ENERGY DECAY OF SOLUTIONS FOR THE WAVE EQUATION WITH A TIME-VARYING DELAY TERM IN THE WEAKLY NONLINEAR INTERNAL FEEDBACKS

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Abstract. In this paper, we investigate the nonlinear wave equation in a bounded domain with a time-varying delay term in the weakly nonlinear internal feedback

\[ \left( |u_t|^{-2}u_t \right)_t - Lu - \int_0^t g(t-s)Lu(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0. \]

The asymptotic behavior of solutions is studied by using an appropriate Lyapunov functional. Moreover, we extend and improve the previous results in the literature.

1. Introduction. This paper is concerned with the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation of the form

\[
\begin{cases}
\left(|u_t|^{-2}u_t\right)_t - Lu - \int_0^t g(t-s)Lu(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0, & \text{in } \Omega \times ]0, +\infty[, \\
u(x,t) = 0, & \text{on } \Gamma \times ]0, +\infty[, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\
u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), & \text{in } \Omega \times ]0, \tau(0)[,
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $(n \in \mathbb{N}^*)$ with a smooth boundary $\partial \Omega = \Gamma$. Moreover, $\tau(t) > 0$ is a time-varying delay and $\mu_1$, $\mu_2$ are positive real numbers.

The initial datum $(u_0; u_1; f_0)$ belongs to a suitable space, where $Lu = -\text{div}(Au) = -\sum_{i,j=1}^N (a_{i,j}(x) \frac{\partial u}{\partial x_i})$ and $A = (a_{i,j}(x))$ is a matrix that will be specified later. In the absence of the delay term ($\mu_2 = 0$), the problem of existence and the decay of the energy has been extensively studied by several authors and many energy estimates have been derived for arbitrary growing feedbacks (see [3], [5], [6], [9], [12], [13], [17], [23]). The decay rate of the energy (when $t$ goes to infinity) depends on the function $H$ which represents the growth at the origin of $\psi$. Time delay is a property of a physical system by which the response to an applied force is delayed in its effect (see [25]). Time delays so often arise in many physical, chemical, biological and economical phenomena. In recent years, the control of PDEs with

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time delay effects has become an intensive area of research (see [1], [26], [28]) and the references therein. In [7], the authors showed that a small delay in a boundary control could turn such well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it also can improve the performance of the systems (see [26]). In order to stabilize a hyperbolic system involving input delay terms, additional control terms are necessary (see [18], [19], [27]). For instance in [18] the authors studied the wave equation with linear internal damping term with constant delay ($\psi$ linear, $\tau(t) = constant$ in the problem (1)). They determined suitable relations between $\mu_1$ and $\mu_2$, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (1) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [18] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting in the boundary. We also recall the result by C.Q. Xu, S.P. Yung and L.K. Li [27], where the authors proved a result similar to the one in [18] for the one-space dimension by adopting the spectral analysis approach. The case of time-varying delay in the wave equation has been investigated recently by S. Nicaise, J. Valein and E. Fridman [22] in one-space dimension and in the linear case ($\psi$ linear in problem (1)) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1 - d}\mu_1,$$

where the constant $d$ satisfies

$$\tau'(t) \leq d < 1, \ \forall t > 0.$$

In [21] S. Nicaise, C. Pignotti and J. Valein extended the above result to higher-space dimension and established an exponential decay results.

Based on the previous works, our purpose in this paper is to give an energy decay estimate of the solution to the problem (1) for a weakly nonlinear damping and in the presence of a time-varying delay term by using a suitable energy and Lyapunov functionals and some properties of convex functions. These arguments of convexity were introduced and developed by I. Lasiecka et al. ([4], [13]) and used by W.J. Liu, E. Zuazua ([15]) and M. Eller et al. [8].

2. Preliminary results. In this section, we present some material for the proof of our main results.

$$(A_1):$$ The matrix $A = (a_{i,j}(x))$, where $a_{i,j} \in C^3(\Omega)$, is symmetric and there exists a constant $a_{01} > 0$ such that for all $x \in \Omega$ and $\delta = (\delta_1, \delta_2, ..., \delta_N) \in \mathbb{R}^N$, we have

$$\sum_{i,j=1}^{N} a_{i,j}(x)\delta_j\delta_i \geq a_{01}|\delta|^2,$$

where

$$Lu = -\text{div}(A\nabla u) = -\sum_{i,j=1}^{N} \left(a_{i,j}(x)\frac{\partial}{\partial x_i}\right), \quad a_{11} = \max \left(\sum_{i=1}^{N} ||a_{i,j}||^2_{\infty}\right),$$

$$a(u(t), v(t)) = \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial v(t)}{\partial x_i}dx = \int_{\Omega} A\nabla u(t), \nabla v(t)dx,$$
(A2) \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) is a bounded \( C^1 \) function satisfying
\[
g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l < 1,
\]
and there exists a non-increasing differentiable function \( \zeta : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( g'(t) \leq -\zeta(t)g(t) \).

