Anderson localization for the completely resonant phases

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Abstract

For the almost Mathieu operator \((H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + \lambda \nu(\theta + n\alpha)u(n)\), Avila and Jitomirskaya guess that for every phase \(\theta \in \mathbb{R} \triangleq \{ \theta \in \mathbb{R} \mid 2\theta + \alpha \mathbb{Z} \in \mathbb{Z} \}\), \(H_{\lambda,\alpha,\theta}\) satisfies Anderson localization if \(|\lambda| > e^{2\beta}\). In the present paper, we show that for every phase \(\theta \in \mathbb{R}\), \(H_{\lambda,\alpha,\theta}\) satisfies Anderson localization if \(|\lambda| > e^{7\beta}\).

1 Introduction

The almost Mathieu operator (AMO) is the quasi-periodic Schrödinger operator on \(\ell^2(\mathbb{Z})\):

\[
(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + \lambda \nu(\theta + n\alpha)u(n), \quad \text{with} \quad \nu(\theta) = 2\cos 2\pi \theta, \quad (1.1)
\]

where \(\lambda\) is the coupling, \(\alpha\) is the frequency, and \(\theta\) is the phase.

AMO is the most studied quasi-periodic Schrödinger operator, arising naturally as a physical model (see [7] for a recent historical account and for the physics background).

We say phase \(\theta \in \mathbb{R}\) is completely resonant with respect to frequency \(\alpha\), if \(\theta \in \mathbb{R} \triangleq \{ \theta \in \mathbb{R} \mid 2\theta + \alpha \mathbb{Z} \in \mathbb{Z} \}\).

Anderson localization (i.e., only pure point spectrum with exponentially decaying eigenfunctions) is not only meaningful in physics, but also relates to reducibility for Aubry dual model(see [5]). In particular, Anderson localization for completely resonant phases is crucial to describe open gaps of \(\Sigma_{\lambda,\alpha}\) (the spectrum of \(H_{\lambda,\alpha,\theta}\) is independent of \(\theta\) for \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), denoted by \(\Sigma_{\lambda,\alpha}\)). See [2], [10] and [11] for details.
It is well known that $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for $\alpha \in \mathbb{Q}$ and all $\lambda$. This implies $H_{\lambda,\alpha,\theta}$ can not satisfy Anderson localization for all $\alpha \in \mathbb{Q}$. Thus we always assume $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ in the present paper.

The following notions are essential in the study of equation (1.1).

We say $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ satisfies a Diophantine condition $DC(\kappa, \tau)$ with $\kappa > 0$ and $\tau > 0$, if

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \kappa|k|^{-\tau}$$

for any $k \in \mathbb{Z}\setminus \{0\}$, where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{e\in\mathbb{Z}} |x - e|$. Let $DC = \bigcup_{\kappa>0,\tau>0} DC(\kappa, \tau)$. We say $\alpha$ satisfies Diophantine condition, if $\alpha \in DC$.

Let

$$\beta = \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},$$

(1.2)

where $\frac{p_n}{q_n}$ is the continued fraction approximants to $\alpha$. Notice that $\beta(\alpha) = 0$ for $\alpha \in DC$.

Avila and Jitomirskaya guess that for any completely resonant phase $\theta$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta}$ (Remark 9.1, [1]). Jitomirskaya-Koslover-Schulteis proves this for $\alpha \in DC$ [6], more concretely, for $\alpha \in DC$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathcal{R}$ and $|\lambda| > 1$. In [2], Avila and Jitomirskaya firstly develop a quantitative version of Aubry duality. By the way, they obtain that for $\alpha$ with $\beta(\alpha) = 0$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathcal{R}$ and $|\lambda| > 1$. The present authors extend the quantitative version of Aubry duality to all $\alpha$ with $\beta(\alpha) < \infty$, and show that for all $\alpha$ with $\beta(\alpha) < \infty$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathcal{R}$ and $|\lambda| > e^{C\beta}$, where $C$ is a large absolute constant [10]. In the present paper, we give a definite quantitative description about the constant $C$, and obtain the following theorem.

**Theorem 1.1.** For $\alpha \in \mathbb{R}\setminus \mathbb{Q}$ with $\beta(\alpha) < \infty$, the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathcal{R}$ and $|\lambda| > e^{\beta}$, where $\mathcal{R} = \{\theta \in \mathbb{R} \mid 2\theta + \alpha \mathbb{Z} \in \mathbb{Z}\}$.

**Remark 1.1.** Avila-Jitomirskaya thinks that $H_{\lambda,\alpha,0}$ does not display Anderson localization if $|\lambda| \leq e^{2\beta}$ (Remark 5.2, [7]), which is still open. Clearly, $0 \in \mathcal{R}$.

Avila and Jitomirskaya guess that for a.e. $\theta$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{\beta}$ (Remark 9.2, [1]), and they establish this for $|\lambda| > e^{\frac{16\beta}{7}}$. This result has been extended to regime $|\lambda| > e^{\frac{3\beta}{2}}$ by the present authors [9]. More precisely, there exists a Lebesgue zero-measure $B$ such that $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \notin B$ and $|\lambda| > e^{\frac{3\beta}{2}}$. Unfortunately, $\mathcal{R} \subset B$. In the present paper, we make some adjustment such that the discussion in [1] and [9] can be applied to completely resonant phase $\theta$.

