LOCAL WHITTAKER-NEWFORMS FOR $GSp(4)$ MATCHING TO LANGLANDS PARAMETERS

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Dedicated to Professor Tomoyoshi Ibukiyama on his 70th birthday

Abstract. We extend the local newform theory of B. Roberts and R. Schmidt for generic, irreducible, admissible representations of $PGSp(4)$ to that for $GSp(4)$. The newform matches to the Langlands parameter.

1. Introduction

Let $F$ be a non-archimedean local field of characteristic 0 and residue characteristic $p$. Let $WD_F$ be the Weil-Deligne group. Let $\phi : WD_F \to GSp(4, \mathbb{C})$ be a $L$-parameter. The local Langlands correspondence for $GSp(4)$ showed by W. T. Gan and S. Takeda [G-T] says that, if $\phi$ is tempered, the $L$-packet of $\phi$ contains a unique generic, irreducible, admissible representation $\pi$ whose $L$- and $\varepsilon$-factors defined by F. Shahidi [Sh] coincide with those of $\phi$ respectively. In the context of noncommutative class field theory, and Shimura type conjectures, for example, Yoshida-Brumer-Kramer conjecture [Y], [B-K] on Abelian surfaces (see also [O-Y] for Siegel threefold varieties), it is natural to quest which vector in $\pi$ possesses the $L$- and $\varepsilon$-factors of $\phi$, and by which subgroup the vector is fixed. For the generic $GL(d)$-case, the answer can be found in the series of the works of H. Jacquet, I. I. Piatetski-Shapiro, J. A. Shalika, and the subsequent works of S. Kondo, S. Yasuda [K-Y], N. Matringe [Ma], and M. Miyauuchi [Mi]. For the generic $PGSp(4)$-case, the answer was provided by B. Roberts and R. Schmidt [R-S] for nondiscrete $L$-parameters (they provided also for some non-generic cases). The ‘paramodular group’ corresponding to the $L$-parameter is the fixing subgroup. After these works, in this paper, we will provide the following answer for the generic $GSp(4)$-case. Let $\mathcal{O}$ be the ring of integers of $F$ and $\mathcal{O} = \mathcal{O}^{\mathfrak{P}}$ be its maximal ideal with a fixed generator $\mathfrak{p}$. Let $q = |\mathcal{O} / \mathfrak{P}| = |\mathfrak{p}|^{-1}$. Let

$$
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
1 & 1
\end{pmatrix}
$$

be the defining matrix for $GSp(4)$. Let $\psi$ be a nontrivial additive character of $F$ such that $\ker(\psi) = \mathcal{O}$. Let $\pi$ be a generic, irreducible, admissible representation of $GSp(4, F)$, and $W_\psi(\pi)$ denote the representation space of consisting of (Whittaker) functions $W$ such that

$$W(\\begin{pmatrix} 1 & -x & * & * \\
1 & y & * & x \\
1 & 1 & 1 \\
\end{pmatrix} g) = \psi(y + x)W(g).$$

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Let $\omega_{\pi}$ be the central character of $\pi$, and $\varepsilon$ its (order of) conductor. For an integer $m \geq 2\varepsilon$, we define $K(m; \varepsilon)$ to be the subgroup of all $k \in GSp(4, F)$ such that $\det(k) \in \mathcal{O}^\times$ and

$$k \in \begin{bmatrix}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{P}^{-l} \\
\mathcal{P}^l & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P}^l & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P}^m & \mathcal{P}^l & \mathcal{P}^l & \mathcal{O}
\end{bmatrix},$$

where $l = m - \varepsilon$. Define $K_1(m; \varepsilon) = \{k \in K(m; \varepsilon) \mid k_{44} \in 1 + \mathcal{P}^\varepsilon\}$. We call these open compact subgroups the quasi-paramodular groups of level $m$. They are contained in the paramodular group $K(m + 1; \varepsilon)$ of level $m + 1 - \varepsilon$. In case of $\varepsilon = 0$, they coincide with $K(m)$. We call $K_1(m; \varepsilon)$-invariant Whittaker functions quasi-paramodular forms of level $m$, including the case of $\varepsilon = 0$. Let $V(m) \subset \mathcal{W}_\psi(\pi)$ denote the subspace consisting of quasi-paramodular forms of level $m$. Observe that if $W \in V(m)$, then $\pi(k)W = \omega_{\pi}(k_{44})W$ for $k \in K(m; \varepsilon)$. Although $K_1(m + 1; \varepsilon) \not\subset K_1(m; \varepsilon)$, there exists an inclusion map $V(m) \hookrightarrow V(m + 1)$. The minimal integer $m$ such that $V(m) \neq \{0\}$ is called the minimal level of $\pi$, and denoted by $m_{\pi}$.

**Main Theorem.** Let $\pi$ be a generic, irreducible, admissible representation of $GSp(4, F)$ with $L$-parameter $\phi_{\pi}$. Fix a nontrivial additive character $\psi$ such that $\ker(\psi) = \mathcal{O}$. Write $\varepsilon(s, \phi_{\pi}, \psi) = \varepsilon_{\pi}q^{-n_{\pi}(s - \frac{1}{2})}$. Then, $m_{\pi} = n_{\pi}$, and $V(m_{\pi})$ is one-dimensional. There exists a unique $W$ in $V(m_{\pi})$ such that

\[(1.2) \quad \int_{F^\times} W(t) \begin{bmatrix} t & & & \varepsilon(t) \\ & t & & \\ & & 1 & \\ & & & 1 \end{bmatrix} |t|^{s - \frac{3}{2}} d^\times t = L(s, \phi_{\pi}),\]

\[(1.3) \quad \int_{F^\times} W(t) \begin{bmatrix} t & & & \varepsilon(t) \\ & t & & \\ & & -t & \\ & & & -t \end{bmatrix} \omega_{\pi}(t)^{-1} |t|^{s - \frac{3}{2}} d^\times t = q' \varepsilon_{\pi} L(s, \phi_{\pi}'),\]

where $d^\times t$ is the Haar measure such that $\text{vol}(\mathcal{O}^\times) = 1$.

The zeta integral (1.2) coincides with Novodovsky’s $Z(s, W)$ ([N]), if $W \in \mathcal{W}_\psi(\pi)$ is quasi-paramodular (Proposition 5.1). As well as in the $PGSp(4, F)$-case, for a tempered representation of $GSp(4, F)$, the genericity is equivalent to the quasi-paramodularity (Theorem 6.10). We now describe our method.

i) We show that if there exists a $W \in V(m)$ satisfying the equalities (1.2), and (1.3) up to a constant multiple, then $m = m_{\pi}$, and $V(m)$ is spanned by this $W$ (Theorem 5.12). Comparing with our functional equation (Theorem 4.3), we find that the existence of such a $W$ means that $m_{\pi}$ equals $n_{\pi}'$, the analytic conductor, and that (1.3) with replacing $\varepsilon_{\pi}$ by $\varepsilon_{\pi}'$, the analytic root number, holds exactly. Here, the functional equation is a modified version of Novodovsky’s [N], and $\varepsilon_{\pi}', n_{\pi}'$ are defined by the $\varepsilon$-factor (4.5). See also the remark in p.82 of [R-S]. Following the idea of B. Roberts and R. Schmidt, we use the $P_3$-representation theory (sect. 3), to prove the functional equation, and Theorem 5.4 that says a quasi-paramodular form vanishing at all diagonal matrices is identically zero. Theorem 5.12 comes from Theorem 5.4.

ii) To show the existence of $W$ as in Theorem 5.12, in sect. 6, we analyze Hecke actions on $V(m_{\pi})$ when $\pi$ is supercuspidal, or when $\pi$ is a constituent of the induction of a supercuspidal representation of the Levi factor of the Klingen parabolic subgroup.
Since the $L$-function defined by $[N]$ of $\pi$ equals 1 in this case, the Kirillov models corresponding to the quasi-paramodular forms have compact supports (Lemma 6.2). This causes the analysis simple, and makes possible to determine all values at diagonal matrices of $W \in V(m_\pi)$ (Theorem 6.1).

iii) For other generic constituents of parabolic inductions, we use the local $\theta$-lift from $GL(2) \times GL(2)$ to $GSp(4)$. It is known by [G-T2] that such constituents are obtained by the $\theta$-lift. In sect. 7, the desired $W$ is constructed explicitly by the $\theta$-lift.

iv) W. T. Gan and S. Takeda [G-T] showed the Langlands correspondence for $GSp(4)$ by observing the local $\theta$-lift from $GSP(4)$ to $GL(4)$, and reducing to that for $GL(4)$ due to M. Harris and R. Taylor [H-R], and G. Henniart [H]. Following this line, in sect. 8, by the $\theta$-lift we construct the newform for $GL(4)$. It matches to $\phi_\pi$, thanks to the newform theory for $GL(d)$ (sect. 2). Seeing that it is constructed by the above $W \in V(m_\pi)$, we obtain the coincidences $\varepsilon'_\pi = \varepsilon_\pi$, and $n'_\pi = n_\pi$.

In the case of $\epsilon > 0$, an elementary argument shows that, if $\pi(k)W = \chi(k)W$ for a quasi-character $\chi$ on a paramodular group, then $Z(s,W) = 0$, different from the case of $GL(d)$. In the case of $\epsilon > 0$, the quasi-paramodular group is not normalized by the Weyl element $j_m$ (c.f (4.3)), and therefore $V(m)$ is not decomposed by the Atkin-Lehner operator defined by $j_m$, different from the case of $\epsilon = 0$. We also consider the $j_m$-conjugate of quasi-paramodular forms, which are called coquasi-paramodular forms.

**Notation** Let $F$ be a non-archimedean local field of characteristic 0, and residue characteristic $p$. Let $\mathcal{O}$ be the ring of integers of $F$ and $\mathcal{P} = \mathfrak{p}\mathcal{O}$ be its maximal ideal with a fixed generator $\mathfrak{p}$. Let $\mathcal{P}^* = \mathcal{P} \setminus \mathcal{P}^2$. Let $q = \mid \mathcal{O}/\mathcal{P} \mid = \mid \mathfrak{p} \mid^{-1}$. Let $\mathfrak{o}(x)$ denote the $p$-adic order of $x \in F$, and let $\nu_s(x) = q^{-\mathfrak{o}(x)s}$ for $s \in \mathbb{C}$. Let $\psi$ denote a nontrivial additive character on $F$. We sometimes assume $\ker(\psi) = \mathcal{O}$. If $G$ is a locally compact totally disconnected group (called an $l$-group), then we let $\text{Alg}(G)$ (resp. $\text{Irr}(G)$) denote the category of smooth (resp. irreducible admissible) complex $G$-modules. Let $\mathcal{X}(G)$ denote the subcategory of $\text{Irr}(G)$ consisting of one-dimensional ones. For $\chi \in \mathcal{X}(F^\times)$, let $c(\chi)$ denote the order of the conductor of $\chi$. If $\pi \in \text{Alg}(G)$, then $\pi'$ denotes the contragredient to $\pi$. Let $L$ and $R$ denote the left and right translations of elements in $G$ on itself, respectively: $L(g)g' = g^{-1}g', R(g)g' = gg'$.

### 2. Newforms for $GL(d)$

We review the newform theory for a generic representation of $GL(d,F)$. We will use the following notation for elements and subgroups of $G_d = GL(d,F)$:

\[
\begin{align*}
N &= \{ n = (n_{ij}) \mid n_{ij} = 0 \text{ for } i > j, \ n_{ii} = 1 \}, \\
\bar{N} &= \{ \bar{n} = \text{the transposition of } n \in N \}, \\
K(m) &= \{ k \in G_d(\mathcal{O}) \mid k_{d1}, \cdots, k_{dd-1} \in \mathcal{P}^m \}, \\
K_1(m) &= \{ k \in K(m) \mid k_{dd} \equiv 1 \pmod{\mathcal{P}^m} \}, \\
A &= \{ a(t) = \begin{bmatrix} t & \ 1_{d-1} \end{bmatrix} \mid t \in F^\times \}, \\
a_i &= a(\mathfrak{p}^i), \\
w_d &= \text{the standard longest Weyl element in } G_d, \\
w_{1,d-1} &= \begin{bmatrix} 1 & w_{d-1} \end{bmatrix}.
\end{align*}
\]
In case of \( r < d \), for an element \( h \in G_r \), let
\[
h' = \begin{bmatrix} h & 0 \\ 0 & 1_{d-r} \end{bmatrix} \in G_d.
\]
For \( h \in G_d \) and \( b \in M_{d \times d}(F) \), let
\[
j(h) = \begin{bmatrix} h & -h^{-1} \\ 0 & 1 \end{bmatrix}, \quad n(b) = \begin{bmatrix} 1_d & b \\ 0 & 1_d \end{bmatrix}, \quad \bar{n}(b) = \begin{bmatrix} 1_d & b \\ cb & 1_d \end{bmatrix} \in SL(2d, F).
\]
The following identities are basic.

\[
\begin{align*}
\bar{n}(h) &= n(h^{-1})j(h)n(h^{-1}), \\
\text{Int}(\bar{n}(c))n(b) &= \begin{bmatrix} 1_d - bc & b \\ cb & 1_d + cb \end{bmatrix}. 
\end{align*}
\]

Let \( \mathcal{W}_\psi = \text{Ind}_{N}^{G_d} \tilde{\psi} \) denote the induced representation consisting of smooth functions \( W : G_d \to \mathbb{C} \) (called Whittaker functions with respect to \( \psi \)) such that \( L(n)W = \tilde{\psi}(n)^{-1}W \) for \( n \in N \), where \( \tilde{\psi} \in \mathcal{X}(N) \) is defined by \( \tilde{\psi}(n) = \prod_{1 \leq i < d-1} \psi(n_{i,i+1}) \). We denote by \( \text{Irr}^{\lambda}(G_d) \) the subcategory consisting of \( \pi \) such that \( \text{Hom}_{G_d}(\pi, \mathcal{W}_\psi) \neq \{0\} \). If \( (\pi, V) \in \text{Irr}^{\lambda}(G_d) \), then \( \text{Hom}_{G_d}(\pi, \mathcal{W}_\psi) = \mathbb{C} \lambda \) for a functional \( \lambda \), unique up to constant multiples, and we identify \( V \) with \( \mathcal{W}_\psi(\pi) := \text{Im}(\lambda) \). Let \( W \in \mathcal{W}_\psi \). For a nonnegative integer \( r \leq d - 2 \), let
\[
Z_r(s, W) = \int \int W(t) dx d^x t
\]
with the integration being over \( t \) in \( F^x \) and \( x \) in the column space \( F^r \), where the Haar measures \( dx \) and \( d^x t \) are chosen so that \( \text{vol}(\mathcal{O}^x) = 1 \) and \( \text{vol}(\mathcal{O}^r) = 1 \) respectively. Let \( \pi \in \text{Irr}^{\lambda}(G_d) \). Let \( L(s, \pi) \) and \( \varepsilon(s, \pi, \psi) \) denote the \( L \)- and \( \varepsilon \)-factors respectively defined in [G-J], which coincide with those of the Rankin-Selberg convolution \( \pi \times 1 \) defined in [J-PS-S3] (c.f. sect. 4 of [J-PS-S2]), where \( \text{Ind} \) indicates the trivial quasi-character of \( G_1 = F^\times \). By the works of M. Harris and R. Taylor [H-R], and G. Henniart [H], these factors also coincide with those of the \( L \)-parameter \( \phi_\pi : WD_F \to GL(d, \mathbb{C}) \), respectively. Define \( W^1 \in \mathcal{W}_{\psi^{-1}} \) by \( W^1(g) = W(w^1 g^{-1}, W_{\psi^{-1}}) \). The \( G_d \)-module \( \pi^1 = \{ W^1 | W \in \mathcal{W}_\psi(\pi) \} \) is equivalent to \( \pi^\nu \) (c.f. [G-Ka]). The functional equation for \( \pi \times 1 \) given in [J-PS-S3] is
\[
(2.3) \quad \frac{Z_0(1-s, W^1)}{L(1-s, \pi^1)} = \varepsilon(s, \pi, \psi) \frac{Z_{d-2}(s, W)}{L(s, \pi)}. 
\]
It holds that \( \varepsilon(s, \pi, \psi) \varepsilon(1-s, \pi^1, \psi^{-1}) = 1 \), and that \( \varepsilon(s, \pi, \psi) = \varepsilon(s, \pi, \psi') \) if \( \ker(\psi) = \ker(\psi') \). Now fix a \( \psi \) such that
\[
\ker(\psi) = \mathcal{O}.
\]
The root number \( \varepsilon_\pi \) and conductor \( n_\pi \) are defined by \( \varepsilon(s, \pi, \psi) = \varepsilon_\pi q^{-n_\pi(s - \frac{1}{2})} \). Let \( V(m) \) denote the subspace consisting of \( K_1(m) \)-invariant vectors in \( \mathcal{W}_\psi(\pi) \). Let \( \omega_\pi \) denote the central character of \( \pi \), and \( \epsilon = \epsilon(\omega_\pi) \) its (order of) conductor. Since \( K(m)/K_1(m) \cong \mathcal{O}^x/(1 + \mathcal{O}^m) \),
\[
(2.4) \quad V(m) = \{ W \in \mathcal{W}_\psi(\pi) | \pi(k)W = \omega_\pi(k_{dd})W, \ k \in K(m) \}.
\]
It is obvious that \( \{0\} = \cdots \subset V(\epsilon) \subset \cdots \subset V(\infty) := \cup_m V(m) \). Observe that \( V(\epsilon - 1) = \{0\} \) in case of \( \epsilon > 0 \). The smallest integer \( m \) such that \( V(m) \neq \{0\} \) is called the minimal level of \( \pi \), and denoted by \( m_\pi \). Then, \( V(m_\pi) \) is one-dimensional, and spanned by a \( W \) such that \( W(1_d) = 1 \), which is called the newform of \( \pi \) and denoted by \( W_\pi \)(called the
essential vector in their original paper [J-PS-S]). The following identity was showed in [Ma], [Mi]:

\[(2.5) \quad Z_0(s, W_\pi) = L(s, \pi). \]

However, by the method of Lemma 4.1.1. of [R-S], \(Z_r(s, W)\) are same for all nonnegative \(r \leq d - 2\), if \(W \in V(\infty)\) as is showed below.

**Proposition 2.1.** With notations as above, if \(W \in V(\infty)\), then \(Z_0(s, W) = Z_r(s, W)\) for any \(0 \leq r \leq d - 2\).

**Proof.** It suffices to show that, for \(x = (x_1, \ldots, x_r) \notin \mathcal{O}^r\),

\[(2.6) \quad W(\begin{bmatrix} t \\ x \\ 1 \end{bmatrix}) = 0. \]

Let \(x_l\) be the last element such that \(x_l \notin \mathcal{O}\). Let \(x_l = (x_1, \ldots, x_{l-1})\). By the \(K_1(m)\)-invariance property,

\[W(\begin{bmatrix} 1 \\ x_l \\ 1_l \end{bmatrix}) = W(\begin{bmatrix} 1 \\ x_l \\ 1_l \end{bmatrix}) = W(\begin{bmatrix} 1 \\ x_l \\ 1_l \end{bmatrix}) = W(\begin{bmatrix} 1 \\ x_l \\ 1_l \end{bmatrix}). \]

By (2.1) and the \(K_1(m)\)-invariance property, this equals

\[W(\begin{bmatrix} x_l^{-1} \\ 1_l \\ 1 \end{bmatrix}) = W(\begin{bmatrix} x_l^{-1} \\ 1_l \\ 1 \end{bmatrix}) = W(\begin{bmatrix} x_l^{-1} \\ 1_l \\ 1 \end{bmatrix}). \]

Since \(x_l \notin \mathcal{O}\), there exists a \(y \in \mathcal{O}\) such that \(\psi(x_l y) \neq 1\). Now (2.6) follows from Lemma 2.2 below combined with

\[\text{Int}^{-1}(\begin{bmatrix} x_l^{-1} \\ 1_l \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1_l \\ 1 \end{bmatrix} \in K_1(m). \]

**Lemma 2.2.** Let \(G\) be a group and \(H, K\) be subgroups of \(G\). Let \(\xi : H \to \mathbb{C}^\times\) and \(\chi : K \to \mathbb{C}^\times\) be homomorphisms. Let \(f : G \to \mathbb{C}\) such that \(L(h^{-1})R(k)f = \xi(h)\chi(k)f\) for \(h \in H, k \in K\). Let \(g \in G\). If there exists an \(h \in H\) such that \(\text{Int}^{-1}(g)h \in K\) and \(\xi(h) \neq \chi(\text{Int}^{-1}(g)h)\), then \(f(g) = 0\).

By [K-Y] it was showed that \(n_{\pi} = m_{\pi}\). Taking into account above results, we obtain the following characterization for the newforms.
Theorem 2.3. Let $\pi \in \text{Irr}^{\text{op}}(G_d)$. An integer $m$ equals $m_{\pi}$, if and only if there exists a $W \in V(m)$ such that $Z_0(s, W) = L(s, \pi)$ and $W(w_d a_m w_{1,d-1}) \neq 0$.

**Proof.** We show only the if-part. Let $W' = \pi'(a_m)W'$. By (2.3) and Proposition 2.1,

\[
\frac{Z_0(1 - s, W')}{L(1 - s, \pi')} = \varepsilon q^{(m-m_{\pi})(s-1/2)}.
\]

Since $W'$ is invariant under the subgroup

\[
\{ \eta \chi \mid \eta \in G_2 \},
\]

and $W'(1) = W(w_d a_m w_{1,d-1})) \neq 0$ is assumed, by Lemma 2.2, $Z_0(1 - s, W')$ is a power series in $q^s$ with a nonzero constant term, and so is the left hand side of the above equation (recall $L(s, \pi')^{-1} = L(s, \pi')^{-1}$ is a polynomial in $q^{-s}$ with constant term 1.). However, the right side is a monomial in $q^s$. Hence, both sides are constant, and $m = m_{\pi}$. \(\square\)

From now on, we concentrate on the argument for the case that $d = 2$ and the central character is ramified, which is an archetype for $GSp(4)$, and will be used repeatedly.

