0. Introduction

In this paper we give a sufficient condition for the tensor product of irreducible finite–dimensional representations of quantum affine algebras to be cyclic. In particular, this proves a generalization of a recent result of Kashiwara [K], and also establishes a conjecture stated in that paper.

We describe our results in some detail. Let $g$ be a complex simple finite–dimensional Lie algebra of rank $n$, and let $U_q$ be the quantized untwisted affine algebra over $C(q)$ associated to $g$. For every $n$–tuple $\pi = (\pi_1, \cdots, \pi_n)$ of polynomials with coefficients in $C(q)[u]$ and with constant term one, there exists a unique (up to isomorphism) irreducible finite–dimensional representation $V(\pi)$ of $U_q$. For each element $w$ in the Weyl group $W$ of $g$, let $v_w$ be the extremal vector defined in [K]. In this paper we compute the action of the imaginary root vectors in $U_q$ on the elements $v_w$. To do this we define in Section 2 an action of the braid group $B$ of $g$ on elements of $(C(q)[u])^n$ and prove that the eigenvalue of $v_w$ is the element $T_w(\pi)$ where $T: W \rightarrow B$ is the canonical section defined in [Bo].

To state our result we assume for simplicity (in the introduction only) that $g$ is simply–laced. We shall again for simplicity, only deal with polynomials in $C(q)[u]$ which split into linear factors. Any such polynomial can be written uniquely as a product

$$\pi(u) = \prod_{r=1}^{k} (1 - a_r q^{m_r - 1}u)(1 - a_r q^{m_r - 3}u) \cdots (1 - a_r q^{m_r - 1}u),$$

where $a_r \in C(q)$ and $m_r \in Z_+$ satisfy

$$\frac{a_r}{a_l} \neq q^{\pm (m_r + m_l - 2m)}, \quad 0 \leq m < \min(m_r, m_l),$$

if $r < l$. Let $S(\pi)$ be the collection of the pairs $(a_r, m_r)$. $1 \leq r \leq k$ defined above. Say that a polynomial $\pi'(u) > \pi(u)$ if

$$\frac{a_r'}{a_l'} \neq q^{m_r' - m_l' - 2p}, \quad 1 \leq p \leq m_l',$n

for all pairs $(a_r', m_r') \in S(\pi')$ and $(a_l, m_l) \in S(\pi)$. Let $s_1, s_2, \cdots, s_n$ be the set of simple reflections in $W$. Our main result says that:

The tensor product $V(\pi') \otimes V(\pi)$ is cyclic on $v_{w} \otimes v_{\pi}$ if, for all $w \in W$ and for all $i = 1, \cdots, n$ with $\ell(s_i w) = \ell(w) + 1$, we have

$$(T_w \pi')_i > \pi_i.$$

More generally, if $V_1, \cdots, V_r$ are irreducible finite–dimensional representations, then $V_1 \otimes \cdots \otimes V_r$ is cyclic, if every pair $V_j \otimes V_i$ is cyclic for all $j < \ell$. 

To make the connection with Kashiwara’s theorem and conjectures, we consider the case
\[ \pi_j(u) = 1 \quad (j \neq i), \quad \pi_i(u) = \prod_{s=1}^{m} (1 - q^{m+1-2s} a u) \quad (a \in C(q)). \]

Denoting this \( n \)–tuple of polynomials as \( \pi^i_{m,a} \) and the corresponding representation by \( V_i(m, a) \), we prove the following: Let \( l \geq 1 \) and let \( i_j \in I, m_j \in \mathbb{Z}_{+}, a_j \in C(q) \) for \( 1 \leq j \leq l \). The tensor product \( V_i(m_1, a_1) \otimes V_i(m_2, a_2) \otimes \cdots \otimes V_i(m_l, a_l) \) is cyclic on the tensor product of highest weight vectors if for all \( r < s \),
\[ \frac{a_r}{a_s} \neq q^{m_r-a_m-p}, \quad \forall \ p \geq 0. \]

The case when \( m_i = 1 \) was originally conjectured and partially proved in [AK] and completely proved in [KV] (and in [KK] for the simply-laced case). The result in the case when the \( m_i \) are arbitrary but \( a_i = 1 \) for all \( i \) was conjectured in [K], [HKOTY].

1. Preliminaries

In this section we recall the definition of quantum affine algebras and several results on the classification of their irreducible finite–dimensional representations.

Let \( q \) be an indeterminate, let \( C(q) \) be the field of rational functions in \( q \) with complex coefficients. For \( r, m \in \mathbb{N}, m \geq r \), define
\[ [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q! = [m]_q[m - 1]_q \cdots [2]_q[1]_q, \quad \left[ m \atop r \right]_q = \frac{[m]_q!}{[r]_q![m-r]_q!}. \]

Let \( g \) be a complex finite–dimensional simple Lie algebra of rank \( n \), let \( I = \{1, 2, \cdots, n\} \), let \( \{\alpha_i : i \in I\} \) be the set of simple roots and let \( \{\omega_i : i \in I\} \) be the set of fundamental weights. Let \( Q^+ \) (resp. \( P^+ \)) be the non–negative root (resp. weight) lattice of \( g \). Let \( A = (a_{ij})_{i,j \in I} \) be the \( n \times n \) Cartan matrix of \( g \) and let \( \hat{A} = (\hat{a}_{ij}) \) be the \((n+1) \times (n+1)\) extended Cartan matrix associated to \( g \). Let \( \hat{I} = I \cup \{0\} \). Fix non–negative integers \( d_i \) (\( i \in \hat{I} \)) such that the matrix \((d_i a_{ij})\) is symmetric. Set \( q_i = q^{d_i} \) and \([m]_i = [m]_q \).

