Memory and Communication Efficient Distributed Stochastic Optimization with Minibatch Prox

Jialei Wang  
*University of Chicago*

Weiran Wang  
*Toyota Technological Institute at Chicago*

Nathan Srebro  
*Toyota Technological Institute at Chicago*

JIALEI@UCHICAGO.EDU  
WEIRANWANG@TTIC.EDU  
NATI@TTIC.EDU

Abstract

We present and analyze statistically optimal, communication and memory efficient distributed stochastic optimization algorithms with near-linear speedups (up to log-factors). This improves over prior work which includes methods with near-linear speedups but polynomial communication requirements (accelerated minibatch SGD) and communication efficient methods which do not exhibit any runtime speedups over a naive single-machine approach. We first analyze a distributed SVRG variant as a distributed stochastic optimization method and show that it can achieve near-linear speedups with logarithmic rounds of communication, at the cost of high memory requirements. We then present a novel method, stochastic DANE, which trades off memory for communication and still allows for optimization with communication which scales only logarithmically with the desired accuracy while also being memory efficient. Stochastic DANE is based on a minibatch prox procedure, solving a non-linearized subproblem on a minibatch at each iteration. We provide a novel analysis for this procedure which achieves the statistical optimal rate regardless of minibatch size and smoothness, and thus significantly improving on prior work.

1. Introduction

Consider the stochastic convex optimization (generalized learning) problem (Nemirovskii and Yudin, 1983; Vapnik, 1995; Shalev-Shwartz et al., 2009):

$$
\min_{\mathbf{w} \in \Omega} \phi(\mathbf{w}) := \mathbb{E}_{\xi \sim D} [\ell(\mathbf{w}, \xi)]
$$

where our goal is to learn a predictor $\mathbf{w}$ from the convex domain $\Omega$ given the convex instantaneous (loss) function $\ell(\mathbf{w}, \xi)$ and i.i.d. samples $\xi_1, \xi_2, \ldots$ from some unknown data distribution $D$. When optimizing on a single machine, stochastic approximation methods such as stochastic gradient descent (SGD) or more generally stochastic mirror descent, are ideally suited for the problem as they typically have optimal sample complexity requirements, and run in linear time in the number of samples, and thus also have optimal runtime. Focusing on an $\ell_2$ bounded domain with $B = \sup_{\mathbf{w} \in \Omega} \|\mathbf{w}\|$ and $L$-Lipschitz loss, the min-max optimal sample complexity is $n(\varepsilon) = O(L^2 B^2 / \varepsilon^2)$, and this is achieved by SGD using $O(n(\varepsilon))$ vector operations. Furthermore, if examples are obtained one at a time (in a streaming setting or through access to a “button” generating examples), we only need to store $O(1)$ vectors in memory.

© J. Wang, W. Wang & N. Srebro.
The situation is more complex in the distributed setting where no single method is known that is optimal with respect to sample complexity, runtime, memory and communication. Specifically, consider \( m \) machines where each machine \( i = 1, \ldots, m \) receives samples \( \xi_i, \xi_2, \ldots \) drawn from the same distribution \( D \). This can equivalently be thought of as randomly distributing samples across \( m \) servers. We also assume the objective is \( \beta \)-smooth, taking \( L, \beta = O(1) \) in our presentation of results. The goal is to find a predictor \( \hat{w} \in \Omega \) satisfying \( \mathbb{E} [\phi(\hat{w}) - \min_{w \in \Omega} \phi(w)] \leq \varepsilon \) using smallest possible number of samples per machine, the minimal elapsed runtime, and the smallest amount of communication, and also minimal memory on each machine (again, when examples are received or generated one at a time). Ideally, we could hope for a method with linear speedup, i.e. \( O(n(\varepsilon)/m) \) runtime, using minimal number of samples \( O(n(\varepsilon)) \) and constant or near-constant communication and memory requirements. Throughout we measure runtime and in terms of vector operations, memory in terms of number of vectors that need to be stored on each machine and communication in terms of number of vectors sent per machine. These resource requirements are summarized in Table 1.

One simple approach for distributed stochastic optimization is minibatch SGD (Cotter et al., 2011; Dekel et al., 2012), where in each update we use a gradient estimate based on \( mb \) examples: \( b \) examples from each of the \( m \) machines. Distributed minibatch SGD attains optimal statistical performance with \( O(n(\varepsilon)/m) \) runtime, as long as the minibatch size is not too large: Dekel et al. (2012) showed that the minibatch size can be as large as \( bm = O(\sqrt{n(\varepsilon)}) \), and Cotter et al. (2011); Lan (2012) showed with acceleration this can be improved to \( bm = O(n(\varepsilon)^{3/4}) \). Using this maximal minibatch size for accelerated minibatch SGD thus yields a statistically optimal method with linear speedup in runtime, \( O(1) \) memory usage, and \( O(n(\varepsilon)^{1/4}) \) rounds of communication—see Table 1. This is the most communication-efficient method with linear speedup we are aware of (Shamir and Srebro, 2014).

An alternative approach is to use distributed optimization to optimize the regularized empirical objective:

\[
\min_w \phi_S(w) + \frac{\nu}{2} \|w\|^2,
\]

where \( \phi_S \) is the empirical objective on \( n(\varepsilon) \) i.i.d. samples, distributed across the machines and \( \nu = O(L/(B\sqrt{n(\varepsilon)})) \). A naive approach here is to use accelerate gradient descent, distributing the gradient computations, but this, as well as approaches based on ADMM (Boyd et al., 2011), are dominated by minibatch SGD (Shamir and Srebro 2014 and see also Table 1). Better alternatives take advantage of the stochastic nature of the problem: DANE (Shamir et al., 2014) requires only \( O(B^2/m) \) rounds of communication of squared loss problems, while DiSCO (Zhang and Lin, 2015) and AIDE (Reddi et al., 2016) reduce this further to \( O(B^{1/2}m^{1/4}) \) rounds of communication. However, these communication-efficient methods usually require expensive computation on each local machine, solving an optimization problem on all local data at each iteration. Even if this can be done in near-linear time, it is still difficult to obtain computational speedup compared with single machine solution, and certainly not linear speedups—see Table 1. Furthermore, since each round of these methods involves optimization over a fixed training set, this training set must be stored thus requiring \( n(\varepsilon)/m \) memory per machine.

Designing stochastic distributed optimization problems with linear, or near-linear, speedups, and low communication and memory requirements is thus still an open problem. We make progress in

---

1. In all methods involved, communication is used to average vectors across machines and make the result known to one or all machines. We are actually counting the number of such operations.
this paper analyzing and presenting methods with near-linear speedups and better communication
and memory requirements. As with the analysis of DANE, DiSCO and AIDE, our analysis is
rigorous only for least squared problems, and so all results should be taken in that context (the
methods themselves are applicable to any distributed stochastic convex optimization problem).

Our contributions

• We first apply the recently proposed distributed SVRG algorithm for regularized loss min-
imization to the distributed stochastic convex optimization problem, and show that on least
square problems it can achieve near-linear speedup with very low communication, but with
high memory cost—see DSVRG in Table 1.

• We propose a novel algorithm that improves the communication, which we call stochastic
DANE. For least square problems it achieves near-linear speedup with communication cost
that is higher than DSVRG but increases only logarithmically with \( n(\varepsilon) \), but with much lower
memory requirements. Moreover, our algorithm is flexible, allowing to trade off between
communication and memory, without affecting the computational efficiency. Our method
is based on careful combinations of inexact minibatch proximal update, communication-
efficient optimization and linearly convergent stochastic gradient algorithms for finite-sums.

• As indicated above, our method is based on minibatch proximal update. That is, a mini-
batch approach where in each iteration a non-linearized problem is solved on a stochastic
minibatch. This can be viewed as a minibatch generalization to the passive-aggressive algo-
rithm (Crammer et al., 2006) and has been considered in various contexts (Kulis and Bartlett,
2010; Toulis and Airoldi, 2014). We show that such an approach achieves the optimal sta-
tistical rate in terms of the number of samples used independent of the number of iterations,
i.e. with any minibatch size. This significantly improves over the previous analysis of Li et al.
(2014), as the guarantee is better, it entirely avoid the dependence on the minibatch size and
does not rely on additional assumptions as in Li et al. (2014). The guarantee holds for any
Lipschitz (even non-smooth) objective. Furthermore, to make the minibatch proximal iterate
more practical and useful in distributed setting, we also extend the analysis to algorithms
which solve each minibatch subproblem inexactly. Our analysis of exact and inexact mini-
batch proximal updates may be of independent interest and useful in other contexts and as a
basis for other methods.

Notations

We denote by \( w_* = \arg \min_{w \in \Omega} \phi(w) \) the optimal solution to (1). Throughout the
paper, we assume the instantaneous function \( \ell(w, \xi) \) is \( L \)-Lipschitz and \( \lambda \)-strongly convex in \( w \) for
some \( \lambda \geq 0 \) on the domain \( \Omega \):

\[
|\ell(w, \xi) - \ell(w', \xi)| \leq L \|w - w'\|,
\]

\[
\ell(w, \xi) - \ell(w', \xi) \geq \langle \nabla \ell(w', \xi), w - w' \rangle + \frac{\lambda}{2} \|w - w'\|^2,
\quad \forall w, w' \in \Omega.
\]

Sometimes we also assume \( \ell(w, \xi) \) is \( \beta \)-smooth in \( w \):

\[
\ell(w, \xi) - \ell(w', \xi) \leq \langle \nabla \ell(w', \xi), w - w' \rangle + \frac{\beta}{2} \|w - w'\|^2,
\quad \forall w, w' \in \Omega.
\]
For distributed stochastic optimization, our analysis focuses on the least squares loss $\ell(w, \xi) = \frac{1}{2}(w^T x - y)^2$ where $\xi = (x, y)$.

2. Distributed SVRG for stochastic convex optimization

Recently, Lee et al. (2015) proposed to simulate the fast randomized optimization algorithms for finite-sums, and in particular the SVRG algorithm, for distributed optimization of (2). The authors noted that, for SVRG, when the the sample size $n(\varepsilon)$ dominates the problem’s condition number $\beta/\nu$ where $\beta$ is the smoothness parameter of $\ell(w, \xi)$, the time complexity mainly comes from computing the batch gradients, and thus one can achieve speedup in total runtime by computing them distributedly. This distributed SVRG algorithm (DSVRG) is further justified in Shamir (2016) using without replacement sampling.

In the distributed stochastic convex optimization setting considered here, DSVRG in fact achieves linear speedup in certain regime as follows. In each iteration of the algorithm, each machine first computes its local gradient and average them with one communication round to obtain the global batch gradient, and then a single machine performs the SVRG stochastic updates by processing its local data once (sampling the $n(\varepsilon)/m$ examples without replacement). By the linear convergence of SVRG, as long as the number of stochastic updates $n(\varepsilon)/m$ is larger than $\beta/\nu = O(\beta B \sqrt{n(\varepsilon)}/L)$, the algorithm converges to $O(\varepsilon)$-suboptimality (in both the empirical and stochastic objective) in $O(\log 1/\varepsilon) = O(\log n(\varepsilon))$ iterations; and this condition is satisfied for $n(\varepsilon) \gtrsim m^2$.

Clearly, in the above regime, each iteration of DSVRG uses two rounds of communications and the total communication complexity is $O(n(\varepsilon))$. On the other hand, the computation for each machine is compute the local gradient (in time $O(n(\varepsilon)/m)$) in each iteration, resulting in a total time complexity of $O(n(\varepsilon) \log n(\varepsilon)/m)$. This explains the DSVRG entry in Table 1.

Being communication- and computation-efficient, DSVRG requires each machine to store a portion of the sample set for ERM to make multiple passes over them, and is therefore not memory-

---

2. If $n(\varepsilon) \gtrsim m^2$ does not hold, we can use a “hot-potato” style algorithm where we process all data once on machine $i$ and pass the predictor to machine $i+1$ until we obtain sufficiently many stochastic updates. But then the computation efficiency deteriorates and we no longer have linear speedup in runtime.
efficient. In fact, this disadvantage is shared by previously known communication-efficient distributed optimization algorithms, including DANE, DiSCO, and AIDE. In order to develop a memory- and communication-efficient algorithm for distributed stochastic optimization, we need to bypass the ERM setting and this is enabled by the following minibatch prox algorithm.

3. The minibatch prox algorithm for stochastic optimization

In this section, we describe and analyze the minibatch prox algorithm for stochastic optimization, which allows us to use arbitrarily large minibatch size without slowing down the convergence rate. We first present the basic version where each proximal objective is solved exactly for each minibatch, which achieves the optimal convergence rate. Then, we show that if each minibatch objective is solved accurately enough, the algorithm still converges at the optimal rate, opening the opportunity for efficient implementations.

3.1. Exact minibatch prox

The “exact” minibatch prox generates the following iterates: for \( t = 1, \ldots, \)

\[
\mathbf{w}_t = \arg \min_{\mathbf{w} \in \Omega} f_t(\mathbf{w}),
\]

where

\[
f_t(\mathbf{w}) := \phi_{I_t}(\mathbf{w}) + \frac{\gamma_t}{2} \| \mathbf{w} - \mathbf{w}_{t-1} \|^2 = \frac{1}{b} \sum_{\xi \in I_t} \ell(\mathbf{w}, \xi) + \frac{\gamma_t}{2} \| \mathbf{w} - \mathbf{w}_{t-1} \|^2,
\]

(3)

\( \gamma_t > 0 \) is the (inverse) stepsize parameter at time \( t \), and \( I_t \) is a set of a \( b \) samples from the unknown distribution \( D \). To understand the updates in (3), we first observe by the first order optimality condition for \( f_t(\mathbf{w}) \) that

\[
\nabla \phi_{I_t}(\mathbf{w}_t) + \gamma_t (\mathbf{w}_t - \mathbf{w}_{t-1}) \in -\mathcal{N}_\Omega(\mathbf{w}_t),
\]

(4)

where \( \nabla \phi_{I_t}(\mathbf{w}_t) \) is some subgradient of \( \phi_{I_t}(\mathbf{w}) \) at \( \mathbf{w}_t \), and \( \mathcal{N}_\Omega(\mathbf{w}_t) = \{ \mathbf{y} | \langle \mathbf{w} - \mathbf{w}_t, \mathbf{y} \rangle \leq 0, \forall \mathbf{w} \in \Omega \} \) is the normal cone of \( \Omega \) at \( \mathbf{w}_t \). Equivalently, the above condition implies

\[
\mathbf{w}_t = P_{\Omega} \left( \mathbf{w}_{t-1} - \frac{1}{\gamma_t} \nabla \phi_{I_t}(\mathbf{w}_t) \right),
\]

(5)

where \( P_{\Omega}(\mathbf{w}) \) denotes the projection of \( \mathbf{w} \) onto \( \Omega \). The update rule (5) resembles that of the standard minibatch gradient descent, except the gradient is evaluated at the “future” iterate.

