Eliminating Higher-Multiplicity Intersections, II.
The Deleted Product Criterion in the $r$-Metastable Range*

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Abstract

Motivated by Tverberg-type problems in topological combinatorics and by classical results
about embeddings (maps without double points), we study the question whether a finite
simplicial complex $K$ can be mapped into $\mathbb{R}^d$ without higher-multiplicity intersections. We
focus on conditions for the existence of almost $r$-embeddings, i.e., maps $f : K \to \mathbb{R}^d$ such that
$f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of $K$.

Generalizing the classical Haefliger-Weber embeddability criterion, we show that a well-
known necessary deleted product condition for the existence of almost $r$-embeddings is suf-
ficient in a suitable $r$-metastable range of dimensions: If $rd \geq (r+1) \dim K + 3$, then
there exists an almost $r$-embedding $K \to \mathbb{R}^d$ if and only if there exists an equivariant map
$K^r_{\Delta} \to e, S^{d(r-1)-1}$, where $K^r_{\Delta}$ is the deleted $r$-fold product of $K$ (the subcomplex of the $r$-fold
cartesian product whose cells are products of pairwise disjoint simplices of $K$) and $S_r$ denotes
the symmetric group. This significantly extends one of the main results of our previous paper
(which treated the special case where $d = rk$ and $\dim K = (r-1)k$ for some $k \geq 3$), and
settles one of the main open questions raised there.

As a corollary, our result together with recent work of Filakovský and Vokřínek on the
homotopy classification of equivariant maps under non-free actions imply that almost $r$-
embeddability of simplicial complexes is algorithmically decidable in the $r$-metastable range
(in polynomial time if $r$ and $d$ are fixed).

Contents

1 Introduction 2

1.1 The Deleted Product Criterion for Almost $r$-Embeddings ............... 2
1.2 Algorithmic Consequences ............................................. 3
1.3 Background and Motivation: Tverberg-Type Problems and Beyond ........... 4

2 Main Lemmas: Reduction & Local Disjunction 6

3 Combining Reduction and Local Disjunction 9

4 The Proof of the Local Disjunction Lemma 11

4.1 Increasing the connectivity of the intersections .................... 12
4.2 Proof of Lemma 10 for balls ............................................ 18
4.3 Deleted Joins ............................................................ 20
4.4 Proof of Lemma 33 ....................................................... 22
4.5 The complete proof of Lemma 10 ....................................... 25

A Block Bundles 28

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1 Introduction

Let $K$ be a finite simplicial complex, and let $f: K \to \mathbb{R}^d$ be a continuous map.\(^1\) Given an integer $r \geq 2$, we say that $y \in \mathbb{R}^d$ is an $r$-fold point or $r$-intersection point of $f$ if it has $r$ pairwise distinct preimages, i.e., if there exist $y_1, \ldots, y_r \in K$ such that $f(y_1) = \ldots = f(y_r) = y$ and $y_i \neq y_j$ for $1 \leq i < j \leq r$. We will pay particular attention to $r$-fold points that are global\(^2\) in the sense that their preimages lie in $r$ pairwise disjoint simplices of $K$, i.e., $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$, where $\sigma_i \cap \sigma_j = \emptyset$ for $1 \leq i < j \leq r$.

We say that a map $f: K \to \mathbb{R}^d$ is an $r$-embedding if it has no $r$-fold points, and we say that $f$ is an almost $r$-embedding if it has no global $r$-fold points.\(^3\)

The most fundamental case $r = 2$ is that of embeddings (=2-embeddings), i.e., injective continuous maps $f: K \to \mathbb{R}^d$. Finding conditions for a simplicial complex $K$ to be embeddable into $\mathbb{R}^d$ — a higher-dimensional generalization of graph planarity — is a classical problem in topology (see [RS99, Sko08] for surveys) and has recently also become the subject of systematic study from a viewpoint of algorithms and computational complexity (see [MTW11, MSTW14, ČKV13]).

Here, we are interested in necessary and sufficient conditions for the existence of almost $r$-embeddings. One motivation are Tverberg-type problems in topological combinatorics (see the corresponding subsection below). Another motivation is that, in the classical case $r = 2$, embeddability is often proved in two steps: in the first step, the existence of an almost embedding (=almost 2-embedding) is established; in the second step this almost embedding is transformed into an embedding, by removing local self-intersections. Similarly, we expect the existence of an almost $r$-embedding to be not only an obvious necessary condition but a useful stepping stone towards the existence of $r$-embeddings and, in a further step, towards the existence of embeddings outside the so-called metastable range (see Remark 3 (c)).

1.1 The Deleted Product Criterion for Almost $r$-Embeddings

There is a well-known necessary condition for the existence of almost $r$-embeddings. Given a simplicial complex $K$ and $r \geq 2$, the (combinatorial) deleted $r$-fold product\(^4\) of $K$ is defined as

$$K^r_\Delta := \{ (x_1, \ldots, x_r) \in \sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \text{ a simplex of } K, \sigma_i \cap \sigma_j = \emptyset \text{ for } 1 \leq i < j \leq r \}.$$  

The deleted product is a regular polytopal cell complex (a subcomplex of the $r$-fold cartesian product), whose cells are products of $r$-tuples of pairwise disjoint simplices of $K$.

**Lemma 1 (Necessity of the Deleted Product Criterion).** Let $K$ be a finite simplicial complex, and let $d, r \geq 2$ be integers. If there exists an almost $r$-embedding $f: K \to \mathbb{R}^d$ then there exists an equivariant map\(^5\)

$$\tilde{f}: K^r_\Delta \to \mathbb{R}, S^{d(r-1)-1},$$

where $S^{d(r-1)-1} = \{ (y_1, \ldots, y_r) \in (\mathbb{R}^d)^r \mid \sum_{i=1}^r y_i = 0, \sum_{i=1}^r ||y_i||^2 = 1 \}$, and the symmetric group $\mathfrak{S}_r$ acts on both spaces by permuting components.

**Proof.** Given $f: K \to \mathbb{R}^d$, define $f^r: K^r_\Delta \to (\mathbb{R}^d)^r$ by $f^r(x_1, \ldots, x_r) := (f(x_1), \ldots, f(x_r))$. Then $f$ is an almost $r$-embedding iff the image of $f^r$ avoids the thin diagonal $\delta_r(\mathbb{R}^d) := \{ (y, \ldots, y) \mid y \in \mathbb{R}^d \} \subset (\mathbb{R}^d)^r$. Moreover, $S^{d(r-1)-1}$ is the unit sphere in the orthogonal complement $\delta_r(\mathbb{R}^d)^\perp \simeq \mathbb{R}^{d(r-1)}$, and there is a straightforward homotopy equivalence $\rho: (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq S^{d(r-1)-1}$. Both $f^r$ and $\rho$ are equivariant, hence so is their composition $\tilde{f} := \rho \circ f^r: K^r_\Delta \to \mathbb{R}, S^{d(r-1)-1}$.

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\(^1\)For simplicity, throughout most of the paper we use the same notation for a simplicial complex $K$ and its underlying polyhedron, relying on context to distinguish between the two when necessary.

\(^2\)In our previous paper [MW14], we used the terminology “$r$-Tverberg point” instead of “global $r$-fold point.”

\(^3\)We emphasize that the definitions of global $r$-fold points and of almost $r$-embeddings depend on the actual simplicial complex $K$ (the specific triangulation), not just the underlying polyhedron.

\(^4\)For more background on deleted products and the broader configuration space/test map framework, see, e.g., [Mat03] or [Ziv96, Ziv98].

\(^5\)Here and in what follows, if $X$ and $Y$ are spaces on which a finite group $G$ acts (all group actions will be from the right) then we will use the notation $F: X \to G, Y$ for maps that are equivariant, i.e., that satisfy $F(x \cdot g) = F(x) \cdot g$ for all $x \in X$ and $g \in G$. 
Our main result is that the converse of Lemma 1 holds in a wide range of dimensions.

**Theorem 2 (Sufficiency of the Deleted Product Criterion in the \( r \)-Metastable Range).** Let \( m, d, r \geq 2 \) be integers satisfying

\[
rd \geq (r + 1)m + 3.
\]

Suppose that \( K \) is a finite \( m \)-dimensional simplicial complex and that there exists an equivariant map \( F : K^r_\Delta \to \varnothing, S^{d(r-1)-1} \). Then there exists an almost \( r \)-embedding \( f : K \to \mathbb{R}^d \).

**Remarks 3.** (a) When studying almost \( r \)-embeddings, it suffices to consider maps \( f : K \to \mathbb{R}^d \) that are piecewise-linear \( (PL) \) and in general position.\(^7\)

(b) Theorem 2 is trivial for codimension \( d - m \leq 2 \). Indeed, if \( r, d, m \) satisfy (1) and, additionally, \( d - m \leq 2 \) then a straightforward calculation shows that \( (r-1)d > rm \), so that a map \( K \to \mathbb{R}^d \) in general position has no \( r \)-fold points.

(c) The special case \( r = 2 \) of Theorem 2 corresponds to the classical Haefliger–Weber Theorem [Hae63, Web67], which guarantees that for \( 2d \geq 3m + 3 \) the existence of an equivariant map \( K^r_\Delta \to \varnothing, S^{d-1} \) guarantees the existence of an almost embedding \( f : K \to \mathbb{R}^d \). The condition \( 2d \geq 3m + 3 \) is often referred to as the metastable range; correspondingly, we call Condition (1) the \( r \)-metastable range.\(^8\)

In the metastable range, an almost embedding can further be turned into an embedding by a delicate geometric construction [Sko98, Web67], which fails outside of the metastable range [SS92].

Turning almost \( r \)-embeddings into \( r \)-embeddings seems to be an even more subtle problem, which we plan to pursue in a future paper.

(d) Theorem 2 significantly extends one of the main results of our previous paper [MW15, Thm. 7] (see also the extended abstract [MW14, Thm. 3]), which treated the special case \( (r-1)d = rm \), \( d - m \geq 3 \), and settles one of the open questions raised there.

### 1.2 Algorithmic Consequences

Very recently, Filakovský and Vokřínek [FV16] obtained the following algorithmic result regarding equivariant maps between \( G \)-spaces.\(^9\) If a group \( G \) acts on a space and \( H \leq G \) is a subgroup, we denote by \( X^H = \{ x \in X : xh = x, \forall h \in H \} \) the set of \( H \)-fixed points of \( X \). Furthermore, we write \( \text{conn}(Y) \) for the connectivity of a space \( Y \) (i.e., the maximum \( k \) such that every map \( S^k \to Y \) can be extended to a map \( B^{k+1} \to Y \)).

**Theorem 4 ([FV16]).** There is an algorithm with the following specifications:

Let \( X \) and \( Y \) be finite simplicial complexes, and let \( G \) be a finite group that acts (simplicially) on \( X \) and \( Y \). Suppose that \( \dim(X^H) \leq 2 \text{conn}(Y^H) + 1 \) for every subgroup \( H \leq G \).\(^10\)

Given \( X \) and \( Y \) (and the simplicial maps specifying the group action) as input, the algorithm decides whether there exists a \( G \)-equivariant map \( X \to Y \).

Moreover, if \( G \) and the dimension of \( X \) are fixed, then the algorithm runs in polynomial time.

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\(^6\)Recall that \( f \) is PL if there is some subdivision \( K' \) of \( K \) such that \( f|_\sigma \) is affine for each simplex \( \sigma \) of \( K' \).

\(^7\)Every continuous map \( g : K \to \mathbb{R}^3 \) can be approximated arbitrarily closely by PL maps in general position, and if \( g \) is an almost \( r \)-embedding, then the same holds for any map sufficiently close to \( g \).

\(^8\)We remark that the terminology "\( k \)-metastable range" also appears in a different setting, namely for links of a finite number of sheres \( S^{p_1}, S^{p_2}, \ldots, S^{p_k} \) smoothly embedded in \( S^m \), see [Hae66, Appendix 9.1]. The definition of the \( r \)-metastable range in the present paper and the \( k \)-metastable range in [Hae66] are not the same, but since the contexts are different, we hope that no confusion will arise.

\(^9\)Their result builds on and significantly generalizes earlier results in the non-equivariant setting [ČKM+14a, ČKM+14b] and for free actions [CKV13], respectively.

\(^10\)Here, we use the convention that \( \dim \emptyset = -\infty = \text{conn}(\emptyset) \). In other words the condition is trivially satisfied if \( X^H = \emptyset \), and if \( Y^H = \emptyset \), then we must have \( X^H = \emptyset \) as well.
As an immediate consequence of Lemma 1 and Theorems 2 and 4, one obtains:

**Corollary 5.** In the $r$-metastable range (1), there is an algorithm that, given as input a finite $m$-dimensional simplicial complex $K$ and parameters $r$ and $d$, decides whether there is an almost $r$-embedding of $K$ to $\mathbb{R}^d$. If $r$ and $d$ are fixed, the algorithm runs in polynomial time.

**Proof.** By Lemma 1 and Theorem 2, there exists an almost $r$-embedding of $K$ to $\mathbb{R}^d$ if there exists an $\mathcal{S}_r$-equivariant map from $X := K^r_d$ to $Y := S^{d(r-1)-1}$. Thus, we only need to check that the assumptions of Theorem 4 are satisfied in this setting. Both the deleted product $X$ and the sphere $Y$ can be triangulated in such a way that $G = \mathcal{S}_r$ acts by simplicial maps. Moreover, the action of $\mathcal{S}_r$ on $X$ is free, hence $X^H = \emptyset$ if $H \leq \mathcal{S}_r$ is not the trivial group $\{e\}$ consisting only of the identity, and $X^e = X$. Thus, the condition $\dim X^H \leq 2 \text{conn}(Y^H) + 1$ is trivially satisfied whenever $H \neq \{e\}$, and for $H = \{e\}$ it is satisfied since $\dim X \leq rm \leq 2 \text{conn}(Y) + 1 = 2d(r-1)-1$ in the $r$-metastable range. \qed

### 1.3 Background and Motivation: Tverberg-Type Problems and Beyond

Methods from algebraic topology and the general framework of configuration spaces and test maps have been very successfully used in discrete mathematics and theoretical computer science, see, e.g., [Ziv96, Ziv98, Mat03, BDB07, Kar08] for surveys. In particular, equivariant obstruction theory and, more generally equivariant homotopy theory, provide powerful tools for deciding whether suitable test maps exist. However, in cases where the existence of a test map does not settle the problem, further geometric ideas are needed. The general philosophy and underlying idea here and in the two companion papers [MW15, AMSW15] is to complement equivariant methods by methods from geometric topology, in particular piecewise-linear topology.

