On a 4-dimensional subalgebra of the 12-tone Equal Tempered Tuning

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Abstract

An operation of associative, commutative and distributive multiplication on Euclidean vector space \( \mathbb{E}_4 \) is introduced by a skew circulant matrix. The resulting algebra \( \mathcal{W} \) over \( \mathbb{R} \) is isomorphic to \( \mathbb{C} \times \mathbb{C} \). The related algebraic, geometrical, and topological properties are given. There are subplanes of \( \mathcal{W} \) isomorphic to the Gauss and Clifford complex number planes. A topology on \( \mathcal{W} \) is given by a norm which is a sum of two norms. A hint how to apply this 4 dimensional algebra over \( \mathbb{R} \) to the 12-tone Equally Tempered Tuning algebra is given.

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1 Musical motivation and introduction

The 12-tone Equal Temperament Tuning, 12-TET, of musical instruments is generally and widely used today in music. It is arose from the European aesthetic models of harmony; from the structure of overtones which are automatically physically generated in the process of vibration of every material
string (or, equivalently, of vibrations of a column of air in every pipe); and of
cause, from the mathematical number theory discoveries. A real multiplicative
number interval \([1, 2)\) is multiplicatively divided by number 12, so there
are 12 tones with frequencies
\[
\{a_0(\frac{\sqrt{2}}{12})^{0+12k}, a_0(\frac{\sqrt{2}}{12})^{1+12k}, \ldots, a_0(\frac{\sqrt{2}}{12})^{p+12k}\},
\]
p = 0, 1, 2, \ldots, 11. This group of tones is called an \((k-)\) octave (in the
musical theory). There are used 8 octaves in music, the lower and upper
octaves are not complete 12 in the audio range of human ear in the standard
keyboard of clavier instruments. The octave names are: Subcontra, Contra,
Great, Small, One-line, Two-line, Three-line, Four-line, (theoretically, Five-
line, etc.). The value \(a_0\) is usually a camertone which has the frequency of
sound 440 Hz today.

A mathematics in this 12-tone Equal Temperament Tuning is locally sim-
ilar to a structure of a vector algebra in the middle area of the audio dia-
pason. The two vector operations have the following musical sense. Melody
is bounded with a role of vector addition, while harmony is bounded with
the operation of multiplication of vectors (tones). An operation expressing
the loudness of the tone plays a role of the mixed operation of multiplication
vectors by scalars. Thus, every European tonal music can be mathematically
understood as a special coding of pictures in an algebra (over \(\mathbb{R}\)) which ap-
pear in the space of 12-TET tone (vector) space. It is an algebra over the
real line acting in time. In other words, it is a kind of "movie" which we can
hear (not to see).

Musical sound (= tone) is mathematically modelled for a string (equiv-
ally, a column of air in the pipe) as a Fourier series. An operation of
addition of tones is reflected as a sum of tones. It is good observable when
combining organ registers.

The definition of the operation of multiplication in music is defined by a
special way, which seems rather artificial from a superficial view. Not going
into details, the definition of it is commonly know among musicians as a
spiral of fifths. Mathematically, an operation of multiplication of vectors is
defined by a skew circulated matrices.

Let us note the following four psycho-acoustical phenomena which are
present/asked from the multiplication operation:

1. The task of freely transposition, every melody may be be started from
an arbitrary tone; Temperaments which are not 12-TET do not have this
property.
2. All tones which have a octave-shifted frequency are mentioned harmonically to be equal; this musical phenomenon is called the octave equivalence.

3. A human ear do not psycho-physically distinguish individual tones $T$ and $(-T)$ although there exists a real physical situation when $T \oplus (-T) = 0$. For instance, two sounding equal organ pipes with opposite standing cutout slits produce together physically a zero sound (the two sound waves annihilate one other). The loudness of tones (multiplication by scalar) is given by amplitude of tone vibrations, it is denoted as a usual multiplication by a real number.

4. In the case of the 12-tone Equal Temperament, the spiral of fifths is approximated (deformed) to the circle of fifth, all semitones are equal, hence the name of the tuning.

These phenomena are taken into account when introducing the definition of the operation of multiplication in the algebra. To compare these requirements with the resulting formalized mathematical working definition, cf. Definition [1].

For a large explanatory mathematical and also music-theoretical context, cf. a book [6]. From this book it follows that all 12-tone Equal Temperament music is about an humanly acceptable appropriate psycho-acoustical mathematical approximation of musical sound.

When considering every tonal music as a movie of objects we need to know the arithmetic in this 12-dimensional algebra over $\mathbb{R}$. Firstly, we are able to describe the algebra of the dimension 4 which is a subalgebra of the 12-TET algebra (in a musical terminology, the basic tones are $c, dis, fis, a$). The further subalgebra of the 12-TET are the whole-tone algebra (in a musical terminology, the basic tones are $c, d, e, fis, gis, ais$), to be complete, there is also a triton subalgebra of 12-TET (in a musical terminology, the basic tones are $c, fis$) which has rather no importance in music.

