The Max $k$-Cut Game: 
On Stable Optimal Colorings *

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Abstract

We study the max $k$-cut game on an undirected and unweighted graph in order to find out whether an optimal solution is also a strong equilibrium. While we do fail to show that, by proving an alternate formula for computing the cut value difference for a strong deviation, we show that optimal solutions are 7-stable equilibria.

Furthermore, we prove some properties of minimal subsets with respect to a strong deviation, showing that each of their nodes will deviate towards the color of one of their neighbors and that those subsets induce connected subgraphs.

1 Introduction

In many contexts, in order to obtain a good income a group of agents might have to choose from a pool of strategies one that is chosen by the least amount of other individuals, or even by no one. This kind of scenario is known as the classic anti-coordination game. The most important aspect of this class of games is related to the fact that some players will end up with a lower profit if there are more choices than players. Generally an individual might only care about the choice of a smaller number of other individuals rather than all of them. Additionally, a given player may behave in different and not necessarily reciprocal way with specific individuals. In order to account for this differentiation, this problem can be modelled via graphs, by attributing to each link the interest of an individual for another.

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Anti-coordination games are very important since they can be fruitfully used to tackle with several real-world problem, such as radio towers communicating via radio signals when only few frequencies are available, companies deciding which type of product to produce, employees learning different skills, miners deciding which area they should drill for resources, airplanes selecting flight paths, and so on.

The anti-coordination game on a graph can also be seen as an extension to game theory of the max $k$-cut problem, i.e. the problem of partitioning the nodes of a graph in $k$ subsets in such a way the sum of the weights of the links between nodes belonging to the same subset is minimized. The max $k$-cut problem has been extensively studied for the great interest on its real-world applications. It has been proven that finding an optimal solution for it is a known NP-complete problem [19]. Modern approaches involve turning the problem into a semi-definite program and either using constraint relaxations alongside suitable search algorithms, such as cut-and-branch and branch-and-bound, to obtain an exact solution. Other approaches use approximation algorithms [24, 25] or a heuristic-based ones [21, 22] in order to find a good enough solution, trading accuracy for computational time. See [23] for a quick excursus on the subject and [20] for a more thorough exploration.

In a sense, the max $k$-cut game asks whether players can achieve a social optimum by themselves (or, equivalently, maximize the cut value by themselves), rather than forcing individuals by an external regulator, or whether an optimal solution found via a search algorithm can be overturned by a joint deviation of some of the nodes, considered as autonomous agents.

It has been proved that this approach fails if the underlying graph is directed or weighted: strategy profiles that maximize the cut value on an undirected weighted graph are not able to withstand “sabotage" by more than 3 players [3], while the same type of strategy profiles on directed graphs might not be a pure Nash equilibrium at all [8].

It is currently unknown whether optimal strategy profiles for undirected unweighted graphs are strong equilibria or not.

**Related work**  Bramoullé worked on an anti-coordination game on an undirected unweighted graph with two strategies where each node plays a two-player anti-coordination game with each of its neighbors, receiving a payoff based on a pre-determined payoff matrix like in the picture below [9].

$$
\begin{pmatrix}
\pi_{AA} & \pi_{AB} \\
\pi_{BA} & \pi_{BB}
\end{pmatrix}
$$

(1)

More specifically, he proved that mixed Nash equilibria minimize frustration, defined as the sum of the number of edges between players that don’t anti-coordinate weighted by the loss induced by not anti-coordinating, that minimizing frustration is NP-hard and some properties on core-periphery graphs, interconnected communities and (somewhat) dense graphs and their complements. In his doctorate dissertation, Höfer
studied many types of games on graphs, among which the max \( k \)-cut game \[^6\]. He showed that the price of stability of this game is 1 and, thus, this game always has a pure Nash equilibrium. Moreover he proved that finding an optimal Nash equilibrium is computationally hard and that the price of anarchy of this game has a lower bound equal to \( \frac{k-1}{k} \).

This direction has been further pushed by Gourvès and Monnot in their articles \[^3\] and \[^4\], where they proved that, for undirected weighted graphs, optimal colorings are 3-stable equilibria but not necessarily 4-stable equilibria, that every Nash equilibrium is a \( \frac{1}{k-1} \)-approximate strong equilibrium, and that the strong price of anarchy of this game is \( \frac{2k-2}{2k-1} \). Additionally, Kun, Powers and Reyzin \[^8\] proved that the bound found by Höfer was tight, that a (not necessarily optimal) Nash equilibrium could be found in polynomial time and that determining whether a max \( k \)-cut game on a directed graph had a Nash equilibrium was NP-hard.

Carosi, Fioravanti, Gualà and Monaco \[^1\] showed that, on undirected unweighted graphs, optimal colorings are 5-stable equilibria and 1-local-strong equilibria (i.e. resistant to deviations by complete subgraphs), that the cut value is not a strong potential function and that, if the graph has some particular properties, an optimal coloring is a strong equilibrium, as it has been defined as in \[^15\] and in \[^16\].

Smorodinski and Smorodinski \[^11\] extended this game to hypergraphs, by examining two possible extensions of the payoff function: in the first one a node \( N \) is rewarded based on how many hyperedges \( N \) is incident to, which contain some nodes that are differently colored with respect to \( N \). In the second extension, the reward of node \( N \) is based on how many hyperedges \( N \) is incident to, which contain only nodes that are differently colored with respect to \( N \). They compute the price of anarchy for games with the first payoff on \( r \)-minimal hypergraphs and bounds for games with the other payoff on \( r \)-uniform hypergraphs.

Looking beyond the topic of max \( k \)-cut games, these games are also related to the coordination games studied by Apt, Simon et alia \[^12\] \[^13\] \[^14\]. In this type of game, a node receives a payoff equal to the number of its neighbors that share its color.

In the first one \[^12\], they showed that, for undirected unweighted graphs, under list colorings, Nash equilibria and 2-stable equilibria always exist, but \( j \)-stable equilibria, with \( j \geq 3 \), might not , that strong equilibria always exist on color forests and pseudoforests and that the problem of determining whether a strategy profile is a \( j \)-stable equilibrium is co-NP-complete.

In the second one \[^13\], they proved that, for directed unweighted graphs, determining whether Nash equilibria exist is NP-complete, while some cases a strong equilibrium always exists and can be found in linear time. In the third one \[^14\], they study many different kinds of directed weighted and unweighted graphs, determining, for each kind, bounds on the improvement paths and c-improvement paths and the complexities of finding a Nash equilibrium and a strong equilibrium.

Höfer and Suri \[^7\] studied the dynamics of network interaction games which are a generalization of the max 2-cut game, and showed that, when each node makes its
move one after another (sequential game), the 2-type interaction game will always converge to a pure Nash equilibrium after at most \((|V| + 1)(|E| + 1)^2\) steps, and that, when all the nodes make their moves at the same time (concurrent game) and adhere to an "inertia"-based approach for convergence, the local interaction game will converge to a Nash equilibrium in expected polynomial time; furthermore, they compare converging times of sequential and concurrent games for the subcases of the coordination game and the max 2-cut game.

Lastly, we want to mention Kearns’ graph coloring games from [17] and [18], which have a payoff function that is similar to the max k-cut’s, except that a node receives a payoff equal to 0 if it shares its color with one of its neighbors.

**Our results**  In this article we will prove some properties of minimal subsets with respect to a strong deviation and that optimal colorings are 7-stable equilibria by exploiting an alternative way to formulate the cut value increase.

More specifically, this article is structured as follows. In Section 2 we state some definitions and some results from [1] that we are going to use in the later sections. Then, in Section 3, we will prove a couple of properties of minimal subsets with respect to a strong deviation and an alternative formula for computing the cut value increase due to a deviation, along with an extension to undirected weighted graphs. In Section 4, by exploiting the formula and (implicitly) the properties, we will prove that optimal colorings are 7-stable equilibria. Lastly, Section 5 deals with issues stemming from the approach used to prove the results in Section 4 and mentions possible work-arounds and future developments.

## 2 Preliminaries

### 2.1 Definitions

An anti-coordination game is a tuple \(\langle V, K, \{f_p\}_{p \in V} \rangle\), where \(V\) is the player set defined as \(V = \{1, \ldots, n(\in \mathbb{N})\}\), with \(n \geq 2\), \(K\) is the strategy set, the same for each player and represented as \(K = \{1, \ldots, k(\in \mathbb{N})\}\), with \(k \geq 2\), and a family of payoff functions (one for each player) defined as follows: \(f_p : K^n \to \mathbb{N}, c \mapsto \left| \{q \in V : q \neq p \text{ and } c_q \neq c_p \} \right|\), where \(c_p\) represents the color chosen by player \(p\). In this sense, the payoff function \(f_p\) is equal to the number of other players who don’t have the same color as \(p\). In the rest of the paper, we will refer to any vector \(c \in K^n\) as a **coloring** of the graph.

