Mathematical modelling of the “Pipeline – pressure sensor” system

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Abstract. A mathematical model of a mechanical system for measuring the pressure of the working medium in an engine is proposed. The system includes an axisymmetric pipe with fluid and a fluid pressure sensor. An integro-differential equation of the dynamics of the elastic element of the sensor is obtained, which relates its deformation to the pressure and temperature of the medium at the entrance to the pipeline (for example, at the exit from the engine combustion chamber). A numerical-analytical method for solving this equation is proposed and examples of numerical calculations of the element dynamics are given.

1. Introduction
Many articles are devoted to theoretical and practical issues of design of pressure sensors [1-10]. All pressure sensors (irrespective of conversion principle) in one degree or another are critical to high temperature and high vibration acceleration [1-3]. Placing a pressure sensor directly on a motor housing in principle provides a higher accuracy of measurement, but, as a rule, it is accompanied by a negative effect upon high temperatures and vibration accelerations sensors. In view of the necessity to weaken such effects, there arises the problem of designing a mechanical “pipeline - pressure sensor” system, when a sensor is located at some distance from an engine and is connected to it by means of a pipeline [9, 10].

The dynamics of the elastic element (plate) of a pressure sensor of a gas-liquid medium (liquid or gas) is investigated on the basis of an axisymmetric mathematical model of “pipeline - pressure sensor”. The aerohydrodynamic load upon the element is determined from the asymptotic equations of aerohydromechanics. In the article, the aerohydrodynamic load upon the element is determined through the deformation function of this element, as a result the solution of the corresponding aerohydroelasticity problem is reduced to the study of the integro-differential equation with partial derivatives for the deformation function. The resulting equation relates the deformation of the elastic element and the pressure at the inlet to the pipeline. The tensile (compressive) element force depends on the temperature distribution over the thickness of the plate, and, consequently, on the temperature of the working medium in the engine. The study of the dynamics of an elastic element is based on the application of the Galerkin method and the numerical experiment.
2. Problem statement of calculating the elastic element dynamics taking into account the thermal and hydrodynamic effects

The vibrations of the elastic element of the sensor, made in the form of a circular plate with thickness \( h \) and diameter \( 2R \) are investigated. The sensor is connected with a pipeline of length \( l - (h/2) \) of the same diameter \( 2R \). Since the problem of oscillations of a circular plate is axisymmetric, we will study the oscillations only as a function of \( r \) - distance from the center of the plate (Figure 1).

![Diagram](image)

**Figure 1.** Diagram of the system "pipeline - pressure sensor": 1 - an engine; 2 - a pipeline, 3 - a sensor; 4 - a working medium; 5 - a plate.

Let \( w = w(r,t) \) be the bending of the plate at the points of a circle with radius \( r \) at time \( t \) \((0 \leq r \leq R, t \geq 0)\); \( \varphi(r,z,t) \) be the potential velocity of the fluid inside the sensor body and the pipeline. Functions \( w(r,t), \varphi(r,z,t) \) are the joint solution of the following initial-boundary problem

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad z \in \left(-l,-\frac{h}{2}\right), \quad r \in (0,R), \quad t \geq 0; \tag{1}
\]

\[
\frac{\partial \varphi}{\partial r}(R,z,t) = 0, \quad z \in \left(-l,-\frac{h}{2}\right), \quad t \geq 0; \tag{2}
\]

\[
\bar{P} - \rho \frac{\partial \varphi}{\partial t}(r,-l,t) = P(t), \quad r \in (0,R), \quad t \geq 0; \tag{3}
\]

\[
\frac{\partial \varphi}{\partial z}(r,-h/2,t) = \frac{\partial w}{\partial t}(r,t), \quad r \in (0,R); \tag{4}
\]

\[
\rho \bar{h} \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w + D \delta_2 \frac{\partial}{\partial t} \left( \nabla^4 w \right) - N(t) \nabla^2 w + \delta_1 \frac{\partial w}{\partial t} + \alpha w = \nabla^2 \left( \frac{w}{r} \right), \quad r \in (0,R), \quad t \geq 0; \tag{5}
\]

\[
w(R,t) = \frac{\partial w}{\partial r}(R,t) = 0, \quad t \geq 0; \tag{6}
\]

\[
w(r,0) = f_1(r), \quad \frac{\partial w}{\partial t}(r,0) = f_2(r), \quad r \in (0,R). \tag{7}
\]

Here (1) there is Laplace equation, which describes the movement of the working medium in the pipeline; (2), (4) are the conditions prohibiting for the medium to pass through the corresponding boundaries; condition (3) sets the law of pressure change at the inlet to the pipeline; (5) is the equation of plate dynamics; (6), (7) are the boundary and initial conditions; \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \) is the Laplace operator, \( E \) is the modulus of elasticity, \( v \) is the Poisson’s ratio, \( D = \frac{E h^3}{12(1-v^2)} \) is the bending stiffness of the plate, \( \rho_0 \) is the plate material density, \( \delta_1, \delta_2 \) are the external and internal damping coefficients, \( \alpha \) is the stiffness coefficient of the base, \( N(t) \) is the tensile (compressive) force, \( P_s(t) \) is the working
pressure at the inlet to the pipeline, i.e. the pressure in the engine \((z = -l)\), \(\rho\) is the density of the medium, \(P_t\) is the pressure in the medium at rest, \(P_0\) is the pressure above the plate \((z = h/2)\).

