ROW CYCLE STRUCTURES AND THE AUTOTOPY GROUP OF A LATIN SQUARE

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ABSTRACT. New bounds for the size of the autotopy group of a Latin square, based on the cycle structure of the rows, are introduced. A new algorithm for computing the autotopy group, that appears to be effective for random Latin squares having a small autotopy group, is shown. It is shown that if a Latin square of order \( n \) has two rows or two columns that map from one to the other by a single cycle permutation, then the autotopy group can be computed in polynomial time on \( n \), and its size is bounded by \( n^2(n - 1) \).

1. Introduction

For a positive integer \( n \) let \([n]\) denote the set \( \{1, \ldots, n\} \). A Latin square of order \( n \) is an \( n \times n \) array of numbers in \([n]\) so that each row and each column of \( L \) is a permutation of \([n]\). A line in \( L \) is either a row or a column.

Let \( LS(n) \) be the set of Latin squares of order \( n \) and let \( S_n \) be the symmetric group of permutations of \([n]\). An isotopism is a triple \((\alpha, \beta, \gamma) \in S_n^3 \) that acts on \( LS(n) \) by permuting the set of rows of a Latin square by \( \alpha \), permuting the set of columns by \( \beta \), and permuting the symbols by \( \gamma \). An autotopism of a Latin square \( L \) is an isotopism \( \Theta \) such that \( \Theta(L) = L \). The autotopy group of \( L \), denoted \( \mathfrak{A}(L) \), is the group of autotopisms of \( L \). Two Latin squares are called isotopic if there is an isotopism that transforms one to the other. If \( L \) and \( L' \) are isotopic, say \( \Theta(L) = L' \), then their autotopy groups are related: \( \mathfrak{A}(L') = \Theta \mathfrak{A}(L) \Theta^{-1} \). For further knowledge about isotopisms and autotopisms the reader is referred to \([4, 8, 11, 12, 13]\), among many others.

The structure and size of autotopism groups, as well as ways to compute them, have been the subject many studies. McKay \([10, 11]\) introduced “nauty”, an algorithm for computing the symmetry groups of a graph, and used it to compute the various symmetry group of a Latin square, including the autotopy group, by mapping the Latin square to a graph. “nauty” can be accelerated for specific classes of Latin squares by using specific vertex invariants (see Wanless \([17]\)).

Although the proportion of Latin squares of order \( n \) which have non-trivial autotopy group tends quickly to zero (McKay and Wanless \([12]\)), some special Latin squares
may have a large autotopy group (see Wanless [17]). For example, if $L$ is the Cayley table of a group $G$ of order $n$ then

$$|\mathfrak{A}(L)| = n^2|\text{Aut}(G)|$$

(as mentioned in [1] and [2] and shown in [13] and [14]). Browning, Stones and Wanless [2] set the following general bound for the size of the autotopy group of a Latin square $L$ of order $n$:

$$|\mathfrak{A}(L)| \leq n^2 \prod_{t=1}^{\lfloor \log_2 n \rfloor} (n - 2^{t-1}).$$

Given a Latin square $L$, we may use easily computable features of $L$ in order to obtain a more accurate bound. For example, viewing the rows and columns of a Latin square as permutations in $S_n$, and assuming that $L$ has $k$ rows of one parity (even or odd) and $n - k$ rows of the opposite parity, it was shown in [9] that

$$|\mathfrak{A}(L)| \leq n(n - k)!k!.$$

The cycle structures of permutations have been considered in the context of Latin squares in different aspects. Cavenagh, Greenhill and Wanless [3] considered the cycle structure of the permutation that transforms one row of a Latin square to another row. Falcón [5] and Stones, Vojtěchovský and I. Wanless [15] considered the cycle structure of the permutations $\alpha$, $\beta$ and $\gamma$ in an isotopism $\Theta = (\alpha, \beta, \gamma)$, in order to derive information on Latin squares for which $\Theta$ is an autotopism. Gałuszka [6] used the cycle structure of rows of Latin squares, viewed as Cayley tables of groupoids, in order to study the quasigroup structure of the groupoids.

The cycle structure of the rows of a reduced Latin square was used in [9] to obtain a bound for the size of the autotopy group. A Latin square is called reduced if its first row and first column are equal to the identity permutation. Since every Latin square can be transformed via an isotopism to a reduced Latin square, we can study the structure and size of autotopism groups of general Latin squares by exploring autotopism groups of reduced Latin squares. It was shown in [9] that if $L$ is reduced and $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of $n$ by the different cycle structures of the rows, then

$$|\mathfrak{A}(L)| \leq n^2 \prod_{i=1}^{s} \lambda_i!.$$

While obtaining this bound an algorithm for finding the elements of $\mathfrak{A}(L)$ was introduced.

