A RATE OF CONVERGENCE OF PHYSICS INFORMED NEURAL NETWORKS FOR THE LINEAR SECOND ORDER ELLIPTIC PDES

YULING JIAO∗, YANMING LAI †, DINGWEI LI‡, XILIANG LU§, FENGRU WANG ¶, YANG WANG∥, AND JERRY ZHIJIAN YANG∗∗

Abstract. In recent years, physical informed neural networks (PINNs) have been shown to be a powerful tool for solving PDEs empirically. However, numerical analysis of PINNs is still missing. In this paper, we prove the convergence rate to PINNs for the second order elliptic equations with Dirichlet boundary condition, by establishing the upper bounds on the number of training samples, depth and width of the deep neural networks to achieve desired accuracy. The error of PINNs is decomposed into approximation error and statistical error, where the approximation error is given in $C^2$ norm with ReLU$^3$ networks (deep network with activations function max$\{0, x^3\}$) and the statistical error is estimated by Rademacher complexity. We derive the bound on the Rademacher complexity of the non-Lipschitz composition of gradient norm with ReLU$^3$ network, which is of immense independent interest.

Key words. PINNs, ReLU$^3$ neural network, B-splines, Rademacher complexity.

1. Introduction. Classical numerical methods such as the finite element method are successful to solve the low-dimensional PDEs, see e.g., [6, 7, 24, 30, 13]. However these methods may encounter some difficulties in both theoretical analysis and numerical implementation for the high-dimensional PDEs. Motivated by the facts that deep learning method for high-dimensional data analysis has been achieved great successful applications in discriminative, generative and reinforcement learning [11, 9, 28], solving high dimensional PDEs with deep neural networks becomes an extremely potential approach and has attracted a lot of attentions [2, 29, 17, 25, 32, 33, 5, 10]. Due to the excellent approximation ability of the deep neural networks, several numerical schemes have been proposed to solve PDEs with deep neural networks including the deep Ritz method (DRM) [32], physics-informed neural networks (PINNs) [25], and deep Galerkin method (DGM) [33]. Both DRM and DGM are applied to variational forms of PDEs, and PINNs are based on residual minimization to the differential equation, see [2, 29, 17, 25], which can be extended to general PDEs [14, 16, 23, 22].

Despite the above mentioned deep PDEs solvers work well empirically, rigorous numerical analysis for these methods are far from complete. The convergence rate of DRM with two layer networks and deep networks are studied in [20, 12, 19, 8], the convergence of PINNs are given in [26, 27, 21]. In this work, we will provide the nonasymptotic convergence rate of the PINNs with ReLU$^3$ networks, i.e., a quantitative error estimation with respect to the topological structure of the neural networks (the depth and width) and the number of the samples. Hence it gives

∗School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, P.R. China. (yulingjiaomath@whu.edu.cn)
†School of Mathematics and Statistics, Wuhan University, P.R. China. (laiyanming@whu.edu.cn)
‡School of Mathematics and Statistics, Wuhan University, P.R. China. (lidingv@whu.edu.cn)
§School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, P.R. China. (xllv.math@whu.edu.cn)
¶School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, P.R. China. (wangfr@whu.edu.cn)
∥Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong (yangwang@ust.hk)
∗∗School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, P.R. China. (zjyang.math@whu.edu.cn)
a rule to determine the hyper-parameters to achieve a desired accuracy. Our contributions are summarized as follows.

Our contributions and main results

- We obtain the approximation results ReLU$^3$ network in $C^2(\bar{\Omega})$, see Theorem 3.1, i.e., \( \forall \bar{u} \in C^3(\bar{\Omega}) \) and for any \( \epsilon > 0 \), there exists a ReLU$^3$ network \( u_\phi \) with depth \( \lceil \log_2 d \rceil + 2 \) and width \( C(d, \| \bar{u} \|_{C^3(\bar{\Omega})}) \left( \frac{1}{\epsilon} \right)^d \) such that
  \[
  \| \bar{u} - u_\phi \|_{C^2(\bar{\Omega})} \leq \epsilon,
  \]
  where \( d, \bar{\Omega}, C(d, \| \bar{u} \|) \) stands for the dimension of \( x \), the closure of the domain \( \Omega \) and some numerical constant that only depends on \( (d, \| \bar{u} \|) \), respectively.

- We establish an upper bound of the statistical error for PINNs by applying the tools of Pseudo dimension, especially we give an upper bound of the Rademacher complexity to the derivative of ReLU$^3$ network which is non-Lipschitz composition with ReLU$^3$ network, via calculating the Pseudo dimension of networks with ReLU, ReLU$^2$ and ReLU$^3$ activation functions, see Theorem 4.13. We prove that \( \forall D, W \in \mathbb{N} \) and \( \epsilon > 0 \), if the number of training samples in PINNs is with the order \( O \left( D^6 W^2 (D + \log W) \left( \frac{1}{\epsilon} \right)^{2+\delta} \right) \), where \( \delta \) is an arbitrarily positive number, then the statistical error
  \[
  E_{\{X_k\}_{k=1}^N, \{Y_k\}_{k=1}^M} \sup_{u \in \mathcal{P}} \left| L(u) - \hat{L}(u) \right| \leq \epsilon,
  \]
  where \( L \) and \( \hat{L} \) are loss functions defined in (2.2) and (2.3) respectively.

- Based on the above bounds on approximation error and statistical error, we establish a nonasymptotic convergence rate to PINNs for second order elliptic equations with Dirichlet boundary condition, see Theorem 5.1, i.e., \( \forall \epsilon > 0 \) if we set the depth and width by
  \[
  D = O(\lceil \log_2 d \rceil + 2), \quad W = O \left( \frac{1}{\epsilon^d} \right)
  \]
  in the ReLU$^3$ network and set the number of training samples used in PINNs as \( O \left( \left( \frac{1}{\epsilon} \right)^{2d+4+\delta} \right) \), where \( \delta \) is an arbitrarily positive number, then
  \[
  E_{\{X_k\}_{k=1}^N, \{Y_k\}_{k=1}^M} \| \hat{u}_\phi - u^* \|_{H^2(\Omega)} \leq \epsilon,
  \]
  where \( \hat{u}_\phi \) is the minimizer in (2.4).

The paper is organized as follows. In Section 2 we describe the problem setting and introduce the error decomposition for the PINNs. In Section 3 the approximation error of ReLU$^3$ networks in Sobolev spaces based on approximation results of B-splines is proved. In Section 4 we provide the statistical error estimation by using the bound of Rademacher complexity. In Section 5 the convergence rate of PINNs is shown. A conclusion and short discussion is given in Section 6.

2. The PINNs Method and Error Decomposition.
2.1. Preliminary and PINNS method. We give the notations of neural networks and function spaces which will be used later. Let $D \in \mathbb{N}$, a function $f$ is called a neural network if it is implemented by:

$$f_0(x) = x,$$

$$f_\ell(x) = g_\ell (A_\ell f_{\ell-1} + b_\ell) = (g_\ell^{(\ell)}((A_\ell f_{\ell-1} + b_\ell))_{i}) \quad \text{for } \ell = 1, \ldots, D-1,$$

$$f := f_0(x) = A_D f_{D-1} + b_D,$$

where $A_\ell = (a_{ij}^{(\ell)}) \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$, $b_\ell = (b_{\ell i}^{(\ell)}) \in \mathbb{R}^{n_\ell}$ and $g_\ell : \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}^{n_\ell}$ is the active function. The hyper-parameters $D$ and $W := \max\{N_\ell, \ell = 0, \ldots, D\}$ are called the depth and the width of the network, respectively. Also $\sum_{\ell=1}^L n_\ell$ is called the number of units of $f$ and $\theta = \{A_\ell, b_\ell\}_{\ell}$ are called the weight parameters. Let $\Phi$ be a set of activation functions and $X$ be a Banach space, we define the neural network function class by

$$\mathcal{N}(D, W, \{\| \cdot \|_X, \mathcal{B}\}, \Phi) := \{f : f \text{ is implemented by a neural network with depth } D \text{ and width } W, \|f\|_X \leq \mathcal{B}, \text{ and } g_\ell^{(\ell)} \in \Phi \text{ for each } i \text{ and } \ell\}.$$

Then we introduce the function spaces and the partial differential equations what we are interested in. The governing equation is defined in an open bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial \Omega$, and without loss of generality we may assume that $\Omega \subset [0, 1]^d$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an $n$-dimensional index with $|\alpha| := \sum_{i=1}^n \alpha_i$ and $s$ be a nonnegative integer. The standard function spaces include continuous function space, $L^p$ space, Sobolev spaces are given below.

