Discrete symmetries and models of flavor mixing

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Abstract.
Evidences of a discrete symmetry behind the pattern of lepton mixing are analyzed. The program of “symmetry building” is outlined. Generic features and problems of realization of this program in consistent gauge models are formulated. The key issues include the flavor symmetry breaking, connection of mixing and masses, ad hoc prescription of flavor charges, “missing” representations, existence of new particles, possible accidental character of the TBM mixing. Various ways are considered to extend the leptonic symmetries to the quark sector and to reconcile them with Grand Unification. In this connection the quark-lepton complementarity could be a viable alternative to TBM. Observational consequences of the symmetries and future experimental tests of their existence are discussed.

1. Flavor of flavor models
Certain features of the leptonic mixing can be considered as an evidence of discrete symmetry. Many different realizations exist [1]. However, in spite of various interesting developments, no convincing model based on discrete symmetries has been proposed so far. The models require new extended structures, many assumptions, ad hoc assignments of charges and selection of the group representations for multiplets. Additional auxiliary symmetries are needed which sometimes are even more powerful than the original one. There are no natural and simple extensions of the leptonic symmetries to the quark sector. In most of the models no connection between mixing and masses exists and different physics (symmetry) is responsible for the mass hierarchies.

So, what to do? Try further using the same context 1. Be less ambitious and explain only some features (e.g., dominant structures of the mass matrices) using the symmetry? Or modify context (some important elements can be still missed). Apply symmetry differently?

For illustration of these statements, several recently proposed models will be discussed. Generic features of the whole approach and its challenges are formulated. For details of specific models see talks [3] at this conference.

The paper is organized as follows. In sect. 2 the experimental evidences in favor of discrete symmetries will be presented. Sect. 3 is devoted to the Tri-Bimaximal (TBM) mixing and “symmetry building”. In sect. 4 attempts to extend the leptonic symmetry to the quark sector are discussed. Sect. 5 is devoted to alternatives to the TBM approach, in particular, to the quark-lepton complementarity (QLC). Sect. 6 outlines perspectives in the field, and conclusions are given in Sect. 7.

1 In attempt to further pursue this approach and to check whether something interesting is overlooked, systematic scanning (including computerized one) of all possible models within certain framework has been performed [2].
2. Data and evidences
The origin of excitement is the neutrino mass and flavor spectrum shown in fig. 1. Regularities of the flavors distribution are obvious: $\nu_\mu$ and $\nu_\tau$ flavors share $\nu_3$ equally (bi-maximal mixing) and $\nu_e$ is absent in $\nu_3$, all three flavors are presented with the same weight in $\nu_2$ (trimaximal mixing). This can be formalized as invariance of the picture with respect to the following transformations:

1. permutation of $\nu_\mu$ and $\nu_\tau$ flavors in all three mass states, which also implies zero 1-3 mixing;
2. dilatation of the $\nu_\nu$-flavor part by factor 2 and shrinking by 2 the rest of the state $\nu_2$, then inverse operation in $\nu_1$: shrinking by factor 2 the $\nu_e$- part, and dilatation of the rest, and finally, permutation of states $\nu_1$ and $\nu_2$. These transformations are the basis of possible discrete symmetry.

The regularities are described by the TBM mixing matrix [4]:

$$ U_{TBM} = U_{23}(\pi/4)U_{12}(\arcsin(1/\sqrt{3})) = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (1) $$

If the length of boxes is 1, the experimental data permit deviations from the pattern shown in the left figure by amount 0.01 - 0.05. The global fit of the oscillation data [5] agrees with the TBM mixing within $(1 - 2)\sigma$. It shows non-zero best fit value for the 1-3 mixing, some deviation of the 2-3 mixing from maximal one with $\theta_{23} < \pi/4$, and the 1-2 mixing slightly below $\sin^2 \theta_{12} = 1/3$:

$$ \sin \theta_{13} = 0.118^{+0.038}_{-0.048}, \quad \sin^2 \theta_{23} = 0.463^{+0.071}_{-0.048}, \quad \sin^2 \theta_{12} = 0.321^{+0.016}_{-0.016} \quad (2) $$

(with $1\sigma$ errors). The deviations from the TBM values are small but robust appearing in the global analyses of different groups. Important feature is that the 1-2 and 1-3 mixing angles correlate in the data, and therefore should be considered simultaneously. The combined analyses of the solar and KamLaND results [6] [7] [8] give non-zero 1-3 mixing at the level $\sin \theta_{13} = 0.02$, $\sin^2 \theta_{12} < 0.33$, and agreement with TBM is at 95% CL. At the same time, the complete $3\nu -$ analysis of the atmospheric neutrino data from SuperKamiokande results in $\sin 2\theta_{13} = 0.0$ (< 0.04) 90%, and $\sin 2\theta_{23} = 0.5$ (0.407 - 0.583) 90% [9]. That is, the best fit value of the 2-3 mixing is maximal in the case of normal mass hierarchy, in spite of the excess of $e-$like sub-GeV events which gave an indication for $\theta_{23} < \pi/4$.