The initial motivation and first application for the predecessor paper [MW15] was the longstanding

**Conjecture 6** (Topological Tverberg Conjecture). Let $r \geq 2$, $d \geq 1$, and $N = (d + 1)(r - 1)$. Then there is no almost $r$-embedding of the $N$-simplex $\sigma^N$ to $\mathbb{R}^d$.

This conjecture, proposed by Bajmoczy and Bárány [BB79] and Tverberg [GS79, Problem 84] as a topological generalization of a classical theorem of Tverberg in convex geometry [Tve66], had been proved by Bajmoczy and Bárány [BB79] for $r = 2$, by Bárány, Shlosman, and Szücs [BSS81] for all primes $r$, and by Özaydin [Ö87] for prime powers $r$, but the case of arbitrary $r$ had remained elusive. An important reason was that Özaydin [Ö87, Thm. 4.2] had shown that for $r$ is not a prime power there does exist an equivariant map $F \colon (\sigma^N)^r_d \rightarrow \mathcal{S}_r S^{d(r-1)-1}$, so that Lemma 1 cannot be directly used to show that there is no almost $r$-embedding.

In [MW14, MW15], we proposed a new approach to the conjecture, namely the idea of constructing counterexamples, i.e., almost $r$-embeddings $\sigma^N \rightarrow \mathbb{R}^d$, when $r$ is not a prime power, by combining Özaydin’s result with the sufficiency of the deleted product product in the special case $\dim K = \frac{d-1}{r}d$, $d - \dim K \geq 3$ of Theorem 2 ([MW14, Thm 3] and [MW15, Thm 7]). At the time the extended abstract [MW14] appeared, there remained what seemed to be a serious obstacle to completing this approach: the assumption of codimension $d - \dim K \geq 3$ is not satisfied for $K = \sigma^N$. Frick [Fri15] observed that this “codimension 3 barrier” can be overcome by a combinatorial trick discovered independently independently by Gromov [Gro10, p. 445-446] and Blagojević-Frick-Ziegler [BFZ14]) and that therefore the results of [Ö87], [Gro10, BFZ14] and [MW14, MW15] combined yields counterexamples to the topological Tverberg conjecture for $d \geq 3r + 1$ whenever $r$ is not a prime power, see [BFZ15]. In the full version [MW15], another solution to the codimension 3 obstacle is given, leading to counterexamples for $d \geq 3r$. In joint work with Avvakumov and Skopenkov [AMSW15], we recently improved this further and obtained counterexamples for $d \geq 2r$, using an extension (for $r \geq 3$) of [MW15, Thm. 7] to codimension 2.

For a more detailed history of the topological Tverberg conjecture and the construction of counterexamples, we refer to the surveys and discussions in [BBZ16, Sko16, BZ16, Sim15, JVZ16].
There are numerous close relatives and other variants of (topological) Tverberg-type problems and results [BFL90, BL92, ZV92, Ziv98, BMZ15, Sar91, Vol96, BFZ14, BZ16, BBZ16]. These can be seen as generalized nonembeddability results or problems and typically state that a particular complex $K$ (or family of complexes) does not admit an almost $r$-embedding to $\mathbb{R}^d$. Theorem 2 provides a general necessary and sufficient condition for such topological Tverberg-type results in the $r$-metastable range.

As an application of a different flavor, Frick [Fri16] recently found a connection between almost $r$-embeddings and chromatic numbers of certain Kneser hypergraphs and used Theorem 2 to prove lower bounds for these chromatic numbers in certain cases.

**Further Questions and Future Research**

(a) **Beyond the $r$-Metastable Range.** Is condition (1) needed in Theorem 2? In the case $r = 2$, it is known that for $d \geq 3$, the Haejjiger–Weber Theorem fails outside the metastable range: for every pair $(m, d)$ with $2d < 3m + 3$ and $d \geq 3$, there are examples [MS67, SS92, FKT94, SSS98, GS06] of $m$-dimensional complexes $K$ such that $K^2_\Delta \to_{\phi_2} S^{d-1}$ but $K$ does not embed into $\mathbb{R}^d$. Moreover, in the case $r = 2$, $m = 2$ and $d = 4$, the examples do not even admit an almost embedding into $\mathbb{R}^4$, see [AMSW15].

On the other hand, as remarked above, in [AMSW15] the following extension of [MW15, Thm. 7] is proved: if $r \geq 3$ $d = 2r$, and $m = 2(r - 1)$, then a finite $m$-dimensional complex $K$ admits an almost $r$-embedding if and only if there exists an equivariant map $K^2_\Delta \to_{\phi_2} S^{d(r-1)-1}$.

It would be interesting to know whether there is analogous extension (for $r \geq 3$) of Theorem 2 that is nontrivial in codimension $d - m = 2$.

(b) **The Planar Case and Hanani–Tutte.** In the classical setting ($r = 2$) of embeddings, the case $d = 2, m = 1$ of graph planarity is somewhat exceptional: the parameters lie outside the (2-fold) metastable range, but the existence of an equivariant map $F: K^2_\Delta \to_{\phi_2} S^1$ is sufficient for a graph $K$ to be planar, by the Hanani–Tutte Theorem\(^{11}\) [CH34, Tut70]. The classical proofs of that theorem rely on Kuratowski’s Theorem, but recently [PSS07, PSS10], more direct proofs have been found that do not use forbidden minors. It would be interesting to know whether there is an analogue of the Hanani–Tutte theorem for almost $r$-embeddings of 2-dimensional complexes in $\mathbb{R}^2$, as an approach to constructing counterexamples to the topological Tverberg conjecture in dimension $d = 2$. We plan to investigate this in a future paper.

**Structure of the Paper**

The remainder of the paper is devoted to the proof of Theorem 2. By Lemma 1, we only need to show that the existence of an equivariant map $K^2_\Delta \to_{\phi_2} S^{d(r-1)-1}$ implies the existence of an almost $r$-embedding $K \to \mathbb{R}^d$. Moreover, by Remarks 3 (b) and (d), we may assume, in addition to the parameters being in the $r$-fold metastable range, that the codimension $d - m$ of the image of $K$ in $\mathbb{R}^d$ is at least 3, and that the intersection multiplicity $r$ is also at least 3. Thus, we will work under the following hypothesis:

$$rd \geq (r+1)m+3, \quad d - m \geq 3, \quad \text{and} \quad r \geq 3. \tag{2}$$

The proof of Theorem 2 is based on two main lemmas: Lemma 8 (Reduction Lemma) reduces the situation to a single $r$-tuple of pairwise disjoint simplices of $K$, and Lemma 10 (Local Disjunction Lemma) solves that reduced situation (this is a generalization of the Weber–Whitney Trick to multiplicity $r$). In Section 2, we give the precise (and somewhat technical) statements of these

\(^{11}\)The existence of an equivariant map implies, via standard equivariant obstruction theory, that there exists a map from the graph $K$ into $\mathbb{R}^2$ such that the images of any two disjoint (independent) edges intersect an even number of times, which is the hypothesis of the Hanani–Tutte Theorem.
lemmas, along with some background, and prove the Reduction Lemma 8. In Section 3, we show how to prove Theorem 3 using these lemmas, before proving the Local Disjunction Lemma 10 (the core of the paper) in Section 4.

2 Main Lemmas: Reduction & Local Disjunction

In this section, we formulate the two main lemmas on which the proof of Theorem 2 rests.

We work in the piecewise-linear (PL) category (standard references are [Zee66, RS82]). All manifolds (possibly with boundary) are PL-manifolds (can be triangulated as locally finite simplicial complexes such that the link of every nonempty face is either a PL-sphere or a PL-ball), and all maps between polyhedra (geometric realizations of simplicial complexes) are PL-maps (i.e., simplicial on sufficiently fine subdivisions). In particular, all balls are PL-ball and all spheres are PL-spheres (PL-homeomorphic to a simplex and the boundary of a simplex, respectively). A submanifold $P$ of a manifold $Q$ is properly embedded if $\partial P = P \cap \partial Q$. The singular set of a PL-map $f$ defined on a polyhedron $K$ is the closure in $K$ of the set of points at which $f$ is not injective.

One basic fact that we will use for the proofs of both Lemmas 8 and 10 is the following version of engulfing [Zee66, Ch. VII]:

**Theorem 7** (Engulfing, [Zee66, Ch. VII, Thm. 20]). Let $M$ be an $m$-dimensional $k$-connected manifold with $k \leq m - 3$. Let $X$ a compact $x$-dimensional subpolyhedron in the interior of $M$. If $x \leq k$, then there exists a collapsible subpolyhedron $C$ in the interior of $M$ with $X \subseteq C$ and $\dim(C) \leq x + 1$.

The collapsible polyhedron $C$ can be thought of as an analogue of a “cone” over $X$.

**Lemma 8** (Reduction Lemma). Let $m, d, r$ be three positive integers satisfying (2). Suppose $f: K \to \mathbb{R}^d$ is a map in general position, and $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint simplices of $K$ of dimension $s_1, \ldots, s_r \leq m$ such that

$$f|_{\sigma_i}^{-1}(f(\sigma_1) \cap \cdots \cap f(\sigma_r))$$

is contained in the interior of each simplex $\sigma_i$. Then there exists a ball $B^d$ in $\mathbb{R}^d$ such that

1. $B^d$ intersects each $f(\sigma_i)$ in a ball that is properly embedded in $B^d$, and that avoids the image of the singular set of $f|_{\sigma_i}$, as well as $f(\partial \sigma_i)$;
2. $B^d$ contains $f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ in its interior; and
3. $B^d$ does not intersect any other parts of the image $f(K)$.

**Proof.** Let us consider $S_i := f^{-1}(f(\sigma_1) \cap \cdots \cap f(\sigma_r)) \cap \sigma_i$. By general position [RS82, Thm 5.4] this is a polyhedron of dimension at most $s_1 + \cdots + s_r - (r - 1)d \leq rm - (r - 1)d$. By Theorem 7, we find $C_i \subseteq \sigma_i$ collapsible, containing $S_i$, and of dimension at most $rm - (r - 1)d + 1$. Figure 1 illustrates the case $r = 3$.

The dimension of the singular set of $f|_{\sigma_i}$ is at most $2s_i - d$. Hence, $C_i$ is disjoint from it since $(rm - (r - 1)d + 1) + (2s_i - d) - s_i \leq (r + 1)m - rd + 1$, which is negative in the metastable range. Thus, $f$ is injective in a neighbourhood of $C_i$.

Again by Theorem 7, we find in $\mathbb{R}^d$ a collapsible polyhedron $C_{r\delta}$ of dimension at most $rm - (r - 1)d + 2$ and containing $f(C_i) \cup \cdots \cup f(C_r)$. Figure 2 illustrates the construction for $r = 3$.

By general position we have the following properties:

1. $C_{r\delta}$ intersects $f(\sigma_i)$ exactly in $f(C_i)$. Indeed, in the metastable range, $rm - (r - 1)d + 2 + s_i - d \leq (r + 1)m - rd + 2 < 0$.
2. $C_{r\delta}$ does not intersect any other part of $f(K)$ (by a similar computation).
Let $d, r \geq 2$ be integers, let $\sigma_1, \ldots, \sigma_r$ be balls of dimensions $s_1, \ldots, s_r$. We define $s := s_1 + \cdots + s_r$.

Let $f$ be a continuous map, mapping the disjoint union of the $\sigma_i$ to a $d$-dimensional ball $B^d$, i.e.,

$$f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d.$$ 

We define the **Gauss map** $\tilde{f}$ associated to $f$

$$\tilde{f} : \sigma_1 \times \cdots \times \sigma_r \to B^d \times \cdots \times B^d,$$

by $(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$.

If, for each $i = 1, \ldots, r$,

$$f\sigma_1 \cap \cdots \cap f\partial\sigma_1 \cap \cdots \cap f\sigma_r = \emptyset,$$

then $\tilde{f}\partial(\sigma_1 \times \cdots \times \sigma_r) \subset B^d \times \cdots \times B^d$, avoids the **thin diagonal** $\delta_r(B^d) = \{(x, \ldots, x) \mid x \in B^d\}$ of $B^d$. Thus,

$$\partial(\sigma_1 \times \cdots \times \sigma_r) \to (B^d \times \cdots \times B^d) \setminus \delta_r(B^d). \quad (3)$$

1²The PL assumption is no loss of generality: if $K$ is a finite simplicial complex and $f : K \to \mathbb{R}^d$ is an almost $r$-embedding then $f$ can be slightly perturbed to a PL map with the same property.
Observe that \( \partial(\sigma_1 \times \cdots \times \sigma_r) \cong S^{s-1} \), where \( s := \sum_i s_i \), and \( (B^d \times \cdots \times B^d) \setminus \delta_r(B^d) \) is homotopy equivalent to \( S^{d(r-1)-1} \). Therefore, the map (3) defines an element
\[
\alpha(f) \in \pi_{s-1}(S^{d(r-1)-1}),
\]
which we call intersection class of \( f \).

**Lemma 10 (Local Disjunction Lemma).** Let \( m, d, r \) be three positive integers satisfying (2).
Let \( \sigma_1, \ldots, \sigma_r \) be balls of dimensions \( s_1, \ldots, s_r \leq m \) properly contained in a \( d \)-dimensional ball \( B \) and with \( \sigma_1 \cap \cdots \cap \sigma_r \) in the interior of \( B \).

1. Let us denote by \( \alpha \) the intersection class of the map \( \sigma_1 \cup \cdots \cup \sigma_r \to B^d \).
   
   If \( \alpha = 0 \), then there exists \( (r - 1) \) proper ambient isotopies of \( B \) that we can apply to \( \sigma_1, \ldots, \sigma_{r-1} \), respectively, to remove the \( r \)-intersection set; i.e., there exist \( (r - 1) \) proper isotopies \( H_1^1, \ldots, H_r^{-1} \) of \( B \) throwing \( \sigma_i \) onto \( H_i^1 \sigma_i \) and such that
   \[
   \sigma_1' \cap \cdots \cap \sigma_{r-1}' \cap \sigma_r = \emptyset.
   \]

2. Let us assume that \( \sigma_1 \cap \cdots \cap \sigma_r = \emptyset \) and \( \sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset \) and let \( z \in \pi_s(S^{d(r-1)-1}) \).
The exists \( J_z \) a proper ambient isotopy of \( B \) such that
   - \( J_z \sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_{r-1} \cap \sigma_r = \emptyset \),
   - The intersection class of \( f \) is \( z \), where
     \[
     f : (\sigma_1 \times I) \cup \sigma_2 \cup \cdots \cup \sigma_r \to B^d
     \]
     is defined as the inclusion on \( \sigma_i \) for \( i \geq 2 \), and for \( (x, t) \in \sigma_1 \times I, f(x, t) = J_z(x) \).

