Hyperbolic positive energy theorem with electromagnetic fields

Yaohua Wang$^1$ and Xu Xu$^2$

$^1$ School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475004, People’s Republic of China
$^2$ School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People’s Republic of China

E-mail: wangyaohua@henu.edu.cn and xuxu2@whu.edu.cn

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Abstract
We establish a type of positive energy theorem for asymptotically anti-de Sitter Einstein–Maxwell initial data sets by Witten’s spinorial techniques.

Keywords: positive energy theorem, anti-de Sitter spacetime, electromagnetic fields, Dirac–Witten operator

1. Introduction

As a fundamental result in general relativity, the positive mass theorem states that an isolated gravitational system satisfying the dominant energy condition must have non-negative total mass. Schoen and Yau first gave a rigorous proof of this theorem for asymptotically flat initial data sets [14–16]. Soon later, Witten provided a different proof by spinorial techniques [20]. Parker and Taubes gave a rigorous proof of the theorem based on Witten’s method [13]. Later on, Gibbons and Hull successfully generalized Witten’s argument to asymptotically flat initial data sets with electromagnetic fields [9]. Gibbons, Hawking, Horowitz and Perry discussed the positive energy theorems for asymptotically flat initial data sets with black holes [8].

For certain spacetimes with nonzero cosmological constant, the positive energy theorem should also be true. Wang [17] and Chruściel and Herzlich [6] gave the definition of the mass for asymptotically hyperbolic manifolds in the time symmetric case and the proof of its positivity respectively. Soon after that, for initial data sets with nonzero second fundamental form, Maerten [12] and Chruściel et al [7] established corresponding positive energy theorems, which gave the upper bounds for angular momentum and center of mass. They [7] pointed out that, if the total energy 4-vector $m_\mu$ was timelike, one can make $SO(3, 1)$ coordinate transformations to get the ‘center of anti-de Sitter (AdS) mass’ coordinates such that
and they proved the energy–momentum inequality

$$m_{(0)} \geqslant \sqrt{\left|\hat{c}\right|^2 + \left|\hat{j}\right|^2 + 2 \left|\hat{c} \times \hat{j}\right|}$$

in the new coordinates. But for general nontrivial initial data sets the total energy 4-vector is not always timelike. Furthermore, the form of (1.1) is not $SO(3, 1)$ invariant under the coordinate transformation back to the non-center of AdS mass coordinates. The first author, Xie and Zhang [19] then established the inequality for Henneaux and Teitelboim’s total energy–momentum in general coordinates. The authors [18] then established the positive energy theorem for (4 + 1)-dimensional asymptotically AdS spacetimes based on their work. We also refer to [1] and [3] for physical discussions.

This paper aims at generalizing the results in [19] to asymptotically AdS spacetimes with electromagnetic fields. Similar to [19], we generalize Witten’s spinorial method to a type of asymptotically AdS spacetimes with electromagnetic fields. Under the modified dominant energy condition

$$\mu \geqslant \frac{1}{4} \max \left\{ \left|\nu \right|^2, \left|\nu^\prime \right|^2 \right\} + (\text{div} E)^2 + (\text{div} B)^2 + \kappa |B|,$$

we give a lower bound of the total energy in terms of the new quantities we define (see definition 2.2).

**Theorem 1.1.** Let $(M, g, p, E, B)$ be a three-dimensional asymptotically AdS Einstein–Maxwell initial data set of the spacetime $(N, \tilde{g})$ which satisfies the modified dominant energy condition (1.2). Then we have the following inequality:

$$E_0 \geqslant \sqrt{L^2 - 2V^2 + 2 \left( \max \left\{ A^4 - L^2 V^2, 0 \right\} \right)^2},$$

where

$$\begin{align*}
L^2 &= b_0^2 + |b|^2 + |c|^2 + |\mathbf{c}^\prime|^2 + |\mathbf{J}|^2 + q^2, \\
V^3 &= b_0 \cdot c + q b \cdot J + e_{ijk} c_i c_j \mathbf{J}_k, \\
A^4 &= |c \times c^\prime|^2 + |c \times J|^2 + |c^\prime \times J|^2 + |b \cdot c|^2 \\
&\quad + |b \cdot J|^2 + |b \cdot c|^2 + b_0^2 |b|^2 + q^2 |b|^2 \\
&\quad + \sum_i \left( b_0 c_i + q l_i \right)^2 + 2b_0 e_{ijk} b_i c_j \mathbf{J}_k + 2q e_{ijk} b_i c_j c_k, \\
\end{align*}$$

and

$$m_{(0)} \to m(0),
m(1), m(2), m(3), c(2), J(1), J(2) \to 0,$$
and
\[ b = (b_1, b_2, b_3), \quad c = (c_1, c_2, c_3), \quad c' = (c'_1, c'_2, c'_3), \quad J = (J_1, J_2, J_3). \]

To prove theorem 1.1, we provide a rigorous proof of the Poincaré inequality which ensures the existence and uniqueness of the solution for the Dirac–Witten equation.

This paper is organized as follows: in section 2, we give some preliminaries needed in the paper, including the explicit Clifford representation, the definition of imaginary Killing spinor on the 0-slice of AdS spacetime, the definition of total energy, total momenta, total electric charge, and total magnetic momenta for asymptotically AdS Einstein–Maxwell initial data sets. In section 3, we derive the Weitzenböck formula and prove the existence and uniqueness of the solution for the Dirac–Witten equation. In section 4, we prove our positive energy theorem for asymptotically AdS Einstein–Maxwell initial data sets and discuss the case with black holes. Finally, in section 5, we present our calculations for the Kerr–Newman–AdS spacetime.

2. Preliminaries

The AdS spacetime with negative cosmological constant \( \Lambda = -3\kappa^2 < 0 \), denoted by \((N, \tilde{g}_{AdS})\), is a static spherically symmetric solution of the vacuum Einstein equations. In polar coordinates, the metric of AdS spacetime could be written as
\[
\tilde{g}_{AdS} = -\cosh^2(\kappa r) dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} \left( d\theta^2 + \sin^2\theta d\psi^2 \right),
\]
where
\[-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 \leq \theta < \pi, \quad 0 \leq \psi < 2\pi.\]

The \( t \)-slice \((\mathbb{H}^3, \tilde{g})\) is the hyperbolic 3-space with constant sectional curvature \( -\kappa^2 \), where \( \kappa > 0 \).

Let the associated orthonormal frame be
\[
\tilde{e}_0 = \frac{1}{\cosh(\kappa r)} \frac{\partial}{\partial t}, \quad \tilde{e}_1 = \frac{\partial}{\partial r}, \quad \tilde{e}_2 = \frac{\kappa}{\sinh(\kappa r)} \frac{\partial}{\partial \theta}, \quad \tilde{e}_3 = \frac{\kappa}{\sinh(\kappa r) \sin \theta} \frac{\partial}{\partial \psi},
\]
and denote \( \tilde{e}^\alpha \) as its dual coframe.

For the following application, we fix the following Clifford representation throughout this paper:
\[
\tilde{e}_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{e}_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\tilde{e}_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{e}_3 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.
\]
For the AdS spacetime, the imaginary Killing spinor $\Phi_0$ is defined as the solution of
\[ \nabla^A \Phi_0 + \frac{\kappa \sqrt{-1}}{2} X \cdot \Phi_0 = 0, \quad \forall X \in TH^3. \]
Its form along the 0-slice could be written down explicitly [19] (see appendix).

Suppose that $(N, \tilde{g})$ is a Lorentzian manifold with the metric $\tilde{g}$ of signature $(-1, 1, 1, 1)$

satisfying the Einstein field equations
\[ \Lambda - \nabla^2 \tilde{g} + \Lambda \tilde{g} = T, \quad \text{(2.3)} \]
where $\nabla^2$, $\Lambda$ are the Ricci and scalar curvatures of $\tilde{g}$ respectively, $T$ is the energy–momentum tensor of matter, and $\Lambda$ is the cosmological constant. Let $M$ be a three-dimensional spacelike hypersurface in $N$ with the induced metric $g$ and $p$ be the second fundamental form of $M$ in $N$. $E$ and $B$ are two vector fields on $M$, representing the electric and magnetic fields respectively. The set $(M, g, p, E, B)$ is called an Einstein–Maxwell initial data set. In this paper, we will focus on the following type Einstein–Maxwell initial data set.

