The induced Cosmological Constant as a tool for exploring geometries

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The cosmological constant induced by quantum fluctuation of the graviton on a given background is considered as a tool for building a spectrum of different geometries. In particular, we apply the method to the Schwarzschild background with positive and negative mass parameter. In this way, we put on the same level of comparison the related naked singularity (−M) and the positive mass wormhole. We use the Wheeler-De Witt equation as a basic equation to perform such an analysis regarded as a Sturm-Liouville problem. The cosmological constant is considered as the associated eigenvalue. The used method to study such a problem is a variational approach with Gaussian trial wave functionals. We approximate the equation to one loop in a Schwarzschild background. A zeta function regularization is involved to handle with divergences. The regularization is closely related to the subtraction procedure appearing in the computation of Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

I. INTRODUCTION

In 1969, Penrose suggested that there might be a sort of “cosmic censor” that forbids naked singularities from forming, namely singularities that are visible to distant observers. Although there is no proof of such conjecture, naked singularities and the cosmic censorship are still a source of interest. A simple and particularly interesting example of naked singularities is the negative mass Schwarzschild spacetime. This is simply obtained by the Schwarzschild solution

\[ ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

replacing \( M \) with \(-M\). This simple substitution gives rise to a naked singularity not protected by a horizon. An immediate consequence of a negative Schwarzschild mass is that if one were to place two bodies initially at rest, one with a negative mass and the other with a positive mass, both will accelerate in the same direction going from the negative mass to the positive one. Furthermore, if the two masses are of the same magnitude, they will uniformly accelerate...
accelerate forever. This feature leads to the problem of stability of such a geometry discussed by Gibbons, Hartnoll and Ishibashi\cite{2} and Gleiser and Dotti\cite{3}. However, even if the stability issue is an open debate which seems to incline more to the unstable behavior than to the stable one\cite{4}, in this paper we wish to study the relation between the Schwarzschild solution for positive and negative masses with the induced cosmological constant. A cosmological constant can be considered “induced” when it appears as a consequence of quantum fluctuations. Since, apparently the Schwarzschild solution, naked singularities and the induced cosmological constant appear to be disconnected, it urges to establish a point of contact. We claim that such a link is in the Wheeler-DeWitt equation (WDW)\cite{5}. This equation can be simply obtained starting by the Einstein field equations without matter fields in four dimensions

\[ G_{\mu\nu} + \Lambda_c g_{\mu\nu} = 0, \]  

(2)

where \( G_{\mu\nu} \) is the Einstein tensor and \( \Lambda_c \) is the cosmological constant. By introducing a time-like unit vector \( u^\mu \), we get

\[ G_{\mu\nu} u^\mu u^\nu = \Lambda_c. \]  

(3)

This is simply the Hamiltonian constraint written in terms of equation of motion. Indeed, if we multiply by \( \sqrt{g}/(2\kappa) \) Eq.\( (3) \), we obtain \( (\kappa = 8\pi G) \)

\[ \frac{\sqrt{g}}{2\kappa} G_{\mu\nu} u^\mu u^\nu = \frac{\sqrt{g}}{2\kappa} R + \frac{2\kappa}{\sqrt{g}} \left( \frac{\pi^2}{2} - \pi^{\mu\nu} \pi_{\mu\nu} \right) = \frac{\sqrt{g}}{2\kappa} \Lambda_c. \]  

(4)

Here \( R \) is the scalar curvature in three dimensions and

\[ \frac{\sqrt{g}}{2\kappa} G_{\mu\nu} u^\mu u^\nu = -\mathcal{H}. \]  

(5)

If we multiply both sides of Eq.\( (4) \) by \( \Psi [g_{ij}] \), we can re-cast the equation in the following form

\[ \left[ \frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda_c) \right] \Psi [g_{ij}] = 0. \]  

(6)

This is known as the Wheeler-DeWitt equation with a cosmological term. Eq.\( (6) \) together with

\[ -2\nabla_i \pi^{ij} \Psi [g_{ij}] = 0, \]  

