Radiating black hole solutions in Einstein-Gauss-Bonnet gravity

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In this paper, we find some new exact solutions to the Einstein-Gauss-Bonnet equations. First, we prove a theorem which allows us to find a large family of solutions to the Einstein-Gauss-Bonnet gravity in \(n\)-dimensions. This family of solutions represents dynamic black holes and contains, as particular cases, not only the recently found Vaidya-Einstein-Gauss-Bonnet black hole, but also other physical solutions that we think are new, such as, the Gauss-Bonnet versions of the Bonnor-Vaidya(de Sitter/anti-de Sitter) solution, a global monopole and the Husain black holes. We also present a more general version of this theorem in which less restrictive conditions on the energy-momentum tensor are imposed. As an application of this theorem, we present the exact solution describing a black hole radiating a charged null fluid in a Born-Infeld nonlinear electrodynamics.

I. INTRODUCTION

In recent years, there has been a renewed interest in theories of gravity in higher dimensions. The motivation arises from string theory. As a possibility, the Einstein-Gauss-Bonnet (EGB) gravity theory is selected by the low energy limit of the string theory \(\mathbb{R}^5 / \mathbb{Z}_2\). In this theory appears corrective terms to Einstein gravity which are quadratic in the curvature of the space-time. For 4-dimensional gravity, these terms result in a topological invariant so they will have no consequences in the field equations of the theory unless a surface term is involved. However, it was shown in \([4, 5]\), that such terms modify the conserved current of the theory in four dimensions. Moreover, the effect of those Gauss Bonnet terms is non-trivial for higher dimensions, so the theory of gravity, which includes Gauss-Bonnet terms, is called Einstein-Gauss-Bonnet (EGB) gravity. Studies in this area were made in \([6]\).

Perhaps black holes are the most striking prediction of any theory of gravity. This issue was also discussed in EGB gravity. Efforts were addressed to the understanding of properties of isolated black holes in equilibrium. In particular, for spherically symmetric space-time, solutions describing static charged black holes for both Maxwell and Born-Infeld electrodynamics and other fields were found in the literature \([7, 8, 9, 10, 11]\). Thermodynamic properties of these solutions were also studied in \([12, 13, 14]\). On physical grounds, one would expect that an interesting solution should be stable under non-spherically symmetric perturbations of the state of the black hole represented by this solution. This subject was investigated by Dotti and Gleiser in \([15, 16, 17]\). Furthermore, the causal structure of these solutions has been studied in detail by Torii and Maeda in \([18, 19]\).

However, nature behaves in more complex way: black holes are dynamic systems, which are seldom in equilibrium. Then, from a physical point of view, the above cited static solutions should represent the eventually steady state of dynamic evolution of black holes. This kind of solutions – dynamical black holes– has a twofold value. First, they allow us to model more realistic physical situations associated with the black hole dynamic, such as processes of formation and evaporation of black holes. Second, these solutions can be used to evaluate the cosmic censorship hypothesis \([20]\), which states that naked singularities are forbidden in physical gravitational collapses.

In the framework of GR, the Vaidya metric \([21]\) represents a radial null fluid which can be used to model dynamic processes associated with black holes. There exists many possible generalizations of the Vaidya metric – see \([22, 23]\) and references therein. At this point, it is important to note that some dynamic solutions describing the collapse of certain kinds of matter seems to evolve to the formation of naked singularities. Unfortunately, no much information is available about the stability of these solutions.

Recently, Dadwood and Ghosh \([22]\) have proved a theorem which is readily seen to generate dynamic solutions of black holes in GR, by imposing some conditions on energy-momentum tensor. This theorem is a generalization of a previous one obtained by Salgado in \([24]\) and that was generalized for \(n\)-dimension by Gallo \([25]\).

Unfortunately, in EGB gravity a few solutions representing dynamic black holes are known. Effectively, a recent Vaidya-type solutions in the context of EGB gravity were independently obtained by Kobayashi and Maeda in \([26, 27]\). The possibility that black hole space-times evolve to the formation of naked singularity were mentioned in \([19, 27]\). On the other hand, Grain, Barrau and Kanti have recently calculated the greybody factors associated to Einstein-Gauss-Bonnet black holes, which are needed to describe the evaporation spectrum by Hawking radiation \([28]\). Konoplya studied in \([29]\) the quasinormal modes of charged EGB black holes, and in \([30]\), he analyzed the evolution of a scalar field in EGB.

The aim of this work is to prove two similar theorems that were presented in \([22]\) but in the framework of EGB.

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gravity. The first one is an extension of that theorem for EGB gravity; the second one is deduced from the first one but relaxing some conditions on energy-momentum tensor, in order to include more realistic physical situations. As a consequence of these theorems, we find the analogous solutions, in EGB gravity, of many well known solutions for GR (Vaidya, Bonnor-Vaidya, Husain, Global Monopole). Also, we obtain the Vaidya-type generalization for static black holes in EGB gravity, such as Born-Infeld solutions.

