Model-independent Study on the Structure of $\Lambda(1405)$

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We present two methods to pin down the internal structure of hadron resonances from the quantities which can in principle be determined by experimental observables in a model-independent manner. One method is to relate the internal structure of a quasibound state with the scattering length and eigenenergy based on the generalized weak-binding relation. The other method is to find a trace of the origin of resonances in the analytic structure of the scattering amplitude. Utilizing both the methods, we conclude that the high-mass pole of $\Lambda(1405)$ is dominated by the $\bar{K}N$ component, and the low-mass pole originates from the dynamics in the $\pi\Sigma$ channel.

KEYWORDS: Exotic hadrons, $\Lambda(1405)$, Compositeness, CDD zero

1. Introduction

Observation of more than 300 species of hadrons [1] shows the rich and complicated spectrum of hadrons in the nonperturbative regime of QCD. While a large part of the hadron spectrum can be understood by the quark-antiquark picture for mesons and three-quark picture for baryons in conventional constituent quark models, there are some exceptions of this classification, called exotic hadrons. Intensive efforts have been put to clarify the structure of such exotic states [2, 3].

A traditional method to study the internal structure of hadrons is to compare the experimental information (mass, width, etc.) and model calculations with specific configurations (multiquarks, hadronic molecule, etc). It is, however, desirable to determine the structure of hadrons from the observable quantities to avoid the ambiguities due to the employed models. In this contribution, we present two recent proposals to discuss the internal structure of hadrons toward this direction.

Because most of exotic hadron candidates are unstable against the strong decay, the quantities we can use are the observables in hadron-hadron scatterings. In particular, here we regard the following quantities as “model independent”:

(i) experimental observables (cross sections, etc.),
(ii) on-shell scattering amplitudes (scattering lengths, phase shifts, etc.), and
(iii) analytically continued scattering amplitudes (position of poles, zeros, etc.).

It is clear that (i) consists of model-independent quantities, but (ii) and (iii) may require an explanation. In principle, with a sufficient accuracy of scattering data, one can determine the on-shell scattering amplitude through the partial wave analysis. In fact, scattering phase shifts have been successfully determined, for instance, in the $NN$, $\pi N$, and $\pi\pi$ sectors. Furthermore, from a given scattering amplitude, the singularities in the complex energy plane can be uniquely determined, thanks to the uniqueness of the analytic continuation. In this way, (ii) and (iii) can be directly related to the observable quantities (i). In contrast, model-dependent quantities are related to the off-shell property of the scattering amplitude [4], such as the wave function of the eigenstate. While the off-shell quantities are sometimes useful to draw an intuitive picture of the eigenstate, in this work, we refrain from
using them to discuss the structure of the eigenstate, because they cannot be determined from the observable quantities.

In practice, however, we should keep in mind that the accuracy of the actual experimental data is limited, and the determination of (ii) and (iii) suffers from sizable uncertainties. There are many cases where the direct scattering experiment is difficult (or impossible), and even the existence of a pole can be controversial. In the case of the $\Lambda(1405)$, (i) consists of the $K^-p$ scattering data (see Ref. [5]) and the measurement of the kaonic hydrogen [6,7]. From these observables, the coupled-channel meson- baryon scattering amplitude (ii) and the pole structure in the $\Lambda(1405)$ region (iii) are successfully determined [8–11]. Currently, the results from different analyses show some systematic uncertainties (see Ref. [1]), but with the refinement of the experimental database, the results should converge in future. In this work, we use the representative results in Refs. [8, 9] to determine the structure of the $\Lambda(1405)$.

There have been a few works on the structure of hadrons from the model-independent quantities mentioned above. In 1965, Weinberg proposed the weak-binding relation [12] which relates the structure of the weakly bound state to the scattering length and binding energy. Another approach is the pole counting method by Morgan and Pennington [13, 14], where the existence of poles in different Riemann sheets (the shadow poles [15]) is used to test the nature of a resonance. The former approach utilizes (ii) and the latter is based on (iii) in the above classification.

In this contribution, we present two theoretical frameworks to study the internal structure of hadron resonances. In section 2, we show the generalization of the weak-binding relation to unstable states based on the effective field theory [16, 17]. While there are several applications of the results in Ref. [12] to hadron resonances [19–25], direct generalization of the weak-binding relation to unstable states has not been established. In section 3, we discuss the origin of resonances from the position of the pole and zero of the scattering amplitude [18]. Compared with the pole counting method, our method is advantageous, because it requires the knowledge of the most adjacent Riemann sheet. We then apply these methods to study the $\Lambda(1405)$ resonance in the $\bar{K}N$ scattering in section 4. The last section is devoted to a summary.

