Padé interpolation and hypergeometric series

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Abstract: We propose a class of Padé interpolation problems whose general solution is expressible in terms of determinants of hypergeometric series.

Key words: Padé interpolation; hypergeometric series; Dodgson condensation; Krattenthaler determinant

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1 Introduction

In this paper we investigate a class of Padé interpolation problems to which the solutions are expressible in terms of determinants of hypergeometric series. Padé interpolation problems have been discussed by Spiridonov–Zhedanov [14] from the viewpoint of biorthogonal rational functions. They are also sources of the Lax pairs for discrete Painlevé equations constructed by Yamada [16], [17], and by Noumi–Tsujimoto–Yamada [11]. The goal of this paper is to clarify how hypergeometric series arise in Padé interpolation problems, by analyzing the determinantal expression of the general solution.

In Section 2, we formulate a general Padé interpolation problem and a universal determinant formula for the general solution (Theorem 2.1). We also show that the determinants expressing the general solution can be condensed to smaller determinants by a variation of the Dodgson condensation (Theorem 2.2). After these preliminaries, we investigate in Section 3 a class of Padé interpolation problems relevant to generalized hypergeometric series $\,_{r+1}F_r$. We propose there two types of formulas expressing the solutions in terms of determinants of generalized hypergeometric series. The first one (Theorem 3.1), derived through Theorem 2.2, is based on the condensation of determinants and Krattenthaler’s determinant formula, while the second (Theorem 3.2) is constructed by means of the Saalschütz summation formula for terminating $\,_{3}F_2$ series. We remark that Padé approximations to generalized hypergeometric functions have been discussed by Luke [7], [8]. It would be an important question to clarify the relationship between interpolations and approximations in the context of generalized hypergeometric functions. Section 4 is devoted to the extension of these results to three types of very well-poised hypergeometric series including basic (trigonometric) and elliptic hypergeometric series. The two determinant formulas of Theorem 4.1 and Theorem 4.2 are obtained by Warnaar’s elliptic extension of the Krattenthaler determinant and by the Frenkel–Turaev summation formula for terminating $\,_{10}V_9$ series.
Two fundamental tools of our approach are the condensation of determinants along a moving core (an identity of Sylvester type), and variations of Krattenthaler’s determinant formula. For the sake of convenience, these subjects are discussed separately in Appendix A and Appendix B respectively. Generalization of Sylvester’s identity on determinants has been developed extensively by Mühlbach–Gasca [9] (see also [1]). The version we use in this paper (Lemma A.2), based on the Neville elimination strategy, is originally due to Gasca–López-Carmona–Ramirez [2]. We also remark that Sylvester’s identity and its extensions play important roles in recent studies of integrable systems (see Spicer–Nijhoff–van der Kamp [13] for example). As to Appendix B, basic references are the works of Krattenthaler [5], [6] and Warnaar [15] (see also Normand [10] for recent works on the evaluation of determinants involving shifted factorials). Although the contents of these appendices are basically found in the literature, we include them as self-contained expositions which might be helpful to the reader.

Throughout this paper we use the following notation of submatrices and minor determinants. For an $m \times n$ matrix $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ (with entries in a commutative ring), we denote by

$$X_{i_1, \ldots, i_r}^{j_1, \ldots, j_s} = \begin{bmatrix} x_{i_1 j_1} & \cdots & x_{i_1 j_s} \\ \vdots & \ddots & \vdots \\ x_{i_r j_1} & \cdots & x_{i_r j_s} \end{bmatrix} = (x_{ia j_b})_{1 \leq a \leq r, 1 \leq b \leq s}$$

(1.1)

the $r \times s$ submatrix of $X$ with row indices $i_1, \ldots, i_r \in \{1, \ldots, m\}$ and column indices $j_1, \ldots, j_r \in \{1, \ldots, n\}$. When $r = s$, we denote by $\det X_{i_1, \ldots, i_r}^{j_1, \ldots, j_s}$ the corresponding minor determinant.

## 2 Padé interpolation problems and their determinant solutions

In this section we formulate general Padé interpolation problems and propose some universal determinant formulas for the solutions.

Let $f_0(x), f_1(x), \ldots, f_m(x)$ and $g_0(x), g_1(x), \ldots, g_n(x)$ be two sequences of linearly independent meromorphic functions in $x \in \mathbb{C}$ and set $N = m + n$. We consider a pair $(P_m(x), Q_n(x))$ of two functions

$$P_m(x) = p_{m0} f_0(x) + p_{m1} f_1(x) + \cdots + p_{mm} f_m(x),$$

$$Q_n(x) = q_{n0} g_0(x) + q_{n1} g_1(x) + \cdots + q_{nn} g_n(x),$$

(2.1)

which are expressed as $\mathbb{C}$-linear combinations of $f_j(x)$ and $g_j(x)$ respectively. Noting that the ratio $P_m(x)/Q_n(x)$ contains $N + 1 = m + n + 1$ arbitrary constants, we investigate the interpolation problem

$$\frac{P_m(u_0)}{Q_n(u_0)} = v_0, \quad \frac{P_m(u_1)}{Q_n(u_1)} = v_1, \ldots, \quad \frac{P_m(u_N)}{Q_n(u_N)} = v_N$$

(2.2)
for a set of $N + 1$ generic reference points $x = u_0, u_1, \ldots, u_N$ and a set of $N + 1$ prescribed values $v_0, v_1, \ldots, v_N$. This problem is equivalently rewritten as

$$P_m(u_k) : Q_n(u_k) = \lambda_k : \mu_k \quad (k = 0, 1, \ldots, N)$$

(2.3)

for $v_k = \lambda_k/\mu_k$ ($k = 0, 1, \ldots, N$). We remark that the Padé interpolation problem defined as above contains the Lagrange interpolation problem as a special case where $n = 0$ and $g_0(x) = 1$.

A general solution of this Padé interpolation problem is given as follows in terms of $(N + 2) \times (N + 2)$ determinants:

$$P_m(x) = \det \begin{bmatrix}
  f_0(x) & \cdots & f_m(x) & 0 & \cdots & 0 \\
  \mu_0f_0(u_0) & \cdots & \mu_0f_m(u_0) & \lambda_0g_0(u_0) & \cdots & \lambda_0g_n(u_0) \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  \mu_Nf_0(u_N) & \cdots & \mu_Nf_m(u_N) & \lambda_Ng_0(u_N) & \cdots & \lambda_Ng_n(u_N)
\end{bmatrix},$$

(2.4)

$$Q_n(x) = -\det \begin{bmatrix}
  0 & \cdots & 0 & g_0(x) & \cdots & g_n(x) \\
  \mu_0f_0(u_0) & \cdots & \mu_0f_m(u_0) & \lambda_0g_0(u_0) & \cdots & \lambda_0g_n(u_0) \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  \mu_Nf_0(u_N) & \cdots & \mu_Nf_m(u_N) & \lambda_Ng_0(u_N) & \cdots & \lambda_Ng_n(u_N)
\end{bmatrix}.$$

(2.5)

**Theorem 2.1** The pair of functions $(P_m(x), Q_n(x))$ defined by (2.4), (2.5) solves the Padé interpolation problem (2.3) if $(P_m(u_k), Q_n(u_k)) \neq (0, 0)$ for $k = 0, 1, \ldots, N$.

In order to prove that this pair $(P_m(x), Q_n(x))$ gives a solution of the interpolation problem, we introduce two parameters $\lambda, \mu$ and consider the $(N + 2) \times (N + 2)$ determinant

$$R_{m,n}(x; \lambda, \mu) = \det \begin{bmatrix}
  \mu f_0(x) & \cdots & \mu f_m(x) & \lambda g_0(x) & \cdots & \lambda g_n(x) \\
  \mu_0f_0(u_0) & \cdots & \mu_0f_m(u_0) & \lambda_0g_0(u_0) & \cdots & \lambda_0g_n(u_0) \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  \mu_Nf_0(u_N) & \cdots & \mu_Nf_m(u_N) & \lambda_Ng_0(u_N) & \cdots & \lambda_Ng_n(u_N)
\end{bmatrix}. $$

(2.6)

By decomposing the top row as

$$\mu \left( f_0(x), \cdots, f_m(x), 0, \cdots, 0 \right) + \lambda \left( 0, \cdots, 0, g_0(x), \cdots, g_n(x) \right),$$

(2.7)

we have

$$R_{m,n}(x; \lambda, \mu) = \mu P_m(x) - \lambda Q_n(x).$$

(2.8)

On the other hand, the determinantal expression of $R_{m,n}(x; \lambda, \mu)$ implies

$$R_{m,n}(u_k; \lambda_k, \mu_k) = \mu_k P_m(u_k) - \lambda_k Q_n(u_k) = 0 \quad (k = 0, 1, 2, \ldots, N),$$

(2.9)
and hence
\[ P_m(u_k) : Q_n(u_k) = \lambda_k : \mu_k \quad (k = 0, 1, \ldots, N) \] (2.10)
as desired.

The \((N + 2) \times (N + 2)\) determinants \((2.4), (2.5)\) representing \(P_m(x)\) and \(Q_n(x)\) can be condensed into an \((m + 1) \times (m + 1)\) and \((n + 1) \times (n + 1)\) determinants respectively, by means of a variation of the Dodgson condensation (see Appendix A).

