Local dynamics for fibred holomorphic transformations

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Received 28 May 2007, in final form 2 October 2007
Published 20 November 2007
Online at stacks.iop.org/Non/20/2939

Abstract

Fibred holomorphic dynamics are skew-product transformations
$F(\theta, z) = (\theta + \alpha, f_\theta(z))$ over an irrational rotation, such that $f_\theta$ is holomorphic for every $\theta$. In this paper we study such a dynamics in a neighbourhood of an invariant curve. We obtain some results analogous to the results in the non-fibred case. In particular, we prove a fibred version of the folklore result stating that Lyapounov stability is equivalent to linearization around a fixed point. We also obtain a fibred version of the Pérez-Marco continua.

Mathematics Subject Classification: 37F50, 34C25

1. Introduction

Let $\alpha$ be a rationally independent vector in $T^d$, which will be fixed for the rest of this work. Let $U \subset \mathbb{C}$ be an open simply connected neighbourhood of the origin. We consider fibred continuous injective transformations

$F : T^d \times U \rightarrow T^d \times \mathbb{C},$

$(\theta, z) \mapsto (\theta + \alpha, f_\theta(z)).$

We will assume that the functions $f_\theta : U \rightarrow \mathbb{C}$ are holomorphic for all $\theta \in T^d$; we call $F$ a fibred holomorphic dynamics, and we denote it by fhd.

The fhds are a special class of the skew-product transformations. A closely related class of skew-product transformations over irrational rotations is fibred circle homeomorphisms, where the fibre is a circle and the $f_\theta$ are circle homeomorphisms. In [2] Herman establishes the basis for the study of such systems, defining the fibred rotation number. Recently this study has been relaunched mainly by the works of Jäger and co-workers (see [4,5]). They have established in particular a Poincaré-like classification of these transformations by means of the fibred rotation number.
number and the presence of invariant graphs. In his doctoral thesis, Sester [14] (see also [15]) has studied hyperbolic fibred polynomials, successfully generalizing the classical notions of the Julia set, Green’s function and the principal cardioid of the Mandelbrot set in the parameter space. Other important contributions to this subject are the works of Jonsson [6, 7].

The notion of periodic or fixed points for $F$ has no sense, since the irrational rotation in the base is minimal. We are interested in the local dynamics of $F$ in a neighbourhood of an invariant curve, that is, a continuous curve $u : \mathbb{T}^d \rightarrow U$ such that

$$F(\theta, u(\theta)) = (\theta + \alpha, u(\theta + \alpha))$$

for all $\theta \in \mathbb{T}^d$, or equivalently $f_\theta(u(\theta)) = u(\theta + \alpha)$. These objects play the role of a centre around which the dynamics of $F$ is organized, generalizing thus the role of a fixed point for the local dynamics of a holomorphic germ $g : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, the so-called non-fibred case. This work is devoted to specifying and proving this last assertion. A complete survey of the non-fibred case can be found in [1].

2. Definitions

Let $F$ be a fhd and $u$ an invariant curve for $F$. We define the infinitesimal characteristics of the curve $u$ by the following definition.

**Definition 2.1.** The fibred multiplicator $\kappa(u)$ of the curve is the real number

$$\kappa(u) = \exp \left( \int_{\mathbb{T}^d} \log |\partial_z f_\theta(u(\theta))| \, d\theta \right).$$

We recall that $F$ is injective and so the differential $\partial_z f_\theta$ is always non-zero. We say $u$ is an attracting curve if $\kappa(u) < 1$, $u$ is a repelling curve if $\kappa(u) > 1$ and an indifferent curve if $\kappa(u) = 1$.

**Definition 2.2.** Suppose $\kappa(u) = 1$ and the application

$$\theta \mapsto \partial_z f_\theta(u(\theta))$$

is homotopic in $\mathbb{C} \setminus \{0\}$ to a constant. We refer to this situation as the indifferent zero degree case. We define a number which represents the average rotation speed of the dynamics around the invariant curve,

$$\varrho_{\text{tr}}(u) = \frac{1}{2\pi i} \int_{\mathbb{T}^d} \log \partial_z f_\theta(u(\theta)) \, d\theta.$$

We call this number the fibred rotation number. Note that the log above is well-defined mod $2\pi i$ and the number $\varrho_{\text{tr}}(u)$ is well-defined mod 1.

In general, we will consider the invariant curve as being the zero section curve $\{z \equiv 0\}_{\mathbb{T}^d}$. We obtain this situation by conjugating $F$ with the changes in coordinates given by $H(\theta) = (\theta, z + u(\theta))$. In this way the resulting transformation $\tilde{F} = H^{-1} \circ F \circ H$ has the zero section as an invariant curve with the same infinitesimal characteristics as the original curve $u$ for $F$. More generally, we will need to consider continuous change in coordinates $H$ defined in a tubular neighbourhood of the invariant curve (the zero section), say

$$(\theta, z) \mapsto (\theta, h_\theta(z)).$$

The functions $h_\theta$ will be biholomorphic transformations between two topological discs fixing the origin, that is $h_\theta(0) = 0$. When conjugating our transformation $F$ we will get a new transformation $\tilde{F} = H^{-1} \circ F \circ H$ having the zero section $\tilde{u} = \{z \equiv 0\}_{\mathbb{T}^d}$ as an invariant curve.
and \( \kappa(\tilde{u}) = \kappa(u) \). That is, the fibred multiplicator is invariant by conjugacy. In the indifferent zero degree case we will admit only those changes in coordinates which have themselves a zero degree, that is, the application \( \theta \mapsto \partial_u h_\theta(0) \) is homotopic in \( \mathbb{C} \setminus \{0\} \) to a constant. In that case we also get \( \partial_u (\tilde{u}) = \partial_u (u) \). That is, in the indifferent zero degree case the fibred rotation number is also invariant by zero degree conjugacy.

We say that \( F \) is linearizable if we can conjugate it in a tubular neighbourhood of the invariant curve to a transformation \( \Lambda_A \) of the form

\[
\Lambda_A(\theta, z) = (\theta + \alpha, A(\theta)z),
\]

for a continuous complex function \( A : \mathbb{T}^d \to \mathbb{C} \). When the absolute value \( |A(\theta)| = \kappa(u) \) for all \( \theta \in \mathbb{T}^d \) we say that \( F \) is modulus linearizable. When \( |A(\theta)| < 1 \) for every \( \theta \in \mathbb{T}^d \) (\( |A(\theta)| > 1 \) for every \( \theta \in \mathbb{T}^d \)) we say that \( F \) is weakly linearizable to an attractive (repelling) linear fhd.

In the indifferent zero degree case we say \( F \) is strongly linearizable if \( A(\theta) = e^{2\pi i \rho_{tr}(u)} \).

We say that an invariant curve \( u \) is stable if there exists an open invariant and bounded tubular neighbourhood \( U \), such that \( F(U) \subset U \). We always suppose that the fibres \( U_\theta \) are topological discs and that \( U \) contains the curve. We call such a neighbourhood an open invariant tube.

3. Statement of the results

In this paper we prove the following results concerning the local dynamics of a fhd \( F \) around an invariant curve \( u \).

**Proposition 3.1.** Let \( F \) be a fhd with an invariant curve \( u \). If \( \kappa(u) < 1 \) then there exists an open tubular neighbourhood of the curve which is attracted to the curve by positive iteration. Moreover, \( F \) is weakly linearizable.

When \( \kappa(u) > 1 \) we obtain the analogous result for the repelling case by considering the inverse \( F^{-1} \). When the curve is indifferent (\( \kappa(u) = 1 \)) we generalize to fibred dynamics a very well-known fact in the non-fibred case: Lyapounov stability is equivalent to linearization.

**Proposition 3.2.** Let \( F \) be a fhd with an invariant curve \( u \). If the curve is indifferent then the stability of the curve is equivalent to modulus linearization.

As we will see, linearizability alone does not imply stability. Then we look at a non-stable situation in the indifferent case. We show that there are still some nearby points with a complete orbit (past and future) which stays near the invariant curve. Indeed, we prove a fibred version of the continua’s theorem by Pérez-Marco [11].

