GEOMETRY OF QUANTUM HOMOGENEOUS VECTOR BUNDLES
AND REPRESENTATION THEORY OF QUANTUM GROUPS I

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Abstract

Quantum homogeneous vector bundles are introduced by a direct description of their sections in the context of Woronowicz type compact quantum groups. The bundles carry natural topologies inherited from the quantum groups, and their sections furnish projective modules over algebras of functions on quantum homogeneous spaces. Further properties of the quantum homogeneous vector bundles are investigated, and their applications to the representation theory of quantum groups are explored. In particular, quantum Frobenius reciprocity and a generalized Borel-Weil theorem are established.

1 INTRODUCTION

The seminal work of Manin [1] and Woronowicz [2] demonstrated that quantum groups play much the same role in noncommutative geometry as that played by Lie groups in classical geometry. This fact has been intensively investigated by several schools of researchers in recent years, considerably advancing our understanding of the underlying geometry of quantum groups. We refer to the articles [3] and [4] for reviews of the current state of the area and for useful references to the subject.

The present paper is the first of a series intending to develop a comprehensive theory of quantum homogeneous vector bundles determined by quantum groups of the Woronowicz type[5], and to explore their applications in a geometrical representation theory of quantum groups. Various versions of quantum deformations of fibre bundles were proposed at the algebraic level (i.e., without any topology) in the literature [6, 7]. We mention in particular reference [6] and subsequent research along a similar line, where the primary aim was to develop a version of deformed gauge theory. Quantum homogeneous vector bundles, in comparison, have been less studied, although they are much more closely related to quantum groups, and have natural applications in representation theory.

In fact, quantum homogeneous vector bundles provide the foundations for developing a geometrical representation theory of quantum groups. There has long been an important interplay between geometry and representation theory in the context of classical Lie groups, e.g., the interaction between representation theory and the Penrose
transforms of twistor theory \[9\]. We expect a similar interaction between representation theory and geometry to carry over to the quantum case.

The main new results of the paper are contained in Sections 3 and 4 where we define quantum homogeneous vector bundles and study their properties and applications. In particular theorem \[9\] is a key result. In subsection \[3.2\] we introduce quantum homogeneous vector bundles by a direct description of their sections. These are defined in terms of Woronowicz type compact quantum groups and their associated quantum homogeneous spaces. Our definition of quantum homogeneous vector bundles is consistent with the general definition of noncommutative vector bundles adopted in Connes’ theory \[8\]. There a noncommutative vector bundle is defined by its space of sections, which is required to be a projective module of finite type over the algebra of functions on the noncommutative (virtual) base space. These noncommutative geometrical structures carry natural topologies inherited from those of the quantum groups. The latter is discussed in subsection \[3.1\]. Projectivity of the space $\mathcal{E}_q^t(V)$ of sections of a quantum homogeneous bundle induced from a $U_q(\mathfrak{l})$-module $V$ is established in Theorem \[3\] of subsection \[3.3\]. In this subsection, it is also shown that $\mathcal{E}_q^t(V)$ forms an induced module over $U_q(\mathfrak{g})$, and a co-module over the associated quantum group, in analogy with the classical situation. Several other classical results are shown to admit quantum analogues. In particular a quantum version of Frobenius reciprocity is established in section \[3.4\] while proposition \[3\] asserts that if the inducing $U_q(\mathfrak{l})$-module is, in fact, the restriction of a $U_q(\mathfrak{g})$-module, then the space of sections $\mathcal{E}_q^t(V)$ is freely generated as a module over the algebra of functions of the quantum homogeneous space, and this means that the quantum homogeneous bundle determined by $\mathcal{E}_q^t(V)$ is trivial. Finally, in section \[4\] a notion of ‘quantum holomorphic’ sections is established and an analogue of the Borel-Weil theorem is established. The reader may note that the approach to the proof there is easily adapted to yield a new proof of the classical Borel-Weil theorem via representative functions and the Peter-Weyl theorem.

The organization of the remainder of the paper is as follows. Section 2 introduces the notation and conventions while reviewing the basic definitions of quantum groups and quantized universal enveloping algebras. It also summaries their main structural and representation theoretical features. While the material, for the most part, is not new, our treatment of real forms, parabolic and reductive quantum subalgebras of quantized universal enveloping algebras , as well as integrals on quantum groups, should be of general interest. The appendix provides a concise and elementary treatment of the classical theory of homogeneous bundles as relevant to the quantum constructions and results as mentioned above. Results are established there in a manner that should shed light on the corresponding arguments for the quantum case and indicate the geometrical nature of our treatment of the latter.

2 QUANTUM GROUPS AND QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS

2.1 Quantized universal enveloping algebras
2.1.1 Generalities

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra of rank $r$. We denote by $\Phi^+$ the set of its positive roots relative to a base $\Pi = \{ \alpha_i \mid i \in \mathbb{N}_r \}$, where $\mathbb{N}_r = \{1, 2, ..., r\}$. Define $E = \bigoplus_{i=1}^{r} \mathbb{R} \alpha_i$. Let $( , ) : E \times E \to \mathbb{R}$ be the inner product induced by the Killing form of $\mathfrak{g}$. Then the Cartan matrix $A$ of $\mathfrak{g}$ is given by $A = (a_{ij})_{ij=1}^{r}$, with $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. We will call $\lambda \in E$ integral if $\lambda_i = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$, $\forall i$, and integral dominant if $\lambda_i \in \mathbb{Z}_+$, $\forall i$. The set of integral elements of $E$ will be denoted by $\mathcal{P}$, and that of the integral dominant elements by $\mathcal{P}_+$.

The Jimbo version \cite{Jimbo1985} of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is defined to be the unital associative algebra over $\mathbb{C}$, generated by $\{k_i^{\pm 1}, e_i, f_i \mid i \in \mathbb{N}_r\}$ subject to relations as below. Here $\left[ \begin{array}{c} s \\ t \end{array} \right]_q$ is the Gauss polynomial, and $q_i = q^{(\alpha_i, \alpha_i)/2}$.

The relations are:

\[
\begin{align*}
    k_i k_j &= k_j k_i, \quad k_i k_i^{-1} = 1, \quad k_i e_j k_i^{-1} = q_i e_j, \\
    k_i f_j k_i^{-1} &= q_i f_j, \quad [e_i, f_j] = \delta_{ij} k_i^2 - k_i^{-2}, \\
    \sum_{t=0}^{1-a_{ij}} (-1)^t \binom{1 - a_{ij}}{t} q_i^{t} (e_i)^t e_j (e_i)^{1-a_{ij}-t} &= 0, \quad i \neq j, \\
    \sum_{t=0}^{1-a_{ij}} (-1)^t \binom{1 - a_{ij}}{t} q_i^{t} (f_i)^t f_j (f_i)^{1-a_{ij}-t} &= 0, \quad i \neq j,
\end{align*}
\]

where $q$ in general is taken to be a complex parameter, which is nonvanishing and not equal to 1. However, in this paper we will assume that $q$ is real positive and different from 1. This restriction is required in order for $U_q(\mathfrak{g})$ to admit a Hopf $*$-algebra structure and for the Haar functional on the corresponding quantum group to be positive definite.

As is well known, $U_q(\mathfrak{g})$ has the structure of a Hopf algebra. We take the following co-multiplication

\[
\begin{align*}
    \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
    \Delta(e_i) &= e_i \otimes k_i + k_i^{-1} \otimes e_i, \\
    \Delta(f_i) &= f_i \otimes k_i + k_i^{-1} \otimes f_i.
\end{align*}
\]

The co-unit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}$ and antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ are respectively given by

\[
\begin{align*}
    \epsilon(e_i) &= \epsilon(f_i) = 0, \quad \epsilon(k_i^{\pm 1}) = \epsilon(1) = 1, \\
    S(e_i) &= -q_i e_i, \quad S(f_i) = -q_i^{-1} f_i, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}.
\end{align*}
\]

The representation theory of $U_q(\mathfrak{g})$ is closely related to that of the corresponding simple Lie algebra $\mathfrak{g}$, and we refer to the many books on the subject, e.g., \cite{Jimbo1985}, for details. Here we mention that all finite dimensional representations are completely reducible. Thus the study of such representations reduces to analyzing the irreducible ones. If $W_\omega(\lambda)$ is a finite dimensional irreducible left $U_q(\mathfrak{g})$-module, then the action of
the $k_i$ can be diagonalized. There also exists a unique vector $v_+$ (up to scalar multiples), called the highest weight vector of $W_\omega(\lambda)$, such that

$$e_i v_+ = 0, \quad k_i v_+ = \omega_i q^{(\lambda, \alpha_i)/2} v_+,$$

$\omega_i \in \{1, -1\}$, $\lambda \in \mathcal{P}_+$,

and the module $W_\omega(\lambda)$ is uniquely determined by $\lambda$ and the $\omega_i$. The existence of the $\omega_i$ is a peculiarity of the Jimbo form of quantized universal enveloping algebra, which stems from the following algebra automorphisms

$$e_i \mapsto \sigma_i e_i, \quad f_i \mapsto \sigma'_i f_i \quad k_i \mapsto \sigma_i \sigma'_i k_i,$$

$\sigma_i, \sigma'_i \in \{1, -1\}$.

When $\omega_i = 1$, $\forall i$, we denote $W_\omega(\lambda)$ by $W(\lambda)$. In this case, a common eigenvector $w \in W(\lambda)$ of the $k_i$ necessarily satisfies

$$k_i w = q^{(\mu, \alpha_i)/2} w,$$

for some $\mu \in \mathcal{P}$. We call $\mu$ the weight of $w$. The maximum weight, relative to the simple root system $\Pi$, is $\lambda$. This will be referred to as the highest weight of $W(\lambda)$. When $\lambda \in \mathcal{P}_+$, we will denote the lowest weight of $W(\lambda)$ by $\bar{\lambda}$, and define

$$\lambda^\dagger = -\bar{\lambda}.$$

Then $\lambda^\dagger$ is integral dominant, and the dual module of $W(\lambda)$ has highest weight $\lambda^\dagger$.

Let $\text{Mod}_q(\mathfrak{g})$ be the set of finite dimensional $U_q(\mathfrak{g})$-modules, which is obviously closed under direct sum and direct product (with respect to the co-multiplication $\Delta$). In fact $\text{Mod}_q(\mathfrak{g})$ forms a tensor category. The following points should be observed:

i). There is a one to one correspondence between the objects of $\text{Mod}_q(\mathfrak{g})$ and finite dimensional representations of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$;

ii). $W(\lambda)$, $\lambda \in \mathcal{P}_+$, has the same weight space decomposition as that of the irreducible $U(\mathfrak{g})$-module with highest weight $\lambda$.

2.1.2 Real forms and parabolic subalgebras

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ admits a variety of Hopf $*$-algebra structures, namely, there exist anti -involutions $*$ satisfying the following relation

$$* S * S = id_{U_q(\mathfrak{g})}. \quad (2)$$

Given an $*$-operation, we set

$$\theta = * S, \quad (3)$$

and call $\theta$ a quantum Cartan involution. Let us define

$$U_q^R(\mathfrak{g}_0) = \{ x \in U_q(\mathfrak{g}) \mid \theta(x) = x \}. \quad (4)$$

It can be readily shown that $U_q^R(\mathfrak{g}_0)$ defines a real associative algebra, which may be regarded as a ‘real form’ of $U_q(\mathfrak{g})$. However, the restriction of $\Delta$ does not lead to
a co-multiplication for $U^\mathbb{R}_q(\mathfrak{g}_0)$, thus $U^\mathbb{R}_q(\mathfrak{g}_0)$ does not possess a natural Hopf algebra structure.

Explicit examples of $*$-operations are the following, each specified by a choice of $\sigma, \sigma' \in \{1, -1\}$:

$$e_i^* = \sigma_i f_i, \quad f_i^* = \sigma'_i e_i, \quad k_i^* = k_i^{\sigma_i \sigma'}. \quad (5)$$

In this paper, we will be interested only in the compact real form of $U_q(\mathfrak{g})$. Thus, henceforth, we will assume that $U^\mathbb{R}_q(\mathfrak{g}_0)$ is defined by using the $*$-operation with

$$\sigma_i = \sigma'_i = 1, \quad \forall i. \quad (5)$$

An important property of this $*$-operation is that every finite dimensional $U_q(\mathfrak{g})$-module $W$ is unitary \[12\] in the sense that there exists a nondegenerate positive definite sesquilinear form $(\ , \ ) : W \times W \to \mathbb{C}$ satisfying

$$(xv, w) = (v, x^* w), \quad \forall v, w \in W, \ x \in U_q(\mathfrak{g}). \quad (6)$$

Denote by $C_q(\mathfrak{g}_0)$ the real vector space spanned by

$$X_i = e_i - q_i f_i, \quad Y_i = \sqrt{-1}(e_i + q_i f_i), \quad Z_i = \sqrt{-1} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad S_i = k_i + k_i^{-1} - 2, \quad i \in \mathbb{N}_r.$$ Then $U^\mathbb{R}_q(\mathfrak{g}_0)$ is generated by $C_q(\mathfrak{g}_0) \cup \{1_{U_q(\mathfrak{g})}\}$. Note that $C_q(\mathfrak{g}_0)$ vanishes under the co-unit. A further property is that $\Delta(C_q(\mathfrak{g}_0)) \subset C_q(\mathfrak{g}_0) \otimes \mathbb{R} U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathbb{R} C_q(\mathfrak{g}_0)$.

That is, we have the following result.

**Lemma 1** $C_q(\mathfrak{g}_0)$ is a two-sided co-ideal of $U_q(\mathfrak{g})$.

Any element $a$ of the complexification of $(U^\mathbb{R}_q(\mathfrak{g}_0))^*$ naturally gives rise to a $\mathbb{C}$-linear functional on $U_q(\mathfrak{g})$ (regarded as the complexification of $U^\mathbb{R}_q(\mathfrak{g}_0)$), by requiring that

$$a(x + \sqrt{-1}y) = a(x) + \sqrt{-1}a(y), \quad \forall x, y \in U^\mathbb{R}_q(\mathfrak{g}_0).$$

Conversely, we can restrict any linear functional on $U_q(\mathfrak{g})$ to one on $U^\mathbb{R}_q(\mathfrak{g}_0)$. This identifies $\mathbb{C} \otimes \mathbb{R} (U^\mathbb{R}_q(\mathfrak{g}_0))^*$ with $(U_q(\mathfrak{g}))^*$. In a similar way, one can easily establish that there exists a one-to-one correspondence between complex representations of $U^\mathbb{R}_q(\mathfrak{g}_0)$ and complex representations of $U_q(\mathfrak{g})$.

