Research Article

Numerical Algorithms of the Discrete Coupled Algebraic Riccati Equation Arising in Optimal Control Systems

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The discrete coupled algebraic Riccati equation (DCARE) has wide applications in robust control, optimal control, and so on. In this paper, we present two iterative algorithms for solving the DCARE. The two iterative algorithms contain both the iterative solution in the last iterative step and the iterative solution in the current iterative step. And, for different initial value, the iterative sequences are increasing and bounded in one algorithm and decreasing and bounded in another. They are all monotonous and convergent. Numerical examples demonstrate the convergence effect of the presented algorithms.

1. Introduction and Preliminaries

The discrete coupled Riccati equation is usually encountered in optimal control and filter design problems in control theory [1–9], particularly in the jump-linear quadratic optimal control problem [10]. Consider the following jump-linear system:

\[
\begin{align*}
    x_{k+1} &= A(r_k)x_k + B(r_k)u_k, \quad 0 \leq k \leq N^*, \\
    y_k &= C(r_k)x_k,
\end{align*}
\]

with initial state \( x(0) = x_0 \), \( r(0) = r_0 \), where \( x_k \in \mathbb{R}^n \) is the plant state, \( u_k \in \mathbb{R}^m \) is the control vector, and \( y_k \in \mathbb{R}^q \) is the process output. Here, \( k \) is the time index, \( r_k \) is the form process taking values in the finite set \( S = \{1, 2, \ldots, s\} \), and \( r_k \) is a finite-state discrete-time Markov chain with transition probabilities.

\[
\Pr\{r_{k+1} = j \mid r_k = i\} = \tilde{e}_{ij}, \quad 1 \leq i, j \leq s, \quad \tilde{e}_{ii} > 0. \tag{2}
\]

Minimizing the cost criterion of system (1) reduces to solving coupled algebraic Riccati-like equations. After some transformation, the coupled algebraic Riccati-like equations turn the following discrete coupled algebraic Riccati equation (DCARE)

\[
P_i = A_i^T F_i A_i - A_i^T F_i B_i \left( I + B_i^T F_i B_i \right)^{-1} B_i^T F_i A_i + Q_i, \tag{3}
\]

where \( A_i \in \mathbb{R}^{m \times m} \) is a constant matrix, \( B_i \in \mathbb{R}^{m \times m} \), \( Q_i \in \mathbb{R}^{m \times m} \) is a symmetric positive definite matrix, \( i \in S \), \( F_i = P_i + \sum_{j \in S} \tilde{e}_{ij} P_j \) is the coupled term, \( \tilde{e}_{ij} \) are real non-negative constants defined as \( \tilde{e}_{ij} = (\tilde{e}_{ij}/\tilde{e}_{ii}) \) with the properties \( \tilde{e}_{ij} \in [0, 1] \), \( \tilde{e}_{ii} > 0 \), and \( \sum_{j \in S} \tilde{e}_{ij} = 1 \), and \( P_i \in \mathbb{R}^{m \times m} \) denotes the symmetric positive definite solution of the DCARE. Applying Woodbury matrix equality

\[
\left( A - BD^{-1}C \right)^{-1} = A^{-1} + A^{-1}B\left( D - CA^{-1}B \right)^{-1}CA^{-1}, \tag{4}
\]

DCARE (3) turns to

\[
P_i = A_i^T \left( F_i^{-1} + B_i B_i^T \right)^{-1} A_i + Q_i. \tag{5}
\]

Because of the importance of Riccati equations in control theory and control engineering, a lot of research studies about Riccati equations have been devoted to this field, such
as solution bounds [11–15], trace and eigenvalue bounds [16–23], and the existence and uniqueness [24–26]. Besides these results, numerical solutions of Riccati equations are very important and have been studied by many scholars [27–34] because the numerical solutions of the Riccati equations are necessary in some practical engineering, such as finding the optimal state feedback controller in the optimal control system. Especially, for the DCARE, fixed point iterative algorithms are given in [24–26]. Stein iterations are presented in [35] which are based on the properties of a Stein equation. Among these results, we find less work has been done to discuss the numerical solution of the DCARE. Considering the importance and necessity of the numerical solutions of the DCARE, we propose two algorithms to discuss the numerical solution of the DCARE.

