A GENERIC IDENTIFICATION THEOREM FOR GROUPS OF FINITE MORLEY RANK

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ABSTRACT

The paper contains a final identification theorem for the ‘generic’ $K^*$-groups of finite Morley rank.

1. Introduction

This paper belongs to a series of publications concerned with the Cherlin–Zil’ber conjecture in model-theoretic algebra.

Conjecture 1.1 (Cherlin–Zil’ber). A simple infinite group of finite Morley rank is a simple algebraic group over an algebraically closed field.

A general discussion of the subject can be found in the book [14] and the survey article [13].

The classification project of infinite simple groups of finite Morley rank came to a fork (called the even–odd fork) after a result by Altinel, Borovik and Cherlin in [3]. In its stronger version proved later by Jaligot [19], it says that Sylow 2-subgroups in a minimal counterexample $G$ to the Cherlin–Zil’ber conjecture are either infinite, or have properties similar to properties of a Sylow 2-subgroup in a simple algebraic group of characteristic 2 (in which case $G$ is said to be of even type), or that of a simple algebraic group of odd or zero characteristic (then we say $G$ is of odd type).

Most of the articles written on the subject after that are either an ‘even-type article’ (such as [1, 4–6, 10]) or an ‘odd-type article’ (such as [9, 12]). In particular, two distinct identification theorems based on different techniques appeared for the ‘generic’ groups of each type [9, 10].

The aim of this work is to show that there can be one identification theorem that combines these two sister theories which are very different in nature. This unifying theorem generalises [9], and considerably shortens some arguments in [12], but it is very different from [10] in the technique used. Besides its uniform approach, it has the advantage of avoiding references to Tits’s classification of buildings, a very difficult result with a long proof which still exists only in Tits’s original written account [25].

A group of finite Morley rank is said to be of $p'$-type, if it contains no infinite abelian subgroup of exponent $p$. Notice that a simple algebraic group over an algebraically closed field $K$ is of $p'$-type if and only if $\text{char } K \neq p$. Other definitions can be found in the next section.

The aim of this work is to prove the following.
Theorem 1.2. Let $G$ be a simple $K^*$-group of finite Morley rank and $D$ a maximal $p$-torus in $G$ of Prüfer rank at least 3. Assume the following.

(A) $(C^G_0(x) \mid x \in D, |x| = p) = G$.

(B) For every element $x$ of order $p$ in $D$, the group $C^G_0(x)$ is of $p'$-type and $C^G_0(x) = F^\circ(C^G_0(x))E(C^G_0(x))$.

Then $G$ is a Chevalley group over an algebraically closed field of characteristic distinct from $p$.

Notice that, under assumption (B) of Theorem 1.2, $C^G_0(x)$ is a central product of $F^\circ(C^G_0(x))$ and $E(C^G_0(x))$.

This theorem can be viewed as an identification theorem for 'generic groups' of finite Morley rank. Indeed, if the Cherlin–Zil’ber conjecture is true and infinite simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields (and therefore, Chevalley groups in view of the classification of simple algebraic groups), then the only series not covered by Theorem 1.2 are $A_1$, $A_2$, $B_2$ and $G_2$.

The proof of Theorem 1.2 is given in Section 3.

The following two corollaries (which will be published elsewhere) put Theorem 1.2 in the context of the previous research on the subject.

Theorem 1.3. Let $G$ be a simple tame $K^*$-group of finite Morley rank and odd type. Assume that the Prüfer 2-rank of $G$ is at least 3. Then $G$ is isomorphic to a Chevalley group over an algebraically closed field of characteristic $\neq 2$.

Theorem 1.3 refines and replaces the ‘trichotomy theorem’ of [12] as the main branching point of the theory of tame groups of odd type.

The maximum of Prüfer $p$-ranks of divisible abelian $p$-subgroups in the normalisers of connected non-trivial 2-subgroups of $G$ is called the 2-local $p$-rank of $G$.

Theorem 1.4. Let $G$ be a simple $K^*$-group of finite Morley rank and even type. Assume that, for an odd prime $p$, the 2-local $p$-rank of $G$ is at least 3. Then $G$ is isomorphic to a Chevalley group over an algebraically closed field of characteristic 2.

Notice that Theorem 1.4 does not use the tameness assumption.

Theorem 1.4 naturally fits in the classification of simple $K^*$-group of finite Morley rank and even type as developed in [4–6, 8]. To describe its context, we need a few definitions. Let $G$ be a simple $K^*$-group of finite Morley rank and even type and $S$ the connected component of a Sylow 2-subgroup in $G$. A minimal parabolic subgroup $P$ is any definable connected subgroup which properly contains $N^G_S$ and is minimal with this property. Let $\mathcal{M}(S)$ be the set of minimal parabolic subgroups. If $|\mathcal{M}(S)| \leq 2$ then, as shown in [7], $G$ is isomorphic to one of the groups $\text{SL}_2(K)$, $\text{SL}_3(K)$, $\text{Sp}_4(K)$, $G_2(K)$ over an algebraically closed field $K$ of characteristic 2. If, on the contrary, $|\mathcal{M}(S)| \geq 3$, it can be shown that the conditions of Theorem 1.4 are satisfied and, consequently, $G$ is a Chevalley group over an algebraically closed field of characteristic 2. This identification theorem is different from that of [10] and, as already mentioned, has the advantage of not using the classification of buildings [25].
1.1. Definitions

All definitions related to groups of finite Morley rank in general can be found in [14].

