Entanglement degradation of cavity modes due to the dynamical Casimir effect

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We study the entanglement dynamics between two cavities when one them is harmonically shaken in the context of relativistic quantum information. We find four different regimes depending on the frequency of the motion and the spectrum of the moving cavity. If the moving cavity is three-dimensional only two modes inside get coupled and the entanglement can either degrade asymptotically with time or oscillate depending on the driving. On the other hand, if the cavity has an equidistant spectrum the entanglement can either vanish asymptotically if it is driven with its fundamental frequency or have a sudden death if it is driven with an uneven harmonic frequency.

I. INTRODUCTION

In the last decade a growing interest has arose in the field of relativistic quantum information (RQI) since a number of interesting novel effects on entanglement between moving observers have been reported [1]. Hence, this area of research has addressed questions related to the relativistic aspects of quantum physics, in particular submerging into the relation between the physics of quantum field theories and quantum information theory [2]. As an example, black holes, once believed to be only mathematical artifacts, are today an established part of the universe and one the greatest mysteries of modern physics. They are very well defined in terms of general relativity but very little is known about their quantum nature. The discovery by Hawking [3] that they emit thermal radiation and the information paradox that this fact had led to is one of the many efforts done in order to get a complete understanding of their relation with quantum mechanics. Particularly, RQI has a crucial role in explaining quantum effects around black hole’s physics.

An interesting approach taken by authors in Ref.[4] investigates the possibility of transmitting signals between inside and outside the event horizon of a black hole. Therein, a pair of entangled particles is considered, each one hold by two different observers: one observer is inertial going into free-fall inside the black hole while the other accelerates away and hovers outside the event horizon on a fixed distance. Authors noticed this study could be approximately described by a scalar quantum field in Minkowski spacetime with two entangled modes. In this framework, they calculated the entanglement between two free modes of the scalar field as seen by an inertial observer detecting one of the modes and a uniformly accelerated observer detecting the second mode. They therefore stated that as the acceleration of this observer increases, the perceived entanglement between the two modes decreases asymptotically to zero, hindering the possibility of signal transmission [5]. This work led to multiple ramifications and numerous investigations on the effect of relativistic motion in quantum information [6–11]. For instance, in Ref.[6], the entanglement shared between observers in relative accelerated motion is analyzed finding that it vanishes between the lowest-frequency modes. In a different approach, authors in Ref.[10] showed that if the scalar field is replaced with a Dirac field the entanglement does not vanish in the infinite acceleration limit meaning that it can be used for quantum information tasks. Sometimes, results can be even more dramatic. In Ref.[11] it has been shown that the entanglement shared between two Unruh-DeWitt detectors (see Ref.[12]) that are being accelerated can suddenly be lost in a finite amount of time. Similarly, in Refs.[8, 13] it has been stated that the entanglement shared by two entangled cavities in accelerated motion oscillates as time elapses, during the time period the motion lasts. In this context, we can notice that the original motivation has led to a more general question about how entanglement changes for observers in relative motion. In particular, entanglement degradation caused by relativistic acceleration is a feature that has received great attention recently and we aim to explore in this manuscript.

In this paper we will continue these investigations by studying how the entanglement between two cavities changes as one of them is rapidly shaken. In the previous cases, the accelerated motion of the observer changed the state of the field due to the Unruh effect [14]. Alternatively, in this manuscript, the state of the field will be altered by the dynamical Casimir effect (DCE) [15], known as being responsible for the creation of photons pairs from the electromagnetic vacuum when mirrors are subjected to ultra fast oscillations [16]. It is important to remark that, from an experimental point of view, making use of the Unruh effect requires either linearly unbounded motion which leads to short integration times or complex rotating setups [17]. On the other hand for the DCE the motion is linear and bounded for arbitrarily long times of integration. Also, while creation of photon pairs out of quantum vacuum is close to being technologically accessible in optomechanical experiments [18, 19] (practical...
optomechanical structures have been created in which the mirror can oscillate as fast as six billion times a second); it is already accessible in superconducting quantum circuits, where the DCE has been measured for the first time. Therefore practical implementations of the kind can be designed in order to get an insight into the consequences of relativistic motion on quantum entanglement. This papers is organized as follows: in Section II we review the physics of a cavity with two oscillating boundaries. We shall show that, in some cases, it can be reduced to the case of a cavity with a single moving mirror, allowing for an analytical solution for small times. Then, in Sec. III we examine some properties of gaussian states in terms of the covariance matrix and see how we can measure the entanglement between two parts in that case. Sec. IV explores the use of these tools to analytically study the entanglement degradation between two parts in that case. Finally, in Sec. V we present our conclusions.

II. DYNAMICAL CASIMIR EFFECT

We consider a rectangular cavity formed by perfectly reflecting mirrors with dimensions \( L_x \), \( L_y \) and \( L_z \). The mirrors oscillate together in the \( x \)-axis, maintaining the distance among them fixed (we will refer to this cavity as a "shaker"), while the two other pairs of mirrors in the \( y \) and \( z \)-axis are at rest for all times \([23][22]\). We shall consider the light as a scalar field satisfying the wave equation

\[
\Box \Phi(x, t) = 0
\]

subjected to time dependent boundary conditions on the mirrors with displacement \( r(t) \),

\[
\begin{align*}
\Phi(x = 0 + r(t), y, z, t) &= \Phi(x = L_x + r(t), y, z, t) = 0 \\
\Phi(x, y = 0, z, t) &= \Phi(x, y = L_y, z, t) = 0 \\
\Phi(x, y, z = 0, t) &= \Phi(x, y, z = L_z, t) = 0.
\end{align*}
\]

