THE PARTIAL INVERSE NODAL PROBLEM FOR DIFFERENTIAL PENCILS ON A FINITE INTERVAL

YU PING WANG, YITENG HU AND CHUNG TSUN SHIEH

Abstract. In this paper, the partial inverse nodal problem for differential pencils with real-valued coefficients on a finite interval [0, 1] was studied. The authors showed that the coefficients \((q_0(x), q_1(x), h, H_0)\) of the differential pencil \(L_0\) can be uniquely determined by partial nodal data on the right(or, left) arbitrary subinterval \([a, b]\) of \([0, 1]\). Finally, an example was given to verify the validity of the reconstruction algorithm for this inverse nodal problem.

1. Introduction

The differential pencil \(L_\xi = L(q_0, q_1, h, H_\xi)\):

\[
\begin{align*}
  ly &:= -y'' + (q_0(x) + 2\lambda q_1(x))y = \lambda^2 y, \quad x \in (0, 1), \\
  U(y) &:= y'(0) - hy(0) = 0, \\
  V_\xi(y) &:= y'(1) + H_\xi y(1) = 0
\end{align*}
\]

is considered, where \(h, H_\xi \in \mathbb{R}, H_0 \neq H_1, q_\xi(x)\) is a real-valued function, \(q_\xi \in W_2^\xi [0, 1], \xi = 0, 1\). We assume that \(q_1(x) \neq \text{const}\) and \(Q_1(1) = 0\), where

\[
Q_1(x) := \int_0^x q_1(t)dt.
\]

Denote \(\tau = |\text{Im}\lambda|, \varphi(x, \lambda)\) and \(\psi_\xi(x, \lambda)\) the solutions of (1.1) associated with initial conditions

\[
\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \quad \psi_\xi(1, \lambda) = 1, \quad \psi_\xi'(1, \lambda) = -H_\xi.
\]

According to [6, 8], we have the following asymptotic formulae

\[
\begin{align*}
  \varphi(x, \lambda) &:= \cos(\lambda x - Q_1(x)) + O\left(e^{\tau x}\right), \\
  \varphi'(x, \lambda) &:= -\lambda \sin(\lambda x - Q_1(x)) + O(e^{\tau x}),
\end{align*}
\]

2010 Mathematics Subject Classification. 34A55, 34K29, 45J05.
Key words and phrases. Inverse nodal problem, differential pencil, potential, twin-dense nodal subset, arbitrary interval.
This paper is dedicated to Professor V.A. Yurko on the occasion of his 70th birthday.
uniformly with respect to $x \in [0, 1]$ for sufficiently large $|\lambda|$. It is easy to obtain the following equality:

$$\int_0^1 (\psi_\xi \phi) - \phi (\psi_\xi) = < \psi_\xi, \phi > (1, \lambda) - < \psi_\xi, \phi > (0, \lambda),$$

where $< \psi_\xi, \phi > (x, \lambda) := \psi_\xi(x, \lambda) \phi'(x, \lambda) - \psi_\xi'(x, \lambda) \phi(x, \lambda)$ is the Wronskian of $\psi_\xi$ and $\phi$. Denote $\Delta_\xi(\lambda) := < \psi_\xi, \phi > (x, \lambda)$ which is independent of $x$. The set of eigenvalues $\sigma(L_\xi)$ of $L_\xi$ consists of the zeros of the characteristic function $\Delta_\xi(\lambda)$ which can be enumerated as $\{\lambda_{\xi n}\}_{n \in \Lambda}$ (counting with their multiplicities), where $\Lambda = \{\pm 0, \pm 1, \pm 2, \cdots\}$. For sufficiently large $|n|$, $\lambda_{\xi n}$ is real and simple, it also satisfies the following asymptotic formula (please refer to [6, 12] for details)

$$\lambda_{\xi n} = n \pi + \frac{\omega_\xi(1)}{n \pi} + o\left(\frac{1}{n}\right)$$

as $|n| \to \infty$, where

$$\omega_\xi(x) := h + H_\xi + \frac{1}{2} \int_0^x (q_0(t) + q_1^2(t)) dt.$$

Suppose that $x_{\xi n}^j$ are the nodal points of the eigenfunction $\phi(x, \lambda_{\xi n})$ of the pencil $L_\xi$, i.e. $\phi(x_{\xi n}^j, \lambda_{\xi n}) = 0$. Gasymov and Guseinov [12] showed that the differential pencil $L_0$ has a discrete spectrum consisting of simple and real eigenvalues with finitely many exceptions, and the $n$-th eigenfunction $u(x, \lambda_{\xi n})$ has exactly $|n|$ nodes in the interval $(0, 1)$ for sufficiently large $|n|$. Then $x_{\xi n}^j$ satisfy the following relations:

$$0 < x_{\xi n}^1 < x_{\xi n}^2 < \cdots < x_{\xi n}^j < \cdots < x_{\xi n}^n < 1, \quad \text{for} \quad n > 0,$$

$$0 < x_{\xi n}^0 < x_{\xi n}^{j+1} < \cdots < x_{\xi n}^{j+1} < \cdots < x_{\xi n}^{n+1} < 1, \quad \text{for} \quad n < 0.$$ 

