Connected sum for modular operads and Beilinson-Drinfeld algebras

Martin Doubek, Branislav Jurčo, and Lada Peksová

Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Prague 186 75, Czech Republic

Ján Pulmann

School of Mathematics, University of Edinburgh, U.K.

Dedicated to the memory of Martin Doubek.

Abstract

Modular operads relevant to string theory can be equipped with an additional structure, coming from the connected sum of surfaces. Motivated by this example, we introduce a notion of connected sum for general modular operads. We show that a connected sum induces a commutative product on the space of functions associated to the modular operad. Moreover, we combine this product with Barannikov’s non-commutative Batalin-Vilkovisky structure present on this space of functions, obtaining a Beilinson-Drinfeld algebra. Finally, we study the quantum master equation using the exponential defined using this commutative product.

1 Introduction

Batalin-Vilkovisky (BV) formalism is a formal integration technique that originated in quantum field theory. Its basic ingredients are an odd, second order differential operator $\Delta$ on the space of functionals and a $\Delta$-closed functional $e^{S/\hbar}$, i.e. a quantum master action. Observables are then computed by taking an integral over a Lagrangian submanifold in the space of fields, weighted by $e^{S/\hbar}$. The closedness of $e^{S/\hbar}$ ensures that the result is independent of the choice of the Lagrangian, generalizing gauge independence.

It was observed by Barannikov that one can obtain a similar algebraic setup for any modular operad $P$ in dg vector spaces. Concretely, for any odd symplectic vector space $V$, Barannikov defined a vector space $\text{Fun}_P(V)$ equipped with a BV operator and a bracket, giving a non-commutative version of a BV algebra. For $P = QC$, the quantum closed modular operad, one recovers the usual BV formalism for $V$.

The goal of this work is to give a construction of the so-far missing commutative product usually present in the BV formalism. To this end, we introduce the notion of a connected sum on a modular operad $P$. For a modular operad $P$ with such connected sum, we define a commutative product on the space $\text{Fun}_P(V)$, compatible with the structure introduced by Barannikov. However, in this way we obtain a Beilinson-Drinfeld algebra, a close relative of a BV algebra. In the examples $QC$ and $QO$ coming from 2D topology, this structure is induced by the actual connected sum of...
surface. Thus, the distinction between BV and BD algebras acquires a topological explanations: see Figure 3. We explicitly describe the commutative products in these two examples, getting the expected products of polynomials and cyclic words built from letters in $V^*$. Instead of the equation $\Delta e^{S/\hbar} = 0$, one usually writes the formally equivalent quantum master equation

$$2\hbar \Delta S + \{S, S\} = 0,$$

which was also the form used in [1]. Using the commutative product, we can now make sense of the exponential $e^{S/\hbar}$, after an appropriate completion. With a simple non-degeneracy condition on the connected sum, we prove that the above quantum master equation is equivalent to $\Delta e^{S/\hbar} = 0$.

 Commutative products and BV algebra structures on $\text{Fun}_p(V)$ coming from the disjoint union of surfaces were considered Kaufmann, Ward and Zúñiga in [18]; see also [24, 23, 19]. Connected sum for the modular operad $\mathcal{QO}$ of surfaces was considered by Kaufmann, Ward and Zúñiga in [18]; see also [24, 23, 19]. Connected sum for the modular operad $\mathcal{QO}$ appeared in the recent work of Berger and Kaufmann [5]. See Remark 4 for a more detailed review.

1.1 Notations and conventions

We consider $\mathbb{Z}$-graded vector spaces over a field with characteristic zero. The degree of a homogeneous element $v$ is denoted $|v|$. Differentials have degree $+1$.

We denote by $\sqcup$ the disjoint union and \setminus the set difference. By $[n]$, we will denote the set $\{1, 2, \ldots, n\}$. The permutation group of $[n]$ is denoted by $\Sigma_n$. The cardinality of a set $A$ is denoted \text{card}A.

1.2 Acknowledgements

We would like to thank Ralph Kaufmann for answering our questions about Feynman categories and connected sums.

The research of B.J. was supported by grant GAČR EXPRO 19–286 28X. The research of J.P. was supported by the NCCR SwissMAP and Postdoc.Mobility 203 065 grants of the SNSF.

This paper gives an extended account of some of the results announced by the third author in [22].

2 Modular operads and the connected sum

Modular operads were introduced by Getzler and Kapranov [15]. We start by recalling a definition of a modular operad in the spirit of [11, 12].

**Definition 1.** Denote $\text{Cor}$ the category of stable corollas: the objects are pairs $(C, G)$ with $C$ a finite set and $G$ a non-negative half-integer such that the stability condition is satisfied:

$$2(G – 1) + \text{card}(C) > 0. \quad (1)$$

Morphisms $(C, G) \to (D, G')$ exist only if $G = G'$, in which case they are bijections $C \xrightarrow{\sim} D$.

**Remark 2.** The condition of stability was introduced in the context of modular operads by Getzler and Kapranov, and its name comes from the theory of moduli spaces of curves. In our context, the stability condition will ensure convergence of certain formal exponentials, see Corollary 35.1

**Definition 3.** A modular operad $\mathcal{P}$ is a functor $\mathcal{P}$ from $\text{Cor}$ to the category of dg vector spaces (with morphisms of degree 0), together with a collection of degree 0 chain maps

$$\{\rho_{\alpha b} : \mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \to \mathcal{P}(C_1 \sqcup C_2, G_1 + G_2) \mid (C_1, G_1), (C_2, G_2) \in \text{Cor} \} \quad \text{and} \quad \{\sigma_{ab} = \rho_{ba} : \mathcal{P}(C \sqcup \{a, b\}, G) \to \mathcal{P}(C, G + 1) \mid (C, G) \in \text{Cor} \}.$$

These data are required to satisfy the following axioms:

- (MO1) $\mathcal{P}(\rho|C_1 \sqcup \sigma|C_2) \circ \rho_{\alpha b} = \rho_{\alpha|C_1} \circ \sigma_{\beta|C_2} (\mathcal{P}(\rho) \otimes \mathcal{P}(\sigma))$,
- (MO2) $\mathcal{P}(\rho|C) \circ \sigma_{ab} = \rho_{\rho(a)|b} \circ \mathcal{P}(\rho)$,
- (MO3) $\rho_{\alpha b}(x \otimes y) = (-1)^{|x||y|} \rho_{\alpha b}(y \otimes x)$ for any $x \in \mathcal{P}(C_1 \sqcup \{a\}, G_1), y \in \mathcal{P}(C_2 \sqcup \{b\}, G_2)$,
• (MO4) $o_{ab} \circ_c o_{cd} = o_{cd} \circ_{ab}$
• (MO5) $o_{ab} \circ_c o_{cd} = a \circ_{ab}$
• (MO6) $a \circ_{ab} (o_{cd} \otimes 1) = o_{cd} \circ_{ab}$ and
• (MO7) $a \circ_{ab} (1 \otimes o_{cd}) = c \circ_{cd} = (a \circ_{ab} \otimes 1)$,
wherever the expressions make sense.

As in [12, 11], we also define odd modular operads, which are special cases of twisted modular operads of [15].

**Definition 4.** An odd modular operad is defined similarly as the modular operad with the only exception of the operadic compositions, now denoted as $a \bullet_{ab}$ and $\bullet_{ab}$, being of degree 1. Correspondingly, the above axioms (MO4)–(MO7) will get an extra minus sign. See [12, Definition 4.] for details.

### 2.1 Connected sum

Let us now define connected sum on a modular operad, motivated by the connected sum operation on surfaces.

**Definition 5.** A modular operad with a connected sum is a modular operad $P$ equipped with two collection of degree 0 chain maps

\[ \#_2: P(C, G) \otimes P(C', G') \to P(C \sqcup C', G + G' + 1) \]

and

\[ \#_1: P(C, G) \to P(C, G + 2) \]

such that

• (CS1) $(P(\sigma \sqcup \sigma'))\#_2 = \#_2(P(\sigma) \otimes P(\sigma'))$, $P(\sigma)\#_1 = \#_1 P(\sigma)$ for all bijections $\sigma: C \to D$, $\sigma': C' \to D'$,
• (CS2) $\#_2 \tau = \#_2$, where $\tau$ is the monoidal symmetry (from the category of graded vector spaces),
• (CS3) $\#_2(1 \otimes \#_2) = \#_2(\#_2 \otimes 1)$, $\#_2(\#_1 \otimes 1) = \#_1 \#_2$
• (CS4) as maps $P(C \sqcup \{a, b\}, G) \to P(C, G + 3)$,
\[ o_{ab} \#_1 = \#_1 o_{ab}, \]
• (CS5a) as maps $P(C, G) \otimes P(C', G') \to P(C \sqcup C' \setminus \{a, b\}, G + G' + 2)$,
\[ o_{ab} \#_2 = \begin{cases} \#_2(o_{ab} \otimes 1) & \text{if } a, b \in C \\ \#_2(1 \otimes o_{ab}) & \text{if } a, b \in C' \\ \#_1 o_{ab} & \text{if } a \in C, b \in C' \\ \#_1 o_{a} & \text{if } b \in C, a \in C', \end{cases} \]
• (CS5b) as maps $P(C \sqcup \{a\}, G) \otimes P(C' \sqcup \{b\}, G') \to P(C \sqcup C', G + G' + 2)$,
\[ o_{ab} \#_1 = \#_1 o_{ab}, \]
• (CS6) as maps $P(C \sqcup \{a\}, G) \otimes P(C', G') \otimes P(C'', G'') \to P(C \sqcup C' \sqcup C'', \{b\}, G + G' + G'' + 1)$,
\[ o_{ab}(1 \#_2) = \begin{cases} \#_2(o_{ab} \otimes 1) & \text{if } b \in C' \\ \#_2(1 \otimes a o_{ab})(\tau \otimes 1) & \text{if } b \in C'', \end{cases} \]
where the map $(\tau \otimes 1): P(C, G) \otimes P(C', G') \otimes P(C'', G'') \to P(C', G') \otimes P(C, G) \otimes P(C'', G'')$ switches the first two tensor factors.

\[1\]The seemingly strange shift of $G$ by 1 and 2 is chosen to match already existing conventions, see Remark [12] for details.
Remark 6. If one considers the disjoint union of surfaces, instead of the connected sum, its compatibility with the operadic compositions $a \circ b$ and $a \circ b$ will look similarly to Definition 5. An important difference will appear in axioms (CS5a): the disjoint union followed by $a \circ b$ is just equal to $a \# b$, and there is no analogue of $\#_1$. Such approach to operads, abstracting the disjoint union, was (to our knowledge) first considered by Schwarz [23, Sec. 2]. There, $\nu$ is used for the disjoint sum, $\sigma$ for the self-composition $a \circ b$; the composition $a \circ b$ can be defined from these two operations.

