Generalization of Bertrand’s Postulate and Upper Bounds on the Number of Primes in Certain Intervals

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Abstract
In this paper, we have proved the Generalization of Bertrand’s Postulate and the Conjecture from Mitra et al.’s work [Arxiv 2011] on the upper bound on the number of primes in the interval \((n, kn)\) for all \(k \geq 1, n \geq 1\). We have also constructed a gap in which there exists a prime \(p\), and this gap is smaller than the gap stated in Bertrand’s Postulate [Journal of the Royal Polytechnic School, 1845] or J Nagura’s Theorem [The Japan Academy, 1952], for large positive integers.

1 Introduction
A branch of number theory studying distribution laws of prime numbers among natural numbers. The Prime Number Theorem [9] describes the asymptotic distribution of prime numbers. It gives us a general view of how primes are distributed amongst positive integers and also states that the primes become less common as they become larger.

In 1845, Joseph Bertrand postulated [1] that, there is always a prime between \(n\) and \(2n\), and he verified this for \(n < 3 \times 106\). Tchebychev gave an analytic proof of the postulate in 1850, and a short but advanced proof was given by Srinivasa Ramanujan [2]. An elementary proof is due to Paul Erdős [3], and totally there are eighteen different proofs of Bertrand Postulate.

As an application of generalized Ramanujan primes [4], Christian Axler proved the conjecture of Mitra, Paul and Sarkar [5] which is named by “Generalization of Bertrand’s Postulate”. Axler proved this result for sufficiently large positive integers. In this paper, as our first contribution, we give a different proof of the above conjecture (Theorem 1 in this draft) for all positive integers \(n \geq 2\).

By the Bertrand Postulate, we know about the minimum number of primes in certain intervals. Now, there is an important question: what about the upper bound on the number of primes
in similar interesting intervals. Prime number theorem formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs. If we notice on the prime gap function, the first, smallest, and only odd prime gap is the gap of size 1 between 2, the only even prime number, and 3, the first odd prime. There is only one pair of consecutive gaps having length 2: the gaps $g_2$ and $g_3$ between the primes 3, 5, and 7. All other prime gaps are even. For any integer $n$, the factorial $n!$ is the product of all positive integers up to and including $n$. Then in the sequence

$$n! + 2, n! + 3, \ldots, n! + n$$

the first term is divisible by 2, the second term is divisible by 3, and so on. Thus, this is a sequence of $n - 1$ consecutive composite integers, and it must belong to a gap between primes having length at least $n - 1$.

It follows that there are gaps between primes that are arbitrarily large, that is, for any integer $N$, there is an integer $m$ with $g_m \geq N$. As of March 2017, the largest known prime gap with identified probable prime gap ends has length 5103138, with 216849-digit probable primes found by Robert W. Smith [6]. The largest known prime gap with identified proven primes as gap ends has length 1113106, with 18662-digit primes found by P. Cami, M. Jansen and J. K. Andersen [7] [8].

In this paper, as our second contribution, we study the upper bound and the lower bound of the prime gap function and how can we apply it to prove the upper bound on the number of prime in the interval $(n, kn)$ for any positive integer $n$ and $k \geq 1$. This was conjectured by Mitra, Paul and Sarkar [5].

As our third contribution, we have constructed an interval between which there exist a prime. This interval is smaller than the interval stated in Bertrand Postulate and J Nagura’s Theorem [16]. In this context, one should note that there exist some results that are better than Bertrand’s Postulate, but they hold for sufficiently large values of $n$. We list them briefly here.

1. From the prime number theorem it follows that for any real $\varepsilon > 0$ there is a $n_0 > 0$ such that for all $n > n_0$ there is a prime $p$ such $n < p < (1 + \varepsilon)n$. It can be shown, for instance, that

$$\lim_{n \to \infty} \frac{\pi((1 + \varepsilon)n) - \pi(n)}{n / \log n} = \varepsilon.$$

which implies that $\pi((1 + \varepsilon)n) - \pi(n)$ goes to infinity [17] (and, in particular, is greater than 1 for sufficiently large $n$).

2. Non-asymptotic bounds have also been proved. In 1952, Jitsuro Nagura [18] proved that for $n \geq 25$, there is always a prime between $n$ and $(1 + 1/5)n$.

3. In 1976, Lowell Schoenfeld [19] showed that for $n \geq 2010760$, there is always a prime between $n$ and $(1 + 1/16597)n$.

