TURBULENT ACCRETION BRAKING TORQUES AND EFFICIENT JETS WITHOUT MAGNETOCENTRIFUGAL ACCELERATION: CORE CONCEPTS.

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ABSTRACT

I discuss three mutually-supportive notions or assumptions regarding jets and accretion. The first is magnetocentrifugal acceleration (MCA), the overwhelmingly favored mechanism for the production of jets in most steady accreting systems. The second is the zero-torque inner boundary condition. The third is that effective viscous dissipation is like real dissipation, leading directly to heating. All three assumptions fit nicely together in a manner that is simple, persuasive, and mutually-consistent. All, I argue, are incorrect. For concreteness I focus on protostars. Magneto-hydrodynamic (MHD) turbulence in accretion is not a sink of energy, but a reservoir, capable of doing mechanical work directly and therefore efficiently, rather than solely through ohmic (“viscous”) heating. Advection of turbulence energy reduces the effective radiative efficiency, and may help solve the missing boundary-layer emission problem. The angular momentum problem, whereby accretion spins up a protostar to breakup, is resolved by allowing direct viscous coupling to the protostar, permitting substantially greater energy to be deposited into the accretion flow than otherwise possible. This goes not into heat, but into a turbulent, tangled, buoyant toroidal magnetic field. I argue that there is neither an angular momentum problem nor an efficiency problem that MCA is needed to solve. Moreover, the turbulent magnetic field has ample strength not just to collimate but to accelerate gas, first radially inwards through tension forces and then vertically through pressure forces, without any MCA mechanism. I suggest then that jets, particularly the most powerful and well-collimated protostellar jets, are not magnetocentrifugally driven.
1. INTRODUCTION

For many years now, the conventional wisdom regarding most astrophysical jets, especially persistent jets such as in protostars, AGN and microquasars, is that they are driven by some type of magnetocentrifugal acceleration (MCA) by large-scale poloidal magnetic fields (Blandford & Payne 1982). Here however, as I have done consistently elsewhere, I take a contrarian view regarding the MCA hypothesis, and I continue the argument against it on theoretical grounds. I focus on the physics of the innermost accretion regions in steady disk-mediated accretion onto a central object, particularly, a protostar.

Conventionally, accretion onto a protostar is thought to occur in one of two scenarios. Either the accretion occurs in a radially-thin boundary layer that does not viscously transmit torque between the protostar and the surrounding disk, or accretion is funneled from a truncated accretion disk onto the protostar by the protostar’s magnetosphere (see e.g. fig. 1 of Hersant et al. 2005).

In the first scenario, accretion in a boundary layer leads to the so-called angular momentum problem, because the boundary layer can not transmit torque and the protostar gains excess angular momentum from pure advection. This halts accretion. Even before then, this scenario also predicts a larger UV or X-ray flux from the boundary layer than is typically seen. Magnetocentrifugal acceleration — either from the protostar (Hartmann & MacGregor 1984) or from the surrounding nearby accretion flow, possibly including the broader disk as well (Pudritz & Norman 1983) — is an appealing solution to these problems, as well as the problem of jet production. The magnetocentrifugal mechanism provides a sink of angular momentum and thereby enables continued accretion of mass onto the protostar, while curbing the attendant accumulation of angular momentum that would otherwise tear the protostar apart. On the other hand it raises new problems, such as the origin of the large-scale poloidal field, and the mechanism by which matter is threaded onto it.

In the second scenario, material is funneled onto the protostar from more radially-distant regions of an accretion disk by a strong large-scale poloidal field anchored in the protostar. See Königl (1991), who extends the neutron-star accretion model of Ghosh & Lamb (1978) (see also Ghosh et al. 1977) to protostars. In that case, it is assumed the disk has an inner truncation radius corresponding to the coupling region, and so there is no boundary-layer. This potentially helps solve the angular momentum problem, because while the material at larger radii has even greater specific angular momentum than material in a boundary-layer would, the poloidal field can transmit torque to the disk, keeping the protostar from gaining excess angular momentum. Jets may still be driven magnetocentrifugally, such as by the disk or by the X-wind process, in which jets originate in this coupling region. On the other hand, this MCA suffers from the problem again of being dependent on assumptions about the reconnection process by which material is threaded onto open field lines, and it also relies upon assumptions about the field topology. As well, the observed speeds of jets, being of order of a few times the escape velocity at the protostellar surface, is suggestive of a jet origin very near the protostar, which tends to argue against disk-winds in particular for the high-velocity (jet) component of outflows.

In both cases, some flavor of MCA has been suggested to help solve the angular momentum problem by providing a sink for angular momentum in the form of a jet. If, alternatively, the disk couples viscously to the protostar, then there is no angular momentum problem to be solved in the first place. Indeed, Pringle (1989) appears to make a similar argument in his objections to Shu et al. (1988), the intellectual precursor to the magnetocentrifugal X-wind model of Shu et al. (1994). Particularly in the case of young or high accretion-rate protostars that are spinning at a substantial fraction (e.g. $\gtrsim 30\%$) of breakup, I therefore suggest a third scenario, in which accretion occurs through a geometrically thick flow that extends all the way down to the surface of the protostar and envelopes it, so that the protostar is essentially fully embedded in the flow (fig. 1). (I will continue to use the terms “star” and “disk,” but with the very important caveat that now there may not actually be a clearly delineable distinction between the two.) In this scenario there is no radially-thin boundary layer and no zero-torque boundary condition on the disk. The protostar is able to exchange torques with the disk viscously, through the tangled, turbulent magnetic field generated by the magnetorotational instability (MRI), possibly augmented by convective or other modes. This viscous coupling of the protostar to the disk allows the protostar to gain mass but not excess angular momentum. No MCA is needed, nor any large-scale poloidal field onto which matter must somehow be coupled. Of course, dynamo processes can be expected to generate some type of large-scale organized field, but the details of this field are not integral to the general schema at this stage and are not addressed here.

It has long been recognized that the inward advection of material in this innermost region in steady accretion onto a protostar is an appealing source for the energy needed to drive jets and related bipolar outflows or winds; for an
Figure 1. I hypothesize a protostar (blue) embedded in a surrounding geometrically- and optically-thick accretion flow or envelope. Streamlines are indicated with continuous arrowed lines. The dashed black line is representative of the effective photosphere of this system in or near optical/IR wavelengths. The envelope includes two large hot ($T ≲ 10^4$ K) recirculation zones above and below the disk (orange). The red streamline is the separatrix; it separates that part of the flow that ultimately accretes onto the protostar (red arrows) from the flow that does not (green streamlines). Veiling is not due to accretion shocks, but the recirculation zones. (NOTE: I am awaiting artwork provided by a professional illustrator for this and other line drawings in this submission.)

early discussion, see Pringle (1989); also see Torbett (1984) and Torbett (1986). The reason for this appeal is that a substantial fraction of all accretion energy can potentially be released there; this is a simple consequence of the virial theorem.

For example, in the case there actually is a thin boundary layer, in the standard thin disk theory of slow ($v_R ≪ v_φ$), cool ($H ≪ R$), radiatively efficient ($q^+ ≃ q^− ≫ q_{adv}$) accretion with negligible vertical mass loss, the radiative luminosity of this boundary layer is\(^1\)

$$L_{BL}^{(rad)} = \frac{GM_∗ \dot{M}}{2R_∗} \left(1 - \frac{1}{Ω_∗ \kappa_*}\right)^2.$$  \(1\)

To the extent that the innermost region of any real disk or other accretion flow is not thin, cool, radiatively efficient, or is otherwise not ideal, this relation may not hold, but it still serves as a useful benchmark for the order of magnitude of energy available there. Boundary layer or not, there is still an enormous amount of energy that potentially may be released in the inner accretion region, of at least comparable order to the relation given above.

The binding energy that must be lost by accreting matter as it falls down the gravitational well towards the star is only part of the energy budget. The second appealing and conventional source of energy to power protostellar jets is the reservoir of rotational energy in the angular momentum of the star, such as again in X-wind models (Shu et al. 1994, et seq.), in which mass loading and torque create a feedback process that maintains the protostellar rotation at some fixed fraction of breakup throughout the accretion process.

I too relied upon torques on the star to power an outflow in my own initial work Williams (2001) on this problem, but in contrast with the X-wind model, I suggested the torques were transmitted viscously. The power available in the form of mechanical luminosity from viscously-coupled braking torque\(^2\) from a fast-rotating protostar is substantial. Generally and roughly, like the boundary-layer luminosity above, the mechanical power available from the braking torque in steady accretion is of order

$$L_{BT}^{(mech)} ≈ \frac{GM_∗ \dot{M}}{R_∗},$$  \(2\)

as in fact was recognized some time ago (see Lynden-Bell & Pringle 1974, p. 634), and may be much larger still than the notional boundary-layer luminosity above. It is a commonplace observation that as much as half of all accretion energy may be released in the innermost regions of accretion (such as the boundary layer); actually, it is potentially

\(^1\) Early work in the field gave a slightly different and subtly incorrect result for this luminosity.

\(^2\) Note that I take care not to call this braking torque a “spin-down” torque. This is potentially an important distinction for a central object with a rather soft equation of state, such as a protostar, as opposed to a neutron star with its very stiff equation of state. Particularly for a protostar, a gain or loss of angular momentum may be simultaneous with a gain or loss of moment of inertia, and so for example what otherwise might have been called a “spin-down” torque can actually result in spinning a body up, such as if the object simultaneously shrinks in effective radius or loses mass, perhaps even due to the same physical effect as that responsible for the torque in the first place.
even far more than that, as much as three times as much or even more in fact in the limiting case of a maximally-rotating central object. (The explanation for this apparently paradoxical result is that the energy nominally “released” or dissipated in the accretion flow can exceed the accretion budget thanks to the additional energy delivered from the star via turbulent stresses.) In either case, whether from the last stages of accretion onto the protostar or from the protostar itself, there is abundantly sufficient energy to power observed jets.

Energy is not the problem. Efficient coupling is, especially if the jet power comes from heat; the Second Law will always take its share. The power that goes into jets is a sizeable fraction of the total accretion power budget; this alone sets a floor on the thermodynamic efficiency of the engine that drives jets. Moreover, if jets were driven thermally, as in a pressure-driven flow through a deLaval nozzle, instead of mechanically such as in the MCA hypothesis, then this would require high temperatures and associated large X-ray fluxes that are not seen; this again argues all the more so for an efficient mechanical means of powering jets. The MCA hypothesis addresses this by proposing a mechanically-efficient mechanism that does not rely upon thermal driving.

So it might appear that a key problem in powering outflows from an alternative mechanism that relies upon either of these two sources appealed to above (the energy of accretion in the innermost regions or the rotation of the protostar itself) is that energy in either case is extracted “viscously” — that is, through the action of the turbulence and its effective “viscosity” — and therefore leads to heating. This is incorrect. As I pointed out in Williams (2001) and later in Williams (2005), the energy that goes into the turbulence creates a tangled, toroidal magnetic field, and the associated hoop-stresses create a radially-inwards force that is able to do real work on the accreting material. In other words, the energy that goes into turbulence can come out in the form of mechanical work, not heat, not even as an intermediary. Here, I take a step towards resolving both turbulence energetics and global energetics in further detail.

The assumption that turbulence always and instantaneously results in heating is present in both in standard thin-disk theory as well as in the theory of ADAFs, where it is common to write that there is a heating term corresponding to a local “viscous dissipation” rate (per area) of

$$Q_+ = \nu_i R^2 \langle \Omega^i \rangle^2 = \int_{-\infty}^{\infty} q_+ \, dz,$$

and where $q_+$, the (per unit volume) viscous heating, ignoring compressibility for simplicity, may be written

$$q_+ = \frac{1}{2 \rho \nu_i} W : W \simeq \rho \nu_i W^2 \langle \Omega^i \rangle = \rho \nu_i R^2 \langle \Omega^i \rangle^2,$$

which is also\(^3\)

$$q_+ = \nu_{\alpha\beta} W^{\alpha\beta} = (\nabla \mathbf{v}) : W$$

where $W$ is the (turbulent) stress tensor\(^4\), $\nu_i$ is the effective turbulent viscosity, $\mathbf{v}$ is the velocity with covariant components $v_\alpha$, and the remaining symbols are standard. See for example eq. 11.3 and also eq. 4.29 in Frank et al. (2002), pages 609–610 in Lynden-Bell & Pringle (1974) and page 344 of Shakura & Sunyaev (1973) in an accretion context, and eqs. 49.2-5 in Landau & Lifshitz (1987) for the fluid-dynamical basis of this in viscous fluids.

Again, this assumes that all of the energy that goes into turbulence comes out in the form of heat, and moreover, that it does so on a time scale (the dissipative time scale) that is short compared to the radial advective time scale. Neither assumption is warranted. In other words, use of these relations for $q_+$ carries the analogy between real viscosity and effective turbulent viscosity too far. Because the distinction is so important, in this paper, I will always designate a turbulent effective kinematic viscosity as $\nu_t$, and reserve $\nu$ exclusively for real molecular kinematic viscosity. The effective dissipation due to turbulence as represented in eqns. 3–5 is not real dissipation following an entropy principle (it is not entirely irreversible), and it does not necessarily lead to heating, and in any case not instantaneously. What has been labeled $Q_+$ or $q_+$ in these equations is actually a turbulence production rate.

Both production and dissipation occur on a finite non-zero time scale, and in any given instant, actual production may exceed dissipation (or vice-versa). In ordinary hydrodynamic turbulence the dissipative time scale $\tau$ is of order

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\(^3\) I use greek indices down and up to refer to components in the covariant/contravariant language of tensors, with a comma indicating a simple partial derivative and a semicolon indicating a covariant derivative. For cartesian index notation, I use general latin indices such as $i, j, k$, and I will use an explicit partial derivative symbol instead of a comma to indicate differentiation, e.g. $\partial_{tj}/\partial x_i$. Angle brackets indicate an actual physical component, which can be related to covariant or contravariant quantities through the square root of the metric, as the cylindrical coordinate system is orthogonal, and no other coordinate system is used.

\(^4\) The thermodynamic pressure $P$ is not included stress tensor $W$ here; otherwise, in compressible flow, $(\nabla \mathbf{v}) : W$ also would include a $P dV$ work contribution, in addition to heating. In principle, $W$ does however include turbulent pressure, but in the interest of simplicity I will largely sidestep the issue of turbulent pressure as it detracts from the main points I am trying to assert in this paper.
the shear time scale, but in MHD turbulence it may be much larger, as I have argued elsewhere, and as seen in, \textit{e.g.}, shearing-box simulations. Plus, while the ratio of the advective rate $v_R/R$ to the shear rate $R\Omega'$ is very small in the outer thin disk, it is not necessarily so in the inner thick region considered here. Altogether then, the ratio of the advective rate $v_R/R$ to the dissipative rate $\tau^{-1}$ may equal or even greatly exceed unity in these inner regions.

Regarding dissipation and heating (or the lack thereof), MHD turbulence also differs from ordinary hydrodynamic turbulence in where its energy ultimately ends up. Much has been made of the fact that MHD turbulence possesses inverse cascades and conserved quantities that ordinary hydrodynamic turbulence does not, but there are other important differences as well. MHD turbulence in a disk creates hoop-stresses completely independently of any inverse cascades or organizing principle; a tangled field plus background shear is sufficient. As mentioned above, these hoop-stresses do real work; this represents a non-dissipative loss channel to the turbulence energy budget. In addition, turbulent energy injected into the field may be buoyantly lost vertically through the Parker instability. The field so lost to a hot, lower-beta region does not just reconnect or evaporate away, but — depending on the radial gradient of magnetic pressure — may snap radially towards the central axis of the system. Both of these two loss channels for turbulent energy may contribute to powering, confining and possibly collimating an outflow.

I suggest then that protostellar jets originate in high accretion-rate systems in which the accretion flow embeds the protostar. The jets will carry away some nonzero fraction of the angular momentum of accretion but the overwhelming sink of angular momentum is the distant regions of the disk. The angular momentum flux of the jets will be far less than what standard MCA theory predicts. Direct turbulent (“viscous”) coupling turns the angular momentum problem into an energy problem. The bulk of the energy discussed above does not go into dissipation and heating, but, as an intermediate stage, into a turbulent, tangled toroidal magnetic field, capable of performing work. The jets are the exhaust for that free energy, explaining the missing boundary-layer luminosity that would otherwise be present in the case of dissipation and heating. The power for the jets comes from a combination of the accretion flow itself and the braking torque on the protostar. The braking torque in turn acts as a natural feedback mechanism, in that, being mediated by shear-driven turbulence, the braking torque increases or decreases if the spin of the protostar increases or decreases respectively.

The most promising venue for jets and related collimated outflow production by the mechanism described here is early (Class 0 or Class I), high accretion-rate protostars. possibly including systems in FU Ori state, although the relation between FU Ori systems and jet production is tentative and not currently known. Nevertheless I argue that in any case it is no coincidence that, possibly outside of FU Ori systems, it is precisely in young, high-\dot{M} protostellar systems that one tends to find the most powerful, well-collimated jets. Nor is it a coincidence that it is also in such systems that the disk tends to be hot and geometrically thick, as opposed to the cool, thin disks that were the original focus of magnetocentrifugal theory. I therefore suggest that jets in such systems are not driven magnetocentrifugally.

