Scaling approach to quantum non-equilibrium dynamics of many-body systems

Vladimir Gritsev\textsuperscript{1,4}, Peter Barmettler\textsuperscript{1,2} and Eugene Demler\textsuperscript{3}

\textsuperscript{1} Physics Department, University of Fribourg, Chemin du Musée 3, 1700 Fribourg, Switzerland
\textsuperscript{2} CPHT, École Polytechnique, 91128 Palaiseau cedex, France
\textsuperscript{3} Lyman Laboratory of Physics, Physics Department, Harvard University, 17 Oxford Street, Cambridge, MA 02138, USA
E-mail: vladimir.gritsev@unifr.ch

\textit{New Journal of Physics} \textbf{12} (2010) 113005 (26pp)

Received 9 July 2010
Published 3 November 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/11/113005

\textbf{Abstract.} Understanding the non-equilibrium quantum dynamics of many-body systems is one of the most challenging problems in modern theoretical physics. While numerous approximate and exact solutions exist for systems in equilibrium, examples of non-equilibrium dynamics of many-body systems that allow reliable theoretical analysis are few and far between. In this paper, we discuss a broad class of time-dependent interacting systems subject to external linear and parabolic potentials, for which the many-body Schrödinger equation can be solved using a scaling transformation. We demonstrate that scaling solutions exist for both local and non-local interactions, and derive appropriate self-consistency equations. We apply this approach to several specific experimentally relevant examples of interacting bosons in one and two dimensions. As an intriguing result, we find that weakly and strongly interacting Bose gases expanding from a parabolic trap can exhibit very similar dynamics.
1. Introduction

Understanding the time evolution of complex quantum systems, often in the presence of strong correlations between constituent particles, is crucial for solving many fundamental problems in physics, from the expansion of the early universe, through heavy ion collisions, to pump and probe experiments in solids. New questions about dynamical evolution arise in recently realized artificial quantum many-body systems, such as ultracold atoms in optical potentials or photons in media with strong optical nonlinearities. These systems are only weakly coupled to external heat baths and have a limited lifetime; thus many experiments require interpretation in terms of coherent quantum dynamics rather than the properties of equilibrium states. On the positive side, these systems allow remarkable control of parameters and open up exciting opportunities for controlled experiments exploring non-equilibrium many-body dynamics.

In the realm of many-body physics, low-dimensional systems have a special place. They have dramatically enhanced quantum and thermal fluctuations and exhibit most surprising manifestations of strong correlations. Rigorous theorems provide strong constraints on long-range order and often such systems cannot be analyzed using mean-field approaches even at zero temperature. Nevertheless, equilibrium properties are well understood using methods specific to low dimensions, such as Coulomb-gas representation of vortices in two dimensions or effective low-energy descriptions of one-dimensional (1D) systems, including the Luttinger liquid and sine-Gordon models (see e.g. [1]). However, such an analysis cannot be straightforwardly extended to non-equilibrium dynamics. Most equilibrium theories focus on the low-energy part of the spectrum, while non-equilibrium dynamics can couple degrees of freedom at very different energy scales [2]–[15]. It would be highly valuable to have examples of many-body dynamics of low-dimensional strongly correlated systems amenable to an unbiased analytical treatment. These examples could be used not only for analyzing experimental systems but also for testing theoretical calculations utilizing effective models or approximations, and for...
checking the validity of new numerical approaches. In this paper, we propose such a class of non-equilibrium quantum problems with time-dependent Hamiltonians that allow for a scaling ansatz of many-body wave functions.

Scaling solutions in quantum dynamics were first discussed in the context of a single harmonic oscillator with a time-dependent frequency [16]–[20]. This problem can be reduced to a time-independent one by properly rescaling space and time. Scaling transformation of variables is possible due to the existence of a dynamical symmetry generated by dynamical invariants of the system [18, 19]. There are also extensions of this approach to single-particle problems with potentials of the Coulomb and inverse square type [17, 21, 22]. In the context of many-body problems, scaling has first been used within mean-field approaches to bosonic systems, for the classical Gross–Pitaevskii equation [23, 24], [27]–[29]. Beyond these effective one-body problems, scaling solutions exist for hard-core bosons in 1D [30] and in the unitary limit of fermionic gases with infinite scattering length [25]; these are problems for which the interaction enters a constraint on the wave function of an otherwise non-interacting system analysis. Away from these specific limits, Pitaevskii and Rosch [26] introduced a scaling ansatz for a 2D many-body system of particles interacting with contact or inverse square interaction and related the existence of such a solution to a hidden $\text{SO}(2, 1)$ symmetry. In this paper, we further extend full many-body scaling solutions to more general types of interaction and arbitrary dimensionality. This generalization can be achieved by allowing three parameters of the system—the mass, the interaction constant and the external potential—to be time dependent. A scaling solution is possible when the interdependence of these parameters is given by an Ermakov-type equation, similar to the one discussed in earlier approaches [25, 26, 30], and an additional self-consistency equation that depends on the dimensionality of the system and the nature of interactions.

Dynamical control over the system parameters is possible in recently developed artificial quantum systems, such as trapped ultracold atomic gases, where the effective interaction can be tuned either using Feshbach resonances or by changing the transverse confining potential, whereas the effective mass can be changed by the application of the weak optical lattice [31]. It is also possible with photons in nonlinear optical devices, where the time-dependent dispersion and Kerr nonlinearity can be achieved using electromagnetically induced transparency [32]–[36]. In this paper, we propose applications of the scaling ansatz that are experimentally relevant in the context of both these systems.

We emphasize that apart from the tunability of the parameters, no specific restrictions on the system properties are imposed. Particles can obey fermionic, bosonic or mixed statistics, interact by pairwise interaction and be subject to parabolic confining potential, to a linear potential and to a complex chemical potential. The basic idea of the scaling solution presented hereafter is to map the non-equilibrium equations of motion to an equilibrium many-body Schrödinger equation. The mapping is based on scaling functions that relate correlation functions of time-dependent systems to correlation functions of systems in equilibrium. Hence, several results known for equilibrium many-body systems can be directly translated to non-equilibrium situations. The reverse conclusion is also true: from a measurement of the system that is out of equilibrium, e.g. a quantum gas after expansion, we can deduce its initial (equilibrium) properties [37].

The paper is organized as follows. In section 2, we introduce a general formalism of scaling transformation for a many-body Schrödinger equation. In section 3, as an example of the application of our approach, we compute momentum distributions for 1D and 2D bosonic
gases with contact interactions released from a parabolic trap. Further details are given in the appendices, where we also discuss the relationship of our work to classical integrability of time-dependent bosonic systems with contact interactions.

2. Scaling transformation—general approach

Our starting point is the many-body Schrödinger equation for \( N \) interacting particles in \( D \) dimensions,

\[
\frac{\partial \Psi(x_1, \ldots, x_N; t)}{\partial t} = H(t) \Psi(x_1, \ldots, x_N; t),
\]

\[
H(t) = -\frac{1}{2m(t)} \sum_{i=1}^{N} \Delta_{x_i}^{(D)} - \mu(t)N + g(t) \sum_{i=1}^{N} x_i + \frac{m(t) \omega^2(t)}{2} \sum_{i=1}^{N} x_i^2 + \sum_{i \neq j} V(x_i - x_j; t),
\]

where \( \Delta_{x_i}^{(D)} \) is a \( D \)-dimensional Laplacian acting on the coordinate \( x_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(D)}) \) of particle \( i \) (\( h = 1 \) here). The external parameters (chemical potential \( \mu(t) \), linear potential \( g(t) \) and trapping frequency \( \omega(t) \)) and the many-body interaction potential \( V(x; t) \) depend explicitly on time. The chemical potential \( \mu(t) = \Re[\mu(t)] + i\Im[\mu(t)] \) can accommodate the effects of dissipation via its imaginary parts\(^5\). While the dependences on the linear and chemical potentials can be removed by the Gallilei transformations and phase shifts, respectively, we note that solving the quantum problem with time dependence of the remaining parameters represents a non-trivial task. For instance, unlike in the non-interacting case, the time dependence of the mass cannot be removed by a simple redefinition of the time variable.

We address the following question: under what conditions can equation (1) (the \( \Psi \)-system) be transformed into the Schrödinger equation for a time-independent (\( \Phi \)-) system?

\[
\frac{\partial \Phi(y_1, \ldots, y_N; \tau)}{\partial \tau} = H_0 \Phi(y_1, \ldots, y_N; \tau),
\]

\[
H_0 = -\frac{1}{2m_0} \sum_{i=1}^{N} \Delta_{y_i}^{(D)} + \frac{m_0 \omega_0^2}{2} \sum_{i} y_i^2 + \sum_{i \neq j} V_0(y_i - y_j).
\]

We emphasize that so far in (2), \( \omega_0 \) and \( m_0 \) are unspecified parameters; in particular, the \( \Phi \)-system can have vanishing confining potential even when the \( \Psi \)-system is confined. We assume that the time dependence of the pairwise interaction potential enters through a single time-dependent coupling \( V(x; t) \equiv V(x) v(t) \) and \( V_0(x) = V(x) v_0 \). We further assume that the interactions have a scaling property and are characterized by the exponent \( \alpha \), which we take to be the same for both \( \Psi \) and \( \Phi \)-systems,

\[
V(\lambda x) = \lambda^\alpha V(x).
\]

Most generic interaction potentials (or pseudo-potentials) satisfy a scaling law (3): s-wave interactions \( V_s(x) \propto \delta(x) \) (\( \alpha = -D \)), any algebraic law, \( V(x) \propto |x|^\alpha \), including Coulomb (\( \alpha = -1 \)), inverse square law (\( \alpha = -2 \)) or dipole–dipole interactions (\( \alpha = -3 \)). Other examples

\(^5\) It is known that the time evolution under a non-Hermitian Hamiltonian in the spirit of stochastic wave function description is equivalent to the description of the open system by the Lindblad master equation, see e.g. [36].
are ultracold fermions interacting via a p-wave channel, which gives rise to the \( \delta' \) pseudo-potential (\( \alpha = D - 1 \)). Also logarithmic potentials can be treated; scaling the logarithmic law produces a time-dependent shift to \( \mu(t) \).

