Svetlichny’s inequality and genuine tripartite nonlocality in three-qubit pure states

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The violation of the Svetlichny’s inequality (SI) [Phys. Rev. D \textbf{35}, 3066 (1987)] is sufficient but not necessary for genuine tripartite nonlocal correlations. Here we quantify the relationship between tripartite entanglement and the maximum expectation value of the Svetlichny operator (which is bounded from above by the inequality) for the two inequivalent subclasses of pure three-qubit states: the GHZ-class and the W-class. We show that the maximum for the GHZ-class states reduces to Mermin’s inequality [Phys. Rev. Lett. \textbf{65}, 1838 (1990)] modulo a constant factor, and although it is a function of the three tangle and the residual concurrence, large number of states don’t violate the inequality. We further show that by design SI is more suitable as a measure of genuine tripartite nonlocality between the three qubits in the \textit{the} W-class states, and the maximum is a certain function of the bipartite entanglement (the concurrence) of the three reduced states, and only when their certain sum attains a certain threshold value, they violate the inequality.

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It is precisely the nonlocality (NL) of quantum correlations which gives an advantage to quantum mechanics over classical theories for certain information processing tasks \cite{1}. The NL not only distinguishes quantum mechanics from a classical theory, but as far as the speed-up of a quantum computational task is concerned, the problem of quantifying the NL of a multipartite quantum state is indispensable. The correlations between outcomes of measurements on two or more spatially separated subsystems are said to be nonlocal, if they cannot be simulated with shared randomness (which is commonly referred as hidden variables) without communication, \textit{i.e.}, one cannot give a classical model which explains the correlations \cite{2}. One defines an appropriate Bell-type inequality (BTI), since it gives an upper bound on correlations which are consistent with any local hidden-variable, or local-realistic, theory \cite{3}. Thus, the amount of violation of such an inequality by an entangled state is said to be a measure of the NL of the correlations between the subsystems.

It is known that all 2-qubit pure states violate the bipartite BTI, known as the CHSH inequality \cite{3}, and the violation increases as the entanglement of the state increases \cite{4}. The extension of this result to 3-qubit pure states is nontrivial. For instance, Mermin’s tripartite BTI is based on absolute local realism \cite{5}, \textit{i.e.}, it is derived on the assumption that all the three qubits are locally but realistically correlated, and its violation is supposed to be a measure of irreducible (genuine) tripartite nonlocal correlations between the qubits. However, the bi-separable states do violate the inequality \cite{3}. This motivated Svetlichny to formulate a hybrid nonlocal-local realism based inequality \cite{6}: a stronger kind of inequality for a three-qubit system where two of the qubits are assumed to be non-locally correlated, but they are locally correlated to the third, and one takes an ensemble average over all such possible combinations. Thus, by construction, the violation of Svetlichny’s inequality (SI) is a signature of genuine tripartite nonlocality, but it is not a necessary requirement. In this letter we use Svetlichny’s inequality (SI) to quantify genuine tripartite nonlocality of the following two subclasses of 3-qubit pure states in terms of their genuine tripartite entanglement \cite{8}— the GHZ-class states

\[
|\psi_{gs}\rangle = \cos \theta |000\rangle + \sin \theta |11\rangle \left\{ \cos \theta_3 |0\rangle + \sin \theta_3 |1\rangle \right\} ,
\]

and the W-class states

\[
|\psi_w\rangle = \alpha |001\rangle + \beta |010\rangle + \gamma |100\rangle ,
\]

where $\alpha$, $\beta$, and $\gamma$ are real.

The monogamy: The reason which complicates the study of nonlocality of 3-qubit pure states is that the entanglement in the two classes are inequivalent \cite{5}. The difference can be quantified by a measure of genuine tripartite entanglement called the three-tangle \cite{10}:

\[
\tau(\psi) = C_{1(23)}^2 - C_{12}^2 - C_{13}^2 ,
\]

