THE SELF-LINKING NUMBER OF A CLOSED CURVE IN $\mathbb{R}^n$

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Abstract. We introduce the self-linking number of a smooth closed curve $\alpha : S^1 \to \mathbb{R}^n$ with respect to a 3-dimensional vector bundle over the curve, provided that some regularity conditions are satisfied. When $n = 3$, this construction gives the classical self-linking number of a closed embedded curve with non-vanishing curvature [5]. We also look at some interesting particular cases, which correspond to the osculating or the orthogonal vector bundle of the curve.

1. Introduction

It is well known that two closed embedded curves $\alpha, \beta : S^1 \to \mathbb{R}^3$ are equivalent as knots if and only if there is a continuous map $H : S^1 \times [0, 1] \to \mathbb{R}^3$ such that for any $u \in [0, 1]$, the curve $H_u : S^1 \to \mathbb{R}^3$ given by $H_u(t) = H(t, u)$ is an embedding and $H_0 = \alpha, H_1 = \beta$ (such a map $H$ is said to be an isotopy between $\alpha, \beta$). For instance, if we look at the two curves shown in Figure 1, it follows that they are equivalent as knots (in fact, they are equivalent to the trivial knot).

![Figure 1](image-url)

However, suppose that we construct these two curves so that they are of class $C^3$ and have non-vanishing curvature at each point. Then, it is not difficult to see that it is not possible to have an isotopy $H$ such that for any $u \in [0, 1]$, $H_u$ has the same property (such a map will be called a non-degenerate isotopy). This is due to the fact that these two curves have different self-linking number. This number was introduced by Călugăreanu [1] and studied with more detail by Pohl [5]. It can be seen as the linking number between the given curve and a curve obtained by slightly pushing the curve along the principal normal. Moreover, it is possible to compute the self-linking number by means of the following integral formula:

$$SL(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\alpha(s) - \alpha(t), \alpha'(s), \alpha'(t))}{\|\alpha(s) - \alpha(t)\|^3} dt \wedge ds + \frac{1}{2\pi} \int_{S^1} \tau dt,$$

where $\tau$ is the torsion of $\alpha$. Recently, Gluck and Pan [2] have shown that there is a non-degenerate isotopy between two embedded closed curves with non-vanishing curvature in $\mathbb{R}^3$ if and only if they have the same knot type and the same self-linking number. Thus,

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the self-linking number is the key invariant if we want to do a curvature sensitive version of the knot theory.

In this paper, we propose a generalization of this invariant for the case of a closed smooth curve \( \alpha : S^1 \rightarrow \mathbb{R}^n \). In our construction, we have to choose a 3-dimensional vector bundle over the curve, so that some regularity conditions hold between the curve and the vector bundle. In the last part of the paper, we analyze the osculating and the orthogonal vector bundle of the curve, respectively. In particular, it seems possible to relate them to some bitangency properties of the curve [4].

A different approach in generalizing the self-linking number can be found in [6], where it is considered a smooth map \( f : M \rightarrow \mathbb{R}^{2n+1} \) from a closed orientable smooth \( n \)-manifold \( M \) into \( \mathbb{R}^{2n+1} \).

2. THE LINKING NUMBER OF TWO CURVES WITH RESPECT TO A VECTOR BUNDLE

The linking number of two disjoint closed curves \( \alpha, \beta : S^1 \rightarrow \mathbb{R}^3 \) is a well known invariant, which is defined as the degree of the map \( e_1 : S^1 \times S^1 \rightarrow S^2 \) given by \( e_1(t, s) = (\beta(s) - \alpha(t))/||\beta(s) - \alpha(t)|| \). In this section, we will generalize this concept for two closed curves in \( \mathbb{R}^n \), by using a vector bundle over one of them.

Let \( \nu : E \rightarrow S^1 \) be a smooth 3-dimensional oriented vector subbundle of the trivial vector bundle \( S^1 \times \mathbb{R}^n \rightarrow S^1 \). Given \( t \in S^1 \), we will denote the fiber by \( \nu_t \), which is a 3-dimensional vector subspace of \( \mathbb{R}^n \). Moreover, we will put \( n_t : \mathbb{R}^n \rightarrow \nu_t \) and \( o_t : \mathbb{R}^n \rightarrow \nu_t^\perp \) for the orthogonal projections, where \( \nu_t^\perp \) is the orthogonal subspace to \( \nu_t \). Using these projections, we define the covariant derivative of a section \( h : S^1 \rightarrow \mathbb{R}^n \) of \( \nu \) by

\[
(Dh)(t) = n_t(h'(t)) = h'(t) - o_t(h'(t)),
\]

and we will say that \( h \) is parallel if \( Dh = 0 \).

Now, if we fix a parameterization of \( S^1 \) in the interval \([0, \ell]\), we can solve the equations of parallel transport and consider \( \{p_1(t), p_2(t), p_3(t)\} \), a parallel oriented orthonormal frame of \( \nu_t \), for \( t \in [0, \ell] \) (so that in general \( p_i(0) \) can be distinct from \( p_i(\ell) \)). This frame allows us to define the linear map \( c_t : \mathbb{R}^n \rightarrow \mathbb{R}^3 \) by

\[
c_t(x) = (\langle x, p_1(t) \rangle, \langle x, p_2(t) \rangle, \langle x, p_3(t) \rangle),
\]

so that the restriction of \( c_t \) to \( \nu_t \) is an oriented isometry.

