REES ALGEBRAS OF TRUNCATIONS OF COMPLETE INTERSECTIONS

KUEI-NUAN LIN AND CLAUDIA POLINI

Abstract. In this paper we describe the defining equations of the Rees algebra and the special fiber ring of a truncation $I$ of a complete intersection ideal in a polynomial ring over a field with homogeneous maximal ideal $m$. To describe explicitly the Rees algebra $R(I)$ in terms of generators and relations we map another Rees ring $R(M)$ onto it, where $M$ is the direct sum of powers of $m$. We compute a Gröbner basis of the ideal defining $R(M)$. It turns out that the normal domain $R(M)$ is a Koszul algebra and from this we deduce that in many instances $R(I)$ is a Koszul algebra as well.

1. INTRODUCTION

In this paper we investigate the Rees algebra $R(I) = R[It]$ as well as the special fiber ring $F(I) = R(I) \otimes k$ of an ideal $I$ in a standard graded algebra $R$ over a field $k$. These objects are important to commutative algebraists because they encode the asymptotic behavior of the ideal $I$ and to algebraic geometers because their projective schemes define the blowup and the special fiber of the blowup of the scheme $\text{Spec}(R)$ along $V(I)$. One of the central problems in the theory of Rees rings is to describe $R(I)$ and $F(I)$ in terms of generators and relations (see for instance \cite{24, 29, 10, 28, 26, 17, 15, 11, 16, 13, 14}). This is a challenging quest which is open for most classes of ideals, even three generated ideals in a polynomial ring in two variables (see for instance \cite{2, 1, 22}). The goal is to find an ideal $\mathcal{A}$ in a polynomial ring $S = R[T_1, \ldots, T_s]$ so that $R(I) = S/\mathcal{A}$.

If the ideal $I$ is generated by forms of the same degree, then these forms define rational maps between projective spaces and the special fiber ring and the Rees ring describe the image and the graph of such rational maps, respectively. By computing the defining equations of these algebras, one is able to exhibit the implicit equations of the graph and of the variety parametrized by the map. This classical and difficult problem in elimination theory has also been studied in applied mathematics, most notably in modeling theory, where it is known as the implicitization problem (see for instance \cite{3, 4, 5, 9}).

If the ideal $I$ is not generated by forms of the same degree, one can consider the truncation of $I$ past its generator degree. In this paper we treat truncations of complete intersection ideals in a polynomial ring. More precisely, let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with homogeneous maximal ideal $m$, let $f_1, \ldots, f_r$ be a homogeneous regular sequence in $R$.
of degrees $d_1 \geq \ldots \geq d_r$, let $d \geq d_1$ be an integer, and write $a_i = d - d_i$. The truncation $I = (f_1, \ldots, f_r)_{d \geq d}$ of the complete intersection $(f_1, \ldots, f_r)$ in degree $d$ is the $R$-ideal generated by the forms in $(f_1, \ldots, f_r)$ of degree at least $d$. In other words, $I = (f_1, \ldots, f_r) \cap m^d = \sum_{i=1}^r m^{a_i} f_i$. The Cohen-Macaulayness of the Rees algebra of such ideals was previously studied in [12], where the authors show that $\mathcal{R}(I)$ is Cohen-Macaulay for all $d > D$ and they give a sharp estimate for $D$. However, the defining equations of $\mathcal{R}(I)$ were unknown. In this paper we describe them explicitly for all $d$ when $r = 2$ and for $d \geq d_1 + d_2$ when $r \geq 3$. Furthermore we prove that the Rees ring and the special fiber ring are Koszul algebras for $d \geq d_1 + d_2$ and for $d \geq d_1 + d_2 - 1$ if $r = 2$.

To determine the defining equations of $\mathcal{R}(I)$ we map another Rees ring $\mathcal{R}(M)$ onto $\mathcal{R}(I)$, where $M$ is the module $m^{a_1} \oplus \ldots \oplus m^{a_r}$. Our aim then becomes to find the defining ideal of $\mathcal{R}(M)$ and the kernel $Q$.

$$0 \to Q \to \mathcal{R}(M) \to \mathcal{R}(I) \to 0.$$ 

The problem of computing the implicit equation of $\mathcal{R}(M)$ is interesting in its own right and it was previously addressed in [23], where the relation type of $\mathcal{R}(M)$ was computed. We solve it in Section 2. It turns out that $\mathcal{R}(M)$ and $\mathcal{F}(M)$ are normal domains whose defining ideals have a Gröbner basis of quadrics; hence, they are Koszul algebras. For $r = 2$, the kernel $Q$ is a height one prime ideal of the normal domain $\mathcal{R}(M)$; therefore it is a divisorial ideal of $\mathcal{R}(M)$. Our goal is then reduced to explicitly describing ideals that represent the elements in the divisor class group of $\mathcal{R}(M)$. The approach is very much inspired by [21] and [20]. For $r = 2$ and $d \geq d_1 + d_2 - 1$ or $r \geq 3$ and $d \geq d_1 + d_2$, the ideal $I$ has a linear presentation and $Q$ turns out to be a linear ideal in the $T'$s. Using this we prove that the defining ideal of $\mathcal{R}(I)$ has a quadratic Gröbner basis and hence $\mathcal{R}(I)$ is a Koszul algebra as well.

2. THE BLOWUP ALGEBRAS OF DIRECT SUMS OF POWERS OF THE MAXIMAL IDEAL

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with homogeneous maximal ideal $m$. Let $0 \leq a_1 \leq \ldots \leq a_r$ be integers. Write $a = a_1, \ldots, a_r$. In this section, we will describe explicitly the Rees algebra and the special fiber ring of the module $M = M_a = m^{a_1} \oplus \ldots \oplus m^{a_r}$ in terms of generators and relations and we will prove that they are Koszul normal domains. We will end the section with a study of the divisor class group of the blowup algebras of $M$.

**Definition 2.1.** Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. Let $a$ be a positive integer, write $J_a$ and $J'_a$ for the two sets of multi-indices in $(\mathbb{N} \cup \{0\})^{n-1}$ and $\mathbb{N}^{n-1}$, respectively, that are defined as follows

$$J_a = \{ j = (j_n-1, \ldots, j_1) \mid 0 \leq j_1 \leq \ldots \leq j_{n-1} \leq a \},$$

$$J'_a = \{ j = (j_n-1, \ldots, j_1) \mid 1 \leq j_1 \leq \ldots \leq j_{n-1} \leq a \}.$$
Write \( x^{a,l} \) and \( x^{a,l,s} \) for the monomials
\[
x^{a,l} = \prod_{i=1}^{n} x_i^{j_i}
\]
with \( j \in J_a, j_0 = 0, j_n = a \) and
\[
x^{a,l,s} = \frac{x_s x^{a,l}}{x_1}
\]
with \( j \in J_a', 1 \leq s \leq n \).

