Vacuum Energy at Apparent Horizon in Conventional Model of Black Holes

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Abstract

For the conventional model of black holes, we show that the back reaction of the ingoing vacuum energy flux is important at the apparent horizon for certain calculations. In particular, for a black hole of Schwarzschild radius \(a\), the vacuum energy density for observers on top of the trapping horizon in vacuum is given by a universal formula: \(\mathcal{E} \simeq -1/2\kappa a^2\), which is of the same order as the classical mass density. This result is independent of the details about the vacuum energy-momentum tensor.
1 Introduction

In the conventional model of black holes, it is often assumed that the back reaction of the vacuum energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) can be ignored at the horizon, and the static Schwarzschild metric provides a good approximation for the geometry of the black hole. A necessary condition for the consistency of this assumption is that the vacuum energy-momentum tensor is finite in generic free-falling frames. In terms of the light-cone coordinates \((u, v)\) defined by the asymptotic Minkowski space at large distance, this implies that \( \langle T_{uu} \rangle \) and \( \langle T_{uv} \rangle \) are extremely small (much smaller than \( O(1/a^4) \)) at the horizon. One must therefore have a negative \( \langle T_{vv} \rangle \) of order \( O(1/a^4) \), as the mass of the black hole should decrease over time during the evaporation. In the conventional model, the ingoing negative vacuum energy dominates at the trapping horizon.

In this work, we first show that the back reaction of the ingoing negative vacuum energy \( \langle T_{vv} \rangle \) is important for certain events. It is well known that for a classical Schwarzschild background, a distant observer never sees anything falling into the horizon within finite
time, but in the conventional model of an evaporating black hole, it is possible for a distant observer to see things crossing the horizon. We consider a collapsing thin shell followed by a second shell of arbitrarily small mass, as a small perturbation to the first shell. It turns out that there would be a moment when the outgoing vacuum energy flux diverges in a generic free-falling frame if the second shell will see the first shell falling under its Schwarzschild radius before the second shell crosses its own Schwarzschild radius. We will show that this divergence can be removed by taking into consideration the back reaction of the ingoing negative vacuum energy $\langle T_{vv} \rangle$.

With the ingoing negative vacuum energy included in the semi-classical Einstein equation:

$$G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle,$$  \hspace{1cm} (1.1)

we study the dynamical geometry of a small neighborhood of the trapping horizon during the gravitational collapse. We show that this trapping horizon (the world-history of the apparent horizon) is time-like outside the collapsing matter (in vacuum), and, remarkably, for observers on top of the trapping horizon, the vacuum energy density is given by a universal formula: $E \simeq - \frac{1}{2\kappa a^2}$, where $a$ is the Schwarzschild radius. This expression is independent of the details of the vacuum energy-momentum tensor, as long as it is dominated by the ingoing negative vacuum energy at the horizon. Note that this gauge-invariant quantity $E$, as it is inversely proportional to $\kappa$, is of the same order as the classical mass density of the black hole.

The energy density $E$ corresponds to a negative energy flux of power $P = -\frac{2\pi}{\kappa}$ falling through the trapping horizon at the speed of light. At this order of approximation, this is the only reason for the decrease in the black hole mass over time. Eventually, the total negative energy that has entered the trapping horizon cancels the initial energy of the black hole. The matter collapsed under the trapping horizon is not really completely evaporated but coexists with an equal magnitude of negative vacuum energy. The holographic principle is not expected to hold in this model since it admits a macroscopic amount of negative energy.

The plan of this paper is as follows. We review the widely applied model of vacuum energy-momentum tensor proposed by Davies, Fulling and Unruh in Sec.2, as a concrete example of the conventional model of black holes. In Sec.3 we review the calculation of a single collapsing shell, and show that if the back reaction of the vacuum energy-momentum tensor (in particular the component $\langle T_{vv} \rangle$) is not properly taken into consideration, there are configurations involving two collapsing thin shells for which a naive treatment leads to a divergence in $\langle T_{uu} \rangle$. In Sec.4 we include the back reaction of $\langle T_{vv} \rangle$ in the semi-classical Einstein equation. Without assuming an explicit expression of the energy-momentum tensor, we compute the vacuum energy density from the viewpoint of observers staying on top of the trapping horizon.

1Here $\kappa = 8\pi G_N$, where $G_N$ is the Newton constant.
trapping horizon. Finally, we comment in Sec. 5 the implications of our results and compare our results for the conventional model with other models of black holes.

2 DFU Model

In this work, we focus on 4D spacetime with spherical symmetry. In general, the metric can be written as

\[ ds^2 = -C(u, v) du dv + r^2(u, v) d\Omega^2, \]  

(2.1)

where \( u, v \) are the light-cone coordinates, and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the metric of the 2-sphere. The sphere at \( r(u, v) = r \) has the area of \( 4\pi r^2 \).

We assume in this section and the next section that the vacuum expectation value of the energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) can be approximated by that of a 2D quantum field theory, as it was proposed by Davies, Fulling and Unruh [1][2]. We refer to the resulting semi-classical theory of black holes as the DFU model. It has been widely used in the studies of black holes.

2.1 Formulation

In this section, we shall review the formulation of the DFU model [2]. Consider the spherical reduction of a 4D quantum field theory, the 4D energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) is related to its dimensionally-reduced 2D counterpart \( \langle T_{\mu\nu}^{(2D)} \rangle \) via

\[
\langle T_{\mu\nu} \rangle = \frac{1}{r^2} \langle T_{\mu\nu}^{(2D)} \rangle \quad (\mu, \nu = 0, 1),
\]

(2.2)

\[
\langle T_{\theta\theta} \rangle = \langle T_{\phi\phi} \rangle = 0.
\]

(2.3)

For the DFU model, \( \langle T_{\mu\nu}^{(2D)} \rangle \) is assumed to be given by that of a 2D massless scalar.