Let \(G = \langle V, E \rangle\), with \(E \subseteq \{s \in \mathcal{P}(V) : |s| = 2\}\), be a undirected and unweighted graph with \(n = |V|\) nodes and \(e = |E|\) links:

**Definition 1.** We define the max k-cut game \(G\) on the graph \(G\) by extending the anti-coordination game as follows:

- the set \(V\) as the player set;
• $K$ again as the strategy set for all players, from which we define a strategy profile or coloring as a vector $\beta \in K^n$ containing the strategies chosen by each node (or, alternatively, a function that maps each node $v$ to its color $\beta_v$);

• for each $v \in V$, the payoff function $\mu_v : K^n \rightarrow \mathbb{N}, \beta \mapsto |\{u \in V : \{u, v\} \in E, \beta_u \neq \beta_v\}|$.

Let $C \subseteq V$ and $\sigma, \gamma \in K^n$: we say that $C$ deviates from the coloring $\sigma$ towards the coloring $\gamma$ if, for any node $d \in C$, $\gamma_d \neq \sigma_d$ and, for any node $b \notin C$, $\gamma_b = \sigma_b$; furthermore, we say that $C$ strongly deviates from $\sigma$ towards $\gamma$ if $C$ deviates from $\sigma$ to $\gamma$ and, for any $d \in C$, $\mu_d(\gamma) > \mu_d(\sigma)$ and that $C$ is minimal with respect to strongly deviating from $\sigma$ towards some other coloring $\eta \in K^n$ if $C$ strongly deviates from $\sigma$ to $\eta$ and, for any $D \subseteq C$, for any coloring $\iota \in K^n$, $D$ does not strongly deviate from $\iota$.

Let $\beta \in K^n$ we define a cut of a graph $G$ as a set $E(G, \beta) := \{(i, j) \in E : \beta_i \neq \beta_j\}$. We define the cut value for a coloring $\beta$ over a graph $G$ as $S(G, \beta) := |E(G, \beta)|$ and an optimal coloring as a coloring that maximizes the cut value on $G$ (or, equivalently, $G$). We define the cut difference between two colorings $\alpha$ and $\beta$ on a graph $G$ as $\Delta S(G, \alpha, \beta) := S(G, \alpha) - S(G, \beta)$. If $C \subseteq V$ deviates from $\sigma$ towards $\gamma$, we say that the set of links $E(G, \sigma) \setminus E(G, \gamma)$ leaves the cut and that the set $E(G, \gamma) \setminus E(G, \sigma)$ enters the cut. We will not mention $G$ whenever it is clear that we are working on $C$.

For $q \in [1, n] \subset \mathbb{N}$, we define a $q$-stable equilibrium as a coloring $\sigma$ such that, for any $C \subseteq V$, if $|C| \leq q$, then, for each coloring $\gamma \in K^n \setminus \{\sigma\}$, $C$ does not strongly deviate from $\sigma$ towards $\gamma$. For $q = 1$, we have the Nash equilibrium (which we will denote by NE) for $G$ and, for $q = n$, we have a special equilibrium which we call strong equilibrium.

For $\alpha, \beta \in K^n$ and $C \subseteq V$, we define:

• $P(\alpha, \beta) := |\{(i, j) \in E : \alpha_i = \alpha_j, \beta_i \neq \beta_j\}|$;

• $P_{int}(C, \alpha, \beta) := |\{(i, j) \in E : i, j \in C, \alpha_i = \alpha_j, \beta_i \neq \beta_j\}|$;

• $P_{bound}(C, \alpha, \beta) := |\{(i, j) \in E : i \in C, j \in V \setminus C, \alpha_i = \alpha_j, \beta_i \neq \beta_j\}|$;

• $P_{ext}(C, \alpha, \beta) := |\{(i, j) \in E : i, j \in V \setminus C, \alpha_i = \alpha_j, \beta_i \neq \beta_j\}|$.

By definition, $P(\alpha, \beta) = |E(\beta) \setminus E(\alpha)|$ and, for each $C \subseteq V$, $P(\alpha, \beta) = P_{int}(C, \alpha, \beta) + P_{bound}(C, \alpha, \beta) + P_{ext}(C, \alpha, \beta)$. Also, like with $G$, we will not mention $C$ whenever it is clear that we are working on $C$.

For $v \in V$, we define the degree of the node $v$ as $\delta_v := |\{j \in V : \{v, j\} \in E\}|$ and, for $\beta \in K^n$ and $a \in K$, the degree of $v$ in $\beta$ with respect to the color $a$ as $\delta_v^a(\sigma) := |\{j \in V : \{v, j\} \in E, \sigma_j = a\}|$. For $C \subseteq V$, we define $K_C(\sigma) := \{c \in K : \exists v \in C \text{ such that } \sigma_v = c\}$ and, for $i = 1, \ldots, n$, $C_i(\sigma) = \{v \in C : \sigma_v = c\}$.

For $C \subseteq V$, we call $G(C)$ the subgraph induced by $C$ and, for $H$ subgraph of $G = \langle V, E \rangle$, we say that $V(H)$ is the node set of $H$. Also, we define the neighborhood of the subset $C$ as $N(C) := \{u \in V \setminus C : \exists d \in C \text{ such that } \{u, d\} \in E\}$. We will call
\( K_p \) the complete graph with \( p \) nodes. Lastly, we say that \( H \) is a connected component of the graph \( G \) if \( H \) is a connected subgraph of \( G \) and, for each node \( i \in V(H) \), for each node \( j \notin V(H) \), \( \{i, j\} \notin E \).

### 2.2 Some previous result

**Proposition 1** (Prop. 1a from [1]). Let \( \sigma \) be a NE for \( G \) and let \( C \subseteq V \) be a minimal subset w.r.t strongly deviating from \( \sigma \) to another coloring \( \gamma \); then \( K_C(\sigma) = K_C(\gamma) \).

This is a very strong result that says that no color can be added or removed from the pool of colors used by a deviating minimal subset, and we will use this proposition alongside the pigeon-hole principle to obtain some of the later results.

**Lemma 1** (Prop. 3 from [1]). Let \( \sigma \) be a NE for \( G \) and let \( C \subseteq V \) be a minimal subset w.r.t strongly deviating from \( \sigma \) (to another coloring \( \gamma \)); then, if \( |K_C(\sigma)| \in \{2, |C| - 1, |C|\} \), \( \Delta S(\sigma, \gamma) > 0 \).

**Theorem 1** (Thm. 4 from [1]). Any optimal coloring for a max k-cut game on an unweighted, undirected graph is a pure 5-stable equilibrium.

This was the previous largest stability result and its improvement is the main topic of this article. Also, because of this result, we can focus on pure equilibria and ignore mixed equilibria, since 5-stability implies \( x \)-stability (\( x < 5 \)); most importantly, a pure 5-stable equilibrium is also a pure Nash equilibrium and we will use this fact in order to prove some results.

**Conjecture 1.** Any optimal coloring for a max k-cut game on an unweighted, undirected graph is a strong equilibrium.

This conjecture also would mean that any solution of the max k-cut problem (on the same type of graph) can’t be “overthrown” by any number of nodes that don’t achieve a profit equal to their degree.

Lastly, we are going to mention a result that gives an “upper bound" to a theorem we will prove in the next section.

**Proposition 2** (Prop. 7 from [1]). The cut value is not a strong potential function for the max k-cut game, even if only minimal subsets are allowed to deviate.

### 3 Results

The following propositions deal with a couple of intuitively obvious properties of max k-cut games on graphs whose proof is deceptively complex. Both properties seem unrelated with the main result, but they help reducing the number of possible “configurations" of deviating minimal subsets, that is, they force certain subgraphs and colorings onto deviating minimal subsets in a way that can’t be replicated with non-minimal subsets.

The first proposition shows that minimal deviations induce a “color rearrangement" inside the minimal subset, which is especially useful when the number of colors used
by that subset is almost as large as the number of nodes in that subset.

**Proposition 3.** Let \( \sigma \) be a NE for \( G \) and let \( C \subseteq V \) be a minimal subset w.r.t strongly deviating from \( \sigma \) (to another coloring \( \gamma \)); then, for each node \( b \in C \) such that \( b \neq d \), \( \{b,d\} \in E \) and \( \gamma_d = \sigma_b \).

**Proof.** The statement says that each node of \( C \) deviates towards the color of one of its neighbors in \( C \). As such, the proof will be split into two halves: “each node in \( C \) deviates towards the color in \( \sigma \) of one of its " and “that/those neighbor/s must be in \( C \).

