MAGNITUDES REBORN: QUANTITY SPACES AS SCALABLE MONOIDS

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Abstract. In ancient Greek mathematics, magnitudes such as lengths were strictly distinguished from numbers. In modern quantity calculus, a distinction is made between quantities and scalars that serve as measures of quantities. The author believes, for reasons apparent from this article, that quantities should play a major rather than a minor role in modern mathematics.

We define scalable monoids and, as a special case, quantity spaces; both can be regarded as universal algebras. Subalgebras and homomorphic images of scalable monoids can be formed, and tensor products of scalable monoids can be constructed as well. We also define and investigate congruence relations on scalable monoids, unit elements of scalable monoids, basis-like substructures of scalable monoids and quantity spaces, and scalar representations of elements of quantity spaces.

As defined, scalable monoids are "siblings" of rings and modules, and while (real or complex) numbers are elements of certain rings, namely certain fields, and vectors are elements of certain modules, namely vector spaces, quantities are elements of certain scalable monoids, namely quantity spaces.

This article supersedes arXiv:1503.00564 and complements arXiv:1408.5024.

Introduction

Formulas such as \( E = \frac{mv^2}{2} \) or \( \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \), used to express physical laws, describe relationships between scalars, commonly real numbers. An alternative interpretation of such equations is possible, however. Since the scalars assigned to the variables in these equations are numerical measures of certain quantities, the equations express relationships between these quantities as well. For example, \( E = \frac{mv^2}{2} \) can also be interpreted as describing a relation between an energy \( E \), a mass \( m \) and a velocity \( v \) – three underlying physical quantities, whose existence and properties do not depend on the scalars that may be used to represent them. With this interpretation, though, \( \frac{mv^2}{2} \) and similar expressions will have meaning only if operations on quantities, corresponding to operations on numbers, are defined. In other words, an appropriate way of calculating with quantities, a quantity calculus, needs to be available.

In a useful survey [3], de Boer describes the development of quantity calculus until the late 20th century, starting with Maxwell’s [19] concept of a physical quantity \( q \) comprised of a unit quantity \( [q] \) of the same kind as \( q \) and a scalar \( \{q\} \) which is the measure of \( q \) relative to \( [q] \), so that we can write \( q = \{q\}[q] \). De Boer argues, however, that the notion of a physical quantity should be a primitive one, not dependent on other notions, and highlights the contributions by Wallot [24], who defined quantities independently of notions of units and measures in 1926. Notable contributions to quantity calculus in the same spirit are found in works by Landolt [16], Fleischmann [7], Quade [21] and Raposo [22].
The roots of quantity calculus go far deeper in the history if mathematics than to Wallot, however, or even to Maxwell or other scientists of the modern era, such as Fourier [8], the origins of quantity calculus can be traced back to ancient Greek geometry and arithmetic, as codified in Euclid’s *Elements* [6].

Of fundamental importance in the *Elements* is the distinction between *numbers* (multitudes) and *magnitudes*. The notion of a number (*arithmos*) is based on that of a "unit" or "monad" (*monas*); a number is "a multitude composed of units". Thus, a number is essentially a positive integer. (A collection of units containing just one unit was not, in principle, considered to be a multitude of units in Greek arithmetic, so 1 was not, strictly speaking, a number.) Numbers can be compared, added and multiplied, and a smaller number can be subtracted from a larger one, but the ratio of two numbers \( m, n \) is not itself a number but just a pair \( m : n \) expressing relative size. Ratios can, however, be compared; \( m : n = m' : n' \) means that \( mn' = nm' \). A bigger number \( m \) is said to be measured by a smaller number \( n \) if \( m = kn \) for some number \( k \); a prime number is a number that is not measured by any other number (or measured only by 1), and \( m, m' \) are relatively prime when there is no number (except 1) measuring both.

Magnitudes (*megethos*), on the other hand, are phenomena such as lengths, areas, volumes or times. Unlike numbers, magnitudes are of *different kinds*, and while the magnitudes of a particular kind correspond loosely to numbers, making measurement of magnitudes possible, the magnitudes form a continuum, and there is no distinguished "unit magnitude". In Greek mathematics, magnitudes of the same kind can be compared and added, and a smaller magnitude can be subtracted from a larger one of the same kind, but magnitudes cannot, in general, be multiplied or divided. One can form the ratio of two magnitudes of the same kind, \( p \) and \( q \), but this is not a magnitude but just a pair \( p : q \) expressing relative size. A greater magnitude \( q \) is said to be measured by a smaller magnitude \( u \) if there is a number \( n \) such that \( q = n \times u \) taken \( n \) times; we may write this as \( q = n \times u \) here.

It is notable that the first three propositions about magnitudes proved by Euclid in the *Elements* are, in the notation used here,

\[
\begin{align*}
  n \times (p_1 + \cdots + p_k) &= n \times p_1 + \cdots + n \times p_k, \\
  (n_1 + \cdots + n_k) \times p &= n_1 \times p + \cdots + n_k \times p, \\
  n \times (m \times p) &= (nm) \times p,
\end{align*}
\]

where \( n,m,n_1,\ldots,n_k \) are numbers (*arithmoi*), \( p \) is a magnitude, \( p_1,\ldots,p_k \) are magnitudes of the same kind, and \( q_1 + \cdots + q_k \) is the sum of magnitudes. We will return to these identities in connection with Proposition 5.2 in Section 5.1.

If \( p \) and \( q \) are magnitudes of the same kind, and there is some magnitude \( u \) of this kind and some numbers \( m,n \) such that \( p = m \times u \) and \( q = n \times u \), then \( p \) and \( q \) are said to be "commensurable"; the ratio of magnitudes \( p : q \) can then be represented by the ratio of numbers \( m : n \).\(^{1}\) However, magnitudes may also be "relatively prime"; it may happen that \( p : q \) cannot be expressed as \( m : n \) for any numbers \( m,n \) because there are no \( m,n,u \) such that \( p = m \times u \) and \( q = n \times u \). In view of the Pythagorean philosophical conviction of the primacy of numbers, the discovery of examples of such "incommensurable" magnitudes created a deep crisis in early Greek mathematics [11], a crisis that also affected the foundations of geometry. If ratios of *arithmoi* do not always suffice to represent ratios of magnitudes, it seems

\(^{1}\)It is natural to assume that if \( p = m \times u = m' \times u' \) and \( q = n \times u = n' \times u' \) then \( m : n = m' : n' \), so that the representation of \( p : q \) is unique.
that it would not always be possible to express in terms of *arithmoi* the fact that two ratios of magnitudes are equal, as are the ratios of the lengths of corresponding sides of similar triangles. This difficulty was resolved by Eudoxos, who realized that a "proportion", that is, a relation among magnitudes of the form "*p* is to *q* as *p*' is to *q'"*, conveniently denoted *p* : *q* :: *p*' : *q'*, can be defined numerically even if there is no pair of ratios of *arithmoi* *m* : *n* and *m*' : *n'* corresponding to *p* : *q* and *p*' : *q'*, respectively, so that *p* : *q* :: *p*' : *q' cannot be inferred from *m* : *n* = *m'* : *n'*. Specifically, Eudoxos invented an ingenious indirect way of determining if *p* : *q* :: *p*' : *q' in terms of nothing but *arithmoi* by means of a construction similar to the Dedekind cut, as described in Book V of the *Elements*. Using modern terminology, one can say that Eudoxos defined an equivalence relation :: between pairs of magnitudes of the same kind numerically, and as a consequence it became possible to conceptualize in terms of positive integers not only ratios of magnitudes corresponding to rational numbers but also ratios of magnitudes corresponding to irrational numbers. Eudoxos thus reconciled the continuum of magnitudes with the discrete *arithmoi*, but in retrospect this feat reduced the incentive to rethink the Greek notion of number, to generalize the *arithmoi*.

To summarize, Greek mathematicians used two notions of muchness, and built a theoretical system around each notion. These systems were connected by relationships of the form *q* = *n* × *u*, where *q* is a magnitude, *n* a number and *u* a magnitude of the same kind as *q*, foreboding from the distant past Maxwell’s conceptualization of a physical quantity, although Euclid did not define magnitudes in terms of units and numbers.

The modern theory of numbers dramatically extends the theory of numbers in the *Elements*. The numbers 1 and 0, negative numbers, rational and irrational numbers, real numbers, complex numbers and so on have been added, and the notion of a number as an element of an algebraic system has come to the forefront.

The modern notion of number was not developed by a straightforward extension of the concept of *arithmos*, however; the development of the new notion of number during the Renaissance was strongly inspired by the ancient theory of magnitudes.

The beginning of the Renaissance saw renewed interest in the classical Greek theories of magnitudes and numbers as codified by Euclid, but later these two notions gradually fused into the notion of number fully developed in 17th century, which combined elements of both. Malet [18] remarks:

> As far as we know, not only was the neat and consistent separation between the Euclidean notions of numbers and magnitudes preserved in Latin medieval translations [...], but these notions were still regularly taught in the major schools of Western Europe in the second half of the 15th century. By the second half of the 17th century, however, the distinction between the classical notions of (natural) numbers and continuous geometrical magnitudes was largely gone, as were the notions themselves.

The force driving this transformation was the need for a continuum of numbers as a basis for computation; the discrete *arithmoi* were not sufficient. As magnitudes of the same kind form a continuum, the idea emerged that numbers should be regarded as an aspect of magnitudes. Number is to magnitude as wetness is to water said Stevin in *L’Arithmétique* [23], published 1585, and defined a number as "cela, par lequel s'explique la quantité de chacune chose" (that by which one can tell the quantity of anything). Thus, numbers were seen to form a continuum by virtue of their intimate association with magnitudes.
Stevin’s definition of number is rather vague, and it is difficult to see how a magnitude can be associated with a definite number, considering that the measure of a magnitude depends on a choice of a unit magnitude. The notion of number was, however, refined during the 17th century. In *La Géométrie* [5], where Descartes laid the groundwork for analytic geometry, he implicitly identified numbers with ratios of two magnitudes, namely lengths of line segments, one of which was considered to have unit length, and in *Universal Arithmetick* [20], Newton, no doubt influenced by Descartes, defined a number as follows:

By *Number* we mean, not so much a Multitude of Unities, as the abstracted *Ratio* of any Quantity, to another Quantity of the same Kind, which we take for Unity.

In modern terminology, a ratio of quantities of any kind $K$ is a "dimensionless" quantity, equipped with a canonical unit quantity, based on a unit for quantities of kind $K$ but independent of the choice of such a unit, and operations on dimensionless quantities produce dimensionless quantities.\(^3\) This means that there is no essential difference between a number and the corresponding dimensionless quantity (see Proposition [10.6]). Quantities, especially "dimensionful" quantities, that is, classical magnitudes, were thus needed only as a scaffolding for the new notion of numbers, and when this notion had been established its origins fell into oblivion and magnitudes fell out of fashion. As a consequence, "quantity calculus" had to be developed more or less *ex novo* in the modern era.

While the Greek theory of magnitudes derived from geometry, the new theory of quantities served the needs of modern mathematical physics, which developed from the second half of the 18th century. In Section IX, Chapter II of *Théorie analytique de la Chaleur* [8], Fourier explained how physical quantities related to the numbers in his equations:

Pour mesurer [des quantités qui entrent dans notre analyse] et les exprimer en nombre on les compare à diverses sortes d’unités, au nombre de cinq, savoir :

l’unité de longueur, l’unité de temps, celle de la température, celle du poids,

et enfin l’unité qui sert à mesurer les quantités de chaleur.

We recognize here the idea that the number associated with a quantity depends on the choice of a unit quantity, and that there are quantities of different kinds; Fourier derived the notion of kinds of quantities from that of sorts of units. In addition, he introduced the idea that quantities of the same or different kinds can be multiplied and divided, going beyond the framework of the Greek theory of magnitudes. Based on this assumption, Fourier also introduced the idea that each unit $u$ needed in the study of idealized heat propagation problems can be expressed as

$$u^{d_{\ell}} u^{d_t} u^{d_T},$$

where $u_{\ell}$, $u_t$, and $u_T$ are units of length, time and temperature, respectively, and the exponents $d_{\ell}$, $d_t$, and $d_T$ are integers, uniquely determined by $u$. (For example, Fourier uses a unit $c$ for heat capacity of the form $u^{-3} u_0^0 u_T^{-1} = u^{-3} u_T^{-1}$.) Quantities

\(^2\)This was a translation of the Latin original *Arithmetica Universalis*, first printed in Cambridge in 1707 and based on lecture notes by Newton for the period 1673 to 1683.

\(^3\)In Greek mathematics, the product of a length and a length was an area, but Descartes argued in *La Geometrie* that it could also be another length. Descartes did not really multiply lengths, however; he multiplied the ratios of two lengths $\ell_1$, $\ell_2$ to a fixed length $\ell_0$ to obtain the ratio of a third length $\ell$ to $\ell_0$, using a geometrical construction with similar triangles such that the number representing the ratio of $\ell$ to $\ell_0$ became equal to the product of the number representing the ratio of $\ell_1$ to $\ell_0$ and the number representing the ratio of $\ell_2$ to $\ell_0$. (See the first figure in *La Geometrie*.)
have the same exponents as the units used to measure them, and terms formed by multiplying and dividing quantities have exponents given by the usual rules. Fourier emphasized that quantity terms can be equal or combined by addition or subtraction only if they have identical arrays of exponents for units, or the same "exposant de dimension", introducing the principle of dimensional homogeneity for equations that contain quantities.

