ACCRETION DISKS AROUND KERR BLACK HOLES:
VERTICAL EQUILIBRIUM REVISITED

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ABSTRACT

We rederive the equation for vertical hydrodynamical equilibrium of stationary thin and slim accretion disks in the Kerr spacetime. All the previous derivations have been unsatisfactory, yielding unphysical singularities. Our equation is nonsingular, more general, and simpler than those previously derived.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — relativity

1. PHYSICAL MEANING OF THE VERTICAL EQUATION

In this paper we derive, in Kerr geometry, the vertical equation for stationary thin and slim accretion disks. The equation governs the balance between the vertical pressure gradient, vertical acceleration, and vertical components of all the inertial forces. Vertical direction means here the direction orthogonal to the plane of the disk, for example, the $z$ direction in cylindrical coordinates or the $\theta$ direction in spherical coordinates. We assume that the disks are stationary and axially symmetric and that the vertical thickness of the disk $H$ is much smaller than the corresponding cylindrical radius $r$.

We derive the equation first in Newtonian theory in order to stress some points that are often overlooked, but the new result presented in this paper is the fully self-consistent derivation of the vertical equation in general relativity, in particular for the Kerr spacetime.

In the reference frame comoving with the matter, the vertical pressure gradient force balances the combined vertical components of all the inertial forces: gravity, centrifugal force, the Euler force due to the vertical acceleration, and (in the case of a rotating black hole) the general relativistic Lense-Thirring force (dragging of inertial frames). The Coriolis force always vanishes in the comoving frame of reference and does not need to be considered.

Previously, the vertical equation has been derived by Novikov & Thorne (1973, hereafter NT), Riffert & Herold (1995, hereafter RH), and Abramowicz et al. (1996, hereafter ACGI). None of these derivations was satisfactory. They all made incorrect assumptions that introduced unphysical singularity into the final form of the vertical equation. Formulas derived by NT and RH are singular at the location of the circular photon orbit, while the formula derived by ACGI is singular at the horizon. NT based their derivation on a discussion of the gravitational tidal forces in the vertical cylindrical direction. This set a standard for several further derivations but (as pointed out by Lasota & Abramowicz 1997) was not a fortunate choice, because a mathematically correct discussion of the vertical tidal forces at the horizon (not yet given) would be very complicated and thus impractical. On the other hand, a direct approach based on the relativistic Euler equation is straightforward and simple. Here we derive the equation directly from the relativistic Euler equation and make no additional simplifying assumptions. Our derivation is based on a general discussion given by Lasota & Abramowicz (1997).

2. THE METHOD: SLIM ACCRETION DISKS

Physical quantities $X(r, z) = X(r, -z)$ that are symmetric with respect to the plane of the disk, $z = 0$, are expanded around the plane according to

$$X(r, z) = \sum_{i=0}^{N=\infty} X_{2i}(r) \left( \frac{z}{r} \right)^{2i}.$$ 

Physical quantities antisymmetric with respect to the plane $X(r, z) = -X(r, -z)$ have expansion

