EXPLORATION IN THE QUASI-GAUSSIAN HJM MODEL

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Abstract. We study the explosion of the solutions of the SDE in the quasi-Gaussian HJM model with a CEV-type volatility. The quasi-Gaussian HJM models are a popular approach for modeling the dynamics of the yield curve. This is due to their low dimensional Markovian representation which simplifies their numerical implementation and simulation. We show rigorously that the short rate in these models explodes in finite time with positive probability, under certain assumptions for the model parameters, and that the explosion occurs in finite time with probability one under some stronger assumptions. We discuss the implications of these results for the pricing of the zero coupon bonds and Eurodollar futures under this model.

1. Introduction

The quasi-Gaussian Heath-Jarrow-Morton (HJM) models \[3, 4, 5, 8, 31\] are frequently used in financial practice for modeling the dynamics of the yield curve \[3\]. They were introduced as a simpler alternative to the HJM model \[16\], which describe the dynamics of the yield curve \( f(t,T) \) as the stochastic differential equation

\[
    df(t,T) = \sigma_f(t,T)dW(t) + \sigma_f(t,T)\left(\int_t^T \sigma_f(t,s)ds\right)dt,
\]

where \( W(t) \) is a vector Brownian motion under the risk-neutral measure \( Q \), and \( (\sigma_f(t,T))_{t \leq T} \) is a family of vector processes. The dynamical variable in the HJM models is the forward rate \( f(t,T) \) for maturity \( T \).

The quasi-Gaussian HJM models assume a separable form for the volatility function \( \sigma_f(t,T) = g(T)h_t \) where \( g \) is a deterministic vector function and \( (h_t) \) is a \( k \times k \) matrix stochastic process. Such models admit a Markov representation of the dynamics of the yield curve in terms of \( k + \frac{1}{2}k(k+1) \) state variables. This simplification aids considerably with the simulation of these models, which can be performed using Monte Carlo or finite difference methods \[6, 11\].

We consider in this paper the one-factor quasi-Gaussian HJM model with volatility specification \( \sigma_f(t,T) = k(t,T)\sigma_r(r_t) \) where \( k(t,T) = e^{-\beta(T-t)} \), and \( \sigma(r_t) \) is the volatility of the short rate \( r_t = f(t,t) \). This model admits a two-state Markov representation.

It has been noted in \[26, 16\] that in HJM models with log-normal volatility specification, that is for which \( \sigma_f(t,T) = \sigma_f(t,T) \), the rates explode to infinity with probability one, and zero coupon bond prices approach zero. See also \[34\] for a general study of the conditions for the existence of strong solutions to stochastic differential equations (SDEs) of HJM type. A similar explosion appears in a two-dimensional model studied in \[17\]. It is natural to ask if such explosions are present also in the quasi-Gaussian HJM models. Models of this type with parametric volatility are used in financial

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practice for modeling the swaption volatility skew [9, 10, 6]. Non-parametric forms have been also considered recently in the literature [15, 7].

We recall that singular behavior is also observed for certain derivatives prices in short rate log-normal interest rates models [2, 3]. It was observed by Hogan and Weintraub [18] that Eurodollar futures prices are infinite in the Dothan and Black-Karasinski models. A milder singularity is also present in finite tenor log-normal models, such as the Black-Derman-Toy model, manifested as a rapid increase of the Eurodollar futures convexity adjustment as the volatility increases above a threshold value [29]. This singularity can be avoided by formulating the models by specifying the distributional properties of rates with finite tenor [32]. This line of argument led to the formulation of the LIBOR Market Models which are free of singularities [3].

In a recent work [30], we studied the small-noise limit of the log-normal quasi-Gaussian model, using a deterministic approximation, and showed rigorously that the short rate may explode to infinity in a finite time. More precisely, it was shown in [30] that for sufficiently small mean-reversion parameter $\beta$, the small-noise approximation for the short rate $r_t$ has an explosion in finite time, and an upper bound is given on the explosion time, which is saturated in the flat forward rate limit.

In this paper, we extend these results in two directions:

(i) We consider a wider class of quasi-Gaussian HJM models with a constant elasticity of variance (CEV)-type volatility specification. This includes the log-normal model as a special case. We also consider the case of the displaced log-normal model. These volatility specifications are widely used by practitioners [9, 3].

(ii) The Brownian noise is taken into account. This requires the study of the explosion of the solutions of a two-dimensional stochastic differential equation. Mathematically, it is well known that for one-dimensional diffusion processes there is the celebrated Feller criterion [14, 25] for explosion/non-explosion, see e.g. [22, 28] for overviews. This is a sufficient and necessary condition under which there is an explosion in finite time. We note that the distribution of the explosion time has been also studied recently [23].

For $d$-dimensional stochastic differential equations with $d > 1$, to the best of our knowledge, there is no sufficient and necessary condition for explosion. Several sufficient conditions for explosions have been presented in the literature for multi-dimensional diffusions [33]. The Khasminskii criterion for explosion is presented in [25, 28]. The method of the Lyapunov function was presented in [12, 24]. This was extended to a non-linear Lyapunov method by [35]. The application of these conditions is non-trivial, and checking that the conditions required hold is sometimes very challenging.

We rely on the sufficient conditions for explosion with positive probability and explosion with probability one given in [12, 24]. The main tool is the construction of some delicate Lyapunov functions that satisfy certain non-trivial conditions [12, 24].

We show rigorously that under certain conditions, in the CEV-type model with exponent in a certain range $\left(\frac{1}{2}, 1\right]$, including the log-normal case, the short rate explodes in finite time with positive probability. We also show rigorously that under additional assumptions, the explosion occurs with probability one.

The explosion phenomenon that we prove rigorously has implications for the practical use of the model for pricing and simulation. Such explosions are observed in practical applications of the model, and we illustrate them on a numerical example in the log-normal quasi-Gaussian HJM model in Section 2.1. This phenomenon implies the collapse of the zero coupon bond prices, similar
to that occurring in the log-normal HJM model \[10\], and an explosion of interest rate derivatives linked to the LIBOR rate, in particular the Eurodollar futures prices. This introduces a limitation in the applicability of the model for pricing these products to maturities smaller than the explosion time.

The paper is organized as follows. In Section 2, we introduce the model, and discuss its use in the literature. In Section 3, we present rigorous results giving sufficient conditions for explosion in finite time with positive probability in the quasi-Gaussian HJM model with CEV-like volatility specification. Furthermore, under stronger assumptions, we can show that the explosion occurs in some finite time with probability one. In Section 4, we discuss the implications of our results to the pricing of the zero coupon bond and the Eurodollar futures. Finally, the proofs are collected in the Appendix.

2. ONE FACTOR QUASI-GAUSSIAN HJM MODEL

We will consider in this paper a class of one-factor quasi-Gaussian HJM models, defined by the volatility specification

\[
\sigma_f(t, T) = \sigma_r(r_t) e^{-\beta(T-t)}.
\]

Several parametric choices for the short rate volatility function \(\sigma_r(x)\) have been considered in the literature, including:

(i) Log-normal model \[6\]: \(\sigma_r(x) = \sigma x\);

(ii) Displaced log-normal model, also known in the literature as the linear Cheyette model \[9, 3\]: \(\sigma_r(x) = \sigma(x + a)\);

(iii) CEV-type model \[6\]: \(\sigma_r(x) = \sigma x^\gamma\), where \(\gamma \in (0, 1]\).

The simulation of the model with the volatility specification \(2\) can be reduced to simulating the stochastic differential equation for the two variables \(\{x_t, y_t\}_{t \geq 0}\)

\[
\begin{align*}
\frac{dx_t}{dt} &= (y_t - \beta x_t)dt + \sigma_r(\lambda(t) + x_t) dW_t, \\
\frac{dy_t}{dt} &= (\sigma^2_r(\lambda(t) + x_t) - 2\beta y_t) dt,
\end{align*}
\]

with initial condition \(x_0 = y_0 = 0\). Here \(\lambda(t) = f(0, t)\) is the forward short rate, giving the initial yield curve. The price of the zero coupon bond with maturity \(T\) is

\[
P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( -G(t, T)x_t - \frac{1}{2}G^2(t, T)y_t \right),
\]

with \(G(t, T) = \frac{1}{\beta}(1 - e^{-\beta(T-t)})\) a non-negative deterministic function \[3\]. The short rate is \(r_t := f(t, t) = \lambda(t) + x_t\).

