Remarks on 2–dimensional HQFTs

MIHAI D STAIC
VLADIMIR TURAEV

We introduce and study algebraic structures underlying 2–dimensional Homotopy Quantum Field Theories (HQFTs) with arbitrary target spaces. These algebraic structures are formalized in the notion of a twisted Frobenius algebra. Our work generalizes results of Brightwell, Turner and the second author on 2–dimensional HQFTs with simply connected or aspherical targets.

57R56

Introduction

A fruitful idea in topology is to construct invariants of manifolds that behave functorially with respect to gluings of manifolds along the boundary. More formally, one associates to every closed oriented \( d \)–dimensional manifold \( M \) a finite dimensional vector space \( V_M \) and to every compact oriented \((d+1)\)–dimensional cobordism \( (W, M, N) \) a homomorphism \( \tau_W : V_M \to V_N \). The conditions satisfied by \( (V, \tau) \) in order to obtain a consistent theory are formalized in the notion of a \((d+1)\)–dimensional Topological Quantum Field Theory (TQFT); see Atiyah [1]. The second author introduced more general Homotopy Quantum Field Theories (HQFTs) [5]. One can think of a \((d+1)\)–dimensional HQFT as a TQFT for a \( d \)–dimensional manifolds and \((d+1)\)–dimensional cobordisms endowed with maps to a fixed space \( X \).

The 1–dimensional HQFTs with target \( X \) are easily classified in terms of finite dimensional representations of \( \pi_1(X) \). A study of 2–dimensional HQFTs is more involved and leads to interesting algebra. For contractible \( X \), the category of 2–dimensional HQFTs with target \( X \) is equivalent to the category of 2–dimensional TQFTs and is known to be equivalent to the category of commutative finite-dimensional Frobenius algebras. If \( X = K(G, 1) \) is an Eilenberg–Mac Lane space determined by a group \( G \), then the category of 2–dimensional HQFTs with target \( X \) is equivalent to the category of so-called crossed Frobenius \( G \)–algebras [5]. If \( X = K(A, 2) \) is an Eilenberg–Mac Lane space determined by an abelian group \( A \), then the category of 2–dimensional HQFTs with target \( X \) is equivalent to the category of Frobenius \( A \)–algebras; see Brightwell and Turner [2].
In this paper we address the case where both groups $G = \pi_1(X)$ and $A = \pi_2(X)$ are allowed to be nontrivial. An important role will be played by the first $k$–invariant $k \in H^3(G, A)$ of $X$. We shall introduce and study algebraic structures underlying 2–dimensional HQFTs with target $X$. We call them twisted Frobenius algebras or, shorter, TF–algebras over $(G, A, k)$. Briefly speaking, a TF–algebra is a $G$–graded algebra of $A$–modules which is associative up to a 3–cocycle representing $k$ and which satisfies a form of commutativity. This work is a step towards classification of 2–dimensional HQFTs with target $X$ (for a different approach, see Porter and Turaev [4]).

The paper is organized as follows. We introduce TF–algebras in Section 1. In Section 2 we recall the notion of an HQFT and the definition of the first $k$–invariant of a topological space. In Section 3 we derive from any 2–dimensional HQFT the underlying TF–algebra. In Section 4 we compute the underlying TF–algebras of the cohomological 2–dimensional HQFTs.

Throughout the paper the symbol $K$ denotes a field and $\otimes = \otimes_K$.

Acknowledgments The work of M Staic was partially supported by the CNCSIS project “Hopf algebras, cyclic homology and monoidal categories”, contract number 560/2009. The work of V Turaev was partially supported by NSF grants DMS-0707078 and DMS-0904262.

1 Twisted Frobenius algebras

1.1 Preliminaries

In this section $G$ is a group with neutral element $\varepsilon$ and $A$ is a left $G$–module with neutral element $1 = 1_A$. We use multiplicative notation for the group operation in $A$. As usual, the group ring of $A$ with coefficients in $K$ is denoted $K[A]$. The action of $\alpha \in G$ on $a \in A$ is denoted by $\alpha a$.

To calculate the cohomology $H^*(G, A)$, we use the standard cochain complex

$$C^0(G, A) \xrightarrow{\delta^0} C^1(G, A) \xrightarrow{\delta^1} C^2(G, A) \xrightarrow{\delta^2} C^3(G, A) \xrightarrow{\delta^3} C^4(G, A) \to \cdots$$

where $C^n(G, A) = \text{Map}(G^n, A)$ for any $n \geq 0$. For small $n$, the coboundary homomorphism $\delta^n: C^n(G, A) \to C^{n+1}(G, A)$ is given by the following formulas:

$$\delta^0(a)(\alpha) = \alpha a a^{-1}$$
Remarks on 2–dimensional HQFTs

for any $a \in A = C^0(G, A)$ and $\alpha \in G$, 

$$\delta^1(\psi)(\alpha, \beta) = \alpha^a \psi(\beta) \psi(\alpha \beta)^{-1} \psi(\alpha)$$

for any $\psi \in C^1(G, A)$ and $\alpha, \beta \in G$, 

$$\delta^2(\omega)(\alpha, \beta, \gamma) = \alpha^a \omega(\beta, \gamma) \omega(\alpha \beta, \gamma)^{-1} \omega(\alpha, \beta \gamma) \omega(\alpha, \beta)^{-1}$$

for any $\omega \in C^2(G, A)$ and $\alpha, \beta, \gamma \in G$, and 

$$\delta^3(\chi)(\alpha, \beta, \gamma, \rho) = \chi(\beta, \gamma, \rho) \chi(\alpha \beta, \gamma, \rho)^{-1} \chi(\alpha, \beta \gamma, \rho) \chi(\alpha, \beta, \gamma \rho)^{-1} \chi(\alpha, \beta, \gamma)$$

for any $\chi \in C^3(G, A)$ and $\alpha, \beta, \gamma, \rho \in G$.

1.2 Definition of TF–algebras

Fix from now on a normalized 3–cocycle $\kappa: G^3 \to A$. Thus, for all $\alpha, \beta, \gamma, \rho \in G$, 

$$a^a \kappa(\beta, \gamma, \rho) \kappa(\alpha \beta, \gamma, \rho)^{-1} \kappa(\alpha, \beta \gamma, \rho) \kappa(\alpha, \beta, \gamma \rho)^{-1} \kappa(\alpha, \beta, \gamma) = 1.$$ 

The word “normalized” means that for all $\alpha, \beta \in G$, 

$$\kappa(\epsilon, \alpha, \beta) = \kappa(\alpha, \epsilon, \beta) = \kappa(\alpha, \beta, \epsilon) = 1.$$ 

A twisted Frobenius algebra (TF–algebra) over the triple $(G, A, \kappa)$ is a $G$–graded $K[A]$–module $V = \bigoplus_{\alpha \in G} V_{\alpha}$ such that the following properties hold:

(a) (Underlying module and action of $A$) The underlying $K$–vector space of $V_{\alpha}$ is finite-dimensional and for all $\alpha \in G$, $u \in V_{\alpha}$ and $a \in A$, 

$$au = a^a u.$$ 

(b) (Multiplication) We have a $K$–bilinear multiplication $V \times V \to V$ carrying $V_{\alpha} \times V_{\beta}$ to $V_{\alpha \beta}$ for all $\alpha, \beta \in G$. For any $u \in V_{\alpha}$, $v \in V_{\beta}$, $w \in V_{\gamma}$ with $\alpha, \beta, \gamma \in G$, 

$$(uv)w = \kappa(\alpha, \beta, \gamma) u(vw).$$

There is a unit element $1_V \in V_{\epsilon}$ such that $1_V u = u 1_V = u$ for all $u \in V$.

(c) (Inner product) We have a nondegenerate symmetric $K$–bilinear form 

$$\eta_{\epsilon}: V_{\epsilon} \otimes V_{\epsilon} \to K$$

such that for all $\alpha \in G$, the pairing 

$$V_{\alpha} \otimes V_{\alpha^{-1}} \to K, \ u \otimes v \mapsto \eta_{\epsilon}(uv \otimes 1_V)$$