Several reservations are in order.
1. Large deviations from the symmetry case are still possible and if correct (from the fundamental physics point of view) measure of mixing is $\sin \theta$, then for 1-3 mixing we have $\sin \theta_{13} = (0.10 - 0.15)$ as the best fit value which should be compared with $\sin \theta_{13} = 0.55$.
2. Mixing is not the RGE invariant, and if theory is formulated at some high mass scales (e.g., GUT scale), one may expect significant deviations from the symmetry case when running to high energies.
3. There are several indications that new light \((m_s \sim 1 \text{ eV}^2)\) and almost sterile neutrino state exists which mixes substantially \((\theta_{as} \sim 0.2)\) with the active neutrinos. Since \(m_s \sin^2 \theta_{as} \sim 0.04 \text{ eV}\) is bigger than the largest element of the mass matrix of active neutrinos (in the case of mass hierarchy), the presence of this state (or states) destroys the present constructions unless further complications and fine tunings are introduced.

4. Existence of the 4th generation of fermions can change the present results completely. All this may lead to different line of developments.

Concerning the mass hierarchy, there are several important features which can be relevant for, but usually not addressed in the models under consideration: (1) The upper quarks have geometrical mass relation: \(m_u m_t = m_2^2\); this hints that masses of all three generation should be considered on the same footing (and not in perturbative manner). (2) The down quark masses satisfy the Gatto-Sartori-Tonin relation \(\sin \theta_C \approx \sqrt{m_d/m_s} [10]\) – explicit relation between masses and mixing. (3) The charged leptons satisfy the Koide relation \([11]\) which indicates certain symmetry, and again, an involvement of all three generations of leptons. (4) Neutrinos have the weakest mass hierarchy among fermions which also shows connection between masses and mixing.

3. TBM and symmetry building
To a large extent relevant symmetry can be systematically derived from the data \([1], [12]\) and this gives the main support of the approach “Mixing from discrete symmetries”.

3.1. Deriving symmetry
1. In assumption that neutrinos are Majorana particles the TBM mass matrix is given by
   \[ m_{TBM} = U_{TBM} m^{diag} U^T_{TBM}, \]
   where \(m^{diag} = \text{diag}(m_1, m_2, m_3)\). In the flavor basis explicitly:
   \[
   m_{TBM} = \begin{pmatrix}
   a & b & b \\
   \ldots & \frac{1}{2}(a+b+c) & \frac{1}{2}(a+b-c) \\
   \ldots & \ldots & \frac{1}{2}(a+b+c)
   \end{pmatrix},
   \]
   \[
   a = \frac{1}{3}(2m_1 + m_2), \quad b = \frac{1}{3}(m_2 - m_1), \quad c = m_3.
   \]
   Immediately one observes the \(S_2\) symmetry of the \(\nu_\mu \leftrightarrow \nu_\tau\) permutations. This symmetry plays a crucial role: it is this symmetry of the dominant structure of the mass matrix that ensures maximal 2-3 mixing and zero (small) 1-3 mixing which are robust features of the lepton mixing. It fixes two out of three angles, and probably, should be a starting point of the symmetry building.

   The TBM-symmetry can be expressed in the form of the TBM-relations:
   \[
   m_{\mu\mu} = m_{\tau\tau}, \quad m_{\mu\tau} = m_{\tau\mu}, \quad m_{ee} + m_{e\mu} = m_{\mu\mu} + m_{\mu\tau},
   \]
   or \(\sum_{\alpha} m_{e\alpha} = \sum_{\beta} m_{\mu\beta}\) instead of the last equality. Notice that in general, fixing three mixing angles leads to three relations between the matrix elements. What is non-trivial is that the relations (5) are simple and of particular type which corresponds to certain symmetry. (In general relations are complicated and no symmetry can be found).

2. The TBM mass matrix (3) in the flavor basis is invariant under transformations
   \[
   V_i m_{TBM} V_i^T = m_{TBM},
   \]
   where
   \[
   V_1 = S = \frac{1}{3} \begin{pmatrix}
   -1 & 2 & 2 \\
   \ldots & -1 & 2 \\
   \ldots & \ldots & -1
   \end{pmatrix}, \quad V_2 = U = \begin{pmatrix}
   1 & 0 & 0 \\
   \ldots & 0 & 1 \\
   \ldots & \ldots & 0
   \end{pmatrix}.
   \]
Two transformations (7) uniquely determine the form of the TBM mass matrix. They do not depend on the mass spectrum, and in fact, can be obtained immediately from the \( U_{TBM} \). At the same time, diagonality of the mass matrix squared of the charged leptons, \( m_{e}^{2} \), can be supported by symmetry

\[
V_{3}^{T}(m_{e}^{2}m_{e})V_{3} = m_{e}^{2}, \quad V_{3}^{T} = \text{diag}(1, e^{i\alpha}, e^{i\beta}),
\]

where \( \alpha \neq \beta \neq \pi k \).

