Abstract

We study the congruence lattices of the multinomial lattices $\mathcal{L}(v)$ introduced by Bennett and Birkhoff [3]. Our main motivation is to investigate Parikh equivalence relations that model concurrent computation. We accomplish this goal by providing an explicit description of the join dependency relation between two join irreducible elements and of its reflexive transitive closure. The explicit description emphasizes several properties and makes it possible to separate the equational theories of multinomial lattices by their dimensions.

In their covering of non modular varieties [16] Jipsen and Rose define a sequence of equations $SD_n(\land)$, for $n \geq 0$. Our main result sounds as follows: if $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$ and $v_i > 0$ for $i = 1, \ldots, n$, then the multinomial lattice $\mathcal{L}(v)$ satisfies $SD_{n-1}(\land)$ and fails $SD_{n-2}(\land)$.

Introduction

Multinomial lattices were introduced in [3] in the context of an order theoretic investigation of rewrite systems associated to common algebraic laws. From this point of view, they form an order theoretic counterpart of the commutativity law.

As a family of finite lattices, multinomial lattices subsume two well known families. The binomial lattices $\mathcal{L}(p)$, $p \in \mathbb{N}^2$, are also known as lattices of lattice-paths, since their elements are paths in the discrete plane from 0 to $p$. Counting properties of paths in the set $\mathcal{L}(p)$ have been intensively investigated, see [22, 18]. Order theoretic properties of $\mathcal{L}(p)$ have been studied in [3, 23]. Among these properties, these lattices are distributive.

The second family of lattices are the permutoedra $\mathcal{Perm}(n)$, $n \geq 0$. Elements of $\mathcal{Perm}(n)$ are permutations on the set $\{1, \ldots, n\}$. It was shown in [15] that this set, endowed with the weak Bruhat order, is a lattice, a result later generalized to all finite Coxeter groups [5]. The lattice structure of $\mathcal{Perm}(n)$ has been deeply investigated as well, see [25, 9, 21, 20, 19, 6].

A multinomial lattice $\mathcal{L}(v)$ has as underlying set the collection of all “discrete” paths from 0 to $v$, where $v = (v_1, \ldots, v_n)$ is a vector in $\mathbb{N}^n$ and $n$, the
dimension, can be an arbitrary positive integer. The paths we consider are discrete in that they add 1 to just one coordinate at each time unit. For this reason we used to refer to multinomial lattices as lattices of paths in higher dimension. These paths are in bijection with words $w$ over an alphabet $\Sigma = \{ a_1, \ldots, a_n \}$ such that the number of letters $a_i$ occurring in $w$ is equal to $v_i$.

It was shown in [3] that $L(v)$, as set, can be endowed with an order structure which turns out to be a lattice structure. If $n$, the dimension, is 2, then $L(v)$ is the usual lattice of lattice paths. Let $1^n$ be the vector in $\mathbb{N}^n$ with just 1's at each coordinate, then $L(1^n)$ is (order isomorphic to) the permutoedron $\mathcal{P}erm(n)$. In this sense, the multinomial lattices are a common generalization of the permutoedra and of lattices of lattice paths.

A main motivation for us to approach multinomial lattices has been investigating Parikh equivalence relations that arise when modeling concurrent computation. If $\Sigma$ is an alphabet, an equivalence relation $\sim$ on $\Sigma^n$ is Parikh if $w \sim u$ implies that the number of occurrences of a letter $\sigma$ in $w$ and in $u$ are the same, for each $\sigma \in \Sigma$. Lattice congruences of $L(v)$ give rise to a class of Parikh's equivalence relations which enjoy a property of interest in concurrency: if $w \sim u$, then we can find a sequence $w = x_0, x_1, \ldots, x_n = u$ such that $x_{i+1}$ is obtained from $x_i$ by switching two contiguous letters. Consequently, our goal has been to understand the congruence lattice of the multinomial lattices $L(v)$.

We accomplish this goal by providing an explicit description of the join dependency relation among join irreducible elements. The explicit description emphasizes several properties of the lattices $L(v)$'s, for example these lattices are bounded. Among the properties a curious one: a sequence of join irreducible elements related by the join dependency relation cannot have length greater than $n-2$, if $v \in \mathbb{N}^n$. This property suffices to separate the equational theories of the $L(v)$'s by dimension. In [16, §4.2] a family of equations $SD_n(\land)$, $n \geq 0$, is introduced, that can be taken as a measure of meet semidistributivity of a finite lattice.\(^1\) Our main result can be phrased as follows: if $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$ and $v_i > 0$ for $i = 1, \ldots, n$, then the multinomial lattice $L(v)$ satisfies $SD_{n-1}(\land)$ and fails $SD_{n-2}(\land)$. For example, assuming that dimensions are not degenerate, lattices of lattice-paths are distributive and not reduced to a point, a lattice $L(v)$ with $v \in \mathbb{N}^3$ is neardistributive but not distributive, and so on.

While the results presented here introduce a notion of dimension for multinomial lattices, it doesn’t appear to exist a direct relation with the order dimension, determined for multinomial lattices in [10].

\(^1\)Analogous considerations and results holds for join semidistributivity and the dual equations $SD_n(\lor)$. 
1 Notation and Background

1.1 Words and functions
With \([n]\) we shall denote the set \(\{1, \ldots, n\}\). Recall that a word over an alphabet \(\Sigma\) is a function \(w : [n] \longrightarrow \Sigma\), \(n\) being the length of the word \(w\). The notations \(w(i)\) and \(w_i\) are therefore equivalent. Similarly, for a permutation \(\sigma : [n] \longrightarrow [n]\), \(\sigma(i) = \sigma_i\). With \((i, j)\) we shall denote the permutation sending \(i\) to \(j\) and viceversa, and fixing all the other members of \([n]\), and \(\sigma^i\) will denote the permutation \((i, i + 1)\).

1.2 Congruences of finite lattices
A lattice is an ordered set with the property that every finite non-empty subset has a least upper bound and a greatest lower bound, see the standard literature on lattices [4, 1, 14, 7]. Let \(x \vee y\) and \(x \wedge y\) denote the least upper bound and the greatest lower bound of the finite set \(\{x, y\}\), respectively. With respect to the binary operations \(\vee\) and \(\wedge\), lattices are algebraic structures, also meaning that the order is equationally definable and determined by the two binary operations.

In this paper we shall be studying finite lattices and, when considering a lattice, we shall assume it is finite unless otherwise stated. It is a standard argument that a finite lattice has a top and a bottom elements. All the structure of a lattice is determined by the order relation restricted to join irreducible elements and meet irreducible elements, see [12]. An element \(j\) is join irreducible if \(j = x \vee y\) implies \(j = x\) or \(j = y\), and an analogous property defines a meet irreducible element. There is an order theoretic characterization of being join/meet irreducible. To this goal, recall that – for an arbitrary ordered set – \(x < y\) is a cover (noted \(x \prec y\) if the closed interval \([x, y]\) has only two elements; then we say that \(x\) is a lower cover of \(y\) and \(y\) is an upper cover of \(x\). An element
A lattice is join irreducible if and only if it has a unique lower cover, which is denoted by \( j \); \( m \) is meet irreducible if it has a unique upper cover \( m^* \).

We recall the definitions/characterizations of standard relations between join/meet, meet/join irreducible elements:

\[
\begin{align*}
 j \not\nearrow m & \iff j \not\leq m \text{ and } j \leq m^*, \\
 m \not\searrow j & \iff j \not\leq m \text{ and } j^* \leq m,
\end{align*}
\]  

(1, 2)

and of the join dependency relation \( D \) between join irreducible elements:

\[
 jDj' \iff j \neq j', j \not\nearrow m \not\searrow j' \text{ for some meet irreducible } m. 
\]  

(3)

The meet dependency relation is defined as expected: \( mD^d m' \iff m \neq m' \), \( m \not\searrow j \not\nearrow m' \) for some \( j \). These relations are central in the theory of finite lattices, as we explain next. Since a lattice is an algebraic structure, we can define a congruence on a lattice \( L \) as an equivalence relation \( \theta \subseteq L \times L \) compatible with the lattice operations, i.e. such that \( x\theta y \) implies \( x \lor z \theta y \lor z \) and \( x \land z \theta y \land z \).

The quotient \( L/\theta \) is then a lattice and the canonical projection is a lattice homomorphism. The following Proposition has motivated us to study the join dependency relation \( D \) and its transitive closure (reflexive and transitive closure) \( \triangleright \) (resp. \( \triangleleft \)) in multinomial lattices.

**Proposition 1.1.** The congruences of a lattice \( L \) are in bijection with subsets \( S \) of join irreducible elements that are closed under the \( D \)-relation. The bijection is an order reversing isomorphism of lattices.