In the accord to the previous motivations, an 4-dimensional algebra $W$ over $\mathbb{R}$ (a subalgebra of 12-TET, isomorphically described) is investigated in this paper. We give the related algebraic, geometrical, and topological properties of this algebra. In Chapter 2 we introduce associative, commutative and distributive multiplication on $E_4$. As in a Theorem [2] this algebra can be seen as a generalization of complex numbers.

A well known extension of the set of Gaussian complex numbers to four dimensions is a algebra of quaternions described by W. R. Hamilton in 1843, [8]. A feature of quaternions in general is that they form a noncommutative division algebra over $\mathbb{R}$. In [10], the subset of commutative quaternions
was studied. Unfortunately, we cannot use the subalgebra of commutative quaternions because a skew circular structure is needed by musicians.

All definitions and notions about algebras over fields used in this paper one can find for example in [5]. We use Hankel, Toeplitz and skew circulant matrices (see e.g. [2]). Recall that $n \times n$ matrix $T = (\tau_{j,k})_{j,k=0}^{n-1}$ is said to be Toeplitz if each descending diagonal from left to right is constant i.e., $\tau_{j,k} = \tau_{j-k}$. The skew-circulant matrix is a special cases of Toeplitz matrix. A $n \times n$ matrix $S = (\sigma_{j,k})_{j,k=0}^{n-1}$ is said to be skew-circulant if $\sigma_{j,k} = \sigma_{j-k}$ and $\sigma_{-l} = -\sigma_{n-l}$ for $1 \leq l \leq n-1$. Skew circulant matrices have applications to various disciplines (see [3, 12, 13, 14, 15], and the references given there).

A square matrix in which each ascending skew-diagonal from left to right is constant is know as Hankel matrix.

In Chapter 2, in order to describe our 4-dimensional algebra $W$ over $\mathbb{R}$, we give its isomorphisms to known mathematical objects. In Theorem 3 we indicate matrix representation of elements $W$ where we use skew circulant matrices. In Theorem 9 it is shown that the algebra $W$ is isomorphic to $\mathbb{C} \times \mathbb{C}$. In Theorem 10 we indicate the method of finding invertible elements and its inverse. Subplanes in $W$ isomorphic to the Gaussian and Clifford complex numbers are indicated in Chapter 3 and Chapter 4.

2 Operation of multiplication. Isomorphisms of algebras.

Let us consider the Euclidean vector space $E_4$ over $\mathbb{R}$ together with the standard vector addition, denoted $\oplus$, and scalar multiplication. So that each element $x \in E_4$ is expressible

\[ x = X_1 \ 1 \oplus X_i \ i \oplus X_j \ j \oplus X_k \ k, \]

as a linear combination of the standard basis elements,

\[ 1 = (1, 0, 0, 0), \ i = (0, 1, 0, 0), \ j = (0, 0, 1, 0), \ k = (0, 0, 0, 1), \]

where $X_1, X_i, X_j, X_k \in \mathbb{R}$.

We use also the notation: $x = (X_1, X_i, X_j, X_k)$ and $\Lambda = (0, 0, 0, 0)$. A sign $\oplus$ denotes the inverse group operation to $\oplus$ in $E_4$.

Let us define an operation of multiplication in $E_4$ as follows:
Definition 1 Let \( x = (X_1, X_i, X_j, X_k), \ y = (Y_1, Y_i, Y_j, Y_k) \in \mathbb{E}_4 \). Then
\[
x \otimes y \overset{\text{def}}{=} (X_1 Y_1 - X_i Y_k - X_j Y_j - X_k Y_j) 1 \oplus (X_1 Y_1 + X_i Y_1 - X_j Y_k - X_k Y_j) i \\
\oplus (X_1 Y_j + X_i Y_i + X_j Y_1 - X_k Y_k) j \oplus (X_1 Y_k + X_i Y_j + X_j Y_1 + X_k Y_1) k \in \mathbb{E}_4.
\]

Let us denote by \( \mathbb{W} \) the space \( \mathbb{E}_4 \) equipped with the operation of multiplication \( \otimes \).

The proof of the following theorem is trivial and therefore omitted.

Theorem 1 The set \( \mathbb{W} \) is an associative, commutative algebra over a field \( \mathbb{R} \) with the multiplicative identity \( 1 = (1, 0, 0, 0) \).

Remark 1 A topology on \( \mathbb{W} \) is given with a standard Euclidean norm
\[
|||x||| = \sqrt{X_1^2 + X_i^2 + X_j^2 + X_k^2} = \sqrt{<x, x>},
\]
implied from the scalar product \( <x, y> = \frac{||x \otimes y||^2 - ||x \otimes y||^2}{4} \), where \( x, y \in \mathbb{W} \).