## 2. Compositeness from weak-binding relation

We consider an unstable quasibound state near the threshold of two-hadron channel 1, which couples to a lower energy decay channel 2. Such system can be described by the following effective field theory:

\[
H = H_{\text{free}} + H_{\text{int}} = \int d^3x (H_{\text{free}} + H_{\text{int}}),
\]

\[
H_{\text{free}} = \sum_{i=1,2} \frac{1}{2M_i} \nabla \psi_i^*(x) \cdot \nabla \psi_i(x) + \sum_{i=1,2} \frac{1}{2m_i} \nabla \phi_i^*(x) \cdot \nabla \phi_i(x) + \frac{1}{2M_0} \nabla B_0^*(x) \cdot \nabla B_0(x) \\
- \omega_\phi \psi_1^*(x) \psi_2(x) - \omega_\phi \phi_1^*(x) \phi_2(x) + \omega_0 B_0^*(x) B_0(x),
\]

\[
H_{\text{int}} = \sum_{i=1,2} g_{0,i} \left( B_0^*(x) \psi_i(x) \phi_i(x) + \psi_1(x) \phi_1^*(x) B_0(x) \right) + \sum_{i,j=1,2} v_{0,ij} \psi_j^*(x) \phi_i(x) \psi_i(x) \phi_j(x),
\]

where $\psi_i, \phi_i$ system forms the two-body channel $i$, which interacts through the four-point contact interactions and couples to the bare state $B_0$ via the three point vertices. The origin of the energy is set at the threshold of channel 1, and the threshold energy of channel 2 is located at $E = -\omega$ with $\omega = \omega_\phi + \omega_0$. We consider the quasibound state $|\Psi\rangle$ with a complex eigenenergy $E_\Psi \in \mathbb{C}$. The compositeness $X$ (of channel 1) of the quasibound state is defined as

\[
X \equiv \int \frac{d^3p}{(2\pi)^3} \langle \bar{\Psi}|p_1\rangle \langle p_1|\Psi\rangle,
\]
where \( |p_1 \rangle \) is the \( \psi_1 \phi_1 \) scattering state with the momentum \( p \). Namely, \( X \) measures the overlap of the quasibound state with the two-hadron scattering state of channel 1. Note however that, because of the unstable nature of the quasibound state, we have to introduce the Gamow vector \( |\tilde{\Psi} \rangle = |\Psi \rangle^* \) to define the compositeness, and therefore, the compositeness \( X \) is obtained as a complex number.

As shown in Refs. [16, 17], the scattering amplitude can be obtained in the closed form. In this procedure, the zero-range interactions in Eq. (3) need to be regularized by the momentum cut-off \( \Lambda \). This scale \( \Lambda \) reflects the finite-range nature of the hadron-hadron interaction of interest. By assuming that the magnitude of the eigenenergy \( |E_h| \) is small, we can expand the scattering length of channel 1 \( a_0 \) to obtain the relation

\[
a_0 = R \left( \frac{2X}{1 + X} + O\left( \frac{R_{\text{typ}}}{R} \right) \right) + O\left( \frac{1}{R} \right) \right),
\]

(5)

\[
R = \frac{1}{\sqrt{-2\mu_1 E_h}}, \quad l = \frac{1}{\sqrt{2\mu_1 \omega}},
\]

(6)

where \( R (l) \) is the length scale associated with the eigenenergy \( E_h \) (threshold energy difference \( \omega \)) and \( R_{\text{typ}} \) is the typical length scale of the two-hadron interaction, which is related to the momentum cut-off as \( R_{\text{typ}} = 1/\Lambda, \mu_1 = m_1 M_1/(m_1 + M_1) \) is the reduced mass in channel 1. The first two terms of Eq. (5) coincide with the original weak-binding relation [12], except for the fact that the scattering length and the eigenenergy are now complex. The last term of Eq. (5) indicates the correction from the decay channel, which can be neglected when \( l \ll |R| \). In other words, the quasi-bound state is said to be “weakly bound”, when the magnitude of \( |E_h| \) is so small that the correction terms in Eq. (5) are safely neglected. In this case, the compositeness \( X \) can be determined only by the scattering length \( a_0 \) and the eigenenergy \( E_h \), which are the model-independent observables.