In this paper, new bounds for $|\mathfrak{A}(L)|$ and an improved version of the algorithm in [9], for finding $\mathfrak{A}(L)$ are introduced. This algorithm appears to perform faster than existing algorithms on random Latin squares. The algorithm’s approach is novel: instead of mapping the Latin square to a graph and then computing the automorphism group of the graph, it views the rows and columns of a Latin square as permutations and uses some basic facts about permutation groups. As a corollary it is shown that for reduced Latin squares having two rows or two columns that map from one to the other by a single cycle, the autotopy group can be computed in polynomial time in $n$ and its size is bounded by $n^2(n - 1)$ (this includes the case that a Latin square has a line that is a single cycle, since the Latin squares in question are reduced).
Notation 1. For a Latin square $L$ of order $n$ let $\{\sigma_i\}_{i=1}^n$ be the rows of $L$, viewed as permutations, and let $\{\pi_j\}_{j=1}^n$ be the columns of $L$.

Convention: When viewing a row or a column of a Latin square as a permutation $\sigma \in S_n$, it is understood that the number $i$ appearing in the $j$th place of the row (column) signifies that $\sigma(j) = i$.

2. Preliminary Results

The following results from [9] are restated here for convenience.

Lemma 1. Let $L$ be a Latin square of order $n$.

(i) The result of permuting the rows (resp. columns) of $L$, by $\alpha \in S_n$, on a column (resp. row) $\pi$ is $\pi\alpha^{-1}$.

(ii) The result of permuting the symbols of $L$, by $\alpha \in S_n$, on a row or column $\pi$ is $\alpha\pi$.

Notation 2. Let $L$ be a reduced Latin square. For any permutation $\alpha \in S_n$ and any column $\pi_j$ of $L$ let $\Theta_{\alpha,j}$ denote the isotopy $(\alpha, \alpha\pi_j^{-1}\sigma_{\alpha^{-1}(1)}, \alpha\pi_j^{-1})$.

The next proposition appears in a different formulation as part of the proof of Theorem 2.1 in [16]. It describes the $n \cdot n!$ isotopies that when applied to a given reduced Latin square produce a reduced Latin square.

Proposition 1. Let $L$ be a reduced Latin square.

(i) For any permutation $\alpha \in S_n$ and any column $\pi_j$ of $L$, The Latin square $\Theta_{\alpha,j}(L)$ is reduced.

(ii) If $\Theta$ is an isotopy such that $\Theta(L)$ is reduced then $\Theta = \Theta_{\alpha,j}$ for some $\alpha \in S_n$ and some column $\pi_j$ of $L$.

The following proposition describes the effect of applying $\Theta_{\alpha,j}$ on a single row of a Latin square.

Proposition 2. Let $L$ be a reduced Latin square of order $n$ with rows $\{\sigma_i\}_{i=1}^n$ and columns $\{\pi_j\}_{j=1}^n$. Let $L' = \Theta_{\alpha,j}(L)$ and let $\{\sigma'_i\}_{i=1}^n$ be the rows of $L'$. Then,

$$\sigma'_i = \alpha\pi_j^{-1}\sigma_{\alpha^{-1}(i)}\sigma_{\alpha^{-1}(1)}^{-1}\pi_j\alpha^{-1}. \quad (2.1)$$

Corollary 1. Let $L$ be a reduced Latin square of order $n$ with rows $\{\sigma_i\}_{i=1}^n$ and columns $\{\pi_j\}_{j=1}^n$. If $\Theta_{\alpha,j} \in \mathcal{A}(L)$, then for each $i = 1, \ldots, n$,

$$\sigma_i = \alpha\pi_j^{-1}\sigma_{\alpha^{-1}(i)}\sigma_{\alpha^{-1}(1)}^{-1}\pi_j\alpha^{-1}, \quad (2.2)$$

and $\sigma_{\alpha^{-1}(i)}\sigma_{\alpha^{-1}(1)}^{-1}$ has the same cycle structure as $\sigma_i$.

Remark 1. It follows from Corollary 1 that for $\alpha$ to be the first component of an autotopism it must satisfy that the set of cycle structures of $\{\sigma_i\}_{i=1}^n$ is the same as the set of cycle structures of $\{\sigma_i\sigma_k^{-1}\}_{i=1}^n$, where $k = \alpha^{-1}(1)$. Since there are at most $n$ possible values of $\alpha^{-1}(1)$ and $n$ possible $\pi_j$, the bound in [13] follows.
3. Some Remarks on Cayley Tables of Finite Groups

The isotopisms $\Theta_{\alpha,j}$ and the results in the previous section can be used to obtain some known and new results on the autotopy group of a Cayley table of a finite group. The first is the known result (1.1) of Sade [13] and Schönhardt [14]. The proof is given here, as it is different from the previous ones:

**Proposition 3.** If $L$ is the Cayley table of a finite group $G$ then $|A(L)| = n^2|\text{Aut}(G)|$.

**Proof.** The set of rows of $L$, as well as the set of columns of $L$, forms a subgroup of $S_n$ isomorphic to $G$. The rows acting by left composition and the columns by right composition. We assume the symbols in $L$ are the elements of $G$ and the rows and columns of $L$ are indexed by the elements of $G$. Now, let $\varphi \in \text{Aut}(G)$ and let $g \in G$. We define a permutation $\alpha$ on the elements of $G$ by

$$\alpha^{-1}(x) = \varphi(x)g.$$ (3.1)

Let $\pi_h$ be a column of $L$. Let $L' = \Theta_{\alpha,h}(L)$. We show that $L'_{k,l} = L_{k,l}$ for all $k, l \in G$. Let $\sigma_k'$ be the $k$-th row of $L'$. Then, by (2.1),

$$L'_{k,l} = \sigma'_k(l) = \alpha\pi_h^{-1}\sigma_{\alpha^{-1}(k)}\pi_h\alpha^{-1}(l) = \alpha\pi_h^{-1}\sigma_{\alpha^{-1}(k)}\pi_h\alpha^{-1}(l)h,$$ (3.2)

since $\pi_h$ acts by right composition. By (3.1) applied to $x = l$ and the fact that the row $\sigma_{\alpha^{-1}(k)}$ acts by left composition by $(\alpha^{-1}(1))^{-1}$ we have from (3.2),

$$L'_{k,l} = \alpha\pi_h^{-1}\sigma_{\alpha^{-1}(k)}\pi_h\alpha^{-1}(l)\varphi(l)g = \alpha\pi_h^{-1}\sigma_{\alpha^{-1}(k)}(\alpha^{-1}(1))^{-1}\varphi(l)gh.$$ (3.3)

Setting $x = 1$ and $x = k$ in (3.1) we obtain from (3.3),

$$L'_{k,l} = \alpha\pi_h^{-1}\varphi(k)gg^{-1}\varphi(l)gh.$$ (3.4)

Since the column $\pi_h^{-1}$ acts as right composition by $h^{-1}$ we have from (3.4)

$$L_{k,l} = \alpha\varphi(k)\varphi(l)ghh^{-1} = \alpha\varphi(k)\varphi(l)g,$$ (3.5)

and, since $\varphi$ is an automorphism, $L'_{k,l} = \alpha\varphi(kl)g = \alpha\alpha^{-1}(kl) = kl = L_{k,l}$.

Since there are $n$ possible values of $g$, $n$ possible columns $\pi_h$, and $|\text{Aut}(G)|$ possible $\varphi$, we have $|A(L)| \geq n^2|\text{Aut}(G)|$.

Now assume that $\Theta_{\alpha,h} \in A(L)$. We have to show that there exists an automorphism $\varphi$ of $G$ such that $\alpha^{-1}(x) = \varphi(x)g$ for some $g \in G$. We define a permutation $\varphi$ on $G$ by $\varphi(x) = \alpha^{-1}(x)\alpha^{-1}(1)\alpha^{-1}(1)$. Note that $\varphi(1) = 1$. Setting $g = \alpha^{-1}(1)$ and $\alpha^{-1}(x) = \varphi(x)\alpha^{-1}(1)$ in (3.2) and (3.5) we obtain $L'_{k,l} = \alpha\varphi(k)\varphi(l)\alpha^{-1}(1)$. On the other hand $L_{k,l} = kl = \alpha\alpha^{-1}(kl) = \alpha\varphi(kl)\alpha^{-1}(1)$ for any $k$ and $l$. Since we assume $L'_{k,l} = L_{k,l}$ it follows that $\varphi(k)\varphi(l) = \varphi(kl)$ and hence $\varphi$ is an isomorphism. Thus, $|A(L)| = n^2|\text{Aut}(G)|$. \hfill $\square$

As a consequence we have a property of the isotopisms $\Theta_{\alpha,j}$ in the case of a Cayley table. It appears in [13] in a different formulation.

**Proposition 4.** Let $L$ be the Cayley table of a finite group $G$. If $\Theta_{\alpha,j} \in A(L)$ for some $\alpha \in S_n$ and a column $\pi_j$ of $L$, then $\Theta_{\alpha,k} \in A(L)$ for all columns $\pi_k$ of $L$. 

Proof. Suppose $\Theta_{a,j} \in \mathcal{A}(L)$. By the second part of the proof of Proposition 3, the map $\varphi(x) = \alpha^{-1}x\alpha^{-1}(1)^{-1}$ is an automorphism of $G$. Let $g = \alpha^{-1}(1)$. By the first part of the proof of Proposition 3, for any column $\pi_h$ of $L$ the isotopism $\Theta_{a,h}$ is an automorphism. \hfill $\Box$

The next proposition can be viewed as the “opposite direction” of Proposition 3.

