A non-rational CFT with central charge 1

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Abstract

Two dimensional conformal field theories with central charge one are discussed. After a short review of theories based on one free boson, a different CFT is described, which is obtained as a limit of minimal models.

We will be concerned with unitary two dimensional conformal field theories. Such theories describe, for example, critical points of equilibrium statistical mechanics systems in two dimensions, or vacua of string theory around which a perturbative expansion of string amplitudes can be computed.

Since the discovery by Belavin, Polyakov and Zamolodchikov of a discrete series of CFTs, the minimal models \cite{1}, much effort has been spent in trying to classify all CFTs. An important parameter in this problem is the central charge $c$ of a CFT, which appears in the two point function of the stress tensor $T(\zeta)$ on the complex plane

$$\langle T(\zeta)T(\zeta') \rangle = \frac{c/2}{(\zeta - \zeta')^4}. \quad (1)$$

Another useful notion is that of a rational CFT. Roughly speaking, a CFT is rational when the algebra generated by all conserved charges is large enough for the Hilbert space to decompose into only finitely many representations.

Here we are only considering unitary CFTs which makes the classification problem somewhat easier, as for example logarithmic CFTs \cite{2} cannot occur. Nonetheless it turned out to be a very difficult problem. A very brief summary of the state of affairs could take the following form

- $c < 1$: All unitary CFTs with central charge less than one are known, they are the minimal models of \cite{1} whose different modular invariants were given in \cite{3}. Their central charges take the values

$$c = 1 - \frac{6}{p(p + 1)}. \quad (2)$$

with $p = 2, 3, 4, \ldots$. For $p = 2$ on has $c = 0$ and the theory is trivial in the sense that its only state is the vacuum. $p = 3$ gives $c = 1/2$ and one obtains the Ising model. For $p \to \infty$ the central charge approaches one. All these theories are rational CFTs.
• $c = 1$: All rational CFTs with central charge one are believed to be known \cite{4, 5}, but no complete list also including all non-rational theories exists. Until recently all examples of non-rational CFTs with $c=1$ were based on the free boson. Here we want to describe another example of such a theory \cite{6}.

• $c > 1$: Here the situation is even more hopeless. The list of known CFTs with central charge greater than one is enormous and includes for example WZW models and cosets thereof, supersymmetric theories, theories with W-symmetry or Liouville theory \cite{7}. The question of a complete classification seems quite out of reach, the hardest part is certainly to get a handle on the non-rational theories.

The value $c = 1$ is special, because it is the smallest central charge where we do not have complete knowledge, and it is also the smallest $c$ for which non-rational theories can occur. From hereon we concentrate on models with $c = 1$, starting with the free boson.

**Free boson**

In \cite{4} Ginsparg presented a list of theories with central charge one based on the free boson. The resulting moduli space is drawn below.

The horizontal axis corresponds to models where the target space of the free boson is periodically identified, $X \sim X + 2\pi r$. T-duality says that the free boson compactified on radius $r$ is not different from the free boson on radius $1/2r$ (in the conventions of \cite{4}). This is the reason the horizontal line ends at $r = 1/\sqrt{2}$. On the right end we have included a point which corresponds to the uncompactified free boson, i.e. to infinite radius.

The vertical line is the moduli space of orbifold models. Those arise when the free boson is modded out by the symmetry $X \rightarrow -X$ of the action. In this procedure the Hilbert space is projected onto invariant states and twisted sectors (corresponding to strings that close up to an action of the symmetry) are added to the Hilbert space.

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* It is customary to insert the notion of “quasi-rational” as another step between rational and non-rational. For the present purposes we can think of non-rational as “not rational”; what is said remains true.
The orbifolded theory at $r = 1/\sqrt{2}$ and the unorbifolded theory at $r = \sqrt{2}$ turn out to be equivalent (in particular their partition functions are equal), and hence the two lines meet at that point. 

