Extremal Indices in the Series Scheme and their Applications*

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Abstract

We generalize the concept of extremal index of a stationary random sequence to the series scheme of identically distributed random variables with random series sizes tending to infinity in probability. We introduce new extremal indices through two definitions generalizing the basic properties of the classical extremal index. We prove some useful properties of the new extremal indices. We show how the behavior of aggregate activity maxima on random graphs (in information network models) and the behavior of maxima of random particle scores in branching processes (in biological population models) can be described in terms of the new extremal indices. We also obtain new results on models with copulas and threshold models. We show that the new indices can take different values for the same system, as well as values greater than one.

Keywords: extremal index, series scheme, random graph, information network, branching process, copula

1 Introduction

The extremal index of a stationary (in a narrow sense) random sequence \( \{\xi_n\} \) is defined as follows [1, Section 3.7].

**Definition A.** Let \( \xi_n, n \geq 1, \) have distribution \( F, \) and let \( M_n = \vee_{k=1}^n \xi_k. \) If for any \( \tau > 0 \) there exists a number sequence \( u_n(\tau) \) such that \( nF(u_n(\tau)) \to \tau \) and \( P(M_n \leq u_n(\tau)) \to e^{-\theta \tau}, \) then \( \theta \) is said to be the extremal index.

Any value of \( \theta \in [0, 1] \) is possible here.

Note that if we take maxima \( \hat{M}_n \) of a sequence of independent random variables with the same distribution \( F, \) then

\[
\lim_{n \to \infty} P(\hat{M}_n \leq u_n(\tau)) = e^{-\tau},
\]

which implies

\[
\lim_{n \to \infty} P(M_n \leq u_n(\tau)) = \left( \lim_{n \to \infty} P(\hat{M}_n \leq u_n(\tau)) \right)^\theta; \quad (1)
\]

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In what follows, by \( \vee \) we denote the maximum; by \( \wedge, \) the minimum; by an overbar above a distribution function, we denote its tail: \( \bar{F}(x) = 1 - F(x); \) by \( f^{-1}, \) the inverse function of \( f, \) and for a distribution function, the generalized inverse: \( F^{-1}(y) = \inf\{x : F(x) \geq y\}; \) by \( f(x)^n \) we denote the \( n \)th power of \( f(x). \)
i.e., limiting distribution functions for $M_n$ and $\hat{M}_n$ have a power-law dependence,

$$\lim_{n \to \infty} P(M_n \leq u_n(\tau)) = \lim_{n \to \infty} P(\hat{M}_n \leq u_n(\tau)), \quad \theta > 0;$$

(2)
i.e., $M_n$ asymptotically grows as the maximum of $[\theta n]$ independent random variables as $n \to \infty$ and

$$\lim_{n \to \infty} P(M_n \leq u_n(\tau)) \geq \lim_{n \to \infty} P(\hat{M}_n \leq u_n(\tau));$$

(3)
i.e., $M_n$ is stochastically not greater than the maximum of independent random variables (in the limit).

The interest to the extremal index is partly due to the fact that its existence preserves the extremal type of the limiting distribution of the maxima. Recall that if for some number sequences $a_n > 0$ and $b_n, n \geq 1$, and for a nondegenerate distribution $G$ there exists a limit

$$\lim_{n \to \infty} P(\hat{M}_n \leq a_n x + b_n) = G(x), \quad \forall x \in \mathbb{R},$$

then $G$ belongs to one of three extremal types, namely: $G_1(x) = G_1(ax + b)$ for some $a > 0$ and $b$, where $G_1(x) = \exp\{-e^{-x}\}$ (Gumbel type), $G_2(x) = \exp\{-x^{-\alpha}\}, x > 0, \alpha > 0$ (Fréchet type), and $G_3(x) = \exp\{-(-x)^\alpha\}, x \leq 0, \alpha > 0$ (Weibull type). Such distributions $G$ are referred to as max-stable or extreme value distributions. For any $s > 0$ there exist $a(s) > 0$ and $b(s)$ such that $G^s(x) = G(a(s)x + b(s))$. Thus, raising the limiting distribution function to the power $\theta > 0$, which arises because of property (1), preserves the extremal type.

One of the interpretations of the extremal index consists in the fact that passages over a high level in a sequence occur not one at a time but in batches (clusters) of average size $1/\theta$. In applications, this can mean natural disasters, failures in technical systems, data losses in information transmission, financial losses, etc. Clearly, if such events happen several times in succession, this is much more dangerous than single occurrences and must be taken into account in risk management.

For more details on this subject, see [1–4].

Since 1980s, active investigations in this field have been made in two main directions: finding the extremal index for various random sequences and constructing statistical estimators for the extremal index based on observations.

For a survey of results and references, one can see, e.g., [3 Section 8.1] and [4 Section 5.5]. Section 1.2 in the dissertation [5] was specially devoted to generalizations of the classical notion of the extremal index and its statistical estimation. In particular, the following definition can be given.

**Definition B.** Let $\xi_n, n \geq 1$, have distribution $F$, and let $M_n = \vee_{k=1}^n \xi_k$. If for each number sequence $u_n, n \geq 1$, such that

$$0 < \lim \inf_{n \to \infty} n\bar{F}(u_n) \leq \lim \sup_{n \to \infty} n\bar{F}(u_n) < \infty,$$

we have $P(M_n \leq u_n) - F(u_n)^{\theta n} \to 0, n \to \infty$, then $\theta$ is called the extremal index.

This definition allows to extend the notion of the extremal index to some stationary sequences of random variables with discrete distributions (for instance, geometric), and for continuous distributions it is equivalent to Definition A.
The extremal index of a random field in various models and applications. In the dissertation [9], new interesting results are obtained concerning extremal indices of sequences of the form

$$X_n = A_n X_{n-1} + B_n,$$

where $(A_n, B_n), n \geq 1,$ are independent random pairs taking values in $\mathbb{R}^2_+$. In some cases, extremal indices and distributions of cluster sizes are explicitly computed, and in the more general case, upper and lower bounds on the extremal index are obtained. Continuity of the extremal indices and distributions of cluster sizes are explicitly computed, and in the more general case, upper and lower bounds on the extremal index are obtained.

Indices of multivariate sequences with heavy tails are introduced and analyzed. A part of the obtained results is presented in [10, 11].

However, in practice, it is necessary to study maxima on more complex structures than the set of natural numbers. Difficulties related to this were discussed as far back as in [2, Sections 3.9 and 3.12]. For example, if we consider lifetimes of components of a compound system (reliability scheme), it is not clear how we can enumerate them so that the model of a stationary sequence can be used, nor whether this is possible in principle. A little simpler is the case of random fields.

The extremal index can naturally be extended from random sequences to random fields on lattices $\mathbb{N}^d$ [12]. Consider, for example, a random field $\{\xi_{n_1, n_2}\}$ in $\mathbb{N}^2$, and let $M_{n_1, n_2} = \vee_{k_1=1}^{n_1} \vee_{k_2=1}^{n_2} \xi_{k_1, k_2}$. If for each $\tau > 0$ there exists $u_{n_1, n_2}(\tau)$ such that $n_1 n_2 F(u_{n_1, n_2}(\tau)) \to \tau$ and $\mathbb{P}(M_{n_1, n_2} \leq u_{n_1, n_2}(\tau)) \to e^{-\theta \tau}$, then $\theta$ is called the extremal index. To the computation of the extremal index of a random field in $\mathbb{N}^2$, the paper [13] is devoted; in [14], the asymptotic location of the maximum of a random field with a certain extremal index was studied.

Papers [6, 7] were devoted to the analysis of extrema and passage over a high level related to telecommunication models, and in [8] distributions and dependence of extrema in network sampling processes were studied, including the extremal indices.

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Consider a collection of random variables $\xi_{n,m}, n \geq 1, m \geq 1,$ with distributions $F_n$ (here $n$ is the series number, and $m$ is the number of the random variable in a series) and also a sequence of integer random variables (series sizes) $\nu_n \overset{P}{\to} +\infty, n \to \infty,$ and let $M_n = \vee_{m=1}^{\nu_n} \xi_{n,m}.$

**Definition 1**

Let for each $s \in (0, 1)$ there exist a sequence $u_n(s)$ such that $\mathbb{E} F_n(u_n(s))^{\nu_n} \to s$ and $\mathbb{P}(M_n \leq u_n(s)) \to \psi(s), n \to \infty.$ The we call $\psi$ the extremal function. If $\psi(s) = s^\theta,$ we call $\theta$ the extremal index.

In the general case, we can define partial indices

$$\theta^+ = \sup_{s \in (0,1)} \log_s \psi(s), \quad \theta^- = \inf_{s \in (0,1)} \log_s \psi(s);$$

then $\theta^+ \geq \theta^-$ and $s^{\theta^+} \leq \psi(s) \leq s^{\theta^-}, s \in (0, 1).$

The essence of Definition [11] consists in comparing the limiting distributions of $M_n$ and of the maxima $M_n$ of $\nu_n$ independent random variables (the number $\nu_n$ being independent of them)

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2 As compared to Definition [14] we have made the change of variables $s = e^{-\tau}$ and accordingly redefined the functions $u_n, n \geq 1.$
under the same normalization given by the condition $\mathbb{P}(\hat{M}_n \leq u_n(s)) \to s, n \to \infty$. Thus, we generalize property (1).

