Explicit Hilbert spaces for certain unipotent representations II.

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0 Introduction

To each real semisimple Jordan algebra, the Tits-Koecher-Kantor theory associates a distinguished parabolic subgroup \( P = LN \) of a semisimple Lie group \( G \). The groups \( P \) which arise in this manner are precisely those for which \( N \) is abelian, and \( P \) is conjugate to its opposite \( \overline{P} \).

Each non-open \( L \)-orbit \( O \) on \( N^* \) admits an \( L \)-equivariant measure \( d\mu \) which is unique up to scalar multiple. By Mackey theory, we obtain a natural irreducible unitary representation \( \pi_O \) of \( P \), acting on the Hilbert space

\[ \mathcal{H}_O = L^2(O, d\mu). \]

In this context, we wish to consider two problems:

1. Extend \( \pi_O \) to a unitary representation of \( G \).
2. Decompose the tensor products \( \pi_O \otimes \pi_O' \otimes \pi_O'' \otimes \cdots \)

If the Jordan algebra is Euclidean (i.e. formally real) then \( G/P \) is the Shilov boundary of a symmetric tube domain. In this case, the first problem was solved in [S1], [S2], where it was shown that \( \pi_O \) extends to a unitary representation of a suitable covering group of \( G \). The second problem was solved in [DS], where we established a correspondence between the unitary representations of \( G \) occurring in the tensor product, and those of a “dual” group \( G' \) acting on a certain reductive homogeneous space. This correspondence agrees with the \( \theta \)-correspondence in various classical cases, and also gives a duality between \( E_7 \) and real forms of the Cayley projective plane.

In this paper we start to consider these two problems for non-Euclidean Jordan algebras. The algebraic groundwork has already been accomplished in

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however the analytical considerations are much more subtle, and here we only treat the case of the representation \( \pi_1 = \pi_{O_1} \) corresponding to the minimal \( L \)-orbit \( O_1 \).

It turns out that in order for the first problem to have a positive solution, one has to exclude certain Jordan algebras of rank 2. This is related to the Howe-Vogan result on the non-existence of minimal representations for certain orthogonal groups.

To each of the remaining Jordan algebras we attach a restricted root system \( \Sigma \) of rank \( n \), where \( n \) is the rank of the Jordan algebra. The root multiplicities, \( d \) and \( e \), of \( \Sigma \) play a decisive role in our considerations. For the reader’s convenience, we include a list of the corresponding groups \( G \) and the multiplicities in the appendix.

For these groups, we show that \( \pi_1 \) extends to a spherical unitary representation of \( G \), and that the spherical vector is closely related to the one variable Bessel \( K \)-function \( K_\tau(z) \), where

\[
\tau = \frac{d - e - 1}{2}.
\]

The function \( K_\tau(z) \) can be characterized, up to a multiple, as the unique solution of the modified Bessel equation

\[
\psi'' + z^{-1} \psi' - \left(1 + \frac{\tau^2}{z^2}\right) \psi = 0
\]

that decays (exponentially) as \( z \to \infty \); and, to us, one of the most delightful aspects of the present consideration is the unexpected and uniform manner in which this classical differential equation emerges from the structure theory of \( G \).

More precisely, we establish the following result:

We identify \( N \) with its Lie algebra \( n = \text{Lie}(N) \) via the exponential map. We also fix an invariant bilinear form on \( \langle \cdot, \cdot \rangle \) on \( g \), which is a certain multiple of the Killing form, normalized as in Definition 1.1 below. We use this form to identify \( N^* \) with \( \overline{n} = \text{Lie}(N) \). For \( y \) in \( \overline{n} \), \( \langle -\theta y, y \rangle \) is positive, and we define

\[
|y| = \sqrt{\langle -\theta y, y \rangle}.
\]

**Theorem 0.1** \( \pi_1 \) extends to a unitary representation of \( G \) with spherical vector \( |y|^{-\tau} K_\tau(|y|) \).

Since \( \pi_1 \) is spherical, its Langlands parameter is its infinitesimal character, and this can be determined via the (degenerate) principal series imbedding described in section 2 below. It is then straightforward to verify that \( \pi_1 \) is the minimal representation of \( G \), with annihilator equal to the Joseph ideal. (For \( G = GL(n) \), the minimal representation is not unique.)
Thus our construction should be compared to other realizations of the minimal representations in \[B\], \[T\], \[H\] etc. Although our construction is for a more restrictive class of groups, it does offer two advantages over the other constructions. The first advantage is that our construction works for a larger class of representations, and the second advantage is that it is well-suited for tensor product computations.

Both of these features will be explored in detail in a subsequent paper. In the present paper, we consider \(k\)-fold tensor powers of \(\pi_1\), where \(k\) is strictly smaller than \(n\) (rank of \(\Sigma\)), and show that the decomposition can be understood in terms of certain reductive homogeneous spaces

\[
G_k/H_k, 1 < k < n.
\]

These spaces are defined in section 3, and are listed in the appendix.

We consider also the corresponding Plancherel decomposition:

\[
L^2(G_k/H_k) = \int_{\hat{G}_k} m(\kappa) d\mu(\kappa),
\]

where \(d\mu\) is the Plancherel measure, and \(m(\kappa)\) is the multiplicity function. Then we have

**Theorem 0.2** For \(1 < k < n\), there is a correspondence \(\theta_k\) between \(\hat{G}_k\) and \(\hat{G}\), such that

\[
\pi_1 \otimes^k = \int_{\hat{G}_k} m(\kappa) \theta_k (\kappa) d\mu(\kappa).
\]

1 Preliminaries

The results of this section are all well-known. Details and proofs may be found in \[S1\], \[KS\] and in the references therein (in particular, \[BK\] and \[Lo\]).

1.1 Root multiplicities

Let \(G\) be a real simple Lie group and let \(K\) be a maximal compact subgroup corresponding to a Cartan involution \(\theta\). We shall denote the Lie algebras of \(G\), \(K\) etc by \(\mathfrak{g}\), \(\mathfrak{k}\) etc. Their complexifications will be denoted by lowercase fraktur letters with subscript \(C\). Fix \(\theta\), and let \(\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\) be the associated Cartan decomposition.

The parabolic subgroups \(P = LN\) obtained by the Tits-Kantor-Koecher construction are those such that \(N\) is abelian, and \(P\) is \(G\)-conjugate to its opposite parabolic

\[
\mathcal{P} = \theta(P) = L\overline{N}.
\]
In this case $N$ has a natural structure of a real Jordan algebra, which is unique up to a choice of the identity element.

In (Lie-)algebraic terms, this means that $P$ is a maximal parabolic subgroup corresponding to a simple (restricted) root $\alpha$ which has coefficient 1 in the highest root, and which is mapped to $-\alpha$ under the long element of the Weyl group.

In this situation, $M := K \cap L$ is a symmetric subgroup of $K$ (this is equivalent to the abelianness of $N$), and we fix a maximal toral subalgebra $t$ in the orthogonal complement of $m$ in $\mathfrak{k}$.

The roots of $t_C$ in $\mathfrak{g}_C$ form a restricted root system of type $C_n$, where $n = \dim \mathfrak{r}$ is the (real) rank of $N$ as a Jordan algebra (this result is essentially due to C. Moore). We fix a basis $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of $t^*$ such that

$$\Sigma(t_C, \mathfrak{g}_C) = \{\pm(\gamma_i \pm \gamma_j)/2, \pm \gamma_j\}.$$ 

The restricted root system $\Sigma = \Sigma(t_C, \mathfrak{g}_C)$ is of type $A_{n-1}$, $C_n$, or $D_n$, and the first of these cases arises precisely when $N$ is a Euclidean Jordan algebra. This case was studied in [S1], therefore we restrict our attention to the last two cases.