Given the Weyl anomaly for a 2D massless scalar field

\[
\langle T_{\mu\mu}^{(2D)} \rangle = \frac{1}{24\pi} R^{(2D)},
\]

(2.4)

the energy-momentum conservation uniquely determines the 2D energy-momentum tensor up to an initial or boundary condition. For the dimensionally reduced metric

\[ ds^2_{(2D)} = -C(u, v) du dv, \]

(2.5)

the vacuum energy-momentum tensor is given by [1]

\[
\langle T_{\mu\nu}^{(2D)} \rangle = \theta_{\mu\nu} + \frac{R^{(2D)}}{48\pi} g_{\mu\nu} + \hat{T}_{\mu\nu},
\]

(2.6)

where

\[
R^{(2D)} = \frac{4}{C^3} \left( C \partial_u \partial_v C - \partial_u C \partial_v C \right)
\]

(2.7)
is the 2D curvature, and

\[ \theta_{uu} \equiv -\frac{1}{12\pi} C^{1/2} \partial_u^2 C^{-1/2}, \quad \theta_{vv} \equiv -\frac{1}{12\pi} C^{1/2} \partial_v^2 C^{-1/2}, \quad \theta_{uv} \equiv 0. \]  

(2.8)

The last term in eq. (2.6) is the integration constant arising from solving the conservation law. It is given by 1-variable functions \( \hat{T}_{uu}(u) \) and \( \hat{T}_{vv}(v) \) as

\[ \hat{T}_{uu} \equiv \hat{T}_{uu}(u), \quad \hat{T}_{vv} \equiv \hat{T}_{vv}(v), \quad \hat{T}_{uv} \equiv 0. \]  

(2.9)

For a given metric, the energy-momentum tensor is uniquely fixed by \( \hat{T}_{uu}(u) \) and \( \hat{T}_{vv}(v) \), which depend on the choice of boundary or initial conditions. For the vacuum state associated with the coordinate system \((U, V)\) with respect to which positive and negative frequency modes (and thus the creation and annihilation operators) are defined, \( \hat{T}_{UU}(U) \) and \( \hat{T}_{VV}(V) \) vanish [1]. Since the expressions (2.8) are not covariant (as the specification of a vacuum state breaks general covariance), \( \hat{T}_{uu}(u) \) and \( \hat{T}_{vv}(v) \) for a different coordinate system \((u(U), v(V))\) is in general non-zero. They are given by the Schwarzian derivatives of the coordinates \( U \) and \( V \) [1]:

\[ \hat{T}_{uu} = \frac{1}{16\pi} \{U, u\}, \quad \hat{T}_{vv} = \frac{1}{16\pi} \{V, v\}, \]  

(2.10)

where the Schwarzian derivative is defined by

\[ \{f, u\} \equiv \left( \frac{d^2 f}{du^2} \right)^2 - \frac{2}{3} \left( \frac{d^3 f}{du^3} \right). \]  

(2.11)

The standard choice of the vacuum state is the Minkowski vacuum of the infinite past, well before the gravitational collapse has started when the energy density is very low everywhere. For a collapsing shell, the empty space inside the shell remains in the Minkowski vacuum, and the light-cone coordinates \((U, V)\) associated with the Minkowski vacuum of the infinite past can be identified with the Minkowski light-cone coordinates inside the shell. The Hawking radiation (2.10) is then determined by the Schwarzian derivative \( \{U, u\} \).

In principle, the metric eq. (2.1) should be solved from the semi-classical Einstein equation (1.1) with the energy-momentum tensor given by eqs. (2.2)–(2.11). However, it is hard to find an analytic solution. Instead, it is often assumed that the classical vacuum solution (e.g. the Schwarzschild solution) is a good approximation outside the collapsing matter, and the \( \hbar \)-expansion is valid. As a consistency check, the vacuum energy-momentum tensor of the DFU model was calculated for a thin shell [2] and it satisfies the regularity conditions [3,4]:

\[ |\langle T^\theta_\theta \rangle| < \infty, \]  

(2.12)

\[ |\langle T_{vv} \rangle| < \infty, \]  

(2.13)

\[ (r-a)^{-1}|\langle T_{uv} \rangle| < \infty, \]  

(2.14)

\[ (r-a)^{-2}|\langle T_{uu} \rangle| < \infty. \]  

(2.15)
These are necessary conditions for the Schwarzschild metric to be a good approximation; yet they may not be sufficient. We will explain below that, strictly speaking, the validity of the perturbative expansion in \( \hbar \) depends on the time scale.

### 2.2 Range of Validity of Constant Background

At the 0-th order in the \( \hbar \)-expansion, when the back reaction of the vacuum energy-momentum tensor is completely ignored, the background Schwarzschild metric outside the collapsing shell (and thus the Schwarzschild radius) is time-independent. We shall show here why this approximation is valid only within a sufficiently short period of time.

