On the $3x + 1$ conjecture.

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Abstract

In this paper, we discuss the well known $3x + 1$ conjecture in form of the accelerated Collatz function $T$ defined on the positive odd integers. We present a sequence of quotient spaces and further, an invertible map, which are intrinsically related to the behavior of $T$. This approach allows to express the $3x + 1$ conjecture in form of equivalent problems, which might be more accessible than the original conjecture.

1 Introduction

Let $\mathbb{N}_0$ stand for the nonnegative integers, $\mathbb{N}_0 = \{0, 1, \ldots\}$, let $U = 2\mathbb{N}_0 + 1$ be the set of odd positive integers, and let

$$T : U \to U, \quad Tx = (3x + 1)2^{-\nu_2(3x+1)},$$

where $\nu_2(y)$ denotes the exponent of the largest power of 2 that divides the integer $y$. The map $T$ is called the reduced or accelerated Collatz function in the literature (see [1]). Hence, $T17 = 13$, $T13 = 5$, and $T5 = 1$.

The $3x + 1$ conjecture, also known as the Collatz conjecture, states that, starting from any $x \in U$, by iterating $T$ we will eventually end up in the number 1. In other words, for every $x \in U$, there exists $k = k(x) \in \mathbb{N}_0$ such that $T^kx = 1$. Here, $T^0$ stands for the identity map, and, for $k \in \mathbb{N}$, $T^k$ is defined recursively by $T^k = T \circ T^{k-1}$. To give an example, $T^317 = 1$. We refer the reader to the comprehensive monograph [1] for details on the $3x + 1$ conjecture and for the numerous aspects that have been studied in this context.

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In this paper, we show how to associate with $T$ an invertible map $T^*$ on a certain quotient space that consists of equivalence classes of odd integers. The properties of $T^*$ reflect the behavior of $T$, see Theorem 2.16. Further, we exhibit several statements that are equivalent to the $3x+1$ conjecture, see Corollary 2.13. In the appendix, we present additional concepts to describe the action of $T$ on $U_0$.

Our approach for the accelerated Collatz map $T$ may be of interest for any such many-to-one map with a unique fixed point.

2 Results

For the sake of better readability, we will write $Tx$ instead of $T(x)$ for the image of $x$ under $T$. The same slight abuse of notation will apply to the functions $S$ and $f$ below. All other functions will be written as usual.

Remark 2.1. The outline of our approach is the following:

1. First, we introduce a map $S : U \to U$ that allows to describe the inverse image $T^{-1}\{y\}$ of a point $y \in U$, $y \not\equiv 0 \pmod{3}$, completely.

2. The next idea is to restrict $T$ to the subset $U_0$ of $U$. $T$ is surjective on $U_0$, hence we may introduce the inverse map $\tau$, inverse in the sense $T \circ \tau$ being the identity map on $U_0$.

3. We then study a sequence of equivalence relations “$\sim_n$”, $n \geq 0$, which yields, for every $x \in U_0$, an increasing sequence of equivalence classes $([x]_n)_{n \geq 0}$ and an associated decreasing sequence of positive integers $(\delta_n(x))_{n \geq 0}$.

4. The next idea is to study an equivalence relation “$\sim_{\infty}$” on $U_0$, which leads to a partition of $U_0$ into equivalence classes $[x]_{\infty}$, $x \in U_0$, and to minimal elements $\delta_{\infty}(x)$ with the property $\delta_{\infty}(x) = \lim_{n \to \infty} \delta_n(x)$. Further, a bijective map $T^*$ that mimics the behavior of $T$ may be defined on the quotient space $U_0/\sim_{\infty}$.

5. The $3x+1$ conjecture is then equivalent to $U_0 = [1]_{\infty}$, and also equivalent to $\delta_{\infty}(x) = 1$, for all $x \in U_0$.