The tensile (compressive) force is determined by the formula

\[
N(t) = P_t - \frac{E\alpha_T}{1-\nu} \int_{-h/2}^{h/2} T_2(z,t) dz,
\]

where \(\alpha_T\) is the coefficient of linear thermal expansion, \(T_2(z,t)\) is the law of change of temperature over the plate thickness, \(P_t\) is the constant force component caused by the design of the system.

3. Solution of the aerohydrodynamic problem

We will seek a solution to problem (1) - (4) in the form

\[
\phi = \frac{1}{\rho_0} \int_0^t \left[ P - P_i(t) \right] dt + \beta(t) + \sum_{m=0}^{m_0} \varphi_m(t) \cdot \sinh \mu_m(z + l) \cdot J_0(\mu_m r),
\]

where \(J_0(x)\) is the Bessel function of zero order, \(\mu_m\) are the positive solutions to the equation \(J_1(\mu R) = 0\), \(J_1(x)\) is the Bessel function of the first order and \(\beta(t), \varphi_m(t)\) are the arbitrary functions. Function (9) satisfies the equation (1) and the conditions (2) and (3).

Equation (5) satisfying the condition (4) can be written as

\[
\rho_0 h \frac{\partial^2 w}{\partial t^2} + D\nabla^4 w + D\delta_2 \frac{\partial}{\partial t} \left( \nabla^2 w \right) - \rho_0 l^2 \frac{\partial^2 w}{\partial t^2} + 2\rho_0 l (h/2)^2 \int_0^R \frac{\partial^2 w}{\partial t^2} (r,t) dr +
+ \alpha w + \sum_{m=0}^{m_0} \frac{\varphi R}{R} \cdot \sinh \mu_m(l/h) \cdot J_0(\mu_m r) \int_0^R r \cdot J_0(\mu_m r) \frac{\partial^2 w}{\partial r^2} (r,t) dr = P_i(t) - P_0.
\]

Equation (10) connects the inlet pressure of the pipeline and the deformation function.

4. Construction of the solution of the dynamics element equation by means of the of Galerkin method

One way of solving equation (10) is to apply the Galerkin method. The deflection plate will be sought in the form \(w(r,t) = \sum_{n=0}^{m_0} w_n(t) \overline{R}_n(r)\), where \(\overline{R}_n(r) = \lambda_n \left[ J_0(\lambda_n r) - \frac{J_0(\lambda_n R)}{I_0(\lambda_n R)} I_0(\lambda_n r) \right], \) \(n = 0,1,2,3...,\),

\(I_0(\lambda_n r) = J_0(\lambda_n r), \) \(\lambda_n\) are the solutions of the equation \(J_0(\lambda R) I_0'(\lambda R) - I_0(\lambda R) J_0'(\lambda R) = 0\).

Conducting the procedure of the Galerkin method, a system of ordinary differential equations for functions \(w_n(t)\) is obtained. This system is solved numerically.

5. Setting the heat problem

To determine the tensile (compressive) force \(N(t)\) it is necessary to know the thickness variation of the plate temperature \(T_2(z,t)\) depending on the fluid temperature \(T_e(t)\) in the engine so the authors have: a) to find the temperature distribution along the pipe used to transfer the working fluid (liquid or gas) from the engine to the pressure sensor and to cool the medium; b) to find the temperature distribution in the plate thickness, that is the elastic pressure sensor element.

Pipeline 2 (Figure 1) is blown in the direction perpendicular to its axis. The side of the plate that is external to the sensor cavity borders on the vacuum region (with a residual discharge of less than 10 mm. Hg).

The mathematical formulation of the heat problem is as follows:
Here $T_1(z,t)$ is the temperature of the working medium in the pipeline, $a_1^2 = \lambda_1 / c_1 \rho$ is the coefficient of thermal diffusivity of the working environment ($\lambda_1$, $c_1$ are the thermal conductivity and heat capacity ratios), $\beta_1$ is the heat transfer coefficient between the pipeline and the gas flow ($\beta_1$ depends on the pipe material, its size, composition and the speed of the cooling gas), $T_0$ is the blowing gas temperature, $T_1^0$ is the temperature of the working medium in the pipeline at the initial time, $a_2^2 = \lambda_2 / c_2 \rho \beta_0$ is the plate thermal diffusivity ($\lambda_2$, $c_2$ are the thermal conductivity and heat capacity ratios), $\alpha$ is the heat transfer coefficient between the plate and the working medium in the sensor cavity, $T_2^0$ is the plate temperature at the initial time. Condition (12) defines the law of temperature change at the engine outlet; conditions (13) and (17) are the effects of the heat insulation of the thin plate of the outer side ($z = h / 2$); (16) is the condition of heat exchange between the plate and the working medium in the sensor cavity; conditions (14), (18) set the initial temperature of the plate and the working medium in the pipeline.

