Periodic quantum graphs with predefined spectral gaps

Andrii Khrabustovskyi

Institute of Applied Mathematics, Graz University of Technology, Austria
Department of Physics, Faculty of Science, University of Hradec Králové, Czech Republic

E-mail: khrabustovskyi@math.tugraz.at and andrii.khrabustovskyi@uhk.cz

Received 18 May 2020, revised 11 July 2020
Accepted for publication 27 July 2020
Published 1 September 2020

Abstract
Let $\Gamma$ be an arbitrary $\mathbb{Z}^n$-periodic metric graph, which does not coincide with a line. We consider the Hamiltonian $H_\varepsilon$ on $\Gamma$ with the action $-\varepsilon^{-1}d^2/dx^2$ on its edges; here $\varepsilon > 0$ is a small parameter. Let $m \in \mathbb{N}$. We show that under a proper choice of vertex conditions the spectrum $\sigma(H_\varepsilon)$ of $H_\varepsilon$ has at least $m$ gaps as $\varepsilon$ is small enough. We demonstrate that the asymptotic behavior of these gaps and the asymptotic behavior of the bottom of $\sigma(H_\varepsilon)$ as $\varepsilon \to 0$ can be completely controlled through a suitable choice of coupling constants standing in those vertex conditions. We also show how to ensure for fixed (small enough) $\varepsilon$ the precise coincidence of the left endpoints of the first $m$ spectral gaps with predefined numbers.

Keywords: periodic quantum graphs, spectral gaps, $\delta$-interactions, $\delta'$-interactions, control of spectrum

Introduction
Traditionally the name quantum graph refers to a pair $(\Gamma, \mathcal{H})$, where $\Gamma$ is a network-shaped structure of vertices connected by edges of certain positive lengths (metric graph) and $\mathcal{H}$ is a second order self-adjoint differential operator on $\Gamma$ (Hamiltonian). Hamiltonians are determined by differential operations on the edges and certain interface conditions at the vertices. We refer to the monograph [5] for a broad overview and an extensive bibliography on this topic.

Quantum graphs arise naturally in mathematics, physics, chemistry and engineering as simplified models of wave propagation in quasi-one-dimensional systems looking like narrow...
neighborhoods of graphs. Typical applications include quantum wires [23, 24], photonic crystals [29, 30], graphene and carbon nanostructures [21, 31], quantum chaos [25, 26] and many other areas. For more details concerning origins of quantum graphs see [27] and [5, chapter 7].

In various applications (for example, to aforementioned graphene and carbon nanostructures, and photonic crystals) periodic infinite graphs are studied. In what follows in order to simplify the presentation (but without any loss of generality) we assume that our graphs are embedded into \( \mathbb{R}^d \) for some \( d \in \mathbb{N} \). An infinite metric graph \( \Gamma \subset \mathbb{R}^d \) is said to be \( \mathbb{Z}^n \)-periodic (\( n \leq d \)) if it invariant under translations through some linearly independent vectors \( \nu_1, \ldots, \nu_n \in \mathbb{R}^d \). The Hamiltonian \( \mathcal{H} \) on a \( \mathbb{Z}^n \)-periodic metric graph \( \Gamma \) is said to be periodic if it commutes with these translations.

It is well-known that the spectrum of a periodic Hamiltonian on a periodic metric graph can be represented as a locally finite union of compact intervals (spectral bands). The bounded open interval is called a gap if it has an empty intersection with the spectrum, but its ends belong to it. The band structure of the spectrum suggests that gaps may exist in principle. In general, however, the presence of gaps is not guaranteed: two spectral bands may overlap, and then the corresponding gap disappears. For instance, if \( \Gamma \) is a rectangular lattice, \( \mathcal{H} \) is defined by the operation \(-d^2/dx^2\) on the edges and the standard Kirchhoff conditions at the vertices, then \( \sigma(\mathcal{H}) \) has no gaps—it coincides with \([0, \infty)\).

Existence and locations of spectral gaps are of primary interest because of various applications, for example in physics of photonic crystals—periodic nanostructures, whose characteristic property is that the light waves at certain optical frequencies fail to propagate in them, which is caused by gaps in the spectrum of the Maxwell operator or related scalar operators. For more details we refer to [29, 30], where periodic high contrast photonic and acoustic media are studied in high contrast regimes leading to appearance of Dirichlet-to-Neumann type operators on periodic graphs.

To create spectral gaps one can use geometrical means. For example, given a fixed graph we ‘decorate’ it changing its geometrical structure at each vertex: either one attaches to each vertex a copy of certain fixed compact graph [28] (see also [39] where similar idea was used for discrete graphs) or in each vertex one disconnects the edges emerging from it and then connects their loose endpoints by a certain additional graph (‘spider’) [8, 37].

Another way to open spectral gaps is to use ‘advanced’ vertex conditions. For example, as we already noted the spectrum of the Kirchhoff Laplacian on a rectangular lattice has no gaps, however (see [9]) if we replace Kirchhoff conditions by the so-called \( \delta \)-conditions of the strength \( \alpha \neq 0 \) one immediately gets infinitely many gaps provided the lattice-spacing ratio is a rational number.

Further results on spectral gaps opening for periodic quantum graphs as well as on various estimates on their location and lengths can be found in [1, 3, 6, 10, 13, 14, 20–22, 31–36].

When designing materials with prescribed properties it is desirable not only to open up spectral gaps, but also to be able to control their location and length—via a suitable choice of operator coefficients or/and geometry of the medium. We addressed this problem for various classes of periodic operators in a series of papers [4, 11, 17–19]. In particular, periodic quantum graphs were treated in [4]. In [4] the required structure for the spectrum is achieved via the combination of two approaches described above: taking a fixed periodic graph \( \Gamma_0 \) we decorate it attaching to each period cell \( m \) compact graphs \( Y_{ij} \); here \( j = 1, \ldots, m \), while the subscript \( i \in \mathbb{Z}^n \) indicates to which period cell we attach \( Y_{ij} \) (see figure 1, here \( m = 2 \)). On \( \Gamma \) we considered the Hamiltonian \( \mathcal{H}_\varepsilon \) defined by the operation \(-\varepsilon^{-1}d^2/dx^2\) on the edges and the Kirchhoff conditions in all its vertices except the points of attachment of \( Y_{ij} \) to \( \Gamma_0 \)—in these points we
Figure 1. Example of a periodic graph utilized in [4]. Reproduced from [4]. © IOP Publishing Ltd. All rights reserved.

pose (a kind of) \( \delta' \)-conditions\(^1\). Note, that the vertex conditions we dealt with in [4] ‘generate’ only Hamiltonians with \( \inf(\sigma(H_\varepsilon)) = 0 \). It was proven that \( \sigma(H_\varepsilon) \) has at least \( m \) gaps for small enough \( \varepsilon \), these gaps converge (as \( \varepsilon \to 0 \)) to some intervals \( (A_j, B_j) \subset [0, \infty) \) whose location and lengths can be nicely controlled by a suitable choice of coupling constants standing in those \( \delta' \)-conditions and a suitable choice of the ‘sizes’ of attached graphs \( Y_{ij} \).

In the current paper we continue the research started in [4]. We will prove that the required structure of the spectrum can be achieved solely by an appropriate choice of vertex conditions without any assumptions on the graph geometry. Namely, let \( \Gamma \) be a \( \mathbb{Z}^n \)-periodic metric graph. The only assumption we impose on it is that \( \Gamma \) does not coincide with a line. On \( \Gamma \) we consider the Hamiltonian \( H_\varepsilon \) defined by the operation \(-\varepsilon^{-1}d^2/dx^2 \) on edges and either Kirchhoff, \( \delta \) or \( \delta' \)-type (different from those treated in [4]) conditions at vertices—see (1.7)–(1.9). We prove that \( \sigma(H_\varepsilon) \) has at least \( m \) gaps; when \( \varepsilon \to 0 \) the first \( m \) gaps (respectively, the infimum of \( \sigma(H_\varepsilon) \)) converge to some intervals \( (A_j, B_j) \subset \mathbb{R} \), \( j = 1, \ldots, m \) (respectively, to some number \( B_0 \in \mathbb{R} \)); the location of \( A_j, j = 1, \ldots, m \) and \( B_j, j = 0, \ldots, m \) depends in an explicit way on the coupling constants standing in \( \delta \) and \( \delta' \)-type vertex conditions; see theorem 1.1. Moreover, choosing these coupling constants in a proper way one can completely control \( A_j \) and \( B_j \) making them coincident with predefined numbers; see theorem 3.1. Note, that in contrast to [4], the limiting intervals and the bottom of the spectrum do not necessarily lie on the positive semi-axis—the numbers \( A_j, B_j \) are also allowed to be negative. Finally we show that for fixed (small enough) \( \varepsilon \) one can guarantee the precise coincidence of the left endpoints of the first \( m \) gaps with prescribed numbers; see theorem 3.2.

The method we use to prove the convergence of spectra is different from the one used in [4], where we utilized Simon’s result [40] about monotonic sequences of forms. In the current work we apply the abstract lemma from [12] serving to compare eigenvalues of two self-adjoint operators acting in different Hilbert spaces. The advantage of this approach is that we are able not only to prove the convergence of spectra, but also to estimate the rate of convergence.

The structure of the paper is as follows. In section 1 we introduce the Hamiltonian \( H_\varepsilon \) and formulate the main convergence result. Its proof is given in section 2. In section 3 we demonstrate how to control the location of spectral gaps.

1. Setting of the problem and main result

1.1. Metric graph \( \Gamma \)

Let \( n \in \mathbb{N} \) and let \( \Gamma \) be an arbitrary connected \( \mathbb{Z}^n \)-periodic locally finite metric graph. The only assumption we impose on the geometry of \( \Gamma \) is that it does not coincide with a line (see

\(^1\) For the definition of \( \delta \) and \( \delta' \)-conditions in the graph context see, e.g., [9].
the footnote\(^2\) explaining the role of this assumptions) and its fundamental domain is compact (see below). W.l.o.g. (cf the discussion after definition 4.1.1 in \cite{5}) one can assume that \(\Gamma\) is embedded into \(\mathbb{R}^d\) with \(d = n\) as \(n \geq 3\) and \(d = 3\) as \(n = 1, 2\). We also assume that \(\Gamma\) has no loops—otherwise one can break them into pieces by introducing a new intermediate vertex.

By \(E_\Gamma\) and \(Y_\Gamma\) we denote the sets of edges and vertices of \(\Gamma\), respectively. By \(I = l(e)\) we denote the function assigning to the edge \(e\) its length \(l(e)\). We assume that \(l(e) < \infty\) for each \(e \in E_\Gamma\). In a natural way we introduce on each edge \(e \in E_\Gamma\) the local coordinate \(x_e \in [0, l(e)]\), so that \(x_e = 0\) and \(x_e = l(e)\) correspond to the endpoints of \(e\). For \(v \in Y_\Gamma\) we denote by \(E(v)\) the set of edges emanating from \(v\).

