CONSTRUCTION OF A BLOW-UP SOLUTION FOR A PERTURBED NONLINEAR HEAT EQUATION WITH A GRADIENT TERM

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Abstract

We consider in this paper a perturbation of the standard semilinear heat equation by a term involving the space derivative and a non-local term. We prove the existence of a blow-up solution, and give its blow-up profile. Our method relies on the two-step method: we first linearize the equation (in similarity variables) around the expected profile, then we use a topological argument to control the positive directions of the spectrum.

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1 Introduction

We are interested in this paper in the following nonlinear parabolic equation

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \Delta u + |u|^{p-1}u + \mu|\nabla u| \int_{B(0,|x|)} |u|^{q-1} \\
    u(0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N),
\end{align*}
\]

(1.1)

where \( u = u(x,t) \in \mathbb{R} \), \( x \in \mathbb{R}^N \) and the parameter \( p, q \) and \( \mu \) are such that

\[
p > 3, \quad \frac{N}{2}(p-1) + 1 < q \leq \frac{N}{2}(p-1) + \frac{p+1}{2}, \quad \mu > 0.
\]

(1.2)
When $\mu = 0$, the blow-up result for the equation (1.1) has been extensively studied. The existence of blow-up solution has been proved by several authors see Fujita [8], Ball [1]. We say that $u$ blows up in finite time $T$ in the sense that

$$\|u(t)\|_{L^\infty} \to 0, \quad t \to T.$$ 

We call $T$ the blow-up time of $u$.

Many works have been describing the asymptotic blow-up behavior near a given blow-up point, see Giga and Kohn [11], [12], Weissler [26], Filippas, Kohn and Liu [6], [7], Herrero and Velázquez [14], [15], [16], [17], Merle and Zaag [18], [19], [20]. Also, lots of results have been devoted to the blow-up profile; see Bricmont and Kupiainen [3], Merle and Zaag [18], Berger and Kohn [2] and Nguyen and Zaag [22], [23].

Particularly, these authors constructed a solution $u$ which approaches an explicit universal profile $f$ depending only on $p$ and independent from initial data as follows

$$\|(T-t)^{\frac{1}{p-1}}u(x,t) - f\left(\frac{x}{\sqrt{(T-t)\log(T-t)}}\right)\|_{L^\infty} \to_{t \to T} 0,$$

where $f$ is the profile defined by

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$ 

Such a construction relies on a two step method:

- The guess of the limiting profile, based on a formal approach in the so-called similarity variables (defined in (2.6) below); this is particularly well explained in Berger and Kohn [2], Filippas and Kohn [6] and Bricmont and Kupiainen [3].

- The rigorous proof performed in similarity variables, where the authors linearize the equation around the introduced profile, and control the negative part of the spectrum thanks to the decaping properties of the linear operator, and use a topological argument for the central of the nonegative directions of the spectrum.

An interesting following the above results is to tell how robust is the construction method?

A first result in that direction was obtained for the following equation with a gradient term:

$$u_t = \Delta u + |u|^{p-1}u + \mu|\nabla u|^q.$$ 

For this equation, we mention that the blow-up profile obtained by Souplet, Tayachi and Weissler [24], when $q = \frac{2p}{p+1}$, $\mu < 0$, Galaktionov and Vazquez [9], [10], when
\( q = 2 \) and \( \mu > 0 \), Ebde and Zaag [3], when \( q < \frac{2p}{p+1} \) and Tayachi and Zaag [25], when \( q = \frac{2p}{p+1}, \mu > 0 \). A numerical result has been proved by Nguyen [21]. Because of the presence of the perturbation including a non linear gradient term, they obtain the convergence in \( W^{1,\infty}(\mathbb{R}^N) \).

We would like to mention that the construction method has proved to be successful in a different class of PDE’s involving non local terms, namely the following, equation modeling Micro Electrical Mechanical Systems (MEMS):

\[
 u_t = \Delta u + \frac{\lambda}{(1-u)^2 (1 + \gamma \int_\Omega \frac{1}{1-u} dx)^2},
\]

see Duong and Zaag [4]. In this paper, we would like to consider a mixed-type equation involving gradient terms together with non local terms, namely, equation (1.1)

We note that the equation (1.1) is a class of perturbed semilinear heat equation but compared to the previous works our perturbation is not trivial since we have a non local gradient term.

The aim of this paper is to construct a solution of the equation (1.1) which approach the same profile \( f \) as for the case \( \mu = 0 \), moreover we prove the following result.

**THEOREM 1.1.** Let \( \mu > 0, \ p > 3, \) and \( \frac{N}{2}(p-1)+1 < q < \frac{N}{2}(p-1)+\frac{p+1}{2} \). There exists \( T > 0 \) such that equation (1.1) has a solution \( u(x,t) \) such that \( u \) and \( \nabla u \) simultaneously blow up at time \( T \) at the point \( a = 0 \). Moreover, For all \( t \in [0,T) \), for all \( x \in \mathbb{R} \)

\[
 |u(x,t)-(T-t)^{-\frac{1}{p-1}}f(\frac{x}{\sqrt{(T-t)|\log(T-t)|}})| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^{\frac{1}{p-1}}} (T-t)^{-\frac{1}{p-1}},
\]

and

\[
 |\nabla u(x,t)-(T-t)^{-\frac{1}{2} - \frac{1}{p-1}} \nabla f(\frac{x}{\sqrt{(T-t)|\log(T-t)|}})| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^{\frac{1}{p-1}}} (T-t)^{-\frac{1}{2} - \frac{1}{p-1}},
\]

where \( f(z) = (p - 1 + b|z|^2)^{-\frac{1}{p-1}}, \ z \in \mathbb{R}^N, \ b = \frac{(p-1)^2}{4p} \).

**REMARK 1.2.** We suspect the origin to be the only blow-up point of \( u \) and \( \nabla u \). Unfortunately, because of the non local term in equation (1.1). We couldn’t apply the localization and iteration method presented by Giga and Kohn in Theorem 2.1 page 85 of [11]. Nevertheless, we could show that for any \( x_0 \in \mathbb{R}^N \) and in some cylinder around \((x_0,T)\) the solution is uniformly negligible with respect to the ODE rate \((T-t)^{-\frac{1}{p-1}}\), which is in our opinion a strong evidence showing that the solution doesn’t blow up at \( x_0 \).
The proof of Theorem 1.1 is based on techniques developed by Bricmont and Kupiainen [3], Merle and Zaag [18] and Tayachi and Zaag [25]. This is reasonable since in similarity variables defined below by (2.6) the new perturbation term comes with an exponentially decreasing term. Although these modifications do not affect the general framework developed in the previous work, we need to perform some crucial modifications with respect to [3], [18], [5], [25] in order to control the new term. Let us mention the crucial modifications:

- We modify the functional space. Since the perturbation contains
  \[ \int_{B(0,|x|)} |u|^{q-1}, \]
  our proofs need some involved argument to control this term. In particular, we need to study the convergence in the new functional space \( W^{1,\infty}_p(\mathbb{R}^N) \) defined by
  \[
  W^{1,\infty}_p(\mathbb{R}^N) = \{ g; \ (1 + |y|^{2/p-1}) g \in L^\infty, \ (1 + |y|^{2/p-1}) \nabla g \in L^\infty \}. \tag{1.5}
  \]
  More specifically, some involved parabolic regularity argument are proved to handle the gradient term.

