THE MULTICRITICAL KONTSEVICH–PENNER MODEL

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Abstract

We consider the hermitian matrix model with an external field entering the quadratic term \( \text{tr} (\Lambda X \Lambda X) \) and Penner–like interaction term \( \alpha N \log (1 + X) - X \). An explicit solution in the leading order in \( N \) is presented. The critical behaviour is given by the second derivative of the free energy in \( \alpha \) which appears to be a pure logarithm, that is a feature of \( c = 1 \) theories. Various critical regimes are possible, some of them corresponds to critical points of the usual Penner model, but there exists an infinite set of multi-critical points which differ by values of scaling dimensions of proper conformal operators. Their correlators with the puncture operator are given in genus zero by Legendre polynomials whose argument is determined by an analog of the string equation.

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1 Introduction

Recently the external field problem for matrix models has achieved a remarkable interest due to the work by Kontsevich [1] who had represented the partition function of 2D topological gravity as that of a hermitian one-matrix model in an external field. Along this line, a direct proof of the Witten’s conjecture [2] about an equivalence of 2D topological and quantum gravities has been obtained [3, 4, 5].

On the other hand, the Kontsevich matrix model can be explicitly solved [6, 7, 8] order by order of genus expansion making use of the standard methods which were first introduced for an analogous unitary matrix problem [9]. This solution has been applied recently by Itzykson and Zuber [10] to calculate explicitly the intersection indices on the moduli space.

The original Kontsevich model is reduced by a linear shift of an integration variable to a generic form

$$\mathcal{F}[\Lambda] = \int DX e^{N \text{tr}(\Lambda X + V(X))}$$

(1.1)

with a cubic potential $V(X)$ and the integral going over $N \times N$ hermitian matrices $X$. For an arbitrary $V(X)$, the partition function (1.1) is associated with the so-called generalized Kontsevich model. As is advocated by Kharchev et al. [11], this model interpolates between arbitrary $(p, q)$ double-scaling limits of the standard multi-matrix model. It is crucial for applications that the exponent in (1.1) depends on the external field $\Lambda$ linearly.

One might think, however, about another way of incorporating the external field by modifying the quadratic term:

$$Z[\Lambda] = \int DX e^{N \text{tr}(\frac{-1}{2} \Lambda XX + V(X))}.$$

(1.2)

A model of this kind was first introduced by Das et al. [12]. In the language of discretized random surfaces, it corresponds to an external field which is connected to curvature of the world sheet.

One may ask a question: which form of the potential $V(X)$ in (1.2) permits such changing the variables that results in an integral of the type (1.1)? The only nontrivial choice is the nonpolynomial $V(X) = \alpha(\log(1 + X) - X)$ which is associated with the Penner model [13]. As is pointed out by Distler and Vafa [14], the double-scaling limit of the Penner model corresponds to $c = 1$ CFT or to the $d = 1$ string at the self dual radius $R = 1$ which is first solved by Gross and Klebanov [15]. Further studies of generalizations of the Penner model at multi-critical points have been done recently in [16, 17].

We solve the model (1.2) in genus zero explicitly using the method of Brézin and Gross [4]. We show that the second derivative of log $Z$ in $\alpha$ is a pure logarithm which is a feature of $c = 1$ theories. We study various multi-critical regimes and relate the traces of inverse powers of $\Lambda$ (the Kontsevich–Miwa variables) to sources for conformal operators. We calculate the correlation functions of the conformal operators with the puncture operator and show that it is given by Legendre polynomials whose argument is determined by an analog of the string equation which is in our case a set of two equations for two variables. We calculate the proper anomalous dimensions which turn out to be different at different multi-critical points.
2 Formulation of the model

We consider the hermitian one-matrix model in an external field:

\[ Z[\Lambda] = \int DX e^{N \text{tr}\{ -\frac{1}{2} \Lambda X \Lambda X + \alpha \log(1 + X) - X \}}, \quad \Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_N). \]  

(2.1)

Our strategy to deal with the model (2.1) is to reduce it to the Kontsevich type integral (1.1). This can be done by a substitution \( \Lambda_1^{1/2} X \Lambda_1^{1/2} \rightarrow X \). We obtain

\[ Z[\Lambda] = e^{-N \text{tr} \Lambda^2 + \alpha N^2 (\det \Lambda)^{-N(\alpha+1)}} \times \int DX e^{N \text{tr}\{ -\frac{X^2}{2} + \eta X \}} (\det X)^{\alpha N}, \quad \eta = \Lambda - \alpha \Lambda^{-1}. \]  

(2.2)

Now our goal is to study the integral

\[ Z[\eta] = \int DX e^{-N \text{tr}\{ \frac{1}{2} X^2 - \eta X \}} (\det X)^{\alpha N}. \]  

(2.3)

There are different techniques to proceed with this integral. In the next sections we shall treat it using the method [9] of Schwinger–Dyson equations written in terms of eigenvalues, which has been applied to hermitian one-matrix models in [6, 7, 8, 4], and discuss Virasoro constraints for this model.