The first statement can be written as: for each node \( u \in C \), \( \delta^{\sigma_u}_u(\sigma) > 0 \). Let then \( t \in C \) be such that \( \delta^{\sigma_t}_t(\sigma) = 0 \); then either \( \delta^{\sigma_u}_t(\sigma) = 0 \) or \( \delta^{\sigma_t}_t(\sigma) > 0 \):

- \( \delta^{\sigma_t}_t(\sigma) = 0 \): in this case \( t \) does not need to improve its payoff as it is already earning its maximum payoff; this contradicts the hypothesis that \( C \) is minimal with respect to strongly deviating from \( \sigma \).
- \( \delta^{\sigma_t}_t(\sigma) > 0 \): since \( \delta^{\sigma_t}_t(\sigma) = 0 \), we have that \( \{u\} \) is minimal with respect to strongly deviating from \( \sigma \), which contradicts the hypothesis that \( \sigma \) is a NE and that \( C \) is minimal with respect to strongly deviating from \( \sigma \).

Therefore, for each node \( u \in C \), \( \delta^{\sigma_u}_u(\sigma) > 0 \). On a sidenote, this holds even if \( t \) deviates towards the color of another node in \( C \) that is not adjacent to \( t \).

As for the other half, let \( h \in C \) such that \( \gamma_h = \sigma_y \), for some \( y \in V \setminus C \), and that, for each node \( b \in C \) such that \( \{b,y\} \in E \), \( \sigma_b \neq \sigma_y \). It is possible that in \( \gamma \) some neighbors of \( h \) might have deviated towards \( \sigma_y \), that is, \( \delta^{\sigma_y}_h(\gamma) \geq 0 \). Since \( C \) strongly deviates, we have that \( \mu_h(\gamma) > \mu_h(\sigma) \), that is, \( \delta^{\sigma_y}_h(\gamma) < \delta^{\sigma_y}_h(\sigma) \).

Now, none of the neighbors of \( h \) have deviated from \( \sigma_y \), which means that \( \delta^{\sigma_y}_h(\gamma) \geq \delta^{\sigma_y}_h(\sigma) \). Furthermore, since \( \sigma \) is a NE, we have that \( \delta^{\sigma_y}_h(\sigma) \geq \delta^{\sigma_u}_u(\sigma) \).

By putting everything together, we have:

\[
\delta^{\sigma_y}_h(\sigma) \geq \delta^{\sigma_y}_h(\sigma) > \delta^{\sigma_y}_h(\gamma) \geq \delta^{\sigma_u}_u(\sigma) \geq 0,
\]

which is impossible.

By uniting the two results, we have that, for each node \( d \in C \), there is a node \( b \in C \) such that \( b \neq d \), \( \{b,d\} \in E \) and \( \gamma_d = \sigma_b \).

The following proposition allows us not to consider non-connected subgraphs in Section 4.

**Proposition 4.** Let \( \sigma \) be a NE for \( G \) and let \( C \subseteq V \) be a minimal subset with respect to strongly deviating from \( \sigma \) (to another coloring \( \gamma \)); then, \( G(C) \) is a connected subgraph of \( G \).

**Proof.** Suppose that \( G(C) \) is not connected and, without loss of generalization, that is split into two connected components \( H \) and \( J \); let \( H \) be such that \( |K_{V(H)}(\sigma)| = 1 \): if \( V(H) = \{h\} \), for some \( h \in C \), then, for each node \( m \in V \) such that \( \{m,h\} \in E \), \( \sigma_m = \gamma_m \), and thus, for each node \( c \in K \), \( \delta^c_h(\sigma) = \delta^c_h(\gamma) \).
If \( \arg\min_{a \in K} \delta_a^h(\sigma) = \sigma_h \), then \( h \) does not actually need to deviate from \( \sigma \), since its payoff would not increase, which means that \( C \setminus (V(H)) \) is minimal with respect to strongly deviating from \( \sigma \); if \( \arg\min_{a \in K} \delta_a^h(\sigma) \neq \sigma_h \), then \( \{ h \} \) is minimal with respect to strongly deviating from \( \sigma \). In both cases we contradict the hypotheses.

Suppose now that \( |V(H)| > 1 \): each node should choose the same color in \( \gamma \) (Prop. 1), yet the previous proposition states that each node deviates towards the color chosen in \( \sigma \) by one of their neighbors in \( C \) (Prop. 3). As \( H \) is a connected component of \( G(C) \), the only color those nodes could deviate towards is their own, which results in a contradiction.

This means that \( |K_{V(H)}(\sigma)| > 1 \).

Suppose now that \( |K_{V(H)}(\sigma)| > 1 \) and \( |K_{V(J)}(\sigma)| > 1 \). The previous proposition states that, for each node \( d \in V(H) \), there is a node \( b \in C \) such that \( b \neq d \), \( \{ b, d \} \in E \) and \( \gamma_d = \sigma_b \), and the same holds for \( V(J) \).

Since \( H \) and \( J \) are connected components of \( G(H) \), there is no link connecting any of the nodes of \( V(H) \) to any of the nodes of \( V(J) \); thus, the statement of the previous proposition becomes: for each node \( d \in V(H)(V(J)) \), there is a node \( b \in V(H)(V(J)) \) such that \( b \neq d \), \( \{ b, d \} \in E \) and \( \gamma_d = \sigma_b \).

This means that both \( V(H) \) and \( V(J) \) could deviate from \( \sigma \) by themselves, that is, \( C \) is not minimal with respect to strongly deviating from \( \sigma \) and both \( V(H) \) and \( V(J) \) are.

This argument can be easily extended to any number of connected components. Thus, \( G(C) \) must be a connected subgraph of \( G \). ■

### 3.1 A formula

The properties of this game leaves one wondering what could happen when a densely connected subgraph were to strongly deviate. It is not hard to imagine that the cut value might not increase because of that (compare with Prop. 2). An example of this is the following lemma, which shows what happens when we discard the assumption of Lemma 1 from [1] that \( \sigma \) is a NE and we extend it to subcliques:

**Lemma 2.** Let \( \sigma \) be a coloring for \( G \) and let \( C \subseteq V \) such that \( K_i \subseteq C \) for some \( i \leq n \) and such that \( C \) strongly deviates from \( \sigma \) to \( \gamma \); then, for \( T := K_i \cap C \), and \( Q := \{ \{ p, q \} \in E : p \in T,q \in N(T) \} \), if \( |T| \leq 2 \), \( |(E(T,\gamma)\setminus E(T,\sigma))| + |(E(Q,\gamma)\setminus E(Q,\sigma))| > 0 \).

**Proof.** Suppose that \( M = |T| \) nodes \((m_1, \ldots, m_M)\) deviate alongside \( C \), that \( \mu_{m_j}(\gamma) - \mu_{m_j}(\sigma) = 1 \), for \( j = 1, \ldots, M \), that each of those \( M \) nodes deviates towards a different color and that, thus, all the links between nodes of \( T \) and nodes of \( K_i \setminus C \); then, for each of those \( M \) nodes, since \( i - 1 \) links enter the cut, \( i - 2 \) links leave the cut. The total number of links between nodes of \( T \) and nodes of \( K_i \setminus C \) that enter the cut
are exactly:

\[
\frac{i(i - 1)}{2} - \frac{(i - M)(i - M - 1)}{2} = \\
= \frac{1}{2}(i^2 - i^2 + iM + iM - M^2 + i - M) = \\
= \frac{1}{2}(2i - M - 1).
\]

Considering that, in total, \(M(i - 2)\) links exit the cut, we have that those \(M\) nodes contribute at least the following number of links to the cut:

\[
\frac{M}{2}(2i - M - 1) = \\
= iM - \frac{M^2}{2} - \frac{M}{2} - iM + 2M = \\
= \frac{M}{2}(3 - M),
\]

which is \(> 0\) if \(M \leq 2\).

While this lemma is not useful by itself, it offers some insight on how deviating densely connected subsets of nodes might affect the cut value difference.

As a sidenote, if we assume that \(C\) is minimal with respect to the deviation from \(\sigma\), then the same result holds if \(M \leq 4\): this is due to the additional nodes and links required to guarantee the minimality of \(C\). If we consider a subset composed only of those \(2M\) nodes, we have that, for \(M \geq 5\), the cut value does not increase; in particular, for \(M = 6\), we get Prop. 2.

The reasoning behind the results of this section can be generalized to arbitrary subsets and arbitrary couples of colorings.