which is invariant under all permutations of subsystem indices; and where the concurrence $C_{1(23)}^2$ is bipartite entanglement between qubit 1 and qubits 2-3, and $C_{12}^2$ is the concurrence of the reduced state $\rho_{12}$ \cite{10}. $\tau \geq 0$ characterizes the generalized GHZ state, whereas $\tau = 0$ for all the W-class states. Since $\tau$ is an entanglement monotone, hence the in-equivalence \cite{5}. The difference arises in the way the bipartite entanglement is distributed among the qubits, \textit{i.e.}, the concurrences are constrained by the monogamy inequality \cite{10}:

\[
C_{1(23)}^2 \geq C_{12}^2 + C_{13}^2 ,
\]

which is saturated by W-class states, while the difference $C_{1(23)}^2 - C_{12}^2 - C_{13}^2$ is maximized by the GHZ-class states.
This implies that the W-class states is determined by the
councurrencies of the three reduced states (modulo local
unitaries), and bigger the sum of the concurrences, more
its tripartite entanglement; in contrast, the GHZ-class
states are fixed by the tangle and the residual concurrencies,
where the latter reduces the genuine tripartite
entanglement of the state [11].

Svetlichny’s inequality: Let the measurements by ob-
servers be spin projections onto unit vectors: \( A = \sigma_1 \cdot \hat{a} \)
or \( A' = \sigma_1 \cdot \hat{a}' \) on qubit 1, \( B = \sigma_2 \cdot \hat{b} \) or \( B' = \sigma_2 \cdot \hat{b}' \) on
qubit 2, and \( C = \sigma_3 \cdot \hat{c} \) or \( C' = \sigma_3 \cdot \hat{c}' \) on the third qubit.
If a theory is consistent with the hybrid nonlocal-local re-
enentanglement of the state \( |\Psi\rangle \), now by setting
where the Svetlichny’s inequality:

\[
\langle \Psi| S|\Psi\rangle \equiv S(\Psi) \leq 4, 
\]

(5)

where the Svetlichny’s operator \( S \) is defined as

\[
S = A(DC + D'C') + A'(D'C - DC') = M + M',
\]

(6)

where \( D = B + B' \) and \( D' = B - B' \) and \( \langle M \rangle \leq 2 \) and
\( \langle M' \rangle \leq 2 \) are Mermin’s inequalities [12].

Note that \( S \) can be further simplified by defining \( \vec{b} + \vec{b}' = 2\hat{d}\cos t \) and \( \vec{b} - \vec{b}' = 2\hat{d}\sin t \), which implies

\[
\vec{d} \cdot \vec{d}' = \cos \theta_d \cos \theta_{d'} + \sin \theta_d \sin \theta_{d'} \cos (\phi_d - \phi_{d'}) = 0. \tag{7}
\]

Now by setting \( D = \vec{d} \cdot \vec{d}_2 \) and \( D' = \vec{d}' \cdot \vec{d}_2 \), gives

\[
S(\Psi) = 2|\cos t \langle ADC \rangle + \sin t \langle AD'C' \rangle |
- \cos t \langle A'D'C' \rangle | + \sin t \langle A'D'C' \rangle |
\leq 2 \left\{ \langle ADC\rangle^2 + \langle AD'C'\rangle^2 \right\}^{\frac{1}{2}}, 
\]

(9)

\[
M = 2 \left\{ \langle ADC\rangle^2 + \langle AD'C'\rangle^2 \right\}^{\frac{1}{2}} \]

\[
\leq 2 \left\{ \sin^2 \theta_a \sin^2 2\theta \left\{ (\cos \theta_3 \cos \phi_{ad} \cos \theta_c + \sin \theta_3 \cos \phi_{adc} \sin \theta_c)^2 + (\cos \theta_3 \sin \phi_{ad} \cos \theta_c + \sin \theta_3 \sin \phi_{adc} \sin \theta_c)^2 \right\} 
+ \cos^2 \theta_a \left( P \cos \theta_{c'} + Q \cos \phi_{c'} \sin \theta_{c'} \right)^2 \right\}^{\frac{1}{2}} \]
\]

\[
\leq \left\{ \frac{2 \sin 2\theta}{2 \left( P^2 + Q^2 \cos^2 \phi_{c'} \right)^{\frac{1}{2}}} \right\} \left\{ \frac{2 \sin^2 \theta_3 \cos^2 \phi_{ad} + \sin^2 \theta_3 \cos^2 \phi_{adc}}{2 \left( P^2 + Q^2 \right)^{\frac{1}{2}}} \right\}^{\frac{1}{2}} \]

\[
\leq \left\{ \frac{2 \sin 2\theta \sqrt{1 + \sin^2 \theta_3}}{2 \sqrt{P^2 + Q^2} = 2(1 - \sin^2 2\theta \sin^2 \theta_3)^{\frac{1}{2}}} \right\}^{\frac{1}{2}}, \]

