QUATERNIONIC BUNDLES AND BETTI NUMBERS OF SYMPLECTIC 4-MANIFOLDS WITH KODAIRA DIMENSION ZERO

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1. INTRODUCTION

For a minimal symplectic 4−manifold \(M\) with symplectic form \(\omega\) and symplectic canonical class \(K_\omega\), the Kodaira dimension of \((M, \omega)\) is defined in the following way ([12], [14]):

\[
\kappa(M, \omega) = \begin{cases} 
  -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\
  0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
  1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
  2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0.
\end{cases}
\]

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

It is shown in [12] that, if \(\omega\) is a Kähler form on a complex surface \((M, J)\), then \(\kappa(M, \omega)\) agrees with the usual holomorphic Kodaira dimension of \((M, J)\).

It is also shown in [12] that minimal symplectic 4−manifolds with \(\kappa = 0\) are exactly those with torsion canonical class, thus can be viewed as symplectic Calabi-Yau surfaces. Known examples of symplectic 4−manifolds with torsion canonical class are either Kähler surfaces with (holomorphic) Kodaira dimension zero or \(T^2\)−bundles over \(T^2\) ([10], [12]). They all have small Betti numbers and Euler numbers: \(b^+ \leq 3, b^- \leq 19\) and \(b_1 \leq 4\); and the Euler number is between 0 and 24. It is speculated in [12] that these are the only ones. In this paper we prove that it is true up to rational homology.

**Theorem 1.1.** Suppose \(M\) is a minimal symplectic 4−manifold with \(\kappa = 0\). Then the rational homology of \(M\) is the same as that of K3 surface, Enriques surface or a \(T^2\)−bundle over \(T^2\). In particular, we have

1. the Euler number of \(M\) is 0, 12 or 24,
2. the signature of \(M\) is \(-16, -8\) or 0, and
3. the Betti numbers of \(M\) satisfy the following bounds:

\[
b^+ \leq 3, \quad b^- \leq 19, \quad b_1 \leq 4.
\]

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The case $b_1 = 0$ follows from [15]. Under the additional assumption that $b_1 \leq 4$, this theorem is proved in [12]. The key is really to bound $b^+$. Our approach here is similar to that in [12], which is to show that, on a closed smooth oriented 4-manifold with $2\chi + 3\sigma = 0$ and $b^+ > 3$, the mod 2 Seiberg-Witten invariant of any reducible Spin$^c$ structure vanishes. In this paper we will call a Spin$^c$ structure reducible if it admits a reduction to a spin structure. We have mentioned that minimal symplectic 4-manifolds with Kodaira dimension zero are exactly those with torsion symplectic canonical class. In addition, a closed symplectic 4-manifold with $b^+ > 1$ and torsion canonical class actually has trivial canonical class, and hence is a spin manifold. For spin manifolds there are stable cohomotopy and stable homotopy/framed bordism refinements of the Seiberg-Witten invariants of spin manifolds in [3], [6] and [9], which take into account of the Pin(2) symmetry of the Seiberg-Witten equations. Such refinements are used in section 3 to construct an unoriented bordism SW invariant when $b^+ \geq 2$ following [8]. The main theorem follows from a rather general vanishing result of the unoriented bordism SW invariant. The proof of the vanishing result relies on a few properties of quaternionic bundles proved in section 2, which certainly are of independent interest.

A basic conjecture of Gompf in [11] is that a symplectic 4-manifold with $\kappa$ at least zero has non-negative Euler number. Theorem 1.1 confirms it when $\kappa = 0$.

**Corollary 1.2.** Any symplectic 4-manifold with $\kappa = 0$ has $b^+ \leq 3, b_1 \leq 4$ and non-negative Euler number.

Notice that the bound for $b_1$ is the same as the dimension. One could speculate whether such a bound continues to hold in higher dimensions.

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## 2. Quaternionic Bundles

Let $J$ be a smooth manifold with an involution $\iota_J$ and with nonempty and isolated fixed point set.

**Example 2.1.** We are interested in the case that $J$ is the torus $T^m = \mathbb{R}^m/\mathbb{Z}^m$ with $\iota_J$ given by $x \to -x$ using the coordinates of $\mathbb{R}^m$. In this case we use $O_J$ to denote the image of the origin in $\mathbb{R}^m$. Notice that there are $2^m$ fixed points including $O_J$. 
Recall that a bundle map between complex bundles is called anti-complex if it anti-commutes with the multiplication by \(i = \sqrt{-1}\).

**Definition 2.2.** A complex bundle \(Q\) on \(J\) with an anti-complex lift \(\iota_Q\) of \(\iota_J\) is called a quaternionic vector bundle if \(\iota_Q \circ \iota_Q = -1\).

Since the fixed point set is nonempty and the fiber over any fixed point is a space over the quaternions \(H = \mathbb{C} \oplus \mathbb{C}j\), the rank is necessarily even. However, we should warn the readers that a quaternionic bundle here is not a bundle over \(H\). In particular, the rank of a quaternionic bundle is its rank as a complex bundle. The Grothendick group of the quaternionic vector bundles is denoted by \(KQ(J)\) (first appeared in \([4]\)).

Let \(H_l\) be the rank \(2l\) quaternionic vector bundle \(J \times H_l\) with the anti-complex map \(\iota_{H_l} : (x, q) \mapsto (\iota_J x, qj)\), where \(qj\) is the right multiplication of \(q\) by the quaternion number \(j\). A rank \(2l\) quaternionic vector bundle \(E\) is called trivial if there is a complex isomorphism \(\Phi : E \to H_l\) with \(\Phi \circ \iota_E = \iota_{H_l} \circ \Phi\). Quaternionic vector bundles over low dimensional tori are classified in \([8]\), and \(KQ(T^m)\) is calculated in \([7]\).

Just as complex vector bundles are acted upon by \(U(1)\) via the complex multiplication, quaternionic vector bundles are naturally acted upon by the group \(Pin(2)\), which is generated by \(U(1)\) and the symbol \(\iota\) with the relations \(\iota^2 = -1, \quad \iota z \iota^{-1} = z^{-1}\) for \(z \in U(1)\).

Clearly \(Pin(2)\) fits into the short exact sequence
\[
1 \to U(1) \to Pin(2) \to \{\pm 1\} \to 1.
\]
Notice that \(Pin(2)\) is isomorphic to the subgroup of \(\mathbb{H}\) generated by \(U(1) = \{\cos \theta + \iota \sin \theta\}\) and \(j\).

We first specify the \(Pin(2)\) action on \(J\): it is simply defined via the surjection of \(Pin(2)\) onto the order 2 group \(\{id, \iota_J\}\). For a quaternionic vector bundle \(E\) over \(J\), since \(\iota_E\) is anti-complex, \(Pin(2)\) acts on \(E\) via the complex multiplication and \(\iota_E\).

**Remark 2.3.** We will also need the simple fact that, for a real vector space \(W\), the trivial real vector bundle \(W = J \times W\) is also \(Pin(2)\)-equivariant via the involution \(\iota_W : (x, a) \mapsto (\iota_J x, -a)\) and the surjection \(Pin(2) \to \{id, \iota_W\}\).

Notice that, since \(Pin(2)\) is compact, there exists a \(Pin(2)\)-invariant Hermitian inner product on any \(Pin(2)\)-equivariant bundle.

The two main results about quaternionic bundles are Theorems 2.7 and 2.14. The first one is about splitting off a trivial summand. We start with the following characterization.

**Lemma 2.4.** Let \(E\) be a quaternionic bundle. Then \(E\) splits into \(\mathbb{H} \oplus E'\) for some quaternionic bundle \(E'\) if and only if there is a nowhere vanishing section \(s\) such that \(s\) and \(\iota_E s\) are complex linearly independent everywhere.
Proof. Suppose $E$ splits into $\mathbb{H} \oplus E'$ for some quaternionic bundle $E'$. The constant section $1 = (x, 1)$ of $\mathbb{H}$ is a nowhere vanishing section of $E$, which we call $s$. Notice that the constant section $j = (x, j)$ can be also written as $\iota_{\mathbb{H}}1$. Therefore, due to the $Pin(2)$—equivariance, the section $\iota_{E}s$ corresponds to $j$. Since 1 and $j$ form a complex basis of $\mathbb{H}$, $s$ and $\iota_{E}s$ are complex linearly independent at every point.

Conversely, we obtain a map from $s$ a quaternionic map
\[ \phi : \mathbb{H} \to E, \quad (x, a + bj) \to as_x + b(\iota_{E}s)_x, \]
where $a, b \in \mathbb{C}$. $\phi$ is an embedding because $s_x$ and $(\iota_{E}s)_x$ are complex linearly independent for any $x$. The required splitting is then obtained by choosing a $Pin(2)$—invariant Hermitian metric and letting $E'$ be the orthogonal complement of $\phi(\mathbb{H})$. \qed

It is certainly not true that if $E$ has a nowhere vanishing section, then it has one such section $s$ such that $s$ and $\iota_{E}s$ are complex linearly independent everywhere. Otherwise, since every rank 2 quaternionic bundle over $T^2$ has a nowhere vanishing section by dimension reason, we would draw the conclusion that every such bundle is isomorphic to $\mathbb{H}$. But by the classification of quaternionic bundles over low dimensional tori\footnote{up to dimension 4} in [8], there is a (unique) non-trivial rank 2 quaternionic bundle over $T^2$.