From now on $G$ is a group of finite Morley rank. The group $G$ is called

1. a $K$-group, if every infinite simple definable and connected section of the
group is an algebraic group over an algebraically closed field;

2. a $K^*$-group, if every proper definable section of $G$ is a $K$-group;

3. tame, if it does not contain a bad group as a proper definable section and
does not interpret bad fields.

Here, a bad group is an infinite connected non-solvable group of finite Morley
rank of which every proper connected definable subgroup is nilpotent. A bad field
is a structure $\langle K; +, \cdot, A \rangle$ of finite Morley rank consisting of an algebraically closed
field $\langle K; +, \cdot \rangle$ and (a unary predicate for) a proper infinite multiplicative subgroup
$A < K^*$. Notice that a recent result by Frank Wagner [26] makes the existence of
bad groups in positive characteristic highly unlikely; if there exists a bad field of
characteristic $p > 0$ then the number of $p$-Mersenne prime numbers (that is, prime
numbers of the form $(p^n - 1)/(p - 1)$) is finite.

A $p$-torus $S$ is a direct product of finitely many copies of the quasi-cyclic group
$\mathbb{Z}_{p^\infty}$. The number of copies is called the Prüfer $p$-rank of $S$ and is denoted by $\text{pr}(S)$.
For a definable group $H$, $\text{pr}(H)$ is the maximum of the Prüfer ranks of $p$-subgroups
in $H$. It is easy to see that the Prüfer $p$-rank of any subgroup of a group of finite
Morley rank is finite.

A group $H$ is called quasi-simple if $H' = H$ and $H/Z(H)$ is simple and non-
abelian. A quasi-simple subnormal subgroup of $G$ is called a component of $G$. The
product of all components of $G$ is called the layer of $G$ and denoted by $L(G)$, and
$E(G)$ stands for $L^2(G)$. It is known (see [14, Lemma 7.6, Lemma 7.10]) that $G$ has
finitely many components and that they are definable and are normal in $E(G)$.

$F(H)$ is the Fitting subgroup of $H$, that is, the maximal normal definable
nilpotent subgroup.

If $H$ is a group of finite Morley rank then $B(H)$ is the subgroup generated by
all definable connected 2-subgroups of bounded exponent in $H$. Note that $B(H)$ is
connected by Assertion 2.3.

$O_2(H)$ is the maximal normal definable connected 2-subgroup of $H$.

$O(H)$ is the maximal normal definable connected subgroup of $H$ without
involutions.

2. Background material

2.1. Algebraic groups

For a discussion of the model theory of algebraic groups, the reader might like
to refer to [9, Section 3.1]. The basic structural facts and definitions related to
algebraic groups can be easily found in the standard references such as [15, 18].

First note that a connected algebraic group $G$ is called simple if it has no
proper normal connected and closed subgroups. Such a group turns out to have
a finite centre, the quotient group being simple as an abstract group. The classical
classification theorem for simple algebraic groups states that simple algebraic groups
over algebraically closed fields are Chevalley groups, that is, groups constructed
from Chevalley bases in simple complex Lie algebras, as described, for example, in [15].

Now fix a maximal torus $T$ in a connected algebraic group $G$ and denote the corresponding root system by $\Phi$, and for each $\alpha \in \Phi$, denote the corresponding root subgroup by $X_\alpha$. The subgroup $\langle X_\alpha, X_{-\alpha} \rangle$ is known to be isomorphic to $\text{SL}_2$ or $\text{PSL}_2$ and is called a root $\text{SL}_2$-subgroup.

If $G$ is simple, the roots can have at most two different lengths, and the terms ‘short root $\text{SL}_2$-subgroup’ and ‘long root $\text{SL}_2$-subgroup’ have the obvious meanings.

A simple algebraic group is generated by its root $\text{SL}_2$-subgroups. In a simple algebraic group, all long root $\text{SL}_2$-subgroups are conjugate to each other, and similarly all short root $\text{SL}_2$-subgroups are conjugate to each other.

**Assertion 2.1.** Suppose that $G$ is a simple algebraic group over an algebraically closed field. Let $T$ be a maximal torus in $G$ and $K$, $L$ closed subgroups of $G$ that are isomorphic to $\text{SL}_2$ or $\text{PSL}_2$ and are normalised by $T$. Then the following hold.

