ON THE SOBOLEV EMBEDDING PROPERTIES FOR COMPACT MATRIX QUANTUM GROUPS OF KAC TYPE

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(Communicated by Quanhua Xu)

Abstract. We study the optimal order of natural analogues of Sobolev embedding properties within the framework of compact matrix quantum groups of Kac type. One of the main results of this paper is that the optimal order is given by the polynomial growth order of dual discrete quantum groups in a broad class, which covers all connected compact Lie groups, duals of polynomially growing discrete groups, $O^+_N$ and $S^+_N$, and that Sobolev embedding properties can be generalized for all compact matrix quantum groups of Kac type whose duals have the rapid decay property. In addition, we generalize sharpened Hausdorff-Young inequalities, compute degrees of the rapid decay property for duals of $O^+_N$, $S^+_N$ and prove sharpness of Hardy-Littlewood inequalities on duals of free groups.

1. Introduction. It is a long tradition to study Fourier multipliers in harmonic analysis, and $L^p - L^q$ multipliers have played major roles in the theory of partial differential equations. A representative example is the Sobolev embedding property. More precisely, the Hardy-Littlewood-Sobolev theorem on tori states that

$$\left\Vert (1 - \Delta)^{-\frac{d}{2}} (\frac{1}{p} - \frac{1}{q}) f \right\Vert_{L^q(\mathbb{T}^d)} \lesssim \left\Vert f \right\Vert_{L^p(\mathbb{T}^d)}$$

(1.1)

for all $1 < p < q < \infty$ and $f \in L^p(\mathbb{T}^d)$, where $\Delta$ is the Laplacian operator. Here, $g_1 \lesssim g_2$ means that $g_1 \leq K g_2$ and $K$ is independent of $g_1$ and $g_2$, and we denote by $g_1 \approx g_2$ if $g_1 \lesssim g_2$ and $g_1 \gtrsim g_2$.

Sobolev embedding properties have been explored in a broad class including Lie groups [18, 7, 17, 45, 46, 14] and, more generally, $L^p - L^q$ multipliers on Lie groups have been extensively studied [15, 4, 5, 3, 2]. In particular, for connected compact Lie groups $G$, it is known that for any $1 < p \leq 2$

$$\left( \sum_{\pi \in \text{Irr}(G)} \frac{n_{\pi}}{(1 + \kappa_\pi)^{\frac{d}{p} - \frac{d}{2}}} \right)^{\frac{1}{2}} \left\Vert \hat{f}(\pi) \right\Vert^2_{S^2_{\kappa_\pi}} \lesssim \left\Vert (1 - \Delta)^{\frac{d}{2}} (\frac{1}{p} - \frac{1}{2}) f \right\Vert_{L^2(G)} \lesssim \left\Vert f \right\Vert_{L^p(G)}.$$

(1.2)

2010 Mathematics Subject Classification. Primary 43A15, 46L52, Secondary 20G42, 81R15.

Key words and phrases. Compact matrix quantum group, noncommutative $L^p$-space, Sobolev embedding property, ultracontractivity, rapid decay degree.

This research was supported by the Natural Sciences and Engineering Research Council of Canada and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1C1C1A01009681).
where \( n \) is the real dimension of \( G \), \( \| A \|_{S_n^2} = \text{tr}(A^* A)^{\frac{1}{2}} \) for any \( A \in M_n \) and \( \Delta : \pi_{i,j} \mapsto -\kappa_\pi \pi_{i,j} \) is the Laplacian operator \([6]\). Since the natural length function \( | \cdot | \) on \( \text{Irr}(G) \) satisfies \( | \pi | \approx \kappa_\pi^\frac{1}{2} \) \([50, \text{Lemma 5.6.6}]\), the above (1.2) is equivalent to
\[
\left( \sum_{\pi \in \text{Irr}(G)} \frac{n_\pi}{(1 + |\pi|)^{n(\frac{2}{p}-1)}} \| \hat{f}(\pi) \|_{S_{n^2_\pi}}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^p(G)}. \tag{1.3}
\]

Note that (1.3) detects the real dimension of \( G \), which is an important geometric quantity. The main purpose of this paper is to study (1.3) within the framework of compact quantum groups by employing geometric information of the underlying quantum group such as growth rates and the rapid decay property. Indeed, for highly important examples, we will show that the polynomial growth order or the degree of rapid decay property replaces the role of the real dimension \( n \).

The theory needed to explore Sobolev embedding properties for compact quantum groups is so-called noncommutative \( L^p \)-analysis. On quantum groups and quantum tori, \( L^p - L^q \) multipliers have been studied from various perspectives \([31, 22, 19, 1, 57]\). In particular, due to \([1, \text{Theorem 4.3}]\) which generalizes a theorem of Hörmander, if \( G \) is a compact matrix quantum group of Kac type whose dual \( \hat{G} \) has a polynomial growth
\[
b_k = \sum_{\alpha \in \text{Irr}(\hat{G}) : |\alpha| \leq k} n_\alpha^2 \leq C(1 + k)^\gamma
\]
for some \( C, \gamma > 0 \), then for any \( 1 < p \leq 2 \) we have
\[
\left( \sum_{\alpha \in \text{Irr}(\hat{G})} \frac{n_\alpha}{(1 + |\alpha|)^{\gamma(\frac{2}{p}-1)}} \| \hat{f}(\alpha) \|_{S_{n^2_\alpha}}^2 \right)^{\frac{1}{2}} \leq (2C)^{\frac{1}{p-\frac{2}{2}}} \| f \|_{L^p(\hat{G})}, \tag{1.4}
\]
where \( | \cdot | \) is the natural length function on \( \text{Irr}(G) \). The above constant \( (2C)^{\frac{1}{p-\frac{2}{2}}} \) comes from the proofs of Theorem 3.1 and Corollary 3.5 of \([59]\).

To our best knowledge, if we exclude compact Lie groups and duals of polynomially growing discrete groups, it is not known whether the above inequalities (1.4) can be improved. One of our main results is that the order \( \gamma \) in (1.4) is optimal in the sense that
\[
\left( \sum_{\alpha \in \text{Irr}(\hat{G})} \frac{n_\alpha}{(1 + |\alpha|)^{\gamma(\frac{2}{p}-1)}} \| \hat{f}(\alpha) \|_{S_{n^2_\alpha}}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^p(\hat{G})} \text{ if and only if } s \geq \gamma
\]
under the following assumptions:

1. (Corollary 3.5) if \( b_k \approx (1 + k)^\gamma \) and there exists a standard noncommutative semigroup \((T_t)_{t>0}\) on \( L^\infty(G) \) whose infinitesimal generator \( L \) satisfies
\[
L(u_{\alpha,j}^\alpha) = -l(\alpha)u_{\alpha,j}^\alpha \text{ with } l(\alpha) \approx |\alpha|, \tag{1.5}
\]
In this case, (1.4) can be generalized to \( L^p-L^q \) Sobolev embedding properties for any \( 1 < p < q < \infty \).

2. (Corollary 3.7) if \( b_k \lesssim (1 + k)^\gamma \) and \( s_k = \sum_{\alpha \in \text{Irr}(\hat{G}) : |\alpha| = k} n_\alpha^2 \gtrsim (1 + k)^{\gamma-1} \).

Note that all connected compact Lie groups, duals of polynomially growing discrete groups, \( O^+_N \) and \( S^+_N \) satisfy one of the above assumptions. Hence, the optimal orders for Sobolev embedding properties are given by the polynomial growth orders of their duals.

Despite the above strong conclusion under the polynomial growth of \( \hat{G} \), it is important to study duals of free groups \( \hat{F}_N \) and free quantum groups \( O^+_N, S^+_N \) whose
duals are exponentially growing. Arguably, these are the most important examples of compact quantum groups in view of operator algebras [51, 43, 52, 49, 10, 48, 24, 20, 11, 25, 12], and noncommutative $L^p$-analysis on $\widehat{\mathbb{F}}_N$, $O_N^+$, $S_N^+$ has been extensively studied [30, 35, 34, 31, 53, 59, 60]. Actually, there are standard arguments to obtain ultracontractivity of the Poisson semigroup of $\widehat{\mathbb{F}}_N$ [27] and the heat semigroup of $O_N^+, S_{N+2}^+$ [19] from the rapid decay property. Some small modifications on the arguments together with [56] provide us with the following Sobolev embedding properties with the order 3:

$$\left( \sum_{\alpha \in \text{Irr}(\mathbb{G})} \frac{1}{(1 + |\alpha|)^{(3/2)-1}} n_\alpha \left\| \hat{f}(\alpha) \right\|_{S^2_{2\alpha}}^2 \right)^{1/2} \lesssim \|f\|_{L^p(\mathbb{G})} \quad (1.6)$$

To prove that the above order 3 in (1.6) is optimal for $\widehat{\mathbb{F}}_N$, $O_{N+1}^+$ or $S_{N+3}^+$ ($N \geq 2$), we compute the rapid decay degrees of duals of $O_N^+$ and $S_{N+2}^+$ (Corollary 4.4), and then apply main results of Section 4 to get the desired conclusion in Section 5. Indeed, for the Poisson or heat semigroups $(T_t)_{t>0}$ of $\mathbb{G} = \widehat{\mathbb{F}}_N$, $O_{N+1}^+$ or $S_{N+3}^+$, we prove that there exists a universal constant $K > 0$ such that

$$\|e^{-tT_t}(f)\|_{L^\infty(\mathbb{G})} \leq \frac{K \|f\|_{L^2(\mathbb{G})}}{t^{1/2}} \quad \text{for all } f \in L^2(\mathbb{G}) \text{ and } t > 0 \quad (1.7)$$

if and only if $s \geq 3$ (Corollary 5.2). Then [56, Theorem 1.1] tells us that the order 3 in (1.6) is optimal for $\widehat{\mathbb{F}}_N$, $O_{N+1}^+$ and $S_{N+3}^+$ with $N \geq 2$. Moreover, these $L^p$-$L^2$ Sobolev embedding properties (1.6) can be generalized to $L^p$-$L^t$ embedding properties (Example 4) thanks to the existence of Poisson or heat semigroups.

As of an effort to find a general approach to study Sobolev embedding properties outside the category of co-amenable compact quantum groups, we establish Sobolev embedding properties for general compact matrix quantum groups $\mathbb{G}$ of Kac type whose duals have the rapid decay property. To do this, we extend [60, Theorem 3.2] under the rapid decay property of $G$ (Theorem 6.2). We call it sharpened Hausdorff-Young inequalities and explain why such a phenomenon does not appear in the category of compact Lie groups, duals of discrete groups, $O_q^+$ or $SU_q(2)$ (Section 6.1). Then, by applying the complex interpolation method between the sharpened Hausdorff-Young inequalities (Theorem 6.2) and Hardy-Littlewood inequalities [59, Theorem 3.8], we prove that

$$\left( \sum_{\alpha \in \text{Irr}(\mathbb{G})} \frac{1}{(1 + |\alpha|)^{(2\beta+1)(3/2)-1}} n_\alpha \left\| \hat{f}(\alpha) \right\|_{S^2_{\alpha}}^2 \right)^{1/2} \leq \left( \frac{2\beta+2D^2}{2\beta+1} \right)^{\frac{\beta}{2}} \|f\|_{L^p(\mathbb{G})} \quad (1.8)$$

for any $1 < p \leq 2$ when a polynomial for the rapid decay property is given as $D(1+k)^\beta$. See Theorem 6.6 for details. As a byproduct of this approach, we can show that Hardy-Littlewood inequalities on $\widehat{\mathbb{F}}_N$ presented in [59, Theorem 4.4] are sharp (Corollary 6.8).

2. Preliminaries.

2.1. Compact quantum groups and the representation theory. A compact quantum group $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital $C^*$-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\text{min}} C(\mathbb{G})$ is a unital $*$-homomorphism such that

1. $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$. 

2. span \{\Delta(a)(b \otimes 1) : a, b \in C(G)\} and span \{\Delta(a)(1 \otimes b) : a, b \in C(G)\} are dense in \(C(G) \otimes \min C(G)\).

For a compact quantum group \(G\), there exists a unique state \(h\) satisfying
\[
(id \otimes h) \circ \Delta = h(\cdot)1 = (h \otimes id) \circ \Delta.
\]

We call \(h\) the Haar state and \(G\) is said to be of Kac type if \(h\) is tracial.