(A3) \( \psi : \mathbb{R} \to \mathbb{R} \) is a non-decreasing function of the class \( C(\mathbb{R}) \) such that there exist \( \epsilon_1, c_1, c_2, c_3 > 0 \) and a convex and increasing function \( H : \mathbb{R}^+ \to \mathbb{R}^+ \) of the class \( C^1(\mathbb{R}^+) \cap C^2([0, \infty[) \) satisfying \( H(0) = 0 \) and \( H \) linear on \([0, \epsilon_1] \) or \( H'(0) = 0 \) and \( H'' > 0 \) on \([0, \epsilon_1]\)), such that
\[
c_1 |s| \leq |\psi(s)| \leq c_2 |s| \quad \text{if}\; |s| \geq \epsilon_1,
\]
\[
s^2 + \psi^2(s) \leq H^{-1}(s\psi(s)) \quad \text{if}\; |s| \leq \epsilon_1,
\]
\[
|\psi'(s)| \leq c_3,
\]
\[
\alpha_1 s \psi(s) \leq G(s) \leq \alpha_2 s \psi(s),
\]
where
\[
G(s) = \int_0^s \psi(r) dr,
\]
with \( \gamma \) satisfying
\[
\gamma - 1 \leq \frac{n + 2}{n - 2}, \quad \text{if}\; n > 2, \quad \text{and}\; \gamma - 1 < \infty, \quad \text{if}\; n \leq 2.
\]

(A4) \( \tau \) is a function such that
\[
\tau \in W^{2,\infty}([0, T]), \forall T > 0,
\]
\[
0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0,
\]
\[
\tau'(t) \leq d < 1, \quad \forall t > 0,
\]
where \( \tau_0 \) and \( \tau_1 \) are two positive constants.

(A5) The weight of dissipation and the delay satisfy
\[
\mu_2 < \frac{\alpha_1(1 - d)}{\alpha_2(1 - \alpha_1 d)} \mu_1.
\]

We now state some Lemmas needed later.

**Lemma 2.1** (Sobolev-Poincaré’s inequality). Let \( q \) be a number with \( 2 \leq q < +\infty \) \((n = 1, 2)\) or \( 2 \leq q \leq 2n/(n - 2) \) \((n \geq 3)\). Then there exists a constant \( c_* = c_*(\Omega, q) \) such that
\[
\|u\|_q \leq c_* \|\nabla u\|_2 \quad \text{for}\; u \in H^1_0(\Omega).
\]

Like in [18] we introduce the auxiliary unknown
\[
z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Omega, \; \rho \in (0, 1), \; t > 0.
\]

Then, we have
\[
\tau(t) z_t(x, \rho, t) + (1 - \tau'(t)\rho) z(x, \rho, t) = 0, \quad \text{in} \; \Omega \times (0, 1) \times (0, +\infty).
\]

Therefore, problem (1) is equivalent to
\[
\begin{aligned}
\frac{\partial}{\partial t} (|u(t)|^{\gamma-2}u(t)) - Lu & - \int_0^t g(t-s)Lu(s) ds \\
+ \mu_1 \psi(u(x,t)) + \mu_2 \psi(u(x,t-\tau(t))) & = 0, \quad \text{in } \Omega \times [0, +\infty[, \\
\tau(t)z_t(x,\rho,t) + (1 - \tau(t)\rho)z_{\rho}(x,\rho,t) & = 0, \quad \text{in } \Omega \times [0, 1] \times [0, +\infty[, \\
u(x, t) = 0, \quad \text{on } \partial \Omega \times [0, +\infty[, \\
z(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \\
z(x,\rho,0) = f_0(x,-\rho\tau(0)), \quad \text{in } \Omega \times [0, 1].
\end{aligned}
\]  

(13)

Let \( \xi \) be a positive constant such that \( \xi \) satisfies

\[
\frac{\mu_2(1-\alpha_1)}{\alpha_1(1-d)} < \xi < \frac{\mu_1-\alpha_2\mu_2}{\alpha_2}.
\]  

(14)

We define the energy associated to the solution of the problem (13) by

\[
E(t) = \frac{\gamma - 1}{\tau}||u_t(t)||^\gamma + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) a(u(t), u(t)) + (g \circ u)(t)
\]

\[+ \alpha(t)\tau(t) \int_0^1 \int_0^1 G(z(x, \rho, t)) d\rho dx,
\]

such that

\[\alpha(t) = \xi \zeta(t).
\]

Lemma 2.2. Let \((u,z)\) be a solution of the problem (13). Then, the energy functional defined by (15) satisfies

\[
E'(t) \leq - (\mu_1 - \alpha(t)\alpha_2 - \mu_2\alpha_2) \int_\Omega u_t \psi(u_t) dx
\]

\[- (\alpha(t)(1 - \tau'(t))\alpha_1 - \mu_2(1-\alpha_1)) \int_\Omega z(x,1,t)\psi(z(x,1,t)) dx
\]

\[+ \frac{1}{2}(g'ou)(t) - \frac{1}{2}g(t)a(u(t), u(t)) \leq 0.
\]  

(16)

Proof. Multiplying the first equation in (13) by \(u_t\), integrating over \(\Omega\) and using parts integration, we get

\[
\frac{1}{2} \frac{d}{dt} (||u_t||^2 \gamma + a(u(t), u(t))) + \mu_1 \int_\Omega \psi(z(x,1,t))u_t(x,t) dx
\]

\[- \int_0^t g(t-s) \int_\Omega A\nabla v(s)\nabla v_t(t) dx ds + \mu_1 \int_\Omega u_t \psi(u_t) dx = 0.
\]  

(17)

Where

\[a(\psi(t), \phi(t)) = \sum_{i,j=1}^N \int_\Omega a_{i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A\nabla \psi(t) \phi(t) dx.
\]

Using hypothesis \((A_1)\), we easily deduce that the bilinear forms \(a(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{R}\) are symmetric and continuous. On the other hand, from (2) for \(\delta = \nabla \psi\), we get

\[a(\psi(t), \psi(t)) \geq a_{01} \int_\Omega \sum_{i,j=1}^N \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx = a_{01} \|\nabla \psi(t)\|^2_2,
\]  