and

$$x_{\xi n}^j = \frac{j - \frac{1}{2}}{n} + \frac{Q_1(x_{\xi n}^j)}{n \pi} + \frac{1}{(n \pi)^2} (\omega_\xi(1)x_{\xi n}^j - H_\xi) + o\left(\frac{1}{n^2}\right)$$

uniformly with respect to $j \in \mathbb{Z}$ (refer to Theorem 4 of [6] for details). Denote $X_\xi := \{x_{\xi n}^j\}$. Clearly the nodal set $X_\xi$ is dense on $[0, 1]$ for $\xi = 0, 1$. Buterin [4] firstly showed that $(q_0(x), q_1(x), h, H_0)$ of $L_0$ can be uniquely determined by the nodal set $X_0$. Analogous result for the case of Dirichlet boundary conditions was proved in [5]. The related results are also found in [6]. However, in [33] a more weak statement was established. By using the results in [16], Guo and Wei [17] proved an alternative result involving specification of nodal points on arbitrarily
small subintervals having the midpoint. Note that the Guo-Wei’s result in [17] excludes the case \( \frac{1}{2} \notin (a, b) \). For the case \( \frac{1}{2} \notin (a, b) \), one need some additional information to reconstruct the coefficients \((q_0(x), q_1(x), h, H_0)\) of \(L_0\), for example, in [17], the authors showed all conditions together with information of eigenfunctions in some interior points can lead to a uniqueness theorem. To the best of our knowledge, this partial inverse nodal problem for differential pencils has been not completely solved.

The aim of this paper is to study a partial inverse nodal problem for differential pencils with real-valued coefficients on a finite interval. The authors show that the coefficients \((q_0(x), q_1(x), h, H_0)\) are uniquely determined by the partial nodal information on the right (or, left) arbitrary subinterval \([a, b]\) of \([0, 1]\). This approach is different from that in [17].

Inverse spectral problems for differential pencils have been studied well (see [6, 9, 12, 13, 15, 16, 20, 21, 25, 37, 38] and references therein). We also note that inverse spectral and nodal problems for other differential pencils were studied in [1, 2, 7, 8, 31, 32, 35]. Define the Weyl m-function of \(L_0\)

\[ m(x, \lambda) = -\frac{q'(x, \lambda)}{q(x, \lambda)}. \]

By the known method in [8, 12], one can obtain

**Theorem 1.1.** The Weyl m-function \(m(a_0, \lambda)\) with \(0 < a_0 \leq 1\) of the boundary value problem (1.1)-(1.3) can uniquely determine \((q_0(x), q_1(x))\) on the interval \([0, a_0]\) and \(h\).

Let \(B_\xi = \{ n_\xi, k \}_{k=\infty}^{\infty}\) be a strictly increasing sequence in \(\mathbb{Z}\) for \(\xi = 0\) and 1 and \(B_\xi\) be almost symmetrical with respect to the origin, i.e. that means

\[ B_\xi \subseteq \mathbb{Z}, \quad n \in B_\xi \Rightarrow -n \in B_\xi, \]

with finitely many possible exceptions, we may also assume \(\lambda_{\xi, n_\xi, k} \neq 0\). The nodal subset

\[ W_{B_\xi}([a, b]) := \{ x_{\xi, n_\xi, k}^{j} : x_{\xi, n_\xi, k}^{j} \in X_\xi, n_\xi, k \in B_\xi, a \leq x_{\xi, n_\xi, k}^{j} \leq b \} \]

is called a twin-dense nodal subset on \([a, b]\), i.e.

1. If \(x_{\xi, n_\xi, k}^{j} \in W_{B_\xi}([a, b])\), then either \(x_{\xi, n_\xi, k}^{j+1} \in W_{B_\xi}([a, b])\) or \(x_{\xi, n_\xi, k}^{j-1} \in W_{B_\xi}([a, b])\),

2. \(W_{B_\xi}([a, b]) = [a, b]\).

Denote

\[ S_\xi := \{ \lambda_{\xi, n_\xi, k} : n_\xi, k \in B_\xi, \lambda_{\xi, n_\xi, k} \in \sigma(L_\xi) \} \]
and define
\[ n_{S_\xi}(t) := \begin{cases} 
0 < \lambda_{\xi,n_{\xi,k}} - t < 1, \quad n_{\xi,k} \in B_\xi, \\
- \sum_{t < \lambda_{\xi,n_{\xi,k}} < 0, \ n_{\xi,k} \in B_\xi} 1, 
\end{cases} \]
\[ w_{B_\xi}(\lambda) := \text{p.v.} \prod_{n_{\xi,k} \in B_\xi} \left( 1 - \frac{\lambda}{\lambda_{\xi,n_{\xi,k}}} \right) = \lim_{N \to +\infty} \prod_{k = -N}^{N} \left( 1 - \frac{\lambda}{\lambda_{\xi,n_{\xi,k}}} \right). \tag{1.9} \]

In addition, we assume that the following two conditions hold for \( \tilde{\xi} \)

(i) \[ \lim_{t \to -\infty} \frac{n_{S_\xi}(t)}{t} = \frac{2\theta_\xi}{\pi}. \tag{1.10} \]

(ii) There exist positive numbers \( t_0, \varepsilon, \delta' \) and \( b_\xi \in \mathbb{Z} \) such that
\[ n_{S_\xi}(t) = \begin{cases} 
geq 2\theta_\xi \left( \frac{|t|}{\pi} \right) + b_\xi + \varepsilon + O \left( t^{-\delta'} \right), & t \geq t_0, \\
\leq -2\theta_\xi \left( -\frac{|t|}{\pi} \right) + b_\xi - 2\theta_\xi + O \left( |t|^{-\delta'} \right), & t \leq -t_0 
\end{cases} \tag{1.11} \]

hold for \( S_\xi \), where \([\cdot]\) denotes the floor function.