Under the CEV-type volatility \(\sigma_r(x) = \sigma x^\gamma\), the equations \[3\] can be expressed in terms of the short rate as

\[
\begin{align*}
\frac{dr_t}{dt} &= \sigma r_t^\gamma dW_t + (y_t - \beta r_t + \beta \lambda(t) + \lambda'(t)) dt, \\
\frac{dy_t}{dt} &= (\sigma^2 r_t^{2\gamma} - 2\beta y_t) dt,
\end{align*}
\]

with the initial condition \(r_0 = \lambda_0 := \lambda(0) > 0\) and \(y_0 = 0\).

One potential complication with the usual CEV volatility specification \(\sigma_r(x) = \sigma x^\gamma\) is related to the non-uniqueness of the solution of the SDE \[5\] for \(0 < \gamma < 1\) \[14\]. Recall that for the usual CEV
model \[13\], given by the SDE with \(0 < \gamma \leq 1\),
\[
dx_t = \sigma x_t^\gamma dW_t + \mu x_t dt,
\]
the origin \(x = 0\) is a regular boundary for \(0 < \gamma < \frac{1}{2}\), and an exit boundary for \(\frac{1}{2} \leq \gamma < 1\). For the
geometric Brownian motion case \(\gamma = 1\), the point zero is a natural boundary. For \(0 < \gamma < \frac{1}{2}\), the
solution of the SDE is not unique, and an additional boundary condition must be imposed at \(x = 0\) in order to ensure uniqueness.

In order to avoid singular behavior near the origin \(r = 0\), practitioners use various modifications of
the quasi-Gaussian model with CEV volatility specification near the \(r = 0\) point, see e.g. Section
4.3 of \[19\]. This work describes three possible modifications: (a) \(\sigma_r(x) \to \sigma |x|^{\gamma}\); (b) \(\sigma_r(x) \to 0\) for
\(|x| \leq \varepsilon\); (c) \(\sigma_r(x) \to \sigma \varepsilon^{\gamma}\) for \(|x| \leq \varepsilon\) with \(\varepsilon > 0\) a small cutoff.

In this paper, following \[1\], we consider the modified quasi-Gaussian HJM model with a CEV-type
volatility specification
\[
\sigma_r(x) = \sigma x \min(x^{\gamma - 1}, \varepsilon^{\gamma - 1}),
\]
with \(\varepsilon > 0\) small and \(0 < \gamma \leq 1\). We call this the \(\varepsilon\)-CEV quasi-Gaussian HJM model.

Note that as \(\varepsilon = 0\), this reduces to the usual CEV volatility specification \(\sigma_r(x) = \sigma x^{\gamma}\), see e.g. \[6\].
The modification \(\varepsilon > 0\) impacts only the region of small \(0 < r_t < \varepsilon\), where the process is identical
with the log-normal model with the volatility \(\sigma \varepsilon^{\gamma - 1}\), and leaves unchanged the behavior of the
process for large \(r_t\), which is relevant for the study of the explosions of \(r_t\). The modification is only
required for \(0 < \gamma < 1\). When \(\gamma = 1\), the equation (7) reduces to \(\sigma_r(x) = \sigma x\), which coincides with
the log-normal model.

With the volatility specification (7), we will study the 2-dimensional SDE with \(\varepsilon > 0\)
\[
dr_t = (y_t - \beta r_t + \beta \lambda(t) + \lambda'(t)) dt + \sigma r_t \min(r_t^{\gamma - 1}, \varepsilon^{\gamma - 1}) dW_t,
\]
\[
dy_t = (\sigma^2 r_t^2 \min(r_t^{2\gamma - 2}, \varepsilon^{2\gamma - 2}) - 2\beta y_t) dt,
\]
with the initial condition \(r_0 = \lambda(0) > \varepsilon\) and \(y_0 = 0\).

In the special case \(\gamma = 1\), (8), (9) reduces to the log-normal model
\[
dr_t = \sigma r_t dW_t + (y_t - \beta r_t + \beta \lambda(t) + \lambda'(t)) dt,
\]
\[
dy_t = (\sigma^2 r_t^2 - 2\beta y_t) dt,
\]
with the initial condition \(r_0 = \lambda_0 := \lambda(0) > 0\) and \(y_0 = 0\).

Assume that \(\lambda'(t) + \beta \lambda(t) \geq 0\) and \(r_0 > 0\). Then the solutions of (10), (11) are positive with probability one
\[
\mathbb{P}(r_t > 0) = 1, \quad \text{for all } t \geq 0.
\]

The result follows by noting that
\[
y_t = \sigma^2 \int_0^t r_s^2 e^{-2\beta(t-s)} ds > 0,
\]
almost surely for every \(t > 0\), and then follows by an application of the comparison theorem
(Theorem 1.1 in \[36\] and Theorem 5.2.18 in \[22\]). See also the Appendix D in \[27\] for a proof of
this result.
This implies that the origin \( r = 0 \) is a natural boundary for this diffusion. For the time-homogeneous case \( \lambda(t) = \lambda_0 \) we use a similar argument to prove the same result for the \( \varepsilon \)-CEV model with general \( \gamma \in (\frac{1}{2}, 1] \), see the argument around Eq. (46).

The SDE for the displaced log-normal model \( \sigma_r(x) = \sigma(x + a) \) reduces to that for the log-normal case by the substitutions \( r_t + a \rightarrow r_t, \lambda(t) + a \rightarrow \lambda(t) \). Expressed in terms of \( r_t, y_t \), this is

\[
\begin{align*}
    dr_t &= \sigma(r_t + a)dW_t + (y_t - \beta r_t + \beta \lambda(t) + \lambda'(t))dt, \\
    dy_t &= (\sigma^2(r_t + a)^2 - 2\beta y_t)dt, \\
\end{align*}
\]

with initial conditions \( r_0 = \lambda(0), y_0 = 0 \). Defining \( \tilde{r}_t = r_t + a \) the shifted short rate, we have

\[
\begin{align*}
    d\tilde{r}_t &= \sigma\tilde{r}_tdW_t + (y_t - \beta \tilde{r}_t + \beta(\lambda(t) + a) + \lambda'(t))dt, \\
    dy_t &= (\sigma^2\tilde{r}_t^2 - 2\beta y_t)dt, \\
\end{align*}
\]

started at \( \tilde{r}_t = \lambda(0) + a, y_0 = 0 \). Redefining \( \lambda(t) + a \rightarrow \lambda(t) \), the shift parameter \( a \) disappears, and the resulting SDE is identical to that for the log-normal \( \gamma = 1 \) model.

Under this model negative values for \( r_t \) can also be accommodated, with a floor on the short rate \( r_t > -a \). We will assume that \( r_0 + a > 0 \), and then \( r_t > -a \) for any \( t > 0 \). All the results for \( \gamma = 1 \) apply also to the displaced log-normal model with minimal substitutions.

In [30], we studied the small-noise deterministic limit of the SDE (8), (9) in the log-normal case \( \gamma = 1 \)

\[
\begin{align*}
    r'(t) &= y(t) - \beta r(t) + \beta \lambda(t) + \lambda'(t), \\
    y'(t) &= \sigma^2(r(t))^2 - 2\beta y(t), \\
\end{align*}
\]

with \( r(0) = \lambda(0) = \lambda_0 > 0 \) and \( y(0) = 0 \). In the small-noise limit, it is proved rigorously in [30] that for sufficiently large \( \beta \) or sufficiently small \( \sigma \), the short rate \( r(t) \) is uniformly bounded, and hence there is no explosion. When \( \beta = 0 \), the short rate explodes in a finite time, and an upper bound is given for the explosion time (Proposition 4 in [30]). Under the further assumption that \( \lambda(t) \equiv \lambda_0 \), the upper bound for the explosion time given in Proposition 4 of [30] is sharp. The \( \beta > 0 \) case is also considered, under the simpler setting of a time homogeneous model \( \lambda(t) \equiv \lambda_0 \). For this case it is shown in [30] that when \( \beta < \beta_C := \sigma \sqrt{2\lambda_0} \), the explosion occurs at a finite time and when \( \beta \geq \beta_C \), we have \( \lim_{t \rightarrow \infty} r(t) = \frac{\beta_0^2}{\sigma^2}(1 - \sqrt{1 - \frac{2\sigma^2\lambda_0}{\beta^2}}) \), and there is no explosion.

In this paper we would like to study directly the original stochastic system (8), (9) in the presence of random noise. We will show rigorously that the solutions of the stochastic system (8), (9) may explode with non-zero probability for \( \gamma \in (\frac{1}{2}, 1] \), and under some additional assumptions with probability one.