We denote by
\[ F = (f_j(u_i))_{0 \leq i \leq n}, 0 \leq j \leq m, \quad G = (g_j(u_i))_{0 \leq i \leq N}, 0 \leq j \leq n \] (2.11)
the matrices defined by the values of the functions \(f_j(x)\) \((0 \leq j \leq m)\) and \(g_j(x)\) \((0 \leq j \leq n)\), respectively, at the reference points \(u_i\) \((0 \leq i \leq N)\). We assume that the configuration of reference points \(u_k\) \((k = 0, 1, \ldots, N)\) is generic in the sense that the minor determinants
\[ \det F_{i_i, \ldots, i_n}^i \quad (0 \leq i \leq m), \quad \det G_{i_0, \ldots, i_m}^i \quad (0 \leq i \leq n) \] (2.12)
of maximal size with consecutive rows are all nonzero.

Assuming that \(\lambda_k \neq 0, \mu_k \neq 0\) for \(k = 0, 1, \ldots, N\), we set
\[ U_{i,j} = \frac{\lambda_i}{\mu_i} \det \left[ \begin{array}{cccc} \frac{\mu_k}{\lambda_k} f_j(u_i) & g_0(u_i) & \cdots & g_n(u_i) \\ \frac{\mu_k+1}{\lambda_k+1} f_j(u_i+1) & g_0(u_i+1) & \cdots & g_n(u_i+1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mu_k+n+1}{\lambda_k+n+1} f_j(u_i+n+1) & g_0(u_i+n+1) & \cdots & g_n(u_i+n+1) \end{array} \right] \left( \det G_{0,1, \ldots, n}^{i+1} \right)^{-1} \] (2.13)
for \(0 \leq i < m, 0 \leq j \leq m\) and
\[ V_{i,j} = \frac{\mu_i}{\lambda_i} \det \left[ \begin{array}{cccc} \frac{\lambda_k}{\mu_k} g_j(u_i) & f_0(u_i) & \cdots & f_m(u_i) \\ \frac{\lambda_k+1}{\mu_k+1} g_j(u_i+1) & f_0(u_i+1) & \cdots & f_m(u_i+1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_k+m+1}{\mu_k+m+1} g_j(u_i+m+1) & f_0(u_i+m+1) & \cdots & f_m(u_i+m+1) \end{array} \right] \left( \det F_{0,1, \ldots, m}^{i+1} \right)^{-1} \] (2.14)
for \(0 \leq i < n, 0 \leq j \leq n\). Then by Lemma A.2 (of condensation along a moving core), the \((N + 2) \times (N + 2)\) determinants \((2.4), (2.5)\) are condensed as follows into \((m + 1) \times (m + 1)\) and \((n + 1) \times (n + 1)\) determinants respectively (see also \(A.18), (A.19)\).

**Theorem 2.2** The two functions \(P_m(x), Q_n(x)\) defined in Theorem 2.1 are expressed as follows
in terms of \((m+1)\times(m+1)\) and \((n+1)\times(n+1)\) determinants respectively:

\[
P_m(x) = \prod_{i=0}^{m-1} \mu_i \prod_{i=0}^{n} \lambda_{m+i} \det G_{0,1,...,n}^{i,...,m+n} \det \begin{bmatrix} f_0(x) & \ldots & f_m(x) \\ U_{0,0} & \ldots & U_{0,m} \\ \vdots & & \vdots \\ U_{m-1,0} & \ldots & U_{m-1,m} \end{bmatrix}, \tag{2.15}
\]

\[
Q_n(x) = \epsilon_{m,n} \prod_{i=0}^{n-1} \lambda_i \prod_{i=0}^{m} \mu_{n+i} \det F_{0,1,...,m}^{n,...,n+m} \det \begin{bmatrix} g_0(x) & \ldots & g_n(x) \\ V_{0,0} & \ldots & V_{0,n} \\ \vdots & & \vdots \\ V_{n-1,0} & \ldots & V_{n-1,n} \end{bmatrix}, \tag{2.16}
\]

where \(\epsilon_{m,n} = (-1)^{mn+m+n}\).

By expanding the determinants \(U_{ij}\) and \(V_{ij}\) along the first column we further obtain the series expansions

\[
U_{ij} = \sum_{k=0}^{n+1} (-1)^k \frac{\mu_{i+k} \lambda_i}{\lambda_{i+k} \mu_i} f_j(u_{i+k}) \frac{\det G_{0,1,...,n}^{i,...,i+k,...,i+n+1}}{\det G_{0,1,...,n}^{i,...,i+k,...,i+n+1}} \quad (0 \leq i < m, \ 0 \leq j \leq m), \tag{2.17}
\]

\[
V_{ij} = \sum_{k=0}^{m+1} (-1)^k \frac{\mu_{i+k} \lambda_i}{\lambda_{i+k} \mu_i} g_j(u_{i+k}) \frac{\det F_{0,1,...,m}^{n,...,n+i+1,...,n+m+1}}{\det F_{0,1,...,m}^{n,...,n+i+1,...,n+m+1}} \quad (0 \leq i < n, \ 0 \leq j \leq n). \tag{2.18}
\]

Hence the problem to determine \(P_m(x)\) and \(Q_n(x)\) is reduced to the computation of minor determinants of the matrices \(F = (f_j(u_i))_{i,j}\) and \(G = (g_j(u_i))_{i,j}\). We remark that these formulas for \(P_m(x)\) and \(Q_n(x)\) hold universally for any choice of the functions \(f_j(x)\) and \(g_j(x)\).

In Sections 3 and 4, we show that these expansion formulas (2.17), (2.18) in fact give rise to hypergeometric series of various types for appropriate choices of the functions \(f_j(x)\), \(g_j(x)\), the reference points \(u_k\) and the prescribed values \(v_k = \lambda_k/\mu_k\).

### 3 Hypergeometric series arising from determinants

We explain below how the series expansions (2.17), (2.18) can be used for generating hypergeometric series. In this section we use the notation of shifted factorials

\[(a)_n = a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a) \quad (n = 0, 1, 2, \ldots). \tag{3.1}\]

As a typical example, we consider the two sequences of rational functions

\[f_j(x) = \frac{(a+x)_j}{(b+x)_j}, \quad g_j(x) = \frac{(c+x)_j}{(d+x)_j} \quad (j = 0, 1, 2, \ldots) \tag{3.2}\]

with four complex parameters \(a\), \(b\), \(c\), \(d\), to form a pair \((P_m(x), Q_n(x))\) of rational functions

\[P_m(x) = \sum_{j=0}^{m} p_{m,j} \frac{(a+x)_j}{(b+x)_j}, \quad Q_n(x) = \sum_{j=0}^{n} q_{n,j} \frac{(c+x)_j}{(d+x)_j}. \tag{3.3}\]
Taking an arithmetic progression \( u_k = u + k \) \((k = 0, 1, 2, \ldots, N)\), we consider the Padé interpolation problem

\[
\frac{P_m(u + k)}{Q_n(u + k)} = \frac{\lambda_k}{\mu_k} \quad (k = 0, 1, \ldots, N).
\]  

**Theorem 3.1** Consider the Padé interpolation problem \((3.3)\), \((3.4)\) for the rational functions \(f_j(x)\), \(g_j(z)\) in \((3.2)\) and the reference points \(u_k = u + k \) \((k = 0, 1, \ldots, N; N = m + n)\). Then the solution \((P_m(x), Q_n(x))\) of Theorem \(2.1\) is explicitly given by

\[
P_m(x) = \frac{\prod_{i=1}^m \frac{1}{i!} (d - c)_i}{\prod_{i=0}^n (d + u_{m+i})_n} \prod_{i=0}^{m-1} \mu_i \prod_{i=0}^{n} \lambda_{m+i} \det \begin{bmatrix} f_0(x) & \ldots & f_m(x) \\ U_{0,0} & \ldots & U_{0,m} \\ \vdots & & \vdots \\ U_{m-1,0} & \ldots & U_{m-1,m} \end{bmatrix},
\]

\[
Q_n(x) = \epsilon_{m,n} \frac{\prod_{i=1}^n \frac{1}{i!} (b - a)_i}{\prod_{i=0}^m (b + u_{n+i})_m} \prod_{i=0}^{n-1} \nu_i \prod_{i=0}^{m} \mu_{n+i} \det \begin{bmatrix} g_0(x) & \ldots & g_n(x) \\ V_{0,0} & \ldots & V_{0,n} \\ \vdots & & \vdots \\ V_{n-1,0} & \ldots & V_{n-1,n} \end{bmatrix},
\]

where

\[
U_{ij} = \frac{(a + u_i)}{(b + u_i)} \sum_{k=0}^{n+1} (-1)^k \frac{(d + u_{i+k})_k}{(a + u_i)_k} \frac{(a + u_{i+j})_k}{(b + u_{i+j})_k} \frac{\mu_{i+k}}{\lambda_{i+k}} \frac{\lambda_i}{\mu_i},
\]

\[
V_{ij} = \frac{(c + u_i)}{(d + u_i)} \sum_{k=0}^{m+1} (-1)^k \frac{(b + u_{i+m})_k}{(c + u_i)_k} \frac{(c + u_{i+j})_k}{(b + u_{i+j})_k} \frac{\lambda_{i+m+k}}{\mu_{i+m+k}} \frac{\mu_i}{\lambda_i}.
\]

As we remarked in the previous section, the functions \(U_{ij}\) in Theorem \(2.2\) are expressed as

\[
U_{ij} = \sum_{k=0}^{n+1} (-1)^k \frac{\mu_{i+k}}{\lambda_{i+k}} \frac{\lambda_i}{\mu_i} f_j(u_{i+k}) \frac{\det G_{i,j}^{0,1,\ldots,i+1,n+1}}{\det G_{0,1,\ldots,n}^{i,j,i+1,n+1}}.
\]

