Riemannian Submersions from Riemannian Manifolds Admitting a Ricci Soliton

Erol Kılıç and Şemsi Eken Meriç

Abstract In this paper, we study the Riemannian submersion from Riemannian manifold admits a Ricci soliton. Here, we characterize any fiber of such a submersion is Ricci soliton or almost Ricci soliton. Indeed, we obtain necessary conditions for which the target manifold of Riemannian submersion is a Ricci soliton. Moreover, we study the harmonicity of Riemannian submersion from Ricci soliton and give a characterization for such a submersion to be harmonic.

Mathematics Subject Classification (2010): 53C25, 53C43.

Keywords: Ricci soliton · Riemannian Submersion · Harmonic map ·

1 Introduction

In [14], Hamilton defined the notion of Ricci flow and showed that the self similar solutions of such a flow are Ricci solitons. According to the definition of Hamilton, a Riemannian manifold \((M, g)\) is said to be a Ricci soliton if it satisfies
\[
\frac{1}{2} \mathcal{L}_V g + Ric + \lambda g = 0,
\]
where \(\mathcal{L}_V g\) is the Lie-derivative of the metric tensor of \(g\) with respect to \(V\), \(Ric\) is the Ricci tensor of \((M, g)\), \(V\) is a vector field (the potential field) and \(\lambda\) is a constant on \(M\). We shall denote a Ricci soliton by \((M, g, V, \lambda)\). The Ricci soliton \((M, g, V, \lambda)\) is said to be shrinking, steady or expanding according as \(\lambda < 0, \lambda = 0\) or \(\lambda > 0\), respectively.

A trivial Ricci soliton is an Einstein metric if the potential field \(V\) is zero or Killing. If the potential field \(V\) is the gradient of some smooth function \(f\) on \(M\), then the \((M, g, V, \lambda)\) is said to be a gradient Ricci soliton which is denoted by \((M, g, f, \lambda)\).

Pigola et al. defined a new class of Ricci solitons by taking \(\lambda\) is a variable function instead of the constant and then, the Ricci soliton \((M, g, V, \lambda)\) is called
an almost Ricci soliton. Hence, the almost Ricci soliton becomes a Ricci soliton, if the function $\lambda$ is a constant. Indeed, the almost Ricci soliton is called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively (17).

In [16], Perelman used the Ricci soliton in order to solve the Poincaré conjecture and then the geometry of Ricci solitons has been the focus of attention of many mathematicians. For example, Chen and et al. studied the Riemannian manifolds endowed with a concurrent, concircular or torqued vector fields admitting a Ricci soliton and gave many characterizations (4, 5, 6 and 7).

Moreover, Ricci solitons have been studied on contact, paracontact and Sasakian manifolds. For instance, Blaga and et al. focused on $\eta$–Ricci solitons and provided some remarks in Sasakian manifolds (1, 2). Also, in [18], Perktas and et al. gave some characterizations on Ricci solitons in 3-dimensional normal almost para-contact metric manifolds.

On the other hand, the concept of Riemannian submersion between Riemannian manifolds was initiated by O’Neill and Gray, independently. They gave some basis formulas on the theory of Riemannian submersions and it has been extended in the last three decades. (For the theory of Riemannian submersions, we refer [12], [15] and [19]).

In recent years, Eken Meriç and et al. established many inequalities for Riemannian submersions to obtain the relationships between the intrinsic and extrinsic invariants for such a submersion (8, 9 and 13).

This work is organized as follows: The section 2 is a very brief of review of Riemannian submersions. In Section 3, a Riemannian submersion $\pi$ from Ricci soliton to a Riemannian manifold is considered and the necessary conditions for which any fiber of $\pi$ is a Ricci soliton or an almost Ricci soliton are given. The last section is devoted to harmonicity. Using the tension field, a necessary and sufficient condition for which a Riemannian submersion from Ricci soliton is harmonic is obtained.