Definition 1 implies that multiplication \( \otimes \) of the basis elements \( \mathbb{E}_4 \) forms a skew circulated Hankel matrix.

\[
\otimes \begin{array}{cccc}
1 & i & j & k \\
1 & i & j & k \\
i & i & j & k \\
j & j & k & -1 & -i & k & k & -1 & -i & -j
\end{array}
\]

Algebra \( \mathbb{W} \) belongs to a class of algebras which are generalization of complex numbers in \( \mathbb{R}^n \) spaces. For example N. Fleury et al. in [4], considered an \( n \)-dimensional commutative algebra generated by \( n \) vectors \( 1, e, \ldots, e^{n-1} \) where the fundamental element satisfies the basis relation \( e^n = -1 \). Detailed investigations for this type of 3-dimensional algebra were carried out, for example in [1, 7, 11]. Algebra \( \mathbb{W} \) is a system of real numbers generated by element \( e \), which satisfies the basic relation \( e^4 = -1 \).

Theorem 2 The algebra \( \mathbb{W} \) is isomorphic to the algebra \( \mathbb{T} \) over \( \mathbb{R} \) with the basis \( \{1_\mathbb{T}, e, e^2, e^3\} \), where \( e^4 = -1_\mathbb{T} \).

\[
\cdot \begin{array}{cccc}
1_\mathbb{T} & e & e^2 & e^3 \\
1_\mathbb{T} & e & e^2 & e^3 \\
e & e & e^2 & e^3 & -1_\mathbb{T} \\
e^2 & e^2 & e^3 & -1_\mathbb{T} & -e \\
e^3 & e^3 & -1_\mathbb{T} & -e & -e^2
\end{array}
\]
Proof. The isomorphism on the elements is as follows: $1 \leftrightarrow 1, i \leftrightarrow e, j \leftrightarrow e^2, k \leftrightarrow e^3$. □

Let $S$ be an algebra of skew circulant $4 \times 4$ Toeplitz matrices over $\mathbb{R}$, i.e.,

$$S = \left\{ \begin{bmatrix} a & b & c & d \\ -d & a & b & c \\ -c & -d & a & b \\ -b & -c & -d & a \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

with standard matrix operations. We can establish matrix representation of algebra $\mathbb{W}$ as follows.

**Theorem 3** Let $x = (X_1, X_i, X_j, X_k) \in \mathbb{W}$, where $X_1, X_i, X_j, X_k \in \mathbb{R}$. Let $\psi$ be a following bijection between $\mathbb{W}$ and $S$:

$$\mathbb{W} \ni (X_1X_i, X_j, X_k) \mapsto \begin{bmatrix} X_1 & X_i & X_j & X_k \\ -X_k & X_1 & X_i & X_j \\ -X_j & -X_k & X_1 & X_i \\ -X_i & -X_j & -X_k & X_1 \end{bmatrix} \in S,$$

Then $\psi$ is an isomorphism of algebras $\mathbb{W}$ and $S$.

Proof. The proof of the theorem is trivial and left to the reader. □

The algebra $\mathbb{W}$ is isomorphic with some polynomial algebra. The following statement establishes this correspondence.

**Theorem 4** Let $x = (X_1, X_i, X_j, X_k) \in \mathbb{W}$ where $X_1, X_i, X_j, X_k \in \mathbb{R}$. Let $\mathbb{R}[y]$ be an algebra of polynomials in one variable $y$ and coefficients from $\mathbb{R}$, and let $R = \mathbb{R}[y]/(y^4 + 1)$ be its homomorphic image. Then $\varphi(x) = X_1 + X_iy + X_jy^2 + X_ky^3$ is an isomorphism of the algebra $\mathbb{W}$ and the algebra $R$.

Proof. Since every element of the ring $R$ can be expressed uniquely as $a + by + cy^2 + dy^3$ where $a, b, c, d \in \mathbb{R}$ the proof follows from Theorem 2 and simple calculations. □
3 ▲-conjugation in \( \mathbb{W} \)

**Definition 2** Let \( x = X_1 \mathbb{1} \oplus X_i \mathbb{i} \oplus X_j \mathbb{j} \oplus X_k \mathbb{k} \in \mathbb{W} \), where \( X_1, X_i, X_j, X_k \in \mathbb{R} \). Then
\[
(x)^\triangle \overset{\text{def}}{=} X_1 \mathbb{1} \oplus X_k \mathbb{i} \oplus X_j \mathbb{j} \oplus X_i \mathbb{k} \in \mathbb{W}.
\]
We will call this unary operation to be a \(^\triangle\text{-conjugation}\). We will write simply: \((x)^\triangle = x^\triangle\).

Another unary function is \( x \mapsto x \otimes x^\triangle \in \mathbb{W} \). Clearly, \( x \otimes x^\triangle = x \).

**Theorem 5** If \( x = X_1 \mathbb{1} \oplus X_i \mathbb{i} \oplus X_j \mathbb{j} \oplus X_k \mathbb{k} \in \mathbb{W} \), then
\[
x \otimes x^\triangle = A(x) \mathbb{1} \oplus B(x) \theta,
\]
where
\[
A(x) \overset{\text{def}}{=} X_1^2 + X_i^2 + X_j^2 + X_k^2,
\]
\[
B(x) \overset{\text{def}}{=} X_1 X_i + X_i X_j + X_j X_k - X_k X_1,
\]
\[
\theta \overset{\text{def}}{=} i \ominus k = (0,1,0,-1),
\]
and \( X_1, X_i, X_j, X_k \in \mathbb{R} \).

