Superconducting Qubit Measurement and Information Conversion from Quantum to Classical

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Abstract. The Josephson bifurcation amplifier (JBA) readout process of a superconducting qubit is analyzed quantum mechanically. We examined the dynamics of the density operator of a detector (a driven nonlinear oscillator) and a qubit coupled system during the measurement process. We have observed the qubit-detector (JBA) entangled state and it is divided into two separable states at the moment the JBA transition begins. Moreover, we discuss the process from the viewpoint of time variations in information quantities, such as mutual information between the qubit and detector. We show that the measurement process corresponds to an information conversion from quantum to classical one.

1. Introduction
A qubit is the most elementary unit of quantum information processing [1]. Qubit readout is one of simple quantum measurements. According to a simple explanation of quantum measurements, it is described as if we can directly obtain an observable eigenvalue of the quantum system by a measurement. In a real measurement, however, we must always use a detector that interacts with the quantum system. A detector is often a macroscopic object. We actually measure a macroscopic quantity in the detector, and merely postulate the physical quantity of the quantum system we want to measure. To discuss what we really measure when we measure a qubit, we have to take account of the detector system. A measurement process is a quantum dynamics of the total system consisting of a qubit and a detector.

A good qubit readout gives a probabilistic projection of a qubit state $|\psi_q(0)\rangle = a |\uparrow\rangle + b |\downarrow\rangle$ into $|\uparrow\rangle$ or $|\downarrow\rangle$ with a probability $|a|^2$ or $|b|^2$, respectively. Here, $|\uparrow\rangle$ and $|\downarrow\rangle$ are two of the eigenstates of the observable in the qubit. The observable is mainly determined by the operator of the qubit appearing in the interaction Hamiltonian between the qubit and the detector. However, the observable is sometimes different from the operator because it is determined by the entire dynamics of the total system during the readout process.

Below, we discuss the readout process for a superconducting qubit. We use a Josephson Bifurcation Amplifier (JBA) as the detector in the measurement [2]. The JBA is an AC-driven SQUID that behaves as a nonlinear oscillator. We find many important quantum characters by investigating the dynamics during the readout process [3].

Usually, the total system is expected to evolve as follows, when the qubit is in a superposition of two of observable eigenstates $|\psi(0)_{q}\rangle = a |\uparrow\rangle + b |\downarrow\rangle$. When the qubit-detector interaction becomes on, an entanglement starts to be formed: $|\psi(0)_{q}\rangle \otimes |C_0\rangle \rightarrow a |\uparrow\rangle \otimes |A_1\rangle + b |\downarrow\rangle \otimes |B_1\rangle$. Since the detector is a macroscopic system where a finite decoherence is inevitable, dephasing occurs.
Then, the total system becomes a classical mixture like $|a|^2|↑⟩|A_1⟩ + |b|^2|↓⟩|B_1⟩$. Here, $|A_1⟩$, $|B_1⟩$ are macroscopic states of the detector, and we can easily distinguish $|A_1⟩$ or $|B_1⟩$. Therefore, when we measure the detector state, we can postulate whether the qubit state is $|↑⟩$ or $|↓⟩$.

2. Superconducting Qubit and Measurement with a DC-SQUID: Setup

There are three types of superconducting qubits: charge, phase, and flux qubits [4, 5, 6, 7, 8, 9, 10]. A superconducting flux qubit is a superconducting ring interrupted by a few Josephson junctions [10, 11, 12, 13, 14, 15]. When the external magnetic field piercing the ring is in the vicinity of the half flux quantum $\Phi_0/2$, the ring becomes a superposition of two macroscopically distinct states. Supercurrent flows clockwise around the ring in one state $|R⟩$, and it flows counterclockwise in the other state $|L⟩$. The qubit Hamiltonian is given by [16, 17, 18]

$$H_q = \frac{1}{2}(\sigma_x + \Delta \sigma_x),$$

where $\sigma_x$ and $\sigma_z$ are two Pauli matrices, and $\sigma_z = |L⟩⟨L| - |R⟩⟨R|$. We can control $\varepsilon$ by changing the externally applied magnetic flux, but $\Delta$ is fixed. The energy-eigenstates of the qubit are given by $|g⟩ = \cos(\theta/2)|L⟩ + \sin(\theta/2)|R⟩$, and $|e⟩ = \sin(\theta/2)|L⟩ - \cos(\theta/2)|R⟩$, where $\tan \theta = \Delta/\varepsilon$.

