Non-Contextual and Local Hidden-Variable Model for the Peres–Mermin and Greenberger–Horne–Zeilinger Systems

Carsten Held

Received: 5 March 2020 / Accepted: 18 December 2020 / Published online: 4 March 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC part of Springer Nature 2021

Abstract
A hidden-variable model for quantum–mechanical spin, as represented by the Pauli spin operators, is proposed for systems illustrating the well-known no-hidden-variables arguments by Peres (Phys Lett A 151:107–108, 1990) and Mermin (Phys Rev Lett 65:3373–3376, 1990) and by Greenberger et al. (Bell’s theorem, quantum theory, and conceptions of the universe, Kluwer, Dordrecht, 1989). Both arguments rely on an assumption of non-contextuality; the latter argument can also be phrased as a non-locality argument, using a locality assumption. The model suggested here is compatible with both assumptions. This is possible because the scalar values of spin observables are replaced by vectors that are components of orientations.

Keywords
No-hidden-variables theorems · Contextuality · Non-locality · Geometric algebra

1 Introduction
Before the creation of quantum mechanics (QM), measurement of a physical system was conceived as faithful in the sense that it consists in ‘the ascertaining of some pre-existing property of some thing, any instrument involved playing a purely passive role’ [1, 2]. As is now well-known, this classical conception of measurement is in conflict with QM, understood as a complete description of the quantum domain. Consider an interpretation of QM that maintains: all observables are faithfully measured in the sense that measurement reveals ‘some pre-existing property’ of the quantum system, ‘any instrument involved playing a purely passive role’. This interpretation is in an immediate conflict with well-known no-hidden-variables arguments of the Bell–Kochen–Specker type [3–6]. Such arguments show that any
hidden-variable model for QM observables is necessarily contextual, given that algebraic relations among operators representing the observables are mirrored in the observables’ values. Contextuality here means that the model must allow some observables to have different values in different contexts, i.e. as elements of different sets of observables.

What sense could be made of such contextuality? A natural idea would be measurement contextuality: the idea that the ‘instrument’ does not play ‘a purely passive role’ in the measurement process. An observable’s value would thus depend upon the process of measurement of a set of observables including it. This idea, of course, conflicts with the original idea of measurement as the recording of pre-existing properties and thus, for the hidden-variable-theorist, is not worth pursuing (see, however, [7, 8]). An alternative is ontological contextuality [9, 10], the idea that instead of one observable with different context-dependent values there are really two different observables (represented by the same operator) with different values. Without further explanation, this idea appears to be entirely ad hoc and, since no explanation has been forthcoming, is no longer pursued in the literature. It is widely agreed that the ideal of faithful measurement implies faithful measurement of non-contextual properties and as such is refuted by the mentioned arguments.

However, abandoning the idea of faithful measurement does not avoid contextuality. Since a QM system’s being in a certain state is equivalent to the state’s being an eigenstate of certain observables, being in a certain state is equivalent to being in a certain context of observables. If it is impossible that all observables have the same values in all states and some observables have values in certain states, others have no values in these states. So, whether an observable has a value depends upon which other observables have values. In this sense, QM is contextual even without faithful measurement. But we have no real understanding of this contextuality.

If there is contextuality it also exists among observables pertaining to space-like separated parts of a QM system [11, 12]. Non-contextuality of such observables is equivalent with their locality, and Bell’s Theorem [13] famously shows such locality to yield statistical predictions at variance with those of QM.¹ Thus, contextuality reappears, in the context of Bell’s Theorem, as non-locality and again it appears as a characteristic of QM that we do not really understand.

Or is there no need here for further understanding, just one for acceptance? The majority of interpreters embraces a fully non-classical world-picture: non-faithful measurement, contextuality and non-locality. There is a minority of dissenters, interpreters trying to restore classicality, but they usually are forced to make extravagant metaphysical assumptions. Here are two examples from the recent literature. The consistent histories approach [14–17] claims that it can save the mentioned three elements of classicality [18, 19] but it is forced to claim that there exists no single description of a QM system that is both exhaustive and true.² This approach, if indeed it saves classicality, does so at an exceedingly high price. We might think that classicality and the possibility of a single true and exhaustive description of any

---

¹ See [10], chap.6.
² See [15] p. 365, [16] Sect. 4.5, [17] Sect. 3.3.
system are parts of the same realist world-view but according to this approach they are not. A second example comes from the group of no-detection theories, i.e. interpretations exploiting the so-called detection loophole. One of these theories, called extended semantic realism [20–22], explicitly claims to save the three classicality features but it must assume that the ensemble of detected systems always is an unfair sample from the one of prepared systems. This suggests that Nature purposefully hides itself from the physicist. (Moreover, there is empirical evidence for the claim that the detection loophole can be closed [23], such that unfair sampling no longer plays an important role in the interpretation of QM.)

In the following, a new attempt to save classicality will be made—but one that avoids counter-intuitive metaphysical consequences. A hidden variable will be suggested that effectively replaces QM observables and their scalar values by vector variables and their vector values. The result will be a conception of properties of QM systems respecting the three features of classicality. The conception will be developed for two related and particularly simple cases: the Peres–Mermin (PM) and Greenberger–Horne–Zeilinger (GHZ) systems consisting, respectively, of two and three spin-½ particles. In the resulting model, the QM observables are replaced by vector variables; these variables jointly have values that are identical across contexts, thus satisfying non-contextuality, and yet they meet the PM and GHZ constraints. Moreover, in the GHZ case the values of variables pertaining to an individual particle can be predicted without interfering with the particle, thus satisfying locality, and yet they meet the GHZ constraints on the values in a particular three-particle state. As a result, arguments from QM against faithful measurement no longer apply and we obtain a possibility to reclaim classicality.