**Proof.**
\[
x \otimes x^\triangle = (X_1^2 + X_i^2 + X_j^2 + X_k^2) \mathbb{1}
\]
\[
\oplus (X_1 X_i + X_i X_j + X_j X_k) \mathbb{i} \oplus (X_k X_1)(-\mathbb{i})
\]
\[
\oplus (X_1 X_j + X_i X_k) \mathbb{j} \oplus (X_1 X_j + X_i X_k)(-\mathbb{j})
\]
\[
\oplus (X_k X_1) \mathbb{k} \oplus (X_1 X_i + X_i X_j + X_j X_k)(-\mathbb{k})
\]
\[
= (X_1^2 + X_i^2 + X_j^2 + X_k^2) \mathbb{1} \oplus (X_1 X_i + X_i X_j + X_j X_k - X_k X_1) \theta.
\]

So,
\[
x \otimes x^\triangle = A(x) \mathbb{1} \oplus B(x) \theta. \quad \square
\]

**Remark 2** From Theorem 5 it follows that the function \( x \mapsto x \otimes x^\triangle \) maps elements of the four dimensional space \( \mathbb{W} \) to a two-dimensional Euclidean space with a basis \( \{ \mathbb{1}, \theta \} \).
The following lemma shows that $\lhd$-conjugation is a homomorphism on $\mathbb{W}$.

**Lemma 1** If $x = (x_1, x_i, x_j, x_k) \in \mathbb{W}$, $y = (y_1, y_i, y_j, y_k) \in \mathbb{W}$ and $\lambda \in \mathbb{R}$, then

$$
(x \oplus y)\lhd = x\lhd \oplus y\lhd, \quad (x \otimes y)\lhd = x\lhd \otimes y\lhd.
$$

and

$$(\lambda x)\lhd = \lambda x\lhd$$

**Proof.** The proof follows from a direct calculations. We omit the details.

A linear subspace in $\mathbb{W}$, a plane spanned by two vectors $1$ and $\theta$, has particular properties from the reduction of operation $\otimes$ to the plane.

**Theorem 6** Let $\pi\lhd \text{ def } \{y = x \otimes x\lhd \mid x \in \mathbb{W}\}$ be a plane defined with the triple of points $(\Lambda, 1, \Theta)$, where

$$\Theta \text{ def } \frac{\theta}{\sqrt{2}}.$$

is a squeezed vector $\theta$. Let the plane $\pi\lhd$ be equipped with the operation $\boxtimes$ which is a reduction of the operation $\otimes$ from the whole space $\mathbb{W}$ to its subplane $\pi\lhd$.

Then the operation $\boxtimes$ is defined by a table

$$
\begin{array}{ccc}
1 & 1 & \Theta \\
1 & 1 & \Theta \\
\Theta & \Theta & 1
\end{array}
$$

on the plane $\pi\lhd$ and the plane $\pi\lhd$ is isomorphic to the hyperbolic (Clifford) complex plane with the „real unit“ $1$ and the „imaginary unit“ $\Theta$, respectively.

**Proof.** For a theory of generalized complex numbers in the plane, see [9].

The points $(\Lambda, 1, \Theta)$ are non-collinear and mutually different. Therefore, they define a non-degenerate plane $\pi\lhd = \text{span}_\mathbb{R}\{1, \Theta\}$.

To prove that the plane $\pi\lhd$ equipped with the operation $\boxtimes$ is isomorphic to the hyperbolic complex plane it is enough to show that $\Theta \boxtimes \Theta = 1$. Indeed,

$$
\Theta \boxtimes \Theta = \left(\frac{i \otimes k}{\sqrt{2}}\right) \boxtimes \left(\frac{i \otimes k}{\sqrt{2}}\right) = \left(\frac{i \otimes k}{\sqrt{2}}\right) \otimes \left(\frac{i \otimes k}{\sqrt{2}}\right) = \frac{1}{2}(j \oplus 1 \oplus 1 \ominus j) = \frac{j \ominus j}{2} \oplus 1 = \Lambda \oplus 1 = 1.
$$

The operation of addition is satisfied trivially in the plane $\pi\lhd$ since it is a linear subspace of the Euclidean space $\mathbb{E}_4$. □
4 Values $A(x) + B(x)$ and $A(x) - B(x)$

Definition 3 Let $x = X_1 1 \oplus X_i i \oplus X_j j \oplus X_k k \in W$, where $X_1, X_i, X_j, X_k \in \mathbb{R}$.

$$A(x) \overset{def}{=} A(x) = X_1^2 + X_i^2 + X_j^2 + X_k^2,$$

$$B(x) = \sqrt{2} B(x) \overset{def}{=} \sqrt{2} (X_1 X_i + X_i X_j + X_j X_k - X_k X_1)$$

and

$$D_\oplus \overset{def}{=} \{x \in W | A(x) - B(x) = 0\},$$

$$D_\ominus \overset{def}{=} \{x \in W | A(x) + B(x) = 0\}.$$

It can be trivially seen that

Lemma 2

If $x \in D_\oplus$ and $x \neq \Lambda$, then $x \notin D_\ominus$. If $x \in D_\ominus$ and $x \neq \Lambda$, then $x \notin D_\oplus$.