In section II, we review the EGB gravity and recalled some of the well known static spherically symmetric black hole solutions. In section III, we prove the former theorem and we show some solutions resulting from this theorem. In section IV, we analyzed the imposition of the energy conditions on these metrics. Finally, in section V we present a more general version of the latter theorem and we apply it to obtain the metric of Born-Infeld dynamic black hole. In the conclusions we discuss possible future works in this area.

II. THE EINSTEIN-GAUSS-BONNET GRAVITY

The action which describe Einstein-Gauss-Bonnet gravity coupled with matter fields reads,

\[
S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left[ R - 2\Lambda + \alpha(R_{abcd}R^{abcd} + \right. \\
\left. + R^2 - 4R_{ab}R^{ab}) \right] + S_{\text{matt}},
\]

where \( S_{\text{matt}} \) is the action associated to the matter fields, and \( \alpha \) is the Gauss-Bonnet coupling constant associated in the string models, with the tension of these strings. This constant introduce a length scale. In fact, the correction that these theory produce to GR, are noted in short distance, given by the scale \( \sqrt{4\alpha} \).

The equations of motion resulting from \( \delta S = 0 \) are

\[
8\pi T_{ab} = G_{ab}^{(0)} + G_{ab}^{(1)} + G_{ab}^{(2)},
\]

where \( T_{ab} \) is the energy-momentum tensor, representing the matter-field distribution resulting from the variation \( \delta S_{\text{matt}}/\delta g^{ab} \), and

\[
G_{ab}^{(0)} = \Lambda g_{ab}, \\
G_{ab}^{(1)} = R_{ab} - \frac{1}{2} R g_{ab}, \\
G_{ab}^{(2)} = -\alpha \left[ \frac{1}{2} g_{ab}(R_{cjk}R^{cjk} - 4R_{c}R^{c} + R^2) - \\
\right. \\
\left. - 2RR_{ab} + 4R_{ac}R^c_b + 4R_{c}R^c_{ab} - 2R_{acje}R^c_{ej} \right].
\]

The static, spherically symmetric metric which satisfies these vacuum equations of fields for n-dimensional was obtained by Boulware and Deser in standard spherical coordinates \((t,r,\theta_1, \ldots, \theta_{n-2})\) reads

\[
ds^2 = -f(r,t)dt^2 + f^{-1}(r,t)dr^2 + r^2d\Omega_{n-2}^2,
\]

where \( f(r,t) \) is given by

\[
f_{\pm}(r,v) = 1 + \frac{r^2}{2\hat{\alpha}} \left\{ 1 \pm \sqrt{1 + \frac{8\hat{\alpha}}{n-2} \left[ \frac{\Lambda}{n-1} \pm \frac{2M}{r^{n-1}} \right]} \right\},
\]

with \( \hat{\alpha} = \alpha(n-3)(n-4) \) and \( d\Omega_{n-2}^2 \) being the metric of the \((n-2)\)-sphere

\[
d\Omega_{n-2}^2 = \frac{\pi^{n-2} i^{-1}}{\sin^2 \theta_j} \delta_{i}^2,
\]

It is straightforward to prove that only the minus-branch solution, \( f_{-}(r,v) \), has limit for \( \alpha \to 0 \), namely n-dimensional GR.

In the next section we will find that this metric is a particular case of a large family of metrics. Although it is not crucial for the proof of the theorem, we adopt \( \alpha \) positive since this condition arises from the strings theory. Hereafter we will assume that both \( \Lambda \) and \( \alpha \) are fixed quantities.

III. RADIATING BLACK HOLES IN EINSTEIN-GAUSS-BONNET GRAVITY

Theorem 1: Let \((M,g_{ab})\) a n-dimensional space-time such that: i) it satisfies the EGB equations, ii) it is spherically symmetric, iii) In the Eddington-Bondi coordinates, where the metric reads \( ds^2 = -\hat{h}^2(r,v)f(r,v)dv^2 + 2\hat{h}(r,v)dvdr + r^2d\Omega_{n-2}^2 \), the energy-momentum tensor \( T^a_b \) satisfies the conditions \( T^v_v = 0 \), and \( T^\theta_\theta = kT'_r \), \( (k = \text{const} \in \mathbb{R}) \) iv) If \( \alpha \to 0 \), the solution converges to the General Relativity limit. Then the metric of the space-time is given by

\[
ds^2 = -f(r,v)dv^2 + 2\epsilon dvdr + r^2d\Omega_{n-2}^2, \quad (\epsilon = \pm1),
\]