Given \( h : S^1 \rightarrow \mathbb{R}^n \) any smooth map, it will be useful to know about the derivative of \( c_t h(t) \). Let \( \{u_i\}_{i=1}^3 \) denote the canonical basis of \( \mathbb{R}^3 \). Then,

\[
(c_t h(t))' = \sum_{i=1}^3 \langle h(t), p_i(t) \rangle' u_i = \sum_{i=1}^3 (\langle h'(t), p_i(t) \rangle + \langle h(t), p'_i(t) \rangle) u_i
\]

\[
= c_t h'(t) + \sum_{i=1}^3 \langle h(t), o_t p'_i(t) \rangle c_t p_i(t) = c_t(h'(t) + A_t h(t)),
\]

where \( A_t : \mathbb{R}^n \rightarrow \nu_t \) is the linear map given by

\[
A_t(x) = \sum_{i=1}^3 \langle x, o_t p'_i(t) \rangle p_i(t).
\]
It is not difficult to see that $A_t$ does not depend on the chosen orthonormal frame $p_i(t)$, parallel or not.

**Definition 2.1.** Let $\alpha, \beta : S^1 \to \mathbb{R}^n$ be two smooth closed curves in $\mathbb{R}^n$ and suppose that $\beta(s) - \alpha(t) \notin \nu^1_t$ for any $(t, s) \in S^1 \times S^1$. We define the map $e_1 : [0, \ell] \times S^1 \to S^2$ by

$$e_1(t, s) = \frac{c(t)(\beta(s) - \alpha(t))}{\|c(t)(\beta(s) - \alpha(t))\|}.$$  

Note that the imposed condition on the curves implies that $c(t)(\beta(s) - \alpha(t)) \neq 0$ and thus, $e_1$ is well defined.

**Lemma 2.2.** Let $\Omega_2$ be the standard volume form on $S^2$. Then $e^*_1 \Omega_2$ does not depend on the frame $p_i(t)$ and it defines a closed smooth 2-form on $S^1 \times S^1$.

**Proof.** To abbreviate, we denote $\delta(t, s) = \beta(s) - \alpha(t)$. Then,

$$e^*_1 \Omega_2 = \frac{\det(ct, \delta(t, s), \partial_t c(t, s), \partial_s c(t, s))}{\|c(t, \delta(t, s))\|^3} dt \wedge ds,$$

where $\partial_t$ and $\partial_s$ denote the partial derivatives with respect to $t$ and $s$ respectively. But, according to the above computation, $\partial_t c(t, s) = c_t(-\alpha'(t) + A_t \delta(t, s))$ and $\partial_s c(t, s) = c_t \beta'(s)$. Therefore,

$$e^*_1 \Omega_2 = \frac{\det(ct, \delta(t, s), c_t(-\alpha' + A_t \delta(t, s)), c_t \beta'(s))}{\|c(t, \delta(t, s))\|^3} dt \wedge ds = \frac{\det(n_t \delta(t, s), n_t(-\alpha' + A_t \delta(t, s)), n_t \beta'(s))}{\|n_t \delta(t, s)\|^3} dt \wedge ds,$$

where the last determinant has to be considered with respect to any oriented orthonormal frame of $\nu_t$.

**Definition 2.3.** Let $\alpha, \beta : S^1 \to \mathbb{R}^n$ be two smooth closed curves in $\mathbb{R}^n$ and suppose that $\beta(s) - \alpha(t) \notin \nu^1_t$ for any $(t, s) \in S^1 \times S^1$. We define its linking number with respect to $\nu$ as

$$L_{\nu}(\alpha, \beta) = \frac{1}{4\pi} \int_{S^1 \times S^1} e^*_1 \Omega_2.$$  

It follows from this definition that if $\nu_t$ is constant, then $L_{\nu}(\alpha, \beta)$ coincides with the classical linking number of the projected curves, $L(c_\nu \circ \alpha, c_\nu \circ \beta)$. In particular, when $n = 3$, we have that $L_{\nu}(\alpha, \beta) = L(\alpha, \beta)$, because then $\nu$ is the trivial bundle.

**Lemma 2.4.** Let $\alpha, \beta, \nu$ be as in **Definition 2.3**. Then, there is $\tilde{\beta} : D^2 \to \mathbb{R}^n$ extension of $\beta$, such that $\tilde{\beta}(z) - \alpha(t) \notin \nu^1_t$ except for a finite number of pairs $(t, z) \in S^1 \times D^2$.

**Proof.** Let $\beta_1 : D^2 \to \mathbb{R}^n$ be an arbitrary extension of $\beta$. Then the map $F : S^1 \times D^2 \times \mathbb{R}^n \to S^1 \times D^2 \times \mathbb{R}^n$ given by $F(t, z, x) = (t, z, \beta_1(z) - \alpha(t) + x)$ is a diffeomorphism. In particular, it is transverse to the submanifold $W = \{(t, z, x) : x \in \nu^1_t\}$. By the Transversality Theorem, it follows that for almost any $x \in \mathbb{R}^n$, the map $F_x : S^1 \times D^2 \to S^1 \times D^2 \times \mathbb{R}^n$ given by $F_x(t, z) = F(t, z, x)$ is also transverse to $W$. Since $W$ has codimension 3, this implies that $F_x^{-1}(W)$ is finite.

To construct the required extension $\tilde{\beta}$, we piece together $\beta_1$ near $S^1$ and $\beta_1 + x$ on the interior as follows. Let $\epsilon, \delta$ be such that $0 < \delta < \epsilon < 1$. Let $g_{\epsilon, \delta} : D^2 \to [0, 1]$ be a smooth function such that $g_{\epsilon, \delta}(z) = 1$ if $\|z\| \leq \delta$ and $g_{\epsilon, \delta}(z) = 0$ if $\epsilon \leq \|z\| \leq 1$ and let
Proof. Putting \( x \) anywhere \( B \) being deg(\( \tilde{z}_2 \)) = \beta_1(z) + x - \alpha(t) \in \nu^1_t. \)

Suppose that the claim is not true. Then, if for each \( n > 2 \) we consider \( \epsilon = 1 - 1/n, \delta = 1 - 2/n \) and \( R = 1/n \), there are \( t_n \in S^1, z_n \in D^2 \) and \( x_n \in \mathbb{R}^n \) with \( \|x_n\| < 1/n \) such that \( \beta_1(z_n) + g_{e, \delta}(z_n)x_n - \alpha(t_n) \in \nu^1_{t_n} \), but \( \beta_1(z_n) + x_n - \alpha(t_n) \notin \nu^1_{t_n} \).