(18)
which implies that \( a(.,.) \) are coercive. Note that
\[
a(u(t), u_t(t)) = \frac{1}{2} \frac{d}{dt} a(u(t), u(t)). \tag{19}
\]
Following the same technique as in [31], we obtain
\[
\int_0^t g(t - s) \int_{\Omega} A \nabla u(s) \nabla u_t(t) dx \, ds
= \sum_{i,j=1}^n \int_0^t \int_{\Omega} g(t - s) a_{i,j}(x) \frac{\partial u(s)}{\partial x_j} \frac{\partial u_t(t)}{\partial x_i} dx \, ds
= \sum_{i,j=1}^n \int_0^t \int_{\Omega} g(t - s) a_{i,j}(x) \left( \frac{\partial u_t(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_j} \right) \frac{\partial u_t(t)}{\partial x_i} dx \, ds
- \frac{1}{2} \int_0^t g(t - s) \left( \frac{d}{dt} a(u(t), u(t)) \right) ds
- \frac{1}{2} \int_0^t g(t - s) \left( \frac{d}{dt} a(u(t) - u(s), u(t) - u(s)) \right) ds
- \frac{1}{2} \int_0^t g(t) a(u(t), u(t)) + \frac{1}{2} \int_0^t g'(t - s) a(u(t) - u(s), u(t) - u(s)) ds
= -\frac{1}{2} \frac{d}{dt} (g \circ u)(t) + \frac{1}{2} \frac{d}{dt} \left[ a(u(t), u(t)) \int_0^t g(s) ds \right]
- \frac{1}{2} g(t) a(u(t), u(t)) + \frac{1}{2} (g' \circ u)(t),
\]
where
\[
(g \circ u)(t) = \int_0^t g(t - s) a(u(t) - u(s), u(t) - u(s)) ds. \tag{21}
\]
We multiply the second equation in (13) by \( \alpha(t) \psi(z) \) and integrate over \( \Omega \times (0,1) \) to obtain
\[
\alpha(t) \tau(t) \int_{\Omega} \int_0^1 z_t \psi(z(x, \rho, t)) \, d\rho \, dx
= -\alpha(t) \int_{\Omega} \int_0^1 (1 - \tau'(t) \rho) \frac{\partial}{\partial \rho} G(z(x, \rho, t)) \, d\rho \, dx. \tag{22}
\]
Consequently,
\[
\frac{d}{dt} \left( \alpha(t) \tau(t) \int_{\Omega} \int_0^1 G(z(x, \rho, t)) \, d\rho \, dx \right)
= -\alpha(t) \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} \left[ (1 - \tau'(t) \rho) G(z(x, \rho, t)) \right] \, d\rho \, dx.
\]
In this way the proof of Lemma 2.2 is completed.

Observing that

From (27) and (28) we easily deduce

We then exploit (17), (20) and (23) to get

Let us denote by \( G^* \) the conjugate function of the convex function \( G \), i.e.,

Then \( G^* \) is the Legendre transform of \( G \), which is given by (see Arnold [2], p. 61-62)

and satisfies the following inequality

Then, from the definition of \( G_2 \), we get

Hence

Making use of (24) and (26), we have

Observing that

From (27) and (28) we easily deduce

In this way the proof of Lemma 2.2 is completed.
3. **Asymptotic behavior.** In this section, we prove the energy decay result by constructing a suitable Lyapunov functional. We denote by $c$ various positive constants which may be different at variant occurrences. Now we define the following functional

$$L(t) = ME(t) + \epsilon\phi(t) + \epsilon\varphi(t) + \epsilon I(t),$$

where

$$\phi(t) = \int_{\Omega} u|u_t|^{\gamma-2}u_t dx,$$

$$\varphi(t) = -\int_{\Omega} |u_t|^{\gamma-2}u_t \int_0^t g(t-s)(u(t) - u(s))ds dx,$$

and

$$I(t) = \int_{\Omega} \int_0^1 e^{-2\rho\tau(t)}G(z(x, \rho, t))d\rho dx.$$  

We need also the following Lemmas

**Lemma 3.1.** Let $(u, z)$ be a solution of problem (13). Then there exists two positive constants $\lambda_1, \lambda_2$ such that

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0,$$

for $M$ sufficiently large.

**Proof.** By applying the Hölder inequality and Young’s inequality and Lemma 2.2, we easily see that

$$\int_{\Omega} u|u_t|^{\gamma-2}u_t dx \leq C_\epsilon \int_{\Omega} |u|^\gamma dx + \epsilon \int_{\Omega} |u_t|^\gamma dx$$

$$\leq C_\epsilon \|\nabla u\|_2^\gamma + \epsilon \|u_t\|_\gamma^\gamma$$

$$\leq C_\epsilon \frac{E^2}{a_01} E^{\frac{\gamma-2}{2}}(0) E(t) + \epsilon \alpha E(t),$$

and

$$\int_{\Omega} u|u_t|^{\gamma-2}u_t dx \geq -C_\epsilon \int_{\Omega} |u|^\gamma dx - \epsilon \int_{\Omega} |u_t|^\gamma dx$$

$$\geq -C_\epsilon \|\nabla u\|_2^\gamma - \epsilon \|u_t\|_\gamma^\gamma$$

$$\geq -C_\epsilon E^2(t) - \epsilon \alpha E(t),$$

hence

$$\varphi(t) = \left| -\int_{\Omega} |u_t|^{\gamma-2}u_t \int_0^t g(t-s)(u(t) - u(s))ds dx \right|$$

