CONFIGURATIONS OF RANK-40r EXTREMAL EVEN UNIMODULAR LATTICES \((r = 1, 2, 3)\)

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Abstract. We show that if \(L\) is an extremal even unimodular lattice of rank 40r with \(r = 1, 2, 3\) then \(L\) is generated by its vectors of norms 4r and 4r+2. Our result is an extension of Ozeki’s analogous result for the case \(r = 1\).

1. Introduction

A lattice of rank \(n\) is a free \(\mathbb{Z}\)-module of rank \(n\) equipped with a positive-definite inner product \((\cdot, \cdot) : L \times L \to \mathbb{R}\). The dual of \(L\), denoted \(L^*\), is the set

\[ L^* = \{ y \in L \otimes \mathbb{R} : \forall x \in L, (x, y) \in \mathbb{Z} \}, \]

which itself forms a lattice of the same rank as \(L\). For a lattice vector \(x \in L\), we call \((x, x)\) the norm of \(x\). A lattice \(L\) is integral if \((x, x') \in \mathbb{Z}\) for all \(x, x' \in L\), i.e. if and only if \(L \subseteq L^*\). An integral lattice is said to be unimodular if it is self-dual \((L = L^*)\).

A lattice \(L\) is called even if and only if every lattice vector has an even integer norm, i.e. \((x, x) \in 2\mathbb{Z}\) for \(x \in L\). An even lattice is automatically integral by the familiar parallelogram identity, \(2(x, x') = (x + x', x + x') - (x, x) - (x', x')\).

Lattices that are simultaneously even and unimodular are especially rare. Indeed, such a lattice’s rank must be divisible by 8. Sloane proved that if \(L\) is an even unimodular lattice of rank \(n\) then the minimal (nonzero) norm in \(L\) is bounded by

\[ \min_{x \in L \setminus \{0\}} (x, x) \leq 2[n/24] + 2 \]

(see [2] p. 194, Cor. 21]). An even unimodular lattice of rank \(n\) is called extremal if it attains the bound (1).

Ozeki [6, 8] showed that if \(L\) is an extremal even unimodular lattice of rank 32 or 48 then \(L\) is generated by its vectors of minimal norm. The first author [5] showed analogous results for extremal even unimodular lattices of ranks 56, 72, and 96. In a similar vein, Ozeki [7] showed that if \(L\) is extremal even unimodular of rank 40, then \(L\) is generated by its vectors of norms 4 and 6. Here, we extend and slightly simplify Ozeki’s methods, recovering Ozeki’s rank-40 result and obtaining analogous results for extremal even unimodular lattices of ranks 80 and 120.
2. Modular Forms and Theta Series

We will use the notation \( \mathcal{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) for the upper half plane of complex numbers. A modular form of weight \( k \) for the group \( \text{PSL}_2(\mathbb{Z}) \) is a holomorphic function \( f : \mathcal{H} \to \mathbb{Z} \) which is holomorphic at \( i\infty \) and satisfies
\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)
\]
for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) \). If a modular form \( f \) vanishes at \( z = i\infty \), it is called a cusp form.

Let \( M_k \) and \( M_k^0 \) be the \( \mathbb{C} \)-vector spaces of modular forms and cusp forms of weight \( k \) respectively. It is known that the Eisenstein series
\[
E_4(z) = 1 + 240e^{2\pi i z} + 2160e^{4\pi i z} + 6720e^{6\pi i z} + \cdots
\]
and
\[
E_6(z) = 1 - 504e^{2\pi i z} - 16632e^{4\pi i z} - 122976e^{6\pi i z} - \cdots,
\]
which are modular forms of weights 4 and 6 respectively, freely generate the spaces \( M_k \) in the sense that any nonzero modular form can be written uniquely as a weighted homogeneous polynomial in \( E_4 \) and \( E_6 \). This implies that \( \dim(M_k) = 0 \) for \( k \) odd, negative, or \( k = 2 \); that \( \dim(M_{2k}) = 1 \) and \( \dim(M_{2k}^0) = 0 \) for \( k = 0, 2 \leq k \leq 5 \) and \( k = 7 \); and that multiplication by the weight-12 modular form \( \Delta = 12^{-3} (E_4^3 - E_6^2) \) defines an isomorphism \( M_{k-12} \cong M_k^0 \). More information on the theory of modular forms for \( \text{PSL}_2(\mathbb{Z}) \) can be found in [9].

The theta function \( \Theta_L : \mathcal{H} \to \mathbb{Z} \) associated to a lattice \( L \) is defined by
\[
\Theta_L(z) = \sum_{x \in L} e^{\pi i (x,x)z},
\]
it is a generating function encoding the norms of \( L \)'s vectors. For a homogeneous harmonic polynomial \( P \in \mathbb{C}[x_1, \ldots, x_n] \), i.e. a homogeneous polynomial for which \( \sum_{j=1}^n \frac{\partial^2 P}{\partial x_j^2} \equiv 0 \), we define the weighted theta series \( \Theta_{L,P} \) by
\[
\Theta_{L,P}(z) = \sum_{x \in L} P(x) e^{\pi i (x,x)z}.
\]
As shown in [9, 10], if \( L \) is an even unimodular lattice of rank \( n \) then \( \Theta_L \) is a modular form of weight \( \frac{n}{2} \), and if in addition \( P \) is a homogeneous harmonic polynomial of degree \( d \), then \( \Theta_{L,P} \) is a modular form of weight \( \frac{n}{2} + d \).

