ON VARIATIONAL MULTIVALUED ELLIPTIC EQUATIONS ON A BOUNDED DOMAIN IN THE PRESENCE OF CRITICAL GROWTH

J. V. Goncalves  M. L. Carvalho

Abstract

We develop arguments on the critical point theory for locally Lipschitz functionals on Orlicz-Sobolev spaces, along with convexity and compactness techniques to investigate existence of solution of the multivalued equation

\[-\Delta_{\Phi} u \in \partial j(.,u) + \lambda h \text{ in } \Omega,\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(\Phi : \mathbb{R} \rightarrow [0, \infty)\) is a suitable \(N\)-function, \(\Delta_{\Phi}\) is the corresponding \(\Phi\)-Laplacian, \(\lambda > 0\) is a parameter, \(h : \Omega \rightarrow \mathbb{R}\) is integrable and \(\partial j(.,u)\) is the subdifferential of a function \(j\) associated with critical growth.

Dedicated to Bernhard Ruf on the occasion of his 60\(^{th}\) birthday.

1. Introduction

We deal with the multivalued equation

\[-\Delta_{\Phi} u \in \partial j(.,u) + \lambda h \text{ in } \Omega,\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega\), \(h : \Omega \rightarrow \mathbb{R}\) is measurable, \(\lambda > 0\) is a parameter, \(\Delta_{\Phi}\) is the \(\Phi\)-Laplacian operator, that is

\[\Delta_{\Phi} u = \text{div}(\phi(|\nabla u|)\nabla u),\]

where \(\phi : (0, +\infty) \rightarrow (0, +\infty)\) is continuous satisfying

\[(\phi_1) \quad \lim_{s \to 0} s\phi(s) = 0, \quad (\phi_2) \quad s \mapsto s\phi(s) \text{ is nondecreasing in } (0, \infty),\]

\[(\phi_3) \quad \text{there exist } \ell, m \in (1, N) \text{ such that } \ell \leq \frac{t^2\phi(t)}{\Phi(t)} \leq m, \quad t > 0\]

and \(s \mapsto s\phi(s)\) is extended to \(\mathbb{R}\) as an odd function. The functions \(\Phi, j\) are given respectively by

\[\Phi(t) = \int_0^t s\phi(s)ds \quad \text{for } t \in \mathbb{R},\]

\[j(x, t) = \sigma(x)[\Phi_s(t) - \Phi_s(a)] \chi_{\{t > a\}}\]

where \(\sigma \in L^\infty(\Omega), \sigma \geq 0, \sigma \neq 0, a > 0\) is a number and \(\Phi_s\) is the inverse of the function

\[t \in (0, \infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{N+1}}ds\]

which extends to \(\mathbb{R}\) by \(\Phi_s(t) = \Phi_s(-t)\) for \(t \leq 0\), while \(\partial j(x, t)\) stands for the subdifferential of \(j\),

\[\partial j(x, t) = \{\mu \in \mathbb{R} \mid j^\circ(x, t; r) \geq \mu r, \quad r \in \mathbb{R}\},\]

\(^1\)Supported in part by CNPq/CAPES/PROCAD/UFG/UnB-Brazil
Remark 1.1

Our main results are,

$$j_0(x, t; r) = \limsup_{y \to t, s \to 0^+} \frac{j(x, y + sr) - j(x, y)}{s}.$$ 

Due to the nature of the differential operator $\Delta_\Phi$, it is natural to work in the framework of Orlicz-Sobolev spaces. It is known, (cf. [14, 25]), that

$$\Phi_*(t) = \int_0^t \phi_*(s) ds,$$

where $\phi_* : [0, \infty) \to [0, \infty)$ satisfies

$$(\phi_*)_1 \quad \phi_*(0) = 0, \quad \phi_*(s) > 0 \quad \text{for } s > 0, \quad \lim_{s \to \infty} \phi_*(s) = \infty,$$

$$(\phi_*)_2 \quad \phi_* \text{ is continuous, nondecreasing,}$$

$$(\phi_*)_3 \quad \ell^* \leq \frac{t \phi_*(t)}{\Phi_*(t)} \leq m^* \text{ for } t > 0,$$

where $p^* := Np/(N - p)$ for $p \in (1, N)$. At this point we notice that

$$j(x, t) = \int_0^t \sigma(x) \chi_{\{r > a\}} \phi_*(r) dr, \quad t \in R.$$ 

The Orlicz space associated with $\Phi$ is

$$L_\Phi(\Omega) := \left\{ u : \Omega \to R \text{ measurable } \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) < +\infty \text{ for some } \lambda > 0 \right\}$$

The Orlicz-Sobolev space, (also denoted $W^{1, L}\Phi(\Omega)$), is

$$W^{1, \Phi}(\Omega) = \left\{ u \in L_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, ..., N \right\}$$

and $W^{1, \Phi}_0(\Omega)$ is the closure of $C^{0, \infty}_0(\Omega)$ with respect to $W^{1, \Phi}(\Omega)$.

**Definition 1.1** Let $h \in L_{\Phi_*}(\Omega)'$. A vector $u \in W^{1, \Phi}_0(\Omega)$ is a solution of (1) if there is an element $\rho := \rho_u \in L_{\Phi_*}(\Omega)'$ such that

$$\rho(x) \in \partial j(x, u(x)) \text{ a.e. } x \in \Omega,$$

$$\int_\Omega \phi(|\nabla u|) \nabla u \nabla v dx = \int_\Omega \rho v dx + \lambda \int_\Omega hv dx, \quad v \in W^{1, \Phi}_0(\Omega).$$

Our main results are,

**Theorem 1.1** Let $a > 0$ and $\ell^* > m$. Assume that $\phi : (0, \infty) \to (0, \infty)$ is continuous, satisfies $(\phi_1) - (\phi_3)$. Let $h \in L_{\Phi_*}(\Omega)'$ be nonnegative with $h \not\equiv 0$. Then there is $\lambda_* > 0$ such that for each $\lambda \in (0, \lambda_*)$, equation (1) admits at least one nonnegative solution, say $u = u_\lambda \in W^{1, \Phi}_0(\Omega)$.

Moreover

$$-\Delta_\Phi u = \rho + \lambda h \text{ a.e. in } \Omega. \quad (3)$$

**Remark 1.1** If $N \geq 3$, $\phi(t) = 2$ and $\sigma \equiv 1$, then by computing, one gets $\Phi_*(t) = t^{N/3}$ and $\phi_*(t) = t^{N/3}$, up to constants. The subdifferential of $j(x, t)$ is shown to be

$$\partial j(x, t) = \begin{cases} 0, & t < a \\ \left[0, a^{\frac{N+2}{2}} \right], & t = a \\ t^{\frac{N+2}{2}}, & t > a. \end{cases}$$
Equation (1) reads as

$$-\Delta u \in \partial j(\cdot, u) + \lambda h$$

in $\Omega$. (4)

A nonnegative solution $u \in H^1_0(\Omega)$ of (4) with $\{|x \in \Omega \mid u(x) = a\}| = 0$ is shown to satisfy

$$-\Delta u = u^{N+2 \chi_{\{u>a\}} + \lambda h} \text{ a.e. in } \Omega.$$  

Equations on bounded domains with jumping nonlinearities have been studied by many authors, see e.g. Badiale & Tarantello [6], Ambrosetti & Turner [5], Chang [9], Motreanu & Tanaka [23], Alves & Bertone [3] and their references.

There is a broad literature on multivalued variational equations, see e.g. Halidias & Naniewicz [17], Fiacca, Matzakos & Papageorgiou [12], Alves, Goncalves & Santos [4], Filippakis & Papageorgiou [13], Kyritsi & Papageorgiou [19], Naniewicz [24] and references therein.