$$\leq \frac{1}{2}\|u_t\|_\gamma^\gamma + \frac{1}{2} \int_{\Omega} \left( \int_0^t g(t-s)(u(t) - u(s))ds \right)^2 dx$$

$$\leq \frac{1}{2} \left( \|u_t\|_\gamma^\gamma + (1-l)c_2^2 \int_0^1 g(t-s)\alpha(u(t) - u(s), u(t) - u(s))ds \right)$$

$$\leq \frac{1}{2} \left( \|u_t\|_\gamma^\gamma + (1-l) \left( \frac{\beta E(0)}{l} \right) \right) c_2^2(gou)(t),$$
it follows from (32) that \( \forall c > 0 \) we have

\[
|I(t)| = \left| \int_0^1 \int_0^t e^{-2\rho(t)} G(z(x, \rho, s)) d\rho dx \right|
\leq c \int_0^1 \int_0^t G(z(x, \rho, s)) d\rho dx.
\] (37)

Hence, combining (35)-(37) we deduce

\[
|L(t) - ME(t)| = c\phi(t) + \varphi(t) + \epsilon I(t)
\]

\[
\leq \frac{C}{a_0} E^{2 - \frac{1}{2}}(0) E(t) + cE(t)\epsilon u_i \|u_i\|_2^2
\]

\[
+ \epsilon(1 - l) \left( \frac{\beta E(0)}{l} \right) c_5^2 (gou)(t)
\]

\[
+ c \int_0^1 \int_0^t G(z(x, \rho, t)) d\rho dx.
\] (38)

Finally, we get

\[
|L(t) - ME(t)| \leq c_5 E(t),
\] (39)

where \( c_5 = \max(c_1, c_2, c_3, c_4) \). From the definition of \( E(t) \) and selecting \( M \) sufficiently large, we obtain

\[
\beta_2 E(t) \leq L(t) \leq \beta_1 E(t),
\] (40)

such that \( \beta_1 = (M - \epsilon c_5) \), \( \beta_2 = (M + \epsilon c_5) \). This completes the proof. \( \square \)

**Lemma 3.2.** Let \((u, z)\) be the solution of (13). Then it holds for any \( \delta > 0 \)

\[
\frac{d}{dt} \phi(t) \leq \left\{ \frac{\mu_1}{a_0} + (\mu_1 + \mu_2)\delta c_5^2 - l \right\} a(u(t), u(t)) + \frac{N}{4a_0} (1 - l)(gou)(t)
\]

\[
+ \frac{\mu_2}{4\delta} \|\psi(z(x, 1, t))\|_2^2 + \|u_i\|_2^2 + \frac{\mu_1}{\delta} \|\psi(u_i)\|_2^2.
\] (41)

**Proof.** We take the derivative of \( \phi(t) \). It follows from (30) that

\[
\frac{d}{dt} \phi(t) = \int_\Omega (|u_i| - \gamma)^2 u_i u \; dx + \|u_i\|_2^2
\] (42)

using the problem (13), then we have

\[
\frac{d}{dt} \phi(t) = \|u_i\|_2^2 - a(u(t), u(t)) + \int_0^t \int_0^t g(t - s)A \nabla u(s) \nabla u(t) d\rho dx
\]

\[
- \mu_1 \int_\Omega \psi(u_i) u(t) dx - \mu_2 \int_\Omega \psi(z(x, 1, t)) u(t) dx
\] (43)

following the same idea in [31], yields

\[
\int_\Omega A \int_0^t g(t - s)(\nabla u(t) \nabla u(s)) d\rho dx
\]

\[
= \sum_{i,j=1}^N \int_0^t g(t - s) \int a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \left( \frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i} + \frac{\partial u(t)}{\partial x_i} \right) dx ds
\]

\[
= \sum_{i,j=1}^N \int_0^t \int a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \frac{\partial u(t)}{\partial x_i} dx ds
\]
Lemma 3.3. This completes the proof.

by applying Hölder and Young’s inequalities for the forth and fifth term in (43) for any \( \delta > 0 \) we get

\[
\left| \int_{\Omega} \psi(u_t)udx \right| \leq \delta \varepsilon^{2}a(u(t), u(t)) + \frac{1}{4\delta} \| \psi(u_t) \|_{2}^{2},
\]

and

\[
\left| \int_{\Omega} \psi(z(x, 1, t))udx \right| \leq \delta \varepsilon^{2}a(u(t), u(t)) + \frac{1}{4\delta} \| \psi(z(x, 1, t)) \|_{2}^{2}.
\]

Using (44)-(46) then (43) becomes

\[
\frac{d}{dt} \varphi(t) \leq \left\{ \frac{\mu_{a_{11}}}{a_{01}} + \left( \mu_{1} + \mu_{2} \right) \delta \varepsilon^{2} - l \right\} a(u(t), u(t)) + \frac{N}{4a_{01} \mu} (1 - l) (gou)(t)
\]

\[
+ \frac{\mu_{2}}{4\delta} \| \psi(z(x, 1, t)) \|_{2}^{2} + \| u_{t} \|_{2}^{2} + \frac{\mu_{1}}{4\delta} \| \psi(u_{t}) \|_{2}^{2}.
\]

This completes the proof. \( \square \)

