INVARIANCE PRINCIPLE FOR LOCAL TIME BY QUASI-COMPACTNESS

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Abstract. The objective of this paper is to prove a functional weak invariance principle for a local time of a process of the form $X_n = \varphi \circ T^n$ where $(X, \mathcal{B}, T, m)$ is a measure preserving system with a transfer operator acting quasi-compactly on a large enough Banach space of functions and $\varphi \in L^2(m)$ is an aperiodic observable.

1. Introduction

To introduce the motivation behind the work in this paper, we first describe the relevant problem and results in the classical case where $(X_n)$ is a sequence of independent identically distributed random variables. In this case, setting $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$, $S_0 = 0$, the classical invariance principle states that the sequence $S_n / \sqrt{n}$ converges in law to the Gaussian distribution. Recall that a càdlàg function is a function that is continuous on the right with finite limits on the left of every point in its domain of definition. We denote by $D$ the space of càdlàg functions on $[0,1]$. Setting $\omega_n (t) = 1 / \sqrt{n} \sum_{k=0}^{[nt]} S_k$, $t \in [0,1]$, where $[x]$ is the integral value of $x$, we obtain a sequence of càdlàg functions and a stronger, functional invariance principle, stating that the random functions $\omega_n (\cdot)$ converge in law to the Brownian motion $\omega (\cdot)$, where $\omega (\cdot)$ is uniquely determined by the equality $E \left( (\omega (1))^2 \right) = E \left( X_i^2 \right)$ (see [Bil]).

For a general function $f \in D$, the occupation measure of $f$ up to time 1 is defined by

\begin{equation}
\nu_f (A) = \int_0^1 1_A (f (t)) \, dt, \quad A \in B (\mathbb{R})
\end{equation}

where $B (\mathbb{R})$ denotes the Borel $\sigma$-field on $\mathbb{R}$. Recall that the occupation measure of the Brownian motion is almost surely absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ [MP]. The (random) density function with respect to the occupation measure, which we
denote by $l(x)$, is the local time of the Brownian motion. Thus
\[
\int_0^1 1_A(\omega(t)) \, dt = \int_A l(x) \, dx
\]
for all $A \in \mathcal{B}(\mathbb{R})$, at almost every sample point of the Brownian motion $\omega$. Moreover, $l(x)$ is almost surely continuous.

Local time $l_n(x)$ of the process $\omega_n$, (which we proceed to define in what follows) may be roughly regarded as the density of the occupation measure $\nu_{\omega_n}$, in the sense that $\nu_{\omega_n} [a,b] - \int_{[a,b]} l_n(x) \, dx$ converges to 0 in law. To define $l_n$ we distinguish between the lattice and non-lattice case, namely between the case when $(X_n)$ is a sequence of random variables taking values in the lattice $\mathbb{Z}$, and when $(X_n)$ is a sequence of random variables taking values in $\mathbb{R}$.

In the lattice case define
\[
l_n(x) := n^{-\frac{1}{2}} \# \{ k \in \{ 1, \ldots, \lfloor nt \rfloor \} : S_k = \lfloor \sqrt{n} x \rfloor \}
\]
where $\lfloor \cdot \rfloor$ is the integral value function. Thus, $l_n(x)$ is the number of arrivals of the random walk $(S_k)$ at the point $((\sqrt{n} x))$ up to time $n$, normalized by $\sqrt{n}$.

In the non-lattice case, let $f : \mathbb{R} \to \mathbb{R}$ be continuous and integrable with integral 1 and define
\[
l_n(x) := n^{-\frac{1}{2}} \sum_{k=1}^{n} f(S_n - \sqrt{n} x).
\]
The invariance principle for local time implies convergence of $l_n(x)$ to the local time of the Brownian motion $l(x)$. More precisely, we say that the invariance principle for local times holds if the sequence $(\omega_n, l_n)$ converges in law to $(\omega, l)$.

The invariance principle for local times in the lattice case, under the assumption of aperiodicity on the random walk was proved in [Bor]. The invariance principle for the non-lattice case, under assumption of aperiodicity and an assumption that the characteristic function of $X_1$ is square integrable was proved in [BI].

There has been a considerable amount of research invested into generalizing the invariance principles in the independent case to the more general settings of various mixing conditions on the processes $(X_n)$, but to the author’s knowledge, no such generalization appeared in literature for the invariance principle for local time until [BK], where the author and Z.Kosloff prove
the invariance principle for local time in the case where \((X_n)\) is a finite state, lattice valued Markov chain. In [Br], a further generalization was proved by the author to the case where \((X_n)\) are of the form \(X_n := \varphi \circ T^n\), where \((X, C, m, T)\) is a Gibbs-Markov system and \(\varphi\) is an aperiodic, square integrable function with values in \(\mathbb{Z}\). The purpose of this paper is to give a proof of the invariance principle for local time in the non-lattice case given that the functional invariance principle holds, under the setting, where the random variables \((X_n)\) are generated by a dynamical system \((X, C, m, T)\) with a quasi-compact transfer operator.

1.1. **Outline of the remaining sections.** Section (2) describes the assumptions used in the proof of the main theorem and develops the basic tools needed for the proof. (3) describes the notions of convergence used in this paper and states the main theorem. Some concrete systems where our assumptions hold, as well as applications of the main theorem are provided in section (4). Section (5) provides the probability estimates needed for the proof. Section (6) proves tightness of the local time process, while section (7) finalizes the proof, by identifying the only possible limit of the local time process.

2. **Characteristic function operators and expansion of the main eigenvalue**

Let \((X, C, m, T)\) be a probability measure preserving dynamical system. Let \(\varphi : X \to \mathbb{R}\) be measurable, and let

\[
X_n := \varphi \circ T^{n-1}, \quad S_n := \sum_{k=1}^{n} X_k, \quad n \in \mathbb{N}, \quad S_0 = 0.
\]

(2.1)

Consider \(T\) as an operator on \(L^\infty (m)\) defined by \(Tf = f \circ T\). Then the transfer operator, also known as the Frobenius-Perron operator, \(\hat{T} : L^1 (m) \to L^1 (m)\) is the pre-dual of \(T\), uniquely defined by the equation

\[
\int f \cdot g \circ T \, d\mu = \int \hat{T} f \cdot g \, d\mu \quad \forall f \in L^1, \quad g \in L^\infty.
\]

We note for future reference that \(\hat{T}\) is a positive operator in the sense that \(\hat{T} f \geq 0\) if \(f \geq 0\) and \(\hat{T}\) real in the sense that if \(f\) is a real valued function, then \(\hat{T} f\) is real valued.

The characteristic function operators associated to \(\varphi\) is a family of operators \(P(t) : L^1 (m) \to L^1 (m)\) defined for all \(t \in \mathbb{R}\) by

\[
P(t) f := \hat{T} \left( e^{it\varphi} f \right).
\]
Note that at \( t = 0 \) we have the equality \( P(0) = \hat{T} \). The reason for the name of \( P(t) \) is due to the fact that \( m(e^{it\varphi}f) = m(P(t)(f)) \) and in particular \( m(P(t)1) \) gives the characteristic function of \( \varphi \). Characteristic function operators may be used to prove convergence theorems and estimates for \((X_n)\) and \((S_n)\) similarly to the way characteristic functions are used for the independent case. In particular powers of \( P(t) \) give rise to characteristic functions of \( S_n \) as shown by the following equality (which is proved by simple induction using definitions).

\[
m(e^{itS_n}) = m(P^n(t)1).
\]

**Definition 1.** An operator \( T \) on a Banach space \( B \) is called quasi-compact with \( s \) dominating simple eigenvalues if

1. There exist \( T \)-invariant spaces \( F \) and \( H \) such that \( F \) is an \( s \) dimensional space and \( B = F \oplus H \).
2. \( T \) is diagonalizable when restricted to \( F \), with all eigenvalues having modulus equal to the spectral radius of \( T \) which we denote by \( \rho(T) \).
3. When restricted to \( H \), the spectral radius of \( T \) is strictly less than \( \rho(T) \).

Quasi-compactness of the characteristic function operator acting on a large enough Banach space of functions, essentially helps in reducing the behavior of the characteristic functions of \( X_n \) to a more familiar i.i.d case. In order to establish an invariance principle for local time we make the following assumptions.

### 2.1. Assumptions.

- **(A1)** There exists a Banach space \( B \subseteq L^\infty(m) \) with norm \( \| \cdot \| \) satisfying \( \| \cdot \|_\infty \leq C \| \cdot \| \) for some \( C > 0 \), such that \( 1 \in B \), and \( f \in B \implies \hat{f} \in B, |f| \in B \).
- **(A2)** (Quasi-Compactness) \( \hat{T} : B \to B \) is quasi compact with one dominating simple eigenvalue equal to 1, and is given by \( \hat{T}(f) = m(f)1 + N(f) \) where the spectral radius of \( N \) satisfies \( \rho(N) < 1 \).
- **(A3)** (Mean zero and finite second moment) \( \varphi \in L^2(m) ; m(\varphi) = 0 \).
- **(A4)** (Continuity) \( t \mapsto P(t) \) is a continuous function from \( \mathbb{R} \) to \( Hom(B,B) \).
- **(A5)** (Differentiability) There exists a neighborhood \( I_0 \) of 0, such that for all \( t \in I_0, P(t) : I_0 \to Hom(B,B) \) is twice continuously differentiable and

\[
P'(0)(f) = \hat{T}(if), \quad P''(0)(f) = \hat{T}(-\varphi^2f).
\]
• (A6) (Aperiodicity) The spectral radius of $P(t)$ as an element of $\text{Hom}(\mathcal{B},\mathcal{B})$, satisfies $\rho(P(t)) < 1, \forall t \neq 0$.

2.2. Remarks. (A2) gives quasi-compactness of the characteristic function operator, with one dominating simple eigenvalue. It follows that the eigenspace corresponding to the dominating eigenvalue is the space of constant functions and the projection onto this eigenspace is given by $f \mapsto m(f) \mathbb{1}$. Note that it is a consequence of the definition of the transfer operator that 1 is always an eigenvalue of $\hat{T}$, since $\hat{T}(\mathbb{1}) = \mathbb{1}$. In applications, the requirement for the eigenvalue 1 to be simple corresponds to assumption of ergodicity of the system $(X, \mathcal{B}, m, T)$, while the lack of other eigenvalues of modulus 1 corresponds to a weak mixing condition on the system (see section 4 for concrete examples). The condition $\mathcal{B} \subseteq L^\infty$ may be replaced by $\mathcal{B} \subseteq L^p, p \geq 1$, with a similar condition on norms.

