Role of causality in ensuring unconditional security of relativistic quantum cryptography

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Abstract

The problem of unconditional security of quantum cryptography (i.e. the security which is guaranteed by the fundamental laws of nature rather than by technical limitations) is one of the central points in quantum information theory. We propose a relativistic quantum cryptosystem and prove its unconditional security against any eavesdropping attempts. Relativistic causality arguments allow to demonstrate the security of the system in a simple way. Since the proposed protocol does not employ collective measurements and quantum codes, the cryptosystem can be experimentally realized with the present state-of-art in fiber optics technologies. The proposed cryptosystem employs only the individual measurements and classical codes and, in addition, the key distribution problem allows to postpone the choice of the state encoding scheme until after the states are already received instead of choosing it before sending the states into the communication channel (i.e. to employ a sort of “antedate” coding).

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The idea of quantum cryptography was first proposed in the paper of Wiesner [1] which, because of the novelty of the developed approach, had not been published for a long time and only existed as a manuscript. The idea of Wiesner became commonly known after the publication of the paper of Bennett and Brassard [2]. An important advance was made in Refs. [3] and [4]. Ekert proposed a cryptosystem based on the EPR-effect [5]. Bennett et al [4] demonstrated that any eavesdropping attempt can be reliably detected employing an arbitrary pair of non-orthogonal states. Later a large number of quantum cryptosystems and their realizations were proposed [6] which are not guaranteed to be unconditionally secure. Currently there exist three proofs of the unconditional security. The proof due to Mayers [7] addresses the so-called BB84 protocol [2] and so does the paper of Biham et al [8]. The proof of Lo and Chau [9] deals with the protocol based on the EPR-effect [3] and, in contrast to Ref. [7], requires the access of legitimate users to a quantum computer. The fact that these proofs have not yet become commonly accepted seems to be due to the lack of a qualitative interpretation of their internal structure. Shor and Preskill [10] attempted to simplify these proofs by explicitly using the quantum codes. The first relativistic quantum cryptosystem was proposed by Goldenberg and Vaidman [11]. The proof of its unconditional security is outlined in Ref. [12]. In our opinion, the major obstacle in proving the unconditional security in Refs. [7–10] arises because the corresponding protocols are formulated as the exchange protocols in the Hilbert state space and do not explicitly employ the causality considerations and the fact that the legitimate users are spatially separated although in real life the transmission of information always implies the preparation of information carriers (quantum systems), their propagation through the communication channel between two distant users, and, finally, performing a measurement of the quantum state of the information carriers at some later time. The restrictions imposed on the measurability of quantum states by special relativity were first considered by Landau and Peierls as early as in 1931 [13]. Further analysis was performed in the paper of Bohr and Rosenfeld [14].

Let us begin with the formulation of the protocol security criterion. Such a criterion seems to have been first explicitly formulated in the work of Mayers [7]. Here we shall adopt a different criterion which is more convenient for our proof. The protocol should satisfy the two requirements which informally can be outlined in the following way: the two strings of $N$ classical bits $s_A(N)$ and $s_B(N)$, possessed by users A and B after the protocol is completed should be (1) identical and (2) known to nobody else. More formally, a protocol is secure if for any $N \geq 1$ and any pair of real numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ its parameters (the employed states, measurements, etc.) can be chosen in such a way that:

1. The probability for the two strings $s_A(N)$ and $s_B(N)$ to differ in at least one bit is less than $\varepsilon_1 > 0$, i.e.

\[
Pr\{s_A(N) \neq s_B(N)\} \leq \varepsilon_1;
\]

in other words (in the language of mutual information between users A and B) for any $\varepsilon_1' > 0$ it is possible to satisfy the inequality

\[
I(A; B) \geq N - \varepsilon_1'.
\]
2. The probability for the eavesdropper E to learn the string \( s_A(N) \) exceeds the probability of a simple guess, \( 2^{-N} \), (remember that error probability associated with simply guessing a particular bit is \( 1/2 \) and represents the worst eavesdropper strategy) by no more than \( \varepsilon_2 \):

\[
\Pr\{s_A(N) = s_B(N)\} \leq 2^{-N} + \varepsilon_2;
\]
equivalently, the eavesdropper has arbitrarily small information on the strings \( s_A(N) \) and \( s_B(N) \) adopted as the key of length \( N \) by legitimate users:

\[
I(A; E) \leq \varepsilon_2, \quad I(B; E) \leq \varepsilon_2.
\]

Here the string of bits adopted as the key should not be understood the bits the original bits sent by user A: each bit of the key is actually a function of the original bits sent by user A which passed the appropriate tests performed by user B aimed at detection of eavesdropping attempts.

