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On the $p$-adic closure of a subgroup of rational points on an Abelian variety
by
Michel Waldschmidt

Abstract
In 2007, B. Poonen (unpublished) studied the $p$–adic closure of a subgroup of rational points on a commutative algebraic group. More recently, J. Bellaïche asked the same question for the special case of Abelian varieties. These problems are $p$–adic analogues of a question raised earlier by B. Mazur on the density of rational points for the real topology. For a simple Abelian variety over the field of rational numbers, we show that the actual $p$–adic rank is at least the third of the expected value.

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1 Introduction
Let $A$ be a simple Abelian variety over $\mathbb{Q}$ of dimension $g$, $\Gamma$ a subgroup of $A(\mathbb{Q})$ of rank $\ell$ over $\mathbb{Z}$, $p$ a prime number, $\log : A(\mathbb{Q}_p) \to T_A(\mathbb{Q}_p)$ the canonical map from the $p$–adic Lie group $A(\mathbb{Q}_p)$ to the $p$–adic Lie algebra $T_A(\mathbb{Q}_p)$ (see § 2.1) and $r$ the dimension of the $\mathbb{Z}_p$–space spanned by $\log \Gamma$ in $T_A(\mathbb{Q}_p)$. We have $r \leq \min \{g, \ell\}$.

Conjecture 1. Under these hypotheses, $r = \min \{g, \ell\}$.

This conjecture trivially holds for an elliptic curve ($g = 1$).
The real analog of this conjecture is related with a conjecture of B. Mazur [13]. See also the conjectures by Yves André [1, 2].
Theorem 2. We have
\[ r \geq \frac{\ell g}{\ell + 2g}. \]

Corollary 3. Under the same assumptions,
\[ r \geq \frac{1}{3} \min\{g, \ell\}. \]

Moreover, if \( \ell > 2g(g - 1) \), then \( r = g \).

Theorem 2 is a special case of Theorem 2.1 of [20], where the simple Abelian variety \( A \) over \( \mathbb{Q} \) is replaced by a commutative algebraic group \( G \) over a number field. Our special case enables us to produce a much simpler proof. In particular, the zero estimate is much easier here, since there is no algebraic subgroup of \( G \) to be taken care of. Also, the main difference between our proof and the two proofs in [20] is that we use an interpolation determinant in place of an auxiliary function (Proposition 2.7 of [20]) or in place of an auxiliary functional (Proposition 2.10 of [20]): we do not need the \( p \)-adic Siegel Lemma (Lemma 3.3 of [19]). The two proofs in [20] are dual to each other, and this duality is just a transposition of the interpolation determinant of the present paper.

2 Further notations and auxiliary results

We keep the notations of §1. We select \( \ell \) elements \( \gamma_1, \ldots, \gamma_\ell \) in \( \Gamma \) linearly independent over \( \mathbb{Z} \).

For \( T \) a positive integer, we denote by \( \mathbb{Z}^g(T) \) the set of tuples \( t = (t_1, \ldots, t_g) \) in \( \mathbb{Z}^g \) with \( 0 \leq t_i < T \) (\( 1 \leq i \leq g \)). Similarly, for \( S \in \mathbb{Z}_{> 0} \), \( \mathbb{Z}^\ell(S) \) denotes the set of tuples \( s = (s_1, \ldots, s_\ell) \) in \( \mathbb{Z}^\ell \) with \( 0 \leq s_j < S \) (\( 1 \leq j \leq \ell \)). Further, \( \Gamma(S) \) will denote the set of \( s_1\gamma_1 + \cdots + s_\ell\gamma_\ell \) with \( s \in \mathbb{Z}^\ell(S) \). Hence \( \Gamma(S) \) is a subset of \( A(K) \) with \( S^\ell \) elements.

2.1 The \( p \)-adic logarithm

We follow the paper by B. Poonen [14] which refers to N. Bourbaki [6] Chap. III, §1 and §7.6.