**Definition 2.1.** An Einstein–Maxwell initial data set $(M, g, p, E, B)$ is asymptotically AdS of order $\tau > \frac{3}{2}$ if:

1. There is a compact set $K$ such that $M_\infty = M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_r$, where $B_r$ is the closed ball of radius $r$ with center at the coordinate origin;
2. under the diffeomorphism, $g_{ij} = g(\tilde{e}_i, \tilde{e}_j) = \delta_{ij} + a_{ij}, \ p_{ij} = p(\tilde{e}_i, \tilde{e}_j), \ E = E \tilde{e}_i, \ B = B \tilde{e}_i$ satisfy
   \[ a_{ij} = O(e^{-\tau r}), \quad \tilde{V}_k a_{ij} = O(e^{-\tau r}), \quad \tilde{V}_k \tilde{V}_k a_{ij} = O(e^{-\tau r}), \]
   \[ p_{ij} = O(e^{-\tau r}), \quad \tilde{V}_k p_{ij} = O(e^{-\tau r}), \]
   \[ |E|_{\tilde{g}} = O(e^{-\frac{\tau r}{2}}), \quad |B|_{\tilde{g}} = O(e^{-\frac{\tau r}{2}}). \quad \text{(2.4)} \]

3. $\text{div} E \in L^1(M)$, and there is a distance function $\rho_2$ such that $T_{00} e^{\rho_2}, T_{0i} e^{\rho_2}, i \text{div} B, e^{\rho_2} B \in L^1(M)$.

**Remark 2.1.** For simplicity, we just assume that the initial data set has only one end. The case of multi-ends is similar. To get our definitions, recall that the AdS spacetime is just the hyperboloid $\{\eta_{\alpha\beta} y^\alpha y^\beta = \frac{3}{4}\}$ of $\mathbb{R}^{3,2}$ with the metric
\[ \eta_{\alpha\beta} dy^\alpha dy^\beta = -\left(dy^0\right)^2 + \sum_{i=1}^{3} \left(dy^i\right)^2 - \left(dy^4\right)^2. \]

The ten Killing vectors
\[ U_{\alpha\beta} = \gamma_\alpha \frac{\partial}{\partial y^\beta} - \gamma_\beta \frac{\partial}{\partial y^\alpha}, \]

$0 \leqslant \alpha < \beta \leqslant 4$, generate rotations for $\mathbb{R}^{3,2}$. By restricting these vectors to the hyperboloid $\{\eta_{\alpha\beta} y^\alpha y^\beta = \frac{3}{4}\}$ with the induced metric, the Killing vectors of AdS spacetime can be derived, denoted as $U_{\alpha\beta}$ also. See appendix for the explicit form of $U_{\alpha\beta}$ along the 0-slice.
Denote
\[ E_i = \frac{\nabla^j g_{ij} - \nabla_i \text{tr}_g (g)}{\kappa} - \kappa (a_{ii} - g_{ii} \text{tr}_g (a)), \]
\[ P_{ki} = p_{ki} - g_{ki} \text{tr}_g (p). \]

Here and henceforth, repeated indices are summed, with the lower-case Latin indices running from 1 to 3. And we can define the following quantities for the asymptotically AdS Einstein–Maxwell initial data set. Some of the quantities have appeared in [7, 10, 19]. For completeness, we still write them down here.

**Definition 2.2.** For the asymptotically AdS Einstein–Maxwell initial data set, the total energy is defined as
\[ E_0 = \frac{\kappa}{16\pi} \lim_{r \to \infty} \int_{S_i} E_i U_{i0}^{(0)} \hat{\omega}, \]
the total momenta are defined as
\[ c_i = \frac{\kappa}{16\pi} \lim_{r \to \infty} \int_{S_i} E_i U_{i0}^{(0)} \hat{\omega}, \]
\[ c_i' = \frac{\kappa}{8\pi} \sum_{j=2}^3 \lim_{r \to \infty} \int_{S_i} P_{j1} U_{i0}^{(0)} \hat{\omega}, \]
\[ J_i = \frac{\kappa}{8\pi} \sum_{j=2}^3 \lim_{r \to \infty} \int_{S_i} P_{j1} V_i^{(0)} \hat{\omega}, \]
the total electric charge is defined as
\[ q = \frac{1}{4\pi} \lim_{r \to \infty} \int_{S_i} E_i \hat{\omega}, \]
and the total magnetic momenta are defined as
\[ b_0 = \frac{\kappa}{2\pi} \lim_{r \to \infty} \int_{S_i} B^1 U_{i0}^{(0)} \hat{\omega}, \]
\[ b_i = \frac{\kappa}{2\pi} \lim_{r \to \infty} \int_{S_i} B^1 U_{i0}^{(0)} \hat{\omega}, \]
where
\[ \hat{\omega} = \tilde{e}^2 \wedge \tilde{e}^3, \quad U_{i0}^{(0)} = U_{i0}^{(0)} \hat{e}^0. \]

**Remark 2.2.** All the quantities come from the boundary terms at infinity of the integral form of the Weitzenböck formula (3.1).

**Remark 2.3.** The quantities we have defined are indeed global geometric invariants. The invariance of energy–momenta can be found in [7, 10, 19]. Now we will consider the total electric charge and the total magnetic momenta. The total electric charge is defined as
where $\iota$ is the interior product and

$$\iota \omega = \hat{\epsilon}^2 \wedge \hat{\epsilon}^3, \quad \eta = \hat{\epsilon}^1 \wedge \hat{\epsilon}^2 \wedge \hat{\epsilon}^3.$$ 

Thus under the $SO(3, 1)$-transformation $\varphi$ of the background metric, we have

$$4\pi (q - \bar{q}) = \lim_{r \to \infty} \int_{S_r} t_E \bar{\eta} - \lim_{\kappa \to \infty} \int_{S_{\kappa}} t_{\varphi(\kappa)} E \bar{\eta},$$

$$= \lim_{r \to \infty} \int_{S_r} t_E \bar{\eta} - \lim_{\kappa \to \infty} \int_{\varphi^{-1}(S_{\kappa})} t_E \bar{\eta}.$$ 

For fixed $r$ and $r_1$, using the divergence theorem, one obtains

$$\int_{S_r} t_E \bar{\eta} - \int_{\varphi^{-1}(S_r)} t_E \bar{\eta} = \int_{M_{r, r_1}} \text{div} E \bar{\eta},$$

where $M_{r, r_1}$ is the domain bounded by $S_r$ and $\varphi^{-1}(S_{r_1})$ in $M$. By virtue of the asymptotic property of the metric and the fact that $\text{div} E \in L^1(M)$, we have $\int_{M_{r, r_1}} \text{div} E \bar{\eta} \to 0$ as $r, \kappa \to \infty$. Therefore, $q = \bar{q}$.