$$C(\Omega) := \{\text{all the continuous functions defined on } \Omega\}, \quad C^s(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \in C(\Omega)\},$$

$$C^{\alpha}(\Omega) := \{\text{all the continuous functions defined on } \Omega\}, \quad \|f\|_{C^{\alpha}(\Omega)} := \max_{x \in \Omega, |\alpha| \leq s} |D^\alpha f(x)|,$$

$$C^{\alpha}(\tilde{\Omega}) := \{f : \tilde{\Omega} \rightarrow \mathbb{R} \mid D^\alpha f \in C(\tilde{\Omega})\}, \quad \|f\|_{C^{\alpha}(\tilde{\Omega})} := \max_{x \in \tilde{\Omega}, |\alpha| \leq s} |D^\alpha f(x)|,$$

$$L^p(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} |f|^p \, dx < \infty\}, \quad \|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f|^p \, dx\right)^{\frac{1}{p}}, \quad \forall p \in [1, \infty),$$

$$L^\infty(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \exists C > 0 \text{ s.t. } |f| \leq C \text{ a.e.}\}, \quad \|f\|_{L^\infty(\Omega)} := \inf\{C \mid |f| \leq C \text{ a.e.}\},$$

$$W^{s,p}(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \in L^p(\Omega), \quad |\alpha| \leq s\}, \quad \|f\|_{W^{s,p}(\Omega)} := \left(\sum_{|\alpha| \leq s} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$
completion space of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$. For $s < 0$, $W^{s,p}(\Omega)$ is the dual space of $W_0^{-s,q}(\Omega)$ with $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. When $p = 2$, $W^{s,p}(\Omega)$ is a Hilbert space and it is also denoted by $H^s(\Omega)$. The constant $C$ may vary from place to place but it is independent to the hyper-parameters of neural networks.

We will consider the following linear second order elliptic equation with Dirichlet boundary condition, where $\frac{\partial u}{\partial x_i}(x)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}(x)$ are denoted by $u_{x_i}$ and $u_{x_i x_j}$, respectively.

\[
\begin{aligned}
&\left\{ \begin{array}{l}
- \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu = f \quad \text{in } \Omega, \\
eu \quad \text{on } \partial \Omega,
\end{array} \right.
\end{aligned}
\tag{2.1}
\]

where $a_{ij} \in C(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $e \in C(\partial \Omega)$, $g \in L^2(\partial \Omega)$ with the strictly elliptic condition, i.e., there exists a constant $\lambda > 0$ such that $\sum_{i,j} a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$, $\forall x \in \Omega, \xi \in \mathbb{R}^d$.

Define the constants by $A = \max_{x \in \Omega} \{\|a_{ij}\|_{C(\Omega)}\}$, $B = \max_{x \in \Omega} \{\|b_i\|_{L^\infty(\Omega)}\}$, $C = \|e\|_{L^2(\Omega)}$, $E = \|g\|_{L^2(\partial \Omega)}$, $\mathcal{F} = \{A, B, C, E, F, G\}$.

Instead of solving problem (2.1), we consider a minimization problem with the loss functional $\mathcal{L}$ on $C^2(\Omega) \cap C(\bar{\Omega})$:

\[
\mathcal{L}(u) := \int_{\Omega} \left( - \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu - f \right)^2 \, dx + \int_{\partial \Omega} (eu - g)^2 \, dx.
\]

**Assumption 2.1.** Assume that (2.1) has a unique strong solution $u^* \in C^2(\Omega) \cap C(\bar{\Omega})$.

If Assumption 2.1 holds, then $u^*$ is also the unique minimizer of loss functional $\mathcal{L}$. Define $|\Omega|$ and $|\partial \Omega|$ be the measure of $\Omega$ and its boundary respectively, i.e., $|\Omega| := \int_{\Omega} 1 \, dx$ and $|\partial \Omega| := \int_{\partial \Omega} 1 \, ds$, then $\mathcal{L}$ can be equivalently written by

\[
\mathcal{L}(u) = |\Omega| \mathbb{E}_{X \sim U(\Omega)} \left( - \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) + \sum_{i=1}^{d} b_i(X) u_{x_i}(X) + c(X) u(X) - f(X) \right)^2 + |\partial \Omega| \mathbb{E}_{Y \sim U(\partial \Omega)} (e(Y) u(Y) - g(Y))^2,
\]

\[
\tag{2.2}
\]

where $U(\Omega)$ and $U(\partial \Omega)$ are uniform distribution on $\Omega$ and $\partial \Omega$, respectively.

To solve minimization of $\mathcal{L}(u)$ numerically, a standard discrete version of $\mathcal{L}$ is given by:

\[
\tilde{\mathcal{L}}(u) \mybox{=\frac{|\Omega|}{N}} \sum_{k=1}^{N} \left( - \sum_{i,j=1}^{d} a_{ij}(X_k) u_{x_i x_j}(X_k) + \sum_{i=1}^{d} b_i(X_k) u_{x_i}(X_k) + c(X_k) u(X_k) - f(X_k) \right)^2 + |\partial \Omega| \mybox{=\frac{M}{M}} \sum_{k=1}^{M} (e(Y_k) u(Y_k) - g(Y_k))^2,
\]

\[
\tag{2.3}
\]

where $\{X_k\}_{k=1}^{N}$ and $\{Y_k\}_{k=1}^{M}$ are i.i.d. random samples according to the uniform distribution $U(\Omega)$ on $\Omega$ and $U(\partial \Omega)$ on $\partial \Omega$, respectively.
The PINNs method considers a minimization problem with respect to $\hat{L}$:

$$
\min_{u_\phi \in \mathcal{P}} \hat{L}(u_\phi),
$$

(2.4)

where the admissible set $\mathcal{P}$ refers to the deep neural network function class parameterized by $\phi$. Assume there exists at least one minimizer to (2.4), which is denote by $\hat{u}_\phi$, and we will give an error estimation to $u^*$ and $\hat{u}_\phi$ for some carefully chosen admissible set $\mathcal{P}$. The definition of $\mathcal{P}$ will be given at beginning of Section 3.

**Remark 2.1.** In practical, it is difficult to compute $\hat{u}_\phi$ precisely due to the nonlinear structure of $\mathcal{P}$. People often call a (random) solver $A$, say SGD, to solve (2.4) and let the output $u_{\phi,A}$ be the final solution. In this work we ignore the optimization error and only consider the error between $u^*$ and $\hat{u}_\phi$.

**Remark 2.2.** The existence of the minimizer to problem (2.4) depends on the choice of the admissible set $\mathcal{P}$. In case of the minimizer may not exist for some admissible set $\mathcal{P}$, we may consider a $\gamma$-optimal solution $\tilde{u}_\phi$, i.e., $\hat{L}(\tilde{u}_\phi) \leq \inf_{u_\phi \in \mathcal{P}} \hat{L}(u_\phi) + \gamma$ for a arbitrary small positive $\gamma$. The main results in this work are also true for the $\gamma$-optimal solution.

### 2.2. Error Decomposition

The a priori estimation to equation (2.1) can be find in [1], and we list it below for completeness.

**Lemma 2.1.** [1] For $u \in H^\frac{1}{2}(\Omega) \cap L^2(\partial \Omega)$,

$$
|u|_{H^\frac{1}{2}(\Omega)} \leq C \left\| \sum_{i,j=1}^d a_{ij} u_{x_i x_j} + \sum_{i=1}^d b_i u_{x_i} + cu \right\|_{H^{-\frac{1}{2}}(\Omega)} + C \|c u\|_{L^2(\partial \Omega)}.
$$

Next we decompose the error into the approximation error and statistical error separately.