For a proof, the reader may consult [11, \$2.34 and \$11.10]. It is convenient to explicit the bijection. Given a congruence \( \theta \), the \( S_\theta \) is defined as the set of join irreducible elements \( j \) such that \( j \theta j^* \) does not hold, i.e. that are not collapsed with their lower cover under the congruence \( \theta \). The latter can be recovered from \( \theta \), since

\[
x\theta y \text{ if and only if } \{ z \in S_\theta \mid z \leq x \} = \{ z \in S_\theta \mid z \leq y \}. 
\]  

(4)

A lattice \( L \) is semidistributive if the conditions

\[
\begin{align*}
x \land y = x \land z \Rightarrow x \land (y \lor z) &= x \land y \quad (SD(\land)) \\
x \lor y = x \lor z \Rightarrow x \lor (y \land z) &= x \lor y \quad (SD(\lor))
\end{align*}
\]

hold in \( L \). Equivalently, a lattice is semidistributive if for each join irreducible there exists a unique \( m \) such that \( j \not\nearrow m \not\searrow j \) and the dual condition hold, see [11, \$2.6]. We shall write \( \kappa(j) \) for such an \( m \) (and \( \kappa^d(m) \), dually).

A lattice is bounded if it is the quotient of a finitely generated free lattice (with top and bottom which usually is infinite), and the quotient map has both a left and a right adjoints. There are several characterizations of the notion of bounded lattice, see in particular [8]. Among them we shall use the following one, see [11, \$2.53]:

**Lemma 1.2.** A lattice is bounded if and only if it is semidistributive and the join dependency relation \( D \) contains no cycle.
1.3 Lattices of lattice paths, i.e. binomial lattices

We shall denote by $L(n,m)$ the set of words $w$ over the alphabet $\Sigma = \{a, b\}$ such that $|w|_a = n$ and $|w|_b = m$. We represent these words as paths in the 2-dimensional space from $(0,0)$ to $(n,m)$: if $w \in L(n,M)$, then the path $f_w : \{0, \ldots, n+m\} \rightarrow \mathbb{N} \times \mathbb{N}$ is defined by induction as follows:

$$f(0) = (0,0), \quad f(i) = f(i-1) + (1,0) \text{ if } w_i = a \quad \text{and} \quad f(i) = f(i-1) + (0,1) \text{ if } w_i = b,$$

for $i = 1, \ldots, n+m$. The following diagram represents the word/path $abaabb \in L(3,3)$:

```
      b
     a  a
    b  a
 a b
```

The rewrite relation $\rightarrow$ on $L(n,m)$ is defined by:

$$w \rightarrow u \text{ iff } w = w_1abw_2 \text{ and } u = w_1baw_2.$$

This terminating and confluent rewrite system gives rise to an order relation $\leq$ which is a distributive lattice. Indeed, we have $f \leq g$ if and only if $f(i) \leq g(i)$ for $i \in \{0, \ldots, n+m\}$, where the pointwise order is defined as follows:

$$(x_1,y_1) \leq (x_2,y_2) \text{ if and only if } x_2 \leq x_1 \text{ and } y_1 \leq y_2.$$

A join irreducible of $L(n,m)$ is a word of the form $a^x b^y a^{-x} b^{-y}$, where $(\bar{x}, \bar{y}) = (n,m) - (x,y)$, $0 < \bar{x} \leq n$, and $0 < \bar{y} \leq m$. Clearly the latter two conditions are equivalent to $0 \leq x < n$ and $0 < y \leq m$. A join irreducible is therefore uniquely determined by a vector $(x,y)$ with this property, and we shall use the notation $\vee (x,y)$ for $a^x b^y a^{-x} b^{-y}$. In [18] a join irreducible is called a turn to NorthEast (a NE turn). The fact that a path can be described uniquely by its NE-turns corresponds to the lattice theoretic property that an element of a finite lattice is the join of the join irreducible elements below it, among which we can retain the antichain of maximal elements.

Similarly, a meet irreducible has the form $b^y a^x b^\bar{y} \bar{a}$ with $0 < x \leq n$ and $0 \leq y < m$ and is uniquely determined by the vector $(x,y)$: we shall use the notation $\wedge (x,y)$ for $b^y a^x b^\bar{y} \bar{a}$. Observe that

$$\vee (x,y) \leq \wedge (z,w) \text{ iff } z \leq x \text{ or } y \leq w. \quad (5)$$

Since $L(n,m)$ is distributive, it is semidistributive as well. Recalling the definitions (1) and (2) of the relations $\nearrow$ and $\searrow$, we observe that in $L(n,m)$

$$\vee (x,y) \nearrow \wedge (z,w) \text{ iff } \wedge (z,w) \searrow \vee (x,y) \text{ iff } z = x + 1 \text{ and } w = y - 1. \quad (6)$$
For example, $\kappa(aabbab) = (baaabb)$ or $\kappa\vee(2, 2) = \wedge(3, 1)$. The join irreducible $\vee(2, 2)$ and $\kappa\vee(2, 2)$ are represented in following diagram:

In a distributive lattice (and more generally in a modular lattice) it is always the case that $j \not\leq m$ if and only if $m \not\leq j$. Indeed, if $j \not\leq m$, then $m^* = j \vee m$. Since $m \wedge j < j$, $m \wedge j \leq j^*$ and if this inequality is strict, then $j, j^*, m, m^*, j \wedge m$ form a pentagon. Recalling that a distributive lattice is a modular semidistributive lattice, we obtain the following well known consequences:

**Lemma 1.3.** If $L$ is a modular lattice, then the reflexive and transitive closure $\triangleright$ of the join dependency relation is an equivalence relation. Consequently, the congruence lattice of $L$ is a Boolean algebra.

**Lemma 1.4.** If $L$ is a distributive lattice, then the join dependency relation, is empty and there is a bijection between congruences on $L$ and subsets of join irreducible elements of $L$.

We illustrate congruences on the lattice path $L(n, m)$, since they have a strong geometrical appealing. Let us identify the join irreducible element $\vee(x, y)$ with a disk within the interior of the square $[x, x + 1] \times [y - 1, y]$: such a disk represents a hole in the square $[0, n] \times [0, m]$ whose goal is to separate paths from $(0, 0)$ to $(n, m)$. Let us fix a set $S$ of join irreducible elements or holes. The formula (4) that extracts the congruence $\theta$ from the set $S$ can be interpreted by saying that two paths $f, g$ are equivalent if and only if there is no hole in $S$ separating them. In the following diagram, we consider $L(3, 3)$, we let $S = \{ (0, 3), (1, 2) \}$, and draw the resulting three equivalence classes as paths up to dihomotopies [13].
1.4 Lattices of permutations

We review some facts on lattices of permutations \( \mathcal{P}erm(n), n \geq 0 \), usually named permutoedra. Elements of \( \mathcal{P}erm(n) \) are permutations on the set \([n]\) and the order – known as the weak Bruhat order – is defined by means of its covering relation: \( \sigma \prec \sigma' \) iff \( \sigma' = \sigma \circ \sigma^i \) and \( \ell(\sigma) < \ell(\sigma') \), where \( \ell(\sigma) \) is the minimum number \( \ell \) such that \( \sigma \) can be written as the product of \( \ell \) exchanges, \( \sigma = \sigma_{j_1} \circ \ldots \circ \sigma_{j_\ell} \).

Let us define an inversion\(^2\) or disagreement as an (unordered) pair \( \{a, b\} \subseteq [n] \). If \( a < b \), then we denote the inversion \( \{a, b\} \) by \( a \backslash b \). The set of all inversions on the set \([n]\) will be denoted \( I_n \) or simply \( I \) if \( n \) is understood. For a permutation \( \sigma \), define

\[
D(\sigma) = \{ a \backslash b \mid \sigma^{-1}(a) > \sigma^{-1}(b) \}, \quad A(\sigma) = \{ a \backslash b \mid \sigma^{-1}(a) < \sigma^{-1}(b) \}.
\]

The first is the set of inversions or disagreements of \( \sigma \), the latter is the set of its agreements. It is well known that

\[
\sigma \leq \sigma' \text{ iff } D(\sigma) \subseteq D(\sigma') \text{ iff } A(\sigma') \subseteq A(\sigma).
\]

Let us say that a subset of inversions \( X \subseteq I \) is closed if \( a \backslash b \in X \) and \( b \backslash c \in X \) implies \( a \backslash c \in X \).

\( X \subseteq I \) is open if and only if it is the complement of a closed, that is if and only if

\[
a < b < c \text{ and } a \backslash c \in X \text{ implies } a \backslash b \in X \text{ or } b \backslash c \in X.
\]

Finally, \( X \) is clopen if it is closed and open. The following Proposition was used in [25] to prove that the weak Bruhat order gives rise to a lattice.

**Proposition 1.5.** A subset \( X \subseteq I \) is clopen if and only if \( X = D(\sigma) \) for some permutation \( \sigma \).

For \( X \subseteq I \), we shall denote by \( l(X) \) its closure. With \( r(X) \) we shall denote the interior of \( X \), defined by \( r(X) = \neg l(\neg X) \).