To express the solution of the time-dependent Schrödinger equation (1) in terms of the solution \( \Phi(y_1, \ldots, y_N; \tau) \) of the static equation (2), we introduce the scaling ansatz,

\[
\Psi(x_1, \ldots, x_N; t) = e^{i\int_0^t \frac{DF}{m(t)} R(t) - \Re[\mu(t)] R(t)} \frac{1}{R^N(t)} \Phi(y_1, \ldots, y_N; \tau),
\]

with \( y_i = (x_i/L(t)) + S(t) \) and \( \tau \equiv t(t) \). Direct calculation shows (see appendix A) that this ansatz is valid if the scaling functions \( R(t), L(t), F(t), \tau(t), G(t), S(t) \) and \( M(t) \) satisfy a set of coupled differential equations,

\[
\ddot{R}(t) = \frac{1}{m(t)} DF(t) R(t) - \Re[\mu(t)] R(t),
\]

\[
\dot{L}(t) = \frac{2}{m(t)} F(t) L(t),
\]

\[
\dot{F}(t) = -\frac{2}{m(t)} F^2(t) - \frac{m(t) \omega^2(t)}{2} + \frac{m_0 \omega_0^2}{2L^4(t)m(t)},
\]

\[
\dot{\tau}(t) = \frac{m_0}{m(t)L^2(t)},
\]

\[
\dot{M}(t) = -\frac{G^2(t)}{2m(t)} - \Re[\mu(t)] + \frac{m_0 \omega_0^2 S^2(t)}{2m(t)L^2(t)},
\]

\[
\dot{S}(t) = -\frac{G(t)}{m(t)L(t)},
\]

\[
\dot{G}(t) = -\frac{2F(t)G(t)}{m(t)} - g(t) + \frac{m_0 \omega_0^2 S(t)}{m(t)L^3(t)},
\]

\[
L^{-(\alpha+2)}(t) = \frac{m(t) v(t)}{m_0 v_0}.
\]

It is not obvious a priori that equations (5)–(12) can be satisfied simultaneously for any reasonable time dependences of system parameters \( m(t) \), \( v(t) \) and \( \omega(t) \). Our next goal is to show that there are a number of non-trivial cases for which equations (5)–(12) are consistent with each other. First of all, we note that equations (5) and (6) imply that \( R(t) = [L(t)]^{D/2} \exp(-\Re[\mu(t)]) \, dt \). In the absence of dissipation (\( \Re[\mu(t)] = 0 \)), this condition is equivalent to the conservation of the norm of the wave function under the scaling transformation. Equation (6) allows us to express \( F(t) \) via \( L(t) \), \( F(t) = (m(t)/2)L/L \), which can then be substituted into equation (7). This leads to the differential equation for \( L(t) \),

\[
\ddot{L}(t) + h(t) \dot{L}(t) + \omega^2(t) L(t) = \frac{m_0 \omega_0^2}{m^2(t)L^3(t)},
\]
where \( h(t) = m_0 m(t)/m(t) \). The term with the first derivative can be removed by the change of variables \( L(t) = \exp[B(t)] y(t) \) with \( B(t) = -h/2 \). For \( y(t) \) we obtain

\[
\ddot{y}(t) + \Omega^2(t) y(t) = \frac{\omega_0^2}{y^3(t)},
\]

where \( \Omega^2(t) = \frac{1}{2} h - \frac{1}{2} \dot{h} + \omega^2(t) \). Equation (14) is the celebrated Ermakov equation [38] first discovered in 1880 [39]. This equation has been used primarily for tracking invariants of the time-dependent harmonic oscillator. In appendix B, we show how one can use the nonlinear superposition principle to reduce equation (14) to the linear equation. Once \( L(t) \) is known, the remaining set of equations for \( S(t), M(t) \) and \( G(t) \) can be solved directly.

In summary, to find time-dependent parameters that admit a scaling solution, one can apply the following recipe: after specifying two time-dependent functions \( \omega(t) \) and \( m(t) \), one obtains a solution of the Ermakov equation (14) from which one determines time-dependent interaction strength \( v(t) \) consistent with equation (12). The solutions of the functions \( M(t), G(t) \) and \( S(t) \) can then be obtained straightforwardly provided that the functions \( g(t) \) and \( \mu(t) \) are explicitly specified. Note that the complexity of our method (e.g. solving the Ermakov equation) does not depend on the number of particles \( N \).

The initial conditions for systems (1) and (2) are related to each other through equation (4) applied at time \( t = 0 \),

\[
\Psi(x_1, \ldots, x_N; 0) = e^{iF(0) \sum_{i=1}^N x_i^2 + G(0) \sum_{i=m+1}^N x_i + M(0) x_n} \frac{1}{R^N(0)} \Phi(x_1/L(0), \ldots, x_N/L(0); \tau(0)).
\]

Generally, at \( t = 0 \) the Hamiltonians controlling the dynamics of \( \Psi \) - and \( \Phi \)-systems do not coincide. For example, they can have different confining potentials, or one system can be in a trap while the other one is in free space \( (\omega_0 = 0) \). In this paper, we focus on a finite initial trapping potential, \( \omega(0) = \omega_0 > 0 \), for which we introduce the additional assumption that at \( t = 0 \) the two systems coincide. This means that we have \( m(t = 0) = m_0, v(t = 0) = v_0, F(t = 0) = G(t = 0) = M(t = 0) = 0 \). At \( t > 0 \), the parameters of the \( \Psi \)-system begin to change in time while the parameters of the \( \Phi \)-system remain constant. Since the two systems coincide for \( t < 0 \), the initial state of the \( \Psi \)-systems at \( t = 0 \) should correspond to the equilibrium state of the \( \Phi \)-system. The existence of the scaling solution in 1D in the hard-core limit \( v_0 \to \infty \) has been established previously [30]. Within our approach, this can be understood as follows: the first equation of (12) is trivially satisfied, whereas other equations do not depend on the interaction strength and remain valid. Another special case is the 2D system with contact interactions studied previously by Pitaevskii and Rosch [26] \( (D = -\alpha = 2) \), for which equation (12) is satisfied by constant mass and interaction.

### 3. Dynamics of a Bose gas with contact interaction released from the trap

In this section, as an example, we apply the scaling approach to an ultracold Bose gas with contact interaction that is prepared in a confined, weakly interacting initial state. The non-trivial dynamics comes from a sudden switching off of the confining potential from \( \omega(t) = \omega_0 \) at \( t = 0 \) to \( \omega(t) = 0 \) at \( t > 0 \). The solution of the scaling equation (13) for constant mass \( m(t) = m_0 \) is then given by \( L(t) = \sqrt{(1 + \omega_0^2 t^2)} \) and consequently \( F(t) = (m_0 \omega_0^2 t/2) L^2(t) \). In appendix B.3, we examine additional scenarios corresponding to varying mass that exhibit similar behavior.
of the scaling functions. Here, we also assume that $\mu(t)$ and $g(t)$ are time-independent constants.

To characterize the non-equilibrium dynamics, it is convenient to deal with correlation functions that can be easily derived within the scaling approach (appendix D). The dynamics of the momentum distribution, for example, can be related to the single-particle density matrix $g_1$ of the initial state,

$$n(p, t) = [L(t)]^D \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' g_1(x, x'; 0)e^{-i[F(t)L^2(t)(x^2-x'^2)+L(t)p(x-x')]}.$$  \hspace{1cm} (15)

From the asymptotic behavior of the scaling functions $L(t) \rightarrow \omega_{0f} \omega_0 t$ and $F(t)L(t) = m(t)L(t)/2 \rightarrow \omega_{0f}^3 m_0 \omega_0^2 / 2$, we can extract the long-time limit of the momentum distribution using the stationary phase approximation (SPA),

$$n(p, t) \rightarrow_{\omega_{0f} \gg 1} \left( \frac{2\pi}{m(t)L(t)} \right)^D g_1 \left( \frac{p}{m(t)L(t)}, \frac{p}{m(t)L(t)}; 0 \right).$$

Hence the momentum distribution becomes fully determined by the density distribution $[\rho(x, t) = g_1(x, x, t)]$ of the initial state.

For a quantitative description of dynamics, we need to specify the initial correlation function, which we take from earlier analyses of effective theories for weakly interacting Bose gases in harmonic traps [40]–[42]. An important characteristic of a condensed state with a sufficiently large number of particles is the Thomas–Fermi shape of the density profile, $\rho(x) = \Theta(R_{TF} - |x|) ((\mu/v_0)(1 - (x/R_{TF})^2)$, where $R_{TF} = \sqrt{2\mu/m_0}/\omega_0$ is the Thomas–Fermi radius.

First, we analyze the 1D case in the low-temperature regime where the coherence length is of the order of the Thomas–Fermi radius (equation (E.1) of appendix D). According to the scaling equation (12), for contact interactions, $V(x, t) = v(t)\delta(x)$ ($\alpha = -D$), the interaction must be tuned inversely proportional to the scattering function, $v(t) = v_0 / L(t)$. In figure 1(a), the results of a numerical evaluation of the momentum distributions (15) for specific initial values are shown together with results from SPA. The behavior of the $p = 0$ component is characterized by a steep decay on a time scale $\omega_0^{-1}$ followed by slowly dephasing oscillations, which are due to the finite extension of the density profile and the quadratic phase factor in (15). The corresponding period of oscillations $P$ is determined by the Thomas–Fermi radius, $P \sim (2\pi m_0 / h R_{TF}^2)$. Oscillations as a function of $|p|$ at constant $t$ can be attributed to the finite Thomas–Fermi radius as well. Here, the quadratic phase factor leads to the oscillation period growing with $|p|$. In agreement with the SPA prediction, the momentum distribution relaxes to a semi-circle law. This is remarkable, since such a behavior has been previously associated with 1D Bose systems in the strongly interacting limit ($v_0 \rightarrow \infty$) [30] only. In our case, the interaction strength is initially small and then even decreases with time. We note that this cannot be understood as the effect of dilution due to expansion of the system because the effective 1D interaction parameter [43], $\gamma \sim v(t)/\rho(t) \sim v(t)L(t)$, remains constant.