The Rees algebra \( R(M) \) of \( M \) is the subalgebra
\[
R(M) = R[\{x^{a,l}t_l | 1 \leq l \leq r, j \in J_a \}] \subset R[t_1, \ldots, t_r]
\]
of the polynomial ring \( R[t_1, \ldots, t_r] \), while the special fiber ring \( F(M) \) is the subalgebra of \( R(M) \)
\[
F(M) = k[\{x^{a,l}t_l | 1 \leq l \leq r, j \in J_a \}] \subset R(M).
\]
To find a presentation of these algebras we consider the polynomial rings
\[
T = T_a = k[\{T_{l,j} | 1 \leq l \leq r, j \in J_a \}],
\]
\[
S = S_a = R \otimes_k T_a = T_a[x_1, \ldots, x_n]
\]
in the new variables \( T_{l,j} \) and the epimorphisms of algebras
\[
\phi : S \to R(M) \quad \psi : T \to F(M)
\]
defined by
\[
\phi(T_{l,j}) = \psi(T_{l,j}) = x^{a,l}t_l.
\]
Notice that \( \psi \) is the restriction of \( \phi \) to \( T \).

We can assume that the \( a_i \)'s are all positive because if \( a_i = 0 \) for \( 1 \leq i \leq s \) with \( s \leq r \) then the Rees algebra \( R(M) \) is isomorphic to a polynomial ring over the Rees algebra of \( m^{a+1} \oplus \cdots \oplus m^{a_r} \)
\[
R(\oplus_{l=1}^{r} m^{a_l}) \cong R(\oplus_{l=s+1}^{r} m^{a_l})[t_1, \ldots, t_s],
\]
and likewise for the special fiber ring. Furthermore, we can treat simultaneously the special fiber ring and the Rees algebra since
\[
R(\oplus_{l=1}^{r} m^{a_l}) \cong F(m \oplus (\oplus_{l=1}^{r} m^{a_l})) \quad \text{and} \quad F(M) = R(M)/mR(M).
\]

**Definition 2.2.** Let \( \tau \) be the lexicographic order on a set of multi-indices in \( (\mathbb{N} \cup \{0\})^n \), i.e. \( p > q \) if the first nonzero entry of \( p - q \) is positive. Let \( 1 \leq a_1 \leq \ldots \leq a_s \) be integers. Write \( a = a_1, \ldots, a_r \). Order the set of multi-indices \( \{ (l, j) | 1 \leq l \leq r, j \in J_a' \} \) by \( \tau \). Write \( T_{l,j,s} \) for the variable \( T_{l,j_0,\ldots,j_s-1,j_s-1,j_{s+1},\ldots,j_r} \).

1. Let \( B_a \) be the \( n \times (\sum_{l=1}^{r} \frac{(a_i + n - 2)}{n-1}) \) matrix whose entry in the \( s \)-row and the \( (l, j) \)-column is the variable \( T_{l,j,s} \) with \( 1 \leq s \leq n, 1 \leq l \leq r \), and \( j \in J_a' \).
(2) Let $C_a$ be the $n \times (1 + \sum_{l=1}^{r} \left( \frac{a_l + n - 2}{n-1} \right))$ matrix

$$C_a = \begin{bmatrix} x_1 & B_a \\ \vdots \\ x_n \end{bmatrix}.$$  

Example 2.3. For instance if $n = 3$, $r = 2$ and $a = 1, 2$ then $B_a$ is the $3 \times 4$ matrix

$$B_a = \begin{bmatrix} T_{1,1,1} & T_{2,1,1} & T_{2,2,1} & T_{2,2,2} \\ T_{1,1,0} & T_{2,1,0} & T_{2,2,0} & T_{2,2,1} \\ T_{1,0,0} & T_{2,0,0} & T_{2,1,0} & T_{2,1,1} \end{bmatrix}.$$  

Write $\mathcal{L}$ for the kernel of the epimorphism $\phi$ defined in (1). Notice that $\phi(T_{l,j,s}) = x_a^{l+j} t_l$ hence one can easily deduce the inclusion

$$I_2(C_a) \subset \mathcal{L}.$$  

Our goal is to show that the above inclusion is an equality. In order to establish this claim it suffices to show that $I_2(C_a)$ is a prime ideal of dimension at most $n + r = \dim \mathcal{R}(M)$ (see for instance [27, 2.2]).

Theorem 2.4. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with homogeneous maximal ideal $m$. Let $1 \leq a_1 \leq \ldots \leq a_r$ be integers and let $M = m^{a_1} \oplus \cdots \oplus m^{a_r}$. The Rees algebra and the special fiber ring of $M$ are Koszul normal domains. Furthermore,

$$\mathcal{R}(M) = S/I_2(C_a) \quad \mathcal{F}(M) = T/I_2(B_a),$$  

where $a$, $C_a$, and $B_a$ are as in Definition 2.2.

An important step in the proof of Theorem 2.4 is to show that the set of 2 by 2 minors of $C_a$ forms a Gröbner basis for $I_2(C_a)$. In the next lemma we will show much more. Indeed, the set of $2 \times 2$ minors of any submatrix $D_a$ of $C_a$ forms a Gröbner basis for $I_2(D_a)$. Notice that $\tau$ induces an ordering of the variables of $T$. With respect to this ordering we consider the reverse lexicographic order on the monomials in the ring $T$, which we also call $\tau$.

Remark 2.5. Let $a' = 1, a_1', \ldots, a_r'$ and denote with $T_{a'}$ the polynomial ring associated with the sequence $a'$. Notice that $T_{a'}/I_2(B_{a'}) \cong S_{a'}/I_2(C_{a'})$ and the matrix $C_a$ is equal (after changing the name of the variables) to the matrix $B_{a'}$.

Lemma 2.6. Adopt assumption 2.2 and let $D_a$ be any submatrix of $C_a$ with the same number of rows. The set of $2 \times 2$ minors of $D_a$ forms a Gröbner basis for $I_2(D_a)$ with respect to $\tau$.

We use a general strategy to compute the Gröbner bases that we outline below.
Strategy 2.7. Let $R$ be a polynomial ring over a field with a fixed monomial order "$>$". Let $D$ be a collection of $2 \times 2$ minors of a matrix with entries in $R$. The Buchberger criterion (see for instance [8, 15.8]) states that a generating set $D$ of an ideal is a Gröbner basis for the ideal if the $S$-polynomials (or $S$-pairs) of any two elements of $D$ reduces to zero modulo $D$. The following strategy will be used to show that the remainders of the $S$-pairs of the elements of $D$ reduces to zero modulo $D$ (see for instance [7, Section 2.9, Definition 1 and Theorem 3]). Let $\text{ad} - \text{bc}$ and $\text{hg} - \text{ef}$ be two elements of $D$. To compute the $S$-polynomial $S(\text{ad} - \text{bc}, \text{hg} - \text{ef})$ we can assume, for instance, that $h = a$ and $\text{ad} > \text{bc}$ and $ag > ef$ because otherwise the leading terms of the two polynomials are relatively prime and therefore $S(\text{ad} - \text{bc}, ag - ef)$ reduces to zero modulo $D$ (see for instance [7, Section 2.9, Proposition 4]). We consider the matrix

$$M = \begin{bmatrix} d & b & 0 \\ c & a & f \\ G & e & g \end{bmatrix}.$$  

We obtain the following equality by computing the determinant of $M$ in two ways

$$(2) \quad \begin{vmatrix} d & a & f \\ e & g & b \\ G & e & f \end{vmatrix} = \begin{vmatrix} c & f \\ G & e \end{vmatrix} - \begin{vmatrix} d & b \\ G & e \end{vmatrix}.$$  

