On Non Commutative Calabi-Yau Hypersurfaces

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Abstract

Using the algebraic geometry method of Berenstein et al. (hep-th/0005087), we reconsider the derivation of the non commutative quintic algebra $A_{nc}(5)$ and derive new representations by choosing different sets of Calabi-Yau charges $\{C^a_i\}$. Next we extend these results to higher $d$ complex dimension non commutative Calabi-Yau hypersurface algebras $A_{nc}(d + 2)$. We derive and solve the set of constraint eqs carrying the non commutative structure in terms of Calabi-Yau charges and discrete torsion. Finally we construct the representations of $A_{nc}(d + 2)$ preserving manifestly the Calabi-Yau condition $\sum_i C^a_i = 0$ and give comments on the non commutative subalgebras.

Key words: Orbifolds of Calabi-Yau Hypersurfaces, Discrete torsion, Non Commutative geometry.
1 Introduction

Since the original work of Connes et al on Matrix model compactification on non commutative (NC) torii \([1]\), an increasing interest has been devoted to the study of non commutative spaces in connection with solitons in NC quantum field \([2,3]\), and string field theories \([4]\). These NC solitons, which have been subject to an intensive interest during the last few years, are involved in the study of \(D(p-4)/Dp\) brane systems \((p > 3)\) of superstrings; in particular in the ADH construction of the \(D0/D4\) system \([5]\), in the determination of the vacuum field solutions of the Higgs branch of supersymmetric gauge theories with eight supercharges \([6,7]\) and in tachyon condensation using the GMS approach \([8]\). However most of the NC spaces considered in these studies involve mainly NC \(R^d_\theta\), NC \(T^d_\theta\) torii \([3,10]\), some cases of \(Z_n\) type orbifolds of NC torii \([11,12]\) and some generalizations to non commutative higher dimensional cycles such as the non commutative Hirzebruch complex surface \(F_0\) used in \([13]\).

Quite recently efforts have been devoted to go beyond these particular manifolds. A special interest has been given to build non commutative Calabi-Yau manifolds containing the commutative ones as subalgebras and a development has been obtained for the case of orbifolds of Calabi-Yau hypersurfaces with discrete torsion. The key point of this construction, using a NC algebraic geometric method \([14]\), is based on solving non commutativity in terms of discrete torsion of the orbifolds \([13]\). More precisely, it has been shown that the \(T^2 \times T^2 \times T^2 / Z_2 \times Z_2\) orbifold of the three elliptic curves with torsion, embedded in the \(C^6\) complex space, define a noncommutative Calabi-Yau threefolds where non commutativity, carried by the discrete torsion, has a remarkable interpretation in terms of closed string states. Moreover on the fixed planes of this non commutative threefolds, branes fractionate and local deformations are no more trivial. This idea was successfully applied for orbifolds of the quintic Calabi-Yau threefolds. In this regards it was shown that the local non commutative quintic, to which we refer herebelow to as \(A_{nc}(5)\), is a subalgebra of \(Mat(5, C)\), the algebra of \(5 \times 5\) complex matrices. The \(A_{nc}(5)\) reads as:

\[
\begin{align*}
Z_1 Z_2 &= \alpha Z_2 Z_1, \\
Z_1 Z_3 &= \alpha^{-1} \beta Z_3 Z_1, \\
Z_1 Z_4 &= \beta^{-1} Z_4 Z_1, \\
Z_2 Z_4 &= \gamma^{-1} Z_4 Z_2, \\
Z_2 Z_5 &= \gamma Z_5 Z_2, \\
Z_3 Z_4 &= \beta \gamma Z_4 Z_3, \\
Z_3 Z_5 &= \beta^{-1} \gamma Z_5 Z_3,
\end{align*}
\]

where \(\alpha, \beta, \gamma\) are fifth roots of the unity generating the \(\mathbb{Z}_5^3\) discrete group and where the \(Z_i\)'s are the generators of \(A_{nc}(5)\). One of the main features of this non commutative algebra
is that its centre $Z$ coincides exactly with the homogeneous polynomial of the quintic Calabi-Yau threefolds.

In this study we reconsider the derivation of this algebra by using the non commutative algebraic geometry approach of [15] and reformulate the non commutativity structure in terms of set of constrains compatible with the Calabi-Yau condition and discrete torsion. Then we construct new representations of $\mathcal{A}_{nc}(5)$. One of the interesting results we have got here is that to each set of Calabi-Yau charges $\{C^a_i\}$, it is associated a representation $R_{\{C^a_i\}}$ of the non commutative algebra $\mathcal{A}_{nc}(5)$. We also give the general solution for the $R_{\{C^a_i\}}$ matrix representations of $\mathcal{A}_{nc}(5)$ preserving manifestly the Calabi-Yau condition and generalizing that derived in [15]. Next we extend these results to higher dimensions by deriving the $d$ complex non commutative orbifolds of Calabi-Yau hypersurfaces with discrete torsion, $\mathcal{A}_{nc}(d+2)$ as well as their representations. We discuss bulk representations and fractional branes at orbifolds points.

