Singular optimal control of stochastic Volterra integral equations

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Abstract

This paper deals with optimal combined singular and regular control for stochastic Volterra integral equations, where the solution $X^{u,\xi}(t) = X(t)$ is given by

$$X(t) = \phi(t) + \int_0^t b(t, s, X(s), u(s)) \, ds + \int_0^t \sigma(t, s, X(s), u(s)) \, dB(s) + \int_0^t h(t, s) \, d\xi(s).$$

Here $\xi$ denotes the singular control and $u$ denotes the regular control. Unless otherwise stated, $\int_a^b h(s) \, d\xi(s)$ means $\int_{[a,b]} h(s) \, d\xi(s)$.

Such systems may for example be used to model for harvesting of populations with memory, where $X(t)$ represents the population density at time $t$, and the singular control process $\xi$ represents the harvesting effort rate. The total income from the harvesting is represented by

$$J(u, \xi) = E[\int_0^T f_0(t, X(t), u(t)) \, dt + \int_0^T f_1(t, X(t)) \, d\xi(t) + g(X(T))],$$

for given functions $f_0, f_1$ and $g$, where $T > 0$ is a constant denoting the terminal time of the harvesting.

Using Hida-Malliavin calculus, we prove sufficient conditions and necessary conditions of optimality of controls. As a consequence, we obtain a new type of backward stochastic Volterra integral equations with singular drift.

Finally, to illustrate our results, we apply them to solve an optimal harvesting problem with possibly density dependent prices.

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1 Introduction

Suppose the density $X(t)$ at time $t$ of a population of a certain type of fish in a lake can be modelled as the solution of the following stochastic Volterra integral equation (SVIE):

$$X(t) = x_0 + \int_0^t b_0(t, s)X(s)ds + \int_0^t \sigma_0(s)X(s)dB(s) - \int_0^t \gamma_0(t, s)d\xi(s),$$

where $X(t)$ is the density of the population at time $t$, the coefficients $b_0, \sigma_0$ and $\gamma_0$ are bounded deterministic functions, and $B(t) = \{B(t)\}_{t \geq 0}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We associate to this space a natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ generated by $B(t)$, assumed to satisfy the usual conditions. The singular process $\xi(t)$ is our control process. It is an $\mathbb{F}$-adapted, non-decreasing left-continuous process representing the harvesting effort. The constant $\gamma_0 > 0$ is the harvesting efficiency coefficient. It turns out that in some cases the optimal process $\xi(t)$ can be represented as the local time of the solution $X(t)$ at some threshold curve.

Volterra equations are commonly used in population growth models, especially when age dependence plays a role. See e.g. Gripenberg et al [7]. Moreover, they are important examples of equations with memory.

We assume that the total expected utility from the harvesting is represented by

$$J(\xi) = \mathbb{E}[\theta X(T) + \int_0^T \log(X(t))d\xi(t)],$$

where $\mathbb{E}$ denotes expectation with respect to $P$. The problem is then to maximise $J(\xi)$ over all admissible singular controls $\xi$.

Control problems for singular Volterra integral equations have been studied by Lin and Yong [12] in the deterministic case. In this paper we study stochastic SVIEs and we present a different approach based on a stochastic version of the Pontryagin maximum principle.

Stochastic control for Volterra integral equations has been studied by Yong [14] and subsequently by Agram et al [3], [5] who used the white noise calculus to obtain both sufficient and necessary conditions of optimality. In the latter, smoothness of coefficients is required. The adjoint processes of our maximum principle satisfy a backward stochastic integral equation of Volterra type and with a singular term coming from the control. In our example one may consider the optimal singular term as the local time of the state process that is keeping it above/below a certain threshold curve. Hence in some cases we can associate this type of equations with reflected backward stochastic Volterra integral equations.

Partial result for existence and uniqueness of backward stochastic Volterra integral equation can be found in Yong [14], [15], and in Agram et al [4], [2] where there are also applications.
The paper is organised as follows: In the next section we give some preliminaries about the the generalised Malliavin calculus, called Hida-Malliavin calculus, in the white noise space of Hida of stochastic distributions. Section 3 is addressed to the study of the stochastic maximum principle where both sufficient and necessary conditions of optimality are proved. Finally, in Section 4 we apply the results obtained in section 3 to solve an optimal harvesting problem with possibly density dependent prices.

2 Hida - Malliavin calculus

Let \( G = \{ G_t \}_{t \geq 0} \) be a subfiltration of \( \mathbb{F} \), in the sense that \( G_t \subseteq \mathcal{F}_t \), for all \( t \geq 0 \). The given set \( U \subset \mathbb{R} \) is assumed to be convex. The set of admissible controls, i.e. the strategies available to the controller, is given by a subset \( \mathcal{A} \) of the cadlag, \( U \)-valued and \( G \)-adapted processes. Let \( \mathcal{K} \) be the set of all \( \mathbb{G} \)-adapted processes \( \xi(t) \) that are nondecreasing and left continuous with respect to \( t \).

Next we present some preliminaries about the extension of the Malliavin calculus into the stochastic distribution space of Hida, for more details, we refer the reader to Aase et al [1], Di Nunno et al [1].

The classical Malliavin derivative is only defined on a subspace \( D_{1,2} \) of \( L^2(P) \). However, there are many important random variables in \( L^2(P) \) that do not belong to \( D_{1,2} \). For example, this is the case for the solutions of a backward stochastic differential equations or more generally the BSVIE.

This is why the Malliavin derivative was extended to an operator defined on the whole of \( L^2(P) \) and with values in the Hida space \( (S)^\ast \) of stochastic distributions. It was proved by Aase et al [1] that one can extend the Malliavin derivative operator \( D_t \) from \( D_{1,2} \) to all of \( L^2(\mathcal{F}_T, P) \) in such a way that, also denoting the extended operator by \( D_t \), for all \( F \in L^2(\mathcal{F}_T, P) \), we have

\[
D_t F \in (S)^\ast \text{ and } (t, \omega) \mapsto \mathbb{E}[D_t F | \mathcal{F}_t] \text{ belongs to } L^2(\lambda \times P), \tag{2.1}
\]

where \( \lambda \) is Lebesgue measure on \([0, T]\). We now explain this in more detail:

**Definition 2.1** (i) Let \( F \in L^2(P) \) and let \( \gamma \in L^2(\mathbb{R}) \) be deterministic. Then the directional derivative of \( F \) in \( (S)^\ast \) (respectively, in \( L^2(P) \)) in the direction \( \gamma \) is defined by

\[
D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(\omega + \varepsilon \gamma) - F(\omega)] \tag{2.2}
\]

whenever the limit exists in \( (S)^\ast \) (respectively, in \( L^2(P) \)).