**Remark 11.**
- The proof of Lemma 10 is the technical core of the paper and will be given in Section 4. For \( r = 2 \), Lemma 10 already appears in Section 4 of Weber’s thesis [Web67]. Our contribution in the present paper is to show that the result holds for any \( r \geq 3 \).
- Roughly speaking, Part 1 of Lemma 10 means that if the intersection class vanishes, then one can resolve the \( r \)-intersection set.
Part 2 means that each element of \( \pi_s(S^{d(r-1)-1}) \) can be obtained by moving from a fixed solution to a new solution.
- In our previous paper [MW15], we consider the special case when all the global \( r \)-intersection points are isolated (i.e., the \( r \)-intersections are 0-dimensional). The “elimination” of these isolated \( r \)-intersections is achieved in two steps:

  (1) First, we obtain the algebraic cancellation of the \( r \)-intersection points by “finger moves”: we modify a given map \( f : K^m \to \mathbb{R}^d \) such that for each \( r \)-tuples of pairwise disjoint cells \( \sigma_1, \ldots, \sigma_r \) of \( K \), the intersection \( f\sigma_1 \cap \cdots \cap f\sigma_r \) consists of pairs of points of opposite intersection signs (hence, algebraically, they “cancel”).

  (2) In a second step, we geometrically cancel each pair of \( r \)-intersection points of opposite sign, and for this, we use an \( r \)-fold version of the Whitney Trick (a special case of the Local Disjunction Lemma). Hence, we obtain \( f\sigma_1 \cap \cdots \cap f\sigma_r = \emptyset \).

In other words, for the special case consider in our previous paper, the proof decomposes naturally into two steps: (1) first a “linking step” when we link cell together to introduce new \( r \)-intersection points (and therefore obtain the “algebraic cancellation” of the \( r \)-intersection points), (2) secondly, in an “unlinking step” we translate that algebraic cancellation into geometry (i.e., from intersection = 0, we obtain intersection = \( \emptyset \)).

In the present paper, these two steps are not so disjoint anymore: multiple cases of global \( r \)-intersection points can occur, resulting in singular set of various dimension (no only isolated
points). Therefore, we will have to merge the two steps (1) and (2): In our construction, we will first “unlink” the \( r \)-intersection points of a given \( r \)-tuple of cells (i.e., remove their \( r \)-intersection points), and immediately after we will “link” this \( r \)-tuple in order to permit the unlinking of \( r \)-tuples of higher dimension. (See both parts of Lemma 10: Part 1 corresponds to the “unlinking”, and Part 2 corresponds to the “linking”).

3 Combining Reduction and Local Disjunction

Here, we show how to use Lemmas 8 and 10 to prove the main theorem. The inductive argument used in the proof mirrors that of Section 5 in Weber’s thesis [Web67], where Theorem 2 is proven for \( r = 2 \).

Let us recall that given a map \( f: K^m \to \mathbb{R}^d \), we can induce

\[
\tilde{f}: K^r_\Delta \to \mathcal{O}_r \mathbb{R}^{r \times d} \quad \text{by} \quad (x_1, \ldots, x_r) \mapsto (f x_1, \ldots, f x_r),
\]

whose image avoids the diagonal \( \{(x, \ldots, x) \mid x \in \mathbb{R}^d\} \) if and only if \( f \) is an \( r \)-almost embedding.

We condense the two parts of Lemma 10 into the following technical statement:

**Lemma 12** (Inductive step). Let \( f: K^m \to \mathbb{R}^d \) be a general position map, and let \( F: K^r_\Delta \to \mathcal{O}_r S^{(r-1)d} \).

Let \( X \subset K^r_\Delta \) be a \( \mathcal{O}_r \)-stable subcomplex such that

- \( \tilde{f}|_X \) avoids the diagonal \( \{(x, \ldots, x) \mid x \in \mathbb{R}^d\} \subset \mathbb{R}^{r\times d} \),
- \( \tilde{f}|_X \) is \( \mathcal{O}_r \)-homotopic to \( F|_X \).

Let \( \sigma_1 \times \cdots \times \sigma_r \) be a cell of \( K^r_\Delta \setminus X \) whose boundary is contained in \( X \), and let us denote by \( Y \) the smallest \( \mathcal{O}_r \)-stable subcomplex of \( K^r_\Delta \) containing \( X \) and \( \sigma_1 \times \cdots \times \sigma_r \).

Then there exists a map \( g: K^m \to \mathbb{R}^d \) such that

- \( \tilde{g}|_Y \) avoids the diagonal \( \{(x, \ldots, x) \mid x \in \mathbb{R}^d\} \subset \mathbb{R}^{r\times d} \),
- \( \tilde{g}|_Y \) is \( \mathcal{O}_r \)-homotopic to \( F|_Y \).

**Proof.** Since the boundary of \( \sigma_1 \times \cdots \times \sigma_r \) is contained in \( X \) and \( \tilde{f}|_X \) avoids the diagonal, we have, for each \( i = 1, \ldots, r \),

\[
f^{-1}((f \sigma_1 \cap \cdots \cap f \sigma_r) \cap \sigma_i) \subset \delta_i.
\]

Furthermore, the map \( \tilde{f}: \partial(\sigma_1 \times \cdots \times \sigma_r) \to S^{(r-1)d} \) is homotopic to \( F \).

We are in position to apply Lemma 8: we find a ball \( B^d \) in \( \mathbb{R}^d \) with the three properties listed in the Lemma. Let us call \( \sigma'_i \) the sub-ball in \( \sigma_i \) properly embedded into \( B^d \), \( i.e., \sigma'_i \hookrightarrow B^d \), and \( f \delta \sigma'_i = \partial B^d \cap f \sigma'_i \).

By the Combinatorial Anulus Theorem [Bry02, 3.10], there exists an isotopy of \( \sigma_i \) in itself that progressively retracts \( \sigma_i \) to \( \sigma'_i \). \( I.e., \) there exists \( G_i^1: \sigma_i \to \sigma_i \) with \( G_0^i \) being the identity and \( G_1^i \) being a homeomorphism between \( \sigma_i \) and \( \sigma'_i \). We define a homotopy by

\[
G: \partial(I \times \sigma_1 \times \cdots \times \sigma_r) \xrightarrow{\partial(t \times I \times \cdots \times I)} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta \mathbb{R}^d \quad \text{and} \quad G^1(t \times x_1, \ldots, x_r) = f(t \times G_i^1 x_1, \ldots, G_i^1 x_r).
\]

Since

\[
\partial(\sigma_1 \times \cdots \times \sigma_r) \xrightarrow{f \times \cdots \times f} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta \mathbb{R}^d
\]

is homotopic to \( F \), and \( F \) is defined over \( \sigma_1 \times \cdots \times \sigma_r \), therefore the homotopy class of

\[
\partial(\sigma'_1 \times \cdots \times \sigma'_r) \xrightarrow{f \times \cdots \times f} B^d \times \cdots \times B^d \setminus \delta \mathbb{R}^d
\]
is trivial. Hence, we can use the first part of the Lemma 10 to find \((r - 1)\) proper ambient isotopies of \(B\), say \(H_1^t, \ldots, H_{r - 1}^t\), such that \(H_1^t(f\sigma_1) \cap \cdots \cap H_{r - 1}^t(f\sigma_r) = \emptyset\). This removes the \(r\)-intersection set.

To finish the proof, we also need to extend the equivariant homotopy between \(\tilde{f}\) and \(F\) on the cell \(\sigma_1 \times \cdots \times \sigma_r\), as the homotopy is already defined on \(\partial(\sigma_1 \times \cdots \times \sigma_r)\). This is when the second part of Lemma 10 becomes useful.

We define a map on \(\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \to \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta \mathbb{R}^d\) in the following way:

1. on \(\{0\} \times \sigma_1 \times \cdots \times \sigma_r\), we use \(F\),
2. on \([0, \frac{1}{3}]\) \times \partial(\sigma_1 \times \cdots \times \sigma_r), we use the homotopy from \(F\) to \((5)\),
3. on \([\frac{1}{3}, \frac{2}{3}]\) \times \partial(\sigma_1 \times \cdots \times \sigma_r), we use \(G\),
4. on \([\frac{2}{3}, 1]\) \times \partial(\sigma_1 \times \cdots \times \sigma_r), we use \((H_1^t \times \cdots \times H_{r - 1}^t \times \text{id}) \circ (fG_1^t \times \cdots \times fG_r^t)\),
5. \(\{1\} \times \sigma_1 \times \cdots \times \sigma_r\), we use \((H_1^t \times \cdots \times H_{r - 1}^t \times \text{id}) \circ (fG_1^t \times \cdots \times fG_r^t)\).

This defines a class \(\theta \in \pi \Sigma_{\dim \sigma_i}(S^{d(r - 1) - 1})\). To conclude, we need to have \(\theta = 0\) (this is the condition to be able to extend to homotopy between \(\tilde{f}\) and \(F\)).

By the second part of Lemma 10, we can perform a “second move” on \(\sigma_1\) with an ambient isotopy \(J_1\) of \(B\) such that

\[
\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \xrightarrow{(J_1 \times \text{id} \times \cdots \times \text{id}) \circ (H_1^t \times \cdots \times H_{r - 1}^t \times \text{id}) \circ (fG_1^t \times \cdots \times fG_r^t)} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta \mathbb{R}^d
\]

represents exactly \(-\theta\). Therefore, by using this last move, we can assume that \(\theta = 0\), i.e., we can extend the equivariant homotopy between \(\tilde{f}\) and \(F\), as needed for the induction.

**Proof of Theorem 2.** We are given \(F : K_\Delta^r \to \varnothing, S^{d(r - 1) - 1}\), and we want to construct \(f : K \to \mathbb{R}^d\) without global \(r\)-intersection points.

We start with a map \(f : K \to \mathbb{R}^d\) in general position. We are going to inductively use Lemma 12 to gradually remove all the global \(r\)-fold intersection of \(f\).

There are two levels in the induction. To describe these, let us fix a total ordering of the simplices of \(K\) that extends the partial ordering by dimension, i.e.,

\[
K = \{\tau_1, \ldots, \tau_N\}, \quad \dim \tau_i \leq \dim \tau_{i+1} \text{ for } 1 \leq i \leq N - 1.
\]

First, we give an informal plan of the “double induction” that we are going to use: we go over the list of simplices \(\tau_1, \ldots, \tau_N\), and for each simplex \(\tau_i\) we consider all the global \(r\)-intersection of \(\tau_i\) with all the simplices before \(\tau_i\) in the list. More precisely, we consider the list \(l_i\) of all \(r\)-tuples of pairwise disjoint simplices containing \(\tau_i\) and simplices before \(\tau_i\) in the list \(\tau_1, \ldots, \tau_N\). For each \(r\)-tuple in \(l_i\), we eliminate its global \(r\)-intersection points, by Lemma 12.

Therefore, once \(\tau_i\) is fixed, we have a new list \(l_i\). We are going to order \(l_i\) (by a notion of dimension), and then inductively scan over it and remove the global \(r\)-intersections points for each \(r\)-tuple in \(l_i\).

More formally, for the first level of the inductive argument, it suffices to prove the following: Suppose we are given a map \(f : K \to \mathbb{R}^d\) in general position with the following two properties:

---

13 We can always obtain the assumption \(\sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset\) by modifying the map \(f\) as follows [MW15, “Finger moves” in the proof of Lemma 43]; we pick \(r - 1\) spheres \(S^{s_2}, \ldots, S^{s_r}\) in the interior of \(B^d\) of dimension \(s_2, \ldots, s_r\) in general position and such that \(S^{s_2} \cap \cdots \cap S^{s_r}\) is a sphere \(S\). Then, for \(i = 2, \ldots, r\), we pipe \(\sigma_i^*\) to \(S^{s_i}\). The resulting map has the desired property.

This “piping” change can be absorbed by a slight modification (and renumbering) of the \(H_i^t\). The support of these modifications is a collection of regular neighborhoods of \(1\)-polyhedra (= paths used for piping).

Also, note that the cases when, by general position, \(\dim S < 0\) corresponds to \((d - s_2) + \cdots + (d - s_r) > d\), i.e., \((r - 1)d + s_1\) and \(\sum s_i\), and since \(s_1 + d \leq -3\), we have \((r - 1)d + 1 \geq \sum s_i\), and so \(\pi_{\dim S}^{s_i}(S^{d(r - 1) - 1}) = 0\).
1. Restricted to the subcomplex $L = \{\tau_1, \ldots, \tau_{N-1}\}$ the map $f|_L$ does not have any $r$-intersections between disjoint $r$-tuples of simplices;

2. $\tilde{f}$ restricted to $L^r_\Delta$ is $\mathcal{S}_r$-equivariantly homotopic to $F$, where $\tilde{f}$ is the map defined in Lemma 1.

Then we can redefine $f$ as to have these two properties on the whole of $K$. This is the first level of induction.

For the second level of the induction, let us define the dimension of a finite set of simplices as the sum of their individual dimensions. For the purposes of this proof, we use the terminology $k$-collection for a set of cardinality $k$. Consider those $(r-1)$-collections $t$ of simplices of $L$ that, together with $\tau_N$, form an $r$-collection of pairwise disjoint simplices. We fix a total ordering of these $(r-1)$-collections that extends the partial ordering given by dimension, i.e., we list them as

$$t_1, \ldots, t_M,$$

with $\dim t_i \leq \dim t_{i+1}$ for $1 \leq i < M$. (Thus, each $t_i$ is an $(r-1)$-collection of simplices of $L$, and $t_i$ joined with $\tau_N$ is a $r$-collection of pairwise disjoint simplices.) Once again, inductively, it suffices to prove the following: Assuming that $f$ has the two properties

1. For each $(r-1)$-collection $t_i$ in the list $t_1, \ldots, t_{M-1}$, the map $f$ does not have any $r$-intersection with preimages in the $r$-collection formed by adjoining $\tau_N$ to $t_i$.

2. the map $\tilde{f}$ is $\mathcal{S}_r$-equivariantly homotopic to $F$ on the complex

$$L^r_\Delta \cup \bigcup_{i \leq M-1} [t_i \cup \{\tau_N\}] \subseteq K^r_\Delta,$$

where the operator $[\cdot]$ converts an unordered $r$-collection of pairwise disjoint simplices of $K$ into the set of its corresponding cells\footnote{E.g., $[\{\alpha, \beta, \gamma\}] = \{\alpha \times \beta \times \gamma, \alpha \times \gamma \times \beta, \beta \times \alpha \times \gamma, \beta \times \gamma \times \alpha, \gamma \times \alpha \times \beta, \gamma \times \beta \times \alpha\}$.} in $K^r_\Delta$.

