Large Constant-Sign Solutions of Discrete Dirichlet Boundary Value Problems with $p$-Mean Curvature Operator

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Abstract: In this paper, we consider the existence of infinitely many large constant-sign solutions for a discrete Dirichlet boundary value problem involving $p$-mean curvature operator. The methods are based on the critical point theory and truncation techniques. Our results are obtained by requiring appropriate oscillating behaviors of the non-linear term at infinity, without any symmetry assumptions.

Keywords: discrete Dirichlet boundary value problem; $p$-mean curvature operator; constant-sign solutions; discrete maximum principle; critical point theory

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of integer numbers, natural numbers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots \}$, and $\mathbb{Z}(a, b) = \{a, a+1, \ldots, b\}$ when $a \leq b$.

Consider the following Dirichlet boundary value problem of the nonlinear difference equation

$$
(D_p^{h, f}) \left\{ \begin{array}{ll}
-\triangle (\phi_{p,c} (\triangle u(k-1))) = \lambda f(k, u(k)), & k \in \mathbb{Z}(1, T), \\
u(0) = u(T+1) = 0,
\end{array} \right.
$$

where $T$ is a given positive integer, $\lambda$ is a positive real parameter, $\triangle$ is the forward difference operator defined by $\triangle u(k) = u(k+1) - u(k)$, $f(k, \cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function for each $k \in \mathbb{Z}(1, T)$ and $\phi_{p,c}(s) := (1 + |s|^2)^{p-2} s$, $p \in [1, +\infty)$. Here, $\triangle (\phi_{p,c} (\triangle u(k-1)))$ may be seen as a discretization of the $p$-mean curvature operator.

We may think problem $(D_p^{h, f})$ as being a discrete analog of one-dimensional case of the following problem

$$
\left\{ \begin{array}{ll}
-\text{div} (\phi_{p,c} (\nabla u)) = \lambda f(x, u), & x \in \Omega \subset \mathbb{R}^n, \\
u = 0, & x \in \partial \Omega,
\end{array} \right. \quad (1)
$$

where $\text{div} (\phi_{p,c} (\nabla u))$ is named $p$-mean curvature operator, which is a generalization of mean curvature operator; see [1,2]. If $p = 1$, it reduces to the mean curvature operator. If $p = 2$, it reduces to the Laplacian operator. The above problem arises from differential geometry and physics such as capillarity; see [3–5] and references therein. When $p = 1$ and $f(x, u) = u$, the above problem describes the free surface of a pendent drop filled with liquid under gravitational field [4]. In the past decades, several authors have discussed the existence and multiplicity of solutions of Problem (1); see [1,6–12]. For example, Chen and Shen in [1] have obtained the existence of infinitely many solutions of Problem (1) with $\lambda = 1$ via a symmetric version of Mountain Pass Theorem. When $p = 1$ and $\Omega = (0, 1)$,
Obersnel and Omari in [11] have established the existence and multiplicity of positive solutions of Problem (1), which depend on the behavior of \( f \) at zero or at infinity. G. A. Afrouzi et al. in [6] have acquired a sequence of nonnegative and nontrivial solutions strongly converging to zero in \( C^1([0, 1]) \), under suitable oscillating behavior of the nonlinear term \( f \) at zero. However, the results on the existence of solutions for problem \((D_p^{\lambda, f})\) are scarce in the literature besides the case of \( p = 1 \).

Nonlinear discrete problems appear in many mathematical models, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics, fluid mechanics and many others; see [13–17]. Many authors have discussed the existence and multiplicity of solutions for difference equations through classical tools of nonlinear analysis: Fixed point theorems, upper and lower solutions techniques; see [7,9] and the references given therein. Since 2003, by starting from the seminal paper [18], variational methods have been used to investigate nonlinear difference equations, which have obtained various results; see [19–34].

In paper [35], the authors have considered problem \((D_1^{\lambda, f})\), obtaining infinitely many positive solutions when \( \lambda \) belongs to a precise real interval. It is worth noticing that the suitable oscillating behaviors of the nonlinear term \( f \) at infinity play a key role. Inspired by [19,32,35–40], the main purpose of this paper is to investigate the existence conditions of infinitely many constant-sign solutions for problem \((D_p^{\lambda, f})\), without any symmetry hypothesis. Here, a solution \( \{u(k)\} \) of \((D_p^{\lambda, f})\) is called a constant-sign solution, if \( u(k) > 0 \) for all \( k \in \mathbb{Z}(1, T) \) or \( u(k) < 0 \) for all \( k \in \mathbb{Z}(1, T) \). Compared to problem \((D_1^{\lambda, f})\), problem \((D_p^{\lambda, f})\) is more difficult to handle. To facilitate the analysis, we have to divide the problem into two categories: \( 1 \leq p < 2 \) and \( 2 \leq p < +\infty \). We believe that this is the first time to discuss the existence of infinitely many solutions for a non-linear second order difference equation with \( p \)-mean curvature operator.