It is clear that the Greek mathematicians’ distinction between numbers and magnitudes is closely related to the modern distinction between scalars and quantities. In view of Fourier’s contribution, it may be said that the foundation of a modern quantity calculus incorporating this distinction and treating quantities as mathematical objects in their own right was laid early in the 19th century. Subsequent progress in this area of mathematics has not been fast and straightforward, however. In his survey from 1994, de Boer noted that the modern theory of quantities had not yet met its Euclid; he concluded that "a satisfactory axiomatic foundation for the quantity calculus" had not yet been formulated [3].

Gowers [9] points out that many mathematical constructs are not defined directly by describing their essential properties, but indirectly by *construction-definitions*, specifying constructions that can be shown to have these properties. For example, an ordered pair \((x, y)\) may be defined by a construction-definition as a set \(\{x, \{y\}\}\); it can be shown that this construction has the required properties, namely that \((x, y) = (x', y')\) if and only if \(x = x'\) and \(y = y'\). Many contemporary formalizations of the notion of a quantity use construction-definitions, typically defining quantities in terms of (something like) scalar-unit pairs, in the tradition from Maxwell. (See Section [3] for some specifics.) However, this is rather like defining a vector as a coordinates-basis pair rather than as an element of a vector space, the modern definition.

Although magnitudes are illustrated by line segments in the *Elements*, the notion of a magnitude is abstract and general. Remarkably, Euclid dealt with this notion in a very modern way. While he carefully defined other important objects such as points, lines and numbers in terms of inherent, characteristic properties, there is no statement about what a magnitude "is". Instead, magnitudes are characterized by how they relate to other magnitudes through their roles in a system of magnitudes, to paraphrase Gowers [10].

In the spirit of modern algebra, quantities are defined in this article simply as elements of *quantity spaces*. Thus, the focus is moved from individual quantities and operations on them to the systems to which the quantities belong, meaning that the notion of quantity calculus will give way to that of quantity spaces.

In quantity calculus, there is agreement that quantities behave like numbers in that quantities can be multiplied, divided and added to quantities of the same kind [3], and like vectors in that quantities can be multiplied by scalars. Furthermore, the *kinds of quantities* can themselves be multiplied in a manner consistent with the multiplication of quantities [3]. A stumbling block on the road to a definite definition of quantity spaces is the problem how to formally define "kind of quantity" objects and "of the same kind" relations in a natural manner.

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4There are mathematical objects for which only construction-definitions are available, so that the mathematician’s task becomes to find the properties of these constructions. A major example is the natural numbers, which were created by God, as Kronecker put it, leaving it to humans to discover their properties.
Quade [21] constructed quantity spaces as collections of one-dimensional vector spaces, thus allowing scalar multiplication of quantities. Quantities are of the same kind if and only if they belong to the same vector space. Naturally, only quantities in the same vector space can be added. Quade also assumed that, for any two one-dimensional vector spaces $U$ and $V$, the set of products of quantities $uv$, where $u \in U$ and $v \in V$, is itself a one-dimensional vector space, denoted $UV$. Thus, kinds of quantities and their products were identified with vector spaces.

Recently, Raposo [22] has proposed a definition of quantity spaces which is similar to Quade’s but more mathematically sophisticated. By this definition, a quantity space $Q$ is an algebraic fiber bundle, with fibers of quantities attached to a base space of dimensions (kinds of quantities) assumed to be a finitely generated free abelian group. Each fiber is again a one-dimensional vector space. Multiplication of quantities and multiplication of dimensions are defined independently, but are assumed to be compatible in the same sense as for Quade. A quantity $1$ such $1q = q1 = q$ for each $q \in Q$ belongs to the fiber attached to the identity element of the group of dimensions.

Quade’s and Raposo’s definitions of quantity spaces may be said to be hybrids of axiomatic definitions and construction-definitions. In [13] and [14], I have presented a simple axiomatic definition, similar to the definitions from the early 20th century of rings, modules and vector spaces. This article elaborates on the treatment of scalable monoids and quantity spaces in these two papers.

In the conceptual framework of universal algebra, a quantity space is just a certain scalable monoid $(Q, \cdot, (\omega_\lambda)_{\lambda \in \mathbb{R}}, 1_Q)$, where $Q$ is the underlying set of the algebra, $(Q, \cdot, 1_Q)$ is a monoid where we write $\cdot(x, y)$ as $xy$, and $\omega_\lambda(x)$ is a scalar product $\lambda \cdot x$ such that $\lambda$ belongs to a fixed ring $R$, $x \in Q$, $\omega_1(x) = x$ for all $x \in Q$, $\omega_\lambda(\omega_\kappa(x)) = \omega_{\lambda \kappa}(x)$ for all $\lambda, \kappa \in R$, $x \in Q$, and $\omega_\lambda(xy) = \omega_\lambda(x)y = x\omega_\lambda(y)$ for all $\lambda \in R, x, y \in Q$. $Q$ is partitioned into orbit classes, which are equivalence classes with respect to the relation $\sim$ defined by $x \sim y$ if and only if $\omega_\alpha(x) = \omega_\beta(y)$ for some $\alpha, \beta \in R$. There is no global operation $(x, y) \mapsto x + y$ defined on $Q$, but within each orbit class addition of its elements is induced by the addition in $R$, and multiplication of orbit classes is induced by the multiplication of elements of $Q$.

A quantity space is a free commutative scalable monoid over a field $K$. Quantities are just elements of quantity spaces, and dimensions are their orbit classes.

The rest of this article is divided into two parts, devoted to scalable monoids and quantity spaces, respectively. Section 1 in Part 1 gives a mathematical context for the notion of a scalable monoid, and scalable monoids are formally defined. Some basic facts about scalable monoids are presented in Section 2. In Section 3.1 the partition of scalable monoids into orbit classes, containing elements (quantities) of the same kind, is introduced. The relation $\sim$ defining this partition of a scalable monoid $X$ turns out to be a congruence on $X$, and there is a corresponding quotient space $X/\sim$, which is effectively a monoid of orbit classes. Congruences of the forms $\sim_M$ and $\sim_M^\perp$, and their corresponding quotient spaces, are investigated in Section 3.2. In Section 4 tensor products of scalable monoids are defined. Section 5.1 contains the main result which links scalable monoids to quantity calculus: an orbit class with a unit element can be regarded as a free module, so that elements (quantities) in the same orbit class can be added and subtracted. In Section 5.2 scalable monoids with sets of unit elements are considered: additive scalable monoids, ordered scalable monoids and coherent sets of unit elements are discussed. Section...
presents a construction-definition of commutative scalable monoids, linking the present definitions to previous work on quantity calculus.

In Part 2, Section 7 motivates and states the definition of quantity spaces; some basic facts about quantity spaces are listed in Section 8. Systems of unit quantities for quantity spaces are discussed in Section 9. The notion of a measure of a quantity is formally defined in Section 10, and ways in which measures serve as proxies for quantities are described. In Section 11, we show that the monoid of dimensions $Q/\sim$ corresponding to a quantity space $Q$ is a free abelian group and that bases in $Q$ and $Q/\sim$ have the same cardinality.

**Part 1. Scalable monoids**

1. **Mathematical background and main definition**

A unital associative algebra $X$ over a (unital, associative) ring $R$ is equipped with three kinds of operations on $X$:

1. **addition** of elements of $X$, a binary operation $+: (x, y) \mapsto x + y$ on $X$ such that $X$ equipped with $+$ is an abelian group;
2. **multiplication** of elements of $X$, a binary operation $*: (x, y) \mapsto xy$ on $X$ such that $X$ equipped with $*$ is a monoid;
3. **scalar multiplication** of elements of $X$ by elements of $R$, a monoid action $(\alpha, x) \mapsto \alpha \cdot x$ where the multiplicative monoid of $R$ acts on $X$ so that $1 \cdot x = x$ and $\alpha \cdot (\beta \cdot x) = \alpha \beta \cdot x$ for all $\alpha, \beta \in R$ and $x \in X$.

These structures are linked pairwise:

(a) addition and multiplication are linked by the distributive laws $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;

(b) addition and scalar multiplication are linked by the distributive laws $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ and $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$;

(c) multiplication and scalar multiplication are linked by the identities $\alpha \cdot xy = (\alpha \cdot x)y$ and $\alpha \cdot xy = x(\alpha \cdot y)$.

Related algebraic structures can be obtained from unital associative algebras by removing one of the three operations and hence the links between the removed operation and the two others. Two cases are very familiar. A **ring** has only addition and multiplication of elements of $X$, linked as described in (a). A **module** has only addition of elements of $X$ and scalar multiplication of elements of $X$ by elements of $R$, linked as described in (b). The question arises whether it would be meaningful and useful to define an “algebra without an additive group”, with only multiplication of elements of $X$ and scalar multiplication of elements of $X$ by elements of $R$, linked as described in (c).

It would indeed. It turns out that this notion, a “sibling” of rings and modules, referred to as scalable monoids in this article, makes sense mathematically and is remarkably well suited for modeling systems of quantities. While (real or complex) numbers are elements of rings, specifically fields, quantities are elements of scalable monoids, specifically quantity spaces.

**Definition 1.1.** Let $R$ be a (unital, associative) ring. A **scalable monoid over $R$**, or $R$-scaloid, is a monoid $X$ equipped with a **scaling action**

$$\cdot : R \times X \to X, \quad (\alpha, x) \mapsto \alpha \cdot x,$$

such that for any $\alpha, \beta \in R$ and $x, y \in X$ we have
(1) \( 1 \cdot x = x \),
(2) \( \alpha \cdot (\beta \cdot x) = \alpha \beta \cdot x \),
(3) \( \alpha \cdot xy = (\alpha \cdot x)y = x(\alpha \cdot y) \).

We denote the identity element of \( X \) by \( 1_X \) or \( 1 \) for any \( x \in X \). An invertible element of a scalable monoid \( X \) is an element \( x \in X \) that has a (necessarily unique) inverse \( x^{-1} \in X \) such that \( xx^{-1} = x^{-1}x = 1 \). The product \( xy \) of invertible elements \( x, y \) is invertible with inverse \( y^{-1}x^{-1} \).

2. Some basic facts about scalable monoids

The following lemma will be used repeatedly.

Lemma 2.1. Let \( X \) be a scalable monoid over \( R \). For any \( x, y \in X \) and \( \alpha, \beta \in R \) we have \((\alpha \cdot x)(\beta \cdot y) = \alpha \beta \cdot xy \) and \( \alpha \beta \cdot x = \alpha (\beta \cdot x) = \beta (\alpha \cdot x) = \beta \alpha \cdot x \).

Proof. By Definition 1.1, \((\alpha \cdot x)(\beta \cdot y) = \alpha \cdot (\beta \cdot xy) = \alpha \beta \cdot xy \). Also, \( \alpha \beta \cdot x = \alpha \cdot (\beta \cdot x) = \alpha \cdot (\beta \cdot 1x) = \alpha \cdot (\beta \cdot 1)x = (\beta \cdot 1)(\alpha \cdot x) = \beta \alpha \cdot 1x = \beta \alpha \cdot x = \beta \cdot (\alpha \cdot x) \) since \((\beta \cdot 1)(\alpha \cdot x) = \beta \alpha \cdot 1x \) by the first part of the lemma.

Let \( R \) be ring and \( X \) a monoid. It is easy to verify that the trivial scaling action of \( R \) on \( X \) defined by \( \lambda \cdot x = x \) for all \( \lambda \in R \) and \( x \in X \) satisfies conditions (1)-(3) in Definition 1.1, so a monoid equipped with this function is scalable monoid, namely a trivially scalable monoid, though effectively just a monoid since any isomorphic monoids are isomorphic as trivially scalable monoids over a fixed ring.

Since every monoid has a unique identity element, the class of all monoids forms a variety of algebras with a binary operation \( \ast : (x, y) \mapsto xy \), a nullary operation \( 1 : () \mapsto 1 \) and identities

\[
(xyz) = (xy)z, \quad 1x = x = x1.
\]

The class of all scalable monoids over a fixed ring \( R \) is a variety of algebras in addition equipped with a set of unary operations \( \{ \omega_\lambda \mid \lambda \in R \} \), corresponding to the external binary operation \( \cdot \) in Definition 1.1 through the law \( \omega_\lambda(x) = \lambda \cdot x \) for all \( \lambda \in R \) and \( x \in X \), and with the additional identities

\[
\omega_1(x) = x, \quad \omega_\lambda(\omega_\kappa(x)) = \omega_{\lambda \kappa}(x), \quad \omega_\lambda(xy) = \omega_\lambda(x)y = x \omega_\lambda(y) \quad (\lambda, \kappa \in R),
\]

corresponding to identities (1) - (3) in Definition 1.1.

A scalable monoid is thus a universal algebra

\[
(X, \ast, (\omega_\lambda)_{\lambda \in R}, 1_X)
\]

with \( X \) as underlying set, here called a unital magma over \( R \) or unital \( R \)-magma. The general definitions of direct products, subalgebras and homomorphisms in the theory of universal algebras apply. In particular, a subalgebra of a unital \( R \)-magma \( X \) is a subset \( Y \) of \( X \) such that \( 1_X \in Y \) and if \( x, y \in Y \) and \( \lambda \in R \) then \( xy, \lambda \cdot x \in Y \). Also, for given unital \( R \)-magnas \( X \) and \( Y \), a unital \( R \)-magma homomorphism \( \phi : X \to Y \) is a function such that \( \phi(xy) = \phi(x)\phi(y) \), \( \phi(\lambda \cdot x) = \lambda \cdot \phi(x) \) and \( \phi(1_X) = 1_Y \) for any \( x, y \in X \) and \( \lambda \in R \).