The main idea is that \( V_{i} \) are the generating elements of certain discrete symmetry group \( G_{f} \) which eventually determines mixing. For instance, selecting \( V_{3}^{T} = T \equiv \text{diag}(1, \omega, \omega^{2}) \), where \( \omega = e^{i\pi/3} \), one finds that \( S, T, U \) are the generating elements of the group \( S_{4} \). Furthermore, it was argued that \( S_{4} \) group is minimal structure which leads to TBM [12] (see discussion in [13]). (Our consideration was in the flavor basis, however neither mixing matrix nor symmetry depend on the basis once the change of basis is described by transformation \( V \) such that \( YY^{T} = I \).)

3. The flavor symmetry \( G_{f} \) (which contains \( V_{i} \)) should be broken. (In fact, no exact flavor symmetry can be introduced in whole theory, see [14]). Neutrino mass matrix is not invariant under \( T \), whereas the charge lepton mass matrix is not invariant with respect to \( S \) and \( U \). The idea is that symmetry \( G_{f} \) is broken \emph{differently and partially} in the sectors which generate the neutrino masses and charged lepton masses. Namely,

\[
G_{f} \rightarrow \text{breaking} \rightarrow \begin{cases} G_{\nu} & (S, U) \text{ neutrinos} \\ G_{l} & (T) \text{ charged leptons} \end{cases}.
\]

The \emph{residual} symmetries \( G_{\nu} \) and \( G_{l} \) in the neutrino and charged lepton sectors are different. Clearly \( G_{f} \) is broken completely in whole theory. Furthermore, the two sectors communicated in high orders, and therefore \( G_{\nu} \) and \( G_{l} \) are broken even in their own sectors being approximate. As a result, the TBM symmetry is broken and the TBM mixing gets corrections.

Thus, the mixing appears as a result of different ways of the flavor symmetry breaking in the neutrino and charge lepton sectors. In turn, the difference of symmetry breaking can originate (1) from different flavor assignments of the right handed (RH) components of \( N^{c} \) and \( l^{c} \), or/and (2) from Majorana nature of the neutrino mass terms (absence of \( N^{c} \)): the neutrino and the charged lepton mass terms (originate from \( LL \) and \( Ll^{c} \) correspondingly) can have different flavor properties.

If the symmetry \( G_{f} \) is broken spontaneously, one should introduce different sets of the Higgs (flavon) fields for neutrinos and charged leptons. Then the form of the elements of residual symmetries, \( S, U \) and \( T \), determines configurations of VEV’s.

The items 1- 3 is a paradigm of the present day flavor model building.

4. It is possible to proceed even further in some particular way. The TBM mass matrix can be presented as the sum of three singular matrices. In the limit of small \( m_{1} \) it becomes as

\[
m_{TBM} \approx A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times (1, 1, 1) + B \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times (0, 1, -1).
\]

This form indicates the see-saw mechanism, for which the mass terms may have the form

\[
\sum_{i} \frac{1}{M_{i}} (L_{f_{i}})(L_{f_{i}})^{T},
\]

where \( M_{i} \) are the masses of RH neutrinos and \( f_{i} \) are the triplets of flavon fields, (see e.g. [15]). The VEV’s of the triplets should be \( \langle f_{2} \rangle (1, 1, 1) \) and \( \langle f_{3} \rangle (0, 1, -1) \), which can be obtained in a SUSY version of this model.

So far so good. Problems appear when we start to realize this program in consistent gauge models.
3.2. Problems

A generic problem originates from the fact that masses of fermions are given by

\[ m = F(\{Y\}, \{v\}), \]  

(12)

where \( F \) is certain functional which depends on the mechanism of neutrino mass generation. In general, it describes several different contributions and includes various corrections. \( \{Y\} \) are the Yukawa couplings or coupling constants of different operators generating masses, and \( \{v\} \) refer to a set of vev’s of Higgs bosons. \( \{Y\} \) and \( \{v\} \) follow from independent sectors: from the Yukawa sector and the scalar potential. Yukawa couplings are determined by symmetry (at least partially), whereas VEV’s are fixed by pattern of symmetry violation. In general, symmetry does not control how it is broken and new ingredients (dynamics, symmetries) should be introduced to fix the VEV alignment. To obtain TBM all these components should be correlated. One step constructions do not work. Essentially this means that TBM is not immediate consequence of symmetry but a result of interplay of different factors, and in this sense – accidental.

