IS THE PERCOLATION PROBABILITY ON $\mathbb{Z}^d$ WITH LONG RANGE CONNECTIONS MONOTONE?

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Abstract. We present a numerical study for the threshold percolation probability, $p_c$, in the bond percolation model with multiple ranges, in the square lattice. A recent Theorem demonstrated by de Lima et al. [B. N. B. de Lima, R. P. Sanchis, R. W. C. Silva, STOCHASTIC PROC APPL 121, 2043-2048 (2011)] states that the limit value of $p_c$ when the long ranges go to infinity converges to the bond percolation threshold in the hypercubic lattice, $\mathbb{Z}^d$, for some appropriate dimension $d$. We present the first numerical estimations for the percolation threshold considering two-range and three-range versions of the model. Applying a finite size analysis to the simulation data, we sketch the dependence of $p_c$ in function of the range of the largest bond. We shown that, for the two-range model, the percolation threshold is a non decreasing function, as conjectured in the cited work, and converges to the predicted value. However, the results to the three-range case exhibit a surprising non-monotonic behavior for specific combinations of the long range lengths, and the convergence to the predicted value is less evident, raising new questionings on this fascinating problem.

1. Introduction

Percolation is one of the most studied phenomena in statistical physics since it is closely related to several phase transitions in a wide range of systems. As a typical interdisciplinary subject, the literature shows contributions of researchers from different areas as mathematics, physics, civil engineering, soil science, hydrology etc. treating a wide range of problems from infiltration and porous flow to jamming and glassy transitions, passing by order-disorder models and magnetism [1, 2]. A definition to percolation could be simply stated as following: considers a regular two-dimensional square lattice ($\mathbb{Z} \times \mathbb{Z}$) in each the vertex are connected by bonds which are present with probability $p$; with we start with one single bond occupied and randomly sort a new bond to connect two neighbors sites, eventually a large cluster will span along the entire system and we say that the system has percolate.

Despite its simplicity, several open problems still challenge any theoretical treatment; for example, the exact value for the threshold percolation probability, $p_c$, for the regular lattice in three dimensions is not known, and even the continuity of the probability density function at $p_c$ still is an open problem (for general facts in percolation see [2]. The major importance in determine the percolation threshold and how it depends on the dimensionality of the space, $D$, is due to the link between percolation and phase transitions in natural systems. Several emergent phenomena arise near to the critical point, where power-law and fractal behaviors are observed. Most of these phenomena could be explained considering the cluster

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size distribution of similar components along the system, and percolation plays a major role to understand the mechanism involved in these natural processes. In the recent years, a particular attention was paid in literature on the study of irregular lattices, or complex networks. A central characteristic of these networks is an inhomogeneous distribution of bonds, or links, by node, and even a wide distribution of bond lengths. Hence, a natural question arises: what is the percolation threshold and how it behaves in function of the network parameters? Several works\cite{5} have studied this problem, but most of them have focused the applications, e. g., the Ising model in complex networks\cite{6}. Here, we present a more generic study of this problem from the percolation theory point of view; we introduce a percolation model with multiple bond ranges extending the usual bond percolation definition in the square lattice: we draw a bond to connect two sites in the lattice randomly chosen among a list of available sites within a given range. Thus, by means of this generalization of the standard model we expect to unveil possible hidden features which arise when we combine several bond ranges, and expect to shed some light in the problem of percolation in more complex networks.

Our primary aim is to verify a very recent conjecture from de Lima et al\cite{3} which states that the percolation threshold should be a decreasing function of the second range, when two bond ranges, (1, k), are available, and Theorem 1 in\cite{3} which states that the \( p_c \) value in this case converges to the bond percolation threshold in a \( d = 4 \) hypercubic lattice in the limit of \( k \) going to infinite. It worth to mention that add a new bond range is equivalent to increase the freedom degrees of the system, thus an operation analogous to increase the spatial dimension \( D \). Here, we present an extensive numerical study of the percolation threshold, \( p_c \), for two-range, (1, k), and three-range, (1, k, m), bond percolation models. In the three-range model, the predicted value when \( k \to \infty \) and \( m/k \to \infty \), is the percolation threshold of the \( d = 6 \) hypercubic lattice. Surprisingly, the numerical results have shown a non-monotonic behavior for specific combinations of \( k \) and \( m \) which are not predicted by the theory.