Let us now consider parabolic and related subalgebras. Take a subset $\Theta$ of $\mathbb{N}_r$. Introduce the following sets of elements of $U_q(\mathfrak{g})$:

$$S_l = \{k_i^{\pm 1}, i \in \mathbb{N}_r; \ e_j, f_j, j \in \Theta\};$$

$$S_p = S_l \cup \{e_j, j \in \mathbb{N}_r \setminus \Theta\}.$$ Clearly $S_l$ and $S_p$ generate Hopf subalgebras of $U_q(\mathfrak{g})$, which we respectively denote by $U_q(l)$ and $U_q(p)$. We call $U_q(l)$ a reductive quantum subalgebra, and $U_q(p)$ a parabolic
quantum subalgebra of $U_q(\mathfrak{g})$, since, in the classical limit, these Hopf subalgebras respectively reduce to the enveloping algebras of a reductive Lie subalgebra $\mathfrak{l}$ and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We may replace the elements $e_j$, $j \in \mathbb{N}_r \setminus \Theta$, by $f_j$ in $\mathcal{S}_p$, and the resulting set generates another Hopf subalgebra, which is the image of $U_q(\mathfrak{p})$ under the quantum Cartan involution. It also deserves the name of a parabolic quantum subalgebra. Results presented in the remainder of the paper can also be formulated using such opposite parabolic Hopf subalgebras. It is important to observe that $U_q(\mathfrak{l})$ is the invariant subalgebra of $U_q(\mathfrak{p})$ under the quantum Cartan involution $\theta$. For later use, we also define

$$U^R_q(\mathfrak{f}) = U_q(\mathfrak{l}) \cap U^R_q(\mathfrak{g}_0).$$

Then $U^R_q(\mathfrak{f})$ is a real subalgebra of $U^R_q(\mathfrak{g}_0)$, and its complexification is $U_q(\mathfrak{l})$. $U^R_q(\mathfrak{f})$ is generated by $1_{U_q(\mathfrak{g})}$ and the set

$$\{X_i, Y_i \mid i \in \Theta \} \cup \{Z_i, S_i \mid i \in \mathbb{N}_r\}.$$ 

We will denote by $C_q(\mathfrak{f})$ the linear span of the elements of this set. Then it can be easily shown that $C_q(\mathfrak{f})$ is a two-sided co-ideal of $U_q(\mathfrak{g})$.

Let $V_\mu$ be a finite dimensional irreducible $U_q(\mathfrak{l})$-module. Then $V_\mu$ is of highest weight type. Let $\mu$ be the highest weight and $\tilde{\mu}$ the lowest weight of $V_\mu$ respectively. We can extend $V_\mu$ in a unique fashion to a $U_q(\mathfrak{p})$-module, which is still denoted by $V_\mu$, such that the elements of $\mathcal{S}_p \setminus \mathcal{S}_l$ act by zero. It is not difficult to see that all finite dimensional irreducible $U_q(\mathfrak{p})$-modules are of this kind.

Consider a finite dimensional irreducible $U_q(\mathfrak{g})$-module $W(\lambda)$, with highest weight $\lambda$ and lowest weight $\bar{\lambda}$. $W(\lambda)$ can be restricted in a natural way to a $U_q(\mathfrak{p})$-module, which is always indecomposable, but not irreducible in general. It can be readily shown that

$$\dim \mathbb{C} \text{Hom}_{U_q(\mathfrak{g})}(W(\lambda), V_\mu) = \begin{cases} 1, & \bar{\lambda} = \tilde{\mu}, \\ 0, & \bar{\lambda} \neq \tilde{\mu} \end{cases}$$ (7)

### 2.2 Quantum groups

#### 2.2.1 Quantum groups

Roughly speaking a quantum group is the dual Hopf algebra of a quantized universal enveloping algebra. However, since $U_q(\mathfrak{g})$ is infinite dimensional, considerable care needs to be exercised in defining the quantum group. The complication stems from the following well known fact. If $A$ is an infinite dimensional algebra, then the dual vector space $A^*$ in general does not admit a co-algebra structure. The way to get around the problem is to consider the so-call finite dual $A^0 \subset A^*$, which is defined by requiring that for any $f \in A^0$, $Ker f$ contains a two-sided ideal $\mathcal{I}$ of $A$ which is of finite co-dimension, i.e., $\dim A/\mathcal{I} < \infty$.

A standard result of Hopf algebra theory states that if $A$ is a Hopf algebra with multiplication $m$, unit $1_A$, co-multiplication $\Delta$, co-unit $\epsilon$ and antipode $S$, then the finite dual $A^0$, when it is not 0, is also a Hopf algebra with a structure dualizing that of $A$. For any $a, b \in A^0$, $x, y \in A$, the multiplication $m_0$ and co-multiplication $\Delta_0$ are
where the direct sum is defined algebraically. The T of the central algebra of W.

Clearly this is independent of the choice of basis for respectively defined by

\[ \langle m_0(a \otimes b), x \rangle = \langle a \otimes b, \Delta(x) \rangle; \]
\[ \langle \Delta_0(a), x \otimes y \rangle = \langle a, m(x \otimes y) \rangle; \]

the unit \( \mathbb{1}_{A^0} \) and co-unit \( \epsilon_0 \) by

\[ \mathbb{1}_{A^0} = \epsilon, \quad \epsilon_0(a) = \langle a, \mathbb{1}_A \rangle; \]

and the antipode \( S_0 \) by

\[ \langle S_0(a), x \rangle = \langle a, S(x) \rangle. \]

Toward defining the quantum group dual to \( U_q(g) \), consider the irreducible objects \( W(\lambda), \lambda \in \mathcal{P}_+ \) of \( \text{Mod}_q(g) \). For each \( W(\lambda) \) of dimension \( d_\lambda \), we choose a basis \( \{w^{(\lambda)}_i | i = 1, 2, ..., d_\lambda\} \), which is arbitrary at this stage. Let \( t^{(\lambda)}(x) = \left(t^{(\lambda)}_{ij}\right)_{i,j=1}^{d_\lambda} \), with \( t^{(\lambda)}_{ij} \) being elements of \( (U_q(g))^* \) defined by

\[ \sum_j t^{(\lambda)}_{ji}(x)w^{(\lambda)}_j = xw^{(\lambda)}_i, \quad \forall x \in U_q(g). \]

We will also denote by \( t^{(\lambda)} \) the irreducible representation of \( U_q(g) \) associated with the module \( W(\lambda) \) relative to the given basis, and call the \( t^{(\lambda)}_{ij} \) matrix elements of the irreducible representation \( t^{(\lambda)} \).

We denote by \( \Pi(\lambda) \) the set of the weights of \( W(\lambda) \). For each \( i \), we defined \( p_i^\pm = \prod_{\rho \in \Pi(\lambda)} \left(k_i^{\pm 1} - q^{\pm (\nu, \alpha_i)/2}\right) \). Then \( t^{(\lambda)}(p_i^\pm) = 0 \), for all \( i \in \mathbb{N}_r \). Let \( \beta(\lambda) = \lambda - \bar{\lambda} \), where \( \bar{\lambda} \) is the lowest weight of \( W(\lambda) \). If \( u \in U_q(g) \) has the property that \( k_i u_{k_i}^{-1} = q^{(\gamma, \alpha_i)/2} u, \forall i \in \mathbb{N}_r \), and \( \gamma > \beta(\lambda) \), or \( \gamma < -\beta(\lambda) \), then \( t^{(\lambda)}(u) = 0 \). Let \( \mathcal{I}_\lambda \) be the two-sided ideal of \( U_q(g) \) generated by all such \( u \) together with the \( p_i^\pm, i \in \mathbb{N}_r \). Then it follows from the quantum analog of PBW theorem that \( \mathcal{I}_\lambda \) has finite co-dimension, because \( W(\lambda) \) is finite dimensional. Hence for all \( \lambda \in \mathcal{P}_+ \), \( t^{(\lambda)}_{ij} \in (U_q(g))^0 \).

The irreducibility of \( W(\lambda) \) together with the so-called Burnside theorem of matrix algebras implies that \( t^{(\lambda)}(U_q(g)) \) coincides with the entire algebra of \( d_\lambda \times d_\lambda \) matrices. Hence for each \( \lambda \) the \( t^{(\lambda)}_{ij} \) are linearly independent. By considering the left action \( \text{[18]} \) of the central algebra of \( U_q(g) \) on them, one can also easily convince oneself that the entire set \( \{t^{(\lambda)}_{ij} | i, j = 1, 2, ..., d_\lambda, \forall \lambda \in \mathcal{P}_+ \} \), is also linearly independent. Denote by \( T^{(\lambda)} \) the subspace of \( (U_q(g))^0 \) defined by

\[ T^{(\lambda)} = \bigoplus_{i,j=1}^{d_\lambda} \mathcal{C}t^{(\lambda)}_{ij}. \]

Clearly this is independent of the choice of basis for \( W(\lambda) \). Let

\[ \mathcal{T}_q(g) = \bigoplus_{\lambda \in \mathcal{P}_+} T^{(\lambda)}, \]

where the direct sum is defined algebraically. The \( \mathcal{T}_q(g) \) is essentially a quantum group of the kind introduced in \( \text{[13]} \). It has the following important property

**Proposition 1** \( \mathcal{T}_q(g) \) is a Hopf subalgebra of \( (U_q(g))^0 \).
This result is of course well known, but several aspects of it are worth mentioning. Note that the multiplication of \((U_q(\mathfrak{g}))^0\) is defined by the co-multiplication of \(U_q(\mathfrak{g})\). Since the tensor product of any two finite dimensional representations of \(U_q(\mathfrak{g})\) can be decomposed into a direct sum of finite dimensional irreducible representations, \(T_q(\mathfrak{g})\) is indeed closed under multiplication. Let \(R^{(\lambda)(\mu)}\) be the \(R\)-matrix associated with the two finite dimensional irreducible representations \(t^{(\lambda)}\) and \(t^{(\mu)}\), then
\[
R^{(\lambda)(\mu)}_{12} t^{(\lambda)}_1 t^{(\mu)}_2 = t^{(\mu)}_2 t^{(\lambda)}_1 R^{(\lambda)(\mu)}_{12}.
\]
The co-product of each \(t^{(\lambda)}_{ij}\) is easy to describe explicitly. We have
\[
\Delta_0(t^{(\lambda)}_{ij}) = \sum_{k=1}^{d_\lambda} t^{(\lambda)}_{ik} \otimes t^{(\lambda)}_{kj}.
\]
It is also useful to have an explicit characterization of the antipode. Introduce for \(W(\lambda^t) = (W(\lambda))^\ast\) the basis \(\{\bar{w}_i^{(\lambda)} | i = 1, 2, ..., d_\lambda\}\), which is dual to the basis chosen for \(W(\lambda)\) in the sense that \(\bar{w}_i^{(\lambda)}(w_j^{(\lambda)}) = \delta_{ij}\). Express the action of \(U_q(\mathfrak{g})\) on \(W(\lambda^t)\) by
\[
x\bar{w}_i^{(\lambda)} = \sum_{j=1}^{d_\lambda} \bar{t}_{ji}^{(\lambda^t)}(x)\bar{w}_j^{(\lambda)}, \quad x \in U_q(\mathfrak{g}),
\]
for some \(\bar{t}_{ji}^{(\lambda^t)}\) in \(T_q(\mathfrak{g})\). The natural action on the dual module is given by \((xv^\ast)(w) = v^\ast(S(x)w)\), for any \(v^\ast \in W(\lambda^t)\), \(w \in W(\lambda)\) and \(x \in U_q(\mathfrak{g})\). It then immediately follows that
\[
\langle \bar{t}_{ji}^{(\lambda^t)}, x \rangle = \langle S_0(t^{(\lambda)}_{ij}), x \rangle, \quad \forall x \in U_q(\mathfrak{g}),
\]
i.e. \(S_0(t^{(\lambda)}_{ij}) = \bar{t}_{ji}^{(\lambda^t)}\).

From here on we will omit the subscript 0 from \(\Delta_0\) and \(S_0\).

In addition \(T_q(\mathfrak{g})\) admits a natural anti-involution \(*\)-operation giving it the structure of a Hopf \(*\)-algebra. The \(*\)-operation is defined by
\[
\langle \ast(a), x \rangle = \langle a, \theta(x) \rangle, \quad \forall a \in T_q(\mathfrak{g}), \ x \in U_q(\mathfrak{g}),
\]
where \(\theta\) is the quantum Cartan involution on \(U_q(\mathfrak{g})\) defined by \((\mathfrak{g})\). Simple computations can show that this indeed gives rise to a \(*\)-operation for \(T_q(\mathfrak{g})\). It takes the simplest form in a unitary basis of \(T_q(\mathfrak{g})\), which we will introduce now. We assume that the basis \(\{w_i^{(\lambda)}\}\) of \(W(\lambda)\) is orthogonal under the sesquilinear form \((\mathfrak{g})\). That is \((w_i^{(\lambda)}, w_j^{(\lambda)}) = \delta_{ij}\). Define \(\tilde{w}_i^{(\lambda)} | i = 1, 2, ..., d_\lambda\) by \(\tilde{w}_i^{(\lambda)}(w_j^{(\lambda)}) = (w_i^{(\lambda)}, w_j^{(\lambda)})\), \(\forall w \in W(\lambda)\), which form a dual basis for \((W(\lambda))^\ast\). Note that \((x\tilde{w}_i^{(\lambda)})(w_j^{(\lambda)}) = (\theta(x)\tilde{w}_i^{(\lambda)}, w_j^{(\lambda)})\), which is equivalent to \(\langle \tilde{t}_{ji}^{(\lambda^t)}, x \rangle = \langle \bar{t}_{ji}^{(\lambda^t)} \theta(x) \rangle\). Thus in this basis, we have
\[
\ast(t_{ij}^{(\lambda)}) = \tilde{t}_{ij}^{(\lambda^t)}. \quad (13)
\]
2.2.2 Quantum Haar measure

The discussions of the last two subsections imply in particular that $\mathcal{T}_q(\mathfrak{g})$ satisfies the conditions of a CQG algebra in the sense of [15]. Therefore the general theory of quantum Haar functionals of [14] [15] is applicable to $\mathcal{T}_q(\mathfrak{g})$. Here we briefly treat the matter for the special case of $\mathcal{T}_q(\mathfrak{g})$. Our treatment should be of general interest.

