The Flanders theorem over division rings

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Abstract

Let \( D \) be a division ring and \( F \) be a subfield of the center of \( D \) over which \( D \) has finite dimension \( d \). Let \( n, p, r \) be positive integers and \( V \) be an affine subspace of the \( F \)-vector space \( M_{n,p}(D) \) in which every matrix has rank less than or equal to \( r \). Using a new method, we prove that \( \dim F V \leq \max(n, p) rd \) and we characterize the spaces for which equality holds. This extends a famous theorem of Flanders which was known only for fields.

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1 Introduction

Throughout the text, we fix a division ring \( D \), that is a non-trivial ring in which every non-zero element is invertible. We let \( F \) be a subfield of the center \( Z(D) \) of \( D \) and we assume that \( D \) has finite dimension over \( F \).

Let \( n \) and \( p \) be non-negative integers. We denote by \( M_{n,p}(D) \) the set of all matrices with \( n \) rows, \( p \) columns, and entries in \( D \). It has a natural structure of \( F \)-vector space, which we will consider throughout the text. We denote by \( E_{i,j} \) the matrix of \( M_{n,p}(D) \) in which all the entries equal 0, except the one at the \((i, j)\)-spot, which equals 1. The right \( D \)-vector space \( D^n \) is naturally identified

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with the space $M_{n,1}(\mathbb{D})$ of column matrices (with $n$ rows). We naturally identify $M_{n,p}(\mathbb{D})$ with the set of all linear maps from the right vector space $\mathbb{D}^p$ to the right vector space $\mathbb{D}^n$. We have a ring structure on $M_n(\mathbb{D}) := M_{n,n}(\mathbb{D})$ with unity $I_n$, and its group of units is denoted by $GL_n(\mathbb{D})$.

Two matrices $M$ and $N$ of $M_{n,p}(\mathbb{D})$ are said to be equivalent when there are invertible matrices $P \in GL_n(\mathbb{D})$ and $Q \in GL_p(\mathbb{D})$ such that $N = PQM$ (this means that $M$ and $N$ represent the same linear map between right vector spaces over $\mathbb{D}$ under a different choice of bases). This relation is naturally extended to whole subsets of matrices.

The rank of a matrix $M \in M_{n,p}(\mathbb{D})$ is the rank of the family of its columns in the right $\mathbb{D}$-vector space $\mathbb{D}^n$, and it is known that it equals the rank of the family of its rows in the left $\mathbb{D}$-vector space $M_{1,p}(\mathbb{D})$: we denote it by $\text{rk}(M)$. Two matrices of the same size have the same rank if and only if they are equivalent.

Given a non-negative integer $r$, a rank-$r$ subset of $M_{n,p}(\mathbb{D})$ is a subset in which every matrix has rank less than or equal to $r$.

Let $s$ and $t$ be non-negative integers. One defines the compression space

$$\mathcal{R}(s, t) := \left\{ \begin{bmatrix} A & C \\ B & 0 \end{bmatrix}_{(n-s) \times (p-t)} \right| A \in M_{s,t}(\mathbb{D}), B \in M_{n-s,t}(\mathbb{D}), C \in M_{s,p-t}(\mathbb{D}) \right\}.$$ 

It is obviously an $\mathbb{F}$-linear subspace of $M_{n,p}(\mathbb{D})$ and a rank-$s+t$ subset. More generally, any space that is equivalent to a space of that form is called a compression space.

A classical theorem of Flanders \[4\] reads as follows.

**Theorem 1** (Flanders’s theorem). Let $\mathbb{F}$ be a field, and $n, p, r$ be positive integers such that $n \geq p > r$. Let $V$ be a rank-$r$ linear subspace of $M_{n,p}(\mathbb{F})$. Then, $\dim V \leq nr$, and if equality holds then either $V$ is equivalent to $\mathcal{R}(0, r)$, or $n = p$ and $V$ is equivalent to $\mathcal{R}(r, 0)$.

