Chirality, charge and spin-density wave instabilities of a two-dimensional electron gas in the presence of Rashba spin-orbit coupling

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We show that a result equivalent to Overhauser’s famous Hartree-Fock instability theorem can be established for the case of a two-dimensional electron gas in the presence of Rashba spin-orbit coupling. In this case it is the spatially homogeneous paramagnetic chiral ground state that is shown to be differentially unstable with respect to a certain class of distortions of the spin-density-wave and charge-density-wave type. The result holds for all densities. Basic properties of these inhomogeneous states are analyzed.

I. INTRODUCTION

Recent interest in the properties of the quasi-two dimensional electron and hole devices in the presence of structural (Rashba-Bychkov) or intrinsic (Dresselhaus) spin orbit has brought to the fore the problem of the interacting chiral electron liquid. It is therefore important to revisit several of the fundamental notions of many-body theory for this intriguing system. The purpose of present paper is to begin a theoretical exploration of the relevance and special properties of a class of spatially non homogeneous spontaneously broken symmetry states of the electron liquid in the presence of Rashba spin-orbit coupling in two dimensions. Specifically we will focus our attention on spin density and charge density wave type states henceforth referred to for simplicity as SDW and CDW. SDW and CDW states, originally conceived by A. W. Overhauser, are generally stabilized by the electron-electron interaction and are characterized by spatial oscillations of the spin density, the charge density, or both. In the absence of spin-orbit coupling, one can begin to describe SDW and CDW states by simply considering the electron number density for both spin projections:

\[ n_{\uparrow} = \frac{n}{2} + A \cos \left( \vec{Q} \cdot \vec{R} + \frac{\phi}{2} \right) , \]

\[ n_{\downarrow} = \frac{n}{2} + A \cos \left( \vec{Q} \cdot \vec{R} - \frac{\phi}{2} \right) . \]

In these expressions the wave vector \( \vec{Q} \) spans the Fermi surface, i.e. does satisfy the condition \( |\vec{Q}| \simeq 2k_F \). A CDW corresponds to \( \phi = 0 \), while a SDW state obtains for \( \phi = \pi \). Mixed state are also possible: one such state is beautifully realized in chromium. As we will show, in the presence of linear Rashba spin orbit the corresponding distorted states are characterized by a more complex spatial dependence of the number density, the spin density and, where appropriate, a chiral density. As a first step towards establishing the fundamental properties of SDW- and CDW-like states in the presence of spin-orbit interaction we present here a generalization of the famous Overhauser’s Hartree-Fock (HF) instability theorem. The latter represents an important exact result in many-body theory for it establishes that, within HF, the homogeneous paramagnetic plane wave state does not represent a minimum of the energy for an otherwise uniform electron gas for it can be always variationally bettered by a suitably constructed distorted chiral SDW or CDW.

The paper is structured as follows: In Section II we discuss the relevant aspects of the theory of a two-dimensional non-interacting electron gas in the presence of Rashba spin-obit coupling. Section III briefly discusses useful notions of the electron-electron interaction within Hartree-Fock approximation. Section IV is dedicated to the actual proof of the theorem and contains the main results. Finally the last Section contains the conclusions while a number of useful mathematical relations are derived in the two Appendices.

II. TWO DIMENSIONAL ELECTRON GAS IN THE PRESENCE OF RASHBA SPIN-ORBIT

In the presence of linear Rashba spin-orbit coupling, the one-particle hamiltonian can be written as follows:

\[ \hat{H}_0 = \frac{\vec{p}^2}{2m} + \alpha \vec{p} \cdot (\vec{\sigma} \times \hat{z}) , \]
where \( \mathbf{\hat{z}} \) is the unit direction along the z-axis, the motion taking place in the \( x, y \) plane.

The non interacting problem can be readily diagonalized to obtain the energy spectrum and the eigenfunctions:

\[
E_{k,\mu} = \frac{\hbar^2 k^2}{2m} - \alpha \mu k ,
\]

and

\[
\psi_{k,+} = \frac{1}{\sqrt{2L}} e^{\mathbf{k} \cdot \mathbf{r}} \left( \begin{array}{c} 1 \\ -ie^{i\phi_k} \end{array} \right),
\]

\[
\psi_{k,-} = \frac{1}{\sqrt{2L}} e^{\mathbf{k} \cdot \mathbf{r}} \left( \begin{array}{c} 1 \\ ie^{i\phi_k} \end{array} \right),
\]

where \( \mu = \pm \) labels a state’s chirality and \( \phi_k \) is the angle spanned by the \( x \)-axis and the two dimensional wave vector \( \mathbf{k} \). A schematic of the lower energy sector of the spectrum is plotted in Fig. 1.

By making use of the states of Eqs. (5) and (6) as a basis for a second quantization representation, the familiar fully interacting electron gas Hamiltonian reads:

\[
\hat{H} = \sum_{\mathbf{k},\mu} E_{k,\mu} \hat{b}_{k,\mu}^{\dagger} \hat{b}_{k,\mu} + \frac{1}{2L^2} \sum_{\mathbf{q},\mathbf{k}_1,\mathbf{k}_2,\mu_1,\mu_2,\mu_3,\mu_4} \sum_{\mu_1,\mu_2,\mu_3,\mu_4} v_q \Phi_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}^{\mu_1,\mu_2,\mu_3,\mu_4} \hat{b}_{\mathbf{k}_1+\mathbf{q},\mu_1}^{\dagger} \hat{b}_{\mathbf{k}_2-\mathbf{q},\mu_2}^{\dagger} \hat{b}_{\mathbf{k}_2,\mu_3} \hat{b}_{\mathbf{k}_1,\mu_4} ,
\]

where the phase factor \( \Phi_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}^{\mu_1,\mu_2,\mu_3,\mu_4} \) is defined as follows:

\[
\Phi_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}^{\mu_1,\mu_2,\mu_3,\mu_4} = i^{\frac{\mu_1+\mu_2-\mu_3-\mu_4}{2}} e^{i\left(\frac{\mu_1+\mu_2}{2} \phi_{\mathbf{k}_1+\mathbf{q}} + \frac{\mu_3+\mu_4}{2} \phi_{\mathbf{k}_2-\mathbf{q}} - \frac{1-\mu_1}{2} \phi_{\mathbf{k}_2} - \frac{1-\mu_3}{2} \phi_{\mathbf{k}_1} \right)} \times \\
\frac{1}{4} \left[ 1 + \mu_1 \mu_4 e^{i(\phi_{\mathbf{k}_1+\mathbf{q}}+\phi)} \right] \left[ 1 + \mu_2 \mu_3 e^{i(\phi_{\mathbf{k}_2-\mathbf{q}}-\phi)} \right].
\]

