Teleparallel formalism of galilean gravity

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A pseudo-Riemannian manifold is introduced, with light-cone coordinates in (4+1) dimensional space-time, to describe a Galilei covariant gravity. The notion of 5-bein and torsion are developed and a galilean version of teleparallelism is constructed in this manifold. The formalism is applied to two spherically symmetric configurations. The first one is an ansatz which is inferred by following the Schwarzschild solution in general relativity. The second one is a solution of galilean covariant equations. In addition, this Galilei teleparallel approach provides a prescription to couple the 5-bein field to the galilean covariant Dirac field.

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I. INTRODUCTION

The Galilei symmetry provides the foundation of non-relativistic classical and non-relativistic quantum mechanics, and phenomena in many-body physics, such as superconductivity and nuclear systems, are restricted to this regime [1]. Beyond these, galilean symmetry finds application even in the ultra-relativistic realm. It has been observed that a Poincaré-covariant quantum field theory in an infinite-momentum frame, in (3+1) dimensions, is reduced to a Galilei covariant theory, in (2+1) dimensions [2,3]. This is a limiting process to describe particles with high velocities, which is equivalent to taking the theory in the light-cone frame, where the kinematical symmetry is the central extension of Galilei group $G_{4,1}$. In order to generalize this result to a Galilei symmetry in (3+1) dimensions, the starting point is a theory in a (4+1) space-time, $G_{4,1}$. Then the Galilei symmetry is written in a manifestly covariant form, and this formalism has been used in various application [4-13]. In particular, such a tensor structure has been associated with a non-relativistic anti-de Sitter/conformal field theory (AdS/CFT), in order to describe strongly coupled fermions, as it is the case of cold fermion atoms at unitarity [4-16].

The metric tensor for the galilean space-time is introduced by [17,18]

$$\eta_{\mu\nu} = \begin{pmatrix} 1_d & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

where $1_d$ is a Euclidian metric in $d$-dimensions. For $d = 3$, this metric leaves the scalar product of vectors invariant in the (4+1) space-time. For instance, consider the 5-momentum $p = (p^1, p^2, p^3, p^4, p^5) = (p^1, p^5)$, where $p$ is the Euclidian linear momentum, $p^4$ is defined by the mass and $p^5$ is the energy. Using $\eta_{\mu\nu}$ and taking $p^4 = m$ and $p^5 = E/m$, up to a constant $v$, that is a scale of velocity characteristic of the system, the Galilei invariant dispersion relations reads $p^2 = p^2 - 2mE$. For $d = 2$, the metric $\eta_{\mu\nu}$ is associated with the light-cone coordinates.

This geometric structure leads to the construction of a galilean covariant theory of gravitation [19,20] based on a geometric approach. It provides solutions, for instance, for the results of post-newtonian approximation of general relativity, with no expansion in powers of $1/c$ while retaining exact symmetry. Experiments, that have been suggested to test Einstein theory of gravity, can be understood equally well within this approach. This provides a strong motivation to pursue the development of galilean symmetry and the theory of gravity, that can also be useful in (2+1)-dimensional infinite-momentum frame theories.

The galilean gravity has a symmetric connection, such that the curvature ensures the dynamics of the field.
On the other hand, this suggests a formalism based in terms of a five-dimensional Weitzenböck space, where the curvature is identically zero and the presence of the field is due to a non-null torsion, corresponding to a 5-dimensional teleparallel approach.

In the context of the (3+1)-dimensional Poincaré symmetry, there are two descriptions of gravity, the geometric approach and the teleparallel formalism, which are equivalent to one another. The teleparallelism is based on a set of 4-bein (vierbein) field, which is introduced by describing the 4-dimensional space-time. Such a field is related to the metric tensor of the geometric formalism, by orthonormal relations [21, 22]. There are at least two appealing aspects to develop the teleparallelism. First, the energy-momentum tensor for the Einstein gravitational theory is defined without ambiguity. Second, a consistent definition of a coupling between the gravitation and the Dirac fields is introduced. An objective here is to show that both of these aspects are established within a Galilei teleparallel formalism.

A proper definition for the energy-momentum tensor of the galilean gravitational field is important for a comprehensive understanding of systems in the weak-field regime. However, in this case, the galilean gravity, developed as a geometric approach, shares the same problem as the Einstein theory of general relativity: there is still no agreement regarding an acceptable symmetric energy-momentum tensor. Then we develop a teleparallel version of the galilean gravity, based on the definition of a 5-bein field. As a result, we show that a symmetric energy-momentum tensor arises naturally from the field equations. In addition, we consider a coupling of the galilean gravitational field with spin-1/2 particles. The equation of motion for the spin-1/2 particles is a manifestly Galilei-covariant version of the Pauli-Schrödinger equation. There is a lack of study in the literature about such a coupling and due to the experimental interest this analysis is addressed here in detail. In these developments the notion of Galilei covariance is crucial.

The paper is organized as follows. In Section II starting with the metric \( \eta_{\mu\nu} \), we introduce the 5-bein field and establish the tangent space. In Section III we derive a teleparallel version field equations of the galilean gravity, showing the equivalence between the geometric and the teleparallel formulations. In addition, the energy-momentum tensor of the galilean gravity is derived. We apply, in Section IV the formalism to two cases, both spherically symmetric systems that represent the same configuration. In Section V the coupling of the galilean covariant Dirac field with the galilean gravity is studied. In the final section we present some concluding remarks. Appendix A gives a derivation of galilean covariant Dirac equation. Appendix B provides a justification for a metric formulation of galilean gravity, in particular the light-cone description. We use natural units with \( G = c = \hbar = 1 \).