For \( t < 0 \) all mirrors are at rest (static cavity) and the field can then be expanded as

\[
\Phi(x, t) = \sum_k \left[ \hat{a}_k^{\text{in}} u_k(x, t) + \text{h.c.} \right],
\]

where \( \hat{a}_k^{\text{in}} \) are the bosonic operators corresponding to different photon modes and the functions \( u_n \) are the positive frequency solutions of the wave equation

\[
u_k(x, t) = \sqrt{\frac{2}{L_x}} \sin(k_x x) \sqrt{\frac{2}{L_y}} \sin(k_y y) \sqrt{\frac{2}{L_z}} \sin(k_z z) \\
\times e^{-i \omega_k t} / \sqrt{2 \omega_k}
\]

with \( k = \left( \frac{n_x \pi}{L_x}, \frac{n_y \pi}{L_y}, \frac{n_z \pi}{L_z} \right) \) and \( \omega_k = |k| \). The mirrors in the \( x \)-axis start to move at \( t = 0 \) and the original basis gets continually deformed into a new one satisfying the boundary conditions \( u_k(x, t) \rightarrow \nu_k(x, t) \). Expanding the field in this new basis as

\[
\hat{\Phi}(x, t) = \sum_n \left[ \hat{a}_n^{\text{out}} v_n(x, t) + \text{h.c.} \right],
\]

we can define new bosonic operators \( \hat{a}_n^{\text{out}} \) corresponding to a new notion of particles. As the mirrors go back to their original position and stop moving, the field can be expressed alternatively in any of the two basis, which means that there exist coefficients \( \alpha_{nk} \) and \( \beta_{nk} \) such that

\[
v_n = \sum_k \left[ \alpha_{nk} u_k + \beta_{nk} \nu_k \right].
\]

Replacing this in Eq. (3) and equating to Eq. (4) we can relate both sets of bosonic operators as

\[
\hat{a}_n^{\text{out}} = \sum_k \left[ \alpha_{nk} \hat{a}_k^{\text{in}} + \beta_{nk} \hat{a}_k^{\text{in}*} \right],
\]

which is known as a Bogoliubov transformation. In the following, we shall obtain explicit analytic solutions for the coefficients \( \alpha_{nk} \) and \( \beta_{nk} \) as a function of time for a harmonic oscillation of the cavity mirrors. This will give us the complete time evolution of the field in the Heisenberg picture.

It is important to remark that, for an initially vacuum field’s state, the number of photons in mode \( k \) at a time \( t > 0 \) can be computed as

\[
\langle \hat{N}_n \rangle = \langle \hat{a}_n^{\text{out}} \hat{a}_n^{\text{out}*} \rangle = \sum_k |\beta_{nk}|^2.
\]

This means that for \( |\beta_{nk}| \neq 0 \) photons in mode \( n \) are created from vacuum. This is commonly known as the Dynamical Casimir Effect (DCE).

Bogoliubov coefficients

We shall focus on obtaining the Bogoliubov coefficients. Hence, we begin by expanding solutions in the comoving basis

\[
v_n(x, t) = \sum_k Q_k^{(n)}(t) \varphi_k(x, t)
\]

where \( Q_k^{(n)}(t) \) are time dependent coefficients and

\[
\varphi_k(x, t) = \sqrt{\frac{2}{L_x}} \sin(k_x (x - r(t))) \sqrt{\frac{2}{L_y}} \sin(k_y y) \\
\times \sqrt{\frac{2}{L_z}} \sin(k_z z).
\]

To find the time dependent coefficients \( Q_k^{(n)}(t) \), we can replace Eq. (3) in Eq. (5) and, then, this field decomposition into the wave equation. Finally, using the orthogonality of \( \varphi_k \) (see [23]) we find the following differential
equation for the coefficients

\[
\dot{Q}_k^{(n)}(t) - 2\dot{r}(t) \sum_j Q_j^{(n)}(t) g_{kj} - r(t) \sum_j Q_j^{(n)}(t) g_{kj} - \dot{r}^2(t) \sum_{jl} Q_j^{(n)}(t) g_{lj} g_{lk} + k_k^2 Q_k^{(n)}(t) = 0, \tag{11}
\]

where we have defined

\[
g_{kj} = \int_{r(t)}^{L + r(t)} \int_{0}^{L_y} \int_{0}^{L_z} \partial \frac{\varphi_k(x)}{\partial x} \varphi_j(x) dz dy dx \quad \tag{12}
\]

\[
= \begin{cases} 
((1)^{j_x + k_x} - 1) \frac{2k_x j_x}{k_x^2 - j_x^2} \delta_{k_x j_x} \delta_{k_x j_x} & k_x \neq j_x \\
0 & k_x = j_x
\end{cases}.
\]

By considering a harmonic oscillation of the mirrors position \(r(t) = \epsilon \sin(\Omega t)\) we further obtain

\[
\dot{Q}_k^{(n)}(t) - 2\epsilon \Omega \cos(\Omega t) \sum_j \dot{Q}_j^{(n)}(t) g_{kj} + \epsilon^2 \Omega^2 \sin(\Omega t) \sum_j Q_j^{(n)}(t) g_{kj} - \epsilon^2 \sin(\Omega t)^2 \sum_{jl} Q_j^{(n)}(t) g_{lj} g_{lk} + \omega_k^2 Q_k^{(n)}(t) = 0.
\tag{13}
\]