1. Either $[K, L] = 1$ or $\langle K, L \rangle$ is a simple algebraic group of rank 2, that is, of type $A_2$, $B_2$ or $G_2$.
2. The subgroups $K$ and $L$ are embedded in $\langle K, L \rangle$ as root $\text{SL}_2$-subgroups.
3. If $\langle K, L \rangle$ is of type $G_2$, then $G = \langle K, L \rangle$.

**Proof.** The proof follows from the description of closed subgroups in simple algebraic groups normalised by a maximal torus [21, 2.5]; see also [22, Section 3.1].

**Assertion 2.2.** Let $G$ be a simple algebraic group over an algebraically closed field of characteristic $\neq p$, and let $D$ be a maximal $p$-torus in $G$. Then $C_G(D)$ is a maximal torus in $G$.

**Proof.** The proof follows from the description of centralisers of subgroups of commuting semisimple elements in simple algebraic groups [24, Theorem 5.5.8].

2.2. Groups of finite Morley rank

**Assertion 2.3** (Zil’ber’s indecomposability theorem). A subgroup of a group of finite Morley rank which is generated by a family of definable connected subgroups is also definable and connected.

**Proof.** See [27] or [14, Corollary 5.28].

**Assertion 2.4** [14, Theorem 8.4]. Let $G \rtimes H$ be a group of finite Morley rank, where $G$ and $H$ are definable, $G$ is an infinite simple algebraic group over an algebraically closed field and $C_H(G) = 1$. Then $H$ can be viewed as a subgroup of the group of automorphisms of $G$, and moreover $H$ lies in the product of the group of inner automorphisms and the group of graph automorphisms of $G$ (which preserve root lengths). In particular, when $H$ is connected, then $H$ consists of inner automorphisms only.
Assertion 2.5 [2]. Suppose that \( G \) is a group of finite Morley rank, \( G = G' \), and \( G/Z(G) \) is a simple algebraic group over an algebraically closed field, and is of finite Morley rank, then \( Z(G) \) is finite and \( G \) is also algebraic.

Lemma 2.6. Let \( G \) be a connected \( K \)-group of \( p' \)-type and \( D \) a maximal \( p \)-torus in \( G \). If \( L \triangleleft G \) is a component in \( G \), then \( D \cap L \) is a maximal \( p \)-torus in \( L \) and \( D = C_D(L)(D \cap L) \).

Proof. As \( G \) is connected, \( L \triangleleft G \). Now the lemma immediately follows from Assertions 2.5, 2.4 and 2.2.

Lemma 2.7. Under the assumptions of Theorem 1.1, we have, for every \( p \)-element \( t \in D \),

\[
C_G^\circ(t) = F \cdot L_1 \ldots L_r,
\]

where \( F = F^\circ(C_G^\circ(t)) \) and each \( L_i \) is a simple algebraic group over an algebraically closed field of characteristic \( \neq p \).

Proof. For every \( p \)-element \( t \) in \( G \), \( C_G^\circ(t) = F \cdot E(C_G^\circ(t)) \), and the \( K^* \)-assumption ensures that \( C_G^\circ(t) \) is a \( K \)-group, hence its components are algebraic groups by Assertion 2.5.

2.3. Lyons’s theorem

A detailed discussion of this particular version of Lyons’s theorem can be found in [9].

Assertion 2.8 (Lyons [16, 17]). Suppose that \( \mathbb{F} \) is an algebraically closed field, \( I \) is one of the connected Dynkin diagrams of the simple algebraic groups of Tits rank at least 3 and \( \tilde{G} \) is the simply connected simple algebraic group of type \( I \) over \( \mathbb{F} \). Let \( G \) be an arbitrary group and for each \( i \in I \), \( K_i \) stand for a subgroup of \( G \) which is centrally isomorphic to \( \text{PSL}_2(\mathbb{F}) \), and \( T_i \subset K_i \) denote a maximal torus in \( K_i \). Also assume that the following statements hold.

1. The group \( G \) is generated by \( K_i \) where \( i \in I \).
2. For all \( i, j \in I \), \([T_i, T_j] = 1\).
3. If \( i \neq j \) and \((i, j)\) is not an edge in \( I \), then \([K_i, K_j] = 1\).
4. If \((i, j)\) is a single edge in \( I \), then \( G_{ij} = [K_i, K_j] \) is isomorphic to \((P)\text{SL}_3(\mathbb{F})\).
5. If \((i, j)\) is a double edge in \( I \), then \( G_{ij} = [K_i, K_j] \) is isomorphic to \((P)\text{Sp}_4(\mathbb{F})\). Moreover, in that case, if \( r_i \in N_{K_i}(T_iT_j) \) and \( r_j \in N_{K_j}(T_iT_j) \) are involutions, then the order of \( r_i r_j \) in \( N_{G_{ij}}(T_iT_j)/T_iT_j \) is 4.
6. For all \( i, j \in I \), \( K_i \) and \( K_j \) are root \( \text{SL}_2 \)-subgroups of \( G_{ij} \) corresponding to the maximal torus \( T_iT_j \) of \( G_{ij} \).