(where $u \in V_{\alpha}$, $v \in V_{\alpha^{-1}}$) is nondegenerate.
(d) (Projective action of $G$) For each $\beta \in G$, we have a $K$–linear isomorphism $\varphi_\beta: V \to V$ carrying $V_\alpha$ to $V_{\beta \alpha \beta^{-1}}$ for all $\alpha$ and such that

\begin{align*}
(4) \quad & \varphi_\beta(au) = \beta a \varphi_\beta(v) \quad \text{for any } a \in A \text{ and } v \in V, \\
(5) \quad & \varphi_\beta|_{V_{\beta}} = \text{id}_{V_{\beta}}: V_{\beta} \to V_{\beta}, \\
(6) \quad & \varphi_\beta(1_{V}) = 1_{V}, \\
(7) \quad & vu = \varphi_\beta(u)v \quad \text{for any } u \in V, v \in V_{\beta}, \\
(8) \quad & \eta_c(\varphi_\beta(u) \otimes \varphi_\beta(v)) = \eta_c(u \otimes v) \quad \text{for any } u, v \in V_{\epsilon}, \\
(9) \quad & \varphi_\beta(u) \varphi_\beta(v) = l^\beta_{\alpha, \gamma} \varphi_\beta(uv) \quad \text{for any } \alpha, \gamma \in G \text{ and } u \in V_{\alpha}, v \in V_{\gamma}, \\
\text{where} \quad & l^\beta_{\alpha, \gamma} = \kappa(\beta \alpha \beta^{-1}, \beta \gamma \beta^{-1}, \beta) \kappa(\beta \alpha \beta^{-1}, \beta, \gamma)^{-1} \kappa(\beta, \alpha, \gamma) \in A, \\
(10) \quad & \varphi_{\gamma} \varphi_\beta|_{V_{\alpha}} = h^\alpha_{\gamma, \beta} (\varphi_{\gamma} \circ \varphi_\beta)|_{V_{\alpha}} \quad \text{for all } \alpha, \beta, \gamma \in G, \\
\text{where} \quad & h^\alpha_{\gamma, \beta} = \kappa(\gamma \beta \alpha (\gamma \beta)^{-1}, \gamma, \beta) \kappa(\gamma, \beta \alpha \beta^{-1}, \beta)^{-1} \kappa(\gamma, \beta, \alpha) \in A.
\end{align*}

(e) (The trace condition) For any $\alpha, \beta \in G$ and $c \in V_{\alpha \beta^{-1} \beta^{-1}}$.

\begin{align*}
(11) \quad & \text{Tr}(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta, \alpha) \mu_c \varphi_\beta: V_{\alpha} \to V_{\alpha}) \\
& \quad = \text{Tr}(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha \beta^{-1}, \beta) \varphi_{\alpha^{-1} \beta} \mu_c: V_{\beta} \to V_{\beta}), \\
\text{where} \quad & \mu_c \text{ is left multiplication } V \to V, v \mapsto cv \text{ by } c \text{ and } \text{Tr} \text{ is the standard trace of linear maps.}
\end{align*}

We have the following elementary consequences of the definition. The $K[A]$–bilinearity of multiplication in $V$ implies that

\begin{align*}
(12) \quad & a(uv) = (au)v = u(av) \\
\text{for all } a \in A \text{ and } u, v \in V. \quad \text{Note that if } u \in V_{\alpha} \text{ and } v \in V_{\beta}, \text{ then for all } a \in A, \\
& a^\alpha a(uv) = (a^\alpha u)v = (au)v = a(uv) \\
\text{and similarly } \beta^\alpha a(uv) = a(uv). \text{ Formula (10) applied to } \gamma = \beta = \epsilon \text{ implies that} \\
& \varphi_{\epsilon} = \text{id}: V \to V.
\end{align*}

In the following lemma and in the sequel the pairing (3) is denoted by $\eta_\alpha$.

**Lemma 1.1** For any $a \in A$, $u \in V_{\alpha}$, $v \in V_{\alpha^{-1}}$ with $\alpha \in G$,

\begin{align*}
\eta_\alpha(au \otimes v) &= \eta_\alpha(u \otimes av), \\
\eta_\alpha(u \otimes v) &= \eta_{\alpha^{-1}}(\kappa(\alpha^{-1}, \alpha, \alpha^{-1})^{-1} v \otimes u).
\end{align*}
Remarks on 2–dimensional HQFTs

For any \( \alpha, \beta \in G \), \( u \in V_\alpha \), \( v \in V_{\alpha^{-1}} \),
\[
\eta_{\beta \alpha^{-1}}(\varphi_\beta(u) \otimes \varphi_\beta(v)) = \eta_\alpha(l^\beta_{\alpha, \alpha^{-1}} u \otimes v).
\]

For any \( \alpha, \beta \in G \), \( u \in V_\alpha \), \( v \in V_\beta \) and \( w \in V_{(\alpha \beta)^{-1}} \),
\[
\eta_{\alpha \beta}(uv \otimes w) = \eta_\alpha(\kappa(\alpha, \beta, (\alpha \beta)^{-1}) u \otimes vw).
\]

**Proof** We check only the first two identities leaving the other two to the reader. For the first identity, we have
\[
\eta_\alpha(au \otimes v) = \eta_\epsilon((au)v \otimes 1_v)
\]
\[
= \eta_\epsilon(u(av) \otimes 1_v)
\]
\[
= \eta_\alpha(u \otimes av).
\]

Formulas (5), (13) and (10) with \( \gamma = \alpha \), \( \beta = \alpha^{-1} \) imply the identity \( \varphi_\alpha(v) = \kappa(\alpha^{-1}, \alpha, \alpha^{-1})^{-1} v \) for all \( v \in V_{\alpha^{-1}} \). Therefore
\[
\eta_\alpha(u \otimes v) = \eta_\epsilon(uv \otimes 1_v)
\]
\[
= \eta_\epsilon(\varphi_\alpha(v)u \otimes 1_v)
\]
\[
= \eta_\epsilon(\kappa(\alpha^{-1}, \alpha, \alpha^{-1})^{-1} vu \otimes 1_v)
\]
\[
= \eta_{\alpha^{-1}}(\kappa(\alpha^{-1}, \alpha, \alpha^{-1})^{-1} v \otimes u).
\]

Given \( z \in K^* \) and a TF–algebra, we can multiply the inner product \( \eta_\epsilon \) by \( z \) (keeping the rest of the data) and obtain thus a new TF–algebra. This operation on TF–algebras is called \( z \)–rescaling.

Let \( V \), \( W \) be TF–algebras over \( (G, A, \kappa) \). A \( K[A] \)–isomorphism \( f: V \to W \) is an isomorphism of TF–algebras if \( f \) is an isomorphism of algebras such that
\[
\eta_\epsilon(f(u) \otimes f(v)) = \eta_\epsilon(u \otimes v)
\]
for all \( u, v \in V_\epsilon \) and \( f \varphi_\beta = \varphi_\beta f \) for all \( \beta \in G \).

### 1.3 Examples

The definition of a TF–algebra over \( (G, A, \kappa) \) generalizes both the notion of a crossed Frobenius \( G \)–algebra [5] and the notion of an \( A \)–Frobenius algebra [2]. Consequently we have the following two sources of examples.

**Example 1.2** If \( L \) is a crossed Frobenius \( G \)–algebra, then \( L \) is a TF–algebra over \( (G, A, \kappa) \) where \( A \) is the trivial group and \( \kappa \) is the trivial cocycle.

**Example 1.3** If \( V \) is an \( A \)–Frobenius algebra, then \( V \) is a TF–algebra over \( (G, A, \kappa) \) where \( G \) is the trivial group and \( \kappa \) is the trivial cocycle.

Further examples of TF–algebras are constructed in Section 4.

*Algebraic \\& Geometric Topology, Volume 10 (2010)*
1.4 Coboundary equivalence

Let $\kappa: G^3 \to A$ be a normalized 3–cocycle. Given a normalized 2–cochain $\omega: G^2 \to A$, the map $\kappa' = \delta^2(\omega)$, $\kappa: G^3 \to A$ is a normalized 3–cocycle cohomological to $\kappa$. Using $\omega$, we can transform a TF–algebra $V$ over $(G, A, \kappa)$ into a TF–algebra $V^{\omega}$ over $(G, A, \kappa')$ as follows. The underlying $G$–graded $K[A]$–modules of $V^{\omega}$ and $V$ are the same. The inner product on $V^{\omega}$ is the same as in $V$. Multiplication on $V^{\omega}$ is defined by

$$u \cdot^{\omega} v = \omega(\alpha, \beta)^{-1} u \cdot v,$$

for $u \in V_\alpha, v \in V_\beta$, where $\cdot$ is multiplication in $V$. Given $\beta \in G$, the automorphism $\psi^\omega_\beta$ of $V^{\omega}$ is defined by

$$\psi^\omega_\beta|_{V_\alpha} = \omega(\beta, \alpha)^{-1} \omega(\beta \alpha \beta^{-1}, \beta) \psi_\beta|_{V_\alpha}$$

for all $\alpha \in G$. Direct computations show that $V^{\omega}$ is a TF–algebra over $(G, A, \kappa')$. We say that $V^{\omega}$ is obtained from $V$ by a coboundary transformation. This transformation defines an equivalence between the category of TF–algebras over $(G, A, \kappa)$ and their isomorphisms and the category of TF–algebras over $(G, A, \kappa')$ and their isomorphisms.

Given $k \in H^3(G, A)$, the coboundary transformations and the isomorphisms generate an equivalence relation on the class of TF–algebras over the triples $(G, A, \kappa)$ where $\kappa: G^3 \to A$ runs over all normalized 3–cocycles representing $k$. This relation is called coboundary equivalence and the corresponding equivalence classes are called TF–algebras over $(G, A, k)$.

2 Preliminaries on HQFTs and $k$–invariants

2.1 HQFTs

We recall the definition of an HQFT from [5; 6]. We say that a topological space is pointed if all its connected components are provided with base points. By maps of pointed spaces we mean continuous maps carrying base points to base points.

Fix a pointed path connected topological space $X$. An $X$–manifold is a pair $(M, g)$, where $M$ is a pointed closed oriented manifold and $g$ is a map $M \to X$. An $X$–homeomorphism between $(M, g)$ and $(M', g')$ is an orientation preserving (and base point preserving) homeomorphism $f: M \to M'$ such that $g = g' f$. An empty set is considered as an $X$–manifold of any given dimension.