Thus \( g_{e, \delta}(z_n) \neq 1 \) so that \( \|z_n\| \geq 1 - 2/n \). By taking subsequences if necessary, we can suppose that \( t_n \to t_0 \in S^1 \) and \( z_n \to s_0 \in S^1 \). Thus, we arrive to \( \beta(s_0) - \alpha(t_0) \in \nu^1_t \), in contradiction with the hypothesis. Now, we can choose \( \tilde{\beta} = \beta_{e, \delta, x} \), where \( x \) is any one of the points with \( \|x\| < R \) for which \( F^{-1}_x(W) \) is finite. \( \square \)

**Proposition 2.5.** Let \( \alpha, \beta, \nu \) be as in Definition 2.3. Then, \( L_\nu(\alpha, \beta) \in \mathbb{Z} \).

**Proof.** Let \( \tilde{\beta} \) be an extension of \( \beta \) such that \( \tilde{\beta}(z) - \alpha(t) \notin \nu^1_t \) for any \( (t, z) \in S^1 \times D^2 \setminus P \), being \( P = \{(t_1, z_1), \ldots, (t_N, z_N)\} \). Then we can extend \( e_1 \) to \( \tilde{e}_1 : [0, \ell] \times D^2 \setminus P \to S^2 \) by putting

\[
\tilde{e}_1(t, z) = \frac{c_t(\tilde{\beta}(z) - \alpha(t))}{\|c_t(\tilde{\beta}(z) - \alpha(t))\|}.
\]

As in Lemma 2.2, it follows that \( \tilde{e}_1^* \Omega_2 \) defines a smooth 2-form on \( S^1 \times D^2 \setminus P \). Moreover, since \( \Omega_2 \) is closed on \( S^2 \), \( d\tilde{e}_1^* \Omega_2 = 0 \) and by Stokes Theorem,

\[
0 = \int_{S^1 \times S^1} \tilde{e}_1^* \Omega_2 + \sum_{i=1}^N \int_{\partial B_i} \tilde{e}_1^* \Omega_2,
\]

where \( B_i \) denotes a small ball centered at \( (t_i, z_i) \) in the interior of \( S^1 \times D^2 \) and such that \( B_i \cap B_j = \emptyset \) if \( i \neq j \). In particular,

\[
L_\nu(\alpha, \beta) = -\frac{1}{4\pi} \sum_{i=1}^N \int_{\partial B_i} \tilde{e}_1^* \Omega_2 = -\sum_{i=1}^N \text{deg}(\tilde{e}_1|_{\partial B_i}) \in \mathbb{Z},
\]

being \( \text{deg}(\tilde{e}_1|_{\partial B_i}) \) the degree of the map \( \tilde{e}_1|_{\partial B_i} \). \( \square \)

An immediate consequence of this, together with the fact that \( L_\nu(\alpha, \beta) \) depends continuously on \( \alpha, \beta, \nu \) (when we consider the corresponding \( C^\infty \) Whitney topologies), is that \( L_\nu(\alpha, \beta) \) is invariant under homotopies of the curves and the vector bundle.

**Corollary 2.6.** Let \( \alpha_u, \beta_u : S^1 \to \mathbb{R}^n \) be 1-parameter families of curves and let \( \nu_u : E_u \to S^1 \) be a 1-parameter family of vector bundles, all of them depending smoothly on the parameter \( u \in [0, 1] \) and such that \( \alpha_u, \beta_u, \nu_u \) satisfy the condition of Definition 2.3 for any \( u \in [0, 1] \). Then, \( L_{\nu_u}(\alpha_u, \beta_u) \) is constant on \( u \).

In the last part of this section, we give a characterization of the linking number that will be used in the next section. Let \( \alpha, \beta, \nu \) be as in Definition 2.3 and suppose that there is a vector field \( \mu : S^1 \to \mathbb{R}^n \) such that \( \mu(t) \notin \nu^1_t \), for any \( t \in S^1 \). Let \( \{f_i(t)\}_{i=1}^n \) be an orthonormal oriented frame of \( \nu^1_t \), that is, the basis \( (p_1(t), p_2(t), p_3(t), f_1(t), \ldots, f_n(t)) \) has the same orientation as the canonical basis of \( \mathbb{R}^n \). We can define the map \( \chi : S^1 \times \mathbb{R} \times \mathbb{R}^{n-3} \to \mathbb{R}^n \) by

\[
\chi(t, \lambda, x_4, \ldots, x_n) = \alpha(t) + \lambda \mu(t) + \sum_{i=4}^n x_i f_i(t).
\]
**Proposition 2.7.** Suppose that $\beta$ meets the map $\chi$ transversely at a finite number of points and let

$$P_i = \beta(s_i) \in \alpha(t_i) + \lambda_i \mu(t_i) + \nu^\perp_{t_i}, \quad i = 1, \ldots, N.$$ 

be those points. Then,

$$L_\nu(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^{N} \text{sgn}(\lambda_i) i(\beta, \chi; P_i),$$

where $i(\beta, \chi; P_i)$ denotes the intersection number of $\beta$ and $\chi$ at $P_i$ and $\text{sgn}(\lambda_i)$ is the sign of $\lambda_i$.

**Proof.** Let $S^0 = [0, \ell] \times S^1 \setminus \{(t_1, s_1), \ldots, (t_N, s_N)\}$. For any $(t, s) \in S^0$, we have that $c_t \delta(t, s) \times c_t \mu(t) \neq 0$, where $\delta(t, s) = \beta(s) - \alpha(t)$. Thus, we can define

$$e_3(t, s) = \frac{c_t \delta(t, s) \times c_t \mu(t)}{\| c_t \delta(t, s) \times c_t \mu(t) \|}$$

and $e_2(t, s) = e_3(t, s) \times e_1(t, s)$, so that $\{e_i(t, s)\}_{i=1}^3$ is a right-handed orthonormal frame of $\mathbb{R}^3$.