The case when $n \leq p$ can be obtained effortlessly by transposing.

Flanders’s theorem has a long history dating back to Dieudonné \[3\], who tackled the case when $n = p$ and $r = n - 1$ (that is, subspaces of singular matrices). Dieudonné was motivated by the study of semi-linear invertibility preservers on square matrices. Flanders came actually second \[4\] and, due to his use of determinants, he was only able to prove his results over fields with more than $r$ elements (he added the restriction that the field should not be of characteristic 2, but a close examination of his proof reveals that it is unnecessary). The extension to general fields was achieved more than two decades later.
by Meshulam [5]. In the meantime, much progress had been made in the classification of rank-$r$ subspaces with dimension close to the critical one, over fields with large cardinality (see [1] for square matrices, and [2] for the generalization to rectangular matrices): the known theorems essentially state that every large enough rank-$r$ linear subspace is a subspace of a compression space.

This topic has known a recent revival. First, Flanders’s theorem was extended to affine subspaces over all fields [8], and the result was applied to generalize Atkinson and Lloyd’s classification of large rank-$r$ spaces [9]. Flanders’s theorem has also been shown to yield an explicit description of full-rank-preserving linear maps on matrices without injectivity assumptions [6, 10].

In a recent article, Šemrl proved Flanders’s theorem in the case of singular matrices over division algebras that are finite-dimensional over their center [11], and he applied the result to classify the linear endomorphisms of a central simple algebra that preserve invertibility.

Our aim here is to give the broadest generalization of Flanders’s theorem to date. It reads as follows:

**Theorem 2.** Let $\mathbb{D}$ be a division ring and $\mathbb{F}$ be a subfield of its center such that $d := [\mathbb{D} : \mathbb{F}]$ is finite. Let $n, p, r$ be non-negative integers such that $n \geq p \geq r$. Let $\mathcal{V}$ be an $\mathbb{F}$-affine subspace of $M_{n,p}(\mathbb{D})$, and assume that it is a rank-$r$ subset. Then,

$$\dim_{\mathbb{F}} \mathcal{V} \leq dnr.$$  \hspace{1cm} (1)

If equality holds in (1), then:

(a) Either $\mathcal{V}$ is equivalent to $\mathcal{R}(0, r)$;

(b) Or $n = p$ and $\mathcal{V}$ is equivalent to $\mathcal{R}(r, 0)$;

(c) Or $(n, p, r) = (2, 2, 1)$, $\#\mathbb{D} = 2$ and $\mathcal{V}$ is equivalent to the affine space

$$\mathcal{U}_2 := \left\{ \begin{bmatrix} x & 0 \\ y & x + 1 \end{bmatrix} \mid (x, y) \in \mathbb{D}^2 \right\}.$$ 

The case when $n = p$, $r = n - 1$, $\mathbb{F} = \mathbb{Z}(\mathbb{D})$, $\mathbb{D}$ is infinite and $\mathcal{V}$ is a linear subspace is Theorem 2.1 of [11].

As in Flanders’s theorem, the case $n \leq p$ can be deduced from our theorem. Beware however that the transposition does not leave the rank invariant! If we
denote by $\mathbb{D}^{\text{op}}$ the opposite division ring\footnote{The ring $\mathbb{D}^{\text{op}}$ has the same underlying abelian group, and its multiplication is defined as $x \times_{\text{op}} y := yx.$}, then $A \in M_{n,p}(\mathbb{D}) \mapsto A^T \in M_{p,n}(\mathbb{D}^{\text{op}})$ is an $\mathbb{F}$-linear bijection that is rank preserving and that reverses products. Thus, the case $n \geq p$ over $\mathbb{D}^{\text{op}}$ yields the case $n \leq p$ over $\mathbb{D}$.