4. In 1998, Pierre Dusart [20] improved the result in his doctoral thesis, showing that for $k \geq 463$, $p_{k+1} \leq (1 + 1/(\ln^2 p_k))p_k$, and in particular for $n \geq 3275$, there exists a prime number between $n$ and $(1 + 1/(2 \ln^2 n))n$.

5. In 2010, Pierre Dusart [21] proved that for $n \geq 396738$ there is at least one prime between $n$ and $(1 + 1/(25 \ln^2 n))n$. 

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6. In 2016, Pierre Dusart \cite{22} improved his result from 2010, showing that for \( n \geq 468991632 \) there is at least one prime between \( n \) and \( (1 + 1/(5000 \ln^2 n))n \).

7. Baker, Harman and Pintz \cite{23} proved that there is a prime in the interval \( [n, n + O(n^{21/40})] \) for all large \( n \).

We have proved that there exists a prime in the interval \( (n, g(n)) \), where \( g(n) = n + \frac{n}{(1.1 \ln(2.5n))} \) for all \( n \geq 2 \). Also, our interval is better than that stated in Bertrand Postulate. The interval \( [n, n + O(n^{21/40})] \) mentioned in \cite{23} is smaller than our interval, but this and all the other existing works hold for large values of \( n \), whereas our result is true for \( n \geq 2 \).

## 2 Preliminaries

In this section we will discuss about some number theoretic results which are already known to prove the Generalization of Bertrand’s Postulate and the upper bound on the number of primes in the interval \( (n, kn) \) for any positive integer \( n \) and \( k \geq 1 \).

**Proposition 1 (Bertrand’s Postulate):** For any integer \( n > 3 \) there always exists at least one prime number \( p \) with, \( n < p < 2n \).

There are many different proofs of this theorem. In this paper, we will prove the generalization of this theorem by inductive argument. To prove this first we will define some sets and functions.

**Definition 1 (Prime gap function):** A prime gap is the difference between two successive prime numbers. The \( n \)-th prime gap, denoted by \( g_n \) or \( g(p_n) \) is the difference between the \((n + 1)\)-th and the \( n \)-th prime numbers, i.e.,

\[
g_n = p_{n+1} - p_n.
\]

The function is neither multiplicative nor additive.

**Definition 2 (Prime counting function):** The prime-counting function is the function counting the number of prime numbers less than or equal to some real number \( x \). It is denoted by \( \pi(x) \).

**Proposition 2 (Prime Number Theorem):** The prime number theorem \cite{9} states that \( \frac{x}{\log x} \) is a good approximation to \( \pi(x) \), in the sense that the limit of the quotient of the two functions \( \pi(x) \) and \( \frac{x}{\log x} \) as \( x \) increases without bound is 1:

\[
\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.
\]

known as the asymptotic law of distribution of prime numbers. Using asymptotic notation this result can be restated as

\[
\pi(x) \sim \frac{x}{\log x}.
\]

**Corollary 1 (Bound for the \( n \)-th prime \( p_n \)):** The \( n \)-th prime number \( p_n \) satisfies, \( p_n \sim n \log n \) and the bound for the \( n \)-th prime number \( p_n \) is \[13\] \cite{14},

\[
n \ln n + n(\ln \ln n - 1) < p_n < n \ln n + n(\ln \ln n)
\]

\[
\Rightarrow n \left( \ln \frac{n \ln n}{e} \right) < p_n < n \ln(n \ln n). \quad (1)
\]
Now, we will state some important results of number theory which will help us to prove our main result. We can state the Upper bound of Prime gap function \([24]\).

**Proposition 3 (Upper bound of the Prime gap function):** The function \( g_n = p_{n+1} - p_n \) satisfies,

\[
g_n < (\log p_n)^2 - \log p_n, \forall n > 4.
\]

**Proposition 4** Let \( p_n \) be the \( n \)-th prime where \( n \geq 1 \). Then the following inequality is true:

\[
n + 1 \leq p_n.
\]

**Proof:** We will proof this inequality, by the method of Mathematical Induction. For \( n = 1 \), \( p_1 = 2 \), then equality holds. For, \( n < 4 \) this inequality holds trivially. Now, let us assume that this inequality is true for \( n - 1 \). So, we can write,

\[
(n - 1) + 1 = n < p_{n-1}, \forall n \geq 4.
\]

Now, for the \( n \)-th prime \( p_n \),

\[
n < n + 1 \leq p_{n-1} < p_n, \forall n \geq 4
\]

\[
\implies n + 1 \leq p_n.
\]

So, by the Induction Hypothesis this inequality holds.