Lemma 3 The following equalities are satisfied for every $x \in W$:

$$4[A(x) - B(x)] = (1 - \sqrt{2})[(X_1 + X_i)^2 + (X_i + X_j)^2 + (X_j + X_k)^2 + (X_k - X_1)^2]$$

$$+ (1 + \sqrt{2})[(X_1 - X_i)^2 + (X_i - X_j)^2 + (X_j - X_k)^2 + (X_k + X_1)^2];$$

and

$$4[A(x) + B(x)] = (1 + \sqrt{2})[(X_1 + X_i)^2 + (X_i + X_j)^2 + (X_j + X_k)^2 + (X_k - X_1)^2]$$

$$+ (1 - \sqrt{2})[(X_1 - X_i)^2 + (X_i - X_j)^2 + (X_j - X_k)^2 + (X_k + X_1)^2].$$

Proof. In both equalities it is sufficient to simplify the right hand side of equality to obtain the left one. The details are left to the reader. \qed

Lemma 4 Let $x = X_1 1 \oplus X_i i \oplus X_j j \oplus X_k k \in W$. Then

$$A(x) - B(x) = \left(\frac{X_1}{\sqrt{2}} - X_i + \frac{X_j}{\sqrt{2}}\right)^2 + \left(\frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} + X_k\right)^2.$$
Proof.

\[
\left( \frac{X_1}{\sqrt{2}} - X_i + \frac{X_j}{\sqrt{2}} \right)^2 + \left( \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} + X_k \right)^2
\]

\[
= \frac{X_1^2}{2} + X_i^2 + \frac{X_j^2}{2} - \sqrt{2}X_1X_i + X_1X_j - \sqrt{2}X_jX_i
\]

\[
+ \frac{X_i^2}{2} + X_k^2 + \frac{X_j^2}{2} - X_1X_j + \sqrt{2}X_1X_k - \sqrt{2}X_jX_k = A(x) - B(x). \quad \square
\]

Lemma 5 Let \( x = X_1 1 \oplus X_i \oplus X_j \oplus X_k \in W \). Then

\[
A(x) + B(x) = \left( \frac{X_1}{\sqrt{2}} + X_i + \frac{X_j}{\sqrt{2}} \right)^2 + \left( \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} - X_k \right)^2.
\]

Proof.

\[
\left( \frac{X_1}{\sqrt{2}} + X_i + \frac{X_j}{\sqrt{2}} \right)^2 + \left( \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} - X_k \right)^2
\]

\[
= \frac{X_1^2}{2} + X_i^2 + \frac{X_j^2}{2} + \sqrt{2}X_1X_i + X_1X_j + \sqrt{2}X_jX_i
\]

\[
+ \frac{X_i^2}{2} + X_j^2 + X_k^2 - X_1X_j - \sqrt{2}X_1X_k + \sqrt{2}X_jX_k = A(x) + B(x) \quad \square
\]

From these two lemmas we obtain explicit general expressions for \( D_\oplus \) and \( D_\ominus \).

Theorem 7 Let

\[
D_\oplus = \left\{ \left( \alpha, \frac{\alpha + \beta}{\sqrt{2}}, \beta, \frac{-\alpha + \beta}{\sqrt{2}} \right) \in W \mid \alpha, \beta \in \mathbb{R} \right\}.
\]

Then

\[
D_\ominus = \text{span}_{\mathbb{R}} \{ 1_{D_\oplus}, i_{D_\oplus} \},
\]

where

\[
1_{D_\oplus} \overset{def}{=} \left( \frac{1}{2}, \frac{1}{\sqrt{8}}, 0, -\frac{1}{\sqrt{8}} \right), \quad i_{D_\oplus} \overset{def}{=} \left( 0, \frac{1}{\sqrt{8}}, \frac{1}{2}, \frac{1}{\sqrt{8}} \right),
\]

and

\[
A(x) = B(x) = 2(\alpha^2 + \beta^2) \text{ for } x \in D_\oplus.
\]
Similarly, let
\[ D_\oplus = \left\{ \left( \gamma, -\frac{\gamma + \delta}{\sqrt{2}}, \delta, \frac{\gamma - \delta}{\sqrt{2}} \right) \in W \mid \gamma, \delta \in \mathbb{R} \right\}. \]

Then \( D_\oplus = \text{span}_\mathbb{R}\{1_{D_\oplus}, i_{D_\oplus}\} \), where
\[ 1_{D_\oplus} \overset{\text{def}}{=} \left( \frac{1}{2}, -\frac{1}{\sqrt{8}}, 0, \frac{1}{\sqrt{8}} \right), \quad i_{D_\oplus} \overset{\text{def}}{=} \left( 0, -\frac{1}{\sqrt{8}}, \frac{1}{2}, -\frac{1}{\sqrt{8}} \right). \]

and
\[ A(x) = -B(x) = 2(\gamma^2 + \delta^2) \text{ for } x \in D_\oplus. \]

