On the Global Bifurcation Diagram for the
One-Dimensional Ginzburg-Landau Model of
Superconductivity

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Abstract

Some new global results are given about solutions to the boundary value
problem for the Euler-Lagrange equations for the Ginzburg-Landau model of a
one-dimensional superconductor. The main advance is a proof that in some pa-
rameter range there is a branch of asymmetric solutions connecting the branch
of symmetric solutions to the normal state. Also, simplified proofs are presented
for some local bifurcation results of Bolley and Helffer. These proofs require
no detailed asymptotics for the solutions of the linear equations. Finally, an
error in Odeh’s work on this problem is discussed.

0.1 Introduction

In 1950 Ginzburg and Landau [16] proposed a model for the electromagnetic prop-
erties of a film of superconducting material of width $2d$ subjected to a tangential
external magnetic field. Under the assumption that all quantities are functions only
of the transverse coordinate, they proposed that the electromagnetic properties of
the superconducting material are described by a pair $(\phi, a)$ which minimizes the free
energy functional

$$G = \frac{1}{2d} \int_{-d}^{d} \left( \phi^2 (\phi^2 - 2) + \frac{2(\phi')^2}{\kappa^2} + 2\phi^2 a^2 + 2(a' - h)^2 \right) dx.$$  

The functional $G$ is now known as the Ginzburg-Landau energy and provides a mea-
sure of the difference between normal and superconducting states of the material.
The variable $\phi$ is the “order parameter” which measures the density of supercon-
ducting electrons, and $a$ is the magnetic field potential. Also, $h$ is the external magnetic

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field, and $\kappa$ is the dimensionless constant distinguishing different superconductors. So-called “type I” superconductors have $0 < \kappa < \frac{1}{\sqrt{2}}$ while $\kappa > \frac{1}{\sqrt{2}}$ for type II superconductors. (But this is really only valid for large $d$; see [2].)

The existence of minimizers for the functional $G$ is proved in a standard way, and such minimizers satisfy the following Euler-Lagrange boundary value problem:

\[
\phi'' = \kappa^2 \phi (\phi^2 + a^2 - 1) \tag{1}
\]

\[
a'' = \phi^2 a \tag{2}
\]

with boundary conditions

\[
\phi'(\pm d) = 0, \quad a'(\pm d) = h. \tag{3}
\]

It is not hard to show that the solutions of physical interest are such that $\phi > 0$ on $[-d, d]$, and this is the only kind of solution we will consider in this paper. Also, $h > 0$. Our goal is to determine for what values of $h$, $d$, and $\kappa$ the problem has solutions, and how many solutions there are in various parameter ranges.

There are two kinds of solutions of interest, so-called symmetric solutions, where $\phi(x)$ is an even function of $x$ while $a(x)$ is odd, and asymmetric solutions, where these conditions are not satisfied. There is a family of trivial solutions, called “normal states” of the form

\[
\phi(x) = 0, \quad a(x) = h(x + c),
\]

which are obviously symmetric when $c = 0$ and asymmetric otherwise. In an early paper [20] Odeh studied when non-trivial solutions may bifurcate off these normal solutions. He concluded that symmetric solutions did bifurcate from the branch of normal solutions, but as we shall see just before Lemma 2, his argument had a flaw. He also considered whether asymmetric solutions could bifurcate off the normal state, but reached no definitive conclusion.

Subsequently, Bolley and Helffer wrote a series of papers on the problem [8], [9], [10], [11], [12] and other references cited in [11]. They gave a quite thorough treatment of the local bifurcations which can occur from the normal state, with these results summarized in [11] and [12]. Among many results, they gave the correct formulation of when and how symmetric solutions bifurcate from the normal state, and did not make the error made by Odeh, though they appear not to have noticed the discrepancy with his assertions. However some of their proofs are complicated, so we will give some simplifications. The proofs below are self-contained, and in particular, we note that at least for the results considered below, it is not necessary to use detailed asymptotics for the parabolic cylinder functions which solve the relevant bifurcation equation.
A problem of particular physical interest is whether, as the strength of the magnetic field is lowered, asymmetric solutions bifurcate from the normal state before the symmetric solutions. This problem is discussed by Boeck and Chapman [7] and by Aftalion and Troy [2]. They relate this question to the formation of vortices in the medium, a phenomenon that can not be seen in the one-dimensional model. According to these authors, if \( d \) is neither too small nor too large, and if the asymmetric solutions bifurcate first, then interference between the two symmetrically placed solutions at either edge of the slab can produce a row of vortices down the center of the slab. The only rigorous result on this problem is by Bolley and Helffer, who show that when \( d \) is sufficiently large, it is indeed the case that \( h_a > h_s \). This is one of the results for which we give a simpler proof below. A related result in two dimensions with radial symmetry appears in [6].

After formulating our results we received a new paper of Aftalion and Troy [2], who did a thorough numerical study of how the bifurcation curves change with \( \kappa \) and \( d \). Based on these computations they make a variety of conjectures, some of which are related to our work. They conjecture in particular that \( h_a > h_s \) for any \( (\kappa, d) \) such that asymmetric solutions exist. (This has not been proved, though it is well accepted by physicists [2].) Subsequently Aftalion and Chapman have used methods of matched asymptotic expansions to study some of the phenomena found by Aftalion and Troy [4][5].

The first rigorous study of the global bifurcation diagram for symmetric solutions was by Kwong [19]. He proved that for any \( (\kappa, d) \) there is a unique curve of symmetric solutions, which can be given in the form \( h = h(\phi(0)) \) for \( 0 < \phi(0) < 1 \). This curve is smooth, and \( h(1) = 0, \ h(0) = h_s \). Hastings, Kwong and Troy studied the nature of this curve for large \( d \), showing that it has at least one minimum, followed by at least one maximum, if \( \kappa > \sqrt{1/2} \). This implies that for some values of \( h \) there will be at least three solutions of the boundary value problem in this range of \( \kappa \) and \( d \). They also showed that for any fixed \( \kappa \in (0, \sqrt{2}) \), if \( d \) is sufficiently large then for some range of \( h \) there will be at least two solutions. More recently, Aftalion and Troy proved that for sufficiently small \( \kappa d \), there is only one symmetric solution, and there are no asymmetric solutions [3]. (Numerically it appears that asymmetric solutions begin when \( \kappa d \) reaches approximately .905 [2].)

Up to now, very little has been done concerning the global structure of asymmetric solutions (in the parameters \( \kappa, d, h \)) or of bifurcations away from the normal states. Some initial conjectures were made by Aftalion [4]. However a numerical study by Seydel [23] shows that the picture can be quite complicated. He considers only a single configuration, namely \( d = 2.5, \ \kappa = 1 \), and presents essentially the graph in Figure 1, in which \( h \) is plotted against the value of \( a \) at the right-hand end of the interval \([-d, d]\). (Seydel uses \( a(-d) \) instead of \( a(d) \). There are a number of possible “bifurcation curves” which one can draw for this problem. For example, we could plot \( h \) vs \( \phi(0) \), as was done for symmetric solutions in [13]. We elect here to follow Seydel and plot \( a(d) \) vs \( h \). Either kind of curve gives the important information of
Figure 1: The horizontal axis is $h$ and the vertical axis is $a(d)$. The solid curve is the branch of symmetric solutions while the dotted curve is the branch of asymmetric solutions. The end points of these curves, other than $(0, 0)$, are bifurcations from a normal state.

Among the features we see here are the existence of up to seven solutions for a given $h$, and bifurcation of asymmetric solutions from the symmetric branch. It must be remembered, though, that asymmetric solutions occur in pairs, and modulo a symmetric reflection, Seydel finds up to two asymmetric solutions and three symmetric solutions for fixed $h$.

There are two obvious questions to ask concerning the Seydel result. How does the picture change as $d$ and $\kappa$ vary, and what are the stability and minimization properties of these solutions?

As stated above, the first of these questions was studied thoroughly in [3]. From their bifurcation diagrams, and results in [12] one can infer results about local stability near bifurcation points. Global minimization was studied by Hastings and Troy [17]. In addition to demonstrating the existence of asymmetric solutions for large $d$, they showed that in some parameter range there are asymmetric solutions but no non-trivial symmetric solutions. They also showed that the energy of the asymmetric solutions can be negative, so that a global minimizer of the Ginzburg-Landau functional $G$ must be asymmetric.

