Physical Properties of Metals from a Renormalization Group Standpoint

Chetan Nayak*

Department of Physics
Joseph Henry Laboratories
Princeton University
Princeton, N.J. 08544

Frank Wilczek†

School of Natural Sciences
Institute for Advanced Study
Olden Lane
Princeton, N.J. 08540

* Research supported in part by a Fannie and John Hertz Foundation fellowship. nayak@puhep1.princeton.edu
† Research supported in part by DOE grant DE-FG02-90ER40542. WILCZEK@IASSNS.BITNET
ABSTRACT

We derive the equilibrium and transport properties of metals using renormalization group equations and finite-size scaling. Particular attention is given to the well-known cases of Fermi and Luttinger liquids. An important subtlety is that the temperature dependence of many transport coefficients is determined by “dangerous” irrelevant operators. We also characterize violations of Fermi or Luttinger liquid behavior in terms of indices, analogous to the critical indices describing phase transitions. We briefly consider the normal-state properties of the cuprates from this standpoint.
1. Introduction

In recent years, Landau’s Fermi liquid theory has been reformulated as the theory of marginal perturbations of the free Fermi gas fixed point in $d > 1$ [1, 2, 3]. Since these perturbations act in restricted regions of phase space, and all other perturbations are irrelevant in the renormalization group sense, one obtains a controlled framework for computing many low-temperature and low-energy properties. In $d = 1$, Fermi liquid theory breaks down because the kinematics is modified, but the special kinematics of one dimension nevertheless allows for solubility. Such a system is called a (one-dimensional) Luttinger liquid [4, 5, 6].

The renormalization group has made possible a unified approach to gapless quantum liquids which are perturbations of the free fermion system. However, the calculation of physical observables – obtained either from the partition function or from correlation functions of two-fermion composite operators – has not been discussed using the full power of this approach. This problem is not so trivial as it seems, even in the case of Fermi liquids. While it is true that equilibrium and some transport properties can be directly computed at the Fermi liquid fixed point, many transport properties are singular at the fixed point and are determined by irrelevant operators. The canonical example is the conductivity, which is given, according to the Kubo formula, by the current-current correlation function. The naive scaling form – which holds for Fermi as well as non-Fermi liquids, as guaranteed by the Ward identity – leads to a conductivity $\sigma \sim \frac{1}{T}$. However, in the case of Fermi and Luttinger liquids, the coefficient is a delta function in frequency, $\sigma(\omega) \sim \delta(\omega)$, as we discuss later. It is irrelevant operators which make the dc conductivity finite. It is not that these operators lead to anomalous dimensions for the current operator. It is simply that terms which are formally corrections to scaling behavior become dominant because the leading term has a vanishing coefficient in Fermi liquid theory. This scenario is familiar from the theory of critical phenomena where the

* Other examples, which may be brought under the same aegis, include the over-screened Kondo model [8, 9] and the non-Fermi liquid models of [10,11,12,13,14,15] which may be relevant to the $\nu = 1/2$ quantum Hall state and the copper-oxides.
hyperscaling relation, $\alpha = 2 - d
u$, is violated in $d > 4$ because the four-point function is not given by the leading scaling term, which has a vanishing coefficient, but by “corrections” to scaling. By a similar mechanism, the conductivity acquires, instead, the familiar temperature dependence $\sigma \sim \frac{1}{T^2}$.

Using renormalization group equations and finite-size scaling, we will find the naive scaling forms as a function of temperature for a number of physical observables of Fermi and Luttinger liquids. These are determined by thermodynamic relations or Kubo formulas. In the case of these two fixed points, direct calculation will show which of these scaling forms have singular coefficients. Physical arguments – which plausibly are more general – will be given as well. One motivation for this work is to provide a context in which the puzzling normal state behavior of the cuprates may be characterized. In particular, we will try to identify which behaviors are incompatible with Fermi liquid theory and what properties a theory must have in order to exhibit them.

2. Basic Framework

The basic renormalization group analysis of Fermi and non-Fermi liquids [1,2,3,10] was concerned with the identification of the correct fixed point. The analysis was, furthermore, restricted to zero temperature. However, many experiments measure the temperature dependence of equilibrium properties and transport coefficients, which are determined from the derivatives of the free energy or from correlation functions of two-fermion composite operators. To extract these physical properties from a given fixed point (which may be Fermi or non-Fermi liquid), we need the scaling forms as a function of temperature of the free energy and composite operator correlation functions. The introduction of finite temperature may be handled in the path integral framework by restricting all time integrals to the finite interval $[0, \beta]$ and requiring all bosonic (fermionic) fields to be periodic (antiperiodic) over this interval. Thus, the inverse temperature, $\beta$, is simply a finite size in the (imaginary) time direction, and the desired scaling forms may be
obtained from the theory of finite-size effects. The fact that response functions are correlation functions of composite operators implies that we cannot focus merely on the anomalous dimensions of the fields in the action, but must also account for the renormalization of composite operators. Since Ward identities often protect composite operators from renormalization, it is possible for the response functions to have no anomalous dimensions even when the fermion fields do.

**RG equations and Finite-Size Effects.** Since renormalization group equations describe the evolution of effective Lagrangians as one integrates out short-distance physics, it is clear that these equations are insensitive to finite-size effects. While the equations themselves are unchanged, the solutions are modified because they depend on an additional dimensionful parameter, namely the size of the system (in our case, $\beta$).

