Research Article

Solving the Modified Regularized Long Wave Equations via Higher Degree B-Spline Algorithm

Pshtiwan Othman Mohammed,1 Manar A. Alqudah,2 Y. S. Hamed,3 Artion Kashuri,4 and Khadijah M. Abualnaja3

1Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Kurdistan Region, Iraq
2Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O.Box 84428, Riyadh 11671, Saudi Arabia
3Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia
4Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania

Correspondence should be addressed to Pshtiwan Othman Mohammed; pshtiwansangawi@gmail.com and Manar A. Alqudah; manarqudah@yahoo.com

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The current article considers the sextic B-spline collocation methods (SBCM1 and SBCM2) to approximate the solution of the modified regularized long wave (MRLW) equation. In view of this, we will study the solitary wave motion and interaction of higher (two and three) solitary waves. Also, the modified Maxwellian initial condition into solitary waves is studied. Moreover, the stability analysis of the methods has been discussed, and these will be unconditionally stable. Moreover, we have calculated the numerical conserved laws and error norms \( L_2 \) and \( L_\infty \) to demonstrate the efficiency and accuracy of the method. The numerical examples are presented to illustrate the applications of the methods and to compare the computed results with the other methods. The results show that our proposed methods are more accurate than the other methods.

1. Introduction

The regularized long wave (RLW) equation is defined by the following nonlinear partial differential equation [1]:

\[
\sigma_\tau + \sigma_\eta + \zeta \sigma_\eta - \mu \sigma_{\eta\eta\eta} = 0, \tag{1}
\]

where \( \mu \) and \( \zeta \) are positive parameters. This equation was first introduced by Peregrine [1] and after that by Benjamin et al. [2] to describe the behavior of the undular bore. It has also a great role in physics science, especially in physics media since it is useful in describing a phenomenon in different disciplines, such as the nonlinear transverse waves in magneto hydrodynamics waves in plasma, ion-acoustic waves in plasma, shallow water, longitudinal dispersive waves in elastic rods, phonon packets in nonlinear crystals, and pressure waves in liquids gas bubbles.

There are many analytical methods to obtain the solution of the RLW equation for certain boundary and initial conditions; for example, see [2, 3]. Also, the numerical solutions of the RLW equation has been studied by many researchers via various methods, such as finite difference methods [4, 5], Fourier pseudospectral methods [6], various models of finite element methods including least square, collocation, and Galerkin methods [7–9], mesh-free method [10], and Galerkin finite element methods [11–13].

The generalized form of the RLW equation is known as the GMRLW equation which is given by

\[
\sigma_\tau + \sigma_\eta + \zeta \sigma_\eta - \mu \sigma_{\eta\eta\eta} = 0, \tag{2}
\]

where \( p \) is a positive parameter. In the extension of nonlinear dispersive waves, this equation has an important role. There are many numerical methods to investigate its solution. For
more details, we advise the reader to visit [14–17]. In the current attempt, we consider a special case of the GMRLW (namely, the MRLW) equation, given by

\[ \sigma_\tau + \sigma_\eta + \zeta \sigma_\eta^2 - \mu \sigma_{\eta\eta\eta\eta} = 0, \quad (3) \]

subject to the boundary conditions (B.Cs):

\[ \begin{align*}
\sigma(\rho_1, \tau) = \beta_1, & \quad \sigma(\rho_2, \tau) = \beta_2, \\
\sigma_\eta(\rho_1, \tau) = 0, & \quad \sigma_\eta(\rho_2, \tau) = 0, \\
\sigma_{\eta\eta}(\rho_1, \tau) = 0, & \quad \sigma_{\eta\eta}(\rho_2, \tau) = 0,
\end{align*} \quad (4) \]

and the initial condition (I.C.) is taken as

\[ \sigma(\eta, 0) = f(\eta), \quad (5) \]

where \( f(\eta) \) is assumed to be localized disturbance inside the given interval. There are many authors who obtained the numerical solution of the MRLW equation; for example, Gardner et al. [18] used the cubic B-spline finite element method, Prenter [19] used variational and spline methods, and Khalifa et al. [20] used finite difference method; in [21], they used the Adomian decomposition method, they also in [22] used the collocation method, and Fazal-i-Haq et al. [23] used the quartic B-Spline collocation method to get an approximate solution of the MRLW equation.

In this study, inspired by the abovementioned studies, we use the sextic B-spline collocation methods to approximate the solution of the MRLW equations (3)–(5). The rest of the paper is organized as follows. In Sections 2.1 and 2.2, we discuss the B-spline collocation methods I and II and their stability analysis on the proposed MRLW equation. Section 3 is dedicated to the numerical implementations and comparison of our obtained results with those obtained in the literature: Section 3.1 is for single solitary wave, and Sections 3.2 and 3.3 are for interactions of multiple solitary waves. A conclusion is subsequently given in Section 3.

2. The Methods of B-Spline Collocation

Let us partition the finite interval \([\rho_1, \rho_2]\) into a uniform mesh by points \(\eta_i, i = -3, -2, \cdots, J + 2\) such a way that \(\rho_1 = \eta_0 < \eta_1 < \cdots < \eta_{J-1} = \eta_J = \rho_2\), where \(\Delta \eta = y = \rho_2 - \rho_1/|\eta_i - \eta_{i-1}|\). Then, we state the sextic B-spline collocation methods I (SBCM1) and II (SBCM2).