6. In the appendix, we analyze a closely related equivalence relation on $U_0$, which results in a partition of $U_0$ into $T$-invariant subsets.

7. Also in the appendix, we extend the map $T$ to an invertible map $f$ on $U_0$. In addition, we provide some concepts for “bookkeeping” concerning the classes $[f^k x]_n$ and the positive integers $\delta_n(f^k x)$, where $k \in \mathbb{Z}$ and $n \geq 0$. 
Remark 2.2. Our abstract approach sheds some light on the behavior of the map $T$. Why do we fail to prove the $3x + 1$ conjecture or, at least, some partial results? What is missing in our study are quantitative results, for example

1. A description of the growth behavior of the classes $[1]_n$, in dependence of $n$. (This might yield a result on the density of $[1]_\infty$ in $\mathbb{U}_0$.)

2. Number-theoretical arguments proving –for an appropriate notion of distance– that the distance between the class $[1]_\infty$ and each class $[x]_\infty$ can be made arbitrarily small. Equivalently, one could try to show that the assumption $\delta_\infty(x) > 1$ leads to a contradiction. (This would yield $[1]_\infty = [x]_\infty$ for all $x \in \mathbb{U}_0$, thereby proving the $3x+1$ conjecture.)

3. (Number-theoretic) Arguments showing that there are no periodic points $[x]_\infty$ of $T^*$ with a period larger or equal to 2. (This would imply that there is no periodic point of $T$ with period larger or equal to 2, which is yet unknown.)

After this outline of concepts and shortcomings in our approach, let us look at the details.

Remark 2.3. The following properties of the map $T$ are well known and elementary to prove.

1. From the definition of $T$, we derive the equivalence
   \[ y = T x \iff 3x + 1 = y2^{\nu_2(3x+1)}. \] (1)

2. For every $y \in \mathbb{U}$ with $y \equiv 0 \pmod{3}$,
   \[ T^{-1}\{y\} = \emptyset. \]
   This follows from (1) for the simple reason that $3x + 1 \equiv 1 \pmod{3}$, whereas $y2^{\nu_2(3x+1)} \equiv 0 \pmod{3}$.

3. The element 1 is the unique fixed point of $T$, i.e.
   \[ \{x \in \mathbb{U} : T x = x\} = \{1\}. \]
   Again, this is a direct consequence of (1).

The following map allows to describe the behaviour of $T$.

Definition 2.1. We define $S : \mathbb{U} \rightarrow \mathbb{U}$ as $S x = 4x + 1$.

The map $S$ permutes the residue classes modulo 3: if $x \equiv a \pmod{3}$, then $S x \equiv a + 1 \pmod{3}$. This simple property will prove to be essential for defining an inverse map associated with $T$, see Definition 2.3.

The next lemma is part of the ‘folklore’ in the $3x + 1$ community. We present a simple proof, for the sake of completeness.
Lemma 2.1. For all $x \in U$, we have
\[ Tx = T(Sx). \] (2)

Proof. We have
\[ T(Sx) = T(4x + 1) = (3x + 1)2^{2 - \nu_2(12x + 4)}. \]
Trivially, $\nu_2(12x + 4) = 2 + \nu_2(3x + 1)$.

Corollary 2.2. Lemma 2.1 implies for all $x \in U$ that $x$ and its iterates $S^kx$ are mapped to $Tx$:
\[ \forall x \in U, \forall k \geq 0 : \quad Tx = T(S^kx). \]

In other words, $T = T \circ S^k$ on $U$, for all $k \geq 0$.

Remark 2.4. By induction for $k$ we see that
\[ \forall x \in U, \forall k \geq 0 : \quad S^kx = 4^kx + (4^k - 1)/3. \]

Definition 2.2. For $y \in U$ with $y \not\equiv 0 \pmod{3}$, let $\xi(y)$ denote the smallest element of $U$ that is mapped to $y$ by $T$:
\[ \xi(y) = \min\{x \in U : Tx = y\}. \]

Lemma 2.3. Let $y \in U$ with $y \not\equiv 0 \pmod{3}$. Then $\xi(y)$ is given as follows.