(ii) Suppose there exists a function \( \psi : \mathbb{R} \mapsto (S)^\ast \) (respectively, \( \psi : \mathbb{R} \mapsto L^2(P) \)) such that

\[
\int_{\mathbb{R}} \psi(t) \gamma(t) dt \text{ exists in } (S)^\ast \text{ (respectively, in } L^2(P) \text{)} \text{ and } \quad D_\gamma F = \int_{\mathbb{R}} \psi(t) \gamma(t) dt, \quad \text{for all } \gamma \in L^2(\mathbb{R}). \tag{2.3}
\]

Then we say that \( F \) is Hida-Malliavin differentiable in \( (S)^\ast \) (respectively, in \( L^2(P) \)) and we write

\[
\psi(t) = D_t F, \quad t \in \mathbb{R}.
\]

We call \( D_t F \) the Hida-Malliavin derivative at \( t \) in \( (S)^\ast \) (respectively, in \( L^2(P) \)) or the stochastic gradient of \( F \) at \( t \).
Let $F_1, ..., F_m \in L^2(P)$ be Hida-Malliavin differentiable in $L^2(P)$. Suppose that $\varphi \in C^1(\mathbb{R}^m)$, $D_tF_i \in L^2(P)$, for all $t \in \mathbb{R}$, and $\frac{\partial \varphi}{\partial x_i}(F)D_tF_i \in L^2(\lambda \times P)$ for $i = 1, ..., m$, where $F = (F_1, ..., F_m)$. Then $\varphi(F)$ is Hida-Malliavin differentiable and

$$D_tF(F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F)D_tF_i. \tag{2.4}$$

We have the following generalized duality formula, for the Brownian motion:

**Proposition 2.2** Fix $s \in [0, T]$. If $t \mapsto \varphi(t, s, \omega) \in L^2(\lambda \times P)$ is $\mathbb{F}$-adapted with $\mathbb{E}[\int_0^T \varphi^2(t, s)dt] < \infty$ and $F \in L^2(\mathcal{F}_T, P)$, then we have

$$\mathbb{E}[F \int_0^T \varphi(t, s)dB(t)] = \mathbb{E}[^T_0 \mathbb{E}[D_tF][\mathcal{F}_t] \varphi(t, s)dt]. \tag{2.5}$$

We will need the following:

**Lemma 2.3** Let $t, s, \omega \mapsto G(t, s, \omega) \in L^2(\lambda \times \lambda \times P)$ and $t, \omega \mapsto p(t) \in L^2(\lambda \times P)$, then the followings hold:

1. The Fubini theorem combined with a change of variables gives

$$\int_0^T p(t)\int_0^T G(t, s)ds dt = \int_0^T (\int_0^T p(s)G(s, t)ds) dt, \tag{2.6}$$

and

$$\int_0^T p(t)\int_0^T G(t, s)ds d\xi(t) = \int_0^T (\int_0^T p(s)G(s, t)ds)d\xi(t). \tag{2.7}$$

2. The generalized duality formula (2.5) together with the Fubini theorem, yields

$$\mathbb{E}[\int_0^T p(t)(\int_0^T G(t, s)dB(s))dt] = \mathbb{E}[^T_0 \mathbb{E}[D_tG][\mathcal{F}_t]G(s, t)dsdt]. \tag{2.8}$$

### 3 Stochastic maximum principles

In this section, we study stochastic maximum principles of stochastic Volterra integral systems under partial information, i.e., the information available to the controller is given by a sub-filtration $\mathcal{G}$. Suppose that the state of our system $X^{u, \xi}(t) = X(t)$ satisfies the following SVIE

$$X(t) = \phi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dB(s) + \int_0^t h(t, s)d\xi(s), \quad t \in [0, T], \tag{3.1}$$

where $b(t, s, x, u) = b(t, s, x, u, \omega) : [0, T]^2 \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}$, $\sigma(t, s, x, u) = \sigma(t, s, x, u, \omega) : [0, T]^2 \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}$

The performance functional has the form

$$J(u, \xi) = \mathbb{E}[\int_0^T f_0(t, X(t), u(t))dt + \int_0^T f_1(t, X(t))d\xi(t) + g(X(T))], \tag{3.2}$$
with given functions \( f(t, x, u) = f(t, x, u, \omega) : [0, T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R} \) and \( g(t, x) = g(t, x, \omega) : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \).

We want to find an optimal pair \((\hat{u}, \hat{\xi})\) such that

\[
J(u, \xi) \leq J(\hat{u}, \hat{\xi}) \text{ i.e., } J(u, \xi) - J(\hat{u}, \hat{\xi}) \leq 0.
\]

We impose the following set of assumptions on the coefficients:

\[
\text{The processes } b(t, s, x, u), \sigma(t, s, x, u), f(s, x, u) \text{ and } h(t, s) \text{ are } \mathcal{F}_s\text{-adapted with respect to } s \text{ for all } s \leq t, \text{ and twice continuously differentiable } (C^2) \text{ with respect to } t, x \text{ and continuously differentiable } (C^1) \text{ with respect to } u \text{ for each } s. \text{ The driver } g \text{ is assumed to be } \mathcal{F}_T\text{-measurable and } (C^1) \text{ in } x. \text{ Moreover, all the partial derivatives are supposed to be bounded.}
\]

Note that the performance functional (3.2) is not of Volterra type.