2 The PM, GHZ and Bell-GHZ Arguments

We consider no-hidden-variables arguments that employ systems consisting of two or three spin-½ particles described by the familiar Pauli spin operators. We first recall the equations defining these operators. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be an orthonormal basis of $\mathbb{R}^3$ that, by stipulation, is right-handed. Let $\sigma_x, \sigma_y, \sigma_z$ be operators associating the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ with values $\pm 1$. Then, QM defines these operators by the equation:

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k,$$

where $i, j, k = x, y, z$ and $\mathbf{1}$ is the unit operator. If spin operators for more than one system appear, they are distinguished by superscripts $1, 2, 3, \ldots$ and QM prescribes, for $a, b, \ldots = 1, 2, 3, \ldots$ and $a \neq b$, for $i, j = x, y, z$, that:

$$\sigma_i^a \sigma_j^b = \sigma_j^b \sigma_i^a.$$

---

3 See e.g. [22], p. 29.
Assume now what is called the Product Rule, i.e. the claim that for commuting observables \( A \) and \( B \), the value of their product equals the product of their values, i.e.

\[
[A B] = [A][B],
\]

where \([A]\) is the value of \( A \) existing in the system, similarly for \([B]\), and we have \([A, B] = 0\). (This rule follows when we require that the algebraic relations among the operators or observables are mirrored in the values.)

From (1–3), we can derive two well-known no-hidden-variables arguments. They operate with systems consisting of two or three spin-½ particles. We refer to the systems measured for certain observables simply as systems. E.g., the Peres–Mermin (PM) system is a two-particle spin-½ system measured for the following nine observables:

\[
\sigma_x^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_z^2, \sigma_x^1 \sigma_z^2, \sigma_y^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_x^2, \sigma_z^1 \sigma_y^2, \sigma_z^1 \sigma_z^2. \quad \text{(See [4, 5].)}
\]

From (1–3), it follows that the values of these observables (all equal to ± 1) satisfy the following six constraints:

\[
\begin{align*}
[\sigma_x^1 \sigma_x^2] [\sigma_x^1] [\sigma_x^2] &= 1, \\
[\sigma_y^1 \sigma_y^2] [\sigma_y^1] [\sigma_y^2] &= 1, \\
[\sigma_z^1 \sigma_z^2] [\sigma_z^1] [\sigma_z^2] &= 1, \\
[\sigma_x^1 \sigma_z^2] [\sigma_y^1] [\sigma_z^2] &= 1, \\
[\sigma_y^1 \sigma_x^2] [\sigma_y^1] [\sigma_z^2] &= 1, \\
[\sigma_z^1 \sigma_z^2] [\sigma_z^1] [\sigma_z^2] &= 1.
\end{align*}
\]

So, QM ((1) and (2) above) and the Product Rule (3) jointly predict that the result of measuring the three observables in any line (4a–f) meets the constraint explicitted in that line. On the other hand, given an assumption of faithful measurement, any such measurement of three observables just reveals their pre-existing values. As a consequence, any of the values appearing on the left of (4) must be the same in both places; this is the non-contextuality assumption. So, all six constraints in (4) must jointly be met. But this is impossible because the product of all the left sides of (4) equals 1, while the one of all the right sides equals − 1. Since this argument operates with the PM system, we call it the PM argument.

The Greenberger–Horne–Zeilinger (GHZ) argument ([6]) can be cast in a similar form ([5]). The GHZ system is a three-particle spin-½ system measured for the
following ten observables: \( \sigma_1^x, \sigma_2^x, \sigma_3^x, \sigma_1^y, \sigma_2^y, \sigma_3^y, \sigma_1^z, \sigma_2^z, \sigma_3^z \). For these observables, (1–3) yield these five constraints on the ten observables’ values:

\[
[\sigma_1^x \sigma_2^y \sigma_3^z][\sigma_1^x][\sigma_2^y][\sigma_3^z] = 1, \quad (5a)
\]

\[
[\sigma_1^y \sigma_2^x \sigma_3^z][\sigma_1^y][\sigma_2^x][\sigma_3^z] = 1, \quad (5b)
\]

\[
[\sigma_1^y \sigma_2^y \sigma_3^x][\sigma_1^y][\sigma_2^y][\sigma_3^x] = 1, \quad (5c)
\]

\[
[\sigma_1^x \sigma_2^x \sigma_3^y][\sigma_1^x][\sigma_2^x][\sigma_3^y] = 1, \quad (5d)
\]

\[
[\sigma_1^x \sigma_2^y \sigma_3^y][\sigma_1^x \sigma_2^y \sigma_3^y][\sigma_1^x \sigma_2^y \sigma_3^y][\sigma_1^x \sigma_2^y \sigma_3^y] = -1. \quad (5e)
\]

Again, each line contains a set of three mutually commuting observables, i.e. QM plus the Product Rule predicts that the result of measuring the three observables in any line meets the constraint explicated in that line. Again, we make an assumption of faithful measurement; we approach the GHZ system picking an arbitrary set of three observables (an arbitrary one of lines (5a–e)) and measurement then faithfully reveals pre-existing values of the observables. In particular, any one of the values appearing on the left of (5) must be the same in both places; this is the non-contextuality assumption, again. So, the five constraints in (5a–e) must jointly be met. Again, this is impossible since the product of all the left sides is 1, but the one of all the right sides is −1.