- In order to study the blow-up in the new functional space, we need to modify the definition of the shrinking set (see Definition 3.1 below). Therefore, some crucial estimates are needed.

- Finally, we linearize the equation around a new profile given by (2.9) below. A good understanding of the linearized operator and which allows to handle the new shrinking set.

**Remark 1.3.** The local Cauchy problem for equation (1.1) can be solved in the functional space \( W^{1,\infty}_p(\mathbb{R}^N) \) using a fixed point argument. For the reader’s convenience we prove this results in Appendix. Our approach is inspired by the method of Bricmont and Kupiainen [3], Merle and Zaag [18].

Note that the solution constructed in Theorem 1.1 satisfies the following result:

**Corollary 1.4.** Let \( u \) be the solution of (1.1) constructed in Theorem 1.1 and \( T \) its blow-up time. For all \( K_0 > 0 \) and \( |x| > K_0 \sqrt{(T - t)\log(T - t)} \), there exist positive constants \( C, C_0 \) such that

\[
1. \quad |u(x, t) - \left( \frac{\log |T - t|}{b|x|^2} \right)^{p-1} | \leq \frac{C}{K_0^2} \left( \frac{|\log(T - t)|}{|x|^2} \right)^{1/p-1},
\]
and

\[ |\nabla u(x,t) - \frac{C_0}{\sqrt{\log |T-t|}} \left( \frac{\log |T-t|}{|x|^2} \right)^{\frac{p+1}{2(p-1)}}| \leq \frac{C}{\sqrt{(T-t) \log |T-t|}} \left( \frac{\log(T-t)}{|x|^2} \right)^{\frac{1}{p+1}}, \]

Let us remark that, the construction method involves the linearisation of the equation (in similarity variables defined below in (2.6)), with a different treatment according to the sign of the eigenvalues

- The infinite dimensional part of the solution, corresponding to the new positive part of the spectrum, is controlled thanks to the decaying properties of the linearized operator; since the positive part of the spectrum is finite dimensional, we call this step a finite dimensional reduction.

- There, the positive part of the spectrum is controlled thanks to a topological argument, based on index theory.

The paper is organized as follows. In Section 2, we give a formulation of the problem. In Section 3, we prove the existence of a solution of equation (2.10). Finally, in Section 4 we prove Theorem 1.1 and corollary 1.4.

2 Formulation of the problem

A fundamental tool for the study of the asymptotic behavior of blow-up solutions is the following similarity variables framework introduced by Giga and Kohn [11], [12], [13]:

\[ y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t) \quad \text{and} \quad w(y, s) = (T-t)^{-\frac{1}{p-1}} u(x, t), \quad (2.6) \]

where \( T \) is the time where we want the solution to blow up. Therefore, if \( u(x, t) \) satisfies (1.1) for all \( (x, t) \in \mathbb{R}^N \times [0, T) \), then \( w(y, s) \) satisfies the following equation:

\[ w_s = \Delta w + |w|^{p-1} w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + \mu e^{-\gamma s} |\nabla w| \int_{B(0,|y|)} |w|^{q-1}, \quad (2.7) \]

where \( \gamma = \frac{p-q}{p-1} + \frac{N-1}{2} \).

REMARK 2.1. We would like to emphasize the fact that \( \gamma > 0 \), which explains the little effect of the gradient term for large times.
The study of $u$ as $t \to T$ is equivalent to the study of the asymptotic behavior of $w$ as $s \to +\infty$.

We would like to find $s_0 > 0$ and an initial data $w_0$ such that the solution $w$ of equation (2.7), $w(s_0) = w_0$, satisfies

$$\|w(y, s) - f\left(\frac{y}{\sqrt{s}}\right)\|_{W^{1,\infty}_p} \to s \to \infty 0,$$

where $f$ is the profile defined by

$$f(z) = (p - 1 + \frac{(p - 1)^2}{4p} |z|^2)^{-\frac{1}{p-1}}.$$  \hfill (2.8)

In order to prove this, we will not linearize equation (2.7) around $f + \frac{\kappa N}{2ps}$ as in [24], [5], but around

$$\varphi(y, s) = f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa N}{2ps} \chi_0\left(\frac{y}{g_\varepsilon(s)}\right),$$ \hfill (2.9)

where $\kappa = (p - 1)^{-\frac{1}{p-1}}$ is a stationary solution for equation (2.7), $\chi_0 \in C^\infty_0([0, \infty))$ with $supp(\chi_0) \subset [0, 2]$ and $\chi_0 \equiv 1$ on $[0, 1]$ and $g_\varepsilon(s) = s^{\frac{1}{2} + \varepsilon}$, $0 < \varepsilon < \min(1, \frac{p-1}{4})$.

We introduce now

$$v(y, s) = w(y, s) - \varphi(y, s).$$

If $w$ satisfies equation (2.7) then $v$ satisfies the following equation

$$v_s = (\mathcal{L} + V)v + B(v) + R(y, s) + N(y, s),$$ \hfill (2.10)

where

- the linear term is

$$\mathcal{L}(v) = \Delta v - \frac{1}{2} y . \nabla v + v \text{ with } V(y, s) = p\varphi^{p-1} - \frac{p}{p - 1},$$

- the nonlinear term is

$$B(v) = |v + \varphi|^{p-1}(v + \varphi) - \varphi^p - p\varphi^{p-1}v,$$

- the rest term involving $\varphi$ is

$$R(y, s) = \Delta \varphi - \frac{1}{2} y . \nabla \varphi - \frac{1}{p - 1} \varphi + \varphi^p - \varphi_s,$$

- and the new term is

$$N(y, s) = \mu e^{-\gamma s} |\nabla v + \nabla \varphi| \int_{B(0,|y|)} |v + \varphi|^{q-1}.$$
In comparison with the case of the equation without gradient \((\mu = 0)\), all the terms in \((2.10)\) were already present in [18], [25] and [3], except the new term \(N(y, s)\) which needs to be carefully studied.

In the following analysis, we will use the following integral form of equation \((2.10)\).

Let \(K\) be the fundamental solution of the operator \(L + V\), then for each \(s \geq \sigma \geq s_0\), we have

\[
v(s) = K(s, \sigma)v(\sigma) + \int_{\sigma}^{s} K(s, t)(B(v(t)) + R(t) + N(t))dt.
\]

(2.11)

Since the linear operator \(L + V\) will play an important role in our analysis, we first need to recall some of these properties (for more details, see [3]).

The operator \(L\) is self-adjoint in \(D(L) \subset L^2_{\rho}(\mathbb{R}^N)\), where

\[
L^2_{\rho}(\mathbb{R}^N) = \{v \in L^2_{loc}(\mathbb{R}^N); \int_{\mathbb{R}^N} (v(y))^2 \rho(y)dy < \infty\}, \quad \rho(y) = \frac{e^{-|y|^2}}{(4\pi)^{\frac{N}{2}}}.
\]

The spectrum of \(L\) consists only in eigenvalues given by

\[
\text{spec}(L) = \{1 - \frac{m}{2}; \quad m \in \mathbb{N}\}.
\]

The eigenfunction of \(L\) are derived from Hermite polynomials.

