AUTOMORPHISM GROUPS OF SMOOTH PLANE CURVES

TAKESHI HARUI

ABSTRACT. The author determines the structure of automorphism groups of smooth plane curves of degree at least four. Furthermore, he gives some upper bounds for the order of automorphism groups of smooth plane curves and classifies the cases with large automorphism groups. This paper also contains a simple proof of the uniqueness of smooth plane curves with the full automorphism group of maximal order for each degree.

1. Background and Introduction

The group of automorphisms of an algebraic curve is an old subject of research in algebraic geometry. Especially there are many works on the order of the group of automorphisms. Among others, Hurwitz [Hu] gave an universal upper bound (see Theorem 3.1 for the precise statement). It is an application of Riemann-Hurwitz formula. Following the same line, Oikawa [O] proved another (and possibly better) upper bound for the order of automorphism groups with invariant subsets. Later Arakawa [A] proceeded further with a similar method (Theorem 3.2). Their results are very useful for our study on smooth plane curves.

There are also many works for the structure of automorphism groups of algebraic curves. In particular, the full automorphism groups of hyperelliptic curves are well-known ([BEM], [BGG]). However, it seems that we have poor knowledge about the determination of the full automorphism groups of non-hyperelliptic curves, except for the cases of low genus ([He], [KKu], [KKi] et al.) and Hurwitz curves.

Even for plane curves, we only have several examples of curves whose group of automorphisms are known, such as Fermat curve. In the joint works with Komeda, Kato and Ohbuchi, the author gave a classification of smooth plane curves with automorphisms of certain type ([HKKO], [HKO]).

We consider the following problem in this article:

Problem. Classify automorphism groups of smooth plane curves.

In the cases of degree one, two and three the answer is classically known. Thus we deal with smooth plane curves of degree at least four.

We shall give a complete answer for this problem in Theorem 2.1. This is the first main result of this article. Roughly speaking, smooth plane curves are divided into five kinds from the viewpoint of automorphism groups. Curves of the first kind is nothing but smooth plane curves whose full automorphism groups are cyclic.
The second kind consists of curves whose full automorphism groups are the central extension of finite subgroups of Möbius group $\text{PGL}(2, \mathbb{C}) = \text{Aut}(\mathbb{P}^1)$. Curves of the third (resp. the fourth) kind are descendants of Fermat (resp. Klein) curves (see Section 2 for the definition of the notion). For curves of the fifth kind, their full automorphism groups are isomorphic to primitive subgroups of $\text{PGL}(3, \mathbb{C})$.

Applying Theorem 2.1, we obtain several by-products on automorphism groups of smooth plane curves. For instance, we give a sharp upper bound of the order of automorphism groups in Theorem 2.2. As for smooth plane curves, it is natural to think that there exists a stronger upper bound of the order of automorphism groups than Hurwitz’s one. Indeed, we show that Fermat curve of degree $d$ is the most symmetric plane curve, that is to say, its group of automorphisms has the largest order among all smooth plane curves of the same degree $d$ unless $d = 4, 6$. Moreover, the most symmetric plane curve is unique for each degree up to projective equivalence.

We remark that Theorem 2.2 is partially proved in [KMP] for $d \leq 20$. Furthermore, Pambianco states the same theorem for $d \geq 8$ in his preprint [P]. However, it seems that he give no proofs of several important facts.

Our last result (Theorem 2.5) is a classification of smooth plane curves with large automorphism groups. We give defining equations of such curves.

### 2. Main results

First of all, we note a simple fact on automorphism groups of smooth plane curves and introduce several notions on polynomials and matrices.

Let $G$ be a group of automorphisms of a smooth plane curve of degree at least four. Then it is naturally considered as a subgroup of $\text{PGL}(3, \mathbb{C}) = \text{Aut}(\mathbb{P}^2)$.

Let $F_d$ be Fermat curve $X^d + Y^d + Z^d = 0$ of degree $d$. In this article we denote by $K_d$ a smooth plane curve defined by the equation $XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$, which is called Klein curve of degree $d$.

For a non-zero monomial $cX^iY^jZ^k$ we define its exponent as $\max\{i, j, k\}$. For a homogeneous polynomial $F$, the core of $F$ is defined as the sum of all terms of $F$ with the greatest exponent. A term of $F$ is said to be low if its exponent is smaller than the greatest one.

Let $C_0$ be a smooth plane curve of degree at least four with a defining polynomial $F_0$. Then a smooth plane curve $C$ is said to be a descendant of $C_0$ if $C$ is defined by a homogeneous polynomial whose core coincides with $F_0$ under a suitable coordinate system and $\text{Aut}(C)$ is conjugate to a subgroup of $\text{Aut}(C_0)$.

In this article, a $3 \times 3$ complex matrix $M$ of the form

$$
\begin{pmatrix}
A & 0 \\
0 & 0 \\
0 & \alpha
\end{pmatrix}
$$

($A$ is a regular $2 \times 2$ matrix, $\alpha \in \mathbb{C}^*$)

is called a regular block diagonal matrix of type $(2, 1)$. We denote by $\text{PGL}(2, 1)$ the subgroup of $\text{PGL}(3, \mathbb{C})$ that consists of all elements representable by a regular block diagonal matrix of type $(2, 1)$. There exists a natural group homomorphism $\rho : \text{PGL}(2, 1) \rightarrow \text{PGL}(2, \mathbb{C})$ ($M \mapsto A$).
Using these notions we state our main result as follows:

**Theorem 2.1.** Let \( C \) be a smooth plane curve of degree \( d \geq 4 \), \( G \) a subgroup of \( \text{Aut}(C) \). Then one of the following holds:

(a-i) \( G \) fixes a point on \( C \) and \( G \) is a cyclic group whose order is not greater than \( d(d-1) \).

(a-ii) \( G \) fixes a point not lying on \( C \) and there exists a commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & \mathbb{C}^* \\
\uparrow & & \uparrow \\
1 & \rightarrow & N \\
\end{array}
\xrightarrow{\rho} \begin{array}{ccc}
PGL(2, 1) & \rightarrow & PGL(2, \mathbb{C}) \\
\uparrow & & \uparrow \\
& & G' \\
\end{array} \rightarrow 1 \quad (\text{exact})
\]

where \( N \) is a cyclic group whose order is a factor of \( d \) and \( G' \) is conjugate to a cyclic group \( C_m \), a dihedral group \( D_{2m} \), the tetrahedral group \( A_4 \), the octahedral group \( S_4 \) or the icosahedral group \( A_5 \), where \( m \) is an integer not greater than \( d-1 \). Moreover, if \( G' \simeq D_{2m} \), then \( m \) divides \( d-2 \) or \( N \) is trivial. In particular \( |G| \leq \max\{2d(d-2), 60d\} \) holds.

(b-i) \( C \) is a descendant of Fermat curve \( F_d : X^d + Y^d + Z^d = 0 \). In this case \( |G| \leq 6d^2 \) holds.

(b-ii) \( C \) is a descendant of Klein curve \( K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0 \). In this case \( |G| \leq 3(d^2 - 3d + 3) \) holds if \( d \geq 5 \). On the other hand, \( |G| \leq 168 \) if \( d = 4 \).

(c) \( G \) is conjugate to a finite primitive subgroup of \( \text{PGL}(3, \mathbb{C}) \), namely, the icosahedral group \( A_5 \), the Hessian group \( H_{216} \) of order 216, the Klein group of order 168, the alternating group \( A_6 \) or a subgroup of \( H_{216} \) of order 36 or 72. In particular \( |G| \leq 360 \) holds.

We remark that the order of \( G \) is bounded above independently of \( d \) in the last case. As a corollary of this theorem, we give an upper bound for the order of automorphism groups of smooth plane curves and classify the extremal cases.

**Theorem 2.2.** Let \( C \) be a smooth plane curve of degree \( d \geq 4 \), \( G \) a subgroup of \( \text{Aut}(C) \). Then \( |G| \leq 6d^2 \) except the following cases:

(i) \( d = 4 \) and \( C \) is projectively equivalent to Klein quartic \( XY^3 + YZ^3 + ZX^3 = 0 \). In this case \( \text{Aut}(C) \) is conjugate to the Klein group of order 168.

(ii) \( d = 6 \) and \( C \) is projectively equivalent to Wiman sextic

\[
10X^3Y^3 + 9X^5Z + 9Y^5Z - 45X^2Y^2Z^2 - 135XY^2Z^4 + 27Z^6 = 0.
\]

In this case \( \text{Aut}(C) \) is conjugate to \( A_6 \), a simple group of order 360.