**Lemma 1.6.** The closure \( l(X) \) of an open \( X \subseteq I \) is open. The interior \( r(X) \) of a closed \( X \subseteq I \) is closed.

The previous Lemma leads to the following representation of the permutoedron, see [5, 2], and to simple formulas to compute in the permutoedron. We shall often make use of this representation later.

Let \( L \) be the Boolean algebra of subsets of \( I \), \( L_r \) be collection of open subsets, and \( L_l \) be collection of closed subsets and call \( L_{rl} \) the collection of all clopens:

\[
\begin{array}{ccc}
L_r & \longrightarrow & L \\
| \downarrow \lor | & | \downarrow \lor | & | \downarrow \lor |
\end{array}
\begin{array}{ccc}
L_l & \longrightarrow & L \\
| \downarrow \land | & | \downarrow \land | & | \downarrow \land |
\end{array}
\]

\( L_{rl} \). Usually an inversion is the inversion of a given permutation. Here we shall use this name coherently with the usage.

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The diagram on the left is meant to show that $L_r$ is a sub-join-semilattice of $L$ and $L_l$ is a quotient-join-semilattice of $L$; $L_{rl}$ a join-quotient of $L_r,$ and a sub-join-semilattice of $L_l$. We obtain an useful formula for computing the join of two clopens in $L_{rl}$: $X \lor Y = l(X \cup Y)$. The diagram on the right is meant to exemplify the dual notions, with $L_{rl}$ replaced by $L_{lr}$ and join homomorphism replaced by meet-homomorphism. We obtain an useful formula for computing the meet of two clopens in $L_{lr}$: $X \land Y = r(X \cap Y)$. However $L_{rl} = L_{lr}$ is simply the collection of clopens, and in both cases the order is subset inclusion: therefore we have $L_{lr} = L_{rl}$ as lattices.

These formulas provide a method for computing meets and joins of permutations, given in set-of-inversions form. An explicit proof of Proposition 1.5 suggests how to recover the string representation of a permutation from its set-of-inversions representation. We recall that efficient algorithms for computing the meet and the join of two permutations given in string representation were proposed in [21].

2 The Lattices Structure of a Set of Multipermutations

We shall consider paths in the space $\mathbb{N}^n$ from 0 to a fixed point $v$. These paths will have the property that each time step increases just one coordinate. We will denote these paths by words over a totally ordered alphabet $\Sigma = \{a_1, a_2, \ldots, a_n\}$ of directions. If $v = (v_1, \ldots, v_n)$ and $k = v_1 + \ldots + v_n$, then we define

$$L(v) = \{ w \in \Sigma^k \mid |w|_{a_i} = v_i, \text{ for } i = 1, \ldots, n \}.$$  

The set $L(v)$ is the set of multipermutations on $v$. The bijection between multipermutations in $L(v)$ and the paths we are considering takes a $w \in L(v)$ to the path $f_w : \{0, \ldots, k\} \rightarrow \mathbb{N}^n$ defined by $f_w(0) = 0$ and $f_w(i) = f_w(i-1) + e_{w_i}$, where $e_l = (0, \ldots, 1, \ldots 0)$ has just the coordinate $l$ different from 0 and equal to 1.

The rewrite relation $\rightarrow$ on $L(v)$ is defined as follows:

$$w \rightarrow u \text{ iff } w = w_1a_ia_jw_2, \text{ } u = w_1a_ja_iw_2, \text{ and } i < j.$$  

The rewrite relation is confluent and terminating, thus its reflexive and transitive closure is a partial order $\leq$ on $L(v)$. W.r.t. this partial order, the word $a_1^{v_1}a_2^{v_2} \ldots a_n^{v_n}$ is the bottom element and $a_n^{v_n}a_{n-1}^{v_{n-1}} \ldots a_1^{v_1}$ is the top. We have seen that in dimension 2 – that is, for $n = 2$ – this partial order is a distributive lattice. In the general case $n \geq 3$, this partial order is also a lattice, as we are going to argue.

It is harmless to assume that $v_i > 0$ for $i = 1, \ldots, n$, so that a word $w \in L(v)$ is a surjective function $w : [k] \rightarrow [n]$ such that $|w^{-1}(i)| = v_i$ for $i = 1, \ldots, n$.

3We shall often make implicit the assumption that $\Sigma = \{1, \ldots, n\}$. 

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Let \( \mu^* : [k] \rightarrow [n] \) be the only order preserving map with this property and observe that for \( w \in \mathcal{L}(v) \) there exists a unique permutation \( \sigma : [k] \rightarrow [k] \) such that \( w = \mu^* \circ \sigma \) and \( \sigma \) is order preserving on every \( w \)-fiber. We shall denote this permutation by \( \iota(w) \). The following Proposition was proved in [3] and we are thankful to Peter McNamara for independently pointing to us its proof.

**Proposition 2.1.** The function \( \iota \) is an order isomorphism from the ordered set \((\mathcal{L}(v), \leq)\) to a principal ideal of the permutoedron \( \text{Perm}(k) \).

Proposition 2.1 suggests that properties of \( \mathcal{L}(v) \) can be reduced to properties of the permutoedron. For example:

**Corollary 2.2.** The lattice \( \mathcal{L}(v) \) is a bounded lattice.

This follows from [6] and [11, §2.14]. Later, our characterization of congruences of \( \mathcal{L}(v) \) will provide us with another proof of this fact. We can also use Proposition 2.1 to argue that \( w \) is join irreducible in \( \mathcal{L}(v) \) if and only if \( \iota(w) \) is join irreducible in \( \text{Perm}(k) \). It looks unnatural, however, to deduce all the properties of \( \mathcal{L}(v) \) from the representation above. For example, \( \iota(w) \) need not to be meet irreducible even if \( w \) is such. Also, observe that permutoedra are complemented lattices, while \( \mathcal{L}(v) \) is not: for \( n = 2 \), \( \mathcal{L}(v) \) is a distributive lattice without necessarily being a Boolean algebra. We can remark differences with lattices of paths in dimension 2 as well, in particular \( \mathcal{L}(v) \) need not be distributive. Therefore, we seek for a direct understanding of \( \mathcal{L}(v) \), the key idea being the equality

\[
\text{Perm}(k) = \mathcal{L}(1, \ldots, 1) \text{,}
\]

We shall consider the lattice \( \mathcal{L}(v) \) as a generalization of the permutoedron. The first step towards understanding its structure is to find a working analogue of the notion of disagreement/agreement.

**Definition 2.3.** Given \( w \in \mathcal{L}(v) \) and \( 1 \leq l < m \leq n \), we define \( \pi_{l,m}(w) \in \mathcal{L}(v_l, v_m) \) as the word that arises by erasing all the symbols different from \( a_l \) or \( a_m \) (and by identifying the letter \( a_l \) with \( a \) and \( a_m \) with \( b \)).

The formal definition of \( \pi_{l,m} \) is that of a monoid morphism by induction on the length of words.

**Proposition 2.4.** Let \( w, u \in \mathcal{L}(v) \), then \( w \leq u \) if and only if \( \pi_{l,m}(w) \leq \pi_{l,m}(u) \) for all \( l, m \) such that \( 1 \leq l < m \leq n \).

**Proof.** Let us compute \( \iota(w) \) for \( w \in \mathcal{L}(n, m) \): if \( w_{\leq j} \) is the prefix of length \( j \) of \( w \), then

\[
\iota(w)(j) = \begin{cases} 
|w_{\leq j}|_a & \text{if } w_j = a \\
|w_{\leq j}|_b & \text{if } w_j = b .
\end{cases}
\]
Clearly, \( \iota(w) \) is a bijection, if we let \( \mu^* \) be the function sending \( x \) to \( a \) if \( x \leq n \) and to \( b \) otherwise then \( \mu^* \circ \iota(w) = w \), and finally if \( i \leq j \) and \( w_i = w_j \) then \( \iota(w)(i) \leq \iota(w)(j) \).

Observe that \( \iota(w)^{-1}(i) \) is the length of least prefix of \( w \) containing \( i \) a’s if \( w_i = a \), and the length of least prefix of \( w \) containing \( i - n \) b’s if \( w_i = b \). We deduce that \( i \in D(\iota(w)) \) iff the \( (j - n) \)-th occurrence of \( b \) in \( w \) precedes the \( i \)-th occurrence of \( a \) in \( w \), and this happens if and only if the join irreducible \( \vee(i - 1, j) \) is below \( w \). Taking into account this bijection between join irreducible elements below \( w \) and inversions in \( D(\iota(w)) \), we conclude that \( w \leq w' \) if and only if \( D(\iota(w)) \subseteq D(\iota(w')) \).

