On the Phase Structure of Many-Flavor QED$_3$

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We analyze the many-flavor phase diagram of quantum electrodynamics (QED) in $2 + 1$ (Euclidean) space-time dimensions. We compute the critical flavor number above which the theory is in the quasi-conformal massless phase. For this, we study the renormalization group fixed-point structure in the space of gauge interactions and pointlike fermionic self-interactions, the latter of which are induced dynamically by fermion-photon interactions. We find that a reliable estimate of the critical flavor number crucially relies on a careful treatment of the Fierz ambiguity in the fermionic sector. Using a Fierz-complete basis, our results indicate that the phase transition towards a chirally-broken phase occurring at small flavor numbers could be separated from the quasi-conformal phase at larger flavor numbers, allowing for an intermediate phase which is dominated by fluctuations in a vector channel. If these interactions approach criticality, the intermediate phase could be characterized by a Lorentz-breaking vector condensate.

I. INTRODUCTION

The competition between screening and anti-screening effects is at the heart of the intriguing diversity of phases occurring in asymptotically free theories. Not only thermal phase transitions governed by parameters such as temperature or chemical potentials, but also quantum phase transitions triggered by the number of active degrees of freedom have recently been of central interest. Most prominently, the number of light fermion degrees of freedom have recently been of central interest. Most prominently, the number of light fermion degrees of freedom $N_f$ often serves as a control parameter to tune the screening–anti-screening competition. While chiral quantum phase transitions of this type have attracted considerable attention in 4-dimensional non-abelian gauge theories because of their potential relevance for embeddings of the Higgs sector in beyond-standard-model scenarios [1–7], similar theoretical mechanisms can be at work in the abelian theory of quantum electrodynamics (QED) in $d = 3$ (Euclidean) spacetime dimensions. Beyond the predominantly conceptual interest, such studies gain significance from layered condensed-matter systems for which dominantly conceptual interest, such studies gain significance

If this reference scale is pushed to asymptotically large energies or momenta. In turn, one expects QED$_3$ to become more strongly coupled at low energies, possibly generating fermion masses through a chiral phase transition. By contrast, increasing the number of fermion flavors enhances the screening properties of fermionic fluctuations. If this screening dominates, the coupling may remain small and the theory can be expected to be in the disordered massless phase. More precisely, the fluctuations can generate an infrared (IR) fixed point, such that the theory remains quasi-conformal: it has a nontrivial RG flow from the Gaussian ultraviolet (UV) to the IR fixed point with the transition scale set by the dimensionful gauge coupling. Scenarios of this type have been suggested and analyzed in many works, and the critical flavor number $N_f^{\chi}$ separating the chirally broken phase for small $N_f$ from the symmetric for large $N_f$ has been estimated by a variety of nonperturbative methods, see, e.g., Refs. [17–38]. Predictions from self-consistent approximations of the Dyson–Schwinger equations (DSE) in their most advanced form yield results near $N_f^{\chi} \approx 4$, see, e.g., [28]. Recently, these studies have been extended to incorporate lattice anisotropies as well as finite temperature in order to approach more realistic applications [39–42]. An early RG study found $N_f^{\chi} \approx 3.1$ [30].

Based on a thermodynamic argument an inequality $N_f^{\chi} \leq 1.5$ has been conjectured [26], but was challenged later [15]. Another upper bound $N_f^{\chi} < 7$ has been claimed recently using an RG monotonicity argument [38]. On the other hand, lattice simulations in QED$_3$ are difficult due to a large separation of scales; however, they appear to agree at least on a lower bound $N_f^{\chi} > 1$ [32, 34]. The actual value of $N_f^{\chi}$ in QED$_3$ is in fact of profound interest for the effective cuprate models, in which the number of four-component Dirac flavors is $N_f = 2$: If $N_f^{\chi} < 2$, then the effective theory predicts a direct transition from the $d$-wave superconducting into the antiferromagnetic state at $T = 0$ as a function of the doping [12, 13]. Otherwise, a small $N_f^{\chi} < 2$ would leave the possibility of an unconventional non-Fermi-liquid phase in the $T = 0$ underdoped cuprates [10, 11, 14].
In the present work, we take a fresh look at the phase structure of QED$_3$ as a function of the fermion number. We pay particular attention to all interaction channels allowed by the large U(2$N_f$) flavor symmetry for Dirac fermions in the reducible representation. Using the functional renormalization group (RG), we find evidence for a more involved structure of the phase diagram. Within our approach, we can straightforwardly identify the “conformal-critical” flavor number $N_{f,cr}^{RE}$ above which the theory is in the quasi-conformal phase. A priori, $N_{f,cr}^{RE}$ can be different from the “chiral-critical” flavor number $N_{f,cr}^{Q}$ below which the theory is in the chirally-broken phase. Our results suggest that $N_{f,cr}^{RE} < N_{f,cr}^{Q}$. This includes the interesting possibility of a third intermediate phase with $N_f$ fermion flavors such that $N_{f,cr}^{RE} < N_f < N_{f,cr}^{Q}$. Our findings suggest that this phase is dominated by vector-channel fluctuations. If they become critical, the model features a Lorentz-breaking vector condensate and a correspondingly mixed spectrum of photonlike massless Goldstone bosons and massive excitations.

The present work is organized as follows: In Sec. II, we discuss the symmetries and fermionic interaction channels of QED$_3$. Corresponding symmetry-breaking patterns are briefly outlined in Sec. III. In Sec. IV, we introduce and apply the functional RG as our central technical tool in order to derive the RG flow equations for the interactions and wave-function renormalizations. Section V is devoted to a fixed-point analysis as a means to identify possible phase structures. An estimate of the conformal-critical flavor number $N_{f,cr}^{RE}$ marking the transition to the disordered quasi-conformal phase is performed in Sec. VI. After illustrating the importance of Fierz completeness of the fermionic interaction channels in Sec. VII, we summarize our findings in the form of a conjectured phase diagram in Sec. VIII and conclude in Sec. IX. Some technical details are summed up in the Appendices.

II. SYMMETRIES AND FERMIONIC INTERACTION CHANNELS

Let us first recapitulate the flavor symmetries of QED$_3$ with many flavors, paying attention to the diversity of interaction channels, see [43, 44] for an extended discussion.

The microscopic (classical) action of QED$_3$ with $N_f$ fermion flavors in $d = 3$ Euclidean space-time is given by

$$S = \int d^3x \left\{ \bar{\psi}^a i \gamma \partial x \psi^a + \bar{\psi}^a i \gamma \cal{A} \psi^a + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right\}, \quad (1)$$

where $\cal{A}$ denotes the bare dimensionful gauge coupling and summation over flavor indices $a$ is tacitly assumed. The fermions $\psi, \bar{\psi}$ are considered to be four-component Dirac spinors, naturally occurring, e.g., in effective theories for electrons on a honeycomb lattice [45–51] or in cuprates [10–16]. They transform under a reducible representation of the Dirac algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu \nu}$, in terms of $4 \times 4$ Dirac matrices

$$\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu \\ i\sigma_\mu & 0 \end{pmatrix}, \quad \mu = 1, 2, 3, \quad (2)$$

where $\{\sigma_\mu\}_{\mu=1,2,3}$ denote the standard Pauli matrices. The Clifford algebra can be spanned with the aid of two further $4 \times 4$ matrices

$$\gamma_4 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad (3)$$

which anticommute with each other as well as with all $\gamma_\mu$. A complete Clifford basis is given by

$$\{\gamma_A\}_{A=1,...,16} = \{I_4, \gamma_\mu, \gamma_\mu \gamma_3, i\gamma_\mu \gamma_4, i\gamma_\mu \gamma_5, \gamma_4, \gamma_5\}, \quad (4)$$

where $\gamma_{45} = i\gamma_4 \gamma_5$ and $\gamma_{24} = \frac{1}{2} \{\gamma_\mu, \gamma_\nu\}$ (in Eq. (4), only those $\gamma_\mu$, with $\mu < \nu$ are counted as independent).

The obvious U($N_f$) flavor symmetry of Eq. (1) together with rotations in the space of irreducible subcomponents of the Dirac spinors leads to an enhanced U(2$N_f$) flavor (or “chiral”) symmetry of QED$_3$, see App. A for details.

From a renormalization group perspective, it is convenient to view the approach from the microscopic theory towards possible symmetry-broken regimes as a two-stage process: first, fluctuations involving gauge-fermion interactions induce effective fermionic self-interactions. Second, further fluctuations may lead to a rapid growth of the fermionic interactions driving the system to criticality and giving rise to possible condensation phenomena.

In the present work, we study the fermionic self-interactions in the pointlike (i.e., the zero-momentum) limit. To this end, we first classify all possible fermionic self-interactions in the pointlike (i.e., the zero-momentum) limit, these four-fermion interactions are compatible with the U($2N_f$) flavor symmetry as well as with the discrete $C$, $P$, and $T$ symmetries of the model. Following [29, 30, 43, 44, 47], these interactions are given by the flavor-singlet channels

$$(V)^2 = \left(\tilde{\psi}^a \gamma_\mu \psi^a\right)^2, \quad (P)^2 = \left(\tilde{\psi}^a \gamma_{45} \psi^a\right)^2, \quad (5)$$

and the flavor-nonsinglet channels

$$(S)^2 = \left(\tilde{\psi}^a \psi^b\right)^2 - \left(\tilde{\psi}^a \gamma_4 \psi^b\right)^2 - \left(\tilde{\psi}^a \gamma_{54} \psi^b\right)^2 + \left(\tilde{\psi}^a \gamma_{45} \psi^b\right)^2, \quad (6)$$

$$(A)^2 = \left(\tilde{\psi}^a \gamma_\mu \psi^b\right)^2 + \frac{1}{2} \left(\tilde{\psi}^a \gamma_{45} \psi^b\right)^2 - \left(\tilde{\psi}^a i \gamma_\mu \gamma_4 \psi^b\right)^2 - \left(\tilde{\psi}^a i \gamma_\mu \gamma_5 \psi^b\right)^2. \quad (7)$$

Here, we have used the convention $(\tilde{\psi}^a \psi^b)^2 \equiv \tilde{\psi}^a \psi^b \tilde{\psi}^b \psi^a$, etc. The corresponding 4-point correlation functions of these fermion interactions can develop largely independent structures in momentum space. By contrast, in the zero-momentum (pointlike) limit, these four-fermion interactions are connected due to Fierz identities,

$$(V)^2 + (S)^2 + (P)^2 = 0, \quad -4(V)^2 - 3(S)^2 + (A)^2 = 0. \quad (8)$$

In this limit, only two four-fermion terms are linearly independent. We choose to work with the flavor singlets
In addition to the vector (Thirring) channel, the bare couplings $\bar{g}, \bar{g}_f$ are set to zero at the initial scale. However, they can be generated dynamically during the RG flow. In any case, the first term $\sim \bar{g}$ corresponds to the interaction known from the Thirring model, whereas the second one $\sim \bar{g}_f$ is similar to a Gross-Neveu interaction.

