"On one Property of one Solution of one Equation"

or

Linear ODEs, Wronskians and Schubert Calculus*

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To the Memory of Vladimir Arnold

Abstract

For a linear ODE with indeterminate coefficients, we exhibit a fundamental system of solutions explicitly, in terms of the coefficients. We show that the generalized Wronskians of the fundamental system are given by an action of the Schur functions on the usual Wronskian, and thence enjoy Pieri’s and Giambelli’s formulae. As an outcome, we obtain a natural isomorphism between the free module generated by the generalized Wronskians and the singular homology module of the Grassmannian.

“The existence of mysterious relations between all these different domains is the most striking and delightful feature of mathematics.” V. Arnold, [1]

Introduction

In this paper we deal with the universal linear ODE

\[ u^{(r+1)} - e_1 u^{(r)} + \ldots + (-1)^{r+1} e_{r+1} u = 0 \]  \hspace{1cm} (1)

where \( e_1, \ldots, e_{r+1} \) are indeterminates.

Let \( E_r = \mathbb{Q}[e_1, \ldots, e_{r+1}] \) be a polynomial \( \mathbb{Q} \)-algebra in the ODE coefficients, and \( E_r[[t]] \) the \( E_r \)-algebra of formal power series of \( t \), supplied with the standard formal derivation with respect to \( t \). In Sec. 1, we solve equation (1) explicitly in \( E_r[[t]] \), as well as a non-homogenous one.

In Sec. 2, we use the obtained universal fundamental system of solutions to express the generalized Wronskians, which are numerated by partitions, in terms of the ordinary Wronskian of (1). It appears, for a given partition the ratio of the corresponding generalized Wronskian with the usual one is a specialization of the Schur function

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associated with the same partition (Theorem 2.2). Thus the product in the free module spanned by the generalized Wronskians obeys the same rule as the product of Schur functions, and hence as the intersection of Schubert cells. In particular, this relation implies a natural isomorphism between the free \( \mathbb{Z}[e_1, \ldots, e_{r+1}] \)-module generated by the generalized Wronskians and the \( H^\ast(G(r, \mathbb{P}^\infty)) \)-module of the singular homology of the Grassmannian \( G(r, \mathbb{P}^\infty) \) of \( r \)-dimensional linear subvarieties of the infinite-dimensional complex projective space.

The title “On one Property of one Solution of one Equation” was mentioned many times by V. Arnold as an example of a meaningless one. Therefore one of the authors, the one who was fortunate to be Arnold’s student and to attend Arnold Seminar, dreamed to publish one day a (preferably interesting) paper with this title, just to surprise Arnold.

In fact, there was one opportunity, when that author, in collaboration with another former student of V. Arnold, prepared for publication a paper dedicated to Arnold’s 65-th birthday. But the collaborator was strict and rejected this idea immediately.

This time the dream nearly comes true, but not really, as no chance to surprise Arnold anymore...

0.1 Universal linear ODE. In Sec. 1, we revise the theory of linear ODE’s in an algebraic context. For the classical theory see e.g. [2].

Denote by \( D : E_r[[t]] \to E_r[[t]] \) the standard formal derivative with respect to \( t \). Throughout the paper we use the following notation:

\[
f = f(t) = \sum_{n \geq 0} a_n t^n := \sum_{n \geq 0} \frac{t^n}{n!}, \quad D^i f = f^{(i)}(t) := \sum_{n \geq 0} \frac{t^{n+i}}{n!}, \quad f^{(i)}(0) := a_i, \tag{2}
\]

for \( f \in E_r[[t]], a_n \in E_r \).

Our one Equation is (1).

Define \( h_j \in E_r \ (j \in \mathbb{Z}) \) by means of the following generating function:

\[
\frac{1}{1 - e_1 t + e_2 t^2 - \ldots + (-1)^{r+1} e_{r+1} t^{r+1}} = \sum_{j \in \mathbb{Z}} h_j t^j. \tag{3}
\]

In particular, \( h_j = 0 \) for \( j < 0 \), \( h_0 = 1 \), \( h_1 = e_1 \), \( h_2 = e_1^2 - e_2 \), etc.

The origin of the present work is the following observation:

The formal power series \( u_0 := \sum_{n \geq 0} h_n \frac{t^n}{n!} \) is a solution of (1).

This is our one Solution.

For each \( 1 \leq j \leq r \), define the formal power series \( u_j = u_j(t) \) as the unique element in the ideal \( (t^j) \) of \( E_r[[t]] \) such that \( D^j u_j = u_0 \). The explicit formulae are

\[
u_j = \sum_{n \geq j} h_{n-j} \frac{t^n}{n!}, \quad 0 \leq j \leq r. \tag{4}
\]
Theorem A. The power series $\left(4\right)$ form a fundamental system of solutions of $\left(1\right)$.

This is our one Property.

We call $\left(4\right)$ the universal fundamental system. The reason is the following. Denote by $E_r[T]$ the $E_r$-polynomial ring in the indeterminate $T$. Equation $\left(1\right)$ may be written then as $U_r+1(D)u = 0$, where

\[ U_{r+1}(T) = T^{r+1} - e_1 T^r + \ldots + (-1)^{r+1} e_{r+1} \]

is the universal polynomial of the degree $r+1$. For any $\mathbb{Q}$-algebra $A$ and for any monic polynomial $P(T) \in A[T]$, we can solve the ODE $P(D)u = 0$ in formal power series $A[[t]]$.

Theorem B. If $P(T) \in A[T]$ is a monic polynomial of degree $r+1$, then there is a unique $\mathbb{Q}$-algebra homomorphism $\psi : E_r \to A$ mapping the universal fundamental system $\left(4\right)$ to a fundamental system of the linear ODE $P(D)v = 0$.

Our version of the theory of linear homogeneous and non-homogeneous ODE’s is described in Sec. $1.0$. Derivatives of the Wronskian and Generalized Wronskians. For a non-decreasing set of $r+1$ integers, $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r \geq 0$, denote $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ the partition of $|\lambda| := \lambda_0 + \lambda_1 + \ldots + \lambda_r$.