In order to readout superconducting flux qubits, a Superconducting Quantum Interference Device (SQUID) is used as the detector. The SQUID detects a very small magnetic flux induced by supercurrent flowing along the qubit ring, which is the only observable in a flux qubit we can measure from the outside (see, Fig. 1). A SQUID can be approximated as a single Josephson junction whose Josephson energy depends on the magnetic flux $\Phi_{SQ}$ piercing the SQUID ring. The Hamiltonian for the SQUID phase $\gamma$ (the Josephson phase of the effective single junction) is given by [19],

$$\frac{1}{2h}\frac{\varepsilon^2}{\gamma^2} = -\bar{E}_J(\Phi_{SQ}) \cos \gamma - \frac{\varepsilon^2}{2}\gamma^2/2 + \gamma^4/24 - \cdots - \frac{\varepsilon^2}{2}\gamma^2/2,$$

where $\gamma$ is the Josephson phase, $\varepsilon$ is the effective capacitance of the junction and $I_b$ is the bias current applied to the SQUID.

A half decade ago, our experimental results, where raw data before averaging behaved as in the right panel of Fig. 2, clearly showed that our flux-qubit measurement with a DC-SQUID is not a $\sigma_z$ projection measurement [20, 21]. Moreover, we theoretically clarified that the raw data behave in the qubit measurement with a SQUID when we increase the SQUID bias current slowly, from a detailed calculation of the dynamics of the qubit-SQUID coupled system [19]. The measurement projects the coupled system to one of its energy-eigenstates. In the state, the...
qubit is approximately in an energy-eigenstate of $H_q$, and the value (signal) obtained from a single measurement approximately corresponds to $\langle g | \sigma_z | g \rangle$ or $\langle e | \sigma_z | e \rangle$.

3. Quantum dynamics of a Josephson Bifurcation Amplifier

When the amplitude of $\gamma$ is large, the oscillator becomes bistable under appropriate operation parameters [22]. One stable state has a small amplitude (low-amplitude state), and the other has a larger amplitude (high-amplitude state).

The critical driving force $f_c$ or the critical detuning $\delta_c$ for the transition between these two states is very sensitive to small changes in the operational parameters of the oscillator. For example, when we increase or decrease the driving force continuously, the amplitude of the oscillation behaves hysteretically as shown in Fig. 3. When using a JBA as a qubit state readout detector, the JBA detects a small change that depends on the qubit state. The bifurcation phenomenon can be discussed only for classical oscillators, and is impossible from the viewpoint of pure quantum mechanics for an isolated system [23].

![Figure 3. Hysteretic behavior of the oscillation amplitude $a(t)$ of a classical JBA. When the driving force $f(t)$ (broken line) is changed with time $t$, the amplitude (thick line) varies as shown (schematic).](image)

![Figure 4. Two macroscopic states of JBA (quasi-distribution representation): $p, x$ are the momentum and amplitude of the oscillation, respectively.](image)

A quantum mechanical analysis is indispensable if we are to understand the readout process [3, 24]. A JBA can be modeled as an anharmonic oscillator in a rotating frame approximation with a Hamiltonian:

$$H_J = (\Omega - \omega)n_a - \alpha a^2 - \frac{1}{2}f(a^\dagger + a)$$

where, $a^\dagger(a)$ is the creation (annihilation) operator of the Josephson plasma oscillation. The normalized amplitude and momentum of an oscillation are expressed by $x = (a^\dagger + a)/2$ and $p = (a - a^\dagger)/(2i)$, respectively. $n_a = a^\dagger a$, and $\Omega$ is the linear resonant frequency of the JBA oscillator. $\omega$ is the driving frequency, which is slightly smaller than $\Omega$ by the detuning $\delta \equiv \Omega - \omega$. $f$ is the driving strength, and $\alpha (> 0)$ is the nonlinearity. In a classical approximation, this model shows the bifurcation in an appropriate parameter region. However, for a quantum-mechanical junction, the transition from $|G\rangle_J$ (low-amplitude state) to $|E\rangle_J$ (high-amplitude state) or, from $|E\rangle_J$ to $|G\rangle_J$ is impossible.

