An action principle for Vasiliev’s four-dimensional higher spin gravity

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Received 29 April 2011, in final form 24 September 2011
Published 18 November 2011
Online at stacks.iop.org/JPhysA/44/495402

Abstract

We provide Vasiliev’s fully nonlinear equations of motion for bosonic higher spin gauge fields in four spacetime dimensions with an action principle. We first extend Vasiliev’s original system with differential forms in degrees higher than 1. We then derive the resulting duality-extended equations of motion from a variational principle based on a generalized Hamiltonian sigma-model action. The generalized Hamiltonian contains two types of interaction freedoms: one, a set of functions that appears in the Q-structure of the generalized curvatures of the odd forms in the duality-extended system; and the other, a set depending on the Lagrange multipliers, encoding a generalized Poisson structure, i.e. a set of polyvector fields of rank 2 or higher in target space. We find that at least one of the two sets of interaction-freedom functions must be linear in order to ensure gauge invariance. We discuss consistent truncations to the minimal type A and B models (with only even spins), spectral flows on-shell and provide boundary conditions on fields and gauge parameters that are compatible with the variational principle and that make the duality-extended system equivalent, on-shell, to Vasiliev’s original system.

PACS numbers: 11.15.–q, 11.10.Ef, 03.50.Kk

1. Introduction

The natural setting for gauge theories with local spacetime symmetries is \textit{unfolded dynamics} [1–5]. This formalism is based on the notion of \textit{exterior differential systems} (see e.g. [6, 7] and references therein). When applied to field theories with local propagating degrees of freedom,
such as gravities, supergravities and higher spin gravities, it yields infinite towers of zero-forms that are independent dynamical fields off-shell. On-shell, their integration constants, or expectation values, represent all the local information of the on-shell curvatures, usually referred to as the Weyl tensors.

In mathematics, an exterior differential system is usually considered as an ideal \( I \) in the graded ring of locally defined differential forms on a smooth manifold \( M \) that is closed under the operation of exterior differentiation. An integral manifold of a differential system is an immersed submanifold of \( M \) on which each form in \( I \) restricts to zero. In unfolded dynamics, the generators of \( I \) are identified as generalized curvatures and the integral manifold becomes a classical solution. Due to Cartan integrability, the curvatures can be integrated and expressed in terms of potentials, providing the fundamental variables in the off-shell formulation.

The canonical framework for the off-shell formulation of exterior differential systems is based on generalized Poisson sigma models \[8–19\] and \[20–25\]. When adapted to unfolded systems with infinite towers of zero-forms, these provide a framework for quantum field theory that one may refer to as unfolded quantum field theory, or deformation quantum field theory. The resulting key physical question is whether this novel framework actually contains standard relativistic quantum fields; see also \[26–28\] for recent developments\(^3\).

Considering retrospectively the works \[30–34\], one sees that these formulations of supergravities are examples of unfolded systems. In these cases, the locality of supergravity implies that all the dynamic content can be accessed in the metric phase by only considering the constraints on the forms in strictly positive degrees, thereby explaining why the authors of \[31–33\] did not consider the constraints on the generalized one-form curvatures for the Weyl tensors.

In this paper, we consider Poisson sigma models for the fully nonlinear and background-independent Vasiliev equations in four spacetime dimensions \[2, 35, 36\]. These equations possess an algebraic structure that enables us to construct a generalized Hamiltonian action with nontrivial Poisson structures, and have geometric structures which allow us to construct additional boundary deformations. In this paper, we focus on the bulk part of the Hamiltonian action, leaving various deformations on submanifolds to future works. In fact, already in \[3\], such an action principle was proposed which did not, however, contain any Poisson structure. In order to achieve non-trivial such structures we use a duality-extended version of Vasiliev’s equations\(^4\).

We wish to stress that, unlike the original Fronsdal programme, which attempts to formulate higher spin gauge theory off-shell in a perturbative expansion around constantly curved spacetime, the work in this paper provides a background-independent formulation in terms of master fields living in the correspondence space, i.e. the local product of a non-commutative phase spacetime containing the commutative spacetime as a Lagrangian submanifold and a non-commutative twistor space. Strictly speaking, the Vasiliev system has a huge classical solution space that admits many different perturbative expansions of which only some reduce to Fronsdal systems (with cosmological constant).

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\(^3\) Note that a relation between the AKSZ formalism and unfolding was not explicitly spelled out before \[20\]. The observation in \[20\] mainly relies on the results of \[29\], where the relation between unfolded and BRST approaches was first established (for linear systems).

\(^4\) The duality extension of linearized unfolded higher spin field equations has been discussed in \[37\], in the context of projective modules, and we thank Vasiliev for pointing this out to us. For the definition of duality extension in the case of nonlinear unfolded systems that we will use here, see appendix D.
2. Duality extension on-shell

2.1. Duality-extended bosonic models

Our starting point is Vasiliev’s unfolded on-shell formulation of higher spin gravities in four spacetime dimensions [2, 35, 36]. Essentially, one may think of this unfolded formulation as being the result of combining the following two key ingredients: (i) the notion of graded associative exterior differential algebras on non-commutative base manifolds (see appendices A and B); and (ii) the on-shell description of massless higher spin fields in four-dimensional anti-de Sitter spacetime in terms of a generating function on a non-commutative twistor space (see appendix C), manifesting the coincidence in four dimensions of conformal masslessness in the manner of Dirac and composite masslessness in the manner of Flato–Fronsdal.

Vasiliev’s equations of motion provide a particular example of an unfolded system, namely one based on a graded associative and quasi-free exterior differential algebra. In general, unfolded systems, that may be based on various types of exterior differential algebras, can be extended by adding forms in higher degrees [37]. In particular, if the underlying differential algebra contains central and closed elements in degrees \( \{0, 2, 4, \ldots\} \), the structure constants can also be extended from the real numbers (in degree zero) to general central elements (see appendix D for a more detailed discussion). If this extension is nontrivial, that is, if it cannot be removed by a field redefinition, then we refer to the resulting extended system as a duality extension of the original system. By its construction, such a duality-extended system contains the original system as a consistent subsystem, and this subsystem sources the duality-extended sector via nontrivial couplings involving central elements of positive degrees.

Vasiliev’s equations (see appendix C for notation and further details) can be extended by adding forms in higher degrees as follows:

\[
A = \sum_{p=1,3,\ldots}^{\infty} A_{[p]}, \quad B = \sum_{p=0,2,\ldots}^{\infty} B_{[p]},
\]

where \( A_{[p]} \) and \( B_{[p]} \) are locally-defined differential forms of total degree \( p \) belonging to the algebra of bosonic forms with generic elements

\[
f = \sum_{p=0}^{\infty} f_{[p]}(X^M, dX^M; \bar{Z}^a, dZ^a; Y^\alpha, k, \bar{k}),
\]

\[
f_{[p]}(\lambda, dX^M; \bar{Z}^a) = \lambda^p f_{[p]}(dX^M; d\bar{Z}^a),
\]

for complex parameters \( \lambda \) (we suppress the irrelevant variables whenever ambiguities cannot arise), where \( X^M \) are commuting coordinates, \( (Y^\alpha, \bar{Z}^a) = (\gamma^\alpha, \bar{\gamma}^a, \bar{\gamma}^a, \bar{\gamma}^a) \) are non-commutative twistor-space coordinates and \( k \) and \( \bar{k} \) are outer Kleinians obeying

\[
k \star f = \pi(f) \star k, \quad \bar{k} \star f = \bar{\pi}(f) \star \bar{k}, \quad k \star k = 1 = \bar{k} \star \bar{k},
\]

with automorphisms \( \pi \) and \( \bar{\pi} \) defined by \( \pi d = d \pi, \quad \bar{\pi} d = d \bar{\pi} \) and

\[
\pi[f(z^a, \bar{z}^a; y^\alpha, \bar{y}^\alpha)] = f(-z^a, \bar{z}^a; -y^\alpha, \bar{y}^\alpha),
\]

\[
\bar{\pi}[f(z^a, \bar{z}^a; y^\alpha, \bar{y}^\alpha)] = f(z^a, -\bar{z}^a; y^\alpha, -\bar{y}^\alpha).
\]

The bosonic projection and irreducibility conditions amount to

\[
\pi \bar{\pi}(f) = f, \quad f = P_+ \star f, \quad \text{where} \quad P_\pm = \frac{1}{2}(1 \pm k \star \bar{k}),
\]

which implies

\[
f = \left[f^{(+)}(X, dX; Z, dZ; Y) + f^{(-)}(X, dX; Z, dZ; Y) \star \frac{(k + \bar{k})}{2}\right] P_+.
\]
The bosonic projection removes all component fields associated with the unfolding of spinorial degrees of freedom in spacetime. Irreducible minimal bosonic models can be obtained by imposing reality conditions and discrete symmetries that remove all odd spins; the Hermitian conjugation $\dagger$ and the relevant anti-automorphism $\tau$ are defined by $d[(\cdot)\dagger] = [d(\cdot)]\dagger$, $d \tau = \tau d$:

$$[f(z^\alpha, z^\bar{\alpha}; y^\theta, \bar{y}^\bar{\theta}; k, \bar{k})] = f(z^\alpha, z^\bar{\alpha}; y^\theta, \bar{y}^\bar{\theta}; k, \bar{k}),\quad (2.8)$$

$$\tau[f(z^\alpha, z^\bar{\alpha}; y^\theta, \bar{y}^\bar{\theta}; k, \bar{k})] = f(-iz^\alpha, -iz^\bar{\alpha}; iy^\theta, iy^\bar{\theta}; k, \bar{k}),\quad (2.9)$$

$$[f_{[\rho]} \ast f'_{[\rho]}] = (-1)^{\rho\bar{\rho}} (f'_{[\rho]})^\dagger \ast (f_{[\rho]})^\dagger,\quad \tau [f_{[\rho]} \ast f'_{[\rho]}] = (-1)^{\rho\bar{\rho}} \tau (f'_{[\rho]}) \ast \tau (f_{[\rho]}).\quad (2.10)$$

We discuss the minimal models below.

The duality extension of the Vasiliev system is based on the following generalized curvature constraints:

$$F + \mathcal{F} = 0, \quad DB = 0,\quad (2.11)$$

with Yang–Mills-like curvature and covariant derivative defined by

$$F = dA + A \ast A, \quad DB = dB + A \ast B - B \ast A,\quad (2.12)$$

and interaction freedom $(I, \bar{I} = 1, 2)$

$$\mathcal{F} = \mathcal{F}_I(B) \ast J^I_{[2]} + \mathcal{F}_{\bar{I}}(B) \ast J^I_{[2]} + \mathcal{F}_{II}(B) \ast J^{II}_{[4]}\quad (2.13)$$

featuring the central elements

$$J^I_{[2]} = -\frac{i}{4} (1, k\kappa) \ast P_+ \ast d^2z,\quad J_{[2]} = -\frac{i}{4} (1, \bar{k}\bar{\kappa}) \ast P_+ \ast d^2\bar{z},\quad (2.14)$$

$$J^{II}_{[4]} = 4i J^I_{[2]} J^I_{[2]},\quad (2.15)$$

and $\ast$-functions $\mathcal{F}_I, \mathcal{F}_{\bar{I}}$ and $\mathcal{F}_{II}$ of $B$ such that $\mathcal{F}_I(\lambda), \mathcal{F}_{\bar{I}}(\lambda)$ and $\mathcal{F}_{II}(\lambda)$ ($I, \bar{I} = 1, 2$), viewed as functions of a single complex variable $\lambda \in \mathbb{C}$, are complex analytic in a finite neighborhood of $\lambda = 0$.

The unfolded equations (2.11) are Cartan integrable because the Yang–Mills-like Bianchi identities $DF = 0$ and $DDB = [F, B]_\ast$ are compatible with the generalized curvature constraints. In other words, defining the generalized curvatures

$$R^A = F + \mathcal{F}, \quad R^B = DB,\quad (2.16)$$

one has the generalized Bianchi identities

$$D R^A = (R^B B_\theta) \ast \mathcal{F} \equiv 0, \quad D R^B = [R^A, B]_\ast \equiv 0.\quad (2.17)$$

The potentials $\{A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \ldots\}$ in positive form degree share one and the same Weyl zero-form $B_{[0]}$, that hence contain all the local perturbative degrees of freedom of the extended system. One may refer to $[B_{[0]}, A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \ldots]$ as a duality extension of the original Vasiliev system consisting of $[B_{[0]}, A_{[1]}]$ in the sense that the presence of the central elements in degree 4 implies that $[B_{[2]}, A_{[3]}, B_{[4]}, \ldots]$ cannot in general be set equal to zero on-shell. Moreover, the extension is massless in the sense that for each $p \in \{1, 2, 3, \ldots\}$ the system of forms with degrees $p' \leq p$ constitutes a closed subsystem, i.e. their curvatures do not depend on the forms with degrees $p' > p$. In particular, this means that any (locally-defined) exact solution to the duality-extended system contains a (locally-defined) exact solution to the original Vasiliev system. The converse statement requires a more careful analysis that we omit here.
A duality-extended spectral flow

The duality-extended system possesses a spectral flow [38] describing the evolution of the system on-shell under changes in a vacuum expectation value $\nu$ and a coupling $g$ defined by the field redefinition

$$B = \nu 1 + gB'. \quad (2.18)$$

We stress that the parameters $(g, \nu)$ are part of the moduli space of the unfolded equations of motion, that is, both $A$ and $B$ depend on $(g, \nu)$ on-shell and in such a way that the differential $d$ commutes with $(\partial_0, \partial_i)$. Letting $f = f(A, dA, B, dB)$ and defining the flow operator

$$L_1 f = \partial_i f - \mu_1 B' \ast \partial_i f - \partial_i f \ast \mu_2 B', \quad \mu_1, \mu_2 \in \mathbb{C}, \quad \mu_1 + \mu_2 = 1, \quad (2.19)$$

one has

$$L_1 F \equiv D L_1 A + \mu_1 D B' \ast \partial_i A - \mu_2 \partial_i A \ast D B', \quad (2.20)$$

$$L_1 DB \equiv D L_1 B + [L_1 A, B], + \mu_1 DB \ast \partial_i B' + \mu_2 \partial_i B' \ast D B, \quad (2.21)$$

$$L_1 \mathcal{F} \equiv (L_1 B \partial_B) \ast \mathcal{F}. \quad (2.22)$$

It follows that the duality-extended equations of motion are compatible with the flow equations

$$L_1 A \approx 0, \quad L_1 B \approx 0, \quad (2.23)$$

where the last flow equation is equivalent to $L_1 B' \approx 0$.