To investigate the range of validity of the assumption of the static Schwarzschild background, consider the collapse of a spherical shell of radius \( R \) and mass \( M \). The spacetime outside the shell is assumed to be the static Schwarzschild metric

\[
d s^2 = - \left( 1 - \frac{a}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{a}{r}} + r^2 d\Omega^2,
\]

(2.16)

with the Schwarzschild radius \( a = 2G_NM \), where \( M \) is the mass of the black hole. Equivalently, the Schwarzschild metric can be written as

\[
d s^2 = - \left( 1 - \frac{a}{r} \right) du^2 - 2dudr + r^2 d\Omega^2,
\]

(2.17)

where \( u = t - r^* \) is an outgoing light-cone coordinate, and \( r^* \) is the tortoise coordinate

\[
r^* \equiv r + a \log \left( \frac{r}{a} - 1 \right).
\]

(2.18)

For simplicity, we assume that the surface of the shell collapses at the speed of light, so the areal radius at the shell \( R(u) \) satisfies

\[
\frac{dR}{du} = - \frac{1}{2} \left( 1 - \frac{a}{R} \right).
\]

(2.19)

Suppose that the shell is very close to the Schwarzschild radius at a certain moment \( u = u_1 \), i.e.

\[
R(u_1) - a = L \ll a,
\]

(2.20)

the equation (2.19) can be approximately solved by

\[
R(u) = a + L e^{- \frac{u - u_1}{2a}}.
\]

(2.21)

Let us assume that, although \( L \) is much smaller than \( a \), it is much larger than the Planck length. At a later time \( u = u_2 \), when the shell is separated from the Schwarzschild radius only by a small distance of Planck length, i.e.

\[
R(u_2) - a \sim \frac{\kappa}{a},
\]

(2.22)
we have
\[ u_2 - u_1 \sim 2a \log \left( \frac{La}{\kappa} \right) \gg 2a. \] (2.23)

On the other hand, over the same period of time \( (u_2 - u_1) \), Hawking radiation leads to a reduction in the black-hole mass and thus a reduction in the Schwarzschild radius
\[ \Delta a \simeq \left| \frac{da}{du} \right| (u_2 - u_1) \sim \frac{\kappa}{a^2} \left[ 2a \log \left( \frac{La}{\kappa} \right) \right] \gg \frac{\kappa}{a}, \] (2.24)
where we have used the conventional formula for Hawking radiation
\[ \frac{da}{du} \sim -\frac{\kappa}{a^2}. \] (2.25)

Eq. (2.24) implies that the estimate of the difference \( (R - a) \) in eq. (2.22) is invalid. In general, the assumption of a constant background is unreliable over the time scale (2.23) for a calculation sensitive to \( (R(u) - a) \), when \( (R(u) - a) \) is of order \( \kappa/a \) or smaller. We will demonstrate this in Sec. 3.3 through a pair of collapsing thin shells.

Incidentally, the study of static solutions to the semi-classical Einstein equation (1.1) also shows that the Schwarzschild metric is in general not necessarily a good approximation for \( r - a \lesssim O \left( \frac{\kappa}{a} \right) \). (2.26)
The reason is the following. Naively, the semi-classical Einstein equation
\[ G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle \] (2.27)
is simplified to the vacuum equation \( G_{\mu\nu} = 0 \) at the 0-th order of the \( h \)-expansion (which is equivalent to the \( \kappa \)-expansion in regions without classical matter). However, as the Schwarzschild metric involves the factor \( (1 - \frac{a}{r}) \), which introduces a factor of \( \kappa/a^2 \) when \( r - a \) is of order \( O(\kappa/a) \), a factor of \( \kappa \) is introduced on the left hand side of the semi-classical Einstein equation, so that it is no longer obvious that it is consistent to ignore the right hand side in the limit \( \kappa \to 0. \)

3 Thin Shells

Thin shells have been widely used as a simple model for the gravitational collapse of matter into black holes. In this section, we first review the collapse of a single thin shell in the DFU model [2] in Sec. 3.1. Then we compute the outgoing vacuum energy flux \( \langle T_{\mu\nu} \rangle \) for a generic system of two shells in Sec. 3.2. In Sec. 3.3 we demonstrate that, for the configuration in which a part of the collapsing matter (the first shell) has already got inside the horizon, ignoring the back reaction of the ingoing vacuum energy flux leads to a divergence in the outgoing vacuum energy flux for the second thin shell in a local free-falling frame.
3.1 A Single Shell

Consider a collapsing thin shell of areal radius \( R_0(u) \). The space inside the shell \((r < R_0(u))\) remains in the Minkowski vacuum. The Minkowski metric is

\[
 ds^2 = -d\tau^2 + dr^2 + r^2d\Omega^2, \tag{3.1}
\]

which can also be written as

\[
 ds^2 = -dUdV + r^2(U,V)d\Omega^2, \tag{3.2}
\]

in terms of the light-cone coordinates \( U, V \). The coordinates \((\tau, r)\) and \((U, v)\) are related via the relations

\[
 \tau = \frac{V + U}{2}, \quad r = \frac{V - U}{2}. \tag{3.3}
\]

In the classical theory of gravity, the space outside the thin shell \((r \geq R_0(u))\) is given by the Schwarzschild metric

\[
 ds^2 = - \left(1 - \frac{a_0}{r(u,v)}\right) dudv + r^2(u,v)d\Omega^2, \tag{3.4}
\]

where \(a_0\) is proportional to the thin shell’s mass. The function \(r(u,v) = r(r^*)\) is the inverse function of

\[
 r^*(r) = r + a_0 \log \left(\frac{r}{a_0} - 1\right), \tag{3.5}
\]

where

\[
 r^* \equiv \frac{v - u}{2} \tag{3.6}
\]

is the tortoise coordinate. From eqs. (3.5) and (3.6), we find

\[
 \frac{\partial r}{\partial u} = -\frac{\partial r}{\partial v} = -\frac{1}{2} \left(1 - \frac{a_0}{r}\right). \tag{3.7}
\]

Assuming the initial condition at \( u = -\infty \) to have no ingoing energy flux outside the thin shell, we have

\[
 \hat{T}_{uv}(v) = 0 \quad \text{for} \quad r \geq R_0(u). \tag{3.8}
\]

The outgoing energy flux \( \hat{T}_{uu}(u) \) is on the other hand to be determined by the Schwarzian derivative \( \{U,u\} \{2.10\} \), which depends on the collapsing process.

