**STABILISATION OF GEOMETRIC DIRECTIONAL BUNDLE FOR A SUBANALYTIC SET**

SATOSHI KOIKE AND LAURENTIU PAUNESCU

**Abstract.** In the previous paper [8] we have introduced the notion of geometric directional bundle of a singular space, in order to introduce global bi-Lipschitz invariants. Then we have posed the question of whether or not the geometric directional bundle is stabilised as an operation acting on singular spaces. In this paper we give a positive answer in the case where the singular spaces are subanalytic sets, thus providing a new invariant associated with the subanalytic sets.

1. **Introduction.**

In a series of papers [3, 4, 5, 6, 7], in order to introduce several local bi-Lipschitz invariants, we investigated several local directional properties. Subsequently, in order to introduce global bi-Lipschitz invariants, we have globalised the local directional properties in [8]. In addition, to further study the global directional properties, we have introduced the notion of the geometric directional bundle of a singular space. We denote by $GD(A)$ the geometric directional bundle of $A$ over $\{0\}$, for $A \subset \mathbb{R}^n$ with $0 \in \overline{A}$ (see Definition 2.2 in §2.1 for the definition of geometric directional bundle). Then we have posed the following question in [8].

**Question 1.** (Question 1 in [8]). Is the operator $GD$ stabilised? Namely, does there exist a natural number $m \in \mathbb{N}$, not depending on $A$ but on $n$, such that $GD^m(A) = GD^{m+1}(A) = GD^{m+2}(A) = \cdots$ ?

Throughout this paper, let us denote by $\mathbb{N}$ the set of natural numbers in the sense of positive integers.

In the case where $n = 2$, we have obtained a positive answer to the above question.

**Proposition 1.1.** (Proposition 5.1 in [8]). Let $A \subset \mathbb{R}^2$ such that $0 \in \overline{A}$. Then we have $GD^3(A) = GD^4(A) = GD^5(A) = \cdots$.

In other words, the operation $GD$ is stabilised at degree 3.

In fact, in the Example 5.2 in [8] we have constructed a singular set $A \subset \mathbb{R}^2$ with $0 \in \overline{A}$ such that the operation $GD$ is stabilised at degree 3 but not degree 2 for $A$. Therefore we can see that 3 is the best degree of stabilisation for the operation $GD$, for a general $A \subset \mathbb{R}^2$ with $0 \in \overline{A}$.

**Date:** May 4, 2021.

2000 Mathematics Subject Classification. Primary 14P15, 32B20 Secondary 57R45.

Key words and phrases. geometric directional bundle, stabilisation, subanalytic set.

This research is partially supported by JSPS KAKENHI Grant Number JP20K03611.
In this paper we give a positive answer to the Question 1 in the case where \( A \subset \mathbb{R}^n \) is a subanalytic set. (See H. Hironaka [2] for the definition of a subanalytic set.) Our main result is the following.

**Theorem 1.2.** Let \( A \subset \mathbb{R}^n \) be a subanalytic set such that \( 0 \in \overline{A} \). Then we have
\[
GD^{n-1}(A) = GD^n(A) = GD^{n+1}(A) = \cdots
\]
if \( n \geq 2 \), and \( GD \) is stabilised at degree 1 if \( n = 1 \).

In §3 we shall prove Theorem 1.2, in fact it is a straightforward consequence of our Proposition 3.20. Concerning this theorem, we may ask whether we can strengthen the statement or not. Namely, we pose the following question.

**Question 2.** Does there exist a natural number \( m \in \mathbb{N} \) such that
\[
GD^m(A) = GD^{m+1}(A) = GD^{m+2}(A) = \cdots
\]
for any natural number \( n \in \mathbb{N} \) and any subanalytic set \( A \subset \mathbb{R}^n \) with \( 0 \in \overline{A} \)?

In §4 we shall give a negative example to Question 2.

2. Preliminaries

2.1. Geometric Directional Bundle. We first recall the direction set.

**Definition 2.1.** Let \( A \) be a subset of \( \mathbb{R}^n \), and let \( p \in \mathbb{R}^n \) such that \( p \in \overline{A} \), where \( \overline{A} \) denotes the closure of \( A \) in \( \mathbb{R}^n \). We define the direction set \( D_p(A) \) of \( A \) at \( p \) by
\[
D_p(A) := \{ a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{p\}, \ x_i \to p \in \mathbb{R}^n \text{ s.t. } \frac{x_i - p}{\|x_i - p\|} \to a, \ i \to \infty \}.
\]
Here \( S^{n-1} \) denotes the unit sphere centred at \( 0 \in \mathbb{R}^n \).

For a subset \( D_p(A) \subset S^{n-1} \), we denote by \( LD_p(A) \) the half-cone of \( D_p(A) \) with \( 0 \in \mathbb{R}^n \) as the vertex, and call it the real tangent cone of \( A \) at \( p \):
\[
LD_p(A) := \{ ta \in \mathbb{R}^n \mid a \in D_p(A), \ t \geq 0 \}.
\]

For \( p \in \overline{A} \), for simplicity, we put \( L_pD(A) := p + LD_p(A) \), and call it the geometric tangent cone of \( A \) at \( p \).

In the case where \( p = 0 \in \mathbb{R}^n \), we write \( D(A) := D_0(A) \) and \( LD(A) := L_0D(A) \) for short.

We next introduce the notion of geometric directional bundle.

**Definition 2.2.** Let \( A \subset \mathbb{R}^n \), and let \( W \subset \mathbb{R}^n \) such that \( \emptyset \neq W \subset \overline{A} \). We define the direction set \( D_W(A) \) of \( A \) over \( W \) by
\[
D_W(A) := \bigcup_{p \in W} (p, D_p(A)) \subseteq W \times S^{n-1} \subseteq \mathbb{R}^n \times S^{n-1}.
\]

We call
\[
GD_W(A) := W \times S^{n-1} \cap \overline{DA(A)},
\]
the geometric directional bundle of $A$ over $W$, where $\overline{D_A(A)}$ denotes the closure of $D_A(A)$ in $\mathbb{R}^n \times S^{n-1}$.

Let $\Pi : \mathbb{R}^n \times S^{n-1} \to S^{n-1}$ be the canonical projections defined by $\Pi(x, a) = a$. In the case where $W = \{p\}$ for $p \in \overline{A}$, we write

$$GD_p(A) := \Pi(GD_{\{p\}}(A)) \subseteq S^{n-1}.$$ 

This is the set of all possible limits of directional sets $D_q(A), q \in A, q \to p$.

We consider the half cone of $GD_p(A)$ with $0 \in \mathbb{R}^n$ as the vertex

$$L_{GD_p}(A) := \{tv \in \mathbb{R}^n \mid v \in GD_p(A), t \geq 0\}.$$ 

We call it the real tangent bundle cone of $A$ at $p$.