**Theorem 2.** Let \(G\) be as before, let \(\sigma, \gamma \in K^n\) and let \(C \subseteq V\); then

\[
\Delta S(\sigma, \gamma) = \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) + P_{int}(\gamma, \sigma) - P_{int}(\sigma, \gamma) + P_{ext}(\sigma, \gamma) - P_{ext}(\gamma, \sigma).
\]

**Proof.** Without loss of generalization, let \(K_C(\sigma) = \{1, \ldots, s\}\), \(K_C(\gamma) = \{1, \ldots, g\}\), for each color \(i = 1, \ldots, l\), \(o_i\) \(i\)-colored nodes \(\in C\) in \(\sigma\) and, for each of those \(o_i\) nodes which we will call \(n_{i,j}\) \((j = 1, \ldots, o_i)\):

- \(v_{i,j}\) \(i\)-colored nodes in \(C\) in \(\sigma\) that are adjacent to it and such that \(b_{i,j}\) of them are *not* \(\gamma_{m_{i,j}}\)-colored in \(\gamma\) (in the following, \(\gamma_{i,j} := \gamma_{n_{i,j}}\));
- \(u_{i,j}\) \(i\)-colored nodes in \(V \setminus C\) in \(\sigma\) (and in \(\gamma\)) that are adjacent to it;
- \(z_{i,j}\) \(\gamma_{i,j}\)-colored nodes in \(C\) in \(\gamma\) that are adjacent to it and such that \(f_{i,j}\) of them are *not* \(i\)-colored in \(\sigma\);
- \(t_{i,j}\) \(\gamma_{i,j}\)-colored nodes in \(V \setminus C\) in \(\gamma\) (and in \(\sigma\)) that are adjacent to it.
By definition, for $i = 1, \ldots, l$ and $j = 1, \ldots, o_i$, $v_{i,j} - b_{i,j} = z_{i,j} - f_{i,j}$: both the quantities $v_{i,j} - b_{i,j}$ and $z_{i,j} - f_{i,j}$ enumerate the number of neighbors of $n_{i,j}$ in $C$ that are both $i$-colored in $\sigma$ and $\gamma_{i,j}$-colored in $\gamma$.

From this, we have that, for $\mu_{i,j}(\gamma) := \mu_{n_{i,j}}(\gamma)$:

$$\mu_{i,j}(\gamma) = \mu_{i,j}(\sigma) + v_{i,j} + u_{i,j} - z_{i,j} - t_{i,j} = \mu_{i,j}(\sigma) + b_{i,j} + u_{i,j} - f_{i,j} - t_{i,j}.$$ 

Since $\sum_{i=1}^{l} o_i = |C|$ and: $\sum_{i=1}^{l} \sum_{j=1}^{o_i} (\mu_{i,j}(\gamma) - \mu_{i,j}(\sigma)) = \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma))$, we have that:

$$\sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) = \sum_{i=1}^{l} \sum_{j=1}^{o_i} (b_{i,j} + u_{i,j} - f_{i,j} - t_{i,j}).$$

Now, by construction, we have that $P_{\text{bound}}(\sigma, \gamma) = \sum_{i=1}^{l} \sum_{j=1}^{o_i} u_{i,j}$ and that $P_{\text{int}}(\sigma, \gamma) = \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{o_i} b_{i,j}$, since each of the $b_{i,j}$ nodes, and, equivalently, links, is counted twice.

From this, it follows that $\Delta S(\sigma, \gamma) = P(\sigma, \gamma) - P(\gamma, \sigma)$:

$$\Delta S(\sigma, \gamma) = S(\gamma) - S(\sigma) = |E(\gamma)| - |E(\sigma)| = |\{\{i, j\} \in E : \gamma_i \neq \gamma_j\}| - |\{\{i, j\} \in E : \sigma_i \neq \sigma_j\}| = |\{\{i, j\} \in E : \gamma_i \neq \gamma_j\}| + |\{\{i, j\} \in E : \sigma_i = \gamma_i\}| - |\{\{i, j\} \in E : \sigma_i = \sigma_j\}| = |E(\gamma) \setminus E(\sigma)| - |E(\sigma) \setminus E(\gamma)| = P(\sigma, \gamma) - P(\gamma, \sigma).$$

Putting everything together:

$$\Delta S(\sigma, \gamma) = P(\sigma, \gamma) - P(\gamma, \sigma) = P_{\text{int}}(\sigma, \gamma) + P_{\text{bound}}(\sigma, \gamma) + P_{\text{ext}}(\sigma, \gamma) - P_{\text{int}}(\gamma, \sigma) - P_{\text{bound}}(\gamma, \sigma) - P_{\text{ext}}(\gamma, \sigma) = \sum_{i=1}^{l} \sum_{j=1}^{o_i} \left( \frac{1}{2} b_{i,j} + u_{i,j} - \frac{1}{2} f_{i,j} - t_{i,j} \right) + P_{\text{ext}}(\sigma, \gamma) - P_{\text{ext}}(\gamma, \sigma) = \sum_{i=1}^{l} \sum_{j=1}^{o_i} \left( b_{i,j} + u_{i,j} - f_{i,j} - t_{i,j} + \frac{1}{2} (f_{i,j} - b_{i,j}) \right) + P_{\text{ext}}(\sigma, \gamma) - P_{\text{ext}}(\gamma, \sigma) = 10.
\[
\sum_{i=1}^{l} \sum_{j=1}^{o_i} (b_{i,j} + u_{i,j} - t_{i,j}) + \sum_{i=1}^{l} \sum_{j=1}^{o_i} \frac{1}{2} (f_{i,j} - b_{i,j}) + P_{\text{ext}}(\sigma, \gamma) - P_{\text{ext}}(\gamma, \sigma) =
\]
\[
= \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) + \sum_{i=1}^{l} \sum_{j=1}^{o_i} \frac{1}{2} f_{i,j} - \sum_{i=1}^{l} \sum_{j=1}^{o_i} \frac{1}{2} b_{i,j} + P_{\text{ext}}(\sigma, \gamma) - P_{\text{ext}}(\gamma, \sigma) =
\]
\[
= \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) + P_{\text{int}}(\gamma, \sigma) - P_{\text{int}}(\sigma, \gamma) + P_{\text{ext}}(\sigma, \gamma) - P_{\text{ext}}(\gamma, \sigma).
\]

While the general theorem is too broad to be useful, it can be further refined into a couple of useful corollaries:

**Corollary 1.** Let \( G \) be as before, let \( \sigma, \gamma \in K^n \) and let \( C \subseteq V \) deviate from \( \sigma \) to \( \gamma \); then
\[
\Delta S(\sigma, \gamma) = \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) + P_{\text{int}}(\gamma, \sigma) - P_{\text{int}}(\sigma, \gamma).
\]

**Proof.** As \( C \) deviates from \( \sigma \) to \( \gamma \), none of the nodes in \( V \setminus C \) change color, which means that \( P_{\text{ext}}(\sigma, \gamma) = P_{\text{ext}}(\gamma, \sigma) \).

**Corollary 2.** Let \( G \) be as before, let \( \sigma \) be a NE for \( G \) and let \( C \subseteq V \) strongly deviate from \( \sigma \) to \( \gamma \in K^n \); then
\[
\Delta S(\sigma, \gamma) \geq |C| - P_{\text{int}}(\sigma, \gamma).
\]

**Proof.** As \( C \) strongly deviates from \( \sigma \) to \( \gamma \), we have that, for each node \( d \in C \), \( \mu_d(\gamma) \geq \mu_d(\sigma) + 1 \), thus, \( \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) \geq \sum_{d \in C} 1 = |C| \).

Thus, we have:
\[
\Delta S(\sigma, \gamma) \geq |C| + P_{\text{int}}(\gamma, \sigma) - P_{\text{int}}(\sigma, \gamma) \geq |C| - P_{\text{int}}(\sigma, \gamma).
\]

Intuitively, this corollary says that, in order to check whether the cut value always increases after some deviation without knowledge of \( N(C) \), we need to check the worst-case scenario, the one in which each node of \( C \) improves its payoff by the least possible margin (i.e. by 1) and as much as possible of that profit is due to color clustering in \( G \) under \( \sigma \).

