How Not To Do Mean-Variance Analysis

Vic Norton
Mathematical Ruminations Inc
622 Morton Avenue
Bowling Green, OH 43402-2223
mailto:vic@norton.name

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Abstract
We use the 2014 market history of two high-returning biotechnology exchange-traded funds to illustrate how ex post mean-variance analysis should not be done. Unfortunately, the way it should not be done is the way it generally is done—to our knowledge.
Ex post mean-variance analysis is a financial application of descriptive statistics. But descriptive statistics, where the sum of square deviations from the mean plays a fundamental role, has a strong geometric flavor. In this paper we emphasize the geometry of mean-variance analysis.

Geometry starts with points. The primary points in our geometric exposition are two biotechnology exchange-traded funds,

- \text{FBT} – First Trust NYSE Arca Biotechnology Indx Fund
- \text{XBI} – SPDR S&P Biotech ETF.

The graphs of the 2013-12-31-normalized adjusted closing prices, \( \mathbf{a}_{\text{FBT}} \) and \( \mathbf{a}_{\text{XBI}} \), of the two funds are shown in Figure 2.1, as well as the graph of an unattended long portfolio, \( \text{UIP} \), in the two funds. \( \text{UIP} \)'s normalized adjusted closing price vector,

\[
\mathbf{a}_{\text{UIP}} = 0.75 \mathbf{a}_{\text{FBT}} + 0.25 \mathbf{a}_{\text{XBI}},
\]

is necessarily a convex combination of \( \mathbf{a}_{\text{FBT}} \) and \( \mathbf{a}_{\text{XBI}} \).

Normalized adjusted closing prices are the horses that drive our mean-variance cart. When the adjusted closing price vectors \( \mathbf{a}_{\text{FBT}}, \mathbf{a}_{\text{XBI}}, \) and \( \mathbf{a}_{\text{UIP}} \) are replaced by their daily return vectors \( \mathbf{r}_{\text{FBT}}, \mathbf{r}_{\text{XBI}}, \) and \( \mathbf{r}_{\text{UIP}} \), you move into a higher dimensional space with all geometry left behind.

The pictures that follow tell the whole story. Thanks to the \text{TikZ} vector-graphics language these pictures are precise, numerical images—not just schematic drawings.
2 Normalized adjusted closing prices – a geometric example

Figure 2.1 shows the 2014 history of an unattended investment portfolio, UIP, in two high-returning exchange traded funds, FBT and XBI. The 2013-12-31 closing composition of UIP was

\[ \text{UIP} = 75\% \text{FBT} + 25\% \text{XBI}, \]  

but these closing proportions never reoccurred in 2014. Indeed UIP closed with more than 25% in XBI (green higher than blue) for most of the first quarter, whereas XBI was less than 25% of UIP (blue higher than green) for much of the remaining three quarters. However Figure 2.1 does shows the geometry of the 3:1 proportions. On every vertical line, the brown UIP point is exactly 3/4 of the way from the green XBI point to the blue FBT point.

There were exactly 253 market days from 2013-12-31 to 2014-12-31 inclusive. Each of the four adjusted closing price graphs in Figure 2.1 represents the changing value of $100 invested at 2013-12-31 closing prices in the corresponding fund or portfolio over the next 252 market days. Each graph corresponds to a point \( a \in \mathbb{R}^{253} \), and the three points, \( a_{\text{FBT}}, a_{\text{XBI}}, \) and \( a_{\text{UIP}} \) of (2.1), are on the line segment

\[ a = t \cdot a_{\text{FBT}} + (1 - t) \cdot a_{\text{XBI}}, \quad 0 \leq t \leq 1, \]  

in \( \mathbb{R}^{253} \)—with \( a_{\text{UIP}} \) corresponding to \( t = 0.75 \). (Appendix A describes how normalized adjusted closing prices can be computed.)
Portfolio UIP represents a completely passive, unattended investment. Think of an investor as having money in FBT and XBI at the close of 2013. Suppose that FBT represents exactly 75% of his total investment at that time and XBI 25%. UIP then simply tracks each $100 of his investment through 2014. The investor does absolutely nothing, and all dividends from the two funds are automatically reinvested.

However the continually reallocated portfolio CRP, mostly hidden by UIP, is an entirely different matter. Here the investor decides, a priori, that 75% FBT and 25% XBI are the right proportions for his investment. Accordingly, before each market day of 2014, he reinvests his money so as to start the day with exactly these proportions in the two funds. Algorithm 2.1 computes the growth of $100 under this scenario.

Algorithm 2.1: To compute the 2014 CRP adjusted closing price vector

\[
a_{0,\text{CRP}} = 100; \quad \% \text{ invest } $100 \text{ in CRP at the 2013-12-31 close}
\]

\[
\text{for } i = 1, \ldots, 252 \quad \% \text{ for each of the 252 market days in 2014 set}
\]

\[
\begin{align*}
  r_F &= a_i,\text{FBT}/a_{i-1,\text{FBT}} - 1; \quad \% = \text{return of FBT on market day } i \\
  r_X &= a_i,\text{XBI}/a_{i-1,\text{XBI}} - 1; \quad \% = \text{return of XBI on market day } i \\
  r_R &= 0.75 \times r_F + 0.25 \times r_X; \quad \% = \text{return of CRP on market day } i \\
  a_{i,\text{CRP}} &= a_{i-1,\text{CRP}} \times (1 + r_R); \quad \% = \text{CRP value at the close of market day } i
\end{align*}
\]

Note. The % signs above begin a comment.

It is difficult to make out the red CRP graph from the brown UIP graph in Figure 2.1. These graphs are very close, and the UIP graph is drawn over the CRP graph, hiding it from view for the most part. It is only toward the end of 2014 that one can really make out the differences.

Figure 2.2 is a blow-up of December 2014. The differences in the graphs are now quite visible. Here we see that the red CRP graph is slightly higher than the brown UIP graph throughout December—and clearly higher on December 31. It follows that CRP returned more than UIP over 2014.