Then we can modify $f$ as to have these two properties on the lists $t_1, \ldots, t_M$.

This inductive step is directly implied by Lemma 12.

4 The Proof of the Local Disjunction Lemma

Our goal in this section is to prove Lemma 10, which was used in the previous section to prove the sufficiency of the deleted product criterion in the $r$-metastable range.

Throughout this section, we assume that $m, d, r$ are positive integers satisfying (2). Furthermore, we will denote the sum of the dimensions $s_i$ of the balls $\sigma_i$ by

$$s := s_1 + \ldots + s_r.$$

The proof of Lemma 10 is essentially inductive: we reduce from $r$ balls to $(r-1)$ balls. The trick is to consider the intersection pattern of the first $(r-1)$ balls $\sigma_1, \ldots, \sigma_{r-1}$ on $\sigma_r$. If each of the intersections $\sigma_i \cap \sigma_r$, $1 \leq i \leq r-1$, were a ball properly embedded in $\sigma_r$, then we could solve the situation first at the level of $\sigma_r$ (i.e., remove the $(r-1)$-intersections between the $\sigma_i \cap \sigma_r$), and then extend the solution to $B$, thus completing the induction.

However, the intersections $\sigma_i \cap \sigma_r$ need not be balls, so our first task is to move $\sigma_1, \ldots, \sigma_{r-1}$ inside $B$ as to modify their intersection with $\sigma_r$. As it will turn out, if we manage to increase sufficiently the connectedness of the intersections $\sigma_i \cap \sigma_r$, then Theorem 7 becomes useful to reduce the situation (as in the proof of Lemma 8) in such a way that the intersections $\sigma_i \cap \sigma_r$ do become balls. For this to work, $\sigma_i \cap \sigma_r$ needs to be $\dim(\sigma_1 \cap \cdots \cap \sigma_r)$-connected.
4.1 Increasing the connectivity of the intersections

**Proposition 13.** With the same notations as in Lemma 10, for each \( i = 1, \ldots, r - 1 \), there exists a proper ambient isotopy \( H_t \) of \( B \) such that \( H_1(\sigma_i \cap \sigma_r) \) is \( \dim(\sigma_1 \cap \cdots \cap \sigma_r) \)-connected, and such that

\[
I \times \partial(\sigma_1 \times \cdots \times \sigma_i \times \cdots \times \sigma_r) \xrightarrow{\text{incl} \times \cdots \times \text{incl}} (B^d \times \cdots \times B^d) \setminus \delta_r(B^d)
\]

is well-defined, i.e., its image is disjoint from the diagonal \( \delta_r(B^d) \).

**Proof.** Proposition 13 follows directly by inductively using the Lemma 14 (below), as in [Mil61, Lemma 2].

**Lemma 14.** (a) With the same notation as above, for all \( 1 \leq k \leq \dim(\sigma_i \cap \cdots \cap \sigma_r) \) and \( S^k \to \sigma_i \cap \sigma_r \) representing a homotopy class in \( \pi_k(\sigma_i \cap \sigma_r) \), there exists a proper ambient isotopy \( H_t \) of \( B \) such that, for \( j < k \),

\[
\pi_j(H_1(\sigma_i \cap \sigma_r) \cong \pi_j(\sigma_i \cap \sigma_r),
\]

and

\[
\pi_k(H_1(\sigma_i \cap \sigma_r) \cong \pi_k(\sigma_i \cap \sigma_r)/a \text{ subgroup containing } [S^k].
\]

(b) An analogous statement holds for \( k = 0 \): If \( \sigma_i \cap \sigma_r \) has more than one connected component, then there exists a proper ambient isotopy \( H_t \) of \( B \) such that \( H_1(\sigma_i \cap \sigma_r) \) has one less connected component.

In both cases (a) and (b) with have the following additional property of \( H_t \): the map (6) defined using \( H_t \) avoids the diagonal.
Here, we only present the proof of the part (a), i.e., for \( k \geq 1 \). For \( k = 0 \), the construction is similar, and is already presented in [MW15] as piping and unpiping.

Our main technique in the proof is to use surgery (as presented by Milnor [Mil61]) to increase the connectivity of \( \sigma_i \cap \sigma_r \). The precise definition of surgery used in our situation is given later (Definition 25).

Figure 3 illustrates the situation, and Figure 4 tries to illustrate how we intend to ‘kill’ a homotopy class of \( S^k \in \pi_k(\sigma_i \cap \sigma_r) \) by surgery.

**Remark 15.** We decompose the proof of Lemma 14 into a series of Lemmas. For the first two Lemmas, we need to use a PL analogous of vector bundles for smooth manifold. In the PL category, this analogous notion is called block bundles. We review the part of this theory that we need in Appendix A. (See also [Bry02] for a rapid introduction, or the original [RS68a]). We will need results from [RS68a, RS68b]. Since we only work in the PL category, we sometimes only say bundle instead of block bundle.

First we render, once and for all, the intersections transverse:

**Lemma 16.** With the same notations as in Lemma 10, we can assume that \( \sigma_r \) is unknotted in \( B^d \), i.e.,
\[
B^d = \sigma_r \times [-1,1]^{d-s_r},
\]
and we can also assume that \( \sigma_i \) intersects \( \sigma_r \) transversely (Definition 65), i.e., for \( \varepsilon > 0 \) small enough, \( \sigma_r \times \varepsilon[-1,1]^{d-s_r} \) is a normal block bundle to \( \sigma_r \) in \( B^d \), and we have
\[
\sigma_i \cap (\sigma_r \times \varepsilon[-1,1]^{d-s_r}) = (\sigma_i \cap \sigma_r) \times \varepsilon[-1,1]^{d-s_r}.
\]

**Proof.** The first statement follows from Zeeman’s Unknotting of balls. The second statement follows by Theorem 66: there exists an \( \varepsilon \)-isotopy of \( B \) carrying \( \sigma_i \) locally transverse to \( \sigma_r \). Using a collar on \( \partial B \), we can furthermore assume that this isotopy is fixed on \( \partial B \). \( \square \)

**Remark 17.** In the sequence of lemmas that follows, \( S^k \to \sigma_i \cap \sigma_r \) represents an homotopy class in \( \pi_k(\sigma_i \cap \sigma_r) \), which we want to “kill”.

**Observation 18.** We have
\[
2 \dim(S^k) + 1 \leq \dim(\sigma_i \cap \sigma_r).
\]
Indeed, this is the case if
\[
2k + 1 \leq s_i + s_r - d
\]
So we have to show
\[
k < \frac{s_i + s_r - d}{2}
\]
Since \( k \leq s - (r - 1)d \), it is sufficient to show
\[
s - (r - 1)d < \frac{s_i + s_r - d}{2}
\]
and, after rearrangement and using \( s_i \leq m \), we get the sufficient condition
\[
2(r - 1)m < (2r - 3)d,
\]
which is the case since \( \frac{2(r - 1)m}{2r - 3m} \leq \frac{r + 1}{r}m < d \), where the first inequality is true for \( r \geq 3 \), and the second follows from the metastable range.

**Lemma 19.** In the situation given by Lemma 16, let \( a : S^k \to \sigma_i \cap \sigma_r \) represents an homotopy class in \( \pi_k(\sigma_i \cap \sigma_r) \). Then there exists an embedded copy of \( S^k \subset \sigma_i \cap \sigma_r \) such that its inclusion map is homotopic to \( a \), and with the two additional properties:

(1) the normal block bundle of \( S^k \subset \sigma_i \cap \sigma_r \) is trivial.
(2) Let $N_{S^k}$ be a regular neighborhood of $S^k$ inside $\sigma_i \cap \sigma_r$. Then

$$N_{S^k} \cong S^k \times [-1, 1]^{s_i + s_r - d - k},$$

containing $S^k$ as $S^k \times 0$.

**Proof.** The existence of the embedded copy of $S^k$ follows by general position (and the above observation).

The first property follows from Theorem 70 from Appendix A: we only need to check that the tangent bundle of $\sigma_i \cap \sigma_r$ is stably trivial. To see it: let us consider the normal bundle of $\sigma_i \cap \sigma_r$ in $\sigma_i$ which is isomorphic to $\varepsilon^{d-s}$ (by hypothesis), hence (using notations defined in Appendix A and Theorem 68)

$$t(\sigma_i \cap \sigma_r) \oplus \varepsilon^{d-s} = t(\sigma_i \cap \sigma_r) = e(\sigma_i \cap \sigma_r).$$

I.e., $t(\sigma_i \cap \sigma_r)/(\sigma_i \cap \sigma_r)$ is stably trivial.

The second property about $N_{S^k}$ follows by the correspondence between regular neighborhoods and normal block bundles (Theorem 54 in Appendix A).

**Lemma 20.** In the situation given by Lemma 19, with $S^k \subset \sigma_i \cap \sigma_r$ and the two additional properties. There exists a ball $D^{k+1}$ in $\sigma_r$ with

$$D^{k+1} \cap \sigma_i = \partial D^{k+1} = S^k,$$

and which avoids the other $\sigma_j$.

Furthermore, the trivialisation of $N_{S^k}$ can be extended to $D^{k+1}$, i.e., there exists in $\sigma_r$

$$N_{D^{k+1}} \cong D^{k+1} \times [-1, 1]^{s_i + s_r - d - k}$$

containing $D^{k+1}$ as $D^{k+1} \times 0$ and with

$$N_{D^{k+1}} \cap \sigma_i = N_{S^k} \cong S^k \times [-1, 1]^{s_i + s_r - d - k},$$

and this last homeomorphism is the restriction of (7).

**Remark 21.** For proving the second part of Lemma 20, we could use the PL-analogue of Stiefel manifolds [RS68b, p. 274]: the obstruction to extending the trivialisation of $S^k$ is always trivial in the metastable range. But, to avoid entering more deeply into the theory of block bundles, we rather use the following unknotting theorem of Hudson\footnote{This step follows a proof of A. Skopenkov in [Sko05, p. 9].}:

**Theorem 22** ([Hud69, Unknotting Theorem Moving the Boundary, 10.2, p. 199]). If $f, g : M^m \to Q^q$ are proper PL embeddings between manifolds $M$ and $Q$. Then $f, g$ homotopic as maps of pairs $(M, \partial M) \to (Q, \partial Q)$ implies that $f, g$ are ambient isotopic provided that

- $M$ is compact
- $q - m \geq 3$
- $(M, \partial M)$ is $(2m - q + 1)$-connected
- $(Q, \partial Q)$ is $(2m - q + 2)$-connected

A consequence of Hudson Theorem is

**Corollary 23.** Let $f, g : S^k \times [-1, 1]^{m-k} \to B^q$ be two proper embeddings, then $f$ and $g$ are ambient isotopic, provided

$$q \geq m + k + 3.$$
Proof. Since \( q - m \geq k + 3 \geq 3 \), and \((B^q, \partial B^q)\) is clearly sufficiently connected, we only need to analyse the connectivity of the pair

\[ (S^k \times [-1, 1]^{m-k}, S^k \times S^{m-k-1}) \]

which we need to be \((2m - q + 1)\)-connected. Let us consider the exact sequence in homotopy for this pair

\[
\cdots \rightarrow \pi_1(S^k \times S^{m-k-1} \subseteq \partial X) \rightarrow \pi_1(S^k \times [-1, 1]^{m-k} \subseteq X) \rightarrow \pi_1(X, \partial X) \rightarrow \cdots
\]

For \( i < m - k - 1 \), the above sequence can be rewritten as

\[
\cdots \rightarrow \pi_i(S^k) \rightarrow \pi_i(S^k) \rightarrow \pi_i(X, \partial X) \rightarrow \cdots
\]

Since the \( \pi_i(S^k) \rightarrow \pi_i(S^k) \) is an isomorphism, we get \( \pi_i(X, \partial X) = 0 \) as long as \( i < m - k - 1 \). So we are left with checking

\[ 2m - q + 1 < m - k - 1, \]

I.e., \( m + k + 2 < q \). \( \square \)

We can now proceed to the proof of Lemma 20:

Proof of Lemma 20. The first statement follows by general position and the metastable range hypothesis.

To prove the existence of \( N_{D^{k+1}} \), let us first take a regular neighborhood \( V \) of \( D^{k+1} \) in \( \sigma_r \). We can assume that

\[ V \cap \sigma_i = N_S \cong S^k \times [-1, 1]^{s_i+s_r-d-k}. \]

If \( N_S \) unknots in \( V \) (in the sense of Theorem 22), then the existence of \( N_{D^{k+1}} \) is immediate: we use an “standard” version of \( N_S \) to construct \( N_{D^{k+1}} \), and move it to our situation by the isotopy given by the unknotting theorem.

So we only need to check the hypothesis of Corollary 23. For us here, \( q := s_r \) and \( m := s_i+s_r-d \), so we need

\[ s_r \geq (s_i + s_r - d) + k + 3. \]

which reduces to \( d - s_i - 3 \geq k \), which is true if \( d - s_i - 3 \geq s - (r - 1)d \), and this is implied by \( rd \geq (r + 1)m + 3 \), i.e., the metastable range hypothesis. \( \square \)

**Lemma 24** (Existence of the surgery-handle). *In the situation given by Lemma 20, there exists in \( B \) a handle

\[ T := D^{k+1} \times [-1, 1]^{s_i+s_r-d-k} \times [-1, 1]^{d-s_r} \]

such that

- \( T \) contains \( D^{k+1} \times [-1, 1]^{s_i+s_r-d-k} \times 0 \),
- \( T \) intersects \( \sigma_i \) as \( D^{k+1} \times [-1, 1]^{s_i+s_r-d-k} \times 0 \),
- \( T \) intersects \( \sigma_r \) as \( S^k \times [-1, 1]^{s_i+s_r-d-k} \times [-1, 1]^{d-s_r} \).

