TAYLOR’S THEOREM FOR FUNCTIONALS ON BMO WITH APPLICATION TO BMO LOCAL MINIMIZERS

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Abstract. In this note two results are established for energy functionals that are given by the integral of $W(x, \nabla u(x))$ over $\Omega \subset \mathbb{R}^n$ with $\nabla u \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$, the space of functions of Bounded Mean Oscillation of John & Nirenberg. A version of Taylor’s theorem is first shown to be valid provided the integrand $W$ has polynomial growth. This result is then used to demonstrate that, for the Dirichlet, Neumann, and mixed problems, every Lipschitz-continuous solution of the corresponding Euler-Lagrange equations at which the second variation of the energy is uniformly positive is a strict local minimizer of the energy in $W^{1,1}(\Omega; \mathbb{R}^N)$ for which the weak derivative $\nabla u \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain. Suppose that $d : D \to \mathbb{R}^N$, $N \geq 1$, is a given Lipschitz-continuous function, where $D \subset \partial \Omega$, the boundary of $\Omega$. We herein consider functionals of the form

$$
\mathcal{E}(u) = \int_{\Omega} W(x, \nabla u(x)) \, dx
$$

(1.1)

for $W$ that satisfy, for some $a > 0$ and $r > 0$,

$$
|D^3 W(x, K)| \leq a(1 + |K|^r),
$$

for all real $N$ by $n$ matrices $K$ and almost every $x \in \Omega$. We take $u = d$ on $D$ and $u \in W^{1,\text{BMO}}(\Omega; \mathbb{R}^N)$, the subspace of the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^N)$ for which the weak derivative $\nabla u$ is of Bounded Mean Oscillation. Our main result shows that any Lipschitz-continuous weak solution $u_e$ of the corresponding Euler-Lagrange equations:

$$
0 = \delta \mathcal{E}(u_e)[w] = \int_{\Omega} D W(x, \nabla u_e(x)) [\nabla w(x)] \, dx \quad \text{for all } w \in \text{Var},
$$

(1.2)

at which the second variation of $\mathcal{E}$ is uniformly positive: for some $b > 0$ and all $w \in \text{Var},$

$$
\delta^2 \mathcal{E}(u_e)[w, w] = \int_{\Omega} D^2 W(x, \nabla u_e(x)) [\nabla w(x), \nabla w(x)] \, dx \geq b \int_{\Omega} |\nabla w(x)|^2 \, dx,
$$

(1.3)

will satisfy, for some $c > 0$,

$$
\mathcal{E}(w + u_e) \geq \mathcal{E}(u_e) + c \int_{\Omega} |\nabla w(x)|^2 \, dx
$$

(1.4)

for all $w \in W^{1,\text{BMO}}(\Omega; \mathbb{R}^N) \cap \text{Var}$ whose gradient has sufficiently small norm in $\text{BMO}(\Omega)$. Here

$$
D^j W(x, K) = \frac{\partial^j}{\partial K^j} W(x, K), \quad \text{Var} := \{w \in W^{1,2}(\Omega; \mathbb{R}^N) : w = 0 \text{ on } D\},
$$

$$
\|\nabla u\|_{\text{BMO}} := |\nabla u|_{\text{BMO}} + |(\nabla u)_\Omega|,
$$

$| \cdot |_{\text{BMO}}$ denotes the standard semi-norm on $\text{BMO}(\Omega)$ (see (2.1)), and $(\nabla u)_\Omega$ denotes the average value of the components of $\nabla u$ on $\Omega$. 

Date: 24 May 2020.
The above result extends prior work\(^1\) of Kristensen & Taheri [19, Section 6] and Campos Cordero [4, Section 4] (see, also, Firoozye [8]). These authors proved that, for the Dirichlet problem, if \(u_e\) is a Lipschitz-continuous weak solution of the Euler-Lagrange equations, (1.2), at which the second variation of \(E\) is uniformly positive, (1.3), then there is a neighborhood of \(\nabla u_e\) in \(\text{BMO}(\Omega)\) in which all Lipschitz mappings have energy that is greater than the energy of \(u_e\).

