Determinant structure for $\tau$-function of holonomic deformation of linear differential equations

Masao Ishikawa, Toshiyuki Mano and Teruhisa Tsuda

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Abstract

In our previous works [15, 16], a relationship between Hermite’s two approximation problems and Schlesinger transformations of linear differential equations has been clarified. In this paper, we study $\tau$-functions associated with holonomic deformations of linear differential equations by using Hermite’s two approximation problems. As a result, we present a determinant formula for the ratio of $\tau$-functions ($\tau$-quotient).

1 Introduction

There are many results concerning determinant formulas for solutions to the Painlevé equations; see [8, 9, 10, 17, 23, 24] and references therein. After pioneering works by D. Chudnovsky and G. Chudnovsky [1, 2], an underlying relationship between the theory of rational approximation for functions and the Painlevé equations has been clarified by several authors [12, 14, 15, 16, 27]. This relationship provides a natural explanation for the determinant structure of solutions to the Painlevé equations.

Among them, the second and third authors of this paper studied the relationship between two approximation problems by Hermite (i.e. the Hermite–Padé approximation and the simultaneous Padé approximation) and isomonodromic deformations of Fuchsian linear differential equations. They constructed a class of Schlesinger transformations for Fuchsian linear differential equations using Hermite’s two approximation problems and a duality between them. As an application, they obtained particular solutions written in terms of iterated hypergeometric integrals to the higher-dimensional Hamiltonian systems of Painlevé type (that were introduced in [25]). For details refer to [15, 16].

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In the present paper, we study using Hermite’s two approximation problems the determinantal structure for \(\tau\)-functions of holonomic deformations of linear differential equations which have regular or irregular singularities of arbitrary Poincaré rank. The main theorem (Theorem 6.2) is stated as follows: fix an integer \(L \geq 2\) and consider a system of linear differential equations of rank \(L\)
\[
\frac{dY}{dx} = \left( \sum_{\mu=1}^{N} \sum_{j=0}^{r_{\mu}} A_{\mu,-j}(x-a_{\mu})^{-j-1} - \sum_{j=1}^{r_{\infty}} A_{\infty,-j}x^{j-1} \right) Y, \tag{1.1}
\]
where \(A_{\mu,-j}\) and \(A_{\infty,-j}\) are \(L \times L\) matrices independent of \(x\). Let \(\tau_{0}\) be Jimbo–Miwa–Ueno’s \(\tau\)-function (see (4.5) and (4.6)) associated with a holonomic deformation of (1.1). We apply the Schlesinger transformation to (1.1) that shifts the characteristic exponents at \(x = \infty\) by

\[
n = ((L-1)n, -n, \ldots, -n) \in \mathbb{Z}^L
\]

for a positive integer \(n\). Let \(\tau_{n}\) denote the \(\tau\)-function associated with the resulting equation. Then the ratio \(\tau_{n}/\tau_{0}\) (\(\tau\)-quotient) admits a representation in terms of an \((L-1)n \times (L-1)n\) block-Toeplitz determinant:

\[
\frac{\tau_{n}}{\tau_{0}} = \text{const.} D_{n}, \quad D_{n} = \left| B_{1}^{n}((L-1)n,n) \cdots B_{L-1}^{n}((L-1)n,n) \right| \tag{1.2}
\]

with \(B_{i}^{n}(k,l)\) being a \(k \times l\) rectangular Toeplitz matrix (see (3.4)) whose entries are specified by the asymptotic expansion of a fundamental system of solutions to (1.1) around \(x = \infty\). It should be noted that our result is valid for general solutions not only for particular solutions such as rational solutions or Riccati solutions.

This paper is organized as follows. In Section 2 we review Hermite’s two approximation problems and a certain duality between them. This duality due to Mahler [13] will be a key point for the construction of Schlesinger transformations in a later section. We remark that the normalization in this paper is slightly different from that in the previous ones [15, 16]. Therefore, we formulate the two approximation problems in a form suitable to the present case. In Section 3 we give determinant representations for the approximation polynomials and the remainder of the approximation problems. In our method, these representations turn out to be the nature of the determinant structure of the \(\tau\)-quotient. In Section 4 we briefly review the theory of holonomic deformation of a linear differential equation following [6, 7]. In Section 5 we construct the Schlesinger transformations of linear differential equations by applying the approximation problems. Section 6 is the main part of this paper. We present the determinant formula for the \(\tau\)-quotient (see (1.2) or Theorem 6.2) based on the coincidence between the Schlesinger transformations and the approximation problems. A certain determinant identity (see (6.7)) plays a crucial role in the proof. Section 7 is devoted to an application of our result. We demonstrate
how to construct particular solutions to the holonomic deformation equations such as the Painlevé equations. We then find some inclusion relations among solutions to holonomic deformations and, typically, obtain a natural understanding of the determinant formulas for hypergeometric solutions to holonomic deformations. In Appendix A we give a proof of the determinant identity applied in the proof of Theorem 6.2. Though this determinant identity can be proved directly, we will prove its Pfaffian analogue in a general setting and then reduce it to the determinant case in order to simplify the proof and to enjoy better perspectives.

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2 Hermite–Padé approximation and simultaneous Padé approximation

In this section, we review Hermite’s two approximation problems in a suitable form, which will be utilized to construct Schlesinger transformations for linear differential equations in a later section.

Let $L$ be an integer larger than one. Given a set of $L$ formal power series

$$f_0(w), f_1(w), \ldots, f_{L-1}(w) \in \mathbb{C}[\lbrack w \rbrack]$$

with the conditions

$$f_0(0) = 1, \quad f_i(0) = 0 \quad (i \neq 0) \quad (2.1)$$

the Hermite–Padé approximation is formulated as follows: find $L^2$ polynomials

$$Q_j^{(i)}(w) \in \mathbb{C}[w] \quad (0 \leq i, j \leq L - 1)$$

such that

$$\deg Q_j^{(i)}(w) \leq n - 1 + \delta_{i,j}, \quad (2.2)$$

$$Q_j^{(i)}(w)f_i(w) + \sum_{j \neq i} wQ_j^{(i)}(w)f_j(w) = w^{Ln}(\delta_{i,0} + O(w)), \quad (2.3)$$

$$Q_j^{(i)}(0) = 1 \quad (i \neq 0). \quad (2.4)$$

There exists a unique set of $L^2$ polynomials $\{Q_j^{(i)}(w)\}$ under a certain generic condition on the coefficients of $f_i(w)$. The precise condition will be later stated in terms of non-vanishing of some block-Toeplitz determinants; see (3.8) in Section 3.
In turn, the **simultaneous Padé approximation** is formulated as follows: find $L^2$ polynomials

$$P_j^{(i)}(w) \in \mathbb{C}[w] \quad (0 \leq i, j \leq L - 1)$$

such that

$$\deg P_j^{(i)}(w) \leq n(L - 1) - 1 + \delta_{i,j}, \quad \text{(2.5)}$$

$$f_0(w)P_j^{(i)}(w) - f_j(w)w^{1-\delta_{i,j}}P_0^{(i)}(w) = O(w^{nL}). \quad \text{(2.6)}$$

Under the same generic condition as above, for each $i$ the polynomials $P_j^{(i)}(w)$ ($0 \leq j \leq L - 1$) are uniquely determined up to simultaneous multiplication by constants.

Interestingly enough, these two approximations are in a dual relation; cf. [13].

**Theorem 2.1** (Mahler’s duality). Let $\{Q_j^{(i)}(w)\}$ and $\{P_j^{(i)}(w)\}$ be the Hermite–Padé approximant and the simultaneous Padé approximant, respectively. Define $L \times L$ matrices $Q(w)$ and $P(w)$ by

$$Q(w) = \left( w^{1-\delta_{i,j}}Q_j^{(i)}(w) \right)_{0 \leq i,j \leq L - 1} \in \mathbb{C}[w]^{L \times L},$$

$$P(w) = \left( w^{1-\delta_{i,j}}P_j^{(i)}(w) \right)_{0 \leq i,j \leq L - 1} \in \mathbb{C}[w]^{L \times L}.$$

Then it holds that

$$Q(w)^T P(w) = w^{nL} \cdot D,$$

where $D$ is a diagonal matrix independent of $w$.

