Deformed quantum harmonic oscillator with diffusion and dissipation

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Abstract

A master equation for the deformed quantum harmonic oscillator interacting with a dissipative environment, in particular with a thermal bath, is derived in the microscopic model by using perturbation theory. The coefficients of the master equation and of equations of motion for observables depend on the deformation function. The steady state solution of the equation for the density matrix in the number representation is obtained and the equilibrium energy of the deformed harmonic oscillator is calculated in the approximation of small deformation.

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1 Introduction

For more than a decade a constant interest has been induced to the study of deformations of Lie algebras – so-called quantum algebras or quantum groups, whose rich structure produced important results and consequences in statistical mechanics, quantum field theory, conformal field theory, quantum and nonlinear optics, nuclear and molecular physics. Their use in physics became intense with the introduction in 1989, by Biedenharn [1] and MacFarlane [2], of the $q$-deformed Heisenberg-Weyl algebra ($q$-deformed quantum harmonic oscillator). Since then the properties of the deformations of the harmonic oscillator have been investigated by many authors. Several kinds of generalized deformed oscillators have been introduced. There are, at least, two properties which make $q$-oscillators interesting objects for physics. The first is the fact that they naturally appear as the basic building blocks of completely integrable theories. The second concerns the connection between $q$-deformation and nonlinearity. In Refs. [3, 4, 5] it was shown that the $q$-oscillator leads to nonlinear vibrations with a special kind of the dependence of the frequency on the amplitude. For example, the $q$-deformed Bose distribution produces a correction to the Planck distribution formula [3, 4, 6].
In the present paper we intend to study the connection of the quantum deformation and quantum dissipation, by setting a master equation for the deformed harmonic oscillator in the presence of a dissipative environment, which is shown to be the deformed version of the master equation obtained in the framework of the Lindblad theory for open quantum systems [7]. When the deformation becomes zero, we recover the Lindblad master equation for the damped harmonic oscillator [8, 9]. We are interested in describing the role of nonlinearities which appear in the master equation, this goal being motivated by the fact that the \(q\)-oscillator can be considered as a physical system with a specific nonlinearity, called \(q\)-nonlinearity [3, 4]. For a certain choice of the environment coefficients, a master equation for the damped deformed oscillator has also been derived by Mancini [10]. In Ref. [11], Ellinas used the \(q\)-deformed oscillator for treatments of dissipation of a two-level atom and of a laser mode.

The paper is organized as follows. In Sec. 2 we remind the basics about the generalized deformed quantum oscillator, in particular the \(f\)-oscillator and \(q\)-oscillator. Using a variant of the Mancini’s model [10], in Sec. 3 we derive a master equation for the \(f\)-deformed oscillator in the presence of a dissipative environment. The equations of motion obtained for different observables present a strong dependence on the deformation. Then in Sec. 4 we write and solve in the stationary state the equation for the density matrix in the number representation. In the particular case when the environment is a thermal bath, we obtain an expression for the equilibrium energy of the oscillator in the approximation of a small deformation parameter. A summary and conclusions are given in Sec. 5.

2 Deformed quantum oscillators

It is known that the ordinary operators \(\{1, a, a^\dagger, N\}\) form the Lie algebra of the Heisenberg-Weyl group and the linear harmonic oscillator can be connected with the generators of the Heisenberg-Weyl Lie group. The generalized deformed quantum oscillators [12, 13] are defined as the algebra generated by the operators \(\{1, A, A^\dagger, N\}\) and the structure function \(F(N)\), which satisfy the following relations:

\[
[A, N] = A, \quad [A^\dagger, N] = -A^\dagger
\]

and

\[
AA^\dagger = F(N + 1), \quad A^\dagger A = F(N),
\]

where \(F(N)\) is a positive analytic function with \(F(0) = 0\) and \(N\) is the Hermitian number operator. It follows that the following commutation and anticommutation relations are
satisfied:

\[[A, A^\dagger] = F(N+1) - F(N), \quad \{A, A^\dagger\} = F(N+1) + F(N).\]  \hfill (3)