By Birkhoff’s theorem 1, varieties are preserved by the operations of forming subalgebras and homomorphic images. Thus, if a unital \( R \)-magma \( X \) is a scalable monoid over \( R \) then a subalgebra of \( X \) is also a scalable monoid over \( R \), and a homomorphic image of \( X \) is also a scalable monoid over \( R \).
3. Congruences and quotients

3.1. On the congruence \( \sim \). In ancient Greek mathematics, the notion of a ratio between magnitudes only applied to magnitudes of the same kind, so only those could be commensurable. In this section, we introduce a more radical idea: quantities are of the same kind if and only if they are commensurable.

Let \( R \cdot x \) denote the orbit of \( x \in X \) with regard to the action \( (\lambda, x) \mapsto \lambda \cdot x \), that is, the set \( \{ \lambda \cdot x \mid \lambda \in R \} \), and let \( \approx \) denote the relation on \( X \) such that \( x \approx y \) if and only if there is some \( t \in X \) such that \( x, y \in R \cdot t \). Note that \( \approx \) is not an equivalence relation; it is reflexive since \( x \in R \cdot x \) for all \( x \in X \) and symmetric by construction but not transitive, meaning that the orbits of a monoid action may overlap.

**Definition 3.1.** Given a scalable monoid \( X \) over \( R \), let \( \sim \) be the relation on \( X \) such that \( x \sim y \) if and only if \( \alpha \cdot x = \beta \cdot y \) for some \( \alpha, \beta \in R \).

Note that \( x \sim y \) if and only if \((R \cdot x) \cap (R \cdot y) \neq \emptyset\). We say that \( x \) and \( y \) are **commensurable** if and only if \( x \sim y \); otherwise \( x \) and \( y \) are **incommensurable**.

**Proposition 3.1.** The relation \( \sim \) on a scalable monoid \( X \) over \( R \) is an equivalence relation.

*Proof.* The relation \( \sim \) is reflexive since \( 1 \cdot x = 1 \cdot x \) for all \( x \in X \), symmetric by construction, and transitive because if \( \alpha \cdot x = \beta \cdot y \) and \( \gamma \cdot y = \delta \cdot z \) for some \( x, y, z \in X \) and \( \alpha, \beta, \gamma, \delta \in R \) then

\[
\gamma \alpha \cdot x = \gamma \cdot (\alpha \cdot x) = \gamma \cdot (\beta \cdot y) = \beta \cdot (\gamma \cdot y) = \beta \cdot (\delta \cdot z) = \beta \delta \cdot z,
\]

where \( \gamma \alpha, \beta \delta \in R \). \( \square \)

An orbit class \( C \) is an equivalence class for \( \sim \). The orbit class that contains \( x \) is denoted \([x]\), and \( X/\sim \) denotes the set \( \{ [x] \mid x \in X \} \).

**Proposition 3.2.** If \( x \sim y \) then \( \lambda \cdot x \sim y, x \sim \lambda \cdot y \) and \( \lambda \cdot x \sim \lambda \cdot y \) for all \( \lambda \in R \).

*Proof.* If \( x \sim y \) then \( \alpha \cdot x = \beta \cdot y \) for some \( \alpha, \beta \in R \), so

\[
\alpha \lambda \cdot x = \alpha \cdot (\lambda \cdot x) = \lambda \cdot (\alpha \cdot x) = \lambda \cdot (\beta \cdot y) = \beta \cdot (\lambda \cdot y) = \beta \lambda \cdot y,
\]

where \( \alpha \lambda, \beta \lambda \in R \). \( \square \)

**Corollary 3.1.** \( \lambda \cdot x \sim x \) and \( x \sim \lambda \cdot x \) for all \( x \in X \) and \( \lambda \in R \).

**Proposition 3.3.** We have \( 0 \cdot x = 0 \cdot y \) if and only if \( x \sim y \).

*Proof.* If \( \alpha \cdot x = \beta \cdot y \) then \( 0 \cdot x = 0 \cdot (\alpha \cdot x) = 0 \cdot (\beta \cdot y) = (0 \beta) \cdot y = 0 \cdot y \). \( \square \)

Thus, for every orbit class \( C \) there is a unique \( 0_C \in C \) such that \( 0_C = 0 \cdot x \) for all \( x \in C \), and if \( C \neq C' \) then \( 0_C \neq 0_{C'} \); \( 0_C \) is the zero element of \( C \). It is clear that \( \lambda \cdot 0_C = 0_C \) for all \( \lambda \in R \), and that \( 0_{[x]} y = 0_{[xy]} \) and \( y 0_{[x]} = 0_{[yx]} \) for all \( x, y \in X \).

If \( x = \alpha \cdot t \) and \( y = \beta \cdot t \) then \( \beta \cdot x = \beta \cdot (\alpha \cdot t) = \alpha \cdot (\beta \cdot t) = \alpha \cdot y \), so if \( x \approx y \) then \( \beta \cdot x \approx \beta \cdot y \). If \( t \in R \cdot x \) then \( t, x \in R \cdot x \), so \( t \approx x \), so \( t \sim x \), so \( t \in [x] \); hence, \( R \cdot x \subseteq [x] \) for all \( x \in X \). As a consequence, \( \bigcup_{t \in [x]} R \cdot t = [x] \).

It is instructive to relate the present notion of commensurability to the classical one. We say that \( x \) and \( y \) are **strongly commensurable** if and only if \( x \approx y \); otherwise, \( x \) and \( y \) are **weakly incommensurable**. Incommensurability of magnitudes in the Pythagorean sense obviously corresponds to weak incommensurability.
We have thus weakened the classical notion of commensurability here, and this makes it possible to reasonably stipulate that two magnitudes (elements of a scalable monoid) are of the same kind if and only if they are commensurable. The deeper significance of the redefinition of commensurability may be said to be that we have shown how to replace the intuitive notion of magnitudes of the same kind by the formally defined notion of commensurable magnitudes.

**Proposition 3.4.** Let $X$ be a scalable monoid over $R$. The relation $\sim$ is a congruence on $X$ with regard to the operations $(x, y) \mapsto xy$ and $(\lambda, x) \mapsto \lambda \cdot x$.

**Proof.** For any $x, x', y, y' \in X$ and $\alpha, \alpha', \beta, \beta' \in R$, we have that if $\alpha \cdot x = \alpha' \cdot x'$ and $\beta \cdot y = \beta' \cdot y'$ then $(\alpha \cdot x)(\beta \cdot y) = (\alpha' \cdot x')(\beta' \cdot y')$, so $\alpha \beta \cdot xy = \alpha' \beta' \cdot x'y'$ by Lemma 2.1. This means that if $x \sim x'$ and $y \sim y'$ then $xy \sim x'y'$. Also, recall that if $x \sim x'$ then $\lambda \cdot x \sim \lambda \cdot x'$ for any $\lambda \in R$. □

We can thus define a binary operation on $X/{\sim}$ by setting $[x][y] = [xy]$ (so that if $A, B \in X/{\sim}$, $a \in A$ and $b \in B$ then $ab \in AB \in X/{\sim}$). We can also set $\lambda \cdot [x] = [\lambda \cdot x]$ and $1_{X/{\sim}} = [1_X]$. Given these definitions, the surjective function $\phi : X \to X/{\sim}$ defined by $\phi(x) = [x]$ satisfies the conditions

$$\phi(xy) = \phi(x)\phi(y), \quad \phi(\lambda \cdot x) = \lambda \cdot \phi(x), \quad \phi(1_X) = 1_{X/{\sim}}.$$  

These identities induce a unital $R$-magma structure on $X/{\sim}$, and by Birkhoff’s theorem $X/{\sim}$ is an $R$-scaloïd, so $\phi$ is a scalable monoid homomorphism. Thus, Proposition 3.4, which is expressed in terms of congruences, leads to Proposition 3.5 expressed in terms of homomorphisms.

**Proposition 3.5.** Let $X$ be a scalable monoid over $R$. The surjective function

$$\phi : X \to X/{\sim}, \quad x \mapsto [x],$$  

is a scalable monoid homomorphism with operations $([x], [y]) \mapsto [xy], (\lambda, [x]) \mapsto \lambda \cdot [x]$ and () $\mapsto 1_{X/{\sim}}$ on $X/{\sim}$ defined by the identities $[x][y] = [xy], \lambda \cdot [x] = [\lambda \cdot x]$ and $1_{X/{\sim}} = [1_X]$.

In this case, $\lambda \cdot [x] = [\lambda \cdot x] = [x]$ by Corollary 3.1, so we have the following fact.

**Proposition 3.6.** If $X$ is a scalable monoid then $X/{\sim}$ is a (trivially scalable) monoid.

3.2. On congruences of the forms $\sim_M$ and $\sim_\#$. In a monoid we have $x(yz) = (xy)z$ and $1x = x = xa$, so a submonoid $M$ of a scalable monoid $X$ can act as a monoid on $X$ by left or right multiplication. In particular, we can define a monoid action $(m, x) \mapsto mx$ on a scalable monoid $X$ by setting $m \cdot x = mx$ for any $m \in M$ and $x \in X$. For any $x \in X$, the orbit of $x$ with regard to this action is the right coset $Mx = \{mx \mid m \in M\}$. Definition 3.2 below is analogous to Definition 3.1 interpreting left multiplication as a monoid action.

**Definition 3.2.** Let $X$ be a scalable monoid and $M$ a submonoid of $X$. Then $\sim_M$ is the relation on $X$ such that $x \sim_M y$ if and only if $mx = ny$ for some $m, n \in M$.

**Proposition 3.7.** If $X$ is a scalable monoid and $M$ a commutative submonoid of $X$ then $\sim_M$ is an equivalence relation on $X$.

**Proof.** The relation $\sim_M$ is reflexive since $1_Xx \sim_M 1_Xx$ for all $x \in X$, symmetric by construction, and transitive because if $mx = ny$ and $m'y = n'z$ for $m, n, m', n' \in M$ then $m'mx = m'ny = nm'y = nn'z$, where $m'm, nn' \in M$. □
We denote the equivalence class \( \{ t \mid t \sim_M x \} \) for \( \sim_M \) by \( [x]_M \), and the set of equivalence classes \( \{ [x]_M \mid x \in X \} \) by \( X/M \).

The center of a scalable monoid \( X \), denoted \( Z(X) \), is the set of elements of \( X \) each of which commutes with all elements of \( X \); clearly, \( 1_X \in Z(X) \). A central submonoid of a scalable monoid \( X \) is a submonoid \( M \) of \( X \) such that \( M \subseteq Z(X) \).

We have the following corollary of Proposition 3.7.

**Corollary 3.2.** If \( X \) is a scalable monoid and \( M \) a central submonoid of \( X \) then \( \sim_M \) is an equivalence relation on \( X \).

Results analogous to Propositions 3.4 and 3.5 hold for central submonoids.

**Proposition 3.8.** Let \( X \) be a scalable monoid and \( M \) a central submonoid of \( X \). Then the relation \( \sim_M \) is a congruence on \( X \) with regard to the operations \( (x, y) \mapsto xy \) and \( (\lambda, x) \mapsto \lambda \cdot x \).

Proof. If \( x, x', y, y' \in X \) and \( m, m', n, n' \in M \) then \( mx = m'x' \) and \( ny = n'y' \) implies \( (mx)(ny) = (m'x')(n'y') \) so that \( (mn)(xy) = (m'n')(x'y') \). Hence, if \( x \sim_M x' \) and \( y \sim_M y' \) then \( xy \sim_M x'y' \) since \( mn, m'n' \in M \).

Also, if \( nx = n'x' \) for some \( n, n' \in M \) then \( \lambda \cdot nx = \lambda \cdot n'x' \) for all \( \lambda \in R \), so \( n(\lambda \cdot x) = n'(\lambda \cdot x') \). Hence, if \( x \sim_M x' \) then \( \lambda \cdot x \sim_M \lambda \cdot x' \).

\( \square \)

**Corollary 3.3.** Let \( X \) be a commutative scalable monoid and \( M \) a submonoid of \( X \). Then the relation \( \sim_M \) is a congruence on \( X \) with regard to the operations \( (x, y) \mapsto xy \) and \( (\lambda, x) \mapsto \lambda \cdot x \).

We can thus define two operations on \( X/M \) by setting \( [x]_M[y]_M = [xy]_M \) and \( \lambda \cdot [x]_M = [\lambda \cdot x]_M \). We also set \( 1_{X/M} = [1_X]_M \). Given these definitions, the surjective function \( \phi_M : X \to X/M \) defined by \( \phi_M(x) = [x]_M \) satisfies the conditions

\[
\phi_M(xy) = \phi_M(x)\phi_M(y), \quad \phi_M(\lambda \cdot x) = \lambda \cdot \phi_M(x), \quad \phi_M(1_X) = 1_{X/M}.
\]

These identities induce a unital \( R \)-magma structure on \( X/M \), and by Birkhoff's theorem \( X/M \) is an \( R \)-scoid, so \( \phi_M \) is a scalable monoid homomorphism. Proposition 3.3 thus corresponds to the following result about homomorphisms.

