MULTIPLIER HERMITIAN-EINSTEIN METRICS ON FANO MANIFOLDS OF KSM-TYPE

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ABSTRACT. In this article we focus on multiplier Hermitian-Einstein metrics introduced by Mabuchi which include Kähler-Einstein metrics, Kähler-Ricci solitons and Mabuchi solitons as special cases. We also focus on KSM-manifolds, which are introduced by the first author as toric bundles, to establish a criterion for the existence of multiplier Hermitian-Einstein metrics in terms of KSM-data. An explicit example for a KSM-manifold admitting a family of multiplier Hermitian-Einstein metrics is constructed by using a continuous path connecting a Kähler-Ricci soliton with a Mabuchi soliton.

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1. INTRODUCTION

Finding a canonical metric on a manifold is a central problem in differential geometry. In particular the existence problem for Kähler-Einstein metrics is one of the central topics. Some generalizations of Kähler-Einstein metrics for Fano manifolds with non-vanishing
Futaki invariant such as Kähler-Ricci solitons and Mabuchi solitons are discussed by many experts. Mabuchi \[21, 22, 23\] introduced the notion of multiplier Hermitian-Einstein metrics which include Kähler-Ricci solitons and Mabuchi solitons as special cases. In this paper we shall focus on multiplier Hermitian-Einstein metrics (see also \[15\] for generalized Kähler-Ricci solitons studied very recently by Han and Li). Let \( M \) be an \( n \)-dimensional Fano manifold. We fix a holomorphic vector field \( V \) on \( M \) and a \( V \)-Im-invariant Kähler metric \( \omega_0 \in 2\pi c_1(M) \), where \( V \)-Im = \( \frac{1}{2\sqrt{-1}}(V - \overline{V}) \) is the imaginary part of \( V \). Let

\[
\mathcal{K} = \{ \omega_\varphi := \omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi \mid \varphi \in C^\infty(M)_R \text{ and } \omega_\varphi > 0 \}
\]

be the set of all Kähler metrics in \( 2\pi c_1(M) \), and put \( \mathcal{K}_V = \{ \omega \in \mathcal{K} \mid L_{V_{\text{im}}} \omega = 0 \} \). Here \( C^\infty(M)_R \) denotes the set of real valued smooth functions on \( M \). For each \( \omega \in \mathcal{K}_V \), there exists a unique real valued function \( \theta_V^{(\omega)} \in C^\infty(M)_R \) on \( M \) such that

\[
(1.1) \quad i_V \omega = \sqrt{-1} \partial \overline{\partial} \theta_V^{(\omega)} \quad \text{and} \quad \int_M \theta_V^{(\omega)} \omega^n = 0,
\]

where \( i_V \omega \) is the interior product of \( V \) and \( \omega \). According to \[14\], two real numbers \( \min_M \theta_V^{(\omega)} \) and \( \max_M \theta_V^{(\omega)} \) are independent of the choice of \( \omega \in \mathcal{K}_V \). Let \( \sigma(s) \) be a real-valued smooth function on the interval \( I = (\alpha, \beta) \) \((-\infty \leq \alpha < \min_M \theta_V^{(\omega)} \leq \max_M \theta_V^{(\omega)} < \beta \leq +\infty) \) satisfying one of the following conditions: (i) \( \dot{\sigma} \leq 0 \leq \ddot{\sigma} \), (ii) \( \ddot{\sigma} > 0 \), where \( \dot{\sigma} \) and \( \ddot{\sigma} \) are the first derivative and the second derivative respectively. We consider the Hermitian form

\[
\tilde{\omega} := \omega \exp \left( -\frac{1}{n} \sigma(\theta_V^{(\omega)}) \right).
\]

Without loss of generality, we can also assume \( \int_M \exp(-\sigma(\theta_V^{(\omega)})) \omega^n = \int_M \omega^n \). Mabuchi \[21, 22, 23\] called a conformally Kähler metric \( \tilde{\omega} \) a multiplier Hermitian metric (of type \( (\sigma, V) \)). Note that the multiplier Hermitian metric \( \tilde{\omega} \) can be seen as an Hermitian metric on the holomorphic tangent bundle \( TM \). Then \( \tilde{\omega} \) defines the Hermitian connection

\[
\tilde{\nabla} := \nabla - \frac{\partial(\sigma(\theta_V^{(\omega)}))}{n} \text{id}_{TM},
\]

where \( \nabla \) is the natural connection with respect to \( \omega \). The Ricci form \( \text{Ric}_V^{(\omega)}(\tilde{\omega}, \tilde{\nabla}) \) is equal to \( \text{Ric}(\omega) + \sqrt{-1} \partial \overline{\partial} \sigma(\theta_V^{(\omega)}) \), where \( \text{Ric}(\omega) \in 2\pi c_1(M) \) is the Ricci form for \( \omega \) defined by \(-\sqrt{-1} \partial \overline{\partial} \log \omega^n \).

**Definition 1.1.** The conformally Kähler metric \( \tilde{\omega} \) is a multiplier Hermitian-Einstein metric (of type \( (\sigma, V) \)) if \( \text{Ric}_V^{(\omega)}(\omega) = \omega \).
Define the Ricci potential $\rho$ for $\omega$ as follows.

\begin{equation}
\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} \rho + \int_X (1 - e^{\rho} \omega) \omega^n = 0.
\end{equation}

In this terminology, $\tilde{\omega}$ is multiplier Hermitian-Einstein if and only if $\rho = -\sigma(\theta^{(\omega)}_V) = 0$.

Multiplier Hermitian-Einstein metrics of type $(\sigma, V)$ give some well-known generalizations of Kähler-Einstein metrics.

(i) When $\sigma$ is a constant function, a multiplier Hermitian-Einstein metric $\tilde{\omega}$ gives a Kähler-Einstein metric.

(ii) When $\sigma(s) = -s + C$, where $C$ is a constant, the metric $\tilde{\omega}$ gives a Kähler-Ricci soliton in the sense that $\text{Ric}(\omega) - \omega = L_V \omega$.

(iii) When $\sigma(s) = -\log(s + C)$, where $C$ is a constant strictly greater than $\min_M \theta^{(\omega)}_V$, the metric $\tilde{\omega}$ gives a Mabuchi soliton (\cite{16, 21, 41}) in the sense that $1 - e^{\rho} \omega$ is a potential function of a holomorphic vector field with respect to $\omega$. In this case the positivity of $\exp(-\frac{1}{n} \sigma(\theta^{(\omega)}_V))$ plays an important role of the existence for Mabuchi solitons on toric Fano manifolds (See \cite{29, 41}).

As well as the theory of Kähler-Einstein metrics, multiplier Hermitian-Einstein metrics do not always exist. Let $\text{Aut}(M)$ be the holomorphic automorphism group of $M$. Futaki \cite{13} (see also \cite{20}) introduced the character

$$\text{Fut}_V^\sigma(X) := \int_M X \left( \rho + \sigma(\theta^{(\omega)}_V) \right) e^{-\sigma(\theta^{(\omega)}_V)} \omega^n$$

on the Lie algebra of the subgroup of $\text{Aut}(M)$ consisting of all elements $g$ such that $\text{Ad}(g)V = V$, and showed that the value $\text{Fut}_V^\sigma(X)$ is independent of the choice of $\omega$.

In this paper we call the invariant $\text{Fut}_V^\sigma$ the $(\sigma, V)$-Futaki invariant. In particular, if $M$ admits a multiplier Hermitian-Einstein metric of type $(\sigma, V)$ then the $(\sigma, V)$-Futaki invariant must vanish identically. Thus the existence for multiplier Hermitian-Einstein metrics is non-trivial.

Integrating the $(\sigma, V)$-Futaki invariant, we get a functional on $\mathcal{K}_V$, called the $(\sigma, V)$-Ding functional in this paper. More explicitly, the $(\sigma, V)$-Ding functional is defined by

$$\text{Ding}_V^\sigma(\varphi) := -E_V^\sigma(\varphi) - \log \left( \frac{1}{\int_M \omega_0^n} \int_M e^{\rho(\omega - \varphi)} \omega_0^n \right)$$
for $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \in \mathcal{K}_V$ (see also [20]), where $E_V^\sigma(\varphi)$ is the modified Monge-Ampère energy ([1]) defined by the derivative

$$\delta E_V^\sigma(\delta\varphi) := \frac{1}{\int_M \omega_0^n} \int_M \delta\varphi e^{-\sigma(\theta^{(\omega)})} \omega_\varphi^n$$

and the normalization $E_V^\sigma(0) = 0$. It is easy to see that a critical point of Ding$_V^\sigma$ defines a multiplier Hermitian-Einstein metric. We shall see the $(\sigma, V)$-Ding functional plays an important role for our results.

Now we state our motivation for our results. In this paper, we also focus on KSM-manifolds which are Fano manifolds having the structure of certain toric bundle. This notion was introduced by the first author [27, 28]. See Section 2 for details about KSM-manifolds. One of the typical examples for KSM-manifolds is a projective bundle $\mathbb{P}(L \oplus \mathcal{O}_W)$ where $L \rightarrow W$ is a line bundle over a Fano Kähler-Einstein manifold $W$ and $\mathcal{O}_W$ is the trivial line bundle over $W$. Such manifold was discussed to establish a characterization of the existence for a non-homogeneous Kähler-Einstein metric or a Kähler-Ricci soliton in [17, 18, 19, 25, 36] in late 1980’s. These results are stated in Section 2.4. The main purpose of this paper is to generalize these results from view points of multiplier Hermitian-Einstein metrics, general KSM-manifolds and both an algebraic and an analytic stability condition associated with the $(\sigma, V)$-Ding functional.

We state our main theorems. Let $\mathfrak{M} = (W; L_1, L_2, \ldots, L_l; P)$ be an $(n, l)$-dimensional KSM-data and $Z_{\mathfrak{M}}$ be the associated KSM-manifold. By definition, $Z_{\mathfrak{M}}$ has the structure of a fiber bundle over an $n$-dimensional Fano Kähler-Einstein manifold $W$ whose fiber is the $l$-dimensional toric Fano manifold associated with an $l$-dimensional Fano polytope $P$. Let $V$ be a holomorphic vector field on $Z_{\mathfrak{M}}$ as above. In the following theorems, we assume $V$ is fiber-directed. Namely, $V$ is assumed to be a holomorphic vector field induced by the natural (fiber-directed) $(\mathbb{C}^*)^l$-action on $Z_{\mathfrak{M}}$.

**Theorem 1.2.** (Theorem 3.4) Suppose a holomorphic vector field $V$ on a KSM-manifold $Z_{\mathfrak{M}}$ is fiber-directed. Then $Z_{\mathfrak{M}}$ admits a multiplier Hermitian-Einstein metric

$$\tilde{\omega} = \omega \exp \left( \frac{-1}{n+l} \sigma(\theta^{(\omega)}) \right),$$

if and only if for any $k = 1, 2, \ldots, l$, we have

$$\int_{P^*} z_k \prod_{\alpha=1}^n (1 + \langle \mu_\alpha, z \rangle) e^{-\sigma(\theta^{(\omega)})} dz = 0.$$
Here the potential function $\theta^{(\omega)}_V$ is written as $-\sum_{k=1}^l c_k z_k + C_V$ on the dual polytope $P^*$ of a Fano polytope $P$, where $z_k$ is the standard coordinate on $P^* \subset \mathbb{R}^l$, and $c_k$ and $C_V$ are constants uniquely determined by $(Z_{2\mathbb{R}}, V)$.