Proof. Let \( x = (X_1, X_i, X_j, X_k) \in D_\oplus \). By definition of \( D_\oplus \) and Lemma 4,
\[ \frac{X_1}{\sqrt{2}} - X_1 + \frac{X_i}{\sqrt{2}} = 0, \quad \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} + X_k = 0. \]

Putting \( \alpha = X_1 \) and \( \beta = X_j \) we obtain
\[ x = \left( \alpha, \frac{\alpha + \beta}{\sqrt{2}}, \beta, -\frac{\alpha + \beta}{\sqrt{2}} \right). \]

In particular, substituting \( \alpha = \frac{1}{2}, \beta = 0 \) and then \( \alpha = 0, \beta = \frac{1}{2} \) we obtain:
\[ D_\oplus = \text{span}_\mathbb{R}\left\{ \left( \frac{1}{2}, \frac{1}{\sqrt{8}}, 0, -\frac{1}{\sqrt{8}} \right), \left( 0, \frac{1}{\sqrt{8}}, \frac{1}{2}, \frac{1}{\sqrt{8}} \right) \right\}. \]

The formula
\[ A(x) = B(x) = 2(\alpha^2 + \beta^2), \quad \alpha, \beta \in \mathbb{R} \]
for \( x \in D_\oplus \) follows directly from Definition 3 and a trivial computation.

Analogous description for \( D_\ominus \) can be obtained by Lemma 5. \( \square \)

Theorem 8

(i) The linear space \( D_\oplus \) with multiplication \( \otimes \) and with an identity element \( 1_{D_\oplus} \) forms a commutative algebra over field \( \mathbb{R} \).

(ii) The linear space \( D_\ominus \) with multiplication \( \otimes \) and with an identity element \( 1_{D_\ominus} \) forms a commutative algebra over field \( \mathbb{R} \).
Proof.

(i) Elements $1_{D\oplus}$ and $i_{D\oplus}$ are basis elements of $D\oplus$ and their multiplication table is as follows:

| ⊗ | $1_{D\oplus}$ | $i_{D\oplus}$ |
|---|---|---|
| $1_{D\oplus}$ | $1_{D\oplus}$ | $i_{D\oplus}$ |
| $i_{D\oplus}$ | $i_{D\oplus}$ | $-1_{D\oplus}$ |

Since multiplication in $D\oplus$ is determined uniquely by multiplication of basis elements, we obtain that $D\oplus$ is multiplicatively closed set. It is evident that $1_{D\oplus}$ is an identity element in $D\oplus$.

(ii) The multiplication of basis elements $1_{D\ominus}$ and $i_{D\ominus}$ in $D\ominus$ is given by the table

| ⊗ | $1_{D\ominus}$ | $i_{D\ominus}$ |
|---|---|---|
| $1_{D\ominus}$ | $1_{D\ominus}$ | $i_{D\ominus}$ |
| $i_{D\ominus}$ | $i_{D\ominus}$ | $-1_{D\ominus}$ |

Hence $D\ominus$ is multiplicatively closed set. It is clear that $1_{D\ominus}$ is an identity element in $D\ominus$. \[\square\]

**Lemma 6** Let $x \in D\oplus$ and $y \in D\ominus$. Then $x \otimes y = \Lambda$.

**Proof.** There exist $\alpha, \beta \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}$ such that

$x = \left( \alpha, \frac{\alpha + \beta}{\sqrt{2}}, \frac{-\alpha + \beta}{\sqrt{2}} \right)$, $y = \left( \gamma, -\frac{\gamma + \delta}{\sqrt{2}}, \frac{\gamma - \delta}{\sqrt{2}} \right)$.

By a simply verification by Definition 1, we obtain $x \otimes y = \Lambda$. \[\square\]

**Theorem 9** The spaces $D\oplus$, $D\ominus$ are ideals in $W$, and $W = D\oplus \times D\ominus$, as a direct sum of ideals. Moreover, $D\oplus$ is isomorphic to $\mathbb{C}$ and also $D\ominus$ is isomorphic to $\mathbb{C}$. Therefore, $W$ is isomorphic to $\mathbb{C} \times \mathbb{C}$.

**Proof.** Since

$$
\begin{vmatrix}
\frac{1}{2} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} \\
0 & \frac{1}{\sqrt{8}} & \frac{1}{2} & \frac{1}{\sqrt{8}} \\
\frac{1}{2} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} \\
0 & -\frac{1}{\sqrt{8}} & \frac{1}{2} & -\frac{1}{\sqrt{8}}
\end{vmatrix} \neq 0,
$$

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this proves that \( \Lambda = D_\oplus \cap D_\ominus \) and \( \mathcal{W} = D_\oplus + D_\ominus \), as a direct sum of linear spaces.