**Proposition 3.9.** Let \( X \) be a scalable monoid and \( M \) a central submonoid of \( X \). The surjective function

\[
\phi_M : X \to X/M, \quad x \mapsto [x]_M
\]

is a scalable monoid homomorphism with operations \( ([x]_M, [y]_M) \mapsto [xy]_M \), \( (\lambda, [x]_M) \mapsto [\lambda \cdot x]_M \) and \( () \mapsto 1_{X/M} \) on \( X/M \) defined by the identities \( [xy]_M = [xy]_M \), \( [\lambda \cdot x]_M = [\lambda \cdot x]_M \) and \( 1_{X/M} = 1_{X/M} \).

Recall that a subalgebra of a scalable monoid \( X \) is itself a scalable monoid, namely, a submonoid \( \mathcal{M} \) of \( X \) such that \( \lambda \cdot x \in \mathcal{M} \) for every \( \lambda \in R \) and \( x \in \mathcal{M} \); we call \( \mathcal{M} \) a scalable submonoid of \( X \). A central scalable submonoid of \( X \) is defined in the same way as a central submonoid of \( X \).

For any central scalable submonoid \( \mathcal{M} \) of \( X \) we can define a congruence \( \sim_{\mathcal{M}} \) on \( X \) in the same way as we defined \( \sim_M \). Hence, we can define \( [x]_{\mathcal{M}}, X/\mathcal{M}, [x]_{\mathcal{M}}[y]_{\mathcal{M}}, \lambda \cdot [x]_{\mathcal{M}} \) and \( 1_{X/\mathcal{M}} \) by just substituting \( \mathcal{M} \) for \( M \).

For any central scalable submonoid \( \mathcal{M} \), we have \( [\lambda \cdot x]_{\mathcal{M}} = [x]_{\mathcal{M}} \) since \( 1(\lambda \cdot x) = \lambda \cdot x = \lambda \cdot 1x = (\lambda \cdot 1)x, \) where \( 1, \lambda \cdot 1 \in \mathcal{M} \), so that \( \sim_{\mathcal{M}} \) is an equivalence relation on \( X \). Hence, \( [x]_{\mathcal{M}} = [\lambda \cdot x]_{\mathcal{M}} = [x]_{\mathcal{M}} \) for any \( \lambda \in R \) and \( [x]_{\mathcal{M}} \in X/\mathcal{M} \), so while \( X/M \) is a scalable monoid, \( X/\mathcal{M} \) is instead a (trivially scalable) monoid.
Furthermore, if $\alpha \cdot x = \beta \cdot y$ for some $\alpha, \beta \in R$ then $(\alpha \cdot 1)x = \alpha \cdot 1x = \beta \cdot 1y = (\beta \cdot 1)y$. Thus $x \sim y$ implies $x \sim_\mathcal{M} y$ for any central scalable monoid $\mathcal{M}$ of $X$ since $\lambda \cdot 1 \in \mathcal{M}$ for any $\lambda \in R$ and any $\mathcal{M}$. Conversely, note that $R \cdot 1$ is a central scalable submonoid of $X$ and if $x \sim_{R \cdot 1} y$ then $\alpha \cdot 1x = (\alpha \cdot 1)x = (\beta \cdot 1)y = \beta \cdot 1y$ for some $\alpha, \beta \in R$, so $x \sim_{R \cdot 1} y$ implies $x \sim y$. Thus, $x \sim_{R \cdot 1} y$ if and only if $x \sim y$, so $x \sim_\mathcal{M} y$ generalizes $x \sim y$.

4. DIRECT AND TENSOR PRODUCTS OF SCALABLE MONOIDS

Let $X$ and $Y$ be scalable monoids. The direct product of $X$ and $Y$, denoted $X \times Y$, is the set $X \times Y$ equipped with the binary operation

$$*: (X \times Y) \times (X \times Y) \to X \times Y,$$

$$(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) \mapsto \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle := \langle x_1x_2, y_1y_2 \rangle$$

and the external binary operation

$$\cdot: R \times (X \times Y) \to X \times Y,$$

$$(\lambda, \langle x, y \rangle) \mapsto \lambda \cdot \langle x, y \rangle := \langle \lambda \cdot x, \lambda \cdot y \rangle.$$

Straight-forward calculations show that $X \times Y$ is a scalable monoid with $*$ as monoid multiplication, $\cdot$ as scaling action and $\langle 1_X, 1_Y \rangle$ as identity element.

The direct product of scalable monoids is a generic product, applicable to any universal algebra. It turns out that another kind of product, which is attuned to the fact that $(\lambda \cdot x)y = \lambda \cdot xy = x(\lambda \cdot y)$ in scalable monoids, namely the tensor product, is often more useful.

**Definition 4.1.** Given scalable monoids $X$ and $Y$ over $R$, let $\sim_\otimes$ be the binary relation on $X \times Y$ such that $(x_1, y_1) \sim_\otimes (x_2, y_2)$ if and only if $(\alpha \cdot x_1, \beta \cdot y_1) = (\beta \cdot x_2, \alpha \cdot y_2)$ for some $\alpha, \beta \in R$.

**Proposition 4.1.** Let $X$ and $Y$ be scalable monoids over $R$. Then $\sim_\otimes$ is an equivalence relation on $X \times Y$.

**Proof.** $\sim_\otimes$ is reflexive since $(1 \cdot x, 1 \cdot y) = (1 \cdot x, 1 \cdot y)$, and symmetric by construction. If $(\alpha \cdot x_1, \beta \cdot y_1) = (\beta \cdot x_2, \alpha \cdot y_2)$ and $(\gamma \cdot x_2, \delta \cdot y_2) = (\delta \cdot x_3, \gamma \cdot y_3)$ then

$$(\gamma \cdot (\alpha \cdot x_1), \delta \cdot (\beta \cdot y_1)) = (\gamma \cdot (\beta \cdot x_2), \delta \cdot (\alpha \cdot y_2)),$$

$$(\beta \cdot (\gamma \cdot x_2), \alpha \cdot (\delta \cdot y_2)) = (\beta \cdot (\delta \cdot x_3), \alpha \cdot (\gamma \cdot y_3)).$$

Thus, we have

$$(\gamma \alpha \cdot x_1, \delta \beta \cdot y_1) = (\gamma \beta \cdot x_2, \delta \alpha \cdot y_2) = (\beta \gamma \cdot x_2, \alpha \delta \cdot y_2) =$$

$$(\beta \delta \cdot x_3, \alpha \gamma \cdot y_3) = (\delta \beta \cdot x_3, \alpha \gamma \cdot y_3),$$

where $\gamma \alpha, \delta \beta \in R$, so $\sim_\otimes$ is transitive as well. \(\square\)

For any $x \in X$ and $y \in Y$, let $x \otimes y$ denote the equivalence class

$$\{(s, t) \mid (s, t) \in X \times Y, (s, t) \sim_\otimes (x, y)\},$$

and let $X \otimes Y$ denote the set $\{x \otimes y \mid x \in X, y \in Y\}$.

**Proposition 4.2.** Let $X, Y$ be scalable monoids over $R$, $x \in X$ and $y \in Y$. Then $(\lambda \cdot x) \otimes y = x \otimes (\lambda \cdot y)$ for every $\lambda \in R$.

**Proof.** We have $(1 \cdot (\lambda \cdot x), \lambda \cdot y) = (\lambda \cdot x, 1 \cdot (\lambda \cdot y))$, so $(\lambda \cdot x, y) \sim_\otimes (x, \lambda \cdot y)$, meaning that $(\lambda \cdot x) \otimes y = x \otimes (\lambda \cdot y)$. \(\square\)
Proposition 4.3. Let $X, Y$ be scalable monoids over $R$, and set $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$ and $\lambda \cdot x \otimes y = (\lambda \cdot x) \otimes y$. Also set $1_{X \otimes Y} = 1_X \otimes 1_Y$. With these definitions, $X \otimes Y$ is a scalable monoid over $R$.

Proof. $X \otimes Y$ is a monoid since
\[(1_X \otimes 1_Y)(x \otimes y) = 1_X x \otimes 1_Y y = x \otimes y = 1_X \otimes y 1_Y = (x \otimes y)(1_X \otimes 1_Y),\]
\[(x_1 \otimes y_1)(x_2 \otimes y_2))(x_3 \otimes y_3) = (x_1 x_2 \otimes y_1 y_2)(x_3 \otimes y_3) = (x_1 x_2 x_3 \otimes (y_1 y_2)y_3) = x_1(x_2 x_3) \otimes y_1(y_2 y_3) = (x_1 \otimes y_1)(x_2 x_3 \otimes y_2 y_3) = (x_1 \otimes y_1)((x_2 \otimes y_2)(x_3 \otimes y_3)).\]
Furthermore,
\[1 \cdot x \otimes y = (1 \cdot x) \otimes y = x \otimes y,\]
\[\alpha \cdot (\beta \cdot x \otimes y) = \alpha \cdot (\beta \cdot x) \otimes y = (\alpha \beta \cdot x) \otimes y = \alpha \beta \cdot x \otimes y,\]
\[\lambda \cdot (x_1 \otimes y_1)(x_2 \otimes y_2) = \lambda \cdot x_1 x_2 \otimes y_1 y_2 = (\lambda \cdot x_1 x_2) \otimes y_1 y_2 =
\]
\[\lambda \cdot x_1 x_2 \otimes y_1 y_2 = ((\lambda \cdot x_1) \otimes y_1)(x_2 \otimes y_2) = (\lambda \cdot x_1 \otimes y_1)(x_2 \otimes y_2),\]
\[\lambda \cdot (x_1 \otimes y_1)(x_2 \otimes y_2) = \lambda \cdot x_1 x_2 \otimes y_1 y_2 = x_1 x_2 \otimes (\lambda \cdot y_1 y_2) =
\]
\[x_1 x_2 \otimes y_1(\lambda \cdot y_2) = (x_1 \otimes y_1)(x_2 \otimes (\lambda \cdot y_2)) = (x_1 \otimes y_1)(\lambda \cdot x_2 \otimes y_2),\]
so $X \otimes Y$ is a scalable monoid.

Corollary 4.1. If $X, Y, Z$ are scalable monoids over $R$ then $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are scalable monoids over $R$.

Proposition 4.4. $\phi : (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$ is a scalable monoid isomorphism $(X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$.

Proof. We have
\[\phi(((x_1 \otimes y_1) \otimes z_1)((x_2 \otimes y_2) \otimes z_2)) = \phi(((x_1 \otimes y_1)(x_2 \otimes y_2)) \otimes z_1 z_2) =
\]
\[\phi((x_1 x_2 \otimes y_1 y_2) \otimes z_1 z_2) = x_1 x_2 \otimes (y_1 y_2 \otimes z_1 z_2) = x_1 x_2 \otimes ((y_1 \otimes z_1)(y_2 \otimes z_2)) =
\]
\[(x_1 \otimes (y_1 \otimes z_1))(x_2 \otimes (y_2 \otimes z_2)) = \phi((x_1 \otimes y_1) \otimes z_1)\phi((x_2 \otimes y_2) \otimes z_2)\]
and
\[\phi(\lambda \cdot (x \otimes y) \otimes z) = \phi((x \otimes y) \otimes (\lambda \cdot z)) = x \otimes (y \otimes (\lambda \cdot z)) =
\]
\[x \otimes (\lambda \cdot y \otimes z) = \lambda \cdot x \otimes (y \otimes z) = \lambda \cdot \phi((x \otimes y) \otimes z).\]

Also,
\[\phi(1_{X \otimes Y}) = \phi((1_X \otimes 1_Y) \otimes 1_Z) = 1_X \otimes (1_Y \otimes 1_Z) = 1_{X \otimes Y \otimes Z}.\]

Thus, $\phi$ is a scalable monoid homomorphism $(X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, and similarly $\phi' : x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z$ is a scalable monoid homomorphism $X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ such that $\phi' \circ \phi = \text{Id}_{(X \otimes Y) \otimes Z}$ and $\phi \circ \phi' = \text{Id}_{X \otimes (Y \otimes Z)}$, so $\phi$ is a scalable monoid isomorphism.

5. SCALABLE MONOIDS WITH UNIT ELEMENTS

5.1. ORBIT CLASSES WITH A UNIT ELEMENT ARE FREE MODULES. Recall the principle that magnitudes of the same kind can be added and subtracted, whereas magnitudes of different kinds cannot be combined by these operations [3]. Also recall the idea that a magnitude $q$ can be represented by a "unit" $[q]$ and a number $\{q\}$ specifying "[how many] times the [unit] is to be taken in order to make up" the magnitude $q$ [19]. As shown below, there is a connection between these two notions.
Specifically, it may happen that \( R \cdot u \supseteq [u] \) for some \( u \in [u] \), and if in addition a natural uniqueness condition is satisfied we may regard \( u \) as a unit of measurement for \([u]\). If such a unit exists then a sum of magnitudes in \([u]\) can be defined by the construction described below.

**Definition 5.1.** Let \( C \) be an orbit class in a scalable monoid over \( R \). A generating element for \( C \) is some \( u \in C \) such that for every \( x \in C \) there is some \( \rho \in R \) such that \( x = \rho \cdot u \). A unit element for \( C \) is a generating element \( u \) for \( C \) such that if \( \rho \cdot u = \rho' \cdot u \) then \( \rho = \rho' \).