Now, we can consider the 1-forms on $S^0$ defined by $\omega_{ij} = \langle de_i, e_j \rangle$, for any $i, j = 1, 2, 3$. Since $\langle e_i, e_j \rangle = \delta_{ij}$, by taking differentials we see that $\omega_{ij} = -\omega_{ji}$. Moreover, we have that

$$e_1^* \Omega_2 = \det(e_1, \partial_t e_1, \partial_s e_1) dt \wedge ds = (\omega_{12}(\partial_t) \omega_{13}(\partial_s) - \omega_{12}(\partial_s) \omega_{13}(\partial_t)) dt \wedge ds = \omega_{12} \wedge \omega_{13}.$$ 

But from the fact that $dd e_i = 0$, we deduce that

$$0 = \langle dde_i, e_j \rangle = d\omega_{ij} - \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.$$ 

In particular, $d\omega_{32} = \omega_{12} \wedge \omega_{13} = e_1^* \Omega_2$ and it is not difficult to see that $\omega_{32}$ defines a 1-form on $S^1 \times S^1 \setminus \{(t_1, s_1), \ldots, (t_N, s_N)\}$. This gives, by Stokes Theorem, that

$$L_\nu(\alpha, \beta) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 = \frac{1}{4\pi} \lim_{\epsilon \to 0} \sum_{i=1}^{N} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32},$$

where $D_\epsilon(t_i, s_i)$ denotes the disk centered at $(t_i, s_i)$ of radius $\epsilon > 0$ in $S^1 \times S^1$. To conclude the proof, we just have to show that for any $i = 1, \ldots, N$,

$$\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32} = \text{sgn}(\lambda_i) i(\beta, \chi; P_i).$$

On one hand, if we put $m(t, s) = c_t \delta(t, s) \times c_t \mu(t)$, it is easy to see that the left hand side is equal to $\pm 1$, in accordance with the sign of $D = \det(\partial_t m(t_i, s_i), \partial_s m(t_i, s_i), c_t \delta(t_i, s_i))$. If we compute this, we get

$$c_t \delta(t_i, s_i) = \lambda_i c_t \mu(t_i)$$

$$\partial_s m(t_i, s_i) = c_t \beta'(s_i) \times c_t \mu(t_i)$$

$$\partial_t m(t_i, s_i) = c_t(-\alpha'(t_i) - \lambda_i \mu'(t_i) + A_t(\delta(t_i, s_i) - \lambda_i \mu(t_i))) \times c_t \mu(t_i),$$

and using the isometry between $\nu_t$ and $\mathbb{R}^3$,

$$D = \lambda_i \| n_t \mu(t_i) \|^2 \det(n_t \beta'(s_i), n_t (\alpha'(t_i) + \lambda_i \mu'(t_i) - A_t(\delta(t_i, s_i) - \lambda_i \mu(t_i))), n_t \mu(t_i)).$$
On the other hand, if we suppose that \( \beta(s_i) = \chi(t_i, \lambda_i, x^i) \) for \( x^i \in \mathbb{R}^{n-3} \), we have that
\[
i(\beta, \chi; P_i) = \pm 1 \text{ depending on the sign of }
\]
\[
E = \det(\beta'(s_i), \partial_t \chi(t_i, \lambda_i, x^i), \partial_\lambda \chi(t_i, \lambda_i, x^i), \partial_{x_4} \chi(t_i, \lambda_i, x^i), \ldots, \partial_{x_n} \chi(t_i, \lambda_i, x^i)).
\]

Now,
\[
\partial_t \chi(t_i, \lambda_i, x^i) = \alpha'(t_i) + \lambda_i \mu'(t_i) + \sum_{j=4}^{n} x_j f_j'(t_i),
\]
\[
\partial_\lambda \chi(t_i, \lambda_i, x^i) = \mu(t_i),
\]
\[
\partial_{x_4} \chi(t_i, \lambda_i, x^i) = f_j(t_i),
\]
and thus,
\[
E = \det(n_t \beta'(s_i), n_t(\alpha'(t_i) + \lambda_i \mu'(t_i) + \sum_{j=4}^{n} x_j f_j'(t_i)), n_t \mu(t_i)).
\]

Finally, note that
\[
c_t \sum_{j=4}^{n} x_j f_j'(t_i) = -c_t A_t(\delta(t_i, s_i) - \lambda_i \mu(t_i)),
\]
which implies the desired result.

\[\square\]

3. The Self-Linking Number of a Curve with Respect to a Vector Bundle

We shall define here the self-linking number of a smooth curve \( \alpha : S^1 \to \mathbb{R}^n \) with respect to a vector bundle \( \nu \), as the linking number of \( \alpha \) and \( \tilde{\alpha} \), where \( \tilde{\alpha} : S^1 \to \mathbb{R}^n \) is close enough to \( \alpha \) and so that the conditions of Definition 2.3 are satisfied. To ensure that there exists such a curve \( \tilde{\alpha} \), we need to assume that \( \alpha(s) - \alpha(t) \notin \nu_t^\perp \), for \( s \neq t \).

Moreover, we also have to put some regularity conditions between the curve and the fiber bundle on the diagonal \( s = t \).

Throughout this section, we will suppose that \( \alpha : S^1 \to \mathbb{R}^n \) be a smooth closed curve in \( \mathbb{R}^n \) and that \( \nu \) is a smooth 3-dimensional oriented vector subbundle of the trivial vector bundle, as in Section II.

