Indefinite causal order is not always a resource for thermodynamic processes

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Indefinite causal order is a key feature involved in the study of quantum higher order transformations. Recently, intense research has been focused on possible advantages related to the lack of definite causal order of quantum processes. Quite often the quantum switch is claimed to provide advantages in information-theoretic and thermodynamic tasks. We address here the question whether indefinite causal order is a resource for quantum thermodynamics. Inspired by previous results in the literature taking free energy and ergotropy as the figures of merit, we propose a framework for properly comparing the thermodynamic value of processes by comparing higher order transformations of the same type, and show that for the tasks considered here indefinite causal order is not necessary for thermodynamic advantages. More specifically, we show that there is a non-Markovian process, a causally ordered higher-order transformation, outperforming the results obtained for the quantum switch. We also discuss a possible way to study the advantages that may arise from indefinite causal order in a general scenario.

I. INTRODUCTION

Classical and quantum physics are causal in the sense that the relative temporal order between two given (causally connected) events are always definite [1]. However, there exist scenarios in which we can locally ascribe an indefinite causal order to quantum events. References [2, 3] introduce indefinite causal order from the perspective of operational probability theories, while a model based on the structure of the Hilbert space is considered in Refs. [4–6]. Many theoretical and experimental developments followed these pioneering studies. Among these achievements, we can cite applications to thermodynamics [7–11], quantum nature of gravity [12–14], relativistic quantum information [15, 16], foundations of quantum mechanics [17–21], communication theory [22–24], quantum computation [25, 26] quantum metrology [27, 28], and other information-theoretic tasks [29–31], just to mention a few recent ones. Recent experiments on these lines were also reported [32–36] (see also the review [37]).

A quantum process is said to have indefinite causal order if it cannot be written as a probabilistic mixture of processes with a fixed causal order, the so-called causally separable processes [3, 38, 39]. An interesting class of higher-order quantum operations violating this condition is the set of processes with quantum control of causal order, including the remarkable example of the quantum switch [4]. This class consists of processes where the order in which events occur is controlled by a quantum system, and it has been shown to be a valuable resource for various information-processing tasks [4, 29, 31, 40–44]. A resource theory approach for quantum control of causal orders was developed in Refs. [45, 46].

The quantum switch was also employed in the context of quantum thermodynamics [7–10]. In Ref. [7], the quantum switch applied to two measurement channels was employed in the study of a thermal device (such as a heat engine, for instance). The authors of Ref. [8] claimed that the lack of causal order is responsible for an advantage in a refrigeration cycle over the ordered sequential use of quantum channels. In Ref. [9], it has been considered that the thermalization process given by two distinct channels taking place in indefinite causal order provided by a quantum switch can enhance work extraction, when compared with the sequential version. In Ref. [10], the ergotropy was employed as a figure of merit in order to provide similar results as [9]. It is interesting to note that these results are based on the same fundamental task, the implementation of quantum control of causal order.

In this work we consider the role of indefinite causally ordered processes in quantum thermodynamics. We address the question whether indefinite causal order implies any advantage for quantum thermodynamic tasks. Taking two figures of merit —namely, free energy and ergotropy—we show that indefinite causal order is not a fundamental resource for the tasks considered here. We address the same setting considered in Refs. [9, 10]. Moreover, we show that non-Markovian quantum processes—that is, causally ordered processes with environment-mediated temporal correlations—imply thermodynamic advantages similar or above to the quantum switch devices.

It is important to observer here that, instead of ruling out the possibility that the lack of causal order can be identified as a thermodynamic resource, our results indicate that a novel approach must be considered in order to identify such contribution in a general scenario. We briefly discuss this issue at the end of the paper.

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II. QUANTUM PROCESSES

There are several equivalent formulations of the process formalism. For our purposes, it is convenient to focus on the supermap formulation, first introduced in Ref. [47]. The word supermap refers to the fact that it is a transformation of maps into maps. Quantum channels are defined as completely positive and trace preserving linear maps between operator spaces [48]. In turn, a quantum process is a multilinear supermap that maps multiple input quantum channels into an output quantum channel.

A process is said to be \( n \)-partite if it is a supermap of type \( \mathcal{W}: (A_1, \cdots, A_n) \to \mathcal{B} \), mapping a sequence of \( n \) quantum channels \( (A_i)_{i=1}^n \) into the quantum channel \( \mathcal{B} \). The input and output spaces of \( \mathcal{B} \) are called the global past and future of the process, respectively.