Later, a similar operation was considered in generality by Borisov and Manin [4] and for modular operads by Kaufmann and Ward [19], under the name of mergers/horizontal compositions. See e.g. [19] Eq. (5.4),(5.5) for the disjoint-union-analogue of (CS5a). The commutative product and the resulting BV algebra was studied by Kaufmann, Ward and Zúñiga in [18]. A notable precursor in string field theory is the work of Sen and Zwiebach [24, Sec. 7.1].

The connected sum of surfaces was considered, for the modular operad $QO$, by Berger and Kaufmann [5, Sec. 5.6]. There, they notice the need for an analogue of (CS5a) [5, Sec. 5.6, “equation 2.9 does not hold”] and remark that such connected sums define Feynman categories, functors out of which are equivalent to our modular operads with connected sum.

Similarly to Definition 5, we can consider an odd modular operad equipped with a connected sum.

Definition 7. An odd modular operad with a connected sum is as in Definition 5 with the black bullet replacing the white one.

Note that $\#_1$ and $\#_2$ are again degree 0 operations. To make the difference between the odd and untwisted cases more explicit, we will write down the axioms (CS5a) and (CS6), evaluated on elements, in both cases

If $p \in \mathcal{P}(C,G), p' \in \mathcal{P}(C', G')$ and $p'' \in \mathcal{P}(C'', G'')$, then in the untwisted case (CS5a),

$$\circ_{ab}(p \#_2 p') = \begin{cases} (a \circ b) \#_2 p' & \ldots a, b \in C \\ p \#_2 (a \circ b) p' & \ldots a, b \in C' \\ \#_1 (p_a \circ b p') & \ldots a \in C, b \in C' \\ \#_1 (p \circ a p') & \ldots b \in C, a \in C'. \end{cases}$$

and in the odd case,

$$\bullet_{ab}(p \#_2 p') = \begin{cases} (a \bullet b) \#_2 p' & \ldots a, b \in C \\ p \#_2 (a \bullet b) (\#_1 |p|) & \ldots a, b \in C' \\ \#_2 (p_a \bullet b) & \ldots a \in C, b \in C' \\ \#_2 (p \bullet a) & \ldots b \in C, a \in C'. \end{cases}$$

Concerning (CS6), in untwisted case, we have

$$p_a \circ b (p' \#_2 p'') = \begin{cases} (p_a \circ b p') \#_2 p'' & \ldots b \in C', \\ p' \#_2 (p_a \circ b p'') & \ldots b \in C'', \end{cases}$$

whereas in the odd case,

$$p \bullet b (p' \#_2 p'') = \begin{cases} (p_a \bullet b p') \#_2 p'' & \ldots b \in C', \\ (\#_1 |p| |p'| + |p'| p' \#_2 (p_a \bullet b p'') & \ldots b \in C''. \end{cases}$$

Remark 8. In all of the examples we will consider, $\#_1$ will be injective. In this case, the axiom (CS5a) determines the operadic compositions $a \circ b$ in terms of $a \circ b$, $\#_2$, and $\#_1$, and similarly for $a \bullet b$.

2.2 Examples of connected sum

We will now describe a connected sum on two basic modular operads: the quantum closed operad $QC$ and the quantum open operad $QO$ (see [12] [11] for their description as modular operads).
Example 9. The quantum closed modular operad $QC$ is the modular envelope of the cyclic commutative operad, but has an explicit description as follows. For each finite set $C$ and each non-negative integer $g$, we define $QC(C, 2g + \text{card}(C)/2 - 1)$ to be a one-dimensional vector space, with generator called $C^g$. This should be seen the homeomorphism class of connected compact oriented surfaces of genus $g$ and with punctures labelled by elements of $C$. The operadic structure corresponds to sewing punctures together. See Remark 12 for the origin of the definition $G = 2g + \text{card}(C)/2 - 1$.

The connected sum is defined simply as

$$C_1^{g_1} \#_2 C_2^{g_2} = (C_1 \sqcup C_2)^{g_1+g_2},$$

$$\#_1 (C^g) = C^{g+1},$$

which increases the “operadic” genus $G = 2g + \text{card}(C)/2 - 1$ by 1 and 2, respectively. Geometrically, $\#_2$ corresponds to the connected sum of surfaces and $\#_1$ to adding a handle, as on Figure 1 involving connected sums of $C_1^1$ and $C_2^2$ with $\text{card}(C_1) = 1$ and $\text{card}(C_2) = 3$.

![Figure 1: Connected sum on the quantum closed operad.](image)

The axioms of the connected sum are satisfied trivially, but they also have a topological interpretation, as on Figure 2.

![Figure 2: Axiom CS5a - cases $o_{ab} \#_2 = \#_2 (o_{ab} \otimes 1)$ and $o_{ab} \#_2 = \#_1 o_{ab}$.](image)

Example 10. The quantum open modular operad $QO$ is defined as follows. A cycle $o$ in a set $O$ is a (possibly empty) cyclic word made of several distinct elements of $O$. The components of the modular operad $QO$ are given as

$$QO(O, G) \equiv \text{Span}_k \left\{ \{o_1, \ldots, o_b\}^g \mid b, g \in \mathbb{N}_0, o_i \text{ cycle in } O, \bigcup_{i=1}^b o_i = O, G = 2g + b - 1 \right\}.$$ 

Geometrically, the generators correspond to homeomorphism class of a compact oriented surface with genus $g$, $b$ boundaries and punctures on the boundaries labelled by elements of $O$. The operadic compositions correspond to sewing/self-sewing of surfaces along punctures.

Similarly to the previous example, the modular operad $QO$ is the modular envelope of the cyclic associative operad $Ass$ by a result of Doubek [10].

The connected sum is defined as

$$\{o_1, \ldots, o_b\}^{g_1} \#_2 \{o_1', \ldots, o_b'\}^{g_2} = \{o_1, \ldots, o_{b_1}, o_1', \ldots, o_{b_2}'\}^{g_1+g_2},$$

$$\#_1 (\{o_1, \ldots, o_b\}^g) = \{o_1, \ldots, o_b\}^{g+1}.$$
with the same geometric interpretation as in the previous example.

These two examples can be combined as follows.

**Example 11.** Although we did not introduce colored modular operads, it is easy to see that we can straightforwardly combine the quantum closed operad and quantum open operad into a two-colored quantum open-closed operad $\mathbb{QOC}$ [12]. It has components $\mathbb{QOC}(C, O, G)$ generated by homeomorphism classes of surfaces with closed punctures labelled by $C$ and open punctures (lying on the boundary) labelled by $O$. In this case, $G = 2g + b + \text{card}(C)/2 - 1$

**Remark 12.** Using the above examples, we can explain the dependence of $G$ on $g$ and the shifts of $G$ in Definition [5]. Already for the operad $\mathbb{QO}$, the operadic self-composition $\circ_{ab}$ can act on punctures on two different boundary components or on same boundary. Since these two cases change $g$ and $b$ differently, we are led to define $G$ as their linear combination.

If we require the operations $\circ_{ij}$ and $\circ_{ij}$ to change $G$ by $+1$ and $0$, then for the more general quantum open-closed operad $\mathbb{QOC}$ we can choose $G = og + \frac{b}{2} + \frac{\text{card}(C)}{2} + \frac{\text{card}(O)}{2} + 1 - \alpha$ for any $\alpha$. Moreover, $\#_2$ will increase $G$ by $\alpha - 1$ and $\#_1$ by $\alpha$. Our choice of $G$ corresponds to $\alpha = 2$, which was used in [12] and ultimately comes from the work of Zwiebach, where he wants $G = 0$ for the disc with three open punctures on the boundary [26, eq. 3.11].

Similarly, there exists colored generalization of these operads coming from string field theory, let us mention the one coming from the type II superstring field theory.

**Example 13.** A four-colored operad relevant relevant to type II superstring field theory was introduced in [17]. The geometric picture here is based on super Riemann surfaces with four kinds of punctures corresponding to the four respective sectors $NS - NS$, $NS - R$, $R - NS$ and $R - R$. The geometric representation of the connected sum would remain the same as above.

### 2.3 Endomorphism operad and the connected sum

In this section, we will describe our main example of an odd modular operad with a connected sum, the endomorphism operad. Let us first recall the definition of the unordered tensor product and positional derivatives.

**Definition 14.** For a finite set $C$ with $\text{card}(C) = n$ and a vector space $V$, we define the **unordered tensor product** as

$$\bigotimes_C V = \bigoplus_{\psi : C \to [n]} V^\otimes n / \sim$$

where the equivalence relation is given by $i_\psi(v_1 \otimes \ldots \otimes v_n) \sim i_{\sigma \psi}(\sigma(v_1 \otimes \ldots \otimes v_n))$, where $\sigma \in \Sigma_n$ and $i_\psi : V^\otimes n \hookrightarrow \bigoplus_{\psi : C \to [n]} V^\otimes n$ is the canonical inclusion into the $\psi$-th summand.

The map $V^\otimes n \xrightarrow{i_\psi} \bigotimes_C V$, the inclusion $i_\psi$ followed by the natural projection, is an isomorphism. Its inverse $\bigotimes_C V \to V^\otimes n$ will be denoted as $w \mapsto (w)_\psi$.

Here are some useful facts about the unordered tensor product, see [12, Definition 10] and [24, Lemma 4].

**Lemma 15.**

1. For an isomorphism $\psi : C \xrightarrow{\sim} [n]$ and a permutation $\sigma : [n] \to [n]$

   $$(w)_{\sigma \psi} = \sigma(w)_\psi, \quad w \in \bigotimes_C V.$$

2. Any isomorphism $\rho : C \xrightarrow{\sim} D$ defines an isomorphism $\rho : \bigotimes_C V \to \bigotimes_D V$ by

   $$(\rho x)_\varphi = (x)_{\varphi \rho}, \quad x \in \bigotimes_D V, ~ \varphi : D \xrightarrow{\sim} [\text{card}(D)].$$

---

2In other words, choosing a linear order of $C$ gives an isomorphism between $\bigotimes_C V$ and $V^\otimes n$, with different isomorphisms related by the corresponding permutation.
3. There is a canonical isomorphism \((\bigotimes C V) \otimes (\bigotimes D V) \cong \bigotimes_{C \sqcup D} (V)\), given by ordering on \(C \sqcup D\) induced on the orderings on \(C\) and \(D\), i.e. by

\[
\left(\bigotimes_{C} V \otimes \bigotimes_{D} V\right) \xrightarrow{(-) \otimes (-)} \bigotimes (\text{card}(C) + \text{card}(D)) \xrightarrow{(-) \otimes \phi} \bigotimes_{C \sqcup D} V
\]

where \(\psi \cup \phi\) is the induced ordering on \(C \sqcup D\) from \(\psi: C \xrightarrow{\sim} [\text{card}(C)]\) and \(\phi: D \xrightarrow{\sim} [\text{card}(D)]\).