**Proposition 1.1.** There is a Hopf algebra \( \hat{U}_q \) over \( C(q) \) which is generated as an algebra by elements \( E_{\alpha_i}, F_{\alpha_i}, K_i^{-1} \) (\( i \in \hat{I} \)), with the following defining relations:
\[ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i E_{\alpha_i} K_{i}^{-1} = q_i^{a_{ij}} E_{\alpha_i}, \]
\[ K_i F_{\alpha_i} K_{i}^{-1} = q_i^{-a_{ij}} F_{\alpha_i}, \]
\[ [E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} K_i - K_{i}^{-1}. \]
\[ \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_i (E_{\alpha_i})^r E_{\alpha_j} (E_{\alpha_i})^{1-a_{ij}-r} = 0 \quad \text{if} \ i \neq j, \]
\[ \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_i (F_{\alpha_i})^r F_{\alpha_j} (F_{\alpha_i})^{1-a_{ij}-r} = 0 \quad \text{if} \ i \neq j. \]
The comultiplication of $\hat{U}_q$ is given on generators by
\[
\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_i \otimes E_{\alpha_i}, \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_i^{-1} + 1 \otimes F_{\alpha_i}, \quad \Delta(K_i) = K_i \otimes K_i,
\]
for $i \in \hat{I}$.

Set $K_\theta = \prod_{i=1}^{n} K_i^{r_i/d_i}$, where $\theta = \sum r_i \alpha_i$ is the highest root in $R^+$. Let $U_q$ be the quotient of $\hat{U}_q$ by the ideal generated by the central element $K_\theta^{-1}$; we call this the quantum loop algebra of $g$.

It follows from [3], [4], [5] that $U_q$ is isomorphic to the algebra with generators $x_{i,r}^\pm (i \in I, r \in \mathbb{Z})$, $K_i^{\pm 1} (i \in I)$, $h_{i,r} (i \in I, r \in \mathbb{Z}\backslash \{0\})$ and the following defining relations:
\[
K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_i h_{j,r} = h_{j,r} K_i, \\
K_i x_{j,r}^\pm K_i^{-1} = q_i^{-a_{ij}} x_{j,r}^\pm, \\
[h_{i,r}, h_{j,s}] = 0, \quad [h_{i,r}, x_{j,r+s}^\pm] = \pm \frac{1}{r} [r \alpha_{ij}] x_{j,r+s}^\pm, \\
x_{i,r+1}^\pm x_{j,s}^\pm - q_i^{a_{ij} - a_{ji}} x_{j,s}^\pm x_{i,r+1}^\pm = q_i^{a_{ij} - a_{ji}} x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm, \\
x_{i,r+s}^\pm = \delta_{ij} \psi_{i,r+s}^+ - \psi_{i,r+s}^-, \quad \psi_{i,r+s}^+ = \psi_{i,r}^+ \psi_{r+s}^-, \quad \psi_{i,r+s}^- = \psi_{i,r}^- \psi_{r+s}^+.
\]

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, $\Sigma_m$ is the symmetric group on $m$ letters, and the $\psi_{i,r}^\pm$ are determined by equating powers of $u$ in the formal power series
\[
\sum_{r=0}^{\infty} \frac{u^{r+s}}{r!} \psi_{i,r}^\pm u^{s} = K_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,s} u^{s} \right).
\]

For $i \in I$, the preceding isomorphism maps $E_{\alpha_i}$ to $x_{i,0}^+$ and $F_{\alpha_i}$ to $x_{i,0}^-$. The subalgebra generated by $E_{\alpha_i}$, $F_{\alpha_i}$, $K_i^{\pm 1}$ ($i \in I$) is the quantized enveloping algebra $U_{q}^{fin}$ associated to $g$. Let $U_q(<)$ be the subalgebra generated by the elements $x_{i,k}$ ($i \in I, k \in \mathbb{Z}$). For $i \in I$, let $U_i$ be the subalgebra of $U_q$ generated by the elements $\{x_{i,k}^\pm : k \in \mathbb{Z}\}$, the subalgebra $U_i^{fin}$ is defined in the same way. Notice that $U_i$ is isomorphic to the quantum affine algebra $U_{q_i}^{sl_2}$. Let $\Delta_i$ be the comultiplication of $U_{q_i}^{sl_2}$.

An explicit formula for the comultiplication on the Drinfeld generators is not known. Also, the subalgebra $U_i$ is not a Hopf subalgebra of $U_q$. However, partial information which is sufficient for our needs is given in the next proposition. Let
\[
X^\pm = \sum_{i \in I, k \in \mathbb{Z}} C(q) x_{i,k}^\pm, \quad X^\pm (i) = \sum_{j \in I \setminus \{i\}, k \in \mathbb{Z}} C(q) x_{j,k}^\pm.
\]

**Proposition 1.2.** The restriction of $\Delta$ to $U_i$ satisfies,
\[
\Delta(x) = \Delta_i(x) \mod (U_q \otimes (U_q \setminus U_i)).
\]

More precisely:
Clearly any highest weight module has a unique irreducible quotient \( V \neq 0 \).