We conclude this section with the Itzykson–Zuber–Mehta technique [18] for integration over angular variables in multi-matrix models. We assume that it is well known and do not intend to engage into any detailed description. In terms of eigenvalues of the matrices \( X \) and \( \eta \), (2.3) can be expressed as follows:

\[ Z[\eta] = Z \int_\infty^{-\infty} N! \prod_{i=1}^N x_i^{\alpha N} dx_i \frac{\Delta[x]}{\Delta[\eta]} \exp \left\{ -N \sum_{i=1}^N \left( \frac{x_i^2}{2} - \eta_i x_i \right) \right\}, \]  

(2.4)

where \( \Delta[x] = \prod_{i>j}^N (x_i - x_j) \) is the Van der Monde determinant. In what follows we assume that \( \alpha N \) is positive integer. Using the relation

\[ \int_\infty^{-\infty} dx x^m e^{-x^2/2 + \eta x} = i^{-m} H_n(i\eta)e^{\eta^2/2} \]  

(2.5)

we may integrate out all \( x_i \)'s in the Eq.(2.4):

\[ Z[\eta] = Z N^{\alpha N^2/2 + N(N+1)/2} \Delta^{-1}[\eta] \prod_{1 \leq i < j \leq N} \| i^{\alpha N - (k-1)} H_{\alpha N + k - 1}(i\sqrt{N} \eta_j) \| \exp \left( N \sum_{i=1}^N \frac{\eta_i^2}{2} \right). \]  

(2.6)

Here \( H_n(i\eta) \) are the Hermite polynomials of an imaginary argument. As is shown by Kharchev et al. [11], the determinant formula (2.6) implies \( Z[\eta] \) to be a \( \tau \)-function.

It is clear from Eq.(2.6) that \( Z[\eta] \) is a symmetric polynomial of degree \( \alpha N^2 \) in variables \( \eta_i \). But it is still unclear how to deal further with the determinant. More effective way to obtain the answer is to use the Schwinger–Dyson equations generating Virasoro constraints and the “master equation” in this model.
3 Virasoro constraints

The Schwinger–Dyson equation for our model (2.1) follows from an invariance of $DX$ under ($X$-independent) variations of $X$:

\[
\int DX \frac{\delta}{\delta X_{ij}} e^{-N \text{tr} \left( \frac{X^2}{2} - \eta X - \alpha \log X \right)} = \left\langle \frac{\delta}{\delta X_{ij}} \text{tr} \left( \frac{X^2}{2} - \eta X - \alpha \log X \right) \right\rangle = 0. \quad (3.1)
\]

Taking into account that $N < \langle x_{ij} F \rangle = \delta_{ij} \langle F \rangle$ and applying one additional external derivative over $\eta$ to eliminate the nonlocal contribution arising from the variation of $\alpha \log X$, we have

\[
\left[ -\frac{\partial^2}{\partial \eta_{ij} \partial \eta_{jk}} + N \left( \eta \frac{\partial}{\partial \eta} \right)_{ik} + (1 + \alpha) N^2 \delta_{ik} \right] Z[\eta] = 0. \quad (3.2)
\]

This equation is discussed in detail below.

One could make alternatively the $X$-dependent variation $\delta X = \epsilon_n X^{n+1}$ which would immediately lead for $n \geq 0$ to a set of Virasoro constraints of the type advocated by Semenoff and one of the authors [8] for the model (1.1) with arbitrary polynomial $V(X)$:

\[
\mathcal{L}_n Z[\eta] = 0 \quad \text{for} \quad n \geq 0 \quad (3.3)
\]

and

\[
\mathcal{L}_n = \sum_i \left( -(\nabla_i)^{n+2} + N(\nabla_i)^{n+1} \eta_i + \alpha N(\nabla_i)^n + \frac{1}{2} \sum_{k=0}^{n} \sum_{j \neq i} (\nabla_i)^k(\nabla_j)^{n-k} \right) \quad (3.4)
\]

where

\[
\nabla_i = \frac{\partial}{\partial \eta_i} + \sum_{j \neq i} \frac{1}{\eta_i - \eta_j}. \quad (3.5)
\]

However, the $n = -1$ equation which corresponds in the polynomial case to $\mathcal{L}_{-1}$ operator is now nonlocal. This seems to be a reflection of the fact that we are dealing with a $c \neq 0$ theory.