II. THE 5-BEIN FIELD

In the absence of the gravitational field, we consider physics as being described in flat space-time, \( \mathcal{G}_{4,1} \) (see the Appendices A and B), whose metric is given in Eq. (1). Let us designate quantities invariant under galilean transformations by Latin indices. Such quantities will be called galilean vectors and tensors. Those ones invariant under coordinate transformations will be marked by using Greek indices. The local reference frame in space-time adapted to an observer is realized by a set of orthogonal 5-vectors called the 5-bein field. It is represented by

\[
e^a = e^a_{\mu} dx^\mu = (e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}, e^{(5)}) ,
\]

where the first three components are chosen following the orientation of cartesian axes of the local reference frame. The fourth component defines the field velocity of the observer and the fifth one fixes the energy of such an observer. Hereafter we shall use the term 5-bein field for \( e^a_{\mu} \).

Considering a coordinate transformation, in flat space-time, \( x^a \to x'^{\mu} \) such as \( x^{(1)} = r \sin \theta \cos \phi \), \( x^{(2)} = r \sin \theta \sin \phi \), \( x^{(3)} = r \cos \theta \), \( x^{(4)} = \frac{1}{\sqrt{2}}(x^4 + x^5) \) and \( x^{(5)} = \frac{1}{\sqrt{2}}(x^4 - x^5) \), the 5-bein field is written as

\[
e^a_{\mu} = \frac{\partial x^a}{\partial x'^{\mu}} = \begin{pmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi & 0 & 0 \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi & 0 & 0 \\ \cos \theta & -r \sin \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.
\]

This coordinate transformation leads to a change in the galilean metric from \( \eta_{ab} \) to the diagonal form \( g_{\mu\nu} = diag(1, r^2, r^2 \sin^2 \theta, -1, 1) \). Although the affine connection is different from zero, the space remains flat since the curvature tensor is null. The form given in Eq. (3) will be used to construct the 5-bein field in section IV.

A coordinate system \( x^{\alpha} \), in which gravitational forces are present locally, can be realized as an inertial frame \( x^{\alpha} \) with acceleration simulating gravitational interaction. At each space-time point there is a flat tangent space. Greek indices run from 1 to 5 and label world tensors \( \eta_{\mu\nu} \). In a geometric description, gravity manifests itself by means of the curvature tensor constructed from the affine connection while in teleparallel version it is realized through the torsion tensor constructed from the Weitzenböck like connection since the curvature tensor is zero in this case.

\[
e^a_{\mu} e^b_{\nu} \eta_{\mu\nu} = g_{\mu\nu} ,
\]

\[
e_a^a e_b^b g_{\mu\nu} = \eta_{ab} .
\]

Therefore the galilean metric raises and lowers the Latin indices. In a geometric description, gravity manifests itself by means of the curvature tensor constructed from the affine connection while in teleparallel version it is realized through the torsion tensor constructed from the Weitzenböck like connection since the curvature tensor is zero in this case.
III. TELEPARALLEL GALILEAN GRAVITY

Two vectors are said to be parallel if their projections on tangent space by the action of the 5-bein field are equal. Thus, considering two vectors, $V^a(x) = e^a_\mu V^\mu(x)$ and $V^a(x+dx) = e^a_\mu V^\mu(x) + (e^a_\lambda \partial_\mu V^\lambda + V^\lambda \partial_\mu e^a_\lambda)dx^\mu$, separated from each other by an infinitesimal displacement, the teleparallelism or distant parallelism is obtained if

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + (e^a_\mu \partial_\nu e^a_\lambda) V^\lambda = 0,$$

where $e^a_\mu \partial_\nu e^a_\lambda$ is called the Weitzenböck connection. The 5-bein field satisfies the same condition.

It is well known that in the Riemannian geometry, the connection (Christoffel symbols) is totally symmetric in its last two indices which implies that the torsion tensor is zero. However, Weitzenböck connection has a torsion tensor given by

$$T^a_{\mu \nu}(e) = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu ,$$

and it is non-zero. On the other hand, the curvature tensor using the Weitzenböck connection is identically zero. It is not difficult to show that both connections are related by

$$\Gamma_{\mu ab} = e^a_\lambda \Gamma_{\mu ab} + K_{\mu ab},$$

where $K_{\mu ab} = \frac{1}{2} e^a_\lambda e^b_\nu (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} + T_{\mu \lambda \nu})$ is the torsion tensor and $e^a_\lambda \Gamma_{\mu ab}$ is the affine connection.

In order to establish the teleparallel version (distant parallelism) of galilean gravity, it is worthwhile to show the relationship between the two descriptions (geometric and teleparallel). Let us first write the curvature scalar, $R(\Gamma)$, constructed from Eq. (6). It is possible to show that such a quantity is related to the curvature scalar obtained from the affine connection in the following way

$$eR(\Gamma) = eR(\Gamma) + e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) - 2\partial_\mu (eT^\mu),$$

where $e$ is the determinant of the 5-bein field $e^a_\mu$ and $T^a = T^b_{ab}$. Since the curvature tensor and all contractions in teleparallelism are identically zero, it follows that

$$eR(\Gamma) \equiv -e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) + 2\partial_\mu (eT^\mu).$$