The composition \((\bigotimes C V) \otimes (\bigotimes D V) \cong \bigotimes_{C \sqcup D} (V) \cong (\bigotimes D V) \otimes (\bigotimes C V)\) is the monoidal symmetry \(\tau: (\bigotimes C V) \otimes (\bigotimes D V) \rightarrow (\bigotimes D V) \otimes (\bigotimes C V)\).

For \(c \in C\), it makes sense to talk about the \(c\)-th element of \(\bigotimes C V\). For example we can contract it with \(v \in V\). This operation is captured in the following definition.

**Definition 16.** For \(v \in V\) and a finite set \(C\) of cardinality \(n\), let us define a **positional derivative**

\[
\partial_v^{(c)}: \bigotimes_{C \sqcup \{c\}} V^* \rightarrow \bigotimes_C V^*
\]

by setting, for \(f \in \bigotimes_{C \sqcup \{c\}} V^*\),

\[
(\partial_v^{(c)} f)_{\psi} = v \otimes 1_{(V^*)^{\otimes n}}(f)_{\psi}
\]

where, for arbitrary \(\psi: C \xrightarrow{\sim} [n]\), the map \(\tilde{\psi}: C \sqcup \{c\} \xrightarrow{\sim} [n + 1]\) is defined by \(\tilde{\psi}(c) = 1\) and \(\tilde{\psi}(c') = \psi(c') + 1\) for \(c' \in C\). On the right hand side, we see \(v \in V\) as a map \(V^* \rightarrow k\) via \(\alpha \mapsto (-1)^{|v||\alpha|} \alpha(v)\).

Here we collect some of the useful properties of the positional derivative.

**Lemma 17.**

1. Under the isomorphism \(\bigotimes_{C \sqcup \{c\} \sqcup D} V^* \cong (\bigotimes_{C \sqcup \{c\}} V^*) \otimes (\bigotimes_D V^*)\), the positional derivative \(\partial_v^{(c)}\) is sent to \(\partial_v^{(c)} \otimes 1_{(D) V^*}\).

2. For \(c \in C\), and \(\rho: C \xrightarrow{\sim} D\) we have \(\rho|_{C \sqcup \{c\}} \partial_v^{(c)} = \partial_v^{(\rho(c))} \rho\).

3. The positional derivatives graded-commute, i.e. \(\partial_v^{(c)} \partial_w^{(d)} = (-1)^{|v||w|} \partial_w^{(d)} \partial_v^{(c)}\).

Now, we can turn to the definition of an endomorphism modular operad.

**Definition 18.** Let \((V, d)\) be a dg vector space which is degree-wise finite-dimensional. An **odd symplectic form** \(\omega: V \otimes V \rightarrow k\) of degree \(-1\) is a nondegenerate graded-antisymmetric bilinear map.

If \(d(\omega) = 0\), i.e.

\[
\omega \circ (d \otimes 1_{V} + 1_{V} \otimes d) = 0,
\]

we call \((V, d, \omega)\) a **dg symplectic vector space**.

If \(\{e_i\}\) is a homogeneous basis of \(V\) and \(\omega_{kl} = \omega(e_k, e_l)\), we define

\[
e_k = \sum_i (-1)^{|e_i|} \omega_{ik} e_l.
\]

Note that \(\omega^{kl}\), defined by \(\sum_l \omega^{kl} \omega_{lm} = \delta_{m}^{k}\), is well defined, for \(V\) degree-wise finite-dimensional. This is because the infinite matrix \(\omega_{ij}\) is nonzero only in finite blocks corresponding to \(V_k\) and \(V_{l-k}\), and we only need to invert those blocks. Similarly, the sum in the definition of \(e_k\) is well defined, since it only has finite number of nonzero terms for fixed \(k\). The fact that \(\omega\) is degree \(-1\) implies \(|e_k| = 1 - |e_k|\).

**Definition 19.** The **odd endomorphism modular operad** \(E_V\) is the odd modular operad defined by

\[
E_V(C, G) = \bigotimes_C V^*,
\]

3Note, this means that \(\omega(u, v) \neq 0\) implies \(|u| + |v| = 1\) and \(\omega(v, u) = (-1)^{|v||u|} + 1 \omega(u, v) = -\omega(u, v)\).

4Note that the tensor powers of \(V^*\) are not degree-wise finite-dimensional, even for degree-wise finite-dimensional \(V^*\).
with an action of $\rho: C \rightarrow D$ given by (4).

The compositions and self-compositions are defined as follows. If $f \in \mathcal{E}_V(C_1 \sqcup \{a\}, G_1)$ and $g \in \mathcal{E}_V(C_2 \sqcup \{b\}, G_2)$, then

$$(f \bullet_b g) = \sum_k (-1)^{|f||e^k|}(\partial^{(a)}_{e^k} f) \otimes (\partial^{(b)}_{e^k} g),$$

where we use the canonical isomorphism $(\otimes C_1 V^*) \otimes (\otimes C_2 V^*) \cong \otimes_{C_1 \sqcup C_2} (V^*)$.

The self-composition for $f \in \otimes_{C\setminus\{a,b\}} V^*$ is given by

$$\bullet_{ab} f = \sum_k \partial^{(a)}_{e^k} \partial^{(b)}_{e^k} f.$$  \tag{6}

This is well defined because $f$ is a finite sum of tensor products of elements of $V^*$.

This operad is equipped with a differential given for $f \in \otimes_C V^*$ by

$$(df)_{\omega} = d((V^*)^\otimes)\circ (f)_{\omega}.$$ 

where the differential on $\alpha \in V^*$ is $(d\alpha)(v) = (-1)^{\alpha+1} \alpha(dv)$ is defined so that the pairing $V^* \otimes V \rightarrow k$ is a chain map.

**Lemma 20.** This defines a structure of an odd modular operad on $\mathcal{E}_V$.

**Proof.** These operations agree with the standard definition of an odd endomorphism modular operad of Markl, i.e. [21] Equations (12b) and (13b)], just that we use $\otimes_C (V^*)$ instead of $(\otimes_C V)^*$. This is because the partial derivative $\partial^{(a)}: V \otimes \otimes_{C\setminus\{a\}} V^* \rightarrow \otimes_C V^*$ is the composition $V \otimes \otimes_{C\setminus\{a\}} V^* \cong V \otimes V^* \otimes \otimes_C V^* \xrightarrow{ev\otimes} \otimes_C V^*$. We get the standard definition because our the degree 1 tensor $e_i \otimes e^i$ comes from left in the definition of $\bullet_{a,b}$ and $\bullet_{ab}$.

Alternatively, the (odd versions of) axioms from Definition 3 follow easily from Lemma 15 and Lemma 17. For example, the axiom (MO3) is

$$a \bullet_b (f \otimes g) = \partial^{(a)}_{c^k} \otimes \partial^{(b)}_{c^k} (f \otimes g) = (-1)^{|f||g|} \partial^{(a)}_{c^k} \otimes \partial^{(b)}_{c^k} \tau(g \otimes f) = (-1)^{|f||g|} (f \otimes g)(\partial^{(a)} \otimes \partial^{(b)}) (g \otimes f)$$

and $\sum_k e^k \otimes e^k = \sum_k (-1)^{|c_k||l_k|} e^k \otimes c^k = \sum c^k \otimes e^k$, which holds in the direct product $\prod_i V_i \otimes V_{i-1}$. Thus, using Item 3 of Lemma 15 we get that the right hand side is indeed $(-1)^{|f||g|} a \bullet_b (g \otimes f)$.

Now, we can define the connected sum on the endomorphism operad.

**Definition 21.** Define

$$\#_1: \mathcal{E}_V(C, G) \rightarrow \mathcal{E}_V(C, G + 2)$$

to be the identity on $\otimes_C V^*$ and define

$$\#_2: \mathcal{E}_V(C_1, G_1) \otimes \mathcal{E}_V(C_2, G_2) \rightarrow \mathcal{E}_V(C_1 \sqcup C_2, G_1 + G_2 + 1)$$

to be the canonical isomorphism $(\otimes C_1 V^*) \otimes (\otimes C_2 V^*) \cong \otimes_{C_1 \sqcup C_2} (V^*)$.

**Lemma 22.** The odd modular operad $\mathcal{E}_V$, with the above defined operations $\#_1$ and $\#_2$, is an odd modular operad with a connected sum.

**Proof.** (CS1) follows easily from the definition of the action of isomorphisms (4). (CS2) follows from item 3 of Lemma 15. (CS3) follows from associativity of the tensor product. (CS4) and (CS5b) are trivial, since $\#_1$ is the identity. The remaining axioms follow from the Lemma 17 for example (CS5a) gives

$$a \bullet_b \#_2 = \sum_k \partial^{(a)}_{e^k} \partial^{(b)}_{e^k} \#_2$$

and commuting the positional derivatives through $\#_2$, we get the four possibilities via Item 1 of Lemma 17.

**Remark 23.** To encode a modular operad $\mathcal{P}$, it is enough to keep the spaces $\mathcal{P}(\{n\}, G)$. The operations then involve a choice of ordering on e.g. $[n_1] \sqcup [n_2]$ for $\#_2$. Choosing the ordering by placing $[n_1]$ before $[n_2]$, the operadic structure map of $\mathcal{E}_V$ acquire a particularly simple form [12] III.D, E]; the connected sum $\#_2$ turns into the identification

$$\mathcal{E}(\{n_1\}, G_1) \otimes \mathcal{E}(\{n_2\}, G_2) = (V^*)^\otimes_{n_1} \otimes (V^*)^\otimes_{n_2} \xrightarrow{\#_2} (V^*)^\otimes_{n_1+n_2} = \mathcal{E}(\{n_1 + n_2\}, G_1 + G_2 + 1).$$

Since we replaced the category of corollas by its skeleton $\{([n], G)\}$, this version of a modular operad is usually called skeletal.
In [1, Section 5], Barannikov introduced a dg Lie algebra structure on the (shifted) space of formal $\mathcal{P}$-functions, for a modular operad $\mathcal{P}$. If $\mathcal{P}$ is endowed with a connected sum, this space of functions acquires a commutative product and becomes a Beilinson-Drinfeld algebra, a close relative of a Batalin-Vilkovisky algebra.

3.1 Beilinson-Drinfeld algebras

Beilinson-Drinfeld algebras, or BD algebras for short, appeared in the work of Beilinson and Drinfeld [4], see also [16, 9].