The organization of this paper is as follows: In section 2, we reconsider the study of the non commutative quintic and give new representations. Then we give the analogue of the NC quintic in weighted projective space $WP^4$. In section 3, we explore the general form of the non commutative algebra $\mathcal{A}_{nc}(d+2)$ associated with complex Calabi-Yau hypersurfaces with discrete torsion embedded in $WP^{d+1}$ while in section 4 we give the explicit form of its representations for both regular and singular points. Our solutions depends manifestly of on the data of the Calabi-Yau. We end this paper by giving our conclusion.

## 2 Local Non Commutative Quintic

To build non commutative quintic according to [14, 15], one starts from the complex homogeneous hypersurface

$$P_5(z_1, ..., z_5) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + a_0 \prod_{i=1}^5 z_i = 0,$$

where $(z_1, z_2, z_3, z_4, z_5)$ are the homogeneous coordinates of $P^4$ and $a_0$ is a complex parameter. This polynomial has a set of discrete isometries acting on the homogeneous coordinates $z_i$ as:

$$z_i \rightarrow z_i \omega^{C^a_i}$$

with $\omega^5 = 1$ and $C^a_i$, to which we refer in this paper as Calabi-Yau charges, are as follows

$$C^1_i = (1, -1, 0, 0, 0)$$
These charges satisfy \( \sum_{i=1}^{5} C_i^a = 0 \) modulo (5); in agreement with the CY condition. Note that under the action (3), the coordinate \( z_5 \) is stable. Using this symmetry, one can build an orbifold of the quintic by identifying \( z_i \) points related by eqs(3). The next step is to consider a coordinate patch of \( P^4 \) where \( z_5 = 1 \), associate to the \( \{z_1, z_2, z_3, z_4\} \) local variables, the set of \( 5 \times 5 \) matrix operators \( \{Z_1, Z_2, Z_3, Z_4\} \); \( Z_5 \) with the \( I_5 \) identity matrix, and construct the non commutative quintic associated with eqs(2-4) by solving the commutative quintic constraint eqs. The result obtained in [22] is given by (1). As one sees, eqs(1) seems to be a more general result as it does not depend explicitly on the Calabi-Yau charges \( C_i^a \) eqs(4). In fact eqs(1) define an abstract non commutative algebra valid for any choice of the set \( \{C_i^a\} \) charges satisfying the Calabi-Yau condition.

\[
\sum_{i=1}^{5} C_i^a = 0, \quad \text{modulo (5).}
\]

In what follows we study this property by considering different choices of sets \( \{C_i^a\} \) and show that they lead indeed to the same non commutative algebra(1). What we want to show throughout this examination is that different choices of sets of Calabi-Yau charges lead to different realizations of the algebra(1) and so constitute different representations of the non commutative quintic. To that purpose, let us first reconsider the example (4) and rederive eqs(1) in our manner by solving constraints on the non commutative quintic structure, then we study the second set of Calabi-Yau charges \( C_i^a \)

\[
C_i^1 = (1, -2, 1, 0, 0) \\
C_i^2 = (0, 1, -2, 1, 0) \\
C_i^3 = (1, 0, 1, -2, 0).
\]

Finally we show how things extend for generic choices of \( \{C_i^a\} \) sets and give the general result.

### 2.1 Representation I

Using the previous analysis, one can define the local non commutative quintic around \( Z_5 = z_5 I \) as follows

\[
Z_i Z_j = \theta_{ij} Z_j Z_i \quad i, j = 1, \ldots, 4, \quad \text{(7)}
\]

\[
Z_i Z_5 = Z_5 Z_i \quad \text{(8)}
\]
together with the constraint eqs
\[
\begin{align*}
[Z_j, Z_i^5] &= 0, \\
[Z_j, \prod_{i=1}^{4} Z_i] &= 0. 
\end{align*}
\] (9)

The \( \theta_{ij} \)'s are non zero complex parameters carrying non commutativity of the algebra \( A_{nc}(5) \). As the monomials \( Z_j^5 \) and \( \prod_{i=1}^{5} (Z_i) \) commute with all \( Z_i \)'s, it follows that the \( \theta_{ij} \)'s should obey the following constraint relations:
\[
\theta_{ij}^5 = 1, \quad \forall i, j (10)
\]
\[
\theta_{ij} \theta_{ji} = 1, \quad \forall i, j (11)
\]
\[
\prod_{i=1}^{4} \theta_{ji} = 1, \quad \forall j. (12)
\]