Algorithms based on proximal steps are less popular than those based on gradient descent steps, probably because solving subproblems with the same form of the original objective is considered more difficult than solving its linear approximation. Crammer et al. (2006) analyzed an algorithm which minimize a margin-based loss from a single example with a quadratic penalty, leads to the “passive-aggressive” update rule; more general loss functions were analyzed in the online learning setting (Cheng et al., 2006; Kulis and Bartlett, 2010). For finite-sum objectives, methods based on incremental/stochastic proximal updates were studied in Bertsekas (2011); Defazio (2016). Needell and Tropp (2014) analyzed a randomized block Kaczmarz method in the context of solving linear systems, which also minimizes the empirical loss on a randomly sampled minibatch. To the best of our knowledge, no prior work has analyzed the general minibatch variant of proximal
updates for stochastic optimization except Li et al. (2014). However, the analysis of Li et al. (2014) assumes a stringent condition which is hard to verify (and is often violated) in practice, which we will discuss in detail in this section.

The following lemma provides the basic property of the update at each iteration.

**Lemma 1** For any \( w \in \Omega \), we have

\[
\frac{\lambda + \gamma_t}{\gamma_t} \| w_t - w \|^2 \leq \| w_{t-1} - w \|^2 - \| w_{t-1} - w_t \|^2 - \frac{2}{\gamma_t} (\phi_{I_t}(w_t) - \phi_{I_t}(w)).
\]

(6)

To derive the convergence guarantee, we need to relate \( \phi_{I_t}(w_t) \) to \( \phi(w) \). The analysis of Li et al. (2014) for minibatch prox made the assumption that for all \( t \geq 1 \):

\[
\mathbb{E}_{I_t} [D_{\phi}(w_t; w_{t-1})] \leq \mathbb{E}_{I_t} \left[ D_{\phi_{I_t}}(w_t; w_{t-1}) \right] + \frac{\gamma_t}{2} \| w_t - w_{t-1} \|^2,
\]

(7)

where \( D_{\phi}(w, w') = f(w) - f(w') - \langle \nabla f(w') , w - w' \rangle \) denotes the Bregman divergence defined by the potential function \( f \). We note that this condition is non-intuitive and hard to verify, and may further constrain the stepsize to be very small. For example, as the authors argued, if \( \ell(w, \xi) \) is \( \beta \)-smooth with respect to \( w \), we have

\[
D_{\phi}(w_t; w_{t-1}) \leq \frac{\beta}{2} \| w_t - w_{t-1} \|^2,
\]

and combined with the fact that \( D_{\phi_{I_t}}(w_t; w_{t-1}) \geq 0 \), one can guarantee (7) by setting \( \gamma_t \geq \beta \).

However, to obtain the optimal convergence rate, Li et al. (2014) needed to set \( \gamma_t = \mathcal{O}(\sqrt{T/b}) \) which would imply \( b = \mathcal{O}(T) \) in order to have \( \gamma_t \geq \beta \). In view of this implicit constraint that the minibatch size \( b \) can not be too large, the analysis of Li et al. (2014) does not really show advantage of minibatch prox over minibatch SGD, whose optimal minibatch size is precisely \( b = \mathcal{O}(T) \).

Our analysis is free of any additional assumptions. The key observation is that, when \( b \) is large, we expect \( \phi_{I_t}(w) \) to be close to \( \phi(w) \). Define the stochastic objective

\[
F_t(w) := \mathbb{E}_{I_t} [f_t(w)] = \phi(w) + \frac{\gamma_t}{2} \| w - w_{t-1} \|^2.
\]

(8)

Then \( w_t \) is the “empirical risk minimizer” of \( F_t(w) \) as it solves the empirical version \( f_t(w) \) with \( b \) samples. Using a stability argument (Shalev-Shwartz et al., 2009), we can establish the “generalization” performance for the (inexact) minimizer of the minibatch objective.

**Lemma 2** For the minibatch prox algorithm, we have

\[
\left| \mathbb{E}_{I_t} [\phi(w_t) - \phi_{I_t}(w_t)] \right| \leq \frac{4L^2}{(\lambda + \gamma_t)b}.
\]

Moreover, if a possibly randomized algorithm \( A \) minimizes \( f_t(w) \) up to an error of \( \eta_t \), i.e., \( A \) returns an approximate solution \( \tilde{w}_t \) such that \( \mathbb{E}_A [f_t(\tilde{w}_t) - f_t(w_t)] \leq \eta_t \), we have

\[
\left| \mathbb{E}_{I_t,A} [\phi(\tilde{w}) - \phi_{I_t}(w_t)] \right| \leq \frac{4L^2}{(\lambda + \gamma_t)b} + \sqrt{\frac{2L^2\eta_t}{\lambda + \gamma_t}}.
\]
Combining Lemma 1 and Lemma 2, we obtain the following key lemma regarding the progress on the stochastic objective at each iteration of minibatch prox.

**Lemma 3**  
For iteration $t$ of exact minibatch prox, we have for any $w \in \Omega$ that

$$\frac{\lambda + \gamma_t}{\gamma_t} \mathbb{E}_{I_t} \| w_t - w \|^2 \leq \| w_{t-1} - w \|^2 - \frac{2}{\gamma_t} \mathbb{E}_{I_t} [\phi(w_t) - \phi(w)] + \frac{8L^2}{\gamma_t(\lambda + \gamma_t) b}.$$  

(9)

We are now ready to bound the overall convergence rates of minibatch prox.

**Theorem 4** (Convergence of exact minibatch prox — weakly convex $\ell(w, \xi)$)  
Set $\gamma_t = \sqrt{\frac{8T}{b}}$.  
for $t = 1, \ldots, T$ in minibatch prox. Then for $w_{\bar{T}} = \frac{1}{T} \sum_{t=1}^T w_t$, we have

$$\mathbb{E} [\phi(w_{\bar{T}}) - \phi(w_*)] \leq \frac{\sqrt{8L}}{\sqrt{bT}} \| w_0 - w_* \|.$$  

**Theorem 5** (Convergence of exact minibatch prox — strongly convex $\ell(w, \xi)$)  
Set $\gamma_t = \frac{\lambda(t-1)}{2}$ for $t = 1, \ldots, T$ in minibatch prox. Then for $w_{\bar{T}} = \frac{2}{T(T+1)} \sum_{t=1}^T t w_t$, we have

$$\mathbb{E} [\phi(w_{\bar{T}}) - \phi(w_*)] \leq \frac{16L^2}{\lambda b(T+1)}.$$  

3.2. Inexact minibatch prox

We now study the case where instead of solving the subproblems $f_t(w)$ exactly, we only solve it approximately to sufficient accuracy. The “inexact” minibatch prox uses a possibly randomized algorithm $A$ for approximately solving one subproblem on a minibatch in each iteration, and generates the following iterates: for $t = 1, \ldots,$

$$\tilde{w}_t \approx \tilde{w}_t := \arg \min_{w \in \Omega} \tilde{f}_t(w) \quad \text{where} \quad \tilde{f}_t(w) := \phi_{I_t}(w) + \frac{\gamma_t}{2} \| w - \tilde{w}_{t-1} \|^2,$$  

(10)

and $$\mathbb{E}_A [\tilde{f}_t(\tilde{w}_t) - \tilde{f}_t(\tilde{w}_t)] \leq \eta_t.$$  

Analogous to Lemma 3, we can derive the following lemma using stability of inexact minimizers.

**Lemma 6**  
Fix any $w \in \Omega$. For iteration $t$ of inexact minibatch prox, we have

$$\mathbb{E}_{I_t, A} [\phi(\tilde{w}_t) - \phi(w)] \leq \frac{\gamma_t}{2} \mathbb{E}_{I_t, A} \| \tilde{w}_{t-1} - w \|^2 - \frac{\lambda + \gamma_t}{2} \mathbb{E}_{I_t, A} \| \tilde{w}_t - w \|^2 + \frac{4L^2}{(\lambda + \gamma_t) b} + \sqrt{\frac{2L^2 \eta_t}{\lambda + \gamma_t}} + \sqrt{2(\lambda + \gamma_t) \eta_t} \cdot \sqrt{\mathbb{E}_{I_t, A} \| \tilde{w}_t - w \|^2}.$$  

(11)

Note that when $\eta_t = 0$, the above guarantee reduces to that of exact minibatch prox.

We now show that when the minibatch subproblems are solved sufficiently accurately, we still obtain the $O(1/\sqrt{bT})$ rate for weakly-convex loss and $O(1/(\lambda bT))$ rate for strongly-convex loss.
Theorem 7 (Convergence of inexact minibatch prox — weakly convex \( \ell(w, \xi) \)) Let \( \delta > 0 \). Set 
\[
\gamma_t = \gamma = \sqrt{\frac{8T}{b}} \cdot \frac{L}{\|w_0 - w_*\|} \text{ for all } t \geq 1 \text{ in inexact minibatch prox. Assume that for all } t \geq 1, \text{ the error in minimizing } \hat{f}_t(w) \text{ satisfies }
\]
\[
\eta_t \leq \min \left( c_1 \left( \frac{T}{b} \right)^{\frac{1}{2}}, c_2 \left( \frac{T}{b} \right)^{\frac{3}{2}} \right) \cdot \frac{L \|w_0 - w_*\|}{\gamma^2 + 28}. 
\]

Then for \( \hat{w}_T = \frac{1}{T} \sum_{t=1}^{T} \hat{w}_t \), we have 
\[
\mathbb{E} [\phi(\hat{w}_T) - \phi(w_\*)] \leq \frac{c_3 L \|w_0 - w_*\|}{\sqrt{bT}}, \text{ where } c_3 \text{ only depends on } c_1, c_2 \text{ and } \delta. 
\]

For example, by setting \( c_1 = 10^{-4}, c_2 = 10^{-4}, \delta = 1/2 \), we have 
\[
\mathbb{E} [\phi(\hat{w}_T) - \phi(w_\*)] \leq \frac{\sqrt{10} L \|w_0 - w_*\|}{\sqrt{bT}}.
\]

Theorem 8 (Convergence of inexact minibatch prox — strongly convex \( \ell(w, \xi) \)) Let \( \delta > 0 \). Set 
\[
\gamma_t = \frac{\lambda(t-1)}{2} \text{ for } t = 1, \ldots \text{ in inexact minibatch prox. Assume that for all } t \geq 1, \text{ the error in minimizing } \hat{f}_t(w) \text{ satisfies }
\]
\[
\eta_t \leq \min \left( c_1 \left( \frac{T}{b} \right)^{\frac{1}{2}}, c_2 \left( \frac{T}{b} \right)^{\frac{3}{2}} \right) \cdot \frac{L^2}{\gamma^{3+28}.} 
\]

Then for \( \hat{w}_T = \frac{2}{T(T+1)} \sum_{t=1}^{T} \hat{w}_t \), we have 
\[
\mathbb{E} [\phi(\hat{w}_T) - \phi(w_\*)] \leq \frac{c_3 L^2}{\lambda \gamma^2}, \text{ where } c_3 \text{ only depends on } c_1, c_2 \text{ and } \delta.
\]

4. Communication-efficient distributed minibatch prox

We now extend the theoretical results of minibatch prox to the distributed stochastic learning setting, and propose a novel algorithm that is both communication and computation efficient.

Assume that we have \( m \) machines in a distributed environment. For each outer loop of our algorithm, each machine \( i \) draws a minibatch \( I_t^{(i)} \) of \( b \) samples independently from other machines, and denote \( I_t = \bigcup_{i=1}^{m} I_t^{(i)} \) which contains \( bm \) samples. To apply the minibatch prox algorithm from the previous section, we need to find an approximate solution to the following problem:

\[
\min_{w \in \Omega} \hat{f}_t(w) := \phi_{I_t}(w) + \frac{2}{2} \|w - w_{t-1}\|^2. \tag{15}
\]

Since the objective (15) involves functions from different machines, we use distributed optimization algorithms for solving it. In Li et al. (2014), the authors proposed a simple algorithm EMSO to approximately solve (15), where each machine first solve its own local objective, i.e.,

\[
w^{(i)}_t = \arg \min_{w \in \Omega} \phi_{I_t^{(i)}} + \gamma \|w - w_{t-1}\|^2, \tag{16}
\]

and then all machines average their local solutions via one round of communication: \( w_t = \frac{1}{m} \sum_{i=1}^{m} w^{(i)}_t \).

We note that this can be considered as the “one-shot-averaging” approach (Zhang et al., 2012) for solving (15). Although this approach was shown to work well empirically, no convergence guarantee for the original stochastic objective (1) was provided in (Li et al., 2014). Here we instead
Algorithm 1 Stochastic DANE for distributed stochastic convex optimization.

Initialize $w_0$.

for $t = 1, 2, \ldots, T$ do

Each machine $i$ draws a minibatch $I_t^{(i)}$ of $b$ samples from the underlying data distribution.