An $X$–cobordism between $X$–manifolds $(M_0, g_0)$ and $(M_1, g_1)$ is an oriented compact cobordism $(W, M_0, M_1)$ endowed with a map $g: W \to X$ such that $\partial W =$
A $(d+1)$–dimensional Homotopy Quantum Field Theory $(V, \tau)$ with target $X$ assigns a finite-dimensional $K$–vector space $V_M$ to any $d$–dimensional $X$–manifold, a $K$–isomorphism $f^\#: V_M \to V_{M'}$ to any $X$–homeomorphism of $d$–dimensional $X$–manifolds $f : M \to M'$ and a $K$–homomorphism $\tau(W) : V_{M_0} \to V_{M_1}$ to any $(d+1)$–dimensional $X$–cobordism $(W, M_0, M_1)$. These vector spaces and homomorphisms should satisfy the following axioms:

1. For any $X$–homeomorphisms of $d$–dimensional $X$–manifolds $f : M \to M'$, $f' : M' \to M''$, we have $(f' f)^\# = f'^\# f^\#$.
2. For any disjoint $d$–dimensional $X$–manifolds $M, N$, there is a natural isomorphism $V_{M \sqcup N} = V_M \otimes V_N$.
3. $V_{\emptyset} = K$.
4. For any $X$–homeomorphism of $(d+1)$–dimensional $X$–cobordisms $F : (W, M_0, M_1, g) \to (W', M'_0, M'_1, g')$, the following diagram is commutative:

\[
\begin{array}{ccc}
V_{(M_0, g)|M_0)} & \xrightarrow{(F|_{M_0})^\#} & V_{(M'_0, g'|M'_0)} \\
\downarrow{\tau(W,g)} & & \downarrow{\tau(W', g')} \\
V_{(M_1, g)|M_1)} & \xrightarrow{(F|_{M_1})^\#} & V_{(M'_1, g'|M'_1)}
\end{array}
\]

5. If a $(d+1)$–dimensional $X$–cobordism $W$ is a disjoint union of $X$–cobordisms $W_1, W_2$, then $\tau(W) = \tau(W_1) \otimes \tau(W_2)$.
6. If an $X$–cobordism $(W, M_0, M_1)$ is obtained from two $(d+1)$–dimensional $X$–cobordisms $(W_0, M_0, N)$ and $(W_1, N', M_1)$ by gluing along an $X$–homeomorphism $f : N \to N'$, then

\[
\tau(W) = \tau(W_1) f^\# \tau(W_0) : V_{M_0} \to V_{M_1}.
\]
We assume that

\[ f(\cdot) = \text{id: } V_M \rightarrow V_M. \]

where we identify \( M \times 0 \) and \( M \times 1 \) with \( M \) in the obvious way and the triple \( (M \times [0, 1], M \times 0, M \times 1, g) \) is the cylinder cobordism over \((M, g)\).

(7) For any \( d \)–dimensional \( X \)–manifold \((M, g)\),

\[ \tau (M \times [0, 1], M \times 0, M \times 1, g) = \text{id: } V_M \rightarrow V_M, \]

(8) For any \((d+1)\)–dimensional \( X \)–cobordism \((W, M_0, M_1, g)\), the homomorphism \( \tau (W) \) is preserved under homotopies of \( g \) constant on \( \partial W = M_0 \sqcup M_1 \).

**Remark** In this definition we follow [6] rather than [5]. The difference is that Axiom (7) in [6] is weakened in comparison with the corresponding axiom in [5].

For shortness, HQFTs with target \( X \) will be also called \( X \)–HQFTs. For examples of \( X \)–HQFTs, see the second author’s papers [5; 6]. Note here that any cohomology class \( \theta \in H^{d+1}(X; K^*) \) determines a \((d+1)\)–dimensional \( X \)–HQFT \((V^\theta, \tau^\theta)\).

An *isomorphism of \( X \)–HQFTs* \( \rho: (V, \tau) \rightarrow (V', \tau') \) is a family of \( K \)–isomorphisms \\{\( \rho_M : V_M \rightarrow V'_M \)\}\_\_M, where \( M \) runs over all \( d \)–dimensional \( X \)–manifolds, \( \rho_\emptyset = \text{id}_K \), \( \rho_{M \sqcup N} = \rho_M \otimes \rho_N \) for all \( M, N \), and the natural square diagrams associated with the \( X \)–homeomorphisms and \( X \)–cobordisms are commutative.

### 2.2 The \( k \)–invariant

Let \( X \) be a path connected topological space with base point \( x_0 \). We recall from [3] the definition of the first \( k \)–invariant of \( X \).

Let \( p: \{0, 1\} \rightarrow S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \) be the map carrying \( t \in \{0, 1\} \) to \(-i \exp(-2\pi it)\). We provide \( S^1 \) with clockwise orientation and base point \(-i = p(0) = p(1)\). Set \( G = \pi_1(X, x_0) \) and recall that \( A = \pi_2(X, x_0) \) is a left \( G \)–module in the standard way.

For each \( \alpha \in G \), fix a loop \( u_\alpha : S^1 \rightarrow X \) carrying \(-i \) to \( x_0 \) and representing \( \alpha \). We assume that the loop \( u_\varepsilon \) representing the neutral element \( \varepsilon \in G \) is the constant path at \( x_0 \). Fix a \( 2 \)–simplex \( \Delta_2 \) with vertices \( v_0, v_1, v_2 \). For every \( \alpha, \beta \in G \), fix a map \( f_{\alpha, \beta} : \Delta_2 \rightarrow X \) such that for all \( t \in \{0, 1\},

\[
\begin{align*}
    f_{\alpha, \beta}((1-t)v_0 + tv_1) &= u_\alpha \ p(t), \\
    f_{\alpha, \beta}((1-t)v_1 + tv_2) &= u_\beta \ p(t), \\
    f_{\alpha, \beta}((1-t)v_0 + tv_2) &= u_{\alpha \beta} \ p(t).
\end{align*}
\]

We assume that \( f_{\varepsilon, \varepsilon} \) is the constant map \( \Delta_2 \rightarrow \{x_0\} \) and that for all \( \alpha \in G \), the maps \( f_{\varepsilon, \alpha} \) and \( f_{\alpha, \varepsilon} \) are the standard “constant” homotopies between two copies of \( u_\alpha \). We call the collection \( \Omega = \{u_\alpha \}_{\alpha \in G}; \{ f_{\alpha, \beta} \}_{\alpha, \beta \in G} \) a basic system of loops and triangles in \( X \).
Figures 1–14 below represent $X$–surfaces using the following conventions. The vertices in each figure are labeled by integers; the vertex labeled by an integer $i$ is denoted $v_i$. With every edge in the figure we associate an element of $G$ called its label. For some edges, the labels are indicated by Greek letters in the picture. The labels of all the other edges can be computed uniquely using the following rule: in any triangle $v_i v_j v_k$ with $i < j < k$ the label of the edge $v_i v_k$ is the product of the labels of $v_i v_j$ and $v_j v_k$. Each figure below represents a compact oriented surface $\Sigma$ endowed with a map $f: \Sigma \to X$. The map $f$ carries all vertices of $\Sigma$ to the base point $x_0 \in X$. The restriction of $f$ to any edge $v_i v_j$ with $i < j$ and with label $\rho \in G$ is given by $f((1-t)v_i + tv_j) = u(1-t)\rho(t)$ for all $t \in [0,1]$. The restriction of $f$ to any triangle $v_i v_j v_k$ with $i < j < k$ is given by

$$f(t_0v_i + t_1v_j + t_2v_k) = f_{\rho,\eta}(t_0v_0 + t_1v_1 + t_2v_2).$$

where $\rho, \eta$ are the labels of $v_i v_j$ and $v_j v_k$ respectively and $t_0, t_1, t_2$ run over nonnegative real numbers whose sum is equal to 1. The map $f: \Sigma \to X$ is considered up to homotopy constant on $\partial \Sigma$.

Let $\Delta_3$ be the standard 3–simplex with vertices $v_0, v_1, v_2, v_3$. Given $\alpha, \beta, \gamma \in G$, Figure 1 describes a map from the 2–sphere $\partial \Delta_3$ to $X$. Here the labels of the edges $v_0v_2, v_1v_3$, and $v_0v_3$ are $\alpha\beta, \beta\gamma$, and $\alpha\beta\gamma$, respectively. We take $v_0$ as the base point of $\partial \Delta_3$. With this choice, the map $\partial \Delta_3 \to X$ in Figure 1 represents an element of $A = \pi_2(X, x_0)$ denoted $\kappa(\alpha, \beta, \gamma)$. This defines a map $\kappa = \kappa(\Omega): G^3 \to A$. It is known that $\kappa$ is a cocycle. It follows from the definitions (and from the choice of the maps $f_{\alpha, \beta}, f_{\beta, \alpha}$) that $\kappa$ is normalized in the sense of Section 1.1. The cohomology class of $\kappa$ in $H^3(G, A)$ is independent of the choice of $\Omega$. This cohomology class is the first $k$–invariant of $X$.