### 3.1 The Hamiltonian and the adjoint equations

Define the **Hamiltonian functional** associated to our control problem (3.1) and (3.2), as

\[
\mathcal{H}(t, x, u, p, q) = H_0(t, x, u, p, q) + H_1(t, x, u, p, q) + \bar{H}_0(t, x, \xi, p) + \bar{H}_1(t, x, \xi, p)
\]

The BSVIE for the adjoint processes \( p(t) \) and \( q(t, s) \) is defined by

\[
p(t) = \frac{\partial p}{\partial x}(X(T)) + \int_t^T \frac{\partial p}{\partial x}(s)ds + \int_t^T \frac{\partial q}{\partial x}(s)d\xi(s) - \int_t^T q(t, s)dB(s),
\]

where we have used the simplified notation

\[
\frac{\partial p}{\partial x}(t) = \frac{\partial p}{\partial x}(t, X(t), u(t), p(t), q(t, t)),
\]

\[
\frac{\partial q}{\partial x}(t) = \frac{\partial q}{\partial x}(t, X(t), \xi(t), p(t)).
\]
Note that from equation \((3.1)\), we get the following equivalent formulation, for each \((t, s) \in [0, T]^2\),

\[
dX(t) = \phi'(t) dt + b(t, t, X(t), u(t)) dt + (\int_0^t \frac{\partial b}{\partial t}(t, s, X(s), u(s)) ds) dt + \sigma(t, t, X(t), u(t)) dB(t) + (\int_0^t \frac{\partial b}{\partial x}(t, s, X(s), u(s)) dB(s) dt + h(t, t, d\xi(t) + (\int_0^t \frac{\partial b}{\partial x}(t, s) d\xi(s)) dt.
\]

We assume that for each \(t \mapsto q(t, s)\) is \(C^1\) for all \(s, \omega\) and moreover,

\[
\mathbb{E} \left[ \int_0^T \int_0^T \left( \frac{\partial q(t, s)}{\partial t} \right)^2 ds dt \right] < \infty,
\]

under which we can write the following differential form of equation \((3.6)_c\):

\[
\begin{align*}
dp(t) &= -\left( \frac{\partial H}{\partial x}(t) \right) dt + \frac{\partial H}{\partial x}(t) d\xi(t) + \int_t^T \frac{\partial H}{\partial t}(t, s) dB(s) dt + q(t, t) dB(t), \\
p(T) &= \frac{\partial H}{\partial x}(X(T)).
\end{align*}
\]  

### 3.2 A sufficient maximum principle

We will see under which conditions the couple \((u, \xi)\) is optimal, i.e. we will prove a sufficient version of the maximum principle approach (a verification theorem).

**Theorem 3.1 (Sufficient maximum principle)** Let \(\hat{u} \in \mathcal{A}\), with corresponding solutions \(\hat{X}(t), (\hat{p}(t), \hat{q}(t, s))\) of \((3.7)\) and \((3.10)\) respectively. Assume that the functions \(x \mapsto g(x)\) and \((x, u, \xi) \mapsto \mathbb{H}(t, x, u, \xi, \hat{p}, \hat{q})\) are concave. Moreover, impose the following optimal conditions for each control:

- **(Maximum condition for \(u\))**
  
  \[
  \sup_{u \in U} \mathbb{E} \left[ \mathbb{H}(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t, t)) | \mathcal{F}_t \right] = \mathbb{E} [\mathbb{H}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t, t)) | \mathcal{F}_t], \text{ for each } t, \text{ P-a.s.}
  \]
  
  where we are using the notation

  \[
  \mathbb{E} [\mathbb{H}(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t, t)) | \mathcal{F}_t] := \mathbb{E} [\mathbb{H}(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t, t)) | \mathcal{F}_t] dt + \mathbb{E} [\mathbb{H}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t, t)) | \mathcal{F}_t] d\xi(t), \text{ for each } t, \text{ P-a.s.}
  \]  

- **(Maximum condition for \(\xi\))** For each \(t \in [0, T]\) we have, in the sense of inequality between random measures,

  \[
  \sup_{\xi \in \mathcal{V}} \mathbb{E} [\mathbb{H}(t, \hat{X}(t), u, \xi, \hat{p}(t), \hat{q}(t, t)) (dt, d\xi(t)) | \mathcal{F}_t] \]

  \[
  \leq \mathbb{E} [\mathbb{H}(t, \hat{X}(t), u, \xi(t), \hat{p}(t), \hat{q}(t, t)) (dt, d\hat{\xi}(t)) | \mathcal{F}_t], \text{ for each } t, \text{ P-a.s.}
  \]

Then \((\hat{u}, \hat{\xi})\) is an optimal pair.
Proof. Choose $u \in \mathcal{A}$ and $\xi \in \mathcal{K}$, we want to prove that $J(u, \xi) - J(\hat{u}, \hat{\xi}) \leq 0$. We set

$$J(u, \xi) - J(\hat{u}, \hat{\xi}) = J(u, \xi) - J(u, \hat{\xi}) + J(u, \hat{\xi}) - J(\hat{u}, \hat{\xi}).$$

Since we have one regular control and one singular, we will solve the problem by separating them, as follows:

First, we prove that $\xi$ is optimal i.e., for all fixed $u \in U$, $J(u, \xi) - J(u, \hat{\xi}) \leq 0$. Then, we plug the optimal $\hat{\xi}$ into the second part and we prove it for $u$, i.e., $J(u, \xi) - J(\hat{u}, \hat{\xi}) \leq 0$. However, the case of regular controls $u$ has been proved in Theorem 4.3 by Agram et al [4]. It rests to prove only for singular ones $\xi$.