On the trajectory of the collapsing shell at \( r = R_0(U) \), we have \( V = U + 2R_0(U) \) according to eq. (3.3). The continuity of the metric across the collapsing shell implies that

\[
 dUdV \bigg|_{r=R_0(U)} = \left(1 - \frac{a_0}{R_0(U)}\right) dudv \bigg|_{r=R_0(U)}, \tag{3.9}
\]

which can be solved by

\[
 \frac{dU}{du} = \frac{1}{1 + 2\frac{dR_0}{dU}} \left[ \frac{dR_0}{dU} + \sqrt{\left(\frac{dR_0}{dU}\right)^2 + \left(1 - \frac{a_0}{R_0}\right) \left(1 + 2\frac{dR_0}{dU}\right)} \right]. \tag{3.10}
\]
(The other solution is discarded because we need \( \frac{dU}{du} > 0 \).)

Expanded in powers of \((R_0 - a_0)\), its leading order terms are

\[
\frac{dU}{du} \simeq -\frac{R_0 - a_0}{2a_0 \frac{dR_0}{dt}} + \left[ \frac{1 + 2 \frac{dR_0}{dt} + 4 \left( \frac{dR_0}{dt} \right)^2}{8a_0^2 \left( \frac{dR_0}{dt} \right)^3} \right] (R_0 - a_0)^2 + \mathcal{O} \left( (R_0 - a_0)^3 \right). \tag{3.11}
\]

For finite non-vanishing \( \frac{dR_0}{dU} \), this reproduces the conventional result for Hawking radiation at the horizon \(^2\)

\[
\hat{T}_{uu}(R_0) = \frac{1}{16\pi} \{ U, u \} = \frac{1}{16\pi} \frac{1}{12a_0^2} + \mathcal{O} \left( (R_0 - a_0)^2 \right). \tag{3.12}
\]

Using eqs. (2.2), (2.6)–(2.9), (3.4), (3.8) and (3.12), we find the vacuum energy-momentum tensor at the horizon \((u = u^*)\) at the 0-th order: \(^2\)

\[
\langle T_{uu}(u^*, r) \rangle = \frac{1}{24\pi r^2} \left( \frac{3a_0^2}{8r^4} - \frac{a_0}{2r^3} + \frac{1}{8a_0^2} \right), \tag{3.13}
\]

\[
\langle T_{vv}(u^*, r) \rangle = \frac{1}{24\pi r^2} \left( \frac{3a_0^2}{8r^4} - \frac{a_0}{2r^3} \right), \tag{3.14}
\]

\[
\langle T_{uv}(u^*, r) \rangle = \frac{1}{24\pi r^2} \left( \frac{a_0^2}{2r^4} - \frac{a_0}{2r^3} \right). \tag{3.15}
\]

One can check that the regularity conditions (2.13)–(2.15) are satisfied. Notice that the vacuum energy-momentum tensor involves an ingoing negative energy flux \( \langle T_{vv} \rangle \) outside the horizon.

### 3.2 Double Shells

Consider a system of two shells of masses \(m_1, m_2\) and radii \(R_1, R_2\) \((R_1 < R_2)\). The metric is

\[
ds^2 = \begin{cases} 
-dUdV + r^2d\Omega^2 & \text{inside the first shell}, \\
- \left( 1 - \frac{a_1}{r} \right) du_1dv_1 + r^2d\Omega^2 & \text{between the two shells}, \\
- \left( 1 - \frac{a_2}{r} \right) du_2dv_2 + r^2d\Omega^2 & \text{outside the second shell},
\end{cases} \tag{3.16}
\]

where we used the coordinates \((U, V)\) inside the first (inner) shell, \((u_1, v_1)\) between the shells, and \((u, v)\) outside the 2nd (outer) shell. The Schwarzschild radii are

\[
a_1 = 2G_N m_1, \quad a_2 = 2G_N (m_1 + m_2). \tag{3.17}
\]

Assuming for simplicity that both shells are falling at the speed of light, for the metric (3.16) the equations of motion for the two shells are

\[
\frac{dR_1}{dU} = -\frac{1}{2}, \quad \frac{dR_1}{du_1} = -\frac{1}{2} \frac{R_1 - a_1}{R_1}, \tag{3.18}
\]

\[
\frac{dR_2}{du_1} = -\frac{1}{2} \frac{R_2 - a_1}{R_2}, \quad \frac{dR_2}{du_2} = -\frac{1}{2} \frac{R_2 - a_2}{R_2}. \tag{3.19}
\]
These equations determine the ratios of various retarded time coordinates

\[
\frac{du_2}{du_1} = \frac{R_2 - a_1}{R_2 - a_2},
\]

\[
\frac{du}{dU} = \frac{du_2}{du_1} \frac{du_1}{dU} = \left(\frac{R_2 - a_1}{R_2 - a_2}\right) \left(\frac{R_1}{R_1 - a_1}\right).
\]

(3.20)
(3.21)

The red-shift factor is thus given by

\[
\psi_0(u) \equiv \log \left(\frac{du}{dU}\right) = \log \left[\left(\frac{R_1}{R_1 - a_1}\right) \left(\frac{R_2 - a_2}{R_2 - a_1}\right)\right].
\]

(3.22)

Consider a moment when both shells are close to their Schwarzschild radii:

\[
R_1 = a_1 + \epsilon_1, \quad R_2 = a_2 + \epsilon_2 = a_1 + \epsilon,
\]