Given an $(r+1)$-tuple $f$ of formal power series,

\[ f := \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_r \end{pmatrix}, \]

and a partition $\lambda$, the generalized Wronskian $W_\lambda(f)$ is defined as the determinant of the matrix whose $i$-th row entries are the derivatives of order $j + \lambda_{r-j}$ of $f_i$, for $0 \leq i, j \leq r$. The usual Wronskian $W(f) = W_0(f)$ corresponds to the trivial partition $\lambda_j = 0, 0 \leq j \leq r$.

\[ W_\lambda(f) := \det(f^{j + \lambda_{r-j}})_{0 \leq i, j \leq r}, \quad W_0(f) := \det(f^{(j)})_{0 \leq i, j \leq r}. \] (5)

It is not surprising that the derivatives of all orders of the Wronskian are positive $\mathbb{Z}$-linear combinations of generalized Wronskians. However the intriguing fact was that the coefficients of the linear combinations have a precise interpretation in terms of enumerative geometry of subspaces of the complex projective space.

For example, the dimension of the Grassmann variety $G(r, \mathbb{P}^d)$ of $r$-dimensional subspaces in the complex projective $d$-dimensional space is $(r+1)(d-r)$. It was observed in [16], that the coefficient of $W_{(d-r,d-r,...,d-r)}(f)$ in the $\mathbb{Z}$-linear combination of the $(r+1)(d-r)$-th derivative of $W(f)$ is the Plücker degree of $G(r, \mathbb{P}^d)$ (cf. [13, Example 14-7-11(iii)]).
Here we put such a remark in a proper framework. Recall that partitions are described by means of Young–Ferrers diagrams, and a standard Young tableau is a filling in the Young–Ferrers diagram of $\lambda$ with numbers $1, \ldots, |\lambda|$ in such a way that the numbers in any column and in any row increase, [12].

**Proposition.** We have

$$D^{k+1}W(f) = \sum_{|\lambda|=k} c_{\lambda} W_\lambda(f),$$

where $c_{\lambda}$ is the number of the standard Young tableaux of the Young–Ferrers diagram $\lambda$. ■

Numbers $c_{\lambda}$'s and their interpretation in terms of Schubert calculus (7) are very well known, see e.g., [12]. In particular, these numbers can be calculated by the hook formula,

$$c_{\lambda} = \frac{|\lambda|!}{k_1! \cdots k_{|\lambda|}!},$$

where $k_j$'s are the hook lengths of the boxes of $\lambda$.

### 0.3 Wronski–Schubert Calculus

In Sec. 2, the $(r+1)$-tuple $f$ is a fundamental system of solutions of the ODE (1), i.e., a basis of $\ker U_{r+1}(D)$. One can re-write the equation (1) in the form

$$\begin{vmatrix}
  u & u' & \cdots & u^{(r+1)} \\
  f_0 & f'_0 & \cdots & f_0^{(r+1)} \\
  f_1 & f'_1 & \cdots & f_1^{(r+1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_r & f'_r & \cdots & f_r^{(r+1)} \\
\end{vmatrix} = 0.$$

In notation (5), we get

$$W_0(f)u^{(r+1)} - W_{(1)}(f)u^{(r)} + W_{(1^2)}(f)u^{(r-1)} + \ldots + (-1)^{r+1}W_{(1^r+1)}(f)u = 0,$$

where $(1^k) := (1, \ldots, 1, 0 \ldots, 0)$ is the primitive partition of $k$.

Thus we have $W_{(1^k)}(f) = c_k W(f)$, $1 \leq k \leq r+1$. In particular, $W_{(1)}(f)$ is the derivative of $W_0(f)$, i.e., the Wronskian solves the first order universal equation $u' - e_1 u = 0$. This is the well-known Liouville theorem, cf. [2, Ch. 3, §27.6].

Relation (3) is satisfied, in particular, by the complete symmetric functions $h_j$'s and the elementary symmetric functions $e_j$'s of $r+1$ variables (see e.g., [27, Ch. I, §2]). As the Referee pointed out, this interpretation of $h_j$'s and $e_j$'s fits the approach of A. Lascoux, [26], within symmetric functions are treated as operators in polynomial spaces.

Being motivated by Schubert Calculus problems, we have chosen to interpret (3) in terms of characteristic classes, as the relation between Chern classes (our $e_j$'s) and Segre classes (our $h_j$'s) of the tautological bundle over the Grassmannian $G(r, \mathbb{P}^\infty)$. We refer to [13] for the intersection theory of complex Grassmannians.

It is well-known that $G = G(r, \mathbb{P}^\infty)$ possesses a cellular decomposition into Schubert cells parameterized by the partitions $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$. The homology classes of
their closures, \(\{\Omega_\lambda}\), form a \(\mathbb{Z}\)-basis of \(H_*(G, \mathbb{Z})\). The \(\cap\) product map turns the homology \(H_*(G, \mathbb{Z})\) into a free module of rank 1 over the cohomology, generated by the fundamental class \([G]\). The Schubert classes \(\sigma_\lambda \in H^*(G, \mathbb{Z})\) are then defined via the equality \(\Omega_\lambda = \sigma_\lambda \cap [G]\), for every partition \(\lambda\). In \(H^*(G, \mathbb{Z})\) the multiplication (or \(\cup\) product, or intersection) of cohomology classes is defined, and the classical Pieri’s formula gives the product of a special Schubert class \(\sigma_k\), i.e., that corresponding to the special partition \((k, 0, \ldots, 0)\) of \(k\), with an arbitrary one,

\[
\sigma_k \sigma_\lambda = \sum_{\mu} c_{\lambda\mu} \sigma_\mu, \quad 1 \leq k \leq r + 1,
\]

where the sum is taken over all the partitions \(\mu = (\mu_0, \mu_1, \ldots, \mu_r)\) such that

\[
|\mu| = k + |\lambda|, \quad \mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r.
\]  

(6)

Notice that \(c_\lambda\)'s from the Proposition above are coefficients in the decomposition of \(\sigma_1^k\) into the sum of Schubert classes:

\[
\sigma_1^k = \sum_{|\lambda|=k} c_\lambda \sigma_\lambda.
\]  

(7)

The special Schubert classes \(\sigma := (\sigma_0, \sigma_1, \ldots, \sigma_k, \ldots)\) are multiplicative generators of the cohomology ring \(H^*(G, \mathbb{Z})\). Recall that the Schur polynomial corresponding to the partition \(\lambda\) is:

\[
\Delta_\lambda(x) := \det(x_{\lambda_r-j, j-i})_{0 \leq i, j \leq r},
\]

where \(x = (x_k)_{k \in \mathbb{Z}}\) are variables, [27].