**Definition 1.1.** We say that a \( U_q \)-module \( V \) is (pseudo) highest weight, with highest weight \((\lambda, h^\pm)\), where \( \lambda = \sum_{i \in I} \lambda_i \omega_i, \ h^\pm = (h^+_i(u), \cdots, h^+_n(u)) \in A^0 \), if there exists \( 0 \neq v \in V_\lambda \) such that \( V = U_q v \) and

\[
x^+_{i,k} v = 0, \quad K_i v = q^{\lambda_i} v, \quad h^+_i(u) v = h^+_i(u) v,
\]
for all \( i \in I, k \in \mathbb{Z} \).

If \( V \) is any highest weight module, then in fact \( V = U_q(\cdot) v \) and so

\[
V_\mu \neq 0 \implies \mu = \lambda - \eta \ (\eta \in Q^+).
\]

Clearly any highest weight module has a unique irreducible quotient \( V(\lambda, h^\pm) \).

The following was proved in \cite{CP2}.

(i) **Modulo** \( U_q X^- \otimes U_q (X^+)^2 + U_q X^- \otimes U_q X^+(i) \), we have

\[
\Delta(x^+_{i,k}) = x^+_{i,k} \otimes 1 + K_i \otimes x^+_{i,k} + \sum_{j=1}^{k} \psi^+_{i,j} \otimes x^+_{i,k-j} \quad (k \geq 0),
\]

\[
\Delta(x^-_{i,k}) = K^{-1}_i \otimes x^+_{i,-k} + x^+_{i,-k} \otimes K^{-1}_i + \sum_{j=1}^{k} \psi^-_{i,-j} \otimes x^+_{i,-k+j} \quad (k > 0),
\]

(ii) **Modulo** \( U_q (X^-)^2 \otimes U_q X^+ + U_q X^- \otimes U_q X^+(i) \), we have

\[
\Delta(x^-_{i,k}) = x^-_{i,k} \otimes K_i + 1 \otimes x^-_{i,k} + \sum_{j=1}^{k-1} x^-_{i,-j} \otimes \psi^+_{i,j} \quad (k > 0),
\]

\[
\Delta(x^-_{i,-k}) = x^-_{i,-k} \otimes K^{-1}_i + 1 \otimes x^-_{i,-k} + \sum_{j=1}^{k} x^-_{i,-k+j} \otimes \psi^-_{i,j} \quad (k \geq 0).
\]

(iii) **Modulo** \( U_q X^- \otimes U_q X^+ \), we have

\[
\Delta(h_{i,k}) = h_{i,k} \otimes 1 + 1 \otimes h_{i,k} \quad (k \in \mathbb{Z}).
\]

**Proof.** Part (iii) was proved in \cite{Da}. The rest of the proposition was proved in \cite{CP2}.

We conclude this section with some results on the classification of irreducible finite-dimensional representations of quantum affine algebras. Let

\[
\mathcal{A} = \{ f \in \mathbb{C}(q)[[u]] : f(0) = 0 \}.
\]

For any \( U_q \)-module \( V \) and any \( \mu = \sum_{i} \mu_i \omega_i \in P \), set

\[
V_\mu = \{ v \in V : K_i v = q^{\mu_i} v, \ \forall i \in I \}.
\]

We say that \( V \) is a module of type 1 if

\[
V = \bigoplus_{\mu \in P} V_\mu.
\]

From now on, we shall only be working with \( U_q \)-modules of type 1. For \( i \in I \), set

\[
h^\pm_i(u) = \sum_{k=1}^{\infty} \frac{h^\pm_{i,k}}{k} u^k.
\]
**Theorem 1.** Assume that the pair \((\lambda, h^\pm) \in P \times A^0\) satisfies the following: \(\lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+\), and there exist elements \(a_{i,r} \in \mathbb{C}(q)\) \((1 \leq r \leq \lambda_i, i \in I)\) such that

\[
h^\pm_i(u) = -\sum_{r=1}^{\lambda_i} \ln(1 - a^{\pm 1}_{i,r} u).
\]

Then, \(V(\lambda, h^\pm)\) is the unique (up to isomorphism) irreducible finite-dimensional \(U_q\)-module with highest weight \((\lambda, h^\pm)\).

**Remark.** This statement is actually a reformulation of the statement in \([\text{CP2}]\). Setting \(\pi(u) = \prod_{i=1}^{\lambda} (1 - a_{i,r} u)\) and calculating the eigenvalues of the \(\psi_{i,k}\) gives the result stated in \([\text{CP2}]\). See also \([\text{CP4}]\).

From now on, we shall only be concerned with the modules \(V(\lambda, h^\pm)\) satisfying the conditions of Theorem \([\text{CP2}]\). In view of the preceding remark, it is clear that the isomorphism classes of such modules are indexed by an \(n\)-tuple of polynomials \(\pi = (\pi_1, \cdots, \pi_n)\), which have constant term 1, and which are split over \(\mathbb{C}(q)\). We shall denote the corresponding module by \(V(\pi)\) and the highest weight vector by \(v_{\pi}\), where \(\lambda = \sum_{i=1}^{\lambda} \deg \pi_i\). For all \(i \in I, k \in \mathbb{Z}\), we have

\[
x^+_{i,k} v_{\pi} = 0, \quad K_i v_{\pi} = q_i^{\deg \pi_i} v_{\pi},
\]

and

\[
\frac{h^\pm_i}{[k]_i} v_{\pi} = h^\pm_i v_{\pi}, \quad (x^-_{i,k})^{\deg \pi_i + 1} v_{\pi} = 0,
\]

where the \(h^\pm_i\) are determined from the functional equation

\[
\exp \left( -\sum_{k \geq 0} h^\pm_i u^k \right) = \pi^\pm_i(u),
\]

with \(\pi^+_i(u) = \pi(u)\) and \(\pi^-_i(u) = u^{\deg \pi_i} \pi_i(u^{-1})/ (u^{\deg \pi_i} \pi_i(u^{-1}))|_{u=0}\).

