Randomness, Integrability and Universality  
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Stationary half-space last passage percolation

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http://wt.iam.uni-bonn.de/ferrari
KPZ stationary models in full-space
TASEP: Totally Asymmetric Simple Exclusion Process

Configurations

\[ \eta = \{ \eta_x \}_{x \in \mathbb{Z}}, \eta_x = \begin{cases} 
1, & \text{if } x \text{ is occupied}, \\
0, & \text{if } x \text{ is empty}.
\end{cases} \]

Dynamics

Independently, particles jump on the right site with rate 1, provided the right is empty.

⇒ Particles are ordered: position of particle \( n \) is \( x_n(t) \) with \( x_n(t) > x_{n+1}(t) \) for all \( n, t \).
Consider independent random variables \( \{\omega_{i,j}\}_{(i,j)\in\mathbb{Z}^2} \) with \( \omega_{i,j} \sim \text{Exp}(1) \).
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The line-to-point LPP from a line \( \mathcal{L} \) to the point \((m, n)\) is given by

\[
L_{m,n} = \max_{\pi: \mathcal{L} \to (m,n)} \sum_{(i,j) \in \pi} \omega_{i,j}
\]

where the maximum is over up-right paths from \( \mathcal{L} \) to \((m, n)\), i.e. paths with increments in \( \{(0, 1), (1, 0)\} \).
The well-known connection between TASEP and LPP is

$$\mathbb{P}(L_{m,n} \leq t) = \mathbb{P}(x_n(t) \geq m - n).$$

where $\mathcal{L} = \{(x_k(0) + k, k), k \in \mathbb{Z} \text{ or } \mathbb{N}\}$. 

Example: Step-initial condition $x_k(0) = -k + 1$, $k \geq 1$, $L = \{(1, k), k \geq 1\}$, equivalent to reduce $L$ to one point.
The well-known connection between TASEP and LPP is

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Example: Step-initial condition \( x_k(0) = -k + 1, k \geq 1, \)
\( L = \{(1, k), k \geq 1\} \), equivalent to reduce \( L \) to one point.
• **Point-to-point LPP:**

\[
\omega_{i,j} = \begin{cases} 
\text{Exp}(1), & i, j \geq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

• **Stationary LPP:** fix \( \alpha \in (-1/2, 1/2) \)

\[
\omega_{i,j} = \begin{cases} 
\text{Exp} \left( \frac{1}{2} + \alpha \right), & i = 0, j \geq 1, \\
\text{Exp} \left( \frac{1}{2} - \alpha \right), & j = 0, i \geq 1, \\
0, & \text{if } i = j = 0, \\
\text{Exp}(1), & \text{otherwise}.
\end{cases}
\]
Some asymptotic results

- **Point-to-point LPP:** GUE Tracy-Widom distribution

\[
\lim_{t \to \infty} \mathbb{P}(L_{N,N} \leq 4N + s2^{4/3}N^{1/3}) = F_{\text{GUE}}(s)
\]

with

\[ F_{\text{GUE}}(s) = \det(1 - K_{\text{Ai}})_{L^2(s,\infty)} \]

with \( K_{\text{Ai}}(x,y) = \int_{\mathbb{R}^+} d\lambda \text{Ai}(x + \lambda)\text{Ai}(y + \lambda) \) is the Airy kernel.

- **Stationary initial condition:** (stated for \( \alpha = 0 \))

Baik-Rains distribution

\[
\lim_{t \to \infty} \mathbb{P}(L_{N+N^2/3,N-N^2/3} \leq 4N + s2^{4/3}N^{1/3}) = F_{\text{BR},w}(s),
\]

with \( F_{\text{BR},w}(s) = \frac{d}{ds}[F_{\text{GUE}}(s + w^2)g(s,w)] \).