(7)

describe the wave function of the universe. The WDW equation represents invariance under time reparametrization in an operatorial form, while Eq.\( (7) \) represents invariance under diffeomorphism. \( G_{ijkl} \) is the supermetric defined as

\[ G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}). \]  

(8)

Note that the WDW equation can be cast into the form

\[ \left[ \frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R \right] \Psi [g_{ij}] = -\frac{\sqrt{g}}{\kappa} \Lambda \Psi [g_{ij}], \]  

(9)

which formally looks like an eigenvalue equation. In this paper, we wish to use the induced cosmological constant argument evaluated to one loop in the different backgrounds as a tool to establish which kind of background induce the larger cosmological constant or, in other words, the larger Zero Point Energy (ZPE). In particular, we will compute the graviton ZPE propagating on the Schwarzschild background which positive and negative mass (naked singularity). This choice is dictated by considering that the Schwarzschild solution represents the only non-trivial static spherical symmetric solution of the Vacuum Einstein equations. Therefore, in this context the ZPE can be attributed only to quantum fluctuations. An example of this method applied in a completely different context without a cosmological term is in Refs.\( (6, 7) \), where the ZPE graviton contribution computed on different metrics is compared. In practice, we desire to compute

\[ \Delta \Lambda_c = \Lambda_c^S - \Lambda_c^N \gtrless 0, \]  

(10)

where \( \Lambda_c^{S,N} \) are the induced cosmological constant computed in the different backgrounds. Moreover, the Schwarzschild solution for both masses, namely \( \pm M \) is asymptotically flat. Therefore we are comparing backgrounds with the same
asymptotically behavior. Nevertheless, in Eq. (6), surface terms never come into play because $H$ as well $\Lambda_c/\kappa$ are energy densities and surface terms are related to the energy (e.g. ADM mass) and not to the energy density. We want to point up that we are neither discussing the problem of forming the naked singularity nor a transition during a gravitational collapse, rather the singularity is considered already existing. The semi-classical procedure followed in this work relies heavily on the formalism outlined in Ref. [8], where the graviton one-loop contribution in a Schwarzschild background was computed, through a variational approach with Gaussian trial wave functionals. A zeta function regularization is used to deal with the divergences, and a renormalization procedure is introduced, where the finite one loop is considered as a self-consistent source for traversable wormholes. Rather than reproducing the formalism, we shall refer the reader to [16] for details, when necessary. The rest of the paper is structured as follows, in section II we show how to apply the variational approach to the Wheeler-De Witt equation and we give some of the basic rules to solve such an equation approximated to second order in metric perturbation, in section III we analyze the spin-2 operator or the operator acting on transverse traceless tensors specified for the Schwarzschild metric with $\pm M$, in section IV we use the zeta function to regularize the divergences coming from the evaluation of the ZPE for TT tensors and we write the renormalization group equation. We summarize and conclude in section V.

II. THE COSMOLOGICAL CONSTANT AS AN EIGENVALUE OF THE WHEELER DE WITT EQUATION

In this section we shall consider the formalism outlined in detail in Ref. [8], where the graviton one-loop contribution in a Schwarzschild background is used. We refer the reader to Ref. [8] for details. The WDW equation (6), written as an eigenvalue equation, can be cast into the form

$$\hat{\Lambda}_\Sigma \Psi [g_{ij}] = -\frac{\Lambda_c}{\kappa} \Psi [g_{ij}],$$

(11)

where

$$\hat{\Lambda}_\Sigma = \frac{2}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} R.$$  

(12)

We, now multiply Eq. (11) by $\Psi^* [g_{ij}]$ and we functionally integrate over the three spatial metric $g_{ij}$, then after an integration over the hypersurface $\Sigma$, one can formally re-write the WDW equation as

$$1/V \frac{\int \mathcal{D}[g_{ij}] \Psi^* [g_{ij}] f_\Sigma d^3x \hat{\Lambda}_\Sigma \Psi [g_{ij}]}{\int \mathcal{D}[g_{ij}] \Psi^* [g_{ij}] \Psi [g_{ij}]} = 1/V \mathcal{D}[g_{ij}] \Psi^* [g_{ij}] \Psi [g_{ij}] = -\frac{\Lambda_c}{\kappa}. $$