Interestingly, Corollary 1 can be extended to max \( k \)-cut games on undirected weighted graphs:

**Corollary 3.** Let \( \hat{G} \) be a max \( k \)-cut on a weighted undirected graph, let \( \sigma, \gamma \in K^n \) and let \( C \subseteq V \) deviate from \( \sigma \) to \( \gamma \); then
\[
\Delta \hat{S}(\sigma, \gamma) = \sum_{d \in C} (\hat{\mu}_d(\gamma) - \hat{\mu}_d(\sigma)) + \hat{P}_{\text{int}}(\gamma, \sigma) - \hat{P}_{\text{int}}(\sigma, \gamma).
\]
Before going through the proof, we need to extend the max k-cut game to undirected weighted graphs:

- the game will be denoted as $\hat{G}$ on a undirected weighted graph $\hat{G} = \langle V, E, w \rangle$, where $w : E \rightarrow \mathbb{R}$ is a weight function over $\hat{G}$;
- both player sets and strategy sets are the same as in the base case;
- the payoff function is slightly different: $\hat{\mu}_v(\sigma) := \sum_{\{p, q\} \in E} w(\{p, v\})$;

$$\hat{S}(\alpha) := \sum_{\{p, q\} \in E: \alpha_p \neq \alpha_q} w(\{p, q\}) \quad \text{and} \quad \hat{P}(\alpha, \beta) := \sum_{\{p, q\} \in E: \alpha_p = \alpha_q \text{ and } \beta_p \neq \beta_q} w(\{p, q\}).$$

$\hat{P}_{\text{int}}, \hat{P}_{\text{bound}}$ and $\hat{P}_{\text{ext}}$ are defined similarly to $\hat{P}$. Also, $\hat{S}(\sigma) = \sum_{e \in E(\sigma)} w(e)$ and $\hat{P}(\alpha, \beta) = \sum_{e \in E(\beta), E(\alpha)} w(e)$.

**Proof (of Corollary 3).** We will extend Thm. 2 first and then prove this corollary.

Like in Thm. 2, for each color $i = 1, \ldots, l$, for $j = 1, \ldots, o_i$, we define for each node $n_{i,j} \in C \subseteq V$ the following quantities:

$$\hat{b}_{i,j} := \sum_{p \in C; \{p, n_{i,j}\} \in E, \sigma_p = i \text{ and } \gamma_p \neq \gamma_{i,j}} w(\{p, n_{i,j}\}), \quad \hat{u}_{i,j} := \sum_{q \in V \setminus C; \{q, n_{i,j}\} \in E, \sigma_q = i} w(\{q, n_{i,j}\}),$$

$$\hat{f}_{i,j} := \sum_{p \in C; \{p, n_{i,j}\} \in E, \sigma_p \neq i \text{ and } \gamma_p = \gamma_{i,j}} w(\{p, n_{i,j}\}), \quad \hat{t}_{i,j} := \sum_{q \in V \setminus C; \{q, n_{i,j}\} \in E, \gamma_q = \gamma_{i,j}} w(\{q, n_{i,j}\}).$$

Again, we have that $\hat{\mu}_i, j(\gamma) = \hat{\mu}_i, j(\sigma) + \hat{b}_{i,j} + \hat{u}_{i,j} - \hat{f}_{i,j} - \hat{t}_{i,j}$ and:

$$\sum_{d \in C}(\hat{\mu}_d(\gamma) - \hat{\mu}_d(\sigma)) = \sum_{i=1}^{l} \sum_{j=1}^{o_i} (\hat{b}_{i,j} + \hat{u}_{i,j} - \hat{f}_{i,j} - \hat{t}_{i,j}).$$

Also, showing that $\Delta \hat{S}(\sigma, \gamma) = \hat{P}(\sigma, \gamma) - \hat{P}(\gamma, \sigma)$ and that

$$\Delta \hat{S}(\sigma, \gamma) = \sum_{d \in C}(\hat{\mu}_d(\gamma) - \hat{\mu}_d(\sigma)) + \hat{P}_{\text{int}}(\gamma, \sigma) - \hat{P}_{\text{int}}(\sigma, \gamma) - \hat{P}_{\text{ext}}(\gamma, \sigma) + \hat{P}_{\text{ext}}(\sigma, \gamma)$$

is similar to Thm. 2, except that we split the sums with respect to the subsets rather than splitting the subsets themselves, that is, instead of

$$S(\gamma) = \left| \{\{i, j\} \in E : \gamma_i \neq \gamma_j \text{ and } \sigma_i = \sigma_j\} \right| + \left\lvert \{\{i, j\} \in E : \gamma_i \neq \gamma_j \text{ and } \sigma_i \neq \sigma_j\} \right\rvert,$$

we have:

$$\hat{S}(\gamma) = \sum_{\{p, q\} \in E: \gamma_p \neq \gamma_q} w(\{p, q\}) - \sum_{\{p, q\} \in E: \gamma_p \neq \gamma_q, \sigma_p = \sigma_q} w(\{p, q\}).$$
Like in Corollary 1, if we assume that $C$ deviates from $\sigma$ to $\gamma$, we have that $P_{\text{ext}}(\sigma, \gamma) = P_{\text{ext}}(\gamma, \sigma) = 0$ and that
\[
\Delta S(\sigma, \gamma) = \sum_{d \in C} (\mu_d(\gamma) - \mu_d(\sigma)) + \hat{P}_{\text{int}}(\gamma, \sigma) - \hat{P}_{\text{int}}(\sigma, \gamma).
\]

Extending Corollary 2 to undirected weighted graphs is possible, but the resulting lower bound is also dependent on the weight function associated with $G$.

Lastly, Thm. 2 can also be extended to directed graphs, but it would not offer much information. We shall only examine the directed unweighted graph, as the directed weighted graph follows from this and from the undirected weighted graph.

Let $\bar{G} = (V, A)$, $\bar{\mu}_d(\alpha) := |\{(p, d) \in A : \alpha_p \neq \alpha_d\}|$, where $A \subseteq V^2 \setminus \{(x, x) : x \in V\}$, $\bar{S}(\alpha) := |\{(p, q) \in A : \alpha_p \neq \alpha_q\}|$ and $\bar{P}(\alpha, \beta) := |\{(p, q) \in A : \alpha_p = \alpha_q, \beta_p \neq \beta_q\}|$; similarly to the undirected unweighted case, we have that $\bar{S}(\gamma) - \bar{S}(\sigma) = \sum_{d \in C} (\bar{\mu}_d(\gamma) - \bar{\mu}_d(\sigma))$. For directed unweighted graphs, $\hat{P}_{\text{int}}(C, \sigma, \gamma) = |\{(p, q) \in A : p, q \in C, \sigma_p = \sigma_q, \gamma_p \neq \gamma_q\}| = \sum_{i=1}^{l} \sum_{j=1}^{o_i} \bar{b}_{i,j}$, where, for each node $n_{i,j}$ ($i = 1, \ldots, l$, $j = 1, \ldots, o_i$), $\bar{b}_{i,j}$ is the number of nodes having an outgoing arc towards $n_{i,j}$ that are $i$-colored in $\sigma$ and are not $\gamma_{i,j}$-colored in $\gamma$; this means that, unlike links in the undirected cases, arcs are not counted twice and, thus, that we cannot use this extension like in the undirected cases.

Lastly, Prop. 2 implies that the left side of the inequality from Corollary 2 can be non-positive, which strongly limits Corollary 2’s usefulness outside this article. Furthermore, even if the left side were positive, we will show in Section 5 some cases where the right side is non-positive.

## 4 Stable equilibria

Before proving the main result of this article, we are going to extend Prop. 1 to minimal subsets $C$ such that $|K_C(\sigma)| \in \{|C| - 3, |C| - 2\}$. In order to do so, we are going to apply the lower bound from Corollary 2 to each possible graph configuration that stems from a minimal subset $C$ using more than $|C| - 4$ colors, improving Lemma 1.

**Theorem 3.** Let $\sigma$ be a NE for $\mathcal{G}$ and let $C \subseteq V$ be a minimal subset w.r.t strongly deviating from $\sigma$ to another coloring $\gamma$; if $|K_C(\sigma)| \geq |C| - 3$, then $\Delta S(\sigma, \gamma) > 0$.

**Proof.** The idea is to find the smallest $C$ with respect to a fixed value of $PI := P_{\text{int}}(\sigma, \gamma)$, both for $|K_C(\sigma)| = |C| - 2$ and for $|K_C(\sigma)| = |C| - 3$.

In both cases, because of Prop. 1 and the pigeon-hole principle, there are only a few subcases, which allows us to prove each subcase individually. In short those subcases are:
\[ |K_C(\sigma)| = |C| - 2: \]
- \(|C_a(\sigma)| = 3, |C_a(\sigma)| = 1 \) (for \(i = 1, \ldots, |C| - 3\), \(PI \in \{0, 1, 2, 3\}\)
- \(|C_d(\sigma)| = |C_e(\sigma)| = 2, |C_a(\sigma)| = 1 \) (for \(i = 1, \ldots, |C| - 4\), \(PI \in \{0, 1, 2\}\)

\[ |K_C(\sigma)| = |C| - 3: \]
- \(|C_a(\sigma)| = 4, |C_a(\sigma)| = 1 \) (for \(i = 1, \ldots, |C| - 4\), \(PI \in \{0, 1, 2, 3, 4, 5, 6\}\)
- \(|C_d(\sigma)| = 3, |C_e(\sigma)| = 2, |C_a(\sigma)| = 1 \) (for \(i = 1, \ldots, |C| - 5\), \(PI \in \{0, 1, 2, 3, 4, 5\}\)
- \(|C_f(\sigma)| = |C_g(\sigma)| = |C_h(\sigma)| = 2, |C_a(\sigma)| = 1 \) (for \(i = 1, \ldots, |C| - 4\), \(PI \in \{0, 1, 2, 3\}\)

Also, because of Prop. 3 and Prop. 4, we can assume that \(C\) induces a connected subgraph in which each node deviates towards the color of one of its neighbor: this further narrows down the possible configurations of \(K_C(\sigma)\) and \(K_C(\gamma)\).