It is clear that the indices, as above, take nonnegative values, but the upper boundary 1 is removed, at least for $\theta^+$, as will be shown below (Examples 5.3, 6.1, 6.2). This happens because inequality (3) can be violated. The maxima over series can grow asymptotically faster than the maxima of independent random variables taken in the same quantity, which corresponds to the case $\psi(s) < s, s \in (0, 1)$.

**Definition 2.** Let for each $s \in (0, 1)$ there exist a sequence $u_n(s)$ such that $\mathbf{E}F_n(u_n(s))^{\theta_n} \to s$ and $\mathbb{P}(M_n \leq u_n(s)) - \mathbf{E}F_n(u_n(s))^{\theta_n} \to 0, n \to \infty$; then $\theta$ is called the extremal index.

The essence of Definition 2 consists in choosing a value of $\theta$ such that the limiting distributions of $M_n$ and of the maxima of $[\theta \nu_n]$ independent random variables (the number $[\theta \nu_n]$ being independent of them) coincide under the same normalization as in Definition 1 (for $\theta > 0$). Thus, property (2) is generalized.

The existence of an extremal index in the sense of Definition 2 actually means that the extremal function from Definition 1 admits the representation

\[ \psi(s) = \lim_{n \to \infty} \mathbf{E}F_n(u_n(s))^{\theta \nu_n}. \]

A question arises of why we give two definitions; cannot we do with only one? Indeed, in many cases both definitions of the index are equivalent: they exist and are equal to each other (Section 3, Example 5.1). However, it also happens that both indices do not exist but in the sense of Definition 1 there exists an extremal function and partial indices (Examples 5.2, 5.3, 6.1, 6.2); it also happens that the index in the sense of Definition 2 exists while the index in the sense of Definition 1 does not exist and both the extremal function and partial indices again exist (Section 1); finally, there can be a surprising situation where both indices do exist but take different values (Example 5.4). Thus, these are indeed two different characteristics of the system which do not reduce to a single one.

Note that maxima in a series scheme were previously considered in [15] for random variables related by IT-copulas (individuated $t$-copulas), and conditions were derived under which the maxima asymptotically grow as in the case of independent variables, i.e., in our terms, $\theta = 1$.

To avoid ambiguity in terminology, below we speak about extremal indices of a system (of random variables), denoted by $\{\xi_{n,m}; \nu_n\}$.

In Section 2 we prove basic properties of the extremal indices; we present their applications to information network models in Section 3 to models of biological populations in Section 4 to models with copulas in Section 5 and to threshold models in Section 6.

### 2 Basic Properties of Extremal Indices

The extremal indices have the following properties.

**Property 1.** Let $\eta_n, \ n \geq 1$, be a stationary sequence with extremal index $\theta$ in the sense of Definition 1. Put $\xi_{n,m} = \eta_m, \ m \geq 1$, and consider an integer sequence $l_n \to +\infty$; then the system $\{\xi_{n,m}; l_n\}$ has extremal index $\theta$ in the sense of Definitions 1 and 2.

**Proof.** Denote by $u^0_n(\tau), \ n \geq 1$, the sequence that exists by Definition 1 and let $u_n(s) = u^0_n(-\ln s)$; then $F(u_n(s))^{\theta_n} \to s, \mathbb{P}(M_n \leq u_n(s)) \to s^\theta$, and $F(u_n(s))^{\theta \nu_n} \to s^\theta$, which gives the same extremal index according to both definitions.

\[ \square \]
Property 2. Let a system \( \{\xi_{n,m}; \nu_n\} \) have an extremal index in the sense of one of Definitions 1 or 2 (or an extremal function), and consider a sequence of functions \( g_n(x), n \geq 1 \), that are continuous and strictly increasing on the set of points of increase of \( F_n \). Put \( \xi_{n,m} = g_n(\xi_{n,m}) \); then the system \( \{\xi_{n,m}; \nu_n\} \) has the same extremal index (extremal function).

**Proof.** For the new system, \( \tilde{F}_n(x) = F_n(g_n^{-1}(x)) \). Let \( \tilde{u}_n(s) = g_n(\tilde{u}_n(s)) \); then \( \tilde{F}_n(\tilde{u}_n(s)) = F_n(\tilde{u}_n(s)) \) and \( P(M_n \leq \tilde{u}_n(s)) = P(M_n \leq u_n(s)) \), so all the limits (in Definitions 1 and 2) do not change.

**Property 3.** Let a system \( \{\xi_{n,m}; \nu_n\} \) have an extremal index in the sense of one of Definitions 1 or 2 and let there exist a sequence \( c_n \to +\infty \) such that \( \nu_n/c_n \to 1 \), \( n \to \infty \); then the system has the same extremal index in the sense of one of Definitions 1 or 2.

**Proof.** In this case \( E F_n(u_n(s))^{\nu_n} = E(F_n(u_n(s))^{\nu_n/c_n} \to s \in (0, 1) \) implies \( F_n(u_n(s))^{c_n} \to s \) and \( E F_n(u_n(s))^{\nu_n} \to s^r, r \geq 0, n \to \infty \). Thus, if \( \theta \) is the extremal index in the sense of Definition 1, then \( P(M_n \leq u_n(s)) \to s^\theta \) implies \( E F_n(u_n(s))^{\theta \nu_n} \to s^\theta \), and therefore \( \theta \) is the extremal index in the sense of Definition 2. Vice versa, if \( \theta \) is the extremal index in the sense of Definition 2, then \( E F_n(u_n(s))^{\theta \nu_n} \to s^\theta \) implies \( P(M_n \leq u_n(s)) \to s^\theta \), and therefore \( \theta \) is the extremal index in the sense of Definition 1.

**Property 4.** Consider a system \( \{\xi_{n,m}; \nu_n\} \) with extremal index \( \theta > 0 \) in the sense of Definition 2 such that:

(a) \( F_n \equiv F \);

(b) For some max-stable law \( G \) and functions \( a(r) > 0 \) and \( b(r), r > 0 \), we have

\[
F^r(a(r)x + b(r)) \to G(x), \quad r \to \infty;
\]

(c) There exists a sequence \( c_n \to +\infty \) such that \( \nu_n/c_n \to 0 \), \( n \to \infty \);

(d) In Definition 2 we may take \( u_n(s) = A_n H^{-1}(s) + B_n \), where \( A_n = a(c_n), B_n = b(c_n) \), and \( H(x) \) is a continuous distribution function.

Then \( H(x) = EG(x)^{\xi} \) and

\[
P(M_n \leq A_n x + B_n) \to H(ax + b), \quad n \to \infty,
\]

where \( a > 0 \) and \( b \) are determined by the identity \( G(x)^{\theta} = G(ax + b) \). Furthermore, the extremal function in the sense of Definition 1 is \( \psi(s) = H(aH^{-1}(s) + b) \).

**Proof.** By [1, Corollary 1.3.2], for any max-stable law there exist \( a > 0 \) and \( b \) such that \( G(x)^{\theta} = G(ax + b) \). Let \( x = H^{-1}(s) \). Since

\[
E F(A_n x + B_n)^{\nu_n} = E(F(a_l x + b_l)^{\nu_n/l_n})^{\nu_n/l_n} \to EG(x)^{\xi}, \quad n \to \infty,
\]

we have \( H(x) = EG(x)^{\xi} \). Then

\[
E F(A_n x + B_n)^{\theta \nu_n} \to EG(x)^{\theta \xi} = EG(ax + b)^{\xi} = H(ax + b), \quad n \to \infty.
\]

By Definition 2 this implies \( P(M_n \leq A_n x + B_n) \to H(ax + b), n \to \infty \), and by Definition 1 we obtain \( \psi(s) = H(aH^{-1}(s) + b) \).

\[3\]If \( \xi \) is degenerate and is equal to a constant \( c \) a.s., then, clearly, \( Eu^\xi = w^e \).
Property 5. Consider a system \( \{\xi_{n,m}; \nu_n\} \) such that:

(a) \( F_n \equiv F \);

(b) There exists a sequence \( c_n \to +\infty \) such that \( \nu_n/c_n \overset{d}{\to} \zeta > 0, n \to \infty \);

(c) For a continuous distribution \( G \) and coefficients \( A_n > 0 \) and \( B_n \), we have

\[
F(A_n x + B_n)^{c_n} \to G(x),
\]
\[
P(M_n \leq A_n x + B_n) \to E G(x)^{\theta}, \quad n \to \infty.
\]

Then \( \theta \) is the extremal index in the sense of Definition 2.