Taking $\mathbb{D}$ as a finite field and $\mathbb{F}$ as its prime subfield, we obtain the following corollary on additive subgroups of matrices:

**Corollary 3.** Let $\mathbb{F}$ be a finite field with cardinality $q$. Let $n, p, r$ be positive integers such that $n \geq p \geq r$. Let $\mathcal{V}$ be an additive subgroup of $M_{n,p}(\mathbb{F})$ in which every matrix has rank at most $r$. Then, $\#\mathcal{V} \leq q^{nr}$, and if equality holds then either $\mathcal{V}$ is equivalent to $\mathcal{R}(0,r)$, or $n = p$ and $\mathcal{V}$ is equivalent to $\mathcal{R}(r,0)$.

Broadly speaking, the proof of Theorem 2 will revive some of Dieudonné’s original ideas from \cite{3} and will incorporate some key innovations. The main idea is to work by induction over all parameters $n, p, r$, with special focus on the rank 1 matrices in the translation vector space of $\mathcal{V}$. In a subsequent article, this new strategy will be used to improve the classification of large rank-$r$ spaces over fields.

The proof of Theorem 2 is laid out as follows: Section 2 consists of a collection of three basic lemmas. The inductive proof of Theorem 2 is then performed in the final section.

## 2 Basic results

### 2.1 Extraction lemma

**Lemma 4** (Extraction lemma). Let $M = \begin{bmatrix} A & C \\ B & d \end{bmatrix}$ be a matrix of $M_{n,p}(\mathbb{D})$, with $A \in M_{n-1,p-1}(\mathbb{D})$. Assume that $\text{rk}(M) \leq r$ and $\text{rk}(M + E_{n,p}) \leq r$. Then, $\text{rk} A \leq r - 1$.

**Proof.** Assume on the contrary that $\text{rk} A \geq r$. Without loss of generality, we can assume that

$$A = \begin{bmatrix} I_r \\ \mathbb{F}^{(p-1-r) \times (n-1-r)} \end{bmatrix} \begin{bmatrix} ?^{(n-1-r) \times (n-1-r)} \\ ?^{r \times (p-1-r)} \end{bmatrix}.$$
We write $C = \begin{bmatrix} C_1 \\ [?]_{(n-1-r)} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ [?]_{1 \times (p-r-1)} \end{bmatrix}$ with $C_1 \in M_{r,1}(\mathbb{D})$ and $B_1 \in M_{1,r}(\mathbb{D})$. Then, by extracting sub-matrices, we find that for all $\delta \in \{0, 1\}$, the matrix

$$H_\delta := \begin{bmatrix} I_r & C_1 \\ B_1 & d + \delta \end{bmatrix}$$

has rank less than or equal to $r$. Multiplying it on the left with the invertible matrix $\begin{bmatrix} I_r \\ -B_1 \end{bmatrix}_{r \times 1}$, we deduce that

$$\forall \delta \in \{0, 1\}, \quad \text{rk} \begin{bmatrix} I_r \\ [0]_{1 \times r} \end{bmatrix}_{r \times 1} C_1 = (d + \delta - B_1 C_1) \leq r.$$

This would yield

$$\forall \delta \in \{0, 1\}, \quad d + \delta - B_1 C_1 = 0,$$

which is absurd. Thus, $\text{rk} A < r$, as claimed.

### 2.2 Range-compatible homomorphisms on $M_{n,p}(\mathbb{D})$

**Definition 1.** Let $U$ and $V$ be right vector spaces over $\mathbb{D}$, and $S$ be a subset of $L(U, V)$, the set of all linear maps from $U$ to $V$. A map $F : S \to V$ is called **range-compatible** whenever

$$\forall s \in S, \quad F(s) \in \text{Im} s.$$

The concept of range-compatibility was introduced and studied in [7]. Here, we shall need the following basic result:

**Proposition 5.** Let $n, p$ be non-negative integers with $n \geq 2$, and $F : M_{n,p}(\mathbb{D}) \to \mathbb{D}^n$ be a range-compatible (group) homomorphism. Then, $F : M \mapsto MX$ for some $X \in \mathbb{D}^p$.