2.1. The SBCM1. We know that the sextic B-splines are usually defined on \(J + 1\) nodes over a given interval \([\rho_1, \rho_2]\) with 12 additional nodes outside the interval \([\rho_1, \rho_2]\). The additional nodes may be given as follows:

\[ \begin{align*}
\eta_{-6} < \eta_{-5} < \eta_{-4} < \eta_{-3} < \eta_{-2} < \eta_{-1} < \eta_0, \\
\eta_J < \eta_{J+1} < \eta_{J+2} < \eta_{J+3} < \eta_{J+4} < \eta_{J+5} < \eta_{J+6}.
\end{align*} \quad (6) \]

The sextic B-splines \(B_6(\eta)\) at the knots \(\eta_i\) are defined as [1]

\[ \begin{align*}
&\left( \eta - \eta_{i-3} \right)^6, & &\left( \eta - \eta_{i-2} \right)^6, \\
&\left( \eta - \eta_{i-3} \right)^6 - 7(\eta - \eta_{i-2})^6, & &\left( \eta - \eta_{i-2} \right)^6, \\
&\left( \eta - \eta_{i-3} \right)^6 - 7(\eta - \eta_{i-2})^6, & +21(\eta - \eta_{i-1})^6, &\left( \eta - \eta_{i-1} \right)^6, \\
&\left( \eta - \eta_{i-3} \right)^6 - 7(\eta - \eta_{i-2})^6, & +21(\eta - \eta_{i-1})^6 - 35(\eta - \eta_{i})^6, &\left[ \eta_i, \eta_{i+1} \right], \\
&\left( \eta - \eta_{i+4} \right)^6, & -7(\eta - \eta_{i+3})^6, &\left[ \eta_{i+1}, \eta_{i+2} \right], \\
&\left( \eta - \eta_{i+4} \right)^6 - 7(\eta - \eta_{i+3})^6, &\left[ \eta_{i+2}, \eta_{i+3} \right], &\left[ \eta_{i+3}, \eta_{i+4} \right], \\
&0, & \text{otherwise,}
\end{align*} \quad (7) \]

for \( i = -3, -2, \cdots, 0 \) and the set \{\(B_{-3}, B_{-2}, \cdots, B_{J+2}\)\} of sextic B-splines can be a basis over the interval \([\rho_1, \rho_2]\).

The approximate solution \(\sigma(\eta, \tau)\) of the GMRLW equation to the exact solution \(\sigma(\eta, \tau)\) will be determined as follows:

\[ \sigma(\eta, \tau) = \sum_{\ell=-3}^{J+2} \vartheta_\ell(\tau) B_6(\eta), \quad (8) \]

where the time dependent parameters \(\vartheta_\ell(\tau)\) will be determined from the sextic B-spline collocation formula of equation (3).

In view of equation (8) and Table 1, the nodal values \(\sigma_i^{(m)}, i = 0, 1, \cdots, 5\) at the knots \(\eta_i\) can be found as

\[ \begin{align*}
\sigma_\ell & = \vartheta_{\ell+2} + 57\vartheta_{\ell+1} + 302\vartheta_\ell + 302\vartheta_{\ell-1} + 57\vartheta_{\ell-2} + \vartheta_{\ell-3}, \\
\sigma_\ell' & = \frac{6}{y^2} (\vartheta_{\ell+2} + 25\vartheta_{\ell+1} + 40\vartheta_\ell - 40\vartheta_{\ell-1} - 25\vartheta_{\ell-2} - \vartheta_{\ell-3}), \\
\sigma_\ell'' & = \frac{30}{y^4} (\vartheta_{\ell+2} + 9\vartheta_{\ell+1} - 10\vartheta_\ell - 10\vartheta_{\ell-1} + 9\vartheta_{\ell-2} + \vartheta_{\ell-3}), \\
\sigma_\ell^{(3)} & = \frac{120}{y^6} (\vartheta_{\ell+2} + \vartheta_{\ell+1} - 8\vartheta_\ell + 8\vartheta_{\ell-1} - \vartheta_{\ell-2} - \vartheta_{\ell-3}), \\
\sigma_\ell^{(4)} & = \frac{360}{y^8} (\vartheta_{\ell+2} - 3\vartheta_{\ell+1} + 2\vartheta_\ell - 2\vartheta_{\ell-1} + 3\vartheta_{\ell-2} - \vartheta_{\ell-3}), \\
\sigma_\ell^{(5)} & = \frac{720}{y^{10}} (-\vartheta_{\ell+2} + 5\vartheta_{\ell+1} - 10\vartheta_\ell + 10\vartheta_{\ell-1} - 5\vartheta_{\ell-2} + \vartheta_{\ell-3}).
\end{align*} \quad (9) \]

Now, we implement the collocation method at the nodes \(\eta_i, i = 0, 1, \cdots, J\). Also, we substitute the nodal variables \(\sigma_\ell\) and its derivatives at the knots \(\eta_i\) in equation (9) into equation
Table 1: The sextic B-spline values at the grid points.

| $\eta$ | $\eta_{t-3}$ | $\eta_{t-2}$ | $\eta_{t-1}$ | $\eta_t$ | $\eta_{t+1}$ | $\eta_{t+2}$ |
|-------|--------------|--------------|--------------|----------|--------------|--------------|
| $Q_t$ | 1            | 57           | 302          | 302      | 57           | 1            |
| $hQ_t$ | 6            | 150          | −240         | −240     | −150         | −6           |
| $h^2Q^\mu_t$ | 30        | 270          | −300         | −300     | 270          | 30           |
| $h^3Q^5_t(3)$ | 120      | 120          | −960         | 960      | −120         | −120         |
| $h^3Q^5_t(3)$ | 360      | −1080        | 720          | 720      | −1080        | 360          |
| $h^3Q^5_t(5)$ | 720      | 23600        | 7200         | −7200    | 3600         | −720         |

(3); then, we get the following system of nonlinear ordinary differential equations:

$$
\begin{align*}
\delta^{(r)}_{t+2} + 57\delta^{(r)}_{t+1} + 302\delta^{(r)}_{t} + 302\delta^{(r)}_{t-1} + 57\delta^{(r)}_{t-2} + \delta^{(r)}_{t-3} \\
+ 6\frac{d}{h}(\delta_{t+2} + 25\delta_{t+1} + 40\delta_{t} - 40\delta_{t-1} - 25\delta_{t-2} - \delta_{t-3}) \\
- 30\mu\frac{\delta}{h^2} (\delta^{(r)}_{t+2} + 9\delta^{(r)}_{t+1} - 10\delta^{(r)}_{t} - 10\delta^{(r)}_{t-1} + 9\delta^{(r)}_{t-2} + \delta^{(r)}_{t-3}) = 0,
\end{align*}
$$

where $d = 1 + \zeta_1 = 1 + \zeta(\delta_{t-3} + 57\delta_{t-2} + 302\delta_{t-1} + 302\delta_{t} + 57\delta_{t+1} + \delta_{t+2})$ and $\delta$ denote derivative with respect to time.