**Proof.** First of all, we have \( EF(A_n x + B_n)^{c_n} \to E G(x)^{\zeta} \). Denote \( H(x) = E G(x)^{\zeta} \); this is a continuous function running over all values in \((0, 1)\). Put \( x = H^{-1}(s), u_n(s) = A_n x + B_n \); then \( EF(u_n(s))^{c_n} \to s \) and \( EF(u_n(s))^{\theta c_n} \to E G(x)^{\theta \zeta} \). Since we also have \( P(M_n \leq u_n(s)) \to E G(x)^{\theta \zeta} \) by the condition, \( P(M_n \leq u_n(s)) - EF(u_n(s))^{\theta c_n} \to 0, n \to \infty \), so \( \theta \) is the extremal index in the sense of Definition 2.

\( \square \)

Property 6. Consider a system \( \{\xi_{n,m}; l_n\} \) with extremal index \( \theta \) in the sense of one of the definitions such that:

(a) \( F_n \equiv F \) is a continuous distribution;

(b) \( l_n, n \geq 1, \) is an integer sequence, \( l_n \to +\infty, l_n \sim n^\alpha L(n), n \to \infty, \alpha > 0, L(x) \) being a slowly varying function on \( \mathbb{R}_+ \).

Let \( \nu_n/l_n \overset{P}{\to} 1, n \to \infty \); then the system \( \{\xi_{n,m}; \nu_n\} \) has the same extremal index in the sense of both definitions.

**Proof.** By Property 3 each of the systems has equal extremal indices in the sense of both definitions. Denote by \( M_n \) the maxima for \( \{\xi_{n,m}; l_n\} \), and by \( \hat{M}_n \), for \( \{\xi_{n,m}; \nu_n\} \).

For any \( \rho > 0 \) we have \( l_{[n\rho]} \sim \rho^\alpha l_n, n \to \infty \). Therefore, \( \nu_n/l_n \overset{P}{\to} 1, n \to \infty, \) implies

\[
P(M_{n(1-\varepsilon)} \leq \hat{M}_n \leq M_{n(1+\varepsilon)}) \to 1, \quad n \to \infty,
\]

for any \( \varepsilon > 0 \). Since \( F(u_n(s))^{l_n} \to s \), we have

\[
u_n(s) = F^{-1}(1 + (1 + o(1))(\ln s)/l_n), \quad n \to \infty,
\]

and for any \( \rho > 0 \) by virtue of Definition 4 we have

\[
P(M_{[n\rho]} \leq u_n(s)) = P(M_{[n\rho]} \leq F^{-1}(1 + (1 + o(1))(\ln s)/l_n))
\]
\[
= P(M_{[n\rho]} \leq F^{-1}(1 + (1 + o(1))(\ln s^{1/\rho \alpha})/l_{[n\rho]})) \to s^{\theta/\rho \alpha}, \quad n \to \infty.
\]

Letting \( \rho = 1 \pm \varepsilon, \varepsilon > 0 \), we obtain

\[
s^{\theta/(1+\varepsilon)\alpha} \leq \liminf_{n \to \infty} P(\hat{M}_n \leq u_n(s)) \leq \limsup_{n \to \infty} P(\hat{M}_n \leq u_n(s)) \leq s^{\theta/(1-\varepsilon)\alpha}.
\]

Passing to the limit as \( \varepsilon \to 0 \), we obtain \( \lim_{n \to \infty} P(\hat{M}_n \leq u_n(s)) = s^{\theta} \); hence, \( \theta \) is the extremal index of \( \{\xi_{n,m}; \nu_n\} \) in the sense of Definition 1.

\( \square \)
Property 7. Consider a system \( \{ \xi_{n,m}; \nu_n \} \) with extremal index \( \theta \) in the sense of Definition 7 for which \( F_n \equiv F \) is a continuous distribution, \( \nu_n/n \to c > 0, n \to \infty \), and consider a random integer sequence \( n \) independent of \( \{ \xi_{n,m}; \nu_n \} \) and a sequence \( \mu_n \to +\infty \) such that \( \eta_n/\mu_n \to \zeta > 0, n \to \infty \). Put \( \tilde{\xi}_{n,m} = \xi_{n,m}, \tilde{\nu}_n = \nu_{n,m} \); then the system \( \{ \tilde{\xi}_{n,m}; \tilde{\nu}_n \} \) has extremal index \( \theta \) in the sense of Definition 2.

Proof. For the sequence \( u_n(s) \), from Definition 1 for the system \( \{ \xi_{n,m}; \nu_n \} \) we have the convergence \( EF(u_n(s))^{\nu_n} \to s \), whence \( F(u_n(s))^{\nu_n} \to s, n \to \infty \), so that

\[
u_n(s) = F^{-1}(1 + (1 + o(1))(\ln n)/(cn)), n \to \infty.\]

For any \( x \in (0, 1) \) we have

\[
EF(u_{[n\mu]}(x))^{\tilde{\nu}_n} = EF(u_{[n\mu]}(x))^{\nu_n} = EF(u_{[n\mu]}(x))^{(\nu_n/\eta_n)(\eta_n/\mu_n)} \to Ex^{\zeta}, n \to \infty.
\]

Denote \( H(x) = Ex^{\zeta}, x = H^{-1}(s) \), and \( \tilde{u}_n(s) = u_{[n\mu]}(x) \); then

\[
EF(\tilde{u}_n(s))^{\tilde{\nu}_n} \to s, \quad EF(\tilde{u}_n(s))^{\nu_n} \to Ex^{\theta\zeta}, n \to \infty.
\]

On the other hand, by Definition 1 for the system \( \{ \xi_{n,m}; \nu_n \} \) we obtain

\[
P(\tilde{M}_n \leq \tilde{u}_n) = P(\tilde{M}_{\nu_n} \leq u_{[n\mu]}(x)) = P(\tilde{M}_{\nu_n} \leq F^{-1}(1 + (1 + o(1))(\ln x)/(c\mu_n))) = P(\tilde{M}_{\nu_n} \leq F^{-1}(1 + (1 + o(1))(\ln x^{\eta_n/\mu_n})/(c\eta_n))) \to Ex^{\theta\zeta}, n \to \infty.
\]

Hence, \( P(\tilde{M}_n \leq \tilde{u}_n) - EF(\tilde{u}_n(s))^{\nu_n} \to 0, n \to \infty \), and \( \theta \) is the extremal index of \( \{ \tilde{\xi}_{n,m}; \tilde{\nu}_n \} \) in the sense of Definition 2.

Let us give some comments on the above-proved properties.

Property 1 means that the introduced indices are indeed generalizations of the classical extremal index (in the sense of Definition 3) and coincide with it if, as series, we take deterministically growing segments of a stationary sequence.

Property 2 means invariance of the extremal indices under continuous strictly increasing transforms of a series. This means, for instance, that in the case of continuous random variables they all can be reduced to the uniform distribution on \([0, 1]\) by a transform with \( g_n = F_n \). A similar property holds for the classical extremal index (in the sense of Definition 4) when we speak about a single continuous strictly increasing transform applied to all elements of the sequence.

Property 3 specifies a restriction on the randomness of series sizes under which both new indices are equivalent. The sizes must asymptotically grow equivalently to a nonrandom sequence.

Property 4 generalizes the well-known statement for the classical extremal index [1, Corollary 3.7.3]: the limiting distribution of maxima of a stationary sequence has the same extremal type as the limiting distribution of maxima of independent random variables with the same marginal distribution. In this case the limiting law need not be max-stable, but its type is preserved. It is max-stable only for a degenerate random variable \( \zeta \) (a constant), i.e., under the conditions of Property 3.
Property 5 allows one to interpret a parameter of the limiting distribution of maxima of dependent random variables as the extremal index in the sense of Definition 2.

Property 6 provides a sufficient condition on the growth rate of series sizes under which the indices for random and nonrandom variables coincide.

Property 7 shows that randomization over a randomly growing series number allows to pass from the extremal index in the sense of Definition 1 to the same index in the sense of Definition 2.

3 Applications to Information Network Models

In the papers [16–18] the author considered maxima of aggregate activity in information networks described by power-law random graphs (also referred to as Internet or Internet-type graphs in the Russian literature).

As examples of recent works of Russian authors on power-law graphs, we refer the reader to [19, 20] and a survey [21]. We also recommend a foreign electronic textbook [22].

Consider the following example of an information network model [17].

Let each network node have an individual random information activity (rate of information production). We assume that activities of the nodes are independent and identically distributed and that their distribution $A$ has a heavy (regularly varying) tail, i.e., $\bar{A}(x) \sim x^{-a}L(x)$, $x \to \infty$, $a > 0$, where $L(x)$ is a slowly varying function. Activities and degrees of vertices (nodes) are assumed to be independent, and this assumption is essential.

