Ergodicity of stochastic differential equations with jumps and singular coefficients

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Overview

1. Background

2. Main results
   - Existence and uniqueness
   - Long time behavior and idea of proof
   - Examples

3. Reference
Consider the following ordinary differential equation (ODE):

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For \( d = 1 \), \( b(x) = 2\text{sign}(x)\sqrt{|x|} \) and \( x_0 = 0 \), the above equation has infinitely many solutions:

\[ X(t) \equiv 0, \quad X(t) = t^2, \quad X(t) = -t^2, \quad \ldots \]
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Note that the function \( b \) is Hölder continuous.
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It is interesting to find that noises may produce some regularization effects.
Consider the following stochastic differential equation (SDE): 
\[ dX_t = \sigma(X_t)dW_t + g(X_{t-})dL_t + b(X_t)dt, \]  
(1.1) 
with \( X_0 = x \in \mathbb{R}^d \).

\((W_t)_{t \geq 0}\) is an \( m \)-dimensional standard Brownian motion.

\((L_t)_{t \geq 0}\) is a \( k \)-dimensional pure jump Lévy process.
In the case $g \equiv 0$:

- **N. V. Krylov and M. Röckner (2005, PTRF):**
  \[
  dX_t = dW_t + b(X_t)dt, \quad X_0 = x. 
  \]
  **Condition:** $b \in L^p(\mathbb{R}^d)$ with $p > d$.

- **X. Zhang (2005, SPA):**
  \[
  dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x. 
  \]
  **Condition:** $\sigma$ is bounded and uniformly elliptic and $\nabla \sigma \in L^p(\mathbb{R}^d)$ with $p > d$. 
There are also many works devoted to study the properties of the unique strong solution:

- **E. Fedrizzi and F. Flandoli (2013, JFA):** The map $x \mapsto X_t(x)$ is Sobolev differentiable.

- **T. Zhang, etc. (2013, Math. Annalen):** The map $\omega \mapsto X_t(\omega)$ is Malliavin differentiable.

- **L. Xie and X. Zhang (2016, AOP):** The strong solution $X_t$ is strong Feller and irreducible.
In the case $\sigma \equiv 0$:

$$dX_t = dL_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d,$$

where $L_t$ is a symmetric $\alpha$-stable process.

- **Tanaka, Tsuchiya and Watanabe (1974, JMKU):**
  When $d = 1$, $\alpha < 1$, $b$ is bounded and $\beta$-Hölder continuous with $\alpha + \beta < 1$, SDE may not have pathwise uniqueness strong solutions.
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- Priola (2012, OJM): Condition: $\alpha \geq 1$, $b$ is bounded and $\beta$-Hölder continuous with $\beta > 1 - \alpha/2$.

- Zhang (2013, Poincare): Condition: $\alpha > 1$, $b \in L^\infty(\mathbb{R}^d) \cap W^{\beta,p}(\mathbb{R}^d)$ with $p > 2d/\alpha$ and $\beta \in (1 - \alpha/2, 1)$.
Main results

We shall consider two cases:

1. SDEs with multiplicative pure jump noise:

   \[ dX_t = \sigma(X_{t^-})dL_t + b(X_t)dt, \]

   where \( L_t \) is a symmetric \( \alpha \)-stable process.

2. SDEs with general Lévy noise:

   \[
   dX_t = \sigma(X_t)dW_t + \int_{|z|\leq1} g(X_{t^-}, z)\tilde{N}(dt, dz) + \int_{|z|>1} g(X_{t^-}, z)N(dt, dz) + b(X_t)dt,
   \]

   where \( N \) is a Poisson random measure.
Consider the following SDE in $\mathbb{R}^d$:

$$dX_t = \sigma(X_t-)dL_t + b(X_t)dt, \quad X_0 = x,$$

(2.2)

where $L_t$ is a symmetric $\alpha$-stable process with $\alpha \in (1, 2)$. 

Conditions:

$\diamond$ $\sigma$ is bounded, uniformly elliptic and $\nabla \sigma \in L^p(\mathbb{R}^d)$ with $p > \frac{2d}{\alpha}$.