Our proof of the above result makes use of a version of Taylor’s theorem on \(\text{BMO}(\Omega)\) that is established herein: Let \(W\) satisfy, for some \(a > 0, r > 0\), and integer \(k \geq 2\),

\[
|D^k W(x, K)| \leq a(1 + |K|^r),
\]

for all real \(N\) by \(n\) matrices \(K\), and almost every \(x \in \Omega\). Fix \(M > 0\) and \(F \in L^\infty(\Omega; \mathbb{R}^{N \times n})\). Then there exists a constant \(c = c(M, ||F||_\infty) > 0\) such that every \(G \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})\) with \(||G - F||_{\text{BMO}} < M\) satisfies

\[
\int_\Omega W(G) \, dx \geq \int_\Omega W(F) \, dx + \sum_{j=1}^{k-1} \frac{1}{j!} \int_\Omega D^j W(F)[H, H, \ldots, H] \, dx - c \int_\Omega |H|^k \, dx,
\]

where \(H = G - F, F = F(x), G = G(x)\), and, e.g., \(W(F) = W(x, F(x))\).

A key ingredient in our proof of (1.5) is the interpolation inequality [22, Theorem 2.5]: If \(1 \leq p < q < \infty\), then there is a constant \(C = C(p, q, \Omega)\) such that, for all \(\psi \in \text{BMO}(\Omega)\),

\[
\int_\Omega |\psi(x)|^q \, dx \leq C(||\psi||_{\text{BMO}} + |\langle \psi \rangle_\Omega|)^{q-p} \int_\Omega |\psi(x)|^p \, dx.
\]

When \(\Omega = \mathbb{R}^n\) and \(\langle \psi \rangle_{\mathbb{R}^n} = 0\) this inequality is due to Fefferman & Stein [7, p. 156], although it is clear from [16, pp. 624–625] that Fritz John was aware of (1.6) when \(||\psi||_{\text{BMO}}\) was sufficiently small and \(\langle \psi \rangle_\Omega = 0\) (for domains \(\Omega\) with bounded eccentricity).

Finally, we note that our main result assumes that the solution \(u_e\) of the Euler-Lagrange equations (1.2) is Lipschitz continuous and has uniformly positive second variation (1.3). It follows that \(u_e\) is a weak relative minimizer of the energy (1.1), that is, a minimizer with respect to perturbations that are small in \(W^{1,\infty}\). Grabovsky & Mengesha [11, 12] give further conditions\(^2\) that they prove imply that \(u_e\) is then a strong relative minimizer of \(E\), that is, a minimizer with respect to perturbations that are small in \(L^\infty\), whereas our result only changes \(W^{1,\infty}\) to \(W^{1,\text{BMO}} \subset \subset L^\infty\). However, as Grabovsky & Mengesha have noted, their results require that \(u_e\) be \(C^1\). Examples of Müller & Šverák [21] demonstrate that not all Lipschitz-continuous solutions of (1.2) need be \(C^1\). Also, the Lipschitz-continuous example of Kristensen & Taheri [19, §7] satisfies both (1.2) and (1.3).

2. Preliminaries

For any domain (nonempty, connected, open set) \(U \subset \mathbb{R}^n, n \geq 2\), we denote by \(L^p(U; \mathbb{R}^N)\), \(p \in [1, \infty)\), the space of (Lebesgue) measurable functions \(u\) with values in \(\mathbb{R}^N, N \geq 1\), whose

\(^1\)The result in [19, Section 6] has been extended to the Neumann and mixed problems in [22, Section 3].

\(^2\)The most significant are quasiconvexity in both the interior and at the boundary. See Ball & Marsden [1].
$L^p$-norm is finite:
\[ \| u \|_{L^p}^p = \| u \|_{p,U}^p := \int_U |u(x)|^p \, dx < \infty. \]

$L^\infty(U; \mathbb{R}^N)$ shall denote those measurable functions whose essential supremum is finite. We write $L^1_{\text{loc}}(U; \mathbb{R}^N)$ for the set of measurable functions that are integrable on every compact subset of $U$.