## 2 Notations and Preliminary Results

In this section we gather notations and results on subdifferential calculus and Orlicz-Sobolev spaces. To begin with, following Chang [9], Clarke [10], Motreanu & Panagiotopoulos [22] and Carl, Le & Motreanu [8], let $X$ be a reflexive real Banach space and let $I : X \to \mathbb{R}$ be a locally Lipschitz continuous ($I \in \text{Lip}_{\text{loc}}(X, \mathbb{R})$ for short).

The generalized directional derivative of $I$ at $u \in X$ in the direction of $v \in X$ is defined as

$$I^0(u;v) = \limsup_{h \to 0, \lambda \downarrow 0} \frac{I(u+h+\lambda v) - I(u+h)}{\lambda}.$$

It is known that $I^0(u;\cdot)$ is convex and continuous, its subdifferential at $z$ is

$$\partial I^0(u;z) = \{\mu \in X' \mid \langle \mu, v - z \rangle \leq I^0(u;v) - I^0(u;z) \quad v \in X\}$$

and the generalized gradient of $I$ at $u \in X$ is

$$\partial I(u) = \{\mu \in X' \mid I^0(u;v) \geq \langle \mu, v \rangle, \quad v \in X\}.$$  

An element $u_0 \in X$ is a critical point of $I$ if $0 \in \partial I(u_0)$.

A main abstract result to be used in this paper is a variant for $\text{Lip}_{\text{loc}}$ functionals, of the Ambrosetti-Rabinowitz Mountain Pass Theorem, to our best knowledge, developed first via the Deformation Lemma, by Chang [9], see also [2] for a proof using the Ekeland Variational Principle and the Ky Fan Minimax Theorem, cf. [7].

If $I \in L_{\text{lip}}(X, \mathbb{R})$ and $u \in X$ then $\partial I(u) \subset X'$ is bounded, nonempty, convex and weak*-closed, in the sense that if $\xi_j \in \partial I(u_j)$, $u_j \to u$ and $\xi_j \rightharpoonup \xi$ then $\xi \in \partial I(u)$. We set

$$m(u) := \min_{w \in \partial I(u)} \|w\|_{X'}, \quad u \in X.$$

**Theorem 2.1** Let $X$ be a Banach space and let $I \in L_{\text{lip}}(X, \mathbb{R})$ with $I(0) = 0$. Suppose there are numbers $\eta, r_1 > 0$ and $e \in X$ such that

(i) $I(u) \geq \eta$ if $\|u\| = r_1$, (ii) $\|e\| > r_1$ and $I(e) \leq 0$.

Let

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$$

3
where
\[ \Gamma = \{ \gamma \in C([0, 1], X) \mid \gamma(0) = 0, \gamma(1) = e \} \]

Then \( c > 0 \) and there is a sequence \((u_n) \subseteq X \) (named a (PS)_c-sequence) satisfying
\[ I(u_n) \to c \quad \text{and} \quad m(u_n) \to 0. \]

The reader is referred to [1, 18, 25, 16] regarding Orlicz-Sobolev spaces. The usual norm on \( L_\Phi(\Omega) \) is (Luxemburg norm),
\[ \|u\|_\Phi = \inf \left\{ \lambda > 0 \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) dx \leq 1 \right\} \]
and the Orlicz-Sobolev norm of \( W^{1,\Phi}(\Omega) \) is
\[ \|u\|_{1,\Phi} = \|u\|_\Phi + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_\Phi. \]

Recall that
\[ \bar{\Phi}(t) = \max_{s \geq 0} \{ ts - \Phi(s) \}, \quad t \geq 0. \]

It turns out that \( \Phi \) and \( \bar{\Phi} \) are N-functions satisfying the \( \Delta_2 \)-condition, (cf. [25, p 22]). In addition, \( L_\Phi(\Omega) \) and \( W^{1,\Phi}(\Omega) \) are separable, reflexive, Banach spaces. By the Poincaré Inequality, (see e.g. [16]),
\[ \int_\Omega \Phi(u) dx \leq \int_\Omega \Phi(2d|\nabla u|) dx \]
where \( d = \text{diam}(\Omega) \), and it follows that
\[ \|u\|_\Phi \leq 2d\|\nabla u\|_\Phi \text{ for } W^{1,\Phi}(\Omega). \]

As a consequence, \( \|u\| := \|\nabla u\|_\Phi \) defines a norm in \( W^{1,\Phi}_0(\Omega) \), equivalent to \( \|\cdot\|_{1,\Phi} \). The imbeddings below (cf. [1, 18, 11]) will be used in this paper:
\[ W^{1,\Phi}_0(\Omega) \overset{\text{cpt}}{\hookrightarrow} L_\Phi(\Omega), \]
\[ L_{\Phi^*}(\Omega) \overset{\text{cont}}{\hookrightarrow} L_{\Phi^*}(\Omega), \]
\[ W^{1,\Phi}_0(\Omega) \overset{\text{cont}}{\hookrightarrow} L_{\Phi^*}(\Omega). \]

Regarding this last case, \( \Phi^* \) is the critical growth function associated to \( \Phi \), and the best constant, labeled \( S \), is positive and given by
\[ S = \inf_{u \in W^{1,\Phi}_0, u \neq 0} \frac{\|u\|^\ell}{\|u\|_\Phi}. \]

**Remark 2.1** We have \( \Delta_\Phi u \in L_{\Phi^*}(\Omega) \) for \( u \in W^{1,\Phi}_0 \). Indeed, set
\[ \langle -\Delta_\Phi u, v \rangle := \int_\Omega \phi(|\nabla u|) \nabla u \nabla v dx \quad \text{for } u, v \in W^{1,\Phi}_0. \]

By [14, p. 263],
\[ \int_\Omega \bar{\Phi}(\phi(|\nabla u|)|\nabla u|) dx \leq \int_\Omega \Phi(2|\nabla u|) dx < \infty, \]
which gives \( \phi(|\nabla u|)|\nabla u| \in L_{\Phi}^s(\Omega) \). By the Hölder inequality,

\[
\int_{\Omega} |\phi(|\nabla u|)|\nabla u|v|dx \leq 2\|\phi(|u|)|\nabla u|\|v\|.
\]

As a consequence of the inequality above and (6), \( \Delta \Phi u \in L_{\Phi}(\Omega) = L_{\Phi^s}(\Omega) \). The energy functional associated with (1) is

\[
I(u) = \int_{\Omega} \Phi(|\nabla u|)dx - \int_{\Omega} j(x,u)dx - \lambda \int_{\Omega} hudx, \; u \in W_0^{1,\Phi}.
\]

Set

\[
Q_{\lambda}(u) = \int_{\Omega} \Phi(|\nabla u|)dx - \lambda \int_{\Omega} hudx \quad \text{and} \quad J(u) := \int_{\Omega} j(x,u)dx.
\]

It is known that

\[
Q_{\lambda} \in C^1(W_0^{1,\Phi}, \mathbb{R}), \quad \langle Q'_{\lambda}(u), v \rangle = \int_{\Omega} \phi(|\nabla u|)|\nabla u|v|dx - \lambda \int_{\Omega} hvdx,
\]

and, (cf. lemma 4.1),

\[
J \in \text{Lip}_{\text{loc}}(W_0^{1,\Phi}, \mathbb{R}) \quad \text{and} \quad \partial J(u) \subseteq \{ \rho \in L_{\Phi^s}(\Omega) \mid \rho(x) \in \partial j(x,u(x)) \text{ a.e. } x \in \Omega \}.
\]

Thus,

\[
I \in \text{Lip}_{\text{loc}}(W_0^{1,\Phi}, \mathbb{R}) \quad \text{and} \quad \partial I(u) = Q'_{\lambda}(u) - \partial J(u).
\]

Moreover, \( u \) is a critical point of \( I \) if \( 0 \in \partial I(u) \) that is, there is some \( \rho \in \partial I(u) \) such that

\[
\langle Q'_{\lambda}(u), v \rangle - \langle \rho, v \rangle = 0 \quad \text{for} \quad v \in W_0^{1,\Phi}.
\]

### 3 The Mountain Pass Geometry of \( I \)

The proof of theorem 1.1 uses theorem 2.1. Items (i)-(ii) in theorem 2.1 are known as the mountain pass geometry for \( I \). In this regard we will present a proof of the result below based on [3].