At this stage, it is important to mention that we shall search for solutions of the form

\[
Q_k^{(n)}(t) = \alpha_{nk}(\tau) e^{-i\omega_k t} \sqrt{2\omega_k} + \beta_{nk}(\tau) e^{i\omega_k t} \sqrt{2\omega_k} \tag{14}
\]

in order to get analytical predictions about the particle creation process (this is a standard procedure known as \textit{multiple scale analysis} \cite{25}). Hence, if we introduce a slow time \(\tau := \frac{t}{\epsilon^2 \omega_k}\), the functions \(\beta_{nk}\) and \(\alpha_{nk}\) are slowly varying and contain the cumulative resonant effects. After averaging over fast oscillations we obtain the following differential equations

\[
\frac{d\alpha_{nk}}{d\tau} = \sum_j \frac{\Omega}{2\omega_k} g_{kj} \alpha_{nj} \left[ (\omega_j + \frac{\Omega}{2}) \delta(\Omega + \omega_j - \omega_k) + (\omega_j - \frac{\Omega}{2}) \delta(-\Omega + \omega_j - \omega_k) \right] + \sum_j \frac{\Omega}{2\omega_k} g_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \alpha_{nj} \delta(\Omega - \omega_j - \omega_k), \tag{15}
\]

\[
\frac{d\beta_{nk}}{d\tau} = \sum_j \frac{\Omega}{2\omega_k} g_{kj} \beta_{nj} \left[ (\omega_j + \frac{\Omega}{2}) \delta(\Omega + \omega_j - \omega_k) + (\omega_j - \frac{\Omega}{2}) \delta(-\Omega + \omega_j - \omega_k) \right] + \sum_j \frac{\Omega}{2\omega_k} g_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \beta_{nj} \delta(\Omega - \omega_j - \omega_k). \tag{16}
\]

We can mention that by replacing Eqs. \eqref{14} and \eqref{10} in Eq. \eqref{9} and comparing the result with that of Eq. \eqref{10}, it is easy to note that these slowly changing \(\alpha_{nk}\) and \(\beta_{nk}\) will be the Bogoliubov coefficients found in the previous section.

There are essentially two distinct cases in which we can solve this set of coupled differential equations depending on whether the cavity’s spectrum is equidistant or non-equidistant. As for a non-equidistant spectrum, it can be achieved for example by considering \(L_x \sim L_y \sim L_z\) (a three-dimensional one) or, in another case by considering a massive scalar field. In such cases, only two modes will couple (except for some exceptional cases described in \cite{24}). Oppositely, if the cavity is one-dimensional (or \(L_x \ll L_y, L_z\) and it behaves as one dimensional); the spectrum is approximately equidistant \(\omega_n = n\pi/L_x\) and then infinitely many modes will couple. In the following, we shall briefly describe both situations.

\textbf{3D cavity and non-equidistant spectrum: Two coupled modes}

Let us consider the case in which \(L_x \sim L_y \sim L_z\). If we assume a driving (shaking) frequency \(\Omega\) that can be formed by two modes \(\omega_s\) and \(\omega_c\), as \(\Omega = \omega_s + \omega_c\); then, because the spectrum is non-equidistant, there is (almost) never a third mode \(\omega_d\) such that \(\Omega = |\omega_s \pm \omega_d|\) or \(\Omega = |\omega_c \pm \omega_d|\). In that case, Eqs. \eqref{15} and \eqref{16} reduce
the spectrum is given by
\[ \omega_k q \]
if \( \Omega = |\omega_1 - \omega_2| \) and it is exponential in time \([25, 26]\). \( \beta \) and since, in this case it is of photons created from the vacuum is given by Eq.(8), also be seen from our previous discussion. The number and \( \omega \) quanta is converted into a pair of photons in modes \( \omega \) with squeezing parameter \( \omega \) This transformation generates a two-mode squeeze state where
\[ j \] +
\[ j \] + \( \Omega \)
occurs when a driving \( \omega \) out \( \omega \) = \( \cosh(\gamma \tau) \omega_{s}^{\dagger} \) \( \sinh(\gamma \tau) \omega_{c}^{\dagger} \) \( \gamma \) \( \gamma \) \( \Omega \) \( \Omega \) \( L_y \) \( L_z \) and the cavity behaves as one dimensional \([27]\). Since the only relevant direction corresponds to the \( x \)-axis we discard the bold face indices to keep only this component. Then, using the coefficients described in Eq.(12), the set of Eqs. (13) and (16) reduces to
\[ \frac{d\beta_{ns}}{d\tau} = \frac{\Omega}{2\omega_s} g_{sc} \left( -\omega_s + \frac{\Omega}{2} \right) \alpha_{nc} \] (17)
\[ \frac{d\alpha_{nc}}{d\tau} = \frac{\Omega}{2\omega_c} g_{sc} \left( -\omega_c + \frac{\Omega}{2} \right) \beta_{ns}. \] (18)

This is a linear system with four coupled differential equations that can be easily solved. When combined with Eq.(7) we obtain
\[ a_{s}^{\text{out}} = \cos(\gamma \tau) a_{s}^{\text{in}} + \sin(\gamma \tau) a_{c}^{\text{in}} \] (19)
\[ a_{c}^{\text{out}} = \cos(\gamma \tau) a_{c}^{\text{in}} - \sin(\gamma \tau) a_{s}^{\text{in}} \] (20)

where
\[ \gamma = \frac{\Omega}{2} g_{sc} \sqrt{\left( \frac{\omega_s - \Omega}{2} \right) \left( \frac{\Omega}{2} - \omega_c \right) \omega_k \omega_c}. \] (21)

This transformation generates a two-mode squeeze state with squeezing parameter \( \gamma \), occurring when a driving quanta is converted into a pair of photons in modes \( \omega_s \) and \( \omega_c \). It is important to remark that this fact can also be seen from our previous discussion. The number of photons created from the vacuum is given by Eq.(8), and since, in this case it is \( \beta_{nk} \neq 0 \), there is creation of particles and it is exponential in time \([25, 26]\).