In 2D, $\alpha = -2$ and equation (12) leads to interactions which are constant in time. When the initial state is weakly interacting (appendix D), we choose an effective theory that incorporates effects of quantum and thermal fluctuations [41]. Results of numerical evaluation of equation (15) are shown in figure 1(b). The momentum distribution evolves very much like in the 1D case and is essentially determined by the initial density distribution and the associated Thomas–Fermi radius. Here, the number of particles ($N = 16$) is set to be smaller than in
Figure 1. Temporal evolution of momentum distribution functions following turning off of the trap at $t = 0$. The insets show the time evolution of the $p = 0$ component. The initial correlation functions are derived from effective theories ([40]–[42], see also appendix D). Dynamical evolution is obtained from numerical integration of equation (15). The SPA represents the asymptotic $t \to \infty$ result. Numerical errors are of the order of the line thickness. In the 1D case (a), the system parameters are $N = 140$, $k_B T = 0.1\hbar\omega_0$, $v_0 = 0.2\sqrt{\hbar^3\omega_0/m_0}$, $R_{TF} = 3.46\sqrt{\hbar/(m_0\omega_0)}$, $v(t) = v_0\sqrt{(1 + \omega_0^2 t^2)}$. In the 2D case (b), the interaction strength is constant, $v(t) = v_0$ and $N = 16$, $k_B T = 0.1\hbar\omega_0$, $v_0 = 0.2\hbar/m_0$, $R_{TF} = 1.41\sqrt{\hbar/(m_0\omega_0)}$.

The analysis of these examples leads to remarkable consequences. We note that the stationary phase regime is reached rather quickly with momentum distribution determined by the initial density distribution. Therefore specially designed initial density distributions (equilibrium or not) can be used to create specific momentum distributions, such as step-like fermionic ones, on demand. It is remarkable that such behaviour, which has been obtained previously in the strongly interacting limit, persists down to arbitrarily weak strength of interaction. This is opposite to what is realized in time-of-flight experiments of ultracold atoms released from a lattice [44], where the expansion at sufficiently large times can be regarded as free and momentum distributions get mapped to density profiles. By contrast, in our case we find that the real space density profile in the trap determines momentum distribution after expansion (see equation (15)). While we do not discuss the appropriate time evolution of $\omega(t)$, $m(t)$ and $v(t)$ here, we point out that the time-of-flight ‘far-field’ limit [44] may also be captured formally by our scaling approach when the asymptotics of $L(t)$ are linear and the contribution of the quadratic phase factor in equation (15), $m(t)\dot{L}(t)$, vanishes in the long-time limit.
4. Conclusions and outlook

We have used scaling ansatz to show that certain quantum non-equilibrium problems with time-dependent parameters can be related to equilibrium problems with constant parameters provided that the time-dependent parameters satisfy a system of self-consistency equations. This approach is valid for rather general types of interactions and is not linked to the integrability of the model. However, an integrable structure, when it exists, is consistent with the scaling transformation. Solvability by the scaling ansatz is a consequence of the non-relativistic dynamical symmetry, which received considerable attention recently in relation to the non-relativistic version of AdS/CFT correspondence [45]–[49]. The appearance of this symmetry in realistic many-body systems, which we discuss in this paper, can open up intriguing connections to the concept of AdS/CFT correspondence.

We have used the scaling approach to analyze the problem of an abrupt switching off of a confining potential for bosonic systems with contact interactions in $d = 1$ and 2. Such experiments can be performed using either ultracold atoms or photons in a nonlinear medium. We find that the asymptotic momentum distribution is essentially given by the initial density profile—a phenomenon that has previously been discussed only in the (Tonks–Girardeau) limit of the infinitely strong repulsive 1D Bose gas [30]. Possible future applications of the scaling ansatz include interaction quenches or transport phenomena (by considering finite linear potentials). Extensions of our method to systems with dissipation are also possible.

In our analysis, we have considered the situation when the scaling ansatz is obeyed exactly. We expect, however, that our results remain qualitatively valid even for systems with small deviations from the exactly scalable Hamiltonians. For example, weak lattice potentials should not have dramatic effects as long as the effective mass approximation is applicable. Therefore one could achieve a full description of time-of-flight experiments if the lattice potential and interactions are tuned accordingly. Moreover, it is conceivable that at a phenomenological level, the ansatz can be used even when the time and space dependences of system parameters do not fully satisfy the consistency equations. The scaling solution could then be seen as a universality class of non-equilibrium systems, very much like a renormalization group fixed point at equilibrium. It would be interesting to address this conjecture in experiments.

Acknowledgments

We thank D Baeriswyl, I Bloch, V Cheianov, D Gangardt, M Lukin, G Morigi, A Polkovnikov and M Zvonarev for useful discussions and remarks. This work was supported by DARPA, MURI, NSF DMR-0705472, Harvard-MIT CUA and the Swiss National Science Foundation.

Appendix A. Derivation of the scaling equations

We consider the ansatz (4)

$$
\Psi(x_1, \ldots, x_N; t) = \frac{1}{R(t)} \exp \left( i \left[ F(t) \sum_{i=1}^{N} x_i^2 + G(t) \sum_{i=1}^{N} x_i + M(t) \right] \right) \Phi \left( \frac{x_i}{L(t)} + S(t); \tau(t) \right)
$$

(A.1)
for the transformation between the many-body Schrödinger equation with time-dependent parameters (equation (1)) and equation (2) with time-independent coefficients. Calculating directly

\[
\Psi = \left( -\frac{\hat{R}}{R^2} + \frac{i \hat{F}}{R} \sum_{i=1}^{N} x_i^2 + \frac{i \hat{G}}{R} \sum_{i=1}^{N} x_i + i \frac{\hat{M}}{R} \right) e^{i \phi(x_i, \tau)} \Phi(y_i, \tau)
\]

\[
+ \frac{1}{R} e^{i \phi(x_i, \tau)} \sum_{i=1}^{N} \frac{\partial \Phi(y_i; \tau)}{\partial y_i} \left[ x_i \left( -\frac{\hat{L}}{L^2} \right) + \hat{S}(t) \right] + \frac{1}{R} e^{i \phi(x_i, \tau)} \frac{\partial \Phi(y_i; \tau)}{\partial \tau} \hat{\tau},
\]  

(A.2)

where for the sake of brevity we introduced \( \phi(x_i, \tau) = F(t) \sum_i x_i^2 + G(t) \sum_i x_i + M(t) \) and where the dot denotes the derivative with respect to \( t \), and

\[
\frac{\partial \Psi(x_i, t)}{\partial x_i} = \frac{1}{R} \left( 2i F \sum_i x_i + G \right) e^{i \phi(x_i, \tau)} \Phi(y_i, t) + \frac{1}{R} e^{i \phi(x_i, \tau)} \frac{\partial \Phi(y_i, \tau)}{\partial y_i},
\]  

(A.3)

\[
\Delta^{(D)}_{y_i} \Phi(x_i, t) = \left\{ \frac{2i F D}{R} + \frac{1}{R} (2i F x_i + i G)(2i F x_i + i G) \right\} \Phi(y_i, \tau)
\]

\[
+ \left( \frac{4i F x_i + 2i G \partial \Phi(y_i; \tau)}{RL} + \Delta^{(D)}_{y_i} \Phi(y_i; \tau) \frac{1}{RL^2} \right) e^{i \phi(x_i, \tau)}.
\]  

(A.4)

Substituting this into the initial Schrödinger equation (1) with time-dependent coefficients, and adding and subtracting the term \( A(t) \sum_i x_i^2 \) with a yet to be determined function \( A(t) \), we regroup the different contributions in front of \( \Phi(y_i, \tau) \), \( \partial \Phi(y_i, \tau)/\partial y_i \) and \( \Delta_{y_i} \). Each group has several contributions proportional to \( x_i^2 \), \( x_i \), \( x_i^2 \) that are linearly independent and must be treated separately. This is how conditions expressed by equation (7) appear. The remaining equation has the form of a Schrödinger equation with time-dependent coefficients

\[
i \frac{\partial \Phi(y_i, \tau)}{\partial \tau} \hat{\tau} = -\frac{1}{2m(t)L^2(t)} \Delta_{y_i} \Phi(y_i, \tau) + \left[ A(t)L^2(t) \sum_i y_i^2 + L^2(t) v(t) V(y_i - y_j) \right] \Phi(y_i, \tau).
\]  

(A.5)

We note that to compensate for the terms appearing after the change \( x_i \rightarrow y_i \) in the quadratic potential, we get terms proportional to \( \omega_0^2 \) in equations (6)–(12). Now, requiring that the three unknown functions \( \tau \), \( L(t) \) and \( A(t) \) satisfy

\[
\hat{\tau} = \frac{m_0}{L^2(t)m(t)}, \quad v_0 \hat{\tau} = v(t)L^2(t), \quad A(t)L^2(t) = \hat{\tau} \frac{m_0 \omega_0^2}{2},
\]  

(A.6)

we obtain the remaining conditions in the set of equations (5)–(12). Under these conditions, the Schrödinger equation for the function \( \Phi(y, \tau) \) has no time-dependent coefficients. From conditions (A.6) above, we determine the function

\[
A(t) = \frac{m_0 \omega_0^2}{2m(t)} \frac{(v(t)m(t))^{4/(\alpha+2)}}{v_0^{4/(\alpha+2)}}.
\]  

(A.7)

Therefore, we find that when pairwise potentials obey equation (3) and the system of equations (5)–(12) is satisfied, equation (1) is indeed mapped to equation (2).
Appendix B. Analysis of the scaling equations and their solutions—the Ermakov equation and dynamical symmetry

B.1. General properties of the Ermakov and related equations

In this appendix, we briefly review some general properties of the Ermakov (sometimes spelled Yermakov) equation that plays such a fundamental role in our formalism. We also point out the relationship of this equation with the Riccati equation and with the linear differential equation with variable coefficients. The Riccati equation directly appears in our approach in some limiting cases.

