A posteriori error bounds for classical and mixed FEM’s for 4th-order elliptic equations with piece wise constant reaction coefficient having large jumps

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Abstract. We present guaranteed, robust and computable a posteriori error bounds for approximate solutions of the equation $\Delta\Delta u + \kappa^2 u = f$ by classical and mixed Ciarlet-Raviart finite element methods. We concentrate on the case when the reaction coefficient $\kappa^2$ is subdomain (finite element) wise constant and chaotically varies between subdomains in the sufficiently wide range. It is proved that the bounds for the classical FEM’s are robust with respect to $\kappa \in [0, c h^{-2}]$, where $c = \text{const}$ and $h$ is the maximal size of finite elements, and possess additional useful features. The coefficients in fronts of two typical norms in their right parts only insignificantly worse than those for $\kappa \equiv \text{const}$, and the bounds can be calculated without resorting to the equilibration procedures. Besides, they are sharp at least for low order methods, if the testing moments and deflection in their right parts are found by accurate recovery procedures. The technique of derivation of the bounds is based on the approach similar to one used in our preceding papers for simpler problems.

1. Introduction

There is a number of papers on the a posteriori error bounds for the finite element solutions of the biharmonic and the thin plate in bending equations, as well as the singularly perturbed equations of the form $\varepsilon^2 \Delta^2 u - \Delta u = f$. One can find such bounds for conforming, nonconforming, mixed, discontinuous Galerkin, and other finite element methods [1]–[14]. However, to the best of the author’s knowledge, no papers were concerned with the problem of the thin plate in bending resting upon Winkler foundation with the subgrade (reaction) modulus, which is piece wise constant and has significant jumps. At the same time, a posteriori bounds for the 2nd-order reaction-diffusion equation with the discontinuous reaction coefficient were paid attention in the literature. However, primarily it was restricted to the case of the subdomain wise or finite element wise constant reaction coefficient that varies "mildly" between neighbouring elements. The typical variant of a mild change is described, e. g., in [15, 16]. In this paper, it is shown that the range of admissible jumps can be considerably widened, and the price for this is an insignificant worsening of the coefficients in front of the typical norms in the right parts of the a posteriori bounds.

We consider the model problem

$$\begin{align*}
\Delta\Delta u + \sigma u &= f(x) \quad \text{in } \Omega, \\
u &= \partial u/\partial \nu = 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(1.1)
with \( f \in L^2(\Omega) \) and the first boundary condition, in which \( \nu \) is the internal normal to the boundary. The main distinction from the preceding papers, including author’s papers [17]–[20], is in the assumption that \( \sigma = k^2 \) is finite element wise constant and otherwise satisfy only one restriction \( \sigma \leq ch^{-4} \), \( c = \text{const} \). Clearly, it is a kind of a weak restriction which admits rather big values of \( \sigma \) as well as its jumps between adjacent finite elements. Somewhat surprising is that the a posteriori error bounds of this paper, do not differ much from those for the case \( \sigma = \text{const} \) in [17, 18] for the 2nd order and in [19, 20] for the 2n\(^{th} \) order elliptic, equations. In their right parts we see the same typical norms whereas the coefficients in front of them are only slightly (in 2 times in the worst case) larger. In addition, their proofs are easy, if one takes into consideration the approximation properties of \( L^2 \) and \( H^1 \) finite element projections of functions.

A posteriori bounds derived in [17]–[20] were termed consistent due to two important properties:

1) accuracy of a consistent bound coincides in the order with that in the corresponding sharp a priori bound,
2) for the proof of the property 1), as well as for the calculation of the bound, it is sufficient to use any testing fields of the flaxes/stresses/stress resultants, which possesses only respective approximation properties, without resorting to the flux equilibration procedure.

Clearly, the so called efficient bounds, i.e., the ones approved by the inverse bounds with the same order of accuracy, are consistent and both are sharp. However, the consistency is much easier to check up.