The other possible coupling between two modes occurs if \( \Omega = |\omega_s - \omega_c| \). In that case, the system of Eqs. (15) and (16) becomes
\[ \frac{d\beta_{ns}}{d\tau} = \frac{\Omega}{2\omega_s} g_{sc} \left( \omega_c + \frac{\Omega}{2} \right) \alpha_{nc} \] (22)
\[ \frac{d\alpha_{nc}}{d\tau} = \frac{\Omega}{2\omega_c} g_{sc} \left( \omega_s + \frac{\Omega}{2} \right) \beta_{ns}. \] (23)

This set of equations can be readily solved and yields
\[ a_{s}^{\text{out}} = \cos(\gamma \tau) a_{s}^{\text{in}} + \sin(\gamma \tau) a_{c}^{\text{in}} \] (24)
\[ a_{c}^{\text{out}} = \cos(\gamma \tau) a_{c}^{\text{in}} - \sin(\gamma \tau) a_{s}^{\text{in}} \] (25)

where
\[ \gamma = \frac{\Omega}{2} g_{sc} \sqrt{\left( \omega_s + \frac{\Omega}{2} \right) \left( \frac{\Omega}{2} - \omega_c \right) \omega_k \omega_c}. \] (26)

It is worth mentioning that in this situation \( \beta_{nk} = 0 \) and no new photons are created. This means that for an initially vacuum state the dynamics is trivial. However, given an initial state with some photons in one of the modes, the number of photons in each mode will oscillate while keeping the total number of photons constant.

1D cavity and equidistant spectrum: Infinite coupled modes

We now consider the case where \( L_x \ll L_y, L_z \) and the cavity behaves as one dimensional \([27]\). Since the only relevant direction corresponds to the \( x \)-axis we discard the bold face indices to keep only this component. Then, using the coefficients described in Eq.(12), the set of Eqs. (13) and (16) reduces to
\[ \frac{d\beta_{nk}}{d\tau} = \sum_{j+k \text{ even}} \frac{\Omega}{2\omega_k} g_{kj} \beta_{nj} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q_\Omega + j - k) + \left( \omega_j - \frac{\Omega}{2} \right) \delta(-q_\Omega + j - k) \right] \]
\[ + \sum_{j+k \text{ odd}} \frac{\Omega}{2\omega_k} g_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \alpha_{nj} \delta(q_\Omega - j - k) \] (27)
\[ \frac{d\alpha_{nk}}{d\tau} = \sum_{j+k \text{ odd}} \frac{\Omega}{2\omega_k} g_{kj} \alpha_{nj} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q_\Omega + j - k) + \left( \omega_j - \frac{\Omega}{2} \right) \delta(-q_\Omega + j - k) \right] \]
\[ + \sum_{j+k \text{ odd}} \frac{\Omega}{2\omega_k} g_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \beta_{nj} \delta(q_\Omega - j - k), \] (28)

where we have defined \( q_\Omega = \Omega/\omega_1 \) and used the fact that the spectrum is given by \( \omega_j = j\omega_1 \) and that \( g_{kj} = 0 \) if \( k + j \) is even. It is important to note that, since \( k + j \) is odd, \( k - j \) is also odd. This means that if \( q_\Omega \) is not an odd number then the RHS of Eqs. (27) and (28) is null and the evolution is trivial.
In order to obtain an analytical solution for the system of Eqs. (27) and (28), we consider the situation in which only one wall (let say the right one) of the cavity oscillates, in the one dimensional approximation. In such a case, an analysis similar to the one performed in the previous section, leads to the following system of equations

\[
\frac{d\tilde{\alpha}_k^{(n)}}{d\tau} = -\frac{\pi^2k^2}{2\omega_k L_x^2} \tilde{\alpha}_k^{(n)} \delta(2k - q\Omega) + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \tilde{\beta}_j^{(n)} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q\Omega - j - k) \right] + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \tilde{\alpha}_j^{(n)} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q\Omega + j - k) \right] + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \tilde{\alpha}_j^{(n)} \delta(q\Omega - j - k)
\]

(29)

\[
\frac{d\tilde{\beta}_j^{(n)}}{d\tau} = -\frac{\pi^2k^2}{2\omega_k L_x^2} \tilde{\beta}_j^{(n)} \delta(2k - q\Omega) + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \tilde{\alpha}_j^{(n)} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q\Omega + j - k) \right] + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \tilde{\alpha}_j^{(n)} \left[ \left( \omega_j + \frac{\Omega}{2} \right) \delta(q\Omega + j - k) \right] + \sum_j \frac{\Omega}{2\omega_k} \tilde{g}_{kj} \left( -\omega_j + \frac{\Omega}{2} \right) \tilde{\beta}_j^{(n)} \delta(q\Omega - j - k)
\]

(30)

with

\[
\tilde{g}_{kj} = \begin{cases} \frac{2k_j}{k^2 - j^2} & k \neq j \\ 0 & k = j. \end{cases}
\]

(31)

This set of equations is very similar to Eqs. (27) and (28), the differences being only two. The first difference among the systems of equations is that Eqs. (29) and (30) have an extra term proportional to \(\delta(2k - q\Omega)\) (which is not present in Eqs. (29) and (30)). This term creates pair of photons in mode \(k\) and is only relevant when \(q\Omega = 2k\). The second difference is that \(\tilde{g}_{kj} \neq \tilde{g}_{kj}\). However, it is easy to see that \(2\tilde{g}_{kj} = \tilde{g}_{kj}\) if \(k + j\) is odd, and this factor 2 can be absorbed in the oscillation amplitude. Therefore, if we excite a cavity with one moving wall at an odd frequency \(q\Omega\), we get that the first term of RHS of Eqs. (29) and (30) vanishes. All others also disappear, except for the case of \(k + j\) being odd (this is due to the \(\delta\) factors). This proves that when \(q\Omega\) is odd Eqs. (29) and (30) are equivalent to Eqs. (29) and (30). Consequently, this means that a cavity being shaken rigidly with an odd frequency behaves just as an identical cavity with only one wall oscillating with the same frequency and double amplitude. This result is very important and, as far as we are aware, it has not been mentioned in the existing literature. We shall take advantage of this equivalence to get analytical predictions on the particle creation process since analytical solutions for the system with one moving wall are well known [28]. In such a case the out operators are given by

\[
a^\text{out}_m = \sum_{n=1}^{\infty} \sqrt{\frac{\rho_{m}^{(n)}}{n}} a^\text{in}_m - \rho^{(n)*}_{-m} a^\text{in}_m.
\]