We have done some numerical work to consider the robustness of an asymmetric global minimizer as we move into the region where both symmetric and asymmetric solutions exist. More precisely, choosing the “Seydel” values $\kappa = 1, d = 2.5$, so that asymmetric solutions exist, we started with $h = h_a$, the asymmetric bifurcation point.
As $h$ is lowered, initially there are only asymmetric solutions, and at points along the branch of asymmetric solutions we evaluated the Ginzburg-Landau functional $G$ and found it to be negative, so that the solutions must be global minimizers. (This is in accord with the result of Hastings and Troy.) Lowering $h$ further we reached the region where there are both symmetric and asymmetric solutions. At a decreasing sequence of $h$ values, we evaluated $G$ at each of the solutions existing for these values of $h$; up to 5 distinct solutions. We found for a considerable distance down the original curve of asymmetric solutions that $G$ takes its minimum on this curve. Thus, the minimization property of this asymmetric branch appears to be very robust.

However, this was a relatively crude examination, and by no means a thorough study of the $(\kappa, d)$ parameter space. In this paper we consider only the existence of solutions of (1) – (3), and not the stability properties of these solutions.

Turning to the structure of the bifurcation diagram as $\kappa$ and $d$ change, we have used the program *Auto* [15] to produce many bifurcation diagrams for different parameter values. (Aftalion and Troy also used Auto, independently.) Here are some samples:

From these pictures we see that there are at least the following possibilities:

1. a single-valued curve of symmetric solutions in the $(h, a(d))$ plane, and no asymmetric solutions,

2. a single-valued curve of symmetric solutions, from which bifurcates a C-shaped curve of asymmetric solutions,
3. a C-shaped curve of symmetric solutions, no asymmetric solutions,

4. a C-shaped curve of symmetric solutions and a C-shaped curve of asymmetric solutions,

5. a C-shaped curve of symmetric solutions, from which there bifurcates a W-shaped curve of asymmetric solutions,

6. an S-shaped curve of symmetric solutions from which there bifurcates a W-shaped curve of asymmetric solutions.

We also see that the asymmetric curves can bifurcate from various parts of the C- or S-shaped symmetric curves. Some of these features, and others, are discussed in more detail in [2].

0.2 Statement of Results

First we consider bifurcations from the normal state. Since the normal state has \( \phi = 0 \), we rescale by letting

\[ \phi = \alpha \psi, \]

where \( \alpha = \phi(-d) \) and \( \psi(-d) = 1 \). Then,

\[ \psi'' = \kappa^2(\alpha^2 \psi^2 + a^2 - 1) \]

\[ a'' = \alpha^2 \psi^2 a \]

\[ \psi(-d) = 1, \psi'(-d) = 0, \quad a'(-d) = h. \]  

For \( \alpha = 0 \) there is a family of solutions \( (\psi_0, h(x + c)) \) where \( h = h(c) \) is chosen so that the linear problem

\[ \psi_0'' = \kappa^2(h^2(x + c)^2 - 1)\psi_0, \]

\[ \psi_0(-d) = 1, \quad \psi_0'(\pm d) = 0. \]

has a unique positive solution \( \psi_0 \). From standard linear theory [13] we have

**Lemma 1** For each \( c \) there is a unique \( h = h(c) > 0 \) such that \( (7) \) – \( (8) \) has a positive solution. When it exists, this solution is unique.
Thus (4) − (6) is degenerate at $\alpha = 0$, in the sense that there is a continuum of solutions, because $c$ is arbitrary.

To remove this degeneracy we reformulate the problem. Consider the equations (4) − (5) with initial conditions $\psi(-d) = 1, \psi'(-d) = 0, a(-d) = h(c - d)$, and $a'(-d) = h$. Then consider $\psi'(d)$ and $a'(d)$ as functions of $(\alpha, c, h)$. We wish to solve the equations

$$
\psi'(d) = 0, a'(d) - h = 0
$$

(9)

for $(c, h)$ as functions of $\alpha$.

Since

$$a'(d) - h = \int_{-d}^{d} \alpha^2 \psi(x)^2 a(x) dx$$

we replace (4) with

$$\psi'(d) = 0, \int_{-d}^{d} \psi(x)^2 a(x) dx = 0$$

(10)

We get a solution at $\alpha = 0$ by setting $h = h(c)$ as given in Lemma 1 and looking for values of $c$ such that

$$I(c) := \int_{-d}^{d} (x + c)\psi_0^2 dx = 0.$$ 

(11)

Suppose that (11) is satisfied for some $c = c_1$. This gives a solution to (10) for $\alpha = 0$, and this solution can be extended to $\alpha > 0$ provided that at $(\alpha, c, h) = (0, c_1, h(c_1))$ the determinant

$$\det\left( \begin{array}{cc}
\frac{\partial \psi_0'(d)}{\partial h} & \frac{\partial \psi_0'(d)}{\partial c} \\
\frac{\partial \psi_0'(d)}{\partial h} & I'(c)
\end{array} \right)$$

is nonzero. We will see later (equation (18)) that $I(c) = 0$ implies that $\frac{\partial \psi_0'(d)}{\partial h} = 0$, and standard theory implies that $\frac{\partial \psi_0'(d)}{\partial c} \neq 0$. Hence, a unique branch of solutions bifurcates from $(0, c_1, h(c_1))$ provided that $I'(c_1) \neq 0$.

For $c = 0$, $a = hx$ is an odd function, and this implies that $\psi_0$ is even. Thus it is automatic that $I(0) = 0$. Further, for $\alpha > 0$ we can consider instead of (4) − (6) the problem for symmetric solutions. This is (4) − (5) on $[0, d]$ with $\psi'(0) = \psi'(d) = 0, a(0) = 0, a'(d) = h$. Kwong’s result in (15) shows that this has a unique solution for each $\alpha \in (0, 1)$, defining $h$ as a function of $\alpha$. If $I'(0) \neq 0$, then this is also the unique solution in a neighborhood of the bifurcation point $(0, c_1, h(c_1))$ to (4) − (6). Hence a unique solution, which is symmetric, bifurcates from $(0, 0, h(0))$ when $I'(0) \neq 0$. But
this condition depends on $d$. When it is not satisfied we expect a more complicated picture.

In [20], Odeh claimed that $I'(0)$ is always positive (for any $d$). He may have thought this was so because he thought that $\frac{\partial \psi}{\partial c}$ was an even function of $x$ when $c = 0$, but in fact, this function is neither even nor odd. We have the following result, also proved by Bolley [9], but with a much longer proof:

**Lemma 2** Suppose that for each positive $\kappa$, and $d$, and each $c, h$ is chosen as in Lemma 1. Then for sufficiently small $\kappa d$, $I'(0) > 0$, while for sufficiently large $\kappa d$, $I'(0) < 0$.

The (relatively short) proof will be given in section 3. As a consequence we have

**Theorem 3** For any $(\kappa, d)$ there is a bifurcation of symmetric solutions from the normal state. For sufficiently large $\kappa d$ and for sufficiently small $\kappa d$ a unique curve of symmetric solutions bifurcates from the normal state. In other words, for sufficiently small $\alpha$, and for some $\delta > 0$, (4)−(6) has a unique solution with $|h - h(0)| < \delta$, and this solution is symmetric.

Further, suppose that $(\kappa_1, d_1)$ and $(\kappa_2, d_2)$ are such that $I'(0) < 0$ if $(\kappa, d) = (\kappa_1, d_1)$ and $I'(0) > 0$ if $(\kappa, d) = (\kappa_2, d_2)$, and assume that $(\kappa(t), d(t))$ is a real analytic curve $C$ joining $(\kappa_1, d_1)$ and $(\kappa_2, d_2)$, with $\kappa(0) = \kappa_1$, $d(0) = d_1$ and $\kappa(1) = \kappa_2$ and $d(1) = d_2$. Then there exists a $t_0 \in (0, 1)$ and asymmetric solutions (which are nearly symmetric) arbitrarily close to $(0, \tilde{h}_0 x)$ with $h$ near $\tilde{h}_0$, $\kappa$ near $\kappa(t_0)$ and $d$ near $d(t_0)$. Here $\tilde{h}_0$ is the eigenvalue found in Lemma 1 for $(\kappa, d) = (\kappa(t_0), d(t_0))$ and $c = 0$.