The flow equations generalize as follows: one first redefines

$$B = v + N(B'), \quad N = v_1 g B' + v_2 g^2 B'^2 + v_3 g^3 B'^3 + \cdots, \quad (2.24)$$

where $v_k (k \geq 1)$ are constants and $g$ is the coupling. The flow operator defined by

$$L f = \partial_i f - \mathcal{M}_1(B') \ast \partial_i f - \partial_i f \ast \mathcal{M}_2(B'), \quad (2.25)$$

where the two $\ast$-functions defined by $(i = 1, 2)$

$$\mathcal{M}_i = \mu_{i,1} g B' + \mu_{i,2} g^2 B'^2 + \cdots, \quad \mu_{i,1} + \mu_{i,2} = k v_k \quad (k \geq 1) \quad (2.26)$$

obey

$$L \mathcal{F} \equiv (L B \partial_B) \ast \mathcal{F}, \quad (2.27)$$

$$L B = v_1 L B' + v_2 g^2 (L B' \ast B' + B' \ast L B') + \cdots, \quad (2.28)$$

$$L F = D L A + D \mathcal{M}_1 \ast \partial_i A - \partial_i A \ast D \mathcal{M}_2, \quad (2.29)$$

$$L D B' = D L B' + [L A, B'], + D \mathcal{M}_1 \ast \partial_i B' + \partial_i B' \ast D \mathcal{M}_2, \quad (2.30)$$

and it follows that one can set the constraints

$$L A = 0, \quad L B' = 0, \quad (2.31)$$

where the latter constraint thus implies that $L B = 0$. One can redefine $N = g B'$ so that $v_1 = 1$ and $v_k = 0$ for $k \geq 1$, leaving the freedom in $\mathcal{M}_i$ that generalizes the two-parameter freedom in having $\mu_1$ and $\mu_2$.
2.3. Consistent truncations

There are two possible reality conditions leading to models with negative cosmological constant $\Lambda < 0$ that we parameterize using $\epsilon_R = \pm 1$ as follows:

$$
(A_{i[p]})^\dagger = -(\epsilon)^{\frac{\tau_+}{\Lambda_1}} A_{i[p]}, \quad (B_{i[p]})^\dagger = (\epsilon)^{\frac{\tau_-}{\Lambda_1}} B_{i[p]},
$$

(2.32)

Moreover, using the map

$$
\pi_k : (k, \bar{k}) \mapsto (-k, -\bar{k}),
$$

(2.34)

there are two possible projections to models without topological (adjoint) zero-forms that we parameterize using $\epsilon_k = \pm 1$ as follows:

$$
\pi_k (A_{i[p]}) = (\epsilon_k)^{\frac{\tau_+}{\Lambda}} A_{i[p]}, \quad \pi_k (B_{i[p]}) = -(\epsilon_k)^{\frac{\tau_-}{\Lambda}} B_{i[p]},
$$

(2.35)

Using the parity transformation $P$ defined by $P d = d P$ and

$$
P \left[ f(X^\mu; \omega, \bar{\omega}; \bar{\tau}^\nu, \eta^\nu, \bar{\eta}^\nu; k, \bar{k}) \right] = (P f)(X^\mu; -\omega, \bar{\omega}; \bar{\tau}^\nu, \eta^\nu, \bar{\eta}^\nu; k, \bar{k}),
$$

(2.37)

which is an automorphism of the $\star$-product algebra and where $P f$ is expanded in terms of parity reversed component fields, there are four ways of fixing parities that we parameterize using $\epsilon, \bar{\epsilon} = \pm 1$ as follows:

$$
P(A_{i[p]}) = (\epsilon \bar{\epsilon})^{\frac{\tau_+}{\Lambda}} A_{i[p]}, \quad P(B_{i[p]}) = (\epsilon \bar{\epsilon})^{\frac{\tau_-}{\Lambda}} (\bar{\epsilon})^{\frac{\tau_+}{\Lambda}} B_{i[p]},
$$

(2.38)

$$
\mathcal{F}_i(\lambda) = \mathcal{F}_i(\epsilon \lambda), \quad \mathcal{F}_{ij}(\lambda) = \epsilon \bar{\epsilon} \mathcal{F}_{ij}(\epsilon \lambda).
$$

(2.39)

Finally, the $\tau$-projection to the minimal models with only even propagating spins reads

$$
\tau(A_{i[p]}) = (-1)^{\frac{\tau_+}{\Lambda}} A_{i[p]}, \quad \tau(B_{i[p]}) = (-1)^{\frac{\tau_-}{\Lambda}} B_{i[p]},
$$

(2.40)

which is the unique choice since $\tau(J_{i[p]}) = (-1)^{\frac{\tau_+}{\Lambda}} J_{i[p]}$ (and there is no condition on $\mathcal{F}$).

In the $(B_{i[0]}, A_{i[1]})$-sector, which forms a closed subsystem, the assignment of $k$-parity combined with the freedom in redefining $A_{i[0]}$ can be used to replace $[2]$

$$
(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F}_1; \mathcal{F}_2) \rightarrow (0, (1 - \mathcal{F}_1)^{(\epsilon - 1)} \star \mathcal{F}_2; 0, (1 - \mathcal{F}_1)^{(\epsilon - 1)} \star \mathcal{F}_2).
$$

(2.41)

Imposing also reality and parity conditions, of which the latter is a multiple choice parametrized by $\epsilon = \pm 1$, the remaining interaction function $(1 - \mathcal{F}_1)^{(\epsilon - 1)} \star \mathcal{F}_2$ becomes real and odd, hence defining the new master field

$$
\Phi \star P_+ = (1 - \mathcal{F}_1)^{(\epsilon - 1)} \star \mathcal{F}_2 \star k \star P_+.
$$

(2.42)

obeying the twisted reality condition $(\Phi)^\dagger = \pi (\Phi)$ and the parity condition $P(\Phi) = \epsilon \Phi$ leading to a physical scalar that is even under parity for $\epsilon = 1$ and odd under parity for $\epsilon = -1$.

Finally, one may project out the odd spins by imposing $\tau (\Phi) = \pi (\Phi)$ yielding the minimal bosonic models.

Assuming linear interaction functions

$$
\mathcal{F}_i = b_i B, \quad \mathcal{F}_{ij} = b_{ij} B, \quad \mathcal{F}_{ij} = c_{ij} B,
$$

(2.43)

and defining a total central element

$$
J = J_{[2]} + J_{[4]}
$$

(2.44)
which in turn implies graded cyclicity:

\[ B \ast J_{[2]} = \mathcal{F}_i \ast J^I_{[2]} + \mathcal{F}_i \ast J^I_{[2]}, \quad B \ast J_{[4]} = \mathcal{F}_{ij} \ast J^I_{[4]}, \]  

(2.45)

\[ J_{[2]} = -\frac{i}{4} [d \bar{z}^2 (b_1 + b_2 k \kappa) + d \bar{z}^2 (b_1 + b_2 \bar{k} \bar{\kappa})] \ast P_+, \]  

(2.46)

\[ J_{[4]} = -\frac{i}{4} d \bar{z} d \bar{z}' [c_{1i} + c_{2i} k \kappa + c_{1i} \bar{k} \bar{\kappa} + c_{2i} \bar{\kappa} \kappa] \ast P_+, \]  

(2.47)

the reality, \( k \)-parity and \( P \)-parity conditions imply

\[ (J_{[p]})^\dagger = - (\bar{\epsilon} k) J_{[p]}, \quad \pi k (J_{[p]}) = - (\bar{\epsilon} k) J_{[p]}, \quad P (J_{[p]}) = (\epsilon \bar{k})(\bar{\epsilon} k) J_{[p]}, \]  

(2.48)

which constrain the parameters \((b_1, b_2, c_{ij})\). These conditions admit nontrivial solutions for \( J_{[p]} \) for all combinations of signs except for \( \epsilon_k = 0 \), since \( \epsilon_k = -1 \) implies that \( \bar{\epsilon} = +1 \).

### 3. Generalized Hamiltonian action principle

#### 3.1. Graded cyclic chiral trace

Vasiliev’s equations are formulated in terms of master fields, which one may think of as functions on a total symplectic manifold called the correspondence space \( \mathcal{C} \); see appendix C. This manifold can be reduced locally to a product space \( \mathcal{C}_k = \mathbb{M}_{\xi} \times 3 \times \mathcal{Y} \), where \( \mathcal{Y} \) and \( \mathcal{Y}_0 \) are two copies of a non-commutative twistor space and \( \mathbb{M}_{\xi} \) denotes a coordinate chart of a commuting base manifold \( \mathbb{M} \); see equation (C.26) in appendix C for more details. In order to build an action principle, we need to integrate over the reduced correspondence space. The integration over \( \mathcal{C} \) of a globally defined \((\hat{p} + 1)\)-form \( \mathcal{L} \) is defined by

\[ \int_{\mathcal{C}} \mathcal{L} = \sum_\xi \int_{\mathbb{M}_\xi} \text{Tr} [f_{\mathcal{L}}], \]  

(3.1)

where \( f_{\mathcal{L}} \) denotes a symbol of \( \mathcal{L} \) and the chiral trace operation is defined by

\[ \text{Tr} [f] = \sum_m \int_{3 \times \mathcal{Y}} \frac{d^2 y d^2 \bar{y}}{(2\pi)^2} f_{[m,2|2]} |_{k=0=k}, \]  

(3.2)

using the decomposition \( f_{[p]} = \sum_{m+q+i-\bar{i}+p} f_{[m,q,\bar{i},\bar{\bar{i}}]} \) with

\[ f_{[m,q,\bar{i},\bar{\bar{i}}]} (\lambda, \mu; dX^M, \tilde{\mu}; d\bar{z}^2) = \lambda^m \mu^q \tilde{\mu}^{\bar{i}} f_{[m,q,\bar{i},\bar{\bar{i}}]} (dX^M; d\bar{z}^2, d\bar{z}^2), \]  

(3.3)

and with the integration domain consisting of real contours for \( \{\vartheta^0, z^\alpha\} \) and \( \{\bar{\vartheta}^0, \bar{z}^\alpha\} \), respectively, that is, one performs separate integrations over the holomorphic and antiholomorphic variables treated as independent real variables (for related discussions, see e.g. appendix G of [40]). The choice of the chiral integration domain (instead of the complex integration domain) implies that

\[ \text{Tr} [\pi (f)] = \text{Tr} [\bar{\pi} (f)] = \text{Tr} [f], \]  

(3.4)

which in turn implies graded cyclicity:

\[ \text{Tr} [f_{[p]} \ast f_{[p]'}] = (-1)^{p'p} \text{Tr} [f_{[p']} \ast f_{[p]}], \]  

(3.5)

as can be seen by expanding \( f_{[p]} = (f_{[p]}^{(\vartheta^0)} + f_{[p]}^{(\bar{\vartheta}^0)} \ast k) \ast P_+ \) idem \( f_{[p']} \) which yields

\[ \text{Tr} [f_{[p]} \ast f_{[p]}'] = \frac{1}{2} \left[ \text{Tr} [f_{[p]}^{(\vartheta^0)} \ast f_{[p']}^{(\vartheta^0)} + f_{[p]}^{(\bar{\vartheta}^0)} \ast \pi (f_{[p']}^{(\bar{\vartheta}^0)})] \right]. \]  

(3.6)

Note that a similar construction for 3D higher spin theory with matter was proposed in [39].
where the second term is graded cyclic by virtue of the chiral integration. Furthermore, the
chiral trace operation commutes to Hermitian conjugation and is invariant under $P$ and $\pi_k$:

$$(\text{Tr}[f])^\dagger = \text{Tr}[\{f\}], \quad \text{Tr}[P(f)] = \text{Tr}[f], \quad \text{Tr}[\pi_k(f)] = \text{Tr}[f].$$  \hspace{1cm} (3.7)

Finally, one may seek to impose boundary conditions in $\mathcal{F} \times \mathcal{G}$, such that the integration
contours can be rotated from real to imaginary axes, in the sense that

$$\text{Tr}[\tau(f)] = \text{Tr}[f].$$  \hspace{1cm} (3.8)

We finally assume that the integration over $\pi$ is non-degenerate such that if $\text{Tr}[f \star g] = 0$
for all $f$, then $g = 0$. An interesting open problem is to understand whether the $\pi, P$ and $\tau$
symmetries could be violated on classical observables evaluated on exact solutions that one
may seek to interpret as describing topology changes of the twistor space, which we leave
for future studies [41]. In what follows we always assume that the discrete symmetries hold
off-shell.

3.2. Odd-dimensional bulk ($\mathbf{p} \in 2\mathbb{N}$)

3.2.1. Action principle. In the case of an odd-dimensional base manifold of dimension
$p + 1 = 2n + 5$ with $n \in \{0, 1, 2, \ldots\}$ such that $\dim(\mathbb{M}) = 2n + 1$, the duality-extended
equations of motion follow from the variational principle based on the generalized Hamiltonian
bulk action

$$S^\text{cl}_{\text{bulk}}([A, B, U, V]_{\epsilon}) = \sum_{\mathcal{I}} \int_{\mathcal{M}_{\mathcal{I}}} \text{Tr}[U \star DB + V \star (F + \mathcal{G}(B, U; J^I, J^I))] ,$$  \hspace{1cm} (3.9)

with interaction freedom $\mathcal{G}$ and locally-defined master fields decomposing under total form
degree into

$$A = A_{[1]} + A_{[3]} + \ldots + A_{[2m-1]}, \quad B = B_{[0]} + B_{[2]} + \ldots + B_{[2m-2]},$$  \hspace{1cm} (3.10)

$$U = U_{[2]} + U_{[4]} + \ldots + U_{[2m]}, \quad V = V_{[1]} + V_{[3]} + \ldots + V_{[2m-1]}, \quad m = n + 2.$$  \hspace{1cm} (3.11)

The function $\mathcal{G}$ must be constrained for the action to be gauge invariant and in order
to avoid systems that are trivial. In what follows we consider the special case

$$\mathcal{G} = \mathcal{F}(B; J^I, J^I) + \tilde{\mathcal{F}}(U; J^I, J^I),$$  \hspace{1cm} (3.12)

$$\mathcal{F} = \mathcal{F}_I(B) \star J^I_{[2]} + \mathcal{F}_I(B) \star J^I_{[2]} + \mathcal{F}_{II}(B) \star J^I_{[4]},$$  \hspace{1cm} (3.13)

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_0(U) + \tilde{\mathcal{F}}_1(U) \star J^I_{[2]} + \tilde{\mathcal{F}}_1(U) \star J^I_{[2]} + \tilde{\mathcal{F}}_{II}(U) \star J^I_{[4]},$$  \hspace{1cm} (3.14)

where the (non-)vanishing of the coupling $\lambda := \partial_0, \tilde{\mathcal{F}}_{0} \mid_{U \rightarrow 0}$ implies that the target space
is equipped with a Poisson (symplectic) structure. In the case of a proper Poisson structure with
$\lambda = 0$ the action cannot be written as a boundary term.