We will further show that the integral condition in the above theorem is equivalent to the vanishing of the $(\sigma, V)$-Futaki invariant for fiber-directed holomorphic vector fields. Theorem 1.2 generalizes the results in [17, 18, 19, 25, 27, 36] stated in Section 2.4 from viewpoints of general KSM-data and multiplier Hermitian-Einstein metrics. The existence for Kähler-Einstein metrics and Kähler-Ricci solitons on homogeneous KSM-manifolds were studied in [9, 33]. More generally, multiplier Hermitian-Einstein metrics on homogeneous KSM-manifolds were studied in [11]. These works are based on the structure of a homogeneous toric bundle, and depends on the representation theory of Lie groups. On the other hand, since our argument depends on convex analysis and combinatorics for toric geometry, we can also deal with non-homogeneous KSM-manifolds. See Section 2.3 for an explicit example of a non-homogeneous KSM-manifold.

Remark 1.3. It is not known whether we can relax the assumption that $V$ is fiber-directed. However, we believe that the extremal vector field for a KSM manifold must be fiber-directed, for example.

The following theorem is an algebraic stability version of Theorem 1.2.

**Theorem 1.4.** Suppose a holomorphic vector field $V$ on a KSM-manifold $Z_{2\mathbb{R}}$ is fiber-directed. Then $Z_{2\mathbb{R}}$ is fiber-directed relative $(\sigma, V)$-D-polystable if and only if the integral condition (1.3) holds.

The fiber-directed relative $(\sigma, V)$-D-polystability is defined by observing the asymptotic slope of the $(\sigma, V)$-Ding functional along geodesics associated with fiber-directed toric test configurations. See Section 4.2. We emphasize that we do not use multiplier Hermitian-Einstein metrics to prove Theorem 1.4. Thus, our argument gives another interpretation for the known results stated in Section 2.4 from viewpoint of the fiber-directed relative $(\sigma, V)$-D-polystability.

The following theorem, which is not used multiplier Hermitian-Einstein metrics to prove, is an analytic stability version of Theorem 1.2.

**Theorem 1.5.** Suppose a holomorphic vector field $V$ on a KSM-manifold $Z_{2\mathbb{R}}$ is fiber-directed. Then the $(\sigma, V)$-Ding functional $D^\sigma_V$ on $Z_{2\mathbb{R}}$ (see (4.1) for the definition of $D^\sigma_V$) is coercive if and only if the integral condition (1.3) holds. (See Definition 5.1 for the definition of the coercivity for $D^\sigma_V$.)
Han-Li [15] established the equivalence among the existence for a multiplier Hermitian-Einstein metric, an algebraic stability condition for general Fano manifolds and the coercivity for an energy functional. In our cases, by focusing on KSM-manifolds, we can prove Theorems 1.2 and 1.4 by a method of toric geometry which is independent of Han-Li’s argument. We also mention that Apostolov-Jubert-Lahdili [1] (see also [10] written by Delcroix-Jubert) discuss multiplier Hermitian-Einstein metrics on semisimple principal toric fibrations as an application of Han-Li’s work.

The structure of this paper is as follows. In Section 2, we review about the notion of KSM-manifolds and the known results on the existence for canonical Kähler metrics on KSM-manifolds. In Section 3, the $(\sigma, V)$-Futaki invariant is discussed to obtain Theorem 1.2. The algebraic and analytic stability is discussed in Section 4 and 5 to prove Theorems 1.4 and 1.5 respectively. In Section 6, inspired by Yao’s work [41, Sections 6 and 7], we discuss the non-uniformly stable case. In this case we have \( \exp(-\frac{1}{n} \sigma(\theta^{(\omega)}_V)) = 0 \) at some point of a manifold. Although there does not necessarily exist a multiplier Hermitian-Einstein metric, we construct a solution of an equation for the Monge-Ampère measure in the sense of Alexandrov under an additional assumption for \( \sigma \) on non-uniform stable KSM-manifolds. Finally, in Section 7, we construct an interesting example of a KSM-manifold satisfying the integral condition (1.3) by using a continuous path between a Kähler-Ricci soliton and a Mabuchi soliton. Moreover we see that the same idea of this construction yields a non-uniformly stable KSM-manifold.

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2. KSM-manifolds

The notion of KSM-manifolds was introduced by the first author [27, 28]. In this section we review definition, properties and examples for KSM-manifolds. We also review known results on the existence for canonical Kähler metrics on KSM-manifolds. The main reference in this section is the article [27] written by the first author.

We should firstly mention that the name of “KSM” comes from three initials, Koiso, Sakane and Mabuchi. They studied non-homogeneous Kähler-Einstein metrics in [17, 18, 19, 36] and toric Kähler-Einstein metrics in [25] in late 1980’s. Each study is one of the pioneering works in the theory of Kähler-Einstein metrics on Fano manifolds.
2.1. Toric Fano manifolds. It is well-known that there is a one to one correspondence between toric Fano manifolds and Fano polytopes (see for instance [31]). A convex polytope $P$ in $\mathbb{R}^l$ is called an $l$-dimensional Fano polytope if it satisfies the following properties:

(i) $P$ is an integrable polytope, that is, the set $V(P)$ of vertices of $P$ is contained in $\mathbb{Z}^l$.

(ii) The origin 0 is contained in the interior $\text{Int}(P)$ of $P$.

(iii) $P$ is a simplicial polytope, that is, each facet (i.e. codimension one face) of $P$ is a simplex.

(iv) For any facet of $P$, the set of its vertices $\{b_1, \ldots, b_l\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^l$.

For an $l$-dimensional Fano polytope $P$, its dual polytope $P^*$ is defined as $P^* := \{ z \in \mathbb{R}^l \mid \langle z, y \rangle \leq 1 \text{ for any } y \in P \}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product for $\mathbb{R}^l$. Let $P^* \cap \mathbb{Z}^l = \{ a_0, a_1, \ldots, a_N \}$ be the set of lattice points in $P^*$. Then the toric Fano manifold associated with $P$, which is expressed as $X_P$, is constructed by the following way. For $t = (t_1, \ldots, t_l) \in (\mathbb{C}^\ast)^l$ and $a = (a_1, \ldots, a_l) \in \mathbb{Z}^l$, set $t^a = t_1^{a_1} \cdots t_l^{a_l} \in \mathbb{C}^\ast$, and define an injective holomorphic map $\varphi_P : (\mathbb{C}^\ast)^l \to \mathbb{P}^N(\mathbb{C})$ by $\varphi_P(t) := [t^{-a_0} : t^{-a_1} : \ldots : t^{-a_N}]$.

Then we have $X_P = \overline{\varphi_P((\mathbb{C}^\ast)^l)}$ where it is the closure of $\varphi_P((\mathbb{C}^\ast)^l)$ in $\mathbb{P}^N(\mathbb{C})$.

2.2. KSM-manifolds. Now we introduce the notion of KSM-manifolds which is a Fano manifold and has the structure of a toric fiber bundle over a Fano Einstein-Kähler manifold.

**Definition 2.1.** The tuple $\mathfrak{M} = (W; L_1, \ldots, L_l; P)$ is called an $(n, l)$-dimensional KSM-data if the following four conditions are satisfied:

(i) $W$ is an $n$-dimensional Fano manifold with a Kähler-Einstein metric $\nu_0$, that is, there exists a Kähler form $\nu_0$ on $W$ satisfying $\text{Ric}(\nu_0) = \nu_0$.

(ii) $L_i$ is a holomorphic line bundle over $W$ for each $i = 1, \ldots, l$, admitting a Hermitian metric $h_i$ whose curvature form $-\sqrt{-1} \partial \overline{\partial} \log h_i$ has constant eigenvalues $\mu_{1}^{(i)}, \ldots, \mu_{n}^{(i)}$ with respect to $\nu$. 
(iii) For any \( w \in W \), there exists a neighborhood \( U \) of \( w \), a holomorphic local coordinate \((z_1, \ldots, z_n)\) of \( U \) and a holomorphic local frame \( e_i \) for \( L_i|_U \) \((i = 1, \ldots, l)\) satisfying

\[
\nu_0 = \sqrt{-1} \sum_{\alpha=1}^n dz_{\alpha} \wedge d\bar{z}_{\alpha},
\]

\[
h_i(e_i, e_i) = 1,
\]

\[
d(h_i(e_i, e_i)) = 0,
\]

\[
-\sqrt{-1} \partial \bar{\partial} \log h_i = -\sqrt{-1} \sum_{\alpha=1}^n \mu_{\alpha}^{(i)} dz_{\alpha} \wedge d\bar{z}_{\alpha}
\]

at \( w \), simultaneously. Here, for all \( \alpha = 1, \ldots, n \), we put

\[
\mu_{\alpha} := (\mu_{\alpha}^{(1)}, \ldots, \mu_{\alpha}^{(l)}) \in \mathbb{R}^l
\]

which are called the curvature vectors of \( M \).

(iv) \( P \) is an \( l \)-dimensional Fano polytope satisfying \(-\mu_{\alpha} \in \text{Int}(P)\) for all \( \alpha = 1, \ldots, n \).

Fix an \((n, l)\)-dimensional KSM-data \( \mathfrak{M} = (W; L_1, \ldots, L_l; P) \) to define a KSM-manifold. Let \( \pi_{\mathfrak{M}} : Q_{\mathfrak{M}} \to W \) be the \((\mathbb{C}^*)^l\)-bundle over \( W \) associated with \( L_1 \oplus \cdots \oplus L_l \). Namely any element \( q \in Q_{\mathfrak{M}} \) satisfies \( q = q_1 \oplus \cdots \oplus q_l \) and \( q_i \in L_i \setminus \text{(zero-section)} \). For \( a = (a_1, \ldots, a_l) \in \mathbb{Z}^l \) and \( q = q_1 \oplus \cdots \oplus q_l \in Q_{\mathfrak{M}} \), we put

\[
q^a := q_1^{a_1} \otimes \cdots \otimes q_l^{a_l} \in L^a \setminus \text{(zero-section)},
\]

where \( L^a := L_1^{a_1} \otimes \cdots \otimes L_l^{a_l} \), and also put

\[
E_{\mathfrak{M}} := \bigoplus_{a \in P^* \cap \mathbb{Z}^l} L^{-a} = L^{-a_0} \oplus L^{-a_1} \oplus \cdots \oplus L^{-a_N}
\]

to define a map

\[
\Phi_{\mathfrak{M}} : Q_{\mathfrak{M}} \to \mathbb{P}(E_{\mathfrak{M}}) := (E_{\mathfrak{M}} \setminus \text{(zero-section)})/\mathbb{C}^*
\]

by \( \Phi_{\mathfrak{M}}(q) := [q^{-a_0} \oplus q^{-a_1} \oplus \cdots \oplus q^{-a_N}] \). Note the map \( \Phi_{\mathfrak{M}} \) is holomorphic and injective. Then we define the KSM-manifold \( Z_{\mathfrak{M}} \) associated with \( \mathfrak{M} \) by

\[
Z_{\mathfrak{M}} := \overline{\Phi_{\mathfrak{M}}(Q_{\mathfrak{M}})} \subset \mathbb{P}(E_{\mathfrak{M}}).
\]

Note the KSM-manifold \( Z_{\mathfrak{M}} \) is an \((n+l)\)-dimensional complex manifold with the structure of an \( X_{\mathfrak{M}}\)-bundle \( \pi_{\mathfrak{M}} : Z_{\mathfrak{M}} \to W \).

