Algebraic Treatment of Compactification on Noncommutative Tori

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ABSTRACT

In this paper we study the compactification conditions of the M theory on $D$-dimensional noncommutative tori. The main tool used for this analysis is the algebra $\mathcal{A}(\mathbb{Z}^D)$ of the projective representations of the abelian group $\mathbb{Z}^D$. We exhibit the explicit solutions in the space of the multiplication algebra of $\mathcal{A}(\mathbb{Z}^D)$, that is the algebra generated by right and left multiplications.

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1 Introduction

In the last few years there has been a renewed interest in string theories mostly motivated by the discovery of string dualities [1]. This fact has induced to conjecture about the existence of a still unknown M theory, which is supposed to underly the known superstring models. A candidate theory has been proposed in [2]. This amounts to a description, for $N \to \infty$, of $N$ interacting $D0$ branes $\mathbb{R}$, that is branes on which strings can end. A completely new feature is that the $D0$ branes are described by coordinates which are valued in the space of the $N \times N$ hermitian matrices $\mathbb{H}$. Since M theory is supposed to describe gravity, one should be able to derive from it the conventional space-time structure. Therefore, given the fact that the $D0$ branes live in 10 dimensions, the usual space-time description will arise only after compactification along various space directions. Among different possibilities, a certain attention has been given to compactifications on circles and on tori, because these give rise in a natural way to strings and membranes. However, the intrinsic noncommuting nature of the coordinates of the $D0$ branes leads to compactification on noncommutative geometries, such as noncommuting tori [4, 5, 6, 7].

In this paper we will present an algebraic study of the noncommutative torus based on the use of projective representations of the abelian group $\mathbb{Z}^D$. These representations form a noncommutative algebra $\mathcal{A}(\mathbb{Z}^D)$ which can be represented on the space of the related multiplication algebra generated by right and left multiplications of $\mathcal{A}(\mathbb{Z}^D)$. Using this fact we will be able to get explicit expressions for the compactified coordinates in terms of a particular set of derivations on the algebra and identifying the gauge field part of the coordinates with the left multiplications. This follows from the simple observation that the associativity requires that left and right multiplications commute.

A solution to the problem presented here has been given in ref. [6] in the framework of the Connes formulation of noncommutative geometry [8]. But our treatment determines in a unique way the derivation part of the compactified coordinates, and allows us to give in explicit terms the more general realization of the gauge part using relatively simple algebraic techniques. In particular we emphasize the relevance of the projective representations for noncommuting compactifications in the framework of the M theory. In fact these techniques can be easily extended to noncommuting geometries more
2 Compactification on noncommuting tori

$D0$ branes are point-like objects described by $N \times N$ hermitian matrices, $X_{i_1,i_2}^\mu$, $\mu = 1, \cdots, 9$, $i_1,i_2 = 1, \cdots N$, moving in a space-time $\mathbb{R}^9 \times \mathbb{R}^D$. Since the usual space-time description is supposed to arise from this theory, several $D0$ brane coordinates need to be compactified. We will study here the compactification on a noncommuting $D$-dimensional torus, $T^D$, ($D < 9$). Since $X^\mu$ are dynamical variables one cannot require directly that they describe a torus geometry. The problem can be solved along the lines outlined in ref. [9]. That is, by observing that $T^D$ is given by the quotient $R^D/Z^D$, where $Z$ is the group of the integers, one can describe the motion of the $D0$ branes on $R^D$, and then take the quotient with respect to the group $Z^D$. Technically this is done by taking infinite copies of the $D0$ branes through the extension of the matrices $X_{i_1,i_2}^\mu$ to $X_{(i_1,a_1)(i_2,a_2)}^\mu$, where $a_1$ and $a_2$ are elements of $Z^D$, that is of the form

$$
\sum_{\mu=1}^{D} m^\mu e_{(\mu)}
$$

(2.1)

where $m^\mu \in Z$, and $e_{(\mu)}$ are a set of linearly independent vectors defining the space lattice. Then we go to the quotient by requiring the theory to be invariant with respect to the compactification condition

$$
U(a)^{-1}X^\mu U(a) = X^\mu + a^\mu
$$

(2.2)

where $a^\mu$ are the components of the vector $a$ (see eq. (2.1)), and $U(a)$ are unitary operators. The operators $U(a)$ act on the group indices of $X^\mu$, and by consistency they must belong to a projective representation of $Z^D$

$$
U(a)U(b) = e^{i\alpha(a,b)}U(a+b)
$$

(2.3)

If the co-cycle $\alpha(a,b)$ is trivial, then we have vector representations and we speak of a commutative torus, otherwise we are in the noncommutative case. The compactification condition (2.2) tells us that the operators $U$, acting on group indices, belong to the regular projective representation of the group $Z^D$. Therefore, the mathematical problem of compactification on a noncommutative torus $T^D$ is now completely defined, and it consists in describing
the projective regular representation of $Z^D$, and in finding operators $X^\mu$ on
this space, such as to satisfy eq. (2.2).