For a nowhere vanishing section $s$, clearly $s$ and $\iota_{E}s$ are complex linearly independent over any fixed point of $J$. On the other hand, if $x$ is not a fixed point of $J$, then $s$ and $\iota_{E}s$ are complex linearly independent over $x$ if and only if $s_x$ is not mapped by $\iota_{E}$ to a point in the complex line generated by $s_{\iota_{J}x}$. To further investigate this problem for a quaternionic bundle of rank $2l$ we introduce some auxiliary bundles.

The (complex) projective space bundle $P(E)$. Let $P(E)$ denote the (complex) projective space bundle of $E$, which is a $\mathbb{C}P^{2l-1}$—bundle over $J$. For any nonzero $u \in E_x$, we use $[u] \in P(E)|_x$ to denote the complex line generated by $u$. Suppose $s$ is a nowhere vanishing section of $E$, then it defines a section $[s]$ of $P(E)$. Notice that $\iota_{E}$ sends a complex line in $E$ to a complex line and therefore induces an action on $P(E)$. This is simply because, for any nonzero $u \in E_x$, we have
\[ \iota_{E}((a + bi)u) = (a - bi)\iota_{E}(u). \]
In other words, $\iota_{E}[u] = [\iota_{E}u]$, where we continue to use $\iota_{E}$ to denote the induced action on $P(E)$. Clearly $\iota_{E}$ is an involution on $P(E)$.

The quaternionic bundle $\hat{E}$. Let $\hat{E}$ be the pull back bundle of $E$ under $\iota_{J}$. The fiber of $\hat{E}$ over $x$ is the fiber of $E$ at $\iota_{J}x$, and vice versa. we can define the quaternionic structure on $\hat{E}$ by
\[ \iota_{\hat{E}}(x, v) = (\iota_{J}x, \iota_{E}|_{\iota_{J}x}(v)), \]
Proof. This is clear from definitions.
Lemma 2.6. Any $\tau$–invariant section of $P(E) \times_J P(\hat{E})$ can be deformed to another one which is transversal to $D$.

Proof. Let $\Gamma$ be a $\tau$–invariant section of $P(E) \times_J P(\hat{E})$. Clearly $\Gamma$ does not intersect $D$ over any fixed point of $J$. Therefore there is a closed invariant neighborhood $V$ of the fixed points set of $J$ over which $\Gamma$ does not intersect $D$. Let $V_0$ be the interior of $V$. Away from $V$, the involution $\tau$ acts freely. Let $P'$ be the quotient of $P(E) \times_J P(\hat{E})$ over $J - V_0$. Then $P'$ is a $P^{2l-1} \times P^{2l-1}$–bundle over $J' = (J - V_0)/\tau$. Since $D$ is $\tau$–invariant, $D' = P' \cap D/\tau$ is a submanifold of $P'$, in fact a $P^{2l-1}$–subbundle over $J'$. Since $\Gamma$ is $\tau$–invariant, it induces a section $\Gamma'$ of $\pi'$ over $J'$ such that $\Gamma'$ is transversal to $D'$. By (the ordinary) transversality applied to the submanifold $D' \subset P'$ and the map $\Gamma' : J' \to P'$, we can deform $\Gamma'$ to another section $\Gamma''$ such that $\Gamma''|_{\partial J'} = \Gamma'|_{\partial J'}$ and $\Gamma''$ is transversal to $D'$. The pull back of $\Gamma''$, together with $\Gamma|_V$, forms a section of $P(E) \times_J P(\hat{E})$, which is a deformation of $\Gamma$ and transversal to $D$.

\[\square\]

Theorem 2.7. Suppose $J$ has (real) dimension $k$ and $E$ is a complex rank $2l$ quaternionic bundle over $J$. If $4l \geq k + 3$ then $E$ splits as $\mathbb{H} \oplus E'$.

Proof. By Lemmas 2.4 and 2.5, we just need to construct a $\tau$–invariant section $l = (l_1, l_2)$ of $P(E) \times_J P(\hat{E})$ with $l \cap D = \emptyset$ and such that $l_1$ has a lift to $E$.

Let $s_0$ be a nowhere vanishing section of $E$. Such a section exists as $4l \geq k + 3$. By Lemma 2.6, we can deform the $\tau$–invariant section $[s_0]$, $[s_0]$ of $P(E) \times_J P(\hat{E})$ to obtain a $\tau$–invariant section $(l_1, l_2)$ which is transversal to the $\tau$–invariant submanifold $D$.

The complex line field $[s_0]$ of $E$ is trivialized by $s_0$. Since deformations of a trivial complex line field remain trivial, $l_1$ is a trivial complex line field of $E$ as well. In particular, $l_1$ lifts to a nowhere vanishing section $s$ of $E$.

Since the dimension of $J$ is $k$, the dimension of $D$ is equal to $k + 4l - 2$, and the dimension of $P$ is equal to $8l - 4 + k$. It follows from the assumption $4l \geq k + 3$ that,

$$\dim D + \dim l_1 = (4l - 2 + k) + k \leq 4l - 2 + 4l - 3 + k = 8l - 5 + k = \dim P - 1.$$ 

As $l_1$ is transverse to $D$, $l_1$ does not intersect $D$. Therefore $s$ is the required section of $E$.

Remark 2.8. The condition $4l \geq k + 3$ in Theorem 2.7 is sharp since, as mentioned, there is a non-trivial rank 2 quaternionic bundle over $T^2$. On the other hand, it follows from Theorem 2.7 that any quaternionic bundle over $T^1$ is trivial, which is already proved in [3].

Corollary 2.9. Suppose $J$ has dimension $4n - \mu$ with $0 \leq \mu \leq 3$ and $E$ is a quaternionic bundle over $J$ with rank $2m \geq 2n$. Then $E$ splits as $Q \oplus \mathbb{H}^{m-n}$ for some rank $2n$ quaternionic bundle $Q$. 
Next we give two types of local trivializations. We first deal with a sufficiently small invariant disk containing only one fixed point.

**Lemma 2.10.** Any quaternionic bundle is trivial near a fixed point.

**Proof.** Let $U$ be an invariant disk containing only one fixed point. Consider a nowhere vanishing section $s$ over $U$. Since $\iota_E s$ and $s$ are complex linearly independent over the fixed point, by possibly shrinking $U$ we can assume they remain so in $U$. Now apply Lemma 2.4 and repeat this process. □

Next we treat certain invariant sets away from fixed points.

**Lemma 2.11.** Let $E$ be complex rank $2\ell$ quaternionic bundle over $J$. Suppose $W$ is a subset of $J$ such that $W$ does not intersect $\iota_J W$ and $E$ is trivial over $W$ as a complex vector bundle. Then $E$ is isomorphic to $\mathbb{H}^\ell$ over $W \coprod \iota_J W$.

**Proof.** Let $\alpha : E|_W \rightarrow \mathbb{H}^\ell$ be a complex trivialization of $E$ over $W$. Then

$$\alpha_J : E|_{\iota_J W} \rightarrow \mathbb{H}^\ell, \quad u \rightarrow -\alpha(\iota_E u)j$$

is a trivialization of $E$ over $\iota_J W$ and is complex linear. As it is assumed that $W \cap W' = \emptyset$, $\alpha \coprod \alpha_J$ is a trivialization of $E$ as a complex bundle over $W \coprod W'$. Moreover, it is a trivialization of $E$ as a quaternionic bundle, since for $u \in E|_{\iota_J W}$, we have,

$$\alpha_J(u)j = [-\alpha(\iota_E u)j]j = (\alpha \circ \iota_E)(u),$$

and for $v \in E|_W$, we have

$$(\alpha_J \circ \iota_E)(v) = -\alpha(-v)j = \alpha(v)j.$$

□

**Proposition 2.12.** For any quaternionic bundle $E$ over $J$, there is an equivariant covering of $J$ such that $E$ is trivial over each open set as a quaternionic bundle.

**Proof.** For each fixed point $x_i$ of $J$, by Lemma 2.10 there exists an open invariant neighborhood $U_i$ of $x_i$ such that $E$ is trivial over $U_i$ as a quaternionic bundle. Let $V_i$ be a smaller closed invariant neighborhood of $x_i$ which is contained in $U_i$. Let $V_0$ be the union of the $V_i$. Then $J - V_0$ is invariant and is covered by disk pairs $(W_j, \iota_J W_j)$, where for each $j$, $W_j$ is a disk contained in $J - V_0$ and $W_j \cap \iota_J W_j = \emptyset$. Then, for each $j$, $E$ is trivial as a complex bundle over the disk $W_j$, and hence trivial as a quaternionic bundle over the invariant open set $W_j \coprod \iota_J W_j$ by Lemma 2.11. Now the $U_i$ and the $W_j \coprod \iota_J W_j$ form a required covering. □

**Example 2.13.** To illustrate Proposition 2.12 we describe an explicit covering of $T^2$. Write $T^2$ as $S^1 \times S^1$. Cover the $p$-th $S^1$ by four disks $A_p^1, A_p^2, B_p^1, B_p^2$. $A_p^1$ and $A_p^2$ are disjoint invariant disks around the two fixed points, and they are called the type $A$ disks; $B_p^1$ and $B_p^2$ are disjoint and...
interchanged by the involution, and they are called the type B disks. Consider the union of the products of the disks where the type of each factor is fixed. Each union consists of $4 = 2^2$ products of disks.