---

**Remarks on 2–dimensional HQFTs**

1375
3 The TF–algebra underlying a 2–dimensional HQFT

Let \( X \) be a path connected topological space with base point \( x_0 \). We derive from any 2–dimensional \( X \)–HQFT \( (V, \tau) \) its underlying TF–algebra.

For shortness, one-dimensional \( X \)–manifolds will be called \( X \)–curves and two-dimensional \( X \)–cobordisms will be called \( X \)–surfaces. We say that two \( X \)–surfaces with the same bases

\[
(W, M_0, M_1, g: W \to X) \quad \text{and} \quad (W', M_0, M_1, g': W' \to X)
\]

are \( h \)–equivalent if there is a homeomorphism \( F: W \to W' \) constant on the boundary and such that \( g'F: W \to X \) is homotopic to \( g \) via a homotopy constant on the boundary.

The axioms of an HQFT imply that then

\[
\tau(W) = \tau(W'): V_{M_0} \to V_{M_1}.
\]

Fix a basic system of loops and triangles \( \Omega = \{u_\alpha\}_{\alpha \in G}; \{f_\alpha, \beta\}_{\alpha, \beta \in G} \) in \( X \), where \( G = \pi_1(X, x_0) \). Let \( \kappa: G^3 \to A = \pi_2(X, x_0) \) be the cocycle constructed in Section 2.2. We construct a TF–algebra \( V^\Omega \) over \((G, A, \kappa)\) in several steps.

**Step 1** (Underlying module and action of \( A \)) For each \( \alpha \in G \), consider the pointed oriented circle \( S^1 \) and the map \( u_\alpha: S^1 \to X \) as in Section 2.2. The pair \((S^1, u_\alpha)\) is an \( X \)–curve. Set \( V_\alpha = V(s^1, u_\alpha) \). By the definition of an HQFT, \( V_\alpha \) is a finite-dimensional \( K \)–vector space.

For all \( \alpha \in G \), the group \( A = \pi_2(X, x_0) \) acts on \( V_\alpha \) by

\[
av = \tau(S(\alpha, a))(v),
\]

where \( a \in A, v \in V_\alpha \), and \( S(\alpha, a) \) is the \( X \)–annulus obtained as a connected sum of the cylinder cobordism over \((S^1, u_\alpha)\) and a map \( S^2 \to X \) representing \( a \). The \( X \)–annulus \( S(\alpha, a) \) is shown in the left part of Figure 2. Here and below we use external arrows to indicate the edges glued to each other (keeping the surface oriented).

Observe that the \( X \)–annulus in the right part of Figure 2 is \( h \)–equivalent to \( S(\alpha, a) \). Therefore, \( av = a' av \) for all \( a \in A \) and \( v \in V_\alpha \).
Step 2 (Multiplication) For \( \alpha, \beta \in G \), consider the \( X \)–cobordism (a disk with two holes) \( D(\alpha, \beta) \) in Figure 3. Composing the map

\[
V_{\alpha} \times V_{\beta} \to V_{\alpha \otimes \beta}, \quad (u, v) \mapsto u \otimes v
\]

with the homomorphism

\[
\tau(D(\alpha, \beta)): V_{\alpha} \otimes V_{\beta} \to V_{\alpha \beta}
\]

we obtain a \( K \)–linear multiplication \( V_\alpha \times V_\beta \to V_{\alpha \beta} \). This extends to multiplication in \( \bigoplus_{\alpha \in G} V_\alpha \) by linearity.

It is clear that for any \( a \in A \), the gluing of \( S(\alpha, a) \) to \( D(\alpha, \beta) \) along the left bottom base of \( D(\alpha, \beta) \) and the gluing of \( S(\alpha \beta, a) \) to \( D(\alpha, \beta) \) along the top base of \( D(\alpha, \beta) \) yield \( h \)–equivalent \( X \)–surfaces. Applying \( \tau \), we obtain \( (au)v = a(uv) \) for any \( u \in V_\alpha \) and \( v \in V_\beta \). Similarly, \( (au)v = u(av) \). Hence, multiplication in \( \bigoplus_{\alpha \in G} V_\alpha \) is \( K[A] \)–bilinear.

To check the twisted associativity (2), note that \( (uv)w \) is computed by applying \( \tau \) to the \( X \)–surface (a disk with three holes) obtained by gluing \( D(\alpha, \beta) \) to \( D(\alpha \beta, \gamma) \). Similarly, \( u(vw) \) is computed by applying \( \tau \) to the \( X \)–surface obtained by gluing \( D(\beta, \gamma) \) to \( D(\alpha, \beta \gamma) \). These \( X \)–surfaces are shown in Figure 4 where the external arrows indicating the gluing of sides are omitted. It is easy to see that the left \( X \)–surface is \( h \)–equivalent to a connected sum of the right \( X \)–surface with the map \( S^2 \to X \) used to define \( \kappa(\alpha, \beta, \gamma) \in A \). This implies Formula (2). Note for future use that the narrow “collar-type” rectangles in Figure 4 play no role in the argument but are necessary to define the \( X \)–surfaces at hand. In some of the figures below we omit these collar rectangles to simplify the figures.

Next consider an oriented \( 2 \)–disk \( B \) mapped to \( \{x_0\} \subset X \) and viewed as an \( X \)–cobordism between an empty set and \( (S^1, u_\epsilon) \). We have a map \( \tau(B): K \to V_\epsilon \), and we set \( 1_\nu = \tau(B)(1) \). One easily sees that \( 1_\nu u = u 1_\nu = u \) for all \( u \in V \).
Step 3 (Inner product) Consider the $X$–annulus $C_{--}(\alpha, \varepsilon)$ shown in Figure 5. The subscript $\varepsilon$ reflects the fact that the orientation on both boundary components of the annulus is opposite to the orientation induced from the annulus. Set

$$\eta_{\alpha} = \tau(C_{--}(\alpha, \varepsilon)) : V_\alpha \otimes V_{\alpha^{-1}} \to K.$$ 

The $X$–annulus $C_{--}(\alpha, \varepsilon)$ can be obtained by gluing the $X$–surfaces $C_{--}(\varepsilon, \varepsilon)$, $D(\alpha, \alpha^{-1})$, and $B$. This yields

$$\eta_{\alpha}(u \otimes v) = \tau(C_{--}(\alpha, \varepsilon))(u \otimes v) = \eta_{\varepsilon}(uv \otimes 1_{\varepsilon})$$

for all $u \in V_\alpha$ and $v \in V_{\alpha^{-1}}$.

The $X$–surfaces shown in Figure 6 are $h$–equivalent. Applying $\tau$ to the surface on the left we get $(id_{\nu_{\alpha^{-1}}} \otimes \eta_{\alpha})(\iota \otimes id_{\nu_{\alpha^{-1}}})$, where $\iota$ is a homomorphism $K \to V_{\alpha^{-1}} \otimes V_\alpha$. Applying $\tau$ to the surface on the right we get $id_{\nu_{\alpha^{-1}}}$. This results in the equality $(id_{\nu_{\alpha^{-1}}} \otimes \eta_{\alpha})(\iota \otimes id_{\nu_{\alpha^{-1}}}) = id_{\nu_{\alpha^{-1}}}$, which implies the nondegeneracy of $\eta_{\alpha}$.

Note finally that the map $C_{--}(\varepsilon, \varepsilon) \to X$ used to define $\eta_{\varepsilon}$ is the constant map with value $\chi_0$. This easily implies that the form $\eta_{\varepsilon}$ is symmetric.
Step 4  (Projective action of $G$) Consider the $X$–annulus $C_{++}(\alpha, \beta)$ shown in Figure 7. The subscript $++$ reflects the fact that the orientation on the boundary component corresponding to $\alpha$ is opposite to the orientation induced from the annulus, while the orientation on the boundary component corresponding to $\beta \alpha \beta^{-1}$ is induced from the orientation of the annulus. Set

$$\varphi_\beta = \tau(C_{++}(\alpha, \beta)): V_\alpha \to V_{\beta \alpha \beta^{-1}}.$$  

The identity $\varphi_\beta(\alpha v) = \beta \alpha \varphi_\beta(v)$ for $\alpha \in A$ and $v \in V_\alpha$ follows from the fact the $X$–annulus obtained by gluing $C_{++}(\alpha, \beta)$ to $S(\alpha, a)$ is $h$–equivalent to the $X$–annulus obtained by gluing $S(\beta \alpha \beta^{-1}, \beta \alpha)$ to $C_{++}(\alpha, \beta)$.

Figure 8 represents two $h$–equivalent $X$–annuli (it is understood that the vertical sides of the right rectangle are glued in the usual way to make an annulus). The left $X$–annulus
is $C_{-+}(\alpha, \varepsilon)$. The right $X$–annulus is obtained by gluing $C_{-+}(\alpha, \beta)$ represented by the lower rectangle to an $X$–annulus $C'(\alpha, \beta)$ represented by the upper rectangle. This proves that $\varphi_\beta$ is invertible and $\varphi_\beta^{-1} = \tau(C'(\alpha, \beta))$.