We shall write $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to denote a Lipschitz domain, that is a bounded domain whose boundary $\partial \Omega$ is (strongly) Lipschitz. (See, e.g., [6, p. 127], [20, p. 72], or [14, Definition 2.5].) Essentially, a bounded domain is Lipschitz if, in a neighborhood of every $x \in \partial \Omega$, the boundary is the graph of a Lipschitz-continuous function and the domain is on “one side” of this graph. $W^{1,p}(\Omega; \mathbb{R}^N)$ will denote the usual Sobolev space of functions $u \in L^p(\Omega; \mathbb{R}^N)$, $1 \leq p \leq \infty$, whose distributional gradient $\nabla u$ is also contained in $L^p$. Note that, since $\Omega$ is a Lipschitz domain, each $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ has a representative that is Lipschitz continuous. We shall write $\mathbb{R}^{N \times n}$ for the space of real $N$ by $n$ matrices with inner product $A : B = \text{trace}(AB^T)$ and norm $\|A\| = \sqrt{\text{trace}(A^T A)}$, where $B^T$ denotes the transpose of $B$.

2.1. Bounded Mean Oscillation. The BMO-seminorm\(^3\) of $F \in L^1_{\text{loc}}(U; \mathbb{R}^{N \times n})$ is given by
\[ [F]_{\text{BMO}(U)} := \sup_{Q \subset U, Q \subset U} \int_U |F(x) - \langle F \rangle_Q| \, dx, \quad (2.1) \]
where the supremum is to be taken over all nonempty, bounded (open) $n$-dimensional hypercubes $Q$ with faces parallel to the coordinate hyperplanes. Here
\[ \langle F \rangle_U := \int_U F(x) \, dx := \frac{1}{|U|} \int_U F(x) \, dx \]
denotes the average value of the components of $F$, $|U|$ denotes the $n$-dimensional Lebesgue measure of any bounded domain $U \subset \mathbb{R}^n$, and we write $Q \subset U$ provided that $Q \subset K_Q \subset U$ for some compact set $K_Q$.

The space $\text{BMO}(U; \mathbb{R}^{N \times n})$ (Bounded Mean Oscillation) is defined by
\[ \text{BMO}(U; \mathbb{R}^{N \times n}) := \{ F \in L^1_{\text{loc}}(U; \mathbb{R}^{N \times n}) : [F]_{\text{BMO}(U)} < \infty \}. \quad (2.2) \]
One consequence of (2.1)–(2.2) is that $L^\infty(U; \mathbb{R}^{N \times n}) \subset \text{BMO}(U; \mathbb{R}^{N \times n})$ with
\[ [F]_{\text{BMO}(U)} \leq 2\|F\|_{\infty,U} \quad \text{for all } F \in L^\infty(U; \mathbb{R}^{N \times n}). \]
We note for future reference that if $U = \Omega$, a Lipschitz domain, then a result of P. W. Jones [18] implies, in particular, that
\[ \text{BMO}(\Omega; \mathbb{R}^{N \times n}) \subset L^1(\Omega; \mathbb{R}^{N \times n}). \]

It follows that\(^4\)
\[ \|F\|_{\text{BMO}} := [F]_{\text{BMO}(\Omega)} + |\langle F \rangle_\Omega| \quad (2.3) \]
is a norm on $\text{BMO}(\Omega; \mathbb{R}^{N \times n})$.

\(^3\)See Brezis & Nirenberg [2, 3], John & Nirenberg [17], Jones [18], Stein [23, §4.1], or, e.g., [13, §3.1] for properties of BMO.

\(^4\)If $F = \nabla w$ with $w = 0$ on $\partial \Omega$ then $\|\nabla w\|_{\text{BMO}} = [\nabla w]_{\text{BMO}(\Omega)}$ since the integral of $\nabla w$ over $\Omega$ is then zero.
2.2. Further Properties of BMO. The main property of BMO that we shall use is contained in the following result. Although the proof can be found in [22], the significant analysis it is based upon is due to Fefferman & Stein [7], Iwaniec [15], and Diening, Růžička, & Schumacher [5].