$$X(r, z) = \sum_{i=0}^{N=\infty} X_{2i}(r) \left( \frac{z}{r} \right)^{2i}.$$ 

Putting these expansions into the partial differential equations that describe stationary accretion disks, one converts the partial differential equations into a set of $N = \infty$ equations with $N = \infty$ unknowns. Truncating the expansions at $N = 0$, thus keeping only zeroth order terms in $(z/r)$, would not give a closed set: in this case there are more unknown functions than the equations. Truncating at $N = 1$ gives a closed set, with the number of unknown functions equal to the number of equations. The same is true for any $N > 1$. Only the terms of the order $(z/r)^0$ and $(z/r)^2$ appear in the $N = 1$ equations. Therefore, the approximation based on truncating all expansions at $N = 1$ offers two advantages: (1) it is mathematically well defined and consistent, and (2) it involves the smallest number of equations to be solved. The precise mathematical meaning of the “slim accretion disks” approach (Abramowicz et al. 1988) is just that: one should keep all the terms of the order $(z/r)^2$ and $(z/r)^0$ in all the equations. This should be contrasted with the “thin accretion disks” approach, in which terms of the order of $(z/r)^2$ are kept in some of the equations, but in others they are rejected according to the tradition established in the influential paper by Shakura & Sunyaev (1973). It is now well understood that this traditional approach often causes serious physical inconsistencies that make the thin disk models inadequate for describing several astrophysically interesting types of accretion disks.
Here we derive one particular equation—the vertical equation—that belongs to the set of the slim accretion disk equations. All the other equations have been derived in the Kerr geometry by Lasota (1994) and Abramowicz et al. (1996). The vertical equation is itself of the order \((z/r)^2\). We keep all the terms of zeroth and second orders in our derivation. Thus, our final “slim disk” equation contains all of the \((z/r)^2\) terms that follow from a mathematically consistent procedure. However, we also discuss here the “thin disk” form of this equation in which some of the \((z/r)^2\) terms have been rejected in a way that is not mathematically consistent but follows the traditional thin disk approach.

3. NEWTONIAN DERIVATION

It will be helpful to derive the vertical equation in Newtonian theory in both cylindrical and spherical coordinates. Of course, the final result does not depend on the coordinates used. However, there are some interesting differences between the equations written in cylindrical and spherical coordinates. When we use cylindrical coordinates, there is no centrifugal force in the “vertical” direction, whereas when we use the spherical coordinates, there is no gravitational force in the “vertical” direction. As we will see, it is exactly this property that makes the spherical coordinates much better adapted for describing the flow near the black hole horizon. All previous derivations have been carried out in cylindrical coordinates, and we feel that it would be proper if we explain our derivation in these coordinates first.

3.1. Cylindrical Coordinates

Let \(H = \text{z}(r)\) describe the location of the disk surface in cylindrical coordinates \((r, z, \phi)\). According to our basic assumption that terms up to the second order should be kept, the pressure \(P(r, z)\), in terms of \((z/r)\), has the expansion

\[
P(r, z) = P_0(r) - \left(\frac{z}{r}\right)^2 P_2(r) + \frac{c^4}{2}\left(\frac{z}{r}\right)^4,
\]

where the small dimensionless parameter \((z/r)\) gives the distance from the plane of the disk. The minus sign before \((z/r)^2 P_2(r)\) has been chosen to make \(P_2(r) > 0\). \(P_0(r)\) is the pressure at the plane of the disk. At the disk surface, \(P(r, z) = 0\); therefore,

\[
P_0(r) = \left(\frac{H}{r}\right)^2 P_2(r).
\]

Note that for \(z \sim H\), higher order terms in equation (1) are not numerically small compared with the lower order ones. This is a weak point of the method, but we have checked that the numerical inaccuracies introduced by this are of the same order as those of other approximate methods.

From equations (1) and (2) it follows that

\[
P(r, z) = P_0(r)\left[1 - \left(\frac{z}{H}\right)^2\right]; \quad \frac{\partial P}{\partial z} = -\frac{2z}{H^2} P_0(r).
\]

We shall denote by \(\rho_0(r)\) the density at the plane of the disk. We do not assume anything about the vertical distribution of the density of matter \(\rho(r, z)\), as all derivations based on the “vertical integration” do. In particular, we do not assume that the density vanishes at the disk surface or that it is in accordance with a polytropic equation of state, \(P = K \rho^{1+1/n}\), as assumed by Hoshi (1977). Hoshi’s formula reduces to our equation (3) for \(n = 0\).