(32)

where

\[
\rho_{j+m,p}^{(j+m)} = \frac{\Gamma(1 + n + j/p)(\sigma\kappa)^{n-m}}{\Gamma(1 + m + j/p)\Gamma(1 + n - m)} \times F(n + j/p, -m - j/p; 1 + n - m; \epsilon^2)
\]

(33)

and

\[
\rho_{j+m,p}^{(k+m)} = 0
\]

(34)

if \(j \neq k\), where

\[
\sigma = (-1)^p
\]

(35)

\[
\kappa = \tanh(p\tilde{\tau}).
\]

(36)

As it has been exposed above, these are also solutions of the rigidly shaken cavity taking \(\tilde{\tau} = 2\tau = \omega_\Omega t\), that is doubling the amplitude \(\epsilon\).

Similarly to what we have noticed before in the three dimensional case, there are two qualitatively different regimes depending on the driving frequency. If \(\Omega = \omega_1 = \omega_{j+1} - \omega_j\) then no photons are created as a whole. However, given an initial number they can “jump” between adjacent modes. For long times, the number of photons on any mode goes to zero as they are lost to higher frequency adjacent modes. For long times, the number of photons on any mode goes to zero as they are lost to higher frequency adjacent modes. Alternatively, if \(\Omega = q\omega_1\), pairs of photons are created from the vacuum on any pair of modes \(j\) and \(k\) such that \(\Omega = \omega_j + \omega_k\).

We have briefly analyzed DCE in different configurations. In all cases, by shaking the cavity, we have altered the state of the scalar field. In addition, we have shown that, in some particular cases, we obtain creation of particles as a further result. In the following we shall study the entanglement process between two of these cavities as one of them is rapidly shaken.
III. ENTANGLEMENT MEASUREMENT

A particularly relevant set of states, from a theoretical viewpoint, are Gaussian states; which include coherent, number, squeezed, and thermal states. Given a set of harmonic oscillators, we can take a basis of quadrature operators \( R = (q_1, p_1, \ldots, q_n, p_n) \), where

\[
q_j = \frac{1}{\sqrt{2}} (a_j + a_j^\dagger) \tag{37}
\]

\[
p_j = -\frac{i}{\sqrt{2}} (a_j - a_j^\dagger) \tag{38}
\]

and completely characterize a Gaussian state \( \rho \) with its displacement vector

\[
d_j = \langle R_j \rangle_{\rho} \tag{39}
\]

and its covariance matrix

\[
V_{ij} = \frac{1}{2} (R_i R_j + R_j R_i) - \langle R_i \rangle_{\rho} \langle R_j \rangle_{\rho}. \tag{40}
\]

Some important properties of these states are that the evolution of a Gaussian state under a quadratic Hamiltonian is also a Gaussian state. Further, given a system in a globally Gaussian state, any subsystem has a state that is also Gaussian and its covariance matrix is given by the restriction of the covariance matrix of the whole to that subsystem. In addition, it is possible to quantify the entanglement in a mixed state \([32]\) by using the logarithmic negativity \([31]\), which is an entanglement monotone given by

\[
\mathcal{N}(\rho) = \log_2 \| \rho^{FA} \|_1, \tag{41}
\]

where \( \| \rho^{FA} \|_1 \) is the trace norm of the partial transpose of \( \rho \) with respect to subsystem \( A \).

Herein, we want to calculate the entanglement between two modes in Gaussian states. In such case, the covariance matrix is of the form

\[
V_{AB} = \begin{pmatrix} V_A & V_{AC} \\ V_{CA} & V_B \end{pmatrix} \tag{42}
\]

and the logarithmic negativity can be calculated as \([32]\)

\[
\mathcal{N} = \max\{0, -\log 2 \nu_-\} \tag{43}
\]

where

\[
\nu_- = \sqrt{\frac{\Sigma}{2} - \frac{1}{2} \sqrt{\Sigma^2 - 4 \det V_{AB}}} \tag{44}
\]

and

\[
\Sigma = \det V_A + \det V_B - 2 \det V_{C}. \tag{45}
\]

In order to determine if there is still information shared between subsystems when the entanglement vanishes, we will use the mutual information \([33][34]\)

\[
I(\rho_{AB}) = S_V(\rho_A) + S_V(\rho_B) - S_V(\rho_{AB}), \tag{46}
\]

which measures the total correlations (quantum and classical), with \( S_V \) being the Von Neumann entropy. In the case of a Gaussian state, it can be easily computed as

\[
I(V_{AB}) = f(\sqrt{\det 2V_A}) + f(\sqrt{\det 2V_B}) - f(\eta^+_{AB}) - f(\eta^-_{AB}) \tag{47}
\]

with

\[
f(x) = \frac{x + 1}{2} \log \left(\frac{x + 1}{2}\right) - \frac{x - 1}{2} \log \left(\frac{x - 1}{2}\right) \tag{48}
\]

where \( \eta^-_{AB}, \eta^+_{AB} \) are the symplectic eigenvalues of \( 2V_{AB} \).

IV. ENTANGLEMENT DEGRADATION

In this section, we shall study the entanglement degradation in two Fabry-Perot cavities. Initially, both cavities are in an entangled two-mode squeezed state comprising mode \( s \) in cavity 1 and \( p \) in cavity 2. A \( t = 0 \), we harmonically shake cavity 1 (as seen in Sec. [11]), keeping the distance between the mirrors fixed, and study how the entanglement between both cavities evolves in time.