Consider the model of a directed random graph where edge directions correspond to directions of information transmission. Assume that we have $n$ vertices and there are independent nonnegative integer random variables $K_1, \ldots, K_n$ with the same distribution defined by the probabilities $p_k \sim ck^{-\beta}$, $k \to \infty$, $\beta > 2$. Let $D_i = \min\{K_i, n-1\}$. For the $i$th vertex, choose at random (equi.probably and independently of the choices for other vertices) $D_i$ different vertices among the others (except for the $i$th vertex) and draw edges from them to the $i$th vertex. The resulting graph can be regarded as a power-law graph in the sense that the number of incoming edges has asymptotically a power-law distribution. The aggregate activity at a node is in this case defined to be the sum of its own activity and activities of all nodes from which information is coming (its incoming neighbors).

We emphasize that the activity has no relation to the notions of “quality” or “weight” of a vertex, used in modern random graphs models. In our case a graph is formed by the above-described algorithm, and individual activities are independent complements to the graph. As regards social networks, it may happen that a user writes much but reads little (or is read by few), or, on the contrary, writes little but reads much (or is read by many). As for the aggregate activity, it may happen to be large because of few incoming neighbors (or even a single neighbor) with large individual activities, or, on the contrary, small but with a large number of incoming neighbors having low individual activities. As is well known, for heavy tails it is typical that large values of a sum are attained at the expense of one large (maximal) summand. This property is extended in this case to sums of randomly many summands as well.

Denote by $M_n$ the maximum of aggregate activities. Let $v(r)$ be a positive nondecreasing function such that $r\bar{A}(v(r)) \to 1$, $r \to \infty$. Note that $v(r)$ definitely exists and is regularly varying with exponent $1/a$ [23, Section 1.5].

Then, for $a < \beta - 2$ if $2 < \beta < 3$ and for $a < (\beta - 1)/2$ if $\beta \geq 3$, the Fréchet limit law holds: $\mathbb{P}(M_n/v(n) \leq x) = \exp\{-x^{-a}\}$, $x > 0$, $n \to \infty$. Note that this limit law is due to the fact that
the maximum of aggregate activities grows asymptotically equivalently (in probability) to the maximum of individual activities over the network, which is proved in [17, Theorem 1].

On the other hand, if the number of incoming neighbors is described by a random variable $K$ independent of the activity, the limiting distribution $F$ of the aggregate activity at each node has a tail

$$
\bar{F}(x) \sim (1 +EK)\bar{A}(x), \quad x \to \infty,
$$

under the condition (which is fulfilled in this case)

$$
EK^{1/(\alpha+\varepsilon)} < \infty, \quad \varepsilon > 0,
$$

according to the results of [24] on the distribution of a sum of randomly many independent random variables with a heavy tail. Therefore,

$$
F(xv(n))^n \to \exp\{-1 + EK\} x^{-\alpha}, \quad x > 0, \quad n \to \infty.
$$

Denoting $s = \exp\{-1 + EK\} x^{-\alpha} \in (0, 1)$, $u_n(s) = xv(n)$, we conclude that the system of individual activities has extremal index $\theta = 1/(1 + EK)$ in the sense of Definition 1 (and thus, also in the sense of Definition 2 by Property 3 since $\nu_n = n$).

The value $\theta \in (0, 1)$ for a sequence means that passages over a high level occur not one at a time but in batches (clusters) of average size $1/\theta$ [3, Section 8.1]. In our case we may also conjecture on forming of such clusters.

In respect to information networks, this may concern batches of nodes with high aggregate activities caused by high individual activity of a single node that is their common incoming neighbor.

Checking the presence of this effect in real-world networks, of course, requires experimental investigation, which is beyond the scope of this theoretical paper.

Also, the author must admit that the choice of a random graph model in [17] was determined not by its advantages in the description of real-world networks against other modern models (e.g., scale-free models) but by a relatively simple proof of the asymptotic equivalence of the growth of maxima of aggregate and individual activities with the use of methods of the author’s paper [25]. Note that merely knowing the power law for the number of incoming vertices is absolutely insufficient, and each random graph model should be analyzed individually. For example, the growth rate of the maximum vertex degree in the graph is of importance. If we enforcingly cut the vertex degrees at a growing (with the number of vertices) threshold, we can obtain a class of models with the same limiting distributions of vertex degrees but with different growth rates of the maximum degree, for which different constraints on $a$ will occur depending on $\beta$. Other nuances also play a role.

Recently, the author has obtained new results for simple models with weights [18]. Similar models were studied in [22, ch. 6] as generalized random graphs. It is assumed that vertices are assigned with independent weights $w_i$, $1 \leq i \leq n$, identically distributed as a nonnegative random variable $W$, $EW^{\beta} < \infty$, $\beta \geq 1$.

In Model 1 we assume $p_i = \varphi(w_i n^{-s/2})$, where $0 < s \leq 1$, and for $\varphi$ on $\mathbb{R}_+$ we have $0 \leq \varphi(x) \leq \min\{1, x\}$, $\varphi(x) \sim x$, $x \to 0$. For known values of $w_i$, $1 \leq i \leq n$, every pair of vertices $i$ and $j$ is joined by an edge with probability $p_ip_j$ independently of other pairs. The graph is assumed to be undirected; information is transmitted along an edge in both directions. In respect to social networks, weights may reflect sociability of users.

In Model 2, under the same assumptions on the weights, we assume $p_i = \varphi(w_i n^{-s})$, $0 < s \leq 1$. For known values of $w_i$, $1 \leq i \leq n$, the $i$th vertex is entered by an edge from any other
vertex with probability \( p_i \) independently of other edges. The graph is assumed to be directed; information is transmitted in the direction of an edge. In respect to social networks, weights may reflect inquisitiveness of users.

In both cases the author has proved asymptotic equivalence of the growth of maxima of aggregate and individual activities under certain restrictions on \( a \) depending on \( \beta \) and \( s \). For \( s = 1 \) there exist limiting distributions of the number of neighbors (incoming neighbors), and the obtained results can be interpreted as existence of extremal indices: \( \theta = 1/(1 + (\mathbf{E}W)^2) \) in Model 1 and \( \theta = 1/(1 + \mathbf{E}W) \) in Model 2 (in the sense of both definitions), similarly to the preceding example.

Application of the developed method to other—more complicated and popular—models of information networks is a subject of further study.

### 4 Application to Models of Biological Populations

As models of biological populations, branching processes are often used. Elements of a population are traditionally referred to as particles. Particles may possess some (quantitative) random scores.

In the case of living organisms, these may be size, weight, or other characteristics such as yield of milk in cows, egg production in hens, crop capacity in plants, sensitivity of organisms to harmful and dangerous factors, etc.

Propagation of computer viruses can also be described by branching processes. Polymorphic computer viruses can not only propagate but also change their codes (similarly to mutations in living organisms). As scores, one may consider certain characteristics of the virus code or its vital activity.

In [26, 27], maxima of independent random scores of particles in branching processes were studied. As applications, [26] considered man height, and in [27] horse racing was mentioned, with prize points as the score.

Let us give one more example. If there is a colony of harmful organisms with different individual sensitivity thresholds to some factor (poison, antibiotic, etc.), then one needs the maximum concentration to exterminate the whole colony, since otherwise it will survive and propagate again.

In a series of works, the author considered maxima of random scores of particles in supercritical branching processes without extinction (with finite mean and variance of the number of descendants). Thus, [28] considered continuous-time processes, and in [29, 30], discrete-time processes were addressed. However, scores of different particles were assumed to be independent. In [31] there was for the first time studied a model with dependence of particle scores in a generation caused by their common heredity.

First, there was considered the case where the scores have a standard normal distribution and the correlation coefficient for scores of a pair is majorized by \( r^k, r \in (0, 1) \), if these particles have the nearest common ancestor \( k \) generations back. It was shown that maxima over generations grow asymptotically in the same way as in the case of independent scores, which corresponds to \( \theta = 1 \).

Next, there was considered the case where the scores have a distribution with a regularly varying tail and the heredity is explicitly described by a linear autoregression process of the first order:

\[
\xi_{n,m} = a\xi_{n-1,n,m} + b\xi_{n,m}^*, \quad a \in (0, 1), \quad b > 0, \tag{4}
\]
where $\xi_{n,m}$ is the score of the $m$th particle in the $n$th generation, $\kappa(n, m)$ is the number of the ancestor of this particle in the preceding generation, and the random variables $\xi^*_n, m \geq 1, n \geq 1$, are independent and have the same distribution $A$ satisfying the conditions

$$\bar{A}(x) \sim x^{-\gamma} L(x), \quad A(-x)/\bar{A}(x) \to p \geq 0, \quad x \to \infty, \quad \gamma > 0, \quad (5)$$

where $L(x)$ is a slowly varying function.

In the model (4), a unique stationary distribution $F$ exists. It is assumed that all particle scores have this stationary distribution. To reveal the role of heredity “in pure form,” it is desirable to ensure independence of the score distribution from the autoregression coefficients (as was the case in the Gaussian framework). Here we can reach this goal only for strictly stable distributions with $0 < \gamma < 2$ by putting

$$a^\gamma + b^\gamma = 1. \quad (6)$$

For arbitrary distributions $A$ satisfying (5), condition (6) ensures asymptotic equivalence of the tails: $\bar{F}(x) \sim \bar{A}(x), x \to \infty$. We assume this condition to be fulfilled.