Furthermore every KSM-manifold is Fano.

**Theorem 2.2.** ([27, Theorem 2.12]) For any \((n, l)\)-dimensional KSM-data \( \mathfrak{M} \), the KSM-manifold \( Z_{\mathfrak{M}} \) has the positive first Chen class.
Proof. We give a proof for reader’s convenience. For \( q_i \in L_i \setminus \text{(zero-section)} \), put
\[
x_i(q_i) := -\log h_i(q_i, q_i).
\]
The pair \( x = (x_1, \ldots, x_l) \) can be seen as a function on \( Q_{\mathfrak{g}} \). We define a function \( u_P \) for \( y = (y_1, \ldots, y_l) \in \mathbb{R}^l \) by
\[
u_P(y) := \log \left( \sum_{a \in P^* \cap \mathbb{Z}^l} e^{(a,y)} \right)
\]
and define a volume form \( \eta_{\text{ref}} \) on \( Q_{\mathfrak{g}} \) by
\[
\eta_{\text{ref}} := \frac{(n+l)!}{n!} e^{-u_P(x)} \bigwedge \frac{d\tau_i}{\tau_i} \wedge \left( \pi_{\mathfrak{g}}^* \nu_0 \right)^n,
\]
where \( \tau_i \) is the fiber coordinate for \( L_i|_U \) satisfying \( q_i = \tau_i e_i \). Note the volume form \( \eta_{\text{ref}} \) naturally extends on \( Z_{\mathfrak{g}} \). Now we consider the real \((1,1)\)-form
\[
\omega_{\text{ref}} := -\sqrt{-1} \partial \bar{\partial} \log \eta_{\text{ref}} \in 2\pi c_1(Z_{\mathfrak{g}})
\]
on \( Z_{\mathfrak{g}} \). Use local coordinates and local frames introduced in Definition 2.1 and the condition \( \text{Ric}(\nu_0) = \nu_0 \) to obtain
\[
\omega_{\text{ref}} = \sqrt{-1} \sum_{i,j=1}^l \frac{\partial^2 u_P}{\partial y_i \partial y_j}(x) \frac{d\tau_i}{\tau_i} \wedge \frac{d\tau_j}{\tau_j} + \sqrt{-1} \sum_{\alpha=1}^n (1 + \langle \mu_\alpha, m_0(x) \rangle) dz_\alpha \wedge d\bar{z}_\alpha,
\]
where \( m_0(x) = \left( \frac{\partial u_P}{\partial y_1}(x), \ldots, \frac{\partial u_P}{\partial y_l}(x) \right) \) is the moment map for the \((\mathbb{C}^*)^l\)-action on the fiber \( \pi_{\mathfrak{g}}^{-1}(w) \subset Z_{\mathfrak{g}} \) for \( w \in W \) with respect to the Kähler form \( \sqrt{-1} \partial \bar{\partial} \left( u_P(x)|_{\pi_{\mathfrak{g}}^{-1}(w)} \right) \). We note that \( m_0(\pi_{\mathfrak{g}}^{-1}(w)) = P^* \) (see for instance [25]). It follows from definition of \( u_P \) and the condition \( -\mu_\alpha \in \text{Int}(P) \) that \( \omega_{\text{ref}} \) is positive. This completes the proof. \( \square \)

2.3. Examples. We see typical examples of KSM-manifolds.

Example 2.3. If \( n = 0 \), that is, \( W \) is the one point space, then a KSM manifold is nothing but a toric Fano manifold \( X_P \).

Example 2.4. If \( l = 1 \), then the Fano polytope is only the closed interval \([-1,1] \subset \mathbb{R} \). For an \((n,1)\)-dimensional KSM-data \( \mathfrak{M} = (W; L; [-1,1]) \), the associated KSM manifold \( Z_{\mathfrak{g}} \) is \( \mathbb{P}(L \oplus O_W) \) which has the structure of a \( \mathbb{P}^1(\mathbb{C}) \)-bundle over \( W \). Here \( O_W \) is the trivial line bundle over \( W \). In particular, the \((1,1)\)-dimensional KSM data \( \mathfrak{M} = (\mathbb{P}^1(\mathbb{C}); O_{\mathbb{P}^1}(1); [-1,1]) \) gives the one point blow up \( Z_{\mathfrak{g}} = \mathbb{P}^2(\mathbb{C}) \# \mathbb{P}^2(\mathbb{C}) \) of the projective plane.
Example 2.5. Let \( P_2 \) be the convex hull of \( \{(1,0), (0,1), (-1,-1)\} \). This is a Fano polytope which gives \( \mathbb{P}^2(\mathbb{C}) \). The \((1,2)\)-dimensional KSM-data \( \mathfrak{M} = (\mathbb{P}^1(\mathbb{C}); \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}; P_2) \) gives the projective bundle \( Z_{\mathfrak{M}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \). On the other hand, the data \((\mathbb{P}^1(\mathbb{C}); \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}; P_2)\) is not a KSM one, since the condition (iv) in Definition 2.1 does not hold. In fact, the associated manifold \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \) is not Fano (see for instance [7, 31]). Note that our definition of projective bundles is different from that in [7, 31]. For example, we define

\[
\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) := ((\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus \text{(zero-section)}) / \mathbb{C}^*.
\]

A KSM-data \( \mathfrak{M} = (W; L_1, \ldots, L_l; P) \) is said to be homogeneous if the following conditions are satisfied:

(i) \( W \) is a simply connected compact complex homogeneous Kähler manifold (i.e. Kähler C-space). Then we have \( W = G/U = G_c/K \), where \( G \) is a simply connected complex semisimple Lie group, \( U \) is a parabolic subgroup of \( G \), \( G_c \) is a maximal compact subgroup of \( G \) and \( K = G_c \cap U \).

(ii) The Kähler-Einstein metric \( \nu_0 \) is \( G_c \)-invariant.

(iii) Each line bundle \( L_i \) admits a \( G \)-action compatible with the \( G \)-action on \( W \), and the Hermitian metric \( h_i \) is \( G_c \)-invariant.

The existence problem for Kähler-Einstein metrics and Kähler-Ricci solitons on homogeneous KSM-manifolds are discussed by Podestà-Spiro [33] and Delcroix [9] from view points of homogeneous toric bundles. More generally, Delcroix-Hultgren [11] discuss multiplier Hermitian-Einstein metrics on homogeneous KSM-manifolds. These works depend on the representation theory of Lie groups. On the other hand, our arguments are based on convex analysis and combinatorics for toric geometry. Thus we can deal with a non-homogeneous KSM-manifold as follows.

Example 2.6. Let \( \mathcal{M}_n \) be the moduli space of smooth hypersurfaces of degree \( n \) in \( \mathbb{P}^{n+1}(\mathbb{C}) \) and let \( \mathcal{M}_n^{\text{EK}} \subset \mathcal{M}_n \) be the moduli space of Kähler-Einstein hypersurfaces. The space \( \mathcal{M}_n^{\text{EK}} \) is non-empty (for instance the Fermat hypersurface \( \{\sum_{i=0}^{n+1} X_i^n = 0\} \) is in \( \mathcal{M}_n^{\text{EK}} \) and open in \( \mathcal{M}_n \). For fixed \( W \in \mathcal{M}_n^{\text{EK}} \), put \( L = \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_W \). Then \( K_W^{-1} \cong L \otimes 2 \) by the adjunction formula. Take an \( l \)-dimensional Fano polytope \( P \) and integers \( k_1, \ldots, k_l \in \mathbb{Z} \) such that \( -\frac{1}{2}(k_1, \ldots, k_l) \in \text{Int}(P) \). Then \( \mathfrak{M} = (W; L^{\otimes k_1}, \ldots, L^{\otimes k_l}; P) \) is an \((n,l)\)-dimensional KSM data, and the KSM manifold \( Z_{\mathfrak{M}} \) is non-homogeneous in general.
Other explicit examples of higher dimensional KSM-manifolds are also discussed by the first author. In [28], Kähler-Einstein KSM-manifolds which are not asymptotically Chow semi-stable are constructed.

2.4. Existence for canonical Kähler metrics on KSM-manifolds. Finally we review known theorems on the existence for canonical Kähler metrics on KSM-manifolds which motivate our results.

**Theorem 2.7.** Let $\mathfrak{M} = (W; L; [-1,1])$ be an $(n,1)$-dimensional KSM-data and $Z_{\mathfrak{M}}$ be the associated KSM-manifold.

(i) (Sakane [36], Koiso-Sakane [18, 19], Mabuchi [25]) The KSM-manifold $Z_{\mathfrak{M}}$ admits a Kähler-Einstein metric if and only if

$$\int_{-1}^{1} z \prod_{\alpha=1}^{n} (1 + \mu_{\alpha} z) dz = 0.$$  

(ii) (Koiso [17]) The KSM-manifold $Z_{\mathfrak{M}}$ always admits a Kähler-Ricci soliton with fiber-directed soliton vector field.

The first author considered general KSM-data to obtain the following theorem which also generalizes a Wang-Zhu’s work [40] on the existence for Kähler-Ricci solitons on toric Fano manifolds.

**Theorem 2.8.** ([27]) The KSM-manifold $Z_{\mathfrak{M}}$ associated to an $(n,l)$-dimensional KSM-data $\mathfrak{M} = (W; L_1, L_2, \ldots, L_l; P)$ always admits a Kähler-Ricci soliton $\omega$ with fiber-directed soliton vector field. Furthermore $\omega$ becomes a Kähler-Einstein metric if and only if for any $k = 1, 2, \ldots, l$, we have

$$\int_{P^*} z_k \prod_{\alpha=1}^{n} (1 + \langle \mu_{\alpha}, z \rangle) dz = 0.$$  

Therefore our result in Theorem 1.2 generalizes the above theorems from view points of general KSM-data and multiplier Hermitian-Einstein metrics.