And there are $4 = 2^2$ such unions, 

$$U_{AA}, U_{AB}, U_{BA}, U_{BB},$$

which form a covering of $T^2$. Since the involution takes a disk to a disk of the same type, each union is an invariant subset and so the covering is equivariant.

We claim that $E$ is trivial as a quaternionic bundle over each union. The 4 products of $A$−disks in $U_{AA}$, called $U_1, \ldots, U_4$, are disjoint invariant disks in $T^2$, each containing precisely one fixed point. In particular, by Lemma 2.10, $E$ is trivial over $U_{AA}$ as a quaternionic bundle if the $A$−disks are sufficiently small. To show that $E$ is trivial as a quaternionic bundle over each of the remaining 3 unions, we notice that each union is a disjoint union of pairs of product of disks interchanged by the involution $\iota_{T^2}$. This is because that two products are disjoint if and only if some factors are disjoint, and distinct disks of the same type are disjoint. Now apply Lemma 2.11.

We could similarly present an explicit equivariant covering of $T^k$, which might be used to give another calculation of $KQ(T^k)$ in [7], and to extend the classification in [8] to all $T^k$.

**Theorem 2.14.** Suppose $E$ is a rank $2l$ quaternionic bundle over a compact $J$. Then there is a $\text{Pin}(2)$−equivariant monomorphism from $E$ to $\mathbb{H}^m$ for $m = l + \lfloor \frac{k+2}{4} \rfloor$. Here $[x]$ denotes the largest integer bounded by $x$ from above.

**Proof.** Consider an equivariant covering $\{U_i, W_j \coprod_{i,j} \iota_j W_j\}_{i,j}$ as in Proposition 2.12. We can assume this covering is finite as $J$ is compact. By possibly shrinking the $U_i$ we can assume that $U_i \cap U_j = \emptyset$ for $i \neq j$. Therefore we can trivialize $E$ as a quaternionic bundle over $U = \coprod_i U_i$. Since $m = l + \lfloor \frac{k+2}{4} \rfloor \geq l$, we can view this trivialization as a $\text{Pin}(2)$−equivariant monomorphism $\Phi_0$ from $E$ to $\mathbb{H}^m \supset \mathbb{H}^l$ over $U$.

Let $W_0 = U$, and for $j \geq 1$, let

$$W_j = W_{j-1} \cup (W_j \coprod_{i,j} \iota_j W_j).$$

We will argue by induction on $j$. Suppose the $\text{Pin}(2)$−equivariant monomorphism $\Phi_j$ has been defined over $W_j$. Over $W_{j+1}$, fix a complex trivialization

$$\Psi_{j+1} : E|_{W_{j+1}} \to \mathbb{H}^l.$$

Let $K = W_j \cap W_{j+1}$. Then for each $x \in K$,

$$\phi_{j+1} = \Phi_j \circ \Psi_{j+1}^{-1} : \mathbb{H}^l \to E|_x \to \mathbb{H}^m$$

is a complex monomorphism, and hence a point in the complex Stiefel manifold $V_{2m,2l}$ of linear maps from $\mathbb{C}^{2l}$ to $\mathbb{C}^{2m}$ of rank $2l$. 

The space $V_{2m,2l}$ naturally lies inside $\mathbb{C}^{2l \times 2m}$. We can use a partition of unity to extend $\phi_{j+1}$ as a map from $K$ to $\mathbb{C}^{2l \times 2m}$ to a map

$$\tilde{\phi}_{j+1} : W_{j+1} \to \mathbb{C}^{2l \times 2m}.$$ 

We would like the extension to actually lie in $V_{2m,2l}$. This is achieved by a transversality argument. The complement of $V_{2m,2l}$ is stratified by linear maps of lower ranks. The stratum $S_{2l-b}$ with rank $2l-b$ is a submanifold with real codimension $2(2m-2l+b)b$. We assume the extension $\tilde{\phi}_{j+1}$ is transversal to all $S_{2l-b}$, $1 \leq b \leq 2l$.

The stratum $S_{2l-1}$ with rank $2l-1$ has the smallest codimension, which is

$$2(2m-2l+1) = 4\left\lfloor \frac{k+2}{4} \right\rfloor + 2 = \begin{cases} k + 2, & \text{if } k \equiv 0 \pmod{4} \\ k + 1, & \text{if } k \equiv 1 \pmod{4} \\ k + 4, & \text{if } k \equiv 2 \pmod{4} \\ k + 3, & \text{if } k \equiv 3 \pmod{4}, \end{cases}$$

because $m = l + \left\lfloor \frac{k+2}{4} \right\rfloor$. Since the dimension of $W_{j+1}$ is $k$, $\tilde{\phi}_{j+1}$ misses each $S_{2l-b}$, $1 \leq b \leq 2l$.

Now at each point $y \in W_{j+1}$,

$$\tilde{\phi}_{j+1} \circ \Psi_{j+1} : E|_y \to \mathbb{H}^l \to \mathbb{H}^m$$

is a complex monomorphism, and it agrees with $\Phi_j$ over $K$. As in Lemma $2.11$ we can canonically extend it $Pin(2)-$equivariantly to $\iota_jW_{j+1}$. Since $\Phi_j$ is assumed to be $Pin(2)-$equivariant, the extension also agrees with $\Phi_j$ over $W_j \cap \iota_jW_{j+1}$. Thus we obtain a $Pin(2)-$equivariant monomorphism

$$\Phi_{j+1} : E|_{W_{j+1}} \to \mathbb{H}^m.$$ 

\[\Box\]

**Example 2.15.** According to Theorem $2.14$ any rank 2 quaternionic bundle over $T^4$ can be embedded into $\mathbb{H}^2$, since $1 + \left\lfloor \frac{4+2}{4} \right\rfloor = 2$. This can be also proved using $[8]$. Indeed, it is shown there that any rank 2 bundle over $T^4$ is of the form $E = \mathbb{H}(S)$, where $S$ is a signed invariant finite set of $T^4$ and $\mathbb{H}(S)$ is obtained from $\mathbb{H}$ by a canonical spinor twisting around $S$. Moreover, if $E' = \mathbb{H}(-S)$, then $E \oplus E' = \mathbb{H}^2$.

3. **Stable homotopy and unoriented bordism Seiberg-Witten invariants**

In this section $M$ is a closed oriented smooth 4-manifold and $c$ is a Spin$^c$ structure. We first review the stable homotopy Seiberg-Witten invariants. Then we construct the unoriented bordism Seiberg-Witten invariants.
3.1. **Seiberg-Witten equations.** Let $S^0$ and $S^1$ be the spinor bundles associated to $c$. The determinant line bundles $\det_{\mathbb{C}} S^0$ and $\det_{\mathbb{C}} S^1$ are isomorphic. Denote the Hermitian line bundle by $L_c$. Fix a Hermitian connection $A_0$ on $L_c$. Let $\mathcal{H}^1(M, \mathbb{R})$ be the space of harmonic 1-forms, and consider the affine space $\mathcal{A}_0$ of Hermitian connections on $L$ of the form $A = A_0 + a \mathbb{I}$ for $a \in \mathcal{H}^1(M, \mathbb{R})$.

Let $\mathcal{H}^0(M, U(1))$ be the group of harmonic maps from $M$ to $U(1)$. Fix a base point $x_0 \in M$ and let $\mathcal{H}^0_0(M, U(1))$ be the subgroup consisting of the harmonic maps sending $x_0$ to the identity. Then $\mathcal{H}^0(M, U(1))$ is the product of $U(1)$ and $\mathcal{H}^0_0(M, U(1))$, where $U(1)$ is the subgroup of constant maps.

Consider the gauge group action of $g \in \mathcal{H}^0(M, U(1))$ on $A \in \mathcal{A}_0$ and let $J$ be the quotient of $A_0$ by $\mathcal{H}^0_0(M, U(1))$. Then $J$ is identified with the quotient of $H^1(M; \mathbb{R})$ by $H^1(M; 2\mathbb{Z})$, and thus a torus of dimension $b_1$.

Let $p_M : \mathcal{A}_0 \times M \to M$ be the projection map. $\mathcal{H}^0(M, U(1))$ acts on the bundle $p_M^*(S^0 \oplus S^1)$ on $\mathcal{A}_0 \times M$ by

$$g : (A, x, q) \mapsto (g(A), x, qg).$$

The action of the subgroup $U(1)$ coincides with the action coming from the complex structure.