Let us begin by defining integrals on a general Hopf algebra $A$ [19]. Let $A^*$ be its dual, which has a natural algebra structure introduced by dualizing the co-algebraic structure of $A$, although, as indicated above, $A^*$ does not admit a co-multiplication in general. An element $\int^l \in A^*$ is called a left integral on $A$ if

$$f \cdot \int^l = \langle f, 1_A \rangle \int^l, \quad \forall f \in A^*.$$ 

Similarly, $\int^r \in A^*$ is called a right integral on $A$ if

$$\int^r \cdot f = \langle f, 1_A \rangle \int^r, \quad \forall f \in A^*.$$ 

A straightforward calculation shows that the defining properties of the integrals are equivalent to the following requirements

$$(\text{id} \otimes \int^l)\Delta(x) = \int^lx, \quad (\int^r \otimes \text{id})\Delta(x) = \int^rx, \quad \forall x \in A. \quad (14)$$

where $\text{id}$ is the identity map $A \to A$.

A normalised Haar measure $\int \in A^*$ on $A$ is an integral on $A$ which is both left and right, and sends $1_A$ to 1, i.e.,

$$(i). \quad (\int \otimes \text{id})\Delta(x) = (\text{id} \otimes \int)\Delta(x) = \int x, \quad \forall x \in A,$$

$$(ii). \quad \int 1_A = 1. \quad (15)$$

When $A$ is a Hopf $*$-algebra, we call a Haar measure positive definite if $\int(x^*x) \geq 0$, and equality holds only when $x = 0$.

Now we go back to $\mathcal{T}_q(\mathfrak{g})$. It is an entirely straightforward matter to establish the following result.

**Theorem 1** The element $\int \in (\mathcal{T}_q(\mathfrak{g}))^*$ defined by

$$\int 1_{\mathcal{T}_q(\mathfrak{g})} = 1; \quad \int t^{(\lambda)}_{ij} = 0, \quad 0 \neq \lambda \in \mathcal{P}_+,$$

gives rise to a Haar measure on $\mathcal{T}_q(\mathfrak{g})$.

Denote by $2\rho$ the sum of the positive roots of $\mathfrak{g}$. Let $K_{2\rho}$ be the product of powers of $k_i^{\pm 1}$'s such that $K_{2\rho}k_{i}K_{2\rho}^{-1} = q^{(2\rho, \alpha_i)}e_i$, $\forall i$. Then it can be easily shown that $S^2(x) = K_{2\rho}xK_{2\rho}^{-1}, \forall x \in U_q(\mathfrak{g})$. Denote by $D_q(\lambda) := \text{tr}\{t^{(\lambda)}(K_{2\rho})\}$ the quantum dimension of the irreducible $U_q(\mathfrak{g})$-module $W(\lambda)$. The Haar measure $\int$ satisfies the following properties.
Lemma 2

\[
\int t^{(\lambda)}_{ij} t^{(\mu^\dagger)}_{rs} = \frac{t^{(\lambda)}_{sj}(K_{2\rho})}{D_q(\lambda)} \delta_{ir} \delta_{\lambda \mu},
\]

\[
\int \tilde{t}^{(\lambda)}_{ij} t^{(\mu)}_{rs} = \frac{\tilde{t}^{(\lambda)}_{ir}(K_{2\rho})}{D_q(\lambda)} \delta_{js} \delta_{\lambda \mu}.
\]

Proof: The proof makes essential use of the fact that \( f \) is a left and right integral. Look at the first equation. The \( \lambda \neq \mu \) case is easy to prove: the integral vanishes because the tensor product \( W(\lambda) \otimes W(\mu^\dagger) \) does not contain the trivial \( U_q(\mathfrak{g}) \)-module. When \( \lambda = \mu \), we introduce the notations

\[
\phi_{ir;sj} = \int t^{(\lambda)}_{ij} t^{(\lambda^\dagger)}_{rs}, \quad \Phi[s, j] = (\phi_{ir;sj})^{d_{\lambda}}_{i,r=1}; \quad \Psi[i, r] = (\phi_{ir;sj})^{d_{\lambda}}_{s,j=1}.
\]

It is clearly true that \( tr(\Psi[i, r]) = \delta_{ir} \).

Note that corresponding to each \( x \in U_q(\mathfrak{g}) \), there exists an \( \tilde{x} \in (\mathcal{T}_q(\mathfrak{g}))^* \) defined by \( \tilde{x}(a) = \langle a, x \rangle, \forall a \in \mathcal{T}_q(\mathfrak{g}) \). The left integral property of \( f \) leads to

\[
\epsilon(x) \phi_{ir;sj} = (\tilde{x} \cdot \int t^{(\lambda)}_{ij} t^{(\lambda^\dagger)}_{rs})
\]

\[
= \sum_{x} \sum_{x'} t^{(\lambda)}_{ix'}(x(1)) \tilde{t}^{(\lambda^\dagger)}_{x'r} (x(2)) \phi_{x'r;sj},
\]

i.e. \( \epsilon(x) \Phi[s, j] = \sum_{x} t^{(\lambda)}(x(1)) \Phi[s, j] t^{(\lambda)}(S(x(2))), \quad \forall x \in U_q(\mathfrak{g}). \)

Schur’s lemma forces \( \Phi[s, j] \) to be proportional to the identity matrix, and we have

\[
\Psi[i, r] = \delta_{ir} \psi;
\]

for some \( d_{\lambda} \times d_{\lambda} \) matrix \( \psi \). The right integral property of \( f \) leads to

\[
\epsilon(x) \phi_{ir;sj} = (\int \tilde{x} \cdot \int t^{(\lambda)}_{ij} t^{(\lambda^\dagger)}_{rs})
\]

\[
= \sum_{x} \sum_{x'} \phi_{ir;s'j'} t^{(\lambda)}_{x'j'} (x(1)) \tilde{t}^{(\lambda^\dagger)}_{s'x} (x(2)).
\]

For \( x = S(y) \), the equation is equivalent to

\[
\epsilon(y) \psi = \sum_{(y)} t^{(\lambda)}(K_{2\rho}) t^{(\lambda)}(y(1)) t^{(\lambda)}(K_{2\rho}^{-1}) \psi t^{(\lambda)}(S(y(2))).
\]

Again by using Schur’s lemma we conclude that \( \psi \) is proportional to \( t^{(\lambda)}(K_{2\rho}) \). Since its trace is 1, we have

\[
\psi = \frac{t^{(\lambda)}(K_{2\rho})}{D_q(\lambda)}.
\]

This completes the proof of the first equation of the lemma. The second equation can be shown in exactly the same way.
Given any $a = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j=1}^{d_\lambda} c_{ij}^{(\lambda)} t_{ij}^{(\lambda)}$, we denote by $C^{(\lambda)}$ the matrix with entries $c_{ij}^{(\lambda)}$. Using the lemma we can easily show that

$$\int a^* a = \sum_{\lambda \in \mathcal{P}_+} \text{tr} \{ \tilde{t}^{(\lambda)} (K_{2\rho}) C^{(\lambda)} C^{(\lambda) \dagger} \} / D_\lambda,$$

which is clearly nonnegative, and vanishes only when $a = 0$. We state this as a lemma.

**Lemma 3** The quantum Haar measure of $T_q(\mathfrak{g})$ is positive definite.

Note that the quantum Haar measure gives rise to a positive definite sesquilinear form $(\cdot, \cdot)_h$ for $T_q(\mathfrak{g})$ defined by

$$(a, b)_h = \int a^* b, \quad a, b \in T_q(\mathfrak{g}).$$

## 3 QUANTUM HOMOGENEOUS VECTOR BUNDLES AND INDUCED REPRESENTATIONS

### 3.1 Completions of $T_q$

Existing treatments, in the literature, of quantum homogeneous spaces are largely worked at the algebraic level, that is, without the introduction of topology. See for example [16, 17, 18]. In such cases the algebra of functions, over the quantum homogeneous space, is defined to be a subset of $T_q(\mathfrak{g})$ satisfying certain homogeneity properties with respect to a two-sided co-ideal of $U_q(\mathfrak{g})$. This is comparable to working with polynomials in a classical analysis situation. Thus, while such studies are instructive, it is ultimately unsatisfactory to remain in this purely algebraic setting. To remedy this, we need to complete $T_q(\mathfrak{g})$ in some way.

Let $|| \cdot \cdot ||_h$ be the norm on $T_q(\mathfrak{g})$ determined by

$$||a||_h = \sqrt{(a, a)_h}, \quad a \in T_q(\mathfrak{g}).$$

This equips $T_q(\mathfrak{g})$ with the structure of a pre-Hilbert space. Let us denote by $L^2_q$ the Hilbert space completion of $T_q(\mathfrak{g})$ in this norm. Denote by $\mathcal{B}(L^2_q)$ the bounded linear operators on $L^2_q$. Then the left regular representation of $T_q(\mathfrak{g})$ can be extended to the completion $L^2_q$, yielding a $\ast$-representation $\pi : T_q(\mathfrak{g}) \to \mathcal{B}(L^2_q)$ in the bounded operators $\mathcal{B}(L^2_q)$. To prove this claim, note that for any $c \in T_q(\mathfrak{g})$,

$$(ca, b)_h = (a, c^* b)_h, \quad a, b \in L^2_q,$$

if $|(ca, b)_h| < \infty$. Also observe that relative to a unitary basis (as discussed above in section 2.1), we have $\sum_k (t_{ki}^{(\lambda)})^* t_{ki}^{(\lambda)} = 1$. Thus, for all $a \in L^2_q$,

$$||a||_h^2 = \sum_k (t_{ki}^{(\lambda)})^* t_{ki}^{(\lambda)} (a, a)_h$$

$$= \sum_k (t_{ki}^{(\lambda)} a, t_{ki}^{(\lambda)} a)_h$$

$$\geq ||t_{ji}^{(\lambda)} a||_h^2.$$
Therefore, the $t_j^{(λ)}$ and their finite linear combinations act on $L_q^2$ by bounded linear operators. Let $\| \cdot \|$ be the operator norm on $B(L_q^2)$. For any $a \in T_q(\mathfrak{g})$, $\|π(a)\| < ∞$. Thus, since $\| \cdot \|$ is a $C^*$-norm on $B(L_q^2)$, the pull back of this under $π$ gives a $C^*$-norm $\| \cdot \|_{op}$ on $T_q(\mathfrak{g})$ defined by

$$||a||_{op} = \sup\{||af||_h; \ f \in L_q^2, \ ||f||_h = 1\}. \ (17)$$

Finally, it is an elementary exercise to check that the completion in this norm extends $T_q(\mathfrak{g})$ to a unital $C^*$-algebra $A_q(\mathfrak{g})$. The $C^*$-algebra $A_q(\mathfrak{g})$ qualifies as a compact quantum group of the Woronowicz type [5], with $T_q(\mathfrak{g})$ a dense subalgebra possessing the structure of a Hopf *-algebra.

However, we should note that it is not possible to extend the co-unit and antipode of $T_q(\mathfrak{g})$ to continuous maps from the entire $A_q(\mathfrak{g})$ to appropriate spaces. Furthermore, an extension of the co-multiplication will necessarily maps $A_q(\mathfrak{g})$ continuously to some completion of $A_q(\mathfrak{g}) \otimes A_q(\mathfrak{g})$ instead of the algebraic tensor product itself.

Let us introduce two types of actions of $U_q(\mathfrak{g})$ on $T_q(\mathfrak{g})$. The first action will be denoted by $\circ$, which corresponds to the right translation in the classical theory of Lie groups. It is defined by

$$x \circ f = \sum_{(f)} f_1 (f_2), \ x \in U_q(\mathfrak{g}), \ f \in T_q(\mathfrak{g}), \ (18)$$

Straightforward calculations show that

$$y \circ (x \circ f) = (yx) \circ f;$$
$$x \circ (f(y)) = f(yx),$$
$$(id_{T_q(\mathfrak{g})} \otimes x) \Delta(f) = \Delta(x \circ f).$$

The other action, which corresponds to the left translation in the classical Lie group theory, will be denoted by $\cdot$. It is defined by

$$x \cdot f = \sum_{(f)} f_1 (S^{-1}(x)) f_2, \ (19)$$

It can be easily shown that

$$(x \cdot f)(y) = f(S^{-1}(x)y),$$
$$x \cdot (y \cdot f) = (xy) \cdot f, \ x, y \in U_q(\mathfrak{g}), \ f \in T_q(\mathfrak{g}).$$

Furthermore, the two actions commute in the following sense

$$x \circ (y \cdot f) = y \cdot (x \circ f), \ \forall x, y \in U_q(\mathfrak{g}), \ f \in T_q(\mathfrak{g}).$$
These actions can only be extended to subspaces of $A_q(g)$. Any $f \in A_q(g)$ is the $n \to \infty$ limit of a Cauchy sequence $\{f_n\}$ with respect to the operator norm $|| \cdot ||_{op}$, where each $f_n \in \mathcal{T}_q(g)$. Let $x \in U_q(g)$. We define
\[
x \circ f = \lim_{n \to \infty} x \circ f_n, \quad \text{if } ||x \circ f_n - x \circ f||_{op} \to 0, \quad n \to \infty,
\]
\[
x \cdot f = \lim_{n \to \infty} x \cdot f_n, \quad \text{if } ||x \cdot f_n - x \cdot f||_{op} \to 0, \quad n \to \infty.
\]
Set
\[
\mathcal{E}_q := \{a \in A_q(g)| x \cdot a, x \circ a \in A_q(g), \quad |a(x)| < \infty, \quad \forall x \in U_q(g)\}.
\]
The $\mathcal{E}_q$ clearly forms a subalgebra of $A_q(g)$. In fact, for all $a, b \in \mathcal{E}_q$, we have
\[
x \circ (ab) = \sum_{(x)} \{x(1) \circ a\} \{x(2) \circ b\},
\]
\[
x \cdot (ab) = \sum_{(x)} \{x(1) \cdot a\} \{x(2) \cdot b\}, \quad \forall x \in U_q(g).
\]
We may regard $\mathcal{E}_q$ as the quantum analog of the algebra of smooth functions over the group.