From definition (3.2), we have

$$J(u, \xi) - J(u, \hat{\xi}) = A_1 + A_2 + A_3,$$  

(3.12)

where we have used hereafter the shorthand notations

$$A_1 = \mathbb{E}[\int_0^T \hat{f}_0(t) \, dt], \quad A_2 = \mathbb{E}[\int_0^T f_1(t) \, dt], \quad A_3 = \mathbb{E}[\hat{g}(T)],$$

with $\hat{f}_0(t) = f_0(t) - \hat{f}_0(t)$, $\hat{g}(T) = g(X(T)) - g(\hat{X}(T))$, and similarly for $b(t, t) = b(t, t, X(t), u(t))$, and the other coefficients. By definition (3.3), we get

$$A_1 = \mathbb{E}[\int_0^T \{ \hat{H}_0(t) - \hat{p}(t) \hat{b}(t, t) + \hat{q}(t, t) \hat{\sigma}(t, t) \} \, dt].$$  

(3.13)

Concavity of $g$ together with the terminal value of the BSVIE (3.6), we obtain

$$A_3 \leq \mathbb{E}[\frac{\partial g}{\partial x}(T) \hat{X}(T)] = \mathbb{E}[\hat{p}(T) \hat{X}(T)].$$

Applying the integration by parts formula to the product $\hat{p}(t) \hat{X}(t)$, we get

$$A_3 \leq \mathbb{E}[\hat{p}(T) \hat{X}(T)]$$

$$= \mathbb{E}[\int_0^T \hat{p}(t) \{ \hat{b}(t, t) + \int_0^t \frac{\partial h}{\partial t}(t, s) \, ds + \int_0^t \frac{\partial h}{\partial x}(t, s) \, dB(s) + \int_0^t \frac{\partial h}{\partial \xi}(t, s) \, d\xi(s) \} \, dt$$

$$+ \int_0^T \hat{p}(t) h(t, t) \, dt - \int_0^T \hat{X}(t) \frac{\partial h}{\partial x}(t, s) \, dt - \int_0^T \hat{X}(t) \frac{\partial h}{\partial \xi}(t, s) \, dB(s) \, dt$$

$$+ \int_0^T \hat{X}(t) \hat{q}(t, t) \, dB(t) + \int_0^T \hat{q}(t, t) \hat{\sigma}(t, t) \, dt].$$  

(3.14)

It follows from formulas (2.6)-(2.8), that

$$\mathbb{E}[\int_0^T \hat{p}(t) (\int_0^t \frac{\partial h}{\partial x}(t, s) \, ds) \, dt] = \mathbb{E}[\int_0^T (\int_0^t \hat{p}(s) \frac{\partial h}{\partial x}(s, t) \, ds) \, dt],$$

$$\mathbb{E}[\int_0^T \hat{p}(t) (\int_0^t \frac{\partial h}{\partial \xi}(t, s) \, ds) \, dt] = \mathbb{E}[\int_0^T (\int_0^t \hat{p}(s) \frac{\partial h}{\partial \xi}(s, t) \, ds) \, dt],$$

$$\mathbb{E}[\int_0^T \hat{p}(t) (\int_0^t \frac{\partial h}{\partial s}(s, t) \, ds) \, dt] = \mathbb{E}[\int_0^T \int_0^T D_{x} p(s) \frac{\partial h}{\partial s}(s, t) \, ds \, dt].$$

Substituting the above into (3.12), we obtain

$$J(\hat{u}, \xi) - J(\hat{u}, \hat{\xi}) \leq \mathbb{E}[\int_0^T (\hat{H}_0(t) + \hat{H}_1(t)) \, dt + \int_0^T f_1(t) \, dt - \int_0^T \hat{f}_1(t) \, dt - \int_0^T \hat{p}(t) h(t, t) \, dt$$

$$+ \int_0^T (\int_0^t \hat{p}(s) \frac{\partial h}{\partial x}(s, t) \, ds) \, dt - \int_0^T \hat{X}(t) \frac{\partial h}{\partial x}(t, s) \, dt - \int_0^T \hat{X}(t) \frac{\partial h}{\partial \xi}(t, s) \, dB(s) \, dt$$

$$= \mathbb{E}[\int_0^T (\mathcal{H}(t) - \mathcal{H}(t)) \, dt + (\mathcal{H}(t) \frac{\partial h}{\partial x}(t, s) \, dt - \mathcal{H}(t) \frac{\partial h}{\partial \xi}(t, s) \, dt)].$$
Since \( \xi \) and \( \hat{\xi} \) are \( \mathcal{G} \)-adapted and \( \hat{\xi} \) maximizes the conditional Hamiltonian, we conclude that

\[
J(\hat{u}, \xi) - J(\hat{u}, \hat{\xi}) \leq \mathbb{E}\left[ \int_0^T \mathbb{E}\left[ (H(t) - \bar{H}(t)) dt + (\bar{H}(t)d\xi(t) - \bar{\xi}(t)d\hat{\xi}(t)) | \mathcal{G}_t \right] \right]
= \mathbb{E}\left[ \int_0^T \mathbb{E}\left[ (H(t)(dt, d\xi(t)) - H(t)(dt, d\hat{\xi}(t)) | \mathcal{G}_t \right] \leq 0. \]

The proof is complete. \( \square \)

### 3.3 A necessary maximum principle

Since the concavity condition is not always satisfied, it is useful to have a necessary condition of optimality where this condition is not required. Suppose that a control \( (\hat{u}, \hat{\xi}) \in \mathcal{A} \times \mathcal{K} \) is an optimal pair and that \( (v, \zeta) \in \mathcal{A} \times \mathcal{K} \). Define \( u^\lambda = u + \lambda v \) and \( \xi^\lambda = \xi + \lambda \zeta \), for a non-zero sufficiently small \( \lambda \). Assume that \( (u^\lambda, \xi^\lambda) \in \mathcal{A} \times \mathcal{K} \). For each given \( t \in [0, T] \), let \( \eta = \eta(t) \) be a bounded \( \mathcal{G}_t \)-measurable random variable, let \( h \in [T - t, T] \) and define

\[
v(s) := \eta 1_{[t, t+h]}(s); s \in [0, T]. \tag{3.15}\]