Denoting $Z' = (A, B, U, V)$, the general variation of the action defines generalized
curvatures $\mathcal{R}$ as follows:

$$\delta S = \sum_{\mathcal{I}} \int_{\mathcal{M}_{\mathcal{I}}} \text{Tr}[\mathcal{R}_I \star \delta Z' / \partial_{\mathcal{I}}] + \sum_{\mathcal{I}} \int_{\mathcal{M}_{\mathcal{I}}} \text{Tr}[U \star \delta B - V \star \delta A],$$  \hspace{1cm} (3.15)

where one thus has

$$\mathcal{R}^A = F + \mathcal{F} + \tilde{\mathcal{F}}, \quad \mathcal{R}^B = DB + (V \partial_B) \star \tilde{\mathcal{F}},$$  \hspace{1cm} (3.16)

$$\mathcal{R}^U = DU - (V \partial_B) \star \mathcal{F}, \quad \mathcal{R}^V = DV + [B, U].$$  \hspace{1cm} (3.17)
with $O_i$ being a constant non-degenerate matrix (defining a symplectic form of degree $\beta + 2$ on the $\mathbb{N}$-graded target space of the bulk theory). Treating $Z^i$ and $dZ^i$ as independent variables, one has the differential identities

$$DR^A = -(R^B \partial_B) \star F - (R^U \partial_U) \star \tilde{F} \equiv \mathcal{A}^A,$$

(3.18)

$$DR^B = [R^A, B]_\star - (R^U \partial_U) \star \tilde{F} - (V \partial_U) \star (V \partial_U) \star \tilde{F} \equiv \mathcal{A}^B,$$

(3.19)

$$DR^U = [R^A, U]_\star + (R^U \partial_U) \star F + (R^B \partial_B) \star (V \partial_U) \star \tilde{F} \equiv \mathcal{A}^U,$$

(3.20)

$$DR^V = [R^A, V]_\star - [R^B, U]_\star + [R^U, B]_\star \equiv \mathcal{A}^V,$$

(3.21)

with $dZ^i$-independent quantities $A^i \equiv A^i(Z^i)$ given by

$$A^A = -(V \partial_U) \star \tilde{F} \partial_B \star F + (V \partial_B) \star (V \partial_U) \star \tilde{F},$$

(3.22)

$$A^B = (V \partial_B) \star F \partial_U \star (V \partial_U) \star \tilde{F},$$

(3.23)

$$A^U = (V \partial_U) \star \tilde{F} \partial_B \star (V \partial_B) \star F,$$

(3.24)

$$A^V \equiv 0,$$

(3.25)

where the last identity follows from

$$[U, (V \partial_U) \star \tilde{F}]_\star \equiv -[V, \tilde{F}]_\star, \quad [B, (V \partial_B) \star F]_\star \equiv -[V, F]_\star.$$

(3.26)

The quantities $A^i$ thus represent obstructions to generalized Bianchi identities off-shell and hence to Cartan integrability of the unfolded equations of motion $R^i \equiv 0$, where in this section we use weak equalities for equations that hold on-shell. These obstructions vanish identically (without further algebraic constraints on $Z^i$) in at least the following two cases:

bilinear $Q$-structure : $F = B \star J, \quad J = J_{[2]} + J_{[4]},$

(3.27)

bilinear $P$-structure : $\tilde{F} = U \star J', \quad J' = J'_{[2]} + J'_{[4]},$

(3.28)

where the central elements are expanded as in equations (2.44)–(2.47).

At this stage it is useful to recall (see appendix B) that if $R^i = dZ^i + Q(Z^i)$ defines a set of generalized curvatures, then one has the following three equivalent statements: (i) $R^i$ obey a set of generalized Bianchi identities $dR^i = -\partial \partial \star \mathcal{Q} \equiv 0$; (ii) $R^i$ transform into each other under Cartan gauge transformations $\delta_i Z^i := d\delta^i := (e^U \partial_U) \star \mathcal{Q}$; and (iii) the quantity $\mathcal{Q} := Q \partial_U$ is a $Q$-structure, i.e. a nilpotent $\star$-vector field of degree 1 in target space, namely $\mathcal{Q} \star \mathcal{Q} \equiv 0$. Furthermore, in the case of differential algebras on commutative base manifolds, one can show that if $R^i$ are defined via a variational principle as in (3.15) (with constant $O_i$), then the action $S$ remains invariant under $\delta_i Z^i$.

In the two Cartan-integrable cases at hand, one thus has the on-shell Cartan gauge transformations

$$\delta_{\epsilon, a} A = De^A - (e^B \partial_B) \star F - (\eta^U \partial_U) \star \tilde{F},$$

(3.29)

$$\delta_{\epsilon, a} B = De^B - [e^A, B]_\star - (\eta^V \partial_U) \star \tilde{F} - (\eta^U \partial_U) \star (V \partial_U) \star \tilde{F},$$

(3.30)

$$\delta_{\epsilon, a} U = D\eta^U - [e^A, U]_\star + (\eta^V \partial_U) \star F + (e^B \partial_B) \star (V \partial_U) \star \tilde{F},$$

(3.31)

$$\delta_{\epsilon, a} V = D\eta^V - [e^A, V]_\star - [e^B, U]_\star + [\eta^U, B]_\star.$$

(3.32)
These transformations remain symmetries off-shell, as can be seen using the following set of identities:

\begin{align}
\text{bilinear } P\text{-structure} : \text{Tr}[J^I \star V \star (V \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}] &\equiv 0, \\
\text{Tr}[V \star (DB \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F} + DB \star (V \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}] &\equiv 0, \\
\text{Tr}[\eta^V \star (DB \partial_B) \star \mathcal{F} - DB \star (\eta^V \partial_B) \star \mathcal{F}] &\equiv 0,
\end{align}

(33)

\begin{align}
\text{bilinear } Q\text{-structure} : \text{Tr}[J^I \star V \star (V \partial_U) \star (\eta^U \partial_U) \star \tilde{\mathcal{F}}] &\equiv 0, \\
\text{Tr}[V \star (DU \partial_U) \star (\eta^U \partial_U) \star \tilde{\mathcal{F}} + DU \star (V \partial_U) \star (\eta^U \partial_U) \star \tilde{\mathcal{F}}] &\equiv 0, \\
\text{Tr}[\eta^V \star (DU \partial_U) \star \tilde{\mathcal{F}} - DU \star (\eta^V \partial_U) \star \tilde{\mathcal{F}}] &\equiv 0.
\end{align}

(34)

(35)

(36)

More precisely, the \((\epsilon^A, \epsilon^B)\)-symmetries leave the Lagrangian invariant, while the \((\eta^U, \eta^V)\)-symmetries transform the Lagrangian into a nontrivial total derivative, namely

\[ \delta_{\eta, \eta} \mathcal{L} \equiv \text{d} \left( \text{Tr}[\eta^U \star K_U + \eta^V \star K_V] \right), \]

(37)

for \((K_U, K_V)\) that are not identically zero. It follows that the Cartan gauge algebra \( \mathfrak{g} \) is of the form

\[ \mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2 \]

with \( \mathfrak{g}_1 \cong \text{span}[\epsilon^A, \epsilon^B] \) and \( \mathfrak{g}_2 \cong \text{span}[\eta^U, \eta^V] \), as one can verify explicitly using the formulae \((B.15)\) given in appendix \(B\).

3.2.2. Global formulation, boundary conditions and embedding of Vasiliev’s original system. Exponentiation of the infinitesimal Cartan gauge transformations leads to locally-defined gauge orbits consisting of elements (see appendix \(A\))

\[ Z_{\lambda^j, \xi^i, Z_0} = \mathcal{G}_{\lambda^j, \xi^i, Z} \big|_{Z^i = Z_0}, \]

(38)

\[ \mathcal{G}_{\lambda^j, \xi^i, Z} := \exp \left[ \mathcal{F}_{\lambda^j, \xi^i, Z} \right], \]

(39)

\[ \mathcal{F}_{\lambda^j, \xi^i, Z} := \left( d\lambda^i - (\lambda^j \partial_j) \right) \frac{\partial}{\partial Z^i}, \]

where \( \lambda^j \) and \( Z_0 \) respectively, are gauge functions and representatives of the orbits defined in coordinate charts of the base manifold. On-shell, one has

\[ dZ^0_0 + Q'(Z^0_0) \sim 0 \implies dZ^j_{\lambda^j, \xi^i, Z_0} + Q'(Z^j_{\lambda^j, \xi^i, Z_0}) \sim 0, \]

(40)

as can be seen by first writing \( d \approx \mathcal{S}_{\lambda^j, \xi^i} - \mathcal{O} \) where \( \mathcal{S}_{\lambda^j, \xi^i} := d\lambda^i / \partial \lambda^j \) and \( \mathcal{O} := Q^0 / \partial Z^0 \), and then using \([\mathcal{S}_{\lambda^j, \xi^i} - \mathcal{O}, \mathcal{F}_{\lambda^j, \xi^i, Z_0}], 0 \) and \([\exp, \mathcal{F}] \star (\mathcal{F} \star \mathcal{F}') = ([\exp, \mathcal{F}] \star \mathcal{F}) \star ([\exp, \mathcal{F}] \star \mathcal{F}') \) for any \( \star \)-vector field \( \mathcal{F} \) and \( \star \)-functions \( \mathcal{F} \) (see appendix \(B\) for details).

In particular, it follows that the space of (locally-defined) classical solutions to the duality-extended \((A, B; U, V)\)-system contains a subspace of (locally-defined) classical solutions to the duality-extended \((A, B)\)-system, obtained simply by setting \( U = 0 = V \). The \((A, B)\)-system contains in turn a subset of the (locally-defined) solutions to the original Vasiliev system in form degrees 0 and 1. The converse issue, whether any given (locally-defined) exact solution to the original Vasiliev system can be uplifted to the \((A, B)\)-system, requires, however, a more careful analysis of the gauge orbits in degrees greater than 1 (due to the non-polynomial dependences on the integration constants for the Weyl zero-form and the zero-form gauge functions).

Turning to the global formulation, it follows from equation \((3.37)\) that the gauge parameters \( (\epsilon^A, \epsilon^B) \in \mathfrak{g}_1 \) can be locally defined on \( \mathcal{M} \), that is, defined independently on
the coordinate charts \( \mathcal{M}_\epsilon \) — provided that the action is not perturbed by impurities that break some of the \((\epsilon^A, \epsilon^B)\)-symmetries, as for example in the soldered phase where perturbations break the local translations in \( e^{\partial_\mu} \). From equation (3.37) it also follows that \((\eta^U, \eta^V) \in \mathfrak{g}_1\) need to be defined globally on \( \mathcal{M}_\epsilon \), that is, \((\eta^U, \eta^V)|_\epsilon\) and \((\eta^U, \eta^V)|_{\epsilon'}\) must be related by transition functions \( \{t^U_\epsilon\} \) across the chart boundary between \( \mathcal{M}_\epsilon \) and \( \mathcal{M}_{\epsilon'} \); in practice, this means that one may take \((\eta^U, \eta^V)|_\epsilon\) to have compact support in \( \mathcal{M}_\epsilon \).

The unbroken phase of the theory thus consists of local representatives \( Z^i_\epsilon = (A, B; U, V)|_\epsilon \) defined up to gauge transformations with parameters \((\epsilon^A, \epsilon^B)\) that are unrestricted on \( \partial \mathcal{M}_\epsilon \) and parameters \((\eta^U, \eta^V)|_\epsilon\) with the aforementioned restrictions on \( \partial \mathcal{M}_\epsilon \), with transitions of the form

\[
Z_{i'}^i = \mathcal{O}_\epsilon^{i'i} \star Z^i_{\epsilon'} \quad \text{defined on } \mathcal{M}_\epsilon \cap \mathcal{M}_{\epsilon'},
\]

(3.41) where \( \mathcal{O}_\epsilon^{i'i} = \exp \int_{\partial \mathcal{M}_\epsilon} \mathcal{A}_\epsilon^{i'i} \) with transition functions \( \mathcal{A}_\epsilon^{i'i} \in \mathfrak{g}_1 \) defined on \( \mathcal{M}_\epsilon \cap \mathcal{M}_{\epsilon'} \).

More generally, softly broken phases of the theory arise by taking the transition functions \( \{t^U_\epsilon\} \) to be generated by various unbroken subalgebras \( \mathfrak{u} \subseteq \mathfrak{g}_1 \). Their moduli spaces \( \mathcal{M}_\mathfrak{u} \) can be coordinatized by classical observables \( \mathcal{O}_\mathfrak{u} \) that are manifestly \( \mathfrak{u} \)-invariant off-shell and diffeomorphism-invariant on-shell (one may thus think of the unbroken phase \( \mathcal{M}_\mathfrak{u} \) as the smallest homotopy phase for a given base manifold; it can be embedded into various broken phases). Of particular interest is the soldered phase in which the action is perturbed as to softly break the gauge symmetries associated with the \( \pi \)-odd projection of \( A_{11} \). The unbroken gauge algebra in this case thus consists of the \( \pi \)-even projection \( \frac{1}{2}(1 + \pi) e^{\partial_\mu} \) together with the remaining \( \epsilon \)-parameters of positive form degree.

