Investigation of surface critical behavior of semi-infinite systems with cubic anisotropy.

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The critical behavior at the special surface transition and crossover behavior from special to ordinary surface transition in semi-infinite \(n\)-component anisotropic cubic models are investigated by applying the field theoretic approach directly in \(d = 3\) dimensions up to the two-loop approximation. The crossover behavior for random semi-infinite Ising-like system, which is the nontrivial particular case of the cubic model in the limit \(n \to 0\), is also investigated. The numerical estimates of the resulting two-loop series expansions for the critical exponents of the special surface transition, surface crossover critical exponent \(\Phi\) and the surface critical exponents of the layer, \(\alpha_1\), and local specific heats, \(\alpha_{11}\), are computed by means of Padé and Padé-Borel resummation techniques. For \(n < n_c\) the system belongs to the universality class of the isotropic \(n\)-component model, while for \(n > n_c\) the cubic fixed point is stable, where \(n_c\) is the marginal spin dimensionality of the cubic model. The obtained results indicate that the surface critical behavior of semi-infinite systems with cubic anisotropy is characterized by new set of surface critical exponents for \(n > n_c\).

I. INTRODUCTION

The investigation of the critical behavior of real systems is a very important task of the condensed matter theory. The critical behavior in such systems as polymers, easy-axis ferromagnets, superconductors, as well as superfluid \(^4\text{He}, \text{Heisenberg ferromagnets and quark-gluon plasma}\) is described by isotropic \(O(n)\) model with \(n = 0, 1, 2, 3\) and 4, respectively, and is very well determined in the framework of different theoretical and numerical approaches.

Investigation of the critical behavior of real cubic crystals is the subject of extensive theoretical work during past decades. In real crystals, due to their complex crystalline structure, some kind of anisotropy is always present. The simplest nontrivial crystalline anisotropy is a cubic one. A most typical model for description of the critical behavior of such systems is the \(O(n)\) model with a cubic interaction term \(\sum_{i=1}^{n}\phi_i^4\) added to the usual \(\frac{u_0}{n}(|\phi|^{2n})\) term. This \(n\)-component cubic model is the particular case of an \(nn\)-component model with cubic anisotropy at \(n = 1\). Anisotropic \(n\)-component cubic model exhibits different types of continuous and first-order phase transitions, depending on the number of spin components \(n\), space dimensionality \(d\), and sign of the cubic coupling constant \(v_0\). Cubic models are widely applied to the study of magnetic and structural phase transitions. In the limiting case of \(n \to 0\), it describes the critical behavior of random Ising-like systems. The case \(m = 1\) and \(n \to \infty\) corresponds to the Ising model with equilibrium magnetic impurities. Depending on the sign of the cubic coupling constant \(v_0\), two types of order are possible: along the diagonals of a hypercube in \(n\) dimensions \((v_0 > 0)\) or along the easy axes \((v_0 < 0)\). In the latter case the system can undergo the first order phase transition, as was confirmed in some experiments. In the present work we are concerned with the case \(v_0 > 0\).

The presence of surfaces, which are inevitable in real systems, leads to additional complications. General reviews on surface critical phenomena are given in Refs.\(^1\). A typical model to study critical phenomena in real physical systems restricted by a single planar surface is the semi-infinite models. There are different surface universality classes, defining the critical behavior in the vicinity of boundaries, at temperatures close to the bulk critical point \((T - T_c)/T_c \to 0)\). Each bulk universality class, in general, divides into several distinct surface universality classes. Three surface universality classes, called ordinary \((\epsilon_0 \to \infty)\), special \((\epsilon_0 = \epsilon_{sp})\) and extraordinary \((\epsilon_0 \to -\infty)\), are relevant for our case; they meet at a multicritical point \((\epsilon_0 = \epsilon_{sp})\) and \((\epsilon_0 \to \infty)\), which corresponds to the special transition and is called the special point. The couplings \(\epsilon_0\) and \(\epsilon_{sp}\) are defined in Eq.\(^\text{}\) below.

Theory of critical behavior of individual surface universality classes is very well developed for pure isotropic systems.\(^6\) The systems with quenched surface-enhancement disorder\(^6\) and the systems with a random quenched bulk disorder at the ordinary and the special surface transitions\(^7\). General irrelevance-relevance criteria of the Harris type for the systems with quenched short-range correlated surface-bond disorder were predicted\(^8\) and confirmed by Monte-Carlo calculations.\(^9\) Moreover, it was established that the surface critical behavior of semi-infinite systems with quenched bulk disorder is characterized by the new set of surface critical exponents in comparison with the case of pure systems.\(^10\)

Experimental systems are typically characterized by the parameters different from the fixed point values. Investigations of the crossover behavior between different surface universality classes, however, have been limited to pure, isotropic systems, in which quite rich and nontrivial crossover behavior has been found.\(^11\) In the anisotropic systems the crossover behavior can be even more complex, because when the rotational symmetry is bro-
ken, the orientation of the surface may affect the surface critical phenomena, and this effect can be different for different surface universality classes.

In the present work we study the critical behavior at the special surface transition and crossover behavior from the special to the ordinary surface transitions, occuring in semi-infinite n-component model with cubic anisotropy.

The effective Landau-Ginzburg-Wilson Hamiltonian of the model under consideration in the semi-infinite space is given by

\[
H(\vec{\phi}) = \int_0^\infty dz \int d^{d-1}r \left[ \frac{1}{2} |\nabla \vec{\phi}|^2 + \frac{1}{2} m_0^2 |\vec{\phi}|^2 + \frac{1}{4!} v_0 \sum_{i=1}^n \phi_i^4 + \frac{1}{4!} u_0 (|\vec{\phi}|^2)^2 \right] + \int d^{d-1}r \frac{1}{2} c_0 \vec{\phi} \cdot \vec{\phi},
\]

(1.1)

where \( \vec{\phi}(x) \) is an n-vector field with the components \( \phi_i(x), i = 1,\ldots,n \). Here \( m_0^2 \) is the "bare mass", representing a linear measure of the temperature difference from the critical point value. The parameters \( u_0 \) and \( v_0 \) are the usual "bare" coupling constants \( u_0 > 0 \) and \( v_0 > 0 \). In the case of replica limit \( n \to 0 \), the Hamiltonian (1.1) describes the critical behavior of semi-infinite Ising-like systems with a random quenched bulk disorder, when \( u_0 < 0 \) and \( v_0 > 0 \) (see 17-22). The constant \( c_0 \) is related to the surface enhancement, which measures the enhancement of the interactions at the surfaces. It should be mentioned that the \( d \)-dimensional spatial integration is extended over a half-space \( \mathbb{R}^d \equiv \{ x=(r,z) \in \mathbb{R}^d \mid r \in \mathbb{R}^{d-1}, z \geq 0 \} \), bounded by a planar free surface at \( z = 0 \). The fields \( \phi_i(r,z) \) satisfy the Dirichlet boundary condition in the case of ordinary transition: \( \phi_i(r,z) = 0 \) at \( z = 0 \) and the Neumann boundary condition in the case of special transition: \( \partial_z \phi_i(r,z) = 0 \) at \( z = 0 \). The model defined in (1.1) is translationally invariant in directions parallel to the external surface, \( z = 0 \). Thus, we shall use mixed representation, i.e. Fourier representation in \( d - 1 \) dimensions and real-space representation in \( z \) direction.