The classical Giambielli’s formula gives any Schubert class \(\sigma_\lambda\) as the specialization of the corresponding Schur polynomial at the special Schubert classes:

\[
\sigma_\lambda = \Delta_\lambda(\sigma),
\]

where one sets \(\sigma_j := 0\) for \(j < 0\).

In Sec. 2 we prove the Giambielli’s and the Pieri’s formulae for generalized Wronskians (Theorem 2.2, Corollary 2.3). We write \(h := (h_j)_{j \in \mathbb{Z}}\) for the sequence \(h_j\)'s defined in (3).

**Theorem C.** We have

\[
W_\lambda(f) = \Delta_\lambda(h)W_0(f), \quad h_k W_\lambda(f) = \sum_{\mu} W_\mu(f), \quad k \geq 1,
\]

where the sum is over all the partitions \(\mu\) satisfying (6).

Denote by \(\mathbb{W}(f)\) the free \(\mathbb{Z}\)-module generated by the generalized Wronskians \(W_\lambda(f)\)'s.

**Corollary.** The correspondence \(\Omega_\lambda \mapsto W_\lambda(f)\) defines an isomorphism between \(\mathbb{W}(f)\) as a \(H^*(G, \mathbb{Z})\)-module and the \(H^*(G, \mathbb{Z})\)-module \(H_*(G, \mathbb{Z})\), the singular homology of the infinite Grassmannian.
0.4 Comments. Wronskians are ubiquitous in mathematics. We especially are interested in their role in algebraic geometry. For example, any linear system on the projective line defines an \((r+1)\)-dimensional subspace of the vector space of complex polynomials in one indeterminate, i.e. an element of the corresponding Grassmannian. The Wronski map sending the polynomial subspace to the class modulo \(C^*\) of the Wronski determinant of its basis appears. In particular, the critical points of a rational function in one variable are roots of the Wronskian of the denominator and the numerator, the ramification points of linear systems on curves occur as the zero locus of a certain Wronskian section of suitable line bundles, etc. In this context, the Wronski map has been studied by many authors, see [8, 10, 22, 31].

The moduli points corresponding to curves of fixed genus possessing special Weierstrass points provide another example; they occur as the zero locus of a suitable derivative of a Wronskian associated to the sheaf of relative differentials, [14, 20]. Further relations between Wronskians of linear systems and Schubert calculus on a Grassmann bundle are investigated in [7, 11, 17].

Discovering of Wronskians in the Bethe Ansatz of the \(sl_n\) Gaudin model was crucial both for study of Bethe vectors and for pure algebro-geometric problems, like transversality of the intersection of Schubert varieties, [29, 30, 32, 34]. In [31], relation to the \(sl_2\) representation theory was used to calculate the intersection number of some Schubert varieties in the Grassmannian of projective lines; later on a purely algebro-geometric proof, based on the Wronskian inspired methods of [18], appeared in [6].

Generalized Wronskians have also appeared within different contexts, e.g. in the theory of Weierstrass points on curves ([35, 36]), and in connection with number theory ([28, 3]).

Recently, D. Laksov and A. Thorup showed that the \(n\)-th exterior power of a polynomial ring in \(n\) indeterminates is a free module over its subring of symmetric functions, and that the module structure is equivalent to the Schubert Calculus for infinite Grassmannians, [24, 25]. Our construction may be considered as a concrete realization of this equivalency.

1 Solution to Linear ODEs in Formal Power Series.

First of all we solve the universal equation (1) with coefficients in \(E_r\), i.e., find \(\ker U_{r+1}(D) \subset E_r[[t]]\). Substitution of (2) into (1) gives the following condition on the coefficients of \(t^n\).

1.1 Proposition. The series \(f = \sum_{n \geq 0} x_n t^n \) is in \(\ker U_{r+1}(D)\) if and only if

\[
x_{n+1} - e_1 x_n + \ldots + (-1)^{r+1} e_{r+1} x_{n-r} = 0, \quad n \geq r.
\]

This recurrence relation on the coefficients \(x_n\)'s imposes no restrictions on \(x_0, x_1, \ldots, x_r\); they are the initial conditions of the solution.
**1.2 Theorem.** The $E_r$-module $\ker U_{r+1}(D) \subset E_r[[t]]$ is free of rank $r+1$ and generated by $u_0, \ldots, u_r$ given by \((3), (4)\).

**Proof.** First we will prove that every $u_j$ is a solution of \((1)\).

Let us begin with $u_0$. The coefficients of $t^n$ on both sides of \((3)\) are the same, that is
\[
h_0 = 1, \quad h_{n+1} - e_1 h_n + \ldots + (-1)^{r+1} e_{r+1} h_{n-r} = 0, \quad n \geq 0, \tag{11}
\]
and Proposition 1.1 immediately implies $u_0 \in \ker U_{r+1}(D)$.