**Remark 4** Note that, unlike [9], we do not need a transversality assumption, and with care we could avoid assuming that the curve is analytic. Note also that Lemma 4 ensures that suitable points $(\kappa_1, d_1)$ and $(\kappa_2, d_2)$ exist.

The asymmetric solutions obtained in Theorem 3 are, at least initially, nearly symmetric, since they start from $c = 0$. A different sort of bifurcation of asymmetric solutions was obtained by Bolley and Helffer [11], and independently, with a different proof, by Hastings and Troy [17]. In this case we consider a fixed large $\kappa d$, and vary $c$, looking for other values of $c$ where $I(c) = 0$. The symmetry in the problem means we only have to consider positive $c$. It is obvious that for $c \geq d$, $I(c) > 0$, so using Lemma 2 we have:

**Corollary 5** For sufficiently large $\kappa d$, there is at least one $c > 0$ where $I(c) = 0$.

In fact, there is only one such $c$ and asymmetric bifurcation occurs at this point. Thus, at this positive $c$ where $I(c) = 0$, we have $h = h_0$. The uniqueness of this positive $c$ was initially shown by Bolley and Helffer [11], but here we will give a simpler, self-contained, proof.
Theorem 6  For sufficiently large $\kappa d$ there is exactly one $c = c_1 > 0$ such that $I(c_1) = 0$. (By symmetry there is also one negative $c$ with this property.) Further, $h(c_1) > h(0)$.

Finally, it is not hard to show that bifurcation does not occur for small $\kappa d$ [11].

Now we turn to more global results. The goal now is to show that bifurcation of asymmetric solutions can occur from the interior of the symmetric branch, rather than just from normal states, and show that the resulting branch of asymmetric solutions can be continued in the $(h,a(-d))$ plane ($d$ large and fixed) to the asymmetric bifurcation point which was found in Theorem [13]. To state this result we must first recall a result of Kwong [19]. This result is about symmetric solutions, and concerns the global bifurcation curve of symmetric solutions, for any fixed $d$ and $\kappa$. Since we are considering only symmetric solutions, we consider (1) with the following initial conditions:

\[
\phi(0) = \alpha, \quad \phi'(0) = 0, \quad a(0) = 0, \quad a'(0) = \delta,
\]

where $\alpha \in (0,1)$ and $\delta > 0$ are to be chosen such that $\phi > 0$ on $[0,d]$ and $\phi'(d) = 0$. By continuing the resulting solution with $\phi$ even and $a$ odd to the entire interval $[-d,d]$ we get a symmetric solution to (1), (2).

Kwong’s result is:

Lemma 7  For each $\alpha \in (0,1]$ there is a unique $\delta > 0$ such that the solution of (1), (11) is positive and satisfies $\phi'(d) = 0$.

Hence, for each $\alpha$ we obtain a unique $h = a'(d)$ and a unique $a(d)$. Plotting $a(d)$ vs $h$ gives the global bifurcation curve for symmetric solutions in the form referred to above. An alternative form as used in [18] is to plot $h$ vs $\alpha$. The main new result of this paper is:

Theorem 8  If the product $\kappa d$ is sufficiently large, then bifurcation of asymmetric solutions occurs somewhere along the curve of symmetric solutions. For any fixed $\kappa$, if $d$ is sufficiently large, there is a continuum of asymmetric solutions which connects the curve of symmetric solutions to an asymmetric normal state.

Remark 9  Here we are identifying points in the $(a(d),h)$ bifurcation diagram corresponding to asymmetric pairs of solutions. The asymmetric normal state referred to in this theorem must be the one discussed in Corollary [12] and Theorem [13], since to within a reflection there is only one asymmetric bifurcation point from the normal state. By real analyticity the continuum in this theorem contains a curve which is parametrized (in some sense) by $h$. 
Remark 10 Also, it is expected that for $\kappa > \frac{1}{\sqrt{2}}$, if $d$ is sufficiently large then the curve of symmetric solutions is S-shaped, so that for some values of $h$ there are three solutions. In [18] it was shown that there are at least three solutions, which is consistent with this conjecture. Theorem 8 shows that bifurcation must occur somewhere along this curve, but we are not able to prove anything about the location of the bifurcation point on this curve. Similarly, for $\kappa < \frac{1}{\sqrt{2}}$ there are at least two solutions for large $d$, and the bifurcation seems to occur on either of the two branches.

Remark 11 A problem which we have not been able to solve is to determine the direction of the bifurcation. As a result, we have not been able to prove that there are some values of the parameters $(d, \kappa, h)$ where there are five distinct solutions, as was seen in Seydel’s original numerical result. In [14] there is a discussion of the stability of solutions bifurcating from the normal states. This involves consideration of the energy functional $G$, and we have not studied this topic here.

1 Proofs

1.1 Local Results

In this section, we will denote $\psi_0$ by $\psi$, since we are only considering the linear equations. Thus, $\psi$ is assumed to satisfy (7)−(8).

Proof of Lemma 2

In (7)−(8) we can make the change of variables $x \to \kappa x$, which removes the term $\kappa^2$ from the differential equation, while replacing $d$ with $D = \kappa d$, and also introducing a new $h$ and $c$. The latter changes are immaterial in this and subsequent results about bifurcation from the normal state, so without loss of generality we will simply assume in (7)−(8) that $\kappa = 1$, and to remind us of this, replace $d$ with $D$. Multiply (7) by $\psi'$ and integrate by parts to get

$$2h^2 I(c) = (h^2(D + c)^2 - 1) \psi(D)^2 - (h^2(-D + c)^2 - 1) \psi(-D)^2.$$  \hfill (13)

Consider (7) with the initial conditions $\psi(-D) = 1, \psi'(-D) = 0$ and denote the solution by $\psi(x, h, c)$. Let $p = \frac{\partial \psi}{\partial h}$ and $q = \frac{\partial \psi}{\partial c}$. Then

$$p'' = (h^2(x + c)^2 - 1) p + 2h(x + c)^2 \psi, \quad p(-D) = p'(-D) = 0,$$  \hfill (14)

and

$$q'' = (h^2(x + c)^2 - 1) q + 2h^2(x + c)^2 \psi, \quad q(-D) = q'(-D) = 0.$$  \hfill (15)
Lemma \[ \text{s} \] tells us that (4) – (8) define \( h \) as a function of \( c \). We can also see this locally by applying the implicit function theorem, solving the equation

\[
\psi'(D, h, c) = 0
\]

for \( h \) as a function of \( c \). We can do this if \( p'(D) \neq 0 \). Multiply (14) by \( \psi \) and (7) by \( p \) and integrate from \(-D\) to \( D\). With the boundary and initial conditions for \( \psi \) and \( q \), this gives

\[
\psi(D)p'(D) = 2h \int_{-D}^{D} (x + c)^2 \psi(x) dx > 0.
\]

Hence \( h \) is a smooth function of \( c \). Further,

\[
\frac{dh}{dc} = -\frac{q'(D)}{p'(D)}.
\]

Now multiply (15) by \( \psi \), (7) by \( q \), subtract and integrate, and use the boundary conditions again to get

\[
\psi(D)q'(D) = 2h^2 \int_{-D}^{D} (x + c)\psi(x)^2 dx = 0.
\]

This shows that \( \frac{dh}{dc} = 0 \) whenever \( I = 0 \). In particular, this is true at \( c = 0 \).

Now differentiate (13) with respect to \( c \) and then set \( c = 0 \). Since, then, \( \psi(\pm D) = 1 \), we get:

\[
h^2 I'(0) = 2h^2 D + (h^2 D^2 - 1)q(D).
\]

From (7) and (14) the equation obtained at \( c = 0 \) is

\[
\left( \frac{q}{\psi} \right)' = \frac{\int_{-D}^{x} 2h^2 s\psi(s)^2 ds}{\psi(x)^2}.
\]

Since \( \psi \) is an even function at \( c = 0 \), the right side of (20) is zero when \( x = D \). Further, the integrand is negative for \( s < 0 \) and positive for \( s > 0 \), and this implies that the integral is strictly negative for \(-D < x < D\).