We consider now \( w \in L(v) \) with \( v \in \mathbb{N}^n \) and \( n \geq 3 \). Consider that the set \( D(\iota(w)) \) is the disjoint union of the sets

\[
D_{l,m}(\iota(w)) = \{ i \in \sigma^{-1}(j) < \sigma^{-1}(j), \mu^*(i) = l, \mu^*(j) = m \},
\]

for \( l, m \) such that \( 1 \leq l < m \leq n \). For \( i \in 1, \ldots, n \) let \( k_i = \sum_{j=1}^{i-1} v_j \). Then \( i \in D_{l,m}(\iota(w)) \) iff \( i - k_i < j - k_m + v_l \in D(\iota(\mu^*_m(w))) \) so that the two sets are in bijection. Therefore

\[
w \leq u \iff D(\iota(w)) \subseteq D(\iota(u)) \iff D_{l,m}(\iota(w)) \subseteq D_{l,m}(\iota(u)) \quad \text{whenever } 1 \leq l < m \leq n,
\]

and, by the bijection, this holds iff \( D(\iota(\pi_{l,m}(w))) \subseteq D_{l,m}(\iota(\pi_{l,m}(u))) \), that is \( \pi_{l,m}(w) \leq \pi_{l,m}(u) \). □

3 Join and Meet Irreducible Elements in \( L(v) \)

A word \( w \in L(v) \) is join irreducible iff there exists a unique \( i \in \{ 1, \ldots, n - 1 \} \) such that \( w_i > w_{i+1} \). Therefore we can write

\[
w = (a_1^{\bar{x}_1} a_2^{\bar{x}_2} \ldots a_n^{\bar{x}_n})(a_1^\bar{x}_1 a_2^\bar{x}_2 \ldots a_n^\bar{x}_n)
\]

where \( \bar{x}_i = v_i - x_i \) for \( i = 1, \ldots, n \). For such join irreducible element \( w \), we let \( x_w \) be the vector \( x_1, \ldots, x_n \), so that \( 0 \leq x_w \leq v \). Now let \( x \) be any vector in the closed interval \([0, v]\) and define

\[
\min_\vee x = \min \{ i \mid x_i < v_i \} \quad \max_\vee x = \max \{ i \mid x_i > 0 \}
\]

where for \( x = v \), we let \( \min_\vee x = \infty \), and for \( x = 0 \) we let \( \max_\vee x = -\infty \). If \( x = x_w \) for a join irreducible element \( w \in L(v) \), then

\[
\min_\vee x < \max_\vee x.
\]

Every join irreducible is uniquely determined by a vector \( 0 \leq x \leq v \) satisfying (8) and we shall use the notation \( \vee x \) for the word \( w \) defined from \( x \) in equation (7). Observe that \( \vee x \) can be defined for vectors \( x \) for which \( \min_\vee x \not< \max_\vee x \), in this case \( \vee x = \bot \).
Definition 3.1. We say that \((\min_{\vee} x, \max_{\vee} x)\) is the principal plan of the join irreducible element \(\vee x\).

Observe that a join irreducible element \(\vee x\) is uniquely determined by the restriction of the vector \(x\) to the closed interval delimited by the principal plan: indeed, \(x_i = v_i\) for \(i < \min_{\vee} x\), and \(x_i = 0\) for \(i > \max_{\vee} x\).

By counting vectors failing (8), the following formula for the number of join irreducible elements in \(L(v)\) was obtained in [3]:

\[
\#\{ w \in L(v) \mid w \text{ is join irreducible} \} = \prod_{i=1}^{n} (v_i + 1) - (1 + \sum_{i=1}^{n} v_i).
\]

Analogous considerations hold for meet irreducible elements: a \(w \in L(v)\) is meet irreducible iff there exists a unique \(i \in \{1, \ldots, n-1\}\) such that \(w_i < w_{i+1}\). We can write

\[
w = (a^x_m a^x_{n-1} \ldots a^x_1)(a^n_m a^n_{n-1} \ldots a^n_1),
\]

and defining

\[
\min_{\wedge} x = \min\{ i \mid x_i > 0 \}, \quad \max_{\wedge} x = \max\{ i \mid x_i < v_i \},
\]

where for \(x = 0\) we let \(\min_{\wedge} x = \infty\) and for \(x = v\) we let \(\max_{\wedge} x = -\infty\), we observe that \(\min_{\wedge} x < \max_{\wedge} x\). Every meet irreducible is uniquely determined by such a vector \(0 \leq x \leq v\) and we use the notation \(\wedge x\) for the \(w\) of (9). Again \(\wedge x\) is well defined even if \(x\) does not satisfy \(\min_{\wedge} x < \max_{\wedge} x\), in this case \(\wedge x = \top\). \((\min_{\wedge} x, \max_{\wedge} x)\) is the principal plan of a meet irreducible \(\wedge x\).

Lemma 3.2. Let \(\vee x \in L(v)\) be join irreducible, and let \((m, M)\) be its principal plan. Then:

- either \(\pi_{i,j}(\vee x)\) is join irreducible, or it is \(\bot\).
- \(\pi_{m,M}(\vee x)\) is join irreducible,
- if \(\pi_{i,j}(\vee x)\) is join irreducible, then \([i, j] \subseteq [m, M]\).

Proof. If \(w\) is the join irreducible of (7), then \(\pi_{i,j}(w) = a^n_i a^n_j a^x_i a^x_j\), and therefore \(\pi_{i,j}(\vee x) = \vee(x_i, x_j)\).

We have \(\pi_{m,M}(\vee x) = \vee(x_m, x_M)\) and by definition \(x_m < v_m\) and \(0 < x_M\).

If \(\vee(x_i, x_j) = \pi_{i,j}(\vee x)\) is join irreducible, then \(x_i < v_i\) and \(0 < x_j\), and hence \(m \leq i\) and \(j \leq M\).

We shall use the characterization of the unique join irreducible \(\wedge x\) such that \(\wedge y \wedge \wedge x \neq \wedge y\) in the distributive lattice \(L(v_i, v_j)\), cf. (6), to characterize the relation \(\vee x \not\leq \wedge y\) in \(L(v_1, \ldots, v_n)\), \(n \geq 3\). We shall use the notation \(x_{i(a,b)}\) for the restriction of the function/vector \(x\) to the open interval \((a, b)\). Hence \(x_{i(a,b)} = x_{i(c,d)}\) iff \(x_i = x_i^x\) for all \(i\) such that \(a < i < b\).
Proposition 3.3. Let \( x \) be join irreducible, \( \wedge y \) be meet irreducible, and let \((a,b)\) and \((c,d)\) be their respective principal plans. The relation \( x \not\succ \wedge y \) holds if and only if \( \pi_{c,d}(x) = \kappa^d(\pi_{c,d}(\wedge y)) \) and \( x_{|c,d} = y_{|c,d} \).

Proof. As a first step we claim that \( x \not\succ \wedge y \) if and only if \( \pi_{c,d}(x) = \kappa^d(\pi_{c,d}(\wedge y)) \) and \( \pi_{i,l}(x) \leq \pi_{i,l}(\wedge y) \) for \((i,l) \neq (c,d)\).

Observe that \( \pi_{c,d}(\wedge y^*) = \pi_{c,d}(\wedge y)^* \) and \( \pi_{i,l}(\wedge y^*) = \pi_{i,l}(\wedge y) \) if \((i,l) \neq (c,d)\). Therefore, the relation \( \nu x \leq \wedge y \) implies \( \pi_{c,d}(\nu x) \leq \pi_{c,d}(\wedge y) \); if \( \pi_{c,d}(\nu x) \leq \pi_{c,d}(\wedge y) \) we would have \( \nu x \leq \wedge y \); hence \( \pi_{c,d}(\nu x) \not\leq \pi_{c,d}(\wedge y) \); overall we obtain \( \pi_{c,d}(\nu x) = \kappa^d(\wedge y) \), i.e. \( x_c = y_c - 1 \) and \( x_d = y_d + 1 \).

As a second step, we claim that the condition \( \pi_{c,d}(\nu x) = \kappa^d(\pi_{c,d}(\wedge y)) \) and \((i,l) \neq (c,d)\) implies \( \pi_{i,l}(\nu x) \leq \pi_{i,l}(\wedge y) \) is equivalent to the condition \( \pi_{c,d}(\nu x) = \kappa^d(\pi_{c,d}(\wedge y)) \) and \((x_{|c,d}) = y_{|c,d}) \).

The condition is necessary. Let \( i \in (c,d) \), then the relation \( \nu(x_c, x_i) = \pi_{c,i}(\nu x) \leq \pi_{c,i}(\wedge y) = \wedge (y_c, y_i) \) holds and is equivalent to \( y_c \leq x_c = y_c - 1 \) or \( x_i \leq y_i \); we deduce \( x_i \leq y_i \). Similarly, we deduce \( y_i \leq x_i \) from \( (x_i, x_d) \leq \wedge (y_i, y_d) \) and therefore \( y_i = x_i \).