The \(\mathbb{Z}^n\)-periodicity of \(\Gamma\) means that

\[
\Gamma + \nu_k = \Gamma, \quad k = 1, \ldots, n
\]

for some linearly independent vectors \(\nu_1, \ldots, \nu_n \in \mathbb{R}^d\). Let us introduce for \(i = (i_1, \ldots, i_n) \in \mathbb{Z}^n\) the mapping \(i : \Gamma \rightarrow \Gamma\) defined by

\[
i \cdot x = x + \sum_{k=1}^n i_k \nu_k, \quad x \in \Gamma.
\]

We denote by \(Y\) a \textit{fundamental domain} of \(\Gamma\), i.e. a compact set (see the assumption above) satisfying

\[
\bigcup_{i \in \mathbb{Z}^n} i \cdot Y = \Gamma, \quad \text{the sets } Y \text{ and } i \cdot Y \text{ may have only finitely many common points as } i \neq 0.
\]

In particular, the above condition implies that the vertices on the boundary of the fundamental domain cannot have any common edges. Evidently a fundamental domain in not uniquely defined. Note that for any

\[
r \in \mathbb{N}_0^n := \{r = (r_1, \ldots, r_n) : \ r_k \in \mathbb{N} \cup \{0\}, \quad k = 1, \ldots, n\}
\]

the graph \(\Gamma\) is also invariant under translations through vectors \(\nu'_1, \ldots, \nu'_n\) defined by \(\nu'_k = r_k \nu_k\). The corresponding fundamental domain is the set \(Y'\) given by

\[
Y' = \bigcup_{i \in \mathcal{S}'} i \cdot Y, \quad \text{where } \mathcal{S}' = \{i \in \mathbb{Z}^n : \ i_k \in [0, r_k], \quad k = 1, \ldots, n\}. \quad (1.2)
\]

Finally, we denote by \(\mathcal{W}_Y\) the set of points of a fundamental domain \(Y\) that simultaneously belong to ‘neighbouring’ fundamental domains, i.e.

\[
\mathcal{W}_Y = \{v \in Y : \ \exists i \in \mathbb{Z}^n \setminus \{0\} \text{ such that } v \in i \cdot Y\}.
\]

An example of a \(\mathbb{Z}^2\)-periodic graph is presented on figure 2(a). This is an equilateral hexagonal lattice in \(\mathbb{R}^2\), which is invariant under translations through vectors \(\mathcal{v}_1 = (\sqrt{3}, 0), \ \mathcal{v}_2 = (-\frac{\sqrt{3}}{2}, \frac{1}{2})\). Its fundamental domain \(Y\) is highlighted in bold lines. On figure 2(b) one sees the fundamental domain \(Y'\) (1.2) for \(r = (2, 2)\). On these figures the bold dots are vertices belonging to \(\mathcal{W}_Y\) and \(\mathcal{W}_{Y'}\), respectively.

\(^2\)In order to achieve decomposition (1.3) and (1.4) we require our initial assumption on \(\Gamma\) that it does not coincide with a line. If \(\Gamma\) is a line, its fundamental domain \(Y'\) would be a compact interval; one can decompose it in such a way that properties (ii)–(v) hold, but then the set \(Y_0\) will be always disconnected.
Figure 2. (a) $\mathbb{Z}^2$-periodic graph $\Gamma$ and its fundamental domain. (b) The fundamental domain $Y^r$ for $r = (2, 2)$. (c) Decomposition of $Y^r$ for $m = 3$.

1.2. Decomposition of a fundamental domain

It is easy to see that for any $m \in \mathbb{N}$ there exists such $r = (r_1, \ldots, r_n) \in \mathbb{N}_0^n$ that the fundamental domain $Y^r$ can be represented as a union

$$Y^r = \bigcup_{j=0}^{m} Y_j$$

(1.3)

of non-empty compact sets $Y_j$, $j = 0, \ldots, m$ satisfying the following conditions:

(i) $Y_j$ are connected, $j = 0, \ldots, m$,

(ii) $\mathcal{U}_{Y} \subset Y_0$, $\mathcal{U}_{Y} \cap Y_j = \emptyset$, $j = 1, \ldots, m$,

(iii) $\forall (j \neq 0, k \neq 0, j \neq k): Y_j \cap Y_k = \emptyset$,

(iv) the sets $\mathcal{V}_j := Y_j \cap Y_0$, $j = 1, \ldots, m$ are non-empty and consist of finitely many points,

(v) $Y_0$ has a vertex $\tilde{v}$ belonging neither to $\mathcal{U}_Y$ nor to $\bigcup_{j=1}^{m} Y_j$.

(1.4)

W.l.o.g. we may assume that the points belonging to $\mathcal{V}_j$ are the vertices of $\Gamma$ (if $v \in \mathcal{V}_j$ lies on the interior of an edge of $\Gamma$ we can regard it as a vertex with two outgoing edges). It is easy to see that such a decomposition is always possible for large enough $r_1, r_2, \ldots, r_n$. Of course such a decomposition is not unique. For example on figure 2(c) the domain $Y^r$ is decomposed in such a way that (1.3) and (1.4) with $m = 3$ holds: $Y_0$ consists of bold solid lines, while $Y_1, Y_2, Y_3$ consist of one dashed edge, the black square is $\tilde{v}$, the white circles indicate vertices belonging to $\mathcal{V}_j$ (on the figure each $\mathcal{V}_j$, $j = 1, 2, 3$ consists of two vertices denoted by $v_{j1}$ and $v_{j2}$).

Now, let $m \in \mathbb{N}$ be given and let us fix such $r = (r_1, \ldots, r_n) \in \mathbb{N}_0^n$ that the fundamental domain $Y^r$ admits representation (1.3) and (1.4). We set for $i \in \mathbb{Z}^n$:

$$\mathcal{V}_{ij} := \vec{i} \cdot \mathcal{V}_j, \quad j = 1, \ldots, m, \quad Y_{ij} := \vec{i} \cdot Y_j, \quad j = 0, \ldots, m, \quad \tilde{v}_i := \vec{i} \cdot \tilde{v},$$

where the mapping $\vec{i} \cdot \Gamma \to \Gamma$ is defined by $\vec{i} \cdot x = x + \sum_{k=1}^{n} i_k r_k \nu_k$. $x \in \Gamma$.

The vertices belonging to $\mathcal{V}_{ij}$ will support $\delta'$-type conditions, the vertices $\tilde{v}_i$ will support $\delta$-conditions, in the remaining vertices the Kirchhoff conditions will be posed.

1.3. Functional spaces

In what follows if $u: \Gamma \to \mathbb{C}$ and $e \in \mathcal{E}_{\Gamma}$ then by $u_e$ we denote the restriction of $u$ onto the interior of $e$. Via a local coordinate $x_e$ we identify $u_e$ with a function on $(0, l(e))$. 

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The space $L^2(\Gamma)$ consists of functions $u : \Gamma \to \mathbb{C}$ such that $u_e \in L^2(0, l(e))$ for each edge $e$ and
\[ \|u\|_{L^2(\Gamma)}^2 := \sum_{e \in \mathcal{E}(\Gamma)} \|u_e\|_{L^2(0, l(e))}^2 < \infty. \]
The space $\tilde{H}^4(\Gamma)$, $k \in \mathbb{N}$ consists of functions $u : \Gamma \to \mathbb{C}$ such that $u_e$ belongs to the Sobolev space $H^4(0, l(e))$ for each edge $e$ and
\[ \|u\|_{\tilde{H}^4(\Gamma)}^2 := \sum_{e \in \mathcal{E}(\Gamma)} \|u_e\|_{H^4_0(0, l(e))}^2 < \infty. \]

By $H^s_0(\Gamma)$ we denote a subspace of $\tilde{H}^4(\Gamma)$ consisting of such function $u \in \tilde{H}^4(\Gamma)$ that
- if $v \in \mathcal{V}(\Gamma \setminus (\bigcup_{i \in \mathbb{Z}} U_{i})$) then $u$ is continuous at $v$, i.e. the limiting value of $u(x)$ when $x$ approaches $v$ along $e \in \mathcal{E}(v)$ is the same for each $e \in \mathcal{E}(v)$. We denote this value by $u(v)$;
- if $v \in \mathcal{V}(\Gamma \setminus (\bigcup_{i \in \mathbb{Z}} U_{i})$) for some $i = (i_1, \ldots, i_m) \in \mathbb{Z}^m$, $j \in \{1, \ldots, m\}$ then $u$ is continuous at $v$, i.e. the limiting value of $u(x)$ when $x$ approaches $v$ along $e \in \mathcal{E}(v)$ is the same for each $e \in \mathcal{E}(v)$.

14. Operator $\mathcal{H}_e$

Let $\varepsilon > 0$ be a small parameter. In $L^2(\Gamma)$ we introduce the quadratic form $\mathcal{H}_e$,
\[ \mathcal{H}_e(u, u) = \varepsilon^{-1} \sum_{e \in \mathcal{E}(\Gamma)} \|u_e\|_{L^2(0, l(e))}^2 + \sum_{i \in \mathbb{Z}} \sum_{j=1}^{m} \alpha_j |u_0(v) - \beta_j u_j(v)|^2 + \sum_{i \in \mathbb{Z}} \gamma |u(v_i)|^2 \]  \hspace{1cm} (1.5)

on the domain $\text{dom}(\mathcal{H}_e) = H^4_0(\Gamma)$. Here $\alpha_j, \beta_j, \gamma$ are real constants, moreover $\alpha_j \neq 0, \beta_j \neq 0$ [this assumption is needed to avoid the decoupling at the vertex $v$, cf (1.9)]. These constants are at our disposal and they will be specified later in section 3. The second and third terms in the right-hand side of (1.5) are indeed finite on $u \in \tilde{H}^4(\Gamma)$, this follows easily from the trace inequality [5, lemma 1.3.8]
\[ |u(0)|^2 \leq 2I^{-1} \|u\|_{L^2(0, l)}^2 + l \|u\|_{L^2(0, l)}^2, \quad \forall u \in H^4(0, l) \]

and periodicity of $\Gamma$. It is also straightforward to verify that the form $\mathcal{H}_e$ is densely defined in $L^2(\Gamma)$, lower semibounded and closed. By the first representation theorem [16, theorem 6.2.1] there exists the unique self-adjoint operator $\mathcal{H}_e$ associated with the form $\mathcal{H}_e$, i.e.

\[ (\mathcal{H}_e, u, w)_{L^2(\Gamma)} = \mathcal{H}_e(u, w), \quad \forall u \in \text{dom}(\mathcal{H}_e) \subset \text{dom}(\mathcal{H}_e), \quad \forall w \in \text{dom}(\mathcal{H}_e), \]  \hspace{1cm} (1.6)

where $\mathcal{H}_e(u, w)$ is the sesquilinear form, which corresponds to the quadratic form (1.5).

