A Graphical Calculus for Multi-Qudit Computations with Generalized Clifford Algebras

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Abstract
In this article, we develop a graphical calculus for multi-qudit computations with generalized Clifford algebras, using the algebraic framework we developed in our previous paper. We build our graphical calculus out of a fixed set of graphical primitives defined by algebraic expressions constructed out of elements of a given generalized Clifford algebra, a graphical primitive corresponding to the ground state, and also graphical primitives corresponding to projections onto the ground state of each qudit. We establish many “superpower”-like properties of the graphical calculus using purely algebraic methods, including a novel algebraic proof of a Yang-Baxter-like equation. We also derive several new identities for the braid elements, which are key to our proofs. In terms of physics, we connect these braid identities to physics by showing the presence of a conserved charge. Furthermore, we demonstrate that in many cases, the verification of involved vector identities can be reduced to the combinatorial application of two basic vector identities. Finally, we show how to explicitly compute various vector states in an efficient manner using algebraic methods.
1 Introduction

The following physics questions motivate this article, which is the second in a series: Can we learn new things about quantum entanglement by studying a graphical calculus for the generalized Clifford algebras\(^1\)? In this setting, braiding operators defined using the generalized Clifford algebra are unitary operations that entangle neighboring qudits (multi-dimensional vector spaces). Thus, when we apply a sequence of braiding operators to the ground (or vacuum) state, we expect different kinds of entangled states to result, depending on the sequence and on the braidings in the sequence. Is there an easy way to classify the resulting kinds of entanglement using the graphical calculus? How does the classification depend on the number of qudits involved?

To treat these questions in a systematic manner, we have developed an algebraic framework in [6].

While the algebraic framework is in itself sufficient for doing calculations and proving identities of various sorts, it turns out to be convenient to consider diagrammatic representations in order to obtain intuition about what kind of algebraic identities might be true. To be more specific, one can follow the following flowchart:

1. Write down an algebraic expression.
2. Convert it to one of the prescribed graphical forms.
3. Guess what graphical identities might be true for the graphical expression.
4. Write down conjectural algebraic identities corresponding to the conjectured graphical identities.
5. Prove the conjectured identities algebraically using explicit calculation with the algebraic framework for the generalized Clifford algebras, or using already proven algebraic identities.
6. Repeat.

It is quite remarkable how far one can get with this approach, once the initial difficulties of getting algebraic identities is overcome. In particular, we show that the algebraic framework, coupled with some new technical innovations of ours, enables us to show algebraically for the first time why one can treat the braiding operator as a braid in the conventional sense (spoiler alert: it satisfies a Yang-Baxter-like equation\(^2\)).

\(^1\)The earliest paper introducing generalized Clifford algebras appears to be [7] in 1952. Other early work included [10] in 1964, [9] in 1966, and [8] in 1967.

\(^2\)The distinction is that the Yang-Baxter equation [1] appears to primarily refer to a morphism from \(A\) to \(A \otimes A\), where \(A\) is an algebra, which embeds in \(A \otimes A \otimes A\); the equation we will prove will have a structural similarity to the Yang-Baxter equation, but one cannot define a tensor product. A different way to express the distinction is that generalized Clifford algebras have an additional time-ordering when one wants to “tensor” elements together. This additional structure could be useful in its own right.
For logical consistency, the reader should consider the graphical calculus as simply a transcription of the algebraic framework into a combination of a few basic building blocks, which aids in intuition. While it may be convenient to imagine that the diagrams mean something, the reader will do well to remember that all our proofs are purely algebraic, and the diagrams are just (very helpful) visual aids.

2 The Graphical Calculus

2.1 Building Blocks

The philosophy that we follow in our graphical calculus is that the diagrams we draw are indivisible. We do not assign any a priori meaning to the subcomponents of the diagrams, i.e. a single strand, or a single cap, or a single cup. In this respect, although the diagrams we draw are all borrowed from the earlier work of [3], the a priori meaning of the diagrams is different. Our philosophy is that our algebraic framework ought to be robust enough that one can derive a posteriori a large number of algebraic relations, and therefore by proving more and more relations, the initially content-free diagrams acquire new, emergent properties. On a technical level, we have opted for a more basic construction which is directly built out of the elements of the generalized Clifford algebra, which is justified by the axiomatic framework in our previous article [6].

The graphical calculus is useful as a guide to doing the algebraic calculations, and down the line, one may use visual inspection to speed up the calculations.

In a previous article [6] I presented the following two axioms as a way to abstract certain high-level properties of the generalized Clifford algebras. I showed that these 2 axioms are satisfied by an explicit construction in the main theorem of [6]. We will now convert these axioms into graphical form.

Fix $N$ a positive integer greater than 1, $n$ a positive integer at least 1, and consider the generalized Clifford algebra $\mathcal{C}_{2n}^{(N)}$ generated by $c_1, c_2, c_3, \ldots, c_{2n}$ subject to $c_i c_j = q c_j c_i$ if $i < j$, and $c_i^N = 1$ for all $i$. Here, $q = \exp(2\pi i/N)$ is a primitive $N$th root of unity. When $N = 2$, one recovers the Clifford algebra with $2n$ generators. As stated in [6], we have the following two axioms.

**Axiom 1:** Let $\mathcal{V}^{N^n}(\mathbb{C})$ be a complex vector space upon which the generalized Clifford algebra is realized as unitary $N^n$ by $N^n$ matrix operators. Assume that there exists a state (which we call the ground state) which is a tensor of states $|\Omega\rangle$, $|\Omega\rangle^\otimes n$, that satisfies the following algebraic identity:

$$c_{2k-1} |\Omega\rangle^\otimes n = \zeta c_{2k} |\Omega\rangle^\otimes n$$

for all $k = 1, 2, \ldots, n$, where $\zeta$ is a square root of $q$ such that $\zeta^{N^2} = 1$.

In addition, for each qudit, the projector $E_k$ onto the $k$th qudit’s ground state $|\Omega\rangle$ is assumed to satisfy

$$c_{2k-1} E_k = \zeta c_{2k} E_k.$$
Axiom 2: Scalar product: The set \( \{ c_2^a c_4^a \ldots c_{2n}^a | \Omega \}^\otimes n : a_i = 0, 1, \ldots, N - 1 \} \) is an orthonormal basis for \( V^N_n(\mathbb{C}) \).

We will now define a series of graphical primitives. These graphical primitives are the only allowed graphical elements in our graphical representations. Any message encoded using this alphabet must be specified by a sequence of hieroglyphs belonging to this alphabet. One may think of each diagram as a hieroglyph in an alphabet of hieroglyphs, and the sequence of hieroglyph as running from top to bottom. (This corresponds to the composition of operators, in which, in terms of the corresponding algebraic objects, our mathematical “messages” are given by a sequence of operations running from right to left.)

Fix \( \delta = \sqrt{N} > 0 \). The following graphical primitives are defined in terms of the ground state via:

**Definition 2.1.**

\[
\begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array}
\cdots \begin{array}{c}
\bigotimes
\end{array} := \delta^{n/2} |\Omega\rangle^\otimes n
\]

\[
\begin{array}{c}
\bigotimes
\end{array}
\cdots \begin{array}{c}
\bigotimes
\end{array} := \delta^{n/2} \langle \Omega |^\otimes n
\]

**Definition 2.2.**

\[
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \begin{array}{c}
a
\end{array} \begin{array}{c}
\vdots
\end{array} := c_{2k-1}^a
\]

\[
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \begin{array}{c}
b
\end{array} \begin{array}{c}
\vdots
\end{array} := c_{2k}^b
\]

\( \forall a, b \in \mathbb{Z} \). Here we mean for the label \( a \) to be placed immediately left of the \( 2k - 1 \)-th strand, and the label \( b \) to be placed immediately left of the \( 2k \)-th strand. There are \( 2n \) total strands in each diagram.

We also define for completion that

\[
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \begin{array}{c}
\vdots
\end{array} \begin{array}{c}
\vdots
\end{array} \begin{array}{c}
\vdots
\end{array} := 1
\]

Note that the identity primitive composed with itself “is” itself, graphically, which is consistent with its definition as being equal to 1. Similarly, the identity primitive composed (in either order) with the primitives for the powers of the generators \( c_k \) again yields those same primitives. In this sense, the diagrammatic definitions are well-behaved.