Let us now turn to Eq.(3.2) which resembles in many details the equation $(\delta^2/\delta \eta^2 + \eta) Z[\eta] = 0$ generating [1,5] the continuum Virasoro constraints in the standard Kontsevich model. For our case we do not know in advance proper expressions for continuum time-variables in terms of the eigenvalues of $\Lambda$. We show below by explicitly solving the model (2.1) in genus zero that those are given by the Kontsevich–Miwa transformation while the problem of constructing continuum Virasoro (or W-type) constraints for our model is beyond the scope of present publication.

However, it is worth mentioning that for the model (2.1) there exists some Virasoro algebra in terms of “reverse times” $g_k = \frac{1}{k N} \text{tr} \eta^k$ while the standard Kontsevich–Miwa times are defined via expansion in negative powers of the matrix $\eta$.

Doing the contraction of Eq.(3.2) with $(\eta^n)_{ki}$ we obtain the set of constraints:

\[
\sum_{s=2}^{\infty} q_{s+n-2} L_s Z[\eta], \quad (3.6)
\]
where

\[ L_s = \sum_{a+b=s} \frac{\partial^2}{\partial q_a \partial q_b} + \sum_{b=1}^{\infty} b q_b \frac{\partial}{\partial q_{s+b}} + \frac{\partial}{\partial q_s} + \frac{\partial}{\partial q_{s-2}} - (1 + \alpha) \delta_{s,2}. \] (3.7)

One may prove that for finite \( N \) these conditions are sufficient to fix uniquely the form of the solution which is the polynomial of the maximum order \( \alpha N^2 \) in \( q_i \) multiplied by a trivial Gaussian factor. The continuous limit is rather nontrivial and will be discussed elsewhere.

We shall consider in what follows not the Eq. (3.2) but the equation for the ratio of the determinants \( Z[\eta] \). \( Z[\eta] \) differs from it by a factor \( e^{N \sum q_i} \). Pulling this factor through Eq. (3.2), we have

\[ \left[ - \left( \frac{\partial^2}{\partial \eta \partial \eta} \right)_{ik} - N \left( \eta \frac{\partial}{\partial \eta} \right)_{ik} + \alpha N^2 \delta_{ik} \right] \frac{\det H_{\alpha N+k-1} (i \sqrt{N} \eta_j)}{\Delta[\eta]} = 0. \] (3.8)

There also exists an interesting representation of these constraints not in Kontsevich–Miwa but rather Macdonald’s like variables [19]:

\[ g_k = \sum_{\{i_1, \ldots, i_k\}} \eta_{i_1} \cdots \eta_{i_k}, \] (3.9)

where the sum runs over all products of \( k \) different \( \eta_i \)'s. Then the symmetric function in Eq. (3.8) is the polynomial of degree \( \alpha N \) (not \( \alpha N^2 \)) in the variables \( g_k \), the higher term being \( g_{N^N} \). Moreover, the coefficient standing by term \( g_{N-k_1} \cdots g_{N-k_{\alpha N}} \) depends only on the set \( \{k_1, \ldots, k_{\alpha N}\} \) but not only \( N \) itself. In this case the explicit solutions for few lowest values of \( \alpha N \) can be found.

4 The “master equation”

In this section we rewrite Eq. (3.2) (or (3.3)) in terms of eigenvalues of the matrix \( \eta \). We obtain an equation (the “master equation”) which is similar to that of Refs. [7, 8, 4] for the Kontsevich model.

We want to find a suitable form of Eq. (3.8) in terms of eigenvalues of the matrix \( \eta \):

\[ \eta = U H U^+, \quad H = \text{diag} (\eta_1, \ldots, \eta_N), \] (4.1)

where \( U \) is a unitary matrix. We contract Eq. (3.8) with the special matrix \( F^{(j)} \) of the form:

\[ F^{(j)} = U f_j U^+, \quad f_j = \text{diag} (0, \ldots, 0, 1(j), 0, \ldots, 0). \] (4.2)

