Inversions of Lévy Measures and the Relation Between Long and Short Time Behavior of Lévy Processes

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Abstract The inversion of a Lévy measure was first introduced (under a different name) in [18]. We generalize the definition and give some properties. We then use inversions to derive a relationship between weak convergence of a Lévy process to an infinite variance stable distribution when time approaches zero and weak convergence of a different Lévy process as time approaches infinity. This allows us to get self contained conditions for a Lévy process to converge to an infinite variance stable distribution as time approaches zero. We formulate our results both for general Lévy processes and for the important class of tempered stable Lévy processes. For this latter class, we give detailed results in terms of their Rosiński measures.

Keywords Inversions of Lévy Measures · Tempered Stable Distributions · Long and Short Time Behavior · Lévy Processes

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1 Introduction

Let \( \{X_t : t \geq 0\} \) be a \( d \)-dimensional Lévy process. The long (short) time behavior of the process is the weak limit of \( X_t \), under appropriate shifting and scaling, as \( t \) approaches infinity (zero). An alternate, but equivalent, definition, in terms of weak convergence of certain time rescaled processes, is also sometimes used (see e.g. [13]). Since Lévy processes are a generalization of sums of iid random variables, it is not difficult to see that the long time behavior of the process corresponds to the stable distribution to whose domain of attraction \( X_1 \) belongs. Necessary and sufficient conditions for this are given in [15] and [11]. On the other hand, short time behavior has no simple analogue with the summation of iid random variables. Never-the-less, in
certain situations, one can construct another Lévy process such that the asymptotic behavior of the new process as time approaches infinity determines the asymptotic behavior of the original process as time approaches zero.

To construct such a process, observe that the long time behavior of a Lévy process is governed by the tails of its Lévy measure. In a similar way, we will see that its short time behavior is governed by the behavior of its Lévy measure near zero. Intuitively, this means that the new process should have a Lévy measure, which inverts the original Lévy measure turning its behavior near zero to behavior near infinity and its behavior near infinity to behavior near zero.

A transformation of this type was introduced in [18] in the context of studying integrals with respect to Lévy processes. There, for any infinitely divisible distribution \( \mu \) with no Gaussian part, the dual distribution of \( \mu \) was defined. This was renamed the inversion of \( \mu \) in [16] and [19]. We will refer to the Lévy measure of the inversion of \( \mu \) as the 0-inversion of the Lévy measure of \( \mu \). We will then generalize this to what we term the \( \beta \)-inversion of the Lévy measure of \( \mu \), where \( \beta \in [0, 2] \).

Inversions of infinitely divisible distributions were used in [19] to derive asymptotic results for Lévy processes. Specifically, it was shown that if \( \{X_t : t \geq 0\} \) and \( \{X'_t : t \geq 0\} \) are Lévy processes such that the distribution of \( X'_1 \) is the inversion of the distribution of \( X_1 \), then short time convergence of \( \{X_t : t \geq 0\} \) to a point mass corresponds to long time convergence of \( \{X'_t : t \geq 0\} \) to a point mass. In other words, [19] uses inversion to show a relationship between the long and short time weak laws of large numbers. In this paper, we will use it to show a relationship between the long and short time central limit theorem for convergence to an infinite variance stable distribution.

There has been particular interest in the study of long and short time behavior in the class of tempered stable Lévy processes. Tempered stable distributions were introduced in [13] as a class of models that (under certain conditions) look like infinite variance stable distributions in some central region, but they have lighter tails. This makes them particularly attractive for a variety of applications, including mathematical finance, physics, computer science, and biostatistics (see the references in [5]). An explanation of why such models appear in applications is given in [6]. Sufficient conditions for the long time behavior of tempered stable Lévy processes to be Gaussian and for the short time behavior to be the stable distribution that is being tempered are given in [13], and (for certain extensions of these models) in [14] and [1].

We will be concerned with the more general class of \( p \)-tempered \( \alpha \)-stable distributions introduced in [5]. The Lévy measure of a \( p \)-tempered \( \alpha \)-stable distribution can be parametrized in terms of its so called Rosiński measure. It is often easier to work with the Rosiński measure than with the Lévy measure directly. For this reason all of our results for \( p \)-tempered \( \alpha \)-stable distributions are given in terms of their Rosiński measures. In fact, it is the particular structure of Rosiński measures that motivates the extension of inversions to \( \beta \)-inversions.

