Identification of Nonlinear Anelastic Models

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Abstract. A useful nonlinear identification technique applied to the anelastic and rheologic models is presented in this paper. First introduced by Feldman, the method is based on the Hilbert transform, and is currently used for identification of the nonlinear vibrations.

1. Introduction

The knowledge on the dynamic properties of materials and structures is very important both in the theoretical analysis and in the experimental investigation, but also in the area of mechanical systems applications. At its highest level, this knowledge is represented by the mathematical model of a dynamical system.

In a large acceptance, the system identification refers to the establishment of the most appropriate system of differential equations describing the behavior of a dynamic system. These equations result from a performance criterion derived from the difference between the actual response of the dynamical system and the theoretical solution for a set of different physical variables or observables. During the last decade, an increasing interest in developing of methods for non-linear systems identification is observed [1-7].

The knowledge of the time dependent anelastic properties of materials is very important due its applications to vibrating systems. The differential equation expressing the relation between stress and strain is known as the constitutive relation; for finding the parameters of the constitutive relations, identification methods have been developed [8].

Due to their specific properties, non-linear anelastic materials are of particular interest in mechanical systems applications. The aim of the present paper is to discuss some non-linear identification methods that may be used in the study of the constitutive properties of materials.

2. Non-linear identification using Hilbert transform

The Hilbert transform of a signal \( x(t) \), where \( t \in (-\infty, \infty) \) is the time, is defined [2-3]:

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\[ \tilde{x} = H(x(t)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{\tau - t} d\tau \quad (1) \]

and provides a representation of the signal in the time domain.

The inverse Hilbert transform is defined as:

\[ x = H^{-1}(\tilde{x}(t)) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{x}(\tau)}{\tau - t} d\tau \quad (2) \]

and gives the initial signal. The Hilbert transform and the inverse Hilbert transform are linear transforms.

It can be shown that the Fourier transform of the signal \( x(t) \) Hilbert transform is [2,3]:

\[ \tilde{X}(\omega) = F(\tilde{x}(t)) = -i \text{sign} \omega X(\omega) , \]

which leads to the idea that the Hilbert transform of \( x(t) \) can be expressed:

\[ \tilde{x}(t) = -i F^{-1}(\text{sign} \omega X(\omega)) , \]

where \( F^{-1} \) is the inverse Fourier transform, thus resulting a practical method for Hilbert transform calculation.

Consider an analytic signal \( z(t) \) defined as:

\[ z(t) = x(t) + i \tilde{x}(t) . \quad (3) \]

Being a complex quantity, the analytic signal can be expressed in the form:

\[ z(t) = A(t) \exp(i\phi(t)) , \quad (4) \]

where

\[ A(t) = \sqrt{x^2(t) + \tilde{x}^2(t)} \quad (5) \]

is the envelope (the instant amplitude) and

\[ \phi(t) = \arctan \left( \frac{\tilde{x}(t)}{x(t)} \right) \quad (5') \]

is the instant phase In these terms, the instant angular frequency \( \omega(t) \) may be expressed as:

\[ \omega(t) = \phi(t) = \frac{x(t)\tilde{x}(t) - \tilde{x}(t)\tilde{x}(t)}{A^2(t)} \quad (6) \]

and the time derivative of the envelope as:
If two signals with non-overlapping spectra, one slow, \( x_s(t) \), of maximal frequency \( \nu_s \), and one fast, \( x_f(t) \), of minimal frequency \( \nu_f > \nu_s \) are considered, a useful property results for the Hilbert transform of their product:

\[
H\{x_s(t)x_f(t)\} = x_s(t)\tilde{x}_f(t)
\]

(8)

3. Identification of non-linear systems

We will further discuss an identification method introduced by Feldman [2,3]. Consider first the free vibration of a one-dimensional system described by the differential equation:

\[
\ddot{x} + 2h_0(A)\dot{x} + \omega_0^2(A)x = 0
\]

(9)

where \( h_0(A) = c(A)/2m \) is the symmetrical damping factor, \( c(A) \) is the damping coefficient, \( m \) is the mass of the system, \( \omega_0^2(A) = k(A)/m \) is the natural angular frequency, \( k(A) \) is the symmetric elastic constant; since \( c \) and \( k \) depend on the instant amplitude of the vibrations, equation (9) describes a non-linear system. It also follows that \( h(A) \) and \( \omega_0(A) \) have a slow time variation compared to \( x(t) \).