Another generic problem is related to the fact that symmetry should be broken differently in the neutrino and charged lepton Yukawa sectors. That is, different Higgs flavon multiplets (typically - several for each sector) should be used. To forbid unwanted couplings of these flavons one is forced to introduce additional symmetries with \emph{ad hoc} charge prescription.

3.3. Flavons versus Flavored Higgses

There are two types of models with broken flavor symmetries:

1. Models with flavons \( f \), singlets of the gauge symmetry group which have non-zero “flavor charges”. In these models usual Higgs doublet(s) \( H \) are the flavor singlets, so that the electroweak symmetry and flavor symmetry breakings are separated. The Yukawa couplings and mass terms are generated by the non-renormalizable interactions:

\[ \frac{1}{\Lambda_f} L e^c H f^m. \]  

(13)

Here \( \Lambda_f \) is the scale of flavor physics which can be above the GUT scale. Clearly it is difficult to test such a scenario directly, unless \( \Lambda_f \) is not far above the electroweak scale.

Typical problem of this scenario is existence of high dimension operators of the type (13) which contribute to masses. Convergence of series is weak, \( \langle f \rangle/\Lambda_f \sim 0.2 - 0.5 \), especially if quarks (top quark) are included in consideration. Further complications (e.g. additional symmetries) are needed to control their effect.

2. Models with flavored Higgses: The Higgs doublets carry flavor charges, and usually large number of such doublets (which form flavor multiplets) should be introduced. The flavor symmetry is broken simultaneously with gauge symmetry at the EW scale. These models are testable, and in fact, strongly restricted by the FCNC, anomalous magnetic moment of muon, etc.. One expects to see many scalar bosons at LHC [16].

3.4. Symmetry groups

The Table 1 presents the list of small groups with irreducible representation \( 3 \), which are used in the TBM model building. Representation \( 3 \) explains existence of three generations of fermions, however all these groups have also singlets (and some - doublets). There is no explanation why (non-trivial) singlet and doublet representations are missing. An alternative is groups, like \( S_3 \), with non-trivial irreducible representation \( 2 \), so that the family structure appears as \( 2 + 1 \) [17].

In the Table 1 we give the order of group (number of elements), irreducible representations and products of the representations which contain invariants. The latter determines the flavor structure of a model.
Table 1. The simplest groups with irreducible representations $3$.

| group $A_4$ | order $12$ | representations $1, 1', 1'', 3$ | invariants $3 \times 3, 1' \times 1''$
|------------|-----------|---------------------------------|----------------|
| $T'$       | $24$      | $1, 1', 1'', 2, 2', 2'', 3$     | $3 \times 3, 1' \times 1'', 2\times$
| $S_4$      | $24$      | $1, 1', 2, 3, 3'$              | $3 \times 3, 3' \times 3', 2 \times 2, 1' \times 1'$
| $T_7$      | $21$      | $1, 1', 1'', 3, 3^*$           | $3 \times 3', 1' \times 1''$
| $\Delta(27)$ | $27$     | $1_1 - 1_9, 3, 3'$            | $3 \times 3', 1_2 \times 1_3, 1_4 \times 1_7, 1_5 \times 1_8, 1_6 \times 1_9$

In what follows we present structures of different models based on various discrete symmetries. The corresponding figures show explicitly ad hoc character of selection of the flavon multiplets and prescription of the flavor charges. An open issue is “missing” representations: not all possible low dimensional representations are used. This may create problem if discrete symmetry originates from breaking of some gauged continuous group [18]. For each model we indicate origins of mixing and shortcomings with criteria based on existence of auxiliary symmetries, presence of new fields, possibility of extension to the quarks sector and further embedding, etc..

3.5. $A_4$ symmetry and a simplest $A_4$ models

The most popular group is $A_4$: the symmetry group of even permutations of 4 elements, or symmetry of the tetrahedron [19]. It has order 12 and two generating elements $S$ and $T$ which are needed to realize the TBM mixing. The presentation of the group:

$$S^2 = 1, \quad T^3 = 1, \quad (ST)^3 = 1.$$  \hspace{1cm} (14)

The most important element, $U = A_{\mu\tau}$, is absent. So, one needs either to introduce this permutation symmetry in addition, thus extending the symmetry group, or obtain it as an accidental symmetry: as a result of particular selection of representation and the VEV alignment.

The flavor structure is determined by properties of products of representations and invariants:

$$3 \times 3 = 3 + 3 + 1 + 1' + 1'', \quad 1' \times 1'' = 1$$  \hspace{1cm} (15)

(and of course, by the VEV alignment).

