BRAUER–MANIN OBSTRUCTIONS ON GENUS-2 K3 SURFACES

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ABSTRACT. We analyze the Brauer–Manin obstruction to rational points on the K3 surfaces over \( \mathbb{Q} \) given by double covers of \( \mathbb{P}^2 \) ramified over a diagonal sextic. After finding an explicit set of generators for the geometric Picard group of such a surface, we find two types of infinite families of counterexamples to the Hasse principle explained by the algebraic Brauer–Manin obstruction. The first type of obstruction comes from a quaternion algebra, and the second type comes from a 3-torsion element of the Brauer group, which gives an affirmative answer to a question asked by Ieronymou and Skorobogatov.

1. Introduction

A variety \( X \) over a number field \( k \) with points in every completion \( k_v \) of \( k \) but no \( k \)-points is said to be a counterexample to the Hasse principle. In 1971, Manin [Man71] identified an obstruction to the existence of rational points using the group \( \text{Br} X : = H^2_{\text{et}}(X, \mathbb{G}_m)_{\text{tors}} \), now known as the Brauer–Manin obstruction, which he used successfully to explain the various counterexamples to the Hasse principle that were known at the time. Since then, groundbreaking work of Skorobogatov [Sko99] has shown that the Brauer–Manin obstruction is not the only one for general varieties over number fields (see also [Poo10], [BBM+16], [CTPS16], [Sme17]) but Colliot-Thélène conjectured [CT03] that the Brauer–Manin obstruction is the only obstruction to rational points for smooth, projective, geometrically rationally connected varieties; that is, if \( X \) is such a variety with points everywhere locally, and there is no Brauer–Manin obstruction to rational points on \( X \), then \( X \) should have a rational point.

1.1. Computational evidence. When \( X \) is a Del Pezzo surface, in addition to the promising theoretical evidence in favor of the conjecture, there is a growing amount of computational evidence for this conjecture in various cases (e.g. [CTKS87], [Cor05], [Cor07], and [Log08]). Computing the Brauer–Manin obstruction on a Del Pezzo surface is relatively fast in practice.

For K3 surfaces, the situation is far more unsettled. Skorobogatov has conjectured that the Brauer–Manin obstruction is the only obstruction to both Hasse principle and weak approximation for K3 surfaces over number fields [Sko]. In light of recent work on diagonal quartics ([Bri02], [IS15]) and various Kummer surfaces ([LvL09], [Cor10], [Arg06], [SZ16]), it seems natural to analyze one of the other main types of K3 surfaces, namely, a double cover of \( \mathbb{P}^2 \) ramified over a sextic which were also studied in [HVA11], [HVA13], [MSTVA17]. In this paper, we study the geometry of one of the simplest such families of surfaces:

(1) \[ X : w^2 = Ax^6 + By^6 + Cz^6 \]

contained in the weighted projective space \( \mathbb{P}[1, 1, 1, 3] = \text{Proj} k[x, y, z, w] \). We study conditions on the coefficients of \( X \) that are necessary in order to give rise to an obstruction.

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1.2. **Supersingular reduction and** \(\text{Pic}(X)\). For \(X\) as in (1), let \(\overline{X}\) be the base change of \(X\) to a fixed algebraic closure \(\overline{k}\) of \(k\). There is a filtration

\[
\text{Br}_0(X) \subseteq \text{Br}_1(X) \subseteq \text{Br}(X),
\]

where \(\text{Br}_1(X) = \ker(\text{Br}(X) \to \text{Br}(\overline{X}))\), and \(\text{Br}_0(X)\) is image of \(\text{Br}(k) \to \text{Br}(X)\). We take advantage of an isomorphism \(\text{Br}_1(X)/\text{Br}_0(X) \cong H^1(k, \text{Pic}(\overline{X}))\) coming from the Hochschild-Serre spectral sequence, to construct elements of \(\text{Br}_1(X)\). Thus our first task is to compute \(\text{Pic}(\overline{X})\) for a surface \(X\) of the form (1). We begin by giving an explicit list of 20 divisors obtained pulling back tangent lines and conics from \(\mathbb{P}^2\), which form a sublattice for \(\text{Pic}(\overline{X})\). We then show that these divisors generate the entire Picard group using supersingular reduction and elliptic fibrations. Recall that over a number field \(k\), the rank of \(\text{Pic}(X)\) is at most 20; we say \(X\) has supersingular reduction at a prime \(p\) of \(k\) if the reduction \(X_p\) has (geometric) Picard rank 22. We use the intersection properties of the extra divisors coming from the supersingular reduction to show the sublattice we constructed is already saturated in \(\text{Pic}(\overline{X})\). To the best of our knowledge, this is the first time supersingular reduction was used to compute the Picard group of the original surface.

Having understood \(\text{Pic}(\overline{X})\) as a Galois module, we find coefficients \(A, B, C\) for which there is a 2 or 3-torsion element in \(\text{Br}_1(X)\) giving an obstruction to rational points.

**Theorem 1.1** (Theorem 5.1). Let \(X/\mathbb{Q}\) be the K3 surface given by

\[
w^2 = 4ax^6 + 2by^6 + 2bc^3z^6
\]

where \(a, b, c\) are nonzero integers. Then the quaternion algebra

\[
\mathcal{A} = \left(\mathbb{Q}(\sqrt{-ac})/\mathbb{Q}, \frac{y^2 + cz^2}{x^2}\right)
\]

is an element of \(\text{Br}(X)\), and if \(a, b, c\) satisfies eight conditions (given in the full statement of the theorem below) then \(\mathcal{A}\) gives a counterexample to the Hasse principle explained by the (algebraic) Brauer–Manin obstruction.

We remark that there are infinitely many triples of integers satisfying the eight conditions—indeed, infinitely many of the form \((-1, 1, c)\).

So far all known examples of Brauer–Manin obstruction for K3 surfaces have been given by a 2-torsion element in the Brauer group. In [IS15], Ieronymou and Skorobogatov ask whether it was possible for the odd torsion part of the Brauer group to obstruct the Hasse principle for a K3 surface over a number field. We show that this is possible:

**Theorem 1.2** (Theorem 6.2). Let \(X/\mathbb{Q}\) be the K3 surface given by

\[
w^2 = -3x^6 + 97y^6 + 97 \cdot 28 \cdot 8z^6
\]

Then the cyclic algebra

\[
\mathcal{A} = \left(\mathbb{Q}(\sqrt[3]{28})/\mathbb{Q}, \frac{w - \sqrt[3]{-32x^3}}{w + \sqrt[3]{-32x^3}}\right)
\]

is a 3-torsion element of \(\text{Br}(X)\), and gives a counterexample to the Hasse principle explained by the (algebraic) Brauer–Manin obstruction.
As with Theorem 1.1 there are infinitely many such examples given by coefficients satisfying certain congruence and primality conditions (Remark 6.3). For these surfaces, we show that \( Br_1(X)/ Br_0(X) \simeq \mathbb{Z}/3\mathbb{Z} \), so there are no obstructions coming from the 2-primary part of the algebraic Brauer group.

