Non-commutative low dimension spaces and superspaces associated with contracted quantum groups and supergroups

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Abstract

Quantum planes which correspond to all one parameter solutions of QYBE for the two-dimensional case of GL-groups are summarized and their geometrical interpretations are given. It is shown that the quantum dual plane is associated with an exotic solution of QYBE and the well-known quantum $h$-plane may be regarded as the quantum analog of the flag (or fiber) plane. Contractions of the quantum supergroup $GL_q(1|2)$ and corresponding quantum superspace $C_q(1|2)$ are considered in Cartesian basis. The contracted quantum superspace $C_h(1|2; \iota)$ is interpreted as the non-commutative analog of the superspace with the fiber odd part.

1 Introduction

In the quantum group theory solutions of Yang-Baxter equation (YBE) have an important meaning. In particular, the Poincaré-Birkhoff-Witt theorem is hold for such quantum group, which mean that the dimension of the space of homogeneous polynomials is the same as in the commutative case. There are several known solutions of YBE. Some of them have clear group theoretical and geometrical meaning of non-commutative (quantum) analogs of simple Lie groups, algebras and corresponding spaces [1], but the other have not. Moreover a contracted quantum group may have such commutation relations for generators which can not be written in the form of RTT-relations [2] with some matrix $R$ in spite of the fact that the initial quantum
group is defined by the solution $R_q$ of YBE. Therefore it is important to analyse the quantum group theory from geometrical point of view and try to find the commutative analogs especially in the case of contraction. In this paper we perform such analysis for the simplest case of the low dimension quantum (super)spaces and corresponding quantum (super)groups associated with all one parameter $4 \times 4$ solutions of YBE and for the case of quantum superspace $C_q(1|2)$.

The paper is organized as follows. The basic concepts of the quantum group theory are briefly reminded in Section 2. Quantum planes and quantum groups generated by the standard and exotic solutions of QYBE as well as their contractions are regarded in Section 3. Section 4 is devoted to the standard and exotic solutions of GYBE and contractions of the superplanes. Quantum superspace $C_q(1|2)$ and its contractions are regarded in Section 5. Our results are summarized in Conclusion.

2 Solutions of YBE, quantum groups and spaces

According to the well-known theory [1] with each solution $R_q$ of quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (1)$$

is connected a function algebra on quantum group $G_q$ (or simply quantum group), which generators $T = (t_{ij}), i, j = 1, \ldots, N$ are subject of the commutation relations

$$R_q T_1 T_2 = T_2 T_1 R_q, \quad (2)$$

and a function algebra on quantum vector space $C^N_R$ (or simply quantum vector space), which generators $X = (x_i)$ commute (in the case of GL-groups) as follows

$$(\hat{R}_q - q I) X \otimes X = 0, \quad (3)$$

where $T_1 = T \otimes I, T_2 = I \otimes T, \hat{R}_q = PR_q, Pu \otimes v = v \otimes u$ for any $u, v \in C^N_R$. A co-action of $G_q$ on $C^N_R$ is given by

$$\delta(X) = T \hat{\otimes} X, \quad \delta(x_i) = \sum_{k=1}^N t_{ik} \otimes x_k. \quad (4)$$

For $N = 2$ one-parameter solutions of YBE are obtained [2] in the form

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}, \quad (5)$$

where $\lambda = q - q^{-1}, \zeta = q$ correspond to the standard solution of QYBE for quantum group $GL_q(2); \zeta = -q^{-1}$ — to the exotic solution of QYBE for $GL_q(2); \zeta = q^{-1}$ — to the standard solution of GYBE for quantum supergroup $GL_q(1|1)$ and $\zeta = -q$ — to the exotic solution of GYBE for $\tilde{GL}_q(1|1)$ [3].
3 Quantum planes and groups

The standard solution of QYBE is given by (5) with \( \zeta = q \). The generators of \( GL_q(2) \) have the commutation relations

\[
T = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
\]

\[ ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb, [a, d] = \lambda bc \] (6)

and corresponding quantum vector plane \( C_q(2) \) is generated by \( X_t = (x, y)^t \) such that \( xy = qyx \).