Definition 24. A BD algebra is a graded commutative associative algebra on a graded module $\mathcal{F}$ over $k[[\varepsilon]]$, flat over $k[[\varepsilon]]$, with a bracket $\{ \}, : \mathcal{F}^{\otimes 2} \to \mathcal{F}$ of degree 1 that satisfies

$$\{ X, Y \} = (-1)^{|X||Y|+1} \{ Y, X \},$$

$$\{ X, \{ Y, Z \} \} = \{ \{ X, Y \}, Z \} + (-1)^{|X|+1} \{ Y, \{ X, Z \} \},$$

$$\{ X, YZ \} = \{ X, Y \} Z + (-1)^{|X||Y|} Y \{ X, Z \},$$

and a square zero operator $\Delta : \mathcal{F} \to \mathcal{F}$ of degree 1 such that

$$\Delta(\Delta Y) = (\Delta X)Y + (-1)^{|X|}X\Delta Y + (-1)^{|X|}\varepsilon \{ X, Y \}.$$  (8)

If $\mathcal{F}$ is also equipped with a differential, we require $\Delta$ and the bracket to commute with it. For algebras with unit 1, we will require $\Delta(1) = 0$.

3.2 Formal functions associated to a modular operad

Let us consider a modular operad $\mathcal{P}$ and an odd modular operad $\mathcal{Q}$. Define

$$\text{Fun}(\mathcal{P}, \mathcal{Q})(n, G) = (\mathcal{P}(n, G) \otimes Q(n, G))^\Sigma_n,$$

$$\text{Fun}(\mathcal{P}, \mathcal{Q}) = \prod_{n \geq 0} \prod_{G \geq 0} \text{Fun}(\mathcal{P}, \mathcal{Q})(n, G).$$

In [1], Barannikov introduced the following operations of degree 1, defined on components

$$\begin{align*}
d & : \text{Fun}(\mathcal{P}, \mathcal{Q})(n, G) \to \text{Fun}(\mathcal{P}, \mathcal{Q})(n, G), \\
\Delta & : \text{Fun}(\mathcal{P}, \mathcal{Q})(n+2, G) \to \text{Fun}(\mathcal{P}, \mathcal{Q})(n, G+1), \\
\{ -,- \} & : \text{Fun}(\mathcal{P}, \mathcal{Q})(n_1 + 1, G_1) \otimes \text{Fun}(\mathcal{P}, \mathcal{Q})(n_2 + 1, G_2) \to \text{Fun}(\mathcal{P}, \mathcal{Q})(n_1 + n_2, G_1 + G_2),
\end{align*}$$

by

$$\begin{align*}
d &= d\varphi \otimes 1 - 1 \otimes d\varphi, \\
\Delta &= (\varphi_{ab} \otimes \varphi_{ab})(\theta \otimes \theta), \\
\{ -,- \} &= (\varphi_{ab} \otimes \varphi_{ab})(\theta_1 \otimes \theta_2 \otimes \theta_1 \otimes \theta_2)(1 \otimes \tau \otimes 1)(X \otimes Y),
\end{align*}$$

for arbitrary bijection $\theta : [n+2] \xrightarrow{\sim} [n] \sqcup \{ a, b \}$; and

$$\begin{align*}
\{ X, Y \} &= (-1)^{|X|}2 \sum_{C_1, C_2} (\varphi_{\theta_1} \otimes \varphi_{\theta_2})(\theta_1 \otimes \theta_2 \otimes \theta_1 \otimes \theta_2)(1 \otimes \tau \otimes 1)(X \otimes Y),
\end{align*}$$

where we sum over all disjoint decompositions $C_1 \sqcup C_2 = [n_1 + n_2]$, such that $\text{card}(C_1) = n_1$, $\text{card}(C_2) = n_2$, the bijections $\theta_1 : [n_1 + 1] \xrightarrow{\sim} C_1 \sqcup \{ n \}$, $\theta_2 : [n_2 + 1] \xrightarrow{\sim} C_2 \sqcup \{ b \}$ are chosen arbitrarily, and $\tau$ is the monoidal symmetry. These operations are then extended to the whole $\text{Fun}(\mathcal{P}, \mathcal{Q})$.

---

$^5$We write $\theta \otimes \theta$ instead of $\mathcal{P}(\theta) \otimes \mathcal{Q}(\theta)$ for brevity.

$^6$No summation over those.
Theorem 25 ([1]). The maps $d$, $\Delta$ and $\{\cdot,\cdot\}$ are well defined, independent of the choice of $\theta$, $\theta_1$, $\theta_2$ and they satisfy the following properties

\[
d^2 = 0, \\
d\{\cdot,\cdot\} + \{\cdot,\cdot\}(d \otimes 1 + 1 \otimes d) = 0, \\
\Delta^2 = 0, \\
\Delta\{\cdot,\cdot\} + \{\cdot,\cdot\}(\Delta \otimes 1 + 1 \otimes \Delta) = 0, \\
\Delta d + d\Delta = 0,
\]

and the bracket satisfies

\[
\{X,Y\} = -(-1)^{|X||Y|+1}\{Y,X\}, \\
\{X,\{Y,Z\}\} = \{\{X,Y\},Z\} + (-1)^{|X|+|Y|+1}\{Y,\{X,Z\}\}.
\]

See [1] Section 5 and also [12] Theorem 20 for a more detailed proof (our bracket $\{X,Y\}$ equals $(-1)^{|X|}|2$ times their bracket). This structure was called a generalized Batalin-Vilkovisky algebra in [12], since it lacks a compatible commutative product.

The motivation for this structure comes from the fact that morphisms from the Feynman transform of $P$ to $Q$ are in bijection with degree-0 elements $S \in \text{Fun}(P,Q)$ that satisfy the quantum master equation $dS + \Delta S + \frac{1}{2}\{S,S\} = 0$, see [1] [12].

3.3 Connected sum and a commutative product

Now we introduce a commutative product and an operation $\sharp$: $\text{Fun}(P,Q) \to \text{Fun}(P,Q)$, coming from $\#_2$ and $\#_1$.

Definition 26. Let $P$ be a modular operad and $Q$ an odd modular operad. Moreover, assume each of them equipped with a connected sum. Define a product

\[
*: \text{Fun}(P,Q)(n_1,G_1) \otimes \text{Fun}(P,Q)(n_2,G_2) \to \text{Fun}(P,Q)(n_1 + n_2,G_1 + G_2 + 1)
\]

as

\[
* = \sum_{C_1,C_2} (\#_2 \otimes \#_2)(\theta_1 \otimes \theta_2 \otimes \theta_1 \otimes \theta_2)(1 \otimes \tau \otimes 1),
\]

where, as before, the sum runs over all disjoint decompositions $C_1 \sqcup C_2 = [n_1 + n_2]$, such that $\text{card}(C_1) = n_1$, $\text{card}(C_2) = n_2$, the bijections $\theta_1: [n_1] \xrightarrow{\cong} C_1$, $\theta_2: [n_2] \xrightarrow{\cong} C_2$ are chosen arbitrarily, and $\tau$ is the monoidal symmetry.

We also define the operator $\sharp$: $\text{Fun}(P,Q)(n,G) \to \text{Fun}(P,Q)(n,G + 2)$ as

\[
\sharp = \#_1 \otimes \#_1.
\]

Finally, we extend $*$ and $\sharp$ on $\text{Fun}(P,Q)$ linearly.

Lemma 27. The maps $*$, $\sharp$ are well defined and $*$ doesn’t depend on the choice of $\theta_1, \theta_2$.

Proof. The product $*$ is well defined since only finite number of terms contribute to the component $\text{Fun}(P,Q)(n,G)$ of the result, i.e. those components $(n_1,G_1)$ and $(n_2,G_2)$ such that $n = n_1 + n_2$ and $G = G_1 + G_2 + 1$. The result is independent of choices of $\theta_i$, because different choices of $\theta_i$ differ by precomposition with a permutation of $[n]$, under which $\text{Fun}(P,Q)(n,G)$ is invariant.

Theorem 28. If $P$ and $Q$ are as in Definition 26 then $\text{Fun}(P,Q)$, with operations $d, \Delta, \{-,-\}$, $\sharp$ and $*$ defined above, satisfies

1. $*$ is a commutative associative product, i.e. on elements:

\[
X*Y = (-1)^{|X||Y|}Y*X \quad \text{and} \quad (X*Y)*Z = X*(Y*Z).
\]

2. $\Delta* = *(\Delta \otimes 1) + *(1 \otimes \Delta) + \sharp\{-,-\}$, i.e. on elements:

\[
\Delta(X*Y) = (\Delta X)*Y + (-1)^{|X|}X*(\Delta Y) + (-1)^{|X|}\sharp\{X,Y\}.
\]
3. \(-, -\)(1 \(\otimes\) \(*\)) = \(*\)(\(-, -\) \(\otimes\) 1) + \(*\)(1 \(\otimes\) \(-, -\))(\(\tau\) \(\otimes\) 1), i.e. on elements:
\[
\{X, Y \star Z\} = \{X, Y\} \star Z + (-1)^{|X||Y|+|Y|} Y \star \{X, Z\}.
\] (15)

4. The maps \(\sharp\) and \(*\) are chain maps with respect to the differential \(d\).

5. The map \(\sharp\) commutes with the other operations, i.e. \(\Delta \sharp = \sharp \Delta\), \(-, -\)(1 \(\otimes\) \(\sharp\)) = \(-, -\)(\(\sharp\) \(\otimes\) 1) = \(\sharp\)(\(-, -\)) and \(*\)(1 \(\otimes\) \(\sharp\)) = \(*\)(\(\sharp\) \(\otimes\) 1) = \(\sharp\)*. On elements, this gives
\[
\Delta(\sharp X) = \sharp(\Delta X),
\] (16)
\[
\{X, \sharp Y\} = \{\sharp X, Y\}.
\] (17)
\[
X \star (\sharp Y) = (\sharp X) \star Y = \sharp(X \star Y).
\] (18)

**Proof.** Let \(X = \sum_i x_P^i \otimes x_Q^i \in \text{Fun}(\mathcal{P}, \mathcal{Q})(n_X, G_X)\), where \(x_P^i \in \mathcal{P}(n_X, G_X)\) and \(x_Q^i \in \mathcal{Q}(n_X, G_X)\). For sake of brevity, we will omit the summation over \(i\) (including the index) from the notation. Hence, we will write \(X = x_P \otimes x_Q\). Similarly, we write \(Y = \sum_i y_P^i \otimes y_Q^i = y_P \otimes y_Q\) and \(Z = \sum_i z_P^i \otimes z_Q^i = z_P \otimes z_Q\) where \(y_P^i \in \mathcal{P}(n_Y, G_Y)\) etc.