**Remark 1.3.** The algebra \( \mathcal{A} \) appearing in Theorem 6.2 can be seen as coming from a double cover of a rational surface. Let \( Y \) be the del Pezzo surface of degree 1 defined by the equation

\[
 u^2 = -3r^6 + 97s^6 + 97 \cdot 28 \cdot 8t^3.
\]

in the weighted projective space \( \mathbb{P}[1, 1, 2, 3] = \text{Proj } k[r, s, t, u] \). Consider the double cover \( X \to Y \) defined by sending \([w : x : y : z] \mapsto [w : x : y : z^2]\). Then the class \( \mathcal{A} \) comes from the pullback of the class

\[
 \left( \mathbb{Q}(\sqrt{28})/\mathbb{Q}, \frac{u - \sqrt{-3r^3}}{u + \sqrt{-3r^3}} \right).
\]

An easy way to see this is to use [ISZ11, Theorem 1.4], which says that if \( \mathcal{A} \) is odd torsion and fixed by the covering automorphism \( z \mapsto -z \), then it comes from a class in \( Br(Y) \).

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3. Geometry

We calculate the geometric Picard group of the surface \( X \) over \( \mathbb{Q} \) given by the equation

\[
 w^2 = x^6 + y^6 + z^6.
\]

Let \( \overline{X} \) be the base change to \( \overline{\mathbb{Q}} \) and \( \overline{P} = \text{Pic } \overline{X} \).

3.1. Generators. Let \( s \) be a cube root of 2 and \( \zeta = \exp(\pi i/6) \). Set

\[
 a_5 = \frac{2\zeta(-\zeta^2 + 2)(\zeta^4)}{3}, \quad c_5 = \frac{\zeta(\zeta^2 - 2)}{3}, \quad r_5 = -\sqrt{3}\zeta(\zeta^2 - 2), \quad v_5 = -2\sqrt{3}\zeta^2.
\]

Consider the following twenty divisors on \( \overline{X} \):

\[
 D_i : w = x^3, \quad y = \zeta^{2i+1}z \quad 1 \leq i \leq 5,
\]

\[
 D_j : w = y^3, \quad z = \zeta^{2j+3}x \quad 6 \leq j \leq 9,
\]

\[
 D_k : w = z^3, \quad x = \zeta^{2k+7}y \quad 10 \leq k \leq 12,
\]

\[
 D_{13} : w = -x^3, \quad y = \zeta^3z,
\]

\[
 D_{14} : w = s(1 - 2\zeta^2)x^2y + (1 - w\zeta^2)y^3, \quad x^2 + s^2y^2 + z^2 = 0,
\]

\[
 D_{15} : w = s(1 - 2\zeta^2)x^2y + (1 - 2\zeta^2)y^3, \quad \zeta^4x^2 + \zeta^4s^2y^2 + z^2 = 0.
\]
\[ D_{16} : w = -3x^3 + s(2 - 4\zeta^2)x^2y + 3s^2xy^2 + (2\zeta^2 - 1)y^3, \]
\[ 2x^2 + s(1 - 2\zeta^2)xy + s^2y^2 + z^2 = 0 \]
\[ D_{17} : w = (1 - 2\zeta^2)x^3 - 3s^2x^2y + s(4\zeta^2 - 2)xy^2 + 3y^3, \]
\[ -s^2\zeta^3x^2 + s(\zeta^2 + 1)xy - 2\zeta^4y^2 + z^2 = 0 \]
\[ D_{18} : w = -3y^3 + s(2 - 4\zeta^2)y^2z + 3s^2yz^2 + (2\zeta^2 - 1)z^3, \]
\[ x^2 + 2y^2 + s(1 - 2\zeta^2)yz + s^2z^2 = 0 \]
\[ D_{19} : w = -3y^3 + s(2 - 4\zeta^2)y^2z + 3s^2yz^2 + (2\zeta^2 - 1)z^3, \]
\[ x^2 + 2\zeta^2y^2 + s(\zeta^2 - 2)yz - s^2\zeta z^2 = 0 \]
\[ D_{20} : w = r_5x^3 + v_5xyz, \]
\[ a_5x^2 + c_5(y^2 + z^2) + yz = 0 \]

The first thirteen divisors \( d_1, \ldots, d_{13} \) each arise as one component of a pullback from a line tangent to \( C \subset \mathbb{P}^2 \) while the last seven are pullbacks of conics tangent to \( C \). For example, the line \( y = \zeta^3z \) on \( \mathbb{P}^2 \) is tritangent to the sextic \( x^6 + y^6 + z^6 = 0 \), and the pullback is giving by the equation
\[ w^2 = x^6, \quad y = \zeta^3z \]
which splits into components corresponding to \( w = x^3 \) or \( w = -x^3 \). The first component is \( D_1 \) while the latter is \( D_{13} \). The last divisor comes from Festi’s thesis [Fes16], which studies the Picard group of the family of K3 surfaces given by \( w^2 = x^6 + y^6 + z^6 + tx^2y^2z^2 \).

Let \( d_i \) be the image of \( D_i \) in \( \overline{\mathcal{P}} \) and \( M \subset \overline{\mathcal{P}} \) be the subgroup generated all the \( d_i \). Computing the intersection pairing on \( M \) shows that the discriminant of this lattice is \( -432 = -2^4 \cdot 3^3 \). Since the rank of \( \overline{\mathcal{P}} \) is at most 20 [BHPVvdV04, VIII.3], the \( M \) must generate a finite-index subgroup \( \overline{\mathcal{P}} \). An easy lattice argument shows that the index \( [\overline{\mathcal{P}} : M] \) must divide 36. In fact, this index is 1:

**Theorem 3.1.** The Picard group of the surface \( \overline{X}/\overline{\mathbb{Q}} \) given by
\[ w^2 = x^6 + y^6 + z^6 \]
is freely generated by the divisor classes \( d_1, \ldots, d_{20} \). In particular \( Pic(\overline{X}) \simeq \mathbb{Z}^{20} \).