(13)

The formal eigenvalue equation is a simple manipulation of Eq. (6). However, we gain more information if we consider a separation of the spatial part of the metric into a background term, $\bar{g}_{ij}$, and a perturbation, $h_{ij}$,

$$g_{ij} = \bar{g}_{ij} + h_{ij}. $$

(14)

The perturbation can be decomposed in a canonical way to give[8, 10, 11, 12]

$$h_{ij} = \frac{1}{3} \left( h + 2 \nabla \cdot \xi \right) g_{ij} + \left( L \xi \right)_{ij} + h^\perp \Sigma_{ij}, $$

(15)

where the operator $L$ maps $\xi_i$ into symmetric tracefree tensors

$$\left( L \xi \right)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} \left( \nabla \cdot \xi \right)$$

(16)

and

$$g^{ij} h^\perp_{ij} = 0, \ \ \ \nabla^i h^\perp_{ij} = 0. $$

(17)

It is immediate to recognize that the trace element

$$\sigma = h + 2 \left( \nabla \cdot \xi \right)$$

(18)
is gauge invariant. We write the trial wave functional as

$$\Psi[h_{ij}(x)] = N \Psi[h^\perp_{ij}(x)] \Psi[h^\parallel_{ij}(x)] \Psi[\sigma(x)],$$

(19)

where

$$\Psi[h^\perp_{ij}(x)] = \exp\left\{-\frac{1}{4} \langle h K^{-1} h \rangle^\perp_{x,y} \right\}$$

$$\Psi[h^\parallel_{ij}(x)] = \exp\left\{-\frac{1}{4} \langle (L \xi)^{K^{-1}} (L \xi) \rangle^\parallel_{x,y} \right\}$$

(20)

$$\Psi[\sigma(x)] = \exp\left\{-\frac{1}{4} \langle \sigma K^{-1} \sigma \rangle^{\text{trace}}_{x,y} \right\}.$$

The symbol “⊥” denotes the transverse-traceless tensor (TT) (spin 2) of the perturbation, while the symbol “∥” denotes the longitudinal part (spin 1) of the perturbation. Finally, the symbol “trace” denotes the scalar part of the perturbation. \(N\) is a normalization factor, \(\langle \cdot, \cdot \rangle_{x,y}\) denotes space integration and \(K^{-1}\) is the inverse “propagator”. We will fix our attention to the TT tensor sector of the perturbation representing the graviton and the scalar sector. Therefore, representation (19) reduces to

$$\Psi[h_{ij}(x)] = N \exp\left\{-\frac{1}{4} \langle h K^{-1} h \rangle^\perp_{x,y} \right\} \exp\left\{-\frac{1}{4} \langle \sigma K^{-1} \sigma \rangle^{\text{trace}}_{x,y} \right\}.$$  

(21)

Actually there is no reason to neglect longitudinal perturbations. However, following the analysis of Mazur and Mottola of Ref.[11] on the perturbation decomposition, we can discover that the relevant components can be restricted to the TT modes and to the trace modes. Moreover, for certain backgrounds TT tensors can be a source of instability as shown in Refs.[13]. Even the trace part can be regarded as a source of instability. Indeed this is usually termed conformal instability. The appearance of an instability on the TT modes is known as non conformal instability. This means that does not exist a gauge choice that can eliminate negative modes. Since the wave functional (21) separates the degrees of freedom, we assume that

$$-\frac{\Lambda_c}{\kappa} = -\frac{\Lambda_c^\perp}{\kappa} - \frac{\Lambda_c^{\text{trace}}}{\kappa},$$

(22)

then Eq.(13) becomes

$$\frac{1}{V} \langle \Psi | \Lambda_{\Sigma} | \Psi \rangle = -\frac{\Lambda_c^\perp}{\kappa}$$

