THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF
ONE-DIMENSIONAL VILENKIN-FOURIER SERIES.

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Abstract. The main aim of this paper is to investigate \((H_p, L_p)\)-type inequalities for
maximal operators of logarithmic means of one-dimensional Vilenkin-Fourier series.

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1. INTRODUCTION

In one-dimensional case the weak type inequality
\[ \mu \left( \sigma^* f > \lambda \right) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0) \]
can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and
in Pál, Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujii [3] and Simon
[13] verified that \( \sigma^* \) is bounded from \( H_1 \) to \( L_1 \). Weisz [17] generalized this result and proved
the boundedness of \( \sigma^* \) from the martingale space \( H_p \) to the space \( L_p \) for \( p > 1/2 \). Simon
[12] gave a counterexample, which shows that boundedness does not hold for \( 0 < p < 1/2 \).
The counterexample for \( p = 1/2 \) due to Goginava ( [7], see also [2]).

Riesz’ s logarithmic means with respect to the trigonometric system was studied by a lot
of autors. We mentioned, for instance, the paper by Szasz [14] and Yabuta [19]. this means
with respect to the Walsh and Vilenkin systems by Simon [12] and Gát [4].

Móricz and Siddiqi [9] investigates the approximation properties of some special Nörlund
means of Walsh-Fourier series of \( L_p \) function in norm. The case when \( q_k = 1/k \) is excluded,
since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means.
In [3] Gát and Goginava proved some convergence and divergence properties of the Nörlund
logarithmic means of functions in the class of continuous functions and in the lebesque space
\( L_1 \). Among there, they gave a negative answer to the question of Móricz and Siddiqi [9]. Gát
and Goginava [6] proved that for each measurable function \( \phi(u) = \phi(u \sqrt{\log u}) \) there exists
an integrable function \( f \), such that
\[ \int_{G_m} \phi(|f(x)|) \, d\mu(x) < \infty \]
and there exist a set with positive measure, such that the Walsh-logarithmic means of the
function diverge on this set.

The main aim of this paper is to investigate \((H_p, L_p)\)-type inequalities for the maximal
operators of Riesz and Nörlund logarithmic means of one-dimensional Vilenkin-Fourier series.

We prove that the maximal operator \( R^* \) is bounded from the Hardy space \( H_p \) to the space
D. We also show that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$, for which

$$\|R^* f\|_{L_p} = +\infty.$$  

For the Nörlund logarithmic means we prove that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$ for which

$$\|L^* f\|_{L_p} = +\infty.$$  

Analogical theorems for Walsh-Paley system is proved in \[8\].

2. DEFINITIONS AND NOTATIONS

Let $N_+$ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots m_k - 1\}$ the addition group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_i}$'s.

The direct product $\mu$ of the measures

$$\mu_k (\{j\}) := 1/m_k,$$  \hspace{1cm} ($j \in Z_{m_k}$)

is the Haar measure on $G_{m_k}$, with $\mu (G_m) = 1$.

If $\sup_{n} m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of $G_m$ represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \hspace{1cm} (x_i \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of $G_m$

$$I_0 (x) : = G_m,$$

$$I_n (x) : = \{y \in G_m \mid y_0 = x_0, \ldots y_{n-1} = x_{n-1}\}, \hspace{1cm} (x \in G_m, n \in N).$$

Denote $I_n := I_n (0)$, for $n \in N_+$.

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \hspace{1cm} M_{k+1} := m_k M_k, \hspace{1cm} (k \in N),$$

then every $n \in N$, can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}, \hspace{1cm} (j \in N_+)$ and only a finite number of $n_j$'s differ from zero.

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system. At first define the complex valued function $r_k (x) : G_m \to C$, The generalized Rademacher functions as
\[ r_k(x) := \exp \left( 2\pi i x_k / m_k \right), \quad (i^2 = -1, x \in G_m, \ k \in N). \]

Now define the Vilenkin system \( \psi := (\psi_n : n \in N) \) on \( G_m \) as:

\[ \psi_n(x) := \prod_{k=0}^{\infty} r_k^n(x), \quad (n \in N). \]

Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \).