where

\[
\hat{h} = \frac{\sqrt{\alpha} \sqrt{\Delta}}{(2\hat{\alpha})^{\frac{n-4}{2}}}, \\
\hat{\alpha} = \alpha(n-3)(n-4), \\
\Delta = \frac{1}{4\alpha} \left[ \frac{\Lambda}{n-1} \pm \frac{2M}{r^{n-1}} \right].
\]
\[
 f(r, v) = \begin{cases} 
 1 + \frac{r^2}{2a} & \left(1 - \sqrt{1 + \frac{8\hat{\alpha}}{n-2} \left[ \frac{\Lambda}{n-1} + \frac{2M(v)}{r^{n-1}} - \frac{8\pi C(v)}{k(n-2)+1} \right]} \right) \text{ if } k \neq -\frac{1}{n-2}, \\
 1 + \frac{r^2}{2a} & \left(1 - \sqrt{1 + \frac{8\hat{\alpha}}{n-2} \left[ \frac{\Lambda}{n-1} + \frac{2M(v)}{r^{n-1}} - \frac{8\pi C(v)\ln(r)}{r^{n-1}} \right]} \right) \text{ if } k = -\frac{1}{n-2}.
\end{cases}
\]

with the diagonal components of \(T^a_b\) given by

\[
 T^a_b(\text{Diag}) = \frac{C(v)}{r^{(n-2)(1-k)}} \text{diag}[1, 1, k, \cdots, k],
\]

and an unique non-vanishing off-diagonal element

\[
 T^r_v = \begin{cases} 
 \frac{1}{4\pi r^{n-2}} \frac{dM}{dv} - \frac{r^{(k-1)(n-2)+1}}{k(n-2)+1} \frac{dC}{dv} & \text{if } k \neq -\frac{1}{n-2} \\
 \frac{1}{4\pi r^{n-2}} \frac{dM}{dv} - \frac{\ln(r)}{r^{n-2}} \frac{dC}{dv} & \text{if } k = -\frac{1}{n-2}
\end{cases}
\]

with \(M(v), C(v)\) two arbitraries functions depending of the distribution of the energy-matter. \(\Box\)

**Proof:** By the hypothesis iii) of the theorem 1, the metric under discussion reads

\[
ds^2 = -h^2(r, v)f(r, v)dv^2 + 2ch(r, v)dvdr + r^2d\Omega^2_n,
\]

which due to hypothesis ii) must satisfy the EGB equations. The only nontrivial components of the EGB tensor are

\[
 G^r_r = (n-2) \left[ r^2 + 2(1-f)\hat{\alpha} \right] \frac{h_r}{r^3ch^2},
\]

\[
 G^r_v = -(n-2) \left[ r^2 + 2(1-f)\hat{\alpha} \right] \frac{f_v}{2r^3},
\]

\[
 G^r_r = \Lambda + (n-2) \left[ r^2 + 2(1-f)\hat{\alpha} \right] \frac{f_r}{2r^3} - (n-2) \left[ (n-3)r^2 + (n-5)(1-f)\hat{\alpha} \right] \frac{1-f}{2r^4}
\]

\[
 + (n-2) \left[ r^2 + 2(1-f)\hat{\alpha} \right] \frac{h_r}{r^3h},
\]

\[
 G^r_v = \Lambda + (n-2) \left[ r^2 + 2(1-f)\hat{\alpha} \right] \frac{f_r}{2r^3} - (n-2) \left[ (n-3)r^2 + (n-5)(1-f)\hat{\alpha} \right] \frac{1-f}{2r^4},
\]

\[
 G^\theta_\theta = \Lambda - \frac{\hat{\alpha}}{r^4} \left[ r^2f_r^2 - 2(n-6)(n-5)(1-f^2) \right] - (n-3)(n-4) \frac{1-f}{2r^2}
\]

\[
 + \left[ r^2 + 2\hat{\alpha}(1-f)r^2 \right] \left[ (fh^2h_{rr} + hh_{rr} - h^3f_{rr}) - eh_rh_v \right] \frac{1}{2r^2h^3}
\]

\[
 + \left[ (n-3)r^2 + 2(n-5)\hat{\alpha}(1-f) \right] \frac{f_r}{r^3}
\]

\[
 + \left\{ 6\alpha^3h + 4\hat{\alpha} [(3-5f)rh + (n-5)(1-f)hf - 4\alpha rf_v] + (n-3) \right\} \frac{f_r}{4r^3h^2}.
\]

By using hypothesis iii), \(T^r_v = 0\), in Eq. (10), we deduce that: either \(h(r, v)\) is independent of \(r\), i.e.,

\[
h(v, r) = h(v),
\]

or \(f(v, r)\) is independent of \(v\), and satisfies

\[
r^2 + 2(1-f)\hat{\alpha} = 0.
\]