Next, let us show $u_r \in \ker U_{r+1}(D)$. Indeed, $u_{r-j} = D^j u_r$ $(0 \leq j \leq r)$, and we get
\[
D^{r+1} u_r - e_1 \cdot D^r u_r + \ldots + (-1)^r e_{r+1} u_r = u_0' - e_1 u_0 + e_2 u_1 + \ldots + (-1)^r e_{r+1} u_r = \sum_{n \geq 0} (h_{n+1} - e_1 h_n + e_2 h_{n-1} + \ldots + (-1)^r e_{r+1} h_{n-r}) \frac{t^n}{n!}.
\]
By \((11)\), all the coefficients of the expansion vanish.

Finally for each $0 \leq i \leq r$, \[
U_{r+1}(D) u_1 = U_{r+1}(D)^{r-i} u_r = D^{r-i} U_{r+1}(D) u_0 = 0,
\]
proving that $u_i$ is a solution as well.

Now we will show that $u_0, \ldots, u_r$ generate $\ker U_{r+1}(D)$ as an $E_r$-module. Assume that $f = \sum_{n \geq 0} x_n \frac{t^n}{n!}$ is a solution of $U_{r+1}(D) u = 0$.

Define $\Lambda_0, \ldots, \Lambda_r \in E_r$ as the unique solution of the linear system
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
h_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
h_r & h_{r-1} & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\Lambda_0 \\
\Lambda_1 \\
\vdots \\
\Lambda_r
\end{pmatrix}
= \begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_r
\end{pmatrix}. \tag{12}
\]
We contend that
\[
f = \Lambda_0 u_0 + \Lambda_1 u_1 + \ldots + \Lambda_r u_r, \tag{13}
\]
or, in terms of coefficients,
\[
x_n = \Lambda_0 h_n + \Lambda_1 h_{n-1} + \ldots + \Lambda_r h_{n-r}, \quad n \geq r+1. \tag{14}
\]
The proof is by induction on $n \geq r+1$. For $n = r+1$ we have, according to \((10)\),
\[
x_{r+1} = e_1 x_r - \ldots + (-1)^r e_{r+1} x_0.
\]
First we express $x_{r+1}$ via $\Lambda_j$'s from \((12)\), and then apply \((11)\):
\[
x_{r+1} = e_1 (h_r \Lambda_0 + \ldots + h_1 \Lambda_{r-1} + \Lambda_r) - \ldots + (-1)^r e_{r+1} \Lambda_0 = \\
= \Lambda_0 (e_1 h_r + e_2 h_{r-1} - \ldots + (-1)^r e_{r+1}) + \ldots + \Lambda_r e_1 = \\
= \Lambda_0 h_{r+1} + \ldots + \Lambda_r h_1.
\]
Suppose now (11) true for all $r \leq m \leq n$. Then

$$x_{n+1} = e_1 x_n - e_2 x_{n-1} + \ldots - (-1)^{r+1} e_r x_{n-r} =$$

$$= e_1 \left( \sum_{i=0}^{r} \Lambda_i h_{n-i} \right) - e_2 \left( \sum_{i=0}^{r} \Lambda_i h_{n-1-i} \right) + \ldots - (-1)^{r+1} e_r \left( \sum_{i=0}^{r} \Lambda_i h_{n-r-i} \right) =$$

$$= \sum_{i=0}^{r} \Lambda_i (e_1 h_{n-i} - e_2 h_{n-1-i} + \ldots - (-1)^{r+1} e_r h_{n-r-i}) =$$

$$= \sum_{i=0}^{r} \Lambda_i h_{n+1-i} = \Lambda_0 h_{n+1} + \Lambda_1 h_n + \ldots + \Lambda_r h_{n-r+1},$$

as desired.

Finally, we conclude that $u_0, \ldots, u_r$ are linearly independent. Indeed, for the trivial solution $x_j = 0$ for all $j \geq 0$; hence $\Lambda_0 = \ldots = \Lambda_r = 0$ is the unique solution of the homogeneous linear system (12).

The inverse matrix of the matrix in (12) is well-known (see e.g. [27, Ch. I, §2]), so we can explicitly find $\Lambda_j$’s in terms of $x_j$’s:

$$\begin{pmatrix}
\Lambda_0 \\
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_r
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
-e_1 & 1 & 0 & \ldots & 0 \\
e_2 & -e_1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^r e_r & (-1)^{r-1} e_{r-1} & (-1)^{r-2} e_{r-2} & \ldots & 1
\end{pmatrix} \begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_r
\end{pmatrix}.$$

Substitution into (13) gives

### 1.3 Corollary (Universal Solution of the Cauchy Problem).

Let $y_0, y_1, \ldots, y_r$ be indeterminates over $E_r$. The unique solution of (1) over $E_r[y_0, y_1, \ldots, y_r]$ satisfying $D^i f(0) = y_i$, $0 \leq i \leq r$, is as follows:

$$g(t) = \Lambda_0(y) u_0(t) + \Lambda_1(y) u_1(t) + \ldots + \Lambda_r(y) u_r(t),$$

$$\Lambda_j(y) = y_j - e_1 y_{j-1} + \ldots + (-1)^j e_j y_0, \quad 0 \leq j \leq r.$$
1.5 **Theorem.** (1) The series \( f(t) = \sum_{n \geq 0} x_n t^n / n! \in A[[t]] \) is a solution of \( P(D)v = 0 \) if and only if
\[
x_{n+1} - a_1 x_n + \ldots + (-1)^{r+1} a_{r+1} x_{n-r} = 0, \quad n \geq r + 1.
\]
(2) \( \ker P(D) \) is a free \( A \)-module of rank \( r + 1 \) generated by \( v_0, \ldots, v_r \), the image of \( (4) \), via the unique homomorphism of \( \mathbb{Q} \)-algebras \( E_r \to A \) mapping \( e_i \mapsto a_i, 1 \leq i \leq r + 1 \). In particular, the fundamental system \( v_j's \) satisfies \( D^j v_r = v_{r-j}, 0 \leq j \leq r \). (3) The unique solution of \( P(D)v = 0 \) with given initial conditions \( x_0, \ldots, x_r \in A \) is
\[
f(t) = V_0(x)v_0(t) + V_1(x)v_1(t) + \ldots + V_r(x)v_r(t),
\]
where \( V_j(x)'s \) are the images of \( \Lambda_j(y)'s \) under the unique \( \mathbb{Q} \)-algebra homomorphism
\[
E_r[y_0, y_1, \ldots, y_r] \to A
\]
mapping \( e_i \mapsto a_i \) and \( y_j \mapsto x_j \).