This argument using the GHZ system and a non-contextuality assumption can also be framed as a non-locality argument that may be referred to as the Bell-GHZ argument. For future reference, we directly quote the argument from Mermin:

[Consider] a system in a particular one of the simultaneous eigenstates of the three operators \( \sigma_1^x \sigma_2^y \sigma_3^z, \sigma_1^y \sigma_2^x \sigma_3^z \), and \( \sigma_1^y \sigma_2^y \sigma_3^x \)—say the state \( \Phi \) in which the three have eigenvalue 1. It follows that \( \Phi \) is also an eigenstate of \( \sigma_1^x \sigma_2^x \sigma_3^x = - (\sigma_1^x \sigma_2^x) (\sigma_1^y \sigma_2^y) (\sigma_1^y \sigma_2^x) \) with eigenvalue −1. We now note that if three mutually well separated particles have their spins in this state, then we can learn in advance the result […] of measuring any component from the result of measuring the other two …], since the product of all three measurements in the state \( \Phi \) must be unity. […] Assuming locality, one is thus] impelled to conclude that the results of measuring either component of any of the three particles must have already been specified prior to any of the measurements—i.e. that any particular system in the state \( \Phi \) must be characterized by numbers \( m_1^x, m_1^y, m_2^x, m_2^y, m_3^x, m_3^y \) which specify the results of whichever of the four different sets (xxy, yxy, yxx, xxx) of three single particle spin measurements one might choose to make on the three far apart particles. Because, however, \( \Phi \) is an eigenstate of \( \sigma_1^x \sigma_2^y \sigma_3^z, \sigma_1^y \sigma_2^x \sigma_3^z \), and \( \sigma_1^y \sigma_2^y \sigma_3^x \), with respective eigenval-
ues 1, 1, 1, and –1, the products of the four trios of 1’s or –1’s must satisfy the relations.

\[ m^1_x m^2_y m^3_z = 1, \] (6a)

\[ m^1_y m^2_x m^3_z = 1, \] (6b)

\[ m^1_z m^2_y m^3_x = 1, \] (6c)

\[ m^1_x m^2_z m^3_y = -1, \] (6d)

which, once again, are mutually inconsistent, the product of the four left sides being necessarily +1.\(^5\)

### 3 Interpretation of the Geometric Algebra of \( \mathbb{R}^3 \)

In the following, our goal is to show how (4a–f), (5a–e), and (6a–d), instead of each implying the falsity \( 1 = -1 \), can each lead to a truth. This will indeed be done for (4a–f), (5a–e), and a proxy for (6a–d), but only given a substantial reinter-pretation of the formalism. The basic idea is to reinterpret the values of observables \( \sigma_x, \sigma_y, \sigma_z \) conventionally conceived to be scalars, as vectors. This will avoid the false equation \( 1 = -1 \).

To prepare for this reinterpretation, we recall basic ideas of geometric algebra (GA), the mathematical theory exploring multivector spaces generated from vector spaces by means of the geometric product. In particular, we consider the multivector space \( \mathbb{G}^3 \) that is generated from the vector space \( \mathbb{R}^3 \), also called the geometric algebra \( \mathbb{G}^3 \), which is described in detail in many places in the literature [24–26]. Let \( \{e_1, e_2, e_3\} \) be an orthonormal and right-handed basis of \( \mathbb{R}^3 \). Given the geometric product, the vectors \( e_1, e_2, e_3 \) instantiate the very structure that was used in (1) to define the Pauli operators, i.e.:

\[ e_i e_j = \delta_{ij} + I \sum_{k=1}^{3} \varepsilon_{ijk} e_k. \] (7)

for \( i, j, k = 1, 2, 3 \), where \( I \) is the unit pseudoscalar in \( \mathbb{G}^3 \). (7) yields:

\[ e_i e_i = 1 \] and

\[ e_i e_j = -e_j e_i, \] for \( i \neq j \). (8b)

Note that (7) is not a stipulation but follows from the properties of our basis \( \{e_1, e_2, e_3\} \) and the definition of the geometric product on \( \mathbb{R}^3 \) [27]. Important

---

\(^5\) [5] p. 3375 (numbering of equations adapted).
multivectors (elements of $G^3$) constituted by the basis vectors with the geometric product are bi- and trivectors. The product $e_i e_j$, for $i \neq j$, is called a unit bivector; the product $e_i e_j e_k$, for $i \neq j, j \neq k, i \neq k$, is called a unit trivector.

### 3.1 Interpreting Bi- and Trivectors

Bi- and trivectors can be interpreted geometrically. We briefly state key ideas that will be used in our model in Sects. 4 and 5. From (8), we can derive:

$$e_i e_j e_j e_i = 1,$$

(9)

$$e_i e_j e_i = -1.$$  

(10)

We interpret $e_i e_j$ and $e_j e_i$ as orientations of systems extended in the $i,j$-plane. We stipulate that an orientation annihilates an orientation iff their product equals 1. Since there are exactly two different orientations $e_i e_j$ (counter-clockwise) and $e_j e_i$ (clockwise) in the $i,j$-plane and they multiply to 1, these two orientations annihilate each other, which is the content of (9). Since $e_i e_j$ and $(-e_j) e_i$ are the same orientation, they do not annihilate each other and their product equals $-1$, which, given (8), is the content of (10).

To interpret trivectors, we multiply (8b) by $e_k$:

$$e_i e_j e_k = - e_j e_i e_k,$$

(11)

where $(i, j, k)$ is any permutation of $(1, 2, 3)$. From (11), by (8):

$$e_i e_j e_k e_k e_j e_i = 1,$$

(12)

$$e_i e_i e_k e_k e_i = -1.$$  

(13)

The product $e_i e_j e_k$ can be interpreted as an orientation of a system extended in all three dimensions of $\mathbb{R}^3$. Again, we stipulate that an orientation annihilates an orientation iff their product equals 1. Since there are exactly two different orientations $e_i e_j e_k$ (right-handed) and $e_j e_i e_k$ (left-handed) and their product equals 1, they annihilate each other, which, given (8), is the content of (12). Since $e_i e_j e_k$ and $(-e_j) e_i e_k$ are the same orientation they do not annihilate each other and their product equals $-1$, which, given (8), is the content of (13). More directly, $e_i e_j e_k$ is identical with itself and does not annihilate itself. Thus, its product with itself does not equal 1 but $-1$, which again is the content of (12).$^6,7$

---

$^6$ arXiv:1907.13073v2 (2019) (an extended version of this paper).