To achieve a globally well-defined variational principle, one considers globally-defined field configurations off-shell consisting of locally-defined representatives \( \{Z^i_\epsilon\} \) related on chart boundaries via transitions (3.41) for a given structure algebra \( \mathfrak{u} \subseteq \mathfrak{g}_1 \). The manifest \( \mathfrak{g}_1 \)-invariance implies that in the general variation (3.15), the contributions from two adjacent boundaries \( \partial \mathcal{M}_\epsilon \) and \( \partial \mathcal{M}_{\epsilon'} \) cancel; on such a boundary one has the transition functions (\( t \equiv t^U_\epsilon \))

\[
\delta_\epsilon(\delta A) = - [t^A, \delta A]_\epsilon - (\delta B \partial B) \star (t^B \partial B) \star \mathcal{F},
\]

(3.42)

\[
\delta_\epsilon(\delta B) = - [t^B, \delta B]_\epsilon + [t^A, \delta A]_\epsilon,
\]

(3.43)

\[
\delta_\epsilon U = - [t^A, U]_\epsilon + (t^B \partial B) \star (V \partial B) \star \mathcal{F},
\]

(3.44)

\[
\delta_\epsilon V = - [t^A, V]_\epsilon - [t^B, U]_\epsilon,
\]

(3.45)

which implies that (\( t \equiv t_\epsilon^{i'i} \))

\[
\delta_\epsilon \left( \int_{\partial \mathcal{M}_\epsilon} \text{Tr} [U \star \delta B - V \star \delta A] \right)
\]

(3.46)

\[
= \int_{\partial \mathcal{M}_\epsilon} \text{Tr} \left[ V \star (\delta B \partial B) \star (t^B \partial B) \star \mathcal{F} - \delta B \star (V \partial B) \star (t^B \partial B) \star \mathcal{F} \right] \equiv 0.
\]

(3.47)

One is thus left with contributions from true boundaries \( \partial \mathcal{M}_\epsilon \subset \partial \mathcal{M} \) (including boundaries of homotopy cylinders surrounding impurities of co-dimension greater than one). It follows that the natural boundary conditions compatible with the locally-defined gauge symmetries are the Dirichlet conditions

\[
(U, V)|_{\partial \mathcal{M}} = 0.
\]

(3.48)

In summary, a classical solution can thus be specified by fixing
(i) the transition functions \([t^a_i] \in \mathfrak{g}_\mathfrak{u} \subseteq \mathfrak{g}_1\);

(ii) an initial datum for the zero-form \(B_0[0]\), say

\[
B_0[0]|_p = C(Y; k, \tilde{\ell}),
\]

at some given point \(p \in \mathbb{M} \times \tilde{\mathbb{E}}\);

(iii) boundary conditions on the gauge functions associated with the softly-broken gauge symmetries, namely

\[
\lambda|_{\partial \mathbb{M}} \quad \text{for} \quad \lambda \in \mathfrak{g}_1/\mathfrak{u};
\]

(iv) the monodromies of the flat connection on \(\mathbb{M}\); and

(v) the boundary conditions (3.48) on the Lagrange multipliers.

### 3.2.3. Duality extended spectral flow with Lagrange multipliers

The equations of motion \(\mathcal{R} \approx 0\) of the extended Lagrangian system \(Z' = (A, B; U, V)\) with bilinear \(P\) and \(Q\) structures (i.e. linear \(\mathcal{F}\) and \(\tilde{\mathcal{F}}\) functions) are compatible with the extended flow equations \(L_1A \approx 0 \approx L_1B\) (or equivalently \(L_1B' \approx 0\)) and

\[
L_1U \approx \mu_1V' \ast (\partial_1A) - \mu_2(\partial_1A) \ast V', \quad L_1V' \approx \mu_1V' \ast (\partial_1B') + \mu_2(\partial_1B') \ast V',
\]

with flow operator \(L_1\) given by (2.19) and the redefinition

\[
B = v + gB', \quad V = gV', \quad v, g \in \mathbb{C}.
\]

We have not found any generalization of the spectral flow to the Lagrangian systems with higher order \(P\)- or \(Q\)-structures (i.e. nonlinear \(\mathcal{F}\) or \(\tilde{\mathcal{F}}\) functions).

### 3.2.4. Consistent truncations off-shell

Reality conditions can be imposed off-shell by requiring the action to be either real or purely imaginary, namely

\[
\left(S_{\text{bulk}}^{\text{cl}}\right)^\dagger = \epsilon_{\text{S}} S_{\text{bulk}}^{\text{cl}}.
\]

leading to the following reality conditions on the Lagrange multipliers and the function \(\tilde{\mathcal{F}}\) appearing in the generalized \(P\)-structure:

\[
(U[p])^\dagger = \epsilon_{\dagger} (\epsilon_{\mathbb{R}})^{n+\frac{1}{2}} U[p], \quad (V[p])^\dagger = -\epsilon_{\dagger} (\epsilon_{\mathbb{R}})^{n+\frac{1}{2}} V[p],
\]

\[
(\tilde{\mathcal{F}}_0(\lambda))^{\dagger} = -\epsilon_{\mathbb{R}} \tilde{\mathcal{F}}_0(\epsilon_{\mathbb{R}})^{n^{\ast}} \lambda, \quad (\tilde{\mathcal{F}}_1(\lambda))^{\dagger} = \tilde{\mathcal{F}}_1(\epsilon_{\mathbb{R}})^{n^{\ast}} \lambda,
\]

\[
(\tilde{\mathcal{F}}_{i,f}(\lambda))^{\dagger} = \epsilon_{\mathbb{R}} \tilde{\mathcal{F}}_{i,f}(\epsilon_{\mathbb{R}})^{n^{\ast}} \lambda.
\]

From \(\text{Tr}[\pi_L(\cdot)] = \text{Tr}[\cdot]\) it follows that in the case of \(\pi_L\)-projection the \(k\)-parities must be correlated as follows:

\[
\pi_L(U[p]) = -\epsilon_L^{n+\frac{1}{2}} U[p], \quad \pi_L(V[p]) = \epsilon_L^{n+\frac{1}{2}} V[p],
\]

\[
\tilde{\mathcal{F}}_0(\epsilon_L)^{n^{\ast}} \lambda = \epsilon_L \tilde{\mathcal{F}}_0(\lambda), \quad \tilde{\mathcal{F}}_1(\epsilon_L)^{n^{\ast}} \lambda = (\epsilon^{1}1)^{\dagger} \tilde{\mathcal{F}}_1(\lambda),
\]

\[
\tilde{\mathcal{F}}_{i,f}(\epsilon_L)^{n^{\ast}} \lambda = \epsilon_L (\epsilon_L)^{1} i^{\dagger} \tilde{\mathcal{F}}_{i,f}(\lambda).
\]

To fix spacetime parity one may impose \((\epsilon, \tilde{\epsilon} = \pm 1)\)

\[
P(U[p]) = \epsilon (\epsilon \tilde{\epsilon})^{n+\frac{1}{2}} U[p], \quad P(V[p]) = (\epsilon \tilde{\epsilon})^{n+\frac{1}{2}} V[p],
\]

\[
\tilde{\mathcal{F}}_0(\epsilon \tilde{\epsilon})^{n^{\ast}} \lambda = \epsilon \tilde{\epsilon} \tilde{\mathcal{F}}_0(\lambda), \quad \tilde{\mathcal{F}}_1(\epsilon \tilde{\epsilon})^{n^{\ast}} \lambda = \epsilon \tilde{\epsilon} \tilde{\mathcal{F}}_1(\epsilon \tilde{\epsilon})^{n^{\ast}} \lambda,
\]

\[
\tilde{\mathcal{F}}_{i,f}(\epsilon \tilde{\epsilon})^{n^{\ast}} \lambda = \epsilon \tilde{\epsilon} \tilde{\mathcal{F}}_{i,f}(\epsilon \tilde{\epsilon})^{n^{\ast}} \lambda.
\]
Finally, assuming $\text{Tr}[\tau(\cdot)] = \text{Tr}[-]$, the projection to the minimal bosonic model takes the form

$$
\tau(U[p]) = (-1)^{n+p} U[p], \quad \tau(V[p]) = (-1)^{n+p} V[p], \quad (3.62)
$$

$$
\bar{\mathcal{F}}_0((-1)^n \lambda) = \bar{\mathcal{F}}_0(\lambda), \quad \bar{\mathcal{F}}_I((-1)^n \lambda) = \bar{\mathcal{F}}_I(\lambda), \quad (3.63)
$$

$$
\bar{\mathcal{F}}_{ij}((-1)^n \lambda) = \bar{\mathcal{F}}_{ij}(\lambda). \quad (3.64)
$$

### 3.3. Even-dimensional bulk ($\hat{p} \in 2\mathbb{N} + 1$)

In the case of an even-dimensional bulk, say of dimension $\hat{p} + 1 = 2n$, one has the action

$$
S_{\text{cl}}[A, B; S, T] = \int_{\mathcal{M}} \text{Tr}[S \star DB + T \star (F + \mathcal{F}) + \mathcal{W}(S, J^i, J^j, J^{ij}) \star T], \quad (3.65)
$$

where $\mathcal{W}$ is an interaction $\star$-function obeying

$$
\mathcal{W}(-\lambda) = \mathcal{W}(\lambda), \quad \mathcal{W}(0) = 0, \quad (3.66)
$$

and the form degrees are assigned as follows:

$$
A = \sum_{m=1,3,\ldots,\hat{p}} A[m], \quad B = \sum_{m=0,2,\ldots,\hat{p} - 1} B[m], \quad (3.67)
$$

$$
S = \sum_{m=1,3,\ldots,\hat{p}} S[m], \quad T = \sum_{m=0,2,\ldots,\hat{p} - 1} T[m]. \quad (3.68)
$$

The variational principle yields the generalized curvatures

$$
\mathcal{R}^A = F + \mathcal{U} + \mathcal{W}(S), \quad \mathcal{R}^B = DB - (T \partial_S) \star \mathcal{W}(S), \quad (3.69)
$$

$$
\mathcal{R}^S = DS + (T \partial_S) \star \mathcal{F}, \quad \mathcal{R}^T = DT + [S, B], \quad (3.70)
$$

The action is gauge invariant and the equations of motion are integrable in the case of

bilinear Q-structure : $\mathcal{F} = J \star B$, \quad (3.71)

for which the integrability of $\mathcal{R}^T$ follows using the identity

$$
[S, (T \partial_S) \star \mathcal{W}] = [T, \mathcal{W}], \quad (3.72)
$$

that holds for general even $\star$-functions $\mathcal{W}$. The Cartan gauge transformations off-shell are given by the on-shell transformations.

### 4. Discussions

Let us summarize our results, speculate on future directions and conclude by trying to place our work and ideas into the more general context.
4.1. Summary

In this paper, we have presented an action principle for interacting higher spin gauge fields, including gravity, in four dimensions. The equations of motion and the boundary conditions following from the variational principle together yield a duality extension of Vasiliev’s equations for bosonic higher spin gravities. These duality-extended equations are locally equivalent to Vasiliev’s original equations in the sense of the unfolded treatment of dynamical systems. The whole construction is nothing but an adaptation of the Poisson sigma-model approach to unfolded dynamics in the specific case of Vasiliev’s system of equations with its inherent properties of the base manifold and zero-form sector, namely their being non-commutative and infinite-dimensional, respectively.

One key ingredient of the proposed off-shell formulation is to embed the four-dimensional spacetime manifold into the boundary of a higher-dimensional bulk manifold\(^6\). Another key ingredient is the introduction of generalized bulk momenta. More precisely, to every variable of the original duality-extended system, viewed as a coordinate-like variable, one assigns a momentum-like variable such that the sum of their form degrees is equal to the dimension of the bulk manifold minus 1. The resulting field content in the bulk thus consists of the bulk extension of the duality extended system and the corresponding set of conjugate bulk momenta, sometimes referred to as auxiliary momenta or Lagrange multipliers. The latter are forced to vanish on the boundary, and hence on spacetime, as a consequence of the variational principle under the single assumption that the variation of the coordinate-like variables are unrestricted everywhere, including the boundary. Hence, to repeat, the variational principle implies bulk equations of motion and boundary conditions. When combined together, they lead to the aforementioned duality-extended equations of motion on the boundary manifold, and thus on its spacetime submanifold.

In our work, we have provided action principles for even- and odd-dimensional bulk manifolds. Indeed, in unfolded dynamics, the dimension of the base manifold is not a crucial ingredient in the classical theory. On-shell, the dependence of the fields on the bulk coordinates, and therefore on the spacetime coordinates, is reconstructed in terms of gauge functions and the values of the zero-forms at a given reference point. More precisely, the independent gauge functions form sections of a bundle associated with the principal bundle of the unbroken gauge group, referred to as the structure group. In other words, the gauge functions take their values in the coset obtained by factoring out the unbroken gauge algebra from the full Cartan gauge algebra. Thus, in this gauge function approach to solving unfolded equations of motion, the coset-valued one-forms are generalized vielbeins that need not be invertible. Moreover, the data associated with the propagating degrees of freedom are encoded in the integration constants for covariantly constant zero-forms. It is in this precise sense that the proposed duality-extended equations are locally equivalent to Vasiliev’s equations: they share the same Weyl zero-forms\(^7\). We stress that the resulting classical moduli space contains, as a special case, solutions with invertible vielbeins. Hence, the gauge-function approach to unfolded dynamics is more general and encompasses the standard formulation of classical field theory. Essentially, this is so because, in both unfolded and standard languages, the notion of what constitutes different types of physical local degrees of freedom can be put in correspondence

---

\(^6\) More generally, the spacetime can be embedded into a submanifold of the bulk manifold. This submanifold is then surrounded by homotopy cylinder. Cutting out this cylinder leaves a truncated bulk manifold whose boundary contains the boundary of the homotopy cylinder. This cylinder can then be shrunk back to the submanifold containing spacetime.

\(^7\) In our models, the bulk momenta have strictly positive form degree. It is true in general, however, that if a generalized Poisson sigma model does contain zero-form bulk momenta, they nonetheless do not bring in any local degrees of freedom since their integration constants have to vanish on the boundary, by virtue of the variational principle.
with sectors of linearized Weyl tensors characterized by various boundary conditions, such as one-particle states or soliton-like solutions.

Off-shell, the key property of the proposed Lagrangian densities that facilitates their construction in rather general dimensions is that they are written using only star-wedge products and exterior derivatives without featuring any inverses of the aforementioned coset-valued vielbein. Besides Yang–Mills-like covariantizations, the Lagrangian contains two other types of nonlinearities in the form of two sets of functions. One set depends on coordinate-like variables, and describes an interaction freedom in the duality-extended Vasiliev system. The other set depends on momentum-like variables, and describes bi-vector and more general Poisson structures on the duality-extended Vasiliev system. These two types of nonlinearities are mutually incompatible, such that requiring nontrivial generalized Poisson structures (that contain more than only bi-vector fields), the interaction freedom in the duality-extended Vasiliev system is fixed to be a linear function.

As naively expected, drawing on properties of Poisson sigma models on commuting base manifolds, the Cartan gauge transformations of the bulk equations of motion also leave invariant the Lagrangian up to a total derivative. As far as total derivatives in the Lagrangian are concerned, we have chosen the kinetic terms such that the general variation yields the aforementioned boundary conditions on the momentum-like variables, and such that the gauge variations imply that the coordinate-like variables are generalized connections with locally defined gauge parameters in the bulk, while the momentum-like variables and their gauge parameters are sections.