Assume that the derivative process \( Y(t) \), defined by \( Y(t) := \frac{d}{d\lambda} X^{u^\lambda, \xi^\lambda}(t) |_{\lambda=0} \) exists. Then we see that

\[
Y(t) = \int_0^t \left( \frac{\partial b}{\partial x}(t, s) Y(s) + \frac{\partial b}{\partial \sigma}(t, s) \sigma Y(s) + \frac{\partial \sigma}{\partial x}(t, s) v(s) \right) ds + \int_0^t \left( \frac{\partial^2 \sigma}{\partial \sigma \partial x}(t, s) \sigma Y(s) + \frac{\partial^2 \sigma}{\partial \sigma \partial \sigma}(t, s) v(s) \right) ds + \int_0^t \frac{\partial \sigma}{\partial \sigma}(t, s) v(s) d\sigma(s),
\]

and hence

\[
dY(t) = \left[ \frac{\partial b}{\partial x}(t, t) Y(t) + \frac{\partial b}{\partial \sigma}(t, t) \sigma Y(t) + \int_0^t \left( \frac{\partial^2 \sigma}{\partial \sigma \partial x}(t, s) \sigma Y(s) + \frac{\partial^2 \sigma}{\partial \sigma \partial \sigma}(t, s) v(s) \right) ds + \int_0^t \frac{\partial \sigma}{\partial \sigma}(t, s) v(s) d\sigma(s) \right] dt + \left( \frac{\partial \sigma}{\partial \sigma}(t, t) Y(t) + \frac{\partial \sigma}{\partial \sigma}(t, t) v(t) \right) dB(t). \tag{3.16}\]

Similarly, we define the derivative process \( Z(t) := \frac{d}{d\lambda} X^{u^\lambda, \xi^\lambda}(t) |_{\lambda=0} \), as follows

\[
Z(t) = \int_0^t \frac{\partial b}{\partial x}(t, s) Z(s) ds + \int_0^t \frac{\partial b}{\partial \sigma}(t, s) \sigma Z(s) dB(s) + \int_0^t h(t, s) d\zeta(s),
\]

which is equivalent to

\[
dZ(t) = \left[ \frac{\partial b}{\partial x}(t, t) Z(t) + \int_0^t \frac{\partial^2 b}{\partial \sigma \partial x}(t, s) Z(s) ds \right] dt + \frac{\partial b}{\partial \sigma}(t, t) Z(t) dB(t)
+ \int_0^t \frac{\partial^2 \sigma}{\partial \sigma \partial x}(t, s) Z(s) dB(s) dt + h(t, t) d\zeta(t) + \int_0^t \frac{\partial \sigma}{\partial \sigma}(t, s) d\zeta(s) dt. \tag{3.17}\]

We shall prove the following theorem:

**Theorem 3.2 (Necessary maximum principle)**

1. For fixed \( \xi \), suppose that \( \hat{u} \in \mathcal{A} \), such that, for all \( \beta \) as in (3.15),

\[
\frac{d}{d\lambda} J(\hat{u} + \lambda \beta, \xi)|_{\lambda=0} = 0 \tag{3.18}
\]

and the corresponding solution \( \hat{X}(t), \hat{\rho}(t), \hat{q}(t, t) \) of (3.11) and (3.14) exists. Then,

\[
\mathbb{E}\left[ \frac{\partial b}{\partial u}(\cdot)| \mathcal{G}_t \right] u=\hat{u}(t) = 0. \tag{3.19}\]

2. Conversely, if (3.19) holds, then (3.18) holds.
3. Similarly, for fixed \( u \), suppose that \( \hat{\zeta} \in \mathcal{K} \) is optimal. Then

\[
\mathbb{E}[\hat{f}_1(t) + \hat{p}(t) h(t, t) + \int_t^T \hat{p}(s) \frac{\partial h}{\partial s}(s, t) ds | \mathcal{G}_t] \leq 0, \]

and

\[
\mathbb{E}[\hat{f}_1(t) + \hat{p}(t) h(t, t) + \int_t^T \hat{p}(s) \frac{\partial h}{\partial s}(s, t) ds | \mathcal{G}_t] d\hat{\zeta}(t) = 0. \]

Proof. For simplicity of notation we drop the "hat".
Part 1 is a direct consequence of Theorem 4.4 in Agram et al [4]. We proceed to prove 2. Set

\[
\frac{d}{dx} J(u, \xi^x) |_{x=0} = \mathbb{E}[\int_0^T (\frac{\partial g}{\partial x}(t, t) Z(t) + \int_0^T \frac{\partial g}{\partial x}(t, t) Z(t) d\xi(t) + \int_0^T f_1(t) d\zeta(t) + \frac{\partial g}{\partial x}(T) Z(T)].
\]

Applying the Itô formula, we get

\[
\mathbb{E}[\int_0^T \frac{\partial g}{\partial x}(t, t) Z(t) + \int_0^T \frac{\partial g}{\partial x}(t, t) Z(t) d\xi(t) + \int_0^T f_1(t) d\zeta(t) + \frac{\partial g}{\partial x}(T) Z(T)] = \mathbb{E}[p(T) Z(T)].
\]

Therefore, from (3.18) and (3.17) we get

\[
\mathbb{E}[p(T) Z(T)] = \mathbb{E}[\int_0^T Z(t) \{ \frac{\partial h}{\partial x}(t, t) p(t) + \frac{\partial^2 h}{\partial x^2}(s, t) p(s) + \frac{\partial^2 \sigma}{\partial x^2}(s, t) q(s, t) ds \} dt + \int_0^T p(t) h(t, t) d\zeta(t) + \int_0^T (\int_0^T p(s) \frac{\partial h}{\partial s}(s, t) ds) d\zeta(t) - \int_0^T Z(t) \frac{\partial^2 h}{\partial x^2}(t, t) dt] - \int_0^T Z(t) \frac{\partial^2 h}{\partial x^2}(t, t) dt + \int_0^T Z(t) \frac{\partial h}{\partial x}(t, t) q(t, t) dt].
\]

Using the definition of \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) in (3.4) – (3.5),

\[
\frac{d}{dx} J(u, \xi^x) |_{x=0} = \mathbb{E}[\int_0^T \{ p(t) h(t, t) + \int_t^T p(s) \frac{\partial h}{\partial s}(s, t) ds \} d\zeta(t)].
\]