Furthermore, for any \( d \neq 6 \), the equality \( |G| = 6d^2 \) holds if and only if \( C \) is projectively equivalent to Fermat curve \( F_d : X^d + Y^d + Z^d = 0 \) and \( G = \text{Aut}(C) \), which is a semidirect product of \( S_3 \) acting on \( C_d^2 \). In particular, for each \( d \geq 4 \), there exists a unique smooth plane curve with the full group of automorphisms of maximal order up to projective equivalence.
Remark 2.3. (1) It is classically known that Klein quartic (resp. Wiman sextic) has the Klein group of order 168 (resp. $A_6$) as its group of automorphisms (see [B]).

(2) When $d = 6$, a smooth plane sextic defined by the equation

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$$

also satisfies $|\text{Aut}(C)| = 216 = 6^3$. In this case $\text{Aut}(C)$ is conjugate to the Hessian group of order 216.

We shall also show the uniqueness of smooth plane curve of degree $d$ whose full automorphism group is of order $3(d^2 - 3d + 3)$.

Proposition 2.4. Let $C$ be a smooth plane curve of degree $d \geq 5$, $G$ a subgroup of $\text{Aut}(C)$. Assume that $|G| = 3(d^2 - 3d + 3)$. Then $C$ is projectively equivalent to Klein curve $K_d$ and $G = \text{Aut}(K_d)$.

As another by-product of Theorem 2.1, we also give a stronger upper bound for the order of automorphism groups of smooth plane curves and classify the exceptional cases when $d \geq 60$:

Theorem 2.5. Let $C$ be a smooth plane curve of degree $d \geq 60$, $G$ a subgroup of $\text{Aut}(C)$. Then $|G| \leq d^2$ unless $C$ is projectively equivalent to one of the following curves:

(i) Fermat curve $F_d : X^d + Y^d + Z^d = 0$ with $|\text{Aut}(F_d)| = 6d^2$.

(ii) Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ with $|\text{Aut}(K_d)| = 3(d^2 - 3d + 3)$.

(iii) a smooth plane curve defined by the equation

$$Z^d + XY(X^{d-2} + Y^{d-2}) = 0,$$

in which case $|\text{Aut}(C)| = 2d(d - 2)$.

(iv) a descendant of Fermat curve defined by the equation

$$X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^m Y^m Z^m = 0,$$

where $d = 3m$ and $\lambda$ is a non-zero number with $\lambda^3 \neq 1$. In this case $|\text{Aut}(C)| = 2d^2$.

(v) a descendant of Fermat curve defined by the equation

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^m Y^m + Y^m Z^m + Z^m X^m) = 0,$$

where $d = 2m$ and $\lambda \neq 0, -1, \pm 2$. In this case $|\text{Aut}(C)| = 6m^2 = \frac{3}{2}d^2$.

3. Preliminary results

Notation and Conventions

We identify a non-zero matrix with the projective transformation represented by the matrix if no confusion occurs. When a projective transformation preserves a smooth plane curve, it is also identified with the automorphism obtained by its restriction on the curve.

We denote by $[H_1(X, Y, Z), H_2(X, Y, Z), H_3(X, Y, Z)]$ a projective transformation defined by $(X : Y : Z) \mapsto (H_1(X, Y, Z) : H_2(X, Y, Z) : H_3(X, Y, Z))$, where $H_1$, $H_2$ and $H_3$ are homogeneous linear polynomials.
A projective transformation of finite order is called a homology if it is defined by $[X, Y, \zeta Z]$ under a suitable coordinate system, where $\zeta$ is a root of unity. A non-trivial homology fixes a unique line pointwise and a unique point not lying the line. They are respectively called the axis and the center of the homology.

The line defined by the equation $X = 0$ (resp. $Y = 0$, $Z = 0$) will be denoted by $L_1$ (resp. $L_2$, $L_3$). We also denote by $P_1$ (resp. $P_2$ and $P_3$) the point $(1 : 0 : 0)$ (resp. $(0 : 1 : 0)$ and $(0 : 0 : 1)$).

For a positive integer $m$, $C_m$ (resp. $C_m^r$) denotes a cyclic group of order $m$ (resp. the direct product of $r$ copies of $C_m$).

A triangle means the set of three non-concurrent lines. Each line is called an edge of this triangle.

Let $C$ be a smooth irreducible projective curve of genus $g$ defined over the field of complex numbers, $G$ a finite subgroup of Aut$(C)$. Then $G$ induces a Galois covering $\pi : C \to B = C/G$. Riemann-Hurwitz formula tells us that

$$2g - 2 \frac{|G|}{g - 1} = 2g(B) - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{e_j}\right), \quad (*)$$

where $r$ is the number of the branch points and $e_j$ is the branch index at the $j$-th branch point ($j = 1, 2, \cdots, r$).

As an application of this formula we have a famous upper bound of the order of automorphism groups of curves, which is known as Hurwitz bound:

**Theorem 3.1.** ([Hu]) Let $G$ be a finite subgroup of Aut$(C)$. Then $|G| \leq 84(g - 1)$ holds. More precisely,

$$\frac{|G|}{g - 1} = 84, 48, 40, 36, 30 \text{ or } \frac{132}{5} \text{ or } \frac{|G|}{g - 1} \leq 24$$

holds.

Oikawa [O] gave another bound in the case where $G$ fixes a finite subset of $C$. The following theorem is an application of Riemann-Hurwitz formula:

**Theorem 3.2.** ([O Theorem 1], [A Theorem 3]) Let $G$ be a finite subgroup of Aut$(C)$.

1. (Oikawa's inequality) If $G$ fixes a finite subset $S$ of $C$ with $|S| = k \geq 1$, then $|G| \leq 12(g - 1) + 6k$ holds.

2. (Arakawa's inequality) If $G$ fixes three finite disjoint subsets $S_i$ ($i = 1, 2, 3$) of $C$ with $|S_i| = k_i \geq 1$, then $|G| \leq 2(g - 1) + k_1 + k_2 + k_3$ holds.

As an application of the former inequality, we can determine the full automorphism groups of Fermat curves and Klein curves.

**Proposition 3.3.** Let $d$ be an integer with $d \geq 4$. Then the full group of automorphisms of Fermat curve $F_d$ is generated by four transformations $[\zeta X, Y, Z]$, $[X, \zeta Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$, where $\zeta$ is a primitive $d$-th root of unity. It is
isomorphic to a semidirect product of $S_3$ acting on $C_d^2$, in other words, there exists a split short exact sequence of groups

$$1 \to C_d^2 \to \text{Aut}(F_d) \to S_3 \to 1.$$  

In particular $|\text{Aut}(F_d)| = 6d^2$.

**Proof.** Let $H$ be a subgroup of $\text{Aut}(F_d)$ generated by four transformations $[\xi X, Y, Z]$, $[X, \xi Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$. This is a semidirect product of $S_3$ acting on $C_d^2$. In particular we have the inequality $|\text{Aut}(F_d)| \geq |H| = 6d^2$. Thus it suffices to verify that $|\text{Aut}(F_d)| \leq 6d^2$.

Recall that Fermat curve $F_d$ has exactly $3d$ total inflection points. They consist a subset of $F_d$ fixed by its full group of automorphisms. Hence it follows from Oikawa’s inequality that

$$|\text{Aut}(F_d)| \leq 12(g - 1) + 6 \cdot 3d = 6d^2.$$

\[\square\]

**Remark 3.4.** It is easy to check that the order of any element of $\text{Aut}(F_d)$ is at most $2d$.

**Proposition 3.5.** If $d \geq 5$ then the full group of automorphisms of Klein curve $K_d : X Y^{d-1} + Y Z^{d-1} + Z X^{d-1} = 0$

is generated by two transformations $[\xi^{-(d-2)} X, \xi Y, Z]$ and $[Y, Z, X]$, where $\xi$ is a primitive $(d^2 - 3d + 3)$-rd root of unity. It is isomorphic to a semidirect product of $C_3$ acting on $C_{d^2 - 3d + 3}$, in other words, there exists a split short exact sequence of groups

$$1 \to C_{d^2 - 3d + 3} \to \text{Aut}(K_d) \to C_3 \to 1.$$  

In particular $|\text{Aut}(K_d)| = 3(d^2 - 3d + 3)$. On the other hand, $|\text{Aut}(K_4)| = 168$.

**Proof.** It is well-known that $|\text{Aut}(K_4)| = 168$. Assume that $d \geq 5$. Let $H'$ be a subgroup of $\text{Aut}(K_d)$ generated by two transformations $[\xi^{-(d-2)} X, \xi Y, Z]$ and $[Y, Z, X]$, where $\xi$ is a primitive $(d^2 - 3d + 3)$-rd root of unity. This is a semidirect product of $C_3$ acting on $C_{d^2 - 3d + 3}$. In particular $|\text{Aut}(K_d)| \geq |H'| = 3(d^2 - 3d + 3)$. Thus we only have to show that $|\text{Aut}(K_d)| \leq 3(d^2 - 3d + 3)$.