The structure function is a characteristic of the deformation. The number operator \(N\) is not equal to \(A^\dagger A\) as in the ordinary case. For \(F(N) = N\) one obtains the relations for the usual harmonic oscillator. Another choice is

\[F(N) = \frac{q^N - q^{-N}}{q - q^{-1}} \equiv [N],\]  \hfill (4)

where the dimensionless \(c\)-number \(q\) is the deformation parameter. Then the operators \(A\) and \(A^\dagger\) are called \(q\)-deformed boson annihilation and creation operators [1, 2]. For the \(q\)-deformed harmonic oscillator the relations (2) become

\[\AA = [N + 1], \quad A^\dagger A = [N],\]  \hfill (5)

with the commutation relation

\[[A, A^\dagger] = [N + 1] - [N].\]  \hfill (6)

If \(q\) is real and positive, then

\[[N] = \frac{\sinh(N \ln q)}{\sinh(\ln q)} \hfill (7)\]

and the condition of Hermitian conjugation \((A^\dagger)^\dagger = A\) is satisfied. In addition to the commutation relation, there exists for the \(q\)-deformed oscillator the reordering relation

\[\AA - q^{\mp 1} A^\dagger A = q^{\pm N},\]  \hfill (8)

which is usually taken as the definition of \(q\)-oscillators.

In the limit \(q \to 1\), \(q\)-operators tend to the ordinary operators because \(\lim_{q \to 1} [N] = N\). Then Eqs. (6) and (8) go to the usual boson commutation relation \([A, A^\dagger] = 1\).

The \(q\)-deformed boson operators \(A\) and \(A^\dagger\) can be expressed in terms of the usual boson operators \(a\) and \(a^\dagger\) (satisfying \([a, a^\dagger] = 1\), \(N = a^\dagger a\) and \([a, N] = a\), \([a^\dagger, N] = -a^\dagger\)) through the relations [14, 15]:

\[A = \sqrt{\frac{N+1}{N+1}} a = a \sqrt{\frac{N}{N}}, \quad A^\dagger = a^\dagger \sqrt{\frac{N+1}{N+1}} = \sqrt{\frac{N}{N}} a^\dagger.\]  \hfill (9)

Using a nonlinear map [16, 17], the \(q\)-oscillator has been interpreted [3, 4] as a nonlinear oscillator with a special type of nonlinearity which classically corresponds to an
energy dependence of the oscillator frequency. Other nonlinearities can also be introduced by making the frequency to depend on other constants of motion, different from energy, through a deformation function $f$ [3, 18]. Let us define the $f$-deformed oscillator operators [18]

$$A = af(N) = f(N + 1)a, \quad A^\dagger = f(N)a^\dagger = a^\dagger f(N + 1),$$ (10)

where $N = a^\dagger a$. They satisfy relations (1) and the commutation relation

$$[A, A^\dagger] = (N + 1)f^2(N + 1) - Nf^2(N).$$ (11)

The function $f$ has a dependence on the deformation parameter such that when the deformation disappears, then $f \to 1$ and the usual algebra is recovered. Without loss of generality, $f$ can be chosen real and nonnegative and it is reasonable from the physical point of view to assume [5] that $f(0) = 1$ and $f(N) = 1$ for a suitable large $N$. A deformation function depending on Laguerre polynomials has been used in [19]. A different type of deformation has been considered by Sudarshan [20], who introduced the so-called harmonious states. Other examples of deformed oscillators are connected to the excited coherent states introduced by Agarwal [21] and Dodonov [22]. The transformation (10) from the operators $a, a^\dagger$ to $A, A^\dagger$ represents a nonlinear non-canonical transformation, since it does not preserve the commutation relation. The notion of $f$-oscillators generalizes the notion of $q$-oscillators. Indeed, if

$$f(N) = \sqrt{\frac{[N]}{N}} = \sqrt{\frac{\sinh(N \ln q)}{N \sinh(\ln q)}},$$ (12)