Note that the 2014-12-31 value of \( a_{\text{UIP}} \) must be

\[
146.90 = 0.75 \times 147.54 + 0.25 \times 144.97
\]

by (1.1) and the terminal values of FBT and XBI shown on Figure 2.2. It follows that UIP had a total return of 46.90% over 2014. In fact CRP returned 47.32% over 2014, more than UIP and just slightly less than FBT.
Here are the adjusted closing prices of the four funds over the month of December 2014.

Table 2.1: The December adjusted closing prices

| date      | FBT   | XBI   | UIP   | CRP   |
|-----------|-------|-------|-------|-------|
| 2014-12-01| 145.736 | 135.477 | 143.171 | 143.490 |
| 2014-12-02| 147.918 | 139.052 | 145.702 | 146.048 |
| 2014-12-03| 148.034 | 139.184 | 145.821 | 146.169 |
| 2014-12-04| 147.123 | 138.121 | 144.873 | 145.215 |
| 2014-12-05| 148.150 | 140.580 | 146.258 | 146.622 |
| 2014-12-08| 151.214 | 141.417 | 148.765 | 149.114 |
| 2014-12-09| 151.807 | 145.504 | 150.231 | 150.630 |
| 2014-12-10| 148.699 | 142.379 | 147.119 | 147.508 |
| 2014-12-11| 148.786 | 142.720 | 147.269 | 147.661 |
| 2014-12-12| 146.878 | 142.480 | 145.778 | 146.179 |
| 2014-12-15| 142.657 | 136.400 | 141.093 | 141.469 |
| 2014-12-16| 140.807 | 135.609 | 139.507 | 139.888 |
| 2014-12-17| 145.851 | 142.076 | 144.907 | 145.314 |
| 2014-12-18| 150.737 | 146.628 | 149.710 | 150.129 |
| 2014-12-19| 152.212 | 147.990 | 151.156 | 151.579 |
| 2014-12-22| 150.463 | 146.886 | 149.569 | 149.990 |
| 2014-12-23| 143.842 | 139.313 | 142.710 | 143.107 |
| 2014-12-24| 145.968 | 141.988 | 144.973 | 145.380 |
| 2014-12-26| 149.554 | 145.261 | 148.481 | 148.897 |
| 2014-12-29| 150.017 | 145.790 | 148.960 | 149.378 |
| 2014-12-30| 148.151 | 144.258 | 147.178 | 147.592 |
| 2014-12-31| 147.544 | 144.973 | 146.901 | 147.321 |

The geometric equation (2.1) holds on every line of Table 2.1, but the proportions of FBT and XBI in UIP,

\[ p_{FBT} = 0.75 \times \text{FBT/UIP} \quad \text{and} \quad p_{XBI} = 0.25 \times \text{XBI/UIP}, \]

are different on every line. The 2013-12-31, 3:1 proportions come closest to being realized on the 2014-12-31 line of Table 2.1, where UIP closes at 75.33% FBT : 24.67% XBI.

As for daily returns,

\[ r_i = a_i / a_{i-1} - 1 \quad (i = 1, \ldots, 252), \]

the 2014 return vector equation,

\[ \mathbf{r}_{CRP} = 0.75 \mathbf{r}_{FBT} + 0.25 \mathbf{r}_{XBI}, \]

is valid in \( \mathbb{R}^{252} \) due to continual reallocation. This insures that the corresponding mean returns satisfy

\[ e_{CRP} = 0.75 e_{FBT} + 0.25 e_{XBI}, \]

as required by “The Standard Mean-Variance Portfolio Selection Model” of Markowitz 1987, pp. 3-4.

The ancillary folder that accompanies this article includes three files, FXUZ7.csv, matlab/FXUC2014.mat, and matlab/FXZC2014.mat, that contain the normalized adjusted closing prices used for this article.
3 How not to do mean-variance analysis

The MathWorks® Financial Toolbox with the MATLAB programming language is perhaps the most popular resource for doing mean-variance analysis. We have computed our mean-variance tables using MATLAB scripts in the matlab subdirectory of the ancillary folder that accompanies this article.

Our MATLAB script hn2mv1a.m illustrates the problem with mean-variance analysis as it is usually practiced. The script begins with the line

```
load FXUC2014.mat; % A dates funds legend A
```

which loads the $253 \times 4$ matrix of adjusted closing prices, $A = [\mathbf{a}_{\text{FBT}}, \mathbf{a}_{\text{XBI}}, \mathbf{a}_{\text{UIP}}, \mathbf{a}_{\text{CRP}}]$, displayed in Figure 2.1. This matrix has rank 3 rather than 4, since $\mathbf{a}_{\text{UIP}}$ is a linear combination of $\mathbf{a}_{\text{FBT}}$ and $\mathbf{a}_{\text{XBI}}$ (1.1).

Next we remove $\mathbf{a}_{\text{CRP}}$ from $A$ and append the columns

$$\mathbf{a}_{\text{UIP2}} = 0.50 \cdot \mathbf{a}_{\text{FBT}} + 0.50 \cdot \mathbf{a}_{\text{XBI}}$$

and

$$\mathbf{a}_{\text{UIP3}} = 0.25 \cdot \mathbf{a}_{\text{FBT}} + 0.75 \cdot \mathbf{a}_{\text{XBI}}$$

so that $A = [\mathbf{a}_{\text{FBT}}, \mathbf{a}_{\text{XBI}}, \mathbf{a}_{\text{UIP}}, \mathbf{a}_{\text{UIP2}}, \mathbf{a}_{\text{UIP3}}]$ becomes a $253 \times 5$ matrix of rank 2. Geometrically, $A$ describes 5 points on a line in $\mathbb{R}^{253}$, and $\text{rank}(A) = 2$ since the line does not pass through the origin.