A special case of our results is the following.

**Theorem 1.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( g(t) t \geq 0 \) for \( t \neq 0 \). Assume that

\[
\liminf_{t \to \infty} \frac{\int_0^t g(\tau) d\tau}{|t|^p} = 0, \quad \text{and} \quad \limsup_{t \to -\infty} \frac{\int_0^t g(\tau) d\tau}{|t|^p} = +\infty.
\]

Then, for every \( \lambda > 0 \), the problem

\[
\begin{align*}
-\triangle \left( \phi_{p,c} (\Delta u(k - 1)) \right) &= \lambda g(u(k)), \quad k \in \mathbb{Z}(1, T), \\
u(0) &= u(T + 1) = 0,
\end{align*}
\]

admits two unbounded sequences of constant-sign solutions (one positive and one negative).

This paper is organized as follows. In Section 2, we introduce the the suitable Banach space and appropriate functional corresponding to problem \((D_p^{\lambda, f})\). To obtain sequences of constant-sign solutions of problem \((D_p^{\lambda, f})\), three basic lemmas are introduced. In Section 3, under suitable hypotheses on \( f \), we obtain the existence of infinitely many constant-sign solutions for problem \((D_p^{\lambda, f})\). In Section 4, we give two examples to demonstrate our results. Finally, conclusions are given for this paper.

2. Mathematical Background

To solve problem \((D_p^{\lambda, f})\), we naturally select the \( T \)-dimensional Banach space

\[
X = \{ u : \mathbb{Z}(0, T + 1) \to \mathbb{R} : u(0) = u(T + 1) = 0 \},
\]

endowed with the norm

\[
\|u\| := \left( \sum_{k=1}^{T} (\Delta u(k))^2 \right)^{\frac{1}{2}} \text{ for all } u \in X.
\]
Another useful norm on $X$ is

$$||u||_\infty := \max_{k \in \mathbb{Z}(1, T)} |u(k)| \text{ for all } u \in X.$$ 

In the sequel, we will use the following inequalities. For $0 < r < s$, $x_k \geq 0, k \in \mathbb{Z}(1, n)$, one has

$$\left( \sum_{k=1}^{n} x_k^s \right)^{1/s} \leq \left( \sum_{k=1}^{n} x_k^r \right)^{1/r}, \quad (2)$$

see [41].

$$||u||_\infty \leq \frac{\sqrt{T+1}}{2}||u||, \quad (3)$$

for every $u \in X$, it can follow from Lemma 2.2 of [42].

For all $u \in X$, let

$$\Phi(u) := \frac{1}{p} \sum_{k=0}^{T} \left( \left( 1 + (\Delta u(k))^2 \right)^{p/2} - 1 \right), \quad \text{and } \Psi(u) := \sum_{k=1}^{T} F(k, u(k)), \quad (4)$$

where $F(k, t) := \int_{k}^{k+1} f(k, \tau) d\tau$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}(1, T)$. Further, let us denote $I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$ for $u \in X$. Through standard arguments, we follow that $I_\lambda \in C^1(S, \mathbb{R})$, and the critical points of $I_\lambda$ are exactly the solutions of problem $(D_{p,f}^\lambda)$. In fact, one has

$$I'_\lambda(u)(v) = \sum_{k=0}^{T} \left( \phi_{p,c}(\Delta u(k)) \right) \Delta v(k) - \lambda \sum_{k=1}^{T} f(k, u(k)) v(k)$$

$$= \sum_{k=0}^{T} \left( \phi_{p,c}(\Delta u(k)) \right) v(k + 1) - \lambda \sum_{k=0}^{T} f(k, u(k)) v(k)$$

$$= \sum_{k=1}^{T} \left( \phi_{p,c}(\Delta u(k)) \right) v(k) - \lambda \sum_{k=1}^{T} f(k, u(k)) v(k)$$

$$= -\sum_{k=1}^{T} \left[ \Delta((\phi_{p,c}(\Delta u(k)) - \lambda f(k, u(k))) v(k),$$

for all $u, v \in X$.

Next, we need to establish the following strong maximum principle to obtain the positive solutions of problem $(D_{p,f}^\lambda)$, i.e., $u(k) > 0$ for each $k \in \mathbb{Z}(1, T)$.

**Lemma 1.** Assume $u \in X$ such that either

$$u(k) > 0 \quad \text{or} \quad -\Delta \left( \phi_{p,c}(\Delta u(k-1)) \right) \geq 0, \quad (5)$$

for any $k \in \mathbb{Z}(1, T)$. Then, either $u > 0$ in $\mathbb{Z}(1, T)$ or $u \equiv 0$.