For any \(U_q\)-module \(V\), let \(V^\ast\) denote its left dual. Let \(- : I \rightarrow I\) be the unique diagram automorphism such that the irreducible \(g\) module \(V(\omega_i) \cong V(\omega_{-i})\). There exists an integer \(c \in \mathbb{Z}\) depending only on \(g\) such that

\[
V(\pi)^\ast \cong V(\pi^\ast), \quad \pi^\ast = (\pi(q^c u), \cdots, \pi(q^c u)).
\]

Analogous statements hold for right duals \([\text{CP3}]\). Recall also, that if a module and its dual are highest weight then they must be irreducible.

Finally, let \(\omega : U_q \rightarrow U_q\) be the algebra automorphism and coalgebra anti-automorphism obtained by extending the assignment \(\omega(x^+_i) = -x^-_{i,-k}\). If \(V\) is any \(U_q\)-module, let \(V^\omega\) be the pull–back of \(V\) through \(\omega\). Then, \((V \otimes V')^\omega \cong (V')^\omega \otimes V^\omega\) and

\[
V(\pi)^\omega = V(\pi^\omega)
\]

where

\[
\pi^\omega = (\pi^-_1(q^c u), \cdots, \pi^-_1(q^c u)),
\]

for a fixed integer \(\kappa\) depending only on \(g\).

We conclude this section with some results in the case when \(g = sl_2\).

**Theorem 2.**
(i) For $a \in \mathbb{C}(q)$, $m \in \mathbb{Z}^+$, set
\[ \pi_{m,a}(u) = \prod_{r=1}^{m}(1 - aq^{m-2r+1}u). \]

The irreducible module $V(m,a)$ with highest weight $\pi_{m,a}$ is of dimension $m$ and is irreducible as a $U_q^{\ell m}$-module.

(ii) For $1 \leq r \leq \ell$, let $a_r \in \mathbb{C}(q)$ and $m_r \in \mathbb{Z}_+$ be such that
\[ r < s \implies \frac{a_r}{a_s} \neq q^{m_r - m_s - 2p} \quad (1 \leq p \leq m_r). \]

The tensor product $V(\pi_{m_1,a_1}) \otimes \cdots \otimes V(\pi_{m_\ell,a_\ell})$ is a highest weight module with highest weight $\pi_{m_1,a_1} \cdots \pi_{m_\ell,a_\ell}$ and highest weight vector $v_{\pi_{m_1,a_1} \cdots \pi_{m_\ell,a_\ell}}$.

(iii) Assume that $a_1, \cdots, a_\ell \in \mathbb{C}(q)$ are such that if $r < s$ then $a_r/a_s \neq q^{-2}$. The module $W(\pi) = V(1,a_1) \otimes \cdots \otimes V(1,a_\ell)$ is the universal finite–dimensional highest weight module with highest weight
\[ \pi(u) = \prod_{r=1}^{\ell}(1 - a_r u), \]

i.e. any other finite–dimensional highest weight module with highest weight $\pi$ is a quotient of $W(\pi)$.

(iv) Assume that $\pi = \prod_{j=1}^{m}(1 - b_j u)$ ($b_j \in \mathbb{C}(q)$) and that $V(m_1,a_1) \otimes \cdots \otimes V(m_\ell,a_\ell)$ is highest weight. Then, the module $W(\pi) \otimes V(m_1,a_1) \otimes \cdots \otimes V(m_\ell,a_\ell)$ is also highest weight if
\[ b_s/a_r \neq q^{1-s}, \]

for all $1 \leq s \leq m$ and $1 \leq r \leq k$.

**Proof.** Part (i) is proved in [CP1]. Part (ii) is proved as in Lemma 4.9 in [CP1]. In fact, the proof given there establishes the stronger result stated here. Part (iii) was proved in [CP4] in the case when $\pi(u) \in \mathbb{C}[q, q^{-1}, u]$. In the general case, choose $v \in \mathbb{C}[q, q^{-1}]$ so that $\tilde{\pi}(u) = \pi(uv)$ has all its roots in $\mathbb{C}[q, q^{-1}]$. Let $\tau_v : U_q \to U_q$ be the algebra and coalgebra automorphism defined by sending $x_k^\pm \to v^k x_k^\pm$. The pull back of $V(\pi)$ through $\tau_v$ is $V(\tilde{\pi})$ and hence $W(\pi(u)) \cong W(\pi(uv))$. This proves (iii). Part (iv) is now immediate.

