel (2010b), while those for our SPH using \( q \) are Equations (61) and (68). For the generalized DISPH, we used Equations (75) and (74) for the run without the grad-\( h \) term and Equations (85) and (86) for the run with the grad-\( h \) term, respectively. Note that unlike the experiments in Hopkins (2013), the equations without the grad-\( h \) term we used here are energy conserving ones, although these equations neglect the gradient of the kernel size. They are different from the case with \( f^{\text{grad}} = 1 \).

Figure 12 shows the density and pressure profiles of three runs. In all of the runs, the grad-\( h \) term is used. From the top row, we can see that the density profiles of all runs are basically
consistent with the analytic solution. When we compare these results, we find that the result of the standard SPH exhibits the smallest scatter in density. Our SPH with $y$ shows the highest peak at the edge of the shell.

These density profiles are drawn using the smoothed density even though our schemes can obtain their intrinsic densities by using EOS. For comparison, we prepared the density profiles of our schemes which used the density derived from the EOS, i.e., $\rho = {mq}/U$ and $\rho = {my}/Z$. These profiles are shown in Figure 13. These profiles follow the analytic solution well. The peak of our SPH with $y$ reaches to 3 and this value is higher than that obtained using the smoothed density.

However, the use of the EOS density has several disadvantages. First, the scatter of density is larger than that obtained using the smoothed density, although the increase of the scatter is not as prominent. Second, at a very early phase, the EOS density profile shows a large error. For this test case, the error at the early phase is several orders of magnitude at the initial step. If radiative cooling is included and the cooling time is comparable to or shorter than the dynamical time, then this may lead to wrong results. If the dynamical time is the shortest one of a system, then this effect is negligible. Thus, this depends strongly on the situation. It would be safer to use the smoothed density for the evaluation of the amount of radiative cooling because the cooling rate depends on the square of the density and would be sensitive to the errors in density.

The logarithmic plots of density in the middle row in Figure 12 indicate differences among runs more clearly. The result of the standard SPH shows that there is a systematic offset from the analytic solution in the central region, since it is hard to reproduce such a much less dense region using a given particle mass. The worst result is provided by DISPH with $q$. It has the largest scatter in density, and in addition it shows the systematic offset in the central region. In the result of DISPH with $y$, we
found that the density profile follows the analytic solution to a much less dense region and the degree of the density scatter is less than that found in DISPH with $q$.

The pressure profiles are shown in the bottom row of Figure 12. We can see that the standard SPH overestimates the pressure in the central, less dense region, whereas our SPH shows improved pressure profiles. This behavior comes from the fact that our SPH adopts the pressure (energy density) or its power as the smoothed value of the formulation. Note, however, that the most central particle in DISPH with $q$ has a very large error which can be observed in Figure 14. The pressure profile of DISPH with $y$ follows the postshock profile of the analytic solution very well. Only this run can reproduce the peak of the pressure profile. This is because we solved the smoothed $y = \rho^{0.5}$, which has a quite shallow gradient in the postshock region.

Figure 14, which shows the profile of the basic quantities, i.e., $\rho$, $q$, and $y$ at $t = 0.01$, clearly demonstrates the advantage of our SPH with $y = \rho^{0.5}$ over others. The density distribution of the standard SPH in the postshock region has a change with several orders of magnitude due to the expansion. In our SPH with $q$, a decrease of $q$ in the postshock region is much more moderate. The range of $y$ in the postshock region is less than 10%. Since the profile of the basic quantity is quite smooth, DISPH with $y$ can follow the peak of the pressure profile very well.