Then there is a homomorphism from \( \tilde{G} \) onto \( G \), under which the root \( \text{SL}_2 \)-subgroups of \( \tilde{G} \) (for some simple root system) correspond to the subgroups \( K_i \).

2.4. Reflection groups

A linear semisimple transformation of finite order is called a reflection if it has exactly one eigenvalue which is not 1.
Theorem 2.9. Let $W$ be a finite group and assume that the following statements hold.

1. There is a normal subset $S \subseteq W$ consisting of involutions and generating $W$.
2. Over $\mathbb{C}$, $W$ has a faithful irreducible representation of dimension $n \geq 3$ in which involutions from $S$ act as reflections.
3. For almost all prime numbers $q$, $W$ has faithful irreducible representations (possibly of different dimensions) over fields $\mathbb{F}_q$. Moreover, for every such representation, involutions in $S$ act as reflections.

Then $W$ is one of the groups $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ for $n \geq 3$.

Proof. A proof, based on the classification of irreducible complex reflection groups [23], can be found in [9].

3. Proof of Theorem 1.2

The strategy is to construct the Weyl group and the root system of $G$, and then to apply Lyons’s theorem. From now on $G$ is a simple $K^*$-group of finite Morley rank and $D$ is a maximal $p$-torus in $G$ of Prüfer rank $\geq 3$. We also assume that $C_G^o(x)$ is of $p'$-type for every element $x \in D$ of order $p$, $C_G^o(x) = F^o(C_G^o(x))E(C_G^o(x))$ and $G = \langle C_G^o(x) \mid x \in D, \ |x| = p \rangle$.

We shall systematically use the following observation.

Lemma 3.1. $F^o(C_G^o(x))$ centralises $D$ for every element $x \in D$ of order $p$.

Proof. The result immediately follows from the fact that a $p$-torus in a definable nilpotent group belongs to the centre of this group [14, Theorem 6.9].

3.1. Root subgroups

From now on, $\text{SL}_2$ will be used instead of $\text{SL}_2(\mathbb{F})$, etc. Denote by $\Sigma$ the set of all definable subgroups isomorphic to $(P)\text{SL}_2$ and normalised by $D$. These are our future root $\text{SL}_2$-subgroups. If $N$ is a subgroup of $G$ which is normalised by $D$, then set $H_N := C_N(D \cap N)$. Note that if $K \in \Sigma$, then $H_K$ is an algebraic torus in $K$.

Lemma 3.2. The set $\Sigma$ is non-empty.

Proof. Assume the contrary. If $L = E(C_G^o(x)) \neq 1$ for some element $x$ of order $p$ from $D$, then $L$ being a central product of simple algebraic groups, contains a definable $\text{SL}_2$-subgroup normalised by $D$. Therefore $C_G^o(x) = F^o(C_G^o(x))$ centralises $D$ by Lemma 3.1. Then $G = \langle C_G^o(x) \mid x \in D, \ |x| = p \rangle$ centralises $D$, which contradicts the assumption that $G$ is simple.

Lemma 3.3. Let $K, L \in \Sigma$ be distinct and set $M = \langle K, L \rangle$. Then the following statements hold.

1. The subgroup $C_D(K) \cap C_D(L) \neq 1$ and $M$ is a $K$-group.
2. Either $K$ and $L$ commute or $M$ is an algebraic group of type $A_2, B_2$ or $G_2$. 
(3) $D \cap M = (D \cap K)(D \cap L)$ is a maximal $p$-torus in $M$.

(4) If $K$ and $L$ do not commute then $H_M$ is a maximal algebraic torus of the algebraic group $M$, and $K$ and $L$ are root $\text{SL}_2$-subgroups of the algebraic group $M$ with respect to the maximal torus $H_M$.

(5) For all $K, L \in \Sigma$, we have $[H_K, H_L] = 1$.

(6) For any $K, L \in \Sigma$, if the $p$-tori $D \cap K$ and $D \cap L$ have intersection of order $> 2$, then $K = L$.

**Proof.** For (1) we refer the reader to the proof of [9, Lemma 9.3].

(2)–(3) For $L \in \Sigma$ set $R_L = C_D^o(L)$. If $n = \text{pr}(D)$, then the Prüfer $p$-rank of $R_L$ is $n - 1$. Note that since $D$ is maximal and $D$ centralises a $p$-torus in $L$, $D \cap L$ is a maximal $p$-torus in $L$. Now let $x$ be an element of order $p$ in $R_K \cap R_L$. Then $K, L \leq E(C_G(x))$ by the assumptions of the theorem. Set $E = E(C_G(x))$.