Throughout this paper, we shall only work with polynomials in $\mathbb{C}(q)[u]$ which are split and have constant term 1. Let $\pi_{m,a} \in \mathbb{C}(q)$ be the polynomial defined in Theorem 3(i). It is a simple combinatorial fact [CP1] that any such polynomial can be written uniquely as a product
\[ \pi(u) = \prod_{j=1}^{s} \pi_{m_j,a_j}, \]

where $m_j \in \mathbb{Z}_+$, $a_j \in \mathbb{C}(q)$ and
\[ j < \ell \implies \frac{a_j}{a_\ell} \neq q^{\pm(m_j + m_\ell - 2p)}, \quad 0 \leq p < \min(m_j, m_\ell). \]
If \( \pi \) and \( \pi' \) are two such polynomials, then we say that \( \pi > \pi' \) if for all \( 1 \leq j \leq s, 1 \leq k \leq s' \), we have
\[
\frac{a_j}{a_k} \neq q^{m_j-m_k'-2p}, \quad 1 \leq p \leq m_j.
\]
This is equivalent to saying that if \( a \) is any root of \( \pi \) and \( 1 \leq k' \leq s' \), then
\[
\frac{a}{a_k'} \neq q^{-1-m_k'}.
\]
Part (iv) of Theorem 2 then says that if \( \pi > \pi' \) the module \( W(\pi) \otimes V(\pi') \) is highest weight.

### 2. Braid group action

Let \( W \) be the Weyl group of \( g \) and let \( B \) be the corresponding braid group. Thus, \( B \) is the group generated by elements \( T_i \ (i \in I) \) with defining relations:
\[
T_iT_j = T_jT_i, \quad \text{if } a_{ij} = 0,
\]
\[
T_iT_jT_i = T_jT_iT_j, \quad \text{if } a_{ij}a_{ji} = 1,
\]
\[
(T_iT_j)^2 = (T_jT_i)^2, \quad \text{if } a_{ij}a_{ji} = 2,
\]
\[
(T_iT_j)^3 = (T_jT_i)^3, \quad \text{if } a_{ij}a_{ji} = 3,
\]
where \( i, j \in \{1, 2, \ldots, n\} \) and \( A = (a_{ij}) \) is the Cartan matrix of \( g \).

A straightforward calculation gives the following proposition.

**Proposition 2.1.** For all \( r \geq 1 \), the formulas
\[
T_i e_j = e_j - q_i^{[ra_{ji}]} e_i
\]
define a representation \( \eta_r : B \to \text{end}(V_r) \), where \( V_r \cong \mathbb{C}(q)^n \) and \( \{e_1, \ldots, e_n\} \) is the standard basis of \( V_r \). Further, identifying
\[
\mathcal{A}^n \cong \prod_{r=1}^{\infty} V_r,
\]
we get a representation of \( B \) on \( \mathcal{A}^n \) given by
\[
(T_i h)_j = h_j(u), \quad \text{if } a_{ji} = 0,
\]
\[
(T_i h)_j = h_j(u) + h_i(q^2u), \quad \text{if } a_{ji} = -1,
\]
\[
(T_i h)_j = h_j(u) + h_i(q^3u) + h_i(qu), \quad \text{if } a_{ji} = -2,
\]
\[
(T_i h)_j = h_j(u) + h_i(q^5u) + h_i(q^3u) + h_i(qu), \quad \text{if } a_{ji} = -3,
\]
\[
(T_i h)_i = -h_i(q_i^2u),
\]
for all \( i, j \in I, h \in \mathcal{A}^n \). \( \square \)

Let \( s_i, i \in I \) be a set of simple reflections in \( W \). For any \( w \in W \), let \( \ell(w) \) be the length of a reduced expression for \( w \). If \( w = s_{i_1}s_{i_2} \cdots s_{i_k} \) is a reduced expression for \( w \), let \( T_w = T_{i_1} \cdots T_{i_k} \). It is well–known that \( T_w \) is independent of the choice of the reduced expression. Given \( h \in \mathcal{A}^n \) and \( w \in W \), we have
\[
T_w h = T_{i_1}T_{i_2} \cdots T_{i_k} h = ((T_w h)_1, \ldots, (T_w h)_n).
\]
We can now prove:
Proposition 2.2. Suppose that \( w \in W \) and \( i \in I \) is such that \( \ell(s_i w) = \ell(w) + 1 \). There exists an integer \( N = N(i, w, h) \geq 0 \) and non-negative integers \( p_{r,j} \) (\( j \in I, 1 \leq r \leq N \)) such that

\[
(T_w h)_i = \sum_{j \in I} \sum_{r=1}^{N} p_{r,j} h_j(q^r u),
\]

Proof. Proceed by induction on \( \ell(w) \): the induction clearly starts at \( \ell(w) = 0 \). Assume that the result is true for \( \ell(w) < k \). If \( \ell(w) = k \), write \( w = s_j w' \) with \( \ell(w') = k - 1 \). Notice that \( j \neq i \) since \( \ell(s_i w) = \ell(w) + 1 \). We get

\[
(T_w h)_i = (T_j T_{w'} h)_i = (T_{w'} h)_i(u) + \sum_{s=0}^{[a_{ij}] - 1} (T_{w'} h)_j(q^{2[a_{ij]} - 2s - 1} u).
\]

If \( \ell(s_j w') = \ell(w') + 1 \), the result follows by induction. If \( \ell(s_j w') = \ell(w') - 1 \), we have \( w = s_j s_i w'' \). Suppose that \( a_{ij} a_{ji} = -1 \). Then, \( \ell(s_j w'') = \ell(w'') + 1 \) and we get

\[
(T_j T_{w''} h)_i = (T_j T_{w''} h)_i + (T_s T_{w''} h)_j(q u) = (T_{w''} h)_j(q u).
\]

The result again follows by induction. The cases when \( a_{ij} a_{ji} = 2, 3 \) are proved similarly. We omit the details. \( \square \)

3. THE MAIN THEOREM

Our goal in this section is to obtain a sufficient condition for a tensor product of two highest weight representations to be highest weight.