These relations constitute actually the defining conditions of local non commutative quintic with discrete torsion and show that the \( \theta_{ij} \)'s parameters carry \( 3 \times 4^2 - (4 - 1) = 3 \) degrees of freedom in agreement with the discrete torsion eq(2). Let us comment briefly these constraints eqs(10-12) and show that the \( \theta_{ij} \)'s are solved as:
\[
\theta_{ij} = \exp i \left( \frac{2\pi}{5} L_{ij} \right) = \omega^{L_{ij}}. (13)
\]

Eq(11) requires that \( L_{ij} \) is a \( 4 \times 4 \) antisymmetric matrix, i.e \( L_{ij} = -L_{ji} \), while eq(12) implies \( \sum_{i=1}^{4} L_{ij} = 0 \) modulo (5). The key idea of our way of solving the constraint \( \sum_{i=1}^{4} L_{ij} = 0 \) modulo (5) is to interpret it as just a manifestation of the Calabi-Yau condition (5). This observation suggests that \( L_{ij} \) can be solved by antisymmetric bilinears of \( C_i^a \) as shown herebelow:
\[
L_{ij} = m_{12} \left( C_i^1 C_j^2 - C_j^1 C_i^2 \right) - m_{23} \left( C_i^2 C_j^3 - C_j^2 C_i^3 \right) + m_{13} \left( C_i^1 C_j^3 - C_j^1 C_i^3 \right), (14)
\]

where \( m_{12} = k_1, m_{23} = k_2 \) and \( m_{13} = k_3 \) are integers modulo 5. In other words
\[
L_{ij} = \begin{pmatrix}
0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\
-k_1 + k_3 & 0 & k_1 & -k_3 \\
k_1 - k_3 & -k_1 & 0 & k_2 \\
-k_3 + k_2 & k_3 & -k_2 & 0
\end{pmatrix}, (15)
\]

Therefore the non commutative quintic associated to eqs(2-4), reads as:
\[
\begin{align*}
Z_1 Z_2 &= \omega^{k_1-k_3} Z_2 Z_1, \quad Z_1 Z_3 = \omega^{-k_1+k_2} Z_3 Z_1, \\
Z_1 Z_4 &= \omega^{k_3-k_2} Z_4 Z_1, \quad Z_2 Z_3 = \omega^{k_1} Z_3 Z_2, \\
Z_2 Z_4 &= \omega^{-k_3} Z_4 Z_2, \quad Z_3 Z_4 = \omega^{k_3} Z_4 Z_3.
\end{align*} (16)
\]
Setting $\alpha = \omega^{k_1-k_3}$, $\beta = \omega^{k_1-k_2}$ and $\gamma = \omega^{k_3}$, one discovers that these relations are identical to the relations (1); so eqs(16) define indeed a representation of the non commutative algebra $A_{nc}(5)$.

Eqs(16) may be realized in terms of $5 \times 5$ matrices in various ways; one of them which was studied in [15] is based on fixing $\beta$ and $\gamma$ in terms of $\alpha$ ($\beta = \alpha^{n+1}$, $\gamma = \alpha^{m+1}$) breaking the way $Z_5^3$ down to $Z_5$. Here we give a solution preserving manifestly the $Z_5^3$ symmetry. This solution involves the following $5 \times 5$ matrices

$$P_\eta = \text{diag}(1, \eta, \eta^2, \eta^3, \eta^4), \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} (17)$$

where $\eta$ stands for $\omega^{k_1}, \omega^{k_2}, \omega^{k_3}$ and their products. In terms of these matrices, the $Z_i$’s read, up to a normalization factor, as:

$$Z_1 = z_1 P_{\omega^{k_1+k_2+k_3}} Q^3$$
$$Z_2 = z_2 P_{\omega^{k_1}} Q^{-1}$$
$$Z_3 = z_3 P_{\omega^{k_2}} Q^{-1}$$
$$Z_4 = z_4 P_{\omega^{k_3}} Q^{-1}, (18)$$

where $\bar{\omega}$ is the complex conjugate of $\omega$. This solution, which extends that given in [15], shows clearly that $Z_5^5$ and the product $\prod_{i=1}^4 Z_i$ are in the centre $Z$ of the local non commutative algebra $A_{nc}(5)$. Actually these relations define the $R\{C_n\}$ representation of eqs(1).