**Proposition 2.4.** Let $\pi \in \text{Irr}^{\text{op}}(G_2)$. If $L(s, \pi) = 1$, then $m_{\pi} > c_{(\omega_{\pi})}$.

Let $c = c(\omega_{\pi})$. In case of $c = 0$, the assertion is obvious, since an unramified representation is a principal series representation. Assume $c > 0$. Let $m \geq c$. Consider the Hecke action $\mathcal{T} : V(m) \to V(m)$ defined by

\[
TW := \sum_{x \in \mathcal{O}/\mathcal{P}} \pi(\tilde{n}(x \omega^m)a_{-1})W = \sum_{x \in \mathcal{O}/\mathcal{P}} \pi(a_{-1}\tilde{n}(x \omega^{m-1}))W.
\]

Observe that $\{\tilde{n}(x \omega^m) \mid x \in \mathcal{O}/\mathcal{P}\}$ is representatives for $K_1(m)/K_1(m) \cap \text{Int}(a_{-1})K_1(m)$. In case of $m = m_{\pi} > c$, we have $TW_{\pi} = 0$ since $TW_{\pi}$ is a constant multiple of $\pi(a_{-1}) \tilde{n}(1)W_{\pi}$. But, this argument does not work in case of $m_{\pi} = c$. To observe $K_{1}(c)$-invariant vectors in $\mathcal{W}_{\psi}$, we need the following Gauss sum and its partial sum. Let $\chi \in \mathcal{B}(\mathcal{O}^\times)$ with $c(\chi) = c > 0$, and extend $\chi$ to $F$ by setting $\chi(x) = 0$ for $x \notin \mathcal{O}^\times$. Let $1 \leq m \leq c$. For $u \in \mathcal{O}^\times$, define

\[
\mathcal{G}(\chi, u) = \int_{\mathcal{O}} \psi\left(\frac{x}{\omega^c u}\right) \chi(x)dx,
\]

\[
\mathcal{S}_m(\chi, u) = \int_{\mathcal{O}} \psi\left(\frac{x}{\omega^c u}\right) \chi(1 + x)dx,
\]

where $dx$ is chosen so that $\text{vol}(\mathcal{O}) = 1$. The mapping $\mathcal{P}^{[\epsilon/2]} / \mathcal{P}^\epsilon \ni x \mapsto \chi(1 + x)$ is an additive character via the isomorphism $\mathcal{P}^{[\epsilon/2]} / \mathcal{P}^\epsilon \simeq (1 + \mathcal{P}^{[\epsilon/2]})/(1 + \mathcal{P}^\epsilon)$. Hence, for $m \geq [\epsilon/2]$, we can define $u_m \in \mathcal{O}^\times$ uniquely modulo $\mathcal{P}^{\epsilon-m}$ by the equation:

\[
\psi\left(\frac{x}{\omega^c u_m}\right)^{-1} = \chi(1 + x), \ x \in \mathcal{P}^m.
\]

Set $u_\chi = u_{[\epsilon/2]}$. Then $u_m$ is congruent to $u_\chi$ modulo $\mathcal{P}^{\epsilon-m}$.

**Lemma 2.5.** $\mathcal{S}_m(\chi, u) \neq 0$, if and only if $u \equiv u_\chi \pmod{\mathcal{P}^m}$.

**Proof.** In case of $m \geq [\epsilon/2]$, the assertion is obvious. In particular, we have

\[
\mathcal{S}_{[\epsilon/2]}(\chi, u) = \begin{cases} q^{-[\epsilon/2]}/ \chi(u) & \text{if } u \equiv u_\chi \pmod{\mathcal{P}^{[\epsilon/2]}}, \\ 0 & \text{otherwise}. \end{cases}
\]
Let $1 \leq l < m \leq [\varepsilon/2]$. Let $Y = \{y\}$ (resp. $Z = \{z\}$) be representatives for $\mathcal{P}^l/\mathcal{P}^m$ (resp. $(\mathcal{O}/\mathcal{P})^\times$). Consider the following decompositions:

\[
S_l(\chi, u) = \sum_{y \in Y} \int_{\mathcal{P}^m} \psi \left( \frac{y + x + yx}{\varepsilon u} \right) \chi((1 + y)(1 + x)) \, dx
\]

\[
= \sum_{y \in Y} \psi \left( \frac{y}{\varepsilon u} \right) \chi(1 + y) \int_{\mathcal{P}^m} \psi \left( \frac{(1 + y)x}{\varepsilon u} \right) \chi(1 + x) \, dx
\]

\[
= \sum_{y \in Y} \psi \left( \frac{y}{\varepsilon u} \right) \chi(1 + y) S_m(\chi, \frac{u}{1 + y}),
\]

and

\[
(0 \neq) G(\chi, 1) = \sum_{z \in Z} \int_{\mathcal{O}} \psi \left( \frac{z + x}{\varepsilon} \right) \chi(z + x) \, dx
\]

\[
= \sum_{z \in Z} \psi \left( \frac{z}{\varepsilon} \right) \chi(z) \int_{\mathcal{O}} \psi \left( \frac{x}{\varepsilon} \right) \chi(1 + z^{-1}x) \, dx
\]

\[
= \sum_{z \in Z} \psi \left( \frac{z}{\varepsilon} \right) \chi(z) S_1(\chi, z^{-1}).
\]

From (2.9) and the former decomposition in case of $m = [\varepsilon/2]$, it follows that $S_l(\chi, u) = 0$ unless $u \equiv u_\chi$ modulo $\mathcal{P}^l$. Hence by the latter decomposition, $S_1(\chi, z^{-1})$ is not zero for a unique $z \in Z$ such that $z^{-1} \equiv u_\chi$ modulo $\mathcal{P}$. Hence $S_m(\chi, u) \neq 0$ if $u \equiv u_\chi$ modulo $\mathcal{P}^m$ by the former decomposition in case of $l = 1$.

We will use the notation:

\[
[t; z] = \begin{bmatrix} t \\ z \\ 1 \end{bmatrix} \in G_2.
\]

**Lemma 2.6.** With the preceding assumption, let $W \in \mathcal{W}_\psi$ such that $R(k)W = \chi(k_{22})W$ for $k \in K(\varepsilon)$.

i) Assume $\varepsilon > 1$. Let $0 < m < \varepsilon$. Then,

\[
\int_{\mathcal{O}^m} W([\omega^i; z]) \, dz = \begin{cases} S_m(\chi, u_\chi) W([\omega^i; \omega^m u_\chi]) & \text{if } i = m - \varepsilon \\
0 & \text{otherwise} \end{cases}
\]

where $dz$ is chosen so that $\text{vol}(\mathcal{O}) = 1$. More precisely, $W([\omega^i; z]) = 0$ for $z \in \mathcal{P}^m$, unless $m = i + \varepsilon$ and $z \equiv \omega^m u_\chi \pmod{\mathcal{P}^{\min(\varepsilon, 2m)}}$. In case of $m = i + \varepsilon$, it holds that

\[
W([\omega^i; (1 + y)^{-1} \omega^m u_\chi]) = \psi \left( \frac{y}{\omega^i u_\chi} \right) \chi(1 + y) W([\omega^i; \omega^m u_\chi])
\]

for $y \in \mathcal{P}^m$.

ii) Assume $\varepsilon = 1$. For $i \geq 0$,

\[
\int_{\mathcal{O}^x} W([\omega^i; z]) \, dz = 0.
\]

**Proof.** i) To show $W([\omega^i; z]) = 0$ in the cases as above, we will use repeatedly Lemma 2.2, and the identity

\[
(2.11) \quad \text{Int}^{-1}([\omega^i; z]) = \begin{bmatrix} 1 + \omega^{-i} z x & \omega^{-i} x \\ -\omega^{-i} x z^2 & 1 - \omega^{-i} z x \end{bmatrix},
\]
which lies in $K(e)$ if $x \in P^\max\{i,i-2m+\ell\}$. Suppose that $i < m - \ell$, and there is an $x \in P^{-1}(\subset P^\max\{i,i-2m+\ell\})$ such that $\psi(x) \neq 1 = \chi(1 - \omega^{-i}z)$. Hence $W([\omega^i; z]) = 0$. Suppose that $i > m - \ell$. Then there is an $x \in P^{\ell-m+i-1}(\subset P^\max\{i,i-2m+\ell\})$ such that $\psi(x) = 1 \neq \chi(1 - \omega^{-i}z)$. Hence $W([\omega^i; z]) = 0$. Suppose that $i = m - \ell$. If $z$ is not congruent to $\omega^m u$, modulo $P^\min(\ell,2m)$, then there is an $x \in P^\max(m-\ell,-\ell)$ such that $\psi(x) \neq \chi(1 - \omega^{-i}z)$ by the definition of $u_m$, and hence $W([\omega^i; z]) = 0$. From the identity

$$n(x)[\omega^i; \omega^m u \chi] = [\omega^i; (\omega^m u \chi - \omega^{t+m} u^2 x \chi) \left(1 + \omega^t u \chi x \right)] \left(1 + \omega^t u \chi x \right)^{-1} n \left(\frac{\omega^{-i} x}{1 + \omega^t u \chi x} \right),$$

the last assertion follows. Now, the first assertion follows from Lemma 2.5.

ii) follows from the computation:

$$\int_\sigma^x W(a_i, \omega^i) dz = \int_\sigma^x W(a_i, n(z^{-1}) j(z) n(z^{-1})) dz = \int_\sigma^x \psi(z^{-1} \omega^i) W(a_i, j(z)) dz = W(a_i, j(1)) \int_\sigma^x \chi(z)^{-1} dz = 0.$$

Now, we can prove Proposition 2.4. By (2.5), $W(1) = 1$ and $W(0) = 0$ for $i \neq 0$. Since $\dim V(m_{\pi}) = 1$, there is a constant $\lambda$ such that $\mathcal{T} W_{\pi} = \lambda W_{\pi}$. From (2.7), and the above Lemma, it follows that

$$\lambda W_{\pi}(a_1) = W_{\pi}(1).$$

This is a contradiction. This completes the proof of the proposition.

3. Representations of $P_3$

Let $P_3$ be the subgroup of $G_3$ of matrices of the form of

$$\begin{bmatrix} g & \beta \\ 1 & 1 \end{bmatrix}, \quad g \in G_2.$$

We need the following notations for subgroups and elements in $P_3$.

- $N_2 = \{n_2(x) := \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \}$, $N_3 = \{n_3(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \}$,
- $N' = \{n'(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \}$,
- $Z_2' = \{z_2'(t) = (t1_2)' | t \in F^\times \}$,
- $M = N' N_3$, $M' = N' N_2 (\simeq M)$.

For $\xi \in \mathscr{X}(F^\times)$, let $\xi_{\psi} \in \mathscr{X}(NA)$ defined by

$$\xi_{\psi}(a(t)n) = \xi(t) \psi(n_{2,3}), \quad n \in N.$$

For $\rho \in \text{Irr}(G_3)$, let $\rho'$ denote the representation of $P_3$ sending elements $g'n \in G_2 N_2 N_3$ to $\rho(g)$, whose representation space is same as $\rho$. Every irreducible smooth representation of $P_3$ is isomorphic to

$$\tau_0 := \text{ind}_{N}^{P_3} \psi, \quad \tau_1(\xi) := \text{ind}_{N,A}^{P_3} \xi_{\psi} \quad \text{or} \quad \tau_2(\rho') := \rho'.$$
where ind indicates the compact induction. For \( \chi \in \mathcal{X}(F^\times) \), and \( b \in F^\times \), let \( \eta_b(\chi) \in \mathcal{X}(MZ'_2), \sigma_b(\chi) \in \mathcal{X}(M^p A) \), and \( \sigma_0(\chi) \in \mathcal{X}(N'Z'_2) \) defined by

\[
\eta_b(\chi)(z'_2(t)m) = \chi(t)\psi(bm_{1,2}), \quad m \in M \\
\sigma_b(\chi)(a(t)m) = \chi(t)\psi(bm_{2,3}), \quad m \in M^p \\
\sigma_0(\chi)(z'_2(t)n) = \chi(t), \quad n \in N'.
\]

For an \( l \)-group \( G \), we say a distribution \( D \) on \( G \) left (resp. right) quasi-invariant with \( \chi \in \mathcal{X}(G) \), if \( \chi(g)D \) equals \( D \circ L(g) \) (resp. \( D \circ d(g) \)) for all \( g \in G \). By the proof of Proposition 1.18 of [B-Z] (taking the family of neighborhoods of \( 1 \) in ker(\( \psi \)), the space of quasi-invariant distributions is one-dimensional. Indeed, there is a constant \( c \) such that \( D(\varphi) = c \int_G \varphi(g)\chi(g)^{-1}dg \) for \( \varphi \in \mathcal{X}(G) \), where \( dg \) is a left (resp. right) Haar measure on \( G \). Following propositions are verified by Bruhat’s distributional technique for induced representations (c.f. section 5 of [W]).

**Proposition 3.1.** With the above notation,

i) The space \( \text{Hom}_{MZ'_2}(\tau_0, \eta_b(\chi)) \) is spanned by the nontrivial functional \( \mu^b_\chi : \tau_0 \to \mathbb{C} \) defined by

\[
\mu^b_\chi(f) = \int_{F^\times} \chi^{-1}(t)f(z'_2(t)a(b))d^\times t.
\]

ii) For any \( \xi \in \mathcal{X}(F^\times) \), \( \text{Hom}_{MZ'_2}(\tau_1(\xi), \eta_b(\chi)) = \{0\} \).

iii) Let \( \rho \in \text{Irr}(G_2) \). Then,

\[
\text{Hom}_{MZ'_2}(\tau_2(\rho), \eta_b(\chi)) = \begin{cases} \mathbb{C} \mu^b_\chi(\neq \{0\}) & \text{if } \chi = \omega_\rho, \text{ and } \rho \in \text{Irr}^{gn}(G_2), \\ \{0\} & \text{otherwise}, \end{cases}
\]

where \( \mu^b_\chi : \tau_2(\rho) \to \mathbb{C} \) is defined by \( \mu^b_\chi(f) = f(a(b)) \).

**Proof.** It suffices to show for the case of ker(\( \psi \)) = \( O \) and \( b = 1 \). i) Let \( \varphi \in \mathcal{X}(P_3) \). Define \( f_\varphi \in \tau_0 \) by

\[
f_\varphi(p) = \int_N \tilde{\psi}(n)^{-1}\varphi(np)dn.
\]

We claim that the linear mapping \( \mathcal{X}(P_3) \ni \varphi \mapsto f_\varphi \in \tau_0 \) is surjective. Let \( f \in \tau_0 \). We will use the following compact subgroups:

\[
\Gamma(m) = \{k \in G_2(O) \mid k \equiv 1_2 \pmod{\mathcal{P}^m}\} \subset G_2, \\
\Upsilon(m) = \{p \in P_3(O) \mid p \equiv 1_3 \pmod{\mathcal{P}^m}\} \subset P_3.
\]

Take \( m \) so that \( f \) is right \( \Upsilon(m) \)-invariant. By the Iwasawa decomposition of \( G_2 \), we have \( P_3 = \bigcup_{l \in \mathbb{Z}^2} N\varpi^lG_2(O)' \). Hence, by a finite subset \( \mathfrak{T} \) of representatives for \( G_2(O)/\Gamma(m) \), we have

\[
P_3 = \bigcup_{l \in \mathbb{Z}^2, t \in \mathfrak{T}} N\varpi^lt\Upsilon(m).
\]

Let \( \varphi^l \in \mathcal{X}(P_3) \) be the characteristic function of the compact orbit \( N(O)\varpi^l\Upsilon(m) \). The function \( f_{\varphi^l} \) vanishes outside of \( N\varpi^lt\Upsilon(m) \), and takes a constant value \( c^l \) on \( N(O)\varpi^lt\Upsilon(m) \).

Any \( \Upsilon(m) \)-invariant \( f' \in \tau_0 \) with \( \text{supp}(f') = N\varpi^lt\Upsilon(m) \) is a constant multiple of \( f_{\varphi^l} \). In particular, \( f'(\varpi^lt) = 0 \) if \( c^l = 0 \). Therefore, for the \( \Upsilon(m) \)-invariant \( f \in \tau_0 \), setting

\[
\varphi = \sum (c^l)^{-1} f(\varpi^lt)\varphi^l
\]
with the sum (finite since \( f \in \tau_0 \)) being over \( l \in \mathbb{Z}^2 \) and \( t \in \mathcal{F} \) such that \( c_l^1 \neq 0 \), we have \( f(\omega^l t') = f_\varphi(\omega^l t') \) for all \( l \in \mathbb{Z}^2, t \in \mathcal{F} \). By the disjoint union (3.2), \( f = f_\varphi \). This proves the claim. Let \( \mu \in \text{Hom}_{\tilde{M}Z'_2}(\tau_0, \eta_\chi(\chi)) \) correspond to the distribution \( D_\mu \) on \( P_3 \) defined by

\[
D_\mu(\varphi) = \mu(f_\varphi).
\]

Since \( \mathcal{S}(P_3) \ni \varphi \mapsto f_\varphi \in \tau_0 \) is surjective, the linear mapping \( \mu \mapsto D_\mu \) to the space of distributions on \( P_3 \) is injective. By definition, if \( D = D_\mu \), then

\[
\begin{align*}
D \circ R(h) &= \eta_\chi(h)D & (h \in MZ'_2), \\
D \circ L(n) &= \tilde{\psi}(n)^{-1}D & (n \in N).
\end{align*}
\]

Now we observe the support of \( D_\mu \) in the sense of 1.10 of [B-Z]. Take representatives for the double coset space \( N \setminus P_3/MZ'_2 \), for example, \( \{a(s) \mid s \in F^\times\} \sqcup \{a(s)w'_2 \mid s \in F^\times\} \). Let \( \varphi_{s,m} \in \mathcal{S}(P_3) \) be the characteristic function of \( a(s)\mathcal{Y}(m) \). For \( k \in \mathcal{Y}(m) \),

\[
L(n'(x))\varphi_{s,m}(a(s)k) = \varphi_{s,m}(n'(-x)a(s)k) = \varphi_{s,m}(a(s)\text{Int}(n'(-sx))kn'(-sx)) = R(n'(-sx))\varphi_{s,m}(a(s)\text{Int}(n'(-sx))k).
\]

If \( s \neq 1 \), then we may take a sufficiently large \( m \) so that there exists an \( x \in F \) such that \( \psi((1-s)x) \neq 1 \) and \( \text{Int}(n'(sx))\mathcal{Y}(m) \subset \mathcal{Y}(m) \), and therefore, by (3.3), (3.4),

\[
\begin{align*}
\psi(-x)D(\varphi_{s,m}) &= D(L(n'(x))(\varphi_{s,m})) = D(R(n'(-sx))(\varphi_{s,m})) = \psi(-sx)D(\varphi_{s,m}).
\end{align*}
\]

Hence \( D(\varphi_{s,m}) = 0 \) and \( a(s) \notin \text{supp}(D) \), unless \( s = 1 \). Similarly, one can see that \( a(s)w'_2 \notin \text{supp}(D) \) by using the identity \( a(s)w'_2n_3(x) = n_2(sx)a(s)w'_2 \). Therefore, \( \text{supp}(D) \subset NZ'_2 \).

By the exact sequence in 1.9 of loc. cit., we may regard \( D \) as a distribution on the closed subgroup \( NZ'_2 \) of \( P_3 \) such that \( D \circ L(n)R(z'_2(t)) = \chi(t)\tilde{\psi}(n)^{-1}D \). Since \( N \cap Z'_2 = \{1\} \), and \( Z'_2 \approx F^\times, \mathcal{S}(NZ'_2) \simeq \mathcal{S}(N) \otimes \mathcal{S}(F^\times) \). Therefore, such a \( D \) lies in the space \( \text{Hom}_{N \times F^\times}(\mathcal{S}(N) \otimes \mathcal{S}(F^\times), C_{\tilde{\psi}^{-1}} \otimes C_\chi) \) where \( C_{\tilde{\psi}^{-1}} \) and \( C_\chi \) indicate the representation spaces of \( \tilde{\psi}^{-1} \) and \( \chi \) respectively. Since the spaces of quasi-invariant distributions on \( N, F^\times \) are one-dimensional, so is \( \text{Hom}_{N \times F^\times}(\mathcal{S}(N) \otimes \mathcal{S}(F^\times), C_{\tilde{\psi}^{-1}} \otimes C_\chi) \) which is isomorphic to \( \text{Hom}_N(\mathcal{S}(N), C_{\tilde{\psi}^{-1}}) \).

Hence, \( \text{Hom}_{\tilde{M}Z'_2}(\tau_0, \eta_\chi(\chi)) \) is 1-dimensional at most. By (3.2) we can define the right \( \mathcal{Y}(\chi(\chi)) \)-invariant \( f_\chi \) in \( \tau_0 \) by

\[
f_\chi(n\omega^lg') = \begin{cases} \tilde{\psi}(n) & \text{if } l = (0,0) \text{ and } ng' \in N\mathcal{Y}(\chi(\chi)), \\ 0 & \text{otherwise.} \end{cases}
\]

Obviously \( \mu_\chi^1(f_\chi) \neq 0 \), and \( \mu_\chi^1 \) spans \( \text{Hom}_{\tilde{M}Z'_2}(\tau_0, \eta_\chi(\chi)) \).

ii) Similar to i). Replace the condition (3.4) with

\[
D \circ L(h) = \eta_\chi(h)(h)^{-1}D & (h \in NA).
\]

By using this condition, (3.3), and representatives for \( NA \setminus P_3/MZ'_2 \), say \( \{1, w'_2\} \), one can see that the supports of corresponding distributions are emptys sets.

iii) follows from the fact that \( \text{Hom}_{G_2}(\rho, \mathcal{M}_\psi) \simeq \text{Hom}_N(\mathcal{S}(\rho), \mathcal{S}(\tilde{\psi})) \) is one-dimensional, if \( \rho \) is generic. \( \square \)

The following proposition is proved similarly (c.f. Lemma 2.5.4., 2.5.5., 2.5.6. of [R-S]).
Proposition 3.2. i) The space Hom$_{AM}$(τ$_0$, σ$_b$(χ)) is spanned by the nontrivial functional λ$_b$ : τ$_0$ → C defined by

\[ \lambda_b^b(f) = \int_{F^\times} \int_{\mathcal{N}'} \chi^{-1}(t)\nu_{-1}(t)f(a(t)nz'_2(b))dn^*t. \]  

(3.5)

ii) Let χ, ξ ∈ \mathcal{X}(F^\times). Then,

\[ \text{Hom}_{AM}^b(\tau_1(\xi), \sigma_b(\chi)) = \begin{cases} \mathbb{C}\lambda_1^b \neq \{0\} & \text{if } \xi = \nu_1\chi, \\ \{0\} & \text{otherwise}, \end{cases} \]

where λ$_1^b$ : τ$_1$(ξ) → C is defined by

\[ \lambda_1^b(f) = \int_{\mathcal{N}'} f(nz'_2(b))dn. \]

iii) For any ρ ∈ Irr(G$_2$), Hom$_{AM}^p(\tau_2(\rho), \sigma_b(\chi)) = \{0\}.

Since both of τ$_2(\xi \circ \det)$ and σ$_0(\chi)$ are one-dimensional, the following is obvious.

Proposition 3.3. Let χ, ξ ∈ \mathcal{X}(F^\times). Then,

\[ \text{Hom}_{Z'_2N'}(\tau_2(\xi \circ \det), \sigma_0(\chi)) = \begin{cases} \mathbb{C}\mu'_2 \neq \{0\} & \text{if } \chi = \xi^2, \\ \{0\} & \text{otherwise}, \end{cases} \]

where µ'_2 : τ$_2$(ξ \circ det) → C is the nontrivial functional defined by µ'_2(f) = f(13).