Let \( V \) be any highest weight finite-dimensional \( U_q \)–module with highest weight \( \pi \) (or \( (\lambda, h^\pm) \) as in Theorem [1]). For all \( w \in W \), we have

\[
\dim V_{w, \lambda} = 1.
\]

If \( s_i, \ldots, s_k \) is a reduced expression for \( w \), and \( \lambda = \sum \lambda_i \omega_i \), set \( m_k = \lambda_k \) and define non–negative integers \( m_j \) (depending on \( w \)), for \( 1 \leq j \leq k \), by

\[
s_{i,j+1} s_{i,j+2} \cdots s_{i,k} \lambda = m_j \omega_j + \sum_{r \neq j} m_r' \omega_r.
\]

Let \( v_\lambda \) be the highest weight vector in \( V \). For \( w \in W \), set

\[
v_{w, \lambda} = (x_{i_1,0}^-)^{m_1} \cdots (x_{i_k,0}^-)^{m_k} v_\lambda.
\]

If \( i \in I \) is such that \( \ell(s_i w) = \ell(w) + 1 \), then

\[
x_{i,k}^+ v_{w, \lambda} = 0, \quad \forall k \in \mathbb{Z}.
\]

To see this, observe that \( w \lambda + \alpha_i \) is not a weight of \( V \), since \( w^{-1} \alpha_i \in R^+ \) if \( \ell(s_i w) = \ell(w) + 1 \). It is now easy to see that \( v_{w, \lambda} \neq 0 \), \( V_{w, \lambda} = C(q)v_{w, \lambda} \) and

\[
V = U_q v_{w, \lambda}.
\]

Since \( h_{i,k} V_{w, \lambda} \subset V_{w, \lambda} \) for all \( i, k \in \mathbb{Z} \) it follows that

\[
\frac{h_{i,k}}{|k|} v_{w, \lambda} = h_{i,k}^w v_{w, \lambda}, \quad \forall i \in I, \ 0 \neq k \in \mathbb{Z}.
\]
where $h_{i,k}^w \in \mathbb{C}(q)$. Set
\[
\hat{h}_i^w(u) = \sum_{k=1}^{\infty} h_{i,k}^w u^k, \quad h^w = (h_1^w(u), \ldots, h_n^w(u)).
\]
Recall that $h^1 = -(\ln \pi_1(u), \ldots, \ln \pi_n(u))$.

**Proposition 3.1.** If $w \in W$, then
\[
h^w = T_w h^1.
\]

**Proof.** We proceed by induction on $\ell(w)$. If $\ell(w) = 0$ then $w = id$ and the result follows by definition. Suppose that $\ell(w) = 1$, say $w = s_j$. Writing $\lambda = \sum_j \lambda_j \omega_j$, we have $v_{s_j \lambda} = (x_{j,0}^-)^{\lambda_j} v_{\lambda}$. We first show that
\[
h_{j,k}(u).v_{s_j \lambda} = -h_j(q^2_j u)v_{s_j \lambda} = (T_j h_j(u))v_{s_j \lambda}.
\]
The subspace spanned by the elements $\{(x_{j}^-)^r.v_{\lambda} : 0 \leq r \leq \lambda_j\}$ is a highest weight module for $U$, hence it is enough to prove (3.2) for highest weight representations of quantum affine $sl_2$. In fact it is enough to prove it for the module $W(\pi)$ of Theorem 2. Using Proposition 3.2, we see that the eigenvalue of $h_{i,k}$ on the tensor product of the lowest (and the highest) weight vectors is just the sum of the eigenvalues in each representation. This reduces us to the case of the two-dimensional representation, which is trivial.

Next consider the case $\ell(w) = s_i$, with $i \neq j$. Recall that
\[
[h_{i,r}, x_{j,0}^-] = -\frac{[r a_{ij}]}{r} x_{j,r}^-, \quad [h_{j,r}, x_{j,0}^-] = -\frac{[2r]}{r} x_{j,r}^-.
\]
Hence,
\[
h_{i,r}(x_{j,0}^-)^{\lambda_j} h_{i,r} + [h_{i,r}, (x_{j,0}^-)^{\lambda_j}]
\]
\[
= (x_{j,0}^-)^{\lambda_j} h_{i,r} + \frac{[r a_{ij}]}{[2r]} [h_{j,r}, (x_{j,0}^-)^{\lambda_j}].
\]
This gives
\[
\frac{h_{i,r}}{[r]_i}(x_{j,0}^-)^{\lambda_j} v_{\lambda} = h_{i,r}(x_{j,0}^-)^{\lambda_j} v_{\lambda} + \frac{[r a_{ij}]}{([q_j^+ + q_j^-][r]_i)} \frac{h_{j,r}}{[r]_j} (x_{i,0}^-)^{\lambda(h_{i,r})} v_{\lambda},
\]
\[
= h_{i,r}(x_{j,0}^-)^{\lambda_j} v_{\lambda} + \frac{[r a_{ij}]}{([q_j^+ + q_j^-][r]_i)} (-h_{j,r}(x_{j,0}^-)^{\lambda_j} v_{\lambda} + \frac{h_{j,r}}{[r]_j} (x_{j,0}^-)^{\lambda_j} v_{\lambda},
\]
\[
= h_{i,r}(x_{j,0}^-)^{\lambda_j} v_{\lambda} - \frac{[r a_{ij}]}{([q_j^+ + q_j^-][r]_i)} (h_{j,r} + q_j^2 h_{j,r}).(x_{j,0}^-)^{\lambda_j} v_{\lambda},
\]
\[
= (h_{i,r} - q_j^r [r a_{ij}]/[r]_i)(x_{j,0}^-)^{\lambda_j} v_{\lambda}.
\]
This proves the result when $\ell(w) = 1$. Proceeding by induction on $\ell(w)$, write $w = s_j w'$ with $\ell(w') = \ell(w) - 1$. Since $v_{w \lambda} = (x_{j,0}^-)^{m_j} v_{w' \lambda}$ for some $m_j \geq 0$, the inductive step is proved exactly as in the case $\ell(w) = 1$, with $v_{\lambda}$ being replaced by $v_{w' \lambda}$. This completes the proof of the proposition. \[\square\]
Lemma 3.1. Let $V, V'$ be finite-dimensional highest weight representations with highest weights $\pi$ and $\pi'$ and highest weight vectors $v_\lambda$ and $v_{\lambda'}$ respectively. Assume that $v_{w_0 \lambda} \otimes v_{\lambda'} \in U_q(v_\lambda \otimes v_{\lambda'})$. Then, $V \otimes V'$ is highest weight with highest weight vector $v_{\lambda} \otimes v_{\lambda'}$ and highest weight $\pi \pi' = (\pi_1 \pi'_1, \cdots, \pi_n \pi'_n)$.