The domain of $\mathcal{H}_e$ consists of functions $u \in H^4_0(\Gamma) \cap \tilde{H}^4(\Gamma)$ satisfying
\[ \sum_{e \in \mathcal{E}(\Gamma)} \frac{d}{dx} u_e \bigg|_{x=0} = 0 \quad \text{at} \ v \in \mathcal{V}(\Gamma \setminus \bigcup_{i \in \mathbb{Z}} U_{i}) \cup \left( \bigcup_{j=1}^{m} \mathcal{V}(\Gamma) \right). \hspace{1cm} (1.7) \]
\[ \sum_{e \in \mathcal{E}(v)} \frac{du_e}{dx_e} \bigg|_{x_e=0} = \gamma \varepsilon u(v) \quad \text{at } v = \tilde{v}_j, \]  
\[ \sum_{e \in \mathcal{E}(v) \cap \mathcal{Y}_j} \frac{du_e}{dx_e} \bigg|_{x_e=0} = \alpha \varepsilon \left( u_0(v) - \beta \mu_j(v) \right), \]  
\[ \sum_{e \in \mathcal{E}(v) \cap \mathcal{Y}_j} \frac{du_e}{dx_e} \bigg|_{x_e=0} = -\alpha \varepsilon \left( u_0(v) - \beta \mu_j(v) \right) \]  
where \( x_e \in [0, l(e)] \) is a natural coordinate on \( e \in \mathcal{E}(v) \) such that \( x_e = 0 \) at \( v \). The action of \( \mathcal{H}_\varepsilon \) is

\[ (\mathcal{H}_\varepsilon u)_{e} = -\varepsilon^{-1} \frac{d^2u_e}{dx_e^2}, \quad e \in \mathcal{E}. \]  

Condition (1.7) is usually referred as Kirchhoff coupling (sometimes the name Neumann coupling is used), condition (1.8) is known as \( \delta \)-coupling of the strength \( \gamma \varepsilon \). We may refer to conditions (1.9) as \( \delta \)-type coupling. The reason for this is as follows. Suppose that \( v \in \mathcal{Y}_j \) has only two outgoing edges \( e \in \mathcal{E}(v) \cap \mathcal{Y}_j \) and \( \tilde{e} \in \mathcal{E}(v) \cap \mathcal{Y}_j \). Also let \( \beta_j = 1 \). Then conditions (1.9) are equivalent to

\[ \frac{du_e}{dx_e} \bigg|_{x_e=0} + \frac{du_{\tilde{e}}}{dx_{\tilde{e}}} \bigg|_{x_{\tilde{e}}=0} = 0, \quad (\alpha \varepsilon)^{-1} \frac{du_e}{dx_e} \bigg|_{x_e=0} = u_e \big|_{x_e=0} = u_{\tilde{e}} \big|_{x_{\tilde{e}}=0}. \]

Taking into account the definition of coordinates \( x_e \) and \( x_{\tilde{e}} \) we conclude that (1.9) coincides with the usual \( \delta \)-conditions of the strength \((\alpha \varepsilon)^{-1}\) at a point on the line [2, section 1.4].

### 1.5. Main results

We denote

\[ l_j := \sum_{e \in \mathcal{E}_j} l(e), \quad j = 0, \ldots, m, \quad N_j := \text{cardinality of } \mathcal{Y}_j, \quad j = 1, \ldots, m. \]  

Then for \( j = 1, \ldots, m \) we set

\[ A_j := \alpha_j \beta_j^2 N_j l_j^{-1}, \]  

where \( \alpha_j, \beta_j \) are real non-zero constants from (1.5). We assume that \( A_j \) are pairwise distinct; in this case we can renumber them in such a way that

\[ \forall j = 1, \ldots, m - 1 : \quad A_j < A_{j+1}. \]  

Finally, we consider the following equation (for unknown \( \lambda \in \mathbb{C} \{ \{ A_1, \ldots, A_m \} \)):

\[ \lambda \left( l_0 + \sum_{j=1}^{m} \frac{A_j l_j}{\beta_j^2 (A_j - \lambda)} \right) = \gamma \]  

where \( \gamma \) is a real constant from (1.5). It is easy to show that this equation has exactly \( m + 1 \) roots \( B_j, j = 0, \ldots, m \), they are real, moreover (after an appropriate renumerations) these roots satisfy

\[ B_0 < A_1 < B_1 < A_2 < B_2 < \ldots < A_m < B_m. \]
We are now in position to formulate the first main result of this work.

**Theorem 1.1.** There exist such positive constants $\Lambda_0$ (depending on $Y$) and $C_A, C_B, \varepsilon_0$ (depending on $\alpha, \beta, \gamma$ and $Y$) that for all $\varepsilon < \varepsilon_0$ the spectrum of $H_\varepsilon$ has the following structure within $(-\infty, \Lambda_0\varepsilon^{-1}]$:

$$\sigma(H_\varepsilon) \cap (-\infty, \Lambda_0\varepsilon^{-1}] = [B_{0,\varepsilon}, \Lambda_0\varepsilon^{-1}] \setminus \bigcup_{j=1}^m (A_{j,\varepsilon}, B_{j,\varepsilon}),$$

(1.16)

where the numbers $A_{j,\varepsilon}, j = 1, \ldots, m$ and $B_{j,\varepsilon}, j = 0, \ldots, m$ satisfy

$$B_{0,\varepsilon} < A_{1,\varepsilon} < A_{2,\varepsilon} < B_{2,\varepsilon} < \ldots < A_{m,\varepsilon} < B_{m,\varepsilon} < \Lambda_0\varepsilon^{-1},$$

(1.17)

moreover

$$0 \leq A_j - A_{j,\varepsilon} \leq C_A\varepsilon^{1/2}, \quad j = 1, \ldots, m, \quad 0 \leq B_j - B_{j,\varepsilon} \leq C_B\varepsilon^{1/2}, \quad j = 0, \ldots, m,$$

(1.18)

the numbers $A_j, B_j$ are specified by (1.12)–(1.14).

**2. Proof of theorem 1.1**

**2.1. Preliminaries**

To simplify the notations we assume that the fundamental domain $Y$ admits representation (1.3) and (1.4) for $r = 0$, i.e., already the initial fundamental domain $Y$ admits such a representation. In the general case one should simply change the notations accordingly.

In the following if $H$ is a self-adjoint lower semi-bounded operator with purely discrete spectrum, we denote by $\{\lambda_k(H)\}_{k \in \mathbb{N}}$ the sequence of its eigenvalues arranged in the ascending order and repeated according to their multiplicity.

The Floquet–Bloch theory [5, chapter 4] establishes a relationship between the spectrum of the operator $H_\varepsilon$, and the spectra of certain operators $H_\varepsilon^j$ in $L^2(Y)$. Namely, let

$$\theta \in \mathbb{T}^n = \{\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n, \quad |\theta_k| = 1 \quad \text{for all} \quad k = 1, \ldots, n\}.$$  

We denote by $H_\varepsilon^{ij}(\Gamma)$ the set of such functions $u : \Gamma \to \mathbb{C}$ that $u_e \in H^1(0, l(e))$ for each $e \in \mathcal{E}_\Gamma$, $u$ satisfy the same conditions at vertices of $\Gamma$ as functions from $H_\varepsilon^i(\Gamma)$, and

$$\forall \ x \in \Gamma, \quad \forall \ i = (i_1, \ldots, i_n) \in \mathbb{Z}^n : \ u(i \cdot x) = \theta^i u(x), \quad \text{where} \quad \theta^i := \left(\prod_{k=1}^n (\theta_k)^{i_k}\right)$$

[recall, that the mapping $i \cdot \Gamma \to \Gamma$ is defined by (1.1)]. We introduce the quadratic form $b_\theta^i$ by

$$b_\theta^i(u, u) = -\varepsilon^{-1} \sum_{e \in \mathcal{E}_\Gamma} ||u_e||^2_{L^2(0, l(e))} + \sum_{j=1}^m \sum_{v \in \mathcal{V}_\Gamma} \alpha_j |u_0(v) - \beta_j u_j(v)|^2 + \gamma |u(\tilde{v})|^2,$$

(2.1)

$$\text{dom}(b_\theta^i) = \left\{ u = f \big|_\Gamma, \quad f \in H_\varepsilon^{ij}(\Gamma) \right\}.$$  

Hereinafter by $\mathcal{E}_\Gamma$ and $\mathcal{V}_\Gamma$ we denote the set of edges and vertices of $Y$, respectively; similar notations will be used for $Y_j$. The form $b_\theta^i$ is densely defined in $L^2(Y)$, lower semibounded and
closed. We denote by $H_{\theta \varepsilon}$ the operator associated with $h_{\theta \varepsilon}$. The spectrum of $H_{\theta \varepsilon}$ is purely
discrete, moreover for each $k \in \mathbb{N}$ the function $\theta \mapsto \lambda_k(H_{\theta \varepsilon})$ is continuous. Consequently, the set

$$L_{k,\varepsilon} = \bigcup_{\theta \in T^n} \{\lambda_k(H_{\theta \varepsilon})\}$$

is a compact interval. (2.2)

According to the Floquet–Bloch theory we have the following representation:

$$\sigma(H_{\varepsilon}) = \bigcup_{k=1}^{\infty} L_{k,\varepsilon}.$$ (2.3)

Along with $h_{\theta \varepsilon}$ we also introduce the forms $h_{N \varepsilon}$ and $h_{D \varepsilon}$ acting on the domains

$$\operatorname{dom}(h_{N \varepsilon}) = \{ u = f \mid f \in H^1_0(\Gamma) \} \quad \text{and} \quad \operatorname{dom}(h_{D \varepsilon}) = \{ u = f \mid f \in H^1_0(\Gamma), \supp(v) \subset Y \}$$

and with the action being again specified by (2.1). By $H_{N \varepsilon}$ and $H_{D \varepsilon}$ we denote the associated operators. The spectra of these operators are purely discrete. It is easy to see that

$$\forall \theta \in T^n : \operatorname{dom}(h_{N \varepsilon}) \supset \operatorname{dom}(h_{\theta \varepsilon}) \supset \operatorname{dom}(h_{D \varepsilon}),$$

whence, using the min–max principle [7, section 4.5], we conclude

$$\forall k \in \mathbb{N}, \forall \theta \in T^n : \lambda_k(H_{N \varepsilon}) \leq \lambda_k(H_{\theta \varepsilon}) \leq \lambda_k(H_{D \varepsilon}).$$ (2.4)

In the following we mostly use two distinguished points of $T^n$,

$$\theta_p := (1, 1, \ldots, 1) \quad \text{and} \quad \theta_a := -(1, 1, \ldots, 1).$$ (2.5)

The subscripts $p$ and $a$ means periodic and antiperiodic, respectively.