More technically speaking, the duality extended version consists of differential forms of degrees \( p \in \{0, 1, 2, \ldots\ \} \) forming two master fields \( B = B_{[0]} + B_{[2]} + \cdots \) and \( A = A_{[1]} + A_{[3]} + \cdots \), and their Lagrange multipliers, which are differential forms of dual form degrees of degrees \( \hat{p} - p \), where \( \hat{p} + 1 \) is the dimension of the base manifold (including the twistor Z-space). The initial and boundary data associated with the Lagrange multipliers are completely fixed by means of boundary conditions following from the variational principle as discussed above. As a result, the Lagrange multipliers can be set equal to zero on-shell, leaving \( A \) and \( B \) subject to the unfolded equations of motion \( dA + A \star A + J \star B \approx 0 \) and \( dB + A \star B - B \star A \approx 0 \) where \( J = J_{[2]} + J_{[4]} \) is a closed and central element. This system contains Vasiliev’s original equations in degrees 0 and 1, namely \( dA_{[1]} + A_{[1]} \star A_{[1]} + J_{[2]} \star B_{[0]} \approx 0 \) and \( dB_{[0]} + A_{[1]} \star B_{[0]} - B_{[0]} \star A_{[1]} \approx 0 \).

An important point that remains to be established is whether the coupling \( J_{[4]} \star B \) is nontrivial in the sense that it cannot be redefined away. In Vasiliev’s original system, the coupling \( J_{[4]} \star B \) (and its Hermitian conjugate) is nontrivial; it is indeed this term that reproduces the nontrivial interactions in the second order in curvature in the effective unfolded equations of motion in the perturbative expansion around a non-degenerate vierbein [42]. The reason \( J_{[2]} \star B \) is nontrivial is that the central term \( J_{[2]} \) contains the inner Kleinian \( \kappa \) (that becomes a Dirac delta function in the Weyl order of the \((Y,Z)\)-oscillator algebra). We also note that \( J_{[4]} \) contains such singular elements, namely \( J_{[4]} \star B \) and its Hermitian conjugate, and \( J_{[4]} \star B \).

Thus, as already stated, the duality-extended \((A,B)\)-system is locally equivalent to Vasiliev’s original \((A_{[1]},B_{[0]})\)-system since they share the same Weyl zero-form \( B_{[0]} \); this master field contains the initial data associated with the Weyl curvature tensors, which contain information of local deformations of the system, such as semi-classical one-particle states and massive parameters of solitonic solutions, such as the black hole-like solutions of [43]. The systems are non-locally inequivalent, however, since the master fields with positive form degrees (including \( A_{[1]} \)) bring gauge functions on-shell that introduce new non-local degrees of freedom in unbroken phases (e.g. monodromies) as well as broken phases (boundary values of the broken gauge functions). In other words, if one has an exact and globally defined solution
to the duality-extended \((A, B)\)-system, then by construction it contains an exact and globally defined solution to the original system.

Whether the converse holds remains to be examined. As known from [44], there exist exact solutions of the original system for which the connections exhibit critical behaviors for finite amplitudes of \(B[0]\) (as can be described invariantly using zero-form invariants). Thus, it is not clear whether a given exact solution to the original system can be uplifted to the duality-extended system, as new critical phenomena may arise in potentials in the duality-extended sector.

Finally, the action principle involves an integration over a base manifold given by the product of an ordinary commuting base manifold, containing four-dimensional spacetime, and the non-commutative twistor \(Z\)-space \(\mathcal{Z}\). The Lagrangian also contains an additional integration over the internal twistor \(Y\)-space \(\mathcal{Y}\) whose role is to create invariants out of various infinite-dimensional representations of an internal higher spin Lie algebra. The latter integrals are the counterparts of index contractions in formulations of field theories involving only finite-dimensional representations of gauge algebras, such as Yang–Mills theories and standard formulations of gravities and supergravities. We note, however, that the salient semiclassical features of the presented action principle do not rely on integrating out the \(Y\)-variables. Indeed, as shown above, the variational principle reproduces the desired field equations, and the classical action vanishes on-shell (contrary to the action proposed in [39]), as expected for (undeformed) generalized Hamiltonian actions\(^8\). In this sense, the generalized Hamiltonian extension of Vasiliev’s equations should be thought of as a formulation of higher spin gravities directly in the correspondence space, given by the product of noncompact base and fiber manifolds. Thus, as stressed already in the introduction, the key issue is not whether there exists a formulation based on four-dimensional Fronsdal tensors, but rather whether the generalized Hamiltonian formulation admits nontrivial deformations that yield physical amplitudes on-shell, such as boundary correlation functions in perturbative expansions around anti-de Sitter backgrounds.

Finally, if one were to take our action principle seriously as a starting point for quantizing higher spin gravity, one would have to address the issue of boundary conditions on the internal connection \((A_\alpha, A_\dot{\alpha})\) in \(Z\)-space. In the standard perturbative expansion in the Weyl zero-form \(B[0]\), it is usually assumed that \((A_\alpha, A_\dot{\alpha})\) is pure gauge in the limit where \(B[0]\) vanishes. However, as found in [44], there are topologically nontrivial exact solutions based on projectors in which \((A_\alpha, A_\dot{\alpha})\) remains nontrivial for vanishing \(B[0]\), whose physical meaning remains to be better understood.

4.2. Outlook: AKSZ-BV quantum action and unfolded quantum field theory

The action principle proposed in this paper is an example of a generalized Hamiltonian action principle for an associative free differential algebra on a noncommutative base manifold. More generally, as far as the off-shell formulation of free differential algebras is concerned, one may think of three different levels of complexity depending on whether the algebra is associative and commutative, or associative and non-commutative, or of strongly homotopy associative type. In the commutative case, the BV quantum action is of the AKSZ-BV type and it has been proposed that the perturbative quantization (with suitable boundary conditions on Lagrange multipliers) yields master theories of the homotopy type (with \(\ell\)-ary products arising via terms in the Hamiltonian that are of \(\ell\)th order in the Lagrange multipliers).

\(^8\) For boundary deformations and more general deformations by lower dimensional impurities, see the subsequent developments in [45].
In our case, there exists a quantum action of AKSZ-BV type which we present elsewhere. Moreover, the classical \((A, B; U, V)\) system extends naturally to the strongly homotopy associative case and there are indications that its completion off-shell leads to an AKSZ-BV-like quantum action (within a suitable Noether procedure). It is thus tempting to speculate that there exist quantum theories based on layers of ’\(n\)-quantized’ unfolded quantum field theories, such that each layer is the master theory of the layer below with radiative corrections interpreted as a topological sum, giving rise to third quantization.

Pursuing these ideas, one is led to attempt to identify Vasiliev’s equations as the master equations for an underlying first-quantized topological open string: the system on the commutative manifold appears related to an underlying A-model; and the system on the noncommutative twistor space appears related to a B-model \([46]\). More generally, one may deform the bulk action with various topological vertex operators inserted on finite-dimensional sub-manifolds; these are gauge-invariant functionals whose variations vanish on-shell (so that the standard first-order action is an example of such a deformation) and whose values on-shell can be interpreted as amplitudes \([11, 17, 18]\). There are many such deformations, each of which one may seek to relate to an underlying first-quantized dual, such as for example the holographic dual in three dimensions for which one may propose a topological vertex operator that is a four-form.

The perturbations of the bulk action by various operators also provide a systematic approach to symmetry-breaking mechanisms: for example, one has topological mechanisms (homotopy phases), spontaneous mechanisms (classical solutions) and dynamical mechanisms (radiative corrections).

More radically, one may go so far as to elevate the aforementioned layered structure of unfolded quantum field theories into a quantum gauge principle, i.e. a set of mathematical rules that are nontrivial in the sense that they are meant to hold for any physical (quantum) system. In particular, the Cartan integrable free differential algebra of the \(n\)th layer, with its exterior derivative \(d\) (on a base manifold) and \(Q\)-structure (in a target space), should arise from the BRST operator of the \((n-1)\)th quantized system (subject to radiative corrections but with trivial topology as the topological sum of the \((n-1)\)th layer should correspond to the radiative corrections of the \(n\)th layer). In other words, the quantum gauge principle is meant to contain Cartan’s version of Weyl’s classical gauge principle.

In other words, the idea is that generic quantum system should not abide by the quantum gauge principle making it nontrivial. We believe, however, that the Vasiliev system is a candidate for (a massless sector of) a system compatible with the quantum gauge principle.

4.3. Conclusions

As far as four-dimensional higher spin gravities are concerned, the only fully nonlinear models that are known to date are those that have been obtained within Vasiliev’s formalism. This formalism provides a general framework for higher spin gravities based on free differential algebras on noncommutative manifolds taking their values in internal associative (super)algebras.

All models arising within this framework are based on one and the same universal equation of motion; different models arise by choosing different base manifolds and associative algebras. In this sense, all models arising within Vasiliev’s framework can be viewed as various Yang–Mills and supersymmetric extensions of a basic minimal bosonic model consisting perturbatively of a scalar field, a metric and a tower of Fronsdal tensors of ranks \(\{4, 6, \ldots\}\).

Strictly speaking, the perturbative formulations in terms of matter fields, a metric and towers of Fronsdal fields arise only under a set of extra assumptions on boundary conditions
in twistor spaces as well as invertibility of a spacetime vierbein; whether the resulting perturbative models exhaust all possibilities within the perturbative Fronsdal programme is an open problem though there are uniqueness theorems to low orders that support the belief that Vasiliev’s formalism is indeed exhaustive. Indeed, remarkably, the perturbative expansions of Vasiliev’s equations around its anti-de Sitter vacuum appear paradigmatic as far as holography is concerned, that is, it reproduces the simplest possible candidates for holographic duals of higher spin gravities [47–50]; see for example the recent works in [51–54].

Vasiliev’s equations admit, however, novel exact solutions that involve moduli that are not visible in the perturbative Fronsdal programme, such as solutions activating only the internal connections in twistor space and having vanishing Weyl tensors. The formalism also admits extensions by differential forms whose exterior derivatives vanish identically in the linearized approximation, which one may think of as analogs of the three-dimensional gauge fields.

Taken altogether, the state of affairs motivates a careful examination of whether Vasiliev’s full master fields, or their duality-extended versions presented here, should be treated as being the actual fundamental fields at the second-quantized level. To this end, the aim becomes to include all unfolded differential-form variables into the action principle and to treat the spacetime and the twistor space on an equal footing. This leads more or less directly to the type of generalized Hamiltonian bulk actions considered in this paper. These lend themselves naturally to the BRST treatment, at which stage one faces the key problem of how to connect back to the perturbative Fronsdal programme with its relatively clear physical interpretation. One natural path to follow is to examine various perturbations of the bulk action by various boundary actions, which we leave for future studies.

Acknowledgments

We acknowledge C Iazeolla and A Sagnotti for collaborations at the earliest stages of this project. We have also benefited from interactions with N Colombo, S Lyakhovich and E Sezgin. We thank I Bandos, G Barnich, M Grigoriev, E Skvortsov, D Sorokin, Ph Spindel and M Vasiliev for discussions. NB acknowledges F Buisseret and E Skvortsov for encouragements. PS acknowledges Ph. Spindel for encouragement. We are both grateful to Scuola Normale Superiore (Pisa) for support at the early stages of the project. The work of NB was supported in parts by an ARC contract no AUWB-2010-10/15-UMONS-1.

Note added. The results in this paper were partly presented by PS at the 4th International Sakharov Conference on Physics, 18–23 May 2009, Lebedev Institute (Moscow), and at the International Workshop on Gauge Theories, Supersymmetry, and Mathematical Physics, 6–10 April 2010, Lyon, France.

Appendix A. Free differential algebras on non-commutative base manifolds

Vasiliev’s on-shell formulation of higher spin gravity makes use of a version of unfolded dynamics that is based on graded-associative quasi-free differential algebras with central and closed terms. Such an algebra encodes the following key structures:

\[(\mathfrak{B}, \mathfrak{A}, \ast, d; J, \mathfrak{Q}, \mathfrak{u}),\]

used to describe the moduli spaces \(\mathcal{M}_u\) consisting of generalized flat connections and covariantly constant sections associated with a principal bundle with structure group generated by a generalized Lie algebra \(u\), referred to as the unbroken (generalized) gauge algebra. The

\(^9\) These forms appear in the \(k\)-independent part of \(B_{[0]}\) and the \(k\)-linear part of \(A_{[1]}\).
connections and sections are grouped into an $\mathfrak{A}$-valued section over a noncommutative base manifold $\mathfrak{B}$. The choice of $u$ breaks the module $\mathfrak{A}$ into a set of representations of $u$ labeled by indices $i$ in an index set $\mathcal{I}$, namely

$$\mathfrak{A} \downarrow u = \bigoplus_{i \in \mathcal{I}} \mathfrak{A}_i.$$  \hfill (A.1)

As a result, the sections can be coordinatized by a set $\{Z^i\}_{i \in \mathcal{I}}$ of differential forms in degrees $p_i \equiv \text{deg}(Z^i) \in \mathbb{N}$ (including zero-forms). These are the fundamental (classical) fields of the unfolded system and we refer to $Z^i$ as the master field\footnote{In the case of theories with manifest Lorentz invariance, i.e. theories for which $u$ contains the Lorentz algebra of some signature, the master fields can thus be broken down further into Lorentz tensors, though for the sake of constructing $u$-covariant theories it is more convenient to work directly with the $u$-tensors $Z^i$.} of flavor $i$. They can be acted upon with the exterior derivative $d$ and composed using the associative and noncommutative product $\star \equiv \star \wedge$ which comprises the product on $\mathfrak{A}$ as well as the composition of differential forms on $\mathfrak{B}$ such that the following rules apply:

$$\text{deg}(Z^i \star Z^j) = \text{deg}(Z^i) + \text{deg}(Z^j), \quad \text{deg}(d) = 1,$$  \hfill (A.2)

$$d(Z^i \star Z^j) - (dZ^i) \star Z^j - (-1)\text{deg}(Z^i)Z^i \star (dZ^j) \equiv 0,$$  \hfill (A.3)

$$(Z^i \star Z^j) \star Z^k - Z^i \star (Z^j \star Z^k) \equiv 0.$$  \hfill (A.4)

In the coordinate charts $\mathfrak{B}_\xi \subset \mathfrak{B}$ labeled here by an additional chart index $\xi \in \mathcal{X}$, which we will drop when ambiguities cannot arise, the sections have local representatives

$$Z^i_\xi \in \Omega(\mathfrak{B}_\xi) \otimes \mathfrak{A},$$  \hfill (A.5)

that can be presented by symbols by going to some definite basis of $\Omega(\mathfrak{B}_\xi) \otimes \mathfrak{A}$. These sections thus obey a set of generalized curvature constraints

$$\mathcal{R}^i_\xi := dZ^i_\xi + Q'(Z^j_\xi, J^j) \approx 0, \quad (i, \xi) \in \mathcal{I} \times \mathcal{X},$$  \hfill (A.6)

which are assumed to be compatible with $d^2 \equiv 0$ on universal base manifolds. The resulting universal Cartan integrability conditions on the structure functions $Q'$ ensure that the constraint surface is invariant under the Cartan gauge transformations

$$\delta_{\xi}Z^i_\xi \equiv T^i_{\epsilon_\xi, \text{dec}} \cdot Z^i_\xi := de^i_\xi - T^i_{\xi} \star Q'(Z^j_{\xi}, J^j), \quad T^i_{\xi} := \epsilon^i_\xi \partial_j,$$  \hfill (A.7)

with locally defined gauge parameters $\epsilon^i_\xi$; for the notation concerning $\star$-vector fields, see appendix B. These transformations close into a field-dependent Cartan gauge algebra $\mathfrak{g}$ on-shell, that is, modulo terms proportional to the generalized curvatures; for details, see equations (B.14) and (B.15). The locally defined configurations can then be glued together into globally defined configurations by making use of transition functions generated by the subalgebra $\mathfrak{u} \subset \mathfrak{g}$, as we describe further below.