### 3.2 Quantum homogeneous spaces and quantum homogeneous vector bundles

Let us now turn to the study of quantum homogeneous spaces. As we will see shortly, the well known fact in classical complex geometry, that any complex analytic function on a compact complex manifold is a constant, also holds in the analogous quantum setting. Therefore, in the first instance, we must work in a category of functions that has a richer family of sections. This family should contain enough information to capture the underlying geometrical aspects of the compact quantum homogeneous spaces. On the other hand we want the class of functions (and ‘bundle sections’) to be closed under operations which generalize classical differentiation. It is natural then to look for the quantum analogs of algebras of smooth functions. As in the classical case (see section [5.2] of the appendix and in particular proposition [7]) this is most easily achieved by working in the ‘real setting’. Thus we consider the compact real form of $U_q(g)$, and regard $\mathcal{T}_q(g)$ as a subset of the complexification of $(U^R_q(g_0))^*$.

Let us introduce the following definition
\[
\mathcal{E}_q^{\mathfrak{t}} := \{f \in \mathcal{E}_q | x \circ f = \epsilon(x)f, \quad \forall x \in U^R_q(\mathfrak{t})\}.
\]
Note that we may replace $U^R_q(\mathfrak{t})$ by $U_q(\mathfrak{t}) = \mathbb{C} \otimes_\mathbb{R} U^R_q(\mathfrak{t})$ in the above equation without altering $\mathcal{E}_q^{\mathfrak{t}}$. To investigate properties of $\mathcal{E}_q^{\mathfrak{t}}$ we consider the action of $\mathcal{C}_q(\mathfrak{t})$ on it. Recall that $\mathcal{C}_q(\mathfrak{t})$ generates the real subalgebra $U^R_q(\mathfrak{t})$ of $U^R_q(g_0)$. Also, it is a two-sided co-ideal of $U_q(g)$ and satisfies $\epsilon(\mathcal{C}_q(\mathfrak{t})) = 0$. For any $a, b \in \mathcal{E}_q^{\mathfrak{t}}$, and $x \in \mathcal{C}_q(\mathfrak{t})$, we have
\[
x \circ (ab) = \sum_{(x)} \{x(1) \circ a\} \{x(2) \circ b\} = 0.
\]
Therefore $ab \in \mathcal{E}_q^{\mathfrak{t}}$, that is,
\( \mathcal{E}_q^f \) is a subalgebra of \( \mathcal{E}_q \).

We will show below that this non-commutative algebra is infinite dimensional. We may regard it as the quantum analog of the algebra of smooth functions on the homogeneous space \( G_{\mathbb{C}}/P \), and will refer to it as the algebra of functions on a (virtual) quantum homogeneous space. It is worth pointing out that in \([18]\), more general quantum homogeneous spaces were considered, the definition of which was similar to \((21)\), but with \( U_q(\mathfrak{t}) \) replaced by a two-sided co-ideal of \( U_q(\mathfrak{g}) \), which vanished under the co-unit and was \( \theta \) invariant. However, the definition presented here is more suitable for the purpose of developing the representation theory of quantum groups.

Let \( V \) be a finite dimensional module over \( U_q(\mathfrak{l}) \), which we will also regard as a \( U_R q(\mathfrak{k}) \)-module by restriction. We extend the actions \( \circ \) and \( \cdot \) of \( U_q(\mathfrak{g}) \) on \( E_q \) trivially to actions on \( E_q \otimes V \): for any \( \zeta = \sum_r f_r \otimes v_r \in E_q \otimes V \)

\[
x \circ \zeta = \sum_r x \circ f_r \otimes v_r, \quad x \cdot \zeta = \sum_r x \cdot f_r \otimes v_r, \quad x \in U_q(\mathfrak{g}).
\]

We now introduce another definition, which will be of considerable importance for the remainder of the paper:

\[
\mathcal{E}_q^f(V) := \{ \zeta \in E_q \otimes V \mid x \circ \zeta = (id_{A_q(\mathfrak{g})} \otimes S(x))\zeta, \forall x \in U_q^R(\mathfrak{g}) \}.
\]

Note that every \( \zeta \in \mathcal{E}_q^f(V) \) satisfies

\[
x \circ \zeta = (id_{A_q(\mathfrak{g})} \otimes S(x))\zeta, \quad \forall x \in U_q(\mathfrak{l}).
\]

Consider the subspace

\[
\mathcal{F}_q(V) := \{ \mathcal{T}_q(\mathfrak{g}) \otimes V \} \cap \mathcal{E}_q^f(V)
\]

of \( \mathcal{E}_q^f(V) \). Since the finite dimensional representations of \( U_q(\mathfrak{l}) \) are completely reducible, the study of its properties reduces to the case when \( V \) is irreducible. Let \( V_\mu \) be a finite dimensional irreducible \( U_q(\mathfrak{l}) \)-module with highest weight \( \mu \) and lowest weight \( \tilde{\mu} \). Any element \( \zeta \in \mathcal{F}_q(V_\mu) \) can be expressed in the form

\[
\zeta = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(t_{ji}^{(\lambda)}) \otimes v_{ij}^{(\lambda)},
\]

for some \( v_{ij}^{(\lambda)} \in V_\mu \). Fix an arbitrary \( \lambda \in \mathcal{P}_+ \). For any nonvanishing \( w \in W(\lambda) \), the following linear map is clearly surjective:

\[
\text{Hom}_\mathbb{C}(W(\lambda), V_\mu) \otimes w \rightarrow V_\mu,
\]

\[
\phi \otimes w \mapsto \phi(w).
\]

Thus there exist \( \phi_i^{(\lambda)} \in \text{Hom}_\mathbb{C}(W(\lambda), V_\mu) \) such that \( v_{ij}^{(\lambda)} = \phi_i^{(\lambda)}(w_j^{(\lambda)}) \), where \( \{w_i^{(\lambda)}\} \) is the basis of \( W(\lambda) \), relative to which the irreducible representation \( t^{(\lambda)} \) of \( U_q(\mathfrak{g}) \) is defined. Now we can rewrite \( \zeta \) as

\[
\zeta = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(t_{ji}^{(\lambda)}) \otimes \phi_i^{(\lambda)}(w_j^{(\lambda)}).
\]
The defining property of $\mathcal{F}_q(V_\mu)$ states that
\[
\ell \circ \zeta = (id_{\mathcal{T}_q(\mathfrak{g})} \otimes S(\ell)) \zeta, \quad \forall \ell \in U_q(\mathfrak{l}).
\]
Thus we have
\[
\sum_{\lambda \in \mathcal{P}_+} \sum_{i,j,k} S(t_{ki}^{(\lambda)}) \otimes t_{jki}^{(\lambda)}(S(\ell)) \phi_i^{(\lambda)}(w_j^{(\lambda)}) = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(t_{ji}^{(\lambda)}) \otimes S(\ell) \phi_i^{(\lambda)}(w_j^{(\lambda)}).
\]
Recalling that the $t_{ki}^{(\lambda)}$ are linearly independent. It follows easily that the $S(t_{ki}^{(\lambda)})$ also form a linearly independent set. So the above is equivalent to
\[
\sum_j t_{jki}^{(\lambda)}(\ell) \phi_i^{(\lambda)}(w_j^{(\lambda)}) = \ell \phi_i^{(\lambda)}(w_j^{(\lambda)}), \quad \forall \ell \in U_q(\mathfrak{l}).
\]
This equation is precisely the statement that the $\phi_i^{(\lambda)}$ be $U_q(\mathfrak{l})$-module homomorphisms,
\[
\phi_i^{(\lambda)} \in \text{Hom}_{U_q(\mathfrak{l})}(W(\lambda), V_\mu) \subset \text{Hom}_\mathbb{C}(W(\lambda), V_\mu), \quad \forall i.
\]
Thus finding sections in $\mathcal{F}_q(V_\mu)$ is equivalent to finding, for all $\lambda \in \mathcal{P}_+$, the homomorphisms $\phi^{(\lambda)} \in \text{Hom}_{U_q(\mathfrak{l})}(W(\lambda), V_\mu)$. Note that each such homomorphism $\phi^{(\lambda)}$ determines $d_\lambda$ linearly independent sections
\[
\zeta_i^{(\lambda)} = \sum_j S(t_{ji}^{(\lambda)}) \otimes \phi^{(\lambda)}(w_j^{(\lambda)}).
\]

Toward constructing such homomorphisms we consider a couple of useful observations. Note that if $W_1 \rightarrow V_1$ and $W_2 \rightarrow V_2$ are each $U_q(\mathfrak{l})$-homomorphism then these induce a $U_q(\mathfrak{l})$-homomorphism on the tensor product in the obvious manner
\[
W_1 \otimes W_2 \rightarrow V_1 \otimes V_2.
\]
Now let $W(\lambda_1)$ and $W(\lambda_2)$ be irreducible $U_q(\mathfrak{g})$-modules of respective highest weights $\lambda_1$ and $\lambda_2$. Let $V_{\mu_1}$ and $V_{\mu_2}$ be irreducible $U_q(\mathfrak{l})$-modules of the highest weights indicated. Then by explicit construction of maximal weights one easily establishes the following:

**Lemma 4** Suppose there are non-trivial $U_q(\mathfrak{l})$-homomorphisms $W(\lambda_1) \rightarrow V_{\mu_1}$ and $W(\lambda_2) \rightarrow V_{\mu_2}$. Then there is an induced non-trivial $U_q(\mathfrak{l})$-homomorphism
\[
W(\lambda_1 + \lambda_2) \rightarrow V_{\mu_1 + \mu_2}.
\]

Let us consider the case $\mu = 0$, then $\mathcal{F}_q(V_{\mu=0}) = \mathcal{T}_q(\mathfrak{g}) \cap \mathcal{E}_q^\mathfrak{l}$. We will show that this has an infinite dimensional vector space of sections. Of course there is a homomorphism from the trivial representation of $U_q(\mathfrak{g})$, $W(0) = \mathbb{C}$, onto $V_0 = \mathbb{C}$. This gives the constant sections of $\mathcal{T}_q(\mathfrak{g}) \cap \mathcal{E}_q^\mathfrak{l}$. Let $\gamma$ be the highest root of $\mathfrak{g}$. Recall that $U_q(\mathfrak{l})$ is reductive and there are $N = r - |\Theta|$ independent central elements in $U_q(\mathfrak{l})$. Thus there are $r$ many linearly independent $U_q(\mathfrak{l})$-homomorphisms $W(\gamma) \rightarrow \mathbb{C}$. As mentioned above each of these corresponds to $d = \dim(\mathfrak{g})$ linearly independent sections. So the representation $W(\gamma)$ determines $Nd$ linearly independent sections. Further linearly independent sections may be obtained using lemma [4]. For example there are $(m|N)$ (partition of $m$ into $\leq N$ parts) linearly independent homomorphisms $W(m\gamma) \rightarrow \mathbb{C}$. It is easily verified that the $d(m|N)$ sections so obtained are precisely the sections obtained by taking $m$-fold products of the $d$ sections arising from the homomorphisms $W(\gamma) \rightarrow V_\mu$. We have proved the following lemma.
Lemma 5 The algebra $E_k^q$ is infinite dimensional.

Now let us consider the case $\mu \neq 0$. It is an elementary exercise to verify that $V_\mu$ is $U_q(l)$-isomorphic to a $U_q(l)$-irreducible part of $W(\lambda')$, where $\lambda'$ is the dominant weight in the Weyl group orbit of $\mu$. Thus there is a non-trivial $U_q(l)$-homomorphism

$$W(\lambda') \to V_\mu,$$

and this determines at least $d_{\lambda'}$ linearly independent sections in $F_q(V_\mu)$.

Further linearly independent sections are obtained by left and right multiplying with sections of $F_q$. As above, the results of such products may alternatively be constructed explicitly using lemma 4 which promises a family of homomorphisms

$$W(\lambda' + m\gamma) \to V_\mu \quad m \in \mathbb{N}_+. $$

Although we have fallen short of a classification of the sections in $E_k^q(V_\mu)$ we have established that $E_k^q(V_\mu)$ is infinite dimensional. This immediately leads to the following result.

**Proposition 2** If the weight of any vector of $V$ is $U_q(g)$-integral, then $E_k^q(V)$ is an infinite dimensional vector space.

$E_k^q(V)$ provides a good candidate for the space of sections of a quantum vector bundle over the quantum homogeneous space corresponding to $E_k^q$. We will discuss this further in the next section. Here we establish the following results.

**Theorem 2** $E_k^q(V)$ furnishes

i). a two-sided $E_k^q$ module under the multiplication of $A_q(g)$;

ii). a left $U_q(g)$-module under $\cdot$; and

iii). $F_q(V)$ forms a left $A_q(g)$ co-module under the co-action $\omega = (\Delta \otimes id_V)$.

**Proof:** Consider arbitrary elements $a \in E_q^t$, $x \in U_q(g)$, $p \in U_q^R(t)$, and $\zeta = \sum_r f_r \otimes v_r \in E_q^t(V)$. The left and right actions of $E_q^t$ on $E_q^t(V)$ are respectively defined by

$$a\zeta = \sum_r af_r \otimes v_r, \quad \zeta a = \sum_r f_r \otimes v_r.$$

Now

$$\begin{align*}
p \circ (a\zeta) &= \sum_{\{p\}} \{p(1) \circ a\}\{p(2) \circ \zeta\} \\
&= \sum_{\{p\}} \epsilon(p(1)) a \{p(2) \circ \zeta\} \\
&= a \{p \circ \zeta\} = (id_{A_q(g)} \otimes S(p)) a\zeta; \\
p \circ (\zeta a) &= \sum_{\{p\}} \{p(1) \circ \zeta\}\{p(2) \circ a\} \\
&= \{p \circ \zeta\} a = (id_{A_q(g)} \otimes S(p)) \zeta a.
\end{align*}$$

This completes the proof of part i). Part ii) follows from

$$\begin{align*}
p \circ (x \cdot \zeta) &= x \cdot (p \circ \zeta) \\
&= (id_{A_q(g)} \otimes S(p))(x \cdot \zeta),
\end{align*}$$
while part iii) is confirmed by

\[(id_{A_q(g)} \otimes p \circ) \omega(\zeta) = (id_{A_q(g)} \otimes p \circ)(\Delta \otimes id_V)\zeta = (\Delta \otimes id_V)(p \circ \zeta) = \omega(id_{A_q(g)} \otimes S(p))\zeta.\]

Note that the left \(U_q(g)\) action of ii) and left \(A_q(g)\) co-action \(\omega\) on \(\mathcal{F}_q(V)\) are closely related. Define a permutation map

\[P_{123} : A_q(g) \otimes A_q(g) \otimes V \rightarrow A_q(g) \otimes V \otimes A_q(g)\]

\[f_1 \otimes f_2 \otimes v \mapsto f_2 \otimes v \otimes f_1. \tag{22}\]

Then \(\tilde{\omega} = P_{123} \omega\) defines a right \(A_q(g)\) co-action on \(\mathcal{F}_q(V)\) which is dual to the left \(U_q(g)\) action. We call \(E^t_q(V)\) an induced \(U_q(g)\) module, and also call \(\mathcal{F}_q(V)\) an induced \(A_q(g)\) co-module.