The Ermakov \[\text{[39]}\] equation is defined as
\[
\ddot{y}(t) + f(t)y(t) = \frac{a}{y(t)^3}.
\] (B.1)

Here, \(a\) is some \(t\)-independent constant. If there is a nontrivial solution of the second-order differential equation
\[
\ddot{x}(t) + f(t)x(t) = 0,
\] (B.2)

then the transformation
\[
\xi(t) = \int_0^t \frac{d\tau}{x^2(\tau)}, \quad z = \frac{y}{x}
\] (B.3)

puts the Ermakov equation into the form
\[
z\dddot{x} = ax^3 - 3,
\] (B.4)

where the subscript denotes the derivative. The solution of the initial equation then follows immediately,
\[
C_1y^2 = ax^2 + x^2 \left(C_2 + C_1 \int \frac{dt}{x^2} \right)^2,
\] (B.5)

where \(C_{1,2}\) are arbitrary constants. If we take two solutions of the linear (Hill) equation to satisfy initial data \(x_1(0) = x_1, \dot{x}_1(0) = \dot{x}_1\) while \(x_2(0) = 0, \dot{x}_2 \neq 0\), then a general solution of the Ermakov equation is given by a nonlinear superposition principle,
\[
y(t) = \sqrt{x_1^2(t) + \frac{1}{w^2}x_2^2(t)},
\] (B.6)

where \(w = x_1\dot{x}_2 - x_2\dot{x}_1\) is a constant Wronskian.

Now, provided the linear equation for \(x(t)\) is satisfied, the function \(u(t)\) defined as
\[
x(t) = \exp \left( - \int_0^t u(t) \, dt \right)
\] (B.7)

satisfies the Riccati equation,
\[
\dot{u} - u^2 = f(t).
\] (B.8)

This demonstrates that all three equations are closely related: the Ermakov equation, the linear second-order differential equation with variable coefficients and the Riccati equation. Other remarkable equations are also connected to the Ermakov equation. For example, taking \(a = 1\) for simplicity in (B.4) and defining \(\xi(t) = z(t)^{-2}\), we obtain \(\xi\dddot{\xi} - (3/2)(\dot{\xi})^2 + 2\xi^4 = 0\).
Now, defining \( w(t) \) via \( \xi(t) = \alpha \dot{w}/w \) with \( \alpha^2 = -1/4 \), we obtain a Kummer–Schwarz equation
\[ \dot{w} \ddot{w} - (3/2)(\dot{w})^2 = 0. \]

In some limiting situations (e.g. \( \omega_0 = 0 \), see the next appendices), the Riccati equation appears naturally in our approach, so we sketch some of its properties here. The general Riccati equation with time-dependent coefficients
\[ \dot{u}(t) = f(t)u^2(t) + g(t)u(t) + h(t) \] (B.9)
can be transformed into the second-order differential equation
\[ f(t)\ddot{y}(t) - [\dot{f}(t) + f(t)g(t)]\dot{y}(t) + f^2(t)h(t)y(t) = 0 \] (B.10)
by the following substitution \( y(t) = \exp(-\int f(t)u(t) \, dt) \). In many cases, a particular solution of (B.10) is easier to find than the one for (B.9).

The Riccati equation has a remarkable property: if there is a known particular solution \( u_0(t) \) of (B.9), then the general solution of (B.9) is given by
\[ u(t) = u_0(t) + \Phi(t) \left[ C - \int f(t)\Phi(t) \, dt \right]^{-1}, \] (B.11)
\[ \Phi(t) = \exp \left[ \int (2f(t)u_0(t) + g(t)) \, dt \right], \] (B.12)
where \( C \) is an arbitrary constant. The particular solution \( u_0(x) \) corresponds to \( C = \infty \).

The property (B.11) allows the construction of many solutions of (B.9) for given functions \( f(t) \), \( g(t) \) and \( h(t) \). If, for example, \( f(t) = 1 \), \( g(t) \) is arbitrary and \( h(t) = -(\alpha^2 + a)g(t) \), a particular solution is \( u_0(t) = a \), and a general solution is then
\[ u(t) = a + \Phi(t) \left[ C - \int \Phi(t) \right]^{-1}, \quad \Phi(t) = \exp \left( 2at + \int g(t) \, dx \right) \] (B.13)
for arbitrary \( C \). For example, for \( f(x) = 1 \), \( g(x) = 0 \), \( h(x) = bx^n \), we obtain
\[ u(t) = -\frac{\dot{w}(t)}{w(t)}, \quad w(t) = \sqrt{t} \left[ C_1 J_{\frac{1}{2}} \left( \frac{1}{k} \sqrt{bt^k} \right) + C_2 Y_{\frac{1}{2}} \left( \frac{1}{k} \sqrt{bt^k} \right) \right], \] (B.14)
\[ k = \frac{1}{2}(n+2), \quad \text{for } n \neq 2, \] (B.15)
\[ u(t) = \frac{\lambda}{t} - t^{2n} \left( \frac{t}{2\lambda + 1} t^{2n} + C \right)^{-1}, \quad \text{for } n = -2, \] (B.16)
where \( \lambda \) is a root of \( \lambda^2 + \lambda + b = 0 \).

B.2. Relation to dynamical symmetry

The Ermakov equation has the symmetry algebra isomorphic to \( SL(2, R) \), which is isomorphic to the algebra \( SO(2, 1) \) of rotations on the surface of a one-sheet hyperboloid. The property (B.11) of the Riccati equation is related to the covariance of the Riccati equation with respect to the fractional-linear transformations generated by the action of \( SL(2, R) \) algebra: the general solution can be expressed as a combination of particular solutions. The same algebra (more explicitly, one of its form, \( SU(1,1) \)) appears as a dynamical symmetry of the
quantum harmonic oscillator, where the Ermakov equation appears as well. This was first found in [16]. There, a single quantum harmonic oscillator with time-dependent frequency has been solved using the methods of (adiabatic) invariants. An adiabatic invariant in this case is a function of a solution of the Ermakov equation. This approach has led to the appearance of the Ermakov–Pinney-type equation [39] in quantum mechanics (see e.g. [20] for a recent review). In [17], the same equation appears as a certain consistency condition on the time-dependent rescaling of coordinate and time in the wave function of the oscillator. It became clear that these two approaches, one based on dynamical invariants and the other on the scaling of dynamical variables, are equivalent. Indeed, the rescaling procedure can be regarded as a transformation, generated by a certain symmetry group, i.e. SL(2, R). The generators of this symmetry are operators corresponding to dynamical invariants. Therefore the successiveness of applicability of scaling transformation implies the presence of dynamical symmetry generated by the dynamical invariants [18, 19]. For this symmetry to hold, one has to have a special class of potential terms in the single-particle Hamiltonian [21]. Physically interesting potentials correspond to the contact interaction, harmonic, Coulomb and inverse square laws. That is why the scaling approach has been applied to a Calogero–Sutherland model [22] and classical Gross–Pitaevski-type systems [23, 24, 26, 27, 29]. The appearance of the SU(1, 1) dynamical symmetry in our non-relativistic systems suggests a possible connection to the non-relativistic version of the AdS/CFT correspondence [45]–[49]. In fact, the Virasoro algebra of any conformal field theory contains SU(1, 1) as subalgebra.

B.3. Specific solutions for $\omega_0 > 0$

We compare examples of decreasing trapping potential and constant, increasing and decreasing masses.

(a) Constant mass. For the case of constant mass $m(t) = m_0$, we choose an exponential decrease of the potential $\omega(t) = \omega_0 e^{-t/\tau_\omega}$. The two independent solutions of the homogeneous equation (B.2) read $x_1(t) = J_0(2\tau_\omega\sqrt{\omega(t)})$ and $x_2(t) = Y_0(2\tau_\omega\omega_0\sqrt{\omega(t)})$. In figure B.1, the resulting scaling functions obeying the initial conditions $L(0) = 1$, $F(0) = 0$ are plotted. For sufficiently small $\tau_\omega$, the functions are well described by the limit $\tau_\omega \to 0$, for which the scaling solution reduces to

$$L(t) = \sqrt{1 + (\omega_0^2 t^2)}, \quad F(t) = \frac{m_0 \omega_0^2 t}{2} \sqrt{L^2(t)}.$$  \hfill (B.17)

(b) Increasing mass. We choose $m(t) = m_0 e^{t/\tau_m}$ and, for the sake of simplicity, $\omega(t > 0) = 0$. The solution then reads

$$L(t) = \sqrt{1 + (1 - e^{-t/\tau_m}) \tau_m^2 m_0 \omega_0^2},$$  \hfill (B.18)

$$F(t) = (1 - e^{-t/\tau_m}) \tau_m^2 m_0 \omega_0^2 / L^2(t)$$

(plotted in figure B.1); this is similar to the scaling functions of case (a), although the time is rescaled and in the limit $t \to \infty$ the functions converge to the values of the functions of case (a) at $t = \tau_m$.

(c) Decreasing mass. For $m(t) = m_0 e^{-t/\tau_m}$, the scaling functions take the form of case (b) when replacing $\tau_m$ by $-\tau_m$ (see figure B.1 for an illustration).
We emphasize that the solutions do not depend on the dimensionality of the system; only the interaction constants, which have to fulfill the consistency equation (12), will do so.

B.4. Specific solutions for $\omega_0 = 0$

Based on two examples, we demonstrate within our formalism that if we relate the non-equilibrium system in the trap to the system without a trap (the case $\omega_0 = 0$ in the main text), we directly obtain a Riccati equation.