The unknown parameters $\delta_0$ and $\delta_1$ are linearly interpolated between $n$ and $n+1$ ($n$ and $n+1$ are two time levels) via the the Crank-Nicolson formula and the usual forward difference formula, respectively, as follows:

$$
\delta_0 = \frac{\delta^{(r)}_{t+1} + \delta^{(r)}_{t}}{2}, \quad \delta_1 = \frac{\delta^{(r)}_{t+1} - \delta^{(r)}_{t}}{\Delta t},
$$

where $\delta^{(r)}_{t}$ denotes the parameters at time $n\Delta t$. Then, by making use of equation (11), it follows that

$$
\begin{align*}
\alpha_{t+1}^{(r)+} + \alpha_{t+2}^{(r)+} + \alpha_{t+3}^{(r)+} + \alpha_{t+4}^{(r)+} + \alpha_{t+5}^{(r)+} + \alpha_{t+6}^{(r)+} = \\
\alpha_{t0}^{(r)+} + \alpha_{t1}^{(r)+} + \alpha_{t2}^{(r)+} + \alpha_{t3}^{(r)+} + \alpha_{t4}^{(r)+} + \alpha_{t5}^{(r)+} + \alpha_{t6}^{(r)+},
\end{align*}
$$

where

$$
\begin{align*}
\alpha_{t0} = h^2 - 3dh\Delta t - 30\mu, \\
\alpha_{t1} = 57h^2 - 75hd\Delta t - 270\mu, \\
\alpha_{t2} = 302h^2 - 120hd\Delta t + 300\mu, \\
\alpha_{t3} = 302h^2 + 120hd\Delta t + 300\mu, \\
\alpha_{t4} = 57h^2 + 75hd\Delta t - 270\mu, \\
\alpha_{t5} = h^2 + 3hd\Delta t - 30\mu.
\end{align*}
$$

The system (12) consisting of the $(J+1)$ equations with $(J+6)$ unknown parameters. It can be solved uniquely if we eliminate the parameters $\delta^{(r)}_{t+3}, \delta^{(r)}_{t+2}, \delta^{(r)}_{t+1}, \delta^{(r)}_{t+0}, \delta^{(r)}_{t+1}$ by using

The five B.Cs $\sigma(\rho_1, \tau) = \beta_1, \sigma(\rho_2, \tau) = \beta_2, \sigma_\eta(\rho_1, \tau) = \sigma_\eta(\rho_2, \tau) = \sigma_\eta(\rho_1, \tau) = 0$; that is,

$$
\begin{align*}
\delta^{(r)}_{t+1} + 57\delta^{(r)}_{t+2} + 302\delta^{(r)}_{t+1} + 302\delta^{(r)}_{t+0} + 57\delta^{(r)}_{t+1} + \delta^{(r)}_{t+1} = \beta_1, \\
\frac{6}{h}(\delta^{(r)}_{t+1} + 25\delta^{(r)}_{t+2} + 40\delta^{(r)}_{t+1} - 40\delta^{(r)}_{t+2} - 25\delta^{(r)}_{t+1} - \delta^{(r)}_{t+1}) = 0, \\
30\mu\frac{\delta}{h^2} (\delta^{(r)}_{t+1} + 9\delta^{(r)}_{t+2} - 10\delta^{(r)}_{t} - 10\delta^{(r)}_{t-1} + 9\delta^{(r)}_{t-2} + \delta^{(r)}_{t-3}) = 0, \\
\delta^{(r)}_{t+3} + 57\delta^{(r)}_{t+2} + 302\delta^{(r)}_{t+1} + 302\delta^{(r)}_{t+0} + 57\delta^{(r)}_{t+1} + \delta^{(r)}_{t+1} = \beta_2, \\
\frac{6}{h}(\delta^{(r)}_{t+1} + 25\delta^{(r)}_{t+2} + 40\delta^{(r)}_{t+1} - 40\delta^{(r)}_{t+2} - 25\delta^{(r)}_{t+1} - \delta^{(r)}_{t+1}) = 0.
\end{align*}
$$

Consequently, we get a system of dimension $(J+1) \times (J+1)$, and one can solve it easily by using a variant of the Thomas algorithm.

To deal with nonlinearity in (12) at each time step, we carry out the following corrector methods:

(1) Approximating $\delta^{(r)+}$ by using the following simple corrector:

$$
(\delta^{(r)+})^{n+1} = \delta^n + \frac{\delta^{(r)+} - \delta^n}{2}
$$

(2) As an approximation for $\delta^{(r)+}$, we use $(\delta^{(r)+})^{n+1}$

(3) Repeating this procedure twice at time step $n+1$ to refine $\delta^{(r)+}$

(4) Repeating this procedure at each time step along the execution of the program

The iterative procedure $\delta_t^n$ in (4) can start by determining the initial parameters $\delta_0$, and it can be determined by making use of the B.Cs (4), I.C. (5), and the following requirements:

$$
\begin{align*}
\alpha'(\rho_1, 0) = \delta_2^n + 25\delta_2^n + 40\delta_0^n - 40\delta_0^n - 25\delta_2^n - \delta_3^n = 0, \\
\alpha''(\rho_1, 0) = \delta_2^n + 9\delta_1^n - 10\delta_0^n + 9\delta_2^n + \delta_3^n = 0, \\
\alpha^{(3)}(\rho_1, 0) = \delta_2^n + \delta_1^n - 8\delta_0^n + 8\delta_2^n - \delta_2^n - \delta_3^n = 0, \\
\alpha^{(j)}(\eta_0, 0) = \delta_0^n + 57\delta_1^n + 302\delta_0^n + 302\delta_2^n + 57\delta_3^n + \delta_4^n = f(\eta_j), \quad j = 0, 1, \cdots, J, \\
\alpha^{(j)}(\rho_1, 0) = \delta_2^n + 25\delta_1^n + 40\delta_0^n - 40\delta_2^n - 25\delta_0^n - \delta_3^n = 0, \\
\alpha''(\rho_1, 0) = \delta_2^n + 9\delta_1^n - 10\delta_0^n + 9\delta_2^n + \delta_3^n = 0.
\end{align*}
$$