In the next section we introduce our notation and give some background. In Section 3, we define \( \beta \)-inversions and give some of their properties. Then, in Sections 4 and 5 we present, in parallel, convergence results for distributions in \( ID_0 \) and those in \( TS^p_\alpha \). Specifically, in Section 4 we relate convergence of a sequence of distributions in \( ID_0 \) (\( TS^p_\alpha \)) with the convergence of a sequence of distributions whose Lévy
Lemma 1 convergence within this class is not closed under weak convergence. However, we can characterize weak divisible distributions with Lévy triplets of the form $c$ of all Lévy measures. If that for any bounded, continuous Borel function $f$ be written as $\hat{g}$ for any $\beta \in [0, 2]$, let $\mathfrak{M}^\beta$ be the class of Borel measures on $\mathbb{R}^d$ such that $M \in \mathfrak{M}^\beta$ if and only if

$$M(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\beta) M(dx) < \infty. \tag{1}$$

Note that if $0 < \beta_1 < \beta_2 < 2$ then $\mathfrak{M}^2 \subseteq \mathfrak{M}^{\beta_2} \subseteq \mathfrak{M}^\beta \subseteq \mathfrak{M}^0$. The class $\mathfrak{M}^0$ is the class of all Lévy measures. If $M_0, M_1, M_2, \cdots \in \mathfrak{M}^0$, we write $M_n \xrightarrow{w} M_0$ on $\mathbb{R}_d$ to mean that for any bounded, continuous Borel function $f : \mathbb{R}^d \to \mathbb{R}$, which vanishes on a neighborhood of zero and on a neighborhood of infinity, we have $\int_{\mathbb{R}^d} f(x) M_n(dx) \to \int_{\mathbb{R}^d} f(x) M_0(dx)$ as $n \to \infty$.

Recall that the characteristic function of an infinitely divisible distribution $\mu$ can be written as $\hat{\mu}(z) = \exp \{ C_\mu(z) \}$ where

$$C_\mu(z) = -\frac{1}{2} \langle z, A z \rangle + i \langle b, z \rangle + \int_{\mathbb{R}^d} \left( e^{i \langle x, z \rangle} - 1 - i \frac{\langle x, z \rangle}{1 + |x|^2} \right) M(dx), \tag{2}$$

$A$ is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and $M \in \mathfrak{M}^0$. The measure $\mu$ is uniquely identified by the Lévy triplet $(A, M, b)$ and we write $\mu = ID(A, M, b)$. If $A = 0$ we write $\mu = ID_0(M, b)$. Let $ID_0$ be the class of all infinitely divisible distributions with Lévy triplets of the form $(0, M, b)$. It is well known that this class is not closed under weak convergence. However, we can characterize weak convergence within $ID_0$ by the following specialization of Theorem 8.7 in [17].

**Lemma 1** If $\mu_n = ID_0(M_n, b_n)$ for $n = 0, 1, 2 \ldots$ then $\mu_n \xrightarrow{w} \mu_0$ if and only if $M_n \xrightarrow{w} M_0$ on $\mathbb{R}_d^d$, $b_n \to b_0$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \int_{|x| < \varepsilon} |x|^2 M_n(dx) = 0, \text{ and } \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} M_n(dx) = 0.$$

An important subclass of $ID_0$ is the class of $p$-tempered $\alpha$-stable distributions introduced in [5]. This is an extension of the tempered stable distributions of [13] and [1]. If we allow these distributions to have a Gaussian part then we would have the
class $J_{\alpha,p}$ defined in [9]. For the remainder of this paper, fix $p > 0$, $\alpha \in (-\infty, 2) \setminus \{0\}$, and define $\gamma = \alpha \vee 0$. A distribution $\mu = ID_0(M, b)$ is called $p$-tempered $\alpha$-stable, and is said to belong to class $TS_{\alpha}^p$, if

$$M(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru)q(r^p, u)r^{-1-\alpha}dr\sigma(u), \ A \in \mathcal{B}(\mathbb{R}^d),$$

(3)

where $\sigma$ is a finite Borel measure on $\mathbb{S}^{d-1}$ and $q : (0, \infty) \times \mathbb{S}^{d-1} \to (0, \infty)$ is a Borel function such that, for all $u \in \mathbb{S}^{d-1}$, $q(\cdot, u)$ is completely monotone and $\lim_{r \to \infty} q(r, u) = 0$. In [5] it is shown that we can write

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx)t^{-1-\alpha}e^{-t}\,drR(dx), \ A \in \mathcal{B}(\mathbb{R}^d)$$

(4)

for some measure $R \in \mathfrak{M}^\gamma$. Moreover, for fixed $p > 0$ and $\alpha < 2$, $R$ and $M$ uniquely determine each other. We call $R$ the Rosiński measure of $\mu$, and we write $\mu = TS_{\alpha}^p(R, b)$.