**Proof.** This can be proved in a procedure similar to Theorem 1.3 in [16].

We can choose the normalization of $P_j^{(i)}(w)$ such that $D = I$ (the identity matrix). We will henceforth adopt this normalization.

### 3 Determinant representation of Hermite–Padé approximants

In this section, we give concrete descriptions of the solution to the Hermite–Padé approximation problem (2.2)–(2.4) in Section 2.

Without loss of generality, we may assume $f_0(w) = 1$ since the approximation conditions remain unchanged if we replace $\{f_0, f_1, \ldots, f_{L-1}\}$ by $\{1, f_1/f_0, \ldots, f_{L-1}/f_0\}$. Therefore, we assume $f_0(w) = 1$ in the sequel. Let us write the power series as

$$f_i(w) = \sum_{k=0}^{\infty} b_k^{(i)} w^k \quad (0 \leq i \leq L - 1).$$
Then we see that $b_0^i = 1$ and $b_k^i = 0 \ (k \neq 0)$ from $f_0(w) = 1$ and that $b_i^0 = 0 \ (i \neq 0)$ from \[(2.1)\]. Besides we set $b_k^i = 0 \ (k < 0)$ for notational convenience. Let us write the polynomials $Q_j^{(i)}(w)$ as

$$Q_j^{(i)}(w) = c_{j,0}^i + c_{j,1}^iw + \cdots + c_{j,n-1+\delta_{i,j}}^iw^{n-1+\delta_{i,j}} \quad (0 \leq i, j \leq L - 1)$$

with $c_{j,k}^i$ being the coefficient of $w^k$. The left-hand side of \[(2.3)\] reads as

$$Q_j^{(i)}f_i + \sum_{j \neq i} wQ_j^{(i)}f_j = \sum_{k=0}^\infty \left( \sum_{l=0}^{n-1} b_{k-l}^i c_{i,l}^j + \sum_{j \neq i}^{n-1} \sum_{l=0} b_{k-l}^j c_{j,l}^i \right) w^k.$$  

Hence the approximation condition \[(2.3)\] can be interpreted as a system of linear equations for the unknowns $c_{j,k}^i$:

$$\sum_{l=0} b_{k-l}^0 c_{0,l}^0 + \sum_{j \neq i}^{n-1} \sum_{l=0} b_{k-l}^j c_{j,l}^0 = 0 \quad (0 \leq k \leq Ln - 1), \quad (3.1)$$

$$\sum_{l=0} b_{Ln-l}^0 c_{0,l}^0 + \sum_{j \neq i}^{n-1} \sum_{l=0} b_{Ln-l}^j c_{j,l}^0 = 1 \quad (3.2)$$

for $i = 0$; and

$$\sum_{l=0} b_{k-l}^i c_{i,l}^j + \sum_{j \neq i}^{n-1} \sum_{l=0} b_{k-l}^j c_{j,l}^i = 0 \quad (1 \leq k \leq Ln) \quad (3.3)$$

for $i \neq 0$.

Let us introduce the column vectors

$$c_i^j = ^T(c_0^i, c_1^i, \ldots, c_{L-1}^i) \in \mathbb{C}^{Ln+1} \quad (0 \leq i \leq L - 1),$$

where

$$c_j^i = (c_{j,0}^i, \ldots, c_{j,n-1+\delta_{i,j}}^i),$$

and introduce the $k \times l$ rectangular Toeplitz matrix

$$B_m^i(k,l) = (b_{m+\alpha-\beta}^j)_{1 \leq \alpha, \beta \leq k} = \begin{pmatrix} b_m^i & b_{m-1}^i & \cdots & b_{m-l+1}^i \\ b_{m+1}^i & b_m^i & \cdots & b_{m-l+2}^i \\ \vdots & \vdots & \ddots & \vdots \\ b_{m+k-1}^i & b_{m+k-2}^i & \cdots & b_{m+k-l}^i \end{pmatrix} \quad (3.4)$$

for the sequence \{$b_j^i$\}. Then the linear equations \[(3.1)\] and \[(3.2)\] are summarized as a matrix form

$$B^0 c^0 = ^T(0, \ldots, 0, 1), \quad (3.5)$$
where \( B^0 \) is an \( Ln + 1 \) square matrix defined by
\[
B^0 = \begin{pmatrix}
B^0_0(Ln + 1, n + 1) & B^1_0(Ln + 1, n) & \cdots & B^L_{-1}(Ln + 1, n)
\end{pmatrix}.
\]

Similarly, (3.3) can be rewritten into
\[
B^i e^i = 0 = T(0, \ldots, 0) \quad (i \neq 0)
\]
where \( B^i \) (\( i \neq 0 \)) are \( Ln \times (Ln + 1) \) matrices defined by
\[
B^i = \begin{pmatrix}
B^0_0(Ln, n) & \cdots & B^i_{-1}(Ln, n) & B^i_1(Ln, n + 1) & B^{i+1}_0(Ln, n) & \cdots & B^{L-1}_0(Ln, n)
\end{pmatrix}.
\]

Solving (3.5) and (3.6) by Cramer’s rule, we have the determinant expressions of the approximants \( Q^i_j(w) \):
\[
Q^0_0(w) = \frac{1}{|B^0|} \begin{vmatrix}
B^0_0(Ln, n + 1) & B^1_0(Ln, n) & \cdots & B^L_{-1}(Ln, n) \\
1, w, \ldots, w^n & 0 & \cdots & 0
\end{vmatrix},
\]
\[
Q^0_j(w) = \frac{1}{|B^0|} \begin{vmatrix}
B^0_0(Ln, n + 1) & \cdots & B^j_{-1}(Ln, n) & \cdots & B^{L-1}_0(Ln, n) \\
0 & \cdots & 1, w, \ldots, w^{n-1} & \cdots & 0
\end{vmatrix} \quad (j \neq 0)
\]

for \( i = 0 \); and
\[
Q^i_j(w) = \frac{(-1)^{L+i}n}{|B|} \begin{vmatrix}
B^0_0(Ln, n) & B^0_0(Ln, n) & \cdots & B^{L-1}_0(Ln, n) \\
0 & 1, w, \ldots, w^{n-1+\delta_{i,j}} & \cdots & 0
\end{vmatrix} \quad (j \neq 0)
\]

for \( i \neq 0 \), where \( B \) is an \( Ln \) square matrix defined by
\[
B = \begin{pmatrix}
B^0_0(Ln, n) & B^0_0(Ln, n) & \cdots & B^{L-1}_0(Ln, n)
\end{pmatrix}.
\]

In the latter case we have used the normalization (2.4). Note that
\[
|B^0| \neq 0 \quad \text{and} \quad |B| \neq 0
\]
are the conditions for the unique existence of \( \{Q^i_j\} \), which we will impose throughout this paper.

Next, we concern
\[
\rho^i(w) = Q^i_i f_i + \sum_{j \neq i} w Q^i_j f_j \quad (0 \leq i \leq L - 1)
\]
which are the reminders of the Hermite–Padé approximation problem (2.2)–(2.4). For \( i = 0 \), we have
\[
\rho^0(w) = w^{Ln}(1 + O(w)).
\]
For \( i \neq 0 \), substituting (3.7) shows that

\[
\rho^i(w) = \frac{(-1)^{(L+i)n}}{|B|} \begin{vmatrix}
B_0^0(Ln, n) & \cdots & B_i^1(Ln, n + 1) & \cdots & B_0^{L-1}(Ln, n) \\
w f_0, \ldots, w^a f_0 & \cdots & f_i, w f_i, \ldots, w^a f_i & \cdots & w f_{L-1}, \ldots, w^a f_{L-1}
\end{vmatrix}
= O(w^{nL+1}).
\]

