Lagrangians Galore

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Abstract

Searching for a Lagrangian may seem either a trivial endeavour or an impossible task. In this paper we show that the Jacobi last multiplier associated with the Lie symmetries admitted by simple models of classical mechanics produces (too?) many Lagrangians in a simple way. We exemplify the method by such a classic as the simple harmonic oscillator, the harmonic oscillator in disguise [H Goldstein, *Classical Mechanics*, 2nd edition (Addison-Wesley, Reading, 1980)] and the damped harmonic oscillator. This is the first paper in a series dedicated to this subject.

1 Introduction

The last multiplier of Jacobi [1, 2, 3, 4, 5] is probably nowadays the generally most forgotten of Jacobi’s contributions to Mathematics. Even after Lie [6] showed that his newly introduced symmetries provided a very direct route to the calculation of the multiplier, its use in practice was slight despite excellent descriptions of its properties and usage in such classics as the text of Bianchi [7]. For a listing of the applications of the Jacobi Last Multiplier see the bibliography of [8]. Although Lie groups became of central importance in some areas of Theoretical Physics, the primary idea of using infinitesimal transformations to elucidate the properties of differential equations fell into disuse apart from the bowdlerised variation known as Buckingham’s Theorem which is widely appreciated by engineers. With the decline in interest in the Lie algebraic properties of differential equations the last multiplier became a mathematical oddity known only to a select few.

The revival of interest in the Lie algebraic analysis of differential equations began some fifty years ago, but its widespread use as a standard tool is only from about half that period ago. One of the obstacles to the use of the techniques of symmetry analysis has always been the high effort in the computation of the symmetries. Their calculation could be characterised as being tedious without the reward of substantial intellectual stimulation.

The advent of symbolic manipulation codes [9, 10] has allowed the tedium to be transferred to the computer and freed the brain of the operator for thinking.

A similar problem of computation besets the calculation of the Jacobi Last Multiplier. Unfortunately there seems not to have been the same incentive to use the multiplier as there was in the application of symmetry methods to the equations of gas dynamics and it is only in the last decade that Jacobi’s Last Multiplier has seen application in its rightful place with the beneficial assistance of the computer [11, 12, 13, 8, 14, 15].

Jacobi’s Last Multiplier is a solution of the linear partial differential equation [3, 4, 5, 16],

\[
\frac{\partial M}{\partial t} + \sum_{i=1}^{N} \frac{\partial (Ma_i)}{\partial x_i} = 0,
\]  

(1)

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where $\partial_t + \sum_{i=1}^N a_i \partial x_i$ is the vector field of the set of first-order ordinary differential equations for the $N$ dependent variables $x_i$. The ratio of any two multipliers is a first integral of the system of first-order differential equations and in the case that this system is derived from the Lagrangian of a one-degree-of-freedom system one has that \[\frac{\partial^2 L}{\partial \dot{q}^2} = M.\] (2)

Consequently a knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system.

We recall that Lie’s method \[6, 17\] for the calculation of the Jacobi Last Multiplier is firstly to find the value of

$$\Delta = \det \begin{bmatrix} e_{ij} & s_{ij} \end{bmatrix},$$

in which the matrix is square with the elements $e_{ij}$ being the vector field of the set of first-order differential equations by which the system is described and the elements, $s_{ij}$, being the coefficient functions of the number of symmetries of the given system necessary to make the matrix square. If $\Delta$ is not zero, the corresponding multiplier is $M = \Delta^{-1}$. Here we consider that the vector fields of the system of equations and symmetries are known and that we seek the multiplier. From another direction one could know the multiplier and all but one of the symmetries. From (3) the remaining symmetry may be determined \[11\]. Moreover one can use equation (1) to raise the order of the system and find nonlocal symmetries of the original equation \[8\].

When one has a Lagrangian, it is natural to consider the Noether symmetries of its Action Integral. These constitute a subset, not necessarily proper, of the Lie symmetries of the corresponding Lagrangian equation of motion. The Lagrangians obtained may be equivalent or inequivalent. The distinction can be variously expressed. We use the Noether point symmetries. If two Lagrangians have the same number and algebra of Noether point symmetries, they are equivalent. Otherwise they are inequivalent.

If we have different Lagrangians, be they equivalent or inequivalent, it is natural to consider the relationships among the Lagrangians, the symmetries which produce them through the Jacobi Last Multiplier, the Noether symmetries of the Lagrangians and the associated Noether integrals. This last group returns us to the Jacobi Last Multipliers since the integrals are ratios of multipliers.

In this paper we explore the connections mentioned above. For the purposes of our investigation we choose some simple one-degree-of-freedom problems with much symmetry. These are the simple harmonic oscillator with equation of motion

$$\ddot{q} + k^2 q = 0,$$

the simple harmonic oscillator in disguise \[18\] [ex 18, p 433] with equation of motion

$$q\ddot{q} - 2\dot{q}^2 - k^2 q^2 = 0$$

and the damped linear oscillator with equation of motion

$$\ddot{q} + 2c^2 \dot{q} + (c^2 + k^2)q = 0,$$

where $c$ and $k$ are constants and in the case of the damped oscillator the coefficient of $q$ has been written as such to simplify later expressions. We have made our selection for several reasons. Firstly the simple harmonic oscillator is ‘too good’ in the sense that one of its multipliers is a constant and so all multipliers are first integrals. This is not the case with \[5\] and \[6\]. Since the three equations are related by point transformations, we do not have to present all of the details for each equation. In fact under a point transformation the Lie point
symmetries of (4) become the corresponding symmetries of (5)/(6). The corresponding multipliers are related according to \[3, 4, 5\]
\[\tilde{M} = M|J|,\]
where \(J\) is the Jacobian of the transformation.