Lemma 3.1. Suppose that \( \alpha, \nu \) satisfy the following conditions:

1. For any \( s \neq t \), \( \alpha(s) - \alpha(t) \notin \nu_t^\perp \).
2. There exists \( 1 \leq k \leq n - 2 \) such that for any \( t \in S^1 \),
   (a) \( \alpha'(t), \ldots, \alpha^{(k+1)}(t) \) are linearly independent;
   (b) \( \alpha'(t), \ldots, \alpha^{(k-1)}(t) \in \nu_t^\perp \);
   (c) \( \langle \alpha^{(k)}(t) \rangle \oplus \langle \alpha^{(k+1)}(t) \rangle \subset \nu_t^\perp \).

Then, there is \( \delta_0 > 0 \) such that \( \alpha(s) - \alpha(t) \notin \nu_t^\perp \), for any \( 0 < \delta < \delta_0 \) and any \( (t, s) \in S^1 \times S^1 \), where \( \alpha_\delta(t) = \alpha(t) + \delta \alpha^{(k)}(t) \).

Proof. Suppose that this is not true. Then, for each \( m \geq 1 \), there are \( \delta_m < 1/m \) and pairs \( (t_m, s_m) \in S^1 \times S^1 \) such that \( \alpha(s_m) - \alpha(t_m) - \delta_m \alpha^{(k)}(t_m) \in \nu_t^\perp \). By taking subsequences if necessary, we can suppose that \( s_m \to s_0 \in S^1 \) and \( t_m \to t_0 \). If \( s_0 \neq t_0 \), we arrive to \( \alpha(s_0) - \alpha(t_0) \in \nu_t^\perp \), in contradiction with condition 1. Otherwise, let \( s_0 = t_0 \). If we denote by \( \{ f_i(t) \}_{i=4}^{n} \) a frame for \( \nu_t^\perp \), we have for any \( m \geq 1 \),
\[
(\alpha(s_m) - \alpha(t_m)) \land \alpha^{(k)}(t_m) \land f_4(t_m) \land \cdots \land f_n(t_m) = 0.
\]
Since
\[ \alpha(s_m) = \alpha(t_m) + \sum_{j=1}^{k+1} \frac{\alpha^{(j)}(t_m)}{j!} (s_m - t_m)^j + O ((s_m - t_m)^{k+2}), \]
we have after substitution and division by \((s_m - t_m)^{k+1}\),
\[ \alpha^{(k+1)}(t_m) \wedge \alpha^{(k)}(t_m) \wedge f_4(t_m) \wedge \cdots \wedge f_n(t_m) + O (s_m - t_m) = 0. \]
This would imply that
\[ \alpha^{(k+1)}(t_0) \wedge \alpha^{(k)}(t_0) \wedge f_4(t_0) \wedge \cdots \wedge f_n(t_0) = 0, \]
in contradiction with condition 2.(c).

**Remark 3.2.** When \( n = 3 \), necessarily \( k = 1 \) and \( \nu^1 = \{0\} \). Thus, conditions 1 and 2 of Lemma 3.1 just say that \( \alpha \) is embedded and that \( \alpha'(t), \alpha''(t) \) are linearly independent, for any \( t \in S^1 \).

**Definition 3.3.** Suppose that \( \alpha, \nu \) satisfy conditions 1 and 2 of Lemma 3.1 and consider \( \alpha_\delta(t) = \alpha(t) + \delta \alpha^{(k)}(t) \). The self-linking number of \( \alpha \) with respect to \( \nu \) is defined as
\[ SL_\nu(\alpha) = \lim_{\delta \to 0} L_\nu(\alpha_\delta, \alpha). \]

Note that since the linking number is invariant under homotopies, Lemma 3.1 ensures that \( L_\nu(\alpha_\delta, \alpha) \) does not depend on \( \delta \), if \( \delta \) is small enough.

**Figure 2.**

We would like now to obtain an integral expression for the self-linking number analogous to the integral expression which defines the linking number of two curves. The first step should be to define the map \( e_1 \). Let \( S \) be the following subset of \( \mathbb{R}^2 \) (see Figure 2):
\[ S = \{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq \ell, \ t \leq s \leq t + \ell \}. \]
We define the map \( e_1 : S \to S^2 \) as follows:
\[
e_1(t, s) = \begin{cases} 
\frac{c_t(\alpha(s) - \alpha(t))}{\|c_t(\alpha(s) - \alpha(t))\|}, & \text{if } t < s < t + \ell, \\
\frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|}, & \text{if } s = t, \\
(-1)^k \frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|}, & \text{if } s = t + \ell.
\end{cases}
\]
Note that $e_1$ is well defined when $\alpha$ satisfies conditions 1 and 2 of Lemma 3.1. Moreover, by taking a Taylor expansion in a neighbourhood of $(t, t)$ or $(t, t + \ell)$, it is easy to see that $e_1$ is smooth.

3.1. **The case $k$ even.** If $k$ is even, the map $e_1$ can be considered as a map from $[0, \ell] \times S^1$ to $S^2$. Moreover, $e_1^* \Omega_2$ defines a closed 2-form on $S^1 \times S^1$, which is the limit when $\delta \to 0$ of the closed 2-form associated to the pair $(\alpha_\delta, \alpha)$ in the definition of the linking number. This gives the following result.

**Proposition 3.4.** Suppose that $\alpha, \nu$ satisfy conditions 1 and 2 of Lemma 3.1 for $k$ even. Then,

$$SL_\nu(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2.$$  

3.2. **The case $k$ odd.** This case is more complicated. Let $S^0$ denote the open subset of $S$ given by the pairs $(t, s)$ such that $t < s < t + \ell$ and $c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t) \neq 0$. Then we can complete $e_1$ on $S^0$ in order to get a frame of $\mathbb{R}^3$ as we did in the proof of Proposition 3.4. If $t < s < t + \ell$, we define

$$e_3(t, s) = \frac{c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t)}{\|c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t)\|}.$$  

Moreover, it is possible to extend $e_3$ smoothly to the boundaries $s = t$ and $s = t + \ell$. In fact, by taking a Taylor expansion in a neighbourhood of $(t, t)$ or $(t, t + \ell)$, we get that

$$e_3(t, s) = e_3(t, t + \ell) = \frac{c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)\|}.$$  

We define $e_2$ in the obvious way, $e_2(t, s) = e_3(t, s) \times e_1(t, s)$. Finally, we define the 1-forms $\omega_{ij} = \langle de_i, e_j \rangle$, for any $i, j = 1, 2, 3$.