4. REPRESENTATIONS OF WHITTAKER TYPES

Let G = GSp(4, F). Subgroups of G will be written in capital boldface. The center of G is isomorphic to F$^\times$, and we identify them. Let Q$^o$ ⊂ G be the subgroup consisting of matrices of the form of 

\[ \begin{bmatrix} * & * & * & * \\ a & b & x \\ c & d & y \\ 1 & & & 1 \end{bmatrix}. \]

The Klingen parabolic subgroup Q is generated by Q$^o$ and F$^\times$. The Jacobi subgroup of Q consists of the above matrices such that ad – bc = 1, and its center is

\[ \mathbb{Z}' = \{z(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} | x ∈ F\}. \]

Let pr : Q$^o$ → P$_3$ be the projection sending the above matrices in Q$^o$ to

\[ \begin{bmatrix} a & b & x \\ c & d & y \\ 1 \end{bmatrix}. \]

Then, pr is a homomorphism with ker(pr) = \mathbb{Z}'$, and thus Q$^o$/\mathbb{Z}' ∼ P$_3$. We will argue about the representations of P$_3$ and Q$^o$. In [R-S], they use the projection sending q = zq$_0$ with z ∈ F$^\times$, q$_0$ ∈ Q$^o$ to pr(q$_0$), and relate the representations of P$_3$ to those of Q/F$^\times$. By using pr, many of their arguments for the representations of PGSp(4) also work for those
of $\mathbf{G}$ having unramified central characters. The following subgroups of $\mathbf{Q}^o$ correspond to those of $P_3$ in the previous section.

$$G'_2 = \{ g' = \begin{bmatrix} \det(g) & g \\ g & 1 \end{bmatrix} \mid g \in G_2 \}, N' = \{ n'(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \mid x \in F \},$$

$$N_2 = \{ n_2(x) = \begin{bmatrix} 1 & -x \\ 1 & x \end{bmatrix} \mid x \in F \}, N_3 = \{ n_3(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \mid x \in F \},$$

$N = N'_2 N'_3 Z'$.

Define $\tilde{\psi} \in \mathcal{X}(N)$ by $\tilde{\psi}(n_2(x)n'(y)n_3(*)z(*)) = \psi(x + y)$. Define $\mathcal{M}_2 = \text{Ind}_{N}^{G} \tilde{\psi}$, $\text{Irr}^{\eta_2}(G)$ and $\mathcal{M}_2(\pi)$ for $\pi \in \text{Irr}(G)$, similar to the $G_{d'}$-case. Let $W \in \mathcal{M}_2$. Via $\text{pr}$, and the embedding

$$G_2 \ni h \mapsto h^2 := \begin{bmatrix} w_2^t h^{-1} w_2 \\ h \end{bmatrix} \in \mathbf{G},$$

we define the function on $P_3$ and that on $G_2$ by

$$f_W(p) = W(\text{pr}^{-1}(p)), \quad \xi_W(h) = W(h^2).$$

They are called the first and second gauge of $W$, respectively. Note that $f_W$ is well-defined since $W$ is left $Z'$-invariant. For the torus subgroups, we will use the following notations:

$$T = \{ t(x, y; z) = \text{diag}(xz, yz, y^{-1}, x^{-1}) \mid x, y, z \in F^x \},$$

$$T_1 = \{ t_1(x) = t(x, 1; 1) \mid x \in F^x \},$$

$$A' = \{ a'(z) = t(1, 1; z) \mid z \in F^x \}.$$

In particular,

$$a_j^i = t(\overline{w}^j, 1; \overline{w}^j), \quad \eta = t_1(\overline{w}^{-1}), \quad a_j = a'(\overline{w}^j).$$

The following Weyl elements are important to our arguments.

$$j'(x) = \begin{bmatrix} 1 & -x^{-1} \\ x & 1 \end{bmatrix}, \quad j''_m = \begin{bmatrix} 1 & -\overline{w}^{-m} \\ \overline{w}^m & 1 \end{bmatrix},$$

$$J_m = \begin{bmatrix} 1 & 0 \\ \overline{w}^m & -1 \end{bmatrix}.$$
Theorem 4.1 ([R-S]). With notations as above, if \( \pi \) is of Whittaker type, then the \( P_3 \)-module \( V_{Z^J} \) has a finite Jordan-Hölder sequence of smooth \( P_3 \)-modules \( 0 \subset V_0 \subset V_1 \subset \cdots \subset V_n = V_{Z^J} \) such that \( V_0 \simeq \tau_0 \) and, for some \( I \leq n-1 \),
\[
V_{i+1}/V_i \simeq \begin{cases} 
\tau_1(\xi_i), & \xi_i \in \mathcal{X}(F^\times) \quad (i \leq I), \\
\tau_2(\rho_i), & \rho_i \in \text{Irr}(G_2) \quad (i > I).
\end{cases}
\]
We have \( V_{Z^J} = V_0 \), if and only if \( \pi \) is supercuspidal.

Proposition 4.2. Fix \( \psi \) and \( b \in F^\times \). Let \( (\pi, V) \) be of Whittaker type. Except for finitely many \( \chi \in \mathcal{X}(F^\times) \), the space of functionals \( \mu : \pi \to \mathbb{C} \) such that
\[
\mu(\pi(t_1)n'(x)n_3(\ast)z(\ast))v = \psi(bx)\chi(t)\mu(v),
\]
and the space of functionals \( \lambda : \pi \to \mathbb{C} \) such that
\[
\lambda(\pi(a'(t)n'(\ast)n_2(x)z(\ast))v) = \psi(bx)\chi(t)\lambda(v)
\]
are both one-dimensional.

Proof. Note that \( \text{pr}(T_1N'N_3Z^J) = MZ_2' \subset P_3 \), and the character \( t_1n'(x)n_3(\ast)z(\ast) \to \psi(bx)\chi(t) \) corresponds to \( \eta_b(\chi) \) defined in previous section. By Theorem 4.1, \( V_{i+1}/V_i \simeq \tau_j(\sigma_i) \) for some \( \sigma_i \in \text{Irr}(G_j) \) for \( j \in \{1, 2\} \). Therefore, the following sequence is exact:
\[
(4.4) \quad \text{Hom}_{MZ^J}(\tau_j(\sigma_i), \eta_b(\chi)) \to \text{Hom}_{MZ^J}(V_{i+1}, \eta_b(\chi)) \to \text{Hom}_{MZ^J}(V_i, \eta_b(\chi)) \to 0.
\]
By Proposition 3.1, \( \dim_{\mathbb{C}} \text{Hom}_{MZ^J}(V_0, \eta_b(\chi)) = 1 \), and \( \text{Hom}_{MZ^J}(\tau_j(\sigma_i), \eta_b(\chi)) = \{0\} \) for all \( i \) except for finitely many \( \chi \in \mathcal{X}(F^\times) \). By (4.4) and induction, \( \dim_{\mathbb{C}} \text{Hom}_{MZ^J}(V_1, \eta_b(\chi)) = \cdots = \dim_{\mathbb{C}} \text{Hom}_{MZ^J}(V_n, \eta_b(\chi)) = \dim_{\mathbb{C}} \text{Hom}_{MZ^J}(V_{Z^J}, \eta_b(\chi)) = 1 \) except for finitely many \( \chi \in \mathcal{X}(F^\times) \). This proves the assertion for the space of \( \mu \). For \( \lambda \), use Proposition 3.2. \( \square \)

For \( \pi \in \text{Irr}(G) \), let \( \pi^r = \pi \otimes (\omega_{\pi}^{-1} \circ \mu) \), which is equivalent to \( \pi^\vee \) by Proposition 2.3 of [T], where \( \mu \) indicates the similitude factor. For \( W \in \mathcal{W}_\psi \), define \( W^r \in \mathcal{W}_\psi \) by
\[
W^r(g) = \omega_{\pi}(\mu(g)^{-1})W(g),
\]
and the zeta integrals:
\[
\Xi(s, W) = \int_{F^\times} W(a'(t))\nu_{s-\frac{1}{2}}(t)d^\times t, \\
Z(s, W) = \Xi(s, \int_{N^\times} \pi(n)Wdn),
\]
where \( d^\times t \) and \( dn \) is chosen so that \( \text{vol}(F^\times) = 1 \) and \( \text{vol}(N'F) = 1 \) respectively. Now, let \( \pi \in \text{Irr}^{\text{gn}}(G) \). Fix \( \psi \). For \( W \in \mathcal{W}_\psi(\pi) \), \( Z(s, W) \) converges absolutely to an element in \( \mathbb{C}(q^{-s}) \) if \( s \in \mathbb{C} \) lies in some right half complex plane, and the \( \mathbb{C} \)-vector subspace \( I(\pi) \subset \mathbb{C}(q^{-s}) \) spanned by all \( Z(s, W) \) is a fractional ideal of the principal ideal domain \( \mathbb{C}[q^{\pm s}] := \mathbb{C}[q^s, q^{-s}] \). Therefore, \( I(\pi) \) admits a generator of the form \( P(q^{-s})^{-1} \) with \( P(X) \in \mathbb{C}[X] \) such that \( P(0) = 1 \). Set \( L(s, \pi) = P(q^{-s})^{-1} \). From Proposition 4.2, we obtain the following functional equation by the standard argument (c.f. [R-S], [J-PS-S3]). We omit the proof.

Theorem 4.3. Let \( \pi \in \text{Irr}^{\text{gn}}(G) \). There exists a monomial \( \varepsilon(s, \pi, \psi) \) in \( q^{-s} \) such that
\[
\frac{Z(1-s, \pi^r(10)W^r)}{L(1-s, \pi^r)} = \varepsilon(s, \pi, \psi) \frac{Z(s, W)}{L(s, \pi)},
\]
for any \( W \in \mathcal{W}_\psi(\pi) \). It holds that \( \varepsilon(s, \pi, \psi)\varepsilon(1-s, \pi^r, \psi) = 1 \).
If \( \ker(\psi) = \ker(\psi') \), then \( \varepsilon(s, \pi, \psi) = \varepsilon(s, \pi, \psi') \). For \( \psi \) such that \( \ker(\psi) = \mathcal{O} \), define the analytic root number \( \varepsilon_\pi' \) and conductor \( n'_\pi \in \mathbb{Z} \) by
\[
\varepsilon(s, \pi, \psi) = \varepsilon'_\pi q^{-n'_\pi(s-1)}.
\]

5. Quasi-paramodular forms

In this and next section, we fix a \( \psi \) such that \( \ker(\psi) = \mathcal{O} \). Let \( \pi \in \text{Irr}^m(G) \). Let \( \omega_\pi \) be the central character of \( \pi \), and \( \varepsilon = \varepsilon(\omega_\pi) \). For \( m \geq 2\varepsilon \), define the quasi-paramodular groups \( K(m; \varepsilon), K_1(m; \varepsilon) \) as in introduction. Define \( K^c(m; \varepsilon) = \text{Int}(\mathfrak{g}_m)(K(m; \varepsilon)) \) and \( K_1(m; \varepsilon) = \{ k \in K^c(m; \varepsilon) \mid k_{33} \in 1 + \mathfrak{p}^+ \} \). Explicitly, \( K^c(m; \varepsilon) \) consists of \( k \in G \) such that \( \det(k) \in \mathcal{O}^\times \) and
\[
k \in \begin{bmatrix}
0 & 0 & \mathcal{P}^{-\varepsilon} & \mathcal{P}^{-m} \\
\mathcal{P}^1 & 0 & 0 & \mathcal{P}^{-\varepsilon} \\
\mathcal{P}^m & \mathcal{P}^\varepsilon & 0 & 0 \\
\mathcal{P}_m & \mathcal{P}^\varepsilon & \mathcal{P}^1 & 0
\end{bmatrix}.
\]

In case of \( \varepsilon = 0 \), these open compact subgroups coincide with the paramodular group \( K(m) \), as well as the quasi-paramodular groups. Let \( V(m) \) denote the space of quasi-paramodular forms of level \( m \) in \( \mathcal{W}_\psi(\pi) \). For each \( W \in V(m) \), define the conjugate \( W^c \) by
\[
W^c = \pi^t(\mathfrak{g}_m)W^t \in \mathcal{W}_\psi(\pi').
\]

Observe that \( \pi^t(k)W^c = \omega_\pi(k_{33})^{-1}W^c \) for \( k \in K^c(m; \varepsilon) \). The image of \( V(m) \) by \( c \) is denoted by \( V^c(m) \). In case of \( \varepsilon = 0 \), \( V(m) = V^c(m) \), and we have a decomposition
\[
V(m) = V(m)_+ \oplus V(m)_-,
\]
where \( V(m)_\pm = \{ W \in V(m) \mid \pi(\mathfrak{g}_m)W = \pm W \} \). In case of \( \varepsilon > 0 \), \( V(m) \neq V^c(m) \), and we call \( K^c_1(m; \varepsilon) \)-invariant Whittaker functions in \( \mathcal{W}_\psi \) coquasi-paramodular forms of level \( m \). We call \( K^c_1(m; \varepsilon) \)-invariant Whittaker functions quasi-paramodular forms of level \( m \) and coquasi-paramodular forms, we assume \( \varepsilon > 0 \). The proof for the existence of nontrivial quasi-paramodular forms (and thus that of coquasi-ones) for the case of \( \varepsilon > 0 \) is easier than that by [R-S] for the case of \( \varepsilon = 0 \). As in Theorem 4.4.1 of loc. cit., one can show that there is a quasi-\( K_l(\mathcal{P}^n) \)-invariant \( W \in \mathcal{W}_\psi(\pi) \) such that \( W(1) \neq 0 \), for a sufficiently large \( n \). Obviously
\[
\int_{K(n; \varepsilon)K(\mathcal{P}^n)} \omega_\pi(k_{44})^{-1}\pi(k)Wdk \in V(n)
\]
is not zero at 1. Quasi- and coquasi-paramodular forms have the following fine property.

**Proposition 5.1.** Let \( W \in \mathcal{W}_\psi \). If \( W \) is \( N_3(\mathcal{P}^{-r}) \)-invariant, then
\[
Z(s, W) = \Xi \left( s, \int_{N(\mathcal{P}^r)} \pi(n)Wdn \right).
\]

In particular, \( Z(s, W) = q^{-r}\Xi(s, W) \), if \( W \) is \( \tilde{N}(\mathcal{P}^r) \)-invariant additionally.

**Proof.** Let \( f = f_W \) be the first gauge of \( W \) (c.f. (4.2)). Then, \( f \) is right \( N_3(\mathcal{P}^{-r}) \)-invaraint, and
\[
Z(s, W) = \int_{F^\times} \int_{F} f(a(t)\tilde{n}(x))\nu_{s-3/2}(t)dxd^\times t.
\]
For $x \not\in \mathcal{P}^r$, $f(a(t)n'(x)) = 0$ is verified similar to Proposition 2.1. Hence the assertion. 

We call $N_3(\mathcal{P}^r)$ and $\mathcal{N}'(\mathcal{P}^r)$-invariant Whittaker functions $r$-balanced. If $W$ is quasi-paramodular, then
\begin{equation}
Z(s, W) = \Xi(s, W), \quad Z(s, W^c) = q^{-e} \Xi(s, W^c).
\end{equation}
Additionally if $\pi \in \text{Irr}^{\text{gm}}(G)$ and $W \in \mathcal{W}_\psi(\pi)$, the functional equation is simplified to
\begin{equation}
q^{-e} \frac{\Xi(1-s, W^c)}{L(1-s, \pi^c)} = \varepsilon_n q^{(m-n_0)(s-1/2)} \frac{\Xi(s, W)}{L(s, \pi)}.
\end{equation}
In case of $e > 0$, we will show that other balanced forms are obtained from quasi-paramodular forms and coquasi-ones of level $m$ by the linear operators $\Gamma_r, \Gamma_r^*$ defined by
\begin{equation}
\Gamma_r : W \mapsto \int_{\mathcal{P}^r \mathcal{E}_1} \pi(\tilde{n}(z))Wdz, \quad \Gamma_r^* : W \mapsto \int_{\mathcal{P}^r \mathcal{E}_1} \pi(\tilde{n}(z))Wdz,
\end{equation}
where $l = m - e$, and
\begin{equation}
\mathcal{C}_a = \left\{ \begin{bmatrix} x \\ y \\ x \end{bmatrix} : x \in \mathcal{O}, y \in \mathcal{P}^a \right\}, \quad \mathcal{B}_a = \left\{ \begin{bmatrix} x \\ y \\ x \end{bmatrix} : x \in \mathcal{O}, y \in \mathcal{P}^{-a} \right\}, a \in \mathbb{Z}.
\end{equation}

**Lemma 5.2.** With notations as above,

i) If $W$ is quasi-paramodular of level $m$, then $\Gamma_r(W)$ is 0-balanced, and $\pi(j_0)\Gamma_r(W)$ is $(-r)$-balanced for $r \geq \max\{m - 2e, e\}$.

ii) If $W$ is coquasi-paramodular of level $m$, then $\Gamma_r^*(W)$ is $(r-l)$-balanced, and $\pi^*(j_0)\Gamma_r^*(W)$ is $(-r)$-balanced for any $r \in \mathbb{Z}$. 

**Proof.** i) The assertion for $r = l$ is obvious, since $\Gamma_lW$ is a constant multiple of $W$. It suffices to show that $\Gamma_r(W)$ is $\mathcal{N}'(\mathcal{O}), N_3(\mathcal{P}^r), N_3(\mathcal{O})$ and $\mathcal{Z}^l(\mathcal{P}^{-r})$-invariant. The $\mathcal{N}'(\mathcal{O}), N_3(\mathcal{P}^r)$-invariance property is obvious. We will show the $N_3(\mathcal{O}), \mathcal{Z}^l(\mathcal{P}^{-r})$-invariance property by induction. We also use identities (2.2), and
\begin{equation}
\text{Int}(A^i)\tilde{n}(C) = \tilde{n}(w_2^iAw_2CA).
\end{equation}
Let $H^2 = \{ A^i | A \in H \}$ be the subgroup of $K_1(m; e)$, where
\begin{equation}
H = \left[ \begin{array}{cc}
\mathcal{O}^\times & \mathcal{O} \\
\mathcal{P}^i & 1 + \mathcal{P}^c
\end{array} \right] \subset G_2.
\end{equation}
If $A \in H$, then the mapping $C \mapsto w_2^iAw_2CA$ induces a translation in the quotient of modules $\mathcal{O}^\times \mathcal{C}_i/\mathcal{O}^\times \mathcal{C}_i$. Therefore, if $\Gamma_{r+1}(W)$ is invariant under $H^2$, then so is $\Gamma_r(W)$. Therefore $\Gamma_r(W)$ is $H^2$-invariant. Now the $N_3(\mathcal{O}), \mathcal{Z}^l(\mathcal{P}^{-r})$-invariance property follows from (2.2), induction hypothesis and the calculation
\begin{align*}
BC &= \begin{bmatrix} ax + by \\ ay \\ ax \end{bmatrix} \begin{bmatrix} bx \\ by + 2axy \\ ax^2 + bxy \end{bmatrix} \in \left[ \begin{array}{cc}
\mathcal{P}^r + \mathcal{P}^t & \mathcal{O} \\
\mathcal{P}^r + \mathcal{P}^t + \mathcal{P}^c & \mathcal{O}
\end{array} \right] \subset \left[ \begin{array}{cc}
\mathcal{P}^c & \mathcal{O} \\
\mathcal{P}^i & \mathcal{O}
\end{array} \right], \\
C^2BC &= \begin{bmatrix} ax + by \\ ay \\ ax \end{bmatrix} \begin{bmatrix} bx^2 \\ by^2 + 2axy \\ ax^2 + bxy \end{bmatrix} \in \left[ \begin{array}{cc}
\mathcal{P}^r + \mathcal{P}^t + \mathcal{P}^c & \mathcal{P}^r \\
\mathcal{P}^r + \mathcal{P}^t + \mathcal{P}^c & \mathcal{P}^r
\end{array} \right] \subset \left[ \begin{array}{cc}
\mathcal{P}^r + \mathcal{P}^t + \mathcal{P}^c & \mathcal{P}^r \\
\mathcal{P}^r + \mathcal{P}^t + \mathcal{P}^c & \mathcal{P}^r
\end{array} \right], \\
BC &= \begin{bmatrix} a \\ b \\ a \end{bmatrix} \in \mathcal{B}_r, \quad C = \begin{bmatrix} x \\ y \\ x \end{bmatrix} \in \mathcal{O}^\times \mathcal{C}_i.
\end{align*}
ii) Similar to i). We only check that
\[ BC = \begin{bmatrix} ax & ay + bx \\ ax & ax \end{bmatrix} \in \begin{bmatrix} \mathcal{P}^l & \mathcal{O} \\ \mathcal{O} & \mathcal{P}^l \end{bmatrix} \subset \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{P}^l \end{bmatrix}, \]
\[ CBC = \begin{bmatrix} ax^2 & 2axy + bx^2 \\ ax^2 & ax^2 \end{bmatrix} \in \begin{bmatrix} \mathcal{P}^l + l & \mathcal{P}^l \\ \mathcal{P}^l + l & \mathcal{P}^l \end{bmatrix} \subset \begin{bmatrix} \mathcal{P}^l + l & \mathcal{P}^l + l \\ \mathcal{P}^l + l & \mathcal{P}^l \end{bmatrix} \]
for
\[ B = \begin{bmatrix} a & b \\ a & a \end{bmatrix} \in \omega^{l-r} \mathcal{B}_l, C = \begin{bmatrix} x & y \\ x & x \end{bmatrix} \in \omega^r \mathcal{B}_l. \]

The proof of the next is similar to that of Theorem 3.1.3 of [R-S], and omitted.

**Theorem 5.3.** Let \((\pi, V) \in \text{Alg}(\mathcal{G})\). Assume that \(V^{Sp(4,F)}\), the subspace of \(Sp(4, F)\)-invariant vectors in \(V\), is \(\{0\}\). Let \(\{m_0 < \cdots < m_r\}\) be a finite set of nonnegative integers, and \(v_i(\neq 0)\) be \(K_1^i(m_i)\)-invariant vectors in \(V\). Then, \(v_i\) are linearly independent.

The next is the main theorem of this section.

**Theorem 5.4.** Let \(\pi \in \text{Ir}^{\eta}(\mathcal{G})\), and \(W \in \mathcal{H}_\psi(\pi)\) be quasi-paramodular. If \(W(T) = 0\), then \(W\) is identically zero.