The invariance of magnetic momenta $(b_0, b_1)$ can also be derived by the property of Killing vector. The total magnetic momenta are defined as

$$b_0 = \frac{\bar{k}}{2\pi r \to \infty} \int_{S_r} B_4 U^{(4)}_{\varphi(0)} \bar{\omega} = \frac{\bar{k}}{2\pi r \to \infty} \int_{S_r} U_{\varphi(0)}^{(4)} \bar{\eta},$$

$$b_1 = \frac{\bar{k}}{2\pi r \to \infty} \int_{S_r} B_4 U^{(4)}_{\varphi(1)} \bar{\omega} = \frac{\bar{k}}{2\pi r \to \infty} \int_{S_r} U_{\varphi(1)}^{(4)} \bar{\eta}. $$

The $SO(3, 1)$ coordinate transformations fixed $t = 0$ slice are reduced to

$$z^a = B^a_{\beta} y^\beta, \quad z^4 = y^4$$

for some $SO(3, 1)$ matrix $B = (B^a_{\beta})$. Denote the lower-bar terms the corresponding quantities in new coordinates. Since $z_\alpha = B_{\alpha}^\beta y^\beta$, $B^a_{\beta} = \eta^{\alpha\beta} B_{\alpha}^\delta \eta_{\delta\beta}$ for the flat metric $\eta$ on $\mathbb{R}^{1, 2}$, we have

$$U_4_\alpha = z_4 \frac{\partial}{\partial z^\alpha} - z_\alpha \frac{\partial}{\partial z^4} = B_{\alpha}^\beta U_{\beta 0}.$$ 

Therefore

$$b_\alpha = B_{\alpha}^\beta b_\beta.$$ 

Also, it is directly to check that the total electric charge and the total magnetic momenta are all invariant under the admissible coordinate transformations

$$\hat{t} = t + o \left( e^{-\frac{3\kappa}{2}} r \right), \quad \hat{\epsilon}_0 (\hat{t}) = \bar{\epsilon}_0 (t) + o \left( e^{-\frac{3\kappa}{2}} r \right),$$

$$\hat{r} = r + o \left( e^{-\frac{3\kappa}{2}} r \right), \quad \hat{\epsilon}_1 (\hat{r}) = \bar{\epsilon}_1 (r) + o \left( e^{-\frac{3\kappa}{2}} r \right),$$

$$\hat{\theta}^A = \theta^A + o \left( e^{-\frac{3\kappa}{2}} r \right), \quad \hat{\bar{\epsilon}}_B (\hat{\theta}^A) = \bar{\epsilon}_B (\theta^A) + o \left( e^{-\frac{3\kappa}{2}} r \right)$$

for sufficiently large $r$. 

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Example 2.1. For Reissner–Nordström AdS spacetime with the metric
\[ \tilde{g} = -f\,dt^2 + f^{-1}\,d\vec{r}^2 + r^2\,d\Omega^2, \quad f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \kappa^2r^2 \]
and the field strength tensor
\[ F = -\frac{Q}{r^2}dt \wedge d\vec{r}, \]
we have \( E^i = F_0^i \), \( B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} \), where \( \epsilon \) is the volume form of the 0-slice. Then the total electric charge \( q = Q \) and the total magnetic momenta \( b_0 = b_j = 0 \).

This is an example with trivial total magnetic momenta, and Kerr–Newman–AdS spacetime is another example with nontrivial total magnetic momenta. As the example is complicated, we put it separately in section 5.

3. The Weitzenböck formula and the Dirac–Witten equation

Let \((N, \tilde{g})\) be a spacetime with the metric \(\tilde{g}\) of signature \((-1, 1, 1, 1)\) and \((M, g, p, E, B)\) be an asymptotically AdS Einstein–Maxwell initial data set of \(N\). As \(M\) is spin, we can choose a spin structure, and thus a spinor bundle \(\mathcal{S}\) over \(M\). This is the same as in [13]. Denote \(\tilde{V}, \nabla\) the Levi-Civita connections with respect to \(\tilde{g}\), \(g\) respectively, and their lifts to the spinor bundle \(\mathcal{S}\). There is a positive definite Hermitian metric \(\langle \cdot, \cdot \rangle\) on \(\mathcal{S}\), with respect to which \(\varepsilon_i\) is skew-Hermitian and \(e_0\) is Hermitian [13, 22, 23]. Furthermore, \(\tilde{\nabla}\) is compatible with \(\langle \cdot, \cdot \rangle\), but \(\nabla\) is not.

For a fixed point \(x \in M\), we choose a suitable local orthonormal basis \(e_0, e_1, e_2, e_3\), with \(V_{\alpha}e_j(x) = 0\) for \(i, j = 1, 2, 3\) and \(V_{\alpha}e_j(x) = 0\) for \(j = 1, 2, 3\), then
\[ \tilde{V}_e e_0(x) = p_0 e_j, \quad \tilde{V}_e e_j(x) = p_j e_0, \]
where \(p_j = \tilde{g}(\tilde{V}_e e_0, e_j)\) is the component of the second fundamental form. Denote \(\{e^\alpha \mid \alpha = 0, 1, 2, 3\}\) as the dual coframe of \(\{e_\alpha\}\). The two connections on the spinor bundle \(\mathcal{S}\) are related by \(\tilde{\nabla}_i = \nabla_i - \frac{1}{2}p_{ij}e_0 \cdot e_j\).

We define the imaginary Einstein–Maxwell connection as
\[ \tilde{\nabla}_i = \nabla_i - \frac{1}{2}p_{ij}e_0 \cdot e_j + \frac{\kappa}{2}\sqrt{-1} e_i \cdot -\frac{1}{2}E \cdot e_i \cdot e_0 \cdot -\frac{1}{4}\epsilon_{ijk}B_{jk} \cdot e_l \cdot e_i \cdot e_l, \]
and define the associated Dirac–Witten operator as
\[ \tilde{D} = \sum_{i=1}^3 e_i \cdot \tilde{\nabla}_i = D + \frac{1}{2}p_{ij}e_0 \cdot \frac{3i}{2}\sqrt{-1} - \frac{1}{2}E \cdot e_0 \cdot -\frac{1}{4}\epsilon_{ijk}B_{jk} \cdot e_l \cdot e_i \cdot e_l, \]
where \(D = \sum_{i=1}^3 e_i \cdot \nabla_i\). Then the adjoints of this two operators with respect to \(\langle \cdot, \cdot \rangle\) are
\[ \tilde{D}^+ = -\tilde{V}_i - \frac{1}{2}p_{ij}e_0 \cdot e_j + \frac{\kappa}{2}\sqrt{-1} e_i \cdot -\frac{1}{2}E \cdot e_0 \cdot -\frac{1}{4}\epsilon_{ijk}B_{jk} \cdot e_l \cdot e_i \cdot e_l, \]
\[ \tilde{D}^* = -\frac{1}{2}p_{ij}e_0 \cdot \frac{3i}{2}\sqrt{-1} - \frac{1}{2}E \cdot e_0 \cdot +\frac{1}{4}\epsilon_{ijk}B_{jk} \cdot e_l \cdot e_i \cdot e_l \cdot e_l. \]

Now we derive the Weitzenböck formulas for \(\tilde{D}\) and \(\tilde{D}^*\).
Theorem 3.1.

\[ \hat{D}^* \hat{D} = \hat{\nabla}^* \hat{\nabla} + \hat{R}, \]

\[ \hat{D} \hat{D}^* = \hat{\nabla}^* \hat{\nabla} + \hat{R}', \]

(3.1)

where

\[ \hat{\nabla}' = V_i - \frac{1}{2} p_{ij} e_0 \cdot e_j - \frac{\kappa}{2} \cdot \frac{1}{2} E \cdot e_i \cdot e_0 \cdot + \frac{1}{4} \varepsilon_{jkl} B_j e_k \cdot e_i \cdot e_0, \]

\[ \hat{R} = \frac{1}{2} (\mu - \nu_i e_0 \cdot e_i \cdot) + \text{div} \: E e_0 \cdot - \text{div} \: B e_1 \cdot e_2 \cdot e_3 \cdot - \frac{\kappa}{2} \varepsilon_{jkl} B_j e_k \cdot e_i \cdot e_0, \]

\[ \hat{R}' = \frac{1}{2} (\mu - \nu'_i e_0 \cdot e_i \cdot) + \text{div} \: E e_0 \cdot + \text{div} \: B e_1 \cdot e_2 \cdot e_3 \cdot - \frac{\kappa}{2} \varepsilon_{jkl} B_j e_k \cdot e_i \cdot e_0, \]

with

\[ \mu = \frac{1}{2} \left( R + \left( \sum_{ij} p_{ij} \right)^2 - \sum_{ij} p_{ij}^2 \right) \]

\[ + 3 \kappa^2 - |E|^2 - |B|^2, \]

\[ \nu_i = V_i p_{ij} - V_i p_{ij} - 2 \varepsilon_{ijk} B_k, \]

\[ \nu'_i = V_i p_{ij} - V_i p_{ij} + 2 \varepsilon_{ijk} B_k. \]