3. Implication from nearby CDD zero

One useful theoretical concept to classify the origin of resonances in coupled-channel scattering is the zero coupling limit (ZCL) [15]. This limit is obtained by multiplying the off-diagonal components of the coupled-channel potential matrix by a factor \( x \) as

\[
V = \begin{pmatrix}
V_{11} & xV_{12} & \cdots \\
xV_{21} & V_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]

(7)

and letting \( x \to 0 \). In the ZCL, the potential matrix is diagonal in channel basis, so that the coupled-channel problem reduces to the sum of independent single-channel scatterings. For a nonzero \( x \), all the components of the scattering amplitude \( T_{ij} \) have an eigenstate pole at the same energy, because the pole condition is given by \( \det T^{-1}(E_h) = 0 \). On the other hand, in the ZCL (\( x = 0 \)), different components are mutually disjoint, so the existence of a pole in one component in general does not imply the existence in the others. Assuming that the pole trajectory is continuous in \( x \), we can classify the behavior of the pole in a diagonal component \( T_{ii} \) toward the ZCL into two cases:

(a) the pole remains in \( T_{ii} \) at the ZCL

(b) the pole decouples from \( T_{ii} \) at the ZCL (it may remain in a different component)

In case (a), the eigenstate can be generated by \( V_{ij} \) in the single-channel calculation, while in case (b), the eigenstate originates in the dynamics other than channel \( i \). Thus, we define that the origin of the pole is in channel \( i \) for case (a), and it is something else for case (b). Note that this definition usually corresponds to the compositeness in channel \( i \) as \( X_i \sim 1 \) for case (a) and \( X_i \sim 0 \) for case (b), but \( X_i \) can be small even for case (a), if \( V_{ji} \) has a strong energy dependence [26]. In addition,
because the Hamiltonian is gradually modified, the nature of the eigenstate may also change during the extrapolation from \( x = 1 \) to 0. It is therefore important to examine the nature of resonances by the method in this section and also by the compositeness in the previous section.

Next, we argue that case (b) is closely related to the existence of the Castillejo-Dalitz-Dyson (CDD) zero [27,28]. The CDD zero is defined as the zero of the scattering amplitude \( T_{ij}(E_{CDD}) = 0 \). In general, the position of the CDD zero is not common among different components. To relate the poles and zeros, we consider the contour integral of the phase of the scattering amplitude along a closed path \( C \) in the complex energy plane. The argument theorem ensures that this integral gives an integer which counts the number of the zeros \((n_Z)\) minus the number of poles \((n_P)\) in the path \(C\):

$$\frac{1}{2\pi i} \oint_C dz \frac{d \arg T_{ij}(z)}{dz} = n_Z - n_P \equiv n_C.$$  

Because this is the topological invariant of \( \pi_i(U(1)) \cong \mathbb{Z} \), the number \( n_C \) does not change under the continuous deformation of the amplitude, such as the variation of \( x \) in Eq. (7). Let us set the path \( C \) to enclose the eigenstate pole near the ZCL. In case (a), the number \( n_C \) is trivially conserved in the \( x \to 0 \) limit. In case (b), however, \( n_P = 1 \) for a small but finite \( x \), while \( n_P = 0 \) at \( x = 0 \). In order to conserve \( n_C \) in the \( x \to 0 \) limit, we must have \( n_Z = 1 \) for \( x \neq 0 \) and \( n_Z = 0 \) at \( x = 0 \). Because we can take an arbitrarily small path \( C \), we find that the pole decouples from the amplitude by encountering the CDD zero. This implies that, the \( i \)-th component of the physical \((x = 1)\) scattering amplitude \( T_{ij} \) has a CDD zero near the eigenstate pole, if this pole decouples from the amplitude at the ZCL.

In summary, we first define that channel \( i \) is not the origin of the eigenstate when the pole decouples from the \( T_{ij} \) component in the ZCL. Next, we show that the decoupling of the pole in the ZCL implies the existence of a nearby CDD zero in \( T_{ij} \). Hence, if the eigenstate pole is accompanied by a CDD zero in \( T_{ij} \), the origin of the eigenstate is not attributed to channel \( i \).

4. Application to \( \Lambda(1405) \)

The methods described in sections 2 and 3 have been applied to physical hadron resonances [16–18]. Here we present the results for \( \Lambda(1405) \). In the energy region between \( \pi \Sigma \) and \( \bar{K}N \) thresholds, there exist two resonance poles corresponding to the \( \Lambda(1405) \) [1,29–31]. Namely, physical “\( \Lambda(1405) \)” is a superposition of two eigenstates. One pole lies near the \( \bar{K}N \) threshold with a narrow width (called high-mass pole), and the other locates at the lower energy with a broad width (called low-mass pole). In the following, we examine the structure of the eigenstate represented by each pole, based on the scattering amplitude determined in Refs. [8,9].