The hn2mv1a.m script continues with the lines

```
%% get asset moments from adjusted closing prices
ptf = Portfolio;
ptf = estimateAssetMoments(ptf, A, 'dataformat', 'prices');
[mn, cv] = getAssetMoments(ptf);
```

Here mn and cv are the $5 \times 1$ and $5 \times 5$ mean daily return and covariance of daily return matrices corresponding to the $252 \times 5$ daily return matrix

$$R = [\mathbf{r}_{\text{FBT}}, \mathbf{r}_{\text{XBI}}, \mathbf{r}_{\text{UIP}}, \mathbf{r}_{\text{UIP2}}, \mathbf{r}_{\text{UIP3}}]$$

derived from $A$ via (2.4). The annualized results are shown in Table 3.1.

Table 3.1: Annualized results of the MATLAB computations

|      | FBT   | XBI   | UIP   | UIP2  | UIP3  |
|------|-------|-------|-------|-------|-------|
| $E$  | 0.4245| 0.4324| 0.4235| 0.4244| 0.4274|
| $\sigma$ | 0.2656| 0.3491| 0.2777| 0.2963| 0.3203|

|      | FBT    | XBI    | UIP    | UIP2   | UIP3   |
|------|--------|--------|--------|--------|--------|
| covariance $V$ |        |        |        |        |        |
| FBT  | 0.0705 | 0.0804 | 0.0729 | 0.0753 | 0.0778 |
| XBI  | 0.0804 | 0.1219 | 0.0905 | 0.1008 | 0.1112 |
| UIP  | 0.0729 | 0.0905 | 0.0771 | 0.0815 | 0.0859 |
| UIP2 | 0.0753 | 0.1008 | 0.0815 | 0.0878 | 0.0942 |
| UIP3 | 0.0778 | 0.1112 | 0.0859 | 0.0942 | 0.1026 |

One problem with Table 3.1 is immediately obvious. How can the mean returns of the long portfolios UIP and UIP2 be less than the mean return of either component fund? A dimensional problem is less obvious but just as troubling. We start with an adjusted closing price history, $A$, which corresponds to five points on a line segment (a one simplex) in $\mathbb{R}^{253}$; but, to do mean-variance analysis on $A$, we must jump to the four simplex in $\mathbb{R}^{252}$ generated by the five linearly independent columns of the daily return matrix $R$ ($\text{rank}(R) = \text{rank}(V) = 5$). It simply doesn’t make sense to us!
This is the \((e, \sigma)\)-picture of what is happening. The pink region is the image of the 4-simplex in \(\mathbb{R}^{252}\) generated by the five columns of \(R\). (The coordinates in the picture are percentages.)

Figure 3.1: Obtainable \((e, \sigma)\)

Figure 3.1 is a graphic representation of Table 3.1. The red, continually reallocated region represents all obtainable \((e_p, \sigma_p)\): all \((e_p, \sigma_p)\) such that
\[
e_p = E p, \quad \sigma_p = \sqrt{v_p}, \quad v_p = p^T V p, \tag{3.1}
\]
with the \(E\) and \(V\) from Table 3.1, and
\[
0 \leq p \leq 1, \quad \sum p = 1, \quad p \in \mathbb{R}^5. \tag{3.2}
\]
The following five portfolios \(p\) are equally \(e\)-spaced on the efficient frontier. They were computed with the line
\[
P = \text{estimateFrontier}(\text{ptf}, 5);
\]
in the MATLAB script \texttt{hn2mv1a.m}.

\[
P = \begin{bmatrix}
1 & 0.75 & 0.50 & 0.25 & 0 \\
0 & 0.25 & 0.50 & 0.75 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Every portfolio in the continually reallocated region, other than the generating funds \(\text{FBT}\), \(\text{XBI}\), \(\text{UIP}\), \(\text{UIP2}\), and \(\text{UIP3}\), must be reallocated at the close each market day of 2014 in order to retain its original 2013-12-31 closing proportions.

On the other hand, the unattended path in Figure 3.1 shows the actual mean returns and standard deviations of return of all unattended portfolios in \(\text{FBT}\) and \(\text{XBI}\) as computed directly from their adjusted closing price vectors \((2.2)\) via MATLAB Example 1 (script \texttt{hn2mv2.m}) below.

MATLAB Example 1: To compute the unattended portfolio path from \(\text{FBT}\) to \(\text{XBI}\) do
\[
T = 0 : 1/(nT - 1) : 1; \quad \% nT \text{ points partitioning } [0, 1]
\]
\[
\text{AT} = \text{A}(:,1) \times T + \text{A}(:,2) \times (1 - T); \quad \% nT \text{ unattended price vectors}
\]
\[
\text{RT} = \text{AT}(2:253,:) ./ \text{AT}(1:252,:); - 1; \quad \% nT \text{ unattended return vectors}
\]
\[
\text{ET} = 252 \times \text{mean}(\text{RT}); \quad \% nT \text{ mean daily returns (annualized)}
\]
\[
\text{SigT} = \text{sqrt}(252) \times \text{std}(\text{RT}, 1); \quad \% nT \text{ standard deviations of return}
\]

We should note that the MATLAB lines of \texttt{hn2mv1a.m},
\[
e\text{CRP} = 252 \times \text{mean} (\text{rCRP}); \quad \% e\text{CRP} = 0.4265
\]
\[
\text{sigCRP} = \text{sqrt}(252) \times \text{std} (\text{rCRP}, 1); \quad \% \text{sigCRP} = 0.2783
\]
produced the coordinates for the two \text{CRP} points of Figure 3.1. The \text{rCRP} in this code corresponds to the \text{r}_\text{CRP} vector of \((2.5)\) or the \text{r}_\text{CRP} from \(\hat{a}_\text{CRP}\) via \((2.4)\). They are the same.
4 The mean periodic return problem

Figure 4.1 on the next page and the hn2domv3.m script below it illustrate a serious problem with mean periodic returns. This is a simple, artificial example, where a fund gains 50% in the first quarter of the year, loses 67% in the second quarter, gains 200% in the third quarter, and loses 17% in the forth quarter.