By Lemma 6 and Theorem 8, \( D_\oplus \) and \( D_\ominus \) are ideals in \( \mathcal{W} \) and \( \mathcal{W} = (D_\oplus \times D_\ominus) \). It remains to prove that \( D_\oplus \) is isomorphic to \( \mathbb{C} \) and analogously that \( D_\ominus \) is isomorphic to \( \mathbb{C} \).

A trivial verification shows that \( (0, -\frac{1}{\sqrt{8}}, -\frac{1}{2}, -\frac{1}{\sqrt{8}}) \) is a (Gaussian) "imaginary unit" in \( D_\oplus \) and \( (0, \frac{1}{\sqrt{8}}, -\frac{1}{2}, \frac{1}{\sqrt{8}}) \) is a (Gaussian) "imaginary unit" in \( D_\ominus \), respectively. Indeed, under the notation in the proof of Theorem 8 we have:

\[
\left(0, -\frac{1}{\sqrt{8}}, -\frac{1}{2}, -\frac{1}{\sqrt{8}}\right)^2 = (-i_{D_\oplus}) \otimes (-i_{D_\oplus}) = -1_{D_\oplus}
\]

and

\[
\left(0, \frac{1}{\sqrt{8}}, -\frac{1}{2}, -\frac{1}{\sqrt{8}}\right)^2 = (-i_{D_\ominus}) \otimes (-i_{D_\ominus}) = -1_{D_\ominus}.
\]

This completes the proof. \( \square \)

In other words, the theorem states that \( x \in \mathcal{W} \) is a divisor of zero if and only if \( x \in D_\oplus \) or \( x \in D_\ominus \). Using isomorphism \( D_\oplus \) to \( \mathbb{C} \) and \( D_\ominus \) to \( \mathbb{C} \) one can determine the inverse elements for each invertible element in \( \mathcal{W} \). But the calculations are quite complicated.

5 Alternative method of finding inverse elements

In this section we find an easier method to compute inverse elements in \( \mathcal{W} \) using the skew circulated structure of the definition of the operation \( \otimes \).

**Theorem 10** Let \( x = X_11 \oplus X_i i \oplus X_j j \oplus X_k k \), where \( X_1, X_i, X_j, X_k \in \mathbb{R} \).

Let \( x \notin D_\oplus \) and \( x \notin D_\ominus \). Then the multiplicative inverse to the element \( x \) exists and it is unique. The element \( x^{-1} \) is defined as follows:

\[
x^{-1} \overset{\text{def}}{=} x^{\bullet} \otimes \left[ A(x) \ 1 \ominus B(x) \ \Theta \right] \frac{[A(x) + B(x)][A(x) - B(x)]}{[A(x) + B(x)][A(x) - B(x)]},
\]

where \( x \otimes x^{-1} = 1 \).

**Proof.** Since \( x \notin D_\oplus \) and \( x \notin D_\ominus \) then \( [A(x) + B(x)] [A(x) - B(x)] \neq 0 \).
By Theorem 5,
\[
x \otimes x^{-1} = x \otimes x^\bullet [A(x) \mathrel{1 \otimes} B(x) \Theta] \over [A(x) + B(x)][A(x) - B(x)]
\]
\[
= \frac{(x \otimes x^\bullet) \otimes [A(x) \mathrel{1 \otimes} B(x) \Theta]}{[A(x) + B(x)][A(x) - B(x)]}
\]
\[
= \frac{[A(x) \mathrel{1 \oplus} B(x) \Theta] \otimes [A(x) \mathrel{1 \oplus} B(x) \Theta]}{[A(x) + B(x)][A(x) - B(x)]}
\]
\[
= \frac{[A^2(x) - B^2(x)] 1}{[A(x) + B(x)][A(x) - B(x)]} = 1.
\]
The uniqueness of \(x^{-1}\) is obvious. \(\square\)

6 Topology on \(W\)

**Definition 4** Under the previous notation of \(A(x)\) and \(B(x)\), let us denote by
\[
\| \cdot \|_\ominus \overset{\text{def}}{=} \sqrt{A(\cdot) - B(\cdot)} : D_\ominus \to [0, \infty)
\]
and
\[
\| \cdot \|_\oplus \overset{\text{def}}{=} \sqrt{A(\cdot) + B(\cdot)} : D_\oplus \to [0, \infty).
\]

**Theorem 11** The function \(\| \cdot \|_\ominus : D_\ominus \to [0, +\infty)\) is a norm.

**Proof.** Let us test the norm definition.
Let \(x = (X_1, X_i, X_j, X_k) \in D_\ominus\). The following properties

(i) \(\|x\|_\ominus = 0 \iff x = \Lambda\);
(ii) \(\|x\|_\ominus \geq 0\);
(iii) \(\|\alpha x\|_\ominus = |\alpha|\|x\|_\ominus\) for every \(\alpha \in \mathbb{R}\)

hold true for every \(x \in D_\ominus\) by Theorem 7 and simple calculations.