In the next section, we present and prove some uniqueness theorems in this paper.

2. Main Results and Proofs

From now on, we denote \( \tilde{L}_\xi = L_\xi(\tilde{a}_0, \tilde{q}_1, h, H_\xi) \) the same form as \( L_\xi = L_\xi(q_0, q_1, h, H_\xi) \) but with different coefficients. If a certain symbol \( \zeta \) denotes an object related to \( L_\xi(q_0, q_1, h, H_\xi) \), then the corresponding symbol \( \tilde{\zeta} \) with tilde denotes the analogous object related to \( L_\xi(\tilde{a}_0, \tilde{q}_1, h, H_\xi) \), and \( \tilde{\zeta} = \zeta - \tilde{\zeta} \). At first, we have

**Lemma 2.1.** The coefficients \((q_0(x) - 2\omega_\xi(1), q_1(x))\) on \([a, b]\) can be reconstructed by the given \( W_{B_\xi}([a, b]) \) for each \( \xi = 0, 1 \).

For each \( \xi, \xi = 0, 1 \), the so-called \( W_{B_\xi}([a, b]) = \tilde{W}_{B_\xi}([a, b]) \) means that for any \( x_{\xi,n_{\xi,k}} \in W_{B_\xi}([a, b]) \), then at least one of two formulae holds, i.e.
\[ x_{\xi,n_{\xi,k}} \in \tilde{x}_{\xi,n_{\xi,k}} \quad \text{and} \quad x_{\xi,n_{\xi,k}} = \tilde{x}_{\xi,n_{\xi,k}}, \]
\[ x_{\xi,n_{\xi,k}} = \tilde{x}_{\xi,n_{\xi,k}} \quad \text{and} \quad x_{\xi,n_{\xi,k}} = \tilde{x}_{\xi,n_{\xi,k}}. \]

where \( x_{\xi,n_{\xi,k}} \in W_{B_\xi}([a, b]) \) and \( \tilde{x}_{\xi,n_{\xi,k}} \in \tilde{W}_{B_\xi}([a, b]) \) for \( j = 0, \pm 1 \) in this paper. Denote
\[ M_0 := \max_{a \leq x \leq b} \{q_1(x)\} \quad \text{and} \quad m_0 := \min_{a \leq x \leq b} \{q_1(x)\}. \]

By using the results in [31], the Weyl \( m \)-function and the theory concerning densities of zeros of entire functions (refer to [22, 23]), we shall show that
\textbf{Theorem 2.2.} Let $\frac{1}{2} < a \leq c_0 < c_1 \leq b \leq 1$. Suppose that $W_{B_1}([a, c_1]) = \overline{W}_{B_1}([a, c_1])$, and $W_{B_0}([c_0, b]) = \overline{W}_{B_0}([c_0, b])$, $\omega_1(1) = \tilde{\omega}_1(1)$, and (1.10) and (1.11) hold for each $S_\xi$, $\xi = 0, 1$, where $\theta_0 = 1 - b$ and $\theta_0 + \theta_1 = a$, and for each $x_{\xi, n_{i,k}}^j = x_{\xi, n_{i,k}}^j$, the corresponding eigenvalues $\lambda_{\xi, n_{i,k}}$ and $\tilde{\lambda}_{\xi, n_{i,k}}$ satisfy the inequalities:

$$
\begin{cases}
\lambda_{\xi, n_{i,k}} + \tilde{\lambda}_{\xi, n_{i,k}} > 2M_0 & \text{for all } n_{\xi,k} > 0, \\
\lambda_{\xi, n_{i,k}} + \tilde{\lambda}_{\xi, n_{i,k}} < 2m_0 & \text{for all } n_{\xi,k} < 0.
\end{cases}
$$

Then

$$
q_1(x) = \tilde{q}_1(x), \quad q_0(x) = \tilde{q}_0(x) \quad \text{on} \quad [0, 1], \quad h = \tilde{h} \quad \text{and} \quad H_\xi = \tilde{H}_\xi, \quad \xi = 0, 1. \tag{2.2}
$$

Note that the length $b - a$ of the right subinterval $[a, b]$ of $[0, 1]$ is arbitrarily small. Furthermore we establish a uniqueness theorem for the right arbitrary subinterval $[a, b]$ as follows:

\textbf{Theorem 2.3.} Let $\frac{1}{2} < a \leq c \leq b \leq 1$. Suppose that $W_{B_1}([a, c]) = \overline{W}_{B_1}([a, c])$, and $W_{B_0}([c, b]) = \overline{W}_{B_0}([c, b])$, $\omega_1(1) = \tilde{\omega}_1(1)$, and (1.10) and (1.11) hold for each $S_\xi$, $\xi = 0, 1$, where $\theta_0 = 1 - b$ and $\theta_0 + \theta_1 = a$, and for each $x_{\xi, n_{i,k}}^j = x_{\xi, n_{i,k}}^j$, the corresponding eigenvalues $\lambda_{\xi, n_{i,k}}$ and $\tilde{\lambda}_{\xi, n_{i,k}}$ satisfy (2.1) and $q_0(x) - \tilde{q}_0(x)$ is continuous at $x = c$, then (2.2) holds.

Note that $0 \leq \theta_0 \leq \frac{1}{2}$, then $\theta_0 < a$ for the case $\frac{1}{2} < a < b \leq 1$. Therefore Theorems 2.2 and 2.3 cannot be valid for the cases $S_1 = \emptyset$ and $\frac{1}{2} < a < b \leq 1$. In addition, one can obtain an analogy of Theorem 2.2 and 2.3 for the case $0 \leq a < b < 1/2$ by symmetry. We omit the details here.

If $q_1(x) \equiv 0$ on $[0, 1]$, then the problem (1.1)-(1.3) becomes a classical Sturm-Liouville operator and such problems are well studied (please refer to [3, 10, 11, 14, 15, 18, 24, 25, 26, 27, 28, 29, 30, 34, 36, 38] and the references therein).

In the remaining of this section, we shall present proofs of Theorems 2.2 and 2.3. At first, we show the proof of Lemma 2.1.

\textbf{Proof of Lemma 2.1.} For each $\xi = 0, 1$ and each fixed $x \in [a, b]$, choose $\{x_{\xi, n_{i,k}}^j\}$ such that

$$
\lim_{|k| \to \infty} x_{n_{i,k}}^j = x.
$$

By virtue of (1.8), then there exist the following finite limits and the corresponding equalities hold:

$$
f_\xi(x) := \lim_{|k| \to \infty} n_{\xi,k} \pi \left( x_{\xi, n_{i,k}}^j - \frac{j - \frac{1}{2}}{n_{\xi,k}} \right) = Q_1(x), \tag{2.3}
$$

$$
g_\xi(x) := \lim_{|k| \to \infty} n_{\xi,k}^2 \pi^2 \left( x_{\xi, n_{i,k}}^j - \frac{j - \frac{1}{2}}{n_{\xi,k}} - \frac{Q_1(x_{\xi, n_{i,k}}^j)}{n_{\xi,k} \pi} \right)
$$
\[ = \omega_\xi(x) - \omega_\xi(1)x - H_\xi. \]  

(2.4)

Thus we find the functions \( f_\xi(x) \) and \( g_\xi(x) \) via (2.3) and (2.4). Moreover we reconstruct \((q_0(x) - 2\omega_\xi(1), q_1(x))\) on \([a, b]\) by

\[
q_1(x) = f'_\xi(x), \quad x \in [a, b],
\]

(2.5)

\[
q_0(x) - 2\omega_\xi(1) = 2g'_\xi(x) - q_1^2(x), \quad x \in [a, b],
\]

(2.6)

This complete the proof of Lemma 2.1.

The following are the proofs of our main results.

**Proof of Theorem 2.2.** Since \( W_{B_i}([a, c_1]) = \tilde{W}_{B_i}([a, c_1]) \), and \( W_{B_0}([c_0, b]) = \tilde{W}_{B_0}([c_0, b]) \), we have

\[
\begin{align*}
\begin{cases}
    f_1(x) = \tilde{f}_1(x), & x \in [a, c_1], \\
    f_0(x) = \tilde{f}_0(x), & x \in [c_0, b], \\
    g_1(x) = \tilde{g}_1(x), & x \in [a, c_1], \\
    g_0(x) = \tilde{g}_0(x), & x \in [c_0, b].
\end{cases}
\end{align*}
\]

(2.7)

(2.8)

(2.9)

(2.10)

Therefore (2.7)-(2.10) together with the assumption \( \omega_1(1) = \tilde{\omega}_1(1) \) lead to that

\[
\begin{align*}
\begin{cases}
    \tilde{q}_1(x) = 0 & \text{on } [a, b], \\
    \tilde{q}_0(x) \overset{a.e.}{=} 0 & \text{on } [a, c_1], \\
    \tilde{q}_0(x) \overset{a.e.}{=} 2\tilde{\omega}_0(1) & \text{on } [c_0, b],
\end{cases}
\end{align*}
\]

which together with \( a \leq c_0 < c_1 \leq b \) imply

\[
\begin{align*}
\begin{cases}
    \tilde{q}_1(x) = 0 & \text{on } [a, b], \\
    \tilde{q}_0(x) \overset{a.e.}{=} 0 & \text{on } [a, b].
\end{cases}
\end{align*}
\]

(2.11)

(2.12)