However, this last equation leads to non-radiating metrics (because Eq. (10) implies \(T^r_v = 0\)), which are iso-metric to AdS/dS Gauss-Bonnet solutions, and naturally contained in the metrics Eq. (6) as particular cases with \(M(v) = C(v) = 0\). We show this in the Appendix A. Then, we conclude that \(h(r, v) = h(v)\), and redefining to the null coordinate \(\tilde{v} = \int h(v)dv\), we can take \(h(v) = 1\), without loss of generality.
Now, from Eq. (11) and Eq. (12), we obtain that \( G^v_v = G^r_r \), and then
\[
T^v_v = T^r_r.
\]
If we impose the conservation laws, \( \nabla_v T^v_v = 0 \), and using again the hypothesis \( iii \), \( T^\theta_\theta = kT^r_r \), we have that
\[
0 = \frac{\partial}{\partial v} T^v_v + \frac{\partial}{\partial r} T^r_r + \frac{1}{2} \left( T^r_r - T^v_v \right) \frac{\partial}{\partial r} f + \left( \frac{n-2}{r} \right) T^r_v,
\]
\[
0 = \frac{\partial}{\partial r} T^r_r + \left( \frac{n-2}{r} \right) (1-k) T^r_r.
\]
Solving Eq. (16) for \( T^r_r \), we obtain:
\[
T^r_r = \frac{C(v)}{r^{(n-2)(1-k)}},
\]
where \( C(v) \) is an arbitrary function.

By joining these results, we can write the diagonal elements of \( T^k_k \) as
\[
T^k_k(\text{Diag}) = \frac{C(v)}{r^{(n-2)(1-k)}} \text{diag}[1,1,k,\cdots,k].
\]

Now, we can find \( f(r,v) \) by replacing Eq. (11) and Eq. (17) in the suitable component of the EGB equations
\[
G^r_r = 8\pi T^r_r,
\]
resulting after solving this differential equation and making some algebraic simplifications

\[
f_{\pm}(r,v) = \begin{cases} 
1 + \frac{\alpha^2}{2\alpha} & \left[ 1 \pm \sqrt{1 + \frac{8\alpha}{n-2} \left( \frac{\Lambda}{n-1} + \frac{2M(v)}{r^{n-1}} - \frac{8\pi C(v)}{k(n-2)+1}) \right)} \right] 
\end{cases} \quad \text{if} \quad k \neq -\frac{1}{n-2},
\]
\[
1 + \frac{\alpha^2}{2\alpha} & \left[ 1 \pm \sqrt{1 + \frac{8\alpha}{n-2} \left( \frac{\Lambda}{n-1} + \frac{2M(v)}{r^{n-1}} - \frac{8\pi C(v)\ln(r)}{r^{n-1}} \right)} \right] 
\end{cases} \quad \text{if} \quad k = -\frac{1}{n-2}.
\]

where \( M(v) \) is another arbitrary function (which can be shown to be proportional to the mass of the underlying matter).

At this point, it is important to note that we have obtained two branches of solutions, namely, \( f_+ \) and \( f_- \), which correspond to \( \pm \) signs in front of the square root term. However, the positive branch, \( f_+ \), does not converge to GR.

Then, if we impose the hypothesis \( iv \), then it can be shown that the only possible solutions are the following:

\[
f(r,v) = \begin{cases} 
1 - \frac{4M(v)}{(n-2)r^{n-1}} & \left[ 1 - \frac{2\alpha^2}{(n-1)(n-2)} + \frac{16\pi C(v)}{k(n-2)+1}) r^{(n-2)-2} \right] 
\end{cases} \quad \text{if} \quad k \neq -\frac{1}{n-2},
\]
\[
1 - \frac{4M(v)}{(n-2)r^{n-1}} & \left[ 1 - \frac{2\alpha^2}{(n-1)(n-2)} + \frac{16\pi C(v)\ln(r)}{(n-2)r^{n-2}} \right] 
\end{cases} \quad \text{if} \quad k = -\frac{1}{n-2}.
\]

In this case, the limit \( \alpha \to 0 \) reduces \( f(r,v) \) to

\[
f(r,v) = \begin{cases} 
1 - \frac{4M(v)}{(n-2)r^{n-1}} & \left[ 1 - \frac{2\alpha^2}{(n-1)(n-2)} + \frac{16\pi C(v)}{k(n-2)+1}) r^{(n-2)-2} \right] 
\end{cases} \quad \text{if} \quad k \neq -\frac{1}{n-2},
\]
\[
1 - \frac{4M(v)}{(n-2)r^{n-1}} & \left[ 1 - \frac{2\alpha^2}{(n-1)(n-2)} + \frac{16\pi C(v)\ln(r)}{(n-2)r^{n-2}} \right] 
\end{cases} \quad \text{if} \quad k = -\frac{1}{n-2}.
\]

These are the expressions that we would have found if we had started with the \( n \)-dimensional Einstein version of the theorem (General Relativity with a cosmological
constant) instead of the Einstein-Gauss-Bonnet gravity. In fact, if \( n = 4 \), these expressions are the same that those found in [22].

Finally, from Eq.(10) and the appropriate EGB equation,

\[ G^\nu_\nu = 8\pi T^\nu_\nu, \]

we obtain that the only non-vanishing off-diagonal element of \( T^\alpha_\beta \) reads

\[
T^\tau_\nu = \begin{cases} 
\frac{1}{4\pi} \frac{dM}{dv} - \frac{r^{k-1}(n-2)+1}{k(n-2)+1} \frac{dC}{dv} & \text{if } k \neq -\frac{1}{n-2} \\
\frac{1}{4\pi r^{n-2}} & \text{if } k \neq -\frac{1}{n-2}
\end{cases}
\]

Q.E.D.