Notice that the left hand side of equation (2) is our $S$-polynomial $S(ag - ef, \text{ad} - \text{bc})$. If by a suitable choice of $G \in R$, the polynomials $cg - fG$ and $de - bG$ are in $D$ and each term on the right hand side has order strictly less than $\text{adg}$, then the $S$-pair $S(ag - ef, \text{ad} - \text{bc})$ has remainder zero modulo $D$ and in particular it reduces to zero modulo $D$ (see for instance [7, Section 2.9, Lemma 2]). The choice of the polynomial $G$ and the matrix $M$ depends on the polynomials we start with as we will explain later.

Proof of Lemma 2.6 Because of Remark 2.5 it is enough to prove the statement for $D_a$ any submatrix of $B_d$ with the same number of rows. Let $D$ denote the set of $2 \times 2$ minors of $D_a$. We use the strategy described above to show that the $S$-polynomials of any two elements of $D$ reduces to zero modulo $D$. To simplify notation, we will use $T_{i,j,s}$ to represent the entry $T_{i,j,s}$ in the matrix $D_a$. The leading terms of two elements of $D$  

$$h_1 = \begin{vmatrix} (l_1, \hat{i}, s) \\ (l_1, \hat{j}, t) \end{vmatrix} \quad \begin{vmatrix} (l_2, \hat{j}, s) \\ (l_2, \hat{j}, t) \end{vmatrix}$$  

and  

$$h_2 = \begin{vmatrix} (l_3, \hat{p}, u) \\ (l_3, \hat{p}, u) \end{vmatrix} \quad \begin{vmatrix} (l_4, \hat{q}, u) \\ (l_4, \hat{q}, v) \end{vmatrix},$$

where $1 \leq s < t \leq n$, $(l_2, \hat{j}, s) > (l_1, \hat{i}, s)$, $1 \leq u < v \leq n$, and $(l_4, \hat{q}, u) > (l_3, p, u)$ are $(l_1, \hat{i}, s)(l_2, \hat{j}, t)$ and $(l_3, \hat{p}, u)(l_4, \hat{q}, v)$. Notice that $T_{l,i,j,s} > T_{l,i,j,t}$ if $s < t$. These will be relatively prime unless $(l_1, \hat{i}, s) = (l_3, \hat{p}, u)$, or $(l_1, \hat{i}, s) = (l_4, \hat{q}, v)$, or $(l_2, \hat{j}, t) = (l_3, \hat{p}, u)$, or $(l_2, \hat{j}, t) = (l_4, \hat{q}, v)$. Since $(l_1, \hat{i}, s) = (l_4, \hat{q}, v)$ and $(l_2, \hat{j}, t) = (l_3, \hat{p}, u)$ are symmetric, it suffices to consider three cases.

Case 1: Set

$$h_1 = \begin{vmatrix} (l_1, \hat{i}, s) = a \\ (l_1, \hat{j}, t) = e \end{vmatrix} \quad \begin{vmatrix} (l_2, \hat{j}, s) = f \\ (l_2, \hat{j}, t) = g \end{vmatrix}$$  

and  

$$h_2 = \begin{vmatrix} (l_1, \hat{i}, s) = a \\ (l_1, \hat{i}, u) = b \end{vmatrix} \quad \begin{vmatrix} (l_3, \hat{p}, s) = c \\ (l_3, \hat{p}, u) = d \end{vmatrix},$$
where we can assume \( s < u \leq t \) and \( l_1 \leq l_2 \leq l_3 \). We use equation (2) and the matrix \( M \) of Strategy [27] with \( G = (l_3, p, t) \). The \( S \)-polynomial \( S(h_1, h_2) \) reduces to zero modulo \( D \) because \( b \) and \( e \) are smaller than any other entries of the two matrices defining \( h_1 \) and \( h_2 \).

**Case 2:** Set

\[
\begin{align*}
    h_1 &= \begin{vmatrix}
        (l_1, \underline{s}, s) = a & (l_2, \underline{j}, s) = e \\
        (l_1, \underline{t}) = f & (l_2, \underline{t}, t) = g
    \end{vmatrix}
    \quad \text{and} \quad
    h_2 &= \begin{vmatrix}
        (l_3, \underline{p}, u) = d & (l_1, \underline{u}, u) = c \\
        (l_3, \underline{p}, s) = b & (l_1, \underline{s}, s) = a
    \end{vmatrix},
\end{align*}
\]

where \( u < s < t \) and \( l_3 \leq l_1 \leq l_2 \). We use equation (2) and the matrix \( M \) of Strategy [27] with \( G = (l_2, j, u) \). The \( S \)-polynomial \( S(h_1, h_2) \) reduces to zero modulo \( D \) because \( b \) and \( f \) are smaller than any other entries of the two matrices defining \( h_1 \) and \( h_2 \).

**Case 3:** Set

\[
\begin{align*}
    h_1 &= \begin{vmatrix}
        (l_1, \underline{s}, s) = d & (l_2, \underline{j}, s) = b \\
        (l_1, \underline{t}) = c & (l_2, \underline{t}, t) = a
    \end{vmatrix}
    \quad \text{and} \quad
    h_2 &= \begin{vmatrix}
        (l_3, \underline{p}, u) = g & (l_2, \underline{j}, u) = e \\
        (l_3, \underline{p}, t) = f & (l_2, \underline{t}, t) = a
    \end{vmatrix},
\end{align*}
\]

where we can assume \( u \leq s < t \) and \( l_1 \leq l_3 \leq l_2 \). We use equation (2) and the matrix \( M \) of Strategy [27] with \( G = (l_1, \underline{u}, u) \). The \( S \)-polynomial \( S(h_1, h_2) \) reduces to zero modulo \( D \) because \( c \) and \( f \) are smaller than any other entries of the two matrices defining \( h_1 \) and \( h_2 \).

**Corollary 2.8.** Adopt assumptions [2, 2] and let \( D_a \) be any submatrix of \( C_a \) with the same number of rows. The ideal \( I_2(D_a) \) is prime in \( S_a \).

**Proof.** Write \( S = S_a \). Notice that by Lemma [2.6] the variable \( u \in S \) appearing in the first column and the last row of the matrix \( D_a \) does not divide any element in the generating set of \( \text{in}_r I_2(D_a) \).

Hence \( u \) is regular on \( S/I_2(D_a) \). After localizing at \( u \) the ideal \( I_2(D_a)_u \) is isomorphic to an ideal generated by variables, which is a prime ideal in the ring \( S_u \). Thus \( I_2(D_a)_u \) is a prime ideal in the ring \( S \).