Similarly we define the geometric bundle cone of $A$ at $p$, to be its translation by $p$

$$L_pGD(A) := p + L_{GD_p}(A).$$

In the case where $p = 0 \in \mathbb{R}^n$, we simply write $GD(A) := GD_0(A)$, and $LGD(A) := L_0GD(A)$.

Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. For $m = 2, 3, \ldots$, we define

$$GD^m(A) := GD(LGD^{m-1}(A)).$$

Remark 2.3. Let $A \subset \mathbb{R}^n$, and let $p \in \mathbb{R}^n$ such that $p \in \overline{A}$. If $A$ is subanalytic, then $D_p(A) \subset GD_p(A)$. In particular, in this case we have

$$D(A) \subseteq GD^m(A) \subseteq GD^{m+1}(A)$$

for any $m \in \mathbb{N}$.

2.2. Basic properties. Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{A}$.

(1) $LGD(A)$ is a subanalytic subset of $\mathbb{R}^n$.

(II) The following equality holds:

$$LGD(A \setminus \{0\}) = LGD(A).$$

(III) Since $GD(A) (= GD_0(A))$ is a notion defined locally around $0 \in \mathbb{R}^n$, it suffices to consider $A \cap U$ for a sufficiently small neighbourhood $U$ of $0 \in \mathbb{R}^n$ when we treat Question. Throughout this paper, let us denote by $B_r(0)$ an $r$-neighbourhood of $0$ in $\mathbb{R}^n$ for $r > 0$, that is an open ball of radius $r$ with $0 \in \mathbb{R}^n$ as the centre.

Let us assume that $\dim A \geq 1$, and express $A \setminus \{0\}$ in a sufficiently small $\epsilon$-neighbourhood $B_\epsilon(0)$ of $0 \in \mathbb{R}^n$ as follows:

$$(A \setminus \{0\}) \cap B_\epsilon(0) = \bigcup_{i=1}^{m} C_i,$$

where $C_i$ is a connected component of $(A \setminus \{0\}) \cap B_\epsilon(0)$ such that $0 \in \overline{C_i}$ ($1 \leq i \leq m$).

Then we have

$$LGD(A \setminus \{0\}) = \bigcup_{i=1}^{m} LGD(C_i).$$
Example 2.4. (1) Let \( A_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 \text{ and } z > 0 \} \), and 
\( A_- = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 \text{ and } z < 0 \} \). Set \( A := A_+ \cup A_- \).

Since \( LGD(A_+) = LGD(A_-) = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq x^2 + y^2 \} \), we have \( LGD(A) = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq x^2 + y^2 \} \). Therefore \( LGD^2(A) = \mathbb{R}^3 \). It follows that \( A \subset \mathbb{R}^3 \) is stabilised at degree 2.

(2) Let \( A_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = 2(x^2 + y^2) \text{ and } z > 0 \} \), and 
\( A_- = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 = 2(x^2 + z^2) \text{ and } y > 0 \} \). Set \( A := A_+ \cup A_- \).

Since \( LGD(A) = \mathbb{R}^3 \), \( A \subset \mathbb{R}^3 \) is stabilised at degree 1.

(IV) Let \( \dim A = k, \) \( 0 \leq k \leq n \). A subanalytic set admits a locally finite Whitney stratification (H. Hironaka [2]). (See H. Whitney [10, 11] for the Whitney stratification.) Therefore, locally at \( 0 \in \mathbb{R}^n \) \( A \) admits a finite Whitney stratification \( S(A) \) such that for any stratum \( Q_i \in S(A) \), \( 0 \in \overline{Q_i} \). For this finite stratification \( S(A) \), we have the following properties:

(i) If \( Q_i \subset \overline{Q_j} \) for \( Q_i, Q_j \in S(A) \), then \( LGD(Q_i) \subset LGD(Q_j) \).

(ii) Let \( \Omega := \{Q_i \in S(A) \mid \#Q_j (j \neq i) \text{ s.t. } Q_i \subset \overline{Q_j} \} \). Then we have 
\[ LGD(A) = \bigcup_{i} LGD(Q_i), \]
where the union is taken over \( \Omega \).

2.3. A notation. In this subsection we prepare for a notation which we will be often used in the sequel.

Notation 1. Let \( m \) be a positive integer such that \( 0 < m < n \), and let \( C \subset \mathbb{R}^n \) be a cone with \( 0 \in \mathbb{R}^n \) as a vertex. If \( C \) is contained in a finite union of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \) (respectively If \( C \) is not contained in any finite union of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \)), we write
\[ C \subset \bigcup_{\text{finite}} \mathbb{R}^m \quad \text{(respectively } C \notin \bigcup_{\text{finite}} \mathbb{R}^m \text{).} \]

We give an example of an algebraic curve in \( \mathbb{R}^n \) which is not contained in any finite union of hyperplanes.

Example 2.5. Let \( \gamma : \mathbb{R} \to \mathbb{R}^n, n \geq 2, \) be an algebraic curve defined by \( \gamma(t) = (t, t^2, \ldots, t^n) \), and let \( A \subset \mathbb{R}^n \) be the image of \( \gamma \). Then even if we take a positive number \( \epsilon \) arbitrarily small, \( A \cap B_\epsilon(0) \) is not contained in any finite union of hyperplanes \( \mathbb{R}^{n-1} \subset \mathbb{R}^n \). In fact, suppose that \( A \cap B_\epsilon(0) \) is contained in a finite union of hyperplanes \( \bigcup_i H_i \). Here, we may assume that each \( H_i \) includes \( 0 \in \mathbb{R}^n \) and infinitely many points of \( A \cap B_\epsilon(0) \). Let one of such hyperplanes \( H_i \) be given by
\[ C_1 x_1 + C_2 x_2 + \cdots + C_n x_n = 0 \quad \text{for} \quad (C_1, C_2, \cdots, C_n) \neq (0, 0, \cdots, 0). \]

By assumption, the equation \( C_1 t + C_2 t^2 + \cdots + C_n t^n = 0 \) has infinitely many roots.
It follows that \( C_1 = C_2 = \cdots = C_n = 0 \). This is a contradiction. Thus we have
\[ A \cap B_\epsilon(0) \notin \bigcup_{\text{finite}} \mathbb{R}^{n-1}. \]
On other hand, we have $LGD^k(A) = \mathbb{R} \subset \mathbb{R}^n$ for any $k \in \mathbb{N}$. Note that $\mathcal{G}D$ is stabilised at degree 1 for $A$, and $\dim A = \dim LGD(A) = 1$.

### 3. Proof of Theorem 1.2

#### 3.1. Decomposition of a subanalytic set

In this subsection we describe a finite decomposition of the subanalytic sets into subanalytic subsets equipped with some dimensional conditions.

We first show an important property on $\mathcal{G}D$.