Now return to (11). We see that \( \psi'' \geq -\psi \), and since \( \psi'(0) = 0 \), this leads to \( \psi(x) \geq \psi(0) \cos x \) on \([0, D]\) for small \( D \). It is therefore seen that as \( D \to 0 \),

\[
\max_{x \in [0, D]} |\psi(x) - 1| \to 0.
\]

Integrating the right side of the differential equation for \( \psi \) then shows that \( \lim_{D \to 0} hD = \sqrt{3} \), so \( h \to \infty \). From (20) it follows that \( \left( \frac{q}{\psi} \right)' = O(h) \) as \( D \to 0 \), so \( q(D) = O(hD) = O(1) \). Hence from (13) we see that for sufficiently small \( D \), \( I'(0) > 0 \).

To complete the proof of Lemma 2 there remains to show that \( I'(0) < 0 \) when \( D \) is sufficiently large. We first need to show that \( h \) is bounded as \( D \to \infty \). In fact, it
is well known, e.g. [14], that \( h \to 1 \) as \( D \to \infty \), but the following technique quickly shows that \( h \) is at least bounded: Let \( \rho = \frac{\psi'}{\psi} \). Then

\[
\rho' = h^2 x^2 - 1 - \rho^2, \quad \rho(0) = \rho(D) = 0,
\]

with \( \rho < 0 \) in \((0, D)\). (Remember that we are only considering \( c = 0 \) here.) It is easy to see from (21) that if \( h \to \infty \), then \( D \to 0 \). Hence, as \( D \to \infty \), \( h \) must remain bounded.

It is also well known that \( h > 1 \), but for completeness here is a quick proof: Compare \( \rho \) from (21) with the solution \( \sigma = -x \) of

\[
\sigma' = x^2 - 1 - \sigma^2, \quad \sigma(0) = 0.
\]

An easy comparison shows that if \( h \leq 1 \) then \( \rho(0) = \sigma(0) = 0 \) implies that \( \rho \leq \sigma \) for all \( x \geq 0 \), which contradicts \( \rho(D) = 0 \).

Lemma 2 now follows from (19) if we can show that \( q(D) \) does not tend to zero as \( D \) tends to infinity. To do this we again use (20), and also (7). Multiply (7) by \( \psi' \) and integrate from \(-D\) to \( x \). This, with (8) and (20), leads to

\[
\left( \frac{q}{\psi} \right)' \leq (h^2 x^2 - 1).
\]

Since \( h \) is bounded for large \( D \), there is some interval around \( x = 0 \) of fixed length \( \mu > 0 \) in which \( \left( \frac{q}{\psi} \right)' \leq -\frac{1}{2} \). In addition, \( \frac{q}{\psi} \) is decreasing on the entire interval \([-D, D]\), so \( \frac{q(D)}{\psi(D)} \leq -\frac{1}{2} \mu \). But \( \psi(D) = 1 \) when \( c = 0 \). Then (13) shows that \( I'(0) < 0 \) for large \( D \). This proves Lemma 2.

Proof of Theorem 3

Note that \( I'(0) \) is a function of \( \kappa \) and \( d \). (We are no longer assuming that \( \kappa = 1 \).) In fact, it is a real analytic function of \( \kappa \) and \( d \), which is seen by observing that one can use the implicit function theorem to prove that \( \psi \) is a real analytic function of \( (\kappa, d) \). Thus \( I'(0) \) is a real analytic function of \( t \) along the curve \( C \), since \( C \) is real analytic. Since \( I'(0) \) does not vanish identically, being nonzero at the endpoints of \( C \), its zeros are isolated. Therefore there is a \( t_0 \in (0, 1) \) such that \( I'(0) \) has a strict change of sign as we cross \((\kappa(t_0), d(t_0))\) on \( C \).

We now look for solutions \( \phi = \alpha(\psi_0 + w) \), \( a = h(x + c) + \rho \), where \( \psi_0 \) is the eigenfunction at \( t = t_0 \), \( h \) is close to \( h_0 \), and the numbers \( \alpha \) and \( c \) and function \( w \) and \( \rho \) are small, and where \( w \) is orthogonal to \( \psi_0 \) and \( \rho \) is orthogonal to 1, over \((-d, d)\). Equation (1) can be written as

\[
-(\psi_0 + w)'' = \kappa^2 (\psi_0 + w)(1 - a^2 - \alpha^2(\psi_0 + w)^2).
\]
We now use Fredholm alternative theory and the implicit function theorem in a standard way ([22], chapter 9) to show that for every \((\alpha, c, h)\) with \(\alpha, c,\) and \((h - h_0)\) sufficiently small, there is a unique pair \((\beta, \gamma)\) such that the equations

\[
-w'' = k^2(w + w)(1 - (h(x + c) + \rho)^2 - \alpha^2(w + w)^2) + \beta w
\]

\[
\rho'' = \alpha^2(w + w)^2(h(x + c) + \rho) + \gamma
\]

have a unique solution \((w, \rho)\) such that \(w'(\pm d) = \rho'(\pm d) = 0\). In other words, we subtract off suitable multiples \(\beta w_0\) of \(w_0\) and \(\gamma \cdot 1\) of 1 in equations (23) - (24) respectively in order to find solutions \((w, \rho)\) in the space

\[
\{w \in C^1[-d, d] : w\text{ is orthogonal to } w_0, w'(-d) = 0\} \times \{\rho \in C^1[-d, d] : \rho\text{ is orthogonal to } 1, \rho'(\pm d) = 0\}
\]

We obtain \(w, \rho, \beta, \gamma\) as smooth functions of \(\alpha, c, h\). Here we are really solving a projected equation. This is a Lyapunov-Schmidt reduction. We want to find solutions of (23) - (24) with \(\beta = \gamma = 0\).

Let \(\hat{\rho}\) denote the solution of

\[
\hat{\rho}'' = \psi_0^2 x, \quad \hat{\rho}'(\pm d) = 0
\]

with mean value zero. Using (2) it is easily shown that

\[
\rho(\alpha, h, c) = h\alpha^2(\hat{\rho} + o(1))
\]

as \((\alpha, h, c) \to (0, h_0, 0)\). Note here that \(s\psi_0^2(s)\) has mean value zero. Hence,

\[
a^2 = (h(x + c) + \alpha^2(\hat{\rho} + o(1)))^2 = h^2(x + c)^2 + 2\alpha^2 h(x + c)\hat{\rho} + \text{terms of higher order.}
\]

(That is, higher order in \(\alpha\).) Note that as \((\alpha, h, c) \to (0, h_0, 0)\), \(w\) and \(\rho\) tend to zero uniformly in \([-d, d]\).

Now multiply (23) by \(\psi_0\) and integrate, using the orthogonality of \(w\) and \(\psi_0\), to get

\[
(h^2 - h_0^2)\int_{-d}^{d} s^2\psi_0(s)^2 ds + o(1)ds + \int_{-d}^{d} o(1)ds = \beta \int_{-d}^{d} \psi_0(s)^2 ds,
\]

where the \(o(1)\) terms are terms in \(w\) and \(\rho\) which are smooth and tend to zero as \((\alpha, h, c) \to (0, h_0, 0)\). (Also, the derivative with respect to \(h\) of the second \(o(1)\) term is small if \(\alpha\) is small, \(h\) is near \(h_0\) and \(c\) is small.) To obtain a solution of (22) we set \(\beta = 0\) in (27) and use the implicit function theorem to solve this equation for \(h\) as a function of \((\alpha, c)\) near \(h = h_0, \alpha = c = 0\). The coefficient of \((h^2 - h_0^2)\) in the first term on the left of (27) is of order 1. For each small \(\alpha \neq 0\) and each small \(c\) there will be a unique \(h = h(\alpha, c)\) near to \(h_0\) for which (27) implies that \(\beta = 0\).
Now integrate (2) over $[-d,d]$ and use the boundary conditions to obtain, upon dividing by $\alpha^2h$ the equation