**Proposition 5.** Let $L$ be a reduced Latin square of order $n$ and suppose that for some $\alpha \in S_n$, $\Theta_{\alpha,j} \in \mathcal{A}(L)$ for all $j = 1, \ldots, n$. Then $L$ is the Cayley table of some group.

**Proof.** Let $k, l \in [n]$. By the assumption,

$$
\Phi = \Theta_{\alpha,k}^{-1}\Theta_{\alpha,l} = (1, \sigma^{-1}_{\alpha^{-1}(1)}\pi_k\pi_l^{-1}\sigma_{\alpha^{-1}(1)}, \pi_k\pi_l^{-1}) \in \mathcal{A}(L).
$$

Let $\sigma = \sigma_{\alpha^{-1}(1)}$, $\beta = \sigma^{-1}\pi_k\pi_l^{-1}\sigma$ and $\gamma = \pi_k\pi_l^{-1} = \sigma\beta\sigma^{-1}$. We have that $\Phi = (1, \beta, \sigma\beta\sigma^{-1}) \in \mathcal{A}(L)$. While applying $\Phi$ to $L$, after permuting the columns by $\beta$, the first row is $\beta^{-1}$, by Lemma 2(ii). Then, by Lemma 2(ii), we have to apply $\beta$ on the symbols in order to transform the first row back into the identity permutation. Thus, $\gamma = \beta$, and $\Phi = (1, \beta, \beta)$. By Proposition 2(ii), $\beta = \pi_j^{-1}$ for some column $j$. We have $\pi_i\pi_k^{-1} = \pi_j$. Since this holds for any two columns, it follows that the columns of $L$ form a subgroup of $S_n$. The result follows from the following lemma. \hfill $\Box$

**Lemma 2.** If $L$ is a reduced Latin square of order $n$ whose rows (or columns), viewed as permutations, form a subgroup $G$ of $S_n$, then $L$ is the Cayley table of a group that is isomorphic to $G$.

**Proof.** Assume the rows of $L$ form a subgroup of $S_n$. Let $g, h$ and $k$ be row indexes such that $\sigma_g\sigma_h = \sigma_k$. We have to show: 1) $L_{g,h} = k$, and 2) the columns satisfy $\pi_h\pi_g = \pi_k$. 1) Since $\sigma_g\sigma_h(1) = \sigma_k(1) = k$ ($L$ is reduced), we have $k = \sigma_g\sigma_h(1) = \sigma_g(h) = L_{g,h}$. 2) Let $x$ be some index. Suppose $\sigma_x\sigma_g = \sigma_t$, $\sigma_x\sigma_h = \sigma_r$, and $\sigma_x\sigma_k = \sigma_s$. Clearly, $r = s$. Now, $\pi_h\pi_g(x) = \pi_h(L_{x,g}) = \pi_h(t) = L_{t,h} = \sigma_t(h) = r$ and $\pi_k(x) = L_{x,k} = \sigma_x(k) = s$. Since $r = s$ we have $\pi_h\pi_g(x) = \pi_k(x)$. Since this holds for any $x$ we have $\pi_h\pi_g = \pi_k$. \hfill $\Box$

4. Bounds for $|\mathcal{A}(L)|$

**Notation 3.** Let $L$ be a reduced Latin square of order $n$. For any $k \in [n]$ let $\phi(L, k)$ denote the number of rows with the same cycle structure as the row $\sigma_k$. Let $\phi(L) = \max_k \phi(L, k)$.

**Example 1.** Consider the following reduced Latin square $L$ of order 8:

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |   |
| 2 | 1 | 6 | 8 | 7 | 5 | 3 | 4 |   |
| 3 | 5 | 1 | 6 | 2 | 8 | 4 | 7 |   |
| 4 | 8 | 7 | 1 | 3 | 2 | 6 | 5 |   |
| 5 | 3 | 2 | 7 | 8 | 4 | 1 | 6 |   |
| 6 | 7 | 4 | 5 | 1 | 3 | 8 | 2 |   |
| 7 | 6 | 8 | 2 | 4 | 1 | 5 | 3 |   |
| 8 | 4 | 5 | 3 | 6 | 7 | 2 | 1 |   |
The cycle representations of the rows of \( L \), grouped by cycle structure, are:

| Row | Cycle representation |
|-----|----------------------|
| 1   | (1)(2)(3)(4)(5)(6)(7)(8) |
| 2   | (1, 2)(4, 8)(3, 6, 5, 7) |
| 3   | (1, 3)(2, 5)(4, 6, 8, 7) |
| 4   | (1, 4)(2, 8, 5, 3, 7, 6) |
| 5   | (2, 3)(1, 5, 8, 6, 4, 7) |
| 6   | (3, 8)(1, 7, 5, 4, 2, 6) |
| 7   | (1, 8)(2, 4, 3, 5, 6, 7) |
| 8   | (2, 7, 8)(1, 6, 3, 4, 5) |

We have \( \varphi(L, 1) = \varphi(L, 6) = 1, \varphi(L, 2) = \varphi(L, 3) = 2, \) and \( \varphi(L, 4) = \varphi(L, 5) = \varphi(L, 7) = \varphi(L, 8) = 4 \). Thus, \( \varphi(L) = 4 \).