The density profiles without the grad-$h$ term are shown in Figure 15. Even though the grad-$h$ term is excluded, the standard SPH can reproduce the density profile regardless of the value of $N_s$. This is because the value of $\partial \rho / \partial h$ in $f^{\text{grad}}$ is rather small in the standard SPH. When $N_s = 256$ is adopted, the profile becomes much smoother compared to that with $N_s = 32$ due to the relatively smooth initial thermal-energy profile. On the other hand, the density profiles of our schemes with $N_s = 32$ have a clear delay in the shock front. This delay can be recovered when we adopt $N_s = 256$. In our SPH with $q$, the gap between Equation (59) and unity becomes quite large at the early phase. Even with our SPH with $y$, the gap in Equation (87) and unity is non-negligible at the early phase. These gaps lead to the delay of the shock front in this test. The amount of the delay in our SPH with $y$ is smaller compared to that in our SPH with $q$. This means $\partial y / \partial h < \partial q / \partial h$ and is reasonable since we used $\xi = 0.1$.

We show the energy errors in the calculations in Table 1. Here, we defined the energy error as $|E(t_{0.05}) - E(0)|/E(0)$, where $E(t)$ is the summation of the kinetic and thermal energy of all particles at the time $t$. The energy errors in the standard SPH runs are around $10^{-4}$. In the runs with $N_s = 32$, the energy error in the run with the grad-$h$ term is an order of magnitude smaller than that without the grad-$h$ term. However, there is no difference in the results of the runs with $N_s = 256$. The contribution of the grad-$h$ term is small for the standard SPH runs.

The energy errors in the DISPH runs with $q$ are comparable to those in the standard SPH runs. However, the grad-$h$ term plays an important role in the DISPH runs with $q$. When the grad-$h$ term is used, the energy error decreases even for $N_s = 256$. For the runs using DISPH with $y$, the energy errors are an order of magnitude larger than the others, however, the values are still acceptable (less than $\sim 0.1\%$). When we increase the iteration count from 3 to 10, the energy errors decrease by a factor of a few or an order of magnitude. The results are insensitive to the use of the grad-$h$ term and the adopted value of $N_s$ in this case.

We note that the energy errors in the runs without the grad-$h$ term are sufficiently low. This is natural since our formulation without the grad-$h$ term is constructed to conserve energy and momentum. Hopkins (2013) pointed out that the runs with $f^{\text{grad}} = 1$ did not conserve the total energy (see Figure 2 in Hopkins 2013). However, he just adopted $f^{\text{grad}} = 1$ and did not use the symmetrized kernel in his tests. Obviously, this reformulation breaks conservations of energy and momentum.

Finally, we investigate the case that the density obtained by the EOS is used for the evaluation of the artificial viscosity term. Instead of the smoothed mass density, we adopted $\rho = mq/U$ and $\rho = my/Z$ to evaluate $\rho_j$ in Equation (46) for DISPH with $q$ and $y$, respectively, and depicted the density profiles in Figure 16. By comparing the left panels in Figure 16 and the middle panels in Figure 15, we can see that the use of the EOS density makes the situation much worse. The scatter of the density profile and

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**Table 1.** Energy Error in the Sedov Tests

| Run  | Iteration | $N_s$ | with grad-$h$ | without grad-$h$ |
|------|-----------|------|---------------|-----------------|
| SPH  | N/A       | 32   | $4.8 \times 10^{-5}$ | $5.3 \times 10^{-4}$ |
| SPH  | N/A       | 256  | $3.6 \times 10^{-4}$ | $3.4 \times 10^{-4}$ |
| DISPH($q$) | N/A | 32   | $6.5 \times 10^{-4}$ | $1.5 \times 10^{-3}$ |
| DISPH($q$) | N/A | 256  | $3.9 \times 10^{-5}$ | $6.2 \times 10^{-4}$ |
| DISPH($y$) | 3    | 32   | $2.7 \times 10^{-3}$ | $6.7 \times 10^{-3}$ |
| DISPH($y$) | 3    | 256  | $4.1 \times 10^{-3}$ | $3.5 \times 10^{-3}$ |
| DISPH($y$) | 10   | 32   | $2.8 \times 10^{-4}$ | $1.1 \times 10^{-3}$ |
| DISPH($y$) | 10   | 256  | $1.1 \times 10^{-3}$ | $6.7 \times 10^{-4}$ |
Figure 14. Comparison of fundamental quantities of three runs. From left to right, the results of the standard SPH and DISPH with $q$ and $y$ at $t = 0.01$ are shown. Note that the vertical axis in the right panel is linear.