**Proposition 3.5.** Suppose that $\alpha, \nu$ satisfy conditions 1 and 2 of Lemma 3.1 for $k$ odd. Then,

$$SL_\nu(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 - \frac{1}{2\pi} \int_{S^1} \phi,$$

where $\phi(t) = \omega_{32}(t, t)$.

**Proof.** Let $\{f_i(t)\}_{i=4}^n$ be an orthonormal oriented frame of $\nu^{-1}_{t}$ and consider the map $\chi : S^1 \times \mathbb{R} \times \mathbb{R}^{n-3} \to \mathbb{R}^n$ given by

$$\chi(t, \lambda, x_4, \ldots, x_n) = \alpha(t) + \lambda \alpha^{(k)}(t) + \sum_{j=4}^n x_j f_j(t).$$

By the Transversality Theorem, we have that for a residual subset of curves $\alpha$ and vector bundles $\nu$ with the corresponding $C^\infty$ Whitney topologies, the curve $\alpha$ meets the hypersurface $\chi$ transversely at a finite number of points:

$$P_i = \alpha(s_i) = \alpha(t_i) + \lambda_i \alpha^{(k)}(t_i) + \sum_{j=4}^n x_j^i f_j(t), \quad i = 1, \ldots, N,$$

with $s_i \neq t_i$ and $\lambda_i \neq 0$. Since $SL_\nu(\alpha)$, $\int_{S^1 \times S^1} e_1^* \Omega_2$ and $\int_{S^1} \phi$ depend continuously on $\alpha$ and $\nu$, we can suppose that $\alpha$ and $\nu$ are generic in the above sense.
In particular, for \( \delta \) small enough, the same can be said if we consider the intersection of \( \alpha \) with \( \chi_\delta \), where
\[
\chi_\delta(t, \lambda, x_4, \ldots, x_n) = \alpha_\delta(t) + \lambda \alpha^{(k)}(t) + \sum_{j=4}^{n} x_j f_j(t).
\]
Then, Proposition 2.4 gives that
\[
L_\nu(\alpha_\delta, \alpha) = \frac{1}{2} \sum_{i=1}^{N} \text{sgn}(\lambda_i + \delta) i(\alpha, \chi_\delta; P_i),
\]
and taking limit when \( \delta \to 0 \),
\[
SL_\nu(\alpha) = \frac{1}{2} \sum_{i=1}^{N} \text{sgn}(\lambda_i) i(\alpha, \chi; P_i).
\]

On the other hand, note that \( S^0 = S \setminus \{(t_1, s_1), \ldots, (t_N, s_N)\} \). By using the same argument as in the proof of Proposition 2.7, \( d\omega_{32} = \omega_{12} \wedge \omega_{13} = e^*_1 \Omega_2 \). If we apply Stokes Theorem,
\[
\frac{1}{4\pi} \int_{S} e^*_1 \Omega_2 = \frac{1}{4\pi} \int_{\partial S} \omega_{32} + \frac{1}{4\pi} \lim_{\epsilon \to 0} \sum_{i=1}^{N} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32},
\]
where \( D_\epsilon(t_i, s_i) \) denotes the disk centered at \((t_i, s_i)\) of radius \( \epsilon > 0 \) in \( S \). Again we refer to the proof of Proposition 2.7 to claim that
\[
\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32} = \text{sgn}(\lambda_i) i(\alpha, \chi; P_i).
\]
In particular,
\[
SL_\nu(\alpha) = \frac{1}{4\pi} \int_{S} e^*_1 \Omega_2 - \frac{1}{4\pi} \int_{\partial S} \omega_{32}.
\]

To conclude the proof, we just have to compute the integral on \( \partial S \). We parameterize \( \partial S \) by considering the curves: \( \gamma_1(u) = (0, u) \), \( \gamma_2(u) = (u, u + \ell) \), \( \gamma_3(u) = (\ell, u + \ell) \) and \( \gamma_4(u) = (u, u) \), for \( u \in [0, \ell] \). Then, we have that
\[
\int_{\partial S} \omega_{32} = \int_{0}^{\ell} \omega_{32} (-\gamma_1' - \gamma_2' + \gamma_3' + \gamma_4') du.
\]
Note that \( \omega_{32}(0, u) = \omega_{32}(\ell, u + \ell) \) and \( \omega_{32}(u, u) = -\omega_{32}(u, u + \ell) \) for any \( u \in [0, \ell] \). This gives that
\[
\int_{\partial S} \omega_{32} = 2 \int_{S^1} \phi.
\]

4. The orthogonal self-linking number

We consider here the case that the vector bundle \( \nu \) is equal to the orthogonal vector bundle of the curve. That is, \( \nu_t \) is the 3-plane orthogonal to the subspace generated by the \( n-3 \) first derivatives of the curve.

**Definition 4.1.** Let \( \alpha : S^1 \to \mathbb{R}^n \) be a closed smooth curve in \( \mathbb{R}^n \) and suppose that:
1. For any \( t \in S^1 \), \( \alpha'(t), \alpha''(t), \ldots, \alpha^{(n-1)}(t) \) are linearly independent. In this way, at each point there is a well defined Frenet frame \( \{f_i(t)\}_{i=1}^n \) and also we have the curvatures \( \{\kappa_i(t)\}_{i=1}^{n-1} \). The orthogonal vector bundle \( \nu \) is defined so that \( \nu_t \) is the 3-plane generated by \( f_{n-2}(t), f_{n-1}(t), f_n(t) \).

2. For any \( s \neq t \) in \( S^1 \), \( \alpha(s) - \alpha(t) \notin \nu_t^\perp \). That is, the \((n-3)\)-osculating plane at \( t \) does not meet the curve at any other point.