Introduce the determinants

\[
D_n = |B| = \begin{vmatrix}
B_0^0(Ln, n) & \cdots & B_0^{L-1}(Ln, n)
\end{vmatrix}
= \begin{vmatrix}
B_i^0((L-1)n, n) & \cdots & B_i^{L-1}((L-1)n, n)
\end{vmatrix}
\]
and

\[
E_{n,j}^i = \begin{vmatrix}
B_0^0(Ln, n) & \cdots & B_i^1(Ln, n + 1) & \cdots & B_0^{L-1}(Ln, n) \\
B_0^{L_n+i-1}(1, n) & \cdots & B_i^{L_n+i}(1, n + 1) & \cdots & B_0^{L_n+i}(1, n)
\end{vmatrix}
= \begin{vmatrix}
B_i^0((L-1)n, n) & \cdots & B_{n+1}^i((L-1)n, n + 1) & \cdots & B_i^{L_n+i}(n, n + 1) \\
B_i^{L_n+i-1}(1, n) & \cdots & B_i^{L_n+i}(1, n + 1) & \cdots & B_i^{L_n+i}(1, n)
\end{vmatrix},
\]

where we have used \( b_0^0 = 1 \) and \( b_k^0 = 0 \) (\( k \neq 0 \)). Thus, the coefficients of \( \rho^i(w) = w^{Ln} \sum_{j=1}^{\infty} \rho_j^i w^j \) are written as

\[
\rho_j^i = (-1)^{(L+i)n} \frac{E_{n,j}^i}{D_n}.
\]

## 4 Holonomic deformation of a system of linear differential equations

In this section, we briefly review the theory of holonomic deformations of linear differential equations following [6, 7].

We consider an \( L \times L \) system of linear differential equations which has regular or irregular singularities at \( x = a_1, \ldots, a_N, a_\infty = \infty \) on \( \mathbb{P}^1 \) with Poincaré rank \( r_\mu \) (\( \mu = 1, \ldots, N, \infty \)), respectively:

\[
\frac{dY}{dx} = A(x) Y,
\]

where

\[
A(x) = \sum_{\mu=1}^{N} \sum_{j=0}^{r_\mu} A_{\mu,-j}(x - a_\mu)^{-j-1} + \sum_{j=1}^{r_\infty} A_{\infty,-j} x^{j-1}
\]

and \( A_{\mu,-j} \) and \( A_{\infty,-j} \) are \( L \times L \) matrices independent of \( x \). We assume that \( A_{\mu,-r_\mu} \) (\( \mu = 1, \ldots, N, \infty \)) is diagonalizable as

\[
A_{\mu,-r_\mu} = G^{(\mu)} T_{-r_\mu}^{(\mu)} G^{(\mu)-1},
\]

7.
where the diagonal matrix $T_{-r_{\mu}} = (t_{-r_{\mu},\alpha}^\mu \delta_{\alpha,\beta})_{0 \leq \alpha, \beta \leq L-1}$ satisfies

\[
t_{-r_{\mu}, \alpha} = t_{-r_{\mu}, \beta} \text{ if } \alpha \neq \beta, \quad r_{\mu} \geq 1,
\]

\[
t_{0, \alpha} = t_{0, \beta} \text{ mod } \mathbb{Z} \text{ if } \alpha \neq \beta, \quad r_{\mu} = 0.
\]

Let us introduce the diagonal matrices

\[
T^\mu(x) = (e^\mu_\alpha(x) \delta_{\alpha,\beta})_{0 \leq \alpha, \beta \leq L-1}
\]

for $\mu = 1, \ldots, N, \infty$ with

\[
e^\mu_\alpha(x) = \sum_{j=1}^{r_{\mu}} t_{-j,\alpha} \frac{z_{\mu}^{-j}}{-j} + t_{0,\alpha}^\mu \log z_{\mu}, \quad z_{\mu} = \begin{cases} x - a_{\mu} & (1 \leq \mu \leq N) \\ x^{-1} & (\mu = \infty). \end{cases}
\]

Then, we can take sectors $S_k^\mu (1 \leq k \leq 2r_{\mu})$ centered at $a_{\mu}$ and there exists a unique fundamental system of solutions to (4.1) having the asymptotic expansion of the form

\[
Y(x) \simeq G^\mu \hat{Y}^\mu(x) e^{T^\mu(x)}, \quad \hat{Y}^\mu(x) = I + Y_{1}^\mu z_{\mu} + Y_{2}^\mu z_{\mu}^2 + \cdots
\]

in each $S_k^\mu$. Note that $\hat{Y}^\mu(x)$ are in general divergent and that even around the same point $z = a_{\mu}$ these power series in two different sectors may differ by a left multiplication of some constant matrix (Stokes phenomena). Without loss of generality, we henceforth assume $G^{(\infty)} = I$.

If we start with the fundamental system of solutions normalized by the asymptotic expansion

\[
Y(x) \simeq \hat{Y}^{(\infty)}(x) e^{T^{(\infty)}(x)}, \quad \hat{Y}^{(\infty)}(x) = I + Y_{1}^{(\infty)} z_{\infty} + Y_{2}^{(\infty)} z_{\infty}^2 + \cdots (4.2)
\]

in the sector $S_1^{(\infty)}$ around $x = \infty$, then the same solution behaves as

\[
Y(x) \simeq G^{(\mu)} \hat{Y}^{(\mu)}(x) e^{T^{(\mu)}(x)} S_{k-1}^{(\mu)} \cdots S_{1}^{(\mu)} C^{(\mu)}
\]

in a different sector $S_k^{(\mu)}$, where $C^{(\mu)}$ and $S_j^{(\mu)}$ are the invertible constant matrices called the connection matrix and Stokes multiplier, respectively.

We consider a deformation of (4.1) by choosing $a_1, \ldots, a_N$ and $t_{-j,\alpha}^\mu (\mu = 1, \ldots, N, \infty; 1 \leq j \leq r_{\mu}; 0 \leq \alpha \leq L-1)$ as its independent variables such that $T_0^{(\mu)}$, $C^{(\mu)}$ and $S_j^{(\mu)}$ are kept invariant. Such a deformation is called a holonomic deformation. Let $d$ denote the exterior differentiation with respect to the deformation parameters $\{a_{\mu}, t_{-j,\alpha}^\mu\}$. The fundamental system of solutions $Y(x)$ specified by (4.2) is subject to the holonomic deformation if and only if it satisfies

\[
dY(x) = \Omega(x) Y(x), \quad (4.3)
\]
where $\Omega(x)$ is a matrix-valued 1-form given as

$$\Omega(x) = \sum_{\mu=1}^{N} B^{(\mu)}(x) da_{\mu} + \sum_{\mu=1,\ldots,N,\infty}^{r_{\mu}} \sum_{j=1}^{L-1} \sum_{\alpha=0}^{\infty} B_{-j,\alpha}^{(\mu)}(x) dt_{-j,\alpha}^{(\mu)},$$

whose coefficients $B^{(\mu)}(x)$ and $B_{-j,\alpha}^{(\mu)}(x)$ are rational functions in $x$. From the integrability condition of (4.1) and (4.3), we obtain a system of nonlinear differential equations for $A(x)$ and $G^{(\mu)}$:

$$dA(x) = \frac{\partial \Omega}{\partial x}(x) + [\Omega(x), A(x)], \quad dG^{(\mu)} = \Theta^{(\mu)} G^{(\mu)} \quad (1 \leq \mu \leq N). \quad (4.4)$$

We remark that $\Omega(x)$ and $\Theta^{(\mu)}$ are computable from $A(x)$ and $G^{(\mu)}$ by a rational procedure; see [7] for details. The 1-form

$$\omega = - \sum_{\mu=1,\ldots,N,\infty} \text{tr Res}_{x=a_{\mu}} \hat{Y}^{(\mu)}(x)^{-1} \frac{\partial \hat{Y}^{(\mu)}(x)}{\partial x} dT^{(\mu)}(x) \quad (4.5)$$

is closed, i.e. $d\omega = 0$, for any solution to (4.4). Hence we can define the $\tau$-function $\tau = \tau(\{a_{\mu}, t_{-j,\alpha}^{(\mu)}\})$ by

$$d \log \tau = \omega. \quad (4.6)$$

5 Construction of Schlesinger transformations

In this section, we construct the Schlesinger transformation that shifts the characteristic exponents at $x = \infty$ of the system of linear differential equations (4.1) as

$$t_{0}^{(\infty)} = (t_{0,0}^{(\infty)}, \ldots, t_{0,L-1}^{(\infty)}) \mapsto t_{0}^{(\infty)} + n,$$

where $n = ((L-1)n, -n, \ldots, -n) \in \mathbb{Z}^{L}$ and $n$ is a positive integer.