Kato proved that Klein curve $K_d$ has exactly three $(d - 3)$-inflection points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$ (see [K Lemma 2.3]). They consist a subset of $K_d$ fixed by its full group of automorphisms. It follows from Oikawa’s inequality again that

$$|\text{Aut}(K_d)| \leq 12(g - 1) + 6 \cdot 3 = 6(d^2 - 3d + 3).$$

Hence we only have to verify that $\text{Aut}(K_d)$ is of odd order.

Suppose that $K_d$ has an involution $\iota$. Then it fixes at least one $(d - 3)$-inflection point. Without loss of generality, we may assume that $\iota$ fixes $P_3$. Then it also fixes the tangent line $L_2 : Y = 0$ to $K_d$ at $P_3$, the set of the remaining two $(d - 3)$-inflection points $\{P_1, P_2\}$ and the set $\{L_1, L_3\}$. Therefore $\iota = [\alpha X, \beta Y, Z] (\alpha, \beta) = (1, -1), (-1, 1)$ or $(-1, -1))$ or $[\gamma Y, \gamma X, Z] (\gamma = \pm 1)$ holds. It is easy to check that such an involution does not fix $K_d$. Hence $K_d$ has no involution. In other words, $\text{Aut}(K_d)$ is of odd order. \[\square\]
The following is a well-known classical result:

**Proposition 3.6.** If a finite subgroup $G$ of $\text{Aut}(C)$ fixes a point on $C$, then $G$ is cyclic.

For cyclic groups of automorphisms of smooth plane curves, we have the following lemma.

**Lemma 3.7.** Let $C$ be a smooth plane curve of degree $d$, $G$ a cyclic subgroup of $\text{Aut}(C)$. Then $|G| \leq d^2$ holds. Furthermore, if $G$ is generated by a homology, then $|G|$ is a factor of $d - 1$ or $d$. The equality $|G| = d - 1$ (resp. $|G| = d$) holds if and only if $C$ has an inner (resp. outer) Galois point and $G$ is the Galois group at the point.

**Proof.** Let $\sigma$ be a generator of $G$. We may assume that $\sigma$ is represented by a diagonal matrix. Then $G$ fixes each of three lines $L_1 : X = 0$, $L_2 : Y = 0$ and $L_3 : Z = 0$ and each of three points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. Set $S_k := C \cap L_k$ for $k = 1, 2$ and 3. They are non-empty subsets of $C$ with $|S_k| \leq d$. Then $G$ fixes at least three distinct subsets of $C$ among $S_1$, $S_2$, $S_3$, $P_1$, $P_2$ and $P_3$, each of which is of order not greater than $d$. It follows from Theorem 3.2 (2) that $|G| \leq 2(g - 1) + d + d + d = d^2$.

Assume that $\sigma$ is a homology. Then we may assume that $\sigma = [1, 1, \zeta]$, where $\zeta$ is a root of unity. Its center is $P_3$ and its axis is $L_3$. Let $\pi : C \to C/G$ be the quotient map, $\pi_{P_3} : C \to \mathbb{P}^1$ the projection from $P_3 (\pi_{P_3}((X : Y : Z)) = (X : Y))$. Then $\psi : C/G \to \mathbb{P}^1 ([x] \mapsto \pi_{P_3}(x))$ is well-defined, where $[x]$ is the equivalence class of $x \in C$. We thus have a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\pi} & C/G \\
\pi_{P_3} \swarrow & & \searrow \psi \\
\mathbb{P}^1.
\end{array}
$$

In particular $|G| = \deg \pi$ is a factor of $\deg \pi_{P_3}$, which is equal to $d - 1$ (resp. $d$) if $P \in C$ (resp. $P \notin C$).

If $|G| = \deg \pi$, then $\pi$ coincides the quotient map, which implies that $P_3$ is a Galois point of $C$ and $G$ is the Galois group at $P_3$. \qed

In the end of this section, we refer to a theorem on finite groups of planar projective transformations. It is a basic tool of the proof of our main results.

**Theorem 3.8.** ([M, Section 1-10], [DI, Theorem 4.8]) Let $G$ be a finite subgroup of $\text{PGL}(3, \mathbb{C})$. Then one of the following holds:

(a) $G$ fixes a line and a point not lying on the line;
(b) $G$ fixes a triangle; or
(c) $G$ is primitive and conjugate to the icosahedral group $A_5$ (of order 60), the Klein group of order 168, the Hessian group of order 216 or its subgroup of order 36 or 72 or the Valentiner group of order 360, which is isomorphic to $A_6$. 

Remark 3.9. To be precise, Mitchell proved that $G$ fixes a point, a line or a triangle unless $G$ is primitive and isomorphic to a group as in the case (c). In fact the first two cases are equivalent. Indeed, if $G$ fixes a point (resp. a line) then $G$ also fixes a line not passing through the point (resp. a point not lying the line). It is a direct consequence of Maschke’s theorem in group representation theory. Combining this fact with Mitchell’s result we obtain the above theorem.

4. Automorphism groups of smooth plane curves: Case (A)

In the following sections $C$ denotes a smooth plane curve of degree $d \geq 4$ defined by a homogeneous polynomial $F$ and let $G$ be a finite subgroup of Aut$(C)$, which is also considered as a subgroup of PGL$(3, \mathbb{C})$. We identify an element $\sigma$ of $G$ with the corresponding projective transformation, which is denoted by the same symbol $\sigma$.

The following two sections are wholly devoted to the proof of Theorem 2.1. First we divide three cases by using Theorem 3.8 as follows:

(A) $G$ fixes a line and a point not lying on the line.
(B) $G$ fixes a triangle and there exist neither a line nor a point fixed by $G$.
(C) $G$ is primitive and conjugate to a group described in Theorem 3.8.

Note that Case (C) is nothing but the statement (c) in Theorem 2.1.

We consider the remaining two cases one by one. In this section we deal with Case (A).

Case (A): $G$ fixes a line $L$ and a point $P$ not lying on $L$.

We prove that the statement (a-i) (resp. (a-ii) or (b-i)) in Theorem 2.1 holds if $P \in C$ (resp. $P \notin C$).

First note that we only have to consider the case where $G = \text{Aut}(C)$. We may also assume that $L$ is defined by $Z = 0$ and $P = (0 : 0 : 1)$. Then $G$ is a subgroup of PGL$(2, 1)$. Hence there exists a short exact sequence of groups

$$1 \to N \to G \xrightarrow{\rho} G' \to 1,$$

where $\rho : \text{PGL}(2, 1) \to \text{PGL}(2, \mathbb{C})$ is a natural map, $N = \text{Ker}\rho$ and $G' = \text{Im}\rho$. If $C$ passes through $P$, then $G$ is cyclic by virtue of Proposition 3.6.

Claim 1. The subgroup $N$ is a cyclic group whose order is a factor of $d - 1$ (resp. $d$) if $C$ passes through $P$ (resp. $C$ does not pass through $P$).

Proof. For each element $\eta$ of $N$, there exists a unique diagonal matrix of the form $\text{diag}(1, 1, \zeta)$ that represents $\eta$. Hence we have an injective homomorphism $\varphi : N \to \mathbb{C}^*$ ($\eta \mapsto \zeta$). Thus $N$ is isomorphic to a finite subgroup of $\mathbb{C}^*$. It follows that $N$ is a cyclic group generated by a homology. Then our assertion on the order of $N$ follows from Lemma 3.7. $\square$

Let $\eta = [1, 1, \zeta]$ be a generator of $N$, where $\zeta$ is a root of unity.

As for $G'$, a finite subgroup of PGL$(2, \mathbb{C})$, it is well-known that $G'$ is isomorphic to $C_m$, $D_{2m}$, $A_4$, $S_4$ or $A_5$. If $C$ passes through $P$, then $G'$ is cyclic since $G$ is cyclic.
In what follows assume that $G' \simeq C_m$ or $D_{2m}$. We give upper bounds for $m$. There exists an element $\sigma$ such that $\rho(\sigma) = \sigma'$ is of order $m$. Let $H = \langle \sigma \rangle$ be the cyclic subgroup of $G$ generated by $\sigma$. We may assume that $\sigma = [\alpha X, \beta Y, Z]$, where $\alpha$ and $\beta$ are roots of unity. Then the fixed points of $\sigma$ on $L$ are $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$. If $G' \simeq C_m$, then $G$ is generated by $\eta$ and $\sigma$. In particular $G$ fixes two points $P_1$ and $P_2$ in this case.

Additionally, when $G' \simeq D_{2m}$, there exists an element $\tau$ such that $\sigma'$ and $\tau' = \rho(\tau)$ generate $G'$ with $\tau'^2 = 1$ and $\tau'\sigma'\tau' = \sigma'^{-1}$. Then $G$ is generated by $\eta$, $\sigma$ and $\tau$. In this case we may also assume that $\tau = [\gamma Y, \gamma X, Z]$ for some $\gamma$, a root of unity.