By this definition, if \( u \) is a generating element for \( C \) then \( R \cdot u \supseteq C \), but recall that \( R \cdot u \subseteq [u] \), so actually \( R \cdot u = [u] \). Also, \([u] = C \) since \( u \in [u] \), and \( u \) belongs to only one orbit class. Furthermore, if \( u \) is a generating element for \([x]\) then \( x \sim u \) since \( 1 \cdot x = \rho \cdot u \) for some \( \rho \in R \), and if \( x \sim u \) then \([x] = [u] = R \cdot u \).

As \( \rho \cdot u = \rho' \cdot u \) implies \( \rho = \rho' \) for unit elements, \( 0 \in C \) cannot be a unit element in a scalable monoid over a non-trivial ring. Also note that if there exists a unit element \( u \) for some orbit class then \( \alpha \beta \cdot u = \beta \alpha \cdot u \) implies \( \alpha \beta = \beta \alpha \), so \( R \) is commutative.

**Proposition 5.1.** Let \( C \) be an orbit class in a scalable monoid over \( R \). If \( u \) and \( u' \) are unit elements for \( C \), \( \rho \cdot u = \rho' \cdot u' \) and \( \sigma \cdot u = \sigma' \cdot u' \) then \((\rho + \sigma) \cdot u = (\rho' + \sigma') \cdot u' \).

**Proof.** As \( u' \in C \), there is a unique \( \tau \in R \) such that \( u' = \tau \cdot u \). Thus,

\[
(\rho' + \sigma') \cdot u' = (\rho' + \sigma') \cdot (\tau \cdot u) = (\rho' + \sigma') \cdot u = (\rho + \sigma) \cdot u,
\]

since \( \rho \cdot u = \rho' \cdot u' = \rho' \cdot (\tau \cdot u) = \rho' \tau \cdot u \) and \( \sigma \cdot u = \sigma' \cdot u' = \sigma' \cdot (\tau \cdot u) = \sigma' \tau \cdot u \), so that \( \rho = \rho' \tau \) and \( \sigma = \sigma' \tau \).

Hence, the sum of two elements of a scalable monoid can be defined as follows.

**Definition 5.2.** Let \( X \) be a scalable monoid over \( R \), and let \( u \) be a unit element for an orbit class \( C \). If \( x = \rho \cdot u \) and \( y = \sigma \cdot u \), we set

\[
x + y = (\rho + \sigma) \cdot u.
\]

Thus, if \( x, y \in C \) then \( x + y = (\rho + \sigma) \cdot u \in R \cdot u = C \), and if \( x \in C \) then \( \lambda \cdot x = \lambda \cdot (\rho \cdot u) = \lambda \rho \cdot u \in R \cdot u = C \). We note that the sum \( x + y \) is given by Definition 5.2 if and only if \( x \) and \( y \) are commensurable and their orbit class has a unit element. This fact motivates that the notion of magnitudes of the same kind is replaced by that of commensurable magnitudes (see Section 5.1).

It follows immediately from Definition 5.2 that

\[
(x + y) + z = x + (y + z), \quad x + y = y + x
\]

for all \( x, y, z \in C \), and that

\[
x + 0_C = 0_C + x = x
\]

for any \( x \in C \) since \( 0_C = 0 \cdot u \).

If \( x = \rho \cdot u \) so that \( \lambda \cdot x = \lambda \rho \cdot u \) and \( \kappa \cdot x = \kappa \rho \cdot u \) then

\[
(\lambda + \kappa) \cdot x = (\lambda + \kappa) \cdot (\rho \cdot u) = (\lambda + \kappa) \rho \cdot u = (\lambda \rho + \kappa \rho) \cdot u = \lambda \cdot x + \kappa \cdot x,
\]

and if \( x = \rho \cdot u \) and \( y = \sigma \cdot u \) so that \( \lambda \cdot x = \lambda \rho \cdot u \) and \( \lambda \cdot y = \lambda \sigma \cdot u \) then

\[
\lambda \cdot (x + y) = \lambda \cdot ((\rho + \sigma) \cdot u) = \lambda (\rho + \sigma) \cdot u = (\lambda \rho + \lambda \sigma) \cdot u = \lambda \cdot x + \lambda \cdot y.
\]

A unital ring \( R \) has a unique additive inverse \(-1 \) of \( 1 \in R \), and we set

\[
-x = (-1) \cdot x
\]
for all \( x \in X \). If \( C \) has a unit element \( u \) and \( x = \rho \cdot u \) for some \( \rho \in R \) then
\[
x + (-x) = -x + x = 0_C
\]
since \( x + (-x) = \rho \cdot u + (\rho) \cdot (-u) = (\rho + (-\rho)) \cdot u = 0 \cdot u \) and \( -x + x = (-\rho) \cdot u + \rho \cdot u = (-\rho + \rho) \cdot u = 0 \cdot u \), using the fact that \( -x = (-1) \cdot (\rho \cdot u) = (-\rho) \cdot u \).

As usual, we may write \( x + (-y) \) as \( x - y \), and thus \( x + (-x) \) as \( x - x \).

We have thus shown the following fact.

**Proposition 5.2.** If \( X \) is a scalable monoid over \( R \) and \( C \in X/\sim \) contains a unit element \( u \) for \( C \) then \( C \) is a free module over \( R \) with \( \{ u \} \) as a basis.

Addition in \( C \) is given by Definition 5.2 and scalar multiplication in \( C \) is inherited from the scalar multiplication in \( X \).

Thus, if every orbit class \( C \in X/\sim \) contains a unit element for \( C \) then \( X \) is the union of disjoint isomorphic free modules over \( R \), namely the orbit classes. This fact may be compared to Quade’s and Raposo’s definitions of quantity spaces [21][22].

Recall that identities corresponding to \( (\lambda + \kappa) \cdot x = \lambda \cdot x + \kappa \cdot x \), \( \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y \) and \( \lambda \cdot (\kappa \cdot x) = \lambda \kappa \cdot x \) were proved in Propositions 1 – 3 in Book V of the *Elements*, so rudiments of Proposition 5.2 were present already in the Greek theory of magnitudes.

### 5.2. Scalable monoids with a set of unit elements.

**Definition 5.3.** A dense set of elements of a scalable monoid \( X \) is a set \( U \) of elements of \( X \) such that for every \( x \in X \) there is some \( u \in U \) such that \( u \sim x \). A sparse set of elements of \( X \) is a set \( U \) of elements of \( X \) such that if \( u, v \in U \) then \( uv \in U \). A closed set of elements of \( X \) is a set \( U \) of elements of \( X \) such that if \( u, v \in U \) then \( uv \in U \). A set of unit elements of a scalable monoid \( X \) is a set of elements of \( X \) each of which is a unit element for some \( C \in X/\sim \).

We call a dense sparse set of unit elements of \( X \) a system of unit elements for \( X \), and a sparse set of unit elements of \( X \) a partial system of unit elements for \( X \).

#### 5.2.1. Additive scalable monoids.

**Definition 5.4.** An additive scalable monoid is a scalable monoid \( X \) where every \( C \in X/\sim \) is equipped with a binary operation
\[
+: C \times C \to C, \quad (x, y) \mapsto x + y
\]
such that \( C \) equipped with + is an abelian group and \( x(y + z) = xy + xz \) and \( (y + z)x = yx + yz \) for all \( x \in C' \) and \( y, z \in C'' \) for all \( C', C'' \in X/\sim \).

**Proposition 5.3.** If a scalable monoid \( X \) is equipped with a dense closed set of unit elements \( U \) then \( X \) is an additive scalable monoid.

**Proof.** By Proposition 5.2 each \( C \in X/\sim \) is a module since \( U \) is dense in \( X \).

For all \( x \in C' \) and \( y, z \in C'' \) there are \( u, v \in U \) such that \( [x] = [u] \) and \( [y] = [z] = [v] \) since \( U \) is dense in \( X \), so \( x = \rho \cdot u, y = \sigma \cdot v \) and \( z = \tau \cdot v \) for some \( \rho, \sigma, \tau \in R \), so
\[
(x + y)z = (\rho \cdot u)((\sigma \cdot v) + (\tau \cdot v)) = (\rho \cdot u)((\sigma + \tau) \cdot v) = \rho(\sigma + \tau) \cdot uv = \\
(\rho \sigma + \rho \tau) \cdot uv = \rho \sigma \cdot uv + \rho \tau \cdot uv = (\rho \cdot u)(\sigma \cdot v) + (\rho \cdot u)(\tau \cdot v) = xy + xz,
\]
\[
(y + z)x = (\sigma \cdot v) + (\tau \cdot v)(\rho \cdot u) = ((\sigma + \tau) \cdot v)(\rho \cdot u) = (\sigma + \tau) \rho \cdot vu = \\
(\sigma \rho + \tau \rho) \cdot vu = \sigma \rho \cdot vu + \tau \rho \cdot vu = (\sigma \cdot v)(\rho \cdot u) + (\tau \cdot v)(\rho \cdot u) = yx + zx,
\]
using the fact that \( uv \) and \( vu \) are unit elements since \( U \) is closed. \( \square \)
5.2.2. **Ordered scalable monoids.** Recall that a total order on a set $S$ is a binary relation $\leq$ such that for all $x, y, z \in S$ we have that

1. $x \leq y$ or $y \leq x$;
2. if $x \leq y$ and $y \leq x$ then $x = y$;
3. if $x \leq y$ and $y \leq z$ then $x \leq z$.

Also recall that a (totally) ordered ring is a (unital) ring $R$ with a total order $\leq$ such that for all $x, y, z \in R$ we also have that

1. if $x \leq y$ then $x + z \leq y + z$;
2. if $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

Well-known facts about inequalities such as $0 \leq 1$ and if $x \leq y$ and $x' \leq y'$ then $x + x' \leq y + y'$ can be derived from the definition of an ordered ring.

Let $U$ be a dense set of unit elements of a scalable monoid $X$ over an ordered ring $R$. For any $C \in X/\sim$, let $\sim_{UC}$ be the relation on $U \cap C$ defined by $u \sim_{UC} v$ if and only if $u = \rho \cdot v$ for some $\rho \in R$ such that $0 \leq \rho$. Then $\sim_{UC}$ is reflexive since $u = 1 \cdot u$, and transitive since if $u = \rho \cdot v$ and $v = \sigma \cdot w$, where $0 \leq \rho, \sigma$, then $u = \rho \sigma \cdot w$, where $0 \leq \rho \sigma$. Also note that if $u \sim_{UC} v$, so that $u = \rho \cdot v$, $v = \tau \cdot u$ and $0 \leq \rho$, then $1 \cdot u = \rho \cdot v = \rho \tau \cdot u$ and $1 \cdot v = \tau \cdot u = \tau \rho \cdot v$, so $\rho \tau = \tau \rho = 1$, and $0 \leq \tau$ since $0 \leq \rho$. Thus, $\tau \cdot u = \tau \rho \cdot v = v$, where $0 \leq \tau$, so $\sim_{UC}$ is symmetric as well. Hence, $\sim_{UC}$ is an equivalence relation.

**Definition 5.5.** A consistent dense set of unit elements is a dense set $U$ of unit elements such for any $C \in X/\sim$ and any $u, v \in U \cap C$ we have $u \sim_{UC} v$.

**Proposition 5.4.** Let $X$ be a scalable monoid over an ordered ring $R$, and let $u, v$ be unit elements for $C \in X/\sim$ such that $u \sim_{UC} v$. For any $x \in C$, if $x = \rho \cdot u = \sigma \cdot v$ for some $\rho, \sigma \in R$ and $0 \leq \rho$ then $0 \leq \sigma$.

**Proof.** There is some $\tau \in R$ such that $u = \tau \cdot v$ and $0 \leq \tau$. Thus, $\sigma \cdot v = x = \rho \cdot u = \rho \cdot (\tau \cdot v) = \rho \tau \cdot v$, so $\sigma = \rho \tau$, so $0 \leq \sigma$ since $0 \leq \rho, \tau$. 

**Definition 5.6.** An ordered scalable monoid is a scalable monoid $X$ over an ordered ring $R$ such that for each $C \in X/\sim$ there is a total order $\leq_C$ on $C$ such that for any $a \in A, b \in B, x, y \in C$ and $\lambda \in R$ we have that

1. if $0_A \leq a$ and $0_B \leq b$ then $0_{AB} \leq AB$ $ab$;
2. if $x \leq_C y$ and $0 \leq \lambda$ then $\lambda \cdot x \leq_C \lambda \cdot y$.

An ordered additive scalable monoid is an additive scalable monoid such that, in addition to (1) and (2), for any $x, y, z \in C$ we have that

3. if $x \leq_C y$ then $x + z \leq_C y + z$.

**Proposition 5.5.** Let $X$ be a scalable monoid over an ordered ring $R$, equipped with a consistent dense closed set $U$ of unit elements of $X$. For every $C \in X/\sim$, there is a unique binary relation $\leq_C$ on $C$ defined by $x \leq_C y$ if and only if $y - x = \rho \cdot u$ for some $\rho \in R$ such that $0 \leq \rho$ and some $u \in U \cap C$, and $X$ with each $C$ ordered by $\leq_C$ is an ordered additive scalable monoid over $R$.

**Proof.** By Proposition 5.3, $X$ is an additive scalable monoid, and by Proposition 5.4, the relation $\leq_C$ does not depend on a choice of unit element in $C$.