Thus,

\[
0 \geq \frac{d}{dx} J(\xi^x) |_{x=0} = \mathbb{E}[\int_0^T \{ p(t) h(t, t) + \int_t^T p(s) \frac{\partial h}{\partial s}(s, t) ds \} d\zeta(t)],
\]

for all \( \zeta \in \mathcal{K}(\hat{\xi}) \).
If we choose \( \zeta \) to be a pure jump process of the form \( \zeta(t) = \sum_{0 \leq t_i \leq T} \alpha(t_i) \) where \( \alpha(t_i) > 0 \) is \( \mathcal{G}_{t_i} \)-measurable for all \( t_i \), then \( \zeta \in \mathcal{K}(\hat{\xi}) \) and (3.21) gives

\[
\mathbb{E}[\hat{f}_1(t) + \hat{p}(t) h(t, t) + \int_t^T \hat{p}(s) \frac{\partial h}{\partial s}(s, t) ds) d\alpha(t_i)] \leq 0 \text{ for each } t_i \text{ a.s.}
\]
Since this holds for all such $\zeta$ with arbitrary $t_i$, we conclude that
\[
\mathbb{E}[(f_1(t) + \dot{p}(t)h(t, t) + \int_t^T \dot{p}(s) \frac{\partial h}{\partial s}(s, t)ds)|\mathcal{G}_t] \leq 0 \text{ for each } t \in [0, T] \text{ a.s.}
\]
Finally, applying (3.21) to $\zeta_1 = \xi \in \mathcal{K}(\xi)$ and to $\zeta_2 = -\xi \in \mathcal{K}(\xi)$, we get for all $t \in [0, T]$
\[
\mathbb{E}[(f_1(t) + \dot{p}(t)h(t, t) + \int_t^T \dot{p}(s) \frac{\partial h}{\partial s}(s, t)ds)|\mathcal{G}_t]d\xi(t) = 0 \text{ for each } t \in [0, T] \text{ a.s.}
\]
\[
\square
\]

4 Applications: Optimal harvesting with memory

4.1 Optimal harvesting with density-dependent prices

Let $X^{u,\xi}(t) = X(t)$ be a given population density (or cash flow) process, modelled by the following
stochastic Volterra equation:
\[
X(t) = x_0 + \int_0^t b_0(t, s)X(s)ds + \int_0^t \sigma_0(s)X(s)dB(s) - \int_0^t \gamma_0(t, s)d\xi(s),
\]
for each $t \geq 0$. We assume that $b_0(t, s)$ and $\sigma_0(s)$ are given deterministic functions of $t, s$, with values in $\mathbb{R}$, and that $b_0(t, s), \gamma_0(t, s)$ are continuously differentiable with respect to $t$ for each $s$ and $\gamma_0(t, s) > 0$. For
simplicity we assume that these functions are bounded, and the initial value $x_0 \in \mathbb{R}$. We want to solve the
following maximisation problem:

**Problem 4.1** Find $\xi \in \mathcal{A}$, such that
\[
\sup_{\xi} J(\xi) = J(\hat{\xi}),
\]
where
\[
J(\xi) = \mathbb{E}[\theta X(T) + \int_0^T \log(X(t))d\xi(t)].
\]
Here $\theta = \theta(\omega)$ is a given $\mathcal{F}_T$-measurable square integrable random variable.

In this case the Hamiltonian $\mathbb{H}$ gets the form
\[
\mathbb{H}(t, x, \xi, \dot{p}, \dot{q}) = [b_0(t, t)x + \sigma_0(t)xq + \int_t^T \frac{\partial b_0}{\partial s}(s, t)x + \int_t^T \frac{\partial \sigma_0}{\partial s}(s, t)x + q]ds - \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s)d\xi(s))dt + \log(x) - \gamma_0(t, t)p\]d\xi(t).
\]
Note that $\mathbb{H}$ is not concave with respect to $(x, \xi)$, so the sufficient maximum principle does not apply. However, we can use the necessary maximum principle as follows: The adjoint equation gets the form
\[
\left\{ \begin{array}{l}
dp(t) = -\left[p(t)b_0(t, t) + \sigma_0(t)q(t, t) + \int_t^T \frac{\partial b_0}{\partial s}(s, t)p(s)dsdt + \frac{1}{X(t)}d\xi(t) \right] + q(t, t)d\xi(t)
\end{array} \right.
\]
\[p(T) = \theta,
\]
equivalently
\[ p(t) = \theta + \int_t^T \{ b_0(t,s)p(s) + \sigma_0(s)q(t,s) \} ds + \int_t^T \frac{T}{X(s)} d\xi(s) - \int_t^T q(t,s) dB(s). \quad (4.6) \]

The variational inequalities for an optimal control \( \hat{\xi} \) and the corresponding \( \hat{p} \) are:
\[
\log(\hat{X}(t)) - \gamma_0(t, t)p(t) - \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds \leq 0,
\]
and
\[
\{ \log(\hat{X}(t)) - \gamma_0(t, t)p(t) - \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds \} d\hat{\xi}(t) = 0.
\]

We have proved:

**Theorem 4.2** Suppose \( \hat{\xi} \) is an optimal control for Problem 4.1, with corresponding solution \( \hat{X} \) of (4.1). Then (4.9) and (4.10) hold, i.e.
\[
\gamma_0(t, t)p(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds \geq \log(\hat{X}(t)) \quad \text{a.s.,} \quad t \in [0, T]
\]
and
\[
\{ \gamma_0(t, t)p(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds - \log(\hat{X}(t)) \} d\hat{\xi}(t) = 0.
\]

**Remark 4.3** The above result states that \( \hat{\xi}(t) \) increases only when
\[
\gamma_0(t, t)p(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds - \log(\hat{X}(t)) = 0.
\]

Combining this with (4.9) we can conclude that the optimal control can be associated to the solution of a system of reflected forward-backward SVIEs with barrier given by (4.11).

### 4.2 Optimal harvesting with density-independent prices

Consider again equation (4.1) but now with performance functional
\[
J(\xi) = \mathbb{E}[\theta X(T) + \int_0^T \rho(t) d\xi(t)],
\]
for some positive deterministic function \( \rho \).

We want to find an optimal \( \hat{\xi} \in A \), such that \( \sup_{\xi} J(\xi) = J(\hat{\xi}) \).