Both sides of Eq. (8) are invariant under galilean transformations. Dropping the divergence term, the Lagrangian density is given by

$$L(e_{\mu \nu}) = -ke(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) - L_M \equiv -k e\Sigma^{abc}T_{abc} - L_M,$$

where $k = 1/(16\pi)$, $L_M$ stands for the Lagrangian density for the matter fields and $\Sigma^{abc}$ is defined by

$$\Sigma^{abc} = \frac{1}{4}(T^{a bc} + T^{b ca} - T^{c ab}) + \frac{1}{2}(\eta^{a c}T^{b} - \eta^{a b}T^{c}).$$

Performing a variational derivative of the Lagrangian density with respect to the 5-bein field $e_{a\lambda}$, we get the Euler-Lagrange equation

$$\partial_\mu (e\Sigma^{a\lambda \nu}) = \frac{1}{4k} e a_\mu (t^{a \lambda \mu} + T^{a \mu}),$$

where

$$t^{a \lambda \mu} = k(\Sigma^{bc\lambda}T_{bc\mu} - \eta^{bc}\Sigma^{bcd}T_{bcd}),$$

and $e^a_\mu T^\mu = \frac{1}{k} e\Delta_M / e_{a\lambda}$ is the energy-momentum tensor of the matter field. It is possible to show that the field equation given in Eq. (11) is equivalent to results in the description of the geometric approach of the Galilei gravity [19]. In addition, the Lagrangian densities are equivalent. Thus it is clear that the teleparallel version of the galilean gravity and its geometric description based on Riemannian geometry are indeed equivalent.

Now let us analyze the meaning of $t^{a \lambda \mu}$. In view of the antisymmetry property $\Sigma^{a\mu \nu} = -\Sigma^{a\nu \mu}$, it follows that

$$\partial_\lambda [e a_\mu (t^{a \lambda \mu} + T^{a \mu})] = 0,$$

which is the local balance equation. Therefore we identify $t^{a \lambda \mu}$ as the galilean gravitational energy-momentum tensor.

The integration of $t^{a \lambda \mu} + T^{a \mu}$ over a hyper-surface of $x^d = constant$ defines the energy-momentum vector due to the galilean gravitational and matter fields. Since the integrand is independent of $x^5$ for systems studied in this paper, the energy-momentum vector reduces to

$$P^a = \int_V a^3 x e a_\mu (t^{4 \mu} + T^{4 \mu}),$$

where $V$ is a volume of the three dimensional space. Using Eq. (11) we obtain

$$P^a = \int_V d^3x dS_j \Pi^{aj} = \oint_S dS_j \Pi^{aj},$$

where $\Pi^{aj} = 4ke \Sigma^{a4j}$. It is interesting to note that the above expression is invariant under coordinate transformation and it is projected on the tangent space, which means that it is frame dependent. In such a formalism, the gravitation is considered as a manifestation of torsion.

IV. ENERGY-MOMENTUM TENSOR OF A SPHERICALLY SYMMETRIC CONFIGURATION

There are two metrics describing spherical symmetry of a galilean gravity field. One of them, analyzed in the following subsection, is obtained when a similar form of Schwarzschild solution of general relativity is assumed [19]. Another one arises from a direct solution of the galilean field equations [20]. Both cases explain the advance of perihelion of Mercury, within galilei invariance and without invoking Lorentz symmetry. We analyze both cases using the energy-momentum tensor and considering the galilean teleparallelism.
A. Case I

Consider the metric introduced in [19]. The line element

\[ ds^2 = f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - 2f(r)(dx^4)(dx^5), \tag{16} \]

where as usual \( f = 1 - 2M/r \). It describes a spherically symmetric system and its form was set as an ansatz keeping in mind the Schwarzschild solution in general relativity [27, 28]. In addition, it leads to the same result as for general relativity for the case of the advance of perihelion of Mercury [19]. Performing a coordinate transformation

\[ x^4 = \frac{1}{\sqrt{2}}(x^4 + x^5), \quad x^5 = \frac{1}{\sqrt{2}}(x^4 - x^5), \]

leaving the others coordinates unaltered, the line element assumes the form

\[ ds^2 = f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - f(r)(dx^4)^2 + f(r)(dx^5)^2. \tag{17} \]

We choose a 5-bein field adapted to a spatial stationary frame which means that \( e_{(4)}^\mu = (0, 0, 0, c_1, c_2) \), where \( c_1 \) and \( c_2 \) are constants. The interpretation of \( e_{(4)}^\mu \) as the field velocity is supported by the fact that such a component lies along a tangent of the world line for an observer. In addition, at spatial infinity, the axes of the coordinate frame should be at rest, which is possible with the choices \( e_{(1)}^\mu = (1, 0, 0, 0, 0) \), \( e_{(2)}^\mu = (0, 1, 0, 0, 0) \) and \( e_{(3)}^\mu = (0, 0, 1, 0, 0) \) in an asymptotic limit, using cartesian coordinates. Therefore the general form of the 5-bein field is

\[
e_{\mu} = \begin{pmatrix}
A \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi & 0 & 0 \\
A \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi & 0 & 0 \\
A \cos \theta & -r \sin \theta & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{2M}{r}} & \frac{\sqrt{2M}}{r} \\
0 & 0 & 0 & \frac{\sqrt{2M}}{r} & -\sqrt{\frac{2M}{r}}
\end{pmatrix}, \tag{18}
\]

where the functions \( A, B \) and \( C \) are defined by the condition \( e_{\mu}^\nu e_{\alpha \nu} = g_{\mu \nu} \), yielding

\[
A^2 = f^{-1}(r), \\
B^2 = f(r), \\
C^2 = f(r). \tag{19}
\]

Then the determinant of the 5-bein field given in Eq. [15] is \( e = f^{1/2}r^2 \sin \theta \).