Since \(u_k = u + k \) \((k = 0, 1, \ldots, N)\), we have

\[
f_j(u_{i+k}) = \frac{(a + u_{i+k})_j}{(b + u_{i+k})_j} = \frac{(a + u_i)_j}{(b + u_i)_j} \frac{(a + u_{i+j})_k}{(b + u_{i+j})_k} \frac{\lambda_{i+k}}{\lambda_i}.
\]

The \((n+1) \times (n+1)\) minor determinants of the matrix

\[
G = \begin{bmatrix} (c + u_i)_j \\ (d + u_i)_j \end{bmatrix}, \quad 0 \leq i \leq n, 0 \leq j \leq n
\]

can be computed by means of a special case of Krattenthaler’s determinant formula \([5]\) (see Appendix B). In fact by \((3.3)\), we have

\[
\det G_{0,1,\ldots,n}^{i,j,i+1,n+1} = \frac{\prod_{l=1}^n (d-c)_l}{\prod_{l=0}^n (d + u_{i+l})_n} \frac{(-1)^k (-n-1)_k (d + u_{i+n})_k}{k! (d + u_i)_k}. \]

\[\text{(3.12)}\]
Hence $U_{ij}$ is computed as

$$U_{ij} = \frac{(a + u_i)_j}{(b + u_i)_j} \sum_{k=0}^{n+1} \frac{(-n - 1)_k (d + u_{i+n})_k (a + u_{i+j})_k \mu_{i+k} \lambda_i}{k! (d + u_i)_k (a + u_i)_k (b + u_{i+j})_k \lambda_{i+k} \mu_i} \tag{3.13}$$

The corresponding formula for $V_{ij}$ is obtained by exchanging the roles of $(m, n)$, $(a, c)$ and $(b, d)$.

If we choose the prescribed values appropriately, the series $U_{ij}$ and $V_{ij}$ in Theorem 3.1 give rise to general hypergeometric series

$$r+1F_r \left[ \frac{\alpha_0, \alpha_1, \ldots, \alpha_r}{\beta_1, \ldots, \beta_r}; \frac{z}{w} \right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_r)_k}{(1)_k (\beta_1)_k \cdots (\beta_r)_k} z^k. \tag{3.14}$$

Consider the case where

$$v_k = \frac{\lambda_k}{\mu_k} = \frac{(s_1)_k \cdots (s_r)_k}{(t_1)_k \cdots (t_r)_k} \left( \frac{z}{w} \right)^k \quad (k = 0, 1, 2, \ldots, N) \tag{3.15}$$

with complex parameters $s_1, \ldots, s_r$ and $t_1, \ldots, t_r$. Since

$$\frac{\mu_{i+k} \lambda_i}{\lambda_{i+k} \mu_i} = \frac{(t_1 + i)_k \cdots (t_r + i)_k}{(s_1 + i)_k \cdots (s_r + i)_k} \left( \frac{w}{z} \right)^k, \tag{3.16}$$

$U_{ij}$ and $V_{ij}$ are determined as

$$U_{ij} = \frac{(a + u_i)_j}{(b + u_i)_j} r+4F_{r+3} \left[ \frac{-n - 1, d + u_{i+n}, a + u_{i+j}, b + u_i, t_1 + i, \ldots, t_r + i}{d + u_i, a + u_i, b + u_{i+j}, s_1 + i, \ldots, s_r + i}; \frac{w}{z} \right],$$

$$V_{ij} = \frac{(c + u_i)_j}{(d + u_i)_j} r+4F_{r+3} \left[ \frac{-m - 1, b + u_{i+n}, c + u_{i+j}, d + u_i, s_1 + i, \ldots, s_r + i}{b + u_i, c + u_i, d + u_{i+j}, t_1 + i, \ldots, t_r + i}; \frac{z}{w} \right]. \tag{3.17}$$

If we choose the prescribed values

$$v_k = \frac{\lambda_k}{\mu_k} = \frac{(b + u)_k (c + u)_k (s_1)_k \cdots (s_r)_k}{(a + u)_k (d + u)_k (t_1)_k \cdots (t_r)_k} \left( \frac{z}{w} \right)^k \quad (k = 0, 1, 2, \ldots, N), \tag{3.18}$$

then $U_{ij}$ and $V_{ij}$ are slightly simplified as

$$U_{ij} = \frac{(a + u_i)_j}{(b + u_i)_j} r+3F_{r+2} \left[ \frac{-n - 1, d + u_{i+n}, a + u_{i+j}, t_1 + i, \ldots, t_r + i}{c + u_i, b + u_{i+j}, s_1 + i, \ldots, s_r + i}; \frac{w}{z} \right],$$

$$V_{ij} = \frac{(c + u_i)_j}{(d + u_i)_j} r+3F_{r+2} \left[ \frac{-m - 1, b + u_{i+n}, c + u_{i+j}, s_1 + i, \ldots, s_r + i}{a + u_i, d + u_{i+j}, t_1 + i, \ldots, t_r + i}; \frac{z}{w} \right]. \tag{3.19}$$

As for the Padé interpolation problem for the rational functions $f_j(x)$ and $g_j(x)$ as in (3.2), one can construct another type of determinant formula for $P_m(x)$ and $Q_n(x)$ involving hypergeometric series.
Theorem 3.2 Consider the Padé interpolation problem \([3,3], [3,1]\) for the rational functions \(f_j(x), g_j(z)\) in \([3,2]\) and the reference points \(u_k = u + k \ (k = 0, 1, \ldots, N; N = m + n)\). Then the solution \((P_m(x), Q_n(x))\) of Theorem 2.1 is expressed as

\[
P_m(x) = K_{m,n}(c,d) \prod_{i=0}^{N} \lambda_i \det \begin{bmatrix} f_0(x) & \cdots & f_m(x) \\ \Phi_{0,0} & \cdots & \Phi_{0,m} \\ \vdots & & \vdots \\ \Phi_{m-1,0} & \cdots & \Phi_{m-1,m} \end{bmatrix},
\]

\[
Q_n(x) = \epsilon_{m,n} K_{m,n}(a,b) \prod_{i=0}^{N} \mu_i \det \begin{bmatrix} g_0(x) & \cdots & g_n(x) \\ \Psi_{0,0} & \cdots & \Psi_{0,n} \\ \vdots & & \vdots \\ \Psi_{n-1,0} & \cdots & \Psi_{n-1,n} \end{bmatrix},
\]

where

\[
\Phi_{ij} = \frac{(a + u_j)(-N_k)(d + u + N - 1 - i)_k}{(b + u_j)(1)_k(c + u - i)_k} \left( (c + u)_k(a + u_j)_k(b + u)_k \mu_k \right),
\]

\[
\Psi_{ij} = \frac{(c + u_j)(-N_k)(b + u + N - 1 - i)_k}{(d + u_j)(1)_k(a + u - i)_k} \left( (a + u_j)_k(c + u)_k(d + u)_k \lambda_k \right).
\]

We remark that if the prescribed values are given by

\[
v_k = \frac{\lambda_k}{\mu_k} = \frac{(s_1)_k \cdots (s_r)_k}{(t_1)_k \cdots (t_r)_k} \left( \frac{z}{w} \right)^k \quad (k = 0, 1, 2, \ldots, N),
\]

then \(\Phi_{ij}\) and \(\Psi_{ij}\) give rise to generalized hypergeometric series

\[
\Phi_{ij} = \frac{(a + u_j)}{(b + u_j)} \, _{r+5}F_{r+4} \begin{bmatrix} -N, d + u + N - 1 - i, c + u, a + u + j, b + u, t_1, \ldots, t_r; w \\ c + u - i, d + u, b + u + j, a + u, s_1, \ldots, s_r \end{bmatrix},
\]

\[
\Psi_{ij} = \frac{(c + u_j)}{(d + u_j)} \, _{r+5}F_{r+4} \begin{bmatrix} -N, b + u + N - 1 - i, a + u, c + u + j, d + u, s_1, \ldots, s_r; z \\ a + u - i, b + u, d + u + j, c + u, t_1, \ldots, t_r \end{bmatrix}.
\]

If we choose

\[
v_k = \frac{\lambda_k}{\mu_k} = \frac{(b + u)_k(c + u)_k (s_1)_k \cdots (s_r)_k}{(a + u)_k(d + u)_k (t_1)_k \cdots (t_r)_k} \left( \frac{z}{w} \right)^k \quad (k = 0, 1, 2, \ldots, N),
\]

then \(\Phi_{ij}\) and \(\Psi_{ij}\) are simplified as

\[
\Phi_{ij} = \frac{(a + u_j)}{(b + u_j)} \, _{r+3}F_{r+2} \begin{bmatrix} -N, d + u + N - 1 - i, a + u + j, t_1, \ldots, t_r; w \\ c + u - i, b + u + j, s_1, \ldots, s_r \end{bmatrix},
\]

\[
\Psi_{ij} = \frac{(c + u_j)}{(d + u_j)} \, _{r+3}F_{r+2} \begin{bmatrix} -N, b + u + N - 1 - i, c + u + j, s_1, \ldots, s_r; z \\ a + u - i, d + u + j, t_1, \ldots, t_r \end{bmatrix}.
\]
In order to obtain the expression of Theorem 3.2, we first rewrite (2.4) as

\[
P_m(x) = \prod_{i=0}^{N} \lambda_i \det \begin{bmatrix} f_0(x) & \cdots & f_m(x) & 0 & \cdots & 0 \\ \frac{\mu_0}{\lambda_0} f_0(u_0) & \cdots & \frac{\mu_0}{\lambda_0} f_m(u_0) & g_0(u_0) & \cdots & g_n(u_0) \\ \vdots & & \vdots & & \vdots & \\ \frac{\mu_N}{\lambda_N} f_0(u_N) & \cdots & \frac{\mu_N}{\lambda_N} f_m(u_N) & g_0(u_N) & \cdots & g_n(u_N) \end{bmatrix}\]

\[
= \prod_{i=0}^{N} \lambda_i \det \begin{bmatrix} f(x) & 0 \\ \bar{F} & G \end{bmatrix}.
\] (3.28)