![Figure 8](image-url) Proof of the invertibility of $\varphi_\beta$

To prove that $\varphi_{\beta}|_{V_\beta} = \text{id}_{V_\beta}$; $V_\beta \rightarrow V_\beta$, consider the $X$–annuli in Figure 9. Using the Dehn twist of an annulus about its core circle, one easily observes that these two $X$–annuli are $h$–equivalent. The $X$–annulus on the right is $C_{-+}(\beta, \varepsilon)$, and the associated map $\varphi_\varepsilon; V_\beta \rightarrow V_\beta$ is the identity by Axiom (7) of an HQFT. It is easy to see that a connected sum of the left $X$–annulus with the map $S^2 \rightarrow X$ used to define $\kappa(\beta, \varepsilon, \beta) \in A$ is $h$–equivalent to $C_{-+}(\beta, \beta)$. Since $\kappa(\beta, \varepsilon, \beta) = 1$, we obtain $\text{id}_{V_\beta} = \varphi_{\beta}|_{V_\beta}$.

![Figure 9](image-url) Proof of the identity $\varphi_{\beta}|_{V_\beta} = \text{id}_{V_\beta}$

The equality $\varphi_{\beta}(1_\nu) = 1_\nu$ follows from the fact that the $X$–disk obtained by gluing the $X$–disk $B$ to the bottom of $C_{-+}(\varepsilon, \beta)$ is $h$–equivalent to $B$.

To prove the identity (7), consider the $X$–surfaces (disks with two holes) in Figure 10 (we omit in the figure the collar rectangles glued to some external edges; they play no role in the argument and the reader may easily recover them). The first $X$–surface is obtained by gluing the $X$–annulus $C'(\alpha, \beta)$ to the right bottom boundary component $(S^1, u_\alpha)$ of $D(\beta, \alpha)$. Since $\kappa(\varepsilon, \beta, \alpha) = 1$, this $X$–surface is $h$–equivalent to the second $X$–surface in Figure 10. The third $X$–surface is obtained from the second one by separating the left face along the $\varepsilon$–labeled edge $v_0v_2$ and gluing it on the
right along the edges indicated by the external arrows. Thus, the second and third \(X\)-surfaces are \(h\)-homeomorphic. The map from the third \(X\)-surface to \(X\) can be easily computed because its restriction to the face \(v_1v_3v_4\) (respectively to \(v_0v_2v_4\)) is the constant homotopy of \(\beta\) to itself (respectively, of \(\beta\alpha\) to itself). This \(X\)-surface is \(h\)-equivalent to \(D(\beta\alpha\beta^{-1}, \beta)\). Applying \(\tau\), we obtain \(v\varphi^{-1}_\beta(w) = wv\) for all \(v \in V_\beta\) and \(w \in V_{\beta\alpha\beta^{-1}}\). This is an equivalent form of (7).

The identity \(\eta_\varepsilon(\varphi_\beta(u) \otimes \varphi_\beta(v)) = \eta_\varepsilon(u \otimes v)\) follows from the fact that the \(X\)-annulus obtained by the gluing of \(C_{+,\varepsilon}^+ (\varepsilon, \beta) \sqcup C_{-,\varepsilon}^- (\varepsilon, \beta)\) to the bottom of \(C_{-,\varepsilon}^- (\varepsilon, \varepsilon)\) is \(h\)-equivalent to \(C_{-,\varepsilon}^- (\varepsilon, \varepsilon)\) (see Figure 11).

To prove that

\[
\varphi_\beta(u) \varphi_\beta(v) = l^\beta_{\alpha,\gamma} \varphi_\beta(uv)
\]

for all \(u \in V_\alpha\) and \(v \in V_\gamma\), consider the \(X\)-disks with three holes \(W_1, W_2, W_3, W_4\) in Figure 12 (again we omit the collar rectangles glued to some external edges). Clearly, the homomorphism \(\tau(W_1)\) carries \(u \otimes v\) to \(\varphi_\beta(u) \varphi_\beta(v)\). The \(X\)-surface \(W_2\) differs from \(W_1\) by two adjacent triangles \(v_0v_1v_5\) and \(v_0v_1'v_5\) both representing \(f_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}:\Delta_2 \to X\). These two copies of \(f_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\) “cancel” each other, and so \(W_2\) is \(h\)-equivalent to \(W_1\). Thus, \(\tau(W_2) = \tau(W_1)\). The \(X\)-surface \(W_3\) is obtained from \(W_2\) by removing the vertices \(v_1\) and \(v_1'\). Hence

\[
\tau(W_3) = \kappa(\beta\alpha\beta^{-1}, \beta\gamma\beta^{-1}, \beta)^{-1} \kappa(\beta\alpha\beta^{-1}, \beta, \gamma) \tau(W_2).
\]
The $X$–surface $W_4$ is obtained from $W_3$ by switching the diagonal in the quadrilateral $v_0v_3v_4v_5$. Therefore $\tau(W_4) = \kappa(\beta, \alpha, \gamma)^{-1} \tau(W_3)$. Combining these formulas, we obtain $\tau(W_4) = (l_{\beta})^{-1} \tau(W_1)$. It remains to observe that the homomorphism $\tau(W_4): V_\alpha \otimes V_\gamma \to V_{\beta\alpha\gamma\beta^{-1}}$ carries $u \otimes v$ to $\varphi_\beta(uv)$.

To show the identity $\varphi_\beta(\varphi_\beta|_{V_\alpha}) = l_{\alpha, \gamma}^{\beta} \varphi_\beta|_{V_\alpha}$, consider the $X$–annuli $W_1$, $W_2$, $W_3$, $W_4$ in Figure 13. Clearly, $W_1 = C_{++}(\alpha, \gamma \beta)$ and therefore $\tau(W_1) = \varphi_\beta|_{V_\alpha}$. The
Remarks on 2–dimensional HQFTs

The $X$–annulus $W_2$ is obtained from $W_1$ by adding two vertices $v_2$ and $v_3$. Hence
\[ \tau(W_2) = \kappa(\gamma\beta\alpha\beta^{-1}\gamma^{-1}, \gamma, \beta)\kappa(\gamma, \beta, \alpha)^{-1} \tau(W_1). \]
The $X$–annulus $W_3$ is obtained from $W_2$ by switching the diagonal in the quadrilateral $v_0v_2v_5v_3$. Therefore \( \tau(W_3) = \kappa(\gamma, \beta\alpha\beta^{-1}, \beta)\tau(W_2) \). Finally, $W_4$ is obtained from $W_3$ by canceling two adjacent copies of the singular triangle $f_{\gamma, \beta}$. Therefore $W_4$ is $h$–equivalent to $W_3$ and $\tau(W_4) = \tau(W_3)$. It is clear that $\tau(W_4) = \varphi_\gamma \varphi_\beta |_{V_\alpha}$. Combining these equalities, we obtain the required identity.

**Step 5** (The trace condition) Consider the three $X$–surfaces (punctured tori) $W_1$, $W_2$, $W_3$ in Figure 14. All three are $X$–cobordisms between $(S^1, \mu_\gamma)$ and $\emptyset$, where $\gamma = \alpha\beta\alpha^{-1}\beta^{-1}$. Clearly,
\[ \tau(W_1) = \kappa(\gamma, \beta\alpha\beta^{-1}, \beta)\tau(W_2) \quad \text{and} \quad \tau(W_3) = \kappa(\gamma, \beta, \alpha)^{-1} \tau(W_2). \]
Consider the $X$–surface obtained by gluing $C_{+}(\alpha, \beta)$ and $D(\alpha, \gamma)$. If we identify the two copies of $(S^1, u_\alpha)$ we obtain $W_1$. Similarly for the $X$–surface obtained by gluing $D(\gamma, \beta)$ and $C'(\beta, \alpha)$ we can identify the two copies of $(S^1, u_\beta)$ in order to obtain $W_3$. This implies the equality

$$\text{Tr}(\kappa(\gamma, \beta, \alpha)\mu_e\varphi_\beta: V_\alpha \to V_\alpha) = \text{Tr}(\kappa(\gamma, \beta\alpha\beta^{-1}, \beta)\varphi_{\alpha}^{-1}\mu_e: V_\beta \to V_\beta).$$

We summarize the results above in the following theorem.

**Theorem 3.1** To every 2–dimensional $X$–HQFT $(V, \tau)$ and to every basic system of loops and triangles $\Omega = \left\{ (\mu_\alpha)_{\alpha \in G}, \left\{ f_{\alpha, \beta} \right\}_{\alpha, \beta \in G} \right\}$ in $X$, we associate a TF–algebra $V^\Omega = V^\Omega(\tau)$ over the triple $(G = \pi_1(X), A = \pi_2(X), \kappa^\Omega: G^3 \to A)$.

It is obvious that the construction of $V^\Omega$ is functorial with respect to isomorphisms of $X$–HQFTs. We now show that $V^\Omega$ does not depend on $\Omega$ at least up to coboundary equivalence (see Section 1.4).

**Lemma 3.2** The TF–algebra $V^\Omega$ considered up to coboundary equivalence does not depend on the choice of $\Omega$. 