**Remark 2.1.** The main ingredients for the proof of theorem 1.1 are two-side estimates for the eigenvalues $\lambda_k(H_{\theta \varepsilon})$, $\lambda_k(H_{N \varepsilon})$, $\lambda_k(H_{D \varepsilon})$, see lemmata 2.1, 2.3–2.6 below. The proof of these estimates is based on the standard min–max principle [7, theorem 4.5.3] and the result [12, lemma 2.1] both requiring no information on the structure of the domains of the operators $H_{\theta \varepsilon}$, $H_{N \varepsilon}$, $H_{D \varepsilon}$ (all calculations are conducted on the level of the associated quadratic forms). However it is interesting to take a more close look on these operators. Let $u \in \operatorname{dom}(H_{\varepsilon})$ with $* \in \{ \theta, N, D \}$. Then

- for each $v \in \mathcal{E}_Y$ one has $u_v \in H^2(0, l(e))$,
- at the vertices from $\mathcal{V}_Y \setminus \mathcal{U}_Y$ $u$ satisfies the same conditions as functions belonging to $\operatorname{dom}(H_{\varepsilon})$.

To describe the behaviour of $u$ on $\mathcal{U}_Y$ we assume for simplicity that points of $\mathcal{U}_Y$ lie on the interior of edges of $\Gamma$ (in fact, one can always choose a period cell in such a way that this assumption fulfills). This assumption on a period cell implies, in particular, that for any $v \in \mathcal{U}_Y$ there is only one edge of $\mathcal{E}_Y$ (we denote it $e_v$) emanating from $v$. Then we get the following boundary conditions at $\mathcal{U}_Y$:

- $u \in \operatorname{dom}(H_{\theta \varepsilon})$ satisfies $\theta$-periodic conditions at $v \in \mathcal{U}_Y$:

$$u_{e_v}(w) = \theta^l u_{e_v}(v), \quad \frac{\partial u_{e_v}}{\partial x_{e_v}}(w) = -\theta^l \frac{\partial u_{e_v}}{\partial x_{e_v}}(v)$$
Lemma 2.1. There exist one has

Recall, that 2.2. Determination of

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0 can be decomposed in a sum

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as above),

there exists a unique w with w = i · v for some i ∈ Z

provided the period cell is chosen as above),

• u ∈ dom(ℋ^N)

satisfies Neumann conditions \( \frac{\partial u}{\partial n} (v) = 0 \)

at v ∈ \( Y \),

• u ∈ dom(ℋ^D)

satisfies Dirichlet conditions \( u(v) = 0 \)

at v ∈ \( Y \). Above \( x_0 \) is defined as above, such that \( x_0 = 0 \) at v; in the same way \( x_0 \) is defined. The action of

all operators above is given by (1.10).

2.2. Determination of \( \lambda_0 \)

Recall, that \( \theta_0 \) is given in (2.5).

Lemma 2.1. There exist \( \lambda_0 > 0 \) and \( \varepsilon_\Lambda > 0 \) such that

\[ \forall \varepsilon < \varepsilon_\Lambda : \lambda_0 \varepsilon^{-1} < \lambda_{m+1}(\mathcal{H}_\varepsilon^0). \] (2.6)

Proof. For \( \theta \in \mathbb{T}^n \) and \( \varepsilon \geq 0 \) we introduce in \( \mathbb{L}^2(Y) \) the form \( h^0_\varepsilon \),

\[ h^0_\varepsilon(u, u) = \sum_{e \in \mathcal{E}_Y} \| u_e \|_{\mathcal{L}^2(0, \varepsilon_0)}^2 + \varepsilon \sum_{j=1}^{m} \sum_{v \in Y_j} \alpha_j |u(v) - \beta_j u_j(v)|^2 + \varepsilon \gamma |u(\widetilde{v})|^2, \]

\[ \text{dom}(h^0_\varepsilon) = \text{dom}(h^0_0). \]

We denote by \( H^0_\varepsilon \) the self-adjoint operator associated with this form. Obviously,

\[ \forall \varepsilon > 0 \quad \forall k \in \mathbb{N} : \varepsilon^{-1} \lambda_k(H^0_\varepsilon) = \lambda_k(\mathcal{H}_\varepsilon^0). \] (2.7)

Also we observe that with respect to the space decomposition \( \mathbb{L}^2(Y) = \oplus_{j=0}^{m} \mathbb{L}^2(Y_j) \) the operator \( H^0_\varepsilon \)

can be decomposed in a sum

\[ H^0_\varepsilon = H^0_{0,0} \oplus H^N_{0,1} \oplus H^N_{0,2} \oplus \cdots \oplus H^N_{0,m}, \] (2.8)

where the operators \( H^0_{0,0}, H^N_{0,j} \) are associated with the forms \( h^0_{0,0}, h^N_{0,j} \) defined as follows,

\[ h^0_{0,0}(u, u) = \sum_{e \in \mathcal{E}_Y} \| u_e \|_{\mathcal{L}^2(0, \varepsilon_0)}^2, \quad \text{dom}(h^0_{0,0}) = \{ u = \varepsilon_{0,0}, \ v \in \text{dom}(h^0_0) \}, \] (2.9)

\[ h^N_{0,j}(u, u) = \sum_{e \in \mathcal{E}_Y} \| u_e \|_{\mathcal{L}^2(0, \varepsilon_0)}^2, \quad \text{dom}(h^N_{0,j}) = \{ u = \varepsilon_{0,j}, \ v \in \text{dom}(h^N_0) \}. \] (2.10)

It is easy to see that

\[ \lambda_1(H^N_{0,j}) = 0, \quad j = 1, \ldots, m, \] (2.11)

and the corresponding eigenspace consists of constant functions. Due to the connectivity of \( Y_j \) one has

\[ \lambda_2(H^N_{0,j}) > 0, \quad j = 1, \ldots, m. \] (2.12)

If \( \lambda_1(H^0_{0,0}) = 0 \), the corresponding eigenfunction would be constant which is possible iff \( \theta = \theta_p \). Thus

\[ \lambda_1(H^0_{0,0}) > 0, \quad \theta \neq \theta_p. \] (2.13)
It follows from \((2.8), (2.11)–(2.13)\) that \(\lambda_k(H^\varepsilon_0) = 0\) for \(k = 1, \ldots, m\), while

\[
\lambda_{m+1}(H^\varepsilon_0) = \min \{ \lambda_1(H^\varepsilon_0); \lambda_2(H^\varepsilon_0); \lambda_2(H^\varepsilon_0); \ldots ; \lambda_2(H^\varepsilon_0) \} > 0, \quad \theta \neq \theta_p. \tag{2.14}
\]

Using the fact that the sequence of forms \(h^\varepsilon\) increases monotonically as \(\varepsilon\) decreases, and moreover \(\lim_{\varepsilon \to 0} h^\varepsilon(u,u) = h^\varepsilon_0(u,u), \forall u \in \text{dom}(h^\varepsilon) = \text{dom}(h^\varepsilon_0)\) we conclude \([40, \text{theorem } 4.1]\):

\[
\forall f \in L^2(Y): \|(H^\varepsilon + I)^{-1} f - (H^\varepsilon_0 + I)^{-1} f\|_{L^2(Y)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.15}
\]

Moreover, since the sequence of resolvents \((H^\varepsilon + I)^{-1}\) decrease monotonically as \(\varepsilon\) decreases, and both resolvents \((H^\varepsilon + I)^{-1}\) and \((H^\varepsilon_0 + I)^{-1}\) are compact one can upgrade \((2.15)\) to the norm resolvent convergence \([16, \text{theorem } 8.3.5]\). As a consequence we get the convergence of spectra, namely

\[
\forall k \in \mathbb{N}: \lambda_k(H^\varepsilon) \to \lambda_k(H^\varepsilon_0) \quad \text{as } \varepsilon \to 0. \tag{2.16}
\]

We set

\[
\Lambda_0 := \frac{\lambda_{m+1}(H^\varepsilon_0)}{2}. \tag{2.17}
\]

Since \(\theta_a \neq \theta_p\), then \(\Lambda_0 > 0\). It follows from \((2.16)\) that there exists \(\varepsilon_\Lambda > 0\) such that

\[
\forall \varepsilon < \varepsilon_\Lambda: \quad \Lambda_0 < \lambda_{m+1}(H^\varepsilon_0). \tag{2.18}
\]

Combining \((2.7)\) and \((2.18)\) we arrive at the desired estimate \((2.6)\). The lemma is proven. \(\square\)

### 2.3. Comparison of eigenvalues

Here we recall a result from \([12]\) serving to compare eigenvalues of two operators acting in different Hilbert spaces. Let \(H\) and \(H'\) be separable Hilbert spaces, \(\mathcal{H}\) and \(\mathcal{H}'\) be non-negative self-adjoint operators in these spaces, and \(h\) and \(h'\) be the associated quadratic forms. We assume that both operators \(\mathcal{H}\) and \(\mathcal{H}'\) have purely discrete spectra.

**Lemma 2.2.** \([12, \text{lemma } 2.1]\) Suppose that \(\Phi : \text{dom}(h) \to \text{dom}(h')\) is a linear map such that

\[
\|u\|_H^2 \leq \|\Phi u\|_{H'}^2 + \delta_1 \left( \|u\|_{H'}^2 + h(u,u) \right),
\]

\[
h'(\Phi u, \Phi u) \leq h(u,u) + \delta_2 \left( \|u\|_{H'}^2 + h(u,u) \right)
\]

for all \(u \in \text{dom}(h)\). Here \(\delta_1, \delta_2\) are some positive constants. Then for each \(j \in \mathbb{N}\) we have

\[
\lambda_j(\mathcal{H}') \leq \lambda_j(\mathcal{H}) + \frac{\lambda_j(\mathcal{H})(1 + \lambda_j(\mathcal{H}))\delta_1 + (1 + \lambda_j(\mathcal{H}))\delta_2}{1 - (1 + \lambda_j(\mathcal{H}))\delta_1} \tag{2.19}
\]

provided the denominator \(1 - (1 + \lambda_j(\mathcal{H}))\delta_1\) is positive.