A (finite dimensional) unitary representation of \(G\) is \(u = (u_{i,j})_{i,j=1}^{n_u} \in M_{n_u} \otimes C(G)\) such that
\[
u^* u = Id_{n_u} \otimes 1 = uu^* \quad \text{and} \quad \Delta(u_{i,j}) = \sum_{k=1}^{n_u} u_{i,k} \otimes u_{k,j} \quad \text{for all} \quad 1 \leq i,j \leq n_u.
\]

We say that a unitary representation \(u\) is irreducible if
\[
\{T \in M_{n_u} : (T \otimes 1)u = u(T \otimes 1)\} = \mathbb{C} \cdot Id_{n_u}
\]
and denote by \(\{\nu^\alpha = (u_{i,j}^\alpha)_{1 \leq i,j \leq n_\alpha}\}_{\alpha \in \text{Irr}(G)}\) the set of irreducible unitary representations up to unitary equivalence. Each \(\alpha \in \text{Irr}(G)\) will be identified with \(\nu^\alpha \in M_{n_\alpha} \otimes C(G)\) and the corresponding character is \(\chi_\alpha = \sum_{i=1}^{n_\alpha} u_{i,i}^\alpha\). The space of polynomials
\[
\text{Pol}(G) = \text{span}\{u_{i,j}^\alpha : \alpha \in \text{Irr}(G) \text{ and } 1 \leq i,j \leq n_\alpha\}
\]
is a dense \(\ast\)-subalgebra of \(C(G)\) and the Haar state \(h\) is faithful on \(\text{Pol}(G)\).

Associated to a compact quantum group \(G\) is the discrete dual quantum group \(\hat{G} = (\ell^\infty(\hat{G}), \Delta, \hat{h})\). Among the structures of \(\hat{G}\), the underlying von Neumann algebra \(\ell^\infty(\hat{G})\) is \(\ell^\infty = \bigoplus_{\alpha \in \text{Irr}(G)} M_{n_\alpha}(\mathbb{C})\) and, if \(G\) is of Kac type, \(\hat{h}\) is a normal semifinite faithful tracial weight on \(\ell^\infty(\hat{G})\) given by
\[
\hat{h}(A) = \sum_{\alpha \in \text{Irr}(G)} n_\alpha \text{tr}(A_\alpha) \quad \text{for all} \quad A = (A_\alpha)_{\alpha \in \text{Irr}(G)} \in \ell^\infty(\hat{G})_+.
\]
See [55, 54, 32, 33, 42] for more details of locally compact quantum groups.

### 2.2. Noncommutative \(L^p\)-spaces

Throughout this paper, we assume that \(G\) is a compact quantum group of Kac type. Since \(h\) is faithful on \(\text{Pol}(G)\), the space \(\text{Pol}(G)\) is canonically embedded into \(B(L^2(G))\) where \(L^2(G)\) is the completion of \(\text{Pol}(G)\) with respect to the inner product \((f,g)_{L^2(G)} = h(g^*f)\) for all \(f,g \in \text{Pol}(G)\).

We define an associated von Neumann algebra \(L^\infty(G) = \text{Pol}(G)^\vee\) in \(B(L^2(G))\), and then the Haar state \(h\) extends to a normal faithful tracial state \(h\) on \(L^\infty(G)\).

For any \(1 \leq p < \infty\), the noncommutative \(L^p\)-space is defined as the completion of \(\text{Pol}(G)\) with respect to the norm structure \(\|f\|_{L^p(G)} = h(|f|^p)^{\frac{1}{p}}\) for any \(f \in \text{Pol}(G)\). Then it is well known that \((L^\infty(G), L^1(G))_{\frac{1}{p}} = L^p(G)\) where \((\cdot, \cdot)_\theta\) is the complex interpolation space and that \(L^p(G) = L^p(G)^*\) for any \(1 \leq p < \infty\) under the dual bracket \((f,g)_{L^p(G), L^{p'}(G)} = h(g(f))\) for all \(f,g \in \text{Pol}(G)\).

On the dual side, for any \(1 \leq p < \infty\), the noncommutative \(\ell^p\)-space is defined as
\[
\ell^p(\hat{G}) = \left\{ A \in \ell^\infty(\hat{G}) : \sum_{\alpha \in \text{Irr}(G)} n_\alpha \text{tr}(|A_\alpha|^p) < \infty \right\}
\]
and the natural norm structure is \(\|A\|_{\ell^p(\hat{G})} = \left(\sum_{\alpha \in \text{Irr}(G)} n_\alpha \text{tr}(|A_\alpha|^p)\right)^{\frac{1}{p}}\) for all \(A \in \ell^p(\hat{G})\). Then \((\ell^\infty(\hat{G}), \ell^1(\hat{G}))_{\frac{1}{p}} = \ell^p(\hat{G})\) and \(\ell^p(\hat{G}) = (\ell^p(\hat{G})^*)^*\) hold for any
1 ≤ p < ∞. The dual bracket between $\ell^p(\hat{G})$ and $\ell^{p'}(\hat{G})$ is given by
\[
\langle A, B \rangle_{\ell^p(\hat{G}), \ell^{p'}(\hat{G})} = \sum_{\alpha \in \text{Irr}(\hat{G})} n_\alpha \text{tr}(B_\alpha A_\alpha).
\] (2.7)

For a matrix $A \in M_n$, its noncommutative $L^p$-norm is called the Schatten $p$-norm, which is given by $\|A\|_{a^p} = \text{tr}(|A|^p)^{\frac{1}{p}}$. We refer to [39, 38] for the general theory of noncommutative $L^p$-spaces.

2.3. Compact matrix quantum groups and the rapid decay property. The tensor product representation of two unitary representations $u$ and $v$ is
\[
u \odot \psi = (u_{i,j} v_{k,l})_{1 \leq i,j \leq n_u, 1 \leq k,l \leq n_v} \in M_{n_u} \otimes M_{n_v} \otimes C(\hat{G})
\] (2.8)
and every unitary representation is decomposed into a direct sum of irreducible representations. If $\sigma$ is an irreducible component of $\nu \odot \psi$, we write $\sigma \subseteq \nu \odot \psi$.

A compact matrix quantum group is a compact quantum group $G$ for which there exists a unitary representation $w$ such that every $w^\alpha \in \text{Irr}(G)$ is an irreducible component of $w^{\odot n}$ for some $n \in \{0\} \cup \mathbb{N}$. In this case, we can define a natural length function on $\text{Irr}(\hat{G})$ by
\[
|\alpha| = \min \left\{ n \in \{0\} \cup \mathbb{N} : u^\alpha \subseteq w^{\odot n} \right\} \text{ for all } \alpha \in \text{Irr}(\hat{G}).
\]

Throughout this paper, we will use the following notations frequently.

**Notation.** Let $G$ be a compact matrix quantum group and $k \in \{0\} \cup \mathbb{N}$.

1. $k$-balls are defined as $B_k = \{ \alpha \in \text{Irr}(\hat{G}) : |\alpha| \leq k \}$ and $b_k$ is defined by $\sum_{\alpha \in B_k} n_\alpha^2$.
2. $k$-spheres $S_k$ are defined as $\{ \alpha \in \text{Irr}(\hat{G}) : |\alpha| = k \}$ and $s_k$ is defined by $\sum_{\alpha \in S_k} n_\alpha^2$.
3. For each $\alpha \in \text{Irr}(\hat{G})$, we define $H_\alpha = \text{span}\{ u^\alpha_i : 1 \leq i, j \leq n_\alpha \}$ and denote by $p_\alpha$ the orthogonal projection from $L^2(\hat{G})$ onto $H_\alpha$.
4. We define $H_k = \bigoplus_{\alpha \in S_k} H_\alpha = \text{span}\{ u^\alpha_i : |\alpha| = k, 1 \leq i, j \leq n_\alpha \}$ and $p_k$ the orthogonal projection from $L^2(\hat{G})$ onto $H_k$.

We say that $\hat{G}$ has a polynomial growth if there exists $C, \gamma > 0$ such that $b_k \leq C(1 + k)^\gamma$ [8] and norm structures of $H_\alpha$ and $H_k$ are inherited from $L^2(\hat{G})$.

In the sense of [47], for a compact matrix quantum group $G$, we say that its discrete dual $\hat{G}$ has the rapid decay property if there exists $D, \beta > 0$ such that
\[
\|f\|_{L^\infty(\hat{G})} \leq D(1 + k)^\beta \|f\|_{L^2(\hat{G})} \text{ for all } f \in H_k.
\] (2.9)

**Notation.** We say that the discrete dual $\hat{G}$ of a compact matrix quantum group $G$ has the rapid decay property with $\|p_k\|_{2 \to \infty} \lesssim (1 + k)^\beta$ if the above inequality (2.9) holds.

2.4. Fourier analysis on compact quantum groups. Within the framework of compact quantum groups, the Fourier transform $\mathcal{F} : L^1(\hat{G}) \to \ell^\infty(\hat{G})$, $\phi \mapsto \hat{\phi} = (\hat{\phi}(\alpha))_{\alpha \in \text{Irr}(\hat{G})}$, is defined by
\[
\hat{\phi}(\alpha)_{i,j} = \phi((u^\alpha)_{i,j}^*) \text{ for all } 1 \leq i, j \leq n_\alpha
\]
under the identification $L^1(\hat{G}) = L^\infty(\hat{G})^*$.

If $G$ is of Kac type, we call
\[
\sum_{\alpha \in \text{Irr}(\hat{G})} n_\alpha \text{tr}(\hat{\phi}(\alpha) u^\alpha) = \sum_{\alpha \in \text{Irr}(\hat{G})} \sum_{i,j=1}^{n_\alpha} n_\alpha \hat{\phi}(\alpha)_{i,j} u^\alpha_{i,j}
\]
the Fourier series of \( \phi \in L^1(\mathbb{G}) \) and denote it by \( \phi \sim \sum_{\alpha \in \text{Irr}(\mathbb{G})} n_{\alpha} \text{tr}(\hat{\phi}(\alpha)u^\alpha) \). Indeed, equality \( f = \sum_{\alpha \in \text{Irr}(\mathbb{G})} n_{\alpha} \text{tr}(\hat{f}(\alpha)u^\alpha) \) holds for all \( f \in \text{Pol}(\hat{\mathbb{G}}) \) since \( \hat{f}(\alpha) = 0 \) for all but finitely many \( \alpha \). Note that the rigorous notation for \( \text{tr}(\hat{f}(\alpha)u^\alpha) \) should be \( (\text{tr} \otimes \text{id})(\hat{f}(\alpha) \otimes 1)u^\alpha \), but we are following conventional notations of Fourier analysis on compact groups.

