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**Title:** Exploring the ultimate limits: Super-resolution enhanced by partial coherence

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Exploring the ultimate limits: Super-resolution enhanced by partial coherence: supplemental document

Mathematical details concerning the calculations of quantum Fisher information for coherent superpositions will be given here.

Coherent superpositions \[|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_+\rangle + e^{i\phi}|\psi_-\rangle)\] play an essential role for understanding of partial coherence. In order to assess the cost needed for preparation of such states assume in the following the state of our system entangled with "spin-like" ancilla states

\[|\phi\rangle = 2^{-1/2} (|\psi_+\rangle \otimes |\uparrow_z\rangle + e^{i\phi}|\psi_-\rangle \otimes |\downarrow_z\rangle) = |\Phi_1\rangle \otimes |\uparrow_z\rangle + |\Phi_2\rangle \otimes |\downarrow_z\rangle, \quad (S1)\]

\[|\Phi_{1,2}\rangle = \frac{1}{2} (|\psi_+\rangle \pm e^{i\phi}|\psi_-\rangle), \quad |\Psi_{\pm}\rangle = \exp(\pm iP/2) |\psi\rangle. \quad (S2)\]

As argued in [1] the state \[|\Phi_{1,2}\rangle\] is generated by projecting the ancilla onto the basis \[|\uparrow_z\rangle, |\downarrow_z\rangle\]. This happens with the probability rate \[C = |\Phi_i| \equiv |\langle \Phi_i|\Phi_j\rangle|, i = 1, 2\]. Similarly, the marginal distribution over the ancilla selects the state of the system in the incoherent mixture. Due to the normalisation used here this happens with the rate equal to one establishing a reference for later comparison. If the rate is included as the multiplicative factor into the CR inequality, the normalisation used here this happens with the rate equal to one establishing a reference.

There are two simple models on how to create partially coherent states that can be constructed directly from coherent superpositions. Model A is defined as incoherent superposition of projectors into the basis of coherent superposition states

\[\rho_A = \frac{p_1}{||\Phi_1||^2} |\Phi_1\rangle \langle \Phi_1| + \frac{p_2}{||\Phi_2||^2} |\Phi_2\rangle \langle \Phi_2|, \quad (S3)\]

\[C = p_1||\Phi_1||^2 + p_2||\Phi_2||^2. \quad (S4)\]

This sorting scheme requires mixing the results of the projections into the ancilla states \[|\uparrow_z\rangle, |\downarrow_z\rangle\] with prior probabilities \[p_1, p_2\]. Yet another mixed superposition of signal state can be projected from the entangled state in Model B, if the ancilla is conditioned by the detection

\[\Pi_B = p_1|\uparrow_z\rangle \langle |\uparrow_z| + p_2|\downarrow_z\rangle \langle |\downarrow_z|, \quad (S5)\]

yielding the state

\[\rho_B = \frac{1}{p_1||\Phi_1||^2 + p_2||\Phi_2||^2} \left[ p_1 |\Phi_1\rangle \langle \Phi_1| + p_2 |\Phi_2\rangle \langle \Phi_2| \right], \quad (S6)\]

with the same rate factor Eq. (S4) as in Model A. The subtle distinction between variants A and B is of fundamental importance. It explains some confusion in the discussions [2, 3] and provides arguments for the enhancement of the super-resolution. Let us note in passing that Model A assumes the summation of the normalised states (as done in quantum optics), whereas in Model B "amplitudes" (like in coherent optics) are added.