Uncorrelated many body wavefunctions for the system at hand can be represented by Slater determinants constructed by occupying any combinations of the chiral states Eqs. (5) and (6).
III. HARTREE-FOCK THEORY OF A TWO-DIMENSIONAL ELECTRON LIQUID IN THE PRESENCE OF RASHBA SPIN-ORBIT COUPLING

An accurate description of a realistic electronic system requires that electron-electron interaction be taken into account. A first step towards developing such a many-body theory is to investigate the results of a mean-field approach. The main idea behind the mean field procedure is to find an effective Hamiltonian which is quadratic in the electron creation and annihilation operators and can therefore be easily diagonalized. Within the HF theory, the ground state is approximated by a single Slater determinant made out of single particle wavefunctions, which, in turn, are determined by imposing the requirement that the expectation value of the Hamiltonian over the Slater determinant be a minimum. Using these wavefunctions as our basis set, a standard Wick decoupling procedure allows us to determine the effective HF potential. It can be easily proved that the non-interacting chiral states are indeed among the solutions of the corresponding HF equations. In this case, the HF potential is diagonal in wave vectors and chiral indices:

\[ V_{HF}^{\mu \mu'} = -\frac{\delta_{\mu \mu' \alpha \beta}}{2L^2} \sum_{\vec{k}} v_{\vec{k} \rightarrow \vec{k}'}^2 \left[ 1 + \mu \nu \cos(\phi_{\vec{k}} - \phi_{\vec{k}'}) \right]. \]  

The corresponding HF eigenvalues are given by:

\[ \epsilon_{\mu} = \frac{\hbar^2 k^2}{2m} - \alpha \mu k - \frac{1}{2L^2} \sum_{\vec{k}} v_{\vec{k}}^2 \left[ 1 + \mu \nu \cos(\phi_{\vec{k}} - \phi_{\vec{k}'}) \right]. \]

An evaluation of the Fermi energy of the two sub-bands leads to an interesting problem. Since one band will, in general, acquire more exchange energy than the other, this may result (in a first iteration) in two different Fermi levels. In order to equalize them (for elementary stability reasons), electrons from one subband will have to be moved to the other. This is the phenomenon of repopulation.

The spatially homogeneous chiral states are just one of the possible Hartree-Fock solutions. A detailed analysis of the possible solutions corresponding to symmetric occupations in momentum space can be done by systematically minimizing the total energy as a function of spin orientation and generalized chirality of the system. More general solutions correspond to non-symmetric occupations of the single particle chiral states. The problem has been studied and the corresponding very interesting phase diagram has been explored. As we will presently discuss there also exists an interesting class of spatially non-homogenous solutions to the problem.

IV. PROOF OF THE INSTABILITY THEOREM

We will proceed by showing that it is always possible to lower the energy of the homogeneous paramagnetic chiral ground state by introducing a suitable real space distortion which is periodic with wave vector \( \vec{Q} = 2k_F \hat{x} \). The general approach follows that of Fedders and Martin and is based on an Ansatz which represents a generalization of that given by Giuliani and Vignale for the case of the three dimensional electron gas.

Let us consider first the putative HF ground state of our many-body system \( |\Phi_S\rangle \). A complete, and, as we shall see convenient, description of this determinantal state can be achieved in terms of the corresponding single-particle density matrix elements here given by:

\[ \rho_{\alpha \beta} = \langle \Phi_S | \hat{a}^\dagger_{\alpha} \hat{a}_{\beta} | \Phi_S \rangle, \]  

where \( \alpha \) and \( \beta \) label the one-particle states which are used to build the Slater determinant.

Now, within the space of Slater determinants, any slightly modification of the state \( |\Phi_S\rangle \) can be described in terms of a corresponding infinitesimal change of the single-particle density matrix elements. Let us indicate such a change by \( \delta \rho_{\alpha \beta} \). At this point the next task consists in trying to evaluate the change of the total HF energy in terms of these quantities.

Since \( |\Phi_S\rangle \) is a solution of the HF equations, the first order variation in the energy must vanish so that the problem at hand is reduced to determining the sign of the energy change to second order in the \( \delta \rho_{\alpha \beta} \)'s. The relevant expression is therefore given by:

\[ \Delta^{(2)} E_{HF} = \frac{1}{2} \sum_{\alpha, \beta} \delta \rho_{\alpha \beta} \frac{\epsilon_{\beta} - \epsilon_{\alpha}}{n_\alpha - n_\beta} \delta \rho_{\beta \alpha} + \frac{1}{2} \sum_{\alpha, \delta} \sum_{\beta, \gamma} (v_{\alpha \beta \gamma \delta} - v_{\alpha \beta \lambda \gamma}) \delta \rho_{\alpha \delta} \delta \rho_{\beta \gamma}, \]  

where \( n_\alpha \) is the number of particles in state \( \alpha \).
where the notation \((\alpha, \beta)\) means that only states situated on opposite sides of the Fermi sea are considered in the summation. In this formula, we indicate the Hartree-Fock eigenvalues as \(\epsilon_\alpha\) and the corresponding occupation numbers as \(n_\alpha\).

The next step in our procedure consists in constructing a Slater determinant for which the HF energy is lower than \(\langle \Phi_S | H | \Phi_S \rangle\). In order to do so, we follow Overhauser’s idea and choose the new one-particle states to be suitable linear combinations of chiral plane waves states situated near opposite points on the Fermi surface, the distortion being limited to a very narrow strip. The width of this strip will play the role of a variational parameter. We then carefully devise an expression for the wave vector dependent amplitude of the coupling between the plane waves and construct the corresponding Slater determinant. In the last step, we calculate the change of the Hartree-Fock energy due to this perturbation to leading order in the distortion amplitude from Eq. (12).

![Figure 2](attachment:image.png)

**FIG. 2:** Case of unit chirality when the states of the lower chiral subband are occupied up to the lower threshold of the upper subband. The arrows indicate the states that are coupled by the distortion along the \(k_x\) axis.

### A. Instability for the case of chirality equal one

Because it presents a formally simpler problem, the first case to be treated is that in which only the lower subband is occupied while the upper one is just about to be filled. This situation is depicted in Fig. 2.