This family of metrics contains many potentially interesting solutions. Some of these space-times are shown in the Table 1. These solutions could be very useful to study the collapse of different matter fields, or the formation of naked singularities. On the other hand, we also think that they can be used to analyzed semiclassical approaches to the evaporation of black holes.

However, the imposed conditions on the energy-momentum tensor are rather restrictive so other important solutions are not allowed. In the section V we generalize the theorem 1 in order to get new exact solutions.

**IV. ENERGY CONDITIONS**

In this section we discuss the restrictions on the energy-momentum tensor based in the dominant energy conditions. These will impose restrictions on \( k, M(v) \) and \( C(v) \), defined in the last section. The case of weak energy conditions is totally analogous to that discussed in [22] for General Relativity theory.

The covariant components of the energy-momentum tensor can be written with the help of two future null vectors, \( l^a \), \( n^a \) (the vector \( l^a \) is tangent to the null surface generated by \( v \), and \( n^a \) is an independent null vector such as \( l^a n_a = -1 \); in Eddington-Bondi coordinates \( \{v, r, \theta\} \), the components of these vectors are \( l_a = (1,0,\cdots,0) \), \( n_a = (f/2, -1, 0, \cdots, 0) \) as follows,

\[ T_{ab} = \epsilon \mu l_a l_b - P_r (l_a n_b + l_b n_a) + P_b (g_{ab} + l_a n_b + l_b n_a), \]

where

\[
\begin{align*}
\mu &= T^v_\nu \\
P_r &= - T^\tau_\nu = - C(v) r^{(2-n)(1-k)} \\
P_b &= kP_r.
\end{align*}
\]

Physically \( \mu \) represents the radiating energy along the null direction given by \( l^a \); \( P_r \) denotes the radial pressure generated for the charges of the fluids and \( P_b \) are the transversal pressures. All these quantities are measured by an observer moving along a time-like direction \( u^a \) given by

\[ u^a = \frac{1}{\sqrt{2}} (l^a + n^a). \]

This observer will measure an energy density given by

\[ \rho = - P_r = - T_{ab} u^a u^b. \]

Note that the energy-momentum tensor corresponds to a null fluid Type II.

As it is well known, the dominant energy conditions implies that for all timelike vector \( t^a \), \( T_{ab} t^b \geq 0 \) and also \( T_{ab} t^b \) is non-spacelike vector. Then, we must require:

\[ \mu \geq 0 \quad \text{and} \quad \rho \geq P_b \geq 0. \]

The first condition, in the case of a radiating fluid \( \mu > 0 \), is equivalent to

\[
\frac{1}{4\pi r^{n-2}} \frac{dM}{dv} - \frac{r^{k-1}(n-2)+1}{k(n-2)+1} \frac{dC}{dv} > 0 \quad \text{if } k \neq -\frac{1}{n-2},
\]

and

\[
\frac{1}{4\pi r^{n-2}} \frac{dM}{dv} - \frac{\ln r}{r^{n-2}} \frac{dC}{dv} > 0 \quad \text{if } k = -\frac{1}{n-2}.
\]

The Eq.(25) is satisfied if \( \frac{dM}{dv} > 0 \), and either \( \frac{dC}{dv} > 0 \) with \( k < -\frac{1}{n-2} \), or \( \frac{dC}{dv} < 0 \) with \( k > -\frac{1}{n-2} \).

On the other hand, the Eq.(26) is satisfied if \( \frac{dM}{dv} > 4\pi \ln r \frac{dC}{dv} \).

Finally, the conditions \( \rho \geq P_b \geq 0 \) are satisfied only if \( C(v) \leq 0 \) and \( -1 \leq k \leq 0 \).

**V. A MORE GENERAL VERSION OF THE THEOREM**

In this section, we present a slightly different version of the theorem, which allows more general distributions of matter fields.

**Theorem 2:** Let \((\mathcal{M}, g_{ab})\) a \( n \)-dimensional space-time such that: i) it satisfies the EGB equations, ii) it is spherically symmetric, iii) In the Eddington-Bondi coordinates, where the metric reads

\[ ds^2 = -h^2(r,v) f(r,v) dv^2 + 2ch(r,v) dv dr + r^2 d\Omega_n^2, \]

the energy-momentum tensor \( T^\nu_\nu \) satisfies the conditions \( T^\tau_\nu = 0 \), and \( T^0_0 = \kappa (r,v) T^\tau_\nu \), with \( \kappa (r,v) \) a real function, iv) If \( \alpha \to 0 \), then the solution converges to the General Relativity limit. Then the metric of the space-time is given by

\[ ds^2 = -f(r,v) dv^2 + 2 edv dr + r^2 d\Omega_n^2, \quad (\epsilon = \pm 1), \]

where
TABLE I: Some space-times satisfying the conditions of the Theorem 1 and 2, obtained with particular values of \(k\), \(C(v)\) and \(M(v)\).

