THE EFFECT OF REPEATING GAMMA RAY BURSTS ON $V/V_{\text{max}}$

David L. Band
Center for Astrophysics and Space Sciences
University of California, San Diego, CA 92093-0111
I: dlbbat@cass09.ucsd.edu

Received 1993 October 6; accepted 1993 December 7

ABSTRACT

I consider the effect of repeating burst sources on the $\langle V/V_{\text{max}} \rangle$ statistic. I find that if the burst sources are distributed homogeneously in a $d$-dimensional space, $\langle V/V_{\text{max}} \rangle$ converges to $d/(d + 3)$ as long as the luminosity function is independent of position. Choosing the brightest event from a cluster of $k$ events shifts the luminosity function to higher luminosities, but if the original luminosity function is position-independent, it remains so. Therefore the treatment of repeating events, if applied consistently, will not affect the effectiveness of $\langle V/V_{\text{max}} \rangle$ as a test of burst homogeneity. The calculation of $\langle V/V_{\text{max}} \rangle$ for apparent repeating and nonrepeating source populations will be biased by the incorrect classification of faint bursts. In conclusion, the current practice of calculating $\langle V/V_{\text{max}} \rangle$ using all bursts, even apparent repeaters, and treating multi-spike bursts as single bursts, is valid and consistent.

Subject headings: gamma rays: bursts

1. Introduction

Before the release of the first gamma ray burst catalog (Fishman et al. 1993) from the Burst and Transient Source Experiment (BATSE) on the Compton Gamma Ray Observatory (GRO), the traditional view had been that the “classical” burst sources did not repeat while the three known repeating gamma ray burst sources constituted a different class, the soft gamma-ray burst repeaters (SGRs), with different spectral properties (see e.g., Higdon & Lingenfelter 1990). Recent studies of this catalog suggest that the classical
bursts do indeed repeat: Wang & Lingenfelter (1993a,b) have identified a repeating source; and Quashnock & Lamb (1993) find an excess of nearest-neighbors at small separation, suggesting a population of repeaters. However, Hartmann et al. (1993) do not find the expected signatures for repeaters in the angular correlation function or the nearest-neighbor statistic. Similarly Narayan & Piran (1993) find only a weak nearest-neighbor excess at small separation, and a comparable excess of antipodal bursts (i.e., farthest neighbors with separations close to 180\(^\circ\)), which they find symptomatic of systematic effects and not of repeaters; Maoz (1993) discusses a possible systematic effect. Nowak (1993) identifies weaknesses in the nearest-neighbor methodology. Therefore, the existence of repeaters is controversial. In addition, Lingenfelter, Wang & Higdon (1993) suggest that complex, multi-spike bursts should be treated as a series of repeating events.

In light of the possibility of repeating burst sources, and the suggestion that each spike within a burst should be considered as a burst in its own right, the effect of repeating bursts on the \(\langle V/V_{\text{max}} \rangle\) test for source density uniformity (Schmidt 1968; Schmidt, Higdon & Hueter 1988) needs to be evaluated. In this study I use “homogeneous” as uniform in flat Euclidean space. I consider a population of events drawn from the same luminosity function without regard to the prior history (i.e., each event is independent). Enough sources are represented so that space is sampled sufficiently. Throughout I deal with the effects of the distance to the burst source and not the projected position on the sky. The events from each source are clustered, and \(\langle V/V_{\text{max}} \rangle\) can be calculated either by including all events, or only the brightest.

The cluster of events might be repeated bursts from the same source (i.e., events separated by a few hours); since rarely can we identify repetitions, \(\langle V/V_{\text{max}} \rangle\) is usually calculated using each event. The assumption that each burst is independent (i.e., characterized by a single luminosity function) is a reasonable assumption in the absence of further information. On the other hand, a cluster of events within a few minutes, which are undoubtedly from one source given the overall burst frequency, is currently considered to be one burst, and \(\langle V/V_{\text{max}} \rangle\) is calculated using the brightest spike within the cluster. The hard-to-soft evolution seen in many bursts (Norris et al. 1986; Band et al. 1992) shows that the burst source retains memory of preceding spikes. Therefore the assumption that the events are drawn independently from a single luminosity function is mediocre. Nonetheless, I assume a single luminosity function for the purpose of the current analysis.