We construct an \((N + 1) \times (N + 1)\) invertible matrix \(L = (L_{ij})_{i,j=0}^{N}\) such that \((LG)_{ij} = 0\) for \(i + j < N\), and define \(M = (M_{ij})_{i,j=0}^{n}\) by \(M_{ij} = (LG)_{m+i,j}\). If we set \(\Phi = L\bar{F}\), we have

\[
\begin{bmatrix} 1 \\ L \end{bmatrix} \begin{bmatrix} f(x) & 0 \\ \bar{F} & G \end{bmatrix} = \begin{bmatrix} f(x) & 0 \\ \Phi & LG \end{bmatrix} = \begin{bmatrix} f(x) & 0 \\ \Phi' & 0 \\ \Phi'' & M \end{bmatrix}.
\] (3.29)

Hence, by taking the determinants of the both sides we obtain

\[
P_m(x) = \lambda_0 \cdots \lambda_N \det \begin{bmatrix} f_0(x) & \cdots & f_m(x) \\ \Phi_{00} & \cdots & \Phi_{0m} \\ \vdots & & \vdots \\ \Phi_{m-1,0} & \cdots & \Phi_{m-1,m} \end{bmatrix} \frac{\det M}{\det L}.
\] (3.30)

which will give formula (3.20) with \(K_{m,n}(c,d) = \det M/\det L\). In view of

\[
g_j(u_k) = \frac{(c + u_k)_{j}}{(d + u_k)_{j}} = \frac{(d + u)_{k} (c + u_j)_{k}}{(c + u)_{k} (d + u_j)_{k}},
\] (3.31)

we recall the Saalschütz sum

\[
\begin{aligned}
\binom{-N, d + u + N - 1 - i, c + u + j}{c + u - i, d + u + j} &= \frac{(d - c)_{N} (-i - j)_{N}}{(c + u - i)_{N} (d + u + j)_{N}},
\end{aligned}
\] (3.32)

\[
\sum_{k=0}^{N} \frac{(-N)_{k} (d + u + N - 1 - i)_{k} (c + u_j)_{k}}{(1)_{k} (c + u - i)_{k} (d + u_j)_{k}} = \frac{(d - c)_{N} (-i - j)_{N}}{(c + u - i)_{N} (d + u + j)_{N}}.
\] (3.33)

With this observation, we define the matrix \(L = (L_{ij})_{i,j=0}^{N}\) by

\[
L_{ij} = \frac{(-N)_{j} (d + u + N - 1 - i)_{j} (c + u)_{j}}{(1)_{j} (c + u - i)_{j} (d + u)_{j}} (0 \leq i, j \leq N).
\] (3.34)
The we have

\[
(LG)_{ij} = \sum_{k=0}^{N} \frac{(-N)_k (d + u + N - 1 - i)_k (c + u j)_k (d + u)_j}{(1)_j (c + u - i)_k (d + u j)_k (d + u)_j}
\]

\[
= \frac{(d - c)N (-i - j)_N (c + u)_j}{(c + u - i)_N (d + u + j)_N (d + u)_j}
\]  

(3.35)

by the Saalschütz sum. In particular \((LG)_{ij} = 0\) \((i + j < N)\). The determinant of the matrix \(M\) is computed as

\[
\det M = (-1)^{\binom{n+1}{2}} \prod_{j=0}^{n} (LG)_{N-j,j} = (-1)^{\binom{n+1}{2}} \prod_{j=0}^{n} \frac{(d - c)N (-N)_N (c + u)_j}{(c + u - N + j)_N (d + u + j)_N (d + u)_j}.
\]  

(3.36)

Also, the entries of \(\Phi = L\tilde{F}\) are expressed as

\[
\Phi_{ij} = \sum_{k=0}^{N} L_{ik} \frac{\mu_k}{\lambda_k} f_j(u_k)
\]

\[
= \sum_{k=0}^{N} \frac{(-N)_k (d + u + N - 1 - i)_k (c + u)_k \mu_k (a + u_k)_j}{(1)_k (c + u - i)_k (d + u)_k \lambda_k (b + u_k)_j}
\]

\[
= \frac{(a + u)_j}{(b + u)_j} \sum_{k=0}^{N} \frac{(-N)_k (d + u + N - 1 - i)_k (c + u)_k (a + u_j)_k (b + u)_k \mu_k}{(1)_k (c + u - i)_k (d + u)_k (b + u)_k (a + u)_k \lambda_k}.
\]  

(3.37)

The determinant of \(L\) can be computed again by Krattenthaler’s formula:

\[
\det L = \det \left( \frac{(d + u + N - 1 - i)_j}{(c + u - i)_j} \right)^N \prod_{j=0}^{N} \frac{(-N)_j (c + u)_j}{(1)_j (d + u)_j}
\]

\[
= (-1)^{\binom{n+1}{2}} \prod_{j=0}^{N} \frac{(c - d - N + 1)_j (-N)_j (c + u)_j}{(c + u - j)_N (d + u)_j}.
\]  

(3.38)

The constant factor in (3.20) is determined as \(K_{m,n}(c,d) = \det M / \det L\).

4 Three types of very well-poised hypergeometric series

In this section we consider three classes of hypergeometric series

(0) rational . . . ordinary hypergeometric series
(1) trigonometric . . . basic (or \(q\)-)hypergeometric series
(2) elliptic . . . elliptic hypergeometric series

corresponding to the choice of a “fundamental" function \([x]\):

(0) rational : \([x] = e^{c_0 x^2 + c_1} x\) \((\Omega = 0)\)
(1) trigonometric : \([x] = e^{c_0 x^2 + c_1} \sin(\pi x/\omega)\) \((\Omega = \mathbb{Z} \omega)\)
(2) elliptic : \([x] = e^{c_0 x^2 + c_1} \sigma(x;\Omega)\) \((\Omega = \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2)\)
where $\sigma(x|\Omega)$ is the Weierstrass sigma function associated with the period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

It is known that these classes of functions $[x]$ are characterized by the so-called Riemann relation: For any $x, \alpha, \beta, \gamma \in \mathbb{C}$,

$$[x + \alpha][x - \alpha][\beta + \gamma][\beta - \gamma] + [x + \beta][x - \beta][\gamma + \alpha][\gamma - \alpha] + [x + \gamma][x - \gamma][\alpha + \beta][\alpha - \beta] = 0. \quad (4.1)$$

By the notation $[x \pm y] = [x + y][x - y]$ of the product of two factors, this relation is expressed as

$$[x + \alpha][\beta + \gamma] + [x + \beta][\gamma + \alpha] + [x + \gamma][\alpha + \beta] = 0. \quad (4.2)$$

In what follows, we fix a nonzero entire function $[x]$ satisfying this functional equation.

Fixing a generic constant $\delta$, we define the $\delta$-shifted factorials $[x]_k$ by

$$[x]_k = [x]_{k,0} = [x + \delta] \cdots [x + (k - 1)\delta] \quad (k = 0, 1, 2, \ldots). \quad (4.3)$$

Then we define the very well-poised hypergeometric series $r + 5V_{r+4}[a_0; a_1 \cdots a_r | z]$ associated with $[x]$ by

$$r + 5V_{r+4}[a_0; a_1 \cdots a_r | z] = \sum_{k=0}^{\infty} \frac{[a_0 + 2k\delta]}{[a_0]} \frac{[a_0]_k [a_1]_k \cdots [a_r]_k}{[\delta]_k [\delta + a_0 - a_1]_k \cdots [\delta + a_0 - a_r]_k} z^k. \quad (4.4)$$

In this paper we use this notation only for terminating series assuming that $a_i$ is of the form $-n\delta$ ($n = 0, 1, 2, \ldots$) for some $i$. When $z = 1$ we also write

$$r + 5V_{r+4}[a_0; a_1 \cdots a_r] = \sum_{k=0}^{\infty} \frac{[a_0 + 2k\delta]}{[a_0]} \frac{[a_0]_k [a_1]_k \cdots [a_r]_k}{[\delta]_k [\delta + a_0 - a_1]_k \cdots [\delta + a_0 - a_r]_k}. \quad (4.5)$$

In this notation, the celebrated Frenkel-Turaev sum is expressed as

$$10V_9[a_0; a_1, a_2, a_3, a_4, a_5] = \frac{[\delta + a_0]_N [\delta + a_0 - a_1 - a_2]_N [\delta + a_0 - a_1 - a_3]_N [\delta + a_0 - a_2 - a_3]_N}{[\delta + a_0 - a_1 - a_2]_N [\delta + a_0 - a_2]_N [\delta + a_0 - a_3]_N [\delta + a_0 - a_1 - a_2 - a_3]_N}, \quad (4.6)$$

under the balancing condition $a_1 + \cdots + a_5 = 2a_0 + \delta$ and the termination condition $a_5 = -N\delta$ ($N = 0, 1, 2, \ldots$). (See for example [3, 4].)