**Proof.** We must show the sublattice \( M \subset \overline{\mathcal{P}} \) is saturated. Recall that \( M \) has discriminant \( -2^4 \cdot 3^3 \), so it suffices to show that the maps
\[ \phi_2 : M/2M \to \overline{\mathcal{P}}/2\overline{\mathcal{P}} \]
\[ \phi_3 : M/3M \to \overline{\mathcal{P}}/3\overline{\mathcal{P}} \]
are injective. Let \( G_2 \) and \( G_3 \) be 2-Sylow and 3-Sylow subgroups respectively of \( G := Gal(K/\mathbb{Q}) \) where \( K \) is the smallest Galois extension where all the divisors \( d_i \) are defined. Now \( G_2 \) acts on both \( M/2M \) and \( \overline{\mathcal{P}}/2\overline{\mathcal{P}} \) so we have an induced map \( \phi_2^{G_2} : (M/2M)^{G_2} \to (\overline{\mathcal{P}}/2\overline{\mathcal{P}})^{G_2} \). Then \( (\ker \phi_2)^{G_2} = \ker \phi_2^{G_2} \) will be nonzero whenever \( \ker \phi_2 \) is nonzero, by [Ser77, Proposition 26]. Let \( d \in M \) whose image \( \overline{d} \in M/2M \) is fixed by \( G_2 \). For \( \overline{d} \) to lie in \( \ker \phi_2^{G_2} \), we must have \( d \cdot \overline{d} \equiv 0 \mod 4 \) and \( d \cdot d' \equiv 0 \mod 2 \) for all \( d' \in M \). A MAGMA computation shows that the only possibility is \( d \equiv d_6 + d_9 + d_{15} \mod 2M \). It suffices to show that \( d_6 + d_9 + d_{15} \notin 2\overline{\mathcal{P}} \).
We use the method of supersingular reduction. For a prime $p$ such that the reduction $X_{\mathbb{F}_p}$ is smooth, there is an embedding $\text{Pic}(X) \to \text{Pic}X_{\mathbb{F}_p}$ (which naturally extends the intersection form on $\text{Pic}(X)$, see proof of Proposition 3.6 of [VA17] for example). If $X_{\mathbb{F}_p}$ is supersingular, i.e., when $\text{Pic}X_{\mathbb{F}_p} \simeq \mathbb{Z}^{22}$ if $p \neq 2$, then intersecting the image of $d_6 + d_9 + d_{15}$ with the extra divisors in Pic$X_{\mathbb{F}_p}$ could yield more information. MAGMA computations show that 5 is a supersingular prime, and
\[d' := w + 2x^3 - (2\alpha + 1)x^2y + 2xy^2 + (2\alpha + 1)y^3, \quad (\alpha + 2)x - z + \alpha y\]
where $\alpha$ is a root of $t^2 + t + 1$, is a divisor on $X_{\mathbb{F}_p}$. Then $(d_6 + d_9 + d_{15}) \cdot d' = 1$ which shows that $d_6 + d_9 + d_{15} \not\equiv 2T$. Similarly for $\phi_3$, we are reduced to showing $d_1 + d_3 + d_5 \not\equiv 3P$. Rewrite (2) as
\[(w + x^3)(w - x^3) = (y^3 + iz^3)(y^3 - iz^3)\]
There is a morphism $X \to \mathbb{P}^3_{[r:s:t:u]}$ given by $[w : x : y : z] \mapsto [w + x^3 : w - x^3 : y^3 + iz^3 : y^3 - iz^3]$, whose image is contained in the quadric $Q$ defined by $rs = uv$. The projection to one of the rulings on $Q$ will define an elliptic fibration on $X$. Indeed, up to a choice of ruling, the fiber above $[\alpha, \beta] \in \mathbb{P}^1$ will be
\[
\alpha(w + x^3) = \beta(y^3 + iz^3) \\
\beta(w - x^3) = \alpha(y^3 - iz^3)
\]
When $\alpha \beta \neq 0$, this is isomorphic to
\[2\alpha \beta x^3 = (\beta^2 - \alpha^2)y^3 + (\beta^2 + \alpha^2)iz^3,
\]
which is a smooth integral curve of genus 1. When $\alpha = 0$, the fiber is precisely $d_1 + d_3 + d_5$. Hence $d_1 + d_3 + d_5$ is a fiber of an elliptic fibration, so it cannot be divisible (see [Huy] Proposition 11.1.5(iii)).

3.2. An alternate approach. We show how one can also compute Pic$(X)$ without the use of the divisor $d_{20}$ from Festi’s thesis. This section is unnecessary for the main results of the paper, but we include it in case it is of use to the reader.

Consider the lattice $M'$ generated by the divisors $d_1, \ldots, d_{19}$ along with
\[d'_{20} : w = z^3, \quad x = \zeta^0 y.
\]
These divisors still generate a rank 20 sublattice in $M' \subset \mathbb{P} = \text{Pic}(X)$. In fact, one can show that $M' \not\equiv \mathbb{P}$, and also abstractly compute the full Picard group without using the divisor $d_{20}$.

By [Nik80] Prop. 1.6.1], $\mathbb{P}'/\mathbb{P} \simeq T'/T$ where $T := \text{NS}(X)^\perp \subset H^2(X(\mathbb{C}), \mathbb{Z})$ is the transcendental lattice. Note in this case the bilinear form on $T$ is given by a positive definite $2 \times 2$ matrix. Hence we can write $\mathbb{P}'/\mathbb{P} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m, n \in \mathbb{Z}$. However the discriminant of $M'$ is $-2^4 \cdot 3^5$ and the discriminant group $M'/M'$ is isomorphic to
\[(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3 \times \mathbb{Z}/9\mathbb{Z}.
\]
This implies that $M' \neq \mathbb{P}$. The method of supersingular reduction used in the proof of Theorem 3.1 can determine the “missing” divisor. For $r$ and $p$ primes, define
\[S_{r,p} = \{d \in M' : d \cdot x \equiv 0 \mod r, \text{ for all } x \in \text{Pic}X_{\mathbb{F}_p}\}.
\]
Then for all \( p \), any \( d \in M' \) which is divisible by \( r \) in \( \overline{\mathbb{F}} \) will lie in \( S_{r,p} \).

For our surface \( X \), recall that 5 is a supersingular prime. Writing down obvious divisors on \( X_{\mathbb{F}_5} \) and computing intersections, we get

\[
S_{2,5} = 2M' \\
S_{3,5} \subseteq 3M' + \mathbb{Z}(d_1 + d_4 + d_6 + d_9 + d_{10} + d'_{20} - (d_2 + d_3 + d_7 + d_8 + d_{11} + d_{12}))
\]

Let \( d = d_1 + d_4 + d_6 + d_9 + d_{10} + d'_{20} - (d_2 + d_3 + d_7 + d_8 + d_{11} + d_{12}) \). Since \( M' \neq \overline{\mathbb{F}} \), the above computations show that \( M' + \mathbb{Z}[d/3] \subseteq \overline{\mathbb{P}} \). One can then show saturation as in proof of Theorem 3.1. This abstract computation of the Picard group is also sufficient for the purpose of doing arithmetic on our surface \( X \), and in particular for computing the group \( H^1(k, \text{Pic} X) \).

4. Arithmetic

4.1. Fields of definition. We are interested in genus-2 K3 surfaces \( X \) over \( \mathbb{Q} \) of the form

(3)

\[ w^2 = Ax^6 + By^6 + Cz^6 (A, B, C \in \mathbb{Q}) \]

Let \( \alpha, \beta, \delta \) be sixth roots of \( A, B, C \) respectively. The results of the previous section show that replacing \( x, y, z \) by \( \alpha x, \beta y, \delta z \) in the divisors \( d_1, \ldots, d_{20} \) gives a set of divisor classes generating \( \overline{\mathbb{P}} = \text{Pic} \overline{X} \). The minimal field of definition of these twenty divisors is the extension \( \mathbb{Q}(s, \zeta, \alpha/\beta, \beta/\delta, \alpha^3) \). Generically this has degree \( 3 \cdot 4 \cdot 6 \cdot 6 \cdot 2 = 864 \) over \( \mathbb{Q} \). Let \( G \) be the generic Galois group of this extension.

4.2. Algebraic Brauer–Manin obstructions. For the definition of the Brauer–Manin obstruction, see [Man71] and also [Sko01]. Our goal is to give some examples of the algebraic Brauer–Manin obstruction on the surfaces (3). More ambitiously, one might hope to eventually carry out a full study of the algebraic Brauer–Manin obstruction on the family (3), as in [CTKS87] and [Cor07].

