Trigonometric approximation of functions $f(x, y)$ of generalized Lipschitz class by double Hausdorff matrix summability method

Abhishek Mishra¹, Vishnu Narayan Mishra² and M. Mursaleen¹,4*

Abstract

In this paper, we establish a new estimate for the degree of approximation of functions $f(x, y)$ belonging to the generalized Lipschitz class $\text{Lip}((\xi_1, \xi_2); r)$, $r \geq 1$, by double Hausdorff matrix summability means of double Fourier series. We also deduce the degree of approximation of functions from $\text{Lip}(\alpha, \beta; r)$ and $\text{Lip}(\alpha, \beta)$ in the form of corollary. We establish some auxiliary results on trigonometric approximation for almost Euler means and $(C, \gamma, \delta)$ means.

Keywords: Double Hausdorff matrix summability; Double Fourier series; Generalized Lipschitz class; Modulus of continuity; Cesàro summability; Almost Euler summability means; Degree of approximation

1 Introduction

The study of various summability means of double Fourier series have been done by several authors, for example, Chow [2], Sharma [11], Lénski [6], and Ustina [15]. Dealing with the first arithmetic means of double Fourier series, Hasegawa [4] obtained the following:

Theorem A If a continuous function $f(x, y)$ of period $2\pi$ with respect to both $x$ and $y$ belongs to $\text{Lip}(\alpha, \beta)$, where $0 < \alpha < 1$ and $0 < \beta < 1$, then

$$\left| \sigma_{m,n}(x, y) - f(x, y) \right| = O(m^{-\alpha} + n^{-\beta})$$

uniformly in $(x, y)$ as $m$ and $n$ independently tend to infinity.

If $\alpha = \beta = 1$, then

$$\left| \sigma_{m,n}(x, y) - f(x, y) \right| = O(m^{-1} \log m + n^{-1} \log n)$$

uniformly in $(x, y)$ as $m$ and $n$ independently tend to infinity.

Siddiqui and Mohammadzadeh [12] investigated the approximation by Cesàro and B means of double Fourier series. Stepanets [13, 14] has established estimates of approximation for certain classes of periodic functions and differentiable periodic functions of two
variables by linear methods of summation of their Fourier sums. Móricz and Shi [8] proved
the following result for the approximation to continuous functions by Cesàro means of
double Fourier series.

**Theorem B** If $f \in E(\alpha, \beta), 0 < \alpha, \beta \leq 1, \gamma, \delta \geq 0,$ then

$$
\| \sigma_{mn}^{\delta}(f, x, y) - f(x, y) \| = O\left( \frac{1}{(m+1)^\alpha} + \frac{1}{(n+1)^\beta} \right) \text{ if } 0 < \alpha, \beta \leq 1,
$$

$$
= O\left( \frac{1}{(m+1)^\alpha} + \frac{\log(n+2)}{(n+1)} \right) \text{ if } 0 < \alpha < \beta = 1,
$$

$$
= O\left( \frac{\log(m+2)}{(m+1)} + \frac{\log(n+2)}{(n+1)} \right) \text{ if } \alpha = \beta = 1.
$$

The degree of approximation using Gauss–Weierstrass integrals was also investigated
by Khan and Ram [5]. Recently, error and bounds of certain bivariate functions by al-
most Euler means of double Fourier series for the functions of Lipschitz and Zygmund
classes was estimated by Rathor and Singh [9]. To find the approximation of functions
of two-dimensional torus, in this paper, we obtain a new estimate for trigonometric ap-
proximation of functions $f(x, y)$ of generalized Lipschitz class by double Hausdorff matrix
summability method of double Fourier series. For other summability methods of approx-
imation, see [1] and [7].

### 2 Definitions and preliminaries

Let $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n}$ be double series with the sequence of $(m, n)$th partial sums

$$
s_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} g_{j,k}.
$$

A double Hausdorff matrix has the entries

$$
h_{j,k}^{m,n} = \binom{m}{j} \binom{n}{k} \Delta_1^{m-j} \Delta_2^{n-k} \mu_{j,k},
$$

where $\{ \mu_{j,k} \}$ is any real or complex sequence, and

$$
\Delta_1^{m-j} \Delta_2^{n-k} \mu_{j,k} = \sum_{w=0}^{m-j} \sum_{z=0}^{n-k} (-1)^{w+z} \binom{m-j}{w} \binom{n-k}{z} \mu_{j+w,k+z}.
$$

If $t_{m,n}^{i,j} = \sum_{j=0}^{m} \sum_{k=0}^{n} h_{j,k}^{i,j} \rightarrow g$ as $m \rightarrow \infty$ and $n \rightarrow \infty$, then $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n}$ is said to be
summable to the sum $g$ by the double Hausdorff matrix summability method [15].