*Algebraic & Geometric Topology, Volume 10 (2010)*
Theorem 3.3 Two 2-dimensional $X$–HQFTs are isomorphic if and only if their underlying TF–algebras are coboundary equivalent.
Proof. We fix a basic system $\Omega$ of loops and triangles in $X$. It is obvious that an isomorphism of $X$–HQFTs $\rho: (V_1, \tau_1) \to (V_2, \tau_2)$ induces an isomorphism $V_1^{\Omega} \simeq V_2^{\Omega}$ of TF–algebras over $(\pi_1(X), \pi_2(X), k^{\Omega})$. Therefore, the underlying TF–algebras are coboundary equivalent. Conversely, assume that we have two $X$–HQFTs $(V_1, \tau_1)$ and $(V_2, \tau_2)$ such that their underlying TF–algebras are coboundary equivalent. If we use the same basic system $\Omega = \{ \{u_\alpha\}_{\alpha \in \pi_1(X)}, \{f_{\alpha, \beta}\}_{\alpha, \beta \in \pi_1(X)} \}$ of loops and triangles in $X$, then the two algebras in question are TF–algebras over the same triple $(\pi_1(X), \pi_2(X), k^{\Omega})$. Since they are coboundary equivalent, there is an isomorphism $\rho: V_1^{\Omega} \to V_2^{\Omega}$. We lift $\rho$ to an isomorphism of the HQFTs. For every $\alpha \in \pi_1(X)$, denote by $\rho_\alpha$ the restriction of $\rho$ to $(V_1^{\Omega})_\alpha = (V_1)(S^1, u_\alpha)$. For every connected $X$–curve $(M, g_M)$, there are $\alpha \in \pi_1(X)$, an $X$–annuli $a = a_M: [0, 1] \times S^1 \to X$, and an $X$–homeomorphism $h = h_M: (S^1, a_M|_{1 \times S^1}) \to (M, g_M)$ such that

$$a_M|_{0 \times S^1} = u_\alpha: S^1 \to X \quad \text{and} \quad a_M([0, 1] \times \{\ast\}) = x_0,$$

where $\ast$ is the base point of $S^1$ and $x_0$ is the base point of $X$. Notice that $\alpha$ is uniquely determined by $(M, g_M)$. We define $\rho_M: (V_1)_M \to (V_2)_M$ by the formula

$$\rho_M = h_{\#_2} \tau_2(a) \rho_\alpha \tau_1(a)^{-1} h_{\#_1}^{-1},$$

where $h_{\#_i}: (V_i)(S^1, a|_{1 \times S^1}) \to (V_i)(M, g_M)$ is the $K$–isomorphism associated to $h$ by the HQFT $(V_i, \tau_i)$. Next we show that $\rho_M$ does not depend on the choice of $a$ and $h$. Indeed, take another pair $\hat{a}$, $\hat{h}$ which satisfies the above properties. Consider $a$ (respectively $\hat{a}$) as an $X$–cobordism between $(S^1, u_\alpha)$ and $(S^1, a|_{1 \times S^1})$ (respectively $(S^1, \hat{a}|_{1 \times S^1})$) and observe that $\hat{h}^{-1}h$ is an $X$–homeomorphism between the top bases of these $X$–cobordisms. Let $C$ be the $X$–cobordism obtained by gluing these two along $\hat{h}^{-1}h$ (the orientation in the second cobordism should be reversed). Then $C$ is a cobordism between two copies of $(S^1, u_\alpha)$. Clearly, $C$ is the trivial $X$–annuli (as in Axiom (7) of an HQFT) up to $h$–equivalence and the action of some $c \in \pi_2(X)$. Hence $\tau_i(C): (V_i^{\Omega})_a \to (V_i^{\Omega})_a$ is multiplication by $c$. Since $\rho$ is an isomorphism of TF–algebras, we have

$$\rho_\alpha \tau_1(C) = \tau_2(C) \rho_\alpha$$

or equivalently

$$\rho_\alpha \tau_1(\hat{a})^{-1} (\hat{h}^{-1}h)_{\#_1} \tau_1(a) = \tau_2(\hat{a})^{-1} (\hat{h}^{-1}h)_{\#_2} \tau_2(a) \rho_\alpha.$$

This implies that

$$\hat{h}_{\#_2} \tau_2(\hat{a}) \rho_\alpha \tau_1(\hat{a})^{-1} \hat{h}_{\#_1}^{-1} = h_{\#_2} \tau_2(a) \rho_\alpha \tau_1(a)^{-1} h_{\#_1}^{-1}.$$

Algebraic & Geometric Topology, Volume 10 (2010)
So $\rho_M$ does not depend on the choice of $a$ and $h$. Denote by $\tilde{\rho}$ the family of $K$–isomorphisms $\{\rho_M: (V_1)_M \to (V_2)_M\}$ where $M$ runs over all $X$–curves. We claim that $\tilde{\rho}$ is an isomorphism of HQFTs.

For any $X$–homeomorphism $f: M \to N$, we have

$$f_{\#} \rho_M = f_{\#} (h_M)_{\#} \tau_2 (a_M) \rho_1 (a_M)^{-1} (h_M)_{\#}^{-1}$$

$$= (f \circ h_M)_{\#} \tau_2 (a_M) \rho_1 (a_M)^{-1} (f \circ h_M)_{\#}^{-1} f_{\#}$$

$$= \rho_N f_{\#}.$$

So, $\tilde{\rho}$ is natural with respect to $X$–homeomorphisms.

Next, notice that any compact oriented $X$–surface splits along a finite family of disjoint simple loops into a disjoint union of $X$–disks with at most two holes. Deforming if necessary the map from the surface to $X$, we can choose the splitting loops to be $X$–curves $X$–homeomorphic to $(S^1, u_\alpha)$, for some $\alpha \in \pi_1(X)$. Moreover, we can make the cuts in such a way that each component adjacent to the boundary is $X$–homeomorphic to some $a_M$ and all the other $X$–surfaces are of types

$$B, \quad S(\alpha, a), \quad D(\alpha, \beta), \quad C_{++}(\alpha, \beta), \quad C_{--}(\varepsilon, \varepsilon), \quad C_{++}(\varepsilon, \varepsilon).$$

We consider the $X$–surface $a_M (id \times h_M^{-1}): [0, 1] \times M \to X$ which is an $X$–cobordism between $M' = (M, u_\alpha \circ h_M^{-1})$ and $(M, g_M)$. Axiom (4) of an HQFT implies that for $i = 1, 2$,

$$\tau_i (a_M (id \times h_M^{-1})) = (h_M)_{\#} \tau_i (a_M) (h_M)_{\#}^{-1}.$$

From this formula and the definition of $\tilde{\rho}$, we have

$$\rho_M \tau_1 (a_M (id \times h_M^{-1})) = (h_M)_{\#} \tau_2 (a_M) \rho_1 (h_M)^{-1}$$

$$= (h_M)_{\#} \tau_2 (a_M) (h_M)^{-1} (h_M)_{\#}^{-1} \rho_1 (h_M)^{-1}$$

$$= \tau_2 (a_M (id \times h_M^{-1})) \rho_M.$$

Therefore the natural square diagram associated to $\tilde{\rho}$ and $a_M (id \times h_M^{-1})$ is commutative. Since $\rho$ is a morphism of TF–algebras, the natural square diagrams associated with the $X$–surfaces listed in (14) are commutative. Finally, since any $X$–surface can be obtained by gluing a finite collection of the above $X$–surfaces along $X$–homeomorphisms, we get that the natural square diagrams associated to all $X$–surfaces are commutative. We conclude that $\tilde{\rho}$ is an isomorphism of HQFTs.

We expect that any TF–algebra can be realized by a 2–dimensional HQFT which if true, together with Theorem 3.3 would yield a complete algebraic characterization of 2–dimensional HQFTs with arbitrary targets.
Given a mapping of pointed topological spaces \( f: Y \to X \), we can pull back a 2–dimensional \( X \)–HQFT along \( f \) to obtain a 2–dimensional \( Y \)–HQFT. If \( f \) induces isomorphisms in \( \pi_1 \) and \( \pi_2 \), then the underlying TF–algebra of this \( Y \)–HQFT is isomorphic to the underlying TF–algebra of the original \( X \)–HQFT.

4 Simple TF–algebras and cohomological HQFTs

In this section we compute the underlying TF–algebras of the cohomological 2–dimensional HQFTs. We begin by introducing a class of simple TF–algebras.

4.1 Simple TF–algebras and \( \kappa \)–pairs

Let \( G \) be a group with neutral element \( e \) and \( A \) be a left \( G \)–module. Let \( \kappa: G^3 \to A \) be a normalized 3–cocycle. A TF–algebra \( V = \bigoplus_{a \in G} V_a \) over \( (G, A, \kappa) \) is simple if \( \dim_K(V_a) = 1 \) for all \( a \in G \). We now classify simple TF–algebras in terms of so-called \( \kappa \)–pairs.

Form now on we endow the multiplicative abelian group \( K^* \) with the trivial action of \( G \). This allows us to apply notation of Section 1.1 to \( K^* \)–valued cochains on \( G \).