$$R(\alpha, h, c) \equiv \frac{1}{\alpha^2h} \int_{-d}^{d} \phi^2a \, dx = \int_{-d}^{d} ((x+c)\psi_0^2(x) + o(1)) \, dx = 0$$

where the small $o(1)$ term is a smooth function of $\alpha, h, c$ and is zero if $\alpha = c = h - h_0 = 0$. Note that $R(\alpha, h, c) = 0$ is equivalent to $\gamma = 0$. Also, $R(0, h_0, c) = I(c)$. Letting $R'_3(\alpha, h, c) = \frac{\partial R}{\partial \kappa}$, we have for large $\kappa$, by Lemma 2, that $R'_3(0, h_0, 0) < 0$, so $R'_3(\alpha, h, c) < 0$ if $\alpha$ and $c$ are small and $h$ is close to $h_0$. Also, $R(\alpha, h, 0) = 0$ because then both $w$ and $\psi_0$ are even functions of $x$ and $a$ is an odd function of $x$. Hence, if $I'(0) < 0$, then $R(\alpha, h, c) < 0$ if $\alpha$ is small, $c$ is positive and small and $h$ is near to $h_0$. Similarly, if $I'(0) > 0$, then $R(\alpha, h, c) > 0$ if $\alpha$ is small, $c$ is small and positive, and $h$ is near to $h_0$.

We now solve (27) for $h$ as a function of $c$ and $\alpha$, for small positive $\alpha$ and $c$. (We do not know the sign of $h - h_0$.) We do this for $(\kappa, d)$ on the curve $C$ with $(\kappa, d)$ close to $(\kappa(t_0), d(t_0))$. On this curve, on one side of $(\kappa(t_0), d(t_0))$, $I'(0) < 0$ and hence $R(\alpha, h(\alpha, c), c) < 0$ if $\alpha$ and $c$ are small and $c > 0$. How small $\alpha$ and $c$ must be depends on the particular point on the curve $C$. For a given $t_1 < t_0$ but close to $t_0$, find positive $\alpha_1$ and $c_1$ such that $R(\alpha, h(\alpha, c), c) < 0$ if $0 < \alpha \leq \alpha_1$ and $0 < c \leq c_1$. Choose a fixed $t_2 > t_0$ but close to $t_0$, and then lower $\alpha_1$ and $c_1$ if necessary so that $R(\alpha, h(\alpha, c), c) > 0$ if $0 < \alpha \leq \alpha_1$ and $0 < c \leq c_1$. Then somewhere between $t_1$ and $t_2$, as we keep $\alpha$ and $c$ nonzero and fixed in $(0, \alpha_1]$ and $[0, c_1]$ and move along $C$, $R$ must equal $0$, which gives the required solution. (Note that $h_0$ varies continuously with $(\kappa, d)$ but this does not affect the argument since everything varies continuously along $C$. ) Since we can choose $c$ arbitrarily small, the solutions will be nearly symmetric.

Proof of Theorem 3.

As previously, for the linearized problem (9) – (11) we can rescale to eliminate $\kappa$, so we will again assume that $\kappa = 1$ and replace $d$ with $D$. We saw in the proof of Lemma 2 that for large $D$, $I(0) = 0$, $I'(0) < 0$, and that consequently Corollary 4 holds. Further, the definition of $I(c)$ shows that $I(c) > 0$ for $c \geq D$.

From (9) – (11), with $\psi > 0$, we see that $\tau(x) = h(x + c)^2 - 1$ must change sign in $(-D, D)$, and from (13) it follows that if $I(c) = 0$, then $\tau(-D)$ and $\tau(D)$ must have the same sign, so $\tau(\cdot)$ has exactly two zeros in $(-D, D)$, with $\psi'(x)$ changing from positive to negative and back to positive as $x$ increases from $-D$ to $D$. Therefore, $\psi$ has a local maximum at some $x_0 \in (-D, D)$, with $\psi' > 0$ on $(-D, x_0)$ and $\psi' < 0$ on $(x_0, D)$.

**Lemma 12** For any $c \in (0, D)$, $h > 1$. 

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Proof: Let $\rho(x) = \frac{\psi'(x)}{\psi(x)}$. Then $\rho(-D) = \rho(x_0) = \rho(D) = 0$. Also,

$$\rho' = h^2(x + c)^2 - 1 - \rho^2. \quad (28)$$

Further, let $\sigma(x) = -x - c$. Then

$$\sigma' = (x + c)^2 - 1 - \sigma^2, \quad \sigma(-c) = 0. \quad (29)$$

But $\rho(-D) < \sigma(-D)$, and an easy comparison of (28) and (29) shows that if $h \leq 1$, then $\rho < \sigma$ on $[-D, D]$, so $\rho(D) < 0$, a contradiction. This proves Lemma 12.

**Lemma 13** For sufficiently large $D$, $(7)$ - $(11)$ has no solution with $\psi > 0$ and $0 < c \leq D/2$.

Proof: Recall that $I(0) = 0$, $I'(0) < 0$. Let $c_1 = \inf \{c > 0 | I(c) = 0 \}$. Then $I'(c_1) \geq 0$. Assume that $c_1 \leq D/2$. We will obtain a contradiction by differentiating (13) with respect to $c$ and setting $c = c_1$.

Recall that $\frac{dh}{dc} = 0$ whenever $I(c) = 0$. Therefore, from (13) we get

$$2h^2I'(c_1) = 2h^2(D + c_1)\psi(D)^2 + 2(h^2(D + c_1)^2 - 1)\psi(D)q(D) + 2h^2(D - c_1)\psi(-D)^2 \quad (30)$$

where $q = \frac{\partial \psi}{\partial c}$, so $q$ satisfies (13). From (13) and (7) - (8) we obtain (20) with $s$ replaced by $(s + c_1)$, and as in the proof of Lemma 2, we then see that $\left(\frac{q}{\psi}\right)' < 0$ in $(-D, D)$ and $\left(\frac{q}{\psi}\right)' \leq h^2(x + c_1)^2 - 1$. Also, as in Lemma 2 it is seen that $h$ is bounded as $D \to \infty$, and this leads to a negative upper bound of the form

$$\left(\frac{q(D)}{\psi(D)}\right) \leq -\eta < 0, \quad (31)$$

where $\eta$ is independent of $D$ and $c_1 \in [0, D/2]$.

From $I(c_1) = 0$ and (13) we obtain

$$\psi(D)^2 = \frac{h^2(D - c_1)^2 - 1}{h^2(D + c_1)^2 - 1}\psi(-D)^2 \geq \frac{h^2(D/2)^2 - 1}{2h^2D^2 - 1}\psi(-D)^2.$$ 

Hence, $\frac{\psi(-D)^2}{\psi(D)^2}$ is bounded as $D \to \infty$. Then (31) and (30) show that $I'(c_1) < 0$ for sufficiently large $D$. This contradiction proves Lemma 13.

Continuing with the proof of Theorem 3, we now assume that $c \geq D/2$. The result will follow if we can show that $I'(c_1) > 0$ for any solution of (7) - (11) in this range of $c$. As before, we assume that $(c_1, h, \psi)$ is a solution, and that $\psi$ has its maximum over
\([-D, D]\) at \(x_0 \in (-D, D)\). Also, as before, we know that \(|x_0 + c_1| < 1\), so \(x_0 \leq -\frac{D}{2} + 1\).

We translate the origin to \(x_0\), letting \(\psi(x) = \chi(x - x_0) = \chi(y)\), so that

\[
\chi''(y) = (h^2(y + x_0 + c_1)^2 - 1)\chi(y).
\]

Now let \(\rho(y) = \frac{\chi'(y)}{\chi(y)}\) (a shift from the previous \(\rho\)) and let \(\omega(y) = -\rho(-y)\). Then

\[
\rho' = h^2(y + x_0 + c_1)^2 - 1 - \rho^2, \quad \rho(0) = \rho(D - x_0) = 0
\]

and

\[
\omega' = h^2(y - x_0 - c_1)^2 - 1 - \omega^2, \quad w(0) = w(D + x_0) = 0.
\]

It is important to recall that \(x_0 < -\frac{D}{2} + 1\).