In (A.6) and (A.7), the structure functions $Q'$ are given by $\star$-power expansions in $Z^i_\xi$ and an additional set $\{J^j\}$ of globally defined elements that are central and closed, namely

$$J^j \in \Omega(\mathfrak{B}) \otimes \mathfrak{A}, \quad dJ^j \equiv 0, \quad J^j \star Z^i_\xi - Z^i_\xi \star J^j \equiv 0,$$  \hfill (A.8)

hence generating a closed and central subalgebra

$$\mathfrak{J} \subset \Omega(\mathfrak{B}) \otimes \mathfrak{A}.$$  \hfill (A.9)

Strictly speaking, one has that the generalized curvature constraints in (A.6) and the relations in (A.8) define a graded-associative quasi-free differential algebra; that is, the set $\{\mathcal{R}^i_\xi, J^j\}$ is an exterior differential system forming an ideal in the graded ring of locally defined differential
forms on \( \mathcal{B} \) that is closed under the operation of the exterior differentiation modulo on a set of algebraic constraints. In the perturbative approach, it is furthermore assumed that the structure functions can be expanded as

\[
Q'(Z', J') = \sum_n Q_{j_1, \ldots, j_n}(J') \star Z^n \star \cdots \star Z^n
\]

with coefficients \( Q_{j_1, \ldots, j_n}(J') \in \mathcal{I} \), that need not be graded symmetric in their lower flavor indices due to the non-commutativity of \( \star \). The universal Cartan integrability of (A.6), which is thus tantamount to compatibility with \( d^2 \equiv 0 \) on base manifolds \( \mathcal{B} \) of arbitrary dimension, amounts to that

\[
\overline{\mathcal{C}} \star \overline{\mathcal{C}} \equiv 0, \quad \overline{\mathcal{C}} := Q'(Z', J') \partial_t, \quad \text{(A.11)}
\]

or equivalently, to that the coefficients obey

\[
\sum_{n_1 + \cdots + n_2 = n - 1} \sum_{m=1}^{n_1} Q_{j_1, \ldots, j_n}(J') \star Q_{j_1, \ldots, j_n}(J') \equiv 0, \quad \text{(A.12)}
\]

where the flavor indices \( j_1, \ldots, j_n \) are not subject to any graded symmetry due to the non-commutativity of \( \star \).

The Cartan gauge transformations (A.7) exponentiate into generalized group elements

\[
\mathcal{G}_{\lambda, Z} := \exp \mathcal{P}_{\lambda, d\lambda, Z}, \quad \text{(A.13)}
\]

generated by finite gauge functions \( \lambda' \). The space \( \mathcal{M}_\xi \) of locally defined solutions to \( \mathcal{R}_\xi := dZ^i + Q'(Z^i', J') \approx 0 \) can be sliced formally into Cartan gauge orbits, namely

\[
\mathcal{M}_\xi = \{ \mathcal{G}_{\lambda, Z} \star Z' : \lambda' = \lambda'_{\xi}, Z' = Z'_{(\xi, C)} \}, \quad \text{(A.14)}
\]

where \( \lambda'_{\xi} \) and \( Z'_{(\xi, C)} \) are locally-defined gauge functions and reference solutions, respectively; the reference solution\(^{11} \) (i) obeys the constraints \( dZ^i_{(\xi, C)} + Q'(Z^i_{(\xi, C)}), J' \approx 0 \); (ii) obeys a physical gauge condition; (iii) has an initial datum \( (Z^i_{(\xi, C)}|_{t=0}) \) \( p \in \mathcal{B}_\xi \) is a base point and \( (\cdot)|_{t=0} \) denotes the projection to zero form degree and (iv) \( \theta \) labels topologically nontrivial sectors of \( \Omega(B_\xi) \otimes \mathfrak{A} \). In other words, setting \( C_\xi = 0 \) there remains nontrivial moduli \( \theta \xi \) associated with generalized flat connections on noncommutative manifolds; for an example in the context of higher spin gravity, see the flat connections constructed from projectors in [55, 44].

The moduli space \( \mathcal{M}_u \) is then obtained by gluing together locally-defined modules \( \mathcal{M}_\xi \) by means of transition functions generated by the unbroken gauge algebra \( u \subseteq g \), namely

\[
\mathcal{M}_\xi \cong \mathcal{G}_{\xi}^u \star \mathcal{M}_{\xi'}, \quad \mathcal{G}_{\xi}^u := \exp \mathcal{P}_{t_{\xi}, d\lambda_{\xi}, Z_{\xi}}, \quad t_{\xi}^u \in u, \quad \text{(A.15)}
\]

where the parameters are defined on cylinders homotopic to the overlaps \( \mathcal{B}_\xi \cap \mathcal{B}_{\xi'} \) (we are assuming that \( \mathcal{B} = \bigcup_\xi \mathcal{B}_\xi \)). The coordinates on \( \mathcal{M}_u \) are gauge-invariant and intrinsically defined classical observables \( O_u \), that is, functionals of the master fields that are manifestly \( u \)-invariant off-shell and invariant under the \( \star \)-morphisms of the base manifold \( \mathcal{B} \) on-shell (which in particular include the diffeomorphisms of commuting submanifolds of \( \mathcal{B} \)). The former condition implies that if \( U_{\xi} \) are gauge transformations, generated by locally defined parameters defined on \( \mathcal{B}_\xi \) and valued in \( u \), then the locally defined group elements \( \mathcal{G}_{\xi} \), generated

\(^{11} \) In the case of Vasiliev’s unfolded formulation of higher spin gravities, there exist interesting sub-sectors of the theory in which the reference solutions can be constructed using exact methods based on Wigner’s deformed oscillator algebra. When combined with the Cartan integrability, this implies that Vasiliev’s higher spin gravity is formally solvable in these sub-sectors.
by the gauge functions \( \lambda^\xi_i \) defined on \( \mathcal{B}_\xi \) and valued in \( \mathfrak{g} \), and the transitions functions \( G^\xi \xi' \) defined above belong to equivalence classes as follows:

\[
G^\xi_\xi' \sim U^\xi_\xi \ast G^\xi_\xi', \quad G^\xi' \xi_\xi' \sim U^\xi_\xi \ast G^\xi_\xi' \ast (U^\xi_\xi)^{-1}.
\] (A.16)

Thus, the gauge functions \( \lambda^\xi_i \) can be taken to be valued in the coset \( \mathfrak{g} / \mathfrak{u} \). Moreover, the transition functions \( G^\xi \xi' \) become elements of generalized characteristic classes which we will not attempt to discuss further here.

In order to extract physical data from the gauge functions, one may, for example, consider homotopy charges given by integrals

\[
O := \oint_{\Sigma'} (\omega^R + k^R), \quad \Sigma' \in [\Sigma]
\] (A.17)

over nontrivial \( p_R \)-cycles [\( \Sigma \)] of \( p_R \)-forms \( \omega^R[Z, J] \) and \( k^R[Z, J] \) that are manifestly \( \mathfrak{u} \)-invariant, i.e.

\[
\delta_\ve (\omega^R, k^R) \equiv 0, \quad \ve \in \mathfrak{u},
\] (A.18)

and defined by the equivariant cohomology system

\[
d\omega^R + f^R(\omega) \approx 0, \quad f^R(\omega)|_{\Sigma_{cy}} \approx d k^R|_{\Sigma_{cy}},
\] (A.19)

where \( \Sigma_{cy} \) is a cylinder of finite thickness containing \( \Sigma \); the homotopy invariance of the de Rham cohomology classes then implies that \( H^{p_R+1}(\Sigma_{cy}) = 0 \) so that \( f^R|_{\Sigma_{cy}} \) must be exact, that is, given by the exterior derivative of some \( \mathfrak{p}_R \)-form \( k^R \) that is globally defined on \( \Sigma \) (and hence gauge invariant). Thus, the integral over \( \Sigma \), which must necessarily be split into several charts, say \( \{ \Sigma_\xi \} \), makes sense and is independent of the choice of \( \Sigma' \). A variation \( \delta_\ve \lambda^\xi_i = \ve^\xi_i \) in the gauge functions thus induces a change in \( (\omega^R + k^R)|_{\Sigma_{cy}} \) given by

\[
\delta_\ve (\omega^R + k^R)|_{\Sigma_{cy}} = dX_\xi (\ve^\xi_i),
\] (A.20)

where \( X_\xi (\ve^\xi_i) \) is a linear functional in \( \ve^\xi_i \). By the \( \mathfrak{u} \)-invariance, one has that \( X_\xi (\ve^\xi_i) \) is invariant under \( \mathfrak{u} \)-transformations that act simultaneously on \( Z^\xi \) and the gauge parameter (cf the BRST treatment where the gauge parameter is promoted into a ghost). It follows that

\[
\delta_\ve O_u = \sum_\xi \oint_{\partial \Sigma_\xi} X_\xi (\ve^\xi_i),
\] (A.21)

which can be split into contributions from chart boundaries in the interior of \( \mathcal{B} \) and from true boundaries of \( \mathcal{B} \). The former must cancel identically if one assumes that the choice of where to cut the interior of \( \mathcal{B} \) into charts should not be of no importance. Taking into account the signs coming from orientation, this is a consequence of the fact that \( \{ \lambda^\xi_i \} \) forms a globally defined section (of the soft \( \mathfrak{u} \)-bundle). One thus has

\[
\delta_\ve O_u = \sum_\xi \oint_{\partial \mathcal{B} \cap \partial \Sigma_\xi} X_\xi (\ve^\xi_i),
\] (A.22)

which is the only physical dependence of the gauge functions that enters via their boundary values, which one may view as an unfolded version of the holographic principle.

Appendix B. Further details: \( \ast \)-vector fields and Cartan integrability

In this appendix, we go into the technical details of \( \ast \)-functions, \( \ast \)-vector fields and Cartan integrability that were introduced in appendix A. Let us recall the general idea of a graded-associative quasi-free differential algebra on a non-commutative base manifold \( \mathcal{B} \). Such an algebra consists of local representatives \( \mathfrak{R}_\xi \) generated by sets \( \{ Z^\xi_i \}_{i \in S} \) of locally-defined
differential forms subject to generalized curvature constraints which we recall here for convenience:

$$\mathcal{R}_\xi := dZ^i_\xi + \mathcal{Q}'(Z_\xi, J) \approx 0,$$

where $$\mathcal{Q} := Q^i \partial_i$$ is a composite $$\star$$-vector field of total degree 1 subject to the Cartan integrability condition

$$\mathcal{Q} \star \mathcal{Q}' \equiv 0.$$  

(B.2)

Here, we use the following notation and conventions:

(i) $$\xi$$ labels charts $$\mathcal{B}_\xi \subset \mathcal{B}$$ with coordinates $$Z^M_\xi$$ of degree zero and differentials $$dZ^M_\xi$$ of degree 1 generating $$\mathbb{N}$$-graded associative $$\star$$-product algebras

$$\Omega_\xi \equiv \text{Env}[Z^M_\xi, dZ^N_\xi]$$ 

(B.3)

modulo the graded $$\star$$-commutators

$$[Z^M_\xi, Z^N_\xi], = 2i\Pi^{MN}, \quad [Z^M_\xi, dZ^N_\xi], = 0, \quad [dZ^M_\xi, dZ^N_\xi], = 0,$$

where $$\Pi^{MN}$$ is a constant matrix (defining a canonical Poisson structure $$\Pi = \Pi^{MN} \partial_M \otimes \partial_N$$);

(ii) the action of the exterior derivative $$d = dZ^M_\xi \partial/\partial Z^M_\xi$$ in $$\Omega_\xi$$ is defined by declaring that

$$d(Z^M_\xi) = dZ^M_\xi, \quad (df) \star g = (df) \star g + (-1)^{\text{deg}f} f \star (dg),$$

for elements $$f, g \in \Omega$$ such that $$f$$ has a fixed form degree $$\text{deg}f$$; one has

$$d^2 \equiv 0.$$  

(B.6)

(iii) the locally-defined differential forms $$Z^j_\xi \in \Omega^{(p)}_\xi \otimes \Theta^j$$, where $$\Omega^{(p)}_\xi$$ is the subspace of $$\Omega_\xi$$ of fixed form degree $$p$$ and $$\Theta^j$$ can be either finite-dimensional internal tensors (such as for example Lorentz tensors) or sectors of an internal associative algebra $$\mathcal{A};$$

(iv) the graded associative $$\star$$-product algebra $$\mathcal{R}_\xi := \text{Env}[Z^j_\xi] \otimes \mathcal{J}$$ where $$\mathcal{J}$$ is a space of central and d-closed elements (including the identity), i.e. if $$\mathcal{F}(Z^j_\xi) \in \mathcal{R}_\xi$$, then

$$\mathcal{F} = \sum_{p \geq 0} \mathcal{F}_{j_1,\ldots,j_k} \star Z^{j_1}_\xi \star \cdots \star Z^{j_k}_\xi, \quad \mathcal{F}_{j_1,\ldots,j_k} \in \mathcal{J};$$

(B.7)

(v) a composite $$\star$$-vector field $$\overline{\mathcal{X}}$$ is a graded inner derivation of $$\mathcal{R}$$, i.e. if $$\mathcal{F}, \mathcal{F}' \in \mathcal{R}$$, then

$$\overline{\mathcal{X}} \star (\mathcal{F} \star \mathcal{F}') = \overline{\mathcal{X}} \star \mathcal{F} \star \mathcal{F}' + (-1)^{\text{deg}(\overline{\mathcal{X}}) \text{deg}(\mathcal{F})} \mathcal{F} \star (\overline{\mathcal{X}} \star \mathcal{F}'),$$