Write the power series part of $Y(x) \simeq \hat{Y}^{(\infty)}(x) e^{T^{(\infty)}(x)}$ (see (4.2)) as

$$\hat{Y}^{(\infty)}(x) = \Phi(w) = \left( \phi_{i,j}(w) \right)_{0 \leq i, j \leq L-1}, \quad \phi_{i,j}(w) = \sum_{k=0}^{\infty} a_{i,j}^{k} w^{k}, \quad (5.1)$$

where

$$w = z_{\infty} = \frac{1}{x}.$$ 

Namely, $\Phi(w)$ is an $L$ square matrix whose entries are formal power series in $w$, and its constant term is the identity matrix, i.e. $\phi_{i,j}(0) = \delta_{i,j}$. Define new power series $f_{i}(w)$ from the first column of $\Phi(w)$ by

$$f_{i}(w) = \sum_{k=0}^{\infty} b_{i}^{k} w^{k} = \frac{\phi_{i,0}(w)}{\phi_{0,0}(w)} \quad (0 \leq i \leq L-1). \quad (5.2)$$
Since it holds that \( f_0(w) = 1 \) and \( f_i(0) = 0 \) \((i \neq 0)\), we can apply the Hermite–Padé approximation problem \((2.2)–(2.4)\) and the simultaneous Padé approximation problem \((2.5)–(2.6)\) considered in Section 2 to the set of \( L \) formal power series \( \{f_0(w), \ldots, f_{L-1}(w)\}\).

Define the matrices
\[
Q(w) = \left( w^{1-\delta_{i,j}} Q_j^{(i)}(w) \right)_{0 \leq i,j \leq L-1} \in \mathbb{C}[w]^{L \times L},
\]
\[
R(x) = x^n Q(x^{-1}) \in \mathbb{C}[x]^{L \times L}.
\]

Recall here that \( \text{deg} Q_j^{(i)}(w) \leq n - 1 + \delta_{i,j} \). The result is stated as follows.

**Theorem 5.1.** The polynomial matrix \( R(x) \) provides the representation matrix of the Schlesinger transformation for \((1.1)\) which shifts the characteristic exponents at \( x = \infty \) by \( \mathbf{n} = ((L-1)n, -n, \ldots, -n) \in \mathbb{Z}^L \).

**Proof.** From Theorem 2.1, we have \(|Q(w)| \cdot |P(w)| = w^{L^2n}\). The conditions for the degrees \((2.2)\) and \((2.5)\) shows that \(|Q(w)|\) is of degree at most \( Ln \) and \(|P(w)|\) at most \( L(L-1)n\), respectively. Consequently, it holds that \(|Q(w)| = cw^{Ln}\) and \(|P(w)| = c^{-1}w^{L(L-1)n}\) for some constant \( c \neq 0\); and thus \(|R(x)| = c\). It implies that \( R(x) \) is an invertible matrix at any \( x \in \mathbb{C} \). Therefore, the transformation \( Y(x) \mapsto R(x)Y(x) \) does not affect the regularity or the singularity of \( Y(x) \) at any \( x \in \mathbb{C} \). Let us observe the influence at \( x = \infty \) of this transformation. It follows from the approximation conditions \((2.3)\) and \((2.4)\) that
\[
R(x) \Phi(w) = w^{-n} Q(w) \Phi(w)
\]
\[
= w^{-n} \left( Q_j^{(i)} \phi_{i,j} + \sum_{k \neq i} w Q_k^{(i)} \phi_{k,j} \right)_{0 \leq i,j \leq L-1}
\]
\[
= (I + O(w)) \text{diag} \left( w^{(L-1)n}, w^{-n}, \ldots, w^{-n} \right). \tag{5.3}
\]

Noticing the expression
\[
e^{T^{(\infty)}(x)} = \text{diag} \left( w^{|i|^{(\infty)}} \right)_{0 \leq j \leq L-1} e^{\sum_{j=1}^{\infty} T^{(\infty)} w^{-j}}
\]
of the exponential part of \( y(x) \), we can conclude that \( Y(x) \mapsto R(x)Y(x) \) induces the Schlesinger transformation that shifts the characteristic exponents at \( x = \infty \) as \( t_0^{(\infty)} \mapsto t_0^{(\infty)} + \mathbf{n} \).

**Remark 5.2.** Taking the determinants of the both sides of \((5.3)\), we have \(|R(x)| \cdot |\Phi(w)| = 1 + O(w)\). Combining this with \(|\Phi(w)| = 1 + O(w)\) yields \( c = |R(x)| = 1 \).
6 Determinant structure of $\tau$-quotients

In this section, we investigate the influence on the $\tau$-function by the Schlesinger transformation.

We consider the Schlesinger transformation of a linear differential equation (4.1), which shifts the characteristic exponents at $x = \infty$ by

$$n = ((L - 1)n, -n, \ldots, -n) \in \mathbb{Z}^L$$

for a positive integer $n$; see Section 5. Let $\tau_n$ denote the $\tau$-function associated with the holonomic deformation of the resulting linear differential equation after the Schlesinger transformation, while $\tau_0$ denotes that of the original (4.1).

First, we shall look at a relation between $\tau_0$ and $\tau_1$. According to [6, Theorem 4.1] it holds that

$$\frac{\tau_1}{\tau_0} = \text{const.}$$

or equivalently

$$a_{i,j} = 0 \text{ for } l < 0.$$ 

Thus we find that

$$\frac{\tau_1}{\tau_0} = \text{const.}$$

Here we note the following elementary fact.

**Lemma 6.1.** Let

$$\sum_{k=1}^{\infty} \alpha_k^i w^k, \sum_{k=1}^{\infty} \beta_k^i w^k (1 \leq i \leq L - 1) \text{ and } \sum_{k=0}^{\infty} \gamma_k w^k$$
be formal power series, where $\gamma_0 = 1$. If the relation
\[
\sum_{k=1}^{\infty} \alpha_k^i w^k = \left( \sum_{k=1}^{\infty} \beta_k^i w^k \right) \left( \sum_{k=0}^{\infty} \gamma_k w^k \right)
\]
among the formal power series holds for each $i$, then the equality
\[
\begin{vmatrix}
\alpha_1^1 & \alpha_1^2 & \cdots & \alpha_1^{L-1} \\
\alpha_2^1 & \alpha_2^2 & \cdots & \alpha_2^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{L-1}^1 & \alpha_{L-1}^2 & \cdots & \alpha_{L-1}^{L-1}
\end{vmatrix}
= \begin{vmatrix}
\beta_1^1 & \beta_1^2 & \cdots & \beta_1^{L-1} \\
\beta_2^1 & \beta_2^2 & \cdots & \beta_2^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{L-1}^1 & \beta_{L-1}^2 & \cdots & \beta_{L-1}^{L-1}
\end{vmatrix}
\]
regarding their coefficients holds.

