ENERGY CORRELATIONS IN THE CRITICAL ISING MODEL
ON A TORUS

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ABSTRACT. We compute rigorously the scaling limit of multi-point energy correlations in the critical Ising model on a torus. For the one-point function, averaged between horizontal and vertical edges of the square lattice, this result has been known since the 1969 work of Ferdinand and Fischer. We propose an alternative proof, in a slightly greater generality, via a new exact formula in terms of determinants of discrete Laplacians. We also compute the main term of the asymptotics of the difference $E(\epsilon_V - \epsilon_H)$ of the energy density on a vertical and a horizontal edge, which is of order of $\delta^2$, where $\delta$ is the mesh size. The observable $\epsilon_V - \epsilon_H$ has been identified by Kadanoff and Ceva as (a component of) the stress-energy tensor.

We then apply the discrete complex analysis methods of Smirnov and Hongler to compute the multi-point correlations. The fermionic observables are only periodic with doubled periods; by anti-symmetrization, this leads to contributions from four “sectors”. The main new challenge arises in the doubly periodic sector, due to the existence of non-zero constant (discrete) analytic functions. We show that some additional input, namely the scaling limit of the one-point function and of relative contribution of sectors to the partition function, is sufficient to overcome this difficulty and successfully compute all correlations.

1. Introduction

The Ising model is a very famous and influential model in statistical physics, mathematical physics, discrete mathematics and computer science. Originally introduced by Wilhelm Lenz in 1920 and named after Ernst Ising who in 1924 solved the 1D case of the model [32], the model was initially introduced to study the magnetic phase change at the so-called Curie temperature. For example, a piece of iron loses its ferromagnetic properties and becomes a paramagnet at 770 °C.

Since its inception, the Ising model has become a very much studied and archetypal model, a “test laboratory”, of statistical physics. This is because the model is rather simple, but still encompasses a lot of the interesting behavior. Because of the wide applicability of the model, the Ising model has been studied in many academic fields ranging from pure mathematics via physics and chemistry to biology and economics.

The most interesting aspect of Ising model is the fact that it has an order-disorder phase transition in dimension 2 and higher dimensions. This fact was established by Peierls in 1936 [47]. The temperature of the phase transition, called critical temperature, was predicted by Kramers and Wannier in 1941 [42]. After it was understood that the model does have a phase transition in dimension 2, finding an exact formula for the partition function of the model in this case became a central question of much interest in statistical physics. The feat was achieved by Lars Onsager in a seminal paper in 1944 [45] using transfer matrices. Due to this exact and rigorous formula being established, it is often said that the Ising model
is exactly solvable or integrable in 2 dimensions. After 1944, the transfer matrix technique was developed further by Onsager and Kaufman [30, 41]. The spinor analysis of Kaufman [40] underlies much of our work.

Besides the algebraic transfer matrix techniques (whose use has continued and developed since the 1940s), Ising model has also been studied by so-called combinatorial approach. This approach was advanced by, among others, Van der Waerden who developed the high temperature expansion in 1941 [53], Kac and Ward who introduced the Kac–Ward determinant in 1952 [35] (see also [38]), Sherman, Hurst-Green, Kasteleyn and Fisher who all developed a technique known as the Pfaffian method in 1960s [51, 31, 37, 38, 39, 26, 27] and McCoy and Wu, who wrote the book [44] summarizing the development of the combinatorial approach in 1970. The results of Van der Waerden and Kac–Ward especially are of key importance to our work. The conjecture of Kac and Ward, namely that the Ising model partition function can be expressed as a determinant of a suitably chosen matrix, the Kac–Ward matrix, was established mathematically in 1999 by Dolbilin, Zinov’ev, Mishchenko, Shtan’ko and Shtogrin [20] and generalized to the form we apply in this paper by David Cimasoni [14, 15].

In 1960s-1970s, another, non-rigorous way to understand Ising model appeared in the physical literature, the renormalization group. This approach postulates that the Ising model has a “continuum limit” described by a “quantum field theory (QFT)” [1]. This theory is thought to be the free fermionic theory. The QFT approach led to multiple seminal hypothesis being proposed by physicists, among them the idea that the continuum limit correlations should be related to the determinants of Laplacians and Dirac operators. We shall show that for the energy density (correlation of 2 neighboring spins) an identity in the discrete setting exists that shows this relation to determinants of Laplacians. It is noteworthy that this identity is exact and rigorous and exists even in the discrete setting, not only in the continuum limit as proposed by QFT approach.

The QFT approach was developed further by Belavin, Polyakov, Zamolodchikov [3] who in 1984 suggested that the limiting QFT for the critical Ising model case (when the fermion theory is “massless”) has conformal symmetry (CFT)(see also [1]). This lead to the famous prediction that the critical Ising model possesses conformal covariance in the scaling limit, a conjecture that has inspired a lot of work in the mathematical community in the last decade.

In the mathematical community, a lot of work has been done to prove the conjectures by physicists, and to further the understanding of the Ising model. Let us here focus only on the study on conformal covariance of the critical Ising model, that is, the conformal covariance of its various correlations. The calculation of the full-plane energy correlations was achieved by Boutillier and de Tilière [4, 5]; this was later extended to some non-integrable models [28, 2]. The full-plane spin correlations were calculated by Palmer [46]. Smirnov [52] introduced a powerful tool to analyze the scaling limit of the Ising model on arbitrary domains, namely, the discrete holomorphic fermionic observables and associated Riemann boundary value problem. This was applied to the calculation of the energy density in simply connected planar domains by Hongler and Smirnov [30]; later extended by Hongler to multi-point energy correlations in his Ph. D. thesis [29]. The case of spin correlations in planar domains was solved by Chelkak, Hongler and Izyurov [10] and recently these results were extended to all mixed correlations of primary fields in [9].
The success of the discrete holomorphicity techniques in these papers naturally leads to the question of whether they can be applied to the analysis of the model on Riemann surfaces. On the positive side, the existing universality results [12, 11, 6] allow one to extend the techniques to families of graphs that are flexible enough to approximate any Riemann surface. However, there are several known difficulties on this path. First, the fermionic correlations are not well defined on the Riemann surface but have non-trivial monodromy properties. Thus, one needs to consider $2^{2g}$ observables on a genus $g$ surface where just one was sufficient in the planar case. This is similar to the $2^{2g}$ Kac–Ward determinants introduced by Cimasoni [14, 15], and in fact this is no coincidence since the observables are related to the inverses of the Kac–Ward matrices, see [43, 7]. What’s more, the observables do not compute the energy or spin correlations per se, but rather their correlations with one of $2^{2g}$ “topological” observables, normalized by the expectation of that observable, given by the corresponding Kac–Ward determinant. Thus, to recover the correlations of interest, one would need to compute scaling limit of the ratio of the square root of each of the $2^{2g}$ Kac-Ward determinants to their sum. Finally, a degeneracy occurs in that one of the observables happens to be a non-zero constant, whose value is not immediately recoverable by the discrete complex analysis methods.

The present paper provides a first step in this program, treating the case of energy correlations on a torus. The advantage of working with the flat torus is that the above-mentioned difficulties can be treated by other methods. Thus, the Kac-Ward determinants can be calculated explicitly, and their limiting asymptotics analyzed. Moreover, as the partition function can be computed for any temperature, one can differentiate it with respect to the temperature to obtain the average energy density at criticality. This has been done by Ferdinand and Fischer [23] and later refined in [49, 50, 33] by using the Onsager–Kaufman expression for the partition function. It turns out that a slight refinement of this result (separating the vertical and the horizontal edges) is sufficient to remedy the above-mentioned degeneracy problem. We propose an alternative computation of the average energy density, based instead on the Kac–Ward solution. The advantage of our approach is that we relate, at the discrete level, the average energy density to a ratio of determinants of the discrete Laplacians. The asymptotics of the determinants of Laplacians has been recently analyzed in great generality [24, 25, 34]. Thus, if a similar “bosonization” relation is found on Riemann surfaces, the asymptotics of the one-points function in the scaling limit would be readily available.

It turns out that once the above-mentioned difficulties are treated, or the missing pieces supplied as an input, the rest of the computation of the arbitrary many-point energy correlations can be done by the discrete complex analysis methods, recovering physicists’ predictions in this regard, see Theorem 5 below. While for even number of marked points, the argument in [29, 9] can be extended almost verbatim, for odd number points we had to modify it, as we have found no combinatorial counterpart for the “propagator” $\zeta(e_n - e_m)$ in (2.5) below. Therefore, rather than reducing everything to the two-point fermionic observables as in [29, 9], we had to work out the convergence result directly for a multi-point observable, and only obtain the Pfaffian formula (2.5) by analysing the resulting scaling limit in the continuum.
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2. Main results

2.1. Setup and notation. We will study the Ising model on tori \(T^\delta := \delta \mathbb{Z}^2/\Lambda^\delta\), where \(\delta > 0\) is the mesh size, \(\Lambda^\delta\) is the lattice 
\[\Lambda^\delta = \{n\omega^\delta + m\omega^\delta : n, m \in \mathbb{Z}\}\]
and \(\omega^\delta_1, \omega^\delta_2 \in \delta \mathbb{Z}^2\) are non-collinear. We will be interested in the scaling limit of the model where \(\delta \to 0\), \(\omega^\delta_i \to \omega_i \in \mathbb{C} \setminus \{0\}\) for \(i = 1, 2\). We will assume that the modular parameter \(\tau := \omega_2 / \omega_1\) satisfies \(\Im \tau > 0\), and denote \(T = \mathbb{C} / \Lambda\), where \(\Lambda = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}\).

The (zero magnetic field) Ising model on \(T^\delta\) is the probability measure on spin configurations \(\sigma : T^\delta \to \{\pm 1\}\) given by 
\[P[\sigma] = \frac{1}{Z} \exp \left(\beta \sum_{x \sim y} \sigma_x \sigma_y\right)\]
where the sum is over all pairs of nearest-neighboring vertices of \(T^\delta\), \(\beta > 0\) is a parameter called the inverse temperature, and 
\[Z = \sum_{\sigma : T^\delta \to \{\pm 1\}} \exp \left(\beta \sum_{x \sim y} \sigma_x \sigma_y\right)\]
is the partition function of the model, which ensures that the configuration probabilities sum up to 1. Our main results will concern the critical temperature \(\beta = \beta_c = \frac{1}{2} \log(\sqrt{2} + 1)\). We will denote by \(P\) and \(E\), respectively, the probability and the expectation with respect to the above measure.

The main object of our interest is the energy observable. If \((xy)\) is an edge of \(T^\delta\), we denote 
\[\epsilon_{(xy)} = \sigma_x \sigma_y - \frac{1}{\sqrt{2}}.\]
The constant \(\frac{1}{\sqrt{2}}\) is the expectation of \(\sigma_x \sigma_y\) in the full-plane, i.e., the thermodynamic limit. Thus, our results will measure how the toric boundary conditions affect the expectation and the correlations of \(\epsilon_{(xy)}\). Clearly, since \(T^\delta\) carries an action of \(\delta \mathbb{Z}^2\) by translations, we have \(E\epsilon_e = E\epsilon_{\hat{e}}\) if the edges \(e, \hat{e}\) are either both horizontal, or both vertical. We thus denote by \(E\epsilon_H\) and \(E\epsilon_V\) the expectation of \(\epsilon\) on any horizontal and vertical edge, respectively.

2.2. Main results. Our first result concerns the sum of vertical and horizontal energy densities. We introduce some notation. Given \(\omega^\delta_1, \omega^\delta_2\) as above and \(i, j \in \mathbb{Z}_2\), denote 
\[V^ij := \{f : \delta \mathbb{Z}^2 \to \mathbb{R} : f(v + \omega^\delta_i) \equiv (-1)^i f(v), \ f(v + \omega^\delta_j) \equiv (-1)^j f(v)\}.\]
Each of $V^{ij}_δ$ is a linear space of dimension $|T^δ|$; $V^{00}_δ$ can be viewed just as the set of functions on $T^δ$. Note that the lattice Laplacian $\Delta_δ f(x) := \sum_{y \sim x} (f(y) - f(x))$ preserves each of these spaces. We denote by $\Delta^{ij}_δ$ the restriction of $\Delta$ to $V^{ij}_δ$.

**Theorem 1.** For the critical Ising model on $T^δ$, we have, for a horizontal edge $H$ and a vertical edge $V$:

$$\mathbb{E} \epsilon_V + \mathbb{E} \epsilon_H = 4 \cdot \sqrt{\det^* \Delta^{00}_δ} \cdot \frac{1}{\sqrt{\det \Delta^{10}_δ + \det \Delta^{01}_δ + \det \Delta^{11}_δ}} \cdot \frac{1}{|T^δ|},$$

where $\det^*$ denotes the product of all non-zero eigenvalues, and $|T^δ|$ denotes the number of vertices of $T^δ$.

The continuous counterpart of this identity has appeared in the CFT literature, see [18], [22], [17], however, the discrete version is, to the best of our knowledge, new. Cimasoni [15] has related the determinants of the discrete Laplacians to the theta constants, and Fischer [23], in a slightly greater generality of arbitrary torus as compared to Ferdinand and Fischer [25], we recover the asymptotics result of Ferdinand in [21], however, it involves more complicated modifications of the original graph and does not seem to lead to (2.1) either. In Section 8, we provide an analog of (2.1) for triangular lattice; we do not know whether such analogs hold true for other lattices.

By combining the above formula with known results on the asymptotics of determinants of discrete Laplacians, we recover the asymptotics result of Ferdinand and Fischer [25], in a slightly greater generality of arbitrary torus as compared to diagonal one:

**Corollary 2.** In the limit $\delta \to 0$, $\omega_{1,2}^{\delta} \to \omega_{1,2}$ with $\omega_2/\omega_1 = \tau$, we have

$$\mathbb{E} \epsilon_V + \mathbb{E} \epsilon_H = 4 \cdot \frac{2(3m\tau)^{\frac{1}{2}}|\theta_2\theta_3\theta_4|}{|\theta_2| + |\theta_3| + |\theta_4|} \cdot \frac{1}{|T|^\frac{1}{2}} \cdot \delta + o(\delta).$$

Hereinafter, our notation for the theta constants is $\theta_i := \theta_i(\tau) := \theta_i(0,q)$, where $q = e^{i\pi\tau}$ and $\theta_i(0,q)$ is as in [19], Chapter 20, and $|T|$ denotes the area of $T$.

Our second result concerns the asymptotics of the difference between energy density on vertical and horizontal edges. This difference happens to be of order $\delta^2$; such an observable was identified [36] as a component of stress-energy tensor in the model, see also [8].

**Theorem 3.** In the limit $\delta \to 0$, $\omega_{1,2}^{\delta} \to \omega_{1,2}$ with $\omega_2/\omega_1 = \tau$, we have

$$\mathbb{E}(\epsilon_H) \mathbb{E}(\epsilon_V) = \frac{\sqrt{2\pi}}{24} \cdot H(\omega_1, \omega_2) \cdot \delta^2 + o(\delta^2),$$

where $H(\omega_1, \omega_2)$ is given by

$$\frac{Z^{(01)}}{Z} \Re \left[ \omega_1^{-2}(\theta_2^2 - 2\theta_3^2) \right] + \frac{Z^{(10)}}{Z} \Re \left[ \omega_1^{-2}(\theta_2^4 + \theta_3^4) \right] + \frac{Z^{(11)}}{Z} \Re \left[ \omega_1^{-2}(\theta_3^4 - 2\theta_3^4) \right]$$

where

$$Z^{(01)} = |\theta_2|, \quad Z^{(10)} = |\theta_4| \quad Z^{(11)} = |\theta_3|,$$

and $Z = Z^{(01)} + Z^{(10)} + Z^{(11)}$. 