**Proof.** For $u \in X$, put $m = \min\{u(k), k \in \mathbb{Z}(0, T+1)\}$, then $m \leq 0$.

If there exists $j \in \mathbb{Z}(1, T)$ such that $u(j) = m$, we claim that $u \equiv 0$. Indeed, since $\Delta u(j-1) = u(j) - u(j-1) \leq 0$ and $\Delta u(j) = u(j+1) - u(j) \geq 0$, $\phi_{p,c}(s)$ is strictly monotone increasing in $s$, and $\phi_{p,c}(0) = 0$, we have

$$\phi_{p,c}(\Delta u(j)) \geq 0 \geq \phi_{p,c}(\Delta u(j-1)). \quad (6)$$
On the other hand, by (5), let $k = j$, we obtain

$$
\varphi_{p,c}(\Delta u(j)) \leq \varphi_{p,c}(\Delta u(j-1)).
$$

(7)

Combining inequalities (6) and (7), we get that $\varphi_{p,c}(\Delta u(j)) = 0 = \varphi_{p,c}(\Delta u(j - 1))$. That is $u(j + 1) = u(j - 1) = u(j) = m$. By iterating this argument, we obtain easily $u(0) = u(1) = u(2) = \ldots = u(T) = u(T + 1)$. Thus $u \equiv 0$.

If $u(j) > m$ for every $j \in \mathbb{Z}(1, T)$, then $u(0) = u(T + 1) = m = 0$. It follows that $u(j) > 0$, for all $j \in \mathbb{Z}(1, T)$. The proof is complete.

In the same way, we have the following result to get negative solutions problem $(D_{p}^{\lambda,f})$, i.e., $u(k) < 0$ for each $k \in \mathbb{Z}(1, T)$.

Lemma 2. Assume $u \in X$ such that either

$$
u(k) < 0 \quad \text{or} \quad - \Delta \big( \varphi_{p,c}(\Delta u(k - 1)) \big) \leq 0,
$$

for any $k \in \mathbb{Z}(1, T)$. Then, either $u < 0$ in $\mathbb{Z}(1, T)$ or $u \equiv 0$.

Truncation techniques are usually used to discuss the existence of constant-sign solutions. To the end, we introduce the following truncations of the functions $f(k, t)$ for every $k \in \mathbb{Z}(1, T)$. If $f(k, 0) \geq 0$ for each $k \in \mathbb{Z}(1, T)$. Set

$$
\Gamma^+(k, t) := \begin{cases} 
  f(k, t), & \text{if } t \geq 0, \\
  f(k, 0), & \text{if } t < 0.
\end{cases}
$$

Clearly, $\Gamma^+(k, \cdot)$ is also continuous, for every $k \in \mathbb{Z}(1, T)$. By Lemma 1, all solutions of problem $(D_{p}^{\lambda,f+})$ are also solutions of problem $(D_{p}^{\lambda,f})$. Therefore, when problem $(D_{p}^{\lambda,f+})$ has non-zero solutions, then problem $(D_{p}^{\lambda,f})$ possesses positive solutions.

If $f(k, 0) \leq 0$ for each $k \in \mathbb{Z}(1, T)$. Set

$$
\Phi^-(k, t) := \begin{cases} 
  f(k, 0), & \text{if } t > 0, \\
  f(k, t), & \text{if } t \leq 0.
\end{cases}
$$

When problem $(D_{p}^{\lambda,f-})$ has non-zero solutions, then problem $(D_{p}^{\lambda,f})$ possesses negative solutions.

Here, we introduce a lemma (Theorem 4.3 of [38]) which is the main tool used to research problem $(D_{p}^{\lambda,f})$.

Lemma 3. Let $X$ be a finite dimensional Banach space and let $I_{\lambda} : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:

(\text{H}) $I_{\lambda}(u) := \Phi(u) - \lambda \Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with $\Phi$ coercive, i.e., $\lim_{\|u\| \to +\infty} \Phi(u) = +\infty$, and such that $\inf_{X} \Phi = \Phi(0) = \Psi(0) = 0$.