Again, one can solve it by a variant of the Thomas algorithm and the approximate solution $\sigma^{(i)}(\eta, \tau), i = 0, 1, \cdots, 5$ that can be obtained from equation (9).
Now, we can apply the von Neumann stability method to establish the stability of the scheme (12), but the von Neumann stability method is applicable to linear schemes; so, we shall line arise the nonlinear term \( \sigma \partial_t \sigma_n \) by taking \( \sigma \) as a constant value \( k \), and thus the nonlinear term becomes \( \sigma^p = z_t = k^p \). Then, by substituting the Fourier mode \( \hat{\sigma}^n = \hat{\sigma}^n \) into our linearized form of equation (12) with writing \( \hat{\sigma}^{n+1} = \hat{\sigma}^{n} \), we get

\[
q = \frac{\mathcal{A}_1 + is\mathcal{A}_2}{\mathcal{A}_3 + is\mathcal{A}_4},
\]

(17)

where \( q \) is growth factor and

\[
\begin{align*}
\mathcal{A}_1 &= \alpha_{\ell_0} \cos (3\varphi) + (\alpha_{\varphi} + \alpha_{\ell_1}) \cos (2\varphi) \\
&+ (\alpha_{\varphi} + \alpha_{\ell_2}) \cos (\varphi) + \alpha_{\ell_3}, \\
\mathcal{A}_2 &= -\alpha_{\ell_0} \sin (3\varphi) - (\alpha_{\varphi} - \alpha_{\ell_1}) \sin (2\varphi) \\
&- (\alpha_{\varphi} - \alpha_{\ell_2}) \sin (\varphi), \\
\mathcal{A}_3 &= \alpha_{\ell_1} \cos (3\varphi) + (\alpha_{\varphi} + \alpha_{\ell_0}) \cos (2\varphi) \\
&+ (\alpha_{\varphi} + \alpha_{\ell_2}) \cos (\varphi) + \alpha_{\ell_4}, \\
\mathcal{A}_4 &= -\alpha_{\ell_1} \sin (3\varphi) - (\alpha_{\varphi} - \alpha_{\ell_0}) \sin (2\varphi) \\
&- (\alpha_{\varphi} - \alpha_{\ell_2}) \sin (\varphi).
\end{align*}
\]

(18)

Thanks to Python software, we obtain same expressions for \( \mathcal{A}_1^2 + \mathcal{A}_2^2 \) and \( \mathcal{A}_3^2 + \mathcal{A}_4^2 \) in the following form:

\[
\begin{align*}
\mathcal{A}_1^2 + \mathcal{A}_2^2 &= \mathcal{A}_3^2 + \mathcal{A}_4^2 = \left[ \sin (\varphi) \left( (245\hbar^2 + 195\hbar d\Delta t + 570\mu) \right) \\
&- \sin (2\varphi) \left( 50\hbar^2 + 78\hbar d\Delta t - 240\mu \right) \\
&+ \sin (3\varphi) \left( -\hbar^2 - 3\hbar d\Delta t + 30\mu \right) \right]^2 \\
&+ \left[ \cos (\varphi) \left( 359\hbar^2 + 45\hbar d\Delta t + 30\mu \right) \\
&+ \cos (2\varphi) \left( 58\hbar^2 + 72\hbar d\Delta t - 300\mu \right) \\
&+ \cos (3\varphi) \left( \hbar^2 + 3\hbar d\Delta t - 30\mu \right) \right]^2 \\
&+ 302\hbar^2 - 120\hbar d\Delta t + 300\mu \right]^2,
\end{align*}
\]

(19)

in order for the magnitude of the growth factor that is \( |q| = 1 \), and thus the linearized numerical algorithm for the GMRLW equation will be unconditionally stable.

2.2. The SBCM2. One can split equation (3) as a system of partial differential equation:

\[
(\sigma - \mu \sigma_{\eta_1})_t + 2\zeta \partial^p \sigma_{\eta_1} = 0, \tag{20}
\]

\[
(\sigma - \mu \sigma_{\eta_j})_t + 2\sigma_{\eta_j} = 0. \tag{21}
\]

To applied the collocation approach for system (20) and (21), we identify the collocation points with the nodes \( \eta_i, i = 0, 1, \cdots, J \). If we put the approximation (7) into equations (20) and (21), we can obtain the following system of 1st order ordinary differential equations:

\[
\begin{align*}
\frac{d\sigma_{\eta_2}}{d\eta} &= \frac{\sigma_{\eta_2}^{n+1} + 57\sigma_{\eta_1}^{n+1} + 302\sigma_{\eta_1}^{n+1} + 57\sigma_{\eta_2}^{n+1} + 302\sigma_{\eta_1}^{n+1} + 57\sigma_{\eta_2}^{n+1}}{30\mu} \\
&- \frac{12\zeta z_t}{h^2} \left( \sigma_{\eta_2}^{n+1} + 9\sigma_{\eta_1}^{n+1} - 10\sigma_{\eta_1}^{n+1} + 9\sigma_{\eta_2}^{n+1} \right) \\
&= 0 + \frac{12\zeta z_t}{h} \left( \sigma_{\eta_2}^{n+1} + 25\sigma_{\eta_1}^{n+1} + 40\sigma_{\eta_1}^{n+1} - 25\sigma_{\eta_2}^{n+1} - 25\sigma_{\eta_2}^{n+1} - 25\sigma_{\eta_2}^{n+1} \right),
\end{align*}
\]

(22)

\[
\begin{align*}
\frac{d\sigma_{\eta_2}}{d\eta} &= \frac{\sigma_{\eta_2}^{n+1} + 57\sigma_{\eta_1}^{n+1} + 302\sigma_{\eta_1}^{n+1} + 57\sigma_{\eta_2}^{n+1} + 302\sigma_{\eta_1}^{n+1} + 57\sigma_{\eta_2}^{n+1}}{30\mu} \\
&- \frac{12\zeta z_t}{h^2} \left( \sigma_{\eta_2}^{n+1} + 9\sigma_{\eta_1}^{n+1} - 10\sigma_{\eta_1}^{n+1} + 9\sigma_{\eta_2}^{n+1} + 9\sigma_{\eta_2}^{n+1} \right) \\
&= 0 + \frac{12\zeta z_t}{h} \left( \sigma_{\eta_2}^{n+1} + 25\sigma_{\eta_1}^{n+1} + 40\sigma_{\eta_1}^{n+1} - 25\sigma_{\eta_2}^{n+1} - 25\sigma_{\eta_2}^{n+1} \right),
\end{align*}
\]