3. Multiplier Hermitian-Einstein metrics and the $(\sigma,V)$-Futaki invariant on KSM-manifolds

The goal in this section is to prove Theorem 1.2. We first prove some lemmas for the $(\sigma,V)$-Futaki invariant for general Fano manifolds to express this invariant in terms of KSM-data.
3.1. The \((\sigma, V)\)-Futaki invariant for Fano manifolds. Let \(M\) be an \(n\)-dimensional Fano manifold with a Kähler metric \(\omega \in 2\pi c_1(M)\). Recall the Ricci potential for \(\omega\) is denoted by \(\rho_\omega\). Fix a holomorphic vector field \(V\) on \(M\) to consider a multiplier Hermitian metric of type-\((\sigma, V)\). As in the Section 1 for any holomorphic vector field \(v\) on \(M\), its potential function \(\theta_\omega\) is defined by

\[
i_v \omega = \sqrt{-1} \partial \overline{\partial} \theta_\omega^{(\omega)} \quad \text{and} \quad \int_M \theta_\omega^{(\omega)} \omega^n = 0.
\]

Lemma 3.1. For any holomorphic vector field \(v\) on \(M\), there exists a unique constant \(c_v\) such that

\[
\Delta_\omega \theta_v^{(\omega)} + \theta_v^{(\omega)} + v(\rho_\omega) = c_v
\]

where \(\Delta_\omega\) is the negative Laplacian for \(\omega\), and \(c_v\) is given by

\[
c_v = \frac{\int_M \theta_v^{(\omega)} e^{\rho_\omega} \omega^n}{\int_M \omega^n}.
\]

Proof. Note \(L_v \omega = \sqrt{-1} \partial \overline{\partial} \theta_v^{(\omega)}\) which comes from the first equation in (3.1) and Cartan’s formula for the Lie derivative. This gives \(L_v(\omega^n) = (\Delta_\omega \theta_v^{(\omega)}) \omega^n\). Then the derivative of the first equation in (1.2) with respect to \(v\) yields

\[
\Delta_\omega \theta_v^{(\omega)} + \theta_v^{(\omega)} + v(\rho_\omega) = c_v
\]

for some constant \(c_v\). Moreover, by Stokes’ theorem, we have

\[
0 = \int_M L_v(e^{\rho_\omega} \omega^n) = \int_M (\Delta_\omega \theta_v^{(\omega)} + v(\rho_\omega)) e^{\rho_\omega} \omega^n.
\]

Thus \(c_v = \int_M \theta_v^{(\omega)} e^{\rho_\omega} \omega^n / \int_M \omega^n\).

\[\square\]

Lemma 3.2. For any holomorphic vector field \(v\) on \(M\), we have

\[
\text{Fut}_V(v) = -\int_M (\theta_v^{(\omega)} - c_v) e^{-\sigma(\theta_v^{(\omega)})} \omega^n
\]

where \(c_v\) is the constant in Lemma 3.1.

Proof. By Lemma 3.1 it suffice to show

\[
\int_M \left( -\Delta_\omega \theta_v^{(\omega)} + v(\sigma(\theta_v^{(\omega)})) \right) e^{-\sigma(\theta_v^{(\omega)})} \omega^n = 0.
\]

By Stokes’ theorem, we have \(\int_M L_v(e^{-\sigma(\theta_v^{(\omega)})} \omega^n) = 0\) which yields (3.3).

\[\square\]
3.2. Multiplier Hermitian-Einstein metrics on KSM-manifolds. We use same notation as in the Section 3. Let $\mathfrak{M} = (W; L_1, L_2, \ldots, L_l; P)$ be an $(n, l)$-dimensional KSM-data and $Z_{2\mathfrak{M}}$ be the associated KSM-manifold with a fiber-directed holomorphic vector field $V$. Let $\omega_{ref}$ be the reference Kähler metric on $Z_{2\mathfrak{M}}$ defined in (2.3).

An $(S^1)^l(\subset \mathbb{C}^*)^l$-invariant function $u$ on $X_P$ can be seen as a function of $y = (y_1, \ldots, y_l) \in \mathbb{R}^l$ on $(\mathbb{C}^*)^l = \{(t_1, \ldots, t_l)\} \subset X_P$, where $y_i = -\log |t_i|^2$. Moreover, by composing with $x = (x_1, \ldots, x_l)$ defined in (2.1), $u(x)$ can be seen as a function on $Q_{2\mathfrak{M}} \subset Z_{2\mathfrak{M}}$. We denote by $\mathcal{F}_{2\mathfrak{M}}$ the set of such functions on $Q_{2\mathfrak{M}}$. For instance, the Ricci potential $\rho_{\omega_{ref}}$ for $\omega_{ref}$ is reduced to an element in $\mathcal{F}_{2\mathfrak{M}}$. Since

$$\omega_{ref}^{n+l} = \frac{(n+l)!}{n!} \det \left( \frac{\partial^2 u_P}{\partial y_i \partial y_j}(x) \right) \prod_{\alpha=1}^{n} (1 + \langle \mu_\alpha, m_0(x) \rangle)$$

by (2.4), then by definition of the Ricci potential, there exists a constant $c \in \mathbb{R}$ such that

$$e^{-\rho_{\omega_{ref}}} = \frac{\omega_{ref}^{n+l}}{\eta_{ref}} = e^{u_P(x)} \det \left( \frac{\partial^2 u_P}{\partial y_i \partial y_j}(x) \right) \prod_{\alpha=1}^{n} (1 + \langle \mu_\alpha, m_0(x) \rangle)$$

on $\pi_{2\mathfrak{M}}^{-1}(w)(\subset Q_{2\mathfrak{M}})$ where $m_0(x) = \left( \frac{\partial u_P}{\partial y_1}(x), \ldots, \frac{\partial u_P}{\partial y_l}(x) \right)$.

The complex torus $(\mathbb{C}^*)^l$ naturally acts on $Z_{2\mathfrak{M}}$ and generates fiber-directed holomorphic vector fields. Let $v_i$ be the fiber-directed holomorphic vector field on $Z_{2\mathfrak{M}}$ which corresponds to $l_i \frac{\partial}{\partial \theta_i} \in \text{Lie}(\mathbb{C}^*)^l$. Then, by definition, its potential function $\theta_{v_i}^{(\omega_{ref})}$ with respect to $\omega_{ref}$ is reduced to an element in $\mathcal{F}_{2\mathfrak{M}}$, and in fact

$$\theta_{v_i}^{(\omega_{ref})}(x) = -\frac{\partial u_P}{\partial y_i}(x) + b_{v_i}$$

on $Q_{2\mathfrak{M}}$ for some constant $b_{v_i} \in \mathbb{R}$.

Lemma 3.3. We have

$$\theta_{v_i}^{(\omega_{ref})}(x) = -\frac{\partial u_P}{\partial y_i}(x) + c_{v_i}$$

on $Q_{2\mathfrak{M}}$, where $c_{v_i}$ is the unique constant in Lemma 3.1.

Proof. We compare $b_{v_i}$ in (3.5) and $c_{v_i}$. It follows from the equation (3.2) in Lemma 3.1 that

$$\int_{Z_{2\mathfrak{M}}} (-\theta_{v_i}^{(\omega_{ref})} + c_{v_i}) e^{\rho_{\omega_{ref}}} \omega_{ref}^{n+l} = 0.$$
Recall each fiber of \( \tilde{\pi}_{\mathbb{R}} : Z_{\mathbb{R}} \to W \) is isomorphic to the toric Fano manifold \( X_P \). Note that in the above integral, the integrals along each fiber are all equal. Thus by (3.4), the above equality is reduced to

\[
0 = \int_{\mathbb{R}^l} \left( -\theta^{(\omega_{\text{ref}})}_{v_i} (y) + c_{v_i} \right) e^{-u_P(y)} dy \\
= \int_{\mathbb{R}^l} \left( \frac{\partial u_P}{\partial y_i} (y) - b_{v_i} + c_{v_i} \right) e^{-u_P(y)} dy \\
= \int_{\mathbb{R}^l} \left( -b_{v_i} + c_{v_i} \right) e^{-u_P(y)} dy.
\]

Therefore \( b_{v_i} = c_{v_i} \). \( \square \)

According to the above lemma, the potential function \( \theta^{(\omega_{\text{ref}})}_V \) for the fiber-directed holomorphic vector field \( V \) on \( Z_{\mathbb{R}} \) is written as

\[
(3.6) \quad \theta^{(\omega_{\text{ref}})}_V (x) = -\sum_{k=1}^l c_k \frac{\partial u_P}{\partial y_k} (x) + C_V = -\langle c, m_0 (x) \rangle + C_V
\]
on \( Q_{\mathbb{R}} \), where \( c = (c_1, \ldots, c_l) \in \mathbb{R}^l \) is the unique constants satisfying \( V = \sum_k c_k v_k \), and \( C_V = \sum_k c_k c_{v_k} \)

**Theorem 3.4.** For a fiber-directed holomorphic vector field \( V \) on \( Z_{\mathbb{R}} \), the followings are equivalent.

(i) The KSM-manifold \( Z_{\mathbb{R}} \) admits an \((S^1)^l\)-invariant multiplier Hermitian-Einstein metric \( \tilde{\omega} = \omega \exp(-\frac{1}{n} \sigma (\theta^{(\omega)}_V)) \) of type \((\sigma, V)\).

(ii) The \((\sigma, V)\)-Futaki invariant vanishes for any fiber-directed holomorphic vector field.

(iii) For any \( k = 1, 2, \ldots, l \), we have

\[
\int_{P^*} z_k \prod_{\alpha=1}^n (1 + \langle \mu_\alpha, z \rangle) e^{-\sigma (\langle c, z \rangle + C_V)} dz = 0.
\]

**Proof.** It suffices to show the two directions (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i). First, we prove (ii) \( \Rightarrow \) (iii). By Lemma 3.2, for the fiber-directed holomorphic vector field \( v_i \) on \( Z_{\mathbb{R}} \) corresponds to \( t_{\theta^{(\omega_{\text{ref}})}_{v_i}} \in \text{Lie}((\mathbb{C}^*)^l) \), we have