**Corollary 2.9.** Adopt assumptions [2, 2] The initial ideal \( \text{in}_r(I_2(B_a)) \) is generated by the monomials \( T_{l_1, \underline{1}, s} T_{l_2, \underline{j}, \underline{1}} \) with \( (l_1, \underline{i}) \prec (l_2, \underline{j}) \) and \( s < t \).

**Proof.** The assertion follows from Lemma [2.6]

**Proof of Theorem [2.4]** We first show that for any sequence of \( r \) positive integers \( a = a_1, \ldots, a_r \), the Rees algebra of \( M \) is defined by the ideal of minors \( I_2(C_a) \),

\[
\mathcal{R}(M) = S/I_2(C_a).
\]

Let \( \mathcal{L} \) be the kernel of the epimorphism \( \phi \) defined in (1). Recall that \( I_2(C_a) \subset \mathcal{L} \), where the first ideal is prime, according to Corollary [2.8] and the second ideal has dimension \( n + r \), according to [27] 2.2]. Hence to show that equality holds it will be enough to prove that the dimension of \( I_2(C_a) \) is at most \( n + r \). By Remark [2.5] \( T_{a_1}/I_2(B_{a_1}) \cong S_a/I_2(C_a) \). Hence it will be enough to prove that the dimension of \( I_2(B_{a_1}) \) is at most \( n + r \) which is equivalent to show that the dimension of \( I_2(B_a) \)
is at most \( n + r - 1 \). As \( \text{ht} \ I_2(B_a) = \text{ht} \ i_r(I_2(B_a)) \), we can compute the dimension of \( i_r(I_2(B_a)) \).

Let \( U \) be the set of \( r + n - 1 \) variables
\[
U = \{ \{ T_l,0,\ldots,0 \}_{1 \leq l \leq r}, T_{r,a_1,\ldots,a_r}, T_{r,a_1,\ldots,a_r,0}, \ldots, T_{r,a_r,0,\ldots,0} \}.
\]

Consider the prime \( \mathfrak{p} \in \text{Spec}(T) \) generated by all the variables of \( T \) that are not in \( U \). We claim that \( \mathfrak{p} \) is a minimal prime over \( \text{in}_r(I_2(B_a)) \). It is clear that the image of \( \text{in}_r(I_2(B_a)) \) in \( T/\mathfrak{p} \) is zero. To show the claim, consider the prime ideal \( \mathfrak{p}' \subset \mathfrak{p} \) obtained by deleting one variable \( T_{l,j} \), with \( j \in J_a \), from the generating set of \( \mathfrak{p} \). If \( l < r \), then \( T_{l,j} T_{r,0} \in \text{in}_r(I_2(B_a)) \setminus \mathfrak{p}' \). Hence the image of \( \text{in}_r(I_2(B_a)) \) in \( T/\mathfrak{p}' \) is not zero. If \( l = r \), then \( 0 < j_u \leq \ldots \leq j_i < a_r \) for some \( u, i \) with \( 1 \leq u \leq i \leq n - 1 \). Thus we can assume that
\[
T_{l,j} = T_{r,a_1,\ldots,a_r,j_1,\ldots,j_a,0,\ldots,0} \notin \mathfrak{p}'.
\]

Therefore the element \( T_{l,j} \) appears at least in two columns of the matrix \( B_a \), namely the columns corresponding to the sequences
\[ r, a_r, \ldots, a_r, j_i, \ldots, j_u, 1, \ldots, 1 \quad \text{and} \quad r, a_r, \ldots, a_r, j_i + 1, \ldots, j_u + 1, 1, \ldots, 1. \]

Hence \( T_{l,j}^2 \in \text{in}_r(I_2(B_a)) \setminus \mathfrak{p}' \). Again the image of \( \text{in}_r(I_2(B_a)) \) in \( T/\mathfrak{p}' \) is not zero. Hence \( \dim T/I_2(B_a) = \dim T/\text{in}_r(I_2(B_a)) = \dim T/\mathfrak{p} = |U| = r + n - 1 \) as claimed, where the second equality follows as \( I_2(B_a) \) is a prime ideal (see for instance [19]).

From the above follows that for any sequence of \( r \) positive integers \( a = a_1, \ldots, a_r \) the special fiber ring of \( M \) is defined by the ideal of minors \( I_2(B_a) \), indeed
\[
\mathcal{F}(M) = k \otimes_R \mathcal{R}(M) = k \otimes_R \mathcal{S}/I_2(C_a) = T/I_2(B_a).
\]

For any sequence \( a \) the ideals \( I_2(C_a) \) and \( I_2(B_a) \) have a Gröbner basis of quadrics according to Lemma 2.6 hence both the Rees algebra and the special fiber ring of \( M \) are Koszul domains. Normality follows because \( \mathcal{F}(M) \) is a direct summand of \( \mathcal{R}(M) \) which in turn is a direct summand of \( \mathcal{R}[t_1, \ldots, t_r] \). The latter claim can be easily seen once we consider \( \mathcal{R}(M) \) as a \( \mathbb{N}^{r+1} \)-graded \( \mathcal{R}[t_1, \ldots, t_r] \)-algebra. \( \square \)

In the rest of this section we study the divisor class group of the normal domain \( A = T_a/I_2(B_a) \) for any sequence \( a \).

**Definition 2.10.** Let \( K \) be the \( A \)-ideal generated by all the variables appearing in the first row of \( B_a \), i.e. all the variables \( T_{l,j} \) with \( 1 \leq l \leq r \) and \( j \in J_a' \).

**Theorem 2.11.** The divisor class group \( \text{Cl}(A) \) is cyclic generated by \( K \).

**Proof.** To compute the divisor class group we use Nagata’s Theorem: If \( W \subset A \) is a multiplicatively closed set, then there is an exact sequence of Abelian groups
\[
0 \rightarrow U \rightarrow \text{Cl}(A) \rightarrow \text{Cl}(A_W) \rightarrow 0,
\]
where $U$ is the subgroup of $Cl(A)$ generated by
\[
\{ [p] \mid p \text{ a height one prime ideal with } p \cap W \neq \emptyset \}.
\]
We use the above theorem with $W = \{ T_{r,a_1,\ldots,a_r}^i = T_{r,a_\underline{\underline{r}}}^i \mid i \in \mathbb{Z} \}$. Notice $T_{r,a_\underline{\underline{r}}} \notin I_2(B_\underline{\underline{r}})$. Hence $T_{r,a_\underline{\underline{r}}}$ is regular on $T/I_2(B_\underline{\underline{r}})$. After localizing at $T_{r,a_\underline{\underline{r}}}$ the ideal $I_2(B_\underline{\underline{r}})T_{r,a_\underline{\underline{r}}}$ is isomorphic to an ideal generated by variables, and the ring $A_{T_{r,a_\underline{\underline{r}}}}$ is a polynomial ring, hence factorial. Thus $Cl(A_W) = 0$ and $Cl(A) = U$.