Let $F$ be a defining polynomial of $C$ and $e_k$ the intersection multiplicity $i_{P_k}(C, L)$ of $C$ and $L$ at $P_k$ ($k = 1, 2$). If $G'$ is a dihedral group, then $e_1 = e_2$ holds.

For the triviality of $N$, we have the following:

**Claim 2.** If $e_1 \geq 2$ or $e_2 \geq 2$, then $N$ is trivial.

*Proof.* Without loss of generality, we may assume that $e_1 \geq 2$. Then, since $C$ is smooth at $P_1 = (1 : 0 : 0)$, its defining polynomial $F$ contains a term of the form $cX^{d-1}Z$ ($c \neq 0$). Then $F$ is written as

\[
F = X^{e_2}Y^{e_1}F_1(X, Y) + cX^{d-1}Z + \text{(other terms)},
\]

where $F_1$ is a homogeneous polynomial of $X, Y$ such that neither $X$ nor $Y$ is its factor. Therefore

\[
F^n = X^{e_2}Y^{e_1}F_1(X, Y) + \zeta^{-1}cX^{d-1}Z + \text{(other terms)}.
\]

Hence $\zeta = 1$ holds, since $F^n$ is equal to $F$ up to constant. That is to say, $N$ is trivial. □

We further divide two subcases as follows:

- **Subcase (A-1):** $C \cap L$ contains a point distinct from $P_1$ and $P_2$.
- **Subcase (A-2):** $C \cap L \subset \{P_1, P_2\}$.

We give a simple remark on these assumptions.

**Remark 4.1.** If $P \notin C$ and $G' \simeq C_m$, we may assume that the former one is the case. The reason is as follows. Suppose that $C \cap L \subset \{P_1, P_2\}$. Then $G$ fixes each of $P_1$ and $P_2$ and at least one of them are lying on $C$, which implies that this case is converted to the case where $G$ fixes a point on $C$.

**Subcase (A-1):** $C \cap L$ contains a point $Q$ distinct from $P_1$ and $P_2$.

We show the following claim:

**Claim 3.** The order $m$ of $\sigma'$ divides $d - e_1 - e_2$. Furthermore, if $P \notin C$ and $m = d$ holds, then $C$ is a descendant of Fermat curve $F_d$.

*Proof.* Suppose that $\sigma^j$ fixes $Q$ for some $j$. Then it fixes three points on $L$, namely, $Q, P_1$ and $P_2$. Hence it fixes $L$ pointwise, that is to say, $\sigma^j \in N$. In other words, $\sigma^j = 1$. Thus $m|j$ holds. On the other hand, it is obvious that $\sigma^m$ fixes $Q$. It follows that the order of the orbit of $Q$ by $H$ is equal to $|H/\langle \sigma^m \rangle| = m$. Therefore we conclude that $m|d - e_1 - e_2$, using Bézout’s theorem.
Assume that $P \notin C$ and $m = d$. Then $e_1 = e_2 = 0$, which implies that neither $P_1$ nor $P_2$ are lying on $C$. It follows that $C$ is defined by a polynomial whose core is $X^d + Y^d + Z^d$ under a suitable coordinate system. Recall that $G = \text{Aut}(C)$ is generated by $\eta$ and $\sigma$ (resp. $\eta$, $\sigma$ and $\tau$) when $G' \simeq C_m$ (resp. $G' \simeq D_{2m}$). Hence $G$ fixes the polynomial $X^d + Y^d + Z^d$ up to constant, in other words, $G$ is a subgroup of $\text{Aut}(F_d)$. Thus $C$ is a descendant of Fermat curve $F_d$.  

We obtain the assertion of Theorem 2.1 using Claim 1 and Claim 3 as follows.

If $P \in C$, then $G$ is cyclic by virtue of Proposition 3.6. Furthermore, $N$ and $G'$ are cyclic groups of order at most $d - 1$ and $d$ respectively. Hence the order of $G$ is at most $d(d - 1)$. Thus (a-i) in Theorem 2.1 holds.

If $P \notin C$, then $N$ is a cyclic group whose order is a factor $d$ by Claim 1. Furthermore, when $G' \simeq C_m$, the inequality $m \leq d - 1$ holds or $C$ is a descendant of Fermat curve $F_d$ by Claim 3. Hence (a-ii) or (b-i) in Theorem 2.1 holds. On the other hand, when $G' \simeq D_{2m}$, note that $e_1 = e_2$. Therefore combining Claim 2 with Claim 3 we come to the following conclusion.

(i) $m|d - 2$ holds if $e_1 = e_2 = 1$.
(ii) $m \leq d - 4$ and $N$ is trivial if $e_1 = e_2 \geq 2$.
(iii) $C$ is a descendant of Fermat curve $F_d$ if $e_1 = e_2 = 0$.

That is to say, (a-ii) or (b-i) in Theorem 2.1 holds also in this case. Thus we complete the proof in this subcase.

Subcase (A-2): $C \cap L \subset \{P_1, P_2\}$.

As we noted in Remark 4.1, we may exclude the case where $P \notin C$ and $G' \simeq C_m$. Thus $P \in C$ and $G$ is cyclic, or $P \notin C$ and $G' \simeq D_{2m}$. By the assumption $e_1 + e_2 = d$ holds and we may assume that $F$ is written as $F = X^{e_2}Y^{e_1} + Z\hat{F}$, where $\hat{F}$ is a homogeneous polynomial of degree $d - 1$. Note that $e_1 \geq 2$ or $e_2 \geq 2$, since $d \geq 4$.

In particular $N$ is trivial by virtue of Claim 2. Without loss of generality, we may assume that $e_1 \geq 2$. Then $P_1 = (1 : 0 : 0)$ is lying on $C$. Since $C$ is smooth, $F$ contains a term of the form $cX^{d-1}Z$ ($c \neq 0$).

First consider the case where $e_2 \geq 2$. Note that this is always the case if $G' \simeq D_{2m}$, since $e_1 = e_2 = \frac{d}{2} \geq 2$ holds.

Claim 4. If $e_2 \geq 2$, then $m|d - 1$ holds.

Proof. By our assumption $C$ passes through both $P_1$ and $P_2$. The same argument on $P_2$ as above implies that $\hat{F}$ contains a term of the form $c'Y^{d-1}Z$ ($c' \neq 0$). We then have the following equalities:

$$F = X^{e_2}Y^{e_1} + cX^{d-1}Z + c'Y^{d-1}Z + (\text{other terms}),$$

$$F^\sigma = \alpha^{-e_2}\beta^{-e_1}X^{e_2}Y^{e_1} + \alpha^{-(d-1)}cX^{d-1}Z + \beta^{-(d-1)}c'Y^{d-1}Z + (\text{other terms}).$$

Since $\sigma$ preserves $F$ up to constant, we obtain the equality $\alpha^{-(d-1)} = \beta^{-(d-1)}$. In other words, $\sigma^{d-1} = 1$. Hence $m|d - 1$ holds.  

Claim 2 and Claim 1 implies the assertion (a-i) (resp. (a-ii)) in Theorem 2.1 if $P \in C$ (resp. $P \notin C$) under the assumption $e_2 \geq 2$. 

\qed
Finally, consider the case where \( e_2 \leq 1 \). In this case \( P \in C \) and \( G \simeq G' \simeq C_m \).

**Claim 5.** If \( e_2 \leq 1 \), then \( m | d(d-1) \) holds.

*Proof.* Since \( P = (0 : 0 : 1) \) is a smooth point of \( C \), the polynomial \( F \) contains a part of the form \((aX + bY)Z^{d-1} \) \((a, b) \neq (0, 0))\).

Suppose that \( e_2 = 1 \). Thus we have the following equalities:

\[
F = XY^{d-1} + cX^{d-1}Z + (aX + bY)Z^{d-1} + \text{(other terms)},
\]

\[
F^\sigma = \alpha^{-1}\beta^{-(d-1)}XY^{d-1} + \alpha^{-(d-1)}cX^{d-1}Z + (\alpha^{-1}aX + \beta^{-1}bY)Z^{d-1} + \text{(other terms)}.
\]

Since \( F^\sigma \) is equal to \( F \) up to constant, we obtain the following equalities:

\[
\alpha^{-1}\beta^{-(d-1)} = \alpha^{-(d-1)} = \alpha^{-1} \quad \text{if } a \neq 0,
\]

\[
\alpha^{-1}\beta^{-(d-1)} = \alpha^{-(d-1)} = \beta^{-1} \quad \text{if } b \neq 0.
\]

In the former case \( \alpha^{d-2} = \beta^{d-1} = 1 \) holds, which implies that \( \alpha^{(d-1)(d-2)} = \beta^{(d-1)(d-2)} = 1 \). Hence \( m | (d-1)(d-2) < d(d-1) \) holds.