*Figure 5 illustrates the handle \( T \).*

Proof. This follows from the construction of \( D^{k+1} \) (Lemma 20) and the transversality of the intersection of \( \sigma_i \) and \( \sigma_r \) (Lemma 16). \( \square \)

**Definition 25** (Ambient surgery). Let \( S^k \) be an embedded sphere in (the interior of) \( \sigma_i \) with a trivialized regular neighborhood \( S^k \times [-1, 1]^{s_i-k} \), and let \( T \) be a handle based on \( S^k \), i.e.,

\[ T = D^{k+1} \times [-1, 1]^{s_i-k} \subseteq B^d \]
for a ball $D^{k+1}$ with

$$T \cap \sigma_i = \partial D^{k+1} \times [-1,1]^{s_i-k} = S^{k+1} \times [-1,1]^{s_i-k}.$$  

Using $T$, we perform a **ambient surgery** on $\sigma_i \subset B^d$ by constructing the new manifold

$$\sigma^*_i := (\sigma_i \setminus (S^k \times [-1,1]^{s_i-k})) \cup (D^{k+1} \times \partial [-1,1]^{s_i-k}) \subset B^d \quad (8)$$

In order to attach the handle $T$ we made several choices: the choice of $S^k$, its regular neighborhood $S^k \times [-1,1]^{s_i-k}$, the ‘core’ $D^{k+1}$, etc. In the next Lemma, we show that, up to isotopy, there is only one way to attach a handle $T$ to $\sigma_i$:

**Lemma 26.** If $S^k$ and $\tilde{S}^k$ are embedded spheres in $\sigma_i$ with a trivialized regular neighborhoods and handles $T$ and $\tilde{T}$ as in Definition 25.

Then performing a surgery on $\sigma_i$, using $T$ or $\tilde{T}$ produces two homeomorphic manifolds $\sigma^*_i$ and $\tilde{\sigma}^*_i$ that are connected by a proper ambient isotopy of $B^d$.

**Proof.** By Irwin’s Theorem [Zee66, Ch. VIII, Theorem 24], there exists a proper isotopy of $\sigma_i$ “throwing” $\tilde{S}^k$ onto $S^k$, so we can assume $S^k = \tilde{S}^k$ (since this isotopy can be extended to $B^d$ [Hud66]).

Then, by the uniqueness of regular neighborhoods, we can assume that the trivialisation of the normal block bundle are identical [RS68a, Theorem 4.4].

We have reached the situation where $T \cap \sigma_i = \tilde{T} \cap \sigma_i$.

Let us take a cone $C$ in $B^d$ over $D^{k+1} \cup \tilde{D}^{k+1}$. By general position\(^\text{16}\), this cone avoids $\sigma_i$ (except on $S^k = \tilde{S}^k$). Let us take a regular neighborhood $V$ of the collapsible space $C$ in $B^d$ relative to $S^k = \tilde{S}^k$. Hence, $V$ is a $d$-ball, and

$$V \cap \sigma_i = S^k, \quad [HZ64, p. 719, (iii)].$$

Inside of $V$, we can find an ambient isotopy (fixed on the boundary) ‘throwing’ $\tilde{D}^{k+1}$ to $D^{k+1}$, hence, we can assume that $D^{k+1} = \tilde{D}^{k+1}$.

We have reached the situation where both $T$ and $\tilde{T}$ are equal on $\sigma_i$ and have the same ‘core’ $D^{k+1} = \tilde{D}^{k+1}$.

To conclude, let us take a regular neighborhood $N$ of $D^{k+1}$. We can assume that

- $N \cap \sigma_i = S^k \times [-1,1]^{s_i-k} = T \cap \sigma_i = \tilde{T} \cap \sigma_i$, and
- $T, \tilde{T} \subseteq N$ (after, possibly, shrinking the handles).

We have that

$$\sigma^*_i \cap N = D^{k+1} \times [-1,1]^{s_i-k} \cong \tilde{\sigma}^*_i \cap N$$

and

$$\partial(\sigma^*_i \cap N) = \partial(\tilde{\sigma}^*_i \cap N) = S^k \times \partial [-1,1]^{s_i-k}.$$  

Hence, to conclude, we only have to check that $D^{k+1} \times \partial [-1,1]^{s_i-k}$ unknits inside of $N$ (keeping the boundary fixed). First, we observe that two proper maps $D^{k+1} \times S^{s_i-k-1} \rightarrow B^d$ that are equal on the boundary are always homotopic (by a straight-line homotopy). Hence, by Irwin’s Theorem [Zee66, Ch. VIII, Theorem 24], we only need to check

$$2s_i - d + 1 \leq s_i - k + 1, \quad \text{i.e.,} \quad k + 3 \leq d - s_i$$

which is true if $rm - (r-1)d + 3 \leq d - s_i$, and this is implied by $(r+1)m + 3 \leq rd$ (the metastable range hypothesis). \(□\)

\(^\text{16}\)We have to check, e.g., $(k+2) + s_r - d < 0$, i.e., $k < d - s_r - 2$, which is true if $s - (r-1)d < d - s_r - 2$, and this is implied by the metastable hypothesis $(r+1)m + 3 \leq rd$.  

16
Figure 6: We perform two complementary surgeries on \( \sigma_i \) such that the resulting manifold \( \sigma_i' \) is a ball homeomorphic to \( \sigma_i \).

**Lemma 27** (Existence of a complementary handle). Let \( S^k \) and \( T \) be as in Definition 25. Then there exists an handle (see Figure 6)

\[ T^c = D^{k+2} \times [-1, 1]^{s_i-k-1} \subseteq B^d \]

with

- \( T^c \cap \sigma_i = S^{k+1} \times [-1, 1]^{s_i-k-1} \) for a \((k+1)\)-sphere \( S^{k+1} \) such that
- \( S^{k+1} \) intersects the cocore of \( T \)

\[ 0^{k+1} \times \partial [-1, 1]^{s_r-k} \subseteq \partial T \]

at exactly one point.

- Furthermore, we can assume that \( S^{k+1} \) is at positive distance of \( \sigma_r \).

**Proof.** By the previous lemma, there exists, up to proper isotopy of \( B^d \), an unique way to to perform a surgery by the handle \( T \) on \( \sigma_i \). From this fact, the existence of the complementary handle \( T^c \) is immediate.

For the last property, we need to shift \( S^{k+1} \) to general position.

**Remark 28.** We call \( T^c \) the ‘complementary handle’ to \( T \).

**Lemma 29.** Let \( \alpha \in \pi(\sigma_i \cap \sigma_r) \). Then there exists a sphere \( S^k \subseteq \sigma_i \cap \sigma_r \) and a handle \( T \) as in Definition 25 such that performing a surgery on \( \sigma_i \) by the handle \( T \), followed by a surgery by the handle \( T^c \) produces a manifold \( \sigma_i^{**} \) which is a \( s_i \)-ball. Furthermore, for \( j < k \)

\[ \pi_j(\sigma_i^{**} \cap \sigma_r) \cong \pi_j(\sigma_i \cap \sigma_r) \]

and

\[ \pi_k(\sigma_i^{**} \cap \sigma_r) \cong \pi_k(\sigma_i \cap \sigma_r)/a \text{ subgroup containing } \alpha. \]

**Proof.** The existence of \( S^k \) is given by Lemma 19. The existence of \( T \) is given by Lemma 24. The existence of \( T^c \) is given by Lemma 29.

To conclude, one notices

- By the first surgery using the handle \( T \), we have ‘killed’ the homotopy class \( \alpha = [S^k] \in \pi_k(\sigma_i \cap \sigma_r) \), i.e., by construction,

\[ \sigma_i^* \cap \sigma_r = ((\sigma_i \cap \sigma_r) \setminus (S^k \times [-1, 1]^{s_i+s_r-d-k} \times 0)) \cup (D^{k+1} \times \partial [-1, 1]^{s_i+s_r-d-k} \times 0) \]

and so we have killed \([S^k]\) in the sense of [Mil61, Lemma 2].

\[^{17}\text{We want } (k+1) + (s_i + s_r - d) - s_r < 0, \text{ i.e., } k < d - s_r - 1. \text{ But } k \leq s - (r-1)d, \text{ so we only need } s - (r-1)d < d - s_r - 1, \text{ i.e., } s + s_r + 1 < rd, \text{ and this is true in the metastable range } (r+1)m + 3 \leq rd.\]
• The effect of two surgeries by complementary handles cancels, hence $\sigma_i^{**}$ is a $s_i$-ball [RS82, Lemma 6.4].

Proof of Lemma 14. One combines Lemma 29 with Zeeman’s Unknotting of balls.

4.2 Proof of Lemma 10 for balls

Proposition 30. The first part of Lemma 10 is true if we add the following hypothesis: for each $i = 1, \ldots, r - 1$,

$$\sigma_i \cap \sigma_r$$

is a $(s_i + s_r - d)$-ball properly contained in $\sigma_r$.

Before proving Proposition 30, we need two Definitions and two Lemmas.

Definition 31. Let $g: \sigma_1 \sqcup \cdots \sqcup \sigma_r \rightarrow B^d$ be balls properly mapped inside $B^d$, with the dimensional restriction of Lemma 10.

We say that $g$ is a suspended map if it has the following structure

• $g\sigma_r$ is an embedded and unknotted ball inside $B^d$, hence we can assume that $B^d = (g\sigma_r) \ast S^{d-s_r-1}$,

for some $S^{d-s_r-1}$.

• For $i = 1, \ldots, r - 1$, the preimage by $g|\sigma_i$ of $g\sigma_r \subset B^d$ is a ball properly embedded and unknotted inside $\sigma_i$. I.e.,

$$\sigma_i = g|\sigma_i^{-1}(g\sigma_r) \ast S^{d-s_r-1},$$

for some $S^{d-s_r-1}$.

Notation: $\tau_i := g|\sigma_i^{-1}(g\sigma_r) \subset \sigma_i$.

• For $i = 1, \ldots, r - 1$, $g$ is defined as follows:

  – the sphere $S^{d-s_r-1} \subset \sigma_i$ is mapped homeomorphically to $S^{d-s_r-1} \subset B^d$,

  – the ball $\tau_i \subset \sigma_i$ is properly map to $g\sigma_r$.

  – $g$ is defined elsewhere on $\sigma_i$ by interpolating in the obvious way between the two joins

$$\sigma_i = \tau_i \ast S^{d-s_r-1} \text{ and } B^d = (g\sigma_r) \ast S^{d-s_r-1}.$$  

Figure 7 shows on the right a suspended map.

Lemma 32 (Suspended maps, Figure 7). Let $g: \sigma_1 \sqcup \cdots \sqcup \sigma_r \rightarrow B^d$ be balls properly embedded inside $B^d$ in general position, with the dimensional restriction of Lemma 10, and with the additional hypothesis of Proposition 30, i.e., for each $i = 1, \ldots, r - 1$, $\sigma_i \cap \sigma_r$ is a $(s_i + s_r - d)$-ball properly contained in $\sigma_r$.

Then there exists a suspended map $g: \sigma_1 \sqcup \cdots \sqcup \sigma_r \rightarrow B^d$, such that

• the intersection classes of $f$ and $g$ are equal

• $f|\sigma_r = g|\sigma_r$.

• For $i = 1, \ldots, r - 1$, we have $f|\sigma_i^{-1}(\sigma_r) = g|\sigma_i^{-1}(\sigma_r) =: \tau_i$.

• $f|\tau_i = g|\tau_i$.

Proof. To simplify notation, during the proof we assume that $g$ is an inclusion map, i.e., $\sigma_i \subset B^d$. The existence of $g$ will follow from the facts that

• $\sigma_r$ unknots in $B^d$,
Figure 7: Using that $\sigma_r$ unknots in $B^d$ and that $\sigma_i \cap \sigma_r$ unknots in $\sigma_r$, we change the setting to a suspension over $\sigma_r$.

- $\sigma_i \cap \sigma_r$ unknots inside of $\sigma_r$,
- the modifications applied during the unknotting on $\sigma_1, \ldots, \sigma_{r-1}$ do not change the homotopy class that we are interested into.

More precisely, since $\sigma_i \cap \sigma_r$ unknots inside of $\sigma_r$, we can represent $\sigma_r$ as

$$\sigma_r = (\sigma_i \cap \sigma_r) \ast S^{d-s_i-1}, \quad \text{and so} \quad B^d = (\sigma_i \cap \sigma_r) \ast S^{d-s_i-1} \ast S^{d-s_r-1}$$

Hence, we define a retraction from

$$B^d \setminus (\emptyset \ast S^{d-s_i-1} \ast \emptyset) \quad \text{onto} \quad (\sigma_i \cap \sigma_r) \ast \emptyset \ast S^{d-s_i-1},$$

and, using this retraction on $\sigma_i$, we can assume that $\sigma_i \subseteq (\sigma_i \cap \sigma_r) \ast S^{d-s_r-1}$.

If $B^{d-s_i}$ is the “standard ball” in $\sigma_r$ with boundary $S^{d-s_i-1}$, then $\sigma_i \cap \sigma_r$ intersects this ball precisely once, and this translates into the fact that $\partial \sigma_i$ is a generator of the homotopy group $\pi_{s_i-1}(\partial(\sigma_i \cap \sigma_r) \ast S^{d-s_r-1}) \cong \mathbb{Z}$.

Hence, we can assume that $\sigma_i = (\sigma_i \cap \sigma_r) \ast S^{d-s_r-1}$, after an homotopy of $\sigma_i$ inside of $(\sigma_i \cap \sigma_r) \ast S^{d-s_r-1}$ (keeping $\partial \sigma_i$ on $\partial B^d$).

**Lemma 33 (Commuting Square for Suspended Maps).** Let $f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be a suspended map. Then there exists a diagram commuting up to homotopy

$$\begin{array}{ccc}
\partial(\sigma_1 \times \cdots \times \sigma_r) & \cong & \Sigma^{d(r-1)-s_r-(r-2)} \partial(\tau_1 \times \cdots \times \tau_{r-1}) \\
\downarrow & & \downarrow \\
B \times \cdots \times B \setminus \delta_r(B) & \cong & \Sigma^{d(r-1)-s_r-(r-2)}((\sigma_r \times \cdots \times \sigma_r) \setminus \delta_{r-1}(\sigma_r))
\end{array}$$

(9)

where

- the map on the left is the obvious one, representing an element $\alpha \in \pi_{s_i-1}(S^{d(r-1)-1})$,
- the map on the right is the suspension $\Sigma$ applied ($d(r-1) - s_r(r-2)$) times to the map $\partial(\tau_1 \times \cdots \times \tau_{r-1}) \to (\sigma_r \times \cdots \times \sigma_r) \setminus \delta_{r-1}(\sigma_r)$,

and this map represents an element

$$\beta \in \pi_{s_1+s_r-d} \cdots (s_{r-1}+s_r-d-1)(S^{s_r(r-2)-1})$$
The two horizontal maps are defined within the proof.

We defer the proof of Lemma 33 to Section 4.4.

Proof of Proposition 30. We apply Lemma 32, to get a suspended map $f: \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ with the same intersection class as our initial map.