An investor in the fund realizes that his fund has returned 25% over the year, but the trip has been terribly rocky; he decides to bail out.

Not so fast, his investment adviser tells him. Just add up the quarterly returns:

\[ 50\% - 67\% + 200\% - 17\% = 166\% \]

You have averaged over 40% per quarter. The fund may seem a bit risky, but, in view of its history, you should expect to average 40% per quarter next year as well. It’s clear from the numbers.

Figure 4.1 tells the whole story. Mean periodic returns tend to accentuate the positive. Mean periodic discounts do just do just the opposite.

**Definition 1** (Effective return and discount). Let \( a_0 \) and \( a_1 \) be the the adjusted closing prices of a security on two different market days with \( a_0 \) occurring before \( a_1 \). Then the *effective return* of the security over that period of time is defined as

\[ r = \frac{a_1 - a_0}{a_0}, \]

and the *effective discount* as

\[ d = \frac{a_1 - a_0}{a_1}. \]

The equation

\[ (1 + r)(1 - d) = 1 \]

always holds, and \( r \) and \( d \) always have the same sign, positive, negative, or zero.

Definition 1 is a paraphrase of definitions in *The Theory of Interest*, Kellison 2009.

The means of periodic changes in year-to-date return and periodic changes in date-to-end-of-year discount are the appropriate measures of effective performance of a security over a year. In the the example of Figure 4.1, the annualized mean of the changes in year-to-date return is

\[ e^0 = 50\% - 00\% + 100\% - 25\% = 25\%, \]

and the annualized mean of changes in date-to-end-of-year discount is

\[ e^1 = 40\% - 80\% + 80\% - 20\% = 20\%. \]

These annualized means do satisfy Definition 1, \((1 + e^0)(1 - e^1) = 1\).

*Note.* The \( e^0 \) and \( e^1 \) above correspond to the e_0 and e_1 in the MATLAB code underneath Figure 4.1. Likewise \( e^r \) and \( e^d \) correspond to e_r and e_d.

On the other hand, the MATLAB code shows that the mean return, \( e^r \), and the mean discount, \( e^d \), have opposite signs, and \((1 + e^r)(1 - e^d) = 5.8667\). These computations show that mean returns and mean discounts are essentially incompatible with the theory of interest. The mean return problem has been noted, for example, in Swensen 2009, pp. 104-105.
Figure 4.1: The mean periodic return/discount problem

adjusted closing price

| Time  | 0/4 | 1/4 | 2/4 | 3/4 | 4/4 |
|-------|-----|-----|-----|-----|-----|
| Value | 100 | 150 | 50  | 150 | 125 |

%% hn2mv3.m – the mean periodic return/discount problem

%% quarterly adjusted closing prices
a = [100, 150, 50, 150, 125];

%% annualized mean quarterly return
r = a(2 : 5) ./ a(1 : 4) - 1; % quarterly returns
e_r = 4 * mean(r); % e_r = 1.6667
sig_r = 2 * std(r, 1); % sig_r = 2.0069

%% annualized mean quarterly discount
d = 1 - a(1 : 4) ./ a(2 : 5); % quarterly discounts
e_d = 4 * mean(d); % e_d = -1.2000
sig_d = 2 * std(d, 1); % sig_d = 2.0580

%% annualized mean quarterly change in year-to-date return
df_0 = diff(a) / a(1); % changes in year-to-date return
e_0 = 4 * mean(df_0); % e_0 = 0.2500
sig_0 = 2 * std(df_0, 1); % sig_0 = 1.5155

%% annualized mean quarterly change in date-to-end-of-year discount
df_1 = diff(a) / a(5); % changes in date_to_end_of_year discount
e_1 = 4 * mean(df_1); % e_1 = 0.2000
sig_1 = 2 * std(df_1, 1); % sig_1 = 1.2124

%% return−discount relationship as per "The Theory of Interest"
(1 + e_0) * (1 - e_1) % = 1

%% meaningless return−discount relationship
(1 + e_r) * (1 - e_d) % = 5.8667
SEC Rule 156 below might apply to the situation illustrated by Figure 4.1 and the hn2mv3.m script which follows it.

Rule 156: Investment Company Sales Literature

Under the federal securities laws, including section 17(a) of the Securities Act of 1933 (15 U.S.C. 77q(a)) and section 10(b) of the Securities Exchange Act of 1934 (15 U.S.C. 78j(b)) and Rule 10b-5 thereunder (17 CFR Part 240), it is unlawful for any person, directly or indirectly, by the use of any means or instrumentality of interstate commerce or of the mails, to use sales literature which is materially misleading in connection with the offer or sale of securities issued by an investment company. Under these provisions, sales literature is materially misleading if it:

1. Contains an untrue statement of a material fact or
2. omits to state a material fact necessary in order to make a statement made, in the light of the circumstances of its use, not misleading.

Securities Act of 1933: Rule 156

Rule 156 raises an interesting question. Is the use of mean-variance analysis, as it appears to be practiced today, “materially misleading” when an investment company tells a client to “expect” a 160% return over the next year based on a 25% total return over the past year? This sort of reasoning reminds us of Mark Twain’s analysis of the expected shortening of the lower Mississippi due to the rounding of its bends over time (Appendix B).