If a single classical bit is to be transmitted, the user A associates with the logical states 0 and 1 of the classical bit the density matrices \( \rho_0 \) and \( \rho_1 \) which can be chosen with the \( a \ priori \) probabilities \( \pi_0 \), \( \pi_0 + \pi_1 = 1 \). Classical information is extracted by performing quantum measurements on the system described by the density matrix \( \rho = \pi_0 \rho_0 + \pi_1 \rho_1 \). The measurements are described by the identity resolutions in the state space, \( \sum_i E_i = I \). Information on the classical bit sent by user A available to user B is defined as the mutual information maximized over all possible measurements which can be performed by user B:

\[
I(A; B, \rho_0; \rho_1) = \max_{\{E_i\}} \sum_i \left\{ \pi_0 \text{Tr}\{\rho_0 E_i\} \log_2 \left( \frac{\text{Tr}\{\rho_0 E_i\}}{\text{Tr}\{\rho E_i\}} \right) + \pi_1 \text{Tr}\{\rho_1 E_i\} \log_2 \left( \frac{\text{Tr}\{\rho_1 E_i\}}{\text{Tr}\{\rho E_i\}} \right) \right\}.
\]

A fundamental upper boundary on the available information is given by the inequality first proved by Holevo [15] (see also [16]):

\[
I(A; B, \rho_0; \rho_1) \leq S_{vN}(\rho) - \sum_{i=0,1} \pi_i S_{vN}(\rho_i), \quad S_{vN}(\rho) = -\text{Tr}\{\rho \log(\rho)\},
\]

where \( S_{vN}(\rho) \) is the von Neumann entropy [17], and the equality is reached if and only if the density matrices \( \rho_0 \) and \( \rho_1 \) commute with each other. For pure states the latter means that the equality in (6) is reached only for orthogonal states \( (\rho_{0,1} = |\psi_{0,1}\rangle\langle\psi_{0,1}| \) and \( (\psi_0|\psi_1) = 0 \). In that case the available information reaches the maximum possible value of

\[
I^{\text{max}}(A; B, \rho_0; \rho_1) = 1 \quad E_0 = |\psi_0\rangle\langle\psi_0|, \quad E_1 = |\psi_1\rangle\langle\psi_1|.
\]

In other words, reliable distinguishability of the orthogonal quantum states allows to transmit the maximum possible classical information. However, just because of their reliable distinguishability they cannot be used in quantum cryptography (at least in the protocols employing the Hilbert state space properties only).

It should be emphasized that it is implicitly assumed in Eqs. (5–7) that the entire Hilbert state space is always available for measurements. The fact that the causality arguments are not explicitly used should be understood as the possibility for user B to perform the measurement over a quantum state \( |\psi_i\rangle \) prepared by user A at time \( t_A \) at arbitrary later time (formally, even at \( t_B = t_A + 0 \)). The measuring operators \( E_i \) at different times are related by the expression \( E_i(t_B) = U(t_B - t_A)E_iU^{-1}(t_B - t_A) \), where \( E_i \) is the measuring operator taken at time \( t_A \). Then the available information (5) does not depend on time and can be obtained immediately after the measurement is performed.

The above formulation of the problem is not only unnatural, but it does not correspond to the real procedure of information transmission between two distant parties. As a matter of fact, the information (to be more precise, the physical quantum objects carrying that information) always propagate from one user to the other. Therefore, it is much more natural to formulate the problem explicitly taking into account the causal relations between the states preparation, propagation, and measurements preformed on these states as they reach the second user and become available to his measurement apparatus.