Consider as an example the equation \( v'' - 3v' + 2v = 0 \) (clearly the real and the complex numbers are \( \mathbb{Q} \)-algebras). The corresponding (characteristic) polynomial is \( T^2 - 3T + 2 = (T - 2)(T - 1) \), and the standard prescription, \( [2] \), gives the fundamental system \( f_1 = e^t, f_2 = e^{2t} \). According to Theorem 1.5, \( f_1 = v_0 - 2v_1 \) and \( f_2 = v_0 - v_1 \).

Of course, one can easily obtain this relation directly. Indeed, the substitution \( e_1 = 3, e_2 = 2 \) into \( (3) \) gives the image of the universal fundamental system,
\[
v_0 = 1 + 3t + \frac{7t^2}{2} + \frac{15t^3}{3!} + \ldots, \quad v_1 = t + \frac{3t^2}{2} + \frac{7t^3}{3!} + \frac{15t^4}{4!} + \ldots,
\]
i.e., \( v_0 = 2f_2 - f_1, v_1 = f_2 - f_1 \) and \( v'_1(t) = v_0(t) \).

1.6 **Universal Euler formula.** For \( x \in A \), define \( \exp(xt) := \sum_{n \geq 0} x^n t^n / n! \). If \( x \) is a root of \( P(T) \), it is not surprising that \( \exp(xt) \in \ker P(D) \). Indeed, the condition of Theorem 1.5 (1) holds:
\[
x^{n+1} - a_1 x^n + \ldots + (-1)^{r+1} a_{r+1} x^{n-r} = x^{n-r} P(x) = 0, \quad n \geq r + 1.
\]
In particular, if \( \epsilon \in \mathbb{C} \) is a primitive root of \( (-1)^{r+1} \) of order \( r + 1 \), then \( \exp(\epsilon t) \) is the solution of the ODE \( u^{(r+1)} + u = 0 \) over \( \mathbb{C} \) with initial conditions \( 1, \ldots, \epsilon^r \). According to Theorem 1.5 (3),
\[
\exp(\epsilon t) = v_0(t) + \epsilon v_1(t) + \ldots + \epsilon^r v_r(t),
\]
\[
v_j(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^j k^{(r+1)}}{(j + kr + k)!}, \quad 0 \leq j \leq r.
\]
For \( r = 1 \), the image of the universal fundamental system \( (u_0, u_1) \) is \( (\cos t, \sin t) \), \( \epsilon = i \), and the classical Euler formula \( \exp(it) = \cos t + i \sin t \) follows.

More generally, let \( \alpha := T + (U_{r+1}(T)) \) be the universal root of \( U_{r+1}(T) \), cf. \([9]\). Define \( E_r[\alpha] := E_r[T]/(U_{r+1}(T)) \). The polynomial \( U_{r+1}(T) \) is defined over \( E_r[\alpha] \) as well, and \( U_{r+1}(\alpha) = 0 \). In fact \( E_r[\alpha] \) is the universal splitting algebra of \( U_{r+1}(T) \) as a
product of two monic polynomials, one of degree 1. It is universal in the sense that for any $E_r$-algebra $B$ where $U_{r+1}(T)$ splits as the product $(T - \beta)q(T)$, with $\beta \in B$ and $q(T) \in B[T]$ monic of degree $r$, there is a unique homomorphism $\phi : E_r[\alpha] \to B$ mapping $\alpha \mapsto \beta$, [9]. Thus $\exp(\alpha t)$ is a solution of $U_{r+1}(D)u = 0$ defined over $E_r[\alpha]$, and we have

$$\exp(\alpha t) = u_0(t) + \Lambda_1(\alpha)u_1(t) + \ldots + \Lambda_r(\alpha)u_r(t),$$

$$\Lambda_j(\alpha) = \alpha^j - e_1\alpha^{j-1} + \ldots + (-1)^je_j, \quad 0 \leq j \leq r. \quad (18)$$

Again, for $r = 1$, one has $\exp(\alpha t) = u_0(t) + (\alpha - e_1)u_1(t).$ Under the unique $\mathbb{Q}$-algebra homomorphism $E_1[\alpha] \to \mathbb{C}$ defined by $e_1 \mapsto 0$, $e_2 \mapsto 1$ and $\alpha \mapsto i$, this formula becomes the classical Euler formula $\exp(it) = \cos t + i \sin t.$

1.7 Remark on fundamental systems. In the algebraic context under consideration, there is a difference between a fundamental system, which is a basis of $\ker P(D)$, and a set $f_0, \ldots, f_r$ of linearly independent solutions of $P(D)u = 0$.

For example, consider $A = \mathbb{Q}[a_1, a_2]$ and the equation $u'' - (a_1 + a_2)u' + a_1a_2u = 0.$ The characteristic polynomial is $P(T) = (T - a_1)(T - a_2)$, and hence $\exp a_1t$, $\exp a_2t$ are two linearly independent solutions of the equation. However these solutions do not form a basis of $\ker P(D)$ over $A$. Indeed, consider $v_0, v_1$, the image of the universal fundamental system $(u_0, u_1)$ of $U_1(D)u = 0$ via the homomorphism $E_2 \to A$, sending $e_1 \mapsto a_1 + a_2$ and $e_2 \mapsto a_1a_2.$ By Theorem 1.5, $(v_0, v_1)$ is a basis of $\ker P(D)$ over $A$. The relation to $\exp a_1t$, $\exp a_2t$ is given by formula (18):

$$\exp a_1t = v_0 - a_2v_1, \quad \exp a_2t = v_0 - a_1v_1.$$ 

Thus $\exp a_1t - \exp a_2t = (a_1 - a_2)v_1$, but $(a_1 - a_2)$ is not invertible in $A!$ We conclude that there is no way to express $v_1$ as an $A$-linear combination of $\exp a_1t$ and $\exp a_2t$.