**Notation 4.** Let \( L \) be a reduced Latin square of order \( n \). Denote by \( \Delta(L) \) the set of integers \( k \in [n] \) such that the set of cycle structures of \( \{\sigma_i \sigma_j^{-1}\}_{i=1}^n \) is the same as the set of cycle structures of \( \{\sigma_i\}_{i=1}^n \). Let \( \delta(L) = |\Delta(L)| \).

(The notation \( \Delta(L) \) was chosen for its association with the translations \( \Delta_a \) in [13].)

**Example 2.** If \( L \) is the Cayley table of a finite group of order \( n \), then \( \{\sigma_i\}_{i=1}^n = \{\sigma_i \sigma_j^{-1}\}_{i=1}^n \) for all \( k \in [n] \). Thus, \( \delta(L) = n \).

**Notation 5.** Let \( L \) be a reduced Latin square of order \( n \) and suppose \( k \in \Delta(L) \). For any \( t \in [n] \) let

\[
R_k(L, t) := \{ i \in [n] : \sigma_i \text{ and } \sigma_i \sigma_k^{-1} \text{ have the same cycle structure} \}.
\]

**Observation 1.** Let \( L \) be a reduced Latin square of order \( n \). If \( k \in \Delta(L) \), then \( R_k(L, 1) = \{ k \} \) and \( |R_k(L, t)| = \varphi(L, t) \) for all \( t \in [n] \).

Also, by [22],

**Observation 2.** Let \( L \) be a reduced Latin square of order \( n \). Let \( \alpha \) be the first component of an autotopism of \( L \). If \( \alpha^{-1}(1) = k \in \Delta(L) \), then for any \( t \in [n] \),

\[
\alpha^{-1}(t) \in R_k(L, t). \quad (4.1)
\]

Since there are \( n \) options for \( \pi_j \) in Proposition [11] and \( \delta(L) \leq n \) the bound in [1.4] follows. By Remark [1] the bound in [1.4] can be slightly improved:

\[
|\mathfrak{A}(L)| \leq n \delta(L) \prod_{i=1}^s \lambda_i!. \quad (4.2)
\]
For $k 
eq 1, 5$ the composition with $\sigma_k^{-1}$ yields a different set of cycle structures. Thus, $\delta(L) = 2$. Also, $R_5(L, 1) = \{5\}$, $R_5(L, 6) = \{8\}$, $R_5(L, 2) = R_5(L, 3) = \{2, 3\}$, and $R_5(L, 4) = R_5(L, 5) = R_5(L, 7) = R_5(L, 8) = \{1, 4, 6, 7\}$. By (4.22), $\vert \mathfrak{A} \vert \leq 2 \cdot 8 \cdot 2! \cdot 4! = 768$.

**Notation 6.** For a permutation $\sigma \in S_n$ let $\nu(\sigma)$ denote the number of cycles in the cycle decomposition of $\sigma$ (including cycles of length 1). For a Latin square $L$ of order $n$ let $\nu(L) = \min_i \nu(\sigma_i)$.

**Example 4.** For the Latin square $L$ in Example 1 $\nu(L) = 2$.

**Theorem 1.** Let $L$ be a reduced Latin square of order $n$. Then,

$$\vert \mathfrak{A}(L) \vert \leq n\delta(L)\lambda(L)^{\nu(L)}.$$  

(4.3)

**Proof.** The identity (2.2) is used to obtain a method for finding all autotopisms of $L$. First, choose a number $l \in [n]$ with minimal value of $\phi(L, l)$ among the indexes $k$ satisfying $\nu(\sigma_k) = \nu(L)$.

**Step 1:** Choose a number $k \in \Delta(L)$ and determine the sets $R_k(L, t)$ for all $t \in [n]$. By Remark 1 set

$$\alpha(k) = 1.$$  

(4.4)

**Step 2:** Choose a column $j$ and consider the permutation $\pi_j$.

**Step 3:** By Observation 2 choose a number $i \in R_k(L, l)$ and set

$$\alpha(i) = l.$$  

(4.5)

Let $\tau_{i,j} = \pi_j^{-1}\sigma_i\sigma_k^{-1}\pi_j$. Note that by (2.2),

$$\sigma_l = \alpha\tau_{i,j}\alpha^{-1}.$$  

(4.6)

By the identity

$$\alpha(a_1, a_2, \ldots, a_t)\alpha^{-1} = (\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_t)),$$  

(4.7)

and the identities (4.4)-(4.6) and $\sigma_l(1) = l$ (since $L$ is reduced), we must have that $\tau_{i,j}(k) = i$. Thus, $\tau_{i,j}$ must contain a cycle $(\ldots, k, i, \ldots)$. If this is not the case, proceed to the next value of $i$.