### 3.3 Projectivity

In classical differential geometry, the space \(\mathcal{H}\) of sections of a vector bundle over a compact manifold \(M\) furnishes a module over the algebra \(\mathcal{A}(M)\) of functions. It then follows from Swann’s theorem that this module must be projective and is of finite type, namely, there exists another \(\mathcal{A}(M)\)-module \(\mathcal{H}'\) such that \(\mathcal{H} \oplus \mathcal{H}'\) is a finitely generated free module, \(\mathcal{A}(M) \oplus \ldots \oplus \mathcal{A}(M)\). (See also theorem 3 of the appendix.) Conversely, any projective module of finite type over \(\mathcal{A}(M)\) is isomorphic to the sections of some vector bundle over \(M\). This result is taken as the starting point for studying vector bundles in noncommutative geometry: one defines a vector bundle over a noncommutative space in terms of the space of sections which is required to be a finite type projective module over a noncommutative algebra which is taken to be the algebra of functions on the virtual noncommutative space.

Let us assume that all the weights of \(V\) are integral, i.e., belonging to \(\mathcal{P}\). In this case, \(E^t_q(V)\) will be called the space of sections of a quantum vector bundle over the quantum homogeneous space associated with \(E^t_q\). To justify this terminology, we need to show that \(E^t_q(V)\) is a projective module over \(E_q\). Let us first prove the following proposition.

**Proposition 3** Let \(W\) be a finite dimensional left \(U_q(g)\)-module, which we regard as a left \(U^\mathcal{P}_q(\mathcal{E})\)-module by restriction. Then \(E^t_q(W)\) is isomorphic to \(E^t_q \otimes W\) either as a left or right \(E^t_q\)-module.

**Proof:** We first construct the right \(E^t_q\)-module isomorphism. Being a left \(U_q(g)\)-module, \(W\) carries a natural right \(A_q(g)\) co-module structure with the co-module action \(\delta : W \rightarrow W \otimes T_q(g) \subset W \otimes A_q(g)\) defined for any element \(w \in W\) by

\[\delta(w)(x) = xw, \quad \forall x \in U_q(g). \tag{23}\]

Define the map \(\eta : E_q \otimes W \rightarrow E_q \otimes W\) by the composition of the maps

\[
E_q \otimes W \xrightarrow{id \otimes \delta} E_q \otimes W \otimes T_q(g) \xrightarrow{P_{123}^{-1}} T_q(g) \otimes E_q \otimes W \rightarrow E_q \otimes W,
\]
where the last map is the multiplication of $\mathcal{A}_q(\mathfrak{g})$, and $P_{123}$ is the permutation map defined by (22). Then $\eta$ defines a right $\mathcal{E}_q^t$-module isomorphism, with the inverse map given by the composition

$$\mathcal{E}_q \otimes W \xrightarrow{id \otimes \delta} \mathcal{E}_q \otimes W \otimes T_q(\mathfrak{g}) \xrightarrow{(S \otimes id \otimes id)P_{123}} T_q(\mathfrak{g}) \otimes \mathcal{E}_q \otimes W \xrightarrow{} \mathcal{E}_q \otimes W,$$

where the last map is again the multiplication of $\mathcal{A}_q(\mathfrak{g})$. It is not difficult to show that

$$x \circ \eta(\zeta) = \sum_{(x)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes x(1)) \eta(x(2) \circ \zeta),$$

$$x \circ \eta^{-1}(\zeta) = \sum_{(x)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes S(x(1))) \eta^{-1}(x(2) \circ \zeta), \quad \forall \zeta \in \mathcal{E}_q \otimes W, \ x \in U_q(\mathfrak{g}).$$

Consider $\zeta \in \mathcal{E}_q^t(W)$. We have

$$p \circ \eta(\zeta) = \sum_{(p)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes p(1)) \eta(p(2) \circ \zeta)$$

$$= \sum_{(p)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes p(1) S(p(2))) \eta(\zeta)$$

$$= \epsilon(p) \eta(\zeta), \quad \forall \ z \in U_q(\mathfrak{g}).$$

Hence $\eta(\mathcal{E}_q^t(W)) \subset \mathcal{E}_q^t \otimes W$. Conversely, given any $\xi \in \mathcal{E}_q^t \otimes W$, we have

$$p \circ \eta^{-1}(\xi) = \sum_{(p)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes S(p(1))) \eta^{-1}(p(2) \circ \xi)$$

$$= \sum_{(p)} (id_{\mathcal{A}_q(\mathfrak{g})} \otimes \epsilon(p(2)) S(p(1))) \eta^{-1}(\xi)$$

$$= (id_{\mathcal{A}_q(\mathfrak{g})} \otimes S(p)) \eta^{-1}(\xi), \quad \forall \ z \in U_q(\mathfrak{g}).$$

Thus $\eta^{-1}(\mathcal{E}_q^t \otimes W) \subset \mathcal{E}_q(W)$. Therefore the restriction of $\eta$ provides the desired right $\mathcal{E}_q^t$-module isomorphism.

The left module isomorphism is given by the restriction of $\kappa : \mathcal{E}_q \otimes W \rightarrow \mathcal{E}_q \otimes W$ defined by the composition of the following maps

$$\mathcal{E}_q \otimes W \xrightarrow{id \otimes \delta} \mathcal{E}_q \otimes W \otimes T_q(\mathfrak{g}) \xrightarrow{id \otimes (S^2 \otimes id) P} \mathcal{E}_q \otimes T_q(\mathfrak{g}) \otimes W \xrightarrow{} \mathcal{E}_q \otimes W,$$

where

$$P : W \otimes \mathcal{A}_q(\mathfrak{g}) \rightarrow \mathcal{A}_q(\mathfrak{g}) \otimes W,$$

$$w \otimes f \mapsto f \otimes w. \quad (24)$$

The inverse map $\kappa^{-1}$ is given by

$$\mathcal{E}_q \otimes W \xrightarrow{id \otimes \delta} \mathcal{E}_q \otimes W \otimes T_q(\mathfrak{g}) \xrightarrow{id \otimes (S^2 \otimes id) P} \mathcal{E}_q \otimes T_q(\mathfrak{g}) \otimes W \xrightarrow{} \mathcal{E}_q \otimes W.$$

Let $V_\mu$ be a finite dimensional irreducible $U_q(\mathfrak{g})$-module with highest weight $\mu$, which is integral with respect to $\mathfrak{g}$. Then $V_\mu$ can always be embedded into an irreducible $U_q(\mathfrak{g})$-module $W(\sigma(\mu))$ with a $\mathfrak{g}$ integral dominant highest weight $\sigma(\mu)$, where $\sigma$ is
some element of the Weyl group \( W \) of \( g \). Such a \( \sigma \) always exists, and belongs to the subgroup \( \mathcal{W}^l \subset W \), which leaves invariant the set of the positive roots of \( l \). Since \( U_q(l) \) is a reductive subalgebra of \( U_q(g) \), all finite dimensional representations of \( U_q(l) \) are completely reducible. Hence, \( W(\sigma(\mu)) \) can be decomposed into a direct sum of \( U_q(l) \)-modules: \( W(\sigma(\mu)) = V_\mu \oplus V^\perp_\mu \). Using the complete reducibility of finite dimensional \( U_q(l) \)-modules again, we conclude that if the weights of the finite dimensional \( U_q(l) \)-module \( V \) are all integral with respect to \( U_q(g) \), then there exist another \( U_q(l) \) module \( V^\perp \) and a finite dimensional \( U_q(g) \) module \( W \) such that

\[
    V \oplus V^\perp = W.
\]

It then immediately follows Proposition 3 that

\[
    \mathcal{E}_q^f(V) \oplus \mathcal{E}_q^f(V^\perp) = \mathcal{E}_q^f \otimes W;
\]

that is,

**Theorem 3** \( \mathcal{E}_q^f(V) \) is projective and of finite type both as a left and right module over the algebra \( \mathcal{E}_q^f \) of functions on the quantum homogeneous space.

### 3.4 Quantum Frobenius reciprocity

We have the following quantum analog of Frobenius reciprocity.

**Proposition 4** Let \( W \) be a \( U_q(g) \) module, the restriction of which furnishes a \( U_q^\mathbb{R}(\mathfrak{t}) \) module in a natural way. Then there exists a canonical isomorphism

\[
    Hom_{U_q(g)}(W, \mathcal{E}_q^f(V)) \cong Hom_{U_q^\mathbb{R}(\mathfrak{t})}(W, V),
\]

where \( U_q(g) \) acts on the left module \( \mathcal{E}_q^f(V) \) via the \( \cdot \) action.

**Proof:** We prove the Proposition by explicitly constructing the isomorphism, which we claim to be the linear map

\[
    F : Hom_{U_q(g)}(W, \mathcal{E}_q^f(V)) \to Hom_{U_q^\mathbb{R}(\mathfrak{t})}(W, V),
\]

\[
    \psi \mapsto \psi(1_{U_q(g)}),
\]

with the inverse map

\[
    \bar{F} : Hom_{U_q^\mathbb{R}(\mathfrak{t})}(W, V) \to Hom_{U_q(g)}(W, \mathcal{E}_q^f(V)),
\]

\[
    \phi \mapsto \tilde{\phi} = (S \otimes \phi)P\delta,
\]

where \( \delta : W \to W \otimes T_q(g) \subset W \otimes A_q(g) \) is the right \( A_q(g) \) co-module action defined by (23), and \( P \) is the permutation map (24).

As for \( F \), we need to show that its image is contained in \( Hom_{U_q^\mathbb{R}(\mathfrak{t})}(W, V) \). Consider \( \psi \in Hom_{U_q(g)}(W, \mathcal{E}_q^f(V)) \). For any \( p \in U_q^\mathbb{R}(\mathfrak{t}) \) and \( w \in W \), we have

\[
    p(F\psi(w)) = ((id_{A_q(g)} \otimes p)\psi(w))(1_{U_q(g)}) = (S^{-1}(p) \circ \psi(w))(1_{U_q(g)}),
\]
where we have used the defining property of $\mathcal{E}_q^f(V)$. Note that

$$(S^{-1}(p) \circ \psi(w))(\mathbb{1}_{U_q(g)}) = (p \cdot \psi(w))(\mathbb{1}_{U_q(g)}) .$$

The $U_q(g)$-module structure of $\mathcal{E}_q^f(V)$ and the given condition that $\psi$ is a $U_q(g)$-module homomorphism immediately leads to

$$p(F\psi(w)) = \psi(pw)(\mathbb{1}_{U_q(g)}) = F\psi(pw), \quad p \in U_q^{\mathbb{R}}(\mathfrak{f}), \ w \in W.$$  

In order to show that $\bar{F}$ is the inverse of $F$, we first need to demonstrate that the image $Im(\bar{F})$ of $\bar{F}$ is contained in $Hom_{U_q(g)}(W, \mathcal{E}_q^f(V))$. Note that $Im(\bar{F}) \subset Hom_{\mathbb{C}}(W, \mathcal{T}_q(g) \otimes V)$. Some relatively simple manipulations lead to

$$(x \cdot \bar{\phi}(w)) = \bar{\phi}(xw),$$
$$ (p \circ \bar{\phi}(w)) = (id_{A_q(q)} \otimes S(p))\bar{\phi}(w), \quad \forall x \in U_q(g), \ p \in U_q^{\mathbb{R}}(\mathfrak{f}), \ w \in W.$$  

Therefore, $Im(\bar{F}) \subset Hom_{U_q(g)}(W, \mathcal{E}_q^f(V))$. Now we show that $F$ and $\bar{F}$ are inverse to each other. For $\psi \in Hom_{U_q(g)}(W, \mathcal{E}_q^f(V))$, and $\phi \in Hom_{U_q^R(g)}(W, V)$, we have

$$ (F\bar{F}\phi)(w) = (\bar{F}\phi)(w)(1_{U_q(g)}) = \phi(w),$$
$$ (\bar{F}F\psi)(w)(x) = (F\psi)(S(x)w) = \psi(S(x)w)(1_{U_q(g)}) = (S(x)\cdot \psi(w))(1_{U_q(g)}) = \psi(w)(x), \quad x \in U_q(g), \ w \in W.$$  

This completes the proof of the Proposition.

4 QUANTUM BOREL-WEIL THEOREM

Let $V_\mu$ be a finite dimensional irreducible $U_q(p)$-module with highest weight $\mu$ and lowest weight $\bar{\mu}$. Recall that any two norms on finite dimensional vector spaces determine the same topology. Thus we may speak of convergence of a sequence in such a space without reference to a particular norm. Let us observe here that there is a similar freedom for a certain class of norms on $\mathcal{E}_q \otimes_{\mathbb{C}} V_\mu$. To each basis $\{v_r\}$ of $V_\mu$ we may define a norm on $\mathcal{E}_q \otimes_{\mathbb{C}} V_\mu$ by

$$\zeta = \sum_r f_r \otimes v_r, \quad ||\zeta||^2 = \sum_r ||f_r||^2_{op}.$$  

It is easily verified that convergence in the norm corresponding to one basis for $V_\mu$ implies convergence in all other norms defined this way. Thus, given $V_\mu$, we simply fix a basis and define $|| \cdot ||$ to be the norm relative to that basis.