(23)

and

$$\frac{1}{V} \langle \Psi | \Lambda_{\Sigma}^{\text{trace}} | \Psi \rangle = -\frac{\Lambda_c^{\text{trace}}}{\kappa}$$

(24)

III. THE TRANSVERSE TRACELESS (TT) SPIN 2 OPERATOR AND THE W.K.B. APPROXIMATION

Extracting the TT tensor contribution from Eq.(23), we get to one loop

$$\frac{\Lambda_c^\perp}{\kappa} \langle \lambda_i \rangle = -\frac{1}{4} \sum_{\tau} \left[ \sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)} \right].$$

(25)

The above expression makes sense only for \(\omega_1^2(\tau) > 0\). To further proceed, we need to compute \(\omega_i^2(\tau)\) \((i = 1, 2)\). To this purpose we write the background metric in the following way

$$ds^2 = -N^2(r) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(26)
with a generic \( b(r) \), to keep the discussion on a general ground, when possible. \( N(r) \) is the “lapse function” playing the role of the “redshift function”, while \( b(r) \) is termed “shape function”. The Spin-two operator for this metric is

\[
(\Delta_2 h^{TT})^i_j := - (\Delta_T h^{TT})^i_j + 2 \left( R h^{TT} \right)^i_j,
\]

where the transverse-traceless (TT) tensor for the quantum fluctuation is obtained with the help of Eq.\((15)\). Thus

\[
- (\Delta_T h^{TT})^i_j = -\Delta_S (h^{TT})^i_j + \frac{6}{r^2} \left( 1 - \frac{b(r)}{r} \right).
\]

\( \Delta_S \) is the scalar curved Laplacian, whose form is

\[
\Delta_S = \left( 1 - \frac{b(r)}{r} \right) \frac{d^2}{dr^2} + \left( \frac{4r - b'(r)r - 3b(r)}{2r^2} \right) \frac{d}{dr} - \frac{L^2}{r^2}
\]

and \( R^a_j \) is the mixed Ricci tensor whose components are:

\[
R^a_j = \left\{ \frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3} \right\},
\]

This implies that the scalar curvature is traceless. We are therefore led to study the following eigenvalue equation

\[
(\Delta_2 h^{TT})^i_j = \omega^2 h^i_j
\]

where \( \omega^2 \) is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity. In particular, our choice for the three-dimensional gravitational perturbation is represented by its even-parity form

\[
h_{ij}^{even}(r, \vartheta, \phi) = \text{diag} \left[ H(r) \left( 1 - \frac{b(r)}{r} \right)^{-1}, r^2 K(r), r^2 \sin^2 \vartheta K(r) \right] Y_{lm}(\vartheta, \phi).
\]

For a generic value of the angular momentum \( L \), representation \((32)\) joined to Eq.\((28)\) lead to the following system of PDE’s

\[
\begin{cases}
-\Delta_S + \frac{6}{r} \left( 1 - \frac{b(r)}{r} \right) + 2 \left( \frac{b'(r)}{r} - \frac{b(r)}{r^2} \right) H(r) = \omega^2_1 H(r), \\
-\Delta_S + \frac{6}{r} \left( 1 - \frac{b(r)}{r} \right) + 2 \left( \frac{b'(r)}{2r} + \frac{b(r)}{2r^3} \right) K(r) = \omega^2_2 K(r).
\end{cases}
\]

Defining reduced fields

\[
H(r) = \frac{f_1(r)}{r}; \quad K(r) = \frac{f_2(r)}{r},
\]

and passing to the proper geodesic coordinate

\[
dx = \pm \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}
\]

the system \((38)\) becomes

\[
\begin{cases}
-\frac{d^2}{dx^2} + V_1(x) f_1(x) = \omega^2_1 f_1(x), \\
-\frac{d^2}{dx^2} + V_2(x) f_2(x) = \omega^2_2 f_2(x).
\end{cases}
\]
with

\[
\begin{align*}
V_1 (r) &= \frac{l(l+1)}{r^2} + U_1 (r) \\
V_2 (r) &= \frac{l(l+1)}{r^2} + U_2 (r)
\end{align*}
\]  

