Vlasov moments, integrable systems and singular solutions

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Abstract
The Vlasov equation for the collisionless evolution of the single-particle probability distribution function (PDF) is a well-known Lie-Poisson Hamiltonian system. Remarkably, the operation of taking the moments of the Vlasov PDF preserves the Lie-Poisson structure. The individual particle motions correspond to singular solutions of the Vlasov equation. The paper focuses on singular solutions of the problem of geodesic motion of the Vlasov moments. These singular solutions recover geodesic motion of the individual particles.

Contents
1 Introduction 2
2 Review of Vlasov moment dynamics 4
  2.1 Dynamics of Vlasov \( q,p \)-Moments 4
  2.2 Dynamics of Vlasov \( p \)-Moments 5
3 Moments and cotangent lifts of diffeomorphisms 6
  3.1 Lagrangian variables and cotangent lifts 6
  3.2 Characteristic equations for the moments 7
4 Applications of the KMLP bracket and quadratic terms 9
  4.1 The Benney equations 9
  4.2 The Vlasov-Poisson system, the wakefield model and singular solutions 10
  4.3 The EPDiff equation and singular solutions 10
5 Geodesic motion and singular solutions 11
  5.1 Quadratic Hamiltonians 11
  5.2 A geodesic Vlasov equation 12
  5.3 Singular geodesic solutions 12
  5.4 Examples of simplifying truncations and specializations 13
6 Open questions for future work 14
1 Introduction

The Vlasov equation. The evolution of $N$ identical particles in phase space with coordinates $(q_i, p_i)$ $i = 1, 2, \ldots, N$, may be described by an evolution equation for their joint probability distribution function. Integrating over all but one of the particle phase-space coordinates yields an evolution equation for the single-particle probability distribution function (PDF). This is the Vlasov equation, which may be expressed as an advection equation for the phase-space density $f$ along the Hamiltonian vector field $X_H$ corresponding to single-particle motion with Hamiltonian $H(q, p)$:

$$\frac{\partial f}{\partial t} = \{ f, H \} = - \text{div}(q, p)(f X_H) = - \mathcal{L}_{X_H} f \quad \text{with} \quad X_H(q, p) = \left( \frac{\partial H}{\partial p}, - \frac{\partial H}{\partial q} \right) \quad (1.1)$$

The solutions of the Vlasov equation reflect its heritage in particle dynamics, which may be reclaimed by writing its many-particle PDF as a product of delta functions in phase space. Any number of these delta functions may be integrated out until all that remains is the dynamics of a single particle in the collective field of the others.

In the mean-field approximation of plasma dynamics, this collective field generates the total electromagnetic properties and the self-consistent equations obeyed by the single particle PDF are the Vlasov-Maxwell equations. In the electrostatic approximation, these reduce to the Vlasov-Poisson (VP) equations, which govern the statistical distributions of particle systems ranging from integrated circuits (MOSFETS, metal-oxide semiconductor field-effect transistors), to charged-particle beams, to the distribution of stars in a galaxy.

A class of singular solutions of the VP equations called the “cold plasma” solutions have a particularly beautiful experimental realization in the Malmberg-Penning trap. In this experiment, the time average of the vertical motion closely parallels the Euler fluid equations. In fact, the cold plasma singular Vlasov-Poisson solution turns out to obey the equations of point-vortex dynamics in an incompressible ideal flow. This coincidence allows the discrete arrays of “vortex crystals” envisioned by J. J. Thomson for fluid vortices to be realized experimentally as solutions of the Vlasov-Poisson equations. For a survey of these experimental cold-plasma results see [11].

Vlasov moments. The Euler fluid equations arise by imposing a closure relation on the first three momentum moments, or $p$--moments of the Vlasov PDF $f(p, q, t)$. The zero-th $p$--moment is the spatial density of particles. The first $p$--moment is the mean fluid momentum. Introducing an expression for the fluid pressure in terms of the density and momentum closes the system of $p$--moment equations, which otherwise would possess a countably infinite number of dependent variables.

The operation of taking $p$--moments preserves the geometric nature of Vlasov dynamics. In particular, this operation is a Poisson map. That is, it takes the Lie-Poisson structure for Vlasov dynamics into another Lie-Poisson system. However, strictly speaking, the solutions for the $p$--moments cannot yet be claimed to undergo coadjoint motion, as in the case of the Vlasov PDF solutions, because the group action underlying the Lie-Poisson structure found for the $p$--moments is not yet understood.

A closure after the first $p$--moment results in Euler’s useful and beautiful theory of ideal fluids which is also Lie-Poisson. As its primary geometric characteristic, Euler’s fluid theory represents fluid flow as Hamiltonian geodesic motion on the space of smooth invertible maps acting on the flow domain and possessing smooth inverses. The (left) action of these smooth maps (called diffeomorphisms) on the fluid reference configuration moves the fluid particles around in their container. And their smooth inverses recall the initial reference configuration.
(or label) for the fluid particle currently occupying any given position in space. Thus, the motion of all the fluid particles in a container is represented as a time-dependent curve in the infinite-dimensional group of diffeomorphisms. Moreover, this curve describing the sequential actions of the diffeomorphisms on the fluid domain is a special optimal curve that distills the fluid motion into a single statement. Namely, “A fluid moves to get out of its own way as efficiently as possible”. Put more mathematically, fluid flow occurs along a curve in the diffeomorphism group which is a geodesic with respect to the metric on its tangent space supplied by its kinetic energy.

