Vector cross product in n-dimensional vector space

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Abstract

The definition of vector cross product (VCP) introduced by Eckmann only exists in the three- and the seven-dimensional vector space. In this paper, according to the orthogonal completeness, magnitude of basis vector cross product and all kinds of combinations of basis vector \( \hat{e}_i \), the generalized definition of VCP in the odd n-dimensional vector space is given by introducing a cross term \( X_{AB} \). In addition, the definition is validated by reducing the generalization definition to the fundamental three- and seven-dimensional vector space.

Keywords: vector space; vector cross product (VCP); the VCP of n-dimensional vector
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1 Introduction

Vector is an important mathematical tool, commonly used in physical field, such as vector inner product, vector cross product (VCP), and vector sensor product. It is well known that vector inner product and tensor product can be existed in n-dimensional Euclidian space (vector space). But for the VCP, it only exists in three- and seven-dimensional vector space [1-4]. The VCP has wide application in physics fields in three-dimensional space, e.g, torque, angular momentum and so on, and in seven-dimensional VCP have been applied at self-dual Yang-Mills fields [5-6] and Supergravity research [7], etc. However, the VCP defined by Eckmann [1] is severely restricted except the three- and seven-dimensional VCP. Consequently, the extension of the definition of VCP to n-dimensional space has important physical significance.

In this paper, a general definition of the VCP of two vectors on odd n-dimensional space is given by employing the method of combination in terms of orthogonal completeness and magnitude of basis vector cross product and all kinds of combinations of basis vector \( \hat{e}_i \). The VCP \( \vec{A} \times \vec{B} \) in odd n-dimensional space satisfies the following two conditions: (i) \( (\vec{A} \times \vec{B}, \vec{A} \) or \( \vec{B} \) = 0; (ii) \( ||\vec{A} \times \vec{B}||^2 = ||\vec{A}||^2||\vec{B}||^2 - (\vec{A}, \vec{B})^2 + X_{AB} \). When \( X_{AB} = 0 \), the generalized definition of VCP corresponds to the three- and seven-dimensional VCP defined by Eckmann [1].

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2 The generalized definition of the VCP in odd n-dimensional vector space

Let V denote an n-dimensional vector space over the real numbers and (,) denote the ordinary (positive definite) vector inner product. The VCP $\vec{A} \times \vec{B}$ of any two vectors on V has been defined by B. Eckmann [1] satisfying the following axioms

\[(\vec{A} \times \vec{B}, \vec{A} \text{ or } \vec{B}) = 0, \tag{1}\]
\[\| \vec{A} \times \vec{B} \|^2 = \| \vec{A} \|^2 \| \vec{B} \|^2 - (\vec{A}, \vec{B})^2 \tag{2}\]

Considering arbitrary two vectors $\vec{A} = a^i \hat{e}_i$ and $\vec{B} = b^j \hat{e}_j$ in n-dimensional vector space, $\hat{e}_i$ and $\hat{e}_j$ are basis vectors from the given orthogonal coordinate system, $a^i$ and $b^j$ are vector components corresponding to $\vec{A}$ and $\vec{B}$, and the Einstein summation convention are adopted.

The VCP $\vec{A} \times \vec{B}$ can be expressed as

\[\vec{A} \times \vec{B} = a^i \hat{e}_i \times b^j \hat{e}_j = a^i b^j \hat{e}_i \times \hat{e}_j \tag{3}\]

Obviously, the magnitude of cross product is determined by these vector components $a^i, b^j$ and the direction is determined by basis vectors $\hat{e}_i \times \hat{e}_j$.

In the following, a generalized definition of VCP is presented based on the orthogonal completeness, magnitude of VCP and all kinds of combinations of basis vector.

Firstly, orthogonal completeness of the VCP requires the VCP only exist in an odd n-dimensional space.

As $\langle \vec{A} \times \vec{B}, \vec{A} \text{ or } \vec{B} \rangle = 0$, the cross product of any two vectors is always perpendicular to both of the vectors being multiplied and a plane containing them (orthogonality of the VCP) and $\hat{e}_i \times \hat{e}_j$ of any two basis vectors must be equal to another basis vector $\hat{e}_k$, i.e., $\hat{e}_i \times \hat{e}_j = \pm \hat{e}_k$ (completeness of the VCP).