Recall the action of $U_q(g)$ on $\mathcal{E}_q \otimes V_\mu$. Since $V_\mu$ is a $U_q(p)$-module the following is a well defined subspace of $\mathcal{E}_q \otimes_{\mathbb{C}} V_\mu$,

$$\mathcal{O}_q(V_\mu) := \{ \zeta \in \mathcal{E}_q \otimes V_\mu | p \circ \zeta = (id_{A_q(g)} \otimes S(p))\zeta, \ \forall p \in U_q(p) \} .$$  

This may be regarded as the quantum analog of the space of holomorphic sections. Recall we use the notation $W(\lambda)$ to denote the irreducible $U_q(g)$ module with highest weight $\lambda$. We have the following result.
Theorem 4  There exists the following $U_q(\mathfrak{g})$ module isomorphism

$$O_q(V_\mu) \cong \begin{cases} W((-\bar{\mu})\dagger), & -\bar{\mu} \in \mathcal{P}_+, \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

Proof of Theorem 4  Let $\zeta \in O_q(V_\mu)$. Let $\{\zeta_n\}$ be a sequence in $\mathcal{T}_q(\mathfrak{g}) \otimes V_\mu$ such that $\zeta_n \to \zeta$ in the norm $\| \cdot \|$ described above. Each $\zeta_n$ can be expressed in the form

$$\zeta_n = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(t_{ij}^{(\lambda)}) \otimes v_{i,j}^{(\lambda),n},$$

for some $v_{i,j}^{(\lambda),n} \in V_\mu$ ($i, j = 1, \ldots, d_\lambda$). Arguing as in the proof of proposition 2 (and lemma 3) one concludes, for each $\lambda \in \mathcal{P}_+$, that there exist $\phi_i^{(\lambda),n} \in \text{Hom}_\mathbb{C}(W(\lambda), V_\mu)$ such that $v_{i,j}^{(\lambda),n} = \phi_i^{(\lambda),n}(w_{j}^{(\lambda)})$, where $\{w_{i}^{(\lambda)}\}$ is the basis of $W(\lambda)$, relative to which the irreducible representation $t^{(\lambda)}$ of $U_q(\mathfrak{g})$ is defined. Now we can rewrite $\zeta_n$ as

$$\zeta_n = \sum_{\lambda \in \mathcal{P}_+} \sum_{i,j} S(t_{ji}^{(\lambda)}) \otimes \phi_i^{(\lambda),n}(w_{j}^{(\lambda)}).$$

It is clear from this that $\zeta$ is determined by the sequences of linear homomorphisms $\phi_i^{(\lambda),n}$. Note that

$$\|\zeta_{n+m} - \zeta_n\| \to 0, \quad n \to \infty.$$ 

Since the $S(t_{ji}^{(\lambda)})$ are linearly independent, this implies that for each $\lambda \in \mathcal{P}_+$ and $i, j \in \{1, 2, \ldots, d_\lambda\}$, $\phi_i^{(\lambda),n}(w_{j}^{(\lambda)})$ is a Cauchy sequence in $V_\mu$. But since $\text{Hom}_\mathbb{C}(W(\lambda), V_\mu)$ is a finite dimensional complex vector space with the basis $\{v_r \otimes \bar{w}_j^{(\lambda)}\}$, it is clear that this further implies that, for each $\lambda \in \mathcal{P}_+$ and $i \in \{1, 2, \ldots, d_\lambda\}$, $\phi_i^{(\lambda),n}$ is a Cauchy sequence in $\text{Hom}_\mathbb{C}(W(\lambda), V_\mu)$ and so

$$\lim_{n \to \infty} \phi_i^{(\lambda),n} = \phi_i^{(\lambda)} \in \text{Hom}_\mathbb{C}(W(\lambda), V_\mu).$$

Now we will now show that this limit $\phi_i^{(\lambda)}$ must in fact be a $U_q(\mathfrak{p})$-module homomorphism. The defining property of $O_q(V_\mu)$ states that

$$p \circ \zeta = (\text{id}_{A_q(\mathfrak{g})} \otimes S(p))\zeta, \quad \forall p \in U_q(\mathfrak{p}).$$

Thus, for each $p$,

$$\|p \circ \zeta_n - (\text{id}_{A_q(\mathfrak{g})} \otimes S(p))\zeta_n\| \to 0.$$

Again using the linear independence of the $S(t_{ji}^{(\lambda)})$’s, we see that this implies that, for each $i, k \in \{1, \ldots, d_\lambda\}$,

$$\sum_j t_{jk}^{(\lambda)}(S(p))\phi_i^{(\lambda),n}(w_j^{(\lambda)}) - S(p)\phi_i^{(\lambda),n}(w_k^{(\lambda)})$$

is a null sequence. Thus in the limit we have

$$\phi_i^{(\lambda)}(pw_j^{(\lambda)}) = p\phi_i^{(\lambda)}(w_j^{(\lambda)}), \quad \forall p \in U_q(\mathfrak{p}).$$
This is precisely the statement that the \( \phi_i^{(\lambda)} \) are \( U_q(p) \)-module homomorphisms,

\[
\phi_i^{(\lambda)} \in \text{Hom}_{U_q(p)}(W(\lambda), V_\mu) \subset \text{Hom}_\mathbb{C}(W(\lambda), V_\mu), \quad \forall i \in 1, \cdots, d_\lambda.
\]

It immediately follows from (7) that

\[
\phi_i^{(\lambda)} = c_i \phi^{(\lambda)}, \quad c_i \in \mathbb{C},
\]

and \( \phi^{(\lambda)} \) may be nonzero only when

\[
\bar{\lambda} = \bar{\mu}.
\]

Hence, if \( -\bar{\mu} \not\in \mathcal{P}_+ \), we have \( \mathcal{O}_q(V_\mu) = 0 \). When \( -\bar{\mu} \in \mathcal{P}_+ \), we set

\[
\nu = (-\bar{\mu})^\dagger.
\]

Then, we may conclude that \( \mathcal{O}_q(V_\mu) \) is spanned by

\[
\zeta_i = \sum_j S(t_{ji}^{(\nu)}) \otimes \phi^{(\nu)}(w_{ij}^{(\nu)}), \quad (27)
\]

which are obviously linearly independent. Furthermore,

\[
x \cdot \zeta_i = \sum_j t_{ji}^{(\nu)}(x) \zeta_j, \quad x \in U_q(\mathfrak{g}).
\]

Thus \( \mathcal{O}_q(V_\mu) \cong W(\nu) \), and this completes the proof of the theorem.

We wish to point out that a possible quantum Borel-Weil theorem for quantum \( GL(n) \) in an algebraic setting (without topology) without the framework of quantum homogeneous vector bundles was elucidated to by Parshall and Wang [20] and Noumi et al [21]. Also in [22], a quantum Borel-Weil theorem for the covariant and contravariant tensor representations of quantum \( GL(m|n) \) was obtained along a similar line as that adopted here but in an algebraic setting. We should also mention that coherent states of compact quantum groups were investigated in [23] from a representation theoretical viewpoint. The results reported in that reference acquire a natural interpretation within the framework of quantum homogeneous vector bundles.

There are several immediate corollaries of the quantum Borel-Weil Theorem 4, which are of considerable interest. First we note that the proof explicitly constructed the isomorphism of the theorem, i.e., (27).

**Corollary 1:** If \( \nu^\dagger = -\bar{\mu} \in \mathcal{P}_+ \), then the following composition of maps defines the \( U_q(\mathfrak{g}) \) module isomorphism \( W(\nu) \cong \mathcal{O}_q(V_\mu) \),

\[
W(\nu) \xrightarrow{(S \otimes \text{id}) P^s} \mathcal{O}_q(W(\nu)) \xrightarrow{id \otimes \phi^{(\nu)}} \mathcal{O}_q(V_\mu), \quad (28)
\]

where \( \phi^{(\nu)} \) is the projection \( W(\nu) \rightarrow V_\mu \).

Recall that in classical geometry, any analytic function on \( G_\mathbb{C}/P \) is constant, as the homogeneous space is a compact complex manifold. A similar result holds in the
quantum homogeneous space setting.

**Corollary 2:**

\[ \mathcal{O}_q(\mathbb{C}) = \mathbb{C} \epsilon. \]

**Proof:** This immediately follows from the \( \mu = 0 \) case of the theorem.

Combining the Corollaries with Proposition 3, we obtain

**Corollary 3:** Let \( W \) be any finite dimensional \( U_q(\mathfrak{g}) \)-module. Then, as \( U_q(\mathfrak{g}) \)-modules,

\[ \mathcal{O}_q(W) \cong \epsilon \otimes W. \]

5 **APPENDIX: THE CLASSICAL CASE**

5.1 **Some generalities on homogeneous structures**

Let \( G \) be a (real or complex) Lie group and \( H \) any subgroup. Corresponding to each representation \( \rho \) of \( H \) on a vector space \( V \) one obtains a homogeneous bundle \( \mathcal{V} = (G \times_H V \to G/H) \) the total space of which is \( G \times_H V \), that is \( G \times V \) factored by the equivalence relation

\[(g, v) = (gh, \rho(h^{-1})v).\]

Sections of \( \mathcal{V} \) are functions

\[ f : G \to V \]

satisfying the homogeneity condition

\[ f(gh) = \rho(h^{-1})f(g). \]

We will use the notation \( \mathcal{E}\mathcal{V} \), or \( \mathcal{E}(\mathcal{V}) \), to mean the sheaf of germs of smooth sections of \( \mathcal{V} \). By a slight abuse of notation we will also use this notation to mean simply local sections.

The space of global smooth sections \( \Gamma \mathcal{E}\mathcal{V} \) is a \( G \)-representation under the action of *left translation*, given by

\[ f \mapsto g \cdot f \ \text{ for } \ g \in G \ \text{ and } \ f \in \Gamma \mathcal{E}\mathcal{V}, \]

where \( g \cdot f \in \Gamma \mathcal{E}\mathcal{V} \) is the left translated section,

\[ g \cdot f(g') = f(g^{-1}g') \ \text{ for all } \ g' \in G. \]

If \( W \) carries a representation \( \mu \) of \( G \) then the homogeneous bundle

\[ \mathcal{W} := G \times_H W \]

is trivial. The mapping giving

\[ (G/H) \times W \cong G \times_H W \]

is

\[ (gH, \tilde{w}) \leftrightarrow (g, w) \ \text{ where } \ \tilde{w} = \mu(g)w. \]
It is easily checked that this is well defined. It follows that local sections can be identified

\[ \mathcal{E}((G/H) \times W) \cong \mathcal{E}(G \times_H W) \]  
(29)

by

\[ \mathcal{E}((G/H) \times W) \ni \tilde{f} \leftrightarrow f \in \mathcal{E}(G \times_H W) \]

where \( \tilde{f}(g) = \rho(g)f(g) \). In particular \( G \times_H W \) has preferred sections of the form \( \rho(g^{-1})w \), where here \( w \) is the constant section of \( (G/H) \times W \) corresponding to \( w \in W \). Note that \( \Gamma\mathcal{E}(G \times_H W) \) and \( \Gamma\mathcal{E}((G/H) \times W) \) are each \( G \)-representations in two different ways; under the left translation action of \( G \) and by left multiplication of \( \rho(g) \) for \( g \in G \).

It is easily verified that (29) is not an isomorphism of \( G \)-representations using any combination of these. Note, however, that if \( f(g) = \rho(g^{-1})w \), for \( w \in W \), then

\[ g' \cdot f(g) = f((g')^{-1}g) = \rho(g^{-1})\rho(g')w \]

and so we have the following proposition.

**Proposition 5** The isomorphism (29) restricts to give an \( G \)-monomorphism

\[ W \hookrightarrow \Gamma\mathcal{E}(G \times_H W) \]

with where, on the left hand side, \( G \) acts via \( \rho \) and, on the right hand side, \( G \) acts via left translation.

Recall that as a vector space the Lie algebra \( \mathfrak{g} \) of \( G \) is just the tangent space to \( G \) at the identity, \( \mathfrak{g} = T_eG \). This tangent space is then identified with the space of left invariant vector fields via

\[(Xf)(g) := \left[ \frac{d}{dt} f(ge^{tx}) \right]_{t=0},\]

for differentiable functions \( f \) on \( G \). The space of left invariant vector fields is closed under commutation and the Lie bracket on \( \mathfrak{g} \) is defined to agree with the commutator of the elements regarded as left invariant vector fields. This is consistent with the adjoint action of \( G \) on \( \mathfrak{g} \). Regarding \( \mathfrak{g} \) as a \( G \)-representation, in this way, allows us to identify the tangent bundle \( TG \) with the homogeneous bundle \( G \times_H \mathfrak{g} \). Note then that the usual identification of \( \mathfrak{g} \) with the right invariant vector fields is a \( G \)-monomorphism \( \mathfrak{g} \hookrightarrow \Gamma\mathcal{E}(G \times_H \mathfrak{g}) \) exactly as in the proposition above.

Functions on \( G/H \) may clearly be identified with functions on \( G \) that are annihilated by \( X \in \mathfrak{h} \), where \( \mathfrak{h} \) is the Lie algebra of \( H \) identified with the appropriate subalgebra of \( \mathfrak{g} \). It follows then that the tangent bundle \( T(G/H) \) to \( G/H \) is the homogeneous bundle,

\[ G \times_H \frac{\mathfrak{g}}{\mathfrak{h}}, \]

where the representation of \( H \) on \( \mathfrak{g}/\mathfrak{h} \) arises from the adjoint action of \( H \) on \( \mathfrak{g} \). Note that the right invariant vector fields on \( G \) determine global sections (although not generally globally non-vanishing) of \( T(G/H) \) via the bundle morphism \( T(G) \to T(G/H) \) determined by the projection \( \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \).
5.2 The Setup

We are interested in studying homogeneous structures on certain complex homogeneous spaces of the form $G^C/Q$ where these groups are described below. For each complex Lie algebra $\mathfrak{g}$ we write $\mathfrak{g}^R$ to mean the same Lie algebra but regarded as a real Lie algebra.

\[
\begin{align*}
G^C &= \text{Connected and simply connected semi-simple complex Lie group. Lie algebra } \mathfrak{g}. \\
Q &= \text{A parabolic subgroup of } G^C \text{ with Levi factor } L. \text{ Lie algebra } \mathfrak{q} \text{ with Levi decomposition } \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}. \\
\theta &= \text{A Cartan involution on } \mathfrak{g}^R \text{ such that } (\theta(\mathfrak{q}^R) \cap \mathfrak{q}^R) = \mathfrak{l}^R \text{ and which fixes a compact real form } \mathfrak{g}_0 \text{ of } \mathfrak{g}. \text{ Then } \mathfrak{q} \text{ is } \theta\text{-stable and } \mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u} \text{ where } \overline{\text{bar}} \text{ denotes conjugation with respect to } \mathfrak{g}_0. \\
G &= \text{The real subgroup of } G^C \text{ with Lie algebra } \mathfrak{g}_0. \\
K &= G \cap Q. \text{ Lie algebra } \mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{l}. 
\end{align*}
\]

Note that given $G^C$ and $Q$ one can find a maximal toral subalgebra of $\mathfrak{g}$ and corresponding root decomposition such that $Q$ is a standard parabolic. In terms of this root decomposition it is an elementary exercise to describe $\theta$ explicitly. In particular $\theta$ exists. (Note that the setup described above is the classical analogue of the quantum situation with which the article except here we have allowed $G^C$ to be semisimple rather than restricting to the simple case.)