(17)

Maximization is over \( \theta_{d'} \) in (14), \( \theta_a \) in (15), and \( \theta_c \) and \( \theta_{c'} \) in (16). Equations (18) and (19) are a particular instance

\[
x \cos \theta + y \sin \theta \leq (x^2 + y^2) \frac{1}{2}, \tag{10}
\]

the equality results when \( \tan \theta = y/x \). The following

\[
x \sin^2 \theta + y \cos^2 \theta \leq \left\{ \begin{array}{ll}
y, & x \leq y \\
x, & x \geq y \end{array} \right.
\]

(11)

will be useful later; the first inequality is realized when
\( \theta = 0 \), and \( \theta = \pi/2 \) gives the second. In the next two
sections, we obtain the maximum value of the expecta-
tion value Svetlichny’s operator, \( S_{\text{max}}(\psi) \), with respect
the GHZ-class states \( |\psi_{gs}\rangle \) and the W-class states
\( |\psi_w\rangle \).
By symmetry in (9), and the residual concurrence of tr
where, as discussed earlier, the entanglement of
and (19). More importantly, both sets of constraints can
be matched; this implies that as far as the GHZ-class states is concerned, SI reduces to Mermin’s inequality, modulo the constant value of 2, which ensures that the violation of SI is sufficient to detect genuine tripartite nonlocality.

Equation (17) implies that $|\psi_{gs}\rangle$ is

$$S_{\text{max}}(\psi_{gs}) = \begin{cases} 
4\sqrt{\frac{1}{2} - \tau}, & 3\tau + C_{12}^2 \leq 1 \\
4\sqrt{\frac{1}{2} - \tau}, & 3\tau + C_{12}^2 \geq 1 
\end{cases}$$

(20)

where, as discussed earlier, the entanglement of $|\psi_{gs}\rangle$ is fixed by its tangle:

$$\tau(\psi_{gs}) = \sin^2 2\theta \sin^2 \theta_3,$$

(21)

and the residual concurrence of $\text{tr}_3(|\psi_{gs}\rangle\langle\psi_{gs}|) = \rho_{12}$:

$$C_{12}^2(\psi_{gs}) = \sin^2 2\theta \cos^2 \theta_3,$$

(22)

and, $C_{23} = C_{31} = 0$.

For instance, the first equality in (20) can be achieved by setting the measurement unit vectors as $\vec{a} = \hat{x}$, $\vec{a}' = \hat{y}$, $\vec{b} = \hat{x} \cos t - \hat{y} \sin t$, $\vec{b}' = \hat{x} \cos t + \hat{y} \sin t$, $\vec{c} = \hat{z} \cos \theta_3 + \hat{x} \sin \theta_3$, and $\vec{c}' = \hat{y}$; and the set $\vec{a} = \hat{z}$, $\vec{a}' = \hat{x} \cos t + \hat{z} \sin t$, $\vec{b}' = \hat{x} \cos t - \hat{z} \sin t$, $\vec{c} = \hat{x}$, and $\vec{c}' = \hat{x}$, where tan $t = \sin \theta_3$, attains the second in (20). The behavior of $S_{\text{max}}(\psi_{gs})$ as a function of tripartite entanglement of $|\psi_{gs}\rangle$ is surprising (see Fig.1). When the state is tri-separable, $\tau = C_{12} = 0$, or bi-separable, $\tau = 0$, $0 < C_{12} \leq 1$, then as expected $S_{\text{max}}(\psi_{gs}) = 4$. In the region where the entanglement of $|\psi_{gs}\rangle$ satisfy $3\tau + C_{12}^2 \leq 1$, as the entanglement increases $S_{\text{max}}(\psi_{gs})$ monotonically decreases below the value of 4 (this was also noted in Ref. [8]). The converse happens in the regime where $3\tau + C_{12}^2 \geq 1$, the value of $S_{\text{max}}(\psi_{gs})$ starts monotonically increasing as the entanglement increases; however only when $C_{12}^2 + 2\tau \geq 1$ do the states violate SI. Note in the latter region one expects the residual bipartite entanglement $C_{12}^2$ to decrease the maximum value, instead of increasing it.