(A color version of this figure is available in the online journal.)

Figure 15. Effects of the grad-$h$ term and the adopted value of $N_s$. The top row shows density profiles without the grad-$h$ term and $N_s = 32$, while bottom row exhibits those without the grad-$h$ term and $N_s = 256$. The smoothed “mass” density is used to draw these density profiles. These profiles are obtained from snapshots at $t = 0.05$.

(A color version of this figure is available in the online journal.)

the delay of the shock front increase. We found that the delay due to the exclusion of the grad-$h$ term is recovered when we used $N_s = 256$. However, in this case, there is still a non-negligible delay. The error in the EOS density induced by the error in the pressure profile distorted the evolution of the expanding shell. The viscosity is the quantity which relates to the inertial force. Thus, the use of the smoothed mass density in the artificial viscosity term is generally suitable and this result supports this idea. Interestingly, in DISPH with $y$, the use of the EOS density for the artificial viscosity makes no difference.

In Appendix A of Read & Hayfield (2012), they discussed the problem in the Ritchie & Thomas formulation. They showed that it could not handle the strong shock (see their Figure A1). We found that the treatment of the artificial viscosity term and the exclusion of the grad-$h$ term are keys to this problem. A careful choice of equations, the density evaluation method for the artificial viscosity term, and the energy input scale provides good results even when the SPH formulation with the smoothed pressure (energy density) is adopted.
5.7. Blob Tests

In this subsection, we performed the blob test proposed by Agertz et al. (2007). This test incorporates both the Kelvin–Helmholtz and Rayleigh–Taylor instabilities.

We used Read’s initial condition of the blob test (Read et al. 2010; Read & Hayfield 2012).1 The computational domain was 0 kpc ≤ x < 2000 kpc, 0 kpc ≤ y < 2000 kpc, and 0 kpc ≤ z < 6000 kpc, and the periodic boundary condition was imposed. A cold cloud of density $\rho_c = 3.13 \times 10^{-7}$ in the mass unit of $2.3 \times 10^7 M_\odot$ and the length unit of 1 kpc, and temperature $T_c = 10^6$ K was centered at ($x, y, z$) = (1000 kpc, 1000 kpc, 2000 kpc). The radius of this cloud was 197 kpc. This cloud was embedded in the diffuse ambient gas the density and temperature of which were $\rho_a = 3.13 \times 10^{-8}$ and $T_a = 10^7$ K, respectively. The ambient gas had a velocity of $v_z = 1000$ km s$^{-1}$. Thus, the Mach number of the flow to the cloud was 2.7. The total number of particles for the system is 4,643,283. We integrated the system up to $t = 5\tau_{kh}$, where $\tau_{kh} = 2$ Gyr is the typical growth timescale of the Kelvin–Helmholtz instability in this test (Agertz et al. 2007).

Figure 17 shows the snapshots of the cloud core. The upper and lower panels are the results with the standard SPH and DISPH, respectively. Their evolutions were quite different. The blob simulated with the standard SPH retained the single cloud structure until the late stage of the simulation. This behavior is consistent with those with the standard SPH shown in Agertz et al. (2007). In contrast, the blob surface was disrupted in the run with our SPH due to the growth of the instabilities on the surface. The blob fragmented into several pieces and eventually mixed with the ambient gas. This behavior is similar to those obtained by the Euler codes.

The evolution of the blob mass is shown in Figure 18. Here, we show the mass of gas with $\rho > 0.64\rho_c$ and $T < 0.9T_a$, following Agertz et al. (2007). At $t = 2.5\tau_{kh}$, the blob mass in the run of DISPH became ∼10% of the initial mass. This result is consistent with the results of the Euler codes (see Figure 6 in Agertz et al. 2007). The evolution of the blob mass in the standard SPH was much slower compared to that in DISPH.