Let us build the new trial wavefunctions as mixtures of the wavefunctions corresponding to wave vector \(\vec{k}\) with those corresponding to wave vector \(\vec{k} \pm \vec{Q}\), i.e.

\[
\Psi_{\vec{k}} \simeq \psi_{\vec{k},+} + A_{\vec{k}+\vec{Q}} \psi_{\vec{k}+\vec{Q},+} + A_{\vec{k}-\vec{Q}} \psi_{\vec{k}-\vec{Q},+}.
\]

Here, as anticipated, \(|\vec{Q}| = 2k_F^+\) as shown in Fig. 2.

In evaluating the HF energy change, only the states situated on opposite sides of the Fermi sea are relevant. We will consider \(|\vec{k} \pm \vec{Q}| > k_F^+\) and \(k < k_F\). Here, the occupation numbers are \(n_{\vec{k}} = \theta(k_F - k)\), while the amplitude satisfies the condition: \(A_{\vec{k} \pm \vec{Q}} = A_{\vec{k} \pm \vec{Q}}\). The only non-zero matrix elements of \(\delta\hat{\rho}\) have the form:

\[
\delta \rho_{\vec{k}+\vec{Q},\vec{k}+\vec{Q}} = \delta \rho_{\vec{k}-\vec{Q},\vec{k}-\vec{Q}} = \left( n_{\vec{k}+\vec{Q}} + n_{\vec{k}-\vec{Q}} \right) A_{\vec{k}}.
\]

These wavefunctions indeed describe SDW/CDW-like states. A simple calculation shows that retaining only the linear order in the amplitude of the distortion, the spin and the charge densities exhibit spatial oscillations with wave vector \(2k_F^+\). Specifically:

\[
\frac{\delta S_x(\vec{r})}{\hbar} = \sum_{\vec{k}} A_{\vec{k}} \left[ \left( \cos \phi_{\vec{k}+\vec{Q}} - \cos \phi_{\vec{k}-\vec{Q}} \right) \sin \vec{Q} \cdot \vec{r} + \left( \sin \phi_{\vec{k}+\vec{Q}} + \sin \phi_{\vec{k}-\vec{Q}} \right) \cos \vec{Q} \cdot \vec{r} \right],
\]

\[
\frac{\delta S_y(\vec{r})}{\hbar} = \sum_{\vec{k}} A_{\vec{k}} \left[ \left( \sin \phi_{\vec{k}+\vec{Q}} - \sin \phi_{\vec{k}-\vec{Q}} \right) \sin \vec{Q} \cdot \vec{r} - \left( \cos \phi_{\vec{k}+\vec{Q}} + \cos \phi_{\vec{k}-\vec{Q}} \right) \cos \vec{Q} \cdot \vec{r} \right],
\]
the same pattern of the proof of reference

The distortion amplitudes. As a first condition, we will perturb only a narrow region near the Fermi surface. Following

electron interaction. By employing Eq. (8), after simple algebraic manipulations, we obtain:

\[
\frac{\delta S_z (\vec{r})}{\hbar} = \sum_k 2A_k \left[ 1 - \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{k}-\vec{Q}} \right) \right] \cos \vec{Q} \cdot \vec{r},
\]

\[
\delta n (\vec{r}) = \sum_k 4A_k \left[ 1 + \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{k}-\vec{Q}} \right) \right] \cos \vec{Q} \cdot \vec{r}.
\]  

(15)

The change in the HF energy is obtained by substituting the expression of the non-zero density matrix elements from Eq. (14) into Eq. (12). The resulting expression can be expressed as:

\[
\Delta_{(2)} H_F = \Delta_{(2)} E_{HF} + \Delta_{(2)} E_{HF} + \Delta_{(2)} E_{HF},
\]

where we have defined

\[
\Delta_{(2)} E_{HF} = \sum_k \left( n_{\vec{k}+\vec{Q}} - n_{\vec{k}-\vec{Q}} \right) \frac{\epsilon_{\vec{k}-\vec{Q}} - \epsilon_{\vec{k}+\vec{Q}}}{n_{\vec{k}+\vec{Q}} - n_{\vec{k}-\vec{Q}}} A_k^2,
\]

\[
\Delta_{(2)} E_{HF} = -\frac{1}{2} \sum_{\vec{k},\vec{p}} \left( \nu_{\vec{k}+\vec{Q}} + \nu_{\vec{k}-\vec{Q}} \right) \left( \nu_{\vec{p}+\vec{Q}} + \nu_{\vec{p}-\vec{Q}} \right) A_{\vec{k}} A_{\vec{p}},
\]

\[
\Delta_{(2)} E_{HF} = \left( \nu_{\vec{k}+\vec{Q}} + \nu_{\vec{k}-\vec{Q}} \right) \left( \nu_{\vec{p}+\vec{Q}} + \nu_{\vec{p}-\vec{Q}} \right) A_{\vec{k}} A_{\vec{p}}.
\]  

(17)

(18)

The Hartree and exchange terms in Eqs. (18)-(19) contain combinations of the matrix elements of the electron-electron interaction. By employing Eq. (10), after simple algebraic manipulations, we obtain:

\[
\frac{\nu_{\vec{k}+\vec{Q}}}{2L^2} \left[ 1 + \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{k}-\vec{Q}} \right) \right] + \cos \left( \phi_{\vec{p}+\vec{Q}} - \phi_{\vec{p}-\vec{Q}} \right) + \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{k}-\vec{Q}} \right) + \cos \left( \phi_{\vec{p}+\vec{Q}} - \phi_{\vec{p}-\vec{Q}} \right) =
\]

\[
\frac{\nu_{\vec{k}-\vec{p}}}{2L^2} \left[ 1 + \cos \left( \phi_{\vec{p}+\vec{Q}} - \phi_{\vec{k}+\vec{Q}} \right) + \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{p}+\vec{Q}} \right) + \cos \left( \phi_{\vec{p}+\vec{Q}} - \phi_{\vec{k}+\vec{Q}} \right) + \cos \left( \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{p}+\vec{Q}} \right) \right] .
\]  

(20)

(21)

In order to explicitly evaluate the change in the Hartree-Fock energy, we need to assume a specific expression for the distortion amplitudes. As a first condition, we will perturb only a narrow region near the Fermi surface. Following the same pattern of the proof of reference, we propose for the present problem the following educated variational guess:

\[
A_k = \begin{cases} 
\left( \frac{b k_p^2}{\sin \theta} \right)^{1/2} \left| \frac{n_{\vec{k}+\vec{Q}} - n_{\vec{k}-\vec{Q}}}{k^2} \right| \sqrt{| \sin \theta |}, & \text{for } b k_p^2 < k < \zeta b k_p^2 \\
0, & \text{otherwise}
\end{cases}
\]

(22)

where \(b \ll 1\) is our small parameter and the second arbitrary \(\zeta > 1\). This expression is intentionally chosen to display singularities for \(k = 0\) and \(\phi_k = 0\). These singularities are crucial to the present proof.