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain. Then, for all $q \in [1, \infty)$,

$$\text{BMO}(\Omega; \mathbb{R}^{N \times n}) \subset L^q(\Omega; \mathbb{R}^{N \times n})$$

with continuous injection, i.e., there are constants $J_1 = J_1(q, \Omega) > 0$ such that, for every $F \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$,

$$\left( \int_{\Omega} |F|^q \, dx \right)^{1/q} \leq J_1 \|F\|_{\text{BMO}}. \quad (2.4)$$

Moreover, if $1 \leq p < q < \infty$ then there exists constants $J_2 = J_2(p, q, \Omega) > 0$ such that every $F \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$ satisfies

$$\|F\|_{p, \Omega} \leq J_2 \left( \|F\|_{\text{BMO}} \right)^{1-p/q} \left( \|F\|_{p, \Omega} \right)^{p/q}. \quad (2.5)$$

Here $\| \cdot \|_{\text{BMO}}$ is given by (2.1) and (2.3).

3. An Implication of Taylor’s Theorem for a Functional on BMO

**Hypothesis 3.1.** Fix $k, N \in \mathbb{Z}$ with $k \geq 2$ and $N \geq 1$. We suppose that we are given an integrand $W : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ that satisfies:

- (H1) $K \mapsto W(x, K) \in C^k(\mathbb{R}^{N \times n})$, for a.e. $x \in \Omega$;
- (H2) $(x, K) \mapsto D^j W(x, K)$, $j = 0, 1, \ldots, k$, are each (Lebesgue) measurable on their common domain $\Omega \times \mathbb{R}^{N \times n}$; and
- (H3) There are constants $c_k > 0$ and $r > 0$ such that, for all $K \in \mathbb{R}^{N \times n}$ and a.e. $x \in \Omega$,

$$|D^k W(x, K)| \leq c_k (1 + |K|^r).$$

Here, and in the sequel,

$$D^0 W(x, K) := W(x, K), \quad D^j W(x, K) := \frac{\partial^j}{\partial K^j} W(x, K)$$

denotes $j$-th derivative of $K \mapsto W(\cdot, K)$. Note that, for every $K \in \mathbb{R}^{N \times n}$, a.e. $x \in \Omega$, and $j = 1, 2, \ldots, k$,

$$D^j W(x, K) \in \text{Lin}(\mathbb{R}^{N \times n} \times \mathbb{R}^{N \times n} \times \cdots \times \mathbb{R}^{N \times n}, \mathbb{R}),$$

that is, $D^j W(x, K)$ can be viewed as a multilinear map from $j$ copies of $\mathbb{R}^{N \times n}$ to $\mathbb{R}$.

**Remark 3.2.** Hypothesis (H3) implies that each of the functions $D^j W$, $j = 0, 1, \ldots, k - 1$, satisfies a similar growth condition, i.e., $|D^j W(x, K)| \leq c_j (1 + |K|^{r+j-k})$. It follows that each of the functions $D^j W$ is (essentially) bounded on $\Omega \times K$ for any compact $K \subset \mathbb{R}^{N \times n}$.

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5 This result, as stated, is valid for a larger class of domains: Uniform domains. (Since BMO $\subset L^1$ for such domains. See P. W. Jones [18], Gehring & Osgood [10], and e.g., [9].) A slightly modified version of this result is valid for John domains. See [22] and the references therein.
Lemma 3.3. Let $W$ satisfy Hypothesis 3.1. Fix $M > 0$ and $F \in L^\infty(\Omega; \mathbb{R}^{N \times n})$. Then there exists a constant $c = c(M, \|F\|_\infty) > 0$ such that every $G \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$ with $\|G - F\|_{\text{BMO}} < M$ satisfies

$$
\int_{\Omega} W(G) \, dx \geq \int_{\Omega} W(F) \, dx + \sum_{j=1}^{k-1} \frac{1}{j!} \int_{\Omega} D^j W(F)[H, H, \ldots, H] \, dx - c \int_{\Omega} |H|^k \, dx, \tag{3.1}
$$

where $H = G - F$, $F = F(x)$, $G = G(x)$, and, e.g., $W(F) = W(x, F(x))$.