To realize this program, we start by considering projective representations
of an abelian group $G$, which will be specialized, later on, to $Z^D$. Let us
take an arbitrary projective representation of the group $G$. This defines an
associative noncommutative algebra $A(G)$

$$x(a)x(b) = e^{i\alpha(a,b)}x(a + b) = \sum_{c \in G} f_{abc}x(c), \quad a, b \in G, \quad x(a), x(b) \in A(G)$$  \hspace{1cm} (2.4)

where $f_{abc} = \delta_{a+b,c}e^{i\alpha(a,b)}$ are the structure constants of the algebra. The
associativity requires the phase $\alpha(a, b)$ to satisfy the co-cycle condition

$$\alpha(a, b) + \alpha(a + b, c) = \alpha(b, c) + \alpha(a, b + c)$$  \hspace{1cm} (2.5)

It is not difficult to show that the co-cycle is an antisymmetric bilinear mapp-
ing $G \times G \to \mathbb{R}$. The regular representation can be evaluated in terms of
the right and left multiplications on $A(G)$. To this end it is convenient to
introduce vectors $\langle x |$ with components $x(a)$ ($\langle x |_a = x(a)$), $a \in G$, and the
corresponding kets $| x \rangle$. We define

$$R_a | x \rangle = | x \rangle x(a), \quad \langle x | L_a = x(a) \langle x | \quad x(a) \in A(G)$$  \hspace{1cm} (2.6)

In the following we will use also $L_a^T | x \rangle = x(a) | x \rangle$. The matrices of the right
and left multiplications can be expressed in terms of the structure constants
of the algebra, and one can easily show that, due to the associativity, the left
and right multiplications give a representation of the algebra itself

$$R_a R_b = \sum_{c \in G} f_{abc} R_c, \quad L_a L_b = \sum_{c \in G} f_{abc} L_c$$  \hspace{1cm} (2.7)

and that

$$[R_a, L_b^T] = 0$$  \hspace{1cm} (2.8)

These matrices can be expressed in terms of the structure constants, obtaining

$$(R_a)_{bc} = f_{bac} = \delta_{a+b,c}e^{i\alpha(b,a)}, \quad (L_a)_{bc} = f_{acb} = \delta_{a+c,b}e^{i\alpha(a,c)}$$  \hspace{1cm} (2.9)

It follows that we can identify the operators $U(a)$ with, for instance, the
matrices $R_a$, for $G = Z^D$. The second step of our problem is to find out
the operators $X^\mu$. This is easily solved by introducing a derivation $D$ on the algebra, that is a linear mapping satisfying the Leibnitz rule. We will call $d$, the matrix of this application, that is $Dx(a) = \sum_{b \in G} d_{ab}x(b)$. Acting with a derivation upon the first of equations (2.6), one can prove the following identity

$$R(x(a))^{-1}dR(x(a)) = d - R(x(a))^{-1}R(Dx(a))$$

(2.10)

where we have used the more explicit notation $R(x(a)) \equiv R_a$. Then, a particular solution to the compactification condition is given by operators $D^\mu$ such that

$$D^\mu x(a) = -a^\mu x(a), \quad d_{ab}^\mu = -a^\mu \delta_{a,b}$$

(2.11)

It can be checked immediately that these operators are indeed derivations (that is they satisfy the Leibnitz rule). This is not the most general solution, since we can always add to $d^\mu$ any operator commuting with $U(a)$. Equation (2.8) gives us such a set of operators. Then, the general solution to eq. (2.2) is given by

$$X^\mu = d^\mu + A^\mu$$

(2.12)

where $A^\mu$ is an arbitrary linear combinations of $L^T_a$

$$A^\mu = \sum_{a \in Z^D} f^\mu_a L^T_a$$

(2.13)

Notice that the operators $L^T_a$ define the so called opposite algebra

$$L^T_a L^T_b = e^{-i\alpha(a,b)} L^T_{a+b}$$

(2.14)

By introducing the algebra valued quantities

$$f^\mu = \sum_{a \in Z^D} f^\mu_a x(a), \quad f^\mu \in \mathcal{A}(Z^D)$$

(2.15)

we can write

$$X^\mu = d^\mu + L^T_{f^\mu}$$

(2.16)

These operators can be considered as connections with a curvature given by

$$[X^\mu, X^\nu] = -L^T_{F^\mu\nu}$$

(2.17)

where

$$F^\mu\nu = D^\mu f^\nu - D^\nu f^\mu - [f^\mu, f^\nu] \in \mathcal{A}(Z^D)$$

(2.18)
and we have used

\[ [L(x(a))^T, d] = L(Dx(a))^T \]  \hspace{1cm} (2.19)