(A4) and (A5) guarantee continuity in $\mathbb{R}$ and differentiability near 0 of the characteristic function operators. Even though we assume that $\varphi \in L^2(m), (A4) and (A5) do not follow, since we do not assume that $\varphi \in \mathcal{B}$, or that $e^{it\varphi} \in \mathcal{B}$ and we make no assumptions about $\mathcal{B}$ being closed under multiplication. Therefore, without (A4) we cannot even conclude that $P(t)$ is $\mathcal{B}$ invariant. In the Gibbs-Markov case for example (see section 4), we do not require that $\varphi \in \mathcal{B}$, but still assumptions (A4) and (A5) are valid. Note, that the formula for $P'(0)$ assumes that the derivative of $P'(0)$ is what one expects it to be, i.e analogous to the derivative of the characteristic function.

Finally (A6) corresponds to an assumption of aperiodicity of the function $f$. The name is derived from references to examples in section 4 where it is shown that this requirement is equivalent to $e^{it\varphi}$ not being cohomologous to a constant. Functions satisfying this last property are usually called aperiodic. This is a standard assumption for proving local limit theorems, but is not required for the central limit theorem. If $(X_n)$ are i.i.d’s then aperiodicity corresponds to the requirement that the modulus of the characteristic function $E(e^{itX_n})$ has modulus strictly less than 1, for all $t \neq 0$. This requirement is satisfied if and only if the random walk $S_n$ does not take values on a lattice in $\mathbb{R}$.

2.3. A perturbation theorem and its implications. The proofs of this section follow the methods that first appeared in [Na] for analytic perturbations (see also [HeH], [GH]). We adapt these to our setting. In what follows $C^m(I, \mathcal{B})$ is used to denote the space of $m$ times continuously differentiable functions from $I$ to a Banach space $\mathcal{B}, \mathcal{B}^*$ is the dual space of $\mathcal{B}$ and
\( \langle \xi, f \rangle \) denotes the action of \( \xi \in \mathcal{B}^* \) on \( f \in \mathcal{B} \). The following proposition is a direct implication of a standard perturbation theorem (see [HeH], Theorem III.8).

**Proposition 2.** Let assumptions (A1) – (A5) be satisfied. Then there exists an open neighborhood of 0 \( I \subseteq I_0 \), and functions \( \lambda (t) \in C^2 (I, \mathbb{C}), \xi (t) \in C^2 (I, \mathcal{B}^*), \eta (t) \in C^2 (I, \mathcal{B}), N (t) \in C^2 (I, \text{Hom} (\mathcal{B}, \mathcal{B})) \) such that for \( t \in I \),

\[
P (t) \eta (t) = \lambda (t) \eta (t), \quad Q (t)^* \xi (t) = \lambda (t) \xi (t) \langle \xi (t), \eta (t) \rangle = 1,
\]

for all \( n \geq 1 \)

\[
P^n (t) (\cdot) = \lambda (t) \langle \xi (t), \cdot \rangle \eta (t) + N^n (t) (\cdot)
\]

and

\[
\rho (N (t)) < q < \inf_{t \in I} |\lambda (t)|, \quad \|N^n (t)\| \leq C q^n
\]

where \( C > 0 \) and \( 0 < q < 1 \) are constants.

Note that by assumption (A2) and by the fact that \( m (\mathbb{1}) = 1 \), we have \( \xi (0) = m, \eta (0) = \mathbb{1} \).

Defining \( \tilde{\xi} (t) := \frac{\xi (t)}{\langle \xi (0), \eta (t) \rangle}, \quad \tilde{\eta} (t) = \frac{\langle \xi (0), \eta (t) \rangle}{\eta (t)} \eta (t) \) we obtain functionals \( \tilde{\xi} (t) \) and eigenvectors \( \tilde{\eta} (t) \) satisfying all the conditions of proposition \( \ref{prop2} \) in some open neighborhood \( I_1 \subseteq I_0 \) of 0, with the extra condition that

\[
\left\langle \tilde{\xi} (0), \eta (t) \right\rangle = m (\eta (t)) = 1.
\]

\( I_1 \) is chosen so that \( \langle \xi (0), \eta (t) \rangle \neq 0 \) for all \( t \in I_1 \) and is non-empty by continuity and the fact that \( \langle \xi (0), \eta (0) \rangle = 1 \). Thus, from now on we may and do assume that \( \xi (t), \eta (t) \) of proposition \( \ref{prop2} \) satisfy the extra condition

\[
\langle \xi (0), \eta (t) \rangle = m (\eta (t)) = 1
\]

for all \( t \in I \). Note that this implies that \( m (\eta' (t)) \equiv 0 \).

In what follows, we need more information on the eigenvectors and eigenvalues of \( P (t) \) in \( B (0, \delta) \) which we summarize in the following lemma.

**Lemma 3.** Let \( I, \lambda (t), \xi (t), \eta (t) \) be as in proposition \( \ref{prop2} \) satisfying \( \ref{2.2} \) and let \( \pi (t) (\cdot) := \langle \xi (t), \cdot \rangle \eta (t) \). Then

\[
\lambda (t) = 1 - \sigma^2 t^2 + o (t^2)
\]
where $\sigma \geq 0$, and there exists $\delta > 0$ and constants $c,C > 0$, such that for all $t \in (-\delta, \delta)$

\[
|\lambda(t)| \leq 1 - ct^2
\]

(2.4)

\[
\|\pi(t) f - m(f) 1\| \leq C |t|
\]

(2.5)

and $\eta'(0)$ is a purely imaginary function.

**Proof.** Since $\varphi(0) = m$ and $\eta(0) = 1$, (2.5) follows immediately from Taylor’s expansion of $\pi(t)$ at 0. To prove the other assertions write

\[
\lambda(t) = m(P(t) \eta(t))
\]

\[
= m \left( \hat{T} \left( e^{it\varphi} \eta(t) \right) \right)
\]

\[
= m \left( e^{it\varphi} \eta(t) \right)
\]

Now $\left| e^{it\varphi} - 1 - it\varphi + \frac{t^2\varphi^2}{2} \right| \leq (\varphi t)^2 \cdot \min(2, |t\varphi|)$. Since $\varphi \in L^2(m)$, by dominated convergence we have $\frac{1}{t} m \left( \varphi^2 t^2 \cdot \min(2, |t\varphi| \cdot \eta(t)) \right) \xrightarrow{t \to 0} 0$. This implies,

\[
\lambda(t) = 1 + m \left( \left( it\varphi - \frac{t^2\varphi^2}{2} \right) \eta(t) \right) + o(t^2).
\]

By Taylor’s expansion $\eta(t) = 1 + \eta'(0) t + \zeta(t)$, where $\|\zeta(t)\| = o(|t|)$ and therefore, $\|\zeta(t)\|_{\infty} = o(|t|)$. Thus, since $m(\varphi) = 0$, $\varphi^2 \in L^2(m)$,

\[
\lambda(t) = 1 - m \left( \varphi^2 \frac{t^2}{2} \right) + m \left( \varphi \eta'(0) \right) it^2 + o(t^2).
\]

It follows that $\lambda'(0) = 0$. Note that $\varphi \eta'(0)$ is $m$ integrable because $\eta'(0) \in B \subseteq L^\infty(m)$.

We prove that $\lambda''(0) \in \mathbb{R}$. Write

\[
\lambda(t) \eta(t) = P(t) \eta(t)
\]

\[
= \hat{T} \left( e^{it\varphi} \eta(t) \right)
\]

\[
= \hat{T} \left( e^{-it\varphi} \eta(t) \right).
\]

It follows that $P(-t) \eta(t) = \hat{\lambda}(t) \eta(t)$, and therefore, $\hat{\lambda}(t)$ is an eigenvalue of $P(-t)$. Since by the perturbation theorem $P(-t)$ has a unique main eigenvalue, and all other eigenvalues
are bounded away from \( \inf_{t \in I} \lambda(t) \), it follows that for \( t \in I \), \( \lambda(-t) = \overline{\lambda(t)} \). This implies that \( \lambda''(0) \) is real.

We prove that \( \lambda''(0) \leq 0 \). The following reasoning is based on the pointwise inequality \( \hat{T}^n f \leq \hat{T}^n |f| \).

\[
|\lambda^n(t) \eta(t)| = |P^n(t) \eta(t)| = \left| \hat{T}^n(e^{it\varphi}\eta(t)) \right| \leq \left| \hat{T}^n(e^{it\varphi}\eta(t)) \right| = m(|\eta(t)|) + N^n(|\eta(t)|).
\]

Since \( \|N^n\| \to 0 \) and \( B \) is continuously embedded in \( L^\infty(m) \), we have

\[
|\lambda^n(t)||\eta(t)| \leq m(|\eta(t)|) + \|N^n(|\eta(t)|)\|_\infty \to m(|\eta(t)|).
\]

This implies \( |\lambda(t)| \leq 1 \). Therefore,

\[
1 \geq |\lambda(t)|^2 = \lambda(t)\overline{\lambda(t)} = (1 + \lambda''(0) t^2 + o(t^2))^2,
\]

and it follows that \( 2\lambda''(0) \leq 0 \) if \( |t| \) is small enough.

We turn to prove that \( \eta'(0) \) is purely imaginary.

\[
P(t) \eta(t) = \lambda(t) \eta(t)
\]

implies

\[
P'(0) \eta(0) + P(0) \eta'(0) = \lambda'(0) \eta(0) + \lambda(0) \eta'(0) = \eta'(0)
\]

where the last equality follows from \( \lambda(0) = 1, \lambda'(0) = 0 \). Since \( \eta(0) = 1 \) we obtain

\[
P'(0) \mathbb{1} = (I - P(0)) \eta'(0).
\]

Now, by (2.2) \( \langle \xi(0), \eta'(0) \rangle = m(\eta'(0)) = 0 \). By the perturbation theorem, \( P(0) \) restricted to the space \( \xi(0)^\perp := \{ v \in B : \langle \xi(0), v \rangle = 0 \} \) satisfies \( P(0) = N(0) \) and \( I - N(0) \) is invertible on this space with inverse given by \( (I - N(0))^{-1} = \sum_{k=0}^\infty N^k(0) = \sum_{k=0}^\infty P^k(0)_{\xi(0)^\perp} \). Since \( P'(0) \mathbb{1} = \hat{T}(i\varphi) \) is purely imaginary and \( m(\varphi) = 0 \) implies that \( P'(0) \mathbb{1} \in \xi(0)^\perp \), we have

\[
\eta'(0) = \left( \sum_{k=0}^\infty P^k(0) \right)(P'(0) \mathbb{1})
\]
Remark. Note that only first order differentiability of $P(t)$ was used in the previous theorem. Nevertheless, we will use derivatives of second order in $\mathcal{T}$.