Based on the definition of $\hat{e}_i \times \hat{e}_j = \hat{e}_k$, a cross product $\hat{e}_i \times \hat{e}_j$ of any two basis vectors is equivalent to a 2-combination of two basis vectors $\hat{e}_i$ and $\hat{e}_j$. There are n basis vectors in n-dimensional vector space, the number of 2-combination is that the number of combination of n basis vectors taken 2 vectors at a time without repetitions. The number of 2-combination of arbitrary two basis vectors in an n-dimensional vector space is $C^2_n = \frac{1}{2} n(n-1)$.

Taking the equality of each basis vector into account, so $C^2_n$ should be averagely distributed to each one basis vector, let K (arbitrary integer) denote the number of 2-combination which is averagely distributed to each one basis vector. The number K is determined by

\[K = \frac{C^2_n}{n} = \frac{1}{2} (n-1) \tag{4}\]

then

\[n = 2K + 1. \tag{5}\]

So the VCP of two vectors there only exist in an odd n-dimensional space.

Secondly, the definition of magnitude of the VCP in Eq.(2) can be generalized to

\[\| \vec{A} \times \vec{B} \|^2 = \| \vec{A} \|^2 \| \vec{B} \|^2 - (\vec{A}, \vec{B})^2 + X_{AB} \tag{6}\]
where $X_{AB}$ is called cross item and it can be expressed as

$$X_{AB} = a_i b_j a^i b^j X_{ij} = a_i b_j a^i b^m [T_{lm} + \delta_i^l \delta_j^m - \delta_i^m \delta_j^l]$$

(7)

$$T_{lm}^i = ((\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k) (\hat{e}_i \times \hat{e}_m) \cdot \hat{e}_k)$$

(8)

where $T_{ij}^{lm}$ is a sign function and $((\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k)$ denote vector inner product of $(\hat{e}_i \times \hat{e}_j)$ and $\hat{e}_k$.

Subsequently, the generalized definition and calculation formula of the VCP in an odd n-dimensional space will be presented.

**Proposition 1:** The VCP $\vec{A} \times \vec{B}$ of any two vectors on an odd n-dimensional space satisfy the following generalized axioms:

1. $(\vec{A} \times \vec{B}, \vec{A}$ or $\vec{B}) = 0$ (9)
2. $\|\vec{A} \times \vec{B}\|^2 = \|\vec{A}\|^2 \|\vec{B}\|^2 - (\vec{A}, \vec{B})^2 + X_{AB}$. (10)

**Proposition 2:** The calculation of any two VCP $\vec{A} \times \vec{B}$ can be expressed by tensor

$$\vec{A} \times \vec{B} = a^i b^j (\vec{e}_i \times \vec{e}_j) = a^i b^j L_{ij} \hat{e}^k$$

(11)

$$L_{ij} \hat{e}^k = (\vec{e}_i \times \vec{e}_j) \cdot \hat{e}^k$$

(12)

where $L_{ij} \hat{e}^k = (\pm 1, 0)$ is a sign function (generalized Levi-Civita symbol) on n-dimensional space, which is determined by a fixed algorithm of cross product of basis vector, the algorithm will be discussed in section 3.

3  Algorithm of VCP in an odd n-dimensional space

As you know, there is only an algorithm of VCP in 3-dimensional space. Although the definition of the VCP has been extended to an odd n-dimensional space ($n > 3$), the algorithm of the VCP is not unique. Owing to the diversity of the combination of basis vector, there are many kinds of algorithm of the VCP in odd n-dimensional space. So-called an algorithm depend on a calculation rule. In the following, it will give a detailed statement about the algorithm of the VCP in an odd n-dimensional space.

3.1 Algorithm of the VCP in 5-dimensional vector space

Obviously, there are 5 basis vectors ($\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5$) in 5-dimensional vector space, one of the basis vectors can be expressed by the cross product of the other two basis vector. From Eq. (4), one can find $k = 2$, one of the basis vectors can be expressed by two 2-combination of basis vector. As is demonstrated in Table 1, e.g., the basis vector $\hat{e}_1$ can be expressed by $\hat{e}_2 \times \hat{e}_3$ (23) and $\hat{e}_4 \times \hat{e}_5$ (45). Here, a double-digit is used to denote the cross product of two basis vectors for simplicity. The (23, 45) of two 2-combination of basis vector is called one kind of distributive combination form, there are 3 kinds of different distributive combination forms under each basis vector $\hat{e}_i$. For example, there are 3 kinds of distributive combination forms (23, 45), (24, 35) and (25, 34) under $\hat{e}_1$ in Table 1.
Table 1. Three kinds of distributive combination forms of basis vector under \( \hat{e}_i \) in 5-dimensional space