From Diagram (9) in Lemma 33
\[ \Sigma^{d(r-1)-s_r(r-2)}\beta = \alpha. \]

But $\alpha = 0$, and we are in the stable range of the suspension homomorphism\(^{18}\), hence $\beta = 0$. Therefore, using the third property in Lemma 32, we have reduced the problem to that of removing the $(r-1)$-intersection set between

$\sigma_1 \cap \sigma_r, \ldots, \sigma_{r-1} \cap \sigma_r \subseteq \sigma_r,$

which are $(r-1)$ balls embedded in $\sigma_r$ in the metastable range\(^{19}\) for $r-1$.

Thus, we are in position to work inductively: since $\sigma_r$ unknots in $B^d$, we have $B^d = \sigma_r \ast S^{d-s_r-1}$, so proper ambient isotopies of $\sigma_r$ can be extended to $B^d$.

The beginning of the induction (for $r = 3$) reduces to the classical case of two balls intersecting inside a third ball, and is solved in Weber [Web67, Prop. 1 & 2].

We are left with proving Lemma 33, this is what the rest of this section is devoted to. Before starting the proof (that will be split into three Lemmas in Section 4.4), we introduce another kind of configuration space that will be useful for us during that proof.

4.3 Deleted Joins

Let $K$ be a simplicial complex. We define the $k$-fold $k$-wise topological deleted join of $K$

\[ K^*r \setminus \delta_r K := K \ast \cdots \ast K \setminus \left\{ \frac{1}{r}x + \cdots + \frac{1}{r}x \mid x \in K \right\}, \]

and the $k$-fold $k$-wise simplicial deleted join of $K$

\[ (K)^*r := \{ \tau_1 \ast \cdots \ast \tau_r \mid \tau_i \in K \text{ and } \tau_1 \cap \cdots \cap \tau_r = \emptyset \}. \]

Both spaces $K^*r \setminus \delta_r K$ and $(K)^*r$ have a natural $S_r$-action by permutation of the coordinates.

Lemma 34. $K^*r \setminus \delta_r K$ can be $S_r$-equivariantly retracted onto $(K)^*r$.

---

\(^{18}\)The suspension $\pi_i(S^n) \to \pi_{i+2}(S^{n+1})$ is an isomorphism if $i < 2n-1$ [Hat02, Corollary 4.24]. For us this translates into

\[ s + (r-2)s_r - d(r-1) - 1 < 2(s_r(r-2)) - 1, \text{ i.e.,} \]

\[ (s_1 - s_r) + \cdots + (s_{r-2} - s_r) + s_{r-1} + 2 < d(r-1), \]

which is trivially true if $m \leq d - 3$.

\(^{19}\)We must have for $i = 1, \ldots, r-1$,

\[ (r-1)s_r \geq r(s_i + s_r - d) + 3, \text{ i.e.,} \]

\[ rd \geq rs_i + s_r + 3, \]

and this is implied by

\[ rd \geq (r+1)m + 3 \geq rs_i + s_r + 3. \]
Proof. Our proof is modelled on the deleted product case [Hu60, Lemma 10.1].

Warm up. We first show the main trick on a very simple case. I.e., assuming ∆ is the simplex on two vertices \{x, y\}, we construct an homeomorphism

\[ \Delta \ast \Delta \cong (\Delta)^2 \ast \delta^2(\Delta), \]  

(see Figure 8).

Once we have this homeomorphism the conclusion is immediate.

First, we name the four vertices of \( \Delta \ast \Delta \) as \( \{x, y, x', y'\} \) (with \( \{x, y\} \in \Delta \ast \emptyset \) and \( \{x', y'\} \in \emptyset \ast \Delta \)). Then every point of \( \Delta \ast \Delta \) is represented as

\[ x = ax + by + a'x' + b'y' \quad \text{with} \quad a, a', b, b' \in [0, 1], \]

Assuming that \( a \geq a' \), \( b \geq b' \) and that \( a' \) or \( b' \) is non-zero, we rewrite \( x \) as

\[ x = \left( a - a' + b - b' \right) \left( \frac{a - a'}{a - a' + b - b'} x + \frac{b - b'}{a - a' + b - b'} y \right) + \]

\[ (2a' + 2b') \left( \frac{a'}{2a' + 2b'} (x + x') + \frac{b'}{2a' + 2b'} (y + y') \right). \]

The other possible orders on \( a, b, a', b' \) can be worked on in a similar way, and will correspond to other faces of \((\Delta)^2\).

The general case. Let \( K \) be a simplicial complex. We can write any simplex of \( K \ast r \) as

\[ (\Delta^1 \ast \omega^1) \ast \cdots \ast (\Delta^r \ast \omega^r) \]

for some simplices \( \Delta^i, \omega^i \in K \), with the condition

\[ \Delta^1 = \cdots = \Delta^r \quad \text{and} \quad \omega^1 \cap \cdots \cap \omega^r = \emptyset. \]

Our goal is to build an homeomorphism

\[ (\Delta^1 \ast \omega^1) \ast \cdots \ast (\Delta^r \ast \omega^r) \cong (\Delta)^{r \ast} \ast \delta^r(\Delta) \ast (\omega^1 \ast \cdots \ast \omega^r). \]

where \( \Delta \) is any of the \( \Delta^i \). Once we have this homeomorphism the conclusion is immediate.

Let us name \( p^i_j \) the vertices spanning \( \Delta^i \), and \( q^i_j \) the vertices spanning \( \omega_i \). Then, any \( x \in (\Delta^1 \ast \omega^1) \ast \cdots \ast (\Delta^r \ast \omega^r) \) can be written as

\[ x = \sum_{i,j} p^i_j(x)p^i_j + \sum_{i,j} q^i_j(x)q^i_j, \quad \text{with} \quad p^i_j(x), q^i_j(x) \geq 0 \quad \text{and} \quad \sum_{i,j} p^i_j(x) + \sum_{i,j} q^i_j(x) = 1. \]
We assume that at least one of the \( q_i^j(x) \) is non-zero (otherwise nothing has to be done). Then, we write \( x \) as

\[
x = \sum p_j^i(x) \left( \frac{1}{\sum p_j^i(x)} \sum p_j^i(x)p_j^i \right) + \sum q_j^i(x) \left( \frac{1}{\sum q_j^i(x)} \sum q_j^i(x)q_j^i \right)
\]

The first term lies in \( \Delta^1 \times \cdots \times \Delta^r \), and the second in \( \omega^1 \times \cdots \times \omega^r \). To further decompose the first term, we name \( p_i \) the minimum of \{\( p_j^i(x) \), ..., \( p_j^i(x) \}\}, then

\[
\sum_{i,j} p_j^i(x)p_j^i = \sum_{i,j} (p_j^i(x) - p_j^i) + \sum_j (p_1p_1 + \cdots + p_rp_r)
\]

Hence, we can write \( \sum_{i,j} p_j^i(x)p_j^i \) as a point in the join of \( (\Delta)^{\sigma} \) and \( \delta^r \Delta \). \( \square \)

### 4.4 Proof of Lemma 33

We split the proof of Lemma 33 in three steps.

**Lemma 35 (A First square).** Let \( \sigma_1, \ldots, \sigma_r \) be balls properly mapped to \( B^d \) by \( f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d \), with the dimensional restrictions of Lemma 10.

Then the diagram

\[
\begin{array}{ccc}
\partial(\sigma_1 \times \cdots \times \sigma_r) & \cong \rightarrow & \partial \sigma_1 \ast \cdots \ast \partial \sigma_r \\
\downarrow & & \downarrow \\
\partial(B \times \cdots \times B) \setminus \delta_r(B) & \cong \rightarrow & \partial B \ast \cdots \ast \partial B \setminus \delta_r(B)
\end{array}
\]  

(10)

commutes up to homotopy. The map on the left is defined as before\(^{\text{20}}\). The map on the right maps

\[
\emptyset \ast \cdots \ast \partial \sigma_1 \ast \cdots \ast \emptyset \to \emptyset \ast \cdots \ast \partial B \ast \cdots \ast \emptyset
\]

and extends linearly. The two horizontal homeomorphisms are obtained in the following way: we represent \( B^d \) as \( I^d = [-1,1]^d \), then \( \partial B \ast \cdots \ast \partial B \) can be formed inside of the cube \( B \times \cdots \times B \), i.e.,

\[
\partial B \ast \cdots \ast \partial B \subseteq B \times \cdots \times B \quad (\text{Figure 9})
\]

and by radial projection from the center of the cube, we get that \( \partial(B \times \cdots \times B) \) is homeomorphic with \( \partial B \ast \cdots \ast \partial B \). This defines the bottom horizontal arrow, and the same construction work with the top horizontal arrow.

**Proof.** The top left-to-right arrow is defined as (where \(|\cdot|\) is the infinity-norm)

\[
(x_1, \ldots, x_r) \to \sum_{\in \partial \sigma_1} |x_1| \frac{x_1}{|x_1|} \oplus \cdots \oplus \sum_{\in \partial \sigma_r} |x_r| \frac{x_r}{|x_r|} \subseteq \partial \sigma_1 \ast \cdots \ast \partial \sigma_r
\]

\(^{\text{20}}\)It is easy to see that \( \partial(\sigma_1 \times \cdots \times \sigma_r) \) maps into

\[
\partial(B \times \cdots \times B) \setminus \delta_r(B) \subseteq B \times \cdots \times B \setminus \delta_r(B)
\]

since the \( \sigma_i \) are properly mapped in \( B^d \). Also, if \( B \) is represented as a cube \( I^d = [-1,1]^d \), then

\[
B \times \cdots \times B \setminus \{0,0,0\}
\]

can be retracted onto its boundary, and this defines a retraction from

\[
B \times \cdots \times B \setminus \delta_r(B) \text{ to } \partial(B \times \cdots \times B) \setminus \delta_r(B).
\]
with the convention that, if \(|x_i| = 0\), then \(\frac{x_i}{|x_i|}\) is undefined (but since its coefficient in the join is 0, this is not a problem). The inverse application divides a point \(p \in \partial \sigma_1 \times \cdots \times \partial \sigma_r \subset \sigma_1 \times \cdots \times \sigma_r\) by its \(|.|-\)norm (as a point in \(\sigma_1 \times \cdots \times \sigma_r\)) to project the point on the boundary \(\partial(\sigma_1 \times \cdots \times \sigma_r)\).

Starting from the top-left corner of Diagram (10), we follow the directions: right, down, left. We obtain a map \(\partial(\sigma_1 \times \cdots \times \sigma_r) \to \partial(B \times \cdots \times B) \setminus \delta_d(B)\) defined as

\[
(x_1, \ldots, x_r) \mapsto \left(\frac{|x_1|}{\sum |x_i|}, \ldots, \frac{|x_r|}{\sum |x_i|}\right).
\]

(11)

To conclude, we must show that (11) is homotopic to \((x_1, \ldots, x_r) \mapsto (f_1 x_1, \ldots, f_r x_r)\).

Let us assume, without loss of generality, that \(x_1 \in \partial \sigma_1\). Then, \(|x_1| = 1\), hence \(\frac{|x_1|}{\sum |x_i|} = \frac{1}{\sum |x_i|}\). Therefore, the denominator in (11) must be \(\frac{1}{\sum |x_i|}\), and so (11) becomes

\[
\left(\frac{|x_1|}{\sum |x_i|}, \ldots, \frac{|x_r|}{\sum |x_i|}\right).
\]

(12)

which is homotopic to \((x_1, \ldots, x_r) \mapsto (f_1 x_1, \ldots, f_r x_r)\) by a straight-line homotopy. Indeed, by contradiction, let us assume that for a given \((x_1, \ldots, x_r) \in \partial(\sigma_1 \times \cdots \times \sigma_r)\) and a given \(t \in (0, 1)\), the straight-line homotopy intersects the diagonal \(\delta_d(B)\). Without loss of generality, \(x_1 \in \partial \sigma_1\). But then, we must have \(f_1 x_2, \ldots, f_r x_r \in \partial B\), which implies that \(x_2 \in \partial \sigma_2, \ldots, x_r \in \partial \sigma_r\). But (12) is the identify on such an \(r\)-tuple \((x_1, \ldots, x_r)\), so it cannot intersect the diagonal. \(\square\)

**Remark 36.**

- We define \(L := d(r-1) - s_r(r-2)\) to shorten the exponent in \(\Sigma^{d(r-1)-s_r(r-2)}\).

- Recall that for \(g\) a suspended map (Definition 31), we define \(\tau_i := g|_{\sigma_i}^{-1}\).

**Lemma 37** (Second square.). Let \(\sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d\) be a suspended map. Then the diagram

\[
\begin{array}{ccc}
\partial \sigma_1 \ast \cdots \ast \partial \sigma_r & \xrightarrow{\cong} & \Sigma^L(\partial \tau_1 \ast \cdots \ast \partial \tau_{r-1}) \\
\downarrow & & \downarrow \\
\partial B \ast \cdots \ast \partial B \setminus \delta_d(B) & \xrightarrow{\Sigma^L(\partial \sigma_1 \ast \cdots \ast \partial \sigma_r \setminus \delta_{r-1}(\sigma_r))} & \Sigma^L(\partial \sigma_1 \ast \cdots \ast \partial \sigma_r \setminus \delta_{r-1}(\sigma_r))
\end{array}
\]

(13)

commutes. The map on the left is the obvious one, and the map on the right is the \((d(r-1) - s_r(r-2))\)-suspension of the map \(\partial \tau_1 \ast \cdots \ast \partial \tau_{r-1} \to \partial \sigma_1 \ast \cdots \ast \partial \sigma_r \setminus \delta_{r-1}(\sigma_r).\)
The horizontal maps are obtained as rearrangements using

\[ B^d = \sigma_r \ast S^{d-s_r-1} \quad \text{and} \quad \sigma_i = \tau_r \ast S^{d-s_r-1}, \quad \text{for} \ i \neq r. \]

More precisely, the top-horizontal homeomorphism is obtained in the following way

\[ \partial \sigma_1 \ast \cdots \ast \partial \sigma_r \ast \frac{(\partial \tau_1 \ast S^{d-s_r-1}) \ast \cdots \ast (\partial \tau_{r-1} \ast S^{d-s_r-1}) \ast \partial \sigma_r}{\partial \sigma_1} \]

\[ \cong \frac{(\partial \sigma_r \ast S^{(r-1)(d-s_r)-1}) \ast \partial \sigma_1 \ast \cdots \ast \partial \tau_{r-1}}{S^{(r-1)(d-s_r)-1}} \]

and this last expression is the suspension applied \(((r-1)d-s_r(r-2))\)-times on \(\partial \tau_1 \ast \cdots \ast \partial \tau_{r-1}\).