**Lemma 4.** Let $K \subseteq \mathbb{R}$ be a compact set such that $0 \not\in K$. Then under assumptions (A1), (A4), (A6), there exist constants $C > 0$, $0 < r < 1$ such that for all $t \in K$ we have $\|P^n(t)\| \leq Cr^n$.

**Proof.** Since the spectral radius is an upper semi-continuous function, by (A6) there exists $r < 1$, such that $\sup_{t \in K} \rho(P(t)) < r < 1$. It follows from Gelfand’s formula for the spectral radius of an operator that $r \geq \rho(P(t)) = \lim_{n \to \infty} \sqrt[n]{\|P^n(t)\|}$. Thus for $\epsilon$ such that $r + \epsilon < 1$, we have $\|P^n(t)\| \leq (r + \epsilon)^n$ if $n$ is large enough. The conclusion of the lemma follows from this. \qed

### 3. Statement of the main theorem

Recall that for a sequence of random variables $(X_n)$ taking values in a complete and separable metric space $(M, d)$ converges in distribution (or in law) to $X$ if for every continuous and bounded $f : M \to \mathbb{R}$

$$
E(f(X_n)) \longrightarrow E(f(X)).
$$

In this case we denote $X_n \overset{d}{\longrightarrow} X$.

Let $(X, \mathcal{C}, m, T)$ a probability preserving system and $\varphi : X \to \mathbb{R}$ a measurable function. Assume that assumptions (A1)-(A6) are satisfied and let $X_n$, $S_n$ be defined by (2.1). By proposition 2 and lemma 3

$$
m\left(e^{it\frac{S_n}{\sqrt{n}}}\right) = m\left(P^n\left(\frac{t}{\sqrt{n}}\right)1\right) = \lambda^n\left(\frac{t}{\sqrt{n}}\right) + m\left(N^n\left(\frac{t}{\sqrt{n}}\right)1\right) = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n + m\left(N^n\left(t1\right)\right)
$$

Therefore, $\lim_{n \to \infty} m\left(e^{it\frac{S_n}{\sqrt{n}}}\right) = e^{-\sigma t^2}$. Thus, setting $a^2 = \frac{\sigma}{2}$ it follows that $\frac{S_n}{\sqrt{n}}$ converges in distribution to the Gaussian distribution with mean 0 and variance $a^2$ ((A6) is not used for the central limit theorem). Note that the limit is degenerate if and only $\lambda''(0) = \sigma = 0$. 

is purely imaginary as claimed. \qed
Let
\[ \omega_n(t) := \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} S_k, \quad t \in [0,1]. \]
It is easily seen by definition that \( \omega_n(t) \) is a càdlàg function on \( [0,1] \) (continuous from the right with limits from the left). As stated in the introduction, we denote by \( D \) the Skorokhod space of càdlàg functions on \( [0,1] \). Recall that endowed with the Skorokhod metric which we denote by \( d_J(\cdot,\cdot) \) (see [Bil]) \( D \) is complete and separable. We denote by \( \omega(t) \) the Brownian motion on \( [0,1] \), uniquely defined by the equalities \( E(\omega(t)) = 0, \forall t \in [0,1], E(\omega(1)^2) = \frac{\sigma}{2} \) (here \( E(\cdot) \) denotes expectation with respect to the Wiener measure). It can be easily seen by arguments similar to the above that for \( 0 \leq t_1 \leq ... \leq t_k \leq 1 \) we have
\[ \left( \omega_n(t_1), \omega_n(t_2) - \omega_n(t_1), ..., \omega_n(t_k) - \omega(t_{k-1}) \right) \]
\[ \xrightarrow{d} (\omega(t_1), \omega(t_2) - \omega(t_1), ..., \omega(t_k) - \omega(t_{k-1})) \]
(3.1)

The functional central limit theorem (or the functional invariance principle) is a statement that \( \omega_n \xrightarrow{d} \omega \), where convergence takes place in the Skorokhod space \( D \). The functional invariance principle does not seem to follow from assumptions (A1)-(A5). To prove it, in addition to (3.1) one has to show that the sequence \( \omega_n \) is tight in \( D \) (for details on tightness see section 5 or [Bil]). If in addition to (A1)-(A5) one assumes that \( \varphi \in \mathcal{B} \) and \( \varphi^2 \in \mathcal{B} \), one can show that for \( r < s < t \),
\[ m\left((S_{[nt]} - S_{[ns]})^2(S_{[ns]} - S_{[nr]})^2\right) \leq C |t-r|^2. \]
which implies tightness (see [Bil]). Instead of assuming these extra conditions, which are not required for our proof of the invariance principle of local time we assume that the functional invariance principle holds. In section 4 we provide references for the functional invariance principle in concrete cases. Thus we add the extra assumption:

- (A7) \( \omega_n \) converges in law to \( \omega \) in the space \( D \), where \( \omega \) is the Brownian motion satisfying \( E(\omega(t)) = 0 \forall t \in [0,1], E(\omega(1)^2) > 0 \).

Note that by (3.1), \( \omega_n \) cannot converge to anything else except \( \omega \), and the requirement that \( E(\omega(1)^2) > 0 \) is equivalent to stating that the limit of \( \omega_n \) is non-degenerate. This in turn happens if and only if \( \lambda''(0) = \sigma > 0 \).
Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth, integrable, symmetric function with compactly supported Fourier transform. In what follows \( \hat{f} \) denotes the Fourier transform of \( f \). Let \( l_n \) be the local time of \( \omega_n \) defined by

\[
  l_n := l_n(x) := n^{-\frac{1}{2}} \sum_{k=1}^{n} f(S_n - \sqrt{nx}).
\]

By continuity of \( f \) it follows that \( m \)-almost surely \( l_n \) takes values in the space of continuous functions on \((\mathbb{R}, \mathbb{R})\) denoted by \( C \). Endow \( C \) with the topology of uniform convergence on compact sets. With respect to the metric \( d(f, g) = \frac{1}{2n} \sup_{[-n, n]} \|f - g\|_{\infty} \), \( C \) is separable and complete. Let \( l \) be the local time of the Brownian motion \( \omega \). We are now in the position to state the main theorem:

**Theorem 5.** Let \((X, C, m, T)\) be a probability preserving system and \( \varphi : X \to \mathbb{R} \) be such that assumptions (A1)-(A7) hold. Then the sequence \((\omega_n, l_n)\) converges in law to \((\omega, \int_{\mathbb{R}} f(x) dx \cdot l)\) in the space \( D \times C \).

**Remark.** Instead of assuming that the function \( f \) has compactly supported Fourier transform, we may assume, in addition to (A6) that \( \limsup_{t \to \infty} \rho(P(t)) < 1 \). This is the so called Cramer’s condition on the function \( \varphi \). It allows to extend the statement of lemma 4 to non-compact intervals that are bounded away from 0, which allows to carry out the estimates in section 5 without the assumption on \( f \) having compactly carried Fourier transform. In this case the theorem would be valid for any symmetric, integrable function \( f \), and in particular for functions of the form \( f = 1_{(-a, a)} \), where \( a > 0 \).

4. Applications and Examples

The theorem is applicable for systems where one can show that the transfer operator acts quasi-compactly on a large enough Banach space. We briefly describe two concrete example of subshifts of finite type and their generalization to a non-compact space via Gibbs-Markov maps and refer the reader to [ADSZ, HeH, LY, Yo] for other examples.

4.1. Subshifts of finite type. We refer the reader to [Bow] as a basic reference for subshifts of finite type. Denote by \( \mathbb{N}_* \) the set \( \mathbb{N} \cup \{0\} \). For \( d \in \mathbb{N} \), let \( S = \{1, \ldots, d\} \). Endow \( S^{\mathbb{N}_*} \) with the (compact) metric \( d_{\theta} (x, y) := \theta^{|x-y|} \) where \( 0 < \theta < 1 \), and \( t(x, y) = \min \{ n : x_n \neq y_n \} \), and let \( \sigma : S^{\mathbb{N}} \to S^{\mathbb{N}} \) be the left shift operator, defined by \( (\sigma x)_n = x_{n+1} \). Let \( A : S \times S \to \{0, 1\} \) be an irreducible, aperiodic matrix, i.e there exists some integer \( n_0 \), such that all entries of
$A^{n_0}$ are strictly positive and let $\Sigma_+ := \{ x \in S^{N+} : A(x_i, x_{i+1}) = 1 \forall i \in \mathbb{N}_+ \}$. Then $\Sigma_+$ is a closed, shift invariant subspace of $S^{N+}$. Let $C(\Sigma_+)$ be the Banach space of all continuous complex valued functions on $\Sigma_+$ endowed with the supremum norm $\| \cdot \|_{\infty}$. Define $F_\theta(\Sigma_+)$ to be the set of all Lipchitz continuous functions on $\Sigma_+$ endowed with the norm $\| \cdot \|_{\infty}+\| \cdot \|_{Lip}$, where $\| f \|_{Lip} = \sup_{x,y \in \Sigma_+} \frac{| f(x) - f(y) |}{d_\theta(x,y)}, F_\theta(\Sigma_+)$ becomes a Banach space.

For a function $\phi \in F_\theta(\Sigma_+)$, there exists a unique, $\sigma$-invariant Borel measure $m_\phi$, called the Gibbs measure with respect to $\phi$ satisfying

$$c_1 \leq m_\phi \{ y \in \Sigma_+, x_i = y_i, i = 1, ..., n \} \exp \left( -Pn + \sum_{i=0}^{n-1} \phi(\sigma^i x) \right) \leq c_2$$

for some constants $c_1 > 0, c_2 > 0, P$ and all $x \in \Sigma_+, n \geq 0$.