Consider a collection of bipartite quantum channels

\[
A_j: A_I^{(j)} \to A_O^{(j)},
\]

with \( j \in \{1, \ldots, n\} \), where \( I \) and \( O \) labels input and output systems, respectively. A process acts trivially on the ancillary systems \( \tilde{A}_I^{(j)}, \tilde{A}_O^{(j)} \) if it is of type \( \mathcal{W} \otimes \mathbb{I}_I \otimes \cdots \otimes \mathbb{I}_n \), with \( \mathbb{I}_i \) being the identity supermap acting on the space of linear maps \( A_I^{(j)} \to A_O^{(j)} \). We refer to Refs. [47, 49] for a discussion on monopartite processes and consider here only bipartite and tripartite processes. The extension of the results presented here to the multipartite case is straightforward.

By means of the Choi representation of channels [48], it is possible to represent a process \( \mathcal{W} \) in terms of an operator \( \mathcal{W} \) called process matrix of \( \mathcal{W} \) [5, 47]. Here, the Choi operator of a linear map \( \mathcal{A} \) is denoted as \( J_{\mathcal{A}} \). Thus, if \( \mathcal{C} = \mathcal{W}(A,B) \) (mapping \( A \) and \( B \) into \( C \)), then it follows that

\[
J_{\mathcal{C}} = T_{\mathcal{A}B}[\mathcal{W}^T_{\mathcal{A}B}(J_{\mathcal{A}} \otimes J_{\mathcal{B}} \otimes \mathbb{I})],
\]

where \( A = (A_I, A_O) \) (with equivalent definitions for \( B \) and \( C \)), \( T_{\mathcal{A}B} \) is the partial transposition of subsystems \( A \) and \( B \), while \( \mathbb{I} \) represents the identity operator acting on \( C \). Figure 1 shows a diagram illustrating such representation of quantum processes.

In order for \( \mathcal{W} \) to be a valid process, i.e., a superchannel that maps quantum channels into quantum channels, its associated process matrix must satisfy some specific conditions, explicitly stated in Ref. [38].

A. Relevant processes

In this work, we will compare the performance of different processes for two thermodynamic tasks: free energy and ergotropy extraction. Our processes of interest are defined in what follows. The particular examples considered in this manuscript have been chosen in order to be compared with the previous results reported in Refs. [9, 10].

1. Channel composition and probabilistic mixtures

Let us consider a simple bipartite process, which is given by a composition of two channels. There are two processes of this type, namely

\[
\mathcal{W}_{A\to B}(A,B) = B \circ A
\]

and

\[
\mathcal{W}_{B\to A}(A,B) = A \circ B.
\]

In the first case, \( A_O \) is a copy of \( B_I \), while in the second one, \( B_O \) is a copy of \( A_I \). See Fig. 2 for a pictorial representation.

Additionally, if all the involved spaces are copies of each other, we can define a probabilistic mixture of such processes as

\[
\mathcal{W} = q \mathcal{W}_{A\to B} + (1 - q) \mathcal{W}_{B\to A},
\]
where $|I\rangle = \sum |i\rangle \otimes |i\rangle$ is the pure Choi representation of the identity operator for an orthonormal basis $\{|i\rangle\}$, while the right-hand side of Eq. (6) is ordered as $C_1A_1A_2B_1B_2C_0$.

Defining $|B\rightarrow A\rangle$ similar to Eq. (6), but with order $C_1B_1B_2O_1A_1A_2O_2C_0$, we can construct the process matrices of the processes $W_{A\rightarrow B}$ and $W_{B\rightarrow A}$ as

$$W_{A\rightarrow B} = |A\rightarrow B\rangle \langle A\rightarrow B|$$

and

$$W_{B\rightarrow A} = |B\rightarrow A\rangle \langle B\rightarrow A|,$$

respectively. A process is called pure if it is represented by a rank-1 process matrix $W = |w\rangle \langle w|$ [38]. Moreover, we say that $|w\rangle$ is the process vector [38].

The process matrix associated with the process defined in Eq. (5) is given by

$$W = qW_{A\rightarrow B} + (1-q)W_{B\rightarrow A}.$$  

Using Eq. (9) together with Eq. (2), we have a matrix representation of the process defined in Eq. (5).

2. Non-Markovian processes

A Markov process is defined by the composition of quantum channels similarly to Eq. (3), but allowing for local operations on the input and output systems of the quantum channels $A$ and $B$.

Let $N_1: C_1 \rightarrow A_1$, $N_2: A_2 \rightarrow B_1$, and $N_3: B_2 \rightarrow C_0$ be quantum channels. A Markov process is a supermap of the type

$$W(A,B) = N_3 \circ B \circ N_2 \circ A \circ N_1.$$  (10)

We note that the composition process is recovered whenever $N_1, N_2$ and $N_3$ are identity channels. See top panel of Fig. 3, and compare it with the top panel of Fig. 2.