(23)

where \( \eta \) is defined in (10), and nonlinearity term is \( z_t = (\sigma_{\eta_2} + 57\sigma_{\eta_1} + 302\sigma_{\eta_1} + 57\sigma_{\eta_2} + 57\sigma_{\eta_1} + 57\sigma_{\eta_2})^p \). Approximating the parameters \( \sigma_t \) between \( n \) and \( n + 1/2 \) by using the Crank-Nicolson formula and \( \sigma_t \) by using the finite difference rule as follows:

\[
\sigma_t = \frac{\sigma_t^{n+1/2} + \sigma_t^n}{2}, \quad \sigma_t = \frac{\sigma_t^{n+1/2} - \sigma_t^n}{\Delta t}.
\]

(24)

Thus, equation (22) becomes

\[
\begin{align*}
\sigma_t^{n+1} &= \alpha_1 \sigma_t^{n+1} + \alpha_2 \sigma_t^{n+1} + \alpha_3 \sigma_t^{n+1} + \alpha_4 \sigma_t^{n+1} + \alpha_5 \sigma_t^{n+1} + \alpha_6 \sigma_t^{n+1} \\
\sigma_t^n &= \alpha_5 \sigma_t^{n-1} + \alpha_6 \sigma_t^{n-1} + \alpha_5 \sigma_t^{n-1} + \alpha_5 \sigma_t^{n-1} + \alpha_5 \sigma_t^{n-1} + \alpha_4 \sigma_t^{n+2},
\end{align*}
\]

(25)

where

\[
\begin{align*}
\alpha_1 &= \hbar^2 - 3h\zeta z_t \Delta t - 30\mu, \\
\alpha_2 &= 57\hbar^2 - 75h\zeta z_t \Delta t - 270\mu, \\
\alpha_3 &= 57h^2 - 120h\zeta z_t \Delta t + 300\mu, \\
\alpha_4 &= 302h^2 + 120h\zeta z_t \Delta t + 300\mu, \\
\alpha_5 &= 57\hbar^2 + 75h\zeta z_t \Delta t - 270\mu, \\
\alpha_6 &= \hbar^2 + 3h\zeta z_t \Delta t - 30\mu.
\end{align*}
\]

(26)

Analogously, in view of the Crank-Nicolson and forward finite difference approaches in time, both parameters \( \sigma_t \) and \( \sigma_t^{n+1} \) are linearly interpolated between two time levels \( n + 1/2 \) and \( n + 1 \), respectively, as follows:

\[
\sigma_t = \frac{\sigma_t^{n+1} + \sigma_t^n}{4}, \quad \sigma_t = \frac{\sigma_t^{n+1} - \sigma_t^n}{\Delta t}.
\]

(27)
Thus, equation (23) becomes
\[
\alpha_7 \theta_{x+1}^{n+1} + \alpha_8 \theta_{x+1}^{n+1} + \alpha_9 \theta_{x+1}^{n+1} + \alpha_{10} \theta_{x+1}^{n+1} + \alpha_{11} \theta_{x+1}^{n+1} + \alpha_{12} \theta_{x+1}^{n+1} = \alpha_{12} \theta_{x-3}^{n+1} + \alpha_{11} \theta_{x-2}^{n+1} + \alpha_{10} \theta_{x-1}^{n+1} + \alpha_9 \theta_{x}^{n+1/2} + \alpha_8 \theta_{x+1}^{n+1/2} + \alpha_7 \theta_{x+2}^{n+1/2},
\]
(28)
where
\[
\begin{align*}
\alpha_7 &= h^2 - 30 \mu - 3h \Delta t, \\
\alpha_8 &= 57h^2 - 75h \Delta t - 270 \mu, \\
\alpha_9 &= 302h^2 + 300 \mu - 120h \Delta t, \\
\alpha_{10} &= 302h^2 + 120h \Delta t + 300 \mu, \\
\alpha_{11} &= 57h^2 - 270 \mu + 75h \Delta t, \\
\alpha_{12} &= h^2 + 3h \Delta t - 30 \mu.
\end{align*}
\]

Equations (25) and (28) constitute the numerical algorithms for the GMRLW equation. We can remove the non-linearity terms occurring in equation (25) by replacing \( \theta_t \) by \( \theta_x^n \) in \( z_t \), and thus the equation (25) will be linearized.

Therefore, we see that the iterative systems of equations (25) and (28) consist of the \((J+1)\) equations in the \((J+6)\) unknown parameters, and one can solve it by eliminating the parameters \( \theta_{j+1}, \theta_{j+2}, \theta_{j+3}, \theta_j, \theta_{j-1}, \theta_{j-2}, \theta_{j-3}, j = n + 1/2, n + 1 \) by making use of the five B.Cs \( \sigma(\rho_1, \tau) = \beta_1, \sigma(\rho_2, \tau) = \beta_2, \sigma(\rho_3, \tau) = \sigma(\eta_1, \tau) = 0, \) then we can obtain
\[

\frac{6}{h} \left( \theta_{j+1} + 25 \theta_j + 40 \theta_{j-1} - 25 \theta_{j-2} - \theta_{j-3} \right) = 0,
\]
\[
\frac{30}{h} \left( \theta_j + 9 \theta_{j-1} - 10 \theta_{j-2} + 9 \theta_{j-3} + 3 \theta_{j-3} \right) = 0,
\]
\[
\theta_{j+1} + 57 \theta_{j+2} + 302 \theta_{j+1} + 302 \theta_j + 57 \theta_{j-1} + 57 \theta_{j-1} + \theta_{j+2} = \beta_2,
\]
\[
\frac{6}{h} \left( \theta_{j+2} + 25 \theta_{j+1} + 40 \theta_{j+1} - 25 \theta_{j-2} - \theta_{j-3} \right) = 0.
\]

Thus, we get a matrix system of dimension \((J+1) \times (J+1)\), and we can easily solve it by using a variant of the Thomas algorithm.