Of course here $M_{n,p}(\mathbb{D})$ is naturally identified with the group of all linear mappings from $\mathbb{D}^p$ to $\mathbb{D}^n$.

**Proof.** We start with the case when $p = 1$. Then, $F$ is simply an endomorphism of $\mathbb{D}$ such that $F(X)$ is (right-)collinear to $X$ for all $X \in \mathbb{D}^n$. Then, it is well-known that $F$ is a right-multiplication map: we recall the proof for the sake of
completeness. For every non-zero vector $X \in \mathbb{D}^n$, we have a scalar $\lambda_X \in \mathbb{D}$ such that $F(X) = X \lambda_X$. Given non-collinear vectors $X$ and $Y$ in $\mathbb{D}^n$, we have

$$X \lambda_X + Y \lambda_Y = (X + Y) \lambda_{X + Y} = F(X + Y) = F(X) + F(Y) = X \lambda_X + Y \lambda_Y,$$

and hence $\lambda_X = \lambda_{X + Y} = \lambda_Y$. Given collinear non-zero vectors $X$ and $Y$ of $\mathbb{D}^n$, we can find a vector $Z$ in $\mathbb{D}^n \setminus \{X\} \setminus \{Y\}$ (since $n \geq 2$), and it follows from the first step that $\lambda_X = \lambda_Z = \lambda_Y$. Thus, we have a scalar $\lambda \in \mathbb{D}$ such that $F(X) = X \lambda$ for all $X \in \mathbb{D}^n \setminus \{0\}$, which holds also for $X = 0$.

Now, we extend the result. As $F$ is additive there are group endomorphisms $F_1, \ldots, F_p$ of $\mathbb{D}^n$ such that

$$F : [C_1 \cdots C_p] \mapsto \sum_{k=1}^p F_k(C_k).$$

By applying the range-compatibility assumption to matrices with only one non-zero column, we see that each map $F_k$ is range-compatible. This yields scalars $\lambda_1, \ldots, \lambda_p$ in $\mathbb{D}$ such that

$$F : [C_1 \cdots C_p] \mapsto \sum_{k=1}^p C_k \lambda_k.$$

Thus, with $X := \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix}$ we find that $F : M \mapsto MX$, as claimed.

2.3 On the rank 1 matrices in the translation vector space of a rank-$r$ affine subspace

**Notation 2.** Let $S$ be a subset of $M_{n,p}(\mathbb{D})$ and $H$ be a linear hyperplane of the $\mathbb{D}$-vector space $\mathbb{D}^p$. We define

$$S_H := \{M \in S : H \subset \text{Ker} \ M\}.$$ 

Note that $S_H$ is an $\mathbb{F}$-linear subspace of $S$ whenever $S$ is an $\mathbb{F}$-linear subspace of $M_{n,p}(\mathbb{D})$.

Let $S$ be an $\mathbb{F}$-affine rank-$k$ space, with translation vector space $S$. In our proof of Flanders’s theorem, we shall need to find a hyperplane $H$ such that the dimension of $S_H$ is small. This will be obtained through the following key lemma.
Lemma 6. Let \( V \) be an \( \mathbb{F} \)-affine rank-\( r \) subspace of \( M_{n,p}(\mathbb{D}) \), with \( p > r > 0 \). Denote by \( V \) its translation vector space. Assume that \( \dim_{\mathbb{F}} V_H \geq dr \) for every linear hyperplane \( H \) of the \( \mathbb{D} \)-vector space \( \mathbb{D}^p \). Then, \( V \) is equivalent to \( \mathcal{R}(r,0) \).

Proof. Denote by \( s \) the maximal rank among the elements of \( V \). Let us consider a matrix \( A \) of \( V \) with rank \( s \). Given a linear hyperplane \( H \) of \( \mathbb{D}^p \) that does not include \( \text{Ker} A \), let us set \( T_H := \sum_{M \in V_H} \text{Im} M \).