Proof. By straightforward computation, we have

\[ \hat{D}^* \hat{D} = V^* V + \frac{1}{4} \left( R + \left( \sum_{ij} p_{ij} \right)^2 + 9 \kappa^2 + |E|^2 + |B|^2 \right) \]

\[ - \frac{1}{2} V_i p_{ij} e_i \cdot e_0 \cdot - \frac{1}{2} V_i E_j e_i \cdot e_j \cdot e_0 \cdot + E_i e_0 \cdot V_i \]

\[ - E \cdot e_0 \cdot D - \frac{1}{4} \varepsilon_{jkl} V_i B_j e_i \cdot e_k \cdot e_l \cdot - \varepsilon_{jkl} B_j e_k \cdot e_l \cdot \cdot \]

\[ \cdot - \frac{3 \kappa}{4} \varepsilon_{jkl} B_j e_k \cdot e_l \cdot - \frac{1}{2} \varepsilon_{jkl} B_j e_k \cdot e_l \cdot e_0 \cdot. \]

(3.2)

Note that

\[ \hat{\nabla}' \hat{V}' = - V_i V_i + \frac{1}{4} \left( \sum_{ij} p_{ij}^2 + 3 \kappa^2 + 3 |E|^2 + 3 |B|^2 \right) \]

\[ - \text{div} \: E e_0 \cdot + \frac{1}{2} V_i p_{ij} e_0 \cdot e_j \cdot + \frac{1}{2} \varepsilon_{jkl} B_j e_k \cdot e_l \cdot e_0 \cdot \]

\[ - \frac{1}{4} \frac{\kappa}{2} \varepsilon_{jkl} B_j e_k \cdot e_l \cdot - E \cdot e_0 \cdot D - \frac{1}{2} V_i E_j e_i \cdot e_j \cdot e_0 \]

\[ + \frac{1}{4} \frac{\kappa}{2} \varepsilon_{jkl} V_i B_j e_k \cdot e_l \cdot e_i \cdot + E_i e_0 \cdot V_i - \varepsilon_{jkl} B_j e_k \cdot V_i. \]

(3.3)
Combining (3.2) and (3.3), we have

\[
\hat{D}^a \hat{D}^b = \hat{\nabla}^a \hat{\nabla}^b + \frac{1}{4} \left( R + \left( \sum p_{ij} \right)^2 - \sum p_{ij}^2 + 6\kappa^2 - 2|E|^2 - 2|B|^2 \right)
\]

\[
= -\frac{1}{2} \left( \nabla_j p_{ij} - \nabla_i p_{jj} - 2\epsilon_{ijk} B_j E_k \right) e_i \cdot e_j + \text{div } E e_0
\]

\[
- \frac{\kappa \sqrt{-1}}{2} \epsilon_{ijkl} B_{ij} e_k \cdot e_l - \frac{1}{4} \epsilon_{ijkl} \nabla_j (e_i \cdot e_k + e_i \cdot e_l).
\]

Note that

\[
\epsilon_{ijkl} \nabla_j (e_i \cdot e_j + e_i \cdot e_l) = 4 \text{ div } B e_1 \cdot e_2 \cdot e_3 .
\]

thus

\[
\hat{D}^a \hat{D}^b = \hat{\nabla}^a \hat{\nabla}^b + \frac{1}{4} \left( R + \left( \sum p_{ij} \right)^2 - \sum p_{ij}^2 + 6\kappa^2 - 2|E|^2 - 2|B|^2 \right)
\]

\[
-\frac{1}{2} \left( \nabla_j p_{ij} - \nabla_i p_{jj} - 2\epsilon_{ijk} B_j E_k \right) e_0 \cdot e_i + \text{div } E e_0 - \text{div } B e_1 \cdot e_2 \cdot e_3 - \frac{\kappa \sqrt{-1}}{2} \epsilon_{ijkl} B_{ij} e_k \cdot e_l .
\]

This proves the first formula in (3.1). The second formula is proved similarly. Q.E.D.

**Remark 3.1.** If \( \kappa = 0 \), the formulas in (3.1) are reduced to the Weitzenböck formulas for the asymptotically flat Einstein–Maxwell initial data set [9]. And if \( E = B = 0 \), the formulas in (3.1) are the same as the formulas for asymptotically AdS spacetimes [17, 21].

The modified dominant energy condition we impose to prove the positive energy theorem is

\[
\frac{1}{2} \mu \geq \frac{1}{4} \max \left\{ |\nu|^2, |\nu'|^2 \right\} + (\text{div } E)^2 + (\text{div } B)^2 + |\kappa | B |.
\]

**Remark 3.2.** The modified dominant energy condition (1.2) is the same as the one used to prove the positive energy theorem for asymptotically flat manifolds with electromagnetic fields if \( \kappa = 0 \), and is reduced to the standard dominant energy condition \( T_{00} \geq \sqrt{\sum T_{ii}^2} \) if \( E = B = 0 \). In fact, the constraint equations for Einstein–Maxwell equations could be written as

\[
\frac{1}{2} \left( R - \sum p_{ij}^2 \right) - \sum p_{ij}^2 + 6\kappa^2 - |E|^2 - |B|^2 = T^\text{matter}_{00},
\]

\[
\nabla_j p_{ij} - \nabla_i p_{jj} + 2\epsilon_{ijk} B_j E_k = T^\text{matter}_{ij},
\]

\[
\text{div } E = J^E_0,
\]

\[
\text{div } B = J^M_0,
\]

where \( T^\text{matter} \) is the energy–momentum tensor of the matter fields, \( \nu \) is the volume form of \( (M, g) \), \( J^E \) and \( J^M \) are the electric current and the magnetic current respectively. Then the
modified dominant energy condition (1.2) is a combination of
\[
T_{00}^{\text{matter}} \geq \sqrt{\sum_{i} \left( T_{00}^{\text{matter}} \right)^{2} + \left( 2J_{j}^{A} \right)^{2} + \left( 2J_{j}^{B} \right)^{2} + 2\kappa |B|}
\]
and itself with $B$ replaced by $-B$.

In order to calculate the boundary term of the Weitzenböck formula, we will define a new connection and see the difference of two connections on the spinor bundle. Most of the results presented here are due to the work in [2, 21, 24]. These are written down here just for completeness. Recall that $g = \bar{g} + a$ with $a = O(e^{-\tau_0})$, $\nabla a = O(e^{-\tau_0})$, $\nabla \nabla a = O(e^{-\tau_0})$. Orthonormalizing $\hat{e}_i$ yields
\[
e_i = \hat{e}_i - \frac{1}{2} a_{ik} \hat{e}_k + o(e^{-\tau_0}).
\]
This provides a gauge transformation $A: SO(\bar{g}) \to SO(g)$
\[
e_i \mapsto e_i
\]
(and in addition $e_0 \mapsto e_0$) which identifies also the corresponding spinor bundles.

A new connection is introduced by $\nabla = \nabla \circ \nabla \circ A^{-1}$. Then we have

**Lemma 3.1.** (propositions 3.2, [25]) Let $(M, g, h)$ be a three-dimensional asymptotically AdS initial data set. Then
\[
\sum_{i,j \neq i} \Re \left\{ \phi, e_i \cdot e_j \cdot \left( V_j - \nabla_j \right) \phi \right\} = \frac{1}{4} \left( \nabla_{\bar{g}} g_{ij} - \nabla_i \nabla_j (g) + o(e^{-\tau_0}) \right) |\phi|^2,
\]
for all $\phi \in \Gamma(S)$.

