Rank probabilities for real random $N \times N \times 2$ tensors

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Abstract

We prove that the probability $P_N$ for a real random Gaussian $N \times N \times 2$ tensor to be of real rank $N$ is $P_N = \frac{\Gamma((N + 1)/2)^N}{G(N + 1)}$, where $\Gamma(x)$, $G(x)$ denote the gamma and Barnes $G$-functions respectively. This is a rational number for $N$ odd and a rational number multiplied by $\pi^{N/2}$ for $N$ even. The probability to be of rank $N + 1$ is $1 - P_N$. The proof makes use of recent results on the probability of having $k$ real generalized eigenvalues for real random Gaussian $N \times N$ matrices. We also prove that $\log P_N = \frac{N^2}{4} \log(\frac{e}{4}) + \frac{\log N - 1}{12} - \zeta'(-1) + O(1/N)$ for large $N$, where $\zeta$ is the Riemann zeta function.

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1 Introduction

The (real) rank of a real $m \times n \times p$ 3-tensor or 3-way array $\mathcal{T}$ is the well defined minimal possible value of $r$ in an expansion

$$\mathcal{T} = \sum_{i=1}^r u_i \otimes v_i \otimes w_i \quad (u_i \in \mathbb{R}^m, v_i \in \mathbb{R}^n, w_i \in \mathbb{R}^p)$$

where $\otimes$ denotes the tensor (or outer) product [1, 3, 4, 8].

If the elements of $\mathcal{T}$ are chosen randomly according to a continuous probability distribution, there is in general (for general $m$, $n$ and $p$) no generic rank, i.e., a rank which occurs with probability 1. Ranks which occur with strictly positive probabilities are called typical ranks. We assume that all elements are independent and from a standard normal (Gaussian) distribution (mean 0, variance 1). Until now, the only analytically known probabilities for typical ranks were for $2 \times 2 \times 2$ and $3 \times 3 \times 2$ tensors [2, 7]. Thus in the $2 \times 2 \times 2$ case the probability that $r = 2$ is $\pi/4$ and the probability that $r = 3$ is $1 - \pi/4$, while in the $3 \times 3 \times 2$ case the probability of the rank equaling 3 is the same as the probability of it equaling 4 which is $1/2$. Before these analytic results the first numerical simulations were performed by Kruskal in 1989, for $2 \times 2 \times 2$
tensors [8], and the approximate values 0.79 and 0.21 obtained for the probability of ranks \( r = 2 \) and \( r = 3 \) respectively. For \( N \times N \times 2 \) tensors ten Berge and Kiers [10] have shown that the only typical ranks are \( N \) and \( N + 1 \). From ten Berge [9], it follows that the probability \( P_N \) for an \( N \times N \times 2 \) tensor to be of rank \( N \) is equal to the probability that a pair of real random Gaussian \( N \times N \) matrices \( T_1 \) and \( T_2 \) (the two slices of \( T \)) has \( N \) real generalized eigenvalues, i.e., the probability that \( \det(T_1 - \lambda T_2) = 0 \) has only real solutions \( \lambda \) [2,5]. Knowledge about the expected number of real solutions to \( \det(T_1 - \lambda T_2) = 0 \) obtained by Edelman et al. [5] led to the analytical results for \( N = 2 \) and \( N = 3 \) in [2]. Forrester and Mays [7] have recently determined the probabilities \( p_{N,k} \) that \( \det(T_1 - \lambda T_2) = 0 \) has \( k \) real solutions, and we here apply the results to \( P_N = p_{N,N} \) to obtain explicit expressions for the probabilities for all typical ranks of \( N \times N \times 2 \) tensors for arbitrary \( N \), hence settling this open problem for tensor decompositions. We also determine the precise asymptotic decay of \( P_N \) for large \( N \) and give some recursion formulas for \( P_N \).