We now notice that in the expressions of the interaction matrix elements, there appear factors of the type:

\[
\cos \left( \phi_{\vec{p}+\vec{Q}} - \phi_{\vec{k}+\vec{Q}} \right) = \frac{\left( p \cos \phi_p + k_p^+ \right) \left( k \cos \phi_k + k_p^+ \right) + pk \sin \phi_p \sin \phi_k}{\sqrt{p^2 + k_p^2} \cos \phi_p \sqrt{k^2 + (k_p^+)^2 + 2k_p^+ k \cos \phi_k}}.
\]  

(23)
Since we are only interested in the leading order expansion with respect to \( b \), these cosines can be simply taken to be equal to unity, since in the region where the amplitude of the distortion is non-vanishing, both \( p/k_F^+ \) and \( k/k_F^+ \) are of order \( b \). Accordingly we will assume

\[
\cos \left( \phi_{\vec{p}+\frac{\vec{Q}}{2}} - \phi_{\vec{k}+\frac{\vec{Q}}{2}} \right) \simeq 1 + O \left( b^2 \right).
\] (24)

At this point, we recall that \( \vec{k} + \frac{\vec{Q}}{2} \) and \( \vec{k} - \frac{\vec{Q}}{2} \) must lie on opposite sides of the Fermi sea (i.e. \( |\vec{k} - \frac{\vec{Q}}{2}| < k_F^+ \) and \( |\vec{k} + \frac{\vec{Q}}{2}| > k_F^+ \)), which implies that \( |\cos \phi_{\vec{k}}| > \frac{b}{2} \).

We can now proceed to the evaluation of the three components of \( \Delta^{(2)} E_{HF} \) from Eqs. (18)-(19) to leading order in \( b \). For \( \Delta^{(2)} E_{HF} \) the first step is to calculate \( \epsilon_{\vec{k}-\frac{\vec{Q}}{2}} - \epsilon_{\vec{k}+\frac{\vec{Q}}{2}} \). Using (10), with energies expressed in Ry, \( x = \frac{k}{k_F} \) and \( x' = \frac{k'}{k_F} \), we have:

\[
\epsilon_{\vec{k}-\frac{\vec{Q}}{2}} - \epsilon_{\vec{k}+\frac{\vec{Q}}{2}} = \left( \frac{4u^2}{r_s} - \frac{4\tilde{a}u}{r_s} - \frac{1}{\pi r_s} \int_0^1 \int_0^{2\pi} \frac{x' (1 + \cos \varphi')}{\sqrt{u^2 + x'^2 - 2ux' \cos \varphi'}} \, dx' \, d\varphi' \right) \left|_{u = \sqrt{1 + x^2 - 2x \cos \phi_{\vec{k}}}} \right.
\]

\[
- \left( \frac{4u^2}{r_s} - \frac{4\tilde{a}u}{r_s} - \frac{1}{\pi r_s} \int_0^1 \int_0^{2\pi} \frac{x' (1 + \cos \varphi')}{\sqrt{u^2 + x'^2 - 2ux' \cos \varphi'}} \, dx' \, d\varphi' \right) \left|_{u = \sqrt{1 + x^2 + 2x \cos \phi_{\vec{k}}}} \right.
\] (25)

The quadratures appearing in this expression can then be manipulated by making use of the results of Appendix A. The result is:

\[
\epsilon_{\vec{k}-\frac{\vec{Q}}{2}} - \epsilon_{\vec{k}+\frac{\vec{Q}}{2}} = -\frac{16x \cos \phi_{\vec{k}}}{r_s^2} \left( 1 - \frac{\tilde{a}r_s}{2} - \frac{r_s}{2\pi} \ln |\xi x \cos \phi_{\vec{k}}| \right),
\] (26)

where the logarithmic term accounts for the divergence of the derivative of the HF single particle energy near the Fermi level. Here, \( \xi \) is a constant approximately equal to 0.51.

By substituting (25) and (22) in (17) we obtain:

\[
\Delta^{(2)} E_{HF} \simeq \frac{16b^3 L^2 k_F^2}{(\ln \frac{\xi}{2})^2 r_s^2 \pi^2} \int_b^c dx \int_0^{2\pi} d\varphi \sin \varphi \cos \varphi \left( 1 - \frac{\tilde{a}r_s}{2} - \frac{r_s}{2\pi} \ln |\xi x \cos \varphi| \right),
\] (27)

an expression that, to leading order in \( b \), reduces to:

\[
\Delta^{(2)} E_{HF} \simeq \frac{48Nb^3 \ln \xi}{r_s \pi^2}.
\] (28)

where \( N \) is the number of particles.

The Hartree term \( \Delta^{(2)} E_{HF} \) (containing \( \nu_Q \)) can in turn be evaluated as follows. By making use of the assumed amplitude in (22), we write:

\[
\Delta^{(2)} H_{HF} \simeq \int_b^c dx \int_0^{2\pi} dy \sqrt{x} \sqrt{y} \int_0^{\frac{\pi}{2}} d\varphi_x \sqrt{\sin \varphi_x \cos \phi_{\vec{k}} x} \int_0^{\frac{\pi}{2}} d\varphi_y \sqrt{\sin \varphi_y \cos \phi_{\vec{k}} y}.
\] (29)

At this point, using the result (15), the leading order in \( b \) of this quantity is given by:

\[
\Delta^{(2)} H_{HF} \simeq \frac{64Nb^4 (\sqrt{\xi} - 1)^2}{\pi^2 r_s}.
\] (30)
The last term of (10), the exchange energy contribution, is clearly negative and therefore will certainly lower the energy. Its evaluation is formidable, for it involves several complicated and seemingly difficult quadratures. Rather than attempting to calculate it, we will establish a lower limit for its magnitude.