In the standard for quantum groups basis (6) in commutative limit \( (q = 1) \) the invariant

\[ \text{inv} = X^t C_0 X \]

is given by the matrix \( (C_0)_{ik} = \delta_{i,3-j} \) with the units on the secondary diagonal, whereas in the really standard in many physical applications Cartesian basis the invariant \( \text{inv} = X^t IX \) is connected with the unit matrix \( I \). Therefore first of all we perform the linear transformation of basis of quantum plane \( C_q(2) \) to the Cartesian one

\[ X = DY, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix}, \] (7)

which induce the similarity transformation \( U = D^{-1}TD \) of the generators of the quantum group \( GL_q(2) \). Commutation relation of Cartesian generators of \( C_q(2) \) are now in the form

\[ [r, p] = iq \frac{q-1}{q+1} (r^2 + p^2) \]

\[ = i(r^2 + p^2) \tanh \frac{z}{2}, \] (8)

where \( q = \exp z \).

Let us substitute \( \iota \hat{r} \) instead of \( r \) and \( \iota v \) instead of \( z \), i.e. \( \exp \iota \nu = 1 + \iota \nu \), where \( \iota \) is nilpotent unit \( \iota^2 = 0 \). Then the commutation relation for the new generators is as follows: \( [\hat{r}, p] = h p^2, h = i \nu \frac{\iota}{2} \) and is coincide with that of \( h \)-plane \( C_h(2) \). In the commutative case the above substitution \( \iota \nu \) correspond to the transition from Euclidean plane to the flag (fiber) plane. Therefore, \( h \)-plane \( C_h(2) \) is the non-commutative (quantum) analog of the flag plane. The associated quantum group \( GL_h(2) \) is obtained from \( GL_q(2) \) by contraction, their Cartesian generators are subject of the commutation relations

\[ U(\iota) = \begin{pmatrix} s & \iota t \\ \iota u & w \end{pmatrix}, \quad [s, w] = 0, \quad [u, t] = h(s + w)(t + u), \]

\[ [s, t] = [u, s] = hs(s - w), \quad [t, w] = [w, u] = hw(s - w) \] (9)

and the following formulae

\[ \delta \left( \begin{pmatrix} p \\ \iota \hat{r} \end{pmatrix} \right) = \left( \begin{pmatrix} s & \iota t \\ \iota u & w \end{pmatrix} \right) \otimes \left( \begin{pmatrix} p \\ \iota \hat{r} \end{pmatrix} \right) = \left( \iota(u \otimes p + w \otimes \hat{r}) \right) \] (10)

describe co-action of \( GL_h(2) \) on \( C_h(2) \).
The exotic solution of QYBE is given by (5) with \( \zeta = -q^{-1} \). The generators of the corresponding quantum plane \( D_q(2) \) commute in the following way [6]

\[
xy = qyx, \quad y^2 = 0.
\] (11)

Dual (or Study) numbers \( x + \iota \tilde{y}, \ x, \tilde{y} \in R \) may be defined by the relations: \( xy = yx, \ y^2 = 0 \), where \( y = \iota \tilde{y} \). Similarly to the complex plane the dual numbers form the dual plane \( D(2) \). Therefore, quantum plane \( D_q(2) \) is the non-commutative (quantum) analog of the dual plane \( D(2) \). If one introduce the generating matrix \( T \) of the associated quantum group \( \tilde{GL}_q(2) \) as in (6), then their commutation relations may be written as

\[
b^2 = c^2 = 0, \ bc = cb, \ ac = qca, \ db = -qbd, \ dc = -qcd, \ [a, d] = \lambda bc.
\] (12)

Let us stress that all generators of \( D_q(2) \) and \( \tilde{GL}_q(2) \) are of the even order and relations \( y^2 = b^2 = c^2 = 0 \) are appear due to their nilpotent nature.