As an example, here, we introduce linear loss in the oscillator (JBA). The time evolution of the JBA is governed by a Liouville equation [25]:

$$\frac{d\rho}{dt} = \frac{1}{i}[H_J, \rho] + \frac{\Gamma}{2}(2\alpha a^\dagger - a^\dagger a + \rho a a^\dagger)$$

where $\rho$ is the density operator of the system, and $\Gamma$ is the relaxation rate due to the linear loss in the JBA.
We carried out a numerical calculations [3]. We set $\delta = 0.007\Omega$, nonlinearity $\alpha = 8 \times 10^{-5}\Omega$, quality factor $Q = 2500$, and the driving force $f$ is operated as $0 \rightarrow 0.025\Omega \rightarrow 0$. These parameters are similar to those used in actual experiments [2]. To order to emphasize quantumness, however, $\delta$ and $\Gamma$ are a factor of $10^{-2}$ times smaller than in real cases. More than one hundred basis states are prepared in order to avoid numerical artifacts. Then we can reproduce hysteretic bifurcation phenomena of our JBA. We found that the purity $\text{Tr}[\rho^2]$ decreases abruptly. This corresponds to the beginning of the transition from $|G\rangle_J$ to $|E\rangle_J$ of the JBA.

The qubit-JBA composite system is approximately expressed by the Hamiltonian

$$H = H_J + k\sigma_z n_a + H_q, \quad H_q = \frac{1}{2}(\epsilon\sigma_z + \Delta\sigma_x)$$

$k$ is the interaction constant between the qubit and the JBA.

4. Quantum Correlation and Quantum Discord
To prepare later discussions, we introduce mutual information in classical information theory and the quantum counterpart.

When there are two information sources A and B, the correlation between A and B is quantified by the mutual information defined by

$$I_C(A, B) = H(\rho_A) + H(\rho_B) - H(\rho_{AB}),$$

where $\rho_A$ and $\rho_B$ are the probability sets for A, B, respectively. $\rho_{AB}$ is the joint probability set of A and B. $H(\rho)$ is the information (Shannon) entropy for $\rho$. This definition implicitly contains the conditional entropy $K_C(\rho_{B|A}) = \sum_j p_j H(\rho_{B|A=j}) = H(\rho_{AB}) - H(\rho_A)$. The Bayes rule makes the equality satisfied.

On the other hand, the quantum counterparts are given as follows. The density operator of a quantum system is defined by $\rho = T^{\dagger}\left(\begin{array}{cccc} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_L \end{array}\right)T$ is defined, where $T$ is a unitary matrix that diagonalizes $\rho$.

For quantum information, we use von Neumann entropy, $S(\rho) = -\text{Tr}[\rho \log_2 \rho] = -\sum_{l=1}^L p_l \log_2 p_l$, instead of Shannon entropy. The quantum mutual information $I_Q$ of a two-body (A and B) system is defined by [26]

$$I_Q(A, B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

This agrees with the classical mutual information $I_C$ if there is only a classical correlation between A and B.

$I_Q$ is different from $I_C$ because the Bayes rule is not always valid for quantum events. For quantum events, the conditional probability should be expressed by [27] $K(\rho_{B|A}) = \min \sum_{\Pi^A_j} p_j S(\rho_{B|\Pi^A_j})$, where $\Pi^A_j$ is an operator that projects the state of A into $j$ and $p_j = \text{Tr}_B[\Pi^A_j \otimes I^B \rho_{AB}]$, $\rho_{B|\Pi^A_j} = \text{Tr}_A[\Pi^A_j \otimes I^B \rho_{AB} \Pi^A_j \otimes I^B]/p_j$, etc.. Then, we obtain, $K(\rho_{B|A}) \geq S(\rho_{AB}) - S(\rho_A)$ and the equality is not always satisfied. Using the conditional entropy $K$, we can define another type of mutual information, $J_Q(A, B) = S(\rho_B) - K(\rho_{B|A})$, which agrees with
5. Qubit Readout with a JBA

The qubit readout process is well understood by employing knowledge of the quantum behavior in the time evolution of the JBA discussed above.

In Fig. 5, we show the dynamics during the qubit readout process ($H_J$ in Eq. (2) replaced with the Hamiltonian $H$ (Eq. (1)) of the qubit-JBA composite system). The JBA parameters are the same as for the above example. The initial state is a separable state; $(\frac{1}{\sqrt{2}}|g\rangle_a + \frac{1}{\sqrt{2}}|e\rangle_q) \otimes |G\rangle_J$. Here, $|g\rangle_a$ and $|e\rangle_q$ are the ground and excited states of the qubit, respectively. The qubit parameters are $\varepsilon = 0.2\Omega$, $\Delta/\varepsilon = 1/2$. The coupling between the qubit and the JBA is set at $k = 0.001\Omega$. The driving force $f$ is increased from 0 to 0.012$\Omega$ (slightly larger than the $f_c$ of the JBA) and maintained. This parameter set provides a typical behavior for a successful qubit readout.