We will restrict ourselves to the region in which both angles $\phi_x$ and $\phi_y$ are in the first quadrant. Since this excludes some contributions of the same sign, the exchange energy will be underestimated. We therefore have:

$$\Delta_{(2)}^X E_{HF} \geq \frac{2Nb^3}{\pi^2 \ln^2 \frac{2}{b}} \int_{b}^{bc} \int_{b}^{bc} \frac{d \varphi_x}{\sqrt{x^2 + y^2 - 2xy \cos(\varphi_x - \varphi_y)}} \sqrt{\frac{x^2 + y^2 - 2xy \cos(\varphi_x - \varphi_y)}{\cos \varphi_x \cos \varphi_y}} \cdot \frac{1}{\sqrt{b}}.$$

(31)

It is simple to see that this expression will turn out to be proportional to $(\ln b)^3$. This is due to the presence of three singularities in the denominator of the integrand. Of these one stems from the divergence of the Coulomb potential, while the other two come from the upper limit of the angular integrations.

Another simplification is provided by the use of the inequality:

$$\frac{1}{\sqrt{x^2 + y^2 - 2xy \cos(\varphi_x - \varphi_y)}} \geq \frac{1}{\sqrt{x^2 + y^2 - 2xy \sin \varphi_x \sin \varphi_y}}.
\quad \quad (32)$$

Using the same changes of variable as in Appendix [11] i.e. $t_x = \tan(\phi_x/2)$ and $t_y = \tan(\phi_y/2)$, we can rewrite this integral as follows:

$$\Delta_{(2)}^X E_{HF} \geq \frac{16Nb^3}{\pi^2 \ln^2 \frac{2}{b}} \int_{b}^{bc} \int_{b}^{bc} \sqrt{\frac{x^2 + y^2 - 2xy (1 - t_x^2)}{s^2}} \sqrt{\frac{1}{x^2 + y^2 - 2xy (1 - t_x^2)}} \sqrt{\frac{1}{1 - t_x^2}} \sqrt{\frac{1}{1 + t_x^2}} \sqrt{\frac{1}{1 - t_y^2}} \sqrt{\frac{1}{1 + t_y^2}} \cdot \frac{1}{\sqrt{b}}.
\quad \quad (33)$$

It is clear now that the main contribution to the integral comes from the region around the upper limit of integration for both $t_x$ and $t_y$. In order to retain the leading order term, a good approximation will be to replace both $t$’s with $1 - b/2$ in the denominator of the first square root. In this way, we can separate the angular integrations in (31) to obtain:

$$\Delta_{(2)}^X E_{HF} \geq \frac{2Nb^3}{\pi^2 \ln^2 \frac{2}{b}} \int_{b}^{bc} \int_{b}^{bc} \sqrt{\frac{x^2 + y^2 - 2xy (1 - b^2/8)}{s^2}} \sqrt{\frac{1}{x^2 + y^2 - 2xy (1 - b^2/8)}} \sqrt{\frac{1}{1 - b^2/8}} \sqrt{\frac{1}{1 + b^2/8}} \cdot \frac{1}{\sqrt{b}}.
\quad \quad (34)$$

The last two integrals are evaluated using (B3) and other changes of variables ($x = ub$, $y = vb$):

$$\Delta_{(2)}^X E_{HF} \geq \frac{8Nb^3}{\pi^2 r_s} \int_{1}^{\sqrt{u}} \int_{1}^{\sqrt{v}} \frac{1}{\sqrt{(u^2 - v^2)^2 + u^2 v^2 b^2}}.
\quad \quad (35)$$

The last quadrature is calculated in (B3), leading us to a very simple inequality for the exchange contribution:

$$\Delta_{(2)}^X E_{HF} \geq \frac{8Nb^3}{\pi^2 r_s} \ln \frac{1}{b}.
\quad \quad (36)$$

This term contains a logarithmic factor $\ln \frac{1}{b}$, which allows the negative change in the exchange contribution to control all the remaining terms. This concludes the proof for this case.

The same chain of arguments does apply to the case in which the generalized chirality is greater than one. The coupling that produces this kind of instability is schematically shown in Fig. 3. All the formulas we derived in the previous case do still apply. The only difference lies in the lower integration limits of Eq. (25), but no relevant contribution ensues from this. The matrix elements related to the Hartree and exchange contributions are the same, and, as a consequence, the leading order approximation is identical.
B. Instability for the case of chirality less than one

The argument of the previous Section can be applied when the chirality is less than one, i.e. when both chiral subbands are occupied. We can try to break the symmetry by coupling states with the same chirality as well as states with different chiralities. When same chirality states are coupled, there is nothing new, as one simply just adds a chirality index to the various quantities. In this case, the wave vectors characterizing the oscillations are given by: $Q_{\mu} = 2k_{F}^\mu$ and the trial wavefunctions can be written as:

$$\Psi_{\vec{k}\mu} \simeq \psi_{\vec{k}\mu} + A_{\vec{k}+\vec{Q},\vec{\mu}} \psi_{\vec{k}+\vec{Q},\mu} + A_{\vec{k}+\vec{Q},\vec{\mu}} \psi_{\vec{k}+\vec{Q},-\mu} + A_{\vec{k}-\vec{Q},\vec{\mu}} \psi_{\vec{k}-\vec{Q},-\mu}.$$ (37)

This type of coupling is depicted in Fig. 4.

The corresponding distortion of the components of the spin density and the number density can be again calculated up to the first order in the amplitudes:

$$\delta S_x(\vec{r}) = \hbar \sum_{k,\mu} A_{\vec{k}\mu} \left( \cos \phi_{\vec{k}+\vec{Q},\mu} - \cos \phi_{\vec{k}-\vec{Q},\mu} \right) \sin \vec{Q}_{\mu} \cdot \vec{r} + \sum_{k,\mu} A_{\vec{k}\mu} \left( \sin \phi_{\vec{k}+\vec{Q},\mu} + \sin \phi_{\vec{k}-\vec{Q},\mu} \right) \cos \vec{Q}_{\mu} \cdot \vec{r}$$, (38)

$$\delta S_y(\vec{r}) = \hbar \sum_{k,\mu} A_{\vec{k}\mu} \left( \sin \phi_{\vec{k}+\vec{Q},\mu} - \sin \phi_{\vec{k}-\vec{Q},\mu} \right) \sin \vec{Q}_{\mu} \cdot \vec{r} - \sum_{k,\mu} A_{\vec{k}\mu} \left( \cos \phi_{\vec{k}+\vec{Q},\mu} + \cos \phi_{\vec{k}-\vec{Q},\mu} \right) \cos \vec{Q}_{\mu} \cdot \vec{r}$$, (39)

$$\delta S_z(\vec{r}) = 2\hbar \sum_{k,\mu} A_{\vec{k}\mu} \left( 1 - \cos \left( \phi_{\vec{k}+\vec{Q},\mu} - \phi_{\vec{k}-\vec{Q},\mu} \right) \right) \cos \vec{Q}_{\mu} \cdot \vec{r}$$, (40)

$$\delta n(\vec{r}) = 4\sum_{k,\mu} A_{\vec{k}\mu} \left( 1 - \cos \left( \phi_{\vec{k}+\vec{Q},\mu} - \phi_{\vec{k}-\vec{Q},\mu} \right) \right) \cos \vec{Q}_{\mu} \cdot \vec{r}.$$ (41)