In case of \(\varepsilon = 0\), this is Corollary 4.3.8. of the ‘\(\eta\)-principle’ of loc. cit. Although they assumed \(\omega_\pi = 1\), their argument works as far as \(\varepsilon = 0\). We will consider the case of \(\varepsilon > 0\), mainly. We need some preparations. Let \(W \in \mathcal{H}_\psi(\pi)\). For \(r \in \mathbb{Z}\), set
\[ W_r = \pi(\eta^{-r})W. \]
If \(W\) is quasi-paramodular, then \(W_r\) is \(\mathcal{N}_2(\mathcal{P}_r)\)-invariant. By using Lemma 2.2 one can show that \(Z(s, W_r) = 0\) if \(r < 0\). We will compute \(Z(s, W_r)\) for \(r \geq 0\). By Proposition 5.1,
\[ Z(s, W_r) = \Xi(s, \int_{\mathcal{P}^{-r}} \pi(n'(z))W_r dz) = \Xi(s, W_r) + \sum_{m=1}^{r} \Xi(s, \int_{\mathcal{P}^{-m}} \pi(n'(z))W_r dz). \]

For \(j \in \mathbb{Z}\) and a Laurent series \(D(X) = \sum c_n X^n\), let
\[ D(X)_j = q^{-1} \left(-c_{j-1}X^{(j-1)} + (q - 1) \sum_{n=j}^{\infty} c_n X^n \right). \]

**Lemma 5.5.** With notation as above, if \(W \in \mathcal{H}_\psi(\pi)\) is quasi-paramodular, then
\[ Z(s, W_r) = \sum_{m=0}^{r} \omega_\pi(\omega)^{-m} q^{2m(s-1)} \Xi(s, W_{r-m})_m, \]
Proof. Let \(m\) be a negative integer. Let \(z \in \mathcal{P}^{-m}\). By using (2.1), and the \(G_2(\mathcal{O})'\)-invariance property of \(W_r\), we compute
\[ W_r(a(t)n'(z)) = W_r\left(a'(t)n'(z^{-1})j'(z)n'(z^{-1})\right) = \psi(tz^{-1})W_r(a'(t)j'(z)) = \psi(tz^{-1})W_r(a'(t)j'(z)) = \psi(tz^{-1})\omega_\pi(\omega^m)W_{r+m}(a'(tz^{-2})). \]
Therefore,
\[
\int_{\mathcal{O}} \int_{\mathbb{R}^*m} W_r(\mathbf{a}'(\varpi^i u)\mathbf{n}'(z))dzdu = c_m \omega(\varpi)^m W_{r+m}(\mathbf{a}'(\varpi^{-2m})),
\]
where \( c_m = \begin{cases} 
q^{-m-1}(q-1) & \text{if } i \geq m, \\
-q^{-m} & \text{if } i = m - 1, \\
0 & \text{otherwise.}
\end{cases} \)

From this, the assertion follows. \( \square \)

**Lemma 5.6.** If \( W \in \mathcal{W}_\psi(\pi') \) is coquasi-paramodular, then
\[
Z(s, W_r) = q^{-e} \Xi(s, W_r) + \sum_{e-r \leq m < e} c_m q^{(e-m)(s-\frac{1}{2})},
\]
where
\[
c_m = \begin{cases} 
\mathbb{S}_m(\omega^{-1}, u_{\omega^{-1}})W_r([\varpi^{-e}; \varpi^m u_{\omega^{-1}}]) & \text{if } m > 0, \\
\omega^{-1}(u_{\omega} \varpi^e)\mathbb{G}(\omega, u_{\omega})W_{r+m}(\mathbf{a}_m - \mathbf{i}'(1)) & \text{if } m \leq 0.
\end{cases}
\]
The notation \([*; *], \mathbb{G}(\omega, u_{\omega}) \) and \( \mathbb{S}_m(\omega^{-1}, u_{\omega^{-1}}) \) are defined in sect. 2. Recall that \( \mathbb{G}(\omega, u_{\omega}) \) and \( \mathbb{S}_m(\omega^{-1}, u_{\omega^{-1}}) \) are not zero.

**Proof.** By Proposition 5.1, \( Z(s, W_r) = q^{-e} \Xi(s, W_r) + \sum_{e-r \leq m < e} \Xi_m, \) where
\[
\Xi_m = \Xi(s, \int_{\mathcal{O}^{*m}} \pi(n')(z)W_rdz).
\]
We will show \( \Xi_m = c_m q^{(e-m)(s-\frac{1}{2})} \) for the constant \( c_m \) in the assertion. Let \( m = \text{ord}(z) \).
Suppose \( 1 \leq m \leq e - 1 \). By the \( K_1(\psi)'(\mathbb{G}) \)-invariance property of \( W_r \), for \( u \in \mathcal{O}^\times \),
\[
W_r(\mathbf{a}'(\varpi^i u)\mathbf{n}'(z)) = W_r(\mathbf{a}'(\varpi^i u)\mathbf{n}'(z)\mathbf{a}'(u^{-1})) = W_r(\mathbf{a}'(\varpi^i u)\mathbf{n}'(u^{-1}z)).
\]
By Lemma 2.6, this is zero unless \( i = m - e \). Therefore, \( \Xi_m = c_m q^{(e-m)(s-\frac{1}{2})} \). Suppose \( e - r \leq m \leq 1 \). By (5.7),
\[
\int_{\mathcal{O}^*m} W_r(\mathbf{a}'(\varpi^i u)\mathbf{n}'(\varpi^m u_{\omega}))dudv = \int_{\mathcal{O}^*m} \psi(\varpi^i u(\varpi^m u_{\omega})^{-1})W_r(\mathbf{a}'(\varpi^i u)\mathbf{j}'(\varpi^m u_{\omega}))dudv = \int_{\mathcal{O}^*m} \psi(\varpi^{-i-m})\omega(\varpi^{-1}u_{\omega})dudv \]
where integrations are over \( u \in \mathcal{O}^\times \) and \( v \in \mathcal{O}^\times \). The last double integral equals the Gauss sum, which is zero unless \( i = m - e \). Since
\[
W_r(\mathbf{a}_m \mathbf{j}'(\varpi^m u_{\omega})) = \omega^{-1}(u_{\omega} \varpi^m)W_{r+m}(\mathbf{a}_m - \mathbf{i}'(1)),
\]
we have \( \Xi_m = c_m q^{(e-m)(s-\frac{1}{2})} \). \( \square \)

**Proposition 5.7.** Let \( \pi \in \text{Irr}^{\psi}(G) \), and \( W \in \mathcal{W}_\psi(\pi) \) be quasi-paramodular.

i) If \( W(T) = 0 \), then \( W(Q) = 0 \).

\( W(T) = W^c(T) = 0, \) then \( W^c(Q) = 0 \).

**Proof.** i) follows from the decomposition \( Q = NT_{\mathcal{O}}' \). In case of \( e = 0 \), \( W^c \) is paramodular, and ii) follows from i). Hence, we may assume \( e > 0 \). Let \( m \) be the level of \( W \). Let \( u = u_{\omega^{-1}} \) be the element in \( \mathcal{O}^\times \) defined in (2.8). We take \( \mathfrak{N} = \{ \mathbf{n}'(\varpi^i u) \mid 0 \leq i \leq e \} \) for the representatives of \( F^x \mathbb{N}A^T_{\mathcal{O}} Q/(Q \cap K^c(m; e)) \). It suffices to show that \( W^c(a_s n) = 0 \) for \( r, s \in \mathbb{Z}, \mathbf{n} \in \mathfrak{N} \). By assumption, \( W^c(a_s) = 0 \). From Lemma 2.2,
it follows that $W^c(\alpha^s_\psi \mathfrak{n}) = 0$ if $r < 0$, and that $W^c(\alpha^s_\psi \mathfrak{j}'(1)) = 0$ if $s < \varepsilon$. By Lemma 2.6, (2.1), our remained task is to show that

\begin{equation}
W^c(\alpha^s_\psi \mathfrak{n}'(\omega^s u)) = W^c(\alpha^s_\psi \mathfrak{j}'(1)) = 0
\end{equation}

for $r \geq 0$, $i \geq -\varepsilon$ and positive $s \leq \varepsilon - 1$. By i) and Lemma 5.5, $Z(s, W_\tau) = 0$. By the functional equation, $Z(s, (W^c)_\tau) = 0$. Since $W^c(\mathbb{T}) = 0$, we have $\Xi(s, (W^c)_\tau) = 0$. Now (5.8) follows from Lemma 5.6.

**Proposition 5.8.** Let $(\pi, V) \in \text{Irr}^m(\mathbb{G})$. If $W \in V^c(m)$ for some $m$ vanishes on $\mathbb{Q}$, then $W \in V(\mathbb{Z}^J)$.

**Proof.** We have constructed a (unique up to a constant multiple) nontrivial functional for each $\tau_j(\sigma)$ with $\sigma \in \text{Irr}(G_j), j = 1, 2$ in section 3. In case of $j = 1$, the functional is $\lambda^b_1$ with $\chi = \sigma \nu_{-1}$, and corresponds to the functional

$$W^c(\pi^*) \ni W \mapsto \int_{F^\times} \int_{\mathbb{N}^\times} W(\mathfrak{n}')(t_1(b))\sigma^{-1} \nu_s(t) d\mathfrak{n}^x t$$

where $s = 1$. In case that $j = 2$, and $\sigma \in \text{Irr}^m(G_2)$ (hence infinite-dimensional), it is $\mu^b_2$ and corresponds to the functional

$$W^c(\pi^*) \ni W \mapsto \int_{F^\times} \int_{\mathbb{N}^\times} W(t_1(u)\mathfrak{n}'(b))\omega^{-1} \nu_s(u) d\mathfrak{n}^x u,$$

where $s = 0$. In case that $j = 2$, and $\sigma = \xi \circ \det$ with $\xi \in \mathcal{X}(F^\times)$, it is $\mu^b_2$ and corresponds to the functional

$$W^c(\pi^*) \ni W \mapsto \int_{F^\times} \int_{\mathbb{N}^\times} W(t_1(u)\mathfrak{n}')\xi^{-1} \nu_s(u) d\mathfrak{n}^x u,$$

where $s = 0$. Since $W(Q) = 0$, all these functionals send $W$ to 0. Now, let $\sigma \in \text{Irr}(G_j)$ and $f \in \tau_j(\sigma)$. If $f$ is sent to 0 by the corresponding functional, and satisfies

\begin{equation}
f(pk) = \omega_\pi(k_{22}^{-1} f(p)) \quad \text{for} \quad k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{P}^{-\varepsilon} \\ \mathcal{O} & \mathcal{O} & 1 \end{bmatrix} (= \text{pr}(K^c(m; \varepsilon))),
\end{equation}

then we have $f = 0$. Indeed, it follows from Lemma 5.9 below in the first case, from the newform theory for $G_2$ with $m_\sigma = \varepsilon$ in the second case, and from the one-dimensionality of $\tau_2(\xi \circ \det)$ in the third case. Therefore, $W \in V_0$ by Theorem 4.1. Let $\mathcal{W}_0 = \{W \in V \mid W(Q) = 0\}$. By the proof of Theorem 4.3.5 of [R-S], $\mathcal{W}_0 \subset V(\mathbb{Z}^J)$. This completes the proof. \qed

**Remark 1.** The last two integrals are absolutely convergent if $\Re(s) >> 0$, and analytically continued to the whole complex plane. They are related to the so-called degree five $L$-function of $\pi$. We will discuss them in a forthcoming paper.

**Lemma 5.9.** Let $\xi \in \mathcal{X}(F^\times)$, and $f \in \tau_1(\xi)$. If $f$ satisfies (5.9), and $\lambda^b_1(f) = 0$ for $\chi = \xi \nu_{-1}$ and any $b \in F^\times$, then $f$ is identically zero.

**Proof.** By (5.9) and the decomposition $P_3 = NAZ_2^0 G_2(\mathcal{O})$, it suffices to show that $f(z_2'(b)\mathfrak{n}'(z)) = 0$ for any $b \in F^\times, z \in F$. In case of $\varepsilon = 0$, the assertion follows immediately from the decomposition. Assume that $\varepsilon > 0$. Let $l = \text{ord}(z) - \varepsilon + 1$, and $x \in \mathcal{O}$,

\begin{align*}
f(z_2'(b)\mathfrak{n}'(z)) &= \xi(\omega)^{-l} f(a_l z_2'(b)\mathfrak{n}(z)) \\
&= \xi(\omega)^{-l} f(n(x) a_l z_2'(b)\mathfrak{n}(z)) \\
&= \xi(\omega)^{-l} f(a_l z_2'(b)\mathfrak{n}(z) k') \\
&= \omega_\pi(1 - \omega^{-l} x) \xi(\omega)^{-l} f(a_l z_2'(b)\mathfrak{n}(z)),
\end{align*}

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where \( k = \text{Int}^{-1}([\omega^l; z])n(x) \in K(e) \subset G_2(\text{c.f. (2.11)}) \). There is an \( x \in \mathcal{O} \) such that \( \omega_\epsilon(1-\omega^{-1}z \bar{z}) \neq 1 \). From Lemma 2.2, \( f(z'_2(b) \bar{n}(z)) = 0 \) follows in this case. Consequently, \( \lambda^0_1(f) = \text{vol}(N(\mathcal{P}^+))f(z'_2(b)) \), and the assertion follows. \( \square \)

Let \((\pi, V) \in \text{Alg}(G)\). For a moment, by abuse of notation, we denote by \( V^c(m) \) the subspace of \( K^c(\varepsilon; e) \)-invariant vectors in \( V \). We will use the following level +2 raising operator \( \eta \), and level +1 one \( \alpha_m \) for \( V^c(m) \) \((\varepsilon \text{ may be zero})\):

\[
\eta : V^c(m) \ni v \mapsto \pi^i(\eta)v \in V^c(m+2),
\]

\[
\alpha_m : V^c(m) \ni v \mapsto \sum_{k \in K^c(m+1; x)/K^c(m;x) \cap K^c(m+1; x)} \pi^i(k)v \in V^c(m+1).
\]

Computing the coset space, we have

\[
\alpha_m v - \sum_{x \in \mathcal{O}/\mathcal{P}} \pi^i(n_3(\omega^{-m-1}x))v = \pi^i(j_{m+1}'' v)
\]

\[
= \pi^i(j_{m+1}'') \pi^i(j_m'')v = \eta v.
\]

**Proposition 5.10.** Let \((\pi, V) \in \text{Alg}(G)\). Assume that \( V^{Sp(4, F)} = \{0\} \). If \( v \in V(Z^l) \) is \( K_c^c(m; \varepsilon) \)-invariant \((\varepsilon \text{ may be zero})\), then \( v = 0 \).

**Proof.** Write the level raising operator \( \alpha_m = \eta + q z_{m+1} \), where \( z_{m+1} \) is the linear operator defined by \( \text{vol}(\mathcal{P}^{-m-1}) \int \pi(n)dn \) with integration over \( n \in \mathcal{N}(\mathcal{P}^{-m-1}) \). At first, we will show by induction that there exist certain linear operators \( \beta_r : V^c(m) \rightarrow V^c(r+m+1) \) and \( \gamma_r : V^c(m) \rightarrow V^c(r+m) \) such that

\[
(5.10) \quad z_{r+m} = \beta_r + \gamma_r.
\]

For \( r = 1 \), this holds obviously. Assume \((5.10)\) for \( r \geq 1 \). Since \( v \) lies in \( V^c(m) \),

\[
egin{align*}
zh_{r+m+1}v &= z_{r+m+1} \circ z_{r+m}v \\
&= q^{-1}(\alpha_{r+m} - \eta) \circ (\beta_r + \gamma_r)v \\
&= q^{-1}(-\eta \gamma_r v + (\alpha_{r+m} \circ \gamma_r + z_{r+m+1} \circ \beta_r)v) \\
&= -q^{-1} \eta \gamma_r v + q^{-1}(\alpha_{r+m} \circ \gamma_r + \beta_r)v
\end{align*}
\]

where the assumption \( \beta_r v \in V^c(r+m+1) \), and \((z_{r+m+1} \circ \beta_r)v = \beta_r v \) are used at the last equality. Therefore, \( \beta_{r+1} := -q^{-1} \eta \gamma_r : V^c(m) \rightarrow V^c(r+m+2) \)

\[
(5.11) \quad \beta_{r+1} := -q^{-1} \eta \gamma_r : V^c(m) \rightarrow V^c(r+m+2)
\]

\[
(5.12) \quad \gamma_{r+1} := q^{-1}(\alpha_{r+m} \circ \gamma_r + \beta_r) : V^c(m) \rightarrow V^c(r+m+1)
\]

are the desired linear operators. This proves \((5.10)\). Next, we will show \( v = 0 \). Since \( v \in V(Z^l) \),

\[
(\alpha_{r+m-1} - \eta) \circ \cdots \circ (\alpha_m - \eta)v = z_{r+m}v = 0
\]

for a sufficiently large \( r \). Since \( \beta_r v \) and \( \gamma_r v \) have different levels and are linearly independent, \( \beta_r v = \gamma_r v = 0 \) by Theorem 5.3. By \((5.11)\), \( \eta \gamma_{r-1} v = 0 \). Since \( \eta \) is obviously injective, \( \gamma_{r-1} v = 0 \). Therefore \( \alpha_{r+m-1} \circ \gamma_{r-1} v = 0 \). By \((5.12)\), \( \beta_{r-1} v = 0 \). Thus \( z_{r+m-1} v = \beta_{r-1} v + \gamma_{r-1} v = 0 \). Hence, \( v = 0 \). \( \square \)

Now, we can prove Theorem 5.4. Suppose that \( W \in V(m) \) with \( W(T) = 0 \). By Proposition 5.7, \( W(Q) = 0 \). Let \( i \) be an arbitrary nonnegative integer. By Lemma 5.5, \( Z(s, W_i) = 0 \). By the functional equation, \( Z(s, (W^c)_i) = 0 \). By Lemma 5.6, \( \Xi(s, (W^c)_i) = \)
0. Thus, \( W^c(T) = 0 \). By Proposition 5.7 again, \( W^c(Q) = 0 \). By Proposition 5.8, \( W^c \in V(Z^I) \). By Proposition 5.10, \( W^c \) is identically zero, and so is \( W \).

**Lemma 5.11.** If \( \pi \in \text{Irr}^m(G) \), then \( m_\pi \geq n'_\pi \).

**Proof.** Let \( W \in V(m_\pi) \). Let \( i_0 \) be the minimal nonnegative integer such that \( \Xi(s, W_{i_0}) \neq 0 \). By Theorem 5.4, such \( i_0 \) exists. By Lemma 5.5, \( Z(s, W_{i_0}) = \Xi(s, W_{i_0}) \neq 0 \). The functional equation for \( W_{i_0} \) is

\[
\frac{Z(1-s, (W^c)_{i_0})}{L(1-s, \pi^r)} = \varepsilon' q^{(m_s-2i_0-n'_\pi)(s-\frac{1}{2})} \Xi(s, W_{i_0}) \frac{L(s, \pi)}{L(s, \pi^r)}.
\]

The right hand side lies in \( q^{(m_s-2i_0-n'_\pi)n} \mathbb{C}[q^{-s}] \). By Lemma 6.1, the left hand side lies in \( q^{-i_0s} \mathbb{C}[q^s] \). In case of \( \varepsilon = 0 \), we may assume \( W^c = \pm W \) by (5.1) and the left hand side lies in \( \mathbb{C}[q^s] \). Therefore, \( m_\pi - n'_\pi - i_0 \geq 0 \) in any case. Thus the assertion. \( \square \)

**Theorem 5.12.** Let \( \pi \in \text{Irr}^m(G) \). Assume that \( V(m) \) contains a \( W_0 \) such that

\[
(5.13) \quad L(s, \pi) = Z(s, W_0), \quad \text{and} \quad c_0 L(s, \pi^r) = Z(s, W_0^c)
\]

for a constant \( c_0 \). Then, \( \varepsilon' = \varepsilon_\pi' \), and \( m = n'_\pi = m_\pi \). Further, \( V(m_\pi) \) is spanned by \( W_0 \).

**Proof.** From (5.13), and the functional equation (5.3) for \( W = W_0 \), it follows that \( \varepsilon' = \varepsilon_\pi' \) and \( m = n'_\pi \). By Lemma 5.11, \( m = m_\pi = n'_\pi \). For the last assertion, we will show that an arbitrary \( W \in V(m) \) is a constant multiple of \( W_0 \). Since \( \Xi(s, W) \) is in \( \mathbb{C}[q^{-s}] \), the ring of formal power series in \( q^{-s} \), and \( \Xi(1-s, W^c) \) is in \( \mathbb{C}[q^s] \),

\[
\Xi(q^s) \ni q^{-\varepsilon_\pi' s} \Xi(1-s, W^c) L(1-s, \pi^r) = \varepsilon'_\pi \Xi(s, W) L(s, \pi^r) \in \mathbb{C}[q^{-s}].
\]

Therefore, these quotients are constants, and there exists a constant \( c_W \) such that

\[
\Xi(s, W - c_W W_0) = \Xi(1-s, (W - c_W W_0)^c) = 0.
\]

Set \( W' = W - c_W W_0 \). We will claim by induction that \( \Xi(s, W'_r) = 0 \) for any \( r \geq 0 \). Assume that \( \Xi(s, W'_r) = \Xi(1-s, (W'^c)_{i_0}) = 0 \) for all \( i < r \). Then, \( Z(s, W'_r) = \Xi(s, W'_r) \) by Lemma 5.5. The functional equation for \( W'_r \) is

\[
\frac{Z(1-s, (W'^c)_{r})}{L(1-s, \pi^r)} = \varepsilon'_r q^{-2r(s-\frac{1}{2})} \Xi(s, W'_r) \frac{L(s, \pi^r)}{L(s, \pi^r)}.
\]

(note that \( (W'^c)_r = \pi^r(j_m-2r)W^c_r \)). The right hand side lies in \( q^{-2rs} \mathbb{C}[q^{-s}] \). In case of \( \varepsilon > 0 \), the left hand side lies in \( q^{-rs} \mathbb{C}[q^s] \) by Lemma 5.6. In case of \( \varepsilon = 0 \), \( W^c \) is also paramodular, and the left hand side lies in \( \mathbb{C}[q^s] \) by Lemma 5.5, again. Hence both sides are zero, and the claim is verified. Now, \( W = c_W W_0 \) by Theorem 5.4. This completes the proof. \( \square \)

We will call \( W_0 \) as in this theorem the newform of \( \pi \), and denote by \( W_\pi \).

6. HECKE OPERATORS

Let \( \chi \in \mathcal{D}(F^\times) \), and \( (\sigma, V) \in \text{Irr}(G_2) \). The Klingen parabolic induction \( \chi \times \sigma \) consists of smooth \( V \)-valued functions \( f \) on \( G \) such that

\[
L(q^{-1})f = |t \det(g)^{-1}| \chi(t) \sigma(g)f,
\]

(6.1) where \( q = \begin{bmatrix} t & * & * \\ g & * & * \\ t^{-1} \cdot \det(g) \end{bmatrix} \in \mathbb{Q}, \quad t \in F^\times, g \in G_2. \)
A Klingen parabolic induction has a unique generic constituent (submodule, c.f. sect. 2.4. of [R-S]). We call $\chi \times \sigma$ a Klingen parabolic induction from supercuspidal when $\sigma$ is supercuspidal. By the work of [T], when $\pi \in \text{Irr}^g(G)$ is supercuspidal, or the constituent of a Klingen parabolic induction from supercuspidal, $L(s, \pi)$ equals 1. In this section, we devote to prove

**Theorem 6.1.** Let $\pi \in \text{Irr}^g(G)$ be supercuspidal, or a constituent of the Klingen parabolic induction from supercuspidal. Assume $\epsilon > 0$. Then, there exists the newform $W_\pi$ in $V(m_\pi)$ ($n_\pi = m_\pi$ and $\dim V(m_\pi) = 1$ by Theorem 5.12). The newform $W_\pi$ and its conjugate $W_\pi^c$ take the following values on $T$:

$$W_\pi(a_i^r) = \begin{cases} 1 & \text{if } (i, r) = (0, 0), \\ 0 & \text{otherwise}. \end{cases}$$

$$W_\pi^c(a_i^r) = \begin{cases} \varepsilon_\pi' & \text{if } (i, r) = (0, 0), \\ -q^2\varepsilon_\pi' & \text{if } (i, r) = (1, 0), \\ 0 & \text{otherwise}. \end{cases}$$

See Corollary 7.4.6. [R-S] for the case of $\epsilon = 0$. In this section, we assume $\epsilon > 0$.

Our proof consists of four steps.

