

2+1 dimensional solution of Einstein Cartan equations

M. Horta¸csu, H.T. ¨Oz¸celik, N. ¨Ozdemir

Istanbul Technical University, Faculty of Science and Letters, 34469 Maslak, Istanbul, TURKEY

In this work a static solution of Einstein-Cartan (EC) equations in 2+1 dimensional space-time is given by considering classical spin-1/2 field as external source for torsion of the space-time. Here, the torsion tensor is obtained from metricity condition for the connection and the static spinor field is determined as the solution of Dirac equation in 2+1 spacetime with non-zero cosmological constant and torsion. The torsion itself is considered as a non-dynamical field.

I. INTRODUCTION

According to the Einstein’s theory of relativity, metric of spacetime represents the gravitational field and the matter-energy forms are related to spacetime curvature. In this theory, spacetime has symmetric metric connection and it is called pseudo-Riemannian spacetime due to its Lorentzian signature. Since the metricity condition determines the connection uniquely in terms of metric components, metric is the unique characteristic of Riemannian spacetime. Since the Einstein field equations relate the geometry with considered energy form, solution of the field equations will determine the metric of spacetime for a given matter-energy form. The field equations are non-linear and therefore, the solution metric may not be the one, instead it may be a solution of a class of solutions.

Existence of non-zero torsion of space-time is based on several speculations in physics literature. For example, in references [1, 2], it is pointed out that one of the reasons of torsion is the spin of matter, which is coupled to the spacetime via a non-Riemannian structure. It may exhibit a further influence of geometry on gravity.

On the other hand, observations show that the present standard model of particle physics does not explain all of the particle spectrum. For example, neutrino oscillations can be result of a more general theory [3]. To overcome this problem new fields or new symmetries can be introduced to theory at low energies. Introducing torsion to the gravity theory may play a role to explain these oscillations [4]. While the metric is the only characteristic of the space-time in Einstein’s theory of gravity, in theories with torsion the connection coefficients are no longer symmetric and space-time is determined by two independent characteristics, metric and torsion. Thus, Einstein’s relativity is generalized to the Einstein-Cartan theory. Theories that include torsion are reviewed in details in references [4, 5, 6] where torsion is considered as both a dynamical and a non-dynamical field. Additional studies, specifying boundary conditions, include the work of Peeters and Waldron [7].

In another line of research, solutions to the Einstein’s equations are investigated in lower dimensions, since it is often easier to obtain solutions on these manifolds. In 2+1 dimensions the gravitational wave does not exist. The Weyl tensor vanishes identically and it is not possible to have a Ricci flat space with non-trivial Riemann tensor components. One has to introduce additional structure to the model to get interesting solutions. Indeed Banados, Teitelboim and Zanelli [8] introduced a nonvanishing cosmological constant to obtain their black hole (BTZ) solution. Deser, Jackiw and Templeton [9] studied the case with additional topological terms (the DJT solution). Two of us have studied a scalar field coupled to gravity in 2+1 dimensions in the past [10]. Torsion is added to the BTZ and DJT models in references [11] and [12] respectively. A recent work of a model with torsion is given by Blagojevi´c and Cvetkovi´c [13] where a Maxwell field is coupled to gravity in 2+1 dimensions.

In this work, we consider torsion as a non-propagating field and solve the field equations in the framework of the Einstein-Cartan theory. The source of torsion is an external static radial classical spin-1/2 field. That is, the torsion is considered to be coupled to the spinor field. An exact solution of the Einstein-Cartan field equations in 2+1 dimensions is obtained. Besides the torsion effect, the spinor field contributes to the energy momentum tensor, right hand side of the Einstein equations, as a source and it may also contribute to the spin (Cartan) equations. Furthermore, the Dirac equation should be satisfied by spinor fields. At the end of our calculations a solution of full Einstein-Cartan and Dirac equations in the background of three dimensional space-time with non-zero curvature and torsion, a metric for $U_3$ spacetime having the metricity condition is obtained. The background spacetime is considered static and it is supposed that the spacetime has positive cosmological constant.