2. ACCRETION PRELIMINARIES

As pointed out above, it has actually long been recognized that an accretion disk can extract angular momentum from a protostar through the torques due to effective turbulent viscosity acting on differential rotation, but this result appears to have been neglected, or at least its importance not fully appreciated, with some noteworthy exceptions (see \textit{e.g.} Popham & Narayan 1991; Popham 1996).\footnote{I have also managed to find this argued pointedly in a graduate thesis somewhere, including an argument similar to what I present below, but I can no longer find this reference. I would be interested to hear from any readers who might know of it, so I might give the author proper credit.}

The commonplace practice rather is to assume a zero-torque boundary condition (that is, zero torque due to effective viscosity; there is still an advective flux of angular momentum, \textit{i.e.} an “advective torque”). For example, Chambers (2009) cites Stepiński (1998), who cites Ruden & Pollack (1991), who cite Bath & Pringle (1981) who explicitly invoke a zero-torque inner disk boundary condition, citing Pringle (1977). But Pringle (1977) only addresses an accretion disk that is thin \textit{by assumption} all the way down to the central star, and cites Shakura & Sunyaev (1973) and Lynden-Bell & Pringle (1974). Shakura & Sunyaev (1973) in turn adopt a zero-torque boundary condition because they are considering thin accretion onto a black hole, not a protostar. Lynden-Bell & Pringle (1974) do indeed discuss boundary-layer emission at some length assuming a zero-torque boundary condition, but in their Appendix I, they explicitly make clear that an alternative scenario is possible, in which the angular velocity does not have a maximum, there is no boundary layer, there is a viscous torque on the star, and the torque provides up to two-thirds of the energy dissipated in the disk. It may be that thin-disk Schwarzschild or Kerr accretion should have a zero-torque inner
boundary condition (Novikov & Thorne 1973) or not (Stoeger 1976, 1980), and in general the issue for general black hole accretion has been controversial (Paczyński 2000, paragraph 2); for protostellar accretion however, there should be no controversy at all: the zero-torque inner boundary condition is unjustifiable, particularly for stars rotating near breakup and accreting from a disk that is not geometrically thin.

Let $\Omega$ be the material angular velocity as a function of cylindrical coordinate $R$, and $\kappa$ be the Keplerian angular velocity as a function of $R$. Assume that the mass of the accretion flow is negligible relative to the mass of the central object, so that $\kappa \sim R^{-3/2}$ for $R > R_s$. Let $\Omega_\ast$ and $\kappa_\ast$ be their corresponding values for the star at the stellar radius $R_\ast$, let $\omega$ be the normalized stellar angular rotation frequency $\omega \equiv \Omega_\ast / \kappa_\ast$, and for simplicity assume the star remains spherical with a well-defined $R_\ast$ regardless of the value of $\omega$. The accretion disk has a (variously defined) vertical geometrical thickness $H$ as a function of $R$. Let $b$ be, loosely speaking for now, the radial thickness of the substantially sub-Keplerian region. If $b$ is small compared to $R_\ast$, it is proper to call this a boundary-layer. If instead $b$ is not small compared to $R_\ast$, then the term “boundary-layer” is no longer appropriate. A boundary layer is necessarily thin by definition; see Schlichting (1955).

The standard argument for the existence of a boundary-layer with a zero-viscous-torque boundary condition is given succinctly by Frank et al. (2002), but, ironically, the argument can be traced essentially unchanged back to Appendix I of Lynden-Bell & Pringle (1974). I say ironically, because again, the weaknesses of the argument are also discussed forthrightly in the very same appendix, and the authors note conditions under which it may fail.

The argument hinges critically upon the assumption of accretion from a thin disk; without that assumption, it falls apart. Vertical thinness guarantees two things. First, it sets a practical upper limit on the magnitude of the turbulent kinematic viscosity, which of course appears in the angular momentum equation. Second, it constrains the magnitude of the sound speed and through it the radial pressure gradient that can be supplied to support the accretion flow in the sup-Keplerian innermost regions. A small effective turbulent kinematic viscosity, as defined in terms of the resultant effective Reynolds number based on the diameter of the star and a fiducial speed (say the Keplerian orbital speed), implies a relatively thin viscous boundary layer; a small sound speed, as compared to the notional speed of Keplerian rotation at a radial coordinate $R$, implies that radial pressure support will not be able to maintain a substantially sub-Keplerian disk rotation profile. The result is that, given that the star’s angular velocity is not itself Keplerian, the angular velocity must have a maximum in a thin boundary layer, and using a simple effective viscous prescription for the stress, this yields a zero-torque condition.

For example, we can take a Shakura-Sunyaev type (Shakura & Sunyaev 1973) local prescription for the effective kinematic viscosity due to turbulence, where $\nu_t$ depends upon $R$. Then in the accretion disk, near the stellar surface,

$$\nu_t = \alpha_{SS} c_s H = \alpha_{SS} \kappa_\ast H^2$$

and the effective Reynolds number $Re_\ast$ for accretion onto the protostar, defined in terms of the stellar radius $R_\ast$ and the Keplerian rotation speed, may be written

$$Re_\ast = \frac{\kappa_\ast R_\ast^2}{\nu_t} = \alpha_{SS}^{-1} (H/R_\ast)^{-2}. \tag{7}$$

Take $\alpha_{SS} \ll 0.1$, and a disk that remains thin all the way down to the star, $(H/R)_\ast \lesssim 1/10$. Then $Re_\ast \gtrsim 10^3$. This suggests a relatively thin ($b \ll R_\ast$) viscous boundary layer.

Alternatively, we can take a global prescription from Lynden-Bell & Pringle (1974), in which $\nu_t$ is fixed. Define $\alpha_{LBP}$ in terms of their $b R_c$ as $\alpha_{LBP} = R_c^{-1} \ll 1$. Note that Lynden-Bell & Pringle (1974) adopt $R_c = 10^3$. Write

$$\nu_t = \alpha_{LBP} R_c^2 \Omega_\ast, \tag{8}$$

and then the effective Reynolds number in terms of the stellar radius and the Keplerian rotation speed there, is just

$$Re_\ast = (\alpha_{LBP} \omega)^{-1} = R_c / \omega = 10^3 / \omega \gtrsim 10^3. \tag{9}$$

All other dynamical arguments regarding $b$ that are rooted in angular momentum conservation essentially reduce to an observation that $Re_\ast$ as defined here is large. One could argue whether it is better to use the star’s actual rotation
speed (as done in Lynden-Bell and Pringle) or the Keplerian speed to define an effective Reynolds number, but for a star that is rotating at anything more than a trivial fraction of breakup, it really does not make much difference.

Turning to radial momentum balance, the dominant terms for slow infall are pressure support, with acceleration of order \(c_s^2/b\), gravity with acceleration \(R\kappa^2\), and centrifugal acceleration with magnitude \(R\Omega^2\). In the inner region of an even marginally substantially sub-Keplerian disk or flow this latter term diminishes in significance. Turbulent pressure and the dynamical support of large-scale coherent motions such as meridional circulations are smaller in magnitude than basic thermal pressure support, all motions being subsonic. Then, following closely the argument laid out in Frank et al. (2002),

\[
\frac{c_s^2}{b} \simeq R_\ast \kappa_\ast^2
\]

(10)

and yet, vertical pressure balance in the disk, with a vertical effective gravity of order \(H\kappa^2\), suggests that just outside the boundary layer,

\[
c_s = H\kappa_\ast.
\]

(11)

The well-known result then is that \(H\) is the geometric mean of \(b\) and \(R\),

\[
bR_\ast \simeq H^2
\]

(12)

so that again, \((H/R)_\ast \lesssim 1/10\) is consistent with

\[
b \ll H \ll R_\ast
\]

(13)

(where, say, \(\ll\) here indicates at least an order of magnitude). Call eq. (13) Assertion A.

Since realistically \(\Omega_\ast < \kappa_\ast\), it is then (the argument goes) suggested that there must be some radius \(R_p > R_\ast\) at which the angular velocity reaches its peak value. Call this Assertion B. Using primes to denote derivatives with respect to \(R\), the boundary layer then corresponds to the region between \(R_\ast\) and \(R_p\), where

\[
R_p = R_\ast + b, \quad b \ll R_\ast, \quad \Omega'(R_p) = 0.
\]

(14)

There is negligible practical difference between this definition of \(b\) and alternative definitions.

Next, it is pointed out that, given such a \(R_p\), the effective viscous flux of angular momentum (due to turbulence) across the \(R = R_p\) surface must vanish. That is a simple result of the prescription tucked into eq. (4) that

\[
W_{(R\phi)} = \nu_\ast R\Omega'.
\]

(15)

Call this Assertion C. The result is the zero-torque boundary condition. This fixes the integration constant \(C_L = 1\) for the total radial angular momentum flux in a steady disk (see section 6.1) and leads to familiar expressions such as

\[
\nu_\ast \Sigma = \frac{M}{3\pi} \left[ 1 - \sqrt{\frac{R_\ast}{R}} \right].
\]

(16)

Note that \(C_L = 1\) is equivalent to setting Novikov-Thorne \(Q = 0\) at \(R_\ast\) (Novikov & Thorne 1973); this leads to divergence in \(v_R\) and the vanishing of \(\Sigma\) at \(R_\ast\). Relaxation of the zero-torque boundary condition addresses these problems; again, see Stoeger (1976, 1980).

Applied to protostars uncritically, the entire argument may not be circular, but it certainly does have a whiff of it. The construction assumes a thin disk to begin with. Along with this, it also assumes that there is a clear and unambiguous distinction between the star, which is supported radially by pressure, and the disk, which is not. If we imagine a thick disk that extends all the way down to (and envelopes) the star, as in fig. (1), then the situation changes, particularly when the (normalized) rotation speed of the protostar, \(\omega\), instead of being vanishingly small, is substantial, say of the order of, say, 20–30% or more as discussed above. The previously convenient fiction of a clearly defined “star” and “disk” now becomes dangerously misleading. Equation (7) says \(Re_\ast \gtrsim 10\), which is not enough to assure us a thin boundary layer. Equation (9) is stronger, but stands on a weaker theoretical foundation. And equation (12), given the assumptions leading to it, really tells us nothing at all.

Moreover, when the disk is no longer thin, Assertion A no longer holds, and with it falls Assertion B, and possibly Assertion C as well. In the usual argument, both B and C must hold to fix the integration constant for the angular
Figure 2. A central object that is rotating above or below Keplerian rotation (KR) can be embedded in a thick disk or envelope that matches to a Keplerian rotation law at some $R_m$, without possessing interior to it a peak in $\Omega$ where $\Omega' = 0$ as in the standard boundary-layer form (BL). Shown is a power-law under-rotation (PUR) for $\omega = 1/4$ and $q = 1/2$, yielding $R_m = 4R_\ast$.

momentum fluid equation, i.e. the total radial angular momentum transport in the disk (advective plus viscous). If either fails, the total flux of angular momentum in the disk is no longer thereby constrained.

Assertion C may be violated because the effective “viscosity” and momentum transport due to turbulence and other unresolved velocity and magnetic fields need not follow Newton’s hypothesis of linearity in the local rate-of-shear, in part because unlike real viscosity, there is not necessarily a large separation of scales between the integral scale and the scale of structures that contribute significantly to turbulent viscosity, particularly when the disk is no longer thin. The Boussinesq hypothesis\(^7\) is, after all, only a model.

Much more importantly though, Assertion B above is particularly dubious because once Assertion A fails, then in a pure kinematic sense, nothing compels us to accept (14), or even that $R_p$ even exists. In principle, the peak angular velocity could be found anywhere interior to the star, including the core if not the dead center $R = 0$ outright. For simplicity in this paper I only consider the central star to undergo simple solid body rotation, rather than the other extreme of core-envelope (as in stellar envelope) decoupling, but even so, the peak angular velocity may not be found interior to the disk, but in the star itself. That is, I assume there is no $R_p$ within the disk at which $\Omega' = 0$, and therefore relations (14) do not hold.

For example, in Williams (2001) I assumed, as I do here, that in the thick part of the flow, in the disk plane ($z = 0$), the angular velocity varied as a power law with cylindrical radial distance $R$ with some exponent $-q$. This extended out to match the Keplerian thin disk at the outermost part of the thick flow, at the radius denoted $R_m$ ($m$ for “match”), or $R_\theta r_m$ in the notation of Williams (2001):

$$\Omega = \begin{cases} 
\Omega_\ast (R/R_\ast)^{-q} & \text{if } R \leq R_m, \\
\kappa & \text{if } R > R_m.
\end{cases}$$

(17)

where $\kappa = \sqrt{GM_\ast/R^3}$ is the usual Keplerian angular velocity. See fig. 2. The match radius $R_m$ and exponent $q$ must satisfy

$$(R_m/R_\ast)^{q-3/2} = \omega$$

(18)

A second relation would be needed to fix $q$ and $R_m$ uniquely, but that was not addressed in Williams (2001).

Consequences of the breakdown of relations (14) will be explored below. The immediate implication is that the disk can couple viscously to the protostar, allowing the protostar to become a source of both mechanical energy (the

\(^7\) That is, turbulence transports momentum like an effective viscosity.
energy of rotation) and angular momentum, and allowing the disk to act as a sink of that angular momentum. The amount of power available due to this coupling is, again, substantial, and possibly much more even than the notional boundary-layer power expressed in eq. (1), reducing the demands on the efficiency of the jet-production process.

Conceptually, I imagine a situation as shown, again, in fig. 1. In the inner thick region of the accretion flow, the accretion still mainly takes the form of a disk, but it is flanked above and below by confining material in recirculation zones that act as reservoirs of mass, energy, and magnetic helicity. I suspect these regions to recirculate because I expect that the magnetic stresses due to viscous coupling to the protostar will tend to generate clockwise meridional vorticity (as drawn) that baroclinicity can not counterbalance, as again discussed in Williams (2001), but ultimately this is just a hypothesis. In the simple treatment given here, I do not consider any coupling of conserved quantities between disk and recirculation zone. They do not exchange mass or angular momentum. The recirculation zones provide a tamper that limits radiative cooling and allows the pressure in the central plane of the disk to rise, and the turbulent viscosity to rise as well.

3. HYDRODYNAMICS PRELIMINARIES

First, I review the energetics of viscosity and viscous dissipation in an ordinary viscous incompressible fluid. Later, I discuss important ways in which turbulence in accretion behaves differently from a simple viscosity.

Let $\sigma$ be the viscous stress tensor, defined here as the Cauchy stress minus that part of the normal stress that is due purely to thermal pressure. For a given scalar or vector $x$, let $D_t(x) = \partial_t x + v \cdot \nabla x$. The momentum equation is

$$\rho D_t v = -\nabla P + \nabla \cdot \sigma$$  \hspace{1cm} (19)

The stress, or equivalently the viscous momentum flux, leads to a force density $f = \nabla \cdot \sigma$, as well as an energy transport due to the viscous energy flux vector $f$ where

$$f = -v \cdot \sigma.$$  \hspace{1cm} (20)

The divergence of (minus) this flux, $-\nabla \cdot f$, represents the local rate $\varepsilon$ at which the energy of the fluid is increased by the action of viscous stresses, so that, e.g., Landau & Lifshitz (1987) eq. 49.2,

$$\partial_t \left( \frac{1}{2} \rho v^2 + \rho e \right) = -\nabla \cdot \left[ \rho v \left( \frac{1}{2} v^2 + w \right) - v \cdot \sigma - \kappa_T \nabla T \right]$$ \hspace{1cm} (21)

i.e., in terms of the advective derivative of the total energy,

$$D_t \left( \frac{1}{2} v^2 + e \right) = -\frac{1}{\rho} \nabla \cdot (v P - v \cdot \sigma - \kappa_T \nabla T),$$ \hspace{1cm} (22)

or in terms of the total (stagnation) enthalpy,

$$D_t \left( \frac{1}{2} v^2 + w \right) = \frac{1}{\rho} \partial_t P + \frac{1}{\rho} \nabla \cdot (v \cdot \sigma + \kappa_T \nabla T),$$ \hspace{1cm} (23)

where $e$ is the specific internal energy, $w$ is the specific enthalpy, and $\kappa_T$ is the thermal conductivity. However, there are two components to this divergence:

$$\varepsilon = -\nabla \cdot f = \nabla \cdot (v \cdot \sigma) = \nabla v : \sigma + v \cdot (\nabla \cdot \sigma) = q + \omega.$$ \hspace{1cm} (24)

The first represents heating, increasing the local entropy, and in keeping with that it is generally positive-definite. (It it sometimes called the dissipation function, although that term has multiple meanings unfortunately.) The general equation of heat transfer (with specific entropy $s$) is then

$$\rho T D_t s = \nabla v : \sigma + \nabla \cdot (\kappa_T \nabla T).$$ \hspace{1cm} (25)

Note that this does not necessarily imply $\text{Tr}(\sigma) = 0$. For a simple incompressible Newtonian viscous fluid this will hold, but it will not hold later when these relations are carried over by analogy to the turbulent stress in accretion.
The second term in eqn. (24) represents work in the form of bulk motion; the fluid here being incompressible, there is no \( PdV \) work in any case:

\[
\rho D_t (v^2/2) = -\mathbf{v} \cdot \nabla P + (\nabla \cdot \sigma).
\]  

(26)

This work can be either positive or negative. It is frequently convenient to re-write the kinetic energy equation as

\[
\rho D_t (v^2/2) = -\mathbf{v} \cdot \nabla P + \nabla \cdot (\mathbf{v} \cdot \sigma) - \nabla \mathbf{v} : \sigma,
\]  

(27)

so that the dissipation \( \nabla \mathbf{v} : \sigma \) appears as a sink in the kinetic energy equation (27) and a source in the heat transfer equation (25).