Consider the two infinite dimensional complex vector bundles over $J$:

$$\tilde{V}^0 = (\mathcal{A}_0 \times \Gamma(p_M^* S^0))/\mathcal{H}^0_0(M, U(1)),
\tilde{V}^1 = (\mathcal{A}_0 \times \Gamma(p_M^* S^1))/\mathcal{H}^0_0(M, U(1)),$$

and the smooth family of $U(1)$-equivariant Dirac operators $\{D_a\}_J$. According to the Atiyah-Singer index theorem we have the formula

$$(3.1) \quad \text{rank}_{\mathbb{C}} \text{Ind}\{D_a\}_J = -\sigma(M)/8.$$

Consider as well the two trivial infinite dimensional real vector bundles over $J$:

$$\hat{W}^0 = J \times d^*(\Gamma(\Lambda^2 TM^*)) \subset J \times \Gamma(TM^*),
\hat{W}^1 = J \times \Gamma(\Lambda^+ TM^*),$$

with trivial $U(1)$-action and the operator $d^+$, which is the self-dual part of $d$.

With this set up, the Seiberg-Witten equations are then a $U(1)$-equivariant bundle map $\tilde{f}_{SW}$ between the infinite dimensional bundles $\tilde{V}^0 \oplus \hat{W}^0$ and $\tilde{V}^1 \oplus \hat{W}^1$, which, at a point $a \in J$, is of the form

$$\tilde{f}_{SW}|_a(s, b) = (D_a s + \frac{1}{2} C(b) s, d^+ b + s \bar{s}).$$

Here $C : TM^* \otimes S^0 \to S^1$ is the Clifford multiplication, and $s \bar{s}$ is a natural algebraic map from $S^0$ to $\Lambda^+ TM^*$ (see e.g. [12]).

**Remark 3.1.** When restricted to $0 \oplus \hat{W}_0$, $\tilde{f}_{SW}$ is the linear (embedding) sending $(0, b)$ to $(0, d^+ b)$ at each $a \in J$, in particular, it is independent of $a \in J$. 
QUATERNIONIC BUNDLES AND BETTI NUMBERS OF SYMPLECTIC 4-MANIFOLDS WITH KODAIRA DIMENSION ZERO

Now let us suppose $c$ is a reducible Spin$^c$ structure. Then $L_c$ is a trivial bundle. Coming with a reduction of $c$ to a spin structure are the involution on $J$ and the enlarged $Pin(2)$ symmetry of $\tilde{f}_{SW}$, which we explain briefly now.

Fixing a trivial connection $A_0$ on $L_c$ corresponding to the spin reduction, and let $D_0$ be the associated Dirac operator. Consider the involution $\iota_{A_0}$ on $A_0$ by sending $a$ to $-a$. We lift $\iota_{A_0}$ to the bundle $p_M^*(S^0 \oplus S^1)$ by

$$\iota_S : (a, x, s) \mapsto (-a, x, sj)$$

to make both $p_M^*S^0$ and $p_M^*S^1$ quaternionic bundles. When dividing out by $\mathcal{H}_0(M, U(1))$, $\iota_{A_0}$ induces the standard involution $\iota_J$ on the torus $J$. Furthermore, $\iota_S$ induces the anti-complex lifts $\iota_{\tilde{V}_0}$ and $\iota_{\tilde{V}_1}$ of $\iota_J$ on the bundles $\tilde{V}_0$ and $\tilde{V}_1$, which make them quaternionic bundles. Therefore $Pin(2)$ acts on both $\iota_{\tilde{V}_0} \oplus \iota_{\tilde{V}_0}$ and $\iota_{\tilde{V}_1} \oplus \iota_{\tilde{V}_1}$ by Remark 2.3.

It is well-known that $D_0$ is a $\mathbb{H}$–linear operator. From which it is not hard to see that the family of linear operators $\{D_a\}_J$ is $Pin(2)$–equivariant. Hence the index bundle of $\{D_a\}_J$, Ind$\{D_a\}_J$, lies in $KQ(J)$. For the $Pin(2)$–equivariance of the remaining terms of the map $f_{SW}$ we refer to [12].

An important property of the Seiberg-Witten equations is that $\tilde{f}_{SW}$ is proper, which implies that we formally have a $Pin(2)$–equivariant map between the two infinite dimensional sphere bundles and thus an element $\tilde{f}_{SW} \in [S(V^0 \oplus \tilde{W}_0), S(V^1 \oplus \tilde{W}_1)]^{Pin(2)}$. Here we follow the notations in [8]: For $G = U(1)$ or $Pin(2)$, Map$(S(V), S(V'))^G$ denotes the set of $G$–equivariant maps between the sphere bundles of the $G$–equivariant bundles $V$ and $V'$. Its quotient divided by the $G$–equivariant homotopy is denoted by $[S(V), S(V')]^G$. To understand this element more explicitly, we need the technique of finite dimensional approximations (initiated in [5]), which leads to the construction of the stable cohomotopy Seiberg-Witten invariants of Bauer and Furuta.

3.2. Finite dimensional approximations. First of all we need the notion of an admissible $Pin(2)$–equivariant triple.

**Definition 3.2.** A $Pin(2)$–equivariant triple is a triple

$$(S(V_0 \oplus W_0), S(V_1 \oplus W_1), f)$$

consisting of

1. finite dimensional quaternionic vector bundles $V_0$ and $V_1$ over $J$,
2. finite dimensional trivialized real vector bundles $W_0$ and $W_1$ over $J$,
3. a $Pin(2)$–equivariant map

$$f : S(V_0 \oplus W_0) \rightarrow S(V_1 \oplus W_1).$$

A $Pin(2)$–equivariant triple is called admissible if, when restricted to $0 \oplus W_0$, $f$ is independent of $a \in J$. In the same way we define admissible $U(1)$–equivariant triples.
Notice that $f$ maps $0 \oplus W_0$ to $0 \oplus W_1$ due to equivariance. And since $W_0$ and $W_1$ are trivialized, it makes sense to require the restriction of $f$ to $0 \oplus W_0$ be independent of $a \in J$.

Let us recall the stabilization process. Given a $Pin(2)$–equivariant triple $(S(V_0 \oplus W_0), S(V_1 \oplus W_1), f)$, a quaternionic vector bundle $V$ and a trivial real vector bundle $W$, define

$$V_0 = V_0 \oplus V, \quad V_1 = V_1 \oplus V, \quad W_0 = W_0 \oplus W, \quad W_1 = W_1 \oplus W.$$ \hspace{1cm} (3.2)

Recall that the sphere bundles $S(V_0 \oplus W_i)$ and $S(V \oplus W)$ can be joined fibrewisely to form the sphere bundle of the direct sum $V_0 \oplus W_1$ by the formula $(1 - t)a_i + ta$ for $t \in [0,1]$, and similarly two $Pin(2)$–equivariant maps

$$f : S(V_0 \oplus W_0) \longrightarrow S(V_1 \oplus W_1) \quad \text{and} \quad g : S(V \oplus W) \longrightarrow S(V \oplus W).$$

can be joined to a $Pin(2)$–equivariant map

$$j(f,g) : S(V_0 \oplus W_0) \longrightarrow S(V_1 \oplus W_1).$$

Thus by taking the join with the identity on $S(V \oplus W)$, we have the stabilization map between the $Pin(2)$–equivariant triples

$$(S(V_0 \oplus W_0), S(V_1 \oplus W_1), f) \longrightarrow (S(V_0 \oplus W_0), S(V_1 \oplus W_1), j(f, id)).$$

It is easy to check that the join of two homotopies is a homotopy. Two triples are called stable homotopic if they become homotopic under stabilization. This is an equivalence relation. We call an equivalence class of triples a $Pin(2)$–equivariant stable homotopy class. Clearly the join of two admissible triples is still admissible.

A finite dimensional approximation to the Seiberg-Witten equations associated to a spin reduction of $c$ is a $Pin(2)$–equivariant triple $(S(V_0 \oplus W_0), S(V_1 \oplus W_1), f_{SW})$ such that

$$[V_0] - [V_1] = \text{ind}\{D_0\}_{J} \in KQ(J),$$

$$[W_0] - [W_1] = [-\mathbb{R}^{b^+}] \in KO(J).$$ \hspace{1cm} (3.3)

And $f_{SW}$ is an approximation of $\tilde{f}_{SW}$ in an appropriate sense, which we do not specify as it will be irrelevant for us (see Proposition 3.6). An admissible finite dimensional approximation is one such that $f_{SW}$ is independent of $a \in J$ when restricted to $S(0 \oplus W_0)$.