For $N_f > 1$, another Fierz basis may be of interest from a physical point of view:

$$\mathcal{L}_{\phi, \text{int}} = -\frac{\bar{g}_V}{2N_f} (V)^2 + \frac{\bar{g}_\phi}{4N_f} (S)^2,$$

where the couplings are related to those of Eq. (9) by

$$\bar{g}_V = \bar{g} - \bar{g}_f,$$

$$\bar{g}_\phi = -2\bar{g}_f.$$ (11)

In addition to the vector (Thirring) channel $\sim (V)^2$, we encounter the nonsinglet channel $\sim (S)^2$ of Eq. (6) reminiscent to the Nambu–Jona-Lasinio (NJL) model. We emphasize that the description of the system in terms of Eq. (9) is completely equivalent to that of Eq. (10) in the pointlike limit. The same is true for any other combination of two linearly independent (“Fierz-complete”) interactions out of the four channels $(V)^2$, $(P)^2$, $(S)^2$, or $(A)^2$.

We conclude this section by critically assessing the pointlike limit: from a more general viewpoint, pointlike interactions are only a special limit of fermionic correlation functions $\Gamma^{(n)}$, i.e.,

$$g_O(\bar{\psi}O\psi)^2 = \lim_{p_i \to 0} \bar{\psi}^{(p_1)}(p_1)\bar{\psi}^{(p_2)}(p_2)\Gamma^{(4,abcd)\text{abcd}}_O(p_1, p_2, p_3, p_4)\psi^{(p_3)}(p_3)\psi^{(p_4)}(p_4).$$ (12)

A priori, the pointlike limit hence ignores a substantial amount of momentum-dependent information. Most importantly, since bound-state formation is encoded in the momentum structure of correlation functions (e.g., as $s$-channel poles in Minkowski space), we cannot expect to obtain reliable information about the mass spectrum of the system. Moreover, the formation of a condensate goes along with a singularity in the fermionic four-point function, such that the fermionic pointlike description cannot access the symmetry-broken regime.

In turn, this implies that the pointlike limit can only be used to study the system within the symmetric regime. In fact, it is adequate to address the large-$N_c$ limit which is expected to lie in the symmetric phase. By lowering the flavor number $N_f$, we can therefore study the approach to the symmetry-broken phase of the theory, as symmetry-breaking inevitably goes along with a break-down of the pointlike description. In this manner, we can determine a conformal-critical flavor number $N_{f,cr}^{\text{sc}}$, below which the pointlike description breaks down, possibly indicating condensate and bound-state formation. In the case that the approach to $N_{f,cr}^{\text{sc}}$ from above exhibits a clear signature for condensation in a particular channel, the conformal-critical flavor number can agree with a specific critical flavor number $N_{f,cr}$ below which the system is in a particular symmetry-broken phase. This reasoning has been used in [52–54] to determine the many-flavor phase diagram of QCD.

However, because of the diversity of possible symmetry-breaking patterns as discussed below, the meaning of $N_{f,cr}^{\text{sc}}$ in QED$_3$ is less obvious. In fact, our results indicate that there may exist more than one critical flavor number corresponding to different symmetry-broken phases. The conformal-critical flavor number $N_{f,cr}^{\text{sc}}$, which we aim to estimate in the present work, provides an upper bound on the potentially existing critical flavor numbers for all kinds of broken phases.

### III. SYMMETRY BREAKING PATTERNS

Let us discuss the various symmetry-breaking patterns that can arise if the fermion self-interactions become critical. Symmetry breaking can give rise to two fundamentally different fermion mass terms: $\psi \bar{\psi} \phi$ and $\psi i \gamma_5 \bar{\psi} \gamma_5 \bar{\psi}$. Further fermion bilinears involving $\gamma_4$ and $\gamma_5$ are $U(2N_f)$ equivalent to these mass terms.

The relation between fermion mass generation and symmetry breaking becomes transparent by means of a Hubbard-Stratonovich transformation [55, 56]. This partial bosonization allows us to treat composites of two fermions in terms of effective bosons, schematically, $\phi \sim \bar{\psi} \psi$. More formally, such a transformation allows us to trade in the four-fermion interaction term for a corresponding term bilinear in bosonic fields and a Yukawa-type interaction term on the level of the path integral:

$$g_O(\bar{\psi}O\psi)^2 \rightarrow g_O^{-1}(\theta_O^2 + \bar{\psi}h_O\phi\psi),$$ (13)

where the Yukawa-type coupling $h_O$ can possibly be flavor- or Dirac-matrix-valued. The quantum numbers and transformation properties of the new bosonic field $\phi = g_O \psi$ depend on the exact definition of the four-fermion interaction associated with the operator $O$. The Yukawa coupling is normalized such that the four-fermion coupling is reproduced upon integrating out the bosonic field.

From Eq. (13), we deduce that the four-fermion couplings are inversely proportional to the mass term $\sim \phi^2$ of the bosonic field. Upon fluctuations, we expect that a full Ginzburg-Landau-type effective potential is generated for the bosonic field. Therefore, a singularity of the pointlike fermionic coupling goes along with the effective potential developing a nontrivial minimum. If so, the expectation value of $\phi$ serves as an order parameter for symmetry breaking. Vice versa, if

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1. If expressed in terms of two-component Weyl spinors, this interaction is indeed identical to the Gross-Neveu interaction, cf. App. A.
2. The functional renormalization group approach used below actually reinserts part of the momentum-dependent information in an effective manner.
we observe a divergence of the fermionic self-interactions at a finite RG scale $k_{SB}$ in the purely fermionic language, this serves as an indication for the possible onset of spontaneous symmetry breaking.

Whereas Fierz completeness can be fully preserved by choosing a suitable basis in the purely fermionic language, simple approximations on the partially bosonized side can actually violate this property. For instance, in mean-field approximations this is known as the “Fierz ambiguity” or “mean-field symmetry” [44].

The present work, anyway study the system by approaching the phase boundary from the symmetric phase, hence the quantitative details of bosonization are not important for our purpose. In order to get a first picture of possible symmetry-breaking patterns, let us take a closer look at the partially bosonized version of Eq. (10) that uses the $(V)^2$ and $(S)^2$ channels, which are considered to be the relevant channels also in the Thirring model [60]. Using the irreducible representation in terms of two-component fermions $\chi$, see App. A, we get for the vector channel

$$-rac{\bar{g}_V}{2N_f}(V)^2 \rightarrow \frac{1}{2}\bar{m}_V^2 V_{\mu} V_{\mu} - \bar{h}_V V_{\mu} \chi^i \sigma_{\mu} \chi^j, \quad i = 1, \ldots, 2N_f,$$

where $V_{\mu}$ denotes a real vector boson, and the $(S)^2$ channel yields

$$\frac{\bar{g}_\phi}{2N_f}(S)^2 \rightarrow \frac{1}{2}\bar{m}_\phi^2 \phi^{ij} \phi^{ji} + \bar{h}_\chi \chi^i \sigma_{ij} \chi^j,$$

where $\phi^i = \phi$ denotes a scalar field represented by a hermitean $2N_f \times 2N_f$ matrix. The equivalence with the fermionic action holds also on the path integral level, if the bare couplings satisfy the constraint

$$\frac{\bar{h}_\phi}{2m_\phi^2} = \frac{\bar{g}_\phi}{2N_f}, \quad \frac{\bar{h}_\chi}{2\bar{m}_\chi} = \frac{\bar{g}_V}{2N_f}.$$

Whereas the vector field $V_{\mu}$ is invariant under $U(2N_f)$ transformations, the scalar field transforms according to the bifundamental representation. Different symmetry-breaking patterns arise depending on which bosonic field component eventually develops a finite vacuum expectation value. For instance, if $\phi^{ij}$ acquires an expectation value $\sim \delta^{ij}$, a fermion mass term $\sim i\bar{m}_\psi \gamma_{ij} \psi \chi$ is generated. As is obvious from the form of the expectation value, this mass term does not break the $U(2N_f)$ symmetry. It breaks parity and time-reversal symmetry [44]. By contrast, an expectation value of the form

$$\phi^{ij} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

gives rise to a mass term $im(\bar{\chi}^a \chi^a - \chi^{a+N_f} \chi^{a+N_f}) = im\bar{\psi}^a \psi^a$ which corresponds to a symmetry-breaking pattern of the form

$$U(2N_f) \rightarrow U(N_f) \otimes U(N_f).$$

This is the pattern expected to occur for small flavor numbers in QEDs. For $N_f > 2$, more breaking patterns arising from the scalar sector are in principle conceivable, but have not been considered in the literature so far and will also be ignored in this work.

Another option is that the vector field $V_{\mu}$ develops an expectation value. This would leave the $U(2N_f)$ flavor symmetry intact, but would break Lorentz invariance. Breaking patterns of this type have already been considered during the heyday of the NJL model and the development of the Higgs mechanism [61–63]. For instance, if the expectation value of $V_{\mu}$ was time-like, the corresponding Goldstone bosons may resemble in some aspects a photon field in temporal gauge. In the present case of QED$_3$, these Goldstone bosons could mix with the photon. In addition, a massive bosonic excitation and Lorentz violating features in correlation functions could be expected to occur. However, the number of non-perturbative studies of this symmetry breaking scenario and the nature of the transition is limited, see, e.g., [64, 65].

IV. Renormalization Group Flow of QED

The preceding sections already anticipated an RG viewpoint on the model. In fact, our quantitative analysis will be based on the functional RG formulated in terms of the Wetterich equation [66] which is a flow equation for the coarse-grained quantum effective action $\Gamma_k$:

$$\partial_t \Gamma_k = \frac{1}{2} \text{Str} \left[ (\partial_t R_k) \cdot [\Gamma^{(2)}_k - R_k]^{-1} \right].$$

Here, $\Gamma^{(2)}_k$ is the second functional derivative of $\Gamma_k$ with respect to the fields, $t = \ln(k/\Lambda)$, and $k$ is a flowing IR cutoff scale which is used to set up the RG flow of the quantum effective action. The regularization is implemented with the aid of the regulator function $R_k$ specifying the details of the Wilsonian momentum shell integrations. In the long-range limit, $k \rightarrow 0$, $R_k$ also vanishes such that all quantum fluctuations have been integrated out. The initial condition of the RG flow is determined by the classical action $S$ in the limit $k \rightarrow \Lambda$: $\Gamma_{k \rightarrow \Lambda} = S$. In an exact solution to Eq. (19), the results for physical observables to be read off for $k \rightarrow 0$ should not depend on our specific choice for the regularization scheme, i.e., the function $R_k$ in our case. In this work, we exploit a variation of the scheme to test the predictive power of our approximations, see Sec. VI.