The paper is structured as following: in order to well state the problem and present the analytical conjectures, we present the model definition and the theoretical predictions in the limit of diverging ranges in the next section. The numerical procedure and simulation results are discussed in the following and we draw some conclusions and perspectives at the final section.

2. Theoretical Aspects

Let \( G = (\mathbb{V}, \mathbb{E}) \) be a graph with a countably infinite vertex set \( \mathbb{V} \). Consider the Bernoulli site percolation model on \( G \) which associates to each vertex the values 1 (“occupied”) and 0 (“vacant”) with probability \( p \) and \( 1 - p \) respectively. This can be done considering the probability space \( (\Omega, \mathcal{F}, \mathbb{P}_p) \), where \( \Omega = \{0, 1\}^{\mathbb{V}} \), \( \mathcal{F} \) is the \( \sigma \)-algebra generated by the cylinder sets in \( \Omega \) and \( \mathbb{P}_p = \prod_{v \in \mathbb{V}} \mu(v) \) is the product of Bernoulli measures with parameter \( p \). We denote a typical element of \( \Omega \) by \( \omega \).

Given two vertices \( v \) and \( u \), we say that \( v \) and \( u \) are connected in the configuration \( \omega \) if there exists a finite path \( (v = v_0, v_1, \ldots, v_n = u) \) of occupied vertices in \( \mathbb{V} \), such that \( v_i \neq v_j, \forall i \neq j \) and \( (v_i, v_{i+1}) \) belongs to \( \mathbb{E} \) for all \( i = 0, 1, \ldots, n - 1 \). We will use the short notation \( \{v \leftrightarrow u\} \) to denote the set of configurations where \( u \) and \( v \) are connected.
For a given vertex $v$, the cluster of $v$ in the configuration $\omega$ is the set $C_v(\omega) = \{u \in V; v \leftrightarrow u \in \omega\}$. We say that the vertex $v$ percolates when the cardinality of $C_v(\omega)$ is infinite; we will use the following standard notation $\{v \leftrightarrow \infty\} \equiv \{\omega \in \Omega; \#C_v(\omega) = \infty\}$. Once $v$ fixed, we define the percolation probability of the vertex $v$ as the function $\theta_v(p) : [0,1] \mapsto [0,1]$ with $\theta_v(p) = \mathbb{P}_p(v \leftrightarrow \infty)$. The percolation threshold (or critical point), $p_v(G)$, is defined by

$$p_v(G) = \sup\{p \in [0,1]; \theta(p) = 0\}.\,$$

From now on, the vertex set $V$ will be $\mathbb{Z}^d$, $d \geq 2$ and for each positive integer $k$ define

$$E_k = \{(v_1,\ldots,v_d),(u_1,\ldots,u_d)\in V \times V; \exists i \in \{1,\ldots,d\}\text{ such that }|v_i - u_i| = k \text{ and } v_j = u_j, \forall j \neq i\}.\,$$

Let’s define the graph $G^k = (V,E_1 \cup E_k)$ that is, $G^k$ is the $\mathbb{Z}^d$ equipped with nearest neighbors bonds and long range bonds with length $k$ parallel to some coordinate axis. Observe that $G^k$ is a transitive graph, hence the function $\theta_v(p)$ does not depend on $v$ and we write only $\theta^k(p)$ to denote $\mathbb{P}_p(0 \leftrightarrow \infty)$ on the graph $G^k$.

Consider the sequence, $(p_v(G^k))_k$, of the percolation thresholds of the graphs $G^k$. In [3] (see Theorem 1), it is proven that

$$\lim_{k \rightarrow +\infty} p_v(G^k) = p_v(\mathbb{Z}^{2d}), \forall d \geq 2,\,$$

that is, the percolation threshold of $G^k$ converges to the percolation threshold of the hypercubic (nearest neighbor) lattice in 2$d$ dimensions. It is also conjectured [3] that the sequence $(p_v(G^k))_k$ is non increasing in $k$. One goal of this work is to simulate long range percolation on the graph $G_k$ in $d = 2$. These simulations, as described in the next section, show that $p_v(G^k)$ goes to $p_v(\mathbb{Z}^d)$ monotonically, confirming the Theorem 1 and the Conjecture in [3].