A necessary and sufficient condition for double Hausdorff matrix summability method
to be regular is there exists a function $\chi(s, t) \in BV[0, 1] \times [0, 1]$ such that

$$
\int_{0}^{1} \int_{0}^{1} |d\chi(s, t)| < \infty
$$

and

$$
\mu_{m,n} = \int_{0}^{1} \int_{0}^{1} s^m t^n \, d\chi(s, t),
$$

where $\Delta_1^{m-j} \Delta_2^{n-k} \mu_{j,k}$
follows:
\[
\chi(x) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \lambda_{m,n} a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny \\
+ c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \cos ny
\]

with \((m, n)\)th partial sums \(s_{m,n}(f; (x, y))\), where

\[
\lambda_{m,n} = \begin{cases} 
1/4 & \text{for } m = n = 0, \\
1/2 & \text{for } m > 0, n = 0 \text{ and } m = 0, n > 0, \\
1 & \text{for } m > 0, n > 0,
\end{cases}
\]

\[
a_{m,n} = \frac{\pi^{-2}}{r} \int_{Q} f(x, y) \cos mx \cos ny \, dx \, dy,
\]

and similar expressions for \(b_{m,n}, c_{m,n}, d_{m,n}\) [3].

We define the \(L^r\) norm by

\[
\|f\|_r = \begin{cases} 
\left( \frac{1}{r^2} \int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^r \, dx \, dy \right)^{1/r}, & r \geq 1, \\
\text{ess sup}_{0 \leq x, y \leq 2\pi} |f(x, y)|, & r = \infty.
\end{cases}
\]

The degree of approximation of a function \(f : \mathbb{R}^2 \to \mathbb{R}\) by a trigonometric polynomial [17]

\[
t_{m,n}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_{m,n} [a_{j,k} \cos mx \cos ny + b_{j,k} \sin mx \cos ny \\
+ c_{j,k} \cos mx \sin ny + d_{j,k} \sin mx \cos ny]
\]

of order \((m + n)\) is defined by

\[
E_{m,n}(f, L^r) = \min_{0 \leq x, y \leq 2\pi} \|t_{m,n} - f\|_r.
\]
A function \( f : \mathbb{R}^2 \to \mathbb{R} \) of two variables \( x \) and \( y \) is said to belong to the class \( \text{Lip}(\alpha, \beta) \) [4] if
\[
|f(x + u, y + v) - f(x, y)| = O(|u|^{\alpha} + |v|^{\beta}), \quad 0 < \alpha \leq 1, 0 < \beta \leq 1,
\]
to the class \( \text{Lip}((\xi_1, \xi_2); r) \) if
\[
\left\{ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} |f(x + u, y + v) - f(x, y)|^r \, dx \, dy \right\}^{1/r} = O(|u|^{\alpha} + |v|^{\beta}), \quad r \geq 1,
\]
and to the class \( \text{Lip}((\xi_1, \xi_2); r) \) if
\[
\left\{ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} |f(x + u, y + v) - f(x, y)|^r \, dx \, dy \right\}^{1/r} = O(\xi_1(u) + \xi_2(v)), \quad r \geq 1,
\]
where \( \xi_1 \) and \( \xi_2 \) are moduli of continuity, that is, nonnegative nondecreasing continuous functions such that \( \xi_1(0) = \xi_2(0) = 0, \xi_1(u_1 + u_2) \leq \xi_1(u_1) + \xi_1(u_2), \) and \( \xi_2(v_1 + v_2) \leq \xi_2(v_1) + \xi_2(v_2). \)