We give mathematical definitions used throughout the work in the following section and use the notation of [14]. Exact solutions of Einstein-Cartan and Dirac equations are given in section 3.

II. EINSTEIN-CARTAN EQUATIONS

A measure of the space-time curvature is the difference between parallel transported vectors along a closed path. In curved space-times, parallel transport of a vector along a closed curve needs the introduction of the symmetric
structure “the connection”. According to Einstein’s theory of relativity, the space-time has symmetric torsion free Levi-Civita connection (or Riemannian connection) $\nabla$. If we allow the space-time to have a metric tensor $g_{\nu\lambda}$, together with the symmetry condition, $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$, the metricity condition $\nabla_{\mu} g_{\nu\lambda} = 0$ is obtained. This symmetry condition determines the connection $\nabla_{\mu}$ uniquely. Then, the connection coefficients, the Christoffel symbols, are given only in terms of metric components

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) ,$$

where sub indices represent the partial derivatives with respect to the coordinates.

When torsion is taken into account, the connection coefficients defined by (1) no longer satisfy the symmetry condition and space-time becomes non-Riemannian. Thus, the Einstein’s theory of gravity is generalized and modified to the Einstein-Cartan (EC) theory. According to the EC theory, affine connection is non-symmetric and the geometry can be determined by two independent properties “the metric” and “the torsion”.

If $\sim$ represents the quantities written in the space-time with torsion, the torsion tensor $T$ is defined in terms of the antisymmetric connection coefficients as follows:

$$T_{\nu\lambda} = \hat{\Gamma}^\mu_{\nu\lambda} - \hat{\Gamma}^\mu_{\lambda\nu} = 2 \hat{\Gamma}_{[\nu\lambda]}^\mu .$$

If we impose the metricity condition, $\hat{\nabla}_{\mu} g_{\nu\lambda} = 0$, additionally, the spacetime is called Riemann-Cartan and the metricity condition enables us to define the connection in terms of metric and torsion in a unique way

$$\hat{\nabla}_{\mu} g_{\nu\lambda} = \partial_{\mu} g_{\nu\lambda} - \hat{\Gamma}^\rho_{\mu\nu} g_{\rho\lambda} - \hat{\Gamma}^\rho_{\mu\lambda} g_{\rho\nu} = 0 .$$

In this equation $\hat{\Gamma}^\mu_{\nu\rho}$ represent non-symmetric connection coefficients defined by

$$\hat{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + K^\mu_{\nu\rho}$$

where $\Gamma^\mu_{\nu\rho}$ are symmetric Riemannian connection coefficients and $K$ is the contortion tensor satisfying the relation $K^\mu_{\nu\rho} = (T^\mu_{\nu\rho} + T^\mu_{\rho\nu} + T^\mu_{\lambda\nu})/2$. The raising and lowering indices are satisfied by the metric tensor $g_{\mu\nu}$. From its definition it is easy to see that the torsion tensor defined by (2) is anti-symmetric with respect to first two indices and by using the metricity condition given by (3) we get the result $K^\mu_{\nu\rho} g_{\rho\lambda} + K^\mu_{\rho\lambda} g_{\rho\nu} = 0$, or $K^\mu_{\nu\lambda} + K^\mu_{\lambda\nu} = 0$ which means the contorsion tensor is anti-symmetric with respect to its last two indices.