Two specific models have been considered recently for construction of partially coherent states. Following the arguments in Larson and Saleh [4], the partially coherent state can be cast as the superposition of fully coherent (here \[\varphi\] dependance can be included) and fully incoherent parts

\[\rho_{LS} = \frac{p}{||\Phi_1||^2} |\Phi_1\rangle \langle \Phi_1| + \frac{1 - p}{2} \left( |\Psi_+\rangle \langle \Psi_+| + |\Psi_-\rangle \langle \Psi_-| \right). \quad (S7)\]
Due to the identity
\[ |Φ₁⟩⟨Φ₁| + |Φ₂⟩⟨Φ₂| = \frac{1}{2} [|Ψ⁺⟩⟨Ψ⁺| + |Ψ⁻⟩⟨Ψ⁻|] \]
the state can cast in the form
\[ ρ_{LS} = \frac{p}{||Φ₁||^2} |Φ₁⟩⟨Φ₁| + (1 - p) |Φ₂⟩⟨Φ₂|. \] (S8)

Such a state corresponds to the mixture between the generic Models A and B.

On the other hand, Tsang and Nair [2] were motivated by an optical definition considering the partially mixed state with the degree of coherence \( γ \)
\[ ρ_{TN} = N₀(|Ψ⁺⟩⟨Ψ⁺| + |Ψ⁻⟩⟨Ψ⁻| + γ |Ψ⁺⟩⟨Ψ⁻| + γ* |Ψ⁻⟩⟨Ψ⁺|). \] (S9)

Using the polar representation for the degree of coherence \( γ = |γ|e^{-iϕ} \) and the definition of in- and anti-phase superposition states, the partially mixed state can be simply rewritten to the form of Model B
\[ ρ_{TN} = N₀[(2 - |γ||) |Φ₁⟩⟨Φ₁| + 2(1 - |γ||) |Φ₂⟩⟨Φ₂|], \] (S10)
\[ N₀^{-1} = 2 - |γ| - |γ|| |Φ₂||^2. \] (S11)

All models including their mixtures are legitimate, but as they correspond to different preparations, they cannot be confused.

The normalisation of the states represents another issue in recent discussions. As argued by Tsang in Ref. [2], the unnormalised function of mutual coherence should be used instead of its normalised form. We agree that this argument can be justified in some cases but the correct usage of QFI, however, requires to use normalised states in order to guarantee the consistent normalisation of probability distributions. Particularly in Model B the factor \( C \) represents just the trace of unnormalised density matrix in the state Eq. (S6). If this factor is known as a function of estimated parameters, it can be exploited for estimation and the total Fisher information carried by such a signal reads
\[ F_{total} = \frac{(∂_iC)^2}{C} + CF_n. \] (S12)

The first term corresponds to (classical) Fisher information attributed to \( C \) and \( F_n \) represents QFI of the (normalised) quantum state. Here and in the following we will use the term Quantum Fisher Information just for normalised states. Before resorting to calculations of QFI for 2-component system let us evaluate the total Fisher information for the pair of in-and anti-phase superpositions \( |Φ⟩_{1,2} \). As derived in [1], the QFI for the normalised state reads (for \( i = 1, 2 \))
\[ F_i = \frac{4}{||Φ_i||^2} \frac{||Φ_i||^2}{4} \frac{||Φ_i||^2}{||Φ_i||^4}, \quad C = ⟨Φ_i|Φ_j⟩, \] (S13)
and therefore the total precision (S12) scales with the Fisher information
\[ F_{total} = 4||Φ_i||^2 + \frac{||Φ_i||^2}{||Φ_i||^2}. \] (S14)

Adopting usual condition \( ⟨P⟩ = 0 \), the Fisher information for separation associated with the in- and anti-phase superpositions can be easily found as (notice the interchange of indices here)
\[ F₁ = ⟨Φ₂|(ΔP)|Φ₂⟩, \quad F₂ = ⟨Φ₁|(ΔP)|Φ₁⟩, \] (S15)
\[ F₁ + F₂ = ⟨Ψ|(ΔP)|Ψ⟩. \] (S16)