**The point 1.** follows from (CS1), (CS2) and (CS3). For commutativity:
\[
X \star Y = \sum_{C_1, C_2} (-1)^{|x_P||y_Q|} (\theta_1 x_P \#_2 \theta_2 y_P) \otimes (\theta_1 x_Q \#_2 \theta_2 y_Q)
\]
\[
Y \star X = \sum_{C_1, C_2} (-1)^{|x_P||y_Q|} (\theta_1 y_P \#_2 \theta_2 x_P) \otimes (\theta_1 y_Q \#_2 \theta_2 x_Q) =
\sum_{C_1, C_2} (-1)^{|x_P||y_Q|+|y_P||y_Q|+|x_P||x_Q|} (\theta_2 x_P \#_2 \theta_1 y_P) \otimes (\theta_2 x_Q \#_2 \theta_1 y_Q) = (-1)^{|X||Y|} X \star Y.
\]

For associativity:
\[
(X \star Y) \star Z = \sum_{C_3, C_4} (-1)^{|x_P||y_Q|} ((\theta_3 x_P \#_2 \theta_4 y_P) \otimes (\theta_3 x_Q \#_2 \theta_4 y_Q)) \star Z =
\sum_{C_3, C_4} (-1)^{|x_P||y_Q|+|y_P||y_Q|+|x_P||x_Q|} (\theta_1 (\theta_3 x_P \#_2 \theta_4 y_P) \#_2 \theta_2 z_P) \otimes (\theta_1 (\theta_3 x_Q \#_2 \theta_4 y_Q) \#_2 \theta_2 z_Q),
\]
where \(C_1 \sqcup C_2 = [n_x + n_y + n_z]\) and \(C_3 \sqcup C_4 = [n_x + n_y]\), \(\theta_1: [n_x + n_y] \xrightarrow{\sim} C_1, \theta_2: [n_z] \xrightarrow{\sim} C_2, \theta_3: [n_x] \xrightarrow{\sim} C_3, \theta_4: [n_y] \xrightarrow{\sim} C_4\) are chosen arbitrarily. From (CS1), we get
\[
\theta_1(\theta_3 x_P \#_2 \theta_4 y_P) = \theta_1(\theta_3 \sqcup \theta_4)(x_P \#_2 y_P),
\]
where \((\theta_1 \sqcup \theta_4): [n_x] \sqcup [n_y + n_z] \xrightarrow{\sim} C_3 \sqcup C_4 = [n_x + n_y]\) and similarly for the \(Q\)-part. Therefore, we can rewrite the sums over decompositions \(C_1 \sqcup C_2\) and \(C_3 \sqcup C_4\) and actions of \(\theta\)'s as
\[
\sum_{E_1 \sqcup E_2 \sqcup E_3} (-1)^A (\psi_1 \sqcup \psi_2 \sqcup \psi_3)((x_P \#_2 y_P) \#_2 z_P) \otimes (\psi_1 \sqcup \psi_2 \sqcup \psi_3)((x_Q \#_2 y_Q) \#_2 z_Q),
\]
where \(A = ([x_P]+|y_P|) \cdot |z_P| + [x_Q] \cdot |y_Q|\), \(\psi_1: [n_x] \xrightarrow{\sim} E_1, \psi_2: [n_y] \xrightarrow{\sim} E_2, \psi_3: [n_z] \xrightarrow{\sim} E_3\) and the sum is over all decompositions \(E_1 \sqcup E_2 \sqcup E_3 = [n_x + n_y + n_z]\). Similarly, one gets
\[
X \star (Y \star Z) = \sum_{D_1, D_2, D_3} (-1)^{|y_P||z_P|} (X \star (\phi_3 y_P \#_2 \phi_4 z_P) \otimes (\phi_3 y_Q \#_2 \phi_4 z_Q)) =
\sum_{D_1, D_2, D_3} (-1)^{|y_P||z_P|+|y_Q||z_Q|} (\phi_1 x_P \#_2 \phi_2 (\phi_3 y_P \#_2 \phi_4 z_P) \otimes (\phi_1 x_Q \#_2 \phi_2 (\phi_3 y_Q \#_2 \phi_4 z_Q)),
\]
where \(\phi_1: [n_x] \xrightarrow{\sim} D_1, \phi_2: [n_y + n_z] \xrightarrow{\sim} D_2, \phi_3: [n_y] \xrightarrow{\sim} D_3, \phi_4: [n_z] \xrightarrow{\sim} D_4\). Rewriting this as a sum over decompositions \(E_1 \sqcup E_2 \sqcup E_3 = [n_x + n_y + n_z]\), this gives
\[
\sum_{E_1 \sqcup E_2 \sqcup E_3} (-1)^A (\psi_1 \sqcup \psi_2 \sqcup \psi_3)(x_P \#_2 (y_P \#_2 z_P)) \otimes (\psi_1 \sqcup \psi_2 \sqcup \psi_3)(x_Q \#_2 (y_Q \#_2 z_Q)).
\]
By (CS3), we finally get \((X \star Y) \star Z = X \star (Y \star Z)\).

**The point 2.** follows from (CS5a). The left side of the required equality is:

\[
\Delta(X \star Y) = \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |},
\]

where \(C_1 \cup C_2 = [n_x + n_y] \) and where we have chosen \( \phi = 1_{[n_x, n_y]} \) (i.e. \( a = n_x + n_y - 1, b = n_x + n_y \)). Now, we split the sum by distinguishing four cases according to positions of \( a, b \) in the decomposition \( C_1 \cup C_2 \) (cf. axiom (CS5a)). See also Figure 3.

\[
\Delta(X \star Y) = \sum_{C_1, C_2} (\phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |} + \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |} + \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |} + \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |}.
\]

It is easy to verify that the third and fourth lines give the same result. We compare the previous calculation with

\[
(\Delta X) \star Y = \sum_{C_1, C_2} (\phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + (1 + | x_1 | + | y_1 | + | y_1 |} + \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |},
\]

where \(C_1 \cup C_2 = [n_x + n_y - 2] \) with the choice \( \phi = 1_{[n_x]} \) and \( a = n_x - 1, b = n_x \).

\[
(-1)^{|X|} X \star (\Delta Y) = \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 | + | y_1 |},
\]

where we take \( \phi = 1_{[n_y]} \) and \( a = n_y - 1, b = n_y \), and

\[
(-1)^{|X|^2} \{X, Y\} = 2 \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{[x_0 || y_1 | + | y_1 | + | y_1 |}.
\]

It is now easy to see that the required equality holds.

**The point 3.** follows from (CS1) and (CS6). First observe that:

\[
\{X, Y \star Z\} = 2 \sum_{C_1, C_2} \phi(x_1 x_2 \theta_2 y_2) \otimes C_{ab} \phi(x_1 x_2 \theta_2 y_2) (-1)^{B}.\]

![Figure 3: Equation (14) pictorially.](image) On the LHS, the operator \(\Delta\) acts on a connected sum of two surfaces, connecting all pairs of punctures. On the RHS, we see three possible cases, depending on whether the punctures are both on the first surface, the second surface or there is one puncture on each surface. In the last case, the result has additional handle, giving the term \(\{X, Y\}^2\) of (14).
where we sum over all decompositions $C_1 \sqcup C_2 = [n_y + n_z, D_1 \sqcup D_2 = [n_x + n_y + n_z - 2]$ and
\(\theta_1: [n_y] \xrightarrow{\sim} C_1, \theta_2: [n_z] \xrightarrow{\sim} C_2, \phi_1: [n_x] \xrightarrow{\sim} D_1 \sqcup \{a\}, \phi_2: [n_y + n_z] \xrightarrow{\sim} D_2 \sqcup \{b\}$ are arbitrary bijections and $B = [yQ] \cdot |zp| + |xQ| \cdot (|yp| + |zp|) + |xp| + |yp| + |zp| + |X|$. We split the sum into two according to position of $b \in \phi_2(C_1)$ or $b \in \phi_2(C_2))$ and compare with the two terms on the right hand side of (15). The first term is
\[
\{X, Y\} \star Z = 2 \sum (\theta_1(\phi_1 x_Q a \circ_b \phi_2 y_Q) \#_2 \theta_2 z_Q) \oplus (\theta_1(\phi_1 x_Q a \circ_b \phi_2 y_Q) \#_2 \theta_2 z_Q) (-1)^C,
\]
where we sum over all decompositions $C_1 \sqcup C_2 = [n_x + n_y + n_z - 2], D_1 \sqcup D_2 = [n_x + n_y]$, and
\(\phi_1: [n_y] \xrightarrow{\sim} C_1, \phi_2: [n_z] \xrightarrow{\sim} C_2, \theta_1: [n_x + n_y] \xrightarrow{\sim} D_1, \theta_2: [n_z] \xrightarrow{\sim} D_2$ are arbitrary bijections, $C = [xQ] \cdot |yp| + |xp| + |yp| + |zp| \cdot (|xQ| + |yQ| + 1) + |X|$ and $a \in D_1, b \in D_2$ are arbitrary. The second term is
\[
Y \star \{X, Z\} = 2 \sum (\theta_1(\phi_1 y_Q \#_2 \theta_1(\phi_1 x_Q a \circ_b \phi_2 y_Q)) \oplus (\theta_1(\phi_1 y_Q \#_2 \theta_1(\phi_1 x_Q a \circ_b \phi_2 y_Q)) (-1)^D,
\]
where we sum over all decompositions $C_1 \sqcup C_2 = [n_x + n_y + n_z - 2], D_1 \sqcup D_2 = [n_x + n_z]$ and
\(\phi_1: [n_y] \xrightarrow{\sim} D_1, \phi_2: [n_z] \xrightarrow{\sim} D_2, \theta_1: [n_x + n_y] \xrightarrow{\sim} C_1, \theta_2: [n_x + n_z] \xrightarrow{\sim} C_2$ are arbitrary bijections, $D = [xQ] \cdot |xp| + |xp| + |zp| + |yQ| \cdot (|xQ| + |yQ| + 1) + |X|$ and $a \in D_1, b \in D_2$ are arbitrary.

Using (CS1) and (CS6) and collecting all the signs, one gets
\[
\{X, Y \star Z\} = \{X, Y\} \star Z + (−1)^{|X|+|Y|} Y \star \{X, Z\}.
\]

The point 4 follows directly from the definition of $\sharp$ and $\star$. In point 5, the compatibility of $\sharp$ with $\Delta$ follows from (CS4). The equation $\{X, Y\} = \sharp\{X, Y\}$ follows directly from (CS5b), the remaining equality follows from the symmetry of the bracket $\{−, −\}$. Similarly, the compatibility of $\sharp$ and $\star$ follows from (CS3) and the symmetry of $\star$.

### 3.4 Beilinson-Drinfeld algebras

Using the action of $\sharp$, we can turn $\text{Fun}(\mathcal{P}, Q)$ to a (non-unital) BD algebra.

**Definition 29.** For $f = \sum_{i \geq 0} f_i x^i \in k[[x]]$ and $p = \sum_{n,G \geq 0} p_{n,G} \in \text{Fun}(\mathcal{P}, Q)$, define the action of $f$ on $p$ by
\[
f p = \sum_{n,G \geq 0} f_i p_{n,G}.
\]

Note that only terms coming from $p_{n,G}$ for $G' \leq G$ contribute to the component $\text{Fun}(\mathcal{P}, Q)(n, G) = (\mathcal{P}(n, G) \otimes Q(n, G))^\Sigma_n$, and thus the result is well-defined.