The root multiplicities in $\Sigma$ play a key role in our considerations. If $\Sigma$ is $C_n$, there are two multiplicities, corresponding to the short and long roots, which we denote by $d$ and $e$, respectively. If $\Sigma$ is $D_n$, and $n \neq 2$, then there is a single multiplicity, which we denote by $d$, so that $D_n$ may be regarded as a special case of $C_n$, with $e = 0$.

The root system $D_2$ is reducible (being isomorphic to $A_1 \times A_1$) and a priori there are two root multiplicities. In what follows, we explicitly exclude the case when these multiplicities are different. This means that we exclude from consideration the groups

$$G = O(p, q), N = \mathbb{R}^{p-1,q-1}(p \neq q);$$

indeed, our main results are false for these groups. When the two multiplicities coincide, we once again denote the common multiplicity by $d$.

The multiplicity of the short roots $\pm(\gamma_i \pm \gamma_j)/2$ in $\Sigma(t_C, \mathfrak{g}_C)$ is equal to $2d$, and the multiplicity of the long roots $\pm \gamma_i$ is $e + 1$.

In the appendix we include a table listing the groups under consideration, as well as the values of $d$ and $e$ for each of these groups.

1.2 Cayley transform

We briefly review the notion of the Cayley transform. Let $C$ be the following element (of order 8) in $SL_2(\mathbb{C})$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

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The Cayley transform of $\mathfrak{sl}_2(\mathbb{C})$ is the automorphism (of order 4) given by
\[
c = \text{Ad } C.
\]
It transforms the “usual” basis of $\mathfrak{sl}_2(\mathbb{C})$
\[
x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
to the basis
\[
X = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \quad H = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
where $X = c(x) = C^{-1}x$, etc. In turn, $c$ can be expressed as
\[
c = \exp \text{ ad } \frac{\pi i}{4} (x + y) = \exp \text{ ad } \frac{\pi i}{4} (X + Y).
\]

The key property of the Cayley transform is that it takes the compact torus (spanned by $iH$) to the split torus spanned by $h$ (cf. [KW]).

We turn now to the Lie algebra $\mathfrak{g}(\mathbb{C})$. By the Cartan-Helgason theorem the root spaces $\mathfrak{p}_{\gamma_j}$ are one-dimensional, and so by the Jacobson-Morozov theorem we get holomorphic homomorphisms $\Phi_j : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}(\mathbb{C})$, $j = 1, \ldots, n$ such that $X_j = \Phi_j(X)$ spans $\mathfrak{p}_{\gamma_j}$.

We fix such maps $\Phi_j$, and denote the images of $x, X, y, Y, h, H$ by $x_j, X_j$, etc. Since the roots $\gamma_j$ are strongly orthogonal, the triples $\{X_j, Y_j, H_j\}$ commute with each other, and the Cayley transform of $\mathfrak{g}$ is defined to be the automorphism
\[
c = \exp \text{ ad } \frac{\pi i}{4} \left( \sum X_j + \sum Y_j \right) = \prod \exp \text{ ad } \frac{\pi i}{4} (X_j + Y_j).
\]

Thus we obtain an $\mathbb{R}$-split toral subalgebra $\mathfrak{a}$ defined by
\[
\mathfrak{a} = c^{-1}(it) = \mathbb{R}h_1 \oplus \cdots \oplus \mathbb{R}h_n.
\]
The roots of $\mathfrak{a}C$ in $\mathfrak{g}(\mathbb{C})$ are
\[
\Sigma(\mathfrak{a}C, \mathfrak{g}C) = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j \} \text{ where } \varepsilon_i = \frac{1}{2} \gamma_i \circ c.
\]
The short roots have multiplicity $2d$ and the long roots have multiplicity $e + 1$.

In fact $\mathfrak{a} \subset \mathfrak{l}$, and we have
\[
\Sigma(\mathfrak{a}, \mathfrak{l}) = \{ \pm (\varepsilon_i - \varepsilon_j) \}, \quad \Sigma(\mathfrak{a}, \mathfrak{n}) = \{ \varepsilon_i + \varepsilon_j, 2\varepsilon_j \}, \quad \Sigma(\mathfrak{a}, \mathfrak{m}) = \{-\varepsilon_i - \varepsilon_j, -2\varepsilon_j\}
\]

**Definition 1.1** The invariant form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is normalized by requiring
\[
\langle x_1, y_1 \rangle = 1.
\]

For $y \in \mathfrak{n}$, we set $|y| \overset{\text{def}}{=} \sqrt{-\langle y, \theta y \rangle}$, as in Introduction.
1.3 Orbits and measures

We now describe the orbits of $L$ in $\mathfrak{p} \cong N^\ast$. For $k = 1, \ldots, n - 1$, define

$$O_k = L \cdot (y_1 + y_2 + \ldots + y_k).$$

Then these, together with the trivial orbit $O_0$, comprise the totality of the singular (i.e., non-open) $L$-orbits in $\mathfrak{p}$.

We define $\nu \in \mathfrak{a}^\ast$ as

$$\nu = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n.$$ 

Then $\nu$ extends to a character of $l$, and we will write $e^\nu$ for the corresponding (spherical) character of $L$.

**Lemma 1.1** The orbit $O_1$ carries a natural $L$-equivariant measure $d\mu_1$, which transforms by the character $e^{2d\nu}$, that is

$$\int_{O_1} g(l \cdot y) d\mu_1(y) = e^{2d\nu(l)} \int_{O_1} g(y) d\mu_1(y).$$

**Proof.** Let $S_1$ be the stabilizer of $y_1$ in $L$. It suffices to show that the modular function of $S_1$ is the restriction, from $L$ to $S_1$, of the character $e^{2d\nu}$. Passing to the Lie algebra $\mathfrak{s}_1$, we need to show that

$$\text{tr} \, \text{ad}_{\mathfrak{s}_1} = 2d\nu|_{\mathfrak{s}_1}.$$ 

To see this, we remark that $\mathfrak{s}_1$ has codimension 1 inside a maximal parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{l}$, corresponding to the stabilizer of the line through $y_1$. The space of characters of $\mathfrak{q}$ is two-dimensional, and it follows that the space of characters of $\mathfrak{s}_1$ is one-dimensional. Hence any character of $\mathfrak{s}_1$ is determined by its restriction to $\mathfrak{a} \cap \mathfrak{s}_1 = \text{Ker} \, \varepsilon_1$. The restriction of $\nu$ to $\mathfrak{s}_1$ is nontrivial, hence

$$\text{tr} \, \text{ad}_{\mathfrak{s}_1} = k\nu$$

for some constant $k$.