If \( \xi_1(u) = u^\alpha \) and \( \xi_2(v) = v^\beta, \) \( 0 < \alpha \leq 1, 0 < \beta \leq 1, \) then the class \( \text{Lip}((\xi_1, \xi_2); r) \) coincides with \( \text{Lip}((\alpha, \beta); r). \) As \( r \to \infty, \text{Lip}((\alpha, \beta); r) \) reduces to \( \text{Lip}(\alpha, \beta). \) Clearly, \( \text{Lip}(\alpha, \beta) \subseteq \text{Lip}((\alpha, \beta); r) \subseteq \text{Lip}((\xi_1, \xi_2); r). \)

We define the forward difference operator \( \Delta \) as \( \Delta \mu_k = \mu_k - \mu_{k+1}; \) also, \( \Delta^{n+1} \mu_k = \Delta(\Delta^n \mu_k), k \geq 0. \) We denote
\[
\phi(u, v) = (1/4) [f(x + u, y + v) + f(x + u, y - v) + f(x - u, y + v) + f(x - u, y - v)
- 4f(x, y)],
\]
\[
M^H_m(u) = \frac{K_1}{2\pi} \sum_{j=0}^{m} \int_0^{1} \binom{m}{j} \phi(1 - s)^{m-j} \, ds \, \frac{\sin(j + \frac{1}{2})u}{\sin \frac{u}{2}},
\]
\[
K^H_n(v) = \frac{K_2}{2\pi} \sum_{k=0}^{n} \int_0^{1} \binom{n}{k} \chi(1 - t)^{n-k} \, dt \, \frac{\sin(k + \frac{1}{2})v}{\sin \frac{v}{2}}.
\]

3 Result

The object of this paper is obtaining the degree of approximation of functions \( f(x, y) \) of generalized Lipschitz class by double Hausdorff matrix summability means of its double Fourier series:

**Theorem 1** If \( f(x, y) \) is a \( 2\pi \) periodic function with respect to both variables \( x \) and \( y, \) Lebesgue integrable in \( (-\pi, \pi) \times (-\pi, \pi) \) and belonging to the class \( \text{Lip}((\xi_1, \xi_2); r) \) \( (r \geq 1), \) then the degree of approximation \( \phi(f(x, y)) \) by double Hausdorff matrix summability means
\[
t^H_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} \int_0^{1} \int_0^{1} \binom{m}{j} \binom{n}{k} \phi(1 - s)^{m-j} \chi(1 - t)^{n-k} \, ds \, dt \, \delta(s, t),
\]
of double Fourier series (1) satisfies

\[
\| \xi_{m,n}^H - f \|_r = O\left( \frac{1}{(m+1)} \int_0^\pi \frac{\xi_1(u)}{u^2} du + \frac{1}{(n+1)} \int_0^\pi \frac{\xi_2(v)}{v^2} dv \right)
\]

for \( m, n = 0, 1, 2, \ldots \).

4 Lemmas

For the proof of our theorems, we need the following lemmas.

Lemma 1 \(|M_{m}^H(u)| = O(m+1)\) for \(0 < u \leq \frac{1}{m+1}\), and \(|K_{n}^H(v)| = O(n+1)\) for \(0 < v \leq \frac{1}{n+1}\).

Proof Since \(|\sin mu| \leq mu\) for \(0 < u \leq \frac{1}{m+1}\) and \(\sin(u/2) \geq (u/\pi)\), we have

\[
|M_{m}^H(u)| = \left| \frac{K_1}{2\pi} \sum_{j=0}^{m} \int_0^1 \left( \frac{m}{j} \right) s^j (1-s)^{m-j} ds \frac{\sin(j + \frac{1}{2})u}{\sin \frac{u^2}{2}} \right|
\]

\[
= \frac{K_1}{2\pi} \sum_{j=0}^{m} \int_0^1 \left( \frac{m}{j} \right) s^j (1-s)^{m-j} ds \frac{|\sin(j + \frac{1}{2})u|}{|\sin \frac{u^2}{2}|}
\]

\[
\leq \frac{K_1}{2\pi} \sum_{j=0}^{m} \int_0^1 \left( \frac{m}{j} \right) s^j (1-s)^{m-j} ds \left( \frac{1}{(j+1)^{\frac{1}{2}}} \right) u
\]

\[
= K_1 \pi \left( m + \frac{1}{2} \right) \int_0^1 \sum_{j=0}^{m} \left( \frac{m}{j} \right) s^j (1-s)^{m-j} ds
\]

\[
= K_1 \pi \left( m + \frac{1}{2} \right) \int_0^1 (1-s)^{m} ds
\]

\[
= O(m+1).
\]