**Lemma 3.1** Let \( h \in L_{\Phi^s}(\Omega) \) be nonnegative, with \( h \neq 0 \), and assume that \( \ell^e > m \). Then there exist \( \lambda_0, \eta, r_1 > 0 \) and \( \epsilon \in W_0^{1,\Phi} \) such that for each \( \lambda \in (0, \lambda_0) \) and \( \alpha > 0 \),

(i) \( I(u) \geq \eta > 0 \) if \( \|u\| = r_1 \),

(ii) \( \|\epsilon\| > r_1 \) and \( I(\epsilon) \leq 0 \).

**Proof** At first we show (i). Indeed, using lemmas 7.1, (cf. Appendix), and the Hölder Inequality we have

\[
I(u) \geq \min\{\|u\|^{e_1}, \|u\|^{m_1}\} - \int_{\Omega} j(x,u)dx - 2\lambda\|h\|_{\Phi^s}\|u\|_{\Phi^s}.
\]

Using lemma 7.2 also in the Appendix, we get

\[
\int_{\Omega} j(x,u)dx + 2\lambda\|h\|_{\Phi^s}\|u\|_{\Phi^s} \leq |\sigma|_{\infty} \max\{\|u\|^{e_1}, \|u\|^{m_1}\} + 2\lambda\|h\|_{\Phi^s}\|u\|_{\Phi^s} \leq |\sigma|_{\infty} \max\left\{\frac{1}{s_{\lambda}}\|u\|^{e_1}, \frac{1}{s_{\lambda}}\|u\|^{m_1}\right\} + \frac{2\lambda}{s_{\lambda}}\|h\|_{\Phi^s}\|u\|.
\]

(11)
Joining estimates (10) and (11) we have
\[
I(u) \geq \min\{\|u\|^{\ell},\|u\|^{m}\} - |\sigma|_{\infty} \max\left\{\frac{1}{S^{\ell}}\|u\|^{\ell}, \frac{1}{S^{m}}\|u\|^{m}\right\} - \frac{2\lambda}{S^{m}}\|h\|_{\Phi^{*}}\|u\|.
\]
Taking \(\|u\| \leq 1\) it follows by the inequality just above that
\[
I(u) \geq \|u\|^{m}\left(1 - \beta\|u\|^{\ell - m} - \alpha\|u\|^{1-m}\right),
\]
where \(\alpha := 2/S^{\ell}\|h\|_{\Phi^{*}}, \beta := |\sigma|_{\infty}/S^{\ell}\). Set \(P(s) := 1 - \beta s^{\ell - m}, s > 0\). Since \(\ell^{*} > m\) one gets
\[
P(s_{0}) - \lambda\alpha s^{1-m} \geq \frac{1}{4} \text{ whenever } \lambda \leq \lambda_{0} := \frac{1}{4\alpha}s_{0}^{m-1}.
\]
Choosing \(r_{1} := \min\{1, s_{0}\}\) it follows that
\[
I(u) \geq \frac{r_{1}^{m}}{4} > 0 \text{ for } u \in W^{1,\Phi}_{0} \text{ with } \|u\| = r_{1}.
\]
This shows (i). In order to show (ii), pick \(\varphi \in C_{0}^{\infty}(\Omega)\) with \(\varphi \geq 0\) such that
\[
\text{meas}\{x \in \Omega \mid \varphi(x) \geq a\} > 0 \text{ and } \|\varphi\| \geq 1.
\]
Taking \(t > 1\) we get
\[
I(t\varphi) \leq t^{m}\|\varphi\|^{m} - t^{m}\int_{\varphi \geq a} |\sigma(x)\Phi^{*}(t\varphi) + |\sigma|_{\infty}\Phi^{*}(a)\Omega|
\leq t^{m}\|\varphi\|^{m} - t^{\ell^{*}}\int_{\varphi \geq a} |\sigma(x)\Phi^{*}(\varphi) + |\sigma|_{\infty}\Phi^{*}(a)\Omega|.
\]
As a consequence,
\[
I(t\varphi) \xrightarrow{t \to \infty} -\infty.
\]
Setting \(e := t_{1}\varphi\) with \(t_{1} > 1\) large enough we have \(I(e) < 0\), showing (ii).

\section{Boundedness of the Palais-Smale Sequence}

The result below is a special case of theorem 1.1 in Le, Motreanu and Motreanu [20] which in turn is a variant for Orlicz-Sobolev spaces, of the Aubin-Clarke Theorem (cf. [10, theorem 2.7.5]). The result itself as well as its proof will be used several times in this paper.

\textbf{Lemma 4.1} Let \(j\) be as in (2). Then
\[
\partial j(x,t) = \begin{cases} 
0, & t < a, \\
[0,\sigma(x)\phi_{\ast}(a)], & t = a, \\
\sigma(x)\phi_{\ast}(t), & t > a,
\end{cases}
\]
and the functional
\[
J(u) := \int_{\Omega} j(x,u(x))dx, \ u \in L_{\phi_{\ast}}(\Omega)
\]
satisfies
\[
J \in Lip_{loc}(L_{\phi_{\ast}}(\Omega), \mathbb{R})
\]
and
\[
\partial J(u) \subseteq \{\rho \in L_{\phi_{\ast}}(\Omega) \mid \rho(x) \in \partial j(x,u(x)) \ a.e. \ x \in \Omega\}.
\]
By lemmas 3.1, 4.1 and theorem 2.1 there is a sequence \((u_n) \subseteq W_0^{1,\Phi}\) such that
\[
I(u_n) \xrightarrow{n} c \quad \text{and} \quad m(u_n) \equiv \min_{w \in \partial I(u_n)} \|w\|_{W^{-1,\Phi}} \xrightleftharpoons{n} 0.
\] (12)

Actually, there is \(w_n \in \partial I(u_n)\) such that
\[
\|w_n\|_{W^{-1,\Phi}} = \min_{w \in \partial I(u_n)} \|w\|_{W^{-1,\Phi}}
\]
and so there is \(\rho_n \in \partial J(u_n)\) such that \(w_n = Q'(\lambda u_n) - \rho_n\). Hence, \(\langle w_n, v \rangle \rightarrow 0, \; v \in W_0^{1,\Phi}(\Omega)\)
so that
\[
\int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla v dx = \lambda \int_{\Omega} hv dx + \int_{\Omega} \rho_n v dx + o_n(1).
\] (13)

The result below is inspired on lemma 1.20 of Willem [26].