Proof. Fix $M > 0$ and $F \in L^\infty(\Omega; \mathbb{R}^{N \times n})$. Let $G \in \text{BMO}(\Omega; \mathbb{R}^{N \times n})$ satisfy $\|G - F\|_{\text{BMO}} < M$. We first note that (2.4) in Proposition 2.1 yields

$$
H := G - F \in L^q(\Omega; \mathbb{R}^{N \times n}) \quad \text{for every } q \geq 1, \tag{3.2}
$$

while (H3) together with the fact that $F$ is in $L^\infty$ yields (see Remark 3.2), for some $C > 0$ and a.e. $x \in \Omega$,

$$
|D^j W(x, F(x))| \leq C, \quad j = 0, 1, \ldots, k - 1. \tag{3.3}
$$

Consequently, (3.2) and (3.3) yield, for every $q \geq 1$,

$$
x \mapsto D^j W(x, F(x))[H(x), H(x), \ldots, H(x)] \in L^q(\Omega; \mathbb{R}^{N \times n}), \tag{3.4}
$$

for $j = 0, 1, \ldots, k - 1$.

Next, by Taylor’s theorem for the function $A \mapsto W(\cdot, A)$, for almost every $x \in \Omega$,

$$
W(G) = W(F) + \sum_{j=1}^{k-1} \frac{1}{j!} D^j W(F)[H, H, \ldots, H] + R(F; H), \tag{3.5}
$$

$$
R(F; H) := \int_0^1 (1 - t)^{k-1} (k-1)! D^k W(F + tH)[H, H, \ldots, H] \, dt.
$$

We note that hypothesis (H3) together with the inequality $|a + b|^r \leq c_r(|a|^r + |b|^r)$, $c_r = \max\{1, 2^{r-1}\}$, and the fact that $t \in [0, 1]$ gives us

$$
|D^k W(F + tH)| \leq c_k (1 + |F + tH|^r) \leq c_k + c_k c_r |F|^r + |H|^r \tag{3.6}
$$

and hence the absolute value of the integrand in (3.5) is bounded by $c_k/(k-1)!$ times

$$
|H|^k (1 + c_r |F|^r) + c_r |H|^{k+r}. \tag{3.7}
$$

We next integrate (3.5) and (3.5) over $\Omega$ to get, in view of (3.4), (3.6), and (3.7),

$$
\int_{\Omega} W(G) \, dx = \int_{\Omega} W(F) \, dx + \sum_{j=1}^{k-1} \frac{1}{j!} \int_{\Omega} D^j W(F)[H, H, \ldots, H] \, dx + \int_{\Omega} R(F; H) \, dx \tag{3.8}
$$

and

$$
\int_{\Omega} R(F; H) \, dx \leq C_1 \int_{\Omega} |H|^k \, dx + C_2 \int_{\Omega} |H|^{k+r} \, dx \tag{3.9}
$$

$$
\leq (C_1 + C_2 J^{k+r} ||H||_{\text{BMO}}^{r}) \int_{\Omega} |H|^k \, dx,
$$

where we have made use of (2.5) of Proposition 2.1 with $p = k$ and $q = k+r$, $C_2 := c_k c_r / (k-1)!$, and $C_1 := c_k(1 + c_r |F|_\infty^r) / (k-1)!$. The desired result, (3.1), now follows from (3.8) and (3.9). \qed
4. The Second Variation and BMO Local Minimizers.

We take \( \partial \Omega = D \cup S \) with \( D \) and \( S \) relatively open and \( D \cap S = \emptyset \).

If \( D \neq \emptyset \) we assume that a Lipschitz-continuous function \( d : D \to \mathbb{R}^N \) is prescribed. We define

\[
W^{1, \text{BMO}}(\Omega; \mathbb{R}^N) := \{ u \in W^{1,1}(\Omega; \mathbb{R}^N) : \nabla u \in \text{BMO}(\Omega; \mathbb{R}^{N \times n}) \}
\]

and denote the set of Admissible Mappings by

\[
\text{AM} := \{ u \in W^{1, \text{BMO}}(\Omega; \mathbb{R}^N) : u = d \text{ on } D \text{ or } \langle u \rangle_{\Omega} = 0 \text{ if } D = \emptyset \}.
\]