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We believe that his result could have been obtained by the methods of [23, 49, 50, 33], by writing down the partition function of the anisotropic Ising model and then differentiating with respect to the coupling constant separately on vertical and horizontal edges. However, we obtain it as a very simple by-product of our analysis of discrete holomorphic fermionic observables.

We record a corollary that will be useful in the study of multi-point energy correlations:

**Corollary 4.** In the limit $\delta \to 0$, $\omega_{1,2}^\delta \to \omega_{1,2}$ with $\omega_2/\omega_1 = \tau$, we have

$$
\mathbb{E}\epsilon_H = \frac{(3\pi \tau)^{\frac{1}{2}} |\theta_2 \theta_3 \theta_4|}{|\theta_2| + |\theta_3| + |\theta_4|} \frac{1}{|T|^\frac{1}{2}} \cdot \delta + o(\delta).
$$

and similarly for $\mathbb{E}\epsilon_V$.

In fact, higher-order terms (up to $\delta^3$) of the expansion of $\mathbb{E}\epsilon_H + \mathbb{E}\epsilon_V$ were computed by Salas and Izmailyan–Hu in [49, 33]. They showed that the $\delta^2$ term is absent from the expansion. Therefore, our results in fact give the expansion of $\mathbb{E}\epsilon_H$ up to order $o(\delta^2)$. Chinta, Jorgenson and Karlsson [13] indicate a way to compute the asymptotic expansion of $\det^* \Delta^{00}$ up to arbitrary order in $\delta$. Combined with our Theorem 3, this could in principle used to the compute $\mathbb{E}\epsilon_H + \mathbb{E}\epsilon_V$ up to arbitrary order.

We also compute the scaling limit of multi-point correlation functions.

**Theorem 5.** In the scaling limit $\delta \to 0$, $T^\delta \to T$, as $e_1, \ldots, e_k$ approach distinct points of $T$, we have, for even $k$,

$$
\pi^k \delta^{-k} \mathbb{E}[\epsilon_{e_1} \ldots \epsilon_{e_k}] \to \frac{Z^{(01)}}{Z} |\text{Pf} [c_{\omega_{1,2}}(e_n - e_m)]|^2 + \frac{Z^{(10)}}{Z} |\text{Pf} [d_{\omega_{1,2}}(e_n - e_m)]|^2 + \frac{Z^{(11)}}{Z} |\text{Pf} [d_{\omega_{1,2}}(e_n - e_m)]|^2,
$$

where $c_{\omega_{1,2}}, d_{\omega_{1,2}}, n_{\omega_{1,2}}$, are Jacobian elliptic functions, see Section 6.

For odd $k$, we have

$$
\pi^k \delta^{-k} \mathbb{E}[\epsilon_{e_1} \ldots \epsilon_{e_k}] \to i^k \cdot \text{Pf} M,
$$

where $M$ is the $2k \times 2k$ anti-symmetric matrix given by $M_{2n-1,2m-1} = c_{\omega_{1,2}}(e_n - e_m)$, $M_{2n,2m} = c_{\omega_{1,2}}(e_n - e_m)$, and

$$
M_{2n-1,2m-1} = -\pi i \cdot \frac{(3\pi \tau)^{\frac{1}{2}} |\theta_2 \theta_3 \theta_4|}{|\theta_2| + |\theta_3| + |\theta_4|} \cdot \frac{1}{|T|^\frac{1}{2}}.
$$

Here $c_{\omega_{1,2}}$ is the Weierstrass $\zeta$–function, see Section 6.

The formula (2.4) has been predicted in the physics literature by di Francesco, Saleur and Zuber [18], see also [17, Section 12]. The formula (2.5), on the other hand, appears to be new. We expect it to be related to the prediction of [18] by an appropriate version of Fay’s formula.

3. **Partition functions and Kac–Ward determinants**

In this Section, we record the necessary results involving Kac–Ward solution to the critical Ising model. This approach was originated in [35]; the first complete proof was given in [20], and in the case of surfaces in [14], with simplified proof in
We now recall the required material in detail in the case of a torus, following [3] Section 4.

Let us denote \( \mathcal{E}(\mathbb{T}^3) \) the set of even subgraphs of \( \mathbb{T}^3 \) (understood as subsets of edges of \( \mathbb{T}^3 \)), that is, subgraphs of \( \mathbb{T}^3 \) such that each vertex is adjacent to an even number of edges. It is well known (see for example [13] Subsection 2.1) for a short exposition) that the partition function \( Z \) can be expressed as (the high temperature expansion)

\[
Z = \cosh(\beta)^{|\text{Edges}(\mathbb{T}^3)|} 2^{|\mathbb{T}^3|} \sum_{\xi \in \mathcal{E}(\mathbb{T}^3)} \tanh(\beta)^{|\xi|}.
\]

To avoid the constant factor appearing in every formula, let us define the partition function of the Ising model without this factor:

\[
(3.1) \quad Z' := \sum_{\xi \in \mathcal{E}(\mathbb{T}^3)} \alpha^{(|\xi|)} \quad \alpha = \tanh(\beta)
\]

We shall refer to this sum also as “partition function”.

Pick a point \( z_0 \in \mathbb{C} \) such that the two lines \( \gamma_{1,2} := \{z_0 + t\omega_1^\delta : t \in \mathbb{R}\} \) do not intersect \( \delta \mathbb{Z}^2 \). We identify \( \gamma_{1,2} \) with their projection onto \( \mathbb{T}^3 \). Given an edge \( e \in \mathbb{T}^3 \) and \( i, j \in \mathbb{Z}_2 \), put

\[
(3.2) \quad \varphi_{ij}(e) = i\mathbb{1}_{e \cap \gamma_{1,2} \neq 0} + j\mathbb{1}_{e \cap \gamma_{1,2} = 0} \mod 2 = i\varphi_{10}(e) + j\varphi_{01}(e) \mod 2
\]

These are four \( \mathbb{Z}_2 \)-valued flat connections on \( \mathbb{T}^3 \), i.e., the function \( \gamma \mapsto \varphi_{ij}(\gamma) := \sum_{e \in \gamma} \varphi_{ij}(e) \) vanishes identically on contractible paths. In fact, these are the only flat connections up to gauge equivalence. As explained in [3] Section 4, this allows one to construct four spin structures \( \lambda_{ij} \), which are, roughly speaking, ways to assign a winding number modulo \( 4\pi \) to a closed lattice path. Namely, if the lattice path \( \gamma \) consist of the edges \( e_1, e_2, \ldots, e_k \), then \( \text{wind}_{\lambda_{ij}}(\gamma) = \text{wind}(\gamma) + 2\pi \varphi_{ij}(\gamma) \), where \( \text{wind} \) is the winding of the lift of \( \gamma \) to the plane. This, in turn, allows one to define four quadratic forms \( q_{ij} \) on \( \mathcal{E}(\mathbb{T}^3) \): given \( \xi \in \mathcal{E}(\mathbb{T}^3) \), decompose it into a collection of loops \( \xi_1, \ldots, \xi_N \) that do not intersect themselves or each other transversally (to this end, for each vertex of degree 4 in \( \xi \), pick any two incident edges forming a right angle, and declare them belong to the same loop, and also other two to belong to the same loop). Then, put \( (-1)^{q_{ij}(\xi)} := \prod_{k=1}^{N} (-\exp(\frac{1}{2}\text{wind}_{\lambda_{ij}}(\xi_k))) \).

Let us calculate \( q_{ij}(\xi) \) concretely. If a loop \( \xi_k \) lifts to a closed loop on \( \delta \mathbb{Z}^2 \), then it crosses \( \gamma_1 \) and \( \gamma_2 \) an even number of times, and thus \( \text{wind}_{\lambda_{ij}}(\xi_k) = \text{wind}(\xi_k) = 2\pi \). Otherwise, it lifts to a path connecting two distinct points in the plane, and we have \( \text{wind}(\xi_k) = 0 \). Thus, we have

\[
(-1)^{q_{ij}(\xi)} = (-1)^{N(\xi)+\varphi_{ij}(\xi)},
\]

where \( N(\xi) \) is a number of non-contractible loops in (the decomposition of) \( \xi \). The lift of a non-contractible loop connects \( z \) and \( z + m_1 \omega_1^\delta + m_2 \omega_2^\delta \); since the loops are simple and non-intersecting, \( m_1 \) and \( m_2 \) are relatively prime and the same for all non-contractible loops in \( \xi \). This means that \( N(\xi) = \varphi_{01}(\xi) + \varphi_{01}(\xi) - \varphi_{01}(\xi) \varphi_{10}(\xi), \) by exclusion-inclusion: configurations with an odd number of non-contractible loops contributes to \( \varphi_{01}(\xi) \) iff \( m_1 \) is odd (respectively, to \( \varphi_{10}(\xi) \) iff \( m_2 \) is odd). Therefore, (3.3)

\[
(-1)^{q_{ij}(\xi)} = (-1)^{(1-i)\varphi_{10}(\xi)+(1-j)\varphi_{01}(\xi)+\varphi_{10}(\xi)\varphi_{01}(\xi)}.
\]

Of course, this is a manifestation of the general fact that \( q_{ij} \) is a quadratic form on the homology space \( H_1(\mathbb{T}^3, \mathbb{Z}_2) \). We summarize the values of \( q_{ij}(\chi) \) in Table 1:
Lemma 7. Kac–Ward matrices.\lowercase{\S}ows, in particular, to express the partition function
\[ (3.4) \]
The first identity easily follows from (3.3): if \( \mathsf{K} \) is the first one by \( \mathsf{K} \), always changes the sign of exactly two terms. The second identity is obtained by
\[ \text{Proof.} \]
\[ (3.6) \]
To each of the four spin structures, one associates a twisted Kac–Ward matrix \( \mathsf{K} \mathsf{W}^{ij} \). That is a matrix indexed by the set \( \mathcal{E}(T^2) \) of oriented edges of \( T^2 \), by putting \( \mathsf{K} \mathsf{W}^{ij} := \text{Id} - T^{ij} \), where
\[ T^{ij}_{\vec{e},\vec{e}'} := \begin{cases} (-1)^{\gamma_{ij}(\vec{e})} \exp \left( \frac{\pi}{2} \text{wind} (\vec{e}, \vec{e}') \right) \alpha, & \text{if } t(\vec{e}) = o(\vec{e}') \text{ but } \vec{e}' \neq -\vec{e} \\ 0, & \text{otherwise} \end{cases} \]
Here \( t(\vec{e}) \) and \( o(\vec{e}) \) denote the end and the beginning of \( e \), respectively.

The following theorem due to Cimasoni relates the Kac–Ward matrices with Ising partition functions:

**Theorem 6.** We have, for \( i, j \in \mathbb{Z}_2 \),
\[ (3.4) \]
\[ \sqrt{\det \mathsf{K} \mathsf{W}^{ij}} = \sum_{\xi \in \mathcal{E}(T^2)} (-1)^{\varphi_{ij}(\xi)} \alpha^{\lvert \xi \rvert} =: Z^{ij}. \]

The following Lemma is a particular case of the equation (4.6) in \[7\]. It allows, in particular, to express the partition function \( Z' \) in terms of determinants of Kac–Ward matrices.

**Lemma 7.** We have, for any \( \xi \in \mathcal{E}(T^2) \),
\[ (3.5) \]
\[ (3.6) \]
\[ 2Z' = -\sqrt{\det \mathsf{K} \mathsf{W}^{00}} + \sqrt{\det \mathsf{K} \mathsf{W}^{01}} + \sqrt{\det \mathsf{K} \mathsf{W}^{10}} + \sqrt{\det \mathsf{K} \mathsf{W}^{11}} \]
\[ Proof. \] The first identity easily follows from (3.3): if \( \varphi_{01}(\xi) = \varphi_{10}(\xi) = 0 \), then all the terms in the left-hand side are equal to 1, and altering \( \varphi_{01}(\xi) \) or \( \varphi_{10}(\xi) \) always changes the sign of exactly two terms. The second identity is obtained by multiplying the first one by \( \alpha^{\lvert \xi \rvert} \) and summing over \( \xi \).

In the case of a torus, the determinant of Kac–Ward matrix can be calculated explicitly. Let us first denote, for \( q \in \mathbb{C}, \ z(q) = \exp(2\pi i \text{Re} q) \) and \( w(q) = \exp(2\pi i \text{Im} q) \), and
\[ (3.7) \]
\[ v(\alpha, q) := (1 + \alpha^2)^2 + \alpha \left( \alpha^2 - 1 \right) \left( z(q) + z(q)^{-1} + w(q) + w(q)^{-1} \right); \]

**Table 1.** The values of \( q_{ij} \) on the four homology classes in \( H_1(T^2, \mathbb{Z}_2) \). Here \( \gamma_1 \) and \( \gamma_2 \) are simple loops in \( T^2 \) that lift to paths on \( \delta \mathbb{Z}^2 \) connecting the origin to \( \omega_2^0 \) and \( \omega_3^0 \) respectively.

| \( \xi \) | \( q_{00}(\xi) \) | \( q_{01}(\xi) \) | \( q_{10}(\xi) \) | \( q_{11}(\xi) \) |
|---|---|---|---|---|
| \( \emptyset \) | 0 | 0 | 0 | 0 |
| \( \gamma_1 \) | 1 | 1 | 0 | 0 |
| \( \gamma_2 \) | 1 | 0 | 1 | 0 |
| \( \gamma_1 + \gamma_2 \) | 1 | 0 | 0 | 1 |

\[ \text{Note that this is real non-negative for all } \alpha \in (0, 1) \text{ and any } q, \text{ and vanishes only at } q \in \mathbb{Z} \text{ and } \alpha = \sqrt{2} - 1. \]
and we define shift vectors (complex numbers) \( s_{(ij)}, i, j \in \{0, 1\} \) as follows:

\[
\Re s_{(ij)} \Re \omega_1^\delta + \Im s_{(ij)} \Im \omega_1^\delta = \frac{i}{2},
\]

\[
\Re s_{(ij)} \Re \omega_2^\delta + \Im s_{(ij)} \Im \omega_2^\delta = \frac{j}{2}.
\]

Note that such \( s_{(ij)} \) with these properties exist and are unique as \( \omega_{1,2}^\delta \) form a basis of the plane.

**Theorem 8.** One has

\[
(3.8) \quad \det \mathcal{KW}^{ij} = \prod_{q \in \Lambda^*/\mathbb{Z}^2 + s_{(ij)}} v(\alpha, q).
\]

**Proof.** The theorem is as in [16] Lemma 4.1. For completeness, let us provide a short argument. This argument originates from Kac and Ward [35].

