The subgroup $\text{PSL}_2(\mathbb{R})$ is spherical in the group of diffeomorphisms of the circle

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We show that the group $\text{PSL}_2(\mathbb{R})$ is a spherical subgroup in the group of $C^3$-diffeomorphisms of the circle. Also, the group of automorphisms of a Bruhat–Tits tree is a spherical subgroup in the group of hierarchomorphisms of the tree.

1. Sphericity. Let $G$ be a topological group, $K$ be a subgroup. An irreducible unitary representation $\rho$ of $G$ in a Hilbert space $H$ is called spherical if there is a unique up to a scalar factor non-zero $K$-invariant vector $v$ in $H$. The matrix element

$$\Phi(g) := \langle \rho(g)v, v \rangle$$

is called a spherical function on $G$. A subgroup $K$ in $G$ is called spherical if for any irreducible unitary representation of $G$ the dimension of the space of $K$-invariant vectors is $\leq 1$.

For various types of spherical pairs in this sense, see [5], [8], [18], [3], [16], [15]. For all known examples the group $K$ is compact or is an infinite-dimensional analog of compact groups as $U(\infty)$, $O(\infty)$, $\text{Sp}(\infty)$, $S(\infty)$ etc. ('heavy groups' in the sense of [13]).

2. Statements. Let $\text{SL}_2(\mathbb{R})$ be the group of $2 \times 2$ real matrices with determinant 1, let $\text{PSL}_2(\mathbb{R})$ be its quotient with respect to the center, $\text{SL}_2(\mathbb{R})^\sim$ be the universal covering group. Denote by $\text{Diff}$ (respectively by $\text{Diff}^3$) the group of $C^\infty$-smooth (resp. $C^3$-smooth) orientation preserving diffeomorphisms of the circle. Denote by $\text{Diff}^\sim$ the universal covering of $\text{Diff}$, we realize $\text{Diff}^\sim$ as the group of smooth diffeomorphisms $q$ of the line $\mathbb{R}$ satisfying the condition

$$q(\varphi + 2\pi) = q(\varphi) + 2\pi.$$ 

The Bott cocycle $c(q_1, q_2)$ on $\text{Diff}^\sim$ is defined by the formula

$$c(q_1, q_2) = \int_0^{2\pi} q_1'(\varphi)q_2'(\varphi) d\varphi.$$ 

Consider the central extension $\widetilde{\text{Diff}}$ of $\text{Diff}^\sim$ determined by the Bott cocycle (see, e.g., [4], §3.4). By $\widetilde{\text{Diff}}^3$ we denote the similar central extension of $\text{Diff}^3$.

**Theorem 1** The subgroup $\text{PSL}_2(\mathbb{R})$ is spherical in the group $\text{Diff}^3$.

**Theorem 2** The subgroup $\text{PSL}_2^\sim(\mathbb{R})$ is spherical in the group $\widetilde{\text{Diff}}^3$.

**Remark.** a) Several series of spherical representations of $\text{Diff}$ and $\text{Diff}^\sim$ were constructed in [11], see also [13], §IX.6. All such spherical representations are continuous in the $C^3$-topology.

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b) Sobolev diffeomorphisms of the circle of the class $s > 3/2$ form a group (see [7], Theorem 1.2 and Appendix B). Our proof survives for the group of Sobolev diffeomorphisms of the class $H^5$.

Next, consider a combinatorial analog of Diff. Fix an integer $n \geq 2$. Consider the Bruhat–Tits tree $T_n$, i.e., the infinite tree such that each vertex is incident to $n + 1$ edges. Let $\text{Abs}(T_n)$ be its boundary (for detailed definitions, see, e.g., [17], [12]). Denote by $\text{Aut}(T_n)$ the group of all automorphisms of the graph $T_n$. It is a locally compact group, stabilizers of finite subtrees form a base of open-closed neighborhoods of unit.