2 Probabilities for typical ranks of \( N \times N \times 2 \) tensors

As above, assume that \( T_1 \) and \( T_2 \) are real random Gaussian \( N \times N \) matrices and let \( p_{N,k} \) be the probability that \( \det(T_1 - \lambda T_2) = 0 \) has \( k \) real solutions. Then Forrester and Mays [7] prove:

**Theorem 1.** Introduce the generating function

\[
Z_N(\xi) = \sum_{k=0}^{N} \xi^k p_{N,k}
\]

where the asterisk indicates that the sum is over \( k \) values of the same parity as \( N \). For \( N \) even we have

\[
Z_N(\xi) = \frac{(-1)^{N(N-2)/4} \Gamma\left(\frac{N+1}{2}\right)^{N/2} \Gamma\left(\frac{N+2}{2}\right)^{N/2}}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{j}{2}\right)^2} \prod_{l=0}^{N-2} (\xi^2 \alpha_l + \beta_l),
\]

while for \( N \) odd

\[
Z_N(\xi) = \frac{(-1)^{(N-1)(N-3)/8} \Gamma\left(\frac{N+1}{2}\right)^{(N+1)/2} \Gamma\left(\frac{N+2}{2}\right)^{(N-1)/2}}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{j}{2}\right)^2} \pi \xi \prod_{l=0}^{\left\lfloor\frac{N-1}{4}\right\rfloor - 1} (\xi^2 \alpha_l + \beta_l) \prod_{l=\left\lfloor\frac{N-1}{4}\right\rfloor}^{N-3} (\xi^2 \alpha_{l+1/2} + \beta_{l+1/2})
\]

Here

\[
\alpha_l = \frac{2\pi}{N - 1 - 4l} \frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)}
\]

and

\[
\alpha_{l+1/2} = \frac{2\pi}{N - 3 - 4l} \frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)}
\]

The expressions for \( \beta_l \) and \( \beta_{l+1/2} \) are given in [7], but are not needed here, and \( \left\lceil \cdot \right\rceil \) denotes the ceiling function.
The method used in [7] relies on first obtaining the explicit form of the element probability density function for

\[ G = T_1^{-1} T_2. \]

A real Schur decomposition is used to introduce \( k \) real and \( (N-k)/2 \) complex eigenvalues, with the imaginary part of the latter required to be positive (the remaining \( (N-k)/2 \) eigenvalues are the complex conjugate of these), for \( k = 0, 2, \ldots, N \) (\( N \) even) and \( k = 1, 3, \ldots, N \) (\( N \) odd). The variables not depending on the eigenvalues can be integrated out to give the eigenvalue probability density function, in the event that there are \( k \) real eigenvalues. And integrating this over all allowed values of the real and positive imaginary part complex eigenvalues gives \( P_{N,k} \).

From Theorem 1 we derive our main result:

**Theorem 2.** Let \( P_N \) denote the probability that a real \( N \times N \times 2 \) tensor whose elements are independent and normally distributed with mean 0 and variance 1 has rank \( N \). We have

\[ P_N = \frac{(\Gamma((N+1)/2))^N}{G(N+1)}, \]

where

\[ G(N+1) := (N-1)!(N-2)! \ldots! \quad (N \in \mathbb{Z}^+) \]

is the Barnes-G function and \( \Gamma(x) \) denotes the gamma function. More explicitly \( P_2 = \pi/4 \), and for \( N \geq 4 \) even

\[ P_N = \frac{\pi^{N/2}(N-1)^{N-1}(N-3)^{N-3} \ldots 3^3}{2^{N^2/2}(N-2)^2(N-4)^4 \ldots 2N^2}, \]

while for \( N \) odd

\[ P_N = \frac{(N-1)^{N-1}(N-3)^{N-3} \ldots 2^2}{2^{N(N-1)/2}(N-2)^2(N-4)^4 \ldots 3^{N-3}}. \]

Hence \( P_N \) for \( N \) odd is a rational number but for \( N \) even it is a rational number multiplied by \( \pi^{N/2} \). The probability for rank \( N+1 \) is \( 1 - P_N \).