- \( w \) measures the distance from the characteristic line.
The Baik-Rains distribution function is

\[ F_{\text{BR},w}(s) = \frac{d}{ds} [F_{\text{GUE}}(s + w^2)g(s, w)]. \]

Let \( \hat{K}_{\text{Ai}}(x, y) = K_{\text{Ai}}(x + w^2, y + w^2) \), and

\[
\mathcal{R} = s + e^{-\frac{2}{3}w^3} \int_{s}^{\infty} dx \int_{0}^{\infty} dy \text{Ai}(x + y + w^2) e^{-w(x+y)},
\]

\[
\Psi(y) = e^{\frac{2}{3}w^3+wy} - \int_{0}^{\infty} dx \text{Ai}(x + y + w^2) e^{-wx},
\]

\[
\Phi(x) = e^{-\frac{2}{3}w^3} \int_{0}^{\infty} d\lambda \int_{s}^{\infty} dy \text{Ai}(x + w^2 + \lambda) \text{Ai}(y + w^2 + \lambda) e^{-wy} - \int_{0}^{\infty} dy \text{Ai}(y + x + w^2) e^{wy}.
\]

Let \( P_s \) be the projection operator \( P_s(x) = \mathbb{1}_{\{x>s\}} \), then the function \( g \) is given by

\[
g(w, s) = \mathcal{R} - \langle (1 - P_s \hat{K}_{\text{Ai}} P_s)^{-1} P_s \Phi, P_s \Psi \rangle.
\]
Step 1: An integrable model with a random shift $\tau$.

For $\alpha, \beta \in (-1/2, 1/2]$ with $\alpha + \beta > 0$:

$$
\omega_{i,j} = \begin{cases} 
\text{Exp} \left( \frac{1}{2} + \alpha \right) & i = 0, j \geq 1, \\
\text{Exp} \left( \frac{1}{2} + \beta \right) & j = 0, i \geq 1, \\
\tau = \text{Exp} \left( \alpha + \beta \right) & \text{if } i = j = 0, \\
\text{Exp}(1) & \text{otherwise}.
\end{cases}
$$

Using a Schur process:

$$
P(L_{\tau m,n} \leq s) = \det(1 - K_{\alpha,\beta})L^2(s,\infty).
$$

$$
L_{\text{stat}}^{m,n} = \lim_{\beta \to -\alpha} (L_{\tau m,n} - \tau)
$$

Step 2: Shift argument.

$$
P(L_{\tau m,n} - \tau \leq s) = \left( 1 + \frac{1}{\alpha + \beta} \frac{d}{ds} \right) P(L_{\tau m,n} \leq s).
$$
Step 3: $K_{\alpha,\beta}$ is a rank-one perturbation:

$$K_{\alpha,\beta}(x, y) = \overline{K}(x, y) + (\alpha + \beta) f_\alpha(x) g_\beta(y)$$

gives

$$\det(\mathbb{1} - K_{\alpha,\beta}) = \det(\mathbb{1} - \overline{K})[1 - (\alpha + \beta)\langle(\mathbb{1} - \overline{K})^{-1} f_\alpha, g_\beta\rangle].$$

Thus

$$\mathbb{P}(L_{m,n}^{\text{stat}} \leq s) = \lim_{\beta \to -\alpha} \frac{d}{ds} \left[ \det(\mathbb{1} - \overline{K}) \left( \frac{1}{\alpha + \beta} - \langle(\mathbb{1} - \overline{K})^{-1} f_\alpha, g_\beta\rangle \right) \right].$$

Step 4: Analytic continuation for $\alpha, \beta \in (-1/2, 1/2)$.

$$\frac{1}{\alpha + \beta} - \langle(\mathbb{1} - \overline{K})^{-1} f_\alpha, g_\beta\rangle = \left[ \frac{1}{\alpha + \beta} - \langle f_\alpha, g_\beta \rangle \right] - \langle(\mathbb{1} - \overline{K})^{-1} \overline{K} f_\alpha, g_\beta\rangle.$$

Step 5: Large time limit:

- $\overline{K}$ converges to $\hat{K}_{\text{Ai}},$
- the term $\lim_{\beta \to -\alpha} \frac{1}{\alpha + \beta} - \langle f_\alpha, g_\beta \rangle$ converges to $\mathcal{R},$
- $\overline{K} f_\alpha$ and $g_{-\alpha}$ converge to $\Phi$ and $\Psi.$
Further stationary KPZ models

**Determinantal systems: one-point distribution**
- Polynuclear growth model  \( \text{Baik,Rains’00,Imamura,Sasamoto’04} \)
- TASEP / last passage percolation  \( \text{Ferrari,Spohn’05} \)

**Determinantal systems: multi-point distributions**
- TASEP  \( \text{Baik,Ferrari,Péché’09} \)
- One-sided reflecting Brownian motion (low density limit of TASEP)  \( \text{Ferrari,Spohn,Weiss’15} \)