Since \( \Gamma \) is a finitely abelian subgroup of \( A(\mathbb{Q}_p) \) of rank \( \ell \), \( \log \Gamma \) is also a finitely generated abelian subgroup of \( T_A(\mathbb{Q}_p) \) of the same rank \( \ell \) over \( \mathbb{Z} \). The closure \( \overline{\log \Gamma} = \log \overline{\Gamma} \) with respect to the \( p \)-adic topology is nothing else than the \( \mathbb{Z}_p \)-submodule of \( T_A(\mathbb{Q}_p) \) spanned by \( \log \Gamma \), hence is a finitely generated \( \mathbb{Z}_p \)-module. The dimension of \( \overline{\Gamma} \) as a Lie group over \( \mathbb{Q}_p \) is
\[ \dim \overline{\Gamma} := \text{rk}_{\mathbb{Z}_p} \overline{\log \Gamma}. \]

2.2 Heights

2.2.1 A projective embedding

We fix an embedding \( \iota \) of the Abelian variety \( A \) into a projective space \( \mathbb{P}_N \) over \( \mathbb{Q} \), with an image which is not contained into the hyperplane \( X_0 = 0 \), and so that
the functions $X_1/X_0, \ldots, X_g/X_0$ are algebraically independent over $A$ (recall that $A$ has dimension $g$). We also assume that for $\overline{s} \in \mathbb{Z}^l$, $\iota(\overline{s})$ does not lie in the hyperplane $X_0 = 0$ and we denote by $(1 : \gamma_{s1} : \cdots : \gamma_{sN})$ the coordinates of $\iota(\overline{s})$ in $\mathbb{P}^N$, so that $\gamma_{s\nu} \in \mathbb{Q}$ for $1 \leq \nu \leq N$ and $s \in \mathbb{Z}^l$. For convenience, we also assume that the zero element of $A$ has projective coordinates $(1 : 0 : \cdots : 0)$.

2.2.2 Absolute logarithmic height

Denote by $P = \{2, 3, 5, \ldots\}$ the set of positive prime numbers and by $M_\mathbb{Q}$ the set of normalized places of $\mathbb{Q}$ indexed by $P \cup \{\infty\}$: for $c \in \mathbb{Q}^\times$ we write $c = \pm \prod_{p \in P} p^{v_p(c)}$ and we have

\[
\begin{cases} 
|c|_v = |c| = \max\{c, -c\} & \text{for } v = v_\infty \\
|c|_p = p^{-v_p(c)} & \text{for } p \in P.
\end{cases}
\]

The product formula, in this very simple case, states that, for $c \in \mathbb{Q} \setminus \{0\}$,

\[
\prod_{v \in M_\mathbb{Q}} |c|_v = 1.
\]

The absolute logarithmic height of $c \in \mathbb{Q}$ is defined as

\[
h(c) = \sum_{v \in M_\mathbb{Q}} \log \max\{1, |c|_v\}.
\]

For $c \in \mathbb{Q}^\times$, we write $c = a/b$ where $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}_{>0}$ are two relatively prime integers. Since $\min\{v_p(a), v_p(b)\} = 0$ for all $p \in P$, we have, for all $p \in P$,

\[
\max\{|a|_p, |b|_p\} = 1, \quad \text{which means} \quad \max\{1, |c|_p\} = |b|^{-1}_p.
\]

Hence, by the product formula,

\[
\prod_{p \in P} \max\{1, |c|_p\} = b.
\]

Multiplying both sides by $\max\{1, |c|\}$ yields

\[
h(c) = \log \max\{|a|, b\},
\]

which can be taken as an alternative definition for the absolute logarithmic height.

Liouville’s inequality is very simple in this context:

Lemma 4. If $c$ is a non–zero rational number and $p$ a prime number, then

\[
\log |c|_p \geq -h(c).
\]
For $N \geq 1$ and $\mathbf{c} = (c_0 : \cdots : c_N) \in P_N(\mathbb{Q})$, we set
\[
h(\mathbf{c}) = \sum_{v \in M_\mathbb{Q}} \log \max \{|c_0|_v, \ldots, |c_N|_v\}.
\]
If $c_0, \ldots, c_N$ are rational integers, not all of which are zero, which are relatively prime, then
\[
h(\mathbf{c}) = \log \max \{|c_0|, \ldots, |c_N|\}.
\]
Notice that for $c \in \mathbb{Q}$, $h(c) = h(1 : c)$.

2.2.3 Néron–Tate height

The projective embedding considered in §2.2.1 is associated with a very ample line bundle on $A$, to which is associated a canonical height which is a quadratic function (see [13] Chap. 3 and [9] §B.5).