**Proposition 3.1.** Let $A, B \subset \mathbb{R}^n$ be subanalytic sets such that $0 \in \overline{A} \cap \overline{B}$. Then the following equality holds.

$$\mathcal{G}D(A \cup B) = \mathcal{G}D(A) \cup \mathcal{G}D(B)$$

**Proof.** One inclusion is trivial hence we concentrate in showing that $\mathcal{G}D(A \cup B) \subseteq \mathcal{G}D(A) \cup \mathcal{G}D(B)$.

In fact we show that $\overline{D_A(A)} \cup \overline{D_B(B)} = \overline{D_{A \cup B}(A \cup B)}$ so we will suffice to show $\overline{D_{A \cup B}(A \cup B)} \subseteq \overline{D_A(A)} \cup \overline{D_B(B)} = \overline{D_A(A)} \cup \overline{D_B(B)}$.

Let $x \in A \cup B$, $D_x(A \cup B) = D_x(A) \cup D_x(B)$. If $x \in A \cap B$ we clearly have that $D_x(A \cup B) = D_x(A) \cup D_x(B) \subseteq D_{A \cup B}(A \cup B)$ and we are done.

The problem is that in general we may have $x \in A \cup B$ but $x \in \overline{B} \setminus B$ for instance. Then $D_x(A \cup B) = D_x(A) \cup D_x(B)$ but $D_x(B)$ does not count in $\mathcal{G}D(B)$.

However in the subanalytic case $D_x(B)$ is approximated by $D_y(B)$ as $y \to x, y \in B$ i.e. $D_x(B) \subseteq \overline{D_y(B)}$ and thus we proved the desired inclusion. \hfill \Box

From the definition of $\mathcal{G}D^n$, we can show the following corollary of the above proposition by induction on $q$.

**Corollary 3.2.** Let $A, B \subset \mathbb{R}^n$ be subanalytic sets such that $0 \in \overline{A} \cap \overline{B}$, and let $q$ be a positive integer. Then the following equality holds.

$$\mathcal{G}D^q(A \cup B) = \mathcal{G}D^q(A) \cup \mathcal{G}D^q(B)$$

We can show a more general statement of the above corollary by induction on the number of subanalytic sets.

**Corollary 3.3.** Let $A_1, \ldots, A_s \subset \mathbb{R}^n$ be subanalytic sets such that $0 \in \overline{A_1} \cap \cdots \cap \overline{A_s}$, and let $q$ be a positive integer. Then the following equality holds.

$$\mathcal{G}D^q(A_1 \cup \cdots \cup A_s) = \mathcal{G}D^q(A_1) \cup \cdots \cup \mathcal{G}D^q(A_s)$$

Let $A \subset \mathbb{R}^n$ be a subanalytic set, and let $p \in A$. Since $A$ admits a locally finite Whitney stratification with analytic strata, there exist a positive number $\epsilon_0 > 0$ and an integer $k$ with $0 \leq k \leq n$ such that for any $0 < \epsilon < \epsilon_0$, $A \cap B_\epsilon(p)$ is a $k$-dimensional subanalytic set, where $B_\epsilon(p)$ denotes an $\epsilon$-neighbourhood of $p$ in $\mathbb{R}^n$. We call this $k$ the local dimension of $A$ at $p$. Using the local dimension, we next prepare for a dimensional condition.
Definition 3.4. Let \( A \subset \mathbb{R}^n \) be a \( k \)-dimensional subanalytic set (\( 0 \leq k \leq n \)). We say that \( A \) is genuinely \( k \)-dimensional, if for any \( p \in A \), the local dimension of \( A \) at \( p \) is \( k \).

Example 3.5. Let \( A \subset \mathbb{R}^3 \) be the Whitney umbrella. Then \( A \) is a 2-dimensional algebraic set. Since the local dimension at the points on the handle of the umbrella is 1, the Whitney umbrella \( A \) is not genuinely 2-dimensional.

Lemma 3.6. Let \( A \subset \mathbb{R}^n \) be a genuinely \( k \)-dimensional subanalytic set, \( 0 \leq k \leq n \), and let \( R_k \) be the set of regular points of \( A \). Then \( L_{GD}(A) \) can be expressed as a union of \( k \)-planes, and \( L_{GD}(A) = L_{GD}(R_k) \).

Proof. Since \( A \) is genuinely \( k \)-dimensional, the closure \( \hat{R}_k \) of \( R_k \) in \( A \) coincides with \( A \). As mentioned in Property (IV), locally at \( 0 \in \mathbb{R}^n \) \( A \) admits a finite Whitney stratification \( \mathcal{S}(A) \) such that for any stratum \( Q_i \in \mathcal{S}(A) \), \( 0 \in \overline{Q_i} \), and we may assume that \( \mathcal{S}(A) \) is compatible with \( R_k \) and \( A \setminus R_k \). Since \( A \) is genuinely \( k \)-dimensional, \( \mathcal{S}(A) \) satisfies the assumption of (ii) in Property (IV). Therefore we have

\[
L_{GD}(A) = \bigcup_{j} L_{GD}(Q_j),
\]

where the union is taken over \( \Omega_k := \{Q_i \in \mathcal{S}(A) \mid \dim Q_i = k \} \). It follows that \( L_{GD}(A) \) can be expressed as a union of \( k \)-planes. In addition, since

\[
\bigcup_{Q_j \in \Omega_k} Q_j \subset R_k,
\]

we have \( L_{GD}(A) = L_{GD}(R_k) \).

Let \( A \subset \mathbb{R}^n \) be a \( k \)-dimensional subanalytic set (\( 0 \leq k \leq n \)) such that \( 0 \in \overline{A} \). Define the following subsets of \( A \),

\[
A_i := \{ p \in A : \text{the local dimension of } A \text{ at } p \text{ is } i \}
\]

for \( 0 \leq i \leq k \). As above, locally at \( 0 \in \mathbb{R}^n \) \( A \) admits a finite Whitney stratification \( \mathcal{S}(A) \). We are assuming the frontier condition (see J. N. Mather [9]) for the Whitney stratification. Therefore the local dimension is constant on each stratum of \( \mathcal{S}(A) \). It follows that if \( A_i \), \( 0 \leq i \leq k \), is non-empty, \( A_i \) is a genuinely \( i \)-dimensional subanalytic set. In particular, since \( A \) is a \( k \)-dimensional subanalytic set, \( A_k \) is non-empty and a genuinely \( k \)-dimensional subanalytic set.

Set

\[
\Lambda_0(A) := \{ 0 \leq i \leq k \mid 0 \in \overline{A_i} \}.
\]

Then we can locally express \( A \) as a disjoint union around \( 0 \in \mathbb{R}^n \) as follows:

\[
(3.1) \quad A \cap B_\epsilon(0) = \bigcup_{i \in \Lambda_0(A)} A_i \cap B_\epsilon(0)
\]

for a sufficiently small \( \epsilon > 0 \). Since \( 0 \in \overline{A} \), \( \Lambda_0(A) \) is non-empty, thus we may consider \( m_0(A) := \min \Lambda_0(A) \).
Remark 3.7. If \(0 \notin A\), \(A_0\) is empty locally at \(0 \in \mathbb{R}^n\). Therefore \(0 \notin \Lambda_0(A)\).