It is clear that the Banach space $B = F_\theta(\Sigma_+)$ satisfies (A1). It is a consequence of the Ruelle-Perron-Frobenius theorem combined with the assumption of irreducibility and aperiodicity of the matrix $A$ that the transfer operator $\hat{\sigma}$ of the system $(\Sigma_+, C, m_\phi, \sigma)$ satisfies (A2) (here $C$ is the Borel $\sigma$-algebra on $\Sigma_+$). Let $\varphi \in B$ such that $m(\varphi) = 0$. Note that $\varphi \in B$ implies that $\varphi$ is bounded and therefore, has finite moments of any order. It is easy to see that $B$ is closed under multiplication. This implies that the characteristic function operator has continuous derivatives of any order with derivatives given by $P(k)(t)(f) = \hat{\sigma}(i^k \varphi^k e^{it\varphi} f)$.

Thus, assumptions (A1)-(A5) are satisfied for $\varphi \in B$.

Assumption (A6) is equivalent to the following: for any $t \in \mathbb{R} \setminus \{0\}$, $e^{it\varphi}$ is not $\sigma$-cohomologous to a constant, i.e. the only solution to the equation

$$e^{it\varphi} = \frac{\lambda f \circ \sigma}{f}, \lambda \in \mathbb{T}, f : \Sigma_+ \to \mathbb{T}, \mathbb{T} = \{ z \in \mathbb{C} : z = |1| \}$$

is $\lambda = 1, f \equiv 1$. A function $\varphi$ satisfying this assumption is called aperiodic.

Finally, (A7) fails if and only if $\varphi$ is a coboundary, i.e. there exists $g : \Sigma_+ \to \mathbb{R}$ measurable, such that $\varphi = g \circ \sigma - g$. For the proof of the functional invariance principle refer to [BS].

As an application in ergodic theory of the invariance principle for local time we refer the reader to [Au] where it is used to prove that the entropy of the scenery is an invariant for random walks in random scenery processes with a subshift of finite type at the base.

4.2. Gibbs-Markov maps. We refer the reader to [AD] as a basic reference for Gibbs-Markov maps. Let $(X, \mathcal{C}, m, T)$ be a probability preserving transformation of a standard probability space. $T$ is a Markov map, if there exists a countable partition $\alpha$ of $X$ such
that $T(\alpha) \subseteq \sigma(\alpha)$ (mod $m$), $T$ when restrict to each element of the partition $\alpha$ is invertible and $\bigcup_{n=0}^{\infty} \{T^{-n}\alpha\}$ generates $\mathcal{C}$ (here $\sigma(\alpha)$ is the $\sigma$-algebra generated by $\alpha$). Write $\alpha = \{a_s : s \in \mathcal{S}\}$ and endow $\mathcal{S}^N$ with the metric $d_\theta (x, y) := \theta^{\ell(x, y)}$ where $0 < \theta < 1$, and $t (x, y) = \min \{n : x_n \neq y_n\}$. Set $\Sigma = \{s \in \mathcal{S}^N : \mu \left(\bigcap_{k=1}^{n} T^{-k+1} a_{s_k} \right) > 0 \forall n \geq 1\}$. Then $\Sigma$ is a closed, shift invariant subset of $\mathcal{S}^N$ and the system $(X, \mathcal{C}, m, T)$ is conjugate to $(\Sigma, \mathcal{B}(\Sigma), m, \sigma)$ by the map $\{\varphi(s_1, s_2, \ldots)\} := \bigcap_{k=0}^{\infty} T^{-k} a_{s_k}$, where $\sigma$ is the left shift, and $m := m \circ \varphi$. Thus, we may assume that $X = \Sigma, \mathcal{C} = \mathcal{B}(\Sigma), T = \sigma$ and $\alpha = \{[s] : s \in \mathcal{S}\}$, where $[s_1, s_2, \ldots, s_n]$ denotes the cylinder $\{x \in \mathcal{S}^N : x_i = s_i \forall i \leq n\}$. A Markov map $(X, \mathcal{B}, m, T, \alpha)$ is Gibbs-Markov if two additional properties are satisfied:

- (Big image property) $\inf_{a \in \alpha} m \left(\{T a\}\right) > 0$.
- (Bounded distortion) For $a \in \alpha$, denote by $f(x)$ the jacobian of the map $T^{-1} : Ta \to a$,

\[
\inf_{x \in A} \frac{f(x) - f(y)}{d(x, y)} < \infty.
\]

For a partition $\tau$ of $X$ let $D_\tau (f) := \sup_{a \in \tau} D_a f$ and let $\operatorname{Lip}_{q,\tau}$ be the space

\[
\{f \in L^q (m) : D_\tau (f) < \infty\}
\]

$\operatorname{Lip}_{q,\tau}$ is a Banach space with respect to the norm $\|f\| := \|f\|_q + D_\rho (f)$. We consider the space $\mathcal{B} = \operatorname{Lip}_{\infty,\beta}$ where $\beta = T \alpha$. Clearly $\mathcal{B}$ satisfies (A1). It is shown in [AD] that if $T$ is mixing then the transfer operator $\hat{T}$ satisfies (A2) and (A4) for $\varphi \in \operatorname{Lip}_{2,\alpha}, m (\varphi) = 0$. To show that (A5) holds note that

\[
P(t)f - P(0)f = \hat{T} \left( (e^{it\varphi} - 1) f \right) = \hat{T} \left( -i \varphi f + \frac{\varphi^2 t^2}{2} f + \varphi t^2 \min (|\varphi t|, 2) \cdot f \right) = \hat{T} (i \varphi f) + \hat{T} (\varphi^2 t^2 f).
\]

As $\left( e^{it\varphi} - 1 - it\varphi + \frac{\varphi^2 t^2}{2} \right) \leq \varphi^2 t^2 \min (|\varphi t|, 2)$ we have that

\[
\hat{T} \left( (e^{it\varphi} - 1) f \right) = \hat{T} (-i \varphi f) + \hat{T} (\varphi^2 t^2 f) + o (t^2).
\]
Proposition 1.4 of [AD] shows that $\hat{T} : Lip_{1,\beta} \to B$. Therefore, $\|\hat{T}(-i\varphi tf)\| < \infty$, $\|\hat{T}(\varphi^2 f)\| < \infty$ and assumption (A5) follows from this. As in the case of subshifts of finite type (A6) holds if and only the only solutions to (4.1) are $\lambda = 1, f \equiv 1$. The functional invariance principle for Gibbs Markov maps follows from a stronger, almost sure invariance principle proved for example in [Gou] for observables in $L^p$ with $p > 2$.

5. Estimates

In this section we obtain the main estimates, used in the proof of theorem 5. Henceforth we assume that assumptions (A1)-(A7) hold and use the notation introduced in section 2. In proofs throughout this section, we use the notation $a \lesssim b$ to mean that there exists a constant $C$ such that $a \leq C b$.

**Proposition 6.** There exists a constant $C$ such that for all $n \in \mathbb{N}$, $m(f(S_n - x)) \leq \frac{C}{\sqrt{n}}$, 

$$\int_{\mathbb{R}} |\lambda^n(t)| \leq \frac{C}{\sqrt{n}}.$$ 

**Proof.** Let $\delta$ be as in lemma 3 and set $C_\delta := (-\delta, \delta), \bar{C}_\delta = \mathbb{R} \setminus (-\delta, \delta)$. By inversion formula for Fourier transform and by definition of the characteristic function operator, we have 

$$m(f(S_n - x)) = m\left(\int_{\mathbb{R}} \hat{f}(t) e^{it(S_n - x)} dt\right)$$

$$= \int_{\mathbb{R}} \hat{f}(t) P^n(t) (\mathbb{1}) e^{-itx} dt$$

$$\leq \int_{C_\delta} \left|\hat{f}(t)\right| \left\|P^n(t)\right\| dt + \int_{\bar{C}_\delta} \left|\hat{f}(t)\right| \left\|P^n(t)\right\| dt.$$ 

Since $\hat{f}(t)$ has compact support, by lemma 4 the second term exponentially tends to 0. We estimate the first term. By the expansion of the characteristic function operator,

$$\int_{C_\delta} \left|\hat{f}(t)\right| \left\|P^n(t)\right\| dt \leq \int_{C_\delta} \left|\hat{f}(t)\right| (|\lambda^n(t)| \left\|\pi(t)\right\| + \left\|N^n(t)\right\|) dt$$

$$\leq \int_{C_\delta} (1 - ct^2)^n dt + \int_{C_\delta} \left\|N^n(t)\right\| dt.$$ 

Since $\left\|N^n(t)\right\|$ exponentially tends to 0, the assertion is satisfied for the second term.
Changing variables $x = \frac{t}{\sqrt{n}}$ in the first term we get

$$\int_{C_\delta} (1 - ct^2)^n dt = \frac{1}{\sqrt{n}} \int_{-\sqrt{n}\delta}^{\sqrt{n}\delta} \left(1 - \frac{x^2}{n}\right)^n dx \leq \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-cx^2} dx \leq \frac{1}{\sqrt{n}}.$$

Thus $m(f(S_n - x)) \leq \frac{C}{\sqrt{n}}$ for some $C > 0$. The proof that $m(|\lambda^n(t)|) \leq \frac{C}{\sqrt{n}}$ is contained in the above proof.

