Energy concentrations and Type I blow-up for the 3D Euler equations

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Abstract

We exclude Type I blow-up, which occurs in the form of atomic concentrations of the \( L^2 \) norm for the solution of the 3D incompressible Euler equations. As a corollary we prove nonexistence of discretely self-similar blow-up in the energy conserving scale.

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1 Introduction

We consider the \( n \)-dimensional Euler equations in \( \mathbb{R}^n \times (0, +\infty) \)

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v &= -\nabla p, & \nabla \cdot v &= 0, \\
v(x, 0) &= v_0(x),
\end{align*}
\]

where \( v = (v_1(x, t), \ldots, v_n(x, t)), (x, t) \in \mathbb{R}^n \times (0, +\infty) \). For the Cauchy problem of the system (1.1) the local well-posedness in the setting of standard Sobolev space \( H^m(\mathbb{R}^n) \), \( m > \frac{n}{2} + 1 \), is proved by Kato in [19]. The question of finite time blow-up of such local in time classical solution, however, is an outstanding open problem in the mathematical fluid mechanics(see e.g. [23, 10] for an introduction and surveys of partial results on the problem, and [16, 17, 20, 22] for the related numerical works). In this direction of study there are also well-known results on the blow-up criterion [2, 12, 13, 21], where the authors deduced various sufficient conditions for the blow-up. We also mention a recent result by Tao [29], which shows the blow-up for a model equation having similar conservation properties to the Euler system.
The aim of the present paper is to study the possibility of the finite time blow-up in terms of the energy concentrations in the 3D Euler equations. The phenomena of $L^2$ norm concentration at the blow-up time is well-known in the other nonlinear evolution equations. For example in the nonlinear Schrödinger equation it is found that there exists a solution which shows that the mass($L^2$ norm of the solution) is concentrating in the form of finite sum of Dirac measures at the blow-up time [24, 25]. Similarly, in the chemotaxis equation the $L^1$ norm of solution is shown to be evolved into Dirac measures in the finite time for a sufficiently large initial data [18]. We also find that there exists a study of the energy concentration for the Navier-Stokes equations, in the context different from ours in [1].

In our case of the 3D Euler system, under Type I condition for the velocity gradient we are able to exclude the atomic concentrations of velocity $L^2$ norm at the possible blow-up time. This means that there exists no concentration of the energy into isolated points in $\mathbb{R}^n$ at the possible blow-up time if we assume Type I condition for the blow-up rate. As an immediate corollary of this result we exclude the discretely self-similar (DSS) blow-up in the energy conserving scale.

Let us denote by $L^2_{\sigma}(\mathbb{R}^n)$ the closure of \{ $\varphi \in C_c^{\infty}(\mathbb{R}^n) \mid \nabla \cdot \varphi = 0$ \} in $L^2(\mathbb{R}^n)$. Given a domain $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{M}(\Omega)$ the space of all bounded Radon measures $\mu \in C_c^0(\Omega)^*$. The space $\mathcal{M}(\Omega)$ will be equipped with the norm

$$\| \mu \|_{\mathcal{M}} = \sup_{\varphi \in C_c^{\infty}(\Omega), \max_{\Omega} |\varphi| \leq 1} \int \varphi d\mu.$$ 

In particular, by $\mathcal{M}^+(\Omega)$ we denote the subspace of all nonnegative $\mu \in \mathcal{M}(\Omega)$, i.e.

$$\int_{\Omega} \phi d\mu \geq 0 \quad \forall \phi \in C_c^0(\Omega), \quad \text{with} \quad \phi \geq 0.$$ 

If $f \in L^\infty(a, b; L^1(\Omega))$, $-\infty < a < b < +\infty$, by $\mathcal{M}_f(b)$ we denote the set of all $\sigma_0 \in \mathcal{M}(\Omega)$ such that there exists a sequence $\{s_k\}$ in the set of Lebesgue points of $f(\cdot)$ such that $s_k \to b$ as $k \to +\infty$ and

$$f(s_k)dx \to \sigma_0 \quad \text{weakly-}^* \text{ in } \mathcal{M}(\Omega) \quad \text{as } \ k \to +\infty.$$ 

Here $t \in (a, b]$ is called a Lebesgue point of $f(\cdot)$ if

$$\frac{1}{h} \int_{t-h}^{t} f(s)ds \to f(t) \quad \text{in } L^1(\Omega) \quad \text{as } \ h \to 0^+.$$ 

Note that due to Lebesgue’s differentiation theorem for the Bochner integrable functions (see e.g. in [32, Theorem 2, pp. 134]) almost every $t \in (a, b]$ is a Lebesgue point of $f(\cdot)$.

For simplicity in the discussion below we consider our time interval $(-1, 0)$, and fix $t = 0$ as the possible blow-up time. Our main theorem of this paper is the following.
Theorem 1.1. Let \( \Omega \subset \mathbb{R}^n \) be a domain. Let \( v \in L^\infty(-1,0;L^2_g(\Omega) \cap L^\infty_{\text{loc}}([-1,0),W^{1,\infty}(\Omega)) \) be a solution to (1.1) in \( \Omega \times (-1,0) \) satisfying the following Type I blow-up condition at \( t = 0 \)

\[
\sup_{t \in (-1,0)} (-t) \| \nabla v(t) \|_{L^\infty(\Omega)} < +\infty.
\]

Then every measure \( \sigma_0 \in \mathcal{M}_{[v]^2}(0) \) has no atoms, i.e.

\[
\sigma_0(\{x\}) = 0 \quad \forall x \in \Omega.
\]

If in addition, \( v(t) \to v_0 \) weakly in \( L^2(\Omega) \) as \( t \to 0^− \) for some \( v_0 \in L^2(\Omega) \), then \( \mathcal{M}_{[v]^2}(0) = \{ \sigma_0 \} \),

\[
|v(t)|^2dx \to \sigma_0 \text{ weakly-* in } \mathcal{M}(\Omega) \text{ as } t \to 0,
\]

and \( \sigma_0 \) has no atoms.

In the case \( \Omega = \mathbb{R}^n \) in the above theorem the fact \( p \in L^3_{\text{loc}}(\mathbb{R}^n \times (-1,0)) \) follows from by the Calderón-Zygmund inequality and the velocity-pressure relation, \( \Delta p = -\sum_{i,k=1}^n \partial_j \partial_k(v_jv_k) \). Therefore, as an immediate consequence of Lemma 2.3 (with \( g = v, f = 0 \)) below the set \( \mathcal{M}_{[v]^2}(0) \) contains only one element \( \sigma_0 \in \mathcal{M}(\mathbb{R}^n) \), which gives the following.

Corollary 1.2. Let \( v \in L^\infty(-1,0;L^2_g(\mathbb{R}^n)) \cap L^\infty_{\text{loc}}([-1,0);W^{1,\infty}(\mathbb{R}^n)) \) be a solution of the Euler equations (1.1) satisfying (1.2) with \( \Omega = \mathbb{R}^n \). Then, there exists \( \sigma_0 \in \mathcal{M}(\mathbb{R}^n) \) such that

\[
|v(t)|^2dx \to \sigma_0 \text{ weakly-* in } \mathcal{M}(\mathbb{R}^n) \text{ as } t \to 0^-,
\]

and \( \sigma_0(\{x\}) = 0 \) for all \( x \in \mathbb{R}^n \).

Remark 1.3. In particular, under Type I condition the limiting measure of the form

\[
\sigma_0 = \sum_{k=1}^\infty c_k \delta_{x_k} + f dx \text{ with a sequence } \{c_k\}_{k=1}^\infty \text{ of nonnegative constants and } f \in L^1_{\text{loc}}(\mathbb{R}^n),
\]

is excluded contrary to the case of the nonlinear Schrödinger equation[24, 25] and the chemotaxis equation[18]. Currently, we are not able to exclude the possibility of energy concentration into a set of positive Hausdorff dimension under Type I condition, which would be an interesting subject for future study.

Remark 1.4. We note that for \( n = 3 \) we have

\[
L^\infty(-1,0;L^2(\mathbb{R}^3)) \cap L^\infty_{\text{loc}}([-1,0);W^{1,\infty}(\mathbb{R}^3)) \subset L^3_{\text{loc}}([-1,0);B^{1+s}_{3,\infty}(\mathbb{R}^3)), \quad \forall s \in \left(0, \frac{2}{3}\right]
\]

which is the energy conserving class for the weak solutions to the Euler equations \( v(\cdot, t) \) for \( t \in [-1,0) \) as studied in [11]. As \( t \to 0 \), however, we cannot say anything about the energy conservation, and the existence of a definite particle trajectory map up to \( t = 0 \). Therefore, the energy concentration to a general measure zero set at the blow-up time cannot be excluded by a naive application of the volume preserving property of the particle trajectory map.
In order to discuss an implication of the above theorem on the scenario of the discretely self-similar blow-up we first recall that a solution \((v,p)\) of the Euler equations is self-similar if there exists \(\alpha \neq -1\) such that

\[
(1.6) \quad v(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \quad \forall (x,t) \in \mathbb{R}^n \times (0, +\infty)
\]

for all \(\lambda > 1\). The discrete self-similarity is a more general concept; a solution \((v,p)\) of the Euler equations is called discretely self-similar (we say \(\lambda\)-DSS), if there exists \(\alpha \neq -1\) and \(\lambda > 1\) such that (1.6) holds. There have been previous studies on the exclusion of the scenario of self-similar blow-up [5, 6, 7] in the Euler equations. Note that discretely self-similar solutions preserve the energy only if \(\alpha = n/2\), which is called the energy conserving scale. The previous studies on the exclusion of discretely self-similar blow-up scenarios were mostly done in the other cases than the energy conserving scale, for which the solution belongs to \(L^q(\mathbb{R}^n)\), \(q \neq 2\), mainly due to the difficulties to prove Liouville type theorems for the corresponding profile equations. The question of nonexistence of self-similar and/or discretely self-similar singularities in the 3D Euler equations has been open only in this case of energy conserving scale, while all the other cases are excluded under suitable decay conditions at infinity on the profiles [5, 6, 7].

As proved in Section 5 below the \(\lambda\)-DSS blow-up in the case \(\alpha = n/2\) is a special case of the one point energy concentration at the time of blow-up. Therefore as a consequence of Corollary 1.2 we exclude the scenario of DSS blow-up in the energy conserving scale as follows.

**Corollary 1.5.** Let \(v \in L^\infty(-1,0; L^2_{\sigma}(\mathbb{R}^n)) \cap L^\infty_{loc}(-1,0; W^{1,\infty}(\mathbb{R}^n))\) be a solution of the Euler equations (1.1) satisfying (1.2). If \(v\) is \(\lambda\)-DSS solution with the energy conserving scale, i.e. if there exists \(\lambda > 1\) such that

\[
(1.7) \quad v(x,t) = \lambda^{\frac{n}{2}} v(\lambda x, \lambda^{\frac{n+2}{2}} t) \quad \forall (x,t) \in \mathbb{R}^n \times (-1, 0).
\]

Then \(v \equiv 0\).

The paper is organized as follows. In Section 2 we recall the notion of local pressure for bounded domains and exterior domains as well, which was previously introduced in [30] for the Navier-Stokes equations. Here the pressure gradient will be written as \(\nabla p = \partial_t \nabla p_h + \nabla p_0\) in the sense of distribution, where \(\nabla p_h\) stands for the harmonic part associated to \(v\), while \(p_0\) represents the part associated to \((v \cdot \nabla)v\). This eventually leads to a local energy inequality in terms of the new localized energy \(\int_{\Omega} |\tilde{v}(t)|^2 \phi dx\) with a cut-off function \(\phi\), where \(\tilde{v} = v + \nabla p_h\). A solution satisfying this form of the local energy inequality will be called *local suitable weak solution* as it has been introduced by Definition 2.1. As an important consequence this notion we show that for such solutions the energy \(|\tilde{v}(t)|^2\) admits a unique measure valued trace, which is in fact weak-* left continuous. Furthermore, we show that the question of concentration of \(|v(t)|^2\) at the possible blow-up time can be reduced to that of concentration of \(|\tilde{v}(t)|^2\). Section 3 is devoted to special case of removing one point concentration of the energy for solution
to the Euler equations in the whole space of \( \mathbb{R}^n \). In particular, we are able to prove Theorem 1.1 for this restricted situation, which is stated in Theorem 3.1. The proof of Theorem 3.1 is based on several space-time decay properties of the velocity field as \( t \to 0^- \). The proof of the decay estimates are presented in Subsections 3.2 and 3.3. In particular, in Subsection 3.3 we show that the energy \( |\tilde{v}(t)|^2 \) of any exterior subdomain excluding the concentration point converges to zero with arbitrary polynomial order. In Subsection 3.4 we complete the proof of Theorem 3.1 based on a local estimate for the function \( w = v((-t)\theta x, t) \) for a suitable \( 0 < \theta < 1 \), which by virtue of Gronwall’s lemma yields triviality of \( w \) in an exterior domain. Next, in Section 4 we will provide the proof our main result, Theorem 1.1. Applying the blow-up argument, we are able to reduce the question of general atomic concentration problem to that of one point concentration in \( \mathbb{R}^n \) treated in Section 3, and applying Theorem 3.1 we conclude the proof. Finally, in Section 5, using Corollary 1.2, we present the proof of Corollary 1.5.

2 Local energy inequalities and the local pressure

In this section we introduce the notion of local suitable weak solution to the Euler equations similarly to the case of the Navier-Stokes equations\(^3\). As we shall prove below any solution satisfying Type I blow-up condition with respect to the velocity gradient is indeed local suitable weak solution before the possible blow-up time.

Let us begin our discussion by recalling the definition of the local pressure in a subdomain \( \Omega \subset \mathbb{R}^n \) with \( C^2 \) boundary. Here we distinguish between the two cases, firstly \( \Omega \) is bounded and secondly, \( \Omega \) is an exterior domain.

1. Local pressure for \( \Omega \) bounded: As in \(^3\) we define the projection \( E^*_\Omega : W^{-1,q}(\Omega) \to W^{-1,q}(\Omega) \) based on the unique solution of the Stokes equation as follows. Let \( f \in W^{-1,q}(\Omega) \) be given. Then we set \( E^*_\Omega(f) := \nabla p \), where \( p \in L^q_0(\Omega) \) stands for the unique pressure from the unique weak solution \( (w,p) \in W^{1,q}_0(\Omega) \times L^q_0(\Omega) \) to the Stokes system

\[
\begin{align*}
-\Delta w + \nabla p &= f, & \nabla \cdot w &= 0 & \text{in } \Omega, \\
 w &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Here \( L^q_0(\Omega) \) stands for a subspace of all \( p \in L^q(\Omega) \) such that \( \int_\Omega p \, dx = 0 \).

(The existence and uniqueness in bounded \( C^2 \) domains is due to Cattabriga\(^4\), while the case of bounded \( C^1 \) domains were treated in \(^1\)). Notice that \( \nabla p \) belongs to \( W^{-1,q}(\Omega) \) by

\[
\langle \nabla p, \varphi \rangle := -\int_\Omega p \nabla \cdot \varphi \, dx, \quad \varphi \in W^{1,q'}(\Omega), \quad 1/q + 1/q' = 1.
\]

Obviously, from this definition it follows that \( E^*(\nabla p) = \nabla p \) for every \( p \in L^q_0(\Omega) \), and thus it holds

\[
(E^*_{\Omega})^2 = E^*_{\Omega}.
\]
Observing the estimate

\[ \| \nabla w \|_{L^q(\Omega)} + \| p \|_{L^q(\Omega)} \leq c \| f \|_{W^{-1,q}(\Omega)}, \tag{2.4} \]

with a constant \( c > 0 \) depending only on \( q \) and the geometric property of \( \Omega \), we see that the operator \( E^*_\Omega \) is bounded, satisfying

\[ \| E^*_\Omega(f) \|_{W^{-1,q}(\Omega)} \leq \| p \|_{L^q(\Omega)} \leq c \| f \|_{W^{-1,q}(\Omega)} \tag{2.5} \]

with the same constant as in (2.4). In case \( f \in L^q(\Omega) \to W^{-1,q}(\Omega) \), by virtue of the elliptic regularity of the Stokes system we find \( E^*_\Omega(f) = \nabla p \in L^q(\Omega) \) together with the estimate

\[ \| \nabla p \|_{L^q(\Omega)} \leq c \| f \|_{L^q(\Omega)}, \tag{2.6} \]

where \( c > 0 \) denotes a constant depending only on \( q \) and the geometric property of \( \Omega \).

We also note that in case \( \Omega \) equals to a ball \( B(x_0, r) \), then the constants in both (2.5) and (2.6) depend neither on \( x_0 \) nor on \( r > 0 \).

In case \( 1 \leq s \leq +\infty \), if the vector valued function \( f \) belongs to the Bochner space \( L^s(a, b; W^{-1,q}(\Omega)) \), we may define \( E^*_\Omega(f) \) pointwise

\[ E^*_\Omega(f)(t) = E^*_\Omega(f(t)) \quad \text{for a.e. } t \in (a, b). \tag{2.7} \]

Clearly, (2.5) and (2.6) imply that \( E^*_\Omega \) is bounded on \( L^s(a, b; W^{-1,q}(\Omega)) \) and \( L^s(a, b; L^q(\Omega)) \) respectively. For \( f = \partial_t g \) in the sense of distributions we define

\[ E^*_\Omega(f) = \partial_t E^*_\Omega(g) \quad \text{in the sense of distributions.} \]

2. Local pressure in case \( \Omega \) is an exterior domain: Since \( \Omega \) is unbounded, it will be more appropriate to replace the usual Sobolev space by the homogenous Sobolev space \( D^{1,q}_0(\Omega) \), which is defined as the closure of \( C_c^\infty(\Omega) \) with respect to the norm

\[ \| u \|_{D^{1,2}(\Omega)} = \left( \int_\Omega |\nabla u|^2 dx \right)^{1/2}. \]

Analogously, the subspace of all divergence free vector functions in \( D^{1,2}_0(\Omega; \mathbb{R}^n) \) will be denoted by \( D^{1,2}_{0,\sigma}(\Omega) \). In what follows by \( D^{-1,2}(\Omega) \) we denote the dual of \( D^{1,2}_0(\Omega) \).

As in the case of bounded domains for \( f \in D^{-1,2}(\Omega) \) we define \( E^*_\Omega(f) = \nabla p \), if \( (w, p) \in D^{1,2}_{0,\sigma}(\Omega; \mathbb{R}^n) \times L^2(\Omega) \) denotes the unique weak solution to Stokes problem (2.1), (2.2). Here the estimate (2.4) for \( q = 2 \) is still valid, which leads to the estimate

\[ \| E^*_\Omega(f) \|_{D^{-1,2}(\Omega)} \leq \| p \|_{L^2(\Omega)} \leq c \| f \|_{D^{-1,2}(\Omega)}. \tag{2.8} \]

This together with (2.3) shows that \( E^*_\Omega \) is a projection in \( D^{-1,2}(\Omega) \) onto the closed subspace of all functionals \( \nabla p \) with \( p \in L^2(\Omega) \). In addition, if \( f \in L^q(\Omega) \) for some \( 1 < q < +\infty \), then \( \nabla p \in L^q(\Omega) \), and there holds

\[ \| \nabla p \|_{L^q(\Omega)} \leq c \| f \|_{L^q(\Omega)}. \tag{2.9} \]
We also wish to remark that in case \( \Omega = B(x_0, r)^c = \mathbb{R}^n \setminus B(x_0, r) \) in both (2.8) and (2.9), the constants are independent of \( x_0 \) and \( r > 0 \), which can be readily seen by a standard scaling argument. For vector valued functions \( f \in L^s(a, b; D^{-1,2}(\Omega)) \) we define \( E_{\Omega}^*(f) \) and \( E_{\Omega}^*(\partial_t f) \) as in the case of bounded domains.