Since the information is carried in the Minkowskii space-time by real physical objects, e.g. photons (rather than the abstract physical systems and the rays in Hilbert state space ascribed to them as frequently assumed in the non-relativistic quantum information theory), the role of the instrument (superoperator) in the quantum field theory is played by the quantum electrodynamical \( \hat{S} \)-matrix. The latter should satisfy the unitarity and causality requirements
first explicitly derived by Bogolubov [18]:

\[
\hat{S}\hat{S}^+ = I, \quad \frac{\delta \hat{S}(g)}{\delta g(y)} \hat{S}^+(g) = 0, \quad \text{for } \hat{x} \leq \hat{y},
\]

which means that the \(\hat{S}\)-matrix does not depend on the behavior of \(g(\hat{x})\) (the function specifying how the interaction is switched on) at point \(\hat{x} \leq \hat{y}\) separated from \(\hat{y}\) by a space-like interval. We do not know whether it is possible to develop a protocol based on the first principles (general structure of the \(\hat{S}\)-matrix) alone. Therefore, in the rest of the paper we shall adopt a simple one-dimensional model containing all the necessary restrictions imposed by the relativistic causality. In addition, the adopted approach is further justified by the fact that the real fiber optics quantum communication channels are actually quasi-one-dimensional systems.

We shall first describe the states and measurements used in the protocol. Legitimate users control the spatially separated domains \(\Omega_A\) and \(\Omega_B\) of size \(L\). When the protocol is started at \(t_A = 0\), user \(A\) prepares with equal probabilities one of the following two orthogonal states corresponding to 0 or 1:

\[
|\psi_{0,1}\rangle = \int_0^\infty dk \mathcal{F}(k) a_{0,1}^\dagger(k)|0\rangle = \int_0^\infty dk \mathcal{F}(k)|k, e_{0,1}\rangle = |\mathcal{F}, e_{0,1}\rangle, |k, e_{0,1}\rangle = a_{0,1}^\dagger(k)|0\rangle, \quad \langle k, e_i|k', e_j\rangle = \delta(k-k')\delta_{ij},
\]

(9)

where \(a_{0,1}^\dagger(k)\) is the operator creating a photon with momentum \(k > 0\) and one of the two orthogonal polarization states \(e_0\) and \(e_1\), \(\mathcal{F}(k)\) is the state amplitude in momentum representation, \(i, j = 0, 1, k \in (0, \infty)\). In the position representation the states are written as

\[
|\psi_{0,1}\rangle = \int_{-\infty}^\infty dx \mathcal{F}(x-t)|x, t\rangle \otimes |e_{0,1}\rangle, \quad \mathcal{F}(x-t) = \int_0^\infty dk \mathcal{F}(k) e^{ik(x-t)}, \quad \langle k|x, t\rangle = \frac{1}{\sqrt{2\pi}} e^{ik(x-t)}, \quad x, t \in (-\infty, \infty).
\]

(10)

The amplitude \(\mathcal{F}(x-t)\) depends on the difference \(x-t\) only, in agreement with the intuitive picture of a packet propagating in the positive direction of the \(x\)-axis with the speed of light and having the spatio-temporal shape described by \(\mathcal{F}(x-t)\).

It should be noted that the basis vectors \(|x, t\rangle\) are not orthogonal. Normalization of the state vector in the position representation can be written as [19]

\[
\int_{-\infty}^\infty dx e^{ik(x-t)} \frac{1}{x-t + a} = i \pi \text{sgn}(k) e^{-ika},
\]

(11)

\[
\langle \psi_{0,1}|\psi_{0,1}\rangle = \langle \mathcal{F}|\mathcal{F}\rangle = \int_{-\infty}^\infty \int_{-\infty}^\infty dx dx' \mathcal{F}(x-t) \mathcal{F}^*(x'-t) \left[ \frac{1}{2} \delta(x-x') + \frac{i}{\pi} \frac{1}{x-x'} \right] = \int_{-\infty}^\infty |\mathcal{F}(x-t)|^2 dx.
\]

(12)

The states are chosen to be almost “monochromatic”, so that the amplitude \(\mathcal{F}(\tau) \approx \text{const} \approx 1/\sqrt{L}\) is actually represented by a constant wide “plateau” (to within the tails at its ends) and

\[
\int |\mathcal{F}(x-t)|^2 dx = 1 - \delta, \quad \delta \to 0.
\]

(13)