The present paper is organized as follows:

In §2, we give some preliminary notions and facts which are taken from Avila-Jitomirskaya [1] or Bourgain [4]. In §3, we set up the regularity of non-resonant $y$. In §4, we set up the
regularity of resonant $y$. In §5, we give the proof of Theorem 1.1 by the regularity of $y$ and block resolvent expansion.

2 Preliminaries

It is well known that Anderson localization for a self-adjoint operator $H$ on $\ell^2(\mathbb{Z})$ is equivalent to the following statements.

Assume $\phi$ is an extended state of $H$, i.e.,

$$H\phi = E\phi \text{ with } E \in \Sigma(H) \text{ and } |\phi(k)| \leq (1 + |k|)^C,$$

(2.1)

where $\Sigma(H)$ is the spectrum of self-adjoint operator $H$. Then there exists some constant $c > 0$ such that

$$|\phi(k)| < e^{-ck} \text{ for } k \to \infty.$$ (2.2)

The above statements can be proved by Gelfand-Maurin Theorem. See [3] for the proof of continuous-time Schrödinger operator. The proof of discrete Schrödinger operator is similar, see [8] for example.

If $\alpha$ satisfies $\beta(\alpha) = 0$, Theorem 1.1 has been proved by Avila-Jitomirskaya [2], which we have mentioned in §1. Thus in the present paper, we fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $0 < \beta(\alpha) < \infty$. Unless stated otherwise, we always assume $\lambda > e^{7\beta}$ (for $\lambda < -e^{7\beta}$, notice that $H_{\lambda,\alpha,\theta} = H_{\lambda,\alpha,-(\theta + \frac{1}{2})}$, and $E \in \Sigma_{\lambda,\alpha}$. Since this does not change any of the statements, sometimes the dependence of parameters $E, \lambda, \alpha, \theta$ will be ignored in the following.

Given an extended state $\phi$ of $H_{\lambda,\alpha,\theta}$, without loss of generality assume $\phi(0) = 1$. Our objective is to show that there exists some $c > 0$ such that

$$|\phi(k)| < e^{-ck} \text{ for } k \to \infty.$$ (2.2)

Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}).$$

Following [6], $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k - 1)\alpha$ and can be written as a polynomial of degree $k$ in $\cos 2\pi(\theta + \frac{1}{2}(k - 1)\alpha)$:

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^j 2\pi(\theta + \frac{1}{2}(k - 1)\alpha) \triangleq Q_k(\cos 2\pi(\theta + \frac{1}{2}(k - 1)\alpha)).$$ (2.3)

Let $A_{k,r} = \{ \theta \in \mathbb{R} | Q_k(\cos 2\pi\theta) \leq e^{(kr+1)r} \}$ with $k \in \mathbb{N}$ and $r > 0$.

**Lemma 2.1.** (p.16, [7]) The following inequality holds

$$\limsup_{k \to \infty} \frac{1}{k} \ln |P_k(\theta)| \leq \ln \lambda.$$ (2.4)
By Cramer’s rule (p. 15, [4]) for given $x_1$ and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has
\[
|G_l(x_1, y)| = \left| \frac{P_{x-l}(\theta + (y + 1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \tag{2.5}
\]
\[
|G_l(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \tag{2.6}
\]

By Lemma 2.1, the numerators in (2.5) and (2.6) can be bounded uniformly with respect to $\theta$. Namely, for any $\varepsilon > 0$,
\[
|P_n(\theta)| \leq e^{(\ln 1 + \varepsilon)n} \tag{2.7}
\]
for $n$ large enough.

**Definition 2.1.** Fix $t > 0$. A point $y \in \mathbb{Z}$ will be called $(t, k)$-regular if there exists an interval $[x_1, x_2]$ containing $y$, where $x_2 = x_1 + k - 1$, such that
\[
|G_{[x_1, x_2]}(y, x_i)| < e^{-ny-x_i} \text{ and } |y - x_i| \geq \frac{1}{t}k, \text{ for } i = 1, 2; \tag{2.8}
\]
otherwise, $y$ will be called $(t, k)$-singular.

It is easy to check that (p. 61, [4])
\[
\phi(y) = -G_{[x_1, x_2]}(x_1, y)\phi(x_1 - 1) - G_{[x_1, x_2]}(y, x_2)\phi(x_2 + 1), \tag{2.9}
\]
where $y \in I = [x_1, x_2] \subset \mathbb{Z}$. Our strategy is to establish the $(t, k(y))$-regular of $y$, then localized property is easy to obtain by (2.9) and the block resolvent expansion.

**Definition 2.2.** We say that the set $\{\theta_1, \cdots, \theta_{k+1}\}$ is $\varepsilon$-uniform if
\[
\max_{x \in [-1, 1]} \max_{i = 1, \cdots, k+1} \prod_{j=1, j \neq i}^{k+1} \left| \frac{x - \cos 2\pi \theta_j}{\cos 2\pi \theta_i - \cos 2\pi \theta_j} \right| < e^{t\varepsilon}. \tag{2.10}
\]

**Lemma 2.2.** (Lemma 9.3, [7]) Suppose $\{\theta_1, \cdots, \theta_{k+1}\}$ is $\varepsilon_1$-uniform. Then there exists some $\theta_j$ in set $\{\theta_1, \cdots, \theta_{k+1}\}$ such that $\theta_j \notin A_{k, \ln t - \varepsilon}$ if $\varepsilon > \varepsilon_1$ and $k$ is sufficiently large.

