DEGENERATE BERNSTEIN POLYNOMIALS

TAEKYUN KIM AND DAE SAN KIM

ABSTRACT. Here we consider the degenerate Bernstein polynomials as a degenerate version of Bernstein polynomials, which are motivated by Simsek’s recent work ‘Generating functions for unification of the multidimensional Bernstein polynomials and their applications’([15,16]) and Carlitz’s degenerate Bernoulli polynomials. We derived their generating function, symmetric identities, recurrence relations, and some connections with generalized falling factorial polynomials, higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

1. Introduction

For \( \lambda \in \mathbb{R} \), the degenerate Bernoulli polynomials of order \( k \) are defined by L. Cartliz as

\[
\left( \frac{t}{1 + \lambda t} \right)^k (1 + \lambda t)^x = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!},
\]

\( \text{(see [4, 5]).} \) \hfill (1.1)

Note that \( \lim_{\lambda \to 0} \beta_{n,\lambda}^{(k)}(x) = B_n^{(k)}(x) \) are the ordinary Bernoulli polynomials of order \( k \) given by

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},
\]

\( \text{(see [1, 12, 13]).} \)

It is known that the Stirling number of the second kind is defined as

\[
x^n = \sum_{l=0}^{n} S_2(n, l)(x)_l, \quad \text{(see [2, 4, 8, 10]),}
\]

\( \text{(1.2)} \)

where \((x)_l = x(x-1) \cdots (x-l+1), \; (l \geq 1), \; (x)_0 = 1.\)

For \( \lambda \in \mathbb{R} \), the \((x)_{n,\lambda} \) is defined as

\[
(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda) \cdots (x-(n-1)\lambda), \; (n \geq 1)
\]

\( \text{(1.3)} \)
In [8,9,10], \((x)_n^\lambda\) is defined as
\[
\binom{x}{n}^\lambda = \frac{x(x-\lambda) \cdots (x-(n-1)\lambda)}{n!}, \quad (n \geq 1), \quad \binom{x}{0}^\lambda = 1. \tag{1.4}
\]
Thus, by (1.4), we get
\[
(1 + \lambda t)^x^\lambda = \sum_{n=0}^{\infty} \binom{x}{n} t^n, \quad \text{(see [7])}. \tag{1.5}
\]
From (1.5), we note that
\[
\sum_{m=0}^{n} \binom{y}{m} \binom{x}{n-m}^\lambda = \binom{x+y}{n}^\lambda, \quad (n \geq 0). \tag{1.6}
\]
The degenerate Stirling numbers of the second kind are defined by
\[
\frac{1}{k!}((1 + \lambda t)^\frac{1}{\lambda} - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [7,8])}. \tag{1.7}
\]
By (1.7), we easily get
\[
\lim_{\lambda \to 0} S_{2,\lambda}(n, k) = S_2(n, k), \quad (n \geq k \geq 0), \quad \text{(see [8,10])}. \tag{1.8}
\]
In this paper, we use the following notation.
\[
(x \oplus \lambda y)^n = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} (y)_{n-k,\lambda}, \quad (n \geq 0). \tag{1.8}
\]
The Bernstein polynomials of degree \(n\) is defined by
\[
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (n \geq k \geq 0), \quad \text{(see [6,11,17])}. \tag{1.9}
\]
Let \(C[0,1]\) be the space of continuous functions on \([0,1]\). The Bernstein operator of order \(n\) for \(f\) is given by
\[
\mathfrak{B}_n(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \tag{1.10}
\]
where \(n \in \mathbb{N} \cup \{0\}\) and \(f \in C[0,1]\), (see [3,6,14]).

A Bernoulli trial involves performing a random experiment and noting whether a particular event \(A\) occurs. The outcome of Bernoulli trial is said to be "success" if \(A\) occurs and a "failure" otherwise. The probability \(P_n(k)\) of \(k\) successes in \(n\) independent Bernoulli trials is given by the binomial probability law:
\[
P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, 1, 2, \cdots,
\]
From the definition of Bernstein polynomials we note that Bernstein basis is probability mass of binomial distribution with parameter \((n, x = p)\).
Let us assume that the probability of success in an experiment is $p$. We wondered if we can say the probability of success in the nineth trial is still $p$ after failing eight times in a ten trial experiment. Because there’s a psychological burden to be successful.