For all $r > 0$, put

$$
\varphi(r) := \frac{\sup_{\|u\|=r} \Psi}{r}, \quad \text{and} \quad \varphi_{\infty} := \lim_{r \to +\infty} \varphi(r).
$$

Assume that $\varphi_{\infty} < +\infty$ and for each $\lambda \in \left(0, \frac{1}{\varphi_{\infty}}\right)$ $I_{\lambda}$ is unbounded from below. Then, there is a sequence $\{u_{n}\}$ of critical points (local minima) of $I_{\lambda}$ such that $\lim_{n \to +\infty} \Phi(u_{n}) = +\infty$. 
3. Main Results

In the following, we will discuss the existence of constant-sign solutions of problem \((D^f_p)\). Our purpose is to apply Lemma 3 to the function \(I^\pm_\lambda : X \to \mathbb{R}, I^\pm_\lambda (u) := \Phi(u) - \lambda \Psi^\pm(u)\), where \(\Psi^\pm(u) = \sum_{k=1}^T F^\pm(k,u(k))\) and \(F^\pm(k,t) := \int_0^T f^\pm(k,\tau) d\tau\) for every \(k \in \mathbb{Z}(1, T)\) and then exploit Lemma 1 or Lemma 2 to get our results.

Let
\[
A_{+\infty} := \lim_{t \to +\infty} \max_{0 \leq s \leq t} \frac{\sum_{k=1}^T F(k,s)}{t^p}, \quad \text{and} \quad B_{+\infty} := \limsup_{t \to +\infty} \frac{\sum_{k=1}^T F(k,t)}{|t|^p}.
\]

Considering the functional \(I^\pm_\lambda\), we have the following conclusions.

Theorem 2. Let \(1 \leq p < 2\) and \(f(k, \cdot) : \mathbb{R} \to \mathbb{R}\) to be a continuous function with \(f(k, 0) \geq 0\) for each \(k \in \mathbb{Z}(1, T)\). Assume that

\[(i_1) A_{+\infty} < \frac{2^{p-1}}{(T+1)^{\frac{p}{2}}} B_{+\infty}.
\]

Then, for each \(\lambda \in \left(\frac{2}{pB_{+\infty}}, \frac{2^p}{p(T+1)^{\frac{p}{2}}A_{+\infty}}\right)\), problem \((D^f_p)\) has an unbounded sequence of positive solutions.

Proof. Consider the auxiliary problem
\[
(D^f_p) \quad \begin{cases}
-\triangle \left( \Phi_{p,c}(\triangle u(k-1)) \right) = \lambda f^+(k,u(k)), & \text{in } \mathbb{Z}(1, T), \\
u(0) = u(T+1) = 0.
\end{cases}
\]

Obviously \(\Phi\) and \(\Psi^+\) satisfy hypothesis required in Lemma 3. For \(\tau > 0\), set
\[
r = \frac{1}{p} \left( \frac{4\tau^2}{T+1} + (T+1)^{\frac{2p-2}{p}} - (T+1)^{\frac{p-1}{p}} \right)^{\frac{p}{2}}.
\]

Assume \(u \in X\) and
\[
\Phi(u) = \frac{1}{p} \sum_{k=0}^T \left( 1 + (\triangle u(k))^2 \right)^{\frac{p}{2}} - 1 \leq r.
\]

Put \(v(k) = (1 + (\triangle u(k))^2)^{\frac{p}{2}} - 1\), for every \(k \in \mathbb{Z}(0, T)\), then \(\sum_{k=0}^T v(k) \leq pr\).

By (2) and Hölder inequality as well, we have
\[
\sum_{k=0}^T (\triangle u(k))^2 = \sum_{k=0}^T \left( (1 + v(k))^\frac{p}{2} \right)^2 - 1 \leq \left( \sum_{k=0}^T v(k) \right)^2 + 2(T+1)^{\frac{p-1}{p}} \left( \sum_{k=0}^T v(k) \right)^{\frac{p}{2}} \leq (pr)^2 + 2(T+1)^{\frac{p-1}{p}} (pr)^{\frac{p}{2}} = \frac{4r^2}{T+1}.
\]

Owing to (3), it follows
\[
||u||_{\infty} \leq \frac{\sqrt{T+1}}{2} \left( \sum_{k=0}^T (\triangle u(k))^2 \right)^{\frac{1}{2}} \leq t.
\]
Thus, one has $\Phi^{-1}[0, r] \subseteq \{ u \in X : ||u||_\infty \leq t \}$.

By the definition of $\varphi$, we obtain

$$
\varphi(r) = \frac{\sup_{u \in \Phi^{-1}[0, r]} \Psi^+}{r} \leq \sup_{||u||_\infty \leq t} \frac{\sum_{k=0}^{T} F^+(k, u(k))}{r} \leq \left( \sqrt{\frac{4t^2}{p} + (T + 1)^\frac{p^2}{2p} - (T + 1)^\frac{p-1}{p}} \right)^p.
$$

Bearing in mind condition $(i_1)$, we follow that $\varphi_\infty \leq \frac{p(T+1)\sqrt{2}}{2p} A_{+\infty} < +\infty$.