Solving the Wetterich equation yields an RG trajectory in theory space, i.e., the space of all action functionals parametrized for instance by all possible field operators compatible with the symmetries of the theory. In the present work, we confine ourselves to an investigation of the RG flow within
a hypersurface of theory space, parametrized by the ansatz
\[ \Gamma_{\vec{\psi}, \psi, A} = \int d^4x \left[ \hat{\rho} \left( i \hat{Z}_\phi \hat{\psi} + \hat{Z}_{\phi A} \hat{\psi} A \right) \right] \psi 
+ \frac{1}{2} A_\mu Z_A \left( -\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu \right) A_\nu 
+ \frac{1}{2k} A_\mu Z_{\phi A} \partial_\mu \partial_\nu A_\nu 
+ \frac{\bar{g}}{2N_f} (\bar{\psi} \gamma_{\mu \nu} \psi)^2 + \frac{\bar{g}}{2N_f} (\bar{\psi} \gamma_{\mu \nu} \gamma_5 \psi)^2, \] (20)

where the couplings \( \bar{g}, \bar{g} \), the wave-function renormalizations \( Z_\phi, Z_A \), and the vertex renormalization \( Z_{\phi A} \) governing the renormalization of \( \bar{\psi} \) are assumed to be functions of the RG scale \( k \). As discussed above, we consider the four-fermion couplings \( \bar{g} \) and \( \bar{g} \) in the pointlike limit. In addition, also the coupling \( Z_{\phi A} \), parametrizing the photon-electron vertex, and the fermionic wave-function renormalization \( Z_\phi \) will be considered in the zero-momentum limit. In fact, as the flow equation is local in momentum space, receiving its dominant contributions from momenta \( p \approx k \) for a given scale \( k \), the dependence of all these couplings can be viewed as an effective momentum dependence of the corresponding vertices and propagators, see also our discussion below.

Within the functional RG approach, the restriction to the pointlike limit is therefore less severe as it may seem: only highly asymmetric momentum dependencies of the vertices are neglected, whereas an overall momentum dependence is effectively parametrized by the \( k \) dependence of the couplings.

The situation is slightly but decisively different for the photon wave-function renormalization, which we a priori consider to be a function of momentum \( Z_A = Z_A(p^2) \). While all qualitative features could still be extracted from the zero-momentum limit, the quantitative description of QED requires rather strongly on the precise form of the momentum dependence of the photon propagator. The reason for this is the qualitative change of the momentum dependence of the polarization tensor \( \Pi_{\mu \nu} \),

\[ \Pi_{\mu \nu}(p) = (p^2 \delta_{\mu \nu} - p^2 \epsilon^2) \Pi(p), \] (21)

across the scale set by the dimensionful QED coupling \( \epsilon^2 \) in three dimensions.\(^3\) For instance, in the large-\( N_f \) limit, the dressing function of the polarization tensor is known to behave as \( 76\)

\[ \Pi(p) \sim \frac{1}{p}, \] (22)

which can have a rather strong effect on the photon wave-function renormalization \( Z_A \),

\[ Z_A(p) = 1 + \Pi(p). \] (23)

We need \( Z_A(p^2) \) mainly in order to extract the running of the gauge coupling. Since the momentum dependence of \( Z_A(p^2) \) is expected to be sensitive to the value of the gauge coupling, it appears quantitatively mandatory to resolve the momentum dependence of \( Z_A(p^2) \) in QED as accurately as possible.

In addition to the kinetic term of the photon, the gauge sector also comes with a gauge fixing term with gauge parameter \( \xi \) and a corresponding wave-function renormalization \( Z_\xi \). In the present work, we work in the Landau gauge \( \xi \to 0 \) which is known to be a fixed point of the RG flow \([77–80]\). This suggest to choose \( Z_\xi = Z_A \) for simplicity.

With these prerequisites, it is in principle straightforward to derive the flow of general action functionals spanned by the ansatz \( 20 \). In order to make proper contact with QED, we have to provide initial conditions for the flow parameters in Eq. \( 20 \). With regard to the classical action Eq. \( 1 \), these initial conditions are given at the microscopic UV scale \( \Lambda \) by

\[ Z_A |_{\Lambda \to \infty} \to 1, \quad Z_{\phi A} |_{\Lambda \to \infty} \to 1, \quad \bar{g} |_{\Lambda \to \infty} \to 0, \quad g |_{\Lambda \to \infty} \to 0. \]

Note that in particular the four-fermion self-interactions are not considered to be independent parameters. If they appear in the RG flow, they are solely generated by quantum fluctuations.

The RG flows for the couplings can conveniently be formulated for the dimensionless renormalized couplings. For the fermionic interactions, these are given by

\[ \bar{g} = Z_{\bar{\psi} \bar{\psi}} k \bar{g} \quad \text{and} \quad g = Z_{\bar{\psi} \psi} k g. \] (25)

The running of the fermionic wave-function renormalization in turn can be parametrized in terms of the fermionic anomalous dimension

\[ \eta_{\bar{\psi} \bar{\psi}} = -\partial_\xi \ln Z_{\bar{\psi} \bar{\psi}}. \] (26)

The calculation of the corresponding fermionic flows is straightforward with standard techniques, see Ref. \([81]\), and the results will be summarized below.

The RG flow of the gauge sector requires a more careful discussion. The corresponding definition of the dimensionless gauge coupling is

\[ \bar{e}^2 = \bar{Z}_{\bar{\psi} \bar{\psi}} e^2 Z_{\bar{\psi} A \bar{\psi}} k, \] (27)

In ordinary perturbation theory, the Ward identity for the photon-electron vertex enforces \( Z_{\phi A} = Z_\phi \) to hold at each order in a coupling expansion, see, e.g. \([82]\). In the Wetlich formulation of the functional RG, the regulator, being introduced as a momentum-dependent mass term, also contributes to the breaking of the gauge symmetry similar to the gauge-fixing procedure. This also affects the Ward identities which are accordingly modified by regulator-dependent terms \([71, 72, 75, 77, 79, 83–86]\). For our case, these terms can be worked out explicitly along the lines of \([74, 87]\), yielding the modified relation

\[ Z_{\phi A} = Z_{\psi} \left( 1 - C_\xi \bar{g} - C_{\bar{\xi}} \bar{g} \right), \] (28)
where $C_{g}$ and $C_{\bar{g}}$ are constants depending on the number of fermion flavors as well as the regularization scheme.

At this point, let us schematically define the photon anomalous dimension analogously to Eq. (26) as $\eta_{A} = -\partial_{t} \ln Z_{A}$ (a more precise definition also accounting for the momentum dependence of $Z_{A}$ will be given below). Then, the flow equation for the gauge coupling (27) reads

$$\partial_{t} e^{2} = (\eta_{A} - 1) e^{2} - \frac{C_{g}(\partial_{t} g) + C_{\bar{g}}(\partial_{t} \bar{g})}{1-C_{g} g - C_{\bar{g}} \bar{g}}. \quad (29)$$

In addition to the first term expected from perturbation theory, we encounter additional terms proportional to the flows of the fermion couplings which diagrammatically correspond to a resummation of a large class of diagrams. Below, we will investigate the approach to possible phase transitions as a function of $N_{f}$ by means of a fixed-point analysis. As fixed points are defined as points in theory space where the RG flow vanishes, i.e., $\partial_{t} g = \partial_{t} \bar{g} = 0$, the additional terms in Eq. (29) vanish identically at the fermionic fixed points and thus are irrelevant for the determination of the fixed point of the full system. For our fixed-point analysis presented below, these additional terms can therefore be ignored.

Finally, we have to give a precise definition of the photon anomalous dimension in order to complete our set of flow equations for our truncation. The evaluation of the photon polarization tensor, corresponding to the diagram in Fig. 1, yields a fully momentum dependent wave-function renormalization $Z_{A}(p^{2})$. Since the integrand of the momentum trace in the flow equation by construction is peaked for loop-momenta $q$ near the regulator scale, $q^{2} \simeq k^{2}$, it is crucial to obtain a reliable estimate of the gauge coupling that parametrizes the photon-fermion interaction strength of the modes interacting at momentum transfer of the order of the scale $k$. As the running of the gauge coupling is dominated by the photon anomalous dimension (at least near fermionic fixed points), we define $\eta_{A}$ with the aid of the scale derivative of $Z_{A}(p^{2})$ at a momentum scale $p^{2}$ evaluated near $k^{2}$. To be more specific, we define

$$\eta_{A} = -\partial_{t} \ln Z_{A}(p^{2} = \xi^{2} k^{2}) \quad (30)$$

where $\xi$ serves as a control parameter that can be used to estimate the dependence of our final results on the details of the definition of $\eta_{A}$ and thus on the definition of the gauge coupling. The parameter $\xi$ fixes the momentum scale $p$ serving as the (re-)normalization point of the photon field amplitude relative to the Wilsonian momentum shell $k$. Large values of $\xi \gg 1$ therefore appear to be artificial, since the physically relevant momenta would then lie far beyond the Wilsonian momentum shell. As a consequence, we expect $\eta_{A}$ to be a decreasing function of $\xi$ for large $\xi$ for purely kinematical reasons. The natural range of physically relevant $\xi$ values hence is $0 \leq \xi \leq 1$, with $\xi \to 0$ corresponding to the pointlike limit. For a more adapted resolution of nontrivial momentum-dependencies of $Z_{A}(p^{2})$, the choice $\xi = 1$ appears a priori preferable.

In the determination of $Z_{A}(p^{2})$ via the polarization tensor, another subtlety is hidden: the standard Ward identity for the polarization tensor $p_{\mu} \Pi(p)_{\nu\mu} = 0$ is also affected by the presence of the regulator, yielding a nonzero regulator-dependent term on the right-hand side that vanishes in the limit $k \to 0$. This is a known peculiarity of the present Wilsonian-type of RG flow, see, e.g., Refs. [71, 72, 73, 85, 86, 88–90] for a more detailed discussion of this issue. In order to avoid a contamination of our gauge coupling definition with these artificial regulator-dependent terms, we subtract the $p \to 0$ limit of $\Pi_{\mu\nu}$ for finite $k$ in the determination of $Z_{A}(p^{2})$. This guarantees that the information entering the anomalous dimension $\eta_{A}$ is not contaminated by contributions that arise in the RG flow only in order to satisfy the regulator-dependent constraint on the (unphysical) longitudinal modes. The technical details of the construction of $\eta_{A}$ are summarized in Appendix C. In any case, the result for $\eta_{A}$ has a comparatively simple form,

$$\eta_{A} = 8v_{3}N_{f}e^{2}L_{1}^{(F)}(\eta_{A}; \xi), \quad (31)$$

where $v_{3} = 1/(8\pi)^{2}$, and $L_{1}^{(F)}$ denotes a threshold function that corresponds to the regularized one-particle irreducible (1PI) Feynman diagram shown in Fig. 1. It depends on the choice of the regulator, thus encoding the RG-scheme dependence, and also on the control parameter $\xi$ introduced above. The dependence on the fermion anomalous dimension $\eta_{\psi}$ signals the “RG-improvement” inherent in the functional RG. The explicit integral representation of $L_{1}^{(F)}(\eta_{A}; \xi)$ can be found in Eq. (C6).