It is known that the Fourier transform \( \mathcal{F} : L^1(\mathbb{G}) \to \ell^\infty(\mathbb{G}) \) is contractive, and the Plancherel theorem states that

\[
\|f\|_{L^2(\mathbb{G})} = \left( \sum_{\alpha \in \text{Irr}(\mathbb{G})} n_{\alpha} \text{tr}(\hat{f}(\alpha)^* \hat{f}(\alpha)) \right)^{\frac{1}{2}} \quad \text{for all } f \in L^2(\mathbb{G}).
\]

Therefore, by the complex interpolation theorem, we obtain the Hausdorff-Young inequalities

\[
\|\hat{f}\|_{\ell^p(\mathbb{G})} \leq \|f\|_{L^p(\mathbb{G})} \quad \text{for all } f \in L^p(\mathbb{G}) \text{ and } 1 \leq p \leq 2.
\]

2.5. Complex interpolation on vector valued \( \ell^p \)-spaces. In this section, we gather some well-known facts on complex interpolation methods for Banach space valued functions from [9, Theorem 5.1.2]. See [58, Section 1] for the operator space version. For a family of Banach spaces \( \{E_k\}_{k \in \mathbb{Z}} \) with a positive measure \( \mu \) on \( \mathbb{Z} \) we define vector valued \( \ell^p \)-spaces by

\[
\ell^p(\{E_k\}_{k \in \mathbb{Z}}, \mu) = \left\{ (x_k)_{k \in \mathbb{Z}} : x_k \in E_k \text{ for all } k \in \mathbb{Z} \text{ and } \|x_k\|_{E_k} \in \ell^p(\mathbb{Z}, \mu) \right\}
\]

and the natural norm structure is

\[
\|(x_k)_{k \in \mathbb{Z}}\|_{\ell^p(\{E_k\}_{k \in \mathbb{Z}}, \mu)} = \left\{ \left( \sum_{k \in \mathbb{Z}} \|x_k\|_{E_k}^p \mu(k) \right)^{\frac{1}{p}} \right\} \quad \text{if } 1 \leq p < \infty
\]

and

\[
\|(x_k)_{k \in \mathbb{Z}}\|_{\ell^\infty(\{E_k\}_{k \in \mathbb{Z}}, \mu)} = \sup_{k \in \mathbb{Z}} \|x_k\|_{E_k} \quad \text{if } p = \infty.
\]

If \( (E_k, F_k) \) is a compatible pair of Banach spaces for all \( k \in \mathbb{Z} \) and \( \mu_0, \mu_1 \) are two positive measures on \( \mathbb{Z} \), then for any \( 0 < \theta < 1 \) we have

\[
(\ell^{p_0}(\{E_k\}_{k \in \mathbb{Z}}, \mu_0), \ell^{p_1}(\{F_k\}_{k \in \mathbb{Z}}, \mu_1))_\theta = \ell^p(\{(E_k, F_k)_\theta\}_{k \in \mathbb{Z}}, \mu) \quad (2.10)
\]

with equal norm, where \( \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p} \) and \( \mu = \frac{\mu_0^\theta \mu_1^{1-\theta}}{\mu_0^{\frac{1-\theta}{p_0}} \mu_1^{\frac{\theta}{p_1}}} \). In particular, for \( p_0 = 2 = p_1 \) and any \( 0 < \theta < 1 \), we have

\[
(\ell^2(\{E_k\}_{k \in \mathbb{Z}}, \mu_0), \ell^2(\{F_k\}_{k \in \mathbb{Z}}, \mu_1))_\theta = \ell^2(\{(E_k, F_k)_\theta\}_{k \in \mathbb{Z}}, \mu_0^{1-\theta} \mu_1^\theta). \quad (2.11)
\]

If \( (E_0, E_1) \) and \( (F_0, F_1) \) are compatible pairs of Banach spaces and if \( T : E_0 + F_0 \to E_1 + F_1 \) is a linear map bounded from \( E_j \) to \( F_j \) for \( j = 0, 1 \), then the complex interpolation theorem states that the norm of \( T : (E_0, E_1)_\theta \to (F_0, F_1)_\theta \) is bounded by \( \|T\|_{(E_0, E_1)_\theta \to (F_0, F_1)_\theta} \leq \|T\|_{E_0 \to F_0}^{1-\theta} \|T\|_{E_1 \to F_1}^\theta \).}

2.6. Examples of compact matrix quantum groups.

2.6.1. Duals of discrete groups. Let \( \Gamma \) be a discrete group and \( C^*_r(\Gamma) \) be the associated reduced group \( C^*\)-algebra generated by left translation operators \( \{\lambda_g\}_{g \in \Gamma} \). The unital \(*\)-homomorphism \( \Delta : C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes_{\min} C^*_r(\Gamma) \), \( \lambda_g \mapsto \lambda_g \otimes \lambda_g \), determines a compact quantum group \( \hat{\Gamma} = (C^*_r(\Gamma), \Delta) \) which we call the dual of the discrete group \( \Gamma \). Then \( \text{Irr}(\hat{\Gamma}) = \{\lambda_g\}_{g \in \Gamma} \) and the Haar state is the vacuum state determined by \( h : \lambda_g \mapsto \delta_{g,e} \). The associated von Neumann algebra \( L^\infty(\hat{\Gamma}) \) is the
group von Neumann algebra $VN(\Gamma)$ and $L^1(\hat{\Gamma}) = A(\Gamma)$ is called the Fourier algebra of $\Gamma$.

Moreover, if $S = \{g_j\}_{j=1}^n \subset \Gamma$ is a generating set, then $w = \sum_{j=1}^n c_{j} \otimes \lambda_{g_j} \in M_n \otimes C^*_v(\Gamma)$ makes $\hat{\Gamma}$ into a compact matrix quantum group and $\Gamma$ is said to be of polynomial growth if $b_k \lesssim (1 + k)^\gamma$ for some $\gamma$ with respect to $w$. Moreover, if $\Gamma$ has a polynomial growth, there exists a non-negative integer $\gamma \in \{0\} \cup \mathbb{N}$ such that $b_k \approx (1 + k)^\gamma$. We call the non-negative integer $\gamma$ the polynomial growth order of $\Gamma$.

If $\Gamma$ is a non-elementary hyperbolic group, it is known that $\Gamma (= \hat{\Gamma})$ has the rapid decay property with $\|p_k\|_{2 \to \infty} \lesssim 1 + k$ [23, 16].

2.6.2. Free orthogonal quantum groups and free permutation quantum groups. Let $N \geq 2$ and $C(O_N^+) \subseteq C(\mathcal{O}_N^+)$ be the universal unital $C^*$-algebra generated by $\{u_{i,j}\}_{i,j=1}^N$ satisfying

$$u_{i,j}^* = u_{i,j} \quad \text{and} \quad \sum_{k=1}^N u_{i,k} u_{k,j} = \delta_{i,j} \sum_{k=1}^N u_{i,k} u_{j,k} \quad \text{for all } 1 \leq i, j \leq N.$$ 

Then $\Delta : u_{i,j} \mapsto \sum_{k=1}^N u_{i,k} \otimes u_{k,j}$ extends to a unital $*$-homomorphism and $O_N^+ = (C(O_N^+), \Delta)$ with $u = (u_{i,j})_{i,j=1}^N$ satisfies the axioms to be a compact matrix quantum group. We call $O_N^+$ free orthogonal quantum groups [51].

On the other hand, we denote by $C(S_N^+)$ the universal unital $C^*$-algebra generated by $\{u_{i,j}\}_{i,j=1}^N$ satisfying

$$u_{i,j}^2 = u_{i,j} = u_{i,j}^* \quad \text{and} \quad \sum_{k=1}^N u_{i,k} = 1 = \sum_{k=1}^N u_{k,j} \quad \text{for all } 1 \leq i, j \leq N.$$ 

Then $\Delta : u_{i,j} \mapsto \sum_{k=1}^N u_{i,k} \otimes u_{k,j}$ again extends to a unital $*$-homomorphism and $(C(S_N^+), \Delta)$ forms a compact matrix quantum group, which we call free permutation quantum groups $S_N^+$ [52].

Let $G$ be either $O_N^+$ or $S_N^+$. Then $\text{Irr}(G) = \{0\} \cup \mathbb{N}$ [47, 8],

$$\left\{ \begin{array}{ll}
\eta_k \approx \left( \frac{N + \sqrt{N^2 - 4}}{2} \right)^k & \text{if } N \geq 3 \\
\|p_k\|_{2 \to \infty} \lesssim 1 + k & \text{if } N = 2 \end{array} \right.$$ 

2.7. Standard noncommutative semigroup and examples. Let $G$ be a compact quantum group of Kac type. We say that a semigroup $(T_t)_{t>0}$ is a standard noncommutative semigroup on $L^\infty(G)$ if $(T_t)_{t>0}$ satisfies the following assumptions [28, 26]:

1. Every $T_t$ is a normal unital completely positive maps on $L^\infty(G)$;
2. For any $t > 0$ and $f, g \in L^\infty(G)$, $h(T_t(f)g) = h(f T_t(g))$;
3. For any $f \in L^\infty(G)$, $\lim_{t \to 0^+} T_t(f) = f$ in the strong operator topology.

Note that the first two conditions imply $h(T_t(f)) = h(f)$ for any $f \in L^\infty(G)$ and that such a semigroup admits an infinitesimal generator $L$ such that $T_t = e^{tL}$.

Example 1. (1). For any connected compact Lie groups, there exists the Poisson semigroup $(T_t)_{t>0} = e^{-t(-\Delta)^{\frac{1}{2}}}$ on $L^\infty(G)$ where $\Delta$ is the Laplacian operator [41, 6].
(2). For duals of non-abelian free groups, there exists the Poisson semigroup \((T_F t)_{t>0}\) on \(VN(F_N)\) such that \(\lambda_g \mapsto e^{-t|g|}\lambda_g\) [23, 29, 34].

(3). For free orthogonal (resp. permutation) quantum groups \(O_N^+\) (resp. \(S_N^+\)) with \(N \geq 2\), there exists the heat semigroup \((T_O t)_{t>0}\) (resp. \((T_S t)_{t>0}\)) on \(L^\infty(O_N^+\)) (resp. \(L^\infty(S_N^+\))) such that \(u_{t}^{k_{i,j}} \mapsto e^{-tck}u_{t}^{k_{i,j}}\) with [19, Lemma 1.7, Lemma 1.8 and Subsection 2.2]

\[
c_k \approx \begin{cases} 
  k & \text{if } N \geq 3 \\
  k^2 & \text{if } N = 2
\end{cases}
\]

3. The optimal orders of Sobolev embedding properties for duals of polynomially growing discrete quantum groups. Recall that [1, Theorem 4.3] implies that, if \(G\) is a compact matrix quantum group of Kac type whose dual has the polynomial growth \(b_k \leq C(1 + k)^\gamma\), then we have

\[
\left( \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^{\gamma(\frac{1}{p} - 1)}} \left\| \hat{f}(\alpha) \right\|_{S_{n_\alpha}^2}^2 \right)^{\frac{1}{2}} \leq (2C)^{\frac{1}{p} - \frac{1}{2}} \| f \|_{L^p(G)} \leq \| f \|_{L^p(G)} \leq (2C)^{\frac{1}{p} - \frac{1}{2}} \| f \|_{L^p(G)}
\]

for any \(1 < p \leq 2\) and \(f \in L^p(G)\). The above constant \((2C)^{\frac{1}{p} - \frac{1}{2}}\) comes from the proofs of Theorem 3.1 and Corollary 3.5 of [59].

In this section, we provide two sufficient conditions in which the order \(\gamma\) of (3.1) is optimal. One strategy requires the existence of a standard noncommutative semigroup \((T_t)_{t>0}\) whose infinitesimal generator \(L\) satisfies

\[
L(u_{t}^{\alpha}) = -l(\alpha)u_{t}^{\alpha} \text{ with } l(\alpha) \approx |\alpha|
\]

and the other one depends on lower bounds of the growth of \(k\)-sphere \(s_k\). These two strategies explain how we are able to obtain Sobolev embedding properties with the optimal order for connected compact Lie groups and duals of polynomially growing discrete groups as already noted in [59, 61]. Furthermore, these methods also apply to the free orthogonal quantum group \(O_2^+\) and the free permutation quantum group \(S_4^+\).

3.1. An approach using semigroups. The purpose of this section is to extend some important techniques of [59, Section 6] to general compact matrix quantum groups of Kac type. Discussions in this section depend on the existence of a standard noncommutative semigroup whose infinitesimal generator behaves like the Poisson semigroup of connected compact Lie groups \(G\).