Observe that we can think of \( \mathcal{KW}^{ij} \) as a non-twisted Kac–Ward matrix \( \mathcal{KW} := \mathcal{KW}^{00} \) acting on the corresponding space of (anti-)periodic function of oriented edges, similar to \( V^{ij} \). The untwisted Kac–Ward operator \( \mathcal{KW} \) commutes with shifts by \( \mathbb{Z}^2 \), hence, it is natural to look for eigenvectors of \( \mathcal{KW}^{ij} \) that are also eigenvectors of these shifts. The latter in general have the form

\[
U_q(\vec{e}) = \hat{U}([\vec{e}]) e^{2\pi i \Re x(\vec{e})} e^{2\pi i \Im y(q(\vec{e}))} = \hat{U}([\vec{e}]) z(q)^x(\vec{e}) w(q)^y(\vec{e}),
\]

where \( \hat{U}([\vec{e}]) \) is some function of the equivalence class of \( \vec{e} \) under shifts, \( x(\vec{e}), y(\vec{e}) \in \mathbb{Z}^2 \subset \mathbb{R}^2 \) are “coordinates” of \( \vec{e} \) (say, of its beginning), and the condition \( q \in \Lambda^*/\mathbb{Z}^2 + s_{(ij)} \) stems from the (anti-)periodicity requirement. The action of \( \mathcal{KW}^{ij} \) on \( U_q \) is then straightforward to compute,

\[
\mathcal{KW}(U_q)(\vec{e}) = (\mathcal{KW}^{(q)}(\hat{U})) ([\vec{e}]) z(q)^x(\vec{e}) w(q)^y(\vec{e}),
\]

where \( \mathcal{KW}^{(q)} \) is the twisted Kac–Ward operator of a one-vertex torus corresponding to the connection \( \varphi^q \) with \( \varphi^q(\vec{e}) = z(q) \) for \( \vec{e} \) pointing to the right, and \( \varphi^q(\vec{e}) = w(q) \) for \( \vec{e} \) pointing upwards. Therefore, we conclude that

\[
\det \mathcal{KW}^{ij} = \prod_{q \in \Lambda^*/\mathbb{Z}^2 + s_{(ij)}} \det \mathcal{KW}^{(q)}.
\]

Finally, \( \mathcal{KW}^{(q)} \) is an explicit \( 4 \times 4 \) matrix, and a straightforward computation yields

\[
\det \mathcal{KW}^{(q)} = v(\alpha, q).
\]

We observe that the same expressions give the determinants of the discrete Laplacians defined in Section 2.2.

**Proposition 9.** The discrete Laplacian \( \Delta^{ij} \) has eigenvalues

\[
(3.9) \quad \frac{1}{2\alpha_c^2} v(\alpha_c, q), \quad q \in \Lambda^*/\mathbb{Z}^2 + s_{(ij)},
\]

where \( \alpha_c = \sqrt{2} - 1 \).

**Proof.** Observe that if \( q \in \Lambda^*/\mathbb{Z}^2 + s_{(ij)} \), then the exponential function \( f_q(z) = \exp(2\pi i (\Re q \Re z + \Im q \Im z)) \), where \( z \in \mathbb{Z}^2 \), belong to \( V^{ij} \), and \( \Delta f_q = \frac{1}{2\alpha_c^2} v(\alpha_c, q) f_q. \)
Thus we find $|\Lambda^* / \mathbb{Z}^2| = |\mathbb{T}^d|$ distinct eigenvalues; as $V_s^{ij}$ has dimension $|\mathbb{T}^d|$, these are all the eigenvalues. $\square$

4. The sum of the horizontal and vertical edge energy densities

We continue with a standard Lemma representing energy density in high-temperature expansion, see e.g. [15, Subsection 2.1] or [36] for background.

**Lemma 10.** Let $e = (xy) \in \text{Edges}(\mathbb{T}^d)$ be any edge. For the critical temperature $\alpha = \sqrt{2} - 1$, one has

$$E(e) = \frac{1}{\sqrt{2}} \frac{1}{Z} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \left( \alpha^{-1} I_{e \in \xi} - \alpha I_{e \notin \xi} \right) \alpha^{[\xi]}.$$\(^1\)

**Proof.** Denote by $\mathcal{E}_e(\mathbb{T}^d)$ the set of configurations $\omega \subseteq \text{Edges}(\mathbb{T}^d)$ such that every vertex except the end-vertices of $e$ have even degree and the end-vertices of $e$ have odd degree in $\omega$. If $\xi \in \mathcal{E}(\mathbb{T}^d)$, define $\omega_e(\xi) \in \mathcal{E}_e(\mathbb{T}^d)$ by $\omega_e(\xi) = \xi \cup e$ if $e \notin \xi$ and $\omega_e(\xi) = \xi \setminus e$ if $e \in \xi$. Then, $\omega_e$ is a bijection between $\mathcal{E}(\mathbb{T}^d)$ and $\mathcal{E}_e(\mathbb{T}^d)$. By high-temperature expansion, we have

$$E(\sigma_x \sigma_y) = \frac{1}{Z} \sum_{\omega \in \mathcal{E}_e(\mathbb{T}^d)} \alpha^{[\omega]} = \frac{1}{Z} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \alpha^{[\omega_e(\xi)]} = \frac{1}{Z} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \left( I_{e \in \xi} \alpha^{[\xi-1]} + I_{e \notin \xi} \alpha^{[\xi]} \right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{Z} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \left( \left( \alpha^{-1} - \frac{1}{\sqrt{2}} \right) I_{e \in \xi} \alpha^{[\xi]} + \left( \alpha - \frac{1}{\sqrt{2}} \right) I_{e \notin \xi} \alpha^{[\xi]} \right),$$

and we finish by noting that $\alpha^{-1} - \frac{1}{\sqrt{2}} = \frac{\alpha-1}{\sqrt{2}}$ and $\alpha - \frac{1}{\sqrt{2}} = -\frac{\alpha}{\sqrt{2}}$. $\square$

For $i, j \in \mathbb{Z}_2$ and $e$ and edge of $\mathbb{T}^d$, define

$$B^{(ij)}(e) := \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \left( \alpha^{-1} I_{e \in \xi} - \alpha I_{e \notin \xi} \right) (-1)^{q_{ij}(\xi)} \alpha^{[\xi]}.$$

In view of Lemma 10 and (3.5), we have

$$E(e) = \frac{1}{2\sqrt{2}} \frac{1}{Z} \left( B^{(01)}(e) + B^{(10)}(e) + B^{(11)}(e) - B^{(00)}(e) \right).$$

It turns out that at the critical temperature, it is possible to express the sum of $B^{(ij)}(e)$ for horizontal and vertical edge in terms of a determinant of the Laplacian:

**Lemma 11.** One has, for $\alpha = \alpha_c = \sqrt{2} - 1$,

$$B^{(ij)}(e_H) + B^{(ij)}(e_V) = \begin{cases} -2^{(|\mathbb{T}^d|+5)/2} \alpha |\mathbb{T}^d| \sqrt{\det \Delta^{[00]} \frac{1}{|\mathbb{T}^d|}}, & i = j = 0; \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** Denote by $Z^{(ij)}(\alpha)$ the right-hand side of (3.4). We can write

$$\frac{d}{d\alpha} Z^{(ij)}(\alpha) = \frac{1}{\alpha} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} |\xi| \alpha^{[\xi]} (-1)^{q_{ij}(\xi)} \frac{1}{\alpha} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} I_{e \in \xi} \alpha^{[\xi]} (-1)^{q_{ij}(\xi)}$$

$$= \frac{|\mathbb{T}^d|}{\alpha} \left( \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} I_{e \in \xi} \alpha^{[\xi]} (-1)^{q_{ij}(\xi)} + \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} I_{e \notin \xi} \alpha^{[\xi]} (-1)^{q_{ij}(\xi)} \right),$$

$\text{The reader may observe that this is essentially the same proof as that of Theorem 8; the latter is a bit more involved since the fundamental domain has only one vertex and four oriented edges.}$

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where we used that $T^\delta$ has exactly $|T^\delta|$ horizontal edges and $|T^\delta|$ vertical edges, and $\sum_{e \in E(T^\delta)} \mathbb{1}_{c \in \alpha}(1-1) \mu(e)$ only depends on whether $e$ is vertical or horizontal. Now, using that $\mathbb{1}_{c \in \alpha} = \frac{1}{\alpha+\alpha^{-1}}(\alpha^{-1} \mathbb{1}_{c \in \alpha} - \alpha \mathbb{1}_{c \in \alpha}) + \frac{\alpha}{\alpha+\alpha^{-1}}$, we arrive at

$$
\frac{d}{d\alpha} Z^{(ij)}(\alpha) = \frac{2 |T^\delta|}{\alpha + \alpha^{-1}} Z^{(ij)}(\alpha) + \frac{|T^\delta|}{\alpha + \alpha^{-1}} \left( B^{(ij)}(e_H) + B^{(ij)}(e_V) \right) 
$$

We now use Theorem 6 and Theorem 8 to compute $\frac{d}{d\alpha} Z^{(ij)}(\alpha)$ at $\alpha = \alpha_c$, starting with the case $(ij) \neq (00)$. It is straightforward to see from (3.7) that

$$
v(\alpha, q) = 2\alpha^2 \left( 4 - z(q) - z(q)^{-1} - w(q) - w(q)^{-1} \right);$$

$$
\frac{d}{d\alpha} v(\alpha, q) = 2\sqrt{2}\alpha^2 \left( 4 - z(q) - z(q)^{-1} - w(q) - w(q)^{-1} \right).
$$

In particular, since $|z(q)| = |w(q)| = 1$, the quantity $v(\alpha_c, q)$ can only vanish when $z(q) = w(q) = 1$, that is, $q \in \mathbb{Z}^2$. If $i = 1$ (respectively, $j = 1$), then, for $q \in \Lambda^* + s_{ij}$, we have $\Re q \delta \omega^i_2 + 3m q \delta \omega^j_2 \in \mathbb{Z} + \frac{1}{2}$ (respectively, $\Re q \delta \omega^j_2 + 3m q \delta \omega^i_2 \in \mathbb{Z} + \frac{1}{2}$). Since $\omega^i_2, \omega^j_2 \in \mathbb{Z}^2$, this cannot happen for $q \in \mathbb{Z}^2$. We conclude that if $(i, j) \neq (00)$, then $Z^{(ij)}(\alpha) \neq 0$. In this case, we have

$$
\frac{d}{d\alpha} \log Z^{(ij)}(\alpha) = \frac{1}{2} \sum_{q \in \Lambda^* / \mathbb{Z}^2 + s_{ij}} \frac{d}{d\alpha} \log v(\alpha, q).
$$

By (4.4) and (4.5), each term in the sum equals $\sqrt{2}$, and $|\Lambda^* / \mathbb{Z}^2 + s_{ij}| = |T^\delta|$. Thus

$$
\frac{d}{d\alpha} Z^{(ij)}(\alpha) = \frac{\sqrt{2} |T^\delta|}{2} Z^{(ij)}(\alpha_c).
$$

Since $2/(\alpha_c + \alpha_c^{-1}) = \sqrt{2}$, plugging this equation into (4.3) at $\alpha = \alpha_c$ yields the desired result $B^{(ij)}(e_H) + B^{(ij)}(e_V) = 0$ if $(ij) \neq (00)$.

We now turn to the the case $(ij) = (00)$. We have $0 \in \Lambda^* / \mathbb{Z}^2 + s_{(00)} = \Lambda^* / \mathbb{Z}^2$ and $v(\alpha_c, 0) = 0$; by (4.5), we also have $\frac{d}{d\alpha} v(\alpha, 0) = 0$, and from (3.7), it is straightforward to compute

$$
\frac{d^2}{d\alpha^2} \bigg|_{\alpha=\alpha_c} v(\alpha, 0) = 16.
$$

By the above discussion, $v(\alpha_c, q) \neq 0$ for other $q \in \Lambda^* / \mathbb{Z}^2$. Therefore,

$$
Z^{(00)}(\alpha) = \left( 8(\alpha - \alpha_c)^2 \prod_{q \in \Lambda^* / \mathbb{Z}^2 \{0\}} v(\alpha_c, q) \right)^{\frac{1}{2}} (1 + o(1)) \quad \text{as } \alpha \to \alpha_c,
$$

and we obtain, taking into account Proposition 9

$$
\frac{d}{d\alpha} Z^{(00)}(\alpha) = -2\sqrt{2} \left( \prod_{q \in \Lambda^* / \mathbb{Z}^2 \{0\}} v(\alpha_c, q) \right)^{\frac{1}{2}} = -2 |T^\delta|^2 \alpha_c |T^\delta|^{-1} \sqrt{\det^* \Delta^{00}}.
$$

The choice of the negative sign of the square root can be justified by looking into the Proof of Theorem 1.1. For in that proof it is shown that $\det \mathcal{K} \mathcal{W}_{00}(\alpha) = Z_{00}(\alpha)$ (Lemma 3.1) behaves as follows: when $\alpha \to 0$, $Z_{00}(\alpha) \to 1$, and when $\alpha \to 1$, $Z_{00}(\alpha) \to -2 |T|$, and $Z_{00}(\alpha) = 0$ has a unique solution, that is, the critical point: $Z_{00}(\alpha_c) = 0$. Thus its derivative at the critical point must be negative.
Since \( Z^{(00)}(\alpha_c) = 0 \), plugging \((4.6)\) into \((4.3)\) proves the desired result. \(\square\)

**Proof of Theorem 1.** By summing \((4.2)\) over a horizontal and vertical edge and taking into account Lemma 11, we get
\[
\mathbb{E}(e_H) + \mathbb{E}(e_V) = -\frac{1}{2\sqrt{2}Z^I} \left( B^{(00)}(e_H) + B^{(00)}(e_V) \right) = 2\frac{\tau^4 + 2}{2\sqrt{2}} \alpha_c |T^I| \sqrt{\det^* \Delta^{00}} \frac{1}{|T^M|}.
\]
Recall that \( Z^{(00)}(\alpha_c) = 0 \). By combining Theorem 6, the equation \((3.4)\) and Proposition 9, we get
\[
Z' = \frac{1}{2} \left( Z^{(01)}(\alpha_c) + Z^{(10)}(\alpha_c) + Z^{(11)}(\alpha_c) \right)
\]
\[
= \frac{1}{2} \left( \sqrt{\det \mathcal{K}_1} + \sqrt{\det \mathcal{K}_2} + \sqrt{\det \mathcal{K}_3} \right)
\]
\[
= 2\frac{\tau^4}{2} \alpha_c |T^I| \left( \sqrt{\det \Delta^{01}} + \sqrt{\det \Delta^{10}} + \sqrt{\det \Delta^{11}} \right).
\]
Plugging this into \((4.7)\) ends the proof of the Theorem 1. \(\square\)

We now turn to the proof of Corollary 2. We scale our lattice and the torus by \(\delta\), as in the introduction. Given a lattice \(\Lambda = n\omega_1 + m\omega_2 \subset \mathbb{R}^2\), we consider the continuous Laplacian \(\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\) acting on the of twice continuously differentiable functions on the torus \(\mathbb{R}^2/\Lambda\). We then define the (Minakshisundaram–Pleijel) zeta function by
\[
\zeta_{\tau}(s) := \sum_{\lambda_n \neq 0} \frac{1}{\lambda_n^s}, \quad \text{Res} > 1
\]
where the sum is over all non-zero eigenvalues of \(\Delta\). This \(\zeta\)-functions can be analytically continued into the vicinity of 0, and the \(\zeta\)-regularized determinant of the Laplacian is defined by
\[
\log \det^* \Delta = -(\zeta_{\tau})'(0).
\]
We will derive Corollary 2 from the following result relating asymptotics of determinants of discrete Laplacian and to \(\det^* \Delta\):

**Proposition 12.** ([13] Theorem 1], see also [33]) As \(\delta \to 0\) and \(\omega_1, \omega_2 \to \omega_1, \omega_2\), one has
\[
\det^* \Delta^{00} = \delta^{-2} \exp \left( C|T^I| \right) \det^* \Delta \cdot (1 + o(1)),
\]
where \(C\) is an explicit constant.