The W-class states: For the W-class states it is convenient to obtain $S_{\text{max}}(\psi_{w})$ by simply adding all the eight terms involved in the Svetlichny’s operator $S$, as all the terms in $S$ contribute differently. This, unlike the GHZ-class, makes SI significantly different from Mermin’s inequality for the W-class states. Let $\cos \phi_{ac} = \cos(\phi_{ad} - \phi_{cd})$, and likewise for similarly defined terms, then the term $\langle ABC \rangle$ in (6) with respect to $|\psi_{w}\rangle$ can be expressed as

$$\cos \theta_b(- \cos \theta_a \cos \theta_c + C_{31} \sin \theta_a \sin \theta_c \cos \phi_{ac})$$

(23)

$$+ \sin \theta_b(C_{12} \cos \theta_a \sin(\theta_c \cos \phi_{bc} + C_{23} \sin \theta_a \cos \theta_c \cos \phi_{ab}),$$

where $C_{23} = 2\alpha \beta$, $C_{23} = 2\beta \gamma$, $C_{31} = 2\gamma \alpha$ are the concurrences of the three reduced states of $|\psi_{w}\rangle$. Due to the inherent symmetry in (23), $S_{\text{max}}(\psi_{w})$ is achieved when all $\phi_i = 0$. Now adding all the terms (6), one obtains for the expectation of Mermin operator:

\[
\langle M \rangle = \frac{1}{4} \left[ (-1 - C_{31} - C_{12} - C_{23}) \left\{ \cos(\theta_a + \theta_b + \theta_v) + \cos(\theta_a' + \theta_b + \theta_v) + \cos(\theta_a + \theta_b + \theta_v) - \cos(\theta_a + \theta_b' + \theta_v) \right\} \\
+ (-1 + C_{31} + C_{12} - C_{23}) \left\{ \cos(\theta_a + \theta_b - \theta_v) + \cos(\theta_a' - \theta_b - \theta_v) + \cos(\theta_a + \theta_b - \theta_v) - \cos(\theta_a + \theta_b' - \theta_v) \right\} \\
+ (-1 - C_{31} + C_{12} + C_{23}) \left\{ \cos(\theta_a - \theta_b + \theta_v) + \cos(\theta_a' - \theta_b + \theta_v) + \cos(\theta_a - \theta_b + \theta_v) - \cos(\theta_a' - \theta_b + \theta_v) \right\} \\
+ (-1 + C_{31} - C_{12} + C_{23}) \left\{ \cos(\theta_a - \theta_b - \theta_v) + \cos(\theta_a' - \theta_b - \theta_v) + \cos(\theta_a - \theta_b - \theta_v) - \cos(\theta_a' - \theta_b - \theta_v) \right\} \right].
\]
The dependence on \( \theta_i \)'s can be suitably expressed by defining \( \vec{\theta}_g = (\theta_g + \theta_g')/2, \theta_g = (\theta_g' - \theta_g)/2, g \in \{a, b, c\} \).

Allowing \( \Sigma = (\vec{\theta}_a + \vec{\theta}_b + \vec{\theta}_c) \), and \( \Sigma_g = \Sigma - 2\vec{\theta}_g \) one obtains,