In order to obtain the energy-momentum tensor, we perform the calculations in Eq. [15] on a spherical hypersurface with infinite radius. We get the following components of \( \Sigma^{\lambda \mu \nu} \), Eq. [10].

\[
\begin{align*}
\Sigma_{141} &= \frac{1}{2}g^{11}g^{44}(g^{22}T_{224} + g^{33}T_{334} - g^{55}T_{545}), \\
\Sigma_{241} &= -\frac{1}{4}g^{22}g^{44}g^{11}(T_{214} + T_{341} + T_{124}), \\
\Sigma_{341} &= -\frac{1}{4}g^{33}g^{44}g^{11}(T_{314} + T_{431} + T_{134}), \\
\Sigma_{441} &= \frac{1}{2}g^{44}g^{11}(g^{22}T_{212} + g^{33}T_{313} + g^{55}T_{515}), \\
\Sigma_{541} &= -\frac{1}{4}g^{55}g^{44}g^{11}(T_{514} + T_{415} - T_{145}). \tag{20}
\end{align*}
\]

The non-null components of the torsion tensor are

\[
\begin{align*}
T_{212} &= r(1 - A), \\
T_{313} &= T_{212} \sin^2 \theta, \\
T_{515} &= \frac{1}{2}\partial_1 (C^2). \tag{21}
\end{align*}
\]

Using Eq. [15], Eq. [20], and Eq. [21], the non-null components of the energy-momentum vector are

\[
\begin{align*}
P^{(4)} &= \frac{\sqrt{2M}}{4}, \\
P^{(5)} &= \frac{\sqrt{2M}}{4}. \tag{22}
\end{align*}
\]

This result is expected since we are dealing with a frame which is at rest. Therefore in such a frame \( P^{(1)}, P^{(2)} \) and \( P^{(3)} \) should be null. The Casimir invariant is given by \( P^2 = P_\alpha P_\alpha = -2P^{(4)}P^{(5)} \), i.e., it is \( P^2 = -M^2/4 \).

B. Case II

Let us now analyze the second case. The following line element

\[ ds^2 = f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - f(r)(dx^4)^2 + (dx^5)^2. \tag{23} \]

describes the invariant interval on the curved manifold with spherical symmetry [29]. It was obtained as an exact solution of the field equations in galilean gravity and applied to analyze the advance of the perihelion of Mercury and the bending of light. Using the coordinate transformations \( x^4 = \frac{1}{\sqrt{r}}(x^4 + x^5) \) and \( x^5 = \frac{1}{r^2}(x^4 - x^5) \), keeping the other variables the same, the line element becomes

\[ ds^2 = f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - f(r)(dx^4)^2 + (dx^5)^2, \tag{24} \]

where \( f = 1 - 2M/r \).

Let us choose a 5-bein field adapted to a spatial stationary frame. It has to obey the same conditions and has the form of Eq. [15], with the functions \( A, B \) and \( C \) chosen as

\[
\begin{align*}
A^2 &= f^{-1}(r), \\
B^2 &= f(r), \\
C^2 &= 1. \tag{25}
\end{align*}
\]
The determinant of the 5-bein field is \( e = \sqrt{\sin^2 \theta} \).

Again, we need the components \( \Sigma^{\mu \nu 1} \) in order to calculate the stress-energy momentum vector, expressing them in terms of the torsion tensor, we find

\[
\Sigma^{141} = \frac{1}{2} \gamma^{11} g^{44} (g^{2 2} T_{224} + g^{3 3} T_{334} - g^{5 5} T_{554}) ,
\]

\[
\Sigma^{241} = -\frac{1}{4} g^{2 2} g^{4 4} (T_{214} + T_{412} + T_{124}) ,
\]

\[
\Sigma^{341} = -\frac{1}{4} g^{3 3} g^{4 4} (T_{314} + T_{413} + T_{134}) ,
\]

\[
\Sigma^{441} = \frac{1}{2} g^{11} g^{44} (g^{2 2} T_{212} + g^{3 3} T_{313} + g^{5 5} T_{513}) ,
\]

\[
\Sigma^{541} = -\frac{1}{4} g^{5 5} g^{4 4} (T_{514} + T_{415} + T_{145}) .
\]

The only non-null components of the torsion tensor appearing in above expressions are

\[
T_{212} = r (1 - A) ,
\]

\[
T_{313} = T_{212} \sin^2 \theta ,
\]

which yield the following non-null components of \( P^a \) when used in Eq. (15)

\[
P^{(4)} = \frac{\sqrt{2} M}{2} ,
\]

\[
P^{(5)} = \frac{\sqrt{2} M}{2} .
\]

This result is also expected once the 5-bein field is chosen with the frame spatially at rest. The difference between the two cases is the factor \( \frac{\sqrt{2}}{2} \) appearing in Eq. (28), while in Eq. (22) it is \( \frac{\sqrt{2}}{2} \). Constructing \( P^2 \), from components (28), it is \( -M^2 \).

By means of the analysis of the Casimir invariant \( \mathcal{P}^a \mathcal{P}_a = P^2 \) for each case, we conclude that the second metric is more appropriate to describe a spherically symmetric configuration in the realm of galilean gravity.