**Theorem 3.3 (Fibred Pérez-Marco’s continua).** Let \( F \) be a fhd with an indifferent invariant curve \( u \). Let \( \mathcal{U} \) be an open neighbourhood of the curve whose fibres are Jordan domains (bounded by a Jordan curve, \( \partial U_\theta \)). We also assume that \( \partial U_\theta \) depends continuously on \( \theta \) and that \( F, F^{-1} \) define fhds which are injective in some neighbourhood of \( \mathcal{U} \). Then there exists a compact connected set \( K \subset \mathcal{U} \) such that

(i) fibres \( K_\theta \) are connected full compact sets for every \( \theta \in \mathbb{T}^d \),
(ii) graph(u) \( \subset K \),
(iii) \( F(K) = F^{-1}(K) = K \), that is, \( K \) is completely invariant by \( F \),
(iv) \( K \cap \partial \mathcal{U} \neq \emptyset \).
We recall that a compact set $K \subset \mathbb{C}$ is called full (or filled) if its complement $\mathbb{C} \setminus K$ is (simply) connected.

We also consider fhds which are analytic with respect to the $\theta$ variable. Let $\delta$ be a positive real number. An analytic fhd is a transformation $(\theta, z) \mapsto F(\theta, z)$ that is defined and holomorphic as a $d + 1$ variables function on the product $B^d_\delta \times U$, where $B_\delta = \{ \theta \in \mathbb{C}/\mathbb{Z} \mid |\text{Im}(\theta)| < \delta \}$. For this class of regularity and under the indifferent zero degree hypothesis one can show a fibred version of the Siegel linearization theorem [16]. The Siegel theorem for holomorphic germs states that under some arithmetical condition on the rotation number, the dynamics is linearizable without additional hypothesis on the stability of the fixed point. More precisely one has the following theorem.

**Theorem 3.4 (Siegel’s theorem for fhd).** Let $\delta > 0$, $F$ be a fhd analytic in the product $B^d_\delta \times \mathbb{D}$. Let $u : B^d_\delta \to \mathbb{C}$ be an indifferent invariant analytic curve with zero degree. If the fibred rotation number $\varphi_{tr}(u) = \beta$ is such that the pair $(\alpha, \beta)$ verifies the arithmetical condition $CD \geq 1(c, \tau)$ for some $c > 0$, $\tau \geq 0$, then $F$ is strongly linearizable in a tubular neighbourhood of the curve.

We denote by $\mathbb{D}$ the unitary open disc in $\mathbb{C}$. The arithmetical condition $CD \geq 1(c, \tau)$ is defined by $CD \geq 1(c, \tau) = \{ (\alpha, \beta) \in \mathbb{T}^d \times \mathbb{T}^1 \mid \|n \cdot \alpha - j\beta\| > \frac{c}{(|n| + j)^{d+1}\tau} \forall j \geq 0, n \in \mathbb{Z}^d \}$, where for an integer vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ we put $|n| = \max_{1 \leq i \leq d} |n_i|$. The proof of this theorem is a straightforward adaptation of the classical proof due to Moser for Siegel’s theorem (see [8]), by using the Newton method. However, this proof will not be given in this paper and can be found in [13].

4. Proofs

We recall that the invariant curve is supposed to be the zero section curve $u = \{ z = 0 \}_{\mathbb{T}_1}$. We use the standard notation in the skew-product dynamics $f^n_\theta(z) := \Pi_{\mathbb{T}} F^n(\theta, z) = f_{\theta+n\alpha}(f_{\theta+(n-1)\alpha}(\cdots (f_\theta(z)) \cdots))$ for the holomorphic second coordinate. We put $\rho_1(\theta) = \partial_z f_\theta(0)$, and in this way the transformation $F$ has the form

$$F(\theta, z) = (\theta + \alpha, \rho_1(\theta)z + z^2 \rho(\theta, z)).$$ (2)

The function $\rho$ is a continuous function, holomorphic in each fibre and $\rho_1 : \mathbb{T}^d \to \mathbb{C} \setminus \{0\}$ is a continuous function.

4.1. Proof of proposition 3.1

Previous to the main proof, we want to point out the following complex analysis result (see [13]).

**Lemma 4.1.** For any constants $A \in (0, 1), B > 0$ there exist constants $R = R(A, B), r = r(A, B) \in (0, 1)$ such that if $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function verifying

(i) $f(0) = 0$,
(ii) $\|f\|_\mathbb{D} := \sup_{\mathbb{D}} |f(z)| \leq B$,
(iii) $A < |\partial_z f(0)| < A^{-1},$

then there exists an open set $U_f \supset D(0, r)$ such that $f : U_f \to D(0, R)$ is a biholomorphism.

As a corollary we obtain the following.
Corollary 4.2. Let $F : \mathbb{T}^d \times \mathbb{D} \to \mathbb{T}^d \times \mathbb{C}$ be a fhd, $\alpha \in \mathbb{T}^d$ its rotation number on the base $\mathbb{T}^d$. Let $u : \mathbb{T}^d \to \mathbb{D}$ be an invariant curve whose multiplicator is $\kappa = \kappa(u)$. Then there exists a positive radius $R \in (0, 1)$ such that the inverse transformation $F^{-1}$ is a fhd which is well defined in the tube $U_R$, of radius $R$ and centred on the invariant curve. The rotation number of $F^{-1}$ on the base is $-\alpha$. The curve $u$ is invariant by $F^{-1}$ and its multiplicator as an invariant curve for $F^{-1}$ is $\kappa^{-1}$.

We now begin the proof of proposition 3.1. We assume $\int_{\mathbb{T}^d} \log |\rho_1(\theta)| \, d\theta < 0$ and we put $\kappa = \kappa(u) \in (0, 1)$. We introduce a first conjugacy in order that the absolute value $|\rho_1(\theta)|$ becomes closer to this mean $\kappa$. Let $l : \mathbb{T}^d \to \mathbb{R}$ be a trigonometric polynomial, $l(\theta) = \sum_{0 \leq |n| < N} \hat{l}(n) e^{2\pi i n \theta}$, satisfying

(i) $\hat{l}(0) = \frac{1}{2\pi} \int_{\mathbb{T}^d} l(\theta) \, d\theta = \log \kappa$,

(ii) $|\log |\rho_1(\theta)| - l(\theta)| < \frac{\log 1 + \kappa^{-1/2}}{2}$ for all $\theta \in \mathbb{T}^d$.

Let $\tilde{u}_1 : \mathbb{T}^d \to \mathbb{R}$ be the zero-mean solution to the cohomological equation

$$\tilde{u}_1(\theta) - \tilde{u}_1(\theta + \alpha) = \log \kappa - l(\theta).$$

The Fourier series method gives that $\tilde{u}_1(\theta) = \sum_{0 \leq |n| < N} \hat{\tilde{u}}_1(n) e^{2\pi i n \theta}$. We put $u_1(\theta) = e^{\tilde{u}_1(\theta)}$.

We conjugate $F$ by the fibred re-scaling $H(\theta, z) = (\theta, u_1(\theta) z)$ getting

$$|\partial_z (H^{-1} \circ F \circ H)_\theta(0)| = \left| \frac{u_1(\theta)}{u_1(\theta + \alpha)} |\rho_1(\theta)| \right| = \kappa \frac{|\rho_1(\theta)|}{\kappa^{1/2}} < \frac{\kappa + \kappa^{1/2}}{2} < 1.$$

We thus may suppose that $|\rho_1(\theta)| < \frac{\kappa^{1/2}}{2} < 1$ for all $\theta \in \mathbb{T}^d$, and the holomorphic functions $f_\theta$ are defined in a disc $D_R$ for some $R > 0$. The following proof is an adaptation of the classical proof of the corresponding non-fibred result (see [1]).