Proof. It is clear from Proposition 1.2 that the element $v_{\lambda} \otimes v_{\lambda'}$ is a highest weight vector with highest weight $\pi \pi'$. It suffices to prove that

$$V \otimes V' = U_q(v_{\lambda} \otimes v_{\lambda'}).$$

Since $x_{i,k}^- v_{w_0 \lambda} = 0$ for all $i \in I$ and $k \in \mathbb{Z}$, it follows from Proposition 1.2 that

$$\Delta(x_{i,k}^-)(v_{w_0 \lambda} \otimes v_{\lambda'}) = v_{w_0 \lambda} \otimes x_{i,k}^- v_{\lambda'}.$$

Repeating this argument we see that $v_{w_0 \lambda} \otimes V' \subset U_q(v_\lambda \otimes v_{\lambda'})$. Now applying the generators $E_r, F_r$ ($i \in I$) repeatedly, we see that $V \otimes V' \subset U_q(v_\lambda \otimes v_{\lambda'})$. This proves the lemma.

Lemma 3.2. Let $w \in W$ and assume that $i \in I$ is such that $\ell(s_i w) = \ell(w) + 1$. Then, $v_{w \lambda} \otimes v_{\lambda'}$ generates a $U_1$-highest weight module with highest weight $(T_w h)_i, h'_i$.

Proof. This is immediate from Proposition 1.2 and Proposition 3.1.

We can now prove our main result. Given $\pi = (\pi_1, \cdots, \pi_n)$, and $w \in W$, set

$$T_w \pi = (\exp( - (T_w \ln \pi_1(u)))_1, \cdots, \exp( - (T_w \ln \pi_n(u)))_n).$$

Theorem 3. The module $V(\pi_1) \otimes \cdots \otimes V(\pi_r)$ is highest weight if for all $w \in W$ and $i \in I$ with $\ell(s_i w) = \ell(w) + 1$, we have

$$j < \ell \implies (T_w \pi)_j > (\pi)_j.$$

Proof. First observe that by Proposition 2.2, $(T_w \pi)_i$ is indeed a polynomial. If $g = sl_2$, then this is the statement of Theorem 2 (ii). For arbitrary $g$, proceed by induction on $r$. If $r = 1$ there is nothing to prove. Let $r > 1$ and let $V' = V(\pi_2) \otimes \cdots \otimes V(\pi_r)$. Then $V'$ is highest weight module with highest weight vector $v' = v_{\pi_2} \otimes \cdots \otimes v_{\pi_r}$ and highest weight $\pi' = \pi_2 \cdots \pi_r$. Setting $\lambda = (\deg \pi_1, \cdots, \deg \pi_n)$, it is enough by Lemma 3.1 to prove that

$$v_{w \lambda} \otimes v' \in U_q(v_{\pi_1} \otimes v').$$

Writing $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ and using Lemma 3.2, it suffices to prove that for all $1 \leq j \leq N$,

$$v_{s_{i_1} s_{i_2} \cdots s_{i_N} \lambda} \otimes v' \in U_{i_j}(v_{s_{i_j+1} \cdots s_{i_N} \lambda} \otimes v').$$

or equivalently that the $U_{i_j}$-module $U_{i_j} v_{s_{i_1} s_{i_2} \cdots s_{i_N} \lambda} \otimes U_{i_j} v_{\lambda}$ is highest weight. Taking $w = id$, we have $(\pi_1)_{i_j} > (\pi_1)_{i_j}$ if $l < s$. It thus follows from Theorem 2 (ii) that

$$U_{i_j} v' = U_{i_j} v_{\pi_2} \otimes \cdots \otimes U_{i_j} v_{\pi_r}.$$

Since $(T_w \pi_1)_j > (\pi_1)_j$, the result follows from Theorem 2 (iv).
4. Relationship with Kashiwara's results and conjectures

Let us consider the special case when $h$ has the following form,

\[ h_j^x(u) = 0, \quad j \neq i, \quad h_i^x(u) = -\sum_{r=1}^{m} \ln(1 - aq_i^{m-2r+1}u), \]

and denote the corresponding $n$–tuple of power series by $h_{i,m,a}$ and the $n$–tuple of polynomials by $\pi_{i,m,a}$. We shall prove the following result.