**Definition 2.3.**

\[
\begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \begin{array}{c}
\bigcup
\end{array} \begin{array}{c}
\vdots
\end{array} := \delta E_k
\]

Here we mean for the “cup-cap” combination to be replacing the \( 2k - 1 \) and \( 2k \)-th strands.

There are \( 2n \) strands in total.

---

3In this respect, in our graphical calculus, we do not allow for the cup-cap combination which is prescribed in [3], i.e. we don’t allow not-in-place placement, i.e. on the \( 2k \) and \( 2k + 1 \)-th strands, which loosely speaking, straddles different qudits.
Definition 2.4. We also define a graphical primitive, which we call the positive braid on strands \( l \) and \( l + 1 \), for \( l = 1, 2, \ldots, 2n - 1 \):

\[
\begin{array}{c}
\backslash / \backslash / \cdots := b_{12} \\
\backslash / \cdots := b_{23} \\
\cdots := b_{k,k+1} \\
\cdots := b_{2n-1,2n}
\end{array}
\]

which defines \( 2n - 1 \) different braid operators.

We also define graphical primitives for the corresponding negative braids.

\[
\begin{array}{c}
\backslash / \backslash / \cdots := b_{21} \\
\backslash / \cdots := b_{32} \\
\cdots := b_{k+1,k} \\
\cdots := b_{2n,2n-1}
\end{array}
\]

The algebraic definition of these braid elements is given by

\[
b_{kl} := \frac{\omega^{1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_k^i c_l^{-i}
\]

and

\[
b_{lk} := \frac{\omega^{-1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_l^i c_k^{-i}
\]

for \( k < l \) in \( \{1, 2, \ldots, 2n\} \). Here,

\[
\omega := \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^i.
\]

Note that this is a general definition of the braid element, which goes beyond the diagrams above, since we allow for \( |k - l| \neq 1 \), which includes the local (nearest-neighbor) braid operators as a special case.

Remark 2.5. \( \omega \) has modulus 1 (see [3] for a proof), implying that \( b_{kl}^\dagger = b_{lk} \) for \( k \neq l \).

Thus, in terms of terminology, we will refer to the positive braids as just braids, and the negative braids as adjoint braids.
2.2 Graphical Representation of the Axioms

The algebraic identities
\[ c_i c_j = q c_j c_i \]
for \( i < j \),
\[ c_i^N = 1 \]
for all \( i = 1, 2, \ldots, 2n \), as well as
\[ c_{2k-1} E_k = \zeta c_{2k} E_k \]
tell us that
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
i.e. when the primitive for \( c_j \) precedes that for \( c_i \), swapping the order of primitives yields a factor of \( q \), for \( i < j \), and also that
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \] and
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
Furthermore, the vector identity
\[ c_{2k-1} |\Omega\rangle^\otimes n = \zeta c_{2k} |\Omega\rangle^\otimes n \]
yields the diagrammatic “identity”
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
\[ \frac{1}{\zeta} \]
An additional identity which is useful is the following:

**Lemma 2.6.**
\[ c^a_i c^b_j = q^{ab} c^b_j c^a_i \] (2.1)
for \( i < j \), \( a, b \) integers.

**Proof.** By double induction on \( a \) and \( b \). □

Another identity, due to [3], is
Lemma 2.7.  \[ c_{2i-1}^a E_i = \zeta^a c_{2i}^a E_{2i} \] for \( i = 1, 2, \ldots, n \), \( a \) an integer.

Proof. By induction.

\[ \square \]

3 Graphical Calculus Superpowers: Algebraic Identities which extend the Set of Graphical “Identities”

3.1 The Golden Rule for the Generalized Clifford Algebra

Proposition 3.1. The generalized Clifford algebra \( C^{(N)} \) has trivial center, i.e. the only elements that commute with all elements of the generalized Clifford algebra are \( \mathbb{C}1 \).

Proof. Any element of the generalized Clifford algebra is a finite sum of elements of the form \( \alpha c_{k_1}^{e_1} c_2^{e_2} \cdots c_{k_m}^{e_m} \) for \( \alpha \in \mathbb{C} \), \( m \) a positive integer, \( k_i \) in the index set \( I_{2n} = \{1, 2, \ldots, 2n\} \), and \( e_i \in \{1, -1\} \) for \( i = 1, 2, \ldots, m \). By repeatedly applying the relations \( c_{k_i}^{-1} = c_{k_i}^{N-1} \) and \( c_i c_j = q c_j c_i \) for \( i < j \) to swap the order of multiplication, we can put each term in the sum into normal form, by which we mean that the term is of the form \( \beta_{r_1 r_2 \cdots r_n} c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}} \), for \( r_i \in \{0, 1, 2, \ldots, N-1\} \). Thus, we obtain that every element of the generalized Clifford algebra is prescribed by a sum given by

\[ x = \sum_{r_1, r_2, \ldots, r_{2n} = 0, 1, \ldots N-1} x_{r_1 r_2 \cdots r_{2n}} c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}}. \]

Now we want to show that \( x = 0 \) in the algebra if and only if \( x_{r_1 r_2 \cdots r_{2n}} = 0 \) for all indices, i.e. the set \( \{c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}} : r_1, r_2, \ldots, r_{2n} = 0, 1, \ldots N-1\} \) is a basis. Here, we must recall that the generalized Clifford algebra is loosely defined to be the “freest” algebra generated by the relations \( c_{k_i}^{-1} = c_{k_i}^{N-1} \) and \( c_i c_j = q c_j c_i \) for \( i < j \).

Hence it suffices to exhibit an algebra satisfying these relations such that \( x = 0 \) if and only if \( x_{r_1 r_2 \cdots r_{2n}} = 0 \) for all indices. Since an explicit vector representation with this property is given in [5], it follows that the set \( \{c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}} : r_1, r_2, \ldots, r_{2n} = 0, 1, \ldots N-1\} \) is a basis.

Using this basis property, it becomes simple to show that the algebra has trivial center. Note that the basis property implies uniqueness of the sum decomposition. Let \( x \) lie in the center of the algebra, and \( x \neq 0 \). Then there is an index label \( r_1, r_2, \ldots, r_{2n} \) such that \( x_{r_1 r_2 \cdots r_{2n}} \neq 0 \). Note that \( xc_1 = c_1 x \) implies that \( x_{r_1 r_2 \cdots r_{2n}} = q^{-\sum i \leq r} x_{r_1 r_2 \cdots r_{2n}} \) by comparing the coefficient of \( c_1^{r_1+1} c_2^{r_2} \cdots c_{2n}^{r_{2n}} \). Thus, \( r_2 + 1 + \cdots + r_{2n} = 0 \). Similarly, \( xc_k = c_k x \) implies that \( q^{-\sum i \leq r} x_{r_1 r_2 \cdots r_{2n}} = q^{-\sum i \neq k} x_{r_1 r_2 \cdots r_{2n}} \) and so \( \sum_i x_{r_i} = 0 \), for \( k \) from 1 to 2n, yielding 2n equations in 2n unknowns. The unique solution is given by \( r_1 = r_2 = \cdots = r_{2n} = 0 \). Thus, \( x \) is a multiple of the identity 1.

\[ \square \]

\[ ^4 \text{We will not attempt to justify this appellation, but will remark that something analogous to the axiomatic construction of the Clifford algebra by quotienting the universal associative algebra on 2n elements by an appropriate two-sided ideal is probably required. There is also some additional work in identifying the construction up to isomorphism, which appears to be somewhat annoying.} \]
3.2 An “Intertwining” Approach: New Identities for the Generalized Clifford Algebra

3.2.1 A Systematic Procedure

The golden rule allows us to give a systematic procedure for proving identities in the algebra. The basis of the procedure is the following proposition:

**Proposition 3.2.** Let \( x, y \) lie in the generalized Clifford algebra, and suppose \( y \) is invertible. Further assume that the constant terms of \( x \) and \( y \) are nonzero. Then \( x = y \) if and only if \( y^{-1}x \) lies in the center of the generalized Clifford algebra, and the constant term in \( x \) agrees with the constant term in \( y \).