**Proposition 2.2.** Assume that $\mathcal{P} \subset H^2(\Omega) \cap C(\Omega)$, then

$$
\|\hat{u}_\phi - u^*\|_{H^\frac{1}{2}(\Omega)}^2 
\leq C \inf_{u_\phi \in \mathcal{P}} \left[ 3 \max\{2d^2A^2, dB^2, C^2\} \|\hat{u} - u^*\|_{H^2(\Omega)}^2 + |\partial \Omega| \|c\|_{L^\infty(\partial \Omega)} \|\hat{u} - u^*\|_{L^2(\partial \Omega)}^2 \right] 
+ C \sup_{u \in \mathcal{P}} \|\mathcal{L}(u) - \hat{\mathcal{L}}(u)\|.
$$

**Proof.** For any $\hat{u} \in \mathcal{P}$, we have

$$
\mathcal{L}(\hat{u}_\phi) - \mathcal{L}(u^*) = \hat{L}(\hat{u}_\phi) - \hat{L}(\hat{u}) + \hat{L}(\hat{u}) - \mathcal{L}(\hat{u}) + \mathcal{L}(\hat{u}) - \mathcal{L}(u^*) 
\leq [\mathcal{L}(\hat{u}) - \mathcal{L}(u^*)] + \sup_{u \in \mathcal{P}} \|\mathcal{L}(u) - \hat{\mathcal{L}}(u)\|,
$$

where the last step is due to the fact that $\hat{L}(\hat{u}_\phi) - \hat{L}(\hat{u}) \leq 0$. Since $\hat{u}$ can be any element in $\mathcal{P}$, we take the infimum for both side and obtain:

$$
\mathcal{L}(\hat{u}_\phi) - \mathcal{L}(u^*) \leq \inf_{u \in \mathcal{P}} [\mathcal{L}(\hat{u}) - \mathcal{L}(u^*)] + \sup_{u \in \mathcal{P}} \|\mathcal{L}(u) - \hat{\mathcal{L}}(u)\|. 
$$

(2.5)
Since \( u^* \) is the strong solution of equation (2.1) and \( \mathcal{L}(u^*) = 0 \), then \( \forall u \in \mathcal{P} \) we have

\[
\mathcal{L}(u) - \mathcal{L}(u^*) = \int_{\Omega} \left( - \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu - f \right)^2 \, dx + \int_{\partial \Omega} (cu - g)^2 \, dx
\]

\[
= \int_{\Omega} \left( - \sum_{i,j=1}^{d} a_{ij} (u - u^*)_{x_i x_j} + \sum_{i=1}^{d} b_i (u - u^*)_{x_i} + c(u - u^*) \right)^2 \, dx + \int_{\partial \Omega} e^2 (u - u^*)^2 \, dx
\]

\[
\leq 3 \int_{\Omega} \left[ \left( \sum_{i,j=1}^{d} a_{ij} (u - u^*)_{x_i x_j} \right)^2 + \left( \sum_{i=1}^{d} b_i (u - u^*)_{x_i} \right)^2 + (c(u - u^*))^2 \right] \, dx
\]

\[
+ \int_{\partial \Omega} e^2 (u - u^*)^2 \, dx
\]

\[
\leq 3 \max \{2d^2 \mathcal{A}, d \mathcal{A}^2, \mathcal{C}^2 \} \|u - u^*\|^2_{H^2(\Omega)} + |\partial \Omega| \mathcal{E}^2 \|u - u^*\|^2_{C(\partial \Omega)}.
\]

On the other hand, by Lemma 2.1 and inequality \( \| \cdot \|_{H^\frac{3}{2}(\Omega)} \leq C \| \cdot \|_{L^2(\Omega)} \), we have \( \forall u \in H^\frac{3}{2}(\Omega) \cap L^2(\partial \Omega) \)

\[
\mathcal{L}(u) - \mathcal{L}(u^*) = \int_{\Omega} \left( - \sum_{i,j=1}^{d} a_{ij} (u - u^*)_{x_i x_j} + \sum_{i=1}^{d} b_i (u - u^*)_{x_i} + c(u - u^*) \right)^2 \, dx + \int_{\partial \Omega} e^2 (u - u^*)^2 \, dx
\]

\[
\geq C \|u - u^*\|^2_{H^\frac{3}{2}(\Omega)}.
\]

Combining (2.5)-(2.7) yields the result. \( \square \)

The approximation error \( \mathcal{E}_{app} \) describes the expressive power of the parameterized function class \( \mathcal{P} \) in \( H^2(\Omega) \) and \( C(\Omega) \) norm, which corresponds to the approximation error in FEM known as the Céa’s lemma [7]. The statistical error \( \mathcal{E}_{sta} \) is caused by the Monte Carlo discretization of \( \mathcal{L}(\cdot) \) defined in 2.2 with \( \tilde{\mathcal{L}}(\cdot) \) in 2.3.

3. Approximation Error. We will choose \( \mathcal{P} \) as a ReLU\(^3 \) networks, to ensure \( \mathcal{P} \subset H^2(\Omega) \cap C(\Omega) \). More precisely,

\[
\mathcal{P} = \mathcal{N}(\mathcal{D}, \mathcal{W}, \{ \| \cdot \|_{C^2(\Omega)}, \mathcal{B} \}, \{ \text{ReLU}^3 \}),
\]

where the hyper-parameters \( \mathcal{B} = 2\|u^*\|_{C^2(\Omega)} \) and \( \{ \mathcal{D}, \mathcal{W} \} \) will be given in later discussions (i.e., Theorem 5.1) to ensure the desired accuracy. The ReLU\(^3 \) network is refer to a neural network where the active function \( \sigma(x) \) is given by

\[
\sigma(x) = \begin{cases} 
 x^3, & x \geq 0, \\
 0, & \text{others.}
\end{cases}
\]

It can be seen that \( \sigma(x) \) is twice differentiable.

**Theorem 3.1.** Given \( \bar{u} \in C^3(\Omega) \) and for any \( \epsilon > 0 \), there exists a ReLU\(^3 \) network \( u \) with depth \( \lfloor \log_d d \rfloor + 2 \) and width \( C(d, \| \bar{u} \|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^d \) such that \( \| \bar{u} - u \|_{C^2(\Omega)} \leq \epsilon \).

**Proof.** A special case in [15]. \( \square \)
4. Statistical Error. In this section we give the statistical error with parameterized function class \( \mathcal{P} \), by establishing the Rademacher complexity of the non-Lipschitz composition of ReLU\(^3 \) network and its partial derivative. The technique used here may be helpful to analysis the statistical errors for other deep PDEs solvers, where the main difficulties are also to estimated the Rademacher complexity of non-Lipschitz composition induced by the derivative operator.

Firstly by carefully computation and triangle inequality, we have the following Lemma.

**Lemma 4.1.**

\[
E_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \hat{\mathcal{L}}(u)| \leq \sum_{j=1}^{13} E_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in \mathcal{P}} |\mathcal{L}_j(u) - \hat{\mathcal{L}}_j(u)|
\]

where

\[
\mathcal{L}_1(u) = |\Omega| E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) \right)^2,
\]

\[
\mathcal{L}_2(u) = |\Omega| E_{X \sim U(\Omega)} \sum_{i=1}^{d} b_i(X) u_{x_i}(X),
\]

\[
\mathcal{L}_3(u) = |\Omega| E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) \right) \cdot c(X) u(X),
\]

\[
\mathcal{L}_4(u) = |\Omega| E_{X \sim U(\Omega)} \left( \sum_{i=1}^{d} b_i(X) u_{x_i}(X) \right) \cdot c(X) u(X),
\]

\[
\mathcal{L}_5(u) = -2|\Omega| E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) \right) \cdot c(X) u(X),
\]

\[
\mathcal{L}_6(u) = -2|\Omega| E_{X \sim U(\Omega)} \left( \sum_{i=1}^{d} b_i(X) u_{x_i}(X) \right) \cdot c(X) u(X),
\]

\[
\mathcal{L}_7(u) = 2|\Omega| E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) \right) \cdot f(X),
\]

\[
\mathcal{L}_8(u) = 2|\Omega| E_{X \sim U(\Omega)} \left( \sum_{i=1}^{d} b_i(X) u_{x_i}(X) \right) \cdot c(X) u(X),
\]

\[
\mathcal{L}_9(u) = -2|\Omega| E_{X \sim U(\Omega)} \left( \sum_{i=1}^{d} b_i(X) u_{x_i}(X) \right) \cdot f(X),
\]

\[
\mathcal{L}_{10}(u) = -2|\Omega| E_{X \sim U(\Omega)} \left( c(X) u(X) f(X) \right),
\]

\[
\mathcal{L}_{11}(u) = |\partial \Omega| E_{Y \sim U(\partial \Omega)} \left( c(Y) u(Y) \right)^2,
\]

\[
\mathcal{L}_{12}(u) = |\partial \Omega| E_{Y \sim U(\partial \Omega)} \left( g(Y) \right)^2,
\]

and \( \hat{\mathcal{L}}_j(u) \) is the discrete sample version of \( \mathcal{L}_j(u) \) by replacing expectation with sample average.

4.1. Rademacher complexity, Covering Number and Pseudo-dimension. By the technique of symmetrization, we can bound the difference between continuous loss \( \mathcal{L}_i \) and empirical loss \( \hat{\mathcal{L}}_i \) via Rademacher complexity.

**Definition 4.2.** [31] The Rademacher complexity of a set \( A \subseteq \mathbb{R}^N \) is defined by

\[
\mathfrak{R}(A) = \mathbb{E}_{(\sigma_i)_{i=1}^{N}} \left[ \sup_{u \in A} \frac{1}{N} \sum_{k=1}^{N} \sigma_k a_k \right]
\]

where, \( \{\sigma_i\}_{i=1}^{N} \) are \( N \) i.i.d Rademacher variables with \( \mathbb{P}(\sigma_i = 1) = \mathbb{P}(\sigma_i = -1) = \frac{1}{2} \).