Among the widely spread in practice there are two types of bounds, namely the residual based a posteriori bounds and the bounds based on the use of the equilibrated flaxes/stresses/ stress resultants. A drawback of the first type is that the derivation of the bounds is essentially based upon the approximation bounds, and the coefficients in the former bounds strongly depend on the constants in the latter. An example of such an efficient bound for the approximate solution of the equation \( \Delta \Delta u = f \) is found, e.g., in [13]. At implementation of the second type bounds, for each numerical solution, e.g., of the second order elliptic equation, one finds a single testing flux. It is evaluated by the "equilibrated" flux recovery procedure, which depends not only on the mesh, but also on the problem. This constricts the universality of the bounds. Besides, the equilibration increases the diffusion type error, and the value of the damage is seen only from the inverse like bound, the constants in which usually are not too optimistic. Thus, bounds of both types, which have been intensively developing in several last decades, have their own restrictions and their real accuracy needs to be checked by practice.

The consistent bounds are more universal, since for providing the sharpness one can use any testing flux possessing the approximation properties similar to those of the numerical flux or better. Recovery procedures, providing such properties, are easy to meet in the literature for the reason that they were thoroughly studied and widely tested in the residual type error estimators popular in the computational practice, see [22]–[24] and references there. Consistent bounds allowed to sharpen the coefficient in front of the \( L^2 \)-norm of the residual type term in the right parts of the bounds with the free flux vector-functions, [17]–[20]. In this paper, the approach of [17]–[20] is expanded to the case of 4th order elliptic equation with the element wise constant reaction coefficient having significant jumps at the inter-element boundary.

The notations \( \| \cdot \|_k \), \( \| \cdot \| \) will stand for the norms and quasi-norms in Sobolev’s spaces \( H^k(\Omega) = W^k_2(\Omega) \) with the agreement that \( \| f \|_0 = \| f \|_0 = \| f \| \). Additionally we introduce the spaces \( H^2_0(\Omega) := \{ v \in H^2(\Omega) : v = \partial \nu / \partial \nu = 0 \text{ on } \partial \Omega \} \), \( H^2_0(\Omega, \Delta \Delta) = \{ v \in H^2_0(\Omega) : \Delta \Delta v \in L^2(\Omega) \} \), and \( L^2(\Omega) = (L^2(\Omega))^4 \). In relation with the problem (1.1), it is helpful to introduce the subspace \( \mathbf{M}(\Omega) = \{ \mathbf{m} = \{ m_{k,l} \}_{k,l=1}^2 \in L^2(\Omega) : m_{1,2} = m_{2,1} \} \) of vector-functions \( \mathbf{m} \) and the operators
\[ \mathcal{D} : H^2(\Omega) \to M(\Omega) \text{ and } \mathcal{D}^* : M(\Omega, \mathcal{D}^*) \to L^2(\Omega), \] defined as
\[ \mathcal{D}_v = \left\{ \frac{\partial^2 v}{\partial x_k \partial x_l} \right\}_{k,l=1}^2, \quad \mathcal{D}^* \mathbf{m} = \sum_{k,l=1}^2 \frac{\partial^2 m_{k,l}}{\partial x_k \partial x_l}, \]
where \( M(\Omega, \mathcal{D}^*) = \{ \mathbf{m} \in M(\Omega) : \mathcal{D}^* \mathbf{m} \in L^2(\Omega) \}. \) If (1.1) is viewed as the thin plate bending problem (at the cylindrical stiffness equal to unity and Poisson coefficient equal to zero), vector-functions \( \mathbf{m} = \mathcal{D} u \) have the meaning of components of the bending and twisting moments acting in the plate. Where it does not cause confusion, for norms \( \| \cdot \| \) in the spaces \( L^2(\cdot) \) we use the notation \( || \cdot || \), so that \( || \mathbf{m} || \) will stand for \( \| \mathbf{m} \|_{L^2(\Omega)} \).