Let us prove the triangle inequality. To do this, let another element \(y = (Y_1, Y_i, Y_j, Y_k) \in D_\ominus\). Further, let us denote by
\[
\|x\|_\ominus = \sqrt{r_x^2 + s_x^2}, \quad r_x = \frac{X_1}{\sqrt{2}} - X_i + \frac{X_j}{2}, \quad s_x = \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} + X_k,
\]
\[\|y\|_\ominus = \sqrt{r^2_x + s^2_x}, \quad r_y = \frac{Y_1}{\sqrt{2}} - Y_i + \frac{Y_j}{\sqrt{2}}, \quad s_y = \frac{Y_1}{\sqrt{2}} - \frac{Y_j}{\sqrt{2}} + Y_k,\]

and
\[\|x \oplus y\|_\ominus = \sqrt{r^2_{x \oplus y} + s^2_{x \oplus y}},\]

where
\[r_{x \oplus y} = \frac{X_1 + Y_1}{\sqrt{2}} - (X_i + Y_i) + \frac{X_j + Y_j}{2}, \quad s_{x \oplus y} = \frac{X_1 + Y_1}{\sqrt{2}} - \frac{X_j + Y_j}{\sqrt{2}} + (X_k + Y_k).\]

We have:
\[r_{x \oplus y} = \frac{X_1 + Y_1}{\sqrt{2}} - (X_i + Y_i) + \frac{X_j + Y_j}{2} = r_x + r_y,\]

and
\[s_{x \oplus y} = \frac{X_1 + Y_1}{\sqrt{2}} - \frac{X_j + Y_j}{\sqrt{2}} + (X_k + Y_k) = s_x + s_y.\]

The inequality
\[0 \leq (r_x s_y - s_x r_y)^2\]

implies
\[r_x^2 r_y^2 + s_x^2 s_y^2 + 2r_x r_y s_x s_y \leq r_x^2 r_y^2 + r_x^2 s_y^2 + s_x^2 r_y^2 + s_x^2 s_y^2\]

which implies
\[(r_x r_y + s_x s_y)^2 \leq (r_x^2 + s_x^2)(r_y^2 + s_y^2)\]

hence we have
\[r_x^2 + r_y^2 + 2r_x r_y + s_x^2 + s_y^2 + 2s_x s_y \leq r_x^2 + s_x^2 + r_y^2 + s_y^2 + 2 \sqrt{(r_x^2 + s_x^2)(r_y^2 + s_y^2)}\]

and, finally,
\[\sqrt{(r_x + r_y)^2 + (s_x + s_y)^2} \leq \sqrt{r_x^2 + s_x^2} + \sqrt{r_y^2 + s_y^2}.\]

In other words,
\[\|x \oplus y\|_\ominus \leq \|x\|_\oplus + \|y\|_\ominus. \quad \square\]

Analogously, we can consider the space \(D_\oplus\) and obtain the following theorem.
Theorem 12  The function $\| \cdot \|_\oplus : D_\oplus \to [0, +\infty)$ is a norm.

Proof. Let $x = (X_1, X_i, X_j, X_k) \in D_\oplus$. The following properties

(i) $\|x\|_\oplus = 0 \iff x = \Lambda$;

(ii) $\|x\|_\oplus \geq 0$;

(iii) $\|\alpha x\|_\oplus = |\alpha|\|x\|_\oplus$ for every $\alpha \in \mathbb{R}$

hold true for every $x \in D_\oplus$ by Theorem 7 and simple calculations.

Let us prove the triangle inequality. To do this, let another element $y = (Y_1, Y_i, Y_j, Y_k) \in D_\oplus$. Further,

For the triangle inequality, i.e.,

$$\|x \oplus y\|_\oplus \leq \|x\|_\oplus + \|y\|_\oplus$$

we put

$$\|x\|_\oplus = \sqrt{(r'_x)^2 + (s'_x)^2}, \quad r'_x = \frac{X_1}{\sqrt{2}} + X_i + \frac{X_j}{2}, \quad s'_x = \frac{X_1}{\sqrt{2}} - \frac{X_j}{\sqrt{2}} - X_k,$$

$$\|y\|_\oplus = \sqrt{(r'_y)^2 + (s'_y)^2}, \quad r'_y = \frac{Y_1}{\sqrt{2}} + Y_i + \frac{Y_j}{2}, \quad s'_y = \frac{Y_1}{\sqrt{2}} - \frac{Y_j}{\sqrt{2}} - Y_k.$$ 

The rest of proving the triangle inequality is similar as in the previous proof.

Since a direct sum of two normed spaces is a normed space, we define:

Definition 5 Let $x \in W = D_\oplus \times D_\ominus$ and $x_\oplus \in D_\oplus$ and $x_\ominus \in D_\ominus$ be two elements such that $x = x_\oplus \oplus x_\ominus$. The function

$$\|x\| \overset{\text{def}}{=} \|x_\oplus\|_\oplus \oplus \|x_\ominus\|_\ominus : W = D_\oplus \times D_\ominus \to [0, +\infty)$$

is a norm equivalent to $||| \cdot |||$. 

Remark 3 This norm is equivalent to the classical Euclidean norm on $\mathbb{E}_4$ since all norms on a finite $n$ dimensional space are mutually equivalent, $n \in \mathbb{N}$. 

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