Obviously, $\text{tr} \, \text{ad}_l = 0$, and the only root spaces missing from $\mathfrak{s}_1$ are the root spaces $\mathfrak{l}_{\varepsilon_j}$, $j \geq 2$ (each of these root spaces has dimension $2d$). Hence, for $a \in \mathfrak{a}$

$$\text{tr} \, \text{ad}_{\mathfrak{s}_1}(a) = -2d \sum_{j=2}^n (\varepsilon_1 - \varepsilon_j)(a),$$ 

and restricting this to $\text{Ker} \, \varepsilon_1$, we obtain $2d\nu|_{\mathfrak{a} \cap \mathfrak{s}_1}$. $\blacksquare$

**Example.** Consider $G = O_{2n,2n}$ realized as the group of all $2n \times 2n$ real matrices preserving the split symmetric form $\begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix}$. Then $P = LN = GL_{2n}(\mathbb{R}) \ltimes \text{Skew}_{2n}(\mathbb{R})$. More precisely,

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} : A \in GL_{2n}(\mathbb{R}) \right\}.$$
and
\[ N = \left\{ \begin{pmatrix} I_{2n} & 0 \\ B & I_{2n} \end{pmatrix} : B + B^t = 0 \right\}. \]

Then
\[ a = \{ \text{diag}(a_1, a_1, a_2, a_2, \ldots, a_n, a_n, -a_1, -a_1, -a_2, -a_2, \ldots, -a_n, -a_n), a_i \in \mathbb{R} \} \]
is the toral subalgebra of \( g \) (and \( l \)) described in the preceding subsection. We can take
\[ y_1 = \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}, \text{ where } B_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
The Lie algebra \( s_1 \) of the stabilizer \( S_1 = \text{Stab}_L y_1 \) can be written as
\[ s_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in \mathfrak{sl}_2, A_{22} \in \mathfrak{gl}_{2n-2} \right\} \]
It is a codimension 1 subalgebra of the parabolic subalgebra \( q \) of \( \mathfrak{gl}_{2n} \), where \( q = (\mathfrak{gl}_2 + \mathfrak{gl}_{2n-2}) + \mathbb{R}^{2,2n-2} \).

Remark. In this example \( \nu = \frac{1}{2} \) tr, \( d = 2 \) and \( e^{2d\nu} = (\det)^2 \).

## 2 Minimal representation of \( G \)

If \( \chi \) is a character of \( l \), we write \( \pi_\chi \) for the (unnormalized) induced representation \( \text{Ind}_{\mathfrak{p}}^{G}(\chi) \). These representations were studied in \([S3]\) in the “compact” picture, by algebraic methods. Among the results established there was the existence of a finite number of “small”, unitarizable, spherical subrepresentations, which occur for the following values of \( \chi \)
\[ \chi_j = e^{-j\nu}, \quad j = 1, \ldots, n - 1. \]

In this paper we use analytical methods, and work primarily with the “non-compact” picture, which is the realization of \( \pi_\chi \) on \( C^\infty(N) \), via the Gelfand-Naimark decomposition
\[ G \approx N\mathbb{F}. \]
In fact, using the exponential map we can identify \( n \) and \( N \), and realize \( \pi_\chi \) on \( C^\infty(n) \).

We will show that the unitarizable subrepresentation of \( \pi_{\chi_1} \) admits a natural realization on the Hilbert space \( L^2(O_1, d\mu) \). Since there is no obvious action of \( G \) on this space, we have to proceed in an indirect fashion. The key is an explicit realization of the spherical vector \( \sigma_{\chi_1} \).
2.1 The Bessel function

We let $d, e$ be the root multiplicities of $\Sigma(t, \mathfrak{f})$ as in previous section, and define

$$\tau_G = \tau = (d - e - 1)/2$$

as in the introduction.

Let $K_\tau$ be the $K$-Bessel function on $(0, \infty)$ satisfying

$$z^2K''_\tau + zK'_\tau - (z^2 + \tau^2)K_\tau = 0. \tag{1}$$

Put $\phi_\tau(z) = \frac{K_\tau(\sqrt{z})}{(\sqrt{z})^\tau}$, then $\phi_\tau$ satisfies the differential equation

$$D\phi_\tau = 0, \text{ where } D\phi = 4z\phi'' + 4(\tau + 1)\phi' - \phi. \tag{2}$$

We lift $\phi_\tau$ to an $M$-invariant function $g_\tau$ on $O_1$, by defining

$$g_\tau(y) = \phi_\tau(-\langle y, \theta y \rangle) = \frac{K_\tau(|y|)}{|y|^\tau}. \tag{3}$$

Remark. If $d = e$ (as is the case for $G = Sp_{2n}(\mathbb{C})$ or $Sp_{n,n}$), then $\tau = -\frac{1}{2}$ and

$$g_\tau(y) = |y|^{1/2} K_{-1/2}(|y|) = |y|^{1/2} \frac{\exp(-|y|)}{|y|^{1/2}} = e^{-|y|}.$$ 

If $d = e + 1$ (this is true for $GL_{2n}(k)$, $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), then

$$g_\tau(y) = K_0(|y|).$$

Proposition 2.1 (1) $g_\tau$ is a (square-integrable) function in $L^2(O_1, d\mu_1)$.

(2) The measure $g_\tau d\mu_1$ defines a tempered distribution on $\mathfrak{g}$.

Proof. (1) We define

$$O' \overset{\text{def}}{=} \{ y' \in O_1 : |y'| = 1 \}.$$

Then $O'$ is compact; the map

$$O' \times (0, \infty) \ni (y', w) \mapsto wy' \in O_1$$

is a diffeomorphism, and the measure $d\mu_1$ can be decomposed as a product

$$d\mu_1(wy') = d\mu'(y')d\mu''(w)$$

We now determine the explicit form of $d\mu''(w)$.
Define \( h = \sum_{i=1}^{n} h_i \), then \((\text{ad} \, h) y = -2y\) for any \( y \in \mathbb{R} \). We take \( y \in \mathcal{O}_1 \), \( z > 0 \), \( a = \ln z \) and calculate
\[
d\mu_1(zy) = z^{dn} d\mu_1(y).
\]
Therefore, for \( z > 0 \)
\[
d\mu_1(zy) = z^{dn} d\mu_1(y) \tag{4}
\]
and it follows that \( d\mu''(zw) = z^{dn} d\mu''(w) \), and so, up to a scalar multiple,
\[
d\mu''(w) = w^{dn-1} dw,
\]
where \( dw \) is the Lebesgue measure.

We can now calculate
\[
\int_{\mathcal{O}_1} |g_\tau(y)|^2 d\mu_1(y) = \int_{0}^{\infty} \int_{\mathcal{O}_1} \frac{K_\tau(w)^2}{w^{2\tau}} d\mu'(y') w^{dn-1} dw
= c \int_{0}^{\infty} \frac{K_\tau(w)^2}{w^{2\tau}} w^{dn-1} dw, \tag{5}
\]
where \( c = \mu'(\mathcal{O}') \) is a positive constant. The function \( K_\tau(w) \) has a pole of order \( \tau \) at 0 (or, in case of \( \tau = 0 \), a logarithmic singularity at 0), and it decays exponentially as \( w \to \infty \) [W, 3.71.15]. Hence \( w^{-2\tau} K_\tau(w)^2 \) has a pole of order
\[
4\tau = 2(d - e - 1) \leq 2d - 2 < dn - 1
\]
(recall that we require \( n \geq 2 \)). Thus the integrand in (5) is non-singular and decays exponentially as \( w \to \infty \). Therefore, the integral (5) converges and \( g_\tau(y) \in L^1_{\text{loc}}(\mathcal{O}_1, d\mu_1) \) and has exponential decay at \( \infty \) (i.e., as \( |y| \to \infty \)). This implies the result.

We can now define the Fourier transform of \( g_\tau \),
\[
\Phi = \hat{g}_\tau d\mu_1
\]
as a (tempered) distribution on \( \mathbb{R} \). The key result is the following

**Proposition 2.2** \( \Phi \) is a multiple of the spherical vector \( \sigma_{\chi_1} \).

The proof of this proposition will be given over the next two subsections.