We remark that, in the rational case where $[x] = x$ and $\delta = 1$, the $r + 5V_{r+4}$ series defined above is expressed in terms of a $r+2F_{r+1}$-series:

$$r + 5V_{r+4}[a_0; a_1, \cdots, a_r | z] = r + 2F_{r+1} \left[ a_0, \frac{1}{2}a_0 + 1, a_1, \ldots, a_r; z \right] \quad (4.7)$$

where $b_i = 1 + a_0 - a_i$ ($i = 1, \ldots, r$). Also, in the trigonometric case where $[x] = e^{cx/2} - e^{-cx/2}$,

$$r + 5V_{r+4}[a_0; a_1, \cdots, a_r | z] = \sum_{k=0}^{\infty} \frac{1 - q^{2k}}{1 - t_0} \frac{(t_0; q)_k (t_1; q)_k \cdots (t_r; q)_k}{(q; q)_k (qt_0/t_1; q)_k \cdots (qt_0/t_r)_k} s^k \quad (4.8)$$

$$= r + 3W_{r+2} \left[ t_0; t_1, \ldots, t_r; q, s \right]$$
in the notation of very well-poised \( q \)-hypergeometric series [3], where \( q = e^{c\delta}, \ t_i = e^{c\delta_i} \ (i = 0, 1, \ldots, r) \) and \( s = (qt_0)^{-\frac{1}{2}} z/t_1 \cdots t_r \). We discuss below a class of Padé interpolation problems that can be formulated in an unified manner in the three types of very well-poised hypergeometric series.

Taking the two sequence of meromorphic functions

\[
\begin{align*}
 f_j(x) &= \frac{[a \pm x]_j}{[b \pm x]_j} = \frac{[a + x]_j[a - x]_j}{[b + x]_j[b - x]_j}, \\
 g_j(x) &= \frac{[c \pm x]_j}{[d \pm x]_j} = \frac{[c + x]_j[c - x]_j}{[d + x]_j[d - x]_j} \quad (j = 0, 1, 2, \ldots)
\end{align*}
\]

(4.9)

and the reference points \( u_k = u + k\delta \ (k = 0, 1, 2, \ldots) \), we consider the Padé interpolation problem

\[
P_m(u_k) = Q_n(u_k) = v_k = \frac{\lambda_k}{\mu_k} \quad (k = 0, 1, \ldots, N)
\]

(4.10)

for a pair of functions

\[
P_m(x) = p_{m,0} f_0(x) + p_{m,1} f_1(x) + \cdots + p_{m,m} f_m(x), \\
Q_n(x) = q_{n,0} g_0(x) + q_{n,1} g_1(x) + \cdots + q_{n,n} g_n(x)
\]

(4.11)

where \( N = m + n \). The prescribed values \( v_k = \lambda_k/\mu_k \ (k = 0, 1, 2, \ldots, N) \) will be specified later.

**Theorem 4.1** Consider the Padé interpolation problem (4.10), (4.11) for the functions \( f_j(x) \), \( g_j(x) \) in (4.9) and the reference points \( u_k = u + k\delta \ (k = 0, 1, \ldots, N; N = m + n) \). Then the solution \((P_m(x), Q_n(x))\) of Theorem 2.1 is explicitly given by

\[
P_m(x) = C_n(c, d) \prod_{l=0}^{n} \left[ 2u_m + l\delta \right] [\delta]_l \prod_{i=0}^{m-1} \mu_i \prod_{i=0}^{n} \lambda_{m+i} \det \begin{bmatrix}
 f_0(x) & \cdots & f_m(x) \\
 U_{0,0} & \cdots & U_{0,m} \\
 \vdots & \vdots & \vdots \\
 U_{m-1,0} & \cdots & U_{m-1,m}
\end{bmatrix},
\]

(4.12)

\[
Q_n(x) = \epsilon_{m,n} C_m(a, b) \prod_{l=0}^{m} \left[ 2u_n + l\delta \right] [\delta]_l \prod_{i=0}^{n-1} \lambda_i \prod_{i=0}^{m} \mu_{n+i} \det \begin{bmatrix}
 g_0(x) & \cdots & g_n(x) \\
 V_{0,0} & \cdots & V_{0,n} \\
 \vdots & \vdots & \vdots \\
 V_{n-1,0} & \cdots & V_{n-1,n}
\end{bmatrix},
\]

(4.13)

where

\[
C_n(c, d) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} [d - c]_k [c + d + (k - 1)\delta]_k, \quad \epsilon_{m,n} = (-1)^{mn+m+n}
\]

(4.14)
and

\[
U_{ij} = \frac{[a \pm u_{ij}]}{[b \pm u_{ij}]} \sum_{k=0}^{n+1} \left[ \begin{array}{c} 2u_i + 2k\delta \\ 2u_i \end{array} \right] \frac{[2u_i]}{[\delta][2u_i + (n+2)\delta]} \frac{[u_i - d + \delta]}{[u_i + d][u_i - d + (1-n)\delta]} \cdot \frac{[u_i - a + \delta]}{[u_i + a + \delta]}
\]

(4.15)

\[
V_{ij} = \frac{[c \pm u_{ij}]}{[d \pm u_{ij}]} \sum_{k=0}^{n+1} \left[ \begin{array}{c} 2u_i + 2k\delta \\ 2u_i \end{array} \right] \frac{[2u_i]}{[\delta][2u_i + (n+2)\delta]} \frac{[u_i - b + \delta]}{[u_i + b][u_i - b + (1-j)\delta]} \frac{[u_i - c + \delta]}{[u_i + c + \delta]}
\]

(4.16)

As before we consider the expansion

\[
U_{ij} = \sum_{k=0}^{n+1} (-1)^k \frac{G_{0,1,\ldots,i+k,\ldots,n+1}}{G_{0,1,\ldots,i+n+1}} f_j(u_{i+k}) \frac{\mu_{i+k}}{\lambda_{i+k} \mu_i}
\]

(4.17)

of the determinant of (2.13). In this case we have

\[
f_j(u_{i+k}) = \frac{[a \pm u_{i+k}]}{[b \pm u_{i+k}]} \frac{[u_i - a + \delta]}{[u_i + a + \delta]}
\]

(4.18)

The \((n+1) \times (n+1)\) minor determinants of the matrix

\[
G = (g_j(u_i))_{0 \leq i \leq N, 0 \leq j \leq n} = \left[ \begin{array}{c} [c \pm u_{ij}] \\ [d \pm u_{ij}] \end{array} \right]_{0 \leq i \leq N, 0 \leq j \leq n}
\]

(4.19)

can be computed by means of an elliptic extension of Krattenthaler’s formula (see Appendix B).

In fact, by (B.14) we have

\[
\det G_{0,1,\ldots,n}^{i,j} = C_n(c,d) \prod_{l=1}^{n} \frac{[2u_i + l\delta]}{[\delta]} \prod_{l=0}^{n} \frac{[d \pm u_{i+l}]}{[d + u_{i+l}]} \cdot (-1)^k \frac{[2u_i + 2k\delta]}{[2u_i]} \frac{[2u_i]}{[\delta][2u_i + (n+2)\delta]} \frac{[u_i - d + \delta]}{[u_i + d][u_i - d + (1-n)\delta]} \cdot \frac{[u_i - a + \delta]}{[u_i + a + \delta]}
\]

(4.20)

Hence \(U_{ij}\) is computed as

\[
U_{ij} = \frac{[a \pm u_{ij}]}{[b \pm u_{ij}]} \sum_{k=0}^{n+1} \left[ \begin{array}{c} 2u_i + 2k\delta \\ 2u_i \end{array} \right] \frac{[2u_i]}{[\delta][2u_i + (n+2)\delta]} \frac{[u_i - d + \delta]}{[u_i + d][u_i - d + (1-n)\delta]} \frac{[u_i - a + \delta]}{[u_i + a + \delta]}
\]

(4.21)

The corresponding formula for \(V_{ij}\) is obtained by exchanging the roles of \((m,n), (a,c)\) and \((b,d)\).
Consider the case where the prescribed values are specified as

\[ v_k = \frac{\lambda_k}{\mu_k} = \left( \frac{z}{w} \right)^k \prod_{s=1}^{r} \frac{[u - e_s + \delta]_k}{[u + e_s]_k} \quad (k = 0, 1, \ldots, N). \] (4.22)

Then we obtain very well-poised series

\[ U_{ij} = \frac{[a \pm u_i]}{[b \pm u_i]} j \cdot 12 V_{r+11} \left[ 2u_i; -(n + 1)\delta, u_i - d + \delta, u_i + d + n\delta, \right. \]
\[ u_i - a + \delta, u_i + a + j\delta, u_i + b, u_i - b + (1 - j)\delta, u_i + e_1, \ldots, u_i + e_r \left| \frac{w}{z} \right], \] (4.23)

\[ V_{ij} = \frac{[c \pm u_i]}{[d \pm u_i]} j \cdot 12 V_{r+11} \left[ 2u_i; -(m + 1)\delta, u_i - b + \delta, u_i + b + n\delta, \right. \]
\[ u_i - c + \delta, u_i + c + j\delta, u_i + d, u_i - d + (1 - j)\delta, u_i - e_1 + \delta, \ldots, u_i - e_r + \delta \left| \frac{z}{w} \right]. \] (4.24)