Using the Hilbert transform of eq. (9), the equation:

\[
\ddot{z} + 2h_0(A)\dot{z} + \omega_0^2(A)z = 0
\]

(10)

results for the analytic signal \( z(t) = x(t) + i\tilde{x}(t) \), due to the property (4). Introducing [2,3]:

\[
z(t) = z(t)\left(\frac{A}{A} + i\omega\right) \quad \dot{z}(t) = z(t)\left[\frac{\dot{A}(t)}{A(t)} - \omega^2(t) + 2i\frac{\dot{A}(t)\omega(t)}{A(t)} + i\ddot{\omega}(t)\right]
\]

(11)

in eq. (9), one obtains:

\[
z \left[\frac{\dot{A}}{A} + \omega^2 + \frac{\dot{\omega}}{\omega} + 2i\frac{\dot{A}}{A} + i\ddot{\omega} + 2h_0\frac{\dot{A}}{A} + i\frac{\dot{\omega}}{\omega}\right] = 0
\]

(12)

Extraction of the real and imaginary parts form (12), leads to:

\[
\omega_0^2(t) = \omega^2 - \frac{\dot{A}}{A} + 2\frac{\dot{A}}{A} + \frac{\dot{A}\dot{\omega}}{A\omega} \quad h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{\omega}
\]

(13)

In the case of one-dimensional non-linear forced vibrations with viscous damping, the motion equation is

\[
\ddot{x} + 2h_0(A)\dot{x} + \omega_0^2(A)x = \frac{F(t)}{m}
\]

(14)

where \( F(t) \) is the driving force. In terms of the analytic signal \( z(t) \), equation (14) can be written as [2,3]:
\[ \ddot{z} + 2h_0(A) \dot{z} + \omega_0^2(A) z = \frac{F(t) + i \dot{F}(t)}{m} \]  \hspace{1cm} (15)

and using eqs (11) the calculus yields:

\[ z \left[ \frac{\dot{A}}{A} - \omega^2 + \omega_0^2 + 2h_0 \dot{A} + i(2 \frac{\dot{A}}{A} \omega + \dot{\omega} + 2h_v \omega) \right] = \frac{F(t) + i \dot{F}(t)}{m}. \hspace{1cm} (17) \]

After separation of the real and imaginary parts of (17) one finds [23]:

\[ \omega_0^2(t) = \omega^2 + \frac{\alpha}{m} + \frac{\beta}{A \omega m} \frac{\dot{A}}{A} - \frac{2}{A} \frac{\dot{A}^2}{A^2} + \frac{A \dot{\omega}}{A \omega}, \hspace{1cm} h_v(t) = \frac{\beta}{2 \omega m} - \frac{\dot{A}}{A} - \frac{\dot{\omega}}{A \omega}, \hspace{1cm} (18) \]

where \( \alpha = \text{Re}\{F(t) + i \dot{F}(t) / z(t)\} \), \( \beta = \text{Im}\{F(t) + i \dot{F}(t) / X(t)\} \), \( \dot{F}(t) \) being the Hilbert transform of the excitation force. In addition, it follows [2,3]:

\[ \frac{F(t) + i \dot{F}(t)}{X(t)} = \alpha(t) + i \beta(t) = \frac{x(t)F(t) + \dot{x}(t)F(t)}{x^2(t)} + i \frac{\dot{x}(t)F(t) - x(t)\dot{F}(t)}{x^2(t)\dot{x}(t)}. \hspace{1cm} (19) \]

In the equations (15)-(19) the mass was supposed to be known. However, it can be expressed from eq (18) as:

\[ m = \left[ (\Delta(\alpha - \frac{\beta \dot{A}}{A \omega} \Delta) - \omega^2 + \frac{\dot{A}}{A} - \frac{2 \dot{A}^2}{A^2} + \frac{A \dot{\omega}}{A \omega} \right]^{-1}, \hspace{1cm} (20) \]

where \( \Delta(\ldots)_m \) means variation of during a time interval \( \Delta t \). Since the third order derivation of the experimental data generates errors, variation instead of derivation was preferred.