Let $C_l$ be the cycle containing 1 and $l$ in the cycle decomposition of $\sigma_l$ and let $C_i'$ be the cycle containing $k$ and $i$ in the cycle decomposition of $\tau_{i,j}$. By (4.7) it must hold that $\vert C_l \vert = \vert C_i' \vert$. If this is not the case, proceed to the next value of $i$. 
Step 4: Given \( (1,5) \), the values \( \alpha(m) \), for all other \( m \) in \( C_1 \), are uniquely determined by \( (1,7) \). If any of these values contradicts \( (1,11) \), proceed to the next value of \( i \) in Step 3.

**Step 5:** For a cycle \( C_2 \) in the cycle decomposition of \( \sigma_j \), for which the values of \( \alpha^{-1} \) of its elements have not been determined yet, choose any number \( s \) with minimal \( |R_k(L,s)| \) among the elements of \( C_2 \). Proceed as in Step 3 with a value \( r \in R_k(L,s) \) for which \( \alpha(r) \) has not been determined yet. Repeat this step with all the cycles in the cycle decomposition of \( \sigma_i \), until \( \alpha \) is completely determined.

**Step 6:** Once all values \( \alpha(m) \), \( m = 1, \ldots, n \), are determined, for a given column \( \pi_j \), we have a candidate autotopism \( \Theta_{\alpha,j} \). Check whether all other rows (besides the row \( l \)) are fixed by \( \Theta_{\alpha,j} \). If this is the case, then \( \Theta_{\alpha,j} \in \mathfrak{A}(L) \).

There are \( \delta(L) \) possible values of \( k \) in Step 1 and there are \( n \) possible columns \( j \) in Step 2. There are at most \( \lambda(L) \) possible values in Step 3, by Observation \( \mathbb{1} \). There are at most \( \nu(L) - 1 \) cycles left in Step 7. For each such cycle there are at most \( \lambda(L) \) values to select for \( \alpha^{-1}(s) \). Each such value determines the value of \( \alpha^{-1} \) for the rest of the elements in that cycle. Thus

\[
|\mathfrak{A}(L)| \leq \delta(L) \cdot n \cdot \lambda(L) \cdot \lambda_r(L)^{\nu(L)-1}.
\]

**Example 5.** Consider the Latin square in Example \( \mathbb{1} \). We have \( \nu(L) = 2 \), \( \lambda(L) = 4 \) and \( \nu(L) = 2 \). Thus, by \( (1,3) \), \( |\mathfrak{A}(L)| \leq 2 \cdot 8 \cdot 4^2 = 256 \).

Now, the rows 4, 5, 6, 7 and 8 all consist of two cycles, which is the minimal number of cycles for the rows of \( L \). We choose \( l = 6 \) (this choice will soon become apparent).

As observed in Example \( \mathbb{1} \) \( \Delta(L) = \{1,5\} \). We consider the case that \( k = 5 \) in Step 1. We have \( R_5(L,1) = \{5\} \), \( R_5(L,6) = \{8\} \), \( R_5(L,2) = R_5(L,3) = \{2,3\} \), and \( R_5(L,4) = R_5(L,5) = R_5(L,7) = R_5(L,8) = \{1,4,6,7\} \). We set \( \alpha(5) = 1 \).

In Step 2, consider first the column \( \pi_1 = \text{id} \) (the identity permutation).

As \( R_5(6) = \{8\} \), there is only one choice, namely \( i = 8 \), in Step 3 (this was the reason for choosing \( l = 6 \)). We set \( \alpha(8) = 6 \). We have

\[
\tau_{8,1} = \text{id}^{-1} \sigma_8 \sigma_5^{-1} \text{id} = (3,4,7)(1,2,5,8,6),
\]

and by \( (1,6) \) we must have \( \sigma_6 = \alpha \tau_{8,1} \alpha^{-1} \). That is,

\[
(2,7,8)(1,6,3,4,5) = \alpha(3,4,7)(1,2,5,8,6) \alpha^{-1}.
\]

This coincides with the values \( \alpha(5) = 1 \) and \( \alpha(8) = 6 \), as \( \tau_{8,1}(5) = 8 \), and we can proceed to Step 4. Now, in Step 4, using \( (1,7) \) we must have \( \alpha(6) = 3 \), \( \alpha(1) = 4 \) and \( \alpha(2) = 5 \). By \( (1,11) \), this would require that \( 2 \in R_5(5) \), \( 6 \in R_5(3) \) and \( 1 \in R_5(4) \). Since the first two don’t hold we can skip Steps 5 and 6 and rule out the case \( j = 1 \).

In Step 2.

