Cutting cycles of rods in space is FPT

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Abstract

In this short note, we show that cutting cycles of rods is fixed-parameter tractable by reducing the problem to computing a feedback vertex set in a mixed graph.

1. Introduction

Let $S$ be a collection of $n$ non-vertical segments in $\mathbb{R}^3$. For two segments $s, s' \in S$, we say that $s \succ s'$ if and only if $s$ lies above $s'$. Formally, if $\ell$ is the vertical line that meets $s$ and $s'$, then $s \succ s'$ if and only if the $z$-coordinate of the point $\ell \cap s$ is larger than the $z$-coordinate of the point $\ell \cap s'$. This induces a relation $\succ$ on the segments in $S$ (which may not be transitive). In general, this relation may contain what we call depth cycles. By cutting the segments of $S$ into multiple smaller segments, we obtain a new collection of segments $S'$. The goal is to cut the segments of $S$ such that the relation for the segments $S'$ is a partial ordering (i.e., there are no depth cycles). The minimum number of such cuts needed is called the cutting number of $S$. See Figure 1. Often segments are referred to as rods, and we use these terms interchangeably.

To summarize, the problem is to make as few cuts as possible to the rods (each rod may be cut in multiple different places) such that the resulting depth ordering is acyclic. In this note, we study the parameterized version of the problem.

Problem 1. Given a set $S$ of $n$ non-vertical segments in $\mathbb{R}^3$ and a parameter $k$, is there an algorithm running in time $f(k)n^{O(1)}$ (where $f$ is some computable function) which decides if all depth cycles can be eliminated using at most $k$ cuts?

This is equivalent to asking if the above problem is fixed-parameter tractable (FPT) by the solution size. See [FG06, CFK+15] for a reference on parameterized algorithms and complexity.

1.1. Previous work

Cutting cycles of rods has applications in hidden surface removal and computer graphics (see discussions in [AKS05] and [dBCvKO08, Chapter 12]). The problem has been studied from both a combinatorial [AKS05, AS18] and computational [Sol98, HS01, AdBG08] viewpoint. Here, we focus on the computational aspect of the problem.

For a given collection of segments $S$ in $\mathbb{R}^3$ with cutting number $\mu$, Solan [Sol98] was the first to develop an output sensitive algorithm for the problem, producing a set of $O(n^{1+\epsilon} \mu^{1/2})$ cuts in time $O(n^{4/3+\epsilon} \mu^{1/3})$,
We prove that Problem 1 is fixed-parameter tractable by the solution size $k$.

**Theorem 1.** Let $S$ be a collection of $n$ non-vertical segments in $\mathbb{R}^3$. Given an integer parameter $k$, there is an algorithm which decides if all depth cycles of $S$ can be eliminated with at most $k$ cuts, and runs in time $O(47.5^k k! n^8)$.

## 2. Proof of Theorem 1

### 2.1. Mixed graphs

A mixed graph $G$ is defined by the tuple $(V, E, A)$, where $V$ is a collection of $n$ vertices, $E$ is a collection of undirected edges, and $A$ is a collection of directed arcs. For each edge $e \in E$ with endpoints $u, v \in V$, we denote $e$ by $\{u, v\}$. Each arc $a \in A$ with tail $u \in V$ and head $v \in V$ is denoted by $(u, v)$. The total degree of a vertex $v$ is the number of edges and arcs adjacent to $v$. A cycle in $G$ consists of a sequence of distinct vertices $v_1, \ldots, v_\ell$ in $G$, where $v_\ell = v_1$ and for each $1 \leq i \leq \ell - 1$ either $\{v_i, v_{i+1}\} \in E$ or $(v_i, v_{i+1}) \in A$. A subset of

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1 $\alpha(n)$ is the inverse Ackermann function.
2 Here, $\omega < 2.37$ is the matrix multiplication constant for multiplying two $n \times n$ matrices.
vertices $S \subseteq V$ is a feedback vertex set for $G$ if and only if $G - S$ is acyclic. We will use the following result of [BL11].

**Theorem 2 ([BL11])**. Let $G$ be a mixed graph on $n$ vertices. Given an integer parameter $k$, there is an algorithm which decides if $G$ contains a feedback vertex set of size at most $k$, and runs in time $O(47.5^k k! n^4)$.

### 2.2. Reducing cutting cycles to FVS in mixed graphs

For a segment $s$ in $\mathbb{R}^3$, let $\bar{s}$ be the projection of $s$ onto the $xy$-plane. Similarly, let $\bar{S} = \{\bar{s} \mid s \in S\}$ for our collection of segments $S$. Given $s \in S$, let $I(s)$ be the set of intersection points in the arrangement $\mathcal{A}(\bar{S})$ along the segment $\bar{s}$. We order these intersection points $I(s)$ along $\bar{s}$ arbitrarily and label them $p^s_1, \ldots, p^s_{|I(s)|} \in \mathbb{R}^2$, where $\ell = |I(s)|$.

Observe that by cutting each segment $s$ at each of its intersection points (with the appropriate $z$-coordinate in $\mathbb{R}^3$), we remove all depth cycles (and make $O(n^2)$ cuts). In particular, it suffices to focus on cutting segments only at intersection points in order to remove depth cycles. Indeed, any solution which cuts a segment $s$ at a point which is a non-intersection can be shifted along $s$ until it meets an intersection, and remains a valid solution.