The paper is structured thusly. In Section 2 we present the Lie point symmetries of (4), the associated multipliers and Lagrangians. In Section 3 we give the Noether point symmetries of these Lagrangians and their associated integrals. The calculations are routine and so in both sections we simply state the results. In §4 we present an analysis of these results.

2 Lie symmetries, Jacobi Last Multipliers and Lagrangians

To determine Jacobi’s Last Multipliers one writes the system under consideration as a set of first-order ordinary differential equations. Thus (4) becomes
\[\dot{u}_1 = u_2,\]
\[\dot{u}_2 = -k^2u_1\]
with associated vector field
\[X_{SHO} = \partial_t + u_2\partial_{u_1} - k^2u_1\partial_{u_2}.\]
We immediately derive that a Jacobi’s last multiplier of (8)/(9) is a constant, namely 1. In fact (1) yields
\[\frac{d\log M}{dt} = - \left[ \frac{\partial u_2}{\partial u_1} + \frac{\partial (-k^2u_1)}{\partial u_2} \right] = 0.\]
As a linear second-order ordinary differential equation (4) possesses eight Lie point symmetries \[17\]. In terms of the variables used in (8) the eight vectors are
\[\Gamma_1 = \cos kt\partial_{u_1} - k\sin kt\partial_{u_2}\]
\[\Gamma_2 = \sin kt\partial_{u_1} + k\cos kt\partial_{u_2}\]
\[\Gamma_3 = u_1\partial_{u_1} + u_2\partial_{u_2}\]
\[\Gamma_4 = \partial_t\]
\[\Gamma_5 = \cos 2kt\partial_t - ku_1\sin 2kt\partial_{u_1} - \left(2k^2u_1\cos 2kt - ku_2\sin 2kt\right)\partial_{u_2}\]
\[\Gamma_6 = \sin 2kt\partial_t + ku_1\cos 2kt\partial_{u_1} - \left(2k^2u_1\sin 2kt + ku_2\cos 2kt\right)\partial_{u_2}\]
\[\Gamma_7 = u_1\cos kt\partial_t - ku_1\sin 2kt\partial_{u_1} - \left(2k^2u_1^2\cos kt + ku_1u_2\cos kt + u_2^2\cos kt\right)\partial_{u_2}\]
\[\Gamma_8 = u_1\sin kt\partial_t + ku_1^2\cos kt\partial_{u_1} - \left(2k^2u_1^2\cos kt - ku_1u_2\cos kt + u_2^2\sin kt\right)\partial_{u_2}\]
in which we have used the ordering: solution symmetries (\(\Gamma_1, \Gamma_2\)), homogeneity symmetry (\(\Gamma_3\)), \(sl(2, R)\) symmetries (\(\Gamma_5, \Gamma_6\)) and adjoint symmetries (\(\Gamma_7, \Gamma_8\)). The last are so called since the time-dependent components of the coefficient functions are solutions of the equation adjoint to (4), which in this case is itself. For a general second-order equation this be not the case.

The possible Jacobi Last Multipliers are the reciprocals of the nonzero determinants of the possible matrices of (3). We denote them by \(JLM_{ij}\), where the \(i\) and \(j\) refer to the symmetries used in the determinants. For
example $JLM_{12}$ is obtained from

$$
\Delta_{12} = \det \begin{bmatrix} 1 & u_2 & -k^2u_1 \\ 0 & \cos kt & -k \sin kt \\ 0 & \sin kt & k \cos kt \end{bmatrix} = k. \tag{12}
$$

In the matrix the second row comes from $\Gamma_1$ and the third from $\Gamma_2$. Then we obtain