(37)

where we have defined \( r \equiv r(x) \) and

\[
\begin{align*}
U_1 (r) &= \frac{6}{\pi^2} \left(1 - \frac{b(1)}{r} \right) + \left[\frac{3}{2\pi^2} b' (r) - \frac{3}{2\pi^2} b (r) \right] \\
U_2 (r) &= \frac{6}{\pi^2} \left(1 - \frac{b(1)}{r} \right) + \left[\frac{1}{2\pi^2} b' (r) + \frac{3}{2\pi^2} b (r) \right]
\end{align*}
\]  

(38)

Note that the coordinate \( x \) is appropriate only for Schwarzschild. Nevertheless, we find convenient use the same variable even in the case of the naked singularity. We choose

\[
b(r_t) = r_t
\]  

(40)

only for Schwarzschild. \( r_t \) is termed the throat and \( r \in [r_t, +\infty) \). Of course, for the negative Schwarzschild mass, \( r \in (0, +\infty) \). The potentials of the Lichnerowicz operator \( \mathcal{L}_G \) simplify into

\[
\begin{align*}
U_1 (r) &= m_1^2 (r) = \frac{6}{\pi^2} \left(1 - \frac{2MG}{r} \right) - \frac{3MG}{r} \\
U_2 (r) &= m_1^2 (r) = \frac{6}{\pi^2} \left(1 - \frac{2MG}{r} \right) + \frac{3MG}{r}
\end{align*}
\]  

(41)

for the Schwarzschild case and

\[
\begin{align*}
\tilde{U}_1 (r) &= \tilde{m}_1^2 (r) = \frac{6}{\pi^2} + \frac{15MG}{r} \\
\tilde{U}_2 (r) &= \tilde{m}_1^2 (r) = \frac{6}{\pi^2} + \frac{9MG}{r}
\end{align*}
\]  

(42)

for the naked singularity. In the Schwarzschild case, we get

\[
m_1^2 (r) \geq 0 \quad \text{when} \quad r \geq \frac{5MG}{2} \\
m_1^2 (r) < 0 \quad \text{when} \quad 2MG \leq r < \frac{5MG}{2} \\
m_2^2 (r) > 0 \quad \forall r \in [2MG, +\infty)
\]  

(43)

The functions \( U_1 (r) \) and \( U_2 (r) \) play the rôle of two \( r \)-dependent effective masses \( m_1^2 (r) \) and \( m_2^2 (r) \), respectively. In order to use the WKB approximation, we define two \( r \)-dependent radial wave numbers \( k_1 (r, l, \omega_{1,nt}) \) and \( k_2 (r, l, \omega_{2,nt}) \)

\[
\begin{align*}
k_1^2 (r, l, \omega_{1,nt}) &= \omega_{1,nt}^2 - \frac{l(l+1)}{r^2} - m_1^2 (r) \\
k_2^2 (r, l, \omega_{2,nt}) &= \omega_{2,nt}^2 - \frac{l(l+1)}{r^2} - m_2^2 (r)
\end{align*}
\]  

(44)

for \( r \geq \frac{5MG}{2} \). When \( 2MG \leq r < \frac{5MG}{2} \), \( k_2^2 (r, l, \omega_{1,nt}) \) becomes

\[
k_1^2 (r, l, \omega_{1,nt}) = \omega_{1,nt}^2 - \frac{l(l+1)}{r^2} + m_1^2 (r)
\]  

(45)

**IV. ONE LOOP ENERGY REGULARIZATION AND RENORMALIZATION**

The total regularized one loop energy density for the graviton is

\[
\rho (\varepsilon) = \rho_1 (\varepsilon) + \rho_2 (\varepsilon)
\]  

