Algorithms and identities for \((p, q)\)-Bézier curves via \((p, q)\)-Blossom

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Abstract

In this paper, a new variant of the blossom, the \((p, q)\)-blossom, is introduced, by altering the diagonal property of the standard blossom. This \((p, q)\)-blossom is has been adapted for developing identities and algorithms for \((p, q)\)-Bernstein bases and \((p, q)\)-Bézier curves. We generate several new identities including an explicit formula representing the monomials in terms of the \((p, q)\)-Bernstein basis functions and a \((p, q)\)-variant of Marsden’s identity by applying the \((p, q)\)-blossom. We also derive for each \((p, q)\)-Bézier curve of degree \(n\), a collection of \(n!\) new, affine invariant, recursive evaluation algorithms. Using two of these new recursive evaluation algorithms, we construct a recursive subdivision algorithm for \((p, q)\)-Bézier curves.

Keywords and phrases: \((p, q)\)-integers; \((p, q)\)-blossom; de Casteljau algorithm; Marsden’s identity; \((p, q)\)-Bernstein polynomials; \(q\)-Bernstein polynomials; \((p, q)\)-Bézier curve

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1 Introduction

P. Simeonova et al. in \cite{simeonova} nicely constructed some algorithms and identities for \(q\)-Bernstein bases and \(q\)-Bézier curves using the method of \(q\)-Blossoming.

Also, Mursaleen et al. \cite{mursaleen} recently introduced first the concept of \((p, q)\)-calculus in approximation theory and studied the \((p, q)\)-analogue of Bernstein operators. Later on, based on \((p, q)\)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, \((p, q)\)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc. have also been introduced in \cite{mursaleen1, mursaleen2, mursaleen3, mursaleen4, mursaleen5}.

Motivated by the work of Mursaleen et al \cite{mursaleen}, the idea of \((p, q)\)-calculus and its importance and the work by P. Simeonova et al. in \cite{simeonova}, we construct \((p, q)\)-Bézier curves based on \((p, q)\)-integers which is further generalization of \(q\)-Bézier curves using the method of \((p, q)\)-Blossoming. For similar works based on \((p, q)\)-integers and \(q\)-integers, one can refer \cite{simeonova1, simeonova2, simeonova3, simeonova4}.

In \cite{mursaleen, mursaleen1, mursaleen2}, we have given an application in computer-aided geometric design and applied these Bernstein basis for construction of \((p, q)\)-Bézier curves and surfaces based on \((p, q)\)-integers which is further generalization of \(q\)-Bézier curves and surfaces \cite{mursaleen3, mursaleen4, mursaleen5}. For other relevant works based on Bézier curves, one can refer \cite{simeonova, simeonova1, simeonova2, simeonova3, simeonova4, simeonova5, simeonova6, simeonova7, simeonova8, simeonova9, simeonova10, simeonova11, simeonova12, simeonova13, simeonova14, simeonova15, simeonova16, simeonova17, simeonova18, simeonova19, simeonova20, simeonova21, simeonova22, simeonova23, simeonova24, simeonova25, simeonova26}. 

It was S.N. Bernstein [1] in 1912, who first introduced his famous operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

and named it Bernstein polynomials to prove the Weierstrass theorem [11]. Later it was found that Bernstein polynomials possess many remarkable properties and has various applications in areas such as approximation theory [11], numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation [22].

In computer aided geometric design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [28] is the classical Bézier curve [2] constructed with the help of Bernstein basis functions.

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in [6, 21, 22].

Thus with the development of $(p, q)$-analogue of Bernstein operators and its variants, one natural question arises, how it can be used in order to preserve the shape of the curves or surfaces. In this way, it opens a new research direction which requires further investigations.

Before proceeding further, let us recall certain notations of $(p, q)$-calculus .

The $(p, q)$ integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad p > q > 0.$$

The formula for $(p, q)$-binomial expansion is as follow:

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{n-k} q^k x^k y^k,$$

$$(x + y)^n_{p,q} = (x + y)((px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

$$(1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x),$$

where $(p, q)$-binomial coefficients are defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}! [k]_{p,q}! [n-k]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}.$$ 

Details on $(p, q)$-calculus can be found in [7, 13, 24, 8, 9].

The $(p, q)$-Bernstein Operators introduced by Mursaleen et al [13] is as follow:

$$B_{n,p,q}(f; x) = \frac{1}{p} \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{k(k+1)} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n} [n]_{p,q}}\right), \quad x \in [0, 1]. \quad (1.2)$$
Note when \( p = 1 \), \((p, q)\)-Bernstein Operators given by (1.2) turns out to be \( q \)-Bernstein Operators. Also, we have

\[
(1 - x)^n_{p, q} = \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2 x)...(p^{n-1} - q^{n-1} x)
\]

\[
= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k} p^k q^k x^k
\]

Recently we apply \((p, q)\)-calculus in computer added geometric design and introduced first the \((p, q)\)-Bézier curves and surfaces based on \((p, q)\)-integers in [8, 9] which is further generalization of \( q \)-Bézier curves and surfaces, for example, [21, 22].