**Lemma 4.2** The \((PS)_c\) - sequence \((u_n) \subseteq W_0^{1,\Phi}\) is bounded. In particular, there is some \(u_1 \in W_0^{1,\Phi}\) such that
\[
u_n \rightharpoonup u \quad \text{in} \; W_0^{1,\Phi}.
\]

**Proof of Lemma 4.1** By the very definition of \(j\), \(j(x,.)\) is differentiable at each \(t \neq a\) and
\[
\partial j(x,t) = j'(x,t) = \sigma(x)\phi_*(t)\chi_{\{t>a\}}.
\]

On the other hand, if \(t = a\), then (cf. [8]),
\[
\partial j(x,a) = \left[ \lim_{t \to a^-} \chi_{\{t>a\}} \sigma(x)\phi_*(t), \lim_{t \to a^+} \chi_{\{t>a\}} \sigma(x)\phi_*(t) \right] = [0,\sigma(x)\phi_*(a)].
\]

In particular, for each \(\rho = \rho(x) \in \partial j(x,t)\) with \(t \geq 0\) we have,
\[
0 \leq t\rho \leq \sigma(x)t\phi_*(t) \leq m^*\sigma(x)\Phi_*(t) \quad \text{a.e.} \quad x \in \Omega.
\]

Actually, if \(t > a\), then
\[
\ell^*\sigma(x)\Phi_*(t) \leq t\rho \leq m^*\sigma(x)\Phi_*(t) \quad \text{a.e.} \quad x \in \Omega.
\] (14)

Notice that if \(\rho := \rho(x) \in \partial j(x,t)\) then
\[
0 \leq \rho \leq \sigma(x)\phi_*(t) \leq |\sigma|_\infty\phi_*(t).
\]

Moreover, using the fact that \(\Phi_*(\phi_*(t)) \leq \Phi_*(2t),\) (cf. [14, p. 263]), we infer that
\[
0 \leq \rho \leq |\sigma|_\infty\Phi_*^{-1} \circ \Phi_*(2t),
\]
which is condition (1.6) in theorem 1.1 of [8]. This proves lemma 5.1.

**Proof of Lemma 4.2.** By (12) we have
\[
|\langle w_n, u_n \rangle| \leq m(u_n)\|u_n\| \leq \ell^*\|u_n\| \quad \text{for} \quad n \quad \text{large enough.}
\]

Set
\[
S_I := \sup_n I(u_n) < \infty.
\]
Estimating using the inequality above, (14), the Hölder Inequality and lemma 7.1 we have

\[ S_I + \|u_n\| \geq I(u_n) - \frac{1}{\ell^*}(w_n, u_n) \]

\[ \geq \left(1 - \frac{m}{\ell^*}\right) \int_{\Omega} \Phi(|\nabla u_n|) dx - \lambda \left(1 - \frac{1}{\ell^*}\right) \int_{\Omega} h u_n dx + \frac{1}{\ell^*} \int_{\{u_n = a\}} \rho_n a dx \]

\[ + \int_{\{u_n > a\}} \left[ \frac{1}{\ell^*} \rho_n u_n - j(x, u_n) \right] dx \]

\[ \geq \left(1 - \frac{m}{\ell^*}\right) \int_{\Omega} \Phi(|\nabla u_n|) dx + \int_{\{u_n > a\}} [\sigma(x) \Phi_*(u_n) - j(x, u_n)] dx \]

\[ - \lambda \left(1 - \frac{1}{\ell^*}\right) \int_{\Omega} h u_n dx \]

\[ \geq \left(1 - \frac{m}{\ell^*}\right) \int_{\Omega} \Phi(|\nabla u_n|) dx - 2 \lambda \left(1 - \frac{1}{\ell^*}\right) \|h\|_{\infty} \|u_n\|_{\Phi^*} \]

\[ \geq \left(1 - \frac{m}{\ell^*}\right) \min\{\|u_n\|^\ell, \|u_n\|^m\} - \frac{2\lambda}{S^*} \left(1 - \frac{1}{\ell^*}\right) \|h\|_{\Phi} \|u_n\|, \]

showing that \(\|u_n\|\) is bounded.

5 On the Convergence of the Palais-Smale Sequence

The result below is crucial, will be proved in detail in this paper, and actually, was motivated by lemma 4.4 by Fukagai, Ito & Narukawa [14].

**Lemma 5.1** Let \((u_n) \subset W^{1,\Phi}_0(\Omega)\) be the sequence in (12). Extend each \(u_n\) to \(\mathbb{R}^N\) by setting \(u_n = 0\) on \(\mathbb{R}^N \setminus \Omega\). Then there are \(x_1, \ldots, x_r \in \mathbb{R}^N\) such that

\[ u_n \xrightarrow{L_{\Phi_*(K)}} u \] \hspace{1cm} (15)

for each compact set \(K \subset \mathbb{R}^N \setminus \{x_1, \ldots, x_r\}\).

At first we gather some notations and remarks, (cf. Willem [26]). Given \(v \in C_0^{\infty}(\Omega)\) we extend it to \(\mathbb{R}^N\) by setting \(v(x) = 0\) if \(x \in \mathbb{R}^N \setminus \Omega\) and denote the extension by \(v\). Then \(v \in C_0^{\infty}(\mathbb{R}^N)\) and \(\text{supp}(v) \subseteq \Omega\). In addition,

\[ \|v\|_{W^{1,\Phi}(\mathbb{R}^N)} = \|v\|_{W^{1,\Phi}(\Omega)} \]

and

\[ W^{1,\Phi}_0(\Omega) = \{ v \in C_0^{\infty}(\mathbb{R}^N) \mid \text{supp}(v) \subseteq \Omega \}^{W^{1,\Phi}(\mathbb{R}^N)} \]

Thus, if \(v \in W^{1,\Phi}_0(\Omega)\) then \(v \in W^{1,\Phi}(\mathbb{R}^N)\). Similar notations for functions in \(L_{\Phi_*}(\Omega)\).

Consider the normed space

\[ C_0 = \{ u \in C(\Omega) \mid \text{supp}(u) \subseteq \mathbb{R}^N \}^{\text{cpt}|_{\infty}} \]
where \( |u|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)| \) and denote by \( \mathcal{M} \) the space of finite measures on \( \mathbb{R}^N \) with the norm

\[
\|\mu\|_{\mathcal{M}} = \sup \left\{ \int ud\mu \mid u \in C_0, \ |u|_\infty = 1 \right\}.
\]

**Remark 5.1** We recall below some notations and results:

(i) \( \mathcal{M} = C_0^* \) and \( \langle \mu, u \rangle = \int ud\mu, \)

(ii) \( \mu_n \overset{\mathcal{M}}{\rightharpoonup} \mu \) means that \( \int ud\mu_n \overset{n \to \infty}{\longrightarrow} \int ud\mu, \ u \in C_0, \)

(iii) if \( \mu_n \subseteq \mathcal{M} \) is bounded then \( \mu_n \overset{\mathcal{M}}{\rightharpoonup} \mu, \) up to subsequence.

By lemma 4.2 the \((PS)_{c}\)-sequence \((u_n) \subseteq W^{1}_0(\Omega)\) is bounded. Consider \( \mu_n, \nu_n : C_0 \to \mathbb{R}, \)

\[
\langle \mu_n, v \rangle = \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) vd\mu \quad \text{and} \quad \langle \nu_n, v \rangle = \int_{\mathbb{R}^N} \Phi_*(|u_n|) vd\mu, \ v \in C_0.
\]

Then there is a constant \( C > 0 \) such that

\[
|\langle \mu_n, v \rangle| \leq C|v|_\infty \quad \text{and} \quad |\langle \nu_n, v \rangle| \leq C|v|_\infty
\]

that is \((\mu_n), (\nu_n) \subseteq \mathcal{M} \) are bounded. It follows that

\[
\Phi(|\nabla u_n|) \rightharpoonup \mu, \quad \Phi_*(|u_n|) \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}.
\]

(16)

We shall need the following variant for Orlicz-Sobolev spaces of the concentration-compactness principle cf. Lions [21], Fukagai, Ito & Narukawa [14].