It seems plausible that the probability is less than $p$. This speculation motivated the study of the degenerate Bernstein polynomials associated with the probability distribution.

In this paper, we consider the degenerate Bernstein polynomials as a degenerate version of Bernstein polynomials. We derive their generating function, symmetric identities, recurrence relations, and some connections with generalized falling factorial polynomials, higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

2. Degenerate Bernstein polynomials

For $\lambda \in \mathbb{R}$ and $k, n \in \mathbb{N} \cup \{0\}$, with $k \leq n,$ we define the degenerate Bernstein polynomials of degree $n$ which are given by

$$B_{k,n}(x|\lambda) = \binom{n}{k}(x)_{\lambda}^k (1-x)_{\lambda}^{n-k}, \ (x \in [0,1]).$$

Note that $\lim_{\lambda \to 0} B_{k,n}(x|\lambda) = B_{k,n}(x), \ (0 \leq k \leq n)$. From (2.1), we derive the generating function of $B_{k,n}(x|\lambda)$, which are given by

$$\sum_{n=k}^{\infty} B_{k,n}(x|\lambda) \frac{t^n}{n!} = \sum_{n=k}^{\infty} \binom{n}{k}(x)_{\lambda}^k (1-x)_{\lambda}^{n-k} \frac{t^n}{n!}$$

$$= \frac{(x)_{\lambda}^k}{k!} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} (1-x)_{\lambda}^{n-k} t^n$$

$$= \frac{(x)_{\lambda}^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)_{\lambda}^n}{n!} t^{n+k}$$

$$= \frac{(x)_{\lambda}^k}{k!} t^k \sum_{n=0}^{\infty} \left( \frac{1-x}{\lambda} \right)^n \frac{t^n}{n!}$$

$$= \frac{(x)_{\lambda}^k}{k!} t^k (1+\lambda t)^{1-x}.$$  

Therefore, by (2.2), we obtain the following theorem.

**Theorem 2.1.** For $x \in [0,1]$ and $k = 0, 1, 2, \cdots$, we have

$$\frac{1}{k!} (x)_{\lambda}^k (1+\lambda t)^{1-x} = \sum_{n=k}^{\infty} B_{k,n}(x|\lambda) \frac{t^n}{n!}.$$
From (2.1), we note that
\[ B_{k,n}(x|\lambda) = \binom{n}{k} (x)_{k,\lambda} (1-x)^{n-k,\lambda} = \binom{n}{n-k} (x)^{k,\lambda} (1-x)^{n-k,\lambda}. \] (2.3)

By replacing \( x \) by \( 1-x \), we get
\[ B_{k,n}(1-x|\lambda) = \binom{n}{n-k} (1-x)^{k,\lambda} (x)^{n-k,\lambda} = B_{n-k,n}(x|\lambda), \] (2.4)

where \( n, k \in \mathbb{N} \cup \{0\} \), with \( 0 \leq k \leq n \).

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.2.** (Symmetric identities) For \( n, k \in \mathbb{N} \cup \{0\} \), with \( k \leq n \), and \( x \in [0,1] \), we have
\[ B_{n-k,n}(x|\lambda) = B_{k,n}(1-x|\lambda). \]

Now, we observe that
\[
\frac{n-k}{n} B_{k,n}(x|\lambda) + \frac{k+1}{n} B_{k+1,n}(x|\lambda)
= \frac{n-k}{n} \binom{n}{k} (x)_{k,\lambda} (1-x)^{n-k,\lambda} + \frac{k+1}{n} \binom{n}{k+1} (x)_{k+1,\lambda} (1-x)^{n-k-1,\lambda}
= \frac{(n-1)!}{k!(n-k-1)!} (x)_{k,\lambda} (1-x)^{n-k,\lambda} + \frac{(n-1)!}{k!(n-k-1)!} (x)_{k+1,\lambda} (1-x)^{n-k-1,\lambda}
= (1-x-(n-k-1)\lambda) B_{k,n-1}(x|\lambda) + (x-k\lambda) B_{k,n-1}(x|\lambda)
= (1+\lambda(1-n)) B_{k,n-1}(x|\lambda).
\]
(2.5)

Therefore, by (2.5), we obtain the following theorem.