**Proof.** Clearly, the only if direction is true since \( x = y \) implies \( y^{-1}x = 1 \).

For the if direction, if \( y^{-1}x \) lies in the center, by the golden rule, \( y^{-1}x \in \mathbb{C}1 \), i.e. \( y = \alpha x \).

In the proof of proposition 3.1, we showed that this implies that all terms of \( y \) and \( \alpha x \) agree, in particular the constant terms. By hypothesis, the constant terms of \( y \) and \( x \) agree and are nonzero, so \( \alpha = 1 \).

We now provide a concrete way to show that an element lies in the center of the generalized Clifford algebra.

**Proposition 3.3.** An element \( x \) lies in the center of the generalized Clifford algebra if and only if it commutes with \( c_i \) for each \( i = 1, 2, \ldots, 2n \).

**Proof.** The only if direction is clearly true.

For the if direction, any element \( y \) in the algebra has a unique decomposition as

\[
y = \sum_{r_1, r_2, \ldots, r_{2n}, r_1, r_2, \ldots, r_{2n} = 0, 1, \ldots, N-1} y r_1 r_2 \ldots r_{2n} c_{r_1}^1 c_{r_2}^2 \cdots c_{2n}^{2n}.
\]

By iterative commutation, using the commutation property of \( x \) with \( c_i \), one can show that \( x c_{r_1}^1 c_{r_2}^2 \cdots c_{2n}^{2n} = c_{r_1}^1 c_{r_2}^2 \cdots c_{2n}^{2n} x \). Multiplying by the constant prefactor and summing over the indices, one obtains that \( xy = yx \), as desired, for arbitrary \( y \) in the algebra.

3.2.2 Intertwining Identities

By intertwining identities, we mean identities of the form \( bx = yb \). In this section, we prove some new intertwining identities.

We have discovered the following new intertwining identity for the braid \( b_{ij} \). We first give a direct proof, and then give an alternate proof which involves some intermediate intertwining identities, which may have more general applications.

This identity significantly generalizes a result of [3], which is the special case for \( a = 0 \).

**Proposition 3.4.**

\[
b_{kl} c_{k}^a b_{i} = q^{a^2 + ab} c_{k}^{2a+b} c_{i}^{-a} b_{kl}
\]

for \( k < l \).
Proof. Since \( b_{kl} = \frac{a^{1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_i^k c_i^{-i} \), it suffices to show that
\[
\left( \sum_{i=0}^{N-1} c_i^k c_i^{-i} \right) c_k^a c_l^b = q^{a^2 + ab} c_k^{2a + b} c_l^{-a} \left( \sum_{i=0}^{N-1} c_i^i c_i^{-i} \right).
\]

Applying lemma 2.6, the LHS becomes
\[
\sum_{i=0}^{N-1} q^{ai} c_i^{a+i} c_l^{-b+i}
\]
and the RHS becomes
\[
\sum_{i=0}^{N-1} q^{a^2 + ab} q^{ai} c_k^{2a + b + i} c_l^{-a - i}.
\]

By shifting the index of summation from \( i \) to \( i + a + b \) in the LHS, the LHS becomes
\[
\sum_{i=0}^{N-1} q^{a(i+a+b)} c_k^{2a+b+i} c_l^{-a-i}
\]
which is just the RHS.

In terms of the graphical calculus, we economically write down the following diagrammatic identity, which is specific to \( b_{12} \) and the generalized Clifford algebra with only 2 generators \( c_1, c_2 \):

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
= q^{a^2 + ab}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 2a + b
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 -a
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

It is convenient to also write down the corresponding identity for the adjoint braid:

**Corollary 3.5.**
\[
b_{lk} c_k^r c_l^s = q^{rs + s^2} c_k^{rs} c_l^{-s} b_{lk}.
\]

for \( k < l \), and \( r,s \) integers.

Proof. The adjoint of the identity in 3.4 is \( c_l^{-b} c_k^{-a} b_{lk} = q^{-a^2 - ab} b_{lk} c_l^a c_k^{-2a - b} \), which becomes \( q^{-ab} c_k^{-a} c_l^{-b} b_{lk} = q^{a^2} b_{lk} c_k^{-2a - b} c_l^a \) upon commutation. Now we let \( r = -2a - b \), \( s = a \), so
\[
b_{lk} c_k^r c_l^s = q^{rs + s^2} c_k^{rs} c_l^{-s} b_{lk}.
\]

The corresponding diagrammatic identity for \( b_{21} \) for the generalized Clifford algebra with two generators \( c_1, c_2 \) is
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 r
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 s
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
= q^{rs + s^2}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 -s
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 r + 2s
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

There is an alternate route to proving 3.4, which is somewhat illuminating: We start with an intertwining identity which is a commutation relation:
Lemma 3.6.

\[(c_k^b c_l^{-b})(c_k^a c_l^{-a}) = (c_k^a c_l^{-a})(c_k^b c_l^{-b})\]

for \(k < l\).

**Proof.** Applying lemma 2.6 to LHS yields \(q^{ab} c_k^{a+b} c_l^{-(a+b)}\); applying lemma 2.6 to RHS yields \(q^{ab} c_k^{a+b} c_l^{-(a+b)}\). Thus, LHS=RHS. \(\square\)

We also note that the following commutation relation holds as well:

**Lemma 3.7.**

\[(c_k^a c_l^{-a}) c_p = c_p (c_k^a c_l^{-a})\]

for \(k < l\) and \(p\) satisfies \(p < k < l\) or \(p > l > k\).

**Proof.** If \(k < l < p\), commuting \(c_p\) past (in front of) \(c_l^{-a}\) in the LHS yields \(q^{-a}\); commuting it past \(c_k^a\) then yields an additional factor \(q^a\). So we obtain the RHS. A similar proof applies for the case \(p < k < l\). \(\square\)

Now comes the exciting part. Since the braid \(b_{kl}\) is a sum of elements of the form \(c_k^i c_l^{-i}\), it follows that

**Lemma 3.8.**

\[b_{kl} c_k^a c_l^{-a} = c_k^a c_l^{-a} b_{kl}\]

for \(k < l\).

**Proof.** By linear extension of Lemma 3.6. \(\square\)

Now we use a simple result due to [3].

**Lemma 3.9.**

\[b_{kl} c_l = c_k b_{kl}\]

for \(k < l\).

**Proof.** It suffices to show that

\[
\left( \sum_{i=0}^{N-1} c_k^i c_l^{-i} \right) c_l = c_k \left( \sum_{i=0}^{N-1} c_k^i c_l^{-i} \right).
\]

Collecting terms, it is equivalent to show that

\[
\sum_{i=0}^{N-1} c_k^i c_l^{-(i-1)} = \sum_{i=0}^{N-1} c_k^{i+1} c_l^{-i}.
\]

It is clear that the two are equal since the RHS is just the LHS with \(i\) shifted to \(i - 1\). \(\square\)

It remains but to combine lemmas 3.8 and 3.9. This gives us an alternate proof of proposition 3.4.
Alternate Proof of Proposition 3.4. We want to show that

\[ b_{kl} c^a c^b_{kl} = q^{a^2 + ab} c^2_{kl} c^a c^b_{kl} \]

for \( k < l \). To use lemmas 3.8 and 3.9, we rewrite \( b_{kl} c^a c^b_{kl} \) as \( b_{kl} c^a c^a_{kl} c^a_{kl} \). This becomes \( c^a_{kl} c^a_{kl} c^a_{kl} c^a_{kl} \) after commuting past the braid, and then \( c^a_{kl} c^a_{kl} c^a_{kl} b_{kl} \) after applying lemma 3.9 \( a + b \) times. Finally, applying lemma 2.6 to the middle two terms yields \( q^{a^2 + ab} c^2_{kl} c^a c^b_{kl} \) as desired.

3.2.3 The Notion of Charge Conservation

We now interpret the previous section’s intertwining identities in terms of physics.

First, we define a charge operator \( C \):

**Definition 3.10.** Define \( C \) by linear extension of its action on the basis:

\[ C(c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}}) := q^{r_1 + r_2 + \cdots + r_{2n}} c_1^{r_1} c_2^{r_2} \cdots c_{2n}^{r_{2n}} \]

for all integer indices \( r_i \).

