Easy Control over Fermionic Computations

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Abstract
Quantum fermionic computations on occupation numbers proposed in quant-ph/0003137 are studied. It is shown that a control over external field and tunneling would suffice to fulfill all quantum computations without valuable slowdown in the framework of such model when an interaction of diagonal type is fixed and permanent. Substantiation is given through a reduction of some subset of this model to the conventional language of quantum computing and application of the construction from quant-ph/0202030.

1 Introduction and background
Quantum computing is an unprecedented examination of the modern physics because it requires a level of control over microscopic sized objects which has never been reached artificially. While a mathematical theory of quantum computations is well developed its physical implementation represents a serious challenge to our understanding of Nature. This is why it is important to look for its simplest possible realization which would depend only on fundamental principles of quantum mechanics and contain minimal technological difficulties. Two requirements can be formulated for such scheme: adequate description of states forming a basis of computational Hilbert space, and realistic method of control. Conventionally a computational element - qubit is represented as some characteristic like spin, charge or position of some elementary particle. This approach works well for one isolated qubit. For a system of many qubits it meets a serious difficulty. It comes from a fundamental physical principle of identity of elementary particles of the same type\[1\]. To control a computation we must be able to address a separated qubit whereas different particles are undistinguishable according to the principle of identity. Of course, we can distinguish particles by their spatial positions placing them fairly far one from another but in this case it will be difficult to keep them in entangled state that is necessary for quantum computations. A solution of this dilemma was found by Kitaev and Bravyi, who proposed to use a Fock space of occupation numbers to the description of quantum computations (see [KB]). It uses in fact a natural identification of qubits with

\[1\]I express my thanks to Sergei Molotkov who attracted my attention to this problem and referred me to the paper [KB]. I am also grateful to Alexander Tsukanov and Alexander Kokin for useful comments to the preliminary version of this paper.
energy levels in Fock space where one is treated as occupied level and zero as free level. This approach
gives a universal quantum computing for a high price: it requires a control over external field, tunneling,
diagonal interaction and changes of numbers of particles (contact with a superconductor) that is to control
over coefficients $\alpha, \beta, \gamma$ in (2) and over additional summand $\delta a_k^+ a_j^+ + \delta^* a_k a_j$.

We shall see how to reduce this price using an idea of fixed and permanent interaction. To do this we
need two things: assumption that the corresponding Hamiltonian consists only of diagonal and tunneling
summands and modified correspondence between states in occupation number representation and Hilbert
spaces. Then to fulfill any quantum computation it is required to control over only an external field
and tunneling. Such kind of control is in principle realizable by lasers. The main scheme is presented in
the section 3 and is in fact based on the idea of computational model with fixed permanent interaction
proposed in works [Oz, OF] adapted to the language of Fock spaces. The next section contains a short
description of the occupation numbers formalism.

2 Formalism of occupation numbers

In this paper we shall consider a system of $n$ identical fermions. At first make nonphysical assumption
that they can be reliably distinguished. Then its state belongs to the Hilbert space of all states with the
basis $\psi(r_1, r_2, \ldots, r_n) = \psi_{j_1}(r_1)\psi_{j_2}(r_2) \ldots \psi_{j_n}(r_n)$ where $\{\psi_j\}$ are some basis for one particle states, where
$j_s$ belongs to the general set of indices $1, 2, \ldots, J$, $r_j$ includes spatial and spin coordinates. A choice of
basis means that the system after measurement can be found in one of the basic states. In a real system
of identical particles they cannot be distinguished. Hence any basic state must contain all summands of
the form $\psi_{j_1}(r_1)\psi_{j_2}(r_2) \ldots \psi_{j_n}(r_n)$ with some factors. Such state must change the sign after permutation
of every two fermions and it is convenient to assume that a basic state for $n$ fermions system is given by

$$\Psi = \frac{1}{\sqrt{N!}} \left| \begin{array}{cccc}
\psi_{j_1}(r_1) & \psi_{j_1}(r_2) & \ldots & \psi_{j_1}(r_n) \\
\vdots & \vdots & & \vdots \\
\psi_{j_n}(r_1) & \psi_{j_n}(r_2) & \ldots & \psi_{j_n}(r_n)
\end{array} \right| . \quad (1)
$$

This state may be considered as a situation when only states $\psi_{j_s}$ for $s = 1, 2, \ldots, n$ are occupied by
particles from our system and all others $\psi_k$ for $k \in \{1, 2, \ldots, J\}$ which have not form $j_s$ are free. If $\psi$ with
indices denotes eigenvectors of one particle Hamiltonian we speak about occupied or free energy levels,
but generally speaking $\psi_k$ may form arbitrary basis in the space of states for one particle. A state of the
form (1) may be represented as a symbol $|\bar{n}\rangle = |n_1, n_2, \ldots, n_J\rangle$ where $n_k$ is one if $k$th energy level is
occupied and zero if it is free. This is a representation of states of fermionic ensemble in form of occupation
numbers. Such vectors $\bar{n}$ form a basis of Fock space and the general form of state of our system will be
$\sum_{\bar{n}} \lambda_{\bar{n}}|\bar{n}\rangle$ with amplitudes $\lambda$.