The Vilenkin system is orthonormal and complete in \( L^2(G_m) \).

[1, 15]

Now we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L^1(G_m) \) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system \( \psi \) in the usual manner:

\[ \hat{f}(k) := \int_{G_m} f \overline{\psi_k} \, d\mu, \quad (k \in N), \]
\[ S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in N_+, S_0 f := 0), \]
\[ \sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \quad (n \in N_+), \]
\[ D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in N_+). \]

Recall that

\[ D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \not\in I_n. \end{cases} \]

The norm (or quasinorm) of the space \( L^p(G_m) \) is defined by

\[ \|f\|_p := \left( \int_{G_m} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}, \quad (0 < p < \infty). \]

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x) : x \in G_m\} \) will be denoted by \( F_n(n \in N) \).

Denote by \( f = (f^{(n)}, n \in N) \) a martingale with respect to \( F_n(n \in N) \). (for details see e.g. [16]).

The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in N} |f^{(n)}|. \]

In case \( f \in L^1(G_m) \), the maximal functions are also be given by

\[ f^*(x) = \sup_{n \in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|. \]
For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingale for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$ 

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{N})$ is a martingale.

If $f = (f^{(n)}, n \in \mathbb{N})$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \Psi_i(x) \, d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in \mathbb{N})$ obtained from $f$.

In the literature, there is the notion of Riesz’ s logarithmic means of the Fourier series. The n-th Riesz’ s logarithmic means of the Fourier series of an integrable function $f$ is defined by

$$R_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{k},$$

where

$$l_n := \sum_{k=1}^{n} (1/k).$$

Let $\{q_k : k > 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund means for the Fourier series of $f$ is defined by

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=1}^{n} q_k.$$

If $q_k = k$, then we get Nörlund logarithmic means

$$L_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{n-k}.$$ 

It is a kind of “reverse” Riesz’ s logarithmic means.

In this paper we call this means logarithmic means.

For the martingale $f$ we consider the following maximal operators of
\[ R^* f(x) := \sup_{n \in \mathbb{N}} |R_n f(x)|, \]
\[ L^* f(x) := \sup_{n \in \mathbb{N}} |L_n f(x)|, \]
\[ \sigma^* f(x) := \sup_{n \in \mathbb{N}} |\sigma_n f(x)|. \]

A bounded measurable function \( a \) is \( p \)-atom, if there exists a dyadic interval \( I \), such that

\[ \begin{cases} 
  a) & \int_I a \, d\mu = 0, \\
  b) & \|a\|_{\infty} \leq \mu(I)^{-1/p}, \\
  c) & \text{supp}(a) \subset I.
\end{cases} \]

3. FORMULATION OF MAIN RESULT

**Theorem 1.** Let \( p > 1/2 \). Then the maximal operator \( R^* \) is bounded from the Hardy space \( H_p \) to the space \( L_p \).

**Theorem 2.** Let \( 0 < p \leq 1/2 \). Then there exists a martingale \( f \in H_p \) such that
\[ \|R^* f\|_p = +\infty. \]

**Corollary 1.** Let \( 0 < p \leq 1/2 \). Then there exists a martingale \( f \in H_p \) such that
\[ \|\sigma^* f\|_p = +\infty. \]

**Theorem 3.** Let \( 0 < p \leq 1 \). Then there exists a martingale \( f \in L_p \) such that
\[ \|L^* f\|_p = +\infty. \]

4. AUXILIARY PROPOSITIONS

**Lemma 1.** [18] A martingale \( f = (f^{(n)}, n \in \mathbb{N}) \) is in \( H_p \) (\( 0 < p \leq 1 \)) if and only if there exist a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of a real numbers such that for every \( n \in \mathbb{N} \):

\[ \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \]

\[ \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \]

Moreover,
\[ \|f\|_{H_p} \sim \inf \left( \sum_{K=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]

where the infimum is taken over all decomposition of \( f \) of the form (1).
5. PROOF OF THE THEOREM

**Proof of theorem 1.** Using Abel transformation we obtain

\[ R_n f(x) = \frac{1}{l_n} \sum_{j=1}^{n-1} \sigma_j f(x) + \frac{\sigma_n f(x)}{l_n}, \]

Consequently,

\[ L^* f \leq c \sigma^* f. \]

On the other hand Weisz proved that \( \sigma^* \) is bounded from the Hardy space \( H_p \) to the space \( L_p \) when \( p > 1/2 \). Hence, from (2) we conclude that \( R^* \) is bounded from the martingale Hardy space \( H_p \) to the space \( L_p \) when \( p > 1/2 \).