Assume without loss of generality that $y > 0$. Define $b_n = q_{n+1}^{8/9}$, where $q_n$ is given by (1.2), and find $n$ such that $b_n \leq y < b_{n+1}$. We will distinguish two cases:

(i) $|y - \ell q_n| \leq b_n$ for some $\ell \geq 1$, called resonance.

(ii) $|y - \ell q_n| > b_n$ for all $\ell \geq 0$, called non-resonance.

Next, we will establish the regularity for resonant and non-resonant $y$ respectively. Given a phase $\theta \in \mathcal{R}$, there exists some $p \in \mathbb{Z}$ such that $2\theta - p\alpha \in \mathbb{Z}$. Without loss of generality, assume $p \leq 0$ below.
3 Regularity for non-resonant $y$

In this section, we will set up the regularity for non-resonant $y$, for this reason, we give some lemmata first. Note that $C$ is a large absolute constant below, which may change through the arguments, even when appear in the same formula. For simplicity, we replace $I = [x_1, x_2] \cap \mathbb{Z}$ with $I = [x_1, x_2]$.

Lemma 3.1. (Lemma 9.7, [7]) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq \ell_0 \leq q_n - 1$ be such that $|\sin \pi (x + \ell_0 \alpha)| = \inf_{0 \leq \ell \leq q_n - 1} |\sin \pi (x + \ell \alpha)|$, then for some absolute constant $C > 0$,

$$-Cq_n \leq \sum_{\ell=0, \ell \neq \ell_0}^{q_n-1} \ln |\sin \pi (x + \ell \alpha)| + (q_n - 1) \ln 2 \leq Cq_n,$$

(3.1)

where $q_n$ is given by (1.2).

Recall that $\{q_n\}_{n \in \mathbb{N}}$ is the sequence of best denominators of irrational number $\alpha$, since it satisfies

$$\forall 1 \leq k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|q_n \alpha\|_{\mathbb{R}/\mathbb{Z}},$$

(3.2)

Moreover, we also have the following estimate.

$$\frac{1}{2q_{n+1}} \leq \Delta_n \doteq \|q_n \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_{n+1}},$$

(3.3)

Now that $y$ is non-resonant. Without loss of generality, let $y = mq_n + y_0$ with $m \leq \frac{q_{n-1}}{q_n}$ and $q_n^{\frac{q_{n-1}}{q_n}} \leq y_0 \leq \frac{q_n}{2}$. Let $s \in \mathbb{N}$ be the largest positive integer such that $4sq_{n-1} - p + 1 \leq y_0$. Notice that $8sq_{n-1} < q_n$.

Set $I_1, I_2$ as follows,

$$I_1 = [-2sq_{n-1}, -1]$$

and

$$I_2 = [mq_n + y_0 - 2sq_{n-1}, mq_n + y_0 + 2sq_{n-1} - 1].$$

The set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $6sq_{n-1}$ elements, where $\theta_j = \theta + j\alpha$ and $j$ ranges through $I_1 \cup I_2$.

Lemma 3.2. For any $\varepsilon > 0$, the set $\{\theta_j\}_{j \in I_1 \cup I_2}$ is $-2 \ln (s/q_n)/q_{n-1} + \varepsilon$-uniform if $n$ is sufficiently large.

Proof: We will first estimate numerator in (2.10). In (2.10), let $x = \cos 2\pi a$ and take the logarithm, one has

$$\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j|$$
\[
\begin{align*}
&= \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin \pi(a + \theta_j)| + \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin \pi(a - \theta_j)| + (6s q_{n-1} - 1) \ln 2 \\
&= \Sigma_+ + \Sigma_- + (6s q_{n-1} - 1) \ln 2, \tag{3.4}
\end{align*}
\]
where
\[
\Sigma_+ = \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin \pi(a + \theta_j)|, \tag{3.5}
\]
and
\[
\Sigma_- = \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin \pi(a - \theta_j)|. \tag{3.6}
\]
Both \(\Sigma_+\) and \(\Sigma_-\) consist of \(6s\) terms of the form of (3.1), plus \(6s\) terms of the form
\[
\ln \min_{j=0,1,\ldots,q_n-1} |\sin \pi(x + j\alpha)|, \tag{3.7}
\]
minus \(\ln |\sin \pi(a \pm \theta_i)|\). Since there exists a interval of length \(q_n\) in sum of (3.5) (or (3.6)) containing \(i\), thus the minimum over this interval is not more than \(\ln |\sin \pi(a \pm \theta_i)|\) (by the minimality). Thus, using (3.1) \(6s\) times of \(\Sigma_+\) and \(\Sigma_-\) respectively, one has
\[
\sum_{j \in I_1 \cup J_2, j \neq i} \ln |\cos 2\pi a - \cos 2\pi \theta_j| \leq -6s q_{n-1} \ln 2 + C \ln q_{n-1}. \tag{3.8}
\]
The estimate of the denominator of (2.10) requires a bit more work. Without loss of generality, assume \(i \in I_1\).