Suppose that \( X \) has points everywhere locally; then \( \text{Br}(k) \) embeds naturally in \( \text{Br}_1(X) \) (elements of the image are called constant algebras) and \( \text{Br}_0(X) \) denotes its image. The idea is to make the key isomorphism \( \text{Br}_1(X)/\text{Br}_0(X) \to H^1(k, \overline{\mathbb{P}}) \) explicit. As in [Cor10], our program proceeds by first analyzing \( H^1(H, \overline{\mathbb{P}}) \) for all the subgroups \( H \) of \( G \), then identifying the \( H \) which will give rise to nonconstant cyclic algebras in \( \text{Br}_1(X) \). Since \( |G| = 864 = 2^5 \cdot 3^3 \), only 6-primary elements appear in \( \text{Br}_1(X)/\text{Br}_0(X) \). We first consider quaternion algebras, and then 3-torsion elements of \( H^1 \) in section 6.

A MAGMA computation shows that \( H^1(G, \overline{\mathbb{P}}) = 0 \), so there is no algebraic Brauer–Manin obstruction on the generic surface (3). This is in marked contrast to the del Pezzo surfaces in [CTKS87] [Cor07], where the generic \( H^1 \) was nontrivial.

In order to construct explicit cyclic algebras, we use the following standard lemma.

**Lemma 4.1.** Let \( k \) be a number field, let \( X/k \) be a smooth projective variety, and let \( (L/k, f) \) be a cyclic algebra in \( \text{Br} k(X) \), where \( L/k \) is cyclic and \( f \in k(X) \). Then \( (L/k, f) \) is in \( \text{Br}(X) \) if and only if the divisor of \( f \) equals the norm from \( L/k \) of a divisor \( D \) on \( X_L \); it is a constant algebra if and only if we can take \( D \) to be principal.

**Proof:** This is a standard result; see [Cor05 Proposition 2.2.3] for a proof.
The lemma can be stated more precisely using [VA08 Theorem 3.3]. Let $N_{L/k} : \text{Div}_L \to \text{Div}_k$ and $\overline{N}_{L/k} : \text{Pic} X_L \to \text{Pic} X_k$ be the usual norm maps. Let $\Delta : \text{Pic}(X_L) \to \text{Pic}(X_L)$ be given as acting by $1 - \sigma$ where $\sigma$ is a generator for $\text{Gal}(L/k)$. Then we have an isomorphism

$$\ker \overline{N}_{L/k} / \text{im} \Delta \simeq \text{Br}_{\text{cyc}}(X, L).$$

where $\text{Br}_{\text{cyc}}(X, L)$ denotes cyclic algebras in $\text{Br}_1(X)$ which are split by $L$. The map is defined by sending $[D]$ to $[(L/f, k)]$ where $f \in k(X)$ is any function such that $\text{div}(f) = N_{L/k}(D)$.

4.3. The explicit quaternion algebra. If we take the short exact sequence of Galois modules

$$0 \to \overline{P} \to P \to \overline{P}/2\overline{P} \to 0$$

and consider its long exact sequence in Galois cohomology, we get

$$H^1(H, \overline{P})[2] \cong \frac{(\overline{P}/2\overline{P})^H}{\overline{P}^H/2\overline{P}^H}.$$  

Hence we search for subgroups $H$ which fix certain elements in $\overline{P}/2\overline{P}$ without fixing any of their representatives in $\overline{P}$. In practice, we search first for groups $H$ with a nonzero element in $H^1(H, \overline{P})[2]$ that becomes zero upon restriction to an index-2 subgroup of $H$. We find four subgroups $H$ of order 288 with this property.

Let $H_1$ be the mod-2 stabilizer of the divisor class

$$d_1 - d_4 = \left( \begin{array}{c}
\alpha^3 x^3 \\
\beta y = \delta z
\end{array} \right) - \left( \begin{array}{c}
\alpha^3 x^3 \\
\beta y = \delta z
\end{array} \right).$$

Then the orbit of $d_1 - d_4$ under $H_1$ is $\{ \pm (d_1 - d_4) \}$.

Let $H_2$ be the stabilizer of the class $d_1 - d_4$ inside $H_1$. Assume that the Galois group of the field of definition of $d_1, \ldots, d_{20}$ is a subgroup of $H_1$ not contained in $H_2$. Then we can construct a quaternion algebra in $\text{Br}_1(X)$ using lemma 4.1.

The first step in applying the lemma is to find a divisor class defined over a suitable quadratic extension whose norm is zero in $\overline{P}$; in this case, we can take $L/k$ to be the field of definition of $d_1 - d_4$, the fixed field of $H_2$, and the class $d_1 - d_4$ has norm zero. The next step (usually the most computationally difficult) is to find a divisor defined over $L$ whose divisor class is $d_1 - d_4$. Note that the divisor $D_1 - D_4$, the difference between the two lines on the right side of (4), is not defined over $L$. In fact, its orbit has four elements, namely

$$D_1 - D_4, D_4 - D_1, D'_1 - D'_4, D'_4 - D'_1,$$

where $D'_i$ is $D_i$ with $x^3$ replaced by $-x^3$. So $D_1 - D_4$ is defined over a quartic extension of the ground field.

It is not hard to see that $H_1$ is the stabilizer inside $G$ of the element $\beta/\delta)^2$, a cube root of $B/C$. Hence the Galois group of the field of definition of the divisor classes is a subgroup of $H_1$ if and only if $C/B$ is a cube in $Q$. So let $C = Be^3$. The quartic extension over which $D_1 - D_4$ is defined is $Q(\alpha^3, \zeta^3 \delta/\beta) = Q(\sqrt{A}, \sqrt{-c})$. Call this extension $M$.

The fixed field of $H_2$ is the extension fixed by the automorphism $\alpha^3 \mapsto -\alpha^3, \zeta^3 \delta/\beta \mapsto \zeta^9 \delta/\beta$ (which sends $D_1$ to $D'_4$ and $D_4$ to $D'_1$—this is because $D_1 - D_4 = D'_4 - D'_1$). So $L$, the fixed field of $H_2$, is the quadratic extension $Q(\sqrt{-Ac})$. 

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In order to find a divisor in the class of \( d_1 - d_4 \) defined over \( L \), we start with \( D_1 - D_4 \), defined over \( M \), and search for a function \( g \in k(X_M) \) such that

(5) \[ D_1 - D_4 + (g) = D'_4 - D'_1 + (\sigma g), \]

where \( \sigma \) is the nontrivial element of \( \text{Gal}(M/L) \).

Simplifying equation (5) gives

\[ \left( \frac{\sigma g}{g} \right) = D'_4 + D_4 - (D'_1 + D_1) = \left( \frac{y - z\zeta^9 \delta / \beta}{y - z\zeta^3 \delta / \beta} \right), \]

which suggests letting \( g = \frac{y - z\zeta^9 \delta / \beta}{x} = \frac{y - z\sqrt{-c}}{x}. \)

Now \( f \) should be a rational function whose divisor is

\[ D_1 - D_4 + (g) + \tau(D_1 - D_4 + (g)), \]

where \( \tau \) is the nontrivial element of \( \text{Gal}(L/Q) \). Extend \( \tau \) to the automorphism fixing \( \sqrt{A} \) and sending \( \sqrt{-c} \) to \( -\sqrt{-c} \); then \( \tau(D_1) = D_4, \tau(D_4) = D_1 \), and \( \tau(g) = \frac{y + z\sqrt{-c}}{x} \). Putting this all together gives the natural choice

\[ f = \frac{y^2 + cz^2}{x^2}. \]

So our quaternion algebra in \( \text{Br}(X) \) is \( \left( \mathbb{Q}(\sqrt{-Ac})/\mathbb{Q}, \frac{y^2 + cz^2}{x^2} \right) \).