For $D = 1$, equation (12) reads $L(t) = m_0(m(t)c(t))^{-1}$ (we define $c(t) = v(t)/v_0$), which we substitute into the equation for $L(t)$ to obtain $F(t) = -(m(t)/2)\frac{d}{dt}\log[c(t)m(t)/m_0]$. Consistency with the equation for $F(t)$ imposes the following relationship between three time-dependent parameters,

$$-\frac{m(t)}{2}\frac{d}{dt}\log[c(t)m(t)] - \frac{m(t)}{2}\frac{d^2}{dt^2}\log[c(t)m(t)]$$

$$= -\frac{m(t)}{2}\left(\frac{d}{dt}\log[c(t)m(t)]\right)^2 - \frac{m(t)\omega(t)}{2}.$$  \hspace{1cm} (B.19)

$$= -\frac{m(t)}{2}\left(\frac{d}{dt}\log[c(t)m(t)]\right)^2 - \frac{m(t)\omega(t)}{2}.$$  \hspace{1cm} (B.20)

By introducing $U(t) = \frac{d}{dt}\log(c(t)m(t))$, it reduces to the Riccati equation

$$\dot{U}(t) = \omega(t) - \frac{d}{dt}(\log[m(t)])U + U^2.$$  \hspace{1cm} (B.21)

The scaling ansatz (4) implies the relationship between initial conditions of the two systems: $\Psi(t = 0) = \exp(iF(0)\sum x_i^2)\Phi(t = 0)$ provided that $L(t = 0) = 1$. The initial condition for the function $U(t)$ is not so important for us because of the special property of the Riccati equation,
related to the Bäcklund symmetry, which allows one to interrelate solutions with different initial conditions via a rational function.

We note that the same equation describes the evolution of spin in a time-dependent magnetic field. A general way to solve it is to note that under some change of variables it can be reduced to the second-order linear differential equation,

\[ \ddot{u} - P(t)\dot{u} + Q(t)u = 0, \quad P(t) = -\frac{d}{dt}\log[m(t)], \quad Q(t) = \omega(t). \]  

(B.22)

Numerous explicit solutions are possible if we specify the functions \( \omega(t), m(t) \).

In the 2D case, we obtain from (5)–(12) that \( R(t) \equiv L(t) \) and time-dependent parameters are connected by the constraint \( c(t)m(t) = c_0 \). Then \( F(t) = (m(t)/2)\frac{d}{dt}\log[L(t)] \). Introducing \( V(t) = \frac{d}{dt}\log L(t) \) and \( h(t) = \frac{d}{dt}\log(m(t)/2) \), we obtain

\[ -\frac{dV(t)}{dt} = \omega(t) + h(t)V(t) + V^2(t), \]  

(B.23)

which is a Riccati equation for the coordinate scaling function \( L(t) \); its solution for given time-dependent parameters \( m(t) \) and \( \omega(t) \) then defines a solution for the time-rescaling function

\[ \frac{dr(t)}{dt} = \frac{m_0}{m(t)L^2(t)}. \]  

(B.24)

To be specific, we list two examples of dynamical parameters:

(a) Increasing mass. From the form of the Riccati equation it is somewhat appealing to take \( m(t) = m_0 e^{\alpha t} \), and constant \( \omega(t) \equiv \Omega \). Then

\[ c(t) = \phi(t)\exp[-\alpha t/2], \]  

(B.25)

where \( \phi(t) = \sin(At + B)/C \) with \( A, B \) and \( C \) related to \( \alpha \) and \( \Omega \). In particular, for \( m(t) = e^{\Omega t} \), where \( \Omega = 1, A = B = C \) and \( C \to 0 \), we obtain \( c(t) = (1 + t)e^{-t} \).

(b) Constant mass. For \( m(t) \equiv m_0 \), the equation can be transformed into the equation for the harmonic oscillator with time dependent-frequency \( \omega(t) \) for which many known solutions exist. Using these solutions we can extract the function \( c(t) \). In particular, for constant \( \omega(t) = \Omega \), the solution for some domain of parameters is

\[ c(t) = \frac{1}{m_0 \cos(\Omega t)}, \]  

(B.26)

In the simplest case of \( m(t) = 1, \omega(t) = 0 \), we obtain \( c(t) = -1/(1 + t) \). This example is a many-body analogue of the solution of the Hamiltonian with potential \( V(x) = c(t)\delta(x) \) found in [21] for a single-particle Schrödinger equation. Direct application of this solution can be found in the ultracold Bose gas close to the confinement-induced resonance [50].

Other examples of solutions of (B.9) can be found in the literature, see e.g. [51].

**Appendix C. Classical integrability of the nonlinear Schrödinger equation (NSE) with time-dependent parameters**

It is instructive to check whether the exact scaling transformation we have studied in this paper is consistent with the property of integrability of the NSE. Here, we address this question for the classical NSE.
In the zero curvature representation, the NSE
\[
\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} + 2c|\Psi|^2 \Psi
\]  
(C.1)
is represented by the system of first-order differential equations
\[
\frac{\partial F}{\partial x} = U(x, t, \lambda) \frac{\partial F}{\partial t} = V(x, t, \lambda), \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]  
(C.2)
such that the matrices \(U(x, t, \lambda)\) and \(V(x, t, \lambda)\) that depend on the spectral parameter \(\lambda\) satisfy
\[
\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0,
\]  
(C.3)
which is equivalent to the compatibility condition of the system,
\[
\frac{\partial^2 F}{\partial x \partial t} = \frac{\partial^2 F}{\partial t \partial x},
\]  
(C.4)
and which is equivalent to the initial Schrödinger equation. In the case of (C.1), one can establish
\[
U = U_0 + \lambda U_1, \quad V = V_0 + \lambda V_1 + \lambda^2 V_2,
\]  
(C.5)
\[
U_0 = \sqrt{c}(\bar{\Psi} \sigma_+ + \Psi \sigma_-), \quad U_1 = \frac{i}{2} \sigma_3,
\]  
(C.6)
\[
V_0 = ic|\Psi|^2 \sigma_3 - \sqrt{c} \left( \frac{\partial \bar{\Psi}}{\partial x} \sigma_+ - \frac{\partial \Psi}{\partial x} \sigma_- \right), \quad V_1 = -U_0, \quad V_2 = -U_1.
\]  
(C.7)
Conserved quantities are constructed from the matrices \(U\) and \(V\) in a known way. This method provides a direct way to various generalizations of NSE. In particular, one can obtain some generalization where the interaction parameter \(c\) and the mass are explicitly time-dependent functions. Introducing generalization of (C.5) as
\[
\tilde{U} = \begin{pmatrix}
-\frac{i}{2} \sigma(x, t) & \gamma(x, t) \tilde{\Psi} \\
\gamma(x, t) \Psi & \frac{i}{2} \beta(x, t)
\end{pmatrix},
\]  
(C.8)
\[
\tilde{V} = \begin{pmatrix}
\frac{i}{A}(|\Psi|^2, \lambda(x, t)) & B \left( \bar{\Psi}, \frac{\partial \bar{\Psi}}{\partial x}, \mu(x, t) \right) \\
B^* \left( \Psi, \frac{\partial \Psi}{\partial x}, \mu(x, t) \right) & -iD(|\Psi|^2, \lambda(x, t))
\end{pmatrix},
\]  
one can look for generalizations of integrable NSE by appropriately choosing the functions \(\alpha(x, t), \beta(x, t), \gamma(x, t), \lambda(x, t), \mu(x, t), A, B\) and \(D\). Analysis of the zero-curvature condition (C.3) in the case of inhomogeneous time-dependent functions leads to a set of equations between those functions and reveals a large class of solutions of the classical equations of motions for NSE with time-dependent coefficients. To get a consistency condition for a zero-curvature representation, we conclude that the spectral parameter should be an inhomogeneous time-dependent function.
A restricted form of this inhomogeneous time-dependent $\tilde{U} - \tilde{V}$ pair has been considered in [52], where it was shown that a combination of space–time transformation together with a $U(1)$ gauge transformation of the linear equations for the $\tilde{U} - \tilde{V}$ pair and corresponding redefinition of the field variables brings the system into the form of a homogeneous time-independent NSE system, thus showing the integrability of a time-dependent system. We note that a similar analysis has been given in [53].

Although it is more difficult to show integrability on the quantum level directly, presumably the property of integrability is not violated in that case for a specific choice of time-dependent parameters that correspond to our scaling equations. A related approach based on the inhomogeneity of spectral parameters for the quantum sine-Gordon model has recently been presented in [54].

Appendix D. Scaling of correlation functions

With the scaling ansatz (4), the relationship between the single-particle correlation functions in the time-dependent and time-independent systems is derived straightforwardly,

$$g_1^{(\psi)}(x, x', t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dx_2 \ldots dx_N \Psi^*(x, x_2, \ldots, x_N; t) \Psi(x', x_2, \ldots, x_N; t)$$

$$= \frac{1}{[L(t)]^D} g_1^{(\phi)} \left( \frac{x}{L(t)}, \frac{x'}{L(t)}; 0 \right) \exp(-iF(t)(x^2 - x'^2)). \quad \text{(D.1)}$$

The labels in the $g_1$-function refer to the time-dependent ($\Psi$) and time-independent ($\Phi$) systems. From this expression, we can readily extract the density: $n^{(\psi)}(x, t) = g_1(x, x, t) = (1/L(t))\rho^{(\phi)}(x/L(t); 0)$. The momentum distribution of a time-dependent system, defined as

$$n^{(\psi)}(p, t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' e^{-ip(x-x')} g_1^{(\psi)}(x, x', t), \quad \text{(D.2)}$$

is then given by

$$n^{(\psi)}(p, t) = [L(t)]^D \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' g_1^{(\phi)}(x, y; 0) \exp[-iF(t)L^2(t)(x^2 - x'^2) - iL(t)p \cdot (x - x')]. \quad \text{(D.3)}$$

Note that because of the quadratic term in the exponent, the integrations are nontrivial.

For the two-particle density matrix, we find analogously

$$g_2^{(\psi)}(x_1, x_2, x_1', x_2'; t) = N(N-1) \int dx_3 \ldots dx_N \Psi^*(x_1, x_2, \ldots, x_N; t) \Psi(x_1', x_2', \ldots, x_N; t)$$

$$= \frac{1}{[L(t)]^2} g_2^{(\phi)} \left( \frac{x_1}{L(t)}, \frac{x_2}{L(t)}, \frac{x_1'}{L(t)}, \frac{x_2'}{L(t)}; 0 \right) \exp(-iF(t)(x_1^2 + x_2^2 - x_1'^2 - x_2'^2)) \quad \text{(D.4)}$$

and the two-particle correlation function reads

$$\rho_2^{(\psi)}(x, y; t) = g_2^{(\psi)}(x, y, x, y; t) = \frac{1}{L^2(t)} \rho_2^{(\phi)} \left( \frac{x}{L(t)}, \frac{y}{L(t)}; 0 \right). \quad \text{(D.5)}$$

Other useful quantities, such as non-equilibrium time-dependent correlation functions (e.g. $n^{(\psi)}(p, t, t')$) or the multi-mode squeezing spectrum ($S(k, k'; t, t') = \langle n^{(\psi)}(p, t)n^{(\psi)}(p', t') \rangle$), can also be easily computed using the scaling approach.