then the operators $A, A^\dagger$ in Eqs. (10) satisfy the $q$-deformed commutation relations (6). This means that a Hamiltonian operator of the form $A^\dagger A = f(N)a^\dagger af(N)$ has a spectrum with the same structure as the spectrum of $a^\dagger a$. The difference is that the eigenvalues in the basis of the Fock space are $nf^2(n)$, $n = 0, 1, 2, ..., n$, instead of $n$. This spectrum associated with $q$-deformation grows with $n$ like $\sinh(n \ln q)$, i.e. exponentially for large occupation numbers $n$, compared to the ordinary case, in which the spectrum is equidistant. The Hamiltonian of the $f$-deformed harmonic oscillator is ($\omega$ is the ordinary frequency)

$$\mathcal{H} = \frac{\hbar \omega}{2}(AA^\dagger + A^\dagger A) = \frac{\hbar \omega}{2}[(N + 1)f^2(N + 1) + Nf^2(N)].$$ (13)

It is diagonal on the eigenstates $|n>$ and in the Fock space its eigenvalues are

$$E_n = \frac{\hbar \omega}{2}[(n + 1)f^2(n + 1) + nf^2(n)].$$ (14)
In the limit $f \to 1$ ($q \to 1$ for $q$-oscillators), we recover the ordinary expression $E_n = \hbar \omega (n + 1/2)$.

Using the operator Heisenberg equation with the Hamiltonian (13)

$$i\hbar \frac{da}{dt} = [a, \mathcal{H}]$$

(15)

or the evolution operator $U(t) = \exp[-(i/\hbar)\mathcal{H}(N)t]$, we obtain the following solutions to the Heisenberg equations of motion for the operators $a$ and $a^\dagger$ [10, 18]:

$$a(t) = \exp[-i\omega \Omega(N)t]a, \quad a^\dagger(t) = a^\dagger \exp[i\omega \Omega(N)t],$$

(16)

where

$$\Omega(N) = \frac{1}{2}[(N+2)f^2(N+2) - Nf^2(N)].$$

(17)

For a $q$-deformed harmonic oscillator,

$$\Omega(N) = \frac{1}{2}([N+2] - [N])$$

(18)

and for a small deformation parameter $\tau$ ($\tau = \ln q$),

$$\Omega(N) = 1 + \frac{\tau^2}{2}(N+1)^2.$$  

(19)

3 Quantum Markovian master equation

In order to discuss the dynamics of the open system S, we use a microscopic description of the composite system S+B. As the subsystem S of interest we take the $f$-deformed harmonic oscillator with the Hamiltonian $\mathcal{H}$ (13), and B is the environment (bath) with the Hamiltonian $H_B$. The coupled system with the total Hamiltonian $H_T = \mathcal{H} + H_B + V$ ($V$ is the interaction Hamiltonian) is described by a density operator $\chi(t)$, which evolves in time according to the von Neumann-Liouville equation

$$\frac{d\chi(t)}{dt} = -\frac{i}{\hbar}[H_T, \chi(t)].$$

(20)

When the Hamiltonian evolution of the total system is projected onto the space of the harmonic oscillator, the reduced density operator of the subsystem is given by

$$\rho(t) = \text{Tr}_B \chi(t).$$

(21)

The derivation of the reduced density operator in which the operators of the environment system have been eliminated up to second order of the perturbation theory can be taken
from literature \([23, 24, 25, 26, 27]\). We assume that the interaction potential \(V\) is linear in the coordinate operator \(s_1 = q\) and momentum operator \(s_2 = p\) in the Hilbert space of the subsystem. Then, following \([26]\), we can write down the master equation for the density operator of the open quantum system in the Born-Markov approximation:

\[
\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho(t)] + \frac{1}{\hbar^2} \sum_{i,j=1,2} \int_0^t dt' \{C_{ij}^*(t')[s_i(s_j(-t')) + C_{ij}(t')][s_j(-t')\rho(t), s_i]\},
\] (22)

where the coefficients \(C_{ij}(t)\) are correlation functions of the environment operators. It is assumed that the correlation functions decay very rapidly on the time scale on which \(\rho(t)\) varies. Ideally, we might take \(C_{ij}(t') \sim \delta(t')\). The Markov approximation relies on the existence of two widely separated time scales: a slow time scale for the dynamics of the system \(S\) and a fast time scale characterizing the decay of environment correlation functions \([26]\).