The quasi-distribution representations of the JBA state $\text{Tr}_q[\rho]$ are shown in Fig. 5, where $\rho$ is the density operator of the qubit-JBA coupled system, and $\text{Tr}_q[\cdots]$ denotes taking partial trace about qubit degrees of freedom. These peaks constitute an incoherent mixture, so they correspond to two possibilities in the measurement result.

This process can be expressed schematically as

$$\rho(0) = |G\rangle_J|G\rangle \otimes \left( \frac{1}{\sqrt{2}}|g\rangle_a + \frac{1}{\sqrt{2}}|e\rangle_q \right) \left( \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|e\rangle \right)$$

$\rightarrow$ $\rho(\tau_1) = \frac{1}{2}(|G\rangle_J|g\rangle_a + |G\rangle_J|g\rangle_q) (|G\rangle_J|e\rangle_a) + |G\rangle_J|e\rangle_q)$

$\rightarrow$ $\rho(\tau_2) = \frac{1}{2}(|G\rangle_J|e\rangle_a) (|G\rangle_J|e\rangle_q) + \frac{1}{2}(|G\rangle_J|g\rangle_a) (|G\rangle_J|g\rangle_q)$

$\rightarrow$ $\rho(\tau_3) = \frac{1}{2}(|G\rangle_J|e\rangle_a) (|G\rangle_J|e\rangle_q) + \frac{1}{2}(|G\rangle_J|g\rangle_a) (|G\rangle_J|g\rangle_q)$. (4)
Figure 5. Time evolution of JBA during readout. The figures show quasi-distribution representations \( \langle \alpha | \text{Tr}_q [\rho] | \alpha \rangle \) of the JBA oscillator states, where \( |\alpha\rangle \) is the coherent state of a complex amplitude \( \alpha = x + ip \) (in the rotating frame). (a) Beginning of the readout. The state of the total system is (schematically) \( \rho = |\psi_0\rangle \langle \psi_0| \), with \( |\psi_0\rangle = 1/\sqrt{2} (|g\rangle_q + |e\rangle_q) \otimes |G\rangle_J \). (b) Starting the transition. Entanglement formation and projection are carried out during this period. (c) During the transition. Entanglement has already been destroyed. (d) The entire system has become a mixture of classically correlated states.

Figure 6. Time evolution of information quantities during the qubit measurement process. \( I_Q \): Quantum mutual information. \( J_Q \): Classical mutual information. \( C \): Concurrence. \( D \): Quantum discord

In Fig. 6, we show the time variations of some of the information quantities introduced in the previous section. The horizontal line corresponds to the time from the onset of the driving, normalized by the linear resonant frequency \( \Omega \), the vertical line is in bit or qubit units. The concurrence \( C(\rho_{AB}) \) grows as the driving strength increases. \( I_Q \) and \( J_Q \) behave in the same manner. \( J_Q \) is kept at almost unity after the concurrence \( C \) vanishes. If we measure the JBA state classically within this time span, we can obtain maximum information about the qubit state. After \( \tau_2 \), the quantum discord \( (I_Q - J_Q) \) disappears. That is, after \( \tau_2 \), a measurement on A or B never destroys the correlation between A and B.

Anyway, this time evolution behavior in information quantities shows that the correlation initially invoked as a quantum correlation is converted into a classical correlation during the process. Then, then we can obtain information about the projected qubit via a classical measurement on the JBA. Moreover, at time \( t = \tau_3 \), the required measurement merely involves distinguishing two peaks in Fig. 5\( (t = \tau_3) \). This is a very easy classical measurement. This situation is established via the separation of the two peaks during the time from \( \tau_2 \) to \( \tau_3 \). The entanglement has already been destroyed at \( \tau_2 \). Therefore, this time variation corresponds to classical information amplification.

6. Conclusion
We analyzed the dynamics of a composite system consisting of a superconducting qubit and a JBA in order to understand the quantum characters in qubit readout process with a JBA, and clarified what happens during the process and how we obtain the qubit information by performing measurements with the detector (JBA). Moreover, we discussed the measurement
process from the viewpoint of time variations in such information quantities as quantum and classical mutual information and entanglement concurrence. We clarified that entanglement formation corresponds to the generation of a quantum correlation between the qubit and the detector and the destruction of the entanglement caused by decoherence constitutes the information conversion from quantum to classical.