(3.23)

where \(\epsilon_1\) and \(\epsilon_2\) are small, and

\[
\epsilon \equiv 2m_2 + \epsilon_2.
\]

(3.24)

For \(r \geq R_2\), according to eq. (2.8), we have

\[
\theta_{\nu\nu} = -\frac{a_2(a_2 + 4\epsilon_2)}{192\pi(a_2 + \epsilon_2)^4} \approx -\frac{1}{192\pi a_2^4} + \frac{\epsilon_2^2}{32\pi a_2^4} + O(\epsilon_2^3)
\]

(3.25)

for the metric (3.16). Assuming that \(|\epsilon_1| \gg |\epsilon|\), the Hawking radiation is

\[
\hat{T}_{\nu\nu} = \frac{1}{48\pi}(\dot{\psi}_0^2 - 2\ddot{\psi}_0)
\]

\[
= \frac{1}{192\pi a_2^4} + \left[-\frac{1}{32\pi a_2^4} + O(\epsilon_1)\right] \frac{\epsilon_2^2}{\epsilon^2} + O\left(\frac{\epsilon_1}{\epsilon}\right) \epsilon_2^2 + O(\epsilon_3^3),
\]

(3.26)

in agreement with the conventional model as the 0-th order term cancels that of \(\theta_{\nu\nu}\) (3.25) and there is no first order term in \(\epsilon_2\).

The vacuum energy flux as \(R_2 \to a_2\) is thus given by

\[
\langle T_{\nu\nu}^{(2D)} \rangle = \theta_{\nu\nu} + \hat{T}_{\nu\nu} = \left[-\frac{1}{32\pi a_2^4} \left(\frac{\epsilon_1}{\epsilon}\right)^2 + O\left(\frac{\epsilon_1}{\epsilon}\right)\right] \epsilon_2^2 + O(\epsilon_3^3),
\]

(3.27)

so that the outgoing energy flux observed in a local free-falling frame is of the same order as

\[
\frac{1}{\left(1 - \frac{a_2}{R_2}\right)^2} \langle T_{\nu\nu}^{(2D)} \rangle = \left[-\frac{1}{32\pi a_2^4} \left(\frac{\epsilon_1}{\epsilon}\right)^2 + O\left(\frac{\epsilon_1}{\epsilon}\right)\right],
\]

(3.28)

as long as both \(\epsilon\) and \(\epsilon_1\) are finite, and the regularity condition (2.15) is satisfied.

### 3.3 Large Energy Flux at Horizon

Let us now consider the following situation. The first shell of mass \(m_1\) has already collapsed into its Schwarzschild radius \(a_1\) so that \(R_1 < a_1\). We take the second shell to have an
infinitesimal mass $m_2 \to 0$ so that this system is in practice indistinguishable from a single shell of mass $m_1$. However, as the 2nd shell approaches $a_1$ (so that $R_2 \sim a_2 \simeq a_1 > R_1$), there are always moments when

$$|\epsilon_1| \gg |\epsilon|,$$

as $|\epsilon| \equiv |R_2 - a_1|$ can be arbitrarily small, so that the outgoing energy flux (3.28) in a free-falling frame is arbitrarily large! In fact, $\langle T_{uu} \rangle$ diverges when $R_2 = a_1$.

There are two ways to interpret this result. One may say that the reason for the divergence in $\langle T_{uu} \rangle$ is that we should have used a time-dependent metric, rather than the constant Schwarzschild metrics (3.16), to properly describe the black-hole evaporation. This means that the back reaction of the vacuum energy-momentum tensor cannot be ignored for this calculation.

Alternatively, one may also say that, with the back reaction of the vacuum energy-momentum tensor ignored, the configuration considered above can never occur if both shells are initially outside the Schwarzschild radius. During the formation process, the geometry is given by the static Schwarzschild metric with constant Schwarzschild radius $a_2$ outside the outer shell and constant Schwarzschild radius $a_1$ between the two shells. One can then see that two shells cross the horizon at $r = a_1$ at the same retarded time $u$. At arbitrary retarded time, we always have $|\epsilon| \geq |\epsilon_1|$, regardless of whether the shells are inside or outside the horizons. It is impossible to have $|\epsilon| \ll |\epsilon_1|$, and there would be no divergence in the outgoing energy flux. This is demonstrated pictorially in Fig.1. The crucial point is that both shells must be both under or above the Schwarzschild radius $a_1$ at any instant of $u$.

![Figure 1](image)

Figure 1: (a) A small neighborhood of the future horizon: The orange curves are constant-$r$ curves. The null shell at $v_2$ has a larger (smaller) areal radius at $u = u_1$ ($u = u_2$) in comparison with the null shell at $v_1$. (b) For a constant Schwarzschild radius, either we have $R_2 > R_1 > a_1$ or $a_1 > R_1 > R_2$. Hence we always have $|\epsilon_1/\epsilon| < 1$. (The orange curves are constant-$r$ curves.)

However, in the conventional model of black holes, it is possible to consider collapsing matters in the black hole geometry which already has the horizon. One may naively assume that the horizon has appeared because a part of the collapsing matter is already under the Schwarzschild radius. If the configurations in which part of the collapsing matter is under
the horizon with the rest still above the horizon is naively considered in the conventional model, a similar situation to $|\epsilon_1| \gg |\epsilon| \sim 0$ in our double-shell configuration would typically occur. Then we would get diverging outgoing energy flux in a local free-falling frame if we still use naively the constant Schwarzschild background. This implies that the conventional model of black holes is incapable of properly describing such a configuration, and that the back reaction of the vacuum energy-momentum tensor should be taken into account.