The condition is sufficient. We only need to prove that \((i,l) \neq (c,d)\) implies \( \pi_{i,l}(\nu x) \leq \pi_{i,l}(\wedge y) \). If \( i < c \) or \( d < l \), then \( \pi_{i,l}(\wedge y) = \top \), see Lemma 3.2. We suppose therefore that \( c \leq i < l \leq d \) with \((i,j) \neq (c,d)\), for example with \( c < i \). Then \( x_i = y_i \) which is enough to ensure the relation \( \pi_{i,l}(\nu x) = \nu(x_i, x_d) \leq \wedge (y_i, y_d) = \pi_{i,l}(\wedge y) \). \( \square \)

The following consequence of Proposition 3.3 is worth remarking:

Corollary 3.4. Let \( \nu x \) be join irreducible, \( \wedge y \) be meet irreducible, and let \((a,b)\) and \((c,d)\) be their respective principal plans. If \( \nu x \not\succ \wedge y \), then \( a < c < d < b \).

The principal plan of \( \wedge y \) is contained in the principal plan of \( \nu x \); the relation \( \pi_{c,d}(\nu x) = \kappa^d(\pi_{c,d}(\wedge y)) \) implies that \( \pi_{c,d}(\nu x) \) is join irreducible, and therefore \([c,d] \subseteq [a,b]\) by Lemma 3.2.

We rephrase explicitly Proposition 3.3 as follows:

\( \nu x \not\succ \wedge y \) if and only if \( y_c = x_c + 1, y_d = x_d - 1 \), and \( x_i = y_i \) for \( i \in (c,d) \).

Dually, \( \wedge y \not\prec \nu x \) holds if and only if \( \pi_{a,b}(\wedge y) = \kappa(\pi_{a,b}(\nu x)) \) and \( y_{|a,b} = x_{|a,b} \), i.e. \( x_a = y_a + 1, x_b = y_b + 1 \), and \( x_i = y_i \) for \( i \in (a,b) \).

Corollary 3.5. The lattice \( \mathcal{L}(v) \) is semidistributive.

Proof. If \( \nu x \not\succ \wedge y \not\succ \nu x \), then \( \nu x \) and \( \wedge y \) have the same principal plan \((a,b)\), \( y_{|a,b} = x_{|a,b} \), \( x_a = x_a + 1 \) and \( y_b = x_b - 1 \), \( y_i = v_i \) for \( i < a \), and \( y_i = v_i \) for \( i > b \). These conditions uniquely determine a vector \( y \) for which \( \min y = a \) (since \( y_a = x_a + 1 > 0 \) ) and similarly \( \max y = b \). Hence \( \wedge y = \kappa(\nu x) \) is the unique meet irreducible with the property that \( \nu x \not\succ \wedge y \not\prec \nu x \).

Similarly, a join irreducible \( \nu x = \kappa^d(\wedge y) \) such that \( \wedge y \not\prec \nu x \not\succ \nu y \) is uniquely determined.

It is easily verified that \( \wedge y = \kappa(\kappa^d(\wedge y)) \) and \( \nu x = \kappa^d(\kappa(\nu x)) \). Therefore \( \mathcal{L}(v) \) is a semidistributive lattice. \( \square \)
Definition 3.6. If $\vee x, \vee z$ are two join irreducible elements, of respective principal plans $(a, b)$ and $(e, f)$, then we say that $\vee x \leq^\bullet \vee z$ if and only if

- $[e, f] \subseteq [a, b]$ and $z_{(e, f)} = x_{(e, f)}$,
- $z_e = x_e - d_e$ where $d_e \in \{0, 1\}$ and $d_e = 0$ if $e = a$,
- $z_f = x_f + d_f$ where $d_f \in \{0, 1\}$ and $d_f = 0$ if $f = b$.

Proposition 3.7. For two join irreducible elements $\vee x, \vee z$, there exists a meet irreducible $\wedge y$ such that $\vee x \nvdash \wedge y$ and $\wedge y \nvdash \vee z$ if and only if $\vee x \leq^\bullet \vee z$.

Proof. The condition is necessary. Let $(c, d)$ be the principal plan of $\wedge y$, then $[e, f] \subseteq [c, d] \subseteq [a, b]$ and similarly $z_{(e, f)} = y_{(e, f)} = (y_{(a,b)})(e, f) = (x_{(a,b)})(e, f) = x_{(e, f)}$.

Let us consider $x_e, y_e, z_e$ and suppose first that $a < c$: if $c < e$, then $z_e = y_e - 1$ and if $c = e$ then $z_e = y_e - 1$ and $y_e = x_e + 1$ imply $z_e = x_e$. Similarly, if $e = c = a$, then $z_e = y_e - 1$ and $y_e = x_e + 1$ imply $z_e = x_e$.

The condition is sufficient. To this goal, we need to define a vector $y$ such that $\vee x \nvdash \wedge y \nvdash \vee z$. If $(c, d)$ is the principal plan of $\wedge y$, then $y$ is determined by the condition $y_{([c, d])} = x_{([c, d])}$, $y_e = x_e + 1$ and $y_d = x_d - 1$. Thus we only need to define the principal plan $(c, d)$ of $\wedge y$, which we do according to four possible cases:

1. $z_e = x_e$ and $z_f = x_f$: we let $(c, d) = (e, f)$,
2. $a < e$ and $z_e = x_e - 1$ and $z_f = x_f$: we let $(c, d) = (a, f)$,
3. $z_e = x_e$, $f < b$ and $z_f = x_f + 1$: we let $(c, d) = (e, b)$,
4. $a < e < f < b$, $z_e = x_e - 1$ and $z_f = x_f + 1$: we let $(c, d) = (a, b)$.

Thus we see that $\vee x D \vee z$ if and only if $x \neq y$ and $\vee x \leq^\bullet \vee z$. The relation $\leq^\bullet$ is clearly antisymmetric, from which we see that if $\vee x D \vee z$, then $[e, f] \subseteq [a, b]$, where $(a, b)$ and $(e, f)$ are the principal plans of $\vee x$ and $\vee z$, respectively. Therefore the $D$-relation contains no cycle and by Lemma 1.2 we obtain:

Corollary 3.8. $\mathcal{L}(v)$ is a bounded lattice.
Corollary 3.9. For two join irreducible elements \( x \) and \( z \), the pair \((x, x_*)\) belongs to the congruence \( \theta_0(x, x_*) \) if and only \( x \leq^* x_0 \).

Indeed, from what we have seen, the relation \( \leq^* \) and the reflexive and transitive closure \( \leq^* \) of the join dependency relation coincide, and it is a general fact for finite lattices that \((x, x_*) \in \theta_0(x, x_*)\) if and only \( x \leq^* x_0 \).

It should also be observed that the explicit description of the relation \( \leq^* \) suffices to compute the dimension monoid of a lattice \( \mathcal{L}(v) \). According to [24] this is the commutative monoid generated by join irreducible elements \( j \) and subject to the relations \( j + k = j \) whenever \( k \leq j \).

When \( v = 1^a \), that is when \( \mathcal{L}(v) = \mathcal{P} erm(n) \), a join irreducible element \( \sigma \) is uniquely described by its principal plan \((a, b)\) and by the subset \( D_\sigma \) of the open interval \((a, b)\) of disagreements of \( a \), \( D_\sigma = \{i \in (a, b) | a \setminus \sigma \in D(\sigma)\} \). In vector notation, if \( \sigma = x \), then \( D_\sigma = \{i \in (a, b) | x_i = 1\} \). Taking the triple \((a, b, D_\sigma)\) as a representation of a join irreducible, we have

**Corollary 3.10.** Let \((a, b, D_a)\) and \((c, d, D_c)\) be two distinct join irreducible elements of \( \mathcal{P} erm(n) \). Then \((a, b, D_a) D(c, d, D_c)\) if and only if \((c, d) \subseteq (a, b)\) and \( D_c = D_a \cap (c, d) \).

Finally, for computational purposes, we study covers of the reflexive transitive closure \( \leq^* \) of the join dependency relation. In a semidistributive lattice, every cover of the \( D \)-relation is either of type \( A \) or of type \( B \). We recall that \( j_1 A j_2 \) if and only if \( j_1 \not\supseteq j_2 \) and \( j_1 \neq j_2 \), and that \( j_1 B j_2 \) if and only if \( j_1 \setminus j_2 \) and \( j_1 \neq j_2 \). We refer the reader to [11, §2.58] for a general background on these relations. In the following Lemma \( d(x, y) = 1 \) if \( x \neq y \) and \( d(x, y) = 0 \) of \( x = y \).

**Lemma 3.11.** Let \( x, z \) be join irreducible elements of \( \mathcal{L}(v) \), with respective principal plans \((a, b)\) and \((e, f)\). Then:

- \( \vee x Ax \) if \( (e, f) \subset (a, b) \) and \( x_{(e, f)} = y_{(e, f)} \),
- \( \vee x By \) if \( (e, f) \subset (a, b) \), \( x_{(e, f)} = y_{(e, f)} \), \( z_e = x_e - d(a, e) \), and \( z_f = x_f + d(b, f) \).