Acting on arbitrary monom \( t_{a_1} \cdots t_{a_s} \) composed from the times \( t_{a_i} = \text{tr} \eta^{a_i} \), where \( a_i \) are no more restricted to be nonnegative integers, we have

\[ \text{tr} \left( F^{(j)} \eta \frac{\partial}{\partial \eta} \right) = \eta_j \frac{\partial}{\partial \eta_j}. \] (4.3)
and for the first term:
\[
\text{tr}\left\{ (U_f U^+)_k \frac{\partial^2}{\partial \eta_{ls} \partial \eta_{kt}} \right\} t_{a_1} \cdots t_{a_s} = \\
= (U_f U^+)_k \frac{\partial}{\partial \eta_{ls}} \left\{ \sum_{i=1}^s a_i (\eta^{a_i-1})_{ik} t_{a_1} \cdots \hat{t}_{a_i} \cdots t_{a_s} \right\} = \\
= (U_f U^+)_k \left\{ \sum_{i=1}^s a_i a_p (\eta^{a_i-1})_{ik} (\eta^{a_p-1})_{sl} t_{a_1} \cdots \hat{t}_{a_i} \cdots t_{a_p} \cdots t_{a_s} + \\
+ \sum_{i=1}^s a_i \left( \frac{\partial}{\partial \eta_{ls}} (\eta^{a_i-1})_{tk} \right) t_{a_1} \cdots \hat{t}_{a_i} \cdots t_{a_s} \right\} = \left\{ \left( \frac{\partial}{\partial \eta_j} \right)^2 + \sum_{i \neq j} \frac{\partial_j - \partial_i}{\eta_j - \eta_i} \right\} t_{a_1} \cdots t_{a_s}. \tag{4.4}
\]

Thus we gave the direct proof of the “master equation” in our theory:
\[
\left\{ \partial_j^2 + \sum_{i \neq j} \frac{\partial_j - \partial_i}{\eta_j - \eta_i} + N \eta_j \partial_j - \alpha N^2 \right\} \tilde{Z}[\eta] = 0, \tag{4.5}
\]

where \( \tilde{Z} \) stands for the ratio of the determinants (2.9).

5 The genus zero solution

Now we concentrate on the solution to Eq. (4.5) in the spherical (or genus zero) limit \( N \to \infty \). In order to describe the solution of Eq. (4.5) at large \( N \), let us define the effective potential
\[
W[\eta] \equiv \frac{1}{N^2} \log Z[\eta] \tag{5.1}
\]
which is normalized to be \( O(1) \) as \( N \to \infty \). The method of solving Eq. (4.5) in the large-\( N \) limit is described in the Appendix. The result for the initial \( W[\eta] \) reads
\[
W[\eta] = \frac{1}{2} \left( \tilde{\alpha} - \frac{1}{2} \right)^2 \log(c - b^2) - \frac{5}{2} b^2 c - \tilde{\alpha} c + \frac{c^2}{4} + 3\tilde{\alpha} b^2 + \frac{9}{4} b^4 + \\
+ \frac{1}{N} \sum_i \left[ \frac{1}{4} \eta_i^2 + \left( \frac{\eta_i}{2} - b \right) \sqrt{\eta_i^2 + b \eta_i + c + \tilde{\alpha} \log \left( \eta_i + 2b + \sqrt{\eta_i^2 + 4b \eta_i + 4c} \right)} \right] - \\
- \frac{1}{4N^2} \sum_{i,j} \log \left[ \frac{\eta_i \eta_j}{4} + \frac{b}{2} (\eta_i + \eta_j) + c + \sqrt{\frac{\eta_i^2}{4} + b \eta_i + c} \sqrt{\frac{\eta_j^2}{4} + b \eta_j + c} \right]. \tag{5.2}
\]

The variables \( b \) and \( c \) are functionals of \( \rho \) and \( \tilde{\alpha} \). They are determined by the nonlinear constraints:
\[
b + \frac{1}{4N} \sum_i \frac{1}{\sqrt{\eta_i^2 + b \eta_i + c}} = 0, \tag{5.3}
\]
and
\[
\frac{1}{4N} \sum_i \frac{\eta_i}{\sqrt{\eta_i^2 + b \eta_i + c}} + c - 3b^2 = \tilde{\alpha}. \tag{5.4}
\]
Due to these conditions the expression Eq.(5.2) is stationary w.r.t. the differentiation over \( b \) and \( c \).