The structure of a simplest $A_4$ model for TBM mixing [20] with charge assignment is presented in fig. 2. The key features of the model include the following. There are three RH neutrinos, $N^c$. Four flavon multiplets participate in generation of masses. The lepton mixing appears due to different flavor assignments of the RH neutrinos and the charged leptons: $N^c$ form triplet $N^c \sim 3$, whereas $l^c$ are all singlets, $1$, of $A_4$. The charged leptons get Dirac masses via non-renormalizable terms, whereas neutrinos - via renormalizable ones. The $U-$symmetry is accidental: due to particular selection of the flavon representations and configuration of VEV’s. The auxiliary symmetry $Z_4$ with ad hoc prescribed charges is introduced; in particular, all $N^c$ have the same whereas $l^c$ have all different $Z_4$ prescriptions. The vacuum alignment is achieved by using SUSY and additional driving fields. The lepton mixing follows to a large extend from structure of mass matrix of the RH neutrinos. The model does not admit simple SO(10) embedding as well as the SU(5) embedding (if quarks also form family symmetry): $l^c$ are singlets of the flavor symmetry group and the rest of the fermions form triplets. To resolve this problem one can introduce additional GUT matter multiplets located in the same multiplets known fermions together with new matter fields. $A_4$ singlet representations of $L$ are missing; the
3.6. Mixing and masses
In majority of the models mixing and masses are unrelated or have indirect relations. The latter may appear as a result of certain choice of the Higgs representation within a given symmetry context. Mixing follows from certain form invariance of the mass matrix. Usually, additional $U(1)$ (Froggatt-Nielsen type) or/and discrete symmetries are used for explanation of the mass hierarchies. Mixing is a consequence of the relations between mass matrix elements (like in eq. (5)), whereas masses depend on the absolute values of the elements.

For particular mass spectrum (set of values of masses), the mass matrix elements are fixed and in some cases this may lead to more symmetric form of the mass matrix - to additional symmetry. Then a covering symmetry group should fix both mixing and masses. (E.g., one can consider the TBM mass matrix with equality of elements $a = b$, which gives the spectrum with normal mass hierarchy and $m_1 \approx 0$, etc.)

4. From leptons to quarks
Do quarks need leptonic discrete symmetries? It is not accidental that in the talks devoted to flavor physics in the quark sector the leptonic symmetries proposed are not even mentioned. Although originally the first discrete symmetries have been applied for flavor in the quark sector [21]. Presently there is no clear attempts to go “from quarks to leptons” (approach which once has failed). The $D_{14}$ symmetry has been proposed for explanation of the Cabibbo angle value without extension to leptons [22]. It is clear that the quark and lepton mixings are strongly different, and probably this difference is directly related to smallness of neutrino mass.

4.1. Extending symmetry to the quark sector
There are two different ways to extend the leptonic symmetries to the quark sector.

1) The first possibility is to use the same representations $3$ and $1$ for quarks as for leptons. In the lowest order one can obtain

$$V_{CKM}^0 = I, \quad U_{PMNS}^0 = U_{TBM}.$$  (16)
This difference of mixings can be attributed to the Majorana nature of neutrinos. As a consequence of symmetry, the Dirac mass matrices in the quark and lepton sectors are the same, both leading to zero mixing. (The Dirac matrices can be responsible for the mass hierarchies of the charged fermions.) The TBM follows via seesaw from certain structure of the Majorana mass matrix of the RH neutrinos. Then corrections from high order operators generate the CKM mixing and the deviations of lepton mixing from the TBM form. Generic problem is that corrections which would explain the Cabibbo angle lead to too large deviations from TBM, so that additional tuning is required.

2) Another way is to use different representations of the flavor symmetry group for quarks and leptons. In particular, one can choose groups which contain not only representations $3$ and $1$ but also $2$, and assign three generations of quarks to the representations $2$ and $1$ instead of $3$ in lepton sector. This implies that family symmetry does not exist and leaves another question: why quarks and leptons have different symmetry properties, that is, fundamentally different.

As an example of realization of the second approach, consider model based on the symmetry $T'$. The $T'$ group has order 24 being the double covering of $A_4$ or binary tetrahedron group. The generating elements of this group are $S$, $T$ and $R$ and presentation of the group:

$$T^3 = I, \quad S^2 = R, \quad R^2 = I, \quad (ST)^3 = 1.$$  \hfill(17)

Here $R = 1$ for the odd-dimensional representations and $R = -1$ for the even-dimensional representations. Again the element $U$ is missing. Irreducible representations of the group include $1$, $1'$, $1''$, $2$, $2'$, $2''$, $3$. The products of representations and invariants,

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3, \quad 1' \times 1'' = 1,$$

(18) coincide with those in the $A_4$ case (see Table 1). New possibilities are related to existence of the doublet representations

$$2^a \times 3 = 2 + 2' + 2'',$$

(19) where $2^a = 2$, $2'$, $2''$ and

$$2 \times 2 = 1 + 3, \quad 1 \times 2 = 2$$

(20) with “conservation” of primes. The singlet which appears in (20) allows one to produce new (in comparison with $A_4$) flavor structures. The mass generation scheme of the model [23] based on $T'$ is shown in fig. 3.