It follows from Lemma 2.6 that the subgroup $D \cap E$ is a maximal $p$-torus of $E$, and the subgroups $K$ and $L$, being $D$-invariant, lie in components of $E$. If $K$ and $L$ belong to different components of $E$, then they commute. Otherwise the component $A$ that contains both $K$ and $L$ is a simple algebraic group, and moreover $D \cap A$ is a maximal $p$-torus in $A$. Hence the results follow from Assertion 2.1.

(4)–(6) These follow by inspecting case by case and Assertion 2.1.

**LEMMA 3.4.** The subgroups in $\Sigma$ generate $G$.

**Proof.** Let $x \in D$ be of order $p$, then by assumption (B) of the theorem

$$C_G^o(x) = F \cdot L_1 \ldots L_n,$$

where $F = F^o(C_G^o(x))$ and $L_i \leq C_G^o(x)$ is a simple algebraic group, for each $i = 1, \ldots, n$.

The first step is to prove that $L_1 \ldots L_n \leq \langle \Sigma \rangle$. Note that $D \leq C_{\Sigma_i}^o(x)$ and $D \cap L_i$ is a maximal $p$-torus in $L_i$ by Lemma 2.6. Let $H_i$ stand for the maximal algebraic torus in $L_i$ containing $D \cap L_i$ and $\Gamma_i$ be the collection of root $\text{SL}_2$-subgroups in $L_i$ normalised by $H_i$. Since $D \cap L_i \leq H_i$, $D \cap L_i$ normalises the subgroups in $\Gamma_i$. By Lemma 2.6, we have $D = C_{\Gamma_i}(L_i)(D \cap L_i)$; hence $D$ normalises $\langle \Gamma_i \rangle = L_i$, that is, $\Gamma_i \subseteq \Sigma$ for each $i = 1, \ldots, n$. This proves the first step.

Hence for each $x \in D$ of order $p$, $C_G^o(x) = F \cdot E(C_G^o(x)) \leq F(\langle \Sigma \rangle) \leq C_G(D)\langle \Sigma \rangle$.

Therefore

$$G = \langle C_G^o(x) \mid x \in D, |x| = p \rangle \leq C_G(D)\langle \Sigma \rangle.$$

Since $C_G(D)$ normalises $\langle \Sigma \rangle$, we have $\langle \Sigma \rangle \leq G$. Now the result follows, since $G$ is simple.

We make $\Sigma$ into a graph by taking $\text{SL}_2$-subgroups $L \in \Sigma$ for vertices and connecting two vertices $K$ and $L$ by an edge if $K$ and $L$ do not commute.

**LEMMA 3.5.** The graph $\Sigma$ is connected.

**Proof.** Otherwise consider a decomposition $\Sigma = \Sigma' \cup \Sigma''$ of $\Sigma$ into the union of two non-empty sets such that no vertex in $\Sigma'$ is connected to a vertex in $\Sigma''$. Then we have

$$G = \langle \Sigma \rangle = \langle \Sigma' \rangle \times \langle \Sigma'' \rangle,$$

which contradicts the assumption that $G$ is simple.
Lemma 3.6. If $L \in \Sigma$ then $L = E(C_G(C_D(L)))$.

Proof. Let $\text{pr}(D) = n$, then $\text{pr}(C_G(C_D(L))) = n$ and $\text{pr}(E(C_G(C_D(L)))) = 1$. Since $L \leq E(C_G(C_D(L)))$ and $E(C_G(C_D(L)))$ is a central product of simple algebraic groups over algebraically closed fields of characteristics $\neq p$, we immediately conclude that $L = E(C_G(C_D(L)))$. \qed

3.2. Weyl group

Recall that when $L \in \Sigma$, $H_L$ stands for the maximal algebraic torus $H_L := C_L(D \cap L)$ in $L \cong \text{SL}_2$. Now set $H = \langle H_L \mid L \in \Sigma \rangle$ and call it the natural torus associated with $D$. It easily follows from Lemma 3.3(5) that $H$ is a divisible abelian group.

For any $L \in \Sigma$, $W(L) := N_L(H)L/H = N_L(H_L)L/H$ is the Weyl group of $\text{SL}_2$ and has order 2; hence $W(L)$ contains a single involution, which will be denoted by $r_L$. Note that the subgroup $L$ is uniquely determined by $r_L$, since $C_D(L) = C_D^p(r_L)$ and $L = E(C_G(C_D(L)))$ by Lemma 3.6.

Lemma 3.7. Consider a graph $\Delta$ with the set of vertices $\Sigma$, in which two vertices $K$ and $L$ are connected by an edge if $[r_K, r_L] \neq 1$. If $K$ and $L$ belong to different connected components of $\Delta$, then $[K, L] = 1$.