$$
\begin{align*}
JLM_{12} &= k \\
JLM_{13} &= \frac{1}{ku_1 \sin kt + u_2 \cos kt} \\
JLM_{14} &= \frac{1}{k} JLM_{23} \\
JLM_{15} &= \frac{1}{k} JLM_{23} \\
JLM_{16} &= \frac{1}{k} JLM_{13} \\
JLM_{17} &= -JLM_{13}^2 = -\frac{1}{(ku_1 \sin kt + u_2 \cos kt)^2} \\
JLM_{18} &= -JLM_{13} \times JLM_{23} = \frac{1}{(ku_1 \sin kt + u_2 \cos kt)(ku_1 \cos kt - u_2 \sin kt)} \\
JLM_{23} &= \frac{1}{-ku_1 \cos kt + u_2 \sin kt} \\
JLM_{24} &= \frac{1}{k} JLM_{13} \\
JLM_{25} &= \frac{1}{k} JLM_{13} \\
JLM_{26} &= \frac{1}{k} JLM_{23} \\
JLM_{27} &= JLM_{18} \\
JLM_{28} &= -JLM_{23}^2 = -\frac{1}{(-ku_1 \cos kt + u_2 \sin kt)^2} \\
JLM_{34} &= \left[ JLM_{13}^{-2} + JLM_{23}^{-2} \right]^{-1} = \frac{1}{u_2^2 + k^2u_1^2} \\
JLM_{35} &= \left[ JLM_{13}^{-2} - JLM_{23}^{-2} \right]^{-1} = \frac{1}{[u_2^2 - k^2u_1^2] \cos 2kt + 2ku_1u_2 \sin 2kt} \\
JLM_{36} &= -\frac{1}{2} JLM_{18} \\
JLM_{37} &= 0 \\
JLM_{38} &= 0 \\
JLM_{45} &= \frac{1}{2k} JLM_{18} \\
JLM_{46} &= \frac{1}{k} JLM_{35} \\
JLM_{47} &= JLM_{13} JLM_{34} = \frac{1}{(u_2^2 + k^2u_1^2)(ku_1 \sin kt + u_2 \cos kt)} \\
JLM_{48} &= JLM_{23} JLM_{34} = \frac{1}{(u_2^2 + k^2u_1^2)(-ku_1 \cos kt + u_2 \sin kt)}
\end{align*}
$$
\[ JLM_{56} = \frac{1}{k} JLM_{34} \]
\[ JLM_{57} = JLM_{13} JLM_{35} = \frac{1}{(ku_1 \sin kt + u_2 \cos kt)((u_2^2 - k^2 u_1^2) \cos 2kt + 2ku_1 u_2 \sin 2kt)} \]
\[ JLM_{58} = JLM_{23} JLM_{45} = \frac{1}{(-ku_1 \cos kt + u_2 \sin kt)((u_2^2 - k^2 u_1^2) \cos 2kt + 2ku_1 u_2 \sin 2kt)} \]
\[ JLM_{67} = \frac{1}{2} JLM_{13} JLM_{25} = \frac{1}{2(ku_1 \sin kt + u_2 \cos kt)^2(-ku_1 \cos kt + u_2 \sin kt)} \]
\[ JLM_{68} = \frac{1}{2} JLM_{13} JLM_{23}^2 = \frac{1}{2(ku_1 \sin kt + u_2 \cos kt)(-ku_1 \cos kt + u_2 \sin kt)^2} \]
\[ JLM_{78} = 0. \]  

(13)

Note that there are 14 different multipliers (multiplicative constants are inessential).

For each of these 14 multipliers we can calculate a Lagrangian using (2). One readily appreciates that two arbitrary functions of integration in the variables \( t \) and \( u_1 \) are introduced with the double quadrature, say \( f_1(t, u_1) \) and \( f_2(t, u_1) \). This means that there is a doubly infinite family of Lagrangians corresponding to each multiplier. There must be a constraint on these functions for the application of the variational principle to give precisely (4). This constraint is listed with the Lagrangians. We use the same subscripts to identify the Lagrangians. The constraint is placed after the Lagrangian to which it applies.

\[ L_{12} = \frac{1}{2} u_2^2 + f_1 u_2 + f_2, \]  
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = k^2 u_1; \]  

(14)

\[ L_{13} = \sec^2 kt \left[ \log(ku_1 \sin kt + u_2 \cos kt)(ku_1 \sin kt + u_2 \cos kt) - u_2 \cos kt - ku_1 \sin kt \right] + f_1 u_2 + f_2, \]  
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]  

(15)

\[ L_{23} = \csc^2 kt \left[ \log(-ku_1 \cos kt + u_2 \sin kt)(-ku_1 \cos kt + u_2 \sin kt) - u_2 \sin kt + ku_1 \cos kt \right] + f_1 u_2 + f_2, \]  
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]  

(16)

\[ L_{17} = -\sec^2 kt \log(ku_1 \sin kt + u_2 \cos kt) + f_1 u_2 + f_2, \]  
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]  

(17)

\[ L_{18} = \frac{1}{ku_1 \sin kt \cos kt} \left[ \sin kt(ku_1 \sin kt + u_2 \cos kt) \log(ku_1 \sin kt + u_2 \cos kt) \right] \]
\[ + \cos kt(-u_2 \sin kt + ku_1) \log(-u_2 \sin kt + ku_1) \] + f_1u_2 + f_2, \quad (18)

\[(u_1 \sin kt \cos kt) \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = 1; \]

\[ L_{28} = \cosec^2 kt \log(ku_1 \cos kt - u_2 \sin kt) + f_1u_2 + f_2, \quad (19) \]

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]

\[ L_{34} = \frac{u_2}{ku_1} \arctan \left( \frac{u_2}{ku_1} \right) - \frac{1}{2} \log \left( \frac{u_2^2}{k^2u_1^2} + 1 \right) + f_1u_2 + f_2, \quad (20) \]

\[ u_1 \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = 1; \]

\[ L_{35} = \frac{1}{2ku_1 \cos 2kt} \left[ 2ku_1 + (u_2 \cos 2kt + ku_1 \sin 2kt - ku_1) \log \left( \left( \sin kt + \cos kt \right)u_2 + (\sin kt - \cos kt)ku_1 \right) \right. \]

\[-(u_2 \cos 2kt + ku_1 \sin 2kt + ku_1) \log \left( \left( \sin kt - \cos kt \right)u_2 - (\sin kt + \cos kt)ku_1 \right) \] + f_1u_2 + f_2, \quad (21)

\[ u_1 \cos 2kt \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = 1; \]

\[ L_{47} = \frac{1}{2k^2u_1^2} \left[ -(u_2 \cos kt + ku_1 \sin kt) \log(u_2^2 + k^2u_1^2) + 2(u_2 \sin kt - ku_1 \cos kt) \arctan \left( \frac{u_2}{ku_1} \right) \right. \]

\[ +2(u_2 \cos kt + ku_1 \sin kt) \log(u_2 \cos kt + ku_1 \sin kt) \] + f_1u_2 + f_2, \quad (22)