For simplicity, let us consider a Euclidean quantum field theory (say, $\phi^4$ theory) with a single relevant coupling, which satisfies a renormalization group equation with a low-energy fixed point:

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{n}{2} \eta(g) \right) G^{(n)}(p_i, g, \mu, L) = 0$$  \hspace{1cm} (2.1)

$G^{(n)}$ is an $n$-point Green function, $L$ is the finite size of the system and $\mu$ is the renormalization scale. We may rescale by $L$, and we find, by dimensional analysis,

$$G^{(n)}(p_i, g, \mu, L) = L^{\delta} G^{(n)}(p_i L, g, \mu L, 1)$$  \hspace{1cm} (2.2)

Here, $\delta$ is the naive, or engineering, dimension of the Green function. We can also use the RG equation to change the renormalization point, $\mu \rightarrow \lambda \mu$, and obtain

$$G^{(n)}(p_i, g, \mu, L) = L^{\delta} G^{(n)}(p_i L, g, \mu L, 1) = L^{\delta} \lambda^{\frac{\eta}{g}} G^{(n)}(p_i L, g(\lambda), \lambda \mu L, 1)$$  \hspace{1cm} (2.3)

Suppose we choose $\lambda = (\mu L)^{-1}$. Then in the large-size limit, $L \rightarrow \infty$, we have
\( \lambda \to 0 \) and \( g(\lambda) \to g^* \). As a result, we have the scaling form:

\[
G^{(n)}(p_i, g, \mu, L) \sim L^{\delta + \frac{n}{2} \eta} G^{(n)}(p_i L, g^*, 1, 1) \tag{2.4}
\]

In the case of metals, the finite size, \( L \), will be the inverse temperature, \( \beta \), so (2.4) will give the temperature dependence of Green functions in the low-temperature limit.

One possible subtlety is that the theory may have an additional dimensionful parameter, \( L' \) which does not run (such as the Fermi wavevector, \( k_F \), in the case of metals). Then the rescaling (2.2) is modified, and the scaling law (2.4) becomes:

\[
G^{(n)}(p_i, g, \mu, L, L') \sim L^{\delta + \frac{n}{2} \eta} G^{(n)}(p_i L, g^*, 1, 1, \frac{L'}{L}) \sim \frac{L^{\delta - \delta' + \frac{n}{2} \eta}}{(L')^{\delta'}} G^{(n)}(p_i L, g^*, 1, 1, 1) \tag{2.5}
\]

The second equality assumes power-law dependence on \( L' \); in general, the dependence on \( L/L' \) could be quite complicated. In the case of metals, the power-law dependence on \( k_F \) (i.e. the exponent \( \delta' \)) can be obtained from simple physical arguments.

**Dangerous Irrelevant Operators and Corrections to Scaling.** The above scaling form is exact at the fixed point, where \( g = g^* \) and all irrelevant couplings are set to zero. However, when any bare coupling (relevant or not) is not equal to its fixed point value, there are subleading corrections to scaling behavior resulting from the flow of the coupling to its fixed point value. These corrections may be expressed as a power series in the irrelevant coupling. For instance, for \( g \neq g^* \), the corrections to scaling may be expressed as a power series in \( g - g^* \):

\[
G^{(n)}(p_i, g, \mu, L) \sim L^{\delta + \frac{n}{2} \eta} C^{(n)}(p_i L, g^*) \left( c_0 + \sum_k c_k (g - g^*)^{k \omega} L^{-k \omega} \right) \tag{2.6}
\]

where \( \omega \) is the scaling dimension of the irrelevant coupling \( g - g^* \) obtained by
linearizing the $\beta$-function about its zero:

$$\mu \frac{\partial}{\partial \mu} (g - g^*) = \omega (g - g^*) \quad (2.7)$$

It is possible, however, that $c_0 = 0$, i.e. the leading scaling term may come with a vanishing coefficient. In this case, a “correction” to scaling behavior may give the true scaling form.

$$G^{(n)}(p_i, g, \mu, L) \sim c_k (g - g^*)^k L^\delta + \frac{n}{2} \omega L^{\delta - k} C^{(n)}(p_i L, g^*) \quad (2.8)$$

where $c_k$ is the first non-vanishing $c_i$.

3. Fermi and Luttinger Liquids

Since Fermi and Luttinger liquids have have been extensively discussed elsewhere, we give here only a highly condensed review sufficient to obtain the desired scaling forms.

Fermi Liquids. Following [1, 2, 3], we consider the free action:

$$S_0 = \int d\omega dl k_{F}^{d-1} d\Omega \left\{ \psi^\dagger (i\omega - v_F l) \psi \right\} \quad (3.1)$$

where $l$ is the distance to the Fermi surface in momentum space and $\Omega$ represents the angular variables. There is no cutoff in frequency, but there is a cutoff, $|l| < \Lambda$ in momentum space. If we integrate out all momenta $s\Lambda < |l| < \Lambda$ and rescale $\omega \to s\omega$, $l \to sl$, and $\psi \to s^{-\frac{d}{2}} \psi$, then the free action is left invariant.