Lemma 3.3. Let \((u, z)\) be the solution of (13). Then \( \varphi(t) \) satisfies for any \( \delta > 0 \)

\[
\varphi'(t) \leq \left\{ \frac{\beta}{a_{01}} + \frac{a_{11}\beta(1 - l)^{2}}{a_{01}} \right\} a(u(t), u(t)) - (g_{0} - \delta) \| u_{t} \|_{2}^{2}
\]

\[+ \mu_{1} \| \psi(u_{t}) \|_{2}^{2} + \frac{1}{4\delta} \varepsilon^{2} \mu_{2} \| \psi(z(x, 1, t)) \|_{2}^{2} + \frac{g(0)\varepsilon^{2}}{4\delta} (-g'ou)(t)
\]

\[+ \left( 1 - l \right) \left\{ \frac{1}{4a_{01} \beta} + \frac{1}{a_{01}} \left( 2\beta a_{11} + \frac{N}{4\beta} \right) + \frac{\mu_{1} + \mu_{2}}{4\delta} \right\} (gou)(t).
\]

Proof. Now taking the derivatives of \( \varphi(t) \) and using the problem (13), we obtain

\[
\frac{d\varphi(t)}{dt} = - \int_{\Omega} (|u_{t}|^{2} - 2u_{t}) \int_{0}^{t} g(t - s)(u(t) - u(s))dsdx
\]

\[= \int_{\Omega} |u_{t}|^{2}u_{t} \int_{0}^{t} g'(t - s)(u(t) - u(s))dsdx - \left( \int_{0}^{t} g(s)ds \right) \int_{\Omega} u_{t}^{2}dx
\]

\[= \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u(t)}{\partial x_{j}} \left( \int_{0}^{t} g(t - s) \left( \frac{\partial u(t)}{\partial x_{i}} - \frac{\partial u(s)}{\partial x_{i}} \right) ds \right) dx
\]
Using Young's inequality and the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, we infer
\begin{equation}
\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \left( \int_0^t g(t-s) \left( \frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right) dx 
\leq \beta \sum_{i,j=1}^{N} \int_{\Omega} \left( \int_0^t g(t-s) \frac{\partial u(s)}{\partial x_i} ds \right)^2 dx 
\end{equation}
and
\begin{equation}
\left| \sum_{i,j=1}^{N} \int_{\Omega} \left( \int_0^t g(t-s) \frac{\partial u(s)}{\partial x_i} ds \right) \left( \int_0^t g(t-s) \left( \frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right) dx \right| 
= \frac{1}{\beta} \sum_{i,j=1}^{N} \int_{\Omega} \left( \int_0^t g(t-s) \left( \frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right)^2 dx 
+ \frac{(1-l)g}{4\beta} \left[ 2\beta a_{11} + \frac{N}{4\beta} \right] \|g \circ u\|_2(t) + \frac{a_{11}\beta}{\beta a_{11}}(1-l)^2 a(u(t),u(t)),
\end{equation}
Next, we will estimate the right hand side of (49). Applying the Hölder inequality and Young's inequality and the assumptions $(A_1) - (A_2)$, we have for any $t_0 > 0$
\begin{equation}
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \quad \forall t \geq t_0.
\end{equation}
Invoking (52) we get the following estimates
\begin{equation}
\int_\Omega |u_t|^{-2} u_t \int_0^t g'(t-s)(u(t) - u(s))ds dx - \left( \int_0^t g(s)ds \right) \int_\Omega u_t^2 dx 
\leq \delta \|u_t\|_2^2 + \frac{g(0)c^2}{4\delta}(-g'ou)(t) - g_0 \|u_t\|_2^2,
\end{equation}
and
\begin{equation}
\left| - \int_\Omega \mu_1 \psi(u_t) \int_0^t g(t-s)(u(t) - u(s)) ds dx \right| 
\leq \mu_1 \|\psi(u_t)\|_2^2 + \frac{\mu_1(1-l)c^2}{4\delta} (gou)(t),
\end{equation}
and
\begin{equation}
\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \left( \int_0^t g(t-s) \left( \frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right) dx 
\leq \beta \sum_{i,j=1}^{N} \int_{\Omega} \left( \int_0^t g(t-s) \frac{\partial u(s)}{\partial x_i} ds \right)^2 dx 
\end{equation}
Lemma 3.4. Let $I$ the functional defined by (32) then it holds
\[
\frac{d}{dt} I(t) \leq -2I(t) - \frac{c_0(t)}{\tau_1} \int_\Omega G(z_1(x,1,t)) dx + \frac{\alpha(t)}{\tau_0} \int_\Omega G(u_t(x,t)) dx,
\]
where $\tau_0$, $\tau_2$ are some positive constants.

Proof. Differentiating (32) with respect to $t$ and using the second equation in (13), we have
\[
\frac{d}{dt} I(t) \leq \frac{\alpha(t)}{\tau_0} \int_\Omega G(u_t(x,t)) dx
\]

\[
\leq \alpha'(t) e^{-\tau(t)} \int_\Omega \int_0^1 z^2(x,\rho,t) \, d\rho dx
\]

\[
= \alpha'(t) e^{-\tau(t)} \int_\Omega \int_0^1 G(z(x,\rho,t)) \, d\rho dx
\]

\[
- \alpha(t) e^{-\tau(t)} \rho \tau'(t) \int_0^1 G(z(x,\rho,t)) \, d\rho dx
\]

\[
+ \frac{1}{\tau(t)} e^{-\tau(t)} \rho \tau(t) \zeta(t) \int_0^1 \frac{d}{dt} G(z(x,\rho,t)) \, d\rho dx
\]

\[
= \alpha'(t) e^{-\tau(t)} \int_\Omega \int_0^1 G(z(x,\rho,t)) \, d\rho dx
\]

\[
- \alpha(t) e^{-\tau(t)} \rho \tau'(t) \int_0^1 G(z(x,\rho,t)) \, d\rho dx
\]