\[ ku_1^2 \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = \sin kt; \]

\[ L_{48} = \frac{1}{2k^2u_1^2} \left[ -(u_2 \sin kt - ku_1 \cos kt) \log(u_2^2 + k^2u_1^2) - 2(u_2 \cos kt + ku_1 \sin kt) \arctan \left( \frac{u_2}{ku_1} \right) \right. \]

\[-2(u_2 \sin kt - ku_1 \cos kt) \log(u_2 \sin kt - ku_1 \cos kt) \] + f_1u_2 + f_2, \quad (23)

\[ ku_1^2 \left( \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} \right) = -\cos kt; \]

\[ L_{57} = \frac{1}{2k^2u_1^2} \left[ (\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1 \right] \log \left( (\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1 \right) \]
\[ + \left( (\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1 \right) \log \left( (\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1 \right) - 2(u_2 \cos kt + ku_1 \sin kt) \log(u_2 \cos kt + ku_1 \sin kt) + f_1 u_2 + f_2, \] (24)

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]

\[ L_{58} = \frac{1}{2k^2u_1} \left[ \left( (\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1 \right) \log \left( (\cos kt - \sin kt)u_2 + (\cos kt + \sin kt)ku_1 \right) - \left( (\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1 \right) \log \left( (\cos kt + \sin kt)u_2 - (\cos kt - \sin kt)ku_1 \right) + 2(u_2 \sin kt - ku_1 \cos kt) \log(u_2 \sin kt - ku_1 \cos kt) \right] + f_1 u_2 + f_2, \] (25)

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]

\[ L_{67} = \frac{ku_1 \cos kt - u_2 \sin kt}{2k^2u_1^2} \left[ \log(ku_1 \sin kt + u_2 \cos kt) - \log(u_2 \sin kt - ku_1 \cos kt) \right] + f_1 u_2 + f_2, \] (26)

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0; \]

\[ L_{68} = \frac{ku_1 \sin kt + u_2 \cos kt}{2k^2u_1^2} \left[ \log(ku_1 \sin kt + u_2 \cos kt) - \log(u_2 \sin kt - ku_1 \cos kt) \right] + f_1 u_2 + f_2, \] (27)

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0. \]

We illustrate the importance of the two functions \( f_1 \) and \( f_2 \) by a consideration of \( L_{12} \). The constraint is

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = k^2 u_1 \]

which is satisfied if we write

\[ f_1 = \frac{\partial g}{\partial u_1}, \quad f_2 = \frac{\partial g}{\partial t} - k^2 u_1, \] (28)

where \( g(t, u_1) \) is an arbitrary function of its arguments. Hence

\[ L_{12} = \frac{1}{2} \left( u_2^2 - k^2 u_1^2 \right) + \frac{\partial g}{\partial t}. \] (29)

We see that for \( L_{12} \) the two arbitrary functions introduced on the integration of the Jacobi Last Multiplier to obtain the Lagrangian are subject to a constraint to obtain compatibility with the Newtonian equation of motion under consideration. The remaining generality can be expressed as the total time derivative of an arbitrary gauge function which has no effect upon the form of the Lagrangian equation of motion. There are
some further considerations. The function, \( g(t, u_1) \), is by its very construction a function only of the indicated variables. Consequently one would expect \( L_{12} \) to have the same number of Noetherian point symmetries for all possible functions, \( g(t, u_1) \), since it may be absorbed into the boundary function of Noether’s Theorem \[19\]. This conflation with the boundary term may be part of the explanation for some of the confused accounts of Noether’s Theorem which one finds in the literature from some twenty years \[20\] after Noether’s seminal work until the almost present day \[21\]. Again we used \( L_{12} \) to demonstrate the result, but we expect that the informed reader appreciates that this is general.

In the case of (5) the corresponding system of first-order equations is

\[
\begin{align*}
\dot{u}_1 &= u_2 \\
\dot{u}_2 &= \frac{1}{u_1} \left( 2u_2^2 + u_1^2 \right)
\end{align*}
\] (30)

with associated vector field

\[ X_{GHO} = \partial_t + u_2 \partial_{u_1} + \frac{1}{u_1} \left( 2u_2^2 + u_1^2 \right) \partial_{u_2}. \] (31)

It is easy to derive \[3\][§18] that a Jacobi’s last multiplier of (30)/(31) is \( u_1^{-4} \). In fact \(1\) yields

\[
\frac{d \log M}{dt} = - \left[ \frac{\partial u_2}{\partial u_1} + \frac{\partial}{\partial u_2} \left( \frac{2u_2^2 + u_1^2}{u_1} \right) \right] = - \frac{4u_2}{u_1}.
\] (32)

The eight Lie point symmetries are

\[
\begin{align*}
\Gamma_1 &= u_1^2 \cos t \partial_{u_1} - (u_1 \sin t - 2u_1 \cos t) \partial_{u_2} \\
\Gamma_2 &= u_1^2 \sin t \partial_{u_1} + (u_1 \cos t + 2u_1 \sin t) \partial_{u_2} \\
\Gamma_3 &= u_1 \partial_{u_1} + u_2 \partial_{u_2} \\
\Gamma_4 &= \partial_t \\
\Gamma_5 &= \cos 2t \partial_t + u_1 \sin 2t \partial_{u_1} + (2u_1 \cos 2t + 3u_2 \sin 2t) \partial_{u_2} \\
\Gamma_6 &= \sin 2t \partial_t - u_1 \cos 2t \partial_{u_1} + (2u_1 \sin 2t - 3u_2 \cos 2t) \partial_{u_2} \\
\Gamma_7 &= \frac{1}{u_1} \left[ u_1 \cos t \partial_t + u_1^2 \sin t \partial_{u_1} + \left( u_1^2 \cos t + u_1 u_2 \sin t + u_2 \cos t \right) \partial_{u_2} \right] \\
\Gamma_8 &= \frac{1}{u_1^2} \left[ u_1 \sin t \partial_t - u_1^2 \cos t \partial_{u_1} + \left( u_1^2 \sin t - u_1 u_2 \cos t + u_2 \sin t \right) \partial_{u_2} \right].
\end{align*}
\] (33)