2.1. Numerical example for \( \gamma = 1 \). Such explosions are indeed observed in numerical simulations of the stochastic system (8), (9). We illustrate this phenomenon for the log-normal model \( \gamma = 1 \) in Figure 1, which shows sample paths for \( \{r_t\}_{t \geq 0} \) for several choices of the model parameters \( \sigma, \lambda(= \lambda_0), \beta \). These results were obtained by numerical simulation of the stochastic differential equations (8), (9) by Euler discretization with time step \( \tau = 0.01 \).

In Figure 1 we fix \( \sigma = 0.2, \lambda = 0.1 \) and consider two values of \( \beta \). The left plot shows sample paths for \( r_t \) with \( \beta = 0 \). The paths explode at various times, which is expected in the presence of the Brownian noise. In the small-noise limit studied in [30], the explosion time is deterministic. The
corresponding explosion time can be found in closed form for \( \lambda(t) = \lambda_0 \), and is given in Proposition 4 of [30]. The prediction is shown in Figure 1 (left) as the red vertical line.

The right plot in Figure 1 shows sample paths for \( \beta = 0.05 \). There is still explosion, but the explosion tends to occur at longer maturities. This is in qualitative agreement with the behavior expected in the small-noise limit [30] where it was shown that increasing \( \beta \) delays the explosion time, and suppresses it completely for \( \beta \geq \beta_C = \sigma \sqrt{2\lambda_0} \). For the parameters considered in Figure 1 the small-noise critical value is \( \beta_C = 0.089 \). In the stochastic case, taking \( \beta = 0.1 \) (not shown) the explosion is further delayed to longer maturities, or completely suppressed.

3. EXPLOSION OF THE CEV-TYPE QUASI-GAUSSIAN HJM MODEL

Assume that the forward rate \( \lambda(t) \) satisfies the inequality,

\[
\lambda'(t) + \beta \lambda(t) \geq \beta \lambda(0).
\]

By a comparison argument, the solutions of (8),(9) are bounded from below by the solutions of the time-homogeneous SDE obtained by replacing \( \lambda(t) \rightarrow \lambda_0 = \lambda(0) \). Thus, for the purpose of studying the explosions of the solutions of the SDE (8),(9) it is sufficient to study the corresponding time-homogeneous SDE with constant \( \lambda(t) = \lambda_0 \)

\[
\begin{align*}
&dr_t = (y_t - \beta r_t + \beta \lambda_0)dt + \sigma r_t \min(r_t^{\gamma-1}, \varepsilon^{\gamma-1})dW_t, \\
&dy_t = (\sigma^2 r_t^2 \min(r_t^{2\gamma-2}, \varepsilon^{2\gamma-2}) - 2\beta y_t)dt,
\end{align*}
\]

with the initial condition \( r_0 = \lambda_0 > \varepsilon \) and \( y_0 = 0 \).

The coefficients of this SDE satisfy a local Lipschitz condition. For \( 0 < \gamma \leq \frac{1}{2} \) they also satisfy a sublinear growth condition and global Lipschitz condition. Thus we can apply the standard result, see for example Theorem 5.2.9 in [22], to conclude that the SDE has a unique strong solution, which is furthermore square integrable and thus non-explosive. On the other hand we show that for \( \frac{1}{2} < \gamma \leq 1 \) the solution can explode to infinity in finite time with non-zero probability.
The infinitesimal generator of this diffusion is

\[
\mathcal{L}_V(r,y) = (\sigma^2 r^2 \min(r^{2\gamma-2},\varepsilon^{2\gamma-2}) - 2\beta y) \partial_y V \\
+ (y - \beta r + \beta r_0) \partial_r V + \frac{1}{2} \sigma^2 r^2 \min(r^{2\gamma-2},\varepsilon^{2\gamma-2}) \partial^2_r V.
\]

We would like to study the explosion time of this diffusion, defined as

\[
\tau := \sup\{t > 0 : y_t < \infty, r_t < \infty\}.
\]

We present a few preliminary results which will be used in our proof. The following theorem was proved in [12], see Theorem 1.

**Proposition 1 (Theorem 1, [12]).** Let \( D \subset \mathbb{R}^d \) be a bounded open set with regular boundary \( \partial D \) and let \( D^c \) be the complement of \( D \). Consider the \( d \)-dimensional diffusion \( dX_t = \sigma(X_t)dW_t + b(X_t)dt \) where the coefficients \( \sigma(\cdot), b(\cdot) \) are Lipschitz continuous on any compact subset of \( \mathbb{R}^d \) for any \( t \geq t_0 \). Moreover, there exists a positive function \( V(t,x) \in C^{1,2}([t_0, \infty) \times D^c) \) and positive constants \( K_1, K_2 < K_3 \) and \( C \) such that

(A.1) \( \sup_{t \geq t_0, x \in D^c} V(t,x) = K_1 < \infty \).

(A.2) \( \sup_{t \geq t_0, x \in \partial D} V(t,x) = K_2 < \inf_{t \geq t_0, x \in \Gamma} V(t,x) = K_3 \), for some set \( \Gamma \subset D^c \).

(A.3) \( \mathcal{L}V(t,x) \geq CV(t,x) \) for every \( t \geq t_0, x \in D^c \), where \( \mathcal{L} \) is the infinitesimal generator of \( X_t \).

Then, the explosion eventually occurs with positive probability if the process starts at a future time \( t_1 \geq t_0 \) at a point \( x \in \Gamma \).

**Proposition 2 (Theorem 2 in [12]).** Assume the conditions in Proposition 1 are satisfied. Then, we have the almost sure explosion provided the additional assumptions hold:

(A.4) \( \inf_{t \geq t_0, x \in D^c} V(t,x) = K_0 > 0 \);

(A.5) For any \( t \geq t_0, x \in \partial D \), \( \mathbb{P}^{D,x}(\tau_T < \infty) = 1 \), where \( \tau_T \) is the first hitting time of the set \( \Gamma \).
Theorem 1 in [12] is a generalization of Theorem 3.6 in [24]. Intuitively, it relates the explosion of the solution of a stochastic differential equation to the behavior of an appropriately defined Lyapunov function for large values in the space domain.

Unlike the case of one-dimensional diffusion processes, where a sufficient and necessary condition for explosion is given by Feller’s criterion [25], for multidimensional diffusions, there are many different theoretical results giving sufficient conditions for explosions [25, 12, 24]. For our purpose, Theorem 1 in [12] suffices. The strategy of the proof will be to construct an appropriate Lyapunov function, and show that for the two-dimensional SDE model (20), (21), explosion occurs with positive probability. The main result of this paper follows.

**Theorem 1.** Assume \( \lambda(t) \equiv r_0 > \varepsilon > 0 \) and \( \beta \geq 0 \).

(a) For \( \gamma \in (0, \frac{1}{2}] \) the solution of the SDE (20), (21) \( \{ r_t, y_t \}_{t \geq 0} \) is non-explosive.

(b) For \( \gamma \in (\frac{1}{2}, 1] \) the solution of the SDE (20), (21) \( \{ r_t, y_t \}_{t \geq 0} \) explodes with non-zero probability \( P(\tau < \infty) > 0 \) provided that any one of the following conditions is satisfied for at least one set of \( (\delta_1, \delta_2) \), where \( \delta_1, \delta_2 > 0 \) are positive constants satisfying \( (1 + \delta_1)(1 + \delta_2) = 2\gamma \).

(i) \( \sup_{R \geq \varepsilon} F(R; \beta, \sigma) > 0 \) where the function \( F(R; \beta, \sigma) \) is defined by

\[
F(R; \beta, \sigma) := R^{\gamma_2} - \left( 2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right) \left( \frac{1}{\delta_1 \sigma^2} (1 + R)^{\delta_1 + 1} + \frac{1}{\delta_2} R^{2\gamma - 1 - (1 + R)^{\delta_2 + 1}} \right).
\]

(ii) \( \sup_{R \geq \varepsilon} (G(R) - (2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1))) \geq 0 \) where the function \( G(R) \) is defined by

\[
G(R) := \frac{\delta_2 R}{(1 + R)^{\delta_{2+1}}},
\]

**Remark 1.** Under the assumptions in Theorem 1, \( V, K_1, K_2, K_3 \) and \( C \) from the conditions in Proposition 4 are given by

\[
V(r, y) = C_1 - \frac{C_2}{(1 + y)^{\delta_1}} - \frac{C_3}{(1 + r)^{\delta_2}},
\]

\[
K_1 = C_1,
\]

\[
K_2 = \sup_{(r, y) \in \partial D} V(r, y) = C_1 - \frac{C_2}{(1 + R)^{\delta_1}} - \frac{C_3}{(1 + R)^{\delta_2}},
\]

\[
K_3 = \inf_{(r, y) \in \Gamma} V(r, y) = C_1 - \frac{C_2}{(1 + 2R)^{\delta_1}} - \frac{C_3}{(1 + 2R)^{\delta_2}},
\]

\[
C = 2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1),
\]

\[
C_1 = C_2 + C_3,
\]

where \( R \geq \varepsilon, \varepsilon > 0 \) is sufficiently small, \( \delta_1, \delta_2 > 0 \) are positive constants satisfying \( (1 + \delta_1)(1 + \delta_2) = 2\gamma \), and \( C_2, C_3 > 0 \) are determined separately for each case as follows (this is a restatement of the inequalities for (a,b) following from Lemma 3).