Now we will show that $U$ is cyclic generated by $K$. Notice that $[K] \in U$ because $K$ is a prime ideal of height one containing $T_{r,a_\underline{\underline{r}}}$. Clearly, $T_{r,a_\underline{\underline{r}}} \in K$. To show that $A/K$ is a domain of dimension $\dim A - 1$, let $R' = k[x_2, \ldots, x_n]$ be the polynomial ring over $k$ in $n - 1$ variables, let $m'$ be its homogeneous maximal ideal, and let $M' = m'_{a_1} \oplus \cdots \oplus m'_{a_r}$. The claim follows by Theorem 2.4 as $A/K \cong \mathcal{F}(M')$ and $\mathcal{F}(M')$ is a domain of dimension $n - 1 + r - 1 = \dim A - 1$.

Let $P$ be the $A$-ideal generated by all the variables $T_{r,a_j}$ with $j \in J_{a_\tau}$. Clearly, $P$ is prime. Indeed, if $r = 1$, then $A/P = k$; if $r > 1$, then, according to Theorem 2.4, $A/P \cong \mathcal{F}(M_{a'})$ with $a' = a_1, \ldots, a_{r-1}$. Furthermore, if $r > 1$, $P$ has height one as $\dim A/K = \dim \mathcal{F}(M_{a'}) = \dim A - 1$.

Next we show that every prime ideal $p$ in $A$ containing $T_{r,a_\underline{\underline{r}}}$ contains either $K$ or $P$. Assume that $K \not\subset p$, then there exists a variable $T_{l,i} \neq T_{r,a_\underline{\underline{r}}}$ with $i \in J_{a_{\underline{\underline{r}}}}$, that is not in $p$. Recall that the set of multi indices $J_{a_\tau}$ is ordered by $\tau$. For all $j \in J_{a_\tau}$ we show that $T_{r,j} \in p$ by descending induction on $J_{a_\tau}$. The base case is trivial since $T_{r,a_\underline{\underline{r}}} \in p$. Assume $T_{r,j} \neq T_{r,a_\underline{\underline{r}}}$. Then there exists a multi-index $\underline{s} \in J_{a_\tau}$ with $\underline{s} > \underline{j}$ such that the equality
\[
T_{r,j}T_{l,i} = T_{r,a_{\underline{j}_t}}
\]
holds in $A$ for some integer $t \geq 2$. Notice that $T_{l,i}T_{l,j}$ is well defined because $i \in J_{a_{\underline{r}}}$, by induction $T_{r,a_{\underline{s}}} \subset p$, thus $T_{r,a_{\underline{s}}} \subset p$ since $p$ is prime and $T_{l,i} \notin p$. Hence $P \subset p$.

If $r = 1$ the above inclusion implies that $ht p > 1$ and hence $U$ is cyclic generated by $K$. If $r > 1$, then $U$ is generated by $P$ and $K$. We conclude by showing that $[P] = a_r[K]$, or equivalently, $P = (T_{r,a_\underline{\underline{r}}}) : A K^{a_r}$. Since both ideals have height one and $P$ is prime it is enough to prove the inclusion $P \subset (T_{r,a_\underline{\underline{r}}}) : A K^{a_r}$. The latter follows from the equation
\[
T_{r,j}K^{a_r-j_1} \in (T_{r,a_\underline{\underline{r}}}) \quad (3)
\]
We prove equation (3) using descending induction on $j_1$ with $0 \leq j_1 \leq a_r$. The base case is trivial since $j_1 = a_r$ implies $T_{r,j} = T_{r,a_\underline{\underline{r}}}$. If $j_1 < a_r$, then the variable $T_{r,j}$ appears in a row $s$ of $B_{a_{\underline{\underline{r}}}}$ with $s > 1$. Thus for any element $\lambda \in K$ there exists $\beta \in S$ such that the equality
\[
T_{r,j}\lambda = T_{r,jn-1,j_{s+1},j_{s+1},\ldots,j_{j_1+1}} \beta
\]
holds in $A$. Now the claim follows by induction. $\square$

According to Theorem 2.11 the classes of the divisorial ideals $K^{(\delta)}$ and $P^{(\delta)}$, the $\delta$-th symbolic power of $K$ and $P$ respectively, constitute $Cl(A)$. In the next theorem we exhibit a monomial
generating set for $K^{(\delta)}$ for $\delta \geq 1$. As in [20] and [23] we identify $K^{(\delta)}$ with a graded piece of $A$. We put a new grading on the ring $T$,
\[ \text{Deg}(T_{i,j}) = j. \]
Notice that $I_2(B_a)$ is an homogeneous ideal with respect to this grading. Thus Deg induces a grading on $A$. Let $A_{\geq \delta}$ be the ideal generated by all monomials $m$ in $A$ with $\text{Deg}(m) \geq \delta$.

**Theorem 2.12.** The $\delta$-th symbolic power of $K$, $K^{(\delta)}$, equals the monomial ideal $A_{\geq \delta}$.

**Proof.** One proceeds as in [21, 1.5].

3. THE BLOWUP ALGEBRAS OF TRUNCATIONS OF COMPLETE INTERSECTIONS

In this section our goal is to compute explicitly the defining equations of the blowup algebras of truncations of complete intersections.

**Assumptions 3.1.** Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with homogeneous maximal ideal $m$. Let $r$ be an integer with $1 \leq r \leq n$. Let $f_1, \ldots, f_r$ be a homogeneous regular sequence in $R$ of degree $d_1 \geq \ldots \geq d_r$. Let $d \geq d_1$ be an integer, write $a_i = d - d_i$. Let $I$ be the truncation of the complete intersection $(f_1, \ldots, f_r)$ in degree $d$, i.e. the $R$-ideal generated by
\[ \{ m^{a_i} f_i \mid 1 \leq i \leq r \}, \]
\[ I = (f_1, \ldots, f_r)_{\geq d} = (m^{a_1} f_1, \ldots, m^{a_r} f_r). \]

**Theorem 3.2.** Adopt assumptions 3.1 and write $M = m^{a_1} \oplus \ldots \oplus m^{a_r}$. There is a short exact sequence
\[ 0 \to Q \to R(M) \to R(I) \to 0 \]
where $Q$ is a prime ideal of height $r - 1$ in the normal domain $R(M)$.

**Proof.** The natural map
\[ (u_1, \ldots, u_r) \mapsto u_1 f_1 + \ldots + u_r f_r \]
induces a surjection on the level of Rees algebras
\[ \Psi : R(M) \to R(I) \]
and Ker $\Psi$ is a prime of height $r - 1$ since $R(M)$ and $R(I)$ are domains and $\dim R(M) = n + r = \dim R(I) + r - 1$. 

\[ \Box \]
Definition 3.3. Adopt assumptions 3.1. Let \( a \) be the sequence \( a_1, \ldots, a_r \) and let \( S \) be the polynomial ring \( S_a \). Consider the algebra epimorphism \( \chi \) obtained by the composition of the two algebra epimorphisms
\[
\phi : S \rightarrow R(M) \quad \text{and} \quad \Psi : R(M) \rightarrow R(I)
\]
with \( \phi \) as in (1) and let \( A \) be the \( S \)-ideal defined by the short exact sequence
\[
0 \rightarrow A \rightarrow S \xrightarrow{\chi} R(I) \rightarrow 0.
\]

Let \( \tau \) be the order on \( T_{l,l,j} \) defined in Section 2. Notice that, through the algebra epimorphism \( \chi \), the order \( \tau \) induces an order on the set
\[
C = \{ x^{a_l,j} f_l | 1 \leq l \leq r, j \in J_a \}
\]
of generators of \( I \). Let \( \varphi \) be the presentation matrix of \( I \) over \( R \) with respect to \( C \). In the following remark we give a resolution of \( I \) when \( r = 2 \). In particular, we describe explicitly \( \varphi \).