The added cubic term breaks the \( O(n) \) invariance of the model, leaving a discrete cubic symmetry. The model (1.1) has four fixed points: the trivial Gaussian, the Ising one in which \( n \) components are decoupled, the isotropic \((O(n))\)-symmetric and the cubic fixed points. The Gaussian and Ising fixed points are never stable for any number of components \( n \). For isotropic systems, the \( O(n) \)-symmetric fixed point is stable for \( n < n_c \), whereas for \( n > n_c \) it becomes unstable. Here \( n_c \) is the marginal spin dimensionality of the cubic model, at which the isotropic and cubic fixed points change stability, i.e. for \( n > n_c \), the cubic fixed point becomes stable. The \( O(n) \)-symmetric fixed point is tricritical. At \( n = n_c \), the two fixed points should coincide, and logarithmic corrections to the \( O(n) \)-symmetric critical exponents are present. The calculation of the critical marginal spin dimensionality \( n_c \) is the crucial point in studying the critical behavior in three-dimensional cubic crystals. Different results for \( n_c \) have been published in a series of works in which different methods have been used. In the framework of the field-theoretical RG analysis the one-loop and three-loop approximations at \( \epsilon = 1 \) lead to the conclusion that \( n_c \) should lie between 3 and 4, and the cubic ferromagnets are described by the Heisenberg model. On the other hand, by using the field theoretic approach directly in \( d = 3 \) dimensions up to the three-loop approximation, it has been found that \( n_c = 2.9 \) (28). Similar conclusions were obtained in Ref. (29) where it was found that \( n_c = 2.3 \). The calculations performed by Newman and Riedel (30) with the help of the scaling-field method, developed by Goldner and Riedel for Wilson’s exact momentum-space RG equations, have given for \( d = 3 \) the value \( n_c = 3.4 \). Field-theoretical analysis, based on the four-loop series in three dimensions (28-32) and results of the five-loop \( \epsilon = 4 - d \) expansion (33) suggest that \( n_c \leq 3 \). Recently a very precise six-loop result for the marginal spin dimensionality of the cubic model, \( n_c = 2.89(4) \), was obtained in the framework of the 3D field-theoretic approach (34). Thus, it was finally established that the critical behavior of the cubic ferromagnets is not described by the isotropic Heisenberg Hamiltonian, but by the cubic model, at the cubic fixed point. However, it was found that the difference between the values of the bulk critical exponents at the cubic and the isotropic fixed points is very small, and it is hard to measure this difference experimentally. The recently obtained results stimulate us to perform the investigation of the surface critical behavior of semi-infinite n-component anisotropic cubic model, and to determine corresponding surface critical exponents.

The calculations are performed by applying the field theoretic approach directly in \( d = 3 \) dimensions, up to the two-loop order approximation. The numerical estimates of the resulting two-loop series expansions for the critical exponents of the special surface transition, and for the surface crossover exponent \( \Phi \) from the special to the ordinary transition and surface critical exponents of the layer, \( \alpha_1 \), and the local specific heat, \( \alpha_{11} \), are computed by means of the Padé and Padé-Borel resummation techniques for the cases \( n = 3,4,8 \) and for the case of \( n \to \infty \), which corresponds to the Ising model with equilibrium magnetic impurities. The crossover behavior from the special to the ordinary transition for random semi-infinite Ising-like system, which is the nontrivial particular case of the cubic model in the limit \( n \to 0 \), is also investigated.
II. RENORMALIZATION

In order to investigate the critical behavior at the special surface transition in the semi-infinite $n$-component anisotropic cubic model and to calculate the crossover exponent $\Phi$, we should consider correlation functions with insertions of the surface operator $\phi^2_s$,

$$G^{\langle N,M,L_1 \rangle}(\{x_i\}, \{r_j\}, \{R_l\}) = \left( \prod_{i=1}^{N} \phi(x_i) \prod_{j=1}^{M} \phi_s(r_j) \prod_{l=1}^{L_1} \frac{1}{2} \phi^2_s(R_l) \right),$$

(2.1)

which involve $N$ fields $\phi(x_i)$ at distinct points $x_i$ ($1 \leq i \leq N$) off the surface, $M$ fields $\phi(r_j, z = 0) \equiv \phi_s(r_j)$ at different surface points with parallel coordinates $r_j$ ($1 \leq j \leq M$), and $L_1$ insertions of the surface operator $\frac{1}{2} \phi^2_s(R_l)$ ($1 \leq l \leq L_1$).

The corresponding parallel Fourier transform of the full free propagator takes the form

$$G(p, z, z') = \frac{1}{2 \kappa_0} \left[ e^{-\kappa_0 |z-z'|} - \frac{\kappa_0 - \kappa_0 e^{-\kappa_0 (z+z')}}{\kappa_0 + \kappa_0} \right],$$

(2.2)

with the standard notation $\kappa_0 = \sqrt{p^2 + m_0^2}$. Here, $p$ is the value of the parallel “momentum”, i.e. the wave-vector, associated with $d-1$ translationally invariant directions in the system.

In the theory of semi-infinite systems the bulk field $\phi(x)$ and the surface field $\phi_s(r)$ should be reparameterized by different $uv$-finite renormalization factors $\zeta_{\phi}$, $Z_0(u,v)$ and $Z_{I}^{sp}(u,v)$. Thus, we have $\phi = Z_{1/2}^{\phi} \phi_R$ and $\phi_s = Z_{1/2}^{\phi^2} (Z_{I}^{sp})^{1/2} \phi_{s,R}$, where $\phi_R$ and $\phi_{s,R}$ are the renormalized bulk and surface fields, respectively. Besides, introduction of the additional surface operator insertions $\frac{1}{2} \phi^2_s(R_l)$ requires additional specific renormalization factor $Z_{\phi^2_s}$,

$$\phi^2_s = [Z_{\phi^2_s}]^{-1} \phi^2_{s,R}.$$  

The corresponding renormalized correlation functions involving $N$ bulk fields, $M$ surface fields and $L_1$ surface operators $\frac{1}{2} \phi^2_s(R_l)$ can be written as

$$G^{\langle N,M,L_1 \rangle}_{R}(0; m, u, v, c) = Z^{-\langle N+M \rangle/2} Z_{1}^{sp} Z_{L_1}^{sp} G^{\langle N,M,L_1 \rangle}(0; m_0, u_0, v_0, c_0).$$

(2.3)

In the present paper we concentrate our attention on the correlation function $G^{\langle 0,2,1 \rangle}(0; m, u, v, c)$ involving two surface fields and a single surface operator insertion $\frac{1}{2} \phi^2_s(R_1)$.