In this case it is in fact shown that $\theta = (1-a^\gamma)/(1-a^\gamma/\mu) \in (0, 1)$ in the sense of Definition 2 (by Property 5), where $\mu > 1$ is the mean number of descendants. Note that $\theta$ tends to 1 both as the dependence parameter $a$ decreases and as the mean number $\mu$ of descendants decreases. An extremal index in the sense of Definition 1 in this case does not exist.

Here we may also expect cluster forming. Clearly, this concerns groups of kindred particles having a common ancestor with an abnormally large score and inheriting this mutation. This conclusion is illustrated by computer simulation [31].

5 Models with Copulas

Recall some notions of copula theory [32; 33, ch. 5 and Section 7.5].

A copula $C$ is a multivariate distribution function on $[0, 1]^d, d \geq 2$, if all marginal distributions are uniform on $[0, 1]$. By Sklar’s theorem, any multivariate distribution function in $\mathbb{R}^d$ can be represented as

$$G(x_1, \ldots, x_d) = C(G_1(x_1), \ldots, G_d(x_d)),$$

where $G_i, 1 \leq i \leq d$, are marginal distribution functions. Thus, to any multivariate distribution there corresponds its copula. If the marginal distributions are continuous, such a representation is unique.

To a vector with independent components there corresponds the independence copula

$$C(y_1, \ldots, y_d) = y_1 \ldots y_d.$$

At present, the mathematical apparatus of copulas is actively used in quite diverse applications and, in particular, spreads into information science and technology. Note the paper [34] on recursive neural networks, where Student, Clayton, and Gumbel copulas were used. Based on them, successful learning of a humanlike robot was performed.

In the general case, copulas may describe dependence in the behavior of components of compound systems caused by their interaction or the influence of common external factors. In models of the preceding sections, dependence of aggregate activities in a network or particle scores in a generation can also be described by some copulas, which, however, are hard to
write out explicitly, making other methods preferable. Note that communication of network users may lead to dependence of their individual information activities, which was not taken into account in [17]. In engineering systems, deterioration or breakage of some parts may affect other parts, and all parts are influenced by a common operation regime (temperature, humidity, etc.).

In financial models, copulas are used to describe the dependence between fluctuations of exchange rates of various shares and currencies [33]. This dependence must be taken into account both in financial arrangements and in programming trading (financial) bots (black boxes).

Below we will assume for simplicity that \( \nu_n = n \) (triangular scheme), \( F_n(x) \equiv x, x \in [0, 1] \), and random variables \( \xi_{n,m} \), \( 1 \leq m \leq n \), are related by an \( n \)-variate copula \( C_n \). Recall that we can pass to the uniform distribution from any continuous distribution by Property 2.

Let for any \( s \in (0, 1) \) the sequence \( u_n(s) \) be such that \( u_n(s) \to s, n \to \infty \); then \( u_n(s) = 1 + (1 + o(1))(\ln s)/n, n \to \infty \).

**Example 5.1. Gumbel–Hougaard copula.** This copula is of the form

\[
C(y_1, \ldots, y_d) = \exp \left\{ - \left( \sum_{i=1}^{d} (\ln y_i)^\alpha \right)^{1/\alpha} \right\}, \quad \alpha \geq 1,
\]

which implies

\[
C(y, \ldots, y) = y^{d^{1/\alpha}}.
\]

Assuming \( C_n \) to be the Gumbel–Hougaard copula with \( \alpha_n \geq 1 \) and \( (\alpha_n - 1) \ln n \to \gamma \geq 0 \), we obtain

\[
P(M_n \leq u_n(s)) = u_n(s)^{n^{1/\alpha_n}} \to s^\theta, \quad \theta = e^{-\gamma} \in [0, 1].
\]

This copula belongs to the class of extreme value (or max-stable) copulas. In the general case, they have the Pickands representation [33, p. 312, Theorem 7.45]:

\[
C(y_1, \ldots, y_d) = \exp \left\{ B \left( \frac{\ln y_1}{\sum_{i=1}^{d} \ln y_i}, \ldots, \frac{\ln y_d}{\sum_{i=1}^{d} \ln y_i} \right) \sum_{i=1}^{d} \ln y_i \right\},
\]

where

\[
B(w_1, \ldots, w_d) = \int_{S^d} \left( \bigvee_{i=1}^{d} x_i w_i \right) dH(x)
\]

and \( H \) is a finite measure on \( S^d = \{ x = (x_1, \ldots, x_d) : x_i \geq 0, \sum_{i=1}^{d} x_i = 1 \} \). Moreover, this measure should be normalized so that \( \int_{S^d} x_i dH(x) = 1 \) for all \( 1 \leq i \leq d \) (which was forgotten to be mentioned in [33]).

Note that the function \( B \) is first-order homogeneous. Thus, in the general case we have

\[
C(y, \ldots, y) = y^{B(1, \ldots, 1)}.
\]

Denote \( \beta_n = B_n(1, \ldots, 1); \) thus, if \( \beta_n/n \to \theta \), then \( \theta \) is the extremal index (in the sense of both definitions). Since \( 0 \leq \beta_n \leq n \), we have \( \theta \in [0, 1] \).
Example 5.2. Clayton copula. This copula is of the form

\[ C(y_1, \ldots, y_d) = \left( \sum_{i=1}^{d} y_i^{-\alpha} - d + 1 \right)^{-1/\alpha}, \quad \alpha \geq 0, \]

where the degenerate case \( \alpha = 0 \) corresponds to the independence copula, arising in the limit as \( \alpha \to 0 \). Hence,

\[ C(y, \ldots, y) = (d(y^{-\alpha} - 1) + 1)^{-1/\alpha}. \]

Let \( C_n \) be the Clayton copula with \( \alpha_n \equiv \alpha > 0 \); then

\[ \mathbf{P}(M_n \leq u_n(s)) = (n(u_n(s)^{-\alpha} - 1) + 1)^{-1/\alpha} \]

\[ \to (1 - \alpha \ln s)^{-1/\alpha} = \psi(s). \]

Here \( \theta^- = 0 \) and \( \theta^+ = 1 \).

Example 5.3. Frank copula. This copula is of the form

\[ C(y_1, \ldots, y_d) = -\frac{1}{\alpha} \ln \left( 1 - \frac{\prod_{i=1}^{d}(1 - e^{-\alpha y_i})}{(1 - e^{-\alpha})^{d-1}} \right), \quad \alpha \geq 0, \]

where the degenerate case \( \alpha = 0 \) corresponds to the independence copula, arising in the limit as \( \alpha \to 0 \). Hence,

\[ C(y, \ldots, y) = -\frac{1}{\alpha} \ln \left( 1 - \frac{(1 - e^{-\alpha y})^d}{(1 - e^{-\alpha})^{d-1}} \right). \]

Let \( C_n \) be the Frank copula with \( \alpha_n \equiv \alpha > 0 \); then, passing to the limit, we obtain

\[ \mathbf{P}(M_n \leq u_n(s)) \to -\frac{1}{\alpha} \ln \left( 1 - (1 - e^{-\alpha})s^{\alpha/(e^\alpha - 1)} \right) = \psi(s). \]

In this case

\[ \lim_{s \to 0} \log_s \psi(s) = \alpha/(e^\alpha - 1), \quad \lim_{s \to 1} \log_s \psi(s) = 1, \]

and in the interval \((0, 1)\) the function attains intermediate values. Therefore, \( \theta^- = \alpha/(e^\alpha - 1) \in (0, 1) \) and \( \theta^+ = 1 \).

All three examples deal with strictly Archimedean copulas. Recall that a copula is said to be strictly Archimedean if it is of the form

\[ C(y_1, \ldots, y_d) = \varphi^{-1} \left( \sum_{i=1}^{d} \varphi(y_i) \right), \quad (7) \]

where \( \varphi \) is a decreasing function on \([0, 1]\), called the generator, \( \varphi(0) = +\infty \), \( \varphi(1) = 0 \). For \( d = 2 \), it suffices that the function is convex. If we require the function \( \varphi^{-1} \) to be completely monotone on \((0, +\infty)\), then equation \( (7) \) defines a copula for any \( d \geq 2 \) [32 Theorem 4.6.2]. Below we assume this condition on \( \varphi \) to be fulfilled.

On the other hand, the function \( f \) is the Laplace–Stieltjes transform of some distribution if and only if \( f \) is completely monotone and \( f(0) = 1 \) [35 ch. 13, Section 4, Theorem 1]. Hence it follows that \( \varphi^{-1} \) must be the Laplace–Stieltjes transform of some distribution, and by the
condition \( \varphi(0) = +\infty \) (and therefore \( \varphi^{-1}(+\infty) = 0 \)), this distribution must have no atoms at zero. Thus, there exists a random variable \( \zeta > 0 \) a.s. such that

\[
\varphi^{-1}(u) = E e^{-u\zeta}, \quad u \geq 0.
\]

Introduce the notation

\[
x_0 = \inf \{ x > 0 : P(\zeta \leq x) > 0 \}, \quad \mu = E\zeta.
\]

For brevity, we denote \( f(u) = \varphi^{-1}(u) \).