In this paper, we first propose an iterative algorithm with a parameter for solving the DCARE and prove its monotonically convergence. Second, we give an upper solution bound of the DCARE, by which another iterative algorithm has been proposed, but there are many restrictions in these algorithms. In this part, we first present an iterative algorithm for DCARE (5) which do not have any restrictions.

2. Main Results

In [25, 26], the authors have derived several solution bounds by which iterative algorithms have been proposed, but there are many restrictions in these algorithms. In this part, we first present an iterative algorithm for DCARE (5) which do not have any restrictions.

Algorithm 1

Step 1: set $P_i(0) = Q_i$, $F_i(0) = Q_i, \sum_{j \neq i} e_{ij}Q_j$, $0 \leq \omega$, $i = 1, 2, \ldots, s$.

Step 2: compute

$$
\begin{align*}
& P_i(k) = A_i^T \left[ F_i^{-1}(k) + B_iB_i^T \right]^{-1} A_i + Q_i,
& F_i(k) = P_i(k) + \sum_{j \neq i} e_{ij} \left[ \omega P_j(k) + (1 - \omega) P_j(k) \right] + \sum_{j \neq i} e_{ij} P_j(k), \quad k = 0, 1, 2, \ldots.
\end{align*}
$$

From Algorithm 1, we get an increasing and bounded iterative sequence, which is convergent to the positive definite solution of DCARE (5).

Theorem 1. Let $P_i(\ast)$ be the positive definite solution of DCARE (5) and $Q_i > 0$. The iterative sequences $\{P_i(k)\}$ and $\{F_i(k)\}$ are generated by the iterative (8) with $0 \leq \omega \leq 1$, and then

$$
\begin{align*}
& P_i(0) \leq P_i(1) \leq P_i(2) \leq \cdots, \quad \lim_{k \to \infty} P_i(k) = P_i(\ast),
& F_i(0) \leq F_i(1) \leq F_i(2) \leq \cdots, \quad F_i(k) \leq P_i(\ast) + \sum_{j \neq i} e_{ij} P_j(\ast).
\end{align*}
$$

Proof. Since $P_i(\ast)$ is positive definite solution of DCARE (5), then

$$
\begin{align*}
& A \geq B, \quad \text{if and only if} \quad B^{-1} \geq A^{-1}.
& A^T(X^{-1} + R)^{-1} A > (\geq) A^T(Y^{-1} + R)^{-1} A,
\end{align*}
$$

Lemma 1 (see [36]). If $A, B \in R^{m \times n}$ are symmetric positive definite matrices, then

$$
\begin{align*}
& A \geq B, \quad \text{if and only if} \quad B^{-1} \geq A^{-1}.
\end{align*}
$$

Lemma 2 (see [22]). Let matrices $A, X, R, Y \in R^{m \times n}$ with $X, Y > 0, R \geq 0$, and $X > (\geq) Y$. Then,

$$
\begin{align*}
& A^T(X^{-1} + R)^{-1} A > (\geq) A^T(Y^{-1} + R)^{-1} A,
\end{align*}
$$

with strict inequality if $A$ is nonsingular, and $X > Y$. 

2. Main Results

In [25, 26], the authors have derived several solution bounds by which iterative algorithms have been proposed, but there are many restrictions in these algorithms. In this part, we first present an iterative algorithm for DCARE (5) which do not have any restrictions.
From (15), (17), (18), we get

\[
F_i(0) = Q_i + \sum_{j \neq i} e_{ij} Q_j = P_i(0) + \sum_{j=1}^{i-1} e_{ij} \left[ \omega P_j(0) + (1 - \omega) P_j(0) \right] + \sum_{j=i+1}^{i} e_{ij} P_j(0)
\]

\[
\leq P_i(1) + \sum_{j=1}^{i-1} e_{ij} \left[ \omega P_j(1) + (1 - \omega) P_j(0) \right] + \sum_{j=i+1}^{i} e_{ij} P_j(1) = F_i(1)
\]

\[
= P_i(1) + \sum_{j=1}^{i-1} e_{ij} \left[ P_j(1) + (\omega - 1) (P_j(1) - P_j(0)) \right] + \sum_{j=i+1}^{i} e_{ij} P_j(1)
\]

\[
\leq P_i(1) + \sum_{j=1}^{i-1} e_{ij} P_j(1) + \sum_{j=i+1}^{i} e_{ij} P_j(1) \leq P_i(*) + \sum_{j \neq i} e_{ij} P_j(*),
\]

that is,
\[
F_i(0) \leq F_i(1) \leq P_i(*) + \sum_{j \neq i} e_{ij} P_j(*). \tag{14}
\]