When

\[ v_k = \frac{\lambda_k}{\mu_k} = \left( \frac{z}{w} \right)^k \prod_{s=1}^{r} \frac{[u - e_s + \delta]_k}{[u + e_s]_k} \] (4.25)

we obtain simpler very well-poised hypergeometric series

\[ U_{ij} = \left[ a \pm u_i \right] j \cdot 10 V_{r+9} \left[ 2u_i; -(n + 1)\delta, u_i - c + \delta, u_i + d + n\delta, \right. \]
\[ u_i + a + j\delta, u_i - b + (1 - j)\delta, u_i + e_1, \ldots, u_i + e_r \left| \frac{w}{z} \right], \] (4.26)

\[ V_{ij} = \left[ c \pm u_i \right] j \cdot 10 V_{r+9} \left[ 2u_i; -(m + 1)\delta, u_i - a + \delta, u_i + b + n\delta, \right. \]
\[ u_i + c + j\delta, u_i - d + (1 - j)\delta, u_i - e_1 + \delta, \ldots, u_i - e_r + \delta \left| \frac{z}{w} \right]. \]

Another type of determinantal expression for \( P_m(x) \) and \( Q_n(x) \) is formulated as follows. We remark that this type of determinant formulas has also been discussed in \([11]\). In what follows, we use the notation

\[ V^{(k)} \left[ a_0; a_1, \ldots, a_r \right] = \frac{[a_0 + 2k\delta]}{[a_0]} \frac{[a_0]_k}{[\delta]_k} \frac{[a_1]_k}{[\delta + a_0 - a_1]_k} \cdots \frac{[a_r]_k}{[\delta + a_0 - a_r]_k}. \] (4.27)

**Theorem 4.2** Consider the Padé interpolation problem \((4.10), (4.11)\) for the functions \( f_j(x), g_j(z) \) in \((4.9)\) and the reference points \( u_k = u + k\delta \) \((k = 0, 1, \ldots, N; N = m + n)\). Then the solution
Then we obtain very well-poised series

\( P_n(x) = K_{m,n}(c, d) \prod_{i=0}^{N} \lambda_i \det \begin{bmatrix} f_0(x) & \cdots & f_m(x) \\ \Phi_{0,0} & \cdots & \Phi_{0,m} \\ \vdots & \ddots & \vdots \\ \Phi_{m-1,0} & \cdots & \Phi_{m-1,m} \end{bmatrix}, \)

(4.28)

\( Q_n(x) = \epsilon_{m,n} K_{n,m}(a, b) \prod_{i=0}^{N} \mu_i \det \begin{bmatrix} g_0(x) & \cdots & g_n(x) \\ \Psi_{0,0} & \cdots & \Psi_{0,n} \\ \vdots & \ddots & \vdots \\ \Psi_{n-1,0} & \cdots & \Psi_{n-1,n} \end{bmatrix}, \)

(4.29)

where

\[ \Phi_{ij} = \left[ \frac{a \pm u}{b \pm u} \right] \sum_{k=0}^{N} V^{(k)} \left[ 2u; -N\delta, u - c + \delta + i\delta, u + d + (N - 1)\delta - i\delta, u + c, u - d + \delta, \\
\lambda_k \right] \frac{\mu_k}{\lambda_k}, \]

(4.30)

\[ \Psi_{ij} = \left[ \frac{c \pm u}{d \pm u} \right] \sum_{k=0}^{N} V^{(k)} \left[ 2u; -N\delta, u - a + \delta + i\delta, u + b + (N - 1)\delta - i\delta, u + a, u - b + \delta, \\
\lambda_k \right] \frac{\mu_k}{\lambda_k}. \]

(4.31)

Consider the case where the prescribed values are specified as

\[ v_k = \frac{\lambda_k}{\mu_k} = \left( \frac{z}{w} \right) \prod_{s=1}^{r} \frac{[u - e_s + \delta]_k}{[u + e_s]_k} \quad (k = 0, 1, \ldots, N). \]

(4.32)

Then we obtain very well-poised series

\[ \Phi_{ij} = \left[ \frac{a \pm u}{b \pm u} \right]_{r+14} V_{r+13} \left[ 2u; -N\delta, u - c + \delta + i\delta, u + d + (N - 1)\delta - i\delta, u + c, u - d + \delta, \\
\lambda_k \right] \frac{\mu_k}{\lambda_k}, \]

(4.33)

\[ \Psi_{ij} = \left[ \frac{c \pm u}{d \pm u} \right]_{r+14} V_{r+13} \left[ 2u; -N\delta, u - a + \delta + i\delta, u + b + (N - 1)\delta - i\delta, u + a, u - b + \delta, \\
\lambda_k \right] \frac{\mu_k}{\lambda_k}. \]

(4.34)

When

\[ v_k = \frac{\lambda_k}{\mu_k} = \left( \frac{z}{w} \right) \prod_{s=1}^{r} \frac{[u - e_s + \delta]_k}{[u + e_s]_k} \]

(4.35)
we obtain simpler very well-poised hypergeometric series

\[
\Phi_{ij} = \left[\frac{a + u}{b + u}\right]_{r+10Vr+9} \left[2u; -N\delta, u - c + \delta + i\delta, u + d + (N - 1)\delta - i\delta, \\
u + a + j\delta, u - b + \delta - j\delta, u + e_1, \ldots, u + e_r\right]_N \left[\frac{w}{z}\right],
\]

(4.36)

\[
\Psi_{ij} = \left[\frac{c + u}{d + u}\right]_{r+10Vr+9} \left[2u; -N\delta, u - a + \delta + i\delta, u + b + (N - 1)\delta - i\delta, \\
u + c + j\delta, u - d + \delta - j\delta, u - e_1 + \delta, \ldots, u - e_r + \delta\right]_N \left[\frac{z}{w}\right].
\]

(4.37)

Theorem 4.2 can be proved by a procedure similar to the one we used in the previous section. In this case we define the matrix \(L = (L_{ij})_{i,j=0}^{N}\) by

\[
L_{ij} = V^{(j)} \left[2u; -N\delta, u - c + (1 + i)\delta, u + d + (N - 1 - i)\delta, u + c, u - d + \delta\right]
\]

(4.38)

for \(0 \leq i, j \leq N\). Then one can show

\[
(LG)_{ij} = \frac{[c + d + (j - i - 1)\delta]_N [-(i + j)\delta]_N [2u + \delta]_N [d - c]_N}{[u + d + j\delta]_N [u - c + (1 - j)\delta]_N [u + c - i\delta]_N [-u + d - (1 + i)\delta]_N [d + u]_N}
\]

by means of the Frenkel-Turaev sum, and hence \((LG)_{ij} = 0\) for \(i + j < N\). Then the series \(\Phi_{ij}\) are obtained by computing the product \(L\overline{F}\) as before. We remark that in this case

\[
\det M = (-1)^{(n+1)} \prod_{j=0}^{n} \left[\frac{c + u}{d + u}\right]_j \prod_{j=0}^{n} \frac{[c + d - (N + 1 - 2j)\delta]_N [-N\delta]_N [2u + \delta]_N [d - c]_N}{[u + d + j\delta]_N [u - c + (1 - j)\delta]_N [u + c - (N - j)\delta]_N [-u + d - (N + 1 - j)\delta]_N}. 
\]

(4.40)

The determinant of \(L\) can also be computed in a factorized form by the elliptic version (B.14) of Krattenthaler’s formula:

\[
\det L = \frac{\prod_{j=1}^{N} \delta_j [c + d + (N - 1 - 2j)\delta]_j [c - d - (N - 1)\delta]_j [2u + j\delta]_j}{\prod_{j=0}^{N} [u + c - i\delta]_N [u - d - (N - 2 - i)\delta]_N} \cdot \frac{\prod_{j=0}^{N} [2u + 2j\delta]_j [2u]_j [u + c + (N - 1)\delta]_j [u - d + \delta]_j [u + d]_j}{\prod_{j=0}^{N} [2u]_j [u + (N + 1)\delta]_j [u + c + \delta]_j [u - d + \delta]_j}. 
\]

(4.41)

The constant in (4.28) is given by \(K_{m,n}(c; d) = \det M / \det L\).

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A Condensation of determinants

In this Appendix A, we give a review on the variation of Dodgson condensation (Sylvester identity) of determinants due to Gasca–López-Carmona–Ramirez [2], which we call the condensation along a moving core. For further generalizations of Sylvester’s identity, we refer the reader to Mühlbach–Gasca [9].

We first recall a standard version of the Dodgson condensation (Sylvester’s identity) for comparison. For a general $m \times n$ matrix $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ (with entries in a commutative ring), we denote by $X_{i_1,\ldots,i_r}^{j_1,\ldots,j_s} = (x_{a,b})_{1 \leq a \leq r, 1 \leq b \leq s}$ the $r \times s$ submatrix of $X$ with row indices $i_1,\ldots,i_r \in \{1,\ldots,m\}$ and column indices $j_1,\ldots,j_s \in \{1,\ldots,n\}$. When $r = s$, we denote by $\det X_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}$ the corresponding minor determinant.