(46)
where the energy densities, \( \rho_i(\varepsilon) \) (with \( i = 1, 2 \)), are defined as

\[
\rho_i(\varepsilon) = \frac{1}{4\pi} \mu^{2\varepsilon} \int_{m_1^2(r)}^{\infty} d\omega_i \frac{\omega_i^2}{[\omega_i^2 - m_i^2(r)]^{\varepsilon-1/2}}
\]

\[
= -\frac{m_i^4(r)}{64\pi^2} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m_i^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right].
\]

(47)

The two \( r \)-dependent effective masses \( m_1^2(r) \) and \( m_2^2(r) \) can be cast in the following form

\[
\begin{cases}
  m_1^2(r) = m_{1,L}^2(r) + m_{1,S}^2(r) \\
  m_2^2(r) = m_{2,L}^2(r) + m_{2,S}^2(r)
\end{cases}
\]

(48)

where

\[
m_{1,L}^2(r) = \frac{6}{r^2} \left( 1 - \frac{b(r)}{r} \right)
\]

(49)

and

\[
\begin{cases}
  m_{1,S}^2(r) = \left[ \frac{d}{dr} b'(r) - \frac{b}{r^2} b(r) \right] \\
  m_{2,S}^2(r) = \left[ \frac{d}{dr} b'(r) + \frac{b}{r^2} b(r) \right]
\end{cases}
\]

(50)

Essentially for the problem we are investigating, the term containing \( m_{L}^2(r) \) is a long range term and will be discarded in this analysis. The zeta function regularization method has been used to determine the energy densities, \( \rho_i \). It is interesting to note that this method is identical to the subtraction procedure of the Casimir energy computation, where the zero point energy in different backgrounds with the same asymptotic properties is involved. In this context, the additional mass parameter \( \mu \) has been introduced to restore the correct dimension for the regularized quantities. Note that this arbitrary mass scale appears in any regularization scheme. Eq. (25) for the energy density becomes

\[
\frac{\Lambda}{8\pi G} = \rho_1(\varepsilon) + \rho_2(\varepsilon).
\]

(51)

Taking into account Eq. (47), Eq. (51) yields the following relationship

\[
\frac{\Lambda}{8\pi G} = \sum_{i=1}^{2} \frac{m_i^4(r)}{64\pi^2} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{4\mu^2}{m_i^2(r)\varepsilon} \right) \right].
\]

(52)

Thus, the renormalization is performed via the absorption of the divergent part into the re-definition of the bare classical constant \( \Lambda_c \)

\[
\Lambda_c \to \Lambda_{0,c} + \Lambda^{div},
\]

(53)

where

\[
\Lambda^{div} = \frac{G}{32\pi}\varepsilon \left( m_1^4(r) + m_2^4(r) \right).
\]

(54)

The remaining finite value for the cosmological constant reads

\[
\frac{\Lambda_{0,c}}{8\pi G} = \sum_{i=1}^{2} \frac{m_i^4(r)}{64\pi^2} \ln \left( \frac{4\mu^2}{m_i^2(r)\varepsilon} \right) = \rho_{T,T}^{eff}(\mu,r).
\]

(55)

The quantity in Eq. (55) depends on the arbitrary mass scale \( \mu \). It is appropriate to use the renormalization group equation to eliminate such a dependence. To this aim, we impose that

\[
\frac{1}{8\pi G} \frac{\partial \rho_{T,T}^{eff}(\mu)}{\partial \mu} = \mu \frac{d}{d\mu} \rho_{T,T}^{eff}(\mu,r).
\]

(56)
Solving it we find that the renormalized constant $\Lambda^TT_0$ should be treated as a running one in the sense that it varies provided that the scale $\mu$ is changing

$$\Lambda_{0,c}(\mu, r) = \Lambda_{0,c}(\mu_0, r) + \frac{G}{16\pi} \left(m^4_1(r) + m^4_2(r)\right) \ln \frac{\mu}{\mu_0}. \quad (57)$$

Substituting Eq.(57) into Eq.(55) we find

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \sum_{i=1}^{2} \frac{m^2_i(r)}{64\pi^2} \ln \left(\left|\frac{4\mu^2_0}{m^2_i(r)/(\sqrt{\epsilon})}\right|\right). \quad (58)$$

Eq.(58) is the expression we shall use to evaluate both the geometries.