It follows that \( \alpha, \nu \) satisfy conditions 1 and 2 of Lemma 3.1 for \( k = n - 2 \). The self-linking number of \( \alpha \) with respect to the orthogonal vector bundle will be called the orthogonal self-linking number and will be denoted by \( SL^\perp(\alpha) \).

With respect to this orthogonal vector bundle, we have that the orthogonal projection \( n_t : \mathbb{R}^n \to \nu_t \) is given by

\[
n_t(x) = \langle x, f_{n-2}(t) \rangle f_{n-2}(t) + \langle x, f_{n-1}(t) \rangle f_{n-1}(t) + \langle x, f_n(t) \rangle f_n(t).
\]

We also need to know about the linear map \( A_t : \mathbb{R}^n \to \nu_t \). To simplify computations, we will suppose that \( \alpha \) is parameterized by arc length. Then,

\[
A_t(x) = \sum_{i=n-2}^n \langle x, \alpha_t' f_i(t) \rangle f_i(t) = -\langle x, f_{n-3}(t) \rangle \kappa_{n-3}(t) f_{n-2}(t).
\]

With this we can easily compute the 2-form \( e_1^* \Omega_2 \) used in the integral formula of the self-linking number. But when \( k = n - 2 \) is odd, we also need to compute the 1-form \( \phi \) on \( S^1 \) given by \( \phi(t) = \omega_{32}(t, t) \). Note that

\[
e_1(t, t) = \frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|} = c_t f_{n-2}(t),
\]

\[
e_3(t, t) = \frac{c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)\|} = -c_t f_n(t),
\]

\[
e_2(t, t) = e_3(t, t) \times e_1(t, t) = -c_t f_{n-1}(t),
\]

\[
\omega_{32}(t, t) = (de_3(t, t), e_2(t, t)) = -\kappa_{n-1}(t) dt.
\]

Thus, we have the following integral expression for the orthogonal self-linking number.

**Corollary 4.2.** Let \( \alpha : S^1 \to \mathbb{R}^n \) be a closed smooth curve in \( \mathbb{R}^n \) satisfying conditions 1 and 2 of Definition 4.1. Then

\[
e_1^* \Omega_2 = \frac{\langle \delta(t, s), f_{n-3}(t) \rangle \kappa_{n-3}(t) \det(n_t \delta(t, s), n_t \alpha'(s), f_{n-2}(t))}{\|n_t \delta(t, s)\|^3} dt \wedge ds,
\]

where \( \delta(t, s) = \alpha(s) - \alpha(t) \). Moreover, the orthogonal self-linking number of \( \alpha \) is equal to

\[
SL^\perp(\alpha) = \begin{cases} 
\frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2, & \text{when } n \text{ is even}, \\
\frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 + \frac{1}{2\pi} \int_{S^1} \kappa_{n-1}(t) dt, & \text{when } n \text{ is odd}.
\end{cases}
\]

Given \( \alpha : S^1 \to \mathbb{R}^n \) a closed smooth curve in \( \mathbb{R}^n \) satisfying conditions 1 and 2 of Definition 4.1, we can consider the osculating developable hypersurface, which is the map
χ^T : S^1 × ℝ^{n-2} → ℝ^n defined by
\[ χ^T(t, x_1, \ldots, x_{n-2}) = α(t) + \sum_{i=1}^{n-2} x_i f_i(t). \]

By condition 1, this is an immersion at those points such that \( x_{n-2} \neq 0 \). Moreover, condition 2 implies that if the curve meets this map at a point \( P = α(s) = χ^T(t, x_1, \ldots, x_{n-2}) \) with \( s \neq t \), then necessarily \( x_{n-2} \neq 0 \). Note that the case \( s = t \) would imply that \( x_1 = \cdots = x_{n-2} = 0 \).

**Corollary 4.3.** Suppose that \( α \) meets the map \( χ^T \) transversely at a finite number of non-diagonal points and let \((t_1, s_1), \ldots, (t_N, s_N)\) be the pairs in \( S^1 \times S^1 \) corresponding to these points. Then, the orthogonal self-linking number of \( α \) is equal to
\[ SL^⊥(α) = \frac{-1}{2} \sum_{i=1}^{N} \text{sgn}(α'(s_i), f_n(t_i)). \]

5. The osculating self-linking number

Here, we look at the self-linking number of a curve with respect to its osculating vector bundle. That is, \( ν_t \) is the 3-plane generated by the 3 first derivatives of the curve.

**Definition 5.1.** Let \( α : S^1 → ℝ^n \) be a closed smooth curve in \( ℝ^n \) and suppose that:
1. For any \( t \in S^1 \), \( α'(t), α''(t), α'''(t) \) are linearly independent. We will denote by \( ν \) the osculating vector bundle of \( α \), that is, \( ν_t = \langle α'(t), α''(t), α'''(t) \rangle \).
2. For any \( s \neq t \) in \( S^1 \), \( α(s) − α(t) \notin ν_t^⊥ \). That is, the \((n−3)\)-orthogonal plane at \( t \) does not meet the curve at any other point.

In this case, \( α, ν \) satisfy conditions 1 and 2 of Lemma 3.1 for \( k = 1 \). Thus, we define the osculating self-linking number, \( SL^T(α) \), as the self-linking number of \( α \) with respect to the osculating vector bundle.

Now, we use \( f_1(t), f_2(t), f_3(t) \) for the (partial) Frenet frame of \( ν_t \) and \( κ_1(t), κ_2(t) \) for the non-vanishing curvatures. Then, \( n_t : ℝ^n → ν_t \) is given by
\[ n_t(x) = \langle x, f_1(t) \rangle f_1(t) + \langle x, f_2(t) \rangle f_2(t) + \langle x, f_3(t) \rangle f_3(t). \]

Again, we will suppose for simplicity that \( α \) is parameterized by arc length. Thus,
\[ A_t(x) = \sum_{i=1}^{3} \langle x, o_i f_i(t) \rangle f_i(t) = k(t) \langle x, o_i α_i^{(4)}(t) \rangle f_3(t), \]

where \( k(t) = 1/κ_1(t)κ_2(t) \).