\[
\text{Fut}_{Z_{\mathbb{R}}} (v_i) = \int_{Z_{\mathbb{R}}} (\theta^{(\omega_{\text{ref}})}_{v_i} - c_{v_i}) e^{-\sigma (\theta^{(\omega_{\text{ref}})}_V)} \omega_{\text{ref}}^{n+l} = 0.
\]
Since the integral along a fiber equals to one another, it follows from Lemma 3.3 that the above equality is reduced to
\[
\int_{\mathbb{R}^l} \frac{\partial u_P}{\partial y_i} e^{-\sigma(-\langle c, \nabla u_P \rangle + CV)} \det(\nabla^2 u_P) \prod_{\alpha=1}^n \left(1 + \langle \mu_\alpha, \nabla u_P \rangle \right) dy = 0.
\]
Recall that the image of \( m_0(x) = \left( \frac{\partial u_P}{\partial y_i}(x), \ldots, \frac{\partial u_P}{\partial y_l}(x) \right) \) on a fiber \( \tilde{\pi}^{-1}(w) \) of \( Z_{2\mathbb{R}} \) is the dual polytope \( P^* \). By taking the Legendre transformation \( z_i = \frac{\partial u_P}{\partial y_i}(y) \) \((i = 1, \ldots, l)\), we have
\[
\int_{P^*} z_k \prod_{\alpha=1}^n \left(1 + \langle \mu_\alpha, z \rangle \right) e^{-\sigma(-\langle c, z \rangle + CV)} dz = 0.
\]
Secondly we prove (iii) \(\Rightarrow\) (i). As in the proof of Theorem 2.2, take any \( \varphi \in F_{2\mathbb{R}} \), and put \( u = u_P + \varphi \) and
\[
\eta_u = \frac{(n+1)!}{n!} e^{-u(x)} \prod_{i=1}^l \left( \sqrt{-1} \frac{d\tau_i}{\tau_i} \wedge \frac{d\bar{\tau}_i}{\bar{\tau}_i} \right) \wedge (\pi_{2\mathbb{R}}^* \nu_0)^n.
\]
Then \( \eta_u \) defines a volume form on \( Q_{2\mathbb{R}} \) and extends that on \( Z_{2\mathbb{R}} \). If \( u(y) \) is strictly convex on \( \mathbb{R}^l \), then \( \omega_u := -\sqrt{-1} \partial \bar{\partial} \log \eta_u \in 2\pi c_1(Z_{2\mathbb{R}}) \) defines a Kähler metric on \( Q_{2\mathbb{R}} \). Note that
\[
\theta_{\omega_u}^V(x) = \theta_{\omega_{\text{ref}}}^V(x) + V(\varphi(x)) = -\sum_{k=1}^l c_k \frac{\partial u}{\partial y_k}(x) + CV
\]
on \( Q_{2\mathbb{R}} \). If \( u = u_P + \varphi (\varphi \in F_{2\mathbb{R}}) \) solves the equation of a multiplier Hermitian-Einstein metric, then by a same calculation as to get the equation (3.4), we have
\[
\det \left( \frac{\partial^2 u}{\partial y_j \partial y_i}(x) \right) \prod_{\alpha=1}^n \left(1 + \sum_{k=1}^l \mu^{(k)}_{\alpha} \frac{\partial u}{\partial y_k}(x) \right) = e^{-u(x) + \sigma(\theta_{\omega_u}^V(x))}.
\]
Therefore, in order to construct a multiplier Hermitian-Einstein metric on \( Z_{2\mathbb{R}} \), it suffice to solve the real Monge-Ampère equation
\[
(3.7) \quad g \left( \nabla u(y) \right) \det \left( \nabla^2 u(y) \right) = e^{-u(y)}
\]
for a convex function \( u \) on \( \mathbb{R}^l \), where
\[
(3.8) \quad g(z) := \prod_{\alpha=1}^n \left(1 + \langle \mu_\alpha, z \rangle \right) e^{-\sigma(-\langle c, z \rangle + CV)}
\]
with \( z = \nabla u(y) \). By a result of Berman-Berndtsson [3] Theorem 1.1, the solvability of (3.7) is guaranteed under the assumption that the barycenter of \( P^* \) with respect to the measure \( g(z) dz \) is equal to the origin. This assumption is nothing but the condition
Moreover the same argument as in [3] Section 3.8 allows us to conclude that \( u \) defines a smooth multiplier Hermitian-Einstein metric on \( Z_{\mathfrak{M}} \) of type \((\sigma, V)\) (see also [35] Theorem 1.1) for another regularity argument by an application of a geometric flow, which is based on \([37, 39]\)). This completes the proof of Theorem 3.4.

4. THE \((\sigma, V)\)-DING FUNCTIONAL AND THE \((\sigma, V)\)-D-POLYSTABILITY FOR KSM-MANIFOLDS

In this section, we introduce the notion of the \((\sigma, V)\)-D-stability for KSM-manifolds to prove Theorem 1.4. Let \( \mathfrak{M} = (W; L_1, L_2, \ldots, L_l; P) \) be an \((n, l)\)-dimensional KSM-data and \( Z_{\mathfrak{M}} \) be the associated KSM-manifold with a fiber-directed holomorphic vector field \( V \). We use same notations as in the previous sections.

4.1. The \((\sigma, V)\)-Ding functional for \( Z_{\mathfrak{M}} \).

As in the proof of Theorem 3.4, the equation to construct a multiplier Hermitian-Einstein metric on a KSM-manifold \( Z_{\mathfrak{M}} \) is reduced to that for convex functions on \( \mathbb{R}^l \). The \((\sigma, V)\)-Ding functional \( \text{Ding}_{\sigma}^V \) should also be reduced to a functional for convex functions. Following [3, 6] (see also [41, Section 3.4]), we define classes of convex functions on \( \mathbb{R}^l \) to discuss \( \text{Ding}_{\sigma}^V \) on \( Z_{\mathfrak{M}} \). Let \( v_P : \mathbb{R}^l \to \mathbb{R} \) be the support function for the dual polytope \( P^* \) of an \( l \)-dimensional Fano polytope \( P \). For a convex function \( u(y) \) on \( \mathbb{R}^l \), let \( u^*(z) := \sup_{y \in \mathbb{R}^l} \langle y, z \rangle - u(y) \) be its Legendre dual. We set

\[
\text{PSH}(P^*) := \left\{ u : \mathbb{R}^l \to \mathbb{R} \mid \text{convex} \mid u \leq v_{P^*} + C \text{ on } \mathbb{R}^l \text{ for some constant } C \right\},
\]

\[
\mathcal{E}^1(P^*) := \left\{ u \in \text{PSH}(P^*) \mid \int_{P^*} u^* dz < +\infty \right\},
\]

\[
\text{PSH}_b(P^*) := \left\{ u : \mathbb{R}^l \to \mathbb{R} \mid \text{convex} \mid |u - v_{P^*}| < C \text{ on } \mathbb{R}^l \text{ for some constant } C \right\}.
\]

An element \( u \in \text{PSH}(P^*) \) corresponds to a torus invariant plurisubharmonic (psh) metric \( e^{-u} \) on the anti-canonical bundle of the toric Fano manifold \( X_P \) associated with \( P \). An element in \( \mathcal{E}^1(P^*) \) is characterized by the finiteness of the Monge-Ampère energy for \( X_P \). An element in \( \text{PSH}_b(P^*) \) corresponds to a bounded psh function on \( X_P \). In these terminology, we have \( \text{PSH}_b(P^*) \subset \mathcal{E}^1(P^*) \subset \text{PSH}(P^*) \). Recall that \( u \in \text{PSH}(P^*) \) defines a psh metric for the KSM-manifold \( Z_{\mathfrak{M}} \) as in the proof of Theorems 2.2 and 3.4. Indeed, we define on \( Q_{\mathfrak{M}} \),

\[
\eta_u = \frac{(n + l)!}{n!} e^{-u(x)} \prod_{i=1}^l \left( -\frac{d\tau_i}{\tau_i} \right) \wedge (\pi_{\mathfrak{M}}^* \nu_0)^n,
\]

and this can be extended to an invariant psh metric on the anti-canonical bundle of \( Z_{\mathfrak{M}} \).
For \( u \in \mathcal{E}^1(P^*) \), we set
\[
D^\sigma_V(u) := \frac{1}{|P^*|_{\mathcal{V}}} \int_{P^*} u^*(z)g(z)dz - \log \int_{\mathbb{R}^l} e^{-u(y)}dy,
\]
where \( g(z) \) is the function defined by (3.3), \( c \) and \( C_V \) are unique constants determined by (3.6) and \( |P^*|_{\mathcal{V}} := \int_{P^*} g(z)dz \) is the volume. It is easy to see that the functional \( D^\sigma_V \) is the restriction of the \((\sigma, V)\)-Ding functional \( \text{Ding}^\sigma_V \) (modulo a positive multiplicative constant and an additive constant) to \( \mathcal{E}^1(P^*) \). Indeed we have
\[
\delta D^\sigma_V(\delta u) = \int_{\mathbb{R}^l} -\delta u \left( \frac{1}{|P^*|_{\mathcal{V}}} g(\nabla u) \det(\nabla^2 u) - \frac{e^{-u}}{\int_{\mathbb{R}^l} e^{-u}} \right) dy
\]
for a smooth strictly convex function \( u \) by using the change of variables \( z = \nabla u(y) \) and the derivative formula \( \delta u = -\delta u^*(\nabla u) \), and we see that a critical point of \( D^\sigma_V \) satisfies the (rescaled) real Monge-Ampère equation to define a multiplier Hermitian-Einstein metric on \( Z_{\mathcal{M}} \). For simplicity, we call the functional \( D^\sigma_V \) the \((\sigma, V)\)-Ding functional for \( Z_{\mathcal{M}} \) if there is no fear of confusion.

4.2. Toric geodesics and the \((\sigma, V)\)-D-stability. Mabuchi [26] introduced the \( L^2 \) metric on the space of smooth Kähler metrics to consider geodesics. Smooth geodesics with respect to this \( L^2 \) metric do not necessarily exist. Darvas [8] introduced a weak notion of geodesics, called the finite energy geodesics. In the case of toric Fano manifolds, the geodesics of invariant metrics on \( \mathcal{E}^1(P^*) \) is considered. These are called the toric geodesics, and are constructed by the Legendre duality as follows. See [34,38] for more details. For \( u_0 \in \mathcal{E}^1(P^*) \), a toric geodesic ray \( u_t \) started from \( u_0 \) is given by
\[
u_t = (u_0^* + t\phi)^* \in \mathcal{E}^1(P^*)
\]
for \( t \geq 0 \), where \( \phi : \text{Int}(P^*) \to \mathbb{R} \) is an integrable convex function. It is known that \( u_t(y) \) is convex in \((y, t)\). In our case, a toric geodesic ray \( u_t \) gives a geodesic ray in the space of metrics on the KSM-manifold \( Z_{\mathcal{M}} \) by considering the metric \( \eta_{u_t} \).

We next review the notion of test configurations. See for instance [2] for more details. Let \( M \) be a Fano manifold. A test configuration \((\mathcal{M}, \mathcal{L})\) for \((M, K_M^{-1})\) is a proper normal variety \( \mathcal{M} \) with a \( \mathbb{Q} \)-line bundle \( \mathcal{L} \) and a morphism \( \pi : \mathcal{M} \to \mathbb{P}^1(\mathbb{C}) \) satisfying the followings. (i) There exists a linearized \( C^* \)-action on \((\mathcal{M}, \mathcal{L})\) such that \( \pi \) is equivariant with respect to the multiplicative \( C^* \)-action on \( \mathbb{P}^1(\mathbb{C}) \). (ii) There exists an equivariant isomorphism with a trivial \( C^* \)-action between \( (\mathcal{M}, \mathcal{L})|_{\mathbb{P}^1(\mathbb{C})\setminus\{0\}} \) and \((M \times \mathbb{P}^1(\mathbb{C}) \setminus \{0\}, \pi^* K_M^{-1}) \). For a test configuration \((\mathcal{M}, \mathcal{L}), \) a bounded psh geodesic ray started from a given bounded psh
metric on $K_{M}^{-1}$ is obtained by an envelope construction \([2]\). The other way to construct such geodesic ray induced from \((\mathcal{M},\mathcal{L})\) is known in \([31,34,32]\).