**Step 1.** For a nontrivial polynomial $\sum c_n X^n \in \mathbb{C}[X^\pm] := \mathbb{C}[X, X^{-1}]$ with $X = q^{-s}$, we call its range the pair of the minimal and maximal integers $n$ such that $c_n \neq 0$. In case of $L(s, \pi) = 1$, for any $W \in \mathcal{H}_q^{\psi}(\pi), i \in \mathbb{Z}$, $Z(s, W_i)$ lies in $\mathbb{C}[X^\pm]$ by definition. In this step, we show that $\Xi(s, W_i) \in \mathbb{C}[X^\pm]$, if $W$ is quasi-paramodular.

**Lemma 6.2.** Let $\pi \in \text{Irr}^g(G)$, and $W \in V(m)$. If $L(s, \pi) = 1$, then $W(a_i^r) = W^c(a_i^r) = 0$ for sufficiently large $i, r$.

**Proof.** Let $i$ be a nonnegative integer. Since $L(s, \pi)$ equals 1, so does $L(s, \pi^i)$. Therefore, both $Z(s, W_i)$ and $Z(s, (W^c)_i)$ are polynomials in $q^{\pm s}$. Let $(c_i, d_i)$ and $(c_i^*, d_i^*)$ be their ranges, respectively. By Lemma 5.5, 5.6, both $\Xi(s, W_i)$ and $\Xi(s, (W^c)_i)$ are polynomials in $q^{\pm s}$. Let $(a_i, b_i)$ and $(a_i^*, b_i^*)$ be their ranges, respectively. From the functional equation for $W_i$,

$$ (c_i + n_\pi^* - m + 2i, d_i + n_\pi^* - m + 2i) = (-d_i^*, -c_i^*). \tag{6.2} $$

Now, assume that $\Xi(s, W_i) \neq 0$ for infinitely many $i$’s. Then, we may take an $i_1 \geq m - n_\pi^*$ so that $\Xi(s, W_{i_1}) \neq 0$, and $b_{i_1} + 2i_1 \geq b_n + 2n$ for all $n < i_1$. By Lemma 5.5, $d_{i_1} = b_{i_1}$. By Lemma 5.6, $c_{i_1}^* \geq -i_1$. By (6.2), $b_{i_1} + n_\pi^* - m + 2i_1 \leq i_1$. Since $b_{i_1} \geq a_{i_1} \geq 0$ by Lemma 2.2, $i_1 \leq m - n_\pi^* - b_{i_1} < m - n_\pi^*$. This is a contradiction. Hence, $\Xi(s, W_i) = 0$ for sufficiently large $i$. Therefore, there is an integer $I$ such that $\Xi(s, W_i) = 0$ for all $i > I$ and $b_i < I$ for all $i < I$. If $i > 2I$, then $Z(s, W_i) = 0$ by Lemma 5.5, $Z(s, (W^c)_i) = 0$ by the functional equation, and $\Xi(s, (W^c)_i) = 0$ by Lemma 5.6. This completes the proof. \hfill \square

**Step 2.** In this and next steps, we assume that $(\pi, V) \in \text{Irr}(G)$ is unitary, and use several Hecke operators. For $h \in G$, let $T_K(h)$ denote the Hecke operator acting on $V^K$ defined by

$$ T_K(h)v = \sum_{t \in K/K \cap \text{Int}(h)K} \pi(th)v. $$
Lemma 6.3. With notations as above, if $(\pi, V)$ is unitary, then the Hecke operator $T_K(h) + T_K(h^{-1})$ on $V^K$ is diagonalizable. In particular, if $T_K(h) = T_K(h^{-1})$, then $T_K(h)$ is diagonalizable.

Proof. Let $\langle *, * \rangle$ denote the inner product in $V$. In general, $\langle T_K(h)v, w \rangle = \langle v, T_K(h^{-1})w \rangle$ for $v, w \in V^K$ (see Lemma 6.5.1 of [R-S]). From this, the assertion follows immediately. \hfill \Box

We use the diagonalities of Hecke operators repeatedly, and therefore the unitarity assumption is needed. In this step, we show the next basic inequality $m_\pi > 2c$, which is an analogue of Proposition 2.4. This inequality is essentially important for the comparison of Hecke operators and level descending $V(m_\pi) \to V(m_\pi - 1)$. Since the quasi-paramodular group $K_1(m;c)$ is defined for $m \geq 2c$, one cannot consider the level descending $V(m_\pi) \to V(m_\pi - 1)$ in case of $m_\pi = 2c$. In [R-S], for the $PGSp(4)$ case, to compute some Hecke operators, the condition $m_\pi \geq 2$ for supercuspidal $\pi$ was used.

Proposition 6.4. Let $\pi \in \text{Irr}^{2n}(G)$ be unitary. If $L(s, \pi) = 1$, then $m_\pi > 2c$.

Let $S = T_K(\eta) + T_K(\eta^{-1}) \circ V(m_\pi)$ with $K = K_1(m_\pi;c)$. This Hecke operator is diagonalizable by Lemma 6.3. Let with $l = m_\pi - c$, we compute

\begin{equation}
S = \sum \pi \left( n_2(x)n_3(y)z(\omega^{c-\ell})\eta \right) + \sum \pi \left( \eta^{-1}\tilde{n}_2(x\omega^{l-1})\tilde{n}_3(y\omega^{l-1})\tilde{z}(z\omega^{m_\pi-2}) \right),
\end{equation}

where both sums are over $x, y \in \mathcal{O}/\mathcal{P}, z \in \mathcal{O}/\mathcal{P}$. In case of $m_\pi > 2c$, one can find that the latter sum is zero, by comparing with the level descending $\int_{K_1(m_\pi-1;\mathcal{E})} \pi(k)dk : V(m_\pi) \to V(m_\pi - 1)$. In case of $m_\pi = 2c$, the sum is not zero, as follows. For $W \in V(2c)$, we set

\[ W' = \sum_{x \in \omega^cE_i / \omega^cE_i} \pi(\tilde{n}(z))W, \quad W'' = \sum_{x \in \mathcal{O}/\mathcal{P}} \pi(\tilde{n}_2(x\omega^{l-1}))W', \]

where $\mathcal{E}_a$ is defined in (5.5).

Lemma 6.5. With notation as above, $W''(a_r^i) = W'(a_r^i) = W(\bar{a}_r^i)$ for $i, r \geq 0$.

Proof. For the first identity, consider the second gauge $\xi$ of $\pi(a_r)W$, which is quasi-invariant on $K(\mathcal{E})$. The mapping $C \mapsto w_2^tAw_2CA$ induces a translation in $\omega^{l-1}E_{i-1}/\omega^{l-1}E_{i}$ if $A \in K(\mathcal{E})$. Hence $R(k)\xi = w_\pi(k_{22})\xi$ for $k \in K(\mathcal{E})$ by the identity (5.6). It suffices to show that

\[ 0 = \sum_{x \in \mathcal{O}^* / \mathcal{P}} \xi(\tilde{n}(x\omega^{c-1})) \]

for $i \geq 0$. This follows from Lemma 2.6. For the second identity in case of $c = 1$, we compute, for $i, r \geq 0$,

\[
\sum W(a_r^i\tilde{n}(C)) = \sum W(a_r^i\tilde{n}(C^{-1})j(C)n(C^{-1})) = \sum W(a_r^i, j(C)) = \sum \omega_\pi(x)W(a_r^i, \tilde{n}_2(y)j(1_{2})) = 0
\]
with the sums being over \( x \in (\mathcal{O}/\mathcal{P})^\times, y \in \mathcal{O}/\mathcal{P}^2 \), where

\[
C = \begin{bmatrix} x & \bar{x} \\ y & x \end{bmatrix}.
\]

By a similar argument,

\[
\sum_{y \in (\mathcal{O}/\mathcal{P}^2)^\times} \pi(z(y))W(a_i^j) = \sum_{y \in (\mathcal{O}/\mathcal{P})^\times} \pi(z(y\varpi))W(a_i^j) = 0.
\]

Now the second identity in case of \( c = 1 \) follows immediately. For the case of \( c > 1 \), if \( c \in \varpi^{e-1} \mathcal{O}_c \setminus \varpi^e \mathcal{O}_c \), then \( W(a_i^j \tilde{n}(c)) = 0 \) follows from Lemma 2.2, and the identity (2.2) with

\[
b \in \left[ \mathcal{O}^\times \mathcal{O}^\times \right] \cup \left[ \mathcal{P}^{*(1-e)} \right].
\]

The second identity follows in this case. This completes the proof. \( \square \)

Now, we can prove Proposition 6.4. By Lemma 6.3, it suffices to show that each eigenform \( W \in V(2c) \) of \( S \) is identically zero. Let \( \lambda_S \) be the eigenvalue of \( W \). From (6.3) and Lemma 6.5, it follows that

\[
\lambda_S W(a_i^j) = q^i W(a_{i+1}) + W(a_{i-1}), \quad i \geq 0.
\]

Fix \( r \geq 0 \). By Theorem 5.4 and Lemma 6.2, we can take the maximal integer \( i_0 \) such that \( W(a_i^{u_0}) \neq 0 \) for some \( r \), if \( W \) is not identically zero. By this recursion formula with \( r = i_0 + 1 \), \( W(a_i^{u_0}) = 0 \). This is a contradiction. Hence, \( W \) is identically zero.

An immediate consequence of this proposition is the next:

**Proposition 6.6.** Let \( \pi \in \text{Int}^{*m}(G) \) be unitary. Assume \( L(s, \pi) = 1 \). If \( W \in V(m_\pi) \) is a nontrivial form, then, \( \Xi(s, W) = 0, i \geq 1 \) and \( \Xi(s, W) \) and \( \Xi(s, W_\pi) \) are nonzero.

**Proof.** Let \( l = m_\pi - c \). Since \( L(s, \pi) = 1, l - 1 = m_\pi - c - 1 > 0 \geq 0 \) by Proposition 6.4. Then, \( W_1 := \eta W \) is invariant under the subgroup

\[
\text{Int}(\eta)K_1(m_\pi; c) = \left[ \begin{array}{cccc} \mathcal{O} & \mathcal{P} & \mathcal{P} & \mathcal{P}^{2-1} \\ \mathcal{P}^{l-1} & \mathcal{O} & \mathcal{P} & \mathcal{P} \\ \mathcal{P}^{l-1} & \mathcal{O} & \mathcal{P} & \mathcal{P} \\ \mathcal{P}^{m_\pi-2} & \mathcal{P}^{l-1} & \mathcal{P}^{l-1} & 1 + \mathcal{P} \end{array} \right],
\]

and \( W'_1 := \sum_{x, y, z \in \mathcal{O}/\mathcal{P}} \pi(n_2(x)n_3(y)z(z\varpi^{e-1})) W_1 \) lies in \( V(m_\pi-1) \). Assume that \( W(a_i^j) \neq 0 \) for some \( i \geq 1 \). Then \( W_1(a_i^{j-1}) \neq 0 \), and \( W_1'(a_i^{j-1}) \neq 0 \). This contradicts the level minimality. Hence \( W(a_i^j) = 0 \) for all \( i \geq 1 \). By Theorem 5.4, \( W(a_i^j) \neq 0 \) for some \( r \). Therefore, \( \Xi(s, W) \) is nonzero, and so is \( \Xi(s, W_\pi) \) by the functional equation (5.3). This completes the proof. \( \square \)

**Step 3.** To show Theorem 6.1, we need the level descending operator

\[
\mathcal{D} := \int_{\mathcal{K}^{(m_\pi-1; t)}} \pi^i(k)dk : V^c(m_\pi) \to V^c(m_\pi - 1)
\]

for \( m_\pi > 2c \), and the Hecke operators

- \( T = T_{K_1(m; c)}(a_1) + T_{K_1(m; c)}(a_{-1}) \)
- \( T' = T_{K_2(m; c)}(a_1) + T_{K_2(m; c)}(a_{-1}) \)
- \( S' = T_{K_3(m; c)}^*(\eta) \)
- \( T'_{+} = T_{K_3(m; c)}^*(a_1) \)
The first three Hecke operators are self-adjoint and diagonalizable by Lemma 6.3. It is not hard to show that $c \circ T = T' \circ c$. First, compare the actions of $S'$ and $D$. We compute
\[ S' = \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi^i (n_2(x)n_3(y)) z(u) \eta^{-1} z(u) \]
\[ + \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi^i (n_2(x)z(u) \eta^{-1}) \]
\[ \mathcal{D} = \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi^i (n_2(x)n_3(y)) \eta^{-1} \]
\[ + \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi^i (n_2(x)z(u) \eta^{-1}) \]
\[ \text{(c.f. Lemma 3.3.7., 6.1.2. of [R-S]).} \]
Comparing their latter sums, we have
\[ \lambda_{S}\omega(x_i) = \begin{cases} q^{2i} \omega(x_i) & \text{if } i = 0, \\ q^{2i} \omega(x_i - 1) & \text{if } i > 0, \end{cases} \]
for an eigenvector $W^c \in V^c(m_\omega)$ with eigenvalue $\lambda_{S}$. By Proposition 6.6, there is a non-negative integer $r$ such that $W^c(a_r) \neq 0$. By Lemma 6.2 and the above recursion formula, $\lambda_{S}$ is equal to 0 or $-q^2$. Assume that $\lambda_{S} = 0$. Then, $\Xi(s, \omega_1) = 0$. By Proposition 6.6, $\Xi(s, W_1) = 0$. By Lemma 5.5, $Z(s, W_1) = \omega_{\pi}(\omega)^{-1} q^{2s-4} \Xi(s, W_1)$. Therefore, the functional equation for $W$, and that for $W_1$ are
\[ q^{-1} \Xi(s, W) = \varepsilon q^{m-n} \Xi(s, W), \]
\[ Z(s, (W^c)_1) = \varepsilon q^{m-n} \Xi(s, W_1). \]
Since $\Xi(s, W)$ and $\Xi(s, W_1)$ are polynomials in $q^{-s}$ with a same range, $\Xi(s, W^c)$ and $Z(s, (W^c)_1)$ have a same range. By Lemma 5.6, $Z(s, (W^c)_1)$ is a constant multiple of $q^{-s}$, since $\Xi(s, W^c)$ is a polynomial in $q^s$. This is a contradiction. Hence,
\[ \lambda_{S} = -q^2, \quad W^c(a_r) = -q^2 W^c(a_r) \quad \text{for all } W^c \in V^c(m_\omega). \]
Next, for $T, T'$, letting $\mathcal{C}_a, \mathcal{B}_b$ be the lattice defined in (5.5), and
\[ \mathcal{B}_a = \mathcal{B}_a \oplus \left[ \begin{array}{c} \mathcal{O} \\ \mathcal{O} \end{array} \right], \quad \mathcal{C}_a = \mathcal{C}_a \oplus \left[ \begin{array}{c} \mathcal{O} \\ \mathcal{O} \end{array} \right], \]
we compute
\[ T_{K_1(m, \varepsilon)}(a_1) = \sum_{B \in \mathcal{B}_1} \pi(n(B)a_1) + \pi(j'(1)n(B)a_1) \]
\[ \text{Since } j'(1) \in K_1(m, \varepsilon), \text{ the latter sum equals} \]
\[ \sum_{B \in \mathcal{B}_1} \pi(j'(1)n(B)a_1) = \omega_{\pi}(\omega) \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi(n_2(x)z(u) \eta^{-1})a_1. \]
Similarly,
\[ T_{K_1(m, \varepsilon)}(a_{-1}) = \sum_{C \in \mathcal{C}_a \mathcal{C}_b} \pi(n(C)a_{-1}j'(1)) + \sum_{C \in \mathcal{C}_a \mathcal{C}_b} \pi(n(C)a_{-1}) \]
\[ = \omega_{\pi}(\omega)^{-1} \sum_{x,y \in \mathcal{O}/\mathcal{O}^2} \pi(n_2(x)z(u) \eta^{-1})a_{-1}. \]
Using (2.1) and the $K_1(m;c)$-invariance property, the sum in the bracket is transformed to
\[
\sum_{C \in \omega^l \mathcal{E}_i / \omega^{l+1} \mathcal{E}_i} \pi(n'(z^{-1})j'(z)n'(z^{-1})\tilde{n}(C)a_{-1}) \\
= \sum_{x,y \in \Theta / \mathcal{P}} \pi(n'(z^{-1})j'(z)n'(z^{-1})a_{-1}\tilde{n}_3(x\varpi^{l^{-1}})\bar{z}(y\varpi^{m^{-1}})) \\
= \sum_{x,y \in \Theta / \mathcal{P}} \pi(n'(z^{-1})j'(z)a_{-1}\tilde{n}_3(x\varpi^{l^{-1}})\bar{z}(y\varpi^{m^{-1}})) \tilde{n}_2(z^{-1}y\varpi^{m^{-1}})\bar{z}(-z^{-1}y^2\varpi^{2m^{-1}})).
\]

Since $K_1(m;c)$ contains $\tilde{n}_2(z^{-1}y\varpi^{m}), \bar{z}(-z^{-1}y^2\varpi^{2m^{-1}})$, this sum equals
\[
\sum_{C \in \omega^{l-1} \mathcal{E}_i / \omega^{l} \mathcal{E}_i} \pi(n'(z^{-1})j'(z)a_{-1}\tilde{n}(C)) = \sum_{C \in \omega^{l-1} \mathcal{E}_i / \omega^{l} \mathcal{E}_i} \pi(n'(z^{-1})j'(z)a_{-1}\tilde{n}(C))j'(z)) \\
= \omega_\pi(\varpi)^{-1} \sum_{x,y \in \Theta / \mathcal{P}} \pi(n'(z^{-1})a_{-1}\tilde{n}_2(x\varpi^{l^{-1}})\bar{z}(y\varpi^{m^{-1}})).
\]

Therefore, for $W \in V(m)$,
\[
\mathcal{T}W(a_i^j) = q^3W(a_{i+1}^j) + q^2\omega_\pi(\varpi)W(a_{i+1}^{j+1}) + \sum_{C \in \omega^{l-1} \mathcal{E}_i / \omega^{l} \mathcal{E}_i} W(a_{i-1}^j\tilde{n}(C)) \\
+ q\omega_\pi(\varpi)^{-1} \sum_{x,y \in \Theta / \mathcal{P}} W(a_{i+1}^{j-1}\tilde{n}_2(x\varpi^{l^{-1}})\bar{z}(y\varpi^{m^{-1}})), \; i, r \geq 0.
\]

By a similar computation, for $W^c \in V^c(m)$,
\[
\mathcal{T}'W^c(a_i^j) = q^3W^c(a_{i+1}^j) + q^3 \sum_{z \in \Theta / \mathcal{P}} W^c(a_{i-1}^j\tilde{n}'(z\varpi^{s^{-1}})) \\
+ \sum_{C \in \omega^{m-1} \mathcal{R}_i} W^c(a_{i-1}^j\tilde{n}(C)) + q \sum_{x,y \in \Theta / \mathcal{P}} W^c(a_{i+1}^{j-1}\tilde{n}_2(x\varpi^{l^{-1}})), \; i, r \geq 0.
\]

Observing the first gauge of $W^c (K_1(c)')(\subset P_3)$-invariant!, one can find by Lemma 2.6 that the second term equals $q^3W^c(a_{i+1}^{j+1})$ if $r \geq 1$. We choose the Haar measures in (5.4) so that the third terms of (6.7) and (6.8) are equal to the values at $a_{i-1}^j$ of $\Gamma_{l-1}W$ and $\Gamma_{m-1}^nW$, respectively.

**Lemma 6.7.** Let $\pi \in \text{Irr}^\text{un}(G)$ be unitary. Assume that $L(s, \pi) = 1$. Let $W \in V(m)$ such that $\Xi(s, W) \neq 0$. Then, for the Haar measures as above, we have the following identities.

\[
\Gamma_{l-1}W(a_r) = \begin{cases} 0 & \text{if } r \geq r_0 \\ qW(a_r) & \text{otherwise} \end{cases}, \quad \Gamma_{m-1}^nW^c(a_r) = qW^c(a_r),
\]

where $l = m - c$, and $r_0$ is the maximal integer such that $W(a_{r_0}) \neq 0$ (such $r_0$ exists by Lemma 6.2).

**Proof.** Since the arguments are similar, we only prove the first identity. We observe the both sides of the functional equation (5.3) and
\[
(6.9) \quad Z(s, \pi'(j_m)(\Gamma_{l-1}W)^s) = q^{(m-n'_c)(s-1/2)}Z(s, \Gamma_{l-1}W).
\]
By Lemma 5.2, Proposition 6.4, $\Gamma_{l-1}W$ is 0-balanced, and $\pi^i(j_m)(\Gamma_{l-1}W)^i$ is $(\epsilon + 1)$-balanced. By Proposition 5.1,

\[ Z(s, \Gamma_{l-1}W) = \Xi(s, \Gamma_{l-1}W), \quad Z(s, \pi^i(j_m)(\Gamma_{l-1}W)^i) = q^{-\epsilon - 1}\Xi(s, \pi^i(j_m)(\Gamma_{l-1}W)^i). \]

Since

\[ \Xi(s, \pi^i(j_m)(\Gamma_{l-1}W)^i) = \Xi(s, \sum_{y, z \in \mathcal{O}/\mathcal{P}} \pi(n_3(yzw^{-\epsilon})n'(zw^{-1})\pi(jm)W^c) \]

we have

\[ q^{1-\epsilon}(\Xi(1 - s, W^c) - W^c(1)) = q^{(m-n_s)(s-1/2)}\Xi(s, \Gamma_{l-1}W) \]

by (6.9). By (5.3), we have

\[ -q^{1-\epsilon}W^c(1) = q^{(m-n_s)(s-1/2)}\Xi(s, \Gamma_{l-1}W - qW), \]

from which the identity follows. □

By this Lemma and (6.6), the second term and third term of (6.8) cancel if $i = 0$ and $r \geq 1$. The last terms of (6.7) and (6.8) vanish if $i = 0$, by the following lemma.