There are many such finite dimensional approximations, all of which are related via the stabilization process. More precisely, it was shown in [3] and [6] that any two finite dimensional approximations become homotopic under stabilization, and moreover, the homotopy itself is well-defined up to homotopy. Notice that it is pointed out in Remark 3.1.1 that the SW map $f_{SW}$ is admissible in the sense that it is independent of $a \in J$ when restricted to $0 \oplus W_0$. Indeed it is further shown in [1] and [9] that it can be assumed that the finite dimensional approximations are admissible.
Therefore there is a well-defined $Pin(2)$–equivariant stable homotopy class of admissible triples. Furthermore, this $Pin(2)$–equivariant stable homotopy class only depends on the oriented diffeomorphism type of $M$ and the reducible Spin$^c$ structure $c$ and the spin reduction $\nu$ (see [6]). Thus, we can write this $Pin(2)$–equivariant stable homotopy class as $SW(M, c, \nu)$ and call it the $Pin(2)$–stable cohomotopy Seiberg-Witten invariant.

3.3. Unoriented bordism SW invariants. In this subsection we construct an unoriented bordism SW invariant of a reducible Spin$^c$ structure when $b^+ \geq 2$ following [8]. Our invariants are simpler, living in the unoriented bordism group rather than the richer Pin bordism group as in [8]. But this would be sufficient for our purpose.

On the other hand, we only need the assumption $b^+ \geq 2$ rather than $b^+ \geq b_1 + 2$. Being able to weaken the assumption on $b^+$ is crucial for us. This is achieved by adding the admissibility as in Definition 3.2.

The construction of the unoriented bordism SW invariant is given in several steps.

3.3.1. The construction of $\gamma'$. Let

$$\text{Map}_{adm}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{U(1)}$$

be the space of $U(1)$–equivariant maps between the sphere bundles which are admissible. Finite dimensional approximations to the Seiberg-Witten equations give rise to such objects.

Given $f_0$ and $f_1$ in $\text{Map}_{adm}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{U(1)}$, we can view them as maps from $S(V_0 \oplus W_0)$ to $V_1 \oplus W_1$. Let $F(f_0, f_1)$ be the set of $U(1)$–equivariant paths

$$\tilde{f}_t : S(V_0 \oplus W_0) \times [0, 1] \rightarrow V_1 \oplus W_1$$

connecting $f_0$ and $f_1$, and satisfying

1. the restriction of $\tilde{f}_t$ to $S(0 \oplus W_0) \times [0, 1]$, which is mapped to $S(0 \oplus W_1)$ due to equivariance, does not vanish and is independent of $a \in J$, i.e. there is a map $\xi$ from $S(\mathbb{R}^{\text{rank}_\mathbb{R} W_0}) \times [0, 1]$ to $\mathbb{R}^{\text{rank}_\mathbb{R} W_1} - 0$ such that $\tilde{f}_t = \xi$ on $S(0 \oplus W_0)|_a \times [0, 1]$ for any $a \in J$.

2. $\tilde{f}_t$ is transverse to the zero section.

From now on we assume that in this section

$$(3.4) \quad \text{rank}_\mathbb{R} W_1 - \text{rank}_\mathbb{R} W_0 \geq 2.$$

This corresponds to $b^+ \geq 2$.

Lemma 3.3. $F(f_0, f_1)$ is non-empty.

Proof. The existence of a $\tilde{f}_t$ is shown by three steps.

Step 1. Since the fibers of $V_1 \oplus W_1$ are linear spaces, we can use simply a linear combination to construct a $U(1)$–equivariant map $\Omega$ from $S(V_0 \oplus W_0) \times [0, 1]$ to $V_1 \oplus W_1$ connecting $f_0$ and $f_1$. 

Lemma 3.4. The unoriented bordism class of such that 

\[(3.5) \dim \mathcal{J} = \text{rank}_{\mathbb{R}} V_0 - \text{rank}_{\mathbb{R}} W_1 - 0\]

Proof. Given (\tilde{f})_0 and (\tilde{f})_1, denote the zero set of \(\tilde{f}^{-1}(0)\) by \(\tilde{\mathcal{M}}\). Then \(\tilde{\mathcal{M}}\) is a smooth, closed submanifold of \(S(V_0 \oplus W_0) \times [0, 1]\). Let \(\tilde{B}\) be the complement of \(S(0 \oplus W_0)\) in \(S(V_0 \oplus W_0)\). Then \(\tilde{\mathcal{M}}\) actually lies in \(\tilde{B} \times (0, 1)\), so it is itself a closed smooth manifold. Denote the \(U(1)\)–quotient \(\tilde{\mathcal{M}}/U(1)\) by \(\mathcal{M}\). Since \(U(1)\) acts freely on \(\tilde{B}\), the quotient \(\mathcal{M}\) is also a closed smooth manifold. Moreover, we can view \(\tilde{f}\) as a section of the bundle 

\[
\mathcal{E} = (S(V_0 \oplus W_0) \times [0, 1]) \times_f (V_1 \oplus W_1)
\]

over \(S(V_0 \oplus W_0) \times [0, 1]\). In particular, the dimension of \(\mathcal{M}\) is easily seen to be

\[
(3.5) \quad \dim J - 1 + 2\text{rank}_{\mathbb{C}} V_0 - 2\text{rank}_{\mathbb{C}} V_1 + \text{rank}_{\mathbb{R}} W_0 - \text{rank}_{\mathbb{R}} W_1.
\]

Lemma 3.4. The unoriented bordism class of \(\mathcal{M}\) does not depend on the choice of \(\tilde{f} \in F(f_0, f_1)\).

Proof. Given (\tilde{f}_t)_0 and (\tilde{f}_t)_1 in \(F(f_0, f_1)\), we can construct a homotopy

\[
\tilde{f}_{t,s} : (S(V_0 \oplus W_0) \times [0, 1]) \times [0, 1] \to V_1 \oplus W_1
\]

such that

1. \(\tilde{f}_{t,0} = (\tilde{f}_t)_0\) and \(\tilde{f}_{t,1} = (\tilde{f}_t)_1\);
2. the restriction of \(\tilde{f}_{t,s}\) to \(S(V_0 \oplus W_0) \times (0 \bigcup 1)\) is independent of \(s \in [0, 1]\), i.e.

\[
(3.6) \quad \tilde{f}_{0,s} = \tilde{f}_{0,0} = \tilde{f}_{0,1} = f_0, \quad \tilde{f}_{1,s} = \tilde{f}_{1,0} = \tilde{f}_{1,1} = f_1,
\]

for any \(s \in [0, 1]\), in particular, \(\tilde{f}_{t,s}\) does not vanish on

\[
S(V_0 \oplus W_0) \times (0 \bigcup 1) \times [0, 1],
\]
(3) the restriction of \( \tilde{f}_{t,s} \) to \( S(0 \oplus W_0) \times [0, 1] \times [0, 1] \) does not vanish,
(4) \( \tilde{f}_t \) is transverse to the zero section.

The zero set of \( \tilde{f}_{t,s} \) then is a compact manifold whose only boundaries are \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \). In addition, \( U(1) \) acts freely on it. The smooth \( U(1)-\)quotient then provides the desired bordism.

The existence of \( \tilde{f}_{t,s} \) is established in the same way as that of \( \tilde{f}_t \). We first construct a \( U(1)-\)equivariant map \( \tilde{\Omega} \) from \( S(V_0 \oplus W_0) \times [0, 1] \times [0, 1] \) to \( V_1 \oplus W_1 \) connecting \( (\tilde{f}_t)_0 \) and \( (\tilde{f}_t)_1 \), and such that (2), or equivalently, (3.6), is satisfied. For example we could use a linear homotopy.

There are maps

\[ \xi_t : S(\mathbb{R}^{\text{rank}_a W_0}) \times [0, 1] \rightarrow \mathbb{R}^{\text{rank}_a W_1} - 0 \]

such that \( \tilde{f}_t = \xi_t \) on \( S(0 \oplus W_0)|_a \times [0, 1] \) for any \( a \in J \). From the assumption (3.4), the maps \( \xi_t \) are homotopic relative to \( S(\mathbb{R}^{\text{rank}_a W_0}) \times (0 \amalg 1) \). Therefore the restrictions of \( \tilde{f}_t \) to \( S(0 \oplus W_0) \times [0, 1] \) are homotopic as maps to \( 0 \oplus (W_1 - 0) \) through a homotopy which is constant on \( S(0 \oplus W_0) \times (0 \amalg 1) \) (and independent of \( a \in J \)). Thus we can perturb \( \tilde{\Omega} \) near

\[ S(0 \oplus W_0) \times [0, 1] \times [0, 1] \]

but away from

\[ S(V_0 \oplus W_0) \times [0, 1] \times (0 \amalg 1) \quad \text{and} \quad S(V_0 \oplus W_0) \times (0 \amalg 1) \times [0, 1] \]

to an \( U(1)-\)equivariant homotopy \( \tilde{\Omega}' \) connecting \( (\tilde{f}_t)_0 \) and \( (\tilde{f}_t)_1 \), and such that both (2) and (3) are satisfied.