A. The Schwarzschild metric

The Schwarzschild background is simply described by the choice $b(r) = 2MG$. In terms of the induced cosmological constant of Eq.(58), we get

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \frac{1}{64\pi^2} \sum_{i=1}^{2} \left(\frac{3MG}{r^3}\right)^2 \ln \left(\left|\frac{4r^3\mu^2_0}{3MG(\sqrt{\epsilon})}\right|\right), \quad (59)$$

where we have used the assumption that $m^2_{i,L}(r)$ can be neglected. We know that an extremum appears, maximizing the induced cosmological constant for

$$\frac{3MG\sqrt{\epsilon}}{4r^3\mu^2_0} = \frac{1}{\sqrt{\epsilon}} \quad (60)$$

and leading to

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \frac{\mu^4_0}{4e^2\pi^2} \quad (61)$$

or

$$\frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} = \left(\frac{3MG}{r^3}\right)^2 \frac{1}{64\pi^2} \quad r \in \left[r_t, \frac{5}{4}r_t\right]. \quad (62)$$

Therefore, it appears that there exists a bound for $\Lambda_{0,c}$

$$\frac{9}{256\pi^2 r_t^4} \leq \frac{\Lambda_{0,c}(\mu_0, r)}{8\pi G} \leq \frac{225}{4096\pi^2 r_t^4} \quad (63)$$

B. The Naked Schwarzschild metric

The energy densities of Eq.(47) can be used also for the negative Schwarzschild mass. The only change is in the range of integration of the energy integral which can be extended to $\omega = 0$. The final result does not change, then in Eq.(58), we can substitute $M$ with $\bar{M}$. Although the equation formally maintains the same expression, the throat is no more there. This means that a piece of $m^2_{i,L}(r)$ cannot be neglected and the two effective masses become

$$\left\{\begin{array}{l}
m^2_1(r) = \frac{6}{r^2} + \frac{15M\bar{M}}{r^3} \\
m^2_2(r) = \frac{6}{r^2} + \frac{9\bar{M}G}{r^3}
\end{array}\right. \quad (64)$$

Since the effective mass grows approaching the singularity, we approximate them close to $r = 0$. Thus, we get

$$\left\{\begin{array}{l}
m^2_1(r) \approx \frac{15M\bar{M}}{r^3} \\
m^2_2(r) \approx \frac{9\bar{M}G}{r^3}
\end{array}\right. \quad (65)$$
In this case, Eq. (58) yields
\[
\frac{\Lambda_{n_{\text{c}}}(\mu_0, r)}{8\pi G} = \frac{1}{64\pi^2} \left( \frac{15MG}{r^3} \right)^2 \ln \left( \frac{4r^3\mu_0^2}{15MG\sqrt{\epsilon}} \right) + \left( \frac{9MG}{r^3} \right)^2 \ln \left( \frac{4r^3\mu_0^2}{9MG\sqrt{\epsilon}} \right) .
\]
(66)

In order to find an extremum, it is convenient to define the following dimensionless quantity
\[
\frac{9MG\sqrt{\epsilon}}{4r^3\mu_0^2} = x ,
\]
(67)
then Eq. (66) becomes
\[
\frac{\Lambda_{n_{\text{c}}}(\mu_0, r)}{8\pi G} = -\frac{\mu_0^4}{4e\pi^2} \left[ x^2 \ln x + \frac{25}{9} x^2 \ln \left( \frac{5x^3}{3} \right) \right] .
\]
(68)