We call \( r_1 + r_2 + \cdots + r_{2n} \) the charge of the basis element, which is well-defined modulo \( N \). This terminology of an element’s charge is also applicable for linear combinations of basis elements with the same charge.

Then, lemma 3.6 tells us that eigenstates of \( C \) of eigenvalue 1 which lie in the subalgebra generated by \( c_k, c_l \) commute. We call eigenstates of \( C \) with eigenvalue 1 neutral.

Graphically, we can describe this commutation relation 3.6 for the algebra generated by \( c_1 \) and \( c_2 \) as

\[
\begin{array}{c|c|c}
  b & -b & -a \\
  a & -a & b \\
\end{array}
\]

and there are analogous diagrams (with additional strands in between, and to the left and right) for the generalized Clifford algebras with more generators.

We now observe that the lemma 3.8 can be reinterpreted in terms of respecting charge conservation, i.e. bringing an element of definite charge across the braid will conserve the charge, which is in this case just 0. Thus, we say that the relation 3.8 provides a physical constraint on the action of the braid. In fact, this physical constraint provides a compelling explanation for why the master intertwining relation 3.4 holds; the latter is essentially forced by the constraint and the additional relation \( b_{kl} c_l = c_k b_{kl} \).

3.2.4 Applications of the Golden Rule

Using the prior sections on the golden rule and various intertwining identities, we can now prove some identities involving the braid in a relatively straightforward manner.
Proposition 3.11 (Unitarity of Braid Elements). Suppose $|k - l| = 1$, then

$$b_{kl} b_{lk} = b_{lk} b_{kl} = 1.$$  

(As was remarked in the definition of the braids, $b^*_{kl} = b_{lk}$, so equivalently, $b_{kl}$ is unitary.)

Proof. Fix $k < l$, so we fix the braid elements. To prove this identity, we rely on propositions 3.2 and 3.3. Thus, we just need to show that a) $b_{kl} b_{lk}$ and $b_{lk} b_{kl}$ lie in the center, and b) the constant terms of $b_{kl} b_{lk}$ and $b_{lk} b_{kl}$ are both 1.

Note that if $p < k < l$ or $p > l > k$, then $c_p$ commutes with $b_{kl}$ since it commutes with $c_k^a c_l^{-a}$ by lemma 3.7. We now note that $c_p b_{kl} = b_{kl} c_p$ implies the adjoint equation $b_{lk} c_p^{-1} = c_p^{-1} b_{lk}$, which further yields $b_{lk} c_p = c_p b_{lk}$ by iterating the commutation relation for $c_p^{-1} N - 1$ times. Thus, $c_p$ commutes with both $b_{kl}$ and $b_{lk}$.

Since $|k - l| = 1$, the only other possibilities for $c_p$ are $p = k$ or $p = l$.

Recall that we have the master braid identity 3.4: $b_{kl} c_k^a c_l^b = q^{a^2 + ab + 2a b - a} b_{kl}$. The corresponding adjoint identity 3.5 is given by $b_{lk} c_k^a c_l^b = q^{-a^2 + ab + 2a b - a} b_{lk}$.

As a result, $b_{kl} b_{lk} c_k = b_{kl} c_k b_{lk} = c_k b_{kl} b_{lk}$, and $b_{lk} b_{kl} c_l = q b_{lk} c_l^{-1} b_{lk} = c_l b_{lk} b_{kl} b_{lk}$ by applying the master braid and adjoint braid identities. Thus, $b_{kl} b_{lk}$ lies in the center. Similarly, $b_{lk} b_{kl} c_l = b_{lk} c_l b_{kl} = c_l b_{lk} b_{kl}$, and $b_{lk} b_{kl} c_k = q b_{lk} c_k^2 c_l^{-1} b_{kl} = c_k b_{lk} b_{kl}$, so $b_{lk} b_{kl}$ lies in the center as well.

We now need to compute the constant terms for $b_{kl} b_{lk}$ and $b_{lk} b_{kl}$. A direct computation shows that $b_{kl} b_{lk}$ has the constant term $\frac{1}{N} \sum_{i=0}^{N-1} (c_i^a c_i^{-a} c_i^b c_i^{-b}) = 1$. Similarly, $b_{lk} b_{kl}$ has the constant term $\frac{1}{N} \sum_{i=0}^{N-1} (c_i^a c_i^{-a} c_i^b c_i^{-b}) = 1$.

Thus, applying proposition 3.2 in the case $x = b_{kl} b_{lk}$ and $y = 1$, we obtain that $b_{kl} b_{lk} = 1$. Again applying proposition 3.2 and setting $x = b_{lk} b_{kl}$ and $y = 1$, we obtain that $b_{lk} b_{kl} = 1$.  

The corresponding graphical identity, for the special case $n = 1$ (only two generators), $b_{21} b_{12} = b_{12} b_{21}$, is

$$\begin{array}{c}
\begin{array}{c}
&\downarrow \quad \uparrow \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.$$  

Analogous graphical identities hold for $b_{k,k+1}$ and for general $n$, where one puts more strands to the left and right of the above diagram.

Furthermore, the above unitarity condition extends to braid elements with no graphical interpretation:

Corollary 3.12.

$$b_{kl} b_{lk} = b_{lk} b_{kl} = 1$$

for all $k \neq l$ in the set $\{1, 2, \ldots, 2n\}$.

Proof. Suppose without loss of generality that $k < l$, and consider the isomorphism of subalgebras $\langle c_1, c_2 \rangle$ and $\langle c_k, c_l \rangle$ given by the linear mapping $\phi$ satisfying $\phi(c_1^a c_2^b) := c_k^a c_l^b$, defining $\phi$ by its action on a basis for the subalgebra $\langle c_1, c_2 \rangle$. This is an isomorphism since $\phi((c_1^a c_2^b)(c_k^a c_l^b)) = \phi(q^{-b_i} c_1^{a_i} c_2^{b+i}) = q^{-b_i} c_k^{a_i} c_l^{b+i} = c_k^{a_i} c_l^{b+i} = \phi(c_1^a c_2^b)\phi(c_k^a c_l^b)$, and the map is
invertible. By double distributivity of multiplication in the two subalgebras, the mapping extends to a homomorphism, and thus is an isomorphism.

The isomorphism maps $b_{12}b_{21}$ to $b_{kl}b_{lk}$ and 1 to 1, so we obtain that $b_{kl}b_{lk} = 1$. Similarly, $b_{ik}b_{kl} = 1$.

The above proof of proposition 3.11 may seem slightly over-kill, since we could have also expanded the product of $b_{kl}$ and $b_{ik}$, and performed the double sum. The strength (and elegance) of the method becomes more apparent when one deals with more complicated products.

We now give one of the main results of this paper, which is an explicit algebraic proof of a Yang-Baxter-like equation, using the golden rule and a systematic application of the master braid and adjoint braid identities.

We first present a special case of the general result:

**Proposition 3.13** (Yang-Baxter-like Equation).

\[ b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23} \]

**Proof.** Since the braid elements are unitary, it suffices to prove the assertion that $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$ lies in the center and that the constant of proportionality between $b_{12}b_{23}b_{12}$ and $b_{23}b_{12}b_{23}$ is 1.

By 3.3, to show that $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$ lies in the center, we just need to show that it commutes with $c_k$ for all $k = 1, 2, \cdots, 2n$. Clearly, for $k > 3$, $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$ commutes with $c_k$, since each braid element commutes with $c_k$. So we want to do case analysis for $k = 1, 2, 3$.

For $k = 1$, $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}c_1 = q b_{32}b_{21}b_{32}b_{12}b_{23}c_1^2 c_2^{-1} b_{12} = q^2 b_{32}b_{21}b_{32}b_{12}c_1^2 c_2^{-2} c_3 b_{23}b_{12} = q^2 b_{32}b_{21}b_{32}b_{12}^2 c_2^{-2} c_3 b_{23} b_{12} b_{12}$ after applying the master braid identity thrice. Applying the adjoint braid identity thrice then yields $q^3 b_{32}b_{21}b_{32}c_1^2 c_2^{-3} c_3 b_{12}b_{23}b_{12} = q b_{32}b_{21}b_{32}c_1^2 c_2^{-1} b_{12}b_{23}b_{12} = b_{32}c_1 b_{21}b_{32}b_{12}b_{23}b_{12} = c_1 b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$, as desired.