We are now in a position to introduce the notion of local suitable weak solution to (1.1) in \( Q = \Omega \times (a, b) \).

**Definition 2.1.** A vector function \( v \in L^\infty(a, b; L^2(\Omega)) \cap L^3(Q) \) with \( \nabla \cdot v = 0 \) in the sense of distributions is said to be a local suitable weak solution to (1.1), if the following two conditions are satisfied.

1. The function \( \bar{v} := v + \nabla p_h := v - E_{\Omega}^*(v) \) solves

\[
\partial_t \bar{v} + (v \cdot \nabla) v = -\nabla p_0 \quad \text{in} \quad Q
\]

in the sense of distributions, where \( \nabla p_h = -E_{\Omega}^*(v) \) and \( \nabla p_0 = -E_{\Omega}^*(v \cdot \nabla v) \).

2. For almost every \( a \leq t < s < b \) and for all \( \phi \in C^\infty_c(\Omega) \) with \( \phi \geq 0 \), the following local energy inequality holds true

\[
\int_\Omega |\bar{v}(t)|^2 \phi dx \leq \int_\Omega |\bar{v}(s)|^2 \phi dx + \int_t^s \int_\Omega (|\bar{v}|^2 v + 2p_0 \bar{v}) \cdot \nabla \phi dxd\tau
\]

\[
+ \int_t^s \int_\Omega v \otimes v : \nabla^2 p_h \phi dxd\tau.
\]

**Remark 2.2.** In [26] the author has introduced the notion of suitable weak solution under the assumption that the pressure \( p \in L^{3/2}(Q) \), and the local energy inequality holds true for almost every \( a \leq t < s < b \) and for all \( \phi \in C^\infty_c(\Omega) \) with \( \phi \geq 0 \),

\[
\int_\Omega |v(t)|^2 \phi dx \leq \int_\Omega |v(s)|^2 \phi dx + \int_t^s \int_\Omega (|v|^2 + 2p)v \cdot \nabla \phi dxd\tau.
\]

In fact, by the same the argument as in the proof of Lemma A.2 in [9], we see that any suitable weak solution satisfying (2.12) is also a local suitable weak solution in the sense of Definition 2.1.

In the following lemma we show that any \( v \), which satisfies the local energy inequality related to the generalized energy inequality (2.11) for local suitable weak solutions, admits a weak measure valued trace in time.

**Lemma 2.3.** Let \(-\infty < a < b < +\infty\). Set \( Q = \Omega \times (a, b) \). Let \( v \in L^\infty(a, b; L^2(\Omega)) \cap L^3(Q) \), \( p \in L^{3/2}(Q) \), \( g \in L^3(Q) \), and \( f \in L^1(a, b; L^2(\Omega)) \). Assume there exists a set
$J \subset (a, b]$ of Lebesgue measure zero such that the following local energy inequality holds true for all nonnegative $\phi \in C_c^\infty(\Omega)$ and for all $s, t \in (a, b) \setminus J, t \leq s$,

$$\int_\Omega |v(t)|^2 \phi dx \leq \int_\Omega |v(s)|^2 \phi dx$$

(2.13) \[ + \int_\Omega \int_t^s (|v|^2 g + 2pv) \cdot \nabla \phi dx d\tau + \int_t^s \int_\Omega f \cdot v \phi dx d\tau. \]

Then there exists a unique trace $\sigma \in L^\infty(a, b; \mathcal{M}^+(\Omega))$ fulfilling the following properties:

1. $\sigma(t) = |v(t)|^2 dx$ for a.e. $t \in (a, b]$.
2. The mapping $t \mapsto \sigma(t)$ is weakly-∗ left continuous, i.e. for every $t \in (a, b]$ it holds

(2.14) \[ \int_\Omega \phi d\sigma(t) = \lim_{s \to t^-} \int_\Omega \phi d\sigma(s) \quad \forall \phi \in C^0_c(\Omega), \phi \geq 0. \]

3. The following generalized local energy inequality holds for all $a < t < s \leq b$ and for all nonnegative $\phi \in C_c^\infty(\Omega)$

(2.15) \[ \int_\Omega \phi d\sigma(t) \leq \int_\Omega \phi d\sigma(s) \]

\[ + \int_\Omega \int_t^s (|v|^2 g + 2pv) \cdot \nabla \phi dx d\tau + \int_t^s \int_\Omega f \cdot v \phi dx d\tau. \]

4. The set $\mathcal{M} |v|^2(b)$ contains only the measure $\sigma_0 = \sigma(b)$.

**Proof:** Let $t \in (a, b]$. By $\mathcal{M}(t)$ we define the set of all measures $\sigma \in \mathcal{M}(\Omega)$, obtained by a weak-∗ limit of the measures $|v(\tau)|^2 dx$ as $\tau \in [a, t] \setminus J \to t$. In fact, since $|v(\tau)|^2 dx \in \mathcal{M}^+(\Omega)$ for all $\tau \in (a, b)$, we have $\mathcal{M}(t) \subset \mathcal{M}^+(\Omega)$.

Let $t_0 \in (a, b]$, and let $\sigma, \widetilde{\sigma} \in \mathcal{M}(t_0)$ be two measures. Let $\{s_k\}$ be a sequence in $(a, t_0) \setminus J$ with $s_k \to t_0$ as $k \to +\infty$ such that

(2.16) \[ |v(s_k)|^2 dx \to \sigma \quad \text{weakly-∗ in } \mathcal{M}(\Omega) \quad \text{as } k \to +\infty. \]

From (2.13) with $s = s_k$ we deduce, after passing $s_k \to t_0$, that for all $t \in (a, t_0) \setminus J$ and for all nonnegative $\phi \in C_c^\infty(\Omega)$

(2.17) \[ \int_\Omega |v(t)|^2 \phi dx \leq \int_\Omega \phi d\sigma \]

\[ + \int_\Omega \int_t^{t_0} (|v|^2 g + 2pv) \cdot \nabla \phi dx d\tau + \int_t^{t_0} \int_\Omega f \cdot v \phi dx d\tau. \]
Analogously, we take a sequence \( \{\tilde{s}_k\} \) in \((a, t_0)\setminus J\) with \( \tilde{s}_k \to t_0 \) as \( k \to +\infty \) such that
\[
|v(\tilde{s}_k)|^2 dx \to \tilde{\sigma} \quad \text{weakly-\ast in} \quad \mathcal{M}(\Omega) \quad \text{as} \quad k \to +\infty.
\]

Then from (2.17) with \( t = \tilde{s}_k \) after passing \( \tilde{s}_k \to t_0 \) together with a standard mollifying argument we obtain
\[
\int_{\Omega} \phi d\tilde{\sigma} \leq \int_{\Omega} \phi d\sigma \quad \forall \phi \in C^0_c(\Omega), \quad \phi \geq 0.
\]

Obviously, we may exchange \( \sigma \) and \( \tilde{\sigma} \) in (2.19), which yields the equality in (2.19).

Since both \( \sigma \) and \( \tilde{\sigma} \) are nonnegative measures, we obtain
\[
\int_{\Omega} \phi d\tilde{\sigma} = \int_{\Omega} \phi d\sigma \quad \forall \phi \in C^0_c(\Omega).
\]

Thus \( \sigma = \tilde{\sigma} \). This shows that for every \( t \in (a, b] \) there exists a unique nonnegative measure \( \sigma(t) \in \mathcal{M}^+(\Omega) \) such that
\[
|v(\tau)|^2 dx \to \sigma(t) \quad \text{weakly-\ast in} \quad \mathcal{M}(\Omega) \quad \text{as} \quad \tau \to t^-.
\]

Furthermore, by the above definition of \( \sigma(t) \) we get for all \( \phi \in C^0_c(\Omega) \) with \( \max_{\Omega} |\phi| \leq 1 \)
\[
\int_{\Omega} \phi \sigma(t) = \lim_{s_k \to t^-} \int_{\Omega} |v(s_k)|^2 \phi dx \leq ||v||^2_{L^\infty(a,b;L^2(\Omega))},
\]

which shows that \( \sigma \in L^\infty(a, b; \mathcal{M}^+(\Omega)) \).

In addition, from (2.20) we deduce that the following local energy inequality holds true for all \( s, t \in (a, b] \) with \( t \leq s \) and for all nonnegative \( \phi \in C_c^\infty(\Omega) \)
\[
\int_{\Omega} \phi \sigma(t) \leq \int_{\Omega} \phi d\sigma(s)
\]
\[
+ \int_{t}^{s} \int_{\Omega} (|v|^2 g + 2pv) \cdot \nabla \phi dx d\tau + \int_{t}^{s} \int_{\Omega} f \cdot v \phi dx d\tau.
\]

By the same reasoning as the above it can be easily checked that
\[
\sigma(t) \to \sigma(s) \quad \text{weakly-\ast in} \quad \mathcal{M}(\Omega) \quad \text{as} \quad t \to s \quad \text{in} \quad (a, s).
\]

This implies that \( \sigma : t \mapsto \sigma(t) \) is weakly-\ast left continuous, and therefore property (2) of the lemma is fulfilled. To verify (1) of the lemma let \( t \in (a, b] \) be chosen so that
\[
\frac{1}{h} \int_{t-h}^{t} |v(\tau)|^2 d\tau \to |v(t)|^2 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad h \to 0^+.
\]
As we have noted in Section 1 due to Lebesgue’s differentiation theorem for the Bochner integrable functions (see e.g. [32, Theorem 2, pp.134]) the property (2.23) holds true for a.e. $t$. It is also readily seen that from (2.23) we get for all $\phi \in C_c^0(\Omega)$

\[(2.24) \quad \frac{1}{h} \int_{t-h}^{t} \int_{\Omega} |v(\tau)|^2 \phi dxd\tau \to \int_{\Omega} |v(t)|^2 \phi dx \quad \text{as} \quad h \to 0^+.\]

We fix $\phi \in C_c^0(\Omega)$. Let $\{h_k\}$ be a sequence in $(0, t - a)$ which converges to zero as $k \to +\infty$. By the mean value theorem for the integrals for every $k \in \mathbb{N}$ we may choose $t_k \in (t - h_k, t) \setminus J$ such that

\[\int_{\Omega} |v(t_k)|^2 \phi dx = \frac{1}{h_k} \int_{t-h_k}^{t} \int_{\Omega} |v(\tau)|^2 \phi dxd\tau.\]

This together with (2.24) and the weakly-∗ left side continuity of $\sigma$ yields

\[\int_{\Omega} \phi d\sigma(t) = \lim_{k \to \infty} \int_{\Omega} |v(t_k)|^2 \phi dx = \int_{\Omega} |v(t)|^2 \phi dx,\]

and therefore (1) of the lemma is satisfied.

Finally, the generalized local energy inequality (2.15) follows immediately from (2.13) together with (2.20), while (4) of the lemma immediately follows from the proof of (1). In fact, we already have proved that $\sigma(t) = |v(t)|^2 dx$ for every Lebesgue point of $|v(\cdot)|^2$, which immediately gives (4), since for every $|v(t)|^2 dx \to \sigma(0)$ as $t \to b^-$ for $t$ in the Lebesgue set of $|v(\cdot)|^2$ in $(a, b)$.

As an important consequence of Lemma 2.3 we are able to study the concentration for the local suitable weak solutions to the Euler equation. In fact we have the following.

**Remark 2.4.** 1. If $v \in L^\infty(a, b, L^2(\Omega)) \cap L^3(Q)$ is a local suitable weak solution to the Euler equations in $Q = \Omega \times (a, b)$, then $\tilde{v} = v + \nabla p_h$ is a distributional solution to

\[(2.25) \quad \partial_t \tilde{v} + (v \cdot \nabla) \tilde{v} = -\nabla p_0 \quad \text{in} \quad Q,
\]

where

\[\nabla p_h = -E^*_{\Omega}(v), \quad \nabla p_0 = -E^*_{\Omega}((v \cdot \nabla) v).\]

Note that

\[(v \cdot \nabla) v = (v \cdot \nabla) \tilde{v} - v \cdot \nabla^2 p_h.\]

Since $\tilde{v}$ fulfills (2.11), the local energy inequality (2.13) holds for $\tilde{v}$ in place of $v$ for a.e. $a < t < s < b$ with

\[g = v, \quad f = v \cdot \nabla^2 p_h.\]
According to Lemma 2.3 there exists a unique $\tilde{\sigma} \in L^\infty(a, b; \mathcal{M}^+(\Omega))$ such that (1)-(4) of the lemma are fulfilled. In particular, we see that $M_{|v|^2}(b) = \{\tilde{\sigma}(b)\}$, and there holds
\begin{equation}
(2.26) \begin{cases} 
|\tilde{v}(t)|^2 dx \to \tilde{\sigma}(b) \text{ weakly-* in } \mathcal{M}(\Omega) \\
\text{as } t \to b^- \text{ for } t \text{ chosen from the Lebesgue set of } |\tilde{v}(\cdot)|^2.
\end{cases}
\end{equation}

While the set $\mathcal{M}_{|v|^2}(b)$ contains only one unique measure, it is not true in general for $\mathcal{M}_{|v|^2}(b)$. The reason is that $v$ may not satisfy the local energy inequality. However, as we shall show below by Lemma 2.5 the concentration set of measures in $\mathcal{M}_{|v|^2}(b)$ coincides with the concentration set of $\tilde{\sigma}(b)$, which is the unique measure in $\mathcal{M}_{|v|^2}(b)$.  

2. In case that $v$ is a solution to the Euler equations (1.1) satisfying Type I blow-up condition with respect to the velocity gradient, then $v$ is a local suitable weak solution in the sense of Definition 2.1. In other words, $\tilde{v} = v + \nabla p_h$ satisfies the energy inequality (2.11) for all $s, t \in [-1, 0)$, $t \leq s$. As we mentioned above, thanks to Lemma 2.3 there exists a unique measure valued trace $\tilde{\sigma} \in \mathcal{M}(\Omega)$. Since every $t \in (0, 1)$ is a Lebesgue point of $|v|^2$ it follows $\tilde{\sigma}(s) = |v(s)|^2 dx$ for all $s \in (-1, 0)$, and there holds
\begin{equation}
(2.27) \quad |\tilde{v}(s)|^2 dx \to \tilde{\sigma}(b) \text{ weakly-* in } \mathcal{M}(\Omega) \text{ as } s \to 0^-.
\end{equation}

**Lemma 2.5.** Let $v \in L^\infty(a, b; L^2(\Omega)) \cap L^3(Q)$ be a local suitable weak solution to (1.1). Let $\nabla p_h$ and $\nabla p_0$ denote the corresponding local pressure (cf. Definition 2.1), and define $\tilde{v} := v + \nabla p_h$. Then each measure $\sigma_0 \in \mathcal{M}_{|v|^2}(b)$ has no atoms if and only if $\tilde{\sigma}(b)$ has no atoms.

**Proof:** Let $\sigma_0 \in \mathcal{M}_{|v|^2}(b)$, then there exists a sequence $\{s_k\}$ in the set of Lebesgue points of $|v(\cdot)|^2$ such that $s_k \to b^-$ and
\begin{equation}
(2.28) \quad |v(s_k)|^2 dx \to \sigma_0 \text{ weakly-* in } \mathcal{M}(\Omega) \text{ as } k \to +\infty.
\end{equation}

Since $\{v(s_k)\}$ is bounded in $L^2(\Omega)$, thanks to the reflexivity, eventually passing to a subsequence, we may assume that there exists $v_0 \in L^2(\Omega)$ such that
\begin{equation}
\begin{aligned}
v(s_k) &\to v_0 \text{ weakly in } L^2(\Omega) \text{ as } k \to +\infty.
\end{aligned}
\end{equation}

By the boundedness of the operator $E_{\Omega}^*$ in $L^2(\Omega)$ we deduce that
\begin{equation}
\begin{aligned}
\nabla p_h(s_k) &\to \nabla p_{h,0} \text{ weakly in } L^2(\Omega) \text{ as } k \to +\infty,
\end{aligned}
\end{equation}
where $\nabla p_{h,0} = -E_{\Omega}^*(v_0)$. By virtue of Lemma A.3, we find that for every $\Omega' \subset \Omega$ the convergence $\nabla p_h(s_k) \to \nabla p_{h,0}$ as $k \to +\infty$ is in fact uniform on $\Omega'$. This together with the weak convergence of $v(s_k) \to v_0$ in $L^2(\Omega)$ implies that
\begin{equation}
(2.29) \begin{cases} 
\left(2v(s_k) \cdot \nabla p_h(s_k) - |\nabla p_h(s_k)|^2\right) dx \to \left(2v_0 \cdot \nabla p_{h,0} - |\nabla p_{h,0}|^2\right) dx \\
\text{weakly-* in } \mathcal{M}(\Omega) \text{ as } k \to +\infty.
\end{cases}
\end{equation}

Furthermore, verifying that $\tilde{v} \in C_w([a, b]; L^2(\Omega))$ and recalling the weakly-* left side continuity of $\tilde{\sigma}$, we have deduce that
\begin{equation}
(2.30) \quad |\tilde{v}(t)|^2 dx \leq \tilde{\sigma}(t) \quad \forall t \in (a, b).
\end{equation}
Combining (2.28) and (2.29), noting \( |v(s_k)|^2 = |\tilde{v}(s_k)|^2 - 2v(s_k) \cdot \nabla p_h(s_k) + |\nabla p_h(s_k)|^2 \), and employing (2.30), we infer that
\[
0 \leq \sigma_0 \leq w^* \lim_{k \to \infty} \left( \bar{\sigma}(t_k) + (-2v_0 \cdot \nabla p_{h,0} + |\nabla p_{h,0}|^2) dx \right) \\
= \bar{\sigma}(b) + \left( -2v_0 \cdot \nabla p_{h,0} + |\nabla p_{h,0}|^2 \right) dx. 
\]
(2.31)

This immediately shows that if \( \bar{\sigma}(b) \) has no atoms, the same also holds true for \( \sigma_0 \).