Let us consider the scaling of four-Fermi interactions under this transformation. Consider the term

$$S_4 = \int d\omega_1 d\omega_2 d\omega_3 d^d k_1 d^d k_2 d^d k_3 u(k_1, k_2, k_3) \psi^\dagger (k_4, \omega_4) \psi^\dagger (k_3, \omega_3) \psi (k_2, \omega_2) \psi (k_1, \omega_1) \quad (3.2)$$

Here, $k_4 = k_1 + k_2 - k_3$ for a momentum conserving process and $k_4 = k_1 + k_2 - k_3 - g$ for umklapp processes corresponding to some reciprocal lattice vector,
g. For $\Lambda \ll k_F$, $u(k_1, k_2, k_3) = 0$ for generic $k_1, k_2, k_3$ because $k_4$ typically does not lie within the cutoff. In the $\Lambda \to 0$ limit, only forward scattering, $u(k_1, k_2, k_1)$, exchange scattering, $u(k_1, k_2, k_2)$, and the Cooper scattering, $u(k, -k, p)$, survive. At small but non-zero $\Lambda$, a small subset of the $u$’s are non-zero. As $\Lambda$ is decreased, some of these are set discontinuously to zero; the rest do not scale. As $\Lambda$ becomes smaller, fewer non-zero $u$’s remain until, finally, at $\Lambda = 0$, only the three mentioned above remain.

It is not difficult to show that these interactions do not contribute any anomalous dimensions to the fermion field operator or any two-fermion composite operators. In other words, the naive scalings, $\psi \to s^{-\frac{2}{3}}\psi$ and $\rho \to s^{-1}\rho$, $j \to s^{-1}j$, $S \to s^{-1}S$, are unchanged, where the density, current density, and spin density are:

$$\rho = \int dl \, k_F^{d-1} d\Omega \, \psi^\dagger(k + q)\psi(q)$$

$$j = \int dl \, k_F^{d-1} d\Omega \, \psi^\dagger(k + q)\psi(q) \left( \frac{\partial}{\partial q} \epsilon(q + 2k) \right)$$

$$S = \int dl \, k_F^{d-1} d\Omega \, \psi^\dagger(k + q) \sigma \psi(q)$$

Thus, the following scaling forms hold:

$$\langle \psi^\dagger(sk_1, s\omega_1) \ldots \psi(-sk_2n, -s\omega_2n) \rangle = s^{n-2} \langle \psi^\dagger(k_1, \omega_1) \ldots \psi(k_{2n}, \omega_{2n}) \rangle$$

$$\langle \rho(sq_1, s\omega_1) \ldots \rho(sq_n, s\omega_n) \rangle = s^{n-2} \langle \rho(k_1, \omega_1) \ldots \rho(k_n, \omega_n) \rangle$$

The scaling forms for current-current and spin-spin correlation functions are identical to (3.7).

The scaling of generic four-fermi interactions is quite awkward for calculations at a finite frequency or temperature scale because the $u$’s don’t scale continuously. Thus, the scaling of a physical quantity which depends on the $u$’s is determined
not by the scaling of the \( u \)'s, which is marginal, but on the \textit{number of non-zero} \( u \)'s, which is scale dependent (except in the important case where the quantity is determined by forward, exchange, or Cooper scattering – which do scale continuously). For such calculations, a different scaling transformation is useful. For processes in the neighborhood of a single point on the Fermi surface, we can also use the scaling \( k_y \to sk_y, \ k_x \to s^{1/2}k_x, \ \omega \to s\omega \), where \( k_y \) and \( k_x \) are local coordinates perpendicular and tangent to the Fermi surface. (We have assumed \( d = 2 \); in \( d > 2 \), there are \( d - 1 \) momenta which scale as \( k_x \).) This scaling was crucial in the study of fermions interacting with gauge fields \([10,11,12,13]\), where it was necessitated by the singular nature of the interaction. Here, it is more of luxury. The same answers are obtained with either scaling transformation; it’s just that some calculations are easier with this one. On the other hand, it’s a less natural renormalization group transformation because it involves selecting a preferred point on the Fermi surface.

Let’s briefly see how this works. The quadratic part of the Lagrangian is of the form:

\[
S_0 = \int d\omega dk_y dk_x \left\{ \psi^\dagger \left( i\omega - v_F l \right) \psi \right\} \tag{3.8}
\]

Hence, the field now scales as \( \psi \to s^{-7/4} \psi \), so four-fermi interactions,

\[
S_4 = \int d\omega_1 d\omega_2 d\omega_3 d^2k_1 d^2k_2 d^2k_3 u(k_1, k_2, k_3) \psi^\dagger(k_4, \omega_4) \psi^\dagger(k_3, \omega_3) \psi(k_2, \omega_2) \psi(k_1, \omega_1) \tag{3.9}
\]

scale as \( s^{1/2} \). The scaling is perfectly continuous. If \( k_1, k_2, k_3, k_4 = k_1 + k_2 - k_3 \) lie within the cutoff \( \Lambda \), then they continue to do so under this renormalization group transformation. If we insert a \( \delta(k_{1x} - k_{3x}) \) or \( \delta(k_{1x} - k_{4x}) \) into the integrand, then we get a marginal interaction, namely forward scattering, as before.\(^\star\) To see why this is a useful scaling, consider the diagram in figure 1. It has a real part, proportional to \( F_f^2 \omega \) which comes from the marginal forward scattering interaction, and

\[^\star\] In \( d > 2 \), a four-fermi interaction generically scales as \( s^{(d-1)/2} \). A \((d - 1)\)-dimensional delta function which restricts to forward scattering gives a marginal operator. A \((d - 2)\)-dimensional delta function which restricts to a plane gives an operator which scales as \( s^{1/2} \), as in \( d = 2 \).
an imaginary part, proportional to \( F_{n_f}^2 \omega^2 \) coming from irrelevant non-forward processes. The above scaling immediately yields the suppression of \( F_{n_f}^2 \) with respect to \( F_f^2 \) by one power of \( \omega \), while this result is more cumbersome to derive with the other RG transformation.