For \(N = 1\), all the eigenvalues are simple, and the eigenfunctions corresponding to \(1 - \frac{m}{2}\) is

\[
h_m(y) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}.
\]

(2.12)

In particular \(h_0(y) = 1, h_1(y) = y\) and \(h_2(y) = y^2 - 2\). Notice that \(h_m\) satisfies

\[
\int_{\mathbb{R}} h_n h_m \rho dx = 2^n n! \delta_{n,m}.
\]

We will note also \(k_m = \frac{h_m}{\|h_m\|_{L^2_{\rho}(\mathbb{R})}}\).

For \(N \geq 2\), the eigenspace corresponding to \(1 - \frac{m}{2}\) is given by

\[
E_m = \{h_{m_1}(y_1) \cdots h_{m_N}(y_N); \quad m_1 + \cdots + m_N = m\}.
\]

In particular,

\[
E_0 = \{1\}, \quad E_1 = \{y_i; \quad i = 1 \cdots N\} \quad \text{and} \quad E_2 = \{h_2(y_i), y_i y_j; \quad i, j = 1, \cdots, N, \ i \neq j\}.
\]

The potential \(V(y, s)\) has two fundamental properties:
• $V(.,s) \to 0$ in $L^2_\rho$ as $s \to +\infty$. In particular the effect of $V$ on the bounded sets or in the ”blow-up” area ($|y| \leq K\sqrt{s}$) is regarded as a perturbation of the effect of $L$.

• Outside of the ”blow-up” area, we have the following property: for all $\varepsilon > 0$ the exist $C_\varepsilon > 0$ and $s_\varepsilon$ such that

$$\sup_{s \geq s_\varepsilon, |y| \geq C_\varepsilon \sqrt{s}} |V(y,s) - (-p/p-1)| \leq \varepsilon.$$  

This means that $L + V$ behaves like $L - \frac{p}{p-1}$ in the region $|y| \geq K\sqrt{s}$. Because 1 is the biggest eigenvalue of $L$, the operator $L - \frac{p}{p-1}$ has a purely negative spectrum, which simplifies greatly the analysis in the outside of the ”blow-up” area.

Since the behavior of $V$ inside and outside the ”blow-up” area are different, we decompose $v$ as follows. We introduce the following cut-off function:

$$\chi(y,s) = \chi_0(\frac{|y|}{K\sqrt{s}}), \quad (2.13)$$

where $K > 0$ is chosen large enough so that various technical estimates hold. We write

$$v(y,s) = v_b(y,s) + v_e(y,s),$$

with

$$v_b(y,s) = v(y,s)\chi(y,s), \quad v_e(y,s) = v(y,s)(1 - \chi(y,s)).$$

We note that $supp v_b(s) \subset B(0,2K\sqrt{s})$ and $supp v_e(s) \subset \mathbb{R}^N \setminus B(0,K\sqrt{s})$.

In order to control $v_b$, we decompose it according to the sign of the eigenvalue of $L$ as follows:

$$v(y,s) = v_b(y,s) + v_e(y,s) = \sum_{m=0}^{2} v_m(s) h_m(y) + v_-(y,s) + v_e(y,s), \quad (2.14)$$

where for $0 \leq m \leq 2$, $v_m = P_m(v_b)$ and $v_-(s) = P_-(v_b)$, with $P_m$ being the $L^2_\rho$ projector on $h_m$, the eigenfunction corresponding to $1 - \frac{m}{2}$, and $P_-$ the projector on $\{h_i; \ i \geq 3\}$, the negative subspace of the operator $L$.

3 Existence

This section is devoted to the proof of the existence of a solution $v$ of (2.10) such that

$$\lim_{s \to +\infty} \|v(s)\|_{W^{1,\infty}_p} = 0.$$  

To do so, we use the framework developed in [18], [25], [23]. We proceed in two steps: Assuming some technical results, we prove in the first step the existence of
a solution \( v \) of \((2.10)\) which converges to 0 in \( W_p^{1,\infty} \). The second step is devoted to the proof of the technical details.

In what follows, we denote by \( C \) a generic positive constant, depending only on \( p, \mu \) and \( K \). Note that \( C \) does not depend on \( A \) and \( s_0 \), the constants that will appear below.

### 3.1 Proof of the existence

Let us explain briefly the general ideas of the proof. First, we define a shrinking set \( V_{A,p}(s) \) and translate our goal of making \( v(s) \) go to 0 in \( W_p^{1,\infty} \) in terms of belonging to \( V_{A,p}(s) \). Reasonably, we choose the initial data such that it starts in \( V_{A,p}(s_0) \).

Using the spectral properties of equation \((2.10)\), we reduce the problem from the control of all the components of \( v(s) \) in \( V_{A,p}(s) \) to the control of its two first components \((v_0(s),v_1(s))\). That is, we reduce an infinite dimensional problem to a finite dimensional one. Finally, we solve the finite dimensional problem using index theory.

#### 3.1.1 Definition of a shrinking set \( V_{A,p}(s) \) and preparation of the initial data

Let first introduce the shrinking set as follows:

**DEFINITION 3.1.** (A set shrinking to zero) For all \( A \geq 1 \) and \( s \geq 1 \), we define \( V_{A,p}(s) \) as the set of all function \( g \) such that

\[
|g_k(s)| \leq \frac{A}{s^2}, \quad k = 0, 1, \quad |g_2(s)| \leq \frac{A^2 \log s}{s^2}, \quad \|g_-(s)\|_{L^\infty} \leq \frac{A}{s^2}, \quad (3.15)
\]

\[
\|g_e(s)\|_{L^\infty} \leq \frac{A^2}{\sqrt{s}}, \quad \|(1 + |y|^{p-1})g_e(s)\|_{L^\infty} \leq \frac{A^2}{s^{\frac{p-1}{2}}} \quad (3.16).
\]

Note that the shrinking set is different from all the previous studies. Therefore, more estimates are needed. Since \( A \geq 1 \), we remark that the set \( V_{A,p}(s) \) is increasing (for fixed \( s, p \)) with respect to \( A \) in the sense of inclusion. We also show the following property of \( V_{A,p}(s) \):

For all \( A \geq 1 \), \( \exists s_{01} > 0 \) such that for all \( s \geq s_{01} \) and \( g \in V_{A,p}(s) \), we have

\[
\|(1 + |y|^{p-1})g(s)\|_{L^\infty} \leq \frac{CA^2}{s^{\frac{p-1}{2}}}, \quad (3.17)
\]

\[
\|g(s)\|_{L^\infty} \leq \frac{CA^2}{\sqrt{s}}. \quad (3.18)
\]

The construction of a solution \( v \) in \( V_{A,p}(s) \) is based on a careful choice of the initial data at a time \( s_0 \). Let us consider the initial data as follows:
**Definition 3.2.** (Choice of the initial data) For $A \geq 1$, $s_0 = -\log(T) > 1$ and $d_0, d_1 \in \mathbb{R}$, we consider the following function as initial data for equation (2.10):

$$
\psi_{s_0,d_0,d_1}(y) = \frac{A}{s_0}(d_0 h_0(y) + d_1 h_1(y)) \chi(2y, s),
$$

where $h_i, i = 0, 1$ are defined in (2.12) and $\chi$ is defined in (2.13).