**Lemma A.1 (Dodgson condensation, Sylvester’s identity)** Let $X = (x_{ij})_{i,j=1}^n$ an $n \times n$ matrix and set $n = r + s$ ($r, s \geq 1$). We define an $r \times r$ matrix $Y = (y_{ij})_{i,j=1}^r$ by using the $(s+1) \times (s+1)$ minor determinants $y_{ij} = \det X_{i,r+1}^{j,r+1,\ldots,n}$ of $X$. Then the determinant of $Y$ is expressed as

$$\det Y = \det X \left( \det X_{r+1,\ldots,n}^{r+1,\ldots,n} \right)^{r-1}, \quad Y = (y_{ij})_{i,j=1}^r, \quad y_{ij} = \det X_{j,r+1,\ldots,n}^{i,r+1,\ldots,n}. \quad (A.1)$$

**Proof**: Define an $n \times n$ upper triangular matrix $Z = (z_{ij})_{i,j=1}^n$ by setting

$$z_{ij} = \begin{cases} \delta_{i,j} \det X_{r+1,\ldots,n}^{r+1,\ldots,n} & (1 \leq i, j \leq r) \\ (-1)^{j-r} \delta_{i,j} \det X_{r+1,\ldots,n}^{r+1,\ldots,n} & (1 \leq i \leq r; \ r + 1 \leq j \leq n) \\ \delta_{i,j} & (\text{otherwise}). \end{cases} \quad (A.2)$$

Then for $i = 1,\ldots,r$, the $(i,j)$-component of the product $ZX$ is given by

$$(ZX)_{ij} = z_{ii} x_{ij} + \sum_{k=r+1}^n z_{ik} x_{kj} = \det X_{r+1,\ldots,n}^{r+1,\ldots,n} x_{ij} + \sum_{k=r+1}^n (-1)^{k-r} \det X_{r+1,\ldots,n}^{r+1,\ldots,n} x_{kj} = \det X_{j,r+1,\ldots,n}^{i,r+1,\ldots,n}. \quad (A.3)$$

namely,

$$\begin{cases} y_{ij} & (j = 1,\ldots,r) \\ 0 & (j = r + 1,\ldots,n) \end{cases} \quad (A.4)$$

This means that

$$ZX = \begin{bmatrix} Y & 0 \\ X_{r+1,\ldots,n}^{r+1,\ldots,n} & X_{r+1,\ldots,n}^{r+1,\ldots,n} \end{bmatrix}. \quad (A.5)$$

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Since \( \det Z = z_{11} \cdots z_{rr} \), we obtain
\[
\det X (\det X_{r+1}^{r+1, \ldots, n})^r = \det Y \det X_{r+1}^{r+1, \ldots, n}.
\] (A.6)

This implies the polynomial identity
\[
\det Y = \det X (\det X_{r+1}^{r+1, \ldots, n})^{r-1}
\] (A.7)
in the variables \( x_{ij} \) (1 ≤ \( i, j \) ≤ \( n \)).

When \((r, s) = (n - 1, 1)\), (A.1) means that
\[
\det (x_{ij}x_{mn} - x_{in}x_{nj})^{n-1}_{i,j=1} = \det X x_{nn}^{n-2}.
\] (A.8)

Another extreme case \((r, s) = (2, n - 2)\) implies
\[
det X_{1,3,\ldots,n}^{1,3,\ldots,n} \det X_{2,3,\ldots,n}^{2,3,\ldots,n} - det X_{1,3,\ldots,n}^{2,3,\ldots,n} \det X_{1,3,\ldots,n}^{1,3,\ldots,n} = det X_{1,2,\ldots,n}^{1,2,\ldots,n} \det X_{3,\ldots,n}^{2,\ldots,n},
\] (A.9)

which is equivalent to
\[
det X_{1,\ldots,n-1}^{1,\ldots,n-1} \det X_{2,\ldots,n}^{2,\ldots,n} - det X_{2,\ldots,n}^{1,\ldots,n-1} \det X_{1,\ldots,n-1}^{2,\ldots,n-1} = det X_{2,\ldots,n}^{1,\ldots,n} \det X_{2,\ldots,n-1}^{2,\ldots,n-1}.
\] (A.10)

These identities (A.9), (A.10) are often referred to as Jacobi’s formula or Lewis–Carroll’s formula.

The variant of Dodgson condensation that we use in this paper is the following identity due to Gasca–López-Carmona–Ramírez [2].

**Lemma A.2 (Condensation along a moving core)**  Let \( X = (x_{ij})^{n}_{i,j=1} \) an \( n \times n \) matrix and set \( n = r + s \) \((r, s ≥ 1)\). We define an \( r \times r \) matrix \( Y = (y_{ij})^{r}_{i,j=1} \) by using the \( (s + 1) \times (s + 1) \) minor determinants \( y_{ij} = \det X_{j,r+1,\ldots,n}^{i,i+1,\ldots,i+s} \) of \( X \). Then the determinant of \( Y \) is expressed as
\[
\det Y = \det X \prod_{i=1}^{r-1} \det X_{r+1,\ldots,n}^{i+1,\ldots,i+s}; \quad Y = (y_{ij})^{r}_{i,j=1}, \quad y_{ij} = \det X_{j,r+1,\ldots,n}^{i,i+1,\ldots,i+s}.
\] (A.11)

**Proof**: We define an \( n \times n \) upper triangular matrix \( Z = (z_{ij})^{n}_{i,j=1} \) as follows by using \( s \times s \) minor determinants of \( X \):
\[
z_{ij} = \begin{cases} (-1)^{j-i} \det X_{r+1,\ldots,n}^{i,\ldots,j+i+s} & (1 ≤ i ≤ r; i ≤ j ≤ i + s) \\ \delta_{ij} & \text{(otherwise)}. \end{cases}
\] (A.12)

Then for \( i = 1, \ldots, r \), we have
\[
(XZ)_{ij} = \sum_{k=i}^{i+s} z_{ik} x_{kj} = \sum_{k=i}^{i+s} (-1)^{k-i} \det X_{r+1,\ldots,n}^{i,i+1,\ldots,j+i+s} x_{kj} = \det X_{j,r+1,\ldots,n}^{i,i+1,\ldots,i+s},
\] (A.13)

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\[ y_{ij} = \det X_{j,r+1,...,n}^{i,i+1,...,i+s} = \begin{vmatrix} \ddots \end{vmatrix} \]

\[ y_{ij} = \begin{cases} y_{ij} & (1 \leq j \leq r) \\ 0 & (r+1 \leq j \leq n). \end{cases} \quad (A.14) \]

This means that

\[
ZX = \begin{bmatrix} Y & 0 \\ X_{r+1}^{i,...,n} & X_{r+1}^{r+1,...,n} \end{bmatrix}.
\]

(A.15)

Since \( \det Z = z_{11} \cdots z_{rr} \), we obtain

\[
\det X \prod_{i=1}^{r} \det X_{r+1}^{i,...,i+s} = \det Y \det X_{r+1}^{r+1,...,n}.
\]

(A.16)

This implies the polynomial identity

\[
\det Y = \det X \prod_{i=1}^{r-1} \det X_{r+1}^{i,...,i+s}
\]

(A.17)

in the variables \( x_{ij} \) (1 ≤ i, j ≤ n). □

We remark that, if we renormalize the matrix \( Y \) by setting

\[
\tilde{Y} = \left( \tilde{y}_{ij} \right)_{i,j=1}^{r}, \quad \tilde{y}_{ij} = \det X_{j,r+1,...,n}^{i,i+1,...,i+s} \left( \det X_{r+1}^{r+1,...,n} \right)^{-1} \quad (i,j = 1, \ldots, r),
\]

(A.18)

then equality (A.11) is rewritten equivalently as

\[
\det X = \det \tilde{Y} \det X_{r+1}^{r+1,...,n}.
\]

(A.19)
B Variations of Krattenthaler’s determinant formula

In this Appendix B, we recall Krattenthaler’s determinant formula \[5\] and its elliptic extension due to Warnaar \[15\]. Although these formulas can be proved in various ways, we remark here that they are consequences of Lemma B.3 below, which can be regarded as an abstract form of Krattenthaler’s determinant formula (for recent works on the evaluation of determinants involving shifted factorials, see Normand \[10\]).

We first recall a typical form of Krattenthaler’s determinan t formula \[5\].

**Lemma B.1** For any set of variables \(x_i (0 \leq i \leq m)\) and parameters \(\alpha_k, \beta_k, \gamma_k, \delta_k (0 \leq k < m)\), one has

\[
\text{det} \left( \prod_{0 \leq k < j} \frac{\alpha_k x_i + \beta_k}{\gamma_k x_i + \delta_k} \right)_{i,j=0}^m = \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{0 \leq k \leq m} (\alpha_k \delta_l - \beta_k \gamma_l) \prod_{0 \leq i \leq m} (\gamma_k x_i + \delta_k).
\]

By specializing the parameters \(\alpha_k, \beta_k, \gamma_k, \delta_k\), we obtain various determinant formulas. We quote below some of them.

(a) Case where \(\alpha_k = \gamma_k = 1\) and \(\beta_k = a_k, \delta_k = b_k\):

\[
\text{det} \left( \prod_{0 \leq k < j} \frac{x_i + a_k}{x_i + b_k} \right)_{i,j=0}^m = \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{0 \leq k \leq m} (b_l - a_k) \prod_{0 \leq i \leq m} (x_i + b_k).
\]

In particular, by setting \(a_k = a + k, b_k = b + k\) one obtains

\[
\text{det} \left( \frac{(a + x_i)_j}{(b + x_i)_j} \right)_{i,j=0}^m = \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{k=1}^m (b - a)_k \prod_{0 \leq i \leq m} (b + x_i)_m
\]

where \((a)_k = a(a + 1) \cdots (a + k - 1)\).