Next we shall show \( \lambda_{\xi, n_{\xi,k}} = \tilde{\lambda}_{\xi, n_{\xi,k}} \) for all \( n_{\xi,k} \in B_\xi \). Note that

\[
\begin{align*}
\begin{cases}
    -\varphi''(x, \lambda_{\xi, n_{\xi,k}}) + (q_0(x) + 2\lambda_{\xi, n_{\xi,k}} q_1(x))\varphi(x, \lambda_{\xi, n_{\xi,k}}) = \lambda_{\xi, n_{\xi,k}}^2 \varphi(x, \lambda_{\xi, n_{\xi,k}}), \\
    \varphi(x_{\xi, n_{\xi,k}}, \lambda_{\xi, n_{\xi,k}}) = \varphi(x_{\xi, n_{\xi,k}}, \lambda_{\xi, n_{\xi,k}}) = 0,
\end{cases}
\end{align*}
\]

(2.13)

(2.14)

and

\[
\begin{align*}
\begin{cases}
    -\tilde{\varphi}''(x, \tilde{\lambda}_{\xi, n_{\xi,k}}) + (\tilde{q}_0(x) + 2\tilde{\lambda}_{\xi, n_{\xi,k}} \tilde{q}_1(x))\tilde{\varphi}(x, \tilde{\lambda}_{\xi, n_{\xi,k}}) = \tilde{\lambda}_{\xi, n_{\xi,k}}^2 \tilde{\varphi}(x, \tilde{\lambda}_{\xi, n_{\xi,k}}), \\
    \tilde{\varphi}(x_{\xi, n_{\xi,k}}, \tilde{\lambda}_{\xi, n_{\xi,k}}) = \tilde{\varphi}(x_{\xi, n_{\xi,k}}, \tilde{\lambda}_{\xi, n_{\xi,k}}) = 0.
\end{cases}
\end{align*}
\]

(2.15)

(2.16)

Equations (2.13)-(2.16) yield to

\[
\int_{x_{\xi, n_{\xi,k}}}^{x_{\xi, n_{\xi,k}}} [\tilde{q}_0(x) + 2(\lambda_{\xi, n_{\xi,k}} q_1(x) - \tilde{\lambda}_{\xi, n_{\xi,k}} \tilde{q}_1(x)) - (\lambda_{\xi, n_{\xi,k}}^2 - \tilde{\lambda}_{\xi, n_{\xi,k}}^2) \varphi(x, \lambda_{\xi, n_{\xi,k}}) \tilde{\varphi}(x, \tilde{\lambda}_{\xi, n_{\xi,k}})] dx = 0.
\]

(2.17)
By virtue of (2.17) together with (2.11) and (2.12), this yields

\[
\left(\lambda_{\xi,n_{\xi,k}} - \bar{\lambda}_{\xi,n_{\xi,k}}\right) \int_{x_{n_{\xi,k}}^{j+1}}^{x_{n_{\xi,k}}^{j}} (2q_1(x) - \lambda_{\xi,n_{\xi,k}} - \bar{\lambda}_{\xi,n_{\xi,k}}) \varphi(x, \lambda_{\xi,n_{\xi,k}}) \bar{\varphi}(x, \bar{\lambda}_{\xi,n_{\xi,k}}) \, dx = 0. \tag{2.18}
\]

Since both \(\varphi(x, \lambda_{\xi,n_{\xi,k}})\) and \(\bar{\varphi}(x, \bar{\lambda}_{\xi,n_{\xi,k}})\) have no zero in the interval \((x_{n_{\xi,k}}^{j}, x_{n_{\xi,k}}^{j+1})\) together with the assumption (2.1), we obtain

\[
\int_{x_{n_{\xi,k}}^{j}}^{x_{n_{\xi,k}}^{j+1}} (2q_1(x) - \lambda_{\xi,n_{\xi,k}} - \bar{\lambda}_{\xi,n_{\xi,k}}) \varphi(x, \lambda_{\xi,n_{\xi,k}}) \bar{\varphi}(x, \bar{\lambda}_{\xi,n_{\xi,k}}) \, dx \neq 0. \tag{2.19}
\]

Therefore (2.18) and (2.19) show that

\[
\lambda_{\xi,n_{\xi,k}} = \bar{\lambda}_{\xi,n_{\xi,k}}, \quad \forall \ n_{\xi,k} \in B_{\xi}. \tag{2.20}
\]

For each \(\lambda_{\xi,n_{\xi,k}}\), by (2.17) and (2.20), we get

\[
\int_{a}^{x_{1,n_{1,k}}^{j}} (\tilde{q}_0(x) + 2\lambda_{1,n_{1,k}}, \tilde{q}_1(x)) \varphi(x, \lambda_{1,n_{1,k}}) \bar{\varphi}(x, \lambda_{1,n_{1,k}}) \, dx

= \langle \bar{\varphi}, \varphi \rangle (x_{1,n_{1,k}}^{j}, \lambda_{1,n_{1,k}}) - \langle \bar{\varphi}, \varphi \rangle (a, \lambda_{1,n_{1,k}}), \tag{2.21}
\]

By virtue of (2.14), (2.16), (2.11) and (2.12), then (2.21) implies

\[
\langle \bar{\varphi}, \varphi \rangle (\alpha_{1,0,n_{0,k}}) = 0, \quad \forall \ n_{1,k} \in B_{1}. \tag{2.22}
\]