In addition, if \(0 \in A_i\) for some positive integer \(i\), then \(0 \notin \Lambda_0(A)\). This shows that \(m_0(A) = 0\) only if \(0\) is an isolated point of \(A\).

Let \(q\) be a positive integer. By (II) and Corollary 3.3 we have the following formula.

\[
LGD^q(A) = LGD^q(A \setminus \{0\}) = \bigcup_{i \in \Lambda_0(A)} LGD^q(A_i \setminus \{0\}) = \bigcup_{i \in \Lambda_0(A)} LGD^q(A_i).
\]

3.2. Decomposition of a geometric bundle cone. In this subsection we discuss some properties concerning a specific decomposition of the geometric bundle cones.

(V) Let \(A \subset \mathbb{R}^n\) be a \(k\)-dimensional subanalytic set, \(0 \leq k \leq n\), such that \(0 \notin \overline{A}\). Write \(A\) as in the equality (3.1) above and denote by \(R_i\) the subset of all regular points in \(A_i\), \(i \in \Lambda_0(A)\).

\[
LGD(A) = \bigcup_{i \in \Lambda_0(A)} LGD(R_i)
\]

and each \(LGD(R_i)\) is a union of \(i\)-planes.

(VI) Let \(A\) be a genuinely \(k\)-dimensional subanalytic set, \(0 \leq k \leq n\), such that \(0 \in \overline{A}\), and let \(R_0\) be the set of regular points of \(A\). By Lemma 3.6 \(B := LGD(A) = LGD(R_k)\) and \(B\) is a union of \(k\)-planes \(B = \bigcup_{i \in I} \pi_i\). In \(B\) we distinguish two kind of points:

\[
B_+ := \{x \in B | \forall V \text{ open in } B, x \in V, V \cap \pi_i \neq \emptyset, \text{ for infinitely many indices } i \in I\},
\]

\[
B_0 := \{x \in B | \exists V \text{ open in } B, x \in V, V \cap \pi_i \neq \emptyset, \text{ only for finitely many indices } i \in I\},
\]

thus \(B = B_+ \cup B_0\). By definition \(B_0\) is open in \(B\) and thus its complement \(B_+\) is closed. Note that, as \(B = \bigcup_{i \in I} \pi_i\), \(B_0\) has to be contained in a finite union of \(k\)-planes \(\pi_i\) from \(B\) (it is a subset of a union of \(k\)-planes and the local dimension is \(k\)). As such, \(LGD(B) = LGD(B_+) \cup LGD(B_0)\) where \(LGD(B_0)\), if not empty, is a finite union of \(k\)-planes, thus of dimension \(k\). In fact we know that \(B_0\) is genuinely \(k\)-dimensional by construction so \(LGD(B_0) = LGD(R)\), \(R\) its regular points, which has to be a finite union of connected components and each component has to be a subset of a participating plane in \(B_0\), so \(LGD(R)\) must be the finite union of those planes. Now if \(B_+\) is not empty it has to have dimension bigger than \(k\) and thus we can apply the decomposition above to conclude that \(LGD(B_+) = \cup LGD(R_j)\) where \(R_j\) are \(j\)-dimensional analytic manifolds with \(j > k\).

Example 3.8. Let \(A \subset \mathbb{R}^3\) be a 2-dimensional algebraic set defined by

\[
A := \{(x, y, z) \in \mathbb{R}^3 \mid xy(x^2 + y^2 - z^2) = 0\}.
\]

Then \(A\) is a genuinely 2-dimensional subanalytic set, and we have

\[
B_0 = \{(x, y, x) \in \mathbb{R}^3 \mid xy = 0 \& z^2 > x^2 + y^2\},
\]

\[
B_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq x^2 + y^2\}.
\]
Let $R_2$ be the set of regular points of $A$, and let $B := \lgd(A) = \lgd(R_2)$. $B = B_0 \cup B_+$ and $B$ is a union of 2-planes. Note that $B_0$ is contained in a union of two 2-planes.

$B_0$ is 2-dimensional and open in $B$. On the other hand, $B_+$ is 3-dimensional and closed in $B$. In addition, $\lgd(B_0) = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$ is a union of two 2-planes.

3.3. Observations. In this subsection we observe that Theorem 1.2 holds in the case where $n \leq 3$. We first show the case $n = 1$.

Assertion 3.9. If $n = 1$, $\mathcal{G}\mathcal{D}$ is stabilised at degree 1.

Proof. Let $A \subset \mathbb{R}$ be a subanalytic set such that $0 \in \overline{A}$. Then $A$ is $\{0\}$, $[0, \epsilon)$, $(0, \epsilon)$, $(-\epsilon, 0]$, $(-\epsilon, 0)$ ($\epsilon$ is a sufficiently small positive number), $(-\epsilon_1, \epsilon_2)$ or $(-\epsilon_1, \epsilon_2) \setminus \{0\}$ ($\epsilon_1$ and $\epsilon_2$ are sufficiently small positive numbers) in a small neighbourhood of $0 \in \mathbb{R}$. In case $A = \{0\}$, $\lgd(A) = \emptyset$. Otherwise, we have $\lgd(A) = \mathbb{R}$. It follows that $\mathcal{G}\mathcal{D}$ is stabilised at degree 1. \qed

We next show the case $n = 2$.

Lemma 3.10. Let $n \geq 2$. Suppose that $\dim A = 1$. Then $\mathcal{G}\mathcal{D}$ is stabilised at degree 1.

Proof. Let us express $A \setminus \{0\}$ in a sufficiently small $\epsilon$-neighbourhood $B_\epsilon(0)$ of $0 \in \mathbb{R}^n$ like in (2.2)(III):

$$(A \setminus \{0\}) \cap B_\epsilon(0) = \bigcup_{i=1}^m C_i,$$

where $C_i$ is a connected component of $(A \setminus \{0\}) \cap B_\epsilon(0)$ such that $0 \in \overline{C_i}$ ($1 \leq i \leq m$). Here $C_i$ is a subanalytic curve. Therefore each $\lgd(C_i) = \mathbb{R} \subset \mathbb{R}^n$ ($1 \leq i \leq m$) in our notation. (Some $\mathbb{R}$’s may coincide.) It follows that $\mathcal{G}\mathcal{D}$ is stabilised at degree 1. \qed

Assertion 3.11. If $n = 2$, $\mathcal{G}\mathcal{D}$ is stabilised at degree 1.

Proof. In the case where $\dim A = 0$, $\lgd(A) = \emptyset$. In the case where $\dim A = 2$, $\lgd(A) = \mathbb{R}^2$. In these cases it is obvious that $\mathcal{G}\mathcal{D}$ is stabilised at degree 1.

By Lemma 3.10 $\mathcal{G}\mathcal{D}$ is stabilised at degree 1 in the case where $\dim A = 1$. \qed

Assertion 3.12. If $n = 3$, $\mathcal{G}\mathcal{D}$ is stabilised at degree 2.