**Figure 3.** Scheme of the mass generation in the $T'$ model [23]. See the caption of fig. 2 for explanation.
Features of the model include the following. The model has an auxiliary group $Z_3$. There is no RH neutrinos, and neutrino masses are generated by $D=6$ operators $LLH_uH_uf$, where $f = \phi_3, \xi$. The origin of mixing is the Majorana nature of neutrinos. In quark sector the two light generations form doublet, whereas the third one is a singlet of $T'$. The RH components of charged leptons are different singlets of $T'$ but they have the same prescription of $Z_3$. Four different flavon fields are introduced: two triplets and one doublet $2''$ (no $2, 2'$). Only doublets $2''$ are used for quarks. $Z_3$ prescription looks random: doublets transform with $\omega$, all RH components have $\omega^2$ transformation, and there are various missing representations.

### 4.2. TBM and GUT's

Generic problem of many models is that the flavor prescriptions required for explanation of difference of mass and mixing of quarks and leptons prevents from their embedding into Grand Unified Theories. The problem can be resolved by increasing number of matter fields and locating the known fermions with new ones in the same multiplets.

One can start immediately from the GUT structure and known matter fields:

$$GUT \times G_{\text{flavor}} + \text{new elements},$$

(21)

where “new elements” may include singlet fermions and additional Higgses or/and pairs of vector-like matter fields, their mixing with usual matter, etc..

As an example, structure of the model based on $SU(5) \times A_4$ [24] is shown in fig. 4. The upper quarks get masses via interactions $T_iT_jH_5\{f_{ij}\}$, where $\{f_{ij}\}$ is product of certain number (from zero to 5) of flavon fields: e.g., $f_{33} = 1$, $f_{32} = \phi_{123} \phi_3 \xi$.

![Figure 4. Scheme of generation of masses of neutrinos, charged leptons and down quarks in the $SU(5) \times A_4$ model [24]. See the caption of fig. 2 for explanation.](image_url)

The following remarks are in order. Extended auxiliary symmetry $Z_2 \times Z_2 \times U(1) \times U(1)$ is imposed. The singlet $N$ and $\Sigma = 24$ - adjoint representation of fermions are introduced apart from usual 10-plets and 5-plets; neutrino masses are generated by a combination of type I and type III seesaw. Only 5-plets form family structure, whereas 10-plets and RH neutrinos are singlets of $A_4$. Matter fields are in $3$, $1$, but $1'$, $1''$ are missing. Four different flavon multiplets with “random” $Z_2 \times Z_2$ prescriptions generate masses.

Actually, $SU(5)$ has enough flexibility: three different representations $10, 5, 1$ allow one to write independent terms for the upper quarks, for down quarks and charged leptons and for neutrinos.
Models based on $SO(10)$ with all known fermions being in the same $16$-plets are more constrained. New elements should be added to the $SO(10) \times G_{\text{flavor}}$ structure. Two different ways are proposed: 1) add singlet fermions and $16_H$ Higgs multiplets with couplings $16_S 16_H$ and flavons. In [25] $G_{\text{flavor}} = T_7$ and screening of the Dirac structures in the see-saw mechanism can be achieved which leads to independent structures of the mass matrices of neutrinos and charged fermions. Incomplete (partial) screening can be the origin of the TBM or bi-maximal mixings.

2) Another way of model building is to introduce $126$ and $\overline{126}$ plets, and thus realize the seesaw type-II mechanism which opens up a possibility to obtain (to a large extent) independent flavor mixing in the quark and lepton sectors. Realistic model proposed in [26] and based on the symmetry $S_4 \times Z_n$ requires also introduction of vector-like pairs $16, \overline{16}$ of matter fields, additional Higgs $10 - 10$plet, and flavons.

4.3. Is TBM accidental?

Experiment still allows relatively large deviation of the mixing parameters from the TBM values: $\Delta \sin^2 \theta_{23} \sim 0.05$, $\Delta \sin^2 \theta_{12} \sim 0.02$, $\Delta \sin \theta_{13} \sim 0.15$. The deviations can lead to strong (maximal) violation of the TBM-conditions (5), and consequently, to significant deviation of $m_{\nu}$ from the TBM form. For instance, instead of the first equality in (5), the equality with changed sign, $m_{\nu_{e}} \sim -m_{\nu_{\tau}}$, is allowed without any modification of two others. Leading structures of the mass matrix are relatively robust, whereas the sub-leading structures can change under these corrections completely. It is therefore not excluded that the approximate TBM is accidental being just an interplay of several independent factors (contributions) [27]. Alternatively it can be a manifestation of some other symmetry which differs from TBM, or other structures. This opens up new approaches to explain the data.