2 Preliminaries

In this section, we recall some basic notions about Riemannian submersions from [11].

A map $\pi : (M, g) \to (B, g')$ is called a $C^\infty$-submersion between Riemannian manifolds $(M^m, g)$ and $(B^n, g')$, if $\pi$ has a maximal rank at any point of $M$. For any $x \in B$, $\pi^{-1}(x)$ is closed $r$-dimensional submanifold of $M$, such that $r = m - n$. For any $p \in M$, denoting $\mathcal{V}_p = ker \pi_p$ and then the distribution $\mathcal{V}$ is integrable. Also, $T_p \pi^{-1}(x)$ are $r$-dimensional subspaces of $\mathcal{V}_p$ and it follows that $\mathcal{V}_p = T_p \pi^{-1}(x)$. Hence, $\mathcal{V}_p$ is called the vertical space of any point $p \in M$. Denote the complementary distribution of $\mathcal{V}$ by $\mathcal{H}$, the one has

$$T_p(M) = \mathcal{V}_p \oplus \mathcal{H}_p$$
where $\mathcal{H}_p$ is called the horizontal space of any point $p \in M$.

Let $\pi : (M, g) \to (B, g')$ be a submersion between Riemannian manifolds $(M, g)$ and $(B, g')$. At any point $p \in M$, we say that $\pi$ is a Riemannian submersion if $\pi_* p$ preserves the length of the horizontal vectors.

Some basic properties about Riemannian submersion are presented as follows:

Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion, and denote by $\nabla$ and $\nabla'$ the Levi-Civita connections of $M$ and $B$, respectively. If $X, Y$ are the basic vector fields, $\pi$-related to $X'$, $Y'$, one has:

\( i \) $g(X, Y) = g'(X', Y') \circ \pi$,

\( ii \) $h[X, Y]$ is the basic vector field $\pi$-related to $[X', Y']$,

\( iii \) $h(\nabla_X Y)$ is the basic vector field $\pi$-related to $\nabla'_X Y'$,

for any vertical vector field $V$, $[X, Y]$ is the vertical.

A Riemannian submersion $\pi : (M, g) \to (B, g')$ determines two tensor fields $T$ and $A$ on the base manifold $M$, which are called the fundamental tensor fields or the invariants of Riemannian submersion $\pi$ and they are defined by defined by

\[
T(E, F) = T_E F = h\nabla^v vE vF + v\nabla^h vF,
\]

\[
A(E, F) = A_E F = v\nabla^h vE hF + h\nabla^v vF,
\]

where $v$ and $h$ are the vertical and horizontal projections, respectively and $\nabla$ is a Levi-Civita connection of $M$, for any $E, F \in \Gamma(TM)$. Indeed, the fundamental tensors $T$ and $A$ satisfy the followings:

\( i \) $T_V W = T_W V$,

\( ii \) $A_X Y = -A_Y X = \frac{1}{2} v[X, Y]$,
Denoting the Riemannian curvature tensors of \((M, g)\), \((B, g')\) and any fibre of \(\pi\) by \(R\), \(R'\) and \(\hat{R}\) respectively. Then,

\[
R(U, V, F, W) = \hat{R}(U, V, F, W) - g(T_UW, T_VF) + g(T_VW, T_UF),
\]

\[
R(X, Y, Z, H) = R'(X', Y', Z', H') \circ \pi + 2g(A_XY, A_ZH)
\]

\[-g(A_YZ, A_XH) + g(A_XZ, A_YH),
\]

for any \(U, V, W, F \in \Gamma(VM)\) and \(X, Y, Z, H \in \Gamma(HM)\).