However, for the same equation considered over the localization $B := A[1/(a_1 - a_2)]$ both $(v_0, v_1)$ and $(\exp a_1t, \exp a_2t)$ are fundamental systems of $U_2(D)y = 0$ over $B$, i.e., bases of $\ker P(D)$ over $B$. Since $(v_0, v_1)$ is also a fundamental system over $A$, it may be considered as “the most economical” one, see [19] for further discussion.

Recall that for any algebra $A$ we have ([5]):

$$g = g(t) \in A[[t]] \text{ is invertible in } A[[t]] \text{ if and only if } g(0) \text{ is invertible in } A.$$ 

Let $P(T) \in A[T]$ be a monic polynomial of degree $r + 1$ and $f_0, \ldots, f_r \in \ker P(D).$ Denote by $C := (f^{(j)}(0))_{0 \leq i < j \leq r}$ the initial condition matrix of the $f_i$’s. Then $\det C \in A$ is the constant term of the determinant of the transition matrix from the image of the universal fundamental system, $v_i$’s, to $f_j$’s.

1.8 Proposition. (1) The solutions $f_0, \ldots, f_r$ of $P(D)y = 0$ form a basis of $\ker P(D)$ over $A$ if and only if $\det C$ is invertible in $A$.

(2) If $\det C \neq 0$ and non-invertible in $A$, then $f_0, \ldots, f_r$ are $A$-linearly independent, but do not form a fundamental system. They do form a fundamental system over any $\mathbb{Q}$-algebra extension of $B := A[1/\det C]$. $\blacksquare$
In particular, if \( P(T) \) has \((r + 1)\) distinct roots \(a_0, \ldots, a_r\), then \(\exp(a_0t), \ldots, \exp(a_r t)\) form a fundamental system only if the Vandermonde determinant
\[
\prod_{0 \leq i < j \leq r} (a_i - a_j)
\]
is invertible in \( A \). On the other hand, the image of the universal fundamental system is always a fundamental system.

1.9 Non-homogeneous case. Here we discuss the case of the non-homogeneous ODE. Let
\[
f = \sum_{n \geq 0} a_n \frac{t^n}{n!} \in E_{r+1}[[[t]]].
\]
Consider the Cauchy problem:
\[
U_{r+1}(D)y = f,
\]
\[
D^k y(0) = b_k \text{ for } k = 0, 1, \ldots, r. \tag{20}
\]

1.10 Proposition. Let \( \sum_{n \geq 0} p_{n+r+1} t^n \) be the formal power series defined by the following generation function:
\[
\sum_{n \geq 0} a_n t^n \left(1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}\right) = \sum_{n \geq 0} p_{n+r+1} t^n.
\]
Then
\[
up = \Lambda_0(b) u_0(t) + \Lambda_1(b) u_1(t) + \ldots + \Lambda_r(b) u_r(t) + \sum_{n \geq r+1} p_n \frac{t^n}{n!},
\]
where \( \Lambda_j(b) = b_j - e_1 b_{j-1} + \ldots + (-1)^j e_j b_0 \) for \( 0 \leq j \leq r \), is the unique solution of the Cauchy problem (20).

Proof. Let us collect coefficients of \( t^n \) on both sides of the given equality,
\[
(p_{r+1} + p_{r+2} t + p_{r+3} t^2 + \ldots)(1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}) = a_0 + a_1 t + a_2 t^2 + \ldots.
\]
Similarly to the homogeneous case (see Theorem 1.2), we get
\[
p_{r+1+n} - p_{r+n} e_1 + \ldots + (-1)^{r+1} p_n e_{r+1} = a_n, \quad n \geq 0, \tag{21}
\]
and this exactly means that \( y_0(t) = \sum_{n \geq r+1} p_n \frac{t^n}{n!} \) is a solution of \( U_{r+1}(D)y = f \) with initial conditions \( y_0^{(k)}(0) = 0 \) for \( k = 0, 1, \ldots, r \).
Therefore the solution of the Cauchy problem will be the sum of \( y_0 \) and the solution \( y_b \) of \( U_{r+1}(D)y = 0 \) that satisfies the initial conditions \( y_b^{(k)}(0) = b_k \) for \( k = 0, 1, \ldots, r \). By Corollary 1.3, one can take \( y_0(t) = \Lambda_0(b) u_0(t) + \Lambda_1(b) u_1(t) + \ldots + \Lambda_r(b) u_r(t) \).

The solution \( u_p = y_0 + y_b \) is clearly unique, as the difference of two solutions of the same Cauchy problem is the (unique) solution of the homogeneous equation that satisfies the zero initial conditions.
2 Generalized Wronskians and Schubert Calculus

Results of this section hold for any fundamental system of solutions of the universal ODE \( U_{r+1}(D)u = 0 \). They are formulated in terms of generalized Wronskians, and therefore it is enough to prove them for one fundamental system. To ease computations, the choice of the universal fundamental system is the more convenient one.

From now on \( u = u(t) \) will denote the “column vector” of the universal solutions (4) of \( U_{r+1}(D)u = 0 \),

\[
\begin{pmatrix}
  u_0 \\
  u_1 \\
  \vdots \\
  u_r
\end{pmatrix},
\]

For any partition \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r) \), the generalized Wronskian \( W_\lambda(u) \) defined by (5), has the form

\[
W_\lambda(u) = D^{\lambda_r}u \land D^{1+\lambda_{r-1}}u \land \ldots \land D^{r+\lambda_0}u.
\]

We use notation (8) and write \( h = (h_k)_{k \in \mathbb{Z}} \) for \( h_k \)'s given by (3).