**Proposition 5 (Lower Bound of the Prime gap function):** Proposition 3 implies a strong form of Cramer’s conjecture \([10]\) \([11]\) \([12]\) but is inconsistent with the heuristics of Granville and Pintz which suggests that,

\[
g_n > 2 - \varepsilon \frac{e}{\gamma} (\log p_n)^2.
\]

infinitely often for any \( \varepsilon > 0 \), where \( \gamma \) denotes the Euler Mascheroni constant.

**Definition 3 (Primorial):** For the \( n \)th prime number \( p_n \), the primorial \( p_n\# \) is defined as the product of the first \( n \) primes

\[
p_n\# = \prod_{k=1}^{n} p_k,
\]

where \( p_k \) is the \( k \)-th prime number.

In the classic proof of Bertrand Postulate by Paul Erdős \([3]\), it is shown that,

\[
x\# < 4^x,
\]

where \( x\# \) is the primorial for \( x \).

**Proposition 6** For all \( n \geq 2 \), \( n\# = \prod_{p \leq n} p \leq 4^n \), where the product is over primes.
Proof: We proceed by induction on $n$. For small values of $n$, the claim is easily verified. For larger even $n$ we have

$$\prod_{p \leq n} p = \prod_{p \leq n} p \leq 4^{n-1} \leq 4^n,$$

the equality following from the fact that, $n$ is even and so not a prime, and the first inequality following from the inductive hypothesis. For larger odd $n$ say $n = 2m + 1$, we have

$$\prod_{p \leq n} p = \prod_{p \leq m+1} p \prod_{m+2 \leq p \leq 2m+1} p \leq 4^{m+1}\left(\frac{2m+1}{m}\right) \leq 4^{m+1}2^{2m} = 4^{2m+1} = 4^n.$$


3 Our Results

In this section, we will describe our main results with proofs. First, let is define some sets and functions.

Definition 4 For the positive integer $a$ and $b$ we denote the set of integers $a, a + 1, \ldots, b$ by $[a, b]$.

Definition 5 Define the set $S_i$, as

$$S_i = \{k : f(k) = i + 1\},$$

where $f(k) = \lceil 1.1 \ln(2.5k) \rceil$ and $k \in \mathbb{N}$.

3.1 Generalization of Bertrand’s Postulate

Now we discuss some intermediate results that will lead to our first main result.

Proposition 7 For all values of $k \geq 2$,

$$f(k + 1) = \begin{cases} \text{either } f(k) \\ \text{or, } f(k) + 1; \end{cases}$$

Proof: By the definition of the set $S_i$, this result is easy to observe. ■

Lemma 1 Let, $f(k) = \lceil 1.1 \ln(2.5k) \rceil$, then the inequality,

$$\left(\log p_{f(k)+k-3}\right)^2 - \left(\log p_{f(k)+k-3}\right) < (k + 1)(f(k) + 1) - p_{f(k)+k-3}. \tag{3}$$

holds for all $k \geq 5$. 

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Proof: Let us consider that \( n = f(k) + k - 3, \forall k \geq 5 \). Now, \( k \) is an integer, so we can reduce the inequality in terms of \( k \) and \( n \), which is,
\[
|\log p_n|^2 - (\log p_n)| < (k + 1)(n + 3 - k) - p_n. \tag{4}
\]

We will use the bound for the \( n \)-th prime \( p_n \), which we have already stated. So from inequality (1), the inequality (4) become,
\[
|\log p_n|^2 - (\log p_n)| < (k + 1)(n + 3 - k) - n \left( \log \frac{n \ln n}{e} \right).
\]

If we can prove the inequality,
\[
\left| (\log (n \ln (n \ln n)))^2 - \left( \log n \left( \log \frac{n \ln n}{e} \right) \right) \right| < (k + 1)(n + 3 - k) - n \left( \log \frac{n \ln n}{e} \right), \forall k \geq 5. \tag{5}
\]

then we are done.