**Proof of theorem 2.** Let \( \{\alpha_k : k \in N\} \) be an increasing sequence of the positive integers such that

\[ \sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty, \]

\[ \sum_{\eta=0}^{k-1} \left( \frac{M_{2\alpha_k}}{\sqrt{\alpha_{\eta}}} \right)^{1/p} < \left( \frac{M_{2\alpha_k}}{\sqrt{\alpha_k}} \right)^{1/p}, \]

\[ \frac{\left( M_{2\alpha_k-1} \right)^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{M_{\alpha_k}}{\alpha_k^{3/2}}. \]

We note that such an increasing sequence \( \{\alpha_k : k \in N\} \) which satisfies conditions (3)-(5) can be constructed.

Let

\[ f^{(A)}(x) = \sum_{\{k : 2\alpha_k < A\}} \lambda_k a_k, \]

where

\[ \lambda_k = \frac{m_{2\alpha_k}}{\sqrt{\alpha_k}} \]

and

\[ a_k(x) = \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} \left( D_{M_{2\alpha_k+1}}(x) - D_{M_{2\alpha_k}}(x) \right). \]

It is easy to show that

\[ \|a_k\|_\infty \leq \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k-1}} \]

\[ \leq (M_{2\alpha_k})^{1/p} = (\text{supp}(a_k))^{-1/p}. \]
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$$S_{MA} a_k (x) = \begin{cases} a_k (x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A. \end{cases}$$

$$f^{(A)} (x) = \sum_{\{ k; 2\alpha_k < A \}} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k S_{MA} a_k (x),$$

$$\text{supp}(a_k) = I_{2\alpha_k},$$

$$\int_{I_{2\alpha_k}} a_k d\mu = 0.$$

from (3) and lemma 1 we conclude that $f = (f^{(n)}, n \in \mathbb{N}) \in H_p$.

Let

$$q_A^s = M_{2A} + M_{2s} - 1, \quad A > S.$$

Then we can write

$$R_{q_{\alpha_k}} f (x) = \frac{1}{l_{q_{\alpha_k}}} \sum_{j=1}^{q_{\alpha_k}^s} S_j f (x)$$

$$= \frac{1}{l_{q_{\alpha_k}}} \sum_{j=1}^{M_{2\alpha_k} - 1} S_j f (x)$$

$$+ \frac{1}{l_{q_{\alpha_k}}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} S_j f (x)$$

$$= I + II.$$

It is easy to show that

$$\hat{f}(j) = \begin{cases} \frac{M^{1/p-1}}{\sqrt{\alpha_k}}, & \text{if } j \in \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}, \quad k = 0, 1, 2, \ldots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{ M_{2\alpha_k}, \ldots, M_{2\alpha_k+1} - 1 \}. \end{cases}$$

Let $j < M_{2\alpha_k}$. Then from (4) and (8) we have
\begin{equation}
|S_j f(x)| \\
\leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} |\hat{f}(v)| \\
\leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \\
\leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \leq c M_{2\alpha_k-1}^{1/p} \sqrt{\alpha_{k-1}}.
\end{equation}

Consequently

\begin{equation}
|I| \leq \frac{1}{l_{q^*_k}} \sum_{j=1}^{M_{2\alpha_k}-1} |S_j f(x)| \\
\leq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p}}{\sqrt{\alpha_{k-1}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{j} \\
\leq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p}}{\sqrt{\alpha_{k-1}}}.
\end{equation}