The curvature tensor in space-time with torsion is defined in terms of spacetime connection coefficients

$$\hat{R}^\mu_{\nu\lambda\rho} = \partial_{\rho} \hat{\Gamma}^\mu_{\nu\lambda} - \partial_{\lambda} \hat{\Gamma}^\mu_{\nu\rho} + \hat{\Gamma}^\mu_{\rho\sigma} \hat{\Gamma}^\sigma_{\nu\lambda} - \hat{\Gamma}^\mu_{\lambda\sigma} \hat{\Gamma}^\sigma_{\nu\rho} ,$$

and the Ricci tensor and the Ricci scalar are given by

$$\hat{R}_{\mu\nu} = \hat{R}_{\sigma\nu}^{\ \ \ \sigma} , \quad \hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu} .$$

Now, the field equations which have to be satisfied in space-times with torsion are called Einstein-Cartan equations

$$\hat{R}_{\mu\nu} - \frac{1}{2} (g_{\mu\nu} \hat{R} - 2\Lambda) = \kappa t_{\mu\nu} ,$$

$$T^\mu_{\nu\lambda} - \delta^\mu_{\nu} T^\rho_{\rho\lambda} - \delta^\mu_{\lambda} T^\rho_{\nu\rho} = \kappa S^\mu_{\nu\lambda} .$$

Here $\kappa$ is the gravitational, $\Lambda$ is the cosmological constant and $t_{\mu\nu}$ is the energy-momentum tensor. The equation (8) means that torsion is related to the classical spin of the matter distribution $S^\mu_{\nu\lambda}$. For brevity we will take $\kappa = 1$ in our further calculations.

### III. 2+1 DIMENSIONAL SOLUTION WITH CLASSICAL SPIN-1/2 SOURCE

Consider the 2+1 dimensional space-time which has the following line element

$$ds^2 = -v^2 dt^2 + w^2 dr^2 + r^2 d\phi^2 .$$

Here $v, w$ are radial coordinate $r$ dependent functions. Let us consider a static external classical spin-1/2 field $\psi$ which is coupled to space-time as source of torsion. Thus, the considered space-time has non-zero torsion.
If we impose the metricity condition (3) for the metric (9), the non-zero connection coefficients \( \tilde{\Gamma}_{\mu \rho}^t \) are obtained as follows:

\[
\begin{align*}
\tilde{\Gamma}_{tt}^t &= \tilde{\Gamma}_{rt}^t = \Gamma_{rt}^t = \frac{v'}{v}, \\
\tilde{\Gamma}_{rr}^r &= \Gamma_{rr}^r = \frac{w'}{w}, \\
\tilde{\Gamma}_{\phi \phi}^\phi &= \tilde{\Gamma}_{\phi \phi}^\phi = \frac{w^2}{u^2}, \\
\tilde{\Gamma}_{t \phi}^t &= -\tilde{\Gamma}_{\phi t}^t = \frac{wu}{u^2}, \\
\tilde{\Gamma}_{\phi r}^\phi &= -\tilde{\Gamma}_{r \phi}^\phi = \frac{wu^2}{r^2}, \\
\tilde{\Gamma}_{\phi \phi}^r &= -\frac{r^2}{w^2} \tilde{\Gamma}_{r \phi}^\phi , \\
\tilde{\Gamma}_{t \phi}^r &= -\tilde{\Gamma}_{\phi t}^r = u.
\end{align*}
\]

Here connection coefficients are obtained in terms of an arbitrary function \( u = u(r) \) which is to be determined by the field equations and the Dirac equation. In these relations we see that \( u(r) \) characterizes the torsion. As \( u \rightarrow 0 \) torsion becomes zero. Then, the connection coefficients become symmetric and thus, the Riemannian case is obtained.

Non-zero components of curvature tensor (6) are:

\[
\begin{align*}
\check{R}_{tt} &= -\frac{1}{r^2 w^3} \left[ 2u^2 w^5 + rru w' - rvw(v' + r^2 vv') \right], \\
\check{R}_{t \phi} &= -\check{R}_{\phi t} = u' - u \left( \frac{v'}{v} - \frac{w'}{w} \right), \\
\check{R}_{rr} &= \frac{1}{r^2 w^2 u^2} \left[ 2u^2 w^5 + rru w' + r^2 v^2 w' - r^2 vv' w' \right], \\
\check{R}_{\phi \phi} &= \frac{1}{v^2 u^2} 2u^2 w^5 - rvw v' + r^2 v^2 w', \\
R &= \frac{2}{r^2 u^2 w^3} \left( 3u^2 w^5 + rvw(v + rv') - rvw(v' + rv') \right),
\end{align*}
\]