Ultimately this is related to an enhancement of the symmetry of the free boson compactified at the self-dual radius $r = 1/\sqrt{2}$, where it is equal to the WZW model $su(2)_1$. This also accounts for the three isolated points, which are obtained by modding out the $su(2)$ theory by the tetrahedral, octahedral and icosahedral subgroups.

The question is now if the moduli space of $c = 1$ theories based on the free boson exhausts already all unitary CFTs with $c = 1$. It has been argued \[5\] that this list does in fact include all rational theories with $c = 1$. The rational theories are those, for which the radius squares to a rational number, in these cases additional conserved charges appear. As we will see in a moment, there is however at least one non-rational theory which is not included in this list.

**Limit of minimal models**

To work out the quantum theory of the free boson, one usually uses canonical quantisation starting from the classical action. Here we would like not to worry whether a CFT can be obtained from an action and rather define it directly in terms of the following data:

- the Hilbert space $H$ of the theory.
- the two– and three–point functions on the complex plane.
- the boundary state for the unit disc.

To see why this is enough, consider for example a four point function on the complex plane. By inserting a complete basis of states, it can be reduced to an integral over three point functions as in

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \int dp \langle \phi(x_1)\phi(x_2) | p \rangle \langle p | \phi(x_3)\phi(x_4) \rangle . \quad (3)$$

The boundary state is needed if one wants to evaluate correlators on surfaces with a boundary (via a conformal mapping to the unit disc).

The model we want to describe is obtained as a limit of minimal models $M_p$. To be more precise, for an allowed value of the central charge $c < 1$, there can be one, two or three distinct CFTs with different modular invariant partition functions \[3\]. $M_p$ denotes the simplest, the charge conjugation modular invariant.

The limit we are interested in is that of $p \to \infty$, which leads to $c \to 1$. The minimal models $M_p$ are known in great detail, in particular there are explicit expressions for the OPE structure constants \[8\]. Using these, one can formulate a limiting theory $M_\infty$ in terms of the three bits of data described above, which has central charge one. The details of that construction can be found in \[6\], here we just present the outcome. The explicit formulas are given mainly to illustrate that everything fits on one page.

The Hilbert space of $M_\infty$ decomposes into representations of $Vir \otimes Vir$, where $Vir$ is the Virasoro algebra, the algebra of modes of the holomorphic (resp. anti-holomorphic) component of the stress tensor. A representation of $Vir$ is characterised by the central charge (which is one) and the lowest conformal weight $h$ in the representation. Paradoxically, the state with lowest conformal weight is usually called a “highest weight state”, in accordance with representation theory of Lie algebras, and we will also stick to that name.
The Hilbert space $\mathcal{H}$ consists of highest weight states $\ket{x}$ and the $\Vir \otimes \Vir$ modules build on them. The parameter $x$ takes any value in the set $\mathbb{R}_{>0} - \mathbb{Z}_{>0}$ and denotes a state with left/right conformal weight equal to $h_x = x^2/4$. A field $\phi_x$ corresponding to a highest weight state $\ket{x}$ in $\mathcal{H}$ is called a primary field.

The two and three point functions of primary fields $\phi_x, \phi_y, \phi_z$ read

$$\langle \phi_x(\zeta_1, \bar{\zeta}_1) \phi_y(\zeta_2, \bar{\zeta}_2) \rangle = \delta(x-y) \cdot |\zeta_1 - \zeta_2|^{-x^2} ,$$

$$\langle \phi_x(\zeta_1, \bar{\zeta}_1) \phi_y(\zeta_2, \bar{\zeta}_2) \phi_z(\zeta_3, \bar{\zeta}_3) \rangle = c(x, y, z) \cdot |\zeta_{12}|(x^2-y^2)^2/2 \times |\zeta_{13}|(y^2-z^2)^2/2 \times |\zeta_{23}|(x^2-z^2)^2/2 ,$$