To deal with nonlinearity in (28) at each time step, we carry out the following corrector procedure:
\[
(\theta^n)_{j}^{t+1} = \theta^n_{j} + \frac{\theta_j^n - \theta_{j+1}^n}{2},
\]
(31)

This iterative scheme is executed two times by determining \((\theta^n)_{j}^{t+1}\) for \( \theta_j^n \), where \( j = n + 1/2, n + 1 \).

We start the time evolution of the \( \theta_j^n, j = n + 1/2, n + 1 \) using (25) and (28) by calculating initial parameters \( \theta_0^n \). Therefore, the approximate solution (8) must agree with the I.C. at the knots, and this leads to \( J+1 \) equations. Also, the further five equations can be obtained by using the derivatives of \( \sigma_1 \) in (8) at the ends:
\[
\begin{align*}
\sigma_1' (\rho_1, 0) &= \theta_1^n + 25 \theta_2^n + 40 \theta_3^n - 25 \theta_2^n - \theta_3^n = 0, \\
\sigma_1'' (\rho_1, 0) &= \theta_1^n + 9 \theta_2^n - 10 \theta_1^n + 9 \theta_2^n + \theta_3^n = 0, \\
\sigma_1^{(3)} (\rho_1, 0) &= \theta_1^n + 8 \theta_2^n + 8 \theta_3^n - \theta_2^n - \theta_3^n = 0, \\
\sigma_1 (\eta_1, 0) &= \theta_{j+1} + 57 \theta_{j+1} + 302 \theta_{j+1} + 302 \theta_j + 57 \theta_{j+1} + \theta_{j+2} = f \left( \eta_1 \right), \quad j = 0, 1, \cdots, J, \\
\sigma_1' (\rho_1, 0) &= \theta_1^n + 25 \theta_1^n + 40 \theta_1^n - 25 \theta_1^n - \theta_1^n = 0, \\
\sigma_1'' (\rho_1, 0) &= \theta_1^n + 9 \theta_1^n - 10 \theta_1^n + 9 \theta_1^n + \theta_1^n = 0.
\end{align*}
\]

(32)

Consequently, the parameters \( \theta_j^n, i = -3, -2, \cdots, J + 2 \) will be determined as solution of a matrix equation.

To establish the stability of the scheme (25), we carry out the von Neumann stability scheme by linearizing the nonlinear term \( \sigma \sigma^* \eta \) by taking \( \sigma \) as a constant \( k \) so that \( \sigma^o \) becomes \( \sigma^o = z_k = k^2 \). By substitution the Fourier mode of \( \theta_j^n = \tilde{\theta}_j e^{i \rho_j} \) into our linearized form of equation (25) and by writing \( \tilde{\theta}_j^{n+1} = \tilde{\theta}_j e^{i \rho_j} \) in the resulting iterative equation, we can deduce
\[
q = \frac{\sigma_1 + i \sigma_2}{\sigma_3 + i \sigma_4},
\]
(33)

where
\[
\begin{align*}
\sigma_1 &= \alpha_3 + (\alpha_1 + \alpha_5) \cos (2 \phi) + (\alpha_2 + \alpha_4) \cos (4 \phi), \\
\sigma_2 &= (\alpha_1 - \alpha_5) \sin (2 \phi) + (\alpha_2 - \alpha_4) \sin (4 \phi), \\
\sigma_3 &= \alpha_4 + (\alpha_2 + \alpha_6) \cos (2 \phi) + \alpha_5 \cos (3 \phi), \\
\sigma_4 &= (\alpha_2 - \alpha_6) \sin (2 \phi) + (\alpha_5 - \alpha_3) \sin (4 \phi) + (\alpha_3 + \alpha_5) \cos (3 \phi).
\end{align*}
\]

(34)

Here, note that the von Neumann condition is fulfilled; that is, \( |q| \leq 1 \). This affects the difference scheme (25) to be unconditionally stable. In the same way, we can show that the difference equation (28) can be unconditionally stable as well.

3. Numerical Calculations

Here, numerical tests are presented to demonstrate the performance of our proposed algorithm for single and
interactions of multiple solitary waves. Also, the modified Maxwellian I.C.s are pointed out to generate a train of solitary waves. Furthermore, the accuracy of the presented schemes is measured in terms of the following discrete error norms $L_2$ and $L_\infty$:

\[
L_2 = \sqrt{\int_0^L \left( \sigma_{\text{exact}} - \sigma_i \right)^2 \, dx}, \quad L_\infty = \| \sigma_{\text{exact}} - \sigma_i \|_\infty
\]

(35)

The conservation properties of the MRLW equation related to energy, mass, and momentum can be determined by finding the three basic invariants [24, 25]:

\[
\chi_1 = \int_{\xi_1}^{\xi_2} \sigma^2 \, dx, \quad \chi_2 = \int_{\xi_1}^{\xi_2} \left[ \sigma^2 + \mu(\sigma) \right] \, dx, \quad \chi_3 = \int_{\xi_1}^{\xi_2} \left[ \sigma^4 - \frac{6}{\zeta} \mu(\sigma) \right] \, dx
\]

(36)

3.1. Single Solitary Wave. Let $\eta_0$ be any arbitrary constant. Then, the exact solution of the solitary wave of the MRLW equation is given as follows [20]:

\[
\sigma(\eta, \tau) = \sqrt{\frac{6\lambda}{\xi}} \sech \left( \sqrt{\frac{\lambda}{\mu(\lambda + 1)}(\eta - (\lambda + 1)\tau - \eta_0)} \right).
\]

(37)

The modified Maxwellian I.C. is defined by

\[
\sigma(\eta, 0) = \sqrt{\frac{6\lambda}{\xi}} \sech \left( \sqrt{\frac{\lambda}{\mu(\lambda + 1)}(\eta - \eta_0)} \right).
\]

(38)

and the B.Cs can be concluded from the exact equation.