In order to prove the opposite direction we assume that each measure in \( M|v|^2(b) \) has no atoms. Let us choose a sequence \( \{s_k\} \) in \((a,b)\) such that \( s_k \to b \) as \( k \to +\infty \) with the property that each \( s_k \) is simultaneously belong to the Lebesgue set of \( |v(\cdot)|^2 \) and \( |\tilde{v}(\cdot)|^2 \). Eventually passing to a subsequence, we may assume there exist a measure \( \sigma_0 \in M^+(\Omega) \) and \( v_0 \in L^2(\Omega) \) having the following convergence properties
\[
|v(s_k)|^2 dx \to \sigma_0 \text{ weakly-* in } M(\Omega),
\]
(2.32)
\[
v(s_k) \to v_0 \text{ weakly in } L^2(\Omega) \text{ as } k \to +\infty.
\]
(2.33)

Thanks to the property (4) of Lemma 2.3 it holds
\[
|\tilde{v}(s_k)|^2 dx \to \bar{\sigma}(b) \text{ weakly-* in } M(\Omega) \text{ as } k \to +\infty.
\]
(2.34)

Arguing as in the first part of the proof, from (2.32) and (2.33) we get the property (2.29). Finally, observing (2.34), we conclude that
\[
\sigma_0 = w^* \lim_{k \to \infty} |v(s_k)|^2 dx \\
= w^* \lim_{k \to \infty} \left( |\tilde{v}(s_k)|^2 - 2v(s_k) \cdot \nabla p_h(s_k) + |\nabla p_h(s_k)|^2 \right) dx \\
= \bar{\sigma}(b) + \left( -2v_0 \cdot \nabla p_{h,0} + |\nabla p_{h,0}|^2 \right) dx.
\]

Since \( \sigma_0 \) has no atoms, the above identity shows that \( \bar{\sigma}(b) \) also has no atoms. \( \blacksquare \)

## 3 Removing one point energy concentration in \( \mathbb{R}^n \)

In this section we restrict ourself to the case \( \Omega = \mathbb{R}^n \). In this case, since any solution which satisfies Type I condition with respect to the velocity gradient enjoys the local energy inequality, the pressure satisfies \( p \in L^{3/2}(\mathbb{R}^n \times (-1,0)) \) due to the Calderón-Zygmund inequality, and thanks to Lemma 2.3 there exists a unique measure \( \sigma \in \mathcal{M}(\mathbb{R}^n) \) such that
\[
|v(t)|^2 dx \to \sigma \text{ weakly-* in } \mathcal{M}(\mathbb{R}^n) \text{ as } t \to 0^-.
\]
(3.1)

Our aim is the proof of Theorem 1.1 for the special case that \( \sigma \) in (3.1) equals to the Dirac measure \( E_0\delta_0 \) for some constant \( 0 \leq E_0 < +\infty \). Namely we shall prove the following:
Theorem 3.1. Let \( v \in L^2(-1, 0; L^2_\sigma(\mathbb{R}^n)) \) be a solution to the Euler equations (1.1). In addition, we assume that \( v \) satisfies the Type I blow up condition (1.2) (cf. Theorem 1.1) and (3.1) with \( \sigma = E_0 \delta_0 \) for some \( 0 \leq E_0 < +\infty \). Then \( v \equiv 0 \).

Remark 3.2. In the proof of Theorem 3.1 we make significant use of several decay properties of the solution to the Euler equations with respect to the space and time variables as we approach the blow-up time. The decay estimate is actually obtained under following more general condition than (1.2).

(3.2) \( \exists \mu \in \left( \frac{n}{n+2}, 1 \right) : \sup_{t \in (-1,0)} (-t)^{\frac{n+2}{n}} \| \nabla v(t) \|_{L^\infty} < +\infty \).

We divide the proof of Theorem 3.1 into four steps, each step being a subsection below.

3.1 Proof of \( E_0 = E \)

The aim of this section is to show that \( E_0 = E = \| v(t) \|_{L^2}^2 \), \(-1 \leq t < 0\), in Theorem 3.1 under the condition (3.1), in other words, the energy cannot escape into infinity at the blow-up time. We begin with the following observation.

Lemma 3.3. Let \( v \in L^\infty(-1, 0; L^2(\mathbb{R}^n)) \), which satisfies (3.2). Then, it holds

(3.3) \( \sup_{t \in (-1,0)} (-t)^\mu \| v(t) \|_{L^\infty} \leq c(E_{\infty}^2 + \sup_{t \in (-1,0)} (-t)^{\frac{(n+2)\mu}{n}} \| \nabla v(t) \|_{L^\infty}) =: C_1 < +\infty \),

where \( E = \| v(-1) \|_{L^2}^2 \).

Proof: This is immediate of the Gagliardo-Nirenberg inequality and the energy conservation \( E = E(t) \) for \( t \in (-1,0) \) (see Remark 1.4),

\[
(-t)^\mu \| v(t) \|_{L^\infty} \leq c(-t)^\mu \| v(t) \|_{L^2}^{\frac{2}{n+2}} \| \nabla v(t) \|_{L^\infty}^{\frac{n}{n+2}} \leq cE^{\frac{1}{n+2}} \left( (-t)^{\frac{(n+2)\mu}{n}} \| \nabla v(t) \|_{L^\infty} \right)^{\frac{n}{n+2}}.
\]

From Lemma 3.3 along with \( v \in L^\infty(-1, 0; L^2(\mathbb{R}^n)) \) we immediately get

(3.4) \( v \in L^1(-1, 0; L^\infty(\mathbb{R}^n)) \cap L^3(-1, 0; L^3(\mathbb{R}^n)) \).

We have the following

Lemma 3.4. Let \( v \in L^\infty(-1, 0; L^2_\sigma(\mathbb{R}^n)) \cap L^\infty_{loc}([-1, 0), W^{1, \infty}(\mathbb{R}^n)) \) be a solution to (1.1) satisfying (3.2) and (3.1) with \( \sigma = E_0 \delta_0 \). Then it holds \( E_0 = E \).

Proof: Given \( 0 < R < +\infty \), we denote by \( \eta_R \in C^\infty_0(\mathbb{R}) \) a cut off function such that \( 0 \leq \eta_R \leq 1 \) in \( \mathbb{R} \), \( \eta_R = 1 \) on \((1, R)\), \( \eta_R = 0 \) in \((-\infty, 0] \cap (2R, +\infty)\) and \( |\eta_R'| \leq \frac{2}{R} \).
in \((R, 2R)\). We multiply the Euler equations by \(-v\eta_R(|x|^2)\), integrate the result over \(\mathbb{R}^n \times (t, \tau), -1 \leq t < \tau < 0\), and apply integration by parts. This gives

\[
\frac{1}{2} \int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|^2) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^n} |v(\tau)|^2 \eta_R(|x|^2) dx + \frac{1}{2} \int_{t}^{\tau} \int_{\mathbb{R}^n} |v(s)|^2 v(s) \cdot \nabla \eta_R(|x|^2) ds dx ds
\]

\[
+ \int_{t}^{\tau} \int_{\mathbb{R}^n} p(s) v(s) \cdot \nabla \eta_R(|x|^2) ds dx ds.
\]

Observing (3.1), we see that \(\int_{\mathbb{R}^n} |v(\tau)|^2 \eta_R(|x|^2) dx \to 0\) as \(\tau \to 0\). In view of (3.4) together with the Calderón-Zygmund estimate, having \(v \in L^3(\mathbb{R}^n \times (-1, 0))\) and \(p \in L^{3/2}(\mathbb{R}^n \times (-1, 0))\), we obtain from the above identity after letting \(\tau \to 0\)

\[
\frac{1}{2} \int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|^2) dx
\]

\[
= \frac{1}{2} \int_{t}^{0} \int_{\mathbb{R}^n} |v(s)|^2 v(s) \cdot \nabla \eta_R(|x|^2) ds dx ds + \int_{t}^{0} \int_{\mathbb{R}^n} p(s) v(s) \cdot \nabla \eta_R(|x|^2) ds dx ds.
\]

We are now in a position to pass \(R \to +\infty\) in the above to get

\[
\int_{\mathbb{R}^n} |v(t)|^2 \eta(|x|^2) dx
\]

\[
= \int_{t}^{0} \int_{\mathbb{R}^n} |v(s)|^2 v(s) \cdot \nabla \eta(|x|^2) ds dx ds + 2 \int_{t}^{0} \int_{\mathbb{R}^n} p(s) v(s) \cdot \nabla \eta(|x|^2) ds dx ds,
\]

where \(\eta \in C^\infty(\mathbb{R})\) stands for the corresponding cut off function such that \(\eta \equiv 1\) on \((1, +\infty)\). Noting that \(1 - \eta(|x|^2) \in C^\infty_c(\mathbb{R}^n)\) and once more appealing to (3.1), from the above identity we deduce

\[
E = \int_{\mathbb{R}^n} |v(t)|^2 \eta(|x|^2) dx + \int_{\mathbb{R}^n} |v(t)|^2 (1 - \eta(|x|^2)) dx
\]

\[
= \int_{\mathbb{R}^n} |v(t)|^2 (1 - \eta(|x|^2)) dx + \int_{t}^{0} \int_{\mathbb{R}^n} |v(s)|^2 v(s) \cdot \nabla \eta(|x|^2) ds dx ds
\]

\[
+ 2 \int_{t}^{0} \int_{\mathbb{R}^n} p(s) v(s) \cdot \nabla \eta(|x|^2) ds dx ds,
\]

\[
\to E_0 \quad \text{as} \quad t \to 0.
\]

Whence, the claim. \(\blacksquare\)
3.2 Decay estimates for energy concentrating solutions

In this subsection our aim is to prove the space-time decay for solutions to the Euler equations satisfying the blow-up rate \((3.2)\) and the energy concentration at \((0, 0)\).

**Lemma 3.5.** Let \(v \in L^2(-\infty, 0; L^2_{\sigma}(\mathbb{R}^n)) \cap L^\infty_{loc}(-1, 0), \quad W^1,\infty(\mathbb{R}^n))\) be a solution to the Euler equations satisfying \((3.2)\) and \((3.1)\) with \(\sigma = E\delta_0\). Then for every \(0 < \beta < n + 2\) there exists a constant \(c\) depending on \(C_1, \mu\) and \(\beta\) such that for every \(t \in [-1, 0)\) it holds

\[
(3.5) \quad \int_{\mathbb{R}^n} |v(t)|^2 |x|^{\beta} dx \leq c(-t)^{\beta(1-\mu)}.
\]

**Proof:** Given \(R > 2\), let \(\eta_R \in C_c^\infty(\mathbb{R})\) denote a cut off function such that \(0 \leq \eta_R \leq 1\) in \(\mathbb{R}\), \(\eta_R \equiv 0\) in \((2R, +\infty)\), \(\eta_R \equiv 1\) in \((-\infty, R)\), and \(|\eta'_R| \leq \frac{2}{R}\) in \(\mathbb{R}\). Observing \((3.1)\) with \(\sigma = E\delta_0\), we get for all \(0 < \beta < +\infty\)

\[
\lim_{t \to 0^-} \int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|)^2 |x|^{\beta} dx = E(\delta_0, \eta_R(|x|)^2 |x|^{\beta}) = 0.
\]

We multiply \((1.1)\) by \(-v\eta_R(|x|)^2 |x|^{\beta}\), \(1 \leq \beta < +\infty\), integrate over \(\mathbb{R}^n \times (t, 0)\), \(-1 < t < 0\), and apply integration by parts. This together with \(\nabla(\eta_R(|x|)^2 |x|^{\beta}) = \beta |x|^{\beta-2} \eta_R(|x|) + 2x \eta_R(|x|) \eta'_R(|x|)|x|^{\beta-1}\) gives

\[
\frac{1}{2} \int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|)^2 |x|^{\beta} dx
\]

\[
= -\frac{\beta}{2} \int_{t}^{0} \int_{\mathbb{R}^n} v(s) \cdot x |x|^{\beta-2} |v(s)|^2 \eta_R(|x|)^2 dx ds
\]

\[
- \int_{t}^{0} \int_{\mathbb{R}^n} v(s) \cdot x \eta_R(|x|) \eta'_R(|x|)|x|^{\beta-1} |v(s)|^2 dx ds
\]

\[
- \beta \int_{t}^{0} \int_{\mathbb{R}^n} p(s)v(s) \cdot x |x|^{\beta-2} \eta_R(|x|)^2 dx ds
\]

\[
- 2 \int_{t}^{0} \int_{\mathbb{R}^n} p(s)v(s) \cdot x \eta_R(|x|) \eta'_R(|x|)|x|^{\beta-1} dx ds
\]

\[
= I + II + III + IV.
\]

In what follows, we will make an extensive use of the following estimate

\[
(3.7) \quad \int_{\mathbb{R}^n} |p(s)|^2 |x|^{\gamma} dx \leq c \int_{\mathbb{R}^n} |v(s)|^4 |x|^{\gamma} dx \leq c(-s)^{-2\mu} \int_{\mathbb{R}^n} |v(s)|^2 |x|^{\gamma} dx
\]
which holds true for all $0 \leq \gamma < n$. Indeed, in case $\gamma = 0$ the estimate (3.7) is an immediate consequence of the well-known Calderón-Zygmund inequality together with (3.3). For $0 < \gamma < n$, noting that $|x|^{\gamma}$ belongs to the class $A_2$, the estimate (3.7) follows by the aid of the weighted Calderón-Zygmund inequality [28, Corollary, p.205] along with (3.3).

We divide the proof of (3.5) into five steps:

1. We consider the case $\beta = 1$: Noting that $\eta'_R(|x|)|x| \leq 4$ and observing (3.3), we immediately get

\[ I + II \leq 5c \int_{t}^{0} (-s)^{-\mu} \int_{\mathbb{R}^n} |v(s)|^2 dx ds \leq cE(-t)^{1-\mu}. \]

For $s \in (-1,0)$ observing that $\Delta p(s) = -\nabla \cdot \nabla \cdot (v(s) \otimes v(s))$, the estimate (3.7) for $\gamma = 0$ gives

\[ \|p(s)\|_{L^2} \leq c\|v(s)\|_{L^2} \leq cE^{1/2}(-s)^{-\mu}. \]

The above inequality along with Cauchy-Schwarz’s inequality yields

\[ III + IV \leq 9 \int_{t}^{0} \int_{\mathbb{R}^n} |p(s)||v(s)|dx ds \leq c \int_{t}^{0} \|p(s)\|_{L^2}\|v(s)\|_{L^2} ds \]

\[ \leq cE^{1/2} \int_{t}^{0} (-s)^{-\mu} ds = cE^{1/2}(-t)^{1-\mu}. \]

Hence, from (3.6) it follows that

\[ \int_{\mathbb{R}^2} |v(t)|^2 \eta_R(|x|)^2 |x| dx \leq cE^{1/2}(-t)^{1-\mu}. \]

After passing $R \to +\infty$ in the above inequality, we get the estimate (3.5) for $\beta = 1$.

2. We consider the case $\beta = 2$: Noting, that $\|v(s)\|_{L^\infty} \leq c(-s)^{-\mu}$ and $|\eta'_R| |x| \leq 4$ together with (3.5) for $\beta = 1$, we easily find

\[ I + II \leq 4 \int_{t}^{0} (-s)^{-\mu} \int_{\mathbb{R}^n} |v(s)|^2 |x| dx ds \leq c(-t)^{2-2\mu}. \]

Applying (3.7) for $\gamma = 1$ and making use of (3.5) for $\beta = 1$, we get

\[ \int_{\mathbb{R}^n} |p(s)|^2 |x| dx \leq c(-s)^{-2\mu} \int_{\mathbb{R}^n} |v(s)|^2 |x| dx \leq c(-s)^{1-3\mu}. \]
Integrating the above inequality over \((t, 0)\), and using once more (3.5) for \(\beta = 1\), we obtain

\[
III + IV \leq c \int_t^0 \left( \int_{\mathbb{R}^n} |p(s)|^2 |x| \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |v(s)|^2 |x| \, dx \right)^{\frac{1}{2}} \, ds
\]

\[
\leq c \int_t^0 (-s)^{-\frac{3}{2} - \frac{2\mu}{3}} \left( \int_{\mathbb{R}^n} |v(s)|^2 |x| \, dx \right)^{1/2} \, ds
\]

\[
\leq c \int_t^0 (-s)^{-2\mu} \, ds = c(-t)^{-2\mu}.
\]

Inserting the estimates of \(I, II, III,\) and \(IV\) into (3.6), and passing \(R \to +\infty\), we obtain

\[
(3.8) \quad \int_{\mathbb{R}^n} |v(t)|^2 |x|^2 \, dx \leq c(-t)^{2(1-\mu)}.
\]

3. Iterating the above argument for \(\beta = 3, \ldots, n\), by using (3.7) for \(0 < \gamma < n\), we find

\[
(3.9) \quad \int_{\mathbb{R}^n} |v(t)|^2 |x|^n \, dx \leq c(-t)^{n(1-\mu)}.
\]

4. Next, we consider the case \(\beta = n + 1\). Arguing as above, in this case we estimate

\[
I + II \leq c(-t)^{(n+1)(1-\mu)}.
\]

For the estimation of \(III\) and \(IV\) we make use of (3.7) for \(\gamma = n - 1\), Cauchy-Schwarz’ inequality and Young’s inequality to get

\[
III + IV \leq c \int_t^0 \int_{\mathbb{R}^n} |p(s)||v(s)| \eta_R(|x|) |x|^n \, dx \, ds
\]

\[
\leq c \int_t^0 \left( \int_{\mathbb{R}^n} |p|^2 |x|^{n-1} \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} \, dx \right)^{1/2} \, ds
\]

\[
\leq c(-t)^{-\mu + \frac{(n-1)(1-\mu)}{2}} \left( \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} \, dx \right)^{1/2} \, ds
\]

\[
\leq c(-t)^{-\mu + \frac{(n-1)(1-\mu)}{2}} \left( \text{ess sup}_{s \in (t, 0)} \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} \, dx \right)^{1/2}
\]

\[
\leq c(-t)^{(n+1)(1-\mu)} + \frac{1}{4} \text{ess sup}_{s \in (t, 0)} \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} \, dx.
\]
Inserting the above estimates of $I, II, III,$ and $IV$ into (3.6), the following inequality holds for all $-1 < t < 0$

$$\int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|)^2 |x|^{n+1} dx \leq c(-t)^{(n+1)(1-\mu)} + \frac{1}{2} \text{ess sup}_{s \in (0,t)} \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} dx.$$

Let $-1 < \tau < 0$. Taking supremum over $t \in (\tau, 0)$ in both sides of the above inequality and noting that function on the right-hand side attains the maximum at $t = \tau$, we get

$$\text{ess sup}_{t \in (\tau, 0)} \int_{\mathbb{R}^n} |v(t)|^2 \eta_R(|x|)^2 |x|^{n+1} dx \leq c(-\tau)^{(n+1)(1-\mu)} + \frac{1}{2} \text{ess sup}_{s \in (0,\tau)} \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^{n+1} dx.$$

Accordingly, for all $-1 < t < 0$ it holds

$$\int_{\mathbb{R}^n} |v(t)|^2 |x|^{n+1} dx \leq c(-t)^{(n+1)(1-\mu)}.$$

5. We now consider the case $n + 1 < \beta < n + 2$. Using Hölder’s inequality together with $\|v(s)\|_{L^2} = E^{\frac{1}{2}}$ and (3.10), we deduce that for all $0 < \gamma \leq n + 1$ and $-1 \leq s < 0$ it holds

$$\int_{\mathbb{R}^n} |v(s)|^2 |x|^\gamma dx \leq c(-s)^{\gamma(1-\mu)}.$$

Applying the estimate (3.7) for $0 < \gamma < n$, and using (3.11), we get

$$\int_{\mathbb{R}^n} |p(s)|^2 |x|^\gamma dx \leq c(-s)^{\gamma(1-\mu)-2\mu}.$$

We now easily estimate $I + II$ by using (3.11) with $\gamma = \beta - 1$. Hence

$$I + II \leq c \int_t^0 (-s)^{-\mu} \int_{\mathbb{R}^n} |v(s)|^2 |x|^\beta dx ds$$

$$\leq c \int_t^0 (-s)^{\beta(1-\mu)-1} ds \leq c(-t)^{\beta(1-\mu)}. $$
In order to estimate $III + IV$ we make use of (3.12) with $\gamma = \beta - 2$ and apply Cauchy-Schwarz’s and Young’s inequality to obtain

\[
III + IV \leq c \int_{t}^{0} \left( \int_{\mathbb{R}^n} |p|^2 |x|^\beta - 2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^\beta \, dx \right)^{1/2} \, ds
\]

\[
\leq c \int_{t}^{0} (-s)^{\beta(1 - \frac{q}{n}) - 1} \left( \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^\beta \, dx \right)^{1/2} \, ds
\]

\[
\leq c(-t)^{\beta(1 - \mu)} + \frac{1}{4} \text{ess sup}_{s \in (t, 0)} \int_{\mathbb{R}^n} |v(s)|^2 \eta_R(|x|)^2 |x|^\beta \, dx.
\]

Inserting the estimates of $I, II, III$ and $IV$ into the right hand side of (3.6), and passing $R \to +\infty$, we get the desired estimate (3.5).