Remark 3.4. To compute a resolution of \( I = (f_1, f_2)_{\geq d} \) we first truncate the Koszul complex of \( f_1, f_2 \) in degree \( d \).

1. If \( d_2 \leq d \leq d_1 + d_2 \) we obtain the short exact sequence:
\[
0 \rightarrow R(-d_1 - d_2 + d) \xrightarrow{\rho} M \xrightarrow{\zeta} I \rightarrow 0,
\]
where \( \zeta \) is the natural surjection described in Theorem 3.2 and \( \rho = \left[ \begin{array}{c} -f_2 \\ f_1 \end{array} \right] \). Set \( \varphi'' \) to be the map that rewrites \( \left[ \begin{array}{c} -f_2 \\ f_1 \end{array} \right] \) in terms of the \( k \)-basis \( B \) of \( M \) ordered with the order induced by \( \tau \). The Eagon-Northcott complex gives us an \( R \)-resolution \( F_* \) of \( M \), while the complex \( G_* \) that is trivial everywhere except in degree 0 gives us an \( R \)-resolution of \( R(-d_1 - d_2 + d) \). The map \( \rho \) can be trivially lifted to a morphism of complexes
\[
\rho_* : G_* \rightarrow F_*
\]
that is trivial in positive degree and is \( \varphi'' \) in degree zero.

2. If \( d \geq d_1 + d_2 + 1 \) we obtain the short exact sequence:
\[
0 \rightarrow m^{d-d_1-d_2} \xrightarrow{\rho} M \xrightarrow{\zeta} I \rightarrow 0,
\]
where \( \zeta \) is the natural surjection described in Theorem 3.2 and \( \rho = \left[ \begin{array}{c} -f_2 \\ f_1 \end{array} \right] \). Write \( E \) for the \( k \)-bases of \( m^{d-d_1-d_2} \) ordered with the order induced by \( \tau \). Set \( \varphi'' \) to be the map that rewrites \( E \left[ \begin{array}{c} -f_2 \\ f_1 \end{array} \right] \) in terms of the \( k \)-basis \( B \) of \( M \) ordered with the order induced by \( \tau \). From the Eagon-Northcott complex we obtain \( R \)-resolutions \( F_* \) and \( G_* \) of \( M \) and \( m^{d-d_1-d_2} \), respectively. The map \( \rho \) can be lifted to a morphism of complexes
\[
\rho_* : G_* \rightarrow F_*
\]
with \( \varphi'' \) in degree zero.
In both cases the mapping cone $C(\rho_\bullet)$ is a non-minimal free resolution of $I$. In particular, the presentation matrix of $I$ with respect to $C$ is $\varphi = [\varphi', \varphi'']$ where $\varphi'$ is the matrix presenting $M$ with respect to $B$.

In the following remark we describe explicitly $\varphi$ for any $r$ when $d \geq d_1 + d_2$.

**Remark 3.5.** Adopt assumptions 3.1. To compute the presentation matrix for $I = (f_1, \ldots, f_r)_d$ we need to truncate the Koszul complex $K_\bullet(f_1, \ldots, f_r) = \bigwedge R \oplus \cdots \oplus R_r$. If $d \geq d_1 + d_2$ then $d \geq d_i + d_j$ for any $1 \leq i < j \leq r$ and we have:

$$\ldots \rightarrow \bigoplus_{1 \leq i < j \leq r} m^{d-d_i-d_j} e_i \wedge e_j \rightarrow M = \bigoplus_{i=1}^r m^{a_i} \rightarrow I \rightarrow 0.$$  

Write $B$ and $E_{i,j}$ for the $k$-bases of $M$ and of $m^{d-d_i-d_j}$, respectively, ordered with the order induced by $\tau$. Set $\varphi''$ to be the map that rewrites $E_{i,j}(-f_je_i + f_ie_j)$ in terms of $B$ for all $1 \leq i < j \leq r$. The presentation matrix of $I$ with respect to $C$ is $\varphi = [\varphi', \varphi'']$ where $\varphi'$ is the matrix presenting $M$ with respect to $B$.

**Definition 3.6.** Adopt assumptions 3.1. Let $h_1, \ldots, h_t \in S$ be the homogeneous polynomials obtained by the matrix multiplication $[T] \varphi''$, where $\varphi''$ is the matrix described in Remark 3.4 and 3.5. Think of $S$ as a naturally bigraded ring with $\deg x_i = (1, 0)$ and $\deg T_j = (0, 1)$. If $r = 2$ and $d_1 \leq d \leq d_1 + d_2$, then $t = 1$ and $h_1$ has bidegree $(\delta, 1)$ with $\delta = d_2 - a_1 = d_1 - a_2 = d_1 + d_2 - d \geq 0$. For any $r$, if $d \geq d_1 + d_2$ then $t = \sum_{1 \leq i < j \leq r} \binom{\sigma_{i,j} + n - 1}{n - 1}$ with $\sigma_{i,j} = d - d_i - d_j \geq 0$ and the $h_k$'s have bidegree $(0, 1)$.

**Proposition 3.7.** Adopt assumptions 3.7 3.6 with $d \geq d_1 + d_2$ then $(h_1, \ldots, h_t) R(M)$ is a non-zero prime ideal of height $\geq r - 1$.

**Proof.** Let $1 \leq i < j \leq r$. Write

$$f_i = \sum_{k \in J_{d_i}} \lambda_k x^{d_i,k} \quad \text{and} \quad f_j = \sum_{k \in J_{d_j}} \alpha_k x^{d_j,k}.$$  