Let \(\{U, V\}\) and \(\{X, Y\}\) be an orthonormal basis of the vertical and horizontal 2-plane, respectively. Then, one has

\[
K(U, V) = \hat{K}(U, V) - \|T_UV\|^2 + g(T_UU, T_VV),
\]

\[
K(X, Y) = K'(\pi_*X, \pi_*Y) + 3\|A_XY\|^2,
\]

where \(K\), \(\hat{K}\) and \(K'\) are the sectional curvature in the total space \(M\), any fiber of \(\pi\) and \((B, g')\), respectively.

The Ricci tensor \(\text{Ric}\) on \((M, g)\) is given by

\[
\text{Ric}(U, W) = \hat{\text{Ric}}(U, W) + g(N, T_UW) - \sum_{i=1}^{n} g((\nabla_{X_i}T)(U, W), X_i) \tag{6}
\]

\[
-\sum_{i=1}^{n} g(A_{X_i}U, A_{X_i}W)
\]

\[
\text{Ric}(X, Y) = \text{Ric}'(X', Y') \circ \pi - \frac{1}{2} \left\{ g(\nabla_XN, Y) + g(\nabla_YN, X) \right\} \tag{7}
\]

\[+2 \sum_{i=1}^{n} g(A_{X_i}X, A_{Y_i}X) + \sum_{j=1}^{r} g(T_{U_j}X, T_{U_j}Y)
\]

\[
-2g((\nabla_{U_j}T)(U_j, U), X)
\]

\[-\sum_{i=1}^{n} \left\{ g((\nabla_{X_i}A)(X_i, X), U) + 2g(A_{X_i}X, T_{U_i}X) \right\}
\]

where \(\{X_i\}\) and \(\{U_j\}\) are the orthonormal basis of \(\mathcal{H}\) and \(\mathcal{V}\), respectively for any \(U, V \in \Gamma(VM)\) and \(X, Y \in \Gamma(HM)\).

On the other hand, the mean curvature vector field \(H\) on any fibre of Riemannian submersion \(\pi\) is given by

\[
N = rH,
\]

such that

\[
N = \sum_{j=1}^{r} T_{U_j}U_j \tag{8}
\]
and \( r \) denotes the dimension of any fibre of \( \pi \) and \( \{U_1, U_2, \ldots, U_r\} \) is an orthonormal basis on vertical distribution. We remark that the horizontal vector field \( N \) vanishes if and only if any fibre of Riemannian submersion \( \pi \) is minimal.

Furthermore, using the equality (8), we get
\[
g(\nabla_E N, X) = \sum_{j=1}^{r} g((\nabla_E T)(U_j, U_j), X)
\]
for any \( E \in \Gamma(TM) \) and \( X \in \Gamma(HM) \).

We denote the horizontal divergence of any vector field \( X \) on \( \Gamma(HM) \) by \( \delta(X) \) and given by
\[
\delta(X) = \sum_{i=1}^{n} g(\nabla X_i X, X_i), \tag{9}
\]
where \( \{X_1, X_2, \ldots, X_n\} \) is an orthonormal basis of horizontal space \( \Gamma(HM) \).

Hence, considering (9), we have
\[
\delta(N) = \sum_{i=1}^{n} \sum_{j=1}^{r} g((\nabla X_i T)(U_j, U_j), X_i).
\]
For details, we refer ([3], pp. 243).

### 3 Riemannian Submersions from Riemannian Manifolds Admitting a Ricci Soliton

In the present section, we study the Riemannian submersion \( \pi : (M, g) \to (B, g') \) from the Ricci soliton to a Riemannian manifold and give some characterizations for any fiber of such a submersion and the target manifold \( B \).

Using the Gauss-Codazzi type equations (2)-(5) for Riemannian submersions, we give the following lemma:

**Lemma 1** Let \( \pi : (M, g) \to (B, g') \) be a Riemannian submersion between Riemannian manifolds. Then, the vertical distribution \( V \) is parallel with respect to \( \nabla \), if the horizontal parts \( T_V W \) and \( A_X V \) of (2) and (4) vanish, identically. Similarly, the horizontal distribution \( H \) is parallel with respect to \( \nabla \), if the vertical parts \( T_V X \) and \( A_X Y \) of (3) and (5) vanish, identically, for any \( X, Y \in \Gamma(HM) \) and \( V, W \in \Gamma(VM) \).