### 2.2 Characterization of spherical vectors

For \( \phi : \mathbb{R} \to \mathbb{R} \), let \( \xi(\phi) \) denote the corresponding vector field:
\[
\xi(\phi) f(x) = \frac{d}{dt} f(x + t\phi(x)) \bigg|_{t=0} \text{ for } f : \mathbb{R} \to \mathbb{C}.
\]

Then we have the following formulas for the action of \( \pi_\chi \) on \( C^\infty(\mathbb{R}) \):
• for \( x_0 \in \mathfrak{n} \), \( \pi_\chi(x_0) = \xi(x_0) \),
• for \( h_0 \in \mathfrak{l} \), \( \pi_\chi(h_0) = \chi(h_0) - \xi([h_0, x]) \),
• for \( y_0 \in \mathfrak{p} \), \( \pi_\chi(y_0) = \chi(x, y_0) - \frac{1}{2} \xi([h, x]) \), where \( h = [x, y_0] \).

We need a Lie algebra characterization of \( \sigma_\chi \):

**Lemma 2.3** The space of \( \pi_\chi(\mathfrak{t}) \)-invariant distributions on \( \mathfrak{n} \) is 1-dimensional (and spanned by \( \sigma_\chi \)).

**Proof.** It is well known (and easy to prove) that the only distributions on \( \mathbb{R}^n \), which are annihilated by \( \frac{\partial}{\partial x_i} \), \( i = 1, \ldots, n \) are the constants. More generally, we can replace \( \mathbb{R}^n \) by a manifold, and \( \left\{ \frac{\partial}{\partial x_i} \right\} \) by any set of vector fields which span the tangent space at each point of the manifold.

For \( \chi = 0 \), the formulas above show that \( \pi_0(\mathfrak{g}) \) acts by vector fields on \( C^\infty(\mathfrak{n}) \). Moreover, using the decomposition \( G = K \mathcal{F} \), we see that \( \pi_0(\mathfrak{t}) \) is a spanning family of vector fields. Thus the result follows in this case.

For general \( \chi \), if \( T \) is a \( \pi_\chi(\mathfrak{t}) \)-invariant distribution, then \( T/\sigma_\chi = T\sigma_{-\chi} \) is \( \pi_0(\mathfrak{t}) \)-invariant, and hence a constant.

**Proposition 2.4** Let \( T \) be an \( M \)-invariant distribution on \( \mathfrak{n} \) such that

\[
\pi_\chi(y + \theta y)T = 0 \quad \text{for some } y \neq 0 \text{ in } \mathfrak{p},
\]

then \( T \) is a multiple of the spherical vector \( \sigma_\chi \).

**Proof.** The \( M \)-invariance of \( T \) implies that

\[
\pi_\chi(m)T = 0
\]

Since \( m \) is a maximal subalgebra of \( \mathfrak{t} \), \( m \) and \( y + \theta y \) generate \( \mathfrak{t} \) as a Lie algebra. Thus

\[
\pi_\chi(\mathfrak{t})T = 0,
\]

and the result follows from the previous lemma.

**2.3 The \( K \)-invariance of the Bessel function**

We now turn to the proof of Proposition 2.2. To simplify notation, we will write \( \pi \) instead of \( \pi_\chi \). Since \( \Phi \) is clearly \( M \)-invariant, by Proposition 2.4 it suffices to show

\[
\pi(y_1 + \theta y_1)\Phi = 0
\]

for \( y_1 \in \mathfrak{p} \). We will prove this through a sequence of lemmas.
It is convenient to introduce the following notation: if \( g_1 \) and \( g_2 \) are functions on \( O_1 \), we define
\[
(g_1, g_2) = \int_{O_1} g_1(y) g_2(y) d\mu_1(y)
\]
provided the integral converges.

If \( g \) is a function on \( O_1 \) and \( h \in \mathcal{I} \), then the action of \( h \) on \( g \) is given by
\[
h \cdot g(y) \overset{\text{def}}{=} \frac{d}{dt} g(e^{th} \cdot y) \bigg|_{t=0}.
\]

In the computation below, we shall work with the expressions of the type
\[
\left( \frac{d}{dt} \int_{O_1} g(e^{th} \cdot y) d\mu(y) \right) \bigg|_{t=0}.
\]

To justify differentiation under the integral sign, we need to impose the standard conditions on \( g \) (e.g. [K99, p.170]), as follows.

Define a class of functions \( \mathcal{I} \subset C^\infty(O_1) \), given by the following conditions:

- A smooth function \( g \) belongs to \( \mathcal{I} \) if
  - \( g \in L^1(O_1, d\mu_1) \) and
  - for any \( h \in \mathcal{I} \) we can find \( c > 0 \) and \( G(y) \in L^1(O_1, d\mu_1) \), such that
    \[
    \left| \frac{d}{dt} g(e^{th} \cdot y) \right|_{t=0} \leq G(y)
    \]
    for all \( y \in O_1 \) and \( |t_0| < c \).

**Lemma 2.5** Suppose \( g_1, g_2 \) are smooth functions on \( O_1 \), such that \( g_1, g_2 \in \mathcal{I} \). Then
\[
(h \cdot g_1, g_2) + (g_1, h \cdot g_2) = 2d\nu(h)(g_1, g_2). \tag{6}
\]

**Proof.** Using the \( L \)-equivariance of \( d\mu_1 \), we obtain
\[
\int_{O_1} g_1(e^{th} y) g_2(e^{th} y) d\mu_1 = e^{2d\nu(h)} \int_{O_1} g_1 g_2 d\mu_1.
\]

Under the assumptions of the lemma, we can differentiate this identity in \( t \), to get
\[
\int_{O_1} h \cdot (g_1 g_2) d\mu_1 = 2d\nu(h) \int_{O_1} g_1 g_2 d\mu_1.
\]

By the Leibnitz rule, the result follows. \( \blacksquare \)
More generally, if \( g_1, g_2 \) are functions on \( n \times O_1 \), then \((g_1, g_2)\) is a function on \( n \). In this notation, for \( g \) in \( L^1(O_1, d\mu_1) \), the Fourier transform of \( gd\mu_1 \) is given by the formula
\[
\hat{gd\mu_1} = (e^{-i(x,y)}, g).
\]

**Lemma 2.6** Let \( g \in L^1(O_1, d\mu_1) \) be a smooth function on \( O_1 \), such that
\[
e^{-i(x,y)}g \in \mathcal{I}.
\]
Suppose \( f = (e^{-i(x,y)}, g) \), then
\[
\pi(y_1)f = -\frac{1}{2}(e^{-i(x,y)}, h \cdot g(y)), \text{ where } h = [x, y_1].
\]

**Proof.** By the formula for the action of \( \pi(y_1) \), we get
\[
-2(\pi(y_1)f + d\nu(h)f) = \xi([h, x]) \cdot (e^{-i(x,y)}, g)
\]
\[
= \frac{d}{dt} \left( e^{-i(x+t[h,x], y)}, g \right)|_{t=0}
\]
\[
= \frac{d}{dt} \left( e^{-i(x,y-t[h,y]), g} \right)|_{t=0}
\]
\[
= -(h \cdot e^{-i(x,y)}, g)
\]
\[
= (e^{-i(x,y)}, h \cdot g) - 2d\nu(h)(e^{-i(x,y)}, g)
\]
where we have used the previous lemma, and the relation
\[
\langle x + t[h, x], y \rangle = \langle x, y \rangle + t \langle [h, x], y \rangle = \langle x, y \rangle - t \langle x, [h, y] \rangle = \langle x, y - t[h, y] \rangle.
\]
The result follows. ■

The pairing \(- \langle \cdot, \theta \cdot \rangle\) gives a positive definite \( M \)-invariant inner product on \( \Pi \), and we now obtain the following