Now the zero set of \( \tilde{\Omega}' \) is away from the closed subsets \( S(0 \oplus W_0) \times [0, 1] \times [0, 1] \) and \( S(V_0 \oplus W_0) \times (0 \amalg 1) \times [0, 1] \). Hence it has a neighborhood \( U \) with the same property. In particular, \( U(1) \) acts freely on \( U \). Hence we can further perturb \( \tilde{\Omega}' \) equivariantly inside \( U \) to make it transverse to the zero section. Notice that \( \tilde{\Omega}' \) is already transverse to the zero section along \( S(V_0 \oplus W_0) \times [0, 1] \times (0 \amalg 1) \), so the perturbation can be chosen to be also away from \( S(V_0 \oplus W_0) \times [0, 1] \times (0 \amalg 1) \). The new perturbation is then a homotopy connecting \( (\tilde{f}_t)_0 \) and \( (\tilde{f}_t)_1 \), and such that (2), (3) and (4) are all satisfied.

Therefore we can make the following definition.

**Definition 3.5.** For \( f_0, f_1 \) in \( \text{Map}_{\text{adm}}(S(V_0 \oplus W_0), S(V_1 \oplus W_1))U(1) \), let \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}(f_0, f_1) = [\mathcal{M}]^{\text{uo}} \in \Omega_n^{\text{uo}} \), where \( n \) is given by (3.5).

3.3.2. *Properties of \( \gamma' \).* We now establish a few properties of \( \gamma' \).

Since (3.4) is invariant under stabilization, \( \gamma'_{-,-} \) is defined on any stabilization of the pair of maps \( f_0 \) and \( f_1 \). Furthermore, the bordism class does not change, as the join of \( \tilde{f} \) and \( \text{id} \) has the same zero set as that of \( \tilde{f} \).

In addition, \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \) satisfies an important additivity property. Given \( \tilde{f} \in F(f_0, f_1) \) and \( \tilde{g} \in F(f_1, f_2) \), they naturally combine to an element
\( \tilde{h} \in F(f_0, f_2) \), defined by
\[
\tilde{h}(t) = \begin{cases} 
\tilde{f}(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\tilde{g}(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
Clearly the zero set of \( \tilde{h} \) is the disjoint union of those of \( \tilde{f} \) and \( \tilde{g} \). Therefore \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \) is additive in the following sense:
\[
(3.7) \quad \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}(f_0, f_1) + \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}(f_1, f_2) = \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}(f_0, f_2).
\]
This additivity immediately implies that \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \) only depends on the homotopy classes of \( f_0 \) and \( f_1 \). Thus, we can and will from now on regard \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \) as a map from
\[
[S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{adm}^{U(1)} \times [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{adm}^{U(1)}
\]
to \( \Omega_n^{uo} \). Obviously the additivity still holds with this new meaning of \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \).

Another important property of \( \gamma' \) is the following.

**Proposition 3.6.** For \( Pin(2) \)–equivariant sections, \( \gamma' \) is independent of homotopy classes of \( Pin(2) \)–equivariant maps, i.e. the composition
\[
\gamma' : [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{adm}^{Pin(2)} \times [S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{adm}^{Pin(2)} \to \Omega_n^{uo}
\]
is a constant map.

**Proof.** Consider two \( Pin(2) \)–equivariant maps \( f_0 \) and \( f_1 \) between the pairs. Notice that, as \( U(1) \), \( Pin(2) \) acts freely away from the \( U(1) \)–fixed point set
\[
\mathcal{F} = S(0 \oplus W_0) \coprod S(0 \oplus W_1).
\]
Notice also that \( \iota \) acts freely on the set \( \mathcal{F} \) as an involution. Applying the dimension assumption (3.4) to the quotient manifolds of \( \mathcal{F} / \iota \), we can actually construct a \( \tilde{f}_t \in F(f_0, f_1) \) which is \( Pin(2) \)–equivariant.

Thus \( \iota \) is a free involution on \( \mathcal{M} \). Let \( p : \mathcal{M} \to \mathcal{M} / \iota \) be the double covering and \( \zeta \) the real line bundle associated to \( p \). Then \( \mathcal{M} \) is diffeomorphic to the sphere bundle of \( \zeta \), hence it bounds the disk bundle of \( \zeta \). Therefore the unoriented bordism class of \( \mathcal{M} \) is zero, that is, \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}([f_0], [f_1]) = 0 \).

Together with the additivity of \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \), we have the proposition. \( \square \)

### 3.3.3. The invariant \( e_1 (V_0 \oplus W_0, V_1 \oplus W_1) \).

We first construct a variation of \( \gamma' \), \( \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1} \), whose input is a single element, rather than a pair, in 
\[
[S(V_0 + W_0), S(V_1 \oplus W_1)]_{adm}^{U(1)}.
\]

Consider constant maps in \( Map(S(V_0 + W_0), S(V_1 \oplus W_1))_{adm}^{U(1)} \). By the \( U(1) \)–equivariance, they must land in \( S(0 \oplus W_1) \). By the assumption (3.4) all such maps are homotopic. Let \( [f_0] \) be this unique homotopy class of constant maps. For any \( [f] \in [S(V_0 + W_0), S(V_1 \oplus W_1)]_{adm}^{U(1)} \) we define
\[
\gamma_{V_0 \oplus W_0, V_1 \oplus W_1}([f]) = \gamma'_{V_0 \oplus W_0, V_1 \oplus W_1}([f_0], [f]).
\]
\begin{align*}
\gamma_{V_0 \oplus W_0, V_1 \oplus W_1} \text{ is also invariant under stabilization since the join of a constant map } f_0 \text{ and } \text{id} \text{ is itself homotopic to a constant map (just observe that the join of a point and a sphere is a disk). By Proposition 3.6 and (3.7), } \gamma \text{ takes a constant value on } \text{Pin}(2)\text{-equivariant sections.}
\end{align*}

**Definition 3.7.** Suppose \([S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{\text{adm}}^{\text{Pin}(2)} \neq \emptyset\). We write the constant image of \(\gamma\) on \([S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{\text{adm}}^{\text{Pin}(2)}\) in \(\Omega_{\text{uo}}^{n}\) as
\begin{align*}
e_1(V_0 \oplus W_0, V_1 \oplus W_1).
\end{align*}

Since \(\gamma_{V_0 \oplus W_0, V_1 \oplus W_1}\) is invariant under stabilization, \(e_1\) satisfies the stabilization property:
\begin{align*}
e_1(V_0 \oplus W_0, V_1 \oplus W_1) = e_1(\hat{V}_0 \oplus \hat{W}_0, \hat{V}_1 \oplus \hat{W}_1),
\end{align*}
where \(\hat{V}_i\) and \(\hat{W}_i\) are given as in (3.2).

### 3.3.4. The unoriented bordism SW invariant

Now let \(M\) be a spin manifold with \(2\chi + 3\sigma = 0\) and \(b^+ \geq 2\). Let \(c\) be a reducible Spin\(c\) structure together with a spin reduction \(\nu\). Then a finite dimensional approximation \((S(V_0 \oplus W_0), (V_1 \oplus W_1), f_{SW})\) is \text{Pin}(2)-equivariant and can be chosen to be admissible. In particular, \([S(V_0 \oplus W_0), S(V_1 \oplus W_1)]_{\text{adm}}^{\text{Pin}(2)}\) is nonempty. Thus we can make the following definition, in view of (3.8).

**Definition 3.8.** Suppose \(M\) is a manifold with \(b^+ \geq 2\) and \(c\) is a reducible Spin\(c\) structure on \(M\) together with a spin reduction \(\nu\). The unoriented bordism Seiberg-Witten invariant \(SW^{\text{uo}}(M, c) : SW(M, c, \nu) \to \Omega_{\text{uo}}^{n(M, c)}\) is defined to be
\begin{align*}
SW^{\text{uo}}(M, c) = e_1(V_0 \oplus W_0, V_1 \oplus W_1)
\end{align*}
where \(V_0, W_0, V_1, W_1\) arise from an admissible finite dimensional approximation of the Seiberg-Witten equations associated to \((c, \nu)\) and \(n(M, c)\) is given by (3.3).

It turns out \(SW^{\text{uo}}(M, c)\) is independent of \(\nu\), and is an invariant of the oriented diffeomorphism type of \(M\) and \(c\). This is because for different spin reductions the admissible finite dimensional approximations are still \(U(1)\)-equivariantly stably homotopic. Hence they will have the same \(\gamma\) invariant due to the stability of \(\gamma\).

Notice that in this case \(J = T^{b_1}\) and \(V_0, W_0, V_1, W_1\) satisfy (3.3). Therefore, we have
\begin{align*}
n(M, c) = b_1 - 1 - \frac{\sigma}{4} - b^+ = \frac{4b_1 - 4 - 5b^+ + b^-}{4},
\end{align*}
by (3.5), (3.3), (3.1), and
\begin{align*}
\sigma = b^+ - b^-.
\end{align*}
Since
\begin{align*}
\chi = 2 - 2b_1 + b^+ + b^-,
\end{align*}
we have
\[
2\chi + 3\sigma = 4 - 4b_1 + 5b^+ - b^-.
\]
Recall that the SW moduli space of the reducible Spin\(^c\) structure \(c\) is
\[
-\frac{2\chi + 3\sigma}{4} + c_1(L_c)^2 = -\frac{2\chi + 3\sigma}{4},
\]
as \(L_c\) is a trivial bundle.