Another ingredient is the classical computation of \(\det^* \Delta\), due to Kronecker:
\[
\det^* \Delta = \Im \tau |T||\eta(\tau)|^4,
\]
where \(\eta(\tau) = \left( \frac{1}{2} \theta_2(\tau) \theta_3(\tau) \theta_4(\tau) \right) \frac{1}{2}\) is the Dedekind eta-function, see e. g. [17] section 10.2.

**Proof of Corollary 2.** In this proof, we will use several lattices, and hence we will use the notation \(V^{ij}_{\omega_1, \omega_2}\) and \(\Delta^{ij}_{\omega_1, \omega_2}\) for \(V^{ij}_{\delta}\) and \(\Delta^{ij}_{\delta}\), respectively, emphasizing the dependence on the periods. Note that
\[
V^{00}_{\omega_1, \omega_2} = V^{00}_{\omega_1, \omega_2} + V^{10}_{\omega_1, \omega_2} + V^{10}_{\omega_1, \omega_2} = V^{00}_{\omega_1, \omega_2} + V^{01}_{\omega_1, \omega_2} + V^{01}_{\omega_1, \omega_2};
\]
\[
V^{01}_{\omega_1, \omega_2} = V^{01}_{\omega_1, \omega_2} + V^{11}_{\omega_1, \omega_2},
\]
\[
V^{10}_{\omega_1, \omega_2} = V^{10}_{\omega_1, \omega_2} + V^{01}_{\omega_1, \omega_2} + V^{01}_{\omega_1, \omega_2}.
\]
and the direct summands are invariant subspaces for the discrete Laplacian. Hence, using the asymptotics \((4.8)\), we obtain

\[
\text{(4.10)} \quad \det \Delta_{01}^{10} = \frac{\det^* \Delta_{00}^{10}}{\det^* \Delta_{01}^{00}} = e^{C|T^\delta| + o(1)} \frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}};
\]

\[
\text{(4.11)} \quad \det \Delta_{01}^{01} = \frac{\det^* \Delta_{00}^{01}}{\det^* \Delta_{01}^{00}} = e^{C|T^\delta| + o(1)} \frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}};
\]

\[
\text{(4.12)} \quad \det \Delta_{01}^{11} = \frac{\det^* \Delta_{00}^{11}}{\det^* \Delta_{01}^{01}} = \frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}};
\]

By plugging \(z = 0\) into the reduction identities for theta functions \([19, 20.7.11-20.7.12]\), we get

\[
\theta_2(2\tau)\theta_3(2\tau)\theta_4(2\tau) = \frac{1}{2} \theta_2^2(\tau)\theta_3^3(\tau)\theta_4^3(\tau)
\]

\[
\theta_2(\tau/2)\theta_3(\tau/2)\theta_4(\tau/2) = \sqrt{2} \theta_2^4(\tau)\theta_3^2(\tau)\theta_4^2(\tau).
\]

Using these identities and \((4.9)\), we get

\[
\frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}} = \frac{|\eta(\tau/2)|^4}{|\eta(\tau)|^4} = \frac{\theta_4(\tau)^4}{|\eta(\tau)|^4};
\]

\[
\frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}} = \frac{4|\eta(2\tau)|^4}{|\eta(\tau)|^4} = \frac{\theta_2(\tau)^4}{|\eta(\tau)|^4};
\]

\[
\frac{\det^* \Delta_{2w_1,2w_2}}{\det^* \Delta_{1,2w_2}} = \frac{\theta_3(\tau)^8}{|\eta(\tau/2)|^4|\eta(2\tau)|} = \frac{\theta_3(\tau)^2}{|\eta(\tau)|^2}.
\]

Combining this with \((1.10)\) and \((4.12)\) and and plugging these asymptotics and \((4.9)\) into the expression obtained in Theorem 3, we get

\[
E(\epsilon_H) + E(\epsilon_V) = 4 \left( \frac{(3m\tau)^{1/2}|T|^3|\eta(\tau)|^3\delta^{-1}}{|\theta_2(\tau)|^2 + |\theta_3(\tau)|^2 + |\theta_4(\tau)|^2} |T^\delta| (1 + o(1)) \right)
\]

\[
= 4 \left( \frac{(3m\tau)^{1/2}|\eta(\tau)|^3}{|T|^3 (|\theta_2(\tau)|^2 + |\theta_3(\tau)|^2 + |\theta_4(\tau)|^2)} \right) \delta + o(\delta),
\]

where we have noticed that \(|T^\delta|\delta^2 \to |T|\). This concludes the proof.

5. Discrete holomorphic fermionic observables on a torus

To prove Theorems 3 and 4, we employ the discrete holomorphic fermionic observables. We follow the definitions and constructions in 9; note that the lattice there is scaled by \(\sqrt{2}\) and rotated by \(\pi/4\). By a corner of the lattice \(\delta\mathbb{Z}^2\), we mean a midpoint of a segment joining a lattice vertex \(v \in \delta\mathbb{Z}^2\) with a vertex of the dual lattice \((\delta\mathbb{Z}^2)^* = \delta\mathbb{Z}^2 + \frac{\delta}{2} + i\frac{\delta}{2}\). The corners thus form another square lattice \(C_5 := \frac{\delta}{2}\mathbb{Z}^2 + \frac{\delta}{4} + i\frac{\delta}{4}\). We also define the corner graph \(C(T^\delta)\) and the dual graph \(T^\delta,*\) of the torus \(T^\delta\) in an obvious way. Given a corner \(z \in C(T^\delta)\), we denote by \(z^o\) (respectively, \(z^*\)) the vertex of \(T^\delta\) (respectively, \(T^\delta,*\)) incident to \(z\).
By the doubling $\hat{T}^\delta$ of the torus $T^\delta$, we mean the torus 
\[\hat{T}^\delta := \delta\mathbb{Z}^2/(2n\omega_1^\delta + 2n\omega_2^\delta : m, n \in \mathbb{Z}^2).\]

Note that $\hat{T}^\delta$ naturally forms a 4-sheet cover of $T^\delta$. Given a corner $a \in \mathcal{C}(\hat{T}^\delta)$, we denote 
\[a_{pq} := a + p\omega_1^\delta + q\omega_2^\delta, \quad p, q \in \{0, 1\},\]
the four point that project to the same image in $T^\delta$.

Given a subset $\gamma$ of edges of $T^{\delta,\ast}$, we define the disorder observable 
\[\mu_\gamma := e^{-2\beta \sum_{(xy) \cap \gamma \neq \emptyset} \sigma_x \sigma_y},\]
where the sum is over the nearest neighbors $x \sim y \in T^\delta$ such that the edge $(xy)$ intersects $\gamma$. We view subsets of edges of $T^{\delta,\ast}$ as chains modulo 2, in particular, $\gamma_1 + \gamma_2 = \gamma_1 - \gamma_2$ will denote the symmetric difference of $\gamma_1$ and $\gamma_2$, and $\partial \gamma$ is boundary of $\gamma$, that is, the set of all vertices of $T^{\delta,\ast}$ incident to an odd number of edges in $\gamma$. It turns out that the correlation of $\mu_\gamma$ with spins only depends on $\partial \gamma$ and a homological class of $\gamma$ modulo 2, as we now describe:

**Lemma 13.** Let $v_1, \ldots, v_{2n}$ be vertices of $T^\delta$, and $\gamma_1, \gamma_2$ two subsets of edges of $T^{\delta,\ast}$ with $\partial \gamma_1 = \partial \gamma_2$. Assume that $[\gamma_1] - [\gamma_2] = 0$ in $H_1(T^{\delta,\ast}, \mathbb{Z}_2)$, so that there exists a collection $F$ of faces of $T^{\delta,\ast}$ such that $\sum_{v \in F} \partial v = \gamma$. Then 
\[E[\sigma_{v_1} \ldots \sigma_{v_{2n}} \mu_{\gamma_1}] = (-1)^N E[\sigma_{v_1} \ldots \sigma_{v_{2n}} \mu_{\gamma_2}],\]
where $N = |\{v_1, \ldots, v_{2n}\} \cap F|$.

**Proof.** Using $\gamma_1$ as a branch cut, one can construct a double cover $\hat{T}_1$ of the graph $T^\delta$. Namely, consider two copies ("sheets") of $T^\delta$, remove in each copy the edges crossing $\gamma_1$, and add instead the two edges connecting the corresponding vertices on different sheets. We can write 
\[E[\sigma_{v_1} \ldots \sigma_{v_{2n}} \mu_{\gamma_1}] = \frac{1}{Z} \sum_{v : T^\delta \rightarrow \{\pm 1\}} \sigma_{v_1} \cdots \sigma_{v_{2n}} e^{\beta \sum_{x \sim y} \sigma_x \sigma_y} e^{-2\beta \sum_{(xy) \cap \gamma \neq \emptyset} \sigma_x \sigma_y} = \frac{1}{Z} \sum_{v : T^\delta \rightarrow \{\pm 1\}} \sigma_{v_1} \cdots \sigma_{v_{2n}} e^{\beta \sum_{x \sim y} \sigma_x \sigma_y},\]
where $v \in T^\delta$ is identified with its copy on the first sheet, and $v^\ast$ denote its copy on the second sheet. We, if we repeat the same construction with $\gamma_2$, since $[\gamma_1] - [\gamma_2] = \sum_{v \in F} \partial v$, then flipping the sheets of all vertices corresponding to $v \in F$ gives an isomorphism of the two double covers, under which exactly $N$ of $v_1, \ldots, v_{2n}$ will move to the second sheet. This gives the desired result. \hfill $\Box$

In view of this lemma, the quantity $E[\sigma_{v_1} \ldots \sigma_{v_{2n}} \mu_\gamma]$ can be understood as a 'multi-valued function' of $v_1, \ldots, v_{2n}$ and $u_1, \ldots, u_{2m}$, where $\partial \gamma = \{u_1, \ldots, u_{2m}\}$, meaning that once the initial condition (i. e., the choice of $\gamma$ for a certain position of $u_1, \ldots, u_{2m}$) is prescribed, there is a natural way to extend its value as the marked points move around in the lattice. Namely, when moving $u_i \mapsto u_i'$, one replaces $\gamma$ by $\gamma + (u_i, u_i')$, and when moving $v_i \mapsto v_i'$, one adds a $-$ sign whenever $(v_i, v_i')$ crosses $\gamma$. With this convention, if all the points move and come back to their initial positions, say, without leaving a fixed fundamental domain of $T^\delta$, then the expression $E[\sigma_{v_1} \ldots \sigma_{v_{2n}} \mu_\gamma]$ changes sign in the same way as $\prod(u_i - v_j)^{\frac{1}{2}}$. 

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Figure 5.1. A piece of the double cover of the corner graph, ramified at every face of that graph, on which the Dirac spinor $\eta_z$ is defined. The white discs are the vertices of the original lattice $\delta \mathbb{Z}^2$, the black discs are vertices of the dual graph, and the small squares are vertices of the corner graph.

For $p, q \in \{0, 1\}$, we denote

$$
\mu_{pq} := \mu_{\gamma_{pq}},
$$

(5.1)

where $\gamma_{pq}$ is a simple loop on $T^{\delta^*}$ that lifts to a path on $\delta \mathbb{Z}^2$ connecting a point $z$ with $z + p\delta_1 + q\delta_2$. In particular, we can choose $\mu_{00} = \emptyset$.

**Definition 14.** We define the Dirac spinor

$$
\eta_z := e^{i\frac{\pi}{4}} \left(\frac{z^\circ - z^\bullet}{|z^\circ - z^\bullet|}\right)^{-\frac{1}{2}},
$$

(5.2)

understood as a two-valued function on the corner lattice $C(T^\delta)$, i.e., the function the double cover of that graph ramified at every face, see Fig. 5.1.

The fermionic observables we use are similar to the to multi-point fermionic observables as in [9, Section 2.4]. Since we will only need those for the energy correlation, we will restrict our consideration to a very specific configuration.

Let $C^{[e_1 \ldots e_k]}(T^\delta)$ denote the graph obtained by deleting from $C(T^\delta)$ the edges crossing $e_1, \ldots, e_k$ and their shifts by $\omega_1^\delta, \omega_2^\delta, \omega_1^\delta + \omega_2^\delta$. The quantity $F_{e_1 \ldots e_k}(a, \cdot)$ defined below in (5.3) is a “spinor” on $C(T^\delta)$ ramified at $a^\bullet, a^\circ$ and their shifts by $\omega_1^\delta, \omega_2^\delta, \omega_1^\delta + \omega_2^\delta$, i.e., a function on the double cover of $C(T^\delta)$ ramified at those points and changing sign between sheets. We prefer to view it as a function rather than spinor, by introducing a branch cut $[a^\circ a^\bullet]$ that divides $a$ into two vertices $a^\pm$, $a^+$ being on the left as seen from $a^\bullet$, and similarly for the shifts. We denote the resulting graph by $C^{[e_1 \ldots e_k]}_{[a]}(T^\delta)$, see Fig. 5.2.

**Definition 15.** Given distinct edges $e_1, \ldots, e_k$ in the torus $T^\delta$ that pairwise have no common incident corner, and a corner $a \in C(T^\delta)$, we define the observable

$$
F_{e_1 \ldots e_k}(a, z) := \eta_z \mathbb{E} \left[ \sigma_{z^\bullet} \sigma_{a^\circ} \mu_{z^\bullet} \mu_{a^\circ} \prod_{i=1}^k \epsilon_{e_i} \right],
$$

(5.3)

$z \in C^{[e_1 \ldots e_k]}_{[a]}(T^\delta)$.
We fix a choice of a square root in the definition of $\eta_a$ and set the “initial conditions” for $F_{e_1...e_k}(a, z)$ of the above observable by $\mu_{a+} = \mu_0$ and $\eta_{a+} = \eta_a$, so that we have

$$F_{e_1...e_k}(a, a^+) = \pm \eta_a \prod_{i=1}^{k} \epsilon_{e_i}.$$  

Recall that a function $F$ defined on a corner graph is called $s$-holomorphic if it satisfies the phase condition $F(z) \in \eta_z \mathbb{R}$, and for every edge $e$, one has

$$F(e_{NE}) + F(e_{SW}) = F(e_{NW}) + F(e_{SE}),$$

where $e_{NE}, e_{SW}, e_{NW}, e_{SE}$ are the four corners neighboring $e$. We will, in fact, only need the following simple properties of $s$-holomorphic functions: if $F$ is $s$-holomorphic, then its restriction to each of the two sub-lattices $\{ z : \eta_{z} \in \eta_z \mathbb{R} \cup \eta_z \mathbb{R} \}$ are discrete holomorphic in the usual sense, that is,

$$F\left(z + \frac{\delta}{2}\right) - F\left(z - \frac{\delta}{2}\right) - \frac{1}{i} \left(F\left(z + i\frac{\delta}{2}\right) - F\left(z - i\frac{\delta}{2}\right)\right) = 0,$$

for every $z \in C$ such that $F$ is $s$-holomorphic at both edges intersecting the segments $[z - \frac{\delta}{2}; z + \frac{\delta}{2}]$ and $[z - i\frac{\delta}{2}; z + i\frac{\delta}{2}]$. As a consequence, if, for an open set $\Omega$, a sequence $F_{\delta}$ of $s$-holomorphic functions defined on refining lattices $\Omega \cap C(\delta \mathbb{Z}^2)$ is uniformly bounded on every compact subset of $\Omega$, then there is a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ and a subsequence $\delta_k$ such that

$$F_{\delta_k}(z) = \text{Proj}_{\eta_z}(f(z)) + o(1),$$

uniformly on compact subsets of $\Omega$. This is a simple consequence of the discrete Cauchy integral formula, see e.g. [12].