Back to Step 2, consider the column \( \pi_2 = (2,1,5,8,3,7,6,4) \) (this is not cycle representation!). Again, in Step 3 we can only choose \( i = 8 \). We have

\[
\tau_{8,2} = \pi_2^{-1} \sigma_8 \sigma_5^{-1} \pi_2 = (5,8,6)(1,3,4,7,2),
\]
and by \((4.6)\) we must have \(\sigma_6 = \alpha \tau_8 \alpha^{-1}\). That is,
\[(2, 7, 8)(1, 6, 3, 4, 5) = \alpha (5, 8, 6)(1, 3, 4, 7, 2) \alpha^{-1}\].
But now the cycle \(C_1\) containing 1 and 6 in \(\sigma_6\) and the cycle \(C'_2\) containing 5 and 8 in \(\tau_{8,2}\) have different lengths (see Step 3). Thus, the case \(j = 2\) in Step 2 is ruled out. We can proceed this way and eventually obtain a trivial autotopism group for \(L\).

**Remark 2.** By Proposition 3 if \(L\) is not the Cayley table of some group then the bound in Theorem 1 is not reached.

As mentioned in the introduction, the importance of the new algorithm is in that it relies on a different approach than the one used so far. Instead of mapping the square to a graph and then computing the graph automorphisms, it uses properties of permutations. For convenience let us denote the algorithm in the proof of Theorem \(H\) by CSA (cycle structure algorithm). CSA was tested on 20,000 randomly generated Latin squares (Jacobson Matthews method \([7]\)) of each of the orders 10, 15, 20, 25, and 30. Its performance was compared, on the same squares, with that of an accelerated version of “nauty” \([11]\), communicated by Ian Wanless, that was designed for a slightly different task and uses vertex invariants based on the indegrees of the train \([17]\). As the results in Table 1 indicate, CSA performed faster than the mentioned version of “nauty”. The significant difference between the times in the two columns of Table 1 is not in the times themselves, which depend on the different implementations of the two algorithms, but in the different rates of growth as the order \(n\) increases.

| Order | “nauty” | CSA  |
|-------|---------|------|
| 10    | 7.2     | 2.4  |
| 15    | 18.8    | 3.9  |
| 20    | 51.5    | 5.5  |
| 25    | 131.3   | 7.4  |
| 30    | 248.3   | 9.3  |

**Table 1.** Time (seconds) for finding the autotopism group of 20,000 random Latin squares.

As these experiments and the illustration in Example 5 indicate, CSA is effective in eliminating cases, and thus it is effective for detecting Latin squares with small or trivial autotopy groups. As such are most Latin squares, it performs well on random samples of Latin squares. When dealing with Latin squares with large autotopy groups, such as Cayley tables of groups, CSA is slower.

The bound in \((4.3)\) can be improved by looking more closely at the algorithm in the proof of Theorem \(H\) and taking into account more specific, easily computable, information about the square \(L\):

**Notation 7.** Let \(L\) be a Latin square and let \(C\) be a in some row permutation of \(L\). Denote \(\phi(L, C) := \min_{s \in C} (\phi(L, s))\).

**Corollary 2.** Let \(L\) be a Latin square of order \(n\) and let \(\sigma_i\) be a row in \(L\), such that \(\phi(L, \sigma_i)\) is minimal among the rows that decompose into \(\nu(L)\) cycles. Suppose
σ_l = (C_1)(C_2)\ldots(C_{\nu(L)}) is the cycle decomposition of σ_l and assume l \in C_1. Then,

|\mathcal{A}(L)| \leq n\delta(L)\phi(L, l) \prod_{i=2}^{\nu(L)} \phi(L, C_i).

(4.8)

Proof. Let k \in \Delta(L). Then, there are \phi(L, l) possible i \in R_k(L, l) to choose in Step 3 of the algorithm in Theorem 1, and for each cycle C_i in Step 5, there are at most \phi(L, C_i) possible choices for i \in R_k(L, s).

□

Example 6. Consider the Latin square L in Example 1. The row σ_6 has minimal number of cycles (2) and \phi(L, 6) = 1 is minimal among the rows with two cycles. Now, 6 \in C_1 = (1, 6, 3, 4, 5) and for C_2 = (2, 7, 8), \phi(L, C_2) = \phi(L, 2) = 2. Thus, by (4.8), \mathcal{A}(L) \leq 2 \cdot 8 \cdot 1 \cdot 2 = 32. Hence, after a simple computation, there are at most 32 isotopies to test in order to find \mathcal{A}(L).

5. Reduced Latin Squares with a Single Cycle Line

In the case that a reduced Latin square has a line whose cycle decomposition consist of one cycle, |\mathcal{A}(L)| has a clear bound and \mathcal{A}(L) can be easily computed:

Corollary 3. Let L be a reduced Latin square of order n such that at least one of its lines is a single cycle, then

|\mathcal{A}(L)| \leq n^2(n-1),

(5.1)

and \mathcal{A}(L) can be computed in polynomial time in n.