Throughout this Section, let us suppose that there exists a standard semigroup \((T_t)_{t>0}\) whose infinitesimal generator \(L\) satisfies

\[
L(u_{t}^{\alpha}) = -l(\alpha)u_{t}^{\alpha} \text{ and } l(\alpha) \approx |\alpha| \text{ for all } \alpha \in \text{Irr}(G).
\]

Indeed, if \((P_t)_{t>0}\) is the Poisson semigroup of a connected compact Lie group \(G\), then the associated infinitesimal generator satisfies

\[
L(u_{t}^{\pi}) = -\kappa_{\pi}^{\frac{1}{2}} u_{t}^{\pi} \text{ and } \kappa_{\pi}^{\frac{1}{2}} \approx |\pi| \text{ for all } \pi \in \text{Irr}(G).
\]

The connection between Sobolev embedding property and ultracontractivity has been explored in [44, 46] for measure spaces and in [27, 56] for noncommutative measure spaces. The following is a part of [56, Theorem 1.1], which is written in accordance with our notation.
Theorem 3.1 (Theorem 1.1, [56]). Let $G$ be a compact quantum group of Kac type and $(T_t)_{t>0}$ be a standard semigroup on $L^\infty(G)$ with the infinitesimal generator $L$. Then, for the semigroup $(S_t)_{t>0} = (e^{-tT_t})_{t>0}$ and $s > 0$, the following are equivalent:

1. For any $1 \leq p < q \leq \infty$,
   \[
   \|S_t(x)\|_{L^p(G)} \lesssim \|x\|_{L^p(G)}^{\frac{1}{t}} \quad \text{for all } x \in L^p(G) \text{ and } t > 0.
   \]

2. There exists $1 \leq p < q \leq \infty$ such that
   \[
   \|S_t(x)\|_{L^q(G)} \lesssim \|x\|_{L^p(G)} \quad \text{for all } x \in L^p(G) \text{ and } t > 0.
   \]

3. For any $1 < p < q < \infty$,
   \[
   \left\| (1 - L)^{-s(\frac{1}{p} - \frac{1}{q})} (x) \right\|_{L^q(G)} \lesssim \|x\|_{L^p(G)} \quad \text{for all } x \in L^p(G).
   \]

4. There exists $1 < p < q < \infty$ such that
   \[
   \left\| (1 - L)^{-s(\frac{1}{p} - \frac{1}{q})} (x) \right\|_{L^q(G)} \lesssim \|x\|_{L^p(G)} \quad \text{for all } x \in L^p(G).
   \]

Our main issue is to find the optimal order $s > 0$ in Theorem 3.1. More precisely, our aim is to find $s_0 > 0$ satisfying that
\[
\left\| (1 - L)^{-s(\frac{1}{p} - \frac{1}{q})} (x) \right\|_{L^q(G)} \lesssim \|x\|_{L^p(G)} \quad \text{for all } x \in L^p(G) \text{ if and only if } s \geq s_0
\]
for any $1 < p < q < \infty$.

Recall that a compact quantum group $G$ is called co-amenable if there exists a contractive approximate identity $(e^i)_i$ in the convolution algebra $L^1(G)$. Such a family $(e^i)_i$ satisfies $\lim_i e^i(\alpha) = Id_{w_n}$ for each $\alpha \in \text{Irr}(G)$. It is known that every dual of a polynomially growing discrete quantum group is co-amenable [8, Proposition 2.1].

Theorem 3.2. Let $G$ be a general compact quantum group of Kac type and $w : \text{Irr}(G) \to (0, \infty)$ be a positive function.

1. If $C_{w} = \sum_{\alpha \in \text{Irr}(G)} w(\alpha)n_{\alpha}^2 < \infty$, then
   \[
   \left\| \sum_{\alpha \in \text{Irr}(G)} \sqrt{w(\alpha)}p_\alpha(f) \right\|_{L^\infty(G)} \leq \sqrt{C_w}\|f\|_{L^2(G)} \quad \text{for all } f \in L^2(G).
   \]

2. If we assume that $G$ is co-amenable and
   \[
   \left\| \sum_{\alpha \in \text{Irr}(G)} w(\alpha)p_\alpha(f) \right\|_{L^\infty(G)} \leq C\|f\|_{L^1(G)} \quad \text{for all } f \in L^1(G),
   \]
   then we have $\|p_\alpha(f)\|_{L^\infty(G)} \leq n_\alpha \text{tr} \left( \tilde{f}(\alpha) \right) \leq n_\alpha^2 \left\| \tilde{f}(\alpha) \right\|_{s^2_{n_\alpha}} = n_\alpha \|p_\alpha(f)\|_{L^2(G)}$

Proof. (1) For a given $f = \sum_{\alpha \in \text{Irr}(G)} p_\alpha(f) \in L^2(G)$, we have
\[
\|p_\alpha(f)\|_{L^\infty(G)} \leq n_\alpha \text{tr} \left( \tilde{f}(\alpha) \right) \leq n_\alpha^2 \left\| \tilde{f}(\alpha) \right\|_{s^2_{n_\alpha}} = n_\alpha \|p_\alpha(f)\|_{L^2(G)}
\]
by the Hausdorff-Young inequality and the Hölder inequality, and hence
\[
\left\| \sum_{\alpha \in \text{Irr}(G)} \sqrt{w(\alpha)} p_\alpha(f) \right\|_{L^\infty(G)} \leq \sum_{\alpha \in \text{Irr}(G)} n_\alpha \sqrt{w(\alpha)} \left\| p_\alpha(f) \right\|_{L^2(G)} \leq \sqrt{C_w} \left\| f \right\|_{L^2(G)}.
\] (3.8)

(2) Let \((e^t)_i\) be a contractive approximate identity in \(L^1(G)\) and then we may assume \(e^t \in \text{Pol}(G)\) for all \(i\). Then we have
\[
1 \geq \left\| e^t \right\|_{L^1(G)} = \sup_{x \in L^\infty(G) : \|x\|_{L^\infty(G)} \leq 1} \left\langle x, e^t \right\rangle_{L^\infty(G), L^1(G)}
\geq C^{-1} \sum_{\alpha \in \text{Irr}(G)} w(\alpha)p_\alpha((e^t)^*), e^t \rangle_{L^\infty(G), L^1(G)}
= C^{-1} \sum_{\alpha \in \text{Irr}(G)} w(\alpha)n_\alpha \text{tr} \left( (\hat{e^t}(\alpha))(\hat{e^t}(\alpha))^* \right).
\]

Hence, by taking limit as \(i \to \infty\), we obtain \(\sum_{\alpha \in \text{Irr}(G)} w(\alpha)n_\alpha^2 \leq C\). \(\quad \square\)

**Remark 3.3.** In Theorem 3.2, note that (2) is a stronger converse of (1). Indeed, by duality and density arguments, (3.5) is equivalent to
\[
\left\| \sum_{\alpha \in \text{Irr}(G)} \sqrt{w(\alpha)} p_\alpha(f) \right\|_{L^2(G)} \leq \sqrt{C_w} \left\| f \right\|_{L^1(G)}\] for all \(f \in L^1(G)\). (3.9)

Then we can obtain
\[
\left\| \sum_{\alpha \in \text{Irr}(G)} w(\alpha)p_\alpha(f) \right\|_{L^\infty(G)} \leq \sqrt{C_w} \left\| \sum_{\alpha \in \text{Irr}(G)} \sqrt{w(\alpha)} p_\alpha(f) \right\|_{L^2(G)} \leq C_w \left\| f \right\|_{L^1(G)}
\] (3.10)

for all \(f \in L^1(G)\) by combining (3.5) and (3.9).

**Corollary 3.4.** Let \(G\) be a compact matrix quantum group of Kac type whose dual satisfies \(b_k \lesssim (1 + k)\gamma\) and \((T_t)_{t > 0}\) be a standard semigroup whose infinitesimal generator \(L\) satisfies
\[
L(u_{\alpha,j}^\gamma) = -l(\alpha)u_{\alpha,j}^\gamma\quad \text{and} \quad l(\alpha) \equiv |\alpha|.
\] (3.11)

Then, for any \(s \geq \frac{\gamma}{2}\), there exists a constant \(K = K(s) > 0\) such that
\[
\left\| e^{-tT_t}(f) \right\|_{L^\infty(G)} \leq K \left\| f \right\|_{L^2(G)} \] for all \(f \in L^2(G)\) and \(t > 0\). (3.12)

Moreover, the converse holds if \(b_k \approx (1 + k)^\gamma\).

**Proof.** (1) Let us assume \(s \geq \frac{\gamma}{2}\). Then, thanks to Theorem 3.2, it is sufficient to show that
\[
\sup_{0 < t < \infty} \left\{ t^{2s} \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{e^{2t(1+l(\alpha))}} \right\} < \infty.
\] (3.13)

Since there exists a constant \(C > 0\) such that \(1 + l(\alpha) \geq C(1 + |\alpha|)\), we have
\[
\sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{e^{2t(1+l(\alpha))}} \leq \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{e^{2tC(1+|\alpha|)}}.
\] (3.14)
Therefore, it is enough to prove that

\[
\sup_{0 < t < \infty} \left\{ t^{2s} \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{e^{2t(1+|\alpha|)}} \right\} < \infty.
\]  

(3.15)

Note that

\[
\sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{e^{2t(1+|\alpha|)}} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{s_k}{e^{2t(1+k)}}
\]

\[
= \lim_{n \to \infty} \left\{ b_ne^{-2t(1+n)} + \sum_{k=0}^{n-1} b_k(e^{-2t(1+k)} - e^{-2t(2+k)}) \right\}
\]

\[
\leq \sum_{k=0}^{\infty} b_k \cdot 2te^{-2t(1+k)} \lesssim t \sum_{k=0}^{\infty} (1+k)\gamma e^{-2t(1+k)}.
\]

Since \( x \mapsto x^{2s+2}e^{-2x} \) has an upper bound \( C'' = C''(s) \) on \([0, \infty)\), we have

\[
\frac{t^{2s+1}(1+k)^{\gamma+2}}{e^{2t(1+k)}} \leq \frac{4}{\gamma} \cdot \frac{t^{2s+2}(1+k)^{2s+2}}{e^{2t(1+k)}} \leq \frac{4C''}{\gamma} \text{ for any } t \geq \frac{\gamma}{4}.
\]

Therefore, \( \sup_{t \leq 0} \left\{ t^{2s+1} \sum_{k=0}^{\infty} (1+k)^{\gamma} e^{-2t(1+k)} \right\} \leq \sum_{k=0}^{\infty} \frac{4C''}{\gamma(1+k)^{\gamma}} < \infty. \) From now on, let us handle the case \( 0 < t < \frac{\gamma}{4} \). Since the function

\[
f(x) = x^\gamma e^{-2x} \text{ is } \begin{cases} \text{increasing} & \text{if } x < \frac{\gamma}{2} \\ \text{decreasing} & \text{if } x > \frac{\gamma}{2}, \end{cases}
\]

we have

\[
t^{\gamma+1} \sum_{k=0}^{\infty} (1+k)^{\gamma} e^{-2t(1+k)} \leq t \left( \sum_{k:(1+k)t \geq \frac{\gamma}{2}} + \sum_{k:(1+k)t \geq \frac{\gamma}{2}} \right) \left[ (1+k)^{\gamma} t^{\gamma} e^{-2t(1+k)} \right]
\]

\[
\leq t \left[ \frac{\gamma^\gamma}{2e^\gamma} \cdot \frac{\gamma}{2t} + \int_{\frac{\gamma}{2} - t}^{\infty} (1+x)^{\gamma} t^{\gamma} e^{-2t(1+x)} \, dx \right]
\]

\[
= t \left[ \frac{\gamma^\gamma}{2e^\gamma} \cdot \frac{\gamma}{2t} + \int_{\frac{\gamma}{2} - t}^{\infty} y^{\gamma} e^{-2y} \left( \frac{dy}{t} \right) \right]
\]

\[
= \left( \frac{\gamma}{2e^\gamma} \right)^{\gamma} \cdot \frac{\gamma}{2} + \int_{\frac{\gamma}{2} - t}^{\infty} y^{\gamma} e^{-2y} \, dy.
\]

Hence, we have

\[
\sup_{0 < t < \frac{\gamma}{4}} \left\{ t^{2s+1} \sum_{k=0}^{\infty} (1+k)^{\gamma} e^{-2t(1+k)} \right\}
\]

\[
\leq \left( \frac{\gamma}{4} \right)^{2s-\gamma} \sup_{0 < t < \frac{\gamma}{4}} \left\{ t^{\gamma+1} \sum_{k=0}^{\infty} (1+k)^{\gamma} e^{-2t(1+k)} \right\}
\]

\[
\leq \left( \frac{\gamma}{4} \right)^{2s-\gamma} \left[ \frac{\gamma}{2e^\gamma} \cdot \frac{\gamma}{2} + \int_{\frac{\gamma}{4}}^{\infty} y^{\gamma} e^{-2y} \, dy \right] < \infty.
\]
(2) Conversely, let us assume \( b_k \approx (1 + k)^\gamma \) and (3.12) hold. Then

\[
\| e^{-tT_t(f)} \|_{L^\infty(G)} \leq \frac{K^2 2^{2s} \| f \|_{L^1(G)}}{t^{2s}}, \tag{3.16}
\]

so that Theorem 3.2 (2) tells us that

\[
\sup_{0 < t < \infty} \left\{ t^{2s} \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}^2}{e^{(1 + |\alpha|)}} \right\} \leq K^2 2^{2s} < \infty \tag{3.17}
\]