Among others, Eq.(5.2) possesses a remarkable property. If we take the double derivative of \( W[\eta] \) over \( \tilde{\alpha} \), then, again due to Eqs.(5.3) and (5.4), we get

\[
\frac{d^2}{d\alpha^2} W[\eta] = \log(b^2 - c),
\]

and the result coincides with that of applying the partial derivative \( \partial^2/\partial \alpha^2 \) to \( W[\eta] \). It is worth to note that the answer is a pure logarithm. We shall show in a moment that the logarithm is immediately connected to the logarithmic scaling violation for \( c = 1 \) models.

To deal with Eqs.(5.3), (5.4), let us expand the l.h.s.’s in \( 1/\eta \). Using the relation

\[
\frac{1}{\sqrt{1 - 2xy + y^2}} = \sum_{n=0}^{\infty} P_n(x)y^n,
\]

where \( P_n(x) \) are the standard Legendre polynomials which are normalized by \( P_n(1) = 1 \), we rewrite Eq.(5.3) as

\[
\sum_{n=1}^{\infty} t_n P_{n-1} \left( -\frac{b}{\sqrt{c}} \right) (2\sqrt{c})^{n-1} = 0
\]

and Eq.(5.4) as

\[
\sum_{n=1}^{\infty} t_n P_n \left( -\frac{b}{\sqrt{c}} \right) (2\sqrt{c})^n = 2\alpha
\]

where

\[
t_n = \frac{1}{N} \sum_i \frac{1}{\eta_i^n} - \delta_{n2} \quad n \geq 1.
\]

Some comments are now in order. The transformation from the variables \( \eta_i \) to \( t_n \) given by Eq.(5.9) is that of the Kontsevich–Miwa type for the (generalized) Kontsevich model. The variables \( t_n \) become independent as \( N \to \infty \) which is just our case. The role of \( t_0 \), the cosmological constant, is now played by \( -\alpha \). Eqs.(5.7), (5.8) resemble the corresponding equation for the Kontsevich model [8], which is in that case nothing but the genus zero string equation, while we have now two variables \( b \) and \( c \). The extra shift of \( t_2 \) in Eq.(5.9) is similar to that in the Kontsevich model (or in 2D topological gravity) where it is associated with a perturbative background. Eqs.(5.7), (5.8) can easily be solved in the following cases:

(i) For \( \eta_i \to \infty \) when \( t_2 = -1, \ t_1 = t_3 = \ldots = 0 \). The solution is \( b = 0, c = \alpha \) so that Eq.(5.3) gives \( \log(-\alpha) \) which recovers the known solution of the Penner model in genus zero.

(ii) If all \( \eta_i \)’s are constant: \( \eta_i \equiv s \). One obtains the one-cut solution of the generalized Penner model with the Gaussian term added [10, 11]. In this case Eqs.(5.3) and (5.4) lead to the relations

\[
c - b^2 = \left( \frac{1}{4b} - b - \frac{s}{2} \right) \left( \frac{1}{4b} + b + \frac{s}{2} \right)
\]

(5.10)
and
\[
\alpha = \left( \frac{1}{4b} - 3b - \frac{s}{2} \right) \left( \frac{1}{4b} + b + \frac{s}{2} \right).
\]  
(5.11)

There are two possible singular points: the first is \(\alpha = 0\) resulting from \(1/4b + b + s/2 = 0\), and it is the auxiliary singularity of the Penner model. The second possibility is \(1/4b - b - s/2 = 0\), and it immediately gives us \(\alpha = -1\), without any reference to the value of \(s\). It is just the true nonperturbative critical point of the standard Penner model.

(iii) For a pure Gaussian case \(\alpha = 0\). One gets \(b = -\sqrt{c}\) (remember that \(P_n(1) = 1\)).

One can construct perturbations of these models by finding the corresponding perturbative solutions of Eqs. (5.7), (5.8) around the discussed ones. One should not take the double-scaling limit of the model (2.1). Similarly to the (generalized) Kontsevich model, it describes already a continuum case.

An example of such a perturbation of the Gaussian case (iii) by the Penner action can be obtained for \(\alpha \to 0\). Let \(b + \sqrt{c} \sim \alpha\). Eq. (5.7) determines then \(\sqrt{c}\) versus \(\{t_n\}\):
\[
\sum_{n=1}^{\infty} t_n \left( 2\sqrt{c} \right)^{n-1} = 0,
\]  
(5.12)

while Eq. (5.8) gives \(b + \sqrt{c}\) versus \(\alpha\):
\[
(b + \sqrt{c}) \sum_{n=1}^{\infty} n(n+1)t_n \left( 2\sqrt{c} \right)^{n-1} = -2\alpha
\]  
(5.13)

(remember that \(P'_n(1) = n(n+2)/2\)). Eq. (5.13) shows again a logarithmic dependence on \(\alpha\).