In the case of accretion, we would not only need to consider compressibility, but more importantly, the energy equations above would need to be modified to include gravity, not to mention radiative losses and so forth. Let us ignore these for the moment however, and focus on turbulent viscosity.

Applied to a standard thin accretion disk with \( v_{(R)} \ll v_{(\phi)} \), where now \( \sigma \) is replaced by the turbulent stress tensor \( W \) (Reynolds stress \( R \) plus turbulent Maxwell stress \( M \)) and \( \mathbf{v} \) is the bulk (mean) flow, the two terms correspond in the following way, in the usual practice. The dissipation function results in local heating as expressed in eqns. (3–5), and is positive. Given eq. (15), the the work due to viscous stress acting on azimuthal rotation (integrated vertically) is

\[
\omega_a = v_{(\phi)} (\nabla \cdot W)_{(\phi)} \quad \omega_a = \int_\infty^{\infty} \nu \Omega a R^2 (\Omega')^2 dz = \frac{\Omega}{R} \nu \Sigma a R^2 (\Omega')^2.
\]  

(28)

and is negative, reflecting the fact that turbulence extracts mechanical energy (kinetic + potential) from the disk material. Combined then, the disk extracts mechanical energy from accreting matter, and converts it to heat and ultimately radiation.

These two results (for heating and for work) share a key hidden assumption that may not actually hold in the inner, thick regions of accretion. Both ignore the relaxation time of the turbulence.

First, the turbulent “dissipation function” is not real dissipation and does not necessarily result in heating, as pointed out above. It is instead a turbulence production (or pumping) rate \( p \), and is the rate of energy going into turbulence, not the rate of energy coming out (via thermal dissipation, either by molecular viscosity or ohmic dissipation):

\[
p = (\nabla \mathbf{v}) : W \quad \nu = \int_{-\infty}^{\infty} p dz = \nu \Sigma a R^2 (\Omega')^2.
\]  

(29)

We now replace eq. (24) with \( \tau = p + \omega \), and there is not necessarily any assurance that \( p = q \). Following a Lagrangian fluid particle, the two quantities will tend to equilibrate on the energetic relaxation time \( \tau \) for the turbulence, but as the pumping \( p \) is steadily increasing as a fluid particle in-spirals, it is not guaranteed that the dissipation (or heating) \( q \) ever catches up.

Second, regarding the mechanical power, the result expressed in eq. (28) ignores the other components of the stress tensor. There are two additional contributions to the mechanical work due to turbulence, due to the normal components \( W_{(RR)} \) and \( W_{(\phi\phi)} \). The azimuthal component or hoop-stress \( W_{(\phi\phi)} \) in particular grows as the stress relaxation time \( s \) grows, and both \( W_{(RR)} \) and \( W_{(\phi\phi)} \) contribute to the \( R \)-component of the divergence of the stress, \( (\nabla W)_{(R)} \). Since \( v_{(R)} \neq 0 \), the mechanical work due to these stresses acting on radial motion, \( \omega_R \), is also not zero:

\[
\omega_R = v_{(R)} (\nabla \cdot W)_{(R)},
\]  

(30)

where

\[
(\nabla \cdot W)_{(R)} = \partial R W_{(RR)} - \frac{W_{(\phi\phi)} - W_{(RR)}}{R}.
\]  

(31)

The first of these last two terms, \( \partial R W_{(RR)} \), is generally positive, reflecting more or less\(^9\) the radially-outward force due to turbulent pressure, including magnetic pressure. The second, \(- (W_{(\phi\phi)} - W_{(RR)})/R \), is negative in the case of MRI-driven MHD turbulence, reflecting the inward-directed force due to the magnetic hoop stresses. The divergence is written in the form above to highlight the significance of the normal stress difference \( W_{(\phi\phi)} - W_{(RR)} \).

\(^9\) A precise statement depends not just on an agreed definition of turbulent pressure, but also on the magnitude of the normal stress component \( W_{(zz)} \).
This normal stress difference and the resultant inwards force, like the production term, is related to the finite non-zero relaxation time of the turbulence, but the connection is perhaps less obvious. The relevant relaxation time scale is not the energetic time scale \( t \) but the stress relaxation (or isotropization) time scale \( s \). This is explored in greater detail in the case of purely hydrodynamic turbulence in the appendix. However, for our purposes, it is sufficient to assume that \( t \) and \( s \) are one and the same, and I will just use the symbol \( s \) for both going forward.

In an ordinary thin disk, \( p \) is positive-definite, and \( \omega \) is negative-definite. As has long been appreciated, the total rate \( \tau \) of energy going into the mean flow due to the turbulent stresses, including heating, can locally be either positive or negative. What has not been appreciated however is, again, that the energy that goes into turbulence might not come out in the form of heat, but through work, in the form of a positive contribution to \( \omega \). To see this, we must discuss stress anisotropy, the energetics of turbulence, and their connection to relaxation in turbulence models. With sincere apologies to the reader, I will now switch to Cartesian index notation to avoid ambiguities that may unfortunately arise with the otherwise cleaner vector notation used above.

4. ENERGETICS IN MHD TURBULENCE WITH ZERO MEAN FIELD

For MHD turbulence, let us perform a Reynolds decomposition, as done for purely hydrodynamic turbulence in Appendix I. To simplify matters, and in keeping with my previous work, let us assume that the mean field is zero. For the sake of avoiding factors of \( 4\pi \), I adopt Heaviside-Lorentz units for the magnetic field. The turbulent Faraday tension is

\[ M_{ij} = B_i^j B_j^i \]  

(32)

which may also be written \( \rho u_i u_j^\prime \) where \( u \) is the Alfvén velocity, highlighting the similarity to the Reynolds stress. The turbulent Maxwell stress is

\[ m_{ij} = M_{ij} - \frac{1}{2} M_{kk} \delta_{ij} = M_{ij} - p_m \delta_{ij} \]  

(33)
where $P_m$ is the magnetic pressure. The full turbulent stress tensor is formed from the Reynolds stress and the turbulent Maxwell stress

$$W_{ij} = R_{ij} + M_{ij}. \quad (34)$$

and the total turbulence energy density is

$$k = \rho k_R + \rho k_M \quad (35)$$

where $\rho k_R = -R_{ij}/2$ and $\rho k_M = -M_{ij} = M_{ij}/2 = P_m$. Here I am borrowing the standard symbol $k$ from studies of hydrodynamic turbulence where it is used to indicate the specific turbulent energy, such as in $k$-$\epsilon$ models, in which $\epsilon$ is the rate of specific turbulent dissipation.

Supposing that the turbulent magnetic stress is much greater than the Reynolds stress, let us now make the simplifying assumption of ignoring the latter. In this section, let us then drop the over-bar for the mean flow $\bar{v}_i$, since there is no longer any need to distinguish between the mean flow and the fluctuations about the mean.

The momentum equation for the mean flow is

$$\partial_t (\rho v_i) + \partial_j (\rho v_i v_j) = -\partial_i P + \partial_j M_{ij} \quad (36)$$

which again is analogous to the momentum equation, eqn. (19).

The turbulent flux of energy $f$ is:

$$f_i = -v_j M_{ij}, \quad (37)$$

and analogous relations follow for the mean kinetic energy, turbulence pumping, etc.

Let us now discuss some physical models for the turbulent Faraday (and Maxwell) stress. First, it is convenient to adopt the notation that

$$\mathcal{D}_t M_{ij} = \partial_t M_{ij} + v_k \partial_k M_{ij} - (\partial_k v_i) M_{kj} - M_{ik} (\partial_k v_j) + 2 (\partial_k v_k) M_{ij} \quad (38)$$

which is just an expression of flux-freezing. The derivative operator $\mathcal{D}_t$ is a modified form of the so-called upper-convected tensor derivative. It reflects the familiar flux-freezing in ideal MHD expressed in a tensor form rather than a vector form; perfect flux-freezing is that

$$\mathcal{D}_t M_{ij} = 0. \quad (39)$$

The simplest model is a Maxwell model for the Faraday stress. A Maxwell viscoelastic model is a model with a single relaxation time, dissipation proportional to the stress, and source proportional to an ordinary Stokes viscous term (note $\mu_t = \rho \tau_t$; the turbulent bulk viscosity $\zeta_t$ is only mentioned here for formal completeness but it is not used elsewhere):

$$s_q \mathcal{D}_t M_{ij} + M_{ij} = \mu_t \left[ \partial_i v_j + \partial_j v_i - \frac{2}{3} (\partial_k v_k) \delta_{ij} \right] + \zeta_t (\partial_k v_k) \delta_{ij}. \quad (40)$$

I will call this the MMF model (for Maxwell Model for Faraday stress). The subscript $q$ for the stress relaxation time scale $s_q$ stands for heating. The reason for this will become clear later. For incompressible flows the model can be written more simply as

$$s_q \mathcal{D}_t M_{ij} + M_{ij} = 2 \mu_t S_{ij} \quad (41)$$

where $S_{ij}$ is the symmetrized velocity gradient, $2S_{ij} = \partial_i v_j + \partial_j v_i$.

Ogilvie (2001) adopts a Maxwell model for the full turbulent stress $W_{ij}$, not the Faraday stress $M_{ij}$. That is a good model for his purposes, but it can lead to problems here if we assume that the turbulence is dominated by magnetic fields so that the Reynolds stress is negligible, $W_{ij} \sim M_{ij}$, because then the magnetic pressure has the wrong sign.

Another simple model is what I have called the $a$-$\delta$ model, which is that

$$s_q \mathcal{D}_t M_{ij} + M_{ij} = a \delta_{ij} \quad (42)$$

The model says this: a statistically isotropic turbulent $B$-field is created at some rate $a/s_q$. It is then passively advected by the mean flow; this is performed by the operator $\mathcal{D}_t$. Finally, it dissipates at the rate $(1/s_q)$. For steady shear, setting the off-diagonal viscous stress equal to $\mu_t \gamma$, then $a = \mu_t / s_q$. This is another simple model that is particularly appropriate if energy is being injected in part from convective turbulence in addition to shear.

The model I will adopt going forward, for concreteness, is the MMF model. Of course, each model leads to different predictions, quantitatively speaking, and the operator $\mathcal{D}_t$ is rather imposing in its full glory to boot, so the reader may
find her or himself skeptical of the developments to ensue. However, qualitatively speaking, the difference between the three models discussed is rather minimal, and again, the operator $D_t$ simply expresses flux-freezing, but in a tensor formalism, rather than the usual vector formalism. The predictions of all three models are dramatically different from a simple viscous model for the turbulent stress, and relative to this, the model-to-model variations are actually surprisingly minimal; furthermore, the viscoelastic models all appear to fit actual shearing-sheet simulations of the MRI far better than a simple viscous model. See for example Table 1 of Williams (2003a). The improved correspondence to simulations of the MRI exists not just in the sense that the models capture the full stress tensor better (and the shear-aligned component of it in particular), but also in the fact that the models, like the simulations, show a relaxation behavior of the turbulence, in that it takes several shear time scales for the turbulence to build, and a corresponding amount of time for it to die as well, again unlike a purely viscous model.

The local loss of turbulent energy due to turbulent dissipation leading to Ohmic (Joule) heating at the magnetic Kolmogorov scale is $\rho \epsilon$. If we neglect any turbulent thermal diffusion, then this is equal to the total local rate of heating $q$. (Since I am not going to discuss cooling, I will henceforth drop the subscript, and the local heating is just $q$.) Then

$$q = \frac{k}{s_q}. \quad (43)$$

To understand energy better, let us re-write the MMF model

$$D_t M_{ij} = (\partial_k v_i) M_{kj} + M_{ik}(\partial_k v_j) - \frac{1}{s_q} M_{ij} + 2 \frac{\mu_t}{s_q} S_{ij} \quad (44)$$

where I assume incompressibility. Taking the trace and dividing by two we have

$$D_t (k) = [(\partial_k v_i) M_{ki} + M_{ik}(\partial_k v_i)] / 2 - \frac{k}{s_q} + 0 = p - q + 0 \quad (45)$$

Suppose study steady uniform linear shear at shear rate $\gamma$: let $v_x = \gamma y$, and $v_y = v_z = 0$. Here, the MMF model predicts (in notation that is hopefully self-explanatory)

$$M_{ij} = \mu_t \gamma \begin{pmatrix} 2s_q\gamma & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad m_{ij} = \mu_t \gamma \begin{pmatrix} s_q\gamma & 1 & 0 \\ 1 & -s_q\gamma & 0 \\ 0 & 0 & -s_q\gamma \end{pmatrix} \quad (46)$$

The energy density is

$$k = -M_{kk} = \frac{1}{2} M_{kk} = P_m = \mu_t \gamma (s_q\gamma) \quad (47)$$

and, starting from eq. (45),

$$0 = D_t k = \mu_t \gamma^2 - \frac{k}{s_q} = p - q \quad (48)$$

Looking at the energetics of the bulk flow, what for a viscous fluid would have been the “heating” term is now the turbulence production term. The frequently-encountered combination $(s_q\gamma)$ is related (up to a factor of 2) to the turbulent Weissenberg number, $W_t$, defined as the ratio of the normal stress difference $W_{xx} - W_{yy}$ to the shear stress $W_{xy}$. Fits are consistent with $W_t \approx 1.5 - 8$ for MRI-driven MHD turbulence.

Production $p$ was a source term in eq. (45) but a loss term in the bulk flow equations, but the bulk flow equations also have $\omega$ as a potential source or sink as well. Here however $\omega = 0$ and $\tau = p$. Just to confirm, looking at the bulk equations we still get (remember $W = M$)

$$p = \nabla v : M = \mu_t \gamma^2 \quad (49)$$

We can also easily verify that

$$\omega = v \cdot (\nabla \cdot M) = 0 \quad (50)$$

and

$$\tau = \nabla \cdot (v \cdot M) = p + \omega = \mu_t \gamma^2. \quad (51)$$
5. MMF MODEL FOR MRI-DRIVEN MHD TURBULENCE IN ACCRETION

The situation changes in accretion, in which $\omega$ is no longer zero.

First however, an important and in fact absolutely key observation to the jet physics I discuss in this paper is that in the case of MHD turbulence in a disk, there is an additional loss channel besides ohmic dissipation resulting in heating, and that is the vertical loss out of the upper and lower disk surfaces. This is due to the Parker instability, i.e. buoyancy, and Alfvén waves, as pointed out earlier. (An analogous situation presents in pure hydrodynamic turbulence as well, in which some power may be lost vertically due to sound waves, but this is far less important effect by comparison to buoyancy and Alfvén waves.) This physics allows a sort of mechanical leverage, if you will, by which a sizeable fraction of the accretion energy can be given to a small fraction of the accreting mass.

Applied to a stream tube in 2D axisymmetry (i.e. a volumetric bundle of streamlines bounded above and below by an axisymmetric stream sheet), the losses (or gains) due to buoyant transport of turbulent magnetic stress into or out of the volume can be found by performing a surface integral of the turbulent energy flux vector $\nu_i$, and taking care to include compressibility effects in it via Favre averaging instead of Reynolds averaging, etc, to capture effects of rising and falling current-carrying blobs of fluid in a density- and pressure-stratified medium. This is best left to another paper. It is impractical here. Instead, perhaps motivated by a bit of intellectual laziness$^{10}$ as well, I adopt an ansatz that the vertical buoyant loss appears as an additional loss channel proportional to $M_\ell$ upon vertical integration, complementing the ohmic loss channel. Then the total loss $\nu$ (verlust) per area $R \, d\phi \, dR$ of a volume element of finite vertical extent bounded below and above by lower and upper stream surfaces $\ell$ and $u$ is

$$\nu = D_\nu^{(\text{loss})} \left( \int_{\ell}^{u} \rho_k M \, dz \right) = \int_{\ell}^{u} q \, dz - B_{\ell} + B_u = \Omega - B_{\ell} + B_u. \quad (52)$$

This loss $\nu$ is loss of turbulent energy, some of which is lost to local heating, $\Omega$, and some of which truly is lost from the volume element. One can adopt the convenient fiction of thinking of this latter as an effective local volumetric loss rate $\bar{\epsilon}$, with the caveat that with greater care it ought to be attributed to unresolved components of the (vertical) turbulent energy flux.

Assume the buoyant rise-time of magnetic flux tubes to be of order the inverse Brunt-Väisälä frequency$^{11}$, which is also of order the frequency $\Omega$, and assume the loss due to Alfvén waves to be roughly comparable. That is, in general on dimensional grounds one expects the time scales of both the loss $\bar{\epsilon}$ and $q$ to be of order $\left(\text{few}\right) \times \Omega^{-1}$, in the context of accretion. The loss term is

$$\frac{1}{s} \rho_k M = \bar{\epsilon} = q + \bar{\epsilon} = \frac{1}{s_q} \rho_k M + \frac{1}{s_\epsilon} \rho_k M = a_q \rho_k M + \frac{1 - a_q}{s} \rho_k M. \quad (53)$$

Here $a_q$ is the heating efficiency parameter. It reflects that fraction of the loss term that goes into actual heating. The remaining fraction $(1 - a_q)$ of the energy is lost to the upper (or lower) boundary of the flow, and may do work on the fluid there.