Denote by $\text{Vert}(T_n)$ the set of vertices of $T_n$. Consider a bijection $\theta : \text{Vert}(T_n) \to \text{Vert}(T_n)$ such that for all but a finite numbers of pairs of adjacent vertices $(a, b)$, vertices $\theta(a), \theta(b)$ are adjacent. Hierarchomorphism of the tree $T_n$ is are homeomorphisms of $\text{Abs}(T_n)$ induced by such maps, see [12], [14]. Denote by $\text{Hier}(T_n)$ the group of all hierarchomorphisms of the tree $T_n$.

Remark. a) For a prime $n = p$ the boundary $\text{Abs}(T_p)$ can be identified with a $p$-adic projective line. The group $\text{Aut}(T_p)$ contains the $p$-adic PSL$_2$ and the representation theory of $\text{Aut}(T_p)$ is similar to the representation theory of $p$-adic and real SL$_2$ (see [2], [17]). The group $\text{Hier}(T_p)$ contains the group of locally analytic diffeomorphisms of the $p$-adic projective line.

b) Richard Thompson groups (see [1]) are discrete subgroups of $\text{Hier}(T_n)$.

c) Several series of $\text{Aut}(T_n)$-spherical representations of $\text{Hier}(T_n)$ were constructed in [10], [12].

We define a topology on the group $\text{Hier}(T_n)$ assuming that $\text{Aut}(T_n)$ is an open subgroup (the coset space $\text{Hier}(T_n)/\text{Aut}(T_n)$ is countable).

**Theorem 3** The subgroup $\text{Aut}(T_p)$ is spherical in $\text{Hier}(T_p)$.

**Theorem 4** Let $G \supseteq K$ be a spherical pair. Assume that $K$ does not admit nontrivial finite-dimensional unitary representations. Let $\Phi_1(g), \Phi_2(g)$ be $K$-spherical functions on $G$. Then $\Phi_1(g)\Phi_2(g)$ is a spherical function.

The both $K = \text{SL}_2(\mathbb{R})^\sim$ and $\text{Aut}(T_n)$ satisfy this condition.

**3. Proof of Theorem 1.** Fix a point $a$ in the circle. Denote by $G_0 \subset \text{Diff}$ the group of diffeomorphisms $q$ such that $q(x) = x$ in a neighborhood of $a$. By $G^*$ we denote the group of diffeomorphisms that are flat at $a$, i.e.,

$$q(a) = a, \quad q'(a) = 1, \quad q''(a) = q'''(a) = \cdots = 0$$

Let $\rho$ be an irreducible unitary representation of $\text{Diff}$ in $H$. Denote by $V$ the subspace of all $\text{PSL}_2(\mathbb{R})$-fixed vectors. Let $P$ be the operator of orthogonal projection on $V$. For $h \in \text{PSL}_2(\mathbb{R})$ we have

$$P\rho(h) = \rho(h)P = P.$$ 

Denote

$$\hat{\rho}(g) := P\rho(g)P.$$
This operator depends only on a double coset of Diff by $\text{PSL}_2(\mathbb{R})$,
$$
\hat{\rho}(h_1 gh_2) := \hat{\rho}(g), \quad h_1, h_2 \in \text{PSL}_2(\mathbb{R}).
$$

**Lemma 5** If $\rho$ is continuous in the $C^3$-topology, then the operators $\hat{\rho}(g)$ pairwise commute.

**Proof.** The following statement is our key argument:

Let a sequence $h_j \in \text{PSL}_2(\mathbb{R})$ converges to infinity. Then $\rho(h_j)$ weakly converges to $P$, see Howe, Moore [6], Theorem 5.1 (this is a general theorem for semisimple group, for $\text{PSL}_2(\mathbb{R})$ it can be easily verified case-by-case).