Proof. It suffices to check this statement in the subgroup $M = \langle K, L \rangle$, where it is obvious by Lemma 3.3(2). \qed

Notice that $D$ is a $p$-torus, the subgroups $N_G(D)$ and $C_G(D)$ are definable and the factor group $N_G(D)/C_G(D)$ is finite. Set $W := N_G(D)/C_G(D)$. The images of involutions $r_L$, for $L \in \Sigma$, in $W$ generate a subgroup which we denote by $W_0$. Since, by their construction, involutions $r_L$ normalise $D$, there is a natural action of $W_0$ on $D$.

Lemma 3.8. The $p$-torus $D$ lies in the natural torus $H$. In particular, $D$ is the Sylow $p$-subgroup of $H$.

Proof. Set $D' = \langle D \cap L \mid L \in \Sigma \rangle$. It suffices to prove that $D' = D$. First note that $D' \leq D \cap H$. If $D' < D$ then, since $[D, r_L] = D \cap L$, all involutions $r_L$ act trivially on the factor group $D/D'$ which is divisible. Let us take an element $d \in D$ which has sufficiently big order so that the image of $d^{|W_0|}$ in $D/D'$ has order at least $p^2$. Then the element

$$z = \prod_{w \in W_0} d^w$$

has the same image in $D/D'$ as $d^{|W_0|}$ and thus $z$ has order at least $p^2$. Since $D = C_D(L)(D \cap L)$ and $|C_D(L) \cap (D \cap L)| \leq |Z(L)| \leq 2$, we see that $|C_D(r_L) : C_D(L)| \leq 2$. Of course, the equality is possible only if $p = 2$. In any case, since $z \in C_D(r_L)$, $z_p \in C_D(L)$ for all $L \in \Sigma$ and $z_p \neq 1$. Hence $z_p \in C_G(\Sigma) = Z(G)$. This contradiction shows that $D = D'$ and $D \leq H$. \qed

Lemma 3.9. $N_G(D) = N_G(H)$. 

Lemma 3.10. \( C_G(D) = C_G(H) \).

Proof. Let \( x \in C_G(D) \), then, for every \( L \in \Sigma \), \( x \) centralises \( C_D(L) \) and thus, by Lemma 3.6, normalises \( L = E(C_G(C_D(L))) \). Since \( x \) centralises a maximal \( p \)-torus \( D \cap L \) of \( L \), it centralises the maximal torus \( H_L = C_L(D \cap L) \). Hence \( x \in C_G(H) \). This proves that \( C_G(D) \leq C_G(H) \). The reverse inclusion follows from Lemma 3.8. \( \square \)

In view of Lemmas 3.9 and 3.10, we can refer to \( W_0 \) either as the subgroup generated by the images of involutions \( r_L \) in the factor group \( N_G(D)/C_G(D) \) or as the subgroup generated by the images of involutions \( r_L \) in \( N_G(H)/C_G(H) \). Also, we now know that the group \( W_0 \) acts on \( D \) faithfully.

3.3. Tate module

Now the aim is to construct a \( \mathbb{Z} \)-lattice on which \( W_0 \) acts as a crystallographic reflection group. For that purpose we shall associate with \( D \) the Tate module \( T_p \).

It is constructed in the following way.

Let \( E_p^k \) be the subgroup of \( D \) generated by elements of order \( p^k \). Notice that every \( r_L \) acts on \( E_p^k \) as a reflection, that is, \( E_p^k = C_{E_p^k}(r_L) \times [E_p^k, r_L], [E_p^k, r_L] \leq D \cap L \) is a cyclic group and \( r_L \) inverts every element in \( [E_p^k, r_L] \).

Consider the sequence of subgroups

\[
E_p^1 \hookrightarrow E_p^2 \hookrightarrow E_p^3 \hookrightarrow \ldots
\]

linked by the homomorphisms \( x \mapsto x^p \). The projective limit of this sequence is the free module \( T_p \) over the ring \( \mathbb{Z}_p \) of \( p \)-adic integers. The action of \( W_0 \) on \( D \) can be lifted to \( T_p \), where it is still an irreducible reflection group. By construction, \( T_p/pT_p \) is isomorphic to \( E_p \), as a \( W_0 \)-module. Notice also that \( W_0 \) acts on the tensor product \( T_p \otimes_{\mathbb{Z}_p} \mathbb{C} \) as a (complex) reflection group, and that the dimension of \( T_p \otimes_{\mathbb{Z}_p} \mathbb{C} \) over \( \mathbb{C} \) coincides with the Prüfer \( p \)-rank of \( D \), hence is at least 3.

3.4. More reflection representations for \( W_0 \)

Now let us focus on odd primes \( q \neq p \). Consider the elementary abelian \( q \)-subgroups \( E_q \) generated in \( H \) by all elements of the fixed prime order \( q \). For the sake of complete reducibility of the action of \( W \) on \( E_q \), one can consider only \( q > |W| \).

Lemmas 3.11, 3.15 and 3.16 below are similar to some results in [9]. We include the proofs here for the sake of completeness of exposition.