In the latter case we may assume that

\[
F = XY^{d-1} + YZ^{d-1} + ZX^{d-1} + \text{(other terms)}
\]

after a suitable change of the coordinate system if necessary. Since \( G = \langle \sigma \rangle \) acts on \( C \), it preserves the polynomial \( XY^{d-1} + YZ^{d-1} + ZX^{d-1} \) up to constant. Thus it also acts on Klein curve \( K_d \), in other words, \( G \) is isomorphic to a subgroup of \( \text{Aut}(K_d) \). From Proposition 3.5 we conclude that \( m \leq d^2 - 3d + 3 < d(d-1) \) holds in this case also, since \( G \) is cyclic.

Finally suppose that \( e_2 = 0 \). We then have the following equalities:

\[
F = Y^d + cX^{d-1}Z + (aX + bY)Z^{d-1} + \text{(other terms)},
\]

\[
F^\sigma = \beta^{-d}Y^d + \alpha^{-(d-1)}cX^{d-1}Z + \alpha^{-1}aXZ^{d-1} + \beta^{-1}bYZ^{d-1} + \text{(other terms)}.
\]

They are equal each other up to constant. Hence

\[
\beta^{-d} = \alpha^{-(d-1)} = \alpha^{-1} \quad \text{if } a \neq 0,
\]

\[
\beta^{-d} = \alpha^{-(d-1)} = \beta^{-1} \quad \text{if } b \neq 0.
\]

In the former case \( \alpha^{d-2} = 1 \) and \( \beta^d = \alpha \), which implies that \( \alpha^{d(d-2)} = \beta^{d(d-2)} = 1 \). Thus \( m | d(d-2) \) holds. In the latter case \( \alpha^{d-1} = \beta \) and \( \beta^{d-1} = 1 \), which implies that \( \alpha^{(d-1)^2} = \beta^{(d-1)^2} = 1 \). Thus \( m | (d-1)^2 \) holds. In particular \( m \leq d(d-1) \) holds in any case.

From Claim 2 and Claim 5 we conclude that the statement (a-i) in Theorem 2.1 holds. Thus we complete the proof of Theorem 2.1 in this case.

5. **Automorphism groups of smooth plane curves: Case (B)**

In this section we show the statement (b-i) or (b-ii) in Theorem 2.1 holds in Case (B).

**Case (B):** \( G \) fixes a triangle \( \Delta \) and there exist neither a line nor a point fixed by \( G \).
We may assume that $\Delta$ consists of three lines $L_1 : X = 0$, $L_2 : Y = 0$ and $L_3 : Z = 0$. Let $V$ be the set of vertices of $\Delta$, i.e., $V = \{P_1, P_2, P_3\}$. Then $G$ acts on $V$ transitively, since otherwise $G$ fixes a line or a point, which conflicts with our assumption. Hence either $C$ and $V$ are disjoint or $C$ contains $V$.

Let $F$ be a defining homogeneous polynomial of $C$. We note a trivial but useful observation:

**Observation.** Each element of $G$ gives a permutation of the set $\{X, Y, Z\}$ of the coordinate functions up to constant. In particular $G$ fixes the core of $F$ up to constant.

If $C$ contains $V$, we denote by $T_i$ the tangent line to $C$ at $P_i$ ($i = 1, 2, 3$). Then $G$ fixes the set $\{T_1, T_2, T_3\}$. It follows that these lines are distinct one another and $G$ acts on the set transitively by the same argument as above. Thus Case (B) is divided into three subcases:

(B-1) $C$ and $V$ are disjoint.

(B-2) $C$ contains $V$ and each of $T_i$'s ($i = 1, 2, 3$) is an edge of $\Delta$.

(B-3) $C$ contains $V$ and none of $T_i$'s ($i = 1, 2, 3$) is an edge of $\Delta$.

**Subcase (B-1):** $C$ and $V$ are disjoint.

We show that $C$ is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$ in this subcase. By our assumption $C$ does not pass through $P_i$ ($i = 1, 2, 3$). Hence the defining polynomial $F$ of $C$ is of the form

$$F = aX^d + bY^d + cZ^d + (\text{low terms}) \quad (a, b, c \neq 0).$$

Furthermore we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Thus the core of $F$ is $X^d + Y^d + Z^d$, which is fixed by $G$ up to constant from the above observation. It implies that $G$ also acts on Fermat curve $F_d$, in other words, $G$ is a subgroup of $\text{Aut}(F_d)$. Hence we conclude that $C$ is a descendant of $F_d$.

**Subcase (B-2):** $C$ contains $V$ and each $T_i$ ($i = 1, 2, 3$) is an edge of $\Delta$.

We show that $C$ is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$ in this subcase. Without loss of generality we may assume that $T_1 = L_3$, $T_2 = L_1$ and $T_3 = L_2$. Then the defining polynomial $F$ of $C$ is of the form

$$F = aXY^{d-1} + bYZ^{d-1} + cZX^{d-1} + (\text{low terms}) \quad (a, b, c \neq 0).$$

Again we may assume that $a = b = c = 1$ after a suitable coordinate change if necessary. Then the core of $F$ is $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$, which is fixed by $G$ up to constant from the above observation. Hence $G$ also acts on Klein curve $K_d$, that is to say, $G$ is a subgroup of $\text{Aut}(K_d)$. Thus $C$ is a descendant of $K_d$.

**Subcase (B-3):** $C$ contains $V$ and no $T_i$ ($i = 1, 2, 3$) is an edge of $\Delta$.

We show that this subcase does not actually occur.

Any element $\sigma \in G$ can be written in the form $\sigma = [\alpha X_i, \beta X_j, \gamma X_k]$, where $\{i, j, k\} = \{1, 2, 3\}$, $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. Hence we have a natural
homomorphism \( \rho : G \to S_3 \) defined by \([\alpha X_i, \beta X_j, \gamma X_k] \mapsto [X_i, X_j, X_k] \). Then \( \text{Im} \rho \) is isomorphic to \( S_3 \) or \( C_3 \), since there exist neither a line nor a point fixed by \( G \). We consider a short exact sequence of groups

\[
1 \to H \to G \xrightarrow{\rho} \text{Im} \rho \to 1,
\]

where \( H \) is the kernel of \( \rho \), which consists of all elements of \( G \) in the form \([\alpha X, \beta Y, Z]\) \((\alpha, \beta \neq 0)\). We show that \( \rho \) is an isomorphism, in other words, \( H \) is trivial.

We may assume that \( G \) contains an element \( \eta = [Y, Z, X] \) after a suitable coordinate change if necessary, since \( G \) acts on the set of the edges of \( \Delta \) transitively. Then \( F \) is fixed by \( \eta \) up to constant. Hence \( F \) can be written in the form

\[
F = (X + cY)Z^{d-1} + \mu(Y + cZ)X^{d-1} + \mu^2(Z + cX)Y^{d-1} + (\text{low terms}),
\]

where \( \mu^3 = 1 \). Note that \( c \neq 0 \) because of our assumption that no \( T_i \) is an edge of \( \Delta \).

Take any element \( \sigma = [\alpha X, \beta Y, Z] \) \((\alpha, \beta \neq 0)\) of \( H \). Then

\[
F^\sigma = (\alpha^{-1}X + c\beta^{-1}Y)Z^{d-1} + \mu(\beta^{-1}Y + cZ)(\alpha^{-1}X)^{d-1} + \mu^2(Z + c\alpha^{-1}X)(\beta^{-1}Y)^{d-1}.
\]

It is equal to \( F \) up to constant. Therefore

\[
\alpha^{-1} = \beta^{-1} = \alpha^{-(d-1)} \beta^{-1} = \alpha^{-(d-1)},
\]

which implies that \( \alpha = \beta = 1 \). It follows that \( H \) is trivial. Therefore \( G \simeq \text{Im} \rho \simeq S_3 \) or \( C_3 \) holds.

If \( G \) is isomorphic to \( C_3 \), then \( G \) fixes a line, which contradicts our assumption. Thus \( G \) is isomorphic to \( S_3 \). Hence \( G \) is generated by \( \eta \) and another element \( \tau \) of order two with \( \tau \eta \tau = \eta^{-1} \). Then we may assume that \( \tau = [\omega Y, \omega^{-1}X, Z] \) \((\omega^3 = 1)\). Both \( \eta \) and \( \tau \) fixes the same point \((1 : \omega^3 : \omega)\). Therefore \( G \) also fixes this point, which conflicts with our assumption. It follows that this subcase is excluded.

Thus we complete the proof of Theorem 2.1 thoroughly.

6. Smooth plane curves with large automorphism groups

In this section we shall prove Theorem 2.2, Proposition 2.4 and Theorem 2.5.

First we give an upper bound for a primitive group as a subgroup of automorphisms of smooth plane curve.