Let us now adopt cylindrical coordinates $(R, \phi, z)$, and let indices run from $R$ to $z$ in that order. Primes no longer indicate fluctuations, but go back here and henceforth to indicate differentiation with respect to $R$. Here, it is sometimes useful to go back and forth between Cartesian index notation and covariant/contravariant tensor notation. For steady azimuthal shear, the MMF model results in a Faraday stress

$$M_{ij} = \mu_t \begin{pmatrix} 0 & R\Omega' & 0 \\ R\Omega' & 2s (R\Omega')^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M^{ij} = \mu_t \begin{pmatrix} 0 & \Omega' & 0 \\ \Omega' & 2s (\Omega')^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (54)$$

from which the full Maxwell stress is $M^{ij} = M^{ij} - P_{\text{mag}} g^{\alpha \beta}$, and $P_{\text{mag}} = \mu_t s (R\Omega')^2$, so

$$M_{ij} = \mu_t \begin{pmatrix} -s (R\Omega')^2 & R\Omega' & 0 \\ R\Omega' & s (R\Omega')^2 & 0 \\ 0 & 0 & -s (R\Omega')^2 \end{pmatrix}, \quad M^{ij} = \mu_t \begin{pmatrix} -s (R\Omega')^2 & \Omega' & 0 \\ \Omega' & s (\Omega')^2 & 0 \\ 0 & 0 & -s (R\Omega')^2 \end{pmatrix}. \quad (55)$$

$^{10}$ I say this because it is actually an important matter to explore this flux a bit more carefully as it will have an associated torque as well (plus helicity fluxes etc), which is important to understand for overall angular momentum balance.

$^{11}$ Of course it is never so simple. Drag forces on small-diameter flux tubes can substantially reduce this rate. See Khaibrakhmanov & Dudorov (2017).
The thermal pressure is $P_{\text{th}} = \beta P_{\text{mag}}$ for a total radial pressure force of $-(\beta + 1)\nabla P_{\text{mag}}$ for fixed $\beta$.

The turbulent magnetic energy density is

$$\rho k_M = \mu_t s (R\Omega')^2$$

(56)

and

$$0 \simeq D_t(\rho k_M) = p - v$$

(57)

where

$$p = v = \mu_t (R\Omega')^2.$$  

(58)

resulting in a heating rate $q$ and a buoyant loss rate $\delta$ of

$$q = a_q \mu_t (R\Omega')^2 \quad \delta = (1 - a_q)\mu_t (R\Omega')^2.$$  

(59)

The force density due to the turbulence is $\nabla \cdot \mathbf{M} = \nabla \cdot \mathbf{M} - \nabla P_{\text{mag}}$. The turbulent radial force density $f_R$ has a pressure term, and a hoop-stress term

$$f_R = (\nabla \cdot \mathbf{M})^R = M^{\alpha R}_{;\alpha} = M^{RR}_{;R} + \frac{M^{RR} - R^2 M^{\phi\phi}}{R}.$$  

(60)

There is, in addition, a torque density

$$R(\nabla \cdot \mathbf{M})^\phi = \frac{1}{R^2} \left( R^3 M^{R\phi} \right)'.$$  

(61)

The overall rate of turbulent energy deposition is

$$\tau = (g_{\gamma\mu} M^{\alpha\gamma} v^{\mu})_{;\alpha} = \frac{1}{R} \left( \mu_t R^3 \Omega' \right)'$$

(62)

which is the sum of turbulence production

$$p = g_{\mu\nu} v^{\mu\nu} M^{\alpha\gamma} = \mu_t (R\Omega')^2$$

(63)

as above in eq. (58), plus the work on the bulk fluid due to turbulence,

$$\omega = g_{\mu\gamma} v^{\mu} M^{\alpha\gamma}_{;\alpha} = \frac{\Omega}{R} \left( R^3 M^{R\phi} \right)'$$

(64)

related to the torque, eq. (61).

For the most part these are the standard results; the exceptions are: 1) The local heating is less than the nominal value by the factor $0 < a_q < 1$, the remainder being buoyantly lost; 2) There is a radially-outward force due to the magnetic-pressure (and thermal pressure) gradient; such forces can be ignored in thin disks but not in geometrically thick accretion; 3) There is a radially-inwards hoop stress due to stress relaxation in the turbulence.

In steady energy balance, there must be, of course, an inwards radial drift that we have neglected, so $v^R < 0$. For a purely viscous model for the turbulence, the radial force (60) would be zero, but this is no longer the case here. There is an additional contribution to $\omega$ then, as discussed before, and the expression above for $\omega$ is just the fraction $\omega_n$ of the total work.

The additional contribution $w_p$ to the work performed on the in-spiralling gas may be approached by breaking the radial force (60) into a term due to the Faraday tension (the hoop term, $\omega_h$ and a term due to the magnetic pressure (the pressure term, $w_p$):

$$f_R = (\nabla \cdot \mathbf{M})^R - \nabla \cdot P_m$$

$$= - \frac{1}{R^2} \left( \mu_t s (R\Omega')^2 \right)'$$

(65)

In general then, the actual total work $\omega$ will be increased (that is, it will still be negative but of less magnitude) due to this additional contribution. This will be studied further below, using the formal solution in Appendix III to a prescribed velocity field in which $|v^R| \propto R^{-1}$. 
Conservation equations are referred to the above control volume diagram, showing a section of the 3D flow in a plane containing the axis of symmetry (vertical). Given a conserved quantity, the total flux from the disk can be broken into a flux $F_1$ from disk $D$ to protostar $P$, and a flux $2F_2$ from disk to jets $J_u$ and $J_ℓ$, assuming $F_{2u} = F_{2ℓ} = F_2$. The fluxes in the jets as determined near or past the critical surfaces are $F_{3u}$ and $F_{3ℓ}$. The total momentum flux (thrust) $F_j$ for a jet is necessarily determined further downstream, after the bulk of acceleration in the nozzle. There is a flow separatrix in the disk (dashed lines in $D$) separating the “working fluid” (material that ultimately accretes onto the protostar) stream tube $D_w$ from the upper and lower “bypass” fluid stream tubes $D_{b,u}$ and $D_{b,ℓ}$. This is viewed as a two-stream compound-compressible channel flow. I ignore fluxes between star and jet, and between upper or lower recirculation zones $R_u$ and $R_ℓ$ and anything else.

6. 1D STEADY DISKS

Let us now consider a 1D steady disk and the angular momentum flux through it. I construct control volumes as shown in figure 3; note that I assume that any jets exist above and below the star (not the disk), and have cylindrical radius $R ≃ R_*$. This vastly simplifies the development below. Additionally, I do not consider any coupling directly between the star and jets, or between the recirculation zones and anything else. The only couplings considered are star-disk, and jet-disk. We could have just as well considered only star-disk and star-jet coupling instead; at this level, the difference is really a matter of accounting, not physics.

As a matter of notation, given any conserved quantity $X$, within the disk there is a radial flux $F_X$ which is a function of $R$, and the total flux into the disk is

$$F_X^{(d)}|_{R=R_*} = F_X^{(1)} + F_X^{(2u)} + F_X^{(2ℓ)} = F_X^{(s)} + 2F_X^{(j)} = -\dot{X}_s - 2\dot{X}_j = -\dot{X}_d$$

(68)

where $j$ and $d$ indicate jet and disk, and for brevity, $F_X^{(d)}$ may just be written as $F_X$. (The outer boundary of the disk is moved to infinity.) The reason for this notation is to serve as a reminder that in general, in the disk, a flux $F_X$ may be a function of radial coordinate $R$, representing the total integrated flux through a cylindrical surface, whereas $\dot{X}$ refers specifically to the flux evaluated at $R_*$ between star and disk or jets and disk.

The fluid equations may be integrated under the simplifying assumptions already addressed above. Conservation of mass yields

$$F_M = 2\pi R \Sigma_d v_\rit < 0, \quad \dot{M}_d = -F_M|_{R=R_*}, \quad \frac{d}{dR} F_M = 0$$

(69)

and

$$\dot{M}_s = \dot{M}_d - 2\dot{M}_j = (1 - \varepsilon) \dot{M}_d$$

(70)
where I note again that I assume no vertical mass gain or loss, all the way down to the radius \( R_* \). Note that the surface density \( \Sigma_d \) is only that contribution to the vertically-integrated density due to the disk material, and does not include the recirculation zones. Inside the thick region, the actual surface density \( \Sigma = 2\Sigma_r + \Sigma_d \) may be much larger than \( \Sigma_d \) alone.

Empirically, it is thought a typical value of the jets’ mass fraction\(^{12}\) \( \varepsilon \) is in the range 2 – 20\%, leaning towards the higher number, although the number can not be measured directly and so that inference is itself model-dependent. Arguably, \( \varepsilon \) may be nearly impossible to determine in Class 0 and Class I objects (Edwards 2008); veiling in particular confounds the analysis.

Integration of the conservation of angular momentum (\( L \)) equation gives:

\[
F_L = -C_L \dot{M}_d \sqrt{GM_* R_*}, \quad \dot{L}_d = -F_L, \tag{71}
\]

(that is, the flux of angular momentum is independent of \( R \) and normalized to a fiducial constant). For future reference, it is convenient to define a notional angular momentum flux

\[
\dot{L}_0 \equiv \dot{M}_d \sqrt{GM_* R_*}. \tag{72}
\]

Then \( F_L = -C_L \dot{L}_0 \).

The total budget for the star is:

\[
\dot{L}_* = \dot{L}_d - 2 \dot{L}_j \tag{73}
\]

where

\[
-\dot{L}_d = F_L = F_L^{(\text{adv})} + F_L^{(\text{turb})}. \tag{74}
\]

Let us assume that the specific angular momentum of the jets is that of the stellar equator, such that

\[
2 \dot{L}_j = 2 \dot{M}_j \cdot R_*^2 \Omega_* = 2 \omega \dot{M}_j R_*^2 \kappa_* = \varepsilon \omega \dot{M}_d R_*^2 \kappa_. \tag{75}
\]

Regarding \( F_L \), the advective part is

\[
F_L^{(\text{adv})} = 2\pi R^3 \nu \Omega = -R^2 \Omega \dot{M}_d, \tag{76}
\]

whereas the turbulent part is

\[
F_L^{(\text{turb})} = -2\pi R^3 \Omega' \nu \Sigma_d. \tag{77}
\]

Nominally, \( F_L^{(\text{adv})} < 0 \) and \( F_L^{(\text{turb})} > 0 \).

The usual practice is to set \( F_L^{(\text{turb})} \) to zero at \( R_* \); that is, zero turbulent (i.e. “viscous”) torque on the star. For accretion to proceed without adding excess angular momentum to the star, it has been asserted elsewhere in the literature, jets provide a sink of angular momentum, effectively allowing \( \dot{L}_* \approx 0 \) in eq. (73) without any turbulent torque on the star by the disk. This amounts to fixing an integration constant for the angular momentum equation, and in the absence of a true boundary layer, it is unwarranted. In fact, the disk is a perfectly fine sink of angular momentum as shown below.

6.1. *Keplerian Inner Envelope*

Let us start first by examining a purely Keplerian flow before proceeding to the more realistic case of an under-rotating disk (or envelope or torus or flow) surrounding a more slowly rotating star. It is fanciful but illustrative for this section only to imagine the central star as a solid body rotating precisely at Keplerian angular velocity at \( R = R_* \), i.e. \( \omega = 1 \), with a purely Keplerian disk extending all the way down to the star’s surface.

In that case, eq. (77) reduces to

\[
F_L^{(\text{turb})} = 3\pi R_*^2 \kappa \nu \Sigma_d \tag{78}
\]

which may be combined with eq. (76) to yield the familiar

\[
\nu \Sigma_d = \frac{\dot{M}_d}{3\pi} \left[ 1 - C_L \sqrt{\frac{R_*}{R}} \right] \tag{79}
\]

\(^{12}\) This is the mass accretion-ejection efficiency, also written as \( \varepsilon_M \) when it is required to distinguish it from the other accretion-ejection efficiencies. See section 10.
Figure 4. The constant $C_L$ shifts $F_L^{(\text{turb})}$ up or down. The usual construction is to set $F_L^{(\text{turb})}$ to zero at $R = R_\ast + b$, so that $C_L = \sqrt{1 + b/R_\ast} \approx 1$, provided $b \ll R_\ast$. Substantially decreasing $C_L$ will shift $F_L^{(\text{turb})}$ up and the total angular momentum flux up as well; for $C_L < 0$, the total flux becomes positive, i.e. there is a net outward flux of angular momentum in the disk, removing angular momentum from the star.

and the total angular momentum of the star changes at the rate (recalling $\omega = 1$ here)

$$\dot{L}_* = (C_L - \varepsilon) \dot{M}_d R_\ast^2 \kappa_* = (C_L - \varepsilon) \dot{L}_0.$$  \hfill (80)

Again, the total disk angular momentum flux is normalized to a fiducial value,

$$C_L = \frac{\dot{L}_\text{disk}}{MR_\ast^2 \kappa_*}.$$  \hfill (81)

The conventional value is $C_L = 1$, corresponding to $F_L^{(\text{turb})} = 0$ at $R_p \approx R_\ast$, and $\nu_\Sigma \to 0$ as $R$ approaches $R_\ast$.

Since in the present case however $R_p$ does not exist, we are not required to set $C_L = 1$, even for a strict viscous prescription for the turbulent stress; $C_L$ could even be negative. All we require for $\nu_\Sigma > 0$ is $C_L < 1$. For example, for $C_L < 0$ we have $F_L^{(\text{turb})} > -F_L^{(\text{adv})}$, and the disk “viscously” transports away at least as much angular momentum as it advects inwards. The central star in this case gains mass but loses angular momentum, even without an outflow ($\varepsilon = 0$).

Strictly speaking, we should also discuss the outer boundary condition, but the assumption here is that if there is an overall inward or outward flux of total angular momentum in the disk, then the outer disk boundary, which is otherwise effectively infinitely far away, will slowly move in or out to compensate.

For future reference, it is convenient to define a notional luminosity $\mathcal{L}_0$ where

$$\mathcal{L}_0 \equiv \frac{GM_\ast \dot{M}_d}{R_\ast} = \frac{\dot{L}_0}{R_\ast^2 \dot{M}_d},$$  \hfill (82)

and a notional luminosity per disk annulus $\mathcal{L}_0'$ where

$$\mathcal{L}_0' = \frac{GM_\ast \dot{M}_d}{R^2} \quad \mathcal{L}_0' = \int_{R_\ast}^{\infty} \mathcal{L}_0' \, dR.$$  \hfill (83)

The local flux of bulk mechanical (PE + KE) energy carried by material\textsuperscript{13} in the disk is, with cylindrical polar coordinates ($R, \phi, z$),

$$F_{E}^{(\text{mech})} = \frac{GM_\ast \dot{M}_d}{R} - \frac{1}{2} \dot{M}_d v_{(\phi)}^2 = \frac{1}{2} \frac{GM_\ast \dot{M}_d}{R} = \frac{1}{2} R^2 \kappa^2 \dot{M}_d,$$  \hfill (84)

\textsuperscript{13} Radial pressure gradients are negligible here.
giving the total flux of mechanical energy to the star, taking account of the flux to the jets, and not counting work due to turbulent “viscous” forces (torques), of
\[ F_{(\text{mech})} = -L_0(1 - \varepsilon) < 0 \] (85)
The (negative) divergence of this radial flux,
\[ -d_RF^{(\text{mech})} = \frac{1}{2} F_{(\text{mech})} \] (86)
is equal and opposite to the work of turbulent torques (below), so that the total rate of change of mechanical energy is zero. This does not include the rate of change in specific disk energy \(-\pi GM_d\Sigma_d\) at a given Eulerian point \(R\) due to the rate \(\dot{M}_d\) of increased mass internal to \(R\), which is equivalent to assuming that the total mass of the disk interior to \(R\) is small compared to \(M_\ast\). This may be a bit borderline for some Class 0 sources but is probably acceptable for Class I sources.

Turbulence energetics is a bit more complicated.

Let us begin by considering a traditional purely viscous model for the turbulent stress, and let us assume \(v_{(R)} \ll v_{(\phi)}\) everywhere.

As before, the turbulent flux of energy can be found by examining the turbulent energy flux vector\(^\text{14}\) \(\mathbf{f}\); in index notation, this is \(-\sigma_{\alpha\beta}\). In physical components in 2D \((R - \phi)\) it is\(^\text{15}\) \(-v_{(\phi)}\sigma_{(R\phi)}\). Integrated vertically and over \(2\pi\) in azimuth, it is
\[
\int f_{(R)} 2\pi R d\phi = F_{E}^{(\text{turb})} = -2\pi v_{\phi}\Sigma_d R^3 \Omega' \Omega' = L_0 \left[ 1 - C_L \sqrt{\frac{R_*}{R}} \right] \] (87)
for a total turbulent flux of energy to the star and jet from the disk of
\[ E_{d}^{(\text{turb})} = -L_0(1 - C_L) = -\Omega F_{E}^{(\text{turb})} \bigg|_{R=R_*} \leq 0 \] (88)
reflecting that for the conventional value \(C_L = 1\), there is no energy drawn from the star (and jet) from turbulent torques, i.e., no “viscous” torque, whereas for \(C_L < 1\) there is a net turbulence-mediated flow of power from the star (and jet) to the disk. Note that this is of order the energy expressed in eq. (2) as claimed.