We shall write for the radial and cylindrical components of velocity,

\[
v_r = v_0(r) + \frac{c^2}{2}\left(\frac{z}{r}\right),
\]

\[
v_z = \left(\frac{z}{r}\right) u_1(r) + \frac{c^3}{2}\left(\frac{z}{r}\right).
\]

For stationary flows, the vertical equation has the general form

\[
\frac{1}{\rho} \left(\frac{\partial P}{\partial z}\right) + \frac{v_r}{r} \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = 0.
\]

Here \(\Phi = \Phi(R)\) is the gravitational potential that depends only on the spherical radius \(R = (r^2 + z^2)^{1/2}\), \((\partial R/\partial z) = z/R\). The Keplerian orbital velocity \(V^2_K(R)\) in the potential \(\Phi(R)\) is given by

\[
V^2_K(R) = \left(\frac{R}{dR} \frac{d\Phi}{dR}\right).
\]

Taking these relations into account, we write equation (7) in the following form:

\[
\left(\frac{z}{r}\right) \left[2\frac{P_0}{\rho_0} + \frac{H^2}{r^2} \left(V^2_K + rv_0 \frac{du_1}{dr} + u_1^2 - u_1 v_0\right)\right] + \frac{c^3}{2}\left(\frac{z}{r}\right) = 0.
\]

Thus, by demanding that the expression in the curly brackets vanishes, we find the most general version of the vertical equation:

\[
-2\frac{P_0}{\rho_0} + \frac{H^2}{r^2} \left(V^2_K + rv_0 \frac{du_1}{dr} + u_1^2 - u_1 v_0\right) = 0.
\]

In this most general version, an additional unknown function \(u_1(r)\) appears that is not present in the standard thin or slim set of equations. The additional equation that closes the set (not needed in the standard treatment, in which \(u_1 = 0\) is assumed ad hoc) is provided by the surface boundary condition, which says that there should be no component of the total velocity orthogonal to the surface of the disk. In cylindrical coordinates this means that

\[
\frac{u_1}{v_0} = \frac{d \ln H}{d \ln r}.
\]

From the above equation it follows that \(u_1\) and \(v_0\) must be of the same order with respect to \((H/r)\) and that \(u_1 = 0\) is only a good approximation if \(H\) varies very slowly with \(r\).

Therefore, in the most general situation, the vertical equation consists of two equations: equation (9), which governs the vertical balance of forces and acceleration, and equation (10), which demands that in a stationary disk there is no component of velocity orthogonal to the surface.

From equation (9) it follows that, in general, \(P_0/\rho_0 \sim C^2_s \sim (H/r)^2\), with \(C_s\) being the sound speed. In addition, it
is obviously $V_k^2 \sim (H/r)^3$. One can make simplifications in equation (9) by considering the $(H/r)^2$ order of each term. In the standard thin disk model the radial flow is very subsonic, and thus $v_0 \sim u_1 \ll C_S$. Therefore, the terms $u_0, du_1/du$, $u_0^2$, and $u_1, v_0$ are of order higher than $(H/r)^2$ and may be dropped. The standard thin disk version of the vertical equation therefore has the form

$$\frac{P_0}{\rho_0} = \frac{1}{2} \left( \frac{H}{r} \right)^2 V_k^2. \quad (11)$$

In the transonic part of the slim disks, the terms containing $u_1$ are exactly of the $(H/r)^2$ order and may also be neglected. From consideration of the vertical equilibrium alone it seems possible that the thin disk version of the vertical equation could also be written in the form of equation (11), without the terms containing $u_1$. However, if these terms have been dropped, then in doing calculations with the full set of equations, one should perform a consistency check and calculate the ratios $(u_1/v_0)/C_S^2$ and $(u_1, u_1)/C_S^2$ using equation (10). If these ratios are approximately 1, one must use the general form of the vertical equation, provided by equations (10) and (9). This is the case close to the horizon of the black hole, in which $v_0 \sim u_1 \sim c \gg C_S$. We shall return to this point later.

