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Some comments on the universal constant in DSR

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Abstract. Deformed Special Relativity is usually presented as a deformation of Special Relativity accommodating a new universal constant, the Planck mass, while respecting the relativity principle. In order to avoid some fundamental problems (e.g. soccer ball problem), we argue that we should switch point of view and consider instead the Newton constant $G$ as the universal constant.

1. Introduction
There are hopes that experiments (like GLAST, or LHC if there are extra dimensions) will soon measure quantum gravitational (QG) effects. Ideally, in order to derive precise predictions for the next experiments, we would like to integrate out all the QG fluctuations (around the flat metric – assuming that the cosmological constant is zero) to obtain an effective action for matter encoding QG effects

\[ S = \int d\phi_M dg \, e^{i \int L_M(\phi_M, g) + L_{GR}(g)} = \int d\phi_M e^{i \int L_M(\phi)} , \]

where $g$ is the metric and $\phi_M$ represent all the matter and gauge fields other than gravity. $L_M(\phi_M, g)$, $L_{GR}(g)$ are respectively the lagrangian for matter and gravity. The new effective lagrangian $L_M(\phi_M)$ takes into account the QG fluctuations. We naturally expect non-trivial effects on the dynamics for matter including a mass renormalisation, higher derivatives, nonlinearities and deformation of the energy conservation law.

This program can be carried out explicitly in three space-time dimensions in the spinfoam formalism. It was shown that the effective dynamics for matter coupled to 3d QG is indeed a Deformed Special Relativity (DSR) \cite{Girelli}. The four-dimensional case is of course much harder, but heuristic arguments again point towards DSR (e.g \cite{Girelli2, Girelli3}). More generally, DSR is usually formulated as implementing the Planck mass $M_P$ as a universal constant the same way that $c$ is a universal speed in Special Relativity (SR) \cite{Girelli4}. However, in the same way that $c$ is a maximal speed in SR, $M_P$ becomes naturally a maximal mass (or more generically an energy bound). This leads to confusion since the Planck mass can not be a maximal bound for macroscopic systems. Even in formulations of DSR where $M_P$ is not a maximal bound, we run into the problem that DSR effects grow with the mass of the system: a macroscopic system with a large mass will not behave classically as expected. This is usually called the “soccer ball” problem and is one of the main obstacles to the physical interpretation of DSR.

We would like to argue here that a proper parallel between DSR and SR tells us that we should consider as the universal constant not $M_P$ but the Newton constant $G$ instead (or more
exactly $G/c^2 = g$). This in particular leads us to consider as fundamental a 5d space which is known to contain the DSR structure [5, 6]. We further argue along the lines that DSR is an effective theory for matter coupled to gravity and show that some DSR features already emerge when taking into account that particles locally deform the flat geometry into the Schwarzschild metric.

2. $G$ as a universal constant
2.1. Renormalisation and choice of units

It is well known that coupling constants run when considering quantum corrections¹ and QFT renormalisation. This applies in particular to the Newton constant $G$. This can be explicitly seen when applying Renormalisation Group technics to gravity [7]: $G(k)$ is a function of the cutoff $k$. The latter is chosen according to the physical situation.

In the search of phenomenology of QG, one would like to construct a theory encoding in an effective way the variations of $G$. When considering the Newton’s potential, we can always change point of view and consider a variable mass with a fixed $G$.

$$G(k) \frac{m}{r} \rightarrow G \frac{m(k)}{r}.$$  

In this sense we can take different points of view: either the mass is fixed with $G$ fluctuating or the mass is fluctuating with $G$ constant. This can be interpreted as a choice of units². The framework describing matter with variable mass has already been studied by Bekenstein [10]. To do physics, one needs to pick up a units system. One can first choose a unit system which is independent of any particle data: the Planck units system, $M_P$, $L_P$, $T_P$. If there is a (spinless) particle present, one can also introduce different units systems depending on some of the particle features. For example if we consider a quantum particle the Compton (or de Broglie if massless) units would be relevant, $L_C$, $M_C = \frac{\hbar}{L_P}$, $T_C = c^{-1} L_C$.