Let $\sigma_{i,j} = d - d_i - d_j \geq 0$. As in Definition 2.1 denote the elements of the basis $E_{i,j}$ of $m^{a_{i,j}}$ with $x^{a_{i,j},s}, t \in J'_{a_{i,j}}$. Since we have a one to one correspondence between the $h_k$'s and the elements of $E_{i,j}$, we write $h_k$ as $h_{a_{i,j},s}$. Let $H$ be the $S$-ideal generated by $\{h_{a_{i,j},s} \mid 1 \leq i < j \leq r, 1 \leq s \leq n\}$. We obtain

$$x^{a_{i,j},s} f_i = \sum_{k \in J_{d_i}} \lambda_k x^{a_{i,j} + s,k} \quad \text{and} \quad x^{a_{i,j},s} f_j = \sum_{k \in J_{d_j}} \alpha_k x^{a_{i,j} + s,k} ,$$  

hence

$$h_{a_{i,j},s} = \sum_{k \in J_{d_i}} \lambda_k T_{i,k} x^{a_{i,j} + s,k} - \sum_{k \in J_{d_j}} \alpha_k T_{j,k} x^{a_{i,j} + s,k} .$$
Let \( \mathcal{H}_j \) be the \( S \)-ideal generated by \( \{h_{\sigma_1,j,L,s} \mid 2 \leq l \leq j, \, L \in \mathcal{J}_{\sigma_1,j}, \, 1 \leq s \leq n \} \). We show by induction on \( j \) with \( 2 \leq j \leq r \) that the ideal \( \mathcal{H}_j R(M) \) is prime of height \( \geq j - 1 \). Let \( j = 2 \). Notice that for each \( L \in \mathcal{J}_{\sigma_1,2} \), the column \( [h_{\sigma_1,2,L,1}, \ldots, h_{\sigma_1,2,L,n}]^t \) is a linear combination of columns of \( C_\alpha \). Write \( E_a \) for the \( n \times (1 + \sum_{q=1}^r (\sigma_{1,2} + n - 2)) \) matrix obtained by \( C_\alpha \) by substituting \( (\sigma_{1,2} + n - 2) \) columns with \( [h_{\sigma_1,2,L,1}, \ldots, h_{\sigma_1,2,L,n}]^t \), \( L \in \mathcal{J}_{\sigma_1,2} \). The two \( S \)-ideals \( I_2(C_\alpha) + \mathcal{H}_2 \) and \( I_2(E_a) + \mathcal{H}_2 \) are equal. The ideal \( \mathcal{H}_2 R(M) \) is prime according to Corollary 2.8 as

\[
S/(I_2(E_a) + \mathcal{H}_2) \cong T'/I_2(D_a)
\]

for some polynomial ring \( T' \) and \( D_a \) a suitable submatrix of \( C_\alpha \) with \( n \) rows. Now degree considerations show that \( h_{\sigma_1,2,L,s} \not\in I_2(C_\alpha) \), hence \( \mathcal{H}_2 R(M) \neq 0 \). Thus the ideal \( \mathcal{H}_2 R(M) \) is prime of height at least one.

Let \( 2 \leq l \leq j \) and assume by induction that \( \mathcal{H}_{j-1} R(M) \) is prime of height \( \geq j - 2 \), we show \( \mathcal{H}_j R(M) \) is prime of height \( \geq j - 1 \). For each \( L \in \mathcal{J}_{\sigma_1,j} \), the column \( [h_{\sigma_1,j,L,1}, \ldots, h_{\sigma_1,j,L,n}]^t \) is a linear combination of columns of \( C_\alpha \). Write \( E_a \) for the \( n \times (1 + \sum_{q=1}^r (\sigma_{1,2} + n - 2)) \) matrix obtained by \( C_\alpha \) by substituting \( \sum_{i=2}^j (\sigma_{1,2} + n - 2) \) columns with \( [h_{\sigma_1,j,L,1}, \ldots, h_{\sigma_1,j,L,n}]^t \), \( 2 \leq l \leq j \) and \( L \in \mathcal{J}_{\sigma_1,j} \). The two \( S \)-ideals \( I_2(C_\alpha) + \mathcal{H}_l \) and \( I_2(E_a) + \mathcal{H}_l \) are equal. The ideal \( \mathcal{H}_l R(M) \) is prime according to Corollary 2.8 as

\[
S/(I_j(E_a) + \mathcal{H}_j) \cong T'/I_2(D_a)
\]

for some polynomial ring \( T' \) and \( D_a \) a suitable submatrix of \( C_\alpha \) with \( n \) rows. Notice that for each \( L \in \mathcal{J}_{\sigma_1,j} \), the column \( [h_{\sigma_1,j,L,1}, \ldots, h_{\sigma_1,j,L,n}]^t \) is a linear combination of a subset of the columns \( \{(i,L) \mid L \in \mathcal{J}_{\sigma_1,i} \} \) and \( \{(j,L) \mid L \in \mathcal{J}_{\sigma_1,j} \} \) of \( C_\alpha \), while the column \( [h_{\sigma_1,j,L,1}, \ldots, h_{\sigma_1,j,L,n}]^t \) is a linear combination of a subset of the columns \( \{(1,L) \mid L \in \mathcal{J}_{\sigma_1,1} \} \) and \( \{(i,L) \mid L \in \mathcal{J}_{\sigma_1,i} \} \) of \( C_\alpha \) with \( 2 \leq l \leq j - 1 \). Hence degree considerations show that \( h_{\sigma_1,j,L,s} \not\in (\mathcal{H}_{j-1}, I_2(C_\alpha)) \), hence \( \mathcal{H}_j R(M) \neq 0 \). Thus the ideal \( \mathcal{H}_j R(M) \) is prime of height \( \geq j - 1 \).

Using the same argument one can show that \( \mathcal{H} R(M) \) is a prime ideal and its height is at least \( r - 1 \) as \( \mathcal{H} R(M) \supset \mathcal{H}_r R(M) \).

Remark 3.8. If \( d \geq d_1 + d_2 - 1 \), then the ideal \( I = (f_1, \ldots, f_r) \geq d \) has a linear resolution. The Rees algebra of linearly presented ideals of height 2 has been described explicitly in terms of generators and relations in [18] under the additional assumption that \( I \) is perfect. However, if \( r < n \), the truncations of codimension \( r \) complete intersections are never perfect. For large \( d \), the Rees algebra \( R(I) \) is Cohen-Macaulay as shown in [12]. But the defining equations of \( R(I) \) were unknown, we give them explicitly in Theorem 3.9. If \( r = 2 \) we prove that for \( d \geq d_1 + d_2 - 1 \) the Rees ring \( R(I) \) is a Koszul domain (see Corollary 3.13). If \( r \geq 3 \), we prove that \( R(I) \) is a Koszul domain for \( d \geq d_1 + d_2 \) (see Corollary 3.13). In addition in [16] we study the depth and regularity of the blowup algebras of \( I \).
Theorem 3.9. Adopt assumptions 3.1 and 3.6 with \( d \geq d_1 + d_2 \) then
\[
\mathcal{R}(I) = \mathcal{R}(M_a)/(h_1, \ldots, h_t) = S_a/(I_2(C_a), h_1, \ldots, h_t).
\]

Proof. Adopt the notation of the proof of Proposition 3.7. Let \( H \) be the \( S \)-ideal generated by \( \{ h_{\sigma, i, j, t} | 1 \leq i < j \leq r, t \in J'_\sigma, 1 \leq s \leq n \} \). According to Remark 3.5, we have \( H \subset A \). Hence \( HR(\mathcal{M}) \subset AR(\mathcal{M}) \) where the first ideal is a non-zero prime ideal of height \( r - 1 \) by Proposition 3.7 and the second one has height \( r - 1 \).