For a given vibrating system, the displacement \( x(t) \), and the driving force \( F(t) \) (in the case of forced vibrations) can be experimentally measured and computer processed via analog-to-digital conversion and data acquisition.

### 4. Identification of non-linear anelastic models

Non-linear anelastic systems may be identified experimentally in different ways [8-10]. Thus, one can use the non-linear free response model (9), or the forced vibrations model (15). If we consider the case of a standard anelastic model (figure 1), consisting in two elastic elements of longitudinal elastic moduli \( E_r \) and \( \delta E \), and in an ideal damper of viscosity \( \eta = \delta E / \tau \), where \( \tau \) is the relaxation constant, we have \( E_u = E_r + \delta E \).

When the system is subjected to a stress \( \sigma(t) \), the relation between the stress and the strain response \( \varepsilon(t) \) will be the constitutive equation:

\[ \frac{d}{dt}(\sigma - E_r \varepsilon) + \frac{1}{\tau}(\sigma - E_r \varepsilon) = \delta E \ddot{\varepsilon}, \hspace{1cm} (21) \]

or, more general system which contain also fractional derivatives can be written [14]:
\[
\frac{d}{dt} (\sigma - E_\varepsilon, \varepsilon) + \alpha \frac{d^\nu}{dt^\nu} (\sigma - E_\varepsilon, \varepsilon) + \frac{1}{\tau} (\sigma - E_\varepsilon, \varepsilon) = \delta E_\varepsilon, 
\]

where \( \nu \) is a fractional derivative order \( 0 < \nu < 1 \) and \( \alpha \) is a small parameter.

In the case of polycrystalline solids, the stress dependence of the relaxation involves non-linearity, and a rigorous model should take into consideration three different relaxation constants.

We can build a non-linear vibrating system (figure 2) described by eq (21) from a mass \( m \) and an anelastic bar of length \( l \) and cross section \( S \); the mass is assumed to move without friction along the \( x \) direction. At any given moment, the stress \( \sigma \) is assumed constant in all the points of the bar, i.e. wave phenomena are assumed absent; in addition, the relation between the displacement of the mass \( m \) and the bar length is \( x = l \varepsilon \). If we denote by \( \Phi(t) \) the total force acting on the mass \( m \), consisting in the superposition of the force produced by the elastic deformation of the bar and an external perturbation force \( F(t) \), acting along the \( x \) direction: \( \Phi(t) = -S \sigma(t) + F(t) \), we can write Newton's law as:

\[
ma = m \ddot{x} = -S \sigma(t) + F(t),
\]

where \( a = l \varepsilon \). Accordingly, if we replace \( \sigma \) from eq. (21) the Newton's law takes the form:

\[
\frac{d^3x}{dt^3} + \frac{1}{\tau} \frac{d^2x}{dt^2} + \frac{E_a S}{ml} \frac{dx}{dt} + \frac{1}{\tau_0} \frac{E_r S}{ml} x = \frac{1}{\tau} \frac{F}{m} + \frac{1}{m} \frac{F}{F}.
\] (22)

We can apply (22) to the non-linear systems considering that the relaxation constant is dependent on the amplitude \( A \), as in (9) and (15), i.e. \( 1/\tau = 1/\tau (A) \). In terms of the analytic signal \( z(t) \), equation (22) may be written as:

\[
\frac{d^3z}{dt^3} + \frac{1}{\tau} \frac{d^2z}{dt^2} + \frac{E_a S}{ml} \frac{dz}{dt} + \frac{1}{\tau_0} \frac{E_r S}{ml} z = \frac{1}{\tau} \frac{F}{m} + \frac{1}{m} \frac{F}{F}.
\] (23)