Initialize $y_0 \leftarrow w_{t-1}$, $x_0 \leftarrow w_{t-1}$.

for $r = 1, 2, \ldots, R$ do

Initialize $z_0 \leftarrow y_{r-1}$, $\alpha_0 = \sqrt{\gamma/(\gamma + \kappa)}$.

for $k = 1, 2, \ldots, K$ do

1. All machines perform one round of communication to compute the average gradient

$$\nabla \phi_{I_t}(z_{k-1}) \leftarrow \frac{1}{m} \sum_{i=1}^{m} \nabla \phi_{I_t^{(i)}}(z_{k-1}).$$

2. Each machine $i$ approximately solves the local objective to $\theta$-accuracy:

apply prox-SVRG to find $z_{k}^{(i)}$ s.t. $\|z_{k}^{(i)} - x_{k}^{(i)}\| \leq \theta \|x_{k-1} - z_{k}^{(i)}\|$.

where $z_{k}^{(i)} = \arg\min_{z \in \Omega} \phi_{I_t^{(i)}}(z) + \langle \nabla \phi_{I_t^{(i)}}(z_{k-1}) - \nabla \phi_{I_t^{(i)}}(z_{k-1}), z \rangle + \frac{\gamma}{2} \|z - w_{t-1}\|^2 + \frac{\kappa}{2} \|z - y_{r-1}\|^2.$ \hfill (12)

3. All machines reach consensus by averaging local updates through another round of communication:

$$z_k \leftarrow \frac{1}{m} \sum_{i=1}^{m} z_{k}^{(i)}.$$

end for

Update $x_r \leftarrow z_K$.

Compute $\alpha_r \in (0, 1)$ such that $\alpha_r^2 = (1 - \alpha_r)\alpha_{r-1}^2 + \gamma \alpha_k/(\gamma + \kappa)$, and compute

$$y_r = x_r + \left(\frac{\alpha_{r-1}(1 - \alpha_{r-1})}{\alpha_r + \alpha_{r-1}^2}\right)(x_r - x_{r-1}).$$ \hfill (14)

end for

Update $w_t \leftarrow x_r$.

end for

Output: $w_T$ is the approximate solution.

use the distributed optimization algorithm DANE (Shamir et al., 2014) and its accelerated variant AIDE (Reddi et al., 2016) for solving (15), which define better local objectives than EMSO and take into consideration the similarity between local objectives.

We detail our algorithm, named stochastic DANE, in Algorithm 1. The algorithm consists of three nested loops, where $t$, $r$ and $k$ are iteration counters for minibatch prox (the outer for-loop), AIDE (the intermediate for-loop) and DANE (the inner for-loop) respectively. Compared to EMSO, DANE adds a gradient correction term to (16) which can be compute efficiently with one round of
communication. On top of that, AIDE uses the idea of universal catalyst (Lin et al., 2015) and adds an extra quadratic term to improve the strong-convexity of the objective for faster convergence, i.e., in order to solve (15), AIDE solves multiple instances of the “augmented large minibatch” problems of the form

$$\min_{w \in \Omega} f_{t,r}(w) := \phi_{I_t}(w) + \frac{\gamma}{2} \|w - w_{t-1}\|^2 + \frac{\kappa}{2} \|w - y_{r-1}\|^2$$

(17)

with carefully chosen extrapolation points $y_{r-1}$. At each DANE iteration, we perform two rounds of communication, one for averaging the local gradients, and one for averaging the local updates, and the amount of data we communicate per round has the same size of the predictor.

To sum up, in Algorithm 1, we have introduced two levels of inexactness. First, we only approximately solve the “large minibatch” subproblem (15) in each outer loop; results from the previous section guarantee the convergence of this approach. Second, we only approximately solve the local subproblems (12) to sufficient accuracy in each inner loop; the analysis of “inexact DANE” (for the non-stochastic setting) provides guarantee for this approach (Reddi et al., 2016), and enables us to use state-of-the-art SGD methods (e.g., SVRG Johnson and Zhang, 2013; Xiao and Zhang, 2014) for solving local subproblems. Overall, we obtain a convergent algorithm for distributed stochastic convex optimization.

We now present detailed analysis for the computation/communication complexity of Algorithm 1 for stochastic quadratic problems, and compare it with related methods in the literature. Throughout this section, we have $\ell(w, \xi) = \frac{1}{2}(w^\top x - y)^2$ where $\xi = (x, y)$. We assume that $\ell(w, \xi)$ is $\beta$-smooth and $L$-Lipschitz in $w$, and that the domain $\Omega$ is bounded with radius $B = \sup_{w \in \Omega} \|w\|$. Note that each $\ell(w, \xi)$ is only weakly convex.

4.1. Efficiency of stochastic DANE

For the distributed stochastic convex optimization problems, we are concerned with the efficiency from four perspectives: sample, communication, computation and memory. Recall for convex $L$-Lipshitz, $B$-bounded problems, in order to learn a predictor $\hat{w}$ with $\varepsilon$-generalization error, i.e., $\mathbb{E}[\phi(\hat{w}) - \phi(w^*)] \leq \varepsilon$, we require the sample size to be at least $n(\varepsilon) = \mathcal{O}((L^2B^2)/\varepsilon^2)$. This sample complexity matches the worst case lower bound, and can be achieved by vanilla SGD.

We now present the main results of this section (full analysis is deferred to Appendix C), which show that with careful choices of the minibatch size and the desired accuracy in each level of approximate solution, stochastic DANE achieves both communication and computation efficiency with the optimal sample complexity. Interestingly, the choices of parameters differ in two regimes which are separated by an “optimal” minibatch size $b^* = \frac{n(\varepsilon)L^2}{128m^2 \beta^2 B^2 \log(md)}$.

**Theorem 9 (Efficiency of stochastic DANE for $b \leq b^*$)** Set the parameters in Algorithm 1 as follows:

- (outer loop) $b \leq b^* = \frac{n(\varepsilon)L^2}{128m^2 \beta^2 B^2 \log(md)}$, $T = \frac{n(\varepsilon)}{bm}$, $\gamma = \frac{\sqrt{2n(\varepsilon)L}}{bmB}$,
- (intermediate loop) $\kappa = 0$, $R = 1$,
- (inner loop) $\theta = \frac{1}{6}$, $K = \mathcal{O}\left(\log \frac{\beta B \cdot n(\varepsilon)}{L}\right)$.

3. We can equivalently assume $\|x\|^2 \leq \beta$ and $y$ is bounded.
Then we have $\mathbb{E} \left[ \phi \left( \frac{1}{T} \sum_{t=1}^{T} w_t \right) - \phi(w_*) \right] \leq \frac{\sqrt{8B}L}{\sqrt{n(\varepsilon)}} = \mathcal{O} \left( \varepsilon \right)$.

Moreover, Algorithm 1 can be implemented with $\tilde{O} \left( \frac{n(\varepsilon)}{bm} \right)$ rounds of communication, and each machine performs $\tilde{O} \left( \frac{n(\varepsilon)}{m} \right)$ vector operations in total, where the notation $\tilde{O}(\cdot)$ hides poly-logarithmic dependences on $n(\varepsilon), \beta, L$ and $B$.

When we choose $b = b^*$, Algorithm 1 can be implemented with $\tilde{O} \left( \frac{m^2 B^2}{L^2} \right)$ rounds of communication, $\tilde{O} \left( \frac{n(\varepsilon)}{m} \right)$ vector operations, and $\mathcal{O} \left( \frac{n(\varepsilon)L^2}{m^2 B} \right)$ memory for each machine.

We comment on the choice of parameters. For sample efficiency, we fix the sample size $n(\varepsilon)$ and number of machines $m$, and so we can tradeoff the local minibatch size $b$ and the total number of outer iterations $T$, maintaining $bT = \frac{n(\varepsilon)}{m}$. For any $b$, the regularization parameters in the “large minibatch” problem is set to $\gamma = \sqrt{\frac{8T}{bm}} \cdot \frac{L}{2B} = \mathcal{O} \left( \frac{\sqrt{2n(\varepsilon)L}}{bmB} \right)$ according to Theorem 7.\(^4\) When $b \leq b^*$, we note that (35) can be satisfied with $\kappa = 0$ and there is no need for acceleration by AIDE ($R = 1$). Then the values of $\theta$ and $K$ follow from Lemma 16.

**Remark 10** The above theorem suggests that in the regime of $b \leq b^*$, we only need to have logarithmic number of DANE iterations for solving each “large minibatch” problem, and logarithmic number of passes over the local data during each DANE iteration. We present experimental results validating our theory in Appendix D.

The next theorem shows that when we use a large minibatch size $b$ in Algorithm 1, we can still satisfy the condition (35) by adding extra regularization ($\kappa > 0$), and then apply accelerated DANE.

**Theorem 11 (Efficiency of stochastic DANE for $b \geq b^*$)** Set the parameters in Algorithm 1 as follows:

- **(outer loop)** $b \geq b^* = \frac{n(\varepsilon)L^2}{128m^2\beta^2B^2 \log(md)}$, $T = \frac{n(\varepsilon)}{bm}$, $\gamma = \sqrt{\frac{2n(\varepsilon)L}{bmB}}$;

- **(intermediate loop)** $\kappa = 16\beta \sqrt{\frac{\log(dm)}{b}} - \gamma$, $R = \mathcal{O} \left( \frac{b^{1/4}m^{1/2} \cdot \beta^{1/2}B^{1/2}}{n(\varepsilon)^{1/4} \cdot L^{1/2}} \log \frac{\beta B \cdot n(\varepsilon)}{L} \right)$;

- **(inner loop)** $\theta = \frac{1}{6}$, $K = \mathcal{O} \left( \log \frac{\beta B \cdot n(\varepsilon)}{L} \right)$.

Then we have $\mathbb{E} \left[ \phi \left( \frac{1}{T} \sum_{t=1}^{T} w_t \right) - \phi(w_*) \right] \leq \frac{\sqrt{8B}L}{\sqrt{n(\varepsilon)}} = \mathcal{O} \left( \varepsilon \right)$.

Moreover, Algorithm 1 can be implemented with $\tilde{O} \left( \frac{n(\varepsilon)^{3/4} \beta^{1/2}B^{1/2}}{b^{3/4}m^{1/2} \cdot L^{1/2}} \right)$ rounds of communication, and each machine performs $\tilde{O} \left( \frac{(b^{1/4}n(\varepsilon))^{3/4} \beta^{1/2}B^{1/2}}{m^{1/2}L^{1/2}} \right)$ vector operations in total, where the notation $\tilde{O}(\cdot)$ hides poly-logarithmic dependences on $n(\varepsilon), \beta, L$ and $B$.

---

\(^4\) Note that Theorem 7 holds with $||w_0 - w_*||$ replaced with an upper bound of it. In this section, we have $||w_0 - w_*|| \leq 2B$. 

11
4.2. Two regimes of multiple resource tradeoffs

From the above analysis, we summarized in Table 2 the resources required by stochastic DANE. We observe two interesting regimes, separated by the minibatch size $b^* \approx n(\varepsilon)/(m^2 B^2)$, that present different tradeoffs between communication, computation and memory.

- When $1 \leq b \leq b^*$, the computation complexity remains $\tilde{O}(n(\varepsilon)/m)$ which is independent of $b$. This means we always achieve near-linear speedup in this regime. Moreover, there is a tradeoff between communication and memory: the communication complexity decreases, while the memory cost increases as the minibatch size $b$ increases, both at the linear rate. Thus in this regime, we can trade communication for memory without affecting computation.

- When $b^* < b \leq b_{\text{max}}$, the computation starts to increase with $b$ at the rate $b^{1/4}$ which is slower than linear, while the communication cost continues to decrease at the rate $b^{3/4}$ which is also slower than linear. Thus in this regime, we can trade communication for computation and memory.

| Samples | Communication | Computation | Memory |
|---------|---------------|-------------|--------|
| $1 \leq b \leq b^*$ | $n(\varepsilon)$ | $n(\varepsilon)/mb$ | $n(\varepsilon)/m$ |
| $b = b^*$ | $n(\varepsilon)$ | $B^2m$ | $n(\varepsilon)/m$ |
| $b^* < b \leq b_{\text{max}}$ | $n(\varepsilon)$ | $B^{1/2}n(\varepsilon)^{3/4}/(m^{1/2}b^{3/4})$ | $B^{1/2}n(\varepsilon)^{3/4}b^{1/4}/m^{1/2}$ |

Table 2: Summary of resources required by stochastic DANE for distributed stochastic convex optimization, in units of vector operations/communications/memory per machine, ignoring constants and log-factors. Here $b^* \approx n(\varepsilon)/(m^2 B^2)$, and $b_{\text{max}} = n(\varepsilon)/m$.

5. Discussion and conclusion

In this paper, we made progress toward linear speedup, communication and memory efficient methods for distributed stochastic optimization, although we still do not have an algorithm that obtains the “ideal” distributed stochastic optimization performance of linear speedup with constant or near-constant communication and memory. There is also no single known algorithm that dominates all others, with different methods being preferable in terms of different resources. These tradeoffs, up to log-factors, are given in Table 1 and the memory, communication and runtime requirements are also schematically depicted in Figure 1. In the figure, the horizontal axis corresponds to the “mini-batch” size, which can be controlled with accelerated minibatch SGD and stochastic DANE, while other methods are batch methods which consider the entire data set.

From Figure 1 we can see that DSVRG dominates the other methods (up to log-factors) in terms of runtime and communication—it has smaller communication requirements than DiSCO/AIDE (and better than DANE) with nearly the same optimal runtime of accelerated minibatch SGD. But like other batch methods, it requires storing and re-accessing the entire data set. Accelerated minibatch SGD is the only one of these methods requiring only $O(1)$ memory per machine, and it achieves true linear speedup, but due to the limit on the maximal allowed minibatch size, has relatively high communication cost. Stochastic DANE with a small minibatch size is dominated by
accelerated minibatch SGD, but it allows significantly increasing the minibatch size while still retaining (near) linear speedup. This allows trading off memory and communication and significantly improves over DSVRG in terms of memory requirements. But even with the maximal minibatch size $b^*$ allowing for linear speedup, stochastic DANE’s communication requirements, although much lower than accelerated minibatch SGD, are still higher than DSVRG’s.

We emphasize that the above discussion is based on guarantees established only for least square problems and ignores log-factors. We are unfortunately not aware of distributed stochastic optimization guarantees that improve over minibatch SGD (i.e., achieve even near-linear speedup with lower communication requirements) for general smooth objectives, or achieve true linear speedup (and improved communication guarantees) even for least-square problems.

Acknowledgement

Research was partially supported by an Intel ICRI-CI award. We would like to thank Ohad Shamir for discussions about distributed SVRG and Tong Zhang for discussions about minibatch prox.
References

Dimitri P Bertsekas. Incremental proximal methods for large scale convex optimization. *Mathematical programming*, 129(2):163, 2011.

Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.