We choose $\zeta = \mu = 1$, $\lambda = 0.1$, $h = 0.125$, $\Delta \tau = 0.1$, $\eta_0 = 0$, $-40 \leq \eta \leq 60$ so that we can compare our results with results in [20, 23]. The program is executed up to times $\tau = 10$ to find error norms and the invariants $\chi_1, \chi_2, \chi_3$ at different times, and the results are given in Table 2. From Table 2, one can observe that the predicted error norms $L_2$ and $L_\infty$ are smaller than those obtained in [20, 23], and also the invariants $\chi_1, \chi_2$, and $\chi_3$ are sanely in good agreement with their

| Time | Method | $L_2 \times 10^4$ | $L_\infty \times 10^4$ | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|------|--------|-----------------|-----------------|--------|--------|--------|
| 0    | Analytical | 7.809875 | 2.129887 | 0.130250
|      | SBCM1   | 0.0         | 0.0            | 7.809702 | 2.129886 | 0.130251
|      | SBCM2   | 0.0         | 0.0            | 7.809702 | 2.129886 | 0.130251
| 1    | SBCM1   | 0.000899    | 0.000497       | 7.809725 | 2.129887 | 0.130251
|      | SBCM2   | 0.001982    | 0.001588       | 7.809725 | 2.129887 | 0.130251
| 2    | SBCM1   | 0.000481    | 0.000681       | 7.809744 | 2.129887 | 0.130251
|      | SBCM2   | 0.000986    | 0.000966       | 7.809744 | 2.129887 | 0.130251
| 3    | SBCM1   | 0.000979    | 0.000381       | 7.809767 | 2.129887 | 0.130251
|      | SBCM2   | 0.001081    | 0.002124       | 7.809767 | 2.129887 | 0.130251
| 4    | SBCM1   | 0.002519    | 0.000994       | 7.809783 | 2.129887 | 0.130251
|      | SBCM2   | 0.006957    | 0.004066       | 7.809783 | 2.129887 | 0.130251
| 5    | SBCM1   | 0.007738    | 0.005744       | 7.809800 | 2.129887 | 0.130251
|      | SBCM2   | 0.009886    | 0.008759       | 7.809800 | 2.129887 | 0.130251
| 6    | SBCM1   | 0.009956    | 0.008964       | 7.809819 | 2.129887 | 0.130251
|      | SBCM2   | 0.011042    | 0.012464       | 7.809819 | 2.129887 | 0.130251
| 7    | SBCM1   | 0.010009    | 0.009979       | 7.809841 | 2.129887 | 0.130251
|      | SBCM2   | 0.021244    | 0.010012       | 7.809841 | 2.129887 | 0.130251
| 8    | SBCM1   | 0.013692    | 0.010210       | 7.809855 | 2.129887 | 0.130251
|      | SBCM2   | 0.033897    | 0.021252       | 7.809855 | 2.129887 | 0.130251
| 9    | SBCM1   | 0.024350    | 0.013776       | 7.809868 | 2.129887 | 0.130251
|      | SBCM2   | 0.039873    | 0.028806       | 7.809868 | 2.129887 | 0.130251
| 10   | SBCM1   | 0.026879    | 0.019698       | 7.809874 | 2.129887 | 0.130251
|      | SBCM2   | 0.035471    | 0.026398       | 7.809874 | 2.129887 | 0.130251
| 10   | [20]    | 6.982800    | 1.995240       | 7.809320 | 2.129880 | 0.130315
| 10   | [23]    | 0.048674    | 0.033611       | 7.807948 | 2.129887 | 0.130251

Table 2: Error norms and invariants for the single solitary wave for the above parameters.
exact values. The solutions at $\tau = 0, 20$ and the motion of the solitary wave along with the interval $-40 \leq \eta \leq 60$ with $0 \leq \tau \leq 20$ to the right are illustrated in Figure 1. Moreover, the error variations are demonstrated for the proposed algorithms SBCM1 and SBCM2 in Figure 2 at time $\tau = 10$. Consequently, we can observe from Table 2 that the results obtained by the SBCM2 are more accurate than those obtained by the SBCM1.

3.2. Two Solitary Waves. Now, we study the interaction of two solitary waves having different amplitudes, which is the sum of two modified Maxwellian $I.C.$:

$$\sigma(\eta, 0) = \sum_{j=1}^{2} \frac{6 \lambda_j}{\zeta} \sech \left( \sqrt{\frac{\lambda_j}{\zeta (\lambda_j + 1)} (\eta - \eta_j)} \right). \quad (39)$$

Here, we take the parameters $\zeta = 6, \lambda_1 = 4, \lambda_2 = 1, \eta_1 = 25, \eta_2 = 55, h = 0.2, \Delta \tau = 0.025, \mu = 1, \eta_0 = 0, 0 \leq \eta \leq 200$ to concur with those used in [22, 23, 26]. The program is executed up to time $\tau = 20$, and the values of invariants $\chi_1, \chi_2, \chi_3$ are shown in Table 3 and compared with those obtained in [22, 23, 26] at time $\tau = 20$. On the other hand, Figure 3 illustrates the interaction of solitary waves at the times $\tau = 0$ and $\tau = 15$, respectively.

3.3. Three Solitary Waves. Here, we study the MRLW equation with the modified Maxwellian $I.C.$ and different amplitudes:

$$\sigma(\eta, 0) = \sum_{j=1}^{3} \frac{6 \lambda_j}{\zeta} \sech \left( \sqrt{\frac{\lambda_j}{\zeta (\lambda_j + 1)} (\eta - \eta_j)} \right). \quad (40)$$

Numerical experiments are executed for the parameters $\zeta = 6, \gamma = 0.2, \mu = 1, \Delta \tau = 0.025, \lambda_1 = 0.03, \lambda_2 = 0.02, \lambda_3 = 0.01, \eta_1 = 18, \eta_2 = 48, \eta_3 = 88$ in the region $-40 \leq \eta \leq 180$ in order to see an interaction of three solitary waves takes place. The program is executed up to time $\tau = 45$. Table 4 compares our obtained values of the invariants of the three solitary waves by SBCM2 with those obtained by [23]. It is clear from the table that our obtained results of the invariants remain