| kind | \( \hat{e}_1 \) | \( \hat{e}_2 \) | \( \hat{e}_3 \) | \( \hat{e}_4 \) | \( \hat{e}_5 \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 23, 45          | 15, 34          | 12, 45          | 13, 25          | 14, 23          |
|      | \( \hat{e}_2 \times \hat{e}_3, \hat{e}_4 \times \hat{e}_5 \) | \( \hat{e}_5 \times \hat{e}_1, \hat{e}_3 \times \hat{e}_4 \) | \( \hat{e}_1 \times \hat{e}_2, \hat{e}_5 \times \hat{e}_4 \) | \( \hat{e}_1 \times \hat{e}_3, \hat{e}_5 \times \hat{e}_2 \) | \( \hat{e}_1 \times \hat{e}_4, \hat{e}_2 \times \hat{e}_3 \) |
| 2    | 24, 35          | 14, 35          | 14, 25          | 12, 35          | 12, 34          |
| 3    | 25, 34          | 13, 45          | 15, 24          | 15, 23          | 13, 24          |

Table 2. Six kinds of algorithm of cross product in 5-dimensional space

| algorithm | \( \hat{e}_1 \) | \( \hat{e}_2 \) | \( \hat{e}_3 \) | \( \hat{e}_4 \) | \( \hat{e}_5 \) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1         | 23, 45          | 15, 34          | 14, 25          | 13, 25          | 12, 34          |
| 2         | 23, 45          | 14, 35          | 14, 25          | 12, 35          | 13, 24          |
| 3         | 24, 35          | 13, 45          | 14, 25          | 15, 23          | 12, 34          |
| 4         | 24, 35          | 15, 34          | 12, 45          | 13, 25          | 14, 23          |
| 5         | 25, 34          | 13, 45          | 15, 24          | 12, 35          | 14, 23          |
| 6         | 25, 34          | 14, 35          | 12, 45          | 15, 23          | 13, 24          |

Furthermore, a pair of double-digit (a distributive combination forms) from each column in Table 1 is taken to constitute a kind of calculation rule, and the calculation rule demands all double-digit of the five pair of double-digit which be extracted is different. Then, let these different double-digit arrange a row to represent a calculation rule. Accordingly, six kinds of different calculation rules of basis vector are obtained in Table 2, that is to say, the VCP in five-dimensional space have six sorts of different algorithms. Thus, One can define a kind of cross product algorithm of basis vector by selecting combination form of a row double-digit from Table 2. For example, the relation of cross product of basis vector from 3th row in Table 2 can be expressed as

\[
\hat{e}_2 \times \hat{e}_4 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_5 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \quad \hat{e}_4 \times \hat{e}_5 = \hat{e}_2, \\
\hat{e}_4 \times \hat{e}_1 = \hat{e}_3, \quad \hat{e}_5 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_5 \times \hat{e}_1 = \hat{e}_4, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_4, \\
\hat{e}_1 \times \hat{e}_2 = \hat{e}_5, \quad \hat{e}_3 \times \hat{e}_4 = \hat{e}_5. \tag{13}
\]

The above relation of cross product of basis vector shows a kind of fixed algorithm of the VCP. Under right-handed orthogonal coordinate frame (or left-handed orthogonal coordinate frame), the cross product of basis vector have to satisfy right-handed rotation rule. So the double-digit in Table 2 must be written into cross product of two basis vector, e.g., “13” under \( \hat{e}_2 \) in Table 2 must be written as \( \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \), or \( \hat{e}_1 \times \hat{e}_3 = -\hat{e}_2 \). Then, the relation of cross product of basis vector can be expressed as

\[
\hat{e}_i \times \hat{e}_j = L_{ij,k} \hat{e}_k, \quad \text{where repeated index } k \text{ does not sum} \tag{14}
\]