We conclude this section by listing the fermion anomalous dimension,

$$\eta_{\psi} = -\frac{16}{3}v_{3}e^{2} \left( n_{2,1}^{(F,B)}(\eta_{A}, \eta_{\psi}) - \tilde{m}_{1,1}^{(F,B)}(\eta_{A}, \eta_{\psi}) \right), \quad (32)$$

with the regulator-dependent threshold functions $n_{2,1}^{(F,B)}$ and $\tilde{m}_{1,1}^{(F,B)}$, as defined, e.g., in Refs. [81, 91, 92]. As the threshold functions are linear in the anomalous dimensions, Eqs. (31) and (32) can unambiguously be solved for $\eta_{\psi}$ and $\eta_{A}$ as functions of the gauge coupling.
The RG $\beta$ functions for the fermion sector read
\[ \partial_t \tilde{g} = (1 + 2 \eta_g) \tilde{g} - 8 \nu_3 \left( \frac{2N_f - 1}{N_f} \tilde{g}^2 - \frac{2}{N_f} \tilde{g} \right)^{(F)}_{1,1} \]
\[ - 8 \nu_3 \left( 2 \tilde{g} e^2 + 4 g e^2 \right)^{(F,B)}_{1,1} + 16 \nu_3 N_I e^4^{(F,B)}_{2,1}, \]  
\[ \partial_t g = g(1 + 2 \eta_g) + 8 \nu_3 \left( \frac{1}{N_f} \tilde{g} g + \frac{2N_f + 1}{3N_f} g^{-1} \right)^{(F)}_{1,1}, \]
\[ - \frac{8}{3} \nu_3 \left( 4 \tilde{g} e^2 - 2 g e^2 \right)^{(F,B)}_{1,1}, \]  
where the threshold functions $l$ again carry the regulator dependence and depend linearly on $\eta_g$ via $l_I^{(F)}$. For the evaluation of the photon exchange diagrams, we neglect the full momentum dependence of the photon propagator, but take the photon field renormalization at the renormalization point $Z_A(p^2) = Z_A(c^2 k^2)$ into account. Hence, the threshold functions $l_{1,1}^{(F,B)}$ and $l_{2,1}^{(F,B)}$ depend also on $\eta_A$. For the so-called sharp-cutoff, Eqs. (33)–(34) are equivalent to the results reported in Ref. [30]. In the limit of large flavor number $N_f$, they also reduce to the large-$N_f$ flow equations found previously within the conventional Wilsonian RG approach [29]. We would like to add that the sharp-cutoff regulator has to be handled with some care. Whereas this type of regulator can be used to compute the flow equations for the pointlike four-fermion couplings without any difficulty, the computation of the flow equations for the wave-function renormalizations suffers from ambiguities which can be traced back to the fact that there is no unique definition for this regulator, see Appendix B. Since the photon wave-function renormalization plays a prominent role in our study of the many-flavor phase structure, we refrain from using this regulator in the following. Instead, we only consider a smeared-out version of this regulator which is free of these difficulties.\(^4\) For the latter we have found that it yields results for the phase structure that are in accordance with those reported in Sect. VI below.

For vanishing gauge coupling $e^2 = 0$, we observe that the fermionic $\beta$ functions (33) and (34) vanish identically if $g, \tilde{g}$ are zero at a particular scale (as, e.g., required by the initial conditions (24)). This obvious fixed point of the flow corresponds to the non-interacting Gaußian fixed point of the theory. For $e^2 \neq 0$, the point of vanishing fermionic couplings is no longer a fixed point due to the last term $- e^4$ in Eq. (33).

Finally, the flow of the gauge coupling is given by Eq. (29) upon insertion of the anomalous dimension $\eta_A$ and the fermionic flows. Near fixed points of the fermionic flow, where $\partial_t g, \partial_t \tilde{g} \approx 0$, the $\beta$ function of the gauge coupling simplifies to
\[ \beta_{e^2} \equiv \partial_t e^2 = (\eta_A - 1) e^2. \]  
(35)

For the fixed-point analysis carried out in the present work, we consider this simplified flow.

We close this section with a few comments on the reliability of the approximations involved in our truncation. In our numerical studies, we indeed find that $|\eta_g| \leq 1$ in the symmetric large-$N_f$ regime where the RG flow is governed by the presence of a fixed point, see also our discussion in the subsequent section. This is a strong support for our implicit assertion that momentum dependencies in the fermion sector are less important, such that higher derivative terms of fermionic operators can safely be dropped in this regime. Moreover, it is worthwhile to point out that in the pointlike limit the RG flow of a Fierz-complete set of four-fermion couplings is completely decoupled from the RG flow of fermionic $n$-point functions of higher order. In particular, 8-fermion interactions do not contribute to the flow of the four-fermion interactions in this limit. This observation corroborates the truncation on the four-fermion level. Further tests of the truncation – particularly of the gauge sector – will actively be pursued in the following sections by studying the amount of artificial regularization-scheme dependence of observables.

V. FIXED-POINT ANALYSIS

The RG fixed-point structure of a theory is intimately related to the phase diagram. Fixed points are defined as common zeros of all $\beta$ functions, in our case by the requirement
\[ \partial_t e^2 |_{\tilde{g}, g, \tilde{g}, g} = \partial_t g |_{\tilde{g}, g, \tilde{g}, g} = \partial_t \tilde{g} |_{\tilde{g}, g, \tilde{g}, g} = 0, \]  
(36)
where $e^2, g, \tilde{g}$ denote the values of the dimensionless couplings at the fixed point. Whereas the fixed-point values themselves are non-universal, i.e., depend on the choice of the regularization scheme, the critical exponents as well as the anomalous dimensions $\eta_{g, \tilde{g}}$ and $\eta_A$ at a fixed point are universal. Summarizing all couplings in $G = (e^2, g, \tilde{g})$, the critical exponents $\theta_I$ are defined in terms of (minus) the eigenvalues of the stability matrix $B^I_I$,
\[ \partial_t G_I = \beta_I(G), \quad B^I_I = \frac{\partial \beta_I}{\partial G_I} \bigg|_{G=G^I}, \]  
(37)
with $-\theta_I$ labeling the eigenvalues of $B^I_I$, and $I$ running from 1 to the number of couplings considered ($I = 1, 2, 3$ in our case). For instance, at the Gaußian fixed point, $G = 0$, we have $\theta_I = \{+1, -1, -1\}$, with the positive exponent $+1$ related to the RG relevant gauge coupling. The negative exponents $-1$ correspond to the RG irrelevant fermionic couplings in QED. At the Gaußian fixed point, the critical exponents simply correspond to the power-counting dimension of the couplings.

In order to illustrate the fixed-point structure of the theory, let us start with the flow of the gauge coupling. Assuming that the fixed-point conditions for the fermion couplings are satisfied, we can use Eq. (35). In addition to the Gaußian fixed point, a non-Gaußian, i.e., interacting, fixed-point exists for
\[ \eta_A = 1, \quad e^2 = \frac{1}{8 \nu_3 N_I L^{(F)}_{1,1}(\eta_{g, \tilde{g}, g})}, \]  
(38)
where the threshold function $L^{(F)}_{1,1}$ with $\eta_{g, \tilde{g}}$ evaluated at the IR fixed point is a regulator-dependent but real-valued positive

\(^4\) This amounts to using a finite value for the parameter $b$ in our definition of the sharp-cutoff regulator, see Eq. (B6).
number. The crucial observation is that the value of the fixed point scales with the flavor number \( N_f \) as \( e^2 \sim 1/N_f \).

Starting the RG flow near the Gaussian fixed point at \( e^2 \ll 1 \), the \( \beta \) function \( \partial e^2 \) is negative, implying that the coupling is asymptotically free towards the UV and increases towards the IR. Hence, the gauge coupling is expected to approach the non-Gaussian fixed-point in the long-range limit, see Fig. 2. As long as no fermion-mass generating phase transition occurs in which case the dynamics of the theory would be governed by a different sector of the theory, the whole system remains massless and the IR fixed point [Eq. (38)] is reached asymptotically at small momentum scales. In that case, the theory is quasi-conformal, i.e., near-conformal in the UV near the Gaussian fixed point as well as near-conformal in the IR near the non-Gaussian fixed point. The two near-conformal regimes are smoothly connected by a crossover occurring at momentum scales near the scale approximately set by the bare coupling \( e^2 \). Note that the maximum coupling strength of the dimensionless coupling is set by the IR fixed-point value, see Eq. (38). In particular, the maximum coupling strength is smaller for larger flavor numbers.

Let us now turn to the fermionic sector with the corresponding flows given in Eqs. (33) and (34), treating the gauge coupling as an external parameter for the moment. As the fixed-point conditions for \( g \) and \( g \) [Eq. (36)] correspond to two coupled quadratic equations, we generically expect up to four distinct fixed-point solutions. Provided that the gauge coupling is sufficiently small, we find four distinct real solutions which thus represent candidates for physically relevant fixed points.

\[ N_f = 3, \quad N_f = 6, \quad N_f = 9 \]

\[ g^2 = 0, \quad g^2 > 0 \]

For finite \( e^2 > 0 \), these points in coupling space are no longer fixed points of the total system, as their positions change with the gauge coupling \( e^2 \). In a slight abuse of language, we still call them fixed points, as for a given value of \( e^2 \) they govern the flow in the fermionic sector. In the limit \( e^2 \rightarrow 0 \), one of the four fixed points is continuously connected to the (true) Gaussian fixed point at \( G = 0 \). For small but finite \( e^2 \), this fixed point is slightly shifted to nonzero couplings \( g, \), but continues to have two RG irrelevant directions. This fixed point, named \( O \) in Fig. 3, is thus IR attractive in the \( (g, \phi) \) plane. Two further fixed points \( A \) and \( C \) have one IR attractive (RG irrelevant) and one IR repulsive (RG relevant) direction, and the fixed point \( B \) exhibits two IR repulsive directions, see Fig. 3.

For vanishing gauge coupling, \( e^2 = 0 \), the Gaussian fixed point \( O \) describes a free theory of non-interacting fermions. The fixed point \( C \) has been extensively studied in [43, 44, 60]. It can be associated with the asymptotically safe three-dimensional Thirring model. For sufficiently small flavor numbers \( N_f < \Lambda^4_{\text{Thirring}} \), the fixed point controls a second-order quantum phase transition, separating the massless phase from the phase of chiral symmetry breaking, see, e.g., [93] for a study of the \( N_f = 1 \) model. In Refs. [44, 60], the crit-
ical flavor number of the Thirring model has been estimated as $N_{f,\text{Thirring}} \approx 5.1$. Lattice studies of the Thirring model with a different realization of the chiral symmetry using staggered fermions found $N_{f,\text{Thirring}} \approx 6.6$ [94].