In this paper we extend these results for \((p, q)\)-Bernstein bases and \((p, q)\)-Bézier curves using \((p, q)\)-blossoming.

The paper has been arranged in the following way. In Section 2 and 3, we introduce the basic definitions, fundamental formulas, and explicit notation for \((p, q)\)-Bernstein bases and \((p, q)\)-Bezier curves. In Section 4 we define the \((p, q)\)-blossom and establish the existence and the uniqueness of this blossom. In Section 5 we invoke \((p, q)\)-blossoming to develop novel evaluation algorithms for \((p, q)\)-Bezier curves and in Section 5 we use the \((p, q)\)-blossom to derive new identities involving the \((p, q)\)-Bernstein basis functions, including a \((p, q)\)-version of Marsdens identity as well as formulas for representing monomials in terms of the \((p, q)\)-Bernstein basis functions.

### 2 \((p, q)\)-Bernstein functions

The \((p, q)\)-Bernstein functions is as follows

\[
B^k,n_{p, q}(t) = \frac{1}{p^{\frac{(n-k)(n-k-1)}{2}}} \binom{n}{k} p^k q^{\frac{k(k-1)}{2}} (1 - t)^{n-k}_{p, q}, \quad t \in [0, 1]
\]

where

\[
(1 - t)^{n-k}_{p, q} = \prod_{s=0}^{n-k-1} (p^s - q^s t)
\]

**Theorem 2.1** [3] Each \((p, q)\)-Bernstein function of degree \( n \) is a linear combination of two \((p, q)\)-Bernstein functions of degree \( n + 1 \).

\[
B^k,n_{p, q}(t) = \left( p^{\frac{k-n}{n+1}}_{p, q} \right) B^{k+1,n}_{p, q}(t) + p^{-n} \left( 1 - \frac{p^{k+1}_{p, q}}{n+1} \right) B^{k+1,n+1}_{p, q}(t)
\]

Throughout the paper onwards, we use \( B_k^n(t, p, q) \) in place of \( B^k,n_{p, q}(t) \).

### 3 \((p, q)\)-Bernstein Bézier curves:

We define the \((p, q)\)-Bézier curves of degree \( n \) using the \((p, q)\)-analogues of the Bernstein functions as follows:
\[ P(t; p, q) = \sum_{i=0}^{n} P_i B^n_i(t, p, q) \]  

(3.1)

where \( P_i \in R^3 \) (\( i = 0, 1, ..., n \)) and \( p > q > 0 \). \( P_i \) are control points. Joining up adjacent points \( P_i, i = 0, 1, 2, ..., n \) to obtain a polygon which is called the control polygon of \((p, q)\)-Bezier curves.

### 3.1 Degree elevation for \((p, q)\)-Bézier curves

\((p, q)\)-Bézier curves have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. Using the technique of degree elevation, we can increase the flexibility of a given curve.

\[ P(t; p, q) = \sum_{k=0}^{n} P_k B^n_k(t, p, q) \]

\[ P(t; p, q) = \sum_{k=0}^{n+1} P'_k B^n_{k+1}(t, p, q), \]

where

\[ P'_k = p^k P' = \left( 1 - \frac{p^k [n+1-k]}{[n+1]_{p,q}} \right) P_{k-1} + p^k \left( \frac{[n+1-k]}{[n+1]_{p,q}} \right) P_k \]  

(3.2)

The statement above can be derived using the identities (??) and (??). Consider

### 3.2 de Casteljau algorithm:

\((p, q)\)-Bézier curves of degree \( n \) can be written as two kinds of linear combination of two \((p, q)\)-Bézier curves of degree \( n - 1 \), and we can get the two selectable algorithms to evaluate \((p, q)\)-Bézier curves. The algorithms can be expressed as:

\[ \hat{P}_i^k(t; p, q) = (p^{n-k} - p^i q^{n-k-i}) \hat{P}_i^{k-1}(t; p, q) + p^i q^{n-k-i} t \hat{P}_i^{k-1}(t; p, q) \]  

(3.3)

\[ P_i^k(t; p, q) = q^i (p^{n-k-i} - q^{n-k-i}) P_i^{k-1}(t; p, q) + p^{n-k} t P_i^{k-1}(t; p, q) \]  

(3.4)

For illustration purpose, these two de Casteljau algorithms are shown in Figs. 1 and 2 for cubic \((p, q)\)-Bézier curves. In order to evaluate value at each node, label on each arrow that enters the node is multiplied by the value at the node from which the arrow emerges and then results get added. In first algorithm the property of affine invariant holds at every intermediate nodes but in second algorithm this property holds only for the final node at the top of diagram.

P. Simeonov et al. [25] has given a new approach to identities and algorithms for \(q\)-Bernstein bases and \(q\)-Bézier curves using \(q\)-blossom. In this paper we extend these results for \((p, q)\)-Bernstein bases and \((p, q)\)-Bézier curves using \((p, q)\)-blossoming.