My argument is as follows: after developing the formalism for calculating \(\langle V/V_{\text{max}} \rangle\) (§2), I show that \(\langle V/V_{\text{max}} \rangle\) is independent of source luminosity functions which are functions of luminosity alone (i.e., are not distance-dependent) if the source population is uniform in some dimension (e.g., a sphere or a disk). Considering all the events from
a repeater, or retaining only the brightest, affects the luminosity function but maintains its spatial independence (§3). Separating bursts into apparent repeaters and nonrepeaters introduces spatial-dependence into the luminosity function, with \( \langle V/V_{\text{max}} \rangle \) biased to lower values for apparent repeaters, and to higher values for nonrepeaters (§4). Finally, I draw the relevant operational conclusions (§5).

2. Functional Dependence of \( V/V_{\text{max}} \)

Assume bursts have the normalized differential luminosity distribution \( \phi[L, r] \) where the peak photon luminosity \( L \) is in photons-sec\(^{-1}\), and the burst sources are distributed as \( n[r] \). Note that I permit the luminosity function to change with radius. The observed peak count rate is \( C_{\text{max}} = L/4\pi r^2 \). At any given time the detector has a count rate threshold \( C_{\text{min}} \) with a normalized distribution \( g[C_{\text{min}}] \). Define \( \xi = C_{\text{max}}/C_{\text{min}} \) and \( v = \xi^{-3/2} \) (this is the transformation that removes the detector threshold to answer specific questions; see Band 1993a). Then the distribution of \( \xi \), \( \Xi[\xi] = dN/d\xi \), is

\[
\Xi[\xi] = \int n[r] g[C_{\text{min}}] \phi[L, r] \delta \left[ C_{\text{max}} - \frac{L}{4\pi r^2} \right] \delta \left[ \xi - \frac{C_{\text{max}}}{C_{\text{min}}} \right] dL dV dC_{\text{min}} dC_{\text{max}}
\]

(1)

In brief, the integral in eqn. (1) considers all the sources which can contribute bursts with a value \( \xi \); as the threshold \( C_{\text{min}} \) varies (described by \( g[C_{\text{min}}] \)), different values of \( C_{\text{max}} \) contribute to \( \Xi \); a given \( C_{\text{max}} \) fixes the ratio of \( L \) to \( r^2 \); the luminosity function \( \phi[L, r] \) therefore weights the distribution of relevant radii; and finally, the source density \( n[r] \) provides the number of burst sources at each radius. To get the average \( \langle V/V_{\text{max}} \rangle = v = \xi^{-3/2} \) I need the distribution of \( v \)

\[
\chi[v] = \Xi[\xi] \frac{d\xi}{dv} = -\frac{2}{3} v^{-5/3} \Xi[\xi]
\]

(2)

where I use \( dN = \Xi d\xi = \chi dv \) (and I drop the minus sign). The average \( \langle V/V_{\text{max}} \rangle \) is

\[
\langle V/V_{\text{max}} \rangle = \int_0^1 dv v \chi[v]/\int_0^1 dv \chi[v]
\]

(3)
Assume \( n(r) = n_0 (r/r_0)^\alpha \); clearly the homogeneous 3-dimensional case corresponds to \( \alpha = 0 \). Then (inserting \( n(r) \) into eqn. [2])

\[
\chi[v] = \frac{n_0 v^{\alpha/3}}{3 (4\pi)^{(\alpha+1)/2} r_0} \int \left( \frac{L}{C_{\text{min}}} \right)^{\frac{\alpha+3}{\alpha}} \phi \left[ L, \left( \frac{L v^{2/3}}{4\pi C_{\text{min}}} \right)^{1/2} \right] g[C_{\text{min}}] dL dC_{\text{min}}.
\]  