One final property of Fermi liquids which we will need is a scaling form for the free energy. This may be obtained by dimensional analysis. The free energy density has dimensions of \((\text{Energy})(\text{Length})^{-d}\) or, modulo a factor of \( v_F \), \((\text{momentum})^{d+1}\). For the free Bose gas the inverse temperature, \( \beta \), is the only dimensionful parameter in the problem. Here, however, there is also \( k_F \) so a little more care is required. We see, by inspection, that the free energy density is proportional to \( k_F^{d-1} \) since there is a Fermi surface; alternatively, we see that the energy density scales as \( s^2 \) under a scaling \( \omega \to s\omega, l \to sl \):

\[
f = f_0 + \int dl k_F^{d-1} d\Omega \epsilon(k) \psi^\dagger \psi + \text{subleading terms} \tag{3.10}
\]

Hence, we have the scaling form for the free energy:

\[
f = f_0 + \frac{k_F^{d-1} Q(.)}{\beta^2} \tag{3.11}
\]

\( Q(.) \) is a function of all of the couplings, but these may be set harmlessly to their fixed point values (ie. zero) since none are dangerous. We will soon need a more general form of the free energy scaling relation which includes the dependence of \( Q \) on the chemical potential and magnetic field. Since the terms in the effective action:

\[
S_\mu = \int d\omega dl k_F^{d-1} d\Omega \mu \psi^\dagger \psi \tag{3.12}
\]

\[
S_H = \int d\omega dl_1 dl_2 k_F^{2d-2} d\Omega_1 d\Omega_1 H \epsilon_{ij} \frac{\partial}{\partial k_{1i}} \psi^\dagger(k_1) \psi(k_2) \left( \frac{\partial}{\partial k_{1j}} \epsilon(k_1 + k_2) \right) \tag{3.13}
\]

both scale as \( s^{-1} \), these enter in the dimensionless combinations \( \delta \mu \beta \) and \( H \beta \):

\[
f = f_0 + \frac{k_F^{d-1}}{\beta^2} Q(\delta \mu \beta, H \beta, \ldots) \tag{3.14}
\]
Luttinger Liquids. We now review the corresponding results of Luttinger liquid theory which will be used below [4, 5, 6, 7]. The action may be written in the form,

\[ S = S_0 + S_{\text{spin–charge separation}} + S_{\text{L–R coupling}} \]  (3.15)

where

\[ S_0 = \int d\omega dk \psi_{iL}^\dagger (\omega - v_F k) \psi_{iL} + \int d\omega dk \psi_{iR}^\dagger (\omega + v_F k) \psi_{iR} \]  (3.16)

\[ S_{s–c} = \int d\omega_1 d\omega_2 d\omega_3 dk_1 dk_2 dk_3 \left( (v_F - v_c^0) \psi_{iL}^\dagger (k_1, \omega_1) \psi_{iL}^\dagger (k_2, \omega_2) \psi_{iL}^\dagger (k_3, \omega_3) \psi_{iL} (k_4, \omega_4) \right. \\
+ \left. (v_F - v_s^0) \psi_{iL}^\dagger (k_1, \omega_1) \sigma_{ij}^\alpha \psi_{jL} (k_2, \omega_2) \psi_{iL}^\dagger (k_3, \omega_3) \sigma_{kl}^\alpha \psi_{jL} (k_4, \omega_4) \right) + L \to R \]  (3.17)

\[ S_{\text{L–R}} = \int d\omega_1 d\omega_2 d\omega_3 dk_1 dk_2 dk_3 g_c \psi_{iL}^\dagger (k_1, \omega_1) \psi_{iL} (k_2, \omega_2) \psi_{jR}^\dagger (k_3, \omega_3) \psi_{jR} (k_4, \omega_4) \]  (3.18)

\( i \) is the spin index, \( i = \pm \), and \( k_4 = k_1 + k_3 - k_2 \). The analogous left-right coupling of spins is marginally irrelevant. It can introduce logarithmic corrections, at best, so we set it to zero. The action (3.15) - (3.18) leads to a Hamiltonian which may be written in terms of the currents

\[ H = v_c^0 (: j_{LJ} : + : j_{JRJ} : ) + v_s^0 (: j_{LJ}^\alpha : + : j_{JRJ}^\alpha : ) \]  (3.19)

\[ + g_c (: j_{LJ} : + : j_{JRJ} : ) \]

This Hamiltonian can be written in the Sugawara form:

\[ H = v_c (: J_{LJ} : + : J_{JRJ} : ) + v_s (: J_{LJ}^\alpha : + : J_{JRJ}^\alpha : ) \]  (3.20)

where

\[ J_{L,R} = j_{L,R} \cosh \alpha + j_{R,L} \sinh \alpha \]  (3.21)

and \( \tanh 2\alpha = \frac{g_c}{2v_c^0} \) and \( v_c = \frac{v_c^0}{\cosh 2\alpha} \). Hyperbolic functions must be used in the redefinition above so that the \( J \)'s form an \( SU(2) \times U(1) \) Kac-Moody algebra just
as the $j$'s do. An abelian bosonic representation in terms of free scalar fields $\chi, \sigma$
exists for the current algebra of $J_L, J_R, J^\alpha_L, J^\alpha_R$:

\[ J_L = i\partial_z \chi_L \]  
\[ J_L^\pm = e^{\pm i\sqrt{2}\sigma_L} : \]  
\[ J_L^3 = i\partial_z \sigma_L \]  

(3.22)  
(3.23)  
(3.24)