| \(T^v_b\)         | Space-Time | \(M(v)\) and \(C(v)\) | \(k\)-index |
|-------------------|------------|-------------------------|-------------|
| \(T^v_b(\text{Diag}) = 0\) , \(T^v_c = \frac{1}{4\pi r^2} \frac{dM}{dv}\) | Kobayashi-Maeda | \(C(v) = 0\) |             |
| \(T^v_b(\text{Diag}) = - \frac{Q^2(v)}{4\pi r^2} \text{diag}[1, 1, -1, \cdots, -1]\) | Bonnor-Vaidya-EGB | \(C(v) = - \frac{Q^2(v)}{8\pi}\) | \(k = -1\) |
| \(T^v_b(\text{Diag}) = - \frac{Q^2(v)}{4\pi r^2} \text{diag}[1, 1, -m, \cdots, -m]\) | Husain-EGB | \(C(v) = - \frac{Q^2(v)}{4\pi}\) | \(k = -m\) |
| \(T^v_b = T^v_c = - \frac{n}{4\pi r^2}\) | Global monopole-EGB | \(M(v) = 0\), \(C(v) = - \frac{n}{2\pi}\) | \(k = 0\) |
| \(T^v_b = 0\) | Boulware-Deser-Wheeler | \(M(v) = M_0\), \(C(v) = 0\) |             |
| \(T^v_b = T^v_c = \frac{G^2}{4\pi L^2 r^{n-2}} \left(r^{n-2} - \sqrt{T^{2n-4} + L^2}\right)\) | BI-Vaidya-EGB | \(C(v) = - \frac{G^2}{4\pi L} k(r, v) = - \frac{r^{n-2}}{\sqrt{T^2(n-2) + L^2}}\) |             |
| \(T^\theta_k = k(r, v) T^r_v\) |             |             |             |
| \(T^v_b = \frac{1}{4\pi r^2} \left(\frac{dM}{dv} - br (\frac{dQ}{dv} F + Q \frac{\partial F}{\partial v})\right)\) |             |             |             |

\[
f(r, v) = 1 + \frac{r^2}{2a} \left(1 - \sqrt{1 + \frac{8a}{n-2} \left[\Lambda \frac{2M(v)}{r^{n-1}} - \frac{8\pi C(v)}{r^{n-1}} I(r, v)\right]}\right) - \frac{8\pi C(v)}{r^{n-1}} I(r, v)
\]

with

\[
I(r, v) = \int_0^r e^{(n-2)\int_0^t k(t, v) t^{-1} dt} ds
\]

and the diagonal components of \(T^b_b\) given by

\[
T^a_b(\text{Diag}) = \frac{C(v) e^{(n-2) \int k(r, v) r^{-1} dr}}{r^{n-2}} \text{diag}[1, 1, k, \cdots, k],
\]

with an unique non-vanishing off-diagonal element

\[
T^v_b = \frac{1}{r^{n-2}} \left(\frac{1}{4\pi} \frac{dM}{dv} - \frac{dC}{dv} I(r, v) - C \frac{\partial I}{\partial v}\right)
\]

where \(M(v)\), \(C(v)\) are two arbitrary functions depending of the distribution of the energy-matter.

The proof is completely analogous to that of the previous theorem. Note that if \(k = \text{const}\), we recover the results of that Theorem. In the next example, we apply this theorem in order to obtain the metric of a null charged fluid in Born-Infeld nonlinear electrodynamics.

VI. AN EXAMPLE: THE RADIATING BORN-INFELD BLACK HOLE

Since Thompson discovered the electron, the physicists were very worried about the divergence of the self-energy of point-like charges in the Maxwell electrodynamics. There were a lot of proposals to solve this problem, and one of the most prominent theories compatible with the spirit of Lorentz invariance was created by Born and Infeld between 1932 and 1934. They proposed a nonlinear and Lorentz invariant Lagrangian, which depends on one parameter \(b\), such as, if \(b \rightarrow \infty\), then its Lagrangian tends to the Maxwell Lagrangian. The electromagnetic field \(F_{ab}\) can be obtained from a 1-form \(A_a\)
in a similar way to Maxwell theory, $F = dA$. In the last years, the interest in this theory was renewed due to the fact that it appears as a low energy limit of string theory [1, 2].

The free Lorentz invariant Lagrangian suggested by Born-Infeld reads

$$\mathcal{L} = \frac{b^2}{4\pi} \left( 1 - \sqrt{1 + \frac{F_{ab}F_{ab}}{2b^2}} \right),$$

where $g$ is the determinant of the metric $g_{ab}$. If we add an interaction term in the form of $J^a A_a$ to the Born Infeld Lagrangian, then the equations of motions that follows from $\mathcal{L}$ are

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left( \sqrt{-g} F_{ab} \right) = 4\pi J^b. \quad (31)$$

In addition to the Bianchi identities

$$F_{ab,c} + F_{ca,b} + F_{bc,a} = 0.$$

Now, we are interested in finding a spherically symmetric Vaidya-like solution with a charged null fluid in Born-Infeld electrodynamics.