In [5] and [9], the class $TS_{\alpha}^p$ is also defined when $\alpha = 0$. However, the conditions on the Rosiński measure are somewhat more complicated and we do not consider this case here. Never-the-less, some (but not all) of our results can be extended to this case, see Remark 1 below.

A probabilistic interpretation of $R$ is given in [9] (see also [8] and Section 4 in [16]). There it is shown that if $\mu = TS_0^p(R, b)$ then under mild conditions (these always hold for $\alpha < 1$) $\mu$ is the distribution of

$$\int_0^{c_{\alpha,p}} G_{\alpha,p}(t)\,dX_t$$

where $G_{\alpha,p}(t)$ is the inverse function of $G_{\alpha,p}(u) = \int_u^\infty x^{-1-\alpha}e^{-tx}\,dx$, $c_{\alpha,p} = G_{\alpha,p}(0)$, and $\{X_t : t \geq 0\}$ is a Lévy process such that the distribution of $X_1$ has Lévy measure $R$.

As with the class $ID_0$, the class $TS_{\alpha}^p$ is not closed under weak convergence. The smallest class that contains $TS_{\alpha}^p$ and is closed under weak convergence is characterized in [4]. The following is a specialization of a result from that paper.

Lemma 2 If $\mu_n = TS_{\alpha}^p(R_n, b_n)$ for $n = 0, 1, 2 \ldots$ then $\mu_n \overset{w}{\to} \mu_0$ if and only if $R_n \overset{v}{\to} R_0$ on $\mathbb{R}^d$, $b_n \to b_0$.

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \int_{|x| < \varepsilon} |x|^2R_n(dx) = 0, \ \text{and} \ \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} |x|^2R_n(dx) = 0.$$

3 Inversions of Lévy Measures

Definition 3 Fix $\beta \in [0, 2]$. If $M \in \mathfrak{M}^\beta$ we call the measure $M^\beta$ the $\beta$-inversion of $M$ if $M^\beta(\{0\}) = 0$ and

$$M^\beta(A) = \int_{\mathbb{R}^d} 1_A(\frac{x}{|x|^\beta}) |x|^{2+\beta}M(dx), \ A \in \mathcal{B}(\mathbb{R}^d).$$
In [18], the dual of an infinitely divisible distribution $\mu = ID_0(M, b)$ was defined to be the distribution $ID_0(M^0, -b)$. Later, in [16] and [19], this was renamed the inversion of $\mu$. Thus the $0$-inversion of $M$ is the Lévy measure of the inversion of $\mu$.

It is straightforward to see that for $M \in M^\beta$ we have $M^\beta \in M^\beta$,
$(M^\beta)^\beta = M$, \hspace{1cm} (5)
and
\[
\int_{|x|>1} |x|^2 M(dx) < \infty \iff \int_{|x|\leq 1} |x|^\beta M^\beta(dx) < \infty. \hspace{1cm} (6)
\]

We now relate the convergence of a sequence of measures in $M^\beta$ to the convergence of the sequence of their $\beta$-inversions.

**Proposition 4** Fix $\beta \in [0, 2]$, and let $M_0, M_1, M_2, \cdots \in M^\beta$.
1. $M_n \overset{v}{\to} M_0$ on $\mathbb{R}^d_0$ if and only if $M_n^\beta \overset{v}{\to} M_0^\beta$ on $\mathbb{R}^d_0$.
2. We have
\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{|x|<\epsilon} |x|^2 M_n(dx) = 0
\]
if and only if
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x|>N} |x|^\beta M_n^\beta(dx) = 0.
\]

**Proof** The second part follows immediately from the definition of $\beta$-inversions. To show the first part let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded continuous function vanishing on a neighborhood of zero and on a neighborhood of infinity. The function $g(x) = f \left( \frac{|x|}{|x|^\beta} \right) |x|^{2+\beta}$ is also a continuous and bounded mapping of $\mathbb{R}^d$ into $\mathbb{R}$, vanishing on a neighborhood of zero and on a neighborhood of infinity. Thus, if $M_n^\beta \overset{v}{\to} M_0^\beta$ on $\mathbb{R}^d_0$ then
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) M_n(dx) = \lim_{n \to \infty} \int_{\mathbb{R}^d} g(x) M_n^\beta(dx) = \lim_{n \to \infty} \int_{\mathbb{R}^d} g(x) M_0^\beta(dx) = \int_{\mathbb{R}^d} f(x) M_0(dx),
\]
and $M_n \overset{v}{\to} M_0$ on $\mathbb{R}^d_0$. The other direction follows by (5). \qed