The right handed currents are completely analogous. The fermion fields then have
the bosonic representation:

\[ \psi_{jL,R} = : e^{\frac{i}{\sqrt{2}}(\chi_{L,R}\cosh\alpha - \chi_{R,L}\sinh\alpha)} \ e^{j \frac{i}{\sqrt{2}}(\sigma_{L,R})} : \]  

(3.25)

The anomalous dimensions of the fermion fields under the scaling $k \to sk$, 
$\omega \to s\omega$ may then be computed, $[\psi] = -\frac{3}{2} - \frac{1}{4}(1 - \cosh 2\alpha)$. However, the currents
$j_L, j_R, J^\alpha_L, J^\alpha_R$ do not receive any anomalous dimensions but have scaling dimension
-1 (or 1, in real space rather than momentum space) as a current in conformal field theory must. Alternatively, the non-renormalization of the currents follows from
the Ward identities. Similarly, the energy-momentum tensor is not renormalized, but has dimension 0 (2 in real space). All scaling forms of conserved currents are
precisely the same as for a Fermi liquid.
4. Physical Properties of Fermi and Luttinger Liquids

Armed with scaling forms for the free energy and correlation functions of Fermi and Luttinger liquids, we can obtain the temperature dependence of many experimentally accessible properties of interest using thermodynamic relations or Kubo formulas. Since the scaling forms are precisely the same, the temperature dependences will be as well, unless the kinematic difference between one and higher dimensions is crucial. We will call attention to these instances; otherwise, all statements will hold equally well for both Fermi and Luttinger liquids. Equilibrium properties which may be obtained in this way include the specific heat, compressibility, and static susceptibilities. Transport properties include the conductivity, thermal conductivity, thermopower, Hall angle, nuclear spin NMR relaxation rate, and dynamic spin susceptibility. Almost all of these properties exhibit anomalous temperature dependencies in the cuprates.

Equilibrium Properties. The specific heat is given by $c \sim T \frac{\partial^2 f}{\partial T^2}$. Hence, using the scaling form (3.11) for the free energy, we find that the specific heat at constant volume goes as:

$$C_V \sim T$$

(4.1)

The compressibility, $\kappa$, and magnetic susceptibility, $\chi$, may be obtained by differentiating the free energy with respect to the chemical potential and magnetic field. They have the temperature dependences:

$$\kappa \sim \frac{\partial^2 f}{\partial \mu^2} \sim T^0$$

(4.2)

$$\chi \sim \frac{\partial^2 f}{\partial H^2} \sim T^0$$

(4.3)

These quantities may also be calculated from the scaling forms for the density-density and spin-spin correlation functions. For example, the scaling form for the
density-density correlation function (3.7) yields:

\[ \langle \rho(q, \omega) \rho(-q, -\omega) \rangle \sim T^0 f_\rho(\omega/T) \] (4.4)

As a result, we recover (4.2).

Conductivity, Thermal Conductivity, and Thermopower. The Kubo formulas relate transport coefficients to current-current Green functions. If we write:

\[ L_{11} = T \lim_{\omega \to 0} \frac{d}{d\omega} \text{Im} \langle j(q = 0, \omega) j(-q = 0, -\omega) \rangle \] (4.5)

\[ L_{12} = T \lim_{\omega \to 0} \frac{d}{d\omega} \text{Im} \langle j_Q(q = 0, \omega) j(-q = 0, -\omega) \rangle \] (4.6)

\[ L_{22} = T \lim_{\omega \to 0} \frac{d}{d\omega} \text{Im} \langle j_Q(q = 0, \omega) j_Q(-q = 0, -\omega) \rangle \] (4.7)

where \( j \) is the current, defined in (3.4), and \( j_Q \) is the heat current,

\[ j_Q = \int d l k^d \epsilon(l) \psi^\dagger(k+q) \psi(k) \left( \frac{\partial}{\partial q} \epsilon(q + 2k) \right) + (\epsilon_F - \mu) j \] (4.8)

then the conductivity, \( \sigma \), thermopower, \( Q \), and thermal conductivity, \( K \), are given by

\[ \sigma = \frac{1}{T} L_{11} \] (4.9)

\[ Q = \frac{L_{12}}{L_{11}} \] (4.10)

\[ K = \frac{1}{T^2} \left( L_{22} - \frac{(L_{12})^2}{L_{11}} \right) \] (4.11)

Hence, from the relative scaling of \( j_Q \) compared to \( j \) we find

\[ K \sim T \sigma \] (4.12)

\[ Q \sim T \] (4.13)

The conductivity itself involves some subtlety. This is a result of the fact that the conductivity is infinite for a free-fermion system. Said differently, the operators
which make the conductivity finite are dangerous irrelevant operators. If we were to proceed naively, we would have the following scaling form for the current-current correlation function:

\[
\langle j(q = 0, \omega) j(-q = 0, -\omega) \rangle = f_{jj}(\omega/T)
\]

which leads to a conductivity:

\[
\sigma \sim \frac{1}{T}
\]