Then, as in the proof of the if part, we have

\[
K^2 2^{2s} \geq t^{2s} \sum_{k=0}^{\infty} \frac{s_k}{e^{(1 + k)}} = t^{2s} \sum_{k=0}^{\infty} b_k(e^{-t(1 + k)} - e^{-t(2 + k)})
\]

\[
\geq t^{2s+1} \sum_{k=0}^{\infty} (1 + k)^\gamma e^{-t(2 + k)}
\]

\[
= t^{2s-\gamma+1} e^{-t} \sum_{k=0}^{\infty} (1 + k)^\gamma t^\gamma e^{-t(1 + k)}
\]

\[
\geq t^{2s-\gamma+1} e^{-t} \sum_{k:(1 + k) t < \gamma} (1 + k)^\gamma t^\gamma e^{-t(1 + k)}.
\]

Since the function \( x \mapsto x^\gamma e^{-x} \) is increasing in \([0, \gamma]\), we have

\[
K^2 2^{2s} \geq t^{2s-\gamma+1} e^{-t} \int_0^{\gamma} \frac{x^{\gamma+1} e^{-t(1 + x)}}{e^{(1 + |\alpha|)}} dx = t^{2s-\gamma} e^{-t} \int_0^{\gamma} y^{\gamma} e^{-y} dy.
\]

By taking \( t \to 0^+ \), we obtain \( \lim_{t \to 0^+} t^{2s-\gamma} < \infty \), and this implies \( 2s - \gamma \geq 0 \). \( \square \)

Now, thanks to Theorem 3.1, we obtain the following Sobolev embedding property with the optimal order:

**Corollary 3.5.** Let \( G \) be a compact matrix quantum group of Kac type whose dual satisfies \( b_k \approx (1 + k)^\gamma \) and let \((T_t)_{t > 0}\) be a standard noncommutative semigroup whose infinitesimal generator \( L \) satisfies

\[
L(u_{\alpha,j}^\alpha) = -l(\alpha) u_{\alpha,j}^\alpha \text{ and } l(\alpha) = |\alpha|.
\]

Then the following are equivalent:

1. For any \( 1 < p < q < \infty \) there exists a constant \( K = K(p, q) > 0 \) such that

\[
\left\| (1 - L)^{-s(\frac{1}{2} - \frac{1}{q})} (f) \right\|_{L^q(G)} \leq K \| f \|_{L^p(G)} \text{ for all } f \in L^p(G). \tag{3.19}
\]

In particular, if \( q = 2 \), we have

\[
\left( \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}}{(1 + |\alpha|)^{s(\frac{3}{2} - 1)}} \left\| \hat{f}(\alpha) \right\|_{S^2_{\alpha}}^2 \right)^{\frac{1}{2}} \leq K \| f \|_{L^p(G)} \text{ for all } f \in L^p(G) \tag{3.20}
\]

2. There exist \( 1 < p < q < \infty \) and a constant \( K > 0 \) such that

\[
\left\| (1 - L)^{-s(\frac{1}{2} - \frac{1}{q})} (f) \right\|_{L^q(G)} \leq K \| f \|_{L^p(G)} \text{ for all } f \in L^p(G). \tag{3.21}
\]

3. \( s \geq \gamma \).
Example 2. For a connected compact Lie group $G$ and the Poisson semigroup $(e^{-t(-\Delta)^{\frac{1}{2}}})_{t>0}$, the optimal order of the Sobolev embedding property coincides with the real dimension $n$ of $G$. Indeed, for any $1 < p < q < \infty$ there exists a constant $K = K(p,q) > 0$ such that
\[
\left\| (1 + (-\Delta)^{\frac{1}{2}})^{-s} \left( \frac{1}{2} \right) (f) \right\|_{L^p(G)} \leq K \| f \|_{L^p(G)} \text{ for all } f \in L^p(G)
\] (3.22)
if and only if $s \geq n$, where $n$ is the real dimension of $G$ and $\Delta$ is the Laplacian operator. Note that if part can be proved by [2, Theorem 4] and growth estimates for duals [5, equation (58)].

3.2. Under the growth rate of spheres. In this section, we provide another sufficient condition in which the order $\gamma$ of (3.1) is optimal without the existence of a standard noncommutative semigroup. The main ingredient is additional information on lower bounds of the growth rate of $k$-spheres. Indeed, for a compact matrix quantum group $G$, Corollary 3.7 tells us that the order $\gamma$ in (3.1) is optimal if its discrete dual $\hat{G}$ satisfies
\[
b_k \lesssim (1 + k)^{\gamma} \text{ and } s_k \gtrsim (1 + k)^{\gamma-1}.
\] (3.23)
Let us begin with looking at $L^2 \rightarrow L^4$ case, which is the dual of $L^4 \rightarrow L^2$ case. The following theorem is motivated by the proof of [61, Theorem 4.5.2], which is for dual of polynomially growing discrete groups.

Theorem 3.6. Let $G$ be a compact matrix quantum group of Kac type whose dual satisfies
\[
b_k \lesssim (1 + k)^{\gamma_1} \text{ and } s_k \gtrsim (1 + k)^{\gamma_2-1} \text{ with } \gamma_1, \gamma_2 \geq 1.
\] (3.24)
Also, suppose that there exists a constant $K > 0$ such that
\[
\left\| \sum_{\alpha \in \text{Irr}(G)} w(|\alpha|) n_{\alpha} \text{tr}(\hat{f}(\alpha)w^\alpha) \right\|_{L^4(G)} \leq K \| f \|_{L^4(G)} \text{ for all } f \in L^2(G)
\] (3.25)
where $w : \{0\} \cup \mathbb{N} \rightarrow (0, \infty)$ is a decreasing function. Then we have
\[
\limsup_{m \to \infty} \left\{ (1 + m)^{\frac{\gamma_2-2\gamma_1}{4}} w(m) \right\} < \infty.
\] (3.26)

Proof. For a given $f \in L^2(G)$, we denote by $T_w(f) = \sum_{\alpha \in \text{Irr}(G)} w(|\alpha|) n_{\alpha} \text{tr}(\hat{f}(\alpha)w^\alpha)$ and take $\xi_m = \frac{1}{\sqrt{m}} \sum_{\alpha \in B_m} n_{\alpha} \chi_\alpha$. Then
\[
1 = \| \xi_m \|_{L^2(G)} \| \xi_m \|_{L^2(G)} \gtrsim \| T_w(\xi_m) \|_{L^4(G)} \| T_w(\xi_m) \|_{L^4(G)} \gtrsim \| T_w(\xi_m) T_w(\xi_m) \|_{L^2(G)}
\]
and
\[
\| T_w(\xi_m) T_w(\xi_m) \|_{L^2(G)}^2 = \left\| \sum_{\alpha_1, \alpha_2 \in B_m} \frac{1}{b_m} w(|\alpha_1|) w(|\alpha_2|) n_{\alpha_1} n_{\alpha_2} \chi_\alpha \chi_\alpha \right\|_{L^2(G)}^2
\]
\[
= \frac{1}{b_m^2} \sum_{\sigma \in \text{Irr}(G)} \left| \sum_{\alpha_1, \alpha_2 \in B_m} h(\sigma) w(|\alpha_1|) w(|\alpha_2|) n_{\alpha_1} n_{\alpha_2} \chi_\alpha \chi_\alpha \right|^2
\]
\[
= \frac{1}{b_m^2} \sum_{\sigma \in \text{Irr}(G)} \left| \sum_{\alpha_1, \alpha_2 \in B_m} w(|\alpha_1|) w(|\alpha_2|) n_{\alpha_1} n_{\alpha_2} \delta_{\alpha_1 \alpha_2} \chi_\alpha \right|^2.
\]
Then the following are equivalent:

**Corollary 3.7.** Let $\mathbb{G}$ be a compact matrix quantum group of Kac type whose dual satisfies

$$b_k \lesssim (1 + k)^\gamma \quad \text{and} \quad s_k \gtrsim (1 + k)^{\gamma - 1}. \quad (3.27)$$

Then the following are equivalent:

1. For any $1 < p < 2$, we have

$$\left( \sum_{\alpha \in \mathcal{Irr}(\mathbb{G})} \frac{n_\alpha}{(1 + |\alpha|)^{s(\frac{2}{p} - 1)}} \left\| f(\alpha) \right\|_{S_{n_\alpha}^2}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^p(\mathbb{G})} \quad \text{for all} \quad f \in L^p(\mathbb{G}). \quad (3.28)$$

2. There exists $1 < p < 2$ such that

$$\left( \sum_{\alpha \in \mathcal{Irr}(\mathbb{G})} \frac{n_\alpha}{(1 + |\alpha|)^{s(\frac{2}{p} - 1)}} \left\| f(\alpha) \right\|_{S_{n_\alpha}^2}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^p(\mathbb{G})} \quad \text{for all} \quad f \in L^p(\mathbb{G}). \quad (3.29)$$

3. $s \geq \gamma$.

**Proof.** (3) $\Rightarrow$ (1) $\Rightarrow$ (2) is clear from the inequality (3.1). Let us prove (2) $\Rightarrow$ (3).

In case $1 < p \leq \frac{4}{3}$, then the Fourier transform $F : f \mapsto (\hat{f}(\alpha))_{\alpha \in \mathcal{Irr}(\mathbb{G})}$ satisfies the following:

1. the map $F : L^p(\mathbb{G}) \rightarrow \ell^2(\mathcal{S}_{n_\alpha}^2, \mu)$ with $\mu(\alpha) = (1 + |\alpha|)^{-s(\frac{2}{p} - 1)}$ is bounded by the assumption.

2. the map $F : L^2(\mathbb{G}) \rightarrow \ell^2(\mathbb{G})$ is an isometry by the Plancherel theorem.

Now, due to equality (2) in [40] or (2.11), we have

$$\langle \ell^2(\mathcal{S}_{n_\alpha}^2, \mu), \ell^2(\mathbb{G}) \rangle_{1-\theta} = \ell^2(\mathcal{S}_{n_\alpha}^2, \mu^\theta) \quad \text{for all} \quad 0 < \theta < 1. \quad (3.30)$$
Therefore, the map \( \mathcal{F} : L^4(G) \to L^2(\{S_{n_\alpha}^2\}_{\alpha \in \text{Irr}(G)}, \mu^{\frac{\theta}{2}}) \) should be bounded at \( \theta = \frac{p}{2(2-p)} \left( \Rightarrow \frac{\theta}{p} + \frac{1-\theta}{2} = \frac{3}{4} \right) \), i.e.

\[
\left( \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}}{(1 + |\alpha|)^{\frac{\theta}{2}}} \|\hat{f}(\alpha)\|_{S_{n_\alpha}^2}^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{L^4(G)}.
\] (3.31)

Then the dual statement is

\[
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}}{(1 + |\alpha|)^{\frac{\theta}{2}}} \text{tr}(\hat{f}(\alpha)u^{\alpha}) \right\|_{L^4(G)} \lesssim \|f\|_{L^2(G)}
\] (3.32)

and we obtain \( s \geq \gamma \) from Theorem 3.6.

In the case \( \frac{3}{4} < p < 2 \), let us choose \( p_0 \in (1, \frac{3}{4}) \). Then we have that

1. \( \mathcal{F} : L^{p_0}(G) \to L^2(\{S_{n_\alpha}^2\}_{\alpha \in \text{Irr}(G)}, \mu^r) \) with \( r = \frac{\gamma(\frac{p}{2}-1)}{\alpha(\frac{3}{4})} \) is bounded by the inequality (3.1) and

2. \( \mathcal{F} : L^p(G) \to L^2(\{S_{n_\alpha}^2\}_{\alpha \in \text{Irr}(G)}, \mu) \) is bounded by the given assumption.