The decay at the ends can be chosen to be arbitrarily sharp and making \(\delta\) arbitrarily small. We shall assume that the latter condition is satisfied and the parameter \(\delta\) is well the smallest parameter in the problem\(^4\). In our one-dimensional model the non-localizability [20–22] can be derived from the Wiener-Paley theorem [23]. Normalization and square integrability conditions in the \(k\)-representation taken together impose restrictions on the asymptotic behavior of the function \(\mathcal{F}(\tau)\):

\[
\mathcal{F}(\tau) = \int_0^\infty \mathcal{F}(k) e^{-ik\tau} dk, \quad \int_{-\infty}^\infty \frac{\ln|\mathcal{F}(\tau)|}{1 + \tau^2} d\tau < \infty.
\]

The function \(\mathcal{F}(\tau)\) cannot have a compact support with respect to \(\tau\) and cannot decay exponentially; however, it can be arbitrarily strongly localized and possess a decay rate arbitrarily close to the exponential one, e.g.

\[
\mathcal{F}(\tau) \propto \exp\{-\alpha \tau/\ln(\ln L)\},
\]

where \(\alpha\) can be any real number. With this in mind, we shall for simplicity use the finite domains when specifying the limits of integration.
Preparation of the extended states when the protocol is started at time \( t_A = 0 \) requires the access to the entire domain \( \Omega_A \) of size \( L \). Intuitively, one can imagine a non-local device of size \( L \) which is simultaneously switched on at all point of the domain. There are no any formal arguments prohibiting such a state preparation procedure at time \( t_A = 0 \). An extended state can also be prepared by a localized (point-like) source which is switched on at \( t_A = 0 \) and produces (emits) a state propagating into the communication channel. For definiteness, it will be more convenient for us to assume that the state is prepared by a non-local source at time \( t_A = 0 \) and is immediately allowed at time \( t = t_A + 0 \) to propagate as a whole into the communication channel.

It is important for the protocol that the length of the quantum communication channel \( L_{ch} \) should not exceed the effective state extent \( L, L_{ch} < L \).

At some later time \( t_B \) when the entire state reaches the domain \( \Omega_B \) of size \( L \) controlled by user B, he performs a measurement described by the following identity resolution:

\[
I = \int_{-\infty}^{\infty} dx |x,t_B\rangle\langle x,t_B| \otimes I_{G2} = P_0(t_B) + P_1(t_B) + P_\perp(t_B), \quad P_{0,1}(t_B) = |F_{t_B},\epsilon_{0,1}\rangle\langle F_{t_B},\epsilon_{0,1}|, \tag{14}
\]

\[
|F_{t_B},\epsilon_{0,1}\rangle = \int_{-\infty}^{\infty} dx F(x-t_B)|x,t_B\rangle \otimes |\epsilon_{0,1}\rangle, \quad x \in \Omega_B, \quad P_\perp(t_B) = I - P_0(t_B) - P_1(t_B). \tag{15}
\]

In other words, user B takes the projection on one of the two states whose amplitude resides entirely in the domain \( \Omega_B \). The probabilities of different outcomes at time \( t_B \) are

\[
\Pr \{i,t_B;j\} = \Tr \{|\psi_i\rangle\langle \psi_i| P_j(t_B)\} = \delta_{ij} \int_{\Omega_B} dx |F(x-t_B)|^2 = \delta_{ij}, \quad x \in \Omega_B. \tag{16}
\]

If the eavesdropper does not delay the states, he never has access to them as a whole. Therefore, the probability of incorrect state identification by the eavesdropper will be different from zero even for orthogonal states. It should be emphasized that due to the orthogonal polarizations our states are even locally orthogonal. Formally, that means that if at certain moment of time the eavesdropper has only access to a spatial domain smaller than the effective extent of the states to be distinguished, the state identification error probability is different from zero. The total error probability is the sum of two terms. The first one describes the situation when the measuring apparatus used by the eavesdropper did not fire at all. These outcomes are inevitable if the entire states are not available as a whole. The error probability in that case is 1/2 (it is actually the error probability for simple guessing strategy). The second term in the total error corresponds to the case when the eavesdropper’s apparatus fired in the spatial domain available to the eavesdropper. The state identification error in that case is strictly zero because of the local orthogonality of the states employed. Therefore, the total error is the product of the error for the case when the outcome occurred in the domain unavailable to the eavesdropper and the fraction of such outcomes. More formally, the total error is