It easily seen that if \( \vee x Dz \) with \( a < e \) and \( f < h \), then \( \vee x Dz y Dz \), where \( y \) is characterized by having principal plan \((e, b)\) or by having principal plan \((a, f)\). We say in the first case that \( \vee x Dz \) is a left move, and in the second case that it is a right move. Say that the width of a join irreducible element is the distance between the two coordinates forming the principal plan. Our next goal is to show that left moves can be factorized through a sequence of left moves \( \vee x Dz \) that decrease the width of the respective principal plans by one. Clearly, an analogous result holds for right moves.

**Lemma 3.12.** Let \( x, z \) be join irreducible elements of principal plans \((a, b)\) and \((e, f)\). If \( \vee x Dz \) with \( a + 1 < e \), then there exists a join irreducible element \( y \), of principal plan \((a + 1, b)\), such that \( \vee x Dz y Dz \).
Proof. If $x_{a+1} = v_{a+1}$, then we let $y_{a+1} = x_{a+1} - 1$, otherwise, we let $y_{a+1} = x_{a+1}$.

**Corollary 3.13.** The set of join irreducible elements of $\mathcal{L}(v)$ ordered by the relation $\succeq$ is a graded poset.

**Corollary 3.14.** Every $D$-path of join irreducible elements in $\mathcal{L}(v)$, $v \in \mathbb{N}^n$, has length at most $n - 2$. If $v_i > 0$ for $i = 1, \ldots, n$, then such length is realized.

Proof. If $x \triangledown D \triangledown y$, then the principal plan of $\triangledown y$ is strictly contained in the principal plan of $\triangledown x$. Conversely, consider the word $a_n^n a_1^1 a_2^2 \ldots a_{n-1}^{n-1}$. Permuting $a_i^i$ with $a_i^i$, $i = 1 \ldots n - 2$ gives a $D$-chain of $n - 2$ elements.

We accomplish our analysis of covers by classifying them in 4 categories. A lower cover $x \triangledown D \triangledown z$ can be $LA$, a left move of type $A$, $LB$, a left move of type $B$, $RA$, a right move of type $A$ and $RB$, a right move of type $B$. We include next some automatically generated examples. The first diagram illustrates the join dependency relation of $\mathcal{P}erm(4)$, which should be compared with Figure 10 in [9].

The following diagram illustrates the join dependency relations for $\mathcal{L}(2,1,1)$ and $\mathcal{L}(1,2,1)$:

The next diagram represents the $D$-relation in $\mathcal{L}(1,1,2,1,1)$:

Finally, the join dependency relation in $\mathcal{L}(2,2,1,1)$ shows that while a join irreducible element can have at most 4 $D$-lower covers, it may have more than
4 upper $D$-covers.

The exact number of upper $D$-covers clearly depends on the multiplicities in the vector $v$.

4 Dimension Equations for Multinomial Lattices

4.1 Pentagons

For a quotient $a/b$ in a lattice we mean a pair of elements $a, b$ such that $a \leq b$. We shall say that a quotient is prime if $a \prec b$, and write $a/b \subseteq c/d$ for $d \leq b \leq a \leq c$.

For a pentagon in a lattice $L$ we mean a triple of elements $a, b, c \in L$ such that $b \leq a$, $a \lor c = a \lor b$, and $a \land c = a \land b$.

We denote such a triple by $N(a/b, c)$ and say that $a/b$ is the central quotient of the pentagon. $N(a/b, c)$ is non degenerate if $b < a$. In a pentagon $N(a/b, c)$, let $1_{N(a/b, c)} = a \lor c = b \lor c$ and $0_{N(a/b, c)} = a \land c = b \land c$ (we shall omit the subscripts if the underlying pentagon $N(a/b, c)$ is understood). As usual, we say that two quotients $x/y$ and $z/w$ transpose to each other if $x = y \lor z$ and $w = y \land z$ or viceversa. We denote this relation by $\sim$, so that $x/y \sim z/w$. In a pentagon $N(a/b, c)$, we have $c/0 \sim 1/b$ and $c/0 \sim 1/a$ (and dually $1/c \sim a/0$ and $1/c \sim 1/a$). Hence, if a pentagon $N(a/b, c)$ is non degenerate, then $\{a, c\}$ and $\{b, c\}$ are antichains: if $c \leq a$ then $c = c \land a = 0$ hence $b \leq 1 = a$, and if $b \leq c$, then $c = b \lor c = 1$, hence $b = 0 \leq a$.

Lemma 4.1. Let $N(a/b, c)$ be a pentagon of a finite lattice and $a'/b' \subseteq a/b$ be a prime quotient. Then we can find a prime quotient $x/y \subseteq b/0$ such that $(a, b) \in \theta(x, y)$.

Proof. We have $(a', b') \in \theta(a, 0) = \theta(1, c) = \theta(b, 0)$, hence $\theta(a', b') \subseteq \theta(b, 0)$. Moreover

$$\theta(0, b) = \bigvee_{b \geq x, y \geq 0} \theta(x, y)$$

and since $\theta(a', b')$ is join prime in the lattice of congruences of $L$, $\theta(b', a') \subseteq \theta(x, y)$ – that is, $a'/b' \in \theta(x, y)$ – for a prime quotient $x/y \subseteq b/0$.

Lemma 4.2. Let $L$ be a lattice and $x/y$ be a prime quotient. Then we can find a join irreducible $j$ such that $j/j* \sim x/y$.
Proof. Consider the set of elements \( z \) such that \( z \lor y = x \). This set is nonempty, let \( j \) be minimal in this set. Observe that if \( z \) is a lower cover of \( j \), then \( y \leq z \lor y \leq x \) and, by minimality, \( z \lor y = y \). It follows that \( j \) is join irreducible, since if \( z_1, z_2 \) are distinct lower covers, then \( z_1 \lor y = z_1 \lor z_2 \lor y = y \), contradicting \( j \lor y = x \).

Let \( j_\ast \) be the unique lower cover of \( j \), we have seen that \( j_\ast \leq y \). It follows that \( j_\ast \leq j \land y \leq j \). Since \( j \nleq y \), it follows that \( j_\ast = j \land y \), showing that the covers \( j/j_\ast \) and \( x/y \) transpose to each other. \( \square \)

**Corollary 4.3.** A lattice \( L \) contains a non degenerate pentagon, then the \( D \) relation is non empty.

**Proof.** Let \( N(a/b,c) \) be a non degenerate pentagon in \( L \), chose a prime quotient \( a'/b' \subseteq a/b \), and use Lemma 4.1 to find a prime quotient \( x/y \subseteq b/0 \) such that \( (a',b') \in \theta(x,y) \). Hence, chose join irreducible elements \( j, j' \) such that \( j/j' \sim a'/b' \) and \( j'/j' \sim x/y \). We have therefore \( (j, j_\ast) \in \theta(j'/j'_\ast) \), i.e. \( j \leq j' \). Notice finally that \( j \neq j' \), since \( j' \leq x \leq b' \) but \( j \nleq b' \), showing that \( j < j' \), hence the join dependency relation \( D \) is not empty. \( \square \)

Together with Lemma 1.4 we obtain a characterization of distributive lattices.

**Corollary 4.4.** A finite lattice is distributive iff the join dependency relation, as well as the dual \( D^d \) are empty.

**Proof.** Since distributivity is autodual, Lemma 1.4 implies that \( D \) and \( D^d \) are empty in a distributive lattice. Conversely, observe that, given \( j \), we can always find an \( m \) and a \( j' \) such that \( j \ntriangleright m \nleq j' \). Then \( j = j' \) since \( D \) is empty, and such an \( m \) is unique since \( D^d \) is empty. The given lattice is therefore semidistributive, modular by Corollary 4.3, hence it is distributive. \( \square \)

### 4.2 Meet semidistributivity at \( n \) and the \( D \)-relation

Recall that a (possibly infinite) lattice \( L \) is meet semidistributive if the relation \( SD(\land) \) holds in \( L \). We are going to investigate such a condition within finite lattices. To this goal, let \( x, y, z \) be three variables, and define two sequences of terms as follows:

\[
\begin{align*}
y_0 &= y \\
y_{n+1} &= y \lor (x \land z_n) \\
z_0 &= z \\
z_{n+1} &= z \lor (x \land y_n). 
\end{align*}
\]

For each \( n \geq 0 \), the equation \( SD_n(\land) \) is

\[
x \land y_n = x \land (y \lor z). \quad SD_n(\land)
\]

If a lattice satisfies \( SD_n(\land) \), then we say that it is **meet semidistributive at \( n \)**. For an \( SD_n(\land) \)-failure we mean a tuple \( (L, x, y, z) \), where \( L \) is a lattice, \( x, y, z \in L \), and \( x \land y_n < x \land (y \lor z) \).