When \( u \) is taken zero, the Riemannian situation is obtained. In the presence of torsion, non-zero components of the totally anti-symmetric torsion tensor, defined by (2), become

\[
\begin{align*}
T_{t \phi}^r &= 2u, \\
T_{r \phi}^t &= \frac{2u w^2}{v^2}, \\
T_{r t}^\phi &= \frac{2uw^2}{r^2}.
\end{align*}
\]

Since the torsion tensor is totally anti-symmetric, spin distribution \( S_{t \phi}^r = 2u, S_{r \phi}^t = \frac{2u w^2}{v^2}, S_{r t}^\phi = \frac{2uw^2}{r^2} \) belonging to matter field, takes the form

\[
\begin{align*}
S_{t \phi}^r &= 2u, \\
S_{r \phi}^t &= \frac{2u w^2}{v^2}, \\
S_{r t}^\phi &= \frac{2uw^2}{r^2}.
\end{align*}
\]

In this work we consider classical spin-1/2 matter field, which is source of the torsion. In addition to contributions coming from ordinary matter forms, energy-momentum tensor in (7), in general, also includes spin-torsion interactions. In 2+1 dimensional space-time, however, spin-torsion interaction term becomes zero; therefore, it does not contribute to \( t_{\mu \nu} \). Thus, energy momentum tensor \( t_{\mu \nu} \) given in (7) consists only of the spin-1/2 field matter part and it is defined in the torsion background by

\[
t_{\mu \nu}^\text{matter} = \frac{i}{4} \left[ \left( \bar{\psi} \gamma_\mu \nabla_\nu \psi + \bar{\psi} \gamma_\nu \nabla_\mu \psi \right) - \left( \nabla_\mu \bar{\psi} \gamma_\nu \psi + \nabla_\nu \bar{\psi} \gamma_\mu \psi \right) \right]
\]

where

\[
\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu^\nu \psi' , \quad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \Gamma_\mu^\nu \bar{\psi}.
\]

Here \( \Gamma_\mu \) is the spinor connection, \( \bar{\psi} \) is conjugate of the spinor field \( \psi \) having the relation \( \bar{\psi} = \psi^\dagger \gamma^0 \).

In an orthonormal frame the metric tensor of space-time can be expressed in the following way:

\[
g_{\mu \nu} = e_\mu^a e_\nu^b \eta_{ab} , \quad e_\mu^a e_\nu^a = \delta_\mu^\nu
\]
where $\eta_{ab} = (-, +, +)$ is 2+1 Minkowski metric. Raising and lowering of indices are made by using the metric tensor $g_{\mu\nu}$.

$$e_{\mu a} = g_{\mu\nu} e^\nu_a. \quad (18)$$

The spinor connections $\Gamma_\mu$ are defined as follows

$$\Gamma_\mu = \frac{1}{8} \{\gamma^a, \gamma^b\} e_a^\nu \left( \partial_\mu e_{\nu b} - \hat{\Gamma}^\rho_{\mu\nu} e_{\rho b} \right), \quad (19)$$

where $\gamma^a$'s ($a = 0, 1, 2$), are 2+1 dimensional Minkowski gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (20)$$

such that $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$, and curved space-time gamma matrices $\gamma^\mu$ can be obtained from flat space gamma matrices by means of orthonormal base transformation $\gamma^\mu = e^\mu_a \gamma^a$ satisfying the relation $\{\gamma^\mu, \gamma^\nu\} = -2g_{\mu\nu}$. Spinor fields in 2+1 dimensions are given in terms of their components as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = (\psi_2^\dagger \; \psi_1^\dagger) \quad (21)$$

where “*” represents the complex conjugate of the functions. In this problem spinor field $\psi$ and conjugate spinor field $\bar{\psi}$ are considered static $r$ dependent functions only. An orthonormal base can be chosen in the following form for the spacetime

$$e^\mu_a = \begin{pmatrix} 1/v & 0 & 0 \\ 0 & 1/w & 0 \\ 0 & 0 & 1/r \end{pmatrix}, \quad e^\nu_a = \begin{pmatrix} v & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & r \end{pmatrix} \quad (22)$$