**Integrable but not determinantal models (only one-point distribution)**
- KPZ equation  \( \text{Imamura,Sasamoto’13} \)
- ASEP and stochastic six-vertex model  \( \text{Borodin,Corwin,Ferrari,Veto’14} \)
- \( q \)-TASEP and Semi-discrete directed polymer  \( \text{Imamura,Sasamoto’17} \)
Half-space stationary models
Fix $\alpha \in (-1/2, 1/2)$ and consider independent random variables $\{\omega_{i,j}\}_{(i,j) \in D}$, $D = \{(i, j) \in \mathbb{N}^2 | 1 \leq j \leq i\}$ and

$$\omega_{i,j} = \begin{cases} 
\text{Exp} \left( \frac{1}{2} + \alpha \right) & i = j \geq 1, \\
\text{Exp} \left( \frac{1}{2} - \alpha \right) & j = 0, i \geq 1, \\
0 & \text{if } i = j = 0, \\
\text{Exp}(1) & \text{otherwise}.
\end{cases}$$
Half-space LPP

- Fix $\alpha \in (-1/2, 1/2)$ and consider independent random variables $\{\omega_{i,j}\}_{(i,j) \in \mathcal{D}}$, $\mathcal{D} = \{(i, j) \in \mathbb{N}^2 | 1 \leq j \leq i\}$ and

$$
\omega_{i,j} = \begin{cases} 
\text{Exp}(\frac{1}{2} + \alpha) & i = j \geq 1, \\
\text{Exp}(\frac{1}{2} - \alpha) & j = 0, i \geq 1, \\
0 & \text{if } i = j = 0, \\
\text{Exp}(1) & \text{otherwise}.
\end{cases}
$$

- A stationary half-space LPP time to the point $(m, n)$ (for $n \leq m$), denoted $L_{m,n}^{\text{stat}}$, is given by

$$
L_{m,n}^{\text{stat}} = \max_{\pi:(0,0) \rightarrow (m,n)} \sum_{(i,j) \in \pi} \omega_{i,j}
$$

where the maximum is over up-right paths in $\mathcal{D}$ from $(1, 1)$ to $(m, n)$, i.e. paths with increments in $\{(0, 1), (1, 0)\}$.

- For TASEP, the boundary random variables are the injection waiting times at the origin.
Why is this model called stationary?

Increments $\{L_{m+1,n}^{\text{stat}} - L_{m,n}^{\text{stat}}, m \geq n\}$ are iid. $\text{Exp}(\frac{1}{2} - \alpha)$.

Also $\{L_{m,n}^{\text{stat}} - L_{m,n-1}^{\text{stat}}, m \geq n\}$ are iid. $\text{Exp}(\frac{1}{2} + \alpha)$

Balázs, Cator, Seppäläinen’06
Case \( \alpha < 0 \): large diagonal weights

Characteristic lines have slopes \(((\frac{1}{2} + \alpha)/(\frac{1}{2} - \alpha))^2 < 1

End-point on characteristics from \((0, 0)\): diagonal visited only \(O(N^{2/3})\) around the origin: like full-space

End-point \((N, N)\): maximizer visits \(O(N)\) times the diagonal: Gaussian fluctuations

\[ Q = N(1, (\frac{1}{2} + \alpha)^2/(\frac{1}{2} - \alpha)^2) \]
Case $\alpha > 0$: small diagonal weights

Characteristic lines have slopes \( \left( \frac{\frac{1}{2} + \alpha}{\frac{1}{2} - \alpha} \right)^2 > 1 \)

End-point \((N, N)\): maximizer visits $O(N)$ times the first row: Gaussian fluctuations in $N^{1/2}$ scale
Critical scaling:

\[ \alpha = \delta 2^{-4/3} N^{-1/3} \]

and end-point \((N, N - \eta N)\) with

\[ \eta = u 2^{5/3} N^{-1/3}. \]

Law of large number gives:

\[ L_{N,N-\eta N}^{\text{stat}} \approx 4N - 4u(2N)^{2/3} + \delta (2u + \delta) 2^{4/3} N^{1/3}. \]
Theorem