Suppose that
\[
\quad
\]

According to (16) and Lemma 2, we get
\[
P_i(m + 1) = A_i^T \left[ F_i^{-1}(k) + B_i B_i^T \right]^{-1} A_i + Q_i \geq A_i^T \left[ F_i^{-1}(k - 1) + B_i B_i^T \right]^{-1} A_i + Q_i = P_i(k). \tag{17}
\]

\[
P_i(k + 1) = A_i^T \left[ F_i^{-1}(k) + B_i B_i^T \right]^{-1} A_i + Q_i \leq A_i^T \left[ P_i(*) + \sum_{j \neq i} e_{ij} P_j(*) \right]^{-1} + B_i B_i^T \right]^{-1} A_i + Q_i = P_i(*). \tag{18}
\]

From (15), (17), (18), we get

\[
F_i(k) = P_i(k) + \sum_{j=1}^{i-1} e_{ij} \left[ \omega P_j(k) + (1 - \omega) P_j(k - 1) \right] + \sum_{j=1}^{i} e_{ij} P_j(k) \leq P_i(k + 1) + \sum_{j=1}^{i-1} e_{ij} \left[ \omega P_j(k + 1) + (1 - \omega) P_j(k) \right]
\]

\[
+ \sum_{j=i+1}^{i} e_{ij} P_j(k + 1) = F_i(k + 1)
\]

\[
= P_i(k + 1) + \sum_{j=1}^{i-1} e_{ij} \left[ P_j(k + 1) + (\omega - 1) (P_j(k + 1) - P_j(k)) \right] + \sum_{j=i+1}^{i} e_{ij} P_j(k + 1) \leq P_i(k + 1) + \sum_{j=1}^{i-1} e_{ij} P_j(k + 1)
\]

\[
+ \sum_{j=i+1}^{i} e_{ij} P_j(k + 1) \leq P_i(*) + \sum_{j \neq i} e_{ij} P_j(*). \tag{19}
\]
Thus, the proof of induction is completed. Because $P_i(k)$ and $F_i(k)$ are monotonically increasing and they are bounded, then $\lim_{k \to \infty} P_i(k)$ and $\lim_{k \to \infty} F_i(k)$ exist. As $k \to \infty$, Algorithm 1 gives

$$P_i(\infty) = A_i^T \left[ F_i^{-1}(\infty) + B_i B_i^T \right]^{-1} A_i + Q_i,$$

$$= A_i^T \left\{ \left[ P_i(\infty) + \sum_{j=1}^{i-1} e_{ij} \left[ \omega P_j(\infty) + (1 - \omega) P_j(\infty) \right] + \sum_{j=i+1}^{s} e_{ij} P_j(\infty) \right]^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i,$$

$$= A_i^T \left\{ P_i(\infty) + \sum_{j \neq i} e_{ij} P_j(\infty) \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i.$$

Thus, $\lim_{k \to \infty} P_i(k) = P_i(*)$. \hfill \Box

**Theorem 2.** Let $P_i$ be the positive definite solution of DCARE (5), then

$$P_i \leq A_i^T \left( B_i B_i^T \right)^{-1} A_i + Q_i.$$  \hspace{1cm} (21)

**Proof.** If $G > P_i + \sum_{j \neq i} e_{ij} P_j$, then by Lemma 1, we get

$$P_i = A_i^T \left\{ P_i + \sum_{j \neq i} e_{ij} P_j \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i,$$

$$\leq A_i^T \left\{ G^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i.$$  \hspace{1cm} (22)

From Algorithm 2, we get a decreasing and bounded iterative sequences, which is convergent to the positive definite solution of DCARE (5).