(4)

If the luminosity function \( \phi[L, r] \) is a function of luminosity alone, \( \phi[L] \), and has no spatial dependence then the dependence of \( \chi[v] \) on \( v = V/V_{\text{max}} \) is decoupled from the luminosity function (i.e., in eqn. [4] the integrand has no \( v \) dependence). In this case

\[
\langle V/V_{\text{max}} \rangle = \frac{\int_0^1 dv v^{(\alpha+3)/3}}{\int_0^1 dv v^{\alpha/3}} = \frac{\alpha + 3}{\alpha + 6}.
\]  

(5)

For source distributions homogeneous in a \( d \)-dimensional space \( n(r) dV \propto r^{d-1} dr \). The equivalent 3-dimensional source density which can be used in the above equations for \( \chi[v] \) is \( n(r) \propto r^{d-3} \) or \( \alpha = d - 3 \); this is the density the spatial distribution would have if averaged spherically. Consequently, if the luminosity function is spatially independent then

\[
\langle V/V_{\text{max}} \rangle = \frac{d}{d+3}
\]  

(6)

which gives \( \langle V/V_{\text{max}} \rangle = 0.5, 0.4 \) and 0.25 for \( d = 3 \) (sphere), 2 (disk—Galactic plane?) and 1 (linear—spiral arm?). An important corollary is that any manipulation that maintains the spatial independence of the luminosity function will not affect \( \langle V/V_{\text{max}} \rangle \).

In addition to its average values for homogeneous source densities, \( V/V_{\text{max}} \) will be distributed uniformly between 0 and 1 for \( d = 3 \), providing a secondary test for this dimension. Variants tailored to dimensions other than \( d = 3 \), such as the \( \langle A/A_{\text{max}} \rangle \) test for \( d = 2 \) (Kluźniak 1992), can be constructed by transforming to \( \xi^{-d/2} \).

3. The Luminosity Function of Burst Clusters

Assume repeating sources all have the same spatially independent luminosity function \( \phi_1(L) \), and that the source sample is large enough so that true averages are observed. The repeating sources can be treated either by considering only the brightest event in calculating \( \langle V/V_{\text{max}} \rangle \), or by including all observed events. I will show that the luminosity function for each case is spatially independent, and will find the effect these choices have on \( \langle V/V_{\text{max}} \rangle \).

First I find the luminosity function \( \phi_k[L] \) for the brightest event in a cluster of \( k \) events. Since the differential luminosity function is normalized, the cumulative luminosity

\[
\phi_k(L) = \int_0^L \phi_{k-1}(L') dL' = \frac{1}{k} \phi(L)
\]
function $\Phi_1[L]$ varies uniformly between 0 and 1 (by definition). Therefore this cumulative
distribution is a mapping of the luminosity into a uniform distribution:

$$g_1[u] = 1 \text{ for } 0 \leq u \leq 1 \text{ where } u = \int_0^L \phi_1[L']dL' = \Phi_1[L] .$$

(7)

If there are $k$ events in a cluster, then the distribution of the value of the single event
cumulative luminosity function corresponding to the brightest event is

$$g_k[u] = ku^{k-1} \text{ for } 0 \leq u \leq 1$$

(8)

because there are $k$ events which could be the brightest and there are $k - 1$ other events
which must be less bright; the probability that an event is less bright is $u$. Note that since
the average value of $u$ is

$$\langle u \rangle = \frac{\int_0^1 g_k(u) u \, du}{k} = \frac{k}{k+1} ,$$

(9)

the average luminosity is shifted to higher values, as expected. The increase is not very
great for flat luminosity functions (e.g., by no more than a factor of 4 for $\phi_1 \propto L^{-1/2}$), but
can be very dramatic for steep luminosity functions (e.g., by more than a factor of 50 for a
ten-event cluster and $\phi_1 \propto L^{-2}$). The luminosity function of the brightest burst is derived
by relating it to the single event luminosity function. The cumulative luminosity function
of the brightest event $\Phi_k$ is also distributed uniformly:

$$G[U] = 1 \text{ for } 0 \leq U \leq 1 \text{ where } U = \int_0^L \phi_k[L']dL' = \Phi_k[L] .$$