**Proof.** It can be verified straightforwardly by using $\alpha_k^i = \sum_{i+m=k} \beta_m^i \gamma_m$. □

Returning to our situation, we have
\[
\begin{vmatrix}
\alpha_{1,0}^1 & \alpha_{2,0}^1 & \cdots & \alpha_{L-1,0}^1 \\
\alpha_{2,0}^1 & \alpha_{2,0}^2 & \cdots & \alpha_{2,0}^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{L-1,0}^1 & \alpha_{L-1,0}^2 & \cdots & \alpha_{L-1,0}^{L-1}
\end{vmatrix} = \begin{vmatrix}
\beta_{1,0}^1 & \beta_{2,0}^1 & \cdots & \beta_{L-1,0}^1 \\
\beta_{2,0}^1 & \beta_{2,0}^2 & \cdots & \beta_{2,0}^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{L-1,0}^1 & \beta_{L-1,0}^2 & \cdots & \beta_{L-1,0}^{L-1}
\end{vmatrix} \quad (6.2)
\]
from Lemma 6.1 since $b_k^i$ and $a_k^{i,0}$ are mutually related by (see (5.1) and (5.2))
\[
f_i(w) = \sum_{k=0}^{\infty} b_k^i w^k = \frac{\phi_{i,0}(w)}{\phi_{0,0}(w)} = \frac{\sum_{k=0}^{\infty} a_k^{i,0} w^k}{1 + O(w)}.
\]

It thus follows from (6.1) that
\[
\frac{\tau_1}{\tau_0} = \text{const.}
\]
\[
\begin{vmatrix}
\beta_{1,0}^1 & \beta_{2,0}^1 & \cdots & \beta_{L-1,0}^1 \\
\beta_{2,0}^1 & \beta_{2,0}^2 & \cdots & \beta_{2,0}^{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{L-1,0}^1 & \beta_{L-1,0}^2 & \cdots & \beta_{L-1,0}^{L-1}
\end{vmatrix}.
\]

Next, we shall track how the entries of $\Phi(w)$ are changed after the Schlesinger transformation. Define
\[
\bar{\Phi}(w) = \left( \bar{\phi}_{i,j}(w) \right)_{0 \leq i, j \leq L-1}, \quad \bar{\phi}_{i,j}(w) = \sum_{k=0}^{\infty} a_k^{i,j} w^k
\]
by
\[
R(w) \Phi(w) = \bar{\Phi}(w) \text{ diag } (w^{L-1}n, w^{-n}, \ldots, w^{-n}).
\]
In particular, the entry $\overline{\phi}_{i,0}(w)$ is obtained from the remainder of the Hermite–Padé approximation as (see (3.9) and (5.2))

$$
\overline{\phi}_{i,0}(w) = w^{-L_n} \left( Q_i^{(i)}(w)\phi_{1,0}(w) + \sum_{j\neq i} w Q_j^{(i)}(w)\phi_{j,0}(w) \right)
$$

$$
= w^{-L_n}\phi_{0,0}(w)\rho^i(w).
$$

Let

$$
\mathcal{T}_i(w) = \sum_{k=0}^{\infty} b_k w^k = \frac{\overline{\phi}_{i,0}(w)}{\phi_{0,0}(w)}
$$

and $\rho^i(w) = w^{L_n}\sum_{k=1}^{\infty} \rho_k^i w^k$ for $1 \leq i \leq L - 1$ as in the previous sections. Namely, overlined symbols denote the quantities after the Schlesinger transformation that shifts the characteristic exponents at $x = \infty$ as $t_0^{(\infty)} \rightarrow t_0^{(\infty)} + \mathbf{n}$. Then, by applying Lemma 6.1 twice, we have

$$
\begin{vmatrix}
\overline{\rho}_1 & \overline{\rho}_2 & \cdots & \overline{\rho}_{L-1} \\
\overline{\rho}_1 & \overline{\rho}_2 & \cdots & \overline{\rho}_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{\rho}_1 & \overline{\rho}_2 & \cdots & \overline{\rho}_{L-1}
\end{vmatrix} = \begin{vmatrix}
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1} \\
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1}
\end{vmatrix} = \begin{vmatrix}
\rho_1 & \rho_2 & \cdots & \rho_{L-1} \\
\rho_1 & \rho_2 & \cdots & \rho_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_1 & \rho_2 & \cdots & \rho_{L-1}
\end{vmatrix}.
$$

(6.4)

Finally, combining (6.3) and (6.4) yields that

$$
\frac{\tau_{n+1}}{\tau_n} = \frac{\tau_1}{\tau_0} = \text{const.}
$$

$$
\begin{vmatrix}
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1} \\
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{b}_1 & \overline{b}_2 & \cdots & \overline{b}_{L-1}
\end{vmatrix} = \text{const.}
$$

$$
\begin{vmatrix}
\rho_1 & \rho_2 & \cdots & \rho_{L-1} \\
\rho_1 & \rho_2 & \cdots & \rho_{L-1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_1 & \rho_2 & \cdots & \rho_{L-1}
\end{vmatrix}.
$$

Substituting (3.12) in the above, we obtain

$$
\frac{\tau_{n+1}}{\tau_n} = \text{const.} D_n^{L+1} \det(E_{n,i,j}^{L})_{1 \leq i,j \leq L-1}.
$$

(6.5)

Now we state the main theorem.

**Theorem 6.2.** Consider a holonomic deformation of (4.1). Let $\tau_0$ be the $\tau$-function associated with (4.1) and let $\tau_n$ be the $\tau$-function associated with the transformed equation from (4.1) by the Schlesinger transformation that shifts the characteristic exponents at $x = \infty$ by

$$
\mathbf{n} = ((L-1)n, -n, \ldots, -n) \in \mathbb{Z}^L
$$

for a positive integer $n$. Then the following determinant formula for the $\tau$-quotient holds:

$$
\frac{\tau_n}{\tau_0} = \text{const.} D_n,
$$

(6.6)
where \( D_n \) is the block Toeplitz determinant defined by (3.4) and (3.10) and its entries \( b_k^i \) are specified by (5.1) and (5.2), i.e. the asymptotic solution to (4.1) at \( x = \infty \).

Proof. We have the equality
\[
D_{n+1} D_n^{L-2} = \det(E_{i,j}^{1,i,j})_{1 \leq i,j \leq L-1},
\]
which will be shown in Appendix A. Therefore, (6.5) implies
\[
\frac{\tau_{n+1}}{\tau_n} = \text{const.} \frac{D_{n+1}}{D_n}.
\]
It is clear from (3.10) and (6.3) that \( \tau_1/\tau_0 = \text{const.} D_1 \). Hence the theorem is proved.

Remark 6.3. In the case of a second-order Fuchsian linear differential equation, their isomonodromic deformations are governed by the Garnier systems and the formula (6.6) has been established in [14].

Remark 6.4. Jimbo and Miwa [6] treat determinant representations of \( \tau \)-quotients for arbitrary Schlesinger transformations and their matrix entries are written in terms of the characteristic matrices. However, the characteristic matrices themselves are, in general, too complicated to compute explicitly. On the other hand, Theorem 6.2 above gives a much simpler representation of \( \tau \)-quotients in terms of block Toeplitz determinants, though the Schlesinger transformations are restricted to a specific direction shifting the characteristic exponents at one point by \( n = ((L - 1)n, -n, \ldots, -n) \). It is expected that more general Schlesinger transformations are related to other types of approximation problems beyond Hermite–Padé type. It would be an interesting problem to explore such relationships.

7 Particular solutions to holonomic deformation

In this section, as an application of results in the previous section, we present a method for constructing particular solutions to holonomic deformation equations such as the Painlevé equations.

Consider the \( L \times L \) system of linear differential equations (4.1). Take a new point \( a_{N+1} \in \mathbb{C} \setminus \{ a_1, \ldots, a_N \} \) where (1.1) is non-singular. The solution (1.2) normalized at \( x = \infty \) can be expanded around \( x = a_{N+1} \) as follows:
\[
Y(x) = Y(a_{N+1})\Psi(w), \quad \Psi(w) = Y(a_{N+1})^{-1} \sum_{n=0}^{\infty} Y^{(n)}(a_{N+1}) \frac{w^n}{n!},
\]
where $w = x - a_{N+1}$ and $Y^{(n)}(x)$ denotes the $n$th derivative of $Y(x)$ with respect to $x$. Write the power series part $\Psi(w)$ as
\[
\Psi(w) = (\psi_{i,j}(w))_{0 \leq i, j \leq L-1}
\]
and put
\[
f_i(w) = \frac{\dot{\psi}_{i,0}(w)}{\psi_{0,0}(w)} \quad (0 \leq i \leq L-1).
\]
We apply the Hermite–Padé approximation problem (2.2)–(2.4) to the set of formal power series \{\(f_0, f_1, \ldots, f_{L-1}\}\}, and introduce the matrices
\[
Q(w) = \left( w^{1-\delta_{j,i}} Q_j^{(i)}(w) \right)_{0 \leq i, j \leq L-1} \in \mathbb{C}[w]^{L \times L},
\]
\[
R(x) = (x - a_{N+1})^{-\mathbf{n}} Q(x - a_{N+1}) \in \mathbb{C}[(x - a_{N+1})^{-1}]^{L \times L}
\]
made from its approximants $Q_j^{(i)}(w)$. Using $R(x)$, we define the rational function matrix
\[
S(x) = Y(a_{N+1}) R(\infty)^{-1} R(x) Y(a_{N+1})^{-1}.
\]
Then $\tilde{Y}(x) = S(x) Y(x)$ satisfies a system of differential equations of the form
\[
\frac{d\tilde{Y}}{dx} = \left( \sum_{\mu=1}^{N} \sum_{j=0}^{r_\mu} \tilde{A}_{\mu,j}(x - a_\mu)^{-j-1} - \sum_{j=1}^{\infty} \tilde{A}_{\infty,-j} x^{-j-1} + \tilde{A}_{N+1}(x - a_{N+1})^{-1} \right) \tilde{Y}.
\]  \quad (7.1)