First let us choose the Planck unit system. The fundamental constants $\hbar$, $G$, $c$ are then in this units system $c = L_P T_P^{-1}$, $\hbar = M_P L_P c$, $G = L_P M_P^{-1} c^2$. Since the Planck units are fixed, the fundamental constants do not run in this unit system. We need also to express the characteristic of the particle in this unit system. The typical scale of the particle is given by $L_C$ which is expressed in Planck unit $L_C = \chi L_P$. Since $L_P$ is the minimum length we need to have $\chi > 1$. It is natural to have the same speed of light in the Planck unit system or the particle unit system³, so that the time unit is just expressed as $T = \chi T_P$. In this sense we are simply doing a conformal transformation which can be local or global. Keeping $\hbar$ also fixed, the mass of the particle is then $m = \chi^{-1} M_P$. If $\chi$ is just a global transformation, we are just doing a general rescaling. However if the transformation is local, we have a particle with a varying mass $m$ with respect to the Planck units. $\chi$ can be also a more complicated function, for example depend on the particle momentum.

$$m = \chi^{-1} M_P = \left( \alpha + \sum_{n=1} \beta_n \frac{p^n}{M_P^n} \right) M_P, \text{ where } \alpha M_P = m_0, \text{ the rest mass.}$$  

Conversely, one can use the Compton unit system. The mass of the particle is then fixed since it is the unit. However if $\hbar$ and $c$ stay fixed in this choice of units, the Newton constant $G$ becomes

¹ $G$ could run also "classically" if one believes in Modified Newtonian Dynamics: $G$ can be seen as dependent on the radial distance $r$ [8].
² Peracci has emphasized that the choice of units is important when dealing with QG and can be related to a 5d picture [9].
³ Note that this could be relaxed, allowing then to have a variable speed of light [11]. The transformation for $G$ would then be also more complicated.
variable: $G' = \chi^2 L_P^3 M_P^{-1} \chi^{-2} T_P^{-2} = \chi^2 G$. In the particle units system, we have a fixed mass but a variable $G$. The idea is therefore to interpret $\chi$ as encoding the QG fluctuations in terms of the cutoff $k$, $G' = G(k) = \chi^2 G$. If this cutoff is related to the particle momentum then we can expect to have $\chi$ as in (2) [12].

The Planck unit system seems to be more natural since it is independent on any matter content, it is universal. In this unit choice, $G$ is universal whereas the mass becomes a variable. We interpret this variability as encoding the QG fluctuations.

An important comment is now of order: $M_P$ is the fundamental unit, however this does not mean that this should be always a maximum mass. Indeed for a quantum particle, $M_P$ can be seen as a maximum bound. On the contrary, it is easy to see that if one considers a classical particle, $M_P$ is then a minimum mass. In this sense, the Planck mass can be seen as encoding the transition from quantum to classical. It surely can not be taken as a universal maximum mass for any body, whereas the Newton constant $G$ is truly universal for any system in this choice of units.

2.2. The effective field theory viewpoint
We can further look at this issue from the effective field theory perspective. Computing loop corrections due to gravity in the standard quantum field theory context, the $\hbar$ and $G$ factors usually combine to give a perturbative expansion in $E/E_P$. The Planck energy/mass appears as a maximal mass scale if one is to make sense of such an expansion. In this context, a DSR theory with a $M_P$ mass/energy bound looks completely natural and the conjecture is indeed that a certain resumming of some of these QG corrections can be interpreted as amplitudes of a DSR field theory with a $\kappa$-deformed Poincaré symmetry for a deformation parameter $\kappa = M_P$.

We should however keep in mind that the apparent cut-off at $E_P$ is not a physical cut-off, but an artefact of the chosen perturbative expansion. For energies larger than $E_P$, we expect non-perturbative effects. For instance, energies larger than $E_P$ will not only induce Planck scale fluctuations around the flat metric but will deform the space-time on larger scales. We would also need to take into account that the systems will actually start to gravitate. Nevertheless $E_P$ is not a physical bound but the energy cut-off of the perturbative expansion. From the effective field theory point of view, one can change the cut-off to a different value $\kappa \geq E_P$ by integrating out the relevant degrees of freedom. Applying this logic to DSR, we could imagine starting from a DSR quantum field theory based at a given deformation parameter $\kappa$ (e.g the Planck mass). Studying bound states with a mass larger than $\kappa$, we can hope that these bound states behave more and more classically as their mass increases and are described by a DSR field theory with a renormalised deformation parameter $\kappa'$. However, extracting the effective physics of bound states of a quantum field theory is not an easy task.