**Theorem 2.3.** For \( k \in \mathbb{N} \cup \{0\} \), \( n \in \mathbb{N} \), with \( k \leq n-1 \), and \( x \in [0,1] \), we have
\[ (n-k) B_{k,n}(x|\lambda) + (k+1) B_{k+1,n}(x|\lambda) = (1+\lambda(1-n)) B_{k,n-1}(x|\lambda). \] (2.6)

From (2.1), we have
\[
\frac{n-k+1}{k} \binom{n-(k-1)\lambda}{1-x-(n-k)\lambda} B_{k-1,n}(x|\lambda)
= \frac{n-k+1}{k} \binom{n-(k-1)\lambda}{1-x-(n-k)\lambda} \binom{n}{k-1} (x)_{k-1,\lambda} (1-x)^{n-k+1,\lambda}
= \frac{n!}{k!(n-k)!} (x)_{k-1,\lambda} (1-x)^{n-k,\lambda} = B_{k,n}(x|\lambda).
\]
(2.7)

Therefore, by (2.7), we obtain the following theorem.
Theorem 2.4. For \( n, k \in \mathbb{N}, \) with \( k \leq n, \) we have

\[
\left( \frac{n - k + 1}{k} \right) \left( \frac{n - (k - 1) \lambda}{1 - x - (n - k) \lambda} \right) B_{k-1,n}(x|\lambda) = B_{k,n}(x|\lambda).
\]

For \( 0 \leq k \leq n, \) we get

\[
(1 - x - (n - k - 1) \lambda) B_{k,n-1}(x|\lambda) + (x - (k - 1) \lambda) B_{k-1,n-1}(x|\lambda)
\]

\[
= (1 - x - (n - k - 1) \lambda) \binom{n - 1}{k} (x)_k \lambda (1 - x)_{n-1-k,\lambda}
\]

\[
+ (x - (k - 1) \lambda) \binom{n - 1}{k-1} (x)_{k-1} \lambda (1 - x)_{n-k,\lambda}
\]

\[
= \left( \binom{n - 1}{k} + \binom{n - 1}{k-1} \right) (x)_k \lambda (1 - x)_{n-k,\lambda} = \binom{n}{k} (x)_k \lambda (1 - x)_{n-k,\lambda}.
\]

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.5. (Recurrence formula). For \( k, n \in \mathbb{N}, \) with \( k \leq n - 1, \) \( x \in [0,1], \) we have

\[
(1 - x - (n - k - 1) \lambda) B_{k,n-1}(x|\lambda) + (x - (k - 1) \lambda) B_{k-1,n-1}(x|\lambda) = B_{k,n}(x|\lambda).
\]

Remark 1. For \( n \in \mathbb{N}, \) we have

\[
\sum_{k=0}^{n} \frac{k}{n} B_{k,n}(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} (x)_k \lambda (1 - x)_{n-k,\lambda}
\]

\[
= \sum_{k=1}^{n} \binom{n - 1}{k-1} (x)_k \lambda (1 - x)_{n-k,\lambda} = \sum_{k=0}^{n-1} \binom{n - 1}{k} (x)_{k+1} \lambda (1 - x)_{n-1-k,\lambda}
\]

\[
= (x - k \lambda) \sum_{k=0}^{n-1} \binom{n - 1}{k} (x)_k \lambda (1 - x)_{n-1-k,\lambda} = (x - k \lambda) (x \oplus_\lambda (1 - x))^{n-1}.
\]

Now, we observe that
\[
\sum_{k=2}^{n} \frac{k}{2} B_{k,n}(x|\lambda) = \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda}
\]
\[
= \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda}
\]
\[
= \sum_{k=2}^{n} \binom{n-2}{k-2} (x)_{k,\lambda} (1-x)_{n-k,\lambda}
\]
\[
= \sum_{k=2}^{n} \binom{n-2}{k} (x)_{k+2,\lambda} (1-x)_{n-2-k,\lambda}
\]
\[
= (x-k\lambda)(x-(k+1)\lambda) \sum_{k=0}^{n-2} \binom{n-2}{k} (x)_{k,\lambda} (1-x)_{n-2-k,\lambda}.
\]

(2.9)

Similarly, we have
\[
\sum_{k=i}^{n} \frac{k}{i} B_{k,n}(x|\lambda) = (x-k\lambda)_{i,\lambda} \sum_{k=0}^{n-i-1} \binom{n-i}{k} (x)_{k,\lambda} (1-x)_{n-i-k,\lambda}
\]
\[
= (x-k\lambda)_{i,\lambda} (x \oplus \lambda (1-x))^{n-i}.
\]

(2.10)

From (2.10), we note that
\[
(x-k\lambda)_{i,\lambda} = \frac{1}{(x \oplus \lambda (1-x))^{n-i}} \sum_{k=i}^{n} \frac{k}{i} B_{k,n}(x|\lambda),
\]

(2.11)

where \(n, i \in \mathbb{N}\), with \(i \leq n\), and \(x \in [0,1]\).