In order to get the time dependence of the operators \(s_1(t) = q(t)\) and \(s_2(t) = p(t)\), we express them through the relations (\(m\) is the oscillator mass)

\[
q(t) = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger(t) + a(t)), \quad p(t) = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger(t) - a(t))
\] (23)

and then insert Eq. (16) for \(a(t)\) and \(a^\dagger(t)\). Then the master equation results

\[
\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho(t)] + \frac{1}{2\hbar^2} \int_0^t dt' \{C_{11}^*(t')[q, \rho(t)](qE_+ + E_+ q - \frac{i}{m\omega}(pE_- - E_+ p))[q, \rho(t)]
\]

\[
+ C_{11}(t')[(qE_+ + E_+ q - \frac{i}{m\omega}(pE_- - E_+ p))\rho(t), q]
\]

\[
+ iC_{22}^*(t')[p, \rho(t)](m\omega(qE_+ - E_+ q) - i(pE_- + E_+ p))[p, \rho(t)]
\]

\[
+ iC_{22}(t')[(m\omega(qE_+ - E_+ q) - i(pE_- + E_+ p))\rho(t), p]
\]

\[
+ C_{12}^*(t')[q, \rho(t)](m\omega(qE_+ - E_+ q) - i(pE_- + E_+ p))[q, \rho(t)]
\]

\[
+ C_{12}(t')[(qE_+ + E_+ q - \frac{i}{m\omega}(pE_- - E_+ p))\rho(t), p]
\]

\[
+ C_{21}^*(t')[p, \rho(t)](qE_+ + E_+ q - \frac{i}{m\omega}(pE_- - E_+ p))[p, \rho(t)]
\]

\[
+ iC_{12}(t')[(m\omega(qE_+ - E_+ q) - i(pE_- + E_+ p))\rho(t), q],
\] (24)

where we have introduced the following notations:

\[
E_+ = \exp[i\omega\Omega(N)t'], \quad E_- = \exp[-i\omega\Omega(N)t'].
\] (25)

If the environment is sufficiently large we may assume that the time correlation functions decay fast enough to zero for times \(t'\) longer than the relaxation time \(t_B\) of the
environment: \( t' \gg t_B \). Therefore, if we are interested in the dynamics of the subsystem over times which are longer than the environment relaxation time, \( t \gg t_B \), we may use the Markov approximation and replace the upper limit of integration \( t \) by \( \infty \). Physically, this amounts to assuming that the memory functions \( C_{ij}(t') E_\pm(t') \) decay over a time which is much shorter than the characteristic evolution time of the system of interest. After certain assumptions [23, 27], one can define the complex decay rates, which govern the rate of relaxation of the system density operator as follows:

\[
\int_0^\infty dt' C_{11}(t') E_+ = \int_0^\infty dt' C^*_{11}(t') E_+ = D_{pp}(\Omega), \tag{26}
\]

\[
\int_0^\infty dt' C_{22}(t') E_+ = \int_0^\infty dt' C^*_{22}(t') E_+ = D_{qq}(\Omega), \tag{27}
\]

\[
\int_0^\infty dt' C_{12}(t') E_+ = \int_0^\infty dt' C^*_{21}(t') E_+ = -D_{pq}(\Omega) + \frac{i\hbar}{2} \lambda(\Omega), \tag{28}
\]

with \( D_{pp}(\Omega) > 0, D_{qq}(\Omega) > 0 \) and

\[
D_{pp}(\Omega)D_{qq}(\Omega) - D_{pq}^2(\Omega) \geq \frac{\hbar^2}{4} \lambda^2(\Omega). \tag{29}
\]