For simplicity, we restrict consideration to the polygonal domain \( \Omega \subset \mathbb{R}^d, \) \( d = 2 \), which is covered by the triangulation \( \mathcal{T}_h \) with the quasi-uniform triangular/rectangular nests \( \tau_r, \) \( r = 1, 2, \ldots, \mathcal{R}, \) of the size \( h \) and
\[ \overline{\Omega} = \bigcup_{r=1}^\mathcal{R} \tau_r, \]
where parameter \( h \) is understood as the maximal diameter of domains \( \tau_r. \) Since we study FEM solutions of the classical and mixed formulations, we have to introduce finite element spaces of the two types, i.e., belonging to the classes \( C^0 \) and \( C^1 \) and defined on the same triangulation \( \mathcal{T}_h. \)

Therefore, in general there are finite element assemblages denoted \( \mathcal{R}_h, \mathcal{R}_h. \), which induce the spaces \( U_h(\Omega), V_h(\Omega), \) respectively, and the subspace \( \hat{U}_h(\Omega) = \{ \phi_h \in U_h(\Omega) : \phi_h = 0 \text{ on } \partial \Omega \}, \) all belonging to \( C(\Omega) \cap H^1(\Omega) \) and used for solution of the mixed formulation. Besides, there is the finite element assemblage \( \mathcal{R}^h, \) which induces the space \( \mathcal{V}_h(\Omega) \subset C^1(\Omega) \cap H^2(\Omega), \) and its subspace \( \mathcal{V}_{h,0}(\Omega) = \{ \phi_h \in \mathcal{V}_h(\Omega) : \phi_h = \partial \nu/\partial v = 0 \text{ on } \partial \Omega \}, \) in which the approximate solution of the primal problem is sought.

Indeed, notwithstanding the assumption that \( \Omega \) is a polygon, the results are easily expanded to arbitrary sufficiently smooth domains. This is for the reason that the techniques for constructing curvilinear \( C^0 \) and \( C^n, \) \( n \geq 2, \) finite elements in [25]–[28] and [26]–[30], respectively, allow one to create the finite element assemblages, which exactly represent \( \Omega \) by means of the special curvilinear finite elements used along curvilinear parts of the boundary. These finite element assemblages satisfy the generalized conditions of quasuniformity [27, Section 3.2], and induce the spaces \( \hat{U}_h(\Omega), U_h(\Omega), V_h(\Omega) \) and \( \mathcal{V}_{h,0}(\Omega), \) which provide the same orders of accuracy of approximation and convergence as in case of the polygonal \( \Omega. \)

2. Preliminaries
Let \( a(w, v) = (\mathcal{D} w, \mathcal{D} v), \) \( \forall(\Omega) \) be the Hilbert space of functions with the scalar product \( [w, v] = a(w, v) + (\sigma w, v) \) and the norm \( ||v||^2 = [v, v], \) and \( \mathcal{V}_0(\Omega) = \{ v \in \mathcal{V}(\Omega) : v = \partial v/\partial v = 0 \text{ on } \partial \Omega \}. \) The weak form of the problem (1.1) reads: find \( u \in \mathcal{V}_0(\Omega) \) such that
\[ a_h(u, v) + (\sigma u, v) = (f, v), \quad \forall v \in \mathcal{V}_0(\Omega), \] where \( \sigma = \sigma_r = \text{const for } x \in \tau_r, \) \( r = 1, 2, \ldots, \mathcal{R}. \) The finite element solution from the space \( \mathcal{V}_{h,0}(\Omega), \) denoted \( u_{\text{fem}}, \) satisfies the integral identity
\[ a_\Delta(u_{\text{fem}}, v) + (\sigma u_{\text{fem}}, v) = (f, v), \quad \forall v \in \mathcal{V}_{h,0}(\Omega). \] The weak mixed Ciarlet-Raviart formulation of the problem (1.1) reads, cf. [28]: find the vector-function \( \mathbf{w} = (v, u)^T \in H^1(\Omega) \times H^1(\Omega), \) \( \hat{H}^1(\Omega) = \{ \phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial \Omega \}, \) satisfying the system of the integral identities
\[ \langle v, q \rangle - \langle \nabla u, \nabla q \rangle = 0, \quad \forall q \in H^1(\Omega), \]
\[ \langle \nabla v, \nabla g \rangle + (\sigma u, g) = (f, g), \quad \forall g \in \hat{H}^1(\Omega), \]
where \( \langle \cdot, \cdot \rangle \) is the scalar product of vector-functions \( \langle z, y \rangle = (z_1, y_1) + (z_2, y_2) \) for \( y = (y_1, y_2)^T \) and \( z = (z_1, z_2)^T \). The mixed finite element solution \( w_h = (v_h, u_h)^T \in V_h(\Omega) \times U_h(\Omega) \) satisfies the system of equations