\[
P_e(t_E) = P_e(\Omega_E,t_E) + P_e(\overline{\Omega_E},t_E); \tag{17}
\]

here \( \Omega_E \) is the domain available to the eavesdropper (accordingly, \( \overline{\Omega_E} \) is the unavailable domain, i.e. the completion of \( \Omega_E \) to the entire position space) and \( t_E \) is the moment of time when the measurement in the domain \( \Omega_E \) was performed. The total identity resolution is

\[
I = I(\Omega_E,t_E) + I(\overline{\Omega_E},t_E), \quad I(\Omega_E,t_E) = \sum_{i=0,1} \int_{\Omega_E} dx |x,t_E,e_i\rangle\langle x,t_E,e_i|, \quad I(\overline{\Omega_E},t_E) = \sum_{i=0,1} \int_{\Omega_E} dx |x,t_E,e_i\rangle\langle x,t_E,e_i|. \tag{18}
\]

Arbitrary strategy with a binary decision function for the outcomes in the domain \( \Omega_E \) is described by an appropriate identity resolution on \( \Omega_E \). The lowest error probability \( P_e(\Omega_E,t_E) \) is found by the minimization over all possible identity resolutions \( I(\Omega_E) \) \cite{16}

\[
P_e(\Omega_E,t_E) = \min_{E_{0,1}} \left\{ \frac{1}{2} \Tr \{|\psi_0\rangle\langle \psi_0| E_1\} + \frac{1}{2} \Tr \{|\psi_1\rangle\langle \psi_1| E_0\} \right\}. \tag{19}
\]

\( E_{0,1} \) is easily found and the total state identification error \( P_e(\Omega_E,t_E) = 0 \) proves to be

\[
E_{0,1} = \int_{\Omega_E} dx |x,t_E;e_{0,1}\rangle\langle x,t_E;e_{0,1}|, \quad P_e(\overline{\Omega_E},t_E) = \frac{1}{2} \|N(\overline{\Omega_E},t_E)\|^2 = \frac{1}{2} \int_{\Omega_E} dx |F(x-t_E)|^2. \tag{20}
\]
Accordingly, if the spatial domain $\Omega_E$ available to the eavesdropper has a fixed size, the probability of correct identification by the eavesdropper of the bit sent by user A is

$$P_{OK}(t_E) = 1 - P_e(\Omega_E t_E) - P_e(\Omega E t_E) = \frac{1}{2} \left( 1 + \int_{\Omega E} dx |F(x - t_E)|^2 \right).$$

(21)

Thus the information on the bit sent by user A available to the eavesdropper depends on the size of the available domain and the choice of the moment when the measurement is performed. This information can be calculated using Eq. (5) taking into account that the measurement is described by the identity resolution $\{E_i\} = \{I_{\Omega_E}, E_0, E_1\}$ (where $E_{0,1}$ are taken from Eq. (20)). The available information is a sum of two terms. The first term describes the part of the mutual information given by the outcomes in the unavailable domain (when the eavesdropper’s apparatus did not fire at all) while the second one originates from the outcomes in the available domain $\Omega_E$:

$$I(A;E, \Omega_E, t_E) = I(A;E, \rho_0, \rho_1, \Omega_E, t_E) + I(A; E, \rho_0, \rho_1, \Omega_E, t_E).$$

(22)

Calculation according to Eq. (5) taking into account that $\{E_i\} = \{I_{\Omega_E}, E_0, E_1\}$ and $\pi_0 = \pi_1 = 1/2$ yields

$$I(A;E, \rho_0, \rho_1, \Omega_E, t_E) = 0, \quad \text{Tr} \{\rho_0 I_{\Omega_E} \} = \frac{1}{2} \text{Tr} \{\rho I_{\Omega_E}\}, \quad \rho = \frac{1}{2} (\rho_0 + \rho_1),$$

(23)

$$I(A;E, \rho_0, \rho_1, \Omega_E, t_E) = \int_{\Omega E} dx |F(x - t_E)|^2, \quad \text{Tr} \{\rho_0 E_0\} = \text{Tr} \{\rho_1 E_1\} = \text{Tr} \{\rho E_{0,1}\}.$$  

(24)