We first show that $\leq_C$ is a total order on $C$. If $x, y \in C$ then $y - x = \rho \cdot u$ for some $\rho \in R, u \in U \cap C$. Thus, if $0 \leq \rho$ then $x \leq_C y$; if $\rho \leq 0$ then $0 \leq (-\rho)$ and $x - y = -(y - x) = -((\rho \cdot u) = (-1) \cdot (\rho \cdot u) = (-\rho) \cdot u$, so $y \leq_C x$. If $x \leq_C y$ and
y ≤ x then 0 ≤ ρ and ρ ≤ 0, so ρ = 0, so y − x = 0, so x = y. Also, if x, y, z ∈ C, 
y − x = ρ · u and z − y = σ · u then z − x = (y − x) + (z − y) = (ρ + σ) · u, so if 
x ≤ y and y ≤ z so that 0 ≤ ρ, σ then x ≤ z since 0 ≤ ρ + σ.

Furthermore, if 0 ≤ x and 0 ≤ y, meaning that x = ρ · u and y = σ · v for some 
ρ, σ ∈ R, where 0 ≤ ρ, σ and u ∈ A, v ∈ B, then xy = (ρ · u)(σ · v) = ρσ · uv, where 
0 ≤ ρσ and uv ∈ AB, so 0 ≤ AB xy. Similarly, if x ≤ y, meaning that x = ρ · u and 
y = σ · u for some ρ, σ ∈ R such that 0 ≤ σ − ρ, then 0 ≤ λ implies 0 ≤ λ(σ − ρ) = 
λσ − λρ, so λ · x ≤ λ · y since λ · y − λ · x = λ · (σ · u) − λ · (ρ · u) = (λσ − λρ) · u.

Also, if x ≤ y then x + z ≤ y since (y + z) − (x + z) = y − x = (σ − ρ) · u. □

Let U and V be systems of unit elements of a scalable monoid X over an ordered 
ring. U and V are said to define the same orientation of X if and only if U ∪ V is a 
consistent set of unit elements, meaning that for all C ∈ X/∼ we have u ∼ (U ∪ V) v, 
where u ∈ U ∩ C and v ∈ V ∩ C.

If U and V define the same orientation on X then U, V and U ∪ V are consistent, 
so by Proposition 5.4 each set uniquely defines, on each C ∈ X/∼, a relation ≤ C 
such that x ≤ y if and only if y − x = ρ · u for some ρ ∈ R such that 0 ≤ ρ and 
some u ∈ U ∩ C, V ∩ C or (U ∪ V) ∩ C, respectively. It is clear that U and V define 
the same relations ≤ C, namely the same relations as U ∪ V defines.

5.2.3. Scalable monoids and coherent systems of unit elements.

Definition 5.7. A coherent system of unit elements for X is a submonoid of X 
which is a system of unit elements for X.

Proposition 5.6. Let X be a scalable monoid, U a coherent system of unit elements 
for X, and V ⊆ U a central submonoid of X. Then X/V is a scalable monoid, 
[v] V = [1] V, for any v ∈ V, and U = {[u] U | u ∈ U} is a coherent system of unit 
elements for X/V.

Proof. As V is a central submonoid of X, X/V is a scalable monoid, and if v ∈ V 
then v ∼ V 1 since 1v = v1 and 1 ∈ V, so [v] V = [1] V.

If u, u′ ∈ U then u ∼ u′ if and only if v · u ∼ V v · u′ for some v, u, u′ ∈ U, and 
uu′ ∈ U since U is a monoid, so uu′ = [uu′] u ∈ U. We also have 

[1] V [u] V = [1u] V = [u1] V = [u] V [1] V

and 

([u][u′])[u″] = ([uu′u″] = [u(u′u″)] = [u][u′][u″]),

so U is a submonoid of X/V.

For any [x] V ∈ X/V there is some u ∈ U and ρ ∈ R such that [x] V = [ρ · u] V = 
ρ · [u] V; thus also [x] V ∼ [u] V since 1 · [x] V = [1] V = [x] V. If [x] V = [ρ · u] V = σ · [u] V 
for some ρ, σ ∈ R then [σ · u] V = σ · [u] V, so ρ · u ∼ V σ · u, so v(ρ · u) = v′(σ · u) 
where v, v′ ∈ V, so v · uu = σ · v′u where uu, v′u ∈ U, so vu ∼ V v′u, so vu = v′u since 
U is sparse, so ρ = σ. Thus, U is a dense set of unit elements of X/V.

Also, if u, u′ ∈ U and uu′ ∼ V u′u then ρ · u ∼ V σ · u′, so σ · uu′ = [σ · u′] V, so 
ρ · u ∼ V σ · uu′, so v(ρ · u) = v′(σ · uu′) for some v, v′ ∈ V, so v · uu = v′ · uu′, so 

vu ∼ V uu′ where vu, uu′ ∈ U, so vu = vu′, so [vu] V = [v′u′] V, so [v] V [u] V = [v′] V [u′] V.

Also, if u, u′ ∈ U and uu′ ∼ V u′u then ρ · u ∼ V σ · uu′, so [uu′] V = [σ · u] V, so 
ρ · uu ∼ V σ · uu′, so v(ρ · uu) = v′(σ · uu′) for some v, v′ ∈ V, so v · uu = v′ · uu′, so 

vu ∼ V uu′ where vu, uu′ ∈ U, so vu = vu′, so [vu] V = [v′uu′] V, so [v] V [u] V = [v′u′] V.

Thus, U is a sparse set of unit elements of X/V.

Note that V is a partial coherent system of unit elements of X, that is, a 
submonoid of X that is a partial system of unit elements of X. For any v, v′ ∈ V and
any \( \lambda \in R \), \( \lambda \cdot v \) and \( \lambda \cdot v' \) in \( X \) correspond to the same element \([\lambda \cdot 1]_V \) of \( X/V \); more generally, for any \( v, v' \in V \), any \( u, u' \in U \) and any \( \lambda \in R \), \( \lambda \cdot uu' \) and \( \lambda \cdot u'u' \) in \( X \) correspond to the element \([\lambda \cdot uu']_V \) of \( X/V \).

The typical application of Proposition 5.6 in physics is described by Raposo [22]:

The mechanism of taking quotients is the algebraic tool underlying what is common practice in physics of choosing “systems of units” such that some specified universal constants become dimensionless and take on the numerical value 1. [...] But it has to be remarked that the mechanism goes beyond a change of system of units; it is indeed a change of space of quantities.

6. Ring-monoids and scalable monoids

**Definition 6.1.** Let \( R \) be a ring and \( M \) a monoid. A ring-monoid \( R \boxtimes M \) is a set \( R \times M \) equipped with a binary operation

\[
\ast : (R \times M) \times (R \times M) \to R \times M, \quad ((\alpha, x), (\beta, y)) \mapsto (\alpha, x) \cdot (\beta, y) := (\alpha \beta, xy)
\]

and an external binary operation

\[
\cdot : R \times (R \times M) \to R \times M, \quad (\lambda, (\alpha, x)) \mapsto \lambda \cdot (\alpha, x) := (\lambda \alpha, x).
\]

**Proposition 6.1.** Let \( R \boxtimes M \) be a ring-monoid. If \( R \) is a commutative ring, then \( R \boxtimes M \) is a scalable monoid over \( R \).

**Proof.** We have

\[
((\alpha, x) \cdot (\beta, y)) (\gamma, z) = ((\alpha \beta), (xy)z) = (\alpha (\beta \gamma), x(yz)) = (\alpha, x) (\beta, y) (\gamma, z),
\]

\[
(1, 1) (\alpha, x) = (\alpha, x) = (\alpha, x) (1, 1)
\]

for any \( \alpha, \beta, \gamma \in R \) and \( x, y, z \in M \), so \( R \boxtimes M \) is a monoid with \( (1, 1) \) as identity element. Furthermore,

\[
1 (\alpha, x) = (1 \alpha, x) = (\alpha, x),
\]

\[
\lambda \cdot (\kappa \cdot (\alpha, x)) = \lambda \cdot (\kappa \alpha, x) = (\lambda \kappa \alpha, x) = (\lambda \kappa \alpha, x),
\]

\[
\lambda \cdot (\alpha, x) (\beta, y) = \lambda \cdot (\alpha \beta, xy) = (\lambda \alpha \beta, xy) = (\lambda \alpha \beta, xy),
\]

\[
(\lambda \alpha \beta, xy) = (\lambda \alpha, x) (\beta, y) = (\lambda \cdot (\alpha, x)) (\beta, y),
\]

\[
(\lambda \alpha \beta, xy) = (\alpha \lambda \beta, xy) = (\alpha, x) (\lambda \beta, y) = (\alpha, x) (\lambda \beta, y)
\]

for any \( \alpha, \beta, \gamma, \lambda, \kappa \in R \) and \( x, y, z \in M \), so \( R \boxtimes M \) is a scalable monoid with \( \cdot \) a scaling action of \( R \) on \( R \times M \).

**Proposition 6.2.** Let a ring-monoid \( R \boxtimes M \) be a scalable monoid over \( R \). Then \( U = \{ (1, x) \mid x \in M \} \) is a coherent system of unit elements of \( R \boxtimes M \).

**Proof.** We have \( (\alpha, x) = \alpha \cdot (1, x) \) for any \( (\alpha, x) \in R \times M \), and if \( \alpha \cdot (1, x) = \alpha' \cdot (1, x) \) then \( (\alpha, x) = (\alpha', x) \), so \( \alpha = \alpha' \). Also, \( 1 \cdot (\alpha, x) = (\alpha, x) \), so \( 1 \cdot (\alpha, x) = \alpha \cdot (1, x) \), so \( (\alpha, x) \sim (1, x) \). Furthermore, if \( (1, x) \sim (1, y) \) so that \( \alpha \cdot (1, x) = \beta \cdot (1, y) \) for some \( \alpha, \beta \in R \) then \( (\alpha, x) = (\beta, y) \), so \( x = y \), so \( (1, x) = (1, y) \). Hence, \( U \) is a system of unit elements of \( R \boxtimes M \).

Finally, if \( (1, x), (1, y) \in U \) then \( x, y \in M \), so \( (1, x) \cdot (1, y) = (1, xy) \in U \), and \( (1, 1) \in U \) since \( 1 \in M \). Hence, \( U \) is a submonoid of \( R \boxtimes M \).

Thus, the fact that a ring-monoid \( R \boxtimes M \) is a free algebraic structure in the sense that \( (\alpha, x) = (\beta, y) \) if and only if \( \alpha = \beta \) and \( x = y \) implies that if \( x \) and \( y \) are distinct elements of \( M \) then \( (1, x) \) and \( (1, y) \) are incommensurable, and that each
\langle 1, x \rangle$, where $x \in M$, is a unit element for $R \cdot \langle 1, x \rangle = \{ \langle r, x \rangle \mid r \in R \}$. In addition, $U$ is a coherent set of unit elements since $M$ is a monoid.

If $R$ is commutative then $R \boxtimes M$ is an additive scalable monoid by Propositions 6.1, 5.3 and 6.2 and if $R$ is in addition an ordered ring then $R \boxtimes M$ is an ordered additive scalable monoid by Proposition 5.5. One may informally regard $R \boxtimes M$ as a "space of quantities" built around "units of measurement" of the form $\langle 1, x \rangle$.

In view of Proposition 6.1, Definition 6.1 is a construction-definition of a scalable monoid in the case when $R$ is commutative. In Part 2, it will become obvious that a ring monoid $R \boxtimes M$, where $R$ is a field and $M$ is a free abelian group, is a quantity space. This is similar to the construction-definition of a quantity space given by Carlson [4], who calls the elements of $M$ "pre-units".

Part 2. Quantity spaces

7. From scalable monoids to quantity spaces

In this section, we specialize scalable monoids in order to obtain a mathematical model suitable for calculation with quantities, a quantity space. The results in Sections 5.1 and 5.2 strongly suggest that a scalable monoid serving this purpose should be equipped with a sufficiently well-behaved set of unit elements. The simplest approach is to require that a quantity space is equipped with a coherent system of unit elements: this is a dense, sparse, closed and consistent set of unit elements.

A coherent system of unit elements of a scalable monoid corresponds to what is called a coherent system of units in metrology. There, coherent systems of units are commonly derived from sets of so-called base units, such as the three base units in the CGS system. The notion corresponding to a set of base units here is a basis in a quantity space, analogous to that of a basis in a vector space or a free abelian group.

Recall that Fourier assumed that each unit needed in the study of idealized heat propagation problems can be uniquely expressed as

$$u^d_{\ell} \cdot u^d_t \cdot u^d_T,$$

where $u_\ell$, $u_t$ and $u_T$ are units of length, time and temperature, respectively, and $d_\ell$, $d_t$ and $d_T$ are integers. In this case, $B = \{ u_\ell, u_t, u_T \}$ is a set of base units, and the set of all quantities of the form $u^d_{\ell} \cdot u^d_t \cdot u^d_T$ is a coherent system of units derived from $B$, provided that 1 and the product of $u^d_{\ell} \cdot u^d_t \cdot u^d_T$ and $u^{d_2} \cdot u^{d_2} \cdot u^{d_2}$ are equal to terms of the form $u^{d_1} \cdot u^{d_2} \cdot u^{d_2}$. It is natural to require that

$$\left( u^d_{\ell_1} \cdot u^d_t \cdot u^d_T \right) \left( u^d_{\ell_2} \cdot u^d_t \cdot u^d_T \right) = u^d_{\ell_1 + \ell_2} \cdot u^d_{t_1 + t_2} \cdot u^d_{T_1 + T_2},$$

such that $G$ is a multiplicatively written vector space over \( \mathbb{Q} \), specifically assumed to be finite-dimensional. This is an unnecessary assumption, however; it suffices to assume that $G$ is a free module over \( \mathbb{Z} \), or equivalently a free abelian group. In Raposo’s definition of a quantity space [22], Carlson’s vector space of pre-units is replaced by a finitely generated free abelian group of dimensions.