In (3.4), let \(a = \theta_i\), we obtain
\[
\begin{align*}
\sum_{j \in I_1 \cup J_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \theta_j| \\
&= \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin (\theta_i + \theta_j)| + \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin (\theta_i - \theta_j)| + (6s q_{n-1} - 1) \ln 2 \\
&= \Sigma_+ + \Sigma_- + (6s q_{n-1} - 1) \ln 2, \tag{3.9}
\end{align*}
\]
where
\[
\Sigma_+ = \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin (2\theta + (i + j)\alpha)|. \tag{3.10}
\]
and
\[
\Sigma_- = \sum_{j \in I_1 \cup J_2, j \neq i} \ln |\sin (i - j)\alpha|. \tag{3.11}
\]
We first estimate \(\Sigma_+\). Set \(J_1 = [-2s, -1]\) and \(J_2 = [0, 4s - 1]\), which are two adjacent disjoint intervals of length \(2s\) and \(4s\) respectively. Then \(I_1 \cup I_2\) can be represented as a disjoint union of segments \(B_j, j \in J_1 \cup J_2\), each of length \(q_{n-1}\). Applying (3.1) on each \(B_j\), we obtain
\[
\Sigma_+ \geq -6s q_{n-1} \ln 2 + \sum_{j \in J_1 \cup J_2} \ln |\sin \pi \theta_j| - C \ln q_{n-1} - \ln |\sin 2\pi (\hat{\theta} + i\alpha)|, \tag{3.12}
\]
where
\[ |\sin \pi \hat{\theta}_j| = \min_{\ell \in B_j} |\sin \pi (2\theta + (\ell + i)\alpha)|. \] (3.13)

We now start to estimate (3.13). Noting that \(2\theta + (\ell + i)\alpha \in (\ell + i + p)\alpha + \mathbb{Z}\), together with the construction of \(I_1\) and \(I_2\), one has

\[ 2\theta + (\ell + i)\alpha = mq_\alpha \alpha + r_1 \alpha \quad \text{mod } \mathbb{Z} \] (3.14)
or

\[ 2\theta + (\ell + i)\alpha = r_2 \alpha \quad \text{mod } \mathbb{Z}, \] (3.15)

where \(1 \leq |r_i| < q_n, i = 1, 2\). By (3.2) and (3.3), we have

\[
\min_{\ell \in I_1 \cup J_2} \|2\theta + (\ell + i)\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|r_j\alpha\|_{\mathbb{R}/\mathbb{Z}} - \frac{\Delta_{n-1}}{2} \\
\geq \Delta_{n-1} - \frac{\Delta_{n-1}}{2} \\
\geq \frac{\Delta_{n-1}}{2},
\] (3.16)
since \(\|mq_\alpha\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{q}{q_n} \Delta_n \leq \frac{\Delta_{n-1}}{2}\).

Next we will estimate \(\sum_{j \in J_1} \ln |\sin \pi \hat{\theta}_j|\). Assume that \(\hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-1}\alpha\) for every \(j, j+1 \in J_1\). Applying the Stirling formula and (3.16), one has

\[
\sum_{j \in J_1} \ln |\sin 2\pi \hat{\theta}_j| > 2 \sum_{j=1}^{s} \ln \frac{j\Delta_{n-1}}{C} \\
> 2 s \ln \frac{s}{q_n} - Cs. \] (3.17)

In the other case, decompose \(J_1\) in maximal intervals \(T_\kappa\) such that for \(j, j+1 \in T_\kappa\) we have \(\hat{\theta}_{j+1} = \hat{\theta}_j + q_{n-1}\alpha\). Notice that the boundary points of an interval \(T_\kappa\) are either boundary points of \(J_1\) or satisfy \(\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-1} \geq \frac{\Delta_{n-1}}{2}\). This follows from the fact that if \(0 < |z| < q_{n-1}\), then \(\|\hat{\theta}_j + q_{n-1}\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-1}\), and \(\|\hat{\theta}_j + (z + q_{n-1})\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|z\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|\hat{\theta}_j + q_{n-1}\alpha\|_{\mathbb{R}/\mathbb{Z}} > \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} - \Delta_{n-1}\). Assuming \(T_\kappa \neq J_1\), then there exists \(j \in T_\kappa\) such that \(\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\Delta_{n-1}}{2} - \Delta_{n-1}\).

If \(T_\kappa\) contains some \(j\) with \(\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-1}}{10}\), then

\[
|T_\kappa| \geq \frac{\Delta_{n-2}}{2} - \Delta_{n-1} - \frac{\Delta_{n-2}}{10} + 1 \\
\geq \frac{1}{4} \frac{\Delta_{n-2}}{\Delta_{n-1}} > s, \] (3.18)
where $|T_x| = b - a + 1$ for $T_x = [a, b]$. For such $T_x$, a similar estimate to (3.17) gives

$$\sum_{j \in T_x} \ln |\sin \pi \hat{\theta}_j| > |T_x| \ln \frac{|T_x|}{q_n} - C s$$

$$> |T_x| \ln \frac{s}{q_n} - C s.$$  \hspace{1cm} (3.19)

If $T_x$ does not contain any $j$ with $\|\hat{\theta}_j\|_{R/Z} < \frac{\Delta n}{10}$, then by (3.3)

$$\sum_{j \in T_x} \ln |\sin \pi \hat{\theta}_j| > -|T_x| \ln q_{n-1} - C|T_x|$$

$$> |T_x| \ln \frac{s}{q_n} - C|T_x|.$$  \hspace{1cm} (3.20)

since $s < \frac{q_n}{q_{n-1}}$.