Therefore, the largest mutual information on the transmitted bit available to the eavesdropper depends on the moment of the measurement and the fraction of the state residing in the available domain. In this way the propagation of information in the space-time is explicitly accounted for, in contrast to the protocols formulated in the state space only. It is not surprising that the mutual information due to outcomes in the unavailable domain is zero since the state identification error probability in that case is 1/2. On the other hand, if the outcome took place in the available domain, the identification error is zero, so that the mutual information is proportional to the fraction of outcomes occurring in the available domain. To increase the mutual information, the eavesdropper should effectively extend the available domain (wait until a larger part of the state travel from the domain controlled by user A to the domain available to his measurements).

Let the effective increase in the domain size available to the eavesdropper (compared to the length of the communication channel which is supposed to be entirely available to him) is $\chi$. The correct identification probability for the eavesdropper is

$$\text{Pr}_E \{\chi\} = \frac{1}{2} \left( 1 + \int_{L - \chi} dx |F(x - t_E)|^2 \right) = \frac{1}{2} \left( 1 + \frac{L_{\text{ch}} + \chi}{L} \right).$$

(25)

For any state $|\tilde{F}\rangle$ delayed by time (distance) $\chi$, the probability of passing a test for possible delay based on the outcomes of the measurement $(14, 15)$ performed by the legitimate user B is

$$\text{Pr}_B \{\chi\} = \text{Tr} \{|\tilde{F}\rangle \langle \tilde{F}| (P_0(t_B) + P_1(t_B))\} =$$

$$\left( \int_{\{L - \chi\}} dx |F(x - t_B)|^2 \right)^{1/2} \left( \int_{\{L - \chi\}} dx |\tilde{F}(x - t_B)|^2 \right)^{1/2} \leq \left( \int_{\{L - \chi\}} dx |F(x - t_B)|^2 \right)^{1/2} \leq \left( 1 - \frac{\chi}{L} \right).$$

(26)

It is sufficient to consider pure delayed states $\tilde{F}$ only, since the linearity arguments can be applied to the case of mixed states. In Eq. (26) we took advantage of the Cauchy-Bunyakowskii inequality. The integration domain is restricted to $L - \chi$ since because of the limited propagation velocity, no state delayed by time $\chi$ cannot reach by the moment of measurement $t_B$ the extreme right boundary of domain $L$; remember also that $F(x - t_B) = 1/\sqrt{L}$.

Thus the probability for the eavesdropper to know the transmitted bit and simultaneously pass the test performed by user B is

$$\text{Pr} \{\text{bit}_E = \text{bit}_A \land \text{E pass test}, \chi\} = \text{Pr}_E \{\chi\} \cdot \text{Pr}_B \{\chi\} = \frac{1}{2} \left( 1 + \frac{L_{\text{ch}} + \chi}{L} \right) \cdot \left( 1 - \frac{\chi}{L} \right), \quad \text{Pr}_{\text{max}} = \frac{1}{2} \left( 1 + \frac{L_{\text{ch}}}{L} \right).$$

(27)
Therefore, for a specified channel length the probability maximum in Eq. (27) is reached at the interval boundary at $\chi = 0$. If the channel length equals the effective state extent ($L_{ch} = L$), this probability is unit. In that case the eavesdropper can reliably know each transmitted state and remain undetected. For $L_{ch} = 0$ the maximum is reached at $\chi = 0$ and the probability is $1/2$, which means that the eavesdropper simply guesses the transmitted state.

Up to this moment, we have not yet taken into account the noise in communication channel. Let us demonstrate now that in a noisy channel the probability (27) cannot exceed the corresponding value for the ideal channel. To do this, we shall take advantage of the relativistic causality arguments. The state modification induced by noise can be described by an appropriate instrument taking into account the relativistic restrictions imposed on it. The instrument can generally be written as [24–27]

$$T[\ldots] = \sum_k S_k[\ldots]S_k^+, \quad S_k = \sqrt{\lambda_k}\langle \varphi_k | \varphi_k \rangle, \quad \sum_k \lambda_k S_k S_k^+ \leq 1, \quad \lambda_k \geq 0,$$