Lemma 3.11 [9, Lemma 9.7]. Let \( N = N_G(H) \), then \( C_N(E_q) = C_G(H) \).

Proof. It is clear that \( C_G(H) \leq C_N(E_q) \). To see the converse, let \( x \in C_N(E_q) \). Since \( x \in N \), it acts on the elements of \( \Sigma \) by conjugation.

First let us prove that \( x \) normalises each subgroup in \( \Sigma \). To get a contradiction, assume that there is some subgroup \( L \in \Sigma \) such that \( L^x \neq L \). Then by Lemma 3.3,
L and \( L^x \) either commute or generate a semisimple group as root \( SL_2 \)-subgroups. Hence \(|L \cap L^x| \leq 2\). This gives a contradiction since \( q \) is an odd prime and \( L \cap E_q = L^x \cap E_q \leq L \cap L^x \).

Hence for each \( L \in \Sigma, L^x = L \) and \( x \) acts on \( H \cap L \) as an element from \( N_L(H \cap L) \), since \( SL_2 \) does not have any definable outer automorphisms. Note that the Weyl group of \( SL_2 \) is generated by an involution which inverts the torus \( H \cap L \). Since \( x \) centralises \( E_q \cap H \), \( x \) centralises \( H \cap L \) for each \( L \in \Sigma \), and hence \( x \) centralises \( H = \langle H \cap L \mid L \in \Sigma \rangle \) and \( x \in C_G(H) \). This proves the equality. \( \square \)

Now notice that \([E_q, r_L] \) is generated by a \( q \)-element in \( H_L \) and thus has order \( q \). Hence \( E_q \) is a finite dimensional vector space over \( \mathbb{F}_q \) on which \( W_0 \) acts as a group generated by reflections.

**Lemma 3.12.** The group \( W_0 \) acts irreducibly on \( E_q \).

**Proof.** Note that \( W_0 \) acts on \( E_q \) faithfully by Lemma 3.11. Since \( q > |W| \), the action of \( W_0 \) on \( E_q \) is completely reducible. If the action is reducible, then we can write \( E_q = E' \oplus E'' \) for two proper \( W_0 \)-invariant subspaces.

Assume that \( W_0 \) acts trivially on one of the subspaces \( E' \) or \( E'' \), say on \( E' \). If \( L \in \Sigma \), then \( E_q = C_{E_q}(L) \times (E_q \cap L) \), and, obviously, \( C_{E_q}(L) = C_{E_q}(r_L) \). Hence all \( L \in \Sigma \) centralise \( E' \) and \( E' \leq C_G(\langle \Sigma \rangle) = Z(G) = 1 \). Therefore \( W_0 \) acts nontrivially on both \( E' \) and \( E'' \).

For \( L \in \Sigma \), the \(-1\)-eigenspace \([E_q, r_L] \) of \( r_L \) belongs to one of the subspaces \( E' \) or \( E'' \) and hence \( r_L \) acts as a reflection on one of the subspaces \( E' \) or \( E'' \) and centralises the other. Set \( \Sigma' = \{ L \in \Sigma \mid [E_q, r_L] \leq E' \} \) and \( \Sigma'' = \{ L \in \Sigma \mid [E_q, r_L] \leq E'' \} \). It is easy to see that \([r_K, r_L] = 1 \) for \( K \in \Sigma' \) and \( L \in \Sigma'' \). By Lemma 3.7, \( K \) and \( L \) commute for all \( K \in \Sigma' \) and \( L \in \Sigma'' \), which contradicts Lemma 3.5. Hence \( W_0 \) is irreducible on \( E_q \). \( \square \)

**Lemma 3.13.** The group \( W_0 \) acts irreducibly on \( T_p \otimes_{\mathbb{Z}_p} \mathbb{C} \).

**Proof.** The proof is analogous to that of the previous lemma. \( \square \)

### 3.5. Root system

The aim of this subsection is to construct a root system on which \( W_0 \) acts as a crystallographic reflection group. The existence of such a root system is guaranteed by the following lemma.

**Lemma 3.14.** There exists an irreducible root system on which \( W_0 \) acts as a crystallographic reflection group.

**Proof.** Recall that \( n \geq 3 \). By Theorem 2.9, the quotient group \( W_0 \) is one of the crystallographic reflection groups \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 \) and acts on the corresponding root system. \( \square \)

Let now \( R = \{ r_i \mid i \in I \} \) be a simple system of reflections in \( W_0 \). We shall identify \( I \) with the set of nodes of the Dynkin diagram for \( W_0 \). It is well known that every reflection in an irreducible reflection group \( W_0 \) is conjugate to a reflection in \( R \).
Lemma 3.15 [9, Lemma 9.9]. Every reflection \( r \in W_0 \) has the form \( r_K \) for some \( \text{SL}_2 \)-subgroup \( K \in \Sigma \).