Thus, a natural question arises: can we choose the initial data such that it starts in $V_{A,p}(s_0)$. For this end, we select the parameter $(d_0, d_1)$ as follows:

**Proposition 3.3.** (Properties of initial data) For each $A \geq 1$, there exists $s_0^2(A) > 1$ such that for all $s_0 \geq s_0^2(A)$:

1. There exists a rectangle $D_{s_0} \subset [-2, 2]^2$ such that the mapping

$$
\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2
$$

$$(d_0, d_1) \mapsto (\psi_0, \psi_1),
$$

(where $\psi := \psi_{s_0,d_0,d_1}$) is linear, one to one from $D_{s_0}$ onto $[-\frac{A}{s_0}, \frac{A}{s_0}]^2$ and maps $\partial D_{s_0}$ into $\partial([-\frac{A}{s_0}, \frac{A}{s_0}]^2)$. Moreover, it has degree one on the boundary.

2. For all $(d_0, d_1) \in D_{s_0}$, $\psi \in V_{A,p}(s_0)$ with strict inequalities except for $(\psi_0, \psi_1)$, in the sense that

$$
\psi_e \equiv 0, \quad |\psi_e(y)| < \frac{1}{s_0}(1 + |y|^3), \quad \forall y \in \mathbb{R},
$$

$$
|\psi_0| \leq \frac{A}{s_0}, \quad k = 0, 1, \quad |\psi_2| < \frac{\log s_0}{s_0^2}.
$$

3. Moreover, for all $(d_0, d_1) \in D_{s_0}$, we have

$$
\|(1 + |y|^{\frac{2}{p-1}}) \nabla \psi\|_{L^\infty} \leq \frac{CA}{s_0^{\frac{1}{p-1}}} \leq \frac{1}{s_0^{\frac{2}{p-1}}},
$$

$$
|\nabla \psi_e(y)| \leq \frac{1}{s_0^2}(1 + |y|^3).
$$

The proof of the previous proposition follows exactly as in [25] except for (3.23). Indeed, the new condition we have in the shrinking set has no influence, since it involves $\psi_e$ and $\psi_e \equiv 0$ by construction in (3.19). That is reason why the proof is omitted except for (3.23). (The interested reader can find details in pages 5915–5918 of [25]). Thus, we only prove (3.23) below.
The following proposition is crucial in the proof of the existence of the blow-up solution. We reduce the problem to a finite dimensional problem. As in [20], [5] and [25], we prove that it is enough to control \((v_0, v_1) \in \left[-\frac{A}{s_0}, \frac{A}{s_0}\right]^2\) in order to control the solution \(v(s) \in V_{A,p}(s)\), which is infinite dimensional.

**PROPOSITION 3.4.** There exists \(A_3 \geq 1\) such that for each \(A \geq A_3\), there exists \(s_{03}(A) \in \mathbb{R}\) such that for all \(s_0 \geq s_{03}(A)\), the following holds:

If \(v\) is a solution of (2.10) with initial data at \(s = s_0\) given by (3.19) with \((d_0, d_1) \in \mathbb{D}_{s_0}\), and \(v(s) \in V_{A,p}(s)\) for all \(s \in [s_0, s_1]\), with \(v(s_1) \in \partial V_{A,p}(s_1)\) for some \(s_1 \geq s_0\), then:

i) (Reduction to a finite dimensional problem) We have:

\[(v_0(s_1), v_1(s_1)) \in \partial \left([-\frac{A}{s_0}, \frac{A}{s_0}]^2\right)\]

ii) (Transverse crossing) There exist \(m \in \{0, 1\}\) and \(\omega \in \{-1, 1\}\) such that

\[\omega v_m(s_1) = \frac{A}{s_1} \quad \text{and} \quad \omega v'_m(s_1) > 0.\]

We give the proof of Proposition 3.4 in subsection 3.2.4.

We remark by (3.17) that if a solution \(v\) stays in \(V_{A,p}(s)\), for \(s \geq s_0\), then \((1 + |y|^\frac{2}{p-1})v(s)\) goes to 0 in \(L^\infty\). As mentioned above, our goal is to get the convergence in \(W_p^{1,\infty}\). Therefore, it remains to show that \(||(1 + |y|^\frac{2}{p-1})\nabla v||_{L^\infty} \to s \to \infty 0\). Thus, we need the following parabolic regularity of equation (2.10):

**PROPOSITION 3.5.** (Parabolic regularity in \(V_{A,p}(s)\))

For all \(A \geq 1\), there exists \(s_{04}(A)\) such that for all \(s_0 \geq s_{04}(A)\), if \(v\) the solution of (2.10) exists on \([s_0, s_1]\), \(s_0 \leq s_1\), with initial data at \(s_0\), given in (3.19) with \((d_0, d_1) \in \mathbb{D}_{s_0}\) define in Proposition 3.3 and \(v(s) \in V_{A,p}(s)\), then, for all \(s \in [s_0, s_1]\), we have

\[||(1 + |y|^\frac{2}{p-1})\nabla v(s)||_{L^\infty} \leq \frac{CA^2}{s_1^{\frac{1}{2} - \frac{2}{p-1}}}. \tag{3.25}\]

The proof of the previous proposition is postponed to subsection 3.2.3.

### 3.1.2 Proof of the existence of a solution in \(V_{A,p}(s)\)

We are going to prove the following existence result using the previous subsections.

**PROPOSITION 3.6.** There exists \(A_5 \geq 1\) such that for \(A \geq A_5\) there exists \(s_{05}(A)\) such that for all \(s_0 \geq s_{05}(A)\), there exists \((d_0, d_1)\) such that if \(v\) is the solution of (2.10) with initial data at \(s_0\), given in (3.19), then \(v(s) \in V_{A,p}(s)\), for all \(s \geq s_0\).
Proof. Let us consider $A \geq 1$, we fix $s_0 \geq \max(s_{01}, s_{02}, s_{03})$ and $(d_0, d_1) \in D_{s_0}$. The problem \eqref{2.10} with initial data at $s = s_0$, $\psi_{s_0,d_0,d_1}$ given in \eqref{3.19} has a solution $v(s)$. Indeed, using a fixed point argument, we prove the wellposedness for equation \eqref{1.1} in $W^{1,\infty}_p(\mathbb{R}^N)$ (we leave the proof to Appendix C).

According to Proposition \ref{3.3}, for each $(d_0, d_1) \in D_{s_0}$, $\psi_{s_0,d_0,d_1} \in V_{A,p}(s_0) \subset V_{A+1,p}(s_0)$ and from the existence theory, starting in $V_{A,p}(s_0)$ the solution $v(s)$ stays in $V_{A,p}(s)$ until some maximal time $s_* = s_0(d_0, d_1)$. We proceed by contradiction and assume that $s_*(d_0, d_1) < \infty$ for any $(d_0, d_1) \in D_{s_0}$. By definition of $s_*$, the solution at the point $s_*$, is on the boundary of $V_{A,p}(s_*)$ and $v(s) \in V_{A,p}(s)$, for all $s \in [s_0, s_*]$.