**Lemma 2.7** Suppose that \( g(y) = \phi(-\langle y, \theta y \rangle) \) for some smooth \( \phi \) on \((0, \infty)\), and \( e^{-i(x,y)}g \in \mathcal{I} \). Put \( f(x) = (e^{-i(x,y)}, g) \), as before. Then
\[
\pi(y_1)f = \left( e^{-i(x,y)}, \langle x, [[\theta y, y_1], y] \rangle \phi'(-\langle y, \theta y \rangle) \right).
\]

**Proof.** Writing \( h = [x, y_1] \) as in the previous lemma, we get
\[
h \cdot g(y) = \int \frac{d}{dt} \phi(-\langle y + t[h, y], \theta(y + t[h, y]) \rangle)|_{t=0}
\]
\[
= \int \frac{d}{dt} \phi(-\langle y, \theta y \rangle - 2t \langle \theta y, [h, y] \rangle + O(t^2))|_{t=0}
\]
\[
= -2 \langle \theta y, [h, y] \rangle \phi'(-\langle y, \theta y \rangle).
\]
Since
\[ \langle \theta y, [h, y] \rangle = \langle \theta y, [[x, y_1], y] \rangle = \langle x, [[\theta y, y_1], y] \rangle , \]
the result follows.

The key lemma is the following computation

**Lemma 2.8** Let \( \phi \) and \( f \) be as in the previous lemma, and suppose for \( x \in \mathfrak{n} \)
\[ e^{-i(x,y)} \phi (- \langle y, \theta y \rangle) \in \mathcal{I}, \quad e^{-i(x,y)} \phi' (- \langle y, \theta y \rangle) \in \mathcal{I}. \] (7)
Then we have
\[ \pi(y_1 + \theta y_1) f(x) = \left( e^{-i(x,y)}, i \langle \theta y_1, y \rangle(D\phi)(- \langle y, \theta y \rangle) \right) , \] (8)
where the differential operator \( D \) is given by the formula (3), i.e.
\[ (D\phi)(- \langle y, \theta y \rangle) = 4 (- \langle y, \theta y \rangle) \phi'' + 2(d + 1 - e)\phi' - \phi. \] (9)

**Proof.** Choose a basis \( l_j \) of \( \mathfrak{l} \) and define functions \( c_j(y) \) by the formula
\[ [\theta y, y_1] = \sum_j c_j(y) l_j. \] Then by the previous lemma
\[ \pi(y_1)f = \sum_j (e^{-i(x,y)}, \langle x, [l_j, y] \rangle c_j \phi') = i \sum_j \frac{d}{dt} \left. \left( e^{-i(x,y+l_j,y)} , c_j \phi' \right) \right|_{t=0} \]
\[ = i \sum_j (l_j \cdot e^{-i(x,y)}, c_j \phi'). \]
Differentiation in this calculation is justified, because \( e^{-i(x,y)} \phi' (- \langle y, \theta y \rangle) \in \mathcal{I} \).

Applying (3) to the last expression, we can write
\[ \pi(y_1)f = -i \sum_j \left( e^{-i(x,y)}, -2d\nu(l_j) c_j \phi' + c_j l_j \cdot \phi' + \phi' l_j \cdot c_j \right). \] (10)
We now calculate each term in this expression.

- First we have
\[ \sum_j \nu(l_j) c_j \phi' = \nu([\theta y, y_1]) \phi'. \]

Since \( \nu \) is a real character of \( \mathfrak{l} \), it vanishes on \( \mathfrak{l} \cap \mathfrak{t} \) and we have \( \nu([\theta y, y_1]) = \nu([\theta y_1, y]) \). Recall that \( \mathfrak{n} \) and \( \mathfrak{p} \) are irreducible \( \mathfrak{l} \)-modules. Therefore, \( \nu([\theta y_1, y]) = k \langle \theta y_1, y \rangle \) for some constant \( k \neq 0 \), independent of \( y \). Setting \( y = y_1 \), we get \( \langle \theta y_1, y_1 \rangle = -x_1, y_1 \rangle = -1 \). Hence \( k = -\nu([\theta y_1, y_1]) = 1 \), and therefore
\[ -\sum_j 2d\nu(l_j) c_j \phi' = -2d \langle \theta y_1, y \rangle \phi'. \] (11)
• Next we compute

\[\sum_j c_j l_j \cdot \phi' = [\theta y, y_1] \cdot \phi'\]

\[= \frac{d}{dt} \phi' (-\langle y + t([\theta y, y_1], y), \theta(y + t([\theta y, y_1], y]) \rangle)|_{t=0}\]

\[= -2 \langle y, [\theta y, y]\rangle \phi'' (-y, \theta y)\]

Since \( y \) is a \( \mathfrak{k} \)-conjugate to a root vector, there is a scalar \( k' \) independent of \( y \) such that \([\theta y, y] = k' (y, \theta y) y\). Setting \( y = y_1 \) we get \( \langle y_1, \theta y_1 \rangle = -1,\]

\([\theta y_1, y_1] = -2y_1\]

and \( k' = 2 \). Also \(-\langle y, [\theta y, y]\rangle = \langle [y, \theta y], [y, \theta y_1] \rangle = \langle [y, \theta y], y_1 \rangle\).

Hence

\[\sum_j c_j l_j \cdot \phi' = 4 \langle y, \theta y \rangle \langle \theta y_1, y \rangle \phi''. \quad (12)\]

• Next we note that \( \sum c_j l_j \cdot c_j \) is independent of the basis \( l_j \), so we may assume that

\(\theta l_j = \pm l_j\) and \(\langle l_j, -\theta l_k \rangle = \delta_{jk}\).

Then \( c_j(y) = \langle [\theta y, y_1], -\theta l_j \rangle\) and

\[\sum_j l_j \cdot c_j = \sum_j \langle [\theta l_j, y], y_1 \rangle - \theta l_j \rangle = \sum_j \langle y_1, [\theta l_j, y], \theta l_j \rangle = -\langle y_1, \Omega y \rangle.\]

Here \(\Omega = \sum_j \text{ad}(\theta l_j)^2 = \Omega_1 - 2\Omega_{\mathfrak{t} \cap \mathfrak{f}},\) where the Casimir elements are obtained by using dual bases with respect to \(\langle , \rangle\).

To continue, we need the following lemma:

**Lemma 2.9** \(\Omega\) acts on \(\mathfrak{n}\) by the scalar \(k'' = 2 - 2e\).

**Proof.** When \(e = 1\) it’s easy to see that the operator \(\Omega\) acts by 0. Indeed, in this case \(\mathfrak{g}\) is a complex semisimple Lie algebra and for each basis element \(l_j \in \mathfrak{t} \cap \mathfrak{f}\) there exists a basis element \(l_j' = \sqrt{-1} l_j \in \mathfrak{p} \cap \mathfrak{f}\). Then \([l_j, [l_j, x]] + [l_j', [l_j', x]] = 0\) and \(k'' = 0\).

When \(e = 0\), \(\mathfrak{g}\) is split and simply laced, and \(\mathfrak{f}\) is the split real form of a complex reductive algebra \(\mathfrak{g}_{\mathfrak{c}}\). Take a root vector \(x_\lambda \in \mathfrak{g}_{\mathfrak{c}}\), where \(\lambda\) is any positive root in \(\mathfrak{n}\). For any positive root \(\alpha\) of \(\mathfrak{c}\) we fix \(e_\alpha \in \mathfrak{t}_0\) and set \(l_\alpha = e_\alpha + \theta e_\alpha \in \mathfrak{t} \cap \mathfrak{f}\) and \(l_\alpha' = e_\alpha - \theta e_\alpha \in \mathfrak{p} \cap \mathfrak{f}\). Then the collection of all \(l_\alpha, l_\alpha'\) together with the orthonormal basis of a Cartan subalgebra \(\mathfrak{f}\) of \(\mathfrak{f}\) forms a basis of \(\mathfrak{f}\).