The $uv$-singularities of the correlation function $G^{\langle N,M,L_1 \rangle}(0; m, u, v, c)$ can be absorbed through a mass shift $m_0^2 = m^2 + \delta m^2$ and a surface-enhancement shift $c_0 = c + \delta c$ (see). The renormalizations of the mass $m$, coupling constants $u, v$ and the renormalization factor $Z_{\phi}$ are defined by standard normalization conditions of the infinite-volume theory. In order to absorb $uv$-singularities located in the vicinity of the surface, a surface-enhancement shift $\delta c$ is required (see Appendix 1). From Eq. (A1.3) and the expression for the renormalized correlation function (2.3) it follows that

$$[Z_{\phi^2_s}]^{-1} = Z_{\parallel} \frac{\partial [G^{\langle 0,2 \rangle}(0; m_0, u_0, v_0, c_0)]^{-1}}{\partial c_0} \bigg|_{c_0 = c_0(m, u, v)}. \quad (2.4)$$

It should be mentioned that the renormalization factor $Z_{\parallel} = Z_{I}^{sp} Z_{\phi}$ is defined via the standard normalization condition (A1.2) (see).

$$Z_{\parallel} = 2 m \frac{\partial}{\partial p^2} [G^{\langle 0,2 \rangle}(p)]^{-1} \bigg|_{p^2 = 0} = \lim_{p \to 0} \frac{m}{p} \frac{\partial}{\partial p} [G^{\langle 0,2 \rangle}(p)]^{-1}. \quad (2.5)$$

It should be mentioned that all the $Z$ factors in the $d < 4$ case have finite limits at $\Lambda \to \infty$ (where $\Lambda$ is the large-momentum cutoff) and depend on the dimensionless variables $u$ and $v$. Besides, the surface renormalization factors $Z_{I}^{sp}$ and $Z_{\phi^2_s}$ depend on both $u$, $v$ and the dimensionless ratio $c/m$. The dependence on the ratio $c/m$ plays a crucial role in the investigation of the crossover behavior from the special surface transition ($c/m \to 0$) to the ordinary surface transition ($c/m \to \infty$).
III. EXPANSION OF THE CORRELATION FUNCTIONS NEAR THE MULTICRITICAL POINT

As was indicated before, the main goal of the present work is to investigate the critical behavior at the special surface transition and to perform the analysis of the scaling critical behavior between the special and the ordinary transition. In this connection let us consider the small deviation $\Delta c_0 = c_0 - c_{sp}^*$ from the multicritical point. The power expansion of the bare correlation functions $G^{(N,M)}(0; m_0, u_0, v_0, c_0)$ in terms of this small deviation $\Delta c_0$ has the form

$$G^{(N,M)}(0; m_0, u_0, v_0, c_0) = \sum_{L_1=0}^{\infty} \frac{(\Delta c_0)^{L_1}}{L_1!} G^{(N,M,L_1)}(0; m_0, u_0, v_0, c_{sp}^*). \quad (3.1)$$

Reexpressing the right-hand side of Eq. (3.1) according to the Eq. (2.3) in terms of the renormalized correlation functions and renormalized variable $\Delta c = [Z\phi^2(u,v)]^{-1}\Delta c_0$, we obtain

$$Z_\phi^{-(N+M)/2}(Z_1^{sp})^{-M/2} G^{(N,M)}(0; m_0, u_0, v_0, c_0) = \sum_{L_1=0}^{\infty} \frac{(\Delta c)^{L_1}}{L_1!} G^{(N,M,L_1)}(0; m, u, v). \quad (3.2)$$

The above equation in a straightforward fashion defines the corresponding renormalized correlation functions, defined in the vicinity of the multicritical point

$$G_R^{(N,M)}(0; m, u, v, \Delta c) = Z_\phi^{-(N+M)/2}(Z_1^{sp})^{-M/2} G^{(N,M)}(0; m_0, u_0, v_0, c_0). \quad (3.3)$$

For dimensional reasons, we can introduce the dimensionless variable $\tilde{c} = \Delta c/m$. Thus, the above correlation functions $G_R^{(N,M)}(0; m, u, v, \tilde{c})$ satisfy the corresponding Callan-Symanzik equations [45,46,47]

$$\left[m \frac{\partial}{\partial m} + \beta_u(u,v) \frac{\partial}{\partial u} + \beta_v(u,v) \frac{\partial}{\partial v} + \frac{N + M}{2} \eta_\phi(u,v) \right. \left. + \frac{M}{2} \eta^{sp}(u,v) - [1 + \eta_c(u,v)] \tilde{c} \frac{\partial}{\partial \tilde{c}} \right] G_R^{(N,M)}(0; m, u, v, \tilde{c}) = \Delta G_R, \quad (3.4)$$

where the inhomogeneous part $\Delta G_R$ should be negligible in the critical region, similarly as this took place in the case of infinite-system field theory. The functions $\beta_u(u,v)$, $\beta_v(u,v)$ and $\eta_\phi(u,v)$, appearing in (3.4), are the usual bulk RG functions. The resulting homogeneous equation differs from the standard bulk Callan-Symanzik equation in that fashion that it involves the additional surface-related function $\eta^{sp}$ and the term $-1 + \eta_c(u,v)\tilde{c}$, where

$$\eta^{sp}_1(u,v) = m \frac{\partial}{\partial m} \ln Z_1^{sp}(u,v) = \beta_u(u,v) \frac{\partial \ln Z_1^{sp}(u,v)}{\partial u} + \beta_v(u,v) \frac{\partial \ln Z_1^{sp}(u,v)}{\partial v} \quad (3.5)$$

and

$$\eta_c(u,v) = m \frac{\partial}{\partial m} \ln Z^{sp}(u,v) = \beta_u(u,v) \frac{\partial \ln Z^{sp}(u,v)}{\partial u} + \beta_v(u,v) \frac{\partial \ln Z^{sp}(u,v)}{\partial v}. \quad (3.6)$$

In the case $\Delta c = 0$ we obtain the analog of the Callan-Symanzik equation for the correlation functions $G_R^{(N,M)}(0; m, u, v, \tilde{c})$ at the special point $c_0 = c_{sp}^*$. The symbol ‘FP’ indicates that the above value should be calculated at the corresponding infrared-stable fixed point (FP) of the underlying bulk theory.

IV. ANALYSIS OF THE SCALING CRITICAL BEHAVIOR

The asymptotic scaling critical behavior of the correlation functions can be obtained through detailed analysis of the CS equation (3.4), as was proposed in [31] and employed to the case of the semi-infinite systems in [34,35]. In the critical region, at $\tau = (T - T_c)/T_c \to 0$, for the renormalization Z factors we obtain

$$Z_\phi \sim m^{\eta_\phi(u,v,v^*)},$$

$$Z_1^{sp} \sim m^{\eta^{sp}_1(u,v,v^*)},$$

$$Z^{sp}_2 \sim m^{\eta_c(u,v,v^*)}, \quad (4.1)$$
where the variable \( m \) is identified with the inverse bulk correlation length \( \xi^{-1} \sim \tau^{\nu} \), as it is usually accepted in the massive field theory.