We define a continuous function $\phi : \mathbb{T}^d \times D_R \to \mathbb{C}$ holomorphic in each fibre, by $\phi(\theta, z) = \rho(\theta, z) \rho_1(\theta)^{-1}$. Let $\{g^n : \mathbb{T}^d \times D_R \to \mathbb{C}\}_{n \geq 1}$ be the sequence of continuous functions defined by

$$g^n(\theta, z) = \left( \prod_{j=0}^{n-1} \rho_1(\theta + j\alpha) \right)^{-1} f_\theta^n(z).$$

One has

$$f_{\theta + 1}(z) = \rho_1(\theta + n\alpha) f_\theta^n(z) \{1 + \phi(\theta + n\alpha, f_\theta^n(z)) f_\theta^n(z)\},$$

$$g_{\theta + 1}(z) = g_\theta^n(z) \{1 + \phi(\theta + n\alpha, f_\theta^n(z)) f_\theta^n(z)\} = z \prod_{j=0}^{n} \{1 + \phi(\theta + j\alpha, f_\theta^j(z)) f_\theta^j(z)\}.$$
From the bound $|\rho_1(\theta)| < \frac{c^{n+1/2}}{2} < 1$ we get a constant $c \in (0,1)$ and a positive radius $r$ such that $|f'_{\theta}(z)| < c^n|z|$ for $|z| < r$, $\theta \in \mathbb{T}^d$ and for all $n$ large enough. This implies the desired convergence. Thus, the limit function $g : \mathbb{T}^d \times D_r \to \mathbb{C}$ is continuous, holomorphic in each fibre and verifies $g_0(0) = 0$, $\partial_z g_0(0) = 1$ for all $\theta \in \mathbb{T}^d$. Lemma 4.1 applies and gives us a tubular neighbourhood of the zero section where $g$ is invertible. There, the inverse $g^{-1}$ is a continuous function; defining

$$\Phi_n(\theta, z) = (\theta, g^n_\theta(z))$$

we have

$$\Phi_n \circ F(\theta, z) = \left( \theta + \alpha, \left( \prod_{i=0}^{n-1} \rho_1(\theta + i\alpha) \right)^{-1} f'_{\theta+n\alpha} (f_{\theta}(z)) \right)$$

$$= \left( \theta + \alpha, \rho_1(\theta) \left( \prod_{i=0}^{n} \rho_1(\theta + i\alpha) \right)^{-1} f'_{\theta+n+1}(z) \right) = \Lambda_{\rho_1} \circ \Phi_{n+1}(\theta, z).$$

Thus in the limit

$$F = \Phi^{-1} \circ \Lambda_{\rho_1} \circ \Phi.$$  

The corresponding result in the repelling case ($\kappa(u) > 1$) is an easy consequence of the attractive one, by considering the curve as an invariant attractive curve for the inverse $F^{-1}$, which is a well-defined hfd in a tubular neighbourhood of the curve.

4.2. Proof of proposition 3.2

If $F$ is modulus linearizable then an open invariant tube is found by the image of $\mathbb{T}^d \times D_r$ by the conjugacy function, for small $r > 0$. Hence, it remains to prove the converse implication. We will call a domain any simply connected open set in the complex plane $\mathbb{C}$, containing the origin and different from $\mathbb{C}$. We need to recall some definitions about domains and their uniformizations. We refer the reader to the book of Pommerenke [12] for further references about the facts below.

**Conformal radius.** Let $\Omega$ be a domain and let $h : \mathbb{D} \to \Omega$ be the uniformization (biholomorphic function) verifying $h(0) = 0$ and $h'(0) > 0$. We call the positive real number $h'(0)$ the conformal radius of $\Omega$ and we denote it by $R(\Omega)$. The following monotonicity property holds: if $\Omega' \subset \Omega$ are domains, then one has $R(\Omega') \leq R(\Omega)$ and equality occurs only in the case of $\Omega' = \Omega$. We associate with a domain $\Omega$ a sequence $\{\Omega_n\}_{n \geq 2}$ of domains defined by $\Omega_n = h(D_{1/2})$. This sequence is a monotone exhausting sequence and the conformal radius verifies $R(\Omega^n) \nearrow R(\Omega)$.

**Carathéodory’s kernel.** Let $\{O_n\}_{n \geq 0}$ be a sequence of domains. The kernel $\nabla (\{O_n\})$ of this sequence consists of the union of the origin and those points $z \in \mathbb{C}$ which satisfy the following property: there exists a domain $\Omega_z$ containing $z$, such that $\Omega_z \subset O_n$ for $n$ large enough. The kernel is either reduced to the origin, a domain, or $\mathbb{C}$.

**Kernel convergence.** We say the sequence $\{O_n\}_{n \geq 0}$ is kernel convergent to a domain $O$ if every subsequence of $\{O_n\}_{n \geq 0}$ has the domain $O$ as kernel.
Theorem 4.3 (Caratheodory). Let \( \{f_n : \mathbb{D} \to \mathbb{C}\}_{n \geq 0} \) be a sequence of univalent functions, with \( f_n(0) = 0 \) and \( f'_n(0) > 0 \) for all \( n \geq 0 \). Let \( O_n = f_n(\mathbb{D}) \). Then the sequence \( \{O_n\} \) is kernel convergent to a domain \( O \) iff the sequence \( \{f_n\} \) is convergent uniformly on compacts of \( \mathbb{D} \). Moreover, in this case the limit function \( f \) is a conformal representation of \( O \). Finally, if \( K \subset O \) is a compact set, then \( K \) is contained in \( O_n \) for large \( n \) and \( f_n^{-1}|_K \) converges uniformly to \( f^{-1}|_K \).

For \( n \geq 0 \) and \( \theta \in \mathbb{T}^d \) we define the Birkhoff sums associated with the invariant curve as being

\[
S_n^u(\theta) = \sum_{i=0}^{n-1} \log |\partial_z f_{\theta+iu}(u(\theta + i\alpha))|.
\]

In the indifferent case we have the uniform limit

\[
\log \kappa(u) = 0 = \lim_{n \to \infty} \frac{1}{n} S_n^u(\theta)
\]
due to the unique ergodicity of the irrational rotation \( \theta \mapsto \theta + \alpha \). Let \( U \) be an invariant open tube containing the invariant curve (the zero section). Each fibre \( U_\theta \) is a domain. We associate with each fibre \( U_\theta \) the uniformization \( h_\theta : \mathbb{D} \to U_\theta \) as before.

Lemma 4.4. Let \( F \) be a fhd with a stable invariant curve \( u \). There exist constants \( B_+ > 0 \), \( B_- < 0 \) and a point \( \tilde{\theta} \in \mathbb{T}^d \) such that \( B_- < S_n^u(\tilde{\theta}) < B_+ \) for all \( n \geq 0 \).

Proof. Since the open set \( U \) is bounded, there exists a positive constant \( W > 0 \) such that \( W^{-1} < R(U_\theta) < W \) for \( \theta \in \mathbb{T}^1 \). Let \( \theta \in \mathbb{T}^1 \) and \( n \geq 0 \). We define the functions \( \tilde{f}_n^\theta : \mathbb{D} \to \mathbb{D} \) by

\[
\tilde{f}_n^\theta(z) = h_{\theta+nu}(f_n^\theta(h_\theta(z))).
\]

The Schwartz lemma yields \( |\partial_z \tilde{f}_n^\theta(0)| \leq 1 \), or in other words

\[
|h'_\theta(0)|^{-1} \left( \prod_{i=0}^{n-1} |\rho_1(\theta + i\alpha)| \right) h'_\theta(0) \leq 1.
\]

From this inequality the existence of \( B_+ \) is evident for any choice of \( \tilde{\theta} \). We show now the existence of \( B_- \). We proceed by contradiction. Suppose \( \inf_{n \geq 0} S_n^u(\theta) = -\infty \) for every \( \theta \in \mathbb{T}^d \). Then for any \( l < 0 \), there would be a smallest natural number \( n_l(\theta) \) with \( S_{n_l}(\theta) < l \). We claim that there exists a uniform upper bound \( N_l \) for \( n_l(\theta) \). Indeed, otherwise, there should exist a sequence \( \{\theta_j\}_{j \in \mathbb{N}} \subset \mathbb{T}^d \) such that \( n_l(\theta_j) > j \) for all \( j \geq 1 \). Let \( \hat{\theta} \) be an accumulation point of this sequence. Since Birkhoff’s sums are continuous functions in the \( \theta \) variable, we know that in an open neighbourhood of that point \( \hat{\theta} \) one has \( S_{n_l(\hat{\theta})}(\hat{\theta}) < l \), a contradiction.