5. Explicit counterexamples to the Hasse principle

By the results of the previous section, the surface \( X \) given by the equation \( w^2 = Ax^6 + By^6 + Bc^3z^6 \) has a quaternion algebra in \( \text{Br}(X) \) given by

\[ A = \left( \mathbb{Q}(\sqrt{-Ac})/\mathbb{Q}, \frac{y^2 + cz^2}{x^2} \right). \]

To find examples where \( A \) exhibits a Brauer–Manin obstruction to rational points, we search for integers \( A, B, c \) such that

\[ \sum_v \text{inv}_v A(P_v) = 1/2 \]

for all \( (P_v) \in X(\mathbb{A}_Q) \). Note that \( \text{inv}_v A(P_v) = [-Ac, f(P_v)]_v \), where \( f = (y^2 + cz^2)/x^2 \) and \([,]_v \) is the Hilbert symbol, written additively: \([a, b]_v = 0 \) if the quadratic form \( x^2 - ay^2 - bz^2 + abw^2 \) represents 0, and \( 1/2 \) otherwise. (This formula holds if \( P_v \) is not in the zero locus of the numerator or denominator of \( f \); otherwise we must use another rational function \( f' \) obtained from \( f \) by multiplying it by the norm of some rational function in \( k(X_L) \).)

**Theorem 5.1.** Let \( a, b, c \) be odd integers satisfying the following conditions:

(i) For every prime \( p > 3 \) dividing \( a \) or \( b \), \( \nu_p(a) \) and \( \nu_p(b) \) lie in \( \{1, 2, 4, 5\} \)
(ii) \( c \) is squarefree
(iii) \( a > 0 \) or \( b > 0 \)
(iv) \( a \) or \( -ac \equiv 1 \mod 3 \)
(v) If \( p \) is a prime divisor of \( a \), then \( p | c \) and \( \left( \frac{2b}{p} \right) = 1 \)
(vi) If \( p \) is a prime divisor of \( b \), then \( \left( \frac{a}{p} \right) = 1 \) and \( \left( \frac{-c}{p} \right) = 1 \).

(vii) If \( 7 | c \), then we do not have \( 4a \equiv 2b \equiv 3, 5, \) or \( 6 \) mod 7.

(viii) The triple \((a \mod 8, b \mod 8, c \mod 8)\) equals \((3, 1, 1), (3, 3, 3), (3, 3, 7), (3, 5, 1), (3, 7, 3),
(3, 7, 7), (5, 1, 1), (5, 1, 5), (5, 1, 7), (5, 3, 1), (5, 3, 5), (5, 3, 7), (5, 5, 1), (5, 5, 5), (5, 5, 7),
(5, 7, 1), (5, 7, 5), (5, 7, 7), (7, 1, 3), (7, 1, 7), (7, 3, 5), (7, 5, 3), (7, 5, 7), \) or \((7, 7, 5)\).

Then the surface \( w^2 = 4ax^2 + 2by^2 + 2bcz^2z^5 \) over \( \mathbb{Q} \) is a counterexample to the Hasse principle explained by the algebraic Brauer–Manin obstruction.

**Proof:** Let \( X \) be the surface given in the theorem. We will show
(a) \( X \) has points everywhere locally
(b) \( \text{inv}_v A(x_v) = 0 \) for all \( v \neq 2 \) and \( x_v \in X(\mathbb{Q}_v) \)
(c) \( \text{inv}_2 A(x_2) = 1/2 \) for all \( x_2 \in X(\mathbb{Q}_2) \)

5.1. **Proof of (a).** At \( v = \infty \), condition (iii) guarantees a real point. Next suppose \( v \) corresponds to a prime \( p \nmid 6ab \). If \( p > 13 \) there is always a point on the genus-2 curve \( w^2 = 4ax^6 + 2by^6 \mod p \) by the Hasse-Weil bound, which we can lift to a \( \mathbb{Q}_p \)-point by Hensel’s lemma. For \( p = 13 \) there is a point on this curve if and only if at least one of \( 4a, 2b, 4a + 2b \), or \( 4a - 2b \) is a square mod 13, which always happens. For \( p = 7 \) there is a point on this curve if and only if at least one of \( 4a, 2b, 4a + 2b \) or \( 4a - 2b \) is a square mod 7, which happens as long as \( 4a \) and \( 2b \) are not congruent to each other and to a nonsquare mod 7, as in condition (vii). If \( 7 \nmid c \), then it is not hard to check that the surface will always have a point mod 7. For \( p = 5 \) and \( p = 11 \), being a sixth power is equivalent to being a square, and the curve \( w^2 = 4ax^2 + 2by^2 \) has \( p + 1 \) points mod \( p \), so \( w^2 = 4ax^6 + 2by^6 \) has \( \mathbb{Q}_p \)-points by Hensel’s lemma.

We finish the proof of (a) by checking primes dividing \( 6ab \). If \( p | a \), then condition (v) guarantees a nontrivial point \((\sqrt{2\alpha} : 0 : 1 : 0)\) on \( X(\mathbb{Q}_p) \). If \( p | b \), then the first part of condition (vi) guarantees a nontrivial point \((2\sqrt{\alpha} : 1 : 0 : 0)\) on \( X(\mathbb{Q}_p) \). For \( p = 3, p \nmid ab \) we either have the point \((2\sqrt{\alpha} : 1 : 0 : 0) \in X(\mathbb{Q}_3) \) if \( a \equiv 1 \mod 3 \), the point \((\sqrt{2b} : 0 : 1 : 0) \) if \( 2b \equiv 1 \mod 3 \), or the point \((\sqrt{4a + 2b} : 1 : 1 : 0) \) otherwise.

For \( p = 2 \), all the triples in condition (viii) satisfy one of the following three conditions:
- \(-c^3 \equiv 1 \mod 8\)
- \(\frac{2}{b} - c^3 \equiv 1 \mod 8\)
- \(\frac{2 - 2a}{b} - c^3 \equiv 1 \mod 8\)

Note that odd rational integers in \( \mathbb{Q}_2 \) are sixth powers if and only if they are \( 1 \mod 8 \). So: if the first condition is satisfied, then there is a point \((0 : 0 : \sqrt[6]{-c^3} : 1) \in X(\mathbb{Q}_2) \). If the second condition is satisfied, then there is a point \((2 : 0 : \sqrt[6]{\frac{2 - 2a}{b}} - c^3 : 1) \in X(\mathbb{Q}_2) \). If the third condition is satisfied, then there is a point \((2 : 1 : \sqrt[6]{\frac{2 - 2a}{b}} - c^3 : 1) \in X(\mathbb{Q}_2) \).