When \((ij,k)\) is even permutation, \(L_{ij,k} = 1\), when \((ij,k)\) is odd permutation, \(L_{ij,k} = -1\), when \((ij,k)\) is else permutation, \(L_{ij,k} = 0\). Intuitively, a simple graphical expression of the right-handed rotation is given in Fig. 1. to determine the sign of \(L_{ij,k}\).
In terms of the relation of basis vectors from Eq. (14), the VCP \( \vec{A} \times \vec{B} \) of any two vectors \( \vec{A} \) and \( \vec{B} \) can be computed by the following

\[
\vec{A} \times \vec{B} = a^i b^j (\vec{e}_i \times \vec{e}_j) = a^i b^j L_{ijk} \hat{e}^k = a^i b^j \hat{e}^k (\vec{e}_i \times \vec{e}_j) \cdot \hat{e}^k
\]

\[
= \begin{bmatrix}
  a_2 & a_3 & a_5 \\
  b_2 & b_3 & b_5
\end{bmatrix} \hat{e}_1 + \begin{bmatrix}
  a_3 & a_1 \\
  b_3 & b_1
\end{bmatrix} \hat{e}_2 + \begin{bmatrix}
  a_4 & a_5 \\
  b_4 & b_5
\end{bmatrix} \hat{e}_3 + \begin{bmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{bmatrix} \hat{e}_4 + \begin{bmatrix}
  a_2 & a_4 \\
  b_2 & b_4
\end{bmatrix} \hat{e}_5
\]

(15)

let \( X_{\alpha\beta} = \begin{vmatrix} a_{\alpha} & a_{\beta} \\ b_{\alpha} & b_{\beta} \end{vmatrix} \) (determinant), then \( \vec{A} \times \vec{B} \) can be expressed as

\[
\vec{A} \times \vec{B} = [X_{24} + X_{35}] \hat{e}_1 + [X_{31} + X_{45}] \hat{e}_2 + [X_{41} + X_{52}] \hat{e}_3 + [X_{23} + X_{51}] \hat{e}_4 + [X_{12} + X_{34}] \hat{e}_5
\]

(16)

Moreover, we can also prove \( \| \vec{A} \times \vec{B} \|^2 = \| \vec{A} \|^2 \| \vec{B} \|^2 - (\vec{A}, \vec{B})^2 + X_{AB} \). where \( X_{AB} = a^i b^j a^l b^m \lambda_{ij}^{lm} \)

is as follows

\[
X_{AB} = a^i b^j a^l b^m \lambda_{ij}^{lm} = a^i b^j a^l b^m [T_{ij}^{lm} + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j]
\]

\[
= a^i b^j a^l b^m \begin{bmatrix}
  (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_1 (\hat{e}^l \times \hat{e}^m) \cdot \hat{e}^1 + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j \\
  + (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_2 (\hat{e}^l \times \hat{e}^m) \cdot \hat{e}^2 + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j \\
  + (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_3 (\hat{e}^l \times \hat{e}^m) \cdot \hat{e}^3 + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j \\
  + (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_4 (\hat{e}^l \times \hat{e}^m) \cdot \hat{e}^4 + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j \\
  + (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_5 (\hat{e}^l \times \hat{e}^m) \cdot \hat{e}^5 + \delta_i^m \delta_j^l - \delta^m_i \delta^l_j
\end{bmatrix}
\]

\[
= 2\left([a_2 b_4 - b_2 a_4](a_3 b_5 - b_3 a_5) + (a_3 b_1 - b_3 a_1)(a_4 b_5 - b_4 a_5) + (a_4 b_1 - b_4 a_1)(a_5 b_2 - b_5 a_2) + (a_2 b_3 - b_2 a_3)(a_3 b_4 - b_3 a_4) + (a_1 b_2 - b_1 a_2)(a_3 b_4 - b_3 a_4)\right)
\]

\[
= 2[X_{24}X_{35} + X_{31}X_{45} + X_{41}X_{52} + X_{23}X_{51} + X_{12}X_{34}] \neq 0
\]

(17)

### 3.2 Algorithm of the VCP in 7-dimensional vector space

Similarly, we can find \( k = 3 \) in 7-dimensional vector space using Eq. (4). As is demonstrated in Table 3, each of basis vector \( e_i \) includes three 2-combination of basis vector and there are 15 kinds of different distributive combination forms.

One can take a combinatorial forms which include 3 double-digit from each column that include 15 kinds of combinatorial forms under each basis vector \( \hat{e}_i \), and these 7 kinds of different 3 double-digit also arrange a row. By means of computer, there are 6240 kinds of no repeating double-digit combinatorial forms. Namely, there are 6240 kinds of algorithms of basis vector in 7-dimensional vector space. Next, we show 30 kinds of no repeating double-digit combinatorial forms in Table 4. From the basis vector of 11th row and the 20th row in Table 4, it is just the previous rule of VCP in 7-dimensional vector space defined by B. Eckmann.