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
Appendix E. Some technical details related to the derivation of one-dimensional (1D) and 2D momentum distributions at equilibrium

E.1. Trapped weakly interacting Bose gases

In order to describe a condensed Bose gas in a harmonic potential, we adopt the results of previous works \[40\]–\[42\] that consider phase fluctuations on top of the mean-field solution while density fluctuations are assumed to be negligible. This is a valid approximation for a sufficiently high number of weakly interacting particles at low temperatures. The temperature range where the density fluctuations are suppressed is \(T_d \gg T \gg T_{\phi}\) where the temperature of quantum degeneracy is \(T_d = N\bar{\rho}^2\hbar\omega_0\) and \(T_{\phi} = T_d\hbar\omega_0/\mu\).

Generically, the single-particle correlation can be represented as

\[
g_1(x, x') = \sqrt{\rho(x)\rho(x')} \exp \left( -\frac{1}{2}(\langle \phi(x) - \phi(x') \rangle^2) \right),
\]

where \(\langle \phi(x) \rangle\) denotes the average over phase fluctuations. We assume the validity of the Thomas–Fermi approximation for the density

\[
\rho(x) \to \rho_{TF}(x) = \frac{\mu}{g} \left( 1 - \left( \frac{x}{R_{TF}} \right)^2 \right) \theta \left( 1 - \left| \frac{x}{R_{TF}} \right| \right),
\]

where \(R_{TF} = \sqrt{2\mu/m_0/\omega_0}\) is the Thomas–Fermi radius.

In a 1D geometry, taking into account thermal fluctuations and neglecting contributions from quantum fluctuations, one obtains the phase average \[40\]

\[
\langle (\phi(x') - \phi(x))^2 \rangle = \frac{4T \mu}{\hbar^2 \omega^2} \ln \left[ \frac{\left( 1 - \frac{x}{R_{TF}} \right) \left( 1 + \frac{x}{R_{TF}} \right)}{\left( 1 + \frac{x}{R_{TF}} \right) \left( 1 - \frac{x}{R_{TF}} \right)} \right].
\]

For the 2D case, an expression similar to the 1D case can be derived. In this work, we used the complete expression obtained by Xia et al (equation (77) in \[41\]), which explicitly accounts for thermal and quantum fluctuations. As a result, at inter-particle distances much smaller than \(2R_{TF}\), the correlations decay exponentially with a decay rate approximately given by \(mk_B T/(2\pi\hbar^2\rho(0))\). However, for the dynamics studied in this paper, we did not find significant effects from quantum corrections.

E.2. 1D and 2D uniform Bose gases

For a 1D Bose gas, it was recently shown \[55\] that the effective field theory (Luttinger liquid) provides an extremely accurate description for a single-body correlation function at distances beyond the inter-particle separation. If we are not interested in its large momentum behavior, it is legitimate to use this effective theory. The single-particle correlation function in time-independent theory is then well known (see e.g. \[1\]). For non-zero temperatures, it is given by (we omit oscillating terms)

\[
g_1^{(\phi)}(x, x'; 0) = \langle \Phi^\dagger(x)\Phi(x') \rangle = \rho_0 B \left[ \frac{\pi/\xi_T}{\rho_0 \sinh(\pi(x - x')/\xi_T)} \right]^{1/2},
\]

\(B = \frac{\pi/\xi_T}{\rho_0 \sinh(\pi/\xi_T)}\).
where $\xi_T = h v_s / T = h^2 \pi \rho / (m_0 KT)$, $\rho_0$ is the uniform equilibrium density, $v_s$ is the sound velocity, $K$ is a Luttinger parameter that is related to the interaction strength $c$ and $\beta = (K / \pi)^{1/2}$ is Popov’s factor.

In the 2D case, we consider a system below the Berezinskii–Kosterlitz–Thouless (BKT) transition. The correlation functions then decay algebraically with a temperature-dependent exponent, which tends to the universal value 1/4 when approaching the BKT transition from below.

Appendix F. Dynamics of initially uniform systems

F.1. Relating systems in the trap and without it

The scaling approach can be used to establish the relationship between correlation functions in the model with time-dependent parameters (the system $\Psi$) and the model with time-independent parameters (the system $\Phi$). As we discussed in the main text, the trapping frequency $\omega_0$ of the time-independent system is not fixed a priori. In particular, it can be set to zero from the very beginning. The scaling transformation therefore will relate the system in the time-dependent trap and a uniform system. The set of differential equations has to be modified accordingly. The aim of this appendix is to look into the behavior of the momentum distribution in this case.

The initial conditions state that the two wave functions are equal at $t = 0$. This means that the density distribution of the trapped system is homogeneous, corresponding to the uniform one. This is possible if we assume the existence of a length scale $l$ on which this condition can be satisfied. Moreover, we assume here that the Thomas–Fermi radius of a trapped system is large enough such that there is a finite region of $x \in [-l, l]^D$ where the density is considered to be constant. In the absence of a trapping potential, this region is equal to the whole observation area. We assume that this region is large enough to contain a relatively large number of particles $N$. Using this length scale $l$ as a sort of cut-off, we evaluate the momentum distribution in the finite window $[-l, l]$ for examples of 1D and 2D systems at finite temperature.

F.2. Evaluation of momentum distributions in 1D for the uniform system

In the Luttinger liquid approximation at finite temperature, we introduce $\xi_{\pm} = \pi (x - y) / \xi_T$ in terms of which $g_1^{(\Phi)}(x, y; 0) \sim (\sinh \xi_-)^{-1/2K}$. This function decays exponentially at large distances and the limits of integration in $\xi_-$ domain can therefore be extended from $[-l, l]$ to $(-\infty, \infty)$ to make analytic progress. One can easily realize that because of the additional structure in the exponent of equation (D.1), the expression for the momentum distribution is essentially different from the one at equilibrium. The corresponding integral is

$$\int_{-\infty}^{\infty} e^{-i C(t) \xi_-} \frac{1}{\sinh |\xi_-|} \xi^- \frac{1}{|\xi^-|} = 2^{-1+1/2K} \Gamma \left(1 - \frac{1}{2K} \right) \left( \frac{\Gamma \left[ \frac{1}{4K} - i \frac{\xi^-}{2} \right]}{\Gamma \left[ 1 - \frac{1}{4K} - i \frac{\xi^-}{2} \right]} + \frac{\Gamma \left[ \frac{1}{4K} + i \frac{\xi^-}{2} \right]}{\Gamma \left[ 1 - \frac{1}{4K} + i \frac{\xi^-}{2} \right]} \right),$$

(F.1)

where $C(t) = F(t) L^2(t) \xi_T^2 \xi_s / \pi^2 + L(t) p \xi_T / \pi$. The integration over $\xi_s$ is then performed in the finite interval $[-l, l]$ corresponding to the size of the selected subsystem. Expression (F.2) is proportional to the equilibrium momentum distribution at $t = 0$ provided that we take $L(0) = 1$. 

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
We find
\[ n(p, t) = 2\rho_0 L(t) \left( \frac{2K}{\xi_T \rho_0} \right)^{1/2K} \Gamma \left( 1 - \frac{1}{2K} \right) \int_{-1/L(t)}^{1/L(t)} d\xi_+ \left\{ \frac{\Gamma[\frac{1}{4K} - \frac{i}{2}(F(t)\tilde{L}^2(t)\xi_+ + p\tilde{L}(t))]}{\Gamma[1 - \frac{1}{4K} + \frac{i}{2}(F(t)\tilde{L}^2(t)\xi_+ + p\tilde{L}(t))] + \frac{\Gamma[\frac{1}{4K} + \frac{i}{2}(F(t)\tilde{L}^2(t)\xi_+ + p\tilde{L}(t))] + \Gamma[\frac{1}{4K} - \frac{i}{2}(F(t)\tilde{L}^2(t)\xi_+ + p\tilde{L}(t))]}{\Gamma[1 - \frac{1}{4K} + \frac{i}{2}(F(t)\tilde{L}^2(t)\xi_+ + p\tilde{L}(t))]} \right\}, \] (F.2)
where \( \tilde{L}(t) = L(t)\xi_T / \pi \).

On the basis of this expression, we have calculated a momentum distribution for various particular functions \( \omega(t) \) and \( m(t) \). Solving the set of consistency equations of section 2, we obtained all the other functions \( v(t) \), \( L(t) \) and \( F(t) \). This is illustrated in figure F.1 for particular choices of time-dependent functions \( \omega(t) \) and \( m(t) \) and corresponds to a particular function \( v(t) \) found from the solution of the Riccati equation. But additional simulations with various other choices of functions \( \omega(t) \) and \( m(t) \) suggest that the resulting momentum distribution defined as above in equation (15) has a step-like form. The formation of an effective momenta scale is associated with the asymptotic emergence of microcanonical-type distribution.