On the other hand, a process is non-Markovian when there are non-trivial system-environment correlations through a sequence of quantum operations [51–58]. We refer the reader to Refs. [59, 60] for review articles, and to Refs. [61–63] for applications in quantum stochastic thermodynamics. In general, a bipartite non-Markov quantum process is of the type

$$W_{A\rightarrow B}(M_1, M_2) = \sum_{i=1}^{n} W_{i} \otimes N_i,$$

for appropriate fixed bipartite quantum channels $N_1$, $N_2$ and $N_3$. The operation $\mathcal{T}$ is the identity channel acting on the environment $E$. The bottom panel of Fig. 3 is a representation of non-Markov processes. Note that whenever $E_1$ and $E_2$ are trivial one-dimensional systems we recover the case of Markov processes.

In Eq. (11), the quantum operation $A$ precedes the operation $B$. Nevertheless, we can define a non-Markovian process such that the opposite situation happens, denoted as $B \prec A$. A quantum process is called causally separable if it can be written as a probabilistic mixture of (possibly non-Markovian) processes with definite, albeit possibly different, causal orders [5]. That is

$$W_{sep} = qW_{A\rightarrow B} + (1-q)W_{B\rightarrow A},$$

with $0 \leq q \leq 1$.

3. Bipartite quantum switch

A more involved example is the quantum control of causal order. The bipartite quantum switch has been introduced in Ref. [29] in order to deal with quantum computation in processes with indefinite causal order.

The quantum switch has a bipartite global past and future denoted as $C^{(1)}_I$ and $C^{(1)}_O$, respectively. The ancilla $C^{(2)}_I$ is a qubit system responsible for controlling the two possible orderings. The space $C^{(1)} := C^{(1)}_I \otimes C^{(1)}_O$ is a copy of the systems $A$ and $B$.

![Diagram of probabilistic mixture of channel compositions](image-url)
Here, we consider a quantum control of the orders \( A \rightarrow B \rightarrow C \) and \( C \rightarrow B \rightarrow A \), given by the process vector
\[
| w_{\text{switch}}^{(3)} \rangle = \frac{1}{\sqrt{2}} \left[ |A \rightarrow B \rightarrow C \rangle \otimes |0\rangle + |C \rightarrow B \rightarrow A \rangle \otimes |1\rangle \right].
\] (16)

The vectors \(|A \rightarrow B \rightarrow C \rangle \otimes |0\rangle\) and \(|C \rightarrow B \rightarrow A \rangle \otimes |1\rangle\) are defined as \(|I\rangle \otimes |I\rangle \otimes |0\rangle\) and \(|I\rangle \otimes |I\rangle \otimes |1\rangle\), with systems ordered as \(D_1^{(1)}ABCD_2^{(1)}D_2^{(2)}\) and \(D_1^{(1)}CBAD_2^{(1)}D_2^{(2)}\), respectively. The systems \(D_1^{(1)}\) and \(D_2^{(2)}\) are the target and the control systems, respectively. Thus, the process represented by \(W_{\text{switch}}^{(3)} = |w_{\text{switch}}^{(3)}\rangle \langle w_{\text{switch}}^{(3)}|\) is tripartite with global bipartite past and future.

5. A tripartite process with indefinite causal order

We consider the tripartite process \(W_{\text{det}}\) defined in Ref. [64]. Each channel's input (output) system \(A_{I(O)}\), \(B_{I(O)}\), \(C_{I(O)}\) is a qubit. The global past and future spaces, \(D_I\) and \(D_O\), consist of three qubits each, and therefore can be written as a triple of qubits \(D_{I(O)} = D_I^{(1)}D_I^{(2)}D_I^{(3)}\). This process has been shown to violate causal inequalities [65], being therefore regarded as a process with indefinite causal order. It is defined by the process matrix
\[
W_{\text{det}} = |w_{\text{det}}\rangle \langle w_{\text{det}}|,
\] (17)

with the associated process vector
\[
|w_{\text{det}}\rangle = \sum_{i,j,k,r,s,t} |r \oplus \neg j \land k, s \oplus \neg k \land i, t \oplus \neg i \land j\rangle \otimes |i, j, k\rangle \otimes |i, j, k\rangle
\] (18)

being ordered as \(A_1B_1C_1D_1D_2A_2B_2C_2D_2\). The summations run over \(\{0,1\}\), while \(\oplus\) represents modulo-2 sum. The logical NOT and AND operations are represented by \(\neg\) and \(\land\), respectively.

The higher-order quantum operation \(W_{\text{det}}\) was first defined in Ref. [65], and is a valuable instance of process violating causal inequalities [65]. The superchannel version presented here appeared in Ref. [64], where it has been shown to be an example of a process violating causal inequalities, and still, preserving unitary quantum operations. This process corresponds to a purely classical process with no causal order that can be interpreted as a closed time-like curve [66, 67].