V. COUPLED GALILEAN GRAVITY AND DIRAC FIELD

Let us consider a galilean covariant Dirac field, \( \Psi(x) \), defined on the five-dimensional manifold \( G_{(4+1)} \) with the galilean metric \( \mathcal{P}^a \mathcal{P}_a = P^2 \). Then the lagrangian density for such a field is given by

\[
\mathcal{L}_M = \overline{\Psi}(x)(i \gamma^\alpha \partial_\alpha - \mu) \Psi(x)
\]

where \( \phi \partial_\alpha \psi \equiv \frac{1}{2} [\phi \partial_\alpha \psi - (\partial_\alpha \phi) \psi] \) and the adjoint field is defined as

\[
\overline{\Psi}(x) = \Psi^\dagger(x) \gamma^0 ,
\]

where

\[
\gamma^{(0)} = \frac{1}{\sqrt{2}} \left( \gamma^{(4)} + \gamma^{(5)} \right) .
\]

The field \( \Psi(x) \) and its adjoint \( \overline{\Psi}(x) \) obey anticommutation relations. The quantity \( \mu \) is the invariant associated with the square momentum on \( G_{(4+1)} \). Thus defining a 5-momentum as \( p_a = i \partial_a = (i \nabla_a, i \partial_a, i \partial_b) = (p, E, m) \), an invariant is \( p^2 = \eta^{ab} p_a p_b = \mu^2 \). The matrices \( \gamma^a \) obey the Clifford algebra

\[
\{ \gamma^a, \gamma^b \} = 2 \eta^{ab} .
\]

In Appendix A the representations of \( \gamma^a \) in 4-dimension are given, for a \((3+1)\) dimensional space-time.

Let us apply the variational principle to the free Lagrangian (29). Then the Euler-Lagrange equations of motion for \( \Psi(x) \) and its adjoint \( \overline{\Psi}(x) \) are respectively

\[
(i \gamma^a \partial_a - \mu) \Psi(x) = 0 \quad \text{and} \quad \overline{\Psi}(x)(i \gamma^a \partial_a + \mu) = 0 ,
\]

where \( \phi \partial_\alpha \psi = (\partial_\alpha \phi) \psi \).

In order to couple the galilean gravity with the galilean-covariant Dirac field, we make use of the Levi-Civita connection \( \omega_{\mu ab} \) which transforms as a vector under coordinate transformations and as a galilean tensor of rank two. Such a connection was introduced in the framework of teleparallelism equivalent to the general relativity to couple gravitation in \((3+1)\) dimension with the Lorentz invariant Dirac field. It leads to the correct field equations and to a vanishing skew-symmetric part of the stress tensor of matter fields [31]. Therefore we shall use the prescription \( \partial_\alpha \to D_\alpha \equiv e_\alpha^\mu D_\mu \), where \( D_\mu \) is given by

\[
D_\mu = (\partial_\mu - \frac{1}{2} \omega_\mu^{ab} \Sigma_{ab}) ,
\]

with \( \Sigma_{ab} = \frac{i}{2}[\gamma^a, \gamma^b] \) and

\[
0_{\omega_{\mu ab}} = -\frac{1}{2} e^c_\mu (\Omega_{abc} - \Omega_{bac} - \Omega_{cab}) ,
\]

\[
\Omega_{abc} = e_{av} (e_b^\nu \partial_\nu e_c^\nu - e_c^\nu \partial_\nu e_b^\nu) ,
\]

the Christoffel symbols \( \Gamma_{\mu \nu}^\lambda \) and the Levi-Civita connection are identically related by \( \partial_\mu e^a_\nu + 0_{\omega^{ab}} 0_{\omega^{ab}} - \partial_\mu e^a_\nu = 0 \), or

\[
0_{\Gamma_{\mu \nu}^\lambda} = e^a_\nu \partial_\mu e_{av} + e^a_\nu (0_{\omega_{\mu ab}}) e^b_\nu .
\]

Taking into account (30) we get \( 0_{\omega_{\mu ab}} = -K_{\mu ab} \). Therefore we can write

\[
D_\mu = (\partial_\mu + \frac{1}{2} K_{\mu ab} \Sigma_{ab}) .
\]

Thus applying this prescription, the modified lagrangian density for the Dirac field is

\[
\mathcal{L}_M = e \overline{\Psi}(x)(i \gamma^\nu D_\nu - \mu) \Psi(x) ,
\]

where \( e \) is the determinant of 5-bein field and \( \gamma^\mu = e_\alpha^\mu \gamma^a \). Applying the variational derivative with respect to \( \Psi(x) \) and to its adjoint respectively we get

\[
(i \gamma^\nu D_\nu - \mu) \Psi(x) = 0 \quad \text{and} \quad \overline{\Psi}(x)(i \gamma^\nu D_\nu + \mu) = 0 .
\]
VI. CONCLUSION

Using light-cone coordinates in a (4+1)-dimensional pseudo-Riemannian manifold, $\mathcal{G}$, the Galilei symmetry, in (3+1)-dimensions, is introduced in a covariant way. This manifold is used to define 5-bein fields in the tangent space. The manifestly Galilei-covariant field equations for the teleparallel formalism of the galilean gravity are derived. We show the equivalence of the teleparallelism and the geometric approach. An energy-momentum tensor is obtained by analyzing conserved quantities from the field equations.