Let \( \Omega \) be a set and \( \mathcal{F} \) be a function class which maps \( \Omega \) to \( \mathbb{R} \). Let \( P \) be a probability distribution on \( \Omega \).
distribution over $\Omega$ and $\{X_k\}_{k=1}^N$ be i.i.d. samples from $P$. The Rademacher complexity of $F$ associated with distribution $P$ and sample size $N$ is defined by

$$\mathcal{R}_{P,N}(F) = \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \left[ \sup_{u \in F} \frac{1}{N} \sum_{k=1}^N \sigma_k w(X_k) \right].$$

For Rademacher complexity, we have the following structural result.

**Lemma 4.3.** Let $\Omega$ be a set and $P$ be a probability distribution over $\Omega$. Let $N \in \mathbb{N}$. Assume that $w : \Omega \to \mathbb{R}$ and $|w(x)| \leq B$ for all $x \in \Omega$, then for any function class $F$ mapping $\Omega$ to $\mathbb{R}$, there holds

$$\mathcal{R}_{P,N}(w(x)F) \leq B\mathcal{R}_{P,N}(F),$$

where $w(x)F := \{u : \bar{u}(x) = w(x)u(x), u \in F\}$.

**Proof.**

\[
\mathcal{R}_{P,N}(w(x)F) = \frac{1}{N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \sup_{u \in F} \sum_{k=1}^N \sigma_k w(X_k)u(X_k)
\]

\[
= \frac{1}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \mathbb{E}_{\{\epsilon_k\}_{k=1}^N} \sup_{u \in F} \left[ w(X_1)u(X_1) + \sum_{k=2}^N \sigma_k w(X_k)u(X_k) \right]
\]

\[
+ \frac{1}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \mathbb{E}_{\{\epsilon_k\}_{k=1}^N} \sup_{u \in F} \left[ -w(X_1)u(X_1) + \sum_{k=2}^N \sigma_k w(X_k)u(X_k) \right]
\]

\[
= \frac{1}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \mathbb{E}_{\{\epsilon_k\}_{k=1}^N} \sup_{u, u' \in F} \left[ w(X_1)[u(X_1) - u'(X_1)] + \sum_{k=2}^N \sigma_k w(X_k)u(X_k) + \sum_{k=2}^N \sigma_k w(X_k)u'(X_k) \right]
\]

\[
\leq \frac{1}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \mathbb{E}_{\{\epsilon_k\}_{k=1}^N} \sup_{u, u' \in F} \left[ B|u(X_1) - u'(X_1)| + \sum_{k=2}^N \sigma_k w(X_k)u(X_k) + \sum_{k=2}^N \sigma_k w(X_k)u'(X_k) \right]
\]

\[
= \frac{1}{2N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \mathbb{E}_{\{\epsilon_k\}_{k=1}^N} \sup_{u, u' \in F} \left[ B|u(X_1) - u'(X_1)| + \sum_{k=2}^N \sigma_k w(X_k)u(X_k) + \sum_{k=2}^N \sigma_k w(X_k)u'(X_k) \right]
\]

\[
\leq \cdots \leq \frac{B}{N} \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \sup_{u \in F} \left[ \sum_{k=1}^N \sigma_k u(X_k) \right] = B\mathcal{R}_{P,N}(F)
\]

\]

**Lemma 4.4.** Let $\{X_k\}_{k=1}^N$ be i.i.d. samples from $U(\Omega)$, then we have

$$\mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in F} \left| \tilde{L}_j(u) - \hat{L}_j(u) \right| \leq C_d \mathfrak{R}_{U(\Omega),N}(F_j), \quad j = 1, \ldots, 13,$$
where

\[ F_1 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i, j, j' \leq d \text{ s.t. } f(x) = u_{x,i}(x)u_{x,j}(x), \}\]

\[ F_2 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i, i' \leq d \text{ s.t. } f(x) = u_i(x)u_{i'}(x), \}\]

\[ F_3 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ s.t. } f(x) = u(x)^2, \}\]

\[ F_4 = \{ f : \Omega \to \mathbb{R} \mid -1, 0, 1, \}\]

\[ F_5 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i, j, i' \leq d \text{ s.t. } f(x) = u_{x,i}(x)u_{x,j}(x), \}\]

\[ F_6 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i, j \leq d \text{ s.t. } f(x) = u_{x,i}(x)u(x), \}\]

\[ F_7 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i, j \leq d \text{ s.t. } f(x) = u_{x,i}(x), \}\]

\[ F_8 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i \leq d \text{ s.t. } f(x) = u_i(x)u(x), \}\]

\[ F_9 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ and } 1 \leq i \leq d \text{ s.t. } f(x) = u_i(x), \}\]

\[ F_{10} = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ s.t. } f(x) = u(x), \}\]

\[ F_{11} = \{ f : \partial \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ s.t. } f = u^2|_{\partial \Omega}, \}\]

\[ F_{12} = \{ f : \partial \Omega \to \mathbb{R} \mid -1, 0, 1, \}\]

\[ F_{13} = \{ f : \partial \Omega \to \mathbb{R} \mid \exists u \in \mathcal{P} \text{ s.t. } f = u|_{\partial \Omega} \}\]

Proof. We only give the proof of \( \mathbb{E} \sum_{k=1}^{N} \sup_{u \in \mathcal{P}} |\mathcal{L}_1(u) - \mathcal{L}_1'(u)| \leq 4|\Omega|d^4\mathcal{R}(F_1) \) since other inequalities can be shown similarly. We take \( \{X_k\}_{k=1}^{N} \) as an independent copy of \( \{X_k\}_{k=1}^{N} \), then

\[
|\mathcal{L}_1(u) - \mathcal{L}_1'(u)| = |\Omega| \mathbb{E}_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X)u_{x,i}(X) \right)^2 - \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{i,j=1}^{d} a_{ij}(X_k)u_{x,i}(X_k) \right)^2
\]

\[
= |\Omega| \mathbb{E}_{X \sim U(\Omega)} \left( \sum_{i,j=1}^{d} a_{ij}(X)u_{x,i}(X) \right)^2 - \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{i,j=1}^{d} a_{ij}(X_k)u_{x,i}(X_k) \right)^2
\]
\[ \leq \frac{|\Omega|}{N} \mathbb{E}(X_k) \sum_{i, i', j, j'=1}^d \left| \sum_{k=1}^N a_{ij}(\overline{X}_k) a_{ij'}(\overline{X}_k) u_{x, x_j}(\overline{X}_k) u_{x_i, x_{j'}}(\overline{X}_k) \right| \]

\[ - a_{ij}(X_k) a_{ij'}(X_k) u_{x, x_j}(X_k) u_{x_i, x_{j'}}(X_k) \right|, \]

Hence

\[ \mathbb{E}(X_k) \sup_{u \in \mathcal{P}} \left| \mathcal{L}_1(u) - \mathcal{L}_1(u) \right| \]

\[ \leq \frac{|\Omega|}{N} \mathbb{E}(X_k) \left( \sum_{i, i', j, j'=1}^d \left| \sum_{k=1}^N a_{ij}(\overline{X}_k) a_{ij'}(\overline{X}_k) u_{x, x_j}(\overline{X}_k) u_{x_i, x_{j'}}(\overline{X}_k) \right| \right) \]

\[ \leq \frac{|\Omega|}{N} \mathbb{E}(X_k) \sum_{i, i', j, j'=1}^d \left| \sum_{k=1}^N a_{ij}(\overline{X}_k) a_{ij'}(\overline{X}_k) u_{x, x_j}(\overline{X}_k) u_{x_i, x_{j'}}(\overline{X}_k) \right| \]

\[ = \frac{|\Omega|}{N} \mathbb{E}(X_k) \left( \sum_{i, i', j, j'=1}^d \left| \sum_{k=1}^N a_{ij}(X_k) a_{ij'}(X_k) u_{x, x_j}(X_k) u_{x_i, x_{j'}}(X_k) \right| \right) \]

\[ \leq \frac{|\Omega|}{N} \mathbb{E}(X_k, \overline{X}_k, \sigma_k) \left( \sum_{i, i', j, j'=1}^d \left| \sum_{k=1}^N \sigma_k \left[ a_{ij}(\overline{X}_k) a_{ij'}(\overline{X}_k) u_{x, x_j}(\overline{X}_k) u_{x_i, x_{j'}}(\overline{X}_k) \right] \right| \right) \]

where the second and eighth steps are from Jensen’s inequality and Lemma 4.3, the third and seventh steps are due to the facts that the insertion of Rademacher variables doesn’t change the distribution and that \( \mathcal{F}_1 \) is symmetric (i.e., if \( f \in \mathcal{F}_1 \), then \( -f \in \mathcal{F}_1 \)), respectively.