Our next result relates the regular variation of a Lévy measure to the regular variation of its $\beta$-inversion. Before giving our results, we define regularly varying Lévy measures. More information on regularly varying measures can be found in e.g. [11], [7], or [12]. However, our formulation is somewhat different from those.
Definition 5 Fix $\rho \leq 0$, $a \in \{0, \infty\}$, and $M \in \mathfrak{M}^0$ such that $M \neq 0$. If $a = \infty$ assume further that $M$ has an unbounded support. $M$ is said to be regularly varying at $a$ with index $\rho$ if there is a finite, non-zero Borel measure $\sigma$ on $\mathbb{S}^{d-1}$ such that for any $t > 0$ and any $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\lim_{r \to a} \frac{M \left( \left| x \right| > r, \frac{x}{\left| x \right|} \in D \right)}{M(\left| x \right| > r)} = t^\rho \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}.$$  

When this holds we write $M \in \text{RV}_\rho^a(\sigma)$. 

Note that, in the above, $\sigma$ is not unique. It is only defined up to multiplication by a positive constant.

Proposition 6 Fix $\beta \in [0, 2]$, $M \in \mathfrak{M}^\beta$, and $\rho \in (-2 - \beta, 0)$. If $\sigma \neq 0$ is a finite Borel measure on $\mathbb{S}^{d-1}$ then

$$M^\beta \in \text{RV}_\rho^a(\sigma) \iff M \in \text{RV}_{-(\rho + 2 + \beta)}^0(\sigma).$$

Moreover if $\ell \in \text{RV}_\rho^a$ then

$$M(||x| > t, x/|x| \in D) \sim \sigma(D) r^{-\rho - 2 - \beta} t^{\ell(1/t)} \text{ as } t \downarrow 0$$

for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ if and only if

$$M^\beta(||x| > t, x/|x| \in D) \sim \frac{D + 2 + \beta}{|\rho|} t^\rho \ell(t) \text{ as } t \to \infty$$

for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$.

Proof Note that the equivalence between (8) and (9) implies (7), thus we just need to show that (8) holds if and only if (9) holds. For every $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$, let

$$V_D(t) = M \left( \left| x \right| > 1/t, \frac{x}{\left| x \right|} \in D \right) \quad \text{and} \quad V_D^\beta(t) = M^\beta \left( \left| x \right| > 1/t, \frac{x}{\left| x \right|} \in D \right).$$

Assume that (8) holds for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$. This means that

$$V_D(t) \sim \sigma(D) t^\rho \ell(t) \text{ as } t \to \infty.$$  

If $\sigma(D) > 0$ then $V_D \in \text{RV}_\rho^{a+2+\beta}$, and since

$$V_D(t) = \int_{\left| x \right| < \frac{1}{t}, \frac{x}{\left| x \right|} \in D} \|x\|^{2+\beta} M^\beta \, dx \quad \text{and} \quad V_D^\beta(t) = \int_{\left| x \right| > \frac{1}{t}, \frac{x}{\left| x \right|} \in D} M^\beta \, dx,$$

Theorem 5.3.11 in [11] (or Theorem 2 in Section VIII.9 of [3]) implies that

$$V_D^\beta(t) \sim t^{-2-\beta} V_D(t) \frac{2 + \beta + \rho}{|\rho|} \text{ as } t \to \infty.$$
If $\sigma(D) = 0$, fix $\varepsilon > 0$, let $M_\varepsilon(dx) = \varepsilon M(dx)$, and $M_D(dx) = 1_D \left( \frac{1}{\varepsilon} \right) M(dx)$. Thus (8) implies
\[
\int_{|x| < \varepsilon} |x|^{2+\beta} \left( M_D^\beta + M_\varepsilon^\beta \right) (dx) \sim \varepsilon \sigma(S^{d-1}) \frac{2+\beta + \rho}{|\rho|} t^\rho \ell(t) \text{ as } t \to \infty,
\]
and, as before, Theorem 5.3.11 in [11] implies that
\[
V_\beta^D(t) + \varepsilon V_{2\beta-1}^\beta(t) \sim \varepsilon \sigma(S^{d-1}) \frac{2+\beta + \rho}{|\rho|} t^\rho \ell(t) \text{ as } t \to \infty.
\]
Thus
\[
\lim_{t \to \infty} \frac{V_\beta^D(t)}{t^\rho \ell(t)} \leq \lim_{t \to 0} \lim_{\varepsilon \to 0} \frac{V_\beta^D(t) + \varepsilon V_{2\beta-1}^\beta(t)}{t^\rho \ell(t)} = \lim_{t \to 0} \varepsilon \sigma(S^{d-1}) \frac{2+\beta + \rho}{|\rho|} = 0.
\]
Hence (9) holds for all $D \in \mathcal{B}(S^{d-1})$ with $\sigma(\partial D) = 0$. The proof of the other direction is similar. \qed