Not surprisingly there are fourteen different Jacobi Last Multipliers, i.e.

\[
\begin{align*}
JLM_{12} &= \frac{1}{u_1^4} \\
JLM_{13} &= \frac{1}{u_1^2 (u_1 \sin t - u_2 \cos t)} \\
JLM_{23} &= \frac{1}{u_1^2 (u_1 \cos t + u_2 \sin t)} \\
JLM_{17} &= \frac{1}{(u_1 \sin t - u_2 \cos t)^2} \\
JLM_{18} &= \frac{1}{(u_1 \sin t - u_2 \cos t)(u_1 \cos t + u_2 \sin t)}
\end{align*}
\]
\[ JLM_{28} = \frac{1}{(u_1 \cos t + u_2 \sin t)^2} \]
\[ JLM_{34} = \frac{1}{u_1^2 + u_2^2} \]
\[ JLM_{35} = \frac{1}{(u_1^2 - u_2^2) \cos 2t + 2u_1 u_2 \sin 2t} \]
\[ JLM_{47} = \frac{u_1^2}{(u_1^2 + u_2^2)(u_1 \sin t - u_2 \cos t)} \]
\[ JLM_{48} = \frac{u_1^2}{(u_1^2 + u_2^2)(u_1 \cos t + u_2 \sin t)} \]
\[ JLM_{57} = \frac{u_1^2}{((u_1^2 - u_2^2) \cos 2t + 2u_1 u_2 \sin 2t)(u_1 \sin t - u_2 \cos t)} \]
\[ JLM_{58} = \frac{u_1^2}{((u_1^2 - u_2^2) \cos 2t + 2u_1 u_2 \sin 2t)(u_1 \cos t + u_2 \sin t)} \]
\[ JLM_{67} = \frac{u_1^2}{((u_1^2 - u_2^2) \sin 2t - 2u_1 u_2 \cos 2t)(u_1 \sin t - u_2 \cos t)} \]
\[ JLM_{68} = \frac{u_1^2}{((u_1^2 - u_2^2) \sin 2t - 2u_1 u_2 \cos 2t)(u_1 \cos t + u_2 \sin t)} \].

(34)

As we have already provided the details of the Lagrangians in the case of (4), we do not repeat the whole listing for (5). However, we do give two Lagrangians and their associated Hamiltonians. In the case of \( JLM_{12} \) the Lagrangian is
\[ L_{12} = \frac{1}{2} u_1^2 + f_1 u_2 + f_2 \]  
subject to the constraint
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = -\frac{1}{u_1^2}. \]  

(36)

It is a simple matter to show that the corresponding Hamiltonian is
\[ H_{12} = \frac{1}{2} q^4 (p - f_1)^2 - f_2 \]  
where we have used the traditional symbols for the canonical variables with \( q = u_1 \) and \( p = \partial L_{12}/\partial u_2 \). A particular solution of (36) is \( f_1 = 0, f_2 = 1/(2u_1^2) \), which substituted into (37) yields the Hamiltonian in [18], i.e.: 
\[ H_G = \frac{1}{2} \left( q^4 p^2 + \frac{1}{q^2} \right). \]  

(38)

In the case of \( JLM_{34} \) the Lagrangian is
\[ L_{34} = \frac{u_2}{u_1} \arctan \left( \frac{u_2}{u_1} \right) - \frac{1}{2} \log \left( \left( \frac{u_2}{u_1} \right)^2 + 1 \right) + f_1 u_2 + f_2 \]  
subject to the constraint
\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = -\frac{1}{u_1}. \]  

(40)
The corresponding Hamiltonian is

\[ H_{34} = \frac{1}{2} \log \left( \tan(q(p - f_1))^2 + 1 \right) - f_2. \]  

(41)

In general, due to the presence of the arbitrary function in the Hamiltonian containing the independent variable, the Hamiltonian is not a constant of the motion. However, the ratio of the multipliers is a first integral. For example we have that both

\[
\frac{JLM_{13}}{JLM_{47}} = \frac{u_2^2 + u_1^2}{u_1^2} \tag{42}
\]

and

\[
\frac{JLM_{23}}{JLM_{58}} = \frac{(u_2^2 - u_1^2) \cos 2t + 2u_1u_2 \sin 2t}{u_1^4} \tag{43}
\]

are first integrals of \((44)\).