This is the "soccer ball" problem of DSR: showing that macroscopic systems behave classically despite the non-commutativity of space-time at the microscopic scale. The parameter $\kappa$, usually set to the Planck mass $M_P$, should only be a cut-off, which gets renormalised according to the considered physical situation. It should not be considered as a universal constant: having allowed $\kappa$ to vary, the "soccer ball" problem simply disappears. The universal constant is then the Newton constant $g = G/c^2 = L_P/M_P$. The issue then becomes how to determine the cut-off $\kappa$ in term of the physical situation. This is similar to determining the renormalisation scale $\mu$ in QFT depending on which experiment we do.

2.3. DSR as a semi-classical regime of QG
In four space-time dimensions, $M_P$ and $L_P$ depend explicitly on the Planck constant: $M_P \sim \sqrt{\frac{\hbar}{G}}$, $L_P \sim \sqrt{\hbar G}$. In the semi-classical limit defined by $\hbar \to 0$, both Planck scales vanish. In
particular, we do not have a universal mass scale in that semi-classical regime\(^4\). Therefore we cannot expect DSR with a fixed universal mass scale \(\kappa = M_P\) to provide an effective description for quantum gravity in the semi-classical limit. DSR would instead correspond to a regime where \(\hbar\) stays of the same order than the Newton constant \(G\), i.e a regime where gravitational effects and quantum fluctuations are of the same magnitude. This behavior is expected at the Planck scale but not at the macroscopic scale. DSR with fixed deformation parameter \(\kappa = M_P\) seems fit to describe quantum gravity effects at a microscopic scale but obviously can not apply to macroscopic objects in the semi-classical limit. The issue then becomes: which is the limit between microscopic and macroscopic, i.e which physical objects/particles are described by DSR?

More precisely, as \(\hbar\) varies and is sent to 0, keeping \(G\) fixed, \(M_P\) and \(L_P\) do not remain fixed and finite but it is their ratio \(M_P/L_P = \frac{\hbar^2}{G}\) does. Indeed \(M_P\) and \(L_P\) are both in \(\sqrt{\hbar}\): as \(\hbar\) varies, the mass scale \(M_P\) goes linearly the length scale \(L_P\). Thus we expect the semi-classical regime of quantum gravity to be described by an extended DSR theory with a variable parameter \(\kappa\) running linearly with the length scale. In this context, the key to the renormalization of the mass scale \(\kappa\) is that it depends on the Planck constant \(\hbar\). This allows an interesting relation between the DSR “soccer ball” problem and the quantum-classical transition. Indeed, for large composite systems, the effective Planck constant gets smaller as the system grows bigger and therefore behaves more and more classically. In DSR, this leads to a renormalisation of \(\kappa\). This can be clearly seen when looking at the non-commutativity for a composite system: DSR predicts a non-commutativity of the space-time coordinates controlled by the mass scale \(\kappa\). Denoting the space-time coordinates \(X\) and the Lorentz generators \(J\), a generic commutation reads as:

\[
\left[ \hat{X}, \hat{X} \right] = i\hbar \left( \frac{1}{\kappa} \hat{X} - \frac{\hbar}{\kappa^2} \hat{J} \right),
\]

where \(a, b\) are arbitrary dimensionless constants. The important is that, when dealing with many copies of the same system, the average space-time coordinates \(\langle X \rangle_N \equiv (X^{(1)} + \ldots + X^{(N)})/N\) do not have the same non-commutativity:

\[
\left[ \langle \hat{X} \rangle_N, \langle \hat{X} \rangle_N \right] = i\hbar \left( \frac{1}{N\kappa} \langle \hat{X} \rangle_N - \frac{\hbar}{N^2\kappa^2} \hat{J} \right).
\]

We therefore obtain a renormalization of the mass scale \(\kappa \rightarrow N\kappa\) with the size \(N\) of the system. This is a crude argument, assuming that the \(N\) components do not strongly interact. Nevertheless, it shows that we do expect the non-commutativity of space-time to get renormalized when considering composite systems: the fundamental non-commutativity, at the Planck scale for instance, will be different from the effective non-commutativity felt by macroscopic objects.