Theorem 1 and the Conjecture in [3] can be generalized for bond percolation with multiple ranges. Given a sequence $\vec{k} = (k_1,\ldots,k_n)$ with $k_i \in \{2,3,\ldots\}, \forall i$, consider now the graph $G^{\vec{k}}$ as $(\mathbb{Z}^d,E_1 \cup (\cup_{i=1}^n E_{k_1\times\cdots\times k_i}))$. That is, $G^{\vec{k}}$ is $\mathbb{Z}^d$ decorated with all bonds parallel to each coordinate axis with lengths $1,k_1,k_1 \times k_2,\ldots,k_1 \times k_2 \times \cdots \times k_n$. In this case Theorem 1 of [3] states that

$$\lim_{k_i \rightarrow +\infty,\forall i} p_v(G^{\vec{k}}) = p_v(\mathbb{Z}^{d(n+1)}), \forall d \geq 2,\,$$

and the Conjecture says that the probability of percolation on the graph $G^{\vec{k}}$ is non decreasing in each variable $k_i$.

In this work, we also perform simulations in the case $n = 3$, considering bonds with three different ranges $1$, $k$ and $m$. The results support the Theorem 1 and the Conjecture in [3] (in this case, $m$ should be a multiple of $k$). One interesting feature displayed by simulations is that the percolation threshold is not monotone in $k$, with $m$ fixed.

3. Numerical Procedure

In this section we present the numerical approach used in the simulations. We consider a square two-dimensional regular lattice as the site substrate of the model. When a link to a given range is open, two sites in a column, or in a line, will be connected to each other forming a bond - see Figure 1. For each bond, besides to
Figure 1. Neighborhood of a bond in the 2-range version of the model. The first neighbors of the central black bond are shown, for \((1, k) = (1, 2)\). Note that there is 6 short-range bonds (A-F) with the same length as the central bond, and 8 long-range bonds (1-8). The sites connected to the central bond are filled with different gray levels, depending on the range of the bond linked to the site.

the six neighbors in the same range, there are eight neighbors consisting of bonds of another length. Thus, in the 2-range version of the model there are 14 neighbors and, in the 3-range, 22 neighbors to be taken into account for every new bond opened.

The algorithm used to estimate the percolation threshold is based on the work of Newman and Ziff [7]. Basically, we label each bond of the lattice assigning the first \(2N\) natural numbers to bonds of unitary length (where \(N = L \times L\), and \(L\) is the system size); the labels \(2N + 1\) to \(4N\) are assigned to the range \(k\), and so on. Thus, we enumerate all the possible bonds to be opened in the system from 0 to \(r 2N\), where \(r\) is the number of ranges considered.

In this way, each realization consists of a random permutation in the list of the bonds to be opened, varying the occupation probability, \(p\), from 0 to 1. After each bond is open, the percolation condition is tested using the modified Hoshen-Kopelman algorithm to identify the clusters [1]. The probability of a given site to belong to the infinite cluster is then estimated from the simulated data using the normalization procedure described in [7].

The code was developed in C and Purebasic languages. Purebasic code developed by M. Schnabel is procedure-oriented and 1000 samples of a system with \(L = 1024\), single range percolation model, takes around two minutes to run in an AMD Athlon X64 1.9 GHz and 3 GBytes DDR2 RAM machine. It is about 20% faster than the similar C implementation.
IS THE PERCOLATION PROBABILITY ON Z^d WITH LONG RANGE CONNECTIONS MONOTONE?

Figure 2. Probability of a site belongs to the infinite cluster in a single range bond percolation model. The plot shows the variation of Φ(p) to 6 values of L = 32, 64, 128, 256, 512, 1024. Note that all curves tend to cross at the same point. Considering two consecutive system sizes, we measure the ordinate value of the crossing point. The inset shows the crossing values obtained in function of the inverse of the geometric mean between the system sizes, 1/√L_i L_j. The independent term of the linear fit correspond to the best estimative for the percolation threshold to a infinite system size, and matches perfectly with the exact value p_c = 1/2.

In order to validate the code developed, we initially consider the bond percolation model limited to a single range, in a square lattice of linear extension L, with periodic boundary conditions in both directions. We simulate 1000 realizations for each system size, 16 < L < 4096, and calculate the probability of a given site belongs to the infinite cluster, Φ(p), Figure 2. In Figure 3, we show the corresponding calculation for the two-range bond model considering k = 7. We denote by (1, k) or (1, k, m) the 2-bond and 3-bond models, respectively, where k, m ∈ N.