More realistically, consider the example of FBT. In Table 3.1 we have seen that the annualized mean daily return of FBT over 2014 was \( e_{FBT}^r = 42.45\% \). An investor with a marginal knowledge of the theory of interest might ask his advisor what the corresponding annualized mean daily discount was. If the advisor were perplexed by this question, the investor could explain that to get the annualized mean discount you simply replace the daily return equation (2.4) by the daily discount equation

\[
d_i = 1 - a_{i-1}/a_i \quad (i = 1, \ldots, 252)
\] (4.1)

and sum the results. The advisor might then be mildly concerned by the annualized average discount, \( e_{FBT}^d = 35.34\% \), if he were told that returns and discounts over the same period of time are supposed to satisfy the equation

\((1 + r)(1 - d) = 1,\)

according to the theory of interest, but, in fact, \((1 + e_{FBT}^r)(1 - e_{FBT}^d) = 0.9211.\)
5 How to do it – the linear model

Our MATLAB script `hn2mv1L.m` is a linear variant of the `hn2mv1a.m` script of Section 3. We again remove \( a_{\text{CRP}} \) from \( A \) and append the unattended portfolio vectors \( a_{\text{UIP2}} \) and \( a_{\text{UIP3}} \) to the result. Then we add the unattended, long-short portfolio \( ZNS \), with normalized adjusted closing price vector
\[
a_{\text{ZNS}} = 1.25515 \cdot a_{\text{FBT}} - 0.25515 \cdot a_{\text{XBI}}
\] (5.1)
to \( A \), so that \( A \) becomes the \( 253 \times 6 \) matrix
\[
A = [a_{\text{FBT}}, a_{\text{XBI}}, a_{\text{UIP}}, a_{\text{UIP2}}, a_{\text{UIP3}}, a_{\text{ZNS}}]
\]
of rank 2. Finally, we put \( a_{\text{CRP}} \) back into \( A \),
\[
A = [a_{\text{FBT}}, a_{\text{XBI}}, a_{\text{UIP}}, a_{\text{UIP2}}, a_{\text{UIP3}}, a_{\text{ZNS}}, a_{\text{CRP}}],
\] (5.2)
and, since \( a_{\text{CRP}} \) is independent of the other six columns of \( A \), the rank of \( A \) increases to 3.

When the lines
\[
\text{%% get asset moments from adjusted closing prices}
\text{ptf = Portfolio;}
\text{ptf = estimateAssetMoments(ptf, A, 'dataformat', 'prices');}
\text{[mn, cv] = getAssetMoments(ptf);}
\]
of `hn2mv1a.m` are replaced with the lines
\[
\text{%% get daily changes in year to date return}
R_0 = \text{diff}(A / 100); \quad \% \text{divide by 100 so that } A(1, :) == 1
\text{%% get (annualized) asset moments}
E_0 = \text{sum}(R_0) \quad \% \text{total return}
\text{Sig_0 = sqrt(252) * std(R_0, 1); \quad \% standard deviation of return}
\text{V_0 = 252 * cov(R_0, 1); \quad \% covariance of return}
\]
in `hn2mv1L.m`, we arrive at the mean-variance results

| Table 5.1: Annualized results from the linear model |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | FBT             | XBI             | UIP             | UIP2            | UIP3            | ZNS             | CRP             |
| \( E^0 \)     | 0.4754          | 0.4497          | 0.4690          | 0.4626          | 0.4562          | 0.4820          | 0.4732          |
| \( \sigma^0 \)| 0.3205          | 0.4050          | 0.3323          | 0.3510          | 0.3756          | 0.3163          | 0.3334          |
| covariance \( V^0 \)
| FBT             | XBI             | UIP             | UIP2            | UIP3            | ZNS             | CRP             |
| 0.1027          | 0.1131          | 0.1053          | 0.1079          | 0.1079          | 0.1105          | 0.1001          | 0.1056          |
| 0.1131          | 0.1641          | 0.1258          | 0.1386          | 0.1513          | 0.1001          | 0.1264          | 0.1108          |
| 0.1053          | 0.1258          | 0.1104          | 0.1156          | 0.1207          | 0.1001          | 0.1108          | 0.1160          |
| 0.1079          | 0.1386          | 0.1156          | 0.1232          | 0.1309          | 0.1001          | 0.1160          | 0.1212          |
| 0.1105          | 0.1513          | 0.1207          | 0.1309          | 0.1411          | 0.1001          | 0.1212          | 0.1003          |
| 0.1001          | 0.1001          | 0.1001          | 0.1001          | 0.1001          | 0.1001          | 0.1001          | 0.1003          |
| 0.1056          | 0.1264          | 0.1108          | 0.1160          | 0.1212          | 0.1003          | 0.1112          |

The covariance matrix \( V^0 \) has rank three, but the upper-left \( 6 \times 6 \) block has rank only two, since the four unattended portfolios UIP through ZNS are affine combinations of the two funds FBT and XBI. Moreover, the corresponding portion of the total return matrix \( E^0 \) mirrors these affine combinations (in contrast to the confusing order of the five mean return values in the \( E \) of Table 3.1).
Figure 5.1 shows the obtainable \((e^0, \sigma^0)\) corresponding to Table 5.1. This nonlinear triangle is the \((e^0, \sigma^0)\)-image of the triangle in \(\mathbb{R}^{252}\) with vertices \(r^0_{\text{XBI}}, r^0_{\text{ZNS}}, \text{ and } r^0_{\text{CRP}}\).

Figure 5.1: Mean-variance analysis – the linear model part 1

\[
Note. \text{ We mentioned the theory of interest in the last section and the relationship between discount and return. To continue this discussion let } a_i (i = 0, 1, \ldots, n) \text{ be adjusted closing prices of a given security over } n \text{ successive investment periods. Then the total return, } e^0, \text{ and the total discount, } e^1, \text{ of the security over this time interval are given by}
\]

\[
e^0 = \sum_{i=1}^{n} (a_i - a_{i-1})/a_0 = (a_n - a_0)/a_0 \quad \text{and} \quad e^1 = \sum_{i=1}^{n} (a_i - a_{i-1})/a_n = (a_n - a_0)/a_n.
\]

Thus \(e^0\) and \(e^1\) conform to the return-discount requirement of the theory of interest,

\[
(1 + e^0)(1 - e^1) = \frac{a_n a_0}{a_0 a_n} = 1,
\]

but a corresponding summand pair only conforms by accident.
5.1 The linear model – part 2

The covariance matrix $V^0$ of Table 5.1 is the Gram matrix, $V^0 = (Z^0)^T Z^0$, of the $252 \times 7$ risk vector matrix

$$Z^0 = R^0 - \mathbf{1}_{252} (E^0 / 252),$$

or

$$Z_0 = R_0 - \text{ones}(252, 1) \ast (E_0 / 252)$$

in MATLAB code. The columns of $Z^0$ represent pure risk in that the sum of each column is zero (= zero total return).