1. If $y \equiv 1 \pmod{3}$, then $\xi(y) = (4y - 1)/3$.
2. If $y \equiv 2 \pmod{3}$, then $\xi(y) = (2y - 1)/3$.

Proof. Suppose that $y \equiv 1 \pmod{3}$. Then equivalence (1) implies that $\nu_2(3x + 1)$ has to be even. The smallest solution in $U$ to (1) is the number $x$ with the property $\nu_2(3x + 1) = 2$. This yields $\xi(y) = (4y - 1)/3$.

The case $y \equiv 2 \pmod{3}$ is treated in the same manner.

Corollary 2.4. Suppose that $y \equiv 0 \pmod{3}$. Then, for $z \in \{Sy, S^2y\}$, the preimage $T^{-1}\{z\}$ is non-void.

Lemma 2.5. The set $T^{-1}\{Tx\}$ of those elements $z$ of $U$ that are mapped to $Tx$ is given by $\xi(Tx)$ and its iterates under $S$:
\[ \{z \in U : Tz = Tx\} = \{S^k\xi(Tx) : k \geq 0\}. \]

Proof. Suppose first that $y = Tx \equiv 1 \pmod{3}$, and assume that $Tz = Tx$. It follows from (1) that $\nu_2(3z + 1) \in \{2, 4, 6, \ldots\}$. If $\nu_2(3z + 1) = 2$, then from Lemma 2.3 Part 1, it follows that $z = \xi(Tx)$. If $\nu_2(3z + 1) = 4$, then
\[ 3z + 1 = 2^4y = 2^2(3\xi(Tx) + 1). \]
This implies \( z = S\xi(Tx) \).

In the general case, if \( \nu_2(3z + 1) = 2 + 2k \), with \( k \geq 1 \), we have
\[
3z + 1 = y2^{2+2k} = (3\xi(Tx) + 1)4^k.
\]

It follows that
\[
z = 4^k\xi(Tx) + (4^k − 1)/3,
\]
from which we derive by Remark 2.4 that \( z = S^k\xi(Tx) \).

**Lemma 2.6.** The set \( \mathbb{U}_0 = \{ x \in \mathbb{U} : x \not\equiv 0 \pmod{3} \} \) has the properties \( T\mathbb{U} = \mathbb{U}_0 \) and \( T\mathbb{U}_0 = \mathbb{U}_0 \). In particular, the map \( T : \mathbb{U}_0 \to \mathbb{U}_0 \) is surjective.

**Proof.** By Remark 2.3(2), we have \( T\mathbb{U} \subseteq \mathbb{U}_0 \), hence \( T\mathbb{U}_0 \subseteq \mathbb{U}_0 \). If \( y \in \mathbb{U}_0 \), then by Lemma 2.3 there exists \( x \in \mathbb{U} \) such that \( Tx = y \). Due to Lemma 2.1, we may assume \( x \in \mathbb{U}_0 \).

**Corollary 2.7.** In order to prove the 3x+1 conjecture, it suffices to restrict the map \( T \) to the set \( \mathbb{U}_0 \).

Hence, from now on, we will study the 3x+1 conjecture for the surjective map \( T : \mathbb{U}_0 \to \mathbb{U}_0 \). We note that the surjectivity of \( T \) implies for all subsets \( B \) of \( \mathbb{U}_0 \),
\[
T(T^{-1}B) = B.
\]

We employ the well-ordering principle to define some sort of inverse map associated with \( T \).