4 Convergence of Sequences in $ID_0$

In this section we extend Proposition 4 to weak convergence of sequences of distribution in $ID_0$ and $TS_\alpha^\beta$.

**Proposition 7** 1. Fix $\beta \in [0, 2]$ and let $M_0, M_1, M_2, \ldots \in \mathcal{M}^\beta$. If $X_n \sim ID_0(M_0, b_n)$ and $X'_n \sim ID_0(M_0, b_n)$ for $n = 0, 1, 2, \ldots$ then
\[
X_n \overset{d}{\to} X_0 \text{ and } \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} |x|^\beta M_n(dx) = 0 \tag{10}
\]
if and only if
\[
X'_n \overset{d}{\to} X_0' \text{ and } \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} |x|^\beta M_n'(dx) = 0. \tag{11}
\]

2. Fix $\beta \in [\gamma, 2]$ and let $R_0, R_1, R_2, \ldots \in \mathcal{M}^\beta$. If $Y_n \sim TS_\alpha^\beta(R_0, b_n)$ and $Y'_n \sim TS_\alpha^\beta(R_0, b_n)$ for $n = 0, 1, 2, \ldots$ then
\[
Y_n \overset{d}{\to} Y_0 \text{ and } \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} |x|^\beta R_n(dx) = 0 \tag{12}
\]
if and only if
\[
Y'_n \overset{d}{\to} Y_0' \text{ and } \lim_{N \to \infty} \limsup_{n \to \infty} \int_{|x| > N} |x|^\beta R'_n(dx) = 0. \tag{13}
\]

**Proof** This result is an immediate consequence of Proposition 4, Lemma 1 (or Lemma 2 in the case of Part 2), and 5. \qed

Combining this with Lemma 1 and Lemma 2 gives the following, the first part of which was previously given in Proposition 2.1 of [16].
Corollary 8 1. If for \( n = 0, 1, 2, \ldots \) we have \( X_n \sim I(D_0(M_n, b_n)) \) and \( X'_n \sim I(D_0(M'_n, b_n)) \) then
\[
X_n \xrightarrow{d} X_0 \iff X'_n \xrightarrow{d} X'_0.
\]

2. If for \( n = 0, 1, 2, \ldots \) we have \( Y_n \sim T_\alpha(M_0, b_n) \) and \( Y'_n \sim T_\alpha(M'_0, b_n) \) then
\[
Y_n \xrightarrow{d} Y_0 \iff Y'_n \xrightarrow{d} Y'_0.
\]

Note that, by definition, \( R \in \mathcal{M}^\gamma \) and thus \( R^\gamma \) is always defined. Assume that for \( n = 0, 1, 2, \ldots \) we have \( Y_n \sim T_\alpha(M_0, b_n) \) such that \( Y_n \xrightarrow{d} Y_0 \). Since \( T_\alpha(M_0, b_n) = I(D_0(M_n, b)) \), where we get \( M_n \) from \( R_n \) by \ref{1}. Corollary \ref{8} implies that if for each \( n = 0, 1, 2, \ldots \) we have \( X_n \sim I(D_0(M'_n, b_n)) \) and \( Y_n \sim T_\alpha(M'_0, b_n) \) then \( X_n \xrightarrow{d} X'_0 \) and \( Y_n \xrightarrow{d} Y'_0 \). However, the distributions of \( X_n \) and \( Y_n \) are, in general, very different. In fact the distribution of \( Y_n \) is necessarily in \( T_\alpha(M^\gamma) \), while the distribution of \( X_n \) is, in general, not an element of this class. This last fact follows from Theorem 4.6 in \cite{16}.