$\diamond$ $b \in W^{\beta,p}(\mathbb{R}^d)$ with $p > \frac{2d}{\alpha}$ and $\beta \in \left(1 - \frac{\alpha}{2}, 1\right)$. 

Main results - Existence and uniqueness
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where $L_t$ is a symmetric $\alpha$-stable process with $\alpha \in (1, 2)$.

**Conditions:**

- $\sigma$ is bounded, uniformly elliptic and
  $$\nabla \sigma \in L^p(\mathbb{R}^d) \text{ with } p > 2d/\alpha.$$

- $b \in W^{\beta,p}(\mathbb{R}^d) \text{ with } p > 2d/\alpha \text{ and } \beta \in (1 - \alpha/2, 1)$. 
Main results - Existence and uniqueness

Theorem 1

SDE (2.2) has a unique strong solution $X_t(x)$ which is strong Feller and irreducible. Moreover, $X_t(x)$ has a density $p(t, x, y)$ with the following estimates:

$$c_1 t (t^{1/\alpha} + |x - y|)^{-d-\alpha} \leq p(t, x, y) \leq c_2 t (t^{1/\alpha} + |x - y|)^{-d-\alpha}.$$
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**Remark:** We drop the boundness condition on the drift $b$, which is new even in the additive noise case.
Main results - Existence and uniqueness

Consider the following SDE in $\mathbb{R}^d$:

\[
dX_t = \sigma(X_t)dW_t + \int_{|z| \leq 1} g(X_{t-}, z) \tilde{N}(dt, dz) \\
+ \int_{|z| > 1} g(X_{t-}, z) \mathcal{N}(dt, dz) + b(X_t)dt. \tag{2.3}
\]
Consider the following SDE in $\mathbb{R}^d$:

$$
\begin{align*}
\mathrm{d}X_t &= \sigma(X_t)\mathrm{d}W_t + \int_{|z| \leq 1} g(X_{t-}, z) \tilde{N}(\mathrm{d}t, \mathrm{d}z)
+ \int_{|z| > 1} g(X_{t-}, z) N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t.
\end{align*}
$$

(2.3)

**Conditions:**

- $\sigma$ is bounded, uniformly elliptic and $\nabla \sigma \in L^p(\mathbb{R}^d)$ with $p > d$.
- $b \in L^p(\mathbb{R}^d)$ with $p > d$.
- For any $0 < \varepsilon < 1$ and some $p > d/2$,

$$
\sup_{x \in \mathbb{R}^d} \left( \int_{|z| \leq 1} |g(x, z)|^2 \nu(\mathrm{d}z) + \int_{\varepsilon < |z| \leq 1} |g(x, z)| \nu(\mathrm{d}z) \right) < +\infty,
$$

$$
\int_{|z| \leq 1} |\nabla g(x, z)|^2 \nu(\mathrm{d}z) \in L^p(\mathbb{R}^d).
$$
Theorem 2

SDE (2.3) has a unique strong solution $X_t(x)$ which is **strong Feller** and **irreducible**. Moreover, for any bounded measurable $\varphi$,

$$\left| E\varphi(X_t(x)) - E\varphi(X_t(y)) \right| \leq \frac{C}{\sqrt{t}} \| \varphi \|_\infty |x - y|.$$
Main results - Existence and uniqueness

**Theorem 2**

SDE (2.3) has a unique strong solution $X_t(x)$ which is strong Feller and irreducible. Moreover, for any bounded measurable $\varphi$,

$$|E\varphi(X_t(x)) - E\varphi(X_t(y))| \leq \frac{C}{\sqrt{t}} \|\varphi\|_{\infty} |x - y|.$$

**Remark:** Notice that we do not make any restrictions on the pure jump Lévy process. In particular, the large jump is allowed.
Consider the following simplest SDE:

\[ dX_t = dL_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d. \]