**3. Extending space-time to a space-time-mass**

The great novelty of Einstein’s Special Relativity is the concept of space-time, which marks the unification of time with the three usual space dimensions. Indeed the speed of light \(c\) is a universal constant defined for all observers and allows the unification of time with space: an event is now described by a single 4-vector \(x_\mu \equiv (ct, x_i)\). Then the interpretation of \(c\) as a maximal speed implies the existence of a light cone, which endows space-time with a non-degenerate flat (pseudo-)metric:

\[
v = \frac{dx}{dt} \leq c \Leftrightarrow c^2 dt^2 - dx^2 \geq 0.
\]

Here, we would like to go one step further. We would like to argue that an effective theory for the kinematics and dynamics of matter coupled to (quantum) gravity can be formulated in

\(^4\) This is an essential difference between 3d quantum gravity where one has \(M_P \sim G^{-1/2}, L_P \sim \hbar G\).
term of an extension of special relativity to a five-dimensional space, where the fifth component can be identified as a function of the mass $m$. In this sense the mass itself would become a variable and relative entity.

We assume the fundamental constant characterizing the gravitational interaction, the Newton constant $G$, to be universal. Using dimensional analysis, $G$ allows to convert a mass $m$ into a quantity with dimension of a length, $x_4 = \frac{G}{c^5} m = gm$. We can then include the mass in a coordinate 5-vector generalizing the standard relativistic 4-vector:

$$x_A \equiv (ct, x_i, gm).$$

We expect this new five-dimensional Space-Time-Mass (STM) to become relevant when taking into account gravitation at an effective level in SR. Similarly to SR, we can further introduce a mass cone structure, that is we put a maximum or minimum bound on $\frac{dx_4}{ds}$, where $s$ is the proper time. Thus we endow our STM with the flat 5d metric $\eta_{AB}$ with signature $(+ - - - -)$:

$$dS^2 \equiv c^2 dt^2 - dx_i dx_i - g^2 dm^2 = ds^2 - dx_4^2.$$

Note that the $g$ is a function of $G$ and so is extremely small, explaining that we do not see any mass fluctuations a priori. As an extension of SR, we propose to call this formalism Extended Special Relativity (ESR). Note also that a similar construction has been proposed in [13].

At this stage the physical interpretation of this mass variable $m$ still has to be decided. A natural candidate is the ADM mass $m$. In this case the mass cone can be interpreted as representing the Schwarzschild ratio or the maximum energy “density” $\frac{c^2}{G} \leq \frac{c^2}{GM} = \frac{1}{2}g^{-1}$. Choosing this interpretation leads to consider a physical object to be time-like with respect in terms of the 5d metric.

On the other hand, a DSR particle is usually formulated as having a space-like trajectory in 5d space. Indeed it was shown in [6] that the different formulations for the DSR particle can all be obtained as different gauge fixings of the same 5d action:

$$S_{5d} = \int \pi^A dy_A - \lambda (\pi^A x_A + \kappa^2) - \mu (x_4 - M),$$

where $\mu$ and $\lambda$ are Lagrange multipliers and $M$ encodes the 4d rest mass. This looks in contradiction with the space-time-mass introduced above. However, the DSR phase space coordinates $(y_A, \pi^A)$ are not straightforwardly related5 to the standard space-time coordinates $(x_\mu, p^\mu)$, so that we can not identify the newly introduced $x_A$ coordinates to the usual DSR coordinates $y_A$: the DSR equivalent of $m$ still has to be deciphered.