We proceed in this case by choosing an amplitude not unlike the one assumed above:

$$A_{\vec{k}\mu} = \begin{cases} \frac{(bk_{F}^\mu)^2}{k_{F}^\mu} \frac{\sin \phi_{\vec{k}+\vec{Q},\mu} - \sin \phi_{\vec{k}-\vec{Q},\mu}}{\sin \phi_{\vec{Q},\mu}} \sqrt{\frac{\sin \phi_{\vec{Q},\mu}}{\cos \phi_{\vec{Q},\mu}}} , & \text{for } \mu = \nu, \text{ and } \mu = \nu < \chi \nu_{b}\mu = k_{F}^\mu \text{ and } \nu_{b}\mu < \chi \nu_{b}\mu \\ 0, & \text{otherwise} \end{cases}$$ (42)

The proof of the corresponding instability theorem proceeds then in exactly the same manner.

As anticipated, there is also not much difference when we try to couple states with opposite chirality (see Fig. 5). Although some of the expressions involved in the derivation do change, the main features of the argument remain unchanged. The coupling vector in this case is given by $Q = k_{F}^+ + k_{F}^-$. Here, we try to find a lower energy state by coupling wavefunctions with wave vector $\vec{k}$ with those with wave vector $\vec{k} \pm \vec{Q}$ and opposite chirality. The trial wavefunctions then read:

$$\Psi_{\vec{k}+} \simeq \psi_{\vec{k}+} + A_{\vec{k}+\vec{Q},\vec{\mu}} \psi_{\vec{k}+\vec{Q},-\mu} + A_{\vec{k}-\vec{Q},\vec{\mu}} \psi_{\vec{k}-\vec{Q},-\mu}.$$ (43)
For $A_{\vec{k}+\vec{Q},\vec{k}} = A_{\vec{k}',\vec{k}+\vec{Q}}$ the only non zero variations of the matrix elements of the single-particle density matrix operator acquire the following form:

$$
\delta \rho_{\vec{k}+\vec{Q},\mu,\vec{k}'+\vec{Q}'} - \rho_{\vec{k}'-\vec{Q}',\mu,\vec{k}-\vec{Q}} = \delta \rho_{\vec{k}'-\vec{Q}',\mu,\vec{k}+\vec{Q}'} = \left( n_{\vec{k}'+\vec{Q}',\mu} + n_{\vec{k}'-\vec{Q}',\mu} \right) A_{\vec{k}} .
$$

(44)

In this case, the new state is characterized by a similar spin and density modulation:

$$
\delta S_x (\vec{r}) = \hbar \sum_k A_{\vec{k}} \left[ \cos \left( \vec{Q} \cdot \vec{r} - \phi_{\vec{k}+\vec{Q}} - \phi_{\vec{k}-\vec{Q}} \right) + \cos \vec{Q} \cdot \vec{r} \right],
$$

(45)

$$
\delta S_y (\vec{r}) = \hbar \sum_k A_{\vec{k}} \left[ \sin \left( -\vec{Q} \cdot \vec{r} + \phi_{\vec{k}+\vec{Q}} + \phi_{\vec{k}-\vec{Q}} \right) + \sin \vec{Q} \cdot \vec{r} \right],
$$

(46)

$$
\delta S_z (\vec{r}) = \hbar \sum_k A_{\vec{k}} \left[ \sin \left( \vec{Q} \cdot \vec{r} - \phi_{\vec{k}+\vec{Q}} \right) + \sin \left( \vec{Q} \cdot \vec{r} - \phi_{\vec{k}-\vec{Q}} \right) \right],
$$

(47)

$$
\delta n (\vec{r}) = 2 \sum_k A_{\vec{k}} \left[ \sin \left( \vec{Q} \cdot \vec{r} - \phi_{\vec{k}+\vec{Q}} \right) - \sin \left( \vec{Q} \cdot \vec{r} - \phi_{\vec{k}+\vec{Q}} \right) \right].
$$

(48)

The corresponding terms in the Hartree-Fock energy change are:

$$
\Delta_{0}^{(2)} E_{HF} = \sum_{\mu} \left( n_{\vec{k}+\vec{Q},\mu} + n_{\vec{k}-\vec{Q},-\mu} \right) \frac{\epsilon_{\vec{k}+\vec{Q},\mu} - \epsilon_{\vec{k},\mu} - \mu}{n_{\vec{k}+\vec{Q},\mu} + n_{\vec{k}-\vec{Q},-\mu}} A_{\vec{k}}^2 ,
$$

(49)

$$
\Delta_{H}^{(2)} E_{HF} = \frac{1}{2} \sum_{\vec{k},\mu,\vec{p},\nu} \left( u_{\vec{k}+\vec{Q},\mu,\vec{p},\nu} - u_{\vec{k},\mu,\vec{p},\nu} + u_{\vec{k}+\vec{Q},-\mu,\vec{p},\nu} - u_{\vec{k},-\mu,\vec{p},\nu} \right) A_{\vec{k}+\vec{Q},\mu} A_{\vec{k},\mu} A_{\vec{p},\nu} A_{\vec{p},\nu},
$$

(50)

$$
\Delta_{X}^{(2)} E_{HF} = \frac{1}{2} \sum_{\vec{k},\mu,\vec{p},\nu} \left( u_{\vec{k}+\vec{Q},\mu,\vec{p},\nu} - u_{\vec{k},\mu,\vec{p},\nu} + u_{\vec{k}-\vec{Q},-\mu,\vec{p},\nu} - u_{\vec{k},-\mu,\vec{p},\nu} \right) A_{\vec{k}+\vec{Q},\mu} A_{\vec{k},\mu} A_{\vec{p},\nu} A_{\vec{p},\nu}. \,
$$

(51)