**Definition 2.3.** For \( x \in \mathbb{U}_0 \), define the (quasi-)inverse function \( \tau \) of \( T \) as follows:
\[
\tau(x) = \min\{ z \in \mathbb{U}_0 : Tz = x \}.
\]

The reader should note that, for \( \xi(x) \in \mathbb{U}_0 \), \( \tau(x) = \xi(x) \), whereas for \( \xi(x) \equiv 0 \pmod{3} \), we have \( \tau(x) = S\xi(x) \). To give an example, \( \tau(5) = 13 \), whereas \( \xi(5) = 3 \). Further, \( T \circ \tau \) is the identity map on \( \mathbb{U}_0 \), whereas, in general, \( \tau(Tx) \neq x \).

The next idea is to generate a series of equivalence relations and, hence, a series of quotient spaces and of partitions of \( \mathbb{U}_0 \).

**Definition 2.4.** For \( x,y \in \mathbb{U}_0 \) and \( n \in \mathbb{N}_0 \), we define the relation \( \sim_n \) on \( \mathbb{U}_0 \) as
\[
x \sim_n y \iff T^n x = T^n y.
\]

Further, we put \( [x]_n = \{ z \in \mathbb{U}_0 : T^n z = T^n x \} \), and \( \delta_n(x) = \min[x]_n \).

For all \( x \in \mathbb{U}_0 \), and all \( n \geq 0 \), we have \( x \in [x]_n \), hence \( [x]_n \neq \emptyset \) and \( \delta_n(x) \leq x \). The set \( [x]_0 \) consists of the single point \( x \).

**Lemma 2.8.** For all \( x \in \mathbb{U}_0 \) and for all \( n \in \mathbb{N}_0 \), the following holds.
1. The relation \( \sim_n \) is an equivalence relation on \( U_0 \) and the set \([x]_n\) is the equivalence class of \( x \) with respect to this equivalence relation. Further, \([x]_n = [\delta_n(x)]_n\).

2. We have strict inclusion \([x]_n \subset [x]_{n+1}\).

3. For all \( n \geq 1 \) and all \( k \in \mathbb{N}_0 \) such that \( S^k x \in U_0 \),

\[
[x]_n = [S^k x]_n.
\]

4. We have

\[
1 \leq \delta_n(x) \leq \delta_{n-1}(x) \leq \cdots \leq \delta_1(x) \leq \delta_0(x) = x.
\]

5. For all \( x \in U_0 \) and all \( n \geq 0 \),

\[
T^{-1}[Tx]_n = [x]_{n+1}.
\]

Proof. Ad 1. This is easy to verify.

Ad 2. The inclusion \([x]_n \subseteq [x]_{n+1}\) is trivial. It follows from the definition of these two sets. In order to prove strict inclusion, put \( y = T^n x \). If \( y \equiv 1 \) (mod 3), then let \( z \in U_0 \) be such that \( T^n z = Sy \). Hence, \( z \notin [x]_n \). On the other hand, \( T^{n+1} x = Ty = T(Sy) = T^{n+1} z \), which implies \( z \in [x]_{n+1} \). If \( y \equiv 2 \) (mod 3), then let \( z \in U \) be such that \( T^n z = S^2 y \). As above, we derive \( z \notin [x]_{n+1} \setminus [x]_n \).

Ad 3. From Corollary 2.2 it follows that, for all \( n \geq 1 \), we have the identity \( T^n = T^n \circ S^k \) on \( U \) and, hence, also on \( U_0 \). This implies \( x \sim_n S^k x \), for all those \( k \geq 0 \) where \( S^k x \in U_0 \).

Ad 4. Trivial.