We have developed a coupling of the galilean-covariant Dirac field with the 5-bein field by introducing a covariant derivative $\partial_{\alpha} \rightarrow D_{\alpha}$. We used a Levi-Civita connection following Coote and Macfarlane [31] when they treated the coupling of the Poincaré-covariant Dirac field with gravity in the 5-bein formalism. They showed that the Levi-Civita connection generates the correct field equations and is the only possibility that leads to a symmetric energy-momentum tensor for the Dirac field. The Galilei covariant form of the lagrangian density for the Dirac field allows the use of the same prescription. It is important to note that the introduction of the galilean metric tensor for the space-time has been a central element to obtain these results.

We apply the formalism to two cases of spherical symmetry and show, by construction of the Casimir invariants, that the second line element presented is more appropriate to describe such a configuration. We address this problem from the point of view of the teleparallel version of the galilean gravity where conserved and frame dependent quantities are allowed to exist.

As a final observation, it is important to emphasize that the galilean gravity reproduces the results of post-newtonian approximation of general relativity, with no expansion in powers of $1/c$, while maintaining the exact symmetry. And it is suitable for an understanding of various experiments that have been considered to establish the Einstein theory of gravity. A galilean-covariant teleparallel formalism is relevant since the coupling of gravity with spin 1/2 particles and the symmetric energy-momentum tensor are introduced without ambiguity.

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Appendix A: Galilei-covariant Dirac equation

In this appendix we describe the Lagrangian formalism for scalar and spinor fields in a galilei covariant version. We use the metric given in Eq. (1) with a dispersion relation given as $2mE = p^2 + \text{const.}$.

The galilean covariance is introduced as an embedding of the space-time point $(x, t)$ into a $(D + 1)$-dimensional space-time. We consider $D = 4$, such that the coordinates are denoted as $x \equiv (x^\mu) = (x^1, x^2, x^3, x^4, x^5) = (x, t, s)$ and the metric given as Eq. (1). This corresponds to a 4 + 1 de Sitter space, $\mathcal{G}_{4,1}$, since the metric can be diagonalized to diag$(1, 1, 1, -1, 1)$. The scalar product $(x|y) = \eta_{\mu \nu} x^\mu y^\nu = x^1 y^1 - x^2 y^2 - x^3 y^3 - x^4 y^4$ is invariant under linear transformations $x^\mu = \Lambda^\mu_{\nu} x^\nu$.

A particular choice of $\Lambda$ is

$$
\Lambda^\mu_{\nu} \equiv \begin{pmatrix} R^1_1 & R^1_2 & R^1_3 & -V^1 & 0 \\
R^2_1 & R^2_2 & R^2_3 & -V^2 & 0 \\
R^3_1 & R^3_2 & R^3_3 & -V^3 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-V_1 R^1_1 & -V_i R^2_i & -V_i R^3_i & 1 & 2V^2 & 1 
\end{pmatrix} \tag{A1}
$$

Then a 5-vector $x^\mu$ is transformed by

$$
x'^i = R^i_j x^j - V^i x^4
$$

$$
x'^4 = x^4
$$

$$
x'^5 = x^5 - V_i (R^5_j x^j) + \frac{1}{2} V^2 x^4
$$

where $i, j = 1, 2, 3$ and $V = (V_1, V_2, V_3) = (V^1, V^2, V^3)$ is the relative velocity. These correspond to the Galilei homogeneous transformations for the components $x^i$, space, and $x^4$, time. The component $x^5$ is associated with the velocity $V^i$, in the following sense. Considering the 5-vector in the light-cone, we have

$$
dx^\mu dx^\mu = (dx^2)^2 - 2dx^4 dx^5 = 0
$$

resulting in $dx_5 = V \cdot dx/2$. Then the Galilei invariant physics is derived by the embedding of the (3+1)-dimensional space-time in the light-cone of the (4+1)-dimensional de Sitter space-time.

An arbitrary Galilei-vector $A^\mu$, that transforms as $A'^\mu = \Lambda^\mu_{\nu} A^\nu$, under a Galilei boost transformation (taking $\mathbf{R} = 1$) reads

$$
A'^i = A^i - V^i A^4
$$

$$
A'^4 = A^4
$$

$$
A'^5 = A^5 - V^i A^i + \frac{1}{2} V^4 V^i A^4 \tag{A3}
$$

Basic vectors are then $x^\mu = (x, t, s)$ and its conjugate momentum $p^\mu = (p, p^4, p^5) = (p, m, E)$. The transformation of $p$ leads to

$$
p'^i = \mathbf{R} p - m V
$$

$$
m' = m
$$

$$
E' = E - (\mathbf{R} p) \cdot V + \frac{1}{2} m V^2
$$

The corresponding five-gradient is $\partial_{\mu} = (\nabla, \partial_t, \partial_\lambda)$ and using the galilean metric, Eq. (1), we have: $x_\mu =$
In this case we have the five-current which is a Galilei relic of the fifth dimension. Now we study the unitary ten dimensional Galilei algebra is a subalgebra of the fifteen dimensional set of affine transformations in $\mathcal{G}$, i.e. $x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$ which leaves the scalar product, defined with the metric $\eta_{\mu\nu}$, invariant, and such that $|A| = 1$. The Lie algebra of this group is generated by $\{M_{\mu\nu}, P_\mu\}$, where $\mu, \nu = 1, \ldots, 5$ and is analogous to the Poincaré algebra, but in a 4+1 dimensional space. Its commutation relations are

$$[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}),$$

$$[P_\mu, M_{\rho\sigma}] = -i (\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho),$$

$$[P_\mu, P_\nu] = 0.$$

The generators of the Galilei group are given by

$$M_{ij} \to \epsilon_{ijk} J_k,$$

$$M_{hi} = -M_{i5} \to K_i,$$

$$P_1 \to -H,$$

$$P_2 \to P_1,$$

$$P_5 \to -m 1,$$

where we have added the central extension $m$ through the generator $P_5$, which is another way to define mass as a relic of the fifth dimension. Now we study the unitary scalar and spinor representations of this algebra.