In this case the Hamiltonian \( \mathbb{H} \) gets the form
\[
\mathbb{H}(t, x, \xi, \hat{p}, \hat{q}) = [b_0(t, s)p + \sigma_0(t)q] + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds - \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) d\xi(s) dt
+ [\rho(t) - \gamma_0] d\hat{\xi}(t).
\]

Note that \( \mathbb{H}(x, \xi) \) is concave in this case. Therefore we can apply the sufficient maximum principle here. The adjoint equation gets the form
\[
\begin{aligned}
\{ dp(t) &= - \left[ p(t)b_0(t, t) + \sigma_0(t)q(t, t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t)p(s) ds \right] dt + q(t, t) dB(t),

p(T) &= \theta,
\end{aligned}
\]
equivalently
\[
p(t) = \theta + \int_t^T \{ b_0(t, s)p(s) + \sigma_0(s)q(t, s) \} ds - \int_t^T q(t, s) dB(s).
\]
In this case the variational inequalities for an optimal control \( \hat{\xi} \) and the corresponding \( \hat{p} \) are:

\[
\gamma_0(t, t) \hat{p}(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t) \hat{p}(s) ds \geq \rho(t) \quad (4.13)
\]

and

\[
\{ \gamma_0(t, t) \hat{p}(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t) \hat{p}(s) ds - \rho(t) \} d\hat{\xi}(t) = 0. \quad (4.14)
\]

We have proved:

**Theorem 4.4** Suppose \( \hat{\xi} \) with corresponding solution \( \hat{p}(t) \) of the BSVIE (4.12) satisfies the equations (4.13) - (4.14). Then \( \hat{\xi} \) is an optimal control for Problem 4.1.

**Remark 4.5** Note that (4.13) - (4.14) constitute a sufficient condition for optimality. We can for example get this equation satisfied by choosing \( (\hat{p}(t), \hat{\xi}(t)) \) as the solution of the BSVIE (4.12) reflected downwards at the barrier given by

\[
\gamma_0(t, t) \hat{p}(t) + \int_t^T \frac{\partial \gamma_0}{\partial s}(s, t) \hat{p}(s) ds - \rho(t) = 0.
\]

## 5 Appendix

**Theorem 5.1** Consider the following linear BSVIE with singular drift

\[
p(t) = \theta + \int_t^T \{ b_0(t, s)p(s) + \sigma_0(s)q(t, s) \} ds + \int_t^T \frac{1}{\lambda(s)} d\xi(s) - \int_t^T q(t, s) dB(s). \quad (5.1)
\]

Therefore, the solution \( p(t) \) can be written on its closed formula as follows

\[
p(t) = \mathbb{E}[\{ \theta + \int_t^T \Psi(t, s) ds + \int_t^T \int_t^T \Psi(t, s) \frac{1}{\lambda(r)} d\xi(r) ds \} K(T)|\mathcal{F}_t],
\]

where

\[
\Psi(t, r) := \sum_{n=1}^{\infty} b_0^{(n)}(t, r) \quad (5.2)
\]

and \( K(T) \) is given by

\[
K(T) = \exp(\int_0^T \sigma_0(s) dB(s) - \frac{1}{2} \int_0^T \sigma_0^2(s) ds).
\]

Proof. The proof is an extension of Theorem 3.1 in Hu and Øksendal [9] to BSVIE with singular drift. Define the measure \( Q \) by

\[
dQ = M(T) dP \text{ on } \mathcal{F}_T,
\]

where \( M(t) \) satisfies the equation

\[
\begin{align*}
\left\{ \begin{array}{l}
dM(t) = M(t) \sigma_0(t) dB(t), \quad t \in [0, T], \\
M(0) = 1,
\end{array} \right.
\end{align*}
\]

which has the solution

\[
M(t) := \exp(\int_0^t \sigma_0(s) dB(s) - \frac{1}{2} \int_0^t \sigma_0^2(s) ds), \quad t \in [0, T].
\]

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Then under the measure $\mathbb{Q}$ the process

$$B_{\mathbb{Q}}(t) := B(t) - \int_0^t \sigma_0(s) \, ds, \quad t \in [0, T]$$

is a $\mathbb{Q}$-Brownian motion.

For all $0 \leq t \leq r \leq T$, define

$$b^{(1)}_0(t, r) = b_0(t, r), \quad b^{(2)}_0(t, r) = \int_t^r b_0(t, s) b_0(s, r) \, ds,$$

and inductively

$$b^{(n)}_0(t, r) = \int_t^r b^{(n-1)}_0(t, s) b_0(s, r) \, ds, \quad n = 3, 4, \ldots .$$

Note that if $|b_0(t, r)| \leq C$ (constant) for all $t, r$, then by induction on $n \in \mathbb{N} : |b^{(n)}_0(t, r)| \leq \frac{C^n t^n}{n!}$, for all $t, r, n$. Hence,

$$\Psi(t, r) := \sum_{n=1}^{\infty} |b^{(n)}_0(t, r)| < \infty,$$

for all $t, r$. By changing of measure, we can rewrite equation (4.6) as

$$\hat{p}(t) = \theta + \int_t^T b_0(t, s) \hat{p}(s) \, ds + \int_t^T X^{-1}(s) \, d\xi(s) - \int_t^T \hat{q}(t, s) dB_{\mathbb{Q}}(s), \quad 0 \leq t \leq T, \tag{5.4}$$

where the process $B_{\mathbb{Q}}$ is defined by (5.3). Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_t$, we get

$$\hat{p}(t) = \mathbb{E}_\mathbb{Q}[\theta + \int_t^T b_0(t, s) \hat{p}(s) \, ds + \int_t^T X^{-1}(s) \, d\xi(s)|\mathcal{F}_t]$$

$$= \hat{F}(t) + \int_t^T b_0(t, s) \mathbb{E}_\mathbb{Q}[\hat{p}(s)|\mathcal{F}_r] \, ds + \mathbb{E}_\mathbb{Q}[\int_t^T X^{-1}(s) \, d\xi(s)|\mathcal{F}_t], \quad 0 \leq t \leq T, \tag{5.5}$$

where

$$\hat{F}(s) = \mathbb{E}_\mathbb{Q}[\theta|\mathcal{F}_s].$$

Fix $r \in [0, t]$. Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_r$ of (5.5), we get

$$\mathbb{E}_\mathbb{Q}[\hat{p}(t)|\mathcal{F}_r] = \hat{F}(r) + \int_t^T b_0(t, s) \mathbb{E}_\mathbb{Q}[\hat{p}(s)|\mathcal{F}_r] \, ds + \mathbb{E}_\mathbb{Q}[\int_t^T X^{-1}(s) \, d\xi(s)|\mathcal{F}_r], \quad r \leq t \leq T.$$