\[
+ \frac{1}{\tau(t)} e^{-\tau(t)} \rho \alpha(t) \int_0^1 \frac{\partial}{\partial \rho} (1 - \tau'(t) \rho) G(z(x,\rho,t)) \, d\rho dx
\]

\[
\leq -\alpha(t) e^{-\tau(t)} \rho \tau'(t) \int_0^1 G(z(x,\rho,t)) \, d\rho dx
\]

\[
+ \alpha(t) \frac{\beta}{\tau(t)} \int_\Omega G(z(x,1,t)) dx
\]

\[
+ \frac{1}{\tau(t)} \left[ \alpha(t) \int_\Omega [G(z(x,0,t)) dx - G(z(x,1,t))] dx \right]
\]

\[
\leq -2c_0(t) I(t) - \frac{c_0(t)}{\tau_1} \int_\Omega G(z(x,1,t)) dx + \frac{\alpha(t)}{\tau_0} \int_\Omega G(u_t(x,t)) dx.
\]
Theorem 3.5. Let the assumptions \((A_1)\) – \((A_5)\) hold and \(u_0 \in H_{01}^1(\Omega)\), \(u_1 \in L^2(\Omega)\) and \(f_0 \in L^2(\Omega \times (0,1))\) be given. Then, we have the following decay estimates

\[ E(t) \leq \omega_1 H_1^{-1}(\omega t + \omega_3), \quad \forall t > 0, \]

where

\[ H_1(t) = \int_t^1 \frac{1}{H_2(s)} \, ds, \quad (59) \]

and \(H_2(t)\) is a function that will be specified later.

Proof. Since \(g\) is positive for any \(t_0 > 0\), we have

\[ \int_0^t g(s) \, ds \geq \int_0^{t_0} g(s) \, ds = g_0, \quad \forall t \geq t_0. \]

Hence we conclude from Lemma 3.2, Lemma 3.3 and Lemma 3.4 that

\[ \frac{dL(t)}{dt} \leq \epsilon \left\{ (1 - l) \left\{ \frac{N}{4a_{01} \mu} + \frac{1}{4a_{01} \beta} + \frac{1}{a_{01}} \left( 2 \beta a_{11} + \frac{N}{4 \beta} \right) \right\} \right\} (g \circ u)(t) \]

\[ + \epsilon \left\{ \frac{\mu_1}{\delta} + \mu_1 \right\} \| \psi(u_t) \|^2_2 + \epsilon(1 - l) \left\{ \left( \mu_1 + \mu_2 \right) c_s^2 + \frac{\mu_2}{\delta} \right\} (g \circ u)(t) \]

\[ - \epsilon \left\{ \frac{\mu a_{11}}{a_{01}} - (\mu_1 + \mu_2) \alpha c_s^2 - \frac{\beta}{a_{01}} - \frac{a_{11} \beta (1 - l)^2}{a_{01}} \right\} a(u(t), u(t)) \]

\[ - \left\{ Mc - \epsilon \frac{c}{\tau_0} \right\} \int_\Omega \psi(u_t) u_t \, dx + \left\{ Mc + \epsilon \left\{ \frac{\mu_1}{\delta} (1 + c_s^2) - \frac{c}{2 \tau_1} \right\} \right\} \]

\[ - 2 \epsilon I(t) + \left\{ \frac{M}{2} - \frac{c g(0) c_s^2}{4 \delta} \right\} (g \circ u)(t) - \epsilon(g_0 - \delta - 1)\| u_t \|^2_2. \]

Choosing carefully \(\epsilon\) sufficiently small and \(M\) sufficiently large such that

\[ \left\{ (1 - l) \left\{ \frac{N}{4a_{01} \mu} + \frac{1}{4a_{01} \beta} + \frac{1}{a_{01}} \left( 2 \beta a_{11} + \frac{N}{4 \beta} \right) \right\} \right\} = \eta_0 > 0, \]

\[ \left\{ \frac{\mu a_{11}}{a_{01}} - (\mu_1 + \mu_2) \alpha c_s^2 - \frac{\beta}{a_{01}} - \frac{a_{11} \beta (1 - l)^2}{a_{01}} \right\} = \eta_1 > 0, \]

\[ (g_0 - \alpha - 1) = \eta_2 > 0, \quad \left\{ \frac{M}{2} - \frac{c g(0) c_s^2}{4 \delta} \right\} = \eta_3 > 0, \]

then (60) takes the form

\[ \frac{dL(t)}{dt} \leq -\theta c E(t) + \epsilon \frac{\eta_1}{2} (gou)(t) + \epsilon c \| \psi(u_t) \|^2_2, \]

(61)

where \(\theta\) is a positive constant. Setting

\[ \lambda_1 = \frac{\theta c}{\beta_2}, \quad \lambda_2 = \frac{\eta_1 \epsilon}{2}, \quad \lambda_3 = \epsilon c, \]

the last inequality becomes

\[ \frac{dL(t)}{dt} \leq -\lambda_1 E(t) + \lambda_2 (gou)(t) + \lambda_3 \| \psi(u_t) \|^2_2, \]

(62)
multiplying (62) by $\alpha(t)$ we easily get

$$
\alpha(t) \frac{dL(t)}{dt} \leq -\lambda_1 \alpha(t)E(t) + \lambda_2 \alpha(t)(gou)(t) + \lambda_3 \alpha(t)\|\psi(u_t)\|_2^2
$$

$$
\leq -\lambda_1 \alpha(t)E(t) - \lambda_2 \alpha(t)(g'ou)(t) + \lambda_3 \alpha(t)\|\psi(u_t)\|_2^2
$$