Let \( M_{2\alpha_k} \leq j \leq q^*_k \). Then we have the following

\begin{equation}
S_j f(x) = \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \hat{f}(v) \psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \hat{f}(v) \psi_v(x) \\
= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x)\right) \\
+ \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \left(D_j(x) - D_{M_{2\alpha_k}}(x)\right).
\end{equation}

This gives that

\begin{equation}
II = \frac{1}{l_{q^*_k}} \sum_{j=M_{2\alpha_k}}^{q^*_k} \frac{1}{j} \left(\sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x)\right)\right) \\
+ \frac{1}{l_{q^*_k}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q^*_k} \frac{1}{j} \left(D_j(x) - D_{M_{2\alpha_k}}(x)\right) \\
= II_1 + II_2.
\end{equation}
To discuss $II_1$, we use (11). Thus we can write:

$$|II_1| \leq c \sum_{\eta=0}^{k-1} \frac{M_{2 \alpha \eta}^{1/p}}{\sqrt{\alpha \eta}} \leq \frac{c M_{2 \alpha k - 1}}{\sqrt{\alpha - 1}}.$$  

Since

$$D_{j+M_{2 \alpha k}} (x) = D_{M_{2 \alpha k}} (x) + \psi_{M_{2 \alpha k}} (x) D_j (x), \text{ when } j < M_{2 \alpha k},$$

for $II_2$ we have

$$II_2 = \frac{1}{l_{q_0 \alpha k}} \frac{M_{2 \alpha k}^{1/p-1}}{\sqrt{\alpha k}} \sum_{j=0}^{M_{2 \alpha k}} \frac{D_{j+M_{2 \alpha k}} (x) - D_{M_{2 \alpha k}} (x)}{j + M_{2 \alpha k}}$$

$$= \frac{1}{l_{q_0 \alpha k}} \frac{M_{2 \alpha k}^{1/p-1}}{\sqrt{\alpha k}} \psi_{M_{2 \alpha k}} \sum_{j=0}^{M_{2 \alpha k} - 1} \frac{D_j (x)}{j + M_{2 \alpha k}}.$$

We write

$$R_{q_0 \alpha k} f (x) = I + II_1 + II_2,$$

Then by (5), (7), (10) and (12)-(15) we have

$$\left| R_{q_0 \alpha k} f (x) \right| \geq |II_2| - |I| - |II_1|$$

$$\geq |II_2| - c \frac{M_{2 \alpha k}}{\alpha k^{3/2}}$$

$$\geq \frac{c}{\alpha k} \frac{M_{2 \alpha k}^{1/p-1}}{\sqrt{\alpha k}} \sum_{j=0}^{M_{2 \alpha k} - 1} \frac{D_j (x)}{j + M_{2 \alpha k}} - c \frac{M_{2 \alpha k}}{\alpha k^{3/2}}.$$

Let $0 < p \leq 1/2$, $x \in I_{2s} \setminus I_{2s+1}$ for $s = \lfloor 2 \alpha k / 3 \rfloor, \ldots, \alpha_k$. Then it is evident

$$\left| \sum_{j=0}^{M_{2s} - 1} \frac{D_j (x)}{j + M_{2 \alpha k}} \right| \geq c \frac{M_{2s}^2}{M_{2 \alpha k}}.$$

Hence we can write
\[ |R_{q_{\alpha_k}} f (x)| \geq \frac{c M_{2\alpha_k}^{1/p-1}}{\alpha_k} \frac{c M_{2s}^2}{\sqrt{\alpha_k}} - c M_0 \frac{\alpha_k}{\alpha_k^3/2} \]
\[ \geq \frac{c M_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^3} \geq c M_{2\alpha_k} M_{2s} \frac{\alpha_k}{\alpha_k^3/2}. \]

Then we have
\[
\int_{G_m} |R^* f (x)|^p d\mu (x) \\
\geq \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} |R_{q_{\alpha_k}} f (x)|^p d\mu (x) \\
\geq \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left( \frac{c M_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^3} \right)^p d\mu (x) \\
\geq c \sum_{s=\lfloor 2\alpha_k/3 \rfloor}^{\alpha_k} \frac{M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1}}{\alpha_k^{3p/2}} \\
\geq \begin{cases} 
\frac{2^{\alpha_k (1-2p)}}{\alpha_k^{3p/2}}, & \text{when } 0 < p < 1/2, \\
ca_{1/4}, & \text{when } p = 1/2, \\
\rightarrow \infty, & \text{when } k \to \infty.
\end{cases}
\]

which complete the proof of the theorem 2.