\section{Quantum superplanes and supergroups}

The standard solution of GYBE is given by [5] with \( \zeta = q^{-1} \). The generators of quantum supergroup \( GL_q(1|1) \) have commutation relations

\[
T = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \quad a^2 = \beta^2 = 0, \quad \{a, \beta\} = 0,
\]

\[
a\alpha = q\alpha a, \quad a\beta = q\beta a, \quad b\alpha = q\alpha b, \quad b\beta = q\beta b, \quad [a, b] = \lambda \beta \alpha
\] (13)

and those for the corresponding quantum superplane \( C_q(1|1) \) are \( x\theta = q\theta x, \ \theta^2 = 0 \). Generators \( a, b, x \) are even and \( \alpha, \beta, \theta \) are odd (Grassmann). Performing the superlinear transformation of generators

\[
x = y + \frac{h}{v}\xi, \quad \theta = \xi + \frac{h}{v}y,
\] (14)

where \( h \) is odd, \( h^2 = 0 \) and \( q = 1 + v \), we obtain \( C_q(1|1) \) in the new basis \( y, \xi \):

\[
[y, \xi] = hy^2 + v\xi y, \quad \xi^2 = -h\xi y.
\] (15)

After contraction \( v \rightarrow 0 \) the quantum \( h \)-superplane \( C_h(1|1) \) is achieved

\[
C_h(1|1) = \{y, \xi|y, \xi\} = hy^2, \quad \xi^2 = -h\xi y
\] (16)

which may be interpreted as non-commutative (quantum) analog of the flag superplane [7]. The quantum supergroup \( GL_h(1|1) \):

\[
T = \begin{pmatrix} m & \psi \\ \varphi & n \end{pmatrix}, \quad \psi^2 = 0, \quad \varphi^2 = h\varphi(n - m), \quad \{\psi, \varphi\} = h\psi(n - m),
\]
[m, ϕ] = h(ϕψ − m(n − m)), [n, ϕ] = h(ϕψ − n(n − m)),

[m, ψ] = [n, ψ] = 0, [n, m] = hψ(n − m) \tag{17}

c-o-act on \(C_h(1\mid 1)\) according with (4).

The exotic solution of GYBE is given by (5) with \(ζ = −q\). The corresponding quantum graded plane is \(\tilde{C}_q(1\mid 1) = \{z, μ \mid zμ = qμz\}\) (but no relation \(μ^2 = 0\)) and the quantum graded group \(\tilde{GL}_q(1\mid 1)\) is given by

\[
T = \begin{pmatrix} c & γ \\ δ & d \end{pmatrix}, \quad \{δ, γ\} = 0, \quad cγ = qγc, \quad cδ = qδc,
\]

\[\gamma d = −qdγ, \quad δd = −qdδ, \quad [c, d] = λδγ, \quad (18)\]

but no relations \(γ^2 = δ^2 = 0\). After the similar to (14) transformation

\[
z = t + \frac{h}{v}ν, \quad μ = ν + \frac{h}{v}t, \quad (19)
\]

where \(h\) is odd, \(h^2 = 0, q = 1 + v\), generator \(t\) is even, \(ν\) is odd and by hand \(ν^2 = 0\), we obtain \(\tilde{C}_q(1\mid 1) = \{t, ν \mid [t, ν] = ht^2 + vvt, \quad ν^2 = 0\}\). Contraction \(v \to 0\) gives in result new exotic \(h\)-superplane \(\tilde{C}_h(1\mid 1) = \{t, ν \mid [t, ν] = ht^2, ν^2 = 0\}\).

5 Quantum superspace \(C_q(1\mid 2)\) and its contraction

The standard \(N = 3\) solution of GYBE associated with the quantum supergroup \(GL_q(1\mid 2)\) and quantum superspace \(C_q(1\mid 2)\) is in the form \[9\]

\[
R_q = \begin{pmatrix}
qu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & λ & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & λ & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & −λ & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}
\end{pmatrix}. \tag{20}
\]

According with \[9\] the quantum superspace \(C_q(1\mid 2)\) is generated by even \(x\) and odd \(θ_1, θ_2\) with commutation relations \[9\]

\[xθ_k = qθ_kx, \quad θ_1θ_2 = −qθ_2θ_1, \quad θ_1^2 = 0, \quad k = 1, 2. \tag{21}\]