where $\zeta_{ij} = \zeta_i - \zeta_j$ and

$$c(x, y, z) = P(x, y, z) \cdot \exp(Q(x, y, z))$$

$$P(x, y, z) = \begin{cases} \frac{1}{2} : & (|x| + |y| + |z| \text{ even}, \text{ and } |x-y| < f_{x-y} < \min(f_{x+y}, 2-f_{y-x}) ) \text{ or } \\
0 : & (|x| + |y| + |z| \text{ odd}, \text{ and } |x-y| < 1-f_{x-y} < \min(f_{x+y}, 2-f_{y-x}) ) \end{cases}$$

$$Q(x, y, z) = \int_{0}^{1} \frac{d\beta}{(-\ln \beta)(1-\beta)^2} \cdot \left\{ 2 + \sum_{\varepsilon = \pm 1} (\beta^{\varepsilon x} + \beta^{\varepsilon y} + \beta^{\varepsilon z}) - \sum_{\varepsilon_x, \varepsilon_y, \varepsilon_z = \pm 1} \beta^{(\varepsilon_x x + \varepsilon_y y + \varepsilon_z z)/2} \right\}$$

Here $[x]$ denotes the largest integer less than or equal to $x$ and $f_x = x - [x]$ is the fractional part of $x$.

Finally, the boundary states which are obtained from the limit of minimal models are

$$\langle a \rangle = 2^{3/4} \int_{0}^{\infty} dx \ (-1)^{[x]} \sin(a \pi x) \langle \langle x \rangle \rangle ,$$

where $a \in \mathbb{Z}_{>0}$ and $\langle \langle x \rangle \rangle$ is the Ishibashi state in the representation with highest weight state $\langle a \rangle$. These boundary conditions and their boundary field content are discussed in detail in [1].

In analogy with Liouville theory [10] one would expect that there are more conformal boundary conditions than these, this remains for future work. In fact the expressions for the two and three point functions above also look very similar to Liouville theory. This raises the interesting question of whether or not $M_\infty$ can be obtained as a limiting form of Liouville theory, where $c = 1$ is approached from above.

Let us briefly discuss two subtleties of the model $M_\infty$ which are also encountered in Liouville theory [11].

The first subtlety is that the Hilbert space $\mathcal{H}$ of $M_\infty$ does not contain a state of conformal weight zero, i.e. there is no vacuum state in $\mathcal{H}$. In rational CFT one has a strict state–field correspondence, i.e. for every field in the theory there is a state in the Hilbert space. In non-rational theories this need no longer be the case. In $M_\infty$ we can use the Ward identities to define correlation functions of the identity $1$ and the stress tensor $T(\zeta)$ consistently, even though the Hilbert space does not contain states corresponding to these fields. E.g. trivially $\langle \phi \cdots \phi \ 1 \rangle = \langle \phi \cdots \phi \rangle$ or

$$\langle \phi \cdots \phi \ T(\zeta) \rangle = \sum_{k} \left( \frac{h_k}{(\zeta - \zeta_k)^2} + \frac{1}{\zeta - \zeta_k} \frac{\partial}{\partial \zeta_k} \right) \langle \phi \cdots \phi \rangle$$

The important property of the Hilbert space is that it is complete in the sense that there is a vector for every possible state of the system. The completeness allows us to insert a basis of states as in [3].
Another subtlety arises from the OPE. Let $\phi_x$ and $\phi_y$ be two primary fields. Their OPE would take the form

$$\phi_x \phi_y = \int \! dz \, c(x, y, z)(\phi_z + \text{descendants}) ,$$

where the integral is over $\mathbb{R}_{>0} - \mathbb{Z}_{>0}$ and we have suppressed the coordinate dependence in that expression. On the other hand the two point function was $\langle \phi_x \phi_y \rangle = \delta(x - y)$. If we were to insert the OPE (8) into the two point function we would obtain zero, since only a field of weight zero can have a nonzero one-point function and there is no such field in $\mathcal{H}$.