First, we use the generalized weak-binding relation (5). Note that this relation is applicable only when \( |E_{kk}| \) is sufficiently small and the correction terms are negligible. From the estimation of the interaction range \( R_{typ} \sim 0.25 \text{ fm} \) with the underlying vector-meson-exchange picture and \( l = 0.76 \text{ fm} \) from the energy difference between \( \pi \Sigma \) and \( \bar{K}N \) thresholds, we find that the \( \bar{K}N \) compositeness of the high-mass pole can be determined. Using the eigenenergy \( E_h = -10 - 26i \text{ MeV} \) and the \( \bar{K}N(I = 0) \) scattering length \( a_0 = 1.39 - 0.85i \), we obtain the compositeness of the \( \bar{K}N \) channel as

$$X_{\bar{K}N} = 1.2 + i0.1$$  

when the correction terms are neglected. The complex compositeness is obtained as a consequence of the unstable nature of the eigenstate. One prescription is proposed to deduce the real-valued compositeness \( \bar{X}_{\bar{K}N} \) which can be interpreted as a probability of finding the \( \bar{K}N \) component [16,17]. Including the estimation of the effect of the correction terms [17], we can determine the real-valued compositeness as

$$\bar{X}_{\bar{K}N} = 1.0^{+0.0}_{-0.4}.$$
The results obtained with the scattering length and eigenenergy in other analyses \cite{10, 11} are consistent within the uncertainties from the correction terms. From this, we conclude that the high-mass pole of the $\Lambda(1405)$ is dominated by the $\bar{K}N$ molecular component.

Next, we investigate the position of the CDD zero in the $\Lambda(1405)$ region. To be specific, here we use the ETW model constructed in Ref. \cite{9} for simplicity. In this case, two poles for the $\Lambda(1405)$ are obtained as

$$W_{\text{Low pole}} = 1375 - 65i, \quad W_{\text{High pole}} = 1423 - 22i.$$ \hspace{1cm} (11)

We then search for zeros in the diagonal components, $T_{\pi\Sigma \rightarrow \pi\Sigma}$ and $T_{\bar{K}N \rightarrow \bar{K}N}$. We find one zero for each component, whose position is

$$W_{\text{Low CDD}} = 1381 - 108i \quad \text{in} \quad T_{\bar{K}N \rightarrow \bar{K}N},$$ \hspace{1cm} (12)

and

$$W_{\text{High CDD}} = 1428 - 0i \quad \text{in} \quad T_{\pi\Sigma \rightarrow \pi\Sigma}.$$ \hspace{1cm} (13)

For the high-mass pole at $W_{\text{pole}}^{\text{High}}$, there is no nearby zero in the $T_{\bar{K}N \rightarrow \bar{K}N}$ amplitude, while the zero at $W_{\text{CDD}}^{\text{High}}$ in $T_{\pi\Sigma \rightarrow \pi\Sigma}$ is close to the pole. The situation of the low-mass pole is opposite; the nearby zero exists at $W_{\text{CDD}}^{\text{Low}}$ in $T_{\bar{K}N \rightarrow \bar{K}N}$ but none in $T_{\pi\Sigma \rightarrow \pi\Sigma}$. According to the discussion in section 3, we conclude that the origin of the high-mass pole (low-mass pole) is in the $\bar{K}N$ ($\pi\Sigma$) channel. In fact, by taking the ZCL within the ETW model, it is shown that the high-mass pole merges with $W_{\text{CDD}}^{\text{High}}$ in $T_{\pi\Sigma \rightarrow \pi\Sigma}$, and the low-mass pole encounters $W_{\text{CDD}}^{\text{Low}}$ in $T_{\bar{K}N \rightarrow \bar{K}N}$ \cite{18}. This reinforces the conclusion of the origin of poles. The conclusion in this section is in accordance with the $\bar{K}N$ dominance of the high-mass pole obtained in the previous section.

5. Summary

We have presented two methods to study the structure of hadron resonances based on model-independent quantities in the hadron-hadron scattering: the generalized weak-binding relation (section 2) and the analysis of the CDD zero around the eigenstate pole (section 3). The constructed methods are applied to the $\Lambda(1405)$ resonance, showing that the high-mass pole is dominated by the $\bar{K}N$ molecular component and the origin of the low-mass pole is in the $\pi\Sigma$ channel.

Currently, hadron scattering amplitudes (in particular, the pole positions) are determined only in limited channels, with sizable uncertainties. We however expect that the accuracy of the amplitude will be increased by the accumulation of forthcoming experimental data, and new development of experimental technique may enlarge the accessible systems in future. It is also worth mentioning that the developments in the lattice QCD allow us to study the hadron-hadron scattering for which the scattering experiments are difficult. Because the methods presented in this contribution do not rely on the specific details of the scattering amplitude, they can be immediately applied to study the nature of hadron resonances for a given scattering amplitude.

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