There are other possible applications of discrete symmetries. In the universal approach to the quark and lepton masses based on certain ansatz for the shape of the mass matrices discrete symmetries are used to get texture zeros. In this context the corrected Fritzsch ansatz has been explored in [28].

5. QLC and quark-lepton symmetries

The Quark-lepton complementarity (QLC) [29] is an alternative to description of the fermion mixings which is based on observation that

$$\theta_{12}^l + \kappa_{12} \theta_{12}^q \approx \pi/4, \quad \theta_{23}^l + \kappa_{23} \theta_{23}^q \approx \pi/4,$$

where $\kappa_{23}, \kappa_{12} \sim 1$, say (0.7 - 1). Qualitatively, the QLC relations mean that

- the 2-3 leptonic mixing is close to maximal because the 2-3 quark mixing is small;
- the 1-2 leptonic mixing deviates from maximal one substantially because the 1-2 quark mixing is relatively large.

In other words,

"Lepton mixing = bi − maximal mixing − quark mixing"

with possible implications being:

1. The quark-lepton symmetry, which, in turn, implies the quark-lepton unification, or GUT, or common family (horizontal) symmetry.
2. Existence of structure which produces the bi-maximal (BM) mixing.

The structure for BM could be related to see-saw with special properties of the RH neutrino mass matrix.
The Bi-maximal mixing, \( U_{\text{BM}} = U_{\text{PMM}}U_{\text{1/2}} \), is characterized by maximal 1-2 and 2-3 rotations, and zero 1-3 rotation. There is no CP-violation. Possible scenario is that in the lowest order
\[
V_{\text{CKM}}^0 = I, \quad U_{\text{PMNS}}^0 = U_{\text{BM}},
\]
and may be \( m_1 = m_2 = 0 \). If the BM structure in the lepton sector is generated by the seesaw mechanism, the corrections from the Dirac mass matrix produce (i) mass splitting, (ii) CKM and (iii) deviation from the bi-maximal mixing. The situation when the deviations and CKM mixing are related by the quark - lepton symmetry (or the same flavor symmetry for quarks and leptons) can be called “strong complementarity”.

Another possibility is Cabibbo “haze” [30] [29] or the weak complementarity [31]. Deviations from BM are due to some corrections which can be of the same order in the quark and lepton sectors but not necessarily related. The corrections can be of the size of the Cabibbo angle. Possible realization is that the corrections are due to high order flavon interactions which generate simultaneously the CKM mixing and deviation from BM. In this case Grand Unification and family symmetries are not necessary.

5.1. BM-symmetry

A discrete symmetry can be behind the BM mixing as the lowest order structure (24). The bi-maximal mass relations are
\[
m_{e\mu} = m_{e\tau}, \quad m_{\mu\mu} = m_{\tau\tau}, \quad m_{ee} = m_{\mu\mu} + m_{\mu\tau}.
\]
The last equality distinguishes the bi-maximal case from TBM (see (5)). The BM mass matrix, \( m_{\text{BM}} \), is invariant under transformations
\[
V_i^T m_{\text{BM}} V_i = m_{\text{BM}}, \quad V_i = S_{\text{BM}}, \quad U,
\]
where
\[
S_{\text{BM}} = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \vdots & -1 & 1 \\ \vdots & \vdots & -1 \end{pmatrix}
\]
with property \( S_{\text{BM}}^2 = I \). One can select the matrix of transformation which keeps the charged leptons mass matrix to be diagonal, in the form \( T_{\text{BM}} = \text{diag}(-1, -i, i) \). In this case \( T_{\text{BM}}^2 = I \), so that \( T \) and \( S_{\text{BM}} \) turn out to be the generating elements of \( S_4 \) symmetry group [32].

5.2. \( S_4 \) symmetry and model

\( S_4 \) has the order 24, it is the permutation group of 4 elements. With generating elements \( S_{\text{BM}} \) and \( T_{\text{BM}} \) it has the following presentation
\[
S_{\text{BM}}^2 = T_{\text{BM}}^4 = (T_{\text{BM}} S_{\text{BM}})^3 = I
\]
(compare with (14)). It has irreducible representations \( 1, 1', 2, 3, 3' \). The products of representations read
\[
3 \times 3 = 3' \times 3' = 1 + 2 + 3 + 3', \quad 3 \times 3' = 1' + 2 + 3 + 3'
\]
\[
1' \times 1' = 1, \quad 1' \times 2 = 2, \quad 2 \times 3 = 2 \times 3' = 3 + 3', \quad 2 \times 2 = 1 + 1' + 2,
\]
and the latter contains singlet, thus leading to new flavor structures.