**Lemma 5.2** There exist a denumerable set \( J, \) a family \( \{x_j\}_{j \in J} \subseteq \mathbb{R}^N \) with \( x_i \neq x_j \) and families of nonnegative numbers \( \{\nu_j\}_{j \in J} \) and \( \{\mu_j\}_{j \in J} \) such that

\[
\nu = \Phi_*(u^1) + \sum_{j \in J} \nu_j \delta_{x_j} \quad \text{and} \quad \mu \geq \Phi(|\nabla u^1|) + \sum_{j \in J} \mu_j \delta_{x_j},
\]

where \( \delta_{x_j} \) is the Dirac measure with mass at \( x_j. \) In addition,

\[
\nu_j \leq \max \left\{ S^{-\frac{\ell - \eta}{\ell}} \frac{\nu_j^\ell}{\mu_j^\ell}, S^{-\frac{m - \eta}{m}} \frac{\nu_j^m}{\mu_j^m} \right\}, \quad j \in J.
\]

**Lemma 5.3** The set \( \bar{J} = \{ j \in J \mid \nu_j > 0 \} \) is finite.

**Proof** We claim that \( \{x_j\}_{j \in \bar{J}} \subseteq \overline{\Omega}. \) Indeed, if on the contrary, \( x_j \in \overline{\Omega^c} \) for some \( j \in \bar{J}, \) there is \( \epsilon > 0 \) such that \( \overline{B_\epsilon(x_j)} \subseteq \overline{\Omega^c}. \) Choose \( \varphi_\epsilon \in C_0^\infty(\mathbb{R}^N) \) such that

\[
\text{supp}(\varphi_\epsilon) \subseteq B_\epsilon(x_j), \quad \varphi_\epsilon \overset{\epsilon \to 0}{\longrightarrow} \chi_{\{x_j\}} \quad \text{a.e.} \quad \mathbb{R}^N.
\]

Now, we extend \( u_n \) to \( \mathbb{R}^N \) by setting \( u_n(x) = 0 \) for \( x \in \mathbb{R}^N - \Omega. \) Take \( \epsilon > 0. \) Using (16), we have

\[
0 = \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) \varphi_\epsilon dx \overset{n \to \infty}{\longrightarrow} \int_{\mathbb{R}^N} \varphi_\epsilon d\mu,
\]
and passing to the limit as $\epsilon \to 0$ we get,

$$0 = \int_{\mathbb{R}^N} \varphi_\epsilon d\mu = \int_{B_\epsilon(x_j)} \varphi_\epsilon d\mu \to \int_{\{x_j\}} d\mu = \mu_j.$$  

Hence, $\mu_j = 0$ and by lemma 5.2 we infer that $\nu_j = 0$, impossible because $j \in \bar{J}$, showing the claim. We claim that

$$(\phi(|\nabla u_n|)|\nabla u_n|) \text{ is bounded in } L_\Phi^\infty(\Omega)$$

Indeed, take $\psi \in C_0^\infty$ such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. Pick $x_j$ with $j \in \bar{J}$, $\epsilon > 0$ and set

$$\psi_\epsilon(x) := \psi\left(\frac{x - x_j}{\epsilon}\right), \quad x \in \mathbb{R}^N.$$  

Notice that $(\psi_i u_n) \subseteq W_0^{1, \Phi}(\Omega)$ is bounded. At this point we recall that

$$w_n = Q_\lambda'(u_n) - \rho_n \text{ for some } \rho_n \in \partial J(u_n). \quad (17)$$  

Since $m(u_n) \to 0$ we infer from (17) that

$$\int_{\Omega} \phi(|\nabla u_n|)\nabla u_n \nabla(\psi_\epsilon u_n) = \lambda \int_{\Omega} hu_n \psi_\epsilon dx + \int_{\Omega} \rho_n u_n \psi_\epsilon dx + o_n(1). \quad (18)$$  

Moreover, by lemma 4.1, $\rho_n \in L_\Phi^\infty(\Omega)$ and $\rho_n(x) \in \partial j(x, u_n(x))$ for $x \in \Omega$. By (18) and lemma 4.1,

$$\int_{\Omega} \phi(|\nabla u_n|)\nabla u_n \nabla(\psi_\epsilon u_n) = \left(\int_{\{u_n < a\}} + \int_{\{u_n \geq a\}}\right) \rho_n u_n \psi_\epsilon dx + \lambda \int_{\Omega} hu_n \psi_\epsilon dx + o_n(1)$$

$$= \int_{\{u_n < a\}} \rho_n u_n \psi_\epsilon dx + \lambda \int_{\Omega} hu_n \psi_\epsilon dx + o_n(1)$$

$$\leq m^* \int_{\{u_n \geq a\}} \sigma(x)\Phi_\Phi(u_n) \psi_\epsilon dx + \lambda \int_{\Omega} hu_n \psi_\epsilon dx + o_n(1)$$

$$\leq m^* |\sigma|_\infty \int_{\Omega} \Phi_\Phi(u_n) \psi_\epsilon dx + \lambda \int_{\Omega} hu_n \psi_\epsilon dx + o_n(1) \quad (19)$$

On the other hand, using the fact that $t^2 \phi(t) \geq \Phi(t)$ we have,

$$\int_{\Omega} \phi(|\nabla u_n|)\nabla u_n \nabla(\psi_\epsilon u_n) = \int_{\Omega} u_n \phi(|\nabla u_n|)\nabla u_n \nabla \psi_\epsilon dx + \int_{\Omega} \psi_\epsilon \phi(|\nabla u_n|)|\nabla u_n|^2 dx$$

$$\geq \int_{\Omega} u_n \phi(|\nabla u_n|)\nabla u_n \nabla \psi_\epsilon dx + \int_{\Omega} \psi_\epsilon \Phi(|\nabla u_n|) dx \quad (20)$$

Using (19), (20) and the inequality $\tilde{\Phi}(t \phi(t)) \leq \Phi(2t)$ it follows that $(\phi(|\nabla u_n|)|\nabla u_n|)$ is bounded in $L_\Phi^\infty(\Omega)$, showing the claim.

As a consequence $(\phi(|\nabla u_n|)\partial u_n/\partial x_i)$ is also bounded in $L_\Phi^\infty(\Omega)$ and so