| Time | Method | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|------|--------|---------|---------|---------|
| 0    | SBCM1  | 11.46798 | 14.629258 | 22.880465 |
| 2    | SBCM2  | 11.46798 | 14.629260 | 22.880477 |
| 4    | SBCM1  | 11.46798 | 14.629616 | 22.882143 |
| 6    | SBCM2  | 11.46798 | 14.627877 | 22.883803 |
| 8    | SBCM1  | 11.46798 | 14.629946 | 22.886081 |
| 10   | SBCM2  | 11.46798 | 14.627551 | 22.888679 |
| 12   | SBCM1  | 11.46798 | 14.628481 | 22.886133 |
| 14   | SBCM2  | 11.46798 | 14.628168 | 22.883663 |
| 16   | SBCM1  | 11.46799 | 14.629275 | 22.885653 |
| 18   | SBCM2  | 11.46799 | 14.628557 | 22.887153 |
| 20   | SBCM1  | 11.46799 | 14.629067 | 22.885269 |
| 20 [22] | SBCM2 | 11.46799 | 14.628562 | 22.888816 |
| 20 [23] | SBCM2 | 11.46799 | 14.629554 | 22.885142 |
| 20 [26] | SBCM2 | 11.46799 | 14.626609 | 22.874989 |
| 20 [26] | SBCM2 | 11.46799 | 14.629903 | 22.874809 |
| 20 [26] | SBCM2 | 11.46799 | 14.629287 | 22.885799 |
| 20 [26] | SBCM2 | 11.46799 | 14.629190 | 22.874809 |
| 20 [26] | SBCM2 | 11.46799 | 14.629292 | 22.8809 |
| 20 [26] | SBCM2 | 11.46799 | 14.583089 | 22.69651 |
| 20 [26] | SBCM2 | 11.4661 | 14.6249 | 22.8631 |

Table 3: Two solitary waves invariants for the above parameters.
almost the same during the computer run, and they are found to be very close to the results given in [23]. In addition, these are all in good agreement with their analytical results.

Also, numerical experiments are carried out for the parameters \( \hbar = 0.2, \mu = 1, k = 0.025, \lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 0.25, \eta_1 = 15, \eta_2 = 45, \eta_3 = 60, 0 \leq \eta \leq 250 \). The computation is

![Figure 3: Plot illustrations for interaction of 2 solitary waves at \( \tau = 0 \) and \( \tau = 15 \).](image)

Table 4: Invariants for 3 solitary waves for \( \zeta = 6, \mu = 1, \hbar = 0.2, \Delta \tau = 0.025, \lambda_1 = 0.03, \lambda_2 = 0.02, \lambda_3 = 0.01, \eta_1 = 18, \eta_2 = 48, \eta_3 = 88, -40 \leq \eta \leq 180 \).

| Time | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) |
|------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0.0  | 9.518251     | 0.904130     | 0.00786304   | 9.517705     | 0.904129     | 0.00786304   |
| 0.1  | 9.518016     | 0.904130     | 0.00786312   | 9.517580     | 0.904130     | 0.00786272   |
| 0.2  | 9.518193     | 0.904130     | 0.00786323   | 9.517585     | 0.904130     | 0.00786286   |
| 0.3  | 9.518101     | 0.904130     | 0.00786338   | 9.517590     | 0.904130     | 0.00786293   |
| 0.4  | 9.518130     | 0.904130     | 0.00786350   | 9.517594     | 0.904130     | 0.00786294   |
| 0.5  | 9.518145     | 0.904130     | 0.00786365   | 9.517598     | 0.904130     | 0.00786296   |
| 0.6  | 9.518159     | 0.904130     | 0.00786388   | 9.517602     | 0.904130     | 0.00786297   |
| 0.7  | 9.518177     | 0.904130     | 0.00786300   | 9.517606     | 0.904130     | 0.00786298   |
| 0.8  | 9.518185     | 0.904130     | 0.00786300   | 9.517610     | 0.904130     | 0.00786299   |
| 0.9  | 9.518197     | 0.904130     | 0.00786300   | 9.517613     | 0.904130     | 0.00786300   |
| 1.0  | 9.518253     | 0.904130     | 0.00786301   | 9.517616     | 0.904130     | 0.00786300   |

Table 5: Invariants for 3 solitary waves for \( \mu = 1, \hbar = 0.2, \Delta \tau = 0.025, \lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 0.25, \eta_1 = 15, \eta_2 = 45, \eta_3 = 60, 0 \leq \eta \leq 250 \).

| Time | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) |
|------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0    | 14.980105    | 15.821789    | 22.992100    | 14.9801      | 15.8375      | 23.0081      |
| 5    | 14.988882    | 15.829547    | 22.914553    | 14.9799      | 15.8365      | 23.0036      |
| 10   | 14.986782    | 15.826933    | 22.935095    | 14.9850      | 15.8453      | 23.0207      |
| 15   | 14.986213    | 15.826005    | 22.940089    | 14.9809      | 15.8367      | 22.9986      |
| 20   | 14.984805    | 15.825383    | 22.954986    | 14.9790      | 15.8340      | 22.9927      |
| 25   | 14.983907    | 15.824426    | 22.962205    | 14.9780      | 15.8323      | 22.9876      |
| 30   | 14.983243    | 15.823337    | 22.977218    | 14.9777      | 15.8311      | 22.9827      |
| 35   | 14.981637    | 15.822498    | 22.999045    | 14.9778      | 15.8299      | 22.9779      |
| 40   | 14.980144    | 15.821066    | 22.997974    | 14.9795      | 15.8291      | 22.9728      |
| 45   | 14.980008    | 15.821781    | 22.996855    | 14.9534      | 15.8290      | 22.9649      |
done until time $\tau = 45$ to find numerical results of the invariants $\chi_1, \chi_2, \chi_3$. The result values of the invariants of the proposed SBCM2 algorithm together with the values of the invariants obtained in [26] are documented in Table 5.

In addition, we demonstrate the interaction of three solitary waves at times $\tau = 1, 5$ and $\tau = 10$, respectively, in Figure 4 and consequently, we can see that at time $\tau = 0, 5$, the three solitary waves interact and then at times $\tau = 20, 35$, the three solitary waves separate and emerging unchanged.

4. Conclusion

The main results of the article can be summarized as follows:

(i) The sextic B-spline collocation methods are presented to approximate a new solution of the MRLW equation

(ii) The unconditionally stability of the methods is derived

(iii) The operations are established by calculating both error norms $L_2$ and $L_\infty$

(iv) The numerical applications are demonstrated through examples of MRLW equations with the modified Maxwellian $I.C.s$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts interests.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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