Ad 5. Let \( z \in T^{-1}[Tx]_n \). Then \( Tz \in [Tx]_n \), which implies \( T^n(Tz) = T^{n+1}z = T^n(Tx) = T^n+1x \). Hence, \( z \in [x]_{n+1} \). This yields \( T^{-1}[Tx]_n \subseteq [x]_{n+1} \). For the converse, if \( z \in [x]_{n+1} \), then \( T^{n+1}z = T^n(Tz) = T^n+1x = T^n(Tx) \), which implies \( Tz \in [Tx]_n \). As a consequence, \( z \in T^{-1}[Tx]_n \). We derive \([x]_{n+1} \subseteq T^{-1}[Tx]_n\). \( \square \)

Corollary 2.9. For all \( x \in U_0 \), and for all \( n \geq 0 \),

\[
T[x]_{n+1} = [Tx]_n, \quad T^{-1}[x]_n = [\tau(x)]_{n+1}, \quad T[\tau(x)]_{n+1} = [x]_n.
\]

This is due to the surjectivity of \( T \), see identity (2). In addition, \( T[x]_0 = [Tx]_0 \).

Corollary 2.10. For all \( n \geq 0 \), we may partition \( U_0 \) as follows:

\[
U_0 = \bigcup_{x \in U_0} [x]_n.
\]
Due to the strict inclusion \([x]_n \subset [x]_{n+1}\), if we pass from \(n\) to \(n+1\), this will result in a ‘reduction’ in the number of different equivalence classes. Hence, if \(n\) increases, we get less and less elements in the partitions \([x]_n\) of \(U_0\). As we will see in Corollary 2.13, the 3x+1 conjecture is equivalent to a collapse of this sequence of nested partitions to a trivial partition of \(U_0\) consisting of a single set.

**Corollary 2.11.** For all \(x \in U_0\), the limit \(\lim_{n \to \infty} \delta_n(x)\) exists. This is due to the fact that the sequence of positive integers \((\delta_n(x))_{n \geq 0}\) is decreasing and bounded from below by 1, hence convergent.

We observe that the 3x+1 conjecture is equivalent to \(\lim_{n \to \infty} \delta_n(x) = 1\) for all \(x \in U_0\). It is also equivalent to \(1 \in \bigcup_{n \geq 0} [x]_n\) for all \(x \in U_0\).

In the next step, we determine \(\lim_{n \to \infty} \delta_n(x)\) and characterize the union of the sets \([x]_n\), \(n \geq 0\).

**Definition 2.5.** For \(x, y \in U_0\), we define the relation \(\sim\) on \(U_0\) as

\[x \sim y \iff \exists n \in \mathbb{N}_0 : T^n x = T^n y.\]

Further, put \([x]_\infty = \{z \in U_0 : \exists n \in \mathbb{N}_0 \text{ such that } T^n z = T^n x\}\), and \(\delta_\infty(x) = \min [x]_\infty\).

**Lemma 2.12.** The following holds.

1. The relation \(\sim\) is an equivalence relation on \(U_0\) and, for all \(x \in U_0\), the set \([x]_\infty\) is the equivalence class of \(x\) with respect to this equivalence relation. Further, \([x]_\infty = [\delta_\infty(x)]_\infty\), and

\[ [x]_\infty = \bigcup_{n \geq 0} [x]_n. \]

2. For all \(x \in U_0\),

\[ \delta_\infty(x) = \lim_{n \to \infty} \delta_n(x). \]

**Proof.** Ad 1. This is easily verified.

Ad 2. Let \(\delta'(x) = \lim_{n \to \infty} \delta_n(x)\). Due to the fact that we are dealing with a convergent integer sequence, there exists an integer \(N\) such that for all \(n \geq N\), \(\delta'(x) = \delta_n(x)\). From the fact \(\delta_n(x) \in [x]_n\), it follows that \(\delta'(x) \in [x]_\infty\). Hence, \(\delta_\infty(x) \leq \delta'(x)\).

On the other hand, for any \(z \in [x]_\infty\), there exists \(n \in \mathbb{N}_0\) such that \(z \in [x]_n\). This implies that \(z \geq \delta_n(x) \geq \delta'(x)\). We note that, by definition, \(\delta_\infty(x) \in [x]_\infty\). Consequently, \(\delta_\infty(x) \geq \delta'(x)\).