We end this paragraph by making three remarks: (i) By an appropriate choice of the values of the $k_1, k_2$ and $k_3$ integers on can get the subalgebras of $A_{nc}(5)$. For example, taking $k_3 = 0$, one gets the following subalgebra

$$Z_1 Z_2 = \omega^{k_1} Z_2 Z_1, \quad Z_1 Z_3 = \omega^{-k_1+k_2} Z_3 Z_1,$$
$$Z_1 Z_4 = \omega^{-k_2} Z_4 Z_1, \quad Z_2 Z_3 = \omega^{k_1} Z_3 Z_2,$$
$$Z_2 Z_4 = Z_4 Z_2, \quad Z_3 Z_4 = Z_4 Z_3. \quad (19)$$

Moreover if we take $k_2 = k_3 = 0$, the algebra (16) reduces to

$$Z_1 Z_2 = \omega^{k_1} Z_2 Z_1, \quad Z_1 Z_3 = \omega^{-k_1} Z_3 Z_1, \quad Z_2 Z_3 = \omega^{k_1} Z_3 Z_2,$$
$$Z_1 Z_4 = Z_4 Z_1, \quad Z_2 Z_4 = Z_4 Z_2, \quad Z_3 Z_4 = Z_4 Z_3. \quad (20)$$
This subalgebra could be linked to the non commutative K3 algebra studied in [16]. (ii) The solution we have given in eq(14) is a particular one. A more general solution involving the toric geometry data of Calabi-Yau manifolds as well as other features can be found in [17]. (iii) Viewing the algebra eq(1) as describing a $\mathbb{Z}_5$ orbifold of the quintic with a $\mathbb{Z}_5$ discrete torsion and considering the singularities in codimension two, it was shown in [15] that branes do fractionate at the orbifold points. In subsection 4.2, we will give explicit details for the example of the $\mathbb{Z}_5^8$ orbifold of the eight-tic.

2.2 Representation II

Here we give another representation of the algebra (1) by using Calabi-Yau charges chosen as in eqs(6). As one sees, these charges are different from those given by eq(4); they are rather similar to the toric data of the blown up of the $\hat{A}_2$ affine singularity of the ALE space. Eqs(6) are used in geometric engineering of fundamental matters in 4D $N = 2$ superconformal theories [18, 19]. An analogous analysis as that used in the previous subsection leads to:

$$L_{ij} = \begin{pmatrix}
0 & L_{12} & L_{13} & L_{14} \\
-L_{12} & 0 & L_{23} & L_{24} \\
-L_{13} & -L_{23} & 0 & L_{34} \\
-L_{14} & -L_{24} & -L_{34} & 0
\end{pmatrix}$$

(21)

with

$$L_{12} = k_1 + k_2 + 2k_3, \quad L_{13} = -2k_1 - 2k_2, \quad L_{14} = k_1 + k_2 - 2k_3, \quad L_{23} = 3k_1 - k_2 - 2k_3, \quad L_{24} = -2k_1 + 2k_2 + 4k_3, \quad L_{34} = k_1 - 3k_2 - 2k_3,$$

(22)

and where $k_1, k_2$ and $k_3$ are integers modulo 5. The new local algebra describing the non commutative quintic reads now as:

$$Z_1Z_2 = Z_2Z_1\omega^{k_1+k_2+2k_3}, \quad Z_1Z_3 = Z_3Z_1\omega^{-2k_1-2k_2},$$

$$Z_1Z_4 = Z_4Z_1\omega^{k_1+k_2-2k_3}, \quad Z_2Z_3 = Z_3Z_2\omega^{3k_1-k_2-2k_3}, \quad Z_2Z_4 = Z_4Z_2\omega^{-2k_1+2k_2+4k_3}, \quad Z_3Z_4 = Z_4Z_3\omega^{k_1-3k_2-2k_3}. \quad (23)$$

Setting $\alpha = \omega^{k_1+k_2+2k_3}$, $\beta = \omega^{2k_3-k_1-k_2}$ and $\gamma = \omega^{2k_1-2k_2-4k_3}$, one discovers once again the non commutative algebra (1). Here also one can derive the various subalgebras of eqs(23) by appropriate choices of the integers $k_1, k_2$ and $k_3$. The corresponding subalgebras define special non commutative geometries. To conclude this section, one should retain that eqs (16) and
(23) define two realizations of the local noncommutative quintic algebra. As such, different choices of sets \( \{ C_a \} \) of Calabi-Yau charges lead to different realizations of the local algebra \( \mathcal{A}_{nc}(5) \). Moreover the special solutions of eq(14) give the subalgebras of eq(1). Such analysis can be extended to other Calabi-Yau manifolds with discrete torsion. For more details see [7].