Lemma 5. For $\mathbf{c} \in \mathbb{Z}^{\ell}(S)$,
\[
h(s_1 \gamma_1 + \cdots + s_\ell \gamma_\ell) = h(1 : \gamma_1 : \cdots : \gamma_N) \leq cS^2.
\]

2.2.4 Upper bound for the height

We shall use the following result, which is a very simple case of Lemma 3.8. in [26] (where $\mathbb{Q}$ is replaced by a number field). We denote by $L(f)$ the length of a polynomial $f$ (sum of the absolute values of the coefficients).

Lemma 6. Let $\nu_1, \ldots, \nu_L$ be positive integers. For $1 \leq i \leq L$, let $\gamma_{i1}, \ldots, \gamma_{i\nu_i}$ be rational numbers. Denote by $\mathbf{\gamma}$ the point $(\gamma_{ij})_{1 \leq j \leq \nu_i, 1 \leq i \leq L}$ in $\mathbb{Q}^{\nu_1 + \cdots + \nu_L}$. Further, let $f$ be a nonzero polynomial in $\nu_1 + \cdots + \nu_L$ variables, with coefficients in $\mathbb{Z}$, of total degree at most $N_i$ with respect to the $\nu_i$ variables corresponding to $\gamma_{i1}, \ldots, \gamma_{i\nu_i}$. Then
\[
h(f(\mathbf{\gamma})) \leq \log L(f) + \sum_{i=1}^{L} N_i h(1 : \gamma_{i1} : \cdots : \gamma_{i\nu_i}).
\]

Proof. Let us write
\[
f(X) = \sum_{\Delta} c_\Delta \prod_{i=1}^{L} \prod_{j=1}^{\nu_i} X_{ij}^{\lambda_{ij}},
\]
where $X$ (resp. $\Delta$) stands for the $\nu_1 + \cdots + \nu_L$-tuple $(X_{ij})_{1 \leq j \leq \nu_i, 1 \leq i \leq L}$ (resp. $(\lambda_{ij})_{1 \leq j \leq \nu_i, 1 \leq i \leq L}$. Lemma 6 follows from the estimates
\[
|f(\mathbf{\gamma})| \leq \sum_{\Delta} |c_\Delta| \prod_{i=1}^{L} \prod_{j=1}^{\nu_i} \max\{1, |\gamma_{ij}|\}^{\lambda_{ij}}
\]
\[
\leq L(f) \prod_{i=1}^{L} \max\{1, |\gamma_{i1}|, \ldots, |\gamma_{i\nu_i}|\}^{N_i}.
\]
\[ |f(z)|_p \leq \max \prod_{i=1}^{L} \prod_{j=1}^{\nu_i} \max\{1, |\gamma_{ij}|_p\}^{\lambda_{ij}} \]
\[ \leq \prod_{i=1}^{L} \max\{1, |\gamma_{i1}|_p, \ldots, |\gamma_{i\nu_i}|_p\}^{N_i} \]
for \( p \in P \).

### 2.3 \( p \)-adic analytic functions

#### 2.3.1 Ultrametric power series

We follow [17]. The field \( \mathbb{Q}_p \) is complete for the \( p \)-adic absolute value. Let
\[ f = \sum_{n_1 \geq 0} \cdots \sum_{n_r \geq 0} a_{n_1, \ldots, n_r} z_1^{n_1} \cdots z_r^{n_r} = \sum_{\mathbb{Z}^r_{\geq 0}} a_{\mathbb{n}} \mathbb{z}^\mathbb{n} \]
be a formal series with coefficients in \( \mathbb{Q}_p \). If \( R \) is a real number > 0, we set
\[ |f|_R = \sup_{\mathbb{n} \in \mathbb{Z}^r_{\geq 0}} R^{\mathbb{z}^\mathbb{n}}|a_{\mathbb{n}}|, \quad \text{where} \quad |\mathbb{n}| = n_1 + \cdots + n_r. \]

We have
\[ |f + g|_R \leq \sup\{|f|_R, |g|_R\}, \quad |\lambda f|_R = |\lambda| \cdot |f|_R \quad \text{and} \quad |fg|_R = |f|_R |g|_R \]
if \( |f|_R \) and \( |g|_R \) are finite. When \( |f|_R \) is finite, the series \( f(z) \) converges in the polydisc \( |z_i| < R \). Moreover, it converges in the closed polydisc \( |z_i| \leq R \) when \( R^{\mathbb{z}^\mathbb{n}}|a_{\mathbb{n}}| \) tends to zero. We have
\[ |f(z)| \leq |f|_R. \]