Figure 5.2. The lattice $\delta \mathbb{Z}^2$ as $\circ$, its dual lattice as $\bullet$, the corner lattice as $\square$. On the right: the modified graph $C_{[a]}^e(\cdot)$. Any closed path on $C_{[a]}^e(\cdot)$ either encircles both $a^*$ and $a^+$, or none; hence $F(a, \cdot)$ is single-valued on the modified graph. The values $F_{e_1}(a, a_L)$ and $F_{e_1}(a, a_R)$ are proportional to $\mathbb{E}(\epsilon_{e_L}, \epsilon_{e_1})$ and $\mathbb{E}(\epsilon_{e_R}, \epsilon_{e_1})$ respectively, see Lemma 16.
Lemma 16. The observable $F_{e_1...e_k}(a, \cdot)$ is a well defined s-holomorphic function on $C_{\{e_1...e_k\}}(\hat{T}_\delta)$. Moreover, it has the following special values:

\begin{equation}
F_{e_1...e_k}(a, a^\pm) = \pm \eta_a E \left[ \prod_{m=1}^{k} \epsilon_{e_m} \right],
\end{equation}

\begin{equation}
F_{e_1...e_k}(a, a^\pm + p\omega_1^d + q\omega_2^d) = \mp \eta_a E \left[ \mu_{pq} \prod_{m=1}^{k} \epsilon_{e_m} \right], \quad (pq) \neq (00).
\end{equation}

If, moreover, we denote $a_L := a + i \cdot (a^* - a^\circ)$ and $a_R = a - i \cdot (a^* - a^\circ)$, and by $e_{L,R}$ the edges incident to both $a$ and $a_{L,R}$ respectively, then

\begin{equation}
F_{e_1...e_k}(a, a_{L,R}) = -i \eta_a \sqrt{2} E \left[ \epsilon_{e_{L,R}} \prod_{m=1}^{k} \epsilon_{e_m} \right],
\end{equation}

\begin{equation}
F_{e_1...e_k}(a, a_{L,R} + p\omega_1^d + q\omega_2^d) = \pm i \eta_a \sqrt{2} E \left[ \mu_{pq} \epsilon_{e_{L,R}} \prod_{m=1}^{k} \epsilon_{e_m} \right], \quad (pq) \neq (00).
\end{equation}

Proof. First, observe that $F_{e_1...e_k}(a, \cdot)$ is locally well-defined, since when $z$ moves around any face of $C(\hat{T})$ except one corresponding to $a_{i,j}$ or $a_{i,j}$, the $-1$ sign acquired by $\eta_z$ cancels the $-1$ sign coming from winding of $z^\circ$ around $z^\circ$. Moving $z$ to $z^\circ$ around $z^\circ$, hence, $F(a, a^-) = -\eta_a$. Moving $z$ around the torus' to $z_{pq} = z + p\omega_1^d + q\omega_2^d$ results in replacing $\gamma$ defining $\mu_{a^\star \mu}$ with $\gamma + \gamma_{pq}$; hence, doing so twice yields the same value. Finally, a simple loop lifting to a path connecting $a$ with $a_{pq}$ must start with $a^+$ and end at $a_{pq}$, hence the $-1$ sign in (5.7) when $(pq) \neq (00)$. The proof of s-holomorphicity in $\mathfrak{H}$ is completely local and thus extend verbatim to our case. To prove (5.8), we start with the value $F_{e_1...e_k}(a, a^+) = \eta_a E[\ldots]$ and move in two steps to $z = a_R$ so that $z^\circ - z^\circ$ rotates counterclockwise. This way, $z^\circ$ does not cross the newly created disorder line $[a^\star a_L^\star]$, and we get

$$\eta_a L_{\sigma^a \sigma_L^a} \mu_{a_L} \mu a^\star = -i \eta_a \sigma_{a^\circ} \sigma_{a_L^\circ} e^{-2\beta \sigma_{a^\circ} \sigma_{a_L^\circ}}$$

$$= -i \eta_a \left( \sigma_{a^\circ} \sigma_{a_L^\circ} \cosh(-2\beta) + \sinh(-2\beta) \right) = -\sqrt{2} \eta_a \epsilon_{a^\circ} \epsilon_{a_L^\circ}$$

inside the correlation, where we have used that $\cosh(-2\beta) = \sqrt{2}$ and $\sinh(-2\beta) = -1$. The computation for $a_R$ is identical, except we start $F_{e_1...e_k}(a, a^-) = -\eta_a E[\ldots]$ and move so that $z^\circ - z^\circ$ rotates clockwise. The proof of (5.9) is similar, taking into account the disorder line $\gamma_{pq}$ created when moving from $a$ to $a_{pq}$. \hfill \Box

It is convenient to anti-symmetrize $F_{e_1...e_k}$ by introducing

$$F_{e_1...e_k}^{(ij)}(a, z) := \frac{1}{4} \sum_{p,q \in \{0,1\}} (-1)^{i+p+j} F(a, z + p\omega_1^d + q\omega_2^d).$$

We summarize their properties in the following Lemma:

Lemma 17. The observables $F_{e_1...e_k}^{(ij)}(a, \cdot)$ are s-holomorphic functions on $C_{\{e_1...e_k\}}(\hat{T}_\delta)$ that satisfy the anti-periodicity condition

\begin{equation}
F_{e_1...e_k}^{(ij)}(a, z + p\omega_1^d + q\omega_2^d) = (-1)^{i+p+j} F_{e_1...e_k}^{(ij)}(a, z).
\end{equation}
Moreover, there exists a constant $c \in \mathbb{R}$ (depending on $\mathbb{T}^d$ and on $a$ but not on $z$) such that $F^{(00)}(a, z) = \text{Proj}_{\eta a}(\eta a c)$. In particular,

$$F^{(00)}(a, a_L) = F^{(00)}(a, a_R).$$

**Proof.** The anti-periodicity (5.10) is manifest from the construction, the s-holomorphicity follows from linearity, and (5.11–5.12) are obtained by simply summing (5.6–5.9). The fact that $F^{(00)}$ is essentially a constant is a discrete analog of the claim that a meromorphic function on a torus with at most one simple pole is constant. Indeed, consider the restriction of $F^{(00)}(a, \cdot)$ onto the index 2 sub-lattice $\mathcal{C}'(\mathbb{T}^d) := \{z \in \mathcal{C}(\mathbb{T}^d) : \eta_z \in e^{i \pi/4} \eta a \mathbb{R} \cup e^{-i \pi/4} \eta a \mathbb{R}\}$ of $\mathcal{C}(\mathbb{T}^d)$. As noticed above, this restriction is discrete holomorphic, that is, the identity (5.5) holds for every $z \in \mathcal{C}(\mathbb{T}^d)$ such that $\eta_z \in \eta a \mathbb{R} \cup i \eta a \mathbb{R}$, except, possibly, for $z = a$. Therefore, (5.5) also holds for $z = a$. Since the restriction of a discrete holomorphic function $F^{(00)}(a, \cdot)$ to each of the two sub-lattices $\{z \in \mathcal{C}(\mathbb{T}^d) : \eta_z = \eta a \mathbb{R}\}$ is discrete harmonic, the maximum principle implies that these restrictions are constant. The s-holomorphicity condition (5.4) now implies that the restrictions of $F^{(00)}(a, \cdot)$ to two other sub-lattices $\{z \in \mathcal{C}(\mathbb{T}^d) : \eta_z = i \eta a \mathbb{R}\}$ and $\{z \in \mathcal{C}(\mathbb{T}^d) : \eta_z = -i \eta a \mathbb{R}\}$ are also constant (and it’s easy to see that the s-holomorphicity implies that in that case, $F^{(00)}(a, z) = \text{Proj}_{\eta a}(C)$. Finally, since $F(a, a^+) = -F(a, a^-)$, it is a simple direct check that the only value consistent with (5.5) vanishing at $a$ is $F(a, a^+) = 0$, hence $F(a, z) \equiv 0$ on the sublattice $\{z \in \mathcal{C}(\mathbb{T}^d) : \eta_z = \pm \eta a\}$, implying the restriction $C \in \eta a \mathbb{R}$.

We now describe the discrete analog of the Cauchy kernel $\frac{1}{z-a}$, tailored for our purposes. Given $a \in \mathcal{C}(\mathbb{Z}^2)$ and a choice of the square root at $\eta a$ this is a unique s-holomorphic function $P_a(\cdot)$ on $\mathcal{C}(\mathbb{Z}^2)$ such that

$$P_a(a^+) = \eta a, \quad P_a(a^-) = -\eta a,$$

and, as $z \to \infty$, one has

$$P_a(z) = \frac{\sqrt{2}}{\pi} \text{Proj}_{\eta a} \left(\bar{\eta a}(z - a)^{-1}\right) + O(|z - a|^{-2}).$$

Moreover, we have

$$P_a(a_L) = P_a(a_R) = 0.$$

We also define the re-scaled version of this function, living on $\mathcal{C}(\delta \mathbb{Z}^2)$:

$$P_a^\delta(z) = \delta^{-1} P_{\delta^{-1}a}(\delta^{-1}z).$$
The function $P_a$ is a multiple of the discrete Cauchy kernel on the square lattice; see ([9, Lemma 4.9]) for the details. Since in ([9]), the lattice is scaled by $\sqrt{2}$ and rotated by $\frac{\pi}{4}$, one must re-define $P_a(z) := e^{\pi} P_{2\alpha i\frac{\pi}{4}} (\sqrt{2} e^{i\frac{\pi}{4}} z)$, which is reflected in the pre-factor in (5.16).

The following lemma elucidates the singularity structure of $F_{e_1...e_k}^{(ij)}(a, z_{pq})$.

**Lemma 18.** The function

$$\tilde{F}_a(\cdot) = F_{e_1...e_k}^{(ij)}(a, \cdot) - \delta \cdot E \left[ \mu_{ij}^{(k)} \prod_{i=1}^k e_{e_i} \right] P_a^{(j)}(\cdot)$$

extends to an s-holomorphic function on $\mathcal{C}(T^\delta)$ in a neighborhood of $a$. If $z_{2m-1}, z_{2m}$ are two corners incident to $e_m$ and symmetric with respect to its centre, then

$$\tilde{F}_m(\cdot) = F_{e_1...e_k}^{(ij)}(a, \cdot) - \frac{\delta}{\sqrt{2}} \cdot \tilde{\eta}_{z_{2m-1}} F_{e_1...e_m...e_k}^{(ij)}(a, z_{2m}) P_{2m-1}^{(j)}(\cdot) - \frac{\delta}{\sqrt{2}} \cdot \tilde{\eta}_{z_{2m}} F_{e_1...e_m...e_k}^{(ij)}(a, z_{2m-1}) P_{2m}^{(j)}(\cdot)$$

extends to an s-holomorphic function on $\mathcal{C}(T^\delta)$ in a neighborhood of $e_m$, where the choice of square root in $\eta_{z_{2m-1}}$ and $P_{2m-1}^{(j)}(\cdot)$ (resp. $\eta_{z_{2m}}$ and $P_{2m}^{(j)}(\cdot)$) are related by (5.15). Moreover, after that extension,

$$\tilde{F}(a) = \tilde{F}_m(z_{2m-1}) = \tilde{F}_m(z_{2m}) = 0.$$

**Proof.** It follows from (5.11) and (5.15) that $\tilde{F}_a(a) = 0$, and hence these two corners of $\mathcal{C}_{[a]}^{[e_1...e_k]}(T^\delta)$ can be glued back together, yielding the first claim.

For the second claim, we use that

$$\sigma_{z_{2m-1}} \sigma_{z_{2m}} \mu_{z_{2m-1}} \mu_{z_{2m}} = \sigma_{z_{2m-1}} \sigma_{z_{2m}} e^{-2\beta \sigma_{z_{2m-1}} \sigma_{z_{2m}}} = \sigma_{z_{2m-1}} \sigma_{z_{2m}} \cosh(-2\beta) + \sinh(-2\beta) = \sqrt{2} e_{z_m},$$

so that we can replace $e_m$ by $2^{-\frac{1}{2}} \sigma_{z_{2m-1}} \sigma_{z_{2m}} \mu_{z_{2m-1}} \mu_{z_{2m}}$ in the definition of $F_{e_1...e_k}(a, \cdot)$.

With this substitution, it becomes a spinor on $\mathcal{C}_{[a]}(T^\delta)$ ramified at $z_{2m-1}, z_{2m-1}, z_{2m}, z_{2m}$, and their shifts by $\omega^\delta_1, \omega^\delta_2, \omega^\delta_3, \omega^\delta_4$. We modify the graph by introducing branch cuts $[z_{2m-1}, z_{2m}]$ and $[\bar{z}_{2m}, \bar{z}_{2m}]$ (and their shifts) which splits the vertices $z_{2m-1}, z_{2m}$ into $z_{2m-1}^\pm$ and $z_{2m}^\pm$, so that $F_{e_1...e_k}(a, \cdot)$ becomes a function on the resulting graph obeying the property $F_{e_1...e_k}(a, z^+_p) = -F_{e_1...e_k}(a, z^-_p)$, $p = 2m - 1, 2m$. More concretely,

$$F_{e_1...e_k}(a, z^+_p) = \frac{1}{\sqrt{2}} \eta_{z_{2m-1}} \tilde{\eta}_{z_{2m}} F_{e_1...e_m...e_k}^{(ij)}(a, z_{2m});$$
where we have, using high-temperature expansion; since \( \mathbb{T}^d \) is self-dual, we can freely pass between primal and dual lattice. For \( x, y \in (\mathbb{T}^d)^* \), \( x \sim y \), recall the notation (3.2). The high-temperature expansion reads

\[
\mathbb{E}\mu_{pq} = \frac{\sum_{\sigma} \exp \left( \beta \sum_{x \sim y} (-1)^{\varphi_{pq}(xy)} \sigma_x \sigma_y \right)}{\sum_{\sigma} \exp \left( \beta \sum_{x \sim y} \sigma_x \sigma_y \right)}
= \frac{\sum_{\sigma} \prod_{x \sim y} \left( 1 + (-1)^{\varphi_{pq}(xy)} \sigma_x \sigma_y \sigma_{y^*} \alpha \right)}{\sum_{\sigma} \prod_{x \sim y} \left( 1 + \sigma_x \sigma_y \sigma_{y^*} \alpha \right)}
= \frac{\sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} (-1)^{\varphi_{pq}(\xi)} \xi^{|\xi|}}{\sum_{\xi \in \mathcal{E}(\mathbb{T}^d)} \xi^{|\xi|}}.
\]

Remark 19. For \( (ij) \neq (00) \), assuming \( \mathbb{E}\mu^{(ij)} \) are known, the formulae (5.18–5.19) identify the functions \( F^{(ij)}_{e_1 \cdots e_k} \) uniquely by recursion. They do not identify \( F^{(00)}_{e_1 \cdots e_k} \) uniquely, because of a possibility of adding a “constant” \( s \)-holomorphic function \( \text{Proj}_{\eta_{ij}}(c) \), \( c \in \mathbb{C} \). However, if \( k \geq 1 \), (5.20) removes this degree of freedom. We were unable to identify this constant and its asymptotics for \( k = 0 \) based on discrete holomorphicity considerations only, hence the input from Corollary 2 is needed to start the induction. For the energy difference of Theorem 3, the value of this constant does not matter as it cancels out, since \( \mathbb{E} \left[ \mu^{(00)} \epsilon_V \right] = \mathbb{E} \left[ \mu^{(00)} \epsilon_H \right] \) by (5.12–5.14).