Next we give an upper bound of Rademacher complexity in terms of the covering number of the corresponding function class.

**Definition 4.5.** [3] Suppose that \( W \subset \mathbb{R}^n \). For any \( \epsilon > 0 \), let \( V \subset \mathbb{R}^n \) be an \( \epsilon \)-cover of \( W \) with respect to the distance \( d_\infty \), that is, for any \( w \in W \), there exists a \( v \in V \) such that \( d_\infty(w, v) < \epsilon \), where \( d_\infty \) is defined by \( d_\infty(u, v) := \max_{1 \leq i \leq n} |u_i - v_i| \). The covering number
\( C(\epsilon, W, d_\infty) \) is defined to be the minimum cardinality among all \( \epsilon \)-cover of \( W \) with respect to the distance \( d_\infty \).

**Definition 4.6.** [3] Suppose that \( \mathcal{F} \) is a class of functions from \( \Omega \) to \( \mathbb{R} \). Given \( n \) sample \( Z_n = (Z_1, Z_2, \cdots, Z_n) \in \Omega^n \), \( \mathcal{F}|_{Z_n} \subset \mathbb{R}^n \) is defined by

\[
\mathcal{F}|_{Z_n} = \{(u(Z_1), u(Z_2), \cdots, u(Z_n)) : u \in \mathcal{N}^3\}.
\]

The uniform covering number \( C_\infty(\epsilon, \mathcal{F}, n) \) is defined by

\[
C_\infty(\epsilon, \mathcal{F}, n) = \max_{Z_n \in \Omega^n} C(\epsilon, \mathcal{F}|_{Z_n}, d_\infty).
\]

**Lemma 4.7.** Let \( \Omega \) be a set and \( P \) be a probability distribution over \( \Omega \). Let \( N \in \mathbb{N} \geq 1 \). Let \( \mathcal{F} \) be a class of functions from \( \Omega \) to \( \mathbb{R} \) such that \( 0 \in \mathcal{F} \) and the diameter of \( \mathcal{F} \) is less than \( B \), i.e., \( \|u\|_{L^\infty(\Omega)} \leq B \), \( \forall u \in \mathcal{F} \). Then

\[
R_{P,N}(\mathcal{F}) \leq \inf_{0 < \delta < B} \left( 4\delta + \frac{12}{\sqrt{N}} \sqrt{\log(2C_\infty(\epsilon, \mathcal{F}, N))} \right).
\]

**Proof.** The proof is based on the chaining method, see [31]. \( \square \)

By Lemma 4.7, we have to bound the covering number, which can be upper bounded via Pseudo-dimension [3].

**Definition 4.8.** Let \( \mathcal{F} \) be a class of functions from \( X \) to \( \mathbb{R} \). Suppose that \( S = \{x_1, x_2, \cdots, x_n\} \subset X \). We say that \( S \) is pseudo-shattered by \( \mathcal{F} \) if there exists \( y_1, y_2, \cdots, y_n \) such that for any \( b \in \{0, 1\}^n \), there exists a \( u \in \mathcal{F} \) satisfying

\[
sign(u(x_i) - y_i) = b_i, \quad i = 1, 2, \ldots, n
\]

and we say that \( \{y_i\}_{i=1}^n \) witnesses the shattering. The pseudo-dimension of \( \mathcal{F} \), denoted as \( \text{Pdim}(\mathcal{F}) \), is defined to be the maximum cardinality among all sets pseudo-shattered by \( \mathcal{F} \).

The following proposition showing a relation between uniform covering number and pseudo-dimension.

**Proposition 4.9 (Theorem 12.2 [3]).** Let \( \mathcal{F} \) be a class of real functions from a domain \( X \) to the bounded interval \([0, B]\). Let \( \epsilon > 0 \). Then

\[
C_\infty(\epsilon, \mathcal{F}, n) \leq \sum_{i=1}^{\text{Pdim}(\mathcal{F})} \binom{n}{i} \left( \frac{B^i}{\epsilon^i} \right),
\]

which is less than \( \left( \frac{enB}{\epsilon^{\text{Pdim}(\mathcal{F})}} \right)^{\text{Pdim}(\mathcal{F})} \) for \( n \geq \text{Pdim}(\mathcal{F}) \).

**4.2. Bound on Statistical error.** By Lemmas 4.1, 4.4, 4.7 and Proposition 4.9, we can bound the statistical error via bounding the pseudo-dimension of \( \mathcal{F}_i \), \( i = 1, \ldots, 13 \). To this end, we show that \( \{\mathcal{F}_i\} \) are subsets of some neural network classes and then bound the pseudo-dimension of associate neural network classes.
In later discussions, a “ReLU² – ReLU³ network” is a network with activation functions be either ReLU² or ReLU³, and other terminology such as “ReLU – ReLU² – ReLU³” are defined similarly.

Proposition 4.10. Let \( u \) be a function implemented by a ReLU² – ReLU³ (ReLU³) network with depth \( D \) and width \( W \). Then for \( i = 1, \ldots, d \), \( D_i u := \frac{\partial u}{\partial x_i} \), can be implemented by a ReLU – ReLU² – ReLU³ (ReLU² – ReLU³) network with depth \( D + 2 \) and width \((D + 2)W\). Moreover, the neural networks implementing \{\( D_i u \)\}₁≤\(i\)≤\(d\) have the same architecture.

Proof. For activation function \( \rho \), we denote \( \tilde{\rho} \) as its derivative, i.e., \( \tilde{\rho}(x) = \rho'(x) \). We then have
\[
\tilde{\rho}(x) = \begin{cases} 
2\text{ReLU}, & \rho = \text{ReLU}^2, \\
3\text{ReLU}^2, & \rho = \text{ReLU}^3.
\end{cases}
\]
Let \( 1 \leq i \leq d \). We deal with the first two layers in details and apply induction for layers \( k \geq 3 \) since there are a little bit difference for the first two layer. For the first layer, we have for any \( q = 1, 2, \ldots, n_1 \)
\[
D_i u_q^{(1)} = D_i \rho_q^{(1)} \left( \sum_{j=1}^{d} a_{qj}^{(1)} x_j + b_q^{(1)} \right) = \tilde{\rho}_q^{(1)} \left( \sum_{j=1}^{d} a_{qj}^{(1)} x_j + b_q^{(1)} \right) \cdot a_{qi}^{(1)}.
\]
Hence \( D_i u_q^{(1)} \) can be implemented by a ReLU – ReLU² – ReLU³ network with depth 2 and width 1. For the second layer,
\[
D_i u_q^{(2)} = D_i \rho_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) = \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \cdot \sum_{j=1}^{n_1} a_{qj}^{(2)} D_i u_j^{(1)}.
\]
Since \( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \) and \( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_i u_j^{(1)} \) can be implemented by two ReLU – ReLU² – ReLU³ subnetworks, respectively, and the multiplication can also be implemented by
\[
x \cdot y = \frac{1}{4} [(x + y)^2 - (x - y)^2] = \frac{1}{4} [\text{ReLU}^2(x + y) + \text{ReLU}^2(-x - y) - \text{ReLU}^2(x - y) - \text{ReLU}^2(-x + y)],
\]
we conclude that \( D_i u_q^{(2)} \) can be implemented by a ReLU – ReLU² – ReLU³ network. We have
\[
\mathcal{D} \left( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \right) = 3W \left( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \right) \leq W
\]
and
\[
\mathcal{D} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_i u_j^{(1)} \right) = 2W \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_i u_j^{(1)} \right) \leq W.
\]
Thus \( \mathcal{D} \left( D_i u_q^{(2)} \right) = 4, \ W \left( D_i u_q^{(2)} \right) \leq \max\{2W, 4\} \). Now we apply induction for layers \( k \geq
3. For the third layer, $D_i u_q^{(3)} = D_i \rho_q^{(3)} \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) = \tilde{\rho}_q^{(3)} \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) \cdot \sum_{j=1}^{n_k} a_{qj}^{(3)} D_i u_j^{(2)}$. Since $D \left( \rho_q^{(3)} \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) \right) = 4, W \left( \rho_q^{(3)} \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) \right) \leq W$ and

$$D \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} D_i u_j^{(2)} \right) = 4, W \left( \sum_{j=1}^{n_k} a_{qj}^{(3)} D_i u_j^{(2)} \right) \leq \max \{2W, 4W\} = 4W,$$

we conclude that $D_i u_q^{(3)}$ can be implemented by a ReLU – ReLU^2 – ReLU^3 network and $D \left( D_i u_q^{(3)} \right) = 5, W \left( D_i u_q^{(3)} \right) \leq \max \{5W, 4\} = 5W$.