3. Main Result

We denote by \( P_{d,x_0}(x) \) the “zonal spherical harmonic polynomial” of degree \( d \), related to the Gegenbauer polynomial by
\[
P_{d,x_0}(x) = G_d((x, x_0), ((x, x_)(x_0, x_0)))^{1/2},
\]
where \( G_d(\cdot, \cdot) \) is the homogeneous polynomial of degree \( d \) such that \( G_d(t, 1) \) is the Gegenbauer polynomial of degree \( d \) evaluated at \( t \) [11].

We let \( L \) be an extremal even unimodular lattice of rank \( 40r \), and adopt the notation used by Ozeki in [7]: For an even unimodular lattice \( L \), we denote by \( \Lambda_{2m}(L) \) the set of vectors in \( L \) having norm \( 2m \). We denote by \( \mathcal{L}_{2m}(L) \) the sublattice of \( L \) generated by \( \Lambda_{2m}(L) \), and similarly denote by \( \mathcal{L}_{2m_1+2m_2}(L) \) the sublattice of \( L \) generated by \( \Lambda_{2m_1}(L) \cup \Lambda_{2m_2}(L) \).
We define \( a(2k, L) := |\Lambda_{2k}(L)| \). It is clear that the theta series \( \Theta_L \) is given by
\[
\Theta_L(z) = \sum_{k=0}^{\infty} a(2k, L) e^{2k \pi i z}.
\]
We note that
\[
4r = 2[5r/3] + 2 = \min\{2k > 0 : a(2k, L) \neq 0\}
\]
is the minimal norm of vectors in \( L \) and use the notation
\[
\begin{align*}
N_j(x) &= |\{y \in \Lambda_{4r}(L) : (x, y) = j\}|, \\
M_j(x) &= |\{y \in \Lambda_{4r+2}(L) : (x, y) = j\}|.
\end{align*}
\]
Using the involution \( y \mapsto -y \) of \( \Lambda_m(L) \), we see that we have \( N_j(x) = N_{-j}(x) \) and \( M_j(x) = M_{-j}(x) \) for any \( j \in \mathbb{R} \) and \( x \in L \oplus \mathbb{R} \).

We will show the following configuration result, which directly extends Ozeki’s [7] result for extremal even unimodular lattices of rank 40:

**Theorem 3.1.** For \( r = 1, 2, 3 \) and \( L \) extremal even unimodular of rank \( 40r \), we have \( L = \mathcal{L}_{4r+(4r+2)}(L) \).

**Proof.** We partition \( L \) into its equivalence classes modulo \( \mathcal{L}_{4r+(4r+2)}(L) \). We need only show that any class \( [x] \in L/\mathcal{L}_{4r+(4r+2)}(L) \) is represented by a vector \( x_0 \in [x] \) with \( (x_0, x_0) \leq 4r + 2 \).

Now, we suppose there exists some equivalence class \( [x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L) \) where \( x_0 \neq 0 \) is a representative of minimal norm with \( (x_0, x_0) = 2t \) for some \( t \geq 2r + 2 \). We have the inequality
\[
|x_0, x| \leq 2r \text{ for all } x \in \Lambda_{4r}(L),
\]
as \( x_0 \) is not minimal in \( L \) whenever \( (x_0, \pm x) > 2r \) since the vector \( x \mp x_0 \) has norm
\[
(x \mp x_0, x \mp x_0) = (x, x) \mp 2(x, x_0) + (x_0, x_0) < (x_0, x_0).
\]
Similarly, we have
\[
|x_0, x| \leq 2r + 1 \text{ for all } x \in \Lambda_{4r+2}(L).
\]
From (3) and (4), we have the equations
\[
\begin{align*}
\sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2k} &= \sum_{j=1}^{2r} 2 \cdot j^{2k} \cdot N_j(x_0), \\
\sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2k} &= \sum_{j=1}^{2r+1} 2 \cdot j^{2k} \cdot M_j(x_0),
\end{align*}
\]
for all \( k > 0 \).

We extract from the theta series \( \Theta_L \) of \( L \) the coefficients \( a(4r, L) \) and \( a(4r+2, L) \).

We observe immediately from (5) and (6) that
\[
\begin{align*}
\sum_{x \in \Lambda_{4r}(L)} (x, x_0)^0 &= a(4r, L), \\
\sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^0 &= a(4r + 2, L).
\end{align*}
\]
Since \( L \) is even unimodular of rank 40, we have \( \Theta_{L, P_d, x_0} \in M_{20r+d}^0 \) for any \( d > 0 \).