By construction $G$ is a compact real form of $G^C$. In view of our connectivity assumptions it follows that $K_C$ is the Levi part $L$ of $Q$. Observe that, since $Q$ is closed, $K$ is compact. Note also that we clearly have a natural inclusion

\[ G/K \hookrightarrow G^C/Q. \]

Now $\dim(\mathfrak{g}_0/\mathfrak{t}) = \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{q})$ so $G/K$ is an open $G$-orbit in $G^C/Q$. On the other hand $G/K$ is compact so,

\[ G/K = G^C/Q. \]

(30)

This identification shows that the symmetric space $G/K$ is naturally endowed with the structure of a complex manifold. Our starting point is $Q := G^C/Q$ and we are interested in using this identification to analyse the complex homogeneous bundles on this in terms of real structures on the left hand side. Meanwhile note that it follows immediately from the above that any element $g \in G^C$ may be written

\[ g = g_0q \]

where $g_0 \in G \subset G^C$ and $q \in Q$. Here $g_0$ is any element of $G$ such that $g_0K$ corresponds to $gQ$ under the identification of (30). Of course $g_0$ is only determined up to right multiplication by elements of $K$. To be precise we have

\[ G^C = G \times_K Q, \]

where the right hand side means $G \times Q$ modulo the equivalence relation $(g_0, q) \sim (g_0k, k^{-1}q)$ for $k \in K$. We will describe $g_0q$ as a $GQ$-decomposition of $g \in G^C$. 

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We wish to study homogeneous bundles on the homogeneous space $Q$. Let $\rho$ be a complex representation of $Q$ on a complex vector space $V$ and denote by $V_Q$ the corresponding induced bundle on $G^C/Q$. By restriction $\rho$ gives a representation of $K$ on $V$. Let us also denote this representation by $\rho$ and denote by $V$ the homogeneous bundle on $G/K$ induced by this representation. In view of (30) both $V_Q$ and $V$ are natural structures on $Q$. We will show that, regarded as a real structure, $V_Q$ may be identified with the ($K$-induced) homogeneous bundle $V$.

First observe that, clearly,

$$G \times V \hookrightarrow G^C \times V.$$ 

If we compose this set inclusion mapping with the onto mapping to equivalence classes,

$$G^C \times V \to G^C \times_Q V,$$

we obtain a mapping

$$G \times V \to G^C \times_Q V.$$ 

Now since $K = G \cap Q$ this clearly factors through the surjective equivalence mapping $G \times V \to G^C \times Q$. That is there is a natural embedding of the total space manifolds

$$G \times_K V \hookrightarrow G^C \times_Q V.$$ 

On the other hand let $(g, v)$ be a representative of any element of $G^C \times_Q V$. According to our observation above there is an element $g_0 \in G$ and an element $q \in Q$ such that $g = g_0q$. Thus $(g_0, \rho(q)v)$ is an element of $G \times V \subset G^C \times V$ representing the same equivalence class as $(g, v)$. It follows that

$$G \times_K V = G^C \times_Q V,$$

as claimed. It follows immediately that we may identify the spaces of sections

$$\Gamma(E)(G \times_K V) \cong \Gamma(E)(G^C \times_Q V).$$

(31)

It is useful to describe this isomorphism explicitly. First observe that a section $v \in \Gamma(E)(G^C \times_Q V)$ determines a section $\tilde{v} \in \Gamma(E)(G \times_K V)$ by restriction to $G \subset G^C$. On the other hand given $\tilde{v} \in \Gamma(E)(G \times_K V)$ we can construct the corresponding function on $G^C = G \times_K Q$ by

$$v(g) := \rho(q^{-1})\tilde{v}(g_0)$$

(32)

where $g_0q$ is a $GQ$-decomposition of $g \in G^C$. It is easily verified that this is invariant under the equivalence $(g, q) \sim (gk, k^{-1}q)$ and satisfies $v(gq') = \rho(q'^{-1})v(g)$ for $q' \in Q$.

Since $K$ acts reductively on $g_0$ this representation splits as a $K$-module

$$g_0 = \mathfrak{k} \oplus \mathfrak{p}.$$ 

Thus, as a $K$-representation, $g_0/\mathfrak{k} = \mathfrak{p}$ and so the tangent bundle is induced from the adjoint action of $K$ on $\mathfrak{p}$. It follows that for any homogeneous bundle $V$, induced by a representation $\rho$ of $K$, there are natural connections $\nabla$,

$$\nabla : \Gamma(E) V \to \Gamma(E)(V \otimes T^*(G/K))$$

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given by
\[(\nabla_x f)(g) = \langle (\nabla f)(g), x \rangle := \langle f(x)g + \mu(x)f(g), \rangle,
\]
for \(x \in \Gamma \mathcal{E}(TG)\), and where \(\mu\) is any function on \(\mathfrak{g}_0\) taking values in \(\text{Aut}(V)\) such that
\[\mu|_{\mathfrak{t}} = \rho|_{\mathfrak{t}} \quad \text{and} \quad \mu(Ad(k^{-1})X) = \rho(k^{-1})\mu(X)\rho(k) \quad \text{if} \quad k \in K \quad \text{and} \quad X \in \mathfrak{g}_0.
\]
(Here, as throughout this appendix, we use the same symbol, \(\rho\) in this case, to represent a representation of a group and the corresponding derivative representation of the Lie algebra.) In certain circumstances there is a natural choice of the linear function \(\rho\). For example if the representation \(\rho\) of \(K\) on \(V\) is the restriction of a representation (that we will also denote \(\rho\)) of \(G\) on \(V\) then it is natural to take \(\mu = \rho\) as then \(\nabla\) annihilates the "constant" sections \(\rho(g^{-1})v, v \in V\). (See section \(\mathfrak{g}_0\) above).

Since \(\mathcal{Q}\) has a complex structure it is natural to extend \(\nabla\) to an operator
\[\nabla : \Gamma \mathcal{E} \mathcal{V} \to \Gamma \mathcal{E}(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{T}^*\mathcal{Q})\]
by complex linearity. That is \(\nabla_x\) is defined as above where now \(x\) is a section of the complexified tangent bundle \(\mathcal{CT}(\mathcal{Q})\). As mentioned above, we are interested in the case that the homogeneous bundle over \(\mathcal{Q}\) arises from a \(Q\)-representation \(\rho\) on a complex vector space \(V\). In this case we have a semi-natural definition of \(\nabla\) where we take \(\mu|_{\mathfrak{q}} = \rho|_{\mathfrak{q}}\). If \(\rho\) extends to a \(G^C\)-representation on \(V\) then we also require \(\mu|_{\mathfrak{p}} = \rho|_{\mathfrak{p}}\); otherwise we simply take \(\mu|_{\mathfrak{p}} = 0\). Henceforth \(\nabla\) refers to such a connection.

Since \(\mathcal{Q}\) is complex, \(\mathcal{T}^*\mathcal{Q}\) splits into \((1,0)\) and \((0,1)\) parts,
\[\mathcal{CT}^*\mathcal{Q} = T^*_{1,0}(\mathcal{Q}) \oplus T^*_{0,1}(\mathcal{Q}).\]
Thus the connection ‘splits’ correspondingly, that is
\[\nabla = \nabla^{1,0} \oplus \nabla^{0,1}\]
where \(\nabla^{1,0} : \Gamma \mathcal{E} \mathcal{V} \to \Gamma(\mathcal{E} \otimes T^*_{1,0}(\mathcal{Q}))\) and \(\nabla^{0,1} : \Gamma \mathcal{E} \mathcal{V} \to \Gamma(\mathcal{E} \otimes T^*_{0,1}(\mathcal{Q}))\). The curvature of the connection \(\nabla\) on \(\mathcal{V}\) is the section \(R\) of \((\wedge^2 \mathcal{CT}^*\mathcal{Q}) \otimes (\mathcal{V} \otimes \mathcal{V}^*)\) defined by
\[R(x,y)v = [\nabla_x, \nabla_y]v - \nabla_{[x,y]}v\]
for \(x, y \in \mathcal{CT}(\mathcal{Q})\) and \(v \in \mathcal{E} \mathcal{V}\).

Now as a \(K\)-representation \(\mathbb{C} \otimes \mathfrak{p} = \mathfrak{g}/\mathfrak{l}\) decomposes
\[\mathfrak{g}/\mathfrak{l} = \mathfrak{p} \oplus \mathfrak{u}\]
and this splitting corresponds precisely to the decomposition of complex tangent vectors into \((1,0)\)- and \((0,1)\)-parts. Consider the curvature of a \(\mathcal{V}\) in the case that \(V\) does not extend to a \(G^C\)-representation. Now, \([\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}\), thus one obtains immediately from the definition of \(\nabla_x\) that, for any \(v \in \mathcal{E} \mathcal{V}\), if \(x, y \in \mathfrak{p}\) then
\[R(x,y)v)(g) = (xyv)(g) - (yxv)(g) - ([x,y]v)(g) = 0 \quad \forall \ g \in G\]
since, recall, \([x,y]\) is precisely the left invariant vector field which acts as \((xy - yx)\). Now if \(x, y \in \mathfrak{u}\), then, using that \([\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{u}\) and that \((x \rho(y)v)(g) = (\rho(y)xyv)(g)\), we obtain
\[R(x,y)v)(g) = (xyv)(g) - (yxv)(g) + [\rho(x), \rho(y)]v(g) - ([x,y]v)(g) - \rho([x,y])v)(g) = 0, \quad \forall g \in G\]
for the same reason as the previous case and also using that \( \rho \) is a \( \mathfrak{q} \) representation.

On the other hand, if \( x \in \mathfrak{u} \) and \( y \in \mathfrak{u} \) then, since \( \mu(y) = 0 \) and \([\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{l}\),
\[
(R(x, y)v)(g) = (xyv)(g) - (yxv)(g) = ([x, y]v)(g) = -\rho([x, y])v.
\]

In particular this shows that the curvature is type \((1, 1)\). It follows that \((\nabla^{0, 1})^2 v = 0\) for any smooth section \(v\) of \(\mathcal{V}\). That is \(\mathcal{V}\) is a holomorphic vector bundle with
\[
\mathcal{J} = \nabla^{0, 1}.
\]

By a similar calculation one easily obtains that \( R = 0 \) for any vector bundle \(\mathcal{W}\) induced from a \(G^C\)-representation \(\mathcal{W}\). Thus again the induced bundle \(\mathcal{W}\) admits a holomorphic structure and \(\mathcal{J} = \nabla^{0, 1}\). Note that for \(x \in \mathfrak{q}\) and \(f \in \Gamma\mathcal{E}\mathcal{W}\),
\[
x(\rho(g)f(g)) = \rho(g)\nabla_x f(g).
\]

Thus the holomorphic sections of \(\mathcal{W}\) correspond to the holomorphic \(\mathcal{W}\)-valued functions on \(\mathcal{Q}\). Using proposition 5 and that \(\mathcal{Q}\) is compact we have then the following.

**Proposition 6** There is an isomorphism of \(G\)-modules
\[
\mathcal{W} \cong \Gamma\mathcal{O}\mathcal{W}
\]
with \(G\)-action as given in the proposition 5.

According to (32) there is no restriction on the \(K\)-homogeneous section \(\tilde{v}\) for it to extend to a \(Q\)-homogeneous function on \(G^C\). The discussion there is implicitly treating the underlying manifold of the group \(G^C\) as a real structure. By construction the function \(v\), obtained from \(\tilde{v}\) as in (32), satisfies the identity
\[
xv + \rho(x)v = 0 \quad \text{for all} \quad x \in \mathfrak{q},
\]
this following immediately from the derivative of (32). This is no condition on \(\tilde{v}\) as each \(x \in \mathfrak{u} \subset \mathfrak{q}\) gives a (real) left invariant vector field on \(G^C\) which, at \(G \subset G^C\), is not tangent to \(G\).

The complex structure of \(\mathcal{Q}\) arises from regarding \(G^C\) as a complex manifold and in this case the story is rather different. As mentioned above it is natural to complexify the tangent space in this case whence each left invariant vector field \(x \in \mathfrak{u}\) along \(G \subset G^C\) can be written as complex linear combination of tangent vectors to \(G\). Thus, at \(G \subset G^C\), (33) becomes an equation on \(\tilde{v}\),
\[
x\tilde{v} + \rho(x)\tilde{v} = 0 \quad \text{for all} \quad x \in \mathfrak{q}.
\]

In other words a section of \(G \times_K V\), that is a function
\[
\tilde{v} : G \to V \quad \text{such that} \quad \tilde{v}(g_0k) = \rho(k^{-1})\tilde{v}(g_0) \quad \text{for} \quad g_0 \in G, \ k \in K,
\]
can only extend to a function
\[
v : G^C \to V \quad \text{such that} \quad v(gq) = \rho(q^{-1})v(g) \quad \text{for} \quad g \in G^C, \ q \in Q
\]
if
\[
\nabla v = 0.
\]

On the other hand if \(\tilde{v}\) satisfies (33) on \(G\) then we can integrate to obtain a function of the form \(\rho(q^{-1})\tilde{v}(g_0)\), which, as discussed above, is a section of \(G \times_Q V\). Thus \(\nabla \tilde{v} = 0\) is also a sufficient condition for \(K\)-homogeneous functions on \(G\) to extend to \(Q\)-homogeneous functions on \(G^C\). In summary then:
Proposition 7 If we regard $V$ as a holomorphic bundle on the complex manifold $Q$ then smooth sections of $V$ extend to $Q$-homogeneous functions on $G^C$ if and only if they are holomorphic.

5.3 The Borel-Weil theorem, Projectivity and Frobenius Reciprocity

We are now poised to give an elementary treatment of the Borel-Weil theorem.

Theorem 5 Let $V$ be an irreducible finite dimensional $Q$-representation. If $V$ denotes the homogeneous bundle on $Q$ induced by $V$. Then, as a $G^C$-representation,

$$\Gamma OV = \begin{cases} W & \text{if there is a } Q\text{-epimorphism } W \to V \\ 0 & \text{otherwise,} \end{cases}$$

where $W$ is an irreducible finite dimensional $G^C$-representation.