Given the beauty and utility of the solution behavior for Euler’s equation for the first $p$—moment, one is intrigued to know more about the dynamics of the other moments of Vlasov’s equation. Of course, the dynamics of the the $p$—moments of the Vlasov-Poisson equation is one of the mainstream subjects of plasma physics and space physics.

**Summary.** This paper formulates the dynamics of Vlasov $p$—moments governed by quadratic Hamiltonians. This dynamics is a certain type of geodesic motion on the symplectomorphisms, rather than on the diffeomorphisms for fluids. The symplectomorphisms are smooth invertible maps acting on the phase space and possessing smooth inverses. The theory of moment dynamics for the Vlasov equation turns out to be equivalent to the theory of shallow water equations, and a particular example is the one-dimensional system of Benney long wave equations, which is integrable [1, 12].

Here we shall consider the singular solutions of the geodesic dynamics of the Vlasov $p$—moments. Remarkably, these equations turn out to be related to other integrable systems governing shallow water wave theory. For example, when the Vlasov $p$—moment equations for geodesic motion on the symplectomorphisms are closed at the level of the first $p$—moment, their singular solutions are found to recover the peaked soliton of the integrable Camassa-Holm (CH) equation for shallow water waves [5]. These singular Vlasov moment solutions also correspond to individual particle motion.

Thus, geodesic symplectic dynamics of the Vlasov $p$—moments is found to possess singular solutions whose closure at the fluid level for the CH equation recovers the peakon solutions of shallow water theory. Being solitons, the CH peakons superpose and undergo elastic collisions in fully nonlinear interactions. The singular solutions for Vlasov $p$—moments presented here also superpose and interact nonlinearly as coherent structures.

The plan of the paper follows:

Section 2 reviews the Vlasov $p$—moment equations and recounts their Lie-Poisson Hamiltonian structure using the Kupershmidt-Manin Lie-Poisson bracket. Variational formulations of the $p$—moment dynamics are also provided.

Section 3 shows how the Lagrangian framework for fluid dynamics is recovered from the Vlasov $p$—moments and establishes connections with some equations for shallow water waves. In this case, the $p$—moment equations are shown to possess singular solutions.

Section 4 establishes the connections between the integrable Benney equations and the physics of charged-particle accelerator beams. To our knowledge, these connections are noted here for the first time. We also point out how the experimental realization of solitary waves in coasting particle beams has its roots in the integrability of the Benney system.

Section 5 formulates the problem of geodesic motion on the symplectomorphisms in terms of the Vlasov $p$—moments and identifies the singular solutions of this problem. This geodesic
motion is related to a geodesic form of the Vlasov equation. Thus the singular solutions are found to originate in the single particle dynamics on phase space. In a special case, the truncation of geodesic symplectic motion to geodesic diffeomorphic motion for the first $p-$moment recovers the singular solutions of the Camassa-Holm equation, and thereby correspond to single particle dynamics.

The geodesic form of the Vlasov equation was introduced in [15], where it was also shown how to extend the treatment to higher dimensions.

2 Review of Vlasov moment dynamics

The Vlasov equation may be expressed as

$$\frac{\partial f}{\partial t} = \left[ f, \frac{\delta h}{\delta f} \right] = \frac{\partial f}{\partial p} \frac{\partial}{\partial q} \frac{\delta h}{\delta f} - \frac{\partial f}{\partial q} \frac{\partial}{\partial p} \frac{\delta h}{\delta f} =: -\text{ad}^*_{\delta h/\delta f} f$$  \hspace{1cm} (2.1)

Here the canonical Poisson bracket $[\cdot, \cdot]$ is defined for smooth functions on phase space with coordinates $(q,p)$ and $f(q,p,t)$ is the evolving Vlasov single-particle distribution function. The variational derivative $\frac{\delta h}{\delta f}$ is the single particle Hamiltonian and the $\text{ad}^*_{\delta h/\delta f} f$ is explained as follows.

A functional $g[f]$ of the Vlasov distribution $f$ evolves according to

$$\frac{dg}{dt} = \iint \frac{\delta g}{\delta f} \frac{\partial f}{\partial t} dq dp = \iint \frac{\delta g}{\delta f} \left[ f, \frac{\delta h}{\delta f} \right] dq dp = -\iint f \left[ \frac{\delta g}{\delta f}, \frac{\delta h}{\delta f} \right] dq dp =: \{ g, h \}$$

In this calculation boundary terms were neglected upon integrating by parts in the third step and the notation $\langle \langle \cdot, \cdot \rangle \rangle$ is introduced for the $L^2$ pairing in phase space. The quantity $\{ g, h \}$ defined in terms of this pairing is the Lie-Poisson Vlasov (LPV) bracket [30]. This Hamiltonian evolution equation may also be expressed as

$$\frac{dq}{dt} = \{ g, h \} = -\langle \langle f, \text{ad}^*_{\delta h/\delta f} \frac{\delta g}{\delta f} \rangle \rangle = -\langle \langle \text{ad}^*_{\delta h/\delta f} f, \frac{\delta g}{\delta f} \rangle \rangle$$

which defines the Lie-algebraic operations $\text{ad}$ and $\text{ad}^*$ in this case in terms of the $L^2$ pairing on phase space $\langle \langle \cdot, \cdot \rangle \rangle$: $\mathfrak{s}^* \times \mathfrak{s} \rightarrow \mathbb{R}$. The notation $\text{ad}^*_{\delta h/\delta f} f$ in (2.1) expresses coadjoint action of $\delta h/\delta f \in \mathfrak{s}$ on $f \in \mathfrak{s}^*$, where $\mathfrak{s}$ is the Lie algebra of single particle Hamiltonian vector fields and $\mathfrak{s}^*$ is its dual under $L^2$ pairing in phase space. This is the sense in which the Vlasov equation represents coadjoint motion on the symplectomorphisms. This Lie-Poisson structure has also been extended to include Yang-Mills theories in [13] and [14].