Four main contributions of this paper are:

**Blossoming:** The \((p, q)\)-blossom, a new variant of the blossom has been introduced which will prove new identities for \((p, q)\)-Bernstein bases and generate new approach for \((p, q)\)-Bezier curves.
Identities: Using \((p,q)\)-blossom, new identities are derived for the \((p,q)\)-Bernstein bases, and a \((p,q)\)-variant of Marsden’s identity and monomials get represented using an explicit formula in terms of the \((p,q)\)-Bernstein basis functions.

Recursive Evaluation Algorithms: Using \((p,q)\)-blossom technique, for a given \((p,q)\)-Bezier curve of degree \(n\), \(n!\) new affine invariant, recursive evaluation algorithms has been constructed.

Subdivision: Subdivision algorithm for \((p,q)\)-Bezier curves has been introduced for first time using two new recursive evaluation algorithms.

4 \((p,q)\)-blossoming

Blossoming has given new approach for deriving identities and developing change of basis algorithms for standard Bernstein bases and Bézier curves \[25\]. In this section, \((p,q)\)-blossoming as an extension of standard \(q\)-blossoming has been achieved.

The \((p,q)\)-blossom or \((p,q)\)-polar form of a polynomial \(S(t)\) of degree \(n\) is the unique symmetric multiaffine function \(s(u_1,\ldots,u_n;p,q)\) that reduces to \(S(t)\) along the \((p,q)\)-diagonal. That is, \(s(u_1,\ldots,u_n;p,q)\) is the unique multivariate polynomial satisfying the following three axioms:

Figure 1: ‘The first de-Casteljau evaluation algorithm for a cubic \((p,q)\)-Bézier curve on the interval \([0,1]\)’
Figure 2: ‘The second de-Casteljau evaluation algorithm for a cubic \((p, q)\)-Bézier curve on the interval \([0,1]\)’

\((p, q)\)-Blossoming axioms:

1. **Symmetry**: \(s(u_1, ..., u_n; p, q) = s(u\sigma(1), ..., u\sigma(n); p, q)\) for every permutation \(\sigma\) of the set \(\{1, ..., n\}\).

2. **Multiaffine**: \(s(u_1, ..., (1 - \alpha)u_k + \alpha v_k, ..., u_n; p, q) = (1 - \alpha) p(u_1, ..., u_k, ..., u_n; p, q) + \alpha s(u_1, ..., v_k, ..., u_n; p, q)\)

3. **Diagonal**: \(s(p^{n-1}t, p^{n-2}tq, ..., tq^{n-1}; p, q) = S(t)\). The multiaffine property is equivalent to the fact that each variable \(u_1, ..., u_n\) appears to at most the first power that is, \(s(u_1, ..., u_n; p, q)\) is a polynomial of degree at most one in each variable. \((p, q)\)-blossoming gets interesting due to Dual functional property, which get derived and that relate \((p, q)\)-blossom of a polynomial to its \((p, q)\)-Bézier control points.

**Dual functional property**

Let \(P(t)\) be a \((p, q)\)-Bézier curve of degree \(n\) over the interval \([0,1]\) with control points \(P_0, ..., P_n\) and let \(s(u_1, ..., u_n; p, q)\) be the \((p, q)\)-blossom of \(S(t)\). Then

\[ P_k = s(0, ..., 0, p^{n-1}t, p^{n-2}tq, ..., tq^{n-1}; p, q), \quad k = 0, ..., n. \]  

This Dual functional property gets proved in Theorem 5.2

Now we establish those functions existence and uniqueness which satisfy \((p, q)\)-blossoming axioms, subject to restrictions on \(p, q\) for all polynomials of degree \(n\). But before proceeding to it let’s get feel of \((p, q)\)-blossom by computing the \((p, q)\)-blossom for some simple cases.

\((p, q)\)-Blossom of cubic polynomials:
Let us consider a cubic polynomial represented by the monomial $1, t, t^2,$ and $t^3$. Now these monomials can be easily $(p, q)$-blossomed for any $q = 0$, since in each case the associated function $s(u_1, u_2, u_3; p, q)$ given below can be easily verified as it is symmetric, multiaffine, and reduces to the required monomial along the $(p, q)$-diagonal:

$$S(t) = 1 \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{1}{p^3}$$

$$S(t) = t \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1 + u_2 + u_3}{p(p^2 + pq + q^2)}$$

$$S(t^2) \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1 u_2 + u_2 u_3 + u_3 u_1}{q(p^2 + pq + q^2)}$$

$$S(t^3) \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1 u_2 u_3}{q^3}$$

In the right hand side of the above equation, it can be seen that functions in numerator are combinations of three variables which are written in symmetrical fashion while in case of denominator function is evaluated in symmetrical order at $p^2, pq$ and $q^2$. Using these results, any cubic polynomial $S(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$ for $p \neq 0, q \neq 0$ can be $(p, q)$-blossom by setting

$$s(u_1, u_2, u_3; p, q) = a_0 \frac{u_1 u_2 u_3}{q^3} + a_1 \frac{u_1 u_2 + u_2 u_3 + u_3 u_1}{q(p^2 + pq + q^2)} + a_2 \frac{u_1 + u_2 + u_3}{p(p^2 + pq + q^2)} + a_3 \frac{1}{p^3}$$