$^7$ See footnote 6, 3.1-3.2 for details of this interpretation.
3.2 Vector Variables

We generalize these notions in two directions: first, from vectors to vector variables and second, from one to several bases of $\mathbb{R}^3$. We begin with the variables. The structure of (9) and (10) suggests the introduction of vector-valued variables. We introduce variables $\sigma_i$, $\sigma_j$, $\sigma_k$ that are two-valued, i.e. can take on values $\pm e_i$, $\pm e_j$, $\pm e_k$, respectively. In (7), $e_i$ can be replaced with $-e_i$, ad libitum, as long as we do it in both occurrences, and similarly for $e_j$; hence, (7) and (10) can be generalized to:

$$\sigma_i \sigma_j \sigma_k \sigma_i = 1,$$

(14)

$$\sigma_i \sigma_j \sigma_i = -1.$$  

(15)

((14) and (15) have the same interpretation as (10) and (11), i.e. one orientation in the $i,j$-plane annihilates the other but does not annihilate itself.) A similar generalization suggests itself for the 3D case. In (12) and (13) $e_i$, $e_j$, and $e_k$ can be freely exchanged with their negatives without affecting the right sides, thus we can again generalize them to the $\sigma$-variables:

$$\sigma_i \sigma_j \sigma_k \sigma_j \sigma_i = 1,$$

(16)

$$\sigma_i \sigma_j \sigma_k \sigma_i = -1.$$  

(17)

(16) expresses that $\sigma_i \sigma_j \sigma_k = \sigma_i \sigma_j \sigma_k$ and (17) expresses that $\sigma_i \sigma_j \sigma_k \neq \sigma_k \sigma_j \sigma_i$. Since there are only two orientations corresponding to odd and even permutations of (1–3), the latter inequality entails $\sigma_i \sigma_j \sigma_k = -\sigma_k \sigma_j \sigma_i$. (16) and (17) have the same interpretation as (12) and (13): different orientations in $\mathbb{R}^3$ annihilate each other and no orientation annihilates itself. Note that the vector value equations, and a fortiori the vector variable equations, are compatible with any choice of orientation in the $i,j$-plane or $\mathbb{R}^3$. The constraints that arise refer to different and identical orientations but whether an individual orientation is right- or left-handed does not have to be specified. This circumstance is what later makes the actual orientation of a system a hidden variable.

3.3 Identities of Orientations

So far, we have considered orientations constituted by elements of the basis $\{e_1, e_2, e_3\}$. Consider now orientations constituted by vectors from different bases. More explicitly, consider $\{e_1, e_2, e_3\}$ and a second orthonormal basis $\{f_1, f_2, f_3\}$ (from now on, the elements of these bases are often briefly called the $e$’s and the $f$’s). Initially, we leave open whether or not the products $e_i e_j e_k$ and $f_i f_m f_n$ (where $(l, m, n)$ is any permutation of $(1, 2, 3)$) are the same orientation. Now assume that these two orientations are in fact identical, i.e. assume:

$$e_i e_j e_k = f_l f_m f_n.$$  

(18)
(l, m, n) is an either odd or even permutation of (1, 2, 3). Choosing the even case, we can rewrite (18) as:

$$e_ie_je_k = f_if_jf_k.$$  \hspace{1cm} (19)

Since the f’s are orthonormal, an analog of (7) holds for them. Hence, (19) is equivalent to:

$$e_ie_je_kf_if_jf_k = 1 \text{ and}$$  \hspace{1cm} (20)

$$e_ie_je_kf_if_jf_k = -1.$$  \hspace{1cm} (21)

Now, what does it mean to say that two orientations are identical? We have interpreted unit bi- and trivectors not as orientations characterizing a certain plane in $\mathbb{R}^3$ or the whole of $\mathbb{R}^3$, but as orientations characterizing systems extended in a plane or in $\mathbb{R}^3$. Hence, we can distinguish two kinds of the identity of orientations: one where two orientations of the same system are identical and another where two orientations of different systems are identical. The distinction is intuitively accessible, though conceptually non-trivial.8

Assume that a system in the i,j-plane or in $\mathbb{R}^3$ has a unique (2D or 3D) orientation. Then the two kinds of identity just considered lead to different consequences. It suffices to consider the 3D case. Consider first two orientations, one constituted by e’s, the other by f’s, and pertaining to the same system. Since the system’s orientation is unique, the two orientations must be identical, i.e. (19) is true by construction. Since the e’s are a basis of $\mathbb{R}^3$, the f’s can be written in terms of the e’s and it is obvious that not all e’s and f’s commute.

Consider second two orientations, one constituted by e’s, the other by f’s, and pertaining to two different systems. In this case, (19) is contingently true or false. Given an orientation $e_ie_je_k$ of one system and a basis of f’s chosen to describe a second system’s orientation, the order of f’s and hence this second orientation is arbitrary. For this case, we would like to assume that the orientations’ constituents, the e’s and f’s, all commute—but will be able to do so only with a certain qualification.

The distinction of e’s and f’s with respect to one system turns out to be superfluous. The orientation $f_if_jf_k$ can be written in terms of the e’s and is identical with $e_ie_je_k$ iff (20) is true. Given our assumptions that $e_ie_je_k$ and $f_if_jf_k$ are orientations of the same system and this system has a unique orientation, (20) is necessarily true. Hence, from now on we stop to refer to components of orientations of the same system via different sets of vectors, the e’s and f’s, and reserve the letters ‘e’ and ‘f’ for constituents of orientations of different systems.

We want to assume that these constituents, the e’s and f’s, generally commute but have to allow for one qualification. The e’s and f’s cannot be assumed to commute in the presence of identities between individual e’s and f’s. Assume (21) and assume, in addition, that $e_k = f_k$, whence it follows that $e_i e_j = f_i f_j$. It is easy to show that in

---

8 See footnote 6, Appendix A4 for details.
this case the e’s and f’s cannot all commute.\(^9\) So, what we finally assume is that all the e’s and f’s commute iff no identity of any e with any f obtains. Below, we will consider also a third orthonormal basis \(\{g_1, g_2, g_3\}\), the g’s. Qualifications analogous to the ones for e’s and f’s hold also for e’s and g’s and for f’s and g’s.