By Proposition \ref{3.3}, we see that $v(s_*)$ can leave $V_{A,p}(s_*)$ only by its first components, $(v_0(s_*), v_1(s_*)) \in \partial([-\frac{A}{s_*^2}, \frac{A}{s_*^2}]^2)$ and the following function is well defined

$$
\Phi : D_{s_0} \rightarrow \partial([-1, 1]^2)
$$

$$(d_0, d_1) \mapsto -\frac{s_*^2}{A}(v_0, v_1)(s_*).
$$

Using the transversality property of $(v_0, v_1)$ given in Proposition \ref{3.4} part $ii)$, we prove that $s_*(d_0, d_1)$ is continuous. Therefore, $\Phi$ is continuous.

From Proposition \ref{3.3}, we have that if $(d_0, d_1) \in \partial D_{s_0}$ then $v(s_0) \in V_{A,p}(s_0)$, $(v_0(s_0), v_1(s_0)) \in \partial([-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2)$ and we have strict inequalities for the other components.

Applying the transverse crossing property of $i)$ in Proposition \ref{3.3} we have that the restriction of $\Phi$ to the boundary is of degree 1.

We conclude that $\Phi$ is continuous and is of degree 1 on the boundary. Therefore, we have a contradiction from the degree theory. Thus, there exists a value $(d_0, d_1) \in D_{s_0}$ such that for all $s \geq s_0$, $v(s) \in V_{A,p}(s)$. This finishes the proof of Proposition \ref{3.6}.

$$
\square
$$

Since $v(s) \in V_{A,p}(s)$, we clearly see from \eqref{3.17} and \eqref{3.25} that

$$
\|(1 + |y|^{\frac{2}{p-1}})v(s)\|_{L^\infty} + \|(1 + |y|^{\frac{2}{p-1}})\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{\frac{4}{p-1}}}.
$$

\section{3.2 Proof of the technical results}

In this section, we prove the technical results used in the previous section. For simplicity in the notation, we give the proof in one dimension ($N = 1$). We proceed in 4 steps:

- In the first step, we prove estimate \eqref{3.23} Proposition \ref{3.3}

- In the second step, we prove that if $v(s) \in V_{A,p}(s)$, then $B(v)$, $R(y, s)$ and $N(y, s)$ given in \eqref{2.10} are trapped in $V_{C,A,p}(s)$ and the potential term $V(v(s) \in V_{C,A,p}(s)$, for some positive constant $C$. 

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In the third step, we prove the parabolic regularity result (Proposition 3.5).

In the last step, we prove the result of the reduction to a finite dimensional problem (Propostion 3.4).

3.2.1 Preparation of the initial data

In this subsection, we prove estimate (3.23) in Proposition 3.3 and refer the reader to pages 5915 – 5918 in [25] for the other items.

First, we give some properties of the shrinking set:

**Proposition 3.7.** For all $A \geq 1$, there exists $s_2$ such that, for all $s \geq s_2$ and $g \in V_{A,p}(s)$, we have

\begin{align*}
&i) \quad \|g\|_{L^\infty(|y| \leq 2Ks)} \leq \frac{CA}{\sqrt{s}} \quad \text{and} \quad \|g\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{\sqrt{s}}, \quad (3.27) \\
&ii) \quad \|(1 + |y|^{\frac{2}{p-1}})g\|_{L^\infty(|y| \leq 2Ks)} \leq \frac{CA}{s^{\frac{1}{2} - \frac{1}{p-1}}} \quad \text{and} \quad \|(1 + |y|^{\frac{2}{p-1}})g\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{\frac{1}{2} - \frac{1}{p-1}}} \quad (3.28)
\end{align*}

**Proof of Proposition 3.7**

Property $i)$ follows exactly as in [25], we refer the reader to Proposition 4.7 of [25] page 5915.

The first inequality of $ii)$ follows from estimate (3.27). For the second inequality of $ii)$, we decompose $g \in V_{A,p}(s)$ as follows

\[
g = \sum_{m=0}^{2} g_m h_m + g_- + g_e = g_b + g_e,
\]

where $g_e = g(1 - \chi)$, with $\chi$ defined in (2.13).

Using the fact that $g \in V_{A,p}(s)$, (2.12) and the fact that supp$(g_b) \subset \{ y, |y| < 2K\sqrt{s} \}$, we obtain

\[
\left| (1 + |y|^{\frac{2}{p-1}})g_b(y) \right| \leq C(1 + |y|^{\frac{2}{p-1}}) \left[ (1 + |y|) \frac{A}{s^2} + (1 + |y|^2) \frac{A^2 \log s}{s^2} + (1 + |y|^3) \frac{A}{s^2} \right] \\
\leq Cs^{\frac{1}{p-1}} \left[ (1 + 2K\sqrt{s}) \frac{A}{s^2} + (1 + (2K\sqrt{s})^2) \frac{A^2 \log s}{s^2} + (1 + (2K\sqrt{s})^3) \frac{A}{s^2} \right].
\]

We choose $s$ large enough, such that

\[
\left| (1 + |y|^{\frac{2}{p-1}})g_b(y) \right| \leq C \frac{A}{s^{\frac{1}{2} - \frac{1}{p-1}}}.
\]
Moreover, since $g \in V_{A,p}(s)$, for all $y \in \mathbb{R}$

$$|(1 + |y|^\frac{2}{p-1})g(y)| \leq \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$ 

Hence,

$$\|(1 + |y|^\frac{2}{p-1})g(y)\|_{L^\infty} \leq C \frac{A}{s^{\frac{1}{2} - \frac{1}{p-1}}} + \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}} \leq C \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$

This concludes the proof of Proposition 3.7.

In the following, we prove estimate (3.23) in Proposition 3.3.

**Proof of Proposition 3.3**

Since the initial data outside the blow-up area satisfies $\psi_e = 0$, we refer the reader to page 5917 of [25] for the proof of i), ii), except for (3.23), for which we give the details. By the definition of initial data and $h_0$, $h_1$, we see that

$$\nabla \psi(y) = d_1 \frac{A}{s_0^2} \chi(2y, s_0) + A \frac{d_0 + d_1 y}{s_0^2} \chi'(\frac{2y}{K\sqrt{s_0}}) \frac{2}{K\sqrt{s_0}}.$$ 

Since $\text{supp}(\psi) \subset \{|y| < 2K\sqrt{s_0}\}$ and $\|z\chi'_0(z)\|_{L^\infty}, \frac{2}{K\sqrt{s_0}}$ are bounded, we see that for $s_0$ large enough

$$\|(1 + |y|^\frac{2}{p-1})\nabla \psi(y)\|_{L^\infty} \leq C \frac{A}{s_0^{\frac{1}{2} - \frac{1}{p-1}}} \leq \frac{1}{s_0^{\frac{1}{2} - \frac{1}{p-1}}}.$$ 

This concludes the proof of Proposition 3.3.

### 3.2.2 Preliminary estimates on various term of equation (2.10)

In this subsection, we give various estimates on different terms appearing in equation (2.10). In particular, We prove that for $s$ large enough and some $C > 0$, the rest term $R(y,s)$ is in $V_{C,p}(s)$. We prove also that if $v(s) \in V_{A,p}(s)$, the nonlinear term $B(v) \in V_{C,p}(s)$ and the potential term $Vv$ is in $V_{C,A,p}(s)$. In addition, we prove that the new term $N(y,s)$ is trapped in $V_{C,p}(s)$ under some additional assumptions in $v$.