In the next step, we need to prove that $I^+_\lambda$ is unbounded from below. To this end, we consider two cases: $B^{+\infty} = +\infty$ and $B^{+\infty} < +\infty$. If $B^{+\infty} = +\infty$, let $\{c_n\}$ be a sequence of positive numbers, with $\lim_{n \to +\infty} c_n = +\infty$, such that

$$
\sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \geq \frac{(2 + p)}{\lambda p} c_n^p - c_n^\rho, \text{ for every } n \in \mathbb{N}.
$$

In the following, we take in $X$ the sequence $\{\omega_n\}$ defined by putting $\omega_n(k) = c_n$, for $k \in \mathbb{N}(1, T)$. Using again (2), one has

$$
I^+_\lambda(\omega_n) = \frac{2}{p} \left( \left( 1 + c_n^2 \right)^\frac{p}{2} - 1 \right) - \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{2}{p} c_n^p - \frac{2 + p}{p} c_n^\rho = -c_n^\rho,
$$

which implies that $\lim_{n \to +\infty} I^+_\lambda(\omega_n) = -\infty$. If $B^{+\infty} < +\infty$, since $\lambda > \frac{2}{p B^{+\infty}}$, we may take $\epsilon_0 > 0$ such that $\frac{2}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 < 0$. Then there exists a sequence of positive numbers $\{c_n\}$ such that $\lim_{n \to +\infty} c_n = +\infty$ and

$$
(B^{+\infty} - \epsilon_0)c_n^p \leq \sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \leq (B^{+\infty} + \epsilon_0)c_n^p.
$$

Arguing as before and by choosing $\{\omega_n\}$ in $X$ as above, we have

$$
I^+_\lambda(\omega_n) = \frac{2}{p} \left( \left( 1 + c_n^2 \right)^\frac{p}{2} - 1 \right) - \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{2}{p} c_n^p - \lambda (B^{+\infty} - \epsilon_0)c_n^p = \left( \frac{2}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 \right)c_n^p.
$$

Since $\frac{2}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 < 0$, it is clear that $\lim_{n \to +\infty} I^+_\lambda(\omega_n) = -\infty$. Considering the above two cases, we follow that $I^+_\lambda$ is unbounded from below.

According to Lemma 3, there exist a sequence $\{u_n\}$ of critical points (local minima) of $I^+_\lambda$ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$. Hence, for every $n \in \mathbb{N}$, $u_n$ is a non-zero solution of problem $(D^+_p f^+)$, by Lemma 1, $u_n$ is a positive solution of problem $(D^+_p f)$. Since $\Phi$ is bounded on bounded sets and $\lim_{n \to +\infty} \Phi(u_n) = +\infty$, $\{u_n\}$ must be unbounded. So Theorem 2 holds and the proof is complete.

**Theorem 3.** Let $2 \leq p < +\infty$ and $f(k, \cdot) : \mathbb{R} \to \mathbb{R}$ to be a continuous function with $f(k, 0) \geq 0$ for each $k \in \mathbb{N}(1, T)$. Assume that

$$(i_2) \ A_{+\infty} < \frac{(\sqrt{2})^p}{(T + 1)^{\frac{p^2}{2p}} B^{+\infty}}.
$$

Then, for each $\lambda \in \left( \frac{(\sqrt{2})^p}{p B^{+\infty}}, \frac{2^p}{(T + 1)^{\frac{p^2}{2p}} A_{+\infty}} \right)$, problem $(D^+_p f)$ has an unbounded sequence of positive solutions.
Proof. We sketch only the differences with the proof of Theorem 2. For \( t > 0 \), make

\[
0 < r = \frac{(2t)^p}{p(T + 1)^{p-1}}.
\]

Assume \( u \in X \) and

\[
\Phi(u) = \frac{1}{p} \sum_{k=0}^{T} \left( \left( 1 + (\triangle u(k))^2 \right)^{\frac{p}{2}} - 1 \right) \leq r.
\]

Denote \( v(k) = (1 + (\triangle u(k))^2)^{\frac{p}{2}} - 1 \), for every \( k \in \mathbb{Z}(0, T) \), then \( \sum_{k=0}^{T} v(k) \leq pr \).