III. QUANTUM PROCESSES AND THERMODYNAMIC TASKS

Now we are ready to present the main results of this study. This section is organized as follows. In Sec. III A

4. Tripartite quantum switch

We can also define a quantum control of the orders of three quantum operations \(A\), \(B\) and \(C\). This quantum process is called tripartite quantum switch, and defined as follows.
we review the scenario in which indefinite causal order has been claimed to provide an advantage for quantum thermodynamics in Refs. [9, 10]. In Sec. III B we present a critique on the comparison of processes previously presented in literature. In Sec. III C we show our main results. In particular, we show that there is a causally ordered non-Markovian quantum process with similar performance as the quantum switch with respect to the thermodynamic tasks addressed in Refs. [9, 10]. Moreover, we show the existence of a quantum process with indefinite causal order for which no thermodynamic advantage can be extracted.

A. Previous work

We consider quantum processes acting on quantum channels whose input and output are qubit systems. The operations examined here are the Generalized Amplitude Damping (GAD) channel $\mathcal{R}_{p,\lambda}$ [68, 69], depending upon $0 \leq p \leq 1$ and $0 \leq \lambda \leq 1$, and the Phase Flip (PF) channel $\mathcal{T}_q$, parametrized by $0 \leq q \leq 1$. The GAD for $\lambda = 1$ is denoted as $\mathcal{R}_p := \mathcal{R}_{p,1}$ [69].

The quantum operations $\mathcal{R}_{p,\lambda}$ and $\mathcal{T}_q$ are among the most studied models of typical noise presented in open quantum systems [70]. Importantly, the composition of $\mathcal{R}_p$ and $\mathcal{T}_q$ in any order results in a thermalization process for input systems with diagonal quantum states in the energy eigenbasis. Let the initial system $S_I$ be in a classical state with respect to the energy eigenbasis, that is, $\rho = r |0\rangle \langle 0| + (1 - r) |1\rangle \langle 1|$ with $0 \leq r \leq 1$. Thus, any of the channel compositions $\mathcal{R}_p \circ \mathcal{T}_q$ and $\mathcal{T}_q \circ \mathcal{R}_p$ results in an output system $S_O$ in the thermal state [9]

$$\tau = \frac{\exp(-\beta H)}{\text{Tr}[\exp(-\beta H)]}, \quad (19)$$

with inverse temperature $\beta = \log_2[p/(1 - p)]$ and Hamiltonian $H = |1\rangle \langle 1|$. That is because $\mathcal{T}_q$ preserves the diagonal states, while $\mathcal{R}_p$ transforms diagonal states into $\tau$. This defines a thermal state and an inverse temperature of reference. Note this is the same setting considered in Refs. [9, 10], so we can properly compare our results.

In order to study quantum processes from the quantum thermodynamics perspective, we consider two figures of merit: free energy, as considered in Ref. [9], and ergotropy, studied in Ref. [10].

The authors of Refs. [9, 10] showed that plugging the channels $\mathcal{R}_p$ and $\mathcal{T}_q$ into the quantum switch, thus implementing a quantum control of the order in which they act, results in a thermodynamic advantage with respect to all probabilistic mixtures of the channel compositions. That is, the quantum switch implies an increase in free energy or ergotropy when compared to the simple compositions of channels, or any probabilistic mixtures of the compositions. Moreover, it has been claimed that the indefinite causal order represented by the quantum switch operation implies in a thermodynamic advantage [9, 10]. Here, we argue that indefinite causal order solely do not result in thermodynamic advantages for quantum processes when considered an appropriate and reasonable comparison of superchannels.

B. Comparison of processes

The quantum switch is defined in Refs. [9, 10] with extra resources when compared to the composition of channels. Namely, the ancillary system responsible for the coherent control of orders in which the quantum operations act on the target system. Therefore, it is not entirely fair to make comparisons of any property of a quantum system based solely on this two particular processes.

In fact, if any causally ordered, bipartite quantum process is available, one can always trivially increase the free energy (or the ergotropy) of a quantum system. We show in Fig. 4 that this can be done by a non-Markovian process. For instance, let $\sigma$ be a quantum state with the desired property (free energy or ergotropy). We can define a causally ordered quantum process such that any input $(\mathcal{A}, \mathcal{B})$ is mapped to the replacement channel $C(X) = \text{Tr}[X|\sigma]$. Thus, independently of the causal order of the action of the channels $\mathcal{A}$ and $\mathcal{B}$, and of the system’s global past state, we can have arbitrary thermodynamic advantages defined by the fixed quantum state $\sigma$.

In order to consider less trivial examples, we show in the following that in both cases there is a causally ordered non-Markovian process not defined through a simple swapping of system and ancilla, and yet with similar results as the quantum switch. Furthermore, we show that the advantage arises in a scheme fully analogous to

![Diagram of Causally ordered bipartite quantum process](image)
the one considered in Refs. [9, 10] for the quantum switch, where the ancillary system is measured and the relevant figure of merit (respectively, free energy and ergotropy) is evaluated for the resulting target state conditioned on the measurement outcome.