(10)

Equating $G[U]dU = g_k[u]du$ gives

$$U = \int_0^L \phi_k[L']dL' = \left( \int_0^L \phi_1[L']dL' \right)^k$$

$$\phi_k[L] = k\Phi_1[L]^{k-1}\phi_1[L] .$$

(11)

Note that $\phi_k[L]$ is the luminosity function regardless of whether all $k$ events can be detected.
Similarly, since each event is assumed to be independent, a cluster of $M$ events can be
considered as two clusters of $N$ and $M - N$ events if the separation into two clusters is
based on a luminosity-independent criterion, e.g., the temporal separation between events.

Let $f_k$ be the fraction of clusters with $k$ events; obviously $\sum_{k=1}^\infty f_k = 1$, and the average
number of events $\sum_{k=1}^\infty k f_k$ must converge. If each event is considered separately,

$$\phi[L] = \phi_1[L] \sum_{k=1}^\infty k f_k$$

(12)
where the average number of events has been inserted to preserve the total number of events observed ($n[r]$ is the density of sources). However, if we consider only the brightest event in each cluster, then

$$\phi[L] = \sum_{k=1}^{\infty} f_k \phi_k[L].$$

Eqn. (13) is applicable if the events from a repeating source are broken into a number of clusters based on a luminosity-independent criteria. For example, events separated by less than some predetermined time (e.g., half an hour) may be considered a single burst, events after a gap greater than this time are considered as part of a new burst.

4. Separating Apparent Repeaters and Nonrepeaters

There is a temptation to separate repeating burst sources from nonrepeaters, and calculate $\langle V/V_{\text{max}} \rangle$ for each class, perhaps to determine whether they originate from different source populations. However, repeaters can only be identified if the second brightest event exceeds the detector threshold. Whether the repeating source is identified as such therefore depends on the distance to the source. This introduces a spatial dependence into the luminosity functions, complicating the determination of the source distribution. Sources observed to repeat will be closer, and have a smaller $\langle V/V_{\text{max}} \rangle$.

Consider a cluster of $k$ events with the luminosity of the brightest event corresponding to a value $u_1$ of the single event cumulative luminosity function (see eqn. [7]). Then $u_2$, the value of the single event cumulative distribution for the second brightest burst, is distributed as

$$g_{k,2}[u_2 | u_1] = (k - 1)u_2^{k-2}u_1^{1-k}, \quad u_2 < u_1;$$

there are $k - 1$ possible second brightest events, and the $k - 2$ less bright events must have values of $u$ less than $u_2$. The dependence on $u_1$ results from restricting all $k - 1$ bursts to the range 0 to $u_1$. Then the fraction of the clusters for which the brightest event has a value $u_1$ and the second event has a luminosity which produces a count rate less than threshold (i.e., $L_2 < L_0 = 4\pi r^2 C_{\text{min}}$) is

$$f_{s,k} = \int_0^{u_0} (k - 1)u_2^{k-2}u_1^{1-k}du_2 = (u_0/u_1)^{k-1}
= \left(\frac{\int_0^{L_0} \phi_1[L']dL'}{\int_0^L \phi_1[L']dL'}\right)^{k-1}
= \left(\frac{\int_0^{L_2} \phi_1[L']dL'}{\int_0^L \phi_1[L']dL'}\right)^{k-1}$$

(15)
where $u_0 = \Phi_1[L_0]$ and $L_0 = 4\pi r^2 C_{min} = 4\pi r^2 C_{max} \xi^{-1} = L v^{2/3}$. Thus $f_{s,k}$ is the fraction of the $k$-event clusters which are counted as nonrepeaters (i.e., misidentified) because the second spike is below threshold. Note that $f_{s,k}$ is a function of both the intensity of the brightest event, and the distance to the source (through $L_0$). The effective differential luminosity functions for apparent single events $\phi_s$ and for apparent multiple events $\phi_m$ are

$$\phi_s = f_1 \phi_1 + \sum_{k=2}^{\infty} f_{s,k} f_k \phi_k \quad \text{and} \quad \phi_m = \sum_{k=2}^{\infty} (1 - f_{s,k}) f_k \phi_k.$$  \hspace{1cm} (16)