**Lemma 30.** The space $\text{Fun}(\mathcal{P}, Q)$ equipped with the action of $k[[x]]$ becomes a graded module over $k[[x]]$ and the operations $d, \Delta, \{−, −\}$ and $\star$ are maps of graded modules.

This module is flat over $k[[x]]$ if and only if the maps $\sharp: (\mathcal{P}(n, G) \otimes Q(n, G))^\Sigma_n \rightarrow (\mathcal{P}(n, G + 2) \otimes Q(n, G + 2))^\Sigma_n$ are injective for all $n, G$.

Thus, if $\sharp$ is injective, $\text{Fun}(\mathcal{P}, Q)$ becomes a non-unital BD algebra. Note that in all of our examples ($Q\mathcal{C}, Q\mathcal{O}$ and $E_V$), all $\#_1$ are injective, which is a stronger condition than the injectivity of $\sharp$.

**Proof.** The first part of the lemma follows from Theorem 28.

Since $k[[x]]$ is a PID, being flat is equivalent to being torsion-free, i.e. no non-zero element of $\text{Fun}(\mathcal{P}, Q)$ is annihilated by a non-zero element of $k[[x]]$ [13 Corollary 6.3]. This is furthermore equivalent to $\text{Ker} x = 0$, since any non-zero element of $k[[x]]$ is equal to $x^i$ up to an invertible element, and if $x^i X = 0$ for minimal $i$, then $x^{i-1} X \neq 0$ is annihilated by $x$. Let us now show the two implications.

If $\sharp(X) = 0$ for some nonzero invariant $X \in \mathcal{P}(n, G) \otimes Q(n, G)$, then $x \in k[[x]]$ annihilates $X$. On the other hand, let us suppose that $x$ annihilates an element $\sum x_{n,G}$. Then each of the summands, an element of $\text{Fun}(\mathcal{P}, Q)(n, G)$, is in the kernel of $\sharp$. 

[13]
3.5 Quantum master equation

To be able to talk about the exponentials $e^{S/\kappa}$, we need to introduce negative powers of $\kappa$. To avoid various convergence issues, we will restrict the possible negative powers of $\kappa$. See Remark 35 at the end of this section explaining the motivation for the following definition.

**Definition 31.** Consider the space of fixed weight $w \in \frac{1}{2}Z$.

\[
\tilde{F}_w \equiv \bigoplus_{n/2+G+2q+1=w} \text{Fun}(P, Q)(n, G) \otimes k\kappa^q
\]

where by $k\kappa^q, q \in \mathbb{Z}$, we mean a 1-dimensional vector space, with a generator $\kappa^q$. Similarly, let

\[
\tilde{F}_{\geq w} = \prod_{w \geq w} \tilde{F}_w
\]

With the multiplication given by $\ast$ and by $\kappa^q \ast \kappa^{q_2} = \kappa^{q_1+q_2}$, this space becomes a graded-commutative algebra, with operations $d$, $\Delta$ and $\{-,-\}$ extended by $\kappa$-linearity (the bracket is possibly defined only partially, since it decreases the weight by 2). The action $\kappa: \kappa^q \mapsto \kappa^{q+1}$ makes $\tilde{F}_{\geq w}$ into a $k[[\kappa]]$-module.

**Definition 32.** Define the space $\text{Fun}_{\exp}(P, Q)$ by the following quotient

\[
\text{Fun}_{\exp}(P, Q) \equiv \tilde{F}_{\geq \frac{1}{2}} / \{\sharp X - \kappa X \mid X \in \tilde{F}_{\geq -\frac{1}{2}}\}.
\]

**Lemma 33.** (1) The space $\text{Fun}_{\exp}(P, Q)$ inherits the algebra structure, action of $k[[\kappa]]$ and the operations $\ast, d, \Delta$ and $\{-,-\}$, with the bracket defined only for arguments of total weight $\geq 5/2$. In the inherited weight grading, the maps $\ast, d, \Delta$ have weight 0, the bracket has weight $-2$ and $\kappa$ has weight 2. As a $k[[\kappa]]$-module, it is flat.

(2) The natural map $\iota: \text{Fun}(P, Q) \rightarrow \text{Fun}_{\exp}(P, Q)$, with the image in weight $> 2$ of $\text{Fun}_{\exp}(P, Q)$, is a map of BD-algebras. It is injective if the condition from Lemma 44 is satisfied, i.e. if the maps $\sharp: (P(n, G) \otimes Q(n, G))^2_n \rightarrow (P(n, G+2) \otimes Q(n, G+2))^2_n$ are injective for all $n, G$.

**Proof.** (1) By Theorem 23, the subspace $J \equiv \{\sharp X - \kappa X \mid X \in \tilde{F}_{\geq -\frac{1}{2}}\}$ is an ideal preserved by the BD-algebra maps. The weight grading is preserved since both $\sharp$ and multiplication by $\kappa$ increase weight by 2.

To show the flatness w.r.t. the action of $k[[\kappa]]$, it is enough to show that multiplication of $\kappa$ is injective. Let $X \in \text{Fun}_{\exp}(P, Q)$ be such that $\kappa X = 0$, i.e. $\kappa(X + J) \in J$, i.e. $\kappa X = \kappa Y - \sharp Y$ for some $Y \in \tilde{F}_{w \geq 12}$. Then $X = \kappa(Y/\kappa) - \sharp(Y/\kappa) \in J$.

(2) The map $\iota$ is well defined, since only elements with $n/2 + G = w - 1$ contribute to the weight $w$ component of $\text{Fun}_{\exp}(P, Q)$. The image of $\iota$ has weight $> 2$ by the stability condition [1].

To show the injectivity of $\iota$, consider an element $Y \in \text{Fun}(P, Q)$ which gets sent to the ideal $J$, i.e. $Y = \sharp X - \kappa X$ for some $X \in \tilde{F}_{\geq -\frac{1}{2}}$. Since the ideal $J$ is compatible with the weight grading, we can assume that $X$ and $Y$ have a definite weight. We expand $X$ in powers of $\kappa$

\[
X = \sum_{-m}^{m} X_m \kappa^m
\]

where the sum is finite thanks to the direct sum in definition of $\tilde{F}_w$. Since $Y = \sharp X - \kappa X$, we get an equality of Laurent polynomials in $\kappa$.

\[
\sharp X_m \kappa^{-m} + (X_m + \sharp X_{m+1})\kappa^{-m+1} + \cdots + (X_{n-1} + \sharp X_n)\kappa^n + X_n \kappa^{n+1} = Y \kappa^0
\]

because $Y \in \text{Fun}(P, Q)$ has no powers of $\kappa$ in itself. If $\sharp$ is injective, then we obtain from this equality that $X_0 = X_{-1} = 0$, which implies $Y = 0$. On the other hand, if $\sharp$ is not injective, an element $K$ of its kernel satisfies $K = \sharp(K/\kappa) - \kappa(K/\kappa)$, which lies in the ideal from Definition 22.

\[\square\]
This allows us to define formal exponentials and logarithms. As an image of the exponential, we will consider \( \text{Fun}^{\text{Exp}}(\mathcal{P}, \mathcal{Q}) \), a multiplicative abelian group of elements \( 1 + X \) of with \( X \in \text{Fun}^{\text{Exp}}(\mathcal{P}, \mathcal{Q}) \), on which the BD-algebra operations can be defined in an obvious way.

**Definition 34.** Define two maps

\[
\exp(X): \text{Fun}^{\text{Exp}}(\mathcal{P}, \mathcal{Q}) \rightleftharpoons \text{Fun}^{\text{Grp}}(\mathcal{P}, \mathcal{Q}): \log(X)
\]

by

\[
\exp(X) = 1 + X + X^2/2! + X^3/3! + \ldots
\]

and

\[
\log(1 + X) = X - X^2/2 + X^3/3 + \ldots
\]

**Lemma 35.** These two maps are well-defined, mutually inverse maps. The exponential behaves with respect to the \( \Delta \) as

\[
\Delta(e^X) = (\Delta X + \frac{1}{2} \{X, X\})e^X.
\]

**Proof.** It is a simple consequence of equations (14) and (15) that

\[
\Delta X^n = nX^{n-1}\Delta X + \frac{n(n-1)}{2} \{X, X\} X^{n-2}.
\]

Thus, for a power series \( f(X) = \sum_{n \geq 0} f_n X^n \), we have in the quotient of Definition (32)

\[
\Delta(f(X)) = \sum_{n \geq 0} f_n \left( nX^{n-1}\Delta X + \frac{n(n-1)}{2} \{X, X\} X^{n-2} \right) = f'(X)\Delta X + \frac{1}{2} f''(X) \{X, X\}.
\]

Thus we arrive at another characterization of morphisms from the Feynman transform of \( \mathcal{P} \).

**Corollary 35.1.** Assume that the condition on \( \ast \) from Lemma (30) is satisfied. Then, a degree 0 element \( S \in \text{Fun}(\mathcal{P}, \mathcal{Q}) \) satisfies the quantum master equation

\[
(d + \Delta)S + \frac{1}{2} \{S, S\} = 0
\]

if and only if

\[
(d + \Delta)e^{\{S, S\}} = 0
\]

holds in \( \text{Fun}^{\text{Exp}}(\mathcal{P}, \mathcal{Q}) \).

**Proof.** Thanks to the stability condition (1), \( \{S, S\} \) has positive weight, and we have

\[
0 = (d + \Delta)e^{\{S, S\}} = \frac{1}{\kappa}(dS + \Delta S + \frac{1}{2} \{S, S\})e^{\{S, S\}}
\]

which is equivalent to the quantum master equation for injective \( \iota \).

**Remark 36.** The weight \( w = n/2 + G + 2q + 1 \) is a generalization of the weight \( 2(g + q) + n \) introduced by Braun and Mauder [7 Def. 2.8]; this choice is motivated by \( \Delta, \ast, \iota/\kappa \) having weight 0. This weight, and the stability condition (1), make it possible to define the expression \( \Delta e^{\{S, S\}}/\kappa \).

See also [11 Sec. 2.2] for similar considerations for the QC case.

The power of \( \kappa \) should be thought of as the geometric genus \( g \), motivated by the relation \( \kappa = \frac{1}{2} \) in Definition (32). Zwiebach uses powers of \( h \) to count \( G \) in the open-closed string theory context [26 Eq. (3.1), (3.11)], which is why we used the letter \( \kappa \) instead.

### 3.6 Examples

We will now describe the BD algebra structure coming from the two modular operads QC and QC. Apart from the connected sum and the induced commutative product, these algebras were described in [12]. Using the commutative product, we obtain a slightly simplified description, since \( d, \Delta \) and the bracket are already specified on generators of the algebra.