Let us realize the circle as the real projective line $\mathbb{RP}^1 = \mathbb{R} \cup \infty$. Without loss of generality we can set $a = \infty$. Let $U_t(x) = x + t$ be a shift on $\mathbb{R}$, we have $U_t \in \text{PSL}_2(\mathbb{R})$. Consider diffeomorphisms $r, q \in G_0$. For sufficiently large $t$ the supports of $r$ and $U_t \circ q \circ U_{-t}$ are disjoint. Therefore, these diffeomorphisms commute. Hence,

$$
\rho(r) \rho(U_t) \rho(q) \rho(U_{-t}) = \rho(U_t) \rho(q) \rho(U_{-t}) \rho(r).
$$

Therefore,

$$
P \rho(r) \rho(U_t) \rho(q) P = P \rho(q) \rho(U_{-t}) \rho(r) P.
$$

Passing to a weak limit as $t \to +\infty$, we get

$$
P \rho(r) P \rho(q) P = P \rho(q) P \rho(r) P.
$$

Thus

$$
\hat{\rho}(r) \hat{\rho}(q) = \hat{\rho}(q) \hat{\rho}(r), \quad \text{where } r, q \in G_0.
$$

But $G_0$ is dense in $G^*$. Therefore the same identity holds for $r, q \in G_*$, Indeed, let $r_j, q_j \in G_0$ be sequences convergent to $r, q$ respectively. Passing to the iterated limit

$$
\lim_{j \to \infty} \left( \lim_{k \to \infty} \hat{\rho}(r_j) \hat{\rho}(q_k) \right) = \lim_{j \to \infty} \left( \lim_{k \to \infty} \hat{\rho}(q_k) \hat{\rho}(r_j) \right)
$$

and keeping in mind the separate weak continuity of the operator product, we get the desired statement.

Our last argument: the set $\text{PSL}_2(\mathbb{R}) \cdot G_* \cdot \text{PSL}_2(\mathbb{R})$ is dense in Diff with respect to the $C^3$-topology.

Let us prove this. Choose a coordinate on $\mathbb{RP}^1$ such that $a = 0$. Let $q \in \text{Diff}$. Consider its Schwarzian derivative,

$$
S(q) = \frac{q' q''' - \frac{3}{2} (q'')^2}{(q')^2}.
$$

Consider a point $b$ such that $S(q)(b) = 0$ (by the Ghys theorem, the Schwarzian derivative of a diffeomorphism of the circle has at least 4 zero, see [19], Theorem 4.2.1). Then for

$$
r := U_{-q(b)} \circ q \circ U_b
$$

2i.e., for any compact subset $B$, we have $h_j \notin B$ starting some number.
we have \( r(0) = 0, S(r)(0) = 0 \). Consider maps
\[
\sigma(x) = \frac{ux}{u^{-1} + vx},
\]
such \( \sigma \in \text{PSL}_2(\mathbb{R}) \) fix 0. Choosing parameters \( u, v \), we can achieve
\[
(q \circ \sigma)'(0) = 1, \quad (q \circ \sigma)''(0) = 0.
\]
Recall the transformation property of the Schwarzian:
\[
S(q \circ \sigma) = (S(q) \circ \sigma) \cdot (\sigma')^2 + S(\sigma).
\]
Since \( \sigma \) is linear fractional, \( S(\sigma) = 0 \). Therefore \( S(q \circ \sigma) = 0, \) and \( q''(0) = 0 \).
Such \( q \) can be approximated in \( C^3 \)-topology by elements of \( G_\ast \). This proves Lemma 5.
\[\square\]

Theorem 1 is a corollary of the lemma. Note, that \( \tilde{\rho}(g^\ast) = \tilde{\rho}(g^{-1}) \). Thus we get a family of commuting operators in \( V \), such that an adjoint operator \( A^\ast \) is contained in the family together with \( A \). If \( \dim V > 1 \), then this family has a proper invariant subspace in \( V \), say \( W \). Consider the Diff-cyclic span of \( W \), i.e.,
the subspace \( Z \) spanned by vectors \( \rho(g)w \), where \( g \in \text{Diff}_3 \) and \( w \in W \). Then
\[
P \rho(g)w = P \rho(g)Pw = \tilde{\rho}(g)w \in W.
\]
Hence, \( PZ = W \) and therefore \( Z \) is a proper subspace in \( H \).