The Luttinger liquid expression for the \( g_1 \) correlation function is a low-energy approximation for the true behavior of the correlation function. However, in the non-equilibrium dynamics, we excite the whole spectrum and therefore the result for our time-dependent theory based on the exact equilibrium theory may appear to be different from the one based on the low-energy approximation. In what follows, we demonstrate that the long-time behavior of the momentum distribution of the time-dependent system has a bounded support in momentum space. Our arguments can be applied to any exactly solvable models.
Suppose the $g_1(x,y)$-correlation function is defined as a ground-state correlator of some field operators $\Psi(x), \Psi^\dagger(x): g_1(x,y) = \langle \Psi^\dagger(x)\Psi(y) \rangle$. We also assume that the matrix elements of the operator $\Psi(x)$ in the eigenbasis of the equilibrium problem are known. This implies that the form factors $F(\lambda, \mu) = \langle \lambda|\Psi(0)|\mu \rangle$ and the norms of the eigenstates $|\lambda\rangle$ and $|\mu\rangle$ are known. Here, $\{\mu\}$ and $\{\lambda\}$ are the sets of numbers that characterize the eigenstates of a system of size $2l$. In particular, these numbers can correspond to the solutions of the Bethe ansatz equations in the exactly solvable problems. We also assume space- and time-translation invariance. Therefore the time-dependent $g_1$ function can be expanded as

$$g_1^{(\Phi)}(x,t;0,0) = \sum_{\{\mu\}} \exp[i(E_\mu - E_\lambda)t - i(P_\mu - P_\lambda)x] \frac{|F(\{\lambda\}, \{\mu\})|^2}{||\lambda||^2||\mu||^2}, \quad (F.3)$$

where $E_\lambda$ and $P_\lambda$ are, respectively, the energy and momentum of the state $|\lambda\rangle$. We assume also that the set $\{\lambda\}$ corresponds to the ground state. By introducing the coordinates $\xi = x - y$ and $\eta = x + y$, the momentum distribution of the time-dependent system (we take for simplicity equal-time correlation function) can be written as

$$n(p,t) = L(t) \sum_{\{\mu\}} \int_{-l/L(t)}^{l/L(t)} d\xi \int_{-l/L(t)}^{l/L(t)} d\eta \exp[-i((P_\mu - P_\lambda) + L(t)p]$$

$$+ F(t)L^2(t)\eta |F(\{\lambda\}, \{\mu\})|^2 \frac{|F(\{\lambda\}, \{\mu\})|^2}{||\lambda||^2||\mu||^2}. \quad (F.4)$$

The $\xi$-integration can be done easily, while after the $\eta$-integration we obtain

$$n(p,t) = \frac{2l}{F(t)L(t)} \sum_{\{\mu\}} \sum_{\sigma = \pm} (\sigma \text{Si}(x_\sigma)) \frac{|F(\{\lambda\}, \{\mu\})|^2}{||\lambda||^2||\mu||^2}. \quad (F.5)$$

where Si(z) is a sine-integral and $x_\sigma = (P_\mu - P_\lambda)/L(t) + lp + \sigma F(t)l^2$. The integrand is essentially proportional to $F^{-1}(t) \sin[(P_\mu - P_\lambda + pL(t))/L(t)]/[((P_\mu - P_\lambda + pL(t))/L(t)]$ and gives the main contribution to the sum when the momentum transfer is equal to $pL(t)$.

**F.3. Evaluation of the momentum distribution function in 2D for the uniform system**

Here, we evaluate the momentum distribution function for the 2D Bose gas below the BKT transition. We consider a system with time-dependent parameters and assume the validity of the long-wavelength approximation.

According to (D.3), the momentum distribution in 2D is given by

$$n(p,t) = L^2(t) \int_{|\mathbf{r}| \leq L(t)} d\mathbf{r} \int_{|\mathbf{r}'| \leq L(t)} d\mathbf{r}' g_1^{(\Phi)}(\mathbf{r}, \mathbf{r}';0)$$

$$\times \exp[-iF(t)L^2(t)(\mathbf{r}^2 - \mathbf{r}'^2) - iL(t)p \cdot (\mathbf{r} - \mathbf{r}')], \quad (F.6)$$

where the integration is restricted to a finite surface of the order of $(2l/L(t))^2$. We choose the density matrix in the scaling form corresponding to temperatures below the BKT transition,

$$g_1^{(\Phi)}(\mathbf{r}, \mathbf{r}';0) = \rho_0 \left( \frac{\xi_T}{|\mathbf{r} - \mathbf{r}'|} \right)^\eta, \quad \eta, (F.7)$$

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
where \( \rho_0 \) is the density and \( \eta = m_0 T/(2\pi\hbar^2 \rho_0(T)) \) \((\eta_{\text{BKT}} = 1/4)\). Introducing the center of mass and relative coordinates

\[
x = x_1 - x_2, \quad y = y_1 - y_2, \quad X = \frac{x_1 + x_2}{2}, \quad Y = \frac{y_1 + y_2}{2}
\]

and assuming the integration from \(-l\) to \(l\), we rewrite the momentum distribution as

\[
n(p, t) = L^2(t) \rho_0 \int_{-l/L(t)}^{l/L(t)} dx \int_{-l/L(t)}^{l/L(t)} dy \frac{\exp[i2F(t)L^2(t)(x + y) + iL(t)(p_x x + p_y y)]}{(x^2 + y^2)^{\eta/2}},
\]

\((p = (p_x, p_y))\), which after integration over \(X\) and \(Y\) and changing variables to

\[
x \to \tilde{x} = 2lF(t)L(t)x, \quad y \to \tilde{y} = 2lF(t)L(t)y, \\
A = 2l^2F(t), \quad p_{x,y} \to \tilde{p}_{x,y} = \frac{p_{x,y}}{2F(t)}
\]

takes the form

\[
n(p, t) = \frac{4l^2 \rho_0 \xi_T^\eta}{(2F(t)L(t))^{2-\eta}} \int_{-A}^{A} dx \int_{-A}^{A} dy \frac{\sin(x) \sin(y) \ e^{i\tilde{p}_x x + i\tilde{p}_y y}}{x y [x^2 + y^2]^{\eta/2}} \sum_{\alpha,\beta = \pm} I_{\alpha,\beta},
\]

\[
I_{\alpha,\beta} = \frac{l^2 \rho_0 \xi_T^\eta}{(2F(t)L(t))^{2-\eta}} \int_{-A}^{A} dx \int_{-A}^{A} dy \frac{\sin(x(1 + \alpha \tilde{p}_x)) \sin(y(1 + \beta \tilde{p}_y))}{x y [x^2 + y^2]^{\eta/2}}.
\]

Now, using the integral \( \int_0^\infty e^{-px^\eta} = p^{-1/\mu} \Gamma(1 + \frac{1}{\mu}) \), we rewrite

\[
\frac{1}{[x^2 + y^2]^{\eta/2}} = \frac{1}{\Gamma(1 + \eta/2)} \int_0^\infty e^{-(x^2 + y^2)^{\eta/2}} \, dt
\]

and substitute it back into equation (F.11). Then the \(x\) and \(y\) integrals are separated and can be performed using

\[
\int_{-\infty}^{\infty} e^{-x^2 t^{2/\eta}} \frac{\sin(Cx)}{x} \, dx = \pi \operatorname{erf}\left(\frac{|B|}{2t^{1/\eta}}\right) \operatorname{sign}(C),
\]

where we assume that the integration region can be effectively extended to infinity. This, in particular, is justified for large times when \(F(t)\) is a growing function of time or for large \(l\) for arbitrary time. We therefore end up with the following integral,

\[
n(p, t) = \frac{\pi^2 l^2 \rho_0 \xi_T^\eta}{(2F(t)L(t))^{2-\eta} \Gamma(1 + \frac{1}{2})} \sum_{\alpha,\beta = \pm} \operatorname{sign}(1 + \alpha \tilde{p}_x) \operatorname{sign}(1 + \beta \tilde{p}_y)
\]

\[
\times \int_0^\infty \operatorname{erf}(|1 + \alpha \tilde{p}_x|/2t^{1/\eta}) \operatorname{erf}(|1 + \beta \tilde{p}_y|/2t^{1/\eta}) \, dt,
\]

which after a change of variables is transformed into

\[
n(p, t) = \sum_{\alpha,\beta = \pm} N_{\alpha,\beta} \int_0^\infty \frac{\operatorname{erf}(a_u |u|) \operatorname{erf}(b_u |u|)}{u^{\eta+1}} \, du,
\]
where
\[ N_{\alpha,\beta} = \frac{\pi^2 l^2 \rho_0 \xi^\eta_t (-\eta) \text{sign}(a_{\alpha}) \text{sign}(b_{\beta})}{(2l F(t) L(t))^{2-\eta} \Gamma(1 + \frac{\eta}{2}) 2^\eta}, \quad (F.16) \]
\[ a_{\alpha} = 1 + \alpha \tilde{p}_x, \quad b_{\beta} = 1 + \beta \tilde{p}_y. \quad (F.17) \]

The last integral is equal to
\[ \tilde{I}(a_{\alpha}, b_{\beta}) = \frac{i}{2\pi} \left( |a_{\alpha}|^\eta B \left( -\frac{b_{\beta}^2}{a_{\alpha}^2}, 1, \frac{\eta}{2} \right) - i^\eta |b_{\beta}|^\eta B \left( -\frac{b_{\beta}^2}{a_{\alpha}^2}, 1 - \eta, \frac{\eta}{2} \right) \right) + \frac{|b_{\beta}|^\eta \sqrt{\pi} \sec\left( \frac{\pi \eta}{2} \right) \Gamma(1 + \frac{\eta}{2})}{\eta \Gamma(\frac{1+\eta}{2})}, \quad (F.18) \]

where \( B(., .) \) is the Euler beta-function. So, finally we obtain
\[ n(p, t) = \frac{\pi^2 l^2 \rho_0 \xi^\eta_t (-\eta)}{(2l F(t) L(t))^{2-\eta} \Gamma(1 + \frac{\eta}{2}) 2^\eta} \sum_{\alpha, \beta = \pm} \text{sign}(1 + \alpha \tilde{p}_x) \text{sign}(1 + \beta \tilde{p}_y) \tilde{I}(a_{\alpha}, b_{\beta}), \quad (F.19) \]

where \( \tilde{I}(a_{\alpha}, b_{\beta}) \) is given in equation \((F.18)\), \( a_{\alpha} \equiv 1 + \alpha p_x/(2l F(t)) \) and \( b_{\beta} \equiv 1 + \beta p_y/(2l F(t)) \). We also introduced \( \tilde{p}_{x,y} = p_{x,y}/2l F(t) \).

In figure F.2, we plot the asymptotic behavior of the momentum distributions for various values of \( \eta \). Similar to the 1D case, we find a step-like distribution that is smeared off when the BKT transition is approached.