### 3.3 Fast decay using the local pressure for exterior domains

Let $0 < r < +\infty$ be fixed. By $B(r)$ we denote the usual ball in $\mathbb{R}^n$ with radius $r > 0$ with respect to the Euclidean norm having its center at the origin. For notational convenience by $E_r^*$ we denote the projection $E_{B(r)}^*$ in $D^{-1,2}(B(r)^c)$ onto the closed subspace containing functionals of the form $\nabla \pi$, which has been introduced in Section 2.

Recalling the definition $E_r^*$, we see that for every functional $f \in D^{-1,2}(B(r)^c)$ there exists a unique $\pi \in L^2(B(r)^c)$ such that $E_r^*(f) = \nabla \pi$.

**Lemma 3.6.** Let $v \in L^2(-1, 0; L^2(\mathbb{R}^n)) \cap L^\infty([0, 1), W^1, \infty(\mathbb{R}^n))$ be a solution to the Euler equations (1.1) satisfying (3.2) for some $\mu \in \left(\frac{n}{n+2}, 1\right)$ and (3.1) with $\sigma_0 = E\delta_0$. Then for all $k \in \mathbb{N} \cup \{0\}$ and $0 < r < +\infty$ it holds

\[
(3.13) \quad \|v(t) - E_r^*(v(t))\|_{L^2(B(r)^c)}^2 \leq C_0^k 4^k (t(1 - \mu) r^{-\frac{1}{2}})(t(1 - \mu) r^{-\frac{1}{2}}) \quad \forall t \in (-1, 0),
\]

where the constant $C_0 > 0$ depends only on $C_1$ of (3.3) and $\mu$.

In the proof of Lemma 3.6 we make use of the following pressure estimate

**Lemma 3.7.** Let $\Omega \subset \mathbb{R}^n$ be an exterior domain. Let $\pi \in L^q(\Omega)$ and $f \in L^q(\Omega; \mathbb{R}^n)$, $\frac{n}{n-1} < q < n$, such that $\Delta \pi = \nabla \cdot \nabla \cdot f$ in $\Omega$ in the sense of distributions. Furthermore, let $\zeta \in C^\infty(\Omega)$ such that $\nabla \zeta \in C^\infty_c(\Omega)$. Then

\[
\|\pi \zeta^n\|_{L^q(K)} \leq c\|f \zeta^n\|_{L^q} + c \left( \max |\nabla \zeta| + \text{meas}(K)^{1/n} \max(|\nabla^2 \zeta| + |\nabla^2 \zeta|^2) \right) \times
\]

\[
\quad \times \left( \|f \zeta^{n-2}\|_{L^\frac{q}{q-1}(K)} + \|\pi \zeta^{n-2}\|_{L^\frac{q}{q-1}(K)} \right),
\]

with a constant $c > 0$ depending only on $n$ and $q$, where $K = \text{supp}(\nabla \zeta)$.

**Proof:** In our discussion below we use the convention that repeated indices imply summation from 1 to $n$. By straightforward calculation we find that

\[
\Delta(\pi \zeta^n) = \nabla \cdot \nabla \cdot (f \zeta^n) + f : \nabla^2 \zeta^n - \partial_i (f_{ij} \partial_j \zeta^n) - \partial_j (f_{ij} \partial_i \zeta^n) + 2 \nabla \cdot (\pi \cdot \nabla \zeta^n) - \pi \Delta \zeta^n.
\]

\[
= G_1 + G_2 + G_3 + G_4 + G_5 + G_6.
\]
We may decompose \( \pi \zeta^n \) into the sum \( \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 \), where
\[
\pi_i = N * G_i, \quad i = 1, \ldots, 6,
\]
and \( N = N(x) \) is the fundamental solution of the Laplace equation in \( \mathbb{R}^n \), given by
\[
N(x) = \begin{cases} 
\frac{1}{c_n|x|^{n-2}} & \text{for } n \geq 3, \\
-\frac{1}{2\pi} \log(|x|) & \text{for } n = 2.
\end{cases}
\]
Using the Calderón-Zygmund estimate, we get
\[
\|\pi_1\|_{L^q} \leq c\|f\zeta^n\|_{L^q},
\]
\[
\|\pi_3\|_{L^q} + \|\pi_4\|_{L^q} + \|\pi_5\|_{L^q} \leq c \max(|\nabla \zeta|\left(\|f\zeta^{n-1}\|_{L^{\frac{nq}{n-q}}(K)} + \|\pi \zeta^{n-1}\|_{L^{\frac{nq}{n-q}}(K)}\right),
\]
\[
\|\pi_2\|_{L^{\frac{nq}{n-q}}} + \|\pi_6\|_{L^{\frac{nq}{n-q}}} \leq c \max(|\nabla^2 \zeta| + |\nabla \zeta|^2) \times \left(\|f\zeta^{n-2}\|_{L^{\frac{nq}{n-q}}(K)} + \|\pi \zeta^{n-2}\|_{L^{\frac{nq}{n-q}}(K)}\right).
\]
Applying Jensen’s inequality, we find
\[
\|\pi_2\|_{L^n(K)} + \|\pi_6\|_{L^n(K)} \leq c \text{meas}(K)^{1/n} \max(|\nabla^2 \zeta| + |\nabla \zeta|^2) \left(\|f\zeta^{n-2}\|_{L^{\frac{nq}{n-q}}(K)} + \|\pi \zeta^{n-2}\|_{L^{\frac{nq}{n-q}}(K)}\right).
\]
Using triangle inequality together with the estimates of \( \pi_i \), \( i = 1, \ldots, 6 \), we obtain (3.14).

**Remark 3.8.** We may apply Lemma 3.7 for the case \( q = 2 \) and \( \Omega = B(r)^c \). If \( \zeta \in C^\infty(\mathbb{R}^n) \) is a cut off function such that \( \zeta \equiv 0 \) in \( B(2r) \), \( \zeta \equiv 1 \) on \( B(4r)^c \) and \( |\nabla^2 \zeta| + |\nabla \zeta|^2 \leq cr^{-2} \). Then the estimate (3.14) becomes
\[
\|\pi \zeta^n\|_{L^2(K)} \leq c\|f\zeta^n\|_{L^2} + cr^{-1}\left(\|f\zeta^{n-2}\|_{L^{\frac{2n}{n+1}}(K)} + \|\pi \zeta^{n-2}\|_{L^{\frac{2n}{n+1}}(K)}\right).
\]
Using Hölder’s inequality and Young’s inequality, we deduce from (3.15)
\[
\|\pi \zeta^n\|_{L^2(K)} \leq c\|f\zeta^n\|_{L^2} + cr^{-\frac{n-1}{2}}\left(\|f\|_{L^{\frac{2n}{n+1}}(K)} + \|\pi\|_{L^{\frac{2n}{n+1}}(K)}\right).
\]
**Proof of Lemma 3.6** We prove (3.13) by induction. Thanks to (2.9) having
\[
\|E_r^*(v(s))\|_{L^2(B(r)^c)} \leq c\|v(s)\|_{L^2} = c\|v(-1)\|_{L^2},
\]
the assertion is true for \( k = 0 \).

We now assume (3.13) is true for \( k \in \mathbb{N} \cup \{0\} \). Let \( 0 < r < +\infty \) be arbitrarily chosen, but fixed. In case \( 0 < r \leq 4(-t)^{1-\mu} \) the assertion is trivially fulfilled. This can be readily seen by
\[
\|v(t) - E_r^*(v(t))\|_{L^2(B(r)^c)}^2 \leq c\|v(t)\|_{L^2}^2 \leq c\|v(-1)\|_{L^2}^2 \left((-t)^{1-\mu}r^{-1}\right)^k.
\]
Thus we only need to prove (3.13) for the opposite case
\[ r > 4(-t)^{1-\mu}. \]

For notational simplicity we set
\[ U := B(r/4)^c, \quad U_1 := B(r)^c, \quad U_2 := B(r/2)^c. \]

Let \( \zeta \in C^\infty(\mathbb{R}^n) \) denote a cut off function such that \( 0 \leq \zeta \leq 1 \) in \( \mathbb{R}^n \), \( \zeta \equiv 0 \) in \( B(r/2) \) and \( \zeta \equiv 1 \) on \( U_1 \). As in Section 2 we define
\[ \nabla p_h = -E^{*}_{r/4}(v), \quad \nabla p_0 = -E^{*}_{r/4}(\nabla \cdot (v \otimes v)), \]
\[ \bar{v} = v + \nabla p_h = v - E^{*}_{r/4}(v). \]

Note that according to (3.5) it holds \( v(s) \in L^{2n/n+2}(U) \subset D^{-1,2}(U) \), and thus
\[ \|p_h(s)\|_{L^2(U)} \leq c\|v(s)\|_{L^{2n/n+2}(U)} \quad \forall s \in (-1, 0). \]

Consulting [31, Theorem A.4] (with \( X = D^{-1,2}_0(U), E^* = E^*_{r} \)), we see that the restriction of \( \nabla p \) to \( U \) equals to \( \partial_t \nabla p_h + \nabla p_0 \) in the sense of the distribution, i.e. the following identity holds true for all \( \varphi \in C^\infty\text{c}(U \times (-1, 0)) \),
\[ -\int_{-1}^{0} \int_U v \cdot \partial_t \varphi + v \otimes v : \nabla \varphi dxdt = \int_{-1}^{0} \int_U \nabla p_h : \nabla \varphi + p_0 \nabla \cdot \varphi dxdt. \]

This shows that \( \bar{v} \) is a solution to
\[ \partial_t \bar{v} + (v \cdot \nabla) v = -\nabla p_0 \quad \text{in} \quad U \times (-1, 0). \]

We compute
\[ (v \cdot \nabla) v = (v \cdot \nabla) \bar{v} - (v \cdot \nabla) \nabla p_h = (v \cdot \nabla) \bar{v} - (\bar{v} - \nabla p_h) \cdot \nabla \nabla p_h \]
\[ = (v \cdot \nabla) \bar{v} - \bar{v} \cdot \nabla \nabla p_h + \frac{1}{2} \nabla |\nabla p_h|^2. \]

Hence, (3.18) implies that \( \bar{v} \) is a solution to the following transformed Euler equations
\[ \partial_t \bar{v} + (v \cdot \nabla) \bar{v} = \bar{v} \cdot \nabla^2 p_h - \nabla p_1 \quad \text{in} \quad U \times (-1, 0), \]
where we set
\[ p_1 = \frac{\|\nabla p_h\|^2}{2} + p_0. \]

Observing that \( E^*_{r/4}(\partial_t \bar{v}) = 0 \) in the sense of distribution, we get
\[ \nabla p_1 = E^*_{r/4}\left(-\nabla \cdot (\bar{v} \otimes \nabla p_h)\right) + E^*_{r/4}\left(\nabla \cdot (v \otimes \bar{v})\right) =: \nabla p_{11} + \nabla p_{12}. \]
Let \( t \leq s < 0 \) be fixed. Since \( \Delta p_{11}(s) = -\nabla \cdot \nabla \cdot (\bar{v}(s) \otimes \nabla p_h(s)) \), using Lemma 3.7 and Remark 3.8, we find

\[
\|p_{11}(s)\zeta^n\|_{L^2(K)} \\
\leq c\left\{ \|(\bar{v}(s) \otimes \nabla p_h(s))\zeta^n\|_{L^2(U)} + r^{-\frac{n-1}{2}} \|\bar{v} \otimes \nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(K)} \right. \\
+ \left. r^{-\frac{n-1}{2}} \|p_{11}(s)\|_{L^{\frac{2n}{n+1}}(K)} \right\} \\
\leq c\left\{ \|(\bar{v}(s) \otimes \nabla p_h(s))\zeta^n\|_{L^2(U)} + r^{-\frac{n-1}{2}} \|\bar{v}(s) \otimes \nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(U)} \right\},
\]

where \( K = \text{supp}(\nabla \zeta) \). Applying Hölder’s inequality, we infer

\[
\|\bar{v} \otimes \nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(U)} \leq c\|\bar{v}(s)\|_{L^2(U)}\|\nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(U)} \\
\leq c\|\bar{v}(s)\|_{L^2(U)}\|v(s)\|_{L^{\frac{2n}{n+1}}(U)} \\
\leq c\|\bar{v}(s)\|_{L^2(U)}\|v(s)\|\|v(s)\|_{L^\infty}^{\frac{1}{n}}.
\]

Furthermore, since \( \mu \geq \frac{n}{n+2} \), we have \( \mu \frac{n-1}{n} \geq (1 - \mu) \frac{n-1}{2} \), and therefore from (3.7) we obtain

\[
\|v(s)\|_{L^\infty}^{\frac{1}{n}} \leq C_1^\frac{1}{n}(-s)^{-\frac{n}{n}} = C_1^\frac{1}{n}(-t)^{\mu}(-s)^{-\mu} \\
\leq C_1^\frac{1}{n}(-t)^{(1-\mu)\frac{n-1}{2}(-s)^{-\mu}} \leq cr^\frac{n-1}{2}(-s)^{-\mu}.
\]

Hence, we estimate

\[
r^{-\frac{n-1}{2}} \|\bar{v} \otimes \nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(U)} \leq cC_1\|\bar{v}(s)\|_{L^2(U)}(-s)^{-\mu}.
\]

Similarly,

\[
\|\bar{v}(s) \otimes \nabla p_h(s)\zeta^n\|_{L^2(U)} \leq cr^{-\frac{n-1}{2}} \|\bar{v}(s)\|_{L^2(U)}\|\nabla p_h(s)\|_{L^{\frac{2n}{n+1}}(U)} \\
\leq cr^{-\frac{n-1}{2}} \|\bar{v}(s)\|_{L^2(U)}\|v(s)\|_{L^\infty}^{\frac{n-1}{2}} \|v(s)\|_{L^\infty}^{\frac{1}{n}} \\
\leq cC_1\|\bar{v}(s)\|_{L^2(U)}(-s)^{-\mu}.
\]

This shows that

\[
\|p_{11}(s)\zeta^n\|_{L^2(K)} \leq cC_1\|\bar{v}(s)\|_{L^2(U)}(-s)^{-\mu}.
\]

Similarly we get

\[
\|p_{12}(s)\zeta^n\|_{L^2(K)} \leq cC_1\|\bar{v}(s)\|_{L^2(U)}(-s)^{-\mu}.
\]

We now assume (3.13) is true for \( \bar{v}(t) = v(t) - E_{r/4}(v(t)) \) with \( k \), then inserting this into the above estimates for \( p_{11} \), \( p_{12} \), we find

\[
(3.21) \quad \|p_1(s)\zeta^n\|_{L^2(K)} \leq cC_1^\frac{1}{2} C_0^{\frac{1}{2}} 4^{-k} + k(-s)^{\frac{1}{2}(1-\mu)-k} F^{-\frac{1}{2}}.
\]

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We multiply (3.19) by \(-\tilde{\zeta}^{2n}\), integrate the result over \(U \times (t, 0), -1 < t < 0\), and apply integration by parts. This yields

\[
\frac{1}{2} \left\| \tilde{v}(t) \zeta^{n} \right\|_{L^{2}(U)}^{2} = - \int_{t}^{0} \int_{U} v(s) \cdot \nabla \zeta |\tilde{v}(s)|^{2} \zeta^{2n-1} dx ds - \int_{t}^{0} \int_{U} \tilde{v}(s) \otimes \tilde{v}(s) : \nabla^{2} p_{h}(s) \zeta^{2n} dx ds
\]

\[
+ 2 \int_{t}^{0} \int_{U} p_{1}(s) \tilde{v}(s) \cdot \zeta^{2n-1} \nabla \zeta dx ds = I + II + III. \tag{3.22}
\]

Applying Cauchy-Schwarz’s inequality, and again using the assumption of (3.13) for \(k\) and \(\frac{1}{4}\) in place of \(r\) together with \(\|v(s)\|_{L^{\infty}} \leq c(-s)^{(1-\mu)}\), we find

\[
I \leq cr^{-1} \int_{t}^{0} \left\| \tilde{v}(s) \right\|_{L^{2}(U)}^{2} \left\| v(s) \right\|_{L^{\infty}(U)} dx ds \leq cr^{-1} 4^{k^{2}} C_{0}^{k} \left( \frac{r}{4} \right)^{-k} \int_{t}^{0} (-s)^{(1-\mu)k-\mu} ds
\]

\[
\leq cC_{1} C_{0}^{k+1}(k+1)^{2} (-t)^{(1-\mu)(k+1)} r^{-k-1}.
\]

Using Lemma A.2, we estimate

\[
\left\| \nabla^{2} p_{h}(s) \zeta^{2n} \right\|_{L^{\infty}(U)} \leq cr^{-\frac{n+1}{2}} \left\| \nabla p_{h} \right\|_{L^{\frac{2n}{n+1}}(U)} \leq cr^{-\frac{n+1}{2}} \left\| v(s) \right\|_{L^{\frac{2n}{n+1}}(U)} \leq cr^{-\frac{n+1}{2}} \left\| v(s) \right\|_{L^{2}(U)} \frac{1}{2} \leq cr^{-1} (-s)^{-\mu},
\]

where for the second inequality we have applied (2.9) with \(q = \frac{2n}{n-1}\), while for the fourth inequality we have used (3.20). Thus, by similar reasoning as we have used for the estimation of \(I\), we get

\[
II \leq \int_{t}^{0} \left\| \tilde{v}(s) \right\|_{L^{2}(U)}^{2} \left\| \nabla^{2} p_{h}(s) \zeta^{2n} \right\|_{L^{\infty}(U)} dx ds
\]

\[
\leq cC_{1} C_{0}^{k} 4^{k^{2}} \left( \frac{r}{4} \right)^{-k} \int_{t}^{0} (-s)^{(1-\mu)k-\mu} ds
\]

\[
\leq cC_{1} C_{0}^{k+1}(k+1)^{2} (-t)^{(1-\mu)(k+1)} r^{-1-k}.
\]

Finally, applying Cauchy-Schwarz’s inequality together with (3.21), and the assumption (3.13) for \(k\), we estimate

\[
III \leq cC_{1} C_{0}^{k+1}(-t)^{(1-\mu)(k+1)} r^{-k-1}.
\]

Inserting the estimates of \(I, II\) and \(III\) into (3.22), we are led to

\[
\left\| v(t) - E_{r/4}(v(t)) \right\|_{L^{2}(U_{1})}^{2} \leq cC_{1} C_{0}^{k}(k+1)^{2} (-t)^{(1-\mu)(k+1)} r^{-k-1}.
\]
Since

\[ v(t) - E_r^*(v(t)) = v(t) - E^*_{r/4}(v(t)) + E^*_{r/4}(v(t)) - E_r^*(v(t)) \]
\[ = v(t) - E^*_{r/4}(v(t)) + E^*_{r/4}(v(t)) - E_r^*(v(t)) \]
\[ = v(t) - E^*_{r/4}(v(t)) + E_r^*(v(t) - E^*_{r/4}(v(t))) \]
\[ = \tilde{v}(t) - E_r^*(\tilde{v}(t)) \]

in \( U_1 \), we estimate

\[
\|v(t) - E_r^*(v(t))\|^2_{L^2(B(r)c)} = \|\tilde{v}(t) - E_r^*(\tilde{v}(t))\|^2_{L^2(B(r)c)} \leq c\|\tilde{v}(t)\|^2_{L^2(B(r)c)} \\
= c\|v(t) - E^*_{r/4}(v(t))\|^2_{L^2(U_1)} \\
\leq cC_1C_0^kA^2(k+1)^2(t)^{(1-\mu)(k+1)}r^{k-1}.
\]

This shows that (3.13) holds for \( k + 1 \) with \( C_0 = cC_1 \).