For a scalar representation, the starting point is the Casimir invariant, $\hat{p}^2 = k^2$, that leads to the equation $\hat{p}^2 \psi(x) = k^2 \psi$. Observe this is consistent with the mass-shell condition

$$p^2 = p^2 - 2mE = k^2.$$

The constant $k'$ may be taken, without loss of generality, to be zero. Then we obtain a galilean Klein-Gordon-like equation,

$$\partial^\mu \partial_\mu \psi(x) = \eta^{\mu\nu} \partial_\mu \partial_\nu \psi(x) = 0$$

which becomes,

$$\nabla^2 \psi(x) - 2 \partial_5 \partial_\nu \psi(x) = 0. \tag{A6}$$

Since $\hat{p}_5 = -m$ is an invariant, we have $\partial_5 \psi(x) = -\frac{i m}{\hbar} \psi(x)$. A solution is

$$\psi(x) = \exp \left( -\frac{im s}{\hbar} \right) \psi(x, t).$$

Then we obtain the Schrödinger equation,

$$\nabla^2 \psi(x, t) - 2 \left( -\frac{im}{\hbar} \right) \partial_t \psi(x, t) = 0.$$

In this case we have the five-current which is a Galilei- 

vector $\{\mathbf{J}(x), J_0(x), \mathcal{E}(x)\}$ such that

$$\mathbf{J}(x) = -\frac{1}{2} \hbar [\psi^\dagger(x) \nabla \psi(x) - (\nabla \psi^\dagger(x)) \psi(x)]$$

$$J_0(x) = m \psi^\dagger(x) \psi(x),$$

$$\mathcal{E}(x) = \frac{k^2}{2m} \nabla \psi^\dagger(x) \cdot \nabla \psi(x). \tag{A7}$$

A spin 1/2 representation is derived by setting $(\gamma^\mu \partial_\mu + k) \Psi = 0$, that in the momentum space reads $(\gamma^\mu p_\mu - ik) \Psi = 0$. The $\gamma$-matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}. \tag{A8}$$

In (4+1) dimension, a $4 \times 4$ representation for the $\gamma$ matrices is

$$\gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The adjoint spinor is defined as $\bar{\Psi} = \Psi^\dagger \zeta$ with

$$\zeta = \frac{-1}{\sqrt{2}} (\gamma^4 + \gamma^5) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A10}$$

The current is given by

$$j^\mu_{\text{Dirac}} = \frac{-i}{\sqrt{2}} \bar{\Psi} \gamma^\mu \Psi. \tag{A11}$$

Using $(\gamma^\mu \partial_\mu - k) \psi$, we get $(\eta^{\mu\nu} \partial_\mu \partial_\nu - k^2) \psi = 0$ which (for $k = 0$) is the Galilei Klein-Gordon like equation.

The Lagrangian density for the galilean-Dirac equation is

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (\gamma^\mu \partial_\mu + k) \Psi.$$

Writing the spinor field as

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{A12}$$

we find

$$\left( \sigma \cdot \mathbf{p} - ik \right) \varphi - ( \sigma \cdot m \mathbf{r} + \sqrt{2m} ) \chi = 0,$$

$$\left( \sigma \cdot \mathbf{p} + ik \right) \chi + ( \sqrt{2E} + im \mathbf{r} ) \varphi = 0. \tag{A13}$$

Subtracting the second equation from the first, we obtain

$$\left( \sigma \cdot \mathbf{p} - ik \right) \varphi + ( \sigma \cdot \mathbf{p} + ik ) \chi - \sqrt{2E} \varphi - \sqrt{2m} \chi = 0. \tag{A14}$$

If we express $\chi$ in terms of $\varphi$ as

$$\chi = \frac{\sqrt{2}}{2m} (\sigma \cdot \mathbf{p} - ik ) \varphi \tag{A15}$$

we find that, with the non-minimal substitution

$$\mathbf{p} \to \mathbf{p} - im \mathbf{r}, \tag{A16}$$
the relation between $\chi$ and $\varphi$ is
\[
\chi = \frac{\sqrt{2}}{2m} (\sigma \cdot p_+ - ik) \varphi
\] (A17)
where we use the notation $p_\pm \equiv p \pm im\omega r$. Then $\varphi$ satisfies the equation
\[
\frac{\sqrt{2}}{2m} (\sigma \cdot p_+ + ik)(\sigma \cdot p_- - ik)\varphi - \sqrt{2}E\varphi = 0
\] (A18)
from which we obtain
\[
E\varphi = \left( \frac{p^2}{2m} + \frac{m\omega r^2}{2} - \frac{3\omega}{2} \right) \varphi - \frac{ik}{2m} (\sigma \cdot p_+ - \sigma \cdot p_-) + \frac{k^2}{2m} \varphi
\] (A19)
or
\[
E\varphi = \left( \frac{p^2}{2m} + \frac{m\omega r^2}{2} - \frac{3\omega}{2} + k\omega\sigma \cdot r + \frac{k^2}{2m} \right) \varphi
\] (A20)
By redefining $E \rightarrow E + \frac{k^2}{m}$, we have
\[
E\varphi = \left( \frac{p^2}{2m} + \frac{m\omega r^2}{2} - \frac{3\omega}{2} + k\omega\sigma \cdot r \right) \varphi.
\] (A21)
This equation therefore describes a non-relativistic harmonic oscillator.