We conclude this section by identifying \( \mathbb{E}\mu^{(ij)} \) with Kac-Ward determinants from Section 3 and deducing their asymptotics:

**Lemma 20.** We have

\[
\mathbb{E}(\mu^{(ij)}) = \frac{Z^{(ij)}}{Z^{(01)} + Z^{(10)} + Z^{(11)}},
\]

where \( Z^{(ij)} \) is defined in (3.4). Therefore, we have, as \( \delta \to 0 \),

\[
(5.21) \quad \mathbb{E}\mu^{(10)} \to \mathcal{E}^{(10)} := \frac{Z^{(10)}}{Z} = \frac{|\theta_1|}{|\theta_2| + |\theta_3| + |\theta_4|},
(5.22) \quad \mathbb{E}\mu^{(01)} \to \mathcal{E}^{(01)} := \frac{Z^{(01)}}{Z} = \frac{|\theta_2|}{|\theta_2| + |\theta_3| + |\theta_4|},
(5.23) \quad \mathbb{E}\mu^{(11)} \to \mathcal{E}^{(11)} := \frac{Z^{(11)}}{Z} = \frac{|\theta_3|}{|\theta_2| + |\theta_3| + |\theta_4|}.
\]
To compute $E\mu^{(ij)}$, we recall that $\varphi_{pq}(\xi) = p\varphi_{10}(\xi) + q\varphi_{01}(\xi) \mod 2$ and note that

$$\sum_{p,q} (-1)^{(1-i)p+(1-j)q+p\varphi_{10}(\xi)+q\varphi_{01}(\xi)} = (-1)^{\eta_{ij}(\xi)+(1-i)(1-j)} \sum_{p,q} (-1)^{(p+\varphi_{10}(\xi)+1-j)(q+\varphi_{01}(\xi)+1-i)}$$

$$= (-1)^{\eta_{ij}(\xi)+(1-i)(1-j)} \sum_{p,q} (-1)^{pq} = 2(-1)^{\eta_{ij}(\xi)}(-1)^{(1-i)(1-j)}.$$

Therefore, plugging the above formula for $E\mu_{pq}$ into (5.13) and taking into account (3.3) yields

$$E\mu^{(ij)} = \frac{1}{4}(-1)^{(1-i)(1-j)} \frac{2}{4} \sum_{\Omega \in \mathcal{T}_1} (-1)^{\eta_{ij}(\xi)} \alpha^{[\xi]} = \frac{1}{4} \frac{2Z^{(ij)}}{Z^{(00)} + Z^{(10)} + Z^{(11)}},$$

where we have used that $Z^{(00)} = 0$ and $(-1)^{(1-i)(1-j)} = 1$ unless $i = j = 0$. For the asymptotics, we use $Z^{(ij)} = \sqrt{\mathcal{K}} \mathcal{W}_{ij} = 2^{-[\mathcal{T}_1]/2} \alpha^{-[\mathcal{T}_1]} \sqrt{\det \Delta_{ij}}$, and then use asymptotics of these determinants computed in the course of the proof of Corollary 2.

6. Scaling limits of the fermionic observables

In what follow, we define the continuous limits of the observable $F^{(ij)}_{e_1...e_k}(a, z)$. These limits will depend on $\eta_{a}$, i.e., the orientation of the corner $a$ and the choice of the sign of the square root in (5.2). Since there are only 8 options for $\eta_{a}$, we will from now on assume it fixed.

It will be convenient to use physics notation for the Pfaffian: for symbols $\mathcal{O}_1, \ldots, \mathcal{O}_{2N}$, and a label $\circ$ to distinguish between different anti-periodicity “sectors”, if a $2N \times 2N$ antisymmetric matrix $M$ is given whose entries are denoted by $M_{n,m} = \langle \mathcal{O}_n \mathcal{O}_m \rangle^{\circ}$, we denote

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_{2n} \rangle^{\circ} := \text{Pf} M = \text{Pf} \langle \mathcal{O}_n \mathcal{O}_m \rangle^{\circ}_{1 \leq n,m \leq 2N}.$$ 

Recall that, given $\omega_{1,2}$, the Weierstrass $\zeta$-function $\zeta_{\omega_1,\omega_2}$ is the unique odd function that has a simple pole of residue 1 at the origin, and such that its derivative is doubly periodic (in fact, $-\zeta'(z) = \wp(z)$, where $\wp(z)$ is the Weierstrass function). Thus, $\zeta(z)$ is not an elliptic function, but has periodicity property $\zeta(z + \omega_{12}) = \zeta(z) + c_{12}$ for some constants $c_{12}$. However, a linear combination $\sum \alpha_i \zeta_{\omega_i} - \beta_{i}$ is an elliptic function provided that $\sum \alpha_i = 0$. We denote by $cs_{\omega_1,\omega_2}(z)$, (respectively, $ns_{\omega_1,\omega_2}(z)$, $ds_{\omega_1,\omega_2}(z)$) the unique meromorphic function on $\hat{T}$ with four simple poles, including one with residue 1 at the origin, and satisfying the anti-periodicity relations (6.7) below with $(ij) = (01)$ (respectively, $(ij) = (10)$, $(ij) = (11)$). We have $cs_{\omega_1,\omega_2}(z) = \frac{2K}{\omega_1} \cdot cs \left( \frac{2K}{\omega_1} (z-a), k \right)$ in the notation of [19, Section 22], and similarly for $ns, ds$, where the elliptic modulus $k$ and the complete elliptic integral $K$ are given by

$$k := \left( \frac{\theta_2(\tau)}{\theta_3(\tau)} \right)^2, \quad K = \frac{\pi}{2} \frac{\theta_3^2(\tau)}{\theta_3(\tau)}.$$

**Definition 21.** Given a continuous torus $\mathbb{T}$ and distinct points $a, e_1, \ldots, e_k \in \mathbb{T}$, for $(ij) \neq (00)$, we define

$$f^{(ij)}_{e_1...e_k}(a, z) = \begin{cases}
\frac{2^{(ij)}}{Z} \eta_{a}^{ik} (\psi_{e_1} \psi_{e_2}^* \ldots \psi_{e_k}^* \psi_{e_k} \psi_{e_1}^* \psi_{e_1})^{(ij)} & \text{k even}, \\
\frac{2^{(ij)}}{Z} \eta_{a}^{ik} (\psi_{e_1} \psi_{e_2}^* \ldots \psi_{e_k}^* \psi_{e_k} \psi_{e_1}^* \psi_{e_1})^{(ij)} & \text{k odd},
\end{cases}$$

where $f^{(ij)}_{e_1...e_k}(a, z) = \frac{2^{(ij)}}{Z} \eta_{a}^{ik} (\psi_{e_1} \psi_{e_2}^* \ldots \psi_{e_k}^* \psi_{e_k} \psi_{e_1}^* \psi_{e_1})^{(ij)}$
where \( \langle \psi_w \psi_w^* \rangle^{(ij)} \equiv 0, \langle \psi_w \psi_w^* \rangle^{(ij)} = \langle \psi_w \psi_w^* \rangle^{(ij)}, \) and

\[
\begin{align*}
(6.2) \quad & \langle \psi_w \psi_w^* \rangle^{(01)} = cs_{\omega_1, \omega_2}(w - \hat{w}), \\
(6.3) \quad & \langle \psi_w \psi_w^* \rangle^{(10)} = ns_{\omega_1, \omega_2}(w - \hat{w}), \\
(6.4) \quad & \langle \psi_w \psi_w^* \rangle^{(11)} = ds_{\omega_1, \omega_2}(w - \hat{w}).
\end{align*}
\]

**Definition 22.** Given data as above, we also define

\[
f_{e_1, ..., e_k}^{(00)}(a, z) = \begin{cases} \bar{n}_i \bar{k} \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_z \psi_a \rangle^{(00)}, & k \text{ odd}, \\ \bar{n}_a \bar{k} \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_z \psi_a \rangle^{(00)}, & k \text{ even}, \end{cases}
\]

where \( \langle \psi_{e_n} \psi_{e_m} \rangle^{(00)} = \zeta(e_n - e_m), \)

\[
\langle \psi_{e_n} \psi_{e_m} \rangle^{(00)} = -\pi i \left( \Im \tau \right)^{1/2} \frac{1}{|\Theta_2|_1},
\]

and \( \langle \psi_{e_n} \psi_{e_m} \rangle^{(00)} = \zeta(e_n - e_m); \) here \( \zeta \) denotes the Weierstrass \( \zeta \)-function.

**Definition 23.** Given data as above, we define for \( (ij) \neq (00), \)

\[
\mathcal{E}_{e_1, ..., e_k}^{(ij)} := \frac{2^{(ij)}}{2} \bar{i}^k \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \rangle^{(ij)} \quad k \text{ even,}
\begin{align*}
\text{even}, & \quad k \text{ odd,}
\end{align*}
\]

and

\[
\mathcal{E}_{e_1, ..., e_k}^{(00)} = \begin{cases} \bar{i}^k \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \rangle^{(00)}, & k \text{ odd,} \\ 0 & k \text{ even.} \end{cases}
\]

The above definitions extend to \( k = 0, \) setting the empty Pfaffian to 1.

**Proposition 24.** The quantity \( f_{e_1, ..., e_k}^{(ij)}(a, \cdot) \) is a meromorphic function on \( \hat{T} \) satisfying, for \( p, q \in \{0, 1\}, \)

\[
f_{e_1, ..., e_k}^{(ij)}(a, z + p\omega_1 + q\omega_2) = (-1)^{ip+jq} f_{e_1, ..., e_k}^{(ij)}(a, z)
\]

Its poles are simple and located at \( e_1, \ldots, e_k, a, \) and

\[
(6.8) \quad f_{e_1, ..., e_k}^{(ij)}(a, \cdot) = \bar{f}_{e_1, ..., e_k, e_m}^{(ij)}(a, e_m) + o(1), \quad z \to e_m,
\]

\[
(6.9) \quad f_{e_1, ..., e_k}^{(ij)}(a, z) = \bar{f}_{e_1, ..., e_k, a}^{(ij)}(z - a) - \bar{n}_a \mathcal{E}_{e_1, ..., e_k}^{(ij)} + o(1), \quad z \to a.
\]

**Remark 25.** The equations \( (6.7), (6.8), (6.9) \) give an overdetermined set of conditions that identify \( f_{e_1, ..., e_k}^{(ij)}(a, z) \) and \( \mathcal{E}_{e_1, ..., e_k}^{(ij)} \) uniquely by induction, given \( \mathcal{E}_{e_1, ..., e_k}^{(ij)} \) for \( (ij) \neq (00) \) and a constant \( \mathcal{E}_{e}^{(00)} \). Indeed, to see that \( f_{e_1, ..., e_k}^{(ij)}(a, \cdot) \) is uniquely determined, note that if two functions \( f, g \) both satisfy \( (6.7), (6.8), (6.9) \), then, by induction hypothesis, their difference is holomorphic everywhere on \( \hat{T} \) and vanishes, say, at \( e_1 \), hence it is zero. In its turn, \( f_{e_1, ..., e_k}^{(ij)} \) uniquely determines \( \mathcal{E}_{e_1, ..., e_k, e_m}^{(ij)} \) by \( (6.8) \). For \( (ij) \neq (00) \), the vanishing of constant terms in \( \mathcal{E}_{e_1, ..., e_k, e_m}^{(ij)} \) is not needed to prove uniqueness; for \( (ij) = (00) \), we need it just for one \( m \). We note that \( f^{(00)}(a, z) = -\bar{n}_a \mathcal{E}_{e}^{(00)} \) (it is an elliptic function with at most one simple pole, hence a constant), and, for \( (i, j) \neq (0, 0), \)

\[
f^{(ij)} = \bar{n}_a \mathcal{E}_{e}^{(ij)}(\psi_z \psi_a)^{(ij)},
\]

where the latter is given by \( (6.2), (6.4) \).
Proof. In all cases, \( \langle \psi_z \psi_w \rangle^\circ = (z - w)^{-1} + o(1) \), since it is an odd function of \( z - w \) with a simple pole of residue 1 at the origin; also, complex conjugating the Pfaffian amounts to replacing \( \psi \leftrightarrow \psi^* \). Expanding the Pfaffian and using the convention \( \langle \psi_{e_1} \psi_{e_2} \rangle^\circ = 0 \), we obtain as \( z \to e_1 \)

\[
\langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_z \psi_a \rangle^\circ = \langle \psi_z \psi_{e_1} \rangle^\circ \langle \psi_{e_1} \cdots \psi_{e_k}^* \psi_{e_k} \psi_a \rangle^\circ + \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k-1} \psi_{e_k-1}^* \psi_z \psi_a \rangle^\circ + o(1),
\]

\[
(z - e_1)^{-1} \langle \psi_{e_1} \cdots \psi_{e_k} \psi_{e_k}^* \psi_a \rangle^\circ + o(1) = (z - e_1)^{-1} (-1)^{k-1} \langle \psi_{e_1} \cdots \psi_{e_k} \psi_{e_k}^* \psi_a \rangle^\circ + o(1),
\]

and similarly for \( m = 2, \ldots, k \) and with \( \psi_a \) replaced with \( \psi_a^* \). This proves (6.8).

Similarly, as \( z \to a \), we have

\[
\langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_z \psi_a \rangle^\circ = \langle \psi_z \psi_a \rangle^\circ \langle \psi_{e_1}^* \cdots \psi_{e_k}^* \psi_z \psi_a \rangle^\circ + o(1),
\]

\[
\langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_z \psi_a \rangle^\circ = \langle \psi_{e_1} \psi_{e_1}^* \cdots \psi_{e_k} \psi_{e_k}^* \psi_a \rangle^\circ + o(1).
\]

The first identity proves (6.9) for \((ij) \neq (00)\), \( k \) even and for \((ij) = 0, k \) odd, while the second identity proves it for \((ij) \neq 00, k \) odd and for \((ij) = (00)\), \( k \) even.

To prove (6.7) for \((ij) \neq (00)\), simply expand the Pfaffian in \( \psi_z \) and note that \( \langle \psi_z \psi_w \rangle^{(ij)} \) satisfies (6.7) for each \( w = e_1, \ldots, e_k, a \). For \((ij) = (00)\), the Weierstrass \( \zeta \)-function does not satisfy (6.7); as discussed above, we need to check that the sum of the residues of \( f_{e_1 \cdots e_k}^{(00)}(a, \cdot) \) is zero. We claim that this result follows by induction from the existence of any solution to (6.7, 6.9) with \(-iE^{(00)}_e\) given by (6.6). Indeed, suppose \( \tilde{f}_{e_1 \cdots e_k}^{(00)}(a, z) \) is such a solution, and suppose by induction hypothesis that \( f_{e_1 \cdots e_k-1}^{(00)}(a, z) = \tilde{f}_{e_1 \cdots e_k-1}^{(00)}(a, z) \) for any \( e_1, \ldots, e_{k-1}, a \). Then, by (6.8, 6.9), the residues of \( f_{e_1 \cdots e_k}^{(00)}(a, z) \) and \( \tilde{f}_{e_1 \cdots e_k}^{(00)}(a, z) \) are the same; since the latter function satisfies (6.7), their sum vanishes; hence also \( f_{e_1 \cdots e_k}^{(00)}(a, z) \) satisfies (6.7), and we have \( f_{e_1 \cdots e_k}^{(00)} \equiv \tilde{f}_{e_1 \cdots e_k}^{(00)} \), completing the induction step. The solution to (6.7, 6.9) exists since it is constructed as the limit of discrete observables in the proof of Theorem 26 below.