This means that the transformation $Y(x) \mapsto \tilde{Y}(x) = S(x) Y(x)$ induces one regular singularity $a_{N+1}$ in \(\tilde{Y}(x)\). It is clear by definition of $S(x)$ that the characteristic exponents of \(\tilde{Y}(x)\) at the additional regular singularity $x = a_{N+1}$ read $\mathbf{n} = ((L - 1)n, -n, \ldots, -n)$. Furthermore, we see that if $Y(x)$ is subject to a holonomic deformation of \(\tilde{Y}(x)\), then $\tilde{Y}(x)$ is also subject to that of \(Y(x)\) since $Y(x)$ and $\tilde{Y}(x)$ have the same monodromy. Consequently, at the level of holonomic deformations, we have a certain *inclusion relation* between solutions as described below.

Suppose for simplicity that \(\tilde{Y}(x)\) is Fuchsian, i.e. $r_\mu = 0$ for any $\mu = 1, \ldots, N, \infty$. One can associate with \(\tilde{Y}(x)\) an \((N + 1)\)-tuple
\[
M = \{(m_{1,1}, m_{1,2}, \ldots, m_{1,k_1}), \ldots, (m_{N,1}, m_{N,2}, \ldots, m_{N,k_N}), (m_{\infty,1}, m_{\infty,2}, \ldots, m_{\infty,k_\infty})\}
\]
of partitions of $L$, called the *spectral type*, which indicates how the characteristic exponents overlap at each of the $N + 1$ singularities $x = a_\mu$ ($\mu = 1, \ldots, N, \infty$). Note that by means of the spectral type the number of accessory parameters in \(\tilde{Y}(x)\) is estimated at
\[
2 + (N - 1)L^2 - \sum_{i=1}^{k_1} \sum_{j=1}^{k_i} m_{i,j}^2.
\]
see e.g. [20]. The argument above provides a procedure to obtain a new system (7.1) of spectral type $\tilde{M} = M \cup (L - 1, 1)$ from the original system (4.1) of spectral type $M$ while keeping the monodromy. Therefore, the general solution to the deformation equation of (4.1) gives rise to a particular solution to the deformation equation of (7.1). This phenomenon is exemplified by the fact that the Garnier system in $N + 1$ variables includes the Garnier system in $N$ variables as its particular solution; cf. [22, Theorem 6.1] It is also interesting to mention that if the original (4.1) is rigid, i.e. having no accessory parameter such as Gauß’s hypergeometric equation, then the deformation equation of (7.1) possesses a solution written in terms of that of the rigid system (4.1) itself. In this case, our procedure gives a natural interpretation to Suzuki’s recent work [21], in which a list of rigid systems or hypergeometric equations appearing in particular solutions to the higher order Painlevé equations is presented.

**Example 7.1 (Case $L = N = 2$).** Let us consider a $2 \times 2$ Fuchsian system of differential equations

$$
\frac{dY}{dx} = \left( \frac{A_0}{x} + \frac{A_1}{x - 1} \right) Y \tag{7.2}
$$

with three regular singularities $x = 0, 1, \infty$, whose spectral type is $\{(1, 1), (1, 1), (1, 1)\}$. We can assume without loss of generality that $|A_0| = 0$ and $A_\infty = -A_0 - A_1$ is diagonal, i.e. $A_\infty = \text{diag}(\kappa_1, \kappa_2)$. It is well known that the entries of a fundamental system of solutions to (7.2) can be written in terms of Gauß’s hypergeometric function. If we take an arbitrary point $t \in \mathbb{C} \setminus \{0, 1\}$ and apply the procedure above, then we obtain a system of differential equations of the form

$$
\frac{d\tilde{Y}}{dx} = \left( \frac{\tilde{A}_0}{x} + \frac{\tilde{A}_1}{x - 1} + \frac{\tilde{A}_t}{x - t} \right) \tilde{Y}; \tag{7.3}
$$

it is a $2 \times 2$ Fuchsian system with four regular singularities $x = 0, 1, \infty, t$, whose spectral type is $\{(1, 1), (1, 1), (1, 1), (1, 1)\}$. We know from the construction that the monodromy of (7.3) is independent of $t$, i.e. (7.3) is subject to an isomonodromic deformation with a deformation parameter $t$. Thus we can derive a particular solution written in terms of Gauß’s hypergeometric functions to the Painlevé VI equation with constant parameters

$$
\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1 - 4n^2}{2},
$$

where $\theta_\infty = \kappa_1 - \kappa_2$, $\theta_i = \text{tr}A_i = \text{tr}\tilde{A}_i$ ($i = 0, 1$) and $n \in \mathbb{Z}_{\geq 0}$. Refer to [3] [4] for the Painlevé VI equation.
A Proof of an identity for determinants

In this appendix we derive the determinant identity (6.7), which is used to verify the main theorem of this paper. We first prove its Pfaffian analogue in a general setting to achieve better perspectives, and then we reduce it to the determinant case. The reader can refer to [3] for various Pfaffian identities and their applications.

Let $A$ be a set of alphabets, which is a totally ordered set. Let $A^*$ denote the set of words over $A$. For a word $I \in A^*$ and its permutation $J$, $\text{sgn}(I,J)$ denotes the sign of the permutation that converts $I$ into $J$ if $I$ has no duplicate letter, and 0 otherwise. Given a word $I = i_1i_2\cdots i_{2n} \in A^*$ of length $I = 2n$, its permutation $J = j_1j_2\cdots j_{2n}$ is called a perfect matching on $I$ if $\sigma(2k-1) < \sigma(2k)$ for $1 \leq k \leq n$ and $\sigma(2k-1) < \sigma(2k+1)$ for $1 \leq k \leq n-1$, where $\sigma \in S_{2n}$ and $j_1j_2\cdots j_{2n} = i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(2n)}$. This perfect matching is designated by the configuration in the $xy$ plane which contains $2n$ vertices $v_k = (k,0)$ ($1 \leq k \leq 2n$) labeled with $i_k$ and $n$ arcs above the $x$ axis connecting the vertices $v_{\sigma(2k-1)}$ and $v_{\sigma(2k)}$ ($1 \leq k \leq n$). Let $\mathcal{F}(I)$ denote the set of all perfect matchings on $I$. For a perfect matching $J = j_1j_2\cdots j_{2n} \in \mathcal{F}(I)$, we call $\mathcal{M}(J) = \{(j_{2k-1},j_{2k}) | 1 \leq k \leq n\}$ the set of arcs in $J$. It is easy to see that the sign $\text{sgn}(I,J)$ equals $(-1)^c$, where $c$ is the number of crossings of the arcs in the configuration of $J$. For example, the set of perfect matchings on a word $I = 1234$ reads

$\mathcal{F}(I) = \{1234, 1324, 1423\}$.

If we take a perfect matching $J = 1423 \in \mathcal{F}(I)$ then we have the set of arcs $\mathcal{M}(J) = \{(1,4),(2,3)\}$ and $J$ is designated by the following configuration:

There is no crossing of the arcs and certainly $\text{sgn}(I,J) = 1$ holds.