Put

$$\tilde{p}(s) = \mathbb{E}_\mathbb{Q}[\hat{p}(s)|\mathcal{F}_r], \quad r \leq s \leq T.$$

Then the above equation can be written as

$$\tilde{p}(t) = \hat{F}(r) + \int_t^T b_0(t, s) \tilde{p}(s) \, ds + \mathbb{E}_\mathbb{Q}[\int_t^T X^{-1}(s) \, d\xi(s)|\mathcal{F}_r], \quad r \leq t \leq T.$$

Substituting $\tilde{p}(s) = \hat{F}(r) + \int_s^T b_0(s, \alpha) \tilde{p}(\alpha) \, d\alpha + \mathbb{E}_\mathbb{Q}[\int_s^T X^{-1}(\alpha) \, d\xi(\alpha)|\mathcal{F}_r]$ in the above equation, we obtain

$$\tilde{p}(t) = \hat{F}(r) + \int_t^T b_0(t, s) \{\hat{F}(r) + \int_s^T b_0(s, \alpha) \tilde{p}(\alpha) \, d\alpha + \mathbb{E}_\mathbb{Q}[\int_s^T X^{-1}(\alpha) \, d\xi(\alpha)|\mathcal{F}_r]\} \, ds$$

$$= \hat{F}(r) + \int_t^T b_0(t, s) \hat{F}(r) \, ds + \int_t^T b_0(t, s) \mathbb{E}_\mathbb{Q}[\int_s^T X^{-1}(\alpha) \, d\xi(\alpha)|\mathcal{F}_r] \, ds$$

$$+ \int_t^T b^{(2)}_0(t, \alpha) \tilde{p}(\alpha) \, d\alpha, \quad r \leq t \leq T.$$
where $\Psi$ is defined by (5.2). Now substituting $\tilde{p}(s)$ in (5.5), for $r = t$, we obtain
\[
\hat{p}(t) = \hat{F}(t) + \int_t^T \Psi(t,s) \hat{F}(t) ds + \int_t^T \Psi(t,s) E_Q [ \int_s^T X^{-1}(\alpha) d\xi(\alpha) | F_t ] ds \\
= E_Q [ \theta + \theta \int_t^T \Psi(t,s) ds + \int_t^T \Psi(t,s) \int_s^T X^{-1}(\alpha) d\xi(\alpha) ds | F_t ] \\
= E_Q [ \theta + \theta \int_t^T \Psi(t,s) ds + \int_t^T \Psi(t,s) ds \int_t^T X^{-1}(\alpha) d\xi(\alpha) ] | F_t ].
\]

References

[1] Aase, K., Øksendal, B., Privault, N., & Ubøe, J. (2000). White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. Finance and Stochastics, 4(4), 465-496.

[2] Agram, N. (2019). Dynamic risk measure for BSVIE with jumps and semimartingale issues. Stochastic Analysis and Applications, 37(3), 361-376.

[3] Agram, N. & Øksendal, B. (2015). Malliavin calculus and optimal control of stochastic Volterra equations. Journal of Optimization Theory and Applications, 167(3), 1070-1094.

[4] Agram, N., Øksendal, B., & Yakhlef, S. (2018). New approach to optimal control of stochastic Volterra integral equations. Stochastics, 1-22.

[5] Agram, N., Øksendal, B. & Yakhlef, S. (2018) Optimal control of forward-backward stochastic Volterra equations. In F. Gesztezy et al (editors): Non-linear Partial Differential equations, Mathematical Physics, and Stochastic Analysis. The Helge Holden Anniversary Volume. EMS Congress Reports, pp. 3-35. [http://arxiv.org/abs/1606.03280v4]

[6] Belbas, S. A. (2007). A new method for optimal control of Volterra integral equations. Applied Mathematics and Computation, 189(2), 1902-1915.

[7] Gripenberg, G., Londen. S.-O. & Staffans, O. (1990): Volterra Integral and Functional Equations. Cambridge University Press.

[8] Hida, T., Kuo, H. H., Potthoff, J. & Streit, L. (1993). White Noise. An Infinite-dimensional Approach. Kluwer.

[9] Hu, Y. & Øksendal, B. (2016). Linear backward stochastic Volterra equations. Stochastic Processes and their Applications (to appear). https://doi.org/10.1016/j.spa.2018.03.016

[10] Malliavin, P. (1978). Stochastic calculus of variations and hypoelliptic operators. In Proc. Internat. Symposium on Stochastic Differential Equations, Kyoto Univ., Kyoto, 1976. Wiley.

[11] Di Nunno, G., Øksendal, B. K., & Proske, F. (2009). Malliavin Calculus for Lévy Processes with Applications to Finance. Second Edition. Springer.

[12] Lin, P., & Yong, J. (2017). Controlled Singular Volterra Integral Equations and Pontryagin Maximum Principle. arXiv preprint [arXiv:1712.05911]
[13] Wang, T., Zhu, Q., & Shi, Y. (2011). Necessary and sufficient conditions of optimality for stochastic integral systems with partial information. In Control Conference (CCC), 2011 30th Chinese (pp. 1950-1955). IEEE.

[14] Yong, J. (2006). Backward stochastic Volterra integral equations and some related problems. Stochastic Processes and their Applications, 116(5), 779-795.

[15] Yong, J. (2008). Well-posedness and regularity of backward stochastic Volterra integral equations. Probability Theory and Related Fields, 142(1-2), 21-77.