Because $f_{s,k}$ is a function of distance to the burst source, the luminosity function for the apparent repeaters and nonrepeaters is also spatially-dependent. Since $f_{s,k}$ increases with $v$, it causes $k$-event clusters to be misclassified as apparently single events with larger values of $\langle V/V_{\max} \rangle$ and correctly classified as repeaters with smaller $\langle V/V_{\max} \rangle$.

5. Conclusions

The $\langle V/V_{\max} \rangle$ statistic is best used to test whether sources are distributed uniformly in Euclidean, 3-dimensional space (Band 1993a), or by extension, in any other $d$-dimensional space. I showed that this diagnostic power assumes the luminosity function is spatially independent. Therefore, transformations which do not introduce a spatial dependence into the luminosity function will not affect the utility of $\langle V/V_{\max} \rangle$.

Using only the brightest event from a cluster of $k$ events modifies the effective luminosity function but maintains its spatial independence. As expected, the average luminosity increases with the number of events, which means that $\langle V/V_{\max} \rangle$ probes greater distances, and is therefore more sensitive to inhomogeneities in the source distribution. However, if the source distribution is indeed homogeneous, the treatment of event clusters does not affect $\langle V/V_{\max} \rangle$.

Apparent repeater and nonrepeater populations do not have meaningful values of $\langle V/V_{\max} \rangle$; $\langle V/V_{\max} \rangle$ is biased towards smaller values for apparent repeaters and towards larger values for apparent nonrepeaters. The second brightest event from a repeater must be detectable, and therefore the fraction of true repeaters which are classified as nonrepeaters increases with distance. Thus multi-spiked bursts should not be separated from single-spike bursts, and repeating sources from nonrepeating.

The burst phenomenon may consist of events with a distribution of separation times peaked at short timescales, but with a tail to large separations (Lingenfelter et al. 1993).
Those events within a few minutes of each other are treated as a single burst, while those separated by hours or more are considered different bursts. Note that the existence of repeating bursts separated by more than a few minutes is controversial, as discussed in the Introduction. This separation into individual multi-spike bursts and repeating bursts does not depend on the distance to the burst source, and should not introduce a spatial dependence into the luminosity function. Therefore the current method of calculating $\langle V/V_{\text{max}} \rangle$ does not compromise the test for homogeneity.

There are two alternatives to the current methodology: calculating $\langle V/V_{\text{max}} \rangle$ using each spike in a burst or including only the strongest burst from a repeating source. If performed consistently, each alternative will not affect the diagnostic power of $\langle V/V_{\text{max}} \rangle$. However, the assumption that in a multi-spike burst each spike is drawn independently from the same luminosity function is invalidated by the observed spectral evolution. Similarly, it is probably impossible to identify distant repeating sources or faint repetitions from nearby sources as a consequence of the large uncertainty in the projected position on the sky of low count rate bursts; incomplete removal of all but the strongest event from a repeater will bias $\langle V/V_{\text{max}} \rangle$ to higher values. Therefore these variants do not increase the effectiveness of $\langle V/V_{\text{max}} \rangle$, and can easily introduce biases.