For a Galilei theory on the light-cone of the (3+1) Minkowski space-time, the vectors are written as $x = (x^\mu) = (x^1, x^2, x^3, x^4) = (x, x^-, x^+)$. The covariant equations we have derived for the scalar and spinor representations are recovered by the change of notations, $x^1 \rightarrow x^-$ and $x^5 \rightarrow x^+$. For the Dirac equation, the Dirac algebra is generated by the four $\gamma$-matrices,
\[
\gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},
\]
\[
\gamma^- = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \end{pmatrix}, \quad \gamma^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (A22)

Appendix B: Metric Formulation of Galilean Gravity

The purpose of this appendix is to recall the ideas developed in Ref. [19] about the geometric approach to galilean gravity. Thus we start with the Galilei transformations written in a covariant form as described in the last appendix and construct a galilean theory of gravity by introducing a Riemannian manifold where locally we have the (4+1)-space-time, $\mathcal{G}_{4,1}$. We carry out the reduction from the (4+1)-dimensional space-time to the (3+1) dimensional manifold in order to provide physical interpretation of field equations.

Taking a non-flat manifold where locally the metric is $\eta_{\mu\nu}$ (defined in A), we define a covariant derivative as
\[
\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\lambda\mu} X^\lambda,
\] (B1)
where $\Gamma^\nu_{\lambda\mu}$ is a connection that stipulates the nature of the galilean space-time.

Let us consider the action of the covariant derivative on a vector field $X^\mu$, introducing the curvature tensor. Hence we write
\[
\nabla_\mu \nabla_\lambda X^\nu = \frac{1}{2} R^\nu_{\gamma\mu\lambda} X^\gamma,
\] (B2)
where $R^\nu_{\gamma\mu\lambda}$ is the curvature tensor in (4+1) dimensions. We assume that when there is no gravitational field, the curvature tensor vanishes.

The metric tensor is a covariant tensor of rank 2. It is used to define distances and lengths of vectors. The infinitesimal distance between two points $x^\mu$ and $x^\mu + dx^\mu$ in the curved manifold defined from $\mathcal{S}_{4,1}$ is
\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu,
\] (B3)
where $g_{\mu\nu}$ is the metric tensor. The relation given in Eq. (B3) represents the line element as well. We emphasize that the metric $\eta_{\mu\nu} = \text{diag}(1, 1, 1, -1, -1)$ defines a flat line element. The imposition that the covariant derivative of metric is zero yields the following expression for the connection
\[
\Gamma^\nu_{\lambda\mu} = \frac{1}{2} g^{\nu\delta}(\partial_\lambda g_{\delta\mu} + \partial_\mu g_{\delta\lambda} - \partial_\delta g_{\lambda\mu}).
\] (B4)
In this case the manifold is said to be affine and the curvature tensor satisfies the following properties:
\[
R_{\mu\nu\lambda\gamma} = -R_{\mu\gamma\nu\lambda} = -R_{\nu\mu\lambda\gamma} = R_{\lambda\gamma\mu\nu},
\]
\[
R_{\mu\nu\lambda\gamma} + R_{\nu\mu\lambda\gamma} + R_{\lambda\mu\gamma\nu} + R_{\gamma\lambda\mu\nu} = 0.
\] (B5)
These properties are derived from Eq. (B2). If we perform a contraction of the indices of the curvature tensor then it is possible to define the Galilei-invariant curvature scalar
\[
R = g^{\mu\nu} g^{\gamma\lambda} R_{\gamma\mu\lambda\nu}.
\] (B6)

The field equations are derived from a Lagrangian invariant under galilean transformations. A natural candidate is the curvature scalar, that gives rise to the action
\[
I = \int d\Omega \sqrt{-g} R + k \mathcal{L}_m,
\] (B7)
where $g = \text{det} g_{\mu\nu}$, $k$ is the coupling constant, $\mathcal{L}_m$ is a matter lagrangian density and $d\Omega$ is the 5-dimensional element of volume. Varying the action with respect to $g_{\mu\nu}$, we obtain
\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = k T^{\mu\nu},
\] (B8)
where $T^\mu_\nu$ is the energy-momentum tensor of matter fields and $R_{\mu\nu} = R^\lambda_\mu_\rho_\lambda_\nu_\rho$. These equations have the same form as those ones describing the general relativity equations in (3+1)-Lorentz space-time. Here, however, Eq. (13) expresses a (3+1)-galilean-transformation invariant gravity theory when analyzed in the light-cone. The quantity $k$ is the coupling constant between the galilean gravity and matter fields. Let us consider an example.

Considering a spherically symmetric line element, we obtain a geodesic motion by analyzing the functional variation of

$$K = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,$$

which takes the possible values 1, -1 or 0. This relation is similar to the galilean mass-shell condition, Eq. (14), and leads to

$$U'' + U = \frac{GM}{r^2} + \frac{3G}{c^2}MU^2,$$  \hspace{1cm} (B10)

where $U$ is the function that describes the trajectory and the prime stands for the derivative with respect to the angle $\phi$. Then we have derived, for instance, the same value for the advance of the Mercury perihelion, as it is obtained in general relativity considering the post-newtonian limit. From this result, we conclude that the galilean gravity theory is, physically, a covariant version of the weak-field limit of the general relativity. This results is appealing, in particular, for experimental purposes and represents a prescription to find such a limit.

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