\(^\text{14}\) Henceforth, except for the turbulent radial force density \(f_{R}^{(R)}\) as well as for the effective “viscosity” \(v_t\) itself, I will drop the cumbersome subscript or superscript \(t\) for turbulent, it being understood that we are no longer discussing an ideal viscous fluid.

\(^\text{15}\) This is the radial physical component \(f_{(R)}\) of \(\mathbf{f}\), again assuming \(v_{(R)} \ll v_{(\phi)}\); otherwise there is an additional term, \(-v_{(R)}\sigma_{(R\phi)}\), where \(\sigma_{(R\phi)} = 2\nu\Sigma_d R v_{(R)}\).
The quantity $\dot{E}^{(\text{turb})}_d$ is of some importance. The notation for it follows the convention adopted above for other fluxes. However, it is also useful to look at turbulence quantities integrated over the entire disk, which I denote with chalkboard bold. The first such quantity is $\mathbb{E}$, which is the total rate of turbulent energy deposition into the disk. Again, the only surfaces through which we allow a flux into or out of control volume $D$ are as shown in fig. (3). Then $\mathbb{E} = -\dot{E}^{(\text{turb})}_d$. Furthermore, the total rate of turbulent energy deposition is equal to the sum of the total integral over the disk of turbulent production, and the total integral over the disk of the work performed by turbulence:

$$\mathbb{E} = \mathbb{P} + \mathbb{W}. \quad (89)$$

The negative divergence of the turbulent flux of energy is the quantity of energy $2\pi R \mathbb{R}$ being deposited (per annulus $dR$) by turbulent stresses at a radial location,

$$2\pi R \mathbb{R} = 2\pi R \int z \, dz = -\frac{d}{dR} F^{(\text{turb})}_E = \mathcal{L}^0 \left[ 1 - \frac{3}{2} C_L \sqrt{\frac{R_*}{R}} \right]. \quad (90)$$

Again, in the conventional case $C_L = 1$, this quantity is negative for $R/R_* < 9/4$ and positive for $R/R_* > 9/4$, reflecting the progressive turbulent re-distribution of the wealth of power from inner regions to outer regions of the disk, as is well-understood. Clearly, for $C_L \leq 0$, it is nowhere negative, and the disk is no longer borrowing energy from the inner regions to power the viscous dissipation in the outer regions.

The local turbulence production is (cf. eqn. 29)

$$2\pi R \mathbb{P} = 2\pi \nu_1 \Sigma R^3 (\Omega')^2 = \frac{3}{2} \frac{GM_* \dot{M}}{R^2} \left[ 1 - C_L \sqrt{\frac{R_*}{R}} \right] = \frac{3}{2} \mathcal{L}^0 \left[ 1 - C_L \sqrt{\frac{R_*}{R}} \right], \quad (91)$$

for a total power going into turbulence production in the disk of

$$\mathbb{P} = \int_{R_*}^\infty 2\pi R \mathbb{P} \, dR = \frac{3}{2} \frac{GM_* \dot{M}}{R_*} \left( 1 - \frac{2}{3} C_L \right) = \frac{3}{2} \mathcal{L}_0 \left( 1 - \frac{2}{3} C_L \right) \quad (92)$$

The quantity $C_L$ is not restricted by the positive-definite constraint on $\mathbb{P}$ (or $\mathbb{E}$) any further than it already has been by mass conservation.

The local turbulent mechanical work due to the off-diagonal “viscous” stress — that is, ignoring the normal stress — is (cf. eqn. 28)

$$2\pi RW = 2\pi \Omega \left( \nu_1 \Sigma R^3 \Omega' \right)' = -\frac{1}{2} \frac{GM_* \dot{M}}{R_*} = -\frac{1}{2} \mathcal{L}_0', \quad (93)$$

and is independent of $C_L$; it is negative reflecting that these forces reduce, rather than increase, the energy of orbiting fluid particles. The total mechanical work of turbulence acting on differential rotation is then

$$\mathbb{W} = \int_{R_*}^\infty 2\pi RW \, dR = -\frac{1}{2} \frac{GM_* \dot{M}}{R_*} = -\frac{1}{2} \mathcal{L}_0 \quad (94)$$

This acts to reduce the total specific mechanical energy $\mathbb{W} = \mathbb{P} + \mathbb{R}$ of fluid particles from zero (at infinity) to $-(1/2)GM/R_*$ at the star’s surface.

It can be seen that the total turbulence production in the disk is equal to minus this quantity, plus the additional turbulence-mediated flux of energy from the star, $\mathbb{P} = -\mathbb{W} + \mathbb{R}$, this latter quantity being 0 when $C_L = 1$:

$$\frac{3}{2} \mathcal{L}_0 \left( 1 - \frac{2}{3} C_L \right) = \frac{1}{2} \mathcal{L}_0 + \mathcal{L}_0(1 - C_L). \quad (95)$$

The total power going into the production of turbulent energy in the disk can be substantially larger than the nominal $(1/2)GM/R_*$. For $C_L = 0$ it is larger by a factor of 3, not coincidentally reminiscent of the famous factor of 3 discrepancy in thin-disk theory between the energy viscously dissipated in a Lagrangian annulus of the disk and the $(\mathbb{W} + \mathbb{P})$ energy lost by the same annulus as it moves radially inwards. In fact, almost all of this increased turbulent pumping occurs in the innermost regions of the disk so that locally, the power delivered into turbulence in these inner
regions is even larger still than 3 times the standard value, as can be seen by setting $C_L = 0$ in eq. (91) and as shown in figure (5) below.

Meanwhile the total losses $\mathcal{V}$, again as integrated over the total volume $D$ in fig. (3), are equal to production $P$ minus turbulence energy advected away into the star (and jets),

$$\mathcal{V} = P - \int \rho k_M \mathbf{v} \cdot dA,$$

where this latter quantity is formally zero if $v_R = 0$. Also,

$$\mathcal{V} = a_q \mathcal{V} + (1 - a_q) \mathcal{V}, \quad a_q \mathcal{V} = \mathcal{Q}, \quad (1 - a_q) \mathcal{V} = \mathcal{B}. \tag{97}$$

The total heating of the integrated volume of the disk (and thus its radiative output) is $\mathcal{Q} < \mathcal{V}$, and if we did not ignore the advection of turbulence energy into the star (and disk), we would have $\mathcal{V} < P$. Of course, turbulence energy advected into the star will eventually be radiated away, but energy lost buoyantly is not (in our accounting), so in general $\mathcal{Q} \lesssim a_q P$. All we can say is that $a_q$ is some fraction of unity, but in any case the total radiated power of the disk is at this point less than the conventional value, although admittedly just by the ad hoc assumption of buoyant loss.

This is the situation, in any case, regarding $\mathcal{R}$, $\mathcal{P}$ and $\mathcal{W}$, if we assume $v_R = 0$, but of course that is internally inconsistent, as then $\dot{M}_d = 0$. Now if we consider small but nonzero $v_R < 0$, then again as mentioned before there is an additional contribution to $\mathcal{W}$ which includes the work of hoop stresses and radial magnetic pressure gradients.

It is convenient now, as before, to break stresses and forces down into that part due to Faraday, $F$, and that part due to the magnetic pressure, $P$. Then the turbulent radial force density $f^R$ has two components, $f^R = f_P^R + f_P^R$, where, assuming the results from the MMF model above,

$$f_P^R = \nabla \cdot M = M^{RR} R + \frac{M^{R\phi} - R^2 M^{\phi \phi}}{R} \simeq -RM^{\phi \phi} = -M_{(\phi \phi)} R \tag{98}$$

and

$$f_P^R = -R P_{mag} = \frac{1}{2} (g_{\alpha \beta} M^{\alpha \beta})_R \simeq -\frac{1}{2} (R^2 M^{\phi \phi})_R = \frac{1}{2} D_R M_{(\phi \phi)} \tag{99}.$$  

Then, using the model results,

$$f_P^R \simeq -\frac{1}{R} (2 \mu_s (R \Omega)^2) \tag{100}$$

and

$$f_P^R \simeq -\left(\mu_s (R \Omega)^2\right)^{\prime} \tag{101}.$$  

The work done by the Faraday hoop stresses in particular is then

$$2 \pi R \dot{W}_{RF} = \int \omega_F \ dz \frac{9}{2} \frac{s v_{\perp}}{R^2} \cdot \mathcal{L}^{\prime}. \tag{102}$$

For a thick disk (and correspondingly high $\nu_t$), and if $s \Omega$ is of order unity or larger, this is not a small quantity. Alone, it is not enough to change the sign of the Bernoulli parameter. However consider if the integral is performed over the region $D_{\nu_t}$. There is now a vertical flux into $D_{b,u}$ and $D_{b,\ell}$ of toroidal field, as well, with a total power of nearly $(1 - a_q)P$. This will also created hoop stresses, which will do comparable work on the fluid in $D_{b,u}$ and $D_{b,\ell}$, but this work is being performed on a fraction $\varepsilon$ of the total accretion mass $M_d$. This is more than sufficient, it is claimed, to ultimately change the sign of the Bernoulli parameter of that gas and make it unbound.

6.2. Non-Keplerian Inner Envelope

There is nothing particularly special about the Keplerian disk in this regard. Of course, a sub- or super-Keplerian disk must have some kind of radial pressure support and so must not be thin, but we are not interested in solving the full problem of such disks here. Effective 1D disk equations for intermediate “slim disks” (Abramowicz et al) may be found by vertical integration which introduces various constants of order unity, but precise values for these constants in turn rely upon further assumptions regarding the full 2D $(R,z)$ disk structure in the case of a truly thick disk, which can not truly be pinned down without a full 2D solution of course. We can still learn a fair bit by making the gross 1D
Figure 5. Power extracted from star and accretion flow and injected into MHD turbulence, $2\pi R \rho$, as a function of $R$ for a steady Keplerian disk accreting onto a star notionally rotating at breakup, for different values of the turbulent torque on the star. Relaxation of the constraint $C_L = 1$ allows the star to dump a substantially larger quantity of energy into turbulence in the inner regions of the disk than otherwise. This energy is largely in the form of a toroidal magnetic field, which is able to deliver energy to an outflow far more efficiently than heat energy would. Magnetic buoyancy allows much of this energy to couple to and accelerate a fraction of the mass, changing the sign of its Bernoulli parameter.

Let us therefore now imagine a more realistically fast-rotating protostar, $\omega = 1/3$, with a thick inner disk or envelope\(^{16}\) with a much softer power-law dependence of $\Omega$ on $R$ than Keplerian, say $q = 1/2$ or 1, leading to $R_m = 3R_*$ or $9R_*$ respectively for the radius of this thick inner disk, and “sew” this envelope onto a Keplerian thin disk for $R > R_m$. This somewhat artificial construction will necessarily result in some complex relations and integrals for the various quantities of interest, which should of course be taken with a grain of salt given the crudeness of the approximations used here, but counterbalancing this is, I hope, the merit of the overall gross conclusions and the qualitative features of the relative magnitude of the various physical quantities.

So long as $q \neq 0$ in the envelope\(^{17}\) we may still define $C_L$ as before, by reference to the same fiducial value for the angular momentum flux as in eq. (81), and

$$\nu \Sigma = \frac{\dot{M}}{2\pi q} \left[ 1 - C_L \frac{R_*^2 \kappa_*}{R^2 \Omega} \right]$$

for this inner envelope, still keeping eq. (79) with the same $C_L$ for the outer thin disk, and eq. (77) still holds. Note however that $C_L$ is no longer the ratio of the total angular momentum flux to the advective flux at the stellar surface; that ratio is now $\omega C_L$, and the total angular momentum of the star changes at the rate (recalling $\omega = 1$ here)

$$\dot{L}_* = (C_L - \omega \varepsilon) \dot{M}_d R_*^2 \kappa_* = (C_L - \omega \varepsilon) \dot{L}_0.$$  \hspace{1cm} (104)

The turbulent energy flux vector becomes

$$\dot{E}^{(visc)} = \omega^2 G M_\star \frac{\dot{M}}{R} \left[ 1 - \frac{C_L R_*}{\omega} \left( \frac{R_*}{R} \right)^{2-q} \right]$$

\hspace{1cm} (105)

\(^{16}\) In previous work I called this inner thick accretion flow a “disk,” but this is a somewhat loaded word with potential connotations (esp. thinness) that I did not intend; here, I will often call this inner accretion region the “envelope.” That is also arguably a loaded word, but the nominal context in which one otherwise discusses an “envelope,” regarding protostars, is at $\gg$ AU scales, which, it should be abundantly clear, is not what is intended here.

\(^{17}\) Actually, even in the case of solid-body rotation $q = 0$, if not turbulent viscosity then velocity structure such as meridional circulations can lead to an effective radial transport of angular momentum so that again we do not require $C_L = 1$, but in that case the transport due to such a velocity field or even turbulence can no longer be represented through a turbulent viscosity model.
Figure 6. Work of turbulence acting on radial velocity \( (w_r) \), broken down into work of Faraday stresses \( (w_{rf}) \) here) and work of magnetic pressure forces \( (w_{rp}) \) in a slow-inspiral with nearly viscous turbulence. This is for \( \vartheta = 0.1 \) and \( \hat{s} = 0.15 \).

in the envelope but retains its value \( (87) \) in the outer thin disk, for a total flux of turbulent energy \textit{from} the star of

\[
\dot{E}^{(\text{visc})}_s = \frac{GM_s\dot{M}}{R_s}(\omega^2 - C_L\omega)
\]  

which reduces to eq. (106) for \( \omega = 1 \).

The local rate of turbulent energy deposition, which includes both turbulence production and work by turbulence, is

\[
2\pi R \dot{R} = -\frac{d}{dR} F_{E}^{(\text{turb})} = \dot{M} R \Omega^2 \left[ (2q - 2) - q \frac{C_L}{\omega} \left( \frac{R_s}{R} \right)^{2-q} \right] = \omega^2 \frac{GM_s\dot{M}}{R^2} \left( \frac{R}{R_s} \right)^{3-2q} \left[ (2q - 2) - q \frac{C_L}{\omega} \left( \frac{R_s}{R} \right)^{2-q} \right]
\]  

(107)

Of this, the viscous production alone is

\[
2\pi R, P = q\dot{M} R \Omega^2 \left[ 1 - \frac{C_L}{\omega} \left( \frac{R_s}{R} \right)^{2-q} \right] = q\omega^2 \frac{GM_s\dot{M}}{R^2} \left( \frac{R}{R_s} \right)^{3-2q} \left[ 1 - \frac{C_L}{\omega} \left( \frac{R_s}{R} \right)^{2-q} \right]
\]  

(108)

for \( R < R_m \); for \( R \geq R_m \), eq. (91) still holds.

Meanwhile, the turbulent work in the envelope is

\[
2\pi R, W_a = -(2 - q)\dot{M} R \Omega^2
\]  

(109)

for \( R < R_m \), and \(-(1/2)\dot{M} R \kappa^2 \) in the thin disk as before.

Again, these are the results for a slow in-spiral in which \( v_{(R)} \ll R \Omega \), and for a purely viscous turbulence model (which is essentially equivalent to \( s\Omega \ll 1 \) in the MMF model). Figure (6) illustrates the small correction to this due to the work of Faraday tension on the radially-inwards motion, for the case \( \vartheta = 0.1 \) and \( s\Omega = 0.15 \) from Appendix III. To satisfy overall energy conservation, \( \mathcal{P} \) (and \( \mathcal{R} \)) also shifts by a small amount to compensate.

The relative fraction of energy going into \( W_{RF} \) increases as \( \vartheta \) and \( \hat{s} \) increase. Consider now a much more elastic type of turbulence, let us say \( s\Omega = 10.0 \), ( but still with slow in-spiral, \( \vartheta = 0.1 \); see fig. (7) and fig. (8). Here \( W_{RF} \) counter-balances much of the work against azimuthal motion, \( W'_a \), reflecting that much of the energy robbed from the gas to allow it to accrete goes back into countering radial pressure gradients.

Again, there are substantial magnetic tension forces capable of performing work, and these magnetic fields can buoyantly rise from region \( D_w \) to regions \( D_{b,u} \) and \( D_{b,\ell} \) as before, and, I argue, change the sign of the Bernoulli parameter in those streamtubes.
Figure 7. Work of turbulence with fairly elastic turbulence (slow relaxation time compared to orbital time) showing the contributions to $W$ due to work against azimuthal motion, $W_a$, and work against radial motion, $W_R$ (here, $w_r$). This is for $\vartheta = 0.1$ and $\dot{s} = 10$.

Figure 8. Work of turbulence acting on radial velocity ($w_r$), broken down into work of Faraday stresses ($w_{rf}$ here) and work of magnetic pressure forces ($w_{rp}$) in a slow-inspiral with elastic turbulence. This is for $\vartheta = 0.1$ and $\dot{s} = 10$.