The 5d point of view allows a simple proposal to avoid the DSR problems such soccer ball problem, spectator problem or non associativity as shown in [14]. Indeed, since we work with a flat five-dimensional space with a theory invariant under the 5d Poincaré symmetry, it is natural to consider the $\pi_A$ as the fundamental momenta and assume the the law of addition of momenta $\pi$ to be trivial. The total momentum of a composite is then simply defined as the sum of the individual 5d momenta. Considering two particles, the total momentum is then

5 The different DSR formulations give different prescriptions for the relation between the $(y, \pi)$ and the $(x, p)$. The Snyder basis is the simplest one and defines the $x$ coordinates as a simple rescaling of the $y$’s [6]:

$$p_\mu = \frac{\pi_\mu}{\pi_4}, \quad x_A = \frac{\pi_A}{\kappa} y_A.$$  

Then the action reads $\pi^A dy_A = (p^\mu dx_\mu - \kappa dx_4) - (p^\mu y_\mu - \kappa x_4) dy_4$ and the space-like condition $\pi_A x^A < 0$ simply means that the rest mass is bounded, $p_\mu p^\mu < \kappa^2$. Work on the precise relation with the space-time-mass is still under investigation [16].
\[ \pi^{\text{tot}} = \pi^{(1)} + \pi^{(2)}. \] Assuming that the two particles have the same “5d mass”, \( \kappa_1 = \kappa_2 = \kappa \), then \( \kappa_{\text{tot}} \) will generically be different from \( \kappa \):

\[
\kappa_{\text{tot}}^2 = -(\pi^{\text{tot}})^2 = -(\pi^{(1)} + \pi^{(2)} )^2 = \kappa_1^2 + \kappa_2^2 - 2\pi^{(1)} \pi^{(2)} .
\]

This is the same as when dealing with the masses of particles in SR. This gives a precise implementation of the \( \kappa \) renormalisation for DSR in the framework of the STM. In this sense an Extended Special Relativity (ESR) based on the five-dimensional space-time-mass and using a 5d momenta seems to be the correct framework to formulate consistently DSR.

4. DSR phenomenology from gravitational effects

The point of view that DSR is an effective theory for (quantum) gravity has been already advocated from many point of views (see for instance [1, 2, 3]). We would like to discuss here how simple gravitational effects naturally lead to DSR-like structures.

In three space-time dimensions, gravity is a topological theory and particles are topological defects deforming space-time as conical singularities. In the quantum gravity context, this feedback of matter on the geometry lead to a renormalisation of the mass, a deformed non-commutative addition of momenta, a deformed Poincaré symmetry, and allowed to show that the effective dynamics of particles (and more generally matter fields) is described by DSR[1].

In four space-time dimensions, gravity can be reformulated as an almost-topological field theory and we could model particles as topological defects in a first order approximation. Maybe this would lead to DSR just as in 3d. Here, we follow the simpler alternative to consider that the particle deforms the flat space-time into the Schwarzschild metric. We will see that makes apparent some DSR features and also explain the renormalisation of the \( \kappa \) deformation parameter.

4.1. Mass

Gravity is an interaction and we need to take its energy into account for the definition of the mass of a system. More generally, the mass of a system is not an easy concept to define in general relativity. We can for example consider the Brown-York formula which computes the energy contained in a region of radius \( R \) around the particle [17]. This takes into account the self-energy of the particle:

\[
m = M(R) - g \frac{M(R)^2}{2R} \Rightarrow M(R) = g^{-1} R \left( 1 - \sqrt{1 - \frac{R_S}{R}} \right),
\]

where \( R_S = 2gm \) is the Schwarzschild radius and \( m \) the ADM mass (seen by an observer at infinity). This formula is valid from \( R \geq R_S \) to \( R \to \infty \). This renormalised mass decreases from \( 2m \) at \( R = R_S \) down to \( m \).

We see that the notion of mass becomes relative. Then which mass is the relevant one? Should we normalize the particle’s momentum to \( m \) or \( M(R) \) depending on the observer?