2.1 Proposition. We have

\[
W_\lambda(u) = \sum_{n \geq 0} \sum_{|\mu| = n} \binom{n}{\mu} \Delta_{\lambda+\mu}(h) \frac{t^n}{n!},
\]

\[
W_\lambda(u)(0) = \Delta_{\lambda}(h),
\]

where \( \mu = (\mu_0, \mu_1, \ldots, \mu_r) \) is a partition of \( |\mu| = \mu_0 + \mu_1 + \ldots + \mu_r \) and

\[
\binom{n}{\mu} = \binom{n}{\mu_0, \mu_1, \ldots, \mu_r} = \frac{n!}{\mu_0! \mu_1! \ldots \mu_r!}.
\]

Proof. By definition of \( u \), see (4),

\[
u = \sum_{n \geq 0} \binom{h_n}{h_{n-1}} \cdots \frac{t^n}{n!},
\]

and then the \( (i + \lambda_{r-i}) \)-th derivative is

\[
D^{i+\lambda_{r-i}}u = \sum_{\mu_{r-i} \geq 0} \binom{h_{\lambda_{r-i}+\mu_{r-i}+i}}{h_{\lambda_{r-i}+\mu_{r-i}+i-1}} \cdots \frac{\mu_{r-i}}{\mu_{r-i}!},
\]

where we put \( \mu_{r-i} \) instead of \( n \). Thus

\[
W_\lambda(u) = D^{\lambda_r}u \land D^{1+\lambda_{r-1}}u \land \ldots \land D^{r+\lambda_0}u =
\]
Proof. Clearly columns. Hence obtains the product of an element of \( \Delta_{\lambda}(h) \) and its consequences. Thus one obtains the product of an element of \( E_r \) with the determinant involving derivatives of \( u \) of orders at most \( r \) only. This determinant is \( W_0(u) \), up to a permutation of the columns. Hence \( W_\lambda(u) = \gamma \lambda W_0(u) \) for some \( \gamma \lambda \in E_r \). But the ratio of the constant terms of \( W_\lambda(u) \) and \( W_0(u) \), according to Proposition 2.1, is

\[ \gamma \lambda = \frac{W_\lambda(u)(0)}{W_0(u)(0)} = \Delta_{\lambda}(h), \]

and the statement follows. 

\[ \square \]

2.2 Theorem. Giambelli’s formula for generalized Wronskians holds, i.e., for every partition \( \lambda \) we have

\[ W_\lambda(u) = \Delta_{\lambda}(h)W_0(u). \]

Proof. Clearly \( W_\lambda(h) \) is an \( E_r \)-multiple of \( W_0(h) \). Indeed, each column of the form \( D^k u \) with \( k \geq r + 1 \) occurring in the expression for \( W_\lambda(u) \) can be replaced with a linear combination of lower derivatives of \( u \), by (1) and its consequences. Thus one obtains the product of an element of \( E_r \) with the determinant involving derivatives of \( u \) of orders at most \( r \) only. This determinant is \( W_0(u) \), up to a permutation of the columns. Hence \( W_\lambda(u) = \gamma \lambda W_0(u) \) for some \( \gamma \lambda \in E_r \). But the ratio of the constant terms of \( W_\lambda(u) \) and \( W_0(u) \), according to Proposition 2.1, is

\[ \gamma \lambda = \frac{W_\lambda(u)(0)}{W_0(u)(0)} = \Delta_{\lambda}(h), \]

and the statement follows. 

\[ \square \]

2.3 Corollary. Pieri’s formula for generalized Wronskians holds, i.e., for every partition \( \lambda \) and \( k \geq 0 \) we have

\[ h_k W_\lambda(u) = \sum_\mu W_{\mu}(u), \]

where the sum is taken over all the partitions \( \mu \) satisfying (6).

Proof. By [13, Lemma A.9.4], we have \( h_k \Delta_{\lambda}(h) = \sum_\mu \Delta_{\mu}(h) \), where the sum is over all the partitions \( \mu \) satisfying (6). Applying Theorem 2.2, we get

\[ h_k W_\lambda(u) = (h_k \Delta_{\lambda}(h))W_0(u) = \sum_\mu \Delta_{\mu}(h)W_0(u) = \sum_\mu W_{\mu}(u), \]
2.4 Schubert Calculus ([13], [21]).

Let \( G := G(r, \mathbb{P}^d) \) be the complex Grassmann variety parameterizing \( r \)-dimensional subspaces of \( \mathbb{P}^d \) and let

\[
0 \rightarrow S_r \rightarrow G \times \mathbb{C}^{d+1} \rightarrow Q_r \rightarrow 0
\]

be the universal exact sequence over \( G \), where \( S_r \) is the rank \( r + 1 \) universal subbundle of \( G \times \mathbb{C}^{d+1} \) and \( Q_r \) the universal quotient bundle of rank \( d - r \).

As it is well known, the ring of symmetric functions in \( r + 1 \) indeterminates surjects onto the cohomology ring of \( G \) by mapping the \( i \)-th complete polynomial to the \( i \)-th Chern class of the universal quotient bundle, and the kernel of the map gives relation between generators, see e.g. [13].

In terms of Schubert Calculus on Grassmannians, the integral cohomology of \( G \) is a \( \mathbb{Z} \)-algebra finitely generated by the Chern classes \((-1)^i \epsilon_i := c_i(S_r) \in H^*(G, \mathbb{Z})\) of the tautological bundle or, due to relation \( c(S_r)c(Q_r) = 1 \), by the Chern classes \( \sigma_i := c_i(Q_r) \in H^*(G, \mathbb{Z}) \) of \( Q_r \). Clearly \( \epsilon_i = 0 \) if \( i > r + 1 \) and \( \sigma_i = 0 \) if \( i > d - r \). Denote \( \sigma_r := (1, \sigma_1, \sigma_2, \ldots, \) and let \( \sigma_{\lambda} := \Delta_{\lambda}(\sigma_r) \in H^{2|\lambda|}(G, \mathbb{Z}) \).