(i) For case (i) of Theorem 1, \( C_2, C_3 > 0 \) satisfy the inequalities

\[
R^{\delta_2(\delta_1+2)} \leq \frac{(\delta_2 C_3)}{\delta_1 C_2 \sigma^2} \left( \frac{R}{1 + R} \right)^{\delta_2 - \delta_1} \leq \frac{R^{2\gamma - \delta_1 - 1} - \kappa_1}{\kappa_2},
\]

where \( \kappa_1, \kappa_2 \) are defined in [73], [74], and \( R \) is in the range allowed by condition (i) of Theorem 1.
Figure 3. Region in the $(\sigma, \beta)$ plane allowed by the condition (ii) in Theorem 1. For given $\gamma \in (\frac{1}{2}, 1]$, this condition is satisfied for $\beta$ below the curves shown.

(ii) For case (ii) of Theorem 1, $C_2, C_3 > 0$ satisfy the inequality

$$
\frac{\delta_2 C_3}{\delta_1 C_2 \sigma^2} \left( \frac{R}{1 + R} \right)^{\delta_2 - \delta_1} \leq \min \left\{ R^{\delta_2 (\delta_1 + 2)}, \frac{\kappa_1}{R^{\delta_2 - \kappa_2}} \right\},
$$

where $\kappa_1, \kappa_2$ are defined in (73), (74), and $R = \frac{1}{\delta_2}$.

Remark 2. The constraints of Theorem 1 can be made stronger by replacing

$$
2 \beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \rightarrow \max \{2\delta_1, \delta_2\} \beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1).
$$

See the discussion around Eq. (67) about the choice of the constant $C$. This gives a wider region for $\beta$.

3.1. Numerical study. We study here the regions for $(\sigma, \beta)$ allowed by Theorem 1. We discuss only the condition (ii) which is more amenable to an analytical treatment. The resulting region for $(\beta, \sigma)$ includes all the typical values of these parameters which are relevant for applications $0 < \sigma < 1.0$ and $0 < \beta < 0.1$, see for example [6]. The constraint on $\beta$ can be weakened further, see Remark 2.

The condition (ii) of Theorem 1 is satisfied in the region below the curves shown in Figure 3. For each $\frac{1}{2} < \gamma \leq 1$ there is one curve, corresponding to $\delta_2$ taking values in $0 < \delta_2 < 2\gamma - 1$.

We outline the main steps in the derivation of these regions. The function $G(R) = \frac{\delta_2 R}{(1 + R)^{\delta_2 + 1}}$ with $\delta_2 > 0$ has the following properties.

(i) $G(R)$ vanishes for $R \rightarrow 0$ and $R \rightarrow \infty$. The function $G(R)$ increases for $R < R_0(\delta_2)$ and decreases for $R > R_0(\delta_2)$, with $R_0(\delta_2) = \frac{1}{\delta_2}$.

(ii) $G(R)$ has a maximum at $R_0(\delta_2)$. At this point the value of the function is

$$
G(R_0) = \left( \frac{\delta_2}{1 + \delta_2} \right)^{\delta_2 + 1}.
$$
Fix the values of $\gamma$ and $\sigma$. By scanning over $\delta_2 \in [0, 2\gamma - 1]$, find the maximum of the expression

$$
\delta_{2*} := \arg \max_{\delta_2 \in [0, 2\gamma - 1]} \left\{ G(R_0(\delta_2)) - \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right\}.
$$

Then the values of $\beta$ allowed by the condition (ii) of Theorem 1 (for given $\sigma, \gamma$) are

$$
0 \leq \beta \leq \frac{1}{2} G(R_0(\delta_{2*})) - \frac{1}{4} \sigma^2 \delta_{2*} (\delta_{2*} + 1),
$$

where $\delta_{2*}$ is given by (36). This region for $\beta$ is shown in Fig. 3 for several values of $\gamma$. The region becomes smaller as $\gamma$ approaches $\frac{1}{2}$ and disappears at this point.

The value $\delta_{2*}$ decreases with $\sigma$, at fixed $\gamma$. (Recall that this determines also the range of allowed values for $R$, which includes the point $R_0 = 1/\delta_{2*}$.) In a range of sufficiently small $\sigma$, the maximum in (36) is realized at the maximally allowed value $\delta_{2*} = 2\gamma - 1$. In this region the curves for maximally allowed $\beta$ with different values of $\gamma$ are distinct, as seen in Figure 3. For $\sigma$ above a certain value, which depends on $\gamma$, the value of $\delta_{2*}$ decreases from $2\gamma - 1$ to zero. In this region the maximal $\beta$ curves are overlapping, since $\delta_{2*}$ is independent of $\gamma$.

There is a maximum value of $\sigma$ for which positive values of $\beta$ are allowed. At this maximum value, which depends on $\beta$, $\delta_{2*}$ reaches zero. For $\beta = 0$, this maximum value is $\sigma_{\text{max}} = \sqrt{2}$. This follows from the small-$\delta_2$ expansion

$$
G(R_0(\delta_2)) = \delta_2 + \delta_2^2 (\log \delta_2 + 1) + O(\delta_2^3).
$$

Substituting into (37) gives

$$
\beta \leq \frac{1}{2} \delta_{2*} \left( 1 - \frac{1}{2} \sigma^2 \right) + O(\delta_{2*}^2).
$$

Requiring the cancellation of the $O(\delta_{2*})$ term gives the maximal value $\sigma_{\text{max}} = \sqrt{2}$.

### 3.2. Almost sure explosion

In Theorem 1 we showed that under certain conditions, the explosion occurs with positive probability. Under some additional assumptions, one can further prove the almost sure explosion, that is, that explosion occurs with probability one.

**Theorem 2.** Suppose the assumptions of Theorem 1 are satisfied. Assume $\beta > 0$. For sufficiently large $r_0$ so that

$$
r_0 > \max \left\{ \frac{e}{\beta} (4\beta R + \beta + \sigma^2), \frac{\sigma^2}{\beta} e^{\frac{2R}{\beta} (4\beta R + \beta + \sigma^2) - 2R - 1} \right\},
$$

where $R$ is determined such that either of the conditions (i) or (ii) of Theorem 1 holds, we have the almost sure explosion, that is, $\mathbb{P}(\tau < \infty) = 1$.

**Remark 3.** Under the assumptions in Theorem 2, $K_0$ from the conditions in Proposition 2 is given by

$$
K_0 = \min \left\{ C_1 - \frac{C_2}{(1 + R)^{\delta_1}} - C_3, C_1 - C_2 - \frac{C_3}{(1 + R)^{\delta_2}} \right\},
$$

where $C_1, C_2, C_3, R, \delta_1, \delta_2$ are defined in Remark 1.
4. IMPLICATIONS FOR ZERO COUPON BOND PRICES AND EURODOLLAR FUTURES

The explosion of \((r_t, y_t)\) is equivalent to the explosion of \(r_t\) due to the explicit form of \(y_t\) in terms of \((r_s)_{0 \leq s \leq t}\). The explosion of \(r_t\) thus implies that the prices of zero coupon bonds \(P(t, T)\) become zero almost surely for all \(t > \tau\), with \(\tau\) the explosion time of \(r_t\).