Therefore, by (2.11), we obtain the following theorem.

**Theorem 2.6.** For \(n, i \in \mathbb{N}\), with \(i \leq n\), and \(x \in [0,1]\), we have
\[
(x-k\lambda)_{i,\lambda} = \frac{1}{(x \oplus \lambda (1-x))^{n-i}} \sum_{k=i}^{n} \frac{k}{i} B_{k,n}(x|\lambda).
\]
From Theorem 2.1, we note that
\[
t_k \frac{x_{k, \lambda} (1 + \lambda t)^{\frac{1}{k}}}{k!} = \frac{(x)_{k, \lambda}}{k!} \left( (1 + \lambda t)^{\frac{1}{k}} - 1 \right)^k \left( \frac{t}{(1 + \lambda t)^{\frac{1}{k}} - 1} \right)^k (1 + \lambda t)^{\frac{1}{k} - x}
\]

\[
= (x)_{k, \lambda} \left( \sum_{m=k}^{\infty} S_{2, \lambda}(m, k) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \beta_{l, \lambda}^{(k)} (1 - x) \frac{t^l}{l!} \right)
\]

\[
= (x)_{k, \lambda} \sum_{n=k}^{\infty} \left( \sum_{m=k}^{n} \binom{n}{m} S_{2, \lambda}(m, k) \beta_{n-m, \lambda}^{(k)} (1 - x) \right) \frac{t^n}{n!}.
\]

(2.12)

On the other hand,
\[
\frac{(x)_{k, \lambda}}{k!} t^k (1 + \lambda t)^{\frac{1}{k} - x} = \sum_{n=k}^{\infty} B_{k, n}(x|\lambda) \frac{t^n}{n!}.
\]

(2.13)

Therefore, by (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.7.** For \(n, k \in \mathbb{N} \cup \{0\}\) with \(n \geq k\), we have

\[
B_{k, n}(x|\lambda) = (x)_{k, \lambda} \sum_{m=k}^{n} \binom{n}{m} S_{2, \lambda}(m, k) \beta_{n-m, \lambda}^{(k)} (1 - x).
\]

Let \(\Delta\) be the shift difference operator with \(\Delta f(x) = f(x + 1) - f(x)\). Then we easily get

\[
\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \quad (n \in \mathbb{N} \cup \{0\}).
\]

(2.14)

Let us take \(f(x) = (x)_{m, \lambda}\), \((m \geq 0)\). Then, by (2.14), we get

\[
\Delta^n (0)_{m, \lambda} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k)_{m, \lambda}.
\]

(2.15)

From (1.7), we note that

\[
\sum_{n=k}^{\infty} S_{2, \lambda}(n, k) \frac{t^n}{n!} = \frac{k}{k!} \left( (1 + \lambda t)^{\frac{1}{k}} - 1 \right)^k = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (1 + \lambda t)^{\frac{l}{k}}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l)_{n, \lambda} \right) \frac{t^n}{n!}.
\]

(2.16)

Thus, by comparing the coefficients on both sides of (2.16), we have

\[
\frac{1}{k!} \Delta^k (0)_{n, \lambda} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l)_{n, \lambda} = \begin{cases} S_{2, \lambda}(n, k) & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases}
\]

(2.17)
By (2.17), we get
\[ \frac{1}{k!} \Delta^k(0)_{n,\lambda} = S_{2,\lambda}(n, k), \text{ if } n \geq k. \] (2.18)

From Theorem 7 and (2.18), we obtain the following corollary.