Let us consider a spherically symmetric and non-stationary electromagnetic field of the form

$$F = E(r,v)dv \wedge dr.$$

In this case, the contravariant components of $F_{ab}$ in the metric $g_{ab}$ (Eq. 31) are

$$F^{\tau v} = -\epsilon F_{\tau v} = \epsilon E(r,v) \quad (32)$$

$$F^{\tau r} = \epsilon F_{\tau r} = -\epsilon E(r,v) \quad (33)$$

and the determinant of the metric reads

$$g = -r^{2(n-2)} \prod_{i=3}^{n-2} [\sin(\theta_i)]^{n-1}. \quad (34)$$

Note, because we are considering a null fluid, there will be a null current given by

$$J^a = J(r,v) \delta^a_v. \quad (35)$$

Having all these expressions into account, the equations of motions for $F_{ab}$ are,

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} \left( \sqrt{-g} E(r,v) \right) = 0, \quad (36)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial v} \left( \sqrt{-g} E(r,v) \right) = 4\pi J(r,v). \quad (37)$$

Solving these equations we find

$$E(r,v) = \frac{Q(v)}{\sqrt{r^{2(n-2)} + L^2(v)}}, \quad (38)$$

where

$$L(v) = \frac{Q(v)}{b}, \quad (39)$$

and $Q(v)$ an arbitrary function of $v$, representing the charge of the fluid. The null current results,

$$J^a = \frac{1}{4\pi r^{n-2}} \frac{dQ}{dv} \delta^a_v.$$

Note that if $b \to \infty$ then $E(r,v)$ tends to its Maxwell expression. Note also that the 1-form $A_a$ from this problem has the form:

$$A = \left[ \int_0^r \frac{dr}{\sqrt{r^{2(n-2)} + L^2}} \right] Q(v)dv.$$

The energy-momentum tensor associated to the field $F_{ab}$ is,

$$T_{ab} = \frac{1}{4\pi} \left( \frac{b^2 g_{ab} - F_{cd}F_{ab}^d - \frac{1}{2}g_{ab}F_{cd}F^{cd}}{\sqrt{1 + F_{cd}F^{cd}}} - b^2 g_{ab} \right).$$

From this and from the form of the current, the non-radiative part of the energy-momentum tensor of the charged null fluid results

$$T^r_r = \frac{Q^2}{4\pi L^2} \left( r^{n-2} - \sqrt{r^{2(n-2)} + L^2} \right), \quad (40)$$

$$T^v_v = T^r_r, \quad (41)$$

$$T^\theta_i = -\frac{r^{n-2}}{\sqrt{r^{2(n-2)} + L^2}} T^r_r, \quad (42)$$

and the other off-diagonal components vanishes (with the exception to $T^\nu_v$).

Then, we can see that this null charged fluid satisfies the conditions of the Theorem 2, with

$$k(r,v) = -\frac{r^{n-2}}{\sqrt{r^{2(n-2)} + L^2}},$$

and

$$C(v) = \frac{Q^2}{4\pi |L|} = -\frac{|bQ(v)|}{4\pi},$$

because it can be easily computed that

$$\epsilon^{(n-2)} f k(r,v)r^{-1} dr = -\frac{1}{|L|} \left( r^{n-2} - \sqrt{r^{2(n-2)} + L^2} \right).$$

If we replace these expressions of $k(r,v)$ and $C(v)$ into $f(r,v)$, we obtain the metric for a Born-Infeld dynamic black hole:
f(r, v) = 1 + \frac{r^2}{2\alpha} \left\{ 1 - \sqrt{1 + \frac{8\alpha}{n-2} \left\{ \frac{\Lambda}{n-1} + \frac{2M(v)}{r^{n-1}} - \frac{2b^2}{(n-1)} \right\} + \frac{2b^2}{r^{n-1}} \int_0^r \sqrt{r^2(n-2) + L^2 dr} } \right\} \quad (43)

In the case of \( Q(v) = Q_0 \), \( M(v) = M_0 \), we recover the static Born-Infeld solution, first obtained by Wiltshire \[10\] and reexamined in \[11\].

It can be shown that the integral factor

\[ I = \int_0^r \sqrt{r^2(n-2) + L^2 dr} , \]

can be written in terms of a generalized hypergeometric function \( F([a_i], [b_j], z) \) (dependent of \( p \) parameters \( a_i \) and \( q \) parameters \( b_j \)) defined by

\[ F([a_i], [b_j], z) = \sum_{k=0}^{\infty} \prod_{i=1}^p \prod_{j=1}^q \frac{\Gamma(a_i + k) \Gamma(b_j + k)}{\Gamma(a_i) \Gamma(b_j + k) k!} z^k. \]

The result is:

\[ I = |L| r F \left( \left[ -\frac{1}{2}, \frac{1}{2n-4} \right], \left[ 2n-3, \frac{2n-4}{2n-4} \right], -\frac{r^2(n-2)}{L^2} \right) . \]

Finally, if we compute the \( T^r_r \) component of the energymomentum tensor using the theorem 2, we get

\[ T^r_r = \frac{1}{4\pi r^{n-2}} \left[ \frac{dM}{dv} - br \left( \frac{dQ}{dv} + Q \frac{\partial F}{\partial v} \right) \right] , \]

where \( F \) denotes the hypergeometric function defined in Eq. (44).