**Theorem 3.** Let $P_i(*)$ be the positive definite solution of DCARE (5) and $Q_i > 0$. The iterative sequences $\{P_i(k)\}$ and $\{F_i(k)\}$ are generated by iterative (23) with $0 \leq \omega \leq 1$, and then

$$P_i(1) = A_i^T \left[ F_i^{-1}(0) + B_i B_i^T \right]^{-1} A_i + Q_i = A_i^T \left\{ A_i^T \left( B_i B_i^T \right)^{-1} A_i + Q_i + \sum_{j \neq i} e_{ij} \left( A_j^T \left( B_j B_j^T \right)^{-1} A_j + Q_j \right) \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i,$$

$$\geq A_i^T \left\{ P_i(*) + \sum_{j \neq i} e_{ij} P_j(*) \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i = P_i(*) .$$  \hspace{1cm} (23)

$$P_i(1) = A_i^T \left[ F_i^{-1}(0) + B_i B_i^T \right]^{-1} A_i + Q_i \leq A_i^T \left( B_i B_i^T \right)^{-1} A_i + Q_i = P_i(0).$$  \hspace{1cm} (24)

**Proof.** According to (21) and (23), we have

$$P_i(1) = A_i^T \left[ F_i^{-1}(0) + B_i B_i^T \right]^{-1} A_i + Q_i = A_i^T \left\{ A_i^T \left( B_i B_i^T \right)^{-1} A_i + Q_i + \sum_{j \neq i} e_{ij} \left( A_j^T \left( B_j B_j^T \right)^{-1} A_j + Q_j \right) \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i,$$

$$\geq A_i^T \left\{ P_i(*) + \sum_{j \neq i} e_{ij} P_j(*) \right\}^{-1} + B_i B_i^T \right\}^{-1} A_i + Q_i = P_i(*) .$$  \hspace{1cm} (25)

$$P_i(1) = A_i^T \left[ F_i^{-1}(0) + B_i B_i^T \right]^{-1} A_i + Q_i \leq A_i^T \left( B_i B_i^T \right)^{-1} A_i + Q_i = P_i(0).$$  \hspace{1cm} (26)
Since \( \omega - 1 \leq 0 \), with (25) and (26) we get
\[
F_i(0) = Q_i + \sum_{j \neq i} e_{ij}Q_j = P_i(0) + \sum_{j \neq i} e_{ij}[\omega P_j(0) + (1 - \omega)P_j(0)]
\]
\[
+ \sum_{j \neq i} e_{ij}P_j(0) \geq P_i(1) + \sum_{j \neq i} e_{ij}[\omega P_j(1) + (1 - \omega)P_j(1)]
+ \sum_{j \neq i} e_{ij}P_j(1)
= P_i(1) + \sum_{j \neq i} e_{ij}[P_i(1) + (1 - \omega)(P_j(0) - P_j(1))]
+ \sum_{j \neq i} e_{ij}P_j(1) \geq P_i(\ast) + \sum_{j \neq i} e_{ij}P_j(\ast),
\]
that is,
\[
F_i(0) \geq F_i(1) \geq P_i(\ast) + \sum_{j \neq i} e_{ij}P_j(\ast).
\]
Suppose that
\[
F_i(k) = P_i(k) + \sum_{j \neq i} e_{ij}[\omega P_j(k) + (1 - \omega)P_j(k - 1)] + \sum_{j \neq i} e_{ij}P_j(k) \geq P_i(k + 1) + \sum_{j \neq i} e_{ij}[\omega P_j(k + 1) + (1 - \omega)P_j(k)]
+ \sum_{j \neq i} e_{ij}P_j(k + 1) = F_i(k + 1)
+ \sum_{j \neq i} e_{ij}P_j(k + 1) \geq P_i(\ast) + \sum_{j \neq i} e_{ij}P_j(\ast).
\]
Thus, the proof of induction is completed. Because \( P_i(k) \) and \( F_i(k) \) are monotonically decreasing and they are bounded, then \( \lim_{k \to \infty} P_i(k) \) and \( \lim_{k \to \infty} F_i(k) \) exist. In a similar way as the proof of (20), as \( k \to \infty \), Algorithm 2 gives \( \lim_{k \to \infty} P_i(k) = P_i(\ast) \).