**Theorem 5.1.** Let \( \mu < \infty \); then we have the extremal function \( \psi(s) = f(-(\ln s)/\mu) = E s^{\zeta/\mu} \), \( \theta^+ = 1, \theta^- = x_0/\mu \).

**Proof.** Since \( 1 - f(u) \sim \mu u, u \to 0 + 0 \), we have \( \varphi(1 - t) \sim t/\mu, t \to 1 - 0 \). Next,

\[
P(M_n \leq u_n(s)) = f(n\varphi(u_n(s)))
\]

\[
= f(n\varphi(1 + (1 + o(1))(\ln s)/n)))
\]

\[
\to f(-(\ln s)/\mu), \quad n \to \infty.
\]

From Jensen’s inequality, we obtain \( \psi(s) = E s^{\zeta/\mu} \geq s^{\zeta/\mu} = s \). On the other hand, since \( \zeta > 0 \) a.s., we have \( \psi(s) \leq s^{\zeta/\mu} \). Hence, \( \theta^+ \leq 1 \) and \( \theta^- \geq x_0/\mu \). Furthermore, we obtain

\[
\lim_{s \to 0} \log_s \psi(s) = x_0/\mu, \quad \lim_{s \to 1} \log_s \psi(s) = 1,
\]

so these estimates are attained at the limit and we have \( \theta^+ = 1 \) and \( \theta^- = x_0/\mu \). \( \square \)

In the case of the Clayton copula, the generator is \( \varphi(t) = t^{-\alpha} - 1 \), and the inverse function \( f(u) = 1/(1 + u)^{1/\alpha} \) corresponds to the gamma distribution with shape parameter \( 1/\alpha \), for which \( x_0 = 0 \), so \( \theta^- = 0 \) and \( \theta^+ = 1 \).

In the case of the Frank copula, the generator is \( \varphi(t) = -\ln((1 - e^{-\alpha t})/(1 - e^{-\alpha})) \), and the inverse function \( f(u) = -(1/\alpha) \ln(1 - (1 - e^{-\alpha})e^{-u}) \) corresponds to the discrete distribution with probabilities \( P(\zeta = k) = (1 - e^{-\alpha})^k/(\alpha k), k \geq 1 \). Then \( x_0 = 1, \mu = f'(0) = (e^\alpha - 1)/\alpha \), whence \( \theta^- = \alpha/(e^\alpha - 1) \) and \( \theta^+ = 1 \).

Among the considered examples, only that of the Gumbel–Hougaard copula does not match the conditions of Theorem 5.1 since it has generator \( \varphi(t) = -(\ln t)^\alpha \) with the inverse function \( f(u) = \exp\{-u^{1/\alpha}\}, \alpha \geq 1 \), which corresponds to an asymmetric \((1/\alpha)\)-stable distribution on \( \mathbb{R}_+ \) without a finite mean.

To analyze such cases, we apply the following modification. Note that if \( \varphi(t) \) is a generator with a completely monotone inverse function, then \( \varphi(t)^\beta, \beta \geq 1 \), is also a generator with a completely monotone inverse function \([32\text{ Lemma 4.6.4}]\).

**Theorem 5.2.** Assume that an \( n \)-variate copula \( C_n \) has generator \( \varphi_n(t) = \varphi(t)^{\beta_n} \) with \( \beta_n \geq 1 \), \( (\beta_n - 1) \ln n \to \gamma \geq 0 \), and for the generator \( \varphi(t) \) we have \( \mu < \infty \). Then \( \psi(s) = f(\exp\{-\gamma(\ln s)/\mu\}), \theta^- = (x_0/\mu)e^{-\gamma}, \text{ and } \theta^+ = e^{-\gamma} \).

**Proof.** From \( \varphi_n(t) = \varphi(t)^{\beta_n} \) it follows that \( f_n(u) = f(u^{1/\beta_n}) \). We have

\[
P(M_n \leq u_n(s)) = f_n(n\varphi_n(u_n(s)))
\]

\[
= f(n^{1/\beta_n}\varphi(1 + (1 + o(1))(\ln s)/n)))
\]

\[
= f(e^{((1-\beta_n)\ln n)/\beta_n}(-(\ln s)/\mu))
\]

\[
\to f(-e^{-\gamma}(\ln s)/\mu), \quad n \to \infty.
\]
Partial indices are obtained from the relation
\[
\log_s \psi(s) = \frac{\ln f(-e^{-\gamma}(\ln s)/\mu)}{\ln s} = e^{-\gamma} \frac{\ln f(-\ln r)/\mu)}{\ln r}, \quad r = s^{e^{-\gamma}} \in (0, 1),
\]
where the fraction on the right-hand side is the logarithm of the extremal function from Theorem 5.1 taking values from \(x_0/\mu\) to 1.

In particular, the result of Example 5.1 for the Gumbel–Hougaard copula is obtained with \(\varphi_n(t) = (-\ln t)^{\alpha_n}\), where \(\varphi(t) = -\ln t\) corresponds to \(\zeta = 1\) a.s. and the independence copula.

Thus, it is seen that in the considered models with copulas there may occur any \(\theta \in [0, 1]\) and any \(0 \leq \theta^- < \theta^+ \leq 1\).

Now let us consider one instructive example with copulas and random series sizes.

**Example 5.4.** Let the series sizes satisfy the condition \(\nu_n/n \xrightarrow{d} \zeta, n \to \infty\), where \(\zeta\) has a stable distribution with the Laplace–Stieltjes transform \(E e^{-u\zeta} = e^{-u\beta}, 0 < \beta < 1\), and assume that in each series the random variables (independent of \(\nu_n\)) are related by the Gumbel–Hougaard copula with \(\alpha_n > 1\), \((\alpha_n - 1) \ln n \to \gamma > 0, n \to \infty\) (see Example 5.1).

First, assume that \(u_n(s)^\alpha \xrightarrow{n} e^{-\tau}, n \to \infty, \tau > 0\); then
\[
E u_n(s)^{\nu_n} = E (u_n(s)^n)^{\nu_n/n} \to E e^{-\tau \zeta} = e^{-\tau \beta}, \quad n \to \infty.
\]
Take \(\tau = (-\ln s)^{1/\beta}\); then \(E u_n(s)^{\nu_n} \to s\), as required.

Next, we have
\[
P(M_n \leq u_n(s)) = u_n(s)^{1/\alpha_n} = \left(u_n(s)^{n^{1/\alpha_n}}\right)^{\nu_n/n^{1/\alpha_n}}
\to E e^{-\tau e^{-\gamma \zeta}} = e^{-(\gamma \tau)^\beta} = s^{e^{-\gamma \beta}}, \quad n \to \infty. \tag{8}
\]
Hence, the extremal index in the sense of Definition 1 is \(e^{-\gamma \beta}\).

On the other hand, for any \(\theta > 0\) we have
\[
E u_n(s)^{\theta \nu_n} \to E e^{-\theta \beta \zeta} = e^{-(\theta \tau \beta)} = s^{e^{-\gamma \beta}}, \quad n \to \infty,
\]
from which together with (8) it follows that the extremal index in the sense of Definition 2 is \(e^{-\gamma}\).

Thus, the system has two different extremal indices according to two different definitions!

In all the previously considered models, the classical property (3) remained valid in the form \(\psi(s) \geq s\) for all \(s \in [0, 1]\). To conclude with, consider an example where it is violated. In this example, the symmetric dependence of random variables in a series can be described by a copula, but it is simpler to do this constructively, using a direct construction.

**Example 5.5.** Let \(\eta_{n,m}, m \geq 1, n \geq 1\), be independent and have the uniform distribution on \([0, 1]\); \(\nu_n = n\); \(\kappa_n\) take values from 1 to \(n\) equiprobably, being independent of \(\eta_{n,m}, 1 \leq m \leq n\); \(\gamma > 0\). Put
\[
\xi_{n,m} = \begin{cases} 
\eta_{n,m}^{1/(\gamma m)}, & m = \kappa_n, \\
\eta_{n,m}, & m \neq \kappa_n.
\end{cases}
\]
Then the joint distribution function of $\xi_{n,m}$, $1 \leq m \leq n$, is of the form

$$F^{(n)}(x_1, \ldots, x_n) = \left( \prod_{m=1}^{n} x_m \right) \left( \frac{1}{n} \sum_{m=1}^{n} x_m \right) \left( 1 + \frac{1}{n} \alpha_n - 1 \right),$$

whence

$$F_n(x) = x \left( 1 + \frac{x^{n-1} - 1}{n} \right), \quad P(M_n \leq x) = x^{(1+\gamma)n-1}.$$

Letting $u_n(s) = 1 - (1 + o(1))\tau/n$, $\tau > 0$, we obtain as $n \to \infty$

$$F_n(u_n(s)) \to e^{-\tau} \exp\{e^{-\gamma \tau} - 1\} = s, \quad P(M_n \leq u_n(s)) \to e^{-(1+\gamma)\tau} = \psi(s),$$

whence we can explicitly find the inverse extremal function

$$\psi^{-1}(u) = u^{1/(1+\gamma)} \exp\{u^{\gamma/(1+\gamma)} - 1\},$$

for which we have $\psi^{-1}(u) > u$ for all $u \in (0, 1)$ and hence $\psi(s) < s$ for all $s \in (0, 1)$. In this case it can be shown that $\theta^- = 1$ and $\theta^+ = 1 + \gamma > 1$.