**Corollary 2.13.** The sets \([x]_\infty, x \in U_0\), form a partition of \(U_0\), \(U_0 = \bigcup_{x \in U_0} [x]_\infty\). The 3x+1 conjecture is equivalent to each of the following statements:
1. \( U_0 = [1]_\infty \).

2. \( \forall x \in U_0 : \ \delta_\infty(x) = 1. \)

Let us study the action of \( T \) on the sets \([x]_\infty\).

**Lemma 2.14.** For all \( x \in U_0 \),

\[
T^{-1}[Tx]_\infty = [x]_\infty.
\]

**Proof.** We have the following chain of equivalences:

\[
z \in T^{-1}[Tx]_\infty \iff Tz \in [Tx]_\infty \iff \exists n \geq 0 : Tz \in [T]_n
\]
\[
\iff \exists n \geq 0 : T^{n+1}z = T^{n+1}x
\]
\[
\iff \exists n \geq 0 : z \in [x]_{n+1} \iff z \in [x]_\infty.
\]

**Corollary 2.15.** In analogy to Corollary 2.9, for all \( x \in U_0 \), we have

\[
T[x]_\infty = [Tx]_\infty, \quad T^{-1}[x]_\infty = [\tau(x)]_\infty, \quad T[\tau(x)]_\infty = [x]_\infty.
\]

**Remark 2.5.** For the analog equivalence classes on the set \( U \) instead of \( U_0 \), one has \( T[x]_\infty \subset [Tx]_\infty \), i.e., strict inclusion, as the inverse image \( T^{-1}\{z\} \) of a point \( z \in [Tx]_\infty \) may be empty.

In view of Lemma 2.14 and Corollary 2.15, we may introduce the following map.

**Definition 2.6.** Let \( U_{0,\infty} \) denote the quotient space \( U_0 / \sim_\infty \), i.e., the set of equivalence classes associated with the equivalence relation \( \sim_\infty \). The induced map \( T^* \) on \( U_{0,\infty} \) is defined as

\[
T^*[x]_\infty = [Tx]_\infty, \quad x \in U_0.
\]

**Theorem 2.16.** The map \( T^* \) on \( U_{0,\infty} \) has the following properties:

1. \( T^* \) is well defined.

2. \( T^* \) is a bijection on \( U_{0,\infty} \).

3. \( T^* \) has the unique fixed point \([1]_\infty\):

\[
[x]_\infty = T^*[x]_\infty \Rightarrow [x]_\infty = [1]_\infty.
\]

4. If \( x \) is a periodic point of \( T \), \( T^kx = x \), with \( k \geq 1 \) the minimum period of \( x \), then \([x]_\infty\) is a periodic point of \( U_{0,\infty} \) with \( T^*[x]_\infty = [x]_\infty \).
Proof. Ad [1] We have to show that the value of $T^*$ is independent of the representative of the equivalence class $[x]_\infty$. Suppose that $[x]_\infty = [z]_\infty$. Then there exists $n \geq 0$ such that $T^n x = T^n z$. Hence, $T^{n+1} x = T^{n+1} z$, which implies $T z \in [Tx]_n \subset [Tx]_\infty$. Thus, $[Tx]_\infty = [Tz]_\infty$.

Ad [2] By Lemma 2.14 $T^*[x]_\infty = T^*[z]_\infty$ implies $[Tx]_\infty = [Tz]_\infty$. Hence, there exists $n \geq 0$ such that $T^n(Tx) = T^n(Tz)$. As a consequence, $z \in [x]_{n+1} \subset [x]_\infty$, which implies $[z]_\infty = [x]_\infty$. Thus, $T^*$ is injective on $U_{0,\infty}$.

Let $[y]_\infty$ be an arbitrary element of $U_{0,\infty}$. Then $[y]_\infty = [T\tau(y)]_\infty = T^*[\tau(y)]_\infty$. Thus, $T^*$ is surjective.