In fact, \( D_{pp}(\Omega), D_{qq}(\Omega), D_{pq}(\Omega) \) and \( \lambda(\Omega) \) play the role of deformed diffusion and, respectively, dissipation coefficients and the relation (29) ensures the positivity of the density operator. The existence of these coefficients reflects the fact that, due to the interaction, the energy of the system is dissipated into the environment, but noise arises also (in particular, thermal noise), since the environment also distributes some of its energy back to the system. In addition, we assume in the following \( \lambda(\Omega) = \lambda = \text{const} \). Then the master equation (24) for the damped deformed harmonic oscillator takes the form

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \frac{1}{2\hbar^2} \left\{ \left( \{ D_{pp}(\Omega), q \} + \frac{i}{m\omega} [D_{pp}(\Omega), p] \right) \rho, q \right\} \\
+ \left\{ \left( \{ D_{pq}(\Omega) - \frac{i\hbar}{2} \lambda, q \} \right) \rho, p \right\} \\
+ \left\{ \left( \frac{1}{m\omega} [iD_{pq}(\Omega) - \frac{\hbar}{2} \lambda, p] \right) \rho, q \right\} \\
- \left\{ \left( \{ D_{pq}(\Omega) + \frac{i\hbar}{2} \lambda, q \} \right) \rho, p \right\} + H.c. \right\}. \tag{30}
\]

We notice that the deformation is present in both the commutator containing the oscillator Hamiltonian \( \mathcal{H} \), as well as in the dissipative part of the master equation, which describes the influence of the environment on the deformed oscillator. This master equation preserves
the Hermiticity property of the density operator and the normalization (unit trace) at all times, if at the initial time it has these properties. In the limit $f \to 1$ ($\Omega \to 1$), the deformation disappears and Eq. (30) becomes the Markovian master equation for the damped harmonic oscillator, obtained in the Lindblad theory for open quantum systems, based on completely positive dynamical semigroups [7, 8, 9].

Expressing the coordinate and momentum operators back in terms of the creation and annihilation operators and introducing the notations

$$D^+ (\Omega) \equiv \frac{1}{2\hbar} [m\omega D_{qq}(\Omega) + \frac{D_{pp}(\Omega)}{m\omega}], \quad D^- (\Omega) \equiv \frac{1}{2\hbar} [m\omega D_{qq}(\Omega) - \frac{D_{pp}(\Omega)}{m\omega}],$$

the master equation (30) for the damped deformed harmonic oscillator takes the form:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \{[[D^+(\Omega)a, \rho], a^\dagger] - [[a^\dagger(D^-(\Omega) + \frac{i}{\hbar} D_{pq}(\Omega)), \rho], a^\dagger] - \frac{\lambda}{2} [a^\dagger, \{a, \rho\}] + H.c.\}.$$  

Mancini considered in Ref. [10] a squeezed bath for the dynamics of the damped deformed harmonic oscillator and his model can be recovered by taking the following coefficients in Eq. (32):

$$D^+ (\Omega) = \gamma (N + \frac{1}{2}), \quad D^- (\Omega) + \frac{i}{\hbar} D_{pq}(\Omega) = -\gamma M, \quad \lambda = \gamma.$$  

In the particular case of a thermal equilibrium of the bath at temperature $T$ ($k$ is the Boltzmann constant), we take the diffusion coefficients of the form (in concordance with Mancini’s results [10])

$$m\omega D_{qq}(\Omega) = \frac{D_{pp}(\Omega)}{m\omega} = \frac{\hbar}{2} \lambda \text{coth} \frac{\hbar\omega \Omega}{2kT}, \quad D_{pq}(\Omega) = 0.$$  

In the limit $\Omega \to 1$, the deformed diffusion coefficients (34) take the known form obtained for the damped harmonic oscillator in the particular case when the asymptotic state is a Gibbs state [8, 9]:

$$D_{pp} = \frac{\hbar m\omega}{2} \lambda \text{coth} \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\hbar}{2m\omega} \lambda \text{coth} \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0.$$  