4. Proof of Theorem 2. It repeats the previous proof with two additional remarks.

1) Consider the homomorphism \( \pi : \text{SL}_2(\mathbb{R})^\sim \rightarrow \text{PSL}_2(\mathbb{R}) \simeq \text{SL}_2(\mathbb{R})^\sim / \mathbb{Z} \). We say that a sequence \( h_j \in \text{SL}_2(\mathbb{R})^\sim \) converges to \( \infty \) if \( \pi(h_j) \rightarrow \infty \). Then the Howe–Moore theorem remains valid.

2) For a pair of diffeomorphisms with disjoint supports \( p, q \) the Bott cocycle \( c(q, p) \) vanishes, hence the diffeomorphisms \( p, q \) commute in the extended group.

5. Proof of Theorem 3. First, there is the following analog of the Howe–Moore theorem: Let a sequence \( h_j \in \text{Aut}(T_n) \) converges to \( \infty \). Then for any unitary representation \( \rho \) of \( \text{Aut}(T_n) \) the sequence \( \rho(h_j) \) converges to the projection to the subspace of \( \text{Aut}(T_n) \)-fixed vectors, see \[9\]; this can be easily verified case-by-case starting the classification theorem of \[17\].

Second, fix a point \( a \in \text{Abs}(T_n) \) and denote by \( G_0 \) the group of hierarchomorphisms that are trivial in a neighborhood of \( a \). Let \( q, r \in G_0 \). Then there is a sequence \( h_j \in \text{Aut}(T_n) \cap G_0 \) such that \( h_j \) tends to \( \infty \) and supports of \( h_jph_j^{-1} \) and \( q \) are disjoint. We omit a proof, since it is easier to understand its self-evidence than to read a formal exposition.

Third,
\[
\text{Aut}(T_n) \cdot G_0 \cdot \text{Aut}(T_n) = \text{Hier}(T_n)
\]
Now we can repeat the proof of Theorem 1.

6. Proof of Theorem 4. The statement is semitrivial.
Lemma 6 Let $\nu_1 \nu_2$ be unitary representations of a group $\Gamma$. If the tensor product $\nu_1 \otimes \nu_2$ contains a nonzero $\Gamma$-invariant vector, then the both $\nu_1$ and $\nu_2$ have finite-dimensional subrepresentations.

Proof of the lemma. Assume that an invariant vector exists. Denote the spaces of representations by $V_1$, $V_2$. We identify $V_1 \otimes V_2$ with the space of Hilbert–Schmidt operators $V'_1 \to V_2$, where $V'_1$ is the dual space to $V_1$. An invariant vector corresponds to an intertwining operator $T: V'_1 \to V_2$. The operator $TT^*$ is an intertwining operator in $V_2$. Since $TT^*$ is compact and nonzero, it has a finite-dimensional eigenspace, and this subspace is $G$-invariant. □

Proof of the theorem. Let $\rho_1$ and $\rho_2$ be $K$-spherical representations of $G$ in $H_1$ and $H_2$. Let $v_1$, $v_2$ be fixed vectors. By the lemma, $v_1 \otimes v_2$ is a unique $K$-fixed vector in $H_1 \otimes H_2$. The cyclic span $W$ of $v_1 \otimes v_2$ is an irreducible subrepresentation. Indeed, let $W = W_1 \oplus W_2$ be reducible. Then the both projections of $v_1 \otimes v_2$ to $W_1$, $W_2$ are $K$-fixed, therefore $v_1 \otimes v_2$ must be contained in one of summands, say $W_1$, and thus the cyclic span of $v_1 \otimes v_2$ is contained in $W_1$, i.e., $W = W_1$.

Now we consider the representation of $G$ in $W$,

$$\langle (\rho_1(g) \otimes \rho_2(g))v_1 \otimes v_2, v_1 \otimes v_2 \rangle_W = \langle \rho_1(g)v_1, v_1 \rangle_{H_1} \cdot \langle \rho_2(g)v_2, v_2 \rangle_{H_2} = \Phi_1(g)\Phi_2(g).$$

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