Ad [3] Suppose that $[x]_\infty = T^*[x]_\infty = [Tx]_\infty$. Then there exists $n \geq 0$ such that $T^n x = T^n(Tx) = T(T^n x)$. As a consequence, the element $T^n x$ is a fixed point in $U_0$ under $T$. This implies $T^n x = 1$ (see Remark 2.3, Part 3.). Due to $T^n 1 = 1$, we have $[x]_\infty = [1]_\infty$.

Ad [4] The property $T^k x = x$ implies $[T^k x]_\infty = T^*[x]_\infty = [x]_\infty$. \qed

3 Appendix

The idea underlying our approach to the $3x+1$ conjecture was to find a suitable metric space $X$ in the form of some quotient space $X = \{\xi : x \in U_0\}$, and a contraction $T^*$ on $X$ that is intrinsically related to the map $T$ in the sense that convergence of the sequences $(T^k x)_{k \geq 0}$, $\xi \in X$, to the unique fixed point of $T^*$ implies the convergence of the sequences $(T^k x)_{k \geq 0}$ to 1, i.e., the validity of the $3x+1$ conjecture.

We were unable to realize this ‘dream’ of applying the Banach fixed-point theorem, because we have not found an appropriate pair $(X, T^*)$. For example, $X = U_{0,\infty}$ can easily be made into a metric space but it is the proof of the contraction property of the induced map $T^*$ with respect to the chosen metric where we failed.

The following concepts allow a somewhat deeper understanding of the dynamics of the map $T$.

Definition 3.1. The invertible accelerated Collatz function $f : U_0 \to U_0$ is defined as follows. For $x \in U_0$ and for $k \geq 0$, define $f^k x = T^k x$. For $k < 0$, put $f^k x = \tau^{-k}(x)$.

Consider the following equivalence relation on $U_0$:

$$x \sim y \iff \exists m, n \geq 0 : T^m x = T^n y.$$  

We write $[x]$ for the equivalence class of $x \in U_0$, and get the following.

Lemma 3.1. Let the relation $\sim'$ be defined as above and write $U'_0$ for the quotient space $U_0/\sim$. Then

1. For all $x \in U_0$, the sets $[x]$ are $T$-invariant in the following sense:

$$T^{-1}[x] = [x], \quad T[x] = [x] = [Tx].$$
2. For all \(x \in U_0\),
\[
[x] = \bigcup_{k \in \mathbb{Z}} [f^k x]_{\infty} = \cdots \cup [\tau(x)]_{\infty} \cup [x] \cup [T x] \cup \cdots
\]

3. We have \([1] = [1]_{\infty}\).

4. The map \(T' : U'_0 \to U'_0\), \(T'[x] = [T x]\) is well-defined and every element
\([x]\) of \(U'_0\) is a fixed point of \(T'\).

**Proof.** The proof is straightforward and employs the techniques introduced in Section 2. \(\square\)

**Remark 3.1.** The reader should note the behavior of the class \([1]\), which is remarkably different from all other classes.

**Remark 3.2.** For \(x \in U_0\), put \(\delta(x) = \min[x]\). The \(3x + 1\) conjecture is equivalent \(U_0 = [1]\). Further, it is equivalent to \(\delta(x) = 1\) for all \(x \in U_0\).

**Remark 3.3.** Lemma 3.1 tells us that every set \([x]\) is \(T\)-invariant, which is to say that \(T^{-1}[x] = [x]\). In addition to this result, Theorem 2.16 shows that \([1] = [1]_{\infty}\) is the only \(T\)-invariant set of the form \([x]_{\infty}\). These two results call out for the application of concepts from the theory of dynamical systems, for example from ergodic theory. Let \((X, \mathcal{B}, m)\) be a probability space. A measure preserving map \(f : X \to X\) is called ergodic if the only \(f\)-invariant elements \(A\) of \(\mathcal{B}\), i.e., \(f^{-1} A = A\), are those with \(m(A) = 0\) or \(m(A) = 1\).