This follows from Eq. (41) for the zero coupon bond price, which gives

\[
P(T, T + \delta) = \frac{P(0, T + \delta)}{P(0, T)} \exp\left(-G(T, T + \delta)x_T - \frac{1}{2}G^2(T, T + \delta)y_T\right),
\]

where we recall that \(x_T = r_T - \lambda(T)\). Suppose the assumptions in Theorem 1 are satisfied, then \(P(\tau < \infty) > 0\), which implies that for sufficiently large \(T\), \(P(\tau < T) > 0\). As a result, with positive probability, and sufficiently large \(T\), the zero coupon bond price \(P(T, T + \delta)\) collapses to zero.

This implies that interest rates \(L(T_1, T_2)\) explode for all \(T_1 > \tau\). Recall that the rate \(L(T_1, T_2)\) is related to \(P(T_1, T_2)\) as \(L(T_1, T_2) = \frac{1}{T_2 - T_1}P^{-1}(T_1, T_2) - 1\), see e.g. [3].

The prices of any derivatives depending on \(L(T_1, T_2)\) such as interest rate caps, swaptions, CMS swaps, and Eurodollar futures also become infinite. We will show this explicitly for the prices of Eurodollar futures contracts. Using Eq. (42) for the zero coupon bond price \(P(T, T + \delta)\) we get

\[
\mathbb{E}^Q[P^{-1}(T, T + \delta)] = \frac{P(0, T)}{P(0, T + \delta)}\mathbb{E}^Q\left[\exp\left(G(T, T + \delta)x_T + \frac{1}{2}G^2(T, T + \delta)y_T\right)\right].
\]

Suppose the assumptions in Theorem 1 are satisfied, then \(x_T = y_T = \infty\) with positive probability for sufficiently large \(T\). It follows that the Eurodollar futures price explodes to infinity, that is, \(\mathbb{E}^Q[P^{-1}(T, T + \delta)] = \infty\), for sufficiently large \(T\).

In practical applications the explosions of the short rate \(r_t\) could be avoided by capping the short rate volatility to a finite value \(c\), possibly using a prescription of the same type as that proposed in [16], \(\sigma_r(r_t) \rightarrow \min\{0, \sigma_r(r_t)\}; c\}. With this change, the diffusion coefficients satisfy the sub-linear growth condition of Theorem 5.2.9 in [22], which ensures that the solution \(r_t\) exists and is non-explosive.

5. APPENDIX: PROOFS

Proof of Theorem 1 (a) For \(0 < \gamma \leq \frac{1}{2}\) the coefficients of the 2-d diffusion (20), (21) satisfy the conditions of Theorem 5.2.9 in [22], which we recall here briefly for convenience.

Consider the SDE for the \(d\)-dimensional vector \(X_t \in \mathbb{R}^d\)

\[
X_t = \sigma(t, x) dW_t + b(t, x) dt
\]

where \(x \in \mathbb{R}^d\) and \(W_t\) is a \(d\)-dimensional Brownian motion. Assume that the coefficients satisfy the global Lipschitz and linear growth conditions

\[
\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|, \\
\|b(t, x)\|^2 + \|\sigma(t, x)\|^2 \leq K^2(1 + \|x\|^2),
\]

for every \(0 \leq t < \infty, x, y \in \mathbb{R}^d\) and \(K\) is a positive constant. Under these conditions, there exists a continuous, adapted process \(X = \{X_t; 0 \leq t < \infty\}\) which is a strong solution of the SDE (44) with initial condition \(X_0\), and is furthermore square-integrable.

\footnote{Note that in Eq. (43) in [20], there is a typo: the factor on the right-hand side of this equation should be \(\frac{P(0, T)}{P(0, T + \delta)}\).}
The SDE (20), (21) with $0 < \gamma \leq \frac{1}{2}$ satisfies the conditions (45), and thus $\{r_t, y_t\}_{t \geq 0}$ does not explode. We study next the case $\frac{1}{2} < \gamma \leq 1$, where the linear growth condition does not hold.

(b) The boundary $r_t = 0, y_t = 0$ is unattainable. Indeed, for $r_t < \varepsilon$, we have $dr_t = \sigma r_t dW_t + (y_t - \beta r_t + \beta r_0) dt$ with

$$
y_t = \sigma^2 \int_0^t r_s^2 \min(r_s^2(\gamma - 1), \varepsilon^2(\gamma - 1)) e^{2\beta(s-t)} ds > 0,
$$

and since the term $\beta r_0 > 0$ and $y_t > 0$ in the drift term of $r_t$, by comparing $r_t$ with a geometric Brownian motion, we have $r_t > 0$. Therefore, $(y_t, r_t) \in \mathbb{R}^+ \times \mathbb{R}^+$. This generalizes to $\gamma \in (\frac{1}{2}, 1]$ the result of (12) and (13). Although this is proved here for the time-homogeneous case $\lambda(t) = \lambda_0$, the result is easily seen to hold also under the weaker assumption $\lambda'(t) + \beta \lambda(t) \geq 0$ by a comparison argument.

Let us take $D := (0, R) \times (0, R)$, with $R \geq \varepsilon$, where we define $D^c = \mathbb{R}_+^+ \times \mathbb{R}_+ \setminus D$. It is clear that $D$ is a bounded open set. The boundary $\partial D$ is regular since $\beta r_0 > 0$ in the drift term of $r_t$. It is also easy to see that, for $\frac{1}{2} < \gamma \leq 1$, on any compact subset of $\mathbb{R}_+^+ \times \mathbb{R}_+$, the coefficients of the SDE (20), (21), are continuous and Lipschitz.

Assume the following form for the Lyapunov function

$$
V(r, y) = C_1 - \frac{C_2}{(1 + y)^{\delta_1}} - \frac{C_3}{(1 + r)^{\delta_2}},
$$

with $C_1, C_2, C_3 > 0$ and $\delta_1, \delta_2 > 0$ are positive constants satisfying the condition

$$
(1 + \delta_1)(1 + \delta_2) = 2\gamma.
$$

We would like to test the conditions (A.1), (A.2) and (A.3) of Proposition 1

1. **Condition (A.1).** For any $(r, y) \in D^c$, we have

$$
V(r, y) \geq \min \left\{ C_1 - \frac{C_2}{(1 + R)^{\delta_1}} - C_3, C_1 - C_2 - \frac{C_3}{(1 + R)^{\delta_2}} \right\} > 0,
$$

provided that we take

$$
C_1 \geq C_2 + C_3.
$$

Thus $V(r, y)$ defined on $D^c$ is a positive function. Since $r, y > 0$, it is clear that $V(r, y) \leq C_1$. Thus the condition (A.1) is satisfied.

2. **Condition (A.2).** Note that $\partial D = \{(r, y) : 0 \leq y \leq R\} \cup \{(r, R) : 0 \leq r \leq R\}$. Thus, we have

$$
K_2 = \sup_{(r, y) \in \partial D} V(r, y) = C_1 - \frac{C_2}{(1 + R)^{\delta_1}} - \frac{C_3}{(1 + R)^{\delta_2}}.
$$

On the other hand, taking $\Gamma = [2R, \infty) \times [2R, \infty) \subset D^c$, we obtain

$$
K_3 = \inf_{(r, y) \in \Gamma} V(r, y) = C_1 - \frac{C_2}{(1 + 2R)^{\delta_1}} - \frac{C_3}{(1 + 2R)^{\delta_2}} > K_2.
$$

Hence, the condition (A.2) holds.
We finally check the condition (A.3). Note that
\[ L_\varepsilon V(r, y) = \delta_1 C_2 \sigma^2 \frac{\min(r^{2\gamma}, r^2 \varepsilon^{2\gamma - 2})}{(1 + y)^{\delta_1 + 1}} - 2\delta_1 C_2 \beta \frac{y}{(1 + y)^{\delta_1 + 1}} + \delta_2 C_3 \frac{y}{(1 + r)^{\delta_2 + 1}} - \delta_2 C_3 \frac{\beta r}{(1 + r)^{\delta_2 + 1}} + \delta_2 C_3 \beta r_0 \frac{1}{(1 + r)^{\delta_2 + 1}} - \frac{1}{2} \delta_2 (\delta_2 + 1) C_3 \sigma^2 \frac{\min(r^{2\gamma}, r^2 \varepsilon^{2\gamma - 2})}{(1 + r)^{\delta_2 + 2}}. \]

Therefore,
\[ L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{\min(r^{2\gamma}, r^2 \varepsilon^{2\gamma - 2})}{(1 + y)^{\delta_1 + 1}} + \delta_2 C_3 \frac{\beta r_0}{(1 + r)^{\delta_2 + 1}} - CC_1 \]