3.4 Proof of Theorem 3.1

Let us fix \( \theta \) so that

(3.23) \[ 0 < \theta < \frac{1}{n + 2} \]

For given solution \((v, p)\) to the Euler equations we define

\[
w(x, t) = v((-t)^\theta x, t), \\
\pi(x, t) = (-t)^{-\theta} p((-t)^\theta x, t), \quad (x, t) \in \mathbb{R}^n \times (-1, 0).
\]

Then, \((w, \pi)\) solves

(3.24) \[ \frac{\partial w}{\partial t} + \theta(-t)^{-1} x \cdot \nabla w + (-t)^{-\theta} (w \cdot \nabla) w = -\nabla \pi, \]
(3.25) \[ \nabla \cdot w = 0. \]

Using the transformation formula, we find

\[
\|w(t)\|^2_{L^2} = (-t)^{-n\theta} \|v(t)\|^2_{L^2} = (-t)^{-n\theta} \|v(-1)\|^2_{L^2}.
\]

On the other hand, by Lemma 3.5 we infer that for any \( 0 < \beta < n + 2 \)

\[
\|w(t)\|^2_{L^2(B(1)c)} = (-t)^{-n\theta} \|v(t)\|^2_{L^2(B((-t)^\theta)c)} \\
\leq (-t)^{-n\theta} \int_{\{|x| > (-t)^\theta\}} |v(t)|^2 \frac{|x|^\beta}{(-t)^{\beta\theta}} dx \\
\leq C(-t)^{-n\theta}(-t)^{-\beta\theta}(-t)^{\frac{\beta}{n+2}} = C(-t)^{-(n+\beta)\theta + \frac{\beta}{n+2}}.
\]

Choosing \( \beta = n \), we get \(- (n + \beta)\theta + \frac{2}{n+2} \beta > 0\) for \( \theta \) satisfying (3.23). Therefore

(3.26) \[ \lim_{t \to 0} \|w(t)\|^2_{L^2(B(1)c)} = 0. \]
By $\mathbb{P}_r$, $0 < r < +\infty$ we denote the Helmholtz projection from $L^2(B(r)^c)$ onto $L^2(O)$. We easily calculate

$$\mathbb{P}_1 w(x, t) = (\mathbb{P}_{(-t)^\theta} v)((-t)^\theta x, t).$$

To see this we only need to check that $w(x, t) - (\mathbb{P}_{(-t)^\theta} v)((-t)^\theta x, t)$ is a gradient field. Indeed,

$$w(x, t) - (\mathbb{P}_{(-t)^\theta} v)((-t)^\theta x, t) = v((-t)^\theta x, t) - (\mathbb{P}_{(-t)^\theta} v)((-t)^\theta x, t)
= \nabla q((-t)^\theta x, t).$$

Appealing to Lemma 3.6 for $\mu = \frac{n}{n+2}$, we see that for every $r > 0$ and $k \in \mathbb{N}$ it holds

$$\|v(t) - E^*_r(v(t))\|_{L^2(B(r)^c)}^2 \leq C(k)(-t)^\frac{2}{n+2} r^{-k},$$

where $C(k)$ depends on $k$ and $C_1$ only. Noting that

$$\mathbb{P}_r v(t) = \mathbb{P}_r (v(t) - E^*_r(v(t))),$$

from the above estimate we deduce

$$\|\mathbb{P}_r v(t)\|_{L^2(B(r)^c)}^2 \leq \|(v(t) - E^*_r(v(t)))\|_{L^2(B(r)^c)}^2 \leq C(k)(-t)^\frac{2}{n+2} r^{-k}.$$

This yields

$$\|\mathbb{P}_1 w(t)\|_{L^2(B(1)^c)}^2 \leq (-t)^{-n\theta} \|\mathbb{P}_{(-t)^\theta} v(t)\|_{L^2(B((-t)^\theta)^c)}^2
\leq C(k)(-t)^\frac{2}{n+2} (-t)^{-n\theta} (-t)^{-k\theta} = C(k)(-t)^k(-t)^{-n\theta}.$$

Since $\theta$ satisfies (3.23), this shows the decay rate of $\|\mathbb{P}_1 (w(t))\|_{L^2(B(1)^c)}$ as $t \to 0$ is of any order $O((-t)^k)$.

Now we set $w_0(t) := \mathbb{P}_1 (w(t))$ on $B(1)^c$ and $\nabla q_h(t) = w(t) - w_0(t)$. Since $\nabla \cdot w = 0$, we see that $\nabla q_h(t)$ is harmonic, and therefore it also solves the system (3.24)-(3.25) with

$$\tilde{\pi} = \partial_t q_h + \theta (-t)^{-1} \left(x \cdot \nabla q_h - q_h + \frac{1}{2} |\nabla q_h|^2\right)$$

in place of $\pi$. Taking the difference of the two equations for $w$ and $\nabla q_h$ respectively, we get

$$\begin{align*}
(3.27) & \quad \frac{\partial w_0}{\partial t} + \theta (-t)^{-1} x \cdot \nabla w_0 + (-t)^{-\theta} (w \cdot \nabla) w - (-t)^{-\theta} \nabla q_h \cdot \nabla^2 q_h = -\nabla (\pi - \tilde{\pi}), \\
(3.28) & \quad \nabla \cdot w_0 = 0,
\end{align*}$$

the both of which are in $B(1)^c \times (-1, 0)$. Note that

$$\begin{align*}
(w \cdot \nabla) w - \nabla q_h \cdot \nabla^2 q_h &= (w_0 \cdot \nabla) w + (\nabla q_h \cdot \nabla) w - \nabla q_h \cdot \nabla^2 q_h \\
&= (w_0 \cdot \nabla) w + (\nabla q_h \cdot \nabla) w_0.
\end{align*}$$
Therefore, (3.27) turns into

\[
(3.29) \quad \frac{\partial w_0}{\partial t} + \theta(-t)^{-1}x \cdot \nabla w_0 + (-t)^{-\theta}(w_0 \cdot \nabla)w + (-t)^{-\theta}(\nabla q_h \cdot \nabla)w_0 = -\nabla(\pi - \tilde{\pi}).
\]

We now multiply \((3.29)\) by \(-w_0(s)\), integrate it over \(B(1)^c \times (t, 0)\), and then apply the integration by parts. Taking into account \((3.26)\), we have the identity

\[
(3.30) \quad \frac{1}{2} \|w_0(t)\|^2_{L^2(B(1)^c)} + \frac{n \theta}{2} \iint_{B(1)^c} (s)^{-1}|w_0(s)|^2 dx ds + \frac{\theta}{2} \iint_{\partial B(1)} (s)^{-1}w_0^2 dS ds
\]

where we used the fact

\[
\iint_{B(1)^c} \nabla q_h(s) \cdot \nabla|w_0(s)|^2 dx = \iint_{B(1)^c} w(s) \cdot \nabla|w_0(s)|^2 dx = - \iint_{\partial B(1)} x \cdot w(s)|w_0(s)|^2 dS
\]

for the second integral of the right-hand side. Since \(-\theta - \frac{n}{n+2} > -1\) due to \((3.23)\), we may choose \(-1 < t_0 < 0\) so that

\[
\max_{x \in \partial B(1)} (-s)^{-\theta}|w(x, s)| \leq (-s)^{-\theta}\|v(s)\|_{L^\infty} \leq c(-s)^{-\theta - \frac{n}{n+2}} \\
\leq \frac{\theta}{2} (-s)^{-1} \quad \forall t_0 \leq s < 0,
\]

which implies that the second term of the right-hand side of \((3.30)\) can be absorbed into the third term of the left-hand side of \((3.30)\). Then, since \((-s)^{-\theta}|\nabla w(s)| \leq \|\nabla v(s)\|_{L^\infty} \leq \frac{a}{2} (-s)^{-1}\) for all \(s \in (-1, 0)\), where we set

\[
a = 2 \sup_{-1 < s < 0} (-s)\|\nabla v(s)\|_{L^\infty},
\]

we obtain

\[
(3.31) \quad \|w_0(t)\|^2_{L^2(B(1)^c)} \leq a \iint_{B(1)^c} (s)^{-1}|w_0(s)|^2 dx ds \quad \forall t \in (t_0, 0).
\]

Let us define

\[
X(t) := \iint_{B(1)^c} (s)^{-1}|w_0(s)|^2 dx ds.
\]

Then, from \((3.31)\) it follows that

\[
(3.32) \quad -(-t)X'(t) \leq aX(t) \quad \forall t \in (t_0, 0),
\]
which is equivalent to \( X'(t) \geq -a(-t)^{-1}X(t) \). If we assume that \( X(t) > 0 \) for all \( t \in (t_0, 0) \), we get

\[
(\log X)' \geq (\log(-t)^a)' \iff \left( \log \frac{X(t)}{(-t)^a} \right)' \geq 0.
\]

Accordingly, \( \log \frac{X(t)}{(-t)^a} \) is nondecreasing, and by the monotonicity of \( \log \) the function \( \frac{X(t)}{(-t)^a} \) is also nondecreasing. However, by the fast decay of \( \|w_0(t)\|_{L^2(B(1)^c)} \) as \( t \to 0 \) \( X(t) \) is decaying faster to zero than \( (-t)^a \). Therefore

\[
\lim_{t \to 0} \frac{X(t)}{(-t)^a} = 0,
\]

which is a contradiction to \( \frac{X(t)}{(-t)^a} \geq \frac{X(t_0)}{(-t_0)^a} > 0 \) for all \( t \in (t_0, 0) \). Consequently, \( X \equiv 0 \).

This shows that \( \nabla \times w(t) = 0 \) on \( B(1)^c \) for all \( t_0 < t < 0 \). This implies that the vorticity \( \omega(t) = \nabla \times v(t) \) also vanishes on \( B((-t)^c) \) for all \( t_0 < t < 0 \), namely

\[
\{ x \in \mathbb{R}^n | |\omega(x,t)| > 0 \} \subset B((-t)^c) \quad \forall t \in (t_0, 0).
\]

Since the measure of the set \( \{ x \in \mathbb{R}^n | |\omega(x,t)| > 0 \} \) is conserved for \( t \in (-1, 0) \) by virtue of the vorticity transport formula (see e.g. [23, Proposition 1.8]), we have

\[
\text{meas}\{ x \in \mathbb{R}^n | |\omega(x,t_0)| > 0 \} = \text{meas}\{ x \in \mathbb{R}^n | |\omega(x,t)| > 0 \} \leq c(-t)^{n\theta} \to 0 \quad \text{as} \quad t \to 0.
\]

Whence, \( \omega(t_0) \equiv 0 \), which implies that \( v(t_0) \) is harmonic. Recalling that \( v(t_0) \in L^2(\mathbb{R}^n) \), we conclude that \( v(t_0) \equiv 0 \), and hence \( v \equiv 0 \).

4 Proof of Theorem 1.1

4.1 Local criterion for the energy non-concentration

In this first subsection we remove one point energy concentration for local weak solution to the Euler equations satisfying the local energy inequality under a weaker condition than the one in Shvydkoy [26]. In our discussion below we make use of the following notation. We define the following space time cylinder

\[
Q(r) = B(r) \times I(r), \quad \text{where} \quad I(r) = (-r^{\frac{n+2}{2}}, 0).
\]

Let \( 0 < R < +\infty \) be fixed. We consider the Euler equations

\[
(4.1) \quad \partial_t v + (v \cdot \nabla)v = -\nabla p, \quad \nabla \cdot v = 0 \quad \text{in} \quad Q(R).
\]

The main result of this subsection is the following
Proof of Theorem 4.1: Let \( v \in L^\infty(I(R); L^2(B(R))) \cap L^3(Q(R)) \) be a local suitable weak solution to \((4.1)\) according to Definition 2.1, such that the local energy inequality \((2.11)\) is fulfilled. Furthermore, we assume that

\[
(4.2) \quad \sup_{0 < r \leq R} r^{-1} \|v\|_{L^3(Q(r))}^3 < +\infty, \quad \lim_{r \to 0^+} r^{-1} \|v\|_{L^3(Q(r))}^3 = 0.
\]

Then, there is no energy concentration at the point \( x = 0 \) as \( t \to 0^- \). More precisely, if \( \sigma_0 \in \mathcal{M}_{|v|^2}(0) \) then

\[
(4.3) \quad \sigma_0(\{0\}) = 0.
\]

Remark 4.2. In \([26]\) Shvydkoy showed that if \( v \in L^q(-1,0; L^\infty(\Omega)) \cap L^\infty(-1,0; L^2(\Omega)) \), 
\( q = \frac{n}{n-1} + 1 \), is a suitable weak solution, then the measure in \( \mathcal{M}_{|v|^2}(0) \) has no atoms in \( \Omega \). This actually follows from the above theorem immediately. Indeed, let \( Q(r) \subset \Omega \times (-1,0) \), then

\[
r^{-1} \|v\|_{L^3(Q(r))}^3 = r^{-1} \int_{-r}^{0} \int_{B(r)} |v|^3 \, dx \, dt \leq \|v\|_{L^\infty(-1,0; L^2(\Omega))} \int_{-r}^{0} \|v\|_{L^\infty(B(r))}^3 \, dt \leq \|v\|_{L^\infty(-1,0; L^2(\Omega))}^3 \left( \int_{-r}^{0} \|v\|_{L^\infty(B(r))}^{\frac{2}{n+2}} \, dt \right)^{\frac{n}{n+2}} \to 0
\]
as \( r \to 0 \).

Proof of Theorem 4.1. Let \( \nabla p_h \) denote the local pressure \(-E^*_{B(R)}(v)\), which has been defined in Definition 2.1. By virtue of Lemma 2.3, there exists a unique measure valued trace \( \tilde{\sigma} \in L^\infty(I(R); \mathcal{M}^+(B(R))) \) of the function \( |\tilde{v}(\cdot)|^2 \), where \( \tilde{v} = v + \nabla p_h \) (cf. also Remark 2.4). Thanks to Lemma 2.3, we only need to show that \( \tilde{\sigma}_0 := \tilde{\sigma}(0) \) has no atoms. In fact, it suffices to prove that \( \tilde{\sigma}_0(\{0\}) = 0 \).

Following the arguments of Section 2, we define the local pressure

\[
\nabla p_h = -E^*_{B(R)}(v), \quad \nabla p_0 = -E^*_{B(R)}((v \cdot \nabla)v).
\]

Recalling Definition 2.1, the function \( \tilde{v} = v + \nabla p_h \) solves the equation

\[
(4.4) \quad \partial_t \tilde{v} + (v \cdot \nabla) \tilde{v} = -\nabla p_0 \quad \text{in} \quad Q(R).
\]

Since \( v \) is a local suitable weak solution to the Euler equations (cf. Definition 2.1) by means of Lemma 2.3, the generalized local energy inequality is satisfied for a.e. \( t \in I(R) \), and for all nonnegative \( \phi \in C^\infty_c(B(R)) \)

\[
\int_{B(R)} |\tilde{v}(t)|^2 \phi \, dx 
\]

\[
\leq \int_{B(R)} \phi d\tilde{\sigma}_0 + \int_0^1 \int_{B(R)} |\tilde{v}|^2 v \cdot \nabla \phi \, dx \, d\tau + \int_0^1 \int_{B(R)} 2p_0 \tilde{v} \cdot \nabla \phi \, dx \, d\tau
\]

\[
(4.5) \quad + \int_0^1 \int_{B(R)} v \otimes v : \nabla^2 p_h \phi \, dx \, d\tau.
\]

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Let $0 < r < \frac{R}{2}$. Let $\zeta \in C_c^\infty(B(R))$ denote a cut off function such that $0 \leq \zeta \leq 1$ in $B(R)$, $\zeta \equiv 1$ on $B(R/2)$. Furthermore, let $\eta \in C_c^\infty(B(r/2))$ denote a cut off function such that $0 \leq \eta \leq 1$ in $B(r/2)$, $\eta \equiv 1$ on $B(r/4)$, and $|\nabla^k \eta| \leq c_k r^{-k}$ for all $k \in \mathbb{N}$. Let $-r + \frac{a+2}{2} < t < 0$. In (4.5) we put $\phi = \zeta(1 - \eta)$. This yields

\begin{align*}
(4.6) \quad \int_{B(R)} |\tilde{v}(t)|^2 \zeta(1 - \eta) dx & \\
& \leq \int_{B(R)} \zeta(1 - \eta) d\bar{\sigma}_0 + \int_0^t \int_{B(R)} |\tilde{v}|^2 v \cdot \nabla \zeta dx d\tau - \int_0^t \int_{B(R)} |\tilde{v}|^2 v \cdot \nabla \eta dx d\tau \\
& \quad + \int_0^t \int_{B(R)} 2p_0 \tilde{v} \cdot \nabla \zeta dx d\tau - \int_0^t \int_{B(R)} 2p_0 \tilde{v} \cdot \nabla \eta dx d\tau \\
& \quad + \int_0^t \int_{B(R)} v \otimes v : \nabla^2 p_h \zeta(1 - \eta) dx d\tau \\
(4.7) \quad & = \int_{B(R)} \zeta(1 - \eta) d\bar{\sigma}_0 + I + II + III + IV + V.
\end{align*}