**Remark 2.2.** The above result was established in \([12]\) under the assumption that \(\text{dim}H = \text{dim}H' = \infty\), however, it is easy to see from its proof that the result remains valid for \(\text{dim}H' < \infty\) as well. In that case \((2.19)\) holds for \(j \in \{1, \ldots, \text{dim}H'\}\).
2.4. Estimates on \( \lambda_1(\mathcal{H}^N_0) \) and \( \lambda_m(\mathcal{H}^N_0) \)

In this subsection we denote by bold letters (e.g., \( \mathbf{u} \)) the elements of \( \mathbb{C}^{m+1} \). Their entries will be enumerated starting from zero, i.e.,

\[
\mathbf{u} \in \mathbb{C}^{m+1} \Rightarrow \mathbf{u} = (u_0, \ldots, u_m) \quad \text{with} \quad u_j \in \mathbb{C}.
\]

Let \( \mathbb{C}^{m+1} \) be the same space \( \mathbb{C}^{m+1} \) equipped with the weighted scalar product

\[
(\mathbf{u}, \mathbf{v})_{\mathbb{C}^{m+1}} = \sum_{j=0}^{m} u_j \overline{v}_j
\]

(recall that \( I_j \) and \( N_j \) are defined by (1.11)). Note that \( \mathbb{C}^{m+1} \) is isomorphic to a subspace of \( L^2(Y) \) consisting of functions being constant on each \( Y_j, \ j = 0, \ldots, m \). In \( \mathbb{C}^{m+1} \) we introduce the form

\[
h^N_0(\mathbf{u}, \mathbf{u}) = \sum_{j=1}^{m} \alpha_j N_j |u_0 - \beta_j u_j|^2 + \gamma |u_0|^2, \quad \text{dom}(h^N_0) = \mathbb{C}^{m+1}.
\]

This form is associated with the operator \( \mathcal{H}^N_0 \) in \( \mathbb{C}^{m+1} \) being given by the symmetric [with respect to the scalar product (2.20)] matrix

\[
\mathcal{H}^N_0 = \begin{pmatrix}
\gamma I_0^{-1} + \sum_{j=1}^{m} \alpha_j N_j I_j^{-1} & \cdots & \alpha_m N_m I_m^{-1} \\
\cdots & \ddots & \cdots \\
-\alpha_1 \beta_1 N_1 I_1^{-1} & \cdots & -\alpha_m \beta_m N_m I_m^{-1}
\end{pmatrix}.
\]

We denote by \( \lambda_1(\mathcal{H}^N_0) \lesssim \lambda_2(\mathcal{H}^N_0) \lesssim \ldots \lesssim \lambda_{m+1}(\mathcal{H}^N_0) \) its eigenvalues. It turns out that

\[
\lambda_j(\mathcal{H}^N_0) = B_j, \quad j = 1, \ldots, m + 1.
\]

Indeed, let \( \lambda \) be the eigenvalue of \( \mathcal{H}^N_0 \) such that \( \lambda \notin \{A_1, A_2, \ldots, A_m\} \), and let \( \mathbf{0} \neq \mathbf{u} = (u_0, \ldots, u_m) \) be the corresponding eigenfunction. The equation \( \mathcal{H}^N_0 \mathbf{u} = \lambda \mathbf{u} \) is a linear algebraic system for \( u_0, \ldots, u_m \). From the last \( m \) equations of this system we infer

\[
u_j = \frac{\alpha_j \beta_j N_j I_j^{-1}}{\alpha_j \beta_j^2 N_j I_j^{-1} - \lambda} u_0, \quad j = 1, \ldots, m.
\]

Note, that the denominator in (2.23) is non-zero since \( \lambda \neq A_j = \alpha_j \beta_j^2 N_j I_j^{-1} \). Inserting (2.23) into the first equation of the system we arrive at

\[
u_0 \left[ \lambda \left( I_0 + \sum_{j=1}^{m} \frac{A_j I_j}{\beta_j^2 (A_j - \lambda)} \right) - \gamma \right] = 0.
\]

\[3\]\text{It is easy to see that for any } \varepsilon > 0 \text{ the form } h^\varepsilon_0 \text{ is the restriction of the form } h^0_0 \text{ onto a subspace of } L^2(Y) \text{ consisting of functions being constant on each } Y_j, \ j = 0, \ldots, m \text{ (as we already noticed this subspace is isomorphic to } \mathbb{C}^{m+1} \).}
Moreover, \( u_0 \neq 0 \) [otherwise, due to (2.23), \( u \) would vanish]. Hence \( \lambda \) is a root of equation (1.14). Evidently, the converse assertion also holds, that is
\[
\lambda \in \sigma(\mathcal{H}_0^\gamma) \setminus \{A_1, A_2, \ldots, A_m\} \quad \iff \quad \lambda \quad \text{is a root of (1.14).}
\] (2.24)
Then (2.22) follows immediately from (1.15) and (2.24).

**Lemma 2.3.** There exist constants \( C_B > 0 \) and \( \varepsilon_B > 0 \) such that
\[
\forall \varepsilon < \varepsilon_B : \quad B_{j-1} \leq \lambda_j(\mathcal{H}_\varepsilon^\gamma) + C_B \varepsilon^{1/2}, \quad j = 1, \ldots, m + 1.
\] (2.25)

**Proof.** W.l.o.g. we may assume that \( \alpha_j \) and \( \gamma \) are non-negative. Evidently, under this assumption the operators \( \mathcal{H}_\varepsilon^\gamma \) are non-negative. Consequently, the operator \( \mathcal{H}_0^\gamma \) is also non-negative, see footnote\(^3\). Thus we are in the framework of lemma 2.2. In the general case we have to consider the shifted operators \( \mathcal{H}_\varepsilon^\gamma - \mu I \) and \( \mathcal{H}_0^\gamma - \mu I \), where \( \mu \) is the smallest eigenvalues of \( \mathcal{H}_\varepsilon^\gamma |_{\varepsilon=1} \) (this eigenvalue could be indeed negative if one of the numbers \( \alpha_j \) and \( \gamma \) is negative). The operator \( \mathcal{H}_\varepsilon^\gamma - \mu I \) is non-negative for each \( \varepsilon \in (0, 1) \) due to the fact that the sequence of forms \( h_\varepsilon^\gamma \) increases monotonically as \( \varepsilon \) decreases; the non-negativity of \( \mathcal{H}_0^\gamma - \mu I \) is again due to footnote\(^3\).

We introduce the operator \( \Phi : \text{dom}(h_\varepsilon^\gamma) \rightarrow \mathbb{C}^{m+1} \) by
\[
(\Phi u)_j = \int_0^{l_j} u_\varepsilon(x) dx, \quad j = 0, \ldots, m.
\] (2.26)
Our goal is to show that the following estimates hold for each \( u \in \text{dom}(h_\varepsilon^\gamma) \):
\[
\|u\|_{\mathcal{L}^2(Y)}^2 \leq \|\Phi u\|_{\mathcal{L}^2(Y)}^2 + C_1 \varepsilon \left(\|u\|_{\mathcal{L}^2(Y)}^2 + h_\varepsilon^\gamma(u, u)\right),
\] (2.27)
\[
h_\varepsilon^\gamma(\Phi u, \Phi u) \leq h_\varepsilon^\gamma(u, u) + C_2 \varepsilon^{1/2} \left(\|u\|_{\mathcal{L}^2(Y)}^2 + h_\varepsilon^\gamma(u, u)\right).
\] (2.28)
with some \( C_1, C_2 > 0 \). By lemma 2.2 (see also remark 2.2 after it) we infer from (2.27) and (2.28) that
\[
B_{j-1} = \lambda_j(\mathcal{H}_0^\rho) \leq \lambda_j(\mathcal{H}_\varepsilon^\rho) + \frac{\lambda_j(\mathcal{H}_\varepsilon^\rho)(1 + \lambda_j(\mathcal{H}_\varepsilon^\rho))C_1 \varepsilon + (1 + \lambda_j(\mathcal{H}_\varepsilon^\rho))C_2 \varepsilon^{1/2}}{1 - (1 + \lambda_j(\mathcal{H}_\varepsilon^\rho))C_1 \varepsilon}
\] (2.29)
provided \( (1 + \lambda_j(\mathcal{H}_\varepsilon^\rho))C_1 \varepsilon < 1 \). Set
\[
\varepsilon_B := \min \left\{ \frac{1}{2C_1(1 + B_m)}, 1 \right\}.
\]
Since \( 0 \leq \lambda_j(\mathcal{H}_\varepsilon^\rho) \leq B_{j-1} \) [the last estimate follows from (2.4) and lemma 2.4 below] and \( B_{j-1} \leq B_m \) for \( j = 1, \ldots, m + 1 \), the denominator in (2.29) is larger than \( 1/2 \) for \( \varepsilon < \varepsilon_B \). Moreover, since \( \varepsilon_B \leq 1 \), one has \( \varepsilon < \varepsilon_B \) for \( \varepsilon < \varepsilon_B \). Taking all above into account we deduce from (2.29):
\[
\forall \varepsilon < \varepsilon_B : \quad B_{j-1} \leq \lambda_j(\mathcal{H}_\varepsilon^\rho) + 2 \left( B_m(1 + B_m)C_1 + (1 + B_m)C_2 \right) \varepsilon^{1/2}.
\]
Thus estimate (2.25) holds for \( \varepsilon < \varepsilon_B \) with \( C_B = 2 \left( B_m(1 + B_m)C_1 + (1 + B_m)C_2 \right) \).
To prove (2.27) we need a Poincaré-type inequality on each \( Y_j \). Namely, let the form \( h_{0,j}^x \) be defined by (2.10), \( j = 1, \ldots, m \); in the same way we define \( h_{0,j}^x \) for \( j = 0 \). By \( H_{0,j}^x \) we denote the associated operators in \( L^2(Y_j) \), \( j = 0, \ldots, m \). One has \( \lambda_1(H_{0,j}^x) = 0 \) (the corresponding eigenspace consists of constants), while \( \lambda_2(H_{0,j}^x) > 0 \). By the max–min principle [38] \( \lambda_2(H_{0,j}^x) \leq h_{0,j}^x(v, v)/\| v \|_{L^2(Y_j)}^2 \) for each \( v \in \text{dom}(H_{0,j}^x) \) such that \( (v, 1)_{L^2(Y_j)} = 0 \). Using the above estimate for \( v := u - (\Phi u) \) we get

\[
\forall \ u \in H_{0,j}^x: \quad \| u - (\Phi u) \|^2_{L^2(Y_j)} \leq C_1 \sum_{e \in \mathcal{E}_{Y_j}} \| u_e' \|^2_{L^2(0, L(e))}, \quad j = 0, \ldots, m, \tag{2.30}
\]

where \( C_1 = (\lambda_2(H_{0,j}^x))^{-1} \). Using (2.30) we obtain

\[
\begin{align*}
\| u \|^2_{L^2(Y_j)} &= \sum_{j=0}^m \| u \|^2_{L^2(Y_j)} = \sum_{j=0}^m \left( \| (\Phi u) \|^2_{L^2(Y_j)} + \| u - (\Phi u) \|^2_{L^2(Y_j)} \right) \\
&\leq \| \Phi u \|^2_{C^{p+1}} + C_1 \sum_{e \in \mathcal{E}_{Y}} \| u_e' \|^2_{L^2(0, L(e))} \leq \| \Phi u \|^2_{C^{p+1}} + C_1 \epsilon h_{e,j}^x(u, u) \\
&\leq \| \Phi u \|^2_{C^{p+1}} + C_1 \epsilon \left( \| u \|^2_{L^2(Y)} + h_{e,j}^x(u, u) \right)
\end{align*}
\]

(on the penultimate step we use the fact that \( \alpha_j \) and \( \gamma \) are non-negative). Inequality (2.27) is checked.