Proof. In the cases where $\dim A = 0$ and $\dim A = 1$, $\mathcal{G}\mathcal{D}$ is stabilised at degree 1 as seen in the proof of Assertion 3.11. In the case where $\dim A = 3$, $\lgd(A) = \mathbb{R}^3$. Therefore $\mathcal{G}\mathcal{D}$ is stabilised at degree 1.

Let us consider the case $\dim A = 2$. As in Lemma 3.10, let us express $A \setminus \{0\}$ in a sufficiently small $\epsilon$-neighbourhood $B_\epsilon(0)$ of $0 \in \mathbb{R}^3$:

$$(A \setminus \{0\}) \cap B_\epsilon(0) = \bigcup_{i=1}^m C_i.$$
We may assume that $C_1, \ldots, C_k$, $1 \leq k \leq m$, are genuinely 2-dimensional, and $C_{k+1}, \ldots, C_m$ are 1-dimensional. In the case where $k+1 \leq i \leq m$, namely in the 1-dimensional case, we have $LGD(C_i) = \mathbb{R} \subset \mathbb{R}^3$ in our notation. Therefore $GD$ is stabilised at degree 1.

We next consider the 2-dimensional case. By Lemma 3.6, for each $1 \leq i \leq k$, $LGD(C_i)$ is a union of 2-planes. Let

$$LGD(C_i) \subseteq \bigcup_{\text{finite}} \mathbb{R}^2, \ 1 \leq i \leq s,$$

and

$$LGD(C_i) \not\subseteq \bigcup_{\text{finite}} \mathbb{R}^2, \ s + 1 \leq i \leq k.$$

In the case where $1 \leq i \leq s$, we have $LGD(C_i) = \bigcup_{\text{finite}} \mathbb{R}^2$. Therefore $GD$ is stabilised at degree 1. On the other hand, in the case where $s + 1 \leq i \leq k$, the subanalytic set $LGD(C_i)$ contains infinitely many 2-planes. Therefore we have $\dim LGD(C_i) = 3$, and $LGD^2(C_i) = \mathbb{R}^3$. Thus $GD$ is stabilised at degree 2.

It follows that $GD$ is stabilised at degree 2 in the case of $n = 3$.

3.4. **Proof in the general case.** We first recall some fundamental properties, seen in 3.3.

**Property 3.13.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{A}$. Then we have the following.

1. If $\dim A = 0$, $GD$ is stabilised at degree 1.
2. If $\dim A = 1$, $GD$ is stabilised at degree 1.
3. If $\dim A = n$, $GD$ is stabilised at degree 1.

Before we start the proof of Theorem 1.2, we prepare for some lemmas. The following lemma is an immediate consequence of Lemma 3.6.

**Lemma 3.14.** Let $A \subset \mathbb{R}^n$ be a genuinely $k$-dimensional subanalytic set, $1 \leq k \leq n$, such that $0 \in \overline{A}$. Then we have $m_0(LGD(A)) \geq k$.

**Lemma 3.15.** Let $A \subset \mathbb{R}^n$ be a genuinely $k$-dimensional subanalytic set, $1 \leq k \leq n$, such that $0 \in \overline{A}$. Suppose that $\dim LGD(A) = k$. Then $LGD(A)$ can be expressed as a union of some finite $k$-planes.

**Proof.** By Lemma 3.6, $LGD(A)$ can be expressed as a union of $k$-planes. Let us assume that the subanalytic set $LGD(A)$ cannot be expressed as a union of finite $k$-planes. Then it is a union of infinitely many different $k$-planes. It implies that $\dim LGD(A) > k$. This contradicts the assumption $\dim LGD(A) = k$. Thus $LGD(A)$ can be expressed as a union of some finite $k$-planes.

Related to the above lemma, we can present a proposition. Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{A}$. Like (3.1), let us locally express $A$ around $0 \in \mathbb{R}^n$ as follows:

$$A \cap B_{\epsilon}(0) = \bigcup_{i \in \Lambda_0(A)} A_i \cap B_{\epsilon}(0)$$

for a sufficiently small $\epsilon > 0$. Then we have the following proposition.
Proposition 3.16. Let $i_0$ be the smallest integer in $\Lambda_0(A)$ such that $\mathcal{LGD}(A_{i_0})$ is not contained in any finite union of hyperplanes. Then we have $\dim \mathcal{LGD}^{n-i_0}(A) = n$.

We next introduce some notations concerning $\mathcal{LGD}(A)$.

**Notation 2.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{\mathcal{A}}$, and let $m_0(A) = k$, $1 \leq k < n$. Then let us denote by $\mathcal{LGD}(A)_{=k}$ (respectively $\mathcal{LGD}(A)_{\geq k+1}$) the set of points of $\mathcal{LGD}(A)$ at which the local dimension of it is $k$ (respectively bigger than or equal to $k+1$).

**Remark 3.17.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in \overline{\mathcal{A}}$, and let $m_0(A) = k$, $1 \leq k < n$. By Lemma 3.14, we have

$$\mathcal{LGD}(A) = \mathcal{LGD}(A)_{=k} \cup \mathcal{LGD}(A)_{\geq k+1}.$$

**Example 3.18.** Let $A \subset \mathbb{R}^6$ be a 5-dimensional algebraic set defined by

$$A := \{(x, y, z, u, v, w) \in \mathbb{R}^6 \mid x (x^2 + y^2 - z^4)^2 + (u^2 + v^2 - w^4)^2 = 0\}.$$

Let us denote by $A_{=5}$ (respectively $A_{=4}$) the set of points of $A$ at which the local dimension of it is 5 (respectively 4) as above. Then we can express $A$ as the union of $A_{=5}$ and $A_{=4}$, where

$$A_{=5} = \{(x, y, z, u, v, w) \in \mathbb{R}^6 \mid x = 0\},$$

$$A_{=4} = \{(x, y, z, u, v, w) \in \mathbb{R}^6 \mid (x^2 + y^2 - z^4)^2 + (u^2 + v^2 - w^4)^2 = 0\} \setminus \{(x, y, z, u, v, w) \in \mathbb{R}^6 \mid x = 0 \& (x^2 + y^2 - z^4)^2 + (u^2 + v^2 - w^4)^2 = 0\}.$$

We can see that $A_{=5} = \mathbb{R}^5 \subset \mathbb{R}^6$ and $\mathcal{LGD}(A_{=4}) = \mathbb{R}^6$. Therefore $\mathcal{GD}$ is stabilised at degree 0 for $A_{=5}$ in some sense, and stabilised at degree 1 for $A_{=4}$. It follows that $\mathcal{GD}$ is stabilised at degree 1 for $A$. We note that the lower dimensional subset $A_{=4}$ of $A$ takes an essential role in this stabilisation for $A$.

We prepare for one more lemma.