Li Cheng, SVN Vishwanathan, Dale Schuurmans, Shaojun Wang, and Terry Caelli. Implicit online learning with kernels. In *Proceedings of the 19th International Conference on Neural Information Processing Systems*, pages 249–256. MIT Press, 2006.

Andrew Cotter, Ohad Shamir, Nati Srebro, and Karthik Sridharan. Better mini-batch algorithms via accelerated gradient methods. In *Advances in Neural Information Processing Systems (NIPS)*, volume 24, pages 1647–1655, 2011.

Koby Crammer, Ofer Dekel, Joseph Keshet, Shai Shalev-Shwartz, and Yoram Singer. Online passive-aggressive algorithms. *Journal of Machine Learning Research*, 7(Mar):551–585, 2006.

Aaron Defazio. A simple practical accelerated method for finite sums. In *Advances In Neural Information Processing Systems*, pages 676–684, 2016.

Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in Neural Information Processing Systems*, pages 1646–1654, 2014.

Ofer Dekel, Ran Gilad-Bachrach, Ohad Shamir, and Lin Xiao. Optimal distributed online prediction using mini-batches. *Journal of Machine Learning Research*, 13(Jan):165–202, 2012.

Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in Neural Information Processing Systems*, pages 315–323, 2013.

Brian Kulis and Peter L Bartlett. Implicit online learning. In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, pages 575–582, 2010.

Simon Lacoste-Julien, Mark Schmidt, and Francis Bach. A simpler approach to obtaining an \(o(1/t)\) convergence rate for the projected stochastic subgradient method. arXiv:1212.2002 [cs.LG], 2012.

Guanghui Lan. An optimal method for stochastic composite optimization. *Mathematical Programming*, 133(1):365–397, 2012.

Jason D Lee, Qihang Lin, Tengyu Ma, and Tianbao Yang. Distributed stochastic variance reduced gradient methods and a lower bound for communication complexity. *arXiv preprint arXiv:1507.07595*, 2015.

Mu Li, Tong Zhang, Yuqiang Chen, and Alexander J. Smola. Efficient mini-batch training for stochastic optimization. In *Proc. of the 20th ACM SIGKDD Int. Conf. Knowledge Discovery and Data Mining (SIGKDD 2014)*, pages 661–670, 2014.
Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. A universal catalyst for first-order optimization. In Advances in Neural Information Processing Systems, pages 3384–3392, 2015.

Deanna Needell and Joel A Tropp. Paved with good intentions: Analysis of a randomized block kaczmarz method. Linear Algebra and its Applications, 441:199–221, 2014.

A Nemirovskii and DB Yudin. Problem complexity and method efficiency in optimization, 1983.

Sashank J Reddi, Jakub Konečný, Peter Richtárik, Barnabás Póczós, and Alex Smola. Aide: Fast and communication efficient distributed optimization. arXiv preprint arXiv:1608.06879, 2016.

Mark Schmidt, Nicolas Le Roux, and Francis Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In Advances in Neural Information Processing Systems (NIPS), volume 24, pages 1458–1466, 2011.

Shai Shalev-Shwartz, Ohad Shamir, Nathan Srebro, and Karthik Sridharan. Stochastic convex optimization. In Proc. of the 22th Annual Conference on Learning Theory (COLT’09), 2009.

Ohad Shamir. Without-replacement sampling for stochastic gradient methods: Convergence results and application to distributed optimization. arXiv preprint arXiv:1603.00570, 2016.

Ohad Shamir and Nathan Srebro. Distributed stochastic optimization and learning. In Communication, Control, and Computing (Allerton), 2014 52nd Annual Allerton Conference on, pages 850–857. IEEE, 2014.

Ohad Shamir, Nathan Srebro, and Tong Zhang. Communication-efficient distributed optimization using an approximate Newton-type method. In Proc. of the 31st Int. Conf. Machine Learning (ICML 2014), pages 1000–1008, 2014.

Panos Toulis and Edoardo M Airoldi. Implicit stochastic gradient descent. arXiv preprint arXiv:1408.2923, 2014.

Vladimir Vapnik. The nature of statistical learning theory. Springer science & business media, 1995.

Lin Xiao and Tong Zhang. A proximal stochastic gradient method with progressive variance reduction. SIAM Journal on Optimization, 24(4):2057–2075, 2014.

Yuchen Zhang and Xiao Lin. Disco: Distributed optimization for self-concordant empirical loss. In Proceedings of the 32nd International Conference on Machine Learning (ICML-15), pages 362–370, 2015.

Yuchen Zhang, Martin J Wainwright, and John C Duchi. Communication-efficient algorithms for statistical optimization. In Advances in Neural Information Processing Systems, pages 1502–1510, 2012.
Appendix A. Analysis of exact minibatch prox

A.1. Proof of Lemma 1

Proof Observe that (4) implies \( \gamma_t(w_{t-1} - w_t) \) is a subgradient at \( w_t \) of the sum of \( \phi_{I_t}(w) \) and the indicator function of \( \Omega \) (which has value 0 in \( \Omega \) and \( \infty \) otherwise), and thus we have for any \( w \in \Omega \) that

\[
\phi_{I_t}(w) - \phi_{I_t}(w_t) \geq \gamma_t \langle w_{t-1} - w_t, w - w_t \rangle + \frac{\lambda}{2} \| w - w_t \|^2. 
\]  

(18)

For any \( w \in \Omega \), we can bound its distance to \( w_{t-1} \) as

\[
\| w_{t-1} - w \|^2 = \| w_{t-1} - w_t + w_t - w \|^2 \\
= \| w_{t-1} - w_t \|^2 + 2 \langle w_{t-1} - w_t, w_t - w \rangle + \| w_t - w \|^2 \\
\geq \| w_{t-1} - w_t \|^2 + \frac{2}{\gamma_t} (\phi_{I_t}(w_t) - \phi_{I_t}(w)) + \frac{\lambda}{\gamma_t} \| w - w_t \|^2 + \| w_t - w \|^2 \\
= \frac{\lambda + \gamma_t}{\gamma_t} \| w_t - w \|^2 + \frac{2}{\gamma_t} (\phi_{I_t}(w_t) - \phi_{I_t}(w)) + \| w_{t-1} - w_t \|^2
\]

where we have used (18) in the first inequality. Rearranging the terms yields the desired result.

A.2. Proof of Lemma 2

The following lemma, which is essentially shown by Shalev-Shwartz et al. (2009, Theorem 6), characterizes the convergence of the empirical loss to the population counterpart for the (approximate) regularized empirical risk minimizer.

Lemma 12 Consider a stochastic convex optimization problem

\[ F(w) := \mathbb{E}_\xi [ \ell(w, \xi) ] + r(w) \]

where the loss \( \ell(w, \xi) \) is \( L \)-Lipschitz and \( \lambda \)-strongly convex in \( w \), the regularization \( r(w) \) is \( \gamma \)-strongly convex. Let \( Z = \{ \xi_1, \ldots, \xi_n \} \) be i.i.d. samples and

\[ \hat{w} = \arg \min_w \hat{F}(w) \quad \text{where} \quad \hat{F}(w) := \frac{1}{n} \sum_{i=1}^n \ell(w, \xi_i) + r(w). \]

Denote by \( G(w) = \mathbb{E}_\xi [ \ell(w, \xi) ] \) and \( \hat{G}(w) = \frac{1}{n} \sum_{i=1}^n \ell(w, \xi_i) \) the expected and the empirical losses respectively.

1. For the regularized empirical risk minimizer \( \hat{w} \), we have

\[
\left| \mathbb{E}_Z \left[ G(\hat{w}) - \hat{G}(\hat{w}) \right] \right| \leq \frac{4L^2}{(\lambda + \gamma)n}.
\]
2. If for any given dataset $Z$, a possibly randomized algorithm $A$ minimizes $\hat{F}(w)$ up to an error of $\eta$, i.e., $A$ returns an approximate solution $\hat{w}$ such that $\mathbb{E}_A \left[ \hat{F}(\hat{w}) - \hat{F}(w) \right] \leq \eta$, we have
\[
\left| \mathbb{E}_{Z,A} \left[ G(\hat{w}) - \hat{G}(\hat{w}) \right] \right| \leq \frac{4L^2}{(\lambda + \gamma)n} + \sqrt{\frac{2L^2\eta}{\lambda + \gamma}}.
\]

**Proof** We prove the lemma by a stability argument.

**Exact ERM** Denote by $Z^{(i)}$ the sample set that is identical to $Z$ except that the $i$-th sample $\xi_i$ is replaced by another random sample $\xi'_i$, by $\hat{F}^{(i)}(w)$ the empirical objective defined using $Z^{(i)}$, i.e.,
\[
\hat{F}^{(i)}(w) := \frac{1}{n} \left( \sum_{j \neq i} \ell(w, \xi_j) + \ell(w, \xi'_j) \right) + r(w),
\]
and by $\hat{w}^{(i)} = \arg \min_w \hat{F}^{(i)}(w)$ the empirical risk minimizer of $\hat{F}^{(i)}(w)$.
By the definition of the empirical objectives, we have
\[
\hat{F}(\hat{w}^{(i)}) - \hat{F}(\hat{w}) = \frac{\ell(\hat{w}^{(i)}, \xi_i) - \ell(\hat{w}, \xi_i)}{n} = \frac{\sum_{j \neq i} \ell(\hat{w}^{(i)}, \xi_j) - \ell(\hat{w}, \xi_j)}{n} + r(\hat{w}^{(i)}) - r(\hat{w})
\]
\[
\leq \frac{\ell(\hat{w}^{(i)}, \xi_i) - \ell(\hat{w}, \xi_i)}{n} + \frac{\ell(\hat{w}, \xi'_i) - \ell(\hat{w}, \xi'_i)}{n} + \left( \hat{F}^{(i)}(\hat{w}^{(i)}) - \hat{F}^{(i)}(\hat{w}) \right)
\]
\[
\leq \frac{2L}{n} \left\| \hat{w}^{(i)} - \hat{w} \right\|.
\]
where we have used the fact that $\hat{w}^{(i)}$ is the minimizer of $\hat{F}^{(i)}(w)$ in the first inequality, and the $L$-Lipschitz continuity of $\ell(w, \xi)$ in the second inequality.
On the other hand, it follows from the $(\lambda + \gamma)$-strong convexity of $\hat{F}(w)$ that
\[
\hat{F}(\hat{w}^{(i)}) - \hat{F}(\hat{w}) \geq \frac{(\lambda + \gamma)}{2} \left\| \hat{w}^{(i)} - \hat{w} \right\|^2.
\]
Combining (19) and (20) yields $\left\| \hat{w}^{(i)} - \hat{w} \right\| \leq \frac{4L^2}{(\lambda + \gamma)n}$.
Again, by the $L$-Lipschitz continuity of $\ell(w, \xi)$, we have that for any sample $\xi$ that
\[
\left| \ell(\hat{w}, \xi) - \ell(\hat{w}^{(i)}, \xi) \right| \leq L \left\| \hat{w}^{(i)} - \hat{w} \right\| \leq \frac{4L^2}{(\lambda + \gamma)n}.
\]
Since $Z$ and $Z^{(i)}$ are both i.i.d. sample sets, we have
\[
\mathbb{E}_Z \left[ G(\hat{w}) \right] = \mathbb{E}_{Z^{(i)}} \left[ G(\hat{w}^{(i)}) \right] = \mathbb{E}_{Z^{(i)} \cup \{\xi_i\}} \left[ \ell(\hat{w}^{(i)}, \xi_i) \right].
\]
As this holds for all $i = 1, \ldots, n$, we can also write
\[
\mathbb{E}_Z \left[ G(\hat{w}) \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{Z^{(i)} \cup \{\xi_i\}} \left[ \ell(\hat{w}^{(i)}, \xi_i) \right].
\]
On the other hand, we have
\[
\mathbb{E}_Z \left[ \hat{G}(\hat{\mathbf{w}}) \right] = \mathbb{E}_Z \left[ \frac{1}{n} \sum_{i=1}^{n} \ell(\hat{\mathbf{w}}, \xi_i) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_Z \left[ \ell(\hat{\mathbf{w}}, \xi_i) \right].
\]  
(23)

Combining (22) and (23) and using the stability (21), we obtain
\[
\mathbb{E}_Z \left[ G(\mathbf{w}) - \hat{G}(\hat{\mathbf{w}}) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{Z \cup \{\xi_i'\}} \left[ \ell(\hat{\mathbf{w}}^{(i)} , \xi_i) - \ell(\hat{\mathbf{w}}, \xi_i) \right] \in \left[ -\frac{4L^2}{(\lambda + \gamma)n}, \frac{4L^2}{(\lambda + \gamma)n} \right].
\]

**Inexact ERM** For the approximate solution \(\tilde{\mathbf{w}}\), due to the \((\lambda + \gamma)\)-strong convexity of \(\hat{F}\), we have
\[
\mathbb{E}_A \|\tilde{\mathbf{w}} - \hat{\mathbf{w}}\|^2 \leq \frac{2}{\lambda + \gamma} \mathbb{E}_A \left[ \hat{F}(\tilde{\mathbf{w}}) - \hat{F}(\hat{\mathbf{w}}) \right] \leq \frac{2\eta}{\lambda + \gamma},
\]
and thus \(\mathbb{E}_A \|\tilde{\mathbf{w}} - \hat{\mathbf{w}}\| \leq \sqrt{\frac{2\eta}{\lambda + \gamma}}\) by the fact that \(\mathbb{E}x^2 \geq (\mathbb{E}x)^2\) for any random variable \(x\).