5.8. Mixing of Two-phase Fluid with Spoon

In this subsection, we discuss the results of simulations of two-fluid mixing with a solid body like a spoon. With this test, we can see the ability of a particle-based scheme to handle fixed and moving boundary conditions. The setup of this test is similar to that in Section 8.9 of Springel (2010a).

The initial setup is as follows. We prepared the two-dimensional computational domain of $-0.025 \leq x \leq 0.025$ and $-0.025 \leq y \leq 0.025$. The lower part of the domain $y < 0.6$ was the dense region with a density of 1 while the upper part of the domain $y > 0.6$ was the less-dense region with a density of 0.5. To make this density distribution, we employed the equal-mass particles the mass of which is $1.87 \times 10^{-6}$ and we placed them on the regular grid with a separation of 1.05/768. Then, we doubled the vertical separation of particles in $y > 0.6$. The pressure was unity in the whole region, with $\gamma = 5/3$. Sound speeds in the dense and less-dense regions were 1.3 and 1.8 in the simulation unit, respectively. Particles which were out of the range $0 \leq x < 1$ and $0 \leq y < 1$ were fixed at the initial boundary.

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1 We obtained the initial condition from the following URL: http://www-theorie.physik.uzh.ch/astrosim/code/.
positions and initial physical quantities. These fixed particles express the fixed boundary condition. The velocities of all the particles were initially set to zero. The number of particles in the dense and less-dense regions in $0 \leq x < 1$ and $0 \leq y < 1$ were 319,185 and 107,604, respectively, and that of the boundary particles were 41,064.

We made the solid-body-like spoon by SPH particles. The detailed procedure is described in the Appendix. The total number of particles of which the spoon consisted was 8652. In Figure 19, we show the shape of the spoon. The physical quantities and relative positions of spoon particles were kept unchanged. The spoon rotates counterclockwise around the rotation center of $(0.5, 0.5)$. The angular velocity is $2\pi/5$. We introduced a repulsive force for the interactions between the fluid particles and solid body particles of the spoon so that the penetration of fluid particles to the spoon is prevented.

Figure 20 shows the snapshots of the representative epoch ($t = 0, 1.6, 3.0, 5.0, 6.5, \text{ and } 8.0$) for the runs with the standard SPH and our SPH. For the run with the standard SPH, the fluid shows the clear sign of unphysical surface tension, resulting in behavior much like that of water and oil. Although the spoon rotated the two-phase fluid, the mixing of the two fluids is prevented by the surface tension. Overall, the result
Figure 20. Evolution of the two-phase fluid forcibly mixed by the spoon up to $t = 8.0$. The top six panels show the evolution using the standard SPH, while the bottom six panels show that using our SPH. The arcuate structure expressed by white points is the spoon. The long white arc (almost circle) found in the panels at $t = 0$ represents the motion of the spoon and the arrow shows the direction of the motion (counterclockwise). The colors predict the density of the fluid and the color bar can be found at the bottom of panels.

(A color version of this figure is available in the online journal.)

is completely different from that obtained by the moving mesh code (see Figure 39 in Springel 2010a).

The situation was drastically improved when we used DISPH. When the spoon lifted-off the dense fluid, a hammer-head-like structure was developed ($t = 1.6$). The fluid spilled away from the edges of the spoon and formed eddies. These prominent structures were not observed in the run with the standard SPH, but they were observed in the moving mesh simulation (Springel 2010a).

6. SUMMARY AND DISCUSSION

In this paper, we described an alternative formulation of SPH in which the energy density (pressure) and its arbitrary function are used as the volume element instead of the mass density. In our formulation, the mass of the particles is not used in the evaluation of the right-hand sides of the energy equation and the equation of motion. As a result, the large error of force estimate at the contact discontinuity, which is unavoidable with
the standard SPH, disappears completely in our SPH. Our new SPH includes the Ritchie & Thomas formulation (Ritchie & Thomas 2001) as a special case. Not surprisingly, our SPH can handle contact discontinuities and the Kelvin–Helmholtz and Rayleigh–Taylor instabilities without difficulty. The behavior of the shock in DISPH is essentially the same as that in the standard SPH, since the equations used in our SPH are almost identical to those in the standard SPH except that a function of energy density (pressure) is used in place of mass density $\rho$. The modification of the existing SPH code to our scheme is simple and straightforward. In particular, there is no increase in the calculation cost, at least for the case of $G(P) = P$. Equations which are not derived in this paper, such as the diffusion equation (Brookshaw 1985), can be derived easily.