**Proof of theorem 3.** We write

\begin{equation}
L_{q_{\alpha_k}} f (x) = \frac{1}{l_{d_{\alpha_k,s}}^{q_{\alpha_k}^s}} \sum_{j=1}^{q_{\alpha_k}^s} S_j f (x) \\
= \frac{1}{l_{q_{\alpha_k}^s}} \sum_{j=1}^{M_{2\alpha_k}-1} S_j f (x) \\
+ \frac{1}{q_{\alpha_k}^s} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s} S_j f (x) = III + IV.
\end{equation}

Since (see 11)
\[ |S_j f (x)| \leq \frac{M_{2\alpha_k-1}^{1/p}}{\sqrt{\alpha_{k-1}}}, \quad j < M_{2\alpha_k}. \]
For III we can write

\begin{equation}
|III| \leq \frac{c}{\alpha_k} \sum_{j=0}^{M_{2\alpha_k - 1}} \frac{1}{q_{\alpha_k}^j - j} \sqrt{\alpha_{k-1}} \leq \frac{c M_{2\alpha_k - 1}^{1/p}}{\sqrt{\alpha_{k-1}}}.
\end{equation}

Using (11) we have

\begin{equation}
IV = \frac{1}{l q_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} \frac{1}{q_{\alpha_k}^j - j} \left( \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_k - 1}^{1/p}}{\sqrt{\alpha_{\eta}}} \left( D_{M_{2\alpha_k + \eta}} (x) - D_{M_{2\alpha_k}} (x) \right) \right)
+ \frac{1}{l q_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}} \frac{M_{2\alpha_k - 1}^{1/p}}{q_{\alpha_k}^j - j} \left( D_j (x) - D_{M_{2\alpha_k}} (x) \right)
= IV_1 + IV_2.
\end{equation}

Applying (4) in IV_1 we have

\begin{equation}
|IV_1| \leq \frac{c M_{2\alpha_k - 1}^{1/p}}{\sqrt{\alpha_{k-1}}}.
\end{equation}

From (14) we obtain

\begin{equation}
IV_2 = \frac{1}{l q_{\alpha_k}} \frac{M_{2\alpha_k - 1}^{1/p}}{q_{\alpha_k}^j - j} \sum_{j=0}^{M_{2\alpha_k - 1}} D_j (x).
\end{equation}

Let \(x \in I_{2s} \setminus I_{2s+1}\). Then \(D_j (x) = j, j < M_s\). Consequently

\[ \sum_{j=0}^{M_{2s-1}} \frac{D_j (x)}{M_{2s} - j} = \sum_{j=0}^{M_{2s-1}} \frac{j}{M_{2s} - j} = \sum_{j=0}^{M_{2s-1}} \left( \frac{M_{2s}}{M_{2s} - j} - 1 \right) \geq csM_{2s}. \]

Then

\begin{equation}
|IV_2| \geq \frac{c M_{2\alpha_k - 1}^{1/p}}{\alpha_k^{3/2}} s M_{2s}, \quad x \in I_{2s} \setminus I_{2s+1}.
\end{equation}

Combining (5), (16)-(21) for \(x \in I_{2s} \setminus I_{2s+1}, s = [2\alpha_k/3] \ldots \alpha_k\) and \(0 < p \leq 1\) we have
Then