An operator of annihilation $a_j$ of a particle on $j$th level and its conjugated $a_j^+$ (creation) are defined
by $a_j|n_1, \ldots, n_J\rangle = \delta_{1,n_j} (-1)^{\sigma_j} |n_1, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_J\rangle$ where $\sigma_j = n_1 + \ldots + n_j$. They possess
the known commutative relations: $a_j^+ a_k + a_k a_j^+ = \delta_{j,k}$, $a_j a_k + a_k a_j = a_j^+ a_k^+ + a_k^+ a_j^+ = 0$.

Assume that any interaction in Nature goes between no more than two particles. Hence any interaction
in many-particle system may be expanded into the sum of one and two particles interactions of the form
$H = H_{\text{one}} + H_{\text{two}}$ with the corresponding potentials $V_1(r)$ and $V_2(r, r')$. Each of them can be represented
by operators of creations and annihilations as
\[ H_{\text{one}} = \sum_{k,l} H_{k,l} a_k^+ a_l, \quad H_{\text{two}} = \sum_{k,l,m,n} H_{k,l,m,n} a_k^+ a_l^+ a_m a_n \]
where
\[
H_{k,l} = \langle \psi_k | H_{\text{one}} | \psi_l \rangle = \int \psi_k^*(r) V_1(r) \psi_l(r) dr,
\]
\[
H_{k,l,m,n} = \langle \psi_1, \psi_k | H_{\text{two}} | \psi_m, \psi_n \rangle = \int \psi_k^*(r) \psi_l(r) V_2(r) \psi_m(r) \psi_n(r') dr dr'.
\]

Hence given potentials of all interactions and all basic states \( \psi_i \), we can in principle obtain its representation in terms of creations and annihilations, e.g. in the language of occupation numbers.

Consider an ensemble with Hamiltonian of the form \( H = \sum_i H_{\text{ext.f.}}^i + \sum_{i,j} (H_{\text{diag.}}^{ij} + H_{\text{tun.}}^{ij}) \) where Hamiltonians of external field, diagonal interaction and tunneling are represented by means of operators of creations and annihilations by
\[
H_{\text{ext.f.}}^i = \alpha_i a_i^+ a_i, \quad \alpha_i \in \mathbb{R},
\]
\[
H_{\text{diag.}}^{ij} = \beta_{i,j} a_i^+ a_j^+ a_j, \quad \beta_{i,j} \in \mathbb{R},
\]
\[
H_{\text{tun.}}^{ij} = \gamma_{i,j} a_i^+ a_j + \gamma_{i,j}^* a_j^+ a_i.
\]

Note that it would not be easy to implement a control over diagonal part of Hamiltonian. Assume that the diagonal interaction is fixed and acts permanently whereas external field and tunneling are subjects of control. Then it is possible to fulfill every quantum computation. This type of control seems to be fairly realistic because a tunneling may be controlled by laser impulses.

### 3 Computation controlled by tunneling

Instead of simple correspondence between occupation numbers and Hilbert spaces described above we now establish another correspondence that makes possible to transfer a universal computing with fixed permanent interaction ([42, 46]) to the language of fermionic computing in Fock space of occupation numbers.

Let us fix some partitioning of all energy levels to two equal parts and choose some one-to-one correspondence between them. Say we can consider \( k \)th level down from Fermi bound \( \epsilon_F \) and agree that it corresponds to \( k \)th level up from \( \epsilon_F \). We shall denote \( j \)th level down from Fermi bound by ordinary letter and \( j \)th level up from Fermi bound by \( j' \). Call the first level \( j \)th lower level and the second one \( j \)th upper level. Fock space \( \mathcal{F} \) can be represented as \( \mathcal{F} = F_1 \otimes F_2 \otimes \ldots \otimes F_k \) where each \( F_j \) corresponds to \( j \)th pair of the corresponding energy levels. Consider a subspace \( F_j \) in \( \mathcal{F} \) which is spanned by two following states. The first one is: ”\( j' \)th level is occupied and \( j \)th level is free”, the second is ”\( j \)th level is occupied and \( j' \)th level is free”. Denote them by \( |1\rangle_j \) and \( |0\rangle_j \) correspondingly. We shall deal with subspace \( F = F_1 \otimes F_2 \otimes \ldots \otimes F_k \) in Fock space \( \mathcal{F} \). Now determine a function \( \theta \) that maps our Hilbert space \( \mathcal{H} \) to \( F \) by the following definition on basic states: \( \theta(|\xi_1, \xi_2, \ldots, \xi_n\rangle) = |\xi_1\rangle_1 \otimes |\xi_2\rangle_2 \otimes \ldots \otimes |\xi_n\rangle_n \) where all \( \xi_j \) are ones and zeroes. Thus \( \theta \) establishes unconventional correspondence between Hilbert and Fock spaces (see Figure 1).