By (3.19) and (3.20), one has

$$\sum_{j \in I_1} \ln |\sin \pi \hat{\theta}_j| \geq 2s \ln \frac{s}{q_n} - C s.$$  \hspace{1cm} (3.21)

Similarly,

$$\sum_{j \in I_2} \ln |\sin \pi \hat{\theta}_j| \geq 4s \ln \frac{s}{q_n} - C s.$$  \hspace{1cm} (3.22)

Putting (3.12), (3.21) and (3.22) together, we have

$$\Sigma_+ > -6s q_{n-1} \ln 2 + 6s \ln \frac{s}{q_n} - C s \ln q_{n-1}.$$  \hspace{1cm} (3.23)

We are now in the position to estimate $\Sigma_-$. Following the discussion of $\Sigma_+$, we have the similar estimate,

$$\Sigma_- > -6s q_{n-1} \ln 2 + 12s \ln \frac{s}{q_n} - C s \ln q_{n-1}.$$  \hspace{1cm} (3.24)

In order to avoid repetition, we omit the proof of (3.24).

By (3.9), (3.23) and (3.24), one obtains

$$\sum_{j \in I_1 \cup I_2, j \neq i} \ln |\cos 2\pi \theta_i - \cos 2\pi \hat{\theta}_j|$$

$$> -6s q_{n-1} \ln 2 + 12s \ln \frac{s}{q_n} - C s \ln q_{n-1}.$$  \hspace{1cm} (3.25)

Combining with (3.8), we have for any $\varepsilon > 0$,

$$\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \hat{\theta}_j|} < e^{(6s q_{n-1} - 1)(-2 \ln(s/q_n)/q_{n-1} + \varepsilon)},$$  \hspace{1cm} (3.26)

for $n$ large enough. \hspace{1cm} \Box
Theorem 3.1. Suppose $y$ is non-resonant. Let $s$ be the largest positive integer such that $4sq_{n-1} - p + 1 \leq \text{dist}(y, \ell q_n \mid \ell \geq 0) \equiv y_0$. Then for any $\varepsilon > 0$ and sufficiently large $n$, $y$ is $(\ln \lambda + 18 \ln (sq_{n-1}/q_n)/q_{n-1} - \varepsilon, 6sq_{n-1} - 1)$-regular if $\ln \lambda > 2\beta$. In particular, $y$ is $(\ln \lambda - 2\beta - \varepsilon, 6sq_{n-1} - 1)$-regular.

Proof: Theorem 3.1 can be derived from Lemma 3.2 directly. See the proof of Lemma 9.4 in [1] (p. 24) for details.

4 Regularity for resonant $y$

In this section, we mainly concern the regularity for resonant $y$. If $b_n \leq y < b_{n+1}$ is resonant, by the definition of resonance, there exists some positive integer $\ell$ with $1 \leq \ell \leq \frac{8}{9}q_{n+1}/q_n$ such that $|y - \ell q_n| \leq b_n$. Fix the positive integer $\ell$ and let $s$ be the largest positive integer such that $7sq_{n-1} \leq q_n + p - 1$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$I_1 = [-4sq_{n-1}, -1]$,  
$I_2 = [\ell q_n - 3sq_{n-1}, \ell q_n + 3sq_{n-1} - 1]$,  

and let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$, the set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $10sq_{n-1}$ elements.

We will use the following three steps to establish the regularity for $y$. Step 1: we set up the $\frac{7}{4}\beta + \varepsilon$-uniformity of $\{\theta_j\}$ for any $\varepsilon > 0$. By Lemma 2.2, there exists some $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{10sq_{n-1} - 1, \ln \lambda - \frac{7}{4}\beta - \varepsilon}$ for any $\varepsilon > 0$. Step 2: we show that $\forall j \in I_1, \theta_j \in A_{10sq_{n-1} - 1, \ln \lambda - \frac{7}{4}\beta - \varepsilon}$ if $\lambda > e^{q\beta}$. Thus there exists $\theta_{j_0} \notin A_{10sq_{n-1} - 1, \ln \lambda - \frac{7}{4}\beta - \varepsilon}$ for some $j_0 \in I_2$. Step 3: we establish the regularity for $y$.

We start with the Step 1.

Lemma 4.1. For any $\varepsilon > 0$, the set $\{\theta_j\}_{j \in I_1 \cup I_2}$ is $(\frac{7}{4}\beta + \varepsilon)$-uniform if $n$ is sufficiently large.