Equipped with these observations, we define the following mixed graph $G_S = (V, E, A)$ (see also Figure 2):

(i) for each $s \in S$, define the vertex set $V_s = \{v^s_i \mid 1 \leq i \leq |I(s)|\}$, where the vertex $v^s_i$ represents the $i$th intersection point $p^s_i$ in $I(s)$. We set $V = \bigcup_{s \in S} V_s$;

(ii) for each $s \in S$, add the undirected edges $\{v^s_i, v^s_j\}$ for $1 \leq i \leq |I(s)| - 1$ to $E$;

(iii) for two segments $s \succ t$ for which $\bar{s} \cap \bar{t} \neq \emptyset$, suppose that $p^s_i = p^t_j$ for some integers $i, j$ (i.e., $\bar{s} \cap \bar{t}$ is the $i$th intersection point along $\bar{s}$, and $\bar{s} \cap \bar{t}$ is the $j$th intersection point along $\bar{t}$). Then we add the directed arc $(v^s_i, v^t_j)$ to $A$.

**Lemma 3.** All cycles in $S$ can be eliminated with at most $k$ cuts if and only if the mixed graph $G_S$ admits a feedback vertex set of size at most $k$.

**Proof:** Let $U \subseteq V$ be a feedback vertex set for $G_S$. Since each vertex in $U$ corresponds to an intersection point $p^s_i$ along some segment $s \in S$, we cut $s$ at the point with $xy$-coordinates $p^s_i$ and the appropriate $z$-coordinate. Let $C$ denote the resulting collection of cuts. Clearly, the number of cuts we make is $|C| = |U| \leq k$. We claim that $C$ eliminates all depth cycles. Suppose not, then there is a cycle consisting of (potentially cut) segments $s_1 \succ s_2 \succ \ldots \succ s_p$, where $s_p = s_1$. Since $s_i \succ s_{i+1}$ for $1 \leq i \leq p - 1$, there is a directed edge $(v^s_{j(i)}, v^{s_{j(i+1)}}_{j'(i+1)})$ for each $i$ and indices $1 \leq j(i), j'(i) \leq |I(s_i)|$. Here, the index $j(i)$ (resp. $j'(i)$) is chosen such that $\bar{s}_i \cap \bar{s}_{i+1}$ (resp. $\bar{s}_i \cap \bar{s}_{i-1}$) is the $j(i)$th (resp. $j'(i)$th) intersection point along $\bar{s}_i$. Since the vertices of $V_{s_{i+1}}$ are connected by an undirected path, we can travel from the vertex $v^{s_{j(i+1)}}_{j'(i+1)}$ to the vertex $v^{s_{j(i+1)+1}}_{j'(i+1)}$ along an undirected path $P_i$. This corresponds to moving along the segment $s_{i+1}$. Note that since each vertex in $V$ is adjacent to
exactly one arc, we have that \( v_{j(i)}^{s_i} \neq v_{j(i)}^{s_i} \) for all \( i \). Next, we can take another directed edge \((v_{j(i+1)}^{s_i+1}, v_{j(i+2)}^{s_i+2})\). Continuing in this fashion, we can stitch together a cycle which concatenates the directed edge \((v_{j(i)}^{s_i}, v_{j(i+1)}^{s_i})\) followed by the undirected path \(P_{i+1}\) from \( v_{j(i+1)}^{s_i+1} \) to \( v_{j(i+1)}^{s_i+1} \), for \( i = 1, \ldots, p - 1 \), to obtain a cycle in \( G_S - U \). A contradiction.

Conversely, let \( C \) be a collection of cuts which eliminates all depth cycles in \( S \). By the above discussion, without loss of generality we can assume that \( C \) cuts the segments of \( S \) at intersection points. Since each intersection point is associated to a vertex in \( G_S \), we add the vertex \( v_i \) to \( U \) if and only if segment \( s \) was cut at the intersection point \( p_i \). Clearly, this collection of vertices \( U \) satisfies \( |U| = |C| \leq k \). The argument that \( U \) is a feedback vertex set for \( G_S \) is similar to above. If \( U \) is not a solution, then there is a cycle in \( G_S \) which corresponds to a depth cycle in \( S \) after making the cuts in \( C \), a contradiction.

**Restatement of Theorem 1.** Let \( S \) be a collection of \( n \) non-vertical segments in \( \mathbb{R}^3 \). Given an integer parameter \( k \), there is an algorithm which decides if all depth cycles of \( S \) can be eliminated with at most \( k \) cuts, and runs in time \( O(47.5^k k! n^8) \).

**Proof:** The algorithm is straightforward. Given \( S \), apply the above reduction to obtain a mixed graph \( G_S \). Note that \( G_S \) has size \( O(n^2) \) (there are \( O(n^2) \) vertices, and each vertex has total degree at most three), and can be easily computed in \( O(n^2) \) time. Next, run the algorithm of Theorem 2 on \( G_S \) to obtain a feedback vertex set \( U \). One can map this set \( U \) back to a cut set \( C \) for \( S \) by Lemma 3. The total running time of the algorithm is \( O(47.5^k k! |G_S|^4) = O(47.5^k k! n^8) \).

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