We find a solution when
\[
x = \frac{1}{\sqrt{e}} \left( \frac{3}{5} \right)^{\frac{2e}{3}} \simeq 0.417
\]
(69)
corresponding to a value of
\[
\frac{\Lambda_{n_{\text{c}}}(\mu_0, r)}{8\pi G} = \frac{\mu_0^4}{4e\pi^2} \left( \frac{17}{75} \right) \left( \frac{3}{5} \right)^{3(\frac{2e}{3})} \simeq 0.328 \frac{\mu_0^4}{4e\pi^2} = 0.328 \frac{\Lambda_{n_{\text{c}}}(\mu_0, r)}{8\pi G} .
\]
(70)

This means that
\[
\frac{\Lambda_{n_{\text{c}}}(\mu_0, r)}{\Lambda_{n_{\text{c}}}(\mu_0, r)} = 0.328 < 1 .
\]
(71)

Finally, we spend few words on the trace part contribution, which essentially confirms what has been found in Ref. [8]. Repeating the same procedure for the trace operator, we find for both values of the Schwarzschild mass ($\pm M$), that the only consistent value of finding extrema is that of vanishing $M$. This happens because solutions (60, 69) create a constraint on $M$, $r$ and $\mu_0$ which cannot be simultaneously satisfied for the graviton and for the trace term.

V. SUMMARY AND CONCLUSIONS

In Ref. [8], we considered how to extract information on the cosmological constant using the Wheeler-De Witt equation. In this paper, even if we have applied the same formalism, we have looked at the reversed idea, namely the induced cosmological constant represents a certain amount of energy density which varies with the choice of the underlying background. It is quite natural of thinking to an arrangement of the various induced constants in such a way to have a classification system which looks like to a spectrum of geometries. Note that this method, in principle can be extended beyond the Schwarzschild sector, in such a way to include all the spherically symmetric metrics. Extensions to modified theories of gravity have also been studied only for positive Schwarzschild mass [17]. It is interesting also to note that it is possible to use such method, represented by Eq. (13), not only for the induced cosmological constant, but even for electric or magnetic charges, simply by replacing
\[
\frac{1}{V} \int \mathcal{D}[g_{ij}] \Psi^* \left[ g_{ij} \right] \int \mathcal{D}[g_{ij}] \Psi \left[ g_{ij} \right] = \frac{1}{V} \left\langle \Psi \left| \tilde{f}_{\Sigma} \int d^3x \tilde{\Lambda}_{\Sigma} \Psi \right| \Psi \right\rangle = -\frac{\Lambda_{\Sigma}}{\kappa} .
\]
(72)

\[
\tilde{\Lambda}_{\Sigma} \rightarrow \tilde{Q}_{\Sigma}
\]
(73)

and
\[
-\frac{\Lambda_{\Sigma}}{\kappa} \rightarrow -\frac{1}{2} \int_{\Sigma} d^3x \sqrt{g} \rho_e
\]
(74)
leading to the following eigenvalue equation:

\[
\frac{\left\langle \Psi \left| \int_{\Sigma} d^3x \hat{Q}_{\Sigma} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = -\frac{1}{2} \int_{\Sigma} d^3x \sqrt{g} \rho_c.
\]  

(75)

\(\hat{Q}_{\Sigma}\) is the charge operator containing only the gravitational field. Coming back to the result (71), we can conclude that the Schwarzschild naked singularity has a lower value of ZPE compared to the positive Schwarzschild mass. This means that, even if the order of magnitude is practically the same, the naked singularity is less favored with respect to the Schwarzschild wormhole. A further progress could be the study of the unstable sector in our formalism to better understand the behavior of the naked Schwarzschild background. Indeed, we have studied the spectrum in a W.K.B. approximation with the following condition \(k_i^2 (r, l, \omega_i) \geq 0, i = 1, 2\). Thus to complete the analysis, we need to consider the possible existence of nonconformal unstable modes, like the ones discovered in Refs. [13]. If such an instability appears, this does not mean that we have to reject the solution. In fact in Ref. [19], we have shown how to cure such a problem. In that context, a model of “space-time foam” has been introduced in a large \(N\) wormhole approach reproducing a correct decreasing of the cosmological constant and simultaneously a stabilization of the system under examination. It could be interesting in the context of the multi-gravity to examine what happens for a large number of naked singularities.

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