(63)

We consider the following partition on $\Omega$

$$
\Omega_{11} = \left\{ x \in \Omega; \ |u_t| \geq \epsilon' \right\}, \quad \Omega_{12} = \left\{ x \in \Omega; \ |u_t| \leq \epsilon' \right\},
$$

then it is clear that $F(t) = L(t) + c\alpha(t)E(t)$ is equivalent to $E(t)$, then

$$
F'(t) \leq -\lambda_1 \alpha(t)E(t) + \lambda_3 \alpha(t)\|\psi(u_t)\|_2^2, \quad \forall t \geq t_0,
$$

(64)

from (3) and (16) it follows that

$$
\int_{\Omega_{12}} |\psi(u_t)|^2 dx \leq \mu_1 \int_{\Omega_{12}} u_t \|\psi(u_t)\|_2^2 dx \leq -\mu_1 E'(t).
$$

(65)

**1. case 1:** $H$ is linear then, according to assumption $(A_3)$, we get

$$
c_1's \leq |\psi(s)| \leq c_2's, \quad \forall s,
$$

and so

$$
\psi^2(s) \leq c_2's\psi(s), \quad \forall s.
$$

$H$ is linear on $[0, \epsilon']$. In this case one can easily check that there exists $\mu_1' > 0$ such that $|\psi(s)| \leq \mu_1'|s|$ for all $|s| \leq \epsilon'$ and thus

$$
\int_{\Omega_{11}} \|\psi(u_t)\|^2 dx \leq \mu_1' \int_{\Omega_{11}} u_t \psi(u_t) dx \leq -\mu_1' E'(t),
$$

(66)

using (65), (66) and the fact that $\alpha'(t) \leq 0$, it is clearly that

$$
\theta = L(t)\alpha(t) + c(\mu_1 + \mu_1')E
$$

is equivalent to $E(t)$ then, from (64) we deduce that

$$
E(t) \leq ce^{-c \int_0^t \alpha(s)ds} = H_1^{-1} \left( \int_0^t \alpha(s)ds \right).
$$

(67)

**2. case 2:** $H'(0) = 0$ and $H'' > 0$ on $[0, \epsilon']$. Since $H$ is convex and increasing then $H^{-1}$ is concave and increasing by Jensen’s inequality, we obtain

$$
\int_{\Omega_{12}} |\psi(u_t)|^2 dx \leq \int_{\Omega_{12}} H^{-1}(u_t \psi(u_t)) dx
$$

$$
\leq |\Omega_{12}| H^{-1} \left( \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} u_t \psi(u_t) dx \right)
$$

$$
\leq c H^{-1}(-c'E'(t)),
$$

(68)

then using (3), (66) and (68) we get

$$
\int_{\Omega} \psi(u_t)^2 dx = \int_{\Omega_{11}} \psi(u_t)^2 dx + \int_{\Omega_{12}} \psi(u_t)^2 dx
$$

$$
\leq \int_{\Omega_{12}} H^{-1}u_t \psi(u_t) dx + \int_{\Omega_{12}} u_t \psi(u_t) dx
$$
\[
\leq |\Omega_{12}| H^{-1} \left( \frac{1}{|\Omega_{12}|} u_t \psi(u_t) dx \right) + \int_{\Omega_{12}} u_t \psi(u_t) dx
\]
\[
\leq c H^{-1}(-c' E'(t)) - \alpha(t) \mu_1 E'(t),
\]
it is clearly that \( F(t) = L(t) + c \mu_1 E(t) \) is equivalent to \( E(t) \). Therefore (64) becomes
\[
F'(t) \leq \lambda_1 \alpha(t) E(t) + c H^{-1}(-c' E'(t)), \quad \forall t \geq t_0.
\]
Let us denote by \( H^* \) the conjugate function of the convex function \( H \), i.e.,
\[
H^* = \sup_{t \in \mathbb{R}_+} (st - H(t)).
\]
Then \( H^* \) is the Legendre transform of \( H \) which satisfies the following inequality
\[
st \leq H^* + H(t), \quad \forall s, t \geq 0,
\]
and
\[
H^* = s(H^{'})^{-1}(s) - H[(H^{'})^{-1}(s)], \quad \forall s \geq 0,
\]
the relation (72) and the fact that \( H'(0) = 0 \) and \((H^{'})^{-1}, H \) are increasing functions, we get
\[
H^*(s) \leq s(H^{'})^{-1}(s), \quad \forall s \geq 0,
\]
using the fact that \( E' \leq 0, H' \geq 0, H'' \geq 0 \) and by differentiating with \( \epsilon_0 > 0 \) small enough, we find that the functional \( F_1 \) defined by
\[
F_1(t) = H'(c_0 E(t))F(t) + c_3 E(t),
\]
satisfies, for some \( \nu_1, \nu_2 > 0 \)
\[
\nu_1 F_1(t) \leq E(t) \leq \nu_2 F_1(t),
\]
taking the derivative of (75) we arrive at
\[
\begin{align*}
F'_1(t) &= \epsilon_0 E'(t)H^{''}(c_0 E(t))(H'(c_0 E(t))F(t) + c_3 E(t)) \\
&\quad + H'(c_0 E(t))(L'(t) + c \mu_1 E'(t)) + c_3 E'(t) \\
&\leq -\lambda_1 \alpha(t) E(t) H'(c_0 E(t)) \\
&\quad + \hat{c} H'(c_0 E(t))H^{-1}(-c' E'(t)) + c \hat{c}' E'(t) \\
&\leq -\lambda_1 \alpha(t) E(t)H'(c_0 E(t)) + \hat{c} H^*(H'(c_0 E(t))) \\
&\quad - \hat{c} \alpha(t) E'(t) + c_3 E'(t) \\
&\leq -\lambda_1 \alpha(t) E(t)H'(c_0 E(t)) + \epsilon_0 \hat{c} \alpha(t) E(t)(H'(c_0 E(t))) \\
&\quad - \hat{c} \alpha(t) E'(t) + c_3 E'(t) \leq -\alpha(t) H_2 E(t),
\end{align*}
\]
where \( H_2(t) = tH'(c_0 t) \). We can observe from Lemma 3.1 that \( L(t) \) is equivalent to \( E(t) \), so, \( F_1(t) \) is also equivalent to \( E(t) \). By the fact that \( H_2 \) is increasing, we obtain
\[
F'_1(t) \leq -\hat{c} \alpha(t) H_2 F_1(t), \quad \forall t \geq 0.
\]
Noting that \( H'_1 = \frac{-1}{H_2} \), we infer from (78)
\[
[F_1(t)H_1(F_1(t))]' \geq \tilde{c} \alpha(t), \quad \forall t \geq 0.
\]
A simple integration over $(0,t)$ yields
\[ H_1(F_1(t)) \geq \hat{c} \int_0^t \alpha(s) ds + H_1(F_1(0)), \] (80)
exploiting the fact that $H_1^{-1}$ is decreasing, we infer
\[ F_1(t) \leq H_1^{-1} \left( \hat{c} \int_0^t \alpha(s) ds + H_1(F_1(0)) \right), \] (81)
the equivalence of $L(t)$, $F_1(t)$ and $E(t)$ yields the estimate
\[ E(t) \leq H_1^{-1} \left( \hat{c} \int_0^t \alpha(s) ds + H_1(F_1(0)) \right). \] (82)
Which completes the proof.