We claim that \( T_H = \text{Im} A \). To support this, we lose no generality in assuming that \( A = J_s := \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} \) and \( H = \mathbb{D}^{p-1} \times \{0\} \).

The elements of \( V_H \) are the matrices of \( V \) whose columns are all zero with the possible exception of the last one. For an arbitrary element \( N \) of \( V_H \), we must have \( \text{rk}(A + N) \leq s \) as \( A + N \) belongs to \( V \). This shows that the last \( n - s \) rows of \( N \) must equal zero. It follows that \( \dim_{\mathbb{F}} V_H \leq ds \), and our assumptions show that we must have \( s = r \) and \( \dim_{\mathbb{F}} V_H = dr \). In turn, this shows that \( V_H \) is the set of all matrices of \( M_{n,p}(\mathbb{D}) \) with non-zero entries only in the last column and in the first \( r \) rows, and it is then obvious that \( T_H = \mathbb{D}^{r} \times \{0\} = \text{Im} A \).

Now, let us get back to the general case. Without loss of generality, we can assume that \( V \) contains \( A = J_r \). Then, taking \( H = \mathbb{D}^{p-1} \times \{0\} \) shows that \( V \) contains \( E_1,p, \ldots, E_{r,p} \). Given \((i,j) \in [1,r] \times [1,p-1] \), taking \( H = \{ (x_1, \ldots, x_p) \in \mathbb{D}^p : x_j = x_p \} \) shows that \( V \) contains \( E_{i,j} - E_{i,p} \), and as the \( \mathbb{F} \)-vector space \( V \) also contains \( E_{i,p} \) we conclude that it contains \( E_{i,j} \). It follows that \( \mathcal{R}(r,0) \subset V \). As \( J_r \in V \), we deduce that \( 0 \in V \), and hence \( V = V \).

Finally, assume that some matrix \( N \) of \( V \) has a non-zero row among the last \( n - r \) ones: then, we know from \( \mathcal{R}(r,0) \subset V \) that \( V \) contains every matrix of \( M_{n,p}(\mathbb{D}) \) with the same last \( n - r \) rows as \( N \); at least one such matrix has rank greater than \( r \), obviously. Thus, \( V \subset \mathcal{R}(r,0) \) and we conclude that \( V = \mathcal{R}(r,0) \).

\[ \square \]

3 The proof of Theorem 2

We are now ready to prove Theorem 2. We shall do this by induction over \( n, p, r \). The case \( r = 0 \) is obvious, and so is the case \( r = p \).
Assume now that \(1 \leq r < p\). Let \(V\) be an \(F\)-affine subspace of \(M_{n,p}(D)\) in which every matrix has rank at most \(r\). Denote by \(V\) its translation vector space.

If \(V\) is equivalent to \(R(r,0)\), then it has dimension \(dpr\) over \(F\), which is less than or equal to \(dnr\). Moreover, if equality occurs then \(n = p\) and we have conclusion (b) in Theorem 2. In the rest of the proof, we assume that \(V\) is inequivalent to \(R(r,0)\).

Thus, Lemma 6 yields a linear hyperplane \(H\) of \(D^p\) such that \(\dim_F V_H < dr\). Without loss of generality, we can assume that \(H = D^{p-1} \times \{0\}\). From there, we split the discussion into two subcases, whether \(V_H\) contains a non-zero matrix or not.

### 3.1 Case 1: \(V_H \neq \{0\}\).

Without further loss of generality, we can assume that \(V_H\) contains \(E_{n,p}\). Let us split every matrix \(M\) of \(V\) into

\[
M = \begin{bmatrix}
K(M) & [?]_{(n-1) \times 1}
\end{bmatrix}
\]

with \(K(M) \in M_{n-1,p-1}(D)\).