Substituting the last equation from (4.1) into the expressions for \( \Delta c \) and for the scaling variable \( \bar{c} \) it is easy to obtain the following asymptotic forms

\[
\Delta c \sim m^{-\eta_c(u^*, v^*)} \Delta c_0, \quad \Delta c \sim \tau^{-\nu\eta_c(u^*, v^*)} \Delta c_0
\]

and

\[
\bar{c} \sim m^{-(1+\eta_c(u^*, v^*))} \Delta c_0, \quad \bar{c} \sim \tau^{-\Phi} \Delta c_0,
\]

where

\[
\Phi = \nu(1 + \eta_c(u^*, v^*))
\]

is the surface crossover critical exponent. The second equation in (4.3) explains the physical meaning of the surface crossover exponent as a value characterising the measure of deviation from the multicritical point. The second equations in (4.2) and (4.3) indicate non-analytic temperature dependence of the renormalized surface-enhancement deviation \( \Delta c \). Thus, from the analysis of the CS equation (3.3) we obtain the following asymptotic scaling form of the surface correlation function \( G(0, 2) \):

\[
G^{(0, 2)}(p; m_0, u_0, v_0, c_0) \sim m^{-\frac{\gamma_{11}^{sp}}{2}} G_R^{(0, 2)}(\frac{p}{m}; 1, u^*, v^*, m^{-\Phi/\nu} \Delta c_0)
\]

\[
\sim \tau^{-\gamma_{11}^{sp}} G(p^{1-\nu}; 1, \tau^{-\Phi} \Delta c_0),
\]

where \( \gamma_{11}^{sp} = \nu(1 - \eta_{||}) \), is the local surface susceptibility exponent and

\[
\eta_{||}^{sp} = \eta_{11}^{sp} + \eta_{\phi}
\]

is the surface correlation exponent at the special surface transition. It is easy to see that the asymptotic scaling critical behavior of the surface correlation function for the semi-infinite \( n \)-component systems with cubic anisotropy is characterized by the new crossover exponent \( \Phi(u^*, v^*) \), calculated at the cubic fixed point \( (u^*, v^*) \).

V. THE FIXED-DIMENSION PERTURBATIVE EXPANSION UP TO TWO-LOOPS.

As was investigated previously\[15,16\], the fixed-dimension \( \phi^4 \) field-theoretic approach\[17\] provides an accurate description of the surface critical behavior of semi-infinite systems. We apply this method to the analysis of the cubic anisotropic model (1.1) at the special surface transition and to the investigation of the crossover behavior in the vicinity of the multicritical point \( c_0 = c_{sp}^* \). The surface correlation exponent \( \eta_{||}^{sp} \) at the special transition can be obtained from Eq. (4.6), where \( \eta_{11}^{sp} \) is defined by Eq. (3.5), and \( \eta_{\phi}(u, v) \) is the standart bulk exponent \( \eta_{\phi} = m \frac{\partial}{\partial m} \ln Z_p \bigg|_{\nu, \nu'} \). After performance of the mass- and surface-enhancement renormalization and carrying out the integration of the corresponding Feynman integrals with subsequent execution of the standart vertex renormalizations of bare dimensionless parameters \( \bar{u}_0 = \bar{u}(1 + \frac{n+8}{6} \bar{u} + \bar{v}) \), \( \bar{v}_0 = \bar{v}(1 + \frac{4}{3} \bar{v} + 2 \bar{u}) \) (where \( \bar{u}_0 = u_0/8\pi m \) and \( \bar{v}_0 = v_0/8\pi m \)), by analogy to that as it was done in \[15\] for random semi-infinite model, we obtain in the case of the cubic anisotropic model the following expression for the surface correlation exponent at the special transition

\[
\eta_{||}^{sp}(u, v) = -\frac{n+2}{2(n+8)} \left( u - \frac{v}{6} \right) + \frac{(n+2)(n+8)}{(n+8)^2} A(n) u^2 + \frac{4}{9} A(1) v^2 + \frac{8}{n+8} A(n)uv.
\]

Here \( A(n) \) is defined as

\[
A(n) = 2A + \frac{n-10}{48} \left( \frac{n+2}{6} (ln^22 - ln2) \right),
\]

and the renormalized coupling constants \( u \) and \( v \), are normalized in the standard fashion, \( u = \frac{n+8}{6} \bar{u} \) and \( v = \frac{3}{2} \bar{v} \).
According to the Eq.(1.4) and Eq.(3.6), the calculation of the crossover critical exponent $\Phi$ is connected with the calculation of the renormalization factor $Z_{\phi^2}$ via (2.4). With that end in view we can rewrite the normalization condition (A.1.1) in the form

$$Z_{\parallel}[G^{(0,2)}(0; m_0, u_0, v_0, c_0)]^{-1} = m + c$$

(5.3)

for the inverse unrenormalized surface correlation function $[G^{(0,2)}(0)]^{-1}$. It is easy to see that by differentiation of the above mentioned normalization condition with respect to $c_0$ and by taking into account Eq.(2.4) we obtain for the renormalization factor $Z_{\phi^2}$ the equation

$$Z_{\phi^2} = \frac{\partial c_0}{\partial c},$$

(5.4)

where $c_0 = c + \delta c$ and

$$\delta c = (Z_{\parallel}^{-1} - 1)(m + c) + \sigma_0(0; m, c_0 = c + \delta c).$$

(5.5)

Here $\sigma_0(0; m, c_0) = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ denotes the sum of loop diagrams of all orders in $[G^{(0,2)}(0; m, u_0, v_0, c_0)]^{-1}$ (see [3.3.4]). Among them $\sigma_1$ corresponds to the one-loop graph, $\sigma_2$ denotes the melon-like two-loop diagrams

$$\sigma_2 = \sigma_2^{(+)} + \sigma_2^{(-)} + \frac{m^2}{2\kappa} \frac{\partial}{\partial k^2} \kappa \mid_{k^2 = 0},$$

(5.6)

$\sigma_3$ and $\sigma_4$ represent the one-particle reducible and one-particle irreducible two-loop diagrams in $[G^{(0,2)}(0; m, u_0, v_0, c_0)]^{-1}$, respectively. These graphs have their corresponding weights (arise from the standard symmetry factors)

$$-\frac{t_1^{(1)}}{2} \quad \text{with} \quad t_1^{(1)} = \frac{n + 2}{3} u_0 + v_0,$$

(5.7a)

$$-\frac{t_2^{(1)}}{6} \quad \text{with} \quad t_2^{(1)} = \frac{n + 2}{3} u_0^2 + v_0^2 + 2v_0 u_0,$$