We now need estimates on \(\rho\) and \(\omega\), which we obtain from the following result:

**Lemma 14** Suppose that for some constants \(\delta\) and \(\Delta\), with \(h|\delta| \leq 1\) and \(\Delta\) large, \(\eta(\cdot)\) solves the boundary value problem

\[
\eta'(y) = h^2(y + \delta)^2 - 1 - \eta^2, \quad \eta(0) = \eta(\Delta) = 0, \quad \eta < 0 \text{ in } (0, \Delta).
\]

Then,

\[
\eta(y) > -hy - \beta
\]

on \([0, \Delta]\), where \(\beta = \sqrt{h - 1 + h^2\delta^2}\), and

\[
\eta(y) < -h(y + \delta) + 1
\]

on \([0, \Delta - 1]\).

Proof:

Inequality \((35)\) follows by assuming equality at some \(y_1 \in (0, \Delta)\) and using \((34)\) to show that \(\eta'(y_1) < -h\). This would imply that \(\eta(y) < -hy - \beta\) for \(y > y_1\), so that \(\eta\) could not vanish at \(\Delta\). Next, observe that \((33)\) holds as long as \(h(y + \delta) \leq 1\), since \(\eta < 0\) on \((0, \Delta)\). If equality holds at some \(y_1 > 0\), then \(h(y_1 + \delta) > 1\) and \(\eta'(y_1) = -2 + 2h(y_1 + \delta) > 0\). Also, \(\eta < 0\) and \(\eta' > 0\) imply that \(\eta'' > 0\). Hence, \(\eta' > -2 + 2h(y_1 + \delta) > 0\) as long as \(\eta < 0\), and if \(\eta < 0\) on \([y_1, y_1 + 1]\), then

\[
\eta(y_1 + 1) \geq \eta(y_1) - 2 + 2h(y_1 + \delta) = -1 + h(y_1 + \delta) > 0.
\]

This contradiction shows that \(\eta = 0\) somewhere in \([y_1, y_1 + 1]\). If \(y_1 \leq \Delta - 1\), then \(\eta = 0\) before \(y = \Delta\), a contradiction which proves Lemma \(14\).

Applying \((33)\) to \(\rho\) with \(\delta = x_0 + c_1\) and \(\Delta = D - x_0\) shows that

\[
\psi(D) = \chi(D - x_0) \leq \psi(x_0)e^{-\frac{1}{2}(D+c_1-1)^2 + \frac{1}{2}(x_0+c_1)^2 + D - x_0 - 1},
\]

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while applying (35) to $\omega$ with $\delta = -x_0 - c_1$ gives

$$\psi(-D) \geq \psi(x_0) e^{-\frac{1}{2}(D+x_0)^2-\beta(D+x_0)}.$$ 

Combining these and noticing that $2Dx_0 < -2x_0^2$ and $\beta < h$, we find that for $c_1 \geq \frac{D}{2}$,

$$\frac{\psi(D)}{\psi(-D)} \leq e^{-rhD^2} \tag{37}$$

for some $r$ which is independent of $c_1$, $h$, and $D$.

We now return to (30). Since $\psi''(-D) \geq 0$, we have $h^2(D - c_1) \geq h$. Further, the proof (following equation (20)) that when $c = 0$, $h$ is bounded as $D \to \infty$ easily extends to $c \geq 0$, since $x_0 \leq -c + 1$. From (37) it follows that a bound of the form

$$q(D) \geq -Lh^mD^a\psi(D) \tag{38}$$

for some $L > 0$ independent of $c_1$, $h$, or $D$ will imply that $I'(c_1) > 0$ for large $D$. There are two cases to consider, namely, $-c_1 - 1 \leq x_0 \leq -c_1$ and $-c_1 \leq x_0 \leq -c_1 + 1$. We consider the first, the two cases being similar. Repeating the derivation of (20) for $c_1 > 0$ we obtain that

$$\frac{q(D)}{\psi(D)} = 2h^2\left(\int_{-D}^{x_0} + \int_{x_0}^{-c_1} + \int_{-c_1}^{D} \frac{1}{\psi(x)^2} \int_{-D}^{x} (s + c_1)\psi(s)^2 dsdx\right). \tag{39}$$

In the first of the three integrals with respect to $x$, $-D \leq s \leq x \leq x_0$, so $\psi(s) \leq \psi(x)$, and this term contributes less than $O(D^3)$ to $\frac{q(D)}{\psi(D)}$ as $D \to \infty$. In the third of the three integrals, we use the fact that $I(c_1) = 0$ to write

$$\int_{-c_1}^{D} \frac{1}{\psi(x)^2} \int_{-D}^{x} (s + c_1)\psi(s)^2 dsdx = -\int_{-c_1}^{D} \frac{1}{\psi(x)^2} \int_{x}^{D} (s + c_1)\psi(s)^2 dsdx, \tag{40}$$

and because $-c_1 \geq x_0$ we again have $\psi(s) \leq \psi(x)$ and get a contribution $O(D^3)$. The second term in (39) is bounded by

$$\int_{0}^{x_{0}+1} \frac{\psi(x_0)^2}{\psi(x_0+1)^2} \int_{-D}^{x} (s + c_1)dsdx.$$ 

Again we let $\rho = \frac{\psi'(x-x_0)}{\psi'(x_0)}$ and note that $\rho' \geq -1 - \rho^2$, $\rho(0) = 0$. This gives a bound

$$\frac{\psi(x_0)}{\psi(x_0+1)} \leq e^\delta$$

for some $\delta$ independent of $D$, $h$, or $c_1$, and so the second term in (39) is $O(D^2)$ as $D \to \infty$.

This proves the desired bound (38). Hence (37) shows that the dominant term in (30) is the last one, proving that $I'(c_1) > 0$ for large $D$, if $D$ is sufficiently large and $I(c_1) = 0$. Hence there is a unique $c_1 > 0$ with $I(c_1) = 0$. It follows from (16) - (18) that $h(c_1) > h(0)$, since $I(c) < 0$ for $0 < c < c_1$. This completes the proof of Theorem 3.
1.2 Proofs of Global Results

We now turn to the global results, about bifurcation from the curve of symmetric solutions. We consider the full problem (1) – (3). The symmetric problem can be studied on the interval \([0, d]\). Let \((\phi(x, \alpha, \delta), a(x, \alpha, \delta))\) denote the solution of (1) – (2) which satisfies the initial conditions \(\phi(0) = \alpha, \phi'(0) = 0, a(0) = 0, a'(0) = \delta\). Recall that Kwong proved in [19] that for each \(\alpha \in (0, 1]\) there is a unique \(\delta = \delta_0(\alpha)\) such that \(\phi\) is positive on \([0, d]\) and \(\phi'(d) = 0\).

Now let

\[
s(x) = \frac{\partial \phi(x)}{\partial \delta} \quad \text{and} \quad z(x) = \frac{\partial a(x)}{\partial \delta}.
\]

Let \(\delta = \delta_0(\alpha)\) and let \((\phi, a)\) be the corresponding solution on \([0, d]\).

**Lemma 15** \(s'(d) > 0\).

**Proof of Lemma 15:**

The pair \((s, z)\) satisfies the system

\[
\begin{align*}
\theta'' &= \kappa^2[(\phi^2 + a^2 - 1)\theta + 2a\phi\mu + 2\phi^2\theta] \\
\mu'' &= \phi^2\mu + 2a\phi\theta
\end{align*}
\]  

(41)

and

\[s(0) = s'(0) = 0, \quad z(0) = 0, \quad z'(0) = 1.\]

It is clear that \(z, z', z''\) are all positive on \((0, d]\) as long as \(s > 0\), since \(a > 0, \phi > 0\). Also, \(s''(0) = s'''(0) = 0\), while \(s'''(0) > 0\), so initially, \(s\) is positive. Suppose that \(s(x_0) = 0\) at some first \(s_0 > 0\) in \((0, d]\), with \(s > 0\) on \((0, x_0)\). Multiply (41) by \(\phi\) and (41) by \(s\), subtract and integrate. We conclude that

\[
\phi s' - s \phi'|_0^{x_0} = \int_0^{x_0} (2a\phi^2 z + 2\phi^3 s)dx > 0
\]

(42)

and applying the boundary conditions shows that \(s'(x_0) > 0\), a contradiction. Hence \(s > 0\) on \((0, d]\) and then from (42) we find that \(s'(d) > 0\), as desired. This proves Lemma 15.