Proof: Notice that for any $i \in I_1 \cup I_2$, there is at most one $\tilde{i} \in I_1 \cup I_2$ such that $|i - \tilde{i}| = \ell q_n$. It is easy to check

$$\ln |\sin \pi (i - \tilde{i})\alpha| = \ln |\sin (\pi \ell q_n \alpha)| > -\ln q_{n+1} - C, \quad (4.1)$$

since $\Delta_n \geq \frac{1}{2q_{n+1}}$. If $j \neq i, \tilde{i}$ and $j \in I_1 \cup I_2$, then $j - i = r + m_j q_n$ with $1 \leq |r| < q_n$ and $|m_j| \leq \ell + 2$. Thus by (3.2) and (3.3),

$$\|\tilde{r}\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_{n-1}$$

and

$$\min_{j \in I_1 \cup I_2, j \neq \tilde{i}} \| (j - i)\alpha \|_{\mathbb{R}/\mathbb{Z}} > \| \tilde{r}\alpha \|_{\mathbb{R}/\mathbb{Z}} - (\ell + 2)\Delta_n$$

$$> \frac{\Delta_{n-1}}{2}, \quad (4.2)$$

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since \((\ell + 2)A_n < \frac{A_n}{2}\) for \(n\) large enough.

Similarly, for any \(i \in I_1 \cup I_2\), there is at most one \(\bar{i} \in I_1 \cup I_2\) such that \(|i + \bar{i} + p| = \ell q_n\). We also have

\[
\ln |\sin \pi (2\theta + (i + \bar{i})a)| > -\ln q_{n+1} - C, \tag{4.3}
\]

and

\[
\min_{j \in I_1 \cup I_2, j \neq \bar{\delta}, \delta} \|2\theta + (j + i)a\|_{\mathbb{R}/\mathbb{Z}} > \frac{A_n - 1}{2}. \tag{4.4}
\]

Replacing (3.16) with (4.1) and (4.2) and following the proof of Lemma 3.2, one has

\[
\Sigma_+ > -10sq_{n-1} \ln 2 + 10s \ln \frac{s}{q_n} - \ln q_{n+1} - C\ln q_{n-1}, \tag{4.5}
\]

since there exists at most one term satisfies (4.1).

Similarly, Replacing (3.16) with (4.3) and (4.4), one has

\[
\Sigma_- > -10sq_{n-1} \ln 2 + 10s \ln \frac{s}{q_n} - \ln q_{n+1} - C\ln q_{n-1}. \tag{4.6}
\]

By (3.8), (4.5) and (4.6), we have for any \(\varepsilon_0 > 0\),

\[
\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2, j \neq \bar{i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos \theta_j - \cos 2\pi\theta_j|} < e^{10sq_{n-1} - \frac{2}{5} - \frac{1}{5} \ln \ln \frac{q_n}{q_{n+1}} + \varepsilon_0}, \tag{4.7}
\]

if \(n\) is large enough.

By the definition of \(s\) and noting that \(\beta = \limsup_{n \to \infty} \frac{\ln q_{n-1}}{q_n}\), one has

\[
-2\ln(s/q_n)/q_{n-1} + \frac{\ln q_{n+1}}{5sq_{n-1}} < \frac{7}{5}\beta + \varepsilon_0, \tag{4.8}
\]

for \(n\) large enough. Combining (4.7) with (4.8), we obtain

\[
\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2, j \neq \bar{i}} \frac{|x - \cos 2\pi\theta_j|}{|\cos \theta_j - \cos 2\pi\theta_j|} < e^{(10sq_{n-1} - 1)(\frac{7}{5}\beta + 3\varepsilon_0)}. \tag{4.9}
\]

By the arbitrariness of \(\varepsilon_0\), we complete the proof. \(\Box\)

Now, we are in the position to undertake Step 2.

**Lemma 4.2.** Assume \(y \in [-2q_n, 2q_n]\) and let \(d = \text{dist}(y, \{jq_n\}, j \geq 0) > \frac{1}{100}q_n\). Then, for any \(\varepsilon > 0\),

\[
|\phi(y)| < \exp(-\ln \lambda - \varepsilon)d \tag{4.10}
\]

if \(n\) is sufficiently large.

**Proof:** Using Theorem 3.1 and block-resolvent expansion, it is easy to obtain Lemma 4.2. See the proof of Lemma 3.4 in [9] for details.
Lemma 4. By the definition of \( n \) is large enough (or equivalently \( y \) is large enough). This implies 

\[ A \]

Theorem 4.1. For any \( \varepsilon > 0 \) and any \( b \in [-9s_{q-1}, -5s_{q-1}] \), we have \( \theta + (b + 5s_{q-1} - 1)\alpha \in A_{10s_{q-1}-1, 4^{\lambda} e} \) if \( n \) is large enough. That is for all \( j \in I_1, \theta_j \in A_{10s_{q-1}-1, 4^{\lambda} e} \).

**Proof:** For any \( b \in [-9s_{q-1}, -5s_{q-1}] \), let \( b_1 = b - 1 \) and \( b_2 = b + 10s_{q-1} - 1 \). Applying Lemma 4.2 one has, for any \( \varepsilon_0 \),

\[ |\phi(b_1)| < e^{-(\ln \lambda - \varepsilon_0)(\ln + b)}, |q_n + b| > \frac{q_n}{100}, \]  \hspace{1cm} (4.11)

and

\[ |\phi(b_2)| \leq \begin{cases} e^{-(\ln \lambda - \varepsilon_0)(10s_{q-1}+b)}, & b \in [-9s_{q-1}, \frac{q_n}{2} - 10s_{q-1}]; \\ e^{-(\ln \lambda - \varepsilon_0)(q_n - 10s_{q-1} - b)}, & b \in [\frac{q_n}{2} - 10s_{q-1}, -5s_{q-1}]. \end{cases} \]  \hspace{1cm} (4.12)