(5.7b)

$$-\frac{t_3^{(1)}}{4} \quad \text{and} \quad t_4^{(1)} \quad \text{with} \quad t_3^{(1)} = t_4^{(1)} = (t_1^{(1)})^2,$$

(5.7c)

The full lines with notes "G" denote the full free propagator (2.2) and signs "-" denote the free bulk propagators, which are associated with the first term in the full free propagator (2.2). The equation (5.4) can be resolved by using the method of sequential iteration. As the result, at the second order of the perturbation theory, we obtain the renormalization factor $Z_{\phi^2}$ in terms of the new renormalized coupling constants $\bar{u}$ and $\bar{v}$,

$$Z_{\phi^2}(\bar{u}, \bar{v}) = 1 + \frac{n + 2}{3} (\ln 2 - \frac{1}{4}) \bar{u} + (\ln 2 - \frac{1}{4}) \bar{v} + \frac{n + 2}{3} C(n) \bar{u}^2 + 2C(n) \bar{u} \bar{v} + C(1) \bar{v}^2,$$

(5.8)

where $C(n)$ is a function defined as

$$C(n) = A - B - \frac{n}{2} \ln 2 + \frac{n + 2}{2} \ln^2 2 + \frac{2n + 1}{12},$$

(5.9)

and $A = 0.202428$, $B = 0.678061$ are integrals originating from the two-loop melon-like diagrams. Combining the renormalization factor $Z_{\phi^2}$ with the one-loop pieces of the $\beta$ functions $\beta_\bar{u}(\bar{u}, \bar{v}) = -\bar{u}(1 - [(n + 8)/6] \bar{u} - \bar{v})$ and $\beta_\bar{v}(\bar{u}, \bar{v}) = -\bar{v}(1 - \frac{2}{3} \bar{v} - 2\bar{u})$, according to Eq.(3.6), we obtain the desired series expansion for $\eta_c$,

$$\eta_c(u, v) = -2 \frac{n + 2}{n + 8} (\ln 2 - \frac{1}{4}) u - \frac{2}{3} (\ln 2 - \frac{1}{4}) v - 8][(n + 2)^2 D(n) u^2 + \frac{2D(n)}{n + 8} u v + \frac{D(1)}{9} v^2],$$

(5.10)

where

$$D(n) = A - B + \frac{n + 2}{3} \ln^2 2 - \frac{n + 1}{2} \ln 2 + \frac{17n + 22}{96}.$$
The knowledge of \( \eta_c \) gives an access to the calculation of the crossover critical exponent \( \Phi \) via the scaling relation \( \left[ \text{11} \right] \). Additionally, we can calculate the critical exponents \( \alpha_1 \) and \( \alpha_{11} \) of the layer and specific heats via the usual scaling relations \( \left[ \text{1} \right] \):

\[
\alpha_1 = \alpha + \nu - 1 + \Phi = 1 - \nu(d - 2 - \eta_c), \quad \alpha_{11} = \alpha + \nu - 2 + 2\Phi = -\nu(d - 3 - 2\eta_c). \tag{5.12}
\]

The above critical exponents should be calculated for different \( n \) (\( n = 3, 4, 8 \)) at the standard infrared-stable cubic fixed (FP) points of the underlying bulk theory, as it is usually accepted in the massive field theory. As it was mentioned above, in the cases \( n < n_c \) the cubic ferromagnets are described by the Heisenberg isotropic Hamiltonian at the \( O(n) \)-symmetric fixed point.

In the case of the replica limit \( n \to 0 \) we can obtain from \( \left[ \text{5.10} \right] \) the series of \( \eta_c \) and corresponding expressions for the surface crossover critical exponent \( \Phi^r \), and critical exponents \( \alpha_1^r \) and \( \alpha_{11}^r \) of the layer and specific heats for semi-infinite random Ising-like systems \( \left[ \text{3} \right] \).

### VI. NUMERICAL RESULTS

In order to obtain the full set of surface critical exponents at the special transition and to calculate the surface crossover exponent \( \Phi \) from the special to the ordinary transition in systems with cubic anisotropy we substitute the expansion \( \left[ \text{5.1} \right] \) for \( \eta_c^p \) into the standard scaling-law expressions for the surface exponents (see Appendix 2) and the expansion \( \left[ \text{5.10} \right] \) for \( \eta_c \) into the scaling relations \( \left[ \text{1.4} \right] \) and \( \left[ \text{5.12} \right] \).

For each of the above mentioned surface critical exponents of the special transition and for the crossover exponent \( \Phi \) we obtained \( d = 3 \) a double series expansion in powers of \( u \) and \( v \), truncated at the second order. As it is known \( \left[ \text{8} \right] \), power series expansions of this kind are generally divergent due to a nearly factorial growth of expansion coefficients at large orders of perturbation theory. In order to perform the analysis of these perturbative series expansions and to obtain accurate estimates of the surface critical exponents a powerful resummation procedure must be used. One of the simplest ways is to perform the double Padé-analysis \( \left[ \text{4} \right] \). This should work well when the series behave in lowest orders “in a convergent fashion”. Another way is to perform for these series the double Padé-Borel analysis \( \left[ \text{5} \right] \). The Padé-Borel resummation procedures are possible to use only in the case when the series are alternating in sign \( \left[ \text{5} \right] \).

The results of our calculations of the surface critical exponents of the special transition for various values of \( n = 3, 4, 8, \infty \) at corresponding cubic fixed points are presented in Tables 1-5. Unfortunately, the second \( (p = 2) \) order analysis of perturbative series \( \left[ \text{3} \right] \) gives the cubic fixed point with coordinates \( u_0 = 1.5347 \) and \( v_0 = -0.0674 \) at \( n = 3 \) for the 3D model. This corresponds to ordering along the easy axes, because \( (v_0 < 0) \). The analysis of the eigenvalues of the stability matrix shows that in the frames of the two-loop approximation the cubic fixed point at \( n = 3 \) is unstable and the \( O(n) \)-symmetric fixed point is stable. But, the estimations of the marginal spin dimensionality of the cubic model \( n_c \) in the frames of three-loop, four-loop, five-loop \( c = 4 - d \) expansion, and six-loop study \( \left[ \text{4} \right] \) show that the cubic ferromagnets are not described by the Heisenberg isotropic Hamiltonian, but by the cubic model at the stable cubic fixed point. Higher precision six-loop field-theoretical analysis \( \left[ \text{3} \right] \) give the value of the marginal spin dimensionality of the cubic model equal to \( n_c = 2.89(4) \). In accordance with this we used the cubic fixed point of the higher \( p = 3 \) order of perturbative series for obtaining the set of surface critical exponents at \( n = 3 \).

For estimation of the reliability of the obtained results we performed calculations at the cubic fixed point of the \( p = 6 \) order in Table 2. The obtained results indicate that the difference in the ways of the \( \beta \) functions resummation have no essential influence on the values of the surface critical exponents and that the results obtained in the frames of the two-loop approximation are stable and reliable. The surface critical exponents of the special transition for \( n = 4, 8 \) and \( n \to \infty \) were calculated at the standard infrared-stable cubic fixed (FP) points of the underlying bulk theory, as it is usually accepted in the massive field theory.