2.3 Other Representations

In this paragraph we want to give a comment on other possible realisations of the non commutative algebra (1) using orbifolds of hypersurfaces in the weighted projective spaces \( \text{WP}^4 \). To that purpose recall that in the weighted projective space \( \text{WP}^4_{\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}} \), the analogue of eq(2) reads as

\[
\sum_{i=1}^{5} u_{i}^{D_{\delta_{i}}} + a_{0} \prod_{i=1}^{5} (u_{i}) = 0.
\]

(24)

where \( D = \sum_{i} \delta_{i} \). To fix the ideas, let us consider the simple example where \( \delta_{1} = 2 \) while \( \delta_{2} = \delta_{3} = \delta_{4} = \delta_{5} = 1 \). In this case, the analogue of the quintic polynomial becomes:

\[
u_{1}^{3} + u_{2}^{6} + u_{3}^{6} + u_{4}^{6} + u_{5}^{6} + a_{0} \prod_{i=1}^{5} (u_{i}) = 0.
\]

(25)

where \( (u_{1}, u_{2}, u_{3}, u_{4}, u_{5}) \) are the quasi-homogeneous coordinates of \( \text{WP}^4_{\{2,1,1,1,1\}} \) and \( a_{0} \) is a complex parameter. This polynomial has also a set of discrete isometries acting on the homogeneous coordinates \( u_{i} \) as:

\[
u_{i} \rightarrow u_{i}^{C_{a_{i}}} \quad i = 1, \ldots, 5
\]

(26)

with \( \zeta_{1}^{3} = 1 \) while the others \( \zeta_{i}^{6} = \omega^{6} = 1 \); i.e \( \zeta_{1} = \omega^{2} \) and \( \zeta_{i} = \omega, \quad i = 2, \ldots, 5 \). The \( C_{a_{i}} \) are chosen as in eqs(4). Note that \( u_{5} \) is stable under the change(26). Using this symmetry and following the non commutative algebraic method, one can build an orbifold of this Calabi-Yau threefolds (25) by identifying \( u_{i} \) points rotated by eqs(26). In the coordinate patch where \( u_{5} = 1 \), the four variables \( \{u_{1}, u_{2}, u_{3}, u_{4}\} \) obey, after performing the correspondence, the non commutative relations.

\[
u_{i}U_{j} = \theta_{ij} U_{j}U_{i},
\]

(27)

where the \( \theta_{ij} \)'s are non zero complex parameters. As the monomials \( U_{i}^{3}, \quad U_{i}^{6} \) and \( \prod_{i=1}^{5} (U_{i}) \) are in the centre \( \mathcal{Z} \) of the algebra, we have:

\[
\left[ U_{j}, U_{i}^{3} \right] = 0,
\]

7
\[
\begin{bmatrix}
U_j, U_i^6 \\
U_j, \prod_{i=1}^{4} U_i
\end{bmatrix} = 0 \quad i = 2, 3, 4, 
\]  \hspace{1cm} (28)

Therefore the \( \theta_{ij} \)'s should obey the following constraints:

\[
\theta_{i1}^3 = 1, \quad i = 2, 3, 4, 
\]  \hspace{1cm} (29)

\[
\theta_{ij}^6 = 1, \quad j \neq 1, i 
\]  \hspace{1cm} (30)

\[
\prod_{i=1}^{4} \theta_{ji} = 1, \quad \forall j 
\]  \hspace{1cm} (31)

\[
\theta_{ij} \theta_{ji} = 1, \quad \forall i, j. 
\]  \hspace{1cm} (32)

The solution of these constraints are built as follows: first use eqs (29-30) and (32) to write \( \theta_{ij} \) as

\[
\theta_{ij} = \exp \left( i \left( \frac{2\pi}{6} L_{ij} \right) \right) = \omega^{L_{ij}}, \quad L_{ij} = -L_{ji}, 
\]  \hspace{1cm} (33)

Eqs (31) implies however that \( \sum_{i=1}^{4} L_{ij} = 0 \) modulo (6). Particular solutions may be learned from eqs (14) using antisymmetric bilinears of \( C^a_i \) (4). Straightforward calculations show that \( L_{ij} \) is given by the following 4 \( \times \) 4 matrix:

\[
L_{ij} = \begin{pmatrix}
0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\
-k_1 + k_3 & 0 & k_1 & -k_3 \\
k_1 - k_2 & -k_1 & 0 & k_2 \\
-k_3 + k_2 & k_3 & -k_2 & 0
\end{pmatrix}
\]  \hspace{1cm} (34)

where now the \( k_i \) integers are such that

\[
k_i - k_j \equiv 2r_{ij} \in 2\mathbb{Z}. 
\]  \hspace{1cm} (35)