The meaning of the master equation becomes clear when we transform it into equations satisfied by the expectation values of observables involved in the master equation, $< O > = \text{Tr}[\rho(t)\mathcal{O}]$, where $O$ is the operator corresponding to such an observable. We give an example, multiplying both sides of Eq. (32) by the number operator $N$ and taking the
trace. In the case of a thermal bath, with the diffusion coefficients (34), the equation of motion for the expectation value of $N$ has the form

$$\frac{d}{dt} < N > = \lambda [< \coth \frac{\hbar \Omega(N)}{2kT} - 1](N + 1) > - < \coth \frac{\hbar \Omega(N - 1)}{2kT} + 1)N >].$$

This equation leads to a time dependence of the number of quanta on dissipation and temperature, compared to the case of an oscillator without dissipation, where the number of quanta is conserved. We remark that in the case of a thermal bath at $T = 0$, Eq. (36) takes the form

$$\frac{d}{dt} < N > = -2\lambda < N >,$$

so that the average number of quanta $< N >$ does not depend on deformation, it only decreases exponentially with dissipation.

We consider another example, taking the simplest case of a thermal bath at $T = 0$, when both diffusion and dissipation coefficients do not depend on the deformation, $D_+ = \lambda/2 = \text{const.}$ Even in this situation, the equations of motion for the expectation values are yet complicated, because they do not form a closed system. Multiplying both sides of Eq. (32) by the operators $a$ and, respectively, $\Omega(N)a$ and taking throughout the trace, we get the following equations for the expectation values of these operators:

$$\frac{d}{dt} < a > = -i\omega < \Omega(N)a > -\lambda < a >,$$

$$\frac{d}{dt} < \Omega(N)a > = -i\omega < \Omega^2(N)a > + \lambda < [2N\Omega(N - 1) - (2N + 1)\Omega(N)]a > .$$

These examples show that the equations of motion contain nonlinearities introduced by the deformed Hamiltonian $H$ and, therefore, depend on the deformation function.

## 4 Equation for the density matrix of the damped deformed oscillator

Let us rewrite the master equation (32) for the density matrix by means of the number representation. Specifically, we take the matrix elements of each term between different number states denoted by $|n>$, and using $N|n> = n|n>$, $a^+|n> = \sqrt{n+1}|n+1>$ and $a|n> = \sqrt{n}|n-1>$, we get
\[
\frac{d\rho_{mn}}{dt} = -\frac{i\omega}{2}[mf^2(m) + (m+1)f^2(m+1) - nf^2(n) - (n+1)f^2(n+1)]\rho_{mn} \\
-[(m+1)D_+(\Omega(m)) + mD_+(\Omega(m-1)) + (n+1)D_+(\Omega(n)) + nD_+(\Omega(n-1))] - \lambda|\rho_{mn} \\
+\sqrt{(m+1)(n+1)}[D_+(\Omega(m)) + D_+(\Omega(n))] + \lambda|\rho_{m+1,n+1} \\
+\sqrt{mn}[D_+(\Omega(m-1)) + D_+(\Omega(n-1))] - \lambda|\rho_{m-1,n-1} \\
-\sqrt{(m+1)n}[D_-(\Omega(m)) + D_-(\Omega(n-1)) - \frac{i}{\hbar}(D_{pq}(\Omega(m)) + D_{pq}(\Omega(n-1)))]|\rho_{m+1,n-1} \\
-\sqrt{m(n+1)}[D_-(\Omega(m-1)) + D_-(\Omega(n)) + \frac{i}{\hbar}(D_{pq}(\Omega(m-1)) + D_{pq}(\Omega(n)))]|\rho_{m-1,n+1} \\
+\sqrt{(m+1)(m+2)}[D_-(\Omega(m+1)) - \frac{i}{\hbar}D_{pq}(\Omega(m+1))]|\rho_{m+2,n} \\
+\sqrt{(n+1)(n+2)}[D_-(\Omega(n+1)) + \frac{i}{\hbar}D_{pq}(\Omega(n+1))]|\rho_{m,n+2} \\
+\sqrt{mn-1}[D_-(\Omega(m-2)) + \frac{i}{\hbar}D_{pq}(\Omega(m-2))]|\rho_{m-2,n} \\
+\sqrt{n(n-1)}[D_-(\Omega(n-2)) - \frac{i}{\hbar}D_{pq}(\Omega(n-2))]|\rho_{m,n-2}. (40)
\]