5.2. **Invariant computations and shortcuts.** Next we prove (b). When we compute invariants, we will avail ourselves of several shortcuts. First, we can certainly substitute any rational function of the form \( \frac{y^2 + cz^2}{\ell(x, y, z)^2} \) for \( f \), since this function differs from \( f \) by a square
in \( k(X) \). So as long as \( y(P_v)^2 + cz(P_v)^2 \) is nonzero, we have
\[
\text{inv}_v \mathcal{A}(P_v) = \left[-ac, y(P_v)^2 + cz(P_v)^2\right]_v.
\]
Since \( y^2 + cz^2 \) is a norm from \( \mathbb{Q}_v(\sqrt{-c}) \) to \( \mathbb{Q}_v \), we can further simplify:
\[
\text{inv}_v \mathcal{A}(P_v) = [a, y(P_v)^2 + cz(P_v)^2]_v.
\]

There is one more formula we will use in our computations. Note that the equation of the surface can be rewritten as \( 2b(y^2 + cz^2)(y^2 - cy^2z^2 + c^2z^4) = w^2 - 4ax^6 \), so
\[
[a, 2b]_v + [a, y(P_v)^2 + cz(P_v)^2]_v + [a, y(P_v)^4 - cy(P_v)^2z(P_v)^2 + c^2z(P_v)^4]_v = [a, w(P_v)^2 - 4ax(P_v)^6]_v = 0
\]
because \( w(P_v)^2 - 4ax(P_v)^6 \) is clearly a norm from \( \mathbb{Q}_v(\sqrt{a}) \) to \( \mathbb{Q}_v \). Since these symbols take values in \( \frac{1}{2}\mathbb{Z}/\mathbb{Z} \), we get
\[
\text{inv}_v \mathcal{A}(P_v) = [a, 2b]_v + [a, y(P_v)^4 - cy(P_v)^2z(P_v)^2 + c^2z(P_v)^4]_v.
\]
We will use this when \( y(P_v)^2 + cz(P_v)^2 = 0 \). Note that if \( y(P_v)^2 + cz(P_v)^2 \) and \( y(P_v)^4 - cy(P_v)^2z(P_v)^2 + c^2z(P_v)^4 \) are both 0, then substituting in \( z(P_v)^2 = -cy(P_v)^2 \) gives \( 3y(P_v)^4 = 0 \), so \( y(P_v) = 0 \), and similarly \( z(P_v)^2 = 0 \). (Note that the same holds mod \( p \) if \( p \nmid 3c \).) This implies that \( w(P_v)^2 = 4ax(P_v)^6 \), so \( a \) would be a square in \( \mathbb{Q}_v \). Hence either \(-ac\) would be a square in \( \mathbb{Q}_v \) or \( y^2 + cz^2 \) would be the norm of \( y + \sqrt{-ac}(z/\sqrt{a}) \) from \( \mathbb{Q}_v(\sqrt{-ac}) \) to \( \mathbb{Q}_v \), so we would automatically have that \( \text{inv}_v \mathcal{A}(P_v) = 0 \) for any \( P_v \). So one of the three formulas above will compute the invariant unless it is automatically 0.

5.3. Proof of (b). At \( v = \infty \), we need either that \(-ac > 0 \) or \( y(P_v)^2 + cz(P_v)^2 > 0 \) for all \( P_v \in X(\mathbb{Q}_v) \). If \( a > 0 \), then the first inequality holds if \( c < 0 \) and the second holds if \( c > 0 \). So (given that condition (iii) holds) we must only check that the second inequality also holds when \( a < 0 \) and \( b > 0 \). But in this case, we have that \( w(P_v)^2 - 4ax(P_v)^6 = 2b(y(P_v)^6 + c^3z(P_v)^6) \), and the left side is positive, so \( y(P_v)^6 + c^3z(P_v)^6 > 0 \); hence \( y(P_v)^2 + cz(P_v)^2 > 0 \) as well.

Next suppose that \( v \) corresponds to a prime \( p \nmid 6abc \). Then \(-ac \) is a unit in \( \mathbb{Q}_p \), and if \( y(P_v)^2 + cz(P_v)^2 \) is a unit in \( \mathbb{Q}_p \), we see that \( \text{inv}_v \mathcal{A}(P_v) = 0 \) automatically. If it is not a unit in \( \mathbb{Q}_p \), then \(-c \) is a square mod \( p \), and we get \( w(P_v)^2 \equiv 4ax(P_v)^6 \) mod \( p \). If \( w(P_v) \) or \( x(P_v) \) is not 0, then \( a \) is a square mod \( p \). So then \(-ac \) is a square mod \( p \), which implies \( \text{inv}_v \mathcal{A}(P_v) = 0 \) as well. If \( w(P_v) = x(P_v) = 0 \), then we can assume \( y(P_v) = 1, z(P_v) = \pm \sqrt{-1/c} \). Thus \( \text{inv}_v \mathcal{A}(P_v) = [a, 2b]_v + [a, 3]_v = 0 \) since \( a, 2b, 3 \) are all units in \( \mathbb{Q}_p \).

Now suppose \( p \mid a, p > 3 \). Then if \( w(P_v) \) is divisible by \( p \), condition (v) shows that \( y(P_v) \) is as well. But then conditions (1), (2), and (5) force \( x(P_v) \) and \( z(P_v) \) to be divisible by \( p \). So we may assume \( w(P_v) \) (and \( y(P_v) \)) are invertible mod \( p \), so \( y(P_v)^2 + cz(P_v)^2 \) is a unit and a square in \( \mathbb{Q}_p \), so the invariant is automatically zero.

Now suppose \( p \mid b, p > 3 \). Then condition (vi) shows that \(-ac \) is a square mod \( p \), so that the invariant is automatically zero.

Suppose \( p \mid c, p > 3 \). We may assume \( p \nmid ab \) as well. There are two cases: if \( y(P_v) \) is divisible by \( p \), then so is \( w(P_v)^2 - 4ax(P_v)^6 \), so that \( a \) is a square mod \( p \) and the invariant is automatically zero (by formula (7)); but if \( y(P_v) \) is a unit in \( \mathbb{Q}_p \), then \( y(P_v)^2 + cz(P_v)^2 \) is a unit and a square in \( \mathbb{Q}_p \), whence the invariant is again automatically zero.

Finally, if \( p = 3 \), then \( a \) or \(-ac \) is a square in \( \mathbb{Q}_3 \), so we are done by formulas (3) and (7).
5.4. **Proof of (c).** We must analyze \( X(\mathbb{Q}_2) \) in order to compute the invariant at any point. So take a point \((2w: y_2: z_2) \in X(\mathbb{Q}_2)\), scaled so that the coordinates are elements of \( \mathbb{Z}_2 \), not all even. (We have written the first coordinate as \( 2w_2, w_2 \in \mathbb{Z}_2 \), because clearly the first coordinate must be even.) Since \( 2b(y_2^6 + c^2z_2^2) = 4w_2^2 - 4ax_2^6 \) is divisible by 4 and \( b \) is odd, \( y_2 \) and \( z_2 \) must have the same parity. Note that if \( y_2 \) and \( z_2 \) are both even, then we get \( w_2^2 \equiv ax_2^6 \mod 2 \), which can only happen if \( x_2 \) is even because \( a \) is never congruent to 1 mod 8 (see condition (viii)). This is a contradiction, so we may assume that \( y_2 \) and \( z_2 \) are both odd.