By our assumption, \( n = f(k) + k - 3, n \geq k \).
Therefore, \( n \ln(n \ln n) < n + 3 - k \) holds, \( \forall n \) and \( k \). Now,
\[
(\log (n \ln (n \ln n)))^2 < \log (n \ln (n \ln n))(k + 1)
\]
\[
< (n + 3 - k)(k + 1).
\]

Again, we have, \( (\log(n \ln(n \ln n))) < n \left( \frac{n \ln n}{e} \right) \). From this we can conclude the inequality (5). □

Lemma 2 Let, \( f(k) = \left\lceil 1.1 \ln(2.5k) \right\rceil \), then the inequality,
\[
\left| \log p_{f(k)+k+r} - \log p_{f(k)+k+r} \right| < (k + 4 + r)(f(k) + 1) - p_{f(k)+k+r}. \tag{6}
\]
holds for all \( k \geq 5 \), where \( r \in [-2, \infty) \in \mathbb{Z} \).

Proof: We will prove this inequality, by induction on \( r \).

For the base case, we have to show that for \( r = -2 \), this inequality,
\[
\left| \log p_{f(k)+k-2} - \log p_{f(k)+k-2} \right| < (k + 2)(f(k) + 1) - p_{f(k)+k-2}. \tag{7}
\]
holds. As we proof the inequality in Lemma 1, we can prove this by the similar argument, where \( n = f(k) + k - 2 \).

Let us assume that the inequality (7) is true for \( r \), where \( n = f(k) + k + r \). Now, for \( r + 1 \), we have \( n + 1 = f(k) + k + r + 1 \). The inequality becomes,
\[
\left| \log ((n + 1) \ln((n + 1) \ln(n + 1)))^2 - \log(n + 1) \ln \frac{(n + 1) \ln(n + 1)}{e} \right| < (k + 4 + r + 1)(n - k - r) - (n + 1)\ln \frac{(n + 1) \ln(n + 1)}{e}.
\]
As we proved the inequality (3), in Lemma 1, we can prove this by the similar argument. So, by the Induction Hypothesis the inequality (6) holds.

Now we are in a position to prove our first result.

**Theorem 1** (Generalization of Bertrand’s Postulate): For any integers \( n \) and \( k \), where \( f(k) = \lceil 1.1 \ln(2.5k) \rceil \), then there are at least \((k - 1)\) primes between \( n \) and \( kn \), when \( n \ge f(k) \).

**Proof:** We will prove this theorem by the inductive method in two parts. In the first part, we will show that this theorem is true for all elements \( k \in S_i \), as we defined the set \( S_i \), in Definition 5. Next, we will prove that this theorem holds for all such set \( S_i \).

**First part:** In this step we will prove that this theorem is true for all \( k \in S_i \).

To prove this, take any set \( S_i \), \( \forall i = 1, 2, \ldots \).

**(Base Case):** The base case is to prove this theorem is true for \( \min\{S_i\} \). Let, \( \min\{S_i\} = k_{\min} \). We have to prove that there are at least \((\min\{S_i\}) - 1\) primes between the gap, 
\[
[f(\min\{S_i\}), (\min\{S_i\} \times f(\min\{S_i\}))].
\]
For this gap we will start our prime counting from \( p_i \), and clearly, 
\[
p_i \in [f(\min\{S_i\}), (\min\{S_i\} \times f(\min\{S_i\}))].
\]
Now, we have to prove that, 
\[
p_{i+(\min\{S_i\})-2} \in [f(\min\{S_i\}), (\min\{S_i\} \times f(\min\{S_i\}))].
\]
Since, we have \( i = f(\min\{S_i\}) - 1 \), so now if we can prove that, 
\[
p_{f(\min\{S_i\})+\min\{S_i\}-3} \in [f(\min\{S_i\}), (\min\{S_i\} \times f(\min\{S_i\}))],
\]
then we are done. By the base case of Lemma 2 which is true. So, we have proved that this theorem is true for \( \min\{S_i\}, \forall i = 1, 2, \ldots \).

**(Inductive Case):** Now we will prove that if this theorem is true for \( \min\{S_i\} \), then it is also true for all elements of \( \{S_i\} \). Let us assume that this theorem is true for an element \( k \in S_i \) and \( k \neq \max\{S_i\} \). Since, by our assumption we can say that there exist \( k - 1 \) primes between the gap, 
\[
[f(k), (k \times f(k))].
\]
By the definition of the set \( S_i \), \( f(k) \) remains same for all \( k \in S_i \). To prove, the first part of this theorem we have to prove that there exist \( k \) primes between the gap 
\[
[f(k+1), (k+1 \times f(k+1))].
\]
So, we can start our prime counting from \( p_i \). We have to prove that, there exists at least one prime between the gap, 
\[
[k \times f(k), k+1 \times f(k)].
\]
By the Lemma 2 which is true.