(28)

$$\text{Tr}\{T[|\psi_{0,1}\rangle\langle \psi_{0,1}|I(\Omega_E, t_e)]\} = \sum_k \text{Tr}\{(|\psi_{0,1}\rangle\langle \psi_{0,1}| (S_k I(\Omega_E, t_e) S_k^+))\} \leq \sum_k \lambda_k \text{Tr}\{|\psi_{0,1}\rangle\langle \psi_{0,1}|(|\varphi_k \langle \varphi_k |\rangle)\} \leq \sum_k \lambda_k |\langle \varphi_k | \varphi_k \rangle|^2 \leq \sum_k \lambda_k |\langle \varphi_k | \varphi_k \rangle|^2 \leq \lambda_k |\langle \psi_{0,1} | \psi_{0,1} \rangle| \leq \langle \psi_{0,1} | \psi_{0,1} \rangle = \int_{\Omega_A} dx |F(x - t_A)|^2 = \int_{\Omega_E} dx |F(x - t_E)|^2.$$

(29)

The last equality in Eq. (29) reflects the fact that the amplitude $F(x - t_A)$ of state $|\psi_{0,1}\rangle$ at time $t_A$ is completely localized in the domain $x \in \Omega_A$, and will not be completely localized in domain $x \in \Omega_E$ earlier than at $t_E = t_A + \text{dist}(\Omega_E, \Omega_A)$. Strictly speaking, in the field theory the problem cannot be treated as a single-particle one in the sense that the operator $\hat{S}$ involves the processes associated with creation of particles and absorption of other photons entering the channel from the environment which can be treated as a sort of noise. However, detection of these external photons can obviously provide no additional information on the transmitted bit. Therefore, the probability for the eavesdropper to know an individual transmitted bit and remain undetected by the legitimate user B does not exceed the corresponding probability $Pr_{max} = 1/2 (1 + L_{ch} / L)$ for the ideal channel.

Let us know describe the protocol.

- At the time moments agreed upon beforehand, user A prepares and sends into the communication channel the states $|\psi_{0,1}\rangle$, while user B performs measurements described by the identity resolution (14,15). Only the bits which pass the test (i.e. those which produced the outcomes in channels $P_{0,1}(t_B)$) are kept, and all the rest transmissions are discarded ($P_{1,0}(t_B)$). If the channel is ideal and eavesdropping is absent, each bit sent by user A is reliably identified by user B and contributes to the key (in contrast to the cryptosystems based on non-orthogonal states).

- Then the noise (identification error probability) is estimated. Users A and B disclose some of the transmissions and obtain an estimate for the error probability $p_{err}$ counting the number of inconsistencies in their data sets. The disclosed bits are discarded from the transmitted string.

- The “antedate” coding is performed. User A divides the remaining transmissions into the groups each consisting of $k$ identical bits (either all zeros or all units) and announces through a public channel which transmission belongs to which group without disclosing the values of the bits occurring in each group. User B performs the error correction for each block employing a simple majority voting principle [28]. The number $k$ is chosen sufficiently large to reduce the error in identification of the block-wise bits $\hat{b}(i)$ ($\hat{b} = \{0,0,...0\}$ and $\hat{b} = \{1,1,...1\}$) below $\approx p_{err}^k \ll p_{err}$. This procedure enhances the probability of survival of the bit string finally accepted both users as the generated key. Then the block-wise bits are assigned their reference numbers.

- The users form $N + M$ parity bits $\hat{b}(j), \quad j = 1..N + M$. To do this, user A in each round chooses a random string $s_l$ of length $N + M - l$ ($l = 1..M$) and announces it to user B through a public channel. Then user A and B check the parities of the subsets of bits in their strings ($\hat{b}(A)$ and $\hat{b}(B)$) comparing the parities with the string $s_l$, since $s_l \cdot \hat{b}(A) = s_l \cdot \hat{b}(B) = (s_l \cdot \hat{b}(A) \oplus (s_l \cdot \hat{b}(B)) = s_l \cdot (\hat{b}(A) \oplus \hat{b}(B))$. If the parities
of the substrings coincide, one bit in the specified position is discarded from the strings Bit_A and Bit_B. If the parities are different, the protocol is aborted. After \( M \) successful rounds the probability for the two remaining strings Bit_A and Bit_B possessed by users A and B, respectively, and each consisting of \( N \) parity bits to be different is \( [29] \)

\[
\Pr\{s_A(N) \neq s_B(N)\} = 2^{-M}.
\]  

(31)

By appropriate choice of \( M \) one can always make this probability sufficiently small. Hence the first part of the protocol security criterion (Eqs. (1) and (2)) is proved.