In our discussion below we frequently make use of the following inequalities for almost every $\tau \in (-1, 0)$

\begin{align*}
(4.8) \quad & \|p_0(\tau)\|_{L^{3/2}(B(R))} \leq c\|v(\tau)\|_{L^3(B(R))}^2, \\
(4.9) \quad & \|\nabla p_h(\tau)\|_{L^3(B(R))} \leq c\|v(\tau)\|_{L^3(B(R))}
\end{align*}

By means of Hölder’s inequality and Young’s inequality, using (4.9), we easily get

$$I + II \leq cR^{-1}\|v\|_{L^3(B(R) \times I(\tau))}^3 + cr^{-1}\|v\|_{L^3(Q(\tau))}^3 + cr^{-1}\|\nabla p_h\|^3_{L^3(Q(\tau))}.$$ 

Recalling that $\nabla p_h(\tau)$ is harmonic in $B(R)$ and employing (4.9), we get for the last term on the right-hand side of the above inequality

$$r^{-1}\|\nabla p_h\|^3_{L^3(Q(\tau))} \leq cr^{-1}\|\nabla p_h\|^3_{L^3(B(R) \times I(\tau))} \leq R^{-1}\|v\|^3_{L^3(B(R) \times I(\tau))}.$$ 

Combining the last two inequalities, we arrive at

$$I + II \leq cR^{-1}\|v\|_{L^3(B(R) \times I(\tau))}^3 + cr^{-1}\|v\|_{L^3(Q(\tau))}^3.$$ 

Applying Hölder’s inequality and using (4.8) for almost every $\tau \in I(\tau)$, we get

$$III \leq cR^{-1}\|v\|_{L^3(B(R) \times I(\tau))}^3.$$
We proceed to estimate $V$. By virtue of Sobolev’s embedding theorem we see that $W^{m,2}(B(R)) \hookrightarrow L^3(B(R))$ for $m \geq \frac{3}{\theta}$. This together with Lemma [A.1] and (4.9) gives

$$
\|\nabla^2 p_h(s)\zeta\|_{L^3(B(R))} \leq \sum_{k=0}^m R^{-\frac{\theta}{2}+k}\|\nabla^k (\nabla^2 p_h(s)\zeta)\|_{L^3(B(R))}
$$

$$
\leq cR^{-\frac{\theta}{2}}\|\nabla p_h(s)\|_{L^2(B(R))} \leq cR^{-\frac{\theta}{2}+1}\|v\|_{L^2(B(R))}
$$

$$
\leq cR^{-1}\|v\|_{L^3(B(R))}.
$$

Using Hölder’s inequality together with the above estimate of $\nabla^2 p_h$ we obtain

$$
V \leq cR^{-1}\|v\|_{L^3(B(R) \times I(r))}^3.
$$

It only remains the estimate the integral $IV$, which contains the pressure $p_0$. Observing the condition (4.2), we find that

$$
\sup_{0<\rho \leq R} \rho^{-1}\|v \otimes v\|_{L^{3/2}(Q(\rho))}^{3/2} < +\infty.
$$

Applying Lemma [A.4] with $f = v \otimes v$, $p = \frac{3}{2}$, and $\lambda = 1$ (cf. also [8, Lemma 2.8]), it can be checked that

$$
(4.10) \quad \sup_{0<\rho \leq R} \rho^{-1}\|p_0\|_{L^{3/2}(Q(\rho))}^{3/2} < +\infty.
$$

Applying Hölder’s inequality along with (4.10), we infer

$$
IV \leq c \sup_{0<\rho \leq R} \rho^{-1}\|p_0\|_{L^{3/2}(Q(\rho))}^{3/2} (r^{-1}\|v\|_{L^3(Q(r))}^3)^{1/3}.
$$

Inserting the estimates of $I, \ldots, VII$ into the right-hand side of (4.7), we arrive at

$$
\begin{align*}
\int_{B(R)} |\tilde{v}(t)|^2 \zeta dx \\
\leq \int_{B(R)} \zeta (1-\eta) d\sigma_0 + \int_{B(r)} |\tilde{v}(t)|^2 \eta dx \\
+ cR^{-1}\|v\|_{L^3(B(R) \times I(r))}^3 + c\left(\frac{r}{R}\right)^{n/2}\|v\|_{L^\infty(-R^{3/2},0;L^2(B(R)))}^3 \\
+ cr^{-1}\|v\|_{L^3(Q(r))}^3 + c \sup_{0<\rho \leq R} \rho^{-1}\|p_0\|_{L^{3/2}(Q(\rho))}^{3/2} (r^{-1}\|v\|_{L^3(Q(r))}^3)^{1/3}.
\end{align*}
$$

(4.11)

Appealing to (4.2), we may choose a sequence $\{r_k\}$ in $(0, R)$ such that $r_k \to 0$ as $k \to +\infty$, and

$$
(4.12) \quad r_k^{-1}\|v\|_{L^3(Q(r_k))}^3 \to 0 \quad \text{as} \quad k \to +\infty.
$$

By means of Jensen’s inequality, having $r_k^{-\frac{n+2}{2}}\|v\|_{L^2(Q(r_k))}^2 \leq cr_k^{-2/3}\|v\|_{L^3(Q(r_k))}^2$, (4.12) gives

$$
(4.13) \quad r_k^{-\frac{n+2}{2}}\|v\|_{L^2(Q(r_k))}^2 \to 0 \quad \text{as} \quad k \to +\infty.
$$

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We take \( t_k \in I(r_k) \) such that \( \|v(t_k)\|_{L^2(B(r))}^2 \leq r_k^{\frac{np}{n+2}} \|v\|_{L^2(Q(r_k))}^2 \) for all \( k \in \mathbb{N} \). We now consider (4.11) with \( t = t_k, r = r_k, \eta = \eta_k \). Thanks to (4.12) and (4.13) all terms except the first and second integral on the right-hand side of (4.11) tend to zero as \( k \to +\infty \). This shows that

\[
\limsup_{k \to \infty} \int_{B(R)} |\bar{v}(t_k)|^2 \zeta dx \leq \limsup_{k \to \infty} \int_{B(R)} \zeta(1 - \eta_k) d\bar{\sigma}_0 = \limsup_{k \to \infty} \int_{B(R)} \zeta(1 - \eta_k) d\bar{\sigma}_0.
\]

On the other hand, by means of the weakly-* left continuity of \( \bar{\sigma} \), using the above inequality, we obtain

\[
\int_{B(R)} \zeta d\bar{\sigma}_0 \leq \limsup_{k \to \infty} \int_{B(R)} \zeta(1 - \eta_k) d\bar{\sigma}_0 = \int_{B(R)} \zeta d\bar{\sigma}_0 - \liminf_{k \to \infty} \int_{B(R)} \eta_k d\bar{\sigma}_0,
\]

which in turn shows that

\[
\bar{\sigma}_0(\{0\}) \leq \lim_{k \to \infty} \int_{B(R)} |\bar{v}(t_k)|^2 \zeta dx \leq \liminf_{k \to \infty} \int_{B(r_k)} \eta_k d\bar{\sigma}_0 = 0.
\]

Whence, the claim. \( \blacksquare \)

### 4.2 Blow-up argument

Let \( \Omega \subset \mathbb{R}^n \). In what follows we use the following notation for the semi-norm for the fractional derivatives of functions in the Sobolev-Slobodeckiï spaces

\[
|f|_{W^{\theta,p}(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\theta p}} dx dy \right)^{1/p}, \quad f \in W^{\theta,p}(\Omega), \quad 0 < \theta < 1, \quad p \geq 1.
\]

**Lemma 4.3.** Let \( v \in L^{\infty}(I(R); L^2(B(R))) \cap L^3(Q(R)), 0 < R < +\infty \), be a local suitable weak solution to the Euler equations (4.1). We assume the following local Type I condition in terms of a fractional Sobolev space norm and energy concentration at time \( t = 0 \).

(i) \( \exists 0 < \theta < \frac{1}{3} : \sup_{r \in (0,R)} r^{3\theta - 1} \int_{I(r)} |v(t)|_{W^{\theta,3}(B(r))}^3 < +\infty \).

(ii) There exists \( \sigma_0 \in \mathcal{M}_{|v|^2}(0) \) with \( \sigma_0(\{0\}) > 0 \).

Then there exists a nontrivial solution \( v^* \in L^{\infty}(-1, 0; L^2(\mathbb{R}^n)) \cap L^3([-1, 0); W^{\theta,3}(\mathbb{R}^n)) \) to the Euler equations which fulfills the following Type I blow-up condition and energy concentration at time \( t = 0 \)

(iii) \( \sup_{r \in (0,R)} r^{3\theta - 1} \int_{I(r)} |v^*(t)|_{W^{\theta,3}(B(r))}^3 < +\infty \).
(iv) $\mathcal{M}_{[v^*]^2}(0) = \{E_0\delta_0\}$ for some constant $E_0 > 0$.

Furthermore, there holds the local energy inequality for all $\phi \in C_c^\infty(\mathbb{R}^n)$ and for a.e. $-1 \leq t \leq s < 0$,

$$
\int_{\mathbb{R}^n} |v^*(t)|^2 \phi dx \leq \int_{\mathbb{R}^n} |v^*(s)|^2 \phi dx + \int_t^s \int_{\mathbb{R}^n} \left( |v|^2 + 2p^* \right) v^* \cdot \nabla \phi dx d\tau.
$$

**Proof:** 1. Scaling invariant $L^3$ estimate. For notational convenience we set

$$K_0 = \|v(t)\|_{L^2(-R^5/2,0:L^2(B(R))}, \quad K_1 = \left( \sup_{r \in (0,R)} r^{3\theta-1} \int_{I(r)} |v(t)|^{3 \frac{\theta}{3-\theta}} v^{3 \frac{\theta}{3-\theta}} (B(r)) \right)^{1/3}.$$

Let $0 < r \leq R$. By means of Hölder’s inequality together with Sobolev’s inequality, we get

$$
\|v(s)\|_{L^3(B(r))}^3 \leq \|v(s)\|_{L^2(B(r))}^{\frac{6\theta}{3-\theta}} \int_{I(r)} |v(t)|^{\frac{3\theta}{3-\theta}} v^{\frac{3\theta}{3-\theta}} (B(r)) \leq cr^{-n/2} \|v(s)\|_{L^2(B(r))}^3 + c\|v(s)\|_{L^2(B(r))} \|v(t)\|_{W^{3,\theta}(B(r))}^\frac{3\theta}{3-\theta}.
$$

Integrating the both sides over $I(r)$, and using the Hölder’s inequality, we obtain

$$
\int_{I(r)} \|v(s)\|_{L^3(B(r))}^3 ds \leq cr^{-n/2} \int_{I(r)} \|v(s)\|_{L^2(B(r))}^3 ds + c \int_{I(r)} \|v(s)\|_{L^2(B(r))} \|v(t)\|_{W^{3,\theta}(B(r))}^\frac{3\theta}{3-\theta} ds
$$

$$
\leq cr^{-n/2} \int_{-I(r)} \|v(s)\|_{L^2(B(r))}^3 ds + \frac{cr}{1+29} \left( r^{-5/2} \int_{I(r)} \|v(s)\|_{L^2(B(r))}^3 ds \right)^\frac{29}{1+29} \left( \int_{I(r)} |v(s)|_{W^{3,\theta}(B(r))}^\frac{3\theta}{3-\theta} ds \right)^\frac{1}{1+29}
$$

$$
\leq cr^{-n/2} \int_{-I(r)} \|v(s)\|_{L^2(B(r))}^3 ds + crK_1^{\frac{3}{1+29}} \left( r^{-\frac{n+2}{2}} \int_{I(r)} \|v(s)\|_{L^2(B(r))}^3 ds \right)^\frac{29}{1+29}.
$$

Multiplying both sides by $r^{-1}$, we are led to

$$
r^{-1} \|v\|_{L^3(\Omega)}^3 \leq cr^{-\frac{n+2}{2}} \int_{-I(r)} \|v(s)\|_{L^2(\Omega)}^3 ds + cK_1^{\frac{3}{1+29}} \left( r^{-\frac{n+2}{2}} \int_{I(r)} \|v(s)\|_{L^2(\Omega)}^3 ds \right)^\frac{29}{1+29}.
$$

(4.15)
Furthermore, from (4.15) we deduce that
\begin{equation}
\sup_{0<r\leq R} r^{-1} \|v\|_{L^2(Q(r))}^2 \leq c(K_0 + K_1)^3. \tag{4.16}
\end{equation}

2. **Blow up argument.** We assume there exists \(\sigma_0 \in \mathcal{M}_{|v|^2}(0)\) with \(\sigma_0(\{0\}) > 0\). Then from (4.15) and (4.16) together with Theorem 4.1, we have a positive constant \(\varepsilon > 0\) such that
\begin{equation}
\sup_{0<r\leq R} r^{-\frac{\alpha+2}{2}} \|v\|_{L^2(Q(r))}^2 \geq \varepsilon \quad \forall 0 < r \leq R. \tag{4.17}
\end{equation}

Otherwise, (4.15) yields \(\lim \inf_{r \to 0} r^{-1} \|v\|_{L^2(Q(r))}^2 = 0\), which by Theorem 4.1 would lead to the contradiction \(0 < \sigma_0(\{0\}) = 0\).

Now we take a decreasing sequence \(\{r_k\}\) in \((0, R)\) such that \(r_k \to 0\) as \(k \to \infty\). We define
\[v_k(x, t) = r_k^\frac{2}{\alpha} v(r_k x, r_k^{-\frac{\alpha+2}{2}} t), \quad (x, t) \in B_k \times (-1, 0), \quad k \in \mathbb{N},\]
where
\[B_k = B(r_k^{-1} R)\]
Clearly, \(v_k \in L^\infty(-1, 0; L^2(B_k)) \cap L^3(B_k \times (-1, 0))\) is a local suitable weak solution to the Euler equations in \(B_k \times (-1, 0)\). Furthermore, for every \(0 < \rho < +\infty\) the sequence \(\{v_k\}_{k \geq N}\) with \(r_N^{-1} R \geq \rho\) is bounded in \(L^\infty(-1, 0; L^2(B(\rho))) \cap L^3(-1, 0; W^{9,3}(B(\rho)))\).

Thus, by means of the reflexivity and the Banach-Alaoglu theorem, using Cantor’s diagonalization argument, eventually passing to a subsequence, we get a function \(v^* \in L^\infty(-1, 0; L^2(\mathbb{R}^n)) \cap L^3(-1, 0; W^{9,3}(\mathbb{R}^n))\) with \(\nabla \cdot v^* = 0\) in \(\mathbb{R}^n \times (-1, 0)\) in the sense of distributions such that for all \(0 < \rho < +\infty\)
\begin{equation}
v_k \to v^* \quad \text{weakly-}^* \quad \text{in} \quad L^\infty(-1, 0; L^2(B(\rho))) \quad \text{as} \quad k \to +\infty, \tag{4.18}
\end{equation}
\begin{equation}v_k \to v^* \quad \text{weakly} \quad \text{in} \quad L^3(-1, 0; W^{9,3}(B(\rho))) \quad \text{as} \quad k \to +\infty. \tag{4.19}
\end{equation}

We now define,
\[\nabla p_{h,k} = -E_{B_k}^*(v_k), \quad \nabla p_{0,k} = -E_{B_k}^*((v_k \cdot \nabla)v_k),\]
Setting \(\tilde{v}_k = v_k + \nabla p_{h,k}\), we see that \(\tilde{v}_k\) solves
\begin{equation}\partial_t \tilde{v}_k + (v_k \cdot \nabla) v_k = -\nabla p_{0,k}, \quad \text{in} \quad B_k \times (-1, 0) \tag{4.20}
\end{equation}
in the sense of distributions. By (4.20) having
\begin{equation}\|\nabla p_{h,k}(t)\|_{L^2(B_k)} \leq c\|v_k\|_{L^\infty(-1,0;L^2(B_k))}, \tag{4.21}
\end{equation}
for a.e. \(t \in (-1, 0)\), and recalling that \(\nabla p_{h,k}\) is harmonic, we can apply the mean value property along with Jensen’s inequality and (4.21) to find
\[\sup_{B_k/2} |\nabla p_{h,k}(t)| \leq c r_k^2 R^{-\frac{\alpha}{2}} \|v_k\|_{L^\infty(-1,0;L^2(B_k))}.\]
Consequently, $\nabla p_{h,k} \to 0$ uniformly on $B(\rho) \times (-1,0)$ as $k \to +\infty$ for all $0 < \rho < +\infty$. Furthermore, employing the identity (A.11), we see that for all $0 < \rho < +\infty$

$$\nabla p_{h,k} \to 0 \text{ strongly in } L^\infty(-1,0;W^{1,2}(B(\rho))) \text{ as } k \to +\infty.$$  

(4.22) 

Hence, together with (2.13) and (2.14) we find for all $0 < \rho < +\infty$

$$\tilde{v}_k \to v^* \text{ weakly-* in } L^\infty(-1,0;L^2(B(\rho))) \text{ as } k \to +\infty,$$

(4.23) 

$$\tilde{v}_k \to v^* \text{ weakly in } L^3(-1,0;W^{0,3}(B(\rho))) \text{ as } k \to +\infty.$$  

(4.24) 

On the other hand, from the estimate

$$\|p_{0,k}\|_{L^{3/2}(B_k \times (-1,0))} \leq \|v_k\|_{L^3(B_k \times (-1,0))}^2,$$

we infer that $\{p_{0,k}\}_{k \geq N}$ is bounded in $L^{3/2}(B(\rho) \times (-1,0))$. Thus, (4.20) shows that $\{\partial_t \tilde{v}_k\}_{k \geq N}$ is bounded in $L^{3/2}(-1,0;W^{-1,3/2}(B(\rho)))$. Taking into account that $\{v_k\}_{k \geq N}$ is bounded in $L^3(-1,0;W^{0,3}(B(\rho)))$, we are in a position to apply the compactness lemma due to Simon[27]. This together with (4.22) yields

$$v_k \to v^* \text{ strongly in } L^2(B(\rho) \times (-1,0)) \text{ as } k \to +\infty.$$  

(4.25) 

In particular, for a.e. $t \in (-1,0)$ and for all $0 < \rho < +\infty$ it holds

$$v_k(t) \to v^*(t) \text{ strongly in } L^2(B(\rho)) \text{ as } k \to +\infty.$$  

(4.26) 

Furthermore, by means of Sobolev’s embedding theorem it can be checked easily that $\{v_k\}_{k \geq N}$ is bounded in $L^q(B(\rho) \times (-1,0))$ for some $3 < q < +\infty$. Thus, (4.25) ensures that

$$v_k \to v^* \text{ strongly in } L^3(B(\rho) \times (-1,0)) \text{ as } k \to +\infty.$$  

(4.27) 

Accordingly, $v^*$ is a weak solution to the Euler equations. Furthermore, since each element of the sequence $\{\tilde{v}_k\}$ satisfies the local energy inequality, after letting $k \to +\infty$, taking into account (4.22), (4.23) and (4.27), we see that $v^*$ also fulfills the local energy inequality (4.14).