We put \( l = -2B_+ \), \( N = N_l \), \( n(\theta) = n_l(\theta) \). Then we have

\[
S_N^u(\theta) = S_{n_l(\theta)}^u(\theta) + S_{n_l(\theta)+n(\theta)}^u(\theta) < l + B_+ = \frac{l}{2} < 0,
\]

but \( \int_{\mathbb{T}^1} S_N^u(\theta) \, d\theta = 0 \), a contradiction. \( \blacksquare \)

Bounded Birkhoff’s sums are an important tool when solving cohomological equations, as showed by the following well-known theorem.
Theorem 4.5 (Gottschalk–Hedlund, see [8]). Assume $X$ is a compact metric space, $w : X \to X$ is a continuous minimal transformation. Let $v : X \to \mathbb{C}$ be a continuous function such that their Birkhoff’s sums verify

$$\sup_{n \in \mathbb{N}} \left| \sum_{i=0}^{n} v \circ w^i(x_0) \right| < \infty$$

for some $x_0 \in X$. Then there exists a continuous solution $y : X \to \mathbb{C}$ for the cohomological equation

$$y \circ w - y = v.$$

Remark 4.6. As shown by Furstenberg, there exist analytic minimal non-uniquely ergodic diffeomorphims of $T^2$. The key to this result is the existence of a real irrational number $\omega$ and an analytic function $\phi : T_1 \to \mathbb{R}$ such that

$$\int_{T_1} \phi \, d\theta = 0$$

and the cohomological equation

$$\psi(\theta + \omega) - \psi(\theta) = \phi(\theta) \quad (4)$$

has a measurable but not continuous solution (see [8]). Let $F(\theta, z) = (\theta + \omega, e^{\phi(\theta)} z)$. For this linear fdh the invariant curve $\{ z \equiv 0 \}$ $T_1$ is indifferent but not stable. Otherwise, lemma 4.4 and theorem 4.5 would provide a continuous solution to (4).

Lemma 4.7. The function $R : T^d \to \mathbb{R}^+$ is a lower semi-continuous function.

Proof. Let $\varepsilon > 0$ and $\tilde{\theta}$ in $T^d$. Let $n^* \geq 2$ such that $R(\Omega_{n^*}(U_{\tilde{\theta}})) > R(\tilde{\theta}) - \varepsilon$. The set $U$ is open, therefore there exists an open neighbourhood $I_{\tilde{\theta}}$ around $\tilde{\theta}$ such that for $s$ in such a neighbourhood, the open set $U_s$ contains the compact set $\Omega_{n^*}(U_{\tilde{\theta}})$ (since this compact set is far away from the complementary set of $U$). One has then

$$R(s) > R(\Omega_{n^*}(U_{\tilde{\theta}})) > R(\tilde{\theta}) - \varepsilon. \quad \blacksquare$$

Lemma 4.8. The conformal radius $R(\theta)$ is independent of $\theta$ in $T^d$.

Proof. The function $R : T^d \to \mathbb{R}^+$ is l.s.c., thus there exists a point $\tilde{\theta}$ in $T^d$ where $R$ reaches its minimum. From the inclusion

$$f_{\tilde{\theta} - \alpha}(U_{\tilde{\theta} - \alpha}) \subset U_{\tilde{\theta}}$$

we deduce that

$$R(f_{\tilde{\theta} - \alpha}(U_{\tilde{\theta} - \alpha})) \equiv R(\tilde{\theta}).$$

The function $f_{\tilde{\theta}}$ preserves the conformal radius since $|\rho_1(\theta)| = 1$ for every $\theta$ in $T^d$. Then we have

$$R(f_{\tilde{\theta} - \alpha}(U_{\tilde{\theta} - \alpha})) = R(\tilde{\theta} - \alpha).$$

The equality

$$R(\tilde{\theta} - \alpha) = R(\tilde{\theta})$$
comes from the minimality of the value \( R(\tilde{\theta}) \). The same argument tells us that \( R(\tilde{\theta} - n\alpha) = R(\tilde{\theta}) \) for all \( n \geq 0 \). We have a l.s.c. function with a dense set of minimal points, therefore a constant function. ■

We define \( R^d_f \) as the conformal radius of some (any) domain \( \mathcal{U}_0 \).

**Lemma 4.9.** Let \( \{\theta_n\}_{n \geq 0} \) be a sequence in the torus converging to a point \( \tilde{\theta} \). The kernel of the sequence of domains \( \{\mathcal{U}_{\theta_n}\} \) is \( \mathcal{U}_{\tilde{\theta}} \). In particular the sequence \( \{\mathcal{U}_{\theta_n}\} \) kernel converges to \( \mathcal{U}_{\tilde{\theta}} \).

**Proof.** Let \( \nabla \) be the kernel of the sequence \( \{\mathcal{U}_{\theta_n}\} \). We first show \( \mathcal{U}_0 \subset \nabla \). Let \( z \) be in \( \mathcal{U}_0 \). Let \( \theta \), \( n^* \geq 2 \) such that \( z \) belongs to \( \Omega_0, \Omega_{\Phi, \theta} \). The set \( \mathcal{U} \) is open, thus there exists an open interval \( I_\theta \) around \( \tilde{\theta} \) such that if \( s \in I_\theta \) then \( \Omega_{\Phi, \theta}(\mathcal{U}_0) \subset \mathcal{U} \). One concludes that the domain \( \Omega_{\Phi, \theta}(\mathcal{U}_0) \) is contained in \( \mathcal{U}_{\theta_n} \) for \( n \) large enough. This shows that \( z \) belongs to the kernel \( \nabla \) of the domains \( \mathcal{U}_{\theta_n} \), i.e.

\[
\mathcal{U}_0 \subset \nabla.
\]

In particular we see that the kernel \( \nabla \) is not reduced to the complex origin. The kernel is bounded since the domains \( \mathcal{U}_{\theta_n} \) are uniformly bounded. Therefore \( \nabla \) is a domain. Its conformal radius verifies

\[
R(\nabla) \geq R(\tilde{\theta}) = R^d_f.
\]

Now we show that \( \nabla = \mathcal{U}_0 \). Let \( \varepsilon > 0 \) and let \( n^* \geq 2 \) such that

\[
R(\nabla) > R(\Omega_{\Phi, \theta}(\mathcal{U}_0)) > R(\nabla) - \varepsilon.
\]

From the definition of the kernel, for all \( z \) which belong to the compact set \( \Omega_{\Phi, \theta}(\mathcal{U}_0) \) there exists a domain \( \Omega_z \) containing the point \( z \) and contained in the domains \( \mathcal{U}_{\theta_n} \) for \( n \) large enough.

Considering a finite cover of \( \Omega_{\Phi, \theta}(\mathcal{U}_0) \) by domains \( \Omega_z \), we see that for \( n \) large enough the compact set \( \Omega_{\Phi, \theta}(\mathcal{U}_0) \) is contained in \( \mathcal{U}_{\theta_n} \) since each \( \Omega_z \) has this property. This implies that for \( n \) large enough

\[
R^d_f = R(\tilde{\theta}) > R(\Omega_{\Phi, \theta}(\mathcal{U}_0)) > R(\nabla) - \varepsilon
\]

and we conclude that the conformal radius \( R^d_f \geq R(\nabla) \), since the last inequality holds for all \( \varepsilon > 0 \). Using (6) and (5) we have the equalities

\[
R^d_f = R(\nabla), \quad \mathcal{U}_0 = \nabla.
\]

Any subsequence from \( \{\theta_n\} \) also converges to \( \tilde{\theta} \), and therefore all subsequences of domains from \( \{\mathcal{U}_{\theta_n}\} \) have the same kernel \( \nabla = \mathcal{U}_0 \). ■

**Proof of proposition 3.2.** Let \( \{h_\theta : \mathbb{D} \to \mathcal{U}_\theta\}_{\theta \in \mathbb{T}^d} \) be the family of uniformizing functions associated with the open tube \( \mathcal{U} \), with \( h_\theta'(0) > 0 \), and let \( \Phi : \mathbb{T}^d \times \mathbb{D} \to \mathcal{U} \) be the bijective function defined by

\[
(\theta, z) \mapsto (\theta, h_\theta(z)).
\]

From lemma 4.9 and theorem 4.3, \( \Phi \) is a homeomorphism between \( \mathbb{T}^d \times \mathbb{D} \) and \( \mathcal{U} \). Let \( \theta \) in \( \mathbb{T}^d \). The function \( l_\theta : \mathbb{D} \to \mathbb{D} \) defined by

\[
l_\theta(z) = \Pi_2(\Phi^{-1} \circ F \circ \Phi(\theta, z)) = h^{-1}_{\theta^{\mathbb{T}^d}}(f_\theta(h_\theta(z)))
\]

is a holomorphic function, is univalent and verifies

\[
l_\theta(0) = 0,
\]

\[
l'_\theta(0) = (h_{\theta^{\mathbb{T}^d}}'(0))^{-1} \rho_1(\theta) h_\theta'(0) = (R^1_{f_\theta})^{-1} \rho_1(\theta) R^1_f = \rho_1(\theta).
\]