**Proposition 6.1.** Let \( C \) be a smooth plane curve of degree \( d \geq 4 \), \( G \) a finite subgroup of \( \text{Aut}(C) \). If \( G \) is primitive, then \( |G| \leq 6d^2 \) except the following cases:

(i) \( d = 4 \) and \( C \) is projectively equivalent to Klein quartic \( XY^3 + YZ^3 + ZX^3 = 0 \) with \( |\text{Aut}(C)| = 168 \).

(ii) \( d = 6 \) and \( C \) is projectively equivalent to Wiman sextic

\[
10X^3Y^3 + 9ZX^5 + 9Y^5Z - 45X^2Y^2Z^2 - 135XYZ^4 + 27Z^6 = 0
\]

with \( |\text{Aut}(C)| = 360 \).

**Proof.** First note that \( |G| \leq 360 \) by Theorem 3.8. Thus we may assume that \( d \leq 7 \), since otherwise \( |G| \leq 360 < 6d^2 \).
If \(d = 5\) or \(7\), then we have the inequality \(|G| < 6d^2\) except for \((d, |G|) = (5, 168), (5, 216), (5, 360)\) or \((7, 360)\) again by Theorem \[3.8\]. It is easy to check by Theorem \[3.1\] that these four exceptional cases do not occur.

Assume that \(d = 4\). We then have the inequality \(|G| \leq 168\) by Hurwitz’s theorem. If \(|G| < 168\), then \(|G| \leq 72 < 6d^2\) holds by Theorem \[3.8\]. Suppose that \(|G| = 168\) and \(C\) is not projectively equivalent to Klein quartic \(K_4\). Then \(G\) is conjugate to the Klein group. Hence we may assume that \(G\) acts on both \(C\) and \(K_4\). In particular \(C \cap K_4\) is fixed by \(G\). This is a non-empty subset of \(C\) of order not greater than \(4^2 = 16\) by virtue of Bézout’s theorem. It follows from Oikawa’s inequality that 168 = \(|G| \leq 12 \cdot 2 + 6 \cdot 16 = 120\), a contradiction.

Next assume that \(d = 6\). If \(|G| < 360\), then \(|G| \leq 216 = 6d^2\) by Theorem \[3.8\]. Suppose that \(|G| = 360\) and \(C\) is not projectively equivalent to Wiman sextic \(W_6\). Then \(G\) is conjugate to \(A_6\). Hence we may assume that \(G\) acts on both \(C\) and \(W_6\). It follows from Bézout’s theorem again that \(C \cap W_6\) is a non-empty subset of \(C\) of order not greater than \(6^2 = 36\), which is fixed by \(G\). Again applying Oikawa’s inequality we come to the conclusion that 360 = \(|G| \leq 12 \cdot 9 + 6 \cdot 36 = 324\), a contradiction. □

We show Theorem \[2.2\] using Theorem \[3.8\] and Oikawa’s inequality.

**Proof of Theorem \[2.2\]** We assume that \(G\) is not primitive by virtue of Proposition \[6.1\]. By Theorem \[3.8\] \(G\) fixes a line or a triangle. First suppose that \(G\) fixes a line \(L\). Then \(S := C \cap L\) is a non-empty set of order not greater than \(d\), which is also fixed by \(G\). Applying Theorem \[3.2\] (1) we obtain the inequality

\[
|G| \leq 12(g - 1) + 6|S| \leq 6d(d - 3) + 6d = 6d(d - 2) < 6d^2.
\]

Next suppose that \(G\) fixes a triangle \(\Delta\). Then \(C \cap \Delta\) is a non-empty set of order not greater than \(3d\), which is also fixed by \(G\). Thus we have the inequality \(|G| \leq 6d^2\) by the same argument as above.

Finally assume that \(|G| = 6d^2\) and \(d \neq 6\). From Theorem \[6.1\] and the above argument \(C\) fixes a triangle. Then \(C\) is a descendant of Fermat curve \(F_d\) by virtue of Theorem \[2.1\]. Comparing the order of two groups we know that \(G = \text{Aut}(F_d)\). Let \(cX^iY^jZ^k\) \((c \neq 0, i + j + k = d)\) be a term of \(F\). Note that \([\zeta X, Y, Z]\) and \([X, \zeta Y, Z]\) \((\zeta\) is a primitive \(d\)-th root of unity\), which are elements of \(G\), preserve \(F\). Hence they also preserve the monomial \(cX^iY^jZ^k\). Then \(\zeta^i = \zeta^j = 1\) holds, which implies that \((i, j, k) = (d, 0, 0), (0, d, 0)\) or \((0, 0, d)\). It follows that \(F = X^d + Y^d + Z^d\). □

Next we give a proof of Proposition \[2.4\] using Proposition \[3.8\].

**Proof of Proposition \[2.4\]** Note that \(|G| = 3(d^2 - 3d + 3)\) is an odd integer greater than \(d^2\). Hence (b-ii) in Theorem \[2.1\] only can occur. Thus \(C\) is a descendant of Klein curve \(K_d\). Then \(C\) is defined by a homogeneous polynomial whose core is \(XY^{d-1} + YZ^{d-1} + ZX^{d-1}\) under a suitable coordinate system. Furthermore \(G = \text{Aut}(K_d)\), since \(|G| = 3(d^2 - 3d + 3) = |\text{Aut}(K_d)|\). Then \(G\) contains an element \(\sigma = [\zeta^{-(d-2)} X, \zeta Y, Z]\) \((\zeta\) is a primitive \((d^2 - 3d + 3)\)-rd root of unity\) from Proposition \[3.5\].
If $F$ contains a term $cX^iY^jZ^k$ ($c \neq 0, i + j + k = d$), then we have following equalities:

$$
F = XY^{d-1} + YZ^{d-1} + ZX^{d-1} + cX^iY^jZ^k + \text{(other low terms)},
$$

$$
F^\sigma = \xi^{-(d-2)i}(XY^{d-1} + YZ^{d-1} + ZX^{d-1}) + \xi^{(d-2)i-j}cX^iY^jZ^k + \text{(other low terms)}.
$$

Since they are equal each other up to constant, $\xi^{(d-2)i-j} = \xi^1$ holds, which implies that $\xi^{(d-2)i-j+1} = 1$. The indices $i, j$ and $k$ are at most $d - 1$ since the maximal exponent of $F$ is $d - 1$. Hence $-d + 2 \leq (d-2)i-j+1 \leq (d-2)(d-1)+1 = d^2 - 3d + 3$ holds. It follows that $(d-2)i-j+1 = 0$ or $(d-2)i-j+1 = d^2 - 3d + 3$. In the former case $(i, j, k) = (0, 1, d-1)$ or $(1, d-1, 0)$. In the latter case $(i, j, k) = (d-1, 0, 1)$.

Thus we conclude that $F = XY^{d-1} + YZ^{d-1} + ZX^{d-1}$.

Finally, we show Theorem 2.5. Before starting our proof, we determine the full automorphism groups of curves in three exceptional cases (iii), (iv) and (v) in the theorem.

**Proposition 6.2.** Assume that $d \geq 4$ and $C$ is a smooth plane curve defined by the equation $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.

(i) If $d \neq 4, 6$, then $\text{Aut}(C)$ is a central extension of $D_{2(d-2)}$ by $C_d$. In particular $|\text{Aut}(C)| = 2d(d-2)$.

(ii) If $d = 6$, $\text{Aut}(C)$ is a central extension of $S_4$ by $C_6$. In particular $|\text{Aut}(C)| = 144$.

(iii) If $d = 4$, then $C$ is isomorphic to Fermat quartic $F_4$. In particular $\text{Aut}(C) \simeq C_4^2 \rtimes S_3$ (|Aut(C)| = 96).

**Proof.** First assume that $d \geq 5$. Note that $G = \text{Aut}(C)$ contains three elements $\sigma = [\xi X, \xi^{-(d-1)}Y, Z]$, $\tau = [Y, X, Z]$ and $\eta = [X, Y, \xi Z]$, where $\xi$ (resp. $\xi$) is a primitive $d(d-2)$-nd (resp. $d$-th) root of unity. Then $H = \langle \sigma, \tau, \eta \rangle$ is a subgroup of $\text{PGL}(2, 1)$. This is a central extension of $H' = \langle \sigma', \tau' \rangle \simeq D_{2(d-2)}$ by $\langle \eta \rangle \simeq C_d$, where $\sigma'$ (resp. $\tau'$) is the image of $\sigma$ (resp. $\tau$) by the natural homomorphism $\rho : \text{PGL}(2, 1) \to \text{PGL}(2, \mathbb{C})$.