Classical results tell us that when \( b \) is locally Lipschitz continuous and dissipative in the sense that there exist \( \kappa_1 > 0 \) and \( \kappa_2 \geq 0 \) such that

\[ \langle x, b(x) \rangle \leq -\kappa_1 |x|^{2+\ell} + \kappa_2, \quad \ell \geq 0, \]

then there exists a unique invariant measure \( \mu \) for \( X_t \).
Main results - Long time behavior

Recall that we consider the following two SDEs:

1. SDEs with multiplicative pure jump noise:

$$dX_t = \sigma(X_{t-})dL_t + b(X_t)dt.$$  \hspace{1cm} (2.4)

2. SDEs with general Lévy noise:

$$dX_t = \sigma(X_t)dW_t + \int_{|z| \leq 1} g(X_{t-}, z)\tilde{N}(dt, dz)$$

$$+ \int_{|z| > 1} g(X_{t-}, z)\tilde{N}(dt, dz) + b(X_t)dt.$$  \hspace{1cm} (2.5)

For simplify, we shall focus on providing conditions in terms of the drift $b$. 
Main results - Long time behavior

Assume that the drift coefficient $b$ is divided into two parts:

$$b = b_1 + b_2.$$ 

Then, SDE (2.4) can be written as

$$dX_t = \sigma(X_{t-})dL_t + b_1(X_t)dt + b_2(X_t)dt, \quad X_0 = x \in \mathbb{R}^d. \quad (2.6)$$
Main results - Long time behavior

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Then, SDE (2.4) can be written as

$$ dX_t = \sigma(X_{t-})dL_t + b_1(X_t)dt + b_2(X_t)dt, \quad X_0 = x \in \mathbb{R}^d. \tag{2.6} $$

**Conditions:**

- The first part $b_1$ is singular and satisfies

  $$ b_1 \in W^{\beta,p}(\mathbb{R}^d) \text{ with } p > 2d/\alpha \text{ and } \beta \in (1 - \alpha/2, 1). $$

- The second part $b_2 \in W^{\beta,p}_{loc}(\mathbb{R}^d)$ is dissipative in the sense that

  $$ \langle x, b_2(x) \rangle \leq -\kappa_1|x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b_2(x)| \leq \kappa_3(1 + |x|)^{1+\ell}. $$
Main results - Long time behavior

Theorem 3

There exists a unique invariant measure $\mu$ for the unique strong solution $X_t$ of SDE (2.6). Moreover,

- If $\ell = 0$, then $\mu$ is $V$-uniformly exponential ergodic.
- If $\ell > 0$, then $\mu$ is uniformly exponential ergodic.

Remark:

1. We do not make any continuous assumptions on the drift $b$.
2. The whole drift $b = b_1 + b_2$ may not be dissipative, since $b_1$ can be unbounded.
Main results - Long time behavior

For SDE

\[ dX_t = \sigma(X_{t-})dL_t + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d, \]

we have the following result:

**Corollary 1**

Suppose that

\[ b \in W_{loc}^{\beta, p}(\mathbb{R}^d) \text{ with } p > 2d/\alpha \text{ and } \beta \in (1 - \alpha/2, 1), \]

and there exists a \( R_0 > 0 \) such that for \( |x| \geq R_0 \),

\[ \langle x, b(x) \rangle \leq -\kappa_1 |x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b(x)| \leq \kappa_3 (1 + |x|)^{1+\ell}. \]

Then, the conclusions in Theorem 3 still hold.
Still, we first assume that the drift

\[ b = b_1 + b_2. \]

Then, SDE (2.5) can be written as

\[
\begin{align*}
\text{d}X_t &= \sigma(X_t)\text{d}W_t + \int_{|z|\leq 1} g(X_{t-}, z)\tilde{N}(\text{d}t, \text{d}z) \\
& \quad + \int_{|z|> 1} g(X_{t-}, z)N(\text{d}t, \text{d}z) + b_1(X_t)\text{d}t + b_2(X_t)\text{d}t. 
\end{align*}
\]

(2.7)
Main results - Long time behavior

Still, we first assume that the drift

\[ b = b_1 + b_2. \]