4.2. Maximum mass

We should also take into account the gravitational potential in the total rest energy of a many-particles system. For two particles (at rest), the resulting total mass is then:

\[
M = m_1 + m_2 - g \frac{m_1 m_2}{r},
\]

where \( r \) is the distance between the two particles. Keeping the length scale \( r \) fixed, this deformed law of addition of masses leads to a maximal mass \( m_{\text{max}} = g^{-1} r \): we derived from simple
principles the existence of a maximal mass scale $m_{\text{max}} \equiv \kappa$. Moreover this mass bound $\kappa$ is not fixed but gets renormalised with the size of the system. Actually it scales linearly with the length scale as expected in the semi-classical regime of DSR. Then if we fix the length scale to the Planck scale, $r = L_P$, we recover as expected a mass bound $\kappa = M_P$.

We can try to make this argument more precise in the context of general relativity. Then the mass bound of DSR-ESR is saturated by the highest density state, i.e black holes. Looking at how to “add” black holes with each other, we can deduce how the maximal bound should get renormalized. Starting with two black holes, the Hawking area theorem states from the resulting total horizon area is at least the sum of the two initial areas. Actually, according to the holographic principle, the area counts the entropy of the black hole. Thus assuming a weak interaction and no additional degrees of freedom, the entropy will be additive. Therefore considering than the horizon area scales as the mass squared $m^2$, the total mass resulting from the merging of two black holes of mass $m$ will be $m_{\text{tot}} = m\sqrt{2}$. This is significatively less than the naive $2m$. The difference of energy is actually due to the gravitational field which absorbs some energy during the process (and deforms the space-time). This is generalizable to a system of $N$ particles of the same size. Assuming that their individual mass is bounded by the same mass scale $\kappa$, the mass bound on the system of $N$ particles will be $\kappa^{(N)} = \kappa\sqrt{N}$ corresponding to $N$ black holes of mass $\kappa$ merging into a single bigger black hole.

At the end of the day, we see that general relativity predicts a natural renormalisation of the mass bound $\kappa$ with the size of the system. In a first approximation, this scaling goes as $\sqrt{N}$. From the 5d point of view, assuming the simple additivity of the 5-momentum $\pi_A$ we see from equation (6) that such a scaling corresponds in the case of a two-particle system to the special configuration when the 5-momenta of the particles will be orthogonal to each other, $\pi^{(1)} \cdot \pi^{(2)} = 0$.

4.3. Modification of momenta addition
Following the same logic as above, gravity will interact with the particles and contribute to the energy/momentum of coupled systems. It will already modify the conserved quantities at the classical level. Therefore, we naturally expect to obtain deformed addition for the energy and momenta if one sticks to the notion of energy/momentum defined without interaction.

To actually derive precisely the addition of momentum, we would need to study the insertion of two massive particles in a (flat) space-time i.e how two Schwarzschild metrics ”merge into” a single metric. This is not an easy task since we do not know how to solve exactly the dynamics of two black holes in general relativity. Thus we need to identify in which regime (strongly or weakly coupled) we would like to work and develop an effective mechanics of black holes in general relativity. More precisely, we should study the merging process in more details and derive the total energy-momentum corresponding to a system of two black holes. It will certainly not be the sum of the energy-momenta of the two black holes and we will obtain a deformation of the law of addition of momenta generalizing the mass addition law which we discussed above. This would provide us with an explicit proposal for a DSR-ESR addition law of momenta.

4.4. 5d formulation
Finally, it would be interesting to be able to rigorously derive the fifth dimension from GR. A natural way to generate a fifth dimension is to consider the renormalization flow of GR, the fifth dimension being the energy scale. As argued for example in [9], it is very natural to endow this resulting 5d space with the AdS metric. Here, we show how a fifth dimension can emerge from a simple point of view.

Let us consider a particle in space-time and call its 4-momentum $p_\mu$, as hypothetically measured by an observer at infinity (if the particle was the sole matter content of the space-time). Then it is natural to assume that a real observer will actually measure a momentum $\tilde{p}_\mu = \alpha p_\mu$
where the coefficient $\alpha$ depends on the observer. $\alpha$ can come from a non-trivial conformal factor or a mass renormalisation or simply the slowing down of the clock due to gravity\cite{16}. In all cases, the parameter $\alpha$ reflects some dynamical effect due to gravity. Thinking of $\alpha$ as a degree of freedom independent from the $p_\mu$'s, it seems natural to introduce a 5-vector:

\begin{equation}
\pi_4 = \alpha \kappa, \quad \pi_\mu = \bar{p}_\mu = \alpha p_\mu,
\end{equation}

where $\kappa$ is a mass scale introduced for the sole purpose to provide $\pi_4$ with the dimension of a moment. This natural parametrisation is similar to the Snyder's basis. Indeed, the 4-momentum is expressed in terms of the 5-momentum as $p_\mu = \kappa \pi_\mu / \pi_4$. In fact on shell we can see that they definitely coincide. Choosing the mass shell to be $\pi_A \pi^A = \pm \kappa^2$, we obtain

\begin{equation}
\alpha = \frac{\pm 1}{\kappa^2}.
\end{equation}

According to the choice of sign of the mass shell condition we have a maximum or minimum mass. Now one needs to further identify the gravitational degree(s) of freedom responsible for $\alpha$ and this will determine the precise addition law for the 5-momentum $\pi_\mu$ and thus the deformed addition law for the 4-momentum $p_\mu$.

References

[1] Freidel L and Livine E R 2006 Effective 3d quantum gravity and non-commutative quantum field theory Phys. Rev. Lett. 96 221301 (Preprint hep-th/0512113)
[2] Amelino-Camelia G, Smolin L and Starodubtsev A 2004 Quantum symmetry, the cosmological constant and Planck scale phenomenology Class. Quant. Grav. 21 3095; Grelli F, Livine E R and Orioli D 2005 Deformed special relativity as an effective flat limit of quantum gravity Nucl. Phys. B 708 411 (Preprint gr-qc/0406100)
[3] Aloisio R, Galante A, Grillo A et al 2005 Deformed special relativity as an effective theory of motion on quantum gravitational backgrounds Preprint gr-qc/0510031
[4] Amelino-Camelia G 2003 Testable scenario for relativity with minimum-length Phys. Lett. B 510 255 (Preprint hep-th/0012238); do. 2002 Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale Int. J. Mod. Phys. D 11 35
[5] Kowalski-Glikman J 2002 De Sitter space as an arena for doubly special relativity Phys. Lett. B 547 291 (Preprint hep-th/0207279)
[6] Grelli F, Konopka T, Kowalski-Glikman J et al 2006 The free particle in deformed special relativity Phys. Rev. D 73 045008 (Preprint hep-th/0512107)
[7] Reuter M 1998 Nonperturbative evolution equation for quantum gravity Phys. Rev. D 57 971 (Preprint hep-th/9805030); Don D and Percacci R 1998 The running gravitational couplings Class. Quant. Grav. 15 3149 (Preprint hep-th/9707239)
[8] Sanders R H and McGaugh S S 2002 Modified Newtonian dynamics as an alternative to dark matter Ann. Rev. Astron. Astrophys. 40 263 (Preprint astro-ph/0204521)
[9] Percacci R 2004 The renormalization group, systems of units and the hierarchy problem Preprint hep-th/0409199
[10] Bekenstein J D 1977 Are particle rest masses variable? Theory and constraints from solar system experiments Phys. Rev. D 15 1458
[11] Albrecht A and Maggiore M 1999 A time varying speed of light as a solution to cosmological puzzles Phys. Rev. D 59 043516 (Preprint astro-ph/9811018)
[12] Grelli F, Liberati S, Percacci R et al 2006 Modified dispersion relations from the renormalization group of gravity Preprint gr-qc/0607030
[13] Wesson P S Space-Time-Mass (Singapore: World Scientific)
[14] Grelli F and Livine E R 2005 Deformed special relativity: problems and solutions Proceeding of the Second International Workshop DICE2004, Bras. J. Phys. 35 432 (Preprint gr-qc/0412079); do. 2004 Physics of deformed special relativity: principle of relativity revisited Preprint gr-qc/0412004
[15] Snyder H 1947 Quantized spacetime Phys. Rev. Lett. 71 38
[16] Grelli F, Livine E R, Orioli D et al work in preparation
[17] Brown J D and York J W 1993 Quasilocal energy and conserved charges derived from the gravitational action Phys. Rev. D 47 1407