References

[1] Giamarchi T 2003 Quantum Physics in One Dimension (Oxford: Clarendon)
[2] Rigol M, Dunjko V and Olshanii M 2008 Thermalization and its mechanism for generic isolated quantum systems Nature 452 854
[3] Barankov R A and Levitov L S 2006 Synchronization in the BCS pairing dynamics as a critical phenomenon Phys. Rev. Lett. 96 230403

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
[4] Yuzbashyan E A, Kuznetsov V B and Altshuler B L 2005 Integrable dynamics of coupled Fermi–Bose condensates Phys. Rev. B 72 144524
Yuzbashyan E A, Altshuler B L, Kuznetsov V B and Enolskii V Z 2005 Solution for the dynamics of the BCS and central spin problems J. Phys. A: Math. Gen. 38 7831
Yuzbashyan E A, Altshuler B L, Kuznetsov V B and Enolskii V Z 2005 Nonequilibrium Cooper pairing in the nonadiabatic regime Phys. Rev. B 72 220503

[5] Faribault A, Calabrese P and Caux J-S 2009 Quantum quenches from integrability: the fermionic pairing model J. Stat. Mech. P03018
Faribault A, Calabrese P and Caux J-S 2009 Bethe ansatz approach to quench dynamics in the Richardson model J. Math. Phys. 50 095212

[6] Calabrese P and Cardy J 2006 Time-dependence of correlation functions following a quantum quench Phys. Rev. Lett. 96 136801
Calabrese P and Cardy J 2007 Quantum quenches in extended systems J. Stat. Mech. P06008

[7] Kollath C, Laeuchli A and Altman E 2007 Quench dynamics and nonequilibrium phase diagram of the Bose–Hubbard model Phys. Rev. Lett. 98 180601

[8] Bistritzer R and Altman E 2007 Intrinsic dephasing in one dimensional ultracold atom interferometers Proc. Natl Acad. Sci. USA 104 9955

[9] Cazalilla M A 2006 Effect of suddenly turning on interactions in the Luttinger Model Phys. Rev. Lett. 97 156403

Iucci A and Cazalilla M A 2009 Quantum quench dynamics of some exactly solvable models in one dimension Phys. Rev. A 80 063619

[10] Daley A J, Kollath C, Schollwoeck U and Vidal G 2004 Time-dependent density-matrix renormalization-group using adaptive effective Hilbert spaces J. Stat. Mech. P04005

[11] Kollath C, Iucci A, Giamarchi T, Hofstetter W and Schollwoeck U 2006 Spectroscopy of ultracold atoms by periodic lattice modulations, Phys. Rev. Lett. 97 050402

[12] Schollwoeck U 2005 Time-dependent density-matrix renormalization-group methods J. Phys. Soc. Japan 74 (Suppl.) 246
Schollwoeck U 2005 The density-matrix renormalization group Rev. Mod. Phys. 77 259

[13] Altland A, Guraie V, Kriecherbauer T and Polkovnikov A 2009 Non-adiabacity and large fluctuations in a many particle Landau–Zener problem Phys. Rev. A 79 042703
Itin A P and Törnä P 2009 Dynamics of a many-particle Landau–Zener model: inverse sweep Phys. Rev. A 79 055602

[14] Barmettler P, Punk M, Gritsev V, Demler E and Altman E 2009 Relaxation of antiferromagnetic order in spin-1/2 chains following a quantum quench Phys. Rev. Lett. 102 130603
Barmettler P, Punk M, Gritsev V, Demler E and Altman E 2010 Quantum quenches in the anisotropic spin-1/2 Heisenberg chain: different approaches to many-body dynamics far from equilibrium New J. Phys. 12 055017

[15] Heidrich-Meisner F et al 2008 Ground-state reference systems for expanding correlated fermions in one dimension Phys. Rev. A 78 013620

[16] Lewis H R 1968 Class of exact invariants for classical and quantum time-dependent harmonic oscillators J. Math. Phys. 9 1976
Lewis H R and Riesenfeld W B 1969 An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field J. Math. Phys. 10 1458

[17] Popov V S and Perelomov A M 1969 J. Exp. Theor. Phys. 29 738
Popov V S and Perelomov A M 1970 J. Exp. Theor. Phys. 30 910
Perelomov A M and Popov V S 1969 Group-theoretical aspects of the variable frequency oscillator problem Theor. Math. Phys. 1 275
Perelomov A M 1986 Generalized Coherent States and Their Applications (New York: Springer)

[18] Malkin I A, Man’ko V I and Trifonov D A 1970 Coherent states and transition probabilities in a time-dependent electromagnetic field Phys. Rev. D 2 1371

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
[19] Selezneva A N 1995 Unitary transformations for the time-dependent quantum oscillator Phys. Rev. A 51 950
[20] Schuch D 2008 Riccati and Ermakov equations in time-dependent and time-independent quantum systems SIGMA 4 43
Cariñena J F, De Lucas J and Rañada M F 2008 Recent applications of the theory of Lie systems in Ermakov systems SIGMA 4 31
[21] Berry M G and Klein G 1984 Newtonian trajectories and quantum waves in expanding force fields J. Phys. A: Math. Gen. 17 1805
[22] Sutherland B 1998 Exact coherent states of a one-dimensional quantum fluid in a time-dependent trapping potential Phys. Rev. Lett. 80 3678
[23] Castin Y and Dum R 1996 Bose–Einstein Condensates in time dependent traps Phys. Rev. Lett. 77 5315
[24] Kagan Yu, Surkov E L and Shlyapnikov G V 1996 Evolution of a Bose-condensed gas under variations of the confining potential Phys. Rev. A 54 R1753
[25] Werner F and Castin Y 2006 Unitary gas in an isotropic trap: Symmetry properties and applications Phys. Rev. A 74 053604
[26] Pitaevskii L P and Rosch A 1997 Breathing modes and hidden symmetry of trapped atoms in two dimensions Phys. Rev. A 55 R853
[27] Son D T and Wingate M 2006 General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas Ann. Phys. 321 197
[28] Ghosh P 2001 Conformal symmetry and the nonlinear Schrödinger equation Phys. Rev. A 65 012103
[29] Garcia-Ripoll J J, Perez-Garcia V M and Torres P 1999 Extended parametric resonances in nonlinear Schrödinger systems Phys. Rev. Lett. 83 1715
Pérez-Garcia V M, Torres P J and Montesinos G D 2007 SIAM J. Appl. Math. 67 990
[30] Minguzzi A and Gangardt D M 2005 Exact coherent states of a harmonically confined Tonks–Girardeau gas Phys. Rev. Lett. 94 240404
[31] Bloch I, Dalibard J and Zwerger W 2008 Many-body physics with ultracold gases Rev. Mod. Phys. 80 885
[32] Fleischhauer M, Imamoglu A and Marangos J P 2005 Electromagnetically induced transparency: optics in coherent media Rev. Mod. Phys. 77 633
[33] Hartmann M J, Brandao F G S L and Plenio M B 2006 Strongly interacting polaritons in coupled arrays of cavities Nat. Phys. 2 849
Fleischhauer M, Otterbach J and Unanyan R G 2008 Bose–Einstein condensation of stationary-light polaritons Phys. Rev. Lett. 101 163601
[34] Shen J-T and Fan S 2007 Strongly correlated multiparticle transport in one dimension through a quantum impurity Phys. Rev. A 76 062709
[35] Chang D E, Gritsev V, Morigi G, Vuletic V, Lukin M and Demler E 2008 Crystallization of strongly interacting photons in a nonlinear optical fiber Nat. Phys. 4 884
[36] Meystre P and Sargent M 1999 Elements of Quantum Optics (New York: Springer)
[37] Lobo C and Gensemer S D 2008 Technique for measuring correlation functions in interacting gases Phys. Rev. A 78 023618
[38] Leach P G L and Andriotopoulos K 2008 The Ermakov equation: a commentary Appl. Anal. Discrete Math. 2 146
[39] Ermakov V P 1880 Transformation of differential equations Univ. Izv. Kiev. 20 1–19
[40] Petrov D S, Gangardt D M and Shlyapnikov G V 2004 Low-dimensional trapped gases J. Physique IV 116 3
[41] Xia X and Silbey R J 2005 Effective Lagrangian approach to the trapped Bose gases at low temperatures Phys. Rev. A 71 063604
[42] Bogoliubov N M, Malyshchev C, Bullough R K and Timonen J 2004 Finite-temperature correlations in the one-dimensional trapped and untrapped Bose gases Phys. Rev. A 69 023619
[43] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas. I. The general solution and the ground state Phys. Rev. 130 1605

New Journal of Physics 12 (2010) 113005 (http://www.njp.org/)
[44] Gerbier F et al 2008 Expansion of a quantum gas released from an optical lattice Phys. Rev. Lett. 101 155303
[45] Son D T 2008 Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry Phys. Rev. D 78 046003
[46] Balasubramanian K and McGreevy J 2008 Gravity duals for nonrelativistic conformal field theories Phys. Rev. Lett. 101 061601
[47] Kachru S, Liu X and Mulligan M 2008 Gravity duals of Lifshitz-like fixed points Phys. Rev. D 78 106005
[48] Adams A, Balasubramanian K and McGreevy J 2008 Hot spacetimes for cold atoms J. High Energy Phys. JHEP11(2008)059 (arXiv:0807.1111)
[49] Goldberger W D 2008 AdS/CFT duality for non-relativistic field theory arXiv:0806.2867
[50] Haller E, Gustavsson M, Mark M J, Danzl J G, Hart R, Pupillo G and Nägerl H-C 2009 Realization of an excited, strongly correlated quantum gas phase Science 325 1224
[51] Polyanin A D and Zaitsev V F 2003 Handbook of Exact Solutions for Ordinary Differential Equations 2nd edn (Boca Raton, FL: Chapman and Hall/CRC)
[52] Kundu A 2008 Integrable inhomogeneous NLS equations are equivalent to the standard NLS arXiv:0809.1924
[53] Ramesh Kumar V, Radha R and Panigrahi P K 2008 Dynamics of Bose–Einstein condensates in a time-dependent trap Phys. Rev. A 77 023611
[54] Kundu A 2007 Integrable inhomogeneous NLS equations are equivalent to the standard NLS Phys. Rev. Lett. 99 154101
[55] Calabrese P, Caux J-S and Slavnov N 2007 One-particle dynamical correlations in the one-dimensional Bose gas J. Stat. Mech. P01008