The fixed point $\mathcal{A}$ corresponds to a variant of the three-dimensional Gross-Neveu model. Different versions of this model exist in $d = 3$, all of which are asymptotically safe because of such a non-Gaussian fixed point [103–105]. This fixed point governs the second-order quantum phase transition of a discrete $\mathbb{Z}_2$ symmetry (parity symmetry in this case) which is known to occur for any $N_f$. By contrast, the fixed point $\mathcal{B}$ has less well been studied, but could equivalently give rise to an asymptotically safe fermionic model potentially exhibiting first-order phase transitions to various phases in the IR.

Returning now back to QED$_3$, the initial conditions (24) put the system into the vicinity of the Gaussian fixed point $O$ at the microscopic scale $k \to \Lambda$, leaving us with one RG relevant parameter, namely the gauge coupling, as it should be. Towards the UV, the full system is asymptotically free. Towards the IR, the gauge coupling increases, shifting the Gaussian fixed point $O$ slightly in the $(\hat{g}, g)$ plane, see blue/bold arrows in Fig. 3. Since $O$ remains IR attractive in the fermionic directions, the flow of $\hat{g}, g$ follows this IR attractive fixed point.

If the gauge coupling approaches a critical value $e_{cr}^2$, the fixed points $C$ and $O$ annihilate, see Fig. 3. If we increase the gauge coupling even further, then the flow of the four-fermion couplings is no longer bounded by the existence of an IR attractive fixed point. On the contrary, the four-fermion interactions start to grow rapidly and diverge at a finite RG scale $\Lambda_{3\pi}$, potentially indicating dynamical symmetry breaking, as discussed above.

From the fixed-point analysis itself, we do not gain immediate insight into the exact type of spontaneous symmetry breaking, as this is a result of the full RG flow towards the IR. Nevertheless, the fixed-point analysis provides for a criterion for symmetry breaking to be possible at all: as long as the fixed point $O$ exists, being IR attractive for the fermionic couplings, no approach to criticality in the fermion sector can occur. Thus, monitoring the existence of this fixed point as a function of $N_f$ provides first information about the structure of the phase diagram as a function of $N_f$.

VI. CONFORMAL-CRITICAL FLAVOR NUMBER

From the preceding discussion, we expect the system to be quasi-conformal as long as the fixed point $O$ in the fermion sector persists and remains IR attractive in the fermionic couplings. The fixed point $O$ vanishes if the gauge coupling exceeds a critical coupling strength $e_{cr}^2$. In the quasi-conformal phase, the IR fixed point $e_{cr}^2$ as given in Eq. (38) is a measure for the maximum coupling strength. Since $e_{cr}^2$ is small for large $N_f$, the quasi-conformal phase occurs at large $N_f$ extending to $N_f \to \infty$. Lowering $N_f$, the annihilation of the fixed points $O$ and $C$ indicate the boundary of the quasi-conformal phase and a possible onset of a different phase. The corresponding value of $N_f$ defines the conformal-critical flavor number $N_{f,\text{cr}}$ which is defined by the criticality condition

$$e_{cr}^2(N_{f,\text{cr}}) = e_{cr}^2(N_{f,\text{cr}}^*)$$

see also Fig. 4. Whereas both $e_{cr}^2$ and $e_{cr}^2^*$ are non-universal and depend on the choice of the regularization scheme, the conformal-critical flavor number $N_{f,\text{cr}}^*$ is expected to be universal. However, the fact that we consider an approximation of the exact RG flow implies that also the universality of $N_{f,\text{cr}}^*$ holds only approximately.

In Tab. I, we list our results for $N_{f,\text{cr}}^*$ as obtained from our computations with three different regulator functions, see App. B for the definitions of these functions. We also consider two different values of the control parameter $\zeta$ which parametrizes the external photon momentum of the vacuum polarization diagram relative to the cutoff scale, cf. Eq. (30). Whereas the choice $\zeta = 1$ appears more adapted to resolve the momentum dependence of the photon wave function, the choice $\zeta = 0$ conforms with the pointlike approximation in the fermion sector. In either case, we obtain the smallest value $N_{f,\text{cr}}^*$ since $N_{f,\text{cr}}^*$ presumably is not an integer, its value might depend on the manner, how theories with non-integer flavor numbers are constructed. Nevertheless, the result that systems with integer $N_f > N_{f,\text{cr}}^*$ have long-range properties substantially different from those with integer $N_f < N_{f,\text{cr}}^*$ is in principle a universal and observable phenomenon.

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6 In the literature, estimates for the critical flavor number of the Thirring model span a wide range of values [94–102]. Many of the analytical estimates show a strong similarity to the corresponding QED$_3$ results.

7 Since $N_{f,\text{cr}}^*$ presumably is not an integer, its value might depend on the manner, how theories with non-integer flavor numbers are constructed. Nevertheless, the result that systems with integer $N_f > N_{f,\text{cr}}^*$ have long-range properties substantially different from those with integer $N_f < N_{f,\text{cr}}^*$ is in principle a universal and observable phenomenon.
of $N_{f,cr}^{qc}$ for the Callan-Symanzik regulator. Since the latter is equivalent to a mass term $\sim k$ without any momentum dependence, it does not entail a UV suppression and therefore is likely to give rise to stronger truncation artifacts, as is also known from many other RG studies. The two other regulators, the exponential and the linear regulator, cf., App. B for details, provide for both a UV and IR regularization and are thus considered as quantitatively more reliable. These two regulators span the range of estimates for $N_{f,cr}^{qc}$ of $N_{f,cr}^{qc} \approx 8, \ldots , 10$ for $\zeta = 1$ and $N_{f,cr}^{qc} \approx 4, \ldots , 5.7$ for $\zeta = 0$ with the largest $N_{f,cr}^{qc}$ value arising from the linear regulator, respectively. Intermediate values of $\zeta$ yield ranges that interpolate between the $\zeta = 0$ and $\zeta = 1$ case.\footnote{Incidentally, a smeared version of the sharp-cutoff with smearing parameter $b \approx 2$ (see App. B) yields values for $N_{f,cr}^{qc}$ within the ranges spanned by the exponential and the linear regulator.} We observe that the variation with respect to the control parameter $\zeta$ is even larger than the regulator dependence. We interpret this as a signature for the importance of the precise resolution of the momentum dependencies of the correlation functions.

In general, these uncertainties indicate a systematic error to be associated with the employed truncation. For example, the inclusion of the full momentum dependence especially of the photon-propagator and the fermion-photon vertex may be required to determine $N_{f,cr}^{qc}$ more precisely. In order to assess the stability of our results for the conformal-critical flavor number, let us discuss the variations of the regulator and the control parameter in more detail: First, the dependence on the regulator is a natural consequence of truncated flows. This dependence can be lifted by identifying "optimized" regularization schemes satisfying a-priori-criteria that can be argued to be equivalent to the pointlike limit, i.e., in the limit of zero external momentum, the vertex enters the flow equations at an asymmetric point, since the internal lines of the diagram carry an in general finite loop momentum. Therefore, potentially asymmetric structures of the vertices are neglected by our approximation. The intrinsic tension between such structures and our estimate for the running coupling could even be amplified by choosing a nonzero $\zeta$.

With this analysis of the regulator and $\zeta$ dependence, we can now summarize our estimates for the location of the conformal-critical flavor number $N_{f,cr}^{qc}$. From a conservative perspective, we have not been able to find estimates of $N_{f,cr}^{qc}$ with values smaller than $N_{f,cr}^{qc} \approx 3.7$ or larger than $N_{f,cr}^{qc} \approx 10.0$ also including extreme regulator choices such as the Callan-Symanzik regulator. We hence conclude $N_{f,cr}^{qc}$ to lie within this interval. Excluding the Callan-Symanzik regulator in order to avoid regulator artifacts, our results span a smaller region. The regulator and $\zeta$ dependence analysis given above suggest the conformal-critical flavor number of QED$_3$ to lie in the region $N_{f,cr}^{qc} \approx 4.1 \ldots 10.0$.\footnote{Incidentally, a smeared version of the sharp-cutoff with smearing parameter $b \approx 2$ (see App. B) yields values for $N_{f,cr}^{qc}$ within the ranges spanned by the exponential and the linear regulator.}

We emphasize, however, that the upper and lower end of this interval should not be viewed as a strict boundary, but may change upon improvements of the approximation. Despite these uncertainties, this estimate represents one of the main results of our study.

### VII. Fierz Completeness

The above given estimate for the conformal-critical flavor number $N_{f,cr}^{qc}$ – though coming with a large uncertainty – appears to include values significantly larger than many results for the critical flavor number for chiral symmetry breaking reviewed in the introduction. While there are many sources that can take a strong influence on the final result (e.g., large finite volume effects in finite-volume studies [39, 109]), we emphasize in this work two issues that have not yet received sufficient attention.

![Figure 5](image_url) **Figure 5.** 1PI diagram contributing to the RG flow of the four-fermion box diagrams as an estimate for the fermion-photon vertex, see also Fig. 5. This estimate can be afflicted with the following problem: As we evaluate the box diagrams in the pointlike limit, i.e., in the limit of zero external momentum, the vertex enters the flow equations at an asymmetric point, since the internal lines of the diagram carry an in general finite loop momentum. Therefore, potentially asymmetric structures of the vertices are neglected by our approximation. The intrinsic tension between such structures and our estimate for the running coupling could even be amplified by choosing a nonzero $\zeta$.
First, we have determined the conformal-critical flavor number \( N_{f_{\text{cr}}}^{\text{eq}} \) above which the system is quasi conformal. While this value is likely to mark a region in the many-flavor phase diagram where a crossover or a phase transition is expected to occur, it does not necessarily have to agree with the critical flavor number for the chiral phase transition \( N_{f_{\text{cr}}}^{\chi} \). As we can only detect the quasi-conformal regime with our pointlike approximation, we can only conclude so far that \( N_{f_{\text{cr}}}^{\text{eq}} \leq N_{f_{\text{cr}}}^{\chi} \). cf. also next section for a discussion. Hence, there is no immediate disagreement with the literature in this respect.

Second, we have emphasized that our ansatz for the effective action is Fierz complete in the sense that it includes all pointlike four-fermion interactions compatible with the symmetries of the model. The significance of Fierz completeness for an appropriate description of an approach to criticality is already obvious from our parametrization. The chiral-symmetry breaking channel \((S)^2\) in the Fierz-transformed Lagrangian in Eq. (10) which, when becoming dominant, generates a mass term \( \sim im\phi^2\bar{\psi}\psi \), is associated with a superposition of both four-fermion channels \( \bar{g}(P)^2 \) and \( g(V)^2 \) used in this work (see dashed line in Fig. 3). Ignoring one of the channels may lead to strong deviations from the Fierz-complete result.