Let $\delta \in \mathbb{R}$, $u > 0$ be fixed. Set

$$\alpha = \delta 2^{-4/3} N^{-1/3}, \quad \eta N = u 2^{5/3} N^{2/3}.$$

Then

$$\lim_{N \to \infty} \mathbb{P} \left( \frac{L_{N,N}^{\text{stat}} - (4N - 4u(2N)^{2/3})}{2^{4/3} N^{1/3}} \leq S \right) = F_{u,\delta}(S),$$

where $F_{u,\delta}(S) = \frac{d}{dS} \{ \text{Pf}(J - \overline{A}) G_{\delta,u}(S) \}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$G_{\delta,u}(S) = e^{\delta,u}(S) - \left\langle -g_{1}^{\delta,u} \ g_{2}^{\delta,u} \left| (1 - J^{-1} \overline{A})^{-1} \begin{pmatrix} -h_{1}^{\delta,u} \\ h_{2}^{\delta,u} \end{pmatrix} \right. \right\rangle.$$
The $2 \times 2$ matrix kernel $\bar{A}$ is the one arising from the model with $\text{Exp}(1)$ also for $j = 0$, instead of $\text{Exp}(\frac{1}{2} - \alpha)$.

Away from the diagonal: \text{Imamura,Sasamoto’04}

General and rigorous case: Baik,Barraquand,Corwin,Suidan’18

For moment computations the derivative is not a problem: denote $F_{u,\delta}(S') = \frac{d}{dS} T(S')$ and $\xi \sim F_{u,\delta}$, then:

- by stationarity: $\mathbb{E}(\xi) = \delta (2u + \delta)$,
- integrating by parts gives

$$
\mathbb{E}(\xi^\ell) = \ell(\ell-1) \int_{\mathbb{R}^+} dSS^{\ell-2}(T(S) - S) + \ell(\ell-1) \int_{\mathbb{R}^-} dSS^{\ell-2}T(S).
$$
The inverse of the operator is not a numerical issue either:

\[
Pf (J - K) \langle c \ d \mid (\mathbb{I} - J^{-1}K)^{-1} \left( \begin{array}{c} a \\ b \end{array} \right) \rangle
\]

\[= Pf (J - K) - Pf \left( J - K \mid \begin{array}{c} b \\ -a \end{array} \right) \langle c \ d \mid \begin{array}{c} c \\ d \end{array} \rangle \langle -b \ a \mid \right).
\]

Then use Bornemann’s method to evaluate the Fredholm determinants (Pfaffians) \cite{Bornemann'08}.
Step 1: An integrable model. Consider the model

\[ \exp\left(1 + \alpha\right) \exp\left(1 + \beta\right) \exp\left(\alpha + \beta\right) \]

The process \( L_{N,1}, L_{N,2}, \ldots, L_{N,N} \) is the marginal of a Pfaffian Schur process.  

Baik, Barraquand, Corwin, Suidan’18
For $\alpha + \beta > 0$ and $\beta > 0$ we a Fredholm Pfaffian expression on $(s, \infty)$

$$\mathbb{P}(L_{N,N-n} \leq s) = \text{Pf}(J - K)$$

with

$$K_{11}(x, y) = -\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{\Phi(x, z)}{\Phi(y, w)} \left[ \frac{1}{2} - z \right] \left[ \frac{1}{2} + w \right] \frac{(z + \beta)(w - \beta)}{(z - \beta)(w + \beta)} \frac{(z + \alpha)(w - \alpha)(z + w)}{4zw(z - w)},$$

$$K_{12}(x, y) = -\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{\Phi(x, z)}{\Phi(y, w)} \left[ \frac{1}{2} - z \right] \left[ \frac{1}{2} + w \right] \frac{z + \alpha}{w + \alpha} \frac{z + \beta}{w + \beta} \frac{w - \beta}{z - \beta} \frac{2z(z - w)}{z + w},$$

$$= -K_{21}(y, x),$$

$$K_{22}(x, y) = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{\Phi(x, z)}{\Phi(y, w)} \left[ \frac{1}{2} + z \right] \left[ \frac{1}{2} - w \right] \frac{1}{(z - \alpha)(w + \alpha)} \frac{1}{z - \beta} \frac{w + \beta}{z - w} + \epsilon(x, y),$$

with $\Phi(x, z) = e^{-xz} \left[ \left( \frac{1}{2} + z \right) / \left( \frac{1}{2} - z \right) \right]^{N-1}$ and

$$\epsilon(x, y) = -\text{sgn}(x - y) \oint \frac{dz}{2\pi i} \frac{2ze^{-z|x-y|}}{(z^2 - \alpha^2) \left( \frac{1}{4} - z^2 \right)^n}.$$

\[\text{Gamma}_{1/2, \alpha}\]
Step 2: Shift argument. We want to get the limit of $\beta = -\alpha$ conditioned on $\omega_{0,0} = 0$.