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[1] Nielsen M and Chuang I 2000 Quantum Computation and Quantum Information, (Cambridge: Cambridge Univ. Press)
[2] Siddiqi I et al. 2004 Phys. Rev. Lett. 93 207002; Siddiqi I et al. 2005 ibid. 94 027005; Lupascu A et al. 2006 ibid. 96 127003; Siddiqi I et al. 2006 Phys. Rev. B 73 054510; Lupascu A et al. 2007 Nature Physics 3 119; Boulant N et al. 2007 Phys. Rev. B 76 014525
[3] Nakano H, Saito S, Sembra K, Takayanagi H 2009 Phys. Rev. Lett. 102 257003
[4] Devoret M and Martinis J 2004 Quantum Information Processing 3 163, and references therein
[5] Friedman J, Patel V, Chen W, Tolpygo S, and Lukens J 2000, Nature 406 43
[6] Vion D, et al. 2002 Science 296, 886
[7] Yu Y, Han S, Chu X, Chu S, and Wang Z 2002 Science 296 889
[8] Nakamura Y, Pashkin Yu, and Tsai J-S 1999 Nature 398, 786
[9] Martinis J, Nam S, Aumentado J, and Urbina C 2002 Phys. Rev. Lett. 89 117901
[10] Makhlin Y, Shnirman A, and Schon G 2001 Phys. Rev. Lett. 85 406
[11] Tian L , Lloyd S, and Orlando T 2002 Phys. Rev. B 65 144516
[12] van der Wal C et al. 2000 Science 290 773; van der Wal C 2001 Quantum superpositions of persistent Josephson currents Ph.D. thesis, (Delft: Delft Univ. Press)
[13] Chiorescu I, Nakamura Y, Harmans C, and Mooij J 2004 Nature 431 159
[14] Chiorescu I, Bertet P, Sembra K, Nakamura Y, Harmans C, and Mooij J 2003 Science 299 1869
[15] Bertet P et al. 2004 cond-mat/0412485.
[16] Crankshaw D and Orlando T 2001IEEE Trans. Appl. Superconductivity 11 1006; You J, Nakamura Y, and Nori F 2003 cond- mat/0309491; van Brink A cond-mat/0310425
[17] van der Wal C, Wilhelm F, Harmans C, Mooij E 2003Euro. Phys. J. B 31 111 (2003)
[18] Orlando T, et al. 1999 Phys. Rev. B 60 15398
[19] Nakano H and Takayanagi H 2003 J. Phys. Soc. Jpn. 72 Suppl. A 1; Nakano H et al. 2004 arXiv:cond-mat/0406622; Nakano H and Takayanagi H 2003, in Toward the Controllable Quantum States (Singapore: World Scientific) p 359
[20] For review, Sembra K, Johansson K, Kakuyanagi K, Nakano H, Saito S, Tanaka H and Takayanagi H 2009 Quantum Information Processing: 8, Issue 2 199.
[21] Tanaka H, Sekine Y, Saito S, Takayanagi H 2002 Physica C 368 300; Takayanagi H, Tanaka H, Saito S, and Nakano H 2002 Physica Scripta T102 95; Tanaka H, Saito S and Takayanagi H 2003 Toward the Controllable Quantum States (Singapore: World Scientific) p 366
[22] Jordan W and Smith P 1999 Nonlinear Ordinary Differential Equations third edition (Oxford: Oxford Univ. Press)
[23] Dykman M and Fistul M 2005 Phys. Rev. B 71 140508 (R); Dykman M 2007Phys. Rev. E 75 011101; Peano V and Thorwart M 2006 Chem. Phys. 322 135; Peano V and Thorwart M 2006 New J. Phys. 8 21
[24] Makhlkin Y, Shnirman A, and Schon G 2001 Rev. Mod. Phys. 73 357
[25] Lindblad G 1976 Commun. Math. Phys. 48 119
[26] Schumacher B and Westmoreland 2010 Quantum Processes, Systems & Information (Cambridge: Cambridge University Press)
[27] Ollivier H and Zurek W 2001 Phys. Rev. Lett. 88 017901
[28] Uhrmann A Phys. Rev. A 62 32307
[29] Horodecki R, Horodecki P, Horodecky M, and Horodecki K 2009 Rev. Mod. Phys. 81 865
[30] Chen K, Alberverio S, and Fei S-M et al. 2005 Phys. Rev. Lett. 95 40504; Chen K, Alberverio S, and Fei S-M et al. 2005 Phys. Rev. Lett. 95 210501