Let \(|K_C(\sigma)| = |C| - 2: \) then either there exists exactly one color \(c \in K\) such that \(|C_c(\sigma)| = 3\) or there are two colors \(a, b \in K\) such that \(a \neq b\) and \(|C_a(\sigma)| = |C_b(\sigma)| = 2\).

When there exists exactly one color \(c \in K\) such that \(|C_c(\sigma)| = 3\), we have that \(PI \in \{0, 1, 2, 3\}\):

- \(PI = 3: \) let \(a \in K\) be such that \(|C_a(\sigma)| = 3\); then, for each node \(d \in C_a(\sigma)\), there is at least another node \(r_d \in C\) such that \(d, r_d \in E, r_d \notin C_a(\sigma), \sigma_{r_d} = \gamma_d\) and, for each node \(e \in C_a(\sigma)\), if \(d \neq e\), then \(r_d \neq r_e\).

The least amount of nodes in \(C\) is 6, when each node in \(C_a(\gamma)\) has exactly one neighbor as above, those three neighbors are colored \(a\) in \(\gamma\) and those 6 nodes are the only ones in \(C\). Therefore \(\Delta S(\sigma, \gamma) \geq |C| - 3 \geq 6 - 3 = 3\).

- \(PI = 2: \) let \(a \in K\) be such that \(|C_a(\sigma)| = 3\); then there exists exactly one node \(m \in C_a(\sigma)\) such that \(\delta_m(G(C)) = 2\). As in the previous subcase, there is at least a neighbor for each \(a\)-colored node that is colored the color the respective \(a\)-colored node deviates to in \(\gamma\).

Unlike the previous case, the \(a\)-colored nodes that are not \(m\) can deviate to the same color in \(\gamma\) (as they are not adjacent) and even have a common neighbor with those properties. Therefore \(\Delta S(\sigma, \gamma) \geq |C| - 2 \geq 5 - 2 = 3\).

- \(PI = 1: \) by reasoning in a similar way as in the previous subcase, it is not hard to show that \(\Delta S(\sigma, \gamma) \geq |C| - 1 \geq 5 - 1 = 4\).

- \(PI = 0: \) in this subcase, each \(a\)-colored node could deviate towards the color of a single common neighbor, which deviates towards \(a\); therefore, the minimum number of nodes in \(C\) is 4 and \(\Delta S(\sigma, \gamma) \geq 4\).

When there are two colors \(a, b \in K\) such that \(a \neq b\) and \(|C_a(\sigma)| = |C_b(\sigma)| = 2\), we have that \(PI \in \{0, 1, 2\}\):
- $PI = 2$: let $a, b \in K$ be such that $|C_a(\sigma)| = |C_b(\sigma)| = 2$; then, for each node $d \in C_a(\sigma) \cup C_b(\sigma)$, there is at least another node $r_d \in C$ such that $\{d, r_d\} \in E$, $\sigma_{r_d} \neq \sigma_d$, $\sigma_{r_d} = \gamma_d$ and, for each node $f \in C_a(\sigma) \cup C_b(\sigma)$, if $\sigma_f = \sigma_d$, then $\gamma_d \neq r_f$.

Let $C_a(\sigma) = \{x, y\}$ and $C_b(\sigma) = \{z, w\}$; by reasoning in a similar way as in "$|C_a(\sigma)| = 3, PI = 2$", we can reduce the minimum number of nodes in $C$ by assuming without loss of generality that $\sigma_y = \gamma_w$ and that $\sigma_w = \gamma_y$. As for $x$ and $z$, if $\{x, z\} \notin E$, then it is possible that $\gamma_x = \gamma_z (= \sigma_i)$, for some $t$ such that $\sigma_t \neq a, b$), otherwise there are at least 2 nodes neither $a$- nor $b$-colored on $C$.

This means that $|C| \geq 5$, from which: $\Delta S(\sigma, \gamma) \geq 5 - 2 = 3$.

- $PI = 1$: by reasoning in a similar way to the previous subcase, it follows that $\Delta S(\sigma, \gamma) \geq |C| - 1 \geq 5 - 1 = 4$.

- $PI = 0$: the smallest $C$ is the one where $a$-colored nodes deviate towards $b$ and $b$-colored nodes deviate towards $a$, which means that $|C| \geq 4$. Therefore, $\Delta S(\sigma, \gamma) \geq 4$.

Let $C$ be such that $|K_C(\sigma)| = |C| - 3$ and, given $\sigma$, let us define the following notations:

- $\sigma : \{(t, u_1, \ldots, u_{|C|-4}), (4, 1, \ldots, 1)\}$ when $\sigma$ is such that there is a color $a \in K_C(\sigma)$ such that $|C_a(\sigma)| = 4$ and, for each other color $b_o \in K_C(\sigma)$ ($o = 1, \ldots, |C| - 4$), $|C_{b_o}(\beta)| = 1$;

- $\sigma : \{(t_1, t_2, u_1, \ldots, u_{|C|-5}), (3, 2, 1, \ldots, 1)\}$ when $\sigma$ is such that there are two colors $a_1, a_2 \in K_C(\sigma)$ such that $|C_{a_1}(\sigma)| = 3, |C_{a_2}(\sigma)| = 2$, $a_1 \neq a_2$ and, for each $b_o \in K_C(\sigma) \setminus \{a_1, a_2\}$ ($o = 1, \ldots, |C| - 5$), $|C_{b_o}(\sigma)| = 1$;

- $\sigma : \{(t_1, t_2, t_3, u_1, \ldots, u_{|C|-6}), (2, 2, 2, 1, \ldots, 1)\}$ when $\sigma$ is such that there are three colors $a_1, a_2, a_3 \in K_C(\sigma)$ such that for $i = 1, 2, 3$, $|C_{a_i}(\sigma)| = 2$, for $j \neq i$, $a_i \neq a_j$ and, for each $b_o \in K_C(\sigma) \setminus \{a_1, a_2, a_3\}$ ($o = 1, \ldots, |C| - 4$), $|C_{b_o}(\sigma)| = 1$.

Now, we examine thoroughly each of the cases, with respect to the possible values of $PI$:

- $\sigma : \{(t, u_1, \ldots, u_{|C|-4}), (4, 1, \ldots, 1)\}$: in this case $PI \in \{0, 1, \ldots, 6\}$. Let $s \in K$ be such that $|C_s(\sigma)| = 4$:
  - $PI = 6$: each node in $C_s(\sigma)$ will deviate towards a color that has not been chosen by the other nodes in $C_s(\sigma)$. By reasoning in a similar way as in "$|K_C(\sigma)| = |C| - 2$", this means that the nodes of $C$ are at least the nodes in $C_s(\sigma)$ and the (at least) four nodes representing the colors chosen by the nodes in $C_s(\sigma)$. Therefore, $|C| \geq 8$ and $\Delta S(\sigma, \gamma) \geq 8 - 6 = 2$.
  - $PI = 5$: there are two cases:
    - $|K_{C_\sigma}(\gamma)| = 4$: like in the previous subcase, $\Delta S(\sigma, \gamma) \geq 8 - 5 = 3$;
\* \(|K_{C_a}(\sigma)| = 3\): this means that the two nodes in \(G(C_a(\sigma))\) that are not adjacent deviated towards the same color; like in "\(|K_C(\sigma)| = |C| - 2\), 1st case, \(PI = 2\)"; it implies that \(|C| \geq 7\) and that \(\Delta S(\sigma, \gamma) \geq 7 - 5 = 2\);

- \(PI = 4\): in this case, \(|K_{C_a}(\sigma)| \in \{2, 3, 4\}:
\* \(|K_{C_a}(\sigma)| = 4\): we get \(\Delta S(\sigma, \gamma) \geq 8 - 4 = 4\);
\* \(|K_{C_a}(\sigma)| = 3\): we get \(\Delta S(\sigma, \gamma) \geq 7 - 4 = 3\);
\* \(|K_{C_a}(\sigma)| = 2\): this one depends on whether \(K_3\) is a subgraph of \(G(C_a(\sigma))\) or not. If it is, this subcase is similar to the previous subcase, which means that the minimum size of \(C\) is 7 and that \(\Delta S(\sigma, \gamma) \geq 7 - 4 = 3\).