Similarly, we can apply $(p, q)$-blossom techniques for polynomials of degree $n$ by first $(p, q)$-blossoming the monomials $t^k$, for $k = 0, ..., n$, and then applying linearity. Indeed, let

$$\phi_{n,k}(u_1, u_2, ..., u_n) = \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n} u_{i_1} u_{i_2} ... u_{i_k}$$

where the sum runs over all subsets $\{i_1, ..., i_k\}$ of $\{1, ..., n\}$, denote the $k$-th elementary symmetric function in the variables $u_1, ..., u_n$. Then we get the result as follows.

**Proposition 4.1** The $(p, q)$-blossom of the monomial $M^k_n(t) = t^k$ (considered as a polynomial of degree $n$) is given by

$$m^k_n(u_1, ..., u_n; p, q) = \frac{\phi_{n,k}(u_1, u_2, ..., u_n)}{\phi_{n,k}(p^{n-1}; p^{n-2}q, ..., q^{n-1})}$$

provided that $\phi_{n,k}(p^{n-1}; p^{n-2}q, ..., q^{n-1}) \neq 0$.

**Proof.** The three blossoming axioms need to be verified now. One can see that the function $m^k_n(u_1, ..., u_n; p, q)$ is symmetric, due to presence of elementary symmetric function divided by a constant in the expression on the right hand side of (4.2). Also, since each variable appears to at most the first power, hence the function on the right hand side of (4.2) is multiaffine. Finally observe that since $\phi_{n,k}(u_1, u_2, ..., u_n)$ is a homogeneous polynomial of total degree $k$ in the variables $u_1, ..., u_n$,

$$\phi_{n,k}(tu_1, ..., tu_n) = t^k \phi_{n,k}(u_1, ..., u_n).$$

Therefore along the $(p, q)$-diagonal

$$m^k_n(p^{n-1} t, p^{n-2}qt, ..., q^{n-1} t; p, q) = \frac{\phi_{n,k}(p^{n-1} t, p^{n-2}qt, ..., q^{n-1} t)}{\phi_{n,k}(p^{n-1}; p^{n-2}q, ..., q^{n-1})} = t^k.$$
We can use Proposition 3.1 to establish the existence of the \( q \)-blossom for arbitrary polynomials of degree \( n \). But before we proceed, we need to determine explicit conditions for which \( \phi_{n,k}(p^{n-1}, p^{n-2}q, ..., q^{n-1}) \neq 0 \), \( k = 1, ..., n \).

**Lemma 4.2.**

\[
\phi_{n,k}(p^{n-1}, p^{n-2}q, ..., q^{n-1}) = p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} \quad k = 1, 2, ..., n. \tag{4.3}
\]

**Proof.** Using induction on \( n \), we get the required result.

**Corollary 4.3.** \( \phi_{n,k}(p^{n-1}, p^{n-2}q, ..., q^{n-1}) = 0 \) if and only if one of the following three conditions is satisfied:

1. \( p = 0 \), \( n > k + 1 \)
2. \( q = 0 \) and \( n > 1, k > 1 \).
3. \( p = -q \) and \( n \) is even, \( k \) is odd.

**Proof.** It can be observed that the only real root of a \((p, q)\)-binomial coefficient can be \( p = -q \) because \( [n]_{p,q} = \frac{p^n - q^n}{p - q} \) when \( p \neq q \). Condition 1 and 2 follows from [5.3] while Condition 3 follows from the observation that \( p = -q \) is a zero of the binomial coefficient \( \binom{n}{k}_{p,q} \) of multiplicity

\[
\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{n-k}{2} \right\rfloor = \begin{cases} 1, & \text{if } n \text{ is even and } k \text{ is odd} \\ 0, & \text{otherwise} \end{cases}
\]

Now the existence and uniqueness of the \((p, q)\)-blossom will be established for all polynomials of degree \( n \) and for all real values of \( p, q \) that satisfy:

(1) \( q \neq 0 \) and \( p \neq q \) for all \( n > 1 \) \( \tag{4.4} \)

(2) \( q \neq -p \) for all even \( n > 1 \) \( \tag{4.5} \)

Conditions [4.4] and [4.5] are now the standard restrictions on the value of \( p, q \).

From now on, whenever there is a talk of \((p, q)\)-blossom or \((p, q)\)-Bernstein basis functions or \((p, q)\)-Bézier curves, the standard restrictions stated above will be applicable for the value of \( p, q \) until it is explicitly mentioned otherwise.