In higher dimensions, particularly $n = 3$, we may take the direct sum of the Vlasov Lie-Poisson bracket, together with with the Poisson bracket for an electromagnetic field (in the Coulomb gauge) where the electric field $E$ and magnetic vector potential $A$ are canonically conjugate. For discussions of the Vlasov-Maxwell equations from a geometric viewpoint in the same spirit as the present approach, see [7], [25], [26] and [30].

2.1 Dynamics of Vlasov $q,p-$Moments

The phase space $q,p-$moments of the Vlasov distribution function are defined by

$$g_{\vec{m}} = \iint f(q,p) q^m p^m \ dq \ dp.$$
The \( q,p \)-moments \( g_{\hat{m}m} \) are often used in treating the collisionless dynamics of plasmas and particle beams \([10]\). This is usually done by considering low-order truncations of the potentially infinite sum over phase space moments,

\[
g = \sum_{\hat{m}, m=0}^{\infty} a_{\hat{m}m} g_{\hat{m}m}, \quad h = \sum_{\hat{n}, n=0}^{\infty} b_{\hat{n}n} g_{\hat{n}n},
\]

with constants \( a_{\hat{m}m} \) and \( b_{\hat{n}n} \), with \( \hat{m}, m, \hat{n}, n = 0, 1, \ldots \). If \( h \) is the Hamiltonian, the sum over \( q,p \)-moments \( g \) evolves under the Vlasov dynamics according to the Poisson bracket relation

\[
\frac{dg}{dt} = \{ g, h \} = \sum_{\hat{m}, m, \hat{n}, n=0} a_{\hat{m}m} b_{\hat{n}n} (\hat{m}m - \hat{n}n) g_{\hat{m}+\hat{n}-1,m+n-1}.
\]

The symplectic invariants associated with Hamiltonian flows of the \( q,p \)-moments were discovered and classified in \([18]\). Finite dimensional approximations of the whole \( q,p \)-moment hierarchy were discussed in \([29]\). For discussions of the Lie-algebraic approach to the control and steering of charged particle beams, see \([10]\).

### 2.2 Dynamics of Vlasov \( p \)-Moments

In contrast to the \( q,p \)-moments, the momentum moments, or “\( p \)-moments,” of the Vlasov function are defined as

\[
A_m(q,t) = \int p^m f(q,p,t) \, dp, \quad m = 0, 1, \ldots
\]

That is, the \( p \)-moments are \( q \)-dependent integrals over \( p \) of the product of powers \( p^m \), \( m = 0, 1, \ldots \), times the Vlasov solution \( f(q,p,t) \). We shall consider functionals of these \( p \)-moments defined by,

\[
g = \sum_{m=0}^{\infty} \int \alpha_m(q) p^m f \, dq dp = \sum_{m=0}^{\infty} \int \alpha_m(q) A_m(q) \, dq =: \sum_{m=0}^{\infty} \langle A_m, \alpha_m \rangle,
\]

\[
h = \sum_{n=0}^{\infty} \int \beta_n(q) p^n f \, dq dp = \sum_{n=0}^{\infty} \int \beta_n(q) A_n(q) \, dq =: \sum_{n=0}^{\infty} \langle A_n, \beta_n \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the \( L^2 \) pairing on position space.

The functions \( \alpha_m \) and \( \beta_n \) with \( m, n = 0, 1, \ldots \) are assumed to be suitably smooth and integrable against the Vlasov \( p \)-moments. To assure these properties, one may relate the \( p \)-moments to the previous sums of Vlasov \( q,p \)-moments by choosing

\[
\alpha_m(q) = \sum_{\hat{m}=0}^{\infty} a_{\hat{m}m} q^{\hat{m}} \quad \text{and} \quad \beta_n(q) = \sum_{\hat{n}=0}^{\infty} b_{\hat{n}n} q^{\hat{n}}.
\]

For these choices of \( \alpha_m(q) \) and \( \beta_n(q) \), the sums of \( p \)-moments will recover the full set of Vlasov \( (q,p) \)-moments. Thus, as long as the \( q,p \)-moments of the distribution \( f(q,p) \) continue to exist under the Vlasov evolution, one may assume that the dual variables \( \alpha_m(q) \) and \( \beta_n(q) \) are smooth functions whose Taylor series expands the \( p \)-moments in the \( q,p \)-moments. These functions are dual to the \( p \)-moments \( A_m(q) \) with \( m = 0, 1, \ldots \) under the \( L^2 \) pairing \( \langle \cdot, \cdot \rangle \) in the spatial variable \( q \). In what follows we will assume homogeneous boundary conditions. This means, for example, that we will ignore boundary terms arising from integrations by parts.
The Poisson bracket among the $p$–moments is obtained from the LPV bracket may be expressed as

$$\{ g, h \}(\{ A \} ) = - \sum_{m,n=0}^{\infty} \int \frac{\delta g}{\delta A_m} \left[ n \frac{\delta h}{\delta A_n} \frac{\partial}{\partial q} A_{m+n-1} + (m+n)A_{m+n-1} \frac{\partial}{\partial q} \frac{\delta h}{\delta A_n} \right] dq$$

$$= - \sum_{m,n=0}^{\infty} \left( A_{m+n-1}, \left[ \frac{\delta g}{\delta A_m}, \frac{\delta h}{\delta A_n} \right] \right)$$