The Tables 6-9 represent the surface crossover exponent \( \Phi \) and the surface critical exponents of the layer \( \alpha_1 \) and local \( \alpha_{11} \) specific heats for different values of \( n = 3, 4, 8, \infty \) at the corresponding cubic fixed points.

The quantities \( O_1/O_2 \) and \( O_{11}/O_{21} \) represent the ratios of magnitudes of first-order and second-order perturbative corrections appearing in direct and inverse series expansions. The larger (absolute) value of these ratios indicate the better apparent convergence of truncated series.

The values \( [p/q] \) (where \( p, q = 0, 1 \)) are simply Padé approximants which represent the partial sums of the direct and inverse series expansions up to the first and the second order. The nearly diagonal two-variable rational approximants of the types \( [11/1] \) and \( [1/11] \) give at \( u = 0 \) or \( v = 0 \) the usual \( [1/1] \) Padé approximant \( \left[ \text{4} \right] \). The results of Padé-Borel analysis of the direct \( R \) and the inverse \( R^{-1} \) series expansions give numerical estimates of the surface critical exponents with a high degree of reliability. As it is easy to see from Tables 1-5, the most reliable estimate is obtained from the inverse series expansion for the surface critical exponent \( \eta_{1L} \), which represent the best convergence properties. Thus
we obtain for $\eta_\perp$: $-0.121$ at $n = 3$; $-0.139$ at $n = 4$; $-0.132$ at $n = 8$; $-0.141$ at $n \to \infty$. Substituting these values of $\eta_\perp$ together with the standard bulk values $\eta$ ($\nu = 0.69$ for $p = 0$; $0.706$ at $p = 0.2$) ($n = 3$); $0.711$ ($n = 4$); $0.717$ ($n = 8$); $0.715$ ($n \to \infty$) and $\eta$ ($\eta = 0.024$ for $p = 0$; $0.0333(26) = 0.033$) at $p = 0.2$ ($n = 3$); $0.026$ ($n = 4$); $0.025$ ($n = 8$); $0.025$ ($n \to \infty$) at $d = 3$ into the surface scaling relations (see Appendix 2) we have obtained the remaining critical exponents that are present in the last columns of Tables 1-5. The deviations of these estimates from the other estimates of the table give a rough measure of the achieved numerical accuracy.

The double Padé and Padé-Borel analysis of the series for $\eta_c, \alpha_1, \alpha_{11}$ and $\Phi$, presented in Tables 6-9, was performed in a similar way. In this case the situation is more complicated, because the series of these critical exponents exhibit bad convergence properties. A very similar situation took place in the analysis of perturbation expansions for the surface critical exponents in isotropic systems. As it is easy to see, we obtain the most reliable estimates for the surface critical exponent $\alpha_{11}$. The result of substituting of this value $\alpha_{11} = -0.331$ ($n = 3$); $-0.374$ ($n = 4$); $-0.384$ ($n = 8$); $-0.380$ ($n \to \infty$)), together with the value of $\nu$, into scaling laws (6.1) and (5.12) is presented in the last columns of Tables 6-9. For evaluation of the reliability of the obtained results we have performed additional calculation of the surface critical exponents on the basis of these $\alpha_{11}$ and the corresponding six-loop perturbation theory result for bulk critical exponent of the correlation length $\nu = 0.706(6)$ ($n = 3$); $0.714(8)$ ($n = 4$); $0.712(6)$ ($n = 8$); $0.708(8)$ ($n \to \infty$). The results of calculation are presented in Table 10. As it is easy to see, these results differ very little from the results presented in the last columns of Tables 6-9. It indicates good stability of the results obtained in the frames of the two-loop approximation scheme.

The obtained results for surface critical exponents of semi-infinite model with cubic anisotropy, calculated at the cubic fixed point, are different from the results for surface critical exponents of standard semi-infinite $n$-component model (see 4.3.3). For example, the difference in the case of $d = 3$ at $n = 3$ is about $7.5\%$ for $\eta_\parallel$ and $6.6\%$ for $\eta_\perp$.

If $n < n_c$, the cubic fixed point is unstable and the cubic term in the Hamiltonian (1.1) becomes irrelevant. In this case the isotropic fixed point is stable and the system is described by the simple $O(n)$ symmetric model in 3D. The corresponding surface critical exponents can be calculated from the series, presented in 4.3.3.

As was indicated previously, in the limit $n \to 0$, the cubic model (1.1) with $u_0 < 0$ and $v_0 > 0$ describes the semi-infinite Ising-like systems with random bulk disorder. The investigation of the surface special transition for such kind of systems was presented in 4.3.3. For the value of $\eta^c_c$, for the surface crossover exponent $\Phi^c$ and for the surface exponents of the layer $\alpha^c_1$ and local $\alpha^c_{11}$ specific heat, in the limit $n \to 0$ at the standard infrared-stable random fixed point $u^* = -0.60509, v^* = 2.39631$ of the underlying bulk theory we obtain:

$$
\begin{align*}
\eta^c_c &= -0.164, \\
\alpha^c_1 &= 0.211, \\
\alpha^c_{11} &= -0.222, \\
\Phi^c &= 0.567.
\end{align*}
$$

These values are different from the surface critical exponents of the pure semi-infinite Ising-like systems and indicate that in a system with random bulk disorder the planar boundary is characterized by a new set of the surface critical exponents.

VII. SUMMARY

We have studied special surface transition and crossover critical behavior in the vicinity of the multicritical point for 3D semi-infinite systems with cubic anisotropy by applying the field theoretic approach directly in $d = 3$ dimensions up to the two-loop approximation. We have performed a double Padé and Padé-Borel analysis of the resulting perturbation series for the surface critical exponents of the special transition, the surface crossover exponent $\Phi$ and the critical exponents of the layer $\alpha_1$ and local specific heats $\alpha_{11}$ for various $n = 3, 4, 8, \infty$, in order to find the best numerical estimates.

We find that at $n > n_c$ the surface critical exponents of the special transition in semi-infinite systems with cubic anisotropy belong to the cubic universality class and the asymptotic scaling critical behavior of the surface correlation function in the vicinity of the multicritical point is characterized by the new crossover exponent of the cubic universality class $\Phi(u^*, v^*)$.

Besides, we present the results of investigation of the crossover critical behavior between special and ordinary transition in random semi-infinite systems, by taking the limit $n \to 1$. In this case we calculate the surface critical exponents $\alpha_1^c$, $\alpha_{11}^c$ and the surface crossover exponent $\Phi^c$ and confirm that they belong to the universality class of the random model.

The further theoretical investigation of the asymptotic surface critical behavior of semi-infinite cubic systems will be highly desirable in the framework of higher-order (three-loop) RG approximations. In particular, it would be very interesting to study the case of the 3D cubic crystal with $n = 3$.