By a \( \kappa \)–pair, we mean a pair of maps \( g_1: G \times G \to K^* \), \( g_2: A \to K^* \) such that \( g_2 \) is a \( \mathbb{Z} \langle G \rangle \)–homomorphism,

\[
\begin{align*}
g_1(\alpha, e) &= g_1(e, \alpha) = 1 \quad \text{for all } \alpha \in G, \\
g_2(\kappa) &= g_2 \circ \kappa: G^3 \to K^*.
\end{align*}
\]

The \( \mathbb{Z}[G] \)–linearity of \( g_2: A \to K^* \) means that \( g_2 \) is a group homomorphism such that for all \( \alpha \in G \), \( a \in A \),

\[
g_2(a) = g_2(a).
\]

For example, for any map \( \psi: G \to K^* \) the pair

\[
(g_1 = \delta^1(\psi): G \times G \to K^*, g_2 = 1: A \to K^*)
\]

is a \( \kappa \)–pair. We call it a coboundary \( \kappa \)–pair.

Lemma 4.1 Let \( (g_1, g_2) \) be a \( \kappa \)–pair. For each \( \alpha \in G \), let \( V_\alpha \) be the one-dimensional vector space over \( K \) with basis vector \( l_\alpha \). We provide \( V = \bigoplus_{a \in G} V_a \) with a structure of an \( A \)–module by \( av = g_2(a)v \) for all \( v \in V \). We provide \( V \) with \( K \)–bilinear multiplication by

\[
l_\alpha l_\beta = g_1(\alpha, \beta)^{-1}l_{\alpha\beta}
\]
for all $\alpha, \beta \in G$. Let $\eta_\varepsilon: V_\varepsilon \otimes V_\varepsilon \to K$ be the $K$–bilinear form such that
\begin{equation}
\eta_\varepsilon(l_\varepsilon \otimes l_\varepsilon) = g_1(\varepsilon, \varepsilon).
\end{equation}

Let $\varphi_\beta: V \to V$ be the $K$–homomorphism defined by
\begin{equation}
\varphi_\beta(l_\alpha) = g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) l_{\beta \alpha \beta^{-1}}
\end{equation}
for all $\alpha, \beta \in G$. Then the $G$–graded vector space $V$ with this data is a TF–algebra over $(G, A, \kappa)$.

The TF–algebra constructed in this lemma is denoted by $V(g_1, g_2)$.

**Proof** The $K[A]$–bilinearity of multiplication in $V$ follows from the definitions. Formula (1) follows from the identity (17). It is sufficient to verify (2) for the basis vectors $u = l_\alpha, v = l_\beta, w = l_\gamma$:
\[
(l_\alpha l_\beta) l_\gamma = (g_1(\alpha, \beta)^{-1} l_{\alpha \beta}) l_\gamma = g_1(\alpha, \beta)^{-1} g_1(\alpha \beta, \gamma) l_{\alpha \beta \gamma} = g_2(\kappa(\alpha, \beta, \gamma)) l_{\alpha \beta \gamma}.
\]

Formula (15) implies that $l_\varepsilon$ is the unit element of $V$. The symmetry and nondegeneracy of $\eta_\varepsilon$ are obvious. The nondegeneracy of the form $\eta_{\alpha}: V_\alpha \otimes V_{\alpha^{-1}} \to K$ defined by (3) follows from the formula $\eta(l_\alpha \otimes l_{\alpha^{-1}}) = g_1(\alpha, \alpha^{-1})^{-1}$. We check (4):
\[
\varphi_\beta(a l_\alpha) = g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) g_2(a) l_{\beta \alpha \beta^{-1}} = g_2(\beta a) g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) l_{\beta \alpha \beta^{-1}} = \beta a \varphi_\beta(l_\alpha).
\]

Formulas (5) and (6) follow from the definitions. We check (7):
\[
\varphi_\beta(l_\alpha) l_\beta = g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) l_{\beta \alpha \beta^{-1}} l_\beta = g_1(\beta, \alpha)^{-1} l_{\beta \alpha \beta^{-1}} l_\beta = g_1(\beta, \alpha)^{-1} l_{\beta \alpha \beta^{-1}} = \beta \alpha \varphi_\beta(l_\alpha).
\]

Similar computations prove (8) and (9). We now check (10). Observe that for $\alpha, \beta, \gamma \in G$,
\[
\varphi_\gamma(\varphi_\beta(l_\alpha)) = g_1(\gamma, \beta \alpha \beta^{-1})^{-1} g_1(\gamma \beta \alpha(\gamma \beta)^{-1}, \gamma) g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) l_{\gamma \beta \alpha(\gamma \beta)^{-1}} = g_1(\gamma, \beta \alpha(\gamma \beta)^{-1})^{-1} g_1(\gamma \beta \alpha(\gamma \beta)^{-1}, \gamma) g_1(\beta, \alpha)^{-1} g_1(\beta \alpha \beta^{-1}, \beta) l_{\gamma \beta \alpha(\gamma \beta)^{-1}}.
\]
Therefore $\varphi_{\gamma \beta}(l_\alpha) = h \varphi_{\gamma}(\varphi_{\beta}(l_\alpha))$, where
\[
h = g_1(\gamma \beta, \alpha)^{-1} g_1(\gamma \beta \alpha(\gamma \beta)^{-1}, \gamma \beta) \\
\quad \times g_1(\gamma, \beta \alpha^{-1}) g_1(\gamma \beta \alpha(\gamma \beta)^{-1}, \gamma)^{-1} g_1(\beta, \alpha) g_1(\beta \alpha^{-1}, \beta)^{-1}.
\]
Now, a direct computation using the definition of $h_{\gamma, \beta}^\alpha$ and the assumption that $g_2$ is a group homomorphism satisfying $g_2 \circ \kappa = \delta^2(g_1)$ shows that $g_2(h_{\gamma, \beta}^\alpha) = h$. Therefore
\[
\varphi_{\gamma \beta}(l_\alpha) = h \varphi_{\gamma}(\varphi_{\beta}(l_\alpha)) = h_{\gamma, \beta}^\alpha \varphi_{\gamma}(\varphi_{\beta}(l_\alpha)).
\]
To check the trace identity (11), observe that
\[
g_2(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta, \alpha)) \\
= (\delta^2(g_1))(\alpha \beta \alpha^{-1} \beta^{-1}, \beta, \alpha) \\
= g_1(\beta, \alpha) g_1(\alpha \beta \alpha^{-1}, \alpha)^{-1} g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha) g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta)^{-1}.
\]
and similarly
\[
g_2(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha^{-1}, \beta)) \\
= (\delta^2(g_1))(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha^{-1}, \beta) \\
= g_1(\beta \alpha^{-1}, \beta) g_1(\alpha, \beta)^{-1} g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha) g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha^{-1} \beta^{-1})^{-1}.
\]
It follows from the definitions that
\[
\text{Tr}(\mu_c \varphi_\beta) = g_1(\beta, \alpha)^{-1} g_1(\beta \alpha^{-1} \beta^{-1}, \beta) g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha^{-1} \beta^{-1})^{-1},
\]
\[
\text{Tr}(\varphi_\alpha^{-1} \mu_c) = g_1(\alpha \beta \alpha^{-1}, \alpha)^{-1} g_1(\alpha, \beta) g_1(\alpha \beta \alpha^{-1} \beta^{-1}, \beta)^{-1}.
\]
Comparing these expressions, we obtain that
\[
g_2(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta, \alpha)) \text{Tr}(\mu_c \varphi_\beta) = g_2(\kappa(\alpha \beta \alpha^{-1} \beta^{-1}, \beta \alpha^{-1} \beta^{-1}, \beta)) \text{Tr}(\varphi_\alpha^{-1} \mu_c).
\]
This formula is equivalent to (11).

**Lemma 4.2** Any simple TF–algebra $V = \bigoplus_{\alpha \in G} V_\alpha$ over $(G, A, \kappa)$ that satisfies $\eta_{\epsilon}(1_V \otimes 1_V) = 1$ is isomorphic to $V(g_1, g_2)$ for a certain $\kappa$–pair $(g_1, g_2)$. This $\kappa$–pair is determined by $V$ uniquely up to multiplication of $g_1$ by $\delta^1(\psi)$ for a map $\psi: G \to K^*$. 

**Proof** For each $\alpha \in G$, fix a nonzero vector $l_\alpha \in V_\alpha$. In the role of $l_\epsilon \in V_\epsilon$ we take $1_V$. For any $a \in A$ and $\alpha \in G$, we have $a l_\alpha = g_2^\alpha(a) l_\alpha$ with $g_2^\alpha(a) \in K$. Clearly,
\[
l_\alpha = a l_\alpha = (a^{-1} a) l_\alpha = g_2^\alpha(a^{-1}) g_2^\alpha(a) l_\alpha
\]
and so $g_2^\alpha(a) \in K^*$.
Given \( \alpha, \beta \in G \), we have \( l_\alpha l_\beta = c(\alpha, \beta) l_{\alpha \beta} \) for some \( c(\alpha, \beta) \in K \). Since the pairing (3) is nondegenerate, \( l_\alpha l_{\alpha^{-1}} \neq 0 \). Thus, \( c(\alpha, \alpha^{-1}) \neq 0 \) for all \( \alpha \).