We end this section by specializing Corollary \ref{8} to the case of convergence to an infinite variance stable distribution. First, recall that, for \( \eta \in (0, 2) \), an \( \eta \)-stable distribution is a distribution \( \mu = I(D_0(M, b)) \) where
\[
M(A) = \int_{S^{d-1}} \int_0^\infty 1_A(ur)r^{-1-\eta}d\sigma(du), \quad A \in \mathcal{B}(\mathbb{R}^d)
\]
for some finite, non-zero Borel measure \( \sigma \) on \( S^{d-1} \). We denote this distribution by \( S_\eta(\sigma, b) \). For details about stable distributions see \cite{20}. Note that for \( \beta \in [0, \eta) \)
\[
M^\beta(A) = \int_{S^{d-1}} \int_0^\infty 1_A(ur)r^{-1-(2+\beta-\eta)}d\sigma(du), \quad A \in \mathcal{B}(\mathbb{R}^d),
\]
Thus \( I(D_0(M^\beta, b)) = S_{2+\beta-\eta}(\sigma, b) \). In \cite{5} it was shown that if \( \eta \in (\gamma, 2) \) then \( S_\eta(\sigma, b) = T_\alpha(M^\gamma(\sigma, b)) \) where \( M^\gamma(\sigma, b) = K_\eta,\alpha,p M(dx) \) with \( M(dx) \) as given by \ref{14} and \( K_\eta,\alpha,p = \int_0^\infty r^\eta-\alpha-1e^{-r}dr \). Thus for \( \eta \in (\gamma, 2) \) we have \( R^\gamma(dx) = K_\eta,\alpha,p M^\gamma(dx) \). Note that, in this case, \( T_\alpha(M^\gamma(\sigma, b)) = S_{2+\gamma-\eta}(\sigma^\gamma, b) \) where
\[
\sigma^\gamma(du) = \frac{K_{\eta,\alpha,p}^{2+\gamma-\eta}}{K_\eta,\alpha,p} \sigma(du).
\]
These facts, combined with Corollary \ref{8}, give the following result.

Corollary 9 1. Fix \( \eta \in (0, 2) \). If for \( n = 0, 1, 2, \ldots \) we have \( X_n \sim I(D_0(M_n, b_n)) \) and \( X'_n \sim I(D_0(M'_n, b_n)) \) then \( X_n \xrightarrow{d} S_\eta(\sigma, b) \) if and only if \( X'_n \xrightarrow{d} S_\eta(\sigma, b) \).

2. Fix \( \eta \in (\gamma, 2) \). If for \( n = 0, 1, 2, \ldots \) we have \( Y_n \sim T_\alpha(M_0, b_n) \) and \( Y'_n \sim T_\alpha(M'_0, b_n) \) then \( Y_n \xrightarrow{d} S_\eta(\sigma, b) \) if and only if \( Y'_n \xrightarrow{d} S_{2+\gamma-\eta}(\sigma^\gamma, b) \), where \( \sigma^\gamma \) is given by \ref{15}.
5 Long and Short Time Behavior For Lévy Processes

In this section we use the tools that we have developed to derive an equivalence between long and short time behavior of certain Lévy processes.

**Theorem 10.** 1. Fix $\eta \in (0, 2)$ and let $\{X_t : t \geq 0\}$ and $\{X'_t : t \geq 0\}$ be Lévy processes with $X_t \sim ID_0(M, c)$ and $X'_t \sim ID_0(M^0, d)$. There exist functions $a_t$ and $\zeta_t$ such that

$$a_t (X_t - \zeta_t) \xrightarrow{d} S_\eta (\sigma, 0) \text{ as } t \to \infty$$

if and only if there exist functions $b_t$ and $\xi_t$ with

$$b_t (X'_t - \xi_t) \xrightarrow{d} S_{2-\eta} (\sigma, 0) \text{ as } t \downarrow 0.$$  

Moreover, when this holds we have $b_t \sim [(1/t) h^{-1}(1/t)]^{1/2}$ as $t \downarrow 0$, where $h(t)$ is any invertible function with $h(t) \sim t^{-1} a_t^{-2}$ as $t \to \infty$.

2. Fix $\eta \in (\gamma, 2)$ and let $\{Y_t : t \geq 0\}$ and $\{Y'_t : t \geq 0\}$ be Lévy processes with $Y_t \sim T_{S_M}(R, c)$ and $Y'_t \sim T_{S_M}(R^0, d)$. There exist functions $a_t$ and $\zeta_t$ such that

$$a_t (Y_t - \zeta_t) \xrightarrow{d} S_\eta (\sigma, 0) \text{ as } t \to \infty$$

if and only if there exist functions $b_t$ and $\xi_t$ with

$$b_t (Y'_t - \xi_t) \xrightarrow{d} S_{2+\gamma-\eta} (\sigma, 0) \text{ as } t \downarrow 0.$$  

Moreover, when this holds we have $b_t \sim \kappa^{-1/\eta} [(1/t) h^{-1}(1/t)]^{1/(2+\gamma)}$ as $t \downarrow 0$, where $h(t)$ is any invertible function with $h(t) \sim t^{-1} a_t^{-2-\gamma}$ as $t \to \infty$ and $\kappa = K_{(2+\gamma-\eta), \alpha, p}/K_{\eta, \alpha, p}$.