By comparing power-series coefficients, we then observe

\begin{align*}
(9) & \quad \Theta_{L, P_d, x_0} \equiv 0 \text{ for } d \in \{2, \ldots, 4r-2, 4r+2\}, \\
(10) & \quad \Theta_{L, P_r, x_0} \equiv c_1 \Delta^{2r} \text{ for a constant } c_1, \\
(11) & \quad \Theta_{L, P_{r+4}, x_0} \equiv c_2 E_4 \Delta^{2r} \text{ for a constant } c_2.
\end{align*}

From (9), we obtain the equations

\begin{align*}
(12) & \quad \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2d} = a(4r, L) \frac{1 \cdot 3 \cdots (2d-1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r)^d t^d \\
(13) & \quad \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2d} = a(4r + 2, L) \frac{1 \cdot 3 \cdots (2d-1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r + 4)^d t^d,
\end{align*}

for \( d \in \{2, \ldots, 4r-2, 4r+2\} \). We obtain from (10)

\begin{align*}
(14) & \quad \sum_{x \in \Lambda_{4r+2}(L)} P_{4r, x_0}(x) = c_{4r} \sum_{x \in \Lambda_{4r}(L)} P_{4r, x_0}(x),
\end{align*}

where \( \Delta^{4r} = e^{(4r)\pi i z} + c_{4r} e^{(4r+1)\pi i z} + O(e^{(4r+2)\pi i z}) \).

Similarly, (11) gives

\begin{align*}
(15) & \quad \sum_{x \in \Lambda_{4r+4}(L)} P_{4r+4, x_0}(x) = c_{4r+4} \sum_{x \in \Lambda_{4r}(L)} P_{4r+4, x_0}(x),
\end{align*}

where \( E_4 \Delta^{4r} = e^{(4r)\pi i z} + c_{4r+4} e^{(4r+1)\pi i z} + O(e^{(4r+2)\pi i z}) \).

Combining the equations (9), (10), (12), (13), (14), and (15) with (11) and (12), we obtain a system of 4r + 4 homogeneous linear equations in the 4r + 3 unknowns

\[ N_0(x_0), \ldots, N_{2r}(x_0), M_0(x_0), \ldots, M_{2r+1}(x_0). \]

At this stage, we diverge from our natural generalization of Ozeki’s original methods and obtain the (extended) determinants of these inhomogeneous linear systems; these determinants must vanish because the system is overdetermined.

For \( r = 1, 2, 3 \), these determinants are respectively

\begin{align*}
(16) & \quad 2^{25} 3^5 5^8 7^4 11^4 13^3 19^6 23^3 \cdot (t - 2) \cdot t \cdot (6t - 13) \cdot (10t^2 - 55t + 77), \\
(17) & \quad 2^{132} 3^{37} 5^{16} 7^{10} 11^{6} 13^{10} 23^{3} 41^{8} 43^{6} 47^{3} \cdot (t - 4) \cdot t \cdot Q_2(t), \\
(18) & \quad 2^{244} 3^{48} 5^{26} 7^{13} 11^{7} 13^{7} 17^{6} 23^{4} 31^{11} 37^{3} 59^{14} 61^{11} 67^{7} 71^{3} 73^{3} \cdot (t - 6) \cdot t \cdot Q_3(t),
\end{align*}

where \( Q_2(t) \) is the irreducible quintic

\[ 10768 t^5 - 242280 t^4 + 2202310 t^3 - 10101795 t^2 + 23361877 t - 21771246 \]

and \( Q_3(t) \) is the irreducible septic

\[ 199898826740569099355 t^7 - 892881426107875310430 t^6 + 172580396012226541515335 t^5 - 187053310321121904306075 t^4 + 1227398249908229181423784 t^3 - 4874010945909263810320032 t^2 + 10840974078436271024624064 t - 10414527769923133690990080. \]
In each case, there are no integer solutions \( t \geq 2r + 2 \). However, we had assumed the existence of an equivalence class

\[
[x_0] \in L/L_{4r+(4r+2)}(L)
\]

with minimal-norm representative \( x_0 \neq 0 \) having \( (x_0, x_0) = 2t \) for integral \( t \geq 2r+2 \); since no such \( t \) exists, all equivalence classes must be generated by vectors having norms \( 4r \) and \( 4r + 2 \).

\[\square\]

### 4. Concluding Remarks

A quick inspection will show that our results are the only possible immediate extensions of Ozeki’s methods. In the cases \( r \geq 4 \), it is not possible to extract sufficiently many linear conditions by these exact techniques, as the dimensions of the relevant spaces of cusp forms grows too large.

However, using different analysis, Elkies [4] has shown a stronger result than our Theorem 3.1 in the \( r = 3 \) case: If \( L \) is an extremal unimodular lattice of rank 120 then \( L = L_{12}(L) \). This result for rank-120 lattices is analogous to Ozeki’s [6] results in dimensions 32 and 48, and to the first author’s [5] results in dimensions 56, 72, and 96.

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