A test configuration \((\mathcal{M},\mathcal{L})\) for the toric Fano manifold $M = X_P$ is a toric test configuration if $M$ is a toric variety itself with a $(\mathbb{C}^*)^l \times \mathbb{C}^*$-action, where the first factor acts on each fiber of $\mathcal{M} \to \mathbb{P}^1(\mathbb{C})$ as the action on $X_P$ and the second factor corresponds to the $\mathbb{C}^*$-action of the test configuration. According to Donaldson \([12]\), a toric test configuration is obtained from a rational piecewise linear (PL) convex function on $P^*$. Let $\phi := \max\{\lambda_1, \ldots, \lambda_p\}$ be a rational PL convex function on $P^*$, where $\lambda_r$ is an affine function with rational coefficients and $p$ is an integer. Take a large integer $R$ such that

$$\Box := \{ (y, y_{l+1}) \in \mathbb{R}^l \oplus \mathbb{R} \mid y \in P^* \text{ and } 0 \leq y_{l+1} \leq R - \phi(y) \}$$

is an $(l+1)$-dimensional polytope, and take a sufficiently divisible integer $r$ such that $r\Box$ is a lattice polytope. Then $r\Box$ defines a toric variety $\mathcal{M}$ with a line bundle $\mathcal{L}^{\otimes r}$, where $\mathcal{L}$ is a $\mathbb{Q}$-line bundle. In fact, $(\mathcal{M},\mathcal{L})$ defines a toric test configuration for the toric Fano manifold $X_P$. According to Song-Zelditch \([38]\), a toric test configuration given by $(P,\phi,R)$ induces a toric geodesic ray in $E_1(P^*)$ defined by

$$u_t = (u_0^* + t(\phi - R))^*.$$ 

In the construction of a KSM-manifold $Z_{\mathfrak{R}}$ associated to $\mathfrak{R} = (W; L_1, \ldots, L_l; P)$, we can use $\tilde{\mathfrak{R}} = (W; L_1, \ldots, L_l, \mathcal{O}_W; (r\Box)^*)$, although $\tilde{\mathfrak{R}}$ is not a KSM-data. Hence, we obtain $\tilde{\mathcal{M}} := Z_{\tilde{\mathfrak{R}}} \subset \mathbb{P}(E_{\tilde{\mathfrak{R}}})$, by abuse of notation, and put

$$\tilde{\mathcal{L}} := \left( \mathcal{O}_{\mathbb{P}(E_{\mathfrak{R}})}(1)|_{\tilde{\mathcal{M}}} \right)^{\otimes (1/r)} \otimes \left( \pi_{\tilde{\mathfrak{R}}}^* K_W^{-1} \right),$$

where $\mathcal{O}_{\mathbb{P}(E_{\mathfrak{R}})}(1)$ is the relative hyperplane-section line bundle over $\mathbb{P}(E_{\mathfrak{R}})$. Then the pair $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ becomes a test configuration for $(Z_{\mathfrak{R}}, K_{Z_{\mathfrak{R}}}^{-1})$. In this paper, we call such a test configuration for $(Z_{\mathfrak{R}}, K_{Z_{\mathfrak{R}}}^{-1})$ a fiber-directed toric test configuration for $Z_{\mathfrak{R}}$. A fiber-directed toric test configuration $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ for $Z_{\mathfrak{R}}$ induces the geodesic ray $\eta_{u_t}$ in the space of metrics on the anti-canonical line bundle for $Z_{\mathfrak{R}}$, where $u_t$ is the geodesic ray \((4.2)\).

Now we observe the limit slope of the $(\sigma,V)$-Ding functional $D_{\sigma}^V$ along a geodesic ray $\eta_{u_t}$ to define an algebraic stability for $Z_{\mathfrak{R}}$.

**Proposition 4.1.** Let $u_t := (u_p^* + t(\phi - R))^*$ be the geodesic ray in $E_1(P^*)$ which is induced by a toric test configuration given by $(P,\phi,R)$. Here $u_p$ is defined in \((2.2)\). Then
we have

\[
\lim_{t \to \infty} \frac{D^\sigma_V(u_t)}{t} = \frac{1}{|P^*|^V} \int_{P^*} \phi(z)g(z)dz - \phi(0)
\]

where \(g(z)\) is the function defined in (3.8).

**Proof.** It is easy to see that

\[
\lim_{t \to \infty} \frac{1}{t} \int_{P^*} u^*_t(z)gdz = \int_{P^*} (\phi(z) - R)g(z)dz.
\]

On the other hand, by [41, Theorem 14], the limit slope of the log term is given by

\[
\lim_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{R}^l} e^{-u_t(z)}dy = \phi(0) - R.
\]

This completes the proof. \(\square\)

For a fiber-directed toric test configuration \((\tilde{M}, \tilde{L})\) for \(Z_{\mathbb{R}}\) given by \((P, \phi, R)\), we define the \((\sigma, V)\)-Ding invariant as

\[
\mathcal{D}^\sigma_V(\tilde{M}, \tilde{L}) := \frac{1}{|P^*|^V} \int_{P^*} \phi(z)g(z)dz - \phi(0),
\]

where \(g(z)\) is the function defined in (3.8).

**Definition 4.2.** The KSM-manifold \(Z_{\mathbb{R}}\) with fiber-directed \(V\) is fiber-directed relative \((\sigma, V)\)-D-polystable if \(\mathcal{D}^\sigma_V(\tilde{M}, \tilde{L}) \geq 0\) for any fiber-directed toric test configuration \((\tilde{M}, \tilde{L})\), and the equality holds if and only if the rational PL convex function defining \((\tilde{M}, \tilde{L})\) is affine linear.

Now we prove Theorem 1.4 which is a counter part of Theorem 1.2 from view point of the fiber-directed relative \((\sigma, V)\)-D-polystability.

**Proof of Theorem 1.4.** We consider the barycenter \(b_g\) of the dual polytope \(P^*\) with respect to the probability measure \(gdz/|P^*|^V\), that is, \(b_g := \frac{1}{|P^*|^V} \int_{P^*} zg(z)dz\). Then the integral condition in Theorem 1.2 is equivalent to \(b_g = 0\). We use Jensen’s inequality which means that

\[
\phi(b_g) \leq \frac{1}{|P^*|^V} \int_{P^*} \phi(z)g(z)dz
\]

for any convex function \(\phi\) on \(P^*\), and that the equality holds if and only if \(\phi\) is affine linear. Thus the condition \(b_g = 0\) implies the fiber-directed relative \((\sigma, V)\)-D-polystability for \(Z_{\mathbb{R}}\).

Conversely, considering fiber-directed toric test configurations given by \((P, \phi, R)\) and \((P, -\phi, R)\) with the coordinate function \(\phi(z) = z_k\), we have the integral condition in Theorem 1.2. This completes the proof. \(\square\)
5. Coercivity for the \((\sigma, V)\)-Ding functional

The coercivity estimate in Theorem 1.5 plays an crucial role in the result of Berman-Berndtsson [3, Theorem 1.1] used in the proof of Theorem 1.2. Theorem 1.5 can be considered as an energy theoretic version of Theorem 1.2. Indeed, we emphasize that we do not use multiplier Hermitian-Einstein metrics to prove Theorem 1.5.

5.1. Reduced \(J\)-functional and \((\sigma, V)\)-\(J\)-functional. First, we define a function \(h(z)\) on \(P^*\) by

\[
h(z) := \prod_{\alpha=1}^{n} \left(1 + \langle \mu_\alpha, z \rangle \right).
\]

Then we have \(g(z) = h(z) \exp(-\sigma(-\langle c, z \rangle + C_V))\), where \(g(z)\) is the function defined by (3.8). Note that both \(h(z)\) and \(g(z)\) are positive functions on \(P^*\). In order to define the coercivity for \(D_{\sigma V}\), we introduce the reduced \(J\)-functional \(J_{\text{red}}\) and the reduced \((\sigma, V)\)-\(J\)-functional \(J_{\sigma V, \text{red}}\) on \(E^1(P^*)\) as follows:

\[
J_{\text{red}}(u) := \inf_{\ell} \left\{ \frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \ell(z)) h(z) dz - \inf_{P^*} (u^* - \ell) \right\},
\]

\[
J_{\sigma V, \text{red}}(u) := \inf_{\ell} \left\{ \frac{1}{|P^*|_V} \int_{P^*} (u^*(z) - \ell(z)) g(z) dz - \inf_{P^*} (u^* - \ell) \right\},
\]

where \(\ell\) runs through arbitrary affine functions on \(P^*\), and \(|P^*|_h := \int_{P^*} h(z) dz\).

Definition 5.1. The \((\sigma, V)\)-Ding functional \(D_{\sigma V}\) is coercive if there exist positive constants \(\delta, C > 0\) such that

\[
D_{\sigma V}(u) \geq \delta J_{\text{red}}(u) - C
\]

for any \(u \in E^1(P^*)\).

For a convex function \(\psi\) on a convex subset \(E\) of \(\mathbb{R}^l\) and \(z_0 \in E\), we put

\[
\partial \psi(z_0) := \left\{ a \in \mathbb{R}^l \mid \psi(z) \geq \langle a, z - z_0 \rangle + u(z_0) \text{ on } E \right\}.
\]

The convexity of \(\psi\) implies \(\partial \psi(z_0) \neq \emptyset\) for any \(z_0 \in E\). Since

\[
\exp(-\sigma(-\langle c, z \rangle + C_V)) > 0
\]
on \(P^*\), there exist positive constants \(A, B > 0\) such that

\[
A \leq \exp(-\sigma(-\langle c, z \rangle + C_V)) \leq B \quad (z \in P^*).
\]
In view of an argument similar to that in [41, Proof of Proposition 27], this estimate (5.2) allows us to obtain

\begin{equation}
\frac{|P^*|_h A}{|P^*|_V} J_{\text{red}}(u) \leq J_{V,\text{red}}^\sigma(u) \leq \frac{|P^*|_h B}{|P^*|_V} J_{\text{red}}(u),
\end{equation}

for any \( u \in \mathcal{E}^1(P^*) \).

Now, we can prove the following lemma:

**Lemma 5.2.** If the integral condition (1.3) holds, then we have

\[ J_{\text{red}}(u) \leq \frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \langle a, z \rangle - u^*(0)) h(z) dz \leq \frac{B}{A} J_{\text{red}}(u), \]

for any \( u \in \mathcal{E}^1(P^*) \) and \( a \in \partial u^*(0) \).