In addition, observing (4.17), it holds

$$\|v_k\|_{L^2(Q(1))}^2 \geq \varepsilon \text{ } \forall k \in \mathbb{N},$$

and thanks to (4.25) this inequality remains true for $v^*$, which shows that $v^* \neq 0$.

It now remains to check that $v^*$ fulfills the properties (iii) and (iv). First, by using the transformation formula of the Lebesgue integral from the definition of $v_k$ it follows that for $0 < \rho < 1$

$$\rho^{3\theta-1}|v_k|^3_{L^3(I(\rho);W^{\theta,3}(B(\rho)))} = (r_k\rho)^{3\theta-1}|v|_{L^3(I(r_k\rho);W^{\theta,3}(B(r\rho)))}^3 \leq K_1^3,$$

By the lower semi continuity of the semi norm $| \cdot |_{L^3(I(\rho);W^{\theta,3}(B(\rho)))}^3$ we find

$$\rho^{3\theta-1}|v^*|^3_{L^3(I(\rho);W^{\theta,3}(B(\rho)))} \leq K_1^3.$$
Now we shall verify (iv). Let \( k \in \mathbb{N} \) be fixed. From (2.11) by using the transformation formula for the Lebesgue integral, we obtain the following local energy inequality for \( \tilde{v}_k \). It holds for almost all \(-r_k^{-\frac{n+2}{2}} R^{\frac{n+2}{2}} < t < s < 0 \) and for all nonnegative \( \phi \in C^\infty(Q(r_k^{\frac{1}{n}} R)) \) with \( \text{supp}(\phi) \subset B_k \times (-r_k^{-\frac{n+2}{2}} R^{\frac{n+2}{2}}, 0] \)

\[
\int_{B_k} |\tilde{v}_k(t)|^2 \phi dx 
\leq \int_{B_k} |\tilde{v}_k(s)|^2 \phi dx + \int_{t}^{s} \int_{B_k} |v_k|^2 v_k \cdot \nabla \phi dx d\tau 
+ \int_{t}^{s} \int_{B_k} 2 p_{0,k} \tilde{v}_k \cdot \nabla \phi dx d\tau + \int_{t}^{s} \int_{B_k} v \cdot \nabla p_{h,k} v_k \cdot \nabla \phi dx d\tau 
+ \int_{t}^{s} \int_{B_k} v_k \otimes v_k : \nabla^2 p_{h,k} \phi dx d\tau.
\]

(4.28)

Next, by \( \tilde{\sigma} \in M^+(B(R)) \) we denote the unique measure valued trace due to Lemma 2.3 (cf. also Remark 2.4). Clearly, from the definition of \( \tilde{v}_k \) the unique trace \( \tilde{\sigma}_k \) of \( |\tilde{v}_k(\cdot)|^2 \), according to Lemma 2.3 is given by the relation

\[
\int_{B_k} \phi d\tilde{\sigma}_k(t) = \int_{B(R)} \phi \left( \frac{x}{r_k} \right) d\tilde{\sigma}(r_k^{\frac{n+2}{2}} t), \quad \phi \in C^0_c(B_k).
\]

We set \( \tilde{\sigma}_{0,k} = \tilde{\sigma}_k(0) \). Clearly, the weakly-* left continuity of \( \tilde{\sigma}_k \) implies

\[
(4.29) \quad \tilde{\sigma}_k(t) \to \tilde{\sigma}_{0,k} \quad \text{weakly-* in} \quad M(B_k) \quad \text{as} \quad t \to 0^-.
\]

Thanks to (4.29) we may pass \( s \to 0 \) in both sides of (4.28). This leads to

\[
\int_{B_k} |\tilde{v}_k(t)|^2 \phi dx 
\leq \int_{B_k} \phi d\tilde{\sigma}_{0,k} + \int_{t}^{0} \int_{B_k} |v_k|^2 v_k \cdot \nabla \phi dx d\tau 
+ \int_{t}^{0} \int_{B_k} 2 p_{0,k} \tilde{v}_k \cdot \nabla \phi dx d\tau + \int_{t}^{0} \int_{B_k} v \cdot \nabla p_{h,k} v_k \cdot \nabla \phi dx d\tau 
+ \int_{t}^{0} \int_{B_k} v_k \otimes v_k : \nabla^2 p_{h,k} \phi dx d\tau.
\]

(4.30)

Obviously, \( \|\tilde{\sigma}_{0,k}\| \leq \|\tilde{\sigma}_0\| \) for all \( k \in \mathbb{N} \). Thus, by virtue of Banach-Alaoglu’s theorem and Cantor’s diagonalization argument we get a measure \( \sigma^*_0 \in M^+(\mathbb{R}^n) \) together with

35
an increasing subsequence \( \{k_j\} \) such that for all \( 0 < \rho < +\infty \)

\[
(4.31) \quad \tilde{\sigma}_{0,k_j} \to \sigma_0^* \text{ weakly-* in } \mathcal{M}(B(\rho)) \text{ as } j \to +\infty.
\]

We claim that \( \sigma_0^* = \tilde{\sigma}_0(\{0\})\delta_0 \). Indeed, let \( \phi \in C_c^0(\mathbb{R}^n) \) be a nonnegative function. We may choose \( 0 < \rho < +\infty \) such that \( \text{supp}(\phi) \subset B(\rho) \). Let \( 0 < \varepsilon < \rho \) be arbitrarily chosen. Take \( \eta_k \in C_c^0(B(2\varepsilon)) \) with \( 0 \leq \eta_k \leq 1 \) and \( \eta_k(0) = 1 \). We find

\[
\int_{\mathbb{R}^n} \phi d\sigma_0^* = \int_{B(2\varepsilon)} \phi \eta_k d\sigma_0^* + \int_{B(\rho) \setminus \{0\}} \phi(1 - \eta_k) d\sigma_0^*
\]

From (4.31) we deduce that

\[
\int_{B(\rho) \setminus \{0\}} \phi(1 - \eta_k) d\sigma_0^* = \lim_{j \to \infty} \int_{B(r_k, \rho) \setminus \{0\}} \phi\left(\frac{x}{r_k^j}\right)\left\{1 - \eta_k\left(\frac{x}{r_k^j}\right)\right\} d\sigma_0
\]

\[
\leq \max \phi \lim_{j \to \infty} \tilde{\sigma}_0(B(r_k, \rho) \setminus \{0\})
\]

\[
= \max \phi \tilde{\sigma}_0\left(\bigcap_{j=1}^{\infty} B(r_k, \rho) \setminus \{0\}\right) = 0.
\]

Hence,

\[
\int_{\mathbb{R}^n} \phi d\sigma_0^* = \int_{B(2\varepsilon)} \phi \eta_k d\tilde{\sigma}_0 \to \phi(0)\tilde{\sigma}_0(\{0\}) \text{ as } \varepsilon \to 0,
\]

which shows that \( \sigma_0^* = a_0\delta_0 \), where \( a_0 := \tilde{\sigma}_0(\{0\}) \).

Observing (4.26) there exists a set \( J \) of Lebesgue measure 0 such that (4.26) is satisfied for all \( t \in [-1, 0] \setminus J \) and the local energy inequalities (4.14) and (4.30) are satisfied for all \( s, t \in [-1, 0] \setminus J \). Taking \( t \in [-1, 0] \setminus J \) in (4.30) with \( k_j \) in place of \( k \), and letting \( j \to \infty \), we obtain the following local energy inequality for all nonnegative \( \phi \in C_c^\infty(\mathbb{R}^n) \) and for all \( t \in [-1, 0] \setminus J \)

\[
(4.32) \quad \int_{\mathbb{R}^n} |v^*(t)|^2 \phi dx \leq a_0\phi(0) + \int_{t}^{0} \int_{\mathbb{R}^n} |v^*|^2 v^* \cdot \nabla \phi dx d\tau + \int_{t}^{0} \int_{\mathbb{R}^n} 2p^* v^* \cdot \nabla \phi dx d\tau.
\]

On the other hand, thanks to Lemma 2.3 there exists a unique measure valued trace \( \sigma^* \in L^\infty(-1, 0; \mathcal{M}^+(\mathbb{R}^n)) \) for \( |v^*(\cdot)|^2 \), which is weakly-* left continuous. Hence, in (4.32) letting \( t \to 0^- \) with an appropriate choice of \( t \) in the Lebesgue set of \( |v^*|^2 \), we obtain for all nonnegative \( \phi \in C_c^\infty(\mathbb{R}^n) \)

\[
\int_{\mathbb{R}^n} \phi d\sigma^*(0) \leq a_0\delta_0.
\]

This shows that \( 0 \leq \sigma^*(0) \leq a_0\delta_0 \). Whence there exists a constant \( 0 \leq E_0 < +\infty \) such that

\[
\sigma^*(0) = E_0\delta_0.
\]
In fact, $E_0 > 0$, otherwise the local energy inequality \((4.14)\) would imply that $v^* \equiv 0$. In fact, letting $s \to 0^-$ in \((4.14)\) we would obtain the inequality

$$
\int_{\mathbb{R}^n} |v^*(t)|^2 \phi dx \leq \int_0^t \int_{\mathbb{R}^n} v(s) \cdot \nabla \phi (|v^*(s)|^2 + 2p^*(s)) dx ds
$$

for all $\phi \in C^\infty_c(\mathbb{R}^n)$. Choosing an appropriate sequence of cut off function approximating 1 we verify the claim. This completes the proof $M |v^*|^2 (0) = E_0 \delta_0$, the property (iv).

4.3 Proof of Theorem 1.1 completed

The proof will be completed by contradiction. To this end, we assume there exist $\sigma_0 \in M |v|^2 (0)$ such that $\sigma_0 \{x\} > 0$ for some $x \in B(R)$. By a simple translation argument without loss of generality we can assume that $x = 0$. In particular, condition (ii) in Lemma 4.3 is satisfied.

In order to apply this lemma it only remains that condition (i) satisfied. Let $0 < r \leq R$ be fixed. From the Gagliardo-Nirenberg inequality we immediately get

$$
\|v(t)\|_{L^\infty(B(r))} \leq cr^{-\frac{n}{2}} \|v(t)\|_{L^2(B(r))} + c \|v(t)\|^\frac{n}{2} \|\nabla v(t)\|^\frac{n}{L^\infty(B(r))}
$$

with an absolute constant $c > 0$.

Let $1 \leq s < \frac{n+2}{n}$. Taking both sides of the above inequality to the $s$-th power, integrate the result over $I(r)$, and using the Type I blow-up condition in terms of the velocity gradient, we obtain

$$
\|v\|^s_{L^s(I(r);L^\infty(B(r)))} \leq cr^{-\frac{n}{2s} + \frac{n+2}{2s}} \|v\|^s_{L^\infty(I(r);L^2(B(r)))}
$$

$$
+ c \left( \|v\|_{L^\infty(I(r);L^2(B(r)))} + \sup_{-r}^0 (-t)^{-\frac{n}{2s}} dt \right)^s
$$

(4.33)

$$
\leq c(K_0 + K_1)^s r^{-\frac{n}{2s} + \frac{n+2}{2s}}
$$

with $c > 0$ depending only on $s$, where

$$
K_0 := \|v\|_{L^\infty(I(r);L^2(B(r)))}, \quad K_1 = \sup_{-r}^0 (-t)^{-\frac{n}{2s}} dt.
$$

By the standard interpolation argument we easily get from (4.33) for every $1 \leq s, q \leq \infty$ with

$$
\frac{n+2}{2s} + \frac{n}{q} > \frac{n}{2}
$$

the inequality

$$
\|v\|_{L^s(I(r);L^q(B(r)))} \leq c(K_0 + K_1)^r \frac{n+2}{2s} + \frac{n}{q} - \frac{n}{2}
$$

(4.34)
where $c$ is a positive constant depending only on $s$ and $q$.

Fix $0 < \theta < \frac{1}{3}$. We choose $1 < p < +\infty$ such that
\begin{equation}
(4.36) \quad p > \frac{3n\theta}{1 - 3\theta}.
\end{equation}
We set
\begin{equation}
q := \frac{3 - 3\theta}{p - 3\theta} \geq 2.
\end{equation}
Clearly, $2 \leq q < 3$ satisfies the relation
\begin{equation}
\frac{1}{3} = \frac{1 - \theta}{q} + \frac{\theta}{p}.
\end{equation}
Furthermore (4.34) ensures that the following inequality holds true
\begin{equation}
(4.37) \quad \frac{(n + 2)(1 - 3\theta)}{2(3 - 3\theta)} + \frac{n}{q} > \frac{n}{2}.
\end{equation}

Using the interpolation theorem between Sobolev-Slobodeckiĭ spaces (cf. Theorem 6.4.5, (7)) and Hölder’s inequality, we get
\begin{align*}
|v(t)|_{W^{3, \theta}(B(r))}^3 & \leq cr^{-3\theta + n - \frac{3n\theta}{q}} |v(t)|_{L^q(B(r))}^3 + c|v(t)|_{L^p(B(r))}^{3 - 3\theta} \|
\nabla v(t)\|_{L^p(B(r))}^{3\theta} \\
& \leq cr^{-3\theta + n - \frac{3n\theta}{q}} |v(t)|_{L^q(B(r))}^3 + cr^{3\theta p - 3\theta} |v(t)|_{L^p(B(r))}^{3\theta} \|
\nabla v(t)\|_{L^p(B(r))}^{3\theta} \\
& \leq cr^{-3\theta + n - \frac{3n\theta}{q}} |v(t)|_{L^q(B(r))}^3 + cK_1^{3\theta} r^{\frac{3\theta p}{p - r}} |v(t)|_{L^p(B(r))}^{3\theta} (t - 3\theta).
\end{align*}

Integrating this inequality over $t \in I(r)$, and applying (4.35) with $s = 3$, we are lead to
\begin{equation}
(4.38) \quad |v|_{L^3(I(r); W^{3, \theta}(B(r)))}^3 \leq cr^{1 - 3\theta} (K_0 + K_1)^3 + cK_1^{3\theta} r^{\frac{3\theta p}{p - r}} \int_{I(r)} |v(t)|_{L^3(B(r))}^{3\theta} (t - 3\theta) dt.
\end{equation}

In view of (4.34), we may choose $\frac{3 - 3\theta}{p - r} < s < +\infty$ such that condition (4.34) is still fulfilled. Applying Hölder’s inequality and appealing to (4.35), we obtain
\begin{align*}
r^{\frac{3\theta p}{p - r}} \int_{I(r)} & |v(t)|_{L^3(B(r))}^{3\theta} (t - 3\theta) dt \\
& \leq r^{\frac{3\theta p}{p - r}} |v|_{L^3(I(r); L^q(B(r)))}^3 \left( \int_{I(r)} (t - \frac{3\theta}{3 - 3\theta}) dt \right)^{\frac{3 - 3\theta}{3 - 3\theta}} \\
& \leq cr^{\frac{3\theta p}{p - r}} |v|_{L^3(I(r); L^q(B(r)))}^3 \left( 1 - 3\theta - \frac{3\theta}{3 - 3\theta} \right)^{\frac{n + 2}{2}} \\
& \leq c(K_0 + K_1)^{3 - 3\theta} r^{\frac{3\theta p}{p - r}} (3 - 3\theta) \left( \frac{n + 2}{2} \frac{3 - 3\theta}{n + 2} \right)^{\frac{2n + 2}{2}} \\
& \leq c(K_0 + K_1)^{3 - 3\theta} r^{\frac{3\theta p}{p - r}} (3 - 3\theta) \left( \theta - \frac{3 - 3\theta}{\theta - \frac{3 - 3\theta}{3 - 3\theta}} \right)^{\frac{n + 2}{2}} (1 - 3\theta) \\
& = c(K_0 + K_1)^{3 - 3\theta} r^{1 - 3\theta}.
\end{align*}
Inserting this inequality into the right-hand side of (4.38) and applying Young’s inequality, we arrive at

\[ |v|^3_{L^3(I(r);W^{3, \theta}(B(r)))} \leq c(K_0 + K_1)^3 r^{1-3\theta}, \]

which shows that condition (i) of Lemma 4.3 is satisfied.

Now, we are in a position to apply Lemma 4.3 to obtain a nontrivial limit \( v^* \in L^\infty(-1, 0; L^2_\sigma(\mathbb{R}^n)) \cap L^{3, \theta}_{\text{loc}}([-1, 0); W^{3, \theta}(\mathbb{R}^n)) \), which is a weak solution to the Euler equations in \( \mathbb{R}^n \times (-1, 0) \) satisfying (iii) and (iv). On the other hand, by the assumption of the theorem \( v \) fulfills the local Type I blow up condition in terms of the velocity gradient. Since this Type I condition is invariant under the scaling, the limit function must enjoy the global Type I blow up condition in terms of the velocity gradient in \( \mathbb{R}^n \). Since \( \sigma_0 \) is a Dirac measure, however, by application of Theorem 3.1, we need to have \( v^* \equiv 0 \), which contradicts to the nontriviality of \( v^* \).