Structure of the model [33] based on \( S_4 \times Z_4 \) and \( U(1)_{FN} \) is shown in fig.5. It resembles the structure of \( A_4 \) model (see Fig. 2). The following are in order. Only \( S_4 \) representations \( 3, 1 \), and \( 1' \) are used for the matter fields, the representations \( 3', 2 \) are missing. Flavons are in \( 3, 3', 1 \) representations, and \( 2, 1' \) are absent. The multiplets have ad hoc \( Z_4 \) prescription. The Froggatt-Nielsen (FN) mechanism is introduced for the mass hierarchies and only RH components of leptons have non-zero FN-charges. Deviation from BM are due to high dimension operators with flavon fields.
6. Perspectives and tests

Minimal and simplest models which lead to the lepton mixing of the TBM or BM type from discrete symmetries, have been systematically explored. The key problem is to check the models or at least some generic features of the whole context. Unfortunately, the majority of the models do not give specific and precise predictions which can be tested. Still one expects certain connections between the low energy observables and also probably observables at high energy accelerators under certain additional conditions.

In this connection several phenomenological directions should be mentioned.

1. Precision measurements of the neutrino parameters. Determination of the 1-3 mixing and the deviation of 2-3 mixing from maximal one are of great importance. Some models predict \( \theta_{13} \); discrimination of models with large and very small \( \theta_{13} \) will be possible. Relations between \( \theta_{13} \) and the deviation of \( \theta_{23} \) from \( \pi/2 \) may reveal certain ways of realizations of the discrete symmetries.

Determination of the absolute scale of neutrino masses and mass hierarchies, establishing possible relations between mass ratios and mixings can give further insight.

2. Double beta decay. Some models lead to certain predictions for the effective Majorana mass of the electron neutrino, \( m_{\nu_e} \), as well as its connections to the effective electron neutrino mass, \( m_e \), and sum of neutrino masses \( \sum m_i \) [34].

3. Rare decays with lepton flavor violation: \( \mu \rightarrow e + \gamma \), \( \tau \rightarrow e + \gamma \), \( \tau \rightarrow e + \gamma \) [35] [36]. Equality of the rates of these decays may testify for certain class of models with discrete symmetries [36]. Interesting predictions for processes like \( \tau^{-} \rightarrow e^{+}\mu^{-}\mu^{-} \), \( \tau^{-} \rightarrow \mu^{+}e^{-}e^{-} \) are given which depend on parameter of violation of the discrete symmetry [35].

4. LHC and high energy accelerators. Models with flavored Higgses can be directly tested in the collider experiments [37]. Even for the SM Higgs one expects modifications of the decay and production rates in the presence of horizontal symmetries ! [38].

5. Leptogenesis. It is affected by discrete symmetries and depends on the way the symmetries are broken [39]. In some cases suppression of the leptonic asymmetry is expected.

6. Dark matter. Particles of the dark matter (e.g. in the multi-Higgs models) can be stabilized by some discrete symmetry which is related to the flavor symmetry [40]. For instance, it may be a residual symmetry after breaking \( A_4 \rightarrow Z_2 \).

7. As already mentioned, some future discoveries can simply reject the described approach or require its strong modification. That includes discoveries of new fermions, like sterile neutrinos,
the 4th generation of fermions, the right handed neutrinos or \( W_R \) of the left-right symmetric models, etc..

7. Conclusions
In recent years, it has been shown that the approximate TBM mixing can be consistently obtained in the context of gauge models with spontaneously broken flavor symmetries and rather extended additional structures. The considered examples of models show the price one should pay for realization of idea “mixing from discrete symmetries”. There are two opposite points of view on the obtained results:

I. The features of experimental data which testify for a symmetry behind lepton mixing are actually accidental. The deviations from TBM can be significant. Realizations are too complicated with the number of assumptions being several times bigger than the number of mixing angles. This indicates that alternative approaches to explanation of the data should be pursued.

II. Some version of broken discrete symmetries give correct explanation of the data. Physics behind neutrino mass and mixing has rich extended structure and it leads to rich phenomenology. (It may happen that still some important elements of the approach are missing.

Preferable scenario? The difference of lepton and quark mixings is related directly to smallness of neutrino mass and probably its Majorana character. GUT’s still look very appealing and there is no point to sacrifice them in favor of the present models with flavor symmetries. The observed symmetry in the lepton mixing is related to a symmetry of Hidden sector at some high mass scales. It communicates with us via the neutrino portal – mixing with neutrinos. No analogy of this in the quark sector exists. Another physics (but the same in \( q^- \) and \( l^- \) sectors) is involved in generation of CKM and deviation of PMNS from the symmetric case. Unfortunately, it is difficult, if possible, to check this possibility, but this is not the problem of Nature...

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