Remark 1. In Algorithm 2, if \( B_iB_i^T \) is singular, we can choose a suitable \( G \) so that \( (G^{-1} + B_iB_i^T)^{-1} \) is nonsingular, as in Theorem 2.

Remark 2. In Algorithm 1, the sequence \( P_i(k) \) in (8) with the initial value \( P_i(0) = Q_i \) converges monotonically to a positive definite solution of DCARE (5), and so does the sequence \( P_i(k) \) in (23) with the initial value \( P_i(0) = A_i^T(B_iB_i^T)^{-1}A_i + Q_i \). But the two positive definite solutions may be different. Whether the positive definite solution of DCARE (5) is unique or not, a problem needs to be discussed further.

Remark 3. When \( e_{ij} = 0 \) (\( j \neq i \)), DCARE (5) changes to the discrete algebraic Riccati equation. And iterative sequences (8) and (23), respectively, in Algorithm 1 and Algorithm 2, become the iterative (17) and iterative (28) in [22], which means that the algorithms of the DCARE in this paper are generalizations of the discrete algebraic Riccati equation. Moreover, when \( \omega = 1 \), the iterative (8) and (23) are extensions on the discrete coupled algebraic Riccati equation of the work of [22].

Remark 4. In this paper, we only prove Algorithms 1 and 2 are convergent under the condition \( 0 \leq \omega \leq 1 \), but we can run the two algorithms with \( \omega > 1 \) in practical computation. And, we have faster convergence speed if appropriate parameters are selected. We will illustrate it in the following examples.

3. Numerical Examples

In this section, the following numerical examples are presented to show the effectiveness of our results.
Example 1 (see [26]). Consider DCARE (5) with

\[
A_1 = \begin{pmatrix} 0.25 & 0.15 \\ 0.25 & -0.1 \end{pmatrix},
A_2 = \begin{pmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{pmatrix},
B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

\[
Q_1 = \begin{pmatrix} 50 & -30 \\ -30 & 40 \end{pmatrix},
Q_2 = \begin{pmatrix} 60 & -20 \\ -20 & 70 \end{pmatrix},
\]

\[
\left(\bar{e}_{ij}\right)_{i,j \in S} = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix},
S = \{1, 2\}.
\]

(34)

Since there are two equations in the DCARE, the superiority of the \(\omega\) in Algorithm 1 is not obvious. So, we choose \(\omega = 1\) here. After 9 steps of iteration of (8), we obtain the solution \(P_i\) of DCARE (5).

\[
P_1(9) = \begin{pmatrix} 54.782858819173867 & -27.163728424128180 \\ -27.163728424128180 & 41.743512895466367 \end{pmatrix},
P_2(9) = \begin{pmatrix} 60.745519196489681 & -21.49103839297359 \\ -21.49103839297359 & 72.982076785958711 \end{pmatrix},
\]

(35)

and the residual \(\|A_1^T\left[P_i(9) + \sum_{i,j} e_{ij} P_i(9)\right]^{-1} + B_i B_i^T\|_2^{-1} A_i + Q_i - P_i(9)\) is 8.246736626915663e – 009.

However, it needs 47 steps of iteration for the algorithm in [26] to get the iteration solution of DCARE (5).

Example 2. Consider DCARE (5) with

\[
A_1 = \begin{pmatrix} 0.8 & 0.07 \\ -0.13 & 2.1 \end{pmatrix},
A_2 = \begin{pmatrix} 1.63 & 0.12 \\ -0.18 & 1.06 \end{pmatrix},
A_3 = \begin{pmatrix} 0.65 & 0.13 \\ 0.14 & -1.08 \end{pmatrix},
Q_1 = \begin{pmatrix} 40 & -2.5 \\ -2.5 & 1.1 \end{pmatrix},
Q_2 = \begin{pmatrix} 6 & -1.5 \\ -1.5 & 0.4 \end{pmatrix},
Q_3 = \begin{pmatrix} 0.48 & -1.4 \\ -1.4 & 48 \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
B_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},
B_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix},
\]

(36)

\[
\left(\bar{e}_{ij}\right)_{i,j \in S} = \begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{pmatrix},
S = \{1, 2, 3\}.
\]

Because the restrictions of the algorithms in [25, 26] are not met for this case, the algorithms in [25, 26] cannot work.