This is the Kupershmidt-Manin Lie-Poisson (KMLP) bracket \[22\], which is defined for functions on the dual of the Lie algebra with bracket

$$\left[ \alpha_m , \beta_n \right] = m \alpha_m \partial_q \beta_n - n \beta_n \partial_q \alpha_m .$$

This Lie algebra bracket inherits the Jacobi identity from its definition in terms of the canonical Hamiltonian vector fields. Thus, we have shown the

**Theorem 2.1** (Gibbons \[12\]) The operation of taking $p$–moments of Vlasov solutions is a Poisson map. It takes the LPV bracket describing the evolution of $f(q,p)$ into the KMLP bracket, describing the evolution of the $p$–moments $A_n(x)$.

A result related to this, for the Benney hierarchy \[1\], was also noted by Lebedev and Manin \[23\].

The evolution of a particular $p$–moment $A_m(q,t)$ is obtained from the KMLP bracket by

$$\frac{\partial A_m}{\partial t} = \{ A_m , h \} = - \sum_{n=0}^{\infty} \left( n \frac{\partial}{\partial q} A_{m+n-1} + m A_{m+n-1} \frac{\partial}{\partial q} \right) \frac{\delta h}{\delta A_n} = \sum_{n=0}^{\infty} \left\{ A_m , A_n \right\} \frac{\delta h}{\delta A_n}$$

These moment equations can also be derived from variational principles, as shown in \[15\], within the Hamilton-Poincaré framework \[8\].

### 3 Moments and cotangent lifts of diffeomorphisms

As explained in the introduction, a first order closure of the moment hierarchy leads to the equations of ideal fluid dynamics. Such equations represent coadjoint motion with respect to the Lie group of smooth invertible maps (diffeomorphisms). This coadjoint evolution may be interpreted in terms of Lagrangian variables, which are invariant under the action of diffeomorphisms. In this section we investigate how the entire moment hierarchy may be expressed in terms of the fluid quantities evolving under the diffeomorphisms and express the conservation laws in this case.

#### 3.1 Lagrangian variables and cotangent lifts

In order to look for Lagrangian variables, we consider the geometric interpretation of the moments, regarded as fiber integrals on the cotangent bundle $T^*Q$ of some configuration manifold $Q$. A $p$–moment is defined as a fiber integral; that is, an integral on the single fiber $T^*_qQ$ with base point $q \in Q$ kept fixed

$$A_n(q) = \int_{T^*_qQ} p^n f(q,p) dp$$

A similar approach is followed for gyrokinetics in \[28\]. Now, the problem is that in general the integrand does not stay on one same fiber under the action of canonical transformations, i.e.
symplectomorphisms are not fiber-preserving in the general case. However, one may avoid this problem by restricting to a subgroup of these canonical transformations whose action is fiber preserving. The transformations in this subgroup (indicated with $T^*\text{Diff}(Q)$) are called point transformations or cotangent lifts of diffeomorphisms and they arise from diffeomorphisms on points in configuration space [24], such that

$$q_t = q_t(q_0)$$

The fiber preserving nature of cotangent lifts is expressed by the preservation of the canonical one-form:

$$p_t dq_t = p_t(q_0, p_0) dq_t(q_0) = p_0 dq_0$$

This fact also reflects in the particular form assumed by the generating functions of cotangent lifts, which are linear in the momentum coordinate, i.e.

$$H(q, p) = \beta(q) \frac{\partial}{\partial q} \int p dq = p \beta(q).$$

Restricting to cotangent lifts represents a limitation in comparison with considering the whole symplectic group. However, this is a natural way of recovering the fluid equations, starting from the full moment dynamics.

### 3.2 Characteristic equations for the moments

Once one restricts to cotangent lifts, Lagrangian moment variables may be defined and conservation laws may be found, as in the context of fluid dynamics. The key idea is to use the preservation of the canonical one-form for constructing invariant quantities. Indeed one may take $n$ times the tensor product of the canonical one-form with itself and write:

$$p_t^n (dq_t)^n = p_0^n (dq_0)^n$$

One then considers the preservation of the Vlasov density

$$f_t(q_t, p_t) dq_t \wedge dp_t = f_0(q_0, p_0) dq_0 \wedge dp_0$$

and writes

$$p_t^n f_t(q_t, p_t) (dq_t)^n \otimes dq_t \wedge dp_t = p_0^n f_0(q_0, p_0) (dq_0)^n \otimes dq_0 \wedge dp_0$$