Noting the inequality \((x + y)^\theta \leq x^\theta + y^\theta\), for \( 0 < \theta \leq 1 \), \( x \geq 0, y \geq 0 \) and Hölder inequality, one has

\[
\sum_{k=0}^{T} (\triangle u(k))^2 = \sum_{k=0}^{T} (1 + v(k))^\frac{2}{p} - 1 \\
\leq \sum_{k=0}^{T} (v(k))^\frac{2}{p} \\
\leq (T + 1)^{\frac{p-2}{2}} \left( \sum_{k=0}^{T} v(k) \right)^{\frac{2}{p}} \\
\leq (T + 1)^{\frac{p-2}{2}} (pr)^{\frac{2}{p}} = \frac{4t^2}{T+1}.
\]

Applying (3), we have

\[
||u||_\infty \leq \frac{\sqrt{T+1}}{2} \left( \sum_{k=0}^{T} (\triangle u(k))^2 \right)^{\frac{1}{2}} \leq t.
\]

By the definition of \( \varphi \), we have

\[
\varphi(r) = \frac{\sup_{[0,k]} \Psi^+(\varphi)}{r} \leq \frac{\sup_{[0,k]} \sum_{l=0}^{T} F^+(k, u(k))}{r} \leq \frac{\sup_{[0,k]} \sum_{l=0}^{T} F(k, u(k))}{r} \leq \frac{p(T + 1)^{p-1} \sum_{k=1}^{T} \max F(k, s)}{2^p 1^p}.
\]

Using condition \((I_2)\), \( \varphi_\infty \leq \frac{p(T+1)^{p-1}}{2^p} A_\infty < +\infty \) holds.

Now, we verify that \( I^+_\lambda \) is unbounded form blow. First, assume that \( B^{+\infty} = +\infty \). Let \( \{c_n\} \) be a sequence of positive numbers, with \( \lim_{n \to +\infty} c_n = +\infty \), such that

\[
\sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \geq \frac{(\sqrt{2})^p + p}{\lambda p} c_n^p, \text{ for } n \in \mathbb{N}.
\]

Picking the sequence \( \{\omega_n\} \) in \( X \) by \( \omega_n(k) = c_n \), for \( k \in \mathbb{Z}(1, T) \). Exploiting the inequality \((x + y)^\theta \leq 2^{\theta-1}(x^\theta + y^\theta)\) for \( \theta \geq 1, x \geq 0, y \geq 0 \), we get

\[
I^+_{\lambda}(\omega_n) = \left( 1 + c_n^\frac{p}{2} \right)^{\frac{p}{2}} - 1 = \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{(\sqrt{2})^p + p}{\lambda p} c_n^p + \frac{(\sqrt{2})^p - 2}{p} c_n^p - \frac{(\sqrt{2})^{p-2}}{p} c_n^p \\
= -c_n^p + \frac{(\sqrt{2})^{p-2}}{p} c_n^p,
\]

which implies that \( \lim_{n \to +\infty} I_{\lambda}(\omega_n) = -\infty \).
Next, assume that \( B^{+\infty} < +\infty \). Since \( \lambda > \frac{(\sqrt{2})^p}{pB^{+\infty}} \), we may take \( \epsilon_0 > 0 \) such that \( \frac{(\sqrt{2})^p}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 < 0 \). Then there exists a sequence of positive numbers \( \{c_n\} \) such that \( \lim_{n \to +\infty} c_n = +\infty \) and

\[
(B^{+\infty} - \epsilon_0)c_n^p \leq \sum_{k=1}^{T} F^+(k, c_n) = \sum_{k=1}^{T} F(k, c_n) \leq (B^{+\infty} + \epsilon_0)c_n^p.
\]

Define the sequence \( \{\omega_n\} \) in \( S \) as above, we obtain

\[
I^+_\lambda(\omega_n) = \frac{2}{p} \left( (1 + \epsilon^2_n)^{\frac{p}{2}} - 1 \right) - \lambda \sum_{k=1}^{T} F^+(k, c_n) \leq \frac{(\sqrt{2})^p}{p} c_n^p + \frac{(\sqrt{2})^p - 2}{p} - \lambda (B^{+\infty} - \epsilon_0)c_n^p
\]

\[
= \left( \frac{(\sqrt{2})^p}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 \right)c_n^p + \frac{(\sqrt{2})^p - 2}{p}.
\]

Since \( \frac{(\sqrt{2})^p}{p} - \lambda B^{+\infty} + \lambda \epsilon_0 < 0 \), it is obvious that \( \lim_{n \to +\infty} I^+_\lambda(\omega_n) = -\infty \).

Thus, we follow that \( I^+_\lambda \) is unbounded from below. According to Lemmas 1 and 3, we have finished the proof of the theorem.

Similarly, considering the functional \( I^-\lambda \), we can achieve the following results.

**Theorem 4.** Let \( 1 \leq p < 2 \) and \( f(k, \cdot) : \mathbb{R} \to \mathbb{R} \) to be a continuous function with \( f(k, 0) \leq 0 \) for each \( k \in \mathbb{Z}(1, T) \). Assume that

\[
(i_3) \quad A_{-\infty} < \frac{2^{p-1}}{(T + 1)^{\frac{p}{2}}} B^{-\infty}.
\]

Then, for each \( \lambda \in \left( \frac{2}{pB^{-\infty}}, \frac{2^p}{p(T + 1)^{\frac{p}{2}} A_{-\infty}} \right) \), problem \((D^{\lambda,f}_p)\) has an unbounded sequence of negative solutions.