By the definition of \( s, (4.11) \) and \( (4.12) \) become

\[ |\phi(b_1)| < e^{-(\ln \lambda - 2\varepsilon)(\ln + b)}|q_n + b| > \frac{q_n}{100}, \]  \hspace{1cm} (4.13)

and

\[ |\phi(b_2)| \leq \begin{cases} e^{-(\ln \lambda - \varepsilon_0)(10s_{q-1}+b)}, & b \in [-9s_{q-1}, \frac{q_n}{2} - 10s_{q-1}]; \\ e^{-(\ln \lambda - \varepsilon_0)(-3s_{q-1} - b)}, & b \in [\frac{q_n}{2} - 10s_{q-1}, -5s_{q-1}]. \end{cases} \]  \hspace{1cm} (4.14)

In (2.9), let \( x = 0 \) and \( I = [b, b + 10s_{q-1} - 2] \), we get for \( n \) large enough,

\[ |G_1(0, b)| > \begin{cases} e^{(\ln \lambda - 3\varepsilon_0)(\ln + b)}, & |q_n + b| > \frac{q_n}{100}, \\ e^{-\varepsilon_0 q_n - 1}, & |q_n + b| \leq \frac{q_n}{100}. \end{cases} \]  \hspace{1cm} (4.15)

or

\[ |G_1(0, b + 10s_{q-1} - 2)| > \begin{cases} e^{(\ln \lambda - 3\varepsilon_0)(10s_{q-1} + b)}, & b \in [-9s_{q-1}, \frac{q_n}{2} - 10s_{q-1}]; \\ e^{(\ln \lambda - 3\varepsilon_0)(-3s_{q-1} - b)}, & b \in [\frac{q_n}{2} - 10s_{q-1}, -5s_{q-1}]. \end{cases} \]  \hspace{1cm} (4.16)

since \( \phi(0) = 1 \) and \( |\phi(k)| \leq (1 + |k|^C) \).

By (2.5), (2.6) and (2.7),

\[ |Q_{10s_{q-1}}(\cos 2\pi(\theta + (b + \frac{10s_{q-1} - 2}{2})\alpha)|
= |P_{10s_{q-1} - 1}(\theta + b\alpha)|
< \min\{|G_1(0, b)|^{-1} e^{(\ln \lambda + \varepsilon_0)(b + 10s_{q-1} - 2)}, |G_1(0, b + 10s_{q-1} - 2)|^{-1} e^{-(\ln \lambda + \varepsilon_0)b}\}
< e^{4(\ln \lambda + 4\varepsilon_0)10s_{q-1}}. \]

This implies \( \theta + (b + 5s_{q-1} - 1)\alpha \in A_{10s_{q-1}-1, 4^{\lambda} e} \). By the arbitrariness of \( \varepsilon_0 \), we have \( \theta + (b + 5s_{q-1} - 1)\alpha \in A_{10s_{q-1}-1, 4^{\lambda} e} \) for any \( b \in [-9s_{q-1}, -5s_{q-1}] \). \( \Box \)

Finally, we will finish the Step 3.

Theorem 4.2. For any \( \varepsilon > 0 \) such that \( t = (\ln \lambda - 7\beta - \varepsilon) > 0 \), \( y \) is \((t, 10s_{q-1} - 1)\)-regular if \( n \) is large enough (or equivalently \( y \) is large enough).
\textbf{Proof:} Let \( y > 0 \) be resonant. By hypothesis \( y = \ell q_n + r \), with \( 0 \leq |r| \leq q_n^\frac{8}{5} \) and \( 1 \leq \ell \leq q_{n+1}/q_n \).

By Lemma 2.2 and Lemma 4.1 (let \( \varepsilon = \varepsilon_0/2 \) in Lemma 4.1), for any \( \varepsilon_0 > 0 \), there exists some \( j \in I_1 \cup I_2 \) such that \( \theta_j \notin A_{10sq_n-1, \ln \lambda - \frac{7}{5}\beta - \varepsilon_0} \). By Theorem 3.1 and noting that \( \ln \lambda > 7\beta \) (i.e. \( \frac{7}{5}\ln \lambda < (\ln \lambda - \frac{7}{5}\beta) \)), we have \( \theta_j \in A_{10sq_n-1, \ln \lambda - \frac{7}{5}\beta - \varepsilon_0} \) for all \( j \in I_1 \) and sufficiently small \( \varepsilon_0 \). Thus, there exists some \( j_0 \in I_2 \) such that \( \theta_{j_0} \notin A_{10sq_n-1, \ln \lambda - \frac{7}{5}\beta - \varepsilon_0} \). Set \( I = [j_0 - 5sq_n + 1, j_0 + 5sq_n - 1] = [x_1, x_2] \). By (2.5), (2.6) and (2.7) again, we have
\[
|G_j(y, x_i)| < e^{(\ln \lambda + \varepsilon_0)(10sq_n - 2 - |y - x_i|) - 10sq_n}(\ln \lambda - \frac{7}{5}\beta - \varepsilon_0).
\]

By a simple computation
\[
|y - x_i| \geq (2sq_n - 1)q_n > \left(\frac{1}{5} - \varepsilon_0\right)10sq_n - 1,
\]
therefore,
\[
|G_j(y, x_i)| < e^{-\varepsilon_0|\ln \lambda - 7\beta - \varepsilon|},
\]
where \( \varepsilon = 20\varepsilon_0 \). Let \( t = \ln \lambda - 7\beta - \varepsilon > 0 \), then for \( n \) large enough, \( y \) is \( (t, 10sq_n - 1) \)-regular. \( \square \)

\section{Proof of Theorem 1.1}

Now that the regularity for \( y \) is established, we will use block resolvent expansion to prove Theorem 1.1.