Proof: First note that if $\Gamma OV \neq 0$ then it contains a section $f$ such that $f(e) \neq 0$ (or else by left translation $\Gamma OV = 0$). Thus evaluation at the identity determines a non-trivial $Q$-homomorphism $\Gamma OV \to V$. Since $V$ is $Q$-irreducible this is a surjection. Now as $Q$ is compact it follows from elliptic theory that $\Gamma OV$ is finite dimensional. Since $G^C$ is reductive this representation decomposes and there is an irreducible component $W$ in $\Gamma OV$ such that there is a $Q$-epimorphism $W \to V$.

Next we observe that given such a $Q$-epimorphism $\pi : W \to V$ then, as $G^C$-representations, $W \hookrightarrow \Gamma EV$. Let $\tilde{\pi}$ be the induced $G^C$-homomorphism

$$\tilde{\pi} : \Gamma OW \to \Gamma OV$$

given by

$$\tilde{\pi} f(g) := \pi(f(g))$$

for $g \in G^C$ and $f \in \Gamma OW$. Note that it follows from proposition [4] that $(\tilde{\pi} f)$ is holomorphic since $\pi$ is a $Q$-homomorphism. Now recall proposition [3] there is a $G^C$-isomorphism

$$W \cong \Gamma OW$$

and each section $\tilde{f} \in \Gamma OW$ is of the form $\tilde{f}(g) = \rho(g^{-1})w$ for some $w \in W$. It follows that for any such non-vanishing section $\tilde{f}$ and given any $w' \in W$ there is some $g \in G^C$ such that $\tilde{f}(g) = w'$. It follows immediately that $\tilde{\pi}$ is injective and so $W$ is a $G^C$-submodule of $\Gamma OV$ as claimed.

Finally we show that $\tilde{\pi}$ is onto. Since $\Gamma OV$ is finite dimensional it follows, by the usual theory of weights, that any element of $\Gamma OV$ is in the $G^C$-orbit of an element that is annihilated by the left action of $\overline{U}$. Since $W$ is irreducible, it suffices to show that a such $\overline{U}$-invariant section $f$ is in the image of $\tilde{\pi}$. Now $\overline{U}$ acts on $G^C/Q$ and the orbit of the base point is an open dense set in $G^C/Q$. By continuity it follows that $f$ is determined by its value $f(e)$ at the identity $e \in G^C$. Now there is an element $w \in W$ such that $\pi(w) = f(e)$ and $\rho(X)w = 0$ for all $x \in \overline{U}$. Let $\tilde{f}(g) := \rho(g^{-1})w$. Then $\tilde{\pi} \tilde{f}$ is $\overline{U}$ invariant and $(\tilde{\pi} f)(e) = f(e)$. Thus $f = \tilde{\pi} \tilde{f} \in \tilde{\pi} W$ as required to be shown.
In summary we have that, if $\Gamma OV \neq 0$, then there is an irreducible $G^C$-representation $W$ such that there is a $Q$-epimorphism $W \to V$, and

$$\Gamma OV \cong W$$

as $G^C$-representations. This proves the theorem. \(\square\)

**Projectivity.** The Borel-Weil theorem, as above, is usually stated in terms of weights since the condition that there be a non-trivial $Q$ (or equivalently $q$) epimorphism $W \to V$ ($W$ and $V$ as in the theorem) is equivalent to $V$ having a highest weight which is dominant for $g$. The quantum version of this theorem, presented in section 4 (theorem 4), is expressed in this manner. Under a weaker condition on the highest weight of $V$ we can establish, in a natural way, the projectivity of the $\mathcal{E}$-module $\Gamma \mathcal{E} V$.

Suppose now, then, that there is a $K$-module epimorphism

$$\phi : W \to V.$$ 

Let $V^\perp := \ker \phi$. Then since $K$ is reductive it follows that as a $K$-module $W = V \oplus V^\perp$. Let $V^\perp$ be the homogeneous bundle over $Q$ induced by the $K$-representation $V^\perp$. Regarding $\mathcal{W}$ and $\mathcal{V}$ also as $K$-induced bundles, it is at once clear that $\mathcal{W}$ may be expressed as a bundle direct sum $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\perp$. It is easily seen that this carries over to the $\mathcal{E}$-module of local smooth sections and the $\Gamma \mathcal{E}$-module of global smooth sections:

$$\mathcal{E} W = \mathcal{E} V \oplus \mathcal{E} V^\perp \quad \text{and} \quad \Gamma \mathcal{E} W = \Gamma \mathcal{E} V \oplus \Gamma \mathcal{E} V^\perp.$$ 

Now recall that $\mathcal{W}$ is a trivial bundle with fibre $W$ and so, in particular, we have

$$\Gamma \mathcal{E} V \oplus \Gamma \mathcal{E} V^\perp = \Gamma \mathcal{E} \otimes_C W.$$ 

This establishes the following theorem.

**Theorem 6** Suppose that $V$ is an irreducible $K$-module such that there is a $K$-module epimorphism, $W \to V$, where $W$ is an irreducible $G$-module. Then $\Gamma \mathcal{E} V$ is a finite type projective module over the algebra $\Gamma \mathcal{E}$ of functions on $Q$.

Of course this theorem is just a special case of Swann’s theorem. However with a view to establishing the corresponding result in the quantum case it is useful to expose, as we have, the mechanics underlying the result. Note also that we can regard $V$ and $W$ as modules for the complexified groups and their Lie algebras. In this picture the condition on $V$ is equivalent to the existence of an $l$-epimorphism $W \to V$. This in turn is equivalent to requiring that the highest weight for $V$ be in the Weyl group orbit of a $g$-dominant weight. That is that this highest weight be integral for with respect to $g$.

**Frobenius Reciprocity.** An early observation in the proof of the Borel-Weil theorem above was that if $\Gamma OV \neq 0$ then there is a $L$-epimorphism $\Gamma OV \to V$. The same argument shows that there is always a $L$-epimorphism $\mathbb{E} \mathcal{V} \mathbb{E} : \Gamma \mathcal{E} V \to V$ determined by evaluation at the identity. It is an elementary exercise to verify that if there is a $G^C$-monomorphism $\iota : W \to \Gamma \mathcal{E} V$ then the composition $\mathbb{E} \mathcal{V} \mathbb{E} \circ \iota$ is a $L$-epimorphism $W \to V$. Here $G^C$ and its subgroup $L$ act on $\Gamma \mathcal{E} V$ by left translation.
On the other hand if we have $\phi \in \text{Hom}_L(W,V)$ then this determines a $G^C$-homomorphism (with respect to the left translation action of $G^C$)
\[ \Gamma E(G^C \times_L W) \to \Gamma E(G^C \times_L V) \]
in the obvious way. Composing this with the $G^C$-equivariant injection $W \hookrightarrow \Gamma E(G^C \times_L W)$ described in proposition 3 we obtain a $G^C$-monomorphism $\tilde{\phi} : W \to \Gamma E(G^C \times_L V)$.

It is easily verified that $Ev \circ \tilde{\phi} = \phi$ and that conversely, $\tilde{Ev} \circ \iota = \iota$. This is just the usual result of Frobenius reciprocity.

**Theorem 7** Let $W$ and $V$ be respectively $G^C$ and $L$ modules and let $V$ denote the homogeneous bundle induced by $V$. Then there is a canonical isomorphism

\[ \text{Hom}_{G^C}(W, \Gamma E \mathcal{V}) \cong \text{Hom}_{G^C}(W, V) \]

where $G^C$-action on $\Gamma E \mathcal{V}$ is by left translation.

### 5.4 Geometry, Analysis and Algebra

Let $G$ be a compact group. Write $K$ to mean either the field $\mathbb{R}$, of real numbers, or the field $\mathbb{C}$, of complex numbers. There is a left action of $G$ on $K$-valued functions given by right translation (c.f. left translation discussed above)

\[ f \mapsto g \circ f \quad \text{where} \quad g \circ f(h) = f(hg) \quad g, h \in G. \]

A $K$-valued representative function $f$ is a continuous function

\[ f : G \to K \]

such that the span of the $G$-orbit of $f$, under right translation, is finite dimensional. That is the representative functions are just the functions which generate the finite dimensional $G$-invariant subspaces of the continuous functions $C(G, K)$ on $G$. We write $T(G, K)$ to denote the space of representative functions. (See, for example, [24] for an introduction to these functions and their role in the theory Lie groups and their representations.)

Suppose that $\rho$ is a finite-dimensional representation of $G$ then the matrix elements satisfy

\[ \rho_{ik}(hg) = \sum_j \rho_{ij}(h)\rho_{jk}(g), \]

demonstrating that the right translate of the function $\rho_{ik}$ is a linear combination of the finite set of functions $\{\rho_{ij}\}$ arising from the matrix elements of the representation $\rho$. Thus the matrix elements of finite-dimensional continuous representations over $K$ are examples of representative functions. It is well known that, conversely, when all irreducible representations are used, such matrix element functions generate $T(G, K)$ as a $K$-vector space. Now considering the matrix elements of dual, direct sum and tensor product of representations quickly reveals that the representative functions $T(G, K)$ admit a natural $K$-algebra structure which, as a subalgebra of $C(G, K)$, is closed under complex conjugation.

From our point of view the importance of these special functions arises from the next two theorems. The proof of these theorems (also see [24]) involves some standard analysis and functional analysis.
The first theorem, which is the celebrated theorem of Peter and Weyl, states that any continuous or $L^2$ function on $G$ can be approximated by representative functions: Recall that a compact Lie group $G$ admits a unique normalised (left and right) invariant Haar integral. Functions are said to be $L^2$ if they are square-integrable with respect to this integral.

**Theorem 8** *(Peter-Weyl)* The representative functions are dense in both $C(G, \mathbb{C})$ and $L^2(G, \mathbb{C})$.

The second theorem, which is a cornerstone of Tannaka-Krein duality theory, implies that the space of representative functions contains all the information of the compact Lie group: Let $G_{\mathbb{R}}$ be the set of $\mathbb{R}$-algebra homomorphisms $T(G, \mathbb{R}) \to \mathbb{R}$. Each $g \in G$ determines an evaluation homomorphism $e_g : T(G, \mathbb{R}) \to \mathbb{R}$ by $t \mapsto t(g)$.

**Theorem 9** The duality map

$$i : G \to G_{\mathbb{R}}, \text{ given by } g \mapsto e_g,$$

is an isomorphism of Lie groups.

Thus the analysis behind these theorems provides the faithful link between the algebra of functions on the group and the group itself, which we regard as a fundamentally geometric object.

These results suggest a programme where one studies the Lie group, its associated structures and representations via the algebra of representative functions or their completion to say $L^2$-functions. For example, the classical results discussed in sections 5.1 to 5.3 above could all be reworked in this picture. (See also [25] where they show that for a closed subgroup $H$ of a compact $G$, $G/H$ may be identified with the algebra of homomorphisms $T(G/H, \mathbb{C}) \to \mathbb{R}$, where $T(G/H, \mathbb{C}) \subset T(G, \mathbb{C})$ is the subring of functions which factor through the map $G \to G/H$.) In the case of quantum groups we are without an underlying concrete group manifold, but these classical results suggest an approach to defining and studying analogous geometric notions in terms of representative type functions on the quantized universal enveloping algebra.

**References**

[1] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, Universite de Montreal, Centre de Recherches Mathematiques, Montreal, PQ (1988).

[2] S. L. Woronowicz, *Differential calculus on compact matrix pseudo groups (quantum groups)*, Commun. Math. Phys. 122 (1989) 125.

[3] C. S. Chu, P. M. Ho and B. Zumino, *Some complex quantum manifolds and their geometry*, Preprint (1996).

[4] S. Majid, *Advances in quantum and braided geometry*, Preprint (1996).

[5] S. L. Woronowicz, *Compact matrix pseudo groups*, Commun. Math. Phys. 111 (1987) 613.
[6] T. Brzezinski and S. Majid, *Quantum group gauge theory on quantum spaces*, Commun. Math. Phys. 157 (1993) 591. Erratum. 167 (1995) 235.

[7] M. Durdevic, *Geometry of principle bundles I*, Commun. Math. Phys. 175 (1996) 457.

[8] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[9] R. J. Baston and M. G. Eastwood, *The Penrose Transform; its interaction with representation theory*, Oxford University Press, Oxford, (1989).

[10] M. Jimbo, *A q-difference analogue of of U(g) and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985) 63.

[11] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge (1994).

[12] Y. Z. Zhang and M. D. Gould, *Unitarity and complete reducibility of certain modules over quantized affine Lie algebras*, J. Math. Phys. 34 (1993) 6045.

[13] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J., 1 (1990) 193.

[14] E. G. Effros and Z. Ruan, *Discrete quantum groups I. The Haar measure*, Internat. J. Math., in press.

[15] M. S. Dijkhuizen and T. H. Koorwinder, *CQG algebras: a direct algebraic approach to compact quantum groups*, Lett. Math. Phys. 32 (1994) 315.

[16] H. J. Schneider, *Principle homogeneous spaces for arbitrary Hopf algebras*, Israel J. Math. 72 (1990) 196.

[17] V. Lakshmibai and N. Yu. Reshetikhin, *Quantum deformation of flag and Schubert schemes*, C. R. Acad. Sci. Paris. Ser. I. Math. 313 No 3. (1991) 121-126.

[18] M. S. Dijkhuizen and T. H. Koorwinder, *Quantum homogeneous spaces, duality and quantum 2-spheres*, Geom. Dedicata 52 (1994) 291.

[19] S. Montgomery, *Hopf algebras and their actions on rings*, Regional Conference Series in Math., 82 (1993).

[20] B. Parshall and J. P. Wang, *Quantum linear groups*, Memoirs Amer. Math. Soc., 89 No. 439 (1991) 1 - 157.

[21] M. Noumi, H. Yamada and K. Mimachi, *Finite-dimensional representations of the quantum group GL_q(n, C) and zonal spherical functions on U_q(n−1)\U_q(n)*, Japanese J. Math., 19 (1993) 31.

[22] R. B. Zhang, *Structure and representation of the quantum general supergroup*, Univ. of Adelaide Preprint (1996).

[23] B. Jurco and P. Stovicek, *Coherent states for compact quantum groups*, Commun. Math. Phys. 182 (1996) 221.

[24] T. Broöcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York (1985).
[25] N. Iwahori and M. Sugiura, *A duality theorem for homogeneous manifolds of compact Lie groups*, Osaka J. Math. 3 (1966) 139–153.

[26] A.W. Knapp and D.A. Vogan, *Cohomology, Induction and Unitary Representations*, Princeton University Press, Princeton (1995).