Then, 2+1 dimensional curved space-time gamma matrices $\gamma^\mu$ become

$$\gamma^t = \begin{pmatrix} 0 & 1/v \\ 1/v & 0 \end{pmatrix}, \quad \gamma^r = \begin{pmatrix} i/w & 0 \\ 0 & -i/w \end{pmatrix}, \quad \gamma^\phi = \begin{pmatrix} 0 & -1/r \\ 1/r & 0 \end{pmatrix} \quad (23)$$

Non-zero components of the Einstein tensor defined by $\Box$ for the metric $\Box$ and torsion tensor $\Box$ will satisfy the field equations

$$G_{tt} - t_{tt} = -\Lambda v^2 + \frac{u'' w^2}{r^2} + \frac{v^2}{u^2} \quad (24)$$

$$G_{tr} - t_{tr} = \frac{i}{4} \left( -w (\bar{\psi}_2 \psi_1' + \bar{\psi}_1 \psi_2') + \psi_2 (v' \bar{\psi}_2 + v \psi_2') + \psi_1 (v' \bar{\psi}_1 + v \psi_1') \right) \quad (25)$$

$$G_{t\phi} - t_{t\phi} = -G_{\phi t} + t_{\phi t} = u' - u \left( \frac{1}{r} - \frac{v'}{v} - \frac{w'}{w} \right) \quad (26)$$

$$G_{rr} - t_{rr} = \Lambda w^2 - \frac{u'^2}{r^2 v^2} + \frac{v'}{rv} + \frac{1}{2v'^2} \left( -rv w^3 (\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2) + 2rv v' + r^2 v^2 w (\bar{\psi}_1 \psi_2' + \psi_2 \bar{\psi}_1') \right) \quad (27)$$

$$G_{r\phi} - t_{r\phi} = G_{\phi r} - t_{\phi r} = \frac{i}{4v} \left( uw w^2 (\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2) + rv (\psi_1 \bar{\psi}_2' + \bar{\psi}_2 \psi_1' - \bar{\psi}_1 \psi_2 + \psi_2 \bar{\psi}_1') \right) \quad (28)$$

$$G_{\phi \phi} - t_{\phi \phi} = \Lambda r^2 - \frac{u'^2}{v^2} + \frac{r^2}{vw^2} (w'' - v' w') \quad (29)$$

Here ‘$\prime$’ denotes the derivative with respect to $r$.

A set of solutions to 2+1 dimensional field equations can be found by solving differential equations $\Box$, but solution of the Dirac equation to be satisfied by spinor field will restrict us to a specific solution.

In general, the 2 dimensional spinor fields satisfy the following Dirac equations:

$$i \gamma^\mu \nabla_\mu \psi - m \psi = 0, \quad i \nabla_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \quad (30)$$
with mass $m$ of the spinor field.

Dirac equations in this background become

\[ i\gamma^\mu \nabla_\mu \psi - m\psi = i\gamma^\mu (\partial_\mu \psi + \Gamma_\mu \psi) - m\psi = \left( \frac{1}{2rvw} [uw^2(\psi_1 - \psi_2) - rv'\psi_2 - 2rv(mw\psi_1 + \psi_1')] \right) = 0, \quad (31) \]

\[ i\tilde{\nabla}_\mu \tilde{\psi}^\mu + m\tilde{\psi} = (\partial_\mu \tilde{\psi} - \Gamma_\mu \tilde{\psi})\gamma^\mu + m\tilde{\psi} \]

\[ = \left( \frac{1}{2rvw} [-uw^2(\tilde{\psi}_1 + \tilde{\psi}_2) + rv'\tilde{\psi}_2 + 2rv(mw\tilde{\psi}_1 + \tilde{\psi}_1')] \right) = 0 \]