We assume that $D_i u_q^{(k)} (q = 1, 2, \ldots, n_k)$ can be implemented by a ReLU-ReLU^2 network and $D \left( D_i u_q^{(k)} \right) = k + 2, W \left( D_i u_q^{(k)} \right) \leq (k + 2)W$. For the $(k + 1)$-th layer, $D_i u_q^{(k+1)} = D_i \rho_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} u_j^{(k)} + b_q^{(k+1)} \right) = \tilde{\rho}_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} u_j^{(k)} + b_q^{(k+1)} \right) \cdot \sum_{j=1}^{n_k} a_{qj}^{(k+1)} D_i u_j^{(k)}$.

Since $D \left( \rho_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} u_j^{(k)} + b_q^{(k+1)} \right) \right) = k+2, W \left( \rho_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} u_j^{(k)} + b_q^{(k+1)} \right) \right) \leq W$ and $D \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} D_i u_j^{(k)} \right) = k + 2, W \left( \sum_{j=1}^{n_k} a_{qj}^{(k+1)} D_i u_j^{(k)} \right) \leq \max \{(k+2)W, 4W\} = (k+2)W$, we conclude that $D_i u_q^{(k+1)}$ can be implemented by a ReLU – ReLU^2 – ReLU^3 network and $D \left( D_i u_q^{(k+1)} \right) = k + 3, W \left( D_i u_q^{(k+1)} \right) \leq \max \{(k+3)W, 4\} = (k+3)W$. Hence we derive that $D_i u = D_i u_1^{d}$ can be implemented by a ReLU – ReLU^2 – ReLU^3 network and $D \left( D_i u \right) = D + 2, W \left( D_i u \right) \leq (D + 2)W$. And through our argument, we know that the neural networks implementing $\{D_i u_q\}_{i=1}^d$ have the same architecture.

We now present the bound for pseudo-dimension of $\mathcal{N}(D, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\})$ with $D, W \in \mathbb{N}$. We need the following Lemma.

**Lemma 4.11. (Theorem 8.3 in [3])** Let $p_1, \ldots, p_m$ be polynomials with $n$ variables of degree at most $d$. If $n \leq m$, then

$$\left| \{ \langle \text{sign}(p_1(x)), \ldots, \text{sign}(p_m(x)) \rangle : x \in \mathbb{R}^n \} \right| \leq 2 \frac{2emd}{n}^n.$$ 

**Proposition 4.12.** For any $D, W \in \mathbb{N}$,

$$\text{Pdim}(\mathcal{N}(D, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\})) = \mathcal{O}(D^2W^2(D + \log W)).$$

**Proof.** The argument is follows from the proof of Theorem 6 in [4]. The result stated here is somewhat stronger than Theorem 6 in [4] since VCdim(sign($\mathcal{F}$)) $\leq$ Pdim($\mathcal{F}$) for any function class $\mathcal{F}$. We consider a new set of functions

$$\tilde{\mathcal{N}} = \{ \tilde{u}(x, y) = \text{sign}(u(x) - y) : u \in \mathcal{N}(D, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\}) \}.$$ 

It is clear that Pdim($\mathcal{N}(D, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\}) \leq \text{VCdim}(\tilde{\mathcal{N}})$. We now bound the VC-dimension of $\tilde{\mathcal{N}}$. Denoting $\mathcal{M}$ as the total number of parameters (weights and biases) in the neural network implementing functions in $\mathcal{N}$, in our case we want to derive the uniform
bound for

\[ K_{\{x_i\},\{y_i\}}(m) := \left| \{ (\text{sign}(f(x_1, a) - y_1), \ldots, \text{sign}(u(x_m, a) - y_m)) : a \in \mathbb{R}^M \} \right| \]

over all \( \{x_i\}_{i=1}^m \subset X \) and \( \{y_i\}_{i=1}^m \subset \mathbb{R} \). Actually the maximum of \( K_{\{x_i\},\{y_i\}}(m) \) over all \( \{x_i\}_{i=1}^m \subset X \) and \( \{y_i\}_{i=1}^m \subset \mathbb{R} \) is the growth function \( \mathcal{G}_N(m) \). In order to apply Lemma 4.11, we partition the parameter space \( \mathbb{R}^M \) into several subsets to ensure that in each subset \( u(x_i, a) - y_i \) is a polynomial with respect to \( a \) without any breakpoints. In fact, our partition is exactly the same as the partition in [4]. Denote the partition as \( \{P_1, P_2, \cdots, P_N\} \) with some integer \( N \) satisfying

\[ N \leq \prod_{i=1}^{D-1} 2 \left( \frac{2emk_i(1 + (i-1)3^{i-1})}{M_i} \right)^{M_i} \tag{4.2} \]

where \( k_i \) and \( M_i \) denotes the number of units at the \( i \)th layer and the total number of parameters at the inputs to units in all the layers up to layer \( i \) of the neural network implementing functions in \( \mathcal{N} \), respectively. See [4] for the construction of the partition. Obviously we have

\[ K_{\{x_i\},\{y_i\}}(m) \leq \sum_{i=1}^{N} \left| \{ (\text{sign}(u(x_1, a) - y_1), \cdots, \text{sign}(u(x_m, a) - y_m)) : a \in P_i \} \right| \tag{4.3} \]

Note that \( u(x_i, a) - y_i \) is a polynomial with respect to \( a \) with degree the same as the degree of \( u(x_i, a) \), which is equal to \( 1 + (D-1)3^{D-1} \) as shown in [4]. Hence by Lemma 4.11, we have

\[ \left| \{ (\text{sign}(u(x_1, a) - y_1), \cdots, \text{sign}(u(x_m, a) - y_m)) : a \in P_i \} \right| \leq 2 \left( \frac{2em(1 + (D-1)3^{D-1})}{M_D} \right)^{M_D} \tag{4.4} \]

Combining (4.2), (4.3) and (4.4) yields

\[ K_{\{x_i\},\{y_i\}}(m) \leq \prod_{i=1}^{D} 2 \left( \frac{2emk_i(1 + (i-1)3^{i-1})}{M_i} \right)^{M_i} \]

We then have

\[ \mathcal{G}_N(m) \leq \prod_{i=1}^{D} 2 \left( \frac{2emk_i(1 + (i-1)3^{i-1})}{M_i} \right)^{M_i} \]

since the maximum of \( K_{\{x_i\},\{y_i\}}(m) \) over all \( \{x_i\}_{i=1}^m \subset X \) and \( \{y_i\}_{i=1}^m \subset \mathbb{R} \) is the growth function \( \mathcal{G}_N(m) \). Doing some algebras as that of the proof of Theorem 6 in [4], we obtain

\[ \text{Pdim}(\mathcal{N}(D, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\})) \leq O(D^2 W^2 (D + \log W)) \]

With the above preparations, we are able to derive our result on the statistical error.
Theorem 4.13. Let $D, W \in \mathbb{N}, B \in \mathbb{R}^+$. For any $\epsilon > 0$, if the number of sample

$$N, M = C(d, \Omega, 3, B)D^6W^2(D + \log W)\left(\frac{1}{\epsilon}\right)^{2+\delta}$$

where $\delta$ is an arbitrarily small number then we have

$$\mathbb{E}_{(X_k)_{k=1}}\left|\sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \tilde{\mathcal{L}}(u)| \right| \leq \epsilon,$$

where $\mathcal{P} = \mathcal{N}\{D, W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \{\text{ReLU}^3\}\}$.

Proof. We need the following Lemma. Let $\Phi = \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\}$.