Since \( |\rho_1(\theta)| = 1 \), the Schwartz lemma implies \( l_\theta(z) = \rho_1(\theta) z \), and the result follows. ■
4.3. Proof of Theorem 3.3

We follow the original proof of Pérez-Marco. We will need a finite version of the Pérez-Marco’s theorem which will allow us to show the theorem for a fibred holomorphic dynamics having a rational (periodic) rotation on the base; an argument of continuity with respect to the Hausdorff distance on compact sets will conclude the proof. Let \( p \in \mathbb{N} \). We say that a \( p \)-tuple of holomorphic transformations \( (f_1, f_2, \ldots, f_p) \) is a good chain for the \( p \)-tuple of sets \((U_1, U_2, \ldots, U_p)\) if for every \( i \in \{1, \ldots, p\} \) we have

(i) \( U_i \) is an open Jordan domain (the interior of a Jordan curve) containing the origin,
(ii) \( f_i \) is a local holomorphic diffeomorphism fixing the origin, \( \partial f_i(0) = e^{\pi i \beta_i}, \beta_i \in \mathbb{T}^1 \),
(iii) \( f_i \) is defined and univalent in a neighbourhood of \( \hat{U}_i \) and \( f_i^{-1} \) is defined and univalent in a neighbourhood of \( U_{p+1} = U_1 \).

In order to fix ideas we will perform the proofs for the case \( p = 2 \), even if the arguments also hold in the general case. Let \((f_1, f_2)\) be a good chain for the pair of sets \((U_1, U_2)\). We put \( f = f_2 \circ f_1 \). The function \( f \) is a local holomorphic diffeomorphism with a fixed point in the origin and multiplicator equal to \( e^{2\pi i (\beta_1 + \beta_2)} \).

Remark 4.10. Since the function \( f = f_2 \circ f_1 \) is a holomorphic function, it may happen that its maximal domain of definition is bigger than (a neighbourhood of) \( \hat{U}_1 \cap f_1^{-1}(U_2) \), the set where the composition is a priori defined, because of analytic continuations. In the arguments that follow, we will consider the compositions as being defined only on (a neighbourhood of) \( \hat{U}_1 \cap f_1^{-1}(U_2) \).

It is a particular case of a simple but important result of Kerekjarto (see [9, 10]) that any connected component of \( U_1 \cap f_1^{-1}(U_2) \) is a Jordan domain. We will denote by \( U_{12} \) (respectively, \( U_{21} \)) the connected component of \( U_1 \cap f_1^{-1}(U_2) \) (respectively, \( U_2 \cap f_2^{-1}(U_1) \)) which contains 0.

Fundamental construction. Let \( K_1 \subset \hat{U}_1 \) and \( K_2 \subset \hat{U}_2 \) be compact, connected and full sets containing the origin and verifying

\[
f_1(K_1) = K_2 = f_2^{-1}(K_1), \quad f_2(K_2) = K_1 = f_1^{-1}(K_2).\]

We also suppose that \( K_1 \) and \( K_2 \) are not reduced to the origin. We will associate with these sets an analytical diffeomorphism of the circle. Indeed, let \( h_1 : \mathbb{C} \setminus \hat{D} \rightarrow \mathbb{C} \setminus K_1 \) be a uniformization with \( h_1(\infty) = \infty \) and \( h_2 \) be defined in an analogous way for \( K_2 \). The composition

\[
h_1^{-1} \circ f_2 \circ h_2^{-1} \circ f_1 \circ h_1 = h_1^{-1} \circ f \circ h_1
\]

is a holomorphic diffeomorphism \( g \) from an open annular set contained in \( \mathbb{C} \setminus \hat{D} \), which has \( \mathbb{T}^1 = \partial \mathbb{D} \) as one of its boundaries, onto a similar annular open set. This diffeomorphism extends continuously to a homeomorphism up to the boundary \( \mathbb{T}^1 \). Indeed, the interior of a cross-cut of the boundary \( \mathbb{T}^1 \) is a simply connected open set and Carathéodory’s extension theorem applies. Schwartz’s reflection principle allows us to extend \( g \) to an annular neighbourhood of \( \mathbb{T}^1 \), producing in this manner a holomorphic diffeomorphism defined on this set. This diffeomorphism preserves the circle and then results in an analytic circle diffeomorphism.

The linearizable case. Assume \( \beta = \beta_1 + \beta_2 \) is a real number verifying a diophantine condition. The Siegel linearization theorem (see [16]) implies that \( f \) is linearizable in an open neighbourhood of the fixed point \( z = 0 \). Let \( S(f) \subset U_{12} \) be the maximal linearization domain for \( f \) in \( U_{12} \), let \( K_1 = S(f) \) be the compact set obtained by filling the closure of \( S(f) \).
and let $K_2 = f_1(K_1)$. As $f_1$ is a homeomorphism in a neighbourhood of $K_1$, $K_2$ is the filling of the closure of $f_1(S(f))$. We apply the fundamental construction to $K_1$ and $K_2$ in order to get an analytic circle diffeomorphism $g$.

Lemma 4.11. Under the above hypothesis the analytic circle diffeomorphism $g$ has a rotation number $\varrho(g) = \beta$.

**Proof.** We follow an argument from [11]. Let $\{\Omega^1_n\}_{n \geq 0}$ be a sequence of regions in $S(f)$ such that

1. $\Omega^1_n \subset \Omega^1_{n+1}$, for every $n \geq 0$,
2. $\bigcup_{n \geq 0} \Omega^1_n = S(f)$,
3. $\Omega^1_n$ is completely invariant under $f$.

Such a sequence may be obtained as the image of radius increasing radii discs by the linearizing application of $f$. We also define a sequence $\{\Omega^2_n\}_{n \geq 0}$ by $\Omega^2_n = f_1(\Omega^1_n)$ which verify analogous exhausting properties for the set $f_1(S(f))$. Let $\{g_n\}_{n \geq 0}$ be the sequence of analytic circle diffeomorphisms resulting from the application of the fundamental construction to $\Omega^1_n$ and $\Omega^2_n$.

Since $g_n$ is linearizable for all $n \geq 0$ (because of $\Omega^2_n \subset S(f)$) we have $\varrho(g_n) = \beta$ for every $n \geq 0$. Let $h^n$ be the uniformizations of $\mathbb{C} \setminus \Omega^1_n$ for all $n \geq 0$ and $i \in \{1, 2\}$. From the definition of $K_i$, the sequence $\{ \mathbb{C} \setminus \Omega^1_n \}_{n \geq 0}$ is kernel convergent to $\mathbb{C} \setminus K_i$, for $i$ in $\{1, 2\}$. This implies that the holomorphic diffeomorphisms $g_n$ are defined simultaneously in some annular neighbourhood $A$ of $\mathbb{T}^1$ for every $n$ large enough. Let $C \subset A$ be a curve, which is homotopic and exterior to $\mathbb{T}^1$. Let $C'$ be its reflection which respect to $\mathbb{T}^1$. The kernel Carathéodory’s theorem gives

$$g_n|_C = (h^n)|_C \circ f \circ h_1^n|_C \xrightarrow{n \to \infty} g|_C = h_1 \circ f \circ h_1$$

uniformly on $C$. We have the same result on $C'$, and by the maximum principle applied to the compact annulus $A'$, bounded by $C$ and $C'$, we have $g_n|_{A'} \to g|_{A'}$ uniformly. Therefore we get that $\varrho(g) = \beta$ since $\varrho(g_n) = \beta$ for all $n \geq 0$.

The global linearization theorem for analytic circle diffeomorphisms (see [3, 17]) implies thus that $g$ is linearizable in an annular neighbourhood of $\mathbb{T}^1$. If $S(f)$ was relatively compact in $U_{12}$, we would get a larger linearization domain in $U_{12}$ in the following way: by gluing $K_1$ to the image by $h_1$ of a small $g$-invariant neighbourhood of $\mathbb{T}^1$, we obtain a Lyapounov stable domain (a simply connected forward invariant open set) which is larger than $S(f)$. From the equivalence between Lyapounov stability and linearization, $f$ is linearizable on this stable domain. As $S(f)$ was the maximal linearization domain, we must have that $\partial S(f)$ meets $\partial U_{12} \subset (\partial U_1 \cup f_1^{-1}(\partial U_2))$. Thus, either $K_1$ meets $\partial U_1$ or $K_2$ meets $\partial U_2$ (or both). We show the following.