Then, by the complex interpolation theorem, the map

\( \mathcal{F} : L^\frac{3}{4}(G) \to L^2(\{S_{n_\alpha}^2\}_{\alpha \in \text{Irr}(G)}, \mu^{\theta(1-\theta)}) \)

should be bounded for \( \theta \in (0,1) \) satisfying \( \frac{\theta}{p_0} + \frac{1-\theta}{2} = \frac{3}{4} \) by (2.11). Then the dual argument shows

\[
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}}{(1 + |\alpha|)^{\frac{\theta}{2}}(1+\theta)} \text{tr}(\hat{f}(\alpha)u^{\alpha}) \right\|_{L^4(G)} \lesssim \|f\|_{L^2(G)}.
\] (3.33)

Now, it is easy to check that

\[
s \left( \frac{1}{p} - \frac{1}{2} \right) \cdot (r\theta + 1 - \theta) = \gamma \left( \frac{1}{p_0} - \frac{1}{2} \right) \theta + s \left( \frac{1}{p} - \frac{1}{2} \right) \cdot (1 - \theta)
\]

\[
= (s - \gamma) \left( \frac{1}{p} - \frac{1}{2} \right) (1 - \theta) + \gamma \cdot \left[ \left( \frac{1}{p_0} - \frac{1}{2} \right) \theta + \left( \frac{1}{p} - \frac{1}{2} \right) (1 - \theta) \right]
\]

\[
= (s - \gamma) \left( \frac{1}{p} - \frac{1}{2} \right) (1 - \theta) + \gamma \left( \frac{3}{4} - \frac{1}{2} \right)
\]

\[
= (s - \gamma) \left( \frac{1}{p_0} - \frac{1}{2} \right) (1 - \theta) + \frac{\gamma}{4}
\]

and Theorem 3.6 asserts that \( (s - \gamma) \left( \frac{1}{p_0} - \frac{1}{2} \right) (1 - \theta) + \frac{\gamma}{4} \geq \frac{7}{2} \). Hence, we can conclude that \( s \geq \gamma \).

**Example 3.** (1). Let \( \Gamma \) be a polynomially growing discrete group with the polynomial growth order \( \gamma \). Then for any \( 1 < p \leq 2 \) we have

\[
\left( \sum_{g \in \Gamma} \frac{|f(g)|^2}{(1 + k)^{\alpha\left(\frac{3}{4}-1\right)}} \right)^{\frac{1}{2}} \lesssim \|\lambda(f)\|_{L^p(\hat{\Gamma})}
\] (3.34)

for all \( \lambda(f) \sim \sum_{g \in \Gamma} f(g)\lambda_g \in L^p(\hat{\Gamma}) \) if and only if \( s \geq \gamma \). For growth rates of \( k \)-spheres, see [13, Corollary 11].
(2). Let $G$ be either $O_{N+1}^+$ or $S_{N+3}^+$. Then for any $1 < p \leq 2$ we have
\[
\left( \sum_{k \geq 0} \frac{n_k}{(1 + k)^{s/(2 - 1)}} \right)^{2/p} \lesssim \|f\|_{L^p(G)}^{2}
\] for all $f \in L^p(G)$ if and only if $s \geq 3$.

4. Rapid decay degree of discrete quantum groups. Arguably, the most important examples of compact quantum groups are duals of free groups $\hat{F}_N$ and free quantum groups such as $O_{N+1}^+, S_{N+3}^+$. Since their discrete duals are exponentially growing if $N \geq 2$, the results from Section 3 are not applicable.

The compact quantum groups $\hat{F}_N, O_{N+1}^+, S_{N+3}^+$ with $N \geq 2$ have unique natures in view of analysis. Those are not co-amenable, the underlying $C^*$-algebras are non-nuclear, etc. However, the duals of these quantum groups satisfy the rapid decay property, which allows us to obtain Sobolev embedding properties though it is not clear what the optimal order is. To get the conclusion in Section 5, we turn our attention to a detailed analysis of the rapid decay property for $\hat{F}_N$ and duals of $O_{N+1}^+, S_{N+3}^+$.

The rapid decay degree, which was suggested in [36], is a way to quantify the rapid decay property of a discrete group, and the notion naturally extends to the framework of duals of compact matrix quantum groups of Kac type. A natural way is to define the degree of rapid decay property $\text{rd}(\hat{G})$ as the infimum of positive numbers $s \geq 0$ satisfying
\[
\|f\|_{L^\infty(G)} \lesssim \left( \sum_{\alpha \in \text{Irr}(G)} (1 + |\alpha|)^{2s} n_\alpha \left\| \hat{f}(\alpha) \right\|^2_{S_{2n}} \right)^{1/2} \text{ for all } f \in \text{Pol}(G). \tag{4.1}
\]
Note that this definition is independent of the choice of a generating unitary representation.

For discrete groups, it has turned out that $\text{rd}(\Gamma) = \frac{\gamma}{2}$ for any finitely generated discrete group $\Gamma$ with the polynomial growth order $\gamma$ [36] and that $\text{rd}(\Gamma) = \frac{3}{2}$ for any non-elementary hyperbolic groups [37].

As quantum analogues of the above results, we aim to compute the rapid decay degree of polynomially growing discrete quantum groups and duals of free quantum groups $O_N^+, S_N^+$. By theorem 3.2, we are ready to extend [36, Proposition 2.2 (2)].

**Proposition 4.1.** Let $G$ be a compact matrix quantum group of Kac type whose dual satisfies $b_k \approx (1 + k)^\gamma$. Then
\[
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha) w^ \alpha) \right\|_{L^\infty(G)} \lesssim \|f\|_{L^2(G)} \text{ for all } f \in L^2(G) \tag{4.2}
\]
if and only if $s > \frac{\gamma}{2}$. In particular, $\text{rd}(\hat{G}) = \frac{\gamma}{2}$.

**Proof.** First of all, we take a positive function $w(\alpha) = (1 + |\alpha|)^{-2s}$ with $s > \frac{\gamma}{2}$ to apply Theorem 3.2 (1). Then
\[
C_w = \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{(1 + |\alpha|)^{2s}} = \lim_{n \to \infty} \frac{b_n}{(1 + n)^{2s}} + \sum_{k=0}^{\infty} b_k \left( \frac{1}{(1 + k)^{2s}} - \frac{1}{(2 + k)^{2s}} \right),
\]
so that $C_w \lesssim \sum_{k=0}^{\infty} (1+k)^{\gamma-(2s+1)} < \infty$.

On the other hand, if we assume the inequality (4.2), the duality and density arguments give us the following equivalent inequality

$$
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha)u^\alpha) \right\|_{L^2(G)} \lesssim \|f\|_{L^1(G)} \quad (4.3)
$$

for all $f \in L^1(G)$. Then, by combining (4.2) and (4.3), we have

$$
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha)u^\alpha) \right\|_{L^\infty(G)} \lesssim \|f\|_{L^2(G)} \lesssim \|f\|_{L^1(G)}.
$$

Then Theorem 3.2 (2) implies $\sum_{k=0}^{\infty} (2+k)^{\gamma-(2s+1)} \lesssim \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha^2}{(1 + |\alpha|)^2} < \infty$ as in the above. Thus, we can conclude that $2s + 1 - \gamma > 1$, equivalently $s > \frac{\gamma}{2}$. \qed

In the case that $\hat{G}$ is exponentially growing, the following approach is valid under the rapid decay property. The proof relies on standard arguments that have been already used in [36, 10, 31, 19].

**Proposition 4.2.** Let $G$ be a compact matrix quantum group of Kac type whose dual has the rapid decay property with $\|p_k\|_{2-\infty} \leq D(1+k)^\beta$ and let $w : \{0\} \cup \mathbb{N} \to (0, \infty)$ be a positive function such that $C_w = \sum_{k \geq 0} w(k)(1+k)^{2\beta} < \infty$. Then we have

$$
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha)u^\alpha) \right\|_{L^\infty(G)} \leq D \sqrt{C_w} \|f\|_{L^2(G)} \quad \text{for all } f \in L^2(G). \quad (4.4)
$$

In particular, for any $s > \beta + \frac{1}{2}$ we have

$$
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha)u^\alpha) \right\|_{L^\infty(G)} \lesssim \|f\|_{L^2(G)} \quad \text{for all } f \in L^2(G). \quad (4.5)
$$

**Proof.** It is enough to see that for any $f \in \text{Pol}(G)$

$$
\left\| \sum_{\alpha \in \text{Irr}(G)} \frac{n_\alpha}{(1 + |\alpha|)^s} \text{tr}(\hat{f}(\alpha)u^\alpha) \right\|_{L^\infty(G)} \leq \sum_{k \geq 0} w(k) \|p_k(f)\|_{L^\infty(G)} \leq D \sum_{k \geq 0} w(k)(1+k)^\beta \|p_k(f)\|_{L^2(G)}
$$

$$
\leq D \left( \sum_{k \geq 0} w(k)^2(1+k)^{2\beta} \right)^{1/2} \|f\|_{L^2(G)}.
$$

\qed

From now on, let us try to detect the rapid decay degree of duals of free quantum groups.
Theorem 4.3. Let $\mathcal{G}$ be a compact matrix quantum group of Kac type and $w : \{0\} \cup \mathbb{N} \to (0, \infty)$ be a positive function. If we suppose that
\[
\left\| \sum_{\alpha \in \text{irr}(\mathcal{G})} w(|\alpha|) n_\alpha \text{tr}(\hat{f}(\alpha) w^\alpha) \right\|_{L^\infty(\mathcal{G})} \leq C \|f\|_{L^2(\mathcal{G})} \quad \text{for all } f \in L^2(\mathcal{G}), \tag{4.6}
\]
then there exists a universal constant $K > 0$ such that $\sum_{k \geq 0} w(k)^2 (1 + k)^2 \leq KC^2$ if $\mathcal{G}$ is one of the following:

- duals of non-elementary hyperbolic groups;
- free orthogonal quantum groups $O_N^+$ with $N \geq 2$;
- free permutation quantum groups $S_N^+$ with $N \geq 4$.

Proof. First of all, let $\Gamma$ be a non-elementary hyperbolic group and $\sigma_k = \sum_{g \in S_k} \lambda_g$. Then for any positive sequence $(a_k)_{k \geq 0}$ the main theorem in [37] states that
\[
\left\| \sum_{k \geq 0} \frac{a_k}{\sqrt{k}} \sigma_k \right\|_{L^\infty(\hat{\Gamma})} \approx \sum_{k \geq 0} (k + 1)a_k. \tag{4.7}
\]
Therefore, from the given assumption, we have
\[
C \left( \sum_{k \geq 0} a_k^2 \right)^{\frac{1}{2}} = C \left\| \sum_{k \geq 0} \frac{a_k}{\sqrt{k}} \sigma_k \right\|_{L^2(\hat{\Gamma})} \geq \left\| \sum_{k \geq 0} \frac{a_k w(k)}{\sqrt{k}} \sigma_k \right\|_{L^\infty(\hat{\Gamma})} \approx \sum_{k \geq 0} a_k w(k)(k + 1).
\]
Since $(a_k)_{k \geq 0}$ is arbitrary, we have $\sum_{k \geq 0} (k + 1)^2 w(k)^2 \lesssim C^2$.

Secondly, let $\mathcal{G}$ be $O_N^+$ (resp. $S_N^{1+}$) with $N \geq 2$. Then for the associated compact Lie group $G = SU(2)$ (resp. $G = SO(3)$), [59, Lemma 4.7] states that for any $1 \leq p \leq \infty$ and $f \sim \sum_{k \geq 0} a_k \chi_k \in L^p(G)$, we have $\|f\|_{L^p(G)} = \|\hat{f}\|_{L^p(G)}$ where $\hat{f} \sim \sum_{k \geq 0} a_k \hat{\chi}_k$ and $\hat{\chi}$ is the $k$-th character of $\mathcal{G}$. Also, note that there exists the Poisson semigroup $(\mu_t)_{t > 0} \subseteq L^1(G)$ satisfying $\mu_t \sim \sum_{k \geq 0} e^{-t \chi_k^2} (k + 1) \chi_k$ (resp. $\mu_t \sim \sum_{k \geq 0} e^{-t \chi_k^2} (2k + 1) \chi_k$) [41, 6]. Let $m_k = k + 1$ (resp. $2k + 1$).