In order to quantify the importance of Fierz completeness, we study the dependence of our result for the conformal-critical flavor number \( N_{f_{\text{cr}}}^{\text{eq}} \) on a one-parameter family of Fierz-incomplete approximations. To be specific, we first introduce a Fierz-complete reparametrization of the couplings as follows:

\[
\begin{align*}
\tilde{s}_\Phi &= g \sin \varphi + \tilde{g} \cos \varphi, \\
\tilde{\tilde{s}}_\Phi &= g \cos \varphi - \tilde{g} \sin \varphi,
\end{align*}
\]

where the angle \( \varphi \) parametrizes a family of couplings \( s_\Phi, \tilde{\tilde{s}}_\Phi \). From here, we arrive at a Fierz-incomplete set by truncating \( \partial_\varphi \tilde{s}_\Phi = 0 \equiv \tilde{\tilde{s}}_\Phi \). The angle \( \varphi \) can now be used to select a specific interaction channel. For example for \( \varphi = \pi/4 \), we have \( \tilde{g} = g \), such that we are left with the chiral channel only, see also Eq. (11) and the dashed line in Fig. 3.

With the \( \varphi \)-dependent Fierz-incomplete approximation at hand, we can now compute the conformal-critical flavor number again. In Fig. 6, we present our results for \( N_{f_{\text{cr}}}^{\text{eq}} \) as a function of the angle \( \varphi \) for \( \zeta = 0 \) (upper panel) and \( \zeta = 1 \) (lower panel). We observe that the predictions for the conformal-critical flavor number strongly vary within this family of Fierz-incomplete approximations. Moreover, we find that a finite range of values for \( \varphi \) exists for which we have \( N_{f_{\text{cr}}}^{\text{eq}} = 0 \). This was to be expected, since for \( \pi/2 \leq \varphi \leq \pi \) we project onto a channel orthogonal to the chiral channel. There is no annihilation of fixed points in this channel for any \( N_c \), since the fixed points \( \mathcal{A} \) and \( \mathcal{B} \) do not approach the Gaußian fixed point \( O \) for any value of \( \epsilon^2 \), see blue/bold lines in Fig. 3. This may be interpreted as a consequence of the Vafa-Witten argument [110], prohibiting the spontaneous breaking of parity symmetry in QED\(_3\). As another specific example, let us consider a projection onto the chiral channel corresponding to \( \varphi = \pi/4 \). Here we find \( N_{f_{\text{cr}}}^{\text{eq}} \approx 5 \) even for all studied regulator functions and \( \zeta \) values. However, this is still significantly different, for instance, from the Fierz-complete result for \( \zeta = 1 \).

Our analysis clearly demonstrates the necessity of a Fierz-complete treatment as one may significantly overestimate by almost a factor of 2 or underestimate (\( N_{f_{\text{cr}}}^{\text{eq}} = 0 \)) the conformal-critical flavor number within a Fierz-incomplete setup, see Fig. 6. This strong ambiguity of \( N_{f_{\text{cr}}}^{\text{eq}} \) within a Fierz-incomplete study represents the second important result of our work. Moreover, any Fierz-incomplete study that is only sensitive to the chiral channel will inevitably identify \( N_{f_{\text{cr}}}^{\chi} \) with \( N_{f_{\text{cr}}}^{\text{eq}} \). In this case, any information about a possibly existing intermediate phase will not be accessible because of Fierz incompleteness.

While Fierz completeness is simple to implement in the present approximation scheme of the exact RG flow, it is less obvious how this issue might affect other methods. Mean-field methods are certainly strongly affected, as the choice of a mean field immediately breaks Fierz completeness [57].

By contrast, lattice simulations are by construction not affected, as no choice of channels is required. Still, our results on Fierz completeness can also be interpreted as a mandate to implement the flavor symmetries exactly. Hence, lattice formulations should be given preference that feature an exact (lattice version of) the \( U(2N_f) \) flavor symmetry.
The largest body of literature on chiral-symmetry breaking in QED$_3$ relies on solutions of Dyson-Schwinger equations for the photon and fermion propagators amended with suitable vertex constructions. For the solution of the equation for the fermion propagator $S_\phi(p)$, an ansatz of the following form is typically used,

$$S_\phi(p)^{-1} = i p A(p^2) + B(p^2),$$

(43)

where $A(p^2)$ is related to the (inverse) wave-function renormalization, and $B(p^2)$ parametrizes the mass function. In particular, $\lim_{p\to 0} B(p^2) \neq 0$ signals fermion mass generation and chiral symmetry breaking. This ansatz is also commonly and successfully used for investigations of the strong-coupling regime of QCD in $d = 4$. Here, we note that the ansatz (43) does not exhaust all possible terms permitted by the special Dirac structure and flavor symmetry of QED in $d = 3$. As suggested by our results, the inclusion of all terms permitted by the symmetries might be an essential ingredient. On the level of the fermion propagator, a complete ansatz would read

$$S_\phi(p)^{-1} = i p A(p^2) + B(p^2) + \gamma_{45} C(p^2) + i p \gamma_{45} D(p^2),$$

(44)

involving two further scalar functions $C$ and $D$. The case of $\lim_{p\to 0} C(p^2) \neq 0$ would signal the generation of a parity-breaking mass term. However, even in the parity-symmetric phase where $\lim_{p\to 0} C(p^2) = 0$, the two further functions might develop a nontrivial momentum dependence at intermediate scales, potentially taking influence on the $B(p^2)$ function and thus on the onset of chiral symmetry breaking.

Let us finally emphasize that there certainly is no one-to-one correspondence between our results for Fierz-incomplete approximations and flavor-symmetry-incomplete DSE ansätze of the type of Eq. (43). It may well be that Eq. (43) is perfectly sufficient to obtain quantitatively reliable results. Our results, however, suggest that an ansatz of the type (44) exhausting the full symmetry could be worthwhile to be studied.

VIII. PHASE STRUCTURE

As our truncation based on pointlike fermion interaction channels is not capable of entering the symmetry-broken regime, the scenario developed in this section is founded only on limited information which we can extract from the RG flow in the symmetric regime. With these reservations in mind, we recall that we have identified a conformal-critical flavor number $N_{\text{f,crit}}^{\text{q}}$ above which we found QED$_3$ to be in the quasi-conformal phase.

So far, we have carefully distinguished between $N_{\text{f,crit}}^{\text{q}}$ and a possible critical flavor number $N_{\text{f,crit}}^{\text{v}}$, indicating the onset of a chirally broken phase. From our results, we can primarily conclude that $N_{\text{f,crit}}^{\text{q}} \leq N_{\text{f,crit}}^{\text{v}}$. For a first attempt to estimate the possible value of $N_{\text{f,crit}}^{\text{v}}$ within our approach, let us take a look at the RG flow trajectories in the plane of fermionic couplings for various flavor numbers below $N_{\text{f,crit}}^{\text{q}}$. For illustrative purposes, we consider the flows obtained with the linear regulator and a control parameter value $\zeta = 1$, which yielded the estimate $N_{\text{f,crit}}^{\text{q}} \approx 10$. Also, we fix the gauge coupling slightly above the critical value $\frac{g^2}{4\pi} = 0$, where the fixed points $O$ and $C$ annihilate, $0 < (\epsilon_1^2 - \epsilon_2^2) \ll 1$.

The resulting fermionic flows in the $(g, \gamma)$ plane are shown in Fig. 7 for the case of $N_f = 1$ (left panel) and $N_f = 9$ (right panel). As before, the dashed line ($g = \gamma$) corresponds to the chiral channel ($\gamma^2$), potentially associated with chiral symmetry breaking when becoming dominant. The solid red line marks the direction of the asymptote of the RG trajectories for large $\gamma$, $g$. Starting the flow for vanishing fermion interactions $\gamma = g = 0$, in general both $\gamma$ and $g$ are generated and will approach this asymptote in the course of the RG flow.

The slope of the RG asymptote thus determines the relative weight of the different possible channels in the IR. For $N_f = 1$ (left panel of Fig. 7), it is fairly close to the dashed line associated with symmetry breaking in the chiral channel; in fact, for $N_f = 1.75$ (not shown) the RG asymptote would lie exactly on top of the chiral channel. By contrast, the $N_f = 9$ asymptote is closer to the pure vector channel $\sim (\sqrt{V})^2$. The fact that this asymptote rotates with increasing $N_f$ towards the vector channel is already known from studies of the Thirring model [43, 60]. In fact, the depicted flows agree with those of the Thirring model for asymptotically large $g$ and $\gamma$, as we have kept the gauge coupling at a fixed finite value. For any $N_f < N_{\text{f,crit}}^{\text{q}}$, the RG asymptote in QED$_3$ thus coincides with the Thirring-model asymptote within our approximation.

On the basis of our pointlike fermionic truncation it is hard to judge which channel ultimately dominates as a function of $N_f$. This is because we do not have a metric in theory (coupling) space available that could provide for a quantitative measure of absolute distance from a certain channel. As a tentative measure for the chiral symmetry-breaking region, we have depicted a gray-shaded region between the angle bisectors between the chiral axis and the $\gamma$ axis and the one between the chiral axis and the $g$ axis.

For small $N_f$ such as $N_f = 1$, the asymptote lies inside this region where we expect chiral symmetry-breaking to occur, cf. Fig. 7 (left panel). For larger $N_f$ such as $N_f = 9$, the asymptote lies outside this region, cf. Fig. 7 (right panel). Taking this rough measure seriously, we find that the asymptote of the four-fermion flows lies within this suspected domain of attraction of the chiral channel for $1 \leq N_f \leq 4$. As a rough estimate, this suggests to identify the maximal value of $N_f$, for which the system is inside this region with a dominant chiral symmetry-breaking when becoming dominant. The solid red line marks the direction of the asymptote of the RG trajectories for large $\gamma$, $g$. Starting the flow for vanishing fermion interactions $\gamma = g = 0$, in general both $\gamma$ and $g$ are generated and will approach this asymptote in the course of the RG flow.

For the linear regulator in the point-like limit $\zeta = 0$ and for all regulators with $\zeta = 1$, we find that the chiral-critical flavor number can in fact be smaller than the conformal-critical flavor number, $N_{\text{f,crit}}^{\text{v}} < N_{\text{f,crit}}^{\text{q}}$. This leaves us with the interesting conclusion that the many-flavor phase diagram of QED$_3$ could be more involved than previously anticipated: in addition to the chiral symmetry-broken phase for $N_f < N_{\text{f,crit}}^{\text{q}}$ and the quasi-conformal phase for $N_f > N_{\text{f,crit}}^{\text{q}}$, there could be another phase in-between for $N_f < N_{\text{f,crit}}^{\text{v}}$, characterized by
Figure 7. RG flow of the four-fermion interactions in the plane spanned by the couplings $\tilde{g}$ and $g$ for $0 < (e^2 - \epsilon_{cr}^2) \ll 1$ and $N_f = 1$ (left panel) and $N_f = 9$ (right panel), as obtained from the linear regulator function with $\zeta = 1$. Recall that $N_{f,cr} = 10.0$ in this case. The dashed line corresponds to the chiral channel ($\tilde{g} = g$). The solid red line represents the asymptotes of the RG trajectories. The gray-shaded area indicates a tentative measure for the chiral symmetry-breaking region, see main text for details.

different low-energy properties.