Since the residue field of \( \mathbb{Q}_p \) is infinite and the group of values of \( \mathbb{Q}_p^\times \) is dense, we also have
\[ |f|_R = \sup |f(z)| \quad \text{for} \quad |z_i| < R. \]

If \( R' \leq R \), we have \( |f|_{R'} \leq |f|_R \) (maximum modulus principle).

#### 2.3.2 Ultrametric Schwarz Lemma

The purpose of the Schwarz’s Lemma is to improve the maximum modulus principle by taking into account the zeros of \( f \) inside the polydisc \( |z_i| < R' \). With the method of interpolation determinants of Laurent [26], we need only to take into account the multiplicity of the zero at the origin. For this reason, the proof reduces to the one variable case (as a matter of fact, we shall use Lemma [7] only for the case of functions of a single variable).
Lemma 7. If \( f \) has a zero of multiplicity \( \geq h \) at the origin, then for \( R' \leq R \) we have
\[
|f|_{R'} \leq \left( \frac{R'}{R} \right)^h |f|_R.
\]

Proof (following [17]). Let \( z \) satisfy \( |f(z)| = |f|_R \) and \( |z| \leq R \). Define \( g(t) = t^{-h} f(tz) \) for \( t \in \mathbb{Q}_p \) with \( |t| \leq 1 \). Since \( R'/R \leq 1 \), we deduce \( |g|_{R'/R} \leq |g|_1 \).

Since \( |g|_1 = |f|_R \) and \( |g|_{R'/R} = (R/R')^h |f|_R \), Lemma 7 follows. \( \square \)

A quantitative version of Lemma 7 is Lemma 3.4.p of [19].

Corollary 8. Let \( f_1, \ldots, f_L \) be power series in \( \mathbb{Q}_p^\mathbb{r} \) with \( |f_\lambda|_R < \infty \) and let \( z_1, \ldots, z_L \) be points in the polydisc \( |z_i| \leq R' \) with \( R' \leq R \). Then the determinant
\[
\Delta = \det \left( f_\lambda(z_\mu) \right)_{1 \leq \lambda, \mu \leq L}
\]
is bounded by
\[
|\Delta| \leq L! \left( \frac{R'}{R} \right)^{L^{1+1/r}} \prod_{\lambda=1}^L |f_\lambda|_R.
\]

Proof. Corollary 8 is an ultrametric version of Lemma 6.3 of [26]; it follows from Lemma 7 by means of Lemmas 6.4 and 6.5 of [26], according to which the function of one variable
\[
\Psi(t) = \det \left( f_\lambda(tz_\mu) \right)_{1 \leq \lambda, \mu \leq L}
\]
has a zero of multiplicity greater than \( (n/e)L^{1+1/n} \) at the origin. \( \square \)

2.3.3 \( p \)-adic theta functions

Since the kernel of the logarithmic map
\[
\log : A(\mathbb{Q}_p) \to T_A(\mathbb{Q}_p)
\]
is the set of torsion points of \( A(\mathbb{Q}_p) \), this map is locally injective near the neutral element of \( A(\mathbb{Q}_p) \). Let \( U \) be an open neighborhood of \( (1 : 0 : \cdots : 0) \) in \( A(\mathbb{Q}_p) \), \( V \) be an open neighborhood of \( 0 \) in \( T_A(\mathbb{Q}_p) \) and \( \theta : V \to U \) be a local inverse of \( \log \):
\[
u \in U \implies \log u \in V \quad \text{and} \quad \theta \log(u) = u,
\]
\[
u \in V \implies \theta(v) \in U \quad \text{and} \quad \log \theta(v) = v.
\]