The percolation criterion used assumes that the infinite cluster arises when there is a path of connected bonds crossing the entire system, vertically and horizontally, not using the periodic boundaries. Although the particular choice used for this criterion, we believe that we would get qualitatively identical results even for other choices, as discussed in the next section.
Figure 3. Probability of a site to belong to the infinite cluster in a 2-range \((1,k)\) bond percolation model, with \(k = 7\). The plot shows the value of \(\Phi(p)\) considering 6 values of \(L = 32, 64, 128, 256, 512, 1024\). The inset shows the finite size analysis used to obtain the best estimative for the percolation threshold at the infinite system (see the caption of Figure 2). This procedure is repeated for each value of \(k\) considered in this version of the model, and the dependence of the threshold percolation in function of \(k\) is shown in Figure 4.

4. Results and Discussion

Figure 3 presents the procedure used estimate the percolation probability to the 2-range model \((1,k)\), with \(k = 7\). Considering different values of \(k\), it is straightforward the calculation of \(p_k^c\) for increasing \(k\), which is shown in Figure 4. The limiting value for \(k \to \infty\) is \(p_c \sim 0.162(2)\), close to the expected value for the bond percolation threshold at the \(d = 4\) hypercubic lattice - \(p_c = 0.160131(1)\) as stated by Theorem 1. Thus, we can conclude from this plot that the percolation threshold exhibits a monotonic dependence on \(k\), supporting the Conjecture in [3], and confirming the prediction for the limit \(k \to \infty\) stated by Theorem 1.

The results for the 3-range model are shown in Figure 5. We consider five different values to \(m\), and all the possible \(k\) between 1 and \(m\). All results are shown in Figure 5. We can observe some striking features: first, there is a clear downward trend of \(p_c\) with increasing \(m\). To \(m\) fixed, generally \(p_c\) decreases with \(k\), except when \(k\) approaches to \(m\), or to the half-value of \(m\). This trend was observed for \(m\) even, odd or prime. For \(m = 32\) a curious feature appeared: the value of \(p_c\) increases about \(k = 11\) and \(k = 21\). We do not have a good explanation for this match so far. The increase to \(k = 8\) can be also be attributed to the correspondence
IS THE PERCOLATION PROBABILITY ON Z^d WITH LONG RANGE CONNECTIONS MONOTONE?

Figure 4. Threshold percolation probability, $p_c$, for 2-range bond models, with increasing range $k$. There is no justification for the fit function used and the line is only to guide the eyes. The limit value of $p_c$ is obtained from the fitting parameter $y_0$.

with $m = 32$, as 32 is multiple of 8; however, for ranges of $k = 4$ and $m = 16$, for example, a similar effect was not observed, discarding this conjecture.

Considering the curves for $m = 10, 13$ and 16, they all behave similarly, with $p_c$ decreasing with $k$ except for $k$ around to $m$ or $m/2$. For $m = 13$, the increasing at $m/2$ is more discreet, but occurs between $k = 6$ and 7, since the half-value of $m$ is not an integer. The curves for $m = 32$ and 33 show different features. We observe that the decrease of $p_c$ with $k$ is more subtle, but the increasing at $m$ and $m/2$ is quite evident. The peaks observed at $k = 8, 11, 21$ are unexpected and not well understood for the moment. The limit value of $p_c \approx 0.0955(7)$ around to $k = 25$, approaches to the bond threshold percolation value at $d = 6$ hypercubic lattice, $p_c = 0.0942019(6)$, which is expected when the two long ranges tend to infinity.

5. CONCLUSIONS AND PERSPECTIVES

We present for the first time the percolation threshold estimation for a multiple range bond percolation model. We study the behavior of $p_c$ in two and three range models in function of the bond range using numerical simulations. We have confirmed the predicted value to the percolation threshold when the long ranges tend to infinity in these models - Theorem 1 in [3]. A remarkable effect was observed
Figure 5. Threshold percolation probability, \( p_c \), for 3-range bond models, in function of the range \( k \); five different curves corresponding to increasing \( m \) are shown. Note the differences when \( m \) is odd or a prime number, smoothing the curves. The striking non-monotonic behavior observed for specific combinations of the long-ranges still is not fully understood to the moment. The dashed line represents a Lorentzian fit to the minima of the plots.

for the 3-range version of the model, with an unexpected increase in the percolation threshold for some values specific values of the second and third ranges which defy any convincing explanation to the present date and could represent an important feature in models with long range interactions or even systems dealing to complex networks.

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