Similarly, for \(0 < v \leq \frac{1}{n+1}\),

\[
|K_{n}^H(v)| = O(n+1).
\]

Lemma 2 \(|M_{m}^H(u)| = O(\frac{1}{(m+1)u})\) for \(\frac{1}{m+1} < u \leq \pi\), and \(|K_{n}^H(v)| = O(\frac{1}{(n+1)v})\) for \(\frac{1}{n+1} < v \leq \pi\).

Proof Since \(\sin(m+1)u \leq 1\) for \(\frac{1}{m+1} < u \leq \pi\) and \(\sin(u/2) \geq (u/\pi)\), we get

\[
\sum_{j=0}^{m} \int_0^1 \left( \frac{m}{j} \right) s^j (1-s)^{m-j} e^{i(j+\frac{1}{2})u} ds = \int_0^1 e^{iu/2} \sum_{j=0}^{m} \left( \frac{m}{j} \right) s^j (1-s)^{m-j} e^{iju} ds
\]

\[
= \int_0^1 e^{iu/2} (1 + s e^{iu})^m ds
\]

\[
= O\left( \frac{1}{(m+1)} \right) \left( \frac{e^{iu/2} (e^{i(m+1)u} - 1)}{e^{iu} - 1} \right).
\]

Equating the imaginary parts of both sides, we get

\[
\sum_{j=0}^{m} \int_0^1 \left( \frac{m}{j} \right) s^j (1-s)^{m-j} \sin \left( k + \frac{1}{2} \right) ds = O\left( \frac{1}{(m+1)u} \right).
\]
Therefore

\[ |M^H_{m,n}(u)| = \left| \frac{K_1}{2\pi} \sum_{j=0}^{m} \int_{0}^{1} \left( \frac{m}{j} \right) s_j(1-s)^{m-j} \frac{\sin(j + \frac{1}{2})u}{\sin \frac{u}{2}} ds \right| \]

\[ = \frac{K_1}{2u} \left| \sum_{j=0}^{m} \int_{0}^{1} \left( \frac{m}{j} \right) s_j(1-s)^{m-j} \sin\left(j + \frac{1}{2}\right)u ds \right| \]

\[ = O\left(\frac{1}{(m+1)u^2}\right). \]

Similarly, for \( \frac{1}{\pi+1} < v \leq \pi \),

\[ |K^H_n(v)| = O\left(\frac{1}{(n+1)v^2}\right). \]

**Lemma 3** If \( f(x,y) \in \text{Lip}((\xi_1, \xi_2); r) \) \((r \geq 1)\), then \( \|\phi(u,v)\|_r = O(\xi_1(u) + \xi_2(v)) \).

**Proof** Clearly,

\[ |\phi(u,v)| = \frac{1}{4} \left| f(x + u,y + v) + f(x - u,y - v) + f(x + u,y - v) + f(x - u,y + v) - 4f(x,y) \right| \]

\[ \leq \frac{1}{4} \left[ |f(x + u,y + v) - f(x,y)| + |f(x + u,y - v) - f(x,y)| + |f(x - u,y + v) - f(x,y)| + |f(x - u,y - v) - f(x,y)| \right], \]

\[ \|\phi(u,v)\|_r \leq \frac{1}{4}\left[ \|f(x + u,y + v) - f(x,y)\|_r + \|f(x + u,y - v) - f(x,y)\|_r + \|f(x - u,y + v) - f(x,y)\|_r + \|f(x - u,y - v) - f(x,y)\|_r \right] \]

\[ = O(\xi_1(u) + \xi_2(v)). \]

**5 Proof of Theorem 1**

The \((m,n)\)th partial sum of the double Fourier series (1) is given by

\[ s_{m,n}(f; (x,y)) - f(x,y) = \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \phi(u,v) \frac{\sin(m + \frac{1}{2})u \sin(n + \frac{1}{2})v}{\sin \frac{u}{2} \sin \frac{v}{2}} du dv. \]