**Proof.** From [2] we know that \( P_N = p_{N,N} \). Hence, by Theorem 1

\[ P_N = p_{N,N} = \frac{1}{N!} \frac{d^N}{d\xi^N} Z_N(\xi) \]

Since

\[ \frac{1}{N!} \frac{d^N}{d\xi^N} \prod_{l=0}^{N-2} (\xi^2 \alpha_l + \beta_l) = \prod_{l=0}^{N-2} \alpha_l \]

and

\[ \frac{1}{N!} \frac{d^N}{d\xi^N} \prod_{l=0}^{\frac{N-1}{2}-1} (\xi^2 \alpha_l + \beta_l) \prod_{l=0}^{N-3} (\xi^2 \alpha_{l+1/2} + \beta_{l+1/2}) = \prod_{l=0}^{\frac{N-1}{2}-1} \alpha_l \prod_{l=0}^{\frac{N-3}{2}} \alpha_{l+1/2} \]

the values of \( \beta_l \) and \( \beta_{l+1/2} \) are not needed for the determination of \( P_N \). By [3] we immediately find

\[ P_N = \frac{(-1)^{N(N-2)/8} \Gamma\left(\frac{N-1}{2}\right)^{N/2} \Gamma\left(\frac{N+1}{2}\right)^{N/2}}{2^{N(N-1)/2} \prod_{l=1}^N \Gamma\left(\frac{l}{2}\right)^2} \prod_{l=0}^{N-2} \alpha_l \]

...
if $N$ is even. For $N$ odd we use (4) to get

$$P_N = \frac{(-1)^{(N-1)(N-3)/8} \Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{N+1}{2} \cdot \frac{(N-1)}{2}\right)}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{N}{2}\right)^2} \pi \prod_{l=0}^{\lfloor \frac{N-1}{4} \rfloor} \alpha_l \prod_{l=0}^{\lfloor \frac{N+3}{4} \rfloor} \alpha_{l+1/2} \quad (16)$$

Substituting the expressions for $\alpha_l$ and $\alpha_{l+1/2}$ into these formulas we obtain, after simplifying, for $N$ even

$$P_N = \frac{(-1)^{N(N-2)/8} (2\pi)^{N/2} \Gamma\left(\frac{N+1}{2}\right)^N}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{N}{2}\right)^2} \prod_{l=0}^{\lfloor \frac{N-2}{4} \rfloor} \frac{1}{N-1-4l}, \quad (17)$$

and for $N$ odd

$$P_N = \frac{(-1)^{(N-1)(N-3)/8} (2\pi)^{N+1/2} \Gamma\left(\frac{N+1}{2}\right)^N}{2^{N(N-1)/2+1} \prod_{j=1}^{N} \Gamma\left(\frac{N}{2}\right)^2} \prod_{l=0}^{\lfloor \frac{N-1}{4} \rfloor} \frac{1}{N-1-4l} \prod_{l=0}^{\lfloor \frac{N+3}{4} \rfloor} \frac{1}{N-3-4l}. \quad (18)$$

Now

$$\prod_{j=1}^{N} \Gamma\left(\frac{j}{2}\right)^2 = \frac{\Gamma(1/2)}{\Gamma((N+1)/2)} \prod_{j=1}^{N} \Gamma\left(\frac{j}{2}\right) \Gamma\left(\frac{(j+1)}{2}\right) = \frac{\Gamma(1/2)}{\Gamma((N+1)/2)} \prod_{j=1}^{N} 2^{1-j} \sqrt{\pi} \Gamma(j) = \frac{\Gamma(1/2)}{\Gamma((N+1)/2)} 2^{-N(N-1)/2} \pi^{N/2} G(N+1), \quad (19)$$

where to obtain the second equality use has been made of the duplication formula for the gamma function, and to obtain the third equality the expression (9) for the Barnes $G$-function has been used. Furthermore, for each $N$ even

$$(-1)^{N(N-2)/8} \prod_{l=0}^{(N-2)/2} \frac{1}{N-1-4l} = \frac{(-1)^{N(N-2)/8}}{(N-1)(N-5) \ldots (N-1-(2N-4))} = \frac{1}{(N-1)(N-3) \ldots 3 \cdot 1} = \frac{\Gamma(1/2)}{2^{N/2} \Gamma((N+1)/2)}, \quad (20)$$

where to obtain the final equation use is made of the fundamental gamma function recurrence