Next we note that $C$ has an outer Galois point $P = (0 : 0 : 1)$. Since $\sigma$ is of order $d(d-2) > 2d$ for $d > 4$, it follows from Remark 3.4 that $C$ is not isomorphic to Fermat curve $F_4$. Hence $P$ is the unique Galois point (see [Y, Theorem 4', Proposition 5']). In particular $G$ fixes $P$. Then it is obvious that $G$ also fixes the line $Z = 0$. It follows that $G \subset \text{PGL}(2, 1)$. Thus we have the short exact sequence

$$
1 \to N = \ker \rho \to G \xrightarrow{\rho} G' = \text{Im} \rho \to 1,
$$

where $\rho : \text{PGL}(2, 1) \to \text{PGL}(2, \mathbb{C})$ is the natural homomorphism. By virtue of Theorem 2.1, the kernel $N$ coincides with $C_d$. On the other hand, $G'$ is a finite subgroup of $\text{PGL}(2, \mathbb{C})$ containing $H' \simeq D_{2(d-2)}$. Hence $G' = H'$ or $G'$ is isomorphic to $A_4, S_4$ or $A_5$ again by Theorem 2.1. We show that $G' = H'$ by excluding the latter case.

Suppose that $G'$ isomorphic to $A_4, S_4$ or $A_5$. Since $G$ fixes the line $L : Z = 0$, the set $S = C \cap L$ is a non-empty subset of $C$ with $|S| \leq d$. It follows from Oikawa’s
Proof. Let \( g = \frac{1}{2}(d-1)(d-2) \) be the genus of \( C \).

The order of an element of \( G' \) is at most four (resp. five) if \( G' \simeq A_4 \) or \( S_4 \) (resp. \( G' \simeq A_5 \)). On the other hand, \( \text{ord} a' = d - 2 \). Hence \( d \leq 6 \) (resp. \( d \leq 7 \)) holds if \( G' \simeq A_4 \) or \( S_4 \) (resp. \( G' \simeq A_5 \)).

If \( G' \simeq A_5 \), then \( 60d = |G| \leq 6d(d-2) \) holds from (*) . It follows that \( d \geq 12 \), a contradiction.

If \( d = 5 \) and \( G' \simeq S_4 \), then \( 24 \cdot 5 = |G| \leq 6 \cdot 5 \cdot 3 \) holds from (*) again, which implies a contradiction.

If \( d = 5 \) and \( G' \simeq A_4 \), then \( H' \simeq D_6 \) is isomorphic to a subgroup of \( A_4 \) of index two. However, \( A_4 \) has no such subgroup. Thus we exclude this case.

We prove that \( G = \text{Aut}(C) \) is a central extension of \( S_4 \) by \( C_6 \) when \( d = 6 \). It suffices to show that \( G' \simeq S_4 \). Since \( G' \) contains \( H' \), a subgroup of order eight, \( G' \) cannot be \( A_4 \). Then we only have to find an element of \( G' \) of order three for verifying that \( G' \simeq S_4 \). Converting slightly the defining polynomial of \( C \), we may assume that \( C \) is defined by \( Z^6 - XY(X^4 - Y^4) = 0 \). Then it is easy to verify that \( G' \) has an element of order three. Indeed, a \( 3 \times 3 \) matrix

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & -\sqrt{-1}
\end{pmatrix}
\]

\( A = c \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix} \)

gives an automorphism \( \epsilon \) of \( C \) for a suitable constant \( c \). Then \( \epsilon' = \rho(\epsilon) \) is of order three.

Finally assume that \( d = 4 \). Set \( F = Z^4 + XY(X^2 + Y^2) \). Substituting \( X + \sqrt{-1}Y \) (resp. \( X - \sqrt{-1}Y \)) for \( X \) (resp. \( Y \)), \( F \) is converted to

\[
F = Z^4 + (X + \sqrt{-1}Y)(X - \sqrt{-1}Y)((X + \sqrt{-1}Y)^2 + (X - \sqrt{-1}Y)^2)
\]

\[
= Z^4 + (X^2 + Y^2) \cdot 2(X^2 - Y^2)
\]

\[
= Z^4 + 2(X^4 - Y^4).
\]

Then it is clear that the curve defined by \( \tilde{F} \) is isomorphic to Fermat quartic \( F_4 \).

\( \Box \)

**Proposition 6.3.** For a positive integer \( d = 3m \), let \( F'_d \) be a smooth plane curve defined by

\[
X^{3m} + Y^{3m} + Z^{3m} - 3\lambda X^{m}Y^{m}Z^{m} = 0,
\]

where \( d = 3m \), \( \lambda \) is a non-zero number with \( \lambda^3 \neq 1 \). It is a descendant of Fermat curve \( F_d \) and \( \text{Aut}(F'_d) \) is generated by five transformations \([\zeta^3 X, Y, Z], [X, \zeta^3 Y, Z], [\zeta X, \zeta^{-1}Y, Z], [Y, Z, X] \) and \([X, Z, Y]\), where \( \zeta \) is a primitive \( d \)-th root of unity. In this case \( |\text{Aut}(C)| = 2d^2 \).

**Proof.** Let \( H \) be a subgroup of \( G = \text{Aut}(F'_d) \) generated by five transformations \([\zeta^3 X, Y, Z], [X, \zeta^3 Y, Z], [\zeta X, \zeta^{-1}Y, Z], [Y, Z, X] \) and \([X, Z, Y]\). Note that \( H \) also acts on Fermat curve \( F_d \). It is easy to check that \( |H| = 3m^2 \cdot 6 = 2d^2 \). Hence \( |G| \) is divided by \( |H| = 2d^2 \). On the other hand, \( |G| \) is a proper factor of \( 6d^2 \) from
Theorem 2.2. Thus $|G| = 2d^2$ holds, which implies that $G = H$. In particular $C$ is a descendant of Fermat curve $F_d$. \hfill $\square$

**Proposition 6.4.** For a positive even integer $d = 2m$, let $F_d''$ be a smooth plane curve defined by

$$X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^mY^m + Y^mZ^m + Z^mX^m) = 0,$$

where $\lambda \neq 0, -1, \pm 2$. It is a descendant of Fermat curve $F_d$ and $\text{Aut}(F_d'')$ is generated by four transformations $[\zeta^2 X, Y, Z]$, $[X, \zeta^2 Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$, where $\zeta$ is a primitive $d$-th root of unity. It is isomorphic to a semidirect product of $S_3$ acting on $C_m^2$, in other words, there exists a split short exact sequence of groups

$$1 \rightarrow C_m^2 \rightarrow \text{Aut}(F_d'') \rightarrow S_3 \rightarrow 1.$$

In particular $|\text{Aut}(F_d'')| = 6m^2 = \frac{3}{2}d^2$.

**Proof.** Let $H$ be a subgroup of $G = \text{Aut}(F_d'')$ generated by four transformations $[\zeta^2 X, Y, Z]$, $[X, \zeta^2 Y, Z]$, $[Y, Z, X]$ and $[X, Z, Y]$. This is a semidirect product of $S_3$ acting on $C_m^2$. In particular $|G|$ is divided by $|H| = 6m^2$. Then it is easy to verify that $G$ is not isomorphic to any primitive group in Theorem 2.1 (c). Furthermore, since $H$ fixes no point, so does $G$. Thus we conclude that $C$ is a descendant of Fermat curve $F_d$ or Klein curve $K_d$ using Theorem 2.1. Since $G$ has an even order, the latter is not the case. Hence $C$ is a descendant of Fermat curve $F_d$.

Suppose that $G$ contains an element of $\text{Aut}(F_d)$ outside $H$. Then it can be converted by $H$ to the transformation $[\zeta X, Y, Z]$. However, this transformation does not act on $C$. It follows that $G = H$. \hfill $\square$

Next we classify descendants of Fermat curve with large automorphism groups:

**Lemma 6.5.** Two curves $F_d'$ and $F_d''$ are the only descendants of Fermat curve $F_d$ whose group of automorphisms has order greater than $d^2$ up to projective equivalence, except $F_d$ itself.

**Proof.** Let $C$ be a descendant of $F_d$ such that $C$ is not isomorphic to $F_d$ and $G = \text{Aut}(C)$ has order greater than $d^2$. Then $G$ is a proper subgroup of $\text{Aut}(F_d)$ from Theorem 2.2 which implies that $|G| = 3d^2, 2d^2, \frac{3}{2}d^2$ or $\frac{5}{2}d^2$. We then have the following commutative diagram:

$$
\begin{array}{c}
1 \rightarrow C_d \times C_d \rightarrow \text{Aut}(F_d) \xrightarrow{\rho} S_3 \rightarrow 1 \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Next we consider the subgroup $H$, which is a subgroup of $C_d \times C_d$ of index less than six. Let $F$ be a defining homogeneous polynomial of $C$. We may assume that $F$ is written as

$$F = X^d + Y^d + Z^d + (\text{low terms}).$$

Let $cX^iY^jZ^k$ ($c \neq 0, i + j + k = d$) be any low term of $F$. Note that $i$, $j$ and $k$ are less than $d$. Consider the projection $\varpi : C_d \times C_d \to C_d((\eta_1^a\eta_2^b) \mapsto \eta_2^b)$ and its restriction $\varpi|_H : H \to C_d$. There are two cases:

(i) $\varpi|_H$ is surjective.