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5 Carlson considers ring-monoids of the form $R \boxtimes G$ where $G$ is an abelian group equipped with an external operation

$$\mathbb{Q} \times G \rightarrow G, \quad (c, x) \mapsto z^c$$

such that $G$ is a multiplicatively written vector space over \( \mathbb{Q} \), specifically assumed to be finite-dimensional. This is an unnecessary assumption, however; it suffices to assume that $G$ is a free module over \( \mathbb{Z} \), or equivalently a free abelian group. In Raposo’s definition of a quantity space [22], Carlson’s vector space of pre-units is replaced by a finitely generated free abelian group of dimensions.
but this identity presupposes, at least, that the units \( u_\ell, u_t \) and \( u_T \) commute with each other. (For example, \( u_\ell u_\ell = (u_\ell^0 u_\ell^1) (u_\ell^0 u_\ell^0) = u_\ell^1 u_\ell^1 = u_\ell u_\ell \).) Thus, if we want to include in our model the feature that a coherent system of units can be derived from a system of base units, there are good reasons to require the specialized scalable monoids to be commutative.

Recall that if \( u \) is a unit element for some orbit class and \( \lambda \cdot u = \lambda' \cdot u \) then \( \lambda = \lambda' \), so \( \alpha \beta \cdot u = \beta \alpha \cdot u \) implies \( \alpha \beta = \beta \alpha \). This suggests that only scalable monoids over commutative rings are of interest. Furthermore, if we want to deal with derived units such as \( u_\ell^1 u_t^{-1} u_0^0 \) (meter per second, etc.) then inverses of units must be admitted, and there is a close connection between inverses of quantities in a scalable monoid over \( R \) and multiplicative inverses in \( R \). This suggests, finally, that quantity spaces should be defined as certain scalable monoids over fields.

We now come to the basic definitions motivated by the considerations above.

**Definition 7.1.** Let \( Q \) be a commutative scalable monoid over \( R \). A finite set of generators for \( Q \) is a set \( B = \{b_1, \ldots, b_n\} \) of elements of \( Q \) such that every \( x \in Q \) has an expansion

\[
x = \mu \cdot \prod_{i=1}^n b_i^{k_i},
\]

where \( \mu \in R \) and \( k_1, \ldots, k_n \) are integers. A finite basis for \( Q \) is a finite set of generators for \( Q \) such that every \( x \in Q \) has a unique expansion of this form.

Note that the uniqueness of the expansion means that the array \( (\mu, k_1, \ldots, k_n) \) is unique given an indexing \( B \rightarrow \{1, \ldots, n\} \) of the basis elements.

**Definition 7.2.** A finitely generated commutative scalable monoid is one equipped with a finite set of generators. A finitely generated free commutative scalable monoid is one equipped with a finite basis. A finitely generated quantity space is a finitely generated free commutative scalable monoid over a field \( K \).

Elements of a quantity space are called quantities, unit elements are called unit quantities, and orbit classes in a quantity space are called dimensions.

It is not very complicated to generalize the notion of a finite basis for a commutative scalable monoid to include possibly infinite bases, and thus to generalize finitely generated quantity spaces accordingly, but in view of the connection to metrology only the special case of finitely generated quantity spaces will be considered below.

### 8. Some Basic Facts About Quantity Spaces

**Proposition 8.1.** Let \( Q \) be a (finitely generated) quantity space with basis \( \{b_1, \ldots, b_n\} \) and \( x, y \in Q \). We have

1. \( 1 = 1 \cdot \prod_{i=1}^n b_i^{0_i} \);
2. if \( x = \mu \cdot \prod_{i=1}^n b_i^{k_i} \) and \( y = \nu \cdot \prod_{i=1}^n b_i^{\ell_i} \) then
   \[
   xy = \mu \nu \cdot \prod_{i=1}^n b_i^{(k_i+\ell_i)};
   \]
3. if \( x = \mu \cdot \prod_{i=1}^n b_i^{k_i} \) and \( \mu \neq 0 \) then \( x^{-1} = \frac{1}{\mu} \cdot \prod_{i=1}^n b_i^{-k_i} \).

**Proof.** To prove (1), note that \( b_i^{0_i} = 1 \) for all \( b_i \). (2) follows from Lemma 2.1 and the fact that \( Q \) is commutative. (3) follows from (1) and (2). \( \Box \)
Proposition 8.2. Every element of a basis for a (finitely generated) quantity space is non-zero.

Proof. We have $0_C = 0 \cdot x$ and $x = 1 \cdot x$ for all $x \in C$, so if $0_C \in B$ then $0_C = 0 \cdot 0_C$ and $0 = 1 \cdot 0_C$ are expansions of $0_C$ in terms of $B$, so $0_C$ does not have a unique expansion in terms of $B$, so $B$ is not a basis for $Q$. \hfill \blacksquare

Proposition 8.3. If $Q$ is a (finitely generated) quantity space with basis $\{b_1, \ldots, b_n\}$ then $x \in Q$ is a non-zero quantity if and only if $x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$ for some $\mu \neq 0$.

Proof. We have $0 \cdot x = 0 \cdot (\mu \cdot \prod_{i=1}^{n} b_i^{k_i}) = 0 \cdot \prod_{i=1}^{n} b_i^{k_i}$, so $0 \cdot x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} = x$ if and only if $\mu = 0$ since the expansion of $0 \cdot x$ is unique. \hfill \blacksquare

In particular, $1$ is a non-zero quantity.

Proposition 8.4. The product of non-zero quantities in a (finitely generated) quantity space is a non-zero quantity.

Proof. Set $x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$ and $y = \nu \cdot \prod_{i=1}^{n} b_i^{k_i}$. Then $xy = \mu \nu \cdot \prod_{i=1}^{n} b_i^{(k_i+\ell_i)}$, and $\mu \nu \neq 0$ since there are no zero divisors in $K$. \hfill \blacksquare

Proposition 8.5. An element $x$ of a (finitely generated) quantity space $Q$ is invertible if and only if it is non-zero.

Proof. Let $\{b_1, \ldots, b_n\}$ be a basis for $Q$ so that $x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$. If $\mu \neq 0$ then $\frac{1}{\mu} \cdot \prod_{i=1}^{n} b_i^{-k_i}$ is an inverse of $x$; conversely, if $\mu = 0$ then there is no $\nu$ such that $\mu \nu = 1$, so $x$ does not have an inverse $\nu \cdot \prod_{i=1}^{n} b_i^{k_i}$. \hfill \blacksquare

Combining Propositions 8.2 and 8.5 we obtain the following result.

Proposition 8.6. Every element of a basis for a (finitely generated) quantity space is invertible.

Lemma 8.1. Let $Q$ be a quantity space over $K$ with a basis $\{b_1, \ldots, b_n\}$, and consider $x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$ and $y = \nu \cdot \prod_{i=1}^{n} b_i^{k_i}$. The following conditions are equivalent:

1. $x \sim y$;
2. $k^i = \ell^i$ for $i = 1, \ldots, n$;
3. $\prod_{i=1}^{n} b_i^{k_i} = \prod_{i=1}^{n} b_i^{\ell_i}$;
4. $\nu \cdot x = \mu \cdot y$.

Proof. The implications (2) $\implies$ (3), (3) $\implies$ (4) and (4) $\implies$ (1) are trivial. To prove (1) $\implies$ (2), note that if $x \sim y$ so that $\alpha \cdot (\mu \cdot \prod_{i=1}^{n} b_i^{k_i}) = \beta \cdot (\nu \cdot \prod_{i=1}^{n} b_i^{k_i})$ for some $\alpha, \beta \in K$ then

$$\alpha \mu \cdot \prod_{i=1}^{n} b_i^{k_i} = z = \beta \nu \cdot \prod_{i=1}^{n} b_i^{k_i}.$$

As the expansion of $z$ is unique, $k^i = \ell^i$ for $i = 1, \ldots, n$. \hfill \blacksquare

Note that Lemma 8.1 implies Fourier’s principle of dimensional homogeneity [8]. If $x = y$ then $x \sim y$, so $k^i = \ell^i$ for $i = 1, \ldots, n$; conversely, if not $k^i = \ell^i$ for $i = 1, \ldots, n$ then $x \not\sim y$, so $x \neq y$. 

9. Systems of Unit Quantities in Quantity Spaces

Proposition 9.1. If $Q$ is a (finitely generated) quantity space then every non-zero quantity $u \in Q$ is a unit quantity for $[u]$.

Proof. Set $u = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$ and $x = \nu \cdot \prod_{i=1}^{n} b_i^{k_i}$. Then $\mu \neq 0$ by Proposition 8.3 and if $u \sim x$ then $\nu \cdot u = \mu \cdot x$ by Lemma 8.1, so $x = \mu^{-1} \cdot (\nu \cdot u) = \mu^{-1} \cdot \nu \cdot u$.

Also, if $\lambda \cdot u = \lambda' \cdot u$ then $\lambda \cdot \mu \cdot \prod_{i=1}^{n} b_i^{k_i} = \lambda' \cdot \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$, so $\lambda \mu = \lambda' \mu$ since the expansion of $z$ is unique, so $\lambda = \lambda'$ since $\mu \neq 0$.

□

Corollary 9.1. If $Q$ is a (finitely generated) quantity space then a dense, sparse set of non-zero elements of $Q$ is a system of unit quantities of $Q$, and a sparse set of non-zero elements of $Q$ is a partial system of unit quantities of $Q$.

In metrology, unit quantities are called measurement units [12]. A set of base units $B$ is a finite set of measurement units each of which cannot be expressed as a product of powers of the other measurement units in $B$ [12]. A finite basis $B$ for a quantity space $Q$ is a set of base units; if $b \in B$ is not a base unit relative to $B$ then $1 \cdot b = b = \prod_{i=1}^{n} b_i^{k_i} = \prod_{i=1}^{n} b_i^{k_i}$, where $b_i \in B$ and $b_i \neq b$ for all $b_i$, so $b$ does not have a unique expansion relative to $B$, so $B$ is not a basis for $Q$.

Proposition 9.2. If $Q$ is a (finitely generated) quantity space over $K$ with basis $B = \{b_1, \ldots, b_n\}$ then

$$\mathcal{U} = \left\{ \prod_{i=1}^{n} b_i^{k_i} \mid k_i \in \mathbb{Z} \right\}$$

is a coherent system of unit quantities for $Q$.

Proof. All elements of $B$ are non-zero by Proposition 8.2 so all elements of $\mathcal{U}$ are non-zero and hence unit quantities by Proposition 9.1. Also, $\mathcal{U}$ is dense in $Q$ since every $x \in Q$ has an expansion $x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$, so $1 \cdot x = \mu \cdot \left( \prod_{i=1}^{n} b_i^{k_i} \right)$. Lastly, if $u = 1 \cdot \prod_{i=1}^{n} b_i^{k_i} \sim 1 \cdot \prod_{i=1}^{n} b_i^{k_i} = v$ then $\prod_{i=1}^{n} b_i^{k_i} = \prod_{i=1}^{n} b_i^{k_i}$ by Lemma 8.1, so $u = v$, meaning that $\mathcal{U}$ is sparse in $Q$.

It remains to prove that $\mathcal{U}$ is a monoid. Clearly, $1 \in \mathcal{U}$ since $1 = 1 \cdot \prod_{i=1}^{n} b_i^{0}$, and we have

$$\left( \prod_{i=1}^{n} b_i^{k_i} \right) \left( \prod_{i=1}^{n} b_i^{l_i} \right) = \prod_{i=1}^{n} b_i^{k_i+l_i},$$

so if $u, v \in \mathcal{U}$ then $uv \in \mathcal{U}$. Thus, $\mathcal{U}$ is a submonoid of $Q$. □

In other words, every (finite) basis $B$ can be extended to a coherent system $\mathcal{U}$ of unit quantities, consisting of basis quantities and other unit quantities that are expressed as products of basis quantities and their inverses.

In metrology, a coherent system of units $U$ is defined essentially as a set of measurement units each of which is either a base unit $b_f \in U$ or a coherent derived unit, a non-base unit of the form $1 \cdot \prod_{i=1}^{n} b_i^{k_i}$, where each $b_i$ is a base unit in $U$ and $k_1, \ldots, k_n$ are integers [12]. By Proposition 9.2, a coherent system of units in this sense is a coherent system of unit quantities in the sense of Definition 5.7.