**Theorem 2** Let \( (M, g, V, \lambda) \) be a Ricci soliton with the vertical potential field \( V \) and \( \pi : (M, g) \to (B, g') \) be a Riemannian submersion between Riemannian manifolds. If the vertical distribution \( V \) is parallel, then any fibre of Riemannian submersion \( \pi \) is a Ricci soliton.
Proof. Since \((M, g)\) is a Ricci soliton, one has
\[
\frac{1}{2}(\mathcal{L}_V g)(U, W) + \text{Ric}(U, W) + \lambda g(U, W) = 0,
\]
for any \(U, W \in \Gamma(VM)\). Using the equality (6), we have
\[
\frac{1}{2}\{g(\nabla_U V, W) + g(\nabla_W V, U)\} + \hat{\text{Ric}}(U, W) + g(N, \mathcal{T}_V W)
- \sum_{i=1}^{n} (g((\nabla_X, T)(U, W), X_i) - g(A_X, U, A_X, W)) + \lambda g(U, W) = 0,
\]
where \(\{X_i\}\) denotes an orthonormal basis of the horizontal distribution \(\mathcal{H}\) and \(\nabla\) is the Levi-Civita connection on \(M\). Using Lemma 1 and the equality (2), it follows
\[
\frac{1}{2}\{\hat{g}(\hat{\nabla}_U V, W) + \hat{g}(\hat{\nabla}_W V, U)\} + \hat{\text{Ric}}(U, W) + \lambda \hat{g}(U, W) = 0,
\]
which means any fibre of the Riemannian submersion \(\pi\) is a Ricci soliton. 

Theorem 3. Let \((M, g, V, \lambda)\) be a Ricci soliton with the vertical potential field \(V\) and \(\pi : (M, g) \to (B, g')\) be a Riemannian submersion between Riemannian manifolds with totally umbilical fibres. If the horizontal distribution \(\mathcal{H}\) is integrable, then any fibre of Riemannian submersion \(\pi\) is an almost Ricci soliton.

Proof. Since the total space \((M, g)\) of Riemannian submersion \(\pi\) admits a Ricci soliton, then using (1) and (6), we have
\[
\frac{1}{2}\{g(\nabla_U V, W) + g(\nabla_W V, U)\} + \text{Ric}(U, W) + \sum_{j=1}^{r} g(\mathcal{T}_U U_j, \mathcal{T}_V W) + \lambda g(U, W) = 0,
\]
for any \(U, W \in \Gamma(VM)\). Also, the Ricci soliton \((M, g, V, \lambda)\) has totally umbilical fibres and putting (2) in (11), one has
\[
\frac{1}{2}\{\hat{g}(\hat{\nabla}_U V, W) + \hat{g}(\hat{\nabla}_W V, U)\} + \hat{\text{Ric}}(U, W) + \lambda \hat{g}(U, W) = 0,
\]
Since the horizontal distribution \(\mathcal{H}\) is integrable, we have
\[
\frac{1}{2}(\mathcal{L}_V \hat{g})(U, W) + \hat{\text{Ric}}(U, W) - \sum_{i=1}^{n} g(\nabla_X, H, X_i) \hat{g}(U, W) + \lambda \hat{g}(U, W) = 0,
\]
where $H$ is the mean curvature vector of any fibre of $\pi$. From (9), we obtain

$$\frac{1}{2} (\mathcal{L}_V \hat{g})(U, W) + \hat{Ric}(U, W) + (r\|H\|^2 - \delta(H) + \lambda) \hat{g}(U, W) = 0,$$

which means any fibre of $\pi$ is an almost Ricci soliton. □

Considering Theorem 3 we get the following:

**Corollary 4** Let $(M, g, V, \lambda)$ be a Ricci soliton and $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds, such that the horizontal distribution $\mathcal{H}$ is integrable. Any fiber of $\pi$ is a Ricci soliton, if one of the following conditions satisfies:

(i) Any fiber of $\pi$ is a totally umbilical and has a constant mean curvature.