Now we use formula (8) to compute the invariant. We get

\[
[a, 2b]_2 + [a, y_2^4 - cy_2^2z_2^2 + c^2z_2^4]_2 = [a, 2b]_2 + [a, 2 - c]_2
\]

because both arguments of the second norm residue symbol are odd, hence we need only know them modulo 8. It is easy to check that the sum of these two symbols is \( 1/2 \) for every triple in condition (viii). So we are done. □

**Remark 5.2.** There are infinitely many triples \((a, b, c)\) satisfying the conditions of the theorem. For instance, let \( \delta \) be any squarefree integer congruent to 7 or 19 mod 24; then the triple \((-1, 1, c)\) satisfies the conditions of the theorem, so the surface

\[
w^2 = -4x^6 + 2(y^6 + c^3z^6)
\]

is a counterexample to the Hasse principle explained by the algebraic Brauer–Manin obstruction.

### 6. Odd torsion obstruction to the Hasse principle

Now we focus on the case of 3-torsion Brauer elements. There are four index-3 subgroups \( H \) of \( G \) for which the 3-torsion in \( H^1(H, \text{Pic}(X)) \) is nontrivial. In each of these cases, \( H^1(H, \overline{\text{Pic}(X)}) \) is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \). When the surface is defined over \( \mathbb{Q} \), each of these four subgroups correspond to one of the following restrictions on the coefficients \( A, B, C \):

1. \( 3ABC \) is a square.
2. \(-3A \) is a square.
3. \(-3B \) is a square.
4. \(-3C \) is a square.

Note that condition (2) is equivalent to saying \( X \) can be written in the form

\[
w^2 = -3x^6 + By^6 + Cz^6
\]

such that \((\zeta, \sqrt{-3}, \beta, \delta)\) is Galois general. We will study the obstruction on the surfaces of the form (9).

**Remark 6.1.** There are in fact 37 maximal subgroups \( H \subset G \) for which there is a nontrivial 3-torsion element in \( H^1(H, \overline{\text{Pic}(X)}) \).

### 6.1. The explicit cyclic algebra

Assume \( X/\mathbb{Q} \) is given by (9) and let \( \mathcal{A} \in \text{Br}(X) \) be a generator for the \( H^1(\mathbb{Q}, \text{Pic}(X)) \). Instead of finding a cyclic algebra representative for \( \mathcal{A} \) over \( \mathbb{Q} \), we find one over the larger field \( K = \mathbb{Q}(\sqrt{-3}) \). The larger field will not be a problem in computing invariants, since \([K : \mathbb{Q}]\) is coprime to 3. Let \( L = K(\delta^2/\beta^2) \) which is a cyclic
extension, and fix a generator \( \sigma \in \text{Gal}(L/K) \). We proceed as before and use Lemma 4.1. A simple computation shows that

\[
\ker N_{L/K}/\text{im} \Delta \simeq \mathbb{Z}/3\mathbb{Z}
\]

with generator \( d_1 + d_4 - d_{13} - d_{13}' \in \text{Pic} X_L \), where \( d_{13}' \) is the class of the divisor

(10) \[
D_{13}' = \begin{pmatrix}
    w = -\sqrt{-3x^3} \\
    y = \zeta^6 \delta / \beta z
\end{pmatrix}.
\]

Then the image of \( D_1 + D_4 \) under \( N_{L/K} \) in \( \text{Div} X_K \) is

(11) \[
\begin{pmatrix}
    w = \sqrt{-3x^3} \\
    y^6 = z^6
\end{pmatrix}.
\]

The image of \( D_{13} + D_{13}' \) under \( N_{L/K} \) is

(12) \[
\begin{pmatrix}
    w = -\sqrt{-3x^3} \\
    y^6 = z^6
\end{pmatrix}.
\]

One easily sees that

\[
f = \frac{w - \sqrt{-3x^3}}{w + \sqrt{-3x^3}}
\]

satisfies \( \text{div}(f) = N_{L/K}(D_1 + D_4 - D_{13} - D_{13}') \) so lemma 4.1 gives \( \mathcal{B} := (L/K, f) \in \text{Br}(X_K) \) and its image generates \( \text{Br}_1(X_K)/\text{Br}(K) \). Since \([K : \mathbb{Q}] = 2\), the natural restriction

\[
H^1(\mathbb{Q}, \overline{\mathbb{P}})[3] \to H^1(K, \overline{\mathbb{P}})[3]
\]

is injective. A MAGMA computation shows \( H^1(K, \overline{\mathbb{P}}) \simeq \mathbb{Z}/3\mathbb{Z} \) also, so the restriction map is in fact an isomorphism. Hence we can assume \( \mathcal{B} = \mathcal{A}_K \).

6.2. The counterexample.

**Theorem 6.2.** The surface \( X \) given by the equation

\[
w^2 = -3x^6 + 97y^6 + 97 \cdot 28 \cdot 8z^6
\]

is a counterexample to the Hasse principle explained by a 3-torsion Brauer element in the algebraic Brauer group.

**Proof.** One easily checks that \( X(\mathbb{A}) \neq \emptyset \). To compute evaluation maps, we first extend the ground field to \( K \) so that we can use the algebra representation \( \mathcal{A}_K = (L/K, f) \). Let \( X_0 \) be the open subset defined by \( w^2 + 3x^6 \neq 0 \). Then the implicit function theorem gives that \( X_0(K_v) \) is dense in \( X(K_v) \). Thus it suffices to compute the evaluation maps on \( X_0 \), where the rational function \( f \) is defined everywhere. Let \( p \in \mathbb{Z} \) be a prime. Since our cyclic algebra representation is only valid on \( K \), we first study the evaluation map \( \text{inv}_v \mathcal{A}_K(-) : X_0(K_v) \to \mathbb{Q}/\mathbb{Z} \) restricted to the image of \( X_0(\mathbb{Q}_p) \to X_0(K_v) \) for some prime \( v \) lying above \( p \). In the case \( K_v = \mathbb{Q}_p \), we immediately get \( \text{inv}_p \mathcal{A}(-) = \text{inv}_v \mathcal{A}_K(-) \). In the case \([K_v : \mathbb{Q}_p] = 2\), we can still discern information on \( \text{inv}_v \mathcal{A}_K(-) \) by using the commutative diagram

(13)

\[
\begin{array}{ccc}
\text{Br}(\mathbb{Q}_p) & \xrightarrow{\text{res}_{K_v/\mathbb{Q}_p}} & \text{Br}(K_v) \\
\downarrow \text{inv}_p & & \downarrow \text{inv}_v \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Q}/\mathbb{Z}
\end{array}
\]
We now compute the invariant maps for all finite primes. For the infinite place, the evaluation map is trivial since \( \text{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z} \). Recall that for any point \( P_v \in X_0(K_v) \) we have \( \text{inv}_v \mathcal{A}_K(P_v) = [f(P), 28 \cdot 8]_v = [f(P), 28]_v \), where \( [\ , \ ]_v \) is the norm residue symbol.

6.2.1. \( p \neq 2, 7 \). Let \( v = v_p \) be a place corresponding to a prime \( p \) lying above \( p \). If 28 is a cube in \( K_v \), then \( N_{K_v/K_v} \) is the identity map, so the norm residue symbol is trivial for any point in \( X(K_v) \). Suppose 28 is not a cube in \( K_v \); in particular this means \( p \neq 3, 97 \). Let \( P \in X(\mathbb{Q}_p) \) where one of \( w, x, y, z \) is a \( p \)-adic unit.