From the standard theory of summation of iid random variables (see e.g. [3] or [11]) it follows that $a \in RV_{-1/\eta}$, thus the functions $t^{-1} a_t^{-2}$ and $t^{-1} a_t^{-2-\gamma}$ are regularly varying at infinity with a positive index of regular variation. This implies that they are asymptotically equivalent to an invertible function and hence $h$ and $h_\gamma$ are well defined. It is straightforward to see that in the first part $b \in RV^{0}_{-1/(2-\eta)}$ and in the second part $b \in RV^{0}_{-1/(2+\gamma-\eta)}$.

**Proof** We only prove the first part as the proof of the second part is similar. By Slutsky’s Theorem it suffices to show that the result holds when $a_t = [(th(t)]^{-1/2}$ and $b_t = [(1/t) h^{-1}(1/t)]^{1/2}$. Note that $b_t = 1/a_{h^{-1}(1/t)}$. Define

$$M'_t (B) = t \int_{R^d} 1_B (x a_t) M(dx) \text{ and } M''_t (B) = t \int_{R^d} 1_B (x b_t) M^0(dx), \quad B \in \mathcal{B}(R^d).$$

These are, respectively, the Lévy measures of $a_t (X_t - \zeta_t)$ and $b_t (X'_t - \xi_t)$. 

Assume that (16) holds. Without loss of generality, we assume that \( \zeta' \) is such that 
\[
a_t (X_t - \zeta') \sim ID_0 (M_t', 0)
\] for every \( t > 0 \). Since 
\[
(M_t')^0 (B) = \int_{\mathbb{R}^d} 1_B \left( \frac{x}{|x|^2} \right) |x|^2 M_t' (dx)
\]
\[
= t a_t^2 \int_{\mathbb{R}^d} 1_B \left( \frac{x}{|x|^2 a_t} \right) |x|^2 M (dx)
\]
\[
= t a_t^2 \int_{\mathbb{R}^d} 1_B \left( s a_t^{-1} \right) M^0 (dx),
\]
Corollary 9 implies that 
\[
\frac{1}{a_t} \left( X_{a_t^2} - q_t \right) \xrightarrow{d} S_{2 - \eta} (\sigma, 0) \text{ as } t \to \infty,
\]
where \( q_t \) is such that \( \frac{1}{a_t} \left( X_{a_t^2} - q_t \right) \sim ID_0 \left( (M_t')^0, 0 \right) \). From here the result follows since 
\[
\lim_{t \to \infty} \frac{1}{a_t} \left( X_{a_t^2} - q_t \right) = \lim_{t \to 0} \frac{1}{a_{1/t}} \left( X_{1/t}^{1 - a_{1/t}^{-1}} - q_{1/t} \right)
\]
\[
= \lim_{t \to 0} \frac{1}{a_{1/t}} \left( X_{1/t}^\prime - q_{1/t} \right)
\]
\[
= \lim_{u \to 0} \frac{1}{a_{b^{-1}(1/u)}} \left( X_u^\prime - q_{b^{-1}(1/u)} \right) = \lim_{u \to 0} b_u \left( X_u^\prime - \zeta_u \right),
\]
where the third line follows by the substitution \( u = 1/h(1/t) \) and \( \zeta_u = q_{b^{-1}(1/u)} \).