For the algorithm of the 11th row in Table 4, the relation of cross product of basis vectors are expressed as
Table 3. Fifteen kinds of distributive combinatorial forms of basis vector under $\hat{e}_i$ in 7-dimensional space

| kind | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| 1    | 24.37,56 | 14.35,67 | 17.25,46 | 12.36,57 | 16.23,47 | 15.27,34 | 13.26,45 |
| 2    | 24.35,67 | 14.36,57 | 17.24,56 | 12.35,67 | 16.24,37 | 15.23,47 | 13.25,46 |
| 3    | 24.36,57 | 14.37,56 | 17.26,45 | 12.37,56 | 16.27,34 | 15.24,37 | 13.24,56 |
| 4    | 23.45,67 | 13.45,67 | 16.24,56 | 13.26,57 | 17.24,36 | 17.24,35 | 14.25,36 |
| 5    | 23.46,57 | 13.46,57 | 16.25,47 | 13.25,67 | 17.23,46 | 17.23,45 | 14.23,56 |
| 6    | 23.47,56 | 13.56,47 | 16.27,45 | 13.27,56 | 17.26,34 | 17.25,34 | 14.26,35 |
| 7    | 25.37,46 | 15.34,67 | 15.24,67 | 15.26,37 | 14.26,37 | 14.25,37 | 15.26,34 |
| 8    | 25.34,67 | 15.36,47 | 15.26,47 | 15.23,67 | 14.23,67 | 14.23,57 | 15.23,46 |
| 9    | 25.36,47 | 15.37,46 | 15.27,46 | 15.27,36 | 14.27,36 | 14.27,35 | 15.24,36 |
| 10   | 26.34,57 | 16.34,57 | 14.25,67 | 14.25,37 | 13.24,67 | 13.24,57 | 16.24,35 |
| 11   | 26.35,47 | 16.35,47 | 14.26,57 | 14.26,37 | 13.25,67 | 13.25,47 | 16.25,35 |
| 12   | 26.37,46 | 16.37,45 | 14.27,56 | 14.27,35 | 13.27,46 | 13.27,45 | 16.25,34 |
| 13   | 27.34,56 | 17.34,56 | 12.45,67 | 17.25,36 | 12.34,67 | 12.34,57 | 12.34,56 |
| 14   | 27.35,46 | 17.35,46 | 12.46,57 | 17.23,56 | 12.36,47 | 12.35,47 | 12.35,46 |
| 15   | 27.36,45 | 17.36,45 | 12.47,56 | 17.26,35 | 12.37,46 | 12.37,45 | 12.36,45 |

\[
\hat{e}_2 \times \hat{e}_4 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_7 = \hat{e}_1, \quad \hat{e}_5 \times \hat{e}_6 = \hat{e}_1,
\]
\[
\hat{e}_4 \times \hat{e}_1 = \hat{e}_2, \quad \hat{e}_3 \times \hat{e}_5 = \hat{e}_2, \quad \hat{e}_6 \times \hat{e}_7 = \hat{e}_2,
\]
\[
\hat{e}_4 \times \hat{e}_6 = \hat{e}_3, \quad \hat{e}_5 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_7 \times \hat{e}_1 = \hat{e}_3,
\]
\[
\hat{e}_5 \times \hat{e}_7 = \hat{e}_4, \quad \hat{e}_6 \times \hat{e}_3 = \hat{e}_4, \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_4,
\]
\[
\hat{e}_6 \times \hat{e}_1 = \hat{e}_5, \quad \hat{e}_7 \times \hat{e}_4 = \hat{e}_5, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_5,
\]
\[
\hat{e}_7 \times \hat{e}_2 = \hat{e}_6, \quad \hat{e}_1 \times \hat{e}_5 = \hat{e}_6, \quad \hat{e}_3 \times \hat{e}_4 = \hat{e}_6,
\]
\[
\hat{e}_1 \times \hat{e}_3 = \hat{e}_7, \quad \hat{e}_2 \times \hat{e}_6 = \hat{e}_7, \quad \hat{e}_4 \times \hat{e}_5 = \hat{e}_7.
\] (18)