So, by the method of Mathematical Induction we have proved that this theorem is true for all \( k \in S_i \).
Second part: In the second part of the proof we will prove that this theorem is true for all such set \(S_i, \forall i = 1, 2, \ldots\). We will prove this by induction on \(i\).

(Base Case): For the base case we have \(S_1 = \{2\}\) and \(f(2) = 2\). Clearly, \(2 < 3 < 4\), here 3 is the required prime.

(Inductive Case): We will start our prime counting for the set \(S_i\), from, \(p_i\), where \(p_i\) is the \(i\)-th prime. Clearly, \(p_i \in [f(k), kf(k)]\).

Now, let us assume that this theorem is true for the set \(S_{i-1}, \forall i \geq 1\). Then by the first part of the proof we can say that for all elements of \(S_{i-1}\) this theorem holds. By the first part of the proof this theorem holds for \(\max\{S_{i-1}\}\). Now, we will show that this also holds for \(\min\{S_i\}\).

For the set \(S_i\),

\[
f(\min\{S_i\}) = i + 1 \implies i = f(\min\{S_i\}) - 1.
\]

We have started, the counting of primes for the set \(S_i\) from \(p_i\), where \(p_i\) is the \(i\)-th prime.

Then for the set \(S_{i-1}\) we count \(\max\{S_{i-1}\} - 1\) primes from \(p_{i-1}\). So, clearly between the gap

\[
[f(\min\{S_i\}) = f(\max\{S_{i-1}\}) + 1, \max\{S_{i-1}\} \times f(\max\{S_{i-1}\})],
\]

there are at least \(\max\{S_{i-1}\} - 2\) primes.

Now, we have to show that there are at least 2-primes between the gap,

\[
[\max\{S_{i-1}\}f(\max\{S_{i-1}\}), (\max\{S_{i-1}\} + 1)f(\max\{S_{i-1}\} + 1)].
\]

We have

\[
\max\{S_{i-1}\} + 1 = \min\{S_i\}
\]

and

\[
f(\max\{S_{i-1}\}) + 1 = f(\min\{S_i\}).
\]

By the prime gap function \(g_n\), and Lemma 1, we have proved that there are at least 2-primes between the gap,

\[
[\max\{S_{i-1}\}f(\max\{S_{i-1}\}), (\max\{S_{i-1}\} + 1)f(\max\{S_{i-1}\} + 1)].
\]

Hence, by the Induction Hypothesis this conjecture is true for all set \(S_i\). From the first and second part of the proof we can say that this theorem is true for all elements of the set \(S_i\), and it is also true for all such set \(S_i, \forall i = 1, 2, \ldots\), which implies that there are at least \((k - 1)\) primes between \(n\) and \(kn\), when \(n \geq f(k)\), where \(f(k) = \lceil 1.1\ln(2.5k) \rceil\) and \(k \in \mathbb{Z}\).

So, we have proved the Generalization of Bertrand’s Postulate.

3.2 Upper bound on the number of primes in the interval \([n, kn]\)

To prove our next result, we first state and prove the following lemma.
Lemma 3 Let \( n \) be a non-zero positive integer, with \( n \geq 2 \), then the inequality
\[
\left( \frac{2n}{9} + 4 \right) \times (\log n \ln(n \ln n))^2 < n
\]
holds.

Proof: We will prove it by induction on \( n \).
(Base Case): For \( n = 2 \), we have,
\[
\left( \frac{2 \times 2}{9} + 4 \right) \times (\log 2 \ln(2 \ln 2))^2
= 4.44 \times 0.034 = 0.151 < 2.
\]
So, the base case holds.
(Inductive Case): Let us assume that this inequality is true for \( n \). Now we have,
\[
\left( \frac{2n}{9} + 4 \right) \times (\log n \ln(n \ln n))^2 < n
\Rightarrow \left( \frac{2n}{9} + 4 \right) \times (\log(n+1) \ln((n+1) \ln(n+1))^2 + \frac{2}{9} \times (\log(n+1) \ln((n+1) \ln(n+1))^2
<n + \frac{2}{9} \times (\log(n+1) \ln((n+1) \ln(n+1))^2
<n + 1.
\]
Hence, by the Induction hypothesis this inequality holds. \( \square \)

Now we prove another related conjecture in [5].