The Grassmann variety \( G \) possesses a cellular decomposition \( \{B_{\lambda}\} \), with respect to some complete flag of linear subvarieties of \( \mathbb{P}^d \), and each affine cell \( B_{\lambda} \) has real codimension \( 2|\lambda| \). The singular homology classes \( \Omega_{\lambda} \in H_{2(r+1)(d-r)-2|\lambda|}(G, \mathbb{Z}) \) of the closures of \( B_{\lambda} \) in \( G \) form a \( \mathbb{Z} \)-basis of \( H_\ast(G, \mathbb{Z}) := \oplus H_{2(r+1)(d-r)-2|\lambda|}(G, \mathbb{Z}) \). The key fact of Schubert calculus on Grassmannian is that \( \sigma_{\lambda} \) is the Poincaré dual of \( \Omega_{\lambda} \):

\[
\sigma_{\lambda} \cap [G] = \Omega_{\lambda},
\]

where \([G]\) is the fundamental class of the Grassmann variety, and the \( \cap \) product turns the homology into a free module of rank 1 over the cohomology, generated by the fundamental class.

2.5 Wronski Calculus. Consider the differential equation

\[
D^{r+1}v - \epsilon_1 D^r v + \ldots + (-1)^{r+1} \epsilon_{r+1} v = 0
\]

(23)

over the \( \mathbb{Q} \)-algebra \( H^*(G, \mathbb{Q}) = H^*(G, \mathbb{Z}) \otimes _{\mathbb{Z}} \mathbb{Q} \). Let \( \psi : E_r \rightarrow H^*(G, \mathbb{Q}) \) denote the unique \( \mathbb{Q} \)-module homomorphism defined by \( \epsilon_i \mapsto \epsilon_i \). By Theorem 1.5, the image under the induced homomorphism \( \tilde{\psi} : E_r[[t]] \rightarrow H^*(G, \mathbb{Q})[[t]] \) of the universal fundamental system \( u \) is a fundamental system \( v \) of solutions of (23):

\[
v = \begin{pmatrix}
v_0 \\
v_1 \\
\vdots \\
v_r
\end{pmatrix}, \quad v_k = \tilde{\psi}(u_k), \quad 0 \leq k \leq r.
\]

In particular, \( \psi(h_k) = \sigma_{\lambda} \), and Theorem 2.2 asserts

\[
W_{\lambda}(v) = \sigma_{\lambda} \cdot W_0(v)
\]
or, more suggestively,
\[ \sigma_\lambda = \frac{W_\lambda(v)}{W_0(v)}. \]

Define \( W(v) := \bigoplus_\lambda W_\lambda(v) \). It is a free \( H^*(G, \mathbb{Z}) \)-module of rank 1 generated by \( W_0(v) \).

### 2.6 Proposition

The \( \mathbb{Z} \)-module isomorphism
\[ wr : H_*(G, \mathbb{Z}) \to W_\lambda(v), \]
mapping \( \Omega_\lambda \mapsto W_\lambda(v) \), is an isomorphism of \( H^*(G, \mathbb{Z}) \)-modules.

#### Proof.

It suffices to show that \( wr(\sigma_k \cap \Omega_\lambda) = \sigma_k W_\lambda(v) \), for each \( k \geq 0 \). According to Corollary (2.3), we have \( \sigma_k \sigma_\lambda = \sum_\mu \sigma_\mu \), where the sum is over all partitions \( \mu \) satisfying (6). Therefore
\[
wr(\sigma_k \cap \Omega_\lambda) = wr(\sigma_k \cap (\sigma_\lambda \cap [G])) = wr((\sigma_k \cup \sigma_\lambda) \cap [G]) = wr(\sum_\mu \sigma_\mu \cap [G]) = \sum_\mu W_\mu(v) = \sigma_k W_\lambda(v). 
\]

Thus the \( \cap \) product can be interpreted as the product of \( h_k \) with a generalized Wronskian, and the \( \cup \)-product in cohomology as the product in the ring \( \mathbb{Z}[\epsilon_1, \ldots, \epsilon_{r+1}] \). Notice that the \( \epsilon_i \)'s are not necessarily algebraically independent. In fact \( \mathbb{Z}[\epsilon_1, \ldots, \epsilon_{r+1}] \) is the quotient of \( E_r \) through the unique ring epimorphism defined by \( e_i \mapsto \epsilon_i \).

In fact, the polynomial \( \mathbb{Q} \)-algebra \( E_r \) can be interpreted as the cohomology of the infinite Grassmannian \( G(r, \mathbb{P}^\infty) \), through the telescoping construction, similarly to [4, p. 302]. More precisely, for each \( d \geq r \), there is a natural inclusion \( G(r, \mathbb{P}^d) \hookrightarrow G(r, \mathbb{P}^\infty) \). The corresponding arrow reversed \( \mathbb{Q} \)-algebra map \( H^*(G(r, \mathbb{P}^\infty)) \to H^*(G(r, \mathbb{P}^d)) \) is nothing else but our unique map \( \psi \) sending \( \epsilon_i \mapsto \epsilon_i \). Hence the induced map \( \tilde{\psi} \) maps the universal generalized Wronskians \( W_\lambda(v) \)'s to the generalized Wronskians \( W_\lambda(v) \)'s. Since \( G(r, \mathbb{P}^\infty) \) is an infinite CW-complex, and since the homology \( H_*(G(r, \mathbb{P}^\infty), \mathbb{Z}) \) is generated by the classes of the closures of the Schubert cells, it is thence clear that \( W(v) \) is a module over \( H^*(G(r, \mathbb{P}^\infty)) = \mathbb{Z}[\epsilon_1, \epsilon_2, \ldots, \epsilon_{r+1}] \) and is \( \mathbb{Z} \)-isomorphic to \( H_*(G(r, \mathbb{P}^\infty), \mathbb{Z}) \). Again, this extends to an isomorphism of \( H^*(G(r, \mathbb{P}^\infty), \mathbb{Z}) \)-modules.

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