Comparing (3.9), (3.12) and (3.13), we find that \(n_{M,c}\) agrees with the
dimension of the SW moduli space of the reducible Spin\(^c\) structure \(c\). This
is certainly expected. Moreover, the following is proved in [9] (see also
similar statements in [1], [3] and [6]).

**Proposition 3.9.** Let \(M\) be a spin manifold with \(2\chi + 3\sigma = 0\) and \(b^+ \geq 2\).
Let \(c\) be a reducible Spin\(^c\) structure. Then, \(n_{M,c} = 0\), and under the natural
isomorphism between \(\Omega^{uo}_{0} \simeq \mathbb{Z}_2\), the unoriented bordism class is equal to
the ordinary SW invariant modulo 2.

**Remark 3.10.** Let \(M\) be a spin manifold with \(2\chi + 3\sigma = 0\) and \(b^+ \geq 2\), and let \(c\) be a reducible Spin\(^c\) structure. It follows from Proposition 3.9 that the Mod
2 Seiberg-Witten invariant of \(c\) depends only on \(b^+\) and \(\text{ind}\{D_a\}_J \in KQ(J)\).

4. **Vanishing of the unoriented bordism Seiberg-Witten invariant**

In this section we prove a vanishing result of the unoriented bordism
Seiberg-Witten invariant.

Suppose \(J = T^{4l-v}\) with \(l \geq 0\) and \(0 \leq v \leq 3\). Suppose \(V_0, V_1\) are
quaternionic bundles over \(J\) with
\[
\text{rank}_C V_0 - \text{rank}_C V_1 = 2p,
\]
and \(W_0, W_1\) are trivial real bundles with
\[
\text{rank}_R W_0 - \text{rank}_R W_1 = -(4p + 4l - v - 1 + \alpha)
\]
for some integer \(\alpha\).

**Proposition 4.1.** Let \(J, V_0, V_1, W_0, W_1\) be as above. If \(p + l + \alpha > 1\) and \(p + l \geq 1\), then
\[
e_1(V_0 \oplus W_0, V_1 \oplus W_1) = 0.
\]

**Proof.** We first apply (3.8), the stability property of \(e_1\), to make the following reduction.

**Lemma 4.2.** \(e_1(V_0 \oplus W_0, V_1 \oplus W_1)\) is the same as
\[
\begin{cases}
   e_1(Q \oplus \mathbb{H}^p \oplus \mathbb{H}^v, \mathbb{H}^{l-p} \oplus \mathbb{R}^{4p+4l-1+\alpha}), & \text{if } p \geq 0, \\
   e_1(Q \oplus \mathbb{R}^v, \mathbb{H}^{f-p} \oplus \mathbb{R}^{4p+4l-1+\alpha}), & \text{if } p < 0,
\end{cases}
\]
where \(Q\) is some quaternionic vector bundle with (complex) rank 2\(l\).
Proof. By possibly stabilizing $V_0$ we can assume that $V_0$ has rank at least $2l$. By Theorem 2.14 we can sum $V_1$ with a quaternionic bundle to make it trivial. Therefore we can assume that $V_1 = \mathbb{H}^a$ for some positive integer $a \geq -p$. Now, by Corollary 2.9 if $p \geq 0$, we can write $V_0 = Q \oplus \mathbb{H}^{a+p-l}$ where $Q$ is a complex rank $2l$ quaternionic bundle. Similarly, if $p < 0$, we can write $V_0 = Q \oplus \mathbb{H}^{a+p-l}$.

\[ \square \]

We assume now that $(V_0 \oplus W_0, V_1 \oplus W_1)$ is of the form as in Lemma 4.2. The next step is to choose a judicious map to compute $e_1(V_0 \oplus W_0, V_1 \oplus W_1)$. Let us first deal with the case that $v = 0$. Notice that in this case the summand $\mathbb{R}^p$ in $V_0$ is trivial.

Since $Q$ has rank $2l$, and

$$l + \left[ \frac{4l + 2}{4} \right] = 2l,$$

by Theorem 2.14 there exists a $Pin(2)$–equivariant monomorphism

$$m = (m_1, ..., m_{2l}) : Q \longrightarrow \mathbb{H}^{2l}.$$

Write

$$\mathbb{R}^{4l+p-l+\alpha} = \text{Im} \mathbb{H}^{l+p} \oplus \mathbb{R}^{l+p+\alpha-1}.$$

Consider the standard $Pin(2)$–equivariant quadratic map

$$h : \mathbb{H} \longrightarrow \text{Im} \mathbb{H}, \quad h(q) = qi\bar{q}.$$

When $p \geq 0$, we define, for $u \in Q$ and $(q_1, ..., q_p) \in \mathbb{H}^p$, a $Pin(2)$–equivariant map

$$g_1 : Q \oplus \mathbb{H}^p \longrightarrow \mathbb{H}^{2l} \oplus \mathbb{H}^p = \mathbb{H}^l \oplus (\mathbb{H}^l \oplus \mathbb{H}^p) \longrightarrow \mathbb{H}^l \oplus \text{Im} \mathbb{H}^{l+p}$$

by

$$g_1(u, q_1, ..., q_p) = (m_1(u), ..., m_l(u), h(m_{l+1}(u)), ..., h(m_{2l}(u)), h(q_1), ..., h(q_p)).$$

And we define $g_0 : Q \oplus \mathbb{H}^p \rightarrow \mathbb{H}^l \oplus \text{Im} \mathbb{H}^{l+p}$ to be the zero map. Then we define, for $i = 0, 1$,

$$f_i : Q \oplus \mathbb{H}^p \longrightarrow \mathbb{H}^l \oplus \text{Im} \mathbb{H}^{l+p} \oplus \mathbb{R}^{l+p+\alpha-1}$$

by $f_i = (g_i, k_i)$ with $k_1 = (0, ..., 0)$ and $k_0 = (1, ..., 1)$. Notice that we can define $k_0$ this way since $l + p + \alpha$ is assumed to be at least 2. Clearly $f_0$ is $U(1)$–equivariant and non-vanishing. Since $g_1$ is $Pin(2)$–equivariant and non-vanishing on the sphere bundle, so is $f_1$. Moreover, we see that the linear homotopy $\tilde{f}_t = tf_0 + (1-t)f_1$ is never 0 in the $\mathbb{R}^{l+p+\alpha-1}$ summand for $t \in (0, 1)$. Therefore $\tilde{f}_t^{-1}(0) = \emptyset$.

When $p \leq 0$, we define, for $u \in Q$, a $Pin(2)$–equivariant map

$$g_1 : Q \longrightarrow \mathbb{H}^{2l} = \mathbb{H}^{l-p} \oplus \mathbb{H}^{l+p} \longrightarrow \mathbb{H}^{l-p} \oplus \text{Im} \mathbb{H}^{l+p}$$

by

$$g_1(u) = (m_1(u), ..., m_{l-p}(u), h(m_{l-p+1}(u)), h(m_{2l}(u))).$$
And we again define \( g_0 : Q \oplus \mathbb{H}^p \to \mathbb{H}^l \oplus \text{Im}\mathbb{H}^{l+p} \) to be the zero map, and define in the same way, for \( i = 0, 1 \),
\[
 f_i : Q \to \mathbb{H}^{l-p} \oplus \text{Im}\mathbb{H}^{l+p} \oplus \mathbb{R}^{l+p+\alpha-1}
\]
by \( f_i = (g_i, k_i) \) with \( k_1 = (0, \ldots, 0) \) and \( k_0 = (1, \ldots, 1) \). It is easy to see that the linear homotopy \( f_i \) has the same property as in the case \( p \geq 0 \).

Notice that in both cases \( f_1 \) is independent of \( a \in J \) when restricted to \( S(0 \oplus W_0) \), i.e. admissible. Thus we can use it to compute \( e_1(V_0 \oplus W_0, V_1 \oplus W_1) \). Since \( f^{-1}_1(0) = 0 \) and \( f_i \) is independent of \( a \in J \) when restricted to \( S(0 \oplus W_0) \times [0, 1] \) we conclude that \( e_1(V_0 \oplus W_0, V_1 \oplus W_1) = 0 \) in the case of \( v = 0 \).

We now use the trick in [12] to reduce the general case to the case \( v = 0 \). Let \( O_{T^v} \) be the point in \( T^v \) coming from the origin of \( \mathbb{R}^v \) and \( B^v \) be an invariant disc of \( T^v \) around \( O_{T^v} \) and \( p : B^v \to \mathbb{R}^v \) be an equivariant diffeomorphism. Consider the projection and the embedding
\[
p : T^{4l-v} \times T^v \to T^{4l-v}, \quad e : T^{4l-v} \to T^{4l-v} \times O_{T^v}.
\]
Via \( \rho \) we identify \( p^*Q |_{T^{4l-v} \times B^v} \) with the bundle \( Q \oplus \mathbb{R}^v \) over \( T^{4l-v} \). Notice that this identification is \( Pin(2) \)-equivariant since \( \rho \) is. Via this identification the monomorphism \( m : p^*Q \to \mathbb{H}^{2l} \) induces a \( Pin(2) \)-equivariant bundle map (not a homomorphism)
\[
m' : Q \oplus \mathbb{R}^v \to \mathbb{H}^{2l}
\]
by the formula
\[
m'|_z(u, s) = m|_{z \times \rho^{-1}(s)}(p^*u),
\]
where \( z \in T^{b_1}, u \in Q \) and \( s \in \mathbb{R}^v \).