Finally, we compute the 1-form \( φ(t) = ω_{32}(t, t) \):
\[ e_1(t, t) = \frac{c_t α'(t)}{∥c_t α'(t)∥} = c_t f_1(t), \]
\[ e_2(t, t) = \frac{c_t α''(t) \times c_t α'(t)}{∥c_t α''(t) \times c_t α'(t)∥} = -c_t f_3(t), \]
\[ e_3(t, t) = e_3(t, t) \times e_1(t, t) = -c_t f_2(t), \]
\[ ω_{32}(t, t) = \langle de_3(t, t), e_2(t, t) \rangle = -κ_2(t) dt. \]

Thus, we have the following integral expression for the osculating self-linking number.
Corollary 5.2. Let $\alpha : S^1 \to \mathbb{R}^n$ be a closed smooth curve in $\mathbb{R}^n$ satisfying conditions 1 and 2 of Definition 5.1. Then

$$e_i^*\Omega_2 = \frac{\det(n_t \delta(t, s), n_t \alpha'(s), \alpha'(t) - k(t) \langle \delta(t, s), \alpha'(4)(t) \rangle f_3(t))}{\| n_t \delta(t, s) \|^3} dt \wedge ds,$$

where $\delta(t, s) = \alpha(s) - \alpha(t)$. Moreover, the osculating self-linking number of $\alpha$ is equal to

$$SL^T(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_i^*\Omega_2 + \frac{1}{2\pi} \int_{S^1} \kappa_2(t) dt.$$

Finally, we can compute the osculating self-linking number by looking at the intersection of the curve with its orthogonal developable. Let $\{f_j(t)\}_{j=4}^n$ be any orthonormal oriented frame that trivializes $\nu^T$. We consider the orthogonal developable hypersurface, which is the map $\chi^\perp : S^1 \times \mathbb{R}^{n-2} \to \mathbb{R}^n$ defined by

$$\chi^\perp(t, x_3, \ldots, x_n) = \alpha(t) + \sum_{i=3}^n x_i f_i(t).$$

Since in this case $k = 1$, we have by Definition 3.3 that $SL^T(\alpha) = \lim_{\delta \to 0} L_\nu(\alpha + \delta f_1, \alpha)$. But it is not difficult to see that we obtain the same number if we change $f_1$ by $f_2$ or $f_3$. Thus, we have the following immediate consequence of Proposition 2.7, for $\mu(t) = f_3(t)$.

Corollary 5.3. Let $\alpha : S^1 \to \mathbb{R}^n$ be a closed smooth curve in $\mathbb{R}^n$ satisfying conditions 1 and 2 of Definition 5.1. Suppose that $\alpha$ meets the map $\chi^\perp$ transversely at a finite number of non-diagonal points and let

$$P_i = \alpha(s_i) = \alpha(t_i) + \sum_{j=3}^n x_j^i f_j(t_i), \quad i = 1, \ldots, N$$

be those points. Then, the osculating self-linking number of $\alpha$ is equal to

$$SL^T(\alpha) = \frac{1}{2} \sum_{i=1}^N \text{sgn}(x_3^i) i(\alpha, \chi^\perp; P_i).$$

6. The examples

In this last section, we will give some examples which show that when $n > 3$, the orthogonal and the osculating self-linking numbers are not trivial and are independent. All the examples are in $\mathbb{R}^4$ and the computations have been done with Mathematica. We compute the intersection of the curve with $\chi^\perp$ or $\chi^\top$ and the corresponding indices. Moreover, we also compute the integral value of $SL^\perp$ or $SL^\top$ in order to ratify the results.

Example 6.1. Let $\alpha : S^1 \to \mathbb{R}^4$ be the curve given by

$$\alpha(t) = \left(\cos(A + t) + \sin^2(t), \cos(A + 2t), \cos(t), \frac{A \sin(3t)}{27}\right).$$

It follows that for $A = 1$ and $A = 1.3$, the curve $\alpha$ satisfies conditions 1 and 2 of Definition 4.1 and Definition 5.1.

When $A = 1$, $\alpha$ meets $\chi^\perp$ transversely at four points with indices $1, 1, 1, -1$. In fact, we compute numerically the integral of Corollary 4.2 and obtain that $SL^\top(\alpha) = 1$. If we look now at the intersection with $\chi^\top$, there are just two points of transverse intersection, both with index 1. In this case, the integral formula of Corollary 5.2 gives $SL^\perp(\alpha) = 1$. 
When $A = 1.3$, the intersection with $\chi^\perp$ gives again four points with indices $1, 1, 1, -1$ and the numerical value of the integral formula is $SL^\top(\alpha) = 1$. However, although there are two points of transverse intersection with $\chi^\top$, this time the indices are $1, -1$ and the integral formula gives in this case $SL^\perp(\alpha) = 0$.

**Example 6.2.** We consider now a different family of curves in $\mathbb{R}^4$:

$$\alpha(t) = \left( -\cos(A + t) + \frac{A \sin(2t)}{8}, -\frac{A^3 \cos(2t)}{8} + \sin(A + t), \frac{\sin(5t)}{125}, \frac{A^2 \sin(3t)}{27} \right).$$

For $A = 1.6$, $\alpha$ satisfies conditions 1 and 2 of Definition 4.1 and Definition 5.1. The intersection with $\chi^\perp$ is equal to six points, all of them having index 1, and the numerical computation of the integral formula gives $SL^\top(\alpha) = 3$. The intersection with $\chi^\top$ is also equal to six points, but in this case two of them have index 1 and the other four $-1$. The integral formula gives $SL^\perp(\alpha) = -1$.

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