By definition of \( r \), \( \overline{\log \Gamma} \) is a \( \mathbb{Z}_p \)-submodule of \( T_A(\mathbb{Q}_p) \) of dimension \( r \) which contains the \( \ell \) elements \( \log \gamma_j \) \( (1 \leq j \leq \ell) \). Let \( e_1, \ldots, e_r \) be a basis. Let \( R > 0 \) be a positive real number such that \( z_1e_1 + \cdots + z_re_r \in V \) for any
\( \mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{Q}_p^r \) with \( |z_i|_p \leq R \). For \( \mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{Q}_p^r \) with \( |z_i|_p < R \), define \( \theta_1(\mathbf{z}), \ldots, \theta_N(\mathbf{z}) \) by

\[
\theta(z_1 e_1 + \cdots + z_r e_r) = \left( 1 : \theta_1(\mathbf{z}) : \cdots : \theta_N(\mathbf{z}) \right).
\]

Then \( \theta_1, \ldots, \theta_N \) are power series in \( r \) variables with coefficients in \( \mathbb{Q}_p \) and radius of convergence \( \geq R \).

Write

\[
\log \gamma_j = \sum_{i=1}^r \eta_{ji} e_i \quad \text{and} \quad y_j = (\eta_{j1}, \ldots, \eta_{jr}) \in \mathbb{Q}_p^r \quad (1 \leq j \leq \ell).
\]

Further, select \( M \in \mathbb{Z}_{>0} \) such that

\[
\max_{1 \leq i \leq \ell} |M \eta_{ji}|_p < R.
\]

Then, for any \( s \in \mathbb{Z}^\ell \) with \( M|s_j \) for \( 1 \leq j \leq \ell \),

\[
s_1 \log \gamma_1 + \cdots + s_\ell \gamma_\ell \in \mathcal{V}
\]

and

\[
\theta(s_1 \log \gamma_1 + \cdots + s_\ell \log \gamma_\ell) = \left( 1 : \gamma_{s_1} : \cdots : \gamma_{s_\ell} \right).
\]

Hence \( \gamma_{s\nu} = \theta_\nu(y_s) \) for all \( s \in \mathbb{Z}^\ell \) with \( M|s_j \) and for all \( \nu \) with \( 1 \leq \nu \leq N \).

3 The zero estimate and the interpolation determinant

The zero–estimate of Masser–Wüstholz (Main Theorem of [12]) is valid for a quasi–projective commutative algebraic group variety over a field \( K \) of zero characteristic. We need it only for a simple Abelian variety, which makes the statement shorter, since there is no algebraic subgroup to worry about.

Let again \( A \) be a simple Abelian variety of dimension \( g \) embedded into a projective space \( \mathbb{P}_N \). When \( P \in K[Y_0, \ldots, Y_N] \) is a non–zero homogenous polynomial, we denote by \( Z(P) \) the hypersurface \( P = 0 \) of \( \mathbb{P}_N \).

**Lemma 9** (Zero estimate). There exists a constant \( c > 0 \) depending only on \( A \) and on the embedding of \( A \) into \( \mathbb{P}_N \) with the following property. Let \( \gamma_1, \ldots, \gamma_\ell \) be \( \mathbb{Z} \)-linearly independent elements in \( A(K) \). Let \( P \in K[Y_0, \ldots, Y_N] \) be a homogenous polynomial of total degree \( \leq D \), such that \( Z(P) \) does not contain \( A(K) \) but contains

\[
\Gamma(S) = \{ s_1 \gamma_1 + \cdots + s_\ell \gamma_\ell \; ; \; \mathbf{s} \in \mathbb{Z}^\ell(S) \}.
\]

Then

\[
D > c(S/g)^{\ell/g}.
\]
Like in [20], §2.b, we could replace the zero estimate by an interpolation lemma due to D.W. Masser ([11] and Theorem 2.1 of [20]). The idea is just to consider the transposed matrix.

Coming back to the notations of §2 (recall in particular the integer $M > 0$ introduced in §2.3.3), we deduce from Lemma 9:

**Corollary 10.** There exist two integers $c_1 > 1$ and $N_0 > 1$, depending on $A$ and $\gamma_1, \ldots, \gamma_\ell$, with the following property: if $N$ is a positive integer with $N \geq N_0$ and if we set
\[
L = N^g, \quad T = N^\ell, \quad S = c_1 N^g,
\]
then there exists a subset $S = \{s_1, \ldots, s_L\}$ of $\mathbb{Z}^\ell(S)$ with $L$ elements $s_\mu = (s_{\mu,j})_{1 \leq j \leq \ell}$ $(1 \leq \mu \leq L)$, such that $M | s_{\mu,j}$ for $1 \leq j \leq \ell$ and $1 \leq \mu \leq L$, and such that the determinant
\[
\Delta = \det \left( \gamma_{s_1}^{t_1} \cdots \gamma_{s_L}^{t_L} \right)_{\Delta \in S, \, \ell \in \mathbb{Z}^g(T)}
\]
does not vanish.