### 6 Threshold Models

Up to now, we considered models where $\nu_n$ was determined by reasons external with respect to the variables $\{\xi_{n,m}\}$. Now we introduce models where $\nu_n$ is the stopping time with respect to the sequence $\{\xi_{n,m}, m \geq 1\}$, where $\xi_{n,m}$, $m \geq 1$, are independent and uniformly distributed on $[0, 1]$, and stopping occurs when a current random variable passes over some threshold value.

Such models may occur, for example, in problems of automated search for objects possessing certain properties by simple exhaustive search.

Note that papers [5,36] considered a model of maxima of random variables where stopping occurred at the time of threshold passage (namely, time $t$) by not the current variable but their accumulated sum. This is a generalization of the classical problem on the longest success series in Bernoulli trials [3, Section 8.5]. However, in this case the stopping time simply grows asymptotically proportionally to the threshold, and there are no such interesting effects as in the models in question.

**Example 6.1.** Consider a number sequence $a_n \in (0, 1)$, $n \geq 1$, with $a_n \to 1$, $n \to \infty$. Denote $\varepsilon_n = 1 - a_n > 0$; then $\varepsilon_n \to 0$, $n \to \infty$. Put $\nu_n = \min\{m \geq 1 : \xi_{n,m} > a_n\}$. Then

$$P(M_n \leq u_n(s)) = P(\xi_{n,\nu_n} \leq u_n(s)) = P(\xi_{1,1} \leq u_n(s)|\xi_{1,1} > a_n) = 0 \vee (u_n(s) - a_n)/\varepsilon_n = 0 \vee (1 - (1 - u_n(s))/\varepsilon_n),$$

where $u_n(s)$ are determined by the condition

$$E u_n(s)^{\nu_n} = \frac{\varepsilon_n u_n(s)}{1 - (1 - \varepsilon_n)u_n(s)} \to s, \quad n \to \infty,$$
since $\nu_n$ have the geometric distribution (starting from 1) with parameter $\varepsilon_n$. Equation (10) implies

$$1 - u_n(s) \sim \varepsilon_n \frac{1 - s}{s}, \quad n \to \infty.$$  \hspace{1cm} (11)

Substituting this into (9) and passing to the limit, we obtain $\psi(s) = 0 \vee (2 - 1/s)$ according to Definition 1. In this case, as in Example 5.5 we have $\psi(s) < s$ for all $s \in (0, 1)$. We have $\psi(s) = 0$, and therefore $\log_\varepsilon \psi(s) = +\infty$ for $s \in [0, 1/2]$, or $\log_\varepsilon \psi(s) > 1$ for $s \in (1/2, 1)$, and also $\log_\varepsilon \psi(s) \to 1$, $s \to 1$. Hence, $\theta^- = 1$ and $\theta^+ = +\infty$.

Surprisingly, the result does not depend on the choice of a sequence $a_n$, $n \geq 1$.

What can be said in this case about the extremal index in the sense of Definition 2?

From (10), taking into account (11), we obtain

$$\mathbf{E} u_n(s)^{\theta_{\nu_n}} = \frac{\varepsilon_n u_n(s)^\theta}{1 - (1 - \varepsilon_n)u_n(s)^\theta} \to \frac{s}{\theta + (1 - \theta)s}, \quad n \to \infty,$$

but the extremal function is not of this form; hence, there is no extremal index in the sense of Definition 2.

Now consider a model with random thresholds $\zeta_n$, $n \geq 1$. Let $0 < \zeta_n < 1$ a.s.; $\zeta_{n,m}$, $m \geq 1$, are independent of $\zeta_n$, and $\nu_n = \min\{m \geq 1 : \zeta_{n,m} > \zeta_n\}$.

**Theorem 6.1.** Let $n(1 - \zeta_n) \xrightarrow{L^1} \zeta > 0$, $n \to \infty$, $\mathbf{E} \zeta = 1$. Then $\psi(s) = g(f^{-1}(s))$, where $f(t) = \mathbf{E}(\zeta/(t + \zeta))$ and $g(t) = \mathbf{E}(\zeta - t + \zeta)$.

**Proof.** Given that $\zeta_n = x \in (0, 1)$, the series size $\nu_n$ has geometric distribution with parameter $1 - x$. Hence it follows that $(1 - \zeta_n)\nu_n \xrightarrow{d} \eta$ and $\nu_n/n \xrightarrow{d} \eta/\zeta$, $n \to \infty$, where $\eta$ has standard exponential distribution and is independent of $\zeta$. Denote $f(t) = \mathbf{E} e^{-\eta/\zeta}$; then $f(t) = \mathbf{E}(\zeta/(t + \zeta))$.

Let $\tau > 0$; then

$$\mathbf{E}(1 - \tau/n)^{\nu_n} = \mathbf{E}((1 - \tau/n)^n)^{\nu_n/n} \to \mathbf{E} e^{-\tau\eta/\zeta} = f(\tau), \quad n \to \infty.$$  

Thus, $\mathbf{E} u_n(s)^{\nu_n} \to s$ implies $u_n(s) = 1 - (1 + o(1))f^{-1}(s)/n$, $n \to \infty$.

We obtain

$$\mathbf{P}(M_n \leq u_n(s)) = \mathbf{P}(\xi_{1,1} \leq u_n(s)|\xi_{1,1} > \zeta_n)$$

$$= \mathbf{P}(\zeta_n < \xi_{1,1} \leq u_n(s))/\varepsilon_n$$

$$= \mathbf{E}(1 - (1 + o(1))f^{-1}(s)/n - \zeta_n)_+/\varepsilon_n$$

$$\to \mathbf{E}(\zeta - f^{-1}(s))_+, \quad n \to \infty,$$

as required. \hfill \Box

Recall a formula

$$\mathbf{E}(\zeta - t)_+ = \int_t^{+\infty} \bar{F}_\zeta(x) \, dx,$$  \hspace{1cm} (12)

convenient for computations, which is obtained via integration by parts.

**Example 6.2.** Let $\zeta$ equiprobably take values $1 - \delta$ and $1 + \delta$, $0 < \delta < 1$ (the case $\delta = 0$ reduces to Example 6.1). Then

$$f(t) = \frac{1}{2} \left( \frac{1}{1 + t/(1 - \delta)} + \frac{1}{1 + t/(1 + \delta)} \right) = \frac{(1 + t) - \delta^2}{(1 + t)^2 - \delta^2},$$
whence
\[ f^{-1}(s) = \frac{1 + \sqrt{1 - 4s(1 - s)\delta^2}}{2s} - 1 \]
and
\[ \psi(s) = \frac{1}{2}(1 - \delta - f^{-1}(s))_+ + \frac{1}{2}(1 + \delta - f^{-1}(s))_. \]

We have \( \psi(s) = 0 \) for \( 0 < s < f(1 + \delta) = (2 - \delta)/4 \), so \( \theta^+ = +\infty \).

In the general case one can analyze the asymptotic behavior of the extremal function as \( s \to 0 \) and \( s \to 1 \). Denote
\[ \theta_0 = \lim_{s \to 0} \log_s \psi(s), \quad \theta_1 = \lim_{s \to 1} \log_s \psi(s). \]

One may claim that \( \theta^- \leq \theta_0 \wedge \theta_1 \) and \( \theta^+ \geq \theta_0 \vee \theta_1 \).

**Corollary 6.1.**

1. If \( \hat{F}_\zeta(x) \sim Cx^{-\alpha}, x \to \infty, C > 0, \alpha > 1 \), then \( \psi(s) \sim Cs^{\alpha-1}/(\alpha - 1), s \to 0 \), and \( \theta_0 = \alpha - 1 \). If \( \hat{F}_\zeta(x) \) decreases faster than any power, then \( \theta_0 = +\infty \).

2. If \( \mathbf{E}\zeta^{-1} < \infty \), then \( \theta_1 = 1/\mathbf{E}\zeta^{-1} \). If \( \mathbf{E}\zeta^{-1} = \infty \), then \( \theta_1 = 0 \).

**Proof.** (1) Note that \( f(t) = \mathbf{E}(\zeta/(\zeta + t)) \sim \mathbf{E}\zeta/t = 1/t, t \to \infty \). Therefore, \( f^{-1}(s) \sim 1/s, s \to 0 \). Now \( \hat{F}_\zeta(x) \sim Cx^{-\alpha}, x \to \infty \), and implies by equation (12) that \( g(t) \sim Ct^{-(\alpha-1)}/(\alpha - 1), t \to \infty \). Hence, \( \psi(s) = g(f^{-1}(s)) \sim Cs^{\alpha-1}/(\alpha - 1), s \to 0 \), and \( \theta_0 = \alpha - 1 \).