Therefore the non commutative algebra associated to eq(25) is

\[
U_1 U_2 = \omega^{k_1-k_3} U_2 U_1, \quad U_1 U_3 = \omega^{k_2-k_1} U_3 U_1, 
\]  \hspace{1cm} (36)

\[
U_1 U_4 = \omega^{k_3-k_1} U_4 U_1, \quad U_2 U_3 = \omega^{k_1} U_3 U_2, 
\]  \hspace{1cm} (36)

\[
U_2 U_4 = \omega^{-k_3} U_4 U_2, \quad U_3 U_4 = \omega^{k_2} U_4 U_3. 
\]  \hspace{1cm} (36)

Moreover taking \( \alpha = \omega^{k_1-k_3}, \beta = \omega^{k_1-k_3} \) and \( \gamma = \omega^{k_3} \), one discovers the non commutative algebra (1), except now the deformation parameters are as follows:

\[
\alpha^3 = \beta^3 = \gamma^6 = 1. 
\]  \hspace{1cm} (37)
Note that for the hypersurface (24) in the weighted projective space $\mathbb{WP}^4_{1,2,1,1,1}$ namely

$$u_1^6 + u_2^3 + u_3^6 + u_4^6 + u_5^6 + a_0 \prod_{i=1}^{5} (u_i) = 0, \quad (38)$$

with the discrete symmetries $u_i \rightarrow u_i \zeta_i^a$; where $\zeta_2 = \omega^2$ and $\zeta_i \neq 2 = \omega$ and where the $C_i^a$ are as in eqs(4), the analogue of the constraint eqs (36) are given by $k_1 \in 2\mathbb{Z}$ and $k_3 \in 2\mathbb{Z}$. Such analysis generalizes naturally other situations.

3 Non Commutative Calabi-Yau

The analysis we have developed for the non commutative quintic (2) and the hypersurfaces (24-25) can be extended straightforwardly to more general $d$ complex dimension Calabi-Yau homogeneous hypersurfaces in $\mathbb{P}^{d+1}$ and $\mathbb{WP}^{d+1}$. Starting from $(d+2)-tic$ in $\mathbb{P}^{d+1}$ [20, 21];

$$z_1^{d+2} + z_2^{d+2} + z_3^{d+2} + z_4^{d+2} + z_5^{d+2} + \ldots + z_{d+2}^{d+2} + a_0 \prod_{i=1}^{d+2} z_i = 0, \quad (39)$$

with the discrete isometries type (3) with Calabi-Yau charges $C_i^a$ charges satisfying the condition

$$\sum_{i=1}^{d+2} C_i^a = 0, \quad a = 1, \ldots, d,$$

one can build a $(d+1) \times (d+1)$ antisymmetric matrix in terms of $C_i^a$ bilinears as:

$$L_{ij} = -L_{ji} = m_{ab} C_i^a C_j^b, \quad (40)$$

where $m_{ab}$ is an antisymmetric $d \times d$ matrix of integers modulo $(d+2)$, satisfying

$$\sum_{i=1}^{d+2} L_{ij} = 0. \quad (41)$$

The non commutative extension of eq(39) is given by the following algebra, to which we refer to as $\mathcal{A}_{nc}(d+2)$;

$$Z_i Z_j = \omega_{ij} Z_j Z_i; \quad i, j = 1, \ldots, (d+1),$$

$$Z_i Z_{d+2} = Z_{d+2} Z_i; \quad i = 1, \ldots, (d+1), \quad (42)$$

where $\omega_{kl}$ is the complex conjugate of $\omega_{kl}$. The non commutativity parameters are realized in terms of the Calabi-Yau charges as follows:

$$\omega_{ij} = \exp \left( \frac{2\pi}{d+2} m_{ab} C_i^a C_j^b \right) = \omega^{m_{ab} C_i^a C_j^b}. \quad (43)$$
Note that this representation involves \( d(d-1)/2 \) integers \( m_{[ab]} \) in agreement with the numbers of degrees of freedom one gets using the generalization of constraint eqs(10-12). Note also that the huge form of the non-commutative algebra (42) describing the non-commutative version of eq(39) has \( d(d-1)/2 \) free parameters generating the \( \mathbb{Z}_{d+2}^{d(d-1)/2} \) discrete symmetry group. By appropriate choices of the integers, one gets all possible subalgebras describing non-commutative special geometries associated with eq(39). For example there are \( d(d-1)/2 \) types of non-commutative subalgebras with discrete symmetry \( \mathbb{Z}_{d+2}^{(d-1)(d-2)/2} \) and \( \frac{d(d-1)}{2} \left( \frac{d(d-1)}{2} - 1 \right) \) subsubalgebras with \( \mathbb{Z}_{d+2}^{(d-2)(d-3)/2} \) symmetry and so on. Note finally that for \( d = 2 \) describing K3, eqs(42) reduce to