Let us fix an odd symplectic vector space \( V \) with a symplectic form \( \omega \) and a differential \( d \). Let \( e_i \) be a basis of \( V \), which determines the dual basis \( \phi^i \) of \( V^* \) and the matrix \( \omega_{ij} = \omega(e_i, e_j) \) with inverse \( \omega^{ij} \).
### 3.6.1 The operad $\mathcal{QC}$

The space of formal functions on $V$, recalled in the following definition, is a BD algebra. We will now show that (up to a few non-stable elements), this BD algebra is isomorphic to $\text{Fun}(\mathcal{QC}, \mathcal{E}_V)$.

**Definition 37.** On $\text{Fun}_{\text{sym}}(V) = \prod_{n \geq 0} \text{Sym}^n(V^*) \otimes k[[x]]$, define $d$ and $\Delta$

$$
\begin{aligned}
d &= (-1)^{\phi^i} (\phi^i \circ d_V) \frac{\partial}{\partial \phi^i}, \\
\Delta &= (-1)^{\phi^i} \cdot |\omega^{ij}| \frac{\partial^2}{\partial \phi^i \partial \phi^j}.
\end{aligned}
$$

The space $(\text{Fun}_{\text{sym}}(V), d, \Delta)$ is a BV algebra, and thus $(\text{Fun}_{\text{sym}}(V), d, \kappa \Delta, \{-,-\})$ is a BD algebra, where $(-1)^{|X|} \{X, Y\} = \Delta(XY) - \Delta(X)Y - (-1)^{|X|} X \Delta(Y)$. For completeness, this gives

$$
\{X, Y\} = (-1)^{|\phi^i| + |X||\phi^i| + 1} |\omega^{ij}| \frac{\partial X}{\partial \phi^i} \frac{\partial Y}{\partial \phi^j}.
$$

Recall from Section 3.2 that the space $\text{Fun}(\mathcal{QC}, \mathcal{E}_V)$ is spanned by $\Sigma_n$-invariant tensors of the form $C_0^n \otimes w$, where $C_0^n$ is the generator of $\mathcal{QC}(n, 2g + n/2 - 1)$ and $w \in (V^*)^\otimes n$.

**Lemma 38.** The map $\Psi: \text{Fun}(\mathcal{QC}, \mathcal{E}_V) \to \text{Fun}_{\text{sym}}(V)$, given by

$$
C_0^n \otimes w \to (n!)^{-1} [w] \mathcal{X}^\theta
$$

is an injective map of BD algebras over $k[[x]]$, with the image given by the elements with $2g + n > 2$. The map $w \mapsto [w]$ is the projection $(V^*)^\otimes n \to \text{Sym}^n(V^*)$ given by $\phi_1 \otimes \ldots \otimes \phi_n \mapsto \phi_1 \ldots \phi_n$, the graded-commutative product of $\phi_i$.

**Proof.** The space $\mathcal{QC}(n, G)$ is the trivial representation of the permutation group $\Sigma_n$, and thus $\text{Fun}(\mathcal{QC}, \mathcal{E}_V)(n, G)$ is the subspace of invariants in $(V^*)^\otimes n$. This implies that $\Psi$ is an injection with the image specified by the stability condition $2(G - 1) + n > 0 \iff 2g + n > 2$.

Compatibility with the action of $\mathcal{X}$ is immediate. To check the compatibility of $\Psi$ with products, note that the terms of the sum in (11) differ only by an action of $\Sigma_{n_1 + n_2}$, as follows from Lemma 13. Thus, after the projection by $[-]$, all the $(n_1 + n_2)$-term gives the same contribution. Concretely, calculating the product, we get

$$
\Psi(C_0^{n_1} \otimes w_1 \star C_0^{n_2} \otimes w_2) = \left(\frac{n_1 + n_2}{n_1} \right) \frac{1}{(n_1 + n_2)!} [w_1 \otimes w_2] \mathcal{X}^{\theta_1 + \theta_2},
$$

which indeed equals

$$
\Psi(C_0^{n_1} \otimes w_1) \cdot \Psi(C_0^{n_2} \otimes w_2) = \frac{1}{n_1 n_2!} [w_1][w_2] \mathcal{X}^{\theta_1 + \theta_2}.
$$

This is thanks to the normalization of $\Psi$ and to the property $[w_1 \otimes w_2] = [w_1][w_2]$. As $\Psi$ is compatible with products, it is enough to check $d$ on linear elements and $\Delta$ on quadratic elements $C_0^2 \otimes (\phi^i \otimes \phi^j + (-1)^{|\phi^i| + |\phi^j|} \phi^j \otimes \phi^i)$. This is because these maps are determined by their values on such elements, possibly after multiplying with a high-enough power of $\mathcal{X}$ to fulfill the stability condition. We discuss only the case of $\Delta$, defined in (15) and (19), which sends the above quadratic element to

$$
C_0^{g+1} \otimes (-1)^{|\phi^i| + |\phi^j|} [\phi^i(e_k) e^k] + (-1)^{|\phi^i| + |\phi^j|} [\phi^j(e_k) e^k] = 2C_0^{g+1} \otimes (-1)^{|\phi^i|} |\omega^{ij}|,
$$

which $\Psi$ sends to $(-1)^{|\phi^i|} 2|\omega^{ij}| \mathcal{X}^{\theta + 1}$. This agrees with the action of $\mathcal{X} \Delta$ from (19) on $\phi^i \phi^j \mathcal{X}^\theta$. \qed

### 3.6.2 The operad $\mathcal{QO}$

Let $V$ be as before. We will now define a BD algebra structure on symmetric powers of cyclic words with letters from $V^*$. Related BV structures appeared for example in the work of Cieliebak, Latschek and Fukaya [8 Section 10.] and Barannikov [2 Section 1.2].
Definition 39. The space of cyclic words in $V^*$ of length $k$ is the space of coinvariants under the action of $\mathbb{Z}_k$ by cyclic permutations

$$C_{\text{yc}}^k(V^*) = ((V^*) \otimes \mathbb{Z}_k)^\perp,$$

with elements denoted by $(\phi_1 \ldots \phi_n) = (-1)^{|\phi_1|}[(\phi_2 \ldots + [\phi_n]) (\phi_2 \ldots \phi_n \phi_1)]$. Then, we define the following algebra

$$\text{Fun}_{\text{yc}}(V) := \prod_{n \geq 0} \text{Sym}^n((\bigoplus_{k \geq 1} C_{\text{yc}}^k(V^*))[[\varkappa, \xi]])$$

This algebra carries a natural BD structure continuous in $\varkappa$ and $\xi$. The Laplacian is defined by

$$\Delta(\phi^1 \ldots \phi^n) = \sum_{k=0}^{n-2} \pm \omega^{i_1 + 2}(\phi^{i_2} \ldots \phi^{i_k + 1})(\phi^{i_k + 2} \ldots \phi^n) + \text{cycl.}$$

where the sign $\pm$ in the first term is equal to $(-1)^{|\phi^{i_2}| + \ldots + |\phi^{i_k + 1}|}$. In the terms $k = 0$ and $k = n - 2$, one of the cyclic words is empty as is replaced by $\xi$. The remainder denoted $\text{cycl.}$ contains the $n - 1$ terms obtained by cyclically permuting the indices $i_1, \ldots, i_n$ in the first term and by multiplying by the Koszul sign of this cyclic permutation.

On products of cycles, $\Delta$ is extended to a BD operator as in [3], using the bracket

$$\{(\phi^1 \ldots \phi^{i_1}), (\phi^{i_2} \ldots \phi^{i_n})\} = \pm 2\omega^{i_1 + 1}(\phi^{i_2} \ldots \phi^{i_1} \phi^{i_2} \ldots \phi^{i_n}) + \text{cycl.} \times \text{cycl.}$$

where the sign $\pm$ in the first term is equal to $(-1)^{|\phi^{i_1}| + |\phi^{i_2}| + \ldots + |\phi^{i_1} + |\phi^{i_2}| + 1}$.

The induced differential is given as in Definition 37.

In contrast to $\text{Fun}_{\text{sym}}(V)$, this BD algebra cannot be induced from a BV algebra by replacing $\Delta_{\text{BV}}$ with $\varkappa \Delta_{\text{BV}}$. This can be seen on the level of the operad $\mathcal{QO}$: the self-composition $\circ_{ab}$ applied on a disk is an annulus, which cannot be written as an image of $\#_1$.

Example 40. To illustrate the above formulas, let us give a few simple examples the BD structure defined above.

$$\Delta(\phi^a \phi^b) = 2(-1)^{|\phi^a|} \omega^{ab} \varkappa \xi,$$

$$\Delta(\phi^a \phi^b \phi^c) = 2((-1)^{|\phi^a|} \omega^{ab} (\phi^c) + \text{cycl.}) \xi$$

$$= 2 \left((-1)^{|\phi^a|} \omega^{ab} (\phi^c) + (-1)^{|\phi^a| + |\phi^b|} \omega^{bc} (\phi^a) + (-1)^{|\phi^a| + |\phi^b|} \omega^{ca} (\phi^b)\right) \xi,$$

$$\{(\phi^a), (\phi^b)\} = 2\omega^{ab} \varkappa \xi.$$

We would like to show that $\text{Fun}_{\text{yc}}(V)$ contains $\text{Fun}(\mathcal{QO}, \mathcal{E}_V)$, with $\varkappa$ counting the geometrical genus of the element of $\mathcal{QO}$ and $\xi$ counting the empty punctures.

In this case, the $\Sigma_n$ invariants in $\mathcal{QO}(n, G) \otimes (V^*) \otimes n$ can be described as follows (see also [12], V.C]): the $\Sigma_n$-representation $\mathcal{QO}(n, G)$ comes from the set of all cycles on letters $1 \ldots n$, with its $\Sigma_n$-action given by renumbering. Orbits of this set-theoretic action are completely specified by sequences $^b\! (b_0, b_1, \ldots) \in \mathbb{N}^N$, where $b_i$ is the number of cycles of length $i$. Choose the following element in each orbit

$$x_b := (\underbrace{() \ldots ()}_{b_0 \text{ times}} (1) \ldots (b_1) (b_1 + 1) (b_1 + 2)) \ldots$$

(20)

For each such admissible $b$ and $w \in (V^*) \otimes n$ invariant under the stabilizer of $x_b$, we have an invariant element

$$\sum_{\sigma \in \Sigma_n/\text{Stab}(b)} \sigma x_b \otimes \sigma w,$$

(21)

The space of invariants $(\mathcal{QO}(n, G) \otimes (V^*) \otimes n)_{\Sigma_n}$ is spanned by such elements.\footnote{These are subject to the obvious conditions $\sum b_i = b$ and $\sum i b_i = n$.}
Define a map $\Theta$: $(\mathcal{O}(n, G) \otimes (V^*)^n) \rightarrow \text{Fun}_{\text{cyc}}(V)$ by

$$\Theta: \sum_{\sigma \in \Sigma_n/\text{Stab}(b)} \sigma x_b \otimes \sigma w_b \mapsto \frac{1}{\prod_{i \geq 1} |b_i|!} \left[ u_b \right] \xi^{b_0},$$

where $w \mapsto [w]$ is the composition

$$(V^*)^n \rightarrow \bigotimes_{i \geq 1} (\text{Cyc}_i(V^*))^{\otimes b_i} \rightarrow \bigotimes_{i \geq 1} \text{Sym}^{b_i} (\text{Cyc}_i(V^*)) \hookrightarrow \text{Fun}_{\text{cyc}}(V).$$

**Lemma 41.** The map $\Theta$: $\text{Fun}(\mathcal{O}, \xi_V) \rightarrow \text{Fun}_{\text{cyc}}(V)$ is an injective map of BD algebras over $k[[\xi]]$, with the image given by elements with $2g + b + n/2 > 2$. Here, $b$ is $b_0$ plus the total number of cyclic words, $n$ is the total number of letters.