We can extend the imaginary Killing spinors $\Phi_0$ in (2.4) on the end to the whole $M$ smoothly. Then corresponding to $g$ we may get the spinors $\bar{\Phi}_0 = A\Phi_0$. Let $\nabla_\chi = \nabla_\chi + \frac{i}{2} \kappa X_\chi$, then
\[
\nabla_\chi \bar{\Phi}_0 = \frac{-1}{4} \kappa a_{ik} e_k \cdot \bar{\Phi}_0 + o(e^{-\tau_0}) \bar{\Phi}_0.
\]

For any compact set $K \subset M$, denote $H^1(K, S)$ as the completion of smooth sections of $\mathbb{S}|_K$ with respect to the norm
\[
\| \phi \|^2_{H^1(K, S)} = \int_{K} \left( |\phi|^2 + |\nabla_\chi \phi|^2 \right) dV_\chi.
\]
And let $H^1(M, S)$ be the completion of compact supported smooth sections $C_0^\infty(S)$ with respect to the norm
\[
\| \phi \|^2_{H^1(M, S)} = \int_{M} \left( |\phi|^2 + |\nabla_\chi \phi|^2 \right) dV_\chi.
\]
Then $H^1(K, S)$ and $H^1(M, S)$ are Hilbert spaces. Furthermore, we have the following Poincaré inequality.
Lemma 3.2. There is a constant $C > 0$ such that
\[
\int_M |\phi|^2 dV_g \leq C \int_M \left| \nabla \phi \right|^2 dV_g, \tag{3.6}
\]
for all $\phi \in H^1(M, S)$.

Proof. As $C_0^\infty(S)$ is dense in $H^1(M, S)$, we just need to prove the inequality for $\phi \in C_0^\infty(S)$. The required inequality is then obtained by continuity.

Separate $M$ into two parts $K$ and $M \setminus K$, where $K$ is a compact set and $M \setminus K$ is the end. Then for any $\phi \in C_0^\infty(S)$,
\[
\int_K |\phi|^2 dV_g \leq C \left( \int_K \left| \nabla \phi \right|^2 dV_g + \int_{\partial(M \setminus K)} |\phi|^2 d\sigma \right), \tag{3.7}
\]
where $C$ is a positive constant. The inequality is obtained by contradiction as follows. Suppose it is not the case, then for any $n \in \mathbb{N}$, there exists $\phi_n \in H^1(K, S)$ with $\|\phi_n\|_{L^2(K, S)} = 1$ such that
\[
\int_K \left| \nabla \phi_n \right|^2 dV_g + \int_{\partial(M \setminus K)} |\phi_n|^2 d\sigma \leq \frac{1}{n}.
\]
Hence $\{\phi_n\}$ is bounded in $H^1(K, S)$, and there is a subsequence, still denoted by $\{\phi_n\}$, converging to some $\phi_\infty \in H^1(K, S)$ weakly in $H^1(K, S)$. Then
\[
\|\phi_\infty\|_{H^1(K, S)} \leq \liminf_{n \to \infty} \|\phi_n\|_{H^1(K, S)} = 1.
\]
By the Rellich theorem, there is a subsequence, still denoted by $\{\phi_n\}$, converging to $\phi_\infty$ strongly in $L^2(K, S)$, thus $\|\phi_\infty\|_{L^2(K, S)} = 1$. Therefore,
\[
\|\nabla \phi_\infty\|_{L^2(K, S)} = 0,
\]
and
\[
\hat{D} \phi_\infty = \nabla \phi_\infty = 0
\]
on $K$. Furthermore, $H^1(K, S)$ is compactly embedded into $L^2(\partial(M \setminus K), S)$ by Rellich theorem, then $\phi_\infty |_{\partial(M \setminus K)} = 0$. Extend $\phi_\infty$ to be zero outside $K$ along $\partial(M \setminus K)$, then $\phi_\infty \in H^1(M, S)$ and $\hat{D} \phi_\infty = 0$. Thus $\phi_\infty \in C^\infty(\text{int} K \cup \partial(M \setminus K))$ by the interior regularity of Dirac-type equation [4]. Then $\phi_\infty \equiv 0$ by the Weitzenböck formula (3.1) of $\hat{D}$ and the modified dominant energy condition (1.2), which contradicts $\|\phi_\infty\|_{L^2(K, S)} = 1$.

Under the asymptotical condition (2.4), the volume element $dV_g$ on $M_\infty = M \setminus K$ is equivalent to $d\mu = e^{2\kappa} r^2 d\Theta$, where $d\Theta$ is the unit volume element on $S^2(1)$. We just need to prove corresponding inequality for the new volume element.
\[
\int_{M_\infty} |\phi|^2 e^{2\kappa} r^2 d\Theta = \frac{1}{2\kappa} \int_{R^3 \setminus B(0, R_0)} |\phi|^2 d\left( e^{2\kappa} \right) d\Theta = \frac{1}{2\kappa} \int_{\partial B(0, R_0)} |\phi|^2 d\sigma - \frac{1}{2\kappa} \int_{R^3 \setminus B(0, R_0)} \partial_j |\phi|^2 d\mu.
\]
Note that
\[
\frac{1}{2\kappa} \int_{\mathbb{R}^4 \setminus B(0, R_0)} \partial_i |\phi|^2 d\mu
\leq \frac{1}{2\kappa} \int_{\mathbb{R}^4 \setminus B(0, R_0)} \left( 2 \left| \langle \nabla_c \phi, \phi \rangle \right| + |p| |\phi|^2 + \kappa |\phi|^2 + |E| |\phi|^2 + |B| |\phi|^2 \right) d\mu
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu + \frac{1}{\kappa} \int_{\mathbb{R}^4 \setminus B(0, R_0)} \left| \langle \nabla_c \phi, \phi \rangle \right| d\mu
\]
\[
+ \frac{C}{2\kappa} e^{-\tau R_0} \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu
\]
\[
\leq \left( \frac{1}{2} + \frac{e}{\kappa} + \frac{C}{2\kappa} e^{-\tau R_0} \right) \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu + \frac{4}{\epsilon \kappa} \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\nabla \phi|^2 d\mu,
\]
thus
\[
\int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu \leq -\frac{1}{2\kappa} \int_{\partial B(0, R_0)} |\phi|^2 d\sigma + \frac{4}{\epsilon \kappa} \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\nabla \phi|^2 d\mu
\]
\[
+ \left( \frac{1}{2} + \frac{e}{\kappa} + \frac{C}{2\kappa} e^{-\tau R_0} \right) \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu.
\]
Choosing \( \epsilon \) small enough and \( R_0 \) large enough gives
\[
\int_{\mathbb{R}^4 \setminus B(0, R_0)} |\phi|^2 d\mu \leq -C_1 \int_{\partial B(0, R_0)} |\phi|^2 d\sigma + C_2 \int_{\mathbb{R}^4 \setminus B(0, R_0)} |\nabla \phi|^2 d\mu,
\tag{3.8}
\]
for some constants \( C_1 > 0, C_2 > 0 \). Combining (3.7) and (3.8), we get the desired Poincaré inequality. Q.E.D.

Lemma 3.2 has the following useful corollary.

**Corollary 3.1.** \( \int_M |\nabla \phi|^2 dV_g \) is an equivalent norm for \( H^1(M, \mathcal{S}) \).

To prove the main result, we shall prove the following key existence and uniqueness theorem for the Dirac–Witten equation.

**Lemma 3.3.** Suppose \( (M, g, p, E, B) \) is an asymptotically \( \text{AdS} \text{ Einstein–Maxwell initial data set of order } \tau > \frac{3}{2} \), then there exists a unique spinor \( \Phi \in H^1(M, \mathcal{S}) \) such that
\[
\bar{D} \left( \Phi + \Phi_0 \right) = 0,
\]
where \( \Phi_0 \) is the imaginary Killing spinor defined by (A.1).