**Lemma 6.8.** Let $\pi \in \text{Irr}^{\text{gm}}(G)$. Let $\epsilon = c(\omega_\pi)$. If $m > 2\epsilon$, and $W \in V(m)$, then, for $x, y \in \mathcal{O}$,

\[ \pi(a_n, n_2(xzw^{-1})z(yzw^{-1})) W(\eta) = \pi(a_n, n_2(xzw^{-1})) W^c(\eta) = 0. \]

**Proof.** By Lemma 2.2, it suffices to see that the second gauges of the above Whittaker functions are $N(\mathcal{O})$-invariant. For the latter Whittaker function, the $N(\mathcal{O})$-invariance property follows from that of the second gauge of $\pi^i(a_r)W^c$, and the identity (2.2). For the former one, use the $j'(1)$-conjugation of the identity (2.2) with

\[ B \in \left[ \begin{array}{c} \mathcal{O} \\ \mathcal{O} \end{array} \right], C \in \left[ \begin{array}{c} \mathcal{P}^{-1} \\ \mathcal{P}^{-1} \end{array} \right] \]

and the calculation in the proof of Lemma 5.2. □

Now suppose that $W(\neq 0) \in V(m_\pi)$ is an eigenvector of $\mathcal{T}$ with eigenvalue $\lambda = \lambda_\mathcal{T}$. Then $W^c \in V^c(m_\pi)$ is also an eigenvector of $\mathcal{T}^c$ with eigenvalue $\lambda$ since $c \circ \mathcal{T} = \mathcal{T}^c \circ c$. By the above argument, we have

\[
\begin{align*}
\lambda W(a_r) & = q^3 W(a_{r+1}) + q W(a_{r-1}), \quad r \leq r_0, \\
\lambda W^c(a_r) & = q^3 W^c(a_{r+1}), \quad r \geq 1,
\end{align*}
\]

where $r_0$ is as in Lemma 6.7. By Lemma 2.2, $W(a_{-1}) = 0$. If we assume that $W(1) = 0$, then, by (6.10), $\Xi(s, W) = 0$, which contradicts to Proposition 6.6. Hence,

\[ W(1) \neq 0. \]

Next, assume that $\lambda \neq 0$. Then, we conclude $W^c(a_1) = 0$ by considering (6.11) and Lemma 6.2. By (6.11) again, $\Xi(1-s, W^c)$ is a constant. By the functional equation (5.3), $\Xi(s, W)$ is a monomial. Since $W(1) \neq 0$, $\Xi(s, W)$ is a constant. In particular, $W(a_1) = 0$. By (6.10), $\lambda = 0$. This is a contradiction. Hence

\[ \lambda = 0. \]

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By (6.10) again, $\Xi(s,W)$ is a constant. Therefore, $W(a^i_r) = 0$ for $(i, r) \neq (0,0)$ by Proposition 6.6. Since $T$ is diagonalizable, by Theorem 5.4,

$$\dim V(m_\pi) = 1.$$ 

Finally, we consider the action $T^*+$. Let $W^c(\neq 0) \in V^c(m_\pi)$. Since $\dim V^c(m_\pi) = 1$, $W^c$ is an eigenvector of $T^*$. Let $\lambda_+$ be the eigenvalue. We compute

$$T^* W^c(a^r_i) = q^3 W^c(a^r_{i+1}) + q \sum_{x \in \Theta^c \backslash \Theta} W^c(a^r_{i+1} n_2(x a^c_{j-1})).$$

By Lemma 6.8, $\lambda_+ W^c(a_r) = q^3 W^c(a^r_{i+1})$. By Proposition 6.6, $W^c(a_r) \neq 0$ for some $r$. By Lemma 6.2,

$$\lambda_+ = 0, \quad W^c(1) \neq 0, \quad W^c(a_r) = 0, \ r \geq 1.$$ 

Now, Theorem 6.1 for the unitary case follows from Theorem 5.12. The values of $W^c$ on $T$ are determined by the recursion formula (6.6).

**Step 4.** Finally, we discuss for the non-unitary case. For a supercuspidal representation, twisting it by a character $\nu_a$, $a \in \mathbb{R}$, we obtain a unitary supercuspidal representation ([Cs2]), where $a$ is unique and called the exponent of the representation. Applying the above argument to the twist, one can show the theorem for this case. For a generic constituent $\pi$ of the Klingen induction $\chi \times \sigma$ from supercuspidal, we apply the argument in sect. 5 of [R-S]. Consider the following facts (c.f. Table A.3., A.4. of [R-S], p. 93-94 [S-T]):

- The Jacquet module of $\chi \times \sigma$ with respect to the unipotent radical $U_P$ of the Siegel parabolic subgroup $P$ vanishes (see p. 29 of [R-S] for the definition of $P$).
- The semisimplification of the Jacquet module of $\chi \times \sigma$ with respect to the unipotent radical $U_Q$ of $Q$ is $\chi \times \sigma + \chi^{-1} \times \chi \sigma$.

Since $\text{pr}(U_P) = N^1 N_3$, $\tau_1$-type does not appear in the Jordan-Hölder sequence in Theorem 4.1. Since $\text{pr}(U_Q) = N_2 N_3$ and $\sigma$ is ramified, $\tau_2$-types in the sequence have no $P_3(\Theta)(= \text{pr}(K_1(m; \epsilon)))$-invariant vector. Hence, non-generic constituents of $\chi \times \sigma$ have no quasi-paramodular vector. Since the generic constituent is a unique constituent, $V(m_\pi)$ and the subspace of $K_1(m_\pi; \epsilon)$-invariant vectors $f$ in $\chi \times \sigma$ have the same dimension. Let $R = \{ r \}$ be representatives for $Q \backslash G/K_1(m_\pi; \epsilon)$. Since $f$ is determined by the values $f(r)$, we should have $\chi(t) \sigma(g) f(r) = f(r)$ for all $r \in R$ and $k \in Q \cap \text{Int}(r) K_1(m_\pi; \epsilon)$, where we write $k = q \in Q$ of the form of (6.1). Then $\det(g) = \mu(k)$ lies in $\Theta^\times$. Since any power of $k$ lies in the compact subgroup $\text{Int}(r) K_1(m_\pi; \epsilon)$, $t$ lies in $\Theta^\times$. Let $a, b$ be the exponents of $\chi, \sigma$, respectively. The generic constituent of $\nu_a \chi \times \nu_b \sigma$ is unitary (c.f. Table A.1. of [R-S]). Denote it by $\pi_1$. We have showed that $\dim V(m_{\pi_1}) = 1$. Since $\det(g), t \in \Theta^\times$, the above condition on $f \in \nu_a \chi \times \nu_b \sigma$ is same for various $a, b$. Hence $\dim V(m_\pi) = \dim V(m_{\pi_1}) = 1$. Now the above argument of Hecke operators for unitary representations works, and therefore, Theorem 6.1 is true also for non-unitary generic constituents. This completes the proof.

For $W \in V(m)$, define

$$W^+ = \pi(H_{m-\epsilon}) W, \quad W^c = \pi'(J_m)(W^-)^t = \pi'(J^t(\varpi^t)) W^c.$$ 

Since $W^-$ is 0-balanced, by Propositon 5.1, the functional equation is

$$\frac{Z(1-s, W^c)}{L(1-s, \pi^s)} = \varepsilon(s, \pi, \psi) \frac{\Xi(s, W^-)}{L(s, \pi)}.$$
Corollary 6.9. For $\pi \in \text{Irr}^{an}(G)$ as in the theorem,
$$\varepsilon_{\pi}^{-1} Z(s, W_{\pi}^{-c}) = \Xi(s, W_{\pi}^{-}) = W_{\pi}^{-}(1) = G(\omega, 1).$$

Proof. Let $f$ be the first gauge of $W_{\pi}^{-c}$. We have $Z(s, W_{\pi}^{-c}) = \int_{F} \tilde{f}(a(t))\nu_{s-3/2}(t)dxdt$ by Proposition 5.1, where $\tilde{f}(p) = \int_{N'} f(pn)dn$. By using the identity (2.2) and the invariance property of $f$ under
$$\left[ \mathcal{O}^{\times}, \mathcal{P}^{\times} \mathcal{R}^{-\epsilon} \right] \subset P_{3},$$
one can show that $\tilde{f}$ is $N'(\mathcal{O})$-invariant. Therefore, $Z(s, W_{\pi}^{-c}) \in \mathbb{C}[q^{-s}]$ by Lemma 2.2. Now, we compute $\tilde{f}(1) = \int_{N'(\mathcal{P}^{\times})} f(n)dn$ which is the constant term of $Z(s, W_{\pi}^{-c})$. By using Lemma 2.2 and the identity (2.2) again, one can show that $f(\tilde{n}'(x)) = 0$ for $x \in \mathcal{P}^{\epsilon+1}$. Therefore,
$$\int_{\mathcal{P}^{\times}} f(\tilde{n}'(x))dx = \int_{\mathcal{P}^{\times}} f(\tilde{n}'(x))dx$$
$$= q^{-\epsilon} \int_{\mathcal{O}^{\times}} f(n'(w^{-\epsilon}u^{-1})j(w^\epsilon u')n'(w^{-\epsilon}u^{-1}))du$$
$$= q^{-\epsilon} \int_{\mathcal{O}^{\times}} \psi(w^{-\epsilon}u^{-1})\omega_{\pi}(u)f(j(w^\epsilon))du$$
$$= q^{-\epsilon} G(\omega, 1)f(j(w^\epsilon)).$$
where (2.1) and the $N'(\mathcal{P}^{-\epsilon})$-invariance property of $f$ are used. Now the assertion follows from (6.14) and the identities $f(j(w^\epsilon)) = W_{\pi}^{-c}(j'(w^\epsilon)) = W_{\pi}^{c}(1) = q^{\epsilon} \varepsilon_{\pi}$. \qed

Now, let $(\pi, V) \in \text{Irr}(G)$ be tempered, non-generic. Then $\pi$ is the representation of Vlb or VIIIb listed in the Table A. 1. of [R-S]. By the proof of Theorem 2.5.3., and Table A.6., A.7. of loc. cit., $V_{2}$ is irreducible and a $\tau_{2}$-type. But, any $\tau_{2}$-type does not have a $\text{pr}(K_{1}(m; \epsilon))-\text{invariant}$ vector by the above argument for the case of $\epsilon > 0$, and by Lemma 3.4.4 of loc. cit. for the case of $\epsilon = 0$. Hence, we have:

Theorem 6.10. A tempered $\pi \in \text{Irr}(G)$ has a quasi-paramodular vector, if and only if $\pi$ is generic.

7. Construction of quasi-paramodular forms

In this section, by local $\theta$-lift from $GSO(2, 2)$ to $G$, we show the existence of the newform (c.f. Theorem 5.12) for generic constituents of Borel and Siegel parabolic inductions, respectively. The proof of the main theorem will be complete, except for the coincidences of root numbers and conductors. Let $X = M_{2 \times 2}(F)$, equipped with the nondegenerate symmetric split form $\frac{1}{2} Tr(x^{\ast}x^{\ast})$, where $x^{\ast}$ indicates the main involution of $x \in X$. Let $GO_{X}$ denote the generalized orthogonal group of $X$ and $\mu_{X}$ the similitude factor. Let $H = GSO_{X} := \ker(\mu_{X}^{-2}\det) \subset GO_{X}$. Letting $G_{2} \times G_{2}$ act on $X$ by $(g_{1}, g_{2}) \cdot x = g_{1}xg_{2}^{\ast}$, we have the isomorphisms
$$H \simeq G_{2} \times G_{2}/\{(z, z^{-1}) \mid z \in F^{\times}\}, SO_{X} \simeq \{(g_{1}, g_{2}) \mid \det(g_{1}g_{2}) = 1\}/\{(z, z^{-1}) \mid z \in F^{\times}\}.$$ 

Via these isomorphisms, we will represent elements and subgroups of $H$ by those of $G_{2} \times G_{2}$, and objects in $\text{Irr}(H)$ by those in $\{\tau_{1} \boxtimes \tau_{2} \mid \omega_{\tau_{1}} = \omega_{\tau_{2}}\} \subset \text{Irr}(G_{2}) \boxtimes \text{Irr}(G_{2})$. Let $B, T$ denote the upper triangular and diagonal matrices in $G_{2}$ respectively. Let $N_{X} = M_{2} \subset H$ and $B_{X}, T_{X}$, similarly. Define $\psi_{X} \in \mathcal{A}(N_{X})$ by $\psi_{X}((n, m)) = \psi(nm^{-1})$. We say $\tau = \tau_{1} \boxtimes \tau_{2} \in \text{Irr}(H)$ is generic, if $\text{Hom}_{N_{X}}(\tau, \psi_{X}) \neq \{0\}$, or equivalently if both
\( \tau_1, \tau_2 \in \text{Irr}(G_2) \) are generic. Let \( Y \) denote the 4-dimensional space equipped with the symplectic form defined by the matrix (1.1). Set \( Z = X \otimes Y \). Let \( Y = Y^+ + Y^-, Z^+ = X \otimes Y^+ = X \oplus X \) be the polarizations. For \( z = x_1 + x_2, z' = x_1' + x_2' \in Z^+, \) write \((z, z') = (\frac{1}{2} Tr(x_i^2 x_i'))\). For \( \varphi \in \mathcal{S}(Z^+) \), let \( \varphi^\sharp \) denote the Fourier transform defined by \( \varphi^\sharp(z') = \int_{X \otimes X} \psi(Tr(z', z)) \varphi(z) dz \) where \( dz \) is chosen so that \((\varphi^\sharp)^\sharp(z) = \varphi(-z)\). The Weil representation \( w_\psi \) of the dual pair \( Sp(4) \times O_X \) can be realized on the Schwartz space \( \mathcal{S}(Z^+) \). The action is given by the following formulas:

\[
\begin{align*}
& w_\psi(1, h) \varphi(z) = \varphi(h^{-1} \cdot z), \quad h \in O_X, \\
& w_\psi(A^2, 1) \varphi(z) = |\det(A)|^{-2} \varphi(z w_2^t A^{-1} w_2), \quad A \in G_2 \\
& w_\psi(n(B), 1) \varphi(z) = \psi(\frac{1}{2} Tr(B w_2(z, z))) \varphi(z), \\
& w_\psi(j(-w_2), 1) \varphi(z) = \varphi^\sharp(z).
\end{align*}
\]

Let \( R = G \times H \), and \( R_0 = \ker(\mu^{-1} \mu_X) \subset R \). For our convenience of the computation below, we adopt the following extension \( w_\psi \) to \( R_0 \) as in [R2]

\[
w_\psi(g, h) \varphi(z) = |\mu_X(h)|^{-2} w_\psi(g_1, 1) \varphi(h^{-1} \cdot z),
\]

where

\[
g_1 = g \begin{bmatrix} 1_2 & \\
& \mu(g)^{-1} 1_2 \end{bmatrix}.
\]

Note that this differs from the normalization used in [G-T2]. Observe that the central elements \((u, u) \in R_0 \) act on \( \mathcal{S}(Z^+) \) trivially. Let \( \Omega = \text{Ind}_{R_0}^R w_\psi \) be the compact induction, which can be realized on the Schwartz space \( \mathcal{S}(Z^+ \times F^+) \) (c.f. [R], [So]). For \( \tau^1 \in \text{Irr}(SO_X) \) define \( w_\psi(\tau^1) = w_\psi / \cap \lambda \in \text{Hom}_{SO_X}(w_\psi, \tau^1) \ker(\lambda) \), and for \( \tau \in \text{Irr}(H) \) define \( \Omega(\tau) \) similarly. By Lemma 2. III. 4. of [M-V-W], there exist \( \Theta(\tau^1) \in \text{Alg}(Sp(4, F)) \) and \( \Theta(\tau) \in \text{Alg}(G) \), such that

\[
\Theta(\tau^1) \simeq \Theta(\tau^1) \otimes \tau^1, \quad \Omega(\tau) \simeq \Theta(\tau) \otimes \tau.
\]

It is known that \( \Theta(\tau^1) \) and \( \Theta(\tau) \) are admissible of finite length. The maximal semi-simple quotients of \( \Theta(\tau) \) and \( \Theta(\tau) \) are denoted by \( \theta(\tau) \) and \( \theta(\tau) \) respectively. Let \( \Omega_{N, \psi} \) be the \( \psi \)-twisted \( N \)-Jacquet module of \( \Omega \). By the Frobenius reciprocity,

\[
\text{Hom}_R(\Omega, \text{Ind}^G_{N, \psi} \psi \otimes \tau) \simeq \text{Hom}_{N \times H}(\Omega|_{N \times H}, \psi \otimes \tau)
\]

\[
\simeq \text{Hom}_{N \times H}(\Omega_{N, \psi}, \psi \otimes \tau).
\]

As in the proof of Proposition 2.4 and its Corollary in [G-R-S], one can prove

\[
\Omega_{N, \psi} \simeq \psi \otimes \text{ind}^H_{N_X} \psi_X
\]

as \( N \times H \)-modules (c.f. Proposition 4.1. of [M-S]). Therefore,

\[
\text{Hom}_{N \times H}(\Omega_{N, \psi}, \psi \otimes \tau) \simeq \text{Hom}_H(\text{ind}^H_{N_X} \psi_X, \tau)
\]

\[
\simeq \text{Hom}_H(\tau, \text{ind}^H_{N_X} \psi_X^{-1}).
\]

Now suppose that \( \tau \) is generic. Then, \( \dim \text{Hom}_R(\Omega, \text{Ind}^G_{N, \psi} \psi \otimes \tau) = \dim \text{Hom}_H(\tau, \text{Ind}^H_{N_X} \psi_X^{-1}) = 1 \). Since the Jacquet module \( (\tau \otimes \tau')_H \) is isomorphic to \( \mathbb{C} \),

\[
(7.2) \quad \dim \text{Hom}_G(\Theta(\tau), \text{Ind}^G_{N, \psi}) = 1
\]

by (7.1). Hence, \( \Theta(\tau) \) has a generic irreducible constituent. By the work of W. T. Gan and S. Takeda [G-T2], this constituent is \( \theta(\tau) \). The (unique) generic constituent of the parabolic induction \( \chi_1 \times \chi_2 \times \chi \) (resp. \( \rho \times \chi \)) (c.f. Table A.1. of [R-S]) for
\( \chi_1, \chi_2, \chi \in \mathscr{A}(F^\times) \) coincides with the small \( \theta \)-lift of \( \pi(\chi_1, \chi_1 \chi_2)^\vee \boxtimes \pi(\chi_1, \chi_2)^\vee \) (resp. \( \pi(\chi, \chi \omega_\rho)^\vee \boxtimes \chi^{-1} \rho^\vee \)) (c.f. [G-T2]), where \( \rho \in \text{Irr}(G_2) \), \( \chi, \chi_i \in \mathscr{A}(F^\times) \). Here, \( \pi(\alpha, \beta) \) for \( \alpha, \beta \in \mathscr{A}(F^\times) \) indicates the principal series induced from the representation of \( B \subset G_2 \) sending \( b \to \nu_{1/2}(b_{11}/b_{22})\alpha(b_{11})\beta(b_{22}) \). Since all \( L \)-functions \( L(s, \pi) \) for \( \pi \in \text{Irr}^g(G) \) are computed by R. Takloo-Bighash [T], and it is known by [G-T2] when the \( \theta \)-lift is a constituent of a parabolic induction, one can show that

\[
L(s, \theta(\tau_1^\vee \boxtimes \tau_2^\vee)) = L(s, \tau_1)L(s, \tau_2)
\]

by case-by-case argument. Let

\[
z_0 = e_0 \oplus 1_2 \in Z^+, \quad e_0 = \left[ \begin{array}{c} 1 \\ \end{array} \right] \in X.
\]

The stabilizer subgroup of \( z_0 \) by \( SO_X \) is \( N_\Delta := \{(n, n) \mid n \in N \} \). Let \( \xi_1 \in \mathcal{H}_\psi(\tau_1), \xi_2 \in \mathcal{H}_\psi^{-1}(\tau_2) \) and \( \xi = \xi_1 \boxtimes \xi_2 \). Let \( \varphi \in \mathcal{S}(X \times X) \). We choose the Haar measure \( dh \) on \( H \) (resp. \( dn \) on \( N_\Delta \)) such that \( \text{vol}(G_2(\theta) \times G_2(\theta)) \) (resp. \( \text{vol}(N_\Delta(\theta)) \)) equals 1. Let \( d\hat{h} = dh/dn \) denote the Haar measure on \( N_\Delta\backslash H \). Consider the function \( \xi_\varphi \) on \( G \) defined by

\[
\xi_\varphi(g) = \int_{N_\Delta \backslash SO_X} w_\psi(g, hh_g)\varphi(z_0)\xi(hh_g)d\hat{h},
\]

where \( h_g \in H \) is chosen so that \( \mu(g) = \mu_X(h_g) \). This integral is independent from the choice of \( h_g \), and converges since the function \( h \to \varphi(h^{-1} \cdot z_0) \) has a compact support modulo \( N_\Delta \). By using the above formulas of \( w_\psi \), one can see that

\[
w_\psi(n'(b)n_3(\ast)z(\ast), 1)\varphi(z_0) = \psi(b)\varphi(z_0),
\]

\[
w_\psi(n_2(b), h)\varphi(z_0) = w_\psi(1, h)\varphi(z_0 - \varepsilon_2 \otimes be_0))
\]

and that \( \xi_\varphi \) is a Whittaker function with respect to \( \psi \). Now, let \( \pi \) denote the (generic) \( G \)-module generated by these \( \xi_\varphi \). We will show that there is a \( G \)-surjection

\[
\Theta(\tau^\vee) \to \pi.
\]

Since the central elements \( (u, u) \in R_0 \) act on \( \mathcal{S}(Z^+) \) trivially, \( \xi_\varphi \) and \( \tau \) have the same central character. Write \( \omega = \omega_\tau = \omega_\pi \). By \( \omega \) and Lemma 2.9 of [B-Z], there is an irreducible \( SO_X \)-submodule \( \tau_0 \) of \( \tau \) and finite subset \( h_1, \ldots, h_r \) of representatives for \( H/F^xSO_X \) such that \( \tau|_{SO_X} = \bigoplus_{i=1}^r \tau_i \) where \( \tau_i \) denotes the \( h_i \)-translation of \( \tau_0 \). For \( 0 \leq i \leq r \), let \( \pi_i \) denote the \( Sp_4 \)-module generated by \( \xi_\varphi \) for \( \xi \in \pi_i \). Let \( g_i \in G \) such that \( \mu(g_i) = \mu_X(h_i) \).

By definition, \( \pi = \bigoplus_{i=1}^r \pi_i \) as \( Sp_4 \)-modules, where \( \pi_i \) denotes the \( g_i \)-translation of \( \pi_0 \). Let \( \lambda_i \in \text{Hom}_{Sp_4 \times SO_X}(w_\psi, \text{Hom}_{C}(\tau_i, \pi_i)) \) denote the mapping \( \varphi \mapsto (\xi \mapsto \xi_\varphi) \). Since \( \text{Hom}_C(\tau_i, \pi_i)^K \simeq (\tau_i^K)^* \boxtimes \pi_i \simeq (\tau_i^K)^* \boxtimes \pi_i \) for any open subgroup \( K \subset SO_X \), \( \lambda_i \) factors through a \( \lambda'_i \in \text{Hom}_{Sp_4 \times SO_X}(w_\psi, \tau_i^\vee \boxtimes \pi_i) \). By (7.1), we have an \( Sp_4 \)-homomorphism \( \Theta_\psi(\tau_i^\vee) \to \pi_i \), which is surjective by construction. Therefore, \( \pi_i \) is admissible, and so is \( \pi \).

Let \( \lambda \in \text{Hom}_{R_0}(w_\psi, \text{Hom}_{C}(\tau, \pi)) \) denote the mapping \( \varphi \mapsto (\xi \mapsto \xi_\varphi) \). Then, \( \lambda \) also factors through a \( \lambda' \in \text{Hom}_{R_0}(w_\psi, \tau^\vee \boxtimes \pi) \). Since \( \tau^\vee \boxtimes \pi \) is \( R \)-admissible, by Proposition 2.15. of [B-Z] and Lemma 7.1 i) below, \( (\tau^\vee \boxtimes \pi)^{|R_0} \simeq \tau^\vee \boxtimes \pi \). From the Frobenius reciprocity,

\[
\text{Hom}_{R_0}(w_\psi, \tau^\vee \boxtimes \pi) \simeq \text{Hom}_{R_0}(w_\psi, (\tau^\vee \boxtimes \pi)^{|R_0}) \simeq \text{Hom}_{R}(\Omega, \tau^\vee \boxtimes \pi).
\]
For any $\xi_\varphi \in \pi$, there exists a $\xi^* \in \tau^\vee$ such that $\lambda(\varphi) = \xi^* \otimes \xi_\varphi$ by construction. Let $\tilde{\lambda} \in \text{Hom}_R(\Omega, \tau^\vee \otimes \pi)$ correspond to $\lambda$. By Lemma 7.1 ii), $\xi^* \otimes \xi_\varphi \in \text{Im}(\tilde{\lambda})$. Since $(\tau \otimes \tau^\vee)_H \simeq \mathbb{C}$, $\tilde{\lambda}$ induces a surjection (7.5).