\textbf{Proof of Theorem 1.1.}

Give some \( k \) with \( k > q_n \) and \( n \) large enough. \( \forall y \in [q_n^\frac{8}{5}, 2k] \), let \( \varepsilon = \varepsilon_0 \) in Theorem 3.1 and 4.2, then there exists an interval \( I(y) = [x_1, x_2] \subset [-4k, 4k] \) with \( y \in I(y) \) such that
\[
\text{dist}(y, \partial I(y)) > \frac{1}{7}|I(y)| \geq \min\left\{ \frac{6sq_n - 1}{7}, \frac{10sq_n - 1}{7} \right\}
\]
\[
\geq \frac{1}{2}q_{n-1}
\]
(5.1)

and
\[
|G_{I(y)}(y, x_i)| < e^{-(\ln \lambda - 7\beta - \varepsilon_0)|y - x_i|}, \quad i = 1, 2.
\]
(5.2)

Denote by \( \partial I(y) \) the boundary of the interval \( I(y) \). For \( z \in \partial I(y) \), let \( z' \) be the neighbor of \( z \), (i.e., \( |z - z'| = 1 \) not belonging to \( I(y) \)).

If \( x_2 + 1 < 2k \) or \( x_1 - 1 > b_n = q_n^\frac{8}{5} \), we can expand \( \phi(x_2 + 1) \) or \( \phi(x_1 - 1) \) as (2.9). We can continue this process until we arrive to \( z \) such that \( z + 1 \geq 2k \) or \( z - 1 \leq b_n \), or the iterating number reaches \( \lfloor \frac{2k}{q_{n-1}} \rfloor \).
By \( (2.9) \),

\[
\phi(k) = \sum_{s\geq s+1} G_{R(k)}(k, z_1)G_{R(z_1')}(z_1', z_2) \cdots G_{R(z_{s+1}')}(z_{s+1}', z_{s+1}) \phi(z_{s+1}) ,
\]

(5.3)

where in each term of the summation we have \( b_n + 1 < z_i < 2k - 1, i = 1, \ldots, s \), and either \( z_{s+1} \notin [b_n + 2, 2k - 2], s + 1 < \left[ \frac{2k}{q_{s+1}} \right] \); or \( s + 1 = \left[ \frac{2k}{q_{s+1}} \right] \).

If \( z_{s+1} \notin [b_n + 2, 2k - 2], s + 1 < \left[ \frac{2k}{q_{s+1}} \right] \), by (5.2) and noting that \( |\phi(z_{s+1})| \leq (1 + |z_{s+1}'|)^C \leq k^C \), one has

\[
|G_{R(k)}(k, z_1)G_{R(z_1')}(z_1', z_2) \cdots G_{R(z_{s+1}')}(z_{s+1}', z_{s+1}) \phi(z_{s+1})| \\
\leq e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)|k - z_i + \sum_{\ell=1}^{s} |z_{s+1}' - z_{s+1}||} k^C \\
\leq e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)(|k - z_i| - (s+1))} k^C \\
\leq \max\{e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)(b_n - 1 - \frac{2k}{q_{s+1}})} k^C, e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)(2k - 2 - \frac{2k}{q_{s+1}})} k^C\}. \\
(5.4)
\]

If \( s + 1 = \left[ \frac{2k}{q_{s+1}} \right] \), using (5.1) and (5.2), we obtain

\[
|G_{R(k)}(k, z_1)G_{R(z_1')}(z_1', z_2) \cdots G_{R(z_{s+1}')}(z_{s+1}', z_{s+1}) \phi(z_{s+1})| \\
\leq e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)\frac{2k}{q_{s+1}} + \frac{\phi(z_{s+1})}{\phi(z_{s+1})}} k^C. \\
(5.5)
\]

Finally, notice that the total number of terms in (5.3) is at most \( 2 \left[ \frac{2k}{q_{s+1}} \right] \). Combining with (5.4) and (5.5), we obtain

\[
|\phi(k)| \leq e^{-\left(\ln l - 7\beta - 2\varepsilon_0\right)\ln l} k \\
(5.6)
\]

for large enough \( n \) (or equivalently large enough \( k \)). By the arbitrariness of \( \varepsilon_0 \), we have for any \( \varepsilon > 0 \),

\[
|\phi(k)| \leq e^{-\left(\ln l - 7\beta - \varepsilon_0\right)k} \text{ for } k \text{ large enough}. \\
(5.7)
\]

For \( k < 0 \), the proof is similar. Thus for any \( \varepsilon > 0 \),

\[
|\phi(k)| \leq e^{-\left(\ln l - 7\beta - \varepsilon_0\right)|k|} \text{ if } |k| \text{ is large enough}. \\
(5.8)
\]

We finish the proof of Theorem 1.1

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