Denoting the double Hausdorff matrix sums of \( s_{m,n} \) by \( t^H_{m,n} \), we have

\[ t^H_{m,n}(x,y) - f(x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} h^j_k \left[ s_j(k; (x,y)) - f(x,y) \right] \]

\[ = \int_{0}^{\pi} \int_{0}^{\pi} \phi(u,v) \sum_{j=0}^{m} \sum_{k=0}^{n} h^j_k \frac{\sin(j + \frac{1}{2})u \sin(k + \frac{1}{2})v}{\sin \frac{u}{2} \sin \frac{v}{2}} du dv \]

\[ = \int_{0}^{\pi} \int_{0}^{\pi} \phi(u,v)M^H_{m,n}(u)K^H_n(v) du dv, \]

(3)
Similarly, we have

\[ \|x_m - f\|_r = \int_0^\pi \left( \int_0^\pi \|\phi(u, v)\| d\nu \right) du \]

\[ = \left( \int_0^{\pi/2} \int_0^{\pi/2} + \int_0^{\pi/2} \int_\pi^{\pi/2} + \int_\pi^{\pi/2} \int_\pi^{\pi/2} + \int_\pi^{\pi/2} \int_\pi^{\pi/2} \right) \]

\[ \|\phi(u, v)\|_r M_m(u, K_H(v)) du dv \]

\[ = I_1 + I_2 + I_3 + I_4, \text{ say.} \]

Using Lemmas 1 and 3, we obtain

\[ |I_1| = \int_0^{\pi/2} \int_0^{\pi/2} \|\phi(u, v)\|_r M_m(u, K_H(v)) du dv \]

\[ = O\left( \int_0^{\pi/2} \int_0^{\pi/2} (\xi_1(u) + \xi_2(v))(m + 1)(n + 1) du dv \right) \]

\[ = O\left( (m + 1)(n + 1) \int_0^{\pi/2} \int_0^{\pi/2} (\xi_1(u) + \xi_2(v)) du dv \right) \]

\[ = O\left[ (m + 1)(n + 1) \left( \int_0^{\pi/2} \int_0^{\pi/2} \xi_1(u) du dv + \int_0^{\pi/2} \int_0^{\pi/2} \xi_2(v) du dv \right) \right] \]

\[ = O\left[ (m + 1)(n + 1) \left( \frac{\xi_1(1/m+1) + \xi_2(1/m+1)}{m+1} \right) \right] \]

\[ = O\left( \xi_1\left( \frac{1}{m+1} \right) + \xi_2\left( \frac{1}{n+1} \right) \right). \]

Again by Lemmas 1–3, we have

\[ |I_2| = O\left[ \int_0^{\pi/2} \int_0^{\pi/2} (\xi_1(u) + \xi_2(v)) \frac{(m + 1)}{(n + 1)v^2} du dv \right] \]

\[ = O\left[ \frac{(m + 1)}{(n + 1)} \left( \int_0^{\pi/2} \int_0^{\pi/2} \xi_1(u) du dv + \int_0^{\pi/2} \int_0^{\pi/2} \xi_2(v) du dv \right) \right] \]

\[ = O\left[ \frac{(m + 1)}{(n + 1)} \left( \xi_1\left( \frac{1}{m+1} \right) \frac{1}{m+1} \left( (n + 1) - \frac{1}{\pi} \right) + \frac{1}{m+1} \int_0^{\pi/2} \xi_2(v) v^2 dv \right) \right] \]

\[ = O\left( \xi_1\left( \frac{1}{m+1} \right) + \frac{1}{n+1} \int_0^{\pi/2} \xi_2(v) v^2 dv \right). \] (6)