$$\phi(|\nabla u_n|)\frac{\partial u_n}{\partial x_i} \rightarrow w_i \text{ in } L_\Phi^\infty(\Omega), \quad i = 1, \ldots, N. \quad (21)$$
Setting \( w = (w_1, \ldots, w_N) \), we claim that
\[
\int_{\Omega} (u_n \phi(|\nabla u_n|) \nabla u_n \nabla \psi - u \, w \cdot \nabla \psi) \, dx = o_n(1) . \tag{22}
\]
Indeed, in a first step applying an easy estimate and in a second step using the the Hölder inequality and applying (21) with test function \( \partial \psi \), we have
\[
\int_{\Omega} \left| \phi(|\nabla u_n|) \frac{\partial u_n}{\partial x_i} \partial \psi \right| u_n \frac{\partial \psi}{\partial x_i} \, dx \leq
\int_{\Omega} \left| \phi(|\nabla u_n|) \frac{\partial u_n}{\partial x_i} (u_n - w_i \frac{\partial \psi}{\partial x_i}) \right| \, dx + \int_{\Omega} \left| \phi(|\nabla u_n|) \frac{\partial u_n}{\partial x_i} \frac{\partial \psi}{\partial x_i} \right| \, dx \leq
2 \left\| \phi(|\nabla u_n|) \frac{\partial u_n}{\partial x_i} \frac{\partial \psi}{\partial x_i} \right\|_{\phi} \| u_n - u \|_{\phi} + o_n(1)
\]
which leads to (22), showing the claim. Replacing (22) in (20) we get
\[
\int \partial \psi \Phi(|\nabla u_n|) \, dx + \int uw \nabla \psi \, dx \leq \int \phi(|\nabla u_n|) \nabla u_n \nabla (\psi \psi u_n) + o_n(1). \tag{23}
\]
It follows from (19) and (23) that
\[
\int \partial \psi \Phi(|\nabla u_n|) \, dx + \int uw \nabla \psi \, dx \leq m^* |\sigma|_{\infty} \int \Phi_\ast(u_n) \psi \, dx + \lambda \int h u_n \psi \, dx + o_n(1).
\]
Passing to the limit in the inequality just above in \( n \), recalling that
\[
\int \partial \psi \Phi(|\nabla u_n|) \, dx \xrightarrow{n \to \infty} \int \psi \, d\mu, \quad \int \Phi_\ast(u_n) \psi \, dx \xrightarrow{n \to \infty} \int \psi \, d\nu
\]
and
\[
\int uh \psi \, dx \xrightarrow{n \to \infty} \int uh \psi \, dx.
\]
we get to
\[
\int \partial \psi \, d\mu + \int uw \nabla \psi \, dx \leq m^* |\sigma|_{\infty} \int \psi \, d\nu + \lambda \int h u \psi \, dx. \tag{24}
\]
We claim that \( (\rho_n) \) is bounded in \( L_{\Phi_\ast}^\infty(\Omega) \). Indeed, using lemma 4.1 we get
\[
\int \Phi_\ast(\rho_n) \, dx \leq \int \Phi_\ast(\sigma(x) \phi_\ast(u_n)) \, dx \leq \int \Phi_\ast(|\sigma|_{\infty} \phi_\ast(u_n)) \, dx
\]
\[
\leq C |\sigma|_{\infty} \int \Phi_\ast(\phi_\ast(u_n)) \, dx \leq C |\sigma|_{\infty} \int \Phi_\ast(2u_n) \, dx \leq C,
\]
showing the claim. Thus there is \( \rho \in L_{\Phi_\ast}^\infty(\Omega) \) such that
\[
\rho_n \to \rho \text{ in } L_{\Phi_\ast}^\infty(\Omega).
\]
Let \( v \in W_0^{1,\Phi}(\Omega) \). Passing to the limit in the expression
\[
\langle w_n, v \rangle = \int_{\Omega} (\phi(|\nabla u_n|)|\nabla u_n| \nabla v - \lambda h v - \rho_n v) \, dx,
\]
and using (21) we get to
\[
\int_{\Omega} (w \nabla v - \lambda h v - \rho v) \, dx = 0. \tag{25}
\]
Setting \( v = u\psi_\epsilon \) in (25) we have
\[
\int_{\Omega} uw \nabla \psi_\epsilon \, dx = \int_{\Omega} (\lambda h u + \rho u - w \nabla u) \psi_\epsilon \, dx.
\]
But
\[
| (\lambda h u + \rho u - w \nabla u) \psi_\epsilon | \leq |\lambda h u| + |\rho u| + |w \nabla u| \in L^1(\Omega)
\]
and
\[
(\lambda h u + \rho u - w \nabla u)\psi_\epsilon \xrightarrow{\epsilon \to 0} 0 \text{ a.e. in } \Omega
\]
By means of Lebesgue’s theorem,
\[
\int_{\Omega} uw \nabla \psi_\epsilon \, dx \xrightarrow{\epsilon \to 0} 0 \text{ and } \int_{\Omega} h u \psi_\epsilon \, dx \xrightarrow{\epsilon \to 0} 0.
\]
Noticing that
\[
\psi_\epsilon \xrightarrow{\epsilon \to 0} \chi_{\{x_j\}} \text{ a.e. in } \mathbb{R}^N \text{ and } \psi_\epsilon(x) \leq \chi_{B_\delta(x_j)}(x) \text{ for } x \in \mathbb{R}^N, \ \epsilon > 0 \text{ small}
\]
we get to
\[
\int_{\mathbb{R}^N} \psi_\epsilon \, d\mu \xrightarrow{\epsilon \to 0} \int_{\{x_j\}} d\mu = \mu(\{x_j\}) = \mu_j \text{ and } \int_{\mathbb{R}^N} \psi_\epsilon \, d\nu \xrightarrow{\epsilon \to 0} \int_{\{x_j\}} d\nu = \nu(\{x_j\}) = \nu_j.
\]
Passing to the limit in (24) we get to
\[
\mu_j \leq m^*|\sigma|_{\infty} \nu_j, \ j \in \tilde{J}. \tag{26}
\]
By lemma 5.2, \( \mu_j \leq c_1 \mu_j^\alpha \), where \( 1 < \alpha \leq \min \{ \ell^*/\ell, m^*/\ell, \ell^*/m, m^*/m \} \).
Thus \( \mu_j \geq c_2 \) for some positive constant \( c_2 \). In addition by (26), \( \nu_j \geq c_3 \), for \( j \in \tilde{J} \) and for some positive constant \( c_3 \). At this point, we infer that if \#(\tilde{J}) = \infty, then
\[
\sum_{j \in \tilde{J}} \nu_j \geq \sum_{j \in \tilde{J}} c_3 = \infty,
\]
which is impossible because \( \nu \) is a finite measure and
\[
\nu = \Phi_\sigma(u) + \sum_{j \in \tilde{J}} \nu_j \delta_{x_j}.
\]
This ends the proof of lemma 5.3.

**Proof of Lemma 5.1** Since \( \tilde{J} \) is finite pick \( \delta > 0 \) such that \( B_\delta(x_i) \cap B_\delta(x_j) = \emptyset \) for \( i \neq j \) with \( i, j \in \tilde{J} \). Next take a compact set \( K_\delta \subset \mathbb{R}^N \setminus \bigcup_{j \in \tilde{J}} B_\delta(x_j) \) and \( \chi \in C_0^\infty \) such that
\[
0 \leq \chi \leq 1, \ \chi = 1 \text{ on } K_\delta, \ \text{supp}(\chi) \cap \left( \bigcup_{j \in \tilde{J}} B_{\frac{\delta}{2}}(x_j) \right) = \emptyset.
\]
Notice that
\[ \Phi_*(u_n - u) \rightharpoonup \nu \quad \text{and} \quad \nu = \Phi_*(0) + \sum_{j \in \tilde{J}} \nu_j \delta_{x_j} \quad \text{in} \quad \mathcal{M}. \]

On the other hand,
\[
0 \leq \int_{K_\delta} \Phi_*(u_n - u) \, dx \leq \int_{\mathbb{R}^N} \Phi_*(u_n - u) \chi \, dx,
\]
\[
\int_{\mathbb{R}^N} \Phi_*(u_n - u) \chi \, dx \rightarrow \int_{\mathbb{R}^N} \chi \, d\nu,
\]
\[
\int_{\mathbb{R}^N} \chi \, d\nu = \sum_{j \in \tilde{J}} \chi(x_j) = 0.
\]

Thus
\[
\int_{K_\delta} \Phi_*(u_n - u) \, dx \rightarrow 0.
\]

Since the argument above holds for each \( \delta > 0 \) we infer that (15) holds for each compact set \( K \subseteq \mathbb{R}^N - \{x_j\}_{j \in \tilde{J}} \).

### 6 Proofs of the Main Results

**Lemma 6.1** \( \rho_n(x) \rightharpoonup \rho(x) \) and \( \rho(x) \in \partial j(x, u(x)) \) a.e. \( x \in \Omega \).

**Proof** We will show, at first that
\( \rho(x) \in \partial j(x, u(x)) \) a.e. \( x \in \Omega \).

Indeed, let \( K \subseteq \mathbb{R}^N \setminus \{x_j\}_{j \in \tilde{J}} \) be a compact set and take \( \varphi \in L_{\overline{\Phi}_*}(K) \).