Case 1: \( p \) divides neither \( w - \sqrt[3]{-3}x^3 \) nor \( w + \sqrt[3]{-3}x^3 \). Then \( f(P) \) is a unit in \( \mathcal{O}_v \), so it must be a cube in \( K_v(\sqrt[3]{28}) \). Hence \( [f(P), 28]_v = 0 \).

Case 2: \( p \) divides either \( w - \sqrt[3]{-3}x^3 \) or \( w + \sqrt[3]{-3}x^3 \). This means \( p \) divides \( 97y^6 + 97 \cdot 28 \cdot 8z^6 \), which can only happen if \( p \mid y, z \). Consequently, \( v_p(97y^6 + 97 \cdot 28 \cdot 8z^6) \equiv 0 \mod 6 \). Moreover, \( p \) must divide exactly one of \( w - \sqrt[3]{-3}x^3 \) or \( w + \sqrt[3]{-3}x^3 \), since otherwise \( p \mid w, x \). Thus \( v_p(f(P)) \equiv 0 \mod 6 \), and \( [f(P), 28]_v = [u, 28]_v \) for some unit \( u \in \mathcal{O}_v \) so must be trivial as in Case 1.

Hence \( \text{inv}_p \mathcal{A}(\cdot) \) is trivial also by (13).

6.2.2. \( p = 2 \). Let \( v = v_2 \) be the unique place lying above 2. Let \( P = [w : x : y : z] \in X_0(\mathbb{Q}_2) \) where one of \( w, x, y, z \) is a 2-adic unit. We can assume either \( w \) or \( x \) is a 2-adic unit, since if \( 2 \mid w, x, x \), then \( 2 \mid y \) so the 2-adic valuation of \( w^2 + 3x^6 - 97y^6 \) is either 2 or \( \geq 6 \). It clearly cannot be 2, so that means 2-adic valuation of \( 97 \cdot 28 \cdot 8z^6 \) is 6 or greater which means \( 2 \mid z \), a contradiction. Now a MAGMA computation shows that \( f(P) \) is a 2-adic unit and is always congruent to 1 mod 8. Then \( [f(P), 28]_2 \) is trivial since \( f(P) \) is a cube in \( K_v \).

6.2.3. \( p = 7 \). Let \( v \) be a place lying above 7. Then \( K_v \cong \mathbb{Q}_7 \). Let \( P = [w : x : y : z] \in X_0(\mathbb{Q}_7) \) where one of \( w, x, y, z \) is a 7-adic unit. It is clear that we can assume either \( w \) or \( x \) is a 7-adic unit. There are two cases:

Case 1: 7 divides neither \( w - \sqrt[3]{-3}x^3 \) nor \( w + \sqrt[3]{-3}x^3 \). A MAGMA computation shows that this is not possible.

Case 2: 7 divides \( w - \sqrt[3]{-3}x^3 \). Note that 7 cannot divide \( w + \sqrt[3]{-3}x^3 \) in this case. Then \( v_7(w - \sqrt[3]{-3}x^3) = v_7(w^2 + 3x^6) = v_7(97y^6 + 97 \cdot 28 \cdot 8z^6) \). Observe that \( v_7(97y^6 + 97 \cdot 28 \cdot 8z^6) \equiv 0, 1 \mod 6 \).

Subcase 1: Valuation is 0 mod 6. This means that \( 97y^6 + 97 \cdot 28 \cdot 8z^6 \equiv 97 \cdot 7^{6n} \mod 7^{6n+1} \). A MAGMA computation also shows that \( w + \sqrt[3]{-3}x^3 \equiv 3, -3 \mod 7 \). Thus

\[
\begin{align*}
f(P) &= \frac{w - \sqrt[3]{-3}x^3}{w + \sqrt[3]{-3}x^3} \\
&= \frac{(w - \sqrt[3]{-3}x^3)(w + \sqrt[3]{-3}x^3)}{(w + \sqrt[3]{-3}x^3)^2} \\
&= \frac{97y^6 + 97 \cdot 28 \cdot 8z^6}{(w + \sqrt[3]{-3}x^3)^2} \\
&\equiv 2 \pmod{7^{6n+1}} \\
&\equiv 3 \cdot 7^{6n} \pmod{7^{6n+1}}
\end{align*}
\]
So we get
\[ [f(P), 28]_7 = [3, 28]_7 = [3, 7]_7 + [3, 4]_7 = [3, 7]_7 = 1/3 \]
by (64) of [CTKS87]. (Note \([3, 4]_7 = 0\) since 3 becomes a cube in \(K_v(\sqrt[4]{4})\))

Subcase 2: Valuation is 1 mod 6. This means that \(97y^6 + 97 \cdot 28 \cdot 8z^6 \equiv 97 \cdot 28 \cdot 8 \cdot 7^n \mod 7^{6n+2}\). We can carry out the same computation as above to get
\[ [f(P), 28]_7 = [28 \cdot 3, 28]_7 = [3, 28]_7 = 1/3. \]

Case 3: 7 divides \(w + \sqrt{-3x^3}\). Note that 7 cannot divide \(w - \sqrt{-3x^3}\) in this case. We carry out the same procedure as case 2 except now
\[
\begin{align*}
 f(P) &= \frac{w - \sqrt{-3x^3}}{w + \sqrt{-3x^3}} \\
 &= \frac{(w - \sqrt{-3x^3})^2}{(w - \sqrt{-3x^3})(w + \sqrt{-3x^3})} \\
 &= \frac{(w - \sqrt{-3x^3})^2}{97y^6 + 97 \cdot 28 \cdot 8z^6}.
\end{align*}
\]

Here again \(w - \sqrt{-3x^3} \equiv 3, -3 \mod 7\), so we get the inverse of the solutions from case 2. Thus
\[ [f(P), 28]_7 = 2/3. \]

Combining the above, we get that \(\text{inv}_p A(-) = 0\) for all \(p \neq 7\) and \(\text{inv}_7 A(-)\) has possible values 1/3 or 2/3. Hence we have a Brauer–Manin obstruction coming from \(\mathcal{A}\). □

**Remark 6.3.** The proof only uses the congruence class of 97 modulo several primes. Hence one can obtain infinitely many examples as follows. For any prime \(p\), let
\[ X_p := w^2 = -3x^6 + py^6 + p \cdot 28 \cdot 8z^6. \]

It is clear that \(X_p\) has real points. Assume that \(p\) is congruent to 97 modulo some sufficiently high enough power of every prime \(q \leq 23\) (The existence of infinitely many such \(p\) follows from Dirichlet’s theorem on primes in arithmetic progressions). This ensures that \(X_p\) has local points for all primes \(q \leq 23\). Then Weil conjectures give that for any prime \(q \geq 23, q \neq p\), there is a smooth \(\mathbb{F}_q\)-point on the curve \(\{w = 0\}\) which can be lifted to a \(\mathbb{Q}_q\)-point. Lastly, \(\sqrt{-3} \in \mathbb{Q}_{97}, \mathbb{Q}_p\) since \(97, p \equiv 1 \mod 3\), so clearly there are \(\mathbb{Q}_{97}\) and \(\mathbb{Q}_p\) points. Hence \(X_p\) is everywhere locally soluble. Furthermore assume \(p \equiv 1 \mod 8\) to ensure that the invariant maps over \(\mathbb{Q}_2\) will carry out same as in the proof. Then \(X_p\) will be a counterexample to the Hasse principle given by a 3-torsion Brauer class in the algebraic Brauer group.

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