Furthermore, for \( 1 < 2\gamma \leq 2 \), we have
\[ 0 < \frac{r^{2\gamma}}{(1 + r)^2} \leq 1, \quad r \geq 0, \]
\[ 0 < \frac{r^2 \varepsilon^{2\gamma - 2}}{(1 + r)^2} \leq 1, \quad 0 \leq r \leq \varepsilon, \]

(\( \varepsilon \) is small, say less than 1) such that we have
\[ L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{\min(r^{2\gamma}, r^2 \varepsilon^{2\gamma - 2})}{(1 + y)^{\delta_1 + 1}} + \delta_2 C_3 \frac{\beta r_0}{(1 + r)^{\delta_2 + 1}} - CC_1. \]

Let us choose \( C \) to be a fixed constant so that
\[ C \geq \max \left\{ 2\delta_1 \beta, \delta_2 \beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right\}. \]

Then we have
\[ L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{\min(r^{2\gamma}, r^2 \varepsilon^{2\gamma - 2})}{(1 + y)^{\delta_1 + 1}} + \delta_2 C_3 \frac{y}{(1 + r)^{\delta_2 + 1}} - CC_1. \]

Recall that for \((r, y) \in D^c\), we have either \( y \geq R \) or \( r \geq R \), and we chose \( \varepsilon < R \).
(I) If \( y \geq R \) and \( r < R \), then we have for both \( 0 \leq r \leq \varepsilon \) and \( \varepsilon < r < R \), by positivity of the first term,

\[
L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{\min(r^{2\gamma}, r^{2\gamma-2})}{(1+y)\delta_1+1} + \delta_2 C_3 \frac{y}{(1+r)^{\delta_2+1}} - CC_1
\]

\[
\geq \delta_2 C_3 \frac{R}{(1+R)^{\delta_2+1}} - CC_1.
\]

(II) If \( r \geq R \) and \( y < R \), then we have (since \( r \geq R > \varepsilon \))

\[
L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{r^{2\gamma}}{(1+y)\delta_1+1} + \delta_2 C_3 \frac{y}{(1+r)^{\delta_2+1}} - CC_1
\]

\[
\geq \delta_1 C_2 \sigma^2 \frac{R^{2\gamma}}{(1+R)^{\delta_1+1}} - CC_1.
\]

(III) If \( y \geq R \) and \( r \geq R \), then we have (again by \( r \geq R > \varepsilon \))

\[
L_\varepsilon V - CV \geq \delta_1 C_2 \sigma^2 \frac{r^{2\gamma}}{(1+y)\delta_1+1} + \delta_2 C_3 \frac{y}{(1+r)^{\delta_2+1}} - CC_1
\]

\[
\geq \delta_1 C_2 \sigma^2 \left( \frac{R}{1+R} \right)^{\delta_1+1} \frac{R^{2\gamma}}{y^{\delta_1+1}} + \delta_2 C_3 \left( \frac{R}{1+R} \right)^{\delta_2+1} \frac{y}{x^{\delta_2+1}} - CC_1
\]

\[
= \delta_1 C_2 \sigma^2 \left( \frac{R}{1+R} \right)^{\delta_1+1} \frac{1}{x} - CC_1,
\]

where we denoted

\[
x = \frac{r^{\delta_1+1}}{y}.
\]

The condition \( 2\gamma = (1+\delta_1)(1+\delta_2) \) was used to reduce the dependence on \( (r, y) \) to a function of \( x \) in the last step.

The sum of the first two terms is bounded from below by the following Lemma.

**Lemma 1.** The infimum of the function \( \hat{F}(x) : (0, \infty) \to (0, \infty) \) defined as

\[
\hat{F}(x) := ax^{\delta_1+1} + b \frac{1}{x}, \quad a, b > 0
\]

is given by

\[
\inf_{x>0} \hat{F}(x) = \kappa_\delta a^{\frac{1}{\delta_1+2}} b^{\frac{\delta_1+1}{\delta_1+2}},
\]

where

\[
\kappa_\delta = (\delta_1 + 2)(\delta_1 + 1)^{-\frac{\delta_1+1}{\delta_1+2}} > 1.
\]

**Proof of Lemma 1** The minimum of \( \hat{F}(x) \) is achieved at \( \hat{F}'(x) = 0 \), which gives

\[
\hat{F}'(x) = a(\delta_1 + 1)x^{\delta_1} - \frac{b}{x^2} = 0,
\]

which implies that

\[
\min_{x>0} \hat{F}(x) = (\delta_1 + 2)(\delta_1 + 1)^{-\frac{\delta_1+1}{\delta_1+2}} a^{\frac{1}{\delta_1+2}} b^{\frac{\delta_1+1}{\delta_1+2}}.
\]
To see that $\kappa_\delta$ is larger than 1 for $\delta_1 \in [0, 1]$, let us define $\hat{G}(\delta_1) := \log(\delta_1 + 2) - \frac{\delta_1 + 1}{\delta_1 + 2} \log(\delta_1 + 1)$.

We will show that $\hat{G}$ is decreasing in $\delta_1 \in [0, 1]$, that is due to

\[(64) \quad \hat{G}'(\delta_1) = -\frac{\log(\delta_1 + 1)}{(\delta_1 + 2)^2} < 0\]

Hence, $\kappa_\delta = e^{\hat{G}(\delta_1)}$ is decreasing in $\delta_1 \in [0, 1]$. We can compute that at $\delta_1 = 1$, we have $\kappa_\delta = (1 + 2)(1 + 1)^{-\frac{1}{2}} = \frac{3}{2} > 1$. Hence, $\kappa_\delta$ is larger than 1.

Therefore, following (57), we have

\[
\mathcal{L}_\varepsilon V - CV \geq \kappa_\delta \left( \delta_1 C_2 \sigma^2 \left( \frac{R}{1 + R} \right)^{\frac{1}{\delta_1 + 2}} \left( \frac{\delta_2 C_3}{\delta_2 + 1} \right) \right) - CC_1.
\]

Hence, from (I), (II) and (III), we conclude that $\mathcal{L}_\varepsilon V \geq CV$ for any $(r, y) \in D^c$ if we have

\[
(65) \quad CC_1 \leq \min \left\{ \frac{R}{(1 + R)^{\delta_2 + 1}}, \frac{R^2 \gamma}{(1 + R)^{\delta_1 + 1}}, \kappa_\delta \left( \delta_1 C_2 \sigma^2 \left( \frac{R}{1 + R} \right)^{\frac{1}{\delta_1 + 2}} \left( \frac{\delta_2 C_3}{\delta_2 + 1} \right) \right) \right\}.
\]

To summarize, in order to have the Lyapunov function $V(r, y)$ to be bounded, positive and satisfy (A.1), (A.2), (A.3), we need the conditions (50), (56) and (65) to hold simultaneously. Taking

\[
(66) \quad C_1 = C_2 + C_3, \quad C = \max \{2\delta_1, \delta_2\} \cdot \beta + \frac{1}{2} \sigma^2 (\delta_2 + 1),
\]

then (50) and (56) are satisfied. This can be simplified by replacing $\max\{2\delta_1, \delta_2\} \to 2$ since for all $0 < \gamma \leq 1$ we have $\delta_1, \delta_2 \leq 1$.

The condition (65) is satisfied as well if we have

\[
(68) \quad \left( 2\beta + \frac{1}{2} \sigma^2 (\delta_2 + 1) \right) (C_2 + C_3) \leq \min \left\{ \frac{R}{(1 + R)^{\delta_2 + 1}}, \frac{R^2 \gamma}{(1 + R)^{\delta_1 + 1}}, \kappa_\delta \left( \delta_1 C_2 \sigma^2 \left( \frac{R}{1 + R} \right)^{\frac{1}{\delta_1 + 2}} \left( \frac{\delta_2 C_3}{\delta_2 + 1} \right) \right) \right\}.
\]

The study of the inequality (68). We study next the conditions for $(\beta, \sigma)$ for which the inequality (68) is satisfied in a region of $(C_2, C_3)$, at least for one value of $R$. We start by writing it in an equivalent way as

\[
(69) \quad \kappa_1 a + \kappa_2 b \leq \min \left\{ \kappa_\delta a^{\delta_1} b^{\delta_2}, a R^{2\gamma - \delta_1 - 1}, b R^{-\delta_2} \right\},
\]
where
\[ (70) \quad \varepsilon_1 = \frac{1}{\delta_1 + 2}, \quad \varepsilon_2 = \frac{\delta_1 + 1}{\delta_1 + 2} \]
satisfying \( \varepsilon_1 + \varepsilon_2 = 1 \), and we defined the new variables
\[ (71) \quad a = \delta_1 C_2 \sigma^2 \left( \frac{R}{1 + R} \right)^{\delta_1 + 1}, \quad b = \delta_2 C_3 \left( \frac{R}{1 + R} \right)^{\delta_2 + 1}, \]
and denoted the constants
\[ (72) \quad \kappa_\delta := (\delta_1 + 2)(\delta_1 + 1)^{-\frac{\delta_1 + 1}{\delta_1 + 2}}, \]
\[ (73) \quad \kappa_1 := \frac{1}{\delta_1 \sigma^2} \left( 2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right) \left( \frac{1 + R}{R} \right)^{\delta_1 + 1}, \]
\[ (74) \quad \kappa_2 := \frac{1}{\delta_2} \left( 2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right) \left( \frac{1 + R}{R} \right)^{\delta_2 + 1}. \]

We would like to obtain the region in the \((a, b) \in \mathbb{R}^2_+\) plane where the inequality \((69)\) holds, and find conditions on \(\kappa_1, \kappa_2\) (or equivalently \(\beta, \sigma\)) for which this region is non-empty, at least for one value of \(R\). These regions are given by the following Lemma.