**Proposition 7. (Potential Kernel Estimate)** There exists a constant $C > 0$, such that for all $y \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} |m(f(S_n) - f(S_n + y))| \leq C |y|.$$

**Proof.** By the inversion formula for Fourier transform,

$$|m(f(S_n) - f(S_n + y))| = \left| m \left( Re \int_{\mathbb{R}} \hat{f}(t) \ e^{iS_n} (1 - e^{ity}) dt \right) \right| = \left| Re \int_{\mathbb{R}} \hat{f}(t) m(P^n(t) \mathbb{1}) (1 - e^{ity}) dt \right|$$

where the first equality follows since the left side is real and the second inequality is valid by Fubini’s theorem. By proposition 2 and lemma 3 there exist a $\delta > 0$ such that for every $t \in (-\delta, \delta),

$$P(t)(\cdot) = \lambda(t) \pi(t)(\cdot) + N(t)(\cdot)$$

where $|\lambda(t)| \leq 1 - ct^2$ for some positive constant $c$, the spectral radius of $N(t)$ satisfies $\rho(N(t)) \leq q < 1$ for all $t \in (-\delta, \delta)$, and $\pi(t) = m \mathbb{1} + \zeta(t)$ with $\|\zeta(t)\| \leq Ct$ for some $C \geq 0$. Write $C_\delta = (-\delta, \delta)$ and $\bar{C}_\delta = \mathbb{R} \setminus (-\delta, \delta)$. Then

$$\left(5.1\right) \left| Re \int_{\mathbb{R}} \hat{f}(t) m(P^n(t) \mathbb{1}) (1 - e^{ity}) dt \right| \leq \left| Re \int_{C_\delta} \hat{f}(t) m(P^n(t) \mathbb{1}) (1 - e^{ity}) dt \right| + \left| \int_{\bar{C}_\delta} \hat{f}(t) m(P^n(t) \mathbb{1}) |1 - e^{ity}| dt \right|$$

Thus $m(f(S_n - x)) \leq \frac{C}{\sqrt{n}}$ for some $C > 0$. The proof that $m(|\lambda^n(t)|) \leq \frac{C}{\sqrt{n}}$ is contained in the above proof. □
Since the support of \( \hat{f} \) is compact by lemma 4 there exists \( 0 < r < 1 \), such that \( \|P^n(t)\| \leq r^n \) on \( \bar{C}_\delta \). This, together with \( |1 - e^{it\gamma}| \leq |\gamma| \) implies
\[
\sum_{n=1}^\infty \left| \int_{C_\delta} \hat{f}(t) m(P^n(t) \mathbb{1}) (1 - e^{it\gamma}) dt \right| \leq \frac{|\gamma|}{1 - r} \leq |\gamma|.
\]
To bound the right hand side of (5.1) use the expansion of the characteristic function operator to get
\[
\left| \Re \int_{C_\delta} \hat{f}(t) m(P^n(t) \mathbb{1}) (1 - e^{it\gamma}) dt \right| \leq \left| \Re \int_{C_\delta} \hat{f}(t) \lambda^n(t) m(\pi(t) \mathbb{1}) (1 - e^{it\gamma}) dt \right| + \int_{C_\delta} |\hat{f}(t)| \|N^n(t)\| |1 - e^{it\gamma}| dt.
\]
(5.2)
Since \( \rho(N(t)) \leq q < 1 \), and \( |1 - e^{it\gamma}| \leq |\gamma| \),
\[
\sum_{n=1}^\infty \int_{C_\delta} \left| \hat{f}(t) \right| \|N(t)\|^n |1 - e^{it\gamma}| dt \leq |\gamma|.
\]
We turn to analyze the first term on the right hand side of the inequality (5.2). Since \( \hat{f}(t) \) is real valued because \( f \) is symmetric,
\[
\left| \Re \int_{C_\delta} \hat{f}(t) \lambda^n(t) m(\pi(t) \mathbb{1}) \cdot (1 - e^{it\gamma}) dt \right|
\]
\[
= \left| \int_{C_\delta} \hat{f}(t) \Re(\lambda^n(t) m(\pi(t) \mathbb{1})) \Re(1 - e^{it\gamma}) dt \right|
\]
(5.3)
\[
+ \left| \int_{C_\delta} \hat{f}(t) \Im(\lambda^n(t) m(\pi(t) \mathbb{1})) \Im(1 - e^{it\gamma}) dt \right|
\]
Since \( |\Re \lambda^n(t)| \leq |\lambda^n(t)| \leq 1 - ct^2 \), and \( \|\pi(t)\| \leq |t| \),
\[
\sum_{n=1}^\infty \left| \int_{C_\delta} \hat{f}(t) \Re(\lambda^n(t) m(\pi(t) \mathbb{1})) \Re(1 - e^{it\gamma}) dt \right|
\]
\[
\leq \sum_{n=1}^\infty \int_{C_\delta} (1 - ct^2)^n |1 - \cos ty| dt
\]
\[
= \int_{C_\delta} \frac{1}{ct^2} |1 - \cos ty| dt
\]
\[
\frac{1}{c} \int_0^{\delta} |1 - \cos ty| \, dt + \frac{1}{t^2} \left| 1 - \cos ty \right| \, dt
\]

Since \( |1 - \cos ty| \leq |ty|^2 \) we have

\[ (5.5) \]

Now if \( \frac{1}{y} \geq \delta \), then \( \frac{2}{c} \int_0^{\delta} \frac{1}{t^2} |1 - \cos ty| \, dt \leq 0 \) and therefore, by \( (5.5) \) and \( (5.4) \)

\[
\sum_{n=1}^{\infty} \left| \int_{C_M} \hat{f}(t) \left( \Re \left( \lambda^n(t) (\pi(t) \mathbb{1}) \right) \Re \left( 1 - e^{it} \right) \right) \, dt \right| \leq |y|.
\]

On the other hand, if \( \frac{1}{y} < \delta \), then

\[
\int_{\frac{1}{y}}^{\delta} \frac{1}{t^2} |1 - \cos ty| \, dt \leq \int_{\frac{1}{y}}^{\delta} \frac{2}{t^2} \, dt = 2 |y|.
\]

Combining this with \( (5.5) \) and \( (5.4) \) again yields

\[
\sum_{n=1}^{\infty} \left| \int_{C_M} \hat{f}(t) \left( \Re \left( \lambda^n(t) (\pi(t) \mathbb{1}) \right) \Re \left( 1 - e^{it} \right) \right) \, dt \right| \leq |y|.
\]

We estimate the sum over the second term in \( (5.3) \). Using \( \pi(t) = m \pm \zeta(t), \|\zeta(t)\| \leq |t| \) we obtain

\[ (5.6) \]

\[
\int_{C_M} \hat{f}(t) \left( \Im \lambda^n(t) (\pi(t) \mathbb{1}) \right) (\sin ty) \, dt \leq \int_{C_M} \left| \Im \lambda^n(t) \right| |\sin ty| \, dt + \int_{C_M} |t \cdot \lambda^n(t) | (\sin ty) \, dt.
\]

Using \( |\lambda(t)| \leq 1 - ct^2 \) we can estimate the second term on the right hand side of the above inequality.

\[
\sum_{n=1}^{\infty} \int_{C_M} |t \cdot \lambda^n(t) | |\sin ty| \, dt \leq \int_{C_M} \frac{1}{ct} |\sin ty| \, dt \leq |y|.
\]

The estimation of the first term on the right hand side of \( (5.6) \) will take up the rest of the proof.

We first note that \( |\Im \lambda^n(t) | \leq n \left| \lambda^{n-1}(t) \right| |\Im \lambda(t)| \). Then

\[
|\Im \lambda(t)| = |m (\Im P(t) \eta(t))| \\
\leq |m (\Im P(t) \mathbb{1})| + |m (\Im P(t) \psi(t))|,
\]
where $\psi(t) = 1 - \eta(t)$. By definition of the characteristic function operator, and the fact that $\hat{T}f$ is real if $f$ is real,

$$|m(ImP(t)\psi(t))| \leq |m(\hat{T} (\cos(t\varphi) \text{Im} \psi(t)))| + |m(\hat{T} (\sin(t\varphi) \text{Re} \psi(t)))|.$$  

Since by (2.2) $m(\psi(t)) = 0$, $m \circ \hat{T} = m$, $|1 - \cos t\varphi| \leq t^2 \varphi^2$, $|\psi(t)| \leq |t|$ and by the positivity of the transfer operator,

$$|m(\hat{T} (\cos(t\varphi) \text{Im} \psi(t)))| = |m((\cos(t\varphi) - 1) \text{Im} \psi(t))| \leq m(t^2 \varphi^2 \|\psi(t)\|) \leq |t|^3$$

where we have used the finiteness of the second moment of $\varphi$.

Since $\psi(0) = 0$, $Re\psi'(0) = 0$ (because $\eta'(0)$ is purely imaginary) and $\psi(t)$ is twice continuously differentiable,

$$|m(\hat{T} (\sin(t\varphi) \text{Re} \psi(t)))| \leq |t|^3.$$  

Therefore,

$$\sum_{n=1}^{\infty} \int_{C_\delta} n |\lambda(t)|^{n-1} |m(ImP(t)\psi(t))||\sin(ty)| \, dt \leq \sum_{n=1}^{\infty} \int_{C_\delta} n (1 - ct^2)^{n-1} |t|^3 |\sin ty| \, dt \leq \int_{C_\delta} \frac{1}{ct^3} |t|^3 |\sin ty| \, dt \leq |y|$$

Finally, since $m(\varphi) = 0$ and $m \circ \hat{T} = m$

$$|m(ImP(t)1)| = |m(\sin(t\varphi))| = |m(\sin(t\varphi) - t\varphi)|$$

We split the last integral into parts where $|t\varphi| \leq 1$ and $|t\varphi| > 1$ to obtain

$$|m(ImP(t)1)| \leq |m(1_{\{|t\varphi|\leq 1\}} (\sin(t\varphi) - t\varphi))| + |m(1_{\{|t\varphi|>1\}} (\sin(t\varphi) - t\varphi))| \leq |m(1_{\{|t\varphi|\leq 1\}} |t\varphi|^3)| + |m(2|t\varphi|1_{\{|t\varphi|>1\}})|$$
Thus, summing over \( n \) and again using \( |\lambda(t)| \leq (1 - ct^2) \) we have

\[
\sum_{n=1}^{\infty} \int_{C_{\delta}} n |\lambda(t)|^{n-1} |m(ImP(t) 1)| |\sin(ty)| \, dt \\
\leq \int_{C_{\delta}} \frac{1}{ct^4} m(|t\varphi|^3 \mathbb{1}_{\{|\varphi| \leq 1\}}) |\sin(ty)| \, dt \\
+ 2 \int_{C_{\delta}} \frac{1}{ct^4} m(|t\varphi| \mathbb{1}_{\{|\varphi| > 1\}}) |\sin(ty)| \, dt
\]

Bounding \(|\sin ty|\) by \(|ty|\) and changing the order of integration in the first term gives

\[
\int_{C_{\delta}} \frac{1}{ct^4} m \left(|t\varphi|^3 \mathbb{1}_{\{|\varphi| \leq 1\}}\right) |\sin ty| \, dt \leq m \left(|\varphi|^3 \int_{-|\varphi|^{-1}}^{\varphi^{-1}} |y| \, dt \right) \\
= m \left(2 |\varphi|^2 \right) |y| \\
\leq |y|
\]

Changing the order of integration in the second term of (5.7) and using the fact that the integrand is an even function of \( t \), gives

\[
2 \int_{C_{\delta}} \frac{1}{ct^4} m \left(|t\varphi| \mathbb{1}_{\{|\varphi| > 1\}}\right) |\sin(ty)| \, dt \leq 4m \left(|\varphi| \int_{|\varphi|^{-1}}^{\delta} \frac{1}{t^2} |y| \, dt \right) \\
\lesssim |y|.
\]

This completes the proof.