Let $f$ be a map which assigns an element of a commutative ring to each pair $(i,j) \in A \times A$ such that $f(j,i) = -f(i,j)$. Such a map is called a skew symmetric map. For each perfect matching $J = j_1j_2\cdots j_{2n} \in \mathcal{F}(I)$, we define the weight $\omega_f(J)$ as

$$\omega_f(J) = \text{sgn}(I,J) \prod_{(i,j) \in \mathcal{M}(J)} f(i,j).$$

The Pfaffian $\text{Pf}_f(I)$ of $f$ corresponding to the word $I = i_1i_2\cdots i_{2n}$ is the sum of the weights $\omega_f(J)$, where $J$ runs over all perfect matchings on $I$, i.e.,

$$\text{Pf}_f(I) = \sum_{J \in \mathcal{F}(I)} \omega_f(J).$$
We use the convention that $\text{Pf}_f(I) = 1$ if $I = \emptyset$. It is known that

$$\text{Pf}_f(K) = \text{sgn}(I, K) \text{Pf}_f(I),$$  \hfill (A.1)

where $K$ is a permutation of $I$. Especially $\text{Pf}_f(I) = 0$ if $I$ has a duplicate letter. For example, the Pfaffian of $f$ corresponding to $I = 1234$ is given as

$$\text{Pf}_f(I) = f(1, 2)f(3, 4) - f(1, 3)f(2, 4) + f(1, 4)f(2, 3).$$

The following identity is the Plücker relation for Pfaffians, which is originally due to Ohta \[19\] and Wenzel \[26\]. Ohta’s proof is by algebraic arguments, and Wenzel employs the Pfaffian form. The proof we present here is more combinatorial one based on the same idea as in \[4\].

**Theorem A.1** (cf. \[3, 19, 26\]). Let $I, J, K \in A^*$ be words such that $\sharp I$ and $\sharp J$ are odd and $\sharp K$ is even. Then it holds that

$$\sum_{i \in I} \text{sgn}(IJ, (I \setminus \{i\})iJ) \text{Pf}_f((I \setminus \{i\})K) \text{Pf}_f(iJK) = \sum_{j \in J} \text{sgn}(IJ, Ij(J \setminus \{j\})) \text{Pf}_f(IjK) \text{Pf}_f((J \setminus \{j\})K).$$ \hfill (A.2)

**Proof.** We put $W_1 = KI$, $W_2 = JK$ and $W = W_1W_2$. Let $\mathcal{G}$ denote the set of perfect matchings on $W$ in which there is exactly one arc connecting a vertex in $W_1$ and a vertex in $W_2$ and all the other arcs are between vertices in $W_1$ or between vertices in $W_2$. For example, if $I = 123$, $J = 456$ and $K = 78$ then $W_1 = KI = 78123$, $W_2 = JK = 45678$ and $W = W_1W_2 = 7812345678$. The following configuration designates such a perfect matching on $W$, $P = 7283154867 \in \mathcal{G}$, in which the arc $(1, 5)$ is the only arc connecting a letter in $W_1$ and a letter in $W_2$:

For $i \in W_1$ and $j \in W_2$, let $\mathcal{G}_{i,j}$ denote the subset of $\mathcal{G}$ having the arc $(i, j)$; thereby, $\mathcal{G} = \bigcup_{i \in W_1, j \in W_2} \mathcal{G}_{i,j}$. Let us consider the sums $\Omega = \sum_{P \in \mathcal{G}} \omega_f(P)$ and $\Omega_{i,j} = \sum_{P \in \mathcal{G}_{i,j}} \omega_f(P)$; thereby,

$$\Omega = \sum_{i \in W_1, j \in W_2} \Omega_{i,j}. \hfill (A.3)$$

**Claim.** It holds that

$$\sum_{j \in W_2} \Omega_{i,j} = \text{sgn}(W, (W_1 \setminus \{i\})iW_2) \text{Pf}_f(W_1 \setminus \{i\}) \text{Pf}_f(iW_2)$$ \hfill (A.4)

for $i \in W_1$. 

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To check the claim, we first associate with each perfect matching \( P \in \mathfrak{G}_{i,j} \) a pair \((P_1, P_2)\) of perfect matchings such that \( P_1 \in \mathfrak{F}(W_1 \setminus \{i\}) \) and \( P_2 \in \mathfrak{F}(iW_2) \) by shifting \( i \) from the original position to the head of \( W_2 \) in the configuration. For the above example \( P \in \mathfrak{G}_{1,5} \) the vertex 1 is shifted and the associated pair \((P_1, P_2)\) is thus illustrated as follows:

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & 7 & 8 & 2 & 3 & 1 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

Since \( \text{sgn}(W, P) = \text{sgn}(W_i(W_1 \setminus \{i\})iW_2) \text{sgn}(W_1 \setminus \{i\}, P_1) \text{sgn}(iW_2, P_2) \), it is then clear that

\[
\omega_f(P) = \text{sgn}(W_i(W_1 \setminus \{i\})iW_2)\omega_f(P_1)\omega_f(P_2),
\]

which proves (A.4).

By the same argument we obtain

\[
\sum_{i \in W_1} \Omega_{i,j} = \text{sgn}(W_i(W_1j(W_2 \setminus \{j\})) \text{Pf}_f(W_1j) \text{Pf}_f(W_2 \setminus \{j\}) \quad (A.5)
\]

for \( j \in W_2 \). Hence (A.3) leads to the desired identity (A.2) via (A.4) and (A.5). Note that if \( i \in K \) or \( j \in K \) then there appears a repeated letter in the word, so we can remove these cases.

**Corollary A.2** (cf. [4, 11]). Let \( I, K \in A^* \) be words such that \( \#I \) and \( \#K \) are even. Then it holds that

\[
\sum_{i \in I, i \neq j} \text{sgn}(I, (I \setminus \{i, j\})i) \text{Pf}_f((I \setminus \{i, j\})K) \text{Pf}_f(ijK) = \text{Pf}_f(IK) \text{Pf}_f(K) \quad (A.6)
\]

for \( j \in I \).

**Proof.** Putting \( \#J = 1 \), i.e. \( J = j \), in Theorem A.1 shows that

\[
\sum_{i \in I} \text{sgn}(Ij, (I \setminus \{i\})ij) \text{Pf}_f((I \setminus \{i\})K) \text{Pf}_f(ijK) = \text{Pf}_f(IjK) \text{Pf}_f(K).
\]

Write \( I = i_1i_2 \cdots i_n \) and \( I' = i_1 \cdots i_{k-1}ji_k \cdots i_n \). Then we have \( \text{sgn}(Ij, (I \setminus \{i\})ij) = \text{sgn}(Ij, I') \text{sgn}(I', (I' \setminus \{i, j\})ij) \) and \( \text{Pf}_f(IjK) = \text{sgn}(Ij, I') \text{Pf}_f(I'K) \) by (A.1). Hence we obtain

\[
\sum_{i \in I', i \neq j} \text{sgn}(I', (I' \setminus \{i, j\})ij) \text{Pf}_f((I' \setminus \{i, j\})K) \text{Pf}_f(ijK) = \text{Pf}_f(I'K) \text{Pf}_f(K),
\]

which coincides with (A.6) if we replace \( I' \) with \( I \).