(ii) $\varpi|_H$ is not surjective.

Case(i) $\varpi|_H$ is surjective. In this case $H$ contains an element $[\zeta^aX, \zeta Y, Z]$ ($0 \leq a \leq d - 1$). Then $H$ also contains $[\zeta X, Y, \zeta Z]$ because $S_3$ acts on $H$. Both of them fixes the term $cX^iY^jZ^k$. It follows that $ai + j \equiv ai + k \equiv 0 \pmod{d}$. Hence $j - k \equiv 0 \pmod{d}$ holds, which implies that $j = k$. In the same way we can show that $i = j$. Therefore $F = X^{3m} + Y^{3m} + Z^{3m} + cX^mY^mZ^m$, where $d = 3m$ and $c \neq 0$. Furthermore, we can write $c = -3$, where $\lambda$ is a non-zero number with $\lambda^3 \neq 1$ because $C$ is non-singular. Thus $C = F''_d$ holds in this case.

Case(ii) $\varpi|_H$ is not surjective. In this case $|C_d \times C_d : H| = 4$. In particular $d$ is an even integer, say $d = 2m$ and $|H| = m^2$. Note that $\text{Ker}(\varpi|_H)$ is a subgroup of $C_d = \langle \eta_1 \rangle$ of index two, that is to say, $\text{Ker}(\varpi|_H) = \langle \eta_1^2 \rangle$. Thus $H$ contains $\eta_1^2$. In the same way we can show that $H$ also contains $\eta_2^2$. Thus $H$ contains the subgroup $\langle \eta_1^2, \eta_2^2 \rangle$, which implies that $H = \langle \eta_1^2, \eta_2^2 \rangle$ since they have the same order $m^2$.

Note both $\eta_1^2$ and $\eta_2^2$ fixes the term $cX^iY^jZ^k$. It follows that $2i \equiv 2j \equiv 0 \pmod{d}$, which implies that $(i, j, k) = (m, m, 0), (m, 0, m)$ or $(0, m, m)$. Thus $F$ can be written as $X^{2m} + Y^{2m} + Z^{2m} + \lambda(X^mY^m + Y^mZ^m + Z^mX^m)$ because $F$ is symmetric. Finally, $\lambda \neq 0, -1, \pm 2$ since $C$ is non-singular. Hence $C = F''_d$ in this case. \hfill $\Box$

Now we are ready to give a proof of Theorem 2.5

**Proof of Theorem 2.5** Let $C$ be a smooth plane curve of degree $d \geq 60$, $F$ a defining homogeneous polynomial of $C$. Assume that a subgroup $G$ of $\text{Aut}(C)$ is of order greater than $d^2$.

Since $d \geq 60$, we have the inequalities $|G| > 60d$ and $|G| > 360$. Then there are only three possibilities from Theorem 2.1

(i) $G$ fixes a point $P$ not lying on $C$ and $G$ is isomorphic to a central extension of $D_{2(d-2)}$ by $C_d$.

(ii) $C$ is a descendant of Fermat curve $F_d : X^d + Y^d + Z^d = 0$.

(iii) $C$ is a descendant of Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$.

Case(i). In this case $G$ also fixes a line $L$ not containing $P$. We may assume that $P = (0 : 0 : 1)$ and $L$ is defined by $Z = 0$. Then $G$ is generated by three elements $\eta = [X, Y, \zeta Z], \sigma = [X, \omega Y, \omega' Z]$ and $\tau = [\gamma Y, \gamma X, Z]$, where $\zeta$, $\omega$, $\omega'$ and $\gamma$ are certain roots of unity and the order of $\zeta$ (resp. $\omega$) is $d$ (resp. $d - 2$). Since $\eta$
preserves $F$ up to constant, $F$ is written as
\[ F = Z^d + \hat{F}(X,Y), \]
where $\hat{F}(X,Y)$ is a homogeneous polynomial of $X$ and $Y$ without multiple factors. Furthermore, $C$ intersects transversally $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$ respectively, by virtue of Claim $[2]$ in Section $[4]$. Hence $\hat{F}(X,Y)$ has a factor of the form $X - cY$ ($c \neq 0$). Since $\sigma$ preserves $\hat{F}(X,Y)$ up to constant, we conclude that $\hat{F}(X,Y) = \lambda XY \Pi_{k=0}^{d-3}(X - \omega^k cY) = \lambda XY(X^{d-2} - c^{d-2}Y^{d-2})$ ($\lambda \in \mathbb{C}^*$). Thus it is clear that $C$ is projectively equivalent to the curve defined by $Z^d + XY(X^{d-2} + Y^{d-2}) = 0$.

Case(ii). From Lemma $[6.5]$ we know that $C$ is projectively equivalent to $F_d$, $F'_d$ or $F''_d$ in this case.

Case(iii). In this case $G$ is a subgroup of $\text{Aut}(K_d)$. Since $\text{Aut}(K_d)$ has an odd order $3(d^2 - 3d + 3)$, we know that $G = \text{Aut}(K_d)$ by our assumption that $|G| > d^2$. It follows from Proposition $[3.5]$ that $C$ is projectively equivalent to Klein curve $K_d$. 

Acknowledgments. The author expresses his sincere gratitude to his professor Kazuhiro Konno and Professor Akira Ohbuchi for their constructive comments and warm encouragement.

References

[A] T. Arakawa, Automorphism groups of compact Riemann surfaces with invariant subsets, Osaka J. Math. 37, No. 4 (2000), 823–846.

[B] H. Blichfeldt, Finite Collineation Groups: With an Introduction to the Theory of Groups of Operators and Substitution Groups, Univ. of Chicago Press, Chicago (1917).

[BEM] E. Bujalance, J. J. Etayo, E. Martínez, Automorphism groups of hyperelliptic Riemann surfaces, Kodai Math. J. 10 (1987), 174–181.

[BGG] E. Bujalance, J. M. Gamboa, G. Gromadzki, The full automorphism groups of hyperelliptic Riemann surfaces, Manuscripta Math. 79 (1993), 267–282.

[Di] I. Dolgachev, V. Iskovskikh, Finite subgroups of the plane Cremona group, Algebra, Arithmetic, and Geometry, Progress in Mathematics Volume 269 (2009), 443–548.

[He] P. Henn, Die Automorphismengruppen der algebraischen Funktionenkörper vom Geschlecht 3, Inaugural-dissertation, Heidelberg (1976).

[Hu] A. Hurwitz, Über algebraische Gebilde mit Eindeutigen Transformationen in sich, Math. Ann. 41, No. 3 (1893), 403–442.

[HKKO] T. Harui, T. Kato, J. Komeda, A. Ohbuchi, Double coverings between smooth plane curves, Kodai Math. J. 31 No. 2 (2008), 257–262.

[HKO] T. Harui, J. Komeda, A. Ohbuchi, Quotient curves of smooth plane curves with automorphisms, Kodai Math. J. 33 No. 1 (2010), 164–172.

[K] T. Kato, A characterization of the Fermat curve, Nonlinear Analysis 47 (2001), 5479–5489.

[KKi] A. Kuribayashi, H. Kimura, Automorphism groups of compact Riemann surfaces of genus five, J. Algebra 134 (1990), 80–103.

[KKu] I. Kuribayashi, A. Kuribayashi, On automorphism groups of compact Riemann surfaces of genus four, Proc. Japan Acad. 62 Ser. A (1986), 65–68.

[KMP] H. Kaneta, S. Marcugini, F. Pambianco, The most symmetric nonsingular plane curves of degree $n \leq 20$, I, Geom. Dedicata 85 (2001), 317–334.
[M] H. H. Mitchell, Determination of the ordinary and modular ternary linear groups, Trans. Amer. Math. Soc. 12, No. 2 (1911), 207–242.

[O] K. Oikawa, Notes on conformal mappings of a Riemann surface onto itself, Kodai Math. Sem. Rep. 8, No. 1 (1956), 23–30.

[P] F. Pambianco, The Fermat curve $x^n + y^n + z^n$: the most symmetric non-singular algebraic plane curve, preprint.

[Y] H. Yoshihara, Function field theory of plane curves by dual curves, J. Algebra 239, Issue 1 (2001), 340–355.

TAKESHI HARUI: ACADEMIC SUPPORT CENTER, KOGAKUIN UNIVERSITY, 2665-1 NAKANO, HACHIOJI, TOKYO 192-0015, JAPAN.

E-mail address: takeshi@cwo.zaq.ne.jp, kt13459@ns.kogakuin.ac.jp