Otherwise, then the couples of nodes in \(G(C_a(\sigma))\) that are not adjacent might choose the same color, making it so that the minimum size of \(C\) is 6 (from which: \(\Delta S(\sigma, \gamma) \geq 6 - 4 = 2\)).

- \(PI < 4\): with the exception of \(PI = 0\), where each node in \(C_a(\sigma)\) might deviate towards the same color, this case is similar to the ones we have mentioned before, which means that in this case \(\Delta S(\sigma, \gamma) > 0\);

\* \(\sigma : \{(t_1, t_2, u_1, \ldots, u_{\lvert C\rvert - 5}), (3, 2, 1, \ldots, 1)\}\): in this case \(PI \in \{0, 1, \ldots, 4\}\).
Let \(a, b \in K\) be such that \(|C_a(\sigma)| = 3\) and \(|C_b(\sigma)| = 2\): this means that, by construction, \(|C| \geq 5\) and that, without delving further, \(\Delta S(\sigma, \gamma) \geq |C| - PI \geq 5 - 4 = 1\).

In this case, we shall examine each case anyway, as the lower bound is actually higher than 1:

- \(PI = 4\): the smallest \(C\) is obtained when an \(a\)-colored node and a \(b\)-colored node deviate towards \(b\) and \(a\) respectively, another \(a\)-colored node and the other \(b\)-colored node deviate towards a third color, say, \(c\), the third \(a\)-colored node deviates towards a fourth color, say, \(d\), the \(c\)-colored node and the \(d\)-colored node deviate towards \(a\) and those 7 nodes are the only nodes in \(C\). Therefore, \(\Delta S(\sigma, \gamma) \geq 7 - 4 = 3\);

- \(PI = 3\): like in the subcase "\(\sigma : \{(t, u_1, \ldots, u_{\lvert C\rvert - 4}), (4, 1, \ldots, 1)\}\), \(PI = 5\)"; either \(G(C_a(\sigma)) \sim K_3\) or \(G(C_a(\sigma)) \sim K_3\):
\* \(G(C_a(\sigma)) \sim K_3\): in this case, each node in \(C_a(\sigma)\) deviates towards a different color. This means that, even taking into account that \(b\)-colored node might deviate towards the same color, the minimum value of \(|K_C(\sigma)|\), that is the least number of colors in \(C\), is 4, like in "\(\sigma : a_1^2 \odot a_2^2 \odot b_1^2 \cdot \ldots \cdot b_{\lvert C\rvert - 5}^2, PI = 4\)".

This means that the least number of nodes in \(C\) is 7, from which: \(\Delta S(\sigma, \gamma) \geq 7 - 3 = 4\);
Let $C$ be a minimal subset w.r.t strongly $\gamma$ such that $|C| \geq 6$, then $\Delta S(\sigma, \gamma) > 0$.

Now, we prove the first half of the main result of this article: the idea is to use Thm. 4 to prove that, for almost all the cases where a minimal subset $C$ has at most 7 nodes, the cut value almost increases, and then to prove separately that the same holds for the subcase $|C| = 7$, $|K_C(\sigma)| = 3$; this allows to say that optimal colorings are resilient to deviations of minimal subsets with at most 7 nodes.

**Theorem 4.** Let $\sigma$ be a NE for $G$ and let $C \subseteq V$ be a minimal subset w.r.t strongly deviating from $\sigma$ to another coloring $\gamma$; if $|C| \leq 7$, then $\Delta S(\sigma, \gamma) > 0$.

**Proof.** Because of the previous theorem, this result holds for $|C| \leq 6$ and for $|C| = 7$ such that $|K_C(\sigma)| \in \{0, 1, 2, 4, 5, 6, 7\}$. This leaves the case where $|C| = 7$ and $|K_C(\sigma)| = |C| - 4 = 3$.

The core idea is to repeat what we have done with $|K_C(\sigma)| \geq |C| - 3$, except we want to prove that, for $|C| = 7$, the maximum values of $PI$ are $< 7$.

Let $C \subseteq V$ be such that $|K_C(\sigma)| = 3$ and, without loss of generalization, assume that $K_C(\sigma) = \{1, 2, 3\}$. Like in Thm. 3, because of Prop. 1 and the pigeon-hole principle, when $|C| = 7$ and $|K_C(\sigma)| = 3$, there are only a few subcases:

- $\sigma : \{(1, 2, 3), (3, 2, 2)\}$ when $\sigma$ is such that $|C_1(\sigma)| = 3$ and $|C_2(\sigma)| = |C_3(\sigma)| = 2$;
- $\sigma : \{(1, 2, 3), (3, 3, 1)\}$ when $\sigma$ is such that $|C_1(\sigma)| = |C_2(\sigma)| = 3$ and $|C_3(\sigma)| = 1$;
- $\sigma : \{(1, 2, 3), (4, 2, 1)\}$ when $\sigma$ is such that $|C_1(\sigma)| = 4$, $|C_2(\sigma)| = 2$ and $|C_3(\sigma)| = 1$;

Therefore, for $|K_C(\sigma)| \geq |C| - 3$, $S(\gamma) > S(\sigma)$.  

\[ \star G(C_3(\sigma)) \cong K_3: \text{like in } \sigma : \{(t, u_1, \ldots, u_{|C|-4}), (4, 1, \ldots, 1)\}, PI = 5' \text{, the non-adjacent nodes in } C_3(\sigma) \text{ might deviate towards the same color. If that happened, by reasoning in a similar way as in } \sigma : \{(t_1, t_2, u_1, \ldots, u_{|C|-5}), (3, 2, 1, \ldots, 1)\}, PI = 4', \text{ the least number of colors in } C \text{ becomes 3, which means that } |C| \geq 6. \text{ From this, it follows that: } \Delta S(\sigma, \gamma) \geq 6 - 3 = 3; \]

- $PI < 3$: like in $\sigma : \{(t, u_1, \ldots, u_{|C|-4}), (4, 1, \ldots, 1)\}$, $PI < 4'$, there is no further need to study this case, as in each case $\Delta S(\sigma, \gamma) > 0$;

- $\sigma : \{(t_1, t_2, t_3, u_1, \ldots, u_{|C|-6}), (2, 2, 2, 1, \ldots, 1)\}$: in this case $PI \in \{0, 1, 2, 3\}$. Let $a, b, c \in K$ be such that $|C_a(\sigma)| = |C_b(\sigma)| = |C_c(\sigma)| = 2$ and $a \neq b \neq c \neq a$:

by construction $|K_C(\sigma)| \geq 3$ and $|C| \geq 6$, which means that, without examining each subcase, $\Delta S(\sigma, \gamma) \geq 6 - 3 = 3$.

Unlike in $\sigma : \{(t_1, t_2, u_1, \ldots, u_{|C|-5}), (3, 2, 1, \ldots, 1)\}$, it suffices to check the $\text{“}PI = 3'\text{”}$ case to show that 3 is the highest upper bound: let $PI = 3$; since $|K_C(\sigma)| \geq 3$, each couple of nodes of, say, $a$ might deviate towards respectively $b$ and $c$ (and so would both the $b$-colored nodes and the $c$-colored nodes). This implies that $|C| \geq 6$ and that $\Delta S(\sigma, \gamma) \geq 6 - 3 = 3$.

Therefore, for $|K_C(\sigma)| \geq |C| - 3$, $S(\gamma) > S(\sigma)$. \[\blacksquare\]
We have already shown that, if \( \sigma \) is such that \( |C_1(\sigma)| = 5 \) and \( |C_2(\sigma)| = |C_5(\sigma)| = 1 \).

Now, we examine each possible "configuration" of \( \sigma \):

- \( \sigma : \{(1, 2, 3), (5, 1, 1)\} \): when \( \sigma \) is such that \( |C_1(\sigma)| = 5 \) and \( |C_2(\sigma)| = |C_5(\sigma)| = 1 \).

As a small sidenote, \( PI \leq 5 \). Suppose that \( PI = 6 \): then either \( G(C_1(\sigma)) \sim K_4 \) or \( G(C_1(\sigma)) \sim K_4' \):
  - \( G(C_1(\sigma)) \sim K_4 \): this one is similar to \( PI = 7 \), except that the 2-colored nodes might deviate towards the same color in \( \gamma \). Therefore, we contradict \( |C| = 7 \), like in \( PI = 7 \);
  - \( G(C_1(\sigma)) \sim K_4' \): this case offers a different problem: by reasoning in a similar way as in \( PI = 7 \), we have that the 1-colored nodes now only need other three colors rather than four, but this contradicts \( |K_C(\sigma)| = 3 \).