**Lemma 3.8.**

1. There exists $s_3$ sufficiently large such that for $s \geq s_3$, the rest term $R \in V_{C,p}(s)$.

2. For all $A \geq 1$, there exists $s_4$ sufficiently large such that for $s \geq s_4$, if $v(s) \in V_{A,p}(s)$, then the nonlinear term $B(v) \in V_{C,p}(s)$ and the potential term $Vv \in V_{C,A,p}(s)$. 

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Proof. These terms are not new, for this reason we need only to give the estimate for the terms outside the blow-up area. For the other terms, we refer to subsection 4.2.2 page 5918 – 5923 in [25].

1. Estimate on the rest term.

We note that, since 

\[- \frac{1}{2}zf'(z) - \frac{1}{p-1}f(z) + f(z)^p = 0,\]

we write \(R(y, s)\) as follows

\[
R(y, s) = \frac{1}{s}f''(z) + \frac{1}{2s}zf'(z) + (f(z) + \chi_0(Z)\frac{\kappa}{2ps})^p - f(z)^p + \frac{\kappa}{2ps} \frac{1}{2} \frac{\chi''(z)}{g_e(s)}Z\chi_0'(Z) + (\frac{1}{s} - \frac{1}{p-1})\chi_0(Z) \]

\[
= R_i + R_{ii} + R_{iii},
\]

where \(z = \frac{y}{\sqrt{s}}\), \(Z = \frac{y}{g_e(s)}\) and \(g_e(s) = s^{\frac{1}{2} + \epsilon}\). Note that \(|y| \geq K\sqrt{s}\) on the support of \(R_e\).

By definition (2.8) of \(f\), we have

\[
f(z) = (p - 1 + \frac{(p-1)^2}{4p}z^2)^{-\frac{1}{p-1}} \sim_{z \to \infty} (\frac{(p-1)^2}{4p})^{-\frac{1}{p-1}} z^{-\frac{2}{p-1}} \] (3.29)

\[
f'(z) \sim_{z \to \infty} \frac{-2}{p-1} \left(\frac{(p-1)^2}{4p}\right)^{-\frac{1}{p-1}} z^{-\frac{p+1}{p-1}} \] (3.30)

\[
f''(z) \sim_{z \to \infty} 2 \frac{p+1}{(p-1)^2} \left(\frac{(p-1)^2}{4p}\right)^{-\frac{1}{p-1}} z^{-\frac{2p}{p-1}}. \] (3.31)

In particular, there exists \(K_0\) such that if \(|z| \geq K_0\), then

\[
|R_i| \leq \frac{C}{s|z|^{\frac{2}{p-1}}}.\]

Hence,

\[
(1 + |y|^{\frac{2}{p-1}})|R_i| \leq \frac{C}{s^{1 - \frac{2}{p-1}}}. \] (3.32)

On the other hand, we write

\[
R_{ii} = f(z)^p \left(1 + \frac{\kappa\chi_0(Z)}{2psf(z)}\right)^p - 1). \]

Using (3.29), we get

\[
\frac{1}{s} \frac{\chi_0(Z)}{f(z)} \sim_{z \to \infty} Cz^{\frac{2}{p-1}} \frac{s^{\frac{1}{2} + \epsilon}}{s} \chi_0(z \sqrt{s} g_e(s)).
\]
Since $\chi_0(z) \sqrt{s}$ is bounded in $\{|z| \leq 2\frac{g(s)}{\sqrt{s}} = 2s^\varepsilon\}$, we deduce that

$$\frac{1}{s} \frac{\chi_0(f(z))}{s} \sim_{z \to \infty} \frac{C}{s^{1-\frac{2\varepsilon}{p-1}}}.$$  

Therefore,

$$(1 + \frac{\kappa \chi_0(f(z))}{2p s f(z)})^p - 1 \sim_{z \to \infty} \frac{Cp}{s^{1-\frac{2\varepsilon}{p-1}}}.$$  

Moreover, if $\varepsilon \leq \frac{p-1}{4}$, then

$$(1 + |y|^{\frac{2}{p-1}})|R_{ii}| \leq \frac{C}{s^{\frac{1}{2} - \frac{2\varepsilon}{p-1}}} \frac{1}{s^{\frac{1}{2} - \frac{1}{p-1}}} \leq \frac{C}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$

Finally, for the last term $R_{iii}$, since $\chi_0(f(z))$ and its derivatives are bounded and $K \sqrt{s} \leq |y| \leq 2K g(s)$, we see that

$$R_{iii} \leq C \left[ \frac{1}{sg^2_e(s)} + \frac{|y|}{sg_e(s)} + \frac{1}{s} + \frac{|y|g'_e(s)}{sg^2_e(s)} + \frac{1}{s^2} \right] \leq \frac{C}{s}.$$  

Hence,

$$(1 + |y|^{\frac{2}{p-1}})|R_{iii}| \leq \frac{C}{s^{\frac{1}{2} - \frac{2\varepsilon}{p-1}}} \frac{1}{s^{\frac{1}{2} - \frac{1}{p-1}}} \leq \frac{C}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$  

Collecting all these bounds yields the bound for $R_e(s)$ as follows

$$(1 + |y|^{\frac{2}{p-1}})|R_e(s)| \leq \frac{C}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$  

2. The nonlinear term.

Since we have the same definition of $B$ as in [18], we have the following estimates (for the proof we refer to Lemma 3.6 page 168 in [18])

$$B(v) \leq C|v|^\tilde{p}, \quad \tilde{p} = \min(p, 2).$$  

Because $p > 3$, we find that

$$|B_e(v)| = (1 - \chi)|B(v)| \leq C|v||v_e|.$$  

From the fact that $v \in V_{A,p}(s)$, we have for $s$ large enough

$$(1 + |y|^{\frac{2}{p-1}})|B_e(v)| \leq C \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}} \frac{CA^2}{\sqrt{s}} \leq \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}}.$$
3. The potential term.
We recall that, by definition of $V$ and a Taylor expansion, we easily prove
that, for $s$ large enough
$$\|V(s)\|_{L^\infty} \leq C.$$ For more details, we refer to Lemma 4.10 page 5918 – 5921 in [25]. Then, using the above inequality and the fact that $v \in \mathcal{V}_{A,p}(s)$, we get,

$$\begin{align*}
(1 + |y|^{\frac{2}{p-1}})|Vv| & \leq \|V\|_{L^\infty}(1 + |y|^{\frac{2}{p-1}})|v| \\
& \leq C \frac{A^2}{s^{\frac{1}{2}} - \frac{1}{p-1}}.
\end{align*}$$

This concludes the proof of Lemma 3.8.

We now estimate the new term. We claim the following Proposition:

**PROPOSITION 3.9.** For all $A \geq 1$, there exists $s_5$, sufficiently large, such that
for all $s \geq s_5$, if $v \in \mathcal{V}_{A,p}(s)$ is such that

$$\begin{align*}
\|\nabla v\|_{L^\infty} & \leq C \sqrt{s}, \\
\|1 + |y|^{\frac{2}{p-1}}\nabla v\|_{L^\infty} & \leq \frac{C}{s^{\frac{1}{2}} - \frac{1}{p-1}},
\end{align*}$$

(3.36)

then $N \in \mathcal{V}_{C,p}(s)$, for some positive constant.