(ii) Any fiber of $\pi$ is a totally geodesic.

Then, we have the following theorem:

**Theorem 5** Let $(M, g, E, \lambda)$ be a Ricci soliton with the potential field $E \in \Gamma(TM)$ and $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds. If the horizontal distribution $\mathcal{H}$ is parallel, then the followings are satisfied:

(i) If the vector field $E$ is vertical, then $(B, g')$ is an Einstein.

(ii) If the vector field $E$ is horizontal, then $(B, g')$ is a Ricci soliton with potential field $E'$, such that $\pi^* E = E'$.

**Proof.** Since the total space $(M, g)$ of Riemannian submersion $\pi$ admits a Ricci soliton with potential field $E \in \Gamma(TM)$, then using (1) and (7), we have

$$\frac{1}{2} \left\{ g(\nabla X E, Y) + g(\nabla Y E, X) \right\} + \hat{Ric}'(X', Y') \circ \pi - \frac{1}{2} (g(\nabla X N, Y) (13)$$

$$+ g(\nabla Y N, X)) + 2 \sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y)$$

$$+ \lambda g(X, Y) = 0,$$

where $X'$ and $Y'$ are $\pi-$related to $X$ and $Y$ respectively, for any $X, Y \in \Gamma(HM)$.

Applying Lemma 1 to above equation (13), one has

$$\frac{1}{2} \left\{ g(\nabla X E, Y) + g(\nabla Y E, X) \right\} + \hat{Ric}'(X', Y') \circ \pi + \lambda g(X, Y) = 0. (14)$$

(i) If the vector field $E$ is vertical, from the equality (14), it follows

$$\frac{1}{2} \left\{ g(\mathcal{A}_X E, Y) + g(\mathcal{A}_Y E, X) \right\} + \hat{Ric}'(X', Y') \circ \pi + \lambda g(X, Y) = 0.$$
Since $\mathcal{H}$ is parallel, one has
\[
\text{Ric}^\prime(X',Y') \circ \pi + \lambda g(X,Y) = 0,
\]
which is nothing but $(B,g')$ is an Einstein.

(ii) If the vector field $E$ is a horizontal, the equation (14) follows
\[
\frac{1}{2}(\mathcal{L}_E g)(X,Y) + \text{Ric}^\prime(X',Y') \circ \pi + \lambda g(X,Y) = 0,
\]
which means the Riemannian manifold $(B,g')$ is a Ricci soliton with potential field $E'$.  

Using Lemma 1 and the equality (7), we get the following:

**Remark 6** Let $(M, g, \xi, \lambda)$ be a Ricci soliton with the horizontal potential field $\xi$ and $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds, such that $\mathcal{H}$ is parallel. Then, the vector field $N$ is Killing on the horizontal distribution $\mathcal{H}$.

**Theorem 7** Let $(M, g, \xi, \lambda)$ be a Ricci soliton with the horizontal potential field $\xi$ and $\pi : (M, g) \to (B, g')$ be a Riemannian submersion from Riemannian manifold to an Einstein manifold. If the horizontal distribution $\mathcal{H}$ is parallel, then the vector field $\xi$ is conformal Killing on $\mathcal{H}$.

**Proof.** Since $(M, g, \xi, \lambda)$ is a Ricci soliton and using (7) in statement (1), we get
\[
\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \text{Ric}^\prime(X',Y') \circ \pi - \frac{1}{2}g(\nabla_X N, Y) + g(\nabla_Y N, X) = 0,
\]
where $\{X_i\}$ denotes an orthonormal basis of $\mathcal{H}$, for any $X, Y \in \Gamma(HM)$. Using Lemma 1, the equation (15) is equivalent to
\[
\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \text{Ric}^\prime(X',Y') \circ \pi + \lambda g(X,Y) = 0.
\]
On the other hand, since the Riemannian manifold $(B,g')$ is an Einstein, one can see that $\xi$ is a conformal Killing.