Let us discuss now an interesting nonperturbative solutions to Eqs. (5.7), (5.8). It looks similar to the multi-critical one-cut solution of the hermitian one-matrix model with a polynomial potential, where for the \(K\)th multi-critical point one puts all \(t_n = 0\) except \(n = 0, K\). Let us take \(t_K \neq 0, t_1 = \ldots = t_{K-1} = t_{K+1} = \ldots = 0\). In this case the solution to Eq. (5.7) reads
\[
-\frac{b}{\sqrt{c}} = x_j^{(K-1)},
\]  
(5.14)

where \(-1 < x_j^{(K-1)} < 1\) is \(j\)th root of equation \(P_{K-1}(x) = 0\) (every \(P_n(x)\) has exactly \(n\) zeros on the interval \([-1, 1]\)). Let us denote \(p_j^{(K)} = P_K(x_j^{(K-1)})\), all \(p_j^{(K)}\) being nonzero. Then for \(c\) and \(b^2 - c\) we have
\[
c = \frac{1}{4} \left[ \frac{2\alpha}{t_K p_j^{(K)}(x_j^{(K-1)})} \right]^{2/K},
\]  
(5.15)

\[
b^2 - c = \frac{1}{4} \left[ (x_j^{(K-1)})^2 - 1 \right] \left[ \frac{2\alpha}{t_K p_j^{(K)}(x_j^{(K-1)})} \right]^{2/K}.
\]  
(5.16)

One substitutes this solution into Eq. (5.3) to see a logarithmic dependence of \(W\) on \(\alpha\). Therefore we conclude that our solution corresponds to a \(c = 1\) theory.
In order to distinguish between different multi-critical points which are labeled by $K$, let us calculate the anomalous dimensions of proper conformal operators. This can be done by calculating their correlation functions with the puncture operator. For the partial derivative of $W$, given by Eq. (5.2), over $\alpha$, one gets

$$\frac{\partial}{\partial \alpha} W[\eta] = \alpha \log(c - b^2) - c + 3b^2 + \frac{1}{N} \sum_i \log \left( \eta_i + 2b + \sqrt{\eta_i^2 + 4b\eta_i + 4c} \right). \quad (5.17)$$

It is easy to verify that the r.h.s. is again stationary w.r.t. $b$ and $c$ due to Eqs. (5.7), (5.8).

The correlation functions can now be obtained expanding the r.h.s. of Eq. (5.17) in $\frac{1}{\eta}$ and representing the result via $t_n$’s. The exact answer reads

$$\frac{\partial}{\partial \alpha} W[\eta] = \frac{1}{N} \sum_i \log 2\eta_i + \alpha \log(c - b^2) - \sum_{n=1}^{\infty} \frac{1}{n} t_n P_n \left( -\frac{b}{\sqrt{c}} \right) (2\sqrt{c})^n. \quad (5.18)$$

We have explicitly verified this formula up to $n = 4$. The proof can be done by showing that after differentiation w.r.t. $b$ and $c$ and using the known rules for the derivative of Legendre polynomials:

$$(1 - x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)], \quad (5.19)$$

one obtains Eqs. (5.7), (5.8).

The first term on the r.h.s. of Eq. (5.18) is due to an extra factor in Eq. (2.2) in front of the integral while the remaining part looks very similar to the corresponding representation of the derivative of the free energy of the Kontsevich model w.r.t. the cosmological constant in genus zero (see, e.g. [20]). Now we have two potentials $b$ and $c$ so that functions of the ratio $\frac{b}{\sqrt{c}}$ appear which are given by Legendre polynomials.

The correlators of proper conformal operators $O_n$ with the puncture operator $P = O_0$ can now be obtained by differentiating Eq. (5.18). One gets

$$\langle O_n P \rangle \equiv \frac{\partial^2}{\partial t_n \partial \alpha} W[\eta] = -\frac{1}{n} P_n \left( -\frac{b}{\sqrt{c}} \right) (2\sqrt{c})^n. \quad (5.20)$$

This formula is an analog of the spherical limit of Gelfand–Dikii differential polynomials in the case of the Kontsevich model. For the limiting case (i) above, only correlators with even $n = 2m$ are nonvanishing and one obtains

$$\langle O_{2n} P \rangle = (-)^{m+1} \frac{(2m - 1)!}{(m!)^2} \alpha^m \quad (5.21)$$

in an agreement with an explicit calculation of Ref. [21].