(B.8)

provided that $$\overline{\mathcal{X}}$$ and $$\mathcal{F}$$ have fixed degrees. In components, one writes $$\overline{\mathcal{X}} := X'(Z) \partial_i$$, where $$X' := \overline{\mathcal{X}} \otimes \partial Z$$ (and $$\partial \equiv \overline{\mathcal{F}}$$). The graded bracket between two composite $$\star$$-vector fields is defined by

$$[\overline{\mathcal{X}}, \overline{\mathcal{X}}]', \star := \overline{\mathcal{X}} \star (\overline{\mathcal{X}}' \star \mathcal{F}) - (-1)^{\text{deg}(\overline{\mathcal{X}}) \text{deg}(\overline{\mathcal{X}}')} \overline{\mathcal{X}}' \star (\overline{\mathcal{X}} \star \mathcal{F}),$$

(B.9)

which is a degree-preserving graded Lie bracket, i.e. $$[\overline{\mathcal{X}}, \overline{\mathcal{X}}]'_*$$ is a graded inner derivation obeying the graded Jacobi identity $$[\overline{\mathcal{X}}, \overline{\mathcal{X}}]' \star, \overline{\mathcal{X}}''], +\text{graded cyclic} \equiv 0.$$ In components, one has

$$[\overline{\mathcal{X}}, \overline{\mathcal{X}}]'_* = (\overline{\mathcal{X}} \star \mathcal{X}' - (-1)^{\text{deg}(\overline{\mathcal{X}}) \text{deg}(\overline{\mathcal{X}}')} \overline{\mathcal{X}}' \star \mathcal{X}') \partial_i.$$  

(B.10)
The Cartan integrability condition (B.2), that can be rewritten as \([ \mathcal{Q}, \mathcal{Q} ] \), \( \equiv 0 \) amounts to that \( \mathcal{Q} \) is a nilpotent composite \( \ast \)-vector field of degree 1. This condition ensures that the generalized curvature constraints \( \mathcal{R}' \approx 0 \) are compatible with \( d^2 \equiv 0 \) without further algebraic constraints on the generating elements \( Z' \). One can also show that the nilpotency of \( \mathcal{Q} \) is separately equivalent to that the generalized curvatures \( \mathcal{R}' \) obey the generalized Bianchi identities

\[
d d \mathcal{R}' - \mathcal{R} \ast Q' \equiv 0, \quad \text{where} \quad \mathcal{R} := \mathcal{R}' \partial_i,
\]

and transform into each other under the following Cartan gauge transformations:

\[
\delta_i Z' := T_i^j := d \epsilon^j - \mathcal{P} \ast Q^i, \quad \text{where} \quad \mathcal{P} := \epsilon^j \partial_j (B.12)
\]

and where \( \epsilon^j \) is an element in \( \Omega \otimes \Theta \) that is considered infinitesimal and independent of \( Z' \), namely

\[
\delta_i \mathcal{R}' = - \mathcal{R} \ast (\mathcal{P} \ast Q^i). \quad \text{(B.13)}
\]

The closure relation reads

\[
[\delta_1, \delta_2] Z^i = \delta_{12} Z^i = - \mathcal{R} \ast \epsilon^{12}, \quad \text{(B.14)}
\]

where the combined parameters \( \epsilon^{12} \) are given by

\[
\epsilon^{12} = - \frac{1}{2} [\mathcal{P}^1, \mathcal{P}^2] \ast Q'. \quad \text{(B.15)}
\]

The above results can easily be obtained upon introducing the even \( \ast \)-vector field

\[
\nabla_{\epsilon} := (\mathcal{P} \ast Q') \partial_i, \quad \text{(B.16)}
\]

and using the following set of identities which are consequences of the first one:

\[
[\mathcal{Q}, \mathcal{Q}] \equiv 0, \quad [\mathcal{Q}, \nabla_{\epsilon}] \equiv 0, \quad [\nabla_{\xi_1}, \nabla_{\xi_2}] \equiv [\mathcal{Q}, \nabla_{\xi_2}], \quad \text{(B.17)}
\]

where we recall that all the commutators are graded commutators.

As discussed above, the local representatives \( \mathcal{R}_i \) are glued together on overlaps \( \mathcal{B}_i \cup \mathcal{B}_j \) by means of the transitions \( Z'_i = \mathcal{G}^j_i \ast Z'_j \), where the transition functions \( \mathcal{G}^j_i \) are soft group elements given by \( \ast \)-exponentials of the Cartan gauge transformations as in (A.13). From the Leibnitz rule (B.8) it follows that these transitions are indeed isomorphisms, namely

\[
\mathcal{G} \ast \mathcal{F}(Z) = \mathcal{F}(\mathcal{G} \ast Z), \quad \mathcal{G} \ast (\mathcal{F} \ast \mathcal{F}') = (\mathcal{G} \ast \mathcal{F}) \ast (\mathcal{G} \ast \mathcal{F}'). \quad \text{(B.18)}
\]

We would like to show that, if \( Z' \) satisfies the star-product equation \( dZ' + Q'(Z') \approx 0 \), then \( Z'_i := (\exp[Z', \lambda]) \ast Z' \) where \( \mathcal{T}_\lambda := d \lambda^j \partial_j - \nabla_\lambda \) (see (B.16)) satisfies the equation \( dZ'_i + Q'(Z'_j, J) \approx 0 \), thereby exhibiting the fundamental integrability of the unfolded equations in the case where the free differential algebra \( \mathcal{A} \) is endowed with a non-commutative star-product. We recall that

Lemma. The following commutation relation is true: \([ \mathcal{T}_\lambda, d ] \equiv 0 \), where the weak equality means an equality on the surface \( \Sigma \equiv \{ dZ' + Q'(Z, J) \} = 0 \).

Proof of the lemma. On the surface \( \Sigma \), the total exterior derivative \( d \approx \nabla - \Lambda' \), where

\[
\Lambda' := d \lambda^j \frac{\partial}{\partial \lambda^j}. \quad \text{(B.19)}
\]

The proof is tantamount to showing that \([ \mathcal{T}_\lambda, \mathcal{Q} - \Lambda' ] \ast Z' = 0 = [ \mathcal{T}_\lambda, \mathcal{Q} - \Lambda ] \ast \lambda^j \) because then, using the fact that \([ \mathcal{T}_\lambda, \mathcal{Q} - \Lambda ] \), \( \ast \)-vector field, it follows that \([ \mathcal{T}_\lambda, \mathcal{Q} - \Lambda ] \ast \mathcal{F}(Z, \lambda) = 0 \) for an arbitrary star-product function \( \mathcal{F}(Z, \lambda) \).
(a) First of all, it is trivial to see that \( [\mathcal{T}_\lambda, \mathcal{Q} - \mathcal{K}]_* \ast \lambda^i = 0 \). Indeed, it gives \( \mathcal{T}_\lambda (d\lambda^i) \) which vanishes.\(^{12}\)  

(b) The fact that \( [\mathcal{T}_\lambda, \mathcal{Q} - \mathcal{K}]_* \ast Z^i = 0 \) is more difficult to show. For that, we write  
\[
Q' = \sum_n Q''_{j_1,\ldots,j_n}(J) \ast Z^{j_1} \ast \ldots \ast Z^{j_n}
\]

where \( Q''_{j_1,\ldots,j_n} \in \mathfrak{g} \) and compute  
\[
\mathcal{Q} \ast (T_\xi \ast Z^i) = - \sum_n \sum_{\alpha < \beta} \sum_{\gamma<\alpha<\beta} \sum_{\gamma<\alpha<\beta} (-1)^{j_0+\cdots+j_{n-1}} Q''_{j_1,\ldots,j_n} \ast Z^{j_1} \ast \ldots \ast Q''_{j_1,\ldots,j_n} \ast \lambda^{j_n} \ast Z^{j_n}.
\]

\[
\mathcal{T}_\lambda \ast \mathcal{Q} \ast Z^i = [dz^k - (\lambda^j \partial_j) \ast Q^i] \partial_k \ast Q^i, \\
\mathcal{K} \ast (T_\xi \ast Z^i) = - \sum_n \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} Q''_{j_1,\ldots,j_n} \ast Z^{j_1} \ast \ldots \ast d\lambda^{j_n} \ast Z^{j_n}, \\
T_\xi \ast (\mathcal{A} \ast Z^i) = 0.
\]

Regrouping all the terms, we find  
\[
[\mathcal{T}_\lambda, \mathcal{Q} - \mathcal{K}]_* \ast Z^i = \mathcal{Q}^i \partial_j \ast [\lambda^j \partial_k] \ast Q^j] - [(\lambda^j \partial_k) \ast Q^j] \partial_j \ast Q^i (B.20)
\]

which vanishes identically due to the second identity of (B.17).

Therefore, since \( [\mathcal{T}_\lambda, \mathcal{Q} - \mathcal{K}]_* \) is a star-product vector field, it follows that \( [\mathcal{T}_\lambda, \mathcal{Q} - \mathcal{K}]_* \ast \mathcal{F}(Z, \lambda) = 0 \) for an arbitrary star-product function \( \mathcal{F}(Z, \lambda) \).

Using the above lemma, we have that \( Z_j^i := (\exp[d(T_\lambda)]) \ast Z^i \) satisfies the equation \( dZ_j^i + [Q^i \ast Z_j^i] \approx 0 \), since \( d[Z_j^i] = d[\exp[d(T_\lambda)] \ast Z_j^i] = (\exp[d(T_\lambda)]) \ast dZ_j^i \approx - (\exp[d(T_\lambda)]) \ast Q^i \ast (Z_j^i) \). This proves the formal Cartan integrability of the star-product unfolded equations.

### Appendix C. The Vasiliev equations

In this appendix, we describe Vasiliev’s unfolded on-shell formulation of four-dimensional bosonic higher spin gravities in the absence of extra outer Kleinians. The corresponding master fields are locally defined operators of the form  
\[
O_\xi (X^M, P^\xi, \xi^M, dX^\xi, dP^\xi, Z^\xi, dZ^\xi, Y^\xi),
\]

where the non-vanishing commutators among the coordinates are  
\[
[X^M, Y^\xi]_* = i\delta^M_\xi, \quad [Y^\xi, Y^\eta]_* = 2i\epsilon^{\alpha\beta}, \quad [Z^\xi, Z^\eta]_* = -2i\epsilon^{\alpha\beta},
\]

with charge conjugation matrix\(^{13}\) \( C^{\alpha\beta} = \epsilon^{\alpha\beta} \) and \( C^{\alpha\beta} = \epsilon^{\alpha\beta} \). The operators are represented by symbols \( f[O_\xi] \) obtained by going to specific bases for the operator algebra which one

\(^{12}\) We consider the algebra where the fields \( \{Z^i\} \) and \( \{\lambda^i\} \) are considered as independent, in accordance with the BRST treatment of gauge systems.

\(^{13}\) We raise and lower quartet and doublet indices using the conventions \( \Lambda^\alpha = C^\alpha\beta \Lambda_\beta \) and \( \lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta \) and \( \lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha} \), and we use the notation \( \Lambda \cdot \Lambda' = \Lambda \Lambda' \), \( \lambda \cdot \lambda' = \lambda \lambda' \) and \( \lambda' \cdot \lambda = \lambda' \lambda \).
may also think of as ordering prescriptions\textsuperscript{14}. One may think of the symbols as forms \( f(X, P, Z; Y) \) (with variables composed using commutative juxtaposition) on a correspondence space

\[
\mathcal{C} = \bigcup_{\xi} \mathcal{C}_{\xi}, \quad \mathcal{C}_{\xi} = \mathcal{M}_{\xi} \times \mathcal{Y}, \quad \mathcal{M}_{\xi} = T^* \mathcal{M}_{\xi} \times \mathcal{Y}
\]

(C.3)
equipped with a composition rule, also denoted by \( \star \), such that

\[
f[O_\xi] \star f[O_\xi'] = f[O_{\xi \xi'}].
\]

(C.4)

Working within a restricted class of orderings, referred to as universal orderings, the exterior derivative on \( \mathcal{B} \) is given by

\[
d = dX^M \partial_M + dP_M \partial^M + dZ^a \partial_a.
\]

(C.5)
The master fields of the (duality-unextended) minimal bosonic model are an adjoint one-form

\[
A = dX^M A_M(X, P, Z; Y) + dP_M A^M(X, P, Z; Y) + dZ^a A_a(X, P, Z; Y),
\]

(C.6)
and a twisted-adjoint zero-form

\[
\Phi = \Phi(X, P, Z; Y);
\]

(C.7)
these fields obey the following projection and reality conditions\textsuperscript{15}:

\[
\tau(A, \Phi) = (-A, \pi(\Phi)), \quad (A, \Phi)^\dagger = (-A, \pi(\Phi)),
\]

(C.8)

where the maps \( \tau, \pi, \bar{\pi} \) and \( \dagger \) are defined by \( d \circ (\tau, \pi, \bar{\pi}, \dagger) = (\tau, \pi, \bar{\pi}, \dagger) \circ d \) and\textsuperscript{16}

\[
\pi(y_a, \bar{y}_a; z_a, \bar{z}_a) = (-y_a, \bar{y}_a; -z_a, \bar{z}_a), \quad \pi(f \star g) = \pi(f) \star \pi(g),
\]

(C.9)

\[
\bar{\pi}(y_a, \bar{y}_a; z_a, \bar{z}_a) = (y_a, -\bar{y}_a; z_a, -\bar{z}_a), \quad \bar{\pi}(f \star g) = \bar{\pi}(f) \star \bar{\pi}(g),
\]

(C.10)

\[
\tau(y_a, \bar{y}_a; z_a, \bar{z}_a) = (iy_a, i\bar{y}_a; -iz_a, -i\bar{z}_a), \quad \tau(f \star g) = (-1)^{fg} \tau(g) \star \tau(f),
\]

(C.11)

\[\begin{aligned}
(y_a, \bar{y}_a; z_a, \bar{z}_a)^\dagger &= (\bar{y}_a, y_a; \bar{z}_a, z_a), \\
(f \star g)^\dagger &= (-1)^{g^*} g^* \star f^\dagger.
\end{aligned}\]

(C.12)
The \( \tau \)-projection removes all terms that are associated with the unfolded description of spacetime fermions as well as spacetime bosons with odd spin.