Integration over the canonical particle momenta yields the following characteristic equations

$$\frac{d}{dt} \left[ A^{(t)}_n(q_t) (dq_t)^n \otimes dq_t \right] = 0 \quad \text{along} \quad \dot{q}_t = \frac{\partial H}{\partial p} = \beta(q) \quad (3.1)$$

which recover the well known conservations for fluid density and momentum ($n = 0, 1$) and can be equivalently written in terms of the Lie-Poisson equations arising from the KMLP bracket. Indeed, if the vector field $\beta$ is identified with the Lie algebra variable $\beta = \beta_1 = \delta h/\delta A_1$ and $h(A_1)$ is the moment Hamiltonian, then the KMLP form of the moment equations is

$$\dot{A}_n + \text{ad}_{\beta_1}^* A_n = 0.$$

In this case, the KM $\text{ad}_{\beta_1}^*$ operation coincides with the Lie derivation $\mathcal{L}_\beta$. Hence, one may also write it equivalently as

$$\dot{A}_n + \mathcal{L}_\beta A_n = 0.$$
This equation is reminiscent of the so called \textit{b-equation} introduced in \cite{20}, for which the vector field \( \beta \) is nonlocal and may be taken as \( \beta(q) = G * A_n \) (for any \( n \)), where \( G \) is the Green’s function of the Helmholtz operator and the star denotes convolution. When the vector field \( \beta \) is sufficiently smooth, this equation is known to possess singular solutions of the form

\[
A_n(q,t) = \sum_{i=1}^{K} P_{n,i}(t) \delta(q - Q_i(t))
\]

where the \( i \)-th position \( Q_i \) and weight \( P_{n,i} \) of the singular solution for the \( n \)-th moment satisfy the following equations

\[
\dot{Q}_i = \beta(q)|_{q=Q_i}, \quad \dot{P}_{n,i} = -nP_{n,i} \frac{\partial \beta(q)}{\partial q} |_{q=Q_i}
\]

Interestingly, for \( n = 1 \) (with \( \beta(q) = G * A_1 \)), these equations recover the pulson solutions of the Camassa-Holm equation, which play an important role in the following discussion. Moreover the particular case \( n = 1 \) represents the single particle solution of the Vlasov equation. However, when \( n \neq 1 \) the interpretation of these solutions as single-particle motion requires the particular choice \( P_{n,i} = (P_i)^n \). For this choice, the \( n \)-th weight is identified with the \( n \)-th power of the particle momentum.

The equations presented in this section provide a geometric interpretation of the moments in terms of the covariant tensor power densities in (3.1). When one restricts to cotangent lifts, one can specify an action of the group of diffeomorphisms on the moments. However, this action is not yet understood at the group level in the general case. The general geometric nature of the Lie algebra would be expressed in terms of contravariant tensors dual to the covariant tensor power densities. For the cotangent lifts, these contravariant tensors reduce to contravariant vector fields. In this case, one is able to characterize their action on the moments as Lie derivatives.

**KMLP bracket and cotangent lifts.** We have seen that restricting to cotangent lifts leads to a Lagrangian fluid-like formulation of the dynamics of the resulting \( p \)-moments. In this case, the moment equations are given by the KMLP bracket when the Hamiltonian depends only on the first moment (\( \beta_1 = \delta h/\delta A_1 \))

\[
\{g, h\} = -\sum_n \left\langle A_n, \left[ \frac{\delta g}{\delta A_n}, \frac{\delta h}{\delta A_1} \right] \right\rangle
\]

If one now restricts the bracket to functionals of only the first moments, one may check that the KMLP bracket yields the well known Lie-Poisson bracket on the group of diffeomorphisms

\[
\{g, h\}[A_1] = \left\langle A_1, \left[ \frac{\delta h}{\delta A_1}, \frac{\partial}{\partial q} \frac{\delta g}{\delta A_1} - \frac{\delta g}{\delta A_1} \frac{\partial}{\partial q} \frac{\delta h}{\delta A_1} \right] \right\rangle
\]

This is a very natural step since diffeomorphisms and their cotangent lifts are isomorphic. In fact, this is the bracket used for ideal incompressible fluids as well as for the construction of the EPDiff equation, which will be discussed later as an application of moment dynamics.
4 Applications of the KMLP bracket and quadratic terms

4.1 The Benney equations

The KMLP bracket (2.2) was first derived in the context of Benney long waves, whose Hamiltonian is

\[ H = \frac{1}{2} \int (A_2(q) + gA_0^2(q)) \, dq. \]

The Hamiltonian form \( \partial_t A_n = \{ A_n, H \} \) with the KMLP bracket leads to the moment equations

\[ \frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial q} + gnA_{n-1} \frac{\partial A_0}{\partial q} = 0 \]

derived by Benney [1] as a description of long waves on a shallow perfect fluid, with a free surface at \( y = h(q, t) \). In his interpretation, the \( A_n \) were vertical moments of the horizontal component of the velocity \( p(q, y, t) \):

\[ A_n = \int_0^h p^n(q, y, t) \, dy. \]

The corresponding system of evolution equations for \( p(q, y, t) \) and \( h(q, t) \) is related by hodograph transformation, \( y = \int_{-\infty}^{\infty} f(q, p', t) \, dp' \), to the Vlasov equation

\[ \frac{\partial f}{\partial t} + p \frac{\partial f}{\partial q} - g \frac{\partial A_0}{\partial q} \frac{\partial f}{\partial p} = 0. \]  (4.1)

The most important fact about the Benney hierarchy is that it is completely integrable. This fact emerges from the following observation. Upon defining a function \( \lambda(q, p, t) \) by the principal value integral,

\[ \lambda(q, p, t) = p + P \int_{-\infty}^\infty \frac{f(q, p', t)}{p - p'} \, dp', \]

it is straightforward to verify [23] that

\[ \frac{\partial \lambda}{\partial t} + p \frac{\partial \lambda}{\partial q} - g \frac{\partial A_0}{\partial q} \frac{\partial \lambda}{\partial p} = 0; \]

so that \( f \) and \( \lambda \) are advected along the same characteristics.