Before proving this Proposition, we need the following Lemma:

**LEMMA 3.10.** Under the assumption of Proposition 3.9, we have, for $s$ sufficiently large

1. $\|N(y, s)\|_{L^\infty} \leq Ce^{-\frac{7}{2}s}$,
2. $\|1 + |y|^{\frac{2}{p-1}}Nv(y, s)\|_{L^\infty} \leq Ce^{-\frac{7}{2}s},$

where $C$ is a positive constant.

**Proof.** Recall that

$$N(y, s) = \mu e^{-\gamma s} \nabla(v + \varphi) \int_{B(0,|y|)} |v + \varphi|^{q-1},$$

where $\varphi(y, s) = f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps} \chi_0(\frac{y}{g_x(s)})$, $\nabla \varphi = -\frac{p - 1}{2p} \frac{1}{\sqrt{s}} \frac{1}{\sqrt{s}} \frac{f'(\frac{y}{\sqrt{s}})}{\sqrt{s} \sqrt{s} f'(\frac{y}{\sqrt{s}})} + \frac{\kappa}{2psg_x(s)} \chi'_0(\frac{g_x(s)}{y})$

and $\gamma = \frac{p - 4}{p - 1} > 0$.

Since $zf(x)$ and $\chi'_0(z)$ are bounded, we get

$$\|\nabla \varphi\|_{L^\infty} \leq \frac{C}{\sqrt{s}}.$$
Thus for
\[ \| \nabla (\varphi + v) \|_{L^\infty} \leq \frac{C}{\sqrt{s}}. \]

Hereafter, we assume \( y \geq 0 \) for simplicity. Therefore, we get
\[ \int_{B(0,|y|)} |v + \varphi|^{q-1} \leq C(\int_{B(0,|y|)} |v|^{q-1} + \int_{B(0,|y|)} |\varphi|^{q-1}) = C(I(v) + I(\varphi)). \]

First, we decompose \( I(v) \) as follows:
\[ I(v) = \int_{B(0,|y|)} |v|^{q-1}1_{\{|y| \leq K \sqrt{s} \}} + \int_{B(0,|y|)} |v|^{q-1}1_{\{|y| \geq K \sqrt{s} \}} = I_b(v) + I_e(v). \]

Since \( \text{supp}(v_b) \subset \{|y| \leq 2K \sqrt{s} \} \) and \( \text{supp}(v_e) \subset \{|y| \geq K \sqrt{s} \} \), we have
\[ I_b(v) \leq \int_{B(0,K \sqrt{s})} |v_b|^{q-1}. \]

Using the fact that \( v \in \mathcal{V}_{A,p}(s) \), we get for \( 0 \leq y' \leq y \leq K \sqrt{s} \)
\[ |v_b(y')|^{q-1} \leq \left( \frac{A}{s^2} (1 + |y'|^3) + \frac{A^2 \log s}{s^2} \right)^{q-1} \]
\[ \leq C \frac{A}{\sqrt{s}} + \frac{A^2 \log s}{s^2} \]

Thus for \( s \) large enough
\[ I_b(v) \leq C \frac{A^{q-1}}{s^{2-q}}. \] (3.38)

On the other hand, we decompose \( I_e(v) \) as follows:
\[ I_e(v) = \left[ \int_{B(0,|y|)} |v(y')|^{q-1}1_{\{|y'| \leq K \sqrt{s} \}} dy' + \int_{B(0,|y|)} |v(y')|^{q-1}1_{\{|y'| \geq K \sqrt{s} \}} dy' \right] 1_{\{|y| \geq K \sqrt{s} \}} \]
\[ = \left[ \int_{B(0,K \sqrt{s})} |v(y')|^{q-1} dy' + \int_{K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^{q-1} dy' \right] 1_{\{|y| \geq K \sqrt{s} \}}. \]

As in \( I_b(v) \), we find that
\[ \int_{B(0,K \sqrt{s})} |v(y')|^{q-1} dy' = \int_{B(0,K \sqrt{s})} |v_b(y')|^{q-1} dy' \leq C \frac{A^{q-1}}{s^{2-q}}. \] (3.39)

For the second term, we need a further decomposition:
\[ \int_{K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^{q-1} dy' = \int_{K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^{q-1} dy' 1_{\{|y| \geq 2K \sqrt{s} \}} \]
\[ + \left[ \int_{K \sqrt{s} \leq |y'| \leq 2K \sqrt{s}} |v(y')|^{q-1} dy' + \int_{2K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^{q-1} dy' \right] 1_{\{|y| \geq 2K \sqrt{s} \}}. \]
If $K \sqrt{s} \leq |y'| \leq |y| \leq 2K \sqrt{s}$, then $v(y') = v_b(y') + v_c(y')$ and from (3.18), we have
\[
\int_{K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^q \, dy' 1_{\{K \sqrt{s} \leq |y'| \leq 2K \sqrt{s}\}} \leq \int_{K \sqrt{s} \leq |y'| \leq 2K \sqrt{s}} |v(y')|^q \, dy' \leq C \frac{A^{2(q-1)}}{s^{\frac{q-1}{2}}}.
\tag{3.40}
\]
If $2K \sqrt{s} \leq |y'| \leq |y|$, then $v(y') = v_c(y')$ and
\[
\int_{2K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^q \, dy' \leq \frac{A^{2(q-1)}}{s^{\frac{q-1}{2}}} \, \frac{1}{(1 + |y'|)^{\frac{q-1}{p-1}}} \, dy'.
\tag{3.41}
\]

Since $2 \frac{q-1}{p-1} > 1$, we have $\int_{\mathbb{R}} \frac{dy'}{(1 + |y'|)^{\frac{q-1}{p-1}}} < \infty$. Therefore,
\[
\int_{2K \sqrt{s} \leq |y'| \leq |y|} |v(y')|^q \, dy' \leq C \frac{A^{2(q-1)}}{s^{\frac{q-1}{2}}}.
\tag{3.40}
\]

From (3.39), (3.40) and (3.41), we deduce that, for $s$ sufficiently large
\[
I_c(v) \leq C \frac{A^{2(q-1)}}{s^{\frac{q-1}{2}}}.
\]

It follows from (3.38) and the above estimates, for $s$ sufficiently large, that
\[
I(v) \leq C \frac{A^{2(q-1)}}{s^{\frac{q-1}{2}}},
\]

It remains to estimate $I(\varphi)$. We see that
\[
I(\varphi) \leq \int_{B(0,|y|)} |f\left(\frac{y'}{\sqrt{s}}\right)|^{q-1} \, dy' + \int_{B(0,|y|)} |\frac{\kappa}{2ps} \chi_0\left(\frac{y'}{g_c(s)}\right)|^{q-1} \, dy'.
\]

On the one hand,
\[
\int_{B(0,|y|)} |f\left(\frac{y'}{\sqrt{s}}\right)|^{q-1} \, dy' = \sqrt{s} \int_{B(0,|z|)} \left( p - 1 + \frac{(p-1)^2}{4p} z^2 \right)^{-\frac{q-1}{p-1}} \, dz', \quad \text{where} \quad z = \frac{y}{\sqrt{s}}.
\]