However, as we mentioned earlier, the coefficient of the $1/T$ term is $\delta(\omega/T)$ for the free Fermi gas. In fact, if we consider a pure, translationally invariant system without phonons, the conductivity is still infinite – irrespective of interactions – because the current is proportional to a conserved quantity, the momentum. In the presence of a periodic lattice, however, there are umklapp processes which violate momentum conservation by a reciprocal lattice vector. In $d > 1$, four-fermi umklapp processes which change the total momentum by a reciprocal lattice vector $\mathbf{g}$ are frozen out at low energy if $g > 4k_F$. Otherwise, these four-fermi interactions scale as $s^{1/2}$, as per (3.9). $g < 4k_F$ is holds generically, but if it does not hold, then there may be six-fermi or higher-order processes which can degrade a current. These are, of course, even more irrelevant. In the case of a Luttinger liquid in $d = 1$, however, we have no angles at our disposal, so umklapp processes are always frozen out at low temperatures, $T < v_F|g - 2k_F|$. Hence, unlike Fermi liquids, clean Luttinger liquids have infinite conductivity at low enough temperatures unless they are nested, $g = 2k_F$, in which case they are insulators. The conductivity of Fermi liquids due to umklapp scattering may be most easily analyzed using the anisotropic scaling (3.9). Under this scaling, these are dangerous irrelevant operators. The contribution from a single lattice vector to the scaling form for the current-current correlation function is:

\[
\langle j(q = 0, \omega) j(-q = 0, -\omega) \rangle = \frac{1}{F_g^2 T} \left( \frac{k_F}{g} \right)^2 f_g^g(\omega/T)
\]

$g$ is the magnitude of the reciprocal lattice vector and $F_g$ is the coupling for the four-
fermion interaction which changes the total electronic momentum by $g$. Since $F_g$ is an irrelevant coupling which scales as $s^{1/2}$, each power of $F_g$ comes with a power of $T^{1/2}$ as per (2.6). $F_g$ enters quadratically because the conductivity is always positive while $F_g$ can take either sign; alternatively, one can see perturbatively that $F_g$ must appear at least quadratically in any current-current diagram. The factor of $(k_F/g)^2$ occurs because the conductivity diverges as $g/k_F \to 0$; by analyticity, $g$ must appear at least as $g^2$. The scaling form (4.16) leads to a conductivity:

$$
\sigma \sim \frac{1}{T^2}
$$

(4.17)

Of course, the system need not be pure. In such a case, scattering is due to a marginal operator, rather than an irrelevant one. However, there are two new dimensionful parameters in the game: the scattering length, $a$, for scattering from a single impurity, and the impurity concentration, $n$. As a result, the scaling form is (compare to (2.5)):

$$
\langle j(q=0,\omega)j(-q=0,-\omega) \rangle = (k_F a)^2 \frac{T}{n} f_j(\omega/T)
$$

(4.18)

so the conductivity is constant at low temperature.

The scaling form (4.14) followed from the nonrenormalization of the current operator in Fermi liquid theory. The same naive scaling will hold for the Luttinger liquid and, in fact, for all sensible theories because the current cannot be renormalized due to the Ward identity. Nevertheless, the conductivity is finite and can have exponents other than the naive one, as we found. Since this has been the source of some confusion, let us take a moment to clarify this point here. In a translationally invariant system, momentum is conserved and satisfies a conservation law, $\frac{\partial}{\partial t} T^0_j + \partial_i T_{ij} = 0$, where $T_{\mu\nu}$ is the energy-momentum tensor. If all particles in the system have the same charge-to-mass ratio, then the current is proportional to the momentum, $j_i = \frac{e}{m} T_{0i}$, and satisfies the same conservation law. In particular, $j_i \propto T_{0i} \propto \frac{q}{\omega}$. As a result, the finite-frequency conductivity vanishes. The
current-current correlation function still has the scaling form (4.14); it's just that the coefficient of the imaginary part vanishes. It is certainly possible, however, that a perturbative calculation will yield a finite conductivity, which is incorrect. Real metals are not translationally invariant, however. The lattice breaks the symmetry to a discrete subgroup and the concomitant umklapp processes degrade currents by violating momentum conservation by a reciprocal lattice vector. Impurity scattering also violates momentum conservation. Furthermore, phonons do not have the same charge-to-mass ratio as electrons, so the current is not proportional to the momentum in their presence.

Any system, whether translationally invariant or not, must conserve charge, however. Hence, \( \frac{\partial}{\partial t} \rho + \partial_i j_i = 0 \), (compare this with the energy-momentum conservation equation) irrespective of the validity of the momentum conservation equation. This conservation law has its expression at the quantum level as the Ward identity:

\[
\partial_\mu \langle j_\mu(y) \psi^\dagger(x_1) \ldots \psi(x_n) \rangle = i \delta(y-x_1) \langle \psi^\dagger(x_1) \ldots \psi(x_n) \rangle + i \delta(y-x_2) \langle \psi^\dagger(x_1) \ldots \psi(x_n) \rangle + \ldots 
\]

(4.19)

Since both sides of this equation must renormalize the same way, the current is not renormalized at all. This dictates the scaling form (4.14). However, the leading scaling piece in (4.14) may vanish as we noted above. Then it is the scaling behavior of the dangerous irrelevant operators that controls the temperature dependence of the conductivity. Incidentally, this cannot happen to the \( \langle j_\mu(y) \psi^\dagger(x_1) \ldots \psi(x_n) \rangle \) correlation function because it must equal the left-hand-side of the Ward identity; this correlation function is more constrained by gauge invariance than the imaginary part of the current-current correlation function.