To continue our study of bifurcation from the symmetric branch, now consider (1) with initial conditions which are possibly asymmetric, namely

\[
\phi(0) = \alpha, \phi'(0) = \beta, a(0) = \gamma, a'(0) = \delta.
\]

(43)

Let
\[ p(x) = \frac{\partial \phi}{\partial \alpha}, \quad q(x) = \frac{\partial \phi}{\partial \beta}, \quad r(x) = \frac{\partial \phi}{\partial \gamma}, \quad s(x) = \frac{\partial \phi}{\partial \delta}, \]

\[ u(x) = \frac{\partial a}{\partial \alpha}, \quad v(x) = \frac{\partial a}{\partial \beta}, \quad w(x) = \frac{\partial a}{\partial \gamma}, \quad z(x) = \frac{\partial a}{\partial \delta}. \]

Then the pairs \((p, u)\), \((q, v)\) , \((r, w)\), and \((s, z)\) all satisfy the system \((41)\). The initial conditions are

\[
\begin{align*}
(p, p', u, u') &= (1, 0, 0, 0) \\
(q, q', v, v') &= (0, 1, 0, 0) \\
(r, r', w, w') &= (0, 0, 1, 0) \\
(s, s', z, z') &= (0, 0, 0, 1)
\end{align*}
\]

at \(x = 0\). (Thus, these four pairs form a fundamental solution for \((41)\).

We wish to solve the three equations

\[ F(\alpha, \beta, \gamma, \delta) = G(\alpha, \beta, \gamma, \delta) = H(\alpha, \beta, \gamma, \delta) = 0 \quad (45) \]

where

\[
\begin{align*}
F(\alpha, \beta, \gamma, \delta) &= \phi'(d) \\
G(\alpha, \beta, \gamma, \delta) &= \phi'(-d) \\
H(\alpha, \beta, \gamma, \delta) &= a'(d) - a'(-d). \quad (46)
\end{align*}
\]

For each \(\alpha \in (0, 1]\), \((\alpha, 0, 0, \delta_0(\alpha))\) is a solution. Starting at \(\alpha = 1\), where \(\delta = 0\), this solution continues uniquely as the smooth curve of symmetric solutions as \(\alpha\) decreases so long as \(J \neq 0\), where

\[
J = \det \begin{bmatrix}
q'(d) & r'(d) & s'(d) \\
q'(-d) & r'(-d) & s'(-d) \\
v'(d) - v'(-d) & w'(d) - w'(-d) & z'(d) - z'(-d)
\end{bmatrix}
\]

where in \((41)\), \((\phi, a)\) is the solution at \((\alpha, 0, 0, \delta_0(\alpha))\).

Since \(\phi\) is even and \(a\) is odd, the initial conditions \((44)\) imply that \(p, v, w, \) and \(s\) are even functions of \(x\), while \(u, q, r, \) and \(z\) are odd functions of \(x\). As a result, \(J\) simplifies to

\[ J = 4s'(d)(q'(d)w'(d) - r'(d)v'(d)). \]

**Lemma 16** For sufficiently large \(kd\), \(J\) changes sign between \(\alpha = 0\) and \(\alpha = 1\).
Proof:

Lemma 15 showed that $s'(d) > 0$. Therefore, to obtain a bifurcation point on the curve of symmetric solutions, we must show that $M(\alpha) = q'(d)w'(d) - r'(d)v'(d)$ changes sign along this curve.

For $\alpha = 1$ we have $\phi \equiv 1$, $a \equiv 0$. The equations (41) and (44) can then be solved explicitly to show that $q'(d) > 0$, $v'(d) = 0$, $r'(d) = 0$, and $w'(d) > 0$ so that $M(1) > 0$.

We now analyze $M(\alpha)$ for small $\alpha$. First, with $(q,v)$ substituted for $(\theta,\mu)$ and $\alpha \psi$ for $\phi$ in (41), we multiply the equation for $q''$ by $\psi$, (4) by $q$, subtract, integrate, and use (6) and (44) to obtain

$$\psi(d)q'(d) = 1 + 2\kappa^2 \alpha \int_0^d (a\psi^2 v + \psi^3 \alpha q) dx.$$

As $\alpha \to 0$, $\psi \to \psi_0$, where

$$\psi_0'' = \kappa^2(h_0^2 x^2 - 1)\psi_0, \quad \psi_0(0) = 1, \quad \psi_0'(\pm d) = 0$$

and $h_0$ is the unique positive number such that (47) has a positive solution. (Earlier we referred to the solution $\psi_0$ of (47) as $\psi$, but now we must distinguish $\psi_0$ from $\psi = \frac{\phi}{\alpha}$ for $\alpha \neq 0$.) Since

$$v'' = \alpha^2 \psi^2 v + 2a\alpha \psi q,$$

(44) implies that $v \to 0$ on $[0,d]$ as $\alpha \to 0$, so we consider, instead, $\frac{v}{\alpha}$, and see that as $\alpha \to 0$,

$$\frac{v'(d)}{\alpha} \to \int_0^d 2h_0 x\psi_0 q_0 dx,$$

where

$$q_0'' = \kappa^2(h_0^2 x^2 - 1)q_0, \quad q_0(0) = 0, \quad q_0'(0) = 1$$

We proceed in the same way with $(r,w)$, to obtain, finally, that

$$\lim_{\alpha \to 0} \frac{M(\alpha)}{\alpha^2} = \frac{1}{\psi_0(d)} \left( \int_0^d \psi_0^2 + 2h_0 x\psi_0 R_0 dx - 4\kappa^2 h_0^2 \int_0^d x\psi_0^2 dx \cdot \int_0^d x\psi_0 q_0 dx \right).$$

In this expression the only term we have not defined is $R_0$, which is $\lim_{\alpha \to 0} \frac{\psi_0}{\alpha}$ and satisfies

$$R_0'' = \kappa^2(h_0^2 x^2 - 1)R_0 + 2\kappa^2 h_0 x\psi_0, \quad R_0(0) = R_0'(0) = 0.$$

To prove that $J$ changes sign it is again convenient to rescale, letting $\psi_0(x) = g(\kappa x)$, so that

$$g'' = (\lambda^2 y^2 - 1)g, \quad g(0) = 1, \quad g'(0) = 0, \quad g'(D) = 0$$

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where $D = \kappa d$ and $\lambda = \frac{h_0}{\kappa}$. It was shown in the proof of Lemma 2 that $\lambda > 1$.

Making the same change of variables in (49), we must show that for large $D = \kappa d$,

$$\frac{1}{\kappa} \int_0^D (g(y))^2 + 2\frac{h_0}{\kappa} yg(y)P(y))dy < 4\frac{h_0^2}{\kappa^2} \int_0^D yg(y)^2 dy \cdot \int_0^D yg(y)Q(y)dy$$  \hspace{1cm} (52)

where $R_0(x) = P(\kappa x)$, and $q_0(x) = Q(\kappa x)$. Hence, we have

$$Q'' = (\lambda^2 y^2 - 1)Q, \quad Q(0) = 0, \quad Q'(0) = \frac{1}{\kappa},$$  \hspace{1cm} (53)

and

$$P'' = (\lambda^2 y^2 - 1)P + 2\lambda yg(y), \quad P(0) = P'(0) = 0$$  \hspace{1cm} (54)

as well as (51). Multiplying (53) by $g$ and (51) by $Q$ and subtracting and integrating gives

$$\left( \frac{Q}{g} \right)' = \frac{1}{\kappa g(y)^2}$$

and similarly from (54) we obtain

$$\left( \frac{P}{g} \right)' = 2\lambda \frac{1}{g(y)^2} \int_0^y s g(s)^2 ds.$$

From these we find that

$$P(y) = g(y) \int_0^y \frac{h_0}{\kappa} \int_0^x s g(s)^2 ds dx, \quad \hspace{1cm} (55)$$

and

$$Q(y) = \frac{g(y)}{\kappa} \int_0^y \frac{1}{g(x)^2} dx.$$

With these substitutions, (52) becomes

$$\int_0^D (g(y))^2 dy < 4\lambda^2 \int_0^D yg(y)^2 dy \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 dt ds dy.$$  \hspace{1cm} (56)

It is tempting to approach this result by studying the asymptotic behavior of $g(y)$ as $D \to \infty$. In fact, one can show that $g(y) \to e^{-\frac{y^2}{2}}$ point-wise, and further effort can refine this result. It turns out to be a mistake, however, to study the result of substituting $e^{-\frac{y^2}{2}}$ for $g$ in (54), because this function does not satisfy the boundary
conditions, and this turns out to make the required estimates much more difficult, or indeed, impossible.