At this point, it is instructive to compare our results with those from the 3d Thirring model which shares with QED$_3$ both its U(2$N_f$) chiral symmetry as well as the corresponding possible symmetry-breaking patterns. In the Thirring model, defined in terms of the non-Gaussian UV fixed point $C$ (for $e^2 = 0$), the long-range chiral properties in the pointlike language are also determined by the competition between the chiral and the vector channel. In [60] the Thirring model was studied in detail using dynamical bosonization techniques that allow to enter the symmetry-broken regime and give direct access to the order-parameter potentials, condensation phenomena and massive excitations. The critical flavor number below which the system is in the chiral symmetry broken phase was determined to be

$$N_{f,cr}^{Thirring} \approx 5.1,$$

which is similar to our rough estimate for $N_{f,cr}^{QED}$ for QED$_3$ given above. In fact the mere quantitative difference between our QED$_3$ flows and those of the Thirring model within the same approximation in the fermion sector are the gauge-coupling terms in the $\beta$ functions. As the approach to criticality is primarily indicated by diverging four-fermion interactions, the following scenario is possible: if the gauge contributions to the fermion self-interactions stay subdominant for the approach to criticality, we conjecture that the critical flavor number of QED$_3$ and the 3d Thirring model are identical.

For this conjecture to hold, the chiral critical flavor number, $N_{f,cr}^{QED} > N_{f,cr}^{Thirring}$, this criterion appears to be satisfied within our approximation for the linear regulator in the pointlike truncation with $\zeta = 0$ and for all regulators with $\zeta = 1$. Otherwise the QED$_3$ system could still be trapped by the IR attractive fixed point $O$ while the analogous Thirring system would already be in the chirally broken phase, such that the conjecture would fail. Whether the gauge-contributions indeed stay subdominant during the approach to criticality is a quantitative question that we cannot resolve within our present simple truncation. For instance, using the simplified $\beta$ function for the gauge coupling (35), the gauge coupling remains bounded by its fixed-point value, $e^2 \leq e_{cr}^2$, and the criterion is satisfied. In the more general case, e.g., using Eq. (29), the situation is less clear and requires a full numerical integration of the flow. Most likely a definite answer requires a dynamically bosonized flow. However, even if the gauge contributions do not stay subdominant, it appears plausible that the chiral-critical flavor numbers for QED$_3$ and the 3d Thirring model would still be similar.

Let us now try to address the new possible phase in-between $N_{f,cr}^{QED}$ and $N_{f,cr}^{Thirring}$ assuming that $N_{f,cr}^{QED} < N_{f,cr}^{Thirring}$. Again, the Thirring model may provide a guideline: in [60], it was observed that for $N_f > N_{f,cr}^{Thirring}$, the system not only is dominated by the vector channel, but moreover the mass term of the vector channel $m_V^2$ approaches zero at a finite scale $k$. This indicates the possibility of the appearance of a Lorentz symmetry breaking condensate $\langle V_\mu \rangle \neq 0$ for $N_{f,cr}^{QED} < N_f < N_{f,cr}^{Thirring}$, going along with two massless Goldstone bosons and a massive “radial” mode.

These considerations suggest a many-flavor phase diagram of QED$_3$ as schematically drawn in Fig. 8 with a chirally
broken small-$N_f$ phase, possibly a phase with spontaneously broken Lorentz symmetry at intermediate $N_f$, and a quasi-conformal massless phase at large $N_f$ extending to $N_f \to \infty$. The nature of the phase transitions at $N_{f,cr}^\chi$ and $N_{f,cr}^{\chi SB}$ cannot be determined within our present approximation. For the Thirring model, the dynamically bosonized study revealed that the chiral phase transition at $N_{f,cr}^\chi$ is of second order [60]. In particular the chirally-broken and Lorentz-broken phases do not overlap, but inhibit one another. This suggests the possibility of a second-order phase transition at $N_{f,cr}^{\chi SB}$ also in QED$_3$, if the gauge coupling does not take a too strong influence on the approach to criticality.

The nature of the transition at $N_{f,cr}^{\chi SB}$ is less clear. On the one hand, the quasi-conformal mode vanishes because of the annihilation of fixed points. This is similar to Berezinsky-Kosterlitz-Thouless (BKT)-type phase transitions, such that one might expect corresponding essential (or Miransky) scaling of observables near the phase transition [111–118] with universal powerlaw corrections [119], see also [41, 53, 54]. On the other hand, the spectra on the two sides of the phase transition share some similarities: on both sides, the fermion and the photon fields are massless; there is a massive (but presumably unstable) vector excitation on the quasi-conformal side, while there are a massive "radial" excitation and massless Goldstone bosons on the Lorentz symmetry-breaking side. Near the transition at $N_{f,cr}^{\chi SB}$ all these vector-like degrees of freedom can possibly mix nontrivially which might influence the nature of the transition.

In order to check the scenario suggested above, it appears highly worthwhile to search for vector condensates $\langle \bar{\psi} \gamma_\mu \psi \rangle$ also with other nonperturbative methods in the region above the chiral phase transition $N_f \gtrsim N_{f,cr}^\chi$. If a vector condensate is found, our work suggests the existence of a further transition to the quasi-conformal phase at $N_{f,cr}^{\chi SB} > N_{f,cr}^\chi$.\n
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_diagram.png}
\caption{Sketch of the conjectured many-flavor phase diagram of QED$_3$. In addition to the phase governed by spontaneous chiral symmetry breaking ($\chi_{SB}$) for small values of $N_f$, an intermediate phase driven by the vector-channel may exist, possibly exhibiting (spontaneous) breaking of Lorentz symmetry, see text for a discussion of the transition lines.}
\end{figure}

\section{Conclusions}

In the present work we have studied the many-flavor phase diagram of QED$_3$ by analyzing the RG fixed-point structure of the theory. In addition to the asymptotically free Gaußian fixed point, the fixed-point structure of QED$_3$ shares similarities with that of the 3-dimensional Thirring model which has the same global chiral/flavor symmetries.

For large flavor numbers $N_f > N_{f,cr}^{\chi SB}$, the screening property of fermionic fluctuations induces an IR attractive, quasi-conformal, fixed point in the gauge sector, which in the fermionic sector corresponds to a slightly shifted Gaußian fixed point, implying that the fermionic system remains attracted by this fixed point. For large $N_f$, the system is in a quasi-conformal phase and remains massless in complete agreement with expectations and literature results. If this large-$N_f$ phase described a condensed-matter system, the existence of the quasi-conformal fixed point would indicate a so-called algebraic-Fermi-liquid phase [11], with striking consequences to the electronic, optical, and thermodynamic experimental observables. Such a material would be one of the very rare examples above $1+1$ dimensions and without disorder or magnetic field, which exhibit genuine non-Fermi liquid behavior. If QED$_3$ is indeed an effective theory for the superconductor-insulator transition in the cuprates, our result of a large $N_{f,cr}^{\chi SB} > 2$, however, supports the scenario that cuprates at $T = 0$ are not in the quasi-conformal phase, and there is no algebraic-Fermi-liquid behavior for any doping of the cuprates.

Lowering $N_f$, the system approaches the lower end of the "quasi-conformal window" at $N_{f,cr}^{\chi SB}$ which is characterized by a merger of the Gaußian and the "BKT" fixed point in the fermionic interactions. This mechanism is similar to the one discovered in 4-dimensional many-flavor QCD [52–54], which gives rise to BKT-type scaling behavior [116, 118, 119]. As an important difference, we observe the possibility in QED$_3$ that the RG flow can remain dominated by the vector channel for $N_f$ slightly below $N_{f,cr}^{\chi SB}$. Only for even smaller $N_f$, the chiral channel eventually takes over such that the theory can definitely be expected to be in the chirally-broken phase with massive fermions.

If these findings persist beyond the approximations underlying our analysis, the phase diagram of QED$_3$ along the many-flavor direction can exhibit more phases than previously anticipated. In between the chirally-broken phase for $N_f < N_{f,cr}^{\chi SB}$ and the quasi-conformal phase for $N_f > N_{f,cr}^{\chi SB}$, there can exist a vector-channel dominated phase provided that $N_{f,cr}^{\chi SB} < N_{f,cr}^{\chi SB}$. If the vector channel becomes critical, this phase could be characterized by a Lorentz-breaking vector condensate and a corresponding excitation spectrum with photonlike Goldstone bosons as well as a massive radial-type mode.

From a technical perspective, we have discovered that a Fierz-complete set of fermionic interactions is a mandatory ingredient for reliably estimating quantities such as $N_{f,cr}^{\chi SB}$. Simple projections onto seemingly physically relevant channels can imply a complete loss of quantitative control. This result may inspire corresponding improvements in other analytic ap-
proximation schemes used in the literature. A similar word of caution applies to lattice approaches: as Fierz completeness is a statement about the exact realization of the U(2Nf) flavor symmetry of the model, a lattice formulation that is not guaranteed to preserve the full continuum flavor symmetry may simply simulate a different continuum model with possibly very different values of N^sc \_f. Indeed, a previous RG approach to such a QED_3 theory in the presence of U(2Nf)-symmetry breaking interactions revealed that those perturbatively irrelevant interactions may become relevant for strong gauge coupling, significantly affecting the corresponding predictions for N^sc \_f. Also, while certainly tempting, it is thus premature to speculate on possible consequences of the new vector-channel-dominated phase, which we predict for N^f \_f < N_f < N^sc \_f, on the cuprate phase diagram: Even if this new phase reached all the way down to the physical flavor number N_f = 2 (i.e., if N^f \_f was smaller than 2, in contrast to most of the previous findings, and also to our estimate), the actual cuprate system does not have the full U(2Nf) symmetry and it is momentarily unclear how the presence of the symmetry-breaking short-range interactions will affect the many-flavor phase diagram in QED_3 and the existence of the vector-channel-dominated intermediate phase. This deserves further investigation.

From a quantitative viewpoint, our result for N^sc \_f is still rather strongly affected by artificial regularization-scheme dependencies. This may hint to the insufficient resolution of momentum dependencies of the vertices which in our work is only estimated by an overall RG scale. We consider Eq. (40) to represent our best estimate: N^sc \_f \approx 4.1 \ldots 10.0.