In the case of \((46)\) the associated system of first-order equations is

\[
u_1 = u_2 \\
u_2 = -(c^2 + k^2) u_1 - 2cu_2 \tag{44}\]

with associated vector field

\[ X = \partial_t + u_2 \partial_{u_1} - \left[ (c^2 + k^2) u_1 + 2cu_2 \right] \partial_{u_2}. \]  

(45)

It is easy to derive \(5\) that a Jacobi’s last multiplier of \((44) - (45)\) is \(\exp[2ct]\). In fact \((1)\) yields

\[
\frac{d \log M}{dt} = -\frac{\partial u_2}{\partial u_1} + \frac{\partial}{\partial u_2} \left[ (c^2 + k^2) u_1 + 2cu_2 \right] = 2c. 
\]

(46)

As a linear second-order equation the damped linear oscillator described by \(6\) possesses eight Lie point symmetries \(17\), which have been given explicitly in \(23\). In terms of the variables introduced in \(44\) the eight vectors are

\[
\begin{align*}
\Gamma_1 &= \exp[-ct] \left[ \cos kt \partial_t - (c \cos kt + k \sin kt) \partial_{u_1} \right] \\
\Gamma_2 &= \exp[-ct] \left[ -\sin kt \partial_t + (c \sin kt - k \cos kt) \partial_{u_1} \right] \\
\Gamma_3 &= u_1 \partial_{u_1} + u_2 \partial_{u_2} \\
\Gamma_4 &= \partial_t \\
\Gamma_5 &= \cos 2kt \partial_t - u_1 \left( c \cos 2kt + k \sin 2kt \right) \partial_{u_1} \\
&\quad + \left[ 2ku_1 \left( c \sin 2kt - k \cos 2kt \right) - u_2 \left( c \cos 2kt - k \sin 2kt \right) \right] \partial_{u_2} \\
\Gamma_6 &= \sin 2kt \partial_t - u_1 \left( c \sin 2kt - k \cos 2kt \right) \partial_{u_1} \\
&\quad + \left[ -2ku_1 \left( c \sin 2kt + c \cos 2kt \right) - u_2 \left( c \sin 2kt + k \cos 2kt \right) \right] \partial_{u_2} \\
\Gamma_7 &= \exp[ct] \left[ u \cos kt \partial_t - u_1^2 \left( c \cos kt + k \sin kt \right) \partial_{u_1} \right. \\
&\quad - \left( u_2^2 \cos kt + u_1 u_2 \left( 3c \cos kt + k \sin kt \right) + (c^2 + k^2) u_1^2 \cos kt \right) \partial_{u_2} \\
\Gamma_8 &= \exp[ct] \left[ -u_1 \sin kt \partial_t + u_2^2 \left( c \sin kt - k \cos kt \right) \partial_{u_1} \\
&\quad + \left( u_2^2 \sin kt - u_1 u_2 \left( c \cos kt - 3c \sin kt \right) + (c^2 + k^2) u_1^2 \sin kt \right) \partial_{u_2} \right]. 
\end{align*} 
\]

(47)
We calculate the Jacobi Last Multipliers as we did for (4). Of course there are fourteen different Jacobi Last Multipliers, namely:

\[
JLM_{12} = \exp[2ct]
\]

\[
JLM_{13} = \frac{\exp[ct]}{u_1(c\cos kt + k\sin kt) + u_2\cos kt}
\]

\[
JLM_{23} = \frac{\exp[ct]}{u_1(c\sin kt - k\cos kt) + u_2\sin kt}
\]

\[
JLM_{17} = \frac{1}{[u_1(c\cos kt + k\sin kt) + u_2\cos kt]^2}
\]

\[
JLM_{18} = \frac{1}{[u_1(c\cos kt + k\sin kt) + u_2\cos kt][u_1(c\sin kt - k\cos kt) + u_2\sin kt]}
\]

\[
JLM_{28} = \frac{1}{[u_1(c\sin kt - k\cos kt) + u_2\sin kt]^2}
\]

\[
JLM_{34} = \frac{1}{(cu_1 + u_2)^2 + k^2u_1^2}
\]

\[
JLM_{35} = \frac{1}{2ku_1(cu_1 + u_2)\sin 2kt + [(cu_1 + u_2)^2 - k^2u_1^2]\cos 2kt}
\]

\[
JLM_{47} = \frac{\exp[ct][(cu_1 + u_2)^2 + k^2u_1^2][u_1(c\cos kt + k\sin kt) + u_2\cos kt]}{\exp[ct][(cu_1 + u_2)^2 + k^2u_1^2][u_1(c\sin kt - k\cos kt) + u_2\sin kt]}
\]

\[
JLM_{48} = \frac{\exp[ct][(cu_1 + u_2)^2 + k^2u_1^2][u_1(c\cos kt + k\sin kt) + u_2\cos kt]}{\exp[ct][(cu_1 + u_2)^2 + k^2u_1^2][u_1(c\sin kt - k\cos kt) + u_2\sin kt]}
\]

\[
JLM_{57} = \frac{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt][2ku_1(cu_1 + u_2)\sin 2kt + [(cu_1 + u_2)^2 - k^2u_1^2]\cos 2kt]}{\exp[ct][u_1(c\sin kt - k\cos kt) + u_2\sin kt][2ku_1(cu_1 + u_2)\sin 2kt + [(cu_1 + u_2)^2 - k^2u_1^2]\cos 2kt]}
\]

\[
JLM_{58} = \frac{\exp[ct][u_1(c\sin kt - k\cos kt) + u_2\sin kt][2ku_1(cu_1 + u_2)\sin 2kt + [(cu_1 + u_2)^2 - k^2u_1^2]\cos 2kt]}{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt][2ku_1(cu_1 + u_2)\sin 2kt + [(cu_1 + u_2)^2 - k^2u_1^2]\cos 2kt]}
\]

\[
JLM_{67} = \frac{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt]^2[u_1(c\sin kt - k\cos kt) + u_2\sin kt]}{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt][u_1(c\sin kt - k\cos kt) + u_2\sin kt]^2}
\]

\[
JLM_{68} = \frac{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt][u_1(c\sin kt - k\cos kt) + u_2\sin kt]^2}{\exp[ct][u_1(c\cos kt + k\sin kt) + u_2\cos kt][u_1(c\sin kt - k\cos kt) + u_2\sin kt]}.
\]