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REFERENCES

[1] C. Abdallah, P. Dorato, J. Benitez-Read and R. Byrne, Delayed Positive Feedback Can Stabilize Oscillatory Systems, ACC, San Francisco, (1993), 3106–3107.
[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.
[3] A. Benaissa and A. Guesmia, Energy decay for wave equations of $\phi$-Laplacian type with weakly nonlinear dissipation, *Electron. J. Differ. Equations*, 2008 (2008), 1–22.
[4] M. M. Cavalcanti, V. D. Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *Jour. Diff. Equa.*, 236 (2007), 407–459.
[5] G. Chen, Control and stabilization for the wave equation in a bounded domain, Part I, *SIAM J. Control Optim.*, 17 (1979), 66–81.
[6] G. Chen, Control and stabilization for the wave equation in a bounded domain, Part II, *SIAM J. Control Optim.*, 19 (1981), 114–122.
[7] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, 24 (1986), 152–156.
[8] M. Eller, J. E. Lagnese and S. Nicaise, Decay rates for solutions of a Maxwell system with nonlinear boundary damping, *Computational. Appl. Math.*, 21 (2002), 135–165.
[9] A. Haraux, Two remarks on dissipative hyperbolic problems, *Research Notes in Mathematics*, Pitman: Boston, MA, 122 (1985), 161–179.
[10] T. Kato, Linear and quasilinear equations of evolution of hyperbolic type, *C.I.M.E. Summer Sch.*, Springer, Heidelberg, 72 (2011), 125–191.
[11] T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems*, Lezioni Fermiane, [Fermi Lectures], Scuola Normale Superiore, Pisa, 1985.
[12] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson-John Wiley, Paris, 1994.
[13] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damps, *Diff. Integr. Equa.*, 6 (1993), 507–533.
[14] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
[15] W. J. Liu and E. Zuazua, Decay rates for dissipative wave equations, *Ricerche di Matematica*, 48 (1999), 61–75.
[16] P. Martinez and J. Vancostenoble, Optimality of energy estimates for the wave equation with nonlinear boundary velocity feedbacks, *SIAM J. Control Optim.*, 39 (2000), 776–797.
[17] M. Nakao, Decay of solutions of some nonlinear evolution equations, *J. Math. Anal. Appl.*, 60 (1977), 542–549.
[18] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, 45 (2006), 1561–1585.
[19] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differ. Integr. Equat.*, 21 (2008), 935–958.

[20] S. Nicaise and J. Valein, Stabilization of second order evolution equations with unbounded feedback with delay, *ESAIM Control Optim. Calc. Var.*, 16 (2010), 420–456.

[21] S. Nicaise; C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Discrete Contin. Dyn. Syst. Ser. S.*, 4 (2011), 693–722.

[22] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, *Discrete Contin. Dyn. Syst. Ser. S.*, 2 (2009), 559–581.

[23] J. Y. Park and S. H. Park, General decay for a nonlinear beam equation with weak dissipation, *J. Math. Phys.*, 51 (2010), 073508, 8pp.

[24] W. Rudin, *Real and Complex Analysis*, second edition, McGraw-Hill, Inc, New York, 1974.

[25] F. G. Shinskey, *Process Control Systems*, McGraw-Hill Book Company, 1967.

[26] I. H. Suh and Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Autom. Control*, 25 (1980), 600–603.

[27] C. Q. Xu, S. P. Yung and L. K. Li, Stabilization of the wave system with input delay in the boundary control, *ESAIM Control Optim. Calc. Var.*, 12 (2006), 770–785.

[28] Q. C. Zhong, *Robust Control of Time-delay Systems*, Springer, London, 2006.

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