It then follows from the Lipschitz continuity of \(G(\mathbf{w})\) that
\[
\mathbb{E}_A |G(\tilde{\mathbf{w}}) - G(\hat{\mathbf{w}})| \leq L \cdot \mathbb{E}_A \|\tilde{\mathbf{w}} - \hat{\mathbf{w}}\| \leq \sqrt{\frac{2L^2\eta}{\lambda + \gamma}}.
\]

Finally, we have by the triangle inequality and the stability of exact ERM that
\[
\left| \mathbb{E}_{Z,A} \left[ G(\mathbf{w}) - \hat{G}(\hat{\mathbf{w}}) \right] \right| \leq \mathbb{E}_Z \left| \mathbb{E}_A [G(\tilde{\mathbf{w}}) - G(\hat{\mathbf{w}})] \right| + \mathbb{E}_Z \left| G(\tilde{\mathbf{w}}) - \hat{G}(\hat{\mathbf{w}}) \right| \\
\leq \sqrt{\frac{2L^2\eta}{\lambda + \gamma}} + \frac{4L^2}{(\lambda + \gamma)n}.
\]

Then Lemma 2 follows from the fact that that our stochastic objective (8) is equipped with \(L\)-Lipschitz, \(\lambda\)-strongly convex loss \(\phi(\mathbf{w})\) and \(\gamma_t\)-strongly convex regularizer \(\frac{1}{2} \|\mathbf{w} - \mathbf{w}_{t-1}\|^2\).

**A.3. Proof of Lemma 3**

**Proof** We have by Lemma 2 that\(^5\)
\[
|\mathbb{E}_I \left[ \phi_I(\mathbf{w}_t) - \phi(\mathbf{w}_t) \right]| \leq \frac{4L^2}{(\lambda + \gamma_t)b}.
\]

\(^5\) Although Lemma 12 is stated for unconstrained objective, a convex constraint can be incorporated by adding the indicator function of the constraint set to the regularizer \(r(\mathbf{w})\) without changing its strong convexity, and the same stability result holds.
Take expectation of (6) over the random sampling of $I_t$ and we obtain
\[
\frac{\lambda + \gamma t}{\gamma t} \mathbb{E}_{I_t} \| w_t - w \|^2 \leq \| w_{t-1} - w \|^2 - \frac{2}{\gamma t} \mathbb{E}_{I_t} [\phi_{I_t}(w_t)] - \phi(w)
\]
\[
= \| w_{t-1} - w \|^2 - \frac{2}{\gamma t} (\mathbb{E}_{I_t} [\phi_{I_t}(w_t) - \phi(w)] + \mathbb{E}_{I_t} [\phi(w_t) - \phi(w)])
\]
\[
\leq \| w_{t-1} - w \|^2 - \frac{2}{\gamma t} \mathbb{E}_{I_t} [\phi(w_t) - \phi(w)] + \frac{2}{\gamma t} |\mathbb{E}_{I_t} [\phi_{I_t}(w_t) - \phi(w_t)]|
\]
\[
\leq \| w_{t-1} - w \|^2 - \frac{2}{\gamma t} \mathbb{E}_{I_t} [\phi(w_t) - \phi(w)] + \frac{8L^2}{\gamma t (\lambda + \gamma t) b}.
\]

\[\text{A.4. Proof of Theorem 4}\]

**Proof** When $\ell(w, \xi)$ is weakly convex (i.e., $\lambda = 0$), we further set $\gamma_t = \gamma$ for all $t \geq 1$. Applying Lemma 3 with $w = w^*$ yields
\[
\mathbb{E}_{I_t} [\phi(w_t) - \phi(w^*]) \leq \frac{\gamma}{2} \left(\| w_{t-1} - w^* \|^2 - \mathbb{E}_{I_t} \| w_t - w^* \|^2 \right) + \frac{4L^2}{\gamma b}.
\]  (24)

Summing (24) for $t = 1, \ldots, T$ yields
\[
\sum_{t=1}^{T} \mathbb{E} [\phi(w_t) - \phi(w^*]) \leq \frac{\gamma}{2} \| w_0 - w^* \|^2 + \frac{4L^2 T}{\gamma b}.
\]

Minimizing the RHS over $\gamma$ gives the optimal choice
\[
\gamma = \sqrt{\frac{8T}{b}} \cdot \frac{L}{\| w_0 - w^* \|},
\]
with a corresponding regret
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} [\phi(w_t) - \phi(w^*]) \leq \frac{\sqrt{8L}}{\sqrt{bT}} \| w_0 - w^* \|.
\]

As a result, by returning the uniform average $\hat{w}_T = \frac{1}{T} \sum_{t=1}^{T} w_t$, we have due to the convexity of $\phi(w)$ that
\[
\mathbb{E} [\phi(\hat{w}_T) - \phi(w^*]) \leq \frac{\sqrt{8L}}{\sqrt{bT}} \| w_0 - w^* \|.
\]
A.5. Proof of Theorem 5

Proof Let $\ell(w, \xi)$ be $\lambda$-strongly convex for some $\lambda > 0$. Applying Lemma 3 with $w = w_*$ yields
\[ \mathbb{E}_{I_t} [\phi(w_t) - \phi(w_*)] \leq \left( \frac{\gamma t}{2} \|w_{t-1} - w_*\|^2 - \frac{\lambda + \gamma t}{2} \mathbb{E}_{I_t} \|w_t - w_*\|^2 \right) + \frac{4L^2}{(\lambda + \gamma t)b}. \] (25)

Setting $\gamma_t = \frac{\lambda(t-1)}{2}$ for $t = 1, \ldots, 6$, the above inequality becomes
\[ \mathbb{E}_{I_t} [\phi(w_t) - \phi(w_*)] \leq \left( \frac{\lambda(t-1)}{4} \|w_{t-1} - w_*\|^2 - \frac{\lambda(t+1)}{4} \mathbb{E}_{I_t} \|w_t - w_*\|^2 \right) + \frac{8L^2}{\lambda b t}, \]
and therefore
\[ t \cdot \mathbb{E}_{I_t} [\phi(w_t) - \phi(w_*)] \leq \frac{\lambda}{4} \left( (t-1)t \|w_{t-1} - w_*\|^2 - t(t+1)\mathbb{E}_{I_t} \|w_t - w_*\|^2 \right) + \frac{8L^2}{\lambda b}. \]

Summing this inequality for $t = 1, \ldots, T$ yields
\[ \sum_{t=1}^T t \cdot \mathbb{E} [\phi(w_t) - \phi(w_*)] \leq \frac{8L^2 T}{\lambda b}. \]

As a result, by returning the weighted average $\tilde{w}_T = \frac{2}{T(T+1)} \sum_{t=1}^T t w_t$, we have due to the convexity of $\phi(w)$ that $\phi(\tilde{w}_T) \leq \frac{2}{T(T+1)} \sum_{t=1}^T t \cdot \phi(w_t)$ and
\[ \mathbb{E} [\phi(\tilde{w}_T) - \phi(w_*)] \leq \frac{2}{T(T+1)} \sum_{t=1}^T t \cdot \mathbb{E} [\phi(w_t) - \phi(w_*)] \leq \frac{16L^2}{\lambda b(T+1)}. \]

\[ \square \]

Appendix B. Analysis of inexact minibatch prox

B.1. Proof of Lemma 6

Proof Due to the $(\lambda + \gamma_t)$-strong convexity of $\tilde{f}_t(w)$, we have
\[ \mathbb{E}_A \|\tilde{w}_t - \tilde{w}_t\|^2 \leq \frac{2}{\lambda + \gamma_t} \mathbb{E}_A \left[ \tilde{f}_t(\tilde{w}_t) - \tilde{f}_t(\tilde{w}_t) \right] \leq \frac{2\eta_t}{\lambda + \gamma_t}. \]

Applying Lemma 1 to the exact minimizer $\tilde{w}_t$ yields
\[ \phi_{I_t}(\tilde{w}_t) - \phi_{I_t}(w) \leq \frac{\gamma_t}{2} \|\tilde{w}_{t-1} - w\|^2 - \frac{\lambda + \gamma_t}{2} \|\tilde{w}_t - w\|^2. \]

6. This choice is inspired by the stepsize rule of Lacoste-Julien et al. (2012) for stochastic gradient descent.
Therefore, for the \( t \)-th iteration, we have
\[
\begin{align*}
\mathbb{E}_{I_t, A} \left[ \phi(\mathbf{w}_t) - \phi(\mathbf{w}) \right] \\
= \mathbb{E}_{I_t, A} \left[ \phi(\mathbf{w}_t) - \phi_{I_t}(\mathbf{w}_t) \right] + \mathbb{E}_{I_t} \left[ \phi_{I_t}(\mathbf{w}_t) - \phi_{I_t}(\mathbf{w}) \right] \\
\leq \frac{4L^2}{(\lambda + \gamma_t)|I_t|} + \sqrt{\frac{2L^2\eta_t}{\lambda + \gamma_t}} + \frac{\gamma_t}{2} \| \mathbf{w}_{t-1} - \mathbf{w} \|^2 - \frac{\lambda + \gamma_t}{2} \mathbb{E}_{I_t} \| \mathbf{w}_t - \mathbf{w} \|^2 \\
\leq \frac{4L^2}{(\lambda + \gamma_t)|I_t|} + \sqrt{\frac{2L^2\eta_t}{\lambda + \gamma_t}} + \frac{\gamma_t}{2} \| \mathbf{w}_{t-1} - \mathbf{w} \|^2 - \frac{\lambda + \gamma_t}{2} \mathbb{E}_{I_t, A} \| \mathbf{w}_t - \mathbf{w} \|^2 \\
\leq \frac{4L^2}{(\lambda + \gamma_t)|I_t|} + \sqrt{\frac{2L^2\eta_t}{\lambda + \gamma_t}} + \frac{\gamma_t}{2} \| \mathbf{w}_{t-1} - \mathbf{w} \|^2 - \frac{\lambda + \gamma_t}{2} \mathbb{E}_{I_t, A} \| \mathbf{w}_t - \mathbf{w} \|^2 \\
+ (\lambda + \gamma_t) \cdot \mathbb{E}_{I_t, A} \| \mathbf{w}_t - \mathbf{w} \| \\
\leq \frac{4L^2}{(\lambda + \gamma_t)|I_t|} + \sqrt{\frac{2L^2\eta_t}{\lambda + \gamma_t}} + \frac{\gamma_t}{2} \| \mathbf{w}_{t-1} - \mathbf{w} \|^2 - \frac{\lambda + \gamma_t}{2} \mathbb{E}_{I_t, A} \| \mathbf{w}_t - \mathbf{w} \|^2 \\
+ \sqrt{2(\lambda + \gamma_t)\eta_t} \cdot \sqrt{\mathbb{E}_{I_t, A} \| \mathbf{w}_t - \mathbf{w} \|^2} \\
\end{align*}
\]
where we have applied Lemma 12 to the approximate minimizer \( \mathbf{w}_t \) in the first inequality, used the triangle inequality \( \| \mathbf{w}_t - \mathbf{w} \| \geq \| \mathbf{w}_t - \mathbf{w} \| - \| \mathbf{w}_t - \mathbf{w} \| \) in the second inequality, dropped a negative term in the third inequality, and used the Cauchy-Schwarz inequality for random variables in the fourth inequality.

\[ \blacksquare \]

B.2. Proof of Theorem 7

When \( \ell(\mathbf{w}, \xi) \) is weakly convex (i.e., \( \lambda = 0 \)), set \( \gamma_t = \gamma \) for all \( t \geq 1 \) as in exact minibatch prox. Then summing (11) for \( t = 1, \ldots, T \) yields
\[
\begin{align*}
\sum_{t=1}^{T} \mathbb{E} [\phi(\mathbf{w}_t) - \phi(\mathbf{w}_*)] + \frac{\gamma_T}{2} \mathbb{E} \| \mathbf{w}_T - \mathbf{w}_* \|^2 & \leq \frac{\gamma}{2} \| \mathbf{w}_0 - \mathbf{w}_* \|^2 + \frac{4L^2T}{\gamma b} + \sum_{t=1}^{T} \sqrt{\frac{2L^2\eta_t}{\gamma}} \\
& + \sum_{t=1}^{T} \sqrt{2\gamma \eta_t} \cdot \sqrt{\mathbb{E} \| \mathbf{w}_t - \mathbf{w}_* \|^2} \\
\end{align*}
\]
where the expectation is taken over random sampling and the randomness of \( \mathcal{A} \) in the first \( T \) iterations. To resolve the recursion, we need the following lemma by Schmidt et al. (2011).

21
Lemma 13  Assume that the non-negative sequence \( \{u_T\} \) satisfies the following recursion for all \( T \geq 1 \):

\[
u_T^2 \leq S_T + \sum_{t=1}^{T} \lambda_t u_t,
\]

with \( S_T \) an increasing sequence, \( S_0 \geq u_0^2 \) and \( \lambda_t \geq 0 \) for all \( t \). Then, for all \( T \geq 1 \), we have

\[
u_T \leq \frac{1}{2} \sum_{t=1}^{T} \lambda_t + \left( S_T + \left( \frac{1}{2} \sum_{t=1}^{T} \lambda_t \right)^{2^{1/2}} \right) \leq \sqrt{S_T} + \sum_{t=1}^{T} \lambda_t.
\]

We are now ready to prove Theorem 7.

Proof Bounding \( \sqrt{\mathbb{E} \| \hat{w}_T - w_* \|^2} \). Dropping the \( \sum_{t=1}^{T} \mathbb{E} [\phi(\hat{w}_t) - \phi(w_*)] \) term from (26) which is non-negative due to the optimality of \( w_* \), we obtain

\[
\mathbb{E} \| \hat{w}_T - w_* \|^2 \leq \| \hat{w}_0 - w_* \|^2 + \frac{8L^2T}{\gamma^2b} + \sum_{t=1}^{T} \sqrt{\frac{8L^2 \eta_t}{\gamma^3}} + \sum_{t=1}^{T} \sqrt{\frac{8 \eta_t}{\gamma}} \cdot \sqrt{\mathbb{E} \| \hat{w}_t - w_* \|^2}.
\]

Now apply Lemma 13 (using \( u_T = \sqrt{\mathbb{E} \| \hat{w}_T - w_* \|^2}, S_T = \| \hat{w}_0 - w_* \|^2 + \frac{8L^2T}{\gamma^2b} + \sum_{t=1}^{T} \sqrt{\frac{8L^2 \eta_t}{\gamma^3}}, \) and \( \lambda_t = \sqrt{\frac{8 \eta_t}{\gamma}} \) and the fact that \( \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \) for \( x, y \geq 0 \), we have

\[
\sqrt{\mathbb{E} \| \hat{w}_T - w_* \|^2} \leq \| \hat{w}_0 - w_* \| + \frac{8L^2T}{\gamma^2b} + \sum_{t=1}^{T} \sqrt{\frac{8 \eta_t}{\gamma}} + \sum_{t=1}^{T} \sqrt{\frac{8L^2 \eta_t}{\gamma^3}}
\]

We have thus bounded the sequence of \( \sqrt{\mathbb{E} \| \hat{w}_T - w_* \|^2} \) by a non-negative increasing sequence.