Assume \( r = 2 \). In the following theorem we express the Rees algebra of \( I \) as defined by a divisor on the Rees algebra of the module \( M = m^{a_1} \oplus m^{a_2} \) that we computed explicitly in the previous section. Indeed, for \( r = 2 \) the prime ideal \( Q \in \text{Spec}(\mathcal{R}(M)) \) of Theorem 3.2 gives rise to an element of the divisor class \( \text{Cl}(\mathcal{R}(M)) \). This group has been studied explicitly in Theorem 2.11: it is cyclic generated by the prime ideal \( L \), where \( L \) be the \( \mathcal{R}(\mathcal{M}) \)-ideal generated by all the variables appearing in the first row of \( C_a \). In the next theorem we identify for which \( s \) the ideal \( L^{(s)} \) is isomorphic to \( Q \).

Definition 3.10. Let \( L \) be the \( \mathcal{R}(\mathcal{M}) \)-ideal generated by all the variables appearing in the first row of \( C_a \).

We will use the convention that the (symbolic) power of any element or ideal with nonpositive exponent is one or the unit ideal, respectively.

Theorem 3.11. Adopt assumptions 3.1, 3.6, and 3.10 with \( r = 2 \) then
\[
(x_1^\delta) \mathcal{A}(\mathcal{M}) = (h_1, \ldots, h_t)L^{(\delta)}.
\]

In particular,
\begin{itemize}
  \item[(1)] If \( d_1 \leq d \leq d_1 + d_2 - 1 \) then the \( \mathcal{R}(\mathcal{M}) \)-ideals \( x_1^\delta \mathcal{A}(\mathcal{M}) \) and \( h_1L^{(\delta)} \) are equal and the bigraded \( \mathcal{R}(\mathcal{M}) \)-modules \( \mathcal{A} \) and \( L^{(\delta)}(0,-1) \) are isomorphic.
  \item[(2)] If \( d \geq d_1 + d_2 \) then
\[
\mathcal{R}(I) = S_a/(I_2(C_a), h_1, \ldots, h_t).
\]
\end{itemize}

Proof. Notice that the first statement follows from (1) and (2) and (2) has been proven in Theorem 3.9. Thus it will be enough to prove (1). Write \( h = h_1 \). One proceeds as in [21, 1.11]. For clarity we rewrite part of the proof here since there are some minor differences and in [21] 1.11 the integer \( \delta \) was assumed to be \( \geq 2 \). Degree considerations show that \( x_1 \) is not in \( I_2(C_a) \). The ideal \( I_2(C_a) \) is prime, so \( x_1^\delta \) is also not in \( I_2(C_a) \). The second assertion in (1) follows from the first as \( x_1^\delta \) has bi-degree \( (\delta, 0) \) and \( h_1 \) has bi-degree \( (\delta, 1) \). Write \( \overline{\cdot} \) to mean image in \( \mathcal{R}(\mathcal{M}) \). We prove the equality \( x_1^\delta \mathcal{A}(\mathcal{M}) = hL^{(\delta)} \) by showing that \( \overline{\mathcal{A}} = (\overline{h}/\overline{x_1^\delta})L^{(\delta)} \), where the fraction is taken in the quotient field \( Q \) of \( \mathcal{R}(\mathcal{M}) \).
Notice that \((\bar{x}_1^i)_Q L^{(i)} = (\bar{x}_1, \ldots, \bar{x}_n)_i^i\). This follows as in the proof of claim (1.12) in [21, 1.11]. Furthermore \(\bar{h} \in \mathfrak{m}^d = (\bar{x}_1^i)_Q L^{(d)}\). Thus, \(\bar{h} L^{(d)} \subseteq \bar{x}_1^d \mathcal{R}(M)\). Define \(D\) to be the ideal \((\bar{h}/\bar{x}_1^d)L^{(d)}\) of \(\mathcal{R}(M)\). At this point, we see that the ideal \(D\) is either zero or divisorial.

To show that \(D\) is not zero and to establish the equality \(\mathcal{A} = D\), it suffices to prove that \(\mathcal{A} \subseteq D\), because \(\mathcal{A}\) is a height one prime ideal of \(\mathcal{R}(M)\). This is the only part where the argument differs from [21, 1.11]. Notice that \(\bar{h} \in D\) as \(\bar{x}_1 \in L\). For every \(w \in m\), one has \(I_w = (f_1, f_2)_w\). Therefore, \(R[It]_w = R[(f_1, f_2)t]_w\) and we obtain \((\bar{h})_w = \mathcal{A}_w\). It follows that \(\bar{h} \neq 0\) and \(\mathcal{A}_w \subseteq D_w\). The rest of the proof follows as in [21, 1.11].

**Corollary 3.12.** Adopt assumptions 3.7, 3.6 and 2.10

(a) If \(d \geq d_1 + d_2\) then

\[
F(I) = F(M_a)/(h_1, \ldots, h_t) = T_a/(I_2(B_a), h_1, \ldots, h_t).
\]

(b) If \(r = 2\) and \(d_1 \leq d \leq d_1 + d_2 - 1\) then

\[
F(I) = T_a/K \quad \text{with} \quad K \cong K^{(d)}(-1).
\]

**Proof.** The proof follows from Theorem 3.9 and Theorem 3.11 and the fact that \(F(I) = k \otimes \mathcal{R}(I)\). Also the same argument as in [21, 4.2] shows the isomorphism \(K \cong K^{(d)}(-1)\).

**Corollary 3.13.** Adopt assumptions 3.7. If \(d \geq d_1 + d_2\) or if \(r = 2\) and \(d = d_1 + d_2 - 1\), then \(\mathcal{R}(I)\) and \(F(I)\) are Koszul algebras.

**Proof.** We will prove the statement for the Rees ring. The same proof works for the special fiber ring using Corollary 3.12. If \(d \geq d_1 + d_2\), then \(\mathcal{R}(I) = S/(I_2(C_a), h_1, \ldots, h_t)\) according to Theorem 3.9 and according to the proof of Corollary 2.8 the latter ring is isomorphic to \(T'/I_2(D_a)\) for some polynomial ring \(T'\) and \(D_a\) a suitable submatrix of \(C_a\) with \(n\) rows. But this ring is Koszul as it has a Gröbner of quadrics by Lemma 2.6.

If \(r = 2\) and \(d = d_1 + d_2 - 1\), then \(\delta = 1\) and \(\mathcal{R}(I) = \mathcal{R}(M)/\mathcal{A} \mathcal{R}(M) \cong S/(I_2(C_a), L)\) according to Theorem 3.11 (1). Let \(R' = k[x_2, \ldots, x_n]\) be the polynomial ring over \(k\) in \(n - 1\) variables, let \(m'\) be its homogeneous maximal ideal, and let \(M' = m'^{a_1} \oplus m'^{a_2}\). The last assertion now follows from Theorem 2.4 as

\[
\mathcal{R}(I) = \mathcal{R}(M)/\mathcal{A} \mathcal{R}(M) \cong S/(I_2(C_a), L) \cong \mathcal{R}(M').
\]

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