$$\Gamma(x+1) = x \Gamma(x), \quad (21)$$
and for $N$ odd
\[
(-1)^{(N-1)(N-3)/8} \prod_{l=0}^{\lfloor (N-1)/2 \rfloor} \frac{1}{N - 1 - 4l} \prod_{l=0}^{\lfloor (N-3)/2 \rfloor} \frac{1}{N - 3 - 4l} = (-1)^{(N-1)(N-3)/8} \begin{cases} 
\frac{1}{(N-1)(N-5) \ldots \ 2 \ (N-3) \ldots \ (N+3)}, & N = 3, 7, 11, \ldots \\
\frac{1}{(N-1)(N-5) \ldots \ 4 \ (N-3) \ldots \ (N+3)}, & N = 5, 9, 13, \ldots 
\end{cases}
\]

Substituting (19) and (20) in (17) establishes (8) for $N$ even, while the $N$ odd case of (8) follows by substituting (19) and (22) in (18), and the fact that
\[
\Gamma(1/2) = \sqrt{\pi}.
\]

The forms (10) and (11) follow from (8) upon use of (9), the recurrence (21) and (for $N$ even) (23).

3 Recursion formulas and asymptotic decay

By Theorem 2 it is straightforward to calculate $P_{N+1}/P_N$ from either (8) or (10) and (11), and $P_{N+2}/P_N$ from either (8) or (10) and (11).

Corollary 3. For general $N$

\[
P_{N+1} = P_N \cdot \frac{\Gamma((N+1)/2)^{N+1}}{\Gamma((N+1)/2)^N \cdot \Gamma(N+1)} \cdot \frac{1}{\Gamma(N+2)\Gamma(N+1)}, \quad P_{N+2} = P_N \cdot \frac{(N+1)^{N+1} \Gamma(N+1/2)^2}{\Gamma((N+1)/2)^{N+2} \cdot \Gamma(N+1)^2} \quad (24)
\]

More explicitly, making use of the double factorial

\[
N!! = \begin{cases} 
N(N-2)\ldots 4 \cdot 2, & N \text{ even} \\
N(N-2)\ldots 3 \cdot 1, & N \text{ odd},
\end{cases}
\]

for $N$ even we have the recursion formulas
\[
P_{N+1} = P_N \cdot \frac{(N!!)^N}{(2\pi)^{N/2}((N-1)!!)^{N+1}} \cdot \pi \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2}, \quad P_{N+2} = P_N \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2} \quad (25)
\]

and for $N$ odd we have
\[
P_{N+1} = P_N \cdot \frac{\pi^{(N+1)/2}(N!!)^N}{2^{3(N+1)/2}((N-1)!!)^{N+1}} \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2}, \quad P_{N+2} = P_N \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2} \quad (26)
\]
We can illustrate the pattern for $P_N$ using Theorem 2 or Corollary 3. One finds

$$P_2 = 1/2 \cdot \pi, \quad P_3 = 1/2$$

$$P_4 = 3/20 \cdot \pi^2, \quad P_5 = 1/32$$

$$P_6 = 5^5 \cdot 3^3 \cdot \pi^3, \quad P_7 = 3^2/5 \cdot 2^5$$

$$P_8 = 7^7 \cdot 5^5 \cdot 3^4 \cdot \pi^4, \quad P_9 = 2^4/7 \cdot 5^4$$

$$P_{10} = 7^7 \cdot 5^5 \cdot 3^{17} \cdot \pi^5, \quad P_{11} = 5^4/74 \cdot 3^6 \cdot 2^5$$

$$P_{12} = 11^{11} \cdot 7^7 \cdot 5^5 \cdot 3^{15} \cdot \pi^6, \quad P_{13} = 5^2/11^{12} \cdot 7^6 \cdot 2^4 \cdots$$ (27)