Bounding function values. Dropping the \( \mathbb{E} \| \hat{w}_T - w_* \|^2 \) term from (26) which is non-negative, we obtain

\[
\sum_{t=1}^{T} \mathbb{E} [\phi(\hat{w}_t) - \phi(w_*)]
\]

\[
\leq \frac{\gamma}{2} \| \hat{w}_0 - w_* \|^2 + \frac{4L^2T}{\gamma b} + \sum_{t=1}^{T} \sqrt{\frac{2L^2 \eta_t}{\gamma}} + \sum_{t=1}^{T} \sqrt{2 \eta_t} \cdot \sqrt{\mathbb{E} \| \hat{w}_t - w_* \|^2}
\]

\[
\leq \frac{\gamma}{2} \| \hat{w}_0 - w_* \|^2 + \frac{4L^2T}{\gamma b} + \sum_{t=1}^{T} \sqrt{\frac{2L^2 \eta_t}{\gamma}} + \left( \sum_{t=1}^{T} \sqrt{2 \eta_t} \right) \cdot \max_{1 \leq t \leq T} \sqrt{\mathbb{E} \| \hat{w}_t - w_* \|^2}
\]

\[
\leq \frac{\gamma}{2} \| \hat{w}_0 - w_* \|^2 + \frac{4L^2T}{\gamma b} + \sum_{t=1}^{T} \sqrt{\frac{2L^2 \eta_t}{\gamma}}
\]

\[
+ \left( \sum_{t=1}^{T} \sqrt{2 \eta_t} \right) \cdot \left( \| \hat{w}_0 - w_* \| + \sqrt{\frac{8L^2T}{\gamma^2b}} + \sum_{t=1}^{T} \sqrt{\frac{8 \eta_t}{\gamma}} + \sum_{t=1}^{T} \sqrt{\frac{8L^2 \eta_t}{\gamma^3}} \right).
\]

(27)
To achieve the same order of regret as in exact minibatch prox, we require that \( \eta_t \) decays with \( t \), and in particular

\[
\eta_t \leq \min \left( c_1 \left( \frac{T}{b} \right)^{\frac{1}{3}}, c_2 \left( \frac{T}{b} \right)^{\frac{1}{2}} \right) \cdot \frac{L \| \hat{w}_0 - w_* \|}{\sqrt{t^{2+2\delta}}} \tag{28}
\]

for some \( \delta > 0 \). Note that \( \eta_t \) has the unit of function value. Let \( c := \sum_{i=1}^{\infty} \frac{1}{i^{1+\delta}} \leq \frac{1+\delta}{\delta} \) which only depends on \( \delta \) (as a concrete example, we have \( c = \frac{2^2}{3} \) when \( \delta = 2 \)).

Using the choice of \( \gamma = \sqrt{\frac{8T}{b}} \cdot \frac{L}{\| w_0 - w_* \|} \), we obtain from (28) that

\[
\begin{align*}
\sum_{t=1}^{T} \sqrt{\frac{8\eta_t}{\gamma}} &= \sum_{t=1}^{T} \sqrt{\frac{8b}{T}} \cdot \| w_0 - w_* \| \cdot \eta_t \leq 8^{\frac{1}{3}} c_1 \| \hat{w}_0 - w_* \| \sum_{t=1}^{T} \frac{1}{t^{1+\delta}} \leq 8^{\frac{1}{3}} c_1 \| \hat{w}_0 - w_* \|, \\
\sum_{t=1}^{T} \sqrt{\frac{8L^2\eta_t}{\gamma^3}} &= \sum_{t=1}^{T} \sqrt{\frac{8b}{T} \cdot \frac{L}{\| w_0 - w_* \|}} \cdot \eta_t \leq 8\frac{1}{3} c_2 \| \hat{w}_0 - w_* \|^2 \sum_{t=1}^{T} \frac{1}{t^{1+\delta}} \leq 8\frac{1}{3} c_2 \| \hat{w}_0 - w_* \|^2.
\end{align*}
\]

Continuing from (27) and substituting in the value of \( \gamma \), we have

\[
\sum_{t=1}^{T} \mathbb{E} [\phi(\hat{w}_t) - \phi(w_*)] \leq \sqrt{\frac{8T}{b}} \cdot L \| \hat{w}_0 - w_* \| + \gamma \frac{T}{2} \sum_{t=1}^{T} \sqrt{\frac{8L^2\eta_t}{\gamma^3}}
\]

\[
+ \gamma^2 \left( \sum_{t=1}^{T} \sqrt{\frac{8\eta_t}{\gamma}} \right) \cdot \left( 2 \| \hat{w}_0 - w_* \| + \sum_{t=1}^{T} \sqrt{\frac{8\eta_t}{\gamma}} + \sum_{t=1}^{T} \sqrt{\frac{8L^2\eta_t}{\gamma^3}} \right)
\]

\[
= \sqrt{\frac{8T}{b}} \cdot L \| \hat{w}_0 - w_* \| + \sqrt{\frac{2T}{b}} \cdot \frac{L}{\| w_0 - w_* \|} \cdot 8\frac{1}{3} c_2 \| \hat{w}_0 - w_* \|^2
\]

\[
+ \sqrt{\frac{2T}{b}} \cdot \frac{L}{\| w_0 - w_* \|} \cdot \frac{1}{2} \cdot \frac{8\frac{1}{3} c_2 \| \hat{w}_0 - w_* \|}{\| w_0 - w_* \|} \times
\]

\[
\left( 2 \| \hat{w}_0 - w_* \| + 8\frac{1}{3} c_1 \| \hat{w}_0 - w_* \| + \sqrt{8\frac{1}{3} c_2 \| \hat{w}_0 - w_* \|^2} \right)
\]

\[
= c_3 \sqrt{\frac{T}{b}} \cdot L \| \hat{w}_0 - w_* \|.
\]

The suboptimality of \( \hat{w}_T \) is then due to the convexity of \( \phi(w) \):

\[
\mathbb{E} [\phi(\hat{w}_T) - \phi(w_*)] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} [\phi(\hat{w}_t) - \phi(w_*)] = \frac{c_3 L \| \hat{w}_0 - w_* \|}{\sqrt{bT}}.
\]
B.3. Proof of Theorem 8

Proof. We have by Lemma 6 that

\[
\mathbb{E}_{I_t,A} [\phi(\tilde{w}_t) - \phi(w_*)] \leq \frac{\lambda(t-1)}{4} \|\tilde{w}_{t-1} - w_*\|^2 - \frac{\lambda(t+1)}{4} \mathbb{E}_{I_t,A} \|\tilde{w}_t - w_*\|^2 \\
+ \frac{8L^2}{\lambda b(t+1)} + \sqrt{\frac{4L^2\eta_t}{\lambda(t+1)}} + \sqrt{\frac{\lambda(t+1)\eta_t}{\lambda} \cdot \sqrt{\mathbb{E}_{I_t,A} \|\tilde{w}_t - w_*\|^2}}.
\]

Relaxing the \(\frac{1}{t+1}\) to \(\frac{1}{t}\) on the RHS, and multiplying both sides by \(t\), we further obtain

\[
t \cdot \mathbb{E}_{I_t,A} [\phi(\tilde{w}_t) - \phi(w_*)] \leq \frac{\lambda(t-1)t}{4} \|\tilde{w}_{t-1} - w_*\|^2 - \frac{\lambda(t+1)t}{4} \mathbb{E}_{I_t,A} \|\tilde{w}_t - w_*\|^2 \\
+ \frac{8L^2 t}{\lambda b} + \frac{4L^2 t\eta_t}{\lambda} + \sqrt{\lambda t\eta_t} \cdot \sqrt{\mathbb{E}_{I_t,A} \|\tilde{w}_t - w_*\|^2}.
\]

Summing this inequality for \(t = 1, \ldots, T\) yields

\[
\sum_{t=1}^T t \cdot \mathbb{E}_{I_t,A} [\phi(\tilde{w}_t) - \phi(w_*)] \leq \frac{8L^2 T}{\lambda b} + \sum_{t=1}^T \frac{4L^2 t\eta_t}{\lambda} + \sum_{t=1}^T \sqrt{\lambda t\eta_t} \cdot \sqrt{\mathbb{E}_{I_t,A} \|\tilde{w}_t - w_*\|^2}.
\] (29)

Bounding \(\sqrt{\mathbb{E} \|\tilde{w}_t - w_*\|^2}\). Dropping the \(\sum_{t=1}^T t \cdot \mathbb{E}_{I_t,A} [\phi(\tilde{w}_t) - \phi(w_*)]\) term from (29) which is non-negative due to the optimality of \(w_*\), we obtain

\[
\mathbb{E} \left[ T(T+1) \|\tilde{w}_T - w_*\|^2 \right] \leq \frac{32L^2 T}{\lambda^2 b} + \sum_{t=1}^T \frac{64L^2 t\eta_t}{\lambda^3} + \sum_{t=1}^T \frac{16t\eta_t}{\lambda} \cdot \sqrt{\mathbb{E} \left[ t(t+1) \|\tilde{w}_t - w_*\|^2 \right]}.
\]

Applying Lemma 13 (using \(u_T = \sqrt{\mathbb{E} \left[ T(T+1) \|\tilde{w}_T - w_*\|^2 \right]}\), \(S_T = \frac{32L^2 T}{\lambda^2 b} + \sum_{t=1}^T \frac{64L^2 t\eta_t}{\lambda^3}\), and \(\lambda_t = \sqrt{\frac{16t\eta_t}{\lambda}}\), we have

\[
\sqrt{\mathbb{E} \left[ T(T+1) \|\tilde{w}_T - w_*\|^2 \right]} \leq \sqrt{\frac{32L^2 T}{\lambda^2 b}} + \sum_{t=1}^T \sqrt{\frac{16t\eta_t}{\lambda}} + \sqrt{\sum_{t=1}^T \frac{64L^2 t\eta_t}{\lambda^3}}.
\]

Bounding function values. Dropping the \(\mathbb{E} \|\tilde{w}_T - w_*\|^2\) term from (29) which is non-negative, we obtain

\[
\sum_{t=1}^T t \cdot \mathbb{E}_{I_t,A} [\phi(\tilde{w}_t) - \phi(w_*)] \leq \frac{8L^2 T}{\lambda b} + \sum_{t=1}^T \sqrt{\frac{4L^2 t\eta_t}{\lambda}} \\
+ \left( \sqrt{\sum_{t=1}^T \lambda t\eta_t} \right) \cdot \left( \sqrt{\frac{32L^2 T}{\lambda^2 b}} + \sum_{t=1}^T \sqrt{\frac{16t\eta_t}{\lambda}} + \sqrt{\sum_{t=1}^T \frac{64L^2 t\eta_t}{\lambda^3}} \right). \tag{30}
\]
To achieve the same order of regret as in exact minibatch prox, we require that \( \eta_t \) decays with \( t \), and in particular

\[
\eta_t \leq \min \left( c_1 \left( \frac{T}{b} \right), c_2 \left( \frac{T}{b} \right)^2 \right) \cdot \frac{L^2}{t^{3+\delta} \lambda} \tag{31}
\]

for some \( \delta > 0 \). Note that \( \eta_t \) has the unit of function value. Let \( c := \sum_{i=1}^{\infty} \frac{1}{i^{1+\delta}} \leq \frac{1+\delta}{\delta} \). Then (31) ensures that

\[
\sum_{t=1}^{T} \sqrt{\frac{t \eta_t}{\lambda}} \leq c \sqrt{c_1} \sqrt{\frac{L^2 T}{\lambda^2 b}}, \quad \text{and} \quad \sum_{t=1}^{T} \sqrt{\frac{L^2 t \eta_t}{\lambda^3}} \leq c \sqrt{c_2} \cdot \frac{L^2 T}{\lambda^2 b}.
\]

Continuing from (30), we have

\[
\sum_{t=1}^{T} t \cdot \mathbb{E} \left[ \phi(\tilde{w}_t) - \phi(w_\star) \right] \leq \frac{8L^2 T}{\lambda b} + 2c \sqrt{c_2} \cdot \frac{L^2 T}{\lambda b} + c \sqrt{c_1} \sqrt{\frac{L^2 T}{\lambda^2 b}} + 4c \sqrt{c_1} \sqrt{\frac{L^2 T}{\lambda^2 b}} + \sqrt{64c^2} \sqrt{\frac{L^2 T}{\lambda^2 b}} = \frac{c_3}{2} \frac{L^2 T}{\lambda b}.
\]

In view of the convexity of \( \phi(w) \), by returning the weighted average \( \tilde{w}_T = \frac{2}{T(T+1)} \sum_{t=1}^{T} t \tilde{w}_t \), we have

\[
\mathbb{E} \left[ \phi(\tilde{w}_T) - \phi(w_\star) \right] \leq \frac{2}{T(T+1)} \sum_{t=1}^{T} t \cdot \mathbb{E} \left[ \phi(\tilde{w}_t) - \phi(w_\star) \right] \leq \frac{c_3 L^2}{\lambda b(T+1)}.
\]

**B.4. Connection to minibatch stochastic gradient descent**

To see the connection between minibatch prox and minibatch SGD, note that if we solve the linearized minibatch problem exactly, we obtain the minibatch stochastic gradient descent algorithm:

\[
\tilde{w}_t = \arg \min_{w \in \Omega} \phi_{I_t}(\tilde{w}_{t-1}) + \nabla \langle \phi_{I_t}(\tilde{w}_{t-1}), w - \tilde{w}_{t-1} \rangle + \frac{\gamma_t}{2} \| w - \tilde{w}_{t-1} \|^2.
\]

Following Cotter et al. (2011), we assume that \( \ell(w, \xi) \) is \( \beta \)-smooth:

\[
\| \nabla \ell(w, \xi) - \nabla \ell(w', \xi) \| \leq \beta \| w - w' \|, \quad \forall w, w' \in \Omega.
\]

Assume that the domain \( \Omega \) is bounded with radius \( B = \sup_{w \in \Omega} \| w \| \). We then have the following guarantee for each iterate of minibatch SGD.