Similarly,

\[ |I_3| = O\left[ \int_0^{\pi/2} \int_0^{\pi/2} (\xi_1(u) + \xi_2(v)) \frac{(n + 1)}{(m + 1)u^2} du dv \right] \]

\[ = O\left[ \frac{(n + 1)}{(m + 1)} \left( \int_0^{\pi/2} \int_0^{\pi/2} \xi_1(u) du dv + \int_0^{\pi/2} \int_0^{\pi/2} \xi_2(v) du dv \right) \right] \]

\[ = O\left( \frac{1}{m+1} \int_0^{\pi/2} \xi_1(u) u^2 du + \xi_2\left( \frac{1}{n+1} \right) \right). \] (7)
Also, using Lemmas 2 and 3, we get

\[
|I_4| = O \left[ \int_{\frac{1}{m+1}}^{\pi} \int_{\frac{1}{n+1}}^{\pi} \left( \xi_1(u) + \xi_2(v) \right) \frac{1}{(m+1)u^2} \frac{1}{(n+1)v^2} \, du \, dv \right]
\]

\[
= O \left[ \frac{1}{(m+1)(n+1)} \left( \int_{\frac{1}{m+1}}^{\pi} \frac{1}{u^2} \, du \int_{\frac{1}{n+1}}^{\pi} \frac{1}{v^2} \, dv + \int_{\frac{1}{n+1}}^{\pi} \frac{1}{u^2} \, du \int_{\frac{1}{m+1}}^{\pi} \frac{1}{v^2} \, dv \right) \right]
\]

\[
= O \left( \frac{1}{m+1} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(u)}{u^2} \, du + \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi_2(v)}{v^2} \, dv \right). \tag{8}
\]

Next,

\[
\frac{1}{m+1} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(u)}{u^2} \, du \geq \frac{1}{(m+1)} \xi_1 \left( \frac{1}{m+1} \right) \int_{\frac{1}{m+1}}^{\pi} \frac{1}{u^2} \, du
\]

\[
= \frac{1}{(m+1)} \xi_1 \left( \frac{1}{m+1} \right) \left\{ \frac{1}{\frac{1}{m+1}} \right\}^\pi
\]

\[
= \xi_1 \left( \frac{1}{m+1} \right) \left\{ 1 - \frac{1}{(m+1)\pi} \right\}
\]

\[
\geq \frac{1}{2} \xi_1 \left( \frac{1}{m+1} \right),
\]

or \( \xi_1 \left( \frac{1}{m+1} \right) = O \left( \frac{1}{m+1} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(u)}{u^2} \, du \right). \tag{9}\)

Similarly,

\( \xi_2 \left( \frac{1}{n+1} \right) = O \left( \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi_2(v)}{v^2} \, dv \right). \tag{10}\)

Combining equations (5)–(10), we have

\[
\left\| t_{m,n}^H f - f \right\|_r = O \left( \frac{1}{(m+1)} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(u)}{u^2} \, du + \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi_2(v)}{v^2} \, dv \right).
\]

This completes the proof of Theorem 1.

### 6 Corollaries

From the main theorem we derive the following corollaries.

**Corollary 1** If \( f(x, y) \) is a 2\( \pi \) periodic function with respect to both variables \( x \) and \( y \), Lebesgue integrable in \((-\pi, \pi) \times (-\pi, \pi)\) and belonging to the class \( \text{Lip}((\alpha, \beta); r) \) \( (r \geq 1) \), then the degree of approximation of \( f(x, y) \) by means \( t_{m,n}^H \) of double Fourier series (1) satisfies

\[
\left\| t_{m,n}^H f - f \right\|_r = \begin{cases} 
O((m+1)^{-\alpha} + (n+1)^{-\beta}), & 0 < \alpha < 1, 0 < \beta < 1, \\
O((m+1)^{-\alpha} + \log^{(m+1)}(n+1)), & 0 < \alpha < 1, \beta = 1, \\
O(\log^{(m+1)}(n+1)^{\alpha} + (n+1)^{-\beta}), & \alpha = 1, 0 < \beta < 1, \\
O(\log^{(m+1)}(n+1)^{\alpha} + \log^{(n+1)}(m+1)), & \alpha = \beta = 1,
\end{cases}
\]

for \( m, n = 0, 1, 2, \ldots \).
Corollary 2 If \( f(x, y) \) is a \( 2\pi \) periodic function with respect to both variables \( x \) and \( y \), Lebesgue integrable in \((-\pi, \pi) \times (-\pi, \pi)\) and belonging to the class \( \text{Lip}(\alpha, \beta) \), then the degree of approximation of \( f(x, y) \) by double Hausdorff matrix summability means \( t_{m,n}^H \) of double Fourier series (1) satisfies