Then, SDE (2.5) can be written as

\[
dX_t = \sigma(X_t) dW_t + \int_{|z| \leq 1} g(X_{t-}, z) \tilde{N}(dt, dz) \\
+ \int_{|z| > 1} g(X_{t-}, z) N(dt, dz) + b_1(X_t) dt + b_2(X_t) dt. \tag{2.7}
\]

**Conditions:**

- The first part \( b_1 \) is singular and satisfies
  \[ b_1 \in L^p(\mathbb{R}^d) \text{ with } p > d. \]

- The second part \( b_2 \) is dissipative in the sense that
  \[ \langle x, b_2(x) \rangle \leq -\kappa_1 |x|^{2+\ell} + \kappa_2 \quad \text{and} \quad |b_2(x)| \leq \kappa_3 (1 + |x|)^{1+\ell}. \]
Theorem 4

There exists a unique invariant measure $\mu$ for the unique strong solution $X_t$ of SDE (2.7). Moreover,

- If $\ell = 0$, then $\mu$ is $V$-uniformly exponential ergodic.
- If $\ell > 0$, then $\mu$ is uniformly exponential ergodic.
Main results - Long time behavior

For SDE

\[ dX_t = \sigma(X_t)\,dW_t + \int_{|z| \leq 1} g(X_{t-}, z) \tilde{N}(dt, dz) \]
\[ + \int_{|z| > 1} g(X_{t-}, z) N(dt, dz) + b(X_t)\,dt, \]

we have the following result:

**Corollary 2**

Suppose that there exists a \( R_0 > 0 \) such that

\[ b \in L^p(B_{R_0}) \text{ with } p > d, \]

and for \( |x| \geq R_0 \),

\[ \langle x, b(x) \rangle \leq -\kappa_1 |x|^{2+\ell} + \kappa_2 \text{ and } |b(x)| \leq \kappa_3 (1 + |x|)^{1+\ell}. \]

Then, the conclusions in Theorem 4 still hold.
Important idea: use partial Zvonkin's transformation to kill only the first part $b_1$ of the drift coefficient.
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Difficulties:

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1. The non-explosion and Krylov estimate of the unique strong solution.
2. The drift $b_2$ will be involved together with the transformation function.
Important idea: use partial Zvonkin’s transformation to kill only the first part $b_1$ of the drift coefficient.

Difficulties:
1. The non-explosion and Krylov estimate of the unique strong solution.
2. The drift $b_2$ will be involved together with the transformation function.
3. Verify that the dissipative property of the new system.
Example 1

Consider the following SDE of OU type:

$$dX_t = -X_t \, dt + b(X_t) \, dt + dL_t, \quad X_0 = x \in \mathbb{R}^d.$$  

- $L_t$ – Brownian motion: we assume $b \in L^p(\mathbb{R}^d), \ p > d$;
- $L_t$ – $\alpha$-stable process with $\alpha \in (1, 2)$: we assume $b \in W^{\theta, p}(\mathbb{R}^d), \ \theta > 1 - \alpha/2$ and $p > 2d/\alpha$.

Then, the above SDE admits a unique strong solution and there exists a unique invariant measure.

Remark: In both cases, the classical Lyapunov condition can not be verified, our result is new even in the existence of invariant measures.
Example 2

Consider the following mixing SDE with jumps:

\[ dX_t = dW_t + \lambda |X_{t-}|^\beta dL_t - X_t |X_t|^{\gamma-1} dt, \quad X_0 = x \in \mathbb{R}^d, \]

where \( \lambda \in \mathbb{R}, \beta \in (0, 1) \) and \( \gamma \in (0, \infty) \), \( L_t \) is a \( d \)-dimensional pure jump Lévy process.

Then, the above SDE admits a unique strong solution and there exists a unique invariant measure which is \( V \)-ergodicity in the case \( \gamma \in (0, 1] \) and exponential ergodicity in the case \( \gamma > 1 \).

**Remark:** The main features of this SDE are that the jump coefficient \( x \mapsto |x|^\beta \) is Hölder continuous and the drift term can be polynomial growth.
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Thank You!