**Theorem 4.1 (Existence and Uniqueness of the \((p, q)\)-Blossom).** Corresponding to every polynomial \( S(t) \) of at most degree \( n \), there exists a unique symmetric multiaffine function \( s(u_1, ..., u_n; p, q) \) that reduces to \( P(t) \) along the \((p, q)\)-diagonal. That is, there exists a unique \((p, q)\)-blossom \( s(u_1, ..., u_n; p, q) \) for every polynomial \( S(t) \) provided that \( p, q \) satisfies the standard restrictions given by [4.4] and [4.5].

**Proof.** By Proposition 4.1 and Corollary 4.3 when \( p, q \) satisfies the constraints given by [5.3] and [5.5] then \((p, q)\)-blossom exists for the monomials \( t^k, k = 0, ..., n \). Since any polynomial can be written as linear combination of monomials and \((p, q)\)-blossom of the sum is actually the sum of the \((p, q)\)-blossoms, so for every given polynomial \( S(t) \), \((p, q)\)-blossom for it always exists while \( p, q \) satisfies the restrictions given by [4.4] and [4.5].

For verifying the uniqueness of the \((p, q)\)-blossom, suppose that a polynomial \( S(t) \) of degree \( n \) has two \((p, q)\)-blossoms \( s(u_1, ..., u_n; p, q) \) and \( r(u_1, ..., u_n; p, q) \). Since every symmetric multiaffine
polynomial of degree \( n \) has a unique representation in terms of the \(( n + 1)\) symmetric polynomials of degree \( n \), there are constants \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_n \) such that

\[
s(u_1, \ldots, u_n; p, q) = \sum_{k=0}^{n} a_k \phi_{n,k}(u_1, u_2, \ldots, u_n; p, q)
\]

and

\[
r(u_1, \ldots, u_n; p, q) = \sum_{k=0}^{n} b_k \phi_{n,k}(u_1, u_2, \ldots, u_n; p, q)
\]

Evaluating on the \((p, q)\)-diagonal \((u_i = p^{n-i}q^{i-1}, i = 1, \ldots, n)\) yields

\[
S(t) = \sum_{k=0}^{n} a_k \phi_{n,k}(p^{n-1}, p^{n-2}q, \ldots, q^{n-1})t^k = \sum_{k=0}^{n} b_k \phi_{n,k}(p^{n-1}, p^{n-2}q, \ldots, q^{n-1})t^k
\]

Thus \( a_k = b_k, k = 0, \ldots, n \), so \( s(u_1, \ldots, u_n; p, q) = r(u_1, \ldots, u_n; p, q) \). Hence the \((p, q)\)-blossom of \( S(t) \) is unique.

From Proposition 4.1, Lemma 4.2, Theorem 4.4, and the linearity of the \((p, q)\)-blossom we deduce the following result.

**Corollary 4.5.** The \((p, q)\)-blossom of the polynomial \( S(t) = \sum_{k=0}^{n} a_k t^k \) is

\[
s(u_1, \ldots, u_n; p, q) = \sum_{k=0}^{n} a_k \phi_{n,k}(u_1, u_2, \ldots, u_n) \cdot \frac{p^{(n-k)(n-k-1)}}{2^k} \cdot q^{k(k-1)}
\]

(4.6)

In this section, study has been done on the \((p, q)\)-blossom of a polynomial using the monomial representation. In the next section investigation will be carried out that how \((p, q)\)-blossom of polynomial is related to the \((p, q)\)-Bernstein representation.

### 5 \((p, q)\)-blossoming and \((p, q)\)-de Casteljau algorithms

In the diagrams below, we use the multiplicative notation \( u_1, \ldots, u_n \) to represent the \((p, q)\)-blossom value \( s(u_1, \ldots, u_n; p, q) \). Though an abuse of notation, this multiplicative notation is highly suggestive. For example, multiplication is commutative and the \((p, q)\)-blossom is symmetric

\[
u_1, \ldots, u_n = u_{\sigma(1)}, \ldots, u_{\sigma(n)} \longleftrightarrow s(u_1, \ldots, u_n) = s(u_{\sigma(1)}, \ldots, u_{\sigma(n)}; p, q)
\]

Moreover, multiplication distributes through addition and the \((p, q)\)-blossom is multiaffine. Thus

\[
u = \frac{b - u}{b - a} + \frac{u - a}{b - a}b
\]

implies both

\[
u_1, \ldots, u_n u = \frac{b - u}{b - a}u_1, \ldots, u_n + \frac{u - a}{b - a}u_1, \ldots, u_n b
\]
Figure 3: ‘Computing $s(p^{n-1} t, p^{n-2} t q, ..., t q^{n-1}; p, q) = S(t)$ recursively from the initial $(p, q)$-blossom values $s(0, ..., 0, p^{k-1}, p^{k-2} q, ..., q^{k-1}; p, q), k = 0, 1, ..., n’$

and

$$s(u_1, ..., u_n; p, q) = \frac{b - u}{b - a} s(u_1, ..., u_n; a; p, q) + \frac{u - a}{b - a} s(u_1, ..., u_n; b; p, q)$$

The diagram represents the symmetry and multiaffinity character while at same time also make the case for multiplication and $(p, q)$-blossoming both. Therefore this multiplicative representation for the $(p, q)$-blossom seems to be natural. Due to similarity between multiplication and $(p, q)$-blossoming, identities corresponding to multiplication expect to give analogous identities for the $(p, q)$-blossom. (See [18] for more explanations to this multiplicative notation.)