We claim that \( g_2^\alpha(a) \) does not depend on \( \alpha \) for every \( a \in A \). Indeed,

\[
(al_\alpha)l_{\alpha^{-1}} = g_2^\alpha(a)l_\alpha l_{\alpha^{-1}} = g_2^\alpha(a)c(\alpha, \alpha^{-1})l_\epsilon,
\]

\[
a(l_\alpha l_{\alpha^{-1}}) = ac(\alpha, \alpha^{-1})l_\epsilon = g_2^\alpha(a)c(\alpha, \alpha^{-1})l_\epsilon.
\]

Since \( (al_\alpha)l_{\alpha^{-1}} = a(l_\alpha l_{\alpha^{-1}}) \) and \( c(\alpha, \alpha^{-1}) \neq 0 \), we have \( g_2^\alpha(a) = g_2^\alpha(a) \). Set \( g_2 = g_2^\alpha : A \to K^* \). Since \( V \) is an \( A \)-module, the map \( g_2 \) is a group homomorphism. Formula (1) implies the identity (17).

Given \( \alpha, \beta \in G \), we have

\[
c(\beta, \beta^{-1})l_\alpha = c(\beta, \beta^{-1})l_\alpha l_\epsilon = l_\alpha(l_\beta l_{\beta^{-1}}) = \kappa(\alpha, \beta, \beta^{-1})^{-1}(l_\alpha l_\beta)l_{\beta^{-1}} = \kappa(\alpha, \beta, \beta^{-1})^{-1}c(\alpha, \beta)l_{\alpha \beta} l_{\beta^{-1}}.
\]

Therefore \( c(\alpha, \beta) \in K^* \). Let \( g_1 : G \times G \to K^* \) be the map defined by \( g_1(\alpha, \beta) = (c(\alpha, \beta))^{-1} \) for all \( \alpha, \beta \). The identity \( l_\alpha l_\beta = g_1(\alpha, \beta)^{-1}l_{\alpha \beta} \) and Formula (2) yield

\[
g_1(\alpha, \beta)^{-1}g_1(\alpha \beta, \gamma)^{-1}l_{\alpha \beta \gamma} = g_2(\kappa(\alpha, \beta, \gamma))g_1(\alpha, \beta \gamma)^{-1}g_1(\beta, \gamma)^{-1}l_{\alpha \beta \gamma}.
\]

This implies that \( g_2 \kappa = \delta^2(g_1) \). The equality (15) follows from the definitions and the choice \( l_\epsilon = 1_V \). Thus, the maps \( g_1, g_2 \) form a \( \kappa \)-pair. It is clear that \( V = V(g_1, g_2) \).

The second claim of the lemma follows from the fact that any two bases \( l = \{l_\alpha\}_{\alpha \in G} \) and \( l' = \{l'_\alpha\}_{\alpha \in G} \) in \( V \) as above are related by \( l_\alpha = \psi_\alpha l'_\alpha \), where \( \psi_\alpha \in K^* \) for all \( \alpha \). The bases \( l, l' \) yield the same map \( g_2 : A \to K^* \) while the associated maps \( g_1 : G \times G \to K^* \) differ by the coboundary of the map \( G \to K^*, \alpha \to \psi_\alpha \).

### 4.2 The group \( H^2(G, A, \kappa; K^*) \)

The \( \kappa \)-pairs \( (g_1 : G \times G \to K^*, g_2 : A \to K^*) \) form an abelian group \( H = H(G, A, \kappa) \) under pointwise multiplication. The neutral element of \( H \) is the \( \kappa \)-pair \( (g_1 = 1, g_2 = 1) \).

The inverse of a \( \kappa \)-pair \( (g_1, g_2) \) in \( H \) is the \( \kappa \)-pair \( (g_1^{-1}, g_2^{-1}) \). The coboundary \( \kappa \)-pairs form a subgroup of \( H \). Let \( H^2(G, A, \kappa; K^*) \) be the quotient of \( H \) by the subgroup of coboundary \( \kappa \)-pairs. Lemma 4.2 yields a bijective correspondence between the set \( H^2(G, A, \kappa; K^*) \times K^* \) and the set of isomorphism classes of simple TF–algebras over \( (G, A, \kappa) \). This correspondence assigns to a pair \( (h \in H^2(G, A, \kappa; K^*), z \in K^*) \) the isomorphism class of the \( z \)-rescaled TF–algebra \( V(g_1, g_2) \), where \( (g_1, g_2) \) is an
We keep the assumptions of Theorem 4.3. The following theorem due to Eilenberg and MacLane computes $H^2(G, A, \kappa; K^*)$ in topological terms.

**Theorem 4.3** [3] Let $X$ be a path connected topological space with base point $x_0$. Let $G = \pi_1(X, x_0)$, $A = \pi_2(X, x_0)$, and $\Omega = \{v_\alpha \in G \colon \{f_\alpha \beta \alpha, \beta \in G\}\}$ be a basic system of loops and triangles in $X$. Let $\theta$ be a $K^*$-valued singular $2$-cocycle on $X$ representing a cohomology class $[\theta] \in H^2(X, K^*)$. Then the pair $(g_1: G \times G \to K^*, g_2: A \to K^*)$ defined by $g_1(\alpha, \beta) = \theta(f_{\alpha \beta})$ for $\alpha, \beta \in G$ and $g_2(a) = \theta(a)$ for $a \in A$ is a $\kappa$-pair, where $\kappa = \kappa \Omega$: $G^3 \to A$ be the $3$-cocycle defined in Section 2.2. Moreover, the formula $[\theta] \mapsto (g_1, g_2)$ defines an isomorphism

$$H^2(X, K^*) \cong H^2(G, A, \kappa; K^*).$$

### 4.3 The underlying TF-algebras of cohomological HQFTs

We keep the assumptions of Theorem 4.3.

**Theorem 4.4** Let $(V^\theta, \tau^\theta)$ be the $2$-dimensional $X$-HQFT determined by the cohomology class $\theta \in H^2(X, K^*)$ [5]. If $\theta$ corresponds to $(g_1, g_2) \in H^2(G, A, \kappa, K^*)$, then the underlying TF-algebra of $(V^\theta, \tau^\theta)$ is isomorphic to $V(g_1, g_2)$.

**Proof** Let $V = \bigoplus_{\alpha \in G} V_\alpha = V^\Omega$ be the underlying TF-algebra of $(V^\theta, \tau^\theta)$. The definition of $V^\theta$ implies that $V_\alpha = V^\theta_{(S^1, \alpha, \alpha)}$ is a $1$-dimensional $K$-vector space. This vector space is generated by a vector $p_\alpha$ represented by the map $p_\alpha: [0, 1] \to S^1$ from Section 2.2 viewed as a fundamental cycle of $S^1$. Multiplication in $V$ is computed by

$$p_\alpha p_\beta = \tau^\theta(D(\alpha, \beta))(p_\alpha \otimes p_\beta) = f^*(\theta)(B) p_{\alpha \beta},$$

where $\alpha, \beta \in G$, the map $f: D(\alpha, \beta) \to X$ is determined by the structure of an $X$-surface in the disk with two holes $D(\alpha, \beta)$, and $B$ is a fundamental singular chain in $D(\alpha, \beta)$ such that $\partial(B) = p_{\alpha \beta} - p_\alpha - p_\beta$. It is easy to see (cf [6]) that

$$g^*(\theta)(B) = \theta(f_{\alpha \beta})^{-1} = g_1(\alpha, \beta)^{-1}.$$

So, $p_\alpha p_\beta = g_1(\alpha, \beta)^{-1} p_{\alpha \beta}$. Similarly, for all $a \in A = \pi_2(X)$,

$$a p_\epsilon = \tau^\theta(S(\epsilon, a))(p_\epsilon) = \theta(a) p_\epsilon = g_2(a) p_\epsilon,$$

where $\theta(a) \in K^*$ is the evaluation of $\theta$ on $a$. Therefore $V = V(g_1, g_2)$.

---

*Algebraic & Geometric Topology, Volume 10 (2010)*
Remarks on 2–dimensional HQFTs

References

[1] M Atiyah, Topological quantum field theories, Inst. Hautes Études Sci. Publ. Math. (1988) 175–186 (1989) MR1001453

[2] M Brightwell, P Turner, Representations of the homotopy surface category of a simply connected space, J. Knot Theory Ramifications 9 (2000) 855–864 MR1780591

[3] S Eilenberg, S MacLane, Determination of the second homology and cohomology groups of a space by means of homotopy invariants, Proc. Nat. Acad. Sci. U. S. A. 32 (1946) 277–280 MR0019307

[4] T Porter, V Turaev, Formal homotopy quantum field theories. I. Formal maps and crossed C–algebras, J. Homotopy Relat. Struct. 3 (2008) 113–159 MR2426178

[5] V Turaev, Homotopy field theory in dimension 2 and group-algebras arXiv: math.QA/9910010

[6] V Turaev, Homotopy Quantum Field Theory, EMS Tracts Math. 10, European Math. Soc., Zürich (2010) With appendices by M Müger and A Virelizier

MDS, VT: Department of Mathematics, Indiana University
Rawles Hall, Bloomington IN 47405, United States

MDS: Institute of Mathematics of the Romanian Academy
PO Box 1-764, RO-70700 Bucharest, Romania

mstaic@indiana.edu, vtouraev@indiana.edu

Received: 14 December 2009 Revised: 30 March 2010