### 4 Riemannian Submersions from Ricci Solitons and their Harmonicity

This section deals with the harmonicity of Riemannian submersion from a Ricci soliton to a Riemannian manifold. As a tool, we use the tension field and provide a necessary and sufficient condition for which such a submersion $\pi$ is harmonic.
Definition 8 Let \((M, g)\) and \((B, g')\) be \(C^\infty\)-Riemannian manifolds of dimension \(m\) and \(n\), respectively and \(\pi : (M, g) \rightarrow (B, g')\) be a smooth map between Riemannian manifolds. Then \(\pi\) is harmonic if and only if the tension field \(\tau(\pi)\) of a map \(\pi\) vanishes at each point \(p \in M\), that is,

\[
\tau(\pi)_p = \sum_{i=1}^{m} \sigma_\pi(e_i, e_i),
\]

where \(\{e_i\}_{1 \leq i \leq m}\) is local orthonormal frame around a point \(p \in M\) and \(\sigma_\pi\) is the second fundamental form of \(\pi\), which is defined by

\[
\sigma_\pi(X, Y) = \nabla_{e_i}^{\pi^{-1}TB} \pi_*(\nabla_X Y),
\]

for any vector fields \(X, Y \in \Gamma(TM)\).

(For the theory of harmonic maps, we refer to [10].)

Now, we assume that \(\pi : (M, g) \rightarrow (B, g')\) is a Riemannian submersion between Riemannian manifolds, such that \((M, g)\) admits a Ricci soliton with the vertical potential field \(V\).

Let \(\{e_1, e_2, ..., e_m\}\) be an orthonormal basis on \(M\), such that \(\{e_i\}_{1 \leq i \leq r}\) are vertical and \(\{e_i\}_{r+1 \leq i \leq m}\) are horizontal. Then, it follows

\[
\sigma_\pi(e_i, e_i) = \sum_{i=1}^{m} \left(\nabla_{e_i}^{\pi^{-1}TB} e_i - \pi_*(\nabla e_i e_i)\right)
= \sum_{i=1}^{r} \left(\nabla_{e_i}^{\pi^{-1}TB} e_i - \pi_*(\nabla e_i e_i)\right) + \sum_{i=r+1}^{m} \left(\nabla_{e_i}^{\pi^{-1}TB} e_i - \pi_*(\nabla e_i e_i)\right)
= -\pi_* \left(\sum_{i=1}^{r} (\nabla e_i e_i)\right),
\]

such that

\[
\nabla_{e_i}^{\pi^{-1}TB} e_i = \pi_*(\nabla e_i e_i), \quad r + 1 \leq i \leq m
\]

and since \(\{e_i\}_{1 \leq i \leq r}\) are vertical,

\[
\nabla_{e_i}^{\pi^{-1}TB} e_i = 0.
\]

Consequently, one has

\[
\tau(\pi) = \sum_{i=1}^{r} \sigma_\pi(e_i, e_i) = -\pi_* \left(\sum_{i=1}^{r} T e_i e_i\right) = -\pi_* N.
\]

Considering Theorem 5 and the statement (17), we get the theorem as follows:
Theorem 9 Let \((M, g, V, \lambda)\) be a Ricci soliton with the vertical potential field \(V\) and \(\pi : (M, g) \to (B, g')\) be a Riemannian submersion between Riemannian manifolds. Then, the Riemannian submersion \(\pi\) is harmonic if any of the following conditions is satisfied:

(i) The horizontal distribution \(H\) is parallel,
(ii) The vertical distribution \(V\) is parallel,
(iii) Any fiber of \(\pi\) is totally geodesic,
(iv) Any fiber of \(\pi\) is minimal.