For an $l$-group, let $\Delta_G$ denote the modulus of $G$.

Lemma 7.1. Let $G$ be an $l$-group, and $G_0$ a closed subgroup of $G$. Let $(\pi, V) \in \text{Alg}(G)$. Assume that $G$ has a system of neighbourhoods $N = \{K\}$ of the unit consisting of open compact subgroups such that $V^K = V^{K \cap G_0}$. Then

i) $(\pi|_{G_0})^\vee = \pi^\vee$.

ii) Let $\rho \in \text{Alg}(G_0)$ and $\lambda \in \text{Hom}_H((\Delta_{G_0}/\Delta_G)\rho, (\pi|_{G_0})^\vee)$. Let $\tilde{\lambda} \in \text{Hom}_G(\text{Ind}_{G_0}^G \rho, \pi^\vee)$ induced by the Frobenius reciprocity. Then $\text{Im}(\lambda) \subset \text{Im}(\tilde{\lambda})$.

If $G_0 \triangleleft G$, then for any $G_0$-admissible $(\pi, V) \in \text{Alg}(G)$, there is a system of neighbourhoods as above.

Proof. i) Let $V^*$ denote the dual of $V$. $\pi|_{G_0}$ and $\pi$ have the same dual $V^*$. By Lemma 2.14 of [B-Z], $(V^*)^{K \cap G_0} = (V^{K \cap G_0})^* = (V^K)^* = (V^*)^K$ for any $K \in N$. Therefore, $(\pi|_{G_0})^\vee = \bigcup_{K \in N} (V^*)^{K \cap G_0} = \bigcup_{K \in N} (V^*)^K = \pi^\vee$.

ii) For $\xi \in \Delta_{G_0}$, take a $K \in N$ so that $\xi$ is $K \cap G_0$-invariant. Then, $\lambda(\xi) \in (V^*)^{K \cap G_0} = (V^{K \cap G_0})^* = (V^K)^*$. Let $\langle , \rangle$ denote the natural pairing for $V, V^*$. By 2.29 of loc. cit., $\tilde{\lambda}$ is given by

$$\langle \tilde{\lambda}(f), v \rangle = \int_{G_0 \backslash G} \langle \lambda(f(g)), \pi(g)v \rangle dg, \ v \in V, f \in \text{Ind}_{G_0}^G \rho.$$ 

Since $\xi$ is $K \cap G_0$-invariant, we can define $f_K \in \text{Ind}_{G_0}^G \rho$ by $f_K(hk) = \Delta_{G_0}/\Delta_G(h)\xi(h)$ for $h \in G_0, k \in K$. By definition, $f_K$ is $K$-invariant, and therefore $\tilde{\lambda}(f_K)$ lies in $(V^*)^K = (V^K)^*$. For $v \in V^K$,

$$\langle \tilde{\lambda}(f_K), v \rangle = \int_{G_0 \backslash G \backslash K} \langle \lambda(f_K(g)), \pi(g)v \rangle dg = \int_{G_0 \backslash G \backslash K} \langle \lambda(\xi), v \rangle dg = \text{vol}(G_0 \backslash G \backslash K) \langle \lambda(\xi), v \rangle.$$ 

Hence $\tilde{\lambda}(\text{vol}(G_0 \backslash G \backslash K)^{-1} f_K) = \lambda(\xi)$. For the last assertion, let $L \subset G$ be an open compact subgroup. Fix an isomorphism $\mu : L/L \cap G_0 \simeq A$ for a compact group $A$. Since $\pi$ is $G_0$-admissible, $V^{L \cap G_0}$ is finite dimensional. Therefore, there is an open subgroup $B \subset A$ such that $V^{L \cap G_0} \subset V^L_B$ for $L_B := \{ k \in L \mid \mu(k) \in B \}$. Then, $L_B \cap G_0 = \{ k \in L \mid \mu(k) = 1 \} = L \cap G_0$, and hence $V^{L_B \cap G_0} = V^{L_B}$. Then $N := \{ L_B \}$ is the desired system of neighbourhoods.

Now, since the generic irreducible $\theta(\tau^\vee)$ is the quotient of $\Theta(\tau^\vee)$, $\tau$ has a generic irreducible quotient isomorphic to $\theta(\tau^\vee)$ by (7.5). By (7.2), $\Theta(\tau^\vee)$ is of Whittaker type. Therefore by Proposition 4.2, for any $\xi$ and $\varphi$, there exists a $W \in \mathcal{W}_\varphi(\theta(\tau^\vee))$ such that

$$Z(s, W) = Z(s, \xi_\varphi), \text{ and } Z(1 - s, \pi'(j_0)W^*) = Z(1 - s, \pi'(j_0)(\xi_\varphi)^*)$$

up to constant multiples. Of course, if $\xi_\varphi$ is quasi-paramodular, then so is $W$ and $Z(s, W^c) = Z(s, \xi_\varphi^c)$. So, for the existence of the newform, we will construct a $\text{K}_1(m; e)$-invariant Whittaker function $\xi_\varphi$ as in Theorem 5.12. Now, fix a $\psi$ such that $\ker(\psi) = \mathcal{O}$. When $L$ is a subgroup of a similitude group, we will denote by $L^1$ the intersection of $L$ and the isometry group. Let $m_i \geq n_{\tau_i} = m_{\tau_i}$ and $K = (K(m_1) \times K(m_2))^1$. Let
\( m = m_1 + m_2, l = m - c \) where \( c = c(\omega) \). Let \( \xi \) be \( K_1(m_i) \)-invariant, and set \( \xi = \xi_1 \otimes \xi_2 \). Define \( \varphi = \varphi_{m_1, m_2} \in \cal{I}(X \oplus X) \) by

\[
\varphi_{m_1, m_2}(u \oplus v) = \text{Ch}(v; M_2(\mathcal{O})) \times \begin{cases} \text{Ch} \left( u; \begin{bmatrix} \mathcal{P}^{m_2} & \mathcal{O} \\ \mathcal{P}^m & \mathcal{P}^{m_1} \end{bmatrix} \right) & \text{if } c = 0, \\
\omega(u_{12}) \text{Ch} \left( u; \begin{bmatrix} \mathcal{P}^{m_2} & \mathcal{O}^x \\ \mathcal{P}^l & \mathcal{P}^{m_1} \end{bmatrix} \right) & \text{if } c > 0,
\end{cases}
\]

so that (7.6). By using the formulas of \( w_\varphi \), one can see that this Schwartz function is \( K_1(m; c)^1 \)-invariant. Therefore \( \xi_\varphi \) is a quasi-paramodular form of level \( m \). Let

\[
\varphi^c = w_\varphi(J_{m_1}, (j_{m_1}, j_{m_2})) \varphi, \quad j_a = \left[ \begin{array}{l} \omega a \\ -1 \end{array} \right]
\]

\[
\xi^c_i(g) = \omega^{-1}(\det(g)) \xi_i(g j_{m_i}).
\]

Then, the conjugate \( (\xi^c)^c \) of \( \xi^c \) equals \( (\xi^c)^c \). In case of \( c = 0 \), \( \varphi^c = \varphi \). In case of \( c > 0 \),

\[
\varphi^c(u \oplus v) = \frac{\mathcal{G}(\omega, 1)}{\omega(\mathcal{P}^{e_{12}})} \text{Ch} \left( u; \begin{bmatrix} \mathcal{P}^{m_2} & \mathcal{O} \\ \mathcal{P}^l & \mathcal{P}^{m_1} \end{bmatrix} \right) \text{Ch} \left( v; \begin{bmatrix} \mathcal{O} \mathcal{P}^{-x} \\ \mathcal{O} \mathcal{P} \end{bmatrix} \right).
\]

For the computation of the zeta integrals, the following lemma and the Bruhat decomposition \( SO_X = \sqcup_{w \in W_X} B^1_X w N_X \) is useful, where \( W_X := \{(1_2, 1_2) , (1_2, j_0), (j_0, 1_2), (j_0, j_0)\} \) is the Weyl group of \( T_X \).

Lemma 7.2. Let \( \xi \in \tau \). Let \( \varphi = \varphi_1 \otimes \varphi_2 \in \cal{I}(X \oplus X) \). Let \( K \) be a open compact subgroup of \( SO_X \) such that

\[
\varphi(k^{-1} \cdot z) \tau(k) \xi = \varphi(z) \xi
\]

for \( z \in Z^+, k \in K \). Assume that

\[
\varphi_1(h^{-1} \cdot e_0) \neq 0 \quad \text{for } h \in SO_X \setminus B^1_X K
\]

(7.7)

\[
\implies \varphi_2(h^{-1} \cdot (1_2 - x e_0)) = \varphi_2(h^{-1} \cdot 1_2) \quad \text{for any } x \in \mathcal{P}^{-1}.
\]

Let \( S \) be representatives for the double coset space \( N_{\Delta} \setminus B^1_X / B^1_X \cap K \). Let \( S' \) be the collection of \( b \in S \) such that \( \varphi(b^{-1} \cdot z_0) = 0 \), or \( \varphi_2(b^{-1} \cdot (1_2 - x e_0)) = \varphi_2(b^{-1} \cdot 1_2) \) for any \( x \in \mathcal{P}^{-1} \). Then,

\[
\int_{N_{\Delta} \setminus SO_X} \varphi(h^{-1} \cdot z_0) \xi(h) \, dh = \frac{\vol_{SO_X}(K)}{\vol_{N_{\Delta}}(\text{Int}(b) K \cap N_{\Delta})} \sum_{b \in S \setminus S'} \xi(b) \varphi(b^{-1} \cdot z_0).
\]

Proof. Let \( h_1 \in SO_X \). For \( x \in F \),

\[
\int \varphi(h^{-1} \cdot z_0) \xi(h) \, dh = \int \varphi \left( ((n(x), 1)h)^{-1} \cdot z_0 \right) \xi((n(x), 1)h) \, dh
\]

\[
= \int \varphi \left( h^{-1} \cdot (e_0 \oplus (1_2 - x e_0)) \right) \psi(x) \xi(h) \, dh
\]

\[
= \int \varphi_1(h^{-1} \cdot e_0) \varphi_2 \left( h^{-1} \cdot (1_2 - x e_0) \right) \psi(x) \xi(h) \, dh,
\]

where integrations are over \( h \) in \( \Delta \setminus N_X h_1 K \). This integral vanishes, if \( \varphi_2(h_1^{-1} \cdot (1_2 - x e_0)) = \varphi_2(h_1^{-1} \cdot 1_2) \) for any \( x \in \mathcal{P}^{-1} \). The definition of \( S' \) and the condition (7.7) mean the noncontributions to the integral in (7.8) of the orbits \( b' K \) for \( b' \in S' \) and the orbits \( B^1_X w N_X K \) for \( w \in W_X \setminus \{1_2, 1_2\} \), respectively. Now the assertion is obvious. \( \square \)
From the definition of $\xi_\varphi$, it follows that

$$\xi_{\varphi}(a_i) = q^{-2i}(\xi^i)_{\varphi_i}(1),$$

where $\xi^i = \tau(1, a_i)\xi$, and

$$\varphi_i(u \oplus v) = \varphi((u \oplus v)a_i)$$

$$= \text{Ch} \left(v; \left[ P^{-i} \oplus \theta \right] \right) \times \begin{cases} \text{Ch} \left(u; \left[ P^{m_2-i} \Theta \otimes \theta \right] \right) & \text{if } \epsilon = 0, \\ \omega(u_{12})\text{Ch} \left(u; \left[ P^{m_2-i} \Theta \otimes \theta \right] \right) & \text{if } \epsilon > 0. \end{cases}$$

From the Bruhat decomposition, we obtain

$$(7.9) \quad SO_X = \bigsqcup_{w \in W_X} N_X T^1_X w N_w K_i,$$

where $K_i = \text{Int}((1, a_i))K$, and

$$N_w = \begin{cases} \{(1, 1)\} & \text{if } w = (12, 12), \\ \{(n(b_1), 1) \mid b_1 \in P^{1-m_1}\} & \text{if } w = (j_0, 12), \\ \{(1, n(b_2)) \mid b_2 \in P^{1-m_2+i}\} & \text{if } w = (12, j_0), \\ \{(n(b_1), n(b_2)) \mid b_1 \in P^{1-m_1}, b_2 \in P^{1-m_2+i}\} & \text{if } w = (j_0, j_0). \end{cases}$$

If $h$ lies in the orbit $N_X T^1_X w N_w$ with $w \neq (12, 12)$, then $h^{-1} \cdot e_0$ is one of the following forms

$$\left[ \begin{array}{c} -b_1 \omega^s \\ \omega^s \end{array} \right], \left[ \begin{array}{cc} -\omega^s & -b_2 \omega^s \\ \omega^s & -b_1 b_2 \omega^s \end{array} \right], \left[ \begin{array}{c} -b_2 \omega^s \\ \omega^s \\ b_1 \omega^s \end{array} \right] (s \in \mathbb{Z}).$$

Now, it is easy to see that (7.7) holds for $\varphi_i$. Let $S = \{h = (\omega^s n(x)a_s, a_t) \mid s + 2r = t, x \in F/P^s\}$. Then, $S$ is the representatives for $N_\Delta \backslash B^1_X \slash B^1_X \cap K_i$, and

$$h^{-1} \cdot z_0 = \left[ \begin{array}{c} \omega^{-s-r} \\ \omega^{-s-r}x \end{array} \right] \oplus \left[ \begin{array}{c} \omega^s \\ \omega^{-s-r}x \end{array} \right], h \in S.$$}

Therefore, $S \setminus S'$ in Lemma 7.2 consists of $(\omega^s n(x)a_s, a_t)$ with $s + r = 0, -i \leq r = t \leq 0, x \in \Theta \slash P^s$, and

$$(7.10) \quad \xi_{\varphi}(a_i) = q^{-i} \text{vol}(K) \sum_{l=0}^{i} \xi_1(a_i) \xi_2(a_{i-l}).$$

By a similar computation and the identity $G(\omega, 1)G(\omega^{-1}, 1) = q^{-\epsilon}$ when $\epsilon > 0$,

$$(7.11) \quad \xi_{\varphi}(a_i) = q^{-i-\epsilon} \text{vol}(K) \sum_{l=0}^{i} \xi_1(a_i) \xi_2(a_{i-l}).$$

**Theorem 7.3.** Let $\tau_1, \tau_2 \in \text{Irr}^m(G_2)$, and let $n_i = n_{\tau_i} (= m_{\tau_i})$. Let $\pi = \theta(\tau_1^\vee \otimes \tau_2^\vee) \in \text{Irr}^m(G)$. Let $r \geq 0$. Then $\pi$ has a quasi-paramodular form $W$ of level $n_1 + n_2 + r$, such that

$$\frac{(\varepsilon'_r)^{-1}(1-s, W^c)}{L(1-s, \tau_1^\vee)L(1-s, \tau_2^\vee)} = \frac{q^{-r(s-\frac{1}{2})}Z(s, W)}{L(s, \tau_1)L(s, \tau_2)} = 1.$$\)

In particular, $m_\pi = n'_\pi = n_1 + n_2$, and $\varepsilon'_\pi = \varepsilon_{\tau_1}\varepsilon_{\tau_2}$, and $V(m_\pi)$ is spanned by this $W$.}

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Proof. Set \( \phi = \varphi_{n_1, r + n_2} \) and \( \xi_1 = W_1, \xi_2 = \tau_2(a_r)W_2 \), where \( W_i \in \tau_i \) are the newforms. Then, \( Z_0(s, \xi_1) = L(s, \tau_1), \) \( Z_0(s, \xi_2) = q^{-r/2}L(s, \tau_2) \), and \( Z_0(s, \xi_i^\vee) = \varepsilon_{\tau_i}L(s, \tau_i^\vee) \) for \( i = 1, 2 \). By (7.3), (7.10), (7.11), and these identities, \( W = \zeta_\phi \) has the desired property. The last assertion follows from Theorem 5.12. \( \square \)

By the proof of Corollary 6.9,

**Corollary 7.4.** With the assumption as in the theorem, assume that \( c(\omega_\pi) > 0 \) additionally. Then, there exists a \( W^- \in V^- (m_\pi + r) \) such that

\[
\frac{\varepsilon_{\pi}^{-1}Z(1-s, W^-)}{L(1-s, \tau_1^\vee)L(1-s, \tau_2^\vee)} = \frac{q^{-r/2}\Xi(s, W^-)}{L(s, \tau_1)L(s, \tau_2)} = \Xi(\omega_\pi, 1).
\]

By the work of [G-T], the \( L \)-parameter \( \phi_\pi : WD_F \to GSp(4, \mathbb{C}) \) of \( \pi = \theta(\tau_1^\vee \otimes \tau_2^\vee) \) is \( \phi_{\tau_1} \oplus \phi_{\tau_2} \). Hence,

**Corollary 7.5.** With the assumption as in the theorem, \( L(s, \pi) = L(s, \phi_\pi) \) and \( \varepsilon(s, \pi, \psi) = \varepsilon(s, \phi_\pi, \psi) \).

8. Construction of newform for \( GL(4) \)

W. T. Gan and S. Takeda [G-T] showed the Langlands correspondence for \( G \), comparing the representations of \( G \) and those of \( G_4 \) by the local \( \theta \)-correspondence for \( G \) and \( GSO(3, 3) \simeq G_4 \times F^*/\{(z, z^{-2}) \mid z \in F^*\} \). In particular, for \( \pi \in \text{Irr}(G) \), the Langlands parameter \( \phi_\pi \) coincides with that of the local \( \theta \)-lift of \( \pi \) to \( G_4 \). In this section, to show the identities of the form \( n_1' = n_\pi \) and \( \epsilon_\pi' = \epsilon_\pi \), we will observe the local \( \theta \)-lift. Let \( U = F^4 \). In this section, let \( X = \wedge^2 U \), which is 6-dimensional. The bilinear form on \( X \) defined by \( x \wedge x' \) is symmetric, non-degenerate and splits, where we identify \( X \wedge X \) with \( F \) naturally. Letting \( G_4 \times F^* \) and \( GSO_X := \text{ker}(\mu_X^3 \text{det}) \) act on \( U \) and \( X \) from the left, respectively, we have an isomorphism

\[
I_a : G_4 \times F^*/\{(z, z^{-2}) \mid z \in F^*\} \simeq GSO_X.
\]

Let \( \{u_1, u_2, u_3, u_4\} \) be the standard basis of \( U \), and set

\[
X^+ = \text{Span}\{e_1, e_2, e_3\}; \quad e_1 = u_2 \wedge u_3, \quad e_2 = u_3 \wedge u_1, \quad e_3 = u_1 \wedge u_2,
\]

\[
X^- = \text{Span}\{e_{-1}, e_{-2}, e_{-3}\}; \quad e_{-1} = u_1 \wedge u_4, \quad e_{-2} = u_2 \wedge u_4, \quad e_{-3} = u_3 \wedge u_4.
\]

We will write the elements of \( GSO_X \) as matrices according to the basis \( \{e_3, e_2, e_{-1}, e_{-2}, e_{-3}\} \).

Then the isomorphism \( I_a \) respects the transpose and sends

\[
P_a \ni \begin{bmatrix} g & s \\ 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & b(s) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_3^{-1}g^{-1}w_3 \\ 1 \end{bmatrix} \in GSO_X,
\]

where \( g \) is an element of \( G_3 \), and

\[
b(s) = b\left( \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \right) = \begin{bmatrix} s_2 & -s_1 \\ -s_3 & s_1 \\ s_3 & -s_2 \end{bmatrix}.
\]

Let \( Z^\pm = X^+ \otimes Y \simeq Y \oplus Y \oplus Y \), and identify \( Z^+ \) with \( M_{3 \times 4}(F) \) via the mapping:

\[
z = \sum e_i \otimes y_i \longleftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in M_{3 \times 4}(F).
\]

For \( z = \sum e_i \otimes y_i, z' = \sum e_i \otimes y_i' \in Z^+ \), we write \( \langle z, z' \rangle = [(y_i, y'_i(-2w))] \in M_3(F) \). For \( \Phi \in \mathcal{S}(Z^+) \), let \( \Phi^z \) denote the Fourier transform defined by \( \Phi^z(z) = \int_{Z^+} \psi^{-1}(Tr(\langle z', z \rangle))\Phi(z')dz \).
where $dz$ is chosen so that $(\Phi^z)^{\sharp}(z) = \Phi(-z)$. Let $G$ act on $Y$ from the right. In this section, we use the Weil representation $w_{\psi^{-1}}$ of $Sp_4 \times O_X$ realized on the space $\mathcal{S}(M_{3 \times 4}(F))$ with the following transformation formulas.

\[ w_{\psi^{-1}}(g, 1)\Phi(z) = \Phi(\psi g), \quad g \in Sp(4), \]

\[ w_{\psi^{-1}}(1, \begin{bmatrix} w_3' a^{-1} w_3 & a \end{bmatrix})\Phi(z) = |\det(a)|^2 \Phi(a^{-1} z), \]

\[ w_{\psi^{-1}}(1, \begin{bmatrix} 1 & b(s) \vspace{1mm} \hline 0 & 1 \end{bmatrix})\Phi(z) = \psi^{-1}(\varepsilon Tr ((z, z)w_3 b(s)))\Phi(z) \]

\[ = \psi(s_1(y_2, y_3) + s_2(y_3, y_1) + s_3(y_1, y_2))\Phi(z), \]

\[ w_{\psi^{-1}}(1, j(-w_3))\Phi(z) = \Phi^z(z). \]

Let $R = G \times GSO_X$, and $R_0 = \ker(\mu^{-1} \mu_X) \subset R$. We extend $w_{\psi^{-1}}$ to $R_0$ via

\[ w_{\psi^{-1}}(g, h)\Phi(z) = |\mu(g)|^{-3} w_{\psi^{-1}}(1, h_1)\Phi(\psi g(z)) \]

so that the central elements $(u, u)$ act on trivially, where

\[ h_1 = h \begin{bmatrix} \mu(g)^{-113} & \vspace{1mm} \hline 13 & 1 \end{bmatrix} \in SO_X. \]

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ denote the standard basis of $Y$. Set

\[ z_0 = \varepsilon_2 \otimes e_1 + \varepsilon_3 \otimes e_2 + \varepsilon_4 \otimes e_3 = [0, 1, 3] \in M_{3 \times 4}(F). \]