Now we define \( g_1 \) in the same way except replacing \( Q \) by \( Q \oplus \mathbb{R}^v \), replacing \( m \) by \( m' \) and adding a monomorphism \( \tau \) from \( \mathbb{R}^v \) to the first \( \text{Im}\mathbb{H} \). We only need to verify that \( g_1 \) is \( Pin(2) \)-equivariant and non-vanishing on the sphere bundle, the remaining arguments are exactly the same as in the case \( v = 0 \). \( g_1 \) is clearly \( Pin(2) \)-equivariant as the linear map \( \tau \) is \( Pin(2) \)-equivariant. Since \( m \) is a monomorphism the \( \mathbb{H}^l \) component of \( g_1 \) vanishes only if \( u = 0 \). And if \( u = 0 \) then the first \( \text{Im}\mathbb{H} \) component of \( g_1 \) vanishes only if \( s = 0 \) as \( h(0) = 0 \) and \( \tau \) is a monomorphism. Thus \( g_1 \) does not vanishes on the sphere bundle. \( \square \)

**Theorem 4.3.** Let \( M \) be a spin 4-manifold with
\[
(4.1) \quad b_1 = 4l - v, \quad 0 \leq v \leq 3, \quad \sigma = -16p, \quad 2\chi + 3\sigma = 4\alpha
\]
and \( b^+ \geq 2 \). Let \( c \) be a reducible \( \text{Spin}^c \) structure. If \( p + l + \alpha > 1 \) and \( p + l \geq 1 \), then \( SW^{uo}(M, c) \) is zero.

**Proof.** By (3.11) and (3.12), we have
\[
-16p = \sigma = (b^+ - b^-) = -4(1 - b_1 + b^+) + 4\alpha,
\]
and hence
\[(4.2) \quad b^+ = 4p + b_1 - 1 + \alpha = 4p + 4l - 1 + \alpha - v.\]
Thus, any \((V_0,W_0,V_1,W_1)\) arising from an admissible finite dimensional approximation of the Seiberg-Witten equations associated to \(c\) satisfies the assumption in Proposition 4.1. \(\square\)

**Remark 4.4.** In [9], in the case \(\alpha = 0\) and \(p + l = 1\), we are able to identify \(SW^{uo}(M,c)\) with the \(\epsilon\) invariant in [8].

5. **Proof of Theorem 1.1**

Let us first recall some relevant facts (see [12], [14], [16]) about minimal symplectic 4−manifold with Kodaira dimension zero.

**Lemma 5.1.** Let \((M,\omega)\) be a minimal symplectic 4−manifold with Kodaira dimension zero, then it has torsion canonical class \(K_\omega\). Moreover, it has the following properties.

1. \(2\chi + 3\sigma = 0\) and \(M\) has even intersection form.
2. \(K_\omega\) is either trivial, or of order two which only occurs when \(M\) is an integral homology Enriques surface. In particular, \(M\) is spin and the spin\(^c\) structure \(K_\omega\) is reducible except when \(M\) is an integral homology Enriques surface.
3. When \(b^+ \geq 2\), the Mod 2 Seiberg-Witten invariant of \(K_\omega\) is nonzero.

We note that \(b^-\) can be expressed via (3.12) as
\[(5.1) \quad b^- = 4 - 4b_1 + 5b^+.\]

Next we list minimal Kähler surfaces with \(\kappa = 0\) and orientable \(T^2\)−bundles over \(T^2\) in the following table according to their homology type.

**Table 1.**

| class | \(b^+\) | \(b_1\) | \(\chi\) | \(\sigma\) | \(b^-\) | known as                        |
|-------|--------|--------|--------|--------|--------|--------------------------------|
| a)    | 3      | 0      | 24     | −16    | 19     | \(K3\)                        |
| b)    | 3      | 4      | 0      | 0      | 3      | 4-torus                        |
| c)    | 2      | 3      | 0      | 0      | 2      | primary Kodaira surface        |
| d)    | 1      | 0      | 12     | −8     | 9      | Enriques surface               |
| e)    | 1      | 2      | 0      | 0      | 1      | hyperelliptic surface if complex|

We now finish the proof of Theorem 1.1

**Proof.** Let \(M\) be a minimal symplectic 4−manifold with Kodaira dimension zero.
Bounds on $b^+, b^-$ and $b_1$. Suppose $M$ is non-spin. In this case $M$ is an integral homology Enriques surface by Lemma 5.1. In particular,
\begin{equation}
(5.2) \quad b^+ = 1, \quad b^- = 9, \quad b_1 = 0.
\end{equation}

Since $M$ is symplectic, $b^+$ is at least 1. Suppose $b^+ = 1$. Then by (5.1) we have $b^- = 9 - 4b_1$. Since $M$ has even intersection form, we have $\sigma = b^+ - b^- = 1 - b^-$ is divisible by 8. Moreover $b^-$ is non-negative, thus we have only two cases:
\begin{equation}
(5.3) \quad b^+ = 1, \quad b^- = 9, \quad b_1 = 0
\end{equation}
or
\begin{equation}
(5.4) \quad b^+ = 1, \quad b^- = 1, \quad b_1 = 2.
\end{equation}

Now we assume that $M$ is spin and has $b^+ \geq 2$. We then can use (4.1) for the homological invariants of $M$. Notice that $\alpha = 0$ by Lemma 5.1. Then by the vanishing from Theorem 4.3 and the non-vanishing from Proposition 3.9 and Lemma 5.1 we conclude that
\begin{equation}
(5.5) \quad p + l \leq 1.
\end{equation}

On the other hand, we have by (4.2)
\begin{equation}
(5.6) \quad b^+ = 4p + 4l - v - 1.
\end{equation}

Since $b^+$ is non-negative, we have
\[
p + l \geq \frac{1 + v}{4}.
\]

Thus, as an integer, we must have $p + l \geq 1$. It then follows from (5.6) that
\begin{equation}
(5.7) \quad p + l = 1.
\end{equation}

It follows from (5.6) and (5.7) that
\begin{equation}
(5.8) \quad b^+ = 4 - v - 1 \leq 3.
\end{equation}

Since $b_1$ is non-negative, we have from (5.1) and (5.8) that
\begin{equation}
(5.9) \quad b^- \leq 4 + 5b^+ \leq 19.
\end{equation}

Since $b^-$ is non-negative, we have by (5.1) and (5.8) that
\[
4b_1 = 4 + 5b^+ - b^- \leq 4 + 5b^+ \leq 19.
\]

Hence we conclude that
\begin{equation}
(5.10) \quad b_1 \leq 4.
\end{equation}

Putting together (5.2), (5.3), (5.4), (5.8), (5.9) and (5.10), we obtain the desired Betti number bounds
\[
b^+ \leq 3, \quad b^- \leq 19, \quad b_1 \leq 4.
\]

Bounds on $\chi$ and $\sigma$. For the signature $\sigma = b^+ - b^-$, it is then between $-19$ and $3$. Since $\sigma$ is divisible by $8$, $\sigma$ can only be $-16$, $-8$, or $0$. It follows from $2\chi + 3\sigma = 0$ that the Euler characteristic $\chi$ can only be $24$, $12$, or $0$. 
Homology type. Comparing with Table I in the case that \( M \) is not spin or has \( b^+ = 1 \), \( M \) is either a homology Enriques surface, or a homology \( T^2 \)-bundle over \( T^2 \). In the case that \( M \) is spin and has \( b^+ \geq 2 \), then \( p + l = 1 \). Since \( l \geq 0 \) and we have shown that \( p = \frac{\chi}{16} \geq 0 \), we have either \( l = 0, p = 1 \) or \( l = 1, p = 0 \). When \( l = 0, p = 1 \), \( M \) is a homology K3. When \( l = 1, p = 0 \), we have \( b_1 \leq 4 \) and \( b^+ = b^- = b_1 - 1 \). In this case, \( M \) is a homology \( T^2 \)-bundle over \( T^2 \) according to Table I.

Finally the proof of Corollary 1.2.

Proof. A non-minimal symplectic 4–manifold with \( \kappa = 0 \) is obtained from blowing up a minimal one. The blow up process keeps \( b^+ \) and \( b_1 \) unchanged and increases \( b^- \). Hence we still have the bounds \( b^+ \leq 3 \) and \( b_1 \leq 4 \), as well as the bound \( \chi \geq 0 \).

Remark 5.2. In the broad context of the geography problem of symplectic 4-manifolds (see the survey [13]). Theorem 1.1 and Corollary 1.2 provide complete answers in the case \( \kappa = 0 \).

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