One qubit state in Hilbert space corresponds to two qubits state in conventional assignment of qubits for Fock space - each occupation number to each qubit. But we shall see that this assignment answers to the task of control over computation better than conventional assignment.

Now the door is open for representation of unitary transformations in Hilbert space required for quantum computing by transformations in Fock space. Consider Hermitian operator \( H \) in one-dimensional Hilbert space \( \mathcal{H} \). It has the form \( H_0 + H_1 \) where
\[
H_0 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}.
\]
It can be straightforwardly verified that for operators $\tilde{H}_0 = d_1 a_k^+ a_k + d_2 a_{k'}^+ a_{k'}$ and $\tilde{H}_1 = d a_k^+ a_{k'} + \bar{d} a_{k'}^+ a_k$ (that is external field and tunneling) we have equalities $\tilde{H}_i \theta = \theta H_i$ for $i = 0, 1$. Using linearity of $\theta$ we obtain $(\tilde{H}_0 + \tilde{H}_1) \theta = \theta H$. Now consider one qubit unitary transformations $U$ in Hilbert space. It has the form $e^{-iH}$ for Hamiltonian $H$ (we choose appropriate time scale to get rid of Plank constant and time). Using linearity of $\theta$ and the equality $\theta^{-1} H^s \theta = (\theta^{-1} H \theta)^s$ for natural $s$ we obtain that for every one qubit unitary transformation $U$ we can effectively find the corresponding Hamiltonian in Fock space containing only external field and tunneling that makes diagram A from the Figure 2 closed.

Take up two qubits transformations in Hilbert space. Since all diagonal matrices commute, for all diagonal transformation in space $F_k \otimes F_j$ we can effectively find the corresponding diagonal transformation on the corresponding Hilbert space that makes diagram B from the Figure 2 closed. Note that for entangling $V$ the transformation $V$ will be entangling as well.

Now all is ready to transfer a trick from [OF] with one qubit controlled universal computations to Fock space. Combination of diagrams from Figure 2 gives diagram from the Figure 3.

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**Figure 1.** Correspondence between Fock and Hilbert spaces

**Figure 2.** Correspondence of transformations in Fock and Hilbert subspaces. $\tilde{F} = F_j \otimes F_k$. 

| $\xi \in \mathcal{H}$ | $|0\rangle_2$ | $|1\rangle_2$ | $|010\rangle$ |
|-----------------------|-------------|-------------|-------------|
| 3'                    | --          | --          | --          |
| 2'                    | -- ⊕ --     | -- ⊕ --     | -- ⊕ --     |
| 1'                    | --          | -- ⊕ --     | -- ⊕ --     |
| $\theta(\xi)$         | 0           | 0           | $\epsilon_F$ |
| 1                     | --          | --          | --          |
| 2                     | -- ⊕ --     | -- ⊕ --     | -- ⊕ --     |
| 3                     | --          | --          | --          |
Let a diagonal part of interaction in Fock space be fixed and act permanently. Then we can find the corresponding diagonal interaction on Hilbert space making all diagonal parts of diagram from Figure 3 closed. By the result from [OF] we can choose 1 qubit transformations implementing arbitrary quantum computations in Hilbert space in the form represented by the lower sequence of the diagram. At last we can find field + tunneling control over states in Fock space making all diagram closed. Note that all operators of creations and annihilations considered in the whole Fock space are not local due to the factor \((-1)^{\sigma_j}\) depending on a given state. For the diagonal operators \(a_j^+a_ja_k^+a_k\) and external fields these factors are compensated. A tunneling operator \(a_j^+a_{j'}\) in space \(F\) brings factor \((-1)^{\sigma'}\) where \(\sigma' = \sum_{s+j}^{j'-1} n_s = j' - j\) that is independent of a given state \(|\vec{n}\rangle \in F\) because for such state exactly a half of levels between \(j\) and \(j'\) are occupied. Hence a sign can be factored out of all state and ignored.

Thus we obtain a universal quantum computer on states in occupation numbers space controlled only by external field and tunneling.

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