Let us now return to our solution (5.15), (5.16). One sees that while the correlator of two puncture operators, given by Eq. (5.3), depends on $\alpha$ logarithmically, the operator $O_n$ is scaled as $\langle O_n \rangle \sim \alpha^\frac{K}{2}$. This dependence of $n$ agrees with the KPZ [22] spectrum for $c = 1$.

We think that further studies of the genus zero solutions to the model (2.1) as well as their extensions to higher genera deserve future investigation.
Appendix A  Explicit solution at large $N$

Eq. (4.5) can be explicitly solved as $N \to \infty$ while $\eta_i = O(1)$. In this standard planar approximation all terms in the exponent (2.4) (as well as $\log \Delta[x]$) are of the same order $O(N^2)$. The solution (5.2) may be obtained along the line of the method of Brézin and Gross [9] introduced for the unitary matrix model in an external field and applied to the hermitian one-matrix model with a cubic potential in [6, 7, 8].

At $N = \infty$ one defines the density of eigenvalues of $\eta$:

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \eta_i), \quad (A.1)$$

so that $W[\eta]$ becomes a functional of $\rho$ while

$$\frac{1}{N} \frac{\partial W[\eta]}{\partial \eta_i} \bigg|_{\eta_i=x} = \frac{d}{dx} \frac{\delta W[\rho]}{\delta \rho(x)} \equiv W(x). \quad (A.2)$$

Noticing that $dW(x)/dx$ is multiplied by $1/N^2$ — the factorization property in the large-$N$ limit — and can be omitted, Eq. (4.3) takes the form of an integral equation

$$W^2(x) + \int_a^d dy \rho(y) \frac{W(y) - W(x)}{y - x} + xW(x) = \alpha \quad \text{for} \quad x \in [a, d] \quad (A.3)$$

where $[a, d]$ is the support of $\rho$.

It is convenient to make a shift $W(x) \to W(x) - \frac{x^2}{2}$. Then Eq. (A.3) transforms into

$$W^2(x) + \int_a^d dy \rho(y) \frac{W(y) - W(x)}{y - x} = \frac{x^2}{4} + \tilde{\alpha} \quad \text{for} \quad x \in [a, d] \quad (A.4)$$

where $\tilde{\alpha} = \alpha + \frac{1}{2}$.

To solve Eq. (A.4) one reduces it to a Riemann–Hilbert problem. Let us define two functions

$$f(z) = \int_a^d dx \frac{\rho(x)}{z - x} \quad (A.5)$$

and

$$F(z) = \int_a^d dx \frac{\rho(x)W(x)}{z - x} \quad (A.6)$$

which are analytic with cuts from $a$ to $d$. It follows from Eqs. (A.5), (A.6) that

$$\text{Im } F = W(x) \text{Im } f \quad \text{for} \quad x \in [-\infty, +\infty]. \quad (A.7)$$

The idea is to choose Re $F$ to satisfy Eq. (A.4), i.e. to have

$$\text{Re } F = W^2(x) + W(x) \text{Re } f - \frac{x^2}{4} - \tilde{\alpha} \quad \text{for} \quad x \in [a, d]. \quad (A.8)$$

This suggests the following analytic ansatz

$$F(z) = W^2(z) + W(z)f(z) - z^2/4 - \tilde{\alpha}, \quad (A.9)$$
where \( W(z) \) is a real analytic function restricted by

\[
\text{Im} W(\text{Im} f + \text{Im} W) = 0 \quad \text{for } x \in [a, d] \tag{A.10}
\]

and

\[
\text{Im} W(\text{Re} f + 2 \text{Re} W) = 0 \quad \text{for } x \in [-\infty, +\infty] \tag{A.11}
\]

in order to satisfy Eqs. (A.8), (A.9). The third restriction on \( W(z) \) is imposed by the asymptotic condition

\[
W(z) \rightarrow z \rightarrow \infty \quad z/2 + O(z^{-1}) \tag{A.12}
\]

which is a consequence of Eq. (A.4).