The bottom horizontal inclusion is derived, in a very similar fashion, as

\[ S^{(r-1)d-s_r(r-2)-1} \ast (\partial \sigma_r \ast \cdots \ast \partial \sigma_r) \]

\[ \cong (\partial \sigma_r \ast S^{d-s_r-1}) \ast \cdots \ast (\partial \sigma_r \ast S^{d-s_r-1}) \ast (S^{s_r-1} \ast \emptyset) \]

\[ \cong \partial B \ast \cdots \ast \partial B \ast (\partial \sigma_r \ast \emptyset) \]

\[ \subseteq \partial B \ast \cdots \ast \partial B \ast \partial B. \]

**Proof.** It follows easily that (13) commutes. We show in the next step that the bottom horizontal inclusion is an homotopy equivalence, using Diagram (15).

**Lemma 38** (Third square). Let \( \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d \) be a suspended map. Then the diagram

\[ \begin{array}{ccc}
\partial B \ast \cdots \ast \partial B \setminus \delta_r(B) & \cong & \Sigma^L (\partial \sigma_r \ast \cdots \ast \partial \sigma_r \setminus \delta_{r-1}(\sigma_r)) \\
\cong & & \cong \\
(\partial B)^{\sigma_r}_{\delta} & \cong & \Sigma^L (\partial \sigma_r)^{\sigma_r-1}_{\delta}
\end{array} \]

(15)

commutes, and the three arrows with the symbol ‘\(\cong\)’ are homotopy equivalences.

**Proof.** Here, \((-)_{\delta}^{\delta_k}\) is the \(k\)-fold \(\text{k-wise (simplicial) deleted join. The definition is given in Section 4.3, where we also prove that both left and right vertical arrows are homotopy equivalences (Lemma 34). Hence, we are left with the bottom-horizontal map.}

We are going to use the two following facts that are easy to check:

1. for any simplicial complexes \(L_1, \ldots, L_n\), \((L_1 \ast \cdots \ast L_n)^{\delta_k}_{\delta} \cong (L_1)^{\delta_k}_{\delta} \ast \cdots \ast (L_n)^{\delta_k}_{\delta}\),

2. \((\partial I)^{\delta_k}_{\delta}\) collapses simplicially onto \(\partial I \ast \cdots \ast \partial I \ast \emptyset \ast \partial I \ast \cdots \ast \partial I\) (i.e., the \(k\)-join of \(\partial I\) where one of the factor is replaced by \(\emptyset\)).

Therefore, if we represent \(\sigma_r\) as \(I^{\sigma_r}\), we have

\[ (\partial \sigma_r)^{\sigma_r-1}_{\delta} \cong (\partial I \ast \cdots \ast \partial I)^{\sigma_r-1}_{\delta} = (\partial I)^{\sigma_r-1}_{\delta} \ast \cdots \ast (\partial I)^{\sigma_r-1}_{\delta}. \]

We collapse each of the \((\partial I)^{\sigma_r-1}_{\delta}\) to \(\partial I \ast \cdots \ast \partial I \ast \emptyset\), hence \((\partial \sigma_r)^{\sigma_r-1}_{\delta}\) collapses to \(\partial \sigma_r \ast \cdots \ast \partial \sigma_r \ast \emptyset\).

The suspension of this space in \((\partial B)^{\sigma_r}_{\delta}\) is, by equation (14),

\[ (\partial \sigma_r \ast S^{d-s_r-1}) \ast \cdots \ast (\partial \sigma_r \ast S^{d-s_r-1}) \ast (\emptyset \ast S^{d-s_r-1}) \ast (S^{s_r-1} \ast \emptyset), \]

We can collapse \((\partial B)^{\sigma_r}_{\delta}\) onto this last space. Indeed,

\[ (\partial B)^{\sigma_r}_{\delta} = (\partial \sigma_r \ast S^{d-s_r-1})^{\sigma_r}_{\delta} = (\partial \sigma_r)^{\sigma_r}_{\delta} \ast (S^{d-s_r-1})^{\sigma_r}_{\delta}. \]

The first term \((\partial \sigma_r)^{\sigma_r}_{\delta}\) can be factors into terms \((\partial I)^{\sigma_r}_{\delta}\), that we all collapse onto \(\partial I \ast \cdots \ast \partial I \ast \emptyset \ast \partial I\).

For the second term \((S^{d-s_r-1})^{\sigma_r}_{\delta}\), we collapse onto \(\partial I \ast \cdots \ast \partial I \ast \emptyset\).

Then, since both \((\partial B)^{\sigma_r}_{\delta}\) and \(\Sigma^L (\partial \sigma_r)^{\sigma_r-1}_{\delta}\) can be collapsed onto the same sub-sphere, it follows that the bottom inclusion in (15) is an homotopy equivalence. \(\square\)
Proof of Lemma 33. Combining the all the previous Lemmas in this section, we get the following commuting diagram

\[
\begin{array}{c}
\partial(\sigma_1 \times \cdots \times \sigma_r) \\
\downarrow \\
\partial(B \times \cdots \times B) \setminus \delta_r(B)
\end{array} \xrightarrow{\sim} \begin{array}{c}
\Sigma^L(\partial \tau_1 \cdots \partial \tau_{r-1}) \\
\downarrow \\
\Sigma^L(\partial \sigma_r \cdots \partial \sigma_1 \setminus \delta_{r-1}(\sigma_r))
\end{array}
\]

Reusing the first square (10) on \(\partial \tau_1, \cdots, \partial \tau_{r-1}\), we form

\[
\begin{array}{c}
\Sigma^L(\partial \tau_1 \cdots \partial \tau_{r-1}) \\
\downarrow \\
\Sigma^L(\partial \sigma_r \cdots \partial \sigma_1 \setminus \delta_{r-1}(\sigma_r))
\end{array} \xrightarrow{\sim} \begin{array}{c}
\Sigma^L(\partial \sigma_r \cdots \partial \sigma_1) \\
\downarrow \\
\Sigma^L(\partial(\sigma_r \times \cdots \times \sigma_1) \setminus \delta_{r-1}(\sigma_r))
\end{array}
\]

Combining the last two diagrams, we get the diagram (9), as wanted. \(\square\)

4.5 The complete proof of Lemma 10

Proof of Lemma 10. First Part. We apply Proposition 13 to make each of the \(\sigma_i \cap \sigma_r\) \((s - d(r - 1))\)-connected. Then there exists, by Theorem 7, for each \(i = 1, \ldots, r - 1\), a collapsible subspace \(C_i\) of \(\sigma_i \cap \sigma_r\) of dimension at most \(s - d(r - 1) + 1\) such that \(C_i \subseteq \sigma_1 \cap \cdots \cap \sigma_r\).

In \(\sigma_r\) there exists a collapsible space \(C\) of dimension at most \(s - d(r - 1) + 2\) containing \(C_1, \ldots, C_r\). Furthermore, by general position\(^{21}\), \(C\) intersects \(\sigma_i \cap \sigma_r\) only on \(C_i\). We take a regular neighbourhood \(N\) of \(C\) in \(B^d\). By construction, \(N\) intersects \(\sigma_i \cap \sigma_r\) in a regular neighbourhood of \(C_i\), which must be a ball (\(C_i\) is disjoint for the other \(\sigma_j\) for \(j \neq i\)) by general position\(^{22}\). Hence, “retracting” from \(B^d\) to the ball \(N\) (as we did in the proof of Theorem 2, equation 4), we are reduced to the situation of Proposition 30, which we can then directly apply.

Second Part. For \(r = 2\), the result already appeared in Weber [Web67, Proposition 3 & proof of Lemma 1]. I.e., if \(\sigma^*\) and \(\tau^*\) are two balls properly contained in \(B^d\) in the metastable range \((m \geq s, t \text{ with } d \geq \frac{3}{2}m + 3)\) and without intersection. Then for every \(\alpha \in \pi_{s+1}(S^{d-1})\) there exists a proper isotopy \(J_1\) of \(B\) such that \(J_1\sigma \cap \tau = \emptyset\), and the homotopy class defined by

\[
\partial(I \times \sigma \times \tau) \xrightarrow{J_1_{incl} \times incl} B^d \times B^d \setminus \delta_2 B^d
\]

represents \(\alpha\) (after identifications).

Hence, we can work inductively: we assume that the part 2 of the Lemma is already true for \((r - 1)\) balls, and we show how construct the isotopy \(J_r : B^d \to B^d\) for \(r\) balls.

\(^{21}\)We must have

\[
(s_1 + \cdots + s_r - d(r - 1) + 2) + (s_1 + s_r - d) - s_r < 0
\]

i.e.,

\[
s_1 + \cdots + s_r + s_1 + 2 < rd,
\]

which is implied by \((r + 1)m + 2 < rd\).

\(^{22}\)This follows, once again by the metastable range hypothesis. Indeed,

\[
(s_1 + \cdots + s_r - d(r - 1) + 1) + (s_1 + s_j + s_r - 2d) - (s_1 + s_r - d) < 0
\]

i.e.,

\[
s_1 + \cdots + s_r + s_j + 1 < rd,
\]

which is implied by \((r + 1)m + 1 < rd\).
Let \( \sigma_1, \ldots, \sigma_r \) be the \( r \) balls properly contained in \( B^d \) as in the hypothesis of part 2 of the Lemma. In particular,

\[
\sigma_1 \cap \cdots \cap \sigma_r = \emptyset \quad \text{and} \quad \sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset,
\]

and we can assume that \( \sigma_r \) is unknotted in \( B^d \), i.e., \( B^d = \sigma_r \ast S^{d-s_r-1} \).

**Claim 39.** We can assume that for \( i = 1, \ldots, (r-1) \), \( \sigma_i \cap \sigma_r \) are balls properly contained inside \( \sigma_r \). Furthermore, we can assume that \( \sigma_2 \cap \cdots \cap \sigma_r \) is also a ball properly contained in \( \sigma_r \).

**Proof.** Let us pick \( x \in \sigma_1 \cap \sigma_r \) and \( y \in \sigma_2 \cap \cdots \cap \sigma_r \), that we join by a path \( \lambda \subseteq \sigma_r \) in general position. We take a regular neighborhood of \( \lambda \) in \( B^d \), and restrict ourselves to this neighborhood. 

By the induction hypothesis applied on

\[
\sigma_1 \cap \sigma_r, \ldots, \sigma_{r-1} \cap \sigma_r \subseteq \sigma_r,
\]

for every homotopy class \( \alpha \in \pi_{s+(r-2)s_r-(r-1)d+1}(S^{(r-2)s_r-1}) \), there exists an isotopy \( J_t \) of \( \sigma_r \) such that \( J_t \) applied to the ball \( \sigma_1 \cap \sigma_r \subseteq \sigma_r \) represents \( \alpha \).

The isotopy \( J_t \) can be extended to \( B^d \) (we still denote it by \( J_t \)), hence this isotopy applied to the ball \( \sigma_1 \subseteq B^d \) represents an homotopy class \( \beta \in \pi_{s}(S^{(r-1)-1}) \). We are done if we can show that \( \beta \) is a suspension of \( \alpha \) (we are in the stable range of the suspension isomorphism).

The problem is similar to Lemma 33. Indeed we have \( r \) balls

\[
J(\sigma_1 \times [-1,1]), \sigma_2, \ldots, \sigma_r
\]

that are mapped into \( B^d \), and we would like to form a diagram as in (9) with the ball \( \sigma_1 \times [-1,1] \) instead of \( \sigma_1 \).

Hence, to conclude, we only need to prove a version of the Suspended Map Lemma 32 for our present situation.

Note that \( \sigma_1 \times [-1,1] \) is not embedded inside \( B^d \), and is not even properly mapped (the boundary is not mapped to the boundary).

**Claim 40.** Let \( \widetilde{\sigma}_1 \) be a \( (s_1 + 1) \)-ball mapped into \( B^d \) with its boundary mapped as follows: \( \partial \widetilde{\sigma}_1 \) is decomposed into two balls \( B_1 \) and \( B_2 \) such that \( B_1 \) is mapped onto \( J_0 \sigma_1 \) and \( B_2 \) is mapped onto \( J_1 \sigma_1 \) (and \( \partial B_1 = \partial B_2 \) mapped onto \( J_0 \partial \sigma_1 = J_1 \partial \sigma_1 \)), then

\[
J(\sigma_1 \times [-1,1]) \ast \sigma_2 \ast \cdots \ast \sigma_r \ast \widetilde{\sigma}_1 \ast \sigma_2 \ast \cdots \ast \sigma_r
\]

define the same element \( \beta \in \pi_{s}(S^{(r-1)-1}) \).

**Proof.** This is immediate by using a straight line homotopy between \( J(\sigma_1 \times [-1,1]) \) and \( \widetilde{\sigma}_1 \) .

So we are reduced with working with a \( (s_1 + 1) \)-ball \( \widetilde{\sigma}_1 \) instead of \( J(\sigma_1 \times [-1,1]) \), and the way that we ‘fill’ this ball does not matter (only the boundary decides of the homotopy class \( \beta \)).

We can decompose \( \partial \widetilde{\sigma}_1 \) as two balls \( B_1 \) and \( B_2 \) both homeomorphic with \( (\sigma_1 \cap \sigma_r) \ast S^{d-s_r-1} \), and with \( B_1 \cap B_2 \simeq \partial(\sigma_1 \cap \sigma_r) \ast S^{d-s-r} \).

**Claim 41.** We can assume that \( B_1 \) is mapped to \( (\sigma_1 \cap \sigma_r) \ast S^{d-s_r-1} \) and that \( B_2 \) is mapped to \( (J_1 \sigma_1 \cap \sigma_r) \ast S^{d-s_r-1} \).

**Proof.** This follows by an argument identical to that of Lemma 32 (we work with the two balls separately).

**Claim 42.** We can assume that \( B_1 = (\sigma_1 \cap \sigma_r) \ast S^{d-s_r-1} \) and \( B_2 = (J_1 \sigma_1 \cap \sigma_r) \ast S^{d-s_r-1} \) are mapped onto the boundary of \( B^d \), and that \( \widetilde{\sigma}_1 = (\widetilde{\sigma}_1 \cap \sigma_r) \ast S^{d-s_r-1} \).
Figure 10: Retraction of the sphere \((B_1 \cap \sigma_r) \cup (B_2 \cap \sigma_r)\) on the boundary of \(\sigma_r\).

Proof. Figure 10 illustrate the construction inside \(\sigma_r\).