\[
\begin{align*}
Z_1 Z_2 &= \alpha Z_2 Z_1; & Z_1 Z_3 &= \overline{\alpha} Z_3 Z_1 \\
Z_2 Z_3 &= \alpha Z_3 Z_2; & Z_i Z_4 &= Z_4 Z_i; & i &= 1, 2, 3.
\end{align*}
\]

(44)

where we have set \( \alpha = \omega_{12} \overline{\omega}_{12} = \omega_{23} \overline{\omega}_{23} = \omega_{31} \overline{\omega}_{31} \) and \( \overline{\alpha} \) is its complex conjugate.

4 Matrix Representation of the \( Z_i \)'s

Finite dimensional representations of the non-commutative algebra are given by matrix subalgebras \( \text{Mat} [n(d + 2), C] \), the algebra of \( n(d + 2) \times n(d + 2) \) complex matrices, with \( n = 1, 2, ... \). To see this property it is enough to take the determinant of eqs(42), which constraint the dimension \( D \) of the representation to be such that:

\[
(\omega_{ij} \overline{\omega}_{ji})^D = 1.
\]

(45)

Using the identity (43), one discovers that \( D \) is a multiple of \( (d + 2) \). In what follows we consider the fundamental \( (d + 2) \times (d + 2) \) matrix representation obtained by extending the solution (17-18). We will distinguish two situations according to whether we are dealing with regular or singular points of the Calabi-Yau orbifold.
4.1 Representations for regular points

Introducing the following set \( \{ Q, P_{\alpha_{ab}}; a, b = 1, \ldots, d \} \) of matrices:

\[
P_{\alpha_{ab}} = \text{diag}(1, \alpha_{ab}, \alpha_{ab}^2, \ldots, \alpha_{ab}^{d+1}); \quad Q =
\begin{pmatrix}
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

(46)
satisfying \( P^d_{\alpha} = 1, P_{\alpha} P_{\beta} = P_{\alpha \beta} \) and \( Q^{d+2} = 1 \); then setting

\[
Z_1 = z_1 \prod_{a,b=1}^d \left( P_{\alpha_{ab}}^{C_{a}^{i}} Q_{\alpha_{ab}}^{C_{b}^{i}} \right)
\]

\[
Z_2 = z_2 \prod_{a,b=1}^d \left( P_{\alpha_{ab}}^{C_{a}^{i}} Q_{\alpha_{ab}}^{C_{b}^{i}} \right),
\]

(47)
and computing \( Z_1 Z_2 \) and \( Z_2 Z_1 \), one discovers

\[
Z_1 Z_2 = \eta_1 Z_2 Z_1,
\]

with \( \eta_1 \) equal to

\[
\eta_1 = \omega^m_{ab} c_{1}^{a} c_{2}^{b}.
\]

(48)
Note that this relation is nothing else than the leading term of a more general tensor given by:

\[
\eta_{ij} = \prod_{a,b} \left( \alpha_{ab} \delta_{ba} \right)^{a_{i}^{a} c_{i}^{b}}
\]

(49)
where \( \alpha_{ab} = \omega^m_{ab} \). The remaining others can be derived as powers of the \( P_{\alpha_{ab}} \) and \( Q \) matrices so that the product \( \prod_{i=1}^{d+1} Z_i \) is in the centre \( \mathcal{Z} \) of the local non commutative algebra \( \mathcal{A}_{nc} [d + 2] \).

Thus the solution for the \( Z_i \)'s reads, up to a normalization factor, as

\[
Z_i = z_i \prod_{a,b=1}^d \left( P_{\alpha_{ab}}^{C_{a}^{i}} Q_{\alpha_{ab}}^{C_{b}^{i}} \right).
\]

(50)
One can check easily that eqs(50) satisfy naturally \( Z^{d+2}_i \sim I_{d+2} \), the Calabi-Yau condition

\[
\prod_{i=1}^{d+1} Z_i = \prod_{a,b=1}^d \left[ \prod_{i=1}^{d+1} \left( z_i P_{\alpha_{ab}}^{C_{a}^{i}} Q_{\alpha_{ab}}^{C_{b}^{i}} \right) \right]
\]

\[
= I_{d+2} \left( \prod_{i=1}^{d+1} z_i \right).
\]

(51)
and so are indeed solutions of the non commutative algebra (42), with

$$\eta_{ij} = \omega_{ij} \omega_{ji}. \quad (52)$$

More details on this derivation as well as other features involving toric geometry of Calabi-Yau manifolds are given in [17].