7. ENERGY FROM THE STAR

The energy must come from the star, which is not without its own constraints. The constraints on $C_L$ due to this aspect of energy conservation may be approximated as follows. Following Shu et al. (1994) who in their section 4.1 cite James (1964), write the angular momentum of the star as

$$L_* = \omega b M_\star R_*^2 \kappa_*$$

where of necessity $b$ is a small fraction of unity; roughly, $b = 1/5$ decreasing to $b = 1/7$ for maximally-rotating stars.

Stellar energy conservation depends critically upon the behavior of $\dot{R}_*$ with respect to $\dot{M}_*$. Define the expansion factor $x_R$ and the speedup factor $x_\omega$ as

$$x_R = \frac{\dot{R}_*}{\dot{R}_*/M_*}, \quad x_\omega = \frac{\dot{\omega}}{\dot{\omega}/M_*}$$

(110)

(111)
Shu et al. (1994) take \( x_R = 1 \); for fixed density, for example, \( x_R = 1/3 \), although in principle, depending on phase of accretion, \( x_R \) might even be negative. It is a purely phenomenological quantity, and is best thought of as descriptive, not prescriptive.

Then

\[
\frac{\dot{L}_*}{M_*} = \left( \frac{3}{2} + \frac{1}{2} x_R + x_\omega \right) \frac{L_*}{M_*}.
\] (112)

This is combined with eq. (104) and eq. (70). For example, for a maximally-rotating \((\omega = 1, \dot{\omega} = 0)\) star with \( \dot{R}_* = 0 \), then

\[
(C_L - \varepsilon) = \frac{3}{2} b (1 - \varepsilon)
\] (113)
yielding \( C_L = 1/4 \) for \( \varepsilon = 0.1 \), and the disk carries away from the star, through turbulence, 75% of the angular momentum that it advects to the star. Even in this fanciful case of a star at nominal breakup and holding \( R_* \) fixed, the star does not overly “spin up” because turbulent torques decrease the star’s angular momentum at nearly the same rate that advection increases it.

More generally,

\[
C_L = \omega \left[ b \left( \frac{3}{2} + \frac{1}{2} x_R + x_\omega \right) (1 - \varepsilon) + \varepsilon \right].
\] (114)

Suppose \( \omega = 1/3, b = 1/5, \varepsilon = 1/10, x_R = 1, \) and \( x_\omega = 0 \), let us say; then \( C_L = 0.153 \ll 1 \). That is, about 85% of the angular momentum advected to the protostar is carried radially back out by turbulent torques acting in the disk.

To do this, the star must yield energy to the disk, powering it (again through turbulent torques) by the quantity given in eq. (88). Is this possible?

The total kinetic energy of rotation of the star is

\[
(K.E.)_* = \frac{1}{2} b R^2_* \Omega^2_* M_* = \frac{1}{2} b \omega^3 R^2_* \kappa^2_* M_* = \frac{1}{2} b \omega^2 \frac{G M^2_*}{R_*}
\] (115)

and this increases at the rate

\[
\frac{d}{dt}(K.E.)_* = b \omega^2 (1 - \varepsilon) \left( 1 - \frac{1}{2} x_R + x_\omega \right) \mathcal{L}_0
\] (116)

The total potential energy of the star however is

\[
(P.E.)_* = -b' R^2_* \kappa^2_* M_* = -b' R^2_* \Omega^2_* M_* = -b' \frac{G M^2_*}{R_*}
\] (117)

for some constant \( b' \) and it changes by the amount

\[
\frac{d}{dt}(P.E.)_* = -b'(1 - \varepsilon)(2 - x_R) \mathcal{L}_0
\] (118)

Unlike \( b \) however, \( b' > 1 \) (e.g. ca. 1.6 for the Sun). Let us take fiducial values as before, plus \( b' = 3/2 \). Then

\[
\frac{d}{dt}(P.E. + K.E.)_* \approx -1.34 \mathcal{L}_0 \quad E_d^{(\text{turb})} = -0.85 \mathcal{L}_0
\] (119)

Evidently, even for \( x_R = 1 \), there is more than enough mechanical energy for the star to provide mechanical (turbulent) energy to the disk in the quantity given in eq. (88) and conserve angular momentum while doing it. This still holds all the way up to \( \omega = 1 \), and lower values of \( x_R \) only improve the situation. Even rotating near nominal breakup, it is only the outermost layers of the star that have a ratio of kinetic energy to potential energy approaching a sizeable fraction of unity in magnitude. The core dominates both the energy balance sheet and the budget, and being gravitationally bound, there is ample energy available for the star to deliver energy to the disk by turbulent stresses as given above even while keeping \( \dot{\omega} = 0 \), so long as the mass efficiency \( \varepsilon \) is not too large. Thermal energy as well as energy of turbulent motions and turbulent and large-scale magnetic fields also enter into the overall accounting, and of course may also provide sinks for the gravitational binding energy released interior to the star, but these do not change the overall conclusion.

Again, as shown in fig. (5), by reducing \( C_L \) from its nominal value \( C_L = 1 \), the energy going into either turbulence production or turbulent work in the inner regions of the disk rises substantially. What is the source of this energy?
Figure 9. Cartoon of schema. Turbulent hoop stresses (plus gravity) behave as a kind of compressor, powered by the “shaft work” of torque on the protostar conveyed by turbulent MHD stresses, compressing fluid against thermal and magnetic pressure gradients. A central “combustion chamber” surrounds the protostar, adding enthalpy to the bypass flows. The effective photosphere of the upper surface has the shape of an inverted Mexican hat (not shown; see fig. 1). Critical surface is either a disk or an annulus (dashed light blue). Upper bypass fluid lies between upper inner separatrix $S_{i,u}$ and upper outer separatrix $S_{o,u}$. The stellar “surface” (dark blue dashed line) is somewhat arbitrary. Beyond critical surface, fluid is accelerated primarily by magnetic and thermal pressure gradients. This requires enormous enthalpy and high ($\sim 10^6 \pm 0.5 \text{ K}$) temperatures deep at the base that would radiate copiously but for blanketing by surrounding flow. A hollow centrifugal funnel flow is avoided by an extremely high rate of turbulent angular momentum extraction from the bypass flow.

Ultimately the energy of course comes from gravitational binding energy, but energy like money being fungible, there are many different ways to describe its flow. It should be clear in this case however that the immediate source of the energy for this increased turbulence production and work in the inner disk is the outward flux of angular momentum from the star. However we do the accounting, it does not make sense to say that the energy comes from the disk, if for no other reason than because the disk doesn’t have enough energy to give.

8. SCHEMA

The overall schema as imagined is shown in figure (9). The core working fluid completely envelopes the protostar and gradually settles onto it at all latitudes. Immediately above the pole is an X-point in the flow, where the inner flow separatrix meets the rotation axis (axis of symmetry). This flow separatrix separates the material that ultimately accretes onto the protostar (the core working flow) from that which does not (the bypass flow). (This separatrix was introduced in Williams (2003b); see fig. 1.1 of that paper.) There is also a separatrix dividing the bypass flow from the recirculating flow. Both separatrices have their lower counterparts.

At some range of radii between $R_s$ and $R_m$, there are recirculation zones above and below the accretion flow (i.e. the working flow and the upper and lower bypass flows). These recirculation zones serve as a tamper, providing vertical and radial pressure gradients that contribute to confining the heated accretion flow. The recirculation zones are also reservoirs of mass, energy, angular momentum, and not of least importance, magnetic helicity. Magnetic helicity is important because being conserved, it contributes to the stability and persistence of the recirculation zones. The magnetic field also provides radial (in the sense of cylindrical $R$, not spherical $r$) confinement of the outflow, as well as the recirculation zone itself. Material in these recirculation zones is near incipient $H$ ionization or recombination. The zones also shield the inner “combustion” region from easy direct observation. The recirculation zones are also hypothesized to be sources of meteoritic chondrules and calcium-aluminum inclusions (CAIs), but that is a subject for another paper.

Inside the working and bypass flows is a tangled, turbulent magnetic field, generated by the MRI. This is shown as alternating $\otimes$ and $\odot$. This field generates hoop-stresses, which create a radially-inwards force, indicated with double-
arrows, \(\leftrightarrow\). The largely toroidal field is transported vertically by buoyancy (the Parker instability) on a buoyant timescale, comparable to the advective timescale, so that the flux tubes move up at the same time they move in (slanted arrows, \(\swarrow\)). One man’s loss is another man’s gain, and what is lost from the working fluid is gained by the bypass fluid. By this mechanism, the working fluid is able to transfer mechanical energy to the bypass flow.

In addition to mechanical energy, the bypass flow is also heated. Squiggled arrows (\(\sim\)) show the thermal transport to the bypass flow both from the working flow as well as from the protostar itself, adding enthalpy to the bypass flow.

Finally, magnetic pressure from the tangled field accelerates the flow vertically through the throat. Additional enthalpy is added from what reconnection occurs in the tangled field. (In principle, \(H\) recombination also adds enthalpy to the flow, but this makes little effect relatively speaking.) The flow becomes supercritical (supersonic and super-Alfvénic) at some height of order a few \(R\) above the star.\(^{18}\) Expansion and acceleration continue in the supercritical region beyond. Radial (in the sense of cylindrical \(R\)) expansion is hindered both by external hoop-stresses acting to some extent as the diverging part of a de Laval nozzle, but the flow is also radially (again in the sense of \(R\)) confined and collimated by the toroidal component of its own internal tangled field.\(^{19}\) It is speculated that this field (internal and external) mitigates the shear-induced Kelvin-Helmholtz instability that would otherwise destroy this radial confinement.

Additional collimation is provided by tangled field that has been lost from the disk by buoyancy (“evaporation”), shown in the upper-right corner. In reality, it is highly unlikely that a flux tube would rise uniformly at all angles in azimuth around the disk at once. Rather, the flux tube can be thought of like a solar prominence, having two footprints in the disk. These footprints will generally not be at precisely the same distance \(R\) from the protostar. As a result, the flux tube may experience differential stretch (if the footprint at smaller \(R\) is also ahead of the other, in the sense of its rotation around the protostar), or “wrap.” Ultimately, the flux tube will “snap” inwards radially (in \(R\)). This motion is against the main flow which as drawn is radially outwards. Either there is a flow reversal (not shown) or there is a relative motion between the flux tubes and the background flow. This “evap, wrap and snap” action has not been addressed in this paper as the focus has been on the region interior to the thick recirculation zones. But, it should be expected to occur at some level, adding both to the collimation and the angular momentum flux of the outflow. Similar notions have been put forth by other researchers. Primarily, this physics is notable in the present context because such additional field wrapping may add a non-negligible quantity of angular momentum to the outflow, in which case that needs to be taken into account, both in the total angular momentum flux of jets and in the angular momentum balance in the disk outer to \(R_m\).

Assume the jet is in a low-beta state, say plasma \(\beta\) of order \(\beta = 0.10\), so \(v_A \simeq 3c_s\). The relevant wave speed is the bulk elastic wave speed, but this is within a factor of order one of the Alfvén speed. Suppose the gas is fully ionized and at a temperature of, let us say, \(10^5\) K, with a sound speed of \(c_s \simeq 40\) km s\(^{-1}\). Then the jet speed at the critical point is

\[
v_0 = 130\ \text{km s}^{-1} \left(\frac{\beta}{1/10}\right)^{-1/2} \left(\frac{T_0}{10^5\ \text{K}}\right)^{-1/2}
\]

and the density in the jet at the throat is of order

\[
\rho_0 = 10^{-13}\ \text{g cm}^{-3} \left(\frac{\xi M}{0.2}\right)^{-1/2} \left(\frac{\beta}{1/10}\right)^{-1/2} \left(\frac{\dot{M}_d}{10^{-8}\ M_\odot\ yr^{-1}}\right) \left(\frac{R_*}{3R_\odot}\right)^{-2} \left(\frac{T_0}{10^5\ \text{K}}\right)^{-1/2}
\]

or an electron number density, for the same fiducial parameters, of about \(n_e \simeq 10^{11}\) cm\(^{-3}\). The jet will continue to accelerate past the critical point, and of course cool adiabatically as well so that \(c_s\) drops and the jet becomes highly supersonic within an order of magnitude \(R_\odot\) or so vertical distance from the protostar, with some additional acceleration provided by the substantial enthalpy of \(H\) recombination.

Given these numbers, the one-sided Bremsstrahlung luminosity might be expected to be of order

\[
\mathcal{L}_x = 2 \times 10^{32}\ \text{erg s}^{-1}.
\]

This is larger than the nominal \(\mathcal{L}_x \sim 10^{29.5\pm0.5}\ \text{erg s}^{-1}\) (Güdel et al. 2017), but the explanation for this is strong absorption by the partially-ionized recirculation zones. For example, the assuming a line-of-sight column density

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]

\[\text{XXX...}\]
through the recirculation zone comparable to $R_\ast n_\ast$, the optical depth just from Thomson scattering alone is roughly $10^{-2}$. In practice, X-ray optical depths from bound-free transitions in passing through a recirculation zone will be substantially higher. Being relatively compact, the column density passing through the recirculation zone can be relatively large without requiring an inordinate mass.

Regarding missing boundary-layer luminosity, the answer I suggest is twofold. First, there is less energy deposited into heating than might be expected. Second, the volume over which this energy is deposited is much smaller than a thin boundary layer on the surface of the protostar.

9. DISCUSSION

Thermodynamics provides both the simplest and most stringent constraints on any jet theory. Energetically, it would appear that if we only channel the energy of accretion back into the accreting material itself, and we assume that all inflowing mass becomes outflowing so that $\varepsilon = 1$, then the best we can do is an outflow that fizzles, meaning it fails to become energetically unbound. Clearly the mass accretion-ejection efficiency $\varepsilon$ is not unity. Even so, the amount of mechanical energy available in the turbulence in accretion onto a rapidly-spinning protostar — again, mechanical, not thermal — is actually much more than the notional accretion energy $L_\ast / 2$. In a more realistic scenario in which $\varepsilon$ is small ($\varepsilon = 0.2$, say), first, the required overall thermodynamic efficiency of the jet-launching engine is actually less than is traditionally claimed, and second, the energy being mechanical rather than thermal in nature, the required thermodynamic efficiency is actually achievable as well.

Regarding the MRI in particular for its role in generating the MHD turbulence that is an intermediary in providing energy to the jet, to quote from Williams (2003a), “That a tangled field can have such important dynamical consequences as angular momentum transport in accretion disks leads us to ask what other consequences such a field might have. We propose that one consequence of this tangled field may be the driving of an axial outflow, i.e. a jet.” As there appears to have been some confusion on the part of the referee for Williams (2001) in this regard, I attempted to clarify in Williams (2005), “To be quite explicit, as a hypothesis, I reject the notion that … magnetocentrifugal mechanisms are responsible for tightly collimated jets.” The confusion likely arose because conventionally, in MCA theory, a far-field toroidal field is invoked to provide collimation, whereas what I have discussed everywhere, consistently, is a near-field toroidal field, up to and including the very mid-plane itself. There is an issue of semantics, as a rocket engine both requires collimation in the supercritical region (the nozzle), and at or near the critical point itself, as well as in the subcritical region (the reservoir and the inlet to the nozzle). Perhaps this latter is best called “confinement,” in which case the title to my paper Williams (2005) should be changed, from “… jet collimation …” to “… jet material confinement and collimation …”; even so, in the case of collimation of supercritical ($v > c_A, c_s$) flow, collimation is only needed before the jet becomes highly supersonic (and super-Alfvénic). Beyond this point the Mach angle is small, and lateral spreading is minimal. From this perspective, in the schema presented here, collimation is mainly needed from $R_\ast$ to 0.1 AU distance in vertical separation from the star (for a fiducial 3 $R_\odot$ and 0.8 $M_\odot$ protostar).

Since the schema I have discussed here and from the very beginning in Williams (2001) is a manifestly non-magnetocentrifugal picture of how jets might be created, it is worthwhile to discuss the grounding of the MCA theory. To date, observations of jets appear consistent with the MCA hypothesis. But consistency is not in itself enough.

At the grossest level, the appeal of MCA theory is that it solves the angular momentum problem, allowing protostars to condense and accrete by dumping excess angular momentum into a jet. This had a definite and considerable appeal before the discovery of the role of the MRI in accretion (Balbus & Hawley 1992), but arguably less so now that the MRI has been (re-)discovered and studied intensely. The MRI does appear to have “dead zones,” locations where it is not active, in protostellar accretion (Gammie 1996). But these are farther out in radius ($R > 0.1$ AU), likely intermittent, and in any case the surface layers still remain active due to cosmic-ray ionization. Some flavor of MCA may indeed “help” accretion proceed by allowing angular momentum to escape to infinity, but it is not at all clear — to me, at least — that this is required. The MRI is quite effective in doing this already, by transporting angular momentum radially outwards, not vertically, and it is fully capable of doing this all the way right down to the protostar itself. This arguably has a much simpler physical appeal, for the same reason that one opens a door at the handle, not at the hinges. Jets do not allow accretion to proceed by providing a place to dump excess angular momentum. Rather, they are a way for the system to discard excess free energy, not just from the disk, but from the star itself as it condenses.