Conversely, assume that (17) holds. Without loss of generality, we assume that \( \xi' \) is such that 
\( b_t (X_t' - \xi') \sim ID_0 \left( (M_t')^0, 0 \right) \) for every \( t > 0 \). As before, since 
\[
(M_t')^0 (B) = t b_t^2 \int_{\mathbb{R}^d} 1_B \left( x b_t^{-1} \right) M (dx),
\]
Corollary 9 implies that 
\[
\frac{1}{b_t} \left( X_{b_t^2} - q_t' \right) \xrightarrow{d} S_0 (\sigma, 0) \text{ as } t \downarrow 0,
\]
where \( q_t' \) is such that \( \frac{1}{b_t} \left( X_{b_t^2} - q_t' \right) \sim ID_0 (M_t', 0) \). The result follows from the fact that 
\[
\lim_{t \downarrow 0} \frac{1}{b_t} \left( X_{b_t^2} - q_t \right) = \lim_{t \to \infty} \frac{1}{b_{1/t}} \left( X_{1/t}^{1 - b_{1/t}^{-1}} - q_{1/t} \right)
\]
\[
= \lim_{t \to \infty} a_{h^{-1}(1/t)} \left( X_{h^{-1}(1/t)} - q_{1/t} \right) = \lim_{u \to \infty} a_u \left( X_u - \zeta_u \right),
\]
where \( \zeta_u = q_{1/h(u)} \) and the third equality follows by the substitution \( u = h^{-1}(t) \). \( \square \)
We now derive self-contained conditions for a Lévy process to converge in distribution to an infinite variance stable distribution when time approaches zero. It does not appear that this has been previously addressed for the multivariate case. However, in the univariate case, conditions in a slightly different form, are given in [10].

**Theorem 11** Fix $\eta \in (0, 2)$ and let $\sigma$ be a finite, nonzero Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t : t \geq 0\}$ be a Lévy Process with $X_1 \sim ID_0(M, b)$.

1. There exist functions $a_\tau > 0$ and $\zeta$ such that

$$a_\tau (X_t - \zeta) \overset{d}{\rightarrow} S_\eta(\sigma, 0) \text{ as } t \downarrow 0 \tag{20}$$

if and only if $M \in RV^0_\eta(\sigma)$.

2. If $\eta \in (\gamma, 2)$ and $X_t \sim TS_\alpha^p(R, b)$ then there exist functions $a_\tau > 0$ and $\zeta$ such that

$$\text{20} \text{ holds if and only if } R \in RV^0_\eta(\sigma).$$

**Proof** First we prove part 1. Theorem [10] implies that (20) holds if and only if the distribution $ID_0(M^0, 0)$ is in the domain of attraction of $S_{2-\gamma}(\sigma, 0)$. By standard results (see e.g. [15] or [11]) this holds if and only if the distribution $ID_0(M^0, 0)$ is an element of $RV^0_{2-\gamma}(\sigma)$. This holds if and only if $M^0 \in RV^0_{2-\gamma}(\sigma)$ (see [7]). From here Part 1 follows by Proposition [5]. The proof of the second part is similar, this time we must find necessary and sufficient conditions for the distribution $TS_\alpha^p(R^0, 0)$ to be an element of $RV^0_{2+\gamma-\eta}(\sigma)$. This holds if and only if $R^0 \in RV^0_{2+\gamma-\eta}(\sigma)$ (see [5]). From here part 2 follows by Proposition [6].

Combining the two parts of this theorem gives the following short time analogue of Theorem 5 in [5].

**Corollary 12** Let $R$ be the Rosiński measure of a $p$-tempered $\alpha$-stable distribution, and let $M$ be the Lévy measure of this distribution (that is, we get $M$ from $R$ by (4)). If $\sigma$ is a finite, nonzero Borel measure on $\mathbb{S}^{d-1}$ and $\eta \in (\gamma, 2)$ then

$$R \in RV^0_\eta(\sigma) \iff M \in RV^0_\eta(\sigma).$$

**Remark 1** We now turn to the case $\alpha = 0$ and $p > 0$. For a measure $R$ to be the Rosiński measure of some $p$-tempered 0-stable distribution, $R$ must satisfy the condition that $R \in \mathcal{M}^0$ and $\int_{|x| > 1} \log |x| R(dx) < \infty$ (see [5]). Let $\mathcal{M}^{\log}$ be the class of measures that satisfy this condition. The natural definition of the inversion of $R \in \mathcal{M}^{\log}$ appears to be to let $R^{\log}((0)) = 0$ and

$$R^{\log}(A) = \int_{\mathbb{R}^d} I_A \left(\frac{x}{|x|}\right) |x|^2 (1 + |\log |x||) \kappa(|x|) R(dx), \ A \in \mathcal{B}(\mathbb{R}^d),$$

where $\kappa(t) = 1$ if $t \geq 1$ and $\kappa(t) = -1$ if $t \in [0, 1)$. It is not difficult to see that $R^{\log} \in \mathcal{M}^{\log}$ and $(R^{\log})^{\log} = R$. All of the results in Sections 3 and 4 have a version for this case. However, we are not able to show the corresponding results from Section 5.
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