**Proof.** We follow the arguments of [21, 25] to prove the lemma. Set
\[
B(\phi, \psi) = \int_M \langle \bar{D} \phi, \bar{D} \psi \rangle,
\]
then corollary 3.1 and the modified dominant energy condition (1.2) imply that \( B(\cdot, \cdot) \) is a coercive bounded bilinear form on \( H^1(M, \mathcal{S}) \). Set
\[ F(\phi) = -\int_M \{ \widehat{D} \mathcal{F}_0, \bar{D} \phi \}. \]

By the definition of \( \widehat{\nabla} \), we have
\[
\widehat{\nabla} \mathcal{F}_0 = (V_i - V_j) \mathcal{F}_0 + \widehat{\nabla} \mathcal{F}_0 - \frac{1}{2} p_{ij} e_0 \cdot e_j \cdot \mathcal{F}_0 - \frac{1}{2} E \cdot e_i \cdot e_0 \cdot \mathcal{F}_0 - \frac{1}{4} \varepsilon_{ijkl} B_{j} e_k \cdot e_i \cdot \mathcal{F}_0.
\]

The expressions (A.1), (2.4), (3.4) and (3.5) imply that \( \widehat{D} \mathcal{F}_0 \in L^2(M, S) \) and then \( \bar{D} \mathcal{F}_0 \in L^2(M, S) \), thus \( F(\cdot) \) is a bounded linear functional on \( H^1(M, S) \).

By the Lax–Milgram theorem, there is a unique \( \Phi_1 \in H^1(M, S) \) satisfying
\[
\bar{D}^* \bar{D} \Phi_1 = -\bar{D}^* \bar{D} \mathcal{F}_0
\]
weakly. Set \( \Phi = \Phi_1 + \mathcal{F}_0, \Psi = \bar{D} \Phi \), then \( \Psi \in L^2(M, S) \) and \( \bar{D}^* \Psi = 0 \) weakly. By the elliptic regularity for Dirac-type equation [4], we have \( \Psi \in H^1(M, S) \) and \( \bar{D}^* \Psi = 0 \) in the classical sense. Then the Weitzenböck formula (3.1) and the modified dominant energy condition (1.2) imply that \( \widehat{\nabla} \Psi = 0 \). We thus have \( \| \Psi \|_2 \leq \kappa + |p| + |E| + |B| \) on the complement of the zero set of \( \Psi \) on \( M \). As a consequence, if there exist \( x_0 \in M \) such that \( \Psi(x_0) \neq 0 \), then
\[
|\Psi(x_0)|^2 \geq |\Psi(x_0)|^2 e^{(\kappa + |p| + |E| + |B|)(|x_0| - |x_1|)}.
\]

This implies that \( |\Psi(x)| \) is not in \( L^2(M, S) \), which is a contradiction. Thus \( \Psi \equiv 0 \), and \( \bar{D} (\Phi_1 + \mathcal{F}_0) = 0 \). Q.E.D.

### 4. Proof of the main theorem

In this section, we prove the main theorem of this paper. The proof of the theorem is largely similar to theorem 4.1 in [19] except for the electric and magnetic part. For completeness, we give some key details here.

**Proposition 4.1.** Let \( \phi = \mathcal{F}_0 + \Phi_1 \), where \( \mathcal{F}_0 \) is of the form (A.1) and \( \Phi_1 \) is the solution given by lemma 3.3, then
\[
\int_M \left| \widehat{\nabla} \phi \right|^2 + \int_M \langle \phi, \bar{R} \phi \rangle = 8\pi \left( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \right) Q \left( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \right)^T,
\]
in which the matrix
\[
Q = \begin{pmatrix} H & G \\ G^T & A \end{pmatrix},
\]
where
\[
H = \begin{pmatrix} E_0 - c_3 & c_1 - \sqrt{1 - c_2} \\ +b_1 - b_2 & +b_1 - \sqrt{1 - b_2} \\ c_1 + \sqrt{1 - c_2} & E_0 + c_3 \\ +b_1 + \sqrt{1 - b_2} & +b_0 + b_3 \end{pmatrix}.
\]
\[ \hat{H} = \begin{pmatrix} E_0 + c_3 & -c_1 + \sqrt{-1} c_2 \\ -b_0 - b_3 & +b_1 + \sqrt{-1} b_2 \\ -c_1 - \sqrt{-1} c_2 & E_0 - c_3 \\ +b_1 - \sqrt{-1} b_2 & -b_0 + b_3 \end{pmatrix}. \]

\[ G = \begin{pmatrix} c'_1 - c'_1 + J_2 \\ +\sqrt{-1} (q - J_3) + \sqrt{-1} (c'_2 + J_1) \\ -c'_1 - J_2 - c'_3 \\ +\sqrt{-1} (J_1 - c'_2) + \sqrt{-1} (J_3 + q) \end{pmatrix}. \]

**Proof.** By integrating the Weitzenböck formula (3.1) for \( \hat{D} \) with \( \phi = \Phi_0 + \Phi_1 \), we get

\[
\int_M \left| \nabla_\phi \right|^2 + \int_M \langle \phi, \hat{R} \phi \rangle = 1
\]

\[
= \lim_{r \to \infty} \text{Re} \int S_r \left\langle \phi, \sum_{j \neq i} e_i \cdot e_j \cdot \nabla_j \phi \right\rangle \div e^i
\]

\[
= \frac{1}{4} \lim_{r \to \infty} \int S_r \left( \nabla^2 \cdot g_{ij} - \nabla_i tr_g \right) \left( \Phi_0 \right)^2 \, \omega
\]

\[
+ \frac{1}{4} \lim_{r \to \infty} \int S_r \kappa \left( a_{k1} - g_{k1} tr_\xi \right) \left( \Phi_0, \sqrt{-1} \tilde{e}_k \cdot \Phi_0 \right) \, \omega
\]

\[
- \frac{1}{2} \lim_{r \to \infty} \int S_r \left( \tilde{e}_{k1} - g_{k1} tr_\xi \right) \left( \Phi_0, \tilde{e}_0 \cdot \tilde{e}_k \cdot \Phi_0 \right) \, \omega
\]

\[
+ \lim_{r \to \infty} \int S_r E^1 \left( \Phi_0, \tilde{e}_0 \cdot \Phi_0 \right) \, \omega
\]

\[
- \lim_{r \to \infty} \int S_r B_i \left( \Phi_0, \tilde{e}_1 \cdot \tilde{e}_2 \cdot \tilde{e}_3 \cdot \Phi_0 \right) \, \omega. \tag{4.2}
\]

By the Clifford representation (2.2) and the explicit form (A.1) of \( \Phi_0 \), the boundary term of (4.2) is equal to

\[
\frac{1}{2} \lim_{r \to \infty} \int S_r E_i \left( \tilde{u}^+ u^+ + \tilde{v}^+ v^+ \right) e^{e_i} \, \omega
\]

\[
+ \lim_{r \to \infty} \int S_r P_{2i} \left( \tilde{u}^+ v^+ + \tilde{v}^+ u^+ \right) e^{e_i} \, \omega
\]

\[
+ \sqrt{-1} \lim_{r \to \infty} \int S_r P_{3i} \left( \tilde{u}^+ v^+ - \tilde{v}^+ u^+ \right) e^{e_i} \, \omega
\]

\[
+ \sqrt{-1} \lim_{r \to \infty} \int S_r E^1 \left( \tilde{u}^+ u^+ - \tilde{u}^+ u^+ + \tilde{v}^+ v^+ \right) \, \omega
\]