**Proposition 8.** Let \( \delta > 0, n \in \mathbb{N} \). Then there exists a constant \( C > 0 \) such that for all \( x, y \in \mathbb{R} \) such that

\[
m \left( \sum_{k=1}^{n} f(S_k - x) - f(S_k - y) \right)^4 \leq C \left( n |x - y|^2 \right).
\]

**Proof.** Opening brackets we obtain

(5.9)

\[
m \left( \sum_{k=1}^{n} f(S_k - x) - f(S_k - y) \right)^4 = \sum_{(k_1, \ldots, k_4) \in \{1, \ldots, n\}^4} m \left( \prod_{l=1}^{4} (f(S_{k_l} - x) - f(S_{k_l} - y)) \right)
\]

It is clear that the left hand term is not greater than \( 4! \) times the same sum over all tuples \((k_1, \ldots, k_4) \in \{1, \ldots, n\}^4\) where \( k_1 \leq k_2 \leq k_3 \leq k_4 \). Thus we assume that \( k_1 \leq k_2 \leq k_3 \leq k_4 \).
By inversion formula for the Fourier transform
\[
\begin{align*}
m\left(\prod_{l=1}^{4} (f(S_{k_l} - x) - f(S_{k_l} - y))\right) &= m\left(\int_{\mathbb{R}^4} \prod_{l=1}^{4} \hat{f}(t_l) e^{it_lS_{k_l}} (e^{it_lx} - e^{it_ly}) dt_1 \ldots dt_4\right) \\
&= \int_{\mathbb{R}^4} \prod_{l=1}^{4} \hat{f}(t_l) (e^{it_lx} - e^{it_ly}) m\left(\prod_{l=1}^{4} e^{it_lS_{k_l}}\right) dt_1 \ldots dt_4
\end{align*}
\]
where the last equality follows by changing order of integration.

Writing \( k_0 = 0 \), by definition of the characteristic function operator we have
\[
m\left(\prod_{l=1}^{4} e^{it_lS_{k_l}}\right) = m\left(\prod_{l=1}^{4} e^{i\sum_{j=l}^{4} t_j (S_{k_l} - S_{k_{l-1}})}\right) = m\left(\left(\prod_{l=1}^{4} P^{k_l-k_{l-1}}\left(\sum_{j=l}^{4} t_j\right)\right)(1)\right).
\]

Thus,
\[
m\left(\prod_{l=1}^{4} (f(S_{k_l} - x) - f(S_{k_l} - y))\right) = \int_{\mathbb{R}^4} m\left(\prod_{l=1}^{4} P^{k_l-k_{l-1}}\left(\sum_{j=l}^{4} t_j\right)\right)(1)) \cdot \prod_{l=1}^{4} \hat{f}(t_l) (e^{it_lx} - e^{it_ly}) dt_1 \ldots dt_4.
\]

Performing a change of variables \( z_i = \sum_{k=1}^{i} t_i \) \( i = 1, \ldots, 4 \), and writing \( t_0 = 0 \), \( n_l = k_l - k_{l-1} \) we obtain
\[
m\left(\prod_{l=1}^{4} (f(S_{k_l} - x) - f(S_{k_l} - y))\right) = \int_{\mathbb{R}^4} m\left(\prod_{l=1}^{4} P^{n_l}(t_l)(1)\right) \prod_{l=1}^{4} \hat{f}(t_l) (e^{i(t_l-t_{l-1})x} - e^{i(t_l-t_{l-1})y}) dt_1 \ldots dt_4
\]

Next, we need to simplify the expression \( \prod_{l=1}^{4} (e^{i(t_l-t_{l-1})x} - e^{i(t_l-t_{l-1})y}) \). Let \( \zeta(x) = y - x \) and \( \zeta(y) = x - y \). We claim that
\[
(5.10) \quad \prod_{l=1}^{4} (e^{i(t_l-t_{l-1})x} - e^{i(t_l-t_{l-1})y}) = \sum_{(z_2,z_4)} e^{it_2z_2} (1 - e^{it_1\zeta(z_2)}) e^{it_4z_4} (1 - e^{it_3\zeta(z_4)})
\]
where the sum is over all \((z_2, z_4) \in \{x, y\}^2\). To see this note that
\[
(e^{it_1 x} - e^{it_1 y}) (e^{i(t_{i+1} - t_i) x} - e^{i(t_{i+1} - t_i) y}) = e^{it_{i+1} x} \left(1 - e^{it_i (x-y)}\right) + e^{it_{i+1} y} \left(1 - e^{it_i (y-x)}\right)
\]
and the claim follows by implementing this on the first two terms and the last two terms in the product on the left hand side of (5.10) separately.

To shorten the writing we write \(\psi(t, z) = 1 - e^{it \zeta(z)}\). Thus,
\[
m\left(\sum_{k=1}^{n} (f(S_k - x) - f(S_k - y))^4\right)
\leq 4! \sum_{n_1, \ldots, n_4} Re \left[ \int_{\mathbb{R}^4} \prod_{l=1}^{4} \hat{f}(t_l) \cdot m \left(\prod_{l=1}^{4} P^{n_l} (t_l) (1)\right) \right] \times \sum_{(z_2, z_4)} e^{it_2 z_2 \psi(t_1, z_2)} e^{it_4 z_4 \psi(t_3, z_4)} dt_1 \ldots dt_4
\]
where the sum is over all tuples \((n_1, \ldots, n_4) \in \{0, \ldots, n\}^4\). The following inequality completes the proof.

\[
(5.11) \sum_{n_1, \ldots, n_4} Re \int_{\mathbb{R}^4} \prod_{l=1}^{4} \hat{f}(t_l) \cdot m \left(\prod_{l=1}^{4} P^{n_l} (t_l) (1)\right) \sum_{(z_2, z_4)} e^{it_2 z_2 \psi(t_1, z_2)} e^{it_4 z_4 \psi(t_3, z_4)} dt_1 \ldots dt_4
\leq n |x - y|^2
\]
To prove this we use the expansion of the characteristic function operator in proposition 2 and lemma 3 and propositions 6, 7. Let \(\delta\) be as in lemma (2.4) and write \(C^i_\delta = [-\delta, \delta]^i\) and \(\bar{C}_\delta = \mathbb{R}^i \setminus C^i_\delta\). Also denote \(\zeta(t)f = m(f) 1 - \pi(t)f\) and recall from 3 that \(\|\zeta(t)\| \leq |t|\) for \(|t| < \delta\). For fixed \((z_2, z_4) \in \{x, y\}^2\),
The proof is conducted similarly for all terms. We continue to expand the products $\prod_{l=1}^{4} P_{l}^{m_{l}}(t_{l})$ using $P^{m}(t)(\cdot) = \lambda^{n}(t) m(\cdot) 1 + \zeta(t)(\cdot) + N^{n}(t)(\cdot)$. After this we split the integrals into a product of integrals. For example we can split the first term on the right of (5.12) into

\[
\int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \cdot \lambda^{n_{l}}(t_{4}) m \left( \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) \right) (1) e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{4}
\]

\[
= \int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \cdot \lambda^{n_{l}}(t_{4}) m \left( \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) \right) 1 e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{4}
\]

\[
+ \int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \lambda^{n_{l}}(t_{4}) \left( \zeta(t_{4}) \left( \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) \right) 1 \right) e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{4}
\]

\[
+ \int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \lambda^{n_{4}}(t_{4}) \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{4}
\]

The proof is conducted similarly for all terms. We continue to expand the products $\prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l})$ using $P^{m}(t)(\cdot) = \lambda^{n}(t) m(\cdot) 1 + \zeta(t)(\cdot) + N^{n}(t)(\cdot)$. After this we split the integrals into a product of integrals. For example we can split the first term on the right of (5.12) into

\[
\int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \cdot \lambda^{n_{l}}(t_{4}) m \left( \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) \right) (1) e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{4}
\]

\[
= \int C_{k} \prod_{l=1}^{4} \hat{f}(t_{l}) \cdot \lambda^{n_{l}}(t_{4}) e^{it_{4}z_{4}} dt_{4}
\]

\[
\times m \left( \prod_{l=1}^{3} P_{l}^{m_{l}}(t_{l}) \right) 1 e^{it_{2}z_{2}} \psi(t_{1}, z_{2}) e^{it_{4}z_{4}} \psi(t_{3}, z_{4}) dt_{1}...dt_{3}.
\]

For the other terms we may not split the integrals right away because we have operators of the form $\zeta(t) \circ \zeta(s)$ or $N^{n}(t) \circ N^{k}(s)$. To handle these kind of terms we continue to expand, and eventually will be able to split the integrals by taking norms (splitting first the terms similar to the above term prior to taking absolute values). Since by the proof of proposition [7] we have

\[
\sum_{k=0}^{\infty} \int C_{k} |\hat{f}(t)| \, |t| \, |\lambda^{k}(t)| \left( 1 - e^{it(x-y)} \right) \, dt \leq \int C_{k} \frac{|t|^{2}}{ct^{2}} |x-y| \, dt \leq |x-y|
\]

\[
\sum_{n=0}^{\infty} \int C_{k} \, \|\hat{f}(t)\| \, \|N^{k}(t)\| \left( 1 - e^{it(x-y)} \right) \, dt \leq |x-y|
\]
and
\[ \sum_{n=0}^{\infty} \left| \int_{C_\delta} \hat{f}(t) \lambda^k(t) \left( 1 - e^{it(x-y)} \right) dt \right| \leq |x - y| \]
the sum of every term involving a function $\psi$ is bounded by a constant multiplied by $|x - y|$.