Applications to coasting accelerator beams. Interestingly, the Vlasov equation (4.1) resulting from the hodograph transformation of the Benney equation is exactly the same as the equation that regulates coasting proton beams in particle accelerators (see for example [32] where a bunching term is also included).

Now, this is a very remarkable fact: the integrability of the Vlasov-Benney equation implies soliton solutions, indications for which seem to have been found experimentally at CERN [21], BNL [2], LANL [9] and FermiLab [27]. (In the last case solitons are shown to appear even when a bunching force is present.) These solutions have attracted the attention of the accelerator community and considerable analytical work has been carried out over the last decade (see for example [31]). Nevertheless to our knowledge, the existence of solitons in coasting proton beams has never been related to the integrability of the governing Vlasov equation via its connection to the Benney hierarchy., although this would explain very naturally why robust coherent structures are seen in these experiments as fully nonlinear excitations. We plan to pursue this direction in future research.
4.2 The Vlasov-Poisson system, the wakefield model and singular solutions

Besides integrability of the Vlasov-Benney equation, there are other important applications of the Vlasov equation that have in common the presence of a quadratic term in $A_0$ within the Hamiltonian:

$$H = \frac{1}{2} \int A_2(q) \, dq + \frac{1}{2} \int A_0(q) G(q, q') A_0(q') \, dq \, dq'.$$

For example, when $G = (\partial_q^2)^{-1}$, this Hamiltonian leads to the Vlasov-Poisson system, which is of fundamental importance in many areas of plasma physics. More generally, this Hamiltonian is widely used for beam dynamics in particle accelerators: in this case $G$ is related to the electromagnetic interaction of a beam with the vacuum chamber. The wake field is originated by the image charges induced on the walls by the passage of a moving particle: while the beam passes, the charges in the walls are attracted towards the inner surfaces and generate a field that acts back on the beam. This affects the dynamics of the beam, thereby causing several problems such as beam energy spread and instabilities. In the literature, the wake function $W$ is introduced so that

$$G(q, q') = \int_{-\infty}^{q} W(x, q') \, dx$$

Wake functions usually depend only on the properties of the accelerator chamber.

An interesting wakefield model has been presented in [31] where $G$ is chosen to be the Green’s function of the Helmholtz operator $\left(1 - \alpha^2 \partial_q^2\right)$: this generates a Vlasov-Helmholtz (VH) equation [6] that is particularly interesting for future work. Connections of this equation with the well known integrable KdV equation have been proposed. However we believe that this is not a natural step since integrability appears already with no further approximations in the Vlasov-Benney system that governs the collective motion of the beam. In particular we would like to understand the VH equation as a special deformation of the integrable Vlasov-Benney case that allows the existence of singular solutions. Indeed, the presence of the Green’s function $G$ above is a key ingredient for the existence of the single-particle solution, which is not allowed in the VB case. In particular, the single-particle solutions for the Vlasov-Helmholtz equation may be of great interest, since these singular solutions arise from a deformation of an integrable system. The Vlasov-Helmholtz solutions may differ considerably from the well known particle behavior of the Vlasov-Poisson case. However, the limit as the deformation parameter (the length-scale $\alpha$) in the Helmholtz Greens function passes to zero ($\alpha \to 0$) in the wake-field equations, one recovers the integrable Vlasov-Benney case.

4.3 The EPDiff equation and singular solutions

Another interesting moment equation is given by the integrable EPDiff equation [5]. In this case, the Hamiltonian is purely quadratic in the first moments:

$$H = \frac{1}{2} \int A_1(q) G(q, q') A_1(q') \, dq \, dq'$$

and the EPDiff equation [16]

$$\frac{\partial A_1}{\partial t} + \frac{\partial A_1}{\partial q} \int G(q, q') A_1(q', t) \, dq' + 2 A_1 \frac{\partial}{\partial q} \int G(q, q') A_1(q', t) \, dq' = 0$$

comes from the closure of the KMLP bracket given by cotangent lifts. (Without this restriction we would obtain again the equations (3.1) with $\beta = G \ast A_1$.) Thus this EPDiff equation is a
geodesic equation on the group of diffeomorphisms. The Camassa-Holm equation is a particular case in which \( G \) is the Green’s function of the Helmholtz operator \( 1 - \alpha^2 \partial_q^2 \). Both the CH and the EPDiff equations are completely integrable and have a large number of applications in fluid dynamics (shallow water theory, averaged fluid models, etc.) and imaging techniques \([19]\) (medical imaging, contour dynamics, etc.).

Besides its complete integrability the EPDiff equation has the important feature of allowing singular delta-function solutions. The connection between the CH and EPDiff equations and the moment dynamics lies in the fact that singular solutions appear in both contexts. The existence of this kind of solution for EPDiff leads to investigate its origin in the context of Vlasov moments. More particularly we wonder whether there is a natural extension of the EPDiff equation to all the moments. This would again be a geodesic (hierarchy of) equation, which would perhaps explain how the singular solutions for EPDiff arise in this larger context.