5 Proof of Corollary 1.5

Let us consider the change of coordinates \((x, t) \mapsto (y, \tau)\) from \(\mathbb{R}^n \times (-1, 0)\) to \(\mathbb{R}^n \times (0, +\infty)\) by

\[ y = \frac{x}{(-t)^{\frac{n}{n+2}}}, \quad \tau = -\log(-t). \]

Given a solution \((v, p)\) of the Euler equations, the profile \((V, P)\) in the energy conserving scale is defined by the relation

\[ v(x, t) = \frac{1}{(-t)^{\frac{n}{n+2}}} V\left( \frac{x}{(-t)^{\frac{n}{n+2}}}, -\log(-t) \right), \]

\[ p(x, t) = \frac{1}{(-t)^{\frac{n}{n+2}}} P\left( \frac{x}{(-t)^{\frac{n}{n+2}}}, -\log(-t) \right), \quad (x, t) \in \mathbb{R}^n \times (-1, 0). \]

We find that the profile \((V, P)\) solves the following system:

\[ \partial_\tau V + \frac{2}{n+2} y \cdot \nabla U + \frac{n}{n+2} V + (V \cdot \nabla)V = -\nabla P, \quad \nabla \cdot V = 0 \quad \text{in} \quad \mathbb{R}^n. \]

One can also check easily that \( v \) is a \( \lambda \)-DSS solution of the Euler equations if and only if \( V(\cdot, \tau) = V(\cdot, \tau + \frac{n+2}{2} \log \lambda) \) for all \( \tau \in (0, +\infty) \). Note that for a solution \( v \in L^\infty(-1, 0; L^2_\sigma(\mathbb{R}^n)) \) to the Euler equation, satisfying \( (1.2) \), satisfies the energy equality

\[ \|v(t)\|_{L^2}^2 = \|v(-1)\|_{L^2}^2 := E \quad \forall t \in (-1, 0), \]

which implies also that

\[ \|V(\tau)\|_{L^2}^2 = \|V(0)\|_{L^2}^2 = E \quad \forall \tau \in (0, +\infty). \]

We first show the following.
Lemma 5.1. Let \( v \in L^\infty(-1, 0; L^2_\sigma(\mathbb{R}^n)) \cap L^\infty_{\text{loc}}([-1, 0), W^{1,\infty}(\mathbb{R}^n)) \) be a \( \lambda \)-DSS solution to the Euler equations for some \( 1 < \lambda < +\infty \). Assume \( v \) satisfies (5.4). Then, for every \( 0 < r < +\infty \) it holds

\[
(5.6) \quad v \in C([−1, 0]; L^2(B(r)^c)), \quad \text{and} \quad \lim_{t \to 0} \|v(t)\|_{L^2(B(r)^c)} = 0.
\]

**Proof:** Let \( 0 < r < +\infty \) be arbitrarily chosen. Using the transformation formula of the Lebesgue integral, we calculate for \( t \in [-1, 0) \)

\[
(5.7) \quad \|v(t)\|_{L^2(B(r)^c)}^2 = \int_{B(-t)^c \setminus \frac{2}{n+2} r} |V(y, -\log(-t))|^2 dy = \int_{B(-t)^c \setminus \frac{2}{n+2} r} |V(y, \tau)|^2 dy,
\]

where \( \tau = -\log(-t) \). Now, let \( (t_k) \) be any sequence in \([-1, 0)\) such that \( t_k \to 0 \) as \( k \to +\infty \). Then, \( \tau_k = -\log(-t_k) \to +\infty \) as \( k \to +\infty \). On the other hand, since \( V \) is \( \lambda \)-DSS the profile, \( V \) satisfies \( V(\cdot, \tau) = V(\cdot, \tau + \frac{n+2}{2} \log \lambda) \) for all \( \tau \in [0, +\infty) \). Accordingly, for every \( k \in \mathbb{N} \) there exists \( \tilde{\tau}_k \in [0, \frac{n+2}{2} \log \lambda] \) such that

\[
V(\tau_k) = V(\tilde{\tau}_k).
\]

Eventually, passing to a subsequence, we may assume \( \tilde{\tau}_k \to \tau_0 \) in \([0, \frac{n+2}{2} \log \lambda] \) as \( k \to +\infty \). Thus, by using triangle inequality we obtain

\[
(5.8) \quad \|v(t_k)\|_{L^2(B(r)^c)} = \|V(\tau_k)\|_{L^2(B(-t_k)^c \setminus \frac{2}{n+2} r)} = \|V(\tilde{\tau}_k)\|_{L^2(B(-t_k)^c \setminus \frac{2}{n+2} r)} = \|V(\tau_0)\|_{L^2(B(-t_k)^c \setminus \frac{2}{n+2} r)} + \|V(\tau_0)\|_{L^2(B(\tau_0)^c)}}.
\]

To argue further we first note that \( V \) solves the profile equation in a weak sense, namely

\[
\int_{\mathbb{R}^n} V(y, \tilde{\tau}_k) \cdot \varphi(y) dy - \int_{\mathbb{R}^n} V(y, \tau_0) \cdot \varphi(y) dy = \frac{2}{n+2} \int_{\tau_0}^{\tilde{\tau}_k} \int_{\mathbb{R}^n} V(s) \cdot (y \cdot \nabla) \varphi(y) dy ds
\]

\[
- \frac{n}{n+2} \int_{\tau_0}^{\tilde{\tau}_k} \int_{\mathbb{R}^n} V(s) \cdot \varphi(y) dy ds + \int_{\tau_0}^{\tilde{\tau}_k} \int_{\mathbb{R}^n} V(s) \cdot (V \cdot \nabla) \varphi(y) dy ds
\]

for all \( \varphi \in C^\infty_0(\mathbb{R}^n) \) with \( \nabla \cdot \varphi = 0 \), from which, taking into account of the fact that \( V \in L^\infty(0, +\infty; L^2_\sigma(\mathbb{R}^n)) \), we find easily that \( V(\tilde{\tau}_k) \to V(\tau_0) \) weakly in \( L^2(\mathbb{R}^n) \) as \( k \to +\infty \). Thus, the norm convergence (5.5) together with weak convergence implies that the first term on the right hand side of (5.8) tends to zero as \( k \to +\infty \). Secondly, by the monotone convergence we see that also the second term on the right hand side of (5.8) tends zero as \( k \to +\infty \). Thus,

\[
\|v(t_k)\|_{L^2(B(r)^c)} \to 0 \quad \text{as} \quad t \to 0.
\]

Since we have shown that \( \|v(t_k)\|_{L^2(B(r)^c)} \to 0 \) and \( v(t_k) \to 0 \) weakly in \( L^2(\mathbb{R}^n) \) as \( k \to \infty \), the conclusion (5.6) follows. \( \blacksquare \)
Corollary 5.2. Let $1 < \lambda < +\infty$ and $v \in L^\infty((-1,0); L^2_\sigma(\mathbb{R}^n)) \cap L^\infty_{{\text{loc}}}([-1,0), W^{1,\infty}(\mathbb{R}^n))$ be a $\lambda$-DSS solution to the Euler equations satisfying (5.4). Then the energy is concentrated at $(0,0)$, i.e. (3.1) is satisfied with $E_0 = E$.

Proof: Let $\varphi \in C^0(\mathbb{R}^n)$ and bounded. Let $\varepsilon > 0$ be arbitrarily chosen. By the continuity of $\varphi$ we may choose $\delta > 0$ such that $\sup_{x \in B(\delta)} |\varphi(x) - \varphi(0)| \leq \varepsilon$. Elementary,

\[
\int_{\mathbb{R}^n} |v(t)|^2 \varphi dx = \int_{\mathbb{R}^n} |v(t)|^2 (\varphi - \varphi(0)) dx + E\varphi(0) \\
= \int_{B(\delta)^c} |v(t)|^2 (\varphi - \varphi(0)) dx + \int_{B(\delta)} |v(t)|^2 (\varphi - \varphi(0)) dx + E\varphi(0).
\]

Thanks to (5.6) and the boundedness of $\varphi$ we get

\[
\limsup_{t \to 0} \int_{\mathbb{R}^n} |v(t)|^2 \varphi dx \leq E\varepsilon + E\varphi(0), \quad \text{and} \quad E\varphi(0) \leq E\varepsilon + \liminf_{t \to 0} \int_{\mathbb{R}^n} |v(t)|^2 \varphi dx.
\]

Therefore,

\[
\limsup_{t \to 0} \int_{\mathbb{R}^n} |v(t)|^2 \varphi dx \leq E\varphi(0) \leq \liminf_{t \to 0} \int_{\mathbb{R}^n} |v(t)|^2 \varphi dx,
\]

which shows

\[
\lim_{t \to 0} \int_{\mathbb{R}^n} |v(t)|^2 \varphi dx = E\varphi(0) = E < \delta_0, \varphi > .
\]

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A Some auxiliary lemmas

Here we prove fundamental properties of a harmonic function used in the proof of the main theorem.

Lemma A.1. Let $p \in L^2(\Omega)$ be a harmonic function on $\Omega$. Then for every $\phi \in C^\infty_c(\Omega)$ and for all $m \in \mathbb{N}$ it holds

\[
(A.1) \quad \int_{\Omega} |\nabla^m p|^2 \phi dx = \frac{1}{2m} \int_{\Omega} p^2 \Delta^m \phi dx.
\]

Proof: We show (A.1) by an inductive argument. First, (A.1) with $m = 1$ is clear by the integration by parts. Assume (A.1) holds for $m \in \mathbb{N}$. Then we have $|\nabla^{m+1} p|^2 =$
\[ \sum_{i=1}^{n} |\nabla^m \partial_i p|^2 \]. Using the assumption that \((A.1)\) holds for \(\partial_i p\) in place of \(p\), and integrating it by parts, we infer

\[
\int_{\Omega} |\nabla^{m+1} p|^2 \phi dx = \sum_{i=1}^{n} \int_{\Omega} |\nabla^m \partial_i p|^2 \phi dx = \frac{1}{2^{m+1}} \int_{\Omega} |\nabla p|^2 \Delta^m \phi dx = \frac{1}{2^{m+1}} \int_{\Omega} p^2 \Delta^{m+1} \phi dx.
\]

**Lemma A.2.** Let \(U = \mathbb{R}^n \setminus B(r), 0 < r < +\infty\). Let \(\zeta \in C^\infty(U)\) denote a cut off function such that \(0 \leq \zeta \leq 1\) in \(U\), and \(|\nabla^k \zeta| \leq cr^{-k}, k = 1, \ldots, n+1\). Then for every \(u \in L^\frac{2n}{n+1}(U)\) which is harmonic in \(U\) it holds

\[(A.2) \quad \|\nabla u\zeta\|_\infty \leq cr^{-\frac{n+1}{2}}\|u\|_{L^\frac{2n}{n+1}(U)}.
\]

**Proof:** First, let \(x \in \mathbb{R}^n \setminus B(2r)\). Applying the mean value property of harmonic functions and Jensen’s inequality, we get

\[
|\nabla u(x)\zeta(x)| \leq |\nabla u(x)| \leq cr^{-n-1} \int_{B(x,r)} |u|dx \leq cr^{-\frac{n+1}{2}}\|u\|_{L^\frac{2n}{n+1}(U)}.
\]

Secondly, let \(x \in U \cap B(2r)\). By \(\eta \in C^\infty_c(B(4r))\) we denote a cut off function such that \(0 \leq \eta \leq 1\) in \(B(4r), \eta \equiv 1\) on \(B(2r)\) and \(|\nabla^k \eta| \leq cr^{-k}, k = 1, \ldots, n+1\). Using Sobolev’s embedding theorem, and applying Lemma \((A.1)\) with \(\phi = |\nabla^j(\zeta \eta)|^2, j = 1, \ldots, n\), we estimate

\[
|\nabla u(x)\zeta(x)| \leq \|\nabla u\zeta\eta\|_{L^\infty(B(4r))} \leq c \sum_{k=0}^{n} r^{-\frac{n}{2}+k} \|\nabla^k (\nabla u\zeta\eta)\|_{L^2(B(4r))}
\]

\[
\leq c \sum_{k=0}^{n} \sum_{j=0}^{k} r^{-\frac{n}{2}+k} \|\nabla^{k-j+1} u\nabla^j (\zeta \eta)\|_{L^2(B(4r))}
\]

\[
\leq cr^{-\frac{n+1}{2}}\|u\|_{L^2(U \cap B(4r))} \leq cr^{-\frac{n+1}{2}}\|u\|_{L^\frac{2n}{n+1}(U)}.
\]

The assertion now follows from the above two estimates. 

**Lemma A.3.** Let \(\{p_k\}\) be a sequence of harmonic functions in \(L^2(\Omega)\), which converges weakly to some limit \(p\) in \(L^2(\Omega)\) as \(k \to +\infty\). Then \(p\) is harmonic and for every compact set \(K \subset \Omega\) and every multi index \(\alpha = (\alpha_1, \ldots, \alpha_n)\) it holds

\[(A.3) \quad D^\alpha p_k \to D^\alpha p \quad \text{uniformly on } K \quad \text{as} \quad k \to +\infty.
\]

**Proof:** By virtue of Weyl’s lemma it is clear that \(p\) is harmonic in \(\Omega\). Let \(x \in \Omega\), and let \(B(x,r) \subset \Omega\) be a ball. By the weak convergence and the mean value property of harmonic functions we obtain

\[
p_k(x) = \int_{B(x,r)} p_k dy \to \int_{B(x,r)} p dy = p(x) \quad \text{as} \quad k \to +\infty.
\]

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This shows that \( p_k \to p \) pointwise as \( k \to +\infty \). In particular, \( p_k \to p \) in \( L^2(\Omega') \) as \( k \to +\infty \) for every \( \Omega' \subseteq \Omega \). Applying the identity (A.1) for a suitable cut off function \( \phi \geq 0 \) it follows that \( p_k \to p \) in \( H^m(\Omega') \) as \( k \to +\infty \) for every \( \Omega' \subseteq \Omega \) and for all \( m \in \mathbb{N} \). The uniform convergences (A.3) is now an immediate consequence of Sobolev’s embedding theorem.

Lemma A.4. For \( 0 < R < +\infty \) define \( Q(R) = B(R) \times I(R) \), \( I(R) = (-R^{2n-1}, 0) \). Let \( f \in L^p(Q(R); \mathbb{R}^n) \), \( 1 < p < +\infty \). Let \( u \in L^p(Q(R)) \), solving the equation

\[
-\Delta u = \sum_{i,j=1}^{n} \partial_i \partial_j f_{ij} \quad \text{in} \quad Q(R)
\]

in the sense of distributions. Assume for some \( \lambda \in (0, n) \) it holds

\[
\sup_{0 < \rho < R} \rho^{-\lambda} \| f \|_{L^p(Q(\rho))}^p < +\infty.
\]

Then there exists a constant \( c > 0 \) depending only on \( n, p \) and \( \lambda \) such that

\[
\sup_{0 < \rho < R} \rho^{-\lambda} \| u \|_{L^p(Q(\rho))}^p \leq c \left( R^{-\lambda} \| u \|_{L^p(Q(R))}^p + \sup_{0 < \rho < R} \rho^{-\lambda} \| f \|_{L^p(Q(\rho))}^p \right).
\]

Proof: By a routine scaling argument we may assume that \( R = 1 \). We extend \( f(t) \) by zero outside \( B(1) \), and denote this extension again by \( f \). Clearly, the family of annulus \( U_j = B(2^{j+1}) \setminus B(2^{j-1}) \), \( j \in \mathbb{Z}, j \leq 0 \), cover \( B(2) \). By \( \{ \psi_j \} \) we denote a corresponding partition of unity of smooth radial symmetric functions, such that \( \sum_{j=-\infty}^{1} \psi_j = 1 \) on \( B(2) \) together with \( |\nabla \psi_j| \leq c 2^{-j} \) and \( |\nabla^2 \psi_j| \leq c 2^{-2j} \) for all \( j \in \mathbb{Z}, j \leq 1 \). Let \( m \in \mathbb{Z} \), with \( m \leq 0 \) be arbitrarily chosen, but fixed. We write \( u = u_1 + u_2 + u_3 \), where

\[
u_1(x, t) = \sum_{j=-\infty}^{m} P.V. \int_{\mathbb{R}^n} f(x - y, t) : \nabla^2 N(y) \psi_j(y) dy,
\]

\[
u_2(x, t) = \sum_{j=m+1}^{1} P.V. \int_{\mathbb{R}^n} f(x - y, t) : \nabla^2 N(y) \psi_j(y) dy,
\]

\[
u_3(x, t) = u(x, t) - \nu_1(x, t) - \nu_2(x, t), \quad (x, t) \in Q(1),
\]

where \( N \) stands for the Newton potential in \( \mathbb{R}^n \).

Our aim will be to estimate the \( L^p \) norm of \( u_1, u_2 \) and \( u_3 \) over \( Q(2^m) \) separately.

First, by triangle inequality we see that for \( x \in B(2^m) \) and \( |x - y| \geq 2^{m+2} \) we get \( |y| \geq 2^{m+1} \). Thus, by Calderón-Zygmund inequality we find for almost every \( t \in (-1, 0) \)

\[
\| u_1(t) \|_{L^p(B(2^m))}^p \leq c \| f(t) \|_{L^p(B(2^{m+2}))}^p.
\]

Integration of both sides over \( I(2^m) \) with respect to time along with (A.5) yields

\[
\| u_1 \|_{L^p(Q(2^m))}^p \leq c \| f \|_{L^p(Q(2^{m+2}))}^p \leq c 2^{m\lambda} \sup_{0 < \rho < R} \rho^{-\lambda} \| f \|_{L^p(Q(\rho))}^p.
\]
Next, fix $x \in B(2^m)$. It is readily seen that for all $j \geq m + 1$ it holds $B(x, 2^{j+1}) \subset B(2^{j+1} + 2^m) \subset B(2^{j+2})$. Noting that $|k(y)| \leq c|y|^{-n}$ it follows $|k|_{\psi_j} \leq c2^{-jn}$. Accordingly, by the aid of Jensen’s inequality, and observing (A.5), we estimate

$$|u_2(x,t)| \leq c \sum_{j=m+2}^{\infty} \int f(x-y,t)2^{-jn}dx \leq \sum_{j=m+2}^{\infty} 2^{jn} \left( \int_{B(2^{j+2})} |f(y,t)|^p dy \right)^{\frac{1}{p}}$$

Taking the ess sup over $x \in B(2^m)$, and taking the $\| \cdot \|_{L^p(I(2^m))}$ of both sides with respect to $t$, using Minkowski’s inequality, and observing (A.5), we are led to

$$\left( \int_{I(2^m)} \|u_2(t)\|_{L^p(B(2^m))}^p dt \right)^{\frac{1}{p}} \leq \left( \sum_{j=m+2}^{\infty} 2^{jn} \int_{Q(2^{j+2})} |f(y,t)|^p dy \right)^{\frac{1}{p}} \leq c \left( \sum_{j=m+2}^{\infty} 2^{-j(n-\lambda)} \sup_{0<\rho<R} \rho^{-\lambda} \|f\|_{L^p(Q(\rho))}^p \right)^{\frac{1}{p}} \leq c2^{-m \frac{\lambda}{p}} \left( \sup_{0<\rho<R} \rho^{-\lambda} \|f\|_{L^p(Q(\rho))} \right)^{\frac{1}{p}}.$$

Consequently,

$$\|u_2\|_{L^p(Q(2^m))} \leq c2^{mn} \int_{I(2^m)} \|u_2(t)\|_{L^p(B(2^m))}^p dt \leq c2^{m\lambda} \sup_{0<\rho<R} \rho^{-\lambda} \|f\|_{L^p(Q(\rho))}^p.$$

In only remains to estimate $u_3$. By the definition of $u_1$ and $u_2$, recalling that $f(t) \equiv 0$ on $\mathbb{R}^n \setminus B(1)$, we see that for almost all $t \in (-1,0)$ and for all $x \in B(1)$ it holds

$$u_1(x,t) + u_2(x,t) = \sum_{j=-\infty}^{1} \text{P.V.} \int_{B(2)} f(x-y,t) : \nabla^2 N(y)\psi_j(y)dy = \int_{\mathbb{R}^n} f(x-y,t) : \nabla^2 N(y)dy.$$

(A.7)

In particular, $u_1 + u_2$ solves (A.4) in the sense of distributions. By Weyl’s lemma we deduce that $u_3(t) = u(t) - u_1(t) - u_2(t)$ is harmonic. Thus,

$$\|u_3\|_{L^p(Q(2^m))} \leq c2^{mn} \|u_3\|_{L^p(I(2^m);B(2^m))} \leq c2^{m\lambda} \left( \|u\|_{L^p(Q(2^m))} + \|f\|_{L^p(Q(1))} \right).$$

Combining the estimates of $u_1, u_2$ and $u_3$ we get for all $m \in \mathbb{Z}, m \leq 0$,

$$2^{-m\lambda} \|u\|_{L^p(Q(2^m))} \leq c \left( \|u\|_{L^p(Q(1))} + \sup_{0<\rho<R} \rho^{-\lambda} \|f\|_{L^p(Q(\rho))} \right).$$

Taking the supremum over all $m \in \mathbb{Z}, m \leq 0$ on the left-hand side, we obtain the assertion (A.6).
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