Table 5.1 and Figure 5.1 are summaries of the 2014 adjusted closing price histories of FBT, XBI, UIP, UIP2, UIP3, ZNS, and CRP. $E^0$ and $Z^0$ are the complete histories split into their return and risk parts.

For example, let $e^0$ and $z^0$ be the total return and risk vector of any one of the seven funds in Table 5.1. Then Algorithm 5.1 will compute the the normalized 2014 adjusted closing price history of this fund (starting from $100 at the close of 2013-12-31).

Algorithm 5.1: To compute the adjusted closing price vector $a$ from $e^0$ and $z^0$

$$a^0 = \begin{bmatrix} z^0 + (e^0 / 252) \mathbf{1}_{252}; \\
1; \\
\vdots; \\
0 \\
\end{bmatrix}$$

Our MATLAB code that illustrates this linear model section is organized into four scripts.

- hn2mv1L.m – compute the seven fund mean-variance table $E^0, V^0$ for the linear model
- hn2mv1L1.m – construct (and save) the orthogonal $U, E^0, Z^0$ system ($UEZ2014.mat$)
- hn2mv1L2.m – generate an adjusted closing price history $A$ from the orthogonal system
- hn2mv1L3.m – construct the seven fund $E^0, Z^0$ table corresponding to the orthogonal system

We have already described the first script, hn2mv1L.m. The second script, hn2mv1L1.m, takes the risk matrix $Z^0$ (5.3) apart orthogonally,

$$Z^0 = U \tilde{Z}^0, \quad \tilde{Z}^0 = U^T Z^0, \quad U^T U = I, \quad U = [u_x, u_y, u_z] \in \mathbb{R}^{252 \times 3},$$

where $U$ is defined by

$$\begin{align*}
v_x &= z^0_{\text{FBT}} - z^0_{\text{XBI}}, & u_x &= v_x / \|v_x\|, \\
v_y &= z^0_{\text{FBT}} - u_x(z^0_{\text{FBT}}), & u_y &= v_y / \|v_y\|, \\
v_z &= z^0_{\text{CRP}} - [u_x, u_y] ([u_x, u_y]^T z^0_{\text{CRP}}), & u_z &= v_z / \|v_z\|,
\end{align*}$$

and the resulting $\tilde{Z}^0$ is

$$\tilde{Z}^0 = \begin{bmatrix}
-0.0514 & -0.2530 & -0.1018 & -0.1522 & -0.2026 & 0 & -0.1030 \\
0.3163 & 0.3163 & 0.3163 & 0.3163 & 0.3163 & 0.3163 & 0.3171 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0024
\end{bmatrix}$$
The orthonormal matrix $U$ and the FBT, XBI, ZSN, CRP columns (1, 2, 6, 7) of $E_0$ (Table 5.1) and $Z_0$ (5.6) are saved as U, E_0, and Z_0 in the MATLAB file UEZ2014.mat. MATLAB Example 2 verifies the contents this file.

MATLAB Example 2: Linear mv-analysis – check UEZ2014.mat

```matlab
%% hn2mv1L1check.m – Linear mv-analysis
% Check UEZ2014.mat

load UEZ2014.mat; % U E_0 Z_0 dates funds legend

legend =
10 x 60 char array
 'U: 252 x 3 orthonormal matrix of risk vectors u: sum(u) = 0'
 'E_0: 1 x 4 matrix of total 2014 returns'
 'Z_0: 3 x 4 matrix of risk vector coordinates'
 'dates: 252 x 10 string array of 2014 market days'
 'funds: 4 x 3 string array of fund symbols'
 ' FBT - First Trust Biotechnology Index Fund'
 ' XBI - SPDR S&P Biotech ETF'
 ' ZNS - Unattended long-short FBT-XBI portfolio'
 ' CRP - Continually reallocated FBT-XBI portfolio'
 'legend: the above description

%% risk and orthogonality of U
%
mean(U) = 1.0e-15 * [0.0039 -0.0007 0.4282] % all risk
norm(U' * U - eye(3)) = 1.7697e-14 % orthonormal
%
%% display E_0, Z_0, Sig_0, and V_0
%
E_0 =
 0.4754 0.4497 0.4820 0.4732
Z_0 =
-0.0514 -0.2530 0 -0.1030
 0.3163 0.3163 0.3163 0.3171
 0 0 0 0.0024
Sig_0 = sqrt(sum(Z_0 .^ 2)) =
 0.3205 0.4050 0.3163 0.3334
V_0 = Z_0' * Z_0 =
 0.1027 0.1131 0.1001 0.1056
 0.1131 0.1641 0.1001 0.1264
 0.1001 0.1001 0.1001 0.1003
 0.1056 0.1264 0.1003 0.1112
%
```

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Table 5.2 below contains all of the information in Table 5.1 in a more compact, geometric form. The computation $V^0 = (Z^0)^T Z^0$ reproduces the $V^0$ of Table 5.1.