The coefficient $\beta_1$ is determined as follows: $\beta_1 = \frac{P_s}{c_1 \rho S_e}$, where $P_s$, $S_e$ are the perimeter and cross-sectional area of the volume and the working environment, $\beta_s$ is the heat transfer coefficient.

The solution of the heat problem is divided into two parts: at first the authors have to find the temperature distribution along the pipe (problems (11) - (14) and then the temperature distribution of the plate thickness (problems (15) - (18).

The solution of (11) - (14) obtained by the separation of variables is as follows:

$$T_1(z,t) = T_s(t) - \sum_{n=0}^{\infty} \chi_n \sin \left( \frac{n \pi z}{l} \right) e^{-n^2 \alpha t} \left[ \frac{\beta T_0}{\gamma_n} - T_1^0 \right] e^{\gamma_n t} + \frac{\beta}{\gamma_n} \int_0^t e^{\gamma_n t} T_s(t) \, dt, $$

$$\gamma_n = a_1^2 \psi_n + \beta_1, \quad \psi_n = \frac{\pi(2n+1)}{2l - h}, \quad \chi_n = \frac{4}{\pi(2n+1)}.$$ Supplementary formula (19) it is possible to calculate the temperature in the pipeline at any point at any given time $t$, if the law temperature change of $T_s(t)$ at the output of the engine is specified.

The solution of (15) - (18), allowing finding the temperature distribution of the plate thickness at any given time is as follows:

$$T_2(z,t) = \tilde{T}(t) + \sum_{n=0}^{\infty} A_n e^{-\delta_n t} \cos \eta_n \left( z - \frac{h}{2} \right) \left[ T_2^0 - T_1^0 \right] e^{\delta_n t} + \int_0^t e^{\delta_n t} \tilde{T}'(t) \, dt.$$
where \( A_n = \frac{(-1)^n 2\alpha}{n \sqrt{\alpha^2 + \lambda_2^n \eta_n^2}} \), \( \delta_n = \alpha \eta_n^2 \), and values \( \eta_n (n = 0 \div \infty) \) are the positive roots of the equation \( \tan \eta_n h = \alpha / (\lambda_2 \eta_n) \). The function \( T(t) = T_1(-h/2, t) \) is defined by (19).

6. Numerical experiments in the problem of the dynamics of the elastic element

The authors give the examples of calculation of the elastic plate deformation \( w(r, t) \), when the pipeline inlet temperature varies according to the law \( T_e(t) = C_0 + C_1 t \). The calculations were performed for different values of coefficients \( C_0, C_1 \) and the constant force component \( P_t \).

The plate material is steel (\( E = 2 \cdot 10^{11}, \nu = 0.3, \rho_0 = 7.8 \cdot 10^3 \)), the working medium is air (\( \rho = 1.3 \)). The pipeline inlet pressure \( P_t(t) = 10^5(20 + \cos(0.1t)) \). The parameter values are as follows \( l = 3, R = 0.025, D = 61.81, \alpha = 0.5, \beta = 0.03, \gamma = 0.2, h = 0.0015, T_0 = 300, T_1^0 = 300, T_2^0 = 300, C_0 = 300, C_1 = 6, \alpha_T = 14 \cdot 10^{-6}, \beta_1 = 0.054, a_1 = 0.0058, a_2 = 0.0028, m_0 = 7, P_0 = 0 \) (all values are in SI). The initial conditions are \( w(r, 0) = 0, \dot{w}(r, 0) = 0 \).

The deflection function graphs \( w(r, t) = \sum_{k=0}^{2} w_k(r) \bar{R}_k(r) \) at the \( r = 0 \) and the force \( N(t) \) at various \( P_t \) are presented in Figures 2–4.

Let engine fluid temperature increases according to the law \( T_e(t) = C_0 + C_2 e^{\theta t} \) (\( C_0 = 250, C_2 = 50, \theta = 0.016 \)), then the graphs of the functions are of the form
7. Conclusion
A mathematical model of a mechanical system “pipeline – pressure sensor” for measuring the pressure of the working medium in an engine is proposed. An integro-differential equation of the dynamics of the elastic element of the sensor is obtained, which relates its deformation to the pressure and temperature of the medium at the entrance to the pipeline (for example, at the exit from the engine combustion chamber). The study of the dynamics of an elastic element is based on the application of the Galerkin method and the numerical experiment. The research results are intended for use at the design stage of pressure sensors.

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