Theorem 2 (Upper Bound on the number of primes in the interval \([n, kn]\)): Given a positive integer \( k \), then the number of primes between \( n \) and \( kn \), for any positive integer \( n \), is bounded by \( \frac{kn}{9} + k^2 \).

Proof: We will prove it by induction on \( k \).
(Base Case): In this case we have to prove that, for \( k = 2 \), the upper bound on the number of primes between the gap \([n, 2n]\) is \( \frac{2n}{9} + 4 \). To prove this we will use the lower bound of the prime gap function \( g_n \), given by Proposition \( \ref{prop:prime-gap} \) as Proposition \( \ref{prop:prime-gap} \) implies a strong form of Cramer’s conjecture \( \ref{conj:cramer} \) \( \ref{conj:cramer-2} \) \( \ref{conj:cramer-3} \), but is inconsistent with the heuristics of Granville and Pintz which suggests the following:
\[
g_n > \frac{2 - \varepsilon}{e^\gamma} (\log p_n)^2.
\]
infinently often for any \( \varepsilon > 0 \), where \( \gamma \) denotes the Euler Mascheroni constant. We can approximate the part
\[
\frac{2 - \varepsilon}{e^\gamma}.
\]
Since, $g_n$ is a prime gap function, so it can not be negative. So, the maximum possible value of $\varepsilon$ is 2.

As we know the value of
\[ e \approx 2.718\ldots, \text{ and } \gamma \approx 1.781062\ldots \]
\[ \Rightarrow \frac{2-\varepsilon}{e^{\gamma}} \approx 0.561\ldots \approx 1. \]

Then we have to prove the inequality,
\[ \frac{2n}{9} + 4 \times (\log p_n)^2 < 2n - n = n. \]
We will use the bound of the $n$-th prime $p_n$, which we have stated in the section of known result. So, the inequality becomes,
\[ \frac{2n}{9} + 4 \times (\log n \ln(n \ln n))^2 < 2n - n = n. \]
Which we have proved in Lemma 3. So, the base case holds.

(Inductive Case): Now let us assume that the inequality,
\[ \left( \frac{kn}{9} + k^2 \right) \times (\log(n \ln(n \ln n)))^2 < kn - n = (k - 1)n. \]
is true. So, we can write,
\[ \left( \frac{kn}{9} + k^2 \right) \times (\log(n \ln(n \ln n)))^2 + \left( \frac{n}{9} + 2k + 1 \right) \times (\log(n \ln(n \ln n)))^2 \]
\[ = \left( \frac{(k+1)n}{9} + (k+1)^2 \right) \times (\log(n \ln(n \ln n)))^2 \]
\[ < ((k-1)n) + \left( \frac{n}{9} + 2k + 1 \right) \times (\log(n \ln(n \ln n)))^2 \]
\[ = (k-1)n + n \]
\[ = kn. \]

Hence, by the Induction Hypothesis for a given positive integer $k$, the number of primes between $n$ and $kn$, for any positive integer $n$, is bounded by $\frac{kn}{9} + k^2$.

### 3.3 After $n$, what is the upper bound before which a prime must exist?

In this section, we will construct an interval $[n, g(n)]$, for some $g(n)$, containing at least one prime. This interval is smaller than those of Bertrand’s Postulate and J. Nagura’s Theorem.

**Theorem 3** There exist at least one prime number $p$ with,
\[ kf(k) < p < k(f(k) + 1), \text{ where } f(k) = \lceil 1.1 \ln(2.5k) \rceil \text{ and } k \in \mathbb{Z}^+. \]

**Proof:** We will prove this theorem by induction on $k$. We will prove this theorem into two parts. For the *first part*, we will use inductive argument on $k$, for which $f(k)$ is same. In the *second part*, we will use inductive argument on $k$, for which $f(k + 1) = f(k) + 1$. 

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**First Part:** In this part, we will use inductive argument on \( k \), for which \( f(k + 1) = f(k) \).

**(Base Case):** For the base case we have \( k = 2 \) and \( f(2) = 2 \). Trivially, we can write, \( 4 < 5 < 6 \), where \( p_3 = 5 \). Hence, the base case holds.