Observe that

\([l_\alpha, [l_\alpha, x_\lambda]] + [l_\alpha', [l_\alpha', x_\lambda]] = [e_\alpha, [e_\alpha, x_\lambda]] + [e_{-\alpha}, [e_{-\alpha}, x_\lambda]] = 0,\]
since $x_\lambda \in \mathfrak{g}_\lambda$ and neither $\lambda + 2\alpha$ nor $\lambda - 2\alpha$ is a root of the simply laced algebra $\mathfrak{g}_\mathbb{C}$.

We choose a basis $\{u_i\}$ of $\mathfrak{f}$, and denote the elements of the dual (with respect to $\langle \cdot , \cdot \rangle$) basis by $\tilde{u}_i$. Then

$$\Omega x_\lambda = \sum_i [u_i, [\tilde{u}_i, x_\lambda]] = \langle \lambda, \lambda \rangle x_\lambda = 2x_\lambda.$$ 

In the remaining two cases $\mathfrak{t} \cap \mathfrak{l}$ acts on $\mathfrak{n}$ irreducibly, therefore $\Omega$ automatically acts by a scalar and it suffices to compute $\sum_j l_j, [l_j, x_1]]$. For $e = 3$ we have $G = GL_{2n}(\mathbb{H}), L = GL_n(\mathbb{H}) \times GL_n(\mathbb{H})$ and $\mathfrak{n} = \mathbb{H}^{n \times n}$. The computation for this group is similar to the case of $G = GL_{2n}(\mathbb{R})$. We reduce the calculation to the summation over the diagonal subalgebra of $\mathfrak{l}$ and obtain

$$\Omega x_\lambda = \langle \lambda, \lambda \rangle x_\lambda + 3 \langle -\sqrt{-1}\lambda, -\sqrt{-1}\lambda \rangle x_\lambda = -4x_\lambda.$$ 

Finally, for $e = 2$ ($G = Sp_{2n,n}$), a direct evaluation of $\sum_j l_j, [l_j, x_1]]$ gives $k'' = -2$. Therefore, we get

$$\sum_j \phi' l_j \cdot c_j = -2(1 - e) \langle \theta y_1, y \rangle \phi'.$$  

(13)

Finally, we have

$$\pi(\theta y_1) f = \frac{d}{dt} \left( e^{-i(x+\theta y_1, y)}, \phi \right) \bigg|_{t=0} = -i \left( e^{-i(x,y)}, \langle \theta y_1, y \rangle \phi \right).$$  

(14)

Putting the formulas (11)–(14) together, we deduce the lemma. 

**Proof of Proposition 2.2.** Recall that we study $\phi_\tau(z) = \frac{K_\tau(\sqrt{z})}{(\sqrt{z})}$, its lift $g_\tau$ to the radial function on $\mathcal{O}_1$,

$$g_\tau(y) = \phi_\tau(- (y, \theta y)) = \frac{K_\tau(|y|)}{|y|^t}$$

and its Fourier transform $\Phi(x) = (e^{-i(x,y)}, g_\tau)$. By Proposition 2.4 it suffices to check that $\pi(y_1 + \theta y_1) \Phi = 0$. This identity would follow immediately from Lemma 2.8, because $D\phi_\tau = 0$ by formula (2) and then the desired result follows from (8).

To complete the proof we have to verify the assumptions (9). In subsection 2.1 we proved that $g_\tau \in L^1(\mathcal{O}_1, d\mu_1)$. It is easy to verify (using the standard facts about the derivatives of $K_\tau$ from (12)), that the lifts to $\mathcal{O}_1$ of the functions $\phi'_\tau(z)$ and $\phi''_\tau(z)$ (we denote them by $g'_\tau(y)$ and $g''_\tau(y)$) both belong to $L^1(\mathcal{O}_1, d\mu_1)$. Observe also that $\phi_\tau(z), \phi'_\tau(z), \phi''_\tau(z)$ are all monotone on $(0, \infty)$.
Moreover, since all these functions tend to zero exponentially as $|y| \to \infty$, the functions $A(y)g_r(y)$, $A(y)g'_r(y)$, $A(y)g''_r(y)$ all belong to $L^1(\Omega_1, d\mu_1)$, for any $A(y)$ bounded in the neighbourhood of $y = 0$ and growing (at most) polynomially with respect to $|y|$ as $|y| \to \infty$.

Fix $h \in I$, $x \in n$ and choose $c > 0$ sufficiently small, such that for all $y \in \Omega_1$ and $|t| < c$

$$|\langle e^{th} \cdot y, \theta(e^{th} \cdot y) \rangle| \geq \frac{|\langle y, \theta y \rangle|}{2}.$$  

We can then estimate the derivative:

$$\left| \frac{d}{dt} \left( e^{-i(x, e^{th} \cdot y)} \phi_r \left( \langle e^{th} y, \theta e^{th} y \rangle \right) \right) \right| \leq |A_1(y)\phi_r \left( |y|^2 / 2 \right)| + |A_2(y)\phi'_r \left( |y|^2 / 2 \right)|,$$

for all $y \in \Omega_1$ and $|t| < c$, where $A_1(y), A_2(y)$ are some functions of polynomial growth. From the discussion above, the right-hand side of this inequality is an $L^1$-function on $\Omega_1$, hence $e^{-i(x,y)} g'_r \in \mathcal{I}$. Proceeding in the same manner, we deduce that $e^{-i(x,y)} g''_r \in \mathcal{I}$. ■

### 2.4 Proof of Theorem 0.1

Denote by $J$ the space of the induced representation $\pi_1 = \text{Ind}_{\mathfrak{n}}^G(e^{-dy})$. By the Gelfand-Naimark decomposition and the exp map, $J$ can be viewed as a subspace of $C^\infty(n)$. Then for $l \in L$ and $\eta \in J$ we have

$$\pi_1(l)\eta(x) = e^{-dy}(l)\eta(l^{-1} \cdot x).$$

It was proved in [S3] that the $(g, K)$-module $J$ has a unitarizable spherical $(g, K)$-submodule $V$, which we also regard as a subspace of $C^\infty(n)$.

**Remark.** It is possible to give a direct description of the elements of the “abstract” Hilbert space $H$, where $H$ is the Hilbert space closure of $V$ with respect to the $(g, K)$-invariant norm on $V$. For that purpose we use the “compact” realization of $\pi_1$ on $C^\infty(K/M)$ from [S3]. It was shown that $\pi_1$ is a representation of ladder type, with all its $K$-types $\{\alpha_m \mid m \in \mathbb{N}\}$ lying on a single line, $\alpha_1$ being a one-dimensional $K$-type. The restriction $\langle \cdot, \cdot \rangle_m$ of a $\pi_1$-invariant Hermitian form to any $K$-type $\alpha_m$ is a multiple of the $L^2(K)$-inner product on $V$, and from the explicit formulas in [S3] it follows that

$$q_m \overset{\text{def}}{=} \langle \cdot, \cdot \rangle_m \overset{\text{def}}{=} O(m^C)$$

for some constant $C > 1$, which can be expressed in terms of parameters $d$, $e$ and $n$. Thus we can identify $H$ with the Hilbert space $L^2(\mathbb{N}, \{q_m\})$, where the constant $q_m$ gives the weight of the point $m \in \mathbb{N}$. That is, any element of $H$ can be viewed as an $M$-equivariant function on $K$, such that its sequence of Fourier coefficients belongs to $L^2(\mathbb{N}, \{q_m\})$. In particular $L^2(\mathbb{N}, \{q_m\}) \subset l^2(\mathbb{N})$, and the elements of $H$ all lie in $L^2(K)$. 