Applying the same arguments as the proof of (2.22), we obtain

\[
\langle \bar{\varphi}, \varphi \rangle (c_{0}, \lambda_{0,n_{0,k}}) = 0, \quad \forall \ n_{0,k} \in B_{0}. \tag{2.23}
\]

By virtue of (2.11), (2.12) and (2.23), this yields

\[
\langle \bar{\varphi}, \varphi \rangle (a_{0,n_{0,k}}) = 0, \quad \forall \ n_{0,k} \in B_{0}. \tag{2.24}
\]

Furthermore (2.11), (2.12), (2.23) and (2.24) show that

\[
\langle \bar{\varphi}, \varphi \rangle (a, \lambda_{\xi,n_{\xi,k}}) = 0, \quad \forall \ n_{\xi,k} \in B_{\xi}, \tag{2.25}
\]

\[
\langle \bar{\varphi}, \varphi \rangle (b, \lambda_{\xi,n_{\xi,k}}) = 0, \quad \forall \ n_{0,k} \in B_{0}. \tag{2.26}
\]

Since the functions \(\varphi(x, \lambda_{0,n_{0,k}})\) and \(\psi_0(x, \lambda_{0,n_{0,k}})\) are both eigenfunctions corresponding to the \(n_{0,k}\)-th eigenvalue \(\lambda_{0,n_{0,k}}\) of \(L_0\), there exists a constant \(\beta_0(\lambda_{0,n_{0,k}}) \neq 0\) such that

\[
\psi_0(x, \lambda_{0,n_{0,k}}) = \beta_0(\lambda_{0,n_{0,k}}) \varphi(x, \lambda_{0,n_{0,k}}), \quad \forall \ x \in [0,1]. \tag{2.27}
\]
Consequently (2.26) and (2.27) imply
\[< \bar{\psi}_0, \psi_0 > (b, \lambda_{0,n_0,k}) = 0, \quad \forall n_0,k \in B_0.\] (2.28)

It is easy to prove
\[|< \bar{\varphi}, \varphi > (a, \lambda)| = \mathcal{O}(e^{2ar})\] (2.29)
for sufficiently large \(\lambda\). Define the function
\[K_1(\lambda) := \frac{< \bar{\varphi}, \varphi > (a, \lambda)}{w_{B_0}(\lambda) w_{B_1}(\lambda)}.\] (2.30)

Note that \(\lambda_{\xi,n_{\xi,k}}\) satisfy (1.6), and
\[
\left(1 - \frac{\lambda}{\lambda_{\xi,n_{\xi,k}}}\right) \left(1 - \frac{\lambda}{\lambda_{\xi,-n_{\xi,k}}}\right) = \left(1 - \frac{\lambda}{n_{\xi,k} \pi + O(1)}\right) \left(1 + \frac{\lambda}{n_{\xi,k} \pi - O(1)}\right)
= 1 - \frac{\lambda^2 + O(1) \lambda + O(1)}{(n_{\xi,k} \pi + O(1))(n_{\xi,k} \pi - O(1))}.
\] (2.31)

Therefore (2.31) implies that the locally uniform convergence of the products (1.9) holds. Since \(H_0 \neq H_1\), then \(\sigma(L_0) \cap \sigma(L_1) = \emptyset\), which guarantees that
\[S_0 \cap S_1 = \emptyset.\] (2.32)