We pick a point \(x\) in the interior of the ball \(\sigma_2 \cap \cdots \cap \sigma_r\). Since this ball unknots in \(\sigma_r\), there exists a retraction \(r_t\) of \(\sigma_r \setminus x\) to \(\partial \sigma_r\) such that \(r_t^{-1} \partial (\sigma_2 \cap \cdots \cap \sigma_r) = \sigma_2 \cap \cdots \cap \sigma_r\).

Using that \(B^d = \sigma_r \ast S^{d-s_r-1}\), we extend \(r_t\) to \(B_d\) (which now retract \(B^d \setminus x\) to \(\partial B^d\)). We can then use \(r_t\) to conclude. (Note that \(B_1\) and \(B_2\) stop to be embedded, but this is not a problem for us).

We can now apply Lemma 33 to the balls \(\tilde{\sigma}_1, \sigma_2, \ldots, \sigma_r\), and thus conclude.
A Block Bundles

In this Appendix, we review the notions of block bundles that are used in Section 4.1. The theory of Block Bundles was developed by Rourke and Sanderson in a series of three papers [RS68a, RS68b, RS68c] (see also [Bry02, p. 236], [RS66, BRS76], and Casson’s chapter in [RCS+97]). The goal of this theory is to establish for the piecewise-linear category a tool analogous to the vector bundle in the differential category. [RS68a, p. 1].

Definition 43 (PL cell complex). A complex $K$ is a finite collection of cells which cover a polyhedron $X$ satisfying: For all $\sigma, \tau \in K$ distinct

1. $\partial \sigma$ and $\sigma \cap \tau$ are unions of cells of $K$,
2. $\hat{\sigma} \cap \hat{\tau} = \emptyset$.

Remarks 44. (a) We write $|K| := X$, when $K$ is a complex associate to a polyhedron $X$.
(b) We often identified $\sigma \in K$ with the subcomplex of $K$ that it determines.

Definition 45 (Product of Complexes). Let $K$ and $L$ be two complexes. The collection of cells $K \times L := \{ \sigma \times \tau \mid \sigma \in K$ and $\tau \in L \}$ is also a complex, whose geometric realization is homeomorphic to $|K| \times |L|$.

Definition 46 (Subdivision). Let $K$ and $K'$ be two cell complex structures of a polyhedron $X$. We say that $K'$ is a subdivision of $K$ if for all $\sigma' \in K'$, there exists $\sigma \in K$ with $\sigma' \subseteq \sigma$.

Definition 47 (Block Bundle). A $q$-block bundle $\xi^q$ is a pair $(E, K)$ of cell complexes such that

1. $|K| \subset |E|$,
2. For each $n$-cell $\sigma \in K$, there exists an $(n + q)$-ball $\beta \in E$ such that $(\beta, \sigma) \cong (I^{n+q}, I^n)$.

We call $\beta$ the block over $\sigma$.

3. $E$ is the union of the blocks $\beta$.
4. The interiors of the blocks are disjoints.
5. Let $\beta, \gamma \in E$ be the block over $\sigma, \tau \in K$, respectively. Let $L = \sigma \cap \tau \subset K$, then $\beta \cap \gamma \subset E$ is the union of the blocks over cells of $L$.

Remarks 48. (a) For a $q$-block bundle $\xi^q = (E, K)$, we denote $E(\xi) := E$, and the whole bundle data is denoted by $\xi^q/K$.

Sometimes, we simply write $\xi/X$, where $X = |K|$.

(b) The trivial $q$-block bundle over $K$ is denoted by $\mathcal{E}^q/K$. For any $\sigma_i \in K$ $\beta_i := \sigma_i \times [-1, 1]^q$.

Definition 49 (Isomorphism of Block Bundles). Two block bundles $\xi, \eta/K$ are isomorphic (written $\xi \cong \eta$) if there exists a homeomorphism $h : E(\xi) \to E(\eta)$, such that
(1) its restriction to $K$ is the identity ($h_K = 1$), and

(2) Let $\sigma \in K$ with $\beta \in E(\xi)$ and $\gamma \in E(\eta)$ the two blocks over $\sigma$, then

$$h(\beta) = \gamma.$$

**Definition 50** (Subdivision). Let $\eta/K$ be a block bundle, and let $K'$ be a subdivision of $K$. A block bundle $\eta'/K'$ is a **subdivision** of $\eta/K$ if for all $\sigma \in K$ (with associated block $\beta$), such that

$$|\sigma| = |\bar{\tau}_1| \cup \cdots \cup |\bar{\tau}_n|$$

where $\tau_1, \ldots, \tau_n \in K'$, then

$$|\beta| = |\bar{\mu}_1| \cup \cdots \cup |\bar{\mu}_n|$$

where $\mu_i$ is the block over $\tau_i$ in $\xi/K'$.

**Definition 51** (Equivalence of Block Bundles). Let $\xi/K$ and $\eta/L$ be two block bundles such that $|K| = |L|$. We say that $\xi/K$ is **equivalent** to $\eta/L$ if there exists subdivisions $\xi'/K'$ and $\eta'/L'$ such that $\xi \cong \eta'$.

**Definition 52.** Let $K$ be a polyhedron.

(1) The set of isomorphism classes of $q$-block bundles over $K$ is denoted $I_q(K)$.

(2) The set of equivalence classes of $q$-block bundles over $|K|$ is denoted $I_q(|K|)$.

**Theorem 53** (Thm 1.10 [RS68a]). Let $K$ be a polyhedron. Then there is a bijection

$$I_q(K) \rightarrow I_q(|K|),$$

defined by associating each block bundle over $K$ to its equivalence class.

We have the following analogue of the 'tubular neighborhood Theorem' for block bundles:

**Theorem 54** (Sec. 4 of [RS68a] or Thm 3.27 in [Bry02]). Suppose $(M, Q)$ is a $(m, q)$-manifold pair with $m - q \geq 3$. Every regular neighborhood of $Q$ in $M$ has the structure of an $(m - q)$-block bundle over $Q$.

**Definition 55** (Abstract Regular Neighborhood). Suppose $M^m \subset Q^q$ is a proper submanifold of $Q$ (both compact). We say that $Q$ is an **abstract regular neighborhood** of $M$ is $Q$ collapses to $M$.

**Theorem 56** (Cor. 4.6 in [RS68a]). Let $M$ be a $m$-manifold, and $R_q(M)$ be the set of homeomorphism (mod $M$) classes of abstract regular neighborhoods of $M$ of dimension $m + q$.

There is a bijection

$$I_q(M) \rightarrow R_q(M)$$

given by associating every block bundle to the corresponding abstract regular neighborhood, i.e., $\xi/M$ is sent to $E(\xi)$.

**Definition 57** (Restriction). Let $L, K$ be two complexes such that $L \subset K$. Given a block bundle $\xi/K$, we define its **restriction** to $L$ (notation: $\xi|L$) which is the block bundle over $L$ obtained from $\xi/K$ by restricting to the block over cells of $L$.

I.e., given $\sigma \in L$, then the block over $\sigma$ in $\xi/L$ is the block over $\sigma$ in $\xi/K$.

**Remark 58.** Let $u/X$ be an equivalence class of block bundle over a polyhedron $X$. Let $Y \subset X$ be a closed subpolyhedron of $X$. Then the restriction $u|Y$ is a well-defined equivalence class of block bundles over $Y$ (see [RS68a, p. 8]).

**Definition 59** (Cartesian Product). Suppose $\xi/K$ and $\eta/L$ are two block bundles. Their **cartesian product** $(\xi \times \eta)/(K \times L)$ is the block bundle over $K \times L$ defined by
(1) \( E(\xi \times \eta) := E(\xi) \times E(\eta) \).

(2) Given \( \sigma \in K \), \( \tau \in L \) with \( \beta \in E(K) \) the block over \( \sigma \) and \( \gamma \in E(L) \) the block over \( \tau \). Then the block over \( \sigma \times \tau \in K \times L \) is \( \beta \times \gamma \).

**Remark 60.** As for the restriction, the cartesian product is also well-defined on equivalence classes of block bundles.

**Definition 61 (Whitney Sum).** Given two equivalences classes of block bundles \( u/X \) and \( v/X \). We define their Whitney sum

\[
(u \oplus v)/X := (u \times v)|\Delta,
\]

where \( \Delta = \{(x, x) \in X \times X\} \) is identified with \( X \) by the diagonal map.

**Theorem 62** (Thm 2.7 [RS68a]). For all \( q \geq 0 \), there exists a locally finite simplicial complex \( B\overline{PL}_q \) such that: For every polyhedron \( X \) there is a bijection

\[
[X, B\overline{PL}_q] \to I_q(X).
\]

**Remark 63.** The above theorem is the PL analogue of the classification of vector bundles over a smooth manifold \( M \): equivalences classes of such vector bundles are in bijection with \( [M, G_n(\mathbb{R}^\infty)] \), where \( G_n(\mathbb{R}^\infty) \) is the infinite \( n \)-Grassmanian.

**Theorem 64** (Stability of Block Bundles, Cor. 5.2–5.4 [RS68b]). Let \( \xi^q/K^k \) be a \( q \)-block bundle over a \( k \)-complex \( K \).

(a) There exists \( \eta^k/K^k \) such that

\[
\xi^q \cong \eta^k \oplus \varepsilon^{q-k}.
\]

(b) There exists \( \eta^k/K^k \) such that

\[
\xi^q \oplus \eta^k \cong \varepsilon^{q+k}.
\]

(c) For \( \xi^{k+1}/K^k \) and \( \eta^{k+1}/K^k \),

\[
\xi^{k+1} \oplus \varepsilon^t \cong \eta^{k+1} \oplus \varepsilon^t \quad \Rightarrow \quad \xi^{k+1} \cong \eta^{k+1}.
\]

**Definition 65** (Transversality). Let \( M, N \subset Q \) be compact proper submanifolds, and \( \xi/M \subset Q \) a block bundle over \( M \) in \( Q \). (In particular, \( E(\xi) \) is a regular neighborhood of \( M \subset Q \).)

(a) We say that \( N \) is transverse to \( M \) with respect to \( \xi \) if there exists a subdivision \( \xi' \) of \( \xi \) such that

\[
N \cap E(\xi) = E(\xi'|N \cap M).\]

(b) We say that \( N \) is locally transverse to \( M \) with respect to \( \xi \) if this is true near \( M \), i.e., there exists a subdivision \( \xi' \) of \( \xi \) and a regular neighborhood \( U \) of \( M \) with

\[
N \cap \xi \cap U = E(\xi'|N \cap M) \cap U.
\]

**Theorem 66** (Thm 1.1 in [RS68b]). Suppose \( N, M \subset Q \) are proper submanifolds, and \( \xi \) is a block bundle on \( M \subset Q \).

(a) There is an \( \varepsilon \)-isotopy of \( Q \) carrying \( N \) locally transverse to \( M \) with respect to \( \xi \).

(b) If \( \partial N \) is already locally transverse to \( \partial M \) with respect to \( \xi|\partial M \), then the isotopy may be taken mod \( \partial Q \).

**Definition 67** (Tangent Bundles). Given a manifold \( M \), the regular neighborhood of \( \Delta_M \subset M \times M \) defines a block bundle. Its equivalence class, denoted \( t(M) \), is called the tangent bundle of \( M \).
We have the following analogue of the decomposition of the tangent vector bundle over a submanifold in the smooth category (it decomposes as the tangent bundle $\oplus$ the normal bundle):

**Theorem 68** (Prop. 5.5 and Cor. 5.6 in [RS68b]). (a) Let $M$ be a manifold with $\xi/M$ a block bundle over $M$. (In particular, $E(\xi)$ is an abstract regular neighborhood of $M$.) Then

$$t(E(\xi))|_M \cong (\xi \oplus t(M))|_M.$$  

(b) Let $M \subset Q$ be a compact submanifold, and $u$ the class of any normal block bundle on $M$ in $Q$, then

$$(t(M) \oplus u)|_M \cong t(Q)|_M.$$  

**Proposition 69** (Tangent bundle of a sphere is stably trivial). Let $S^k$ be a $k$-sphere, then

$$t(S^k) \oplus \varepsilon^1 \cong \varepsilon^{k+1}.$$  

**Proof.** Let us consider $S^k \subset \mathbb{R}^{k+1}$. By the existence of bicollars, any normal bundle of $S^k \subset \mathbb{R}^{k+1}$ is isomorphic to $\varepsilon^1/S^k$.

Then, by Theorem 68,

$$\varepsilon^1 \oplus t(S^k) = t(\mathbb{R}^{k+1})|S^k = \varepsilon^{k+1}/S^k.$$  

**Proposition 70** (PL analogue of Thm 2 in [Mil61]). Let $S^k$ be a $k$-sphere embedded in an $m$-manifold $M^m$, with $2k + 1 \leq m$.

Assume that the tangent bundle of $M$ is stably trivial, i.e., there exists $l \geq 0$ such that

$$t(M) \oplus \varepsilon^l \cong \varepsilon^{m+l}.$$  

Then any normal bundle over $S^k$ in $M$ is trivial.

**Proof.** Let $\xi^{m-k}/S^k$ be a normal bundle of $S^k \subset M$. Let us form $M \times [-1,1]^l$ (i.e., $\varepsilon^l/M$, note that we can always assume $l \geq 1$).

A normal bundle of $S^k \subset M \times [-1,1]^l$ is

$$\xi^{m-k} \oplus \varepsilon^l.$$  

By Theorem 68 (for the first and second equalities),

$$(t(S^k) \oplus \xi^{m-k} \oplus \varepsilon^l)/S^k = t(M \times [-1,1]^l)|S^k = t(M) \oplus \varepsilon^l)|S^k = \varepsilon^{m+l}|S^k.$$  

By Theorem 69

$$(\xi^{m-k} \oplus t(S^k) \oplus \varepsilon^l)/S^k = (\xi^{m-k} \oplus \varepsilon^{k+1})/S^k.$$  

Combining the last two equations (we do not write $u/S^{kn}$ anymore)

$$\xi^{m-k} \oplus \varepsilon^{k+l} = \varepsilon^{m+l}.$$  

By Theorem 64, there exists $\eta^k/S^k$ such that

$$\xi^{m-k} = \eta^k \oplus \varepsilon^{m-2k}.$$  

So, by the last two equations,

$$\eta^k \oplus \varepsilon^{m-2k+k+l} = \varepsilon^{m+l}.$$  

Again by Theorem 64, this implies

$$\eta^k \oplus \varepsilon^1 = \varepsilon^{k+1}.$$  

So

$$\xi^{m-k} = \eta^k \oplus \varepsilon^{m-2k} = \varepsilon^{m-k},$$  

since $m - 2k \geq 1$.  

31
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