Using this multiplicative notation, Fig. 3 shows (for $n = 3$) how to compute an arbitrary value of $s(p^{n-1} t, p^{n-2} t q, ..., t q^{n-1}; p, q) = S(t)$ recursively from the initial $(p, q)$-blossom values $s(0, ..., 0, p^{k-1}, p^{k-2} q, ..., q^{k-1}; p, q)$, with exactly $n - k$ blossom values set to 0 for $k = 0, ..., n$, by applying the multiaffine and symmetry properties at each node.

Now compare the $(p, q)$-blossoming algorithm in Fig. 3 to the de-Casteljau algorithm in Fig. 1 for $(p, q)$-Bézier curves. For arbitrary $n$, Figs. 1 and 3 are similar, and Fig. 3 is this de Casteljau algorithm for $s(p^{n-1} t, p^{n-2} t q, ..., q^{n-1} t; p, q) = S(t)$ with control points $s(0, ..., 0, p^{n-1}, p^{n-2} q, ..., q^{n-1}; p, q), k = 0, ..., n$. In next three theorems certain observations having some important consequences for arbitrary values of the degree $n$ has been done. Standard restrictions given by 4.4 and 4.5 on the value of $p, q$ will be applicable until mentioned otherwise.

**Theorem 5.1** Any polynomial represents $(p, q)$-Bézier Curve. In other words, let $S(t)$ be a polynomial of degree $n$ with $(p, q)$-blossom $s(u_1, ..., u_n; p, q)$. Then using de Casteljau algorithm (2.3), $S(t)$ can be generated by control points $P_k = s(0, ..., 0, p^{n-1}, p^{n-2} q, ..., q^{n-1}; p, q), k = 0, ..., n$. 

Proof. Let \( S(t) \) be a polynomial of degree \( n \) and let \( s(u_1, ..., u_n; p, q) \) be the \((p, q)\)-blossom of \( S(t) \). Set \( P_k = \tilde{P}_k = s(0, ..., 0, p^{n-1}, p^{n-2}q, ..., q^{n-1}; p, q) \), \( k = 0, ..., n \) and apply the \((p, q)\)-de Casteljau algorithm. The \((p, q)\)-Bézier curve is given by

\[
\tilde{P}_0^n = \sum_{i=0}^{n} P_i B_i^n(t; p, q) \tag{5.1}
\]

On the other hand applying induction on \( k \) and using multiaffine property of the \((p, q)\)-blossom, it can be shown that points \( \tilde{P}_k(t) \) generated by the \((p, q)\)-de Casteljau algorithm satisfy

\[
\tilde{P}_k(t) = s(0, ..., 0, p^{i-1}, p^{i-2}q, ..., q^{i-1}, tp^{k-1}q^{n-k}, ..., tq^{n-k}; p, q), \quad i = 0, ..., n - k, \quad k = 0, ..., n.
\]

In particular,

\[
\tilde{P}_0^n = s(p^{n-1}t, p^{n-2}qt, ..., q^{n-1}t; p, q) = S(t). \tag{5.2}
\]

The theorem now follows from (5.1) and (5.2).

Corollary 4.2. On interval \([0, 1]\), \( n \) degree \((p, q)\)-Bernstein basis functions form the basis for \( n \) degree polynomial, except when \( q = p \) and for even \( n \).

Proof. Result can be drawn directly from Theorem 4.1 when \( p \) and \( q \) satisfies the standard restrictions given by (4.4) and (4.5). Further, when \( p = 1, q = 0 \), this result can be obtained explicitly by using formula for basis in (2.1) since

\[
B_i^n(t; 0) = t^i - t^{i+1}, \quad i = 0, ..., n - 1
\]

\[
B_0^n(t; 0) = t^n
\]

Corollary 4.3. On interval \([0, 1]\), \((p, q)\)-Bézier curve’s control points are unique.

Theorem 5.2 (Dual Functional Property of the \((p, q)\)-Blossom). Let \( S(t) \) be a \((p, q)\)-Bézier curve of degree \( n \) and let \( s(u_1, ..., u_n; p, q) \) be the \((p, q)\)-blossom of \( S(t) \). Then the \((p, q)\)-Bézier control points of \( S(t) \) are given by

\[
P_k = s(0, ..., 0, p^{n-1}, p^{n-2}q, ..., q^{n-1}; p, q), \quad k = 0, ..., n. \tag{5.3}
\]

Proof. By Theorem 4.1

\[
S(t) = \sum_{k=0}^{n} s(0, ..., 0, p^{n-1}, p^{n-2}q, ..., q^{k-1}; p, q)B_k^n(t; p, q) \tag{5.4}
\]

Now (5.3) follows from (5.4) and the uniqueness of the \((p, q)\)-Bézier control points.