**Proposition 14**  For iteration \( t \) of minibatch SGD, we have

\[
\mathbb{E}_{I_t} \left[ \phi(\tilde{w}_t) - \phi(w_\star) \right] \leq \frac{2L^2}{(\gamma_t - \beta)b} + \frac{\gamma_t - \lambda}{2} \| w_\star - \tilde{w}_{t-1} \|^2 - \frac{\gamma_t}{2} \mathbb{E}_{I_t} \| w_\star - \tilde{w}_t \|^2. \tag{32}
\]
**Proof** Our proof closely follows that of Cotter et al. (2011). Due to the smoothness of \( \phi \), we have that

\[
\phi(\tilde{w}_t) \leq \phi(\tilde{w}_{t-1}) + \left< \nabla \phi(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> + \frac{\beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 \\
\leq \phi(\tilde{w}_{t-1}) + \left< \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> + \frac{\beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 \\
+ \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> \\
= \phi(\tilde{w}_{t-1}) + \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \| \cdot \| \tilde{w}_t - \tilde{w}_{t-1} \| + \frac{\beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 \\
+ \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> \\
\leq \phi(\tilde{w}_{t-1}) + \frac{1}{2(\gamma_t - \beta)} \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \|^2 + \frac{\gamma_t - \beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 \\
+ \frac{\beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 + \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> \\
= \phi(\tilde{w}_{t-1}) + \frac{1}{2(\gamma_t - \beta)} \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \|^2 + \frac{\gamma_t - \beta}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 \\
+ \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> \tag{33}
\]

where we have used the Cauchy-Schwarz inequality in the second inequality, and the inequality \( xy \leq \frac{x^2}{2\alpha} + \frac{\alpha y^2}{2} \) in the third inequality.

Now, since \( \tilde{w}_t \) is the minimizer of the \( \gamma_t \)-strongly convex function

\[
\frac{\gamma_t}{2} \| w - \tilde{w}_{t-1} \|^2 + \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), w - \tilde{w}_{t-1} \right>
\]

in \( \Omega \), we have according to Lemma 1 (replacing the local objective with its linear approximation) that

\[
\frac{\gamma_t}{2} \| w_* - \tilde{w}_{t-1} \|^2 + \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), w_* - \tilde{w}_{t-1} \right> \\
\geq \frac{\gamma_t}{2} \| \tilde{w}_t - \tilde{w}_{t-1} \|^2 + \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), \tilde{w}_t - \tilde{w}_{t-1} \right> + \frac{\gamma_t}{2} \| w_* - \tilde{w}_t \|^2 .
\]

Substituting this into (33) gives

\[
\phi(\tilde{w}_t) \leq \phi(\tilde{w}_{t-1}) + \frac{1}{2(\gamma_t - \beta)} \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \|^2 + \frac{\gamma_t - \beta}{2} \| w_* - \tilde{w}_{t-1} \|^2 \\
+ \left< \nabla \phi_{I_t}(\tilde{w}_{t-1}), w_* - \tilde{w}_{t-1} \right> - \frac{\gamma_t}{2} \| w_* - \tilde{w}_t \|^2 .
\]

Taking expectation of this inequality over the random sampling of \( I_t \) further leads to

\[
\mathbb{E}_{I_t} \left[ \phi(\tilde{w}_t) \right] \leq \phi(\tilde{w}_{t-1}) + \frac{1}{2(\gamma_t - \beta)} \mathbb{E}_{I_t} \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \|^2 + \frac{\gamma_t}{2} \| w_* - \tilde{w}_{t-1} \|^2 \\
+ \left< \nabla \phi(\tilde{w}_{t-1}), w_* - \tilde{w}_{t-1} \right> - \frac{\gamma_t}{2} \mathbb{E}_{I_t} \| w_* - \tilde{w}_t \|^2 \\
\leq \phi(w_*) + \frac{1}{2(\gamma_t - \beta)} \mathbb{E}_{I_t} \| \nabla \phi(\tilde{w}_{t-1}) - \nabla \phi_{I_t}(\tilde{w}_{t-1}) \|^2 \\
+ \frac{\gamma_t - \lambda}{2} \| w_* - \tilde{w}_{t-1} \|^2 - \frac{\gamma_t}{2} \mathbb{E}_{I_t} \| w_* - \tilde{w}_t \|^2 \tag{34}
\]

26
where in the second inequality we have used the fact that

\[ \phi(w_*) \geq \phi(\tilde{w}_{t-1}) + \langle \nabla \phi(\tilde{w}_{t-1}), w_* - \tilde{w}_{t-1} \rangle + \frac{\lambda}{2} \|w_* - \tilde{w}_{t-1}\|^2 \]

due to the convexity of \( \phi(w) \).

On the other hand, let \( I_t = \{\xi_1, \ldots, \xi_b\} \), we have

\[
\begin{align*}
\mathbb{E}_{I_t} \|\nabla \phi(w) - \nabla \phi_{I_t}(w)\|^2 &= \mathbb{E}_{I_t} \left\| \frac{1}{b} \sum_{i=1}^{b} \nabla \ell(w, \xi_i) \right\|^2 \\
&= \mathbb{E}_{I_t} \left\| \frac{1}{b} \sum_{i=1}^{b} (\nabla \phi(w) - \nabla \ell(w, \xi_i)) \right\|^2 \\
&= \frac{1}{b^2} \sum_{i=1}^{b} \mathbb{E}_{\xi_i} \|\nabla \phi(w) - \nabla \ell(w, \xi_i)\|^2 + \frac{1}{b^2} \sum_{i \neq j} \mathbb{E}_{I_t} (\nabla \phi(w) - \nabla \ell(w, \xi_i), \nabla \phi(w) - \nabla \ell(w, \xi_j)) \\
&= \frac{1}{b} \cdot \mathbb{E}_{\xi} \|\nabla \phi(w) - \nabla \ell(w, \xi)\|^2 \\
&\leq \frac{4L^2}{b}
\end{align*}
\]

where we used the fact that the samples are i.i.d. in the fourth equality, and that \( \|\nabla \phi(w)\|, \|\nabla \ell(w, \xi)\| \leq L \) in the last inequality. Continuing from (34) yields the desired result.

Comparing this result to (24) and (25), we observe that minibatch SGD has a similar recursion to that exact minibatch prox, except the appearance of \( \beta \) in the denominator of the “stability” term. We now show that this difference leads to significant difference in convergence rate.

Let \( \ell(w, \xi) \) be weakly convex (\( \lambda = 0 \)), and \( \gamma_t = \gamma \) for all \( t \geq 1 \). Summing (32) over \( t = 1, \ldots, T \) gives

\[
\begin{align*}
\sum_{t=1}^{T} \mathbb{E} [\phi(\tilde{w}_t) - \phi(w_*)] &\leq \frac{2L^2T}{(\gamma - \beta)b} + \frac{\gamma}{2} \|w_* - \tilde{w}_0\|^2.
\end{align*}
\]

Minimizing the RHS over \( \gamma \) gives

\[
\gamma = \beta + \sqrt{\frac{4T}{b}} \cdot \frac{L}{\|w_* - \tilde{w}_0\|},
\]

which leads to

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} [\phi(\tilde{w}_t) - \phi(w_*)] \leq \frac{2L \|w_* - \tilde{w}_0\|}{\sqrt{bT}} + \frac{\beta \|w_* - \tilde{w}_0\|^2}{2T}.
\]

So we obtain the familiar \( O\left(\frac{1}{\sqrt{bT}} + \frac{1}{T}\right) \) rate for minibatch SGD.
Appendix C. Analysis of stochastic DANE

In order to fully analyze Algorithm 1, we need several auxiliary lemmas that characterize the iteration complexity of solving the local problem (12) by prox-SVRG (Xiao and Zhang, 2014), the large minibatch problem (15) by DANE (Shamir et al., 2014) and AIDE (Reddi et al., 2016).

C.1. Some auxiliary lemmas

First, we apply prox-SVRG to the local problem (12), pushing all terms but \( \phi_{t_i'(z)} \) in to the proximal operator; this does not complicate the computation since the proximal step simply computes the Euclidean projection onto \( \Omega \). The benefit of this approach (as opposed to using plain SVRG Johnson and Zhang, 2013) is that the smoothness parameter that determines the iteration complexity is simply \( \beta \), same results hold when applying prox-SAGA (Defazio et al., 2014) as well.

**Lemma 15 (Iteration complexity of prox-SVRG for (12))** For any target accuracy \( \theta > 0 \), with initialization \( z_{k-1} \), prox-SVRG outputs \( z_k \) such that \( \| z_k - z_k^* \| \leq \theta \) \( z_{k-1} - z_k^* \) after

\[
O \left( b + \frac{\beta}{\gamma + \kappa} \cdot \log \left( \frac{\beta + \gamma + \kappa}{(\gamma + \kappa)\theta^2} \right) \right)
\]

vector operations.

**Proof** Observe that the objective (12) by \( f_k'(z) \), which is an quadratic function of \( z \) with the Hessian matrix \( H_i = \nabla^2 \phi_{t_i'(z)} + (\kappa + \gamma)I \geq (\kappa + \gamma)I \). As a result, the suboptimality of \( z_k^i \) is

\[
\epsilon_{\text{final}} = f_k'(z_k^i) - f_k'(z_k^i)^* = \frac{1}{2} \left( z_k^i - z_k^i \right)^\top H_i \left( z_k^i - z_k^i \right) \geq \gamma + \kappa \frac{\| z_k^i - z_k^i \|^2}{2}.
\]

To satisfy the requirement of \( \| z_k^i - z_k^i \| \leq \theta \) \( z_{k-1} - z_k^i \), we require

\[
\epsilon_{\text{final}} \leq \frac{(\gamma + \kappa)\theta^2}{2} \| z_{k-1} - z_k^i \|^2.
\]

On the other hand, when initializing from \( z_{k-1} \), the initial suboptimality is

\[
\epsilon_{\text{init}} = f_k'(z_{k-1}) - f_k'(z_k^i)^* \leq \sigma_{\text{max}}(H_i) \frac{2}{\theta} \| z_{k-1} - z_k^i \|^2 \leq \frac{\beta + \gamma + \kappa}{2} \| z_{k-1} - z_k^i \|^2.
\]

Therefore, it suffices to have

\[
\frac{\epsilon_{\text{init}}}{\epsilon_{\text{final}}} = \frac{(\beta + \gamma + \kappa)}{(\gamma + \kappa)\theta^2}.
\]

Noting that \( \phi_{t_i'(z)} \) is the sum of \( b \) components, and each component is \( \beta \)-smooth while the overall function \( f_k'(z) \) is \( (\gamma + \kappa) \)-strongly convex, the lemma follows directly from the convergence guarantee of prox-SVRG (Xiao and Zhang, 2014, Corollary 1).

Next, we state the convergence rates of “inexact DANE” and AIDE, which can be easily derived from Reddi et al. (2016). At the outer loop \( t \) and intermediate loop \( r \), let \( x_t^r = \arg \min_{w \in \Omega} \hat{f}_{t,r}(w) \) be the exact minimizer of the “augmented large minibatch” problem (17), which is approximately solved by the inner DANE iterations.
Lemma 16 (Iteration Complexity of inexact DANE) Let \( \theta = \frac{1}{b} \), and assume that
\[
b(\gamma + \kappa)^2 \geq 256\beta^2 \log(dm/\delta). \tag{35}\]

By initializing from \( y_{r-1} \), and setting the number of inner iterations in Algorithm 1 to be
\[
K = \lceil \frac{1}{2} \log_{4/3} \left( \frac{\beta + \gamma + \kappa}{(\gamma + \kappa) \eta} \right) \rceil,
\]
we have with probability \( 1 - \delta \) over the sample set \( I_t \) that
\[
\tilde{f}_{t,r}(x_r) - \tilde{f}_{t,r}(x^*_r) \leq \eta \left( \tilde{f}_{t,r}(y_{r-1}) - \tilde{f}_{t,r}(x^*_r) \right).
\]

Proof Denote by \( H_i = \nabla^2\phi_i(z) + (\gamma + \kappa)I \) the Hessian matrix of the local objective (12) for machine \( i \). Let \( H = \frac{1}{m} \sum_{i=1}^m H_i \) be the Hessian matrix of the global objective (17), and \( \tilde{H}^{-1} = \frac{1}{m} \sum_{i=1}^m H_i^{-1} \). As our objective is quadratic, \( H_i, \tilde{H}, \tilde{H}^{-1} \) remain unchanged during the inner iterations. By Reddi et al. (2016, Theorem 1), we have
\[
\|z_k - x^*_r\| \leq \left( \left\| \tilde{H}^{-1} H - I \right\| + \frac{\theta}{m} \sum_{i=1}^m \left\| H_i^{-1} H \right\| \right) \|z_{k-1} - x^*_r\|. \tag{36}
\]

Since \( \nabla^2\ell(w, \xi) \leq \beta \), by Shamir et al. (2014, Lemma 2), we have with probability at least \( 1 - \delta \) over the sample set \( I_t \) that
\[
\|H_i - H\| \leq \sqrt{\frac{32\beta^2 \log(dm/\delta)}{b}} =: \rho, \quad i = 1, \ldots, m.
\]

On the other hand, we have \( H_i \succeq (\gamma + \kappa)I \) and
\[
\frac{4\rho^2}{(\gamma + \kappa)^2} = \frac{128\beta^2 \log(dm/\delta)}{b(\gamma + \kappa)^2} \leq \frac{1}{2}
\]
by our assumption (35). By Shamir et al. (2014, Lemma 1), we have
\[
\left\| \tilde{H}^{-1} H - I \right\| \leq \frac{1}{2}. \tag{37}
\]

Moreover, we have
\[
\frac{\theta}{m} \sum_{i=1}^m \left\| H_i^{-1} H \right\| \leq \frac{\theta}{m} \sum_{i=1}^m (1 + \left\| H_i^{-1} H - I \right\|)
\]
\[
\leq \frac{\theta}{m} \sum_{i=1}^m (1 + \left\| H_i^{-1} \left\| \| H - H_i^{-1} \right\| \right) \leq \frac{\theta}{m} \sum_{i=1}^m \left( 1 + \frac{\rho}{\gamma + \kappa} \right) \leq \frac{\theta}{m} \sum_{i=1}^m \left( 1 + \frac{1}{2\sqrt{2}} \right) \leq \frac{3\theta}{2} \leq \frac{1}{4}. \tag{38}
\]
Plugging (37) and (38) into (36) yields
\[
\|z_k - x^*_r\| \leq \frac{3}{4} \|z_{k-1} - x^*_r\|,
\]
and thus \(\|z_K - x^*_r\| \leq (3/4)^K \|y_{r-1} - x^*_r\|\). To guarantee the suboptimality in the objective \(\bar{f}_{t,r}(w)\), we note that
\[
\bar{f}_{t,r}(z_K) - \bar{f}_{t,r}(x^*_r) = \frac{1}{2}(z_K - x^*_r)^\top H(z_K - x^*_r) \leq \frac{\beta + \gamma + \kappa}{2} \|z_K - x^*_r\|^2
\]
\[
\leq \left(\frac{3}{4}\right)^{2K} \frac{\beta + \gamma + \kappa}{2} \|y_{r-1} - x^*_r\|^2
\]
\[
\leq \left(\frac{3}{4}\right)^{2K} \frac{\beta + \gamma + \kappa}{\gamma + \kappa} (\bar{f}_{t,r}(y_{r-1}) - \bar{f}_{t,r}(x^*_r))
\]
where we have used the fact that \(f_{t,r}(w)\) is \((\gamma + \kappa)\)-strongly convex in the last inequality. Setting \(\left(\frac{3}{4}\right)^{2K} \frac{\beta + \gamma + \kappa}{\gamma + \kappa} = \eta\), and noting \(x_r = z_K\), we obtain the desired iteration complexity.