We suggest that the obtained results could stimulate further experimental and numerical investigations of the surface critical behavior of random systems and systems with cubic anisotropy.
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Appendix 1

We require that the following surface normalization conditions take place:

\[ G_R^{(0,2)}(0; m, u, v, c) = \frac{1}{m + c}, \]  \hspace{1cm} (A1.1)

\[ \frac{\partial G_R^{(0,2)}(p; m, u, v, c)}{\partial p^2} \bigg|_{p=0} = -\frac{1}{2m(m + c)^2}. \]  \hspace{1cm} (A1.2)

\[ G_R^{(0,2,1)}(0; m, u, v, c) = \frac{1}{(m + c)^2}, \]  \hspace{1cm} (A1.3)

where the correlation function \( G^{(0,2,1)} \) contains the insertion of the surface operator \( \frac{1}{2}\phi_s^2 \).

The Eq. (A1.3) is motivated by the fact that the bare correlation function \( G^{(0,2,1)}(0; m_0^2, u_0, v_0, c_0) \) may be written as a derivative \( -\frac{\partial}{\partial c_0} G^{(0,2)}(0; m_0^2, u_0, v_0, c_0) \). This equation simplifies considerably the calculation of the correlation function with insertions of the surface operator \( \frac{1}{2}\phi_s^2 \).

It is easy to see from (A1.1) that the special point is located at \( m = c = 0 \), because at this point the divergence of the bulk and the surface correlation length and susceptibility is observed. At \( c = 0 \) the surface normalization conditions are simplified and yield \( c_0 = c_{sp}^* \). This point corresponds to the multicritical point \( (m_0^2, c_{sp}^*) \) at which special transition takes place. On the other hand, the above mentioned equation implies also that the surface correlation length and the susceptibility are finite at the ordinary transition, because in this case we have \( c > 0 \) when \( m \rightarrow 0 \). This latter case corresponds to the situation when the surface remains "noncritical" at the bulk transition temperature.

Appendix 2

The individual RG series expansions for other critical exponents can be derived through standard surface scaling relations with \( d = 3 \)

\[ \eta_\perp = \frac{\eta + \eta_\parallel}{2}, \]
\[ \beta_1 = \frac{\nu}{2}(d - 2 + \eta_\parallel), \]
\[ \gamma_{11} = \nu(1 - \eta_\parallel), \]
\[ \gamma_1 = \nu(2 - \eta_\perp), \]
\[ \Delta_1 = \nu\left(\frac{d}{2} - \eta_\parallel\right), \]
\[ \delta_1 = \frac{\Delta_1}{\beta_1} = \frac{d + 2 - \eta_\parallel}{d - 2 + \eta_\parallel}, \]
\[ \delta_{11} = \frac{\Delta_{11}}{\beta_1} = \frac{d - \eta_\parallel}{d - 2 + \eta_\parallel}. \]  \hspace{1cm} (A1.1)

Each of these critical exponents characterizes certain properties of the cubic anisotropic system near the external surface, in the vicinity of the critical point. The values \( \nu, \eta, \) and \( \Delta = \nu(d + 2 - \eta)/2 \) are the standard bulk exponents.
| $\exp \frac{\partial}{\partial u}$ | $\exp \frac{\partial}{\partial \nu}$ | $\exp \frac{\partial}{\partial n}$ | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ | $R$ | $R^{-1}$ | $f(\eta_1, \nu, n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\eta_1$ | -4.0 | 14.7 | 0.00 | -0.319 | -0.240 | -0.237 | -0.252 | -0.254 | -0.253 | -0.258 | — | -0.275 |
| $\Delta_1$ | 10.8 | -4.4 | 0.75 | 1.066 | 1.212 | 1.095 | 1.075 | 1.098 | 1.099 | — | 1.106 | 1.156 |
| $\eta_1$ | -3.1 | -6.0 | 0.00 | -0.159 | -0.137 | -0.107 | -0.117 | -0.121 | -0.120 | -0.124 | -0.121 | — | -0.121 |
| $\beta_1$ | 0.0 | 0.0 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\gamma_11$ | 11.8 | -4.3 | 0.50 | 0.819 | 0.968 | 0.846 | 0.823 | 0.848 | 0.849 | — | 0.857 | 0.882 |
| $\gamma_1$ | 12.6 | -3.1 | 1.00 | 1.398 | 1.662 | 1.430 | 1.372 | 1.433 | 1.433 | — | 1.450 | 1.478 |
| $\delta_1$ | -3.8 | -1.7 | 5.00 | 6.593 | 7.339 | 6.236 | 5.763 | 6.262 | 6.266 | 6.287 | 6.359 | 6.779 |
| $\delta_11$ | -5.8 | -1.7 | 3.00 | 4.275 | 5.217 | 4.057 | 3.622 | 4.090 | 4.095 | 4.101 | 4.186 | 4.450 |

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TABLE IV: Surface critical exponents of the special transition for $d = 3$ up to two-loop order at the cubic fixed point (of order $p = 2$) $u^* = 0.525, v^* = 1.146$ at $n = 8$.

| $\exp \frac{\partial}{\partial n}$ | $\frac{\partial \ln Z}{\partial n}$ | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ | $R$ | $R^{-1}$ | $f(n, \nu, \eta)$ |
|-----------------|-----------------|---------|---------|--------|--------|--------|--------|--------|--------|--------|-----------------|
| $\eta_1$ | -5.2 | 6.2 | 0.0 | -0.355 | -0.262 | -0.286 | -0.292 | -0.304 | -0.292 | -0.308 | — | -0.311 |
| $\Delta_1$ | 6.5 | -5.0 | 0.75 | 1.105 | 1.300 | 1.160 | 1.146 | 1.164 | 1.178 | — | 1.187 |
| $\eta_2$ | -3.8 | -11.6 | 0.00 | -0.177 | -0.151 | -0.131 | -0.140 | -0.145 | -0.142 | -0.148 | -0.143 |
| $\beta_1$ | 0.0 | 0.0 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | — | — | — | 0.247 |
| $\gamma_1$ | 6.4 | -5.1 | 0.50 | 0.855 | 1.050 | 0.911 | 0.898 | 0.913 | — | 0.940 | 0.940 |
| $\delta_1$ | 7.1 | -3.3 | 1.00 | 1.444 | 1.798 | 1.506 | 1.447 | 1.500 | 1.532 | — | 1.552 |
| $\delta_{11}$ | -9.6 | -1.7 | 3.00 | 4.420 | 5.695 | 4.272 | 3.750 | 4.302 | 4.382 | — | 4.310 |

TABLE V: Surface critical exponents of the special transition for $d = 3$ up to two-loop order at the cubic fixed point (of order $p = 2$) $u^* = 0.201, v^* = 1.508$ at $n \to \infty$.