Fig. 4 illustrates a recursive evaluation algorithm for computing an arbitrary \((p, q)\)-blossom value \( s(u_1, ..., u_n; p, q) \) from the \((p, q)\)-blossom values \( s(0, ..., 0, p^{n-1}, p^{n-2}q, ..., q^{n-1}; p, q) \), \( k = 0, ..., n \) by blossoming the de Casteljau evaluation algorithm i.e. by substituting \( u_k \) for \( tq^{n-k} \) on the \( k \)-th level of the de Casteljau evaluation algorithm in Fig. 3.

From fig. 4 it can also be observed that for \((p, q)\)-Bézier curves the recursive evaluation algorithm is not unique, as \( p^2t, ptq, tq^2 \) can be substituted in any order for the values of parameters \( u_1, u_2, u_3 \). These observations are summarized in the next two theorems.
Theorem 5.3  Let \( S(t) = \sum_{i=0}^{n} P_i B_i^n(t; p, q) \) be a \((p, q)\)-Bézier curve of degree \( n \) with \((p, q)\)-Blossom \( s(u_1, \ldots, u_n; p, q) \). Define recursively a set of multiaffine functions by setting \( Q_i^0 = P_i, \ i = 0, 1, \ldots, n \) and

\[
Q_{i+1}^{k+1}(u_1, \ldots, u_{k+1}) = (1 - u_{k+1} p^i q^{-i})Q_i^k(u_1, \ldots, u_k) + u_{k+1} p^i q^{-i}Q_{i+1}^k(u_1, \ldots, u_k) \quad (5.5)
\]

\( i = 0, 1, \ldots, n-k-1 \) and \( k = 0, \ldots, n-1 \). Then

\[
Q_i^k(u_1, \ldots, u_k) = s(0, \ldots, 0, p^{n-1}, p^{n-2}q, \ldots, q^{n-1}, u_1, \ldots, u_n; p, q)
\]

\( i = 0, 1, \ldots, n-k \) and \( k = 0, \ldots, n \). In particular,

\[
Q_i^0(u_1, \ldots, u_n) = s(u_1, \ldots, u_n; p, q) = u_1 u_2 u_3
\]

![Recursive evaluation algorithm for the \((p, q)\)-blossom of a Cubic \((p, q)\) Bézier curve](image)

**Proof.** By the dual functional property,

\[
Q_i^0(u_1, \ldots, u_n) = s(0, \ldots, 0, p^{i-1}, p^{i-2}q, \ldots, q^{i-1}; p, q)
\]

\( i = 0, \ldots, n \).

By applying induction on \( k \), rest of the proof can be easily done. The case \( n = 3 \) is illustrated by Fig. 4. ......

Theorem 5.4  Let \( S(t) = \sum_{i=0}^{n} P_i B_i^n(t; p, q) \) be a \((p, q)\)-Bézier curve of degree \( n \) with \((p, q)\)-Blossom \( s(u_1, \ldots, u_n; p, q) \). There are \( n! \) affine invariant, recursive evaluation algorithms for \( S(t) \) defined as follows: Let \( \sigma \) be a permutation of \( 1, \ldots, n \) and let \( P_i^\sigma(t) = P_i, \ i = 0, \ldots, n \). Define
\[ P_{i}^{k+1}(t) = (p^{(k+1)-1} - t \ p^{i} \ q^{(k+1)-1-i}) P_{i}^{k}(t; p, q) + p^{i} \ q^{(k+1)-1-i}) t P_{i+1}^{k}(t; p, q) \]  
(5.6)

\[ i = 0, 1, \ldots, n - k - 1 \text{ and } k = 0, \ldots, n - 1. \text{ Then} \]

\[ P_{i}^{k}(t) = s(0, \ldots, 0, p^{i-1} q, \ldots, q^{1-i}, t q^{(k-1) \ -1}, \ldots, t q^{(k-1) \ -1}; p, q) \]  
(5.7)

\[ i = 0, 1, \ldots, n - k \text{ and } k = 0, \ldots, n. \text{ In particular} \]

\[ P_{0}^{n}(t) = s(t q^{(n-1) \ -1}, \ldots, t q^{(n-1) \ -1}; p, q) = S(t) \]  
(5.8)

**Proof:** Theorem 5.4 follows from 5.3 substituting value \[ u_{k} = t p^{(k-1) \ -1} q^{(k-1) \ -1}. \]

6 Identities for \((p, q)\)-Bernstein basis functions based on \((p, q)\)-blossoming

Three identities has been derived for the \((p, q)\)-Bernstein basis functions in this section. Each of these identities can be expressed into standard \(q\)-Bernstein basis functions after putting \( p = 1 \). Standard restrictions on \( p \) and \( q \) given by 4.4 and 4.5. Starting from new variant of Marsden’s identity.