5 Geodesic motion and singular solutions

5.1 Quadratic Hamiltonians

The previous examples show how quadratic terms in the Hamiltonian produce interesting behaviour in various contexts. This suggests that a deeper analysis of the role of quadratic terms may be worthwhile particularly in connections between Vlasov \( p \)-moment dynamics and the EPDiff equation, with its singular solutions. Purely quadratic Hamiltonians are considered in \([15]\), leading to the problem of geodesic motion on the space of \( p \)-moments.

we are interested in the problem of geodesic motion on the space of \( p \)-moments. In this problem the Hamiltonian is the norm on the \( p \)-moment given by the following metric and inner product,

\[
 h = \frac{1}{2} \| A \|^2 = \frac{1}{2} \sum_{n,s=0}^{\infty} \int \int A_n(q) G_{ns}(q, q') A_s(q') \, dq \, dq'
\]  

(5.1)

The metric \( G_{ns}(q, q') \) in (5.1) is chosen to be positive definite, so it defines a norm for \( \{ A \} \in g^\ast \). The corresponding geodesic equation with respect to this norm is found as in the previous section to be,

\[
 \frac{\partial A_m}{\partial t} = \{ A_m, h \} = -\sum_{n=0}^{\infty} \left( n \beta_n \frac{\partial}{\partial q} A_{m+n-1} + (m+n) A_{m+n-1} \frac{\partial}{\partial q} \beta_n \right)
\]  

(5.2)

with dual variables \( \beta_n \in g \) defined by

\[
 \beta_n = \frac{\delta h}{\delta A_n} = \sum_{s=0}^{\infty} \int G_{ns}(q, q') A_s(q') \, dq' = \sum_{s=0}^{\infty} G_{ns} \ast A_s.
\]  

(5.3)

Thus, evolution under (5.2) may be rewritten as coadjoint motion on \( g^\ast \)

\[
 \frac{\partial A_m}{\partial t} = \{ A_m, h \} = -\sum_{n=0}^{\infty} \text{ad}^*_n A_{m+n-1}
\]  

(5.4)

This system comprises an infinite system of nonlinear, nonlocal, coupled evolutionary equations for the \( p \)-moments. In this system, evolution of the \( n \)th moment is governed by the potentially infinite sum of contributions of the velocities \( \beta_n \) associated with \( n \)th moment sweeping the \( (m+n-1) \)th moment by a type of coadjoint action. Moreover, by equation (5.3), each of the \( \beta_n \) potentially depends nonlocally on all of the moments.
5.2 A geodesic Vlasov equation

Importantly, geodesic motion for the $p$–moments is equivalent to geodesic motion for the Euler-Poincaré equations on the symplectomorphisms (EPSymp) given by the following Hamiltonian

$$H[f] = \frac{1}{2} \int \int f(q,p) G(q,p,q',p') f(q',p') dq dp dq' dp'$$  \hspace{1cm} (5.5)

The equivalence with EPSymp emerges when the function $G$ is written as

$$G(q,q',p,p') = \sum_{n,m} p^n G_{nm}(q,q') p'^m.$$ 

and the corresponding Vlasov equation reads as

$$\frac{\partial f}{\partial t} + \{ f, G \ast f \} = 0$$

where $\{ \cdot, \cdot \}$ denotes the canonical Poisson bracket.

Thus, whenever the metric $G$ for EPSymp has a Taylor series, its solutions may be expressed in terms of the geodesic motion for the $p$–moments. More particularly the geodesic Vlasov equation presented here is nonlocal in both position and momentum and is equivalent to the vorticity equation in two-dimensions and for a particular choice of the metric. However this equation extends to more dimensions and to any kind of geodesic motion, no matters how the metric is expressed explicitly.

5.3 Singular geodesic solutions

We have now clarified the geometric meaning of the moment equations and we can therefore characterize singular solutions, since the geodesic Vlasov equation (EPSymp) essentially describes advection in phase space. Indeed, the geodesic Vlasov equation possesses the single particle solution

$$f(q,p,t) = \sum_j \delta(q - Q_j(t)) \delta(p - P_j(t))$$

which is a well known singular solution that is admitted whenever the phase-space density is advected along a smooth Hamiltonian vector field. This happens, for example, in the Vlasov-Poisson system and in the general wakefield model. On the other hand, these singular solutions do not appear in the Vlasov-Benney equation.

Equation \[5.2\] admits singular solutions of the form

$$A_n(q,t) = \sum_{j=1}^N P_j^n(t) \delta(q - Q_j(t))$$ \hspace{1cm} (5.6)

In order to show this is a solution in one dimension, one checks that these singular solutions satisfy a system of partial differential equations in Hamiltonian form, whose Hamiltonian couples all the moments

$$H_N = \frac{1}{2} \sum_{n,s=0}^{\infty} \sum_{j,k=1}^N P_j^s(t) P_k^n(t) G_{ns}(Q_j(t),Q_k(t))$$
Explicitly, one takes the pairing of the coadjoint equation
\[ \dot{A}_m = - \sum_{n,s} \text{ad}_{G_n}^* A_s A_{m+n-1} \]
with a sequence of smooth functions \( \{ \varphi_m(q) \} \) and finally obtains the equations for \( Q_j \) and \( P_j \) in canonical form,
\[ \frac{dQ_j}{dt} = \frac{\partial H_N}{\partial P_j}, \quad \frac{dP_j}{dt} = - \frac{\partial H_N}{\partial Q_j}. \]

These singular solutions of EPSymp are also solutions of the Euler-Poincaré equations on the diffeomorphisms (EPDiff). In the latter case, the single-particle solutions reduce to the pulson solutions for EPDiff \([5]\). Thus, the singular pulson solutions of the EPDiff equation arise naturally from the single-particle dynamics on phase-space. A similar result also holds in higher dimensions \([15]\).