**Proof.** For any \( u \in \mathcal{E}^1(P^*) \) and \( a \in \partial u^*(0) \), the same argument as that in [41, Proposition 27] allows us to obtain

\begin{align*}
J_{\text{red}}(u) &= \frac{1}{|P^*|_h} \int_{P^*} u^*(z) h(z) dz - u^*(b_h), \\
J_{V,\text{red}}^\sigma(u) &= \frac{1}{|P^*|_V} \int_{P^*} u^*(z) g(z) dz - u^*(b_g),
\end{align*}

where \( b_h := \frac{1}{|P^*|_h} \int_{P^*} z h(z) dz \) and \( b_g := \frac{1}{|P^*|_V} \int_{P^*} z g(z) dz \) are the barycenters of \( P^* \) with respect to the probability measures \( h dz/|P^*|_h \) and \( g dz/|P^*|_V \), respectively. Note that the condition (1.3) is equivalent to \( b_g = 0 \). Since \( u^*(z) \geq \langle a, z \rangle + u^*(0) \) on \( P^* \), we have

\[ u^*(b_h) \geq \langle a, b_h \rangle + u^*(0) \]

\[ = \frac{1}{|P^*|_h} \int_{P^*} (\langle a, z \rangle + u^*(0)) h(z) dz. \]

Therefore, we have

\begin{align*}
J_{\text{red}}(u) &= \frac{1}{|P^*|_h} \int_{P^*} u^*(z) h(z) dz - u^*(b_h), \\
&\leq \frac{1}{|P^*|_h} \int_{P^*} u^*(z) h(z) dz - \frac{1}{|P^*|_h} \int_{P^*} (\langle a, z \rangle + u^*(0)) h(z) dz, \\
&= \frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \langle a, z \rangle - u^*(0)) h(z) dz.
\end{align*}
On the other hand, in view of the condition (1.3) and the inequalities (5.2) and (5.3), we obtain
\[
\frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \langle a, z \rangle - u^*(0)) h(z) dz \\
\leq \frac{1}{|P^*|_{hA}} \int_{P^*} (u^*(z) - \langle a, z \rangle - u^*(0)) g(z) dz \\
= \frac{1}{|P^*|_{hA}} \int_{P^*} (u^*(z) - u^*(b_g)) g(z) dz \\
= \frac{|P^*|_V^{\sigma}}{|P^*|_{hA}} J_{V, \text{red}}(u) \\
\leq B \frac{A}{J_{\text{red}}(u)}.
\]
This completes the proof. \(\square\)

Finally we prove Theorem 1.5 to conclude this section.

**Proof of Theorem 1.5.** First, we assume that \(\int_{P^*} z_k g(dz) = 0\) for any \(k\), that is, \(b_g = 0\), where \(b_g\) is the barycenter of \(P^*\) with respect to the measure \(\frac{1}{|P^*|_V} g(dz)\). For an arbitrary \(u \in E_1(P^*)\), since \(b_g = 0\), a simple calculation allows us to obtain
\[
D_{V, \sigma}((u^* + \ell)^*) = D_{V, \sigma}(u),
\]
where \(\ell\) is an any affine function. Fix \(a \in \partial u^*(0)\) and we put
\[
\tilde{u}^*(z) := u^*(z) - \langle a, z \rangle - u^*(0).
\]
Then we have \(\tilde{u}^*(z) \geq \tilde{u}^*(0) = 0\) and \(D_{V, \sigma}^\sigma(u) = D_{V, \sigma}^\sigma((\tilde{u}^*)^*)\). In view of (5.2), an argument similar to that in [29, Proposition 4.2] allows us to obtain
\[
D_{V, \sigma}^\sigma((\tilde{u}^*)^*) \geq \delta \frac{1}{|P^*|_h} \int_{P^*} \tilde{u}^*(z) h(z) dz - C
\]
for some positive constants \(\delta, C > 0\). In view of Lemma 5.2, we have
\[
\frac{1}{|P^*|_h} \int_{P^*} \tilde{u}^*(z) dz = \frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \langle a, z \rangle - u^*(0)) h(z) dz \geq J_{\text{red}}(u).
\]
Hence, we have the coercivity for \(D_{V, \sigma}^\sigma\).

Conversely, we assume that for some \(k\) (\(1 \leq k \leq l\)),
\[
a_k := \frac{1}{|P^*|_V} \int_{P^*} z_k g(z) dz \neq 0.
\]
For an affine function \( \ell \), by the simple calculation, we have
\[
D_V^\sigma(\ell^*) = -\log N_0 + \frac{1}{|P^*|} \int_{P^*} \ell(z) g(z) dz,
\]
where \( N_0 \) is the number of vertices of \( P^* \). We put \( \ell(z) := -r a_k z_k \) for a positive constant \( r > 0 \). If we consider the case \( r \to +\infty \), then we have
\[
D_V^\sigma(\ell^*) = -\log N_0 - ra_k^2 \to -\infty,
\]
while \( J_{\text{red}}(u) \geq 0 \) for any \( u \in E^1(P^*) \). This completes the proof of Theorem 1.5.

A KSM-manifold \( Z_{2R} \) is said to be fiber-directed uniformly relative D-stable with respect to \((\sigma,V)\), where \( V \) is a fiber-directed holomorphic vector field on \( Z_{2R} \), if there exist a positive constant \( \lambda > 0 \) such that
\[
D_V^\sigma(\phi) \geq \lambda J_{\text{red}}(\phi^*),
\]
for any rational PL convex function \( \phi \) on \( P^* \). Here \( D_V^\sigma \) is the \((\sigma,V)\)-Ding invariant defined in Section 4.

Since the function \( g(z) \) satisfies \( g(z) \geq C \) on \( P^* \) for some positive constant \( C > 0 \), Lemma 5.2 also allows us to obtain the following theorem:

**Theorem 5.3.** Suppose a holomorphic vector field \( V \) on a KSM-manifold \( Z_{2R} \) is fiber-directed. Then \( Z_{2R} \) is fiber-directed uniformly relative D-stable with respect to \((\sigma,V)\) if and only if the integral condition (1.3) holds.

**Remark 5.4.** The fiber-directed uniformly relative D-stability also follows from the coercivity estimate of \( D_V^\sigma \). Indeed we can consider the geodesic ray (1.2) to calculate the limit slope.

### 6. Non-uniformly stable case

In this section, we consider the non-uniformly stable case, inspired by [41, Sections 6 and 7]. Namely, let \( \sigma(s) \) be a real-valued smooth function on the interval \( I = (\alpha, \beta) \) satisfying
\[
-\infty < \alpha = \min_M \theta_V^{(\omega)} \leq \max_M \theta_V^{(\omega)} < \beta \leq +\infty
\]
\[
\dot{\sigma} \leq 0 \leq \ddot{\sigma},
\]
\[
\lim_{t \to \alpha+0} \sigma(t) = +\infty.
\]
In this case we have \( \exp(-\sigma(\theta_V^\omega)) = 0 \) at some point of \( M \). Furthermore, for a KSM-manifold \( Z_{2\mathfrak{M}} \) with a fiber-directed holomorphic vector field \( V \), if the integral condition (1.3) holds, then we also have
\[
\mathcal{D}_V^\sigma(\phi) \geq 0,
\]
for any rational PL convex function \( \phi \) on \( P^* \), and \( Z_{2\mathfrak{M}} \) is said to be non-uniformly relative D-stable with respect to \((\sigma, V)\).

In this situation, we can prove the following existence of a subsolution of the equation for the multiplier Hermitian-Einstein metric:

**Lemma 6.1.** Suppose a holomorphic vector field \( V \) on \( Z_{2\mathfrak{M}} \) is fiber-directed. Moreover, we assume that
\[
(6.1) \quad f(t) := \exp(-\sigma(t)) \geq A_0(t - \alpha) \quad (\alpha \leq t < \beta),
\]
for some positive constant \( A_0 > 0 \). Then, there exists a smooth and strictly convex subsolution \( u \in \text{PSH}_b(P^*) \) of the equation for the multiplier Hermitian-Einstein metric in the sense that
\[
Cg(\nabla u) \det(\nabla^2 u) \geq e^{-u}
\]
for some positive constant \( C > 0 \), where \( g(z) \) is the non-negative function on \( P^* \) defined by (3.8).

**Proof.** We put \( \bar{t} := t - \alpha \ (\bar{t} \geq 0) \), and \( \bar{f}(\bar{t}) := f(\bar{t} + \alpha) = f(t) \). Then we have, for \( \bar{t} \geq 0 \),
\[
\bar{f}(\bar{t}) \geq \bar{f}(0) = f(\alpha) = 0,
\]
\[
\bar{f}(\bar{t}) \geq A_0\bar{t}.
\]

We put \( k(z) := -\langle c, z \rangle + C_V \), where \( c \) and \( C_V \) are the unique constants defined by (3.6), and \( \tilde{k}(z) := k(z) - \alpha \). Then we have \( g(z) = h(z)f(k(z)) \) and \( \tilde{k}(z) \geq 0 \) on \( P^* \), where \( h(z) \) is the positive function on \( P^* \) defined by (5.1). Hence, we have
\[
\log f \left( \frac{\sum_{p \in V(P^*)} k(p)e^{\langle p, z \rangle}}{\sum_{p \in V(P^*)} e^{\langle p, z \rangle}} \right) = \log \tilde{f} \left( \frac{\sum_{p \in V(P^*)} \tilde{k}(p)e^{\langle p, z \rangle}}{\sum_{p \in V(P^*)} e^{\langle p, z \rangle}} \right) \geq \log \left( A_0 \left( \frac{\sum_{p \in V(P^*)} \tilde{k}(p)e^{\langle p, z \rangle}}{\sum_{p \in V(P^*)} e^{\langle p, z \rangle}} \right) \right) \]
\[
= \log A_0 + \log \left( \sum_{p \in V(P^*)} \tilde{k}(p)e^{\langle p, z \rangle} \right) - \log \left( \sum_{p \in V(P^*)} e^{\langle p, z \rangle} \right),
\]
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where $V(P^*)$ is the set of vertices of $P^*$. In view of this inequality, the similar argument as that in [41, Proof of Lemma 41] allows us to prove Lemma 6.1.

In view of this lemma, by the same argument as that in [41, Proof of Theorem 41] (see also [3, Theorem 2.16]) we can prove the following partial coercivity for $D^\sigma_V$:

**Theorem 6.2.** Assume the same condition as in Lemma 6.1 and the integral condition (1.3). Then for any $\varepsilon \in (0, 1)$, there exists a positive constant $C > 0$ such that

$$D^\sigma_V(u) \geq \varepsilon \frac{1}{|P^*|^\sigma_V} \int_{P^*} u^*(z)g(z)dz - C$$

for any $u \in \mathcal{E}^1(P^*)$ satisfying $u \geq u(0) = 0$. Moreover, we also have

$$D^\sigma_V(u) \geq \varepsilon J_{V,\text{red}}^\sigma(u) - C$$

for any $u \in \mathcal{E}^1(P^*)$. In particular, $D^\sigma_V$ is bounded from below.

Moreover, by the same argument as that in [30, Theorem 1.4] due to the second author, we can prove the following pseudo-boundedness for $D^\sigma_V$ without assuming the condition (6.1):

**Theorem 6.3.** If the integral condition (1.3) holds, then for an arbitrary $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that

$$D^\sigma_V(u) \geq -\varepsilon J_{\text{red}}^\sigma(u) - C_\varepsilon,$$

for any $u \in \mathcal{E}^1(P^*)$.

For any $u \in \mathcal{E}^1(P^*)$ and $K \subset \mathbb{R}^l$, we put $\partial u(K) := \bigcup_{z \in K} \partial u(z)$. The Monge-Ampère measure $\text{MA}_g(u)$ on $\mathbb{R}^l$ in the sense of Alexandrov is defined by

$$\text{MA}_g(u)(B) := \frac{1}{|P^*|^\sigma_V} \int_{\partial u(B)} g(z)dz,$$

for a Borel set $B \subset \mathbb{R}^l$ (see for instance [41, Section 3.3.1]).