4.2 Fractional Branes

The solutions we gave earlier correspond to regular points of non commutative Calabi-Yau. The $A_{nc}(d + 2)$ representations eq(51) are irreducibles and the branes do not fractionate. Similar solutions may be worked out as well for orbifold points with discrete torsions where we expect to get fractional branes. Due to the richness of possibilities, we will focus our attention herebelow on giving a particular solution for the $\mathbb{Z}_8^6$ orbifold of the eight-tic; general solutions will be reported in [17], see also [22] for the resolution of stringy singularities by non commutative algebras. To that purpose consider the orbifold of eq(39), $d = 6$, with respect to $\mathbb{Z}_8^6$ and take the Calabi-Yau vector charges as:

$$C_1^i = (1, -1, 0, 0, 0, 0, 0, 0)$$
$$C_2^i = (1, 0, -1, 0, 0, 0, 0, 0)$$
$$C_3^i = (1, 0, 0, -1, 0, 0, 0, 0)$$
$$C_3^i = (1, 0, 0, 0, -1, 0, 0, 0)$$
$$C_3^i = (1, 0, 0, 0, 0, -1, 0, 0)$$
$$C_3^i = (1, 0, 0, 0, 0, 0, -1, 0). \quad (53)$$

For regular points, the matrix representation of eqs(42) is irreducible as shown on eqs (51). For fixed points however, the situation is more subtle as there exists situations where representations are reducible. One way to deal with the singularity of the orbifold with respect to $\mathbb{Z}_8^6$ is to interpret the algebra as describing a $\mathbb{Z}_3^3$ orbifold with $\mathbb{Z}_3^3$ discrete torsions having singularities in codimension four. Starting from eqs(42) and choosing $Z_5$, $Z_6$ and $Z_7$ in the centre of the algebra by setting

$$(\omega_{ij} \omega_{ji}) = 1, \quad \text{for } i = 5, 6, 7, 8; \quad \forall j = 1, ..., 8, \quad (54)$$

the algebra reduces to

$$Z_1Z_2 = \alpha_1 \alpha_2 Z_2Z_1$$
\begin{align}
Z_1Z_3 &= \alpha_1^{-1}\alpha_3Z_3Z_1 \\
Z_1Z_4 &= \alpha_2^{-1}\alpha_3^{-1}Z_4Z_1 \\
Z_2Z_3 &= \alpha_1Z_3Z_2 \\
Z_2Z_4 &= \alpha_2Z_4Z_2, \\
Z_3Z_4 &= \alpha_3Z_4Z_3
\end{align}

and all remaining other relations are commuting. In this equation, the \( \alpha_i \)'s are such that \( \alpha_i^8 = 1 \); these are the phases of the \( \mathbb{Z}_3^8 \) discrete torsions. In the singularity where the \( z_1, z_2, z_3, \) and \( z_4 \) moduli of eq(51) go to zero, one ends with the familiar result for orbifolds with discrete torsion. Therefore the branes fractionate in the codimension four singularities. Actually this result extends naturally the one on the non commutative quintic obtained in [15] and can be generalized straightforwardly [17].

5 Conclusion

In this paper we have studied the building of non commutative Calabi-Yau hypersurface orbifolds using the algebraic geometry approach of [15] and the toric data of the Calabi-Yau manifolds with discrete torsion. Our main results are summarized as follows: (i) We have reviewed the construction of the non commutative quintic \( \mathcal{A}_{nc}(5) \), its subalgebras and built new representations using methods of toric geometry. We have also studied the analogue of the non commutative quintic in the weighted projective space \( \mathbb{WP}(4) \). (ii) We have given the generalization of these results to higher dimensional Calabi-Yau hypersurface orbifolds and derived the explicit form of the corresponding non commutative algebra \( \mathcal{A}_{nc}(d+2) \) and its bulk and orbifold representations. As an example we have studied the fractional branes for the \( \mathbb{Z}_3^8 \) orbifold of the eight-tic with \( \mathbb{Z}_3^8 \) discrete torsions. One of the lessons we learnt in this work is that to each Calabi-Yau data \( \{C_i^a\} \) corresponds a representation \( \mathcal{R}\{C_i^a\} \) of the non commutative algebra \( \mathcal{A}_{nc}(d+2) \).

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