Then, by the extraction lemma, we see that \(K(V)\) is an \(F\)-affine rank-\(r-1\) subspace of \(M_{n-1,p-1}(D)\). By induction,

\[
\dim_F K(V) \leq d(n-1)(r-1).
\]

On the other hand, by the rank theorem

\[
\dim_F V \leq \dim_F K(V) + d(p-1) + \dim_F V_H.
\]

Hence,

\[
\dim_F V < d(n-1)(r-1) + d(p-1) + dr = d(nr + p - n) \leq dnr.
\]

Thus, in this situation we have proved inequality (1), and equality cannot occur.

### 3.2 Case 2: \(V_H = \{0\}\).

Here, we split every matrix \(M\) of \(V\) into

\[
M = \begin{bmatrix}
A(M) & [?]_{n \times 1}
\end{bmatrix}
\]

with \(A(M) \in M_{n,p-1}(D)\).
Then, $A(V)$ is an $\mathbb{F}$-affine rank-$r$ subspace of $M_{n,p-1}(\mathbb{D})$, and as $V_H = \{0\}$ we have
\[
\dim_{\mathbb{F}} A(V) = \dim_{\mathbb{F}} V.
\]
By induction we have
\[
\dim_{\mathbb{F}} A(V) \leq dn_r
\]
and hence
\[
\dim_{\mathbb{F}} V \leq dn_r.
\]
Assume now that $\dim_{\mathbb{F}} V = dn_r$, so that $\dim_{\mathbb{F}} A(V) = dn_r$. As $n > p - 1$, we know by induction that $A(V)$ is equivalent to $R(0, r)$ (cases (b) and (c) in Theorem 2 are barred). Without loss of generality, we may now assume that $A(V) = R(0, r)$. Then, as $V_H = \{0\}$ we have an $\mathbb{F}$-affine map $F : M_{n,r}(\mathbb{D}) \to \mathbb{D}^n$ such that
\[
V = \left\{ \begin{bmatrix} N & [0]_{n \times (p-1-r)} & F(N) \end{bmatrix} \mid N \in M_{n,r}(\mathbb{D}) \right\}.
\]

Claim 1. The map $F$ is range-compatible unless $\#\mathbb{D} = 2$ and $(n, p, r) = (2, 2, 1)$.

Proof. Throughout the proof, we assume that we are not in the situation where $\#\mathbb{D} = 2$, $n = p = 2$ and $r = 1$.

Set $G_1 := \mathbb{D}^{n-1} \times \{0\}$. Let us prove that $F(N) \in G_1$ for all $N \in M_{n,r}(\mathbb{D})$ such that $\text{Im} N \subset G_1$. We have an $\mathbb{F}$-affine mapping $\gamma : M_{n-1,r}(\mathbb{D}) \to \mathbb{D}^n$ such that
\[
\forall R \in M_{n-1,r}(\mathbb{D}), \quad F \left( \begin{bmatrix} R & [0]_{1 \times r} \end{bmatrix} \right) = \begin{bmatrix} ? & \gamma(R) \end{bmatrix}_{(n-1) \times 1}
\]
and we wish to show that $\gamma$ vanishes everywhere on $M_{n-1,r}(\mathbb{D})$. Assume that this is not true and choose a non-zero scalar $a \in \mathbb{D} \setminus \{0\}$ in the range of $\gamma$. Then $W := \gamma^{-1}\{a\}$ is an $\mathbb{F}$-affine subspace of $M_{n-1,r}(\mathbb{D})$ with codimension at most $d$. Moreover, as $\mathcal{V}$ is a rank-$r$ space it is obvious that every matrix in $W$ has rank at most $r - 1$. Thus, by induction we know that $\dim_{\mathbb{F}} W \leq d(n - 1)(r - 1)$. This leads to $(n - 1)(r - 1) \geq (n - 1)r - 1$, and hence $n \leq 2$. Assume that $n = 2$, so that $p = 2$ and $r = 1$. Then, $\#\mathbb{D} \neq 2$. Moreover, we must have $\dim_{\mathbb{F}} W \leq d(n - 1)(r - 1) = 0$ and hence $\gamma$ is one-to-one. As $\mathbb{D}$ has more than two elements this yields a non-zero scalar $b \in \mathbb{D}$ such that $\gamma(b) \neq 0$, yielding a rank 2 matrix in $\mathcal{V}$. Therefore, in any case we have found a contradiction. Thus, $\gamma$ equals 0.