The cases $k = 2, 3$ are similarly shown to satisfy $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}c_k = c_k b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$ in like manner. Thus, we conclude that $b_{32}b_{21}b_{32}b_{12}b_{23}b_{12}$ lies in the center.

It remains to show that the constant of proportionality between $b_{12}b_{23}b_{12}$ and $b_{23}b_{12}b_{23}$ is 1. First focus on the constant terms. Since $b_{kl} = \frac{\omega^{1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_i^k c_i^{-i}$, it suffices to compare the constant terms of $\sum_{i,j,k=0}^{N-1} (c_i^1 c_2^{-i})(c_i^2 c_3^{-j})(c_i^k c_2^{-k})$ and $\sum_{i,j,k=0}^{N-1} (c_i^2 c_3^{-i})(c_i^j c_2^{-j})(c_i^k c_3^{-k})$. Note that in the first sum, the constant term only includes terms with $i + k = 0$ and $j = 0$, so the constant is given by $\sum_{i=0}^{N-1} (c_i^1 c_2^{-i})(c_1^i c_2^{-i}) = \sum_{i=0}^{N-1} q^{-i^2}$. In the second sum, the constant term only includes terms with $j = 0$ and $i + k = 0$, so the constant is given by $\sum_{i=0}^{N-1} (c_i^2 c_3^{-i})(c_1^i c_2^{-i}) = \sum_{i=0}^{N-1} q^{-i^2}$. Clearly the constant terms agree. However, this is not sufficient to conclude the constant of proportionality is 1, since the constant term may vanish. In fact, for $N = 2 \mod 4$, it does vanish, while it does not vanish for other $N$. This fact is due to the following formulas corresponding to Gauss’ classical result for quadratic sums, which are tabulated in [2]:

13
\[
\sum_{k=0}^{n-1} \sin \left( \frac{2\pi k^2}{n} \right) = \frac{\sqrt{n}}{2} \left( 1 + \cos(n\pi/2) - \sin(n\pi/2) \right)
\]
\[
\sum_{k=0}^{n-1} \cos \left( \frac{2\pi k^2}{n} \right) = \frac{\sqrt{n}}{2} \left( 1 + \cos(n\pi/2) + \sin(n\pi/2) \right)
\]

Applying these formulas to \( \sum_{i=0}^{N-1} q^{-i^2} = \sum_{k=0}^{N-1} \exp(-2\pi i k^2/N) \) yields that the real part of the sum vanishes if \( 1 + \cos(N\pi/2) + \sin(N\pi/2) \) vanishes, and the imaginary part vanishes if \( 1 + \cos(N\pi/2) - \sin(N\pi/2) \) vanishes. Thus, we require that \( \cos(N\pi/2) = -1 \) and \( \sin(N\pi/2) = 0 \), so \( N\pi/2 = \pi + 2m\pi \) and \( N\pi/2 = l\pi \), i.e. \( N = 2 + 4m \) and \( N = 2l \), i.e. \( N = 2 \pmod{4} \). This shows that the constant term does not vanish unless \( N = 2 \pmod{4} \).

Now focus on the term with \( c_2 c_3^{-1} \). In the first sum, this term is \( \left( \sum_{i=0}^{N-1} q^{i-i^2} \right) c_2 c_3^{-1} \). In the second sum, this term is \( \sum_{i=0}^{N-1} (c_2 c_3^{-i}) (c_2^{1-i} c_3^{-1}) = \left( \sum_{i=0}^{N-1} q^{i-i^2} \right) c_2 c_3^{-1} \), so the two terms are identical. The multiplicative factor \( \sum_{i=0}^{N-1} q^{i-i^2} = q^{1/4} \sum_{k=0}^{N-1} q^{-(k-1/2)^2} \), which equals \( q^{1/4} \sum_{k=0}^{N-1} e^{-2\pi i (2k-1)^2/4N} \), vanishes only for \( N = 0 \pmod{4} \).\(^5\)

Thus, the constant term and the \( c_2 c_3^{-1} \) term agree and their sum can never vanish. Hence, we conclude that the constant of proportionality must be 1, as desired.

The corresponding graphical identity for the Yang-Baxter-like equation \( b_{12} b_{23} b_{12} = b_{23} b_{12} b_{23} \) is given economically for the algebra with 3 generators \( c_1, c_2, c_3 \), as

\[
\begin{array}{c}
\includegraphics{diagram}
\end{array}
= \begin{array}{c}
\includegraphics{diagram}
\end{array}
\]

For \( 2n \) generators, one needs to put \( 2n - 3 \) strands to the right of the diagram for completeness.

Similar to the case of the unitarity condition, a more general Yang-Baxter-like equation holds for braid elements which do not admit a graphical interpretation:

**Corollary 3.14.** Suppose \( i < j < k \), then

\[
b_{ij} b_{jk} b_{ij} = b_{jk} b_{ij} b_{jk}.
\]

**Proof.** Again, we define an isomorphism, this time between the subalgebras \( \langle c_1, c_2, c_3 \rangle \) and \( \langle c_1, c_j, c_k \rangle \). Specifically, define \( \phi \) by its action on a basis for the subalgebra \( \langle c_1, c_2, c_3 \rangle \) via \( \phi(c_1^p c_2^q c_3^r) := c_1^p c_j^q c_k^r \) for all \( p, q, r \in \{0, 1, \ldots, N-1\} \). Clearly, \( \phi(1) = 1 \). Furthermore, \( \phi \) is a homomorphism since \( \phi((c_1^u c_2^v c_3^w)(c_1^p c_2^q c_3^r)) = \alpha \phi(c_1^{u+p} c_2^{v+q} c_3^{w+r}) = \alpha c_1^{u+p} c_j^{v+q} c_k^{w+r} = \)

\(^5\)I have not been able to find the corresponding Gauss sum identity in the literature, but have been able to verify this numerically using Mathematica, which shows that the half-integer-shifted quadratic Gauss sum multiplied by \( 1/\sqrt{N} \) is periodic in \( N \pmod{4} \).
\((c^w_i c_j^r c_k^w)(c^r_j c^w_k)\), where \(\alpha\) collects all the phase factors from commuting the \(c\)'s around. It is clear that \(\phi\) is a one-to-one mapping. Then applying \(\phi\) to the product formula \(b_{32}b_{21}b_{32}b_{12}b_{23}b_{12} = 1\) yields \(b_{kj}b_{ji}b_{kj}b_{ij}b_{kj}b_{ij} = 1\), which implies the desired result by taking the adjoint braids to the other side to become braids.

### 3.3 Vector Identities for the Algebraic Framework

The fact that the Yang-Baxter-like equation holds for the braid elements is quite suggestive. It suggests that perhaps some kind of graphical identities hold for the vector representation we have chosen for the generalized Clifford algebra as well. While this is in fact the case, it turns out that the graphical identities are superseded by more general algebraic identities. In terms of our results, we will show that in some sense, two basic vector identities give rise to a plethora of identifications between different vectors generated from the ground state by braidings.

First, we begin by proving a general projection-braid identity and two basic vector identities which uniformly apply to a multi-qudit space of an arbitrary number of qudits. The second vector identity, which we call the “slip” move, appears to be new. In their full generality, our two vector identities go beyond a graphical representation. We then show by example that these identities can be thought of as representing combinatorial moves that one can perform on braided states without changing the state. We conclude with an example in which we show, rigorously and without any computations, that two entangled vector states can be shown to be equal using these combinatorial moves in combination.

Thus, our main breakthrough in this section is a proof of concept that in many cases, one may reduce the problem of showing equivalence of two different sequences of braidings applied to the ground state, to that of a tractable combinatorial problem, instead of one of explicit algebraic computation.

The essential ingredient for these vector identities is the identity 2.7.

For convenience, we will sometimes work with projections instead of vectors.

In this section, we will start drawing the diagrams first, and then writing out the algebraic expressions, since it is the diagrams which are the motivation.