Now let us prove the estimate (2.28). One has:

\[
\begin{align*}
h_{e,j}^x(\Phi u, \Phi u) &= h_{e,j}^x(u, u) - \epsilon^{-1} \sum_{e \in \mathcal{E}_{Y}} \| u_e' \|^2_{L^2(0, L(e))} \\
&\quad + \sum_{j=1}^m \sum_{\epsilon \in \mathcal{V}_j} \alpha_j \left[ \| (\Phi u) - \beta_j(\Phi u) \|^2 - \| u_0(v) - \beta_j u_j(v) \|^2 \right] \\
&\quad + \gamma \left[ \| (\Phi u)_0 \|^2 - \| u_0(v) \|^2 \right] \\
&= R_x. \tag{2.31}
\end{align*}
\]

We estimate the remainder \( R_x \) as follows (below we use the estimate \( |a|^2 - |b|^2 \leq |a - b| (|a| + |b|) \)):

\[
|R_x| \leq \sum_{j=1}^m \sum_{\epsilon \in \mathcal{V}_j} \alpha_j \left[ \| (\Phi u)_0 - u_0(v) \| + \| \beta_j \cdot (\Phi u) - u_j(v) \| \right] \\
\times \left( \| (\Phi u)_0 \| + \| u_0(v) \| + \| \beta_j \cdot (\Phi u) \| + \| \beta_j \cdot u_j(v) \| \right) \\
+ \gamma \left[ \| (\Phi u)_0 \| - \| u_0(v) \| \right] \cdot \left( \| (\Phi u)_0 \| + \| u_0(v) \| \right). \tag{2.32}
\]

To proceed further we need a standard trace estimate

\[
\| w \|_{L^\infty(Y_j)} \leq \tilde{C} \| w \|_{H^1(Y_j)}, \quad w \in H^1(Y_j), \quad j = 0, \ldots, m, \tag{2.33}
\]
where $\tilde{C} > 0$ depends on $Y$. Applying it for $w := u|_{Y_j} - (\Phi u)|_{Y_j}$ and then using (2.30) we obtain

$$\|u - (\Phi u)|_{L^\infty(Y_j)} \leq \tilde{C} \left( \|u - (\Phi u)|_{L^2(Y_j)} + \sum_{e \in \delta Y_j} \|u_e^\prime\|_{L^2(0, \kappa e)}^2 \right)^{1/2}$$

$$\leq \tilde{C}(C_1 + 1) \left( \sum_{e \in \delta Y_j} \|u_e^\prime\|_{L^2(0, \kappa e)}^2 \right)^{1/2}$$

$$\leq \tilde{C}(C_1 + 1) \varepsilon^{1/2} \left( \|u\|_{L^2(Y_j)}^2 + h^N_\varepsilon(u, u) \right)^{1/2}, \quad j = 0, \ldots, m. \quad (2.34)$$

Also, using the Cauchy–Schwarz inequality and (2.33) and taking into account that $\varepsilon \leq 1$, one gets

$$|\Phi u_j| \leq I_j^{-1/2} \|u\|_{L^2(Y_j)} \leq \max_{I_j} I_j^{-1/2} \left( \|u\|_{L^2(Y_j)}^2 + h^N_\varepsilon(u, u) \right)^{1/2}, \quad j = 0, \ldots, m. \quad (2.35)$$

$$\|u\|_{L^\infty(Y_j)} \leq \tilde{C} \left( \|u\|_{L^2(Y_j)}^2 + h^N_\varepsilon(u, u) \right)^{1/2}, \quad j = 0, \ldots, m. \quad (2.36)$$

Combining (2.32), (2.34)–(2.36) we arrive at the estimate

$$|R_{\varepsilon}| \leq C_2 \varepsilon^{1/2} \left( \|u\|_{L^2(Y_j)}^2 + h^N_\varepsilon(u, u) \right) \quad (2.37)$$

with some constant $C_2$ depending on $\alpha_j, \beta_j, \gamma, Y$. The required estimate (2.28) follows from (2.31), (2.37); this ends the proof of lemma 2.3.

**Lemma 2.4.** One has:

$$\lambda_j(\mathcal{H}^0_{p_h}) \leq B_j^{-1}, \quad j = 1, \ldots, m + 1.$$

**Proof.** By the min–max principle [7, section 4.5] we have

$$\lambda_j(\mathcal{H}^0_{p_h}) = \min_{V \in \mathcal{F}_j} \max_{u \in \mathcal{V}_j \setminus \{0\}} \frac{h^0_\varepsilon(u, u)}{\|u\|_{L^2(Y_j)}^2} \quad (2.38)$$

where $\mathcal{F}_j$ is a set of all $j$-dimensional subspaces in $\text{dom}(h^0_\varepsilon)$.

We introduce the operator $\Psi : C^{m+1}_Y \rightarrow L^2(Y)$ by

$$\Psi u = \sum_{j=0}^m u_j \chi_{Y_j}, \quad \text{\(\chi_{Y_j}\) is the indicator function of \(Y_j\).} \quad (2.39)$$

It is easy to see that the image of $\Psi$ is contained in $\text{dom}(h^0_\varepsilon)$, and

$$\|\Psi u\|_{L^2(Y)} = \|u\|_{C^{m+1}_Y}, \quad h^0_\varepsilon(\Psi u, \Psi u) = h^0_\varepsilon(u, u) \quad (2.40)$$

[recall, that the form $h^0_\varepsilon$ is given by (2.21), by $\mathcal{H}^0_0$ we denote the associated operator].
Let \( \{e_1, e_2, \ldots, e_m\} \) be an orthonormal system of eigenvectors of \( \mathcal{H}_0^N \) such that \( \mathcal{H}_0^N e_1 = B_{j-1} e_1 \) [see (2.22)]. For \( j = 1, \ldots, m + 1 \) we set \( W^j := \text{span}(e_1, \ldots, e_j) \). It is easy to see that

\[
\max_{u \in W^j(0)} \frac{b^N_0(u, u)}{\|u\|_{L^2_j}^2} = B_{j-1}.
\]  

(2.41)

Finally, we set \( V^j := \Psi W^j \), obviously \( V^j \in V^j \). Then using (2.38)–(2.41) we obtain:

\[
\lambda_j(\mathcal{H}_0^N) \leq \max_{u \in V^j(0)} \frac{b^N_0(u, u)}{\|u\|_{L^2_j}^2} = \max_{u \in W^j(0)} \frac{b^N_0(u, u)}{\|u\|_{L^2_j}^2} = B_{j-1}.
\]

The lemma is proven. \(\square\)

2.5. Estimates on \( \lambda_A(\mathcal{H}_0^N) \) and \( \lambda_A(\mathcal{H}_0^P) \)

Let \( C^m \) be the subspace of \( C^{m+1} \) consisting of vectors of the form \( u = (0, u_1, \ldots, u_m) \) with \( u_j \in \mathbb{C} \) with the scalar product generated by (2.20), i.e.

\[
(u, v)_C^m = \sum_{j=1}^m u_j v_j.
\]

In this space we introduce the quadratic form

\[
b_0^N(u, u) = \sum_{j=1}^m \alpha_j \beta_j^2 N_j |u_j|^2.
\]

It is easy to see that \( b_0^N = b_0^N |_C^m \). The operator associated with this form is given by the matrix

\[
\mathcal{H}_0^N = \text{diag} (\alpha_1 \beta_1^2 N_1 l_1^{-1}, \alpha_2 \beta_2^2 N_2 l_2^{-1}, \ldots, \alpha_m \beta_m^2 N_m l_m^{-1}).
\]

Evidently, the eigenvalues of this matrix are the numbers \( A_1 < A_2 < \ldots < A_m \).

Lemma 2.5. There exist such constants \( C_A > 0 \) and \( \varepsilon_A > 0 \) that

\[
\forall \varepsilon < \varepsilon_A : A_j \leq \lambda_j(\mathcal{H}_0^N) + C_A \varepsilon^{1/2}, \quad j = 1, \ldots, m.
\]  

(2.42)

Proof. The proof is similar to the proof of lemma 2.3. There is only one essential difference: instead of the operator \( \Phi \) (2.26) one should use the operator \( \Phi_0 : \text{dom}(h_0^N) \to C^m \) defined by

\[
(\Phi_0 u)_0 = 0, \quad (\Phi_0 u)_j = (\Phi u)_j, \quad j = 1, \ldots, m.
\]

and, as a consequence, instead of the Poincaré inequality (2.30) on \( Y_0 \) one should use the inequality

\[
\|u\|_{L^2_j(Y_0)}^2 \leq C_1 \sum_{e_j \in \mathcal{F}_0} \|u_{e_j}'\|_{L^2(0, e_j)}^2
\]

where \( C_1 = (\lambda_1(\mathcal{H}_0^N)^{-1})^{-1} \) [recall, that the operator \( \mathcal{H}_0^N \) is introduced in the proof of lemma 2.1, and its first eigenvalue is non-zero provided \( \theta \neq \theta_p \)]. \(\square\)
Lemma 2.6. One has:
\[ \lambda_j(\mathcal{H}^D) \leq A_j, \quad j = 1, \ldots, m. \]

Proof. The proof is similar to the proof of lemma 2.4. Namely, one has to replace everywhere in the proof of lemma 2.4 the superscript \( \theta_D \) by \( D \), the superscript \( N \) by \( \theta_D \), \( B_{j-1} \) by \( A_j \), and to use instead of the mapping \( \Psi \) (2.39) its restriction to \( \mathbb{C}_1 \) (the image of this restriction is contained in \( \text{dom}(h^D) \)). \( \square \)

2.6. Proof of theorem 1.1

It follows from (2.2), (2.4) and lemmata 2.3 and 2.4 that
\[ \forall \varepsilon < \varepsilon_B: \quad B_{j-1} - C_B \varepsilon^{1/2} \leq \inf(L_{j,\varepsilon}) \leq B_{j-1}, \quad j = 1, \ldots, m + 1. \] (2.43)

Similarly, using (2.2), (2.4) and lemmata 2.5 and 2.6 we get
\[ \forall \varepsilon < \varepsilon_A: \quad A_j - C_A \varepsilon^{1/2} \leq \sup(L_{j,\varepsilon}) \leq A_j, \quad j = 1, \ldots, m \] (2.44)

Finally, we infer from (2.2) and lemma 2.1 that
\[ \forall \varepsilon < \varepsilon_A: \quad \Lambda_0 \varepsilon^{-1} \leq \sup(L_{m+1,\varepsilon}). \] (2.45)

Set
\[ A_{j,\varepsilon} := \sup(L_{j,\varepsilon}), \quad j = 1, \ldots, m, \quad B_{j,\varepsilon} := \inf(L_{j+1,\varepsilon}), \quad j = 0, \ldots, m. \] (2.46)

Combining (1.15), (2.43)–(2.46) we conclude that there exists such \( \varepsilon_0 > 0 \) that properties (1.16)–(1.18) hold for \( \varepsilon < \varepsilon_0 \), \( \Lambda_0 \) being defined by (2.17), \( C_A \) being defined in lemma 2.5, \( C_B \) being defined in lemma 2.3. Evidently, \( \Lambda_0 \) depends only on \( Y \), while \( \varepsilon_0 \), \( C_A \), \( C_B \) depend also on \( \alpha_j, \beta_j, \gamma \). theorem 1.1 is proven.