**Hall angle.** The Hall angle is defined in the following way. We introduce an electric field, \( \mathbf{E} = E \cos \theta_H \hat{x} + E \sin \theta_H \hat{y} \), in the presence of a magnetic field, \( \mathbf{A} = H y \hat{x} \). The Hall angle, \( \theta_H \), is the angle such that \( j_x = \sigma_{xx}(H) E \cos \theta_H = \sigma_{yx}(H) E \sin \theta_H \) and \( j_y = 0 \). The latter condition has the following statement in
terms of correlation functions:

\[ \langle j_y \rangle = \langle j_y \ e^{\int (j_x E \cos \theta_H + j_y E \sin \theta_H)} e^{\int j_x A_x} \rangle = 0 \] (4.20)

To lowest order in \( E \) and \( H \), this is:

\[ \frac{\partial}{\partial q_y} \langle j_x j_y j_x \rangle H \cos \theta_H + \langle j_y j_y \rangle \sin \theta_H = 0 \] (4.21)

Hence, the Hall angle is given by:

\[ \tan \theta_H = -\frac{\frac{\partial}{\partial q_y} \langle j_x j_y j_x \rangle}{\langle j_y j_y \rangle} H \] (4.22)

Using (3.7) and simple kinematics, we find the following naive scaling form for the triple current correlator:

\[ \langle j_x(q_1+q_2, \omega) j_y(-q_1, -\omega) j_x(-q_2, 0) \rangle = \frac{1}{T} k_F f_{jjj}(\omega/T, v_F q_i/T) + \frac{1}{T} q_{2y} g_{jjj}(\omega/T, v_F q_i/T) \] (4.23)

Then

\[ \frac{\partial}{\partial q_y} \langle j_x(0, \omega) j_y(0, -\omega) j_x(q, 0) \rangle_{q=0} = \frac{1}{T^2} v_F k_F f_{jjj}(\omega/T, 0) + \frac{1}{T} g_{jjj}(\omega/T, 0) \] (4.24)

while the current-current correlation function has the naive scaling form (4.14). Hence, we find a Hall angle which naively scales as:

\[ \tan \theta_H \sim \frac{v_F k_F}{T^2} + \frac{c}{T} \] (4.25)

Semiclassically, \( \theta_H \sim \omega_c \tau_H \), where \( \tau_H \) is the lifetime of excitations contributing to the Hall current. From a dimensional standpoint, we would expect \( \tau_H \sim \frac{1}{\tau} \). The leading \( v_F k_F/T^2 \) term is, therefore, a surprise. In Fermi liquid theory, the coefficient of this term vanishes and the second term is, as in the case of the conductivity, modified by a dangerous irrelevant operator, \( \frac{c}{T} \rightarrow \frac{c}{T} \frac{1}{T} \). Hence, \( \theta_H \sim \frac{1}{T} \) in Fermi liquid theory. A more exotic theory could certainly have both terms of (4.25).
**NMR Relaxation Rate.** Experimental probes of the spin dynamics measure the dynamic spin susceptibility or, in other words, the spin-spin correlation function. The nuclear spin relaxation rate, $T_1$, which is due to the coupling of nuclear spins to the conduction electrons and is measured in NMR experiments is given by:

$$\frac{1}{T_1 T} = \int d^d q \ A(q) \lim_{\omega \to 0} \text{Im} \left\{ \frac{\chi(q, \omega)}{\omega} \right\} \quad (4.26)$$

where

$$\chi(q, \omega) \delta_{ij} = \langle S_i(q, \omega) S_j(-q, -\omega) \rangle \quad (4.27)$$

According to (3.7),

$$\chi(q, \omega) = f_{ss}(\omega/T, q/T) \quad (4.28)$$

so

$$\frac{1}{T_1 T} = \int d^d q \ A(q) \lim_{\omega \to 0} \text{Im} \left\{ \frac{1}{\omega} f_{ss}(\omega/T, q/T) \right\} = \int d^d q \ A(q) \frac{1}{T} \lim_{\omega \to 0} g(\omega/T, q/T) \quad (4.29)$$

where $g(x, y) = \frac{1}{x} \text{Im} f_{ss}(x, y)$. Then, changing variables to $Q = q/T$ and taking the $\omega \to 0$ limit, we have:

$$\frac{1}{T_1 T} = \int d^d Q \ A(QT) \frac{1}{T} g(0, Q) \quad (4.30)$$

So long as $A$ is slowly varying, this leads to the scaling form:

$$\frac{1}{T_1} \sim T \quad (4.31)$$

This is the well-known leading behavior of Fermi liquid theory. Subleading corrections – which are truly subleading in this case – due to irrelevant operators are given by (2.6).
5. Metallic Indices and the Cuprates

The preceding analysis has shown that there is very little possibility for variation in the asymptotic temperature dependence of most physical properties of a metal. Indeed, since the Ward identities constrain the operator dimensions of conserved currents, the main freedom comes from the dimensions of the (possibly irrelevant) operators which give the leading contribution to dissipative processes and in the relative scaling of frequency and momentum.

Let us be a little more precise about this. First, let us restrict attention to systems of gapless excitations about a Fermi surface (and, possibly, other degrees of freedom, as well). This excludes, for instance, BCS superconductors, which are metals in the sense of conducting at $T = 0$. More exotic metals may need a vastly different renormalization group analysis. Let us also restrict attention, for the moment, to clean systems. One might worry that the above restrictions have limited us to Fermi and Luttinger liquids. However, the non-Fermi liquid gauge theory models of [10,11] are a proof in principle that there are other theories in this class. Generically, such models might have other gapless modes that interact with the fermions.