In (20) and (21), we cannot replace arbitrary e’s or f’s with their negatives on pain of incoherence with (19). Thus, (20) and (21) can be generalized by means of possible values ±1 for \(e_i\), ±1 for \(e_j\), ±1 for \(e_k\), and ±1 for \(f_i\), ±1 for \(f_j\), ±1 for \(f_k\), respectively. The variables’ values must be chosen so as to cohere with (19), which is the case, e.g., when the values of (\(\sigma^1\))’s and (\(\sigma^2\))’s reproduce the LHS of (20). Given this choice, (20) and (21) can be generalized to:

\[
(\sigma^1_i \sigma^1_j \sigma^1_k)(\sigma^2_j \sigma^2_k) = 1 \quad \text{and} \quad (22)
\]

\[
(\sigma^1_i \sigma^1_j \sigma^1_k)(\sigma^2_j \sigma^2_k) = -1 \quad \text{(23)}
\]

(with round brackets added for clarity). We assume that there are no identities between e’s and f’s and thus may assume that they commute with each other. If they do, then so do the (\(\sigma^1\))’s and the (\(\sigma^2\))’s and we can rewrite (22) and (23) thus:

\[
(\sigma^1_i \sigma^1_j \sigma^1_k)(\sigma^1_j \sigma^1_k)(\sigma^1_k \sigma^1_k) = 1, \quad \text{(24)}
\]

\[
(\sigma^1_i \sigma^1_j \sigma^1_k)(\sigma^2_j \sigma^2_k)(\sigma^2_k \sigma^2_k) = -1. \quad \text{(25)}
\]

We have considered the identity of 3D orientations pertaining to different systems and now turn to the 2D case. Consider the e’s and f’s again, plus a third orthonormal basis \(\{g_1, g_2, g_3\}\), the g’s. Since they all obey (7), they also all obey (9) and (10), i.e. \(e_i e_i e_j = -(f_i f_j f_j) = g_i g_j g_j = -1\), which yields:

\[
(e_i e_i e_j)(f_i f_j f_j)(g_i g_j g_j) = -1. \quad \text{(26)}
\]

We assume that there are no identities between the e’s, f’s, and g’s and thus may assume that they all commute. Hence, from (26):

\[
(e_i f_i g_i)(e_i f_i g_j)(e_j f_j g_j)(e_j f_j g_i) = -1. \quad \text{(27)}
\]

Since any one of the e’s, f’s, and g’s appearing in (26) appears twice, we can generalize (26) to \(\sigma\)-variables as:

\[
(\sigma^1_i \sigma^1_j \sigma^1_j)(\sigma^2_j \sigma^2_k \sigma^2_j)(\sigma^2_k \sigma^2_k \sigma^3 \sigma^3 \sigma^3) = -1. \quad \text{(28)}
\]

\(^9\) See footnote 6, Appendix A3 for a proof.
Since the e’s, f’s and g’s all mutually commute, so do the (σ^1)'s, (σ^2)'s and (σ^3)'s. Thus, from (27) by generalization to σ-variables or from (28) by commutation of the σ’s pertaining to different systems:

\[(σ^1_1σ^2_1σ^3_1)(σ^1_1σ^2_2σ^3_2)(σ^1_1σ^2_3σ^3_3) = −1.\]  

(29)

In our considerations of orientations pertaining to two or three different systems, we have explicitly assumed that there are no identities between the e’s, f’s, and g’s. We now finally drop this assumption and consider a triple of orientations with the additional property of identical components. As emphasized above, this excludes the claim that the components of different systems’ orientations commute. We consider the e’s, f’s and g’s lying in the i,j-plane and their products; in particular, we are interested in these four multivectors: e_i f_j g_j, e_j f_i g_j, e_j f_j g_i, e_i f_i g_i. We choose the most natural identities: e_i = f_i = g_i and e_j = f_j = g_j. From this assumption and using (8) again, we immediately get:

\[e_i f_j g_j = e_i e_j e_j = e_i,\]  

(30a)

\[e_j f_i g_j = e_j e_i e_j = −e_i,\]  

(30b)

\[e_j f_j g_i = e_j e_j e_i = e_i,\]  

(30c)

\[e_i f_i g_i = e_i e_i e_i = e_i.\]  

(30d)

We note that all three columns in (30a–d) multiply to −1, in contrast with the similar, yet inconsistent, system of equations (6a–d).

4 Deriving the PM and GHZ Value Constraints

Evidently, the equations we constructed—(24), (25), (29) and (30)—are structurally similar to Eqs. (4e), (4f), (5e) and (6) characterizing the PM and GHZ systems. In deriving the former, we have made use of the idea that bi- and trivectors are orientations attached to systems that are extended in two or three spatial dimensions. This suggests an interpretation of the examples in terms of such orientations.

Our hidden variable model consists solely in the introduction of an orientation characterizing individually each subsystem of the PM and GHZ systems. More specifically, we replace the values of the spin operators by components of orientations, either vectors or their geometric products, and rule that these components obey the GA equations ((7) and its implications) for such multivectors. Formally, we replace the quantum–mechanical operators, which are scalar-valued, by vector variables, which are vector-valued, but we indiscriminately write both (the

10 See the discussion in footnote 6, Appendix A4.
operators and the variables) as $\sigma$’s; moreover, we write the values of the variables in square brackets, as we did with the ones of the operators. For the elementary vector variables, consisting of only one of the $\sigma$’s, we assume that $[\sigma^1_x] = \pm e_1$, $[\sigma^2_x] = \pm e_3$; similarly, $[\sigma^1_y] = \pm f_1$, $[\sigma^2_y] = \pm f_3$, and $[\sigma^3] = \pm g_1$, $[\sigma^3_y] = \pm g_2$, $[\sigma^3_z] = \pm g_3$. For simplicity, we set all these values to positive vectors ($[\sigma^1_x] = e_1$, and so on), since, as will become apparent, the choice of these values, and thus the choice of one particular orientation, is arbitrary. Call this ascription of exclusively positive vectors to the elementary $\sigma$-variables the value assumption. Concerning all multivector variables, we assume a counterpart of the Product Rule (3), i.e. if for two such variables $A$, $B$ we have $[A, B] = 0$, then $[AB] = [A][B]$. Call this the product assumption. Finally, assume the commutativity of $e$’s, $f$’s, and $g$’s and call it the commutativity assumption or briefly: commutativity. Given these three assumptions, the PM and GHZ value constraints, i.e. equation systems (4) and (5) above, are readily derived from GA. We begin with the PM system. (4a) above is derived as follows:

$$
[\sigma^1_x \sigma^2_x][\sigma^1_x][\sigma^2_x]
= (e_1 f_1)(e_1) (f_1)
= (e_1 e_1)(f_1 f_1)
= 1.
$$