Thus (2.25), (1.9) and (2.32) show that the function \(K_1(\lambda)\) is an entire function in \(\lambda\). Next we shall prove \(K_1(\lambda) \equiv 0\). By the classical estimate of Levinson in [23] together with the assumptions in Theorem 2.2 and (1.9), there exists a constant \(C_\xi\) such that
\[
\frac{1}{|w_{B_\xi}(\lambda)|} = \mathcal{O}\left(e^{-2(\theta_0 + \theta_1 - a) \tau + 2\varepsilon r}\right), \quad \forall \lambda \in G_{C_\xi}, \quad r = |\lambda|,
\] (2.33)
where \(\varepsilon > 0\), \(G_{C_\xi} := \{\lambda : |\lambda - \lambda_{n_{\xi,k}}| \leq \frac{1}{8} C_\xi, \lambda_{\xi,n_{\xi,k}} \in S_\xi\}\). Consequently (2.33) and (2.29) imply
\[|K_1(\lambda)| = \mathcal{O}\left(e^{-2(\theta_0 + \theta_1 - a) \tau + 2\varepsilon r}\right), \quad \forall \lambda \in G_{C_\xi} \bigcap G_{C_1}\]
for sufficiently large \(|\lambda|\), where \(\varepsilon\) is arbitrary. Since \(\theta_0 + \theta_1 - a = 0\), the maximum modulus principle shows that
\[|K_1(\lambda)| \leq c_2 e^{2\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C},\] (2.34)
where \(c_2\) is constant. Therefore (2.34) implies that \(K_1(\lambda)\) is of zero exponential type. We say that the notation \(\equiv\) means that both \(
\frac{|w_{B_\xi}(\lambda)|}{|w_{B_\xi}^*(\lambda)|}, \quad \frac{|w_{B_\xi}^*(\lambda)|}{|w_{B_\xi}(\lambda)|}\)
are bounded, where
\[w_{B_\xi}^*(\lambda) = \text{p.v.} \prod_{n_{\xi,k} \in B_\xi} \left(1 - \frac{\lambda}{n_{\xi,k}}\right).\]
By Lemmas 2.5-2.7 in [19], for each \( \delta > 0 \), we have
\[
|w_{B_1}(\lambda)| = |w_{B_1}^*(\lambda)| \quad \text{if} \quad |\lambda - \lambda_{\xi,k}| \geq \delta, \quad |\lambda - n_{\xi,k}| \geq \delta \quad \text{for} \quad n_{\xi,k} \in B_1.
\]
This implies
\[
|w_{B_1}(i y)| = |w_{B_1}^*(i y)|, \quad |y| \to \infty. \tag{2.35}
\]
By calculating (refer to [16, 31] for details), we obtain
\[
\ln |w_{B_1}^*(i y)| = \int_{-\infty}^{\infty} \frac{n_S(t)}{t} \frac{y^2}{y^2 + t^2} \, dt + \int_{1}^{\infty} \frac{n_S(t)}{t} \frac{y^2}{y^2 + t^2} \, dt + O(1) = 2\theta_1 |y| + \varepsilon \ln |y| + O(1).
\]
This shows that
\[
|w_{B_1}^*(i y)| \approx |y| e^{2\theta_1 |y|}. \tag{2.36}
\]
Therefore (2.34)-(2.36) show that
\[
|K_1(i y)| = O\left(\frac{1}{|y|^{2\varepsilon}}\right). \tag{2.37}
\]
By the Phragmén-Lindelöf-type result in [22] together with (2.34) and (2.37), we get
\[
K_1(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}.
\]
This implies
\[
m(a, \lambda) = \tilde{m}(a, \lambda). \tag{2.38}
\]
By virtue of Theorem 1.1 together with (2.38), we have
\[
\tilde{q}_1(x) = 0, \quad \text{and} \quad \tilde{q}_0(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0, a] \quad \text{and} \quad h = \tilde{h}. \tag{2.39}
\]
Define the function
\[
K_2(\lambda) := \frac{<\tilde{\psi}_0, \psi_0> (b, \lambda)}{w_{B_0}(\lambda)}, \tag{2.40}
\]
Consequently (2.23), (2.28) and (2.40) show that the function \( K_2(\lambda) \) is an entire function in \( \lambda \).
Applying the same arguments as the proof of (2.34) and (2.37), we obtain
\[
\begin{cases} 
|K_2(\lambda)| \leq ce^{\varepsilon |\lambda|}, & \lambda \in \mathbb{C}, \\
|K_2(i y)| = O\left(\frac{1}{|y|^{\varepsilon}}\right). 
\end{cases} \tag{2.41}
\]
\[
|K_2(i y)| = O\left(\frac{1}{|y|^{\varepsilon}}\right). \tag{2.42}
\]
By the Phragmén-Lindelöf-type result in [22] together with (2.41) and (2.42) again, we have

\[ K_2(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}. \]

This implies

\[ < \tilde{\psi}_0, \psi_0 > (b, \lambda) \equiv 0, \quad \lambda \in \mathbb{C}. \quad (2.43) \]

The function

\[ m_+ (b, \lambda) = \frac{\psi'_0 (b, \lambda)}{\psi_0 (b, \lambda)}, \quad 0 \leq b < 1 \]

is called the Weyl \( m \)-function of \( L_0 \). Thus (2.43) shows that

\[ m_+ (b, \lambda) = \frac{\psi'_0 (b, \lambda)}{\psi_0 (b, \lambda)} = \tilde{m}_+ (b, \lambda), \quad (2.44) \]

Similar to Theorem 1.1, then it follows from (2.44)

\[ \tilde{q}_1 (x) = 0, \quad \tilde{q}_0 (x) \xrightarrow{a.e.} 0 \quad \text{on } [b, 1] \quad \text{and} \quad H_0 = \tilde{H}_0. \quad (2.45) \]

Moreover we have

\[ H_1 = \tilde{H}_1. \]

This together with (2.11), (2.12), (2.39) and (2.45) implies Theorem 2.2 holds.

Next we prove Theorem 2.3.

**Proof of Theorem 2.3.** By virtue of \( W_{B_1} ([a, c]) = \tilde{W}_{B_1} ([a, c]), \) and \( W_{B_0} ([c, b]) = \tilde{W}_{B_0} ([c, b]), \) this together with Lemma 2.1 and the assumption \( \omega_1 (1) = \tilde{\omega}_1 (1) \) yields

\[
\begin{aligned}
\tilde{q}_1 (x) &= 0 \quad \text{on } [a, b], \\
\tilde{q}_0 (x) &\xrightarrow{a.e.} 0 \quad \text{on } [a, c], \\
\tilde{q}_0 (x) &\xrightarrow{a.e.} 2\tilde{\omega}_0 (1) \quad \text{on } [c, b].
\end{aligned}
\]