The initial state of the system can be written in terms of the destruction operators as

\[
a^s_0 = \cosh(r)a_s + \sinh(r)a^\dagger_s
\]

\[
a^p_0 = \cosh(r)a_p + \sinh(r)a^\dagger_p
\]

where \( a_s \) and \( a^\dagger_s \) are the destruction operators of the uncorrelated modes, satisfying \( \langle a^\dagger_j a_j \rangle = 0 \) and \( \langle a_j a^\dagger_j \rangle = 0 \), for \( j = s, p \). As has been mentioned before, there are two qualitatively different behaviors for the evolution of the field in the shaken cavity depending on whether the spectrum is approximately equidistant or unevenly spaced. We therefore analyze all cases separately.

3D case and non-equidistant spectrum: Two coupled modes

In the case that the shaking cavity is considered as three dimensional \( L_s \sim L_y \sim L_z \), the spectrum is not evenly spaced. The excited mode \( s \) will then result coupled to at most one more mode, which we hereafter call mode \( c \). As explained in the preceding Sec[11] the altered field state will result qualitatively different when the driving frequency is \( \Omega = \omega_s + \omega_c \) or \( \Omega = |\omega_s - \omega_c| \).

Firstly, we begin by analyzing the system when the cavity 1 is shaken with a frequency \( \Omega = \omega_s + \omega_c \). We further assume these two modes do not couple to any other mode. Shaking cavity 1 with this frequency creates pairs of photons in modes \( c \) and \( s \). Accordingly to the derivations of the previous section, the new operators \( a_s^{out} \) and \( a_c^{out} \) are related to \( a_s^0 \) and \( a_c^0 \) by Eqs. [19] and...
FIG. 1: Three dimensional cavity and a non-equidistant spectrum with driving frequency $\Omega = \omega_s + \omega_c$. (a) The exponential creation of pairs of particles, as a function of time, in modes $s$ and $c$ for different values of the squeezing parameter $r$. (b) Entanglement degradation measured by the logarithmic negativity as time evolves, vanishing for long times. (c) Mutual information as function of time. Even though quantum correlations disappear for long times, classical correlations among the modes persist in the long time limit.

[20] By considering the relation between creation operators and quadratures [37, 38] we are able to calculate the covariance matrix, which in the basis $(q_p, p_p, q_s, p_s)$, is

$$V_{p|s} = \frac{1}{2} \begin{vmatrix} \cosh(2r) & 0 & \cosh(\gamma \tau) \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\cosh(\gamma \tau) \sinh(2r) \\ \cosh(\gamma \tau) \sinh(2r) & 0 & (\cosh^2(r) \cosh(2\gamma \tau) + \sinh^2(r)) & 0 \\ 0 & -\cosh(\gamma \tau) \sinh(2r) & 0 & (\cosh^2(r) \cosh(2\gamma \tau) + \sinh^2(r)) \end{vmatrix}.$$  \hspace{1cm} (50)

We can further compute the long time properties of this state. We note that

$$\Sigma = \frac{1}{4} \cosh(2r)^2 + \frac{1}{4} (\cosh^2(r) \cosh(2\gamma \tau) + \sinh^2(r))^2$$

$$+ \frac{1}{2} (\cosh(\gamma \tau) \sinh(2r))^2 \xrightarrow{\tau \to \infty} \infty.$$ \hspace{1cm} (51)

Before continuing, it is important to mention that the following limit remains finite

$$\frac{\det V_{p|s}}{\Sigma} \xrightarrow{\tau \to \infty} \frac{1}{4}. \hspace{1cm} (52)$$

Hence, it is easy to see that the negativity ($\mathcal{N} = -\log 2 \nu_-$) vanishes in the long time limit

$$\mathcal{N} = -\log 2 \left( \sqrt{\frac{\det V_{p|s}}{\Sigma}} \right) \xrightarrow{\tau \to \infty} 0.$$ \hspace{1cm} (53)

In addition, we can also study the mutual information between the cavities by calculating the symplectic eigenvalues of $2V_{p|s}$. They are

$$\eta_- = 1,$$

$$\eta_+ = |\cosh(r)^2 \cosh(2\gamma \tau) - \sinh(r)^2|.$$ \hspace{1cm} (54)

In Fig. we compile all properties computed for this case. In panel (a), we show the number of particles in mode $s$. As can be seen, there is an exponential creation of pairs of photons in modes $s$ and $c$ as expected. In panel (b), we can see the time evolution of the entanglement as measured by the logarithmic negativity which is evidently degraded as time evolves, vanishing for long times and destroying the initial entanglement between both cavities. In panel (c), we plot the mutual information that evidences that classical correlations among the modes persist for long times. This means that even though the entanglement vanishes, there are still classical correlations contained in the mutual information between the modes $p$ and $s$ in the long time limit.

We now consider the distinct case of the cavity 1 being driven with a frequency $\Omega = |\omega_s - \omega_c|$. As mentioned before, this frequency does not create new photons but only redistributes the existing ones between the modes. In that case, the out operators are given by the Eqs. [24]
FIG. 2: Three dimensional cavity and a non-equidistant spectrum with driving frequency $\Omega = |\omega_s - \omega_c|$. In (a) the system behaves periodically with photons, originally in mode $s$, switching back and forth between modes $s$ and $c$. This leads to an oscillatory behavior of the entanglement (b) and the mutual information (c). They oscillate between a maximum, when the photons are in $s$ and zero, when they are in $c$.

and (25) and the covariance matrix can then be written as

$$V_{ps}|s = \frac{1}{2} \begin{vmatrix} \cosh(2r) & 0 & \cos(\gamma\tau) \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\cos(\gamma\tau) \sinh(2r) \\ \cos(\gamma\tau) \sinh(2r) & 0 & (\cos^2(\gamma\tau) \cosh(2r) + \sin^2(\gamma\tau)) & 0 \\ 0 & -\cos(\gamma\tau) \sinh(2r) & 0 & \cos^2(\gamma\tau) \cosh(2r) + \sin^2(\gamma\tau) \end{vmatrix}.$$ 