**VII. CONCLUSIONS**

In this work, we have proved two theorems, which allow us to obtain exact solutions to the Einstein-Gauss-Bonnet equations. These solutions represent dynamic black holes, and we thought that these solutions are of interest, because they generalize some known solutions of GR to the context to Einstein-Gauss-Bonnet gravity which has been proved, is the first nontrivial term of low energy limit of string theory.

It is important to note that with plus sign in front of the square root, there are black hole solution even when the \((n-2)\)-dimensional submanifold is plane or hyperbolically symmetric. The analogous theorem seems to be hold in these cases.

In a future work, we will make a detailed study of these metrics, analyzing the causal structure and its thermodynamics.

Also, it should be very interesting to apply these metrics to study the matter collapse, the naked singularities formations, and semiclassical analysis of evaporation of black holes. These works, are now in progress.

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**APPENDIX A**

In this appendix, we will analyze the consequences to impose the Eq. (14) as solution to the Eq. (9).

We will use the conservation laws, \( \nabla_a T^a_v = 0 \), which, as we saw, implies that

\[ T^r_r = C(v)r^{(\alpha-\beta)}(n-2), \quad T^\theta_{\theta_i} = kC(v)r^{(\alpha-\beta)}(n-2). \]

The solution of the Eq. (14) reads,

\[ f(r, v) = 1 + \frac{r^2}{2\alpha} . \]

By replacing Eqs. (A1), (A2) and (A3) in the EGB equations:

\[ G^r_r - 8\pi T^r_r = 0, \quad G^\theta_{\theta_i} - 8\pi T^\theta_{\theta_i} = 0, \quad G^r_r - 8\pi T^r_r = 0, \]

we get respectively

\[ T^r_r = 0, \quad T^\theta_{\theta_i} = \frac{\Lambda}{8\pi} + \frac{(n-1)(n-2)}{64\pi \hat{\alpha}}, \]

and

\[ \Lambda + \frac{(n-1)(n-2)}{8\hat{\alpha}} - 8\pi kC(v)r^{(\alpha-\beta)}(n-2) = 0. \]

Now, Eq. (A9) has two possible solutions:

**Case 1:** \( C(v) = 0 \). In this case, the Eq. (A9) can be satisfied if and only if the constants \( \Lambda \) and \( \hat{\alpha} \) are related in the following form:

\[ \Lambda = \frac{(n-1)(n-2)}{8\hat{\alpha}}, \]

which, when is replaced in the \( r-r \) component of the EGB equations,

\[ G^r_r - 8\pi T^r_r = 0, \]
gives a differential equation for \( h(r, v) \):

\[
(n - 2)(1 + \frac{r^2}{2\alpha}) \frac{h_r}{r} = 0. \tag{A11}
\]

This implies again, that \( h(r, v) = h(v) \), which can be taken as 1 by an appropriated coordinate transformation. Finally, collecting the previous results, we obtain that the energy-momentum tensor is zero,

\[ T^a_b = 0. \]

The resulting metric, is given by:

\[
ds^2 = - \left(1 + \frac{r^2}{2\alpha}\right) dv^2 + 2edvdr + r^2d\Omega^2_n
\]

which are the known solutions de Sitter/anti-de Sitter (dS/AdS) metrics in EGB gravity.

**Case 2:** \( C(v) = \text{const.} \), and \( k = 1 \). In this case, from Eq. \((A9)\) we get that

\[
C(v) = \frac{\Lambda}{8\pi} + \frac{(n-1)(n-2)}{64\pi\alpha}, \tag{A12}
\]

which, when it is replaced in the \( r-r \) component of the EGB equations,

\[
G^r_r - 8\pi T^r_r = 0,
\]

gives the same differential equation for \( h(r, v) \) that in the Case 1, Eq. \((A11)\), and again we can take \( h(r, v) = 1 \).

By replacing the expressions of \( C(v) \) and \( f(r) \) in the EGB equations, we conclude that the only non-zero components of \( T^a_b \) are in its diagonal part,

\[
T^a_b = \left[\frac{\Lambda}{8\pi} + \frac{(n-1)(n-2)}{64\pi\alpha}\right] \delta^a_b. \tag{A13}
\]

This energy-momentum tensor, can be thought as an effective vacuum energy, and from this, resulting a cosmological effective term \( \Lambda \) given by

\[
\Lambda = - \frac{(n-1)(n-2)}{8\alpha}.
\]

These metrics are again dS/AdS solutions to the EGB equations. Note that these two cases are contained in the expression of the metric in the theorem, if we take \( M(v) = C(v) = 0 \).

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