We note that for small separations the anti-phase component \( Φ₂ \) carries the dominant information about separation, whereas information embedded in the in-phase component is negligible. However, sum of both contributions saturates the limit of incoherent mixtures. In addition to this, the dominant contribution to \( F₂ \) stems from the normalisation term
\[ F_C = \frac{(∂_iC)^2}{C}. \] (S17)
This behaviour is illustrated in Fig. 2 of the main text. Notice that the Fisher information for anti-phase superposition is constant even for vanishing signal if a Poissonian noise model is assumed. Similar argumentation with complementary results can be applied to the estimation of the centroid position $c_0$ induced by the overall unitary transformation $e^{-i\delta_P}$. The Fisher information terms for individual channels are given as

$$\mathcal{F}_1 = 4\langle \Phi_1 \rangle^2 \langle \delta \rangle^2, \quad \mathcal{F}_2 = 4\langle \Phi_2 \rangle^2 \langle \delta \rangle^2.$$

Obviously, the dominant contribution is carried by the in-phase superposition $\Phi_1$. It is intriguing to note that a beam splitter creating both superpositions acts like the device sorting the components with (dominant) information dedicated to different parameters. Such a distribution of information may provide another potential advantage to metrology inspired by the quantum information protocols.

The analysis of partial coherent fields requires to use the concept of quantum Fisher information matrix and symmetric logarithmic derivative $L^\dagger$

$$\partial \rho = \frac{1}{2}[\rho L + L^\dagger \rho].$$

This is not an easy task in the case of generic multi-parameter estimation. The theoretical background, existing calculation techniques and applications in physics has been addressed in the topical review [5]. The analysis provided here simplifies to the case of rank-two states (see Eq. (14) of Ref. [5]) evaluated in the diagonal basis of two-component mixed state

$$\rho = \lambda_1 |u_1\rangle \langle u_1| + \lambda_2 |u_2\rangle \langle u_2|.$$  

QFI can be written in the form of decomposition into two operator - $S$, acting in the 2-dimensional space spanned by the eigenbasis, and its orthogonal part $P$

$$L_0 = S + P \quad (S21)$$

$$\mathcal{F} = \text{Tr}(\rho L_0^2) \quad (S22)$$

where

$$S = \sum_k \frac{\partial \lambda_k}{\lambda_k} |u_k\rangle \langle u_k| + 2(\lambda_1 - \lambda_2) \langle \langle u_2 | \partial u_1 \rangle |u_2\rangle |u_1\rangle + \text{h.c.}$$

$$P = 2(|u_1^+\rangle \langle u_1| + |u_2^+\rangle \langle u_2| + \text{h.c.}),$$

$$|u_i^+\rangle = \partial u_i - (|u_1\rangle \langle u_1| + |u_2\rangle \langle u_2|) \partial u_i.$$  

Here $|\partial u_i\rangle$ denotes the projection of the derivative of the eigenstate into the subspace orthogonal to the eigenbasis $|u_i\rangle$. The optimal measurement is given by projections into the eigen-basis of logarithmic derivative; this will not be discussed here as we simply adress ultimate limits. Finally QFI can be brought to the form

$$\mathcal{F}_{2-\text{rank}} = \sum_i \frac{(\partial \lambda_i)^2}{\lambda_i} + 4(\lambda_1 - \lambda_2)^2 \langle \partial u_1 | u_2 \rangle^2 + 4\lambda_1 ||\partial u_1||^2 + 4\lambda_2 ||\partial u_2||^2,$$

$$||u_i^+||^2 = ||\partial u_i||^2 - |\langle u_1 | \partial u_1 \rangle|^2 - |\langle u_2 | \partial u_1 \rangle|^2.$$

In the following we will specialize to the common choice $\varphi = 0$ and such states for which $\langle \psi | e^{-i\hat{P}} | \psi \rangle$ is real (including e.g. Gaussian PSF). This assumption simplifies the analysis considerably since the states are orthogonal $\langle \Phi_2 | \Phi_1 \rangle = \frac{1}{2} \Im(\langle \Phi_2 | \Phi_1 \rangle) = 0$ providing the eigenstate basis

$$|u_{1,2}\rangle = \frac{1}{\sqrt{||\Phi_{1,2}||^2}} |\Phi_{1,2}\rangle.$$

QFI can be simplified to the final form

$$\mathcal{F}_{2-\text{rank}} = \sum_{i=1,2} \frac{(\partial \lambda_i)^2}{\lambda_i} + 4 \sum_{i=1,2} \lambda_i \left( \frac{||\partial \Phi_i||^2}{||\Phi_i||^2} - \frac{||\Phi_i||^2}{||\Phi_i||^4} \right).$$

The terms can be easily interpreted. The first term is attributed to the dependance coded in the eigenvalues, whereas the second sum represents the linear combination of the QFI for coherent superpositions $|\Phi_{1,2}\rangle$ given by expression (S13).