We conclude that $F(N) \in G_1$ for all $N \in M_{n,r}(\mathbb{D})$ such that $\text{Im} N \subset G_1$. Using row operations, we generalize this as follows: for every linear hyperplane
G of the right \( \mathbb{D} \)-vector space \( \mathbb{D}^n \) and every matrix \( N \in M_{n,r}(\mathbb{D}) \), we have the implication

\[
\text{Im} \ N \subset G \Rightarrow F(N) \in G.
\]

Let then \( N \in M_{n,r}(\mathbb{D}) \). We can find linear hyperplanes \( G_1, \ldots, G_k \) of the right \( \mathbb{D} \)-vector space \( \mathbb{D}^n \) such that \( \text{Im} \ N = \bigcap_{i=1}^k G_i \), and hence

\[
F(N) \in \bigcap_{i=1}^k G_i = \text{Im} \ N.
\]

Thus, \( F \) is range-compatible.

Now we can conclude. Assume first that we are not in the special situation where \( (n, p, r, \#\mathbb{D}) = (2, 2, 1, 2) \). Then, we have just seen that \( F \) is range-compatible. In particular this shows that \( F(0) = 0 \), and as \( F \) is \( \mathbb{F} \)-affine we obtain that \( F \) is a group homomorphism. Proposition 5 yields that \( F : N \mapsto NX \) for some \( X \in \mathbb{D}^r \). Setting

\[
P := \begin{bmatrix}
I_r & [0]_{r \times (p-1-r)} & -X \\
[0]_{(p-1-r) \times r} & I_{p-1-r} & [0]_{(p-1-r) \times 1} \\
[0]_{1 \times r} & [0]_{1 \times (p-1-r)} & 1
\end{bmatrix},
\]

we see that \( P \) is invertible and that \( \mathcal{V} P = \mathcal{R}(0, r) \), which completes the proof in that case.

Assume finally that \( (n, p, r, \#\mathbb{D}) = (2, 2, 1, 2) \). Then,

\[
F : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \alpha x + \beta y + \gamma \\ \delta x + \epsilon y + \eta \end{bmatrix}
\]

for fixed scalars \( \alpha, \beta, \gamma, \delta, \epsilon, \eta \). As every matrix in \( \mathcal{V} \) is singular, we deduce that, for all \( (x, y) \in \mathbb{D}^2 \),

\[
\begin{vmatrix} x & \alpha x + \beta y + \gamma \\ y & \delta x + \epsilon y + \eta \end{vmatrix} = 0,
\]

and hence

\[
(\epsilon + \alpha)xy + (\delta + \eta)x + (\beta + \gamma)y = 0.
\]

Thus, \( \epsilon = \alpha \), \( \delta = \eta \) and \( \beta = \gamma \). Performing the column operation \( C_2 \leftarrow C_2 - \alpha C_1 \) on \( \mathcal{V} \), we see that no generality is lost in assuming that \( \alpha = 0 \). Then,

\[
\mathcal{V} = \left\{ \begin{bmatrix} x & \beta y + 1 \\ y & \delta x + 1 \end{bmatrix} \mid (x, y) \in \mathbb{D}^2 \right\}
\]

10
and there are four options to consider:

- If $\beta = \delta = 0$, then $V = \mathcal{R}(0, 1)$;
- If $\beta = 0$ and $\delta = 1$, then $V = U_2$;
- If $\beta = 1$ and $\delta = 0$, then the row swap $L_1 \leftrightarrow L_2$ takes $V$ to $U_2$;
- Finally, if $\beta = \delta = 1$, then the column operation $C_2 \leftarrow C_2 + C_1$ followed by the row operation $L_1 \leftarrow L_1 + L_2$ takes $V$ to $U_2$.

This completes the proof of Theorem 2.

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