For the chiral-critical flavor number, our results are compatible with those of the most advanced DSE studies, suggesting N^\chi \_f \approx 4. Hence, the window of theories in the vector-channel-dominance phase could be finite and include theories with integer N_f.

However, under the assumption that the gauge contributions to the approach to criticality stay subdominant, we conjecture the chiral-critical flavor number of QED_3 and the 3d Thirring model to be identical. A recent study of the 3d Thirring model suggests that N^\chi \_\text{Thirring} \approx 5.1, see Ref. [60]. In the light of our QED_3-Thirring conjecture and the approximation involved in our computation, we can therefore not exclude the possibility that N^\chi \_f and N^sc \_f are so close to each other that the vector-dominance phase does not include a system with integer N_f. While it is certainly not inconceivable that N^\chi \_f and N^sc \_f are in fact identical, we see no natural reason for this coincidence to hold. Of course, a verification and exact determination of the phase boundaries of the many-flavor phase diagram requires more elaborate studies in the future, ideally by using various different theoretical approaches. In any case, the present work points to a so far overlooked new intermediate phase and may therefore help to better our understanding of the dynamics underlying low-dimensional fermionic field theories and the corresponding strongly-correlated condensed-matter systems.

### Table II

| N^f \_f | R_{CS} | R_{exp} | R_{inf} | R_{sc} |
|---------|--------|---------|---------|--------|
| 1 \_A | N \_f | -1 | -0.298558 | -0.261313 | -0.243833 | -0.153062 | -0.126032 | -0.102494 |
| 1 \_B | N \_f | -1 | -0.443434 | -0.214602 | -0.243833 | -0.214602 | -0.194340 | -0.170823 |
| 1 \_C | N \_f | -1 | -0.243833 | -0.243833 | -0.243833 | -0.243833 | -0.208436 | -0.194340 |
| 1 \_D | N \_f | -1 | -0.243833 | -0.243833 | -0.243833 | -0.243833 | -0.208436 | -0.194340 |
| 1 \_E | N \_f | -1 | -0.243833 | -0.243833 | -0.243833 | -0.243833 | -0.208436 | -0.194340 |

Table II. Numerical values for the threshold functions as obtained from the various regulators employed in this work and listed in App. B. Depending on the type of internal lines in the 1PI diagram underlying the different threshold functions, these functions can be written as sum of three terms: a pure (real-valued) number (N), a number times \eta_0 (2nd row), and a number times \eta_0 (3rd row). Values with an asterisk * depend on the details of the definition of the non-analytic sharp cutoff.

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### Appendix A: Irreducible representation

Though the reducible representation using 4-component Dirac spinors \psi^a, a = 1, \ldots, N_f, has its merits from the viewpoint of applications in condensed-matter systems, some aspects become more transparent in the irreducible representation using 2-component spinors \chi^i, i = 1, \ldots, 2N_f. In our conventions, the transition between these representations can be defined using the projector

\[ P_{LR}^{(4\chi)} = \frac{1}{2} (1 \pm \chi a). \]  

Decomposing \chi^i into \(\chi^a \chi^{a+N_f}\), for a = 1, \ldots, N_f, we introduce the \chi subcomponents by

\[ P_L \psi^a = \frac{1}{\sqrt{2}} \chi^a \otimes \left( \begin{array}{c} 1 \\ i \end{array} \right), \ h^a P_L = \frac{1}{\sqrt{2}} \chi^a \otimes (1, -i), \]  

and

\[ P_R \psi^a = \frac{1}{\sqrt{2}} \chi^{a+N_f} \otimes \left( \begin{array}{c} 1 \\ -i \end{array} \right), \ h^a P_R = -\frac{1}{\sqrt{2}} \chi^{a+N_f} \otimes (1, i). \]
In the irreducible representation, the enhanced U(2Nf) symmetry of QED becomes obvious, since
\[ \tilde{\psi}^a \gamma_\mu \psi^\mu = \tilde{\chi}^i \sigma_\mu \chi^i, \quad i = 1, \ldots, 2Nf, \] (A4)
and \( \sigma_\mu \) denote the Pauli matrices. Similarly, it is straightforward to show that
\[ \tilde{\psi}^a \gamma_\mu \psi^\mu = \tilde{\chi}^i \sigma_\mu \chi^i \Rightarrow \tilde{\chi}^i \chi^i. \] The latter implies that a mass term of the form \( \tilde{m} \tilde{\psi}^a \gamma_\mu \psi^\mu \) actually preserves the U(2Nf) symmetry. Also, the interaction term \( (P)^2 \) introduced in the main text in Eq. (5) in this notation indeed becomes the standard Gross-Neveu interaction for two-component spinors.

In the same spirit the nonsinglet interaction channel \( (S)^2 \) as used in Eq. (10) can be shown to read
\[ (S)^2 = 2(\tilde{\chi}^i \chi^i)^2 \equiv 2\tilde{\chi}^i \chi_i, \] (A5)
where the factor of two on the right-hand side motivates the different coupling normalization between the \( (V)^2 \) and the \( (S)^2 \) term in Eq. (10).

Appendix B: Regulator functions

In this appendix, we summarize the regulator functions employed in the present work. For the definition of the regulator functions, it is convenient to introduce so-called regulator shape functions \( R_{R,B} \) for the fermions (F) and bosons (B), respectively:
\[ R_F(p) = -pF(y) \quad \text{and} \quad R_B(p^2) = p^2B(y), \quad (B1) \]
where \( y = p^2/k^2 \). Overall, we have used four different regulator functions, namely the Callan-Symanzik regulator \( R_{CS} \) with
\[ r_F(y) = \frac{y+1}{y} - 1, \quad r_B(y) = \frac{1}{y}, \quad (B2) \]
the exponential regulator \( R_{exp} \) with
\[ r_F(y) = \frac{1}{\sqrt{1-e^{-y}}}, \quad r_B(y) = \frac{1}{e^y-1}, \quad (B3) \]
the linear regulator \( R_{lin} \), see Refs. [106–108], with
\[ r_F(y) = \left( \frac{1}{\sqrt{y}} - 1 \right) \theta(1-y), \quad \theta(1-y) \]
\[ r_B(y) = \left( \frac{1}{y} - 1 \right) \theta(1-y), \quad (B5) \]
and the so-called sharp-cutoff regulator with
\[ r_F(y) = \lim_{b \to \infty} \sqrt{1 + \frac{1}{y_b} - 1}, \quad r_B(y) = \lim_{b \to \infty} \frac{1}{y_b}. \quad (B6) \]

Note that the sharp-cutoff regulator has to be handled with care as it requires a definite prescription of the order of the various limiting processes involved, in order to avoid ambiguities in the evaluation of the loop integrals. In particular, this is the case for the threshold function \( \tilde{m}_{1,1}^{(R,F)} \), cf. also the RG equations in Ref. [29]. These artifacts of the sharp-cutoff scheme are well known, see, e.g., the discussion of the BKT-phase transition in [90, Chapter 6.4]. In Tab. II, we list the numerical values for the threshold functions as obtained from the various employed regulators.

Appendix C: RG flow of \( Z_A \)

We briefly summarize the derivation of the equation for the anomalous dimension of the photon, \( \eta_A = -\partial_t \ln Z_A \). We begin by rewriting the Wetterich equation (19) as follows:
\[ \partial_t \Gamma_k = \frac{1}{2} \text{STr} \tilde{\partial}_t \ln \left( \Gamma^{(2)} + R_k \right), \quad (C1) \]
where \( \tilde{\partial}_t \) denotes a formal derivative acting only on the of the regulator function \( R_k \). The representation (C1) of the Wetterich equation is a convenient starting point for the computation of both the fermionic RG flows (see, e.g., Ref. [81] for a detailed introduction) as well as for the anomalous dimensions. In order to calculate the flow equation for \( Z_A \), we decompose the inverse regularized propagator \( \Gamma^{(2)}_k \) on the right-hand side of the flow equation into a field-independent \( (P_k) \) and a field-dependent \( (F_k) \) part,
\[ \Gamma^{(2)}_k + R_k = (P_k + F_k). \quad (C2) \]
The flow equation can then be decomposed in powers of the fields:
\[ \partial_t \Gamma_k = \frac{1}{2} \text{STr} \left( \tilde{\partial}_t \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \frac{1}{n} \left( P_k^{-1} F_k \right)^n \right). \quad (C3) \]
On the right-hand side we have dropped a field-independent term which is of no relevance for our present study. The powers of \( P_k^{-1}F_k \) can be calculated by straightforward matrix multiplications. It is then straightforward to project the various terms from the expansion appearing on the right-hand side of Eq. (C3) onto our ansatz for the effective action. To the flow of \( Z_A \) only the second term of the expansion contributes and we find
\[ \eta_A = - \frac{1}{2Z_A} \left\{ \frac{\pi^\nu_{\mu}(p)}{p^\nu} \left[ \int \frac{d^3q}{(2\pi)^3} \tilde{\delta} \delta A_\mu(-p) \frac{1}{2} \text{STR} \left[ \frac{\tilde{\delta}}{2} \left( \frac{1}{2} \left( \pi^\nu_{-\mu} F_k \right) \right) \right] \frac{\tilde{\delta}}{2} \delta A_\lambda(q) \right] \right\}, \]  

where we have used the transversal projector \( \pi^\nu_{\mu}(p) = \delta_{\nu\mu} - \frac{p_{\nu} p_{\mu}}{p^2} \). The second term corresponds to the subtraction of the zero-momentum limit of the regularized flow which is constrained by the regulator-modified Ward identity. In this way, the transversal projection entering the definition of \( \eta_A \) satisfies the standard Ward identity at all scales. This construction is based on the implicit assumption that the longitudinal and the transversal part of the photon propagator do not differ by non-analyticities at small momenta. From this expression, we then obtain

\[ \eta_A = 8 v_3 N_F e^2 \zeta L_1^{(F)} \],

where \( v_3 = 1/(8\pi^2) \) and

\[ L_1^{(F)}(\eta_0; \zeta) \equiv L_1^{(F)} = \frac{1}{2 \pi} \int_0^\infty dy \left\{ \frac{2}{3} \frac{\partial}{\partial y} r_\mu(y) \eta_0 r_\mu(y) \right\} - \frac{1}{2} \int_1^\infty \frac{dx}{x} \left[ \frac{\partial}{\partial y} r_\mu(y) \eta_0 r_\mu(y) \right] \left[ \frac{1 + r_\mu(y)}{1 + r_\mu(y) + \sqrt{y + \zeta^2}} \right] \]

\[ \left[ \frac{1 + r_\mu(y)}{1 + r_\mu(y)} \right] \left[ \frac{1 + r_\mu(y)}{1 + r_\mu(y) - 2 \sqrt{y + \zeta^2}} \right] \right\} \]  

Here, we have introduced \( y = q^2 / k^2 \) for convenience and \( x = \cos \theta \).  

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