**Lemma 2.** The inequality
\[ (75) \quad \kappa_1 a + \kappa_2 b \leq \min \left\{ \kappa_{\delta} \varepsilon_1 b^{\varepsilon_2}, a R^{2\gamma - \delta_1 - 1}, b R^{-\delta_2} \right\}, \]
with \(\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2 > 0, \kappa_{\delta} > 1, \varepsilon_1 (\delta_1 + 2) = 1\) and \(\varepsilon_1 + \varepsilon_2 = 1\), holds in two regions of the \((a, b)\) plane:

(i) A wedge-like region of the positive quadrant of the \((a, b)\) plane, contained between the two straight lines passing through origin
\[ (76) \quad a R^{\delta_2 (\delta_1 + 2)} \leq b \leq a R^{2\gamma - \delta_1 - 1} \frac{\kappa_1}{\kappa_2}. \]

(ii) A wedge-like region of the positive quadrant of the \((a, b)\) plane, below a straight line passing through origin given by
\[ (77) \quad b \leq a \min \left\{ R^{\delta_2 (\delta_1 + 2)}, \frac{\kappa_1}{R^{-\delta_2} - \kappa_2} \right\}. \]

**Proof of Lemma 2.** We prove that the inequality \((69)\) is satisfied in the regions \((76)\) and \((77)\).

The line
\[ (78) \quad b = a R^{2\gamma - \delta_1 - \delta_2 - 1} = a R^{\delta_2 (\delta_1 + 2)} \]
divides the first quadrant of the \((a, b) \in \mathbb{R}^2_+\) plane into two regions:

(i) **Region 1** with \(b > a R^{\delta_2 (\delta_1 + 2)}\);
(ii) **Region 2** with \(b < a R^{\delta_2 (\delta_1 + 2)}\).

We show that the inequality \((69)\) simplifies in each of these regions as follows.

(i) **Region 1** with \(b > a R^{\delta_2 (\delta_1 + 2)}\).
In this region we have clearly
\begin{equation}
\min \left\{ bR^{-\delta_2}, aR^{2\gamma-\delta_1-1} \right\} = aR^{2\gamma-\delta_1-1}.
\end{equation}
Furthermore, we have
\begin{equation}
\kappa_\delta a^1 \delta_2 > \kappa_\delta aR^{\delta_2(\delta_1+2)} = \kappa_\delta aR^{\delta_2(\delta_1+1)} = \kappa_\delta aR^{\delta_2(\gamma-\delta_1-1)} \geq aR^{\delta_2(\gamma-\delta_1-1)},
\end{equation}
since \( \kappa_\delta > 1 \) as noted above. Thus the inequality \( (69) \) reduces in this region to a linear inequality
\begin{equation}
\kappa_1 a + \kappa_2 b \leq aR^{2\gamma-\delta_1-1}.
\end{equation}
This gives the upper bound on \( b \) in \( (76) \).

\(\text{(ii) Region 2 with } b < aR^{\delta_2(\delta_1+2)}.\)

In this region we have clearly
\begin{equation}
\min \left\{ bR^{-\delta_2}, aR^{2\gamma-\delta_1-1} \right\} = bR^{-\delta_2}.
\end{equation}
Furthermore, we have the lower bound
\begin{equation}
\kappa_\delta a^1 \delta_2 > \kappa_\delta (bR^{-\delta_2(\delta_1+2)})(1+R)\delta_2 b = \kappa_\delta bR^{-\delta_2(\delta_1+2)} > bR^{-\delta_2}
\end{equation}
since \( \kappa_\delta > 1 \). Thus the inequality \( (69) \) reduces in this region to the linear inequality
\begin{equation}
\kappa_1 a + \kappa_2 b \leq bR^{-\delta_2}.
\end{equation}
This gives an upper bound on \( b \)
\begin{equation}
b \leq \frac{\kappa_1}{R^{-\delta_2} - \kappa_2} a
\end{equation}
which is useful only if \( R^{-\delta_2} > \kappa_2 \), or equivalently if
\begin{equation}
\left( 2\beta + \frac{1}{2} \sigma^2 \delta_2(\delta_2 + 1) \right) (1 + R)\delta_2 + 1 < R\delta_2.
\end{equation}
This is obtained using the expression \( (74) \) for \( \kappa_2 \).

If this condition is satisfied, then we get that \( (69) \) is satisfied in the subset of region 2
\begin{equation}
b \leq \min \left\{ R^{\delta_2(\delta_1+2)}, \frac{\kappa_1}{R^{-\delta_2} - \kappa_2} a \right\}.
\end{equation}
This is either the entire region 2, or a subset, bounded by the real axis and the line \( b = \frac{\kappa_1}{R^{-\delta_2} - \kappa_2} a \). \(\square\)

Finally, let us get back to the proof of Theorem \(1\) In order for the region \( (76) \) to be non-empty, the following inequality must hold
\begin{equation}
\kappa_2 R^{\delta_2(\delta_1+2)} \leq R^{2\gamma-\delta_1-1} - \kappa_1.
\end{equation}
Substituting here the expressions \( (73), (74) \) for \( \kappa_1, \kappa_2 \); this becomes
\begin{equation}
R^{2\gamma} \geq \left( 2\beta + \frac{1}{2} \sigma_2^2 \delta_2(\delta_2 + 1) \right) \left( \frac{1}{\delta_1 \sigma_2^2} (1 + R)\delta_1 + 1 + \frac{1}{\delta_2} R^{\delta_1 \delta_2 + \delta_1 + \delta_2} (1 + R)\delta_2 + 1 \right).
\end{equation}
In order for the region \( \mathbb{D} \) to be non-empty one requires \( R^{-\delta_2} > \kappa_2 \) which gives the inequality
\[
R \geq \frac{1}{\delta_2} \left( 2\beta + \frac{1}{2} \sigma^2 \delta_2 (\delta_2 + 1) \right) (1 + R)^{\delta_2 + 1}.
\]

The inequality (\ref{ineq:delta2}) yields the statement (i) of Theorem 1 and the inequality (\ref{ineq:R0}) the statement (ii). This completes the proof of Theorem 1. \( \square \)

Proof of Theorem 2. We would like to test the conditions (A.4) and (A.5) of Proposition 2.

(1) Condition (A.4). Let \( V(r, y) \) be the Lyapunov function (\ref{lyapunov}) defined in our Theorem 1. We have to check that its infimum on \( \mathbb{D}^c \) is positive. We can compute that
\[
K_0 := \inf_{(r, y) \in \mathbb{D}^c} V(r, y) = C_1 - \sup_{(r, y) \in \mathbb{D}^c} \left( \frac{C_2}{(1 + y)\delta_1} + \frac{C_3}{(1 + r)\delta_2} \right) = \min\left\{ C_1 - \frac{C_2}{(1 + R)\delta_1}, C_1 - C_2 - \frac{C_3}{(1 + R)\delta_2} \right\}.
\]
By (\ref{ineq:C1}) we have \( C_1 \geq C_2 + C_3 \) which gives \( K_0 > 0 \). Thus, (A.4) holds.