Let $dg, dz$ be Haar measures on $Sp_4$, and $Z^I$, respectively. We choose $dz$ such that $\text{vol}(Z^I(\mathcal{O})) = 1$. Let $dg = dg/dz$ denote the Haar measure on $Z^I \backslash Sp_4$. Let $\pi \in \text{Irr}^{m}(G)$. For $W \in \mathcal{U}_\psi(\pi)$, and $\Phi \in \mathcal{S}(M_{3 \times 4}(F))$, we define a function $W_\Phi$ on $G_4$ by

\[ W_\Phi(h) = \int_{Z^I \backslash Sp_4} w_{\psi^{-1}}(g_1 g_h, h)\Phi(z_0)W(g_1 g_h)dg, \]

where $g_h$ is an element in $G$ such that $\mu(g_h) = \det(h)$. By the above formulas of $w_{\psi^{-1}}$, for $n \in \mathbb{N} \subset G_4$,

\[ w_{\psi^{-1}}(1, n)\Phi(z_0) = \psi(n_{34})\Phi(z_0 n_{2}(-n_{23})n_{3}(-n_{13})n(-n_{12})), \]

from which one can find that $W_\Phi$ is a Whittaker function on $G_4$ with respect to $\psi$. Let $\Pi$ be the $G_4$-module generated by these $W_\Phi$. Since the central elements $(u, u) \in R_0$ act on $\mathcal{S}(M_{3 \times 4}(F))$ trivially, $\omega_\Pi = \omega_\pi^2$. Define the big theta $\Theta(\pi)$ and the small theta $\theta(\pi)$, similar to the previous section. By the work of [G-T2], $\tilde{\pi} := \theta(\pi^\vee)$ is generic. By the similar argument, and the proof of Lemma 2.10 of [J-PS-S3], instead of Proposition 4.2, for any $W \in \pi$ and $\Phi \in \mathcal{S}(M_{3 \times 4}(F))$, there exists a $\widetilde{W} \in \mathcal{U}_{\widetilde{\psi}}(\tilde{\pi})$, such that

\[ Z_2(s, \tilde{W}) = Z_2(s, W_\Phi), \quad \text{and} \quad Z_0(1 - s, \tilde{W}^\pi) = Z_0(1 - s, (W_\Phi)^\pi). \]

Now we will construct a $K_1(m)$-invariant $W_\Phi$ using $W \in V(m)$ for $m \geq 2\epsilon$ where $\epsilon = \frac{\pi}{2}(\omega_\pi)$. Set

\[ \mathcal{L}_m = \begin{bmatrix} \mathcal{P}^m & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^m & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^m & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}, \]

which is a $K(m)$-invariant lattice. According to $\epsilon$, we define $\Phi_m \in \mathcal{S}(M_{3 \times 4}(F))$ by

\[ \Phi_m(z) = \begin{cases} \text{Ch}(\mathcal{L}_m) & \text{if } \epsilon = 0 \\ \omega_\pi(\det(\tilde{z}))\text{Ch}(a_{m-\epsilon}G_3(\mathcal{O}); \tilde{z})\text{Ch}(M_{3 \times 1}(\mathcal{O}); z_4) & \text{if } \epsilon > 0 \end{cases} \]
where $z_4$ and $\dot{z}$ indicate the the right $M_{3\times 1}$-part and the left $M_{3\times 3}$-part of $z$, respectively. Note that the support of $\Phi_m$ is contained in $L_{m-4}$. Let $W \in V(m)$. By the formulas of $w_{\psi^{-1}}$ and definition, for $k \in K_1(m) \cap SL_4$ and $k' \in K_1(m; e)^1$,

\[(8.1) \quad w_{\psi^{-1}}(k', k)\Phi(z)\pi(k')W = \Phi(z)W,\]

and $W_\Phi$ is $K_1(m)$-invariant. We will compute $W_\Phi(a_r)$. By definition, $W_\Phi(a_r) = q^{-r}(W^{(r)})_{\Phi(r)}(1)$, where

\[W^{(r)} = \pi(a_r)W, \text{ and } \Phi^{(r)}(z) = \Phi(a_r, z).\]

(Computation for $W_\Phi(a_r)$ in the case of $e = 0$.) Let $K = \text{Int}(a_r)K(m)$. Let $W' = \{1, w_{2}, w_{3}, j(w_2)\}$. For $w \in W'$, let $N_w$ be the following finite subset of $N$:

\[N_w = \\begin{cases} \{n_2(\omega^i)\}, & 1 - m \leq i \leq 0 \\ \{n_3(\omega^h)\}, & 1 + r - m \leq h \leq r - 1 \\ \{n_2(\omega^i)n_3(\omega^h)\}, & r - m - h < i < h - r \leq -1 \end{cases} \text{ if } w = 1, \quad w = w^3, \quad w = j_0, \quad w = j(w_2).\]

Then, $wn$ with $w \in W'$, $n \in N_w$ are representatives for $B^1 \setminus S_{p_4}/K^1$ (c.f. Proposition 5.1.2 of [R-S]). Therefore, we may write for representatives for $Z^1 \setminus S_{p_4}/K^1$ of the form of $g = n_2(-x)n_3(y)n'(z)t(\alpha, \beta)wn$ with $n \in N_w$. By (8.1),

\[\int_{Z^1 \setminus S_{p_4}} \Phi^{(r)}_m(z_0g)W^{(r)}(g)dg = \tilde{q}^{-m}\text{vol}(K^1) \sum_{w \in W'} \int_{Z^3 \setminus B^1 \setminus wN_w} \Phi^{(r)}_m(z_0g)W^{(r)}(g)dg\]

where

\[\Phi^{(r)}_m = \text{Ch}(\begin{bmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{bmatrix}).\]

Now we observe the integral $\int_{Z^1 \setminus B^1 \setminus wN_w} \ldots dg$. In case of $w = w^3$,

\[z_0g = \begin{bmatrix} \beta & \beta \omega^i & \alpha^{-1}y & \beta^{-1}z + \alpha^{-1}\omega^i y \\ 0 & 0 & \alpha^{-1}x & \beta^{-1} + \alpha^{-1}\omega^i x \\ 0 & 0 & \alpha^{-1} & \alpha^{-1}\omega^i \end{bmatrix} .\]

If the $(2, 3)$-coefficient $\alpha^{-1}x$ lies in $\mathcal{O}$, and the $(1, 1)$-coefficient lies in $\mathcal{P}^m$, then $\alpha^{-1}\omega^i x$ lies in $\mathcal{P}^{1-m}$, and the $(2, 4)$-coefficient $(\beta^{-1} + \alpha^{-1}\omega^i x)$ has order $\leq -m$ and is not in $\mathcal{O}$. Hence, $\Phi(z_0g)$ is 0, and so is the integral. In case of $w = j_0$,

\[z_0g = \begin{bmatrix} \beta^{-1}z & \alpha^{-1}y & \beta^{-1}\omega^h z & \beta + \alpha^{-1}\omega^h y \\ \beta^{-1} & \alpha^{-1}x & \beta^{-1}\omega^h x & \alpha^{-1}\omega^h x \\ 0 & 0 & \alpha^{-1} & \alpha^{-1}\omega^h \end{bmatrix} .\]

If the $(1, 2)$-coefficient $\alpha^{-1}y$ lies in $\mathcal{O}$, and the $(2, 1)$-coefficient $\beta^{-1}$ lies in $\mathcal{P}^{1-m-r}$, then $\alpha^{-1}\omega^h y$ lies in $\mathcal{P}^{1-m-r}$, and the $(1, 4)$-coefficient $(\beta + \alpha^{-1}\omega^h y)$ has order $\leq r - m$, and is not in $\mathcal{O}$. Hence, $\Phi(z_0g)$ is 0, and so is the integral. In case of $w = j(w_2)$,

\[z_0g = \begin{bmatrix} \alpha^{-1}y & \beta^{-1}z + \alpha^{-1}\omega^i y & \beta + \alpha^{-1}\omega^h y & \beta\omega^i + \beta^{-1}\omega^h z \\ \alpha^{-1}x & \beta^{-1} + \alpha^{-1}\omega^i x & \alpha^{-1}\omega^h x & \alpha^{-1}\omega^h x \\ \alpha^{-1} & \alpha^{-1}\omega^i & \alpha^{-1}\omega^h & 0 \end{bmatrix} .\]

If the $(1, 1)$-coefficient $\alpha^{-1}y$ lies in $\mathcal{P}^m$, and the $(2, 4)$-coefficient $\beta^{-1}\omega^h$ lies in $\mathcal{O}$, then $\alpha^{-1}\omega^h y$ lies in $\mathcal{P}^{m+h}$, and $\beta^{-1}$ lies in $\mathcal{P}^{-h}$, and therefore, the $(1, 3)$-coefficient $\beta + \alpha^{-1}\omega^h y$
has order $\leq h$ and is not in $\mathcal{P}^\gamma$. Hence, $\Phi(z_0g)$ is 0, and so is the integral. In case of $w = 1$,

$$z_0g = \begin{bmatrix} 0 & \beta & \beta^{-1}z & \alpha^{-1}y \\ 0 & 0 & \beta^{-1} & \alpha^{-1}x \\ 0 & 0 & 0 & \alpha^{-1} \end{bmatrix}.$$  

(8.2)

From the $(1, 2), (2, 3), (3, 4)$-coefficients, it follows that $\Phi(z_0g) = 0$ unless $\beta \in \mathcal{O}^x$, $\alpha^{-1} \in \mathcal{O}$. By Lemma 2.2, $W^{(\gamma)}(t(\alpha, \beta)) = 0$ if $\alpha \not\in \mathcal{O}$. Therefore $\Phi \gamma(z_0g)W^{(\gamma)}(g) = 0$ unless $g \in \mathbb{Z}^J K^1$, and

$$W_\Phi(a_r) = q^{-m\text{vol}(K)}W(a_r).$$  

(Computation for $W_\Phi(a_r)$ in the case of $c > 0$.) Let $l = m - c$. Noting that $\text{supp}(\Phi_m) \subset \mathcal{L}_l$, we find that $\Phi(z_0g) = 0$ unless $g \in \mathcal{B}^1 \text{Int}(a_r)K(l)^1$ by the above computation. Therefore, we may assume that $g \in \mathbb{Z}^J \backslash \mathcal{B}^1 / \text{Int}(a_r)K(m; c)^1$ is of the form of $n_2(x)n_3(y)n'(z)t(\alpha, \beta)j''_{x-r}$ or $n_2(x)n_3(y)n'(z)t(\alpha, \beta)z(\mu)$ with $\mu \in \mathcal{P}^{l-r+1}$. If $g$ is of the latter form, then

$$z_0g = \begin{bmatrix} \mu \alpha^{-1}y & \beta & \beta^{-1}z & \alpha^{-1}y \\ \mu \alpha^{-1}x & 0 & \beta^{-1} & \alpha^{-1}x \\ \mu \alpha^{-1} & 0 & 0 & \alpha^{-1} \end{bmatrix}.$$  

Assume $\Phi(z_0g) \neq 0$. From the $(1, 2), (2, 3)$-coefficients, $\beta \in \mathcal{O}^x$. From the $(3, 4)$-coefficient, $\alpha^{-1} \in \mathcal{O}$, and therefore $\mu \alpha^{-1} \in \mathcal{P}^{l-r+1}$. But, $\det(z_0g) = \mu \alpha^{-1} \in \mathcal{P}^{(l-r)}$ by the definition of $\Phi_m$. Hence, we may assume $g$ is of the first form. Then

$$z_0g = \begin{bmatrix} \omega^{l-r} \alpha^{-1}y & \beta & \beta^{-1}z & 0 \\ \omega^{l-r} \alpha^{-1}x & 0 & \beta^{-1} & 0 \\ \omega^{l-r} \alpha^{-1} & 0 & 0 & 0 \end{bmatrix},$$

and it is easy to see that $W_\Phi(a_r) = q^{-l\text{vol}(K(m;c)^1)}W(a_j'^{l}) = q^{-l\text{vol}(K(m;c)^1)}W^-(a_r)$ ($W^-$ is defined at (6.13)). We have showed:

**Proposition 8.1.** With notations as above, if $W \in V(m)$, then

$$Z_0(s, W_\Phi) = q^{-m\text{vol}(K(m;c)^1)}Z(s, W^-).$$

**Computation for $W_\Phi(w_4a_m w_{1,3})$.** By (2.4),

$$W_\Phi(w_4a_m \begin{bmatrix} 1 \\ w_3 \end{bmatrix}) = \omega^2(-1) W_\Phi(\begin{bmatrix} 1 \\ \omega^m \end{bmatrix} w_4 \begin{bmatrix} 1 \\ w_3 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}) = W_\Phi(\begin{bmatrix} 1 \\ \omega^m \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}) = W_\Phi(\begin{bmatrix} 1 \\ \omega^m \end{bmatrix} \begin{bmatrix} 1 \\ w_4 \end{bmatrix}).$$

The isomorphism $I_a$ sends

$$\begin{bmatrix} 1 \\ \omega^m \end{bmatrix} \begin{bmatrix} w_2 \\ w_2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ \omega^m \end{bmatrix} \begin{bmatrix} 1 \\ w_4 \end{bmatrix} =: u_m.$$  

Set

$$\Phi^c = w_\psi^{-1}(j_m, u_m)\Phi.$$  

By definition of $W_\Phi$, $W_\Phi(w_4a_m w_{1,3}) = (\pi(j_m)W)_{\Phi^c}(1).$
Lemma 8.2. With the notations as above, we have the followings.

i) In the case of $c = 0$,

$$\Phi_c^m = \text{Ch}( \begin{bmatrix} P^m & 0 & 0 & 0 \\ P^m & 0 & 0 & 0 \\ P^m & P^m & P^m & 0 \end{bmatrix} ).$$

ii) In the case of $c > 0$, the support $\text{supp}(\Phi_c^m)$ is contained in the lattice

$$\begin{bmatrix} \mathcal{O} & 0 & 0 & \mathcal{O}^- \\ \mathcal{O} & 0 & 0 & \mathcal{O}^- \\ \mathcal{O} & \mathcal{O}^- & \mathcal{O}^- & 0 \end{bmatrix}.$$

If

$$u \in \begin{bmatrix} P^m & P^c & 0 & 0 \\ P^m & P^c & 0 & 0 \\ P^{m+c} & P^m & P^m & P^c \end{bmatrix},$$

then $\Phi_c^m(z + u) = \Phi_c^m(z)$.

Proof. Suppose that $\Phi \in \mathcal{I}(Z^+)$ is of the form of $\phi_1 \otimes \phi_2 \otimes \phi_3$ with $\phi_r \in \mathcal{I}(Y \otimes e_r)$. Since

$$(j_m, u_m) = (1, u_0)(j_m, \begin{bmatrix} 1 & \omega^m 1_2 \\ \omega^m 1_2 & \omega^m \end{bmatrix}),$$

by the formulas of $w_{\psi^{-1}}$,

$$\Phi_c = (j_m \cdot (\phi_1)^2) \otimes (j_m \cdot (\phi_2)^2) \otimes (\omega^{-m} j_m \cdot \phi_3),$$

where $g \cdot \phi$ is defined by $g \cdot \phi(z) = \phi(zg)$, and $\phi^\sharp$ is the Fourier transform defined by $\phi^\sharp(y) = \int_Y \psi^{-1}((y, y'j(-w_2))) \phi(y')dy$ where $dy$ is chosen so that $(\phi^\sharp)^2(y) = \phi(-y)$. Now, i) is a direct calculation. For ii), we write $\Phi_m = \sum_i \phi_i \otimes \phi_i^\sharp \otimes \phi_i^\sharp$ so that

$$\text{supp}(\phi_i^\sharp) \subset \hat{L} \oplus L_4 := [\mathcal{O}, \mathcal{O}, 0] \oplus [0, 0, 0, \mathcal{O}],$$

$$\phi_i^\sharp(y + u) = \phi_i^\sharp(y) \text{ for } u \in \omega^t \hat{L} \oplus L_4.$$

Then,

$$\text{supp}(j_m \cdot \phi_r^\sharp) \subset L' \oplus L_4' := [\mathcal{O}, 0, \mathcal{O}^-] \oplus [0, 0, 0],$$

$$j_m \cdot \phi_r^\sharp(y + u) = j_m \cdot \phi_r^\sharp(y) \text{ for } u \in \omega^t L' \oplus L_4',$$

and

$$\text{supp}((j_m \cdot \phi_r^\sharp)^2) \subset L'' \oplus L_4'' := [\mathcal{O}, 0, \mathcal{O}^-] \oplus [0, 0, 0],$$

$$(j_m \cdot \phi_r^\sharp)^2(y + u) = (j_m \cdot \phi_r^\sharp)^2(y) \text{ for } u \in \omega^t L'' \oplus L_4''.$$

From this, ii) follows. \qed

At the computation for $W_\phi(a_m)$ in the case of $c = 0$, we do not use the third row of $z_0g$ for the condition $\Phi_m(z_0g) \neq 0$. Therefore, by Lemma 8.2 i), the same argument can be applied for $\Phi_c^m$, and we have $W_\phi(w_4 a_m w_{1,3}) \neq 0$. 

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Let $\varepsilon > 0$. By Lemma 8.2 ii), supp($\Phi_m^c$) is contained in the lattice
\[
\mathcal{L}_1 \cdot \mathbf{a}_- = \begin{bmatrix}
\mathcal{P}_1^{l} & \mathcal{P}_1^{r} & \mathcal{P}_1^{s} \\
\mathcal{P}_1^{l} & \mathcal{P}_1^{r} & \mathcal{P}_1^{s} \\
\mathcal{P}_1^{l} & \mathcal{P}_1^{r} & \mathcal{P}_1^{s}
\end{bmatrix},
\]
which is Int($\mathbf{a}_-$)$\mathbf{K}(l)$-invariant. By the above argument for the case of $\varepsilon = 0$,
\[
\Phi_m^c(z_0 g) = 0, \text{ unless } g \in \mathbf{B}_1 \text{Int}(\mathbf{a}_-)$\mathbf{K}(l)$.\]
Therefore, since Int($\mathbf{a}_-$)$\mathbf{K}(l) \supset \mathbf{K}^c(m; \varepsilon)$, we may assume that $g \in \mathbb{Z}^d \setminus \mathbb{S}_{p,q}/\mathbf{K}^c(m; \varepsilon)$ is written of the form of $\mathbf{n}_2(-x)\mathbf{n}_3(y)\mathbf{n}'(u)\mathbf{t}(\alpha, \beta)j'(1)\mathbf{w}(\omega^i)$ or $\mathbf{n}_2(-x)\mathbf{n}_3(y)\mathbf{n}'(u)\mathbf{t}(\alpha, \beta)$. Assume that $g$ is of the former form. Then,
\[
z_0 g = \begin{bmatrix}
0 & \beta^{-1}z & \alpha^{-1}y & -\beta + \omega^i \beta^{-1}z \\
0 & 0 & \alpha^{-1} & 0 \\
0 & \beta^{-1} & \alpha^{-1}x & \omega^i \beta^{-1}
\end{bmatrix}.
\]
By Lemma 8.2 ii), for the condition $\Phi(z_0 g) \neq 0$, we need the $(3, 2)$-coefficient $\beta^{-1} \in \mathcal{P}_m$. Then, $\Phi(z_0 g \mathbf{n}'(u)) = \Phi(z_0 g)$ for $u \in \mathcal{P}_m$. Therefore, the integration $\int_{\mathbf{z}_1 \setminus \{g\}} \Phi(z_0 g) W^c(g) dg$ over the set of the former forms vanishes. Hence, we may assume that $g$ is of the latter form. By (8.2), and Lemma 8.2,
\[
W_\Phi(\mathbf{w}_4 \mathbf{a}_m \mathbf{w}_{1,3}) = c_m \Phi_m^c(z_0) W^c(1),
\]
where $c_m$ is a constant depending only on $\Phi_m^c$.

**Lemma 8.3.** In case of $\varepsilon > 0$,
\[
c_m \Phi_m^c(z_0) = \mathcal{G}(\omega_\pi, 1).
\]

**Proof.** There exist principal series $\tau_1, \tau_2 \in \text{Irr}^m(G_2)$ such that $\omega_\tau_1 = \omega_\tau_2 = \omega$ and $n_{\tau_1} = n_{\tau_2} = \varepsilon$. Let $\pi = (\tau_1^\prime \otimes \tau_2^\prime)$, and $\Pi = \theta(\pi^\prime)$. By [G-T], the $L$-parameter of $\Pi$ is $\phi_{\tau_1} \otimes \phi_{\tau_2}$. By Corollary 7.5, $L(s, \Pi) = L(s, \pi)$. Let $r \geq 0$. Let $W \in \mathcal{M}_\psi(\pi)$ be quasiparamodular of level $2\varepsilon + r$ be as in Theorem 7.3. Let $\Phi = \Phi_{2\varepsilon + r}$. By Proposition 8.1, $Z_0(s, W_\Phi) = \mathbf{q}^{-r(s-\frac{1}{2})} \mathcal{G}(\omega, 1)L(s, \Pi)$. By the functional equation (2.3),
\[
Z_0(1-s, \Pi^\prime(a_{-2\varepsilon-r})(W_\Phi)^\prime) = \mathcal{E}_\pi \mathcal{G}(\omega_\pi, 1)L(1-s, \Pi^\prime).
\]
Comparing the constant terms of both sides, we obtain the assertion from (8.4). $\square$

Now, we prove the coincidences $L(s, \phi_\pi) = L(s, \pi)$ and $\varepsilon(s, \phi_\pi, \psi) = \varepsilon(s, \pi, \psi)$. We have showed this for generic constituents of Borel and Siegel parabolic inductions in the previous section. Hence, we may assume $L(s, \pi) = 1$. Let $\tilde{\pi} = \theta(\pi^\prime)$. If we write $L(s, \tilde{\pi})^{-1} = \prod_{i=1}^d (1-\alpha_i q^{-s})$ by some $\alpha_i \in \mathbb{C}$, then $L(1-s, \tilde{\pi})^{-1} = \prod_{i=1}^d (1-\alpha_i^{-1} q^{s-1})$ (recall that $\tilde{\pi}$ is equivalent to $\tilde{\pi}$). By the above argument, $\mathcal{E}_\pi \mathcal{G}(\omega_\pi, 1)$ is the constant term of $Z_0(1-s, \tilde{\pi}^\prime(a_{-m_\pi})(W_\Phi)^\prime)$. From Theorem 2.3, 6.1, the functional equations, and the above argument, it follows that
\[
\frac{Z_0(1-s, \tilde{\pi}^\prime(a_{-m_\pi})(W_\Phi)^\prime)}{L(1-s, \tilde{\pi}^\prime)} = q^{(m_\pi-m_{\tilde{\pi}})(s-\frac{1}{2})} \mathcal{E}_\pi \mathcal{G}(\omega_\pi, 1).\]
Comparing zeros of these polynomials in $\mathbb{C}[X, X^{-1}]$ with $X = q^s$, we conclude that $\{\alpha_i\}_{i=1}^d = \{q \alpha_i\}_{i=1}^d$ as sets. Hence, $L(s, \tilde{\pi}) = L(1-s, \tilde{\pi}) = 1$. Therefore, $L(s, \phi_\pi) = L(s, \tilde{\pi}) = L(s, \pi)$. Now, the zeta integral $Z_0(1-s, \tilde{\pi}^\prime(a_{-m_\pi})(W_\Phi)^\prime)$ is constant, and equals $\mathcal{E}_\pi \mathcal{G}(\omega_\pi, 1)$. Thus, $m_\pi = m_\tilde{\pi}, \mathcal{E}_\pi = \mathcal{E}_\pi$, and $\varepsilon(s, \phi_\pi, \psi) = \varepsilon(s, \pi, \psi)$. This completes the proof.
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