\( ^4 \)Clearly, we are overloading notation, since we should make clear when we are dealing with terms, and when we are dealing with their interpretations in a given lattice.
We explicit some properties of the sequences \( y_n, z_n \).

**Lemma 4.5.** For \( n, m \geq 0 \)

\[
y_n \leq y_{n+1}, \quad z_n \leq z_{n+1}, \quad \text{and} \quad y \lor z = y_n \lor z_m.
\]

**Proof.** Clearly \( y_0 \leq y_1 \) and \( z_0 \leq z_1 \). If \( y_n \leq y_{n+1} \) and \( z_n \leq z_{n+1} \), then \( y_{n+1} = y \lor (x \land z_n) \leq y \lor (x \land z_{n+1}) = y_{n+2} \), and similarly \( z_{n+1} \leq z_{n+2} \).

For \( y \lor z = y_n \lor z_m \), observe that the previous property implies \( y \lor z \leq y_n \lor z_m \), and similarly, \( z_n \leq y \lor z \). Therefore \( y_n \leq y \lor z \) and \( z_m \leq y \lor z \) for each \( n, m \geq 0 \), implying \( y_n \lor z_m = y \lor z \). \( \square \)

**Fact 4.6.** If \( SD_\infty(\land) \) holds, then \( SD_k(\land) \) holds for \( k \geq n \).

Indeed, we can use the previous Lemma to see that if \( SD_\infty(\land) \) holds, then \( x \land (y \lor z) = x \land y_n \leq x \land y_{n+1} \) while \( x \land y_{n+1} \leq x \land (y \lor z) \) holds for each \( n \).

**Fact 4.7.** A finite lattice is meet semidistributive if and only if it is meet semidistributive at \( n \) for some \( n \geq 0 \).

In [16, §4.2] it is shown that a semidistributive lattice generates a variety whose lattices are all meet semidistributive if and only if one among the equations \( SD_\infty(\land) \) holds. It is straightforward to generalize the argument to obtain the above statement. To this goal, let us define

\[
x_k = (x \land y_{k-1}) \lor (x \land z_{k-1}), \quad \text{for } k \geq 1.
\]

Let \( x, y, z \) be fixed elements of a meet semidistributive lattice \( L \), we claim that if \( y_k = y_{k+1} \) and \( z_k = z_{k+1} \), then \( (L, x, y, z) \) is not an \( SD_{k+1}(\land) \)-failure. Indeed

\[
y_k \land x = y_{k+1} \land x \geq x_{k+1} = (y_k \land x) \lor (y_k \land x) \geq y_k \land x.
\]

Therefore \( y_k \land x = x_{k+1} \) and similarly \( z_k \land x = x_{k+1} \). Hence \( y_k \land x = z_k \land x = x_{k+1} \) and therefore

\[
(y \lor z) \land x = (y_k \lor z_k) \land x = x_{k+1} \leq y_{k+1}.
\]

Therefore if \( L \) is finite and \( x, y, z \in L \) we can let \( \mu(x, y, z) \) be the least integer for which \( y_{n-1} = y_n \) and \( z_{n-1} = z_n \). Then \( L \) satisfies \( SD_M(\land) \) where \( M = \max\{\mu(x, y, z) \mid (x, y, z) \in L^3\} \).

The following Lemma shows that, when figuring out the configuration given by \( x \) and the sequences \( y_n \) and \( z_n \), we can assume that \( x \leq y \lor z \). While such representation has some interest for heuristics, it plays no role in the following exposition.

**Lemma 4.8.** Let \( x, y, z \) be given, let \( \bar{x} = x \land (y \lor z) \), and define the sequences \( \tilde{y}_n, \tilde{z}_n \) consequently out of the triple \( \bar{x}, y, z \). Then \( \tilde{y}_n = y_n, \tilde{z}_n = z_n, x \land y_n = \bar{x} \land y_n, \) and \( x \land z_n = \bar{x} \land z_n \).

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Proof. The relation $\tilde{y}_n = y_n$ and $\tilde{z}_n = z_n$ hold for $n = 0$. By induction,

$$\tilde{x}_n \land \tilde{y}_n = x \land (y \lor z) \land y_n = x \land y_n$$

since $y_n \leq y \lor z$, hence $\tilde{z}_{n+1} = z_{n+1}$.

**Proposition 4.9.** Let $x_k, k \geq 1$ be defined as in (10). Then

$$x \land y_{k-1} \leq x_k \leq x \land y_k \leq y_k$$

and moreover $x \land y_k/x_k$ and $y_{k-1}$ form a possibly degenerate pentagon, with $x \land y_{k-1} = 0$ and $y_k = 1$.

**Proof.** Clearly $x \land y_{k-1} \leq x_k$ and $x_k \leq x$. To argue that $x_k \leq y_k$, observe that actually $y \lor x_k = y_k$:

$$y \lor x_k = y \lor (x \land y_{k-1}) \lor (x \land z_{k-1}) = y \lor (x \land z_{k-1}) \lor (x \land y_{k-1}) = y \lor (x \land y_{k-1}) = y_k.$$  

It follows that $y_{k-1} \lor x_k = y_k$, and of course $(x \land y_k) \lor y_{k-1} = x_k \lor y_{k-1} = y_k$, since $y_{k-1} \leq y_k$. 

Thus, for $k \geq 1$, we let $Y_k = N(x \land y_k/x_k, y_{k-1})$. Since the roles of $y$ and $z$ are symmetric, $x \land z_k/x_k$ and $z_{k-1}$ from a possibly degenerate pentagon which we shall denote by $Z_k$.

The pentagons $Y_k, Z_k$ might be degenerate, however, they respect precise patterns:

**Lemma 4.10.** If $Z_k$ is non degenerate, then $Y_{k-1}$ is non degenerate.

The Lemma is best proved with the following diagram at hand:

![Diagram of pentagons](image)

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Suppose that $Y_{k-1}$ is degenerate, i.e. that $x_{k-1} = x \wedge y_k$. Observe that $x_k/0_{Z_k}$ weakly projects down to $x \wedge y_{k-1}/x_{k-1}$, i.e.
\[
x_k = (x \wedge y_{k-1}) \lor (x \wedge z_{k-1}) ,
\]
\[
x_{k-1} = (x \wedge y_{k-2}) \lor (x \wedge z_{k-2}) \leq x \wedge z_{k-1} = 0_{Z_k} .
\]
Therefore, if the quotient $x \wedge y_k/x_{k-1}$ collapses, then $x_k = 0_{Z_k}$. Since $x_k/0_{Z_k} \sim 1_{Z_k}/z_{k-1}$ and $1_{Z_k}/z_{k-1} \sim x \wedge z_k/0$, then $x \wedge z_k = x_k$.

**Lemma 4.11.** Let $x, y, z$ be an $SD_n(\wedge)$-failure in a meet semidistributive lattice $L$. Then, either $Y_k$ is non degenerate, or $Z_k$ is non degenerate, for $1 \leq k \leq n$.

**Proof.** If $x_k = x \wedge y_k$ and $x_k = x \wedge z_k$, then $x \wedge y_k = x \wedge z_k$ implying $x \wedge (y \lor z) = x \wedge y_k$. Therefore $SD_k(\wedge)$ holds and $k > n$.

Consequently, if $1 \leq k \leq n$, then either $x_k < x \wedge y_k$, or $x_k < x \wedge z_k$.

**Proposition 4.12.** Let $L$ be a finite meet semidistributive lattice such that $SD_n(\wedge)$ fails in $L$. Then $L$ contains a simple $D$-path of length $n$.

**Proof.** Let $(L, x, y, z)$ be an $SD_n(\wedge)$-failure, and define the pentagons $Y_k, Z_k$ as usual. We have seen that either $Y_n$ is non degenerate, or $Z_n$ is non degenerate. We shall suppose that $Y_n$ is non degenerate, so that $Y_{n-2k}$ is non degenerate, and $Z_{n-2k+1}$ is non degenerate, by Lemma 4.10.

Since $Y_n$ is non degenerate, then we can chose a prime quotient $u_n/v_n$ contained in $x \wedge y_n/x_n$, the central quotient of the pentagon $Y_n$.

Suppose that we have constructed a sequence of prime quotients
\[
u_n/v_n, u_{n-1}/v_{n-1}, \ldots, u_k/v_k
\]
where $u_i/v_i \leq x \wedge y_i/x_i$ for $i$ even and otherwise $u_i/v_i \leq x \wedge z_i/x_i$; moreover $(u_i/v_i) \in \theta(u_{i-1}, v_{i-1})$ for $i = n, \ldots, k + 1$.