### 3.2. Spherical Coordinates

In spherical coordinates, $(R, \theta, \phi)$, the small parameter that characterizes the distance from the plane of the disk, should now be $\cos \theta$, and the equation for the surface of the disk is given by $\theta = \Theta(R)$. Obviously, in this case $H/r \simeq \cos \Theta$. According to this, a formula for the pressure, similar to equation (3), should be written as

$$P(R, \theta) = P_0(R) \left[ 1 - \left( \frac{\cos^2 \theta}{\cos^2 \Theta} \right) \right],$$

$$\frac{1}{R} \frac{\partial P}{\partial \theta} = \frac{2R \cos \theta}{H^2} P_0(R). \quad (12)$$

For the radial, azimuthal and polar components of the velocity we again write

$$v_\phi(R, \theta) = v_\phi(R) + \theta^2(\cos \theta), \quad (13)$$

$$v_\theta(R, \theta) = v_\theta(R) + \theta^2(\cos \theta), \quad (14)$$

Note that $v_\phi$ does not have the physical dimension of velocity. This is because, in spherical coordinates, $d\phi/dt = \Omega$, with $\Omega$ being the angular velocity, and $v_\phi \equiv \theta \omega \phi \equiv R^2 \sin^2 \theta \Omega = R^2 \Omega = L^2$, with $L^2$ being the angular momentum per unit mass. Similarly, $\theta^2$ has the physical dimensions of angular velocity, and both $v_\theta$ and $u_1$ have the dimensions of angular momentum per unit mass. Of course, $\theta^3$, $v_\theta$, and $v_\phi$ have the dimensions of velocity. One should bear this in mind when checking the physical dimensions of our formulae in spherical coordinates or when comparing our formulae with those given e.g., by Tassoul (1978, pp. 479–480). Tassoul's azimuthal velocity "$v_\phi$", which we denote here (to avoid confusion) by $V_{(\phi)}$, is given by $V_{(\phi)} = R \sin \theta \omega \theta = v_\phi/R \sin \theta$ and has the physical dimensions of velocity.

In spherical coordinates, the condition that there should be no component of velocity orthogonal to the surface of the disk, similar to that given by equation (10), takes the form

$$\frac{1}{R} \frac{u_1}{v_0} = \frac{d \ln |\pi/2 - \Theta|}{d \ln R} \cdot (15)$$

The general form of the vertical equilibrium is now given by a formula similar to equation (7) but is written in spherical coordinates,

$$\frac{1}{R} \frac{\partial P}{\partial \theta} + R v_\theta \frac{\partial (v_\phi/R)}{R} + \frac{v_\theta^2}{2} \frac{\partial (v_0/R)}{R} + \frac{v_\phi^2}{R} \frac{\partial \Theta}{R} \cot \theta = 0. \quad (16)$$

From these formulae, using the same procedure as before, we derive the general form of the vertical equation in spherical coordinates, similar to its cylindrical version (eq. [9]),

$$-2 \frac{P_0}{\rho_0} + \left( \frac{H^2}{R^2} \right) \left[ V_{(\phi)}^2 + \frac{u_1^2}{R^2} - \frac{du_1}{dR} \right] = 0. \quad (17)$$

Note the significant difference between the two versions of the same vertical equation, written in cylindrical coordinates (eq. [9]) and in spherical coordinates (eq. [17]). While in equation (9) the Keplerian velocity $V_k^2$ appears, in equation (17) we have the rotational velocity $V_{(\phi)}^2$. The difference is due to the fact that in the $z$ direction there is gravity force $(\sim V_k^2/R)$ but no centrifugal force, while in the $\theta$ direction there is centrifugal force $(\sim V_{(\phi)}^2/R)$ but no gravity. Does this matter? One may argue that it does not, because

$$V_k^2 - V_{(\phi)}^2 \sim H^2 \frac{d^2}{r^2} = \cos^2 \Theta; \quad (18)$$

therefore, with accuracy to the higher order terms, these two versions of the vertical equation are identical [for the standard thin disk $V_k^2 \equiv V_{(\phi)}^2$].