\[
(v_h, q_h) - \langle \nabla u_h, \nabla q_h \rangle = 0, \quad \forall q_h \in V_h(\Omega),
\]

\[
\langle \nabla v_h, \nabla g_h \rangle + (\sigma u_h, g_h) = (f, g_h), \quad \forall g \in U_h(\Omega). \tag{2.4}
\]

Obviously the error of the finite element solution, denoted \( e_{\text{fem}} = (e_v, e_u)^T \), where

\[
e_u = u_h - u, \quad e_v = v_h + \Delta u,
\]

satisfies the integral identities

\[
(e_v, q_h) - \langle \nabla e_u, \nabla q_h \rangle = 0, \quad \forall q_h \in V_h(\Omega),
\]

\[
\langle \nabla e_v, \nabla g_h \rangle + (\sigma e_u, g_h) = 0, \quad \forall g \in U_h(\Omega),
\]

We list basic some properties of the finite element spaces and the primal boundary problem, which, in part, we rely upon in what follows. Let \( H^l(\Omega) := H^l(\Omega) \cap H^0(\Omega), \ l \geq 2 \).

(A1) For any \( w \in H^l(\Omega) := H^l(\Omega) \cap H^0(\Omega) \) the space \( U_h(\Omega) \) provides the approximation \( \tilde{w} = G_{h,u}w \) that at \( k = 0, 1 \) and \( 1 \leq l \leq p_u + 1 \) we have

\[
|\tilde{w} - w|_k \leq c_{k,l}h^{l-k}|w|_l, \quad c_{k,l} = \text{const}, \tag{2.5}
\]

where \( G_{h,u} : H^l(\Omega) \to \hat{U}_h(\Omega) \) is a linear operator. Similar approximation estimates hold for \( w \in H^l(\Omega), \ \tilde{w} \in V_h(\Omega), \ k = 0, 1, \) and \( 1 \leq l \leq p_u + 1 \).

(A2) For any \( w \in H^l_0(\Omega) \) the space \( V_{h,0}(\Omega) \) provides the approximation \( \tilde{w} = Q_hw \) that at \( k = 0, 1, 2 \) and \( 2 \leq l \leq p + 1 \) we have

\[
|\tilde{w} - w|_k \leq c_{k,l}h^{l-k}|w|_l, \quad c_{k,l} = \text{const}, \tag{2.6}
\]

where \( Q_h : H^l_0(\Omega) \to V_{h,0}(\Omega) \) is a linear operator.

(B) The domain \( \Omega \) is such that if \( \sigma \equiv 0 \), then at any \( f \in L^2(\Omega) \) the solution \( u \) of (1.1) belongs to \( H^4(\Omega) \) and for some \( c_0 = \text{const} \)

\[
|u|_4 \leq c_0 \|f\|. \tag{2.7}
\]

Let us note, that if \( \sigma = \text{const} \geq 1 \), then the constant \( c_0 \) is not bigger than doubled \( c_0 \) for the case \( \sigma \equiv 0 \), [19]. The condition (B) of the regularity of the problem (1.1) can be dropped, but at the price of some worsening in the coefficients of the bounds.