On the other hand, by proposition 6, each term of the form
\[ \int_{C_\delta} \left| \hat{f}(t) \lambda^k(t) \right| e^{itx} dt \]
is bounded by a constant multiplied by $\frac{1}{\sqrt{n}}$. It follows that the sum from 0 to $n$ of such terms is bounded by $\sqrt{n}$. Since the integral in 5.11 has precisely two terms involving functions $\psi$ it follows that
\[ \sum \prod_{l=1}^{4} \hat{f}(t_l) \cdot m \prod_{l=1}^{4} P^{n_l}(t_l)(1) e^{it_2z_2\psi(t_1,z_2)} e^{it_4z_4\psi(t_3,z_4)} dt_1...dt_4 \leq |x - y|^2. \]
Estimating
\[ \sum \int_{C_\delta} \prod_{l=1}^{4} \hat{f}(t_l) \cdot m \prod_{l=1}^{4} P^{n_l}(t_l)(1) e^{it_2z_2\psi(t_1,z_2)} e^{it_4z_4\psi(t_3,z_4)} dt_1...dt_4 \]
is easier since at least one of the four integrals at hand is over $\mathbb{R} \setminus (-\delta,\delta)$ and therefore, exponentially tends to 0 by lemma 4. Expanding this integral similarly to the integral over $C_\delta$ we obtain a similar estimate
\[ \sum \int_{C_\delta} \prod_{l=1}^{4} \hat{f}(t_l) \cdot m \prod_{l=1}^{4} P^{n_l}(t_l)(1) e^{it_2z_2\psi(t_1,z_2)} e^{it_4z_4\psi(t_3,z_4)} dt_1...dt_4 \leq |x - y|^2 \]
whence the proposition follows. \(\square\)

**Corollary 9.** There exists a constant $C$ such that for all $n \in \mathbb{N}$, $x, y \in \mathbb{R}$,
\[ m \left( ((l_n(x) - l_n(y)))^4 \right) \leq C |x - y|^2. \]
Proof. By proposition 9

\[
m \left( \left( (l_n(x) - l_n(y))^4 \right) \right) = \frac{1}{n^2} m \left( \left( \sum_{k=0}^{n} f \left( S_k - \sqrt{n}x \right) - f \left( S_k - \sqrt{n}y \right) \right)^4 \right)
\]
\[
\leq \frac{C}{n^2} \left( n^2 |x - y|^2 \right)
\]
\[
= C |x - y|^2 .
\]
\[\square\]

Proposition 10. There exists a constant C such that for all \( x \in \mathbb{R}, n \in \mathbb{N}, \)

\[
m \left( \left( \sum_{k=1}^{n} f (S_k - x) \right)^2 \right) \leq Cn .
\]

Proof. Using similar methods to proposition 8 we have

\[
m \left( \left( \sum_{k=1}^{n} f (S_k - x) \right) \right) = \sum_{k,l=1}^{n} m (f (S_l - x) f (S_k - x))
\]
\[
\leq 2! \sum_{l=1}^{n} \sum_{k=1}^{n} m (f (S_l - x) f (S_k - x))
\]
\[
\leq \sum_{l=1}^{n} \sum_{k=1}^{n} \int_{\mathbb{R}^2} \hat{f} (t_1) \hat{f} (t_2) P^{k-l} (t_1) P^l (t_2 + t_1) (1) e^{-i(t_1 + t_2)x} dt_1 dt_2
\]
\[
\leq \sum_{l=1}^{n} \sum_{k=1}^{n} \int_{\mathbb{R}^2} |\hat{f} (t_1) \hat{f} (t_2)| \left\| P^{k-l} (t_1) \right\| \left\| P^l (t_2) \right\| dt_1 dt_2
\]

where the last inequality follows by a change of variables and taking absolute values. Writing
\( C_\delta = (-\delta, \delta)^2, \tilde{C}_\delta = \mathbb{R}^2 \setminus (-\delta, \delta)^2 \) and taking \( \delta \) as in lemma 3 we obtain by proposition 8

\[
\sum_{l=1}^{n} \sum_{k=1}^{n} \int_{C_\delta} |\hat{f} (t_1) \hat{f} (t_2)| \left\| P^{k-l} (t_1) \right\| \left\| P^l (t_2) \right\| dt_1 dt_2 \leq \sum_{l=1}^{n} \sum_{k=1}^{n} \int_{-\delta}^{\delta} |\lambda^{k-l} (t_1) | dt_1 \int_{-\delta}^{\delta} |\lambda^k (t_2) | dt_2 \leq n .
\]
By the use of lemma \([\text{4}]\) and proposition \([\text{6}]\)

\[
\sum_{l=1}^{n} \sum_{k=1}^{n} \int_{C_{\delta}} \left| \hat{f}(t_{1}) \hat{f}(t_{2}) \right| \left\| P^{l-k}(t_{1}) \right\| \left\| P^{k}(t_{2}) \right\| dt_{1}dt_{2} \lesssim n,
\]

which completes the proof. \(\square\)

**Corollary 11.** There exists a constant \(C\) such that for all \(n \in \mathbb{N}, \ x \in \mathbb{R}, \ m \left( (l_{n}(x))^{2} \right) \leq C.\)

**Proof.** By proposition \([\text{10}]\)

\[
m \left( (l_{n}(x))^{2} \right) = \frac{1}{n} m \left( \left( \sum_{k=1}^{n} f \left( S_{n} - \sqrt{n}x \right) \right)^{2} \right) \leq C.
\]

To prove tightness of the process \(l_{n}(x)\), additionally to the above estimates we need the following estimate for maxima of continuous processes. \(\square\)

**Proposition 12.** Let \(\gamma(t)\) be an almost surely continuous process on an interval \(I\) of length \(\delta\).

Assume that for every \(\epsilon > 0\), \(t, s \in I\) we have \(P( |\gamma(t) - \gamma(s)| \geq \epsilon ) < C \frac{|t-s|^{\alpha}}{\epsilon^{\beta}}\), where \(\alpha > 1, \ \beta \geq 0, \ C > 0\). Then \(P( \sup_{t,s \in I} |\gamma(t) - \gamma(s)| \geq \epsilon ) \leq \tilde{C} \frac{\epsilon^{\beta}}{\epsilon^{\alpha}}.\)

**Proof.** Assume without loss of generality that \(I = [0, \delta]\) and let \(D_{k} := \left\{ 0, \frac{1}{2^{k}} \delta, \frac{2}{2^{k}} \delta, ..., \frac{2^{k}-1}{2^{k}} \delta, \delta \right\}\). Let \(B_{k} = \max_{t_{1}, t_{2} \in D_{k}} |\gamma(t_{1}) - \gamma(t_{2})|\) and let \(A_{k} = \max_{t_{1}, t_{2} \in B_{k}, |t_{1}-t_{2}|=\frac{1}{2^{k}} \delta} \). For \(t \in D_{k}\), define a point \(t' \in D_{k-1}\) by

\[
t' = \begin{cases} 
  t & t \in D_{k-1}, \\
  t - 2^{-k} & t \notin D_{k-1}.
\end{cases}
\]

Then for every \(t \in D_{k}\), \(|\gamma(t) - \gamma(t')| \leq A_{k}\) and therefore, for \(t_{1}, t_{2} \in D_{k}\) we have

\[
|\gamma(t_{1}) - \gamma(t_{2})| \leq |\gamma(t_{1}) - \gamma(t'_{1})| + |\gamma(t'_{1}) - \gamma(t'_{2})| + |\gamma(t'_{2}) - \gamma(t_{2})| \leq |\gamma(t'_{1}) - \gamma(t'_{2})| + 2A_{k}.
\]

Since \(t'_{1}, t'_{2}\) are in \(D_{k-1}\), it follows that \(B_{k} \leq B_{k-1} + 2A_{k}\). Since \(A_{0} = B_{0}\), we conclude by induction that \(B_{k} \leq 2 \sum_{i=1}^{k} A_{i}\). By continuity of the paths of \(\gamma\), we get that \(\lim_{k \to \infty} B_{k} = \sup_{t,s \in I} |\gamma(t) - \gamma(s)|\) and therefore,

\[
\sup_{t,s \in I} |\gamma(t) - \gamma(s)| \leq 2 \sum_{k=1}^{\infty} A_{k}.
\]
Suppose that \( \theta \in (0, 1) \) and let \( r \) be such that \( r \cdot \sum_{k=1}^{\infty} \theta^k = \frac{1}{2} \). Then
\[
P \left( \sup_{t,s \in I} |\gamma(t) - \gamma(s)| \geq \epsilon \right) \leq P \left( 2 \sum_{k=1}^{\infty} A_k \geq \epsilon \right)
\leq \sum_{k=1}^{\infty} P \left( A_k \geq r \epsilon \theta^k \right)
\leq \sum_{k=1}^{\infty} C 2^k \left( \frac{\delta}{2^k} \right)^{\alpha} \frac{1}{(r \epsilon \theta^k)^{\beta}}
= C \delta^\alpha \sum_{k=1}^{\infty} \frac{1}{(2^{\alpha-1} \theta^\beta)^k}.
\]
Since \( \alpha - 1 > 0 \) and \( \beta \geq 0 \), there exists \( \theta \) for which the sum converges and the claim follows. \( \square \)

6. Tightness of \( l_n \) in \( D \).

A sequence \( \{X_n\} \) of random variables taking values in a complete and separable metric space \( (X, d) \) is tight if for every \( \epsilon > 0 \) there exists a compact \( K \subset X \) such that for every \( n \in \mathbb{N} \),
\[
P_n(K) > 1 - \epsilon,
\]
where \( P_n \) denotes the distribution of \( X_n \). By Prokhorov’s Theorem (see [Bil]) relative compactness of \( t_n(x) \) in \( C \) is equivalent to tightness. Therefore, we are interested in characterizing tightness in \( C \).

For \( h > 0 \), denote by \( C_{[-h,h]} \) the space of continuous functions on \( C_{[-h,h]} \). For \( x(t) \) in \( C_{[-h,h]} \), set
\[
\omega_x(\delta) := \sup_{|s-t|<\delta} \{|x(s)-x(t)| : s,t \in C_{[-h,h]}, |s-t|<\delta\}.
\]
\( \omega_x(\delta) \) is called the modulus of continuity of \( x \). Due to the Arzela - Ascoli theorem, the modulus of continuity plays a central role in characterizing precompactness in the space \( C_{[-h,h]} \), with the Borel \( \sigma \)-algebra generated by the topology of uniform convergence.

The next theorem is a characterization of tightness in the space \( C \).