Now, we will prove the Inductive case.

**(Inductive Case):** Let us assume that, there exist a prime \( p_m \) between the gap

\[ [kf(k), k(f(k) + 1)], \]

for some integer \( m \), and \( p_m \) is the \( m \)-th prime. Then we have to prove that, there exist at least one prime between the gap,

\[ [(k + 1)f(k), (k + 1)(f(k) + 1)]. \]

Trivially, we can write \( m < kf(k) \). So, if we can prove that,

\[ p_{m+r} \in [(k + 1)f(k), (k + 1)(f(k) + 1)], \]

for some integer \( r \geq 1 \) then we are done. By our assumption,

\[ kf(k) < p_m < kf(k) + k, \]

this inequality holds. Then we have to prove that there exist at least one prime between the gap,

\[ [(k + 1)f(k), (k + 1)(f(k) + 1)]. \]

Now, there are two possible cases.

**Case (i):** If

\[ p_m > (k + 1)f(k), \]

then there is nothing to prove.

**Case (ii):** If

\[ p_m < (k + 1)f(k), \]

then we have to show that

\[ p_{m+r} < (k + 1)(f(k + 1)), \]

for some integer \( r \geq 1 \). So we can write,

\[
p_{m+r} = p_m + \sum_{i=1}^{r} g_{m+i} < (k + 1)f(k) + \sum_{i=1}^{r} \left( (\log p_{m+i})^2 - \log p_{m+i} \right)
\]

\[
\implies p_m + \sum_{i=1}^{r} g_{m+i} < (k + 1)f(k) + \sum_{i=1}^{r} (\log p_{m+i})^2 - \sum_{i=1}^{r} (\log p_{m+i})
\]

\[
= (k + 1)f(k) + (\log p_{m+1} + \ldots + \log p_{m+r})^2 - (\log p_{m+1} + \ldots + \log p_{m+r})
\]

\[
= (k + 1)f(k) + \left( \log \frac{p_{m+r}}{p_m} \right)^2 - \left( \log \frac{p_{m+r}}{p_m} \right).
\]
From the proposition 6, we can write,

\[(k + 1)f(k) + \left(\log \frac{p_{m+r}^#}{p_m^#}\right)^2 - \left(\log \frac{p_{m+r}^#}{p_m^#}\right)\]

\[< (k + 1)f(k) + \left(\log \frac{4p_{m+r}}{4p_m}\right)^2 - \left(\log \frac{4p_{m+r}}{4p_m}\right)\]

\[< (k + 1)f(k) + \left((p_{m+r} - p_m)\log 4\right)^2 - \left((p_{m+r} - p_m)\log 4\right)
= (k + 1)f(k) + \left((p_{m+r} - p_m) \times 0.6020\right)^2 - \left((p_{m+r} - p_m) \times 0.6020\right)
< (k + 1)f(k) + \left((p_{m+r} - (k + 1)f(k)) \times 0.6020\right)^2 - \left((p_{m+r} - (k + 1)f(k)) \times 0.6020\right)
< (k + 1)f(k) + (k + 1) = (k + 1)(f(k) + 1).

Hence, by the Induction Hypothesis we have proved that there exists a prime between the gap

\[[(k + 1)f(k), (k + 1)(f(k) + 1)]\).

**Second Part:** In this part, we will use inductive argument on those \(k\), for which \(f(k+1) = f(k) + 1\).

**(Base Case):** For the base case, we have \(k = 2\) and \(f(2) = 2\). Trivially, we can write, \(4 < 5 < 6\), where \(p_3 = 5\). Hence, the base case holds.

Now, we will prove the Inductive case.

**(Inductive Case):** In this case first let us assume that there exists a prime \(p_m\) between the gap \(kf(k)\) and \(k(f(k) + 1)\), for some integer \(m\), and \(p_m\) (the \(m\)-th prime). Then we have to show that there exist one prime between the gap,

\[([(k + 1)(f(k) + 1), (k + 1)(f(k) + 2)]\).

So, if we can prove that

\(p_{m+r} \in [(k + 1)(f(k) + 1), (k + 1)(f(k) + 2)]\),

for some integer \(r \geq 1\), then we are done. By our assumption,

\(kf(k) < p_m < kf(k) + k\),

holds. Then we have to prove that there exist at least one prime between the gap

\[([(k + 1)(f(k) + 1), (k + 1)(f(k) + 2)]\).