\[
S(\psi_w) = \frac{1}{2} \left\{ (-1 - C_{31} - C_{12} - C_{23}) \sin(\vec{\theta}_a + \vec{\theta}_b + \vec{\theta}_c) \{ G - 2 \sin(\vec{\theta}_a - \vec{\theta}_b - \vec{\theta}_c) \} \\
+ (-1 + C_{31} + C_{12} - C_{23}) \sin(\vec{\theta}_a - \vec{\theta}_b + \vec{\theta}_c) \{ G - 2 \sin(\vec{\theta}_a - \vec{\theta}_b + \vec{\theta}_c) \} \\
+ (-1 - C_{31} + C_{12} + C_{23}) \sin(\vec{\theta}_a - \vec{\theta}_b + \vec{\theta}_c) \{ G - 2 \sin(\vec{\theta}_a + \vec{\theta}_b - \vec{\theta}_c) \} \\
+ (-1 + C_{31} - C_{12} + C_{23}) \sin(\vec{\theta}_a - \vec{\theta}_b - \vec{\theta}_c) \{ G - 2 \sin(\vec{\theta}_a + \vec{\theta}_b + \vec{\theta}_c) \} \right\} \\
= \{ - \sin \Sigma + \sin \Sigma_a + \sin \Sigma_b + \sin \Sigma_c \} + C_{13} \{ \sin \Sigma + \sin \Sigma_a - \sin \Sigma_b + \sin \Sigma_c \} \\
+ C_{12} \{ \sin \Sigma - \sin \Sigma_a + \sin \Sigma_b + \sin \Sigma_c \} + C_{23} \{ \sin \Sigma + \sin \Sigma_a + \sin \Sigma_b - \sin \Sigma_c \} \\
\equiv 4(p_1 + p_2 C_{13} + p_3 C_{12} + p_4 C_{23}) , \\
G = \{ \sin(\vec{\theta}_a + \vec{\theta}_b + \vec{\theta}_c) + \sin(\vec{\theta}_a + \vec{\theta}_b - \vec{\theta}_c) + \sin(\vec{\theta}_a - \vec{\theta}_b + \vec{\theta}_c) + \sin(\vec{\theta}_a - \vec{\theta}_b - \vec{\theta}_c) \} .
\]

and where the second equality \([24]\) is achieved when \( \vec{\theta}_a = \vec{\theta}_b = \vec{\theta}_c = \pi/2 \). By symmetry, the global maximum of \( S_{\text{max}}(\psi_w) \) occurs when \( C_{31} = C_{12} = C_{23} = 2/3 \), for which \( \vec{\theta}_a = \vec{\theta}_b = \vec{\theta}_c = \vec{\theta} \). Then, \( |S_{\text{max}}(\psi_w)| = \sin 3\vec{\theta} + 5 \sin \vec{\theta} \).

The maximum occurs at \( \theta = 54.736^\circ \) giving \( S_{\text{max}}(\psi_w) = 4.354 \) \([12]\), which can be obtained when the measurement directions are \( \vec{a} = \vec{b} = \vec{c} = \hat{x} \cos \vec{\theta} + \hat{z} \sin \vec{\theta} \), and \( \vec{a}' = \vec{b}' = \vec{c}' = \hat{x} \cos \vec{\theta} - \hat{z} \sin \vec{\theta} \). As expected, \( S_{\text{max}}(\psi_w) \) is 4 for the tri-separable states (then only the first term survives in \([23]\), and for the bi-separable states when the first two terms \( (C_{13} \neq 0) \) survives in \([23]\)). For arbitrary tripartite entangled states (see Fig. 2), the only states which violates SI is when \( (p_1 + p_2 C_{13} + p_3 C_{12} + p_4 C_{23}) \geq 1 \).

**Conclusion:** In this letter we quantified the genuine tripartite nonlocality of the subclass of 3-qubit pure states, which can be generalized to all the pure states. Our main results showed that by construction SI is a suitable measure of tripartite nonlocal for the W-class states, for the GHZ-class states it reduces to Mermin’s inequality, and gives counter intuitive results. A large number of states in both classes don’t violate the inequality, which implies that perhaps the Svetlichny’s kind of hybrid local-nonlocal theory is too strong by assumption, and thus can simulate the genuine tripartite correlations in such states. Elsewhere, we show how the inequality should be appropriately modified such that the resulting inequality completely quantifies the nonlocality of all the 3-qubit pure states.

PR thanks S. Ghose for introducing him to SI. This letter is dedicated to the memory of Jharana Rani Samal.

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FIG. 2: Maximum of the Svetlichny operator for varying sum \( (C_{12} + C_{23} + C_{31}) \leq 2 \), for three values of \( C_{12} = \{0.35, 0.45, 1/2\} \).

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