The energy of \( u \in \text{AM} \) is defined by

\[
E(u) := \int_{\Omega} W(x, \nabla u(x)) \, dx,
\]

where \( W \) is given by Hypothesis 3.1 with \( k = 3 \). We shall assume that we are given a \( u_e \in \text{AM} \) that is a weak solution of the Euler-Lagrange equations corresponding to (4.1), i.e.,

\[
0 = \int_{\Omega} D W(x, \nabla u_e(x)) [\nabla w(x)] \, dx,
\]

for all variations \( w \in \text{Var} \), where

\[
\text{Var} := \{ w \in W^{1,2}(\Omega; \mathbb{R}^N) : w = 0 \text{ on } D \text{ or } \langle w \rangle_{\Omega} = 0 \text{ if } D = \emptyset \}.
\]

**Theorem 4.1.** Let \( W \) satisfy Hypothesis 3.1 with \( k = 3 \). Suppose that \( u_e \in \text{AM} \cap W^{1,\infty}(\Omega; \mathbb{R}^N) \) is a weak solution of (4.2) that satisfies, for some \( a > 0 \),

\[
\int_{\Omega} D^2 W(\nabla u_e)[\nabla z, \nabla z] \, dx \geq 4a \int_{\Omega} |\nabla z|^2 \, dx \text{ for all } z \in \text{Var}.
\]

Then there exists a \( \delta > 0 \) such that any \( v \in \text{AM} \) that satisfies

\[
||\nabla v - \nabla u_e||_{\text{BMO}} < \delta
\]

will also satisfy

\[
E(v) \geq E(u_e) + a \int_{\Omega} |\nabla v - \nabla u_e|^2 \, dx.
\]

In particular, any \( v \neq u_e \) that satisfies (4.4) will have strictly greater energy than \( u_e \).

**Remark 4.2.** 1. The theorem’s conclusions remain valid if one subtracts \( \int_{\Omega} b(x) \cdot u(x) \, dx \) and \( \int_{S} s(x) \cdot u(x) \, dS_x \) from \( E \). 2. Fix \( q > 2 \). Then inequality (2.5) in Proposition 2.1 together with (4.5) yields a constant \( \hat{j} = \hat{j}(q) \) such that any \( v \in \text{AM} \) that satisfies (4.4) will also satisfy

\[
E(v) \geq E(u_e) + \hat{a} \hat{j} \delta^{2-q} \int_{\Omega} |\nabla v - \nabla u_e|^q \, dx.
\]

**Remark 4.3.** The conclusions of Theorem 4.1 remain valid if we replace the assumption that \( u_e \) is a weak solution of (4.2) by the assumption that \( u_e \) is a weak relative minimizer of \( E \), i.e., \( E(v) \geq E(u_e) \) for all \( v \in \text{AM} \cap W^{1,\infty}(\Omega; \mathbb{R}^N) \) with \( ||\nabla v - \nabla u_e||_{\infty} \) sufficiently small.
Proof of Theorem 4.1. Let $u_e \in AM$ be a weak solution of the Euler-Lagrange equations, (4.2), that satisfies (4.3). Suppose that $v \in AM$ satisfies (4.4) for some $\delta > 0$ to be determined later and define $w := v - u_e \in \text{Var} \cap W^{1,\text{BMO}}$. Then Lemma 3.3 yields a constant $c > 0$, such that

$$E(v) \geq E(u_e) + 2\hat{k} \int_{\Omega} |\nabla w|^2 \, dx - c \int_{\Omega} |\nabla w|^3 \, dx,$$

where we have made use of (4.1)–(4.3).

We next note that inequality (2.5) in Proposition 2.1 (with $q = 3$ and $p = 2$) gives us

$$J^3 ||\nabla w||_{\text{BMO}} \int_{\Omega} |\nabla w|^2 \, dx \geq \int_{\Omega} |\nabla w|^3 \, dx.$$  

The desired inequality, (4.5), now follows from (4.4), (4.6), and (4.7) when $\delta$ is sufficiently small. Finally, $E(v) > E(u_e)$ is clear from (4.5) since $\Omega$ is a connected open region and either $\langle w \rangle_{\Omega} = 0$ or $w = 0$ on $D \subset \partial \Omega$. \hfill \square

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