Numerically, it is clear that $P_N \to 0$ as $N \to \infty$. Some qualitative insight into the rate of decay can be obtained by recalling $P_N = p_{N,N}$ and considering the behaviour of $p_{N,k}$ as a function of $k$. Thus we know from [5] that for large $N$, the mean number of real eigenvalues $E_N := \langle k \rangle_{p_{N,k}}$ is to leading order equal to $\sqrt{\pi N/2}$, and from [7] that the corresponding variance $\sigma_N^2 := \langle k^2 \rangle_{p_{N,k}} - E_N^2$ is to leading order equal to $(2 - \sqrt{2})E_N$. The latter reference also shows that $\lim_{N \to \infty} \sigma_N p_{N,k} \sigma_N x + E_N = 1/\sqrt{2\pi}$, and is thus $p_{N,k}$ is a standard Gaussian distribution after centering and scaling in $k$ by appropriate multiples of $\sqrt{N}$. It follows that $p_{N,N}$ is, for large $N$, in the large deviation regime of $p_{N,k}$. We remark that this is similarly true of $p_{N,N}$ in the case of eigenvalues of $N \times N$ real random Gaussian matrices (i.e. the individual matrices $T_1, T_2$ of (7)), for which it is known $p_{N,N} = 2^{-N(N-1)/4}$ [5, [6, Section 15.10].

In fact from the exact expression (8) the explicit asymptotic large $N$ form of $P_N$ can readily be calculated. For this, let

$$A = e^{-\zeta'(-1)+1/12} = 1.28242712...$$ (28)

denote the Glaisher-Kinkelin constant, where $\zeta$ is the Riemann zeta function [11].

**Theorem 4.** For large $N$,

$$P_N = N^{1/12} (e^{N^2/4} \cdot A e^{-1/6} (1 + O(1/N))$$ (29)

or equivalently

$$\log P_N = (N^2/4) \log (e/4) + (\log N - 1)/12 - \zeta'(-1) + O(1/N).$$ (30)

**Proof.** We require the $x \to \infty$ asymptotic expansions of the Barnes $G$-function [12] and the gamma function

$$\log G(x + 1) = \frac{x^2}{2} \log x - \frac{3}{4} x^2 + \frac{x}{2} \log 2\pi - \frac{1}{12} \log x + \zeta'(-1) + O\left(\frac{1}{x}\right),$$ (31)

$$\Gamma(x + 1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + O\left(\frac{1}{x^2}\right)\right)$$ (32)
For future purposes, we note that a corollary of (32), and the elementary large $x$ expansion

$$\left(1 + \frac{c}{x}\right)^x = e^c \left(1 - \frac{c^2}{2x} + O\left(\frac{1}{x^2}\right)\right)$$  \hspace{1cm} (33)$$

is the asymptotic formula

$$\frac{\Gamma(x + 1/2)}{\Gamma(x)} = \sqrt{x} \left(1 - \frac{1}{8x} + O\left(\frac{1}{x^2}\right)\right).$$  \hspace{1cm} (34)$$

To make use of these expansions, we rewrite (8) as

$$P_N = \frac{(\Gamma(N/2 + 1))^N}{G(N + 1)} \frac{\Gamma((N + 1)/2)}{\Gamma(N/2 + 1)}.$$

(35)

Now, (34) and (33) show that with

$$y := N/2$$

and $y$ large we have

$$\left(\frac{\Gamma(y + 1/2)}{\Gamma(y + 1)}\right)^N = e^{-y \log y} e^{-1/4} \left(1 + O\left(\frac{1}{y}\right)\right).$$

(37)

Furthermore, in the notation (36) it follows from (31) and (32) and further use of (33) (only the explicit form of the leading term is now required) that

$$\frac{\Gamma((N + 1)/2)^N}{G(N + 1)} = e^{-y^2 \log(4/e)} e^{y \log y + \frac{1}{12} \log 2} e^{1/6 - \zeta'(1)} \left(1 + O\left(\frac{1}{y}\right)\right).$$

(38)

Multiplying together (37) and (38) as required by (35) and recalling (36) gives (29).

Recalling (28), the second stated result (30) is then immediate.

Corollary 5. For large $N$,

$$\frac{P_{N+1}}{P_N} = \left(\frac{e}{4}\right)^{(2N+1)/4} \left(1 + O(N^{-1})\right)$$

(39)

This corollary follows trivially from Theorem 4. It can however also be derived directly from the recursion formulas in Corollary 3 without use of Theorem 4.

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