For either interpretation for the divergence in $\langle T_{uu} \rangle$, the conclusion is that the configuration we considered cannot be consistently described in a background geometry where the back reaction of the vacuum energy-momentum tensor is completely ignored.

It turns out that it is possible to find a consistent description of the space-time geometry by taking into consideration only the back reaction of the ingoing vacuum energy flux $\langle T_{vv} \rangle$. The vacuum state has ingoing negative energy outside the collapsing matters. We can simply use thin shells of tiny negative mass to represent the ingoing negative vacuum energy, so that the Schwarzschild radius outside a thin shell of negative-mass is slightly smaller than the one inside, and the decrease in the black hole mass can be properly described (although in a discretized fashion).

For instance, for the configuration considered above, we can introduce a thin shell with a negative mass between the two shells of collapsing matter to describe approximately the negative vacuum energy between the shells. See Fig.2. In this case, the distance between $a_1$ and the radius at the negative mass shell cannot be smaller than $|\epsilon_1|$ because of the same reason depicted in Fig.1. There is no divergence at the negative mass shell. For the second shell of collapsing matter, $|\epsilon|$ can be smaller than $|\epsilon_1|$ (as we claimed above), however, the outgoing energy flux no longer diverges in a local free-falling frame due to the back reaction of the shell of negative mass at $v = v'$.

![Figure 2: Part of the Penrose diagram with two null shells at $v_1$ and $v_2$, with a negative shell in the middle at $v'$. The Schwarzschild radius is $a_1$ for $v \in (v_1, v')$, and it is $a'$ for $v \in (v', v_2)$. The orange curve is the constant-$r$ curve for $r = R_1$. The blue curve is the constant-$r$ curve for $r = a_1$, which is taken to be very close to $R_2$ at the instant $u$. At the instant $u$, we have $|\epsilon|$ arbitrarily close to 0, so $|\epsilon_1|/|\epsilon|$ is arbitrarily large as we considered in eq.(3.29).]
The lesson we learned from this exercise is that while the ignorance of back reaction of the vacuum energy-momentum tensor is not good for generic configurations, it is possible to find a good approximate description of the near-horizon geometry by including the ingoing negative vacuum energy flux alone.

In Sec. 4 we will discuss the continuous version including the ingoing vacuum energy flux. We shall present an approximate solution around the trapping horizon to the semi-classical Einstein equation in which the back reaction of the ingoing vacuum energy flux \( \langle T_{vv} \rangle \) is incorporated, although the other two components \( \langle T_{uu} \rangle \) and \( \langle T_{uv} \rangle \) of the vacuum energy-momentum tensor are still ignored. The vacuum energy-momentum tensor is of course not exactly the same as a continuous ingoing negative energy flux from past infinity, yet this may still provide a good approximation of the geometry in a sufficiently small neighborhood around the trapping horizon. The geometry far away from the trapping horizon is still expected to be well approximated by a time-dependent Schwarzschild background.

4 Generic Null Shell

Recall that both \( \langle T_{uu} \rangle \) and \( \langle T_{uv} \rangle \) vanish at the horizon according to eqs. (3.13) and (3.15) in the DFU model. More generally, they must vanish at the horizon as consistency conditions (2.14), (2.15) for the static Schwarzschild background to be a valid 0-th order approximation, even if we do not assume the DFU model. In the following, we will assume that \( \langle T_{uu} \rangle \) and \( \langle T_{uv} \rangle \) are negligible in the semi-classical Einstein equation for a small neighborhood of the trapping horizon, but we will not assume that the vacuum energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) is given by any specific expression.

Setting \( \langle T_{uu} \rangle = \langle T_{uv} \rangle = 0 \) for the background, \( \langle T_{vv} \rangle \) is a conserved ingoing energy flux, so we have the ingoing Vaidya metric

\[
\text{ds}^2 = -\left(1 - \frac{a(v)}{r}\right)dv^2 + 2dvdr + r^2d\Omega^2
\]

(4.1)

for a generic, spherically symmetric collapsing sphere. This should be a good approximation of a sufficiently small neighborhood of the trapping horizon.

The ingoing null energy-momentum tensor

\[
T_{vv} = \frac{1}{\kappa} \frac{a'(v)}{r^2}
\]

(4.2)

includes both the collapsing matter of positive energy and the negative vacuum energy flux. Outside the collapsing matter where there is only negative vacuum energy flux, we should have \( a'(v) < 0 \).

The ingoing Vaidya metric can also be expressed in terms of the light-like coordinates \((u, v)\) as

\[
\text{ds}^2 = -C(u, v)du dv + R^2(u, v)d\Omega^2,
\]

(4.3)
where \( C(u, v) \) and \( R(u, v) \) satisfy

\[
\frac{\partial R(u, v)}{\partial u} = -\frac{1}{2} C(u, v), \tag{4.4}
\]

\[
\frac{\partial R(u, v)}{\partial v} = \frac{1}{2} \left( 1 - \frac{a(v)}{R(u, v)} \right). \tag{4.5}
\]

The consistency of eqs. (4.4) and (4.5) demands that

\[
\frac{\partial}{\partial v} C(u, v) = a(v) \frac{2r^2(u, v)}{C(u, v)}. \tag{4.6}
\]

As a result,

\[
C(u, v') = C(u, v) e^{\int v' dv'' \frac{a(v'')}{2r^2(u, v')}}. \tag{4.7}
\]

Using the boundary condition for \( C(u, v) \)

\[
C(u, \infty) = 1 \tag{4.8}
\]

for the asymptotic Minkowski space at large \( v \), we find

\[
C(u, v) = e^{- \int_v^\infty dv'' \frac{a(v'')}{2r^2(u, v')}}. \tag{4.9}
\]

4.1 Around Trapping Horizon

The trapping horizon is sometimes considered as the geometric feature that characterizes a black hole. Since we have assumed spherical symmetry, it is convenient to define the trapping horizon by using the foliation of the space-time which is given by the symmetric 2-spheres.