Now, there are two possible cases.

**Case (i):** If

\(p_m > (k + 1)(f(k) + 1)\),

then there is nothing to prove.

**Case (ii):** If

\(p_m < (k + 1)(f(k) + 1)\),
then we have to show that,

\[ p_{m+r} < (k + 1)(f(k + 2)), \]

for some integer \( r \geq 1 \). So, we can write,

\[
\begin{align*}
p_{m+r} &= p_m + \sum_{i=1}^{r} g_{m+i} < (k + 1)(f(k) + 1) + \sum_{i=1}^{r} \left( (\log p_{m+i})^2 - \log p_{m+i} \right) \\
\implies p_m + \sum_{i=1}^{r} g_{m+i} &< (k + 1)(f(k) + 1) + \sum_{i=1}^{r} (\log p_{m+i})^2 - \sum_{i=1}^{r} (\log p_{m+i}) \\
&= (k + 1)(f(k) + 1) + (\log p_{m+1} + \ldots + \log p_{m+r})^2 - (\log p_{m+1} + \ldots + \log p_{m+r}) \\
&= (k + 1)(f(k) + 1) + \left( \log \frac{p_{m+r} \#}{p_m \#} \right)^2 - \left( \log \frac{p_{m+r} \#}{p_m \#} \right).
\end{align*}
\]

By the Proposition 6 we can write,

\[
\begin{align*}
(k + 1)(f(k) + 1) + \left( \log \frac{p_{m+r} \#}{p_m \#} \right)^2 - \left( \log \frac{p_{m+r} \#}{p_m \#} \right) < (k + 1)(f(k) + 1) + \left( \log \frac{4p_{m+r} \#}{4p_m} \right)^2 - \left( \log \frac{4p_{m+r} \#}{4p_m} \right) \\
&< (k + 1)(f(k) + 1) + ((p_{m+r} - p_m) \log 4)^2 - ((p_{m+r} - p_m) \log 4) \\
&= (k + 1)(f(k) + 1) + ((p_{m+r} - p_m) \times 0.6020)^2 - ((p_{m+r} - p_m) \times 0.6020) \\
&< (k + 1)(f(k) + 1) + ((p_{m+r} - (k + 1)(f(k) + 1)) \times 0.6020)^2 \\
&\quad - ((p_{m+r} - (k + 1)(f(k) + 1)) \times 0.6020) \\
&< (k + 1)(f(k) + 1) + (k + 1) = (k + 1)(f(k) + 2).
\end{align*}
\]

Hence, by the Induction Hypothesis we have proved that there exists a prime between the gap,

\[ [(k + 1)(f(k) + 1), (k + 1)(f(k) + 2)]. \]

Hence, by the first and second part of the proof and by Induction Hypothesis theorem 3 is true.

**Remark 1** The statement in the Theorem 3 states about the gap, between which there exist at least one prime. This gap is smaller than the gap which is in Bertrand’s Postulate and J Nagura’s Theorem for all \( k \geq 38 \). To see this, let

\[ n = k \times f(k), \text{ where } f(k) = \lceil 1.1 \ln(2.5k) \rceil \implies n \geq k, \]

because for all \( k \geq 2, f(k) \geq 2 \). In Bertrand’s Postulate, the length of the gap is \( n \) and in our result the length of the gap is \( k \). Similarly, we can prove that our gap is smaller than that of J Nagura’s Theorem.
In this graph, we have compared the different intervals \((n, g(n))\), where \(g(n) = 2n\) for Bertrand Postulate (red), \(g(n) = \frac{6}{5}n\) for J. Nagura’s Theorem (blue) and \(g(n) = n + \left\lceil \frac{n}{1.1 \ln(2.9n)} \right\rceil\) for our Theorem (green). Note that, for \(n = 240\), the value of \(g(n)\) is 480, 288 and 280 respectively for the three cases.

4 Conclusion

In this paper, we have proved one Generalization of Bertrand’s Postulate and one upper bound on the number of primes in the interval \(n\) and \(kn\), for any positive integer \(n\) and \(k \geq 1\). To derive these results, we have used the lower bound of the prime gap function, which plays an important role in the distribution of primes. Finally, we have also been able to tighten the upper bound beyond \(n\), before which there must exit at least one prime. It is an interesting open question whether our theorem can be used to prove Legendre’s Conjecture or not.

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