Since $\{z \in P^* \mid g(z) = 0\} = \{z \in P^* \mid -\langle c, z \rangle + C_V = \alpha\}$, by the same argument as that in [41, Proof of Theorem 44], Theorem 6.2 allows us to prove the following theorem:

**Theorem 6.4.** Assume the same condition as in Lemma 6.1 and the integral condition (1.3). Then there exists $u \in \text{PSH}_b(P^*)$ such that

$$\text{MA}_g(u) = \frac{e^{-u}}{\int_{\mathbb{R}^n} e^{-u(y)}dy}$$
on \( \mathbb{R}^l \), in the sense of Alexandrov. The Legendre dual \( u^* \) of \( u \) is Hölder continuous on \( P^* \) for any Hölder exponent \( \gamma \in (0, 1) \). Moreover, if \( \dim \{ z \in P^* \mid -\langle c, z \rangle + C_V = \alpha \} \leq \frac{1}{2} \), then \( u \) is smooth and strictly convex on \( \mathbb{R}^l \), and \( \nabla u \) induces a diffeomorphism from \( \mathbb{R}^l \) to \( \text{Int}(P^*) \).

7. Examples

In this section, we shall give an example of a KSM-manifold which admits non-trivial family of multiplier Hermitian-Einstein metrics, and also an example of a non-uniformly relative D-stable KSM-manifold.

First, we put

\[
\sigma_0(t) := -t \quad (-\infty < t < +\infty), \\
\sigma_1(t) := -\log(t + 1) \quad (-1 < t < +\infty),
\]

which correspond to Kähler-Ricci solitons and Mabuchi solitons, respectively. Moreover, for any \( 0 \leq \tau \leq 1 \), we put

\[
\sigma_\tau(t) := (1 - \tau)\sigma_0(t) + \tau\sigma_1(t) = -(1 - \tau)t - \tau \log(t + 1).
\]

Then \( \sigma_\tau(t) \) is defined on \((-1, +\infty)\) for \( 0 \leq \tau \leq 1 \).

**Example 7.1.** Let \( Z_1 := \mathbb{P}(\mathcal{O}_\mathbb{P}^1(1) \oplus \mathcal{O}_\mathbb{P}^1) \) be the KSM-manifold associated to the \((1, 1)\)-dimensional KSM-data \( \mathcal{M} = (\mathbb{P}^1(\mathbb{C}); \mathcal{O}_\mathbb{P}^1(1); [-1, 1]) \). Now, fix any \( \tau \in [0, 1] \) and let \( k^{(1)}_\tau(z) = b_1 z + b_2 \) be an affine function on \([-1, 1]\) corresponding to a fiber-directed holomorphic vector \( V^{(1)}_\tau \) on \( Z_1 \). In this case, the normalization condition in (1.1) becomes

\[
\int_{-1}^{1} (b_1 z + b_2) \left( 1 + \frac{1}{2} z \right) dz = \frac{1}{3} b_1 + 2b_2 = 0.
\]

Therefore, we have \( b_1 = -6b_2 \). Furthermore, on \([-1, 1]\), \( k^{(1)}_\tau(z) \) must be greater than \(-1\). Hence, \( b_2 \) is necessarily satisfied \(-1/7 < b_2 < 1/5\). In this case, the integral condition (1.3) is equivalent to

\[
I^{(1)}_\tau(b_2) := \int_{-1}^{1} z \left( 1 + \frac{1}{2} z \right) (-6b_2 z + b_2 + 1)^\tau e^{(1-\tau)(-6b_2 z + b_2)} = 0.
\]
We want to show that we can find a solution \( b_2 \in (-1/7, 1/5) \) for the equation \( I_\tau^{(1)}(b_2) = 0 \). Simple calculation allows us to have

\[
I_\tau^{(1)} \left( -\frac{1}{7} \right) = \left( \frac{6}{7} \right)^\tau e^{-\frac{1}{\bar{\tau}}(1-\tau)} \int_{-1}^{1} z \left( 1 + \frac{1}{2}z \right) (z+1)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} dz \\
= \left( \frac{6}{7} \right)^\tau e^{-\frac{1}{\bar{\tau}}(1-\tau)} \int_{-1}^{1} z \left\{ \left( 1 + \frac{1}{2}z \right) (1+z)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} + (1-\frac{1}{2}z) (1-z)^\tau e^{-\frac{\bar{\tau}}{2}(1-\tau)} \right\} dz.
\]

We want to prove \((1 + \frac{1}{2}z)(1 + z)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} + (1 - \frac{1}{2}z)(1 - z)^\tau e^{-\frac{\bar{\tau}}{2}(1-\tau)} \geq 0\) for \(0 \leq z \leq 1\). This is equivalent to

\[
F_\tau(z) := \left( \frac{2 + z}{2 - z} \right) \left( \frac{1 + z}{1 - z} \right)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} z \geq 1,
\]

for \(0 \leq z \leq 1\), which clearly holds. Hence, we conclude that \(I_\tau^{(1)}(-1/7) > 0\) for \(0 \leq \tau \leq 1\).

On the other hand, for \(b_2 = 1/5\), simple calculation also implies that

\[
I_\tau^{(1)} \left( \frac{1}{5} \right) = \left( \frac{6}{5} \right)^\tau e^{1/\bar{\tau}} \int_{-1}^{1} z \left( 1 + \frac{1}{2}z \right) (1-z)^\tau e^{-\frac{\bar{\tau}}{2}(1-\tau)} dz \\
= \left( \frac{6}{5} \right)^\tau e^{1/\bar{\tau}} \int_{0}^{1} z \left\{ \left( 1 + \frac{1}{2}z \right) (1-z)^\tau e^{-\frac{\bar{\tau}}{2}(1-\tau)} \right\} dz.
\]

Now, we want to prove \((1 - \frac{1}{2}z)(1 + z)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} - (1 + \frac{1}{2}z)(1 - z)^\tau e^{-\frac{\bar{\tau}}{2}(1-\tau)} \geq 0\), which is equivalent to

\[
G_\tau(z) := \left( \frac{2 - z}{2 + z} \right) \left( \frac{1 + z}{1 - z} \right)^\tau e^{\frac{\bar{\tau}}{2}(1-\tau)} z \geq 1,
\]

for \(0 \leq z \leq 1\). We have \(G_\tau(0) = 1\) and

\[
G'_\tau(z) = \frac{2e^{\frac{\bar{\tau}}{2}(1-\tau)}z}{5(2+z)^2(1-z)^2} \left( \frac{1 + z}{1 - z} \right)^{\tau-1} \left\{ 5\tau(2 + z^2) + 2(1 - \tau)(1 - z^2)(7 - 3z^2) \right\} \geq 0,
\]

for \(0 \leq z < 1\). Hence, for \(0 \leq \tau \leq 1\), we get \(G_\tau(z) \geq 1\), which implies \(I_\tau^{(1)}(1/5) < 0\). By the continuity of \(I_\tau^{(1)}(b_2)\) for \(-1/7 \leq b_2 \leq 1/5\), we can find a solution \(b_2 \in (-1/7, 1/5)\) for the equation \(I_\tau^{(1)}(b_2) = 0\). Therefore, by Theorem 1.2, the KSM-manifold \(Z_1 = \mathbb{P} (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})\) admits a multiplier Hermitian-Einstein metric of type \((\sigma_\tau, V_\tau^{(1)})\) for any \(0 \leq \tau \leq 1\).
Example 7.2. First of all, we note that $\sigma_\tau(t)$, defined as above, satisfies the assumption (6.1) for any $0 \leq \tau \leq 1$, that is,

$$e^{-\sigma_\tau(t)} = (t + 1)^\tau e^{(1-\tau)t} \geq t + 1 \quad (-1 \leq t < +\infty).$$

Let $Z_2 := \mathbb{P}(O_{\mathbb{P}^2}(2) \oplus O_{\mathbb{P}^2})$ be the KSM-manifold associated to the $(2, 1)$-dimensional KSM-data $\mathfrak{M} = (\mathbb{P}^2(\mathbb{C}); O_{\mathbb{P}^2}(2); [-1, 1])$. Since $Z_2$ is a toric Fano manifold, $Z_2$ admits a Kähler-Ricci soliton, of course. However, Mabuchi proved that $Z_2$ admits no Mabuchi solitons [24, Example 6.6]. Now, for any $\tau \in [0, 1]$, let $k_\tau^{(2)}(z) = b_1 z + b_2$ be an affine function on $[-1, 1]$ corresponding to a fiber-directed holomorphic vector $V_\tau^{(2)}$ on $Z_2$. In this case, the normalization condition in (1.1) becomes

$$\int_{-1}^{1} (b_1 z + b_2) \left( 1 + \frac{2}{3} z \right)^2 \left( b_1 z - \frac{12}{31} b_1 + 1 \right)^\tau e^{(1-\tau)(b_1 z - \frac{12}{31} b_1)} \, dz = 0.$$

Therefore, we have $b_2 = -\frac{12}{31} b_1$. Furthermore, the condition $k_\tau^{(2)}(z) > -1$ on $[-1, 1]$ is equivalent to $-31/19 < b_1 < 31/43$. In this case, the integral condition (1.3) becomes

$$I_\tau^{(2)}(b_1) := \int_{-1}^{1} z \left( 1 + \frac{2}{3} z \right)^2 \left( b_1 z - \frac{12}{31} b_1 + 1 \right)^\tau e^{(1-\tau)(b_1 z - \frac{12}{31} b_1)} \, dz = 0.$$

For $b_1 = -31/19$, by a simple calculation, we have

$$I_0^{(2)}\left(\frac{-31}{19}\right) = \frac{19}{9 \cdot 31^4} \left( -\frac{2268214 \cdot e^{-\frac{31}{19}} + 80048 \cdot e^{\frac{31}{19}}}{9 \cdot 31^4} \right) < 0,$$

$$I_1^{(2)}\left(\frac{-31}{19}\right) = \frac{62}{855} > 0.$$

By the continuity of $I_\tau^{(2)}(-31/19)$ for $\tau \in [0, 1]$, we can find a $\tau_0 \in (0, 1)$ satisfying $I_\tau^{(2)}(-31/19) = 0$. For this $\tau_0$ and $b_1 = -31/19$, we have

$$k_\tau^{(2)}(z) = -\frac{31}{19} z + \frac{12}{19},$$

$$k_\tau^{(2)}(1) = -1,$$

$$k_\tau^{(2)}(-1) = \frac{43}{19}.$$

Therefore, $k_\tau^{(2)}(z)$ satisfies the integral condition (1.3) and $\exp\left(-\sigma_\tau_0 k_\tau^{(2)}(z)\right) \geq 0$ on $[-1, 1]$. Hence, $Z_2 = \mathbb{P}(O_{\mathbb{P}^2}(2) \oplus O_{\mathbb{P}^2})$ is non-uniformly relative D-stable with respect to $(\sigma_\tau_0, V_\tau^{(2)})$. 
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