For a wide range of approximations of the solution \( u \), including obtained by numerical methods, a weak norm of the error is less, or even much less than a stronger norm. This implies that, if \( \phi \in H^2_0(\Omega) \) is an approximation of \( u \) and \( e = \phi - u \), then for some positive \( \lambda_\phi \gg 1 \), the error \( e \) satisfies the inequality

\[
|e|^2 \leq \lambda_\phi^{-1}\|De\|^2. \tag{2.8}
\]

As it was shown, the accuracy of a posteriori bounds strongly depends upon an adequate assessment of the value of \( \lambda_\phi \). For finite element solutions, \( i.e. \) at \( \phi = u_{\text{fem}} \) this inequality holds with \( \lambda_{\text{fem}}^{-1} = O(h^4) \), for instance, when \( \sigma = \text{const} \), cf. [19, (5)]. Unfortunately, for many problems and numerical methods it is not easy to estimate \( \lambda_{\text{fem}} \) adequately. On the contrary, in some cases it is possible to replace \( \lambda_\phi \) by the value satisfying much simpler condition

\[
\|e_\Delta\| \leq \lambda_\phi^{-1}\|De_\Delta\|, \tag{2.9}
\]

where \( e_\Delta = u_\Delta - u, \ u_\Delta \) is the finite element function, which minimizes the norm \( \|D(\phi - u)\| \) on the space of functions \( \phi \in V_{h,0}(\Omega) \).
3. A posteriori error bounds for FEM solutions of primal and mixed problems

First, we present the a posteriori error bound for FEM solutions of problem (1.1). This bound has its own significance, the additional reason for its presentation is that a posteriori error bounds for mixed methods can be based on the existing a posteriori bounds for approximate solutions of (1.1).

**Theorem 1.** Let \( u \in H^2_0(\Omega, \Delta\Delta) \), \( u_{\text{fem}} \in \mathcal{V}_{h,0}(\Omega) \) where \( u_{\text{fem}} \) is the finite element solution, and \( \mathbf{m} \in \mathbf{M}(\Omega, D^*) \). Let also the conditions (A2) with \( p \geq 3 \) and (B) be satisfied and \( 0 \leq \sigma \leq \sigma_\ast = \text{const} \leq \lambda_\Delta \). Then

\[
\|u_{\text{fem}} - u\|^2 \leq \Theta \mathcal{M}_1(\sigma_\ast, u_{\text{fem}}, \mathbf{m}),
\]

where \( \Theta = 1 + (\sigma_\ast - \sigma_{\text{min}})/\lambda_\Delta \), \( \sigma_{\text{min}} = \min_{x \in \Omega} \sigma \), and it can be adopted \( \lambda_\Delta = C h^{-4} \), \( C = 1/(c_s c_{2.4})^2 \).

**Proof.** For \( e = e_{\text{fem}} := u_{\text{fem}} - u \) and \( \forall w \in \mathcal{V}_{h,0}(\Omega) \), by using the Galerkin property, integrating by parts and by application of the Cauchy inequalities

\[
(\phi, \psi) \leq \|\phi\| \|\psi\| \quad \text{and} \quad (\phi_1, \psi_1) + (\phi_2, \psi_2) \leq \|\phi_1\|^2 + \sigma_\ast^{-1}\|\phi_2\|^2 \|\psi_1\|^2 + \sigma_\ast\|\psi_2\|^2, \]