Therefore (2.47) and (2.48) together with the function \( q_0 (x) - \tilde{q}_0 (x) \) is continuous at \( x = c \), this yields

\[ \tilde{\omega}_0 (1) = 0. \quad (2.49) \]

Consequently (2.47), (2.48) and (2.49) imply that

\[ \tilde{q}_0 (x) \xrightarrow{a.e.} 0 \quad \text{on } [a, b]. \quad (2.50) \]

Similar to the proof of (2.20) in Theorem 2.2, we have

\[
\lambda_{\xi, n_{\xi, k}} - \tilde{\lambda}_{\xi, \tilde{n}_{\xi, k}} = 0, \quad \forall \ n_{\xi, k} \in B_{\xi}, \quad \xi = 0, 1.
\quad (2.51)
\]
Modifying the proof in Theorem 2.2 simply together with (2.46), (2.50) and (2.51), we obtain

\[ q_1(x) = 0, \quad q_0(x) \equiv 0 \quad \text{on} \quad [0, 1], \quad h = \tilde{h}, \quad \text{and} \quad H_\xi = \tilde{H}_\xi, \quad \xi = 0, 1. \]

Finally, we shall present an example for reconstruction of \((q_0(x), q_1(x), h, H_0)\) from the twin-dense nodal subset \(W_{B_0}([0, 1])\).

**Example 2.4.** Let \(W_{B_0}([0, 1]) = \{x_{n_k}^j\}, B_0 \subseteq \mathbb{Z}\), be the twin-dense nodal subset of the pencil \(L(q_0, q_1, h, H_0)\) and \(\int_0^1 q_0(t)dt = \frac{1}{2}\), where

\[ x_{n_k}^j = \frac{j - \frac{1}{2} - \frac{j - \frac{1}{2}}{n_k^2 \pi} + \left(\frac{j - \frac{1}{2}}{n_k \pi}\right)^2 + o\left(\frac{1}{n_k^2}\right)}{2}, \quad \forall n_k \in B_0, \tag{2.52} \]

where

\[ \omega_0(x_{n_k}^j) = \frac{2(j - \frac{1}{2})^3}{3n_k^3} - \frac{3(j - \frac{1}{2})^2}{4n_k^2} - \frac{23(j - \frac{1}{2})}{12n_k} + 1, \]

reconstruct \((q_0(x), q_1(x), h, H_0)\).

For each fixed \(x \in [0, 1]\), we choose \(x_{n_k}^j\) such that \(\lim_{k \to \infty} j - \frac{1}{2} n_k = x\). By (2.52), we have

\[ f(x) := \lim_{|k| \to \infty} n_k \pi \left( x_{n_k}^j - \frac{j - \frac{1}{2}}{n_k} \right) = \lim_{|k| \to \infty} \left( \frac{(j - \frac{1}{2})^2}{n_k^2} - \frac{j - \frac{1}{2}}{n_k} + O\left(\frac{1}{n_k}\right) \right) = x^2 - x = \int_0^x q_1(x)dt, \]

which implies

\[ q_1(x) = 2x - 1, \quad x \in [0, 1]. \tag{2.53} \]

By (2.53), we obtain

\[ g(x) := \lim_{|k| \to \infty} n_k^2 \pi^2 \left( x_{n_k}^j - \frac{j - \frac{1}{2}}{n_k} + \frac{j - \frac{1}{2}}{n_k^2 \pi} - \frac{(j - \frac{1}{2})^2}{n_k^3 \pi} \right) = \lim_{|k| \to \infty} \left( \frac{2(j - \frac{1}{2})^3}{3n_k^3} - \frac{3(j - \frac{1}{2})^2}{4n_k^2} - \frac{23(j - \frac{1}{2})}{12n_k} + 1 + o(1) \right) = \frac{2}{3} x^3 - \frac{3}{4} x^2 - \frac{23}{12} x + 1 = \omega_0(x) - \omega_0(1)x - H_0. \tag{2.54} \]

Therefore (2.54) shows that

\[ h = g(0) = 1, \quad \text{and} \quad H_0 = -g(1) = 1. \tag{2.55} \]
By (2.53)-(2.55) together with $\int_0^1 q_0(t) \, dt = \frac{1}{2}$, we get
\[
q_0(x) \overset{a.e.}{=} 4 + 2g'(x) - (2x - 1)^2 + \int_0^1 (2x - 1)^2 \, dx + \frac{1}{2} = x, \quad x \in [0, 1].
\] (2.56)
Thus the coefficients $(q_0(x), q_1(x), h, H_0)$ are reconstructed by (2.53), (2.55) and (2.56).

**Acknowledgements**

The authors would like to thank the anonymous referees for valuable suggestions, which help to improve the readability and quality of the paper.

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Department of Applied Mathematics, Nanjing Forestry University, Nanjing, 210037, Jiangsu, People's Republic of China.

E-mail: ypwang@njfu.com.cn

Department of Applied Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, Jiangsu, China.

E-mail: huyiteng6@163.com

Department of Mathematics, Tamkang University, Danshui Dist., New Taipei City, 25137, Taiwan(R.O.C.).

E-mail: ctshieh@mail.tku.edu.tw