**Further remarks on singular solutions.** Another kind of singular solution for the moments may be obtained by considering the *cold-plasma solution* of the Vlasov equation
\[ f(q,p,t) = \sum_j \rho_j(q,t) \delta(p - P_j(q,t)) \]
Indeed exchanging the variables \( q \leftrightarrow p \) in the single particle PDF leads to the following expression
\[ f(q,p,t) = \sum_j \psi_j(p,t) \delta(q - \lambda_j(p,t)) \]
which is always a solution of the Vlasov equation because of the symmetry in \( q \) and \( p \). This leads to the following singular solutions for the moments:
\[ A_n(q,t) = \sum_j \int dp \, p^n \psi_j(p,t) \delta(q - \lambda_j(p,t)) \]
At this point, if one considers a Hamiltonian depending only on \( A_1 \) (i.e. one considers the action of cotangent lifts of diffeomorphism), then it is possible to drop the \( p \)-dependence in the \( \lambda \)'s and thereby recover to the singular solutions previously found for eq. (3.1).

### 5.4 Examples of simplifying truncations and specializations.

The problem presented by the coadjoint motion equation \([5,4]\) for geodesic evolution of \( p \)-moments under EPDiff needs further simplification. One simplification would be to modify the (doubly) infinite set of equations in \([5,4]\) by truncating the Poisson bracket to a finite set. These moment dynamics may be truncated at any stage by modifying the Lie-algebra in the KMLP bracket to vanish for weights \( m + n - 1 \) greater than a chosen cut-off value.

For example, if we truncate the sums to \( m, n = 0, 1, 2 \) only, then equation \([5,4]\) produces the coupled system of partial differential equations,
\[ \frac{\partial A_0}{\partial t} = - \partial_q (A_0 \beta_1) - 2A_1 \partial_q \beta_2 - 2 \beta_2 \partial_q A_1 \]
\[ \frac{\partial A_1}{\partial t} = -A_0 \partial_q \beta_0 - 2A_1 \partial_q \beta_1 - \beta_1 \partial_q A_1 - 3A_2 \partial_q \beta_2 - 2 \beta_2 \partial_q A_2 \]
\[ \frac{\partial A_2}{\partial t} = -2A_1 \partial_q \beta_0 - 3A_2 \partial_q \beta_1 - \beta_1 \partial_q A_2 \]
We specialize to the case that each velocity depends only on its corresponding moment, so that \( \beta_s = G * A_s, \ s = 0, 1, 2 \). If we further specialize by setting \( A_0 \) and \( A_2 \) initially to zero, then these three equations reduce to the single equation

\[
\frac{\partial A_1}{\partial t} = -\beta_1 \partial_q A_1 - 2A_1 \partial_q \beta_1.
\]

Finally, if we assume that \( G \) in the convolution \( \beta_1 = G * A_1 \) is the Green’s function for the operator relation

\[
A_1 = (1 - \alpha^2 \partial_q^2) \beta_1
\]

for a constant lengthscale \( \alpha \), then the evolution equation for \( A_1 \) reduces to the integrable Camassa-Holm (CH) equation \([5]\) in the absence of linear dispersion. This is the one-dimensional EPDiff equation, which has singular (peakon) solutions. Thus, after these various specializations of the EPDiff \( p \)-moment equations, one finds the integrable CH peakon equation as a further specialization of the coadjoint moment dynamics of equation \((5.4)\).

That such a drastic restriction of the \( p \)-moment system still leads to such an interesting special case bodes well for future investigations of the EPSymp \( p \)-moment equations. Further specializations and truncations of these equations will be explored elsewhere. Before closing, we mention one or two other open questions about the solution behavior of the \( p \)-moments of EPSymp.

## 6 Open questions for future work

**Emergence of singular solutions.** Several open questions remain for future work. The first of these is whether the singular solutions found here will emerge spontaneously in EPSymp dynamics from a smooth initial Vlasov PDF. This spontaneous emergence of the singular solutions does occur for EPDiff. In fact, integrability of EPDiff in one dimension by the inverse scattering transform shows that only the singular solutions (peakons) are allowed to emerge from any confined initial distribution in that case \([5]\) (this also happens in higher dimensions as it is shown by numerical simulations). In contrast, the point vortex solutions of Euler’s fluid equations (which are isomorphic to the cold plasma singular solutions of the Vlasov-Poisson equation) while comprising an invariant manifold of singular solutions, do not spontaneously emerge from smooth initial conditions. Nonetheless, something quite analogous to the singular solutions is seen experimentally for cold plasma in a Malmberg-Penning trap \([11]\). Therefore, one may ask which outcome will prevail for the singular solutions of EPSymp. Will they emerge from a confined smooth initial distribution, or will they only exist as an invariant manifold for special initial conditions? One might argue that in two dimensions, the EPSymp equation encompasses the equation of vorticity and thus spontaneous emergence of point vortices should not occur. However it is possible that the choice of the metric plays an important role in this matter. Of course, the interactions of these singular solutions for various metrics and the properties of their collective dynamics is a question for future work.

**Possible connections with the Bloch-Iserles system** The EPSymp equation is surprisingly similar in construction to another important integrable geodesic equation on the linear Hamiltonian vector fields (Hamiltonian matrices), which has recently been proposed \([3]\). This is a finite dimensional equation whose dynamical variables are symmetric matrices. Now it has been shown that this system may be written as the geodesic equation on the group of the linear canonical transformations \( \text{Sp}(\mathbb{R}; 2n) \) \([4]\). This association to canonical transformations raises the
question whether it is possible to establish connections with the geodesic Vlasov equation that was introduced here.

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