we get

\[
\|e\|^2 = \langle \mathcal{D}e, \mathcal{D}e \rangle + \langle \sigma e, e \rangle = \langle \mathcal{D}e, \mathcal{D}(e + w) \rangle + \langle \sigma e, e + w \rangle - \langle \mathcal{D}v - \mathbf{m}, \mathcal{D}(e + w) \rangle + \langle \mathcal{D}u - \mathbf{m}, \mathcal{D}(e + w) \rangle + \langle f - \sigma v - \mathcal{D}^* \mathbf{m}, e + w \rangle \
\]

\[
\leq \left\{ \|\mathcal{D}v - \mathbf{m}\|^2 + \frac{1}{\sigma_\ast}\|f - \sigma v - \mathcal{D}^* \mathbf{m}\|^2 \right\}^{1/2} \left\{ \|\mathcal{D}e + w\|^2 + \sigma_\ast\|e + w\|^2 \right\}^{1/2} \
\leq \left\{ \|\mathcal{D}v - \mathbf{m}\|^2 + \frac{1}{\sigma_\ast}\|f - \sigma v - \mathcal{D}^* \mathbf{m}\|^2 + \|f - \hat{f}\|^2 \right\}^{1/2} \left\{ \|\mathcal{D}(e + w)\|^2 + \sigma_\ast\|e + w\|^2 \right\}^{1/2}. \tag{3.2}
\]

Let \( Q_\Delta : H^2_0(\Omega) \rightarrow \mathcal{V}_{h,0} \) be the orthogonal projection operator defining \( u_\ast \) for each function \( u \in H^2_0(\Omega) \). Having set \( w = -Q_\Delta e_{\text{fem}} \), we see that \( e + w = (I - Q_\Delta) e = e_\Delta \) and further

\[
\|\mathcal{D}(e + w)\|^2 + \sigma_\ast\|e + w\|^2 = \|\mathcal{D}e_\Delta\|^2 + \sigma_\ast\|e_\Delta\|^2 \leq \|\mathcal{D}e_\Delta\|^2 + (\sigma_\ast - \sigma_{\text{min}})\|e_\Delta\|^2 + \|e\|^2. \tag{3.3}
\]

The key fact of the proof is the bound

\[
\lambda_\Delta^{-1} \leq 1/(c_s c_{2.4} h^2)^2, \tag{3.4}
\]

which is a consequence of the convergence bound in \( H^2(\Omega) \), stemming from (2.6), (2.7), and the Aubin-Nitsche trick [32, 33]. For more general problems the proof of (3.4) is found in [19, Lemma 1]. Applying (3.4) to the second term in (3.3) and taking into account the obvious inequality

\[
\|\mathcal{D}e_\Delta\| \leq \|\mathcal{D}e_{\text{fem}}\|, \tag{3.5}
\]

we come to the bound

\[
\|\mathcal{D}(e + w)\|^2 + \sigma_\ast\|e + w\|^2 \leq \Theta\|e_{\text{fem}}\|^2, \quad \Theta = 1 + \sigma_\ast/\lambda_\Delta, \tag{3.6}
\]

which together with (3.2) concludes the proof.
Remark 1. The estimate (2.6) for \( k = 2, \ l = 4 \) only was needed in the proof. Under some conditions, the bound (3.1) can be derived with the regularity condition (B) omitted. For instance, if \( Q_h \) is \( L_2 \)-projector. The corresponding constants \( \lambda_\alpha \) and \( \Theta \) became slightly more complex and depend on constants in (2.6) for \( k \leq l = 0, 2 \).