At the heart of our analysis is the following convergence result:

**Theorem 26.** One has, as \( \delta \to 0 \),

\[
\delta^{-(k+1)} E_{e_1 \cdots e_k}^{(ij)}(a, z) = C_k \cdot \text{Proj}_{\eta_k} \left( f_{e_1 \cdots e_k}^{(ij)}(a, z) \right) + o(1),
\]

uniformly in \( z, e_1, \ldots, e_k, a \) away from each other, where \( C_k = \frac{\sqrt{\pi}}{\sqrt{2k+1}} \). Moreover,

\[
\delta^{-k} E \left[ \mu^{(ij)} \prod_{m=1}^k \epsilon_{e_m} \right] \to \pi^{-k} E_{e_1 \cdots e_k}^{(ij)}.
\]

**Proof.** We prove this result by induction: \( (6.11)_k \implies (6.10)_k \implies (6.11)_{k+1} \), first separately for each \((ij) \neq (00)\) and then for \((ij) = (00)\). For the base of induction, for \((ij) \neq 0\), we have (5.21, 5.23), and for \((ij) = 0\), the induction starts with \( k = 1 \). Namely, we have

\[
\delta^{-1} E \left[ \mu^{(00)} \epsilon_e \right] = \delta^{-1} E \epsilon_e - \sum_{(ij) \neq (00)} \delta^{-1} E \left[ \mu^{(ij)} \epsilon_e \right] \to \pi^{-1} E^{(00)}_e - \pi^{-1} \sum_{(ij) \neq (00)} E^{(ij)}_e,
\]

where the convergence of the first term is by Corollary 4 and the definition of \( E^{(00)}_e \), and the convergence of other terms will be by that point have been proven. It remains to notice that \( E^{(ij)}_e \equiv 0 \) for \((ij) \neq 0\). Note that the proof of Corollary 4 relies
on Theorem 3 whose proof below does use (6.10), but only in the case \((ij) \neq (00)\). Thus, the base of induction is established.

To prove (6.10), we follow a general scheme used in [9], where we first identify the scaling limit assuming precompactness, and then justify precompactness. Let us first assume that for all \(r > 0\), there is a constant \(C_r\) such that the functions \(\delta^{-(k+1)} F_{\epsilon_1...\epsilon_k}^{(ij)} (a, \cdot)\) are bounded by \(C_r\) on the set \(\mathbb{T}_r := \mathbb{T} \setminus B_r(a) \cup B_r(e_1) \cup \cdots \cup B_r(e_k)\), uniformly in \(\delta\). We claim that in this case,

\[
\delta^{-(k+1)} F_{\epsilon_1...\epsilon_k}^{(ij)} (a, \cdot) = C_k \cdot \text{Proj}_{\eta_z} \left( F_{\epsilon_1...\epsilon_k}^{(ij)} (a, z) \right) + o(1)
\]

uniformly on compact subsets of \(\mathbb{T} \setminus \{a_{pq}\}\). Indeed, as noted above, the \(s\)-holomorphicity of \(F_{\epsilon_1...\epsilon_k}^{(ij)} (a, z)\), together with uniform boundedness, implies that (6.12) holds along a subsequence, and with some holomorphic function \(f : \mathbb{T} \setminus \{a_{pq}\} \to \mathbb{C}\) instead of \(F_{\epsilon_1...\epsilon_k}^{(ij)} (a, z)\). Therefore, it is enough to show that any sub-sequential limit \(f\) must be equal to \(F_{\epsilon_1...\epsilon_k}^{(ij)} (a, z)\). Clearly, \(f\) must satisfy the (anti)-periodicity condition \(f(z + p\omega_1 + q\omega_2) = (-1)^{qp+jq} f(z)\), because of (5.10).

Denote \(U = F_{\epsilon_1...\epsilon_m...\epsilon_k}^{(ij)} (a, \epsilon_m)\). We have by induction hypothesis

\[
\begin{align*}
\delta^{-k} F_{\epsilon_1...\epsilon_m...\epsilon_k}^{(ij)} (a, \epsilon_{2m}) &= C_{k-1} \text{Proj}_{\eta_z} (U) + o(1), \\
\delta^{-k} F_{\epsilon_1...\epsilon_m...\epsilon_k}^{(ij)} (a, \epsilon_{2m-1}) &= C_{k-1} \text{Proj}_{\eta_z} (U) + o(1),
\end{align*}
\]

so that, taking into account (5.16) and the identity \(\text{Proj}_{\eta_z} (U) = \frac{1}{\pi} (U + \eta_z U)\), in a fixed annulus around \(B_R(e_m) \setminus B_r(e_m)\), we have, for the “corrective” term in the definition of \(F_k(\cdot)\):

\[
\begin{align*}
&i\eta_{z_{2m-1}} \delta^{-k-1} \frac{\delta}{\sqrt{2}} F_{\epsilon_1...\epsilon_m...\epsilon_k}^{(ij)} (a, \epsilon_{2m}) P_{\eta_z^{2m-1}}^\delta (z) + i\eta_{z_{2m}} \delta^{-k-1} \frac{\delta}{\sqrt{2}} F_{\epsilon_1...\epsilon_m...\epsilon_k}^{(ij)} (a, \epsilon_{2m-1}) P_{\eta_z^{2m}}^\delta (z) \\
&= C_{k-1} \cdot \frac{\delta}{\sqrt{2}} \text{Proj}_{\eta_z} (U) \text{Proj}_{\eta_z} \left( \eta_{z_{2m-1}} \right) \\
&+ C_{k-1} \cdot \frac{\delta}{\sqrt{2}} \text{Proj}_{\eta_z} (U) \text{Proj}_{\eta_z} \left( \frac{\eta_{z_{2m}}}{z - \epsilon_m} \right) + o(1)
\end{align*}
\]

Recall \(\delta^{-(k+1)} \hat{F}_m(\cdot)\) (5.19) extends to an \(s\)-holomorphic function near \(e_m\). We can express its values by discrete Cauchy integral formula with contour in \(B_R(e_m) \setminus B_r(e_m)\) and pass to the limit in that formula; this shows that \(f(z) - C_k \cdot \frac{\delta}{\epsilon_{z-e_m}}\) extends to a holomorphic function at \(B_{2r}(e_m)\); moreover, (5.20) shows that this function vanishes at \(e_m\). Hence, \(f\) must satisfy (6.8).

The analysis near \(a\) is similar. Since \(\delta^{-k} \cdot \mathbb{E} \left[ \mu^{(ij)} \prod_{i=1}^k \epsilon_{e_i} \right] = \mathcal{E}_{\epsilon_1...\epsilon_k}^{(ij)} + o(1)\) by induction hypothesis, we have the expansion

\[
\delta^{-k} \mathbb{E} \left[ \mu^{(ij)} \prod_{i=1}^k \epsilon_{e_i} \right] P_{a}^\delta (\cdot) = \frac{\sqrt{2}}{\pi^{k+1}} \mathcal{E}_{\epsilon_1...\epsilon_k}^{(ij)} \cdot \text{Proj}_{\eta_z} \left( \frac{\eta_{a}}{z - a} \right) + o(1)
\]

uniformly in \(B_R(e_m) \setminus B_r(e_m)\). Therefore, by the same argument as above, \(f(z) - \eta_z \frac{\mathcal{E}_{\epsilon_1...\epsilon_k}^{(ij)}}{z - a}\) can be analytically continued in \(B_R(a)\), and thus satisfies (6.9). In other
words, \( f(\cdot) \) satisfies the defining conditions of \( f^{(ij)}_{e_1 \ldots e_k}(a, \cdot) \), and therefore, due to Remark 25 we have \( f(\cdot) \equiv f^{(ij)}_{e_1 \ldots e_k}(a, \cdot) \).

We now turn to justifying the uniform (in \( \delta \)) boundedness away form \( a, e_1, \ldots, e_k \). Let us assume towards a contradiction that there exist a (small fixed) \( R > 0 \) such that
\[
C^\delta_R := \max_{z, |z|^2 < R} \left| \delta^{-1-k} F^{(ij)}_{e_1 \ldots e_k}(a, \cdot) \right|
\]
tends to infinity (at least along some sequence of \( \delta \)). We claim that in that case, the functions \( \left| (C^\delta_R)^{-1} \delta^{-1-k} F^{(ij)}_{e_1 \ldots e_k}(a, \cdot) \right| \) are uniformly bounded on any \( \hat{T}^\delta_r \) with \( r < R \). Indeed,
\[
(C^\delta_R)^{-1} \left( \delta^{-1-k} F^{(ij)}_{e_1 \ldots e_k}(a, \cdot) - \delta^{-k-1} E \left[ \mu^{(ij)} \prod_{i=1}^k \epsilon_{e_i} \right] P^\delta_{a}(\cdot) \right)
\]
is uniformly bounded on \( B(a, 2R) \setminus B(a, R) \) and discrete holomorphic in \( B(a, 2R) \), and hence, by maximum principle, it is uniformly bounded in the whole of \( B(a, 2R) \). As \( P^\delta_{a}(z) \) is also uniformly bounded on compact subsets of \( B(a, 2R) \setminus \{a\} \) and \( (C^\delta_r)^{-1} \to 0 \), we get the claim. Similarly, we justify the uniform boundedness on each \( B_r(e_m) \setminus B_r(e_m) \) for \( m = 1, \ldots, k \).

Therefore, the above convergence argument can be applied verbatim to
\[
(C^\delta_R)^{-1} \delta^{-1-k} F^{(ij)}_{e_1 \ldots e_k}(a, \cdot)
\]
with the conclusion that it converges uniformly on compact subsets of \( \hat{T} \) to a function that is analytic in the whole \( \hat{T} \) and vanishing, say, at \( e_1 \), that is, to 0. This contradicts the choice of \( C^\delta_R \).

To derive (6.11), note that we have shown above that \( \tilde{F}_a(\cdot) \) extends to an s-holomorphic function in a neighborhood of \( a \), and that
\[
\delta^{-(k+1)} \tilde{F}_a(z) = C_k \cdot \text{Proj}_{\eta_a} \left( f^{(ij)}_{e_1 \ldots e_k}(a, z) - E^{(ij)}_{e_1 \ldots e_k} \frac{\eta_a}{z-a} \right) + o(1),
\]
uniformly in a neighborhood of \( a \), where the function inside the projection is analytically continued. It remains to take into account (5.8), (5.9) and (5.17) to note that
\[
\tilde{F}_a(a_{L,R}) = -\text{ir}_{\eta_a} \cdot \sqrt{2} \cdot \mathbb{E} \left[ \sum_{m=1}^k \epsilon_{e_{L,R}} \right].
\]

\( \Box \)

Remark 27. The planar domain counterpart of this theorem is a particular case of (9, Theorem 2), asserting that
\[
\delta^{-k-1} \mathbb{E} \left[ \epsilon_{e_1} \ldots \epsilon_{e_k} \psi^\eta_{a} \psi_z \right] = C_k^2 \eta_{a} \text{Proj}_{\eta_a} \left( \langle \epsilon_{e_1} \ldots \epsilon_{e_k} \psi^\eta_{a} \psi_z \rangle \right) + o(1).
\]
The expansions of the right-hand side at \( z = e_m \) and \( z = a \) can be read off the fusion rules (9, (6.2–6.4)). In the case \( (ij) \neq (00) \), the same convergence proof as in (9, by expressing \( F^{(ij)}_{e_1 \ldots e_k}(a, z) \) as a Pfaffian of two-point correlations, and then passing to the limit term-by-term, could have been carried on. We were unable to do it for the (00) sector, due to the lack of combinatorial analog of \( \langle \psi_z \psi_a \rangle^{(00)} \). By the formalism of (9, Section 5.2), in the RHS of (6.13) we can expand \( \psi^\eta_{a} = \bar{\eta}_{a} \psi_a + \eta_a \psi^*_{a} \); in the torus case, only one of these terms will contribute for each sector depending on parity of \( k \), which is reflected in (6.1, 6.5).
7. Proofs of Theorem 3 and Theorem 5

In order to prove Theorem 3 we need the one more Lemma. Denote
\[ g^{(ij)}(a, z) := f^{(ij)}(a, z) - \frac{\mathcal{E}^{(ij)}(\eta_a)}{z - a}; \]
\[ G^{(ij)}(a, z) := \delta^{-1} F^{(ij)}(a, z) - \mathbb{E} \left[ \mu^{(ij)} \right] P^a(z). \]

It is a standard fact that if discrete holomorphic functions converge uniformly in a ball \( B(a, r) \), then so do their discretized derivatives. This is what it means for our case:

**Lemma 28.** One has, as \( \delta \to 0 \), for \( (ij) \neq (00) \),
\[ \delta^{-2} \left( F^{(ij)}(a, a_L) - F^{(ij)}(a, a_R) \right) \to \frac{2}{\pi} \mathfrak{H}_a \Re \left[ i \eta_a^3 \partial_z g^{(ij)}(a, z) \right] \bigg|_{z = a}. \]

**Proof.** First of all, note that by (5.17), we have
\[ \delta^{-2} \left( F^{(ij)}(a, a_L) - F^{(ij)}(a, a_R) \right) = \delta^{-1} \left( G^{(ij)}(a, a_L) - G^{(ij)}(a, a_R) \right). \]

Now, by (5.18), \( G^{(ij)}(a, \cdot) \) is discrete holomorphic in a (fixed) neighborhood of \( a \), its restriction to the sub-lattice \( \{ u \in \mathcal{C} : \eta_u = \eta_{a_L} = \eta_{a_R} = \eta_a \} \) is a discrete harmonic function, as it was shown in the proof of Theorem 26 that converges to
\[ \sqrt{\frac{2}{\pi}} \text{Proj}_a \left( g^{(ij)}(z) \right). \]

It is well known that this implies convergence of its (normalized) finite difference to the corresponding derivative. Denote \( a_L - a_R := a_{LR} = -\sqrt{2}\delta \eta_a^2 \), we thus have
\[ G^{(ij)}(a, a_L) - G^{(ij)}(a, a_R) \]
\[ = \frac{\sqrt{2}}{\pi} \left( a_{LR} \partial_z \text{Proj}_a \left( g^{(ij)}(a, z) \right) \bigg|_{z = a} + a_{LR} \partial_z \text{Proj}_a \left( g^{(ij)}(a, z) \right) \bigg|_{z = a} \right) + o(\delta). \]