As we have already provided the details of the Lagrangians in the case of (4), we do not repeat the whole listing for (6). We show just three Lagrangians, and corresponding constraint, i.e.:

\[
L_{12} = \frac{1}{2}\exp[2ct]u_2^2 + f_1u_2 + f_2,
\]

\[
\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = \exp[2ct](c^2 + k^2)u_1;
\]

\[
L_{34} = \left(\frac{u_2}{ku_1} + \frac{c}{k}\right)\arctan\left(\frac{u_2}{ku_1} + \frac{c}{k}\right) - \frac{1}{2}\log\left(\left(\frac{u_2}{ku_1} + \frac{c}{k}\right)^2 + 1\right) + f_1u_2 + f_2,
\]

\[
u_1\left(\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1}\right) = 1;
\]
\[ L_{67} = -\frac{u_2 \sin 2kt + (c \sin 2kt + 2k \sin^2 kt)u_1}{2 \exp[ct] k^2 u_1^2 \sin kt} \log \left[ \frac{u_2 \cos kt + (c \cos kt + k \sin kt)u_1}{u_2 \sin kt + (c \sin kt - k \cos kt)u_1} \right] + f_1u_2 + f_2, \quad (51) \]

\[ \frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial u_1} = 0 \]

3 Noether Symmetries and Integrals

For the purposes of this Section we confine our interest to Noetherian point symmetries being well aware that Noether did not such thing [19]. It is known that the number of Noetherian point symmetries of the Action Integral for a Lagrangian of standard form for a one-degree-of-freedom mechanical system is 0, 1, 3 or 5 [24, 22]. We list the symmetries and integrals in classes according to the number of Noetherian point symmetries, which turn to be 5, 3, or 2. Since the results are the same for each of the equations under consideration, we give examples only for (4) but in the notation of (8).

Five Noetherian point symmetries

\[ L_{12} \quad \Gamma_1 \Rightarrow -u_2 \cos kt - ku_1 \sin kt \]
\[ \Gamma_2 \Rightarrow u_2 \sin kt - ku_1 \cos kt \]
\[ \Gamma_4 \Rightarrow \frac{1}{2} (u_2^2 + k^2 u_1^2) \]
\[ \Gamma_5 \Rightarrow \frac{1}{2} (u_2^2 - k^2 u_1^2) \cos 2kt + ku_1 u_2 \sin 2kt \]
\[ \Gamma_6 \Rightarrow -\frac{1}{2} (u_2^2 - k^2 u_1^2) \sin 2kt + ku_1 u_2 \cos 2kt \]

Three Noetherian point symmetries

\[ L_{13} \quad \Gamma_1 \quad \Rightarrow \quad \log (u_2 \cos kt + ku_1 \sin kt) \]
\[ \Gamma_4 + \Gamma_5 \quad \Rightarrow \quad 2 (u_2 \cos kt + ku_1 \sin kt) \]
\[ -k \Gamma_3 + \Gamma_6 \quad \Rightarrow \quad 2 (-ku_1 \cos kt + u_2 \sin kt) \]

\[ L_{23} \quad \Gamma_2 \quad \Rightarrow \quad \log (-ku_1 \cos kt + u_2 \sin kt) \]
\[ -\Gamma_4 + \Gamma_5 \quad \Rightarrow \quad 2 (ku_1 \cos kt - u_2 \sin kt) \]
\[ k \Gamma_3 + \Gamma_6 \quad \Rightarrow \quad 2 (u_2 \cos kt + ku_1 \sin kt) \]

\[ L_{28} \quad \Gamma_2 \quad \Rightarrow \quad \frac{1}{ku_1 \cos kt - u_2 \sin kt} \]
\[ \Gamma_3 \quad \Rightarrow \quad \frac{1}{k} \frac{ku_1 \sin kt + u_2 \cos kt}{ku_1 \cos kt - u_2 \sin kt} \]
\[ -\Gamma_4 + \Gamma_5 \quad \Rightarrow \quad 2 (\log (ku_1 \cos kt - u_2 \sin kt)) - 1 \]
\[ L_{67} \quad k \Gamma_3 + \Gamma_6 \Rightarrow -ku_1 \cos kt + u_2 \sin kt \]
\[ -\frac{1}{2} \left( (u_2^2 - k^2 u_1^2) \sin 2kt + ku_1 u_2 \cos 2kt \right) \]
\[ \Gamma_7 \Rightarrow \frac{1}{2k} \log \left( \frac{-ku_1 \cos kt + u_2 \sin kt}{u_2 \cos kt + ku_1 \sin kt} \right) + \frac{1}{2k} \]
\[ \Gamma_8 \Rightarrow \frac{1}{2k} \log \left( \frac{-ku_1 \cos kt + u_2 \sin kt}{u_2 \cos kt + ku_1 \sin kt} \right) - \frac{1}{2k} \]