**Lemma 3.19.** Let $A \subset \mathbb{R}^n$ be a genuinely $k$-dimensional subanalytic set, $1 \leq k \leq n$, such that $0 \in \overline{\mathcal{A}}$. Then $\mathcal{LGD}(A)_{=k}$ is contained in a union of some finite $k$-planes.

**Proof.** By Lemma 3.6, $B := \mathcal{LGD}(A)$ is a union of $k$-planes $B = \cup_{i \in I} \pi_i$. We consider the decomposition of $B$ in (VI), namely $B = B_0 \cup B_+$. On the other hand, we have another decomposition $B = \mathcal{LGD}(A)_{=k} \cup \mathcal{LGD}(A)_{\geq k+1}$. By definition, we can easily see that $B_0 \subset \mathcal{LGD}(A)_{=k}$. Pick a point $P \in \mathcal{LGD}(A)_{=k}$. Then there exists a subanalytic neighbourhood $W \subset \mathbb{R}^n$ of $P$ such that the $k$-dimensional subanalytic set $W \cap B$ admits a finite Whitney stratification $\mathcal{S}_P(B)$ satisfying the following conditions:

1. Any $k$-dimensional stratum of $\mathcal{S}_P(B)$ is analytically diffeomorphic to a $k$-plane.
2. For any $k$-dimensional stratum $\sigma \in \mathcal{S}_P(B)$, $P$ is in the closure of $\sigma$.

Suppose that $P \in B_+$. Then there exists an infinite subset $I_1$ of $I$ such that $W \cap \pi_i \neq \emptyset$ for $i \in I_1$ and $W \cap B = W \cap \cup_{i \in I_1} \pi_i$ ($= \cup_{i \in I_1} (W \cap \pi_i)$). Note that $\cup_{i \in I_1} \pi_i$ cannot yield a point in $W$ at which the local dimension of $B$ is bigger than $k$. 
Then the union of strata of $S_P(B)$ does not coincide with $\cup_{i \in I_i} (W \cap \pi_i)$. Therefore $P \in B_0$. It follows that $LGD(A)_{=k} = B_0$. Thus, by (VI), $LGD(A)_{=k}$ is contained in a union of some finite $k$-planes.

Now let us start the proof of Theorem 1.2. By Assertions 3.9 and 3.11 it suffices to show the theorem in the case where $n \geq 3$. After this, we assume $n \geq 3$. Under this assumption, we have the following result.

**Proposition 3.20.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in A$. Suppose that $2 \leq m_0(A) \leq n$. Then $GD$ is stabilised at degree $n - (m_0(A) - 1)$.

**Proof.** Let us show this proposition by downward induction on $m_0(A)$. By Property 3.13 (3), $GD$ is stabilised at degree 1 for any genuinely $n$-dimensional subanalytic set $A$ with $0 \in A$. Therefore the proposition holds in the case where $m_0(A) = n$.

Let $i$ be a positive integer with $2 \leq i \leq n-1$. Let us assume that this proposition holds for any subanalytic set $B$ with $0 \in B$ such that $m_0(B) \geq i + 1$. Let $A \subset \mathbb{R}^n$ be an arbitrary subanalytic set with $m_0(A) = i$ such that $0 \in A$.

We first consider the case where $LGD(A)_{=i} = \emptyset$. Then $m_0(LGD(A)) \geq i + 1$. By assumption on induction, for the subanalytic set $LGD(A)$, $GD$ is stabilised at degree $n - (m_0(LGD(A)) - 1)$. Therefore, for $LGD(A)$, $GD$ is stabilised at degree $n - i$. It follows that for $A$, $GD$ is stabilised at degree $n - i + 1 = n - (i - 1) = n - (m_0(A) - 1)$.

We next consider the case where $LGD(A)_{=i} \neq \emptyset$. If $LGD(A)_{\geq i+1} = \emptyset$, then $\cup_{j \in A_0(A) \setminus \{i\}} A_j = \emptyset$. Therefore $A = A_i$ is a genuinely $i$-dimensional subanalytic set. Since $dimLGD(A) = i$, by Lemma 3.15 $LGD(A)$ can be expressed as a union of some finite $\mathbb{R}^i$. Therefore, for $A$, $GD$ is stabilised at degree 1. Since $n - (m_0(A) - 1) = n - i + 1 \geq n - (n - 1) + 1 = 2 > 1$,

$GD$ is stabilised at degree $n - (m_0(A) - 1)$.

If $LGD(A)_{\geq i+1} \neq \emptyset$, then by Remark 3.17 we have

$$LGD(A) = LGD(A)_{=i} \cup LGD(A)_{\geq i+1}.$$ 

Note that this union is a disjoint one. By Lemma 3.14 $LGD(A)_{=i} \subset LGD(A_i)$. It follows that $LGD(A)_{=i} = LGD(A_i)_{=i}$. Since $A_i$ is a genuinely $i$-dimensional subanalytic set, by Lemma 3.19 $LGD(A)_{=i}$ is contained in a union of some finite $\mathbb{R}^i$. Therefore for $LGD(A)_{=i}$, $GD$ is stabilised at degree 1. On the other hand, $m_0(LGD(A)_{\geq i+1}) \geq i+1$. Using the same argument as in the case of $LGD(A)_{=i} = \emptyset$, we can see that for $LGD(A)_{\geq i+1}$, $GD$ is stabilised at degree $n - i$. Therefore for $LGD(A)$, $GD$ is stabilised at degree $\max\{1, n - i\} = n - i$. It follows that for $A$, $GD$ is stabilised at degree $n - i + 1 = n - (m_0(A) - 1)$.

**Proof of Theorem 1.2.** By Proposition 3.20 if $2 \leq m_0(A) \leq n$, then $GD$ is stabilised at degree $n - (m_0(A) - 1)$. Since $n - (m_0(A) - 1) \leq n - (2 - 1) = n - 1$,

$GD$ is stabilised at degree $n - 1$ under the assumption that $2 \leq m_0(A) \leq n$.

We next consider the case where $m_0(A) = 1$. If $A$ is a 1-dimensional subanalytic set, then by Property 3.13 (2), $GD$ is stabilised at degree 1. Since $n \geq 2$, $GD$ is
stabilised at degree $n - 1$. If $A$ is a subanalytic set of dimension bigger than or equal to 2, we have a partition of $A$ into two subanalytic subsets $A_1$ and $A_2$, where $A_1$ consists of finite subanalytic curves and $A_2$ consists of points of $A$ at which the local dimension of it is bigger than or equal to 2. Therefore for $A_1$, $GD$ is stabilised at degree 1, and for $A_2$, $GD$ is stabilised at degree $n - 1$. It follows that for $A$, $GD$ is stabilised at degree $\max\{1, n - 1\} = n - 1$.