Lemma 4.14. Let $\{F_i\}_{i=1}^{13}$ be defined in Lemma 4.4. There holds

$$\begin{align*}
F_1 \subset N_1 &:= \mathcal{N}\{D + 5, 2(D + 2)(D + 4)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_2 \subset N_2 &:= \mathcal{N}\{D + 3, 2(D + 2)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_3 \subset N_3 &:= \mathcal{N}\{D + 4, (D + 2)(D + 4)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\} \\
F_4 \subset N_4 &:= \mathcal{N}\{D + 5, (D + 2)(D + 5)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_5 \subset N_5 &:= \mathcal{N}\{D + 3, (D + 3)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_6 \subset N_6 &:= \mathcal{N}\{D + 4, (D + 2)(D + 4)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_7 \subset N_7 &:= \mathcal{N}\{D + 5, (D + 2)(D + 5)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_8 \subset N_8 &:= \mathcal{N}\{D + 2, (D + 2)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\} \\
F_9 \subset N_9 &:= \mathcal{N}\{D + 3, (D + 3)W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\} \\
F_{10} \subset N_{10} &:= \mathcal{N}\{D, W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\} \\
F_{11} \subset N_{11} &:= \mathcal{N}\{D + 1, W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B^2\}, \Phi\} \\
F_{12} \subset N_{12} &:= \mathcal{N}\{D, W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\} \\
F_{13} \subset N_{13} &:= \mathcal{N}\{D, W, \{\| \cdot \|_{C^2(\bar{\Omega})}, B\}, \Phi\}.
\end{align*}$$

Proof. By Proposition 4.10, we know that for $u$ is a ReLU$^3$ network with depth $D$ and width $W$, $u_{x_i}$ can be implemented by a ReLU$^2$ – ReLU$^3$ network with depth $D + 2$ and width $(D + 2)W$. Then by Proposition 4.10 again we have that $u_{x_i x_j}$ can be implemented by a ReLU$^2$ – ReLU$^3$ network with depth $D + 4$ and width $(D + 2)(D + 4)W$. These facts combining with (4.1) yields the results. \[\Box\]

By Lemma 4.7 and Proposition 4.9, we have for $i = 1, 2, ..., 10$,

$$\begin{align*}
\mathfrak{R}_{U(\Omega), N}(F_i) &\leq \frac{4\delta + 12}{\sqrt{N}} \int_{\delta}^{B_i} \sqrt{\log(2C(\epsilon, F_i, N))} \left(\frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)}\right)^{\text{Pdim}(F_i)} \frac{de}{\epsilon} \tag{4.5}
\end{align*}$$

Now we calculate the integral. Let $t = \sqrt{\log(\frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)})}$, then $\epsilon = \frac{eNB_i}{\text{Pdim}(F_i)} \cdot e^{-t^2}$. Denoting
t_1 = \sqrt{\log(\frac{eNB_i}{B_i \cdot \text{Pdim}(F_i)})}, \quad t_2 = \sqrt{\log(\frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)})}, \text{ we have}
\int_{B_i} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)} \, de = \frac{2eNB_i}{\text{Pdim}(F_i)} \int_{t_1}^{t_2} t^2 e^{-t^2} \, dt
= eNB_i \text{Pdim}(F_i) \left[ t_1 e^{-t_1^2} - t_2 e^{-t_2^2} + \int_{t_1}^{t_2} e^{-t^2} \, dt \right]
\leq eNB_i \text{Pdim}(F_i) \left[ t_1 e^{-t_1^2} - t_2 e^{-t_2^2} + (t_2 - t_1)e^{-t_1^2} \right]
\leq eNB_i \cdot t_2 e^{-t_1^2} = \|u\| \sqrt{\log \left( \frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)} \right)} \quad (4.6)

Combining (4.5) and (4.6) and choosing \( \delta = B_i \left( \frac{\text{Pdim}(F_i)}{N} \right)^{1/2} \leq B_i \), we get for \( i = 1, 2, 3, 5, 6, 7, 8, 9, 10 \),
\[ \mathcal{R}_{U(\Omega), N}(F_i) \leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12B_i}{\sqrt{N}} + \frac{12}{\sqrt{N}} \int_{B_i} \sqrt{\text{Pdim}(F_i) \cdot \log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)} \, de \right) \]
\[ \leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12B_i}{\sqrt{N}} + \frac{12B_i \text{Pdim}(F_i)}{\sqrt{N}} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)} \right) \]
\[ \leq 28 \left( \frac{3}{2} B_i \frac{\text{Pdim}(F_i)}{N} \right)^{1/2} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)} \quad (4.7) \]
\[ \leq 28 \left( \frac{3}{2} B_i \frac{\text{Pdim}(N)}{N} \right)^{1/2} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(N)} \right)} \]
\[ \leq 28 \left( \frac{3}{2} \max\{B, B^2\} \frac{\mathcal{H}}{N} \right)^{1/2} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)}, \quad (4.8) \]

with
\[ \mathcal{H} = 4C(D + 2)^2(D + 4)^2(D + 5)^2W^2(D + 5 + \log((D + 2)(D + 4)W)). \]

where in the forth step we apply Lemma 4.14 and we use Proposition 4.12 in the last step. Similarly for \( i = 11, 13 \),
\[ \mathcal{R}_{U(\Omega), N}(F_i) \leq 28 \sqrt{\frac{3}{2} \max\{B, B^2\} \left( \frac{\mathcal{H}}{M} \right)^{1/2} \sqrt{\log \left( \frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)} \right)}} \quad (4.8) \]

Obviously, \( \mathcal{R}_{U(\Omega), N}(F_9) \) and \( \mathcal{R}_{U(\Omega), N}(F_{12}) \) can be bounded by the right hand side of (4.7) and (4.8), respectively. Combining Lemma 4.1 and 4.4 and (4.7) and (4.8), we have
\[ \mathbb{E}_{(x_k)_{k=1}^N, (\gamma_k)_{k=1}^M} \sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \hat{\mathcal{L}}(u)| \leq 13 \mathbb{E}_{(x_k)_{k=1}^N, (\gamma_k)_{k=1}^M} \sup_{u \in \mathcal{P}} |\mathcal{L}_i(u) - \hat{\mathcal{L}}_i(u)| \]
\[ \leq 28 \sqrt{\frac{3}{2} \max\{B, B^2\} \left( 40\Omega C_1 \left( \frac{\mathcal{H}}{N} \right)^{1/2} \sqrt{\log \left( \frac{eN}{\mathcal{H}} \right)} + 12\sqrt{\Omega} C_2 \left( \frac{\mathcal{H}}{M} \right)^{1/2} \sqrt{\log \left( \frac{eM}{\mathcal{H}} \right)} \right)} \]
where
\[
C_1 = \max\{A^2d^4, B^2d^2, C^2d^2, ABD, ACd^2, AFd, BDd, C^4\},
\]
\[
C_2 = \max\{C^2, D^2, E^4\}.
\]

Hence for any \( \epsilon > 0 \), if the number of sample
\[
N, M = C(d, \Omega, \mathcal{B})D^6W^2(D + \log W) \left( \frac{1}{\epsilon} \right)^{2+\delta}
\]
with \( \delta \) being an arbitrarily small number, then we have
\[
\mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in \mathcal{P}} |L(u) - \hat{L}(u)| \leq \epsilon.
\]

5. Convergence rate for the PINNs. With the preparation in last two sections on the bounds of approximation and statistical errors, we will give the main results in this section.

Theorem 5.1. Let Assumption 2.1 holds true and further assume \( u^* \in C^3(\Omega) \). For any \( \epsilon > 0 \), we choose the parameterized neural network class
\[
\mathcal{P} = \mathcal{N}' \left( \lceil \log_2 d \rceil + 2, C(d, \Omega, 3, \|u^*\|_{C^3(\Omega)}) \left( \frac{1}{\epsilon} \right)^d, \{\| \cdot \|_{C^3(\Omega)}, 2\| u^* \|_{C^3(\Omega)}\}, \{\text{ReLU}^3\} \right)
\]
and let the number of samples be
\[
N, M = C(d, \Omega, 3, \|u^*\|_{C^3(\Omega)}) \left( \frac{1}{\epsilon} \right)^{2d+4+\delta},
\]
where \( \delta \) is an arbitrarily positive number, then we have
\[
\mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \|u_\phi - u^*\|_{H^4(\Omega)} \leq \epsilon.
\]

Proof. For any \( \epsilon > 0 \), by Theorem 3.1, there exists an neural network function \( \bar{u} \) with depth \( \lfloor \log_2 d \rfloor + 2 \) and width \( C(d, \Omega, 3, \|u^*\|_{C^3(\Omega)}) \left( \frac{1}{\epsilon} \right)^d \) such that
\[
\| u^* - \bar{u}\|_{C^2(\Omega)} \leq \left( \frac{\epsilon^2}{3C(d^2 + 3d + 4)|\Omega| \max\{2d^2, d^2, C^2\} + 2C|\partial \Omega| C^2} \right)^{1/2}.
\]