**Proposition 4.12.** Let $(f_1, f_2, \ldots, f_p)$ be a good chain for the open sets $(U_1, \ldots, U_p)$. Assume the sum of the rotation numbers $\beta = \sum \beta_i$ verifies a diophantine condition and let $S(f) \subset U_1$ be the maximal linearization domain for $f = f_p \circ \cdots \circ f_1$ in $U_1 \cap f_1^{-1}(U_2) \cap \cdots \cap (f_{p-1} \circ \cdots \circ f_1)^{-1}(U_p)$. We consider the compact, connected full sets containing the origin given by $K_1 = \overline{S(f)}$, $K_2 = f_1(K_1)$, $\ldots$, $K_p = f_{p-1}(K_{p-1})$. There exists an index $i$ in $\{1, 2, \ldots, p\}$ such that $K_i \cap \partial U_i \neq \emptyset$.

In the following paragraphs we want to eliminate the arithmetical hypothesis on $f$. We will show indeed proposition 4.13.
Proposition 4.13. Let \((f_1, f_2, \ldots, f_p)\) be a good chain for the open sets \((U_1, \ldots, U_p)\). Then there exists compact, connected full sets containing the origin \(K_1 \subset \tilde{U}_1, K_2 \subset \tilde{U}_2, \ldots, K_p \subset \tilde{U}_p\), verifying

\[
f_i(K_i) = K_{i+1}, \quad f_i^{-1}(K_{i+1}) = K_i
\]

for all \(i \in \{1, 2, \ldots, p\}\) and an index \(i^*\) such that \(K_{i^*} \cap \partial U_{i^*} \neq \emptyset\).

The proof of this proposition is obtained by showing that the conclusion is a dense and closed property in an adequate space. We will need some topology about compact sets.

Hausdorff distance. Let \((X, d)\) be a compact metric space. Let \(\epsilon > 0\) and \(A \subset X\). We denote by \(V_\epsilon(A)\) the \(\epsilon\)-neighbourhood of \(A\) in \(X\)

\[
V_\epsilon(A) = \{ x \in X \mid d(x, A) < \epsilon \}.
\]

The Hausdorff distance between two non-empty compact sets \(K_1, K_2\) contained in \(X\) is defined by

\[
d_H(K_1, K_2) = \inf\{\epsilon > 0 \mid K_1 \subset V_\epsilon(K_2) \text{ and } K_2 \subset V_\epsilon(K_1)\}.
\]

The \(d_H\) is a distance on the space \(K(X)\) of compact non-empty subsets of \(X\). Moreover, \((K(X), d_H)\) is a compact metric space. The space \(K_n(X)\) of connected, non-empty compact sets is a compact subspace of \(K(X)\). Let \((Y, d')\) be another compact metric space. We denote by \(C^0(Y)\) the space of continuous functions from \(Y\) to \(Y\) and we endow it with the topology of the uniform convergence.

Lemma 4.14 (See [11]). The application

\[
C^0(X, Y) \times K(X) \longrightarrow K(Y)
\]

\[
(f, K) \longmapsto f(K)
\]

is continuous. ■

Lemma 4.15. Let \(\{A^n\}_{n \in \mathbb{N}}\) be a sequence of compact sets of \(X\) and \(A\) be a compact set such that \(A^n \to A\) in the Hausdorff distance. Let \(x \in X\). There exists a sequence \(x_n \in A^n\) such that \(x_n \to x\) iff \(x \in A\). ■

Lemma 4.16. Let \(A \subset \mathbb{T}^1 \times \mathbb{C}\) be a compact set, whose fibres are connected. Let \(\hat{A}_0\) be the filling of the fibre \(A_0\), that is, the filling \(\hat{A}_0\) of the fibre \(A_0\) is the smaller filled compact set containing \(A_0\). Then the set \(\hat{A} = \bigcup_{\theta \in \mathbb{T}^1 \setminus \{0\}} \hat{A}_\theta\) is compact.

Proof. We must show that \(\hat{A}\) is closed. Otherwise, suppose \((\theta_n, z_n) \in \{\theta_n\} \times \hat{A}_\theta\) is a sequence converging to a point \((\theta, z) \notin \{\theta\} \times \hat{A}_\theta\). The set \(\hat{A}_\theta\) is a compact full set, therefore there exists a continuous path \(\gamma \subset \mathbb{C} \cup \{\infty\}\) which joins \(z\) to \(\infty\) and that does not touch \(\hat{A}_\theta\). Define a sequence of continuous paths \(\gamma_n\) joining \(z_n\) to \(\infty\) by \(\gamma_n = [z_n, z] \cup \gamma\). We have \(\gamma_n \to \gamma\) in the Hausdorff topology on \(\mathbb{C} \cup \{\infty\}\). Since \(z_n \in \hat{A}_\theta\), for every \(n \in \mathbb{N}\) there exists a point \(\hat{z}_n \in \hat{A}_\theta \cap \gamma_n\). Choose a subsequence \(\{\hat{z}_{n_i}\}_{i \geq 0}\) and a point \(\hat{z}\) such that \(\hat{z}_{n_i} \to \hat{z}\). Thus \(\hat{z} \in \hat{A}_\theta \cap \gamma\), but \(\gamma \cap \hat{A}_\theta = \emptyset\), a contradiction. ■

Let \(\mathcal{F}\) be the set of all the \(p\)-tuples \((f_1, \ldots, f_p)\) verifying the hypothesis of proposition 4.13 for the open sets \((U_1, \ldots, U_p)\). We embed \(\mathcal{F}\) in the product space

\[
C(\tilde{U}_1, \mathbb{C}) \times C(\tilde{U}_2, \mathbb{C}) \times C(\tilde{U}_3, \mathbb{C}) \times \cdots \times C(\tilde{U}_p, \mathbb{C}) \times C(\tilde{U}_1, \mathbb{C})
\]

by the inclusion

\[
(f_1, \ldots, f_p) \longmapsto (f_1, f_1^{-1}, f_2, f_2^{-1}, \ldots, f_p, f_p^{-1}).
\]

The topology on \(\mathcal{F}\) is the induced topology from the uniform convergence topology in the product.
Lemma 4.17. The set of \( p \)-tuples in \( \mathcal{F} \) verifying the conclusions of proposition 4.13 is closed.

Proof. Let \( \{(f^n_1, \ldots, f^n_p)\}_{n \geq 0} \) be a sequence of \( p \)-tuples in \( \mathcal{F} \) verifying the conclusions of proposition 4.13 and converging to the \( p \)-tuple \( (f_1, \ldots, f_p) \) in \( \mathcal{F} \). So there exists a sequence of connected full compact sets containing the origin \( \{K^n_i\}_{n \geq 0} \subset \mathcal{K}_{c}(U_i), i \in \{1, \ldots, p\} \), satisfying

\[
f^n_i(K^n_i) = K^n_i,\]

\[
(f^n_i)^{-1}(K^n_i) = K^n_i.
\]

Moreover, for all \( n \geq 0 \) one has \( K^n_i \cap \partial U_i \neq \emptyset \) for at least one integer \( i \in \{1, \ldots, p\} \). Since \( \mathcal{K}_{c}(U_i) \) is compact for every \( i \), passing to a subsequence if necessary, we get compact connected sets \( \tilde{K}_i \subset \mathcal{K}_{c}(U_i) \) for \( i \in \{1, \ldots, p\} \) verifying for every \( i \) that

\[
K^n_i \xrightarrow{n \to \infty} \tilde{K}_i
\]

in the Hausdorff distance. Furthermore, there exists an index \( i^* \in \{1, \ldots, p\} \) such that \( K^n_{i^*} \cap \partial U_{i^*} \neq \emptyset \) for infinitely many values of \( n \) and therefore \( \tilde{K}_{i^*} \cap \partial U_{i^*} \neq \emptyset \). In the limit we have \( f_{i^*}(\tilde{K}_{i^*}) = \tilde{K}_{i^*} \) and \( f_{i^*}^{-1}(\tilde{K}_{i^*}) = \tilde{K}_{i^*} \) for every \( i \in \{1, \ldots, p\} \), \( mod \ p \).