Recall that a symmetric 2-sphere \( S \) is a trapped surface if both ingoing and outgoing null geodesics orthogonal to \( S \) have negative expansion. This means that \( \partial_u R < 0 \) and \( \partial_v R < 0 \) in terms of the areal radius \( R(u, v) \) as a function of \( u \) and \( v \). (In contrast, \( \partial_u R < 0 \) and \( \partial_v R > 0 \) for the Minkowski space-time.) The boundary of a trapped region — a 3D region composed of trapped surfaces — is called a trapping horizon, where \( \partial_v r = 0 \). A 2D space-like section of the trapping horizon is an apparent horizon.

Let us assume that there is a trapped region and thus a trapping horizon for the black hole under consideration. The Penrose diagram with a trapping horizon in the absence of singularity is schematically shown in Fig.3. We leave out the upper part of the Penrose diagram which may involve UV physics at least near the origin.\(^2\)

The trapping horizon is divided into two parts by the point with the minimal value of the \( u \)-coordinate. (It is marked by \( A \) in Fig.3.) We will show below that the branch of the trapping horizon to the right of \( A \) is time-like and has \( T_{vv} < 0 \), while the branch of the

\(^2\) In the absence of singularity, the trapping horizon should be a closed curve \(^5\). But it is possible that a regular geometric description is no longer valid at the origin.
trapping horizon to the left of A is space-like and has $T_{vv} > 0$. We shall refer to the former as the “trapping horizon in vacuum” and the latter as the “trapping horizon in matter”.

We will focus on the trapping horizon in vacuum. It is absent for a classical black hole because it is cut off by the space-like singularity at the origin. On this part of the trapping horizon, we have

$$\partial_v^2 R > 0$$

(4.10)

(unless there is degeneracy) because $\partial_v R$ is positive (negative) at slightly larger (smaller) $v$ (with $u$ fixed).

The outer trapping horizon defined in Ref. [5], includes both branches of the trapping horizon in Fig.3. On the outer trapping horizon, one has $\partial_u \partial_v R < 0$.

Figure 3: This is the Penrose diagram with a trapping horizon. The solid and dashed curves (in blue) represent the trapping horizon in vacuum and that in matter, respectively. These two curves meet at A, which is the point on the trapping horizon with the lowest value of the $u$-coordinate. The dotted curves (in orange) are constant $r$-curves, whose tangents are light-like on the trapping horizon.

Let the $u$-coordinate of the point A in Fig.3 be denoted $u_A$. For $u > u_A$, a constant $u$-curve intersects the trapping horizon at two points. The trapping horizon in vacuum has the larger $v$-coordinate, which will be denoted $v_0(u)$. On the apparent horizon at $(u, v_0(u))$, we have

$$\frac{\partial R}{\partial v}(u, v_0(u)) = 0.$$  

(4.11)

The expansions of $R(u, v)$ and $a(v)$ in powers of $(v - v_0(u))$ are thus

$$R(u, v) = R_0(u) + \frac{1}{2} R_2(u)(v - v_0(u))^2 + \cdots ,$$

(4.12)

$$a(v) = a(v_0(u)) + a'(v_0(u))(v - v_0(u)) + \cdots ,$$

(4.13)

in a small neighborhood of $v = v_0$. (We use primes and dots to indicate derivatives with respect to $v$ and $u$.)
With the expansions (4.12) and (4.13), we deduce from eq.(4.5) that

\[ R_2(u)(v - v_0) \simeq \frac{R_0 - a(v_0)}{2R_0} - \frac{a'(v_0)}{2R_0} (v - v_0), \]

which implies that

\[ R_0(u) = a(v_0(u)), \quad \text{and} \quad R_2(u) = -\frac{a'(v_0(u))}{2a(v_0(u))}. \]

As we have mentioned above (see eq.(4.10)), on the trapping horizon in vacuum we have \( R_2(u) > 0 \), so we deduce that

\[ a'(v_0) < 0. \]

As \( T_{vv} \) is proportional to \( a'(v) \), the trapping horizon exists only if the null energy condition is violated. The null energy condition can in principle be violated by quantum correction in vacuum, hence we have referred to this branch of the trapping horizon as the trapping horizon in vacuum.

Plugging eqs.(4.15) into eq.(4.4), we find

\[ \dot{v}_0(u) \simeq - \frac{C(u, v_0(u))}{2a'(v_0(u))}, \]

where \( C(u, v) \) is given in eq.(4.9). Due to eq.(4.16), we must have \( \dot{v}_0 > 0 \), so that the trapping horizon in vacuum is always time-like.

Incidentally, the same analysis can be applied to the trapping horizon in matter, and one would find \( R_2 < 0 \), \( a' > 0 \) and \( \dot{v}_0 < 0 \) instead. The trapping horizon in matter is hence space-like, and \( T_{vv} \propto a' > 0 \). In conventional models, typically the vacuum has negative energy and collapsing matter has positive energy, so this part of the trapping horizon resides inside the collapsing matter, and that is why we have referred to it as the trapping horizon in matter.