The full equations of motion for the minimal bosonic model with the simplest interaction freedom amount to the statement that the full curvature \( F := dA + A \star A \) is proportional to \( \Phi \), namely \( F + \Phi \star J = 0 \), where \( J \) is a globally defined two-form on the correspondence space obeying

\[
\tau(J) = J^\dagger = -J, \quad \pi(J) = J,
\]

(C.13)
and

\[
dJ = 0, \quad [J, f]_\pi = 0,
\]

(C.14)

\textsuperscript{14}The symbols are thus defined modulo similarity transformations generated by inner automorphisms (related to the higher spin gauge transformations) as well as changes of the order prescription, that is, changes of basis of the operator algebra. These types of transformations may have a drastic effect on the mathematical nature of the symbols, that may change from being a smooth or real analytic into being singular or even distributions. Thus, in order to extract physically meaningful information from the master fields, one needs to develop the notion of observables \( \mathcal{O} \), namely functionals of the locally-defined master fields that are invariant under both gauge transformations and re-orderings. The construction of such functionals introduces various geometric concepts into the theory, such as flat connections, covariantly constant sections (going into decorated Wilson loops), equivariantly closed forms (used to define homotopy charges) and metrics (that yield minimal areas of closed cycles).

\textsuperscript{15}Here we are focusing on the models containing spacetimes with Lorentzian signature and negative cosmological constant; for other signatures and signs of the cosmological constant, see [44].

\textsuperscript{16}The rule \( (f \star g)^\dagger = (-1)^{g^*} g^* \star f^\dagger \) holds for both real and chiral integration domain.
for any \( f \) obeying \( \pi \bar{\pi} (f) = f \), and where we have defined

\[
[f, g]_\pi = f \ast g - (-1)^{\ell_f} g \ast \pi (f).
\]

(C.15)

In the minimal model,

\[
J = -\frac{i}{4} (b d \bar{z}^2 \kappa + \bar{b} d \bar{z}^2 \bar{\kappa}),
\]

(C.16)

where the chiral Klein operators are given in the normal ordering by

\[
\kappa = \exp (i y^\alpha z_\alpha), \quad \bar{\kappa} = \kappa^\dagger = \exp (-i \bar{y}^\alpha \bar{z}_\alpha).
\]

(C.17)

By making use of field redefinitions \( \Phi_1 \rightarrow \lambda \Phi_1 \) with \( \lambda \in \mathbb{R} \), \( \lambda \neq 0 \), the parameter \( b \) in \( J \) can be taken to obey

\[
|b| = 1, \quad \arg (b) \in [0, \pi].
\]

(C.18)

The phase breaks parity except in the following two cases:

- type A model (parity-even physical scalar) : \( b = 1 \),
- type B model (parity-odd physical scalar) : \( b = i \).

(C.19)

The integrability of \( D \Phi_1 = 0 \) implies that \( D \Phi_1 \ast J = 0 \), that is, \( D \Phi = 0 \), where the twisted-adjoint covariant derivative \( D \Phi = d \Phi + [A, \Phi]_\pi \). This constraint is integrable since

\[
D^2 \Phi = F \ast \Phi - F \ast \pi (\Phi) = -\Phi \ast J \ast \Phi + \Phi \ast \pi (\Phi) \ast J = 0,
\]

(C.21)

using the constraint on \( F \) and (C.14). Thus, in summary, the unfolded system describing the minimal bosonic higher spin gravity with simplest possible interaction term is given by the following graded associative and quasi-free exterior differential algebra\(^{17}\):

\[
F + \Phi \ast J = 0, \quad D \Phi = 0, \quad dJ = 0,
\]

(C.22)

\[
F := dA + A \ast A, \quad D \Phi := d \Phi + [A, \Phi]_\pi,
\]

(C.23)

\[
[A, J]_\pi = 0, \quad [\Phi, J]_\pi = 0,
\]

(C.24)

together with the kinematic constraints (C.8) and (C.13). The integrability follows from the associativity of the \( \ast \)-product. It implies the Cartan gauge transformations\(^{18}\)

\[
\delta_\epsilon A = D \epsilon, \quad \delta_\epsilon \Phi = -[\epsilon, \Phi]_\pi,
\]

(C.25)

with closure relations \( [\delta_\epsilon, \delta_\zeta] = \delta_{\epsilon \zeta} \), where \( \epsilon_{12} = [\epsilon_1, \epsilon_2]_\pi \), for zero-form gauge parameters \( \epsilon (X, P, Z; Y) \) obeying the same kinematic constraints as the master one-form, i.e. \( \tau (\epsilon) = -\epsilon \) and \( (\epsilon) ^\dagger = -\epsilon \). These gauge transformations define the algebra \( \mathfrak{hs}(4) \) and can be identified as the subalgebra of the infinitesimal canonical transformations of \( \mathcal{C} \) subject to the supplementary kinematic constraints.

The quantities \( \kappa \) and \( \bar{\kappa} \) are the Klein operators of the chiral Heisenberg algebras generated by \( (y^\alpha, z_\alpha) \) and \( (\bar{y}_\bar{\alpha}, \bar{z}_{\bar{\alpha}}) \). Their symbols are distributions on the doubled twistor space whose functional form depends on the choice of ordering scheme that can thus be adapted to different physical problems; for example, in overall Weyl order they localize to Dirac delta functions that are useful in trace calculations, while in overall normal order they become Gaussians

\(^{17}\) The format applies also to Yang–Mills extended or supersymmetric models; for example, see [56–58].

\(^{18}\) These transformations are the canonical transformations of the \( \ast \)-product algebra generated by (C.2) containing the diffeomorphisms of Lagrangian submanifolds of the unifold.
that are useful in perturbation theory. The singular nature of the Kleinians implies that the twistor-space term $\Phi \ast J$ cannot be absorbed into a field redefinition [36].

The system spelled out so far can be projected to the Lagrangian submanifold $\mathcal{M}_\xi$ of $T^*\mathcal{M}_\xi$ by imposing

$$A^M = 0, \quad \delta^M(\Phi, A_M, A_\alpha) = 0,$$

which is the formulation that we use in the main body of this paper.

The spacetime formulation is then obtained by further restriction to $Z^\alpha = 0$ and to a four-dimensional submanifold of $\mathcal{M}_\xi$. The details of this projection have been spelled out in a number of places [36, 57]. Here, we simply state the basic steps. First, $A$ and $\Phi$ are expressed in terms of $\tilde{A} := A|_{Z^\alpha = 0}$ and $\Phi := \Phi|_{Z^\alpha = 0}$. This is done in a perturbative $\Phi$-expansion under the assumption that $\tilde{A}$ has trivial boundary conditions in $Z$-space. The twistor-space source term implies nontrivial $Z$-dependences leading to nontrivial reduced generalized curvatures on $\mathcal{M}_\xi$. These are given by $F|_{Z^\alpha = 0} = \tilde{F} + \sum_{n \geq 1} P^{(n)}(\tilde{A}, \tilde{A}, \Phi, \ldots, \Phi)$ and $D\Phi|_{Z^\alpha = 0} = \tilde{D}\Phi + \sum_{n \geq 1} P^{(n)}(\tilde{A}, \tilde{A}, \Phi, \ldots, \Phi)$, where $\tilde{F} := d\tilde{A} + \tilde{A} \ast \tilde{A}$ and $\tilde{D}\Phi := d\tilde{\Phi} + \tilde{A} \ast \tilde{\Phi} - \tilde{\Phi} \ast \pi(\tilde{A})$ are standard Yang–Mills-like curvatures. The deformations $J^{(n)}$ and $P^{(n)}$ are bilinear and linear functionals in $A$, respectively, and $n$-linear functionals in $\Phi$. One then decomposes $\Lambda = W + K$ and $W = e + W'$, where $K$ is linear in the Lorentz connection $(\omega^{ab\beta}, \tilde{\omega}^{ab\beta})$, $e$ consists of the vierbein $e^{a\alpha}$ and $W'$ contains Fronsdal fields and associated auxiliary connections and St"uckelberg connections. Treating the vierbein and spin-connection non-perturbatively and both $W$ and $\Phi$ as weak fields yields a perturbative expansion in which the linear piece of $J^{(1)}$ consists of the linearized source terms for unfolded gauge fields in accordance with Vasiliev’s central on-mass-shell theorem [3] (these linearized spacetime source terms are bilinear in $e^{a\alpha}$ and linear in primary Weyl tensors and appear in the constraints on the spin-two Riemann tensor and the generalized Riemann tensors in $\nabla W' + [e, W']_\alpha$).

Assuming that the vierbein is invertible on a four-dimensional submanifold of $\mathcal{M}_\xi$, the reduced unfolded field content in $\langle \tilde{\Phi}; e, \omega; W' \rangle$ can be expressed in terms of a set of dynamical fields forming a higher spin multiplet: a set of matter fields, a metric and a tower of Fronsdal fields. These fields obey equations of motion with standard second-order kinetic terms and non-local interactions that can be calculated order by order in the weak-field expansion. As a result, the standard general covariance is manifest while the higher spin gauge symmetry is blurred. Another conceptual problem is the fact that the original full system contains various moduli that have either problematic or no description at all in terms of the perturbatively defined higher spin multiplet, such as classical solutions with degenerate vierbeins and topological degrees of freedom contained in the internal connection $\Lambda_\alpha$ [44] and $A^M$.

Over and above their formal Cartan integrability, the Vasiliev equations exhibit the following more powerful integrable structures.

- The Maurer–Cartan integrability facilitates the explicit construction of solutions using gauge functions [41, 44, 59–61] and the formal construction of gauge-invariant observables [62].
- The zero-forms $S_\alpha := z_\alpha - 2iA_\alpha$ and $\bar{S}_\alpha := \bar{z}_\alpha - 2iA_\alpha$ yield the following generalization of Wigner’s deformed oscillator algebra with a local anyonic deformation parameter $\Phi$, namely

$$[S_\alpha, S_\beta] = -2i\epsilon_{a\beta}(1 - \Phi \ast \kappa), \quad [S_\alpha, S_\beta] = -2i\epsilon_{a\beta}(1 - \Phi \ast \bar{\kappa}),$$

$$[S_\alpha, S_\beta] = 0, \quad S_\alpha \ast \Phi + \Phi \ast \pi(S_\alpha) = 0, \quad S_\alpha \ast \Phi + \Phi \ast \pi(S_\alpha) = 0,$$

which one may also think of as describing the deformation of the symplectic structure on a submanifold of complex dimension 2 of the doubled twistor space (of complex dimension 4).

These properties have been used to construct classical solutions in [41, 43, 44, 53, 63], for perturbative calculations of the twistor-space vertices $P(W; \Phi)$ and $J(W, W; \Phi)$ in [57]
and direct verification of the conjectured correspondence between Vasiliev’s four-dimensional higher spin gravities and three-dimensional conformal field theories [49, 50], first in [64] at the level of cubic scalar self-couplings, and recently for the complete cubics in [51, 52].

Finally, let us comment on the following subtlety. The two-dimensional complexified Heisenberg algebra \([u, v] = 1\) has the Klein operator \(\hat{\kappa} = c \cos_4(\pi \star u)\), which anti-commutes with \(u\) and \(v\) and squares to 1. Hence, \(\hat{\kappa}\) remains invariant under \(SL(2; \mathbb{C})\) generated by \(u \star u\), \([u, v]\), and \(u \star v\). This symmetry is manifest in Weyl order where the symbol of \(\hat{\kappa}\) is proportional to \(\delta(u) \delta(v)\). It follows that \((\kappa, \hat{\kappa})\) are invariant under \(SL(4; \mathbb{C}) \times \overline{SL}(4; \mathbb{C})\), where \(SL(4; \mathbb{C})\) acts on the quartet \((y_\alpha, z_\alpha)\) in the fundamental representation. Its intersection with the group of canonical transformations is given by \(Sp(4; \mathbb{C}) \times \overline{Sp}(4; \mathbb{C})\) generated by \(y_{(\alpha \star \beta)}, y_\alpha \star z_\beta\) and their Hermitian conjugates. In the non-minimal bosonic models, these symmetries are part of the Cartan gauge algebra.

Restricting to the minimal bosonic models, the further intersection to \(\mathfrak{hs}(4)\) leaves \(SL(2; \mathbb{C})_1 \times SL(2; \mathbb{C})_2 \times \overline{SL}(2; \mathbb{C})_1 \times \overline{SL}(2; \mathbb{C})_2\) generated by \(y_{(\alpha \star \beta)}\) and \(z_{(\alpha \star \beta)}\), and their Hermitian conjugates. Indeed, this symmetry is a manifest symmetry of the two-form \(J\) (and its diagonal subalgebra can be made into a manifest local symmetry of the Vasiliev equations by means of the aforementioned field redefinition and identified with the Lorentz group in four-dimensional spacetime). In the deformed-oscillator formulation, however, the fact that there are no longer any differentials in \(Z\)-space implies that there is actually a symmetry enhancement corresponding to the exchange \((y_\alpha, z_\alpha) \leftrightarrow (iz_\alpha, iy_\alpha)\). The resulting \(GL(1; \mathbb{C}) \times SL(2; \mathbb{C})_1 \times SL(2; \mathbb{C})_2 \times GL(1; \mathbb{C}) \times \overline{SL}(2; \mathbb{C})_1 \times \overline{SL}(2; \mathbb{C})_2\)-symmetry is thus hidden in the formulation of the minimal bosonic model using differential forms in \(Z\)-space, whereas it is part of the gauge algebra in the corresponding formulation of the non-minimal bosonic model.

Appendix D. Duality extension

We consider a graded-associative quasi-free differential algebra consisting of master fields \(Z^i\) and structure coefficients \(Q^i_{j_1 \ldots j_k} (J^k)\) of fixed degrees, say \(\deg(Z^i) = p_i \in \mathbb{N}\) and \(\deg(Q^i_{j_1 \ldots j_k}) = p'_{j_1 \ldots j_k} \in 2\mathbb{N}\). This system can always be duality extended (without adding any new local degrees of freedom) by (i) replacing \(Z^i\) by \(\overline{Z}^i := \sum_k Z^i_{(p + 2k)}\); and (ii) exploiting field redefinitions to introduce coupling constants \(g_{[0]}\) and then replace these by \(g_{[J]} := \sum_k g_{[2k]}\). It follows that the extended system \([\overline{Z}^i, g_{[J]}]\) contains the original system \([Z^i_{(p)}, g_{[0]}]\) as a consistent subsystem, though the added master fields \(Z^i_{(p + 2k)}\) with \(k > 0\) cannot in general be set equal to zero, since they are sourced from \([Z^i_{(p)}]\) via terms involving the new couplings \(g_{[2k]}\) with \(k > 0\).

One may refer to the duality extension as non-trivial if the central elements cannot be removed by redefining the master fields; we are not aware of any general condition that guarantees non-triviality.

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28
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