**Theorem 13.** [Bil] The sequence \( l_n \) is tight in \( C \) if and only if its restriction to \( [-h, h] \) is tight in \( C_{[-h,h]} \) for every \( h \in \mathbb{R}_+ \). The sequence \( l_n \) is tight in \( C_{[-h,h]} \) if and only if the following two conditions hold:
(i) \( \forall x \in [-h, h], \lim_{a \to \infty} \limsup_{n \to \infty} m(|l_n(x)| \geq a) = 0. \)

(ii) \( \forall \epsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} m(\omega_{l_n}(\delta) \geq \epsilon) = 0. \)

**Proposition 14.** The sequence \( \{l_n\}_{n=1}^{\infty} \) is tight.

**Proof.** Condition (i) of theorem 13 easily follows from corollary 11 and Chebychev’s inequality, since for all \( x \in \mathbb{R}, n \in \mathbb{N}, \)

\[
m(|l_n(x)| \geq a) \leq \frac{m((l_n(x))^2)}{a^2} \leq \frac{C}{a^2} \xrightarrow{a \to \infty} 0.
\]

We prove that condition (ii) of theorem 13 holds. In order to do that, we have to show that for fixed \( h > 0, \)

\[
(6.1) \quad \forall \epsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} m \left( \sup_{x, y \in [-h, h]; |x - y| < \delta} |l_n(x) - l_n(y)| \geq \epsilon \right) = 0.
\]

Fix \( \epsilon > 0. \) Then by 9 and Chebychev’s inequality there exists a constant \( C \) such that for all \( x, y \in [-h, h], \)

\[
m(|l_n(x) - l_n(y)| \geq \epsilon) \leq \frac{m((l_n(x) - l_n(y))^2)}{\epsilon^2} \leq \frac{C|x - y|^2}{\epsilon^2}.
\]

Thus, for \( x, y \in [-h, h] \) we have,

\[
m(|l_n(x) - l_n(y)| \geq \epsilon) \leq \frac{C|x - y|^2}{\epsilon^2}
\]

Let \( \delta > 0 \) and \( n > \delta^{-2}. \) Then by proposition 12

\[
m \left( \sup_{x, y \in [-h, h]; |x - y| < \delta} |l_n(x) - l_n(y)| \geq 4\epsilon \right) \leq \sum_{|k\delta| \leq h} m \left( \sup_{k\delta \sqrt{n} \leq x, y \leq (k+1)\delta \sqrt{n}} |l_n(x) - l_n(y)| \geq \epsilon \right)
\]

\[
\leq \sum_{|k\delta| \leq h} \frac{\tilde{C}}{\epsilon^2} \delta^2 \leq 2h\tilde{C}\delta \xrightarrow{\delta \to 0} 0
\]

whence 6.1 follows. \( \square \)

## 7. Proof of the main theorem

In this section we identify \( (\omega, \int_{\mathbb{R}} f(x) \, dx \cdot l) \) as the unique distributional limit of \( (\omega_n, l_n) \)
and complete the proof of theorem 3.
Proof of theorem 5: By assumption (A7) and proposition 14, 

\((\omega_n, l_n)\) is tight in \(D[0, 1] \times C\). Let \((p, q)\) be a distributional limit of some subsequence \((\omega_{n_k}, l_{n_k})\). We must show that \((p, q) \overset{d}{=} \omega, \int_{\mathbb{R}} f(x) \, dx \cdot l\). In what follows, we assume without loss of generality that the convergent subsequence is \((\omega_n, l_n)\) itself. By Skorokhod’s representation theorem there exists a probability space \((\Omega, \mathcal{B}, P)\) with random functions \((\omega'_n, l'_n, p', q')\) defined on it, such that \((\omega'_n, l'_n) \overset{d}{=} (\omega_n, l_n), (p', q') \overset{d}{=} (p, q)\) and \((\omega'_n, l'_n)\) almost surely converge to \((p, q)\); i.e., \(d_J(\omega'_n, p) \rightarrow 0, d(l'_n, l) \rightarrow 0\) almost surely, where \(d_J\) is the metric of \(D\) and \(d\) is the metric of \(C\). By assumption that \(\omega_n \overset{d}{=} \omega\), we have \(p \overset{d}{=} \omega\). Let \(G_k = \{a_1, b_1, \ldots, a_k, b_k : a_i < b_i, a_i, b_i \in \mathbb{Q}, i = 1, \ldots, k\}\) and \(G = \bigcup_{k=1}^{\infty} G_k\). For

\[ g = \{a_1, b_1, \ldots, a_k, b_k : a_i < b_i, i = 1, \ldots, k\} \in G_k \]

define the transformation \(\pi_g : C \rightarrow \mathbb{R}^k\) by

\[ \pi_g (h) = \left( \int_{a_1}^{b_1} h(x) \, dx, \ldots, \int_{a_n}^{b_n} h(x) \, dx \right) \]

and \(\Pi_g : D \rightarrow \mathbb{R}^k\) by

\[ \Pi_g (h) = \left( \int_0^1 \mathbb{1}_{[a_1, b_1]} (h) \, dt, \ldots, \int_0^1 \mathbb{1}_{[a_k, b_k]} (q(t)) \, dt \right). \]

Note that \(\pi_g\) is continuous and therefore \(\lim_{n \rightarrow \infty} \pi_g (l'_n) = \pi_g (g)\). The following fact is proved in [KS]: for all \(k \in \mathbb{N}, g \in G_k\), the function \(\pi_g(\cdot)\) is continuous in the Skorokhod topology at almost every sample point of the Brownian motion, i.e., if \(h_n \in D\) converges to \(h \in D\) where \(h\) is a generic sample point of a Brownian motion, then \(\Pi_g (h_n) \rightarrow \Pi_g (h)\) (here, convergence is of vectors in \(\mathbb{R}^k\)). Since \(G\) is a countable set, this implies that almost surely, for all \(g \in G\),

\[ \Pi_g (\omega'_{n_k}) \rightarrow \Pi_g (p) \]

To complete the proof, it is enough to show that for \(g \in G_k\), almost surely

\[ \pi_g (q') = \int_{\mathbb{R}} f(x) \, dx \cdot \pi_g (h) \]

where \(h\) is the local time of the Brownian motion \(p\). Since by definition of local time the equality

\[ \pi_g (h) = \Pi_g (p) \]
is almost surely satisfied for all \( q \in G \) and since the transformations \( \pi_g(h), g \in G \) uniquely determine the function \( h \), it would follow that almost surely \( q \) coincides with \( h \), i.e that \( q \) is the local time of the Brownian motion \( p \) as claimed. Setting \( S_k^n := \sqrt{n} \sum_{i=0}^{k} \omega_n \left( \frac{i}{n} \right) \) we have \( l_n'(x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(S_k^n) \). Fix a positive \( \epsilon \in \mathbb{Q} \), and \( a, b \in \mathbb{Q} \), auch that \( a < b \). By straightforward calculations using a change of variable \( y = S_k - \sqrt{n} x \), we have

\[
\int_{a}^{b} f(S_k^n - \sqrt{n} x) \, dx = \frac{1}{\sqrt{n}} \int_{S_k - \sqrt{n} a}^{S_k - \sqrt{n} b} f(x) \, dx
\]

\[
\leq n^{-\frac{1}{2}} \cdot 1_{[a-\epsilon, b+\epsilon]}(n^{-\frac{1}{2}} S_k^n) \cdot \int_{-\infty}^{\infty} f(x) \, dx
\]

\[
= n^{-\frac{1}{2}} + 1_{\mathbb{R} \setminus [a-\epsilon, b+\epsilon]}(n^{-\frac{1}{2}} S_k^n) \cdot \int_{-\infty}^{\infty} f(x) \, dx
\]

\[
\leq n^{-\frac{1}{2}} 1_{[a-\epsilon, b+\epsilon]}(n^{-\frac{1}{2}} S_k^n) \cdot \int_{-\infty}^{\infty} f(x) \, dx + n^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \, dx.
\]

This implies

\[
\int_{a}^{b} l_n'(x) \, dx \leq \frac{1}{n} \# \left\{ k \in \{1, \ldots, n\} : n^{-\frac{1}{2}} S_k^n \in [a-\epsilon, b+\epsilon] \right\} + \int_{-\infty}^{\infty} f(x) \, dx.
\]

On the other hand

\[
\int_{0}^{1} 1_{[a-\epsilon, b+\epsilon]}(\omega_n'(t)) \, dt = \frac{1}{n} \# \left\{ k \in \{1, \ldots, n\} : S_k^n \in [a-\epsilon, b+\epsilon] \right\}
\]

and by (7.1) we have

\[
\lim_{n \to \infty} \int_{0}^{1} 1_{[a-\epsilon, b+\epsilon]}(\omega_n(t)) \, dt = \int_{0}^{1} 1_{[a-\epsilon, b+\epsilon]}(p(t)) \, dt = \int_{a-\epsilon}^{b+\epsilon} h(x) \, dx \quad \text{a.s}
\]

Since \( \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \, dx = 0 \), it follows that

\[
\lim_{n \to \infty} \int_{a}^{b} l_n'(x) \, dx \leq \int_{-\infty}^{\infty} f(x) \, dx \int_{a-\epsilon}^{b+\epsilon} h(x) \, dx
\]
and since this holds for every rational \( \epsilon > 0 \), we have
\[
\lim_{n \to \infty} \int_a^b l_n'(x) \, dx = \int_a^b q'(x) \, dx \leq \int_a^b h(x) \, dx.
\]

To obtain a lower bound, imitating the calculations for the upper bound, we have
\[
\int_a^b l_n'(x) \, dx \geq \frac{1}{n} \# \left\{ k \in \{1, \ldots, n\} : n^{-\frac{1}{2}} S_k \in [a + \epsilon, b - \epsilon] \right\} \int_{-\epsilon \sqrt{n}}^{\epsilon \sqrt{n}} f(x) \, dx
\]
and therefore
\[
\lim_{n \to \infty} \int_a^b l_n(x) \, dx = \int_a^b q(x) \, dx \geq \int_{\mathbb{R}} f(x) \, dx \cdot \int_{a+\epsilon}^{b-\epsilon} l(x) \, dx.
\]

It follows that
\[
\lim_{n \to \infty} \int_a^b l_n'(x) \, dx \geq \int_{\mathbb{R}} f(x) \, dx \int_a^b l(x) \, dx
\]
and therefore,
\[
\lim_{n \to \infty} \int_a^b l_n'(x) \, dx = \int_a^b q(x) = \int_{\mathbb{R}} f(x) \, dx \int_a^b h(x) \, dx \quad \text{a.s.}
\]

This proves that (7.2) holds almost surely for \( g \in G_1 \). The proof for \( g \in G_k \) is performed using similar calculations coordinatewise. Thus the proof is complete.

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