**Proposition 5.1 (Marsden’s Identity).**

\[ \prod_{i=1}^{n} (p^{i-1} x - q^{i-1} t) = \sum_{j=0}^{n} \binom{n}{j} \ p^{j(n-j)} q^{j(n-j)} B_{n-j}^{n}(x, \frac{1}{p}, \frac{1}{q}) B_{j}^{n}(t, p, q) \]  
(6.1)

**Proof**

Let \( S(t) \) denote the left hand side of Eq. 6.1. The \((p, q)\)-blossom of \( S(t) \) is given by

\[ s(u_{1}, \ldots, u_{n}; p, q) = \prod_{i=1}^{n} (p^{i-1} x - u_{i}) \]

where \( u_{i} = t q^{i-1} \)

Thus by the dual functional property 5.2 and which after factoring out powers of \( p \) and \( q \) gives the right hand side of 6.1. Monomials can also be expressed in terms of the \((p, q)\)-Bernstein basis functions

**Proposition 5.2 (Monomial Representation).**

\[ t^{i} = \sum_{k=i}^{n} p^{i(n-k)} \binom{k}{i} B_{k}^{n}(t, p, q) \]  
(6.2)

**Proof.** Using dual functional property 5.2 Eqs. 4.2 and 5.3 the above result will be obtained and the fact that

\[ \phi_{n,i}(0, \ldots, 0, p^{k-1} q, \ldots, q^{k-1}) = \phi_{k,i}(p^{k-1} q, \ldots, q^{k-1}) \]

for \( k \geq i \).
Reparametrization formula for \((p, q)\)-Bernstein basis functions is last identity of this section which has its use in subdivision algorithms for standard Bezier curves. Before proceeding to proof this change of basis formula, first there is a need to know a lemma.

**Lemma 5.3.** Let \(b^n_i(u_1, ..., u_n; p, q)\) denote the \((p, q)\)-blossom of \(B^n_i(t; p, q)\), \(i = 0, ..., n\). Then

\[
b^n_i(u_1, ..., u_{n-1}, 0; p, q) = b^n_{i-1}(u_1, ..., u_{n-1}; p, q), \quad i = 0, ..., n - 1.
\]

**Proof.** First apply the \((p, q)\)-blossom algorithm from Theorem 4.5 to the polynomials \(B^n_i(t; p, q)\) and \(B^{n-1}_i(t; p, q)\). Through dual functional property, the first \(n - 1\) initial \((p, q)\)-blossom values for these two polynomials as defined in Theorem 4.5 are the same: \(Q^n_j = 0, j = 0, ..., n - 1, j \neq i\) and \(Q^n_i = 1\). Therefore the functions \(Q^n_j, j = 0, ..., n - 1 - k, k = 0, ..., n - 1\) generated by Eq. (4.5) of the recursive evaluation algorithms for the \((p, q)\)-blossoms of \(B^n_i(t; p, q)\) and \(B^{n-1}_i(t; p, q)\). Thus the function \(Q^n_{n-1}\) for \(B^n_i(t; p, q)\) is the same as the \(q\)-blossom of \(B^{n-1}_i(t; p, q)\). On the other hand by (4.5), substituting \(u_n = 0\) in the function \(Q^n_0\) for \(B^n_i(t; p, q)\), which is precisely the \((p, q)\)-blossom of \(B^n_i(t; p, q)\), also gives exactly the function \(Q^n_{n-1}\) for \(B^n_i(t; p, q)\).

**Proposition 5.4** (Reparametrization Formula).

\[
B^n_k(rt; p, q) = \sum_{i=k}^{n} B^i_k(r; p, q) B^n_i(t; p, q).
\]

**Proof.** Let \(F\) and \(G\) be polynomials of degree \(n\) with \((p, q)\)-blossoms \(f\) and \(g\). If \(F(t) = G(rt)\), then \(f(u_1, ..., u_n; p, q) = g(ru_1, ..., ru_n; p, q)\). This property holds because the three \((p, q)\)-blossoming axioms for \(f(u_1, ..., u_n; p, q)\) are satisfied by \(g(ru_1, ..., ru_n; p, q)\). Therefore, \(b^i_k(ru_1, ..., ru_n; p, q)\) is the \((p, q)\)-blossom of \(B^n_i(rt; p, q)\). Hence by the dual functional property, Corollary 3.5, Lemma 5.3, and the \((p, q)\)-diagonal property

\[
B^n_k(rt; p, q) = \sum_{i=0}^{n} b^i_k(0, ..., 0, p^{i-1}r, p^{i-2}rq, ..., rq^{i-1}; p, q) B^n_i(t; p, q)
\]

\[
= \sum_{i=k}^{n} b^i_k(p^{i-1}r, p^{i-2}rq, ..., rq^{i-1}; p, q) B^n_i(t; p, q)
\]

\[
= \sum_{i=k}^{n} B^i_k(r; p, q) B^n_i(t; p, q).
\]

## 7 Future Work

Currently, we are working on \((p, q)\)-blossoming of identities and algorithms for \((p, q)\)-Bernstein bases over arbitrary intervals as given in [25]. In near future, we will construct \((p, q)\)-blossoming of identities and algorithms for derivatives of \((p, q)\)-Bernstein bases and \((p, q)\)-Bezier curves.
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