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If the eigenvalues are independent of the estimated parameter as in Model A, the first sum is zero and QFI is just the sum of particular contributions saturating the convexity condition for QFI. Such a partially coherent state does not offer any advantage for enhancing the precision.

An extra term appears in Model B as a consequence of the modulation of eigenvalues $\lambda_{1,2}$, and this term can be large for small separations

$$
\mathcal{F}_A = \sum_i \left( \delta \lambda_i / \lambda_i \right)^2 / \lambda_i = \frac{3}{1 - c^2} \frac{p(1 - p)}{\langle \Phi_2 \rangle^2 + cp}.
$$

The parameters here are denoted for brevity as $p_1 = p, p_2 = 1 - p, c = \Re \langle \Phi_2 \rangle$, $\langle \Phi_2 \rangle^2 = \frac{1}{2}(1 - c) \approx \frac{1}{2}s^2(\Delta p)^2$. However, the term $\mathcal{F}_A$ is large but when normalised with respect to the rate factor $C$. This will be limited by the resolution of incoherent mixture $(\Delta P)^2$. This is an expected result since any detection on the entangled state Eq. S1 is limited by its QFI, which equals to $(\Delta P)^2$ [1]. Moreover the complementary measurement to $\Pi_B$

$$
\Pi_B = p_2|\uparrow_z\rangle\langle\uparrow_z| + p_1|\downarrow_z\rangle\langle\downarrow_z|
$$

sorts the complementary state $\rho_B$ and the complementary rate $C_B$ obtained by simple interchange $p_1 \leftrightarrow p_2$. It is intriguing to note that combining those normalised states with their rate factors (what is effectively the combination of un-normalised states) gives again the incoherent mixture of separated states, and henceforth

$$
C_B\mathcal{F}_B + C_B\mathcal{F}_B \leq (\Delta P)^2.
$$

These arguments shows that (partial) coherence by itself does not bring any advantage with respect to precision when normalised to detection probabilities. For single channel the optimal regime approaching the resolution of incoherent mixture requires the adjustment of the weight $p$ for the given separation $s$. As can be shown by simple analysis the maximum of the product $C_F\mathcal{F}_3$ this is approximately achieved for $p \approx \sqrt{\|\Phi_1\|^2}$ with the rate $C = \sqrt{\|\Phi_2\|^2}$. For instance, for $s = 0.1(0.5)c$ the rate factor $C$ shows that just 2% (12%) of the intensity of incoherent signal is enough for reaching the limiting precision!

This is the clue to enhance the resolution as established in Model E. What is only needed is to increase the rate factor $C$ for the state as in Model B. This can be done if the state of Model B is prepared according to the recipe of Model A! In other words, such a state can be prepared as the mixture with weights dependent on the separation

$$
\hat{p}_i = \frac{p_i}{C} |\Phi_i|^2.
$$

This will give effectively the same state as in Model B but with the enhanced rate

$$
C_E = \frac{1}{C} [p_1 |\Phi_1|^4 + p_2 |\Phi_2|^4].
$$

Straightforward analysis shows the enhancement of precision for $p, s \to 0$. QFI normalised with respect to the strength of the signal scales as

$$
\mathcal{F}_A C_E \propto \frac{1}{s^2}.
$$

The procedure of Model E shows how to enhance the information about the estimated parameter high above the level of incoherent mixtures. However such a state preparation is not "blind" in the sense that prior knowledge about the estimated parameter is required and used in the state preparation stage in controlled experiments.

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