The connected component of the origin in \( U_1 \cap f_{i^*}^{-1}(U_2) \cap \cdots \cap (f_{p-1} \circ \cdots \circ f_1)^{-1}(U_p) \) is a Jordan domain. This set is contained in the connected component of the origin in \( U_1 \cap f_{i^*}^{-1}(U_2) \cap \cdots \cap (f_{p-1} \circ \cdots \circ f_1)^{-1}(U_p) \), which is a filled set and contains \( \tilde{K}_1 \). Similarly for \( \tilde{K}_i \). Therefore, the filled-in sets \( \tilde{K}_i = \tilde{K}_i \) verify the conclusions of proposition 4.13 for the \( p \)-tuple \( (f_1, \ldots, f_p) \).

Now we can conclude the proof of proposition 4.13. If \( (f_1, \ldots, f_p) \) belongs to \( \mathcal{F} \) and has multipliers \( (e^{2\pi i \beta_1}, \ldots, e^{2\pi i \beta_p}) \), for \( i \in \{1, \ldots, p\} \), there exists a sequence of real numbers \( \{\eta_n^i\}_{n \geq 0} \) converging to zero such that the \( \sum_i (\beta_i + \eta_n^i) \) is a real number verifying a diophantine condition for all \( n \geq 0 \). The \( p \)-tuples

\[
(e^{2\pi i \eta_n^1} f_1, \ldots, e^{2\pi i \eta_n^p} f_p)
\]

belong to \( \mathcal{F} \) for \( n \) large enough and converges to \( (f_1, \ldots, f_p) \). Moreover, they verify the conclusions of proposition 4.13 by proposition 4.12. The preceding lemma allows us to conclude in remark 4.18.

Remark 4.18. Proposition 4.13 can be obtained in an alternative way as a direct application of Pérez-Marco’s theorem (which corresponds to the case \( p = 1 \)). Indeed, suppose \( p = 2 \). Under the hypothesis of the proposition, the composition \( f = f_2 \circ f_1 \) and its inverse \( f^{-1} = f_1^{-1} \circ f_2^{-1} \) are well defined and univalent in a neighbourhood of the set

\[
\mathcal{V} = \bigcap \{U_1 \cap f_1^{-1}(U_2) \cap f_2(U_2)\},
\]

where \( \bigcap \) denotes the connected component containing the origin. Then \( \mathcal{V} \) is a Jordan domain and Pérez-Marco’s theorem applies. We have thus a connected, compact set \( K \) containing the origin, completely invariant by \( f \), touching the boundary of \( \mathcal{V} \). Suppose for instance that \( K \cap \partial f_1^{-1}(U_2) \neq \emptyset \). The sets \( K_1 = K, K_2 = f_1(K) \) satisfy the inclusion properties and \( K_2 \cap \partial U_2 \neq \emptyset \).

We will show theorem 3.3 as a consequence of proposition 4.13. The actual proof is done in a very analogous way to the proof of proposition 4.13 (and also very analogous to the original proof of Pérez-Marco’s theorem). We will allow fhdts having a base rotation vector different from \( \alpha \) on the torus \( \mathbb{T}^n \) (in fact, a rational rotation vector). We recall that the invariant curve is supposed to be the zero section curve \( u = \{z \equiv 0\}_{\mathbb{T}^n} \).
Let $\mathcal{F}_U$ be the set of all fhds $F$ (with free rotation vector over $\mathbb{T}^d$), having the zero section as an invariant curve and such that $F$, $F^{-1}$ define injective fhds in a neighbourhood of $\bar{U}$. Thus, if a fhd $F$ belongs to $\mathcal{F}_U$ then it has the form

$$F(\theta, z) = (\theta + \alpha_F, \rho_1(\theta)z + z^2\rho(\theta, z)), \quad (9)$$

where $\alpha_F$ is a vector in $\mathbb{T}^d$, $\rho_1$ is a non-vanishing continuous function and $\rho$ is a continuous function holomorphic in each fibre. We endow this set with the topology of the uniform convergence of $F$ and $F^{-1}$ on $\bar{U}$.

**Lemma 4.19.** The set of fhds in $\mathcal{F}_U$ verifying the conclusions of theorem 3.3 is closed.

**Proof.** Let $\{F_i\}_{i \geq 0}$ be a sequence in $\mathcal{F}_U$ whose elements verify the conclusions of theorem 3.3. We suppose that the sequence converges to the fhd $F$. Let $\{K^i\}_{i \geq 0}$ be an associated sequence of compact connected sets verifying the conclusions of the theorem. For every $i \geq 0$ the invariant curve is included in the compact set $K^i$, and there exists a point $(\theta_i, z_i)$ in $K^i \cap \partial U$. The set $\partial U$ is a compact set and the space $\mathcal{K}_c(\bar{U})$, formed by all connected compact sets of $\bar{U}$, is also compact. Considering a subsequence if necessary, there exists a connected compact set $K \subset \bar{U}$ containing the invariant curve and a point $(\tilde{\theta}, \tilde{z})$ such that

$$d_H(K^i, K) < \frac{1}{i}, \quad (10)$$

$$(\theta_i, z_i) \xrightarrow{i \to \infty} (\tilde{\theta}, \tilde{z}) \in K \cap \partial U. \quad (11)$$

Lemma 4.14 implies that $K$ is completely invariant. Let us show that the fibres $K_\theta$ are connected compact sets. We will use the following notation: let $\tilde{\theta} \in \mathbb{T}^d$, $\varepsilon > 0$ and $A \subset \bar{U}$ a compact set. We define a compact set $A_{[\tilde{\theta} ; \varepsilon]}$ by

$$A_{[\tilde{\theta} ; \varepsilon]} = \bigcup_{|\theta - \tilde{\theta}| \leq \varepsilon} \{\theta\} \times A_\theta. \quad (12)$$

Let $\theta \in \mathbb{T}^d$. Inequality (10) implies

$$\{\theta\} \times K_\theta \subset V^1_i(K^i_{[\theta ; 1]}),$$

$$K^i_{[\theta ; 1]} \subset V^1_i(K_{[\theta ; 1]}). \quad (13)$$

The sets $K^i_{[\theta ; 1]}$ are compact connected sets, therefore there exists a subsequence $\{K^n_{[\theta ; 1]}\}_{n \geq 0}$ converging in the Hausdorff topology to a connected compact set $[\theta] \times \hat{K}_\theta$. We will show that $K_\theta = \hat{K}_\theta$. From (12), we have that $K_\theta \subset \hat{K}_\theta$. Conversely, let $z \in \hat{K}_\theta$. There exists a sequence $(\theta_n, z_n) \in K^n_{[\theta ; 1]}$ such that $\theta_n \to \theta$ and $z_n \to z$. From (13) we obtain a sequence $(\tilde{\theta}_n, \tilde{z}_n) \in K_{[\theta ; 1]}$ such that $|\tilde{\theta}_n - \tilde{\theta}| < \frac{1}{n}$. Therefore $(\tilde{\theta}_n, \tilde{z}_n)$ converges to $(\theta, z)$ and $z \in K_\theta$. Finally, the compact set $\hat{K}_\theta$ obtained by filling the fibres of $K$ as defined in lemma 4.16, is still completely invariant and verifies all the conclusions of theorem 3.3.

The proof of theorem 3.3 follows from lemma 4.20.