The way out of this problem is to think of the fields $\phi_x$ as distributions and to work with the analogue of test functions instead, i.e. with “smeared fields”

$$\phi_f(\zeta, \bar{\zeta}) = \int \! dx \, f(x) \phi_x(\zeta, \bar{\zeta}) .$$

Note that the fields are smeared in representation space, not over position. Further we can formally define $|0\rangle = \lim_{x \to 0} x^{-1} |x\rangle$ and think of $|0\rangle$ as shorthand notation for this limit. In particular it is not a state in $\mathcal{H}$. The OPE can now be formulated in terms of smeared fields and inserting this OPE in the two point function gives

$$\langle 0 | \phi_f \phi_g | 0 \rangle = \int \! dx \, f(x) g(x) ,$$

where the position dependence has again been suppressed. This is precisely the result one obtains when inserting the definition (9) in the two point function $\langle \phi_x \phi_y \rangle = \delta(x - y)$.

**Open questions and conclusion**

We have defined the CFT $M_\infty$ with central charge one as a limit of minimal models. While its partition function is proportional to that of an uncompactified free boson, its operator algebra is entirely different. In $M_\infty$, the OPE of any two fields contains continuous intervals of primary fields, whereas the generic OPE of the free boson contains only one primary field (two for the orbifolded free boson). The exception is the OPE of two twist fields in the uncompactified, orbifolded free boson, which contains all untwisted representations, but otherwise the operator algebra (and the partition function) is different also in this case.

Thus the question remains what the interpretation of the quantum field theory $M_\infty$ should be. We do not know the answer to that, but let us make two suggestive remarks.

First note that there is a qualitative Landau-Ginzburg description of the minimal model $M_p$ \cite{12} which consists of one free scalar field $X(\zeta, \bar{\zeta})$ perturbed by a potential $gX^{2(p-1)}$. Taking $p$ to infinity means that the potential approaches the form of a square well with walls at $X = \pm 1$, forcing $X$ to take values only in the interval $[-1, 1]$. The resulting action is that of a membrane fluctuating freely between two walls\cite{13}. However since the Landau Ginzburg action is not yet conformal, one has to still find the end point of the renormalisation group flow, and it is not clear that the limit of minimal models leading to $M_\infty$ has anything to do with the limit of the potentials in the classical action. This has been investigated further in the context of a different limit of the minimal models $M_p$ \cite{13} to that we study.

The second remark concerns the target space interpretation of coset WZW models. The minimal model $M_p$ can be written as the coset $su(2)_{p-2} \oplus su(2)_1/su(2)_{p-1}$. In the limit $p \to \infty$ the target

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\* The Landau Ginzburg interpretation was pointed out by J. Cardy.
space should approach the quotient of classical group manifold. If we interpret the factor $su(2)_1$ as some internal degree of freedom, we are left with $su(2)/su(2)$ as target space, where the quotient $su(2)$ acts by conjugation. $su(2)$ has the topology of an $S^3$ and the conjugacy classes are $S^2$’s, so that the quotient space has the topology of an interval $\mathbb{I}$.

Both points indicate that the theory might have something to do with a sigma model taking values in an interval, but any detailed connection remains to be worked out. In particular there has to be an effect which allows the spectrum to be continuous, even though the target space is compact.

Other interesting open problems are the precise connection between $M_\infty$ and Liouville theory and whether the calculation can be repeated for other series with a limit point in the central charge, such as super minimal models, WZW models and cosets thereof. This might also help to find the correct interpretation for $M_\infty$. It would also be worth investigating if theories obtained in this way can be used as string theory vacua.

Acknowledgments – This work has been presented by IR at the RTN meeting “The quantum structure of space-time and the geometric nature of the fundamental interactions”. The authors would like to thank T. Gannon and C. Schweigert for helpful comments.

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‡ The target space interpretation for minimal models seen as cosets was pointed out to us by V. Schomerus.