**Proof.** The injectivity of $\Theta$ follows from the fact that the map $w \mapsto [w]$ is an isomorphism from invariants to coinvariants for the stabilizer subgroup of $b$.

Let us check the compatibility of $\Theta$ with the products. In the product of two elements

$$\sum_{\sigma} \sigma x_b(1) \otimes \sigma w(1) \ast \sum_{\sigma'} \sigma' x_b(2) \otimes \sigma' w(2)$$

there are $\prod_{i \geq 1} \left( \binom{b_1 + b_2}{b_1} + \binom{b_1 + b_2}{b_2} \right)$ contributions to the term $x_b(1) \ast b \otimes W$, and their contributions to the tensor $W$ all differ by a permutation only among cycles of the same length, i.e. a permutation stabilizing $x_b(1) \ast b \otimes W$. Thus, in $[W]$, they are all equal, and the combinatorial factor $\prod_{i \geq 1} \left( \binom{b_1 + b_2}{b_1} + \binom{b_1 + b_2}{b_2} \right)$ cancels thanks to $\prod 1/b_i!$ in the definition of $\Theta$.

Using the compatibility of $\Theta$ with products, it is now enough to check $d$, $\Delta$ and the bracket on elements with only one cycle.

Let us calculate $\Delta(\phi^{b_1} \cdots \phi^{b_n})$. The cyclic word $(\phi^{b_1} \cdots \phi^{b_n})$ can be obtained by $\Theta$ from the element

$$(1 \cdots n) \otimes [(1 + \tau + \cdots + \tau^{n-1}) \phi^{b_1} \otimes \cdots \otimes \phi^{b_n}) + \ldots,$$

where $\tau$ is the cyclic permutation of $1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1$ and the ... at the end denote the $(n-1)! - 1$ remaining terms. In (23), we choose $\theta$ as

$$\theta(1) = a, \theta(2) = b, \theta(k) = k - 2$$

for $k > 2$.

The operator $\Delta$ then cuts the relabeled cycle $(1 \cdots a \cdots b \cdots n)$ at $a$ and $b$ into two (possibly empty) cycles. To calculate $\Theta$, we need to find which terms contribute to the terms $x_b \otimes \ldots$, i.e. which cuts result in two cycles in the standard form (20). There are $2k(n - 2 - k)$ such contributions to each possible length of cycles, coming from term labeled $(a|1 \cdots k|b|k+1 \cdots 2*n - 2|)$ and $(a|k+1 \cdots n - 2|b|1 \cdots k|)$ for any $0 \leq k \leq n/2 - 1$. Using the symmetry of $(-1)^{\delta a b} \omega_{a b}$, re-expressing the tensor $[1 + \tau + \cdots + \tau^{n-1}] \phi^{b_1} \cdots \phi^{b_n}$ using a cyclic permutation exchanging $a \leftrightarrow b$ and collecting the signs, one obtains that these contributions are equal.

The factor $k(n - 2 - k)$ cancels with the normalisation $\prod_{i > 0} \delta^{b_i}$ of the map $\Theta$. The factor 2 can be removed by expanding the possible values of $k$ to $n - 2$; if $k = n - 2 - k$ and the two cycles are of the same length, this factor of 2 instead cancels the 2! from the normalisation of $\Theta$. Together, we thus obtain

$$\Delta(\phi^{b_1} \cdots \phi^{b_n}) = \sum_{k=0}^{n-2} \left[ (-1)^{1 + \ldots + 1} \phi^{b_1} \otimes \cdots \otimes \phi^{b_n} + \text{cycl} \right],$$

with the convention that an empty cycle (for $k = 0$ or $k = n - 2$) is replaced by $\xi$. The $n - 1$ terms in + cycl are obtained from $\tau^m \phi^{b_1} \otimes \cdots \otimes \phi^{b_n}$ for $1 \leq m \leq n - 1$, i.e. they also contain the sign from permuting the graded covectors $\phi$.

The bracket, defined in (10), is computed similarly. Looking at

$$\{(\phi^{b_1}, \ldots, \phi^{b_n}), (\phi^{b_1}, \ldots, \phi^{b_n})\}$$

If $k = 0$ or $n - 2 - k = 0$, there are still two contributions; let us not mention this technicality again.
for $n_1, n_2 > 1$, the only terms from the sum over decompositions which contribute to the cycle $(1 \ldots n_1 + n_2 - 2)$ are those where $C_1$ is an interval $\{l, \ldots, l + n_1 - 1\} \mod (n_1 + n_2 - 2)$. Moreover, for each such decomposition, only one permutation from the sum (21) contributes. There are $n_1 + n_2 - 2$ choices for $l$, which are all equal after the projection $[\cdot]$; this cancels the normalization of $\Theta$.

The case $n_2 = 1$ is different: there is only one choice of a decomposition, and $n_1 - 1$ different permutations from (21) contribute, namely the cyclic permutations of the interval $\{1, \ldots, n_1 - 1\}$.

References

[1] S. Barannikov, “Modular operads and Batalin-Vilkovisky geometry”, International Mathematics Research Notices, Oxford University Press, Vol. 2007 (2006). arXiv:1710.08442

[2] S. Barannikov, “Noncommutative Batalin–Vilkovisky Geometry and Matrix Integrals.” Comptes Rendus Mathematique, vol. 348, no. 7–8, Apr. 2010, pp. 359–62. Crossref, https://doi.org/10.1016/j.crma.2010.02.002

[3] I. Batalin, G. Vilkovisky, “Gauge algebra and quantization,” Physics Letters B 102 no. 1, (June, 1981) 27–31.

[4] A. Beilinson, V. Drinfeld, “Chiral Algebras”, American Mathematical Society (2004).

[5] C. Berger, R. Kaufmann “Trees, graphs and aggregates: a categorical perspective on combinatorial surface topology, geometry, and algebra” (2022) arXiv:2201.10537.

[6] D. Borisov, and Y. Manin. “Generalized operads and their inner cohomomorphisms.” In Geometry and dynamics of groups and spaces, pp. 247-308. Birkhäuser Basel, (2007).

[7] C. Braun, J. Maunder, “Minimal models of quantum homotopy Lie algebras via the BV-formalism”, Journal of Mathematical Physics, Vol.59, Issue 6 (2018). arXiv:1703.00082

[8] K. Cieliebak, K. Fukaya, J. Latschev, “Homological algebra related to surfaces with boundary”, (2015). arXiv:1508.02741

[9] K. Costello, O. Gwilliam, “Factorization Algebras in Quantum Field Theory” (New Mathematical Monographs). Cambridge University Press. doi:10.1017/9781316678626 (2016).

[10] M. Doubek, “The Modular Envelope of the Cyclic Operad Ass”, Appl. Categor. Struct. 25, 1187–1198 (2017).

[11] M. Doubek, B. Jurčo, M. Markl, I. Sachs, “Algebraic Structure of String Field Theory”, Lecture Notes in Physics 937, Springer (2020).

[12] M. Doubek, B. Jurčo, K. Münster, “Modular operads and the quantum open-closed homotopy algebra”, Journal of High Energy Physics, Vol. 2015, No. 12 (2015). arXiv:1308.3223v2

[13] D. Eisenbud, “Commutative Algebra: with a view toward algebraic geometry” Vol. 150. Springer Science & Business Media, (2013).

[14] M. Doubek, B. Jurčo, J. Pulmann, “Quantum $L_\infty$ Algebras and the Homological Perturbation Lemma”, Communications in Mathematical Physics, Vol 367 (2017). arXiv:1712.02696

[15] E. Getzler, M. M. Kapranov, “Modular operads”, Compositio Mathematica, Vol. 110 (1998). arXiv:dg-ga/9408003

[16] O. Gwilliam, “Factorization algebras and free field theories,” (2013). https://ncatlab.org/nlab/files/GwilliamThesis.pdf

[17] B. Jurčo, K. Münster, “Type II Superstring Field Theory: Geometric Approach and Operadic Description”, J. High Energ. Phys., Vol. 2013, 4 (2013). arXiv:1303.2323

[18] R. M. Kaufmann, B. C. Ward, J. J. Zuniga, “The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators, and master equations”, J. of Mathematical Physics, Vol. 56 Issue 10 (2015). arXiv: 1208.5543
[19] R. M. Kaufmann, B. C. Ward, Feynman categories, Astérisque 387 (2017), vii+161pp
arxiv:1312.1269

[20] M. Markl, “Loop Homotopy Algebras in Closed String Field Theory,” Communications in
Mathematical Physics 221 no. 2, (July, 2001) 367–384, arXiv:hep-th/9711045

[21] M. Markl, “Odd structures are odd”, Advances in Applied Clifford Algebras, Vol. 27 (2017).
arXiv:1603.03184

[22] L. Peksová “Modular operads with connected sum and Barannikov’s theory.” Archivum Math-
ematicum, vol. 56 (2020), issue 5, pp. 287-300 http://dx.doi.org/10.5817/AM2020-5-287

[23] A. Schwarz, “Grassmannian and String Theory”, Communications in Mathematical Physics,
Vol. 199 (1998). arXiv:hep-th/9610122

[24] A. Sen, B. Zwiebach. “Quantum background independence of closed-string field theory. Nuclear
Physics B 423.2-3 (1994): 580-630. https://arxiv.org/abs/hep-th/9311009

[25] B. Zwiebach, “Closed string field theory: Quantum action and the Batalin-Vilkovisky master
equation,” Nuclear Physics B 390 no. 1, (June, 1993) 33–152 arXiv:hep-th/9206084

[26] B. Zwiebach, “Oriented Open-Closed String Theory Revisited” Annals of Physics, Volume 267,
Issue 2 (1998). arXiv:hep-th/9705241