For any two vectors $\vec{A} = a^i\hat{e}_i$ and $\vec{B} = b^j\hat{e}_j$ in 7-dimensional vector space, the $\vec{A} \times \vec{B}$ can be expressed as

\[
\vec{A} \times \vec{B} = [X_{24} + X_{37} + X_{56}]\hat{e}_1 + [X_{41} + X_{35} + X_{67}]\hat{e}_2 + [X_{71} + X_{52} + X_{46}]\hat{e}_3 \\
+ [X_{12} + X_{63} + X_{57}]\hat{e}_4 + [X_{61} + X_{23} + X_{74}]\hat{e}_5 + [X_{15} + X_{72} + X_{34}]\hat{e}_6 \\
+ [X_{13} + X_{26} + X_{45}]\hat{e}_7
\] (19)

Certainly, one can verify

\[
X_{AB} = 2a^ib^ja_{li}b_mT_{ij}^{lm} = 2\{[X_{24}X_{37} + X_{24}X_{56} + X_{37}X_{56}] + [X_{41}X_{35} + X_{41}X_{67} + X_{35}X_{67}] \\
+ [X_{71}X_{52} + X_{71}X_{46} + X_{52}X_{46}] + [X_{12}X_{63} + X_{12}X_{57} + X_{63}X_{57}] \\
+ [X_{61}X_{23} + X_{61}X_{74} + X_{23}X_{74}] + [X_{15}X_{72} + X_{15}X_{34} + X_{72}X_{34}] \\
+ [X_{13}X_{26} + X_{13}X_{45} + X_{26}X_{45}]\} = 0.
\] (20)

Therefore, the magnitude of vector cross product satisfy axiom $\| \vec{A} \times \vec{B} \|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - (\vec{A}, \vec{B})^2$, i.e., $X_{AB} = a^ib^ja_{li}b_mT_{ij}^{lm} = 0$
| Algorithm | $\hat{e}_1$ | $\hat{e}_2$ | $\hat{e}_3$ | $\hat{e}_4$ | $\hat{e}_5$ | $\hat{e}_6$ | $\hat{e}_7$ |
|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 01        | 23 45 67    | 13 46 57    | 12 47 56    | 15 26 37    | 14 27 36    | 17 24 35    | 16 25 34    |
| 02        | 23 45 67    | 13 47 56    | 12 46 57    | 15 27 36    | 14 26 37    | 17 25 34    | 16 24 35    |
| 03        | 23 46 57    | 13 45 67    | 12 47 56    | 16 25 37    | 17 24 36    | 14 27 35    | 15 26 34    |
| 04        | 23 46 57    | 13 47 56    | 12 45 67    | 16 27 35    | 17 26 34    | 14 25 37    | 15 24 36    |
| 05        | 23 47 56    | 13 45 67    | 12 46 57    | 17 25 36    | 16 24 37    | 15 27 34    | 14 26 35    |
| 06        | 23 47 56    | 13 46 57    | 12 45 67    | 17 26 35    | 16 27 34    | 15 24 37    | 14 25 36    |
| 07        | 24 35 67    | 14 36 57    | 15 26 47    | 12 37 56    | 13 27 46    | 17 23 45    | 16 25 34    |
| 08        | 24 35 67    | 14 37 56    | 15 27 46    | 12 36 57    | 13 26 47    | 17 25 34    | 16 23 45    |
| 09        | 24 36 57    | 14 35 67    | 16 25 47    | 12 37 56    | 17 23 46    | 13 27 45    | 15 26 34    |
| 10        | 24 36 57    | 14 37 56    | 16 27 45    | 12 35 67    | 17 26 34    | 13 25 47    | 15 23 46    |
| 11        | 24 37 56    | 14 35 67    | 17 25 46    | 12 36 57    | 16 23 47    | 15 27 34    | 13 26 45    |
| 12        | 24 37 56    | 14 36 57    | 17 26 45    | 12 35 67    | 16 27 34    | 15 23 47    | 13 25 46    |
| 13        | 25 34 67    | 15 36 47    | 14 26 57    | 13 27 56    | 12 37 46    | 17 23 45    | 16 24 35    |
| 14        | 25 34 67    | 15 37 46    | 14 27 56    | 13 26 57    | 12 36 47    | 17 24 35    | 16 23 45    |
| 15        | 25 36 47    | 15 34 67    | 16 24 57    | 17 23 56    | 12 37 46    | 13 27 45    | 14 26 35    |
| 16        | 25 36 47    | 15 37 46    | 16 27 45    | 17 26 35    | 12 34 67    | 13 24 57    | 14 23 56    |
| 17        | 25 37 46    | 15 34 67    | 17 24 56    | 16 23 57    | 12 36 47    | 14 27 35    | 13 26 45    |
| 18        | 25 37 46    | 15 36 47    | 17 26 45    | 16 27 35    | 12 34 67    | 14 23 57    | 13 24 56    |
| 19        | 26 34 57    | 16 35 47    | 14 25 67    | 13 27 56    | 17 23 46    | 12 37 45    | 15 24 36    |
| 20        | 26 34 57    | 16 37 45    | 14 27 56    | 13 25 67    | 17 24 36    | 12 35 47    | 15 23 46    |
| 21        | 26 35 47    | 16 34 57    | 15 24 67    | 17 23 56    | 13 27 46    | 12 37 45    | 14 25 36    |
| 22        | 26 35 47    | 16 37 45    | 15 27 46    | 17 25 36    | 13 24 67    | 12 34 57    | 14 23 56    |
| 23        | 26 37 45    | 16 34 57    | 17 24 56    | 15 26 37    | 14 27 36    | 12 35 47    | 13 25 46    |
| 24        | 26 37 45    | 16 35 47    | 17 25 46    | 15 27 36    | 14 23 67    | 12 34 57    | 13 24 56    |
| 25        | 27 34 56    | 17 35 46    | 14 25 67    | 13 26 57    | 16 23 47    | 15 24 37    | 12 36 45    |
| 26        | 27 34 56    | 17 36 45    | 14 26 57    | 13 25 67    | 16 24 37    | 15 23 47    | 12 35 46    |
| 27        | 27 35 46    | 17 34 56    | 15 24 67    | 16 23 57    | 13 26 47    | 14 25 37    | 12 36 45    |
| 28        | 27 35 46    | 17 36 45    | 15 26 47    | 16 25 37    | 13 24 67    | 14 23 57    | 12 34 56    |
| 29        | 27 36 45    | 17 34 56    | 16 24 57    | 15 23 67    | 14 26 37    | 13 25 47    | 12 35 46    |
| 30        | 27 36 45    | 17 35 46    | 16 25 47    | 15 26 37    | 14 23 67    | 13 24 57    | 12 34 56    |
For the algorithm of the 20th row in Table 4, the relation of cross product of basis vectors are expressed as