**Theorem 5.** Let \( 2 \leq p < +\infty \) and \( f(k, \cdot) : \mathbb{R} \to \mathbb{R} \) to be a continuous function with \( f(k, 0) \leq 0 \) for each \( k \in \mathbb{Z}(1, T) \). Assume that

\[
(i_4) \quad A_{-\infty} < \frac{(\sqrt{2})^p}{(T + 1)^{p-1}} B^{-\infty}.
\]

Then, for each \( \lambda \in \left( \frac{(\sqrt{2})^p}{pB^{-\infty}}, \frac{2^p}{p(T + 1)^{\frac{p}{2}} A_{-\infty}} \right) \), problem \((D^{\lambda,f}_p)\) has an unbounded sequence of negative solutions.

Combining Theorems 2 and 4, we have the following corollary.

**Corollary 1.** Let \( 1 \leq p < 2 \) and \( f(k, \cdot) : \mathbb{R} \to \mathbb{R} \) to be a continuous function with \( f(k, 0) = 0 \) for each \( k \in \mathbb{Z}(1, T) \). Assume that

\[
(i_5) \quad \max \{A_{+\infty}, A_{-\infty}\} < \frac{2^{p-1}}{(T + 1)^{\frac{p}{2}}} \min \{B^{+\infty}, B^{-\infty}\}.
\]

Then, for each \( \lambda \in \left( \frac{2}{p \min \{B^{+\infty}, B^{-\infty}\}}, \frac{2^p}{p(T + 1)^{\frac{p}{2}} \max \{A_{+\infty}, A_{-\infty}\}} \right) \), problem \((D^{\lambda,f}_p)\) admits two unbounded sequences of constant-sign solutions (one positive and one negative).

Combining Theorems 3 and 5, we have the following corollary.

**Corollary 2.** Let \( 2 \leq p < +\infty \) and \( f(k, \cdot) : \mathbb{R} \to \mathbb{R} \) to be a continuous function with \( f(k, 0) = 0 \) for each \( k \in \mathbb{Z}(1, T) \). Assume that

\[
(i_6) \quad \max \{A_{+\infty}, A_{-\infty}\} < \frac{(\sqrt{2})^p}{(T + 1)^{p-1}} \min \{B^{+\infty}, B^{-\infty}\}.
\]
Then, for each \( \lambda \in \left( \frac{2p}{p \min \{B_{+\infty}, B_{-\infty}\}}, \frac{2p}{p (T+1) \max \{A_{+\infty}, A_{-\infty}\}} \right) \), problem (\( D_{p,f}^\lambda \)) admits two unbounded sequences of constant-sign solutions (one positive and one negative).

**Remark 1.** If we let \( p \to 2^- \) in Theorem 2, we find that the conditions and consequence of Theorem 2 is the same as those of Theorem 3 for \( p = 2 \). Moreover the results are consistent with results in [37]. For the special case, \( p = 1 \), Theorem 2 reduces to Corollary 2.1 of [35].

**Remark 2.** We note that, if for each \( k \in \mathbb{Z}(1, T) \), \( f(k, \cdot) : \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying \( f(k, t) \geq 0 \) for all \( t \in \mathbb{R} \setminus \{0\} \), then

\[
A_{+\infty} = \liminf_{t \to +\infty} \frac{\sum_{k=1}^T F(k, t)}{t^p}, \quad \text{and} \quad A_{-\infty} = \liminf_{t \to -\infty} \frac{\sum_{k=1}^T F(k, t)}{|t|^p}.
\]

Consequently, Theorem 1 immediately follows by Corollaries 1 and 2.

### 4. Two Examples

**Example 1.** For \( 1 \leq p < 2 \), we consider the boundary value problem (\( D_{p,f}^\lambda \)) with

\[
f(k, t) = p|t|^{p-1}\text{sign}(t) \left( \frac{T + 1}{T} + \sin \left( \frac{1}{2T} \ln(|t|^p + 1) \right) + \frac{1}{2T} \cos \left( \frac{1}{2T} \ln(|t|^p + 1) \right) \right), \quad (9)
\]

for \( k \in \mathbb{Z}(1, T) \), then

\[
F(k, t) = \int_0^t f(k, \tau) d\tau = \frac{T + 1}{T} |t|^p + (|t|^p + 1) \sin \left( \frac{1}{2T} \ln(|t|^p + 1) \right), \quad \text{for} \ t \in \mathbb{R}.
\]