**Proof.** Consider the matrix
\[
\left( \gamma_{s_1}^{t_1} \cdots \gamma_{s_L}^{t_L} \right)_{\Delta \in S, \, \ell \in \mathbb{Z}^g(T)},
\]
where the index of rows is $\Delta \in \mathbb{Z}^g(T)$, while the index of columns $s_\mu$ runs over the elements in $\mathbb{Z}^\ell(S)$ for which $M$ divides $s_{\mu,j}$. Our goal is to prove that this matrix has maximal rank $L$. Consider a system of relations among the rows of the matrix
\[
\sum_{\ell \in \mathbb{Z}^g(T)} p_{\ell} \gamma_{s_1}^{t_1} \cdots \gamma_{s_L}^{t_L} = 0 \quad (s_\Delta \in \mathbb{Z}^\ell(S), \, M | s_{\mu,j})
\]
with $p_{\ell} \in k$ for all $\ell \in \mathbb{Z}^g(T)$. The polynomial
\[
\sum_{\ell \in \mathbb{Z}^g(T)} p_{\ell} X_1^{t_1} \cdots X_g^{t_L}
\]
has degree $\leq T$ in each of the variables $X_1, \ldots, X_g$ and vanishes at all points of $\gamma_{s_\Delta} \in \Gamma(S)$ for which $M | s_{\mu,j}$ $(1 \leq j \leq \ell)$. Use Lemma 9 with $\gamma_1, \ldots, \gamma_\ell$ replaced by $M \gamma_1, \ldots, M \gamma_\ell$. Taking $c_1 > Mg/c^g/\ell$, so that $gN^\ell < c(c_1 N^g/gM)^{\ell/g}$, it follows that this polynomial is $0$, hence $p_{\ell} = 0$ for all $\ell \in \mathbb{Z}^g(T)$.

**4 Upper bound for the height and lower bound for the absolute value of the interpolation determinant**

Under the assumptions of Theorem 2, we give an upper bound for the height of the determinant $\Delta$ introduced in Corollary 10.
Proposition 11. There exists a positive integer $c_2 > 1$, depending on $A$ and $\gamma_1, \ldots, \gamma_\ell$, such that, for all $N \geq N_0$,

$$h(\Delta) \leq c_2 LT S^2.$$ 

Proof. From Lemma 5 we deduce, for any $s \in \mathbb{Z}^L(S)$,

$$h (1 : \gamma_{s1} : \cdots : \gamma_{sN}) \leq cS^2.$$ 

Proposition 11 now follows from Lemma 6 with

$$\nu_1 = \cdots = \nu_L = g, \quad N_1 = \cdots = N_L = T \quad \text{and} \quad L(f) \leq L!$$

$$\square$$

Liouville’s inequality (Lemma 4) implies:

Corollary 12. With the notations of Proposition 11,

$$\log |\Delta|_p \geq -c_2 LT S^2.$$ 

5 Analytic estimate: upper bound for the absolute value of the interpolation determinant

Proposition 13. There exists a positive integer $c_3 > 1$, depending on $A$ and $\gamma_1, \ldots, \gamma_\ell$, such that, for all $N \geq N_0$,

$$\log |\Delta|_p \leq -c_3 L^{1+1/r}.$$ 

Proof. Proposition 13 follows from Corollary 8 with the set of functions

$$\{f_1, \ldots, f_L\} = \{\theta_{t1}^{\mu_1} \cdots \theta_{g}^{\mu_L} ; L \in \mathbb{Z}_{\geq 0}(T)\}$$

and the points $z_{\mu} = s_{\mu 1} y_1 + \cdots + s_{\mu \ell} y_\ell (1 \leq \mu \leq L)$. 

$$\square$$

6 Proof of the main transcendence result

Proof of Theorem 2. Since $TS^2 = c_1^2 L^{(1/g)+(2/\ell)}$, the conclusions of corollary 12 and proposition 13 imply

$$\frac{1}{r} \leq \frac{1}{g} + \frac{2}{\ell}.$$ 

$$\square$$
7 Remarks

• 7.1. In place of the rational number field and the prime number $p$, one may work with an algebraic number field and a finite place $v$, replacing $\mathbb{Q}_p$ with the completion $k_v$. One main difference is in §2.2.2, where, in the case of a number field, one needs to introduce height functions on the field of algebraic numbers in place of the rational number field. See [26] Chap. 3 § 2, [9], § B.2, [18], Chap. 2, [4] Chap. 1, [10] Chap. 4.