**Corollary 2.8.** For \( n, k \in \mathbb{N} \cup \{0\} \) with \( n \geq k \), we have
\[ B_{k,n}(x|\lambda) = (x)_{k,\lambda} \sum_{m=k}^{n} \binom{n}{m} \beta^{(k)}_{n-m,\lambda} (1-x) \frac{1}{k!} \Delta^k(0)_{m,\lambda}. \]

Now, we observe that
\[ (1 + \lambda t)^x = \left((1 + \lambda t)^{\frac{x}{\lambda}} - 1 + 1\right)^x = \sum_{k=0}^{\infty} \binom{x}{k} \left((1 + \lambda t)^{\frac{x}{\lambda}} - 1\right)^k \]
\[ = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{1}{k!} \left((1 + \lambda t)^{\frac{x}{\lambda}} - 1\right)^k \]
\[ = \sum_{k=0}^{\infty} (x)_{k,\lambda} \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (x)_{k,\lambda} S_{2,\lambda}(n, k) \right) \frac{t^n}{n!}. \] (2.19)

On the other hand,
\[ (1 + \lambda t)^x = \sum_{n=0}^{\infty} \binom{x}{n} \lambda^n t^n = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}. \] (2.20)

Therefore, by (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.9.** For \( n \geq 0 \), we have
\[ (x)_{n,\lambda} = \sum_{k=0}^{n} (x)_{k,\lambda} S_{2,\lambda}(n, k). \]

By Theorem 2.9, we easily get
\[ (x - k\lambda)_{i,\lambda} = \sum_{l=0}^{i} (x - k\lambda)_{i} S_{2,\lambda}(i, l). \] (2.21)

From Theorem 2.6, we have the following theorem.

**Theorem 2.10.** For \( n, i \in \mathbb{N} \), with \( i \leq n \), and \( x \in [0, 1] \), we have
\[ \sum_{i=0}^{n} (x - k\lambda)_{i} S_{2,\lambda}(i, l) = \frac{1}{(x \oplus \lambda (1-x))^{n-i}} \sum_{k=0}^{n} \binom{k}{l} B_{k,n}(x|\lambda). \]
References

1. S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 399-406.

2. M. Arató, A. Rényi, *Probabilistic proof of a theorem on the approximation of continuous functions by means of generalized Bernstein polynomials*, Acta Math. Acad. Sci. Hungar. **8** (1957), 91-98.

3. V. A. Baskakov, *A generalization of the Bernstein polynomials*, (Russian) Izv. Vysš. Učebn. Zaved. Matematika **1960** (1960), no. 3 (16), 48-53.

4. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51-88.

5. L. Carlitz, *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel) **7** (1956), 28-33.

6. T. Kim, *A note on q-Bernstein polynomials*, Russ. J. Math. Phys. **18** (2011), no. 1, 73-82.

7. T. Kim, *λ-analogue of Stirling numbers of the first kind*, Adv. Stud. Contemp. Math. (Kyungshang) **27** (2017), no. 3, 423-429.

8. T. Kim, *A note on degenerate Stirling polynomials of the second kind*, Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 319-331.

9. T. Kim, D. S. Kim, *Degenerate Laplace Transform and degenerate gamma functions*, Russ. J. Math. Phys. **24** (2017), 241–248.

10. T. Kim, Y. Yao, D. S. Kim, G.-W. Jang, *Degenerate r–Stirling numbers and r–Bell polynomials*, Russ. J. Math. Phys. **25** (2018), no. 1, 44–58.

11. G. G. Lorentz, *Bernstein polynomials*, Second edition. Chelsea Publishing Co., New York, 1986.

12. S.-H. Rim, J. Joung, J.-H. Jin, S.-J. Lee, *A note on the weighted Carlitz’s type q-Euler numbers and qBernstein polynomials*, Proc. Jangjeon Math. Soc. **15** (2012), no. 2, 193-201.

13. C. S. Ryoo, *Some relations between twisted q-Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **21** (2011), no. 2, 217-223.

14. M.A. Siddiqui, R. R. Agrawal, N. Gupta, *On a class of modified new Bernstein operators*, Adv. Stud. Contemp. Math. (Kyungshang) **24** (2014), no. 1, 97-107.

15. Y. Simsek, *Combinatorial identities associated with Bernstein type basis functions*, Filomat **30** (2016), no. 7, 1683-1689.

16. Y. Simsek, *Generating functions for unification of the multidimensional Bernstein polynomials and their applications*, Math. Methods Appl. Sci. **2018**, 1–12.

17. O. Szasz, *Generalization of S. Bernstein’s polynomials to the infinite interval*, J. Research Nat. Bur. Standards **45** (1950), 239-245.