Then, from the given assumption and [59, Lemma 6.3 (2)], we have
\[
\sum_{k \geq 0} e^{-t \chi_k^2} w(k)^2 (1 + k)^2 \leq \left\| \sum_{k \geq 0} e^{-t \chi_k^2} w(k)^2 m_k \chi_k \right\|_{L^\infty(G)} = C \left\| \sum_{k \geq 0} e^{-t \chi_k^2} w(k)^2 m_k \chi_k \right\|_{L^2(G)} \leq C^2 \|\mu_t\|_{L^1(G)} = C^2 \|\mu_t\|_{L^1(G)} = C^2
\]
The last equality comes from the fact that $\mu_t$ is a probability distribution. Since the right hand side is independent of $t$, by taking the limit $t \to 0^+$, we obtain $\sum_{k \geq 0} w(k)^2 (1 + k)^2 \leq KC^2$ for a universal constant $K > 0$. \hfill \Box

By combining Proposition 4.2 and Theorem 4.3, we can compute the rapid decay degrees of $O_N^+$ and $S_N^{1+}$. 
Corollary 4.4. Let \( G \) be \( O^+_N \) or \( S^+_N \) with \( N \geq 2 \) and \( s \geq 0 \). Then
\[
\left\| \sum_{k \geq 0} \frac{n_k}{(1 + k)^s} \frac{\tr(\hat{f}(k)a^k)}{1 + t^s} \right\|_{\mathbb{L}^\infty(\mathbb{G})} \lesssim \|f\|_{\mathbb{L}^2(\mathbb{G})} \quad \text{for all } f \in \mathbb{L}^2(\mathbb{G})
\] (4.8)
if and only if \( s > \frac{3}{2} \). In particular, \( \text{rd}(\mathbb{G}) = \frac{3}{2} \).

Proof. It is sufficient to see the only if part. By Theorem 4.3, the given assumption implies \( \sum_{k \geq 0} \frac{1}{(1 + k)^s} < \infty \), so that \( s > \frac{3}{2} \). \( \square \)

5. The optimal orders of Sobolev embedding properties for \( \hat{F}_N, O^+_N, S^+_N \).

Let \( T_t \) be the Poisson semigroup of \( \hat{F}_N \) or the heat semigroup of \( O^+_N \) or \( S^+_N \) with \( N \geq 2 \). Then some standard arguments from \([27, 19]\) together with some slight modifications to apply \([56, \text{Theorem 1.1}]\), we have the following Sobolev embedding properties of the order 3: For any \( 1 < p < q < \infty \) we have
\[
\left\| (1 - L)^{-3t}e^{\frac{1}{4}t}f(t) \right\|_{\mathbb{L}^q(\mathbb{G})} \lesssim \|f\|_{\mathbb{L}^q(\mathbb{G})}
\] (5.1)
where \( \mathbb{G} \) is one of the above mentioned compact quantum groups and \( L \) is the associated infinitesimal generator of \( T_t \).

The main question in this section is whether the above order 3 in (5.1) is optimal or not. To get the affirmative answer (See Example 4), we will handle the question associated infinitesimal generator of \( T_t \) where \( G \) is one of the above mentioned compact quantum groups and \( L \) is the associated infinitesimal generator of \( T_t \).

We begin with the following computational lemma.

Lemma 5.1. (1) For any \( t > 0 \),
\[
\sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1 + k)}} = e^{-2t}(1 - e^{-2t})^{-3}(1 + e^{-2t}).
\] (5.2)

(2) If \( a_k \approx k \), then \( \sup_{0 < t < \infty} \left\{ t^s \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1 + k)}} \right\} < \infty \) holds if and only if \( s \geq 3 \).

Proof. (2) Set \( f(t) = \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1 + k)}} \) and \( g(t) = \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1 + k)}} \). Since there exist \( D_1, D_2 > 0 \) such that \( D_1 \cdot (1 + k) \leq 1 + a_k \leq D_2 \cdot (1 + k) \), we have \( f(D_2 t) \leq g(t) \leq f(D_1 t) \). Thus, it is enough to show that \( \sup_{0 < t < \infty} \{ t^s f(t) \} \) if and only if \( s \geq 3 \). Due to the explicit form (5.2) of \( f(t) \), we have \( \lim_{t \to 0^+} t^s f(t) = \frac{1}{4} \) and \( \lim_{t \to \infty} t^s f(t) = 0 \) for any \( x \geq 0 \), which gives us the conclusion. \( \square \)

Now, we are ready to compute the optimal order of ultracontractivity of \( S_t = e^{-t}T_t \). Note that ultracontractivity of \( T_t \) has been also studied in \([19]\).

Corollary 5.2. (1) Let \( \mathbb{G} = \hat{F}_N \) with \( N \geq 2 \) and \( T_t = T_t^F \). Then there exists a universal constant \( K > 0 \) such that
\[
\|S_t(\lambda f)\|_{\mathbb{L}^\infty(\mathbb{G})} \leq \frac{K}{t^2} \left\| \frac{f}{t^2} \right\|_{\hat{F}_N} \quad \text{for all } f \in \hat{F}_N \text{ and } t > 0
\] (5.3)
if and only if \( s \geq 3 \).

(2) Let \( N \geq 3 \), \( \mathbb{G} = O^+_N \) (resp. \( S^+_N \)) and \( T_t = T_t^O \) (resp. \( T_t^S \)). Then there exists a universal constant \( K > 0 \) such that
\[
\|S_t(f)\|_{\mathbb{L}^\infty(\mathbb{G})} \leq \frac{K}{t^2} \left\| \frac{f}{t^2} \right\|_{\mathbb{L}^2(\mathbb{G})} \quad \text{for all } f \in \mathbb{L}^2(\mathbb{G}) \text{ and } t > 0
\] (5.4)
if and only if $s \geq 3$.

Proof. Recall that, in the case of (2), $T_t : u_{i,j}^k \mapsto e^{-tc_k}u_{i,j}^k$ with $c_k \approx k$.

Now, in all cases, if we suppose $s \geq 3$, then

$$C_w = \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1+k)}} \left( \text{resp.} \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1+c_k)}} \right) \lesssim \frac{1}{t^s}$$

by Lemma 5.1 (2), so Proposition 4.2 is applicable.

Conversely, from the assumed inequalities, we obtain

$$\sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1+k)}} \left( \text{resp.} \sum_{k \geq 0} \frac{(1 + k)^2}{e^{2t(1+c_k)}} \right) \lesssim \frac{1}{t^s}$$

by Theorem 4.3, so Lemma 5.1 (2) tells us $s \geq 3$.

Finally, since we have sharp estimates for the ultracontractivity of $(S_t)_{t>0} = (e^{-tT_t})_{t>0}$ (Corollary 5.2), we reach the following Sobolev embedding properties with the optimal order 3 for $\hat{F}_N$, $O_N^+$ and $S_N^+$ by Theorem 3.1:

**Example 4.** (1). Let $N \geq 2$ and $L$ be the infinitesimal generator of the Poisson semigroup $(T_t^F)_{t>0}$ of the free group $\mathbb{F}_N$. Then for any $1 < p < q < \infty$ we have

$$\left\| (1 - L)^{-s(\frac{1}{p} - \frac{1}{q})} (f) \right\|_{L^r(\mathbb{F}_N)} \lesssim \| f \|_{L^p(\mathbb{F}_N)}$$

(5.5)

for all $f \in L^p(\mathbb{F}_N)$ if and only if $s \geq 3$. Note that $1 - L : \lambda_g \mapsto (1 + |g|)\lambda_g$.

(2). Let $\mathbb{G}$ be either $O_N^+$ or $S_N^+$ with $N \geq 3$ and let $L$ be the infinitesimal generator of the corresponding heat semigroup $T_t^O$ or $T_t^S$. Then for any $1 < p < q < \infty$ we have

$$\left\| (1 - L)^{-s(\frac{1}{p} - \frac{1}{q})} (f) \right\|_{L^r(\mathbb{G})} \lesssim \| f \|_{L^p(\mathbb{G})}$$

(5.6)

for all $f \sim \sum_{k \geq 0} n_k \text{tr} (\hat{f}(k) u^k) \in L^p(\mathbb{G})$ if and only if $s \geq 3$. In particular, if $q = 2$, the above (5.6) is equivalent to

$$\left( \sum_{k \geq 0} \frac{n_k}{(1 + k)^{\frac{s}{2} - 1}} \left\| \hat{f}(k) \right\|_{S_{n_k}^2}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^p(\mathbb{G})}$$

(5.7)

6. Sobolev embedding properties under the rapid decay property. To go beyond the class of co-amenable compact quantum groups and concrete examples $\mathbb{F}_N, O_{N+1}^+, S_{N+3}^+$, we establish a general approach to explore Sobolev embedding properties for all compact matrix quantum groups of Kac type whose duals satisfy the rapid decay property (without the assumption of the existence of Poisson or heat semigroups).

Our first strategy is to generalize [60, Theorem 3.2], which we call sharpened Hausdorff-Young inequalities, to general compact matrix quantum groups of Kac type whose duals have the rapid decay property (Theorem 6.2). Then we interpolate Theorem 6.2 and [59, Corollary 3.9] to establish Sobolev embedding properties under the rapid decay property (Theorem 6.6).
6.1. **Sharpened Hausdorff-Young inequalities.** For general compact quantum groups, Hausdorff-Young inequalities state that the Fourier transform \( \mathcal{F} : L^p(\mathbb{G}) \to \ell^p(\widehat{\mathbb{G}}) \) is contractive for all \( 1 \leq p \leq 2 \). Moreover, all bounded multipliers of the form

\[
\mathcal{F}_w : L^p(\mathbb{G}) \to \ell^p(\widehat{\mathbb{G}}), \quad f \mapsto (w(\alpha)\widehat{f}(\alpha))_{\alpha \in \operatorname{Irr}(\mathbb{G})},
\]

(6.1)

arise from bounded sequences \( w = (w(\alpha))_{\alpha \in \operatorname{Irr}(\mathbb{G})} \) in many cases as follows:

**Proposition 6.1.** Let \( 1 \leq p \leq 2 \) and \( w : \operatorname{Irr}(\mathbb{G}) \to \mathbb{C} \) be a sequence. If we suppose that \( \mathcal{F}_w : L^p(\mathbb{G}) \to \ell^p(\widehat{\mathbb{G}}), \quad f \mapsto (w(\alpha)\widehat{f}(\alpha))_{\alpha \in \operatorname{Irr}(\mathbb{G})} \), is bounded and that \( \mathbb{G} \) is one of the compact quantum groups listed below:

- connected semisimple compact Lie groups
- duals of discrete groups \( \Gamma \)
- free orthogonal quantum group \( O_2^+ \)
- quantum \( SU(2) \) group

Then the given sequence \( w = (w(\alpha))_{\alpha \in \operatorname{Irr}(\mathbb{G})} \) is bounded.

**Proof.** The following proof is motivated by [60, Section 4]. For connected semisimple compact Lie groups, [21, Main theorem] gives a family of matrix elements \( \{u_{i,i}^\alpha\}_{\alpha \in \operatorname{Irr}(\mathbb{G})} \) such that \( \|u_{i,i}^\alpha\|_{L^p(\mathbb{G})} \approx n^{-\frac{1}{p}} \), so that

\[
\sup_{\alpha \in \operatorname{Irr}(\mathbb{G})} |w(\alpha)| \leq \sup_{\alpha \in \operatorname{Irr}(\mathbb{G})} \left( \frac{|w(\alpha)|}{n^{-\frac{1}{p}}} \right) \leq \sup_{\alpha \in \operatorname{Irr}(\mathbb{G})} \left( \frac{\|w(\alpha)\|_{\ell^p(\widehat{\mathbb{G}})}}{\|u^\alpha_{i,i}\|_{L^p(\mathbb{G})}} \right) < \infty.
\]

This idea applies to duals of discrete groups \( \widehat{\Gamma} \) and the free orthogonal quantum group \( O_2^+ \) similarly. It is enough to choose families of matrix elements \( \{\lambda_g\}_{g \in \Gamma} \) and \( \{u_{0,0}^n\}_{n \geq 0} \) with respect to canonical choices of orthonormal bases.