Let us continue our study of the trapping horizon in vacuum. Its tangent vector on the \((u, v)\)-plane is

\[ \xi = (\xi^u, \xi^v) = \frac{1}{\sqrt{C\dot{v}_0}} (1, \dot{v}_0) \simeq \left( \frac{\sqrt{2|a'(v_0)|}}{C(u, v_0(u))}, \frac{1}{\sqrt{2|a'(v_0)|}} \right). \]

This is the unit time-like vector for an observer staying on top of the trapping horizon. An orthonormal space-like vector is

\[ \chi = (\chi^u, \chi^v) \simeq \left( \frac{\sqrt{2|a'(v_0)|}}{C(u, v_0(u))}, -\frac{1}{\sqrt{2|a'(v_0)|}} \right). \]

In this approximate solution to the semi-classical Einstein equation, \( \langle T_{uu} \rangle = \langle T_{uv} \rangle = 0 \), so the energy density on the trapping horizon in vacuum is

\[ \mathcal{E} \equiv \langle T_{vv} \rangle \xi^v \xi^v \simeq -\frac{1}{2\kappa a^2(v_0)} < 0. \]
Notice that this is a quantity independent of the details of the vacuum energy-momentum tensor. The only assumption in addition to spherical symmetry is the existence of the trapping horizon, if the approximation scheme in use is valid. Notice also that $E$ is proportional to $\kappa^{-1}$, which is of the same order as the (naive) mass density of the classical matter. It diverges in the limit $\kappa \to 0$, signaling the non-perturbative nature of this effect. Furthermore, as the trapping horizon is independent of the choice of coordinate system for spherically symmetric configurations, $E$ is gauge-invariant.

The amount of negative vacuum energy flowing into the trapping horizon in vacuum (at the speed of light) per unit time is thus the area of the apparent horizon times $E$, that is,

$$P = 4\pi a^2(v_0)E c = -\frac{2\pi}{\kappa},$$

(4.21)

where $c = 1$ is the speed of light. This is equivalent to the negative mass of $-10^{36}$ kg per second!

### 4.2 Under Trapping Horizon

Although the formulation above is strictly speaking only valid around the trapping horizon, the region under the trapping horizon vacuum is essentially “frozen” by a huge red-shift factor, i.e. it changes extremely slowly with changes in $u$, since $C(u, v)$ is extremely small. We are therefore allowed to sketch the $R - v$ relation under the trapping horizon in vacuum using our approximate description, in particular eq.(4.5). With a hypothetical profile of energy distribution $a(v)$ in Fig[4](a), a schematic behavior of $R$ is shown in Fig[4](b), by numerically solving eq.(4.5). The function $a(v)$ goes to zero at the minimum of $v$ where $R = 0$, and is an increasing function for small $v$ where the collapsing matters exist. For larger $v$, outside the collapsing matters, $a(v)$ is an decreasing function because of the negative vacuum energy.

In the $R$-$v$ diagram in Fig[4](b), the value of $R$ at a large $v$ is given as a boundary condition for each curve. The local minimum of $R$ (the “neck”) is where the apparent horizon is. Due to eq.(4.4), a curve with a narrower “neck” corresponds to a larger value of $u$. Fig[4](b) is in agreement with the numerical simulation of the DFU model for a fully dynamical collapsing process.

The areal radius at the neck should be approximately equal to the Schwarzschild radius. Although the outgoing energy flux $\langle T_{uu}\rangle$ approximately vanishes at the trapping horizon in vacuum, the total energy under the neck decreases over time because negative energy accumulates under the trapping horizon with the power (4.21). Hence the areal radius at the neck shrinks towards 0, and the geometry becomes reminiscent of the “Wheeler’s bag of gold”. The low-energy effective theory breaks down before the areal radius at the neck is of Planck length. It cannot determine whether or not the neck eventually shrinks to zero, or whether the Wheeler’s bag of gold detaches.
Figure 4: (a) Schematic $a$-$v$ diagram: For small $v$, $a$ increases with $v$ in the region covered by collapsing matter of positive energy. At larger $v$, $a$ decreases with $v$ due to the negative vacuum energy. (The amount of negative energy in vacuum is exaggerated for demonstration.) (b) Schematic $R$-$v$ plots over a sequence of $u$’s: The apparent horizon is located at the local minimum of $R$, where $\partial R/\partial v = 0$. The areal radius of the apparent horizon shrinks as $u$ increases, while the internal region (at small $v$) is essentially frozen by the large red shift factor.

5 Comments

5.1 Information Loss Paradox

It is widely believed that there is no high-energy event around the horizon of a black hole. In this work, for the conventional model of black holes, we found a gauge-invariant quantity $E \simeq -1/2\kappa a^2$ (4.20) (the energy-density for an observer on top of the trapping horizon in vacuum) that is inversely proportional to $\kappa$. This implies that the quantum correction is at least comparable to the classical energy. While this may not immediately justify the need of a high-energy theory, it opens such a possibility and motivates further investigation.

Furthermore, the ingoing negative vacuum energy (with the power $P = -2\pi \kappa$ (4.21)) is accumulated over time so that eventually it cancels the energy of the collapsed matter in the black hole. With such a macroscopic negative energy in the conventional model, one should not expect the holographic principle to hold.

5.2 Comparison With Other Models

If the holographic principle should hold for any consistent theory of quantum gravity, one should rule out all models of vacuum energy-momentum tensor in which a black hole loses energy mainly due to ingoing negative vacuum energy. An alternative is to choose models in which the vacuum energy-momentum tensor around a dense collapsing matter is dominated by Hawking radiation. In fact, the implication of this assumption about vacuum energy-momentum tensor has been investigated in the KMY model [8]. (See also its follow-up works [9]–[14].) It was found that there would be no apparent horizon, but there can be Planck-scale pressure at the surface of the collapsing matter which signals the breakdown of low-energy effective theories.
It would be interesting to see more rigorously how different assumptions about the vacuum energy-momentum tensor is associated with the existence of trapping horizon and the accumulation of macroscopic negative energy. We leave this question for future works.

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