Acknowledgements: This work is supported by 1001-Scientific and Technological Research Projects Funding Program of TÜBİTAK project number 117F434.
References

[1] Blaga AM, Perktaš SY. Remarks on almost $\eta$–Ricci solitons in $(\varepsilon)$-para Sasakian manifolds. 2018; arXiv:1804.05389v1.

[2] Blaga AM, Perktaš SY, Acet BE, Erdoğan FE. $\eta$–Ricci solitons in $(\varepsilon)$-almost paracontact metric manifolds. 2017; arXiv: 1707.07528v2.

[3] Besse AL. Einstein manifolds, Berlin-Heidelberg-New York: Springer-Verlag, 1987.

[4] Chen BY, Concircular vector fields and pseudo-Kähler manifolds. Kragujevac J. Math. 2016; 40(1): 7-14.

[5] Chen BY, Deshmukh S. Ricci solitons and concurrent vector fields, Balkan J. Geom. Its Appl. 2015; 20(1): 14-25.

[6] Chen BY, Some Results on Concircular Vector Fields and their Applications to Ricci solitons, Bull. Korean Math. Soc. 2015; 52(5): 1535-1547.

[7] Chen, BY. Classification of torqued vector fields and its applications to Ricci solitons, Kragujevac J. Math. 2017; 41(2): 239-250.

[8] Eken Meriç Ş, Gülbahar M, and Kılıç E, Some inequalities for Riemannian submersions, An. Ştiint. Univ. Al. I. Cuza Iaşi Mat. (N.S.) 2017; 63(3): 471482.

[9] Eken Meriç Ş, Kılıç, Erol; Sağroğlu, Yasemin Scalar curvature of Lagrangian Riemannian submersions and their harmonicity. Int. J. Geom. Methods Mod. Phys. 2017; 14(12): 1750171, 16pp.

[10] J. Eells and H. J. Sampson, Harmonic mappings of Riemannian manifolds, Am. J. Math. 1964; 86: 109-160.

[11] M. Falcitelli, S. Ianus and A. M. Pastore, Riemannian Submersions and Related Topics (World Scientific Publishing Co. Pte. Ltd., 2004).

[12] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Metch. 1967; 16: 715-737.

[13] Gülbahar, M.; Eken Meriç, Ş.; Kılıç, E. Sharp inequalities involving the Ricci curvature for Riemannian submersions. Kragujevac J. Math. 2017; 41(2): 279293.

[14] Hamilton RS. The Ricci flow on surfaces, Mathematics and General Relativity(Santa Cruz, CA, 1986) Contemp. Math. Amer. Math. Soc. 1988; 71: 237-262.

[15] B. O’Neill, The fundamental equations of a Riemannian submersions, Mich. Math. J. 1966; 13: 459-469.
[16] Perelman G. The Entropy formula for the Ricci flow and its geometric applications. 2002; arXiv math/0211159

[17] Pigola S, Rigoli M, Rimoldi M, Setti AG. Ricci almost solitons. Ann. Scuola Norm. Sup. Pisa. Cl. Sci. 2011; 10(4): 757-799.

[18] Perktaş SY, Keleş S. Ricci solitons in 3-dimensional normal almost para-contact metric manifolds, Int. Elect. J. Geom. 2015; 8(2): 34-45.

[19] Şahin, B. Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and their Applications, Elsevier, Academic, Amsterdam, 2017.

Erol KILIÇ
Address: İnönü University, Faculty Sciences and Arts, Department of Mathematics, Malatya, TURKEY.
e-mail: erol.kilic@inonu.edu.tr

Şemsi EKEN MERİÇ
Address: Mersin University, Faculty of Sciences and Arts, Department of Mathematics, Mersin, TURKEY.
e-mail: semsicken@hotmail.com