The relevant interaction matrix elements which appear in the expression of the Hartree term become:

$$
\frac{2v_Q}{L^2} \left( \sin \phi_{\vec{p}+\vec{Q},\mu} - \sin \phi_{\vec{p}-\vec{Q},\mu} \right) \left( \sin \phi_{\vec{k}+\vec{Q},\mu} - \sin \phi_{\vec{k}-\vec{Q},\mu} \right),
$$

(52)

while those determining the exchange contribution reduce to:

$$
\frac{v_{\vec{k}+\vec{Q}}}{L^2} \left( \cos \phi_{\vec{p}+\vec{Q},\mu} - \cos \phi_{\vec{p}-\vec{Q},\mu} \right) + \cos \phi_{\vec{p}-\vec{Q},\mu} - \cos \phi_{\vec{k}+\vec{Q},\mu} + \cos \phi_{\vec{p}+\vec{Q},\mu} - \cos \phi_{\vec{p}-\vec{Q},\mu} + \cos \phi_{\vec{k}+\vec{Q},\mu} - \cos \phi_{\vec{p}+\vec{Q},\mu} + \cos \phi_{\vec{k}-\vec{Q},\mu} - \cos \phi_{\vec{p}-\vec{Q},\mu} + \cos \phi_{\vec{k}-\vec{Q},\mu} - \cos \phi_{\vec{p}-\vec{Q},\mu} + 1\right).
$$

(53)

As in the previous cases, we assume the following coupling amplitude:

$$
A_{\vec{k}}^\mu = \begin{cases} 
\left( \frac{\sin \phi_{\vec{k}+\vec{Q},\mu}}{\cos \phi_{\vec{k}+\vec{Q},\mu}} \right) \sqrt{\left| \frac{n_{\vec{k}+\vec{Q},\mu} - n_{\vec{k}-\vec{Q},\mu}}{k^2} \right|}, & b k^2 < k < \zeta b k^2, \cos \phi_{\vec{k}} < \frac{b}{2} \\
0, & \text{otherwise}.
\end{cases}
$$

(54)

The first term in the Hartree-Fock energy (17) has a positive contribution which is proportional to $b^3$. This originates from the same logarithmic factor associated with the divergence of the derivative of the single-particle energy at the
Fermi level. The Hartree term introduces higher order terms in $b$ due to the presence of the sine factors in its matrix elements. Finally, the leading order contribution to the exchange matrix elements is $4v|\mathbf{k} - \mathbf{p}|$. By evaluating integrals similar to those in Eq. (31) we again obtain a negative energy change of order $b^3 \ln(1/b)$.

V. CONCLUSIONS

We have been able to formally construct a number of distorted chiral states which, irrespective of the electron density, have, within mean field, a lower energy than the spatially homogeneous paramagnetic chiral HF ground state, thereby affording a rigorous proof of a generalization of Overhauser’s Hartree-Fock instability theorem to a two dimensional electron liquid in the presence of linear Rashba-Dresselhaus spin-orbit coupling. It is important to notice that, as mentioned in Section III to establish the instability we have not allowed for momentum space repopulation: inclusion of this phenomenon would have further lowered the energy of the trial states while greatly increasing the difficulty of the analysis.

It is worth mentioning that the states that have been analyzed in this paper differ in a number of ways from the original spin/charge density waves proposed by Overhauser. Our states are chiral density states, that display both spin and charge modulations. The presence of charge modulations cannot be ignored (as commonly done in the case without spin-orbit) and the Hartree term has to be evaluated explicitly. The exchange gain is shown to be larger than the kinetic energy plus the ensuing Hartree terms. In this respect the difference with the plain electron liquid is quite marked for it is precisely the electrostatic term that is believed to be fatal to the charge density waves in that case. We prove that this is not the case for the chiral waves. The main reason is that the exchange energy behaves quite differently as compared to that in the absence of spin-orbit interaction. In particular it does not necessarily favor a homogeneous chiral instability as shown in Ref. 11. Our calculations show that in a vast class of inhomogeneous states, the exchange energy can induce a chiral instability for all densities of the electron liquid. Of particular importance is that the chiral states coupled by the type of mean fields explored here do always have opposite spins. The fact that
such couplings are possible for all values of the spin-orbit coupling is another consequence of the non analyticity of
the properties of the electron liquid on this parameter.

Our results only represent the first step in understanding non homogeneous states in this interesting many-body
system. How the inclusion of correlations will modify the HF scenario is of course a most important question. Advances
in this respect can in principle be pursued by following the program outlined in Reference 16. Such a study will require
a vastly more complicated analysis. An alternative route is to make use of perturbative techniques to establish the
role of correlation effects in the high density limit. Part of this program has been carried out in Reference 17 for
homogeneous states.

Acknowledgments

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Appendix A: Elliptic integral expansion

For the purpose of our calculations the following expression must be evaluated in the limit of small $u$:

$$
\frac{2\pi}{1} \begin{array}{c}
\int_0^{2\pi} \int_0^1 \frac{x'(1 + \cos \varphi) dx' d\varphi}{\sqrt{z^2 + x'^2 - 2zx' \cos \varphi}} \\
\int_0^{2\pi} \int_0^1 \frac{x'(1 + \cos \varphi) dx' d\varphi}{\sqrt{z^2 + x'^2 - 2zx' \cos \varphi}}
\end{array} + u \frac{\partial A(z)}{\partial z}
$$

(A1)

An obvious problem is the presence of singularities in the integrand for $z = 1$. Let us define:

$$
A(z) = \frac{2\pi}{1} \begin{array}{c}
\int_0^{2\pi} \int_0^1 \frac{x'(1 + \cos \varphi) dx' d\varphi}{\sqrt{z^2 + x'^2 - 2zx' \cos \varphi}} \\
\int_0^{2\pi} \int_0^1 \frac{x'(1 + \cos \varphi) dx' d\varphi}{\sqrt{z^2 + x'^2 - 2zx' \cos \varphi}}
\end{array}.
$$

(A2)

Of course, $A$ is not differentiable in $z = 1$. Still we have to evaluate $\frac{\partial A(z)}{\partial z}$ and expand it in an asymptotic series
around $z = 1$.

$$
\frac{\partial}{\partial z} A(z) = \frac{\partial}{\partial z} \left( \int_0^{2\pi} \int_0^1 \frac{y(1 + \cos \varphi) dy d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}} \right)
$$

$$
= \int_0^{2\pi} \int_0^1 \frac{y(1 + \cos \varphi) dy d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}}
$$

$$
+ z \frac{\partial}{\partial z} \left( \int_0^{2\pi} \int_0^1 \frac{y(1 + \cos \varphi) dy d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}} \right).
$$