Price (2008) improved the treatment of the Kelvin–Helmholtz instability of the standard SPH by applying artificial conductivity at the contact discontinuity. Unlike the artificial viscosity, artificial conductivity introduces the dissipation that was not in the original set of equations. Our SPH does not need such additional dissipation, and thus the contact discontinuity is kept sharp.

One might think our result contradicts the requirement that all quantities in SPH must be smooth (Monaghan 1997). However, it is obvious that in our SPH, all quantities on the right-hand side of the equations are smooth. Thus, our results do not contradict Monaghan’s requirement.

In this paper, we discuss the treatment of ideal gas only. We are currently working on the extension to non-ideal fluid, and the result will be given in a forthcoming paper.

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Figure 21. Shape of the spoon. The dotted red curve exhibits the boundary of the spoon. The circles consisting of the closed surface are also shown. Left: whole region of the spoon. Right: close-up of the left edge of the spoon. (A color version of this figure is available in the online journal.)

APPENDIX

SETUP OF THE SOLID BODY LIKE SPOON

Here, we describe the procedure to make the spoon that was used in Section 5.8. To express the spoon in the fluid simulation, we first chose particles in a region for which the shape is an arcuate with smooth edges. Particles in the region evolve like a solid body. We enforced that the relative positions and other physical quantities of the particles composing the spoon were kept unchanged during the simulations. To avoid the penetration of fluid particles to the spoon, we added a repulsive force which acted on the fluid particles. We explain these in detail below.

As shown in Figure 21, we combined arcs of four circles to determine the boundary of the spoon. We picked up a ringed region which is located between two circles ($C1$ and $C2$):

\[(x - 0.75)^2 + (y - 0.7)^2 = (0.19)^2, \quad (A1)\]

\[(x - 0.75)^2 + (y - 0.7)^2 = (0.2)^2. \quad (A2)\]

Then, we cut the ringed region and made edges by putting two small circles, $C3$ and $C4$, which contact the circles $C1$ and $C2$. The centers of the two small circles are $(x_1, y_1) = (0.66, 0.52)$ and $(x_2, y_2) = (0.84, 0.52)$, respectively. Their radii are 0.01. The right (left) side of $C3$ ($C4$) makes the smooth edge.

The particle found in the enclosed region was selected and converted into four smaller SPH particles conserving the center of mass with a separation of one-half of the original particle separation. The total number of the spoon particles was 6608. The spoon rotated counterclockwise around the rotation center of $(0.5, 0.5)$ and with an angular velocity of $2\pi/5$.

The spoon faced strong pressure from the fluid particles. When we only considered the hydrodynamical force from the particles consisting of the spoon, a little but non-negligible amount of fluid particles penetrated through the spoon. We thus introduced a repulsive force for the interactions between fluid particles and spoon particles. In Monaghan (1994), the Lenard–Jones potential was used for interactions between boundary particles and fluid particles. Here, we choose a more...
simple one. The form of the repulsive force between particles \(i\) and \(j\), \(F_{ij}^{\text{rep}}\), is the same as the gravitational force with the Plummer potential:

\[
F_{ij}^{\text{rep}} = C \frac{m_i m_j}{|r_{ij}|^{3/2} + \epsilon_{\text{soft}}^{3/2} r_{ij}}, \tag{A3}
\]

where \(\epsilon_{\text{soft}}\) is the softening length of the repulsive force and is set to \(2h\) of the spoon particles. We chose \(C = 100\). This force worked only when fluid particles were in the kernel size of the spoon particles.

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