(b) Case where \(\beta_k = \delta_k = 1, \alpha_k = -a_k, \gamma_k = -b_k\):

\[
\text{det} \left( \prod_{0 \leq k < j} \frac{1 - a_k x_i}{1 - b_k x_i} \right)_{i,j=0}^m = \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{0 \leq k \leq m} (b_l - a_k) \prod_{0 \leq i \leq m} (1 - b_k x_i)
\]

\[\text{(B.1)}\]
By setting $a_k = p^k a$ and $b_k = q^k b$, one has

$$\det \left( \prod_{0 \leq k < j} \frac{(ax_i; p)_j}{(bx_i; q)_j} \right)_{i,j=0}^{m} = \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{0 \leq k < l \leq m} (q^l b - p^k a).$$  \hspace{1cm} (B.5)

where $(a;p)_k = (1 - a)(1 - pa) \cdots (1 - p^{k-1}a)$. In particular,

$$\det \left( \prod_{0 \leq k < j} \frac{(ax_i; q)_j}{(bx_i; q)_j} \right)_{i,j=0}^{m} = a^{(m+1)} q^{(m+1)} \prod_{0 \leq i < j \leq m} (x_j - x_i) \prod_{k=1}^{m} (b/a; q)_k$$  \hspace{1cm} (B.6)

(c) Noting that

$$(az; q)_n(ac/z; q)_n = \prod_{0 \leq k < n} (1 + a^2 cq^2k - aq^k(z + c/z))$$  \hspace{1cm} (B.7)

set

$$x_i = z_i + c/z_i; \quad \alpha_k = -ap^k, \quad \beta_k = 1 + a^2 cp^2k, \quad \gamma_k = -bq^k, \quad \delta_k = 1 + b^2 cq^2k.$$  \hspace{1cm} (B.8)

Then we have

$$\det \left( \frac{(az; p)_j(ac/z; p)_j}{(bz; q)_j(bc/z; q)_j} \right)_{i,j=0}^{m} = \prod_{0 \leq i < j \leq m} (z_j - z_i)(1 - c/z_i z_j) \prod_{0 \leq k < l \leq m} (bq^l - ap^k)(1 - p^k q^l abc)$$  \hspace{1cm} (B.9)

In particular,

$$\det \left( \frac{(az; q)_j(ac/z; q)_j}{(bz; q)_j(bc/z; q)_j} \right)_{i,j=0}^{m} = a^{(m+1)} q^{(m+1)} \prod_{0 \leq i < j \leq m} (z_j - z_i)(1 - c/z_i z_j) \prod_{k=1}^{m} (b/a; q)_k(q^{2(m-1-k)} abc; q)_k$$  \hspace{1cm} (B.10)

Let $[x]$ a nonzero entire function in $x \in \mathbb{C}$ and suppose that $[x]$ satisfies the so-called Riemann relation: For any $x, \alpha, \beta, \gamma \in \mathbb{C},$

$$[x \pm \alpha][\beta \pm \gamma] + [x \pm \beta][\gamma \pm \alpha] + [x \pm \gamma][\alpha \pm \beta] = 0,$$  \hspace{1cm} (B.11)
where \([x \pm \alpha] = [x + \alpha][x - \alpha]\). This functional equation is equivalent to

\[
[x \pm u][y \pm v] - [x \pm v][y \pm u] = [x \pm y][u \pm v].
\] (B.12)

The following lemma is the elliptic extension of Lemma B.1 due to Warnaar [15].

**Lemma B.2** For any set of variables \(x_i\) \((0 \leq i \leq m)\) and parameters \(a_k, b_k\) \((0 \leq k < m)\), one has

\[
\det \left( \prod_{0 \leq k < j} \frac{[a_k \pm x_i]}{[b_k \pm x_i]} \right)_{i,j=0}^{m} = \prod_{0 \leq i < j \leq m} \frac{[x_j \pm x_i]}{[b_k \pm x_i]} \prod_{0 \leq i \leq m} \prod_{0 \leq k \leq l < m} [a_k \pm b_l].
\] (B.13)

As a special case where \(a_k = a + k\delta, b_k = b + k\delta \) \((0 \leq k < m)\), we obtain

\[
\det \left( \frac{[a \pm x_i]}{[b \pm x_i]} \right)_{i,j=0}^{m} = \prod_{0 \leq i < j \leq m} \frac{[x_j \pm x_i]}{[b \pm x_i]} \prod_{0 \leq i \leq m} \prod_{0 \leq k \leq l < m} [a \pm b].
\] (B.14)

where \([a]_k = [a][a + \delta] \cdots [a + (k-1)\delta]\) and \([a \pm b]_k = [a + b][a - b]\).

Lemma B.1 and Lemma B.2 can be proved as consequences of the following abstract form of Krattenthaler’s determinant formula.

Let \(a_{ik}, b_{ik}\) \((0 \leq i \leq N; 0 \leq k < N)\) be elements a field \(\mathbb{K}\) with \(b_{ik} \neq 0\) for all \(i, k\), and consider the matrix

\[
X_m = \left( \prod_{0 \leq k < j} \frac{a_{ik}}{b_{ik}} \right)_{i,j=0}^{m}.
\] (B.15)

for \(m = 0, 1, \ldots, N\). Suppose that there exist elements \(p_{ij}\) \((0 \leq i, j \leq N)\), \(q_{kl}\) \((0 \leq k \leq l < N)\) of \(\mathbb{K}\) such that

\[
a_{ik}b_{jl} - a_{jk}b_{il} = p_{ij}q_{kl}, \quad p_{ji} = -p_{ij}
\] (B.16)

for all \(i, j \in \{0, 1, \ldots, N\}\) and \(k, l \in \{0, 1, \ldots, N - 1\}\).

**Lemma B.3** Under the assumption (B.16), the determinant \(\det X_m\) is factorized as

\[
\det X_m = \det \left( \prod_{0 \leq k < j} \frac{a_{ik}}{b_{ik}} \right)_{i,j=0}^{m} = \prod_{0 \leq i < j \leq m} p_{ij} \prod_{0 \leq k \leq l < m} q_{kl} \prod_{m}^{m-1} \prod_{i=0}^{m-1} b_{i,k}.
\] (B.17)

for \(m = 0, 1, \ldots, N\).
Set $\tau_m = \det X_m$ for $m = 0, 1, 2, \ldots, N$, so that

$$
\tau_0 = 1, \quad \tau_1 = \det \begin{bmatrix} a_{00} & \frac{a_{10}}{b_{00}} \\ b_{00} & \frac{b_{10}}{a_{10}} \end{bmatrix} = \frac{a_{10}}{b_{10}} - \frac{a_{00}}{b_{00}}. \tag{B.18}
$$

The first nontrivial case is guaranteed by the assumption (B.16):

$$
\tau_1 = \frac{a_{10}b_{00} - a_{00}b_{10}}{b_{00}b_{10}} = \frac{p_{10}q_{00}}{b_{00}b_{10}}. \tag{B.19}
$$

Then (B.17) can be proved by means of the Lewis–Carroll formula. In fact from (A.10), we obtain the bilinear identities

$$
\frac{a_{m+1,1}}{b_{m+1,1}} \tau_m T_C T_R(\tau_m) - \frac{a_{11}}{b_{11}} T_C(\tau_m) T_R(\tau_m) = \tau_{m+1} T_C T_R(\tau_{m-1}), \tag{B.20}
$$

for $\tau_m$, where $T_R$ and $T_C$ stand for the symbolic shift operator for the row indices and the column indices:

$$
T_R(a_{ij}) = a_{i+1,j}, \quad T_R(b_{ij}) = b_{i+1,j}, \quad T_R(p_{ij}) = p_{i+1,j+1}, \quad T_R(q_{kl}) = q_{kl},
\quad T_C(a_{ij}) = a_{i,j+1}, \quad T_C(b_{ij}) = b_{i,j+1}, \quad T_C(p_{ij}) = p_{ij}, \quad T_C(q_{kl}) = q_{k+1,l+1}. \tag{B.21}
$$

Thanks to the bilinear identities, one can prove

$$
\tau_m = \prod_{0 \leq i < j \leq m} p_{ji} \prod_{0 \leq k \leq l \leq m-1} q_{kl} \prod_{i=0}^{m-1} \prod_{k=0}^{b_{i,k}} \quad (m = 0, 1, \ldots, N) \tag{B.22}
$$

by the induction on $m$.

We remark that Lemma (B.1) is the case where $a_{ik} = \alpha_k x_i + \beta_k$, $b_{ik} = \gamma_k x_i + \delta_k$. Since

$$
(\alpha_k x_i + \beta_k)(\gamma_l x_j + \delta_l) - (\alpha_k x_j + \beta_k)(\gamma_l x_i + \delta_l) = (x_i - x_j)(\alpha_k \delta_l - \beta_k \gamma_l) \tag{B.23}
$$

the factorization condition (B.16) is verified with $p_{ij} = x_i - x_j$ and $q_{kl} = \alpha_k \delta_l - \beta_k \gamma_l$. Lemma (B.2) is the case where $a_{ik} = [a_k \pm x_i]$, $b_{ik} = [b_k \pm x_i]$. Since

$$
[a_k \pm x_i][b_l \pm x_j] - [a_k \pm x_j][b_l \pm x_i] = [a_k \pm b_l][x_i \pm x_j] \tag{B.24}
$$

the condition (B.16) is satisfied with $p_{ij} = [x_i \pm x_j]$ and $q_{kl} = [a_k \pm b_l]$. One can prove in fact that generic solutions to the system of equations (B.16) reduce to the case of (B.23). It would be worthwhile, however, to recognize the role of bilinear equations (B.21) which lead to the factorization of determinants.
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