One prediction of the MCA hypothesis, both in the standard self-similar disk-wind theory and in the X-wind theory in which the locus of initial magnetocentrifugal acceleration is near the protostar, is that the far-field material velocities

\[ \varepsilon \]
are roughly of the order of the escape velocity near the protostar. That is, the asymptotic specific energy of the material is of the same magnitude but opposite sign, roughly, as it had near the star. On the one hand, this is an important constraint, and tends to argue against jets being created by the acceleration of material coming from more remote \((R \gg R_*)\) regions of the accretion flow. On the other hand, it is not a terribly restrictive constraint, as any mechanism that operates in the immediate neighborhood of the protostellar surface might be expected to eject material at similar speeds. However difficult it may be to construct a scenario that will take material near a stellar surface and make it energetically unbound, once this has been accomplished, it would require a delicate balancing act to do so without giving such material much more than a trivial fraction, at infinity, of the (negative) energy it formerly had (but with opposite sign). Besides, as discussed below, energy is not the problem. As emphasized by X-wind theory adherents as well, there is abundant mechanical energy not just from the accretion flow but from the star in the form of rotation.

The problem, again as emphasized in the introduction, rather, is one of efficiency, first and foremost thermodynamic efficiency, and secondly, mass efficiency.\(^\text{20}\) Indeed, MCA theory is appealing here because it provides a theoretically thermodynamically efficient method to accelerate material. But as I have discussed above, neither is this as restrictive a constraint as might be thought. The notion that “viscous dissipation” in the disk can not be thermodynamically efficient at driving jets rests on a mistake: “viscous dissipation” as treated in standard disk theory is neither viscous nor dissipation at all, but turbulent production (pumping), leading to mechanical energy – notably tangled, turbulent, buoyant, encircling and confining magnetic fields – not heat, at least not directly, and not necessarily on a time scale short compared with the advective time scale.

Nor is MCA theory without its problems. Among these are of course the problem of the origin of the large-scale poloidal field (is it created by flux conservation as part of the process of condensing of the initial gravitational collapse of a potential star-forming cloud, advected from afar, or created on-site by the disk, or by the star, or some combination?), and more importantly, the problem of on the one hand threading disk material onto the magnetic field while on the other hand ensuring that the field is strong enough to centrifugally accelerate the same material once it is so ensnared.

Another interesting problem that arises in standard MCA theories of jets and indeed with any MHD theory of jets that relies upon large-scale magnetic fields is the persistence of the conserved quantity magnetic flux. As mentioned \(\text{e.g.}\) by Hartigan (2008), to quote, “for a magnetized disk wind the field will be mostly toroidal at large distances and will scale as \(r^{-1}\), while the density drops as \(r^{-2}\) for a collimated jet.” The implication is that the Alfvén speed remains constant, and the jet, if initially strongly magnetized at its base, should remain so at large distances. This is at odds with measurements (Morse et al. 1992) of the magnetic field in the working surface of jets (as inferred from shock compression ratios as determined by spectroscopic measurements of the free electron density), which indicate that fields at large distances (e.g. \(\gtrsim 10^{3.5}\) AU for HH 34) are not dynamically significant. This is also consistent with the observation of shocks in jets, which would otherwise not form in the presence of a dynamically-significant field.

One solution as mentioned by Hartigan (2008), see Hartigan et al. (2007), is that the jet launching process itself is strongly time-dependent.

Acceleration by a turbulent, tangled field, as I have discussed here and elsewhere beginning with Williams (2001), offers another solution. A tangled field is not encumbered by a magnetic flux constraint. The total magnetic flux through any surface perpendicular to a stream tube in the outflow can be identically \(\text{zero}\) at the same time that the Alfvén speed is large (\(\text{e.g.}\) of the same order of magnitude as the ejecta speed, larger or smaller depending on if we have yet crossed the critical surface), only it is a turbulent Alfvén speed corresponding to effective elastic modes (Gruzinov & Diamond 1996; Schekochihin et al. 2002; Williams 2004) due to the turbulent elasticity of the bulk fluid that is the relevant quantity, not the Alfvén speed corresponding to the mean field. In this sense we can have our cake and eat it too, by having both a large magnetic field (in magnitude) that can accelerate material though magnetic tension forces and magnetic pressure forces, and at the same time a negligible magnetic field (in terms of total flux), so that the magnetic field can reconnect and dissipate at large distances.

For example, as cited by Cabrit (2007), Ferreira (2007) suggests a poloidal enclosed magnetic flux at 1 AU of \(\Phi_B \simeq 10^{26}\) G cm\(^2\), whereas we have \(B_{\text{coll}} \neq 0\) while \((\Phi_B)_{\text{coll}} = 0\).

Another way of saying this is that in the schema I have presented, one does not care at all about closing the global circuit. No Biermann battery or the like or indeed any global current at all is required to maintain the large-scale

\(^{20}\) That is, once you have figured out a way to give energy to an outflow, how to you find a way to give that energy only to part of \(\dot{M}_d\) and not all of it?
field, as there is no such field to begin with. (Of course in reality there will be some nonzero mean field, but this does not provide a meaningful constraint to the gross picture at this point, in any case.)

Yet another appeal of MCA is its analytical tractability. This is rooted in self-similarity (in the original theory). Symmetry is a phenomenal tool when you can use it, such as in the Sedov-Taylor blast wave theory. Nature is not necessarily so kind, however. Already it is clear that scaling invariance is broken first by the presence of the star itself (assuming, again, that jets are launched near the star), and secondly by microphysics, such as incipient H ionization and its effect upon opacities, such as in the recirculation zones I have posited. It should not be surprising at all if, as claimed here, the jet launching process is not at all respective of self-similarity.

The most stringent and direct support for the MCA hypothesis to date, rather, rests in observations of jet rotation, and indeed observations of such rotation have been widely seen as supportive of the MCA hypothesis (Cabrit, Bacciotti). In contrast, the mechanism described here tends to create jets with a much smaller specific angular momentum flux than should be expected from MCA-type models. Concerning this, let us look at some fiducial numbers from possible observations of jet rotation. From Bacciotti et al 2007, observing DG Tau and several other T Tauri stars, let us take azimuthal velocities in the range 6–25 km s⁻¹, at radial separations of 20–30 AU. Taking the geometric mean of both quantities yields an observed specific angular momentum \( \ell_{\text{obs}} \) of

\[
\ell_{\text{obs}} = 300 \text{ AU km/s}.
\]

In contrast, the specific angular momentum of the jet-launching process described here is much less, of order

\[
\ell = 1.05 \text{ AU km/s} \left( \frac{\omega}{0.33} \right) \left( \frac{M_*}{0.8 M_\odot} \right)^{1/2} \left( \frac{R_*}{3 R_\odot} \right)^{1/2}.
\]

However, the specific angular momentum just due to Keplerian rotation at 20–30 AU is of order 100–150 AU km s⁻¹. If such observations of rotation hold up, there remains the question of whether the material observed is the true jet core itself, or is rather surrounding entrained or evaporated material. It is not just the angular momentum of material in or around the jet that is of interest, but the flux of angular momentum in particular, and the ratio of this to the turbulent flux in the disk itself. It is claimed here that that ratio is low, whereas MCA says otherwise. Outside of observations at the sub-0.1 AU scale, this ratio is the critical discriminant and test of MCA theory.

10. APPENDIX I: ACCRETION EJECTION EFFICIENCIES

The discussion here is based heavily upon Cabrit (2007) with some additional comments. There are three traditional measures of the efficiency of jets, related to the three conserved quantities of mass, momentum, and energy, and to which we might add a fourth, related to angular momentum.

The first is the mass efficiency, defined in terms of that fraction of accreting matter that ends up in a jet (or outflow). It should be noted from the outset that it is not at all clear that such a quantity even exists however. If material circulates, then there may be no clear way of saying what matter is actually accreting; furthermore, outflows may happen at various locations (e.g. radii \( R \)), as may inflows; should these all be counted?

Caveats aside, the definition (assuming two opposing jets that are in all other senses equal) is

\[
\varepsilon_M \equiv \frac{2 \dot{M}_j}{\dot{M}_d} \simeq 0.2.
\]

(this and other typical values are taken from Cabrit (2007)). This quantity being of prime importance, in the main body of the text, I have simply denoted this \( \varepsilon \), dropping the subscript. Actually, it may be worth noting that \( \varepsilon_M \) is usually defined in terms of the accretion rate \( \dot{M}_\text{acc} \), so one could say that I am assuming here that \( \dot{M}_\text{acc} = \dot{M}_d = \dot{M}_* \); the difference between \( \dot{M}_d \) and \( \dot{M}_* \), given other uncertainties, though, is almost certainly negligible in most if not all cases. As an amusing if somewhat strained analogy, thinking of astrophysical jets as turbofan engines, \( \varepsilon_M \) is related to a quantity in turbofans known as the bypass ratio, \( \varepsilon_M/(1 - \varepsilon_M) \). In this sense, protostellar jets are like low-bypass turbofan engines.

Momentum (or thrust) efficiency relates the thrust to the accretion luminosity \( \mathcal{L}_\text{acc} \). The one-sided jet thrust is \( F_j = \dot{M}_j v_j \); it is assumed that \( F_j \) is taken here at a location far enough from the star that it has approached its asymptotic value (that is, gravity, thermal pressure, and magnetic pressure are no longer decelerating or accelerating the outflow; for an observer, this is a non-issue). This is then compared to the notional momentum flux \( \mathcal{L}_\text{acc}/c \), which
is the momentum that would be obtained if all the photons from $\mathcal{L}_{\text{acc}}$ were pointed in one direction like a flashlight. Then

$$\varepsilon_F \equiv \frac{2\dot{M}_j v_j}{\mathcal{L}_{\text{acc}}/c} \simeq 300. \quad (126)$$

The fact that $\varepsilon_F$ is so large argues strongly against scattering of light from $\mathcal{L}_{\text{acc}}$ being a substantial source of the momentum of jets. This is a very large number for an “efficiency” however; an alternative sort of efficiency might normalize to the thermal velocity of gas at the stellar surface, or to the escape velocity. The escape velocity from the stellar surface is

$$v^{(\star)}_{\text{esc}} = 320 \text{ km s}^{-1} \left( \frac{M_\star}{0.8 M_\odot} \right)^{0.5} \left( \frac{R_\star}{3 R_\odot} \right)^{-0.5}. \quad (127)$$

Defined in terms of this, an alternative thrust efficiency measure is

$$\varepsilon'_F \equiv \frac{2\dot{M}_j v_j}{\mathcal{L}_{\text{acc}}/v^{(\star)}_{\text{esc}}} = 0.32. \quad (128)$$

This is a quantity of more theoretical interest than observational, clearly.

The energy efficiency is defined in terms of the mechanical luminosity of the jets; the one-sided mechanical jet luminosity is $\mathcal{L}_j = \dot{M}_j v_j^2 / 2$. For this quantity, it is important that one determines $\mathcal{L}_j$ close enough to the source that there has been little entrainment of surrounding material that would otherwise reduce the jet speed (and increase its mass loading). The momentum flux of jets is generally unchanged by entrainment but the energy flux is, as it must be; entrainment requires dissipative structures in order to satisfy conservation. Then

$$\varepsilon_E \equiv \frac{2\mathcal{L}_j}{\mathcal{L}_{\text{acc}}} \simeq 0.15. \quad (129)$$

In principle $\varepsilon_F$ and $\varepsilon_E$ are determined independently, but in practice this may be difficult.

The fourth and final efficiency is the angular momentum efficiency. To define this, we need a notional angular momentum flux. One choice is

$$\dot{L}_0 = \dot{M}_d \sqrt{GM_\star R_\star}. \quad (130)$$

Then

$$\varepsilon_L \equiv \frac{2\dot{L}_j}{\dot{L}_0}. \quad (131)$$

Under this definition, $\varepsilon_L$ for the scenario I have described, ignoring entrainment, is $\varepsilon_L = \varepsilon_M \simeq 0.2 < 1$. This ignores the “evaporate, wrap and snap” action of toroidal field components from $R \gg R_\star$; these will carry angular momentum as well, increasing the total angular momentum flux in the outflow, and possibly in the jet itself. It does not take much entrainment to vastly inflate $\varepsilon_L$. Still, offhand one expects the efficiency $\varepsilon_L$ to be much less than it would be for MCA theories, in which the entire point is that the jet enables accretion by carrying away a substantial angular momentum flux.

11. APPENDIX II: ENERGY AND STRESS RELAXATION IN HYDRODYNAMIC TURBULENCE

Although our interest is MHD turbulence, this section discusses relaxation time and anisotropy in the context of purely hydrodynamic turbulence. This is done to compare and contrast with MHD turbulence, and in order to provide a bit more intellectual continuity with studies of purely hydrodynamic turbulence, as it is more mature and well-developed topic than the theory of MHD turbulence. As the modeling minutiae may distract from the overall thrust of this paper however, readers should feel free to skip over some of the more weedy details presented here.

Like all crude models, closer inspection reveals finer distinctions. There is the one hand the time scale $s$ on which the anisotropy of the turbulence stress relaxes, and there is on the other hand the time scale $\tau$ on which the kinetic (or magnetic) energy in turbulence is ultimately dissipated by molecular viscosity or resistivity into actual heat. In previous work I have ignored the distinction, as the two time scales were essentially synonymous for the simple models I considered, and I largely focused on the stress relaxation in any case. I claimed that given a shear rate $\gamma$, the product $s\gamma$ is of order unity for hydrodynamic turbulence, but potentially much larger for MHD turbulence. (As always, I am assuming a zero mean-field in the MHD case.) In hydrodynamics, anisotropy is carried on the largest scales; small
eddy stresses contribute only an isotropic component to the stress. A key implication is that the normalized first normal stress difference for hydrodynamic turbulence is small, of order unity, whereas it is potentially much larger for MHD turbulence, as appears confirmed by shearing-sheet simulations of MRI-driven turbulence in particular.

For a simple shear in Cartesian geometry in which \( v_x = \gamma y \) and \( v_y = v_z = 0 \), the normalized first normal stress difference is \( (\sigma_{xx} - \sigma_{yy})/\sigma_{xy} \). For a Newtonian viscous fluid — and for standard theory of accretion disk turbulence and indeed any model of turbulence that adopts the Boussinesq hypothesis — this quantity is zero. Effectively, and loosely, this quantity is related to the relaxation time of the anisotropy of the stress. For example, for purely hydrodynamic turbulence, the notion that eddies on the largest scale have a coherency time of order of the eddy-turnover time implies that the normalized first normal stress difference is of order or unity. On the other hand, it takes very many large scale eddy-turnover times (or shear times) for energy to cascade all the way down to the Kolmogorov scale and thermally dissipate. In general, it seems reasonable to expect that \( \tau \) may be much greater than \( s \), but certainly not much less. This is discussed further below.

Consider purely hydrodynamic turbulence in an incompressible fluid of fixed density \( \rho \). In this section, I adopt Cartesian index notation, with \( \delta_{ij} \) as the Kronecker delta. For this and the next section only, primes indicate fluctuations, not derivatives with respect to radial coordinate \( R \). These fluctuations correspond to a Reynolds decomposition with standard Reynolds rules of averaging. Over-lines indicate ensemble mean quantities, so that a given field variable \( x \) may be written

\[
x = \bar{x} + x'.
\]

The intermediate results presented below are well-known and the various symbols adopted, while not universal, are fairly standard. See Tennekes & Lumley (1972), Pope (2000), Davidson (2004), or especially Wilcox (2006); for an excellent introduction, see ch. 3 of Versteeg & Malalasekera (1996).

The nomenclature chosen here is that the Reynolds stress is \( R_{ij} = -\rho \bar{v}_i \bar{v}_j' \), and the specific Reynolds stress is \( \tau_{ij} = -\bar{v}_i' \bar{v}_j' \) (in many engineering contexts the term “Reynolds stress” refers to this latter quantity.) The momentum equation for the mean flow, assuming that the mean transport of momentum due to molecular viscosity is negligible compared to the transport due to turbulence, is

\[
\partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i = -\frac{1}{\rho} \partial_i \bar{P} + \partial_j \tau_{ij},
\]

analogous to the momentum equation, eqn. (19). Also analogously, there is a turbulent flux of energy, \( f_{ti} \), with components

\[
f_{ti}^{(t)} = -\bar{v}_j R_{ij} \tag{134}
\]

The specific kinetic energy of the mean flow is \( K = \bar{v}_i \bar{v}_i/2 \). For sufficiently intense turbulence – i.e. at high Reynolds number when the molecular viscosity makes a negligible contribution to the mean stress compared to the turbulence – this obeys the transport equation

\[
\partial_t K + \bar{v}_j \partial_j K = -\bar{v}_j \partial_j \bar{P} + \partial_j (\bar{v}_i \tau_{ij}) - S_{ij} \tau_{ji},
\]

which is analogous to eq. (27). Here \( S_{ij} \) is the symmetrized velocity gradient of the mean flow: \( S_{ij} = (\partial_i \bar{v}_j + \partial_j \bar{v}_i)/2 \).

Without any modeling to simplify matters, the transport equation for the specific Reynolds stress \( \tau_{ij} \) is fairly involved. Using standard symbols:

\[
\partial_t \tau_{ij} + \bar{v}_k \partial_k \tau_{ij} + \tau_{ik} \partial_k \bar{v}_j + \tau_{jk} \partial_k \bar{v}_i = \epsilon_{ij} - \Pi_{ij} + \partial_k \left[ \nu \partial_k \tau_{ij} + C_{ijk} \right] \tag{136}
\]

where

\[
\Pi_{ij} = \frac{\bar{P}}{\rho} \left( \partial_i v_i' + \partial_j v_j' \right),
\]

\[
\epsilon_{ij} = 2\nu (\partial_i v_i')(\partial_j v_j'),
\]

Notes:

21 In the literature on continuum mechanics and non-Newtonian fluids, the shear rate is often denoted \( \dot{\gamma} \), but here and elsewhere I simply use \( \gamma \).