By Propositions 9.1 and 5.3, every (finitely generated) quantity space is an additive quantity space in the sense of Definition 5.4.
Also, Propositions 9.2 and 5.5 imply that if $Q$ is a (finitely generated) quantity space over an ordered field $K$ then $Q$ is an ordered additive quantity space over $K$ with $C$ ordered by $\leq_C$ defined by $x \leq_C y$ if and only if $y - x = \rho \cdot u$ for some $\rho \in K$ such that $0 \leq \rho$ and some $u \in U \cap C$, where $U$ is a coherent system of unit quantities derived from a basis for $Q$. Thus, every quantity space over $Q$ or $\mathbb{R}$ can be regarded as an ordered additive quantity space since $Q$ and $\mathbb{R}$ are ordered. We normally want quantity spaces to be ordered, and since any Dedekind-complete ordered field is isomorphic to $\mathbb{R}$, it is natural to let $K$ be the real numbers $\mathbb{R}$.

10. Measures of Quantities

Definition 10.1. Let $Q$ be a (finitely generated) quantity space over $K$, and let $B = \{b_1, \ldots, b_n\}$ be a basis for $Q$. The uniquely determined scalar $\mu \in K$ in the expansion

$$x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$$

is called the measure of $x$ relative to $B$ and will be denoted by $\mu_B(x)$. □

For example, $1 = 1 \cdot \prod_{i=1}^{n} b_i^0$ for any $B$, so we have the following simple but useful fact.

Proposition 10.1. Let $Q$ be a (finitely generated) quantity space. For any basis $B$ for $Q$ we have $\mu_B(1) = 1$.

Relative to a fixed basis, measures of quantities can be used as proxies for the quantities themselves. The following fact follows immediately from Proposition 8.1.

Proposition 10.2. If $Q$ is a (finitely generated) quantity space with basis $B$ and $x, y \in Q$ then $\mu_B(xy) = \mu_B(x)\mu_B(y)$.

Proposition 10.3. Let $Q$ be a (finitely generated) quantity space with basis $B$. A quantity $x \in Q$ is invertible if and only if $\mu_B(x) \neq 0$, and for any invertible $x \in Q$ we have $\mu_B(x^{-1}) = \mu_B(x)^{-1}$.

Proof. The first part of the assertion follows from Propositions 8.3 and 8.5, the second part follows Propositions 8.1, 8.3 and 8.5. □

Proposition 10.4. If $Q$ is a (finitely generated) quantity space with basis $B$ then $\mu_B(\lambda \cdot x) = \lambda \mu_B(x)$ for all $\lambda \in K$ and $x \in Q$.

Proof. If $x = \mu \prod_{i=1}^{n} b_i^{k_i}$ then $\lambda \cdot x = \lambda \cdot (\mu \cdot \prod_{i=1}^{n} b_i^{k_i}) = \lambda \mu \cdot \prod_{i=1}^{n} b_i^{k_i}$, so $\mu_B(\lambda \cdot x) = \lambda \mu = \lambda \mu_B(x)$. □

Proposition 10.5. If $Q$ is a (finitely generated) quantity space with basis $B$ then $\mu_B(x) + \mu_B(y) = \mu_B(x + y)$ for all $x, y \in X$ such that $x \sim y$.

Proof. Let $x = \mu_B(x) \cdot \prod_{i=1}^{n} b_i^{k_i}$ and $y = \mu_B(y) \cdot \prod_{i=1}^{n} b_i^{k_i}$ be the expansions of $x$ and $y$ relative to $B = \{b_1, \ldots, b_n\}$. As $\prod_{i=1}^{n} b_i^{k_i}$ is non-zero, and thus a unit quantity for $\prod_{i=1}^{n} b_i^{k_i}$ by Proposition 8.1, we have

$$x + y = \mu_B(x) \cdot \prod_{i=1}^{n} b_i^{k_i} + \mu_B(y) \cdot \prod_{i=1}^{n} b_i^{k_i} = (\mu_B(x) + \mu_B(y)) \cdot \prod_{i=1}^{n} b_i^{k_i},$$

proving the assertion. □
In general, the measure of a quantity depends on a choice of basis, but there is an important exception to this rule.

**Proposition 10.6.** Let \( Q \) be a (finitely generated) quantity space. For every \( x \in [1] \), the measure \( \mu_B(x) \) of \( x \) relative to a basis \( B \) for \( Q \) does not depend on \( B \).

**Proof.** \( 1 \) is a unit quantity for \([1]\) by Proposition 9.1 so there is a unique \( \lambda \in K \) such that \( x = \lambda \cdot 1 \), so \( \mu_B(x) = \lambda \mu_B(1) \) by Proposition 10.4 and \( \mu_B(1) \) does not depend on \( B \) by Proposition 10.1. \( \square \)

It is common to refer to a quantity \( x \in [1] \) as a “dimensionless quantity”, although \( x \) is not really “dimensionless” – it belongs to, or “has”, the dimension \([1]\). The so-called Buckingham \( \Pi \) theorem and hence dimensional analysis depends on the fact stated in Proposition 10.6 [13].

11. Groups of dimensions: cardinality of bases

Recall that a trivially scalable monoid \( Q/\sim \) may also be regarded as a plain monoid. The definition of a basis for a commutative monoid differs slightly from that for a scalable commutative monoid.

**Definition 11.1.** Let \( M \) be a commutative monoid. A finite basis for \( M \) is a set \( B = \{b_1, \ldots, b_n\} \) of elements of \( M \) such that every \( x \in M \) has a unique expansion

\[
x = \prod_{i=1}^{n} b_i^{k_i},
\]

where \( k_1, \ldots, k_n \) are integers.

In this section, every quotient of the form \( Q/\sim \), where \( Q \) is a quantity space, will be regarded as a monoid, which means that Definition 11.1 will be used instead of Definition 7.1 in these cases.

**Proposition 11.1.** If \( Q \) is a (finitely generated) quantity space then \( Q/\sim \) is an abelian group.

**Proof.** \( Q/\sim \) is commutative since \([x][y] = [xy] = [yx] = [y][x]\) for all \([x], [y] \in Q/\sim\). To prove that \( Q/\sim \) is a group it suffices to show that for every \( x \in X/\sim \) there is a dimension \([x]^{-1} \in X/\sim \) such that \([x][x]^{-1} = [x]^{-1}[x] = [1]\). Let \( B = \{b_1, \ldots, b_n\} \) be a basis for \( Q \) and let \( x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} \) be the unique expansion of \( x \) relative to \( B \). Also set \( y = 1 \cdot \prod_{i=1}^{n} b_i^{-k_i} \). Then \([x][y] = [xy] = [\mu \cdot 1] = [yx] = [y][x]\), so \([y]\) is an inverse \([x]^{-1}\) of \([x]\) since \([\mu \cdot 1] = [1]\). \( \square \)

By a finite basis for an abelian group we mean a finite basis for the underlying commutative monoid, and a finitely generated free abelian group is an abelian group for which such a finite basis exists.

**Proposition 11.2.** If \( Q \) is a (finitely generated) quantity space over \( K \) and \( B = \{b_1, \ldots, b_n\} \) is a basis for \( Q \), then \( B = \{|b_1|, \ldots, |b_n|\} \) is a basis for \( Q/\sim \) with the same cardinality as \( B \).

**Proof.** The unique expansions of \( b_i, b_\nu \in B \) relative to \( B \) are \( b_i = 1 \cdot b_i \) and \( b_\nu = 1 \cdot b_\nu \). Hence, \(|b_i| = |b_\nu| \) implies \( b_i = b_\nu \) since \( b_i \sim b_\nu \) implies \( 1 \cdot b_i = 1 \cdot b_\nu \) by Lemma 8.1 so the surjective mapping \( \phi : B \to B \) given by \( \phi(b_i) = |b_i| \) is injective as well and hence a bijection.
There exists a unique similar basis for a finitely generated commutative scalable monoid, we have the following much stronger result.

Consider the function \( \psi : B \to \psi(B) \) given by \( \psi(b_i) = b_i \). We have \( \psi(B) = \{b_1, \ldots, b_n\} \), and \( \psi \) is surjective. Also, if \( [b_i] \neq [b_j] \), then \( b_i \neq b_j \), since dimensions are disjoint, meaning that \( \psi \) is injective as well and hence a bijection.

Let \( x \) be an arbitrary quantity in \( Q \). As \( B \) is a basis for \( Q/\sim \), we have \( x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} \) for some \( \mu \in K \) and some integers \( k_1, \ldots, k_n \), so \( x = \left[ \prod_{i=1}^{n} b_i^{k_i} \right] \). Also, if \( x = \prod_{i=1}^{n} b_i^{k_i} = \prod_{i=1}^{n} [b_i]^{k_i} \), then \( \prod_{i=1}^{n} b_i^{k_i} = \left[ \prod_{i=1}^{n} b_i^{k_i} \right] \). Hence, \( B \) is a basis for \( Q/\sim \).

We say that a basis \( \{b_1, \ldots, b_n\} \) for \( Q \) and a basis \( \{b_1, \ldots, b_m\} \) for \( Q/\sim \) are similar when \( m = n \) and \( [b_i] = b_i \) for \( i = 1, \ldots, n \).

**Corollary 11.1.** Let \( Q \) be a (finitely generated) quantity space. For every basis for \( Q \) there exists a unique similar basis for \( Q/\sim \).

Hence, corresponding to the fact that if \( X \) is a scalable monoid then \( X/\sim \) is a monoid, we have the following much stronger result.

**Proposition 11.3.** If \( Q \) is a (finitely generated) quantity space then \( Q/\sim \) is a (finitely generated) free abelian group.

The idea that the set of dimensions of a quantity space forms a free abelian group is present in articles by Krystek [13] and Raposo [22]. This is actually an assumption built into the definition of quantity spaces in [22]; here it is a fact derived from the definitions of quantity spaces and commensurability relations on quantity spaces.

A finitely generated abelian group may have no finite basis; in this case, a corresponding finitely generated trivially scalable commutative monoid over a field cannot have a finite basis since this would contradict Proposition 11.2. Thus, a finitely generated commutative scalable monoid over a field need not be a finitely generated quantity space. (This may be generalized to the case of infinite bases.)

**Proposition 11.4.** If \( Q \) is a (finitely generated) quantity space and \( B = \{b_1, \ldots, b_n\} \) is a basis for \( Q/\sim \) such that for each \( b_i \in B \) there is is a non-zero quantity \( b_i \in Q \) such that \( b_i = [b_i] \), then \( B = \{b_1, \ldots, b_n\} \) is a basis for \( Q \) with the same cardinality as \( B \).

**Proof.** Consider the function \( \psi : B \to \psi(B) \) given by \( \psi(b_i) = b_i \). We have \( \psi(B) = \{b_1, \ldots, b_n\} \), and \( \psi \) is surjective. Also, if \( [b_i] \neq [b_j] \) then \( b_i \neq b_j \), since dimensions are disjoint, meaning that \( \psi \) is injective as well and hence a bijection.

Let \( x \) be an arbitrary quantity in \( Q \). As \( B \) is a basis for \( Q/\sim \), we have \( x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} \) for some integers \( k_1, \ldots, k_n \), and if \( b_i \neq 0 \) for each \( b_i \) then \( \prod_{i=1}^{n} b_i^{k_i} \) is non-zero and thus a unit quantity for \( [x] \) by Proposition 9.4, so there exists a unique \( \mu \in K \) such that \( x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} \). Also, if \( x = \mu \cdot \prod_{i=1}^{n} b_i^{k_i} = \nu \cdot \prod_{i=1}^{n} b_i^{k_i} \), then \( \mu \cdot \prod_{i=1}^{n} b_i^{k_i} = \left[ \nu \cdot \prod_{i=1}^{n} b_i^{k_i} \right] \), so \( \prod_{i=1}^{n} b_i^{k_i} = \left[ \prod_{i=1}^{n} b_i^{k_i} \right] \), so \( \prod_{i=1}^{n} b_i^{k_i} = \left[ \prod_{i=1}^{n} b_i^{k_i} \right] \). Hence, \( B \) is a basis for \( Q/\sim \), so \( \nu = \mu \) by the uniqueness of \( \mu \). We have thus shown that \( B \) is a basis for \( Q \).

Let \( \{b_1, \ldots, b_n\} \) be a basis for \( Q \) and \( B \) a basis for \( Q/\sim \). For every \( b_i \in B \), we have \( b_i = \left[ \mu_i \cdot \prod_{j=1}^{n} b_j^{k_{ij}} \right] = \left[ \prod_{j=1}^{n} b_j^{k_{ij}} \right] \) for some \( \mu_i \in K \) and integers \( k_1, \ldots, k_n \), where \( \prod_{j=1}^{n} b_j^{k_{ij}} \neq 0 \). Thus, Proposition 11.3 implies the following fact.
Corollary 11.2. Let $Q$ be a (finitely generated) quantity space. For every basis for $Q/\sim$ there exists a similar basis for $Q$.

Propositions 11.2 and 11.4, the fact that if $B = \phi(B)$ then we can set $\psi = \phi^{-1}$, so that $\psi(B) = B$, since each $b_i \in B$ is non-zero, and the fact that any two bases for a free abelian group have the same cardinality give the following result.

**Proposition 11.5.** If $Q$ is a (finitely generated) quantity space then any two bases for $Q/\sim$ have the same cardinality, any basis for $Q$ has the same cardinality as any basis for $Q/\sim$, and any two bases for $Q$ have the same cardinality.

Recall that a free module of rank $n$ is a module with a basis and such that all bases have the same cardinality $n$. Defining the rank of a commutative scalable monoid analogously, we can say that finitely generated quantity spaces are free of finite rank, as are finite-dimensional vector spaces.

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