(2) Condition (A.5). According to Theorem 3.9. and the discussion at the beginning of Chapter 3.7. in [24], it suffices to show that there exists a non-negative function \( V_0(r, y) \) for \((r, y) \in \Gamma^c\) that is twice differentiable in \((r, y)\) such that
\[
\mathcal{L} \mathcal{V}_0(r, y) \leq -\alpha, \quad \text{for any } (r, y) \in \Gamma^c,
\]
where \( \alpha > 0 \) is some constant. We use the notation \( V_0 \) to distinguish it from the Lyapunov function \( V \) defined in Theorem 1.

Let us recall from the proof of Theorem 1 that \( \Gamma = [2R, \infty) \times [2R, \infty) \). Therefore, \( \Gamma^c = \{ (r, y) : 0 < y < 2R \text{ or } 0 < r < 2R \} \). Let us define
\[
V_0(r, y) = e^{-r} + e^{-y}.
\]
Then \( V_0 \) is non-negative and twice differentiable. We can compute that
\[
\mathcal{L} \mathcal{V}_0(r, y) = (-\sigma^2 \min(r^{2\gamma}, r^2 e^{2\gamma - 2}) + 2\beta y)e^{-y} + \left( \beta r - y - \beta r_0 + \frac{1}{2} \sigma^2 \min(r^{2\gamma}, r^2 e^{2\gamma - 2}) \right) e^{-r}.
\]
For \((r, y) \in \Gamma^c\), either one of the inequalities \( 0 < y < 2R \) or \( 0 < r < 2R \) holds.

(a) If \( 0 < r < 2R \), we distinguish between \( 0 < r \leq \varepsilon \) and \( \varepsilon < r < 2R \). In the latter case we have
\[
\mathcal{L} \mathcal{V}_0(r, y) \leq -\beta r_0 e^{-2R} + \sup_{0 < y < \infty, \varepsilon < r < 2R} \left\{ (-\sigma^2 r^{2\gamma} + 2\beta y)e^{-y} + \left( \beta r - y + \frac{1}{2} \sigma^2 r^{2\gamma} \right) e^{-r} \right\}
\leq -\beta r_0 e^{-2R} + \sup_{0 < y < \infty, \varepsilon < r < 2R} \left( \beta r + \frac{1}{2} \sigma^2 r^{2\gamma} \right) e^{-r}
\leq -\beta r_0 e^{-2R} + 2\frac{\beta}{\varepsilon} e^{-r} + (2\beta R + 2\sigma^2 R^{2\gamma}) < 0,
\]
for any sufficiently large \( r_0 \) such that
\[
r_0 > \frac{2e^{-2R}}{\beta} \left[ \frac{2\beta}{\varepsilon} + (2\beta R + 2\sigma^2 R^{2\gamma}) \right],
\]
where we assumed that $\beta > 0$.

For $0 < r \leq \varepsilon$ we get by a similar argument

$$L_\varepsilon V_0(r, y) \leq -\beta_0 e^{-\varepsilon} + \sup_{0 < y < \infty, 0 < r \leq \varepsilon} \left\{ (-\sigma^2 r^2 \varepsilon^{2\gamma-2} + 2\beta y) e^{-y} + \left( \beta r - y - \beta \sigma^2 r^2 \varepsilon^{2\gamma-2} \right) e^{-r} \right\}$$

$$\leq -\beta_0 e^{-\varepsilon} + \sup_{0 < y < \infty} 2\beta y e^{-y} + \sup_{0 < r \leq \varepsilon} \left( \beta r + \frac{1}{2} \sigma^2 r^2 \varepsilon^{2\gamma-2} \right) e^{-r}$$

$$\leq -\beta_0 e^{-\varepsilon} + \frac{2\beta}{e} + \left( \beta \varepsilon + \frac{1}{2} \sigma^2 \varepsilon^{2\gamma} \right) < 0,$$

for any sufficiently large $r_0$ such that

$$r_0 > \frac{e^{\varepsilon}}{\beta} \left[ \frac{2\beta}{e} + \left( \beta \varepsilon + \frac{1}{2} \sigma^2 \varepsilon^{2\gamma} \right) \right].$$

Since $R \geq \varepsilon$, the condition $(95)$ implies $(96)$. Thus $L_\varepsilon V_0(r, y)$ for $0 < r < 2R$ as long as $(95)$ holds.

(b) If $r \geq 2R$, then we must have $0 < y < 2R$. Then, we have

$$L_\varepsilon V_0(r, y) = (-\sigma^2 r^{2\gamma} + 2\beta y) e^{-y} + \left( \beta r - y - \beta \sigma^2 \varepsilon^{2\gamma} \right) e^{-r}$$

$$\leq (-\sigma^2 r^{2\gamma} + 4\beta R) e^{-y} + \left( \beta r + \frac{1}{2} \sigma^2 \varepsilon^{2\gamma} \right) e^{-r} - \beta_0 e^{-r}.$$}

For $\frac{1}{2} < \gamma \leq 1$ we have

$$L_\varepsilon V_0(r, y) \leq -\sigma^2 r^{2\gamma} e^{-2R} + 4\beta R + \left( \beta r + \frac{1}{2} \sigma^2 (r^2 + r) \right) e^{-r} - \beta_0 e^{-r},$$

where we used the fact that $r^{2\gamma} \leq r^2 + r$ for every $r \geq 0$ and $\frac{1}{2} < \gamma \leq 1$. Moreover, for any $r \geq 0$, we have $e^r \geq r + \frac{1}{2} r^2$ so that

$$\left( \beta r + \frac{1}{2} \sigma^2 (r^2 + r) \right) e^{-r} \leq \frac{\beta + \frac{1}{2} \sigma^2) r + \frac{1}{2} \sigma^2 r^2}{r + \frac{1}{2} r^2} \leq \beta + \sigma^2.$$

Hence, by plugging $(98)$ into $(97)$, we get

$$L_\varepsilon V_0(r, y) \leq -\sigma^2 r^{2\gamma} e^{-2R} + 4\beta R + \beta + \sigma^2 - \beta_0 e^{-r}.$$

Denote $H(r) := \sigma^2 r^{2\gamma} e^{-2R} + \beta_0 e^{-r}$, $r \geq 0$. Let us give a lower bound of $H(r)$ over $r \geq 0$. For $0 \leq r \leq 1$, we have $H(r) \geq \frac{\beta}{e} r_0$, and for $r > 1$, we have $H(r) \geq \sigma^2 r e^{-2R} + \beta r_0 e^{-r}$ since $\gamma \in (\frac{1}{2}, 1]$. Denote $\tilde{H}(r) := \sigma^2 r e^{-2R} + \beta r_0 e^{-r}$, and we can compute that

$$\tilde{H}(r) = \sigma^2 e^{-2R} - \beta_0 e^{-r},$$

which is negative for $r < \log(\frac{\beta r_0}{\sigma^2}) + 2R$ and positive for $r > \log(\frac{\beta r_0}{\sigma^2}) + 2R$. Thus

$$\tilde{H}(r) \geq \sigma^2 e^{-2R} \left[ \log \left( \frac{\beta r_0}{\sigma^2} \right) + 2R \right] + \beta r_0 e^{-\log(\frac{\beta r_0}{\sigma^2}) - 2R} = \sigma^2 e^{-2R} \left[ \log \left( \frac{\beta r_0}{\sigma^2} \right) + 2R + 1 \right].$$

Hence,

$$H(r) \geq \sigma^2 r^{2\gamma} e^{-2R} + \beta_0 e^{-r} \geq \min \left\{ \frac{\beta}{e} r_0, \sigma^2 e^{-2R} \left[ \log \left( \frac{\beta r_0}{\sigma^2} \right) + 2R + 1 \right] \right\}.$$
Hence, we conclude that
\[
\max_{r, y \geq 0} \mathcal{L}_x V_0(r, y) < 0,
\]
if \( r_0 \) is sufficiently large so that
\[
\min_{\beta, \sigma^2} \left\{ \frac{\beta}{\epsilon} r_0, \sigma^2 e^{-2R} \left( \log \left( \frac{\beta r_0}{\sigma^2} \right) + 2R + 1 \right) \right\} > 4\beta R + \beta + \sigma^2,
\]
which holds if
\[
r_0 > \max \left\{ \frac{e}{\beta} (4\beta R + \beta + \sigma^2), \frac{\sigma^2}{\beta} e^{\frac{2R}{\sigma^2}} (4\beta R + \beta + \sigma^2) - 2R - 1 \right\}.
\]
For both cases (a) and (b), for sufficiently large \( r_0 \) the inequality \( \max_{r, y \geq 0} \mathcal{L}_x V_0(r, y) < 0 \) is satisfied. The proof is complete. \( \square \)

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