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We write $H$ for the space of those tempered distributions on $\mathfrak{n}$ which are Fourier transforms of $\psi d\mu_1$ for some $\psi \in L^2(O_1, d\mu_1)$. If $\eta$ is the Fourier transform of a distribution of the form $\psi d\mu_1$, i.e.,

$$\eta(x) = \int_{O_1} e^{-i(x,y)} \psi(y) d\mu_1(y) = \left(e^{-i(x,y)}, \psi(y)\right),$$

then

$$\pi_1(l)\eta(x) = e^{-d\nu(l)}\eta(l^{-1} \cdot x) = \int_{O_1} e^{-i(l^{-1}x,y)} \psi(y) e^{-d\nu(l)} d\mu_1(y)$$

$$= \int_{O_1} e^{-i(l^{-1}x,l^{-1}y)} \psi(l^{-1} \cdot y) e^{-d\nu(l)} d\mu_1(l^{-1} \cdot y)$$

$$= \left(e^{-i(x,y)}, e^{d\nu(l)} \psi(l^{-1} \cdot y)\right).$$

It follows from the calculation above that $P$ acts unitarily on $H$ (it is convenient to identify this action with its realization on $L^2(O_1, d\mu_1)$ via the Fourier transform). We denote this unitary representation of $P$ by $\pi'$. Observe that $(\pi', H)$ is an irreducible representation of $P$.

According to Proposition 2.1, $\Phi(x) = (e^{-i<x,y>}, |y|^{-\tau} K_\tau(|y|))$ belongs to $H$.

**Theorem 2.10** $V$ is a dense subspace of $H$, and the restriction of the norm is $(g, K)$-invariant.

**Proof.** Let $C^\infty(K)_V$ be the subspace of $C^\infty(K)$, consisting of those smooth functions on $K$, whose $K$-isotypic components belong to $V$. Since $V$ is a submodule of $J$, $C^\infty(K)_V$ is obviously $G$-invariant.

Denote by $C(G)$ the convolution algebra of smooth $L^1$ functions on $G = PK$, and consider

$$W = \pi_1(C(G))\Phi \subset C^\infty(K)_V.$$  

So all elements of $W$ are continuous functions on $K$, hence continuous on $G$, and therefore are determined by their restrictions to $N$. Moreover, $W = \pi_1(C(P))\Phi$ and $K$ fixes $\Phi$, therefore

$$W = \pi_1(C(P))\Phi = \pi'(C(P))\Phi.$$  

This shows that $W$ is a $\pi'(P)$-invariant subspace of $H$, and from the irreducibility of $\pi'$ we conclude that $W$ is dense in $H$.

We can now put two $\pi_1(P)$-invariant norms on $W$ – one from $H$ and another from $V$, as follows. If $f = \sum c_m v_m$, with $v_m$ in the $K$-isotypic component with highest weight $\alpha_m$ (occurring in $V$) and $\|v_m\|_{L^2(K)} = 1$, then

$$\|f\|_V^2 = \sum |c_m|^2 q_m.$$  

(15)
Since \( f \) is smooth, it follows that \( |c_m| \) decays rapidly, so the series in (13) converges, thus giving a \( \pi_1(P) \)-invariant norm on \( W \).

Then it follows from [3] (cf. [5], p.417]), that we can find a (dense) \( C(P) \)-invariant subspace \( W' \subset W \), such that these two forms are proportional on \( W' \).

Considering the closure of \( W' \) we obtain an isometric \( P \)-invariant imbedding of \( H \) into \( H \).

Then \( W \) is:

1. a \( G \)-invariant subspace of the irreducible module \( H \), hence dense in \( H \); 
2. a dense subspace of the Hilbert space \( H \).

It follows that \( H = H \). \( \blacksquare \)

This concludes the proof of Theorem 0.1.

### 3 Tensor powers of \( \pi_1 \)

#### 3.1 Restrictions to \( P \)

In the previous section we constructed a unitary representation \( \pi_1 \) of \( G \) acting on the Hilbert space \( L^2(\mathcal{O}_1, d\mu_1) \), where \( \mathcal{O}_1 \) is the minimal \( L \)-orbit in a non-Euclidean Jordan algebra \( N \). Define the \( k \)-th tensor power representation

\[ \Pi_k = \pi_1 \otimes^k (2 \leq k < n) \]

As we shall show, the techniques developed in [DS] allow us to establish a duality between the spectrum of this tensor power and the spectrum of a certain homogeneous space. We omit the proofs of the several propositions below, because the proofs of the corresponding statements from [DS] can be used without any substantial modification.

Observe that the orbit \( \mathcal{O}_k \) is dense in \( \mathcal{O}_1 + \mathcal{O}_1 + \ldots + \mathcal{O}_1 \). The representation \( \Pi_k \) acts on \( [L^2(\mathcal{O}_1, d\mu_1)]^\otimes_k \simeq L^2(\mathcal{O}'_k, d\mu') \), where \( \mathcal{O}'_k = \mathcal{O}_1 \times \ldots \times \mathcal{O}_1 \) and \( d\mu' \) is the product measure on \( \mathcal{O}'_k \). We fix a generic representative \( \xi' = (\xi_1, \xi_2, \ldots, \xi_k) \in \mathcal{O}'_k \), such that

\[ \xi = \xi_1 + \xi_2 + \ldots + \xi_k \in \mathcal{O}_k. \]

Denote by \( S'_k \) and \( S_k \) the isotropy subgroups of \( \xi' \) and \( \xi \), respectively, with respect to the action of \( L \) on \( \mathcal{O}'_k \) and \( \mathcal{O}_k \). Observe that the Lie algebras \( s'_k \) and \( s_k \) of \( S'_k \) and \( S_k \), respectively, can be written as

\[ s'_k = (\mathfrak{h}_k + l_k) + u_k \]
\[ s_k = (\mathfrak{g}_k + l_k) + u_k. \]

Here \( l_k, g_k \) and \( h_k \) are reductive, \( h_k \subset g_k \) and \( u_k \) is a nilpotent radical common for both \( s'_k \) and \( s_k \). Let \( G_k \) and \( H_k \) be the corresponding Lie groups.
Example. Take \( G = O_{2n,2n} \) and \( k < n \). Then \( \xi_i = E_{2i-1,2i} - E_{2i,2i-1} \) \((1 \leq i \leq k)\), \( \xi = \sum_{i=1}^{k} \xi_i \) and

\[
\mathfrak{s}_k = (\mathfrak{sp}_{2k}(\mathbb{R}) + \mathfrak{gl}_{2(n-k)}(\mathbb{R})) + \mathbb{R}^{2k,2(n-k)}.
\]

Then \( G_k = Sp_{2k}(\mathbb{R}) \) and it’s easy to check that \( H_k = SL_2(\mathbb{R})^k \).

The following Lemma can be verified by direct calculation (cf. [DS, Lemma 2.1]).