We lastly consider the case where $m_0(A) = 0$. In this case, the dimension of $A$ locally at $0 \in \mathbb{R}^n$ is 0. Because if the dimension of $A$ locally at $0 \in \mathbb{R}^n$ is bigger than 0, then $0 \in \mathbb{R}^n$ is a point of some genuinely positive dimensional subanalytic subset of $A$ or $0 \notin A$. Therefore the condition $m_0(A) = 0$ is equivalent to the condition $A = \{0\}$ locally at $0 \in \mathbb{R}^n$. Then, by Property 3.13 (1), $GD$ is stabilised at degree 1. It follows that $GD$ is stabilised at degree $n - 1$. 

\[ \square \]

4. Negative example to Question 2

Let us recall Question 2.

**Question 2.** Does there exist a natural number $m \in \mathbb{N}$ such that

\[ GD^m(A) = GD^{m+1}(A) = GD^{m+2}(A) = \ldots \]

for any natural number $n \in \mathbb{N}$ and any subanalytic set $A \subset \mathbb{R}^n$ with $0 \in \overline{A}$?

We first prepare for some lemmas.

**Lemma 4.1.** Let $A \subset \mathbb{R}^n$ be a $k$-dimensional subanalytic set, $0 \leq k \leq n$, such that $0 \in \overline{A}$. Then we have

\[ k \leq \dim L(GD)(A) \leq \min(2k, n). \]

**Proof.** By definition, it is obvious that $k \leq \dim L(GD)(A) \leq n$. Therefore let us show the statement that $\dim L(GD)(A) \leq 2k$ if $2k < n$.

We locally express $A$ around $0 \in \mathbb{R}^n$ as (3.1):

\[ A \cap B_\epsilon(0) = \bigcup_{i \in \Lambda_0(A)} A_i \cap B_\epsilon(0) \]

for a sufficiently small $\epsilon > 0$, where each $A_i$, $i \in \Lambda_0(A)$, is a genuinely $i$-dimensional subanalytic set. If we can show that $\dim L(GD)(A_i) \leq 2i$, $i \in \Lambda_0(A)$, then the statement follows. Therefore, from the beginning, we may assume that $A$ is a genuinely $k$-dimensional subanalytic set. Let $R_k$ be the set of regular points of $A$. By Lemma 3.6, $L(GD)(A) = L(GD)(R_k)$. Since $GD_{\{0\}}(R_k) = \{0\} \times S^{n-1} \cap \overline{D_{R_k}(R_k)}$, $\dim GD_{\{0\}}(R_k) \leq 2k - 1$.

Therefore we have

\[ \dim GD(R_k) = \dim \Pi(GD_{\{0\}}(R_k)) \leq 2k - 1, \]

where $\Pi : \mathbb{R}^n \times S^{n-1} \to S^{n-1}$ is the canonical projection by definition. But in this case, since $GD_{\{0\}}(R_k) \subset \{0\} \times S^{n-1}$, we can regard $\Pi|_{\{0\} \times S^{n-1}}$ as the identification map between $\{0\} \times S^{n-1}$ and $S^{n-1}$. It follows that $\dim L(GD)(A) = \dim L(GD)(R_k) \leq (2k - 1) + 1 = 2k$. 

\[ \square \]
Remark 4.2. Let $A \subset \mathbb{R}^n$ be a $k$-dimensional subanalytic set. Then $\dim \operatorname{LGD}^2(A) \leq 4k - 1$ for instance. Indeed

$$\operatorname{LGD}^2(A) = \operatorname{LGD}(\operatorname{LGD}(A)) = \operatorname{LGD}(\text{Cone}(B)) = \text{Cone}(\operatorname{LGD}(B)),$$

where $\operatorname{LGD}(A) = \text{Cone}(B)$ and $\operatorname{LGD}(B) := \cup_{b \in B} \operatorname{LGD}(B)$, the union of the corresponding geometric bundles. Now $\dim \operatorname{LGD}(A) \leq 2k$ so $\dim B \leq 2k - 1$, thus by induction $\dim \operatorname{LGD}(B) \leq 2(2k-1)$ thus finally $\dim \text{Cone}(\operatorname{LGD}(B)) \leq 2(2k-1)+1 = 4k - 1$, a bit of improvement. Using the same observation we may get an estimation for

$$\dim \operatorname{LGD}'(A) \leq 2^r k - 2^{-1} + 1, \ r \geq 1.$$

Here we used the formula $\operatorname{LGD}(\text{Cone}(B)) = \text{Cone}(\operatorname{LGD}(B))$. Thus starting with $A$, two dimensional in $\mathbb{R}^8$, such that $\operatorname{LGD}(A)$ is not contained in any finite union of hyperplanes, then $\operatorname{LGD}^4(A) \neq \operatorname{LGD}^2(A)$. If we start with $A$, 2-dimensional such that $\operatorname{LGD}(A)$ is 3-dimensional, then $\operatorname{LGD}^2(A)$ has dimension $\leq 1 + 4 = 5$ thus it cannot stabilise at 2 in $\mathbb{R}^6$. Can we find such examples in $\mathbb{R}^4, \mathbb{R}^5$?

Example 4.3. Let $A_0$ be a 2-dimensional algebraic set in $\mathbb{R}^3$ defined by

$$A_0 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^4\}.$$

Then it is easy to see that $\operatorname{LGD}(A_0) = \mathbb{R}^3$ thus it stabilises at degree 1.

Let $A$ be a 4-dimensional algebraic set in $\mathbb{R}^6$ defined by

$$A = \{(x, y, z, u, v, w) \in \mathbb{R}^6 \mid x^2 + y^2 = z^4 \ \& \ u^2 + v^2 = w^4\} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^4\} \times \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 = w^4\}.$$

Then we have $\operatorname{LGD}(A) = \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$, thus it stabilises at degree 1. In this way we can create examples of subspaces $A$ in any codimension, such that $\operatorname{LGD}(A)$ is the entire space.

Lemma 4.4. Let $A \subset \mathbb{R}^n$ be a $k$-dimensional subanalytic set, $0 < k < n$, such that $0 \in \overline{A}$. Suppose that $\operatorname{LGD}(A)$ is not contained in any finite union of hyperplanes. Then $\operatorname{LGD}(A) \notin A \cup F$, where $\operatorname{LGD}(F)$ is contained in a finite union of hyperplanes.

Proof. Consider a decomposition of $A$ into a $k$-dimensional subanalytic set $A_k$ and a smaller dimensional subanalytic set $B$. Here we take a decomposition in $(3,1)$ and $B = \cup_{i<k} A_i$. In this case $A_k$ is a genuinely $k$-dimensional. If $\operatorname{LGD}(A) = \operatorname{LGD}(B) \cup \operatorname{LGD}(A_k) \subset A \cup F$ such that $\operatorname{LGD}(F)$ is contained in a finite union of hyperplanes, then there are two cases to consider.

Case I: Let $\operatorname{LGD}(A_k)$ be a finite union of $k$-planes. Then $\operatorname{LGD}(B) \subset B \cup A_k \cup F$ where $\operatorname{LGD}(A_k \cup F)$ is contained in a finite union of hyperplanes and $\operatorname{LGD}(B)$ is not contained in any finite union of hyperplanes, which is a contradiction by induction on dimension.