At the outer iteration \(t\) of Algorithm 1, we are trying to approximately minimize the objective (15) by iteratively (approximately) solving \(R\) instances of the “augmented” problem (17). Let \(w^*_t\) be the exact minimizer of the “large minibatch” subproblem (15):
\[
w^*_t = \arg \min_{w \in \Omega} \bar{f}_t(w).
\]
The following lemma characterizes the accelerated convergence rate.

**Lemma 17 (Acceleration by universal catalyst, Theorem 3.1 of Lin et al. (2015))** Assume that for all \(r \geq 1\), we have
\[
\bar{f}_{t,r}(x_r) - \bar{f}_{t,r}(x^*_r) \leq \frac{2}{9} \left(1 - \frac{9}{10} \sqrt{\frac{\gamma}{\gamma + \kappa}}\right)^R \left(\bar{f}_t(x_0) - \bar{f}_t(w^*_t)\right),
\]
then
\[
\bar{f}_t(x_R) - \bar{f}_t(w^*_t) \leq \frac{800(\gamma + \kappa)}{\gamma} \left(1 - \frac{9}{10} \sqrt{\frac{\gamma}{\gamma + \kappa}}\right)^{R+1} \left(\bar{f}_t(x_0) - \bar{f}_t(w^*_t)\right).
\]

**C.2. Proof of Theorem 9**

**Proof** First of all, because \(R = 1\), our algorithm collapses into two nested loops.

On the one hand, as we choose \(\gamma\) as Theorem 7 suggested, we just need to verify the inexactness conditions in Theorem 7 is satisfied, i.e., for \(t = 1, \ldots, T\), we require (recall that \(w^*_t = \arg \min_{w \in \Omega} \bar{f}_t(w)\))
\[
\bar{f}_t(w_t) - \bar{f}_t(w^*_t) \leq \frac{1}{10^4} \cdot \min \left(\left(\frac{T}{bm}\right)^{1/2}, \left(\frac{T}{bm}\right)^{3/2}\right) \cdot \frac{2LB}{t^3},
\]
where we have used the fact that \(f_{t,r}(w)\) is \((\gamma + \kappa)\)-strongly convex in the last inequality. Setting \(\left(\frac{3}{4}\right)^{2K} \frac{\beta + \gamma + \kappa}{\gamma + \kappa} = \eta\), and noting \(x_r = z_K\), we obtain the desired iteration complexity.
On the other hand, we can bound the initial suboptimality by the $(\beta + \gamma)$-smoothness of $\tilde{f}_t(w)$:

$$\tilde{f}_t(\tilde{w}_{t-1}) - \tilde{f}_t(w^*) \leq \frac{\beta + \gamma}{2} \|\tilde{w}_{t-1} - w^*_t\| \leq 2(\beta + \gamma)B^2.$$ 

Combining the above two inequalities, it suffices to set the initial versus final error for the $K$ inexact DANE iterations to be

$$\frac{1}{\eta} = 10^4 \cdot \max \left( \left( \frac{bm}{T} \right)^{1/2}, \left( \frac{bm}{T} \right)^{3/2} \right) \cdot \frac{\rho^3 \cdot (\beta + \gamma)B}{L} \leq 10^4 \cdot \max \left( \left( \frac{bm}{T} \right)^{1/2}T^{5/2}, \left( \frac{bm}{T} \right)^{3/2}T^{3/2} \right) \cdot \frac{(\beta + \gamma)B}{L} \leq 10^4 \cdot \max \left( \left( \frac{n(\varepsilon)^{5/2}}{b^2m^2}, \left( \frac{n(\varepsilon)^{3/2}}{b^2m^2} \right) \cdot \frac{(\beta + \gamma)B}{L} \right.$$

where we have substituted in $T = \frac{n(\varepsilon)}{bm}$ in the last step. Using Lemma 16, we know as long as the inequality (35) is satisfied, we have the desired suboptimality in $\tilde{f}_t(w)$ using

$$K = \frac{1}{2} \log \frac{1}{\delta} \cdot 10^4 \cdot \max \left( \left( \frac{n(\varepsilon)^{5/2}}{b^2m^2}, \left( \frac{n(\varepsilon)^{3/2}}{b^2m^2} \right) \cdot \frac{(\beta + \gamma)^2B}{L} \right.$$\n
$$= \frac{1}{2} \log \frac{1}{\delta} \cdot 10^4 \cdot \max \left( \left( \frac{n(\varepsilon)^2}{bm}, \frac{bm \cdot n(\varepsilon)}{bm \cdot n(\varepsilon)} \right) \cdot \frac{(\beta + \gamma)^2B^2}{L^2} \right.$$\n
$$= \mathcal{O} \left( \log \frac{\beta B \cdot n(\varepsilon)}{L} \right)$$

rounds of communication, where we have plugged in the value of $\gamma$ in the second step. It remains to verify the condition (35), by our choice of $\gamma$ and $b$, we have

$$b\gamma^2 = \frac{2n(\varepsilon)L^2}{bm^2B^2} \geq \frac{2n(\varepsilon)L^2}{b^2m^2B^2} = 256\beta^2 \log(md),$$

as desired.

Next we summarize the communication, computation, and memory efficiency.

**Communication:** the total rounds of communication required by Algorithm 1 is

$$KRT = \mathcal{O} \left( \frac{n(\varepsilon)}{mb} \log \frac{\beta B \cdot n(\varepsilon)}{L} \right).$$

**Computation:** For each communication round, we need to solve the local problem (12) using prox-SVRG. Now, in view of (39), we have $\beta = \mathcal{O}(\sqrt{b}\gamma)$. This implies that $\frac{\beta}{\gamma} = \mathcal{O}(\sqrt{b})$ and thus by Lemma 15, the dominant term of the iteration complexity of prox-SVRG is

$$\mathcal{O} \left( b \log \frac{\beta + \gamma}{\gamma} \right) = \mathcal{O} \left( b \log \frac{\beta B \cdot n(\varepsilon)}{L} \right).$$

Multiplying this with the number of communication rounds yields the desired computation complexity.

**Memory:** It is straightforward to see each machine only need to maintain $b$ samples.
C.3. Proof of Theorem 11

Proof First, it is straightforward to verify the condition (35):

\[ b(\gamma + \kappa)^2 = 256\beta^2 \log(dm). \]

Similarly to Theorem 9, we need the ratio between final versus initial error for the \( R \) AIDE iterations to be

\[ \text{ratio} = \left( \frac{10^4 \cdot \max \left( \frac{n(\varepsilon)^{5/2}}{b^2 m^2}, \frac{n(\varepsilon)^{3/2}}{l^2} \right)}{\beta L} \right)^{-1}. \]

Equating this ratio to be \( \frac{800(\gamma + \kappa)}{\gamma} \left( 1 - \frac{9}{10} \sqrt{\frac{\gamma}{\gamma + \kappa}} \right)^{R+1} \), we have

\[ R = \frac{10}{9} \sqrt{\frac{\gamma + \kappa}{\gamma}} \log \left( \frac{800(\gamma + \kappa)}{\gamma} \cdot \frac{1}{\text{ratio}} \right) \]
\[ = \mathcal{O} \left( \frac{b^{1/4} m^{1/2} \cdot \beta^{1/2} B^{1/2}}{n(\varepsilon)^{1/4} \cdot L^{1/2}} \cdot \log \frac{\beta B \cdot n(\varepsilon)}{L} \right). \]

Now according to Lemma 17, the final suboptimality for \( \tilde{f}_{t,r}(w) \) need to be

\[ \epsilon_{\text{final}} = \frac{2}{9} \left( 1 - \frac{9}{10} \sqrt{\frac{\gamma}{\gamma + \kappa}} \right)^{R} \cdot \left( \tilde{f}_t(x_0) - \tilde{f}_t(w^*_t) \right). \]

Let us initialize \( \min_{w \in \Omega} \tilde{f}_{t,r}(w) \) by \( x_0 \). By definition, we have \( \tilde{f}_{t,r}(w) \geq \tilde{f}_t(w) \) and thus

\[ \epsilon_{\text{init}} = \tilde{f}_t(x_0) - \tilde{f}_{t,r}(x^*_r) \leq \tilde{f}_t(x_0) - \tilde{f}_t(x^*_r) \leq \tilde{f}_t(x_0) - \tilde{f}_t(w^*_t) \]

where we have used the fact that \( w^*_t \) is the minimizer of \( \tilde{f}_t(w) \) in the second inequality. This means we only need the initial versus final suboptimality of solving \( \tilde{f}_{t,r}(w) \) to be

\[ \frac{1}{\eta} = \frac{\epsilon_{\text{init}}}{\epsilon_{\text{final}}} = \frac{9}{2} \left( 1 - \frac{9}{10} \sqrt{\frac{\gamma}{\gamma + \kappa}} \right)^{-R}, \]

which, according to Lemma 16, is achieved by inexact DANE with

\[ K = \mathcal{O} \left( \log \frac{1}{\eta} + \log \frac{\beta + \gamma + \kappa}{\gamma + \kappa} \right) \]
\[ = \mathcal{O} \left( R \sqrt{\frac{\gamma}{\gamma + \kappa}} \right) \]
\[ = \mathcal{O} \left( \log \frac{\beta B \cdot n(\varepsilon)}{L} \right). \]

iterations.
Next we analyze the communication and computation efficiency of our algorithm. 

**Communication:** The total rounds of communication is

\[ KRT = \mathcal{O}\left( \log \frac{\beta B \cdot n(\varepsilon)}{L} \cdot \frac{b^{1/4} m^{1/2} \cdot \beta^{1/2} B^{1/2}}{n(\varepsilon)^{1/4} \cdot L^{1/2}} \log \frac{\beta B \cdot n(\varepsilon)}{b m} \right) \]

\[ = \mathcal{O}\left( \frac{n(\varepsilon)^{3/4} \cdot \beta^{1/2} B^{1/2}}{b^{3/4} m^{1/2} \cdot L^{1/2}} \log^2 \frac{\beta B \cdot n(\varepsilon)}{L} \right). \]

**Computation:** Similar to the case of \( b \leq b^* \), for each DANE local subproblem (12), the sample size \( b \) is larger than its condition number. Therefore, the total computational cost is

\[ \mathcal{O}(bKRT) = \mathcal{O}\left( \frac{b^{1/4} n(\varepsilon)^{3/4} \cdot \beta^{1/2} B^{1/2}}{m^{1/2} \cdot L^{1/2}} \log^2 \frac{\beta B \cdot n(\varepsilon)}{L} \right). \]

Appendix D. Experiments

| Name      | #Samples | #Features | loss   |
|-----------|----------|-----------|--------|
| codrna    | 271,617  | 8         | logistic |
| covtype   | 581,012  | 54        | logistic |
| kddcup99  | 1,131,571| 127       | logistic |
| year      | 463,715  | 90        | squared |

In this section we present empirical results to support our theoretical analysis of stochastic DANE. We perform least squares regression and classification on several publicly available datasets\(^7\); the statistics of these datasets and the corresponding losses are summarized in Table 3. For each dataset, we randomly select half of the samples for training, and the remaining samples are used for estimating the stochastic objective.

For stochastic DANE, we use SAGA (Defazio et al., 2014) to solve each local DANE subproblem (12) and fix the number of SAGA steps to \( b \) (i.e., we just make one pass over the local data), while varying the number of DANE rounds \( K \) over \( \{1, 2, 4, 8, 16\} \). For simplicity, we do not use catalyst acceleration and set \( R = 1 \) and \( \kappa = 0 \) in all experiments. Our experiments simulate a distributed environment with \( m \) machines, for \( m = 4, 8, 16 \). We conduct a simple comparison with minibatch SGD. Stepsizes for SAGA and minibatch SGD are set based on the smoothness parameter of the loss.

We plot in Figure 2 the estimated population objective vs. minibatch size \( b \) for different parameters. We make the following observations.

\(^7\)https://www.csie.ntu.edu.tw/~cjlin/libsvm/
Figure 2: Illustration of the convergence properties of stochastic DANE (SDANE), for different minibatch size $b$, number of machines $m$, and number of DANE iterations $K$.

- For minibatch SGD, as $b$ increases, the objective often increases quickly, this is because minibatch SGD cannot use large minibatch sizes while preserving sample efficiency.
- For stochastic DANE, the objective increases much more slowly as $b$ increases. This demonstrates the effectiveness of minibatch prox for using large minibatch sizes.
Running more iterations of DANE often helps, but with diminishing returns. This validates our theory that only a near-constant number of DANE iterations is needed for solving the large minibatch objective, without affecting the sample efficiency.