Thus, \( PI \leq 5 \) and \( \Delta S(\sigma, \gamma) \geq 7 - \max\{PI\} = 7 - 5 = 2 \);

- \( \sigma : \{(1, 2, 3), (5, 1, 1)\} \): while, if we were to work like in the previous theorem, we would get a more thorough proof, we just need to realize that, like in the previous case, the nodes of \( C_1(\sigma) \) are only allowed to choose between 2 and 3 when deviating. This means that \( |K_{G(C_1(\sigma))}(\gamma)| \leq 2 \), that is \( G(C_1(\sigma)) \) must be either a bipartite graph or a graph without links.

As such, we have that max \( PI = 6 \), when \( G(C_1(\sigma)) \sim K_{3.2} \), the complete bipartite graph with 3 nodes in a part and 2 in the other. This means that \( \Delta S(\sigma, \gamma) \geq 7 - 6 = 1 \).

Therefore, we have that, in each subcase, \( \Delta S(\sigma, \gamma) > 0 \), that is, \( S(\gamma) > S(\sigma) \).

In order to prove that optimal colorings are 7-stable equilibria, we need to prove that the statement of Thm. 4 holds for non-minimal subsets as well:

**Theorem 5.** Let \( \sigma \) be an optimal coloring for \( \mathcal{G} \); then \( \sigma \) is also a 7-stable equilibrium.

**Proof.** Suppose first that \( \sigma \) is a 5-stable equilibrium and let \( C \subseteq V \) be such that \( |C| = 6 \) and \( C \) is not minimal with respect to strongly deviating from \( \sigma \) (towards some coloring \( \gamma \)); we want to show that this leads to a contradiction.

We have already shown that, if \( F \subseteq V \) is minimal with respect to strongly deviating from \( \sigma \) (towards some coloring \( \eta \)) and \( |F| \leq 7 \), then \( S(\eta) > S(\sigma) \). Since \( C \) is not minimal with respect to strongly deviating from \( \sigma \), there is a subset \( D \subset C \) such that \( D \) is minimal with respect to strongly deviating from \( \sigma \) (towards some other
coloring $\iota$). Since we assumed that $\sigma$ is an optimal coloring and since $D$ is minimal with respect to strongly deviating from $\sigma$ (towards $\iota$) and $|D| < 6$, this leads to a contradiction because $S(\iota) > S(\sigma) = \max_{\alpha \in K^n} S(\alpha)$.

Therefore, optimal colorings are also 6-stable equilibria.

By repeating the same reasoning with $|C| = 7$ and with $\sigma$ as a 6-stable equilibrium, we have that the same holds and, from this, that optimal colorings are 7-stable equilibria. 

In a similar way, we can extend Thm. 3 to non-minimal subsets:

**Corollary 4.** Let $\sigma$ be an optimal coloring for $G$ played on the graph $G$; then, for each $C \subseteq V$, if $|K_C(\sigma)| \geq |C| - 3$, then, for each coloring $\iota$, $C$ does not strongly deviate towards $\iota$.

**Proof.** Because of Thm. 3, we already know that in a max $k$-cut game $H$ on a graph $H$, if $D \subseteq V(H)$ is minimal with respect to strongly deviating from a NE $\alpha$ towards some coloring $\beta$, then $S(\alpha) < S(\beta)$.

Suppose now that $C \subseteq V(G)$ strongly deviates from an optimal coloring $\sigma$ towards some coloring $\gamma$, that $|K_C(\sigma)| \geq |C| - 3$, and that $C$ is not minimal with respect to doing so. Then there is a subset $F \subset C$ such that $F$ itself is minimal with respect to strongly deviating from $\sigma$. Since $F \subset C$ and, thus, for any $a \in K$, $F_a(\sigma) \subseteq C_a(\sigma)$, we have that $|D| - 3 \leq |K_D(\sigma)| \leq |D|$, which contradicts Thm. 3.

Therefore, if any subset $C$ of $V(G)$ were to strongly deviate from $\sigma$, then it must be that $|K_C(\sigma)| < |C| - 3$.

As a last note, since in Thm. 5 we only use the fact that any optimal coloring is resistant to strong deviation done by minimal subsets of at most 6 nodes, the same result also holds for non-minimal subsets (with respect to a strong deviation) containing at most 8 nodes: indeed, a non-minimal subset of 8 nodes that deviates from an optimal coloring would have to have a minimal subset deviating from the same coloring and containing at most 7 nodes, which we know it’s impossible.

As we shall see in the next section, proving the same for a minimal subset containing at most 8 nodes will require a different approach.

### 5 Remarks

**Issues with Thm. 4** Because of how we have stated Thm. 4 (and, likewise, Thm. 3), its statement is equivalent to saying that strong deviations from a $r$-stable equilibria ($r \leq 7$) always induce an increase of the cut value and an eventual extension to strong equilibria would imply Conj. 1.

Unfortunately, the way Corollary 2 works gets around having to understand yet unknown properties of optimal colorings and it has already been proven that $S(G, \cdot)$
is not a strong potential function (Prop. 2); this means that the lower bound in Corollary 2 can be non-positive. This means that it is not true that strong deviations from strong equilibria always induce an increase of the cut value. In the following, we will show a couple of subgraphs that contain a deviating subset that does not induce a cut increase.

**Remark:** throughout this section, we will denote in purple the edges that enter the cut after a deviation, in orange the edges that leave the cut after a deviation and in black the edges that neither enter the cut nor enter the cut after a deviation; furthermore, we will denote in light blue the deviating nodes, in light green the non-deviating nodes whose profit increases after a deviation and in red the non-deviating nodes whose profit decreases after a deviation. Lastly, each of the black edges shown in the following graphs guarantees the minimality of the deviating subgraphs and that each node deviates as shown in the pictures.

![Prototype Subgraph](image)

**Fig. 1:** prototype subgraph for $\Delta S(\sigma, \gamma) = 0$.

Note that $|K_C(\sigma)| = 6 = |C| - 4$.

In the graph above, by applying the formula from Corollary 2, we have that the lower bound is exactly 0, i.e. the cut value might not increase after the strong deviation of this subset. An actual example of this happening is the following:
Fig. 2: subgraph that includes a deviating subset such that $\Delta S(\sigma, \gamma) = 0$.

The graph above is similar to the one used in the proof of Prop. 2: like in that proof, we assume that each nondeviating node has no need to deviate and that each deviating node has enough neighbors of the colors towards which does not deviate that it could only deviate towards the color it is deviating towards. Also, this example disproves an extension of Thm. 3 to $|K_C(\sigma)| = |C| - 4$ (for $\sigma$ NE).
In this case, by applying the formula from Corollary 2, we have that the lower bound is -1, which means that, depending on the graph and on the starting coloring $\sigma$, the cut value might actually decrease after the strong deviation of the subset.

For this subgraph, we make the same assumptions as for the graph in Fig. 2. This example disproves that a NE is a 8-stable equilibrium even when only minimal subsets are allowed to deviate.

**Future developments** Both examples show that, in order to prove Conj. 1, it is necessary to find out more about the properties of optimal colorings. For example, one could study how the neighborhood of a deviating subset is colored in an optimal coloring or how clustered colors are in an optimal coloring; another approach would be to study the graph induced by nodes whose utility is lower than its degree.

Alternatively, one could try to exploit defective colorings [26]: a $(k, m)$ defective coloring (or $(k, m)$-coloring) for a graph $G$ is a coloring of $G$ with $k$ colors such that each node has at most $m$ neighbors of the same color as itself. We conjecture that any optimal coloring that is a $(k,1)$-coloring (respectively, $(k,2)$-coloring) for a $(k,1)$-colorable graph that is not $k$-chromatic (respectively, for a $(k,2)$-colorable graph that is not $(k,1)$-colorable) is a strong equilibrium. If it were so, it would also imply that Conj. 1 is true for both planar graphs and toroidal graphs [27].

Also, one could try to find a strong potential function for the class of max $k$-cut game on undirected unweighted graphs: Gourvès and Monnot proved that this game on undirected weighted graphs does not have a strong potential function and Corollary 2 only confirms that the cut value is not a strong potential function (being an extension...
of Prop. 2), not that max $k$-cut games on undirected unweighted graphs do not have one at all.

It is known that, for undirected weighted graphs, an optimal coloring is not necessarily a 4-stable equilibrium [3] and, for directed graphs, an optimal coloring might not be a Nash equilibrium at all [8]; a possible development would be to find a characterization of 4-stable equilibria for the former and a characterization of Nash equilibria for the latter.

Lastly, one could study how stable an optimal coloring is in other games similar to the max $k$-cut game, like the generalized graph $k$-coloring games [2] or the ones from [10] and [5].
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