We can express the third order derivative of \( z(t) \) as:

\[
\frac{d^3z}{dt^3} = z(t) \left[ \frac{1}{A} \frac{d^3A}{dt^3} - 3 \frac{\omega^2 A}{A} - 3\omega \omega + i \left( 3 \frac{\omega A}{A} + 3 \frac{\omega A}{A} - \omega - \omega^3 \right) \right].
\] (24)

Introducing (23) and (24) in (21) and separating the real and the imaginary parts, in the case of the free vibrations \( (F=0) \) we find:

\[
\frac{1}{A} \frac{d^3A}{dt^3} - 3 \frac{\omega^2 A}{A} - 3\omega \omega + i \left( \frac{A}{A} - \omega^2 \right) + \frac{E_a S}{ml} \frac{A}{A} + \frac{1}{\tau} \frac{E_r S}{ml} = 0
\] (25)
and

$$3 \frac{\omega A}{A} + 3 \frac{\omega A}{A} - \omega - \omega^3 + 2 \frac{\omega A}{A} + \frac{E_s A}{m l} \omega = 0.$$  \tag{26}$$

Then, we can identify $E_u$ and $\tau$ as:

$$E_u = - \left( 3 \frac{\omega A}{A} + 3 \frac{\omega A}{A} - \omega - \omega^3 + 2 \frac{\omega A}{A} + \omega \right) \frac{m l}{S} \omega.$$  \tag{27}$$

and

$$\frac{1}{\tau} = \left( \frac{1}{A} \frac{d^3 A(t)}{dt^3} - 3 \frac{\omega^2 A}{A} - 3 \omega A + \frac{E_s A}{m l} \frac{A}{A} + \frac{E_s A}{m l} \left( \frac{A}{A} - \omega^2 \right) \right)^{-1}, \tag{27'}$$

which contains the previously identified $E_u$. We note that parameters $E_u$ and $\tau$ are time dependent. It follows then that the dependence of $\tau$ on $\sigma$ may be derived if we represent $1/\tau$ as a function of $\sigma(t) = -m \ddot{x}/S$.

We see that eq. (27) contains the third order derivative of $A$. In order to avoid numerical errors which appear in the third order derivative calculations from experimental data processing, we can calculate the third order derivative by using finite differences as in (26), or by using the properties of the Fourier transform. If we denote by $A(\omega)$ the Fourier transform of $A(t)$, the third order derivative will be:

$$F\left( \frac{d^3 A(t)}{dt^3} \right) = (i \omega)^3 A(\omega), \quad \frac{d^3 A(t)}{dt^3} = F^{-1} \left[ (i \omega)^3 A(\omega) \right],$$

where $F^{-1}$ is the inverse Fourier transform of the quantity $(i \omega)^3 A$. Practically, we have the response $x(t)$ in a numerical (sampled) form $x(t_i)$ (where $t_i = 0, T, 2T, \ldots, N T$) and $\nu_e = 1/T$ is the sampling frequency of the data acquisition device. The direct and the inverse Fourier transforms may be calculated using the well-known FFT (fast Fourier transform) procedures. The sampling frequency must be taken in accord with the sampling theorem, i.e. $\nu_e \geq 2 \nu_{\text{max}}$, where $\nu_{\text{max}}$ is the maximal frequency from the spectrum of $x(t)$.

5. Conclusions

A useful nonlinear identification technique applied to the anelastic and rheologic models is presented in this paper. Such nonlinearity occurs in the relaxation phenomena in polycrystalline solids and also in magnetic relaxation. The method based on the Hilbert transform was first introduced by Feldman, for identification of the nonlinear vibrations.

In the last years in a lot of nonlinear identification problems the Volterra series method was used. This method can be useful in identification of the systems on which harmonic excitation acts [5-7].

It is also an extremely efficient and convergent method to obtain the solutions for these nonlinear models by a variational method [11,12]. It is important to note that this method can be extended to the rheologic systems with fractional order [18,19].

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