**Proposition 3.15** (Projection-Braid Identity, or the “Twist” Move).

\[
\begin{align*}
\bigcup & = \omega^{-1/2} \bigcup \\
\text{Equivalently (by scaling the graphical identity by } \delta), \quad & b_{12}E_1 = \omega^{-1/2}E_1 \\
& \text{More generally,} \\
& b_{2k-1,2k}E_k = \omega^{-1/2}E_k
\end{align*}
\]

for \(k = 1, 2, \ldots, n\).
Proof. By definition, \( b_{12}E_1 = \frac{\omega^{1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_i^1 c_i^2 E_1 \). Recall that the axioms for the projectors imply via lemma 2.7 that \( c_i^1 E_1 = \zeta^a c_i^2 E_1 \). So we obtain that the sum equals
\[
\frac{\omega^{1/2}}{\sqrt{N}} \left( \sum_{i=0}^{N-1} \zeta^{-i^2} \right) E_1 = \omega^{1/2} \omega^* E_1 = \omega^{-1/2} E_1.
\]

The general statement \( b_{2k-1,2k}E_k = \omega^{-1/2} E_k \) follows similarly since the same lemma tells us that \( c_{2k-1}^a E_k = \zeta^a c_{2k}^a E_k \).

\[ \square \]

**Proposition 3.16** ("Slide" Move).

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{slide_move_diagram.png}
\end{array}
\]

More generally (i.e. for \( n \) (where \( 2n \) is the number of strands) not necessarily equal to 2),

\[ b_{23}b_{34}b_{12}b_{23} \ket{\Omega}^\otimes n = \ket{\Omega}^\otimes n. \]

Proof. Graphically, it is wisest to expand the braids on the 2nd and 3rd strands, since we may use existing algebraic graphical identities to simplify the result.

This yields
\[
b_{23}b_{34}b_{12}b_{23} \ket{\Omega}^\otimes n = \frac{\omega}{N} \sum_{i,j=0}^{N-1} c_i^i c_j^j b_{34}b_{12}c_i^i c_j^j \ket{\Omega}^\otimes n.
\]

Note that \( b_{12}, b_{34} \) commute by linear extension of lemma 3.7 so the order doesn’t matter.

In terms of a diagram, expanding the middle braids yields

\[
\frac{\omega}{N} \sum_{i,j=0}^{N-1} \left[ \begin{array}{c}
\zeta^{-i^2} \\
\zeta^{-j^2}
\end{array} \right] = \frac{\omega}{N} \sum_{i,j=0}^{N-1} \left[ \begin{array}{c}
\zeta^{-i^2} \\
\zeta^{-j^2}
\end{array} \right] = \frac{\omega}{N} \sum_{i,j=0}^{N-1} \zeta^{-i^2} \zeta^{-j^2}
\]

where we have applied axiom 1 to bring the charge -i over to the 4th strand, yielding the phase factor \( \zeta^{-i^2} \), and then commutted it over the braid back to the 3rd strand. Similarly, the charge i can be brought over the braid. Note that no additional phase accumulates, since overall the relative vertical positions of the charges are unchanged. Now apply the twist move in proposition 3.15 to get the diagram

\[
\frac{1}{N} \sum_{i,j=0}^{N-1} \zeta^{-i^2} \zeta^{-j^2} \left[ \begin{array}{c}
\zeta^{-i^2} \\
\zeta^{-j^2}
\end{array} \right]
\]

Following the logic of the diagram, we can perform the same operations to obtain that
\[
b_{23}b_{34}b_{12}b_{23} \ket{\Omega}^\otimes n = \frac{1}{N} \sum_{i,j=0}^{N-1} c_i^i c_j^j c_i^j c_j^i \ket{\Omega}^\otimes n.
\]

16
By unitarity of the braids, it suffices to show that 
\[ \langle \Omega | \otimes^n b_{23} b_{34} b_{12} b_{23} | \Omega \rangle \otimes^n = 1. \]

Note that the projection onto the ground state yields
\[ \frac{1}{N} \sum_{i,j=0}^{N-1} \zeta^2 \langle \Omega | \otimes^n c_1 c_2 c_3^{-j} c_4^i | \Omega \rangle \otimes^n = \frac{1}{N} \sum_{i,j=0}^{N-1} \zeta^2 \langle \Omega | \otimes^n c_1 c_2 c_3^{-j} | \Omega \rangle \otimes^n \]
by commuting \( c_1 \) past the neutral \( c_2 c_3^{-j} \). By orthonormality of \( c_2 c_4 \langle \Omega \rangle \otimes^n \) states, and equivalently, the orthonormality of \( c_1 c_4 \langle \Omega \rangle \otimes^n \) states, only the terms with \(-i - j = 0\) survive. Thus, the sum reduces to
\[ \frac{1}{N} \sum_{i,j=0}^{N-1} \zeta^2 \langle \Omega | \otimes^n c_1 c_2 c_4^{-j} | \Omega \rangle \otimes^n, \]
and this is simply equal to 1 by lemma 2.7.

Thus, it follows by unitarity of the braids that
\[ b_{23} b_{34} b_{12} b_{23} | \Omega \rangle \otimes^n = | \Omega \rangle \otimes^n. \]

In terms of the diagram, for \( n = 2 \), we have
\[ \begin{array}{ccc}
    \text{ } & \cdot & \cdot \\
    \text{ } & \cdot & \cdot \\
\end{array} \]

= \[ \begin{array}{ccc}
    \text{ } & \cdot & \cdot \\
    \cdot & \cdot & \cdot \\
\end{array} \]

In terms of combinatorial moves, this identity gives us a way to “slide” one cap over the other.

**Corollary 3.17.**
\[ b_{12} b_{23} | \Omega \rangle \otimes^n = b_{43} b_{32} | \Omega \rangle \otimes^n. \]

**Proof.** By taking \( b_{34} \) and \( b_{23} \) to the right hand side in 3.16. □

The above “slide” move generalizes to the general result:

**Proposition 3.18 (General “Slide” Move).**
\[ b_{2k,2l-1} b_{2l-1,2l} b_{2k-1,2k} b_{2k,2l-1} | \Omega \rangle \otimes^n \]
for \( k < l \) in \( \{1, 2, \ldots, n\} \).

**Note that this result does not generally have a graphical interpretation unless \( l = k + 1 \).**

**Proof.** Again, by expansion,
\[ b_{2k,2l-1} b_{2l-1,2l} b_{2k-1,2k} b_{2k,2l-1} | \Omega \rangle \otimes^n = \frac{\omega}{N} \sum_{i,j=0}^{N-1} c_j c_{2l-1} b_{2l-1,2l} b_{2k-1,2k} c_i c_{2k}^{-1} | \Omega \rangle \otimes^n. \]

The same proof as before works in this general case since we can apply the braid intertwining identities and also the twist moves (for braids \( b_{2l-1,2l} \) and \( b_{2k-1,2k} \)), and then apply the axioms to simplify the vacuum expectation value. So we conclude that
\[ b_{2k,2l-1} b_{2l-1,2l} b_{2k-1,2k} b_{2k,2l-1} | \Omega \rangle \otimes^n = | \Omega \rangle \otimes^n. \] □
We would also like to be able to “slip” one cap in and out of another cap.

**Proposition 3.19 (“Slip” Move).**

\[
\begin{array}{c}
\hline
\hline
\end{array}
\]

*More generally, for \( n \) a positive integer not necessarily 1,*

\[
b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^\otimes n = |\Omega\rangle^\otimes n.
\]

**Proof.** As demonstrated in the proof of the “slide” move, this kind of proof doesn’t depend on \( n \), so long as \( n \geq 2 \), so let’s specialize to \( n = 2 \) for convenience. The previous proposition gave a clear handle on how to manipulate the algebraic computations, so we’ll stick with the algebra.

\[
b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^\otimes n = \frac{1}{N} \sum_{i,j=0}^{N-1} c_2^i c_3^{-j} b_{34}b_{21}c_3^i c_2^{-j} |\Omega\rangle^\otimes n.
\]

In terms of a diagram, multiplying the state by \( \delta \) (every cap contributes an extra factor of \( \sqrt{\delta} \)) yields

\[
LHS = \frac{1}{N} \sum_{i,j=0}^{N-1} c_2^i c_3^{-j} b_{34}b_{21}c_3^i c_2^{-j} |\Omega\rangle^\otimes n.
\]

since the factors of \( \zeta^{i^2} \) and \( \zeta^{-j^2} \) cancel.