\[
\| t_{m,n}^H - f \|_\infty = \begin{cases} 
O\left((m+1)^{-\alpha} + (n+1)^{-\beta}\right), & 0 < \alpha < 1, 0 < \beta < 1, \\
O\left((m+1)^{-\alpha} + \frac{\log(n+1)}{(n+1)}\right), & 0 < \alpha < 1, \beta = 1, \\
O\left(\frac{\log(m+1)}{m+1} + (n+1)^{-\beta}\right), & \alpha = 1, 0 < \beta < 1, \\
O\left(\frac{\log(m+1)}{m+1} + \frac{\log(n+1)}{n+1}\right), & \alpha = \beta = 1,
\end{cases}
\]

for \( m, n = 0, 1, 2, \ldots \).

Corollary 3 If \( f(x, y) \) is a \( 2\pi \) periodic function with respect to both variables \( x \) and \( y \), Lebesgue integrable in \((-\pi, \pi) \times (-\pi, \pi)\) and belonging to the class \( \text{Lip}(\xi_1, \xi_2); r) \), then the degree of approximation of \( f(x, y) \) by almost Euler summability means of double Fourier series (1) satisfies

\[
\| t_{m,n}^E - f \|_r = O\left(\frac{1}{m+1}\int_{-\pi}^{\pi} \frac{\xi_1(u)}{u^2} \, du + \frac{1}{n+1}\int_{-\pi}^{\pi} \frac{\xi_2(v)}{v^2} \, dv\right)
\]

for \( m, n = 0, 1, 2, \ldots \).

Corollary 4 For \( \gamma, \delta \geq -1 \), the Cesàro means \( \sigma_{m,n}^{\gamma,\delta} \) of order \( \gamma \) and \( \delta \), that is, \( (C, \gamma, \delta) \) means of double Fourier series, are given by

\[
\sigma_{m,n}^{\gamma,\delta} = \frac{1}{\pi^m A_m} A_n \sum_{j=0}^m \sum_{k=0}^n A_{m-j}^{-1} A_{n-k}^{-1} \xi_{j,k},
\]

where \( A_m = \binom{\gamma + m}{m} \) and \( A_n = \binom{\delta + n}{n} \).

If \( f(x, y) \) is a \( 2\pi \) periodic function with respect to both variables \( x \) and \( y \), Lebesgue integrable in \((-\pi, \pi) \times (-\pi, \pi)\) and belonging to the class \( \text{Lip}(\xi_1, \xi_2); r) \), then the degree of approximation of \( f(x, y) \) by \( (C, \gamma, \delta) \) means of double Fourier series (1), satisfies

\[
\| \sigma_{m,n}^{\gamma,\delta} - f \|_r = O\left(\frac{1}{m+1}\int_{-\pi}^{\pi} \frac{\xi_1(u)}{u^2} \, du + \frac{1}{n+1}\int_{-\pi}^{\pi} \frac{\xi_2(v)}{v^2} \, dv\right)
\]

for \( m, n = 0, 1, 2, \ldots \).

7 Conclusion

We established the degree of approximation of a function \( f(x, y) \) belonging to the generalized Lipschitz class by double Hausdorff matrix summability means of its double Fourier
series in the form of equation (2). If \( \xi_1 = u^\alpha \) and \( \xi_2 = v^\beta \), then Theorem 1 reduces to Corollary 1, and as \( r \to \infty \), Corollary 1 reduces to Corollary 2. Independent proofs of Corollaries 1–4 can be developed along the same lines as that of Theorem 1. Results similar to Corollaries 3 and 4 can be derived for \((E,1,1)\) means and \((C,1,1)\) means of its double Fourier series. In this way, we can obtain some more different results by changing \( \xi_1, \xi_2, \) and \( \mu_{mn} \) under given conditions. For functions \( f(x,y) \) belonging to the Zygmund classes \( \text{Zyg}(\alpha, \beta) \) and \( \text{Zyg}(\alpha, \beta; p) \) discussed in [9], the degree of approximation using double Hausdorff matrix summability means and hence almost Euler means of its double Fourier series can be obtained similarly to Theorem 1.

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Author details
1Department of Mathematics, Netarhat Vidyalaya, Netarhat, Jharkhand, India. 2Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak, Madhya Pradesh, India. 3Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan. 4Department of Mathematics, Aligarh Muslim University, Aligarh, India.

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