Case II: Let $\operatorname{LGD}(A_k)$ be not contained in any finite union of hyperplanes. Then $\operatorname{LGD}(A_k) = A^+ \cup A^0$ such that $A^0$ is a finite union of hyperplanes and $A^+$ of dimension bigger than $k$ at any point. Accordingly $\operatorname{LGD}(A^+)$ is not contained in any finite union of hyperplanes (same as $\operatorname{LGD}(A_k)$). We have $A^+ \subset A \cup F$, so $A^+ \setminus A \subset F$. 

Note that $A^+ \setminus A$ is dense in $A^+$ and thus $LGD(A^+) = LGD(A^+ \setminus A) \subseteq LGD(F)$ is contained in a finite union of hyperplanes by assumption, which is a contradiction again.

We conclude that $LGD(A) \not\subseteq A \cup F$ where $LGD(F)$ is contained in a finite union of hyperplanes. \hfill \square

As corollaries of Lemma 4.4 we have the following.

**Corollary 4.5.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \notin \overline{A}$. Suppose that $LGD(A)$ is not contained in any finite union of hyperplanes. If $A = LGD(A)$, then $A = \mathbb{R}^n$.

**Corollary 4.6.** Let $A \subset \mathbb{R}^n$ be a subanalytic set such that $0 \notin \overline{A}$. Let $m$ be a positive integer such that $1 < m < n - 1$. Suppose that $LGD(A)$ is not contained in any finite union of hyperplanes, and that

\[ LGD^{m-1}(A) \neq LGD^m(A) \text{and } LGD^m(A) = LGD^{m+1}(A). \]

Then $LGD^m(A) = \mathbb{R}^n$.

**Example 4.7.** Let $n$ be an an natural number with $n \geq 3$, and let $J := (-\frac{1}{2}, \frac{1}{2})$. We define a regular curve $\gamma : J \to S^{n-1}$ by

\[ \gamma(t) := (t, t^2, \ldots, t^{n-1}, \sqrt{1 - (t^2 + \cdots t^{2(n-1)})}). \]

Let us denote by $C$ the image of $\gamma$.

We call the intersection of $S^{n-1}$ and a hyperplane in $\mathbb{R}^n$ passing through $0 \in \mathbb{R}^n$ a hyperplane of $S^{n-1}$. Let us show that $C \subset S^{n-1}$ is not contained in any finite union of hyperplanes $S^{n-1}$. We assume that $C$ is contained in a finite union of hyperplanes $\bigcup_i (H_i \cap S^{n-1})$ of $S^{n-1}$. Here, each $H_i$ is a hyperplane in $\mathbb{R}^n$ passing through $0 \in \mathbb{R}^n$, and some of $H_i$’s include infinitely many points of $C$. Let one of such hyperplanes $H_i$ be given by

\[ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0 \quad \text{for} \quad (a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0). \]

By assumption, the equation

\[ a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} + a_n \sqrt{1 - (t^2 + \cdots t^{2(n-1)})} = 0 \]

has infinitely many roots. Therefore the polynomial equation

\[ (a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1})^2 = (-a_n \sqrt{1 - (t^2 + \cdots t^{2(n-1)})})^2 \]

also has infinitely many roots. It follows that $a_n = 0$ and $a_1 = a_2 = \cdots = a_{n-1} = 0$. This is a contradiction. Thus $C \subset S^{n-1}$ is not contained in any finite union of hyperplanes $S^{n-1}$.

Let $A$ be a cone of $C$ with $0 \in \mathbb{R}^n$ as the vertex. Then $\dim A = 2$ and $\dim LGD(A) = 3$. By construction, we can see that $LGD(A) \cap S^{n-1} \supset C$. Therefore $LGD(A)$ is not contained in any finite union of hyperplanes in $\mathbb{R}^n$.

Suppose that there exists a natural number $m \in \mathbb{N}$ such that $GD$ is stabilised at degree $m$ for any natural number $n \in \mathbb{N}$ and any subanalytic set $A \subset \mathbb{R}^n$ with $0 \in \overline{A}$. 

Then we choose a natural number $n \in \mathbb{N}$ so that $n > 3 \cdot 2^{m-1}$. We construct $A \subset \mathbb{R}^n$ in the above way. By Lemma 4.1 we have $\dim \text{LGD}^m(A) \leq 3 \cdot 2^{m-1} < n$, which contradicts Corollary 4.6. Thus there does not exist a natural number $m \in \mathbb{N}$ such that $\text{GD}$ is stabilised at degree $m$ for any natural number $n \in \mathbb{N}$ and any subanalytic set $A \subset \mathbb{R}^n$ with $0 \in A$.

References

[1] A. Fish and L. Paunescu, Unwinding spirals, I. Methods Appl. Anal. (Special issue dedicated to the memory of John Mather) 25 (2018), 3, 225–232.
[2] H. Hironaka, Subanalytic sets, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, pp. 453–493, Kinokuniya, Tokyo, 1973.
[3] S. Koike and L. Paunescu, The directional dimension of subanalytic sets is invariant under bi-Lipschitz homeomorphisms, Ann. Inst. Fourier 59 (2009), 2448–2467.
[4] S. Koike, Ta Lê Lôï, L. Paunescu and M. Shiota, Directional properties of sets definable in $o$-minimal structures, Ann. Inst. Fourier 63 (2013), 2017–2047.
[5] S. Koike and L. Paunescu, On the geometry of sets satisfying the sequence selection property, J. Math. Soc. Japan 67 (2015), 721–751.
[6] S. Koike and L. Paunescu, (SSP) geometry with directional homeomorphisms, J. Singularity 13 (2015), 169–178.
[7] S. Koike and L. Paunescu, Applications of the sequence selection property to bi-Lipschitz geometry, European Journal of Mathematics, 5 (2019), 1202–1211.
[8] S. Koike and L. Paunescu, Global directional properties of singular spaces, Revue Roumaine de Mathématiques Pures et Appliquées, 64 (2019), 479–501.
[9] J. N. Mather: Notes on topological stability, Bull. Amer. Math. Soc. 49 (2012), 475–506.
[10] H. Whitney, Local topological properties of analytic varieties, Differential and Combinatorial Topology (ed. S.S. Cairns), A Symposium in Honor of M. Morse, pp. 205–244, Princeton Univ. Press, 1965.
[11] H. Whitney, Tangents to an analytic variety, Ann. of Math. 81 (1965), 496–549.

Department of Mathematics, Hyogo University of Teacher Education, 942-1 Shimokume, Kato, Hyogo 673-1494, Japan
Email address: koike@hyogo-u.ac.jp

School of Mathematics and Statistics, University of Sydney, Sydney, NSW, 2006, Australia
Email address: laurentiu.paunescu@sydney.edu.au