For providing high accuracy it is important to pick up the testing vector-function \( m \) with the components as close as possible to their exact values. It is usually done with the use of the respective recovery procedures, in particular, the same as used for the derivation of the residual type a posteriori error bounds. As it is noted in the book [21] and several papers, flux recovery procedures demonstrated very high efficiency at the use for evaluation of a posteriori error bounds for the finite element solutions of the 2\textsuperscript{nd} order elliptic equations. If attended for evaluation of (3.1), on the basis of the finite element solution \( u_{\text{fem}} \) they produce \( m \) as an element of some appropriate finite element space \( M(\Omega) \subset M(\Omega, D^*) \). The most popular in the practice is called averaging procedure exemplified in [34, 13]. In it, any nodal parameter of any node is defined by the averaging with weights of the values of \( \partial^2 u_{\text{fem}}/\partial x_k \partial x_l \), or the respective derivative of \( \partial^2 u_{\text{fem}}/\partial x_k \partial x_l \) corresponding to the nodal parameter, over finite elements containing the node. The weights can be the relations of the measure of the area of every finite element to the measure of the area of the patch of the finite elements, containing the node. For understandable reasons it is convenient to take \( M(\Omega) = [V_h(\Omega)]^4 \cap M(\Omega, D^*) \), where \( V_h(\Omega) = V_h(\Omega) \) is the space of the used finite element method, see (B), however for maintaining the order of accuracy it can be admissible to employ the space of a lesser degree \( p - 2 \). Another efficient and optimal in the computational cost procedure for finding \( m \) is the least squares procedure. In it, moments \( m_{k,l} \) are defined as \( L^2(\Omega) \) orthogonal projections of the derivatives \( \partial^2 u_{\text{fem}}/\partial x_k \partial x_l \) upon the corresponding subspaces \( M_{k,l}(\Omega) \) of the space \( M(\Omega) \), e.g., \( M_{k,l}(\Omega) = V_h^{\text{pol}}(\Omega) \).

Turning to the error bound for the Ciarlet-Raviart mixed finite element method, we assume for simplicity \( p = p_u = p_v \geq 2 \) and introduce the operator \( \pi_h \) of \( L^2 \) orthogonal projection on the space \( V_{h,0}(\Omega) \). For the error of the solution by the mixed finite element method we will use the norm

\[
\|e_{\text{fem}}\| = \frac{1}{\sqrt{2}} \left\{ \|e_v\|^2 + \|\Delta_h e_u\|^2 + 2\|\kappa e_u\|^2 \right\}^{1/2}, \quad \kappa = \sqrt{\sigma}. \tag{3.7}
\]

Lemma 1. Let the assumptions \((A_\alpha)\) with \( p = p_u = p_v \geq 2, \alpha = 1, 2 \), assumption \((B)\) and the inequalities \( 0 \leq \sigma \leq \sigma_\alpha \leq \lambda_u \) be fulfilled. Let also \( w_h = (v_h, u_h)^T \in V_h(\Omega) \times U_h(\Omega) \) be the solution to the system (2.4), \( \bar{u} \) be any function from \( V_{h,0}(\Omega) \), and \( m \) be any vector-function belonging to \( M(\Omega, D^*) \). Then the a posteriori error bound

\[
\|e_{\text{fem}}\|^2 \leq \|\Delta_h (u_h - \bar{u})\|^2 + \|v_h - \Delta_h \bar{u}\|^2 + 2\|\kappa (u_h - \bar{u})\|^2 + 2\Theta M(\sigma_u, \bar{u}, m), \tag{3.8}
\]

holds where \( \Theta = 1 + (\sigma_u - \sigma_\text{min})/\lambda_u \).

Proof. Two first summands in the figure brackets we transform to the form

\[
\|\Delta_h e_u\|^2 + \|\kappa e_u\|^2 = (\Delta_h (u_h - \bar{u}), \Delta_h e_u) + (\sigma (u_h - \bar{u}), e_u) + (\Delta (\bar{u} - u), \Delta_h e_u) + (\sigma (\bar{u} - u), e_u), \tag{3.9}
\]

and in a similar way transform the rest terms:

\[
\|e_v\|^2 + \|\kappa e_u\|^2 = ((v_h - \Delta \bar{u}), e_v) + (\sigma (u_h - \bar{u}), e_u) + (\Delta (\bar{u} - u), e_v) + (\sigma (\bar{u} - u), e_u). \tag{3.10}
\]
For \( e = \tilde{e} := \tilde{u} - u \), we use the inequality (2.8) in the form
\[
\|\tilde{e}\|^2 \leq \lambda_{\tilde{u}}^{-1} \|\Delta \tilde{e}\|^2.
\]
The function \( \tilde{u} \) can be considered as an approximation of the of the solution of the problem (1.1) and, therefore, \( \|\Delta \tilde{e}\|^2 + \|k \tilde{e}\|^2 \) we can use the a posteriori bound similar to (3.1)
\[
\|\Delta \tilde{e}\|^2 + \|k \tilde{e}\|^2 \leq \Theta M(\sigma, \tilde{u}, m),
\]
where \( \Theta = 1 + (\sigma - \sigma_{\min}) / \lambda_{\tilde{u}} \). The proof of it follows the same path with the proof of (3.1). Combining (3.9), (3.10) and (3.7), the use of Cauchy inequality and the bound (3.11), result in the inequality
\[
\|e_{\text{fem}}\|^2 \leq \{\|u_h - \tilde{u}\|^2 + \|v_h - \Delta_h \tilde{u}\|^2 + 2\|k(u_h - \tilde{u})\|^2 + 2\theta M(\sigma, \tilde{u}, m)\}^{1/2} \times
\]
\[
\frac{1}{\sqrt{2}} (\|\Delta_h e_u\|^2 + \|e_u\|^2 + 2\|k e_u\|^2)^{1/2},
\]
from which the bound (3.8) follows.

The value \( \lambda_{\tilde{u}}^{-1} \) is calculated by the constant \( c_F^{-1} \) from the Friedreich type inequality
\[
c_F \|\phi\|^2 \leq \|\Delta \phi\|^2, \quad \forall \phi \in \tilde{H}^1(\Omega).
\]
The testing moments \( m \) for the use in (3.8) can be defined by means of the recovery procedure of the same type as for moments \( m \) in (3.1). Additionally, it is necessary to define \( \tilde{u} \) by smoothing the finite element solution \( u_h \). This can be done at least in two ways: by the least squares projection of \( u_h \) upon the space \( V_{h,0}(\Omega) \), i.e., by setting \( \tilde{u} = \pi_h u_h \), or as \( \tilde{u} = E_h u_h \), where \( E_h : \mathcal{U}_h(\Omega) \rightarrow V_{h,0}(\Omega) \) is the recovery operator, based on averaging. If \( V_{h,0}(\Omega) \) is Hsieh-Clough-Tocher (HCT) finite element space, then some properties for the corresponding recovery operator can be found in [13, 34].

4. Conclusion
To the best of the author’s knowledge, for the first time, the a posteriori error bounds for solutions by the mixed Ciarlet-Raviart type finite element method of the 4th-order elliptic equations with the nonzero reaction term. There is an additional feature in the problem, which is taken into consideration in the paper and is new even for the studies related to the conform methods for the 2nd-order elliptic equations. The reaction coefficient, assumed for simplicity to be finite element wise constant, is allowed to change chaotically between elements in a wide range. In the initial step we considered solutions by the \( C^1 \)-compatible finite element method and proved that the a posteriori bound was robust in case of the reaction coefficient varying chaotically between finite elements on the segment \([0, \mathcal{O}(h^{-4})]\). This a posteriori estimate is sharp (exact in the order of accuracy), if the testing moments accuracy is equal in the order to the accuracy of the finite element moments. And the testing moments do not require equilibration as in a number of other papers on the a posteriori error bounds for the mesh methods for the equation with thebiharmonic operator. The a posteriori error bound for the mixed Ciarlet-Raviart method was derived with the use of the presented result on the conform methods. One of the open questions, related to the mixed methods and left for future to answer, is about the most accurate evaluation of the coefficient before the norm of the residual.
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