In the case of quantum \( SU(2) \) group, let us take \( \{u_{n,n}^\alpha\}_{n \geq 0} \) with respect to the canonical orthonormal basis. Then \( \|u_{n,n}^\alpha\|_{L^p(\mathbb{G})} \approx 1 \) and \( \|\widehat{u}_{n,n}^\alpha\|_{\ell^p(\widehat{\mathbb{G}})} \approx 1 \), so we obtain

\[
\sup_{\alpha \in \operatorname{Irr}(\mathbb{G})} |w(\alpha)| \approx \sup_{\alpha \in \operatorname{Irr}(\mathbb{G})} \left( \frac{\|w(\alpha)\widehat{u}_{n,n}^\alpha\|_{\ell^p(\widehat{\mathbb{G}})}}{\|u_{n,n}^\alpha\|_{L^p(\mathbb{G})}} \right) < \infty.
\]

\( \square \)

Nevertheless, [60, Theorem 3.2] has established the existence of an unbounded (exponentially) increasing sequence for the case of \( O_N^+ \) with \( N \geq 3 \), and we will adapt the proof of [60, Theorem 3.2] to general compact matrix quantum groups under the rapid decay property.

**Theorem 6.2** (A sharpened Hausdorff-Young inequality). Let \( \mathbb{G} \) be a compact matrix quantum group of Kac type whose dual \( \widehat{\mathbb{G}} \) has the rapid decay property with \( \|p_k\|_{2 \to \infty} \leq D(1 + k)^\beta \). Then we have

\[
\left( \sum_{k \geq 0} \frac{1}{(1 + k)^{p(p-2)}} \left( \sum_{\alpha \in \mathcal{S}_k} n_\alpha \left\| \widehat{f}(\alpha) \right\|_{S^2_{2n}}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq D^{\frac{1}{\beta}-1} \|f\|_{L^p(\mathbb{G})}
\]

(6.2)

for any \( 1 \leq p \leq 2 \) and \( f \in L^p(\mathbb{G}) \).
Corollary 6.4. Under the same assumption of Theorem 6.2, for any \(1 \leq p \leq 2\), we have

\[
\left( \sum_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}^{p'}}{(1 + |\alpha|)^{3(p-2)}} \cdot n_{\alpha} \left\| \hat{f}(\alpha) \right\|_{S_{n_{\alpha}}^{p'}} \right)^{\frac{1}{p'}} \lesssim \|f\|_{L^p(G)} \quad \text{for all } f \in L^p(G) \tag{6.7}
\]

Proof. It is enough to note that \(\sum_{\alpha \in S_k} n_{\alpha}^{p'} \left\| A(\alpha) \right\|_{S_{n_{\alpha}}^{p'}} \lesssim (\sum_{\alpha \in S_k} n_{\alpha} \left\| A(\alpha) \right\|^{2}_{S_{n_{\alpha}}^{p'}})^{\frac{1}{p'}}\) and \(\|A\|_{S_k^{p'}} \lesssim |A|_{S_k^{p'}}\).

Remark 6.5. In view of Corollary 6.4, if \(\sup_{\alpha \in \text{Irr}(G)} \frac{n_{\alpha}^{p'}}{(1 + |\alpha|)^{3(p-2)}} = \infty\), we are able to find an unbounded sequence \(w = (w(\alpha))_{\alpha \in \text{Irr}(G)}\) such that

\[
L^p(G) \rightarrow \ell^p(G), \quad f \mapsto (w(\alpha) \hat{f}(\alpha))_{\alpha \in \text{Irr}(G)},
\]

is bounded. \(\tag{6.8}\)

This happens when \(G\) is one of the following:

- Free orthogonal quantum groups \(O_N^+\) with \(N \geq 3\);
- Free unitary quantum groups \(U_N^+\) with \(N \geq 3\);
- Quantum automorphism group \(\mathbb{G}_{\text{aut}}(B, \psi)\) with a \(\delta\)-trace \(\psi\) and \(\dim(B) \geq 5\).

In these cases, (6.2) and (6.7) are stronger than Hausdorff-Young inequalities up to constant.

6.2. Sobolev embedding properties under the rapid decay property. In this section, we will present Sobolev embedding properties under the rapid decay property by interpolating Theorem 6.2 and Hardy-Littlewood inequalities [59, Theorem 3.8].
Theorem 6.6. Let \( G \) be a compact matrix quantum group of Kac type whose dual \( \hat{G} \) has the rapid decay property with \( \|p_k\|_{2\to\infty} \leq D(1+k)^{\beta} \). Then we have

\[
\left( \sum_{\alpha \in \text{Irr}(\hat{G})} \frac{n_{\alpha}}{(1+|\alpha|)^{(2\beta+1)(\frac{\beta}{2}-1)}} \|\hat{f}(\alpha)\|_{S_{\alpha}}^2 \right)^{\frac{1}{2}} \leq \left( \frac{2^{\beta+2}D^2}{2\beta+1} \right)^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(G)} \tag{6.9}
\]

for any \( 1 < p \leq 2 \) and \( f \in L^p(G) \).

Proof. Let \( K_k \) be the image of \( H_k \) through the Fourier transform in \( \ell^2(G) \). By [59, Corollary 3.9] and Theorem 6.2, the Fourier transform \( F: f \mapsto (\hat{f}(\alpha))_{\alpha \in \text{Irr}(\hat{G})} \) satisfies the following:

1. The norm of \( F: L^p(G) \to \ell^p(\{K_k\}_{k \geq 0}, \mu_0) \) with \( \mu_0(k) = (1+k)^{-\beta}2^{-\beta+1}(2-p) \) is bounded by \( \left( \frac{2^{\beta+2}D^2}{2\beta+1} \right)^{\frac{1}{2}-\frac{1}{p}} \). This constant comes from the proofs of Theorem 3.8 and Corollary 3.9 of [59].

2. The norm of \( F: L^p(G) \to \ell^p(\{K_k\}_{k \geq 0}, \mu_1) \) with \( \mu_1(k) = (1+k)^{-\beta}2^{\beta-2} \) is bounded by \( D\frac{1}{p-1} \).

Since \( \frac{1}{2} = \frac{1}{p} + \frac{1}{2} \cdot \frac{1}{2} \), we have \( (\mu_0^\frac{1}{p} \mu_1^\frac{1}{2}) \) and the norm of \( F: L^p(G) \to \ell^p(\{K_k\}_{k \geq 0}, \mu_0^\frac{1}{p} \mu_1^\frac{1}{2}) \) is bounded by \( \left( \frac{2^{\beta+2}D^2}{2\beta+1} \right)^{\frac{1}{2}-\frac{1}{p}} \). \( \square \)

Corollary 6.7. Let \( s \geq 2\beta + 1 \) and \( G \) be a compact matrix quantum group of Kac type whose dual has the rapid decay property with \( \|p_k\|_{2\to\infty} \leq (1+k)^{\beta} \). Suppose that there exists a standard noncommutative semigroup \( (T_t)_{t>0} \) whose infinitesimal generator \( L \) satisfies

\[
L(u_{i,j}^0) = -l(\alpha)u_{i,j}^\alpha \quad \text{and} \quad l(\alpha) \approx |\alpha|. \tag{6.10}
\]

Then for any \( 1 < p < q < \infty \) we have

\[
\left\| (1-L)^{-\frac{1}{p}}(f) \right\|_{L^q(G)} \lesssim \|f\|_{L^p(G)} \quad \text{for all} \quad f \in L^p(G). \tag{6.11}
\]

Proof. It is enough to note that

\[
\left( \sum_{\alpha \in \text{Irr}(\hat{G})} \frac{n_{\alpha}}{(1+|\alpha|)^{(\frac{2\beta}{p}-1)}} \|\hat{f}(\alpha)\|_{S_{\alpha}}^2 \right)^{\frac{1}{2}} \approx \left\| (1-L)^{-\frac{1}{p}}(f) \right\|_{L^q(G)}. \tag{6.12}
\]

Then Theorem 6.6 and Theorem 3.1 complete the proof. \( \square \)

As a byproduct of the idea to interpolate Hardy-Littlewood inequalities and sharpened Hausdorff-Young inequalities, we can show that Sobolev embedding properties with the optimal order 3 for \( \hat{F}_N, O^+_{N+1}, S^+_{N+3} \) are applicable to prove the following:

- Hardy-Littlewood inequalities on \( \hat{F}_N \) [59, Theorem 4.4] are sharp;
- Theorem 6.2 for \( \hat{F}_N \) and \( O^+_{N+1}, S^+_{N+3} \) with \( N \geq 2 \) is sharp.

More precisely, we have the following corollary:

Corollary 6.8. 1. Let \( N \geq 2 \) and \( 1 < p \leq 2 \). Then

\[
\left( \sum_{k \geq 0} \frac{1}{(1+k)^s} \left( \sum_{g \in \mathbb{F}_N: |g|=k} |f(g)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \lesssim \|\lambda(f)\|_{L^p(\mathbb{F}_N)} \tag{6.13}
\]
for all \( \lambda(f) \sim \sum_{g \in \mathcal{F}_N} f(g) \lambda_g \in L^p(\mathcal{F}_N) \) if and only if \( s \geq 4 - 2p \).

2. Let \( 1 < p \leq 2 \) and \( \mathbb{G} \) be \( \mathbb{F}_N, O_{N+1}^+ \) or \( S_{N+3}^+ \) with \( N \geq 2 \). Then

\[
\left( \sum_{k \geq 0} \frac{1}{(1 + k)^s} \left( \sum_{\alpha \in \mathbb{S}_k} n_\alpha \left\Vert \hat{f}(\alpha) \right\Vert^2_{S^2_{n_\alpha}} \right) \right)^{\frac{1}{p'}} \lesssim \| f \|_{L^p(\mathbb{G})} \tag{6.14}
\]

for all \( f \in L^p(\mathbb{G}) \) if and only if \( s \geq p' - 2 \).

**Proof.** In both cases, it is sufficient to show the only if parts. Let \( K_k \) be the image of \( H_k \) through the Fourier transform in \( L^2(\mathbb{G}) \).

1. From the assumption and Theorem 6.2, we know that the Fourier transform \( \mathcal{F} \) satisfies that

   (a) \( \mathcal{F} : L^p(\mathcal{F}_N) \to L^p(\{K_k\}_{k \geq 0}, \mu_0) \) with \( \mu_0(k) = (1 + k)^{-s} \) is bounded and

   (b) \( \mathcal{F} : L^p(\mathcal{F}_N) \to L^p(\{K_k\}_{k \geq 0}, \mu_1) \) with \( \mu_1(k) = (1 + k)^{2 - p'} \) is bounded.

Then, by applying (2.10), we have

\[
\| \mathcal{F} \|_{L^p(\mathcal{F}_N) \to L^p(\{K_k\}_{k \geq 0}, \mu)} < \infty \quad \text{with} \quad \mu(k) = (1 + k)^{-\frac{s}{p} + \frac{2}{p'}} - 1
\]

and Example 4 (1) tells us that \( \frac{s}{p} - \frac{2}{p'} + 1 \geq 3 \left( \frac{2}{p} - 1 \right) \) \( \leftrightarrow s \geq 4 - 2p \).

2. [59, Corollary 3.9] and the given assumption tell us that

   (a) \( \mathcal{F} : L^p(\mathbb{G}) \to L^p(\{K_k\}_{k \geq 0}, \mu_0) \) with \( \mu_0(k) = (1 + k)^{2p - 4} \) is bounded and

   (b) \( \mathcal{F} : L^p(\mathbb{G}) \to L^p(\{K_k\}_{k \geq 0}, \mu_1) \) with \( \mu_1(k) = (1 + k)^{-s} \) is bounded.

Then, by (2.10), we have

\[
\| \mathcal{F} \|_{L^p(\mathbb{G}) \to L^p(\{K_k\}_{k \geq 0}, \mu)} < \infty \quad \text{with} \quad \mu(k) = (1 + k)^{-\frac{2}{p} - 1}
\]

and Example 4 tells us that \( -2 + \frac{4}{p} + \frac{s}{p'} \geq 3 \left( \frac{2}{p} - 1 \right) \leftrightarrow s \geq p' - 2 \). \qed

**Remark 6.9.** The proof of Corollary 6.8 (1) is available for \( O_{N}^+ \) and \( S_{N+3}^+ \) with \( N \geq 3 \) by similar arguments. This partially reconfirms sharpness of [59, Theorem 4.5 (2)].

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Received September 2019; revised December 2019.

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