For Algorithm 1, the steps of iteration and the residual are presented in Table 1 with different parameter \(\omega\). Although we only prove the convergence of Algorithm 1 with \(0 \leq \omega \leq 1\), from Table 1, we find the convergence rapid is the fastest when \(\omega = 1.8\). After 31 steps of iteration of (8) with \(\omega = 1.8\), we obtain the solution \(P_i\) of DCARE (5).

\[
P_1(31) = \begin{pmatrix} 1103.165003916018 & 101.279835324595 \\ 101.279835324595 & 15.487684241437 \end{pmatrix},
P_2(31) = \begin{pmatrix} 9664.052810636161 & 713.129010075344 \\ 713.129010075344 & 53.563508057380 \end{pmatrix},
P_3(31) = \begin{pmatrix} 7325.056498352728 & 1445.922335403706 \\ 1445.922335403706 & 773.201080130236 \end{pmatrix},
\]

(37)

and the residual \(\|A_1^T\left[P_i(31) + \sum_{i,j} e_{ij} P_i(31)\right]^{-1} + B_i B_i^T\|_2^{-1} A_i + Q_i - P_i(31)\) is 8.307893040182535e – 009.

Example 3 (see [26]). Consider the DCARE (5) with

\[
A_1 = \begin{pmatrix} 0.07 & 0.8 \\ -1.3 & 0.1 \end{pmatrix},
A_2 = \begin{pmatrix} 0.12 & 1.63 \\ 0.18 & 1.06 \end{pmatrix},
A_3 = \begin{pmatrix} 0.13 & 0.65 \\ -1.08 & 0.14 \end{pmatrix},
Q_1 = \begin{pmatrix} -2.5 & 40 \\ 1.1 & -25 \end{pmatrix},
Q_2 = \begin{pmatrix} -1.5 & 6 \\ 0.4 & -1.5 \end{pmatrix},
Q_3 = \begin{pmatrix} -1.4 & 0.48 \\ 48 & -1.4 \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
B_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},
B_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix},
\]

(38)

\[
\left(\bar{e}_{ij}\right)_{i,j \in S} = \begin{pmatrix} 0.05 & 0.13 \\ 0.14 & 0.13 \end{pmatrix},
S = \{1, 2, 3\}.
\]

Q_1, Q_2, Q_3, B_1, B_2, B_3, \left(\bar{e}_{ij}\right)_{i,j \in S} and S are the same as Example 2.

For Algorithm 2, since \(B_i B_i^T\) is singular, by choosing \(G = I\), then Algorithm 2 can work now. After 4 steps of iteration of (23) with \(\omega = 1\), we obtain the solution \(P_i\) of DCARE (5):

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix},
\begin{pmatrix} 0 \\ 2 \end{pmatrix},
\begin{pmatrix} 0 \\ 3 \end{pmatrix}.
\]

The remaining solutions are the same as Example 2.
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Table 1: Numerical results.

| $\omega$ | Iterations | Residual | $\omega$ | Iterations | Residual |
|----------|------------|----------|----------|------------|----------|
| 0        | 46         | 9.0841e-009 | 1.8      | 31         | 8.3079e-009 |
| 0.4      | 44         | 8.3616e-009 | 2        | 32         | 8.0616e-009 |
| 0.8      | 42         | 6.2660e-009 | 4        | 41         | 5.6463e-009 |
| 1        | 41         | 4.7366e-009 | 6        | 49         | 3.9612e-009 |
| 1.4      | 37         | 8.0066e-009 | 8        | 58         | 6.7581e-009 |

and the residual $\|A^T_1[\{P_1,4\}+\sum_{\mu\epsilon i,j}P_1(4)]^{-1}+B_iB_j^{-1}\|A_i+Q_i-P_1(4)\|$ is 9.070366799608115.

However, it needs 18 steps of iteration for the algorithm in [26] to get the iteration solution of DCARE (5).

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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