If $k > 0$, then we extend the sequence as follows. Since the roles of $Y_i$ and $Z_i$ are symmetric, we shall suppose that $u_k/v_k$ belongs to the central quotient of $Z_k$, i.e. $u_k/v_k \leq x \wedge z_k/x_k$. By Lemma 4.1 there is a prime quotient $u'/v' \subseteq x_k/x_k \wedge z_{k-1}$ such that $(u_k/v_k) \in \theta(u'/v')$.

If $k = 1$, then we let $u_0/v_0 = u'/v'$. Otherwise, recall that the quotient $x_k/x_k \wedge z_{k-1}$ weakly projects down to $x \wedge y_{k-1}/x_{k-1}$. Consequently, $(u', v') \in \theta(x \wedge y_{k-1}/x_{k-1})$, hence as in Lemma 4.1 we can find a prime quotient $u_{k-1}/v_{k-1} \subseteq x \wedge y_{k-1}/x_{k-1}$ such that $(u', v') \in \theta(u_{k-1}, v_{k-1})$. Since $(u_k, v_k) \in \theta(u'/v')$, we have $(u_k, v_k) \in \theta(u_{k-1}/v_{k-1})$ as well.

Finally, for each prime quotient $(u_i/v_i)$ let $j_i$ be a join irreducible element such that $j_i \lor v_i = x_i$ and $j_i \leq v_i$, so that $j_i/j_{i-1} \sim u_i/v_i$. Since $(j_i, j_{i-1}) \in \theta(j_i-1, j_{i-1})$, we have $j_i \leq j_{i-1}$.

Also observe that $j_i \neq j_k$ for $i \neq k$. Indeed, let us suppose that $i < k$, then $j_i \leq u_i \leq (x \wedge y_i) \lor (x \wedge y_k) = x_{i+1} \leq x_k$ while $j_k \leq u_k$ implies $j_k \leq x_k$. In particular we have $j_i < j_{i-1}$ for $i = 1, \ldots, n$.

**Corollary 4.13.** If $L$ is a meet distributive lattice whose maximal length of a simple $D$-path is $k$, then $SD_n(\wedge)$ holds in $L$ for $n > k$. 

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Corollary 4.14. If $L$ is a finite bounded lattice whose maximal length of a $D$-path is $k$, then $SD_n(\wedge)$ holds in $L$ for $n > k$.

Corollary 4.15. For $v \in \mathbb{N}^n$, the multinomial lattice $\mathcal{L}(v)$ is meet semidistributive at $k \geq n - 1$.

Indeed, the longest $D$-chain in $\mathcal{L}(v)$ has length at most $n - 2$. We shall show later that this measure is tight, meaning that multinomial lattices $\mathcal{L}(v)$ fail to be meet semidistributive at $n - 2$ provided that $v_i > 0$ for $i = 1, \ldots, n$.

Finally, we exemplify the kind of problems arising when analysing further Proposition 4.12.

It could seem natural generalize Corollary 4.14 to non bounded lattice by the following statement statement “if every chain of join irreducible elements in the congruence lattice of a meet semidistributive $L$ has at most $k$ elements, then $SD_n(\wedge)$ holds in $L$ for $n \geq k$”. The statement does not hold: let us consider $\mathcal{JN}$, see [17], the canonical example of a non bounded semidistributive lattice, and its join dependency graph:

![Diagram of the join dependency graph of JN](image)

It is easily argued that every chain of join irreducible elements in the congruence lattice of $\mathcal{JN}$ has at most 2 elements. On the other hand, $SD_2(\wedge)$ fails in $\mathcal{JN}$: let $x = p_0$, $z = q$, and $y$ as in the diagram, then $x \wedge y_2 = p_1 < p_0 = x = x \wedge (y \vee z)$.

The following example exemplifies the problems found when trying to find some converse to Proposition 4.12:

![Diagram of a partial order](image)
Even if this bounded lattice contains simple $D$-paths of length 3 (and simple $A$-paths of length 2) we cannot construct $SD_n(\land)$ failures out of these paths. Actually, this lattice satisfies $SD_2(\land)$.

### 4.3 Height of meet semidistributivity for $\mathcal{L}(v)$

Let us say that the dimension of $\mathcal{L}(v)$ is the number of indexes $i$ such that $v_i > 0$. Every lattice $\mathcal{L}(v)$ of dimension $n$ is clearly isomorphic to a lattice $\mathcal{L}(v')$ with $v' \in \mathbb{N}^n$.

**Lemma 4.16.** If $\mathcal{L}(v)$ has dimension $n$, then $\text{Perm}(n)$ embeds into $\mathcal{L}(v)$.

**Proof.** We can assume that $v \in \mathbb{N}^n$, so that $v_i > 0$ for $i = 0, \ldots, n$. Let $\psi : \text{Perm}(n) \rightarrow \mathcal{L}(v)$ such that $\psi(\sigma_1 \ldots \sigma_n) = w_{\sigma_1} \ldots w_{\sigma_n}$ where $w_j = a^v_j$.

Using Lemma 2.4 it is easily seen that this mapping is order preserving. Now suppose that $\psi(\sigma) \lor \psi(\tau) \leq w$: therefore, if $i \not< j$ is an inversion of either $\sigma$ or $\tau$, then all the $a_j$'s precede all the $a_i$'s in $w$. By an easy induction, if $i \not< j$ belongs to the closure of $I(\sigma) \cup I(\tau)$, then all the $a_j$'s precedes all the $a_i$'s in $w$, that is $\psi(\sigma \lor \tau) \leq w$. Clearly, we can argue similarly to prove the relation $\psi(\psi \land \tau) = \psi(\psi) \land \psi(\tau)$. \hfill $\Box$

**Proposition 4.17.** If $n$ is the dimension of $\mathcal{L}(v)$, then the equation $SD_{n-2}(\land)$ fails in $\mathcal{L}(v)$.

**Proof.** Since $\text{Perm}(n)$ is a sublattice of $\mathcal{L}(v)$ it is enough to show that $SD_{n-2}(\land)$ fails in $\text{Perm}(n)$. We shall use the representation of the permutoedron as a lattice of clopen sets. Hence, let us define

$$y = \{ i \not< i + 1 \mid i \text{ even} \}, \quad z = \{ i \not< i + 1 \mid i \text{ odd} \}, \quad x = \{ 1 \not< i \mid i = 2, \ldots, n \}.$$  

We claim that $x, y, z$ form an $SD_{n-2}(\land)$-failure in $\text{Perm}(n)$. To ease the verification, let $w_0 = \emptyset$ and $w_k = \{ 1 \not< i \mid i = 2, \ldots, k + 1 \}$, where $1 \leq k \leq n - 1$, so that $x = w_{n-1}$. We claim that $w_k = x \land y_k$ if $k$ is even, and $w_k = x \land z_k$ if $w_k$ is odd.

We remark first that $x \land y_0 = \emptyset = w_0$ and $x \land z_1 = x \land z_0 = \{ 1 \not< 2 \} = w_1$, where we use the fact that $z_0 \geq (x \land y_0)$ implies that $z_1 = z_0$. We suppose therefore that $x \land y_{2k} = w_{2k}$ and $x \land z_{2k+1} = w_{2k+1}$. We deduce

$$y_{2k+2} = y \lor (x \land z_{2k+1}) = y \lor (w_{2k+1}) = l(y \cup w_{2k+1})$$
$$= l(\{ i \not< i + 1 \mid i \text{ even} \} \cup \{ 1 \not< i \mid i = 2, \ldots, 2k + 2 \})$$
$$= \{ i \not< i + 1 \mid i \text{ even} \} \cup \{ 1 \not< i \mid i = 2, \ldots, 2k + 3 \},$$

hence

$$x \land y_{2k+2} = r(\{ 1 \not< i \mid i = 2, \ldots, 2k + 3 \})$$
$$= \{ 1 \not< i \mid i = 2, \ldots, 2(k + 1) + 1 \} = w_{2(k+1)},$$
since this set is already open. Similarly:

\[ z_{2(k+1)+1} = z \lor (x \land y_{2k+2}) = z \lor w_{2k+2} = l(z \lor w_{2k+2}) = l(\{ i \mid i \text{ odd} \} \cup \{ 1 \mid i = 2, \ldots, 2k + 3 \}) \]
\[ = \{ i \mid i \text{ odd} \} \cup \{ 1 \mid i = 2, \ldots, 2k + 4 \} \]

hence

\[ x \land z_{2(k+1)+1} = r(\{ 1 \mid i = 2, \ldots, 2(k+1) + 2 \}) \]
\[ = \{ 1 \mid i = 2, \ldots, 2(k+1) + 2 \} = w_{2(k+1)+1} . \]

Finally, observe that \( y \lor z = \top \), hence \( x \land (y \lor z) = x \). Hence, if \( n = 2k \) is even, then:

\[ x \land y_{n-2} = w_{n-2} < w_{n-1} = x \land (y \lor z) , \]

and if \( n = 2k + 1 \) is odd, then

\[ x \land z_{n-2} = w_{n-2} < w_{n-1} = x \land (y \lor z) . \]

□

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