22 It might seem preferable to characterize the stress anisotropy given by the normal stress difference in terms of the principle stresses instead. There is no added benefit of this here however, and the first normal stress difference for a steady shear is a quantity of historical and practical significance. Note that in this section I have dropped the angle brackets in tensor components as the analysis here is entirely in Cartesian coordinates and so no distinction is required between covariant, contravariant, or physical components.
and
\[ \rho C_{ijk} = \rho \nu_i \nu_j \nu_k + \overline{P} \nu_i \nu_j \delta_{jk} + \overline{P} \nu_j \delta_{ik}. \]  

(139)

The transport equation for the specific turbulent kinetic energy \( k = \nu_i \nu_i / 2 \) is obtained upon taking the trace, simplifying matters somewhat:
\[ \partial_t k + \bar{v}_j \partial_j k = \tau_{ij} \partial_j \bar{v}_i - \nu (\partial_k \nu_i) (\partial_k \nu_j) + \partial_j \left[ \nu \partial_j k - \frac{1}{2} \nu_i \nu_j - \frac{1}{2} \rho \overline{P} \nu_j \right]. \]  

(140)

Let us consider one of the simplest practical turbulence models, a one-equation model. Modeling begins when we then invoke the turbulent viscosity assumption
\[ \tau_{ij} = 2 \nu_i S_{ij} - \frac{2}{3} k \delta_{ij} \]  

(141)

and model the sum of the turbulent (self-)transport and pressure diffusion terms (which arose from \( \Pi_{ij} \) and \( C_{ijk} \)) as a simple diffusion term
\[ \frac{1}{2} \nu_i \nu_j \nu_k + \frac{1}{3} \rho \overline{P} \nu_j = - \frac{\nu_i}{\sigma_k} \partial_j k \]  

(142)

where \( \sigma_k \) is a constant of order unity and indeed is often set equal to 1 identically.

The final ingredients of the model are to assume that
\[ \epsilon_{jj} = 2 \nu (\partial_k \nu_i) (\partial_k \nu_j) \simeq 2 \nu \overline{S}_{ik} \overline{S}_{kj} = \epsilon \]  

(143)

where \( \epsilon \) is the familiar turbulence dissipation rate (and \( s_{ij} \) is the symmetrized turbulent velocity gradient), and further to assume that \( \epsilon \) is related to \( \nu_j \) and \( k \) through
\[ \epsilon = C_D \frac{k^2}{\nu_i}. \]  

(144)

Note that \( \epsilon \) is a true dissipation, representing the action of molecular viscosity on small-scale eddies. In other words, \( \rho \epsilon \) here is identically equal to the local heating rate \( q \).

From this we arrive at the one-equation turbulence model
\[ \partial_t k + \bar{v}_j \partial_j k = \tau_{ij} \partial_j \bar{v}_i - C_D \frac{k^2}{\nu_i} \partial_j (\nu_i \partial_j k) = \frac{1}{\rho} (p - \rho \epsilon + \delta) \]  

(145)

where \( p \) is the turbulence production rate
\[ p = \rho S_{ij} \tau_{ji} \]  

(146)

and \( \delta \) is a diffusion term. Eqn. (146) should be compared with eq. (5) and eq. (29). Note that the stress tensor is symmetric in all cases so it does not matter if we symmetrize the velocity gradient in these contractions. Eqn. (145) is a slight variation of the standard Prandtl model (Prandtl 1945), also proposed independently by Emmons (1954), according to Wilcox (2006).

In steady shear, \( p = \rho \epsilon - \delta \).\(^{23}\) Note that in general however, whether we ignore the diffusive term \( \delta \) or not, there is no reason to expect the local production \( p \) to instantaneously equal the local dissipation \( \rho \epsilon \), and in realistic applications they differ. In fact, if they didn’t, the transport equation would be pretty useless.

Modulo diffusive effects, \( p \) and \( \rho \epsilon \) equilibrate on the time scale
\[ \tau = k/\epsilon, \]  

(147)

which can be compared to the shear time scale \( \gamma^{-1} \) by forming the product \( \tau \gamma \). Suppose rectilinear shear in which \( \nu_x = \gamma y \) and all other components of \( \mathbf{v} \) vanish. Then from eq. (146) and eq. (141) the production rate is
\[ p = \rho \nu_t \gamma^2. \]  

(148)

\(^{23}\) For homogeneous free shear flows \( p/(\rho \epsilon) \neq 1 \) even though \( \delta = 0 \), but this is because \( k \) grows in time \( t \) in such flows. See Pope (2000, sec. 5.4.5). We could imagine a steady solution to homogeneous shear in which \( p = \rho \epsilon \), but actually this is an artifice, since the length scale of the largest eddies will continue to grow until either boundary effects or diffusion effects become important.
Suppose as well for the moment that, suspending objections, \( p = \rho \epsilon \), then combining eq. (144) and eq. (147), we have
\[
\tau \gamma = C_D^{-1/2}. \tag{149}
\]
Empirical fits to laboratory turbulence suggest \( 0.07 \lesssim C_D \lesssim 0.09 \). Then \( \tau \gamma \simeq 3.5 \). A similar analysis based on the \( k-\epsilon \) model would have given a nearly identical result.

In fact, let us turn to the \( k-\epsilon \) model now. In standard form, it consists of two coupled advection-diffusion equations, one for \( k \),
\[
D_t k = \frac{1}{\sigma_k} \partial_j (\nu_t \partial_j k) + \tau_{ij} S_{ji} - \epsilon \tag{150}
\]
and one for \( \epsilon \),
\[
D_t \epsilon = \frac{1}{\sigma_\epsilon} \partial_j (\nu_t \partial_j \epsilon) + C_1 \frac{\epsilon}{k} \tau_{ij} S_{ji} - C_2 \frac{\epsilon^2}{k}, \tag{151}
\]
plus the prescription
\[
\nu_t = C_\mu \frac{k^2}{\epsilon}, \tag{152}
\]
just as for the one-point model (eq. 144). Standard values are
\[
C_\mu = 0.09 \quad \sigma_k = 1 \quad \sigma_\epsilon = 1.3 \quad C_1 = 1.44 \quad C_2 = 1.92. \tag{153}
\]
Let us focus on the two terms on the right of eqs. (150) and (151). Note that \( \tau_{ij} S_{ji} = p/\rho \). Then, ignoring diffusive effects, the turbulent kinetic energy density \( \rho k \) grows at the rate
\[
D_t (\rho k) = \text{diffusion} + p - \rho \epsilon, \tag{154}
\]
and the dissipation \( \rho \epsilon \) grows at the rate
\[
D_t (\rho \epsilon) = \text{diffusion} + C_1 \frac{p}{\tau} - C_2 \frac{\rho \epsilon}{\tau}. \tag{155}
\]
Given a Heaviside step-function increase in \( p \) at some time \( t_0 \) then, \( k \) and \( \epsilon \) respond in qualitatively similar manner, both growing asymptotically to a new equilibrium at a relaxation time of order \( \tau \). Again, the same results obtain with the one-equation Prandtl model as well.

In the field of laboratory studies of hydrodynamic turbulence, \( \tau \) is known as the turbulence time or the turbulence dissipation time. Note also that the shear rate \( \gamma \) is conventionally denoted \( \delta \) in turbulence studies, and \( \dot{\gamma} \) in the theory of non-Newtonian (e.g. viscoelastic) fluids. The product \( \gamma \tau \), i.e. \( \delta k/\epsilon \), is an important quantity in turbulence modeling; depending on the model, the value in steady shear typically lies in the range \( 3.5 \lesssim \delta k/\epsilon \lesssim 9 \). Models such as the one-equation model above and the \( k-\epsilon \) model tend to have values of \( \delta k/\epsilon \) on the low end of the range. Full stress-transport models tend towards the higher end.

The relationship of the normal stress difference \( (\tau_{yy} - \tau_{xx}) \) to the relaxation time can not be studied in simple models such as eq. (145) or the \( k-\epsilon \) or \( k-\omega \) models because the normal stress difference in such models is identically zero. Let us turn now therefore to a popular Reynolds stress model of turbulence, the Launder-Reece-Rodi (LRR) model (Launder et al. 1975).

The LRR model (and related similar models) model the dissipation term (138) as proportional to the simple scalar \( \epsilon \) times \( \delta_{ij} \), and just as for \( k-\epsilon \), there is a transport equation for \( \epsilon \). The transport term \( C_{ijk} \) is modeled as a symmetrized gradient of the stress \( \tau_{ij} \) and is essentially turbulent self-diffusion. The most troublesome term is \( \Pi_{ij} \), the pressure-strain redistribution term.

The LRR model for \( \Pi_{ij} \) includes a so-called slow pressure strain term that follows a Rotta (1951) return-to-isotropy prescription
\[
\Pi_{ij}^{(\text{slow})} = C_1 \frac{\epsilon}{k} \left( \tau_{ij} + \frac{2}{3} k \delta_{ij} \right). \tag{156}
\]
Launder et al. (1975) give \( C_1 = 1.5 \). It is not so simple as to say that \( \Pi_{ij} \) just dissipates or creates anisotropy however; rather, it re-orient the anisotropy, reducing the shear-aligned component but increasing other components.
The Reynolds anisotropy tensor \( a_{ij} \) is here defined as
\[
a_{ij} = -\frac{\tau_{ij} - \frac{2}{k} k \delta_{ij}}{k}.
\] (157)

Hamlington & Dahm (2008) note an approximate evolution equation for the anisotropy is
\[
\frac{D a_{ij}}{D t} = -\frac{1}{\Lambda_m} a_{ij} + \alpha_2 S_{ij}
\] (158)
where \( \Lambda_m \) is the turbulence memory time scale,
\[
\Lambda_m = \frac{1}{k} \frac{k}{\alpha_1 \epsilon}
\] (159)
where
\[
\alpha_1 = \frac{p}{\rho \epsilon} - 1 + C_1.
\] (160)
Using \( C_1 = 1.5 \) and \( p/\rho/\epsilon = 1.8 \), \( \Lambda_m \approx \tau/2.3 \). For LRR, \( \gamma \tau = 4.83 \) (Abid & Speziale 1993). Then, we have
\[
\gamma \Lambda_m \approx 2.1.
\] (161)

In other words, the isotropization rate is of order the shear rate. Qualitatively similar conclusions might be gleaned from the magnitude of the Bradshaw’s (or Townsend’s) constant \( \beta_r = \tau_{xy}/k \); empirically, \( \beta_r \approx 0.3 \), according to Wilcox (2006), citing Townsend (1976). One can identify \( s \) with \( \Lambda_m \), which then justifies the statement made earlier that \( s \gamma \) is of order unity (actually 2.1 here). The resultant anisotropy leads to
\[
\frac{\tau_{xx} - \tau_{yy}}{\tau_{xy}} = 1.46
\] (162)
for steady shear in the LRR model, versus 2.19 as determined experimentally (Abid & Speziale 1993).

The point of this section is not to discuss these points ad nauseam, but to put some substance behind the claim made in previous work (Williams 2004) that, in contrast to MHD turbulence, the stress anisotropy in hydrodynamic turbulence relaxes on a time scale comparable to the shear time scale, i.e. \( \gamma s \) is of order unity, and that this result is tied to the result that the normalized first normal stress difference is also of order unity. These two results, in other words, are connected. Further, turbulent energy also decays on a nonzero timescale \( \tau \approx s \). Ultimately, these are all empirical statements, but it is good to see how they are related to aspects of the modeling process.

12. APPENDIX III: MMF MODEL FOR MHD TURBULENCE IN STEADY FAST IN-SPIRAL

Here I discuss the purely kinematical problem of a steady solution to the MMF model for fast incompressible in-spiral. Having solved this, it is possible to discuss the resultant forces, but of course there is no guarantee that the system represents a viable solution to the full fluid equations with realistic EOS and overall force balance. Still, the example helps elucidate some important physics.

Assume an incompressible flow-field. This assumption is adopted because it allows for an exact solution that helps clarify the different roles of the normal stresses versus the off-diagonal stresses; the gross conclusions regarding the importance of the relaxation time carry over, it is expected, to the more general case of a compressible flow. The imposed flow-field is \( \nu(\phi) = R v^\phi = R \Omega \) where \( 0 < \Omega = \Omega_s (R/R_*)^{-\eta} \), and \( \nu(R) = v^R = -\theta R^2 \Omega_s / R \) where \( \theta \) is a constant, equal to the ratio \(-\nu(R)/\nu(\phi)\) evaluated at \( R_* \). Assume \( \nu_z = 0 \), and that the system has azimuthal and vertical symmetry so that all quantities depend only on \( R \). One can imagine the accretion flow is of some fixed vertical thickness, and within the flow, the density \( \rho \) is fixed.

The solution to the Maxwell model for the Faraday stress for the flow-field, in cylindrical coordinates, and applying an outer boundary condition that the stress asymptotes to the purely azimuthal flow solution (and therefore approaches zero at radial infinity), is that the off-diagonal stress \( M^{R \phi} \) and the two normal stresses \( M^{R R} \) and \( M^{\phi \phi} \) are nonzero; the remaining stresses are zero: \( M^{R z} = M^{\phi z} = M^{z z} = 0 \). The non-zero stresses satisfy the differential equations
\[
sv^R (R^2 M^{R R})' + R^2 M^{R R} = -2 \mu_4 R v^R
\] (163)
\[
sv^R (RM^{R \phi})' - s R \Omega^2 M^{R R} + RM^{R \phi} = \mu_4 R \Omega v^R
\] (164)
\[
sv^R R^2 (M^{\phi \phi})' - 2 s \Omega^2 (RM^{R \phi}) + R^2 M^{\phi \phi} = 2 \mu_4 \frac{v^R}{R}
\] (165)
where again, primes indicate differentiation with respect to $R$. Note that the physical components are $M_{(RR)} = M^{RR}$, $M_{(R\phi)} = RM^{R\phi}$, and $M_{(\phi\phi)} = R^2M^{\phi\phi}$. Now assume that the relaxation rate $s^{-1}$ is proportional to the shear rate $q\Omega$, and define the constant $\hat{s} = s\Omega$. Also write

$$x \equiv \frac{R}{R_*} \quad w_q(x; \vartheta, \hat{s}) \equiv \frac{M^{RR}}{\mu_t \Omega_*} \quad y_q(x; \vartheta, \hat{s}) \equiv \frac{-RM^{R\phi}}{\mu_t \Omega_*} \quad z_q(x; \vartheta, \hat{s}) \equiv \frac{R^2M^{\phi\phi}}{\mu_t \Omega_*}. \quad (166)$$

Up to sign convention, the functions $w_q$, $y_q$ and $z_q$ are the physical stresses scaled to the fiducial value $\mu_t \Omega_*$. Now set $q = 1$. Actually, it turns out that $w$ is independent of $q$, once we zero out the constant of integration, i.e. toss out the unphysical homogeneous solution, which is ruled out by the imposed boundary conditions. In any case,

$$w_1(x; \vartheta) = 2\vartheta x^{-2}. \quad (167)$$

Define $\tilde{x} \equiv x/(\hat{s} \vartheta)$. The remaining solutions may be found by using integrating factors to turn the differential equations into exact differentials. The solution for $y_1$ is that

$$y_1(x; \vartheta, \hat{s}) = \frac{1}{x} \left[ 2 - \tilde{x} e^{\tilde{x}} E_1(\tilde{x}) \right] \quad (168)$$

where $E_1$ is the exponential integral,

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt. \quad (169)$$

The solution for $z_1$ is

$$z_1(x; \vartheta, \hat{s}) = 2\vartheta^{-1} \tilde{x}^2 e^{\tilde{x}} \left[ -\frac{1}{x} E_1(\tilde{x}) + \Gamma(-1, \tilde{x}) + 2\Gamma(-2, \tilde{x}) - \frac{1}{\tilde{x}^2} \Gamma(-3, \tilde{x}) \right], \quad (170)$$

where $\Gamma(a, x)$ is the upper incomplete Gamma function,

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \, dt. \quad (171)$$

Again, in the cases of $y_1$ and $z_1$ just as for $w_1$, there is also a homogeneous solution that comes from the constant of integration that arises upon integrating the exact differential obtained by multiplying the differential equations by the integrating factor. These homogeneous solutions can again be tossed out, using the imposed boundary conditions at large $R$.

Note that for $R \gg R_*$ and for small $\hat{s}$ and vanishing $\vartheta$, the solution above approaches the purely azimuthal flow-field ($v_z = 0$) solution

$$M^{RR} = M_{(RR)} = 0, \quad (172)$$

$$RM^{R\phi} = M_{(R\phi)} = \mu_t R\Omega', \quad (173)$$

$$R^2M^{\phi\phi} = M_{(\phi\phi)} = 2\mu_t s R^2 (\Omega')^2. \quad (174)$$

Because of the particular flow-field chosen in which $v^z = 0$, upper and lower streamsurfaces of this flow are parallel to the $z = 0$ plane, plus, vertical integration to find $R$, $\vartheta$ and $\Omega$ is straightforward. These quantities are discussed and shown in the main text.

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