\[
\begin{align*}
\hat{e}_2 \times \hat{e}_6 &= \hat{e}_1, & \hat{e}_3 \times \hat{e}_4 &= \hat{e}_1, & \hat{e}_5 \times \hat{e}_7 &= \hat{e}_1, \\
\hat{e}_6 \times \hat{e}_1 &= \hat{e}_2, & \hat{e}_3 \times \hat{e}_7 &= \hat{e}_2, & \hat{e}_4 \times \hat{e}_5 &= \hat{e}_2, \\
\hat{e}_4 \times \hat{e}_1 &= \hat{e}_3, & \hat{e}_2 \times \hat{e}_2 &= \hat{e}_3, & \hat{e}_5 \times \hat{e}_6 &= \hat{e}_3, \\
\hat{e}_1 \times \hat{e}_3 &= \hat{e}_4, & \hat{e}_5 \times \hat{e}_2 &= \hat{e}_4, & \hat{e}_6 \times \hat{e}_7 &= \hat{e}_4, \\
\hat{e}_7 \times \hat{e}_1 &= \hat{e}_5, & \hat{e}_2 \times \hat{e}_4 &= \hat{e}_5, & \hat{e}_6 \times \hat{e}_3 &= \hat{e}_5, \\
\hat{e}_1 \times \hat{e}_2 &= \hat{e}_6, & \hat{e}_3 \times \hat{e}_5 &= \hat{e}_6, & \hat{e}_3 \times \hat{e}_4 &= \hat{e}_6, \\
\hat{e}_1 \times \hat{e}_5 &= \hat{e}_7, & \hat{e}_2 \times \hat{e}_3 &= \hat{e}_7, & \hat{e}_4 \times \hat{e}_6 &= \hat{e}_7. 
\end{align*}
\]

(21)