Previously I (and others) showed that $\langle V/V_{\text{max}} \rangle$ is appropriate for testing the hypothesis of source homogeneity, but not for investigating inhomogeneous source distributions (Hartmann & The 1992; Petrosian 1993; Band 1993a). Maximum-likelihood methods using the observed peak flux and the known detection threshold distributions (Band 1993b; Loredo & Wasserman 1993), survival analysis (Efron & Petrosian 1993), and moment methods (Horack et al. 1993) are more powerful for studying inhomogeneous sources. Since BATSE has found that $\langle V/V_{\text{max}} \rangle$ is definitely not 1/2, and the intensity distribution is not a power law (e.g., Meegan et al. 1993; Fishman et al. 1993), the burst source distribution is clearly not homogeneous. Therefore, $\langle V/V_{\text{max}} \rangle$ should be retired in favor of methods which utilize all the information in the burst distribution. For these superior methods the treatment of repeaters discussed here is just as relevant. Since the luminosity function is unknown and must be modeled, manipulations which alter the luminosity function will not complicate significantly an already difficult problem. However, it is preferable to keep the luminosity function distance-independent.

In summary, the current practice of treating all events which are separated by less than a reasonable time (e.g., $\sim$half-an-hour) as one burst and retaining all the bursts from an apparent repeater produces a burst database appropriate both for the calculation of $\langle V/V_{\text{max}} \rangle$ and for studies of inhomogeneous source models.
It is a pleasure to thank M. Briggs, L. Ford, D. Gruber, R. Lingenfelter, R. Rothschild, and V. Wang for incisive discussions and assistance in improving this text. This work was supported by NASA contract NAS8-36081.

REFERENCES

Band, D., et al., 1992, in *Gamma Ray Bursts: AIP Conference Proceedings 265*, eds. W. S. Paciesas and G. J. Fishman (New York: AIP), 169

Band, D. 1993a, ApJ, 400, L63

Band, D. 1993b, in *Compton Gamma-Ray Observatory, AIP Conference Proceedings 280*, eds. M. Friedlander, N. Gehrels and D. J. Macomb (New York: AIP), 734

Efron, B., & Petrosian, V. 1993, ApJ, submitted

Fishman, G. J., et al. 1993, ApJS, accepted

Hartmann, D. H., Blumenthal, G. R., Linder, E. V., & Hurley, K. 1993, ApJ, submitted

Hartmann, D. H., & The, L. S. 1993, Ap&SS, 201, 347

Higdon, J. C. & Lingenfelter, R. E. 1990, ARA&A, 28, 401

Horack, J., et al. 1993, ApJ, accepted

Kluzniak, W. 1992, in *Gamma Ray Bursts: AIP Conference Proceedings 265*, eds. W. S. Paciesas and G. J. Fishman (New York: AIP), 105

Kouveliotou, C., Meegan, C. A., Fishman, G. J., Bhat, N. P., Briggs, M. S., Koshut, T. M., Paciesas, W. S., & Pendleton, G. N. 1993, ApJ, 413, L101

Lingenfelter, R., Wang, V., & Higdon, J. 1993, ApJ, submitted

Loredo, T. & Wasserman, I. 1993, in *Compton Gamma-Ray Observatory, AIP Conference Proceedings 280*, eds. M. Friedlander, N. Gehrels and D. J. Macomb (New York: AIP), 749

Maoz, E. 1993, ApJ, submitted

Meegan, C., Fishman, G., Wilson, R., Brock, M., Horack, J., Paciesas, W., Pendleton, G., & Kouveliotou, C. 1993, in *Compton Gamma-Ray Observatory, AIP Conference Proceedings 280*, eds. M. Friedlander, N. Gehrels and D. J. Macomb (New York: AIP), 681

Narayan, R. & Piran, T. 1993, ApJ, submitted

Norris, J., Share, G. H., Messina, D. C., Dennis, B. R., Desai, U. D., Cline, T. L., Matz, S. M., & Chupp, E. L. 1986, ApJ, 301, 213
Nowak, M. A. 1993, ApJ, submitted
Petrosian, V. 1993, ApJ, 402, L33
Quashnock, J. M. & Lamb, D. Q. 1993, MNRAS, submitted
Schmidt, M. 1968, ApJ, 151, 393
Schmidt, M., Higdon, J. C., & Hueter, G. 1988, ApJ, 329, L85
Wang, V. & Lingenfelter, R. E. 1993a, in 23rd International Cosmic Ray Conference (Calgary), 1-93
——— 1993b, ApJ, 416, L13