Table 5.2: Annualized results from the linear model – part 2

|       | FBT | XBI | UIP | UIP2 | UIP3 | ZNS | CRP |
|-------|-----|-----|-----|------|------|-----|-----|
| $E^0$ | 0.4754 | 0.4497 | 0.4690 | 0.4626 | 0.4562 | 0.4820 | 0.4732 |
| $\sigma^0$ | 0.3205 | 0.4050 | 0.3323 | 0.3510 | 0.3756 | 0.3163 | 0.3334 |
| $Z^0$ | -0.0514 | -0.2530 | -0.1018 | -0.1522 | -0.2026 | 0 | -0.1030 |
|       | 0.3163 | 0.3163 | 0.3163 | 0.3163 | 0.3163 | 0.3163 | 0.3171 |
|       | 0     | 0     | 0     | 0     | 0     | 0     | 0.0024 |

The $E^0$ and $\sigma^0$ rows of Table 5.2 are exactly the same as those of Table 5.1, but the $Z^0$ of Table 5.2 replaces the $V^0 = (Z^0)^T Z^0$ of Table 5.1 and, in each column, $\sigma^0 = ||z||$.

Figure 5.2 shows the $xy$-plane in the risk hyperplane $\{z \in \mathbb{R}^{252} : \text{mean}(z) = 0\}$. It exactly reflects the $Z^0$ data in Table 5.2. Of course $z^0_{\text{CRP}}$ is not in $xy$-plane as evidenced by its nonzero $z$-coordinate and the fact that its projection onto the $xy$-plane is linearly incompatible with its total 2014 return, $e^0_{\text{CRP}}$.

The illustrative unattended portfolio $p = 0.39946 \times \text{FBT} + 0.60054 \times \text{XBI}$ (at 2013-12-31 closing prices) in Figure 5.1 and 5.2 had total 2014 return $e^0_p = 0.4600$ with $x_p = -0.1725$ and $\sigma^0_p = 0.3603$. 
6 History: adjusted closing prices revisited

Let us close this article with a revised version of Figure 2.1. The revision, Figure 6.1, shows the same adjusted-closing-price history of the exchange traded funds FBT and XBI and the continually reallocated portfolio CRP, but now the unattended long-short portfolio, ZNS, has replaced UIP. Of the four funds and portfolios, ZNS (purple) had the highest 2014 return with the least volatility.

These normalized adjusted closing prices were generated from UEZ2014.mat by the MATLAB script hn2mv1L3.m. They are recorded, to 5-decimal places (along with the prices of UIP, UIP2, and UIP3), in the comma-separated-value file FXUZC7.csv.

Figure 6.1: 2013-12-31-normalized adjusted closing prices of two biotechnology ETFs and two portfolios in these ETFs

FBT = First Trust Biotechnology Index ETF
XBI = SPDR S&P Biotech ETF
ZNS = Unattended long-short FBT-XBI portfolio
CRP = Continually reallocated FBT-XBI portfolio
Now the MATLAB script `hn2mv1b.m`, with the A of 5.2 in the
```matlab
tp = estimateAssetMoments(ptf, A, 'dataformat', 'prices');
```
code produces

Table 6.1: Annualized results from the traditional nonlinear model

|        | FBT   | XBI   | UIP   | UIP2  | UIP3  | ZNS   | CRP   |
|--------|-------|-------|-------|-------|-------|-------|-------|
| $\mu$  | 0.4245| 0.4324| 0.4235| 0.4244| 0.4274| 0.4275| 0.4265|
| $\sigma$| 0.2656| 0.3491| 0.2777| 0.2963| 0.3203| 0.2605| 0.2783|

covariance $V$

|        | FBT   | XBI   | UIP   | UIP2  | UIP3  | ZNS   | CRP   |
|--------|-------|-------|-------|-------|-------|-------|-------|
| $V_{FBT}$ | 0.0705| 0.0804| 0.0729| 0.0753| 0.0778| 0.0682| 0.0730|
| $V_{XBI}$  | 0.0804| 0.1219| 0.0905| 0.1008| 0.1112| 0.0703| 0.0908|
| $V_{UIP}$   | 0.0729| 0.0905| 0.0771| 0.0815| 0.0859| 0.0686| 0.0773|
| $V_{UIP2}$  | 0.0753| 0.1008| 0.0815| 0.0878| 0.0942| 0.0691| 0.0817|
| $V_{UIP3}$  | 0.0778| 0.1112| 0.0859| 0.0942| 0.1026| 0.0697| 0.0862|
| $V_{ZNS}$   | 0.0682| 0.0703| 0.0686| 0.0691| 0.0697| 0.0678| 0.0687|
| $V_{CRP}$   | 0.0730| 0.0908| 0.0773| 0.0817| 0.0862| 0.0687| 0.0774|

with the corresponding TikZ image

Figure 6.2: How not to do MV-analysis – a last look

Here the continually-reallocated black-dotted portfolio points are exactly 1/2 and 3/4 of the $e$-way from FBT to XBI on the red, continually-reallocated, FBT-to-XBI path.
7 Conclusion

The growth in value of an unattended investment portfolio $P$ over a given interval of time can be completely described by a normalized adjusted closing price equation

$$a_P = \sum_j p_j a_j,$$  \hspace{1cm} (7.1)

where the $p_j$ are the proportions of the securities in the portfolio $P$ at the close of the day of normalization. The corresponding mean periodic return equation,

$$e_P = \sum_j p_j e_j,$$  \hspace{1cm} (7.2)

does not follow when $e = \text{mean}(r)$ and periodic return vectors $r$ are defined by

$$r_i = a_i / a_{i-1} - 1.$$  \hspace{1cm} (7.3)

The mean periodic return equation (7.2) does hold with (7.3) when one restricts his attention to continually reallocated portfolios. Unfortunately continually reallocated investment portfolios are more numerical artifact than financial reality.
A An adjusted closing price primer

The adjusted closing prices of a security are artificial “closing prices” that are adjusted to incorporate all dividends and splits. The day-to-day growth of a security or an unattended investment portfolio of securities is completely described by its adjusted closing prices. If the adjusted closing price of the security/portfolio is \( a_0 \) on market day 0 and \( a_1 \) on a later market day 1, then its total return from day 0 to day 1 is \( r = \Delta a/a_0 \) (\( \Delta a = a_1 - a_0 \)). Two adjusted closing price vectors for a given security that cover the same time interval must be positive scalar multiples of each other. Thus the returns, \( r \), of the security from one market day to another do not depend on any particular adjusted closing price representation.