Remark 2.3. The proof of theorem 1.1 relies, in particular, on some properties of the eigenvalues of the operator \( \mathcal{H}_\varepsilon^D \)—see the estimates (2.6), (2.42). In fact, the only specific property of \( \theta_D \) we use is that \( \theta_D \neq \theta_D \). Thus, instead of \( \mathcal{H}_\varepsilon^D \) one can utilize any other \( \mathcal{H}_\varepsilon^D \) with \( \theta \neq \theta_D \)—the above estimates are still valid for its eigenvalues (but, of course, with another constants \( \Lambda, \varepsilon_A, C_A, \varepsilon_A \)).

3. Control over the endpoints of spectral gaps

Our first goal is to show that under a suitable choice of coupling constants \( \alpha_j, \beta_j, \gamma \) the numbers \( A_j, B_j \) (cf theorem 1.1) coincide with prescribed ones.

Throughout this section we will use the notation \( \mathcal{H}_\varepsilon[\alpha, \beta, \gamma] \) for the operator \( \mathcal{H}_\varepsilon \), defined in subsection 1.4 [recall that this operator is associated with the form given by (1.5)]; here let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m, \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, \gamma \in \mathbb{R} \) be such that \( \alpha_j \neq 0, \beta_j \neq 0 \) and, moreover, (1.13) holds (so, we are in the framework of theorem 1.1). For the numbers \( A_j \) and \( B_j \) defined by (1.12), (1.14), (1.15) we will use the notations \( A_j[\alpha, \beta, \gamma] \) and \( B_j[\alpha, \beta, \gamma] \), respectively.

Theorem 3.1. Let \( \tilde{A}_j, j = 1, \ldots, m \) and \( \tilde{B}_j, j = 0, \ldots, m \) be arbitrary numbers satisfying
\[ \tilde{B}_0 < \tilde{A}_1 < \tilde{B}_1 < \tilde{A}_2 < \tilde{B}_2 < \ldots < \tilde{A}_m < \tilde{B}_m, \quad \tilde{A}_j \neq 0, \quad j = 1, \ldots, m. \] (3.1)
We set
\[
\tilde{\alpha}_j = \frac{\tilde{r}_j(A_j - B_0)l_0}{N_j}, \quad \tilde{\beta}_j = \sqrt{\frac{A_jl_j}{\tilde{r}_j(A_j - B_0)l_0}}, \quad \tilde{\gamma} = \tilde{B}_0 \left(1 + \sum_{j=1}^{m} \tilde{r}_j \right) l_0, \tag{3.2}
\]
where \(\tilde{r}_j, j = 1, \ldots, m\) is defined by
\[
\tilde{r}_j = \frac{\tilde{B}_j - \tilde{A}_j}{\tilde{A}_j} \prod_{i=1, i \neq j}^{m} \left(\frac{\tilde{B}_i - \tilde{A}_j}{\tilde{A}_i - \tilde{A}_j}\right). \tag{3.3}
\]
Then
\[
A_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}] = \tilde{A}_j, \quad j = 1, \ldots, m, \quad B_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}] = \tilde{B}_j, \quad j = 0, \ldots, m.
\]

**Remark 3.1.** The equality \(A_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}] = \tilde{A}_j, \quad j = 1, \ldots, m\) is straightforward—one just needs to insert \(\tilde{\alpha}_j, \tilde{\beta}_j\) and \(\tilde{\gamma}\) defined by (3.2) into the definition of the numbers \(A_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}]\) (1.12).

Now, let us prove that \(B_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}] = \tilde{B}_j\) as \(j = 0, \ldots, m\). For this purpose, we consider the following system of linear algebraic equations (for unknown \(z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m\)):
\[
\sum_{i=1}^{m} \frac{\tilde{A}_i z_i}{\tilde{A}_i - \tilde{B}_j} = -1, \quad j = 1, \ldots, m.
\]
It was proven in [17] that \(z = (\tilde{r}_1, \ldots, \tilde{r}_m)\) with \(\tilde{r}_j\) being defined by (3.3) is the solution to this system. Thus for \(j = 1, \ldots, m\) one has \(\sum_{i=1}^{m} \tilde{A}_i (\tilde{A}_i - \tilde{B}_j)^{-1} \tilde{r}_i = -1\) or, equivalently,
\[
\forall j \in \{0, \ldots, m\} : \quad \langle \tilde{B}_j - \tilde{B}_0 \rangle \sum_{i=1}^{m} \frac{\tilde{A}_i}{\tilde{A}_i - \tilde{B}_j} \tilde{r}_i = \tilde{B}_0 - \tilde{B}_j. \tag{3.4}
\]
It is straightforward to check that (3.4) implies\(^4\)
\[
\forall j \in \{0, \ldots, m\} : \quad \tilde{B}_j \left(l_0 + \sum_{i=1}^{m} \frac{\tilde{A}_i l_i}{\tilde{\beta}_i (\tilde{A}_i - \tilde{B}_j)}\right) = \tilde{\gamma}. \tag{3.5}
\]
Using \(A_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}] = \tilde{A}_j\) we conclude from (3.5) that \(\tilde{B}_j, j = 0, \ldots, m\) are the roots of (1.14) in which \(\alpha_j = \tilde{\alpha}_j, \beta_j = \tilde{\beta}_j, \gamma = \tilde{\gamma}\) are set. Hence \(\tilde{B}_j = B_j[\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}]\) as \(j = 0, \ldots, m\). Theorem 3.1 is proven. \(\Box\)

Theorems 1.1, 3.1 yield that for all \(\varepsilon < \varepsilon_0 \sigma(H, [\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}])\) has \(m\) gaps within \((-\infty, \Lambda_0 \varepsilon^{-1}]\), moreover the endpoints of these \(m\) gaps and the bottom of the spectrum converge to prescribed numbers as \(\varepsilon \to 0\). Our next goal is to improve this result: we show that under a proper choice

\(^4\)Indeed, inserting into (3.5) \(\tilde{\beta}\) and \(\tilde{\gamma}\) defined by (3.2) and then performing simple calculations one arrives at (3.4). Using these calculations in the reverse order one gets the required implication (3.4) \(\Rightarrow\) (3.5).
of \( \alpha_j \) one can ensure the precise coincidence of the left endpoints of the spectral gaps of \( \mathcal{H}_x[\alpha, \tilde{\beta}, \tilde{\gamma}] \) with prescribed numbers.

**Theorem 3.2.** Let \( \tilde{A}_j, j = 1, \ldots, m \) and \( \tilde{B}_j, j = 0, \ldots, m \) be arbitrary numbers satisfying (3.1), and let \( \tilde{\beta}, \tilde{\gamma} \) be defined by (3.2). Then there exists such \( \tilde{\varepsilon} > 0 \) and \( C_0 > 0 \) that

\[
\forall \varepsilon < \tilde{\varepsilon} \quad \exists \alpha = \alpha(\varepsilon) \in \mathbb{R}^m : \sigma(\mathcal{H}_x[\alpha, \tilde{\beta}, \tilde{\gamma}]) \cap (-\infty, \Lambda_0 \varepsilon^{-1}] = [B_{0\varepsilon}, \Lambda_0 \varepsilon^{-1}] \bigcup_{j=1}^{m} (\tilde{A}_j, \tilde{B}_{j\varepsilon}),
\]

where \( \Lambda_0 \) is defined by (2.17), \( B_{0\varepsilon} < \tilde{A}_1 < B_{1\varepsilon} < \tilde{A}_2 < B_{2\varepsilon} < \ldots < \tilde{A}_m < B_{m\varepsilon} < \Lambda_0 \varepsilon^{-1} \), moreover

\[
0 < \tilde{B}_j - B_{j\varepsilon} \leq C_0 \varepsilon^{-1/2}, \quad j = 0, \ldots, m.
\]

The proof of theorem 3.2 is based on the following multi-dimensional version of the intermediate value theorem established in [15].

**Lemma 3.3.** [15, lemma 3.5] Let \( \mathcal{D} = \Pi_{k=1}^m [a_k, b_k] \) with \( a_k < b_k, k = 1, \ldots, m \), and suppose we are given a continuous function \( F : \mathcal{D} \to \mathbb{R}^m \) such that each component \( F_k \) of \( F \) is monotonically increasing in each of its arguments. Let us suppose that \( F_k^- < F_k^+ \), \( i = 1, \ldots, m \), where

\[
F_k^- = F(b_1, b_2, \ldots, b_{k-1}, a_k, b_{k+1}, \ldots, b_m),
\]

\[
F_k^+ = F(a_1, a_2, \ldots, a_{k-1}, b_k, a_{k+1}, \ldots, a_m).
\]

Then for any \( F^* \in \Pi_{k=1}^m [F_k^-, F_k^+] \) there exists a point \( x \in \mathcal{D} \) such that \( F(x) = F^* \).