Since
\( \rho_n \in \partial J(u_n), \quad \rho_n \rightharpoonup \rho \quad \text{in} \quad L_{\overline{\Phi}_*}(\Omega) \quad \text{and} \quad \rho_n(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^N \setminus \Omega \)

then
\[ \rho_n \rightharpoonup \rho \quad \text{in} \quad L_{\overline{\Phi}_*}(K) \]

and so
\[ \rho_n \overset{\ast}{\rightharpoonup} \rho, \quad \text{em} \quad L_{\overline{\Phi}_*}(K). \]

On the other hand, by lemma 5.1,
\[ u_n \overset{L_{\Phi_*(K)}}{\rightharpoonup} u \]

and by [10, Proposition 2.1.5], \( \rho \in \partial J(u) \). By the Aubin-Clarke theorem (cf. lemma 4.1 above ),
\[ \rho \in L_{\overline{\Phi}_*}(K) \quad \text{and} \quad \rho(x) \in \partial j(x, u(x)) \quad \text{a.e.} \quad x \in K. \]

Since
\[ \mathbb{R}^N - \{x_j\}_{j \in \tilde{J}} = \bigcup_{\nu=1}^{\infty} K_{\nu}, \tag{27} \]

where \( \{K_{\nu}\}_{\nu=1}^{\infty} \) is a sequence of compact sets, it follows that \( \rho(x) \in \partial j(x, u(x)) \) a.e. \( x \in \Omega \).

Next we will show that
\[ \rho_n(x) \rightharpoonup \rho(x) \quad \text{a.e.} \quad x \in \Omega. \]
Indeed, take \( \varphi_\nu \in C_0^\infty(\mathbb{R}^N) \) such that \( \text{supp}(\varphi_\nu) = K_\nu \). Then
\[
\int_\Omega (\rho_n - \rho) \varphi_\nu \, dx = \int_{K_\nu} (\rho_n - \rho) \varphi_\nu \, dx \xrightarrow{n \to \infty} 0,
\]
As a consequence,
\[
(\rho_n - \rho) \varphi_\nu \xrightarrow{n \to \infty} 0 \text{ a.e. in } K_\nu,
\]
so that
\[
\rho_n - \rho \xrightarrow{n \to \infty} 0 \text{ a.e. in } K_\nu.
\]
Therefore
\[
\rho_n - \rho \xrightarrow{n \to \infty} 0 \text{ a.e. in } \mathbb{R}^N.
\]
Since \( \rho_n = 0 \) on \( \mathbb{R}^N - \Omega \), it follows that
\[
\rho_n - \rho \xrightarrow{n \to \infty} 0 \text{ a.e. in } \Omega.
\]
This ends the proof of lemma 6.1.

The proof of the next lemma is based on lemma 4.5 in [14].

**Lemma 6.2** \( \nabla u_n(x) \xrightarrow{n \to \infty} \nabla u(x) \) a.e. \( x \in \Omega \).

**Proof** Let \( \{K_\nu\}_{\nu=1}^\infty \) be a family of compact sets such that (27) holds. Pick an integer \( \nu \geq 1 \) and a function \( \chi \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \chi \leq 1, \chi = 1 \text{ on } K_\nu \) and \( \text{supp}(\chi) \cap \{x_j\}_{j \in \tilde{J}} \neq \emptyset \).

Set \( v_n = \chi(u_n - u) \). It follows that \( v_n \) is bounded in \( W_0^1,\Phi(\mathbb{R}^N) \) and since \( \langle m(u_n), v_n \rangle \to 0 \) we infer that
\[
\int_{\mathbb{R}^N} \phi(|\nabla u_n|) \nabla u_n \nabla v_n \, dx - \lambda \int_{\mathbb{R}^N} h v_n \, dx - \int_{\mathbb{R}^N} \rho_n v_n \, dx = o_n(1). \tag{28}
\]

Setting \( S_\chi = \text{supp}(\chi) \) we get
\[
\int_{S_\chi} \phi(|\nabla u_n|) \nabla u_n (\nabla u_n - \nabla u) \, dx + \int_{S_\chi} \phi(|\nabla u_n|) \nabla u_n \nabla \chi(u_n - u) \, dx
\]
\[= \int_{S_\chi} h v_n \, dx + \int_{S_\chi} \rho_n v_n \, dx + o_n(1). \tag{29}\]

Notice that
\[
\int_{S_\chi} |\phi(|\nabla u_n|) \nabla u_n \nabla \chi(u_n - u)| \, dx \leq \|\phi(|\nabla u_n|) \nabla u_n \|_{L_\Phi(S_\chi)} |\nabla \chi|_\infty \|u_n - u\|_{L_\Phi(S_\chi)} = o_n(1),
\]
\[
\int_{S_\chi} h v_n \, dx = o_n(1),
\]
and since \( (\rho_n) \) is bounded in \( L_{\tilde{\Phi}_\ast}(\Omega) \),
\[
\int_{\mathbb{R}^N} |\rho_n v_n| \, dx \leq \|\rho_n\|_{\tilde{\Phi}_\ast} |\chi|_\infty \|u_n - u\|_{L_{\tilde{\Phi}_\ast}(S_\chi)} = o_n(1),
\]
which shows via (29) that
\[
\int_{K_\nu} \phi(|\nabla u_n|) \nabla u_n (\nabla u_n - \nabla u) \, dx \xrightarrow{n \to \infty} 0.
\]
Using the well known fact that $-\Delta \Phi$ is a map of type $(S_+)$,
\[
||\nabla u_n - \nabla u||_{L^p(K_\nu)}^n \to 0.
\]
It follows that
\[
\nabla u_n \nabla u \text{ a.e. on } K_\nu
\]
and as a consequence,
\[
\nabla u_n \to \nabla u \text{ a.e. on } \mathbb{R}^N.
\]
Recalling that $u_n(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$, we get to
\[
\nabla u_n \to \nabla u \text{ a.e. in } \Omega,
\]
ending the proof of lemma 6.2.

\begin{lemma}
\textbf{Lemma 6.3} \quad \phi(||\nabla u_n||) \nabla u_n \rightharpoonup \phi(||\nabla u||) \nabla u \text{ in } \prod L_{\tilde{\Phi}}(\Omega).
\end{lemma}

\begin{proof}
By lemma 6.2,
\[
\nabla u_n \to \nabla u \text{ a.e. in } \Omega.
\]
Since $\phi$ is continuous,
\[
\phi(||\nabla u_n||) \nabla u_n \rightharpoonup \phi(||\nabla u||) \nabla u \text{ a.e. in } \Omega.
\]
Applying lemma 2 in Gossez [16, p 88], ends the proof of lemma 6.3.
\end{proof}

\begin{proof}[Proof of Theorem 1.1]
By lemma 6.3,
\[
\int_{\Omega} \phi(||\nabla u_n||) \nabla u_n \nabla v dx \to \int_{\Omega} \phi(||\nabla u||) \nabla u \nabla v dx, \quad v \in W^{1,\Phi}_0.
\]
On the other hand,
\[
\int_{\Omega} \rho_n v dx \to \int_{\Omega} \rho v dx, \quad v \in W^{1,\Phi}_0,
\]
where
\[
\rho \in L_{\tilde{\Phi}_*}(\Omega) \quad \text{and} \quad \rho_n(x) \in \partial j(x,u_n(x)) \text{ a.e. } x \in \Omega.
\]
Passing to the limit in (13) we get to
\[
\int_{\Omega} \phi(||\nabla u||) \nabla u \nabla v dx - \lambda \int_{\Omega} h v dx - \int_{\Omega} \rho v dx = 0, \quad v \in W^{1,\Phi}_0.
\]
Thus $u \in W^{1,\Phi}_0$ is a solution of (1), in the sense of Definition 1.1 and since $h \neq 0$, we get $u \neq 0$.

\begin{claim}
$u \geq 0$. Indeed, note that
\[
u_n = v_n^+ - u_n^-, \quad \nabla u_n = \nabla u_n^+ - \nabla u_n^- \quad \text{and} \quad |\nabla u_n|^2 = |\nabla u_n^+|^2 + |\nabla u_n^-|^2.
\]
Thus
\[
\int_{\Omega} \phi(||\nabla u_n^-||) v dx \leq \int_{\Omega} \phi(||\nabla u_n^-||^2 + ||\nabla u_n^+||^2) dx
\]
\[= \int_{\Omega} \phi(||\nabla u_n||) dx
\]
\end{claim}
so that \((u_n^-)\) is bounded in \(W_0^{1,\Phi}\). Noting that \(\langle w_n, -u_n^- \rangle = o_n(1)\) we have
\[
o_n(1) = -\int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla u_n^- \, dx + \lambda \int_{\Omega} h u_n^- \, dx + \int_{\Omega} \rho_n u_n^- \, dx \\
= \int_{\Omega} \phi(|\nabla u_n^-|) |\nabla u_n^-|^2 \, dx + \lambda \int_{\Omega} h u_n^- \, dx + \int_{\Omega} \rho_n u_n^- \, dx \\
\geq \ell \int_{\Omega} \Phi(|\nabla u_n^-|) \, dx.
\]
Thus
\[
\int_{\Omega} \Phi(|\nabla u_n^-|) \, dx \to 0,
\]
and hence \(u_n^- \to 0\) in \(W_0^{1,\Phi}\), showing that \(u \geq 0\).