Here, we expand the class of superchannels considered to all processes of the same type, and hence accounting for equivalent resources in the comparison. We consider all bipartite processes with bipartite global past and future systems. This is precisely the class of processes of the same form as the quantum switch, considered previously in [9, 10]. See Fig. 5 for an illustration. Here, we also consider tripartite processes with bipartite global past and future systems.

The state of the system \( S_I \) is initially uncorrelated with a pure state of the ancilla \( Q_I \), i.e., the initial global state of \( S_I Q_I \) is \( \rho \otimes |\varphi\rangle \langle \varphi| \), for some vector \( |\varphi\rangle \).

After the action of the process on the channels, the global state of the output system \( S_O Q_O \) is possibly entangled—depending upon the particular process. Then, a local projective measurement in a particular orthonormal basis \( \{|z_0\}, \{|z_1\} \} \) is performed on the ancillary system. The measurement is defined by the operators \( M_j = I \otimes Z_j \), with \( Z_j = |z_j\rangle \langle z_j| \) and \( j = 0, 1 \). Measuring the outcome \( j \in \{0, 1\} \) with probability (with similar definition for tripartite processes)

\[
p_j = \text{Tr}[M_j \mathcal{W}[A, B](\rho \otimes |\varphi\rangle \langle \varphi|)],
\]

results in the conditional post-measurement state

\[
\sigma_j = \frac{1}{p_j} M_j \mathcal{W}[A, B](\rho \otimes |\varphi\rangle \langle \varphi|) M_j^{\dagger}.
\] (20)

Here, our study focus on the average output free energy and ergotropy defined as

\[
T(\mathcal{W}) := \sum_{j=0,1} p_j T(\sigma_j),
\] (21)

where \( T \) generically denotes free energy \( F \) or ergotropy \( E \) of a quantum system. Note that the quantity defined in Eq. (21) is also a function on the initial state of \( S_I \), input quantum channels \( (A, B) \), and measurement operators \( \{M_0, M_1\} \).

C. Non-Markovian processes as a thermodynamic resource

As we have seen, it is easy to reproduce an advantage in a thermodynamic process (increasing free energy or ergotropy) when an additional system, beyond the original one of interest is available. The simplest example is the process shown in Fig. 4. No matter what state one starts with, and regardless of what the channels do, the output target state is always \( \sigma \), which can have arbitrary thermodynamic properties.

We consider here a less trivial example, for which the external system interact with the one of interest in a non-trivial way, and such that a final measurement on the external system and classical communication with the target is necessary to obtain an increase of free energy. This is closer in spirit to how the bipartite quantum switch was used to extract free energy and ergotropy in Refs. [9, 10].

Let our quantum system of interest \( S_I \) interact with an ancillary qubit system \( Q_I \) through the Ising Hamiltonian

\[
H_{\text{ising}} = -\sigma_x \otimes \sigma_x - (I \otimes \sigma_z + \sigma_z \otimes I),
\]

where \( \sigma_z = |0\rangle \langle 1| + |1\rangle \langle 0| \) is a Pauli matrix.

That defines a bipartite unitary quantum channel

\[
\mathcal{U}(\rho) = U_{\text{ising}} \rho U_{\text{ising}}^{\dagger},
\]

with \( U_{\text{ising}} := \exp (-iH_{\text{ising}}) \). We set the Plank constant \( \hbar \) and time displacement equal to unit in the evolution operation \( U_{\text{ising}} \).

Let us define the bipartite non-Markovian process

\[
\mathcal{W}_{\text{ising}}^{(2)} (R, T) := \mathcal{U} \circ (T \otimes I) \circ \mathcal{U} \circ (R \otimes I) \circ \mathcal{U},
\] (22)

where the identity map acts on the ancillary system. The top panel of Fig. 6 shows a diagram for the process above. Analogously, we define the tripartite non-Markovian process (see the bottom panel in Fig. 6)

\[
\mathcal{W}_{\text{ising}}^{(3)} (R, T, T) := \mathcal{U} \circ (T \otimes I) \circ \mathcal{U} \circ (R \otimes I) \circ \mathcal{U} \circ (R \otimes I) \circ \mathcal{U}.
\] (23)

For the case of bipartite processes we compare the following examples.

- **Composition of channels.** We consider the consecutive application of the channels \( R_p \) and \( T_q \).

- **Bipartite quantum switch \( W_{\text{switch}}^{(2)} \).** We consider \( (R_p, T_q) \) as input channels. The control system is initially prepared in the state \( |+\rangle \), thus, providing maximal control of the causal orders.
Bipartite non-Markovian process $W_{\text{ising}}^{(2)}$. We consider the input channels $(R_p, T_q)$. The ancillary qubit is prepared in the state $|+\rangle$.