Here, we have used the abbreviated notation \(\rho_{mn} = <m|\rho(t)|n>\). This equation gives an infinite hierarchy of coupled equations for the matrix elements. When

\[D_-(\Omega(n)) = 0, \quad D_{pq}(\Omega(n)) = 0, \quad (41)\]

the diagonal elements are coupled only amongst themselves and not coupled to the off-diagonal elements. In this case the diagonal elements (populations) satisfy a simpler set of master equations:

\[
\frac{dP(n)}{dt} = -[2(n+1)D_+(\Omega(n)) + 2nD_+(\Omega(n-1))] - \lambda|P(n) \\
+(n+1)[2D_+(\Omega(n)) + \lambda]P(n+1) + n[2D_+(\Omega(n-1))] - \lambda|P(n-1), \quad (42)
\]

where we have set \(P(n) \equiv \rho_{nn}\). We define the transition probabilities

\[t_+(n) = (n+1)[2D_+(\Omega(n))] - \lambda], \quad t_-(n) = n[2D_+(\Omega(n-1))] + \lambda]. \quad (43)\]

With these notations Eq. (42) becomes:

\[
\frac{dP(n)}{dt} = t_+(n-1)P(n-1) + t_-(n+1)P(n+1) - [t_+(n) + t_-(n)]P(n). \quad (44)
\]

The steady state solution of Eq. (44) is found to be

\[P_{ss}(n) = P(0) \prod_{k=1}^{n} \frac{2D_+(\Omega(k-1)) - \lambda}{2D_+(\Omega(k-1)) + \lambda}. \quad (45)\]
We note that in the steady state the detailed balance condition holds:

\[ t_-(n)P(n) = t_+(n - 1)P(n - 1). \]  

(46)

In the particular case of a thermal state, when the diffusion coefficients have the form (34), the stationary solution of Eq. (44) takes the following form:

\[ P_{ss}^{th}(n) = Z_f^{-1} \exp\{-\frac{\hbar \omega}{2kT}[(n + 1)f^2(n + 1) + nf^2(n)]\}, \]  

(47)

where

\[ Z_f^{-1} = P(0) \exp \frac{\hbar \omega f^2(1)}{2kT} \]  

(48)

and \( Z_f \) is the partition function:

\[ Z_f = \sum_{n=0}^{\infty} \exp\{-\frac{\hbar \omega}{2kT}[(n + 1)f^2(n + 1) + nf^2(n)]\}. \]  

(49)

In his model, Mancini obtained a result similar with Eq. (47).

Using Eq. (14), the distribution (47) can be written

\[ P_{ss}^{th}(n) = Z_f^{-1} \exp(-\frac{E_n}{kT}). \]  

(50)

Expression (47) represents the Boltzmann distribution for the deformed harmonic oscillator. In the limit \( f \to 1 \) the probability \( P_{ss}^{th}(n) \) becomes the Boltzmann distribution for the ordinary harmonic oscillator with the well-known partition function

\[ Z = \frac{1}{2 \sinh \frac{\hbar \omega}{2kT}}. \]  

(51)

For the \( q \)-oscillator described by the Hamiltonian \( \mathcal{H} \) (13) and weakly coupled to a reservoir kept at the temperature \( T \), the \( q \)-deformed partition function can be obtained as a particular case of the partition function \( Z_f \) (49), by taking the deformation function (12):

\[ Z_q = \sum_{n=0}^{\infty} \exp\{-\frac{\hbar \omega}{2kT} \frac{\sinh(\tau(n + 1)) + \sinh(\tau n)}{\sinh \tau}\}. \]  