In the case of \( D = 1 \) and of the commuting \( D \)-torus, the representations of the algebra are one-dimensional and they are given by the characters

\[ x(m^\mu e(\mu)) = \chi(\vec{m}) = e^{i\vec{m} \cdot \vec{q}} \]  \hspace{1cm} (2.20)

where \( 0 \leq q_\mu \leq 2\pi \) are the coordinates on the torus \( T^D \). In this case, the derivation introduced before is essentially \( \partial/\partial q_\mu \). From this point of view it is interesting to notice that one could retain the parameterization of eq. (2.20) also in the case of a non-commutative \( D \)-torus, by requiring

\[ [q_\mu, q_\nu] = i\epsilon_{\mu\nu} \]  \hspace{1cm} (2.21)

with \( \epsilon_{\mu\nu} \) commuting with \( q_\mu \). In fact from

\[ e^{im^\mu q_\mu} e^{-im^\mu q_\mu} = q_\nu - m^\mu \epsilon_{\mu\nu} \]  \hspace{1cm} (2.22)

we get

\[ e^{im^\mu q_\mu} e^{in^\nu q_\nu} e^{-im^\mu q_\mu} = e^{-im^\mu n^\nu \epsilon_{\mu\nu}} e^{in^\nu q_\nu} \]  \hspace{1cm} (2.23)

allowing us to make the following identifications

\[ x(a) = e^{im^\mu q_\mu}, \quad \alpha(a, b) = -m^\mu n^\nu \epsilon_{\mu\nu} \]  \hspace{1cm} (2.24)

where \( m^\mu \) and \( n^\mu \) are the components of \( a \) and \( b \). Therefore, one could think to characterize the representations of the projective algebra by the noncommuting operators \( \vec{q} \), writing

\[ x(a) \equiv x_{\vec{q}}(\vec{a}) \]  \hspace{1cm} (2.25)

Then, the generic element of the algebra \( \mathcal{A}(G) \)

\[ \tilde{f}(\vec{q}) = \sum_{a \in \mathbb{Z}^D} f(\vec{a}) x_{\vec{q}}(\vec{a}) \]  \hspace{1cm} (2.26)

can be regarded as a generalized Fourier transform (GFT) of the function on the group \( \mathbb{Z}^D, f(\vec{a}) \), with respect to the non-commuting variables \( \vec{q} \). The product of two such GFT’s

\[ \tilde{h}(\vec{q}) = \tilde{f}(\vec{q}) \tilde{g}(\vec{q}) = \sum_{a \in \mathbb{Z}^D} h(\vec{a}) x_{\vec{q}}(\vec{a}) \]  \hspace{1cm} (2.27)
gives rise to a deformed convolution product

\[ h(\vec{a}) = \sum_{\vec{b} \in \mathbb{Z}^{D}} f(\vec{b})g(\vec{a} - \vec{b})e^{-i\alpha(\vec{a}, \vec{b})} \] (2.28)

On the contrary, eq. (2.26) shows that the GFT of the deformed convolution product is equal to the product of the GFT's. However, defining the usual Fourier transform (FT) of the function \( f(\vec{a}) \) in terms of the characters of the vector representations of \( \mathbb{Z}^{D} \) (see eq. (2.20))

\[ \tilde{f}_{V}(\vec{q}) = \sum_{\vec{a} \in \mathbb{Z}^{D}} f(\vec{a})\chi_{\vec{q}}(\vec{a}) \] (2.29)

we find that the FT of \( h(\vec{a}) \), that is of the deformed convolution product, is the Moyal product of the FT's of \( \tilde{f}_{V}(\vec{q}) \) and \( \tilde{g}_{V}(\vec{q}) \)

\[ \tilde{h}_{V}(\vec{q}) = \sum_{\vec{a} \in \mathbb{Z}^{D}} h(\vec{a})\chi_{\vec{q}}(\vec{a}) = e^{i\alpha_{\mu\nu}\partial^{\mu}_{\vec{q}_{1}}\partial^{\nu}_{\vec{q}_{2}}} \tilde{f}_{V}(\vec{q}_{1})\tilde{g}_{V}(\vec{q}_{2}) \bigg|_{\vec{q}_{1}=\vec{q}_{2}=\vec{q}} \equiv \tilde{f}_{V}(\vec{q}) \ast \tilde{g}_{V}(\vec{q}) \] (2.30)

This is a strong indication for the use of the GFT in the harmonic analysis on the noncommuting torus. This analysis is completed by the use of the integration theory on generic algebras that we have introduced in \[10\], and discussed in \[11\] in the case of the projective group algebras. In fact this theory allows us to invert the GFT. Following ref. \[11\] one has the integration formula (depending only on the algebraic structure and not from its representation)

\[ \int_{(x)} x(\vec{a}) = \delta_{\vec{a}, \vec{0}} \] (2.31)

which generalizes the integration over the coordinates \( \vec{q} \) of the commuting torus, giving

\[ \int_{(x)} \tilde{f}(\vec{q})x(-\vec{a}) = f(\vec{a}) \] (2.32)

In conclusion we have shown that the the projective representations of \( \mathbb{Z}^{D} \) are a powerful and simple tool to find the solutions to the M theory compactification conditions on a noncommutative torus and to study its geometrical properties. Also, the present methods can be easily extended to other compactification geometries.

\textit{Note added in proof:} After this work was completed the paper \[12\] appeared in hep-th, which also emphasizes the relevance of the projective representations in the study of the compactification in the framework of the M theory.
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