Proof. Without loss of generality we may assume that L contains a single cycle row. Thus, \nu(L) = 1. By (4.7) we have

|\mathcal{A}(L)| \leq n\delta(L)\lambda(L) \leq n^2(n-1).

By the algorithm described in Theorem 1, the process of computing the group \mathcal{A}(L) consists of at most O(n^2(n-1)) iterations, each consisting of elementary computations (such as products of permutations and finding the inverse of a permutation).

□

In order to estimate the proportion of reduced Latin squares with a single cycle line we state the following conjecture:

Conjecture 1. The distribution of the cycle structures of the rows and columns of a random Latin squares tends asymptotically to the distribution of the cycle structures of randomly and independently generated permutations.

Now, let P_1(n) be the probability that a randomly chosen reduced Latin square will have at least one single cycle line. There are approximately n!/e derangements (permutations without fixed points) of order n, among which (n-1)! are single cycles. Thus, the probability of each line, that is not the first row or the first column, of being a single cycle is approximately e/n. If Conjecture 1 holds, then the probability that none of the rows 2,\ldots,n and columns 2,\ldots,n is a single cycle can be approximated by (1 - (e/n))^{2(n-1)} \approx e^{-2e} \approx 0.0044 for large n. Thus, P_1(n) \approx 0.9956. Indeed, 99,580 of 100,000 randomly generated reduced Latin
squares of order 20 had at least one single cycle line. Similar results were obtained for other samples of Latin square of different cycle orders. These results support the validity of Conjecture [1].

Given two rows \( k \) and \( l \) of a Latin square, Cavenagh, Greenhill and Wanless [3] defined \( \sigma_{k,l} \) as the permutation that transforms \( \sigma_k \) to \( \sigma_l \). That is, \( \sigma_{k,l} = \sigma_l \sigma_k^{-1} \) (permutations being applied from right to left). It is said that the rows \( k \) and \( l \) consist of the cycle structure \( c \) if the cycle structure of \( \sigma_{k,l} \) is \( c \). We have:

**Lemma 3.** Let \( L \) and \( L' \) be isotopic reduced Latin squares. Let \( c \) be a fixed cycle structure. Then, \( L' \) has a row with cycle structure \( c \) if and only if \( L \) has two rows \( l \) and \( k \) that consist of the cycle structure \( c \).

**Proof.** Suppose \( L' = \Theta_{\alpha,j}(L) \). Denote the \( i \)-th row of \( L \) by \( \sigma_i \) and the \( i \)-th row of \( L' \) by \( \sigma'_i \). Let \( k = \alpha^{-1}(1) \) and \( l = \alpha^{-1}(i) \) in (2.1). Thus, \( \sigma'_i \) and \( \sigma_{k,l} \) have the same cycle structure. \( \square \)

**Theorem 2.** Let \( L \) be a reduced Latin square, such that two of its lines (either two rows or two columns) consist of a single cycle. Then,

\[
|\mathfrak{A}(L)| \leq n^2(n-1),
\]

and \( \mathfrak{A}(L) \) can be computed in polynomial time.

**Proof.** Given two such rows, we can define \( \alpha \) using the values in (2.1) and then \( \Theta_{\alpha,j} \), as described in Proposition [1] such that \( L' = \Theta_{\alpha,j}(L) \) is reduced and has a line that is a single cycle. Then compute \( \mathfrak{A}(L') \) using the algorithm in Theorem [1]. \( \square \)

Let \( \mathbb{P}_1(n) \) be the probability that a randomly chosen reduced Latin square will have two lines (either two rows or two columns) that consist of a single cycle. We can try to approximate this probability. It was proved in [3] that the probability that two rows of a Latin square consist of a single cycle is at least \( \frac{2}{e^2} \). Making the same (unproven) independence assumption as above, and since there are \( n(n-1)/2 \) pairs of rows and \( n(n-1)/2 \) pairs of columns, the minimum value for \( \mathbb{P}_1(n) \) is approximately \( 1 - \frac{1}{e^2} \approx 0.865 \). Furthermore, it was conjectured in [3] that \( \sigma_{k,l} \) shares the asymptotic distribution of a random derangement. This means that the probability that \( \sigma_{k,l} \) is a single cycle would be approximately \( e/n \). Assuming this and the independence assumption we obtain that \( \mathbb{P}_1(n) \approx 1 - 1/e^{2(e(n-1))} \), which tends very quickly to 1. This agrees with the result of McKay and Wanless [12] who proved that the proportion of order \( n \) Latin squares which have a non-trivial symmetry tends very quickly to zero. However, if the conjecture in [3] holds, then we have a slightly different statement, namely, that the proportion of Latin squares for which the computation of \( \mathfrak{A}(L) \) is fast tends very quickly to 1.

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