\( \square \)
Corollary A.3 (cf. [11]). Let $I, K \in A^*$ be words such that $\sharp I$ and $\sharp K$ are even with $\sharp I = 2n$. Let $F_{f,K}$ be a skew symmetric map on $A \times A$ defined by $F_{f,K}(i,j) = Pf_f(ijK)$. Then it holds that

$$Pf_{F_{f,K}}(I) = \sum_{J \in \mathcal{H}(I)} \omega_{F_{f,K}}(J) = Pf_f(IK) Pf_f(K)^{n-1}. \quad (A.7)$$

Proof. Let $I = i_1i_2 \cdots i_{2n}$. We proceed by induction on $n$. If $n = 1$, it is trivial. (If $n = 2$, (A.7) is implied by Corollary A.2.) Assume the $n-1$ case holds for some $n > 1$. In view of $\mathcal{H}(I) = \bigcup_{k=2}^{2n} J \in \mathcal{H}(I \setminus \{i_1, i_k\}) \{i_1i_kJ\}$, we observe by definition that

$$Pf_{F_{f,K}}(I) = \sum_{k=2}^{2n} \sum_{J \in \mathcal{H}(I \setminus \{i_1, i_k\})} \text{sgn}(I, i_1i_kJ) \ F_{f,K}(i_1, i_k) \prod_{(i,j) \in M(J)} F_{f,K}(i,j).$$

Using $\text{sgn}(I, i_1i_kJ) = \text{sgn}(I, i_1i_k(I \setminus \{i_1, i_k\})) \text{sgn}(I \setminus \{i_1, i_k\}, J)$, we have

$$Pf_{F_{f,K}}(I) = \sum_{k=2}^{2n} \text{sgn}(I, i_1i_k(I \setminus \{i_1, i_k\})) \ Pf_f(i_1i_kK) \ Pf_{F_{f,K}}(I \setminus \{i_1, i_k\}).$$

Using the induction hypothesis, we have $Pf_{F_{f,K}}(I \setminus \{i_1, i_k\}) = Pf_f((I \setminus \{i_1, i_k\})K) Pf_f(K)^{n-2}$. By virtue of Corollary A.2 it is immediate to verify (A.7) for any $n$. \hfill \Box

From here we consider identities for determinants. Assume the set $A$ of alphabets is a disjoint union of $\overline{A}$ and $\underline{A}$, i.e. $A = \overline{A} \cup \underline{A}$. Let $R$ and $C$ be any sets of alphabets which possess injections $R \to \overline{A}$ and $C \to \underline{A}$, denoted by $i \mapsto \overline{i}$ and $j \mapsto \underline{j}$, respectively. For instance, we let $A = R = C$ be the set of positive integers, and $\overline{A}$ and $\underline{A}$ the sets of odd and even integers, respectively. Then we may put $\overline{i} = 2i - 1$ and $\underline{j} = 2j$, which define the injections $R \to \overline{A}$ and $C \to \underline{A}$. For a pair $I = i_1i_2 \cdots i_n \in R^*$ and $J = j_1j_2 \cdots j_n \in C^*$ of words of length $n$, we introduce the word $m(I, J) = \overline{i_1} \underline{j_1} \overline{i_2} \underline{j_2} \cdots \overline{i_n} \underline{j_n} \in A^*$ of length $2n$. Let $g$ be a map which assigns an element of a commutative ring to each pair $(i, j) \in R \times C$. We then define a skew symmetric map $f_g$ on $A \times A$ as follows:

$$f_g(i, j) = \begin{cases} g(k, l) & \text{if } i = \overline{k} \in \overline{A} \text{ and } j = \underline{l} \in \underline{A}, \\ -g(l, k) & \text{if } i = \underline{k} \in \underline{A} \text{ and } j = \overline{l} \in \overline{A}, \\ 0 & \text{otherwise.} \end{cases} \quad (A.8)$$

We also use the notation $\text{det}_g(I, J)$ of determinant

$$\text{det}_g(I, J) = \text{det}(g(i, j))_{i \in I, j \in J} = \text{det}(g(i_k, j_l))_{1 \leq k, l \leq n},$$

where $I = i_1i_2 \cdots i_n \in R^*$ and $J = j_1j_2 \cdots j_n \in C^*$.

A determinant can be expressed as a Pfaffian.
Proposition A.4 (cf. [3, 11, 18]). Let $I \in R^*$ and $J \in C^*$ be words such that $\sharp I = \sharp J$. Then it holds that

$$\text{Pf}_{f_g}(m(I,J)) = \det_g(I,J).$$

Proof. Let $I = i_1i_2\cdots i_n$ and $J = j_1j_2\cdots j_n$. To compute $\text{Pf}_{f_g}(m(I,J))$, we need to consider only perfect matchings on $m(I,J) = \overline{i_1j_1i_2j_2\cdots i_nj_n} \in A^*$ whose arcs are all between $\overline{A}$ and $\overline{A}$; recall (A.8). The set of such perfect matchings is in one-to-one correspondence with $S_n$. To simplify the description, we first rearrange the word $m(I,J)$ to be

$$m(I,J)' = \overline{i_1i_2\cdots i_nj_nj_2j_1}$$

and then consider its perfect matching $P_{\sigma} = \overline{i_1j_{\sigma(1)}i_2j_{\sigma(2)}\cdots i_nj_{\sigma(n)}}$ for each $\sigma \in S_n$. Because $\text{sgn}(m(I,J), m(I,J)) = 1$, we have

$$\text{Pf}_{f_g}(m(I,J)) = \text{Pf}_{f_g}(m(I,J)) = \sum_{\sigma \in S_n} \text{sgn}(m(I,J)', P_{\sigma}) \prod_{k=1}^{n} g(i_k, j_{\sigma(k)})$$

(see (A.11)) and

$$\text{sgn}(m(I,J)', P_{\sigma}) = \text{sgn}(m(I,J), P_{\sigma}) = \text{sgn} \sigma,$$

which complete the proof. \qed

Combining Corollary A.3 and Proposition A.4 leads to the following determinant identity, which we may call Sylvester’s identity.

Corollary A.5. Let $I, K \in R^*$ and $J, M \in C^*$ be words such that $\sharp I = \sharp J = n$ and $\sharp K = \sharp M$. Let $G_{g,K,M}$ be a map on $R \times C$ defined by $G_{g,K,M}(i,j) = \det_g(iK,jM)$. Then it holds that

$$\det G_{g,K,M}(I,J) = \det (\det_g(iK,jM))_{i \in I, j \in J} = \det_g(IK,JM) \det_g(K,M)^{n-1}. \tag{A.9}$$

Finally, let us derive the determinant identity (6.7) from Corollary A.5. For notation, recall (3.4), (3.10) and (3.11). Let $R = C = \{1, 2, \ldots, (L - 1)(n + 1)\}$ and put

$$g(i,j) = b^*_i - s(n+1) \quad \text{with} \quad s = \left\lfloor \frac{j}{n+1} \right\rfloor + 1$$

for $(i,j) \in R \times C$, where $\lfloor x \rfloor$ denotes the largest integer which does not exceed $x$. We take the words $I = i_1i_2\cdots i_{L-1} \in R^*$ and $J = j_1j_2\cdots j_{L-1} \in C^*$ of length $L - 1$ given by

$$i_k = (L - 1)n + k \quad \text{and} \quad j_k = (k - 1)(n+1) + 1 \quad \text{for} \quad 1 \leq k \leq L - 1.$$
Let \([i, j]\) denote the word \(i(i + 1) \cdots j\) for \(i < j\); e.g. \(I = [(L - 1)n + 1, (L - 1)(n + 1)]\). We take the words \(K = [1, (L - 1)n] \in \mathbb{R}^*\) and \(M = [1, (L - 1)(n + 1)] \setminus J \in \mathbb{C}^*\) of length \((L - 1)n\). Then it holds that \(\det_g(K, M) = D_n\) and

\[
\det_g(IK, JM) = (-1)\frac{L(L-1)n}{2} \det_g([(1, (L - 1)(n + 1)], [1, (L - 1)(n + 1)])
\]

\[
= (-1)\frac{L(L-1)n}{2} D_n+1
\]

since both \(IK\) and \(JM\) can be rearranged to be \([1, (L - 1)(n + 1)]\) and

\[
\sgn(IK, [1, (L - 1)(n + 1)]) \sgn(JM, [1, (L - 1)(n + 1)]) = (-1)^\frac{L(L-1)n}{2}.
\]

In a similar manner, it holds that

\[
\det_g(i_kK, j_lM) = (-1)^{(L-l)n}E_{n}^{k,l}
\]

for \(1 \leq k, l \leq L - 1\). Hence we obtain (6.7) from (A.9) with \(n\) replaced by \(L - 1\).

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Masao Ishikawa  
Department of Mathematics  
Okayama University  
Okayama 700-8530, Japan  
e-mail: mi@math.okayama-u.ac.jp

Toshiyuki Mano  
Department of Mathematical Sciences  
University of the Ryukyus  
Okinawa 903-0213, Japan  
e-mail: tmano@math.u-ryukyu.ac.jp

Teruhisa Tsuda  
Department of Economics  
Hitotsubashi University  
Tokyo 186-8601, Japan  
e-mail: tudateru@econ.hit-u.ac.jp