Similarly, for any two vectors \(\vec{A}\) and \(\vec{B}\), there exist \(\| \vec{A} \times \vec{B} \|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - (\vec{A}, \vec{B})^2\). i.e., \(X_{AB} = a^ib^ja_l b_m T^{lm}_{ij} = 0\)

Importantly, we pay more attention to the case of \(X_{AB} \neq 0\), and we take the algorithm of the 2th row in Table 4 as an example, the relation of cross product of basis vectors are expressed as

\[
\begin{align*}
\hat{e}_2 \times \hat{e}_3 &= \hat{e}_1, & \hat{e}_4 \times \hat{e}_5 &= \hat{e}_1, & \hat{e}_6 \times \hat{e}_7 &= \hat{e}_1, \\
\hat{e}_3 \times \hat{e}_1 &= \hat{e}_2, & \hat{e}_4 \times \hat{e}_7 &= \hat{e}_2, & \hat{e}_5 \times \hat{e}_6 &= \hat{e}_2, \\
\hat{e}_1 \times \hat{e}_2 &= \hat{e}_3, & \hat{e}_4 \times \hat{e}_6 &= \hat{e}_3, & \hat{e}_5 \times \hat{e}_7 &= \hat{e}_3, \\
\hat{e}_5 \times \hat{e}_1 &= \hat{e}_4, & \hat{e}_7 \times \hat{e}_2 &= \hat{e}_4, & \hat{e}_6 \times \hat{e}_3 &= \hat{e}_4, \\
\hat{e}_4 \times \hat{e}_5 &= \hat{e}_6, & \hat{e}_6 \times \hat{e}_2 &= \hat{e}_5, & \hat{e}_7 \times \hat{e}_3 &= \hat{e}_5, \\
\hat{e}_1 \times \hat{e}_6 &= \hat{e}_7, & \hat{e}_2 \times \hat{e}_4 &= \hat{e}_7, & \hat{e}_3 \times \hat{e}_5 &= \hat{e}_7. 
\end{align*}
\]

(22)

For any two vectors \(\vec{A}\) and \(\vec{B}\) in 7-dimensional space, the \(\vec{A} \times \vec{B}\) can be expressed as

\[
\vec{A} \times \vec{B} = [X_{23} + X_{45} + X_{67}]\hat{e}_1 + [X_{32} + X_{46} + X_{57}]\hat{e}_2 + [X_{12} + X_{47} + X_{56}]\hat{e}_3 + [X_{51} + X_{46} + X_{73}]\hat{e}_4 + [X_{14} + X_{57} + X_{63}]\hat{e}_5 + [X_{71} + X_{24} + X_{35}]\hat{e}_6 + [X_{16} + X_{25} + X_{34}]\hat{e}_7
\]

(23)

In addition, one can verify

\[
X_{AB} = a^ib^ja_l b_m T^{lm}_{ij} = 2\{[X_{23}X_{45} + X_{23}X_{67} + X_{45}X_{67}] + [X_{31}X_{46} + X_{31}X_{57} + X_{46}X_{57}] \\
+ [X_{12}X_{47} + X_{12}X_{56} + X_{47}X_{56}] + [X_{51}X_{62} + X_{51}X_{73} + X_{62}X_{73}] \\
+ [X_{14}X_{72} + X_{14}X_{63} + X_{72}X_{63}] + [X_{71}X_{24} + X_{71}X_{35} + X_{24}X_{35}] \\
\} \neq 0
\]

(24)

From the above discussion, a kind of algorithm of the VCP is determined once the relation of cross product in an odd n-dimensional vector is chosen. And, the VCP satisfies axiom \(\| \vec{A} \times \vec{B} \|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - (\vec{A}, \vec{B})^2\) is only a special instance of \(\| \vec{A} \times \vec{B} \|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - (\vec{A}, \vec{B})^2 + X_{AB}\) proposed in this paper. Of course, different algorithms will yield different results. Probably, different algorithms of the VCP correspond to different physical problems.
4 Conclusion

In this paper, the definition of the VCP defined by Eckmann has been extended to an odd n-dimensional space by introducing a cross term $X_{AB}$, and the results show that the new generalized definition can be reduced to Eckmann’s definition in three- and seven-dimensional vector space. It should be noted that the result of VCP $\vec{A} \times \vec{B}$ depend on the choice of relation of basis vector cross product, and different algorithms may correspond to different physical problems, this need to be studied in the future. Its potential applications in quantum physics [8,9], also deserves to be further investigated.

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