To construct \( W(z) \) that satisfies (A.10)–(A.12), one assumes that \( \text{Im} W \neq 0 \) for \( x \in [-\beta_-, -\beta+] \), where \( -\beta_+ < a \) to satisfy Eq. (A.10). Let \( -\beta_- \) and \( -\beta_+ \) are the roots of equation \( \beta^2/4 + b\beta + c = 0 \). Now Eq. (A.11) implies

\[
\text{Im} \left( W(x) \sqrt{\frac{x^2}{4} + bx + c} \right) = -\frac{1}{2} f(x) \sqrt{-\frac{x^2}{4} - bx - c} \quad \text{for } x \in [-\beta_-, -\beta_] \tag{A.13}
\]

since \( f(x) \) is real on the real axis outside \([a, d]\). Then Cauchy’s theorem unambiguously determines \( W(x) \sqrt{\frac{x^2}{4} + bx + c} \) to be

\[
W(x) \sqrt{\frac{x^2}{4} + bx + c} = \frac{x^2}{4} + \frac{1}{2} bx + c - \frac{b^2}{2} + \frac{\tilde{\alpha}}{2} - \frac{1}{2} \int_a^d dy \frac{\rho(y)}{\sqrt{\frac{y^2}{4} + by + c}} \sqrt{\frac{x^2}{4} + by + c} \tag{A.14}
\]

where one has used Eq. (A.5) and the additive polynomial has been determined to satisfy the asymptotic condition (A.12).

To fix \( b \) and \( c \) we impose the requirement that \( W^2(z) \) does not have poles at \( z = -\beta_\pm \) which have been assumed in obtaining Eq. (A.11) from Eq. (A.9). As follows from Eq. (A.14), the poles are eliminated providing

\[
b + \frac{1}{4} \int_a^d dy \frac{\rho(y)}{\sqrt{\frac{y^2}{4} + by + c}} = 0
\]

\[
\frac{1}{4} \int_a^d dy \frac{y\rho(y)}{\sqrt{\frac{y^2}{4} + by + c}} + c - 3b^2 = \tilde{\alpha}. \tag{A.15}
\]

Finally, using Eq. (A.15), \( W(x) \) can be represented as

\[
W(x) = \sqrt{\frac{x^2}{4} + bx + c} + \frac{1}{2} \int_a^d dy \frac{\rho(y)}{\sqrt{\frac{y^2}{4} + by + c}} \frac{\sqrt{\frac{x^2}{4} + by + c} - \sqrt{\frac{y^2}{4} + by + c}}{x - y}, \tag{A.16}
\]

and the absence of undesirable poles is explicit in this form.

This form of solution is also convenient to show of how the obtained \( W(x) \) satisfies Eq. (A.2). When \( W(x) \) given by (A.16) is substituted into the l.h.s. of Eq. (A.2), the terms
which do not contain $\rho$ as well as those linear and bilinear in $\rho$ emerge. The key point is to show that the bilinear in $\rho$ term splits into the product of two functionals linear in $\rho$. Namely, after symmetrization this term became

$$\int_a^d dy \rho(y) \frac{\sqrt{y^2 + 4by + c}}{4}. \quad (A.17)$$

Other terms can be adjusted to eliminate the $\rho$-dependence. At this point the restrictions (A.15) arose.

The obtained $b$ and $c$ are functionals sophisticatedly depended on $\rho(x)$. To obtain $W[\rho]$ one should integrate $W(x)$ over $x$ and $\rho$. The simplest way to do it is to find such expression for $W(x)$ which is stationary w.r.t. the partial differentiation over $b$ and $c$. Then we can obtain $W[\rho]$ since the total derivative in $\rho$ will coincide with the partial derivative. It appears that the expression (A.14) possesses the necessary property and after a little algebra we obtain $W[\rho]$:

$$W[\rho] = f(\tilde{\alpha}) + \frac{1}{2} \left( \tilde{\alpha} - \frac{1}{2} \right)^2 \log(c - b^2) - \frac{5}{4} b^2 c - \tilde{\alpha} c + \frac{c^2}{4} + 3\tilde{\alpha} b^2 + \frac{9}{4} b^4 +$$

$$+ \int_a^d dx \rho(x) \left[ \frac{x^2}{4} + bx + c + \tilde{\alpha} \log \left( x + 2b + \sqrt{x^2 + 4bx + 4c} \right) \right] -$$

$$- \frac{1}{4} \int_a^d dx dy \rho(x) \rho(y) \log \left[ \frac{xy}{4} + \frac{b}{2} (x + y) + c + \sqrt{x^2 + bx + c} \sqrt{y^2 + by + c} \right] \quad (A.18)$$

where the integration "constant" $f(\tilde{\alpha})$ is not essential for applications. Similarly to the strong-coupling solution of Brézin and Gross [9], this $W[\rho]$ is stationary w.r.t. $b$ and $c$.

Coming back to Eq. (A.3), the solution to it differs from (A.18) by an irrelevant to the critical behaviour factor $-\frac{1}{4} \int_a^d dx x^2 \rho(x)$.

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