| $\exp \frac{\partial}{\partial n}$ | $\frac{\partial \ln Z}{\partial n}$ | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ | $R$ | $R^{-1}$ | $f(n, \nu, \eta)$ |
|-----------------|-----------------|---------|---------|--------|--------|--------|--------|--------|--------|--------|-----------------|
| $\eta_1$ | -5.0 | 6.5 | 0.00 | -0.352 | -0.260 | -0.282 | -0.289 | -0.301 | -0.288 | -0.306 | — | -0.307 |
| $\Delta_1$ | 6.7 | -4.9 | 0.75 | 1.102 | 1.293 | 1.154 | 1.140 | 1.157 | 1.173 | — | 1.182 |
| $\eta_2$ | -3.7 | -10.7 | 0.00 | -0.176 | -0.150 | -0.129 | -0.138 | -0.144 | -0.140 | -0.147 | -0.141 |
| $\beta_1$ | 0.0 | 0.0 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | — | — | — | 0.248 |
| $\gamma_1$ | 6.6 | -5.0 | 0.50 | 0.852 | 1.043 | 0.905 | 0.891 | 0.906 | 0.926 | — | 0.935 |
| $\gamma_i$ | 7.4 | -3.3 | 1.00 | 1.440 | 1.785 | 1.499 | 1.440 | 1.501 | 1.526 | — | 1.546 |
| $\delta_1$ | -4.7 | -1.8 | 5.00 | 6.759 | 7.714 | 7.204 | 5.906 | 6.492 | 6.564 | 6.515 | 6.669 |
| $\delta_{11}$ | -9.1 | -1.7 | 3.00 | 4.407 | 5.651 | 4.252 | 3.738 | 4.287 | 4.376 | — | 4.483 |

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TABLE VI: Surface critical exponents involving the RG function $\eta_\epsilon$ for the case $n = 3, d = 3$ at the cubic fixed point (of order $p = 3$) $u^* = 1.348, v^* = 0.074$.

| $\exp \frac{\partial}{\partial n}$ | $\frac{\partial \ln Z}{\partial n}$ | $[0/0]$ | $[1/0]$ | $[0/1]$ | $[2/0]$ | $[0/2]$ | $[11/1]$ | $[1/11]$ | $R$ | $R^{-1}$ | $f(\sigma_{i1}, \nu, \eta)$ |
|-----------------|-----------------|---------|---------|--------|--------|--------|--------|--------|--------|--------|-----------------|
| $\eta_\epsilon$ | -1.0 | -2.1 | 0.00 | -0.565 | -0.361 | 0.022 | -0.229 | -0.278 | -0.280 | -0.311 | -0.287 |
| $\alpha_1$ | 2.4 | 56.3 | 0.50 | 0.58 | 0.194 | 0.246 | 0.190 | 0.190 | 0.188 | 0.176 | — |
| $\alpha_{11}$ | -1.4 | -6.4 | 0.00 | -0.565 | -0.361 | -0.158 | -0.323 | -0.329 | -0.331 | -0.356 | -0.331 |
| $\Phi$ | -0.6 | -0.6 | 0.50 | 0.377 | 0.390 | 0.597 | 0.589 | 0.455 | 0.455 | 0.446 | 0.447 |

0.532
TABLE VII: Surface critical exponents involving the RG function $\eta_c$ for the case $n = 4, d = 3$ at the cubic fixed point (of order $p = 2$) $u^* = 1.064, v^* = 0.520$.

| Exponent | $\alpha_1$ | $\alpha_{11}$ | $\Phi$ |
|----------|------------|---------------|-------|
| $\eta_c$ | -1.0 -2.4  | 0.00 -0.625 -0.385 0.023 -0.269 -0.315 -0.329 -0.362 -0.323 | -0.263 |
| $\alpha_1$ | -2.6 9.7  | 0.50 0.011 0.172 0.200 0.150 0.150 0.137 0.125 | 0.102 |
| $\alpha_{11}$ | -1.5 -17.2 | 0.00 -0.625 -0.385 -0.198 -0.371 -0.373 -0.392 -0.417 -0.374 | -0.374 |
| $\Phi$ | -0.6 -0.6  | 0.50 0.364 0.380 0.602 0.591 0.443 0.442 0.434 0.436 | 0.524 |

TABLE VIII: Surface critical exponents involving the RG function $\eta_c$ for the case $n = 8, d = 3$ at the cubic fixed point (of order $p = 2$) $u^* = 0.525, v^* = 1.146$.

| Exponent | $\alpha_1$ | $\alpha_{11}$ | $\Phi$ |
|----------|------------|---------------|-------|
| $\eta_c$ | -1.0 -2.7  | 0.00 -0.629 -0.386 -0.001 -0.284 -0.328 -0.354 -0.384 -0.335 | -0.268 |
| $\alpha_1$ | -2.8 7.4  | 0.50 0.008 0.170 0.183 0.141 0.143 0.120 0.109 | 0.091 |
| $\alpha_{11}$ | -1.6 -72.9 | 0.00 -0.629 -0.386 -0.225 -0.383 -0.384 -0.419 -0.442 -0.384 | -0.384 |
| $\Phi$ | -0.6 -0.7  | 0.50 0.363 0.379 0.592 0.579 0.435 0.434 0.427 0.429 | 0.525 |

TABLE IX: Surface critical exponents involving the RG function $\eta_c$ for the case $n \to \infty, d = 3$ at the cubic fixed point (of order $p = 2$) $u^* = 0.201, v^* = 1.508$.

| Exponent | $\alpha_1$ | $\alpha_{11}$ | $\Phi$ |
|----------|------------|---------------|-------|
| $\eta_c$ | -1.0 -2.7  | 0.00 -0.624 -0.384 0.001 -0.279 -0.327 -0.356 -0.385 -0.333 | -0.266 |
| $\alpha_1$ | -2.8 8.0  | 0.50 0.012 0.172 0.189 0.146 0.148 0.123 0.111 | 0.095 |
| $\alpha_{11}$ | -1.5 -38.2 | 0.00 -0.624 -0.384 -0.218 -0.378 -0.380 -0.419 -0.441 -0.380 | -0.380 |
| $\Phi$ | -0.6 -0.7  | 0.50 0.364 0.380 0.593 0.580 0.434 0.433 0.426 0.429 | 0.525 |
TABLE X: Surface critical exponents calculated on the basis of $\alpha_{11}$ for $n=3, 4, 8$ and $n \to \infty$ (see Tables 6-9) and correspondent six-loop perturbation theory results for bulk critical exponent of the correlation length $\nu$.

| exp  | $n=3$ | $n=4$ | $n=8$ | $n \to \infty$ |
|------|-------|-------|-------|-----------------|
| $\alpha_{11}$ | -0.331 | -0.374 | -0.384 | -0.380 |
| $\eta_0$ | -0.234 | -0.262 | -0.270 | -0.268 |
| $\alpha_1$ | 0.129 | 0.099 | 0.096 | 0.102 |
| $\Phi$ | 0.541 | 0.527 | 0.520 | 0.518 |

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