**Proof of theorem 3.2.** Let \( \delta > 0 \) and \( \mathcal{D} := \Pi_{k=1}^m [\tilde{\alpha}_k - \delta, \tilde{\alpha}_k + \delta] \), where \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m \) be defined by (3.2). We assume that \( \delta \) is so small that

\[
\forall \alpha \in \mathcal{D} : \quad \alpha_j \neq 0, \quad j = 1, \ldots, m \quad \text{and} \quad A_j[\alpha, \tilde{\beta}, \tilde{\gamma}] < A_{j+1}[\alpha, \tilde{\beta}, \tilde{\gamma}], \quad j = 1, \ldots, m - 1.
\]

(3.6)

This could be indeed achieved since (3.6) holds for \( \alpha = \tilde{\alpha} \). Thus theorem 1.1 is applicable for each \( \alpha \in \mathcal{D} \). Also, analyzing the proof of theorem 1.1, it is easy to see that the constants \( \varepsilon_0, C_0 \) in theorem 1.1 can be chosen the same for all \( \alpha \in \mathcal{D} \); the proof of this fact relies on the compactness of \( \mathcal{D} \). Hence there exists \( \varepsilon_0 > 0 \) and \( C_0 > 0 \) such that

\[
\forall \varepsilon < \varepsilon_0 \quad \forall \alpha \in \mathcal{D} : \quad \sigma(\mathcal{H}_x[\alpha, \tilde{\beta}, \tilde{\gamma}]) \cap (-\infty, \Lambda_0 \varepsilon^{-1}] = [B_{0\varepsilon}, \Lambda_0 \varepsilon^{-1}] \bigcup_{j=1}^{m} (\tilde{A}_j, \tilde{B}_{j\varepsilon}),
\]

(3.7)

where \( A_{j\varepsilon}, B_{j\varepsilon} \) satisfy (1.17) and (1.18) (with \( A_j[\alpha, \tilde{\beta}, \tilde{\gamma}], B_j[\alpha, \tilde{\beta}, \tilde{\gamma}] \) instead of \( A_j \) and \( B_j \)). Further, for these \( A_{j\varepsilon}, B_{j\varepsilon} \) we will use the notations \( A_{j\varepsilon}[\alpha, \tilde{\beta}, \tilde{\gamma}], B_{j\varepsilon}[\alpha, \tilde{\beta}, \tilde{\gamma}] \), respectively. We denote

\[
\alpha_j^\pm := \tilde{\alpha}_j \pm \delta, \quad \alpha^\pm := (\alpha_1^\pm, \alpha_2^\pm, \ldots, \alpha_{j-1}^\pm, \alpha_j^\pm, \alpha_{j+1}^\pm, \ldots, \alpha_{m-1}^\pm, \alpha_m^\pm), \quad A_{j\varepsilon}^\pm := A_{j\varepsilon}[\alpha^\pm, \tilde{\beta}, \tilde{\gamma}].
\]

It is easy to see that there exists such \( \tilde{\varepsilon} \in (0, \varepsilon_0) \) that

\[
\forall \varepsilon < \tilde{\varepsilon} : \quad A_{j\varepsilon}^\pm < \tilde{A}_j < A_{j\varepsilon}^+, \quad j = 1, \ldots, m.
\]

(3.8)
Indeed, since \( \alpha_j^+ < \tilde{\alpha}_j < \alpha_j^+ \) and \( \tilde{A}_j^{Th}B_j^3 \tilde{A}_j^\dagger[\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}] = \tilde{\alpha}_j \tilde{\beta}_j N_j^{-1} \), then
\[
\forall \ j = 1, \ldots, m : \quad A_j^+ < \tilde{A}_j < A_j^+ ,
\] (3.9)
where \( A_j^\pm := \tilde{A}_j^\dagger[\alpha_j, \beta_j, \gamma_j] = \alpha_j^\pm \tilde{\beta}_j^\pm N_j^{-1} \). Moreover for \( \varepsilon < \varepsilon_0 \) we have
\[
0 \leq A_j^+ - A_j^+ \leq C \varepsilon^{1/2}.
\] (3.10)
Property (3.8) follows immediately from (3.9) and (3.10).

Now, let us fix \( \varepsilon \in (0, \varepsilon) \). We introduce the function \( F = (F_1, \ldots, F_m) : \mathcal{D} \to \mathbb{R}^m \) by
\[
\alpha_j \rightarrow F_j \mathcal{A}_k[\alpha, \tilde{\beta}, \tilde{\gamma}], \quad k = 1, \ldots, m.
\] (3.11)
The functions \( F_j \) are continuous. Indeed, let \( \alpha, \alpha' \in \mathcal{D} \). To simplify the presentation we assume that \( \alpha_j, \alpha_j', \gamma_j \geq 0 \) (and consequently \( \mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] \geq 0, \mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}] \geq 0 \)); general case need slight modifications. By virtue of (1.6) one has for \( f, g \in L^2(\Gamma) \),
\[
\left( (\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] + I)^{-1} f - (\mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}] + I)^{-1} f, g \right)_{L^2(\Gamma)} = h_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}](u, w) - h_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}](u, w)
= \sum_{i \in \mathcal{I}} \sum_{j=1}^m \sum_{\gamma_j \in \mathcal{Y}} (\alpha_j^\gamma_j - \alpha_j)(u_\gamma_j(v) - h_j')u_j(v)(w_\gamma_j(v) - h_j'w_j(v)),
\] (3.12)
where \( u = (\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] + I)^{-1} f, w = (\mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}] + I)^{-1} g, h_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] \) is a form associated with \( \mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] \). Using (2.33) and taking into account that \( \alpha_j, \alpha_j', \gamma_j \geq 0 \) we continue (3.12) as follows,
\[
\left| (\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] + I)^{-1} f - (\mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}] + I)^{-1} f, g \right|_{L^2(\Gamma)} \leq C_{\varepsilon^{1/2}} |\alpha - \alpha'|^{1/2} \left( \sum_{i \in \mathcal{I}} \sum_{j=1}^m \sum_{\gamma_j \in \mathcal{Y}} (\alpha_j^\gamma_j - \alpha_j)(u_\gamma_j(v) - h_j')u_j(v)(w_\gamma_j(v) - h_j'w_j(v)) \right)^{1/2}
\leq C_{\varepsilon^{1/2}} |\alpha - \alpha'| \left( \sum_{i \in \mathcal{I}} \sum_{j=1}^m \sum_{\gamma_j \in \mathcal{Y}} (\alpha_j^\gamma_j - \alpha_j)(u_\gamma_j(v) - h_j')u_j(v)(w_\gamma_j(v) - h_j'w_j(v)) \right)^{1/2}
\leq C_{\varepsilon^{1/2}} (f, w)_{L^2(\Gamma)} \leq C_{\varepsilon^{1/2}} |\alpha - \alpha'| \left( \sum_{i \in \mathcal{I}} \sum_{j=1}^m \sum_{\gamma_j \in \mathcal{Y}} (\alpha_j^\gamma_j - \alpha_j)(u_\gamma_j(v) - h_j')u_j(v)(w_\gamma_j(v) - h_j'w_j(v)) \right)^{1/2},
\] (3.13)
where \( C > 0 \) is a constant. It follows from (3.13) that
\[
\| (\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}] + I)^{-1} - (\mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}] + I)^{-1} \| \to 0 \quad \text{as} \quad \alpha - \alpha' \to 0,
\]
whence for an arbitrary compact set \( \mathcal{I} \subset \mathbb{R} \) one has
\[
\text{dist}_\mathcal{I}(\sigma(\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}]) \cap \mathcal{I}, \sigma(\mathcal{H}_\varepsilon[\alpha', \tilde{\beta}, \tilde{\gamma}]) \cap \mathcal{I}) \to 0 \quad \text{as} \quad \alpha - \alpha' \to 0,
\] (3.14)
where \( \text{dist}_\mathcal{I}(\cdot, \cdot) \) stands for the Hausdorff distance. Taking into account a special structure of \( \sigma(\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}]) \) (3.7) we conclude from (3.14) that \( \mathcal{A}_k[\alpha, \tilde{\beta}, \tilde{\gamma}] - \mathcal{A}_k[\alpha, \tilde{\beta}, \tilde{\gamma}] \to 0 \quad \text{as} \quad \alpha - \alpha' \to 0, \) i.e. \( F_k \) is continuous. The number \( \mathcal{A}_k[\alpha, \tilde{\beta}, \tilde{\gamma}] \) is the right endpoint of the kth spectral band:
\[
\mathcal{A}_k[\alpha, \tilde{\beta}, \tilde{\gamma}] = \max_{\beta \in \mathcal{I}} \lambda_k(\mathcal{H}_\varepsilon[\alpha, \tilde{\beta}, \tilde{\gamma}]),
\] (3.15)
where $\mathcal{H}_\theta^\beta[^\alpha, \tilde{\beta}, \tilde{\gamma}]$ denotes the operator $\mathcal{H}_\zeta^\beta$ with $\beta_j = \tilde{\beta}_j$, $\gamma = \tilde{\gamma}$. Since $\mathcal{H}_\theta^\beta[^\alpha, \tilde{\beta}, \tilde{\gamma}] \leq \mathcal{H}_\zeta^\beta[^\alpha', \tilde{\beta}, \tilde{\gamma}]$ (in the form sense) as $\alpha_j \leq \alpha_j'$, $\forall j = 1, \ldots, m$, by min–max principle we conclude for $k = 1, \ldots, m$:

$$\forall \theta \in \mathbb{T}^n: \lambda_k(\mathcal{H}_\theta^\beta[^\alpha, \tilde{\beta}, \tilde{\gamma}]) \leq \lambda_k(\mathcal{H}_\zeta^\beta[^\alpha', \tilde{\beta}, \tilde{\gamma}]) \text{ provided } \alpha_j \leq \alpha_j', \forall j = 1, \ldots, m.$$  

(3.16)

It follows from (3.15) and (3.16) that the functions $F_k$ increase monotonically in each of their arguments. Taking into account (3.8) we infer that the function $F$ satisfy all the requirements of lemma 3.3. Applying this lemma we conclude that there exists such $\alpha \in D$ that

$$F_k(\alpha) = \tilde{\Lambda}_k, \quad k = 1, \ldots, m.$$  

(3.17)

Combining (3.7), (3.11), (3.17) we arrive at the statement of theorem 3.2.

□

Remark 3.2. The assumption $\tilde{A}_j \neq 0, j = 1, \ldots, m$ in (3.1) is essential—one cannot avoid it when using the Hamiltonians $\mathcal{H}_\zeta$ introduced in subsection 1.4, since the numbers $A_j$ (1.12) are always non-zero. To overcome this restriction one can add to $\mathcal{H}_\zeta$ a constant potential, which shift the spectrum accordingly. Another option is to to pick in each $Y_j$, $j = 0, \ldots, m$ an internal point $\tilde{v}_j$, and then to add at $\tilde{v}_j$ the $\delta$-coupling of the strength $\tilde{\gamma}_l$, where $l_j$ is defined by (1.11) and $\tilde{\gamma} \in \mathbb{R}$. Denote by $\mathcal{H}_\zeta^{\tilde{\gamma}}$ the modified Hamiltonian. Repeating verbatim the arguments we use in the proof of theorem 1.1 one can show that the spectrum of $\mathcal{H}_\zeta^{\tilde{\gamma}}$ satisfies (1.16)–(1.18), but with $A_j + \tilde{\gamma}$ and $B_j + \tilde{\gamma}$ instead of $A_j$ and $B_j$.

Acknowledgments

The author is supported by Austrian Science Fund (FWF) under the Project M 2310-N32. Also he thanks the anonymous referees for useful comments which improved the paper considerably.

ORCID iDs

Andrii Khrabustovskyi © https://orcid.org/0000-0001-6298-9684

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