The components of this matrix oscillate harmonically in time and for $\gamma\tau_n = \frac{\pi(2n+1)}{2}$ we obtain

$$V_{ps} = \frac{1}{2} \text{diag}(\cosh(2r), \cosh(2r), 1, 1)$$

which means that the logarithmic negativity, the number of photons in mode $s$ and mutual information between $s$ and $p$ all vanish. Contrarily, for $\gamma\tau_m = \frac{m\pi}{\omega_1}$, these magnitudes oscillate returning to their initial maximum values, as can be seen in Fig.2. In Fig.2(a) we show the number of particles. As can be inferred, there is no creation of new ones, but an oscillatory redistribution of particles already in the cavity mode. Particles get transferred from mode $s$ to mode $c$, until no photons remain in $s$ at $\tau_m$. In Fig.2(b) we show the entanglement temporal evolution which also results in a null entanglement at $\tau_m$. In Fig.2(c), we show the temporal evolution of the mutual information between $s$ in the first cavity and mode $p$ in the second.

1D case and equidistant spectrum: Infinite coupled modes

If the cavity is considered as one-dimensional (see Sec.1) the spectrum is equidistant. This means that infinitely many modes get coupled as the cavity is shaken. In this situation the out operators are described by a more general Bogoliubov transformation Eq.(7). Similarly as with the three dimensional cavities, we can see that there are two distinctly interesting cases to consider as for the driving frequency of cavity 1: $\Omega = \omega_1$ and $\Omega = q\omega_1$.

Firstly, we start by considering $\Omega = \omega_1$. As it has already been pointed out, in this case there is no particle creation in cavity 1. Using Eq.(7), we can compute the covariance matrix for this case as

$$V_{sp}|s = \frac{1}{2} \begin{vmatrix} |\alpha_{ss}|^2 \cosh(2r) + \sum_{j \neq s} |\alpha_{sj}|^2 & 0 & \sinh(2r)\alpha_{ss} & 0 \\ 0 & |\alpha_{ss}|^2 \cosh(2r) + \sum_{j \neq s} |\alpha_{sj}|^2 & 0 & -\sinh(2r)\alpha_{ss} \\ \sinh(2r)\alpha_{ss} & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r)\alpha_{ss} & 0 & \cosh(2r) \end{vmatrix}.$$
There we have already used that $\beta_{sj} = 0$ and that $\alpha_{sj}(\tau)$, is a real function given by Eq. (53). In order to understand the long time behavior of this state, we can use the Bogoliubov relation

$$1 = \sum_{j=0}^{\infty} |\alpha_{sj}|^2 - |\beta_{sj}|^2 = \sum_{j=0}^{\infty} |\alpha_{sj}|^2$$

(57)

to reduce the problem in terms of a unique Bogoliubov coefficient:

$$(V_{s|p})_{11} = (V_{s|p})_{22} = \frac{1}{2} \left( |\alpha_{ss}|^2 \cosh(2r) + \sum_{j \neq s} |\alpha_{sj}|^2 \right)$$

$$= \frac{1}{2} \left( |\alpha_{ss}|^2 (\cosh(2r) - 1) + 1 \right).$$

(58)

As a matter of fact, the explicit expression for this coefficient $|\alpha_{ss}|$ is already known to be [35]

$$\alpha_{ss}(\tau) = \sum_{j=1}^{s} \left[ (s-1)! (s+j-1)! (-1)^{s-j} \right]$$

$$\times \left[ (s-1)! j! (s-j)! \right]^{-1} (\cosh \tau)^{-2j} \xrightarrow{\tau \to \infty} 0.$$  

This means that not only the entanglement but the mutual information as well vanish in the long time limit. In Fig. 3 we compile the results obtained when $\Omega = \omega_1$ in a one dimensional cavity behavior. In Fig. 3(a), we show the particle creation process for different initial values of the squeezing parameter $r$. The number of particles in mode $s = 1$ is lost to higher frequency modes since all cavity modes are coupled. The smaller the initial value of $r$, the sooner the particles are spread into other modes. In Fig. 3(b), we show that the entanglement evolution behaves accordingly to the number of particles $N_s$: the entanglement is lost very rapidly as time goes on and decrease asymptotically to zero. In Fig. 3(c), the mutual information exhibits a similar behavior in concordance to the rest of the quantities considered. It is important to note that the entanglement degradation in a one dimensional cavity is qualitatively different to that occurring in a three dimensional cavity. In the former case, the loss of entanglement is due to the many cavity modes available after shaking the cavity. This implies a redistribution of the existing particles in the cavity since no creation process takes place for this case. We can even think of a big environment to which information is lost. In the case of a three dimensional cavity, the apparent responsible for the loss of entanglement is the exponential creation of particles in each mode.

Finally, we consider cavity 1 to be driven with a frequency $\Omega = q \omega_1$. This excites a parametric creation of photon pairs which is then redistributed along higher frequency modes. In this case the out operators are given by Eq. (32), from which we can calculate the covariance matrix as
the number of photons from the vacuum 
with one oscillating mirror is driven with this frequency
ref [28], authors have already proved that when a cavity
since
Therefore, we proceed to study the asymptotic behavior
where
for
we conclude that
We can simplify this expression by noticing that $V_{11} = V_{22}$. This is because $\beta_{1,k+np} = -\sqrt{1/(k+np)}\rho_{k-1} = 0$ for $k \neq p - 1$ while $\alpha_{1,k+np} = \sqrt{1/(k+np)}\rho_{k+1} = 0$ for $k \neq 1$ and so $\alpha_{1,k+np}\beta_{1,k+np} = 0$, from which we conclude that $|\alpha_{1,k+np} \pm \beta_{1,k+np}|^2 = |\alpha_{1,k+np}|^2 + |\beta_{1,k+np}|^2$. By writing explicitly the covariance elements with Eq. (40), we can see that

$$V_{11} + V_{22} = 1 + 2N_s,$$

where $N_s = \langle a^\text{out}_s a^\text{out}_s \rangle$ is the number of photons in mode $s$ and so $V_{11} = N_s + 1/2$. The covariance matrix is then reduced to only 3 independent components $V_{11}, V_{33}, V_{24}$. Therefore, we proceed to study the asymptotic behavior of the state for long times. We must note that

$$\lim_{t \to \infty} V_{13} = -\lim_{t \to \infty} V_{24} < \infty,$$

since $V_{13}, V_{24} \propto F(a, b, c; \kappa^2)$ and for $t \to \infty$ we have $\kappa \to 1$ and $F(a, b, c; \kappa^2) \to F(a, b, c; 1) < \infty$.