\[
+ 2 \lim_{r \to \infty} \int S_r B_i \left( \tilde{u}^+ u^+ - \tilde{v}^+ v^+ \right) e^{e_i} \, \omega. \tag{4.3}
\]
Substituting the explicit form of $u^+$, $u^−$, $v^+$ and $v^−$ into (4.3), we have
\[
\overline{u^+}u^+ + \overline{v^+}v^+ = \frac{1}{2}(\lambda_1\lambda_1 + \lambda_2\lambda_2 + \lambda_3\lambda_3 + \lambda_4\lambda_4) \\
+ \frac{1}{2} \sin \theta \cos \psi (\lambda_1\lambda_2 + \lambda_2\lambda_1 - \lambda_3\lambda_4 - \lambda_4\lambda_3) \\
- \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \theta \sin \psi (\lambda_1\lambda_2 - \lambda_2\lambda_1 - \lambda_3\lambda_4 + \lambda_4\lambda_3) \\
- \frac{1}{2} \cos \theta(\lambda_1\lambda_1 - \lambda_2\lambda_2 - \lambda_3\lambda_3 + \lambda_4\lambda_4),
\]
(4.4)
\[
\overline{u^+}v^+ + \overline{v^+}u^+ = -\frac{1}{2} \sin \theta (\lambda_1\lambda_3 - \lambda_2\lambda_4 + \lambda_3\lambda_1 - \lambda_4\lambda_2) \\
+ \frac{1}{2} \cos \psi (\lambda_1\lambda_4 - \lambda_2\lambda_3 - \lambda_3\lambda_2 + \lambda_4\lambda_1) \\
- \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \psi (\lambda_1\lambda_4 + \lambda_2\lambda_3 - \lambda_3\lambda_2 - \lambda_4\lambda_1) \\
- \frac{1}{2} \cos \psi \cos \theta (\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_3\lambda_2 + \lambda_4\lambda_1) \\
+ \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \psi \cos \theta (\lambda_1\lambda_4 - \lambda_2\lambda_3 - \lambda_3\lambda_2 - \lambda_4\lambda_1),
\]
(4.5)
\[
\overline{u^+}v^+ - \overline{v^+}u^+ = -\frac{1}{2} \sin \theta (\lambda_1\lambda_3 - \lambda_2\lambda_4 - \lambda_3\lambda_1 + \lambda_4\lambda_2) \\
+ \frac{1}{2} \cos \psi (\lambda_1\lambda_4 - \lambda_2\lambda_3 + \lambda_3\lambda_2 - \lambda_4\lambda_1) \\
- \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \psi (\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_3\lambda_2 + \lambda_4\lambda_1) \\
- \frac{1}{2} \cos \psi \cos \theta (\lambda_1\lambda_4 + \lambda_2\lambda_3 - \lambda_3\lambda_2 + \lambda_4\lambda_1) \\
+ \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \psi \cos \theta (\lambda_1\lambda_4 - \lambda_2\lambda_3 - \lambda_3\lambda_2 + \lambda_4\lambda_1),
\]
(4.6)
\[
\overline{u^+}u^+ - \overline{u^+}u^+ - \overline{v^+}v^+ + \overline{v^+}v^+ = \lambda_1\lambda_3 + \lambda_2\lambda_4 - \lambda_3\lambda_1 - \lambda_4\lambda_2,
\]
(4.7)
\[
\overline{u^+}u^+ - \overline{v^+}v^+ = \frac{1}{2}(\lambda_1\lambda_1 + \lambda_2\lambda_2 - \lambda_3\lambda_3 - \lambda_4\lambda_4) \\
+ \frac{1}{2} \sin \theta \cos \psi (\lambda_1\lambda_2 + \lambda_2\lambda_1 + \lambda_3\lambda_4 + \lambda_4\lambda_3) \\
- \frac{\sqrt{1-\sin^2 \theta}}{2} \sin \theta \sin \psi (\lambda_1\lambda_2 - \lambda_2\lambda_1 + \lambda_3\lambda_4 - \lambda_4\lambda_3) \\
- \frac{1}{2} \cos \theta(\lambda_1\lambda_1 - \lambda_2\lambda_2 + \lambda_3\lambda_3 - \lambda_4\lambda_4).
\]
(4.8)

Combining (4.2)–(4.8), we obtain (4.1). Q.E.D.

**Remark 4.1.** By the dominant energy condition (1.2), the term $\langle \psi, \hat{R}\psi \rangle$ in (4.1) is non-negative, which implies that the Hermitian matrix $Q$ is non-negative.
Proof of theorem 1.1. The non-negativity of the first-order principal minors ensures \( E_0 \geq 0 \).
The sum of the second-order principal minors yields
\[
E_0^2 \geq L^2/3.
\]

Therefore
\[
V^3 \leq V^2L/\sqrt{3} \leq V^2E_0.
\]
The sum of the third-order principal minors yields
\[
0 \leq E_0 \left( E_0^2 - L^2 \right) + 2V^3.
\]
If \( E_0 = 0 \), (1.3) becomes trivial. If \( E_0 > 0 \), one may derive
\[
E_0^2 \geq L^2 - 2V^2.
\]
The inequality above also implies
\[
E_0^2 \geq L^2 - 2V^2 \geq L^2 - 2L^2/3 = L^2/3.
\]
Now we use the non-negativity of the determinant of \( Q \) to prove (1.3).
\[
\det Q = \left( E_0^2 - L^2 \right)^2 + 8E_0V^3 - 4A^4.
\]
Since
\[
2E_0V^3 \leq \left( E_0^2 + V^2 \right) V^2,
\]
we obtain
\[
0 \leq \det Q \leq \left( E_0^2 - L^2 + 2V^2 \right)^2 - 4 \left( A^4 - L^2 V^2 \right).
\]
The inequality (1.3) follows immediately. Q.E.D.

Remark 4.2. Under the condition that the spacetime is stationary and antisymmetric, Kostelecky and Perry [11] obtained similar inequality for the total energy.

Remark 4.3. If \( b_0 = b_1 = q = 0 \), the inequality (1.3) is just (4.2) in theorem 4.1 of [19]. If \( \kappa = 0 \), the inequality (1.3) is reduced to Gibbons and Hull’s inequality [9].

Remark 4.4. Suppose \( M \) has inner boundary \( \Sigma = \Sigma^+ \cup \cup \Sigma^- \), where \( \Sigma^+ \) and \( \Sigma^- \) are future and past trapped surfaces, defined as
\[
\Sigma^+ = \left\{ \text{tr} (h) - \text{tr} \left( p_{\Sigma^+} \right) \geq 0 \right\},
\]
\[
\Sigma^- = \left\{ \text{tr} (h) + \text{tr} \left( p_{\Sigma^-} \right) \geq 0 \right\}.
\]
If we take \( e_3 \) as the outer unit normal of \( \Sigma \) in \( M \) and take the boundary condition \( e_0 \cdot e_3 \cdot \phi = \pm \phi \) on \( \Sigma^\pm \), then
\[
\int_\Sigma \left\{ \phi, e_1 \cdot \hat{\nabla} \phi + \hat{V}_e \phi \right\}
\]
is non-positive. In such situation, lemmas 3.2 and 3.3 are all valid. Similar arguments appear in [22]. This verifies theorem 1.1 for black holes.

**Remark 4.5.** If \( E_0 = 0 \), there are four linearly independent spinors satisfying \( \hat{V} \phi = 0 \). The characterization of the manifold \( M \) in such case will be addressed elsewhere.

## 5. Kerr–Newman–AdS case

In this section, we will calculate our definitions for time slices in the Kerr–Newman–AdS spacetime [5].

For Kerr–Newman–AdS spacetime, the metric in the Boyer–Lindquist coordinates \((\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})\) is

\[
\tilde{g} = \frac{\Delta_\rho}{\rho^2} \left[ d\hat{t} - \frac{a \sin^2 \hat{\theta}}{\Sigma} d\hat{\phi} \right]^2 + \frac{\rho^2}{\Delta_\rho} d\hat{r}^2 + \frac{\rho^2}{\Delta_\rho} d\hat{\theta}^2 + \frac{\Delta_\rho \sin^2 \hat{\theta}}{\rho^2} \left[ a d\hat{t} - \frac{\hat{r}^2 + a^2}{\Sigma} d\hat{\phi} \right]^2,
\]

where

\[
\Delta_\rho = \left( \hat{r}^2 + a^2 \right) \left( 1 + \kappa^2 \hat{r}^2 \right) - 2m \hat{r} + e^2,
\]

\[
\Delta_\rho = 1 - \kappa^2 a^2 \cos^2 \hat{\theta},
\]

\[
\rho^2 = \hat{r}^2 + a^2 \cos^2 \hat{\theta},
\]

\[
\Sigma = 1 - \kappa^2 a^2.
\]