- The probability for the eavesdropper to reliably know one of the parity bits and remain undetected is calculated in the following way. The total number of ways in which every parity bit can be built from the block-wise bits is \( [30] \)

\[
\frac{1}{2} \sum_{i=0}^{n} C_{n,k}^i = \frac{2^{n-k}}{2k} \sum_{l=1}^{k} \cos^{n-k} \left( \frac{l\pi}{k} \right) \cos (nl\pi) \approx \frac{1}{2k} 2^{n-k}.
\]  

(32)

The Hartley information of the set of block-wise strings is (to within the rounding error) the number of binary symbols required to identify the string parity which practically coincides with the string length \( n \cdot k \):

\[
I = \log_2 \left( \frac{2^{n-k}}{2k} \sum_{l=1}^{k} \cos^{n-k} \left( \frac{l\pi}{k} \right) \cos (nl\pi) \right) \approx \eta \cdot n \cdot k, \quad \eta \approx 1,
\]  

(33)

that is almost all the bits in the string should be known. The probability of knowing every bit and passing the test is does not exceed (27) so that the conditional probability for the eavesdropper to know \( N \) final bits (key) transmitted by user A and accepted by both users as the key is (remember that \( 1 + L_{ch}/L < 1 \))

\[
\Pr\{s_A(N) = s_E(N)\} = 2^{-N} \{1 + 2 \cdot 2^{-\eta n \cdot k} [(1 + L_{ch}/L)]^{\eta n \cdot k}\}^N = 2^{-N} (1 + 2\zeta)^N, \quad \zeta = 2^{-\eta n \cdot k} [(1 + L_{ch}/L)]^{\eta n \cdot k}. \]  

(34)

Mutual information on the string of final bits of length \( N \) possessed by user A available to the eavesdropper is

\[
I(A;E) = I(A) - I(A|E), \quad I(A) = -\log_2 2^{-N}, \quad I(A|E) = -\log_2 \Pr\{s_A(N) = s_E(N)\},
\]  

(35)

where \( I(A) \) is the proper information of the final string \( s_A(N) \) of length \( N \), \( I(A|E) \) is the conditional information on the string \( s_A(N) \) available to E. Since all possible bit strings arise with the same probability and the conditional probability for all strings is the same, we use the information which is not yet averaged over the string distribution. The conditional information has a natural interpretation as the number of additional bits required for E to reliably recover the bit string of length \( N \). As to the mutual information, it is interpreted as the number of bits measuring the information on string \( s_A(N) \) of length \( N \) available to the eavesdropper [31]. Taking into account Eqs. (34,35) one obtains for the mutual information between A and E:

\[
I(A;E) = N - N + N \log_2 (1 + 2\zeta) \approx \frac{2N \cdot \zeta}{\ln 2} = 2N \cdot 2^{-\eta n \cdot k} [(1 + L_{ch}/L)]^{\eta n \cdot k}/\ln 2 \ll 1.
\]  

(36)

For any specified \( N \), \( L_{ch} \), and \( L \) \( (L_{ch} < L) \) this information can be made exponentially small in the parameter \( n \cdot k \).

- Let us now show that the mutual information available to E on the string \( s_B(N) \) is also exponentially small. Mutual information between A and B is

\[
I(A;B) = I(A) - I(A|B) = -\log_2 2^{-N} + \log_2 \Pr\{s_A(N) = s_B(N)\} = N + \log_2 (1 - 2^{-M}) \approx N - \frac{2^{-M}}{\ln 2}.
\]  

(37)

Taking advantage of the triangle inequality for the information [32] one finally obtains

\[
I(A;E) \leq I(A|B) + I(B|E), \quad I(B|E) \geq N - (2N \cdot \zeta - 2^{-M})/\ln 2, \quad I(B;E) \leq (2N \cdot \zeta - 2^{-M})/\ln 2 \ll 1.
\]  

(38)

Thus the second part of (Eqs. (3) and (4)) of the security criterion is also proved.

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