\[
\int_{G_m} |L^{*} f (x)|^p \, d\mu (x) \\
\geq \sum_{s=[2\alpha_k/3]}^{m_k} \int_{I_2s \setminus I_{2s+1}} |L^{*} f (x)|^p \, d\mu (x) \\
\geq c \sum_{s=[2\alpha_k/3]}^{m_k} \int_{I_2s \setminus I_{2s+1}} \left( \frac{M^{1/p-1}_{2\alpha_k-1}}{\alpha_k^{3/2}} sM_{2s} \right) d\mu (x) \\
\geq c \sum_{s=[2\alpha_k/3]}^{m_k} \frac{M^{1-p}_{2\alpha_k-1}}{\alpha_k^{p/2}} M^{p-1}_{2s} \\
\geq \begin{cases} 
\frac{2^{\alpha_k(1-p)}}{\alpha_k^{p/2}}, & \text{when } 0 < p < 1, \\
c\sqrt{\alpha_k}, & \text{when } p = 1, \\
\rightarrow \infty, & \text{when } k \rightarrow \infty.
\end{cases}
\]

Theorem 3 is proved.

REFERENCES

[1] G. N. AGAEV, N. Ya. VILENKIN, G. M. DZHAFA RLY and A. I. RUBINSHEITN, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, Baku, Elim, 1981 (in Russian).
[2] I. BLAHOTA, G. GÁT and U. GOGINAVA, Maximal operators of Fejér means of Vilenkin-Fourier series. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), 1-7.
[3] N. J. FUJII, A maximal inequality for \( H_1 \) functions on the generalized Walsh-Paley group, Proc. Amer. Math. Soc. 77 (1979), Ill-116.
[4] G. GÁT, Investigations of certain operators with respect to the Vilenkin systems, Acta Math. Hunger. N 1-2.61(1993), 131-149.
[5] G. GÁT, U. GOGINAVA, Uniform and L-convergence of logarithmic means of Walsh-Fourier series. Acta Math. Hunger. 22 (2006), no. 2, 497-506.
[6] G. GÁT, U. GOGINAVA, On the divergence of Nörlund logarithmic means of Walsh-Fourier series. Acta Math. Sin. (Engl. Ser.) 25 (2009), no 6, 903-916.
[7] U. GOGINAVA, The maximal operator of Marcinkiewicz-Fejér means of the \( d \)-dimensional Walsh-Fourier series. East J. Approx. 12 (2006), no. 3, 295–302.
[8] U. GOGINAVA, The maximal operator of logarithmic means of Walsh-Fourier series. Rendiconti del Circlo Matematico di Palermo Serie II, 82(2010), pp. 345-357.
[9] F. MóRICZ, A. SIDDIQI, Approximation by Nörlund means of Walsh-Fourier series, Journal of approximation theory, 70 (1992), 375-389.
[10] J. PÁL and P. SIMON, On a generalization of the concept of derivative, Acta Math. Hung., 29 (1977), 155-164.
[11] F. SCHIPP, Certain rearrangements of series in the Walsh series, Mat. Zametki, 18 (1975), 193-201.
[12] P. SIMON, Cesáro summability with respect to two-parameter Walsh systems, Monatsh. Math., 131 (2000), 321-334.
[13] P. SIMON, Investigations with respect to the Vilenkin system, Annales Univ. Sci. Budapest Eotv., Sect. Math., 28 (1985), 87-101.
[14] O. SZASZ, On the logarithmic means of rearranged partial sums of Fourier series, Bull. Amer. Math. Soc., 48 (1942), 705-711.
[15] N. Ya. VILENKIN, A class of complete orthonormal systems, Izv. Akad. Nauk. U.S.S.R., Ser. Mat., 11 (1947), 363-400.
[16] F. WEISZ, Martingale Hardy spaces and their application in Fourier analysis, Springer, Berlin-Heidelberg-New York, 1994.
[17] F. WEISZ, Cesáro summability of one and two-dimensional Fourier series, Anal. math. Studies, 5 (1996), 353-367.
[18] F. WEISZ, Hardy spaces and Cesáro means of two-dimensional Fourier series, Bolyai Soc. Math. Studies, (1996), 353-367.
[19] K. YABUTA, Quasi-Tauberian theorems, applied to the summability of Fourier series by Riesz’ s logarithmic means, Tohoku Math. Journ. 22 (1970), 117-129.
[20] A. ZYGMUND, Trigonometric Series, Vol. 1, Cambridge Univ. Press, 1959.

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