If we take

\[
e^0 = \frac{\sqrt{\Delta_\rho}}{\rho} \left( d\hat{t} - \frac{a \sin^2 \hat{\theta}}{\Sigma} d\hat{\phi} \right), \quad e^1 = \frac{\rho}{\sqrt{\Delta_\rho}} d\hat{r},
\]

\[
e^2 = \frac{\rho}{\sqrt{\Delta_\rho}} d\hat{\theta}, \quad e^3 = \frac{1}{\rho} \sqrt{\Delta_\rho} \sin \hat{\theta} \left( a d\hat{t} - \frac{\hat{r}^2 + a^2}{\Sigma} d\hat{\phi} \right),
\]

then the field strength tensor is

\[
F = -\frac{1}{\rho^2} e \left( \hat{r}^2 - a^2 \cos^2 \hat{\theta} \right) e^0 \wedge e^1 - \frac{2}{\rho^4} e^0 a \cos \hat{\theta} e^2 \wedge e^3.
\]

Similar to the process in [10], after the coordinate transformations

\[
t = \hat{t}, \quad \varphi = \hat{\phi} + \kappa \hat{r}, \quad \sinh (\kappa r) \cos \theta = \kappa \hat{r} \cos \hat{\theta},
\]

\[
\sqrt{\Sigma} \sinh (\kappa r) \sin \theta = \kappa \sqrt{\hat{r}^2 + a^2} \sin \hat{\theta},
\]

the Kerr–Newman–AdS metric can be written as

\[
\tilde{g} = -\cosh^2 (\kappa r) dt^2 + dr^2 + \frac{\sinh^2 (\kappa r)}{\kappa^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + a_{\mu\nu} dx^\mu dx^\nu,
\]
where the nonzero components $a_{\mu\nu}$ have the following asymptotic behaviors

\[
a_{rr} = \frac{2mk}{\sinh (kr)} \Theta^{-5/2} + O\left(e^{-3kr}\right),
\]

\[
a_{\theta\phi} = -\frac{2ma^2k}{\sinh (kr)} \sin^2 \theta \Theta^{-5/2} + O\left(e^{-3kr}\right),
\]

\[
a_{\Phi\Phi} = \frac{2ma^2k^3}{\sinh (kr)} \sin^4 \theta \Theta^{-5/2} + O\left(e^{-3kr}\right),
\]

\[
a_{\tau\tau} = \frac{2mk}{\sinh (kr)} \cosh (kr)^2 \Theta^{-3/2} + O\left(e^{-7kr}\right),
\]

\[
a_{r\phi} = -\frac{2ma^2k^2}{\sinh (kr)^4} \cosh (kr) \sin \theta \cos \theta \Theta^{-5/2} + O\left(e^{-6kr}\right),
\]

\[
a_{\theta\theta} = \frac{2ma^3k^3}{\sinh (kr)^3} \sin^2 \theta \cos^2 \theta \Theta^{-7/2} + O\left(e^{-5kr}\right),
\]

\[
\Theta = 1 - a^2 k^2 \sin^2 \theta.
\]

Simple calculations show that for $t$-slices, the quantities with the order not higher than $e^{-3kr}$ are

\[
a_{11} = 16ma^3 \Theta^{-3/2} e^{-3kr} + o\left(e^{-3kr}\right),
\]

\[
a_{33} = 16ma^3 \Theta^{-3/2} \sin^2 \theta e^{-3kr} + o\left(e^{-3kr}\right),
\]

\[
p_{13} = p_{31} = 24ma^3 \Theta^{-5/2} \sin \theta e^{-3kr} + o\left(e^{-3kr}\right),
\]

and

\[
P_{31} = 24ma^3 \Theta^{-5/2} \sin \theta e^{-3kr} + o\left(e^{-3kr}\right).
\]

\[
E_1 = 2\kappa a_{11} + \partial_{\tau}a_{33}.
\]

We also have

\[
E^1 = 4k^2 e \Theta^{-3/2} e^{-2kr} + o\left(e^{-2kr}\right),
\]

\[
B^1 = 16k^2 e a \Theta^{-5/2} e^{-3kr} + o\left(e^{-3kr}\right).
\]

Finally we get for Kerr–Newman–AdS spacetime

\[
E_0 = \frac{m}{\Sigma^2}, \quad J_1 = \frac{mka}{\Sigma^2}, \quad q = \frac{e}{\Sigma}, \quad b_3 = \frac{4kae}{3\Sigma},
\]

\[
J_i = J_i = b_0 = b_1 = b_2 = c_i = c_i' = 0, \quad i = 1, 2, 3.
\]

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Appendix

The imaginary Killing spinors along the 0-slice of AdS spacetime are of the form

\[
\Phi_0 = \begin{pmatrix}
  u^+ e^{i\frac{\psi}{2}} + u^- e^{-i\frac{\psi}{2}} \\
  v^+ e^{i\frac{\psi}{2}} + v^- e^{-i\frac{\psi}{2}} \\
  -\sqrt{-1} u^+ e^{i\frac{\psi}{2}} + \sqrt{-1} u^- e^{-i\frac{\psi}{2}} \\
  \sqrt{-1} v^+ e^{i\frac{\psi}{2}} - \sqrt{-1} v^- e^{-i\frac{\psi}{2}}
\end{pmatrix},
\]

(A.1)

where

\[
\begin{align*}
  u^+ &= \lambda_1 e^{-\frac{i\psi}{2}} \sin \frac{\theta}{2} + \lambda_2 e^{-\frac{i\psi}{2}} \cos \frac{\theta}{2}, \\
  u^- &= \lambda_3 e^{-\frac{i\psi}{2}} \sin \frac{\theta}{2} + \lambda_4 e^{-\frac{i\psi}{2}} \cos \frac{\theta}{2}, \\
  v^+ &= -\lambda_3 e^{-\frac{i\psi}{2}} \cos \frac{\theta}{2} + \lambda_4 e^{-\frac{i\psi}{2}} \sin \frac{\theta}{2}, \\
  v^- &= -\lambda_1 e^{-\frac{i\psi}{2}} \cos \frac{\theta}{2} + \lambda_2 e^{-\frac{i\psi}{2}} \sin \frac{\theta}{2}.
\end{align*}
\]

Here \(\lambda_1, \lambda_2, \lambda_3,\) and \(\lambda_4\) are arbitrary complex numbers.

The Killing vectors of AdS spacetime are

\[
\begin{align*}
  U_{40} &= \kappa^{-1} \frac{\partial}{\partial t}, \\
  U_{10} &= \kappa^{-1} \sin \theta \cos \psi \frac{\partial}{\partial r} + \coth (kr) \left( \cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi \frac{\partial}{\partial \theta}}{\sin \theta \frac{\partial}{\partial \varphi}} \right), \\
  U_{20} &= \kappa^{-1} \sin \theta \sin \psi \frac{\partial}{\partial r} + \coth (kr) \left( \cos \theta \sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi \frac{\partial}{\partial \theta}}{\sin \theta \frac{\partial}{\partial \varphi}} \right), \\
  U_{30} &= \kappa^{-1} \cos \theta \frac{\partial}{\partial r} - \coth (kr) \sin \theta \frac{\partial}{\partial \theta}, \\
  U_{14} &= \kappa^{-1} \tanh (kr) \sin \theta \cos \psi \frac{\partial}{\partial t}, \\
  U_{24} &= \kappa^{-1} \tanh (kr) \sin \theta \sin \psi \frac{\partial}{\partial t}, \\
  U_{34} &= \kappa^{-1} \tanh (kr) \cos \psi \frac{\partial}{\partial t}, \\
  U_{23} &= -\sin \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi \frac{\partial}{\partial \varphi}}{\sin \theta}, \\
  U_{31} &= \cos \psi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \psi \frac{\partial}{\partial \varphi}}{\sin \theta}, \\
  U_{12} &= \frac{\partial}{\partial \varphi}.
\end{align*}
\]

We set \(V_1 = U_{23}, V_2 = U_{31}\) and \(V_3 = U_{12}\) for convenience.
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