We claim, now, that the physical properties of metals in this class can be parametrized by a small number of indices, just as three indices $\nu$, $\eta$, $z$ characterize dynamic critical fixed points. The simplest possibility is to have an index $\lambda$ characterizing the dissipation of currents and a set of indices $\eta^{i}_{\nu F}$ which are the anomalous dimensions of the Fermi velocity determining the relative rescalings of space and time for various types of excitations. In particular, the anomalous dimensions of the fermion field, which is not gauge-invariant, is unimportant for measurable properties. In Fermi liquid theory, there is only one such index, $\eta_{\nu F} = 0$, but this is not always the case. When fermions interact with gauge fields, there are two indices, $\eta^{l}_{\nu F}$, which is non-vanishing and negative [10,11], and $\eta^{g}_{\nu F}$, which vanishes[16]. $\eta^{l}_{\nu F}$ is the anomalous dimensions of the Fermi velocity which determines the relative scaling of frequency and momentum for processes which occur in the
vicinity of an arbitrary point on the Fermi surface (in a sense that can be made precise). The fermion two-point function, for instance, is then a function of the scaling variable $k/\omega^{1+\eta^l_{vF}}$. $\eta^q_{vF}$, on the other hand, is the anomalous dimensions of the Fermi velocity for processes which are averaged over the Fermi surface, such as density-density correlation functions, $\langle \rho(q,\omega)\rho(-q,-\omega) \rangle = f(v_F q/\omega^{1+\eta^q_{vF}})$. The crossover between these two behaviors has been studied in [17]. The reason that two indices arise where, naively, only one is expected is that both gauge fields and fermion bilinears at finite wavevector pick out preferred directions in momentum space (the former pick out the tangent to the Fermi surface, while the latter pick out the direction of the wavevector $q$ even in the $q \to 0$ limit). If these two directions are not the same, the effects of the gauge field are suppressed; of course, for quantities arising from averages over the whole Fermi surface, the isolated points where they agree are unimportant. On the other hand, there is no preferred direction in the two-point function or free energy so the gauge field effects are seen in, for instance, the specific heat. We can imagine a theory with gauge fields of the type just considered and another interaction without the same kinematic limitations. Such a theory could have both $\eta^l_{vF}$ and $\eta^q_{vF}$ non-vanishing and not equal to each other. By a straightforward generalization of the analysis of the previous section, we would find $C_V \sim T^{1+\eta^l_{vF}}$, $Q \sim T^{1+\eta^q_{vF}}$, $K \sim T^{1+2\eta^q_{vF}}$, and $1/T_1 \sim T^{1+\eta^q_{vF}}$.

$\lambda$ is the dimension (in units of time) of the leading operator which can degrade currents. According to the arguments which we gave earlier, the conductivity has the temperature dependence:

$$\sigma \sim \left(\frac{k_F}{g(T)}\right)^2 \frac{1}{T^{1+\lambda+\eta_{vF}}}$$

(5.1)

Here $g(T)$ is the typical momentum by which currents are degraded; it can be a function of temperature. If the operator in question is a dangerous irrelevant operator, then $\lambda > 0$. If the operator is marginally irrelevant, then we expect logarithmic corrections to a $1/T$ conductivity. Suppose the operator is relevant, however. There are two possibilities. First, the coupling can flow to some new
fixed point where a gap develops or the metallic state is destabilized in some other way. Alternatively, the coupling can flow to some new metallic fixed point (as in [10,11,13,12]). Then, the coupling is not dangerous and may be set to its fixed point value. In such a case (5.1) holds with $\lambda = 0$. In particular, a relevant or marginal operator which degrades currents by some fixed momentum $g$ (eg. an umklapp process mediated by a gapless mode) leads to $\sigma \sim 1/T$, unless it destabilizes the metallic state. Furthermore, such an operator would make it possible for the naive scaling (4.25) to hold for the Hall angle. It is remarkable that $\sigma \sim 1/T$, $\tan \theta_H \sim 1/T^2$ is, in some sense, the most natural scaling beavior possible.

The NMR relaxation rate of the copper nuclei in the cuprate materials exhibits a more complicated temperature dependence than the simple power law behavior of the conductivity. $T_1 T \sim T + T_x$, so $T_1 \sim \frac{1}{T}$ in the $T \to 0$ limit as in Fermi liquid theory. However, at $T > T_x$, $T_1 \sim \text{const.}$, which is highly anomalous. One possible interpretation is that $T_1$ crosses over from some unstable fixed point where $T_1 \sim \text{const.}$ to a fixed point where $T_1 \sim \frac{1}{T}$. Such a rich structure could emerge in the presence of other low-energy modes (antiferromagnetic fluctuations, gauge fields, etc.) which could invalidate the simple scaling form (4.28) by selecting preferred wavevectors such as $(\pi, \pi)$.

As has been much discussed, the cuprates exhibit striking $\sigma \sim 1/T$, $\tan \theta_H \sim 1/T^2$ behavior and $T_1 T \sim A + BT$. As we have seen above, this is qualitatively consistent with a theory of excitations about a Fermi surface subject to relevant interactions with other gapless modes. At least one of these interactions would have to degrade currents by a temperature-independent momentum and one of these – not necessarily the same – would have to give the correction to simple scaling of the NMR relaxation rate.
6. Discussion

The renormalization group has facilitated the identification of universality classes of low temperature behavior of interacting fermion systems. As we have shown above, the concomitant technology of the renormalization group such as scaling forms for correlation functions, composite operator renormalization, and finite-size effects provides a simple means of obtaining the physical behavior characteristic of these universality classes. Our analysis leads to a characterization of more interesting possible non-Fermi liquid behaviors in terms of a small number of indices, analogous to the characterization of critical phenomena.

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7. Figure Caption

Figure 1. This two-loop diagram is the simplest diagram which contributes to the imaginary part of the fermion self-energy.
Figure 1.