- For $\alpha + \beta > 0$, we have

$$
\mathbb{P}(L_{N,N-n} \leq s | \omega_{0,0} = 0) = \left(1 + \frac{1}{\alpha + \beta} \frac{d}{ds}\right) \mathbb{P}(L_{N,N-n} \leq s).
$$
Step 3: Rank one decomposition.

- By deforming contours such that the expressions are analytic at $\alpha + \beta = 0$ we get

$$K = \overline{K} + (\alpha + \beta)R$$

with $R$ of the form

$$R = \begin{pmatrix}
|g_1\rangle \langle f^\beta| & -|f^\beta\rangle \langle g_1| & |f^\beta\rangle \langle g_2| \\
-g_2 \langle f^\beta| & 0 & 0
\end{pmatrix}$$

with $f^\beta(x) \sim e^{-\beta x}$.

- Thus we have

$$\mathbb{P}(L_{N,N-n}^{\text{stat}} \leq s) = \lim_{\beta \to -\alpha} \frac{d}{ds} \left[ \text{Pf}(J - \overline{K}) \left( \frac{1}{\alpha + \beta} - \langle Y | (1 - \overline{G})^{-1} X \rangle \right) \right]$$

with $X = \begin{pmatrix} 0 \\ f^\beta \end{pmatrix}$ and $Y = \langle -g_1 \ g_2 |$ and $\overline{G} = J^{-1} \overline{K}$. 
Step 4: Analytic continuation.

- Let $G = J^{-1}K$, then the idea is to use

$$\frac{1}{\alpha + \beta} \langle Y | (\mathbb{1} - \overline{G})^{-1}X \rangle = \frac{1}{\alpha + \beta} \langle Y | X \rangle - \langle Y | (\mathbb{1} - \overline{G})^{-1}G X \rangle$$

- Problem: $\langle Y | \overline{G} X \rangle$ is a sum of 4 terms, some of which diverge for $\beta \leq 0$, due to the $f^{\beta}$ term. A term-by-term limit $\beta \to -\alpha$ for $\alpha \geq 0$ is not possible.

- Solution: The diverging terms exactly cancels for any $\beta > 0$, namely we show that

$$\langle Y | (\mathbb{1} - \overline{G})^{-1}G X \rangle = \langle Y | (\mathbb{1} - \overline{G})^{-1}\tilde{G} X \rangle$$

where $\tilde{G}$ is without the problematic terms. The result is then analytic on $(\alpha, \beta) \in (-1/2, 1/2)^2$.

Step 5: Large time asymptotics. Standard steep descent method.
Full-space vs. half-space stationary models
| Full-space                                      | Half-space                             |
|------------------------------------------------|----------------------------------------|
| One-parameter family                           | Two-parameter family                   |
| Determinantal structure                        | Pfaffian structure                     |
| Simple analytic continuation                    | Tricky analytic continuation            |

Is the full-space distribution a limit of half-space one?
Taking $\delta \to -\infty$, the characteristic line has direction far away from the diagonal. Thus the maximizer of the LPP will touch less and less the diagonal away from a $O(N^{2/3})$-neighborhood of the origin, so one might expect to recover the Baik-Rains distribution.

**Theorem**

Let $S = s + \delta (2u + \delta)$ and $u = w - \delta$ (for $w = 0$ we are on the characteristic line). Then,

$$\lim_{u \to \infty} F_{u,\delta}(S) = F_{BR,w}(s).$$
In arXiv:2012.10337 we extended the result to multi-point distributions.

In arXiv:2204.06782 we get some results on the time-time covariance close to the characteristic direction (compare with Alessandra Occelli’s talk a few weeks ago).

The general stationary process in TASEP has two parameters: one for the input rate and one for the density at infinity. This is reflected into the LPP setting as well (see (maybe) Barraquand’s talk next week).

Barraquand-Krajenbrink-Le Doussal’22; Barraquand-Corwin’22