Since \( f(k, t) \geq p t^{p-1} \left( \frac{T + 1}{T} - 1 - \frac{1}{2T} \right) = \frac{p}{2T} t^{p-1} > 0 \), for \( t > 0 \) and \( f(k, 0) = 0 \), we follow that for each fixed \( k \in \mathbb{Z}(1, T) \), \( F(k, t) \) is strictly monotone increasing on \([0, +\infty)\). One has \( \max_{0 \leq s \leq 1} F(k, s) = F(k, t) \), for each \( t \geq 0 \). Clearly,

\[
A_{+\infty} = \liminf_{t \to +\infty} \frac{TF(k, t)}{t^p} = \liminf_{t \to +\infty} \frac{(T+1)t^p + (t^p + 1) \sin\left(\frac{1}{2T} \ln(t^p + 1)\right)}{t^p} = 1,
\]

and

\[
B^{+\infty} = \limsup_{t \to +\infty} \frac{TF(k, t)}{t^p} = \limsup_{t \to +\infty} \frac{(T+1)t^p + (t^p + 1) \sin\left(\frac{1}{2T} \ln(t^p + 1)\right)}{t^p} = 2T + 1.
\]

In view of \( 1 \leq p < 2 \), we follow that \( A_{+\infty} < \frac{2p-1}{(T+1)^2} B^{+\infty} \). Applying to Theorem 2, problem (\( D_{p,f}^\lambda \)) admits an unbounded sequence of positive solutions.

Let us consider another example.

**Example 2.** Let \( T = 4 \), \( p = 3 \) and \( f \) be a function defined as follows

\[
f(k, t) = \frac{5}{4} |t| \left( \frac{1}{8} + \sin \left( \frac{1}{8} \ln(|t|^3 + 1) \right) + \frac{1}{8} \cos \left( \frac{1}{8} \ln(|t|^3 + 1) \right) \right), \quad k \in \mathbb{Z}(1, 4)
\]
Then, for every \( \lambda \in (\frac{3\sqrt{3}}{2}, \frac{8}{3}) \), the problem

\[
\begin{cases}
-\Delta (\varphi_{3,c}(\Delta u(k-1))) = \lambda f(k,u(k)), & k \in \mathbb{Z}(1,4), \\
u(0) &= u(5) = 0,
\end{cases}
\] (10)

Admits an unbounded sequence of positive solutions and an unbounded sequence of negative solutions.

Indeed, \( f(k,t) \geq 3t^2 \left( \frac{5}{4} - 1 - \frac{1}{8} = \frac{3}{8}t^2 \right) > 0 \), for \( t > 0 \) and \( f(k,0) = 0 \).

\[
F(k, t) = \int_0^t f(k, \tau)d\tau = \frac{5}{4}|t|^3 + (|t|^3 + 1) \sin \left( \frac{1}{8} \ln (|t|^3 + 1) \right), \text{ for } t \in \mathbb{R}.
\]

Since \( f(k,t) \geq 3t^2 \left( \frac{5}{4} - 1 - \frac{1}{8} = \frac{3}{8}t^2 > 0 \), for \( t > 0 \), we follow that for each fixed \( k \in \mathbb{Z}(1,4) \), \( F(k,t) \) is strictly monotone increasing on \([0, \infty)\). Thus, \( \max F(k,s) = F(k,t) \), for each \( t \geq 0 \). Obviously,

\[
A_{\pm \infty} = \lim_{t \to +\infty} \inf \lim_{t \to +\infty} \frac{4F(k,t)}{t^3} = \lim_{t \to +\infty} \inf \frac{5t^3 + 4(t^3 + 1) \sin \left( \frac{1}{8} \ln (t^3 + 1) \right)}{t^3} = 1,
\]

and

\[
B_{\pm \infty} = \lim_{t \to +\infty} \sup \lim_{t \to +\infty} \frac{4F(k,t)}{t^3} = \lim_{t \to +\infty} \sup \frac{5t^3 + 4(t^3 + 1) \sin \left( \frac{1}{8} \ln (t^3 + 1) \right)}{t^3} = 9.
\]

Through simple computation, \( \max \{A_{+ \infty}, A_{- \infty}\} < \frac{(\sqrt{2})^p}{(1+1)^p} \min \{B_{+ \infty}, B_{- \infty}\} \) holds. Corollary 2 ensures our claim.

5. Conclusions

In this paper, we have discussed the Dirichlet boundary value problem of the difference equation with \( p \)-mean curvature operator. Some sufficient conditions are derived for the existence of sequences of constant-sign solutions to the problem. Two examples are given to show the effectiveness of our results.

To solve problem \( (D_p^{\lambda,f}) \), we further develop the methods adopted in \([23]\). The approaches can be used for the boundary value problems of differential equations involving \( p \)-mean curvature operator. Therefore, our work has both theoretical and practical significance.

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