Undoing the twists yields factors of \( \omega^{1/2} \) and \( \omega^{-1/2} \), respectively, which cancel, so we are left with

\[
LHS = \frac{1}{N} \sum_{i,j=0}^{N-1} c_2^i c_3^{-j} b_{34}b_{21}c_3^i c_2^{-j} |\Omega\rangle^\otimes n.
\]

Converting back to the algebraic form, one has

\[
b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^\otimes n = \frac{1}{N} \sum_{i,j=0}^{N-1} c_2^i c_3^{-j} c_3^i c_2^{-j} |\Omega\rangle^\otimes n.
\]

Note that the \( |00\rangle \) component has norm 1, since setting \( i = j \) yields the \( |00\rangle \) component. Thus, by unitarity of the braid elements, the other basis state projections vanish, so

\[
b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^\otimes n = |\Omega\rangle^\otimes n
\]
as desired.

As with the “slide” move, there is again an algebraic generalization to braid elements with no graphical interpretation:

**Proposition 3.20** (General “Slip” Move).

\[ b_{2k,2l-1}b_{2l-1,2l}b_{2k,2k-1}b_{2l-1,2k} |\Omega\rangle^\otimes n = |\Omega\rangle^\otimes n \]

for \( k < l \) in \( \{1, 2, \ldots, n\} \).

**Proof.** By expansion,

\[ b_{2k,2l-1}b_{2l-1,2l}b_{2k,2k-1}b_{2l-1,2k} |\Omega\rangle^\otimes n = \frac{1}{N} \sum_{i,j=0}^{N-1} c_{2i}^j c_{2l-1}^{-j} b_{2l-1,2l}b_{2k,2k-1}c_{2l-1}^i c_{2k}^{-i} |\Omega\rangle^\otimes n, \]

and the same proof follows through as before.

**Corollary 3.21.**

\[ b_{21}b_{32} |\Omega\rangle^\otimes n = b_{43}b_{32} |\Omega\rangle^\otimes n \]

**Proof.** By taking \( b_{23} \) and \( b_{34} \) to the right hand side in proposition 3.19.

**Proposition 3.22.**

\[ \begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array} \]

i.e.

\[ b_{34}b_{23} |\Omega\rangle^\otimes n = b_{43}b_{32} |\Omega\rangle^\otimes n \]

**Proof.** It suffices to show that \( b_{23}b_{34}b_{34}b_{23} |\Omega\rangle^\otimes n = |\Omega\rangle^\otimes n \), using the fact that \( b_{jk}b_{kj} = 1 \).

Note that this relation does **not** follow immediately from the Yang-Baxter-like equation, since the Yang-Baxter-like equation does not know about the vector structure, or even about the behavior of the ground state.

First recall that proposition 3.16 says that the ground state \( |\Omega\rangle^\otimes n \) is invariant under a “slide” move via

\[ |\Omega\rangle^\otimes n = b_{23}b_{34}b_{12}b_{23} |\Omega\rangle^\otimes n \]

and so we have that

\[ b_{32}b_{43}b_{21}b_{32} |\Omega\rangle^\otimes n = |\Omega\rangle^\otimes n. \]

Thus,

\[ b_{23}b_{34}b_{34}b_{23} |\Omega\rangle^\otimes n = b_{23}b_{34}b_{34}b_{32}b_{43}b_{21}b_{32} |\Omega\rangle^\otimes n = b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^\otimes n \]

which equals \( |\Omega\rangle^\otimes n \) by proposition 3.19, as desired.
Now we prove something quite nontrivial using the above braiding relations in combination.

**Proposition 3.23.**

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{prop323.png}
\end{array}
\]

i.e.

\[ b_{56}b_{45}b_{34}b_{23} |\Omega\rangle^{\otimes n} = b_{65}b_{54}b_{43}b_{32} |\Omega\rangle^{\otimes n} \]

**Proof.** Equivalently, we will show that

\[ b_{23}b_{34}b_{45}b_{56}b_{56}b_{45}b_{34}b_{23} |\Omega\rangle^{\otimes n} = |\Omega\rangle^{\otimes n} . \]

We first substitute \( b_{32}b_{43}b_{21}b_{32} |\Omega\rangle^{\otimes n} \) for \( |\Omega\rangle^{\otimes n} \) following 3.16. This kills off the \( b_{34} \) and \( b_{23} \) braids and we are left with

\[ b_{23}b_{34}b_{45}b_{56}b_{56}b_{45}b_{21}b_{32} |\Omega\rangle^{\otimes n} . \]

Now we commute the braids which do not overlap so we get

\[ b_{23}b_{34}b_{21}b_{32}b_{45}b_{56}b_{56}b_{45} |\Omega\rangle^{\otimes n} . \]

We now substitute \( b_{34}b_{43}b_{54} |\Omega\rangle^{\otimes n} \) for \( |\Omega\rangle^{\otimes n} \) to get

\[ b_{23}b_{34}b_{21}b_{32}b_{45}b_{56}b_{43}b_{54} |\Omega\rangle^{\otimes n} \]

upon braid and adjoint braid cancellation. Now we apply the slip move in reverse to get

\[ b_{23}b_{34}b_{21}b_{32} |\Omega\rangle^{\otimes n} \]

and then apply the slip move in reverse again to get \( |\Omega\rangle^{\otimes n} \), as desired.

\[ \square \]

### 4 Explicit Computation of Some Entangled Vector States

This section is devoted to explicit algebraic computations of some entangled vector states, to demonstrate some of the variety of entangled states that can arise by braid element actions.

Whereas the previous section was devoted to proof methods for showing that two vector states are equal, it did not resolve the question of what those states were, which is clearly a more complicated matter, from the computational standpoint. In proving vector identities, we were able to cleverly chain together two basic moves, the “slide” and “slip” moves, which enable one to maneuver neighboring caps over and under, as well as in and out of each other. Clearly, different methods are needed for explicit computation of the states.
In this section, we develop computational techniques which enable one to reduce vector state computation in various cases to the evaluation of a single explicit inner product, i.e. a single vacuum expectation value. Thus, the novelty here, compared with [4], for example, which also studies state computations, is that we show that state computation of entangled states emerges from the application of braid elements to the ground state, which is neutral.

Proposition 4.1. The sum in normal order, we put each term into “pairwise” normal order, so \[ \sum N \] to the axiom \[ \zeta \] to the strategy one should employ to reduce the state computation to the evaluation of a single explicit inner product, i.e. the novelty here, compared with [4], is that some algebraic structure may be expected to emerge from the application of braid elements to the ground state, which is neutral.

For example, we have the following identity:

Proposition 4.3. 

Proof. By direct expansion, \[ b_{34} b_{23} |\Omega\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^2 c_3^{-1} c_{34}^{-i} |\Omega\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} q^{-i} c_3^{-i} c_{34}^{-i} |\Omega\rangle^{\otimes n} \]

Remark 4.2. Note that if we restrict to the case of the 2-qudit ground state, then up to phase redefinition of the basis, the resulting state is of the form \[ \sum_{j=0}^{N-1} \zeta^{-(i+j)^2} q^2 \] (as noted in [4]). More generally, we have (up to phase redefinitions) \[ \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i, -i, 0, 0, \ldots, 0\rangle \]

There is actually an easier way to get this state algebraically, using \[ b_{42} \], one of the nonlocal braids we defined.

Proposition 4.3.

Proof. Since \[ b_{42} = \frac{\omega^{-1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} c_4^{-i} c_2 \] \[ = \frac{\omega^{-1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} q^{-i} c_2 c_4^{-i} \] if we apply it to \[ |\Omega\rangle^{\otimes n} \] we get \[ \frac{\omega^{-1/2}}{\sqrt{N}} \sum_{i=0}^{N-1} q^{-i} \zeta^{-i} c_2 c_4^{-i} |\Omega\rangle^{\otimes n} \] by bringing the charge \(i\) from the fourth strand over to the third strand using the property of the ground state. Thus, \[ b_{42} |\Omega\rangle^{\otimes n} = \omega^{-1/2} b_{34} b_{23} |\Omega\rangle^{\otimes n} \]
We can also get rid of the extra constant factor by the following corollary:

**Corollary 4.4.**

\[ b_{42} |\Omega\rangle^\otimes n = b_{34} b_{23} |\Omega\rangle^\otimes n \]

**Proof.** It follows from \( b_{34} |\Omega\rangle^\otimes n = \omega^{-1/2} |\Omega\rangle^\otimes n \) by proposition 3.15.