Considering tripartite processes, we compare the following examples.

- Composition of channels. We consider the consecutive application of two GAD and one PF channel.
- Tripartite quantum switch $W_{\text{switch}}^{(3)}$. We consider the input channels $(R_p, R_p, T_q)$. The orders considered are $A < B < C$ and $C < B < A$. The control system is initially prepared in the state $|+\rangle$.
- Tripartite pure quantum process $W_{\text{det}}$. We consider the input channels $(R_p, R_p, T_q)$. The second ancillary system is defined with input in the quantum state $|0\rangle \langle +|)$ for free energy (ergotropy) calculations and then the output system is discarded, thus, implementing a tripartite quantum superchannel with bipartite global past and future.
- Tripartite non-Markovian process $W_{\text{ising}}^{(3)}$. We consider the input channels $(R_p, R_p, T_q)$. The ancillary system is started in the quantum state $|+\rangle$.

1. Free energy

We consider here the free energy of a qubit system $S$ after the action of different quantum processes. We take a system with initial and final Hamiltonian $H = |1\rangle \langle 1|$. With respect to a thermal reservoir at inverse temperature $\beta$, we define the free energy of the quantum system $S$ in the state $\rho$ as \cite{71, 72}

$$F_{\beta}(\rho) = \text{Tr}[H\rho] - \beta^{-1} \ln(2)S(\rho),$$

where $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy of a quantum state.

Let us start considering bipartite quantum processes.

Figure 7 shows that we can use the non-trivial interaction between $S_Q$ and $Q_O$ in order to increase the output free energy. As reported in \cite{9}, the quantum switch has an associated average free energy higher than the one for the channel composition for any value $r > 0$. This allows the output system $S_Q$ to extract work, situation which is not possible with the channel composition solely. Nevertheless, this result is not exclusive for processes with indefinite causal order. The non-Markovian process defined in Eq. (22) has similar behaviour. In fact, it results in higher average free energy when compared with the bipartite quantum switch for any value of $r$. Furthermore, this examples show that indefinite causal order is not a necessary condition on the thermodynamic advantages considered here.

Now, let us consider the case of tripartite processes with bipartite global past and future. Similar conclusions with respect to the case of bipartite processes can be made. That is, the tripartite causally ordered non-Markovian process in Eq. (23) implies a greater average output free energy when compared with the tripartite quantum switch defined in Eq. (16).

Now, we examine the existence of a tripartite process with indefinite causal order, but with no thermodynamic advantage. We can use the process defined in Eq. (17) in order to build a tripartite process with bipartite global past and future. That can be achieved by feeding the overall channel $W_{\text{det}}(A, B, C)$ with an input state $|0\rangle \langle 0|$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image1.png}
\caption{Non-Markov processes defined through Ising model interactions. The bipartite and tripartite processes defined in Eqs. (22) and (23) are represented in top and bottom panels, respectively. Here, we consider the ancillary system $Q_I$ in the pure quantum state $|+\rangle$, and perform a projective measurement on $Q_O$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image2.png}
\caption{Free energy for different quantum processes. The global past state is the classical state $\rho = r |0\rangle \langle 0| + (1 - r) |1\rangle \langle 1|$. The ancillary global past state is $\varphi = |+\rangle \langle +|$, with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The local projective measurement on the ancillary global future state is defined by the basis $\{|+: -\rangle\}$. The channel parameters considered here are $\lambda = 1$, $p = q = 0.8$.}
\end{figure}
of the system $C^{(3)}_f$ and tracing out the resulting output system $C^{(3)}_O$. In the following discussion we also denote this process with the symbol $W_{\text{det}}$. Figure 7 shows that for the parameters considered, $p = q = 0.8$, the process $W_{\text{det}}$ do not imply in a thermodynamic advantage when compared with the composition of channels. Thus, indefinite causal order solely is not a sufficient condition on the advantage addressed here.

2. Ergotropy

In contrast to free energy, representing the maximum work that can be extracted from the system while it is in contact with a thermal bath (i.e. at a constant temperature), ergotropy refers to work extractable from a system using cyclic unitaries. In principle, certain unitaries can be used for maximum work extraction which transform the initial state of the system into a thermal state, called completely passive, from which no work can be extracted. Therefore, this quantity – ergotropy – provides a bound on the extractable work from a system using cyclic unitaries. The formal definition, for an initial state $\rho$ is given as [73] follows:

$$E(\rho) = \max_{U} \text{Tr} \left[ H(\rho - U\rho U^\dagger) \right]. \quad (24)$$

Here, the cyclic unitary process can be achieved by a time-dependent Hamiltonian $H$ which drives the system such that $H(t) = H(0)$, where $t$ is the driving period. During the driving period, $H(s) = H(0) + V(s)$, where $V(s)$ can be identified as the driving Hamiltonian, for $0 < s < t$. While calculating the ergotropy of a system, by definition, requires a maximization over all possible unitaries, here we have considered a single qubit system for which the ergotropy can be analytically computed from its state $\rho$.