As pointed out in [14] (Remark 6.4), one cannot deduce the general case of a number field from the special case of the rational numbers by means of the restriction of scalars.

• 7.2. As mentioned in [21] (§ 6a p. 643), similar results hold when the simple Abelian variety $A$ is replaced by a commutative algebraic group $G$. There is a condition in [21] for the ultrametric case that a subgroup of finite index of $\Gamma$ is contained in a compact subgroup of $A(k_v)$ – for an Abelian variety $A$, the group $A(k_v)$ is compact and this condition is always satisfied.

Let us write, like in [21], $G = G_{d_0} \times G_{d_1} \times G'$, where $G'$ has dimension $d_2$ (and therefore $G$ has dimension $d = d_0 + d_1 + d_2$). Roughly speaking, in this general sitting, one replaces

$$\frac{\ell g}{\ell + 2g} \text{ by } \frac{\ell d}{\ell + d_1 + 2d_2}.$$  

However, one needs to take into account possible degeneracies occurring from the algebraic subgroups of $G$. We refer to [21] for precise statements.

In the case of a power of the multiplicative group $G = G_m^d$, the transcendency result yields lower bounds for the $p$–adic rank of the units of an algebraic number field (namely partial results towards Leopoldt’s Conjecture).

• 7.3. Following [14], consider a commutative algebraic group $G$ over $\mathbb{Q}$ and a finitely generated subgroup $\Gamma$ of $G(\mathbb{Q})$ contained in the union of compact subgroups of $G(\mathbb{Q}_p)$. The number $\dim(\Gamma)$ can be defined exactly like in §2.1 as the dimension of the $\mathbb{Z}_p$–submodule of the tangent space at the origine $\text{Lie}(G)$ spanned by the image of $\Gamma$ under the logarithmic map. Another function $d(\Gamma)$ of $\Gamma$ is introduced by B. Poonen in [14]:

$$d(\Gamma) := \min_{H \subset G} \{\dim H + \text{rk}(\Gamma \cap H)\},$$

where the minimum is over all algebraic subgroups $H$ of $G$ over $\mathbb{Q}$. The inequality $\dim(\Gamma) \leq d(\Gamma)$ is always true. Here is an example where this inequality is strict (compare with Langevin’s example in [23] p. 1201 and 1209 for $\mathbb{G}_m^3$).

Consider an elliptic curve $E$ over $\mathbb{Q}$ with three linearly independent algebraic points $\gamma_1, \gamma_2, \gamma_3$ in $E(\mathbb{Q})$. Let $\Gamma$ be the subgroup of $E^3(\mathbb{Q})$ generated by $(0, \gamma_3, -\gamma_2), (-\gamma_3, 0, \gamma_1), (\gamma_2, -\gamma_1, 0)$. Then $\dim(\Gamma) = 2$, while $d(\Gamma) = 3$.

To produce a lower bound for the $p$–adic rank amounts to produce lower bounds for the rank of certain matrices whose entries are $p$–adic logarithms of algebraic points. From a conjectural point of view, the answer is given by the
structural rank introduced by D. Roy. See [26] for the case of linear algebraic groups.

*7.4.* Further applications of the algebraic subgroup theorem in the ultrametric case are given by D. Roy in [15].

*7.5.* Our $p$-adic result Theorem 2 is an ultrametric version of [21, 22] (see also [16]). In the Archimedean case, quantitative refinements are given in [24]. See also [8]. Since the method is “effective”, it is also possible to produce quantitative refinements of Theorem 2.

*7.6.* An alternative proof of the main result (Theorem 2) can be given by means of Arakelov’s geometry and Bost slope inequality. See the papers by J.B. Bost [5] and A. Chambert–Loir [7].

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