Table A.1 shows how one can compute adjusted closing prices for the exchange traded fund XBI over the period 2013-12-31 through 2014-12-31. The required input data are all closing prices for the fund over this period as well as the dividends it made during the period with their ex-dividend dates. On each line the adjusted closing price is computed by

\[
\text{adjusted closing price} = \text{closing price} \times \text{adjusted closing shares}.
\]

The adjusted closing shares in the table increase on each ex-dividend day and are constant in between. If the closing price on the market day prior to an ex-dividend day is \( c_0 \) and the dividend on the ex-dividend day is \( d_1 \), then the adjusted closing shares on the ex-dividend day must be increased by a factor of \( c_0/(c_0 - d_1) \) in order that an investor who has his dividends reinvested maintains the value of his investment.

The adjusted closing prices in Table A.1 are “normalized” at 100.000 on 2013-12-31. To compute the adjusted closing prices for the 243 missing days just fill in the missing closing prices and multiply them by the corresponding adjusted closing shares. Also note that these closing prices and distributions have not been adjusted for the 3:1 split in 2015.

Table A.1: How to generate normalized adjusted closing prices for XBI – SPDR S&P Biotech ETF

| market day | distribution | closing price | adjusted closing price | adjusted closing shares |
|------------|--------------|---------------|------------------------|-------------------------|
| 2013-12-31 |              | 130.20        | 0.768049               | 100.000                 |
| ...        |              | 160.17        | 0.768049               | 123.018                 |
| 2014-03-20 |              | 0.333023      | 153.15                 | 0.769649               |
| 2014-03-21 |              | 0.616142      | 153.42                 | 0.772754               |
| 2014-06-19 |              | 0.562774      | 158.27                 | 0.775483               |
| 2014-09-18 |              | 159.94        | 0.772754               | 123.594                 |
| 2014-09-19 |              | 0.490997      | 190.34                 | 0.777502               |
| 2014-12-18 |              | 189.08        | 0.777502               | 146.628                 |
| 2014-12-19 |              | 0.490997      | 190.34                 | 0.777502               |
| 2014-12-31 |              | 186.46        | 0.777502               | 144.973                 |

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A.1 Yahoo!Finance

Yahoo!Finance is a good source for adjusted closing prices of an individual security, like our XBI. Simply download the daily, historical prices over the time interval desired as a CSV (comma-separated-value) file and open the file in a spreadsheet program (e.g., Excel).

This spreadsheet will have seven labeled columns:

Date, Open, High, Low, Close, Adj Close, Volume.

Delete all but the “Date” and the “Adj Close” columns. We will assume these are now columns A and B, respectively, as in the spreadsheet image (Figure A.1) below.

The Yahoo adjusted closing prices are not normalized at any particular date. Yahoo simply sets the adjusted closing price of a security at the close of the latest market day equal to its closing price on that day. Then previous adjusted closing prices must be rescaled if the day is an ex-distribution or an ex-split day.

To normalize your Yahoo adjusted prices at say 100 on a particular date (i.e., 2013-12-31 in Figure A.1) start a new normalized adjusted closing price column on your spreadsheet, say column C, by putting

$$= B[\text{date row}] * 100 / B\$[\text{date row}]$$  \hspace{1cm} (A.1)

in the cell of that date. The number 100 should appear in this cell (i.e., in cell C2). Now you need only fill up and/or down from this cell to get all normalized adjusted closing prices in column C. (We only filled down in in Figure A.1.)

Figure A.1: XBI data from Yahoo on 2018-09-19

| A     | B           | C     | D  | E  | F   |
|-------|-------------|-------|----|----|-----|
| 1     | Date        | Adj Close | A_0 | XBI | error |
| 2     | 2013-12-31  | 42.348824 | 100.000 | 100.000 | 0.000 |
| 3     | 2014-01-02  | 42.556995 | 100.492 | 100.492 | 0.000 |
| 4     | 2014-01-03  | 42.420387 | 100.169 | 100.169 | 0.000 |
| 5     | 2014-01-06  | 41.932495 | 99.017 | 99.017 | 0.000 |
| 6     | 2014-01-07  | 42.940807 | 101.398 | 101.398 | 0.000 |
| 250   | 2014-12-24  | 60.130085 | 141.988 | 141.988 | 0.000 |
| 251   | 2014-12-26  | 61.516277 | 145.261 | 145.261 | 0.000 |
| 252   | 2014-12-29  | 61.740173 | 145.790 | 145.790 | 0.000 |
| 253   | 2014-12-30  | 61.091537 | 144.258 | 144.258 | 0.000 |
| 254   | 2014-12-31  | 61.394451 | 144.973 | 144.973 | 0.000 |
| 255   |             |         |      |     | 0.007 |

The normalized adjusted closing prices in column C of Figure A.1 were generated from the prices in column B as described above. These prices were then rounded to 3 decimal places. The numbers in the D column come from our anc/FXUZ7.csv file. They were generated by the process used to generate Table A.1 from the closing prices and distributions of XBI. Out of the 253 market days considered, the C (A_0) price was 0.001 greater than the D (XBI) price on 7 days. Otherwise the two columns of adjusted closing prices were exactly the same. (These 7 “errors” occurred in the 243 rows that have been collapsed in Figure A.1.)
B  Life on the Mississippi

In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upwards of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.

– Mark Twain
References

Kellison, Stephen G. (2009). *The Theory of Interest, 3rd Edition*. McGraw-Hill.
Markowitz, Harry M. (1987). *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Blackwell.
Swensen, David F. (2009). *Pioneering Portfolio Management*. FREE PRESS.