**Proof of (3)** Since \(u\) is a solution of (1), there is \(\rho := \rho_u \in L_{\Phi^*}(\Omega)\) such that
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx = \int_{\Omega} \rho v \, dx + \lambda \int_{\Omega} h v \, dx, \quad v \in C_0^\infty(\Omega).
\]
By Remark 2.1, \(\Delta \Phi u \in L_{\Phi^*}(\Omega)\). Since also \(h \in L_{\Phi^*}(\Omega)\) it follows that
\[
\int_{\Omega} [-\Delta \Phi u - \rho - \lambda h] v \, dx = 0, \quad v \in C_0^\infty(\Omega).
\]
Hence
\[
-\Delta \Phi u = \rho + \lambda h \quad \text{a.e. in } \Omega.
\]

7 Appendix

The results below are elementary and can be found in [14, 15].

**Lemma 7.1** Assume \((\phi_1) - (\phi_3)\). Let
\[
\zeta_0(t) = \min\{t^\ell, t^m\} \quad \text{and} \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \geq 0.
\]
Then
\[
\zeta_0(\rho) \Phi(t) \leq \Phi(\rho t) \leq \zeta_1(\rho) \Phi(t), \quad \rho, t > 0,
\]
\[
\zeta_0(\|u\|_{\Phi}) \leq \int_{\Omega} \Phi(u) \, dx \leq \zeta_1(\|u\|_{\Phi}), \quad u \in L_{\Phi}(\Omega).
\]

**Lemma 7.2** Assume \((\phi_1) - (\phi_3)\). Let
\[
\zeta_2(t) = \min\{t^{\ell'}, t^{m'}\} \quad \text{and} \quad \zeta_2(t) = \max\{t^{\ell'}, t^{m'}\}, \quad t \geq 0.
\]
Then
\[
\zeta_2(\rho) \Phi_*(t) \leq \Phi_*(\rho t) \leq \zeta_3(\rho) \Phi_*(t), \quad \rho, t > 0,
\]
\[
\zeta_2(\|u\|_{\Phi_*}) \leq \int_{\Omega} \Phi_*(u) \, dx \leq \zeta_3(\|u\|_{\Phi_*}), \quad u \in L_{\Phi_*}(\Omega).
\]
References

[1] Adams, R., *Sobolev Spaces*, Academic Press, New York, (1975).

[2] Alves, C. O., Bertone, A. M. & Gonçalves, J. V. A., *A variational approach to discontinuous problems with critical Sobolev exponents*, J. Math. Anal. App. 265 (2002) 103-127

[3] Alves, Claudianor Oliveira & Bertone, Ana Maria, *A discontinuous problem involving the p-Laplacian operator and critical exponent in RN*, Electron. J. Differential Equations (2003).

[4] Alves, C. O., Goncalves, J. V. & Santos, J. A., *On multiple solutions for multivalued elliptic equations under Navier boundary conditions*, J. Convex Analysis 18 (2011) 627-644.

[5] Ambrosetti, A. & Turner, R. E. L., *Some discontinuous variational problems*, Diff. Int. Equns., 3 (1988) 341-349.

[6] Badiale, M. & Tarantello, G., *Existence and Multiplicity results for elliptic problems with critical growth and discontinuous nonlinearities*, Nonlinear Anal. 29 (1997) 639-677.

[7] Brézis, H, Nirenberg, L. & Stampacchia, G., *Remarks on Ky Fan's Min-max Theorem*, Boll. U. M. I. 6 (1972) 293-300.

[8] Carl, S., Le, Vy Khoi & Montreanu, D., *Nonsmooth Variational Problems and Their Inequalities - Comparison Principles and Applications*, Springer, New York, (2007)

[9] Chang, K. C., *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. 80 (1981) 102-129.

[10] Clarke, F.H., *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, (1990).

[11] Donaldson, T. K. & Trudinger, N. S., *Orlicz-Sobolev spaces and imbedding theorems*, J. Functional Analysis 8 (1971) 52-75.

[12] Fiacca, A., Matzakos, N., Papageorgiou, N. S. & Servadei, R., *Nonlinear elliptic differential equations with multivalued nonlinearities*, Czechoslovak Math. J. 53 (2003) 135-159.

[13] Filippakis, M. & Papageorgiou, N., *Multiple solutions for nonlinear elliptic problems with a discontinuous nonlinearity*, Anal. Appl. (2006) 1-18.

[14] Fukagai, N., Ito, M. & Narukawa, K., *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on RN*, Funkcialaj Ekvacioj, 49 (2006) 235-267.

[15] Fukagai, N. & Narukawa, K., *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Annali di Matematica, 186 (2007) 539-564.

[16] Gossez, Jean-Pierre, *Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems. Nonlinear analysis, function spaces and applications*, (Proc. Spring School, Horni Bradlo, 1978), Teubner, Leipzig, (1979) 59-94.

[17] Halidias, N. & Naniewicz, Z., *On a class of hemivariational inequalities at resonance*, J. Math. Anal. Appl. 289 (2004) 584-607.

[18] Kufner, A., John, O. & Fučík, S., *Function spaces*, Noordhoff, Leyden, (1977).
[19] Kyritsi, S. Th. & Papageorgiou, N., *Multiple solutions for strongly resonant nonlinear elliptic problems with discontinuities*, Proc. Amer. Math. Soc. 133 (2005) 2369-2376.

[20] Le, Vy Khoi, Motreanu, D. & Motreanu, V., *On a non-smooth eigenvalue problem in Orlicz-Sobolev spaces*, Appl. Anal. 89 (2010) 229-242.

[21] Lions, P. L. *The concentration-compactness principle in the calculus of variations. The limit case. I* Rev. Mat. Iberoamericana 1 (1985) 145-201.

[22] Motreanu, D. & Panagiotopoulos, P. D., *Minimax Theorems and Qualitative Properties of Solutions of Hemivariational Inequalities*, Kluwer Academic Publishers (1999).

[23] Motreanu, D. & Tanaka, M., *Existence of solutions for quasilinear elliptic equations with jumping nonlinearities under Neumann boundary conditions*, Calc. Var. 43 (2012) 231-264.

[24] Naniewicz, Z., *On economics equilibrium type problems with applications*, Set Valued Analysis 19 (2011) 417-456.

[25] Rao, M. N. & Ren Z. D., *Theory of Orlicz Spaces*, Marcel Dekker, New York, (1985).

[26] Willem, M., *Minimax Theorems*, Birkhauser, (1996).

J. V. Gonçalves
Universidade Federal de Goiás
Instituto de Matemática e Estatística
74001-970 Goiânia, GO - Brasil

M. L. Carvalho
Universidade Federal de Goiás
Departamento de Matemática
75804-020 Jataí, GO - Brasil