It is well known that ergodicity of \(f\) is equivalent to each of the following properties: (i) for every \(A \in \mathcal{B}\) with \(m(A) > 0\) we have \(m(\bigcup_{k=1}^{\infty} f^{-k} A) = 1\), or (ii) for every \(A, B \in \mathcal{B}\) with \(m(A) > 0\) and \(m(B) > 0\), there exists \(k > 0\) with \(m(f^{-k} A \cap B) > 0\) (see, for example, Walters[4, Theorem 1.5]). In our case, we would have to prove ergodicity for \(f = T\), where \(U_0\) would have to be equipped with an appropriate probability space structure. We would then be able to derive the \(3x + 1\) conjecture for almost all \(x\).

The following two notions allow some kind of “bookkeeping” when we iterate the map \(f\). With every \(x \in U_0\), we may associate two infinite matrices as follows.

**Definition 3.2.** Let \(x \in U_0\). We define the matrix of equivalence classes associated with \(x\) as \(C(x) = (c_{k,n})_{k \in \mathbb{Z}, n \geq 0}\), where \(c_{k,n} = [f^k x]_n\).

In addition, we define the matrix of minimal elements associated with \(x\) as \(M(x) = (\mu_{k,n})_{k \in \mathbb{Z}, n \geq 0}\), with \(\mu_{k,n} = \min c_{k,n}\). Let \(\delta^*(x)\) denote the minimal element of the matrix \(M(x)\).

Clearly, we have \(\mu_{k,n} = \delta_n(f^k x)\), and \(\delta^*(x) = \delta(x)\). Note that if we fix the row index \(k\), then the row \((\mu_{k,n})_{n \geq 0}\) in \(M(x)\) has a constant tail eventually, because the convergent sequence \((\delta_n(f^k x))_{n \geq 0}\) is constant from some index \(N = N(k)\) onwards, with every element then being equal to \(\delta_{\infty}(f^k x)\).

There is even further ‘tail’-structure in \(M(x)\): suppose that \(\delta^*(x)\) is equal to \(\mu_{k,n}\), where \(k\) and \(n\) are minimal with this property (in this order). Then
\[ \delta^*(x) = \mu_{k,m} \text{ for all } m \geq n. \] That is to say, the \( k \)-th row becomes eventually constant.

From the discussion above it follows that it is sufficient to prove the \( 3x+1 \) conjecture for the subset \( \{ \delta_\infty(x) : x \in U_0 \} \) of \( U_0 \) or, alternatively, \( \{ \delta_1(x) : x \in U_0 \} \). These facts suggest the following notion.

**Definition 3.3.** A subset \( V \) of \( U_0 \) is called sufficient if the validity of the \( 3x+1 \) conjecture for every element of \( V \) implies the validity of the \( 3x+1 \) conjecture for every element of \( U_0 \).

**Lemma 3.2.** The set \( \{ x \in U_0 : 1 \leq \nu_2(3x+1) \leq 4 \} \) is sufficient.

**Proof.** Let \( x \in U_0 \) be arbitrary. Trivially, we have \( [x]_\infty = [\delta_1(x)]_\infty \). As a consequence, the set \( \{ \delta_1(x) : x \in U_0 \} \) is sufficient. Further, \( \delta_1(x) = \tau(Tx) \).

From Lemma 2.3 it follows that \( \nu_2(3\xi(x)+1) \in \{1, 2\} \), for all \( x \in U_0 \). Due to the fact that either \( \tau(x) = \xi(x) \), or \( \tau(x) = S\xi(x) \), we have \( \nu_2(3\tau(x)+1) \in \{1, 2, 3, 4\} \).

The reader might want to compare this result with Sander [3, Theorem 1]. For further, very extensive results on sufficient sets we refer the reader to Monks[2].

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