Two Noetherian point symmetries

\[ L_{17} \quad \Gamma_1 \Rightarrow -\frac{1}{u_2 \cos kt + ku_1 \sin kt} \]
\[ k \Gamma_4 + \Gamma_5 \Rightarrow \frac{2}{u_2 \cos kt + ku_1 \sin kt} \left( -\log (u_2 \cos kt + ku_1 \sin kt) + 1 \right) \]

\[ L_{18} \quad \Gamma_3 \Rightarrow \frac{1}{k} \log \left( \frac{ku_1 \cos kt - u_2 \sin kt}{ku_1 \sin kt + u_2 \cos kt} \right) \]
\[ \Gamma_6 \Rightarrow -\log \left( -\frac{1}{2} (u_2^2 - k^2 u_1^2) \sin 2kt + ku_1 u_2 \cos 2kt \right) \]

\[ L_{34} \quad \Gamma_3 \Rightarrow -\frac{1}{k} \arctan \left( \frac{u_2}{ku_1} \right) - t \]
\[ \Gamma_4 \Rightarrow \frac{1}{2} \log \left( \frac{u_2^2 + k^2 u_1^2}{k^2} \right) \]

\[ L_{35} \quad \Gamma_3 \Rightarrow \frac{1}{2k} \log \left( \frac{-u_2 + ku_1 \cos kt + (u_2 - ku_1) \sin kt}{(u_2 - ku_1) \cos kt + (u_2 + ku_1) \sin kt} \right) \]
\[ \Gamma_5 \Rightarrow \frac{1}{2} \log \left( -\frac{u_2^2 - k^2 u_1^2}{\cos 2kt - 2ku_1 u_2 \sin 2kt} \right) - 1 \]
\[ L_{47} \quad \Gamma_7 \Rightarrow \frac{1}{k} \arctan\left( \frac{u_2}{ku_1} \right) + t \]
\[
\Gamma_8 \Rightarrow \frac{1}{2k} \log \left( \frac{u_2^2 + k^2u_1^2}{(ku_1 \sin kt + u_2 \cos kt)^2} \right)
\]

\[ L_{48} \quad \Gamma_7 \Rightarrow -\frac{1}{2k} \log \left( \frac{u_2^2 + k^2u_1^2}{(-ku_1 \cos kt + u_2 \sin kt)^2} \right) + \frac{1}{k} \]
\[
\Gamma_8 \Rightarrow \frac{1}{k} \arctan\left( \frac{u_2}{ku_1} \right) + t
\]

\[ L_{57} \quad \Gamma_7 \Rightarrow \frac{1}{2k} \log \left( \frac{(u_2 - ku_1) \cos kt + (u_2 + ku_1) \sin kt}{(u_2 + ku_1) \cos kt - (u_2 - ku_1) \sin kt} \right)
\]
\[
\Gamma_8 \Rightarrow \frac{1}{2k} \log \left( \frac{(ku_1 \sin kt + u_2 \cos kt)^2}{[(u_2 - ku_1) \cos kt + (u_2 + ku_1) \sin kt][(u_2 + ku_1) \cos kt - (u_2 - ku_1) \sin kt]} \right)
\]

\[ L_{58} \quad \Gamma_7 \Rightarrow \frac{1}{2k} \log \left( \frac{(u_2 \sin kt - ku_1 \cos kt)^2}{[(u_2 - ku_1) \cos kt + (u_2 + ku_1) \sin kt][(u_2 + ku_1) \cos kt - (u_2 - ku_1) \sin kt]} \right)
\]
\[
\Gamma_8 \Rightarrow \frac{1}{2k} \log \left( \frac{(u_2 - ku_1) \cos kt + (u_2 + ku_1) \sin kt}{(u_2 + ku_1) \cos kt - (u_2 - ku_1) \sin kt} \right)
\]

4 Discussion

In Sections 2 and 3 we provided the information necessary for the discussion of properties of Lagrangians for (6) given by the method of Jacobi’s Last Multiplier in conjunction with the Lie point symmetries of the equation of motion (3). It is apparent that Lie’s method for the calculation of the Jacobi Last Multiplier provides a direct route to the determination of many Lagrangians for a system the equation of motion of which is richly endowed with Lie point symmetries. It takes no effort to infer that even more Lagrangians could be obtained if one were to expand the class of symmetries to include generalised symmetries. The disadvantage of this generalisation is that the calculation of generalised symmetries for second-order equations is not as algorithmic as is the calculation of point symmetries. One must make assumptions about the nature of the dependence of the symmetry on the first derivative. Another route which one can take to obtain more Lagrangians is to make use of the fact that the ratio of two multipliers is a first integral. Once an integral is obtained, further multipliers can be generated by making use of the fact that an arbitrary function of an integral is itself an integral.

An important consequence of the relationship, (2), between the last multiplier and a Lagrangian is that one obtains a whole class of Lagrangians from a given multiplier. Each member of the class is an element of an equivalence class of Lagrangians in the sense that each possesses the same number of Noether point symmetries. This is not necessarily the case for Lagrangians obtained from different multipliers. As we saw in §3, Lagrangians leading to the same Euler-Lagrange equation can have all possible numbers of Noether point symmetries. Consequently they are inequivalent Lagrangians. In the classical case the existence of inequivalent Lagrangians is a matter of some curiosity. Naturally it is more than merely curious if one happens upon a
Lagrangian which gives a lower number of Noether point symmetries for then one could be misled on the
subject of the integrability of the underlying differential equation.

Although in this paper we have emphasised the approach of Lie for the determination of the multipliers, we
remind the reader that the Jacobi Last Multiplier can be obtained in a variety of ways as we indicated in the
Introduction. The method to be employed must be gauged by the utility of the results which it produces.

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