But actually it does matter a lot. First, in both Newtonian and general relativistic theories, the stationary accretion flows calculated in realistic two- and three-dimensional simulations resemble quasi-spherical flows ($\Theta \approx$ constant) much more than quasi-horizontal flows ($H \approx$ constant). This has been noticed by several authors, e.g., Papaloizou & Szuszkiewicz (1994) and Narayan & Yi (1995). Thus, while $u_1 = 0$ may be a reasonable approximation in spherical coordinates, it is not so in the cylindrical ones. Second, anticipating the relativistic discussion in the next section, the Keplerian velocity $V_k^2(R)$ is singular at the location of the circular photon orbit, while the azimuthal velocity $v_\phi$ is nonsingular everywhere, including the horizon. Third, we know exactly what happens on the horizon in spherical coordinates. A general theorem (Carter 1973) ensures that in spherical coordinates, $u_1(R_{\text{g}}) = 0$. The use of cylindrical coordinates to describe the flow brings artificial, and technically complicated, difficulties near the horizon. These are generated only by a bad choice of coordinates (as, e.g., in Riffert & Herold 1995).

### 4. RELATIVISTIC DERIVATION

We shall now derive the vertical equation in Kerr geometry, using the same method as in the previous section.
It is convenient to work at the beginning with a general metric,
\[ ds^2 = g_{\alpha \beta} dt^2 + 2g_{\alpha \phi} dt d\phi + g_{\phi \phi} d\phi^2 + g_{R R} dR^2 + g_{\theta \theta} d\theta^2 , \]
(19)
and specify the Kerr metric functions \( g_{\alpha \beta} \) in Kerr (R, \( \theta \)) at the end of the calculations.

The stress-energy tensor for a perfect fluid has the form
\[ T^i_k = \rho u^i u_k - \delta^i_k \rho \]
(20)
where \( u^i = dx^i/ds \) is the four-velocity of the fluid, and \( \rho = \epsilon + P \), where \( \epsilon \) is the energy density (we use units in which \( c = 1 \)). The four-velocity is a unit vector, which means that
\[ 1 = g^{ik} u_k u_k = g^{ii} u_i u_i + 2g^{i\phi} u_i u_\phi + g^{\phi \phi} u_\phi u_\phi + g^{R R} u_R u_R + g^{\theta \theta} u_\theta u_\theta . \]
(21)

The inverse metric \( g^{ik} \) obeys \( g^{ik} g_{jk} = \delta^i_j \).

We derive the vertical equation taking the \( j = \theta \) component of the general equation \( h^i_j V_k T^i_k = 0 \). Here \( V_k \) is the covariant derivative operator, and \( h^i_j = \delta^i_j - u^i u_j \) is the projection tensor. This gives, at the end of simple standard calculations,
\[ \frac{1}{\rho} \frac{\partial P}{\partial \theta} = u^k \frac{\partial u_\theta}{\partial x^k} - \Gamma^i_{\theta k} u_i u^k , \]
(22)
where \( \Gamma^i_{\theta k} \) is the Christoffel symbol that should be computed from the metric components and their first derivatives. This relativistic equation provides the replacement for the Newtonian equation (16). As in the Newtonian case, the leading order of this equation is \( (\cos \theta)^3 \). Thus, in the calculations that follow, we shall keep only these terms and neglect the \( \cos^3 \) and higher order terms.

The \( -\Gamma^i_{\theta k} u_i u^k \) term should be computed with the help of equation (21). The result in Kerr geometry is
\[ -\Gamma^i_{\theta k} u_i u^k = -u_\theta u_\phi (g^{\phi \phi} \Gamma^i_{\phi \theta} + g^{\phi \phi} \Gamma^i_{\theta \phi} - g^{\phi \phi} \Gamma^i_{\theta \theta}) \]
\[ - u_\phi u_\theta (g^{\phi \phi} \Gamma^i_{\phi \phi} - g^{\phi \phi} \Gamma^i_{\phi \theta} + g^{\phi \phi} \Gamma^i_{\phi \phi} - 2g^{\phi \phi} \Gamma^i_{R R}) \]
\[ - u_i u_\phi (g^{\phi \phi} \Gamma^i_{\phi \phi} - g^{\phi \phi} \Gamma^i_{\phi \theta} - g^{\phi \phi} \Gamma^i_{\phi \rho} R_{R \theta} R_{R \theta} (\cos^2 \theta) - g^{\phi \phi} \Gamma^i_{R R}) . \]
(23)

In deriving this equation we took into account that \( u_\phi u_\phi \sim \cos^2 \theta \) should be dropped and that in Kerr geometry \( g^{\phi \phi} \Gamma^i_{\phi \theta} + g^{\phi \phi} \Gamma^i_{\theta \phi} = 0 \).