In general, the formalism of work extraction using cyclic unitary processes can also be extended to quantum channels where the system is governed by open quantum dynamics during the driving period. Here, we consider the initial and final states of a qubit, which undergoes transformation through the quantum channels based on quantum processes, as discussed in the previous section. The ergotropy vested in the initial and final states of the qubit can be compared and the changes, if any, can be calculated.

Another important concept that we have used extensively henceforth is that of Daemonic ergotropy [74]. This is the ergotropy obtained upon the projective measurement of the ancilla ($A$) such that the post measurement outcome gives some information about the target system ($S$). The gain due to access to this information, by virtue of correlation between ancilla and the target is called daemonic gain on account of its similarity to the Maxwell’s daemon, where thermodynamic work is enhanced by access to extra information about the state of the system. For defining the Daemonic ergotropy, the probabilities associated with measurement of the ancilla as required, which are given as follows:

$$p_\alpha = \text{Tr} \{ \pi_\alpha \rho_{SA} \}, \quad (25)$$

where $\alpha$ indexes the measurement outcome obtained through projective measurement operator $\pi_\alpha$ on the joint state of system and ancilla, given as $\rho_{SA}$. Upon the post-selection of the measurement outcome of the ancilla, the state of the target qubit (of which the ergotropy is measured) is given as:

$$\rho_{SA}^\alpha = \frac{1}{p_\alpha} \text{Tr}_A \{ \pi_\alpha \rho_{SA} \} \quad (26)$$

As such, the average Daemonic ergotropy is given as:

$$E_{\text{DA}}^D = \sum_\alpha p_\alpha E(\rho_{SA}^\alpha) \quad (27)$$

It is important to note that the states $\rho_{SA}$, and $\rho_{SA}^\alpha$ that we consider here are the joint output states, and the conditional state of the target respectively, obtained after application of channels based on quantum processes. The post selection of the ancilla, therefore, ensures that an appropriate evolution is selected for the target qubit, which results in a certain ergotropy. Since the task is to maximise the energetic content of the (post selected) state (see Eq. 24), it translates to finding the appropriate orthogonal basis $\{ \pi_\alpha \}$ of the measurement of ancilla, which maximises the daemonic ergotropy in Eq. (27).

Here, the generalized measurement bases (orthogonal) considered for the ancillae in different cases are as follows:

$$|M_1\rangle = \sqrt{p(0)} + e^{i\phi} \sqrt{1 - p}|1\rangle, \quad |M_2\rangle = -e^{i\phi} \sqrt{1 - p}|0\rangle + \sqrt{p}|1\rangle. \quad (28)$$

This measurement basis is used for the control qubit in the case of switch (bipartite/tripartite) and the ancilla in the process $W_{\text{det}}$, and the (bipartite/tripartite) non-Markovian processes. The probabilities obtained after the measurement in this basis are used as weights for the ergotropy of the target (which is calculated analytically from reduced state of the target system, after the measurement of the control), as outlined above in Eq. (27).

Additionally, we also allow an added flexibility on the initial (input) state of the ancilla/control. This is given as: $\rho_A = \cos(x/2)|0\rangle\langle 0| + e^{i\chi} \sin(x/2)|1\rangle\langle 1|$, where $x \in [0, \pi]$, and $\chi \in [0, 2\pi]$. Therefore, in all the cases that we consider, the optimization is also performed over the input state of the ancilla, i.e., the parameters $x$, and $\chi$, in addition to the measurement basis in Eq. (28).

Effectively, the task reduces to the optimization over the parameters $p$, $x$, $\chi$ and $\phi$ for maximum Daemonic ergotropy in Eq. (27). We perform this optimization using the basinhopping routine in Scipy, with sequential least square programming (SLSQP) as the core algorithm. As a benchmark, we consider the bipartite quantum switch setup which has been considered previously in [10], and
find that this algorithm is capable of recovering the optimized parameters found analytically therein.