(A3)

The derivative in the second term can be evaluated in the following way:

$$
\frac{\partial}{\partial z} \left( \int_0^{2\pi} \int_0^1 \frac{y(1 + \cos \varphi) dy d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}} \right)
$$
\[
\begin{align*}
&= - \frac{1}{z^2} \int_0^{2\pi} \frac{y (1 + \cos \varphi) \, d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}} \\
&= - \frac{1}{z^2} \int_0^{2\pi} \frac{(1 + \cos \varphi) \, d\varphi}{\sqrt{1 + z^2 - 2z \cos \varphi}}. \tag{A4}
\end{align*}
\]

We can integrate this expression over its angular part so that the result is expressed as:

\[
\int_0^{2\pi} \frac{(1 + \cos \varphi) \, d\varphi}{\sqrt{1 + z^2 - 2z \cos \varphi}} = - \frac{2 |z - 1|}{z} E \left( \frac{-4z}{(z - 1)^2} \right) + 2 \left( \frac{z + 1}{z} \right)^2 K \left( \frac{-4z}{(z - 1)^2} \right), \tag{A5}
\]

where \( E \) and \( K \) are the elliptic integrals of first and second kind defined as in \(^{18,19}\).

The asymptotic expansion of the elliptic integrals around \( z = 1 \) leads to:

\[
\frac{\partial A(z)}{\partial z} = \rho - 12 \ln 2 + 4 + 4 \ln |z - 1|, \tag{A6}
\]

where \( \rho = \int_0^{2\pi} \int_0^1 \frac{y (1 + \cos \varphi) \, dy \, d\varphi}{\sqrt{1 + y^2 - 2y \cos \varphi}} \simeq 5.6639. \)

Integrating over \( z \) we finally obtain:

\[
A(1 + u) - A(1 - u) \simeq 2 (\rho - 12 \ln 2 + 4 \ln |u|) u = \simeq 8u (\ln |\xi u|), \tag{A7}
\]

with \( \xi \simeq 0.51. \)

**Appendix B: Evaluation of useful quadratures**

We begin by calculating here the leading order term in the expansion of the expression

\[
I = \arccos \frac{1}{2} \int_0^{\arccos \frac{1}{2}} \frac{y \sin \varphi}{\cos \varphi} \, d\varphi \quad \text{for } b \ll 1, \tag{B1}
\]

Expanding the upper limit and setting \( t = \tan \frac{\varphi}{2} \), the integral becomes:

\[
I \simeq 2\sqrt{2} \int_0^{1 - \frac{1}{b}} \frac{dt}{1 - t^2} \sqrt{\frac{t}{1 + t^2}} \simeq \sqrt{2} \int_0^{1 - \frac{1}{b}} \frac{dt}{1 + t^2} \left( \frac{1}{1 - t} + \frac{1}{1 + t} \right), \tag{B2}
\]

The divergence in the limit \( b \to 0 \) stems only from the first term and we therefore proceed to try isolating the singularity:
\[ I \simeq \sqrt{2} \int_0^{1 - \frac{b}{2}} dt \sqrt{\frac{t}{1 + t^2}} + \sqrt{2} \int_0^{1 - \frac{b}{2}} \sqrt{\frac{1}{1 + t^2}} + O(b) \simeq \]
\[ \simeq -\sqrt{2} \int_0^{1 - \frac{b}{2}} dt \sqrt{\frac{t}{1 + t^2}} (\ln (1 - t))' + 0.5256 + O(b) . \]

An integration by parts is used in the remaining integral to further isolate the singular contribution:

\[ I \simeq -\sqrt{2} \int_0^{1 - \frac{b}{2}} dt \sqrt{\frac{t}{1 + t^2}} (\ln (1 - t))' \]
\[ + \sqrt{2} \int_0^{1 - \frac{b}{2}} dt \frac{d}{dt} \left( \sqrt{\frac{t}{1 + t^2}} \right) \ln (1 - t) + 0.5256 + O(b) . \]

At this point the non singular second term is calculated numerically so that the final result is:

\[ I \simeq \ln b + 0.9475 . \]

The following integral is used in the proof of the HF instability theorem:

\[ J = \int_1^{\sqrt{\frac{\varsigma}{2}}} \int_1^{\sqrt{\frac{\varsigma}{2}}} \frac{dudv}{\sqrt{(u^2 - v^2)^2 + \frac{u^2v^2}{8}}} , \]

for \( b \ll 1. \)

Because the singular behavior originates from the region where \( u \simeq v, \) in order to find the leading order term, we approximate \( u^2 - v^2 \) with \( 2u(u - v) \). Since the function is symmetric with respect to the interchange of \( u \) and \( v \) we can use the relation \( \int_1^{\sqrt{\frac{\varsigma}{2}}} \int_1^{\sqrt{\frac{\varsigma}{2}}} f(u,v) = 2 \int_1^{\sqrt{\frac{\varsigma}{2}}} \int_1^{\sqrt{\frac{\varsigma}{2}}} f(u,v) \).

Then:

\[ J \simeq \int_1^{\sqrt{\frac{\varsigma}{2}}} \int_1^{u} \frac{dv}{\sqrt{(u - v)^2 + \frac{u^2v^2}{8}}} \simeq -\int_1^{\sqrt{\frac{\varsigma}{2}}} \frac{du}{u} \ln \left( \frac{bu}{2\sqrt{2}} \right) \]
\[ + \int_1^{\sqrt{\frac{\varsigma}{2}}} \frac{du}{u} \ln \left( u - 1 + \sqrt{(u - 1)^2 + \frac{u^2b^2}{8}} \right) . \]

The integrand of the first term has a logarithmic singularity in the limit of small \( b \) while the second one is instead analytic in this limit. The main contribution to the integral is therefore:

\[ J \simeq -\int_1^{\sqrt{\frac{\varsigma}{2}}} \frac{du}{u} \ln \left( \frac{bu}{2\sqrt{2}} \right) \simeq -\ln \varsigma \ln b . \]

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Stability of a CDW state can only be achieved if the associated Hartree term is cancelled by a corresponding deformation of the neutralizing background. The simplest such case being represented by the case of the so called deformable jellium. A rather interesting case is discussed in Ref. 16.

That the states of Eqs. (5) and (6) are solutions of the Hartree-Fock problem was shown in Ref. 11. Notice that in this case, the generalized chirality, as defined for instance in Ref. 11, also equals one.

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