If \( \hat{F}_\zeta(x) = o(x^N), x \to \infty \), then \( g(t) = o(t^{N-1}), t \to \infty \), and \( \psi(s) = o(s^{N-1}), s \to 0 \), for any \( N > 0 \), whence \( \theta_0 = +\infty \).

(2) For \( \mathbf{E}\zeta^{-1} < \infty \) we have \( 1 - f(t) = \mathbf{E}(t/(\zeta + t)) \sim t\mathbf{E}\zeta^{-1}, t \to 0 \). Hence, \( f^{-1}(s) \sim (1 - s)/\mathbf{E}\zeta^{-1}, s \to 1 \). Furthermore, \( 1 - g(t) \sim t, t \to 0 \). Therefore, \( 1 - \psi(s) \sim (1 - s)/\mathbf{E}\zeta^{-1}, s \to 0 \), whence \( \theta_1 = 1/\mathbf{E}\zeta^{-1} \). The result for \( \mathbf{E}\zeta^{-1} = \infty \) is obtained by passing to the limit. 

By Corollary 6.1 in Example 6.2 we obtain \( \theta_0 = +\infty \) and \( \theta_1 = 1 - \delta^2 \in (0, 1) \).

It is clear that if one of the exponents \( \theta_0 \) and \( \theta_1 \) is greater than 1 and the other is smaller, then the graph of \( \psi(s) \) necessarily crosses the diagonal. Then property (3) in the form \( \psi(s) \geq s \) holds for some \( s \in (0, 1) \) and is violated for some other.

### 7 Conclusion

We have generalized the notion of the extremal index of a stationary random sequence to a series scheme with random lengths (by two definitions). We have studied properties of the new extremal indices. We have considered various applications of these indices to models of information networks and biological populations, models with copulas, and threshold models. We gave examples where there exist both extremal indices, only one of them, or none. In cases where the extremal index in the sense of the first definition does not exist, we have found partial indices. Thus, we have made a number of important steps in constructing a new mathematical apparatus, which is of both theoretical and applied importance in the description of extremal behavior of various systems.

Of course, the research of the new extremal indices, their properties, and their applications cannot be covered by a single paper. This paper is rather intended to open a cycle of papers—or perhaps a whole scientific direction—that other researchers can join, as it happens in the study of the classical extremal index.
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References

[1] Leadbetter, M.R., Lindgren, G., Rootzen, H. 1986. *Extremes and related properties of random sequences and processes*. Spinger. 336 pp.

[2] Galambos, J. 1978. *The asymptotic theory of extreme order statistics*. N.Y.: Wiley. 352 pp.

[3] Embrechts, P., Klüppelberg, C., Mikosh, T. 2003. *Modelling extremal events for insurance and finance*. Springer. 638 pp.

[4] de Haan, L., Ferreira, A. 2006. *Extreme value theory. An introduction*. Springer. 420 pp.

[5] Novak, S.Yu. 2014. Predel’nye teoremy i otsenki skorosti skhodimosti v teorii ekstremal’nykh znachenii [Limit theorems and convergence rate estimation in the extreme value theory]. Dr. Sci. Diss. Saint Petersburg. 230 pp.

[6] Markovich, N.M. 2013. Modeling clusters of extreme values. *Extremes* 17(1): 97–125.

[7] Markovich, N.M. 2013. Quality assessment of the packet transport of peer-to-peer video traffic in high-speed networks. *Perform. Evaluation* 70(1): 28–44.

[8] Avrachenkov, K., Markovich, N.M., Sreedharan, J.K. 2014. Distribution and dependence of extremes in network sampling processes. *INRIA Research report N 8578*. 25 pp. Available at http://arxiv.org/abs/1408.2529

[9] Goldaeva, A.A. 2014. Tyazhelye hvosty, ekstremumy i klastery lineinykh stokhasticheskikh rekurrentnykh posledovatelnostei [Heavy tails, extremes and clusters of linear stochastic recursive sequences] PhD Diss. Moscow. 94 pp.

[10] Goldaeva, A.A. 2012. Uniform estimator of the extremal index of stochastic recurrent sequences. *Moscow Univ. Math. Bull.* 67(2): 82–85.

[11] Goldaeva, A.A. 2013. Extremal indices and clusters in the linear recursive stochastic sequences. *Theory Probab. Appl.* 58(4): 689–698.

[12] Choi, H. 2002. Central limit theory and extremes of random fields. PhD Dissertation in Univ. of North Carolina and Chapel Hill.

[13] Ferreira, H., Pereira, L. 2008. How to compute the extremal index of stationary random fields. *Statistics and Probability Letters*. 78: 1301–1304.

[14] Pereira, L. 2009. The asymptotic location of the maximum of a stationary random field. *Statistics and Probability Letters*. 79: 2166–2169.

[15] Savinov, E.A. 2014. Limit theorem for the maximum of random variables connected by IT-copulas of Student’s distribution *Theory Probab. Appl.* 59(3): 508–516.
[16] Lebedev, A.V. 2008. Activity maxima in random networks in the heavy tail case. *Problems of Information Transmission*. 44(2): 156–160.

[17] Lebedev, A.V. 2011. Maksimumy aktivnosti v bezmashtabnykh sluchainykh setyakh s tyazhelymi khvostami [Activity maxima in free-scale random networks with heavy tails]. *Informatika i ee primeneniya [Informatics and applications]* 5(4): 25–28.

[18] Lebedev, A.V. 2015. Activity maxima in some models of information networks with random weights and heavy tails. *Problems of Information Transmission*. 51(1): 66–74.

[19] Pavlov, Yu.L. 2009. On the limit distributions of the vertex degrees of conditional Internet graphs. *Discrete Mathematics and Applications*. 19(4): 349–359.

[20] Leri, M.M. 2011. Ob odnoi statisticheskoi zadache dlya sluchainykh grafov Internet-tipa [On a statistical problem for random Internet-type graphs] *Informatika i ee primeneniya [Informatics and applications]* 5(3): 34–40.

[21] Raigorodskii A.M. 2010. Modeli sluchainykh grafov i ikh primeneniya [Models of random graphs and their applications]. *Trudy MFTI [MIPT Proceedings]*. 2(4): 130–140.

[22] van der Hofstad, R. 2014. *Random graphs and complex networks*. V. 1. Eindhoven Univ. of Technology. 328 pp. Available at http://www.win.tue.nl/~rhofstad/NotesRGCN.pdf

[23] Seneta, E. 1976. *Regularly varying functions*. Springer. 116 pp.

[24] Stam, A.J. 1973. Regular variation of the tail of a subordinated probability distribution. *Adv. Appl. Prob.* 5: 308–327.

[25] Lebedev, A.V. 2005. General scheme of maxima of sums of independent random variables and its applications. *Mathematical Notes*. 77(4): 503–509.

[26] Arnold, B.C., Villasenor, J.A. 1996. The tallest man in the world. *Statistical theory and applications*. Papers in honor of H.A.David. Springer. 81–88.

[27] Pakes, A.G. 1998. Extreme order statistics on Galton-Watson trees. *Metrika*. 47(1): 95–117.

[28] Lebedev, A.V. 2008. Maxima of random particles scores in Markov branching processes with continuous time. *Extremes* 11(2): 203–216.

[29] Lebedev, A.V. 2008. Maxima of random properties of particles in supercritical branching processes. *Moscow Univ. Math. Bull.* 63(5): 175–178.

[30] Lebedev, A.V. 2013. Multivariate extremes of random properties of particles in supercritical branching processes. *Theory Probab. Appl.* 57(4): 678–683.

[31] Lebedev A.V. 2010. Asimptoticheskoe povedenie ekstremumov sluchainykh priznakov chastits v vetvyshashchikhsya protsessakh s nasledstvennost’yu [The asymptotic behaviour of extremes of random particles scores in branching processes with a heredity]. *Yaroslavskii pedagogicheskii vestnik. Ser. Fiziko-matematicheskie i estestvennye nauki [Yaroslavl pedagogical bulletin. Physics and mathematics and natural sciences]* (1): 7–14.
[32] Nelsen, R. 2006. *An introduction to copulas*. Springer. 276 pp.

[33] McNeil, A.J., Frey, R., Embrechts, P. 2005. *Quantitative risk management*. Princeton University Press. 538 pp.

[34] Chatzis, S.P., Demiris, Y. 2012. The copula echo state network. *Pattern Recognition* 45: 570–577.

[35] Feller, W. 1971. *An introduction to probability and its applications*. V. 2. N.Y.: Wiley. 668 pp.

[36] Novak, S.Yu. 1992. On the distribution of the maximum of a random number of random variables. *Theory Probab. Appl.* 36(4): 714–721.

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