Since \( \text{Proj}_a(x) = \frac{1}{2} (x + \eta_a^2 \bar{x}) \), \( a_{LR} = -\sqrt{2}\delta \eta_a^2 \), and \( g^{(ij)}(a, \cdot) \) is holomorphic, using the notation \( X = \partial_z g^{(ij)}(a, z) \bigg|_{z = a} \), this can be further simplified as
\[ a_{LR} \partial_z \text{Proj}_a \left( g^{(ij)}(a, z) \bigg|_{z = a} \right) + a_{LR} \partial_z \text{Proj}_a \left( g^{(ij)}(a, z) \bigg|_{z = a} \right) \]
\[ = -\frac{\sqrt{2}}{2} \delta \left( \eta_a^2 X - \eta_a^4 \bar{X} \right) = \sqrt{2} \delta \mathfrak{H}_a \frac{1}{2i} \left( \eta_a^3 X - \eta_a^5 \bar{X} \right) \]
\[ = \sqrt{2} \delta \mathfrak{H}_a \Re \left[ i \eta_a^3 X \right]. \]

\( \square \)

We are in the position to prove Theorem 3:

**Proof of Theorem 3.** Note that the correlations \( \mathbb{E} \left[ \mu^{(ij)}(\epsilon_V - \epsilon_H) \right] \) can be expressed, using (5.12), as
\[ \mathbb{E} \left[ \mu^{(ij)}(\epsilon_{el} - \epsilon_{er}) \right] = -\frac{1}{i\sqrt{2}\eta_a} \left( F^{(ij)}(a, a_L) - F^{(ij)}(a, a_R) \right). \]

In fact, this is zero for \( (ij) = (00) \) by (5.14). Choose \( a \) so that \( \frac{a^* - a_0}{|a|^2 - \eta_a^2} = e^{i\pi} \), i. e., \( \eta_a = e^{i\pi} \); then \( \epsilon_{el} = \epsilon_V, \epsilon_{er} = \epsilon_H \). Since \( \sum_{i,j \in \{0,1\}} (-1)^{(1-i)p + (1-j)q} 4_{p+q=0} = 4 \mathbb{1}_{p=q=0} \) we
have
\[
(7.2) \quad \sum_{(ij)} \mu^{(ij)} = \mu_{00} = 1.
\]
Therefore, summing (7.1) and applying Lemma 28 yields
\[
(7.3) \quad \mathbb{E}(\epsilon_H) - \mathbb{E}(\epsilon_V) = \delta^2 \sum_{(ij) \neq (00)} \frac{\delta^{-2}}{i\sqrt{2}\eta_a} \left( F^{(ij)}(a, a_L) - F^{(ij)}(a, a_R) \right).
\]
\[
= \delta^2 \cdot \frac{\sqrt{2}}{\pi} \sum_{(i,j) \neq (0,0)} \left( \Re \left[ i\eta_a \partial_z g^{(ij)}(a, z) \right] \right) + o(1).
\]
It remains to compute explicitly the derivatives. Jacobian elliptic functions have series expansions
\[
\begin{align*}
\text{cs} (u, k) &= \frac{1}{u} + \frac{1}{3} k^2 u + \mathcal{O} \left( u^3 \right), \\
\text{ns} (u, k) &= \frac{1}{u} + \frac{1}{6} (1 + k^2) u + \mathcal{O} \left( u^3 \right), \\
\text{ds} (u, k) &= \frac{1}{u} + \frac{1}{6} - \frac{1}{3} k^2 u + \mathcal{O} \left( u^3 \right)
\end{align*}
\]
and plugging them into the formula for \( f^{(ij)}(a, ) \) yields
\[
\partial_z g^{(ij)}(a, z) = \eta_a E^{(ij)} \frac{4K^2}{\omega^2} \left( \frac{k^2 - \frac{1}{3}}{\frac{1}{6} (1 + k^2)} \right) = \eta_a E^{(ij)} \frac{\pi^2}{24\omega^2} \left( \begin{array}{c}
\theta_1^4 - 2\theta_2^4, \\
\theta_2^4 + \theta_3^4, \\
\theta_3^4 - 2\theta_2^2
\end{array} \right), \quad (ij) = (01), (10), (11).
\]
Plugging into (7.3) concludes the proof.

Proof of Theorem 3. By (7.2) and (6.11), we have
\[
\delta^{-k} \mathbb{E} \left[ \prod_{m=1}^{k} \epsilon_m \right] = \sum_{(ij)} \delta^{-k} \mathbb{E} \left[ \mu^{(ij)} \prod_{m=1}^{k} \epsilon_m \right] \rightarrow \pi^{-k} \sum_{(ij)} \mathcal{E}^{(ij)}.
\]
By Definition 23, only \( \mathcal{E}^{(00)}_{e_1 \ldots e_k} \) contributes for odd \( k \); chasing the definitions leads directly to (2.5). For even \( k \), only \( \mathcal{E}^{(01)}_{e_1 \ldots e_k}, \mathcal{E}^{(10)}_{e_1 \ldots e_k}, \mathcal{E}^{(11)}_{e_1 \ldots e_k} \) contribute, and we have
\[
\mathcal{E}^{(ij)}_{e_1 \ldots e_k} = \frac{Z^{(ij)}}{Z} \left( -1 \right)^{\frac{k}{2}} \langle \psi_{e_1} \psi_{e_1}^* \ldots \psi_{e_k} \psi_{e_k}^* \rangle^{(ij)}
\]
\[
= \frac{Z^{(ij)}}{Z} \left( -1 \right)^{\frac{k}{2}} \left( -1 \right)^{\frac{k(k-1)}{2}} \langle \psi_{e_1} \ldots \psi_{e_k} \psi_{e_1} \ldots \psi_{e_k}^* \rangle^{(ij)}
\]
\[
= \frac{Z^{(ij)}}{Z} \langle \psi_{e_1} \ldots \psi_{e_k} \rangle^{(ij)} \langle \psi_{e_1}^* \ldots \psi_{e_k}^* \rangle^{(ij)} = \frac{Z^{(ij)}}{Z} \left| \langle \psi_{e_1} \ldots \psi_{e_k} \rangle^{(ij)} \right|^2.
\]

8. Theorem 11 For the Triangular Lattice

In this section, we prove an analog of Theorem 1 for triangular lattice \( \mathbb{H} := \{ m + e^{2\pi i n} : m, n \in \mathbb{Z} \} \); a similar argument (or duality) can be applied to treat the hexagonal lattice. We follow the argument for the case of the square lattice. The energy observable on the triangular lattice is defined by
\[
\epsilon_{xy} = \sigma_x \sigma_y - \bar{c},
\]
where $\bar{\varepsilon} = \frac{2}{3}$. The critical value of $\alpha$ is $\alpha_{\text{tri}} = 2 - \sqrt{3}$. There are three types of edges on the lattice, of which we choose representatives $e_0, e_1, e_2$, where $e_k$ is aligned with $e^{\frac{2\pi}{3}}k$. Our result reads as follows:

**Theorem 29.** For the critical Ising model on $T^\delta = \delta \mathbb{H}/\Lambda^\delta$, we have

$$E_{e_0} + E_{e_1} + E_{e_2} = 6\sqrt{2} \frac{\sqrt{\det^* \Delta^0_0}}{\sqrt{\det \Delta^1_0} + \sqrt{\det \Delta^0_1} + \sqrt{\det \Delta^1_1}} \frac{1}{|T^\delta|},$$

where $\det^*$ denotes the product of all non-zero eigenvalues.

**Remark 30.** Since the asymptotics of the determinant of the Laplacian has been worked out for arbitrary doubly-periodic lattices [34], this does lead immediately to the analog of Corollary 2, obtained by other methods in [50]. Other part of the paper extend to $\mathbb{H}$ as well: the discrete holomorphicity techniques of Section 5 are known to extend to a larger class of isoradial lattices with critical weights, see [12, 11].

As in the square lattice case, we start with the high-temperature expansion identity

$$(8.1) \quad E_{e} = \frac{1}{Z} \sum_{\xi \in \mathcal{E}(T^\delta)} b(e, \xi) \alpha^{|\xi|},$$

where $b(e, \xi) = (\alpha^{-1} - \varepsilon)I_{e \in \xi} + (\alpha - \varepsilon)I_{e \notin \xi}$.

Denote the triangular lattice by $\mathbb{H}$, and define $v^{\text{tri}}(\alpha, q) := (1 + 3\alpha^2 + 8\alpha^3 + 3\alpha^4 + \alpha^6)$

$$+ (3\alpha^3 - \alpha - \alpha^5) \left( z + w + \frac{1}{z} + \frac{1}{w} + \frac{z}{w} + \frac{w}{z} \right),$$

where, as in section 3, $z(q) = \exp(2\pi i \Re q)$ and $w(q) = \exp(2\pi i \Im q)$. The Kac–Ward determinants in this case read, as before

$$\det \mathcal{K}W^{ij} = \prod_{q \in \Lambda^* / \mathbb{H} + s_{ij}} v^{\text{tri}}(\alpha, q).$$

As in the case of a square lattice, the determinant of the Laplacian is related to the critical Kac–Ward determinant: for the Laplacian $\Delta^{ij}$ on the triangular lattice, the eigenvalues are

$$c_{\text{tri}} v^{\text{tri}}(\alpha_{\text{tri}}, q), \quad q \in \Lambda^*/\mathbb{H} + s_{ij},$$

with $c_{\text{tri}} = \frac{1}{12(-26 + 15\sqrt{3})}$.

Denoting

$$B^{(ij)}(e) := \sum_{\xi \in \mathcal{E}(T^\delta)} b(e, \xi) \alpha^{|\xi|} (-1)^{a_{ij}(\xi)},$$

we have the following analog of Lemma [11]

**Lemma 31.** One has, for $\alpha = \alpha_{\text{tri}}$,

$$B^{(ij)}(e_0) + B^{(ij)}(e_1) + B^{(ij)}(e_2) = \begin{cases} 12(11 - 5\sqrt{3})c_{\text{tri}}^2 \frac{1 - |T^\delta|}{|T^\delta|} \sqrt{\det^* \Delta^0_0} \frac{1}{|T^\delta|}, & i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$
Proof. As in the square lattice case, we have

\[
\frac{d}{da} Z_{ij}^{(ij)} = \frac{\lvert T^d \rvert}{\alpha} \sum_{\xi \in \mathcal{E}(T^d)} (1 + \frac{1}{\sqrt{3}}) d_{\alpha}^{\xi}(1)\eta_j(\xi)
\]

The following identities are straightforward to check:

\[
\frac{d}{da} v_{\text{tri}}(\alpha, q) \bigg|_{\alpha = \alpha_{\text{tri}}} = (2 + \frac{1}{\sqrt{3}}) v_{\text{tri}}(\alpha_{\text{tri}}, q).
\]

\[
\frac{d^2}{da^2} v_{\text{tri}}(\alpha, 0) \bigg|_{\alpha = \alpha_{\text{tri}}} = 144(2 - \sqrt{3}).
\]

on the other hand, for \((ij) \neq (00)\), we have \(\Delta_{\alpha} \neq 0\), and therefore \((8.3)\) gives

\[
\frac{d}{da} Z_{ij}^{(ij)} \bigg|_{\alpha = \alpha_{\text{tri}}} = \frac{d}{da} \sqrt{\det \Delta_{\alpha}^{00}} \bigg|_{\alpha = \alpha_{\text{tri}}} = \frac{1}{2\sqrt{\det \Delta_{\alpha}^{00}}} \frac{d \det \Delta_{\alpha}^{00}}{da} \bigg|_{\alpha = \alpha_{\text{tri}}} = \frac{c}{2} \lvert T^d \rvert Z_{ij}^{(00)} \bigg|_{\alpha = \alpha_{\text{tri}}}
\]

where \(c = 2 + \frac{1}{\sqrt{3}}\). Subtracting \((8.2)\) and \((8.5)\), we get

\[
\sum_{\xi \in \mathcal{E}(T^d)} \left( \hat{b}(e_0, \xi) + \hat{b}(e_1, \xi) + \hat{b}(e_2, \xi) \right) \alpha^{\xi}(-1)^{\eta_j(\xi)} = 0,
\]

where \(\hat{b}(e, \xi) = \left( \frac{\xi}{6} - \frac{1}{2} \right) I_e \in \mathcal{E} + \left( \frac{\xi}{6} - \frac{1}{2} \right) I_e \in \mathcal{E}.\) It is now a matter of simple algebra to check that \(b(e, \xi) = (6 - 4\sqrt{3})\hat{b}(e, \xi),\) concluding the proof in the case \((i, j) \neq (0, 0)\).

For \((i, j) = (0, 0)\), we have \(Z^{(00)} = 0\), therefore, subtracting \(0 = \frac{1}{2} \lvert T^d \rvert Z^{(00)}\) from the right-hand side of \((8.2)\) yields

\[
\frac{d}{da} Z_{ij} \bigg|_{\alpha = \alpha_{\text{tri}}} = - \sum_{\xi \in \mathcal{E}(T^d)} \left( \hat{b}(e_0, \xi) + \hat{b}(e_1, \xi) + \hat{b}(e_2, \xi) \right) \alpha^{\xi}(-1)^{\eta_j(\xi)}.
\]

Now, since \(Z_{00}(\alpha) = \sqrt{\det \Delta_{\alpha}^{00}(\alpha)}\), and because of \((8.3)\) \((8.4)\), we have

\[
\det \Delta_{\alpha}^{00}(\alpha) = \frac{1}{2} 144(2 - \sqrt{3})(\alpha - \alpha_c)^2 \prod_{q \in \Lambda^* \cap H \setminus \{0\}} v(\alpha, q) + o((\alpha - \alpha_c)^2), \quad \alpha \to \alpha_c,
\]

therefore, we arrive at

\[
\frac{d}{da} Z_{ij} \bigg|_{\alpha = \alpha_{\text{tri}}} = 12 \sqrt{1 - \frac{\sqrt{3}}{2}} \left( \prod_{q \in \Lambda^* \cap H \setminus \{0\}} v(\alpha, q) \right)^{\frac{1}{2}} = 6(\sqrt{3} - 1) \left( \prod_{q \in \Lambda^* \cap \mathbb{R} \setminus \{0\}} v(\alpha, q) \right)^{\frac{1}{2}}.
\]

Putting everything together, we arrive at

\[
\sum_{\xi \in \mathcal{E}(T^d)} (b(e_0, \xi) + b(e_1, \xi) + b(e_2, \xi)) \alpha^{\xi}(-1)^{\eta_j(\xi)} = (4\sqrt{3} - 6)(\sqrt{3} - 1) \frac{1}{\sqrt{\det \Delta_{\alpha}^{00}}},
\]

as required.$\square$

Proof of Theorem 29. As in the the case of the square lattice, \((8.1)\) implies that

\[
\mathbb{E}_{\epsilon_c} = \frac{1}{2Z^2} (B^{01}(\epsilon) + B^{10}(\epsilon) + B^{11}(\epsilon) - B^{00}(\epsilon)),
\]

29
and summing this over $e_0, e_1, e_2$ and applying Lemma 31 yields
\[ E_{e_0} + E_{e_1} + E_{e_2} = \frac{1}{2Z^I} 12(11 - 5\sqrt{3})c_{\text{tri}}^2 \sqrt{\det^* \Delta^{00}} \frac{1}{|\Delta^I|}. \]

Now, we can recall that
\[ 2Z^I = Z_{01} + Z_{10} + Z_{11} = \sqrt{\det K\mathcal{W}^{01} + \det K\mathcal{W}^{10} + \det K\mathcal{W}^{11}} = c_{\text{tri}}^{-\frac{1}{2}} \left( \sqrt{\det \Delta_0^0 + \det \Delta_0^1 + \det \Delta_0^1} \right), \]
and putting all together, after some tedious algebra to simplify the constant in front, yields the result. \qed

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