In (31), equation 1 is due to the value and product assumptions, equation 2 to the commutativity of $e$’s and $f$’s, and equation 3 to GA, i.e. (8a) above. Equations (4b–d) follow on the same lines. We can derive (4e) as follows:

$$
[\sigma^1_x \sigma^2_y][\sigma^1_y][\sigma^2_x]
= (e_1 f_2)(e_2 f_1) (e_3)
= (e_1 e_2 e_3)(f_2 f_1 f_3)
= 1.
$$

The equations in (32) again hold due to the value and product assumptions, commutativity of $e$’s and $f$’s, and finally GA, i.e. (20) above. (4f) is obtained in the same way, using (21). The heart of each one of these simple derivations is its last equation. The last step of (32) is an instance of (20) and in a parallel derivation of (4f) the last equation would be an instance of (21): $e_1 e_2 e_3 f_1 f_2 f_3 = -1$. Multiplication of all value equations in the PM argument boils down to multiplying these two instances, giving us (with the help of commutativity):

$$
(e_1 e_2 e_3)(e_1 e_2 e_3)(f_2 f_1 f_3)(f_1 f_2 f_3) = -1.
$$

Clearly, (33) could not be satisfied if the $e$’s and $f$’s were real numbers, but since we assume them to be vectors obeying (8), (33) follows directly (from (20), (21) and commutativity).

We can treat the GHZ system in the same way. We derive (5a) as follows:
\[ [\sigma_x^1 \sigma_y^2 \sigma_z^3][\sigma_x^1][\sigma_y^2][\sigma_z^3] \]
\[ = (e_1 f_2 g_2)(e_1 f_2)(g_2) \]
\[ = (e_1 e_1)(f_2 f_2)(g_2 g_2) \]
\[ = 1. \tag{34} \]

Here, we have used the value assumption, the commutativity of \( e \)'s, \( f \)'s, and \( g \)'s, and GA, i.e. (8) above. Similarly for (5b–d). Finally, for (5e):

\[ [\sigma_x^1 \sigma_y^2 \sigma_z^3][\sigma_x^1 \sigma_y^2 \sigma_z^3][\sigma_x^1 \sigma_y^2 \sigma_z^3][\sigma_x^1 \sigma_y^2 \sigma_z^3] \]
\[ = (e_1 f_1 g_1)(e_1 f_2 g_2)(e_2 f_1 g_2)(e_2 f_2 g_1) \]
\[ = (e_1 e_1 e_2 e_2)(f_1 f_2 f_2 f_2)(g_1 g_2 g_2 g_1) = -1. \tag{35} \]

Again, we have used the value assumption, commutativity, and GA, i.e. (8).

Our derivations have not required an assumption of non-contextuality, but obviously such an assumption is respected. We have effectively assumed that in (4a–f) all value expressions in square brackets are vectors or their products, but not scalars, and we have arbitrarily restricted ourselves to vectors with a positive sign. These vectors are the same for every occurrence of any vector variable. E.g., we have \([\sigma_1^1]=e_1\) in both (4a) and (4c) and \([\sigma_1^1 \sigma_2^1]=e_1 f_2\) in both (4c) and (4e), and similarly for all other expressions on the LHS of (4). The values of all variables of the PM system are the same in different contexts, i.e., across different lines of (4), and thus are manifestly non-contextual. The same holds for (5a–e) and the variables of the GHZ system.

### 5 Deriving a Proxy for the Bell-GHZ Constraints

Defusing the Bell-GHZ argument is less straightforward. Mermin’s version of the argument ends with the false equation \( 1 = -1 \), as did the PM and GHZ arguments, but Mermin’s \( m \)'s are assumed to individually equal \( \pm 1 \), i.e. they are real numbers, not vectors. However, we can drop Mermin’s assumption in line with our basic idea to re-interpret the values of QM observables as vectors. We thus assume the values of elementary observables to be vectors and the values of their products to be geometric products of vectors. Our proxy for (6a–d) then is the system of four equations that are the rightmost equations of (36a–d) below. Multiplying these four equations and using (8), we get \(-1 = -1\), a trivial truth replacing the falsity \( 1 = -1 \).

We derive (36) as follows. Consider again the vector variables \( \sigma_x^1, \sigma_y^1, \sigma_z^2, \sigma_y^2, \sigma_z^3, \sigma_z^3 \) and let all but one of them have positive vector values, i.e.: \([\sigma_1^1]=e_1, [\sigma_1^1]=e_2, [\sigma_2^1]=-f_1, [\sigma_2^2]=f_2, [\sigma_3^1]=g_1, [\sigma_3^3]=g_2\). Assuming again the Product Rule for vector variables, we get: \([\sigma_1^1 \sigma_2^2 \sigma_3^3]=e_1 f_2 g_2\), and so on (see the leftmost equations of (36a–d) below). So far, we have not assumed any connection between the orientations formed by the \( e \)'s, \( f \)'s and \( g \)'s. Now we identify the \( e \)'s, \( f \)'s and \( g \)'s, just as we did above in the general discussion of (30). (Commutativity of vectors from the relevant orientations is thereby excluded.) We choose \( i=1, j=2 \) and demand that
\( \mathbf{e}_1 = \mathbf{f}_1 = \mathbf{g}_1 \) and \( \mathbf{e}_2 = \mathbf{f}_2 = \mathbf{g}_2 \). (Compare the natural identities chosen above for (30).) In this case, we get \( \mathbf{e}_1 \mathbf{f}_2 \mathbf{g}_2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 \), and so on (the middle equations of (36a–d) below). Finally, by using (8), we get the identities that are the rightmost equations of (36a–d). All in all, we have:

\[
\sigma_x^1 \sigma_y^2 \sigma_y^3 = \mathbf{e}_1 \mathbf{f}_2 \mathbf{g}_2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 = \mathbf{e}_1, \tag{36a}
\]

\[
\sigma_y^1 \sigma_y^2 \sigma_x^3 = \mathbf{e}_2 (-\mathbf{f}_1) \mathbf{g}_2 = \mathbf{e}_2 (-\mathbf{e}_1) \mathbf{e}_2 = \mathbf{e}_1. \tag{36b}
\]

\[
\sigma_y^1 \sigma_x^2 \sigma_x^3 = \mathbf{e}_2 \mathbf{f}_2 \mathbf{g}_1 = \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_1 = \mathbf{e}_1, \tag{36c}
\]

\[
\sigma_x^1 \sigma_x^2 \sigma_x^3 = \mathbf{e}_1 (-\mathbf{f}_1) \mathbf{g}_1 = \mathbf{e}_1 (-\mathbf{e}_1) \mathbf{e}_1 = -\mathbf{e}_1. \tag{36d}
\]

Here our choice of identities among the \( \mathbf{e} \)'s, \( \mathbf{f} \)'s and \( \mathbf{g} \)'s is the counterpart of specifying a certain QM state. As quoted above, Mermin chooses a state \( \Phi \) with the property \( [\sigma_x^1 \sigma_y^2 \sigma_y^3] = [\sigma_y^1 \sigma_x^2 \sigma_x^3] = [\sigma_x^1 \sigma_x^2 \sigma_y^3] = 1 \), which choice implies the property \( [\sigma_x^1 \sigma_y^2 \sigma_x^3] = -1 \). We, on the other hand, have assumed identities such that \( [\sigma_x^1 \sigma_y^2 \sigma_y^3] = [\sigma_y^1 \sigma_x^2 \sigma_x^3] = [\sigma_x^1 \sigma_x^2 \sigma_y^3] = \mathbf{e}_1 \), where \( [\sigma_x^1] = -[\sigma_y^2] = [\sigma_x^3] = \mathbf{e}_1 \), which equations imply, via the Product Rule, the property \( [\sigma_x^1 \sigma_x^2 \sigma_x^3] = -\mathbf{e}_1 \). This parallelism confirms the earlier claim that (36) is a consistent replacement of Mermin’s inconsistent (6). Moreover, a simple geometric interpretation of (36) suggests itself: \( \Phi \) is a state where the three systems in question all have a fixed (clockwise or counter-clockwise) orientation in the 1,2-plane.¹¹

Our construction, again, does not explicitly use a locality assumption but it respects (Mermin’s version of) such an assumption. Given that we have measured any two of the components of any multivector variable in the left equations of (36a–d), we can with certainty predict the third component without measuring it. E.g., assume for (36a) that we have found \( [\sigma_x^1] = \mathbf{e}_1 \) and \( [\sigma_y^2] = \mathbf{f}_2 = \mathbf{e}_2 \). Then, because the product of all three vectors must be \( \mathbf{e}_1 \), we can predict that \( [\sigma_y^3] = \mathbf{g}_2 = \mathbf{e}_2 \) without any measurement of the third system, and similarly for all other components. With Mermin we assume that the three systems are ‘mutually well separated’ such that there is no influence from the two measured systems onto the third, unmeasured one. Given this locality assumption we are, as Mermin writes, ‘impelled to conclude that the results of measuring either component of any of the three particles must have already been specified prior to any of the measurements’ (which is Merman’s formulation of the faithful measurement assumption). Thus, the chosen vector variables must possess all their values jointly, where we have, by our choice of identities between the \( \mathbf{e} \)'s, \( \mathbf{f} \)'s and \( \mathbf{g} \)'s, constraints on these values as specified in the middle equations of (36a–d). In particular, the first three values (listed in the left equations of (36a–c)) must each be identical to \( \mathbf{e}_1 \), while the last must be identical to \(- \mathbf{e}_1 \).

¹¹ See footnote 6, Appendix B.
But satisfying these constraints is no problem. The equations on the right of (36a–d) are trivial consequences of (8), which dictates, moreover, that their respective LHS and RHS both multiply to – 1.

6 Discussion

We have seen that for the PM and GHZ systems a non-contextual or non-contextual and local model can be provided—by exploiting the multivector structure of $G^3$. Of course, we have only shown the general possibility of such a model but not developed it in detail. Accordingly, many questions about it remain. We record three salient ones.

First, one may ask why orientations are introduced as hidden variables. Why does it not suffice to replace the scalar values of QM observables by vectors? Clearly, the basic idea of our GA approach is to interpret values of magnitudes as vectors in $G^3$ obeying commutation relations like (8b) and (11). These relations or their equivalents are necessary for nearly all derivations, but it always remains open which orientations they originate from. E.g., the last equation of (35) follows from (10) and (11) independently of whether the $e$’s, $f$’s and $g$’s constitute clockwise or counterclockwise orientations in the 1,2-plane. Or compare (36). From the chosen identities, it is immediately clear that the $e$’s, $f$’s and $g$’s constitute the same orientation in the 1,2-plane, but which one it is remains undetermined because it plays no role in the ensuing constructions. So, the orientations, on which the single vector values depend, are truly hidden within the systems. Presumably, they play a key role for a plausible interpretation, within the model, of entanglement, e.g., of common QM states for two- and three-particle systems like Mermin’s state $\Phi$, referred to in (6) and (36) above, and the state $\Phi'$, referred to in (30) above.\(^{12}\)