At this point, it is important to mention the fact that in ref [28], authors have already proved that when a cavity with one oscillating mirror is driven with this frequency the number of photons from the vacuum $N_s(r = 0)$ grows linearly with time. Then using the equivalence between this setup and the shaken cavity, we see that

$$V_{11} = \frac{1}{2} |\alpha_{ss}|^2 \cosh(2r) - 1 + \frac{1}{2} + N_s(r = 0) \propto t.$$
\[
\eta_{\pm} = \frac{1}{\sqrt{2}} \sqrt{V_{11}^2 + V_{33}^2 + 2V_{13}V_{24} \pm \sqrt{V_{11}^4 + V_{33}^4 + 4V_{13}^2V_{24}^2 - 2V_{11}^2(V_{33}^2 - 2V_{13}V_{24}) + 4V_{11}V_{33}(V_{13}^2V_{24}^2)}}. \]  

(67)

By use of Eqs. (61) and (62), we have

\[
\eta_+ \approx V_{11}, \quad \text{for } \tau \gg 1 \\
\eta_- \approx V_{33}, \quad \text{for } \tau \gg 1
\]

from which we conclude that the mutual information also vanishes in the long-time limit

\[
I = f(V_{11}) + f(V_{33}) - f(\eta_-) - f(\eta_+) \xrightarrow{\tau \to \infty} 0. \]  

(69)

In Fig. 4(a) we present the numerical results for \( q = 3 \). In Fig. 4(a), we compute the numerical evolution of \( N_s \), while in Fig. 4(b), we show the entanglement degradation as time evolves. It is easy to note that the entanglement is qualitatively different from the previous cases as it dies suddenly in a finite time. This is due to the fact that two different effects are now combined and affect the entanglement: photon redistribution and pair creation. In Fig. 4(c), we can note that the mutual information, however, decreases slowly and asymptotically to zero. This last case is completely different to all others. It implies many infinite modes coupled in a cavity in addition to particle creation process. Initially, it can be seen that the number of particles \( N_s \) decreases because they are redistributed to higher modes available. This is much evident for \( r > 1 \). For a critical time the particle creation rate starts to gain importance and the number of particles in mode \( s \) starts increasing. Surprisingly, this critical time is similar to the time the entanglement suddenly dies (or becomes zero). This combination of factors in not present in the other cases described above.

V. CONCLUSIONS

In this work we have studied how the entanglement and classical correlations between modes \( s \) and \( p \) in two different cavities change when one of them is in relative oscillatory motion. In achieving so, we have first reviewed the DCE. We have presented the reigning equations for a two-moving wall ("shaker") cavity and computed the Bogoliubov transformation. Further we have analyzed the particle creation process for a non-equidistant and an equidistant spectrum and stressed their similarities and differences. It is important to mention that even though the analysis of the DCE has been performed previously, the equivalence between a rigid translational movement of a two-wall moving cavity and a single moving one with twice the amplitude of movement is a new contribution which allowed to achieve a complete analytical description of the system.

We have found that there are four qualitatively different behaviors for the system, depending on whether the spectrum is equidistant or not. The other feature that determines the behavior is whether the driving frequency \( \Omega \) is able to create new photons or just redistribute the initial existing ones. If the spectrum is unevenly spaced and there is an additional mode \( c \) such that \( \Omega = |\omega_s - \omega_c| \) then photons oscillate between \( s \) and \( c \). This causes the entanglement and the mutual information among the cavities to oscillate in time. However, if \( c \) has a frequency such that \( \Omega = |\omega_s + \omega_c| \) then pairs of photons are created in these modes, this degrades the entanglement between the cavities which goes asymptotically to zero. In spite of this, classical correlations persist as the mutual information converges to a positive value in the long-time limit. Our results show that the situation becomes qualitatively different when the spectrum of the cavity is evenly spaced, since this causes infinitely many modes the get coupled. In this case, if the moving cavity is shaken with its fundamental frequency, no photons are produced. However, as time goes on the photons in the initially excited mode are eventually lost which causes the entanglement and mutual information between the cavities to vanish as well. On the other hand, if the cavity oscillates with a frequency that is an uneven harmonic, there is also production of photons that forces the entanglement to have a sudden death in a finite time, while the mutual information goes asymptotically to zero.

This setup captures many of the results previously found for observers and cavities in accelerated motion. In both cases we have a Bogoliubov transformations that generate photon pairs. However, in previous results of the existing literature, the alteration of the field state was due to the Unruh Effect. In this work we exploit the non trivial structure of quantum vacuum and the effects derived from time dependent boundaries conditions. The result obtained is an apparententanglement degradation and information loss for mostly cases considered. We believe that this setup has more promising experimental qualities since it relies on a bounded motion in an optomechanical system. While, there are still technological challenges that must be overcome for it to be tested in this exact setting, as the frequency of nanoresonators is

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1 An exception for this behavior occurs when the frequency of the initially excited mode \( s \) coincides with the shaking frequency \( \omega_s = q\omega_1 \). In this case the solutions behave as for \( \Omega = \omega_1 \), which we have already analyzed.
not high enough, a simulation in a superconducting cavity is within experimental reach, since DCE has already been tested there.

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