The term \( u^k \frac{\partial u_\theta}{\partial x^k} \) equals
\[ u^k \frac{\partial u_\theta}{\partial x^k} = \cos \theta (g^{R R} V_0 \frac{dU_1}{dR} - g^{\theta \theta} U_1^2) , \]
(24)
where \( U_1 \) and \( V_0 \) are defined in terms of the following expansions (see eqs. [13] and [14]):
\[ u_R = V_0 (R) + \mathcal{O}^2 ( \cos \theta) , \quad u_\theta = \cos \theta U_1 (R) + \mathcal{O}^3 ( \cos \theta) . \]
(25)

One also needs to use the relations \( u^\theta = g^{\theta \theta} u_\theta \) and \( u^R = g^{R R} u_R \).

The terms containing pressure gradient should be treated the same way as in the Newtonian derivation.

To the desired orders, the components of the inverse metric and the Christoffel symbols that appear in equations (23) and (24) are in Kerr geometry,
\[ g^{tt} = \frac{2a^2 (R + 2M) + R^3}{\Delta} + \Delta ; \quad g^{\phi \phi} = \frac{2aM}{\Delta} , \]
\[ g^{R R} = - \frac{\Delta - \xi}{R^2} ; \quad g^{\theta \theta} = \frac{R^2}{\Delta} \cos \theta , \]
\[ \Gamma^\phi_{\theta \phi} = \frac{2aM}{R^3} \cos \theta ; \quad \Gamma^R_{\theta \theta} = \frac{a^2 M}{R^3} \cos \theta \]
\[ \Gamma^\phi_{\phi \phi} = \left( 1 + \frac{2a^2 M}{R^3} \right) \cos \theta ; \quad \Gamma^R_{\theta \theta} = - \frac{a^2}{R^3} \cos \theta . \]
(26)

By inserting these into equations (22), (23), and (24), we obtain the final result:
\[ -\frac{2 P_0}{\rho_0} + \left( \frac{H}{R} \right)^2 \left( \mathcal{L}_0^2 + U_1^2 - \Delta V_0 \frac{dU_1}{dR} \right) = 0 . \]
(27)

Here \( \mathcal{L}_0^2 = \mathcal{L}^2 - a^2 (\beta - 1) \), where \( \mathcal{L} = -u_\phi \) is the conserved angular momentum for a geodesic motion, and \( \beta = u_t \) is the conserved energy for such a motion. Realistic matter almost free-falls (moves along geodesics lines) when it crosses the horizon. Thus, \( \mathcal{L}_0 \approx 0 \) near the horizon.

Equation (27) above gives the most general version of the vertical equation for stationary flow. On the horizon, \( U_1 = 0 \), and \( \mathcal{L}_0 \) is finite, \( V_0 \to \infty \), but \( \Delta \to 0 \), in such a way that \( V_0 \Delta \) is a finite quantity. Thus, our equation is nonsingular on the horizon.

As in Newtonian theory, equation (27) should be considered together with equation (15), which, obviously, has the same form in relativity. However, in the spirit of the “thin disk approach,” we suggest that in practical applications, involving stationary flow, it is always safe to drop the terms containing \( U_1 \). As we have already explained, they cannot be significant anywhere in the flow: neither far from the black hole, because the flow there is subsonic or transonic, nor close to the hole, because the flow there is quasi-spherical. Thus, we conclude that the vertical equation could be taken in practical applications in the following form:
\[ -\frac{2 P_0}{\rho_0} + \left( \frac{H}{R} \right)^2 \mathcal{L}_0^2 = 0 . \]
(28)

The Schwarzschild version of this equation was first derived by Lasota & Abramowicz (1997).

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