**Lemma 3.1** Let \( \chi_\xi \) be the character of \( N \) corresponding to \( \xi \in N^* \). Then

\[
\Pi_k|_P = \text{Ind}^{P_{S_k N}}_{S_k N}(1 \otimes \chi_\xi) = \text{Ind}^{P_{S_k N}}_{S_k N} \left( (\text{Ind}^{S_k}_{S_k'} 1) \otimes \chi_\xi \right) \quad (L^2\text{-induction}).
\]

Let \( \gamma' = \text{Ind}^{H_k}_{G_k} 1 \) be the quasiregular representation of \( G_k \) on \( L^2(G_k/H_k) \); then it can be decomposed using the Plancherel measure \( d\mu \) for the reductive homogeneous space \( X_k = G_k/H_k \) and the corresponding multiplicity function \( m : \hat{G}_k \to \{0,1,2,\ldots\} \), i.e.,

\[
\gamma' \simeq \int_{\hat{G}_k} \Theta(\kappa) d\mu(\kappa).
\]

Each irreducible representation \( \kappa \) of \( G_k \) can be extended to an irreducible representation \( \kappa'' \) of \( S_k \), and the decomposition of the Lemma above can be rewritten as

\[
\Pi_k|_P = \int_{\hat{G}_k} m(\kappa) \Theta(\kappa) d\mu(\kappa),
\]

where \( \Theta(\kappa) = \text{Ind}^{P_{S_k N}}_{S_k N}(\kappa'' \otimes \chi_\xi) \).

Moreover, by Mackey theory all representations \( \Theta(\kappa) \) are unitary irreducible representations of \( P \), and \( \Theta(\kappa') \simeq \Theta(\kappa'') \) if and only if \( \kappa' \simeq \kappa'' \).

### 3.2 Low-rank theory

In [DS] we extended the theory of low-rank representations ([13]) to the conformal groups of euclidean Jordan algebras. Inspection of the argument in [DS] shows that the analogous theory can be developed in exactly the same manner for the conformal groups of non-euclidean Jordan algebras.

For any unitary representation \( \eta \) of \( G \), we decompose its restriction \( \eta|_N \) into a direct integral of unitary characters, where the decomposition is determined by a projection-valued measure on \( \hat{N} = N^* \). If this measure is supported on a single non-open \( L \)-orbit \( \mathcal{O}_m, 1 \leq m < n \) we call \( \eta \) a low-rank representation, and write \( \text{rank} \eta = m \). Proceeding by induction on \( m \), as in [13], [DS, Sect 3], we can prove the following
Theorem 3.2 Let $\eta$ be a low-rank representation of $G$. Write $A(\eta, P)$ for the von Neumann algebra generated by $\{\eta(x) | x \in P\}$ and $A(\eta, G)$ for the von Neumann algebra generated by $\{\eta(x) | x \in G\}$. Then $A(\eta, G) = A(\eta, P)$.

Proof of Theorem 3.2. Now consider the restriction of $\Pi_k$ to $N$. Its restriction to $P$ is given by the direct integral decomposition (16), and we can further restrict it to $N$. The rank of the induced representation $\Theta(\kappa) = \text{Ind}_{S_k N}^G (\kappa \vee \chi \xi)$ is $k$ (the $N$–spectrum is supported on the $L$–orbit of $\xi$, i.e. $\mathcal{O}_k$). Therefore $\Pi_k$ can be decomposed over the irreducible representations of $G$ of rank $k$.

It follows from the theorem above that any two non-isomorphic representations from the spectrum of $\Pi_k$ restrict to non-isomorphic irreducible representations of $P$. Hence the representation $\Pi_k$ can be decomposed as

$$\Pi_k = \int_{\hat{G}_k} m(\kappa) \theta(\kappa) d\mu(\kappa),$$

where for almost every $\kappa$ the unitary irreducible representation $\theta(\kappa)$ is obtained as the unique irreducible representation of $G$ determined by the condition $\theta(\kappa)|_P = \Theta(\kappa)$.

Therefore, the map $\kappa \mapsto \theta(\kappa)$ gives a (measurable) bijection between the spectrum of $\Pi_k = \pi \otimes \pi$ and the unitary representations of $G_k$ occurring in the quasiregular representation on $L^2(G_k/H_k)$. 

Example. Take $G = E_7(7)$. It is the conformal group of the split exceptional real Jordan algebra $N$ of dimension 27. Consider the tensor square of the minimal representation $\pi_1$ of $G$ ($k = 2$). Then $L = \mathbb{R}^* \times E_6(6)$, $S'_2$ is the stabilizer of $y_1$ and $y_2$ and $S_2$ is the stabilizer of $y_1 + y_2 \in O_2$. One can see that in this case $g_2 = \text{Stab}_{SO(5,5)}(y_1 + y_2) = so(4,5)$ and $h_2 = \text{Stab}_{SO(5,5)}(y_1) \cap \text{Stab}_{SO(5,5)}(y_2) = so(4,4)$ (cf. [A, 16.7]). Hence the decomposition (17) establishes a duality between the representations of $E_7(7)$ occurring in $\Pi_2 = \pi_1 \otimes \pi_1$ and the unitary representations of $SO_9(\mathbb{C})/SO_8(\mathbb{C})$. The homogeneous space $Spin(4,5)/Spin(4,4)$ is a (pseudo-riemannian) symmetric space of rank 1, and it is known to be multiplicity free. Therefore, $\pi_1 \otimes \pi_1$ has simple spectrum.

Similarly, for $G = E_7(\mathbb{C})$ we obtain a duality between $E_7(\mathbb{C})$ and the symmetric space $SO_9(\mathbb{C})/SO_8(\mathbb{C})$. 

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A Groups associated to non-Euclidean Jordan algebras

| $G$          | $K/M$                          | $d$ | $e$ | $G_k/H_k$ for $2 \leq k < n$                  |
|--------------|--------------------------------|-----|-----|-----------------------------------------------|
| $GL_{2n}(\mathbb{R})$ | $O_{2n}/(O_n \times O_n)$     | 1   | 0   | $GL_k(\mathbb{R})/GL_1(\mathbb{R})^k$       |
| $O_{2n,2n}$  | $(O_{2n} \times O_{2n})/O_{2n}$| 2   | 0   | $Sp_{2k}(\mathbb{R})/SL_2(\mathbb{R})^k$    |
| $E_{7(7)}$   | $SU_8/Sp_4$                    | 4   | 0   | $Spin(4,5)/Spin(4,4)$                        |
| $O_{p+2,p+2}$| $[O_{p+2}/[O_1 \times O_{p+1}]]$|$p$ | 0   |                                               |
| $Sp_n(\mathbb{C})$ | $Sp_n/U_n$                     | 1   | 1   | $O_k(\mathbb{C})/[O_1(\mathbb{C})]^k$       |
| $GL_{2n}(\mathbb{C})$ | $U_{2n}/(U_n \times U_n)$     | 2   | 1   | $GL_k(\mathbb{C})/[GL_1(\mathbb{C})]^k$     |
| $O_{4n}(\mathbb{C})$ | $O_{4n}/U_{2n}$                | 4   | 1   | $Sp_{2k}(\mathbb{C})/[SL_2(\mathbb{C})]^k$  |
| $E_7(\mathbb{C})$  | $E_7/(E_6 \times U_1)$      | 8   | 1   | $SO_9(\mathbb{C})/SO_5(\mathbb{C})$         |
| $O_{p+4}(\mathbb{C})$ | $O_{p+4}/(O_{p+2} \times U_1)$ | 1   |     |                                               |
| $Sp_{n,n}$    | $(Sp_n \times Sp_n)/Sp_n$     | 2   | 2   | $O_k^*/[O_1]^k$                               |
| $GL_{2n}(\mathbb{H})$ | $Sp_{2n}/(Sp_n \times Sp_n)$  | 4   | 3   | $GL_k(\mathbb{H})/[GL_1(\mathbb{H})]^k$     |

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