Proof. Let \( r \in W_0 \) be a reflection. Working our way back through the construction of the module \( T_p \), one can easily see that the Prüfer \( p \)-rank of \([D, r]\) is 1.

Let \( r_L \in W_0 \) be a reflection which corresponds to a \( \text{SL}_2 \)-subgroup \( L \in \Sigma \). By comparing the Prüfer \( p \)-ranks of the groups \( C_D(r_L) \) and \( C_D(r) \), we see that \( Z = (C_H(r_L) \cap C_H(r))^\circ \) has Prüfer \( p \)-rank at least 1. Hence the subgroup \( \langle L, H, r \rangle \) contains a non-trivial central \( p \)-torus; also note that \( \langle L, H, r \rangle \) is a \( K \)-group. It is well known that a finite irreducible reflection group contains at most two conjugate classes of reflections. Therefore, after replacing \( r \) and \( r_L \) by their appropriate conjugates in \( W_0 \), we can assume without loss of generality that the images of \( r_L \) and \( r \) in \( W_0 \) correspond to adjacent nodes of the Dynkin diagram. Now we can easily see that \( \langle L, H, r \rangle = Y * Z \) for some simple algebraic group \( Y \) of Lie rank 2, and that \( r = r_K \) for some root \( \text{SL}_2 \)-subgroup \( K \) of \( Y \) such that \( K \in \Sigma \). \( \square \)

3.6. Final step

The next task is to prove that the conditions of Lyons’s theorem (Assertion 2.8) are satisfied.

Lemma 3.16 [9, Lemma 9.10]. Attach an \( \text{SL}_2 \)-subgroup \( L_i \in \Sigma \) to each vertex of the Dynkin diagram \( I \) in such a way that the simple reflection \( r_i \) corresponding to this vertex is \( r_{L_i} \) in \( W_0 \). Then the following statements hold.

(1) \( |L_i, L_j| = 1 \) if and only if \( |r_i r_j| = 2 \).

(2) \( \langle L_i, L_j \rangle \) is isomorphic to \( (P)\text{SL}_3 \) if and only if \( |r_i r_j| = 3 \).

(3) \( \langle L_i, L_j \rangle \) is isomorphic to \( (P)\text{Sp}_4 \) if and only if \( |r_i r_j| = 4 \).

(4) \( L_i \) and \( L_j \) are embedded in \( \langle L_i, L_j \rangle \) as root \( \text{SL}_2 \)-subgroups.

Proof. It is well known that for each \( i, j \in I \), the order \( |r_i r_j| \) takes the values 2, 3 or 4, in a Dynkin diagram of type \( A_n, B_n, C_n, D_n, E_6, E_7, E_8 \) or \( F_4 \). By Lemma 3.3, \( L_i \) and \( L_j \) either commute or generate \( (P)\text{SL}_3, (P)\text{Sp}_4 \) or \( G_2 \). However \( \langle L_i, L_j \rangle \cong G_2 \) is not possible in our case since \( |r_i r_j| = 6 \) does not occur in \( I \).

The ‘only if’ parts of (1) and (2) are easy to see. In the case of part (3), that is when \( L_i \) and \( L_j \) generate \( (P)\text{Sp}_4 \), we have to show that \( |r_i r_j| \neq 2 \). To get a contradiction, assume that \( L_i \) and \( L_j \) generate \( (P)\text{Sp}_4 \) and \( |r_i r_j| = 2 \). However, then \( L_i \) and \( L_j \) are both short root \( \text{SL}_2 \)-subgroups. Note that \( r_i \) and \( r_j \) are simple reflections, and it can be checked by inspection that one of them must be a long reflection. This proves the ‘only if’ part of (3). Now parts (1), (2) and (3) follow from Lemma 3.3 and the previous discussion. Part (4) is a direct consequence of Lemma 3.3. \( \square \)

Lemma 3.17. Each subgroup in \( \Sigma \) is isomorphic to \( (P)\text{SL}_2(\mathbb{F}) \) for the same field \( \mathbb{F} \).

Proof. By Lemma 3.5 any two subgroups of \( \Sigma \) are connected by a sequence of edges. Note that each pair which is connected by a single edge generates a simple group of Lie rank 2 by Lemma 3.3(2), hence their underlying fields coincide. Thus the underlying fields of any two subgroups in \( \Sigma \) coincide. \( \square \)
Finally, we are in a position to apply Lyons’s theorem. Set $G_0$ to be the subgroup of $G$ generated by the subgroups $L_i$ for $i \in I$. By Lyons’s theorem, $G_0$ is a simple algebraic group over $\mathbb{F}$ with the Dynkin diagram $I$. Its Weyl group, with respect to the torus $T$, is $W_0$, hence $G_0$ contains all subgroups from $\Sigma$. Therefore by Lemma 3.4, $G_0 = G$ is a Chevalley group over $\mathbb{F}$. Since $G$ is of $p'$-type, $\mathbb{F}$ is of characteristic different from $p$. □

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