Instead, we proceed directly, and this is possible primarily because using (11) we can evaluate integrals of the form $\int y g(y)^2 \, dy$. In particular, multiplying (11) by $g'$ and integrating by parts, we find that

$$\lambda^2 \int y g(y)^2 \, dy = (\lambda^2 y^2 - 1) \frac{g(y)^2}{2} - \frac{g'(y)^2}{2}.$$ 

Substituting this in (56) and using the boundary conditions gives

$$\lambda^2 \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 \, dt \, ds = \frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^y \frac{1}{g(s)^2} \, ds - \frac{1}{2} \frac{g'(y)^2}{g(y)}.$$  

(57)

Substituting this in the right side of (56) gives

$$\lambda^2 \int_0^D y g(y)^2 \, dy \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 \, dt \, ds \, dy = \frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^y \frac{1}{g(s)^2} \, ds \, dy + \frac{1}{2} \int_0^D g(y)^2 \, dy - \frac{D g(D)^2}{4}.$$  

(58)

Using this in (56) implies that the following inequality is sufficient for our result:

$$\frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^D y g(y)^2 \, dy \int_0^y \frac{1}{g(s)^2} \, ds \, dy - \frac{D g(D)^2}{4} > 0$$  

(59)

(for large $D$).

We need only one additional fact about $g(y)$; namely, that $1 \geq g(y) \geq e^{-y^2}$. We already know that $g' \leq 0$ giving the first inequality, and for the second, differentiate $\left(\frac{g}{e^{-y^2/2}}\right)$ twice and use (51) to see that this expression, which is 1 at $y = 0$, then increases on $[0, D]$.

Using this information it is seen that the double integral on the left of (59) does not tend to zero with $D$, and this proves (59) for large $D$. This implies (52) and completes the proof of Lemma 16.

We also need a global bound on the solutions, for fixed $\kappa, d$.

**Lemma 17** For given $\kappa$ and $d$, there is an $M$ such that if $(\phi, a, h)$ is a solution of (1) with $\phi > 0$ on $[-d, d]$, then $|\phi| + |\phi'| + |a| + |a'| \leq M$ on $[-d, d]$.

**Proof:** Letting $\psi = \frac{\phi}{\max_{-d \leq x \leq d} \phi(x)}$, we see that $\psi'(\pm d) = 0$ and $\psi'' \geq -\kappa$. This implies that $\psi'$ is bounded independent of which solution is being considered. Further, for any solution there is an $x_0 \in (-d, d)$ with $a(x_0) = 0$. If $a'(x_0)$ is bounded then we are done, so we can assume that there are solutions with $a'(x_0)$ arbitrarily large. Since $a'$ is a minimum at $x_0$, this implies that for any $A$, and $\varepsilon$, there is a $\Lambda$ such that if $a'(x_0) \geq \Lambda$, then the length of the interval in which $|a| \leq A$ is less than $\varepsilon$. 

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Since $\psi'(\pm d) = 0$ we must have
\[ \int_{-d}^{d} (\psi - \psi^3) \, dx = \int_{-d}^{d} \psi a^2 \, dx. \] (60)

Since $\max_{[-d,d]} \psi(x) = 1$, and $\psi' \geq -\kappa$, there must be an $\varepsilon > 0$ such that for any solution of (1) − (3), $\psi \geq \frac{1}{2}$ in some interval $\Omega$ of length $\varepsilon$. For solutions with $a'(x_0)$ sufficiently large we must have $|a|$ large in at least half of $\Omega$, which means that the right side of (60) can be arbitrarily large, while the left side is bounded by $2d$. This contradiction proves Lemma 17.

We remark that M. K. Kwong gave a proof using Sturmian methods [?].

Completion of proof of Theorem 8:

With $M$ as in Lemma 17, truncate (1) − (2) by letting
\[ g(x) = \min(x, M^2) \]
and considering
\[ \phi'' = \kappa^2(g(a^2 + \phi^2) - 1)\phi \] (61)
and
\[ a'' = g(\phi)^2 a. \]

with boundary conditions (3). With the notation of (10) let $\tilde{F} = (F, G, H) = \tilde{F}(\alpha, \tilde{\beta})$, where $\tilde{\beta} = (\beta, \gamma, \delta)$. We use a truncation to ensure that whatever the initial condition at $x = 0$ the solutions exist up to $x = d$.

Since $J > 0$ when $\alpha$ is close to 1, (when $\tilde{F}(\alpha, (0, 0, \delta_0(\alpha))) = 0$), this solution $t(\alpha) = (0, 0, \delta_0(\alpha))$ is non-degenerate and isolated and has Brouwer degree $\sgn J = 1$ (for the map $\tilde{F}$ with fixed $\alpha$). Similarly, if $\alpha$ is small and positive, then $J < 0$ and $t(\alpha)$ has degree -1. Hence there is a change of degree along the branch of symmetric solutions between $\alpha$ small and $\alpha = 1$. Hence, by a slight variant of Theorem 1.16 of Rabinowitz in [21], there is a connected set of solutions of $\tilde{F}(\alpha, \tilde{\beta}) = 0$ branching off the symmetric solution $(\alpha, t(\alpha))$ at a point $\alpha \in (0, 1)$ where $J = 0$, and this branch will either be unbounded in $R^4$, or return to $(0,0,0,0)$, or meet the symmetric branch again.

Note that there are no solutions with $h = 0$, since then $a$ would have to be of constant sign and we could not have $\int_{-d}^{d} \psi a^2 \, dx = 0$. Also, solutions cannot bifurcate from the symmetric branch near $\alpha = 0$ or $\alpha = 1$, because $J \neq 0$ there.

The solutions of interest are positive. (That is, $\phi$ is positive.) Solutions on the bifurcating branch of asymmetric solutions start off positive as the branch leaves the symmetric branch, from continuity. From (1) it follows that solutions can only fail to be positive along this branch if $\phi \to 0$ (uniformly on $[-d, d]$). Also, Lemma 17

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shows the solutions must remain bounded, and in fact, remain within the truncated region where (1) — (2) apply. If \( \phi \to 0 \), then \( a \to h(x + c) \).

It is conceivable that there are several points \( \alpha \) where \( J = 0 \). However, the change of degree ensures that one branch does not return to the branch of symmetric solutions, and so, by Rabinowitz’s global result, must tend to \((0, h(x + c))\) for some \( c \neq 0 \), that is, to a normal solution. In fact, by the symmetry of the problem, there will be two branches bifurcating from the same point, one tending to the unique bifurcation point from the normal solution with \( c > 0 \) and the other to the symmetric reflection of this solution around 0. (The uniqueness of this bifurcation point follows from Theorem 8.) This completes the proof of Theorem 8.

1.3 Conclusion:

The initial motivation for this paper was Seydel’s bifurcation diagram, Figure 1. Our goal was to prove that in some parameter range the problem could have as many as seven solutions (five essentially distinct). Unfortunately we have not achieved this goal. There are at least two features of Seydel’s curve that seem important in obtaining such a proof. We would like to determine where on the symmetric branch the bifurcation to asymmetric solutions does exist, and we want to know the direction of bifurcation at this point. These remain challenges for future work. However, we have verified that for large \( \kappa d \) the desired bifurcation from the symmetric branch occurs, and furthermore, there is a curve of asymmetric solutions going from the symmetric branch to the normal state. We have also shown how earlier results on bifurcation from the normal state can be obtained without the use of detailed asymptotics for the linear problem.

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