(52)

In the limit of a small deformation \( \tau \) we can write [4] \( Z_q = Z + b\tau^2 \), where

\[ b = \frac{-\beta Z}{12} (2\bar{n}^3 + 3\bar{n}^2 + \bar{n}), \]  

(53)
with
\[ \bar{n} = \frac{1}{e^\beta - 1}, \quad \bar{n}^2 = \frac{e^\beta + 1}{(e^\beta - 1)^2}, \quad \bar{n}^3 = \frac{e^{2\beta} + 4e^\beta + 1}{(e^\beta - 1)^3}, \quad \beta = \frac{\hbar \omega}{kT}. \] (54)

We can calculate the equilibrium energy by using the formula
\[ E(\infty) = -\hbar \omega \frac{1}{Z_q} \frac{\partial Z_q}{\partial \beta} \] (55)
and obtain
\[ E(\infty) = \frac{\hbar \omega}{2} (\coth \frac{\hbar \omega}{2kT} + \tau^2 c), \] (56)
where
\[ c = \frac{e^\beta}{(e^\beta - 1)^2} \left[ \frac{e^\beta + 1}{e^\beta - 1} - \beta \frac{e^{2\beta} + 4e^\beta + 1}{(e^\beta - 1)^2} \right]. \] (57)

We note that in the approximation of a small deformation parameter \( \tau \), the energy of the deformed damped oscillator depends on the oscillator ground state energy \( \hbar \omega/2 \) and on the temperature \( T \). Evidently, when there is no deformation (\( \tau \to 0 \)), one recovers the energy of the ordinary harmonic oscillator in a thermal bath \([8, 9]\). In the limit \( T \to 0 \), one has \( c \to 0 \), \( E(\infty) = \hbar \omega/2 \) and the deformation does not play any role.

## 5 Summary and conclusions

Our purpose was to study the dynamics of the deformed quantum harmonic oscillator in interaction with a dissipative environment, in particular with a thermal bath. We derived in the Born-Markov approximation a master equation for the reduced density operator of the damped \( f \)-deformed oscillator. The one-dimensional \( f \)- or \( q \)-oscillator is a nonlinear quantum oscillator with a specific nonlinearity and, consequently, the diffusion and dissipation coefficients which model the influence of the environment on the deformed oscillator depend strongly on the introduced nonlinearities. The equations of motion for the observables of the considered system are also nonlinear. In the limit of zero deformation, the master equation for the deformed damped oscillator takes the form of a master equation for the damped oscillator obtained in the framework of the Lindblad theory of open quantum systems based on quantum dynamical semigroups. We have also derived the equation for the density matrix in the number representation. In the case of a thermal bath we obtained the stationary solution, which is the Boltzmann distribution for the deformed harmonic oscillator. In the approximation of a small deformation, we obtained the expression of the
equilibrium energy of the deformed harmonic oscillator, which depends on the oscillator
ground state energy and on the temperature.

The master equation for the damped deformed harmonic oscillator is an operator equa-
tion. It could be useful to study its consequences for the density operator by transforming
this equation into more familiar forms, such as partial differential equations of Fokker-
Planck type for the Glauber, antinormal ordering and Wigner quasiprobability distribu-
tions or for analogous deformed quasiprobabilities [5] associated with the density operator.
It could also be interesting to find the states which minimize the rate of entropy produc-
tion for the damped deformed harmonic oscillator. In the case of the undeformed damped
oscillator such states are represented by correlated coherent states. For the damped de-
formed oscillator the corresponding states could be deformed (nonlinear) coherent states
[18], which may play an important role in the description of the phenomenon of environment
induced decoherence. The dissipative dynamics of deformed coherent states superposition
and the related coherence properties have been studied recently by Mancini and Man’ko
[28].

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