In the previous literature [10], the bipartite quantum switch configuration with GAD and PF has been shown to provide an advantage in ergotropy over the composition of channels, with a particular choice of measurement basis of the ancilla (control). There, it was established that the maximum Daemonic ergotropy is found when the control is prepared in the $|+\rangle$ basis and subsequently measured in the orthogonal basis $\{|+\rangle, |-\rangle\}$ on the output side. Consequently, the Daemonic ergotropy is:

$$E_{\rho_S|\pm}^D = p_+ E(\rho_{S|+}) + p_- E(\rho_{S|-})$$

wherein the ergotropies $E(\rho_{S|\pm})$ can be calculated analytically from the post selected states $\rho_{S|\pm}$. In the aforementioned work, the input target state and the channel parameters considered were $\sqrt{\tau}|0\rangle + \sqrt{\tau^*}|1\rangle$, and $p = 1/3$, $q = 0$, $\lambda = 1/2$ respectively. The gain in ergotropy, compared over population imbalance of the input state $\delta\rho = \rho_{22} - \rho_{11}$, where $\rho_{22}$, $\rho_{11}$ are the diagonal entries of $\rho$—had been attributed to coherent control over the order of application of maps. Here we expand the paradigm of process-induced ergotrophic gains to include the Det process and also exemplary non-markovian processes. We adopt the same channel parameters, and the parametric initial state of the target, and also base our results on the population imbalance in the initial state: $\delta\rho$.

In Fig. 8, we show that for the non-markovian processes, the maximized daemonic ergotropy over ancilla preparation and measurement could be more than the bi- and tri-partite switch, det process, and the separable configuration of channels’ composition. Though the optimized parameters defining the ancilla preparation/measurement bases in all the cases are interesting in their own right, we have skipped their discussion to focus only on the ergotropic gains. Here, we also show in Fig. 8 that for the tri-partite process $W_{det}$ (with the same channel parameters and input target state), the ergotropy may or may not be more than that in the composition process. Since the process $W_{det}$ violates causal inequalities and the quantum switch does not, we prove that the ergotropic gains cannot be simply attributed to indefinite causal order, or violation of the causal inequality. Through the trends obtained in various cases, these results void the possibility of establishing a simple connection between the nature of quantum processes and the daemonic gain in ergotropy possible through preparation and measurement of ancilla in an optimal scenario.

IV. CONCLUSIONS

We considered the role of indefinite causal order for quantum thermodynamics. The results presented in Refs. [9, 10] claimed the indefinite causal order of the switch to be an advantage in output free energy and ergotropy, respectively, when compared with the composition processes for a fixed strategy. We claim here that, under the reasonable comparison of processes of the same type, and with the assistance of the same type of resources, indefinite causal order plays no clear advantage for the above mentioned tasks.

Specifically, we studied the influence of indefinite causal order over the free energy and ergotropy generated by the action of higher order transformations on channels of interest (GAD and PF) and compared it with causally ordered processes with comparable resources—namely, an additional ancillary system. Allowing for measurements on the ancillary systems, we found that the causally ordered processes generally perform similarly or better than those with indefinite causal order. A reasonable interpretation is that the advantages are due to the non-trivial interaction between system and ancilla, rather than to indefinite causal order, as the latter does not necessarily imply higher output free energy and ergotropy. We have considered here examples of bipartite and tripartite processes supporting this claim. Although indefinite causal order is shown not to be an advantage for quantum thermodynamics in the tasks considered here, we do not discard this possibility for other protocols. Finding such protocols for which an advantage is associated with indefinite causal order is left for future studies. Indeed, studies conducted in this direction have been successful for quantum information problems in Refs. [75, 76].

We can further develop such reasoning. In order to talk about thermodynamic advantages arising from indefinite causal order, we must first define a measure of such property. Let us consider a situation where, under certain set $X$ of conditions, the free energy (or some other thermodynamic property) is less than a given value $T_0$ as

![FIG. 8. Maximum ergotropy obtained after optimizing over preparation and measurement of ancilla, as a function of population imbalance $\delta\rho$. The input state of the target considered in all the cases is $[\sqrt{\tau}|0\rangle + \sqrt{\tau^*}|1\rangle]$. The channel parameters considered are $p = 1/3$, $q = 0$, $\lambda = 1/2$. Since there are two ancillae in the Det process, one of them is discarded at the output while the other is optimized over in measurement basis. This is done for a fair comparison between all of the cases, where only one ancilla is considered uniformly in all the other setups.](image-url)
long as we have definite order. In this case, if we have the considered property larger than $T_0$, and the set $X$ is satisfied, then we can conclude that there was indefinite causal order. In other words, $T(W) > T_0 \Rightarrow f(W) > 0$, for some real-valued function $f$ on the space of bipartite processes with bipartite global past and future, quantifying the extent to which the process is non-causal.

Our examples show that this is not the case for the tasks considered here. Any reasonable acausality measure $f$ would necessarily satisfy $f(W_{\text{switch}}) > f(W)$, as $W$ is any causally ordered process; strict inequality must hold since the quantum switch in not causally ordered. Thus the condition given above is not satisfied in the examples considered in our work, as can be seen from Figs. 7 and 8.

We, however, do not discard the possibility of indefinite causal order to be a resource for some other thermodynamic task, with respect to some specific acausality measure.

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