After the linear transformation of the generators

\[x = x, \quad θ_1 = \frac{1}{\sqrt{2}}(ξ_1 − iξ_2), \quad θ_2 = \frac{1}{\sqrt{2}}(ξ_1 + iξ_2) \tag{22}\]
quantum superspace in Cartesian basis looks as follows

\[ C_q(1|2) = \{ x, \xi | x\xi_k = q\xi_k, \, k = 1, 2, \, \xi_1, \xi_2 \} = 0, \, \xi_1^2 = \xi_2^2 = i\frac{q-1}{q+1}\xi_1\xi_2 \}. \] (23)

A superspace with flag (fiber) odd subspace is obtained \[ \text{by substitution } \xi_2 = i\xi_2 \] and transformation \( q = e^\alpha = e^{iv} \) of deformation parameter need be added in quantum case. As the result we have quantum flag superspace

\[ C_h(1|2; \iota) = \{ x, \xi | x\xi_k = 0, \, \xi_1, \xi_2 \} = 0, \, \xi_1^2 = 0, \xi_2^2 = h\xi_1\xi_2, \, h = i\frac{v}{2} \} \] (24)

with the fiber odd subspace. Generator \( x \) commute with \( \xi_1, \xi_2 \), therefore we may put \( x = 1 \) and the quotient algebra

\[ \hat{C}_h(2) = C_h(1|2; \iota) \{ x = 1 \} = \{ \xi_1, \xi_2 | \xi_1, \xi_2 \} = 0, \, \xi_1^2 = 0, \xi_2^2 = h\xi_1\xi_2 \} \] (25)

is geometrically interpreted as the flag superplane with both odd generators. It is interesting to note that commutation relations \[ (25) \] are the same as for the differentials of \( h \)-plane \( C_h(2) \) (see [10], (6.11–14)). Co-action of consistent with \( C_h(1|2; \iota) \) quantum group \( SL_h(1|2; \iota) \) is given by

\[ U(\iota) \otimes X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & iv \\ 0 & m & k^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \xi_1 \\ \iota\xi_2 \end{pmatrix} = \begin{pmatrix} 1 \otimes 1 \\ k \otimes \xi_1 \\ \iota(m \otimes \xi_1 + k^{-1} \otimes \xi_2) \end{pmatrix}, \] (26)

where generators of \( SL_h(1|2; \iota) \) are subject of commutation relations

\[ [r, k] = [k, m] = h(k^2 - 1), \quad [k^{-1}, r] = [m, k^{-1}] = h(1 - k^{-2}), \]

\[ [r, m] = h(k + k^{-1})(r + m). \] (27)

### 6 Conclusion

We have investigated quantum (super)groups and quantum (super)planes, which correspond to the simplest one-parameter \( GL \)-type solutions of YBE from geometrical point of view. After linear transformation of generators to the Cartesian basis and standard contraction the well-known \( h \)-plane \( C_h(2) = \{ [\hat{r}, \hat{p}] = h\hat{p}^2 \} \) with both even generators is interpreted as the non-commutative flag plane. Quantum plane \( D_q(2) = \{ xy = qyx, \, y^2 \} \), associated with exotic solution of QYBE, may be regarded as the non-commutative dual plane. The standard and exotic solutions of GYBE lead after contractions to the non-commutative flag superplane \( C_h(1|1) = \{ [y, \xi] = h\xi^2, \, \xi^2 = -h\xi y \} \) and new exotic flag superplane \( \hat{C}_q(1|1) = \{ [t, \nu] = ht^2, \, \nu^2 = 0 \} \) with one even and one odd generators. New non-commutative flag superplane \( C\hat{q}(2) = \{ \xi_1, \xi_2 \} = 0, \, \xi_1^2 = 0, \xi_2^2 = h\xi_1\xi_2 \} \) with both odd generators is obtained from \( C_q(1|2) \) by the linear transformation of the odd generators, contraction and quotient on even generator.
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