(32)

for the metric given in (9). And a solution of Einstein-Cartan equations can be found as

\[ v = r, \quad w = \frac{1}{r\sqrt{c^2 - \Lambda}}, \quad u = cr^3 \sqrt{c^2 - \Lambda}, \quad (33) \]

\[ \psi_1 = \tilde{\psi}_2 = d_1 \cos \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r + d_2 \sin \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r \]

\[ \psi_2 = \frac{1}{c + \sqrt{c^2 - \Lambda}} \left[ d_3(c - 2m) - d_4(\sqrt{\Lambda - c^2 + 4m(c - m)}) \right] \cos \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r \]

\[ + \frac{1}{c + \sqrt{c^2 - \Lambda}} \left[ d_4(c - 2m) - d_3(\sqrt{\Lambda - c^2 + 4m(c - m)}) \right] \sin \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r, \]

\[ \tilde{\psi}_2 = \frac{1}{c + \sqrt{c^2 - \Lambda}} \left[ -d_3(c - 2m) + d_4(\sqrt{\Lambda - c^2 + 4m(c - m)}) \right] \cos \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r \]

\[ + \frac{1}{c + \sqrt{c^2 - \Lambda}} \left[ -d_4(c - 2m) + d_3(\sqrt{\Lambda - c^2 + 4m(c - m)}) \right] \sin \frac{\sqrt{\Lambda - c^2 + 4m(c - m)}}{2\sqrt{c^2 - \Lambda}} \log r \]

(34)

with $d_1, d_2, d_3, d_4$ real. Furthermore, $c > \sqrt{\Lambda}$ is a positive constant and subject to the constraint $4m(c - m) > c^2 - \Lambda$. Then, the metric (9) takes the form

\[ ds^2 = -r^2 dt^2 + \frac{1}{(c^2 - \Lambda)^2} dr^2 + r^2 d\phi^2. \quad (35) \]

In this solution the non-zero curvature tensor components are

\[ \tilde{R}_{tt} = -2r^2 \Lambda, \quad \tilde{R}_{rr} = \frac{2\Lambda}{r^2(c^2 - \Lambda)}, \quad \tilde{R}_{\phi\phi} = 2r^2 \Lambda \]

(36)

and the scalar curvature for (35) becomes $\tilde{R} = 6\Lambda$. We note that it is essential to have a non-vanishing cosmological constant to have a metric with non-zero curvature.

It is also worthwhile to mention that the Laplace operator does not change its form for the torsion spacetime with metric (35). The Laplace operator in torsion spacetime is

\[ \tilde{\nabla}_\mu \tilde{\nabla}^\mu = \nabla_\mu \nabla^\mu - K^\mu_\nu g^{\sigma\rho} \partial_\sigma = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^\mu_\nu \partial_\nu) - K^\mu_\nu g^{\sigma\rho} \partial_\sigma \]

(37)

where $K^\mu_\nu$ is the contorsion tensor defined by (4). Since the contorsion tensor is zero for the spacetime (35) having torsion (14), the Laplace equation reduces to Riemannian case.
IV. CONCLUSION

By considering Dirac type static classical spin-1/2 field as the source of torsion of the space-time, a solution of 2+1 dimensional Einstein-Cartan equations is obtained. Here, torsion is taken a non-propagating field and couples to the spacetime via spinor field. In the beginning, we started with a general type of the spinor, it could either be Dirac or Majorana. But, during the calculations we see that the torsion tensor is determinative of the type of the spinor and only Dirac type source fulfills the equations. Therefore, the Dirac equation determines the final form of the metric.

In the further works, it will be interesting to work with 3+1 and higher dimensional spacetime and to examine whether the torsion tensor forms a certain structure and restricts the type of the spinor field to the Dirac or Majorana type in general.

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