Second, a more philosophical question suggests itself, i.e., whether assuming the values of physical magnitudes to be vectors instead of scalars is plausible. Here, we must bear in mind that no claim for all magnitudes corresponding to QM observables is made and the magnitudes in question here are spin components, and assuming their values to be vectors, not scalars, is indeed very plausible.\(^{13}\)

Finally, the question arises whether the model can be developed so as to be transferrable to probabilistic no-HV arguments like the one based on the Bell-CHSH inequality. Some progress in this question has been made recently.\(^{14}\)

7 Conclusion

We have considered the geometric algebra $G^3$, which is generated from the vector space $\mathbb{R}^3$. An immediate consequence of the definition of $G^3$ is Eq. (7), the fundamental equation of GA for orthonormal vectors. We have interpreted equations

\(^{12}\) See again footnote 6, Appendix B.
\(^{13}\) See footnote 6, Appendix C.
\(^{14}\) See footnote 6, Appendix D.
concerning unit bi- and trivectors that follow from (7) in terms of 2D and 3D orientations pertaining to different systems. Moreover, we have assumed that elements of two or three different orientations mutually commute iff no identities between any elements of these orientations obtain. From these assumptions plus commutativity, we derived the equations characterizing the PM and GHZ arguments ((4a–f) and (5a–e)), respecting non-contextuality. From these assumptions plus certain identities between elements of different orientations, we derived a proxy for the equations characterizing the Bell-GHZ argument ((6a–d)), respecting locality. Non-contextuality and locality are consequences of the idea of faithful measurement and by salvaging the former we can rebut the arguments against the latter.

Contextuality, non-locality and non-faithful measurement are generally embraced as the key non-classical features of QM. Interpretations trying to restore classicality usually pay a high price in plausibility, as they make extravagant metaphysical assumptions. We have seen, however, that classicality can be recovered at lesser costs, by invoking simple mathematics rather than implausible metaphysics. Moreover, our use of GA suggests that the mathematical structure of QM can be further elucidated in terms of geometry. Of course, only after the interpretation proposed here has been spelled out in more detail, we will be able to judge its merits for our understanding of the theory as a whole.

References

1. Bell, J.S.: On the impossible pilot wave. Found. Phys. 12, 989 (1982), quoted in [2], 166
2. Bell, J.S.: Speakable and Unspeakable in Quantum Mechanics. Cambridge University Press, Cambridge (1987)
3. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. J. Math. Mech. 17, 59 (1967)
4. Peres, A.: Incompatible results of quantum measurements. Phys. Lett. A 151, 107–108 (1990)
5. Mermin, N.D.: Simple unified form of the major no-hidden variables theorems. Phys. Rev. Lett. 65, 3373–3376 (1990)
6. Greenberger, D.M., Horne, M., Zeilinger, A.: Going beyond Bell’s theorem. In: Kafatos, M. (ed.) Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, pp. 69–72. Kluwer, Dordrecht (1989)
7. Dewdney, C., Holland, P.R., Kyprianidis, A.: What happens in a spin measurement? Phys. Lett. A 119, 259–267 (1986)
8. Norsen, T.: The pilot-wave perspective on spin. Am. J. Phys. 82, 337–348 (2014)
9. van Fraassen, B.C.: Semantic analysis of quantum logic. In: Hooker, C.A. (ed.) Contemporary Research in the Foundations and Philosophy of Quantum Theory, pp. 80–113. Reidel, Dordrecht (1973)
10. Redhead, M.L.G.: Incompleteness, Nonlocality, and Realism. A Prolegomenon to the Philosophy of Quantum Mechanics, p. 135. Clarendon Press, Oxford (1987)
11. Heywood, P., Redhead, M.L.G.: Non-locality and the Kochen-Specker paradox. Found. Phys. 13, 481–499 (1983)
12. Stairs, A.: Quantum logic, realism and value definiteness. Philos. Sci. 50, 578–602 (1983)
13. Bell, J.S.: On the Einstein–Podolsky–Rosen paradox. Physics 1, 195–200 (1964). Reprinted in [2], 14–21
14. Griffiths, R.B.: Consistent histories and the interpretation of quantum mechanics. J. Stat. Phys. 36, 219–272 (1984)
15. Griffiths, R.B.: Consistent Quantum Theory. Cambridge University Press, Cambridge (2002)
16. Griffiths, R.B.: The new quantum logic. Found. Phys. 44, 610–640 (2014)
17. Wallace, D.: Philosophy of quantum mechanics. In: Rickles, D. (ed.): The Ashgate Companion to Contemporary Philosophy of Physics, pp. 16–98. Ashgate Publishing, Aldershot (2008)
18. Griffiths, R.B.: Quantum measurements and contextuality. Phil. Trans. Roy. Soc. A 377, 2019033 (2019)
19. Griffiths, R.B.: Nonlocality claims are inconsistent with Hilbert space quantum mechanics. Phys Rev. A 101, 022117 (2020)
20. Garola, C., Sozzo, S.: Extended representations of observables and states for a noncontextual reinterpretation of QM. J. Phys. A 45, 075303 (2012)
21. Garola, C.: A survey of the ESR model for an objective reinterpretation of quantum mechanics Int. J. Theor. Phys. 54, 4410–4422 (2015)
22. Garola, C., Sozzo, S., Wu, J.: Outline of a generalization and a reinterpretation of quantum mechanics recovering objectivity, arXiv:1402.4394 (2015)
23. Hensen, B., et al.: Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. Nature 526, 682–686 (2015)
24. Hestenes, D., Sobczyk, G.: Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics. Reidel, Dordrecht (1984)
25. Baylis, W.E.: Electrodynamics: A Modern Geometric Approach. Birkhäuser, Boston (1999)
26. Doran, C.J.L., Lasenby, A.N.: Geometric Algebra for Physicists. Cambridge University Press, Cambridge (2003)
27. Macdonald, A.: A survey of geometric algebra and geometric calculus. Adv. Appl. Cliff. Alg. 27, 853–891 (2017), Sect.1.2.1

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.