**Theorem 4.** Let $i_1, i_2, \ldots, i_l \in I$, $a_1, \ldots, a_l \in C(q)$, $m_1, \ldots, m_l \in Z_+$, and assume that

\[ r < s \quad \Rightarrow \quad \frac{\alpha_r}{\alpha_s} \neq q^{d_{i_1,m_1} - d_{i_2,m_2} - \cdots - d_{i_s,m_s} - p} \quad \forall \quad p \geq 0. \]

Then, the tensor product $V(\pi_{m_1,a_1}^{i_1}) \otimes \cdots \otimes V(\pi_{m_l,a_l}^{i_l})$ is a highest weight module.

Assume the theorem for the moment.

**Remark.** In the special case when $m_j = 1$ for all $j$, it was conjectured in $[\text{AK}]$ that such a tensor product is cyclic if $d_{i_j}/a_l$ does not have a pole at $q = 0$ if $j < l$, and this was proved when $g$ is of type $A_n$ or $C_n$; subsequently, a geometric proof of this conjecture was given in $[\text{VV}]$ when $g$ is simply–laced; a complete proof was given using crystal basis methods in $[\text{K}].$

The following corollary to Theorem 4 was conjectured in $[\text{K}], [\text{HKOTY}].$

**Corollary 4.1.** The tensor product $V = V(\pi_{m_1,a_1}^{i_1}) \otimes \cdots \otimes V(\pi_{m_l,a_l}^{i_l})$ is an irreducible $U_q$–module.

**Proof.** First observe that if $d_{i_1,m_1} \leq d_{i_2,m_2} \leq \cdots \leq d_{i_l,m_l}$ then the tensor product is cyclic by Theorem 4. We claim that it suffices to prove the corollary in the case when $\ell = 2$. For then, by rearranging the factors in the tensor product we can show that both $V$ and its dual are highest weight and hence irreducible. To see that $V = V(\pi_{m_1,a_1}^{i_1}) \otimes V(\pi_{m_2,a_2}^{i_2})$ is cyclic if $d_{i_1,m_1} > d_{i_2,m_2}$, it is enough to show that $V^\omega$ is cyclic, since $\omega$ is an algebra automorphism. Now, $V^\omega = V(\pi_{m_2,a_2}^{i_2}) \otimes V(\pi_{m_1,a_1}^{i_1})$ for some fixed $v$ depending only on $g$, and this is cyclic by the theorem. This proves the result.

\[ \square \]

It remains to prove the theorem, for which we must show that, if $j < l$ and $\ell(s_i u) = \ell(u) + 1$, then

\[ (T_{w_i} \pi_{m_1,a_1}^{i_j})_i \geq (\pi_{m_1,a_1}^{i_j})_i. \]

Using Proposition 2.2, we see that

\[ (T_{w_i} \pi_{m_1,a_1}^{i_j})_i = \prod_{r \geq 0} \pi_{m_1,i}^{r} (q^r u), \]

where $r$ varies over a finite subsubset of $Z_+$ with multiplicity. This means that any root of $(T_{w_i} \pi_{a_1,a_1})_i$ has the form $q^{d_{i_j}(m_j - 2p + 1) + r} a_j$ where $r \geq 0$ and $1 \leq p \leq m_j$, and hence, using the assumption on $a_j/a_l$, that

\[ \frac{q^{d_{i_j}(m_j - 2p + 1) + r} a_j}{a_l} \neq q^{1-m_l}. \]

This proves (4.1) and the proof of the theorem is complete.
REFERENCES

[AK] T. Akasaka and M. Kashiwara, Finite-dimensional representations of quantum affine algebras, *Publ. Res. Inst. Math. Sci.* **33** (1997), no. 5, 839-867.

[B] J. Beck, Braided group action and quantum affine algebras, *Commun. Math. Phys.* **165** (1994), 555-568.

[BCP] J. Beck, V. Chari and A. Pressley, An algebraic characterization of the affine canonical basis, *Duke Math. J.* **99** (1999), no. 3, 455-487.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4,5,6, Hermann, Paris (1968).

[CP1] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991), 261-283.

[CP2] V. Chari and A. Pressley, Quantum affine algebras and their representations, in Representations of Groups, (Banff, AB, 1994), 59-78, *CMS Conf. Proc.* **16**, AMS, Providence, RI, 1995.

[CP3] V. Chari and A. Pressley, Minimal affinizations of representations of quantum groups: the simply-laced case, *J. Alg.* **184** (1996), no. 1, 1-30.

[CP4] V. Chari and A. Pressley, Weyl modules for classical and quantum affine algebras, preprint [math.qa/0004174].

[Da] I. Damiani, *La R-matrice pour les algèbres quantiques de type affine non tordu*, preprint.

[Dr] V.G. Drinfeld, A new realization of Yangians and quantum affine algebras. *Soviet Math. Dokl.* **36** (1988), 212-216.

[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on the fermionic formula, *Contemp. Math* **248** (1999).

[J] N. Jing, On Drinfeld realization of quantum affine algebras. The Monster and Lie algebras (Columbus, OH, 1996), pp. 195-206, *Ohio State Univ. Math. Res. Inst. Publ.* **7**, de Gruyter, Berlin, 1998.

[K] M. Kashiwara, On level zero representations of quantized affine algebras, [math.qa/0010293].

[VV] M. Varagnolo and E. Vasserot, Standard modules for quantum affine algebras, preprint.

VYJAYANTHI CHARI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521.