Without loss of generality we assume that \( \epsilon \) is small enough such that
\[
\| \bar{u}\|_{C^2(\Omega)} \leq \| u^* - \bar{u}\|_{C^2(\Omega)} + \| u^*\|_{C^2(\Omega)} \leq 2\| u^*\|_{C^2(\Omega)}.
\]
Hence \( \bar{u} \) belongs to the function class
\[
\mathcal{P} = \mathcal{N}' \left( \lceil \log_2 d \rceil + 2, C(d, \Omega, 3, \|u^*\|_{C^3(\Omega)}) \left( \frac{1}{\epsilon} \right)^d, \{\| \cdot \|_{C^3(\Omega)}, 2\| u^* \|_{C^3(\Omega)}\}, \{\text{ReLU}^3\} \right).
\]
And
\[
E_{\text{app}} \leq \frac{3}{2} (d^2 + 3d + 4) |\Omega| \max \{2d^2 \mathbb{A}^2, d \mathbb{A}^2, \mathbb{E}^2 \} \| \hat{u} - u^* \|_{C^2(\Omega)}^2 + |\partial \Omega| \| \mathbb{E}^2 \| \| \hat{u} - u^* \|_{C^2(\Omega)}^2 \leq \frac{\epsilon^2}{2C}. \tag{5.1}
\]
By Theorem 4.13, when the number of samples
\[
N, M = C(d, \Omega, 3, \mathcal{B}) D^6 W^2 (D + \log W) \left( \frac{1}{\epsilon} \right)^{4+\delta} = C(d, \Omega, 3, \| u^* \|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^{2d+4+\delta}
\]
with \( \delta \) being an arbitrarily positive number, we have
\[
E_{\text{sta}} = \mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in P} \left| \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right| \leq \frac{\epsilon^2}{2C}. \tag{5.2}
\]
Combining Proposition 2.2, (5.1) and (5.2) yields the result. \( \square \)

**Remark 5.1.** The asymptotic convergence results for PINNs have been studied in [26, 27, 21], i.e., when the number of parameters in the neural networks and number of training samples go to infinity, the solution of PINNs with converges to the solution of the PDEs’. Our work establishes a nonasymptotic convergence rate of PINNs. According to our results in Theorem 5.1, the influence of the depth and width in the neural networks and number of training samples are characterized quantitatively. It gives an answer on how to choose the hyperparameters to archive the desired accuracy, which is missing in [26, 27, 21].

**Remark 5.2.** As shown in Theorem 5.1, the PINNs suffers from the curse of dimensionality. However, if no additional smoothness or low-dimensional compositional structure on the underlying solutions of PDEs are imposed, the curse is unavoidable. Recently, the minimax lower bound for PINNs are proved in [18], where they showed that to achieve error of \( \epsilon \) for PINNs to solve elliptical equation defined on \([0,1]^d\) whose solution lives in \(H^1([0,1]^d)\), the number of samples \( n \) are lower bounded by \(\mathcal{O}((1/\epsilon)^d)\). In [19], the authors reduce the curses by assuming the underlying solutions living in Barron spaces which is much smaller than \(H^1\). One can also utilize the structures of the solution to reduce the curse, for example by considering PDEs whose solution are composition of functions with number of variables much smaller than \(d\).

**6. Conclusion.** This paper provided an analysis of convergence rate for PINNs. Our results give a way about how to set depth and width of networks to achieve the desired convergence rate in terms of number of training samples. The estimation on the approximation error of deep ReLU\(^3\) network is established in \(C^2\) norms. The statistical error can be derived technically by the Rademacher complexity of the non-Lipschitz composition with ReLU\(^3\) network. It is interesting to extend the current analysis of PINNs to PDEs with other boundary conditions or the optimal control (inverse) problems.

**Acknowledgments.** The authors would like to thank the anonymous referees for their useful suggestions that help improve the manuscript.

**REFERENCES**
[1] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Communications on pure and applied mathematics, 12 (1959), pp. 623–727.

[2] C. Antitescu, E. Atroschenko, N. Alajlan, and T. Rabczuk, Artificial neural network methods for the solution of second order boundary value problems, Cmc-computers Materials & Continua, 59 (2019), pp. 345–359.

[3] M. Anthony and P. L. Bartlett, Neural network learning: Theoretical foundations, Cambridge University press, 2009.

[4] P. L. Bartlett, N. Harvey, C. Liaw, and A. Mehrabian, Nearly-tight vc-dimension and pseudo-dimension bounds for piecewise linear neural networks., Journal of Machine Learning Research, 20 (2019), pp. 1–17.

[5] J. Berner, M. Dablander, and P. Grohs, Numerically solving parametric families of high-dimensional kolmogorov partial differential equations via deep learning, in Advances in Neural Information Processing Systems, vol. 33, Curran Associates, Inc., 2020, pp. 16615–16627.

[6] S. Brenner and R. Scott, The mathematical theory of finite element methods, vol. 15, Springer Science & Business Media, 2007.

[7] P. G. Ciarlet, The finite element method for elliptic problems, SIAM, 2002.

[8] C. Duan, Y. Jiao, Y. Lai, X. Lu, and Z. Yang, Convergence rate analysis for deep ritz method, Communications in Computational Physics, (2022).

[9] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, Generative adversarial networks, Advances in Neural Information Processing Systems, 3 (2014).

[10] J. Han, A. Jentzen, and E. Weinan, Solving high-dimensional partial differential equations using deep learning, Proceedings of the National Academy of Sciences, 115 (2018), pp. 8505–8510.

[11] K. He, X. Zhang, S. Ren, and J. Sun, Delving deep into rectifiers: Surpassing human-level performance on imagenet classification, in Proceedings of the IEEE international conference on computer vision, 2015, pp. 1026–1034.

[12] Q. Hong, J. W. Siegel, and J. Xu, Rademacher complexity and numerical quadrature analysis of stable neural networks with applications to numerical pdes, arXiv preprint arXiv:2104.02903, (2021).

[13] T. J. Hughes, The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Courier Corporation, 2012.

[14] A. D. Jagtap, E. Kharazmi, and G. E. Karniadakis, Conservative physics-informed neural networks on discrete domains for conservation laws: Applications to forward and inverse problems, Computer Methods in Applied Mechanics and Engineering, 365 (2020), p. 113028.

[15] Y. Jiao, L. Kang, Y. Lai, and Y. Xu, Approximation of deep ReLU^k neural network in sobolev spaces, (2022).

[16] I. E. Lagaris, A. Likas, and D. I. Fotiadis, Artificial neural networks for solving ordinary and partial differential equations, IEEE transactions on neural networks, 9 (1998), pp. 987–1000.

[17] L. Lu, X. Meng, Z. Mao, and G. E. Karniadakis, Deepxde: A deep learning library for solving differential equations, SIAM Review, 63 (2021), pp. 208–228.

[18] Y. Lu, H. Chen, J. Lu, L. Ying, and J. Blanchet, Machine learning for elliptic pdes: Fast rate generalization bound, neural scaling law and minimax optimality, in ICLR, 2021.

[19] Y. Lu, J. Lu, and M. Wang, A priori generalization analysis of the deep ritz method for solving high dimensional elliptic partial differential equations, in Conference on Learning Theory, PMLR, 2021, pp. 3196–3241.

[20] T. Luo and H. Yang, Two-layer neural networks for partial differential equations: Optimization and generalization theory, arXiv preprint arXiv:2006.15733, (2020).

[21] S. Mishra and R. Molinaro, Estimates on the generalization error of physics-informed neural networks for approximating a class of inverse problems for pdes, IMA Journal of Numerical Analysis, (2021).

[22] G. Pang, M. D’Elia, M. Parks, and G. Karniadakis, npinns: Nonlocal physics-informed neural networks for a parametrized nonlocal universal laplacian operator. algorithms and applications, Journal of Computational Physics, 422 (2020), p. 109760.

[23] G. Pang, L. Lu, and G. E. Karniadakis, fpinns: Fractional physics-informed neural networks, SIAM Journal on Scientific Computing, 41 (2019), pp. A2603–A2626.
[24] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, vol. 23, Springer Science & Business Media, 2008.

[25] M. Raissi, P. Perdikaris, and G. E. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational Physics, 378 (2019), pp. 686–707.

[26] Y. Shin, J. Darbon, and G. E. Karniadakis, *On the convergence of physics informed neural networks for linear second-order elliptic and parabolic type pdes*, arXiv preprint arXiv:2004.01806, (2020).

[27] Y. Shin, Z. Zhang, and G. E. Karniadakis, *Error estimates of residual minimization using neural networks for linear pdes*, arXiv preprint arXiv:2010.08019, (2020).

[28] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. Van Den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, et al., *Mastering the game of go with deep neural networks and tree search*, nature, 529 (2016), pp. 484–489.

[29] J. A. Sirignano and K. Spiliopoulos, *Dgm: A deep learning algorithm for solving partial differential equations*, Journal of Computational Physics, 375 (2018), pp. 1339–1364.

[30] J. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, vol. 22, Springer Science & Business Media, 2013.

[31] A. W. Van Der Vaart and J. A. Wellner, *Weak convergence*, in Weak convergence and empirical processes, Springer, 1996.

[32] E. Weinan and B. Yu, *The deep ritz method: A deep learning-based numerical algorithm for solving variational problems*, Communications in Mathematics and Statistics, 6 (2017), pp. 1–12.

[33] Y. Zang, G. Bao, X. Ye, and H. Zhou, *Weak adversarial networks for high-dimensional partial differential equations*, Journal of Computational Physics, 411 (2020), p. 109409.