A more “entangled” state is given by

**Proposition 4.5.**

\[ b_{56} b_{45} b_{34} b_{23} |\Omega\rangle^\otimes n = \frac{1}{N} \sum_{j,l=0}^{N-1} q^{-jl} q^{j^2 + j^2} c_2^l c_4^{-j} c_6^{-j} |\Omega\rangle^\otimes n \]

**Proof.** We give a direct computation analogous to that of proposition 4.1. Expanding all of the braids yields

\[ \frac{\omega^2}{N^2} \sum_{j,l=0}^{N-1} q^{-jl} c_2^l \left( \sum_k q^{-k^2} \zeta^{(k-l)^2} \right) c_4^j \left( \sum_i q^{-i^2} \zeta^{(i-j)^2} \right) c_6^{-j} |\Omega\rangle^\otimes n \]

which yields

\[ \frac{\omega^2}{N^2} \sum_{j,l=0}^{N-1} q^{-jl} c_2^l \left( \sqrt{N} \omega^{-1} q^{j^2} \right) c_4^j \left( \sqrt{N} \omega^{-1} q^{j^2} \right) c_6^{-j} |\Omega\rangle^\otimes n \]

which is just

\[ \frac{1}{N} \sum_{j,l=0}^{N-1} q^{-jl} q^{j^2 + j^2} c_2^l c_4^{-j} c_6^{-j} |\Omega\rangle^\otimes n \]

as desired.

As a simple example, suppose we take \( N = 3 \), so there are nine terms on the right-hand-side, yielding

\[ b_{56} b_{45} b_{34} b_{23} |\Omega\rangle^\otimes n = \frac{1}{3} \sum_{j=0}^{2} \left( q^{j^2} c_4^j c_6^{-j} + q^{-j} q^{1+j^2} c_2 c_4^{-j} c_6^{-j} + q^{-2j} q^{4+j^2} c_2 c_4^{-j} c_6^{-j} \right) |\Omega\rangle^\otimes n \]

Interestingly, we can write the coefficient term as \( \zeta a_1^2 + a_2^2 + a_3^2 \) where the sum is given by \( \frac{1}{N} \sum_{a_1+a_2+a_3=0 \mod N} \zeta a_1^2 + a_2^2 + a_3^2 |\Omega\rangle^\otimes n \). We thus conjecture that the general case is given by

\[ b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34} b_{23} |\Omega\rangle^\otimes n = \frac{1}{N^{(k-1)/2}} \sum_{\sum_{i=1}^{k} a_i = 0} \zeta^{\sum_{i=1}^{k} a_i^2} c_2^a c_4^a \cdots c_2^k |\Omega\rangle^\otimes n \]
Clearly, the case $k = 2$ and $k = 3$ hold.

It turns out that this is indeed the case in general:

**Proposition 4.6.** Suppose $k \leq n$. Then

$$b_{2k-1,2k}b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n} = \frac{1}{N^{(k-1)/2}} \sum_{i=0}^{\ell} \zeta^{\sum_{i=1}^{k} a_i^2} c_2^{a_2} \cdots c_{2k}^{a_{2k}} |\Omega\rangle^{\otimes n}.$$ 

Equivalently,

$$b_{2k-1,2k}b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n} = \frac{1}{N^{(k-1)/2}} \sum_{i=0}^{\ell} c_1^{a_1} c_3^{a_3^2} \cdots c_{2k-1}^{a_{2k-1}} |\Omega\rangle^{\otimes n}.$$ 

**Proof.** By unitarity of the braid element, it suffices to show that

$$\langle \Omega |^{\otimes n} c_{2k-1}^{a_{2k-1}} c_{2k-3}^{a_{2k-3}} \cdots c_3^{a_3} c_1^{a_1} b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n} = \frac{1}{N^{(k-1)/2}}$$

whenever $\sum_{i=1}^{\ell} a_i = 0$. The norm of the sum over these states is already 1, so this would imply that there cannot be components in addition to these neutral states.

First, observe that we can change the $c_1^{a_1}$ to $\zeta^{-a_1^2} c_2^{a_2}$ by commuting past the other $c_i$’s to act on the bra vector and then commuting back to its original position. Then we can commute $c_2^{a_2}$ past the braids until we get $c_2^{a_2} b_{23} |\Omega\rangle^{\otimes n}$, which is just $b_{23} c_2^{a_2} |\Omega\rangle^{\otimes n} = b_{23} \zeta^{a_2^2} c_4^{a_4} |\Omega\rangle^{\otimes n}$. This phase factor cancels the previous $\zeta^{-a_1^2}$ so we are left with the $b_{34}b_{23} c_4^{a_4} |\Omega\rangle^{\otimes n}$, acted on by a product of $c_i$’s and braids. We can then move $c_4^{a_4}$ past $b_{23}$ and then apply $b_{34} c_4^{a_4} = c_4^{a_4} b_{34}$. After commuting this $c_3$ past the other braids we finally get

$$\langle \Omega |^{\otimes n} c_{2k-1}^{a_{2k-1}} c_{2k-3}^{a_{2k-3}} \cdots c_3^{a_3} c_1^{a_1} b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n}.$$ 

Applying this same procedure iteratively, the end result is

$$\langle \Omega |^{\otimes n} c_{2k-1}^{a_{2k-1}+a_{k-1}+ \cdots + a_1} b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n}.$$ 

By assumption $a_k + a_{k-1} + \cdots + a_1 = 0$, so we just need to compute

$$\langle \Omega |^{\otimes n} b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n}.$$ 

Since $b_{l,l+1} = \frac{\omega^{l/2}}{\sqrt{N}} \sum_{m=0}^{N-1} c_l^m c_{l+1}^m$, the only terms that contribute to the projection onto the ground state are from the constant component of $b_{23}$, and similarly, the constant component of $b_{45}$, $b_{67}$, etc. So we are left to evaluate

$$\frac{\omega^{(k-1)/2}}{N^{(k-1)/2}} \langle \Omega |^{\otimes n} b_{2k-1,2k} b_{2k-2,2k-1} \cdots b_{34}b_{23} |\Omega\rangle^{\otimes n}.$$ 

Applying the twist move $k-1$ times to get rid of the braids yields $\omega^{-(k-1)/2}$, so this expression evaluates to $\frac{1}{N^{(k-1)/2}}$, as desired.

---

6This series of manipulations is motivated by drawing the diagram for this vacuum expectation value, and trying to transfer the charge on the first strand over to the third strand.

7This fact is justified by the axiom that the $c_2^{a_1} c_4^{a_2} \cdots c_{2n}^{a_{2n}} |\Omega\rangle^{\otimes n}$ form a basis. Drawing the diagram for the expanded braid sums makes the deduction apparent.
Remark 4.7. As seen in numerous computations for vector states, the key is to latch onto a symmetry (which may be more readily deduced from the diagram) of the vector state under the action of a neutral product of generators $c_{2k-1}$ (which act on the vacuum state to form a basis; it is important that we project onto a basis). For a complete set of such symmetries (i.e. enough so that the square norm of the sum of projections onto the corresponding states is 1), the computation of a normalized vector state reduces to the computation of the projection onto a single vector state. Thus, in the end, only one explicit computation (expanding braid elements) must be performed.

5 Conclusion

In this paper, we showed that the algebraic framework we developed in [6] allows us to construct a purely definitional graphical calculus for multi-qudit computations with the generalized Clifford algebra. We established many “superpower”-like properties of the graphical calculus using purely algebraic methods, including a novel algebraic proof of a Yang-Baxter-like equation. We also derived several new identities for the braid elements, which are key to our proofs. We demonstrated that in many cases, the verification of involved vector identities can be reduced to the combinatorial application of two basic vector identities. Finally, we showed how to explicitly compute various vector states in an efficient manner using algebraic methods.

In addition, in terms of physics, we connected these braid identities to physics by showing the presence of a conserved charge.

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