**Lemma 4.20.** Let $F$ be a fhd verifying the hypothesis of theorem 3.3. There exists a sequence in $\mathcal{F}_U$ converging to $F$, such that its elements verify the conclusions of theorem 3.3.
Proof. We write $F$ as in (9). Let $\{l_n\}_{n \geq 0}$ be a sequence of zero-mean trigonometric polynomials, converging uniformly to the function $\log |\rho_1|$ (Féjer’s theorem). Since $\alpha$ is a rationally independent vector we can pick a sequence of vectors $\alpha_n$ with rational components $(p_n/q_n)_{1, \ldots, d}$ converging to $\alpha$. Furthermore, we may assume that this sequence verifies that the denominators $(q_n)_j, j \in \{1, \ldots, d\}$ are different prime numbers bigger than the degree of the trigonometric polynomial $l_n$, for every $n \geq 0$. We put

$$\rho_n^i = \frac{\rho_1}{|\rho_1|}. \quad (14)$$

These functions converge uniformly to $\rho_1$. We define a sequence of fhds $\{F_n\}_{n \geq 0}$ by

$$F_n(\theta, z) = (\theta + \alpha_n, \rho_n^i(\theta)z + z^2 \rho(\theta, z)). \quad (15)$$

For every $n$ large enough the transformation $F_n$ belongs to $F_{\mathcal{U}}$ and moreover $F_n$ converges to $F$. We can solve the cohomological equation

$$\tilde{u}_n^i(\theta + \alpha_n) - \tilde{u}_n^i(\theta) = l_n(\theta) \quad (16)$$

since the conditions over the denominators avoid falling in a resonance using the Fourier coefficients method. We put $u_n^i = \exp(\tilde{u}_n^i)$. This function satisfies

$$\frac{u_n^i(\theta + \alpha_n)}{u_n^i(\theta)} = \epsilon_\theta(\theta). \quad (17)$$

The change in coordinates $\tilde{F}_n = H_n^{-1} \circ F_n \circ H_n$, with

$$H_n(\theta, z) = (\theta, u_n^i(\theta)z),$$

gives us the following normal form for $F_n$

$$\tilde{F}_n(\theta, z) = (\theta + \alpha_n, \tilde{\rho}_n^i(\theta)z + z^2 \tilde{\rho}(\theta, z)), \quad \text{with } |\tilde{\rho}_n^i(\theta)| = 1 \text{ for every } \theta \text{ in } \mathbb{T}^1.$$

We put $\mathcal{U}^i = H_n^{-1}(\mathcal{U})$. Let $\tilde{\theta}$ in $\mathbb{T}^d$ and $Q_n = \prod_{1}^{d}(q_n)_j$. We consider the holomorphic transformations

$$\gamma_n^i(z) = \tilde{f}_n(\tilde{\theta} + i\alpha_n, z)$$

for $i \in \{0, \ldots, Q_n-1\}$. Thus, the $Q_n$-tuple $(\gamma_0^i, \ldots, \gamma_{Q_n-1}^i)$ is a good chain for the fibres $\mathcal{U}_n^i$, respectively. Applying proposition 4.13 we get connected, full and compact sets $K_n^i \subset \mathcal{U}_n^i$, $i \in \{0, \ldots, Q_n-1\}$, containing the origin and verifying

$$\gamma_n^i(K_n^i) = K_n^{i+1},$$

$$\gamma_n^{i+1}(K_n^{i+1}) = K_n^i.$$

We also have an index $i = \bar{\ell}_n$ in $\{0, \ldots, Q_n-1\}$ such that $K_n^\bar{\ell}_n \cap \partial \mathcal{U}_n^\bar{\ell}_n \neq \emptyset$. The set

$$K_n^\bar{\ell}_n = H_n(\{z \equiv 0\}_{2\pi} \bigcup_{j=0}^{Q_n-1} \{\tilde{\theta} + j\alpha_n \times K_n^i\})$$

verifies the conclusions of theorem 3.3 for $F_n$. 

Corollary 4.21. Let $F$ be a fhd and $u$ be an indifferent invariant curve. Then there exists orbits, other than those situated on the curve, that are in past and future completely contained in a neighbourhood of the curve. In fact, there actually exists such an orbit in any tubular neighbourhood of the invariant curve.

Proof. As usual we suppose $u = \{z \equiv 0\}_{2\pi}$ and $F(\theta, z) = (\theta + \alpha, \rho_1(\theta)z + z^2 \rho(\theta, z))$, with $\rho_1(\theta) \neq 0$ for every $\theta$ in $\mathbb{T}^d$. There exists a constant $C > 1$ such that $C > |\rho_1(\theta)| > C^{-1}$ for every $\theta$ in $\mathbb{T}^d$. Applying lemma 4.1 we get a uniform positive radius $\tilde{r}$ such that $f_\tilde{r}^{-1}$ and $f_{\tilde{r}}$ are well defined and univalent in some neighbourhood of $D_r$ for every $r \in (0, \tilde{r})$. Taking $\mathcal{U} = \mathbb{T}^d \times D_r$, theorem 3.3 applies. 

\[\square\]
An intrinsic definition of the invariant continua. Let $F$, $u$, $\mathcal{U}$ be as in the hypothesis of theorem 3.3. We define the stable set associated with the curve by

$$\mathcal{L} = \bigcap_{j \in \mathbb{Z}} F^{-j}(\mathcal{U}).$$

This is a compact set and consists of those points for which the complete orbit by $F$ (past and future) is contained in $\bar{\mathcal{U}}$. Clearly, the set $K$ provided by theorem 3.3 is contained in $\mathcal{L}$. For any fibre $L_0$, we denote by $L_0^0$ the connected component which contains $u(\theta)$ (the point over the invariant curve). Thus, $L_0^0$ is a connected compact set and is full by the maximum principle (since $f_\theta$ and $f_\theta^{-1}$ are holomorphic functions). We construct an intrinsically defined invariant set

$$L_0^0 = \bigcup_{\theta \in \mathbb{T}^d} \{\theta\} \times L_0^0.$$

This set contains $K$. Indeed, the set $L_0^0$ contains any set verifying the conclusions of theorem 3.3. We obtain as a corollary of theorem 3.3 the following.

**Corollary 4.22.** Under the notation and hypothesis of theorem 3.3, set $L_0^0$ verifies that

1. $L_0^0$ contains the invariant curve,
2. $L_0^0$ is a compact connected set,
3. each fibre $L_0^0$ is connected and full,
4. $L_0^0$ is completely invariant by $F$,
5. $L_0^0 \cap \partial \mathcal{U} \neq \emptyset$.

**Proof.** We only need to show that $L_0^0$ is a closed set. Let $(\theta_i, z_i)$ be a convergent sequence in $L_0^0$, say $(\theta_i, z_i) \to (\theta, z) \in \bar{\mathcal{U}}$. The sets $L_0^0$ are connected compact sets, therefore, passing to a subsequence if necessary, there exists a connected compact set $\tilde{L}_0$ such that $\{\theta_i\} \times L_0^0 \to \{\theta\} \times \tilde{L}_0 \subset \{\theta\} \times L_0$ in the Hausdorff topology. In particular, the points $u(\theta)$ and $z$ belong to $\tilde{L}_0$, so we have $z \in L_0^0$.

In general, the maximal invariant compact set provided by corollary 4.22 has at least one (and so infinitely many) fibre which is not reduced to the point over the invariant curve alone. The following example shows that we cannot hope that for every fibre.

**Example.** Let $A : \mathbb{T}^d \to \mathbb{C}$ be a non-vanishing continuous function, such that $\int_{\mathbb{T}^d} \log |A(\theta)| \, d\theta = 0$. We consider the linear fhd given by $F(\theta, z) = (\theta + \alpha, A(\theta)z)$. Let $R > 0$ be a positive radius. The invariant curve $\{z = 0\}$ is indifferent and the tube $\mathbb{T}^d \times D(0, R)$ verifies the hypothesis of theorem 3.3. Let $K$ be the maximal invariant compact set as defined by corollary 4.22. By radial symmetry, we have that fibres $K_\theta$ are discs (with their radius taking its values in $[0, R]$). Let us take $A(\theta) = e^{\phi(\theta)}$ and $\alpha = \omega$ as in remark 4.6. Let $\theta \in \mathbb{T}^d$ be a point for which the Birkhoff sums of $\phi$ are unbounded from above (below). Therefore, the positive (negative) orbit of a point $(\theta, z)$ is unbounded unless $z = 0$. That is, the fibre $K_\theta$ is reduced to the origin alone, and we call it a singular fibre.

Note that this example is obtained by using the very wild oscillations of the absolute value $|A(\theta)|$. We can ask the following.

**Question** Does there exist an example of a fhd $F$, with an indifferent invariant curve $u$, such that $|D_1 u(\theta)| = 1$ for every $\theta$, and where the maximal compact invariant set has singular fibres?
Acknowledgments

This work is part of the Thèse de Doctorat of the author. The author thanks J-C Yoccoz for suggesting and guiding this thesis, for interesting discussions and useful comments. The author is partially supported by CONICYT-Chile and French government. The author would like to thank the referees for the careful reading of the manuscript and for many suggestions.

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