Since $(p - 1 + \frac{(p-1)^2}{4p} z^2)^{-\frac{q-1}{p-1}} \sim_{+\infty} C z^{-2 \frac{q-1}{p-1}}$ and $2 \frac{q-1}{p-1} > 1$, we get
\[
\int_{B(0,|y|)} |f\left(\frac{y'}{\sqrt{s}}\right)|^{q-1} \, dy' \leq C \sqrt{s}.
\]

On the other hand, making the change of variable $z' = \frac{y'}{g_c(s)}$ and using the boundedness of $\chi_0$, we obtain
\[
\int_{B(0,|y|)} \left|\frac{\kappa}{2ps} \chi_0\left(\frac{y'}{g_c(s)}\right)\right|^{q-1} \, dy' \leq \frac{g_c(s)}{s^{q-1}}.
\]

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Using the fact that $g_\varepsilon(s) = s^{\frac{1}{q}-\varepsilon}$, with $\varepsilon < 1$ and $q > 2$, we obtain for $s$ large enough
\[ I(\varphi) \leq C\sqrt{s}. \]

This yields
\[ \int_{B(0,|y|)} |v + \varphi|^{q-1} \leq C(I(v)I(\varphi)) \leq C\left[ \frac{A^{2(q-1)}}{s^{\frac{q-1}{p-1}} + \sqrt{s}} \right]. \tag{3.42} \]

Collecting all these bounds yields the bound for $N(y,s)$ as follows
\[ |N(y,s)| \leq C\frac{e^{-\gamma s}}{\sqrt{s}} \left[ \frac{A^{2(q-1)}}{s^{\frac{q-1}{p-1}} + \sqrt{s}} \right]. \]

Exploiting the fact that $\gamma > 0$ and $s$ sufficiently large, we obtain the desired estimate. Let us now estimate $\| (1 + |y|^{\frac{2}{p-1}})N_\varepsilon(y,s) \|_{L^\infty}$. We see that
\[ |(1 + |y|^{\frac{2}{p-1}})N_\varepsilon(y,s)| \leq \mu e^{-\gamma s}(1 + |y|^{\frac{2}{p-1}})\nabla(\varphi + v)(1 - \chi)(I(v) + I(\varphi)). \]

Therefore, we write
\[ (1 + |y|^{\frac{2}{p-1}})\nabla \varphi = (1 + |y|^{\frac{2}{p-1}}) \left( -\frac{2(p-1)}{4p} \frac{1}{\sqrt{s}} \frac{f'(y/s)}{\sqrt{y/s}} + \frac{\kappa}{2psg_\varepsilon(s)} \chi'(y/g_\varepsilon(s)) \right). \]

We use a Taylor expansion, we obviously obtain
\[ (1 + |y|^{\frac{2}{p-1}}) \frac{1}{\sqrt{s}} \left( \frac{y}{\sqrt{s}} f'(y/s) \right) \sim \frac{C}{s^{\frac{1}{2}-\frac{1}{p-1}}} \left( \frac{y}{\sqrt{s}} \right)^{\frac{p+1}{p-1}} \sim \frac{C}{s^{\frac{1}{2}-\frac{1}{p-1}}} \left( \frac{y}{\sqrt{s}} \right)^{-1}, \]

which yields
\[ |(1 + |y|^{\frac{2}{p-1}}) \frac{1}{\sqrt{s}} \left( \frac{y}{\sqrt{s}} f'(y/s) \right)| \leq \frac{C}{s^{\frac{1}{2}-\frac{1}{p-1}}}. \]

By the definition of $\chi_0$ and $g_\varepsilon$, we derive
\[ |(1 + |y|^{\frac{2}{p-1}}) \frac{\kappa}{2psg_\varepsilon(s)} \chi'(y/g_\varepsilon(s))| \leq C \frac{g_\varepsilon(s)^{\frac{p+1}{p-1}}}{s} \leq C \frac{1}{s^{\frac{1}{2}-\frac{1}{p-1}}} \frac{1}{s^{\frac{1}{2}-\frac{1}{p-1}}}. \]

Since $\varepsilon < \frac{p-1}{4}$, we have for $s$ sufficiently large
\[ |(1 + |y|^{\frac{2}{p-1}}) \frac{\kappa}{2psg_\varepsilon(s)} \chi'(y/g_\varepsilon(s))| \leq \frac{C}{s^{\frac{1}{2}-\frac{1}{p-1}}}. \]

Hence
\[ |(1 + |y|^{\frac{2}{p-1}})\nabla \varphi| \leq \frac{C}{s^{\frac{1}{2}-\frac{1}{p-1}}}. \tag{3.43} \]

From (3.42) and (3.43), we deduce that
\[ |(1 + |y|^{\frac{2}{p-1}})N_\varepsilon(y,s)| \leq C \frac{e^{-\gamma s}}{s^{\frac{1}{2}-\frac{1}{p-1}}} \left[ \frac{A^{2(q-1)}}{s^{\frac{q-1}{p-1}} + \sqrt{s}} \right]. \]

Using again the fact $\gamma > 0$ and $s$ large enough, we conclude the proof of Lemma 3.10. \qed
We now give the proof of Proposition 3.9.

**Proof of Proposition 3.9**

By definition of $N_m$, $0 \leq m \leq 2$, we have

$$|N_m(s)| = \left| \int_{\mathbb{R}} N(y, s)k_m(y)\rho dy \right| \leq C e^{-\frac{\gamma}{2}s} \int_{\mathbb{R}} |k_m(y)|\rho dy \leq C_m e^{-\frac{\gamma}{2}s}.$$ 

Since $\gamma > 0$ and $s$ is sufficiently large, we obtain

$$|N_m(s)| \leq \frac{C}{s^2}, \text{ for } 0 \leq m \leq 1,$$

and

$$|N_2(s)| \leq C^2 \log \frac{s}{s^2}.$$ 

Moreover, we have that

$$|N_b(y, s)| = |N(y, s)\chi(y, s)| \leq C|N(y, s)| \leq C e^{-\frac{\gamma}{2}s}.$$ 

Furthermore, by the definition of $N_{-}(y, s)$, we see that

$$|N_{-}(y, s)| = |N_b(y, s) - \sum_{m=0}^{2} N_m(s)h_m(y)| \leq C e^{-\frac{\gamma}{2}s}(1 + |y|^3).$$

Hence,

$$\left\| \frac{N_{-}(y, s)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{C}{s^2}.$$ 

On the other hand, for $s$ sufficiently large, we have

$$|N_e(y, s)| = |N(y, s)(1 - \chi(y, s))| \leq 2C e^{-\frac{\gamma}{2}s} \leq \frac{C^2}{\sqrt{s}}.$$ 

Finally by Lemma 3.10, for $s$ sufficiently large, we obtain

$$\left| (1 + |y|^\frac{2}{p-1})N_e(y, s) \right| \leq C e^{-\frac{\gamma}{2}s} \leq \frac{C^2}{s^{\frac{2}{p-1}}}. $$

This finishes the proof of Proposition 3.9. 

3.2.3 Parabolic regularity

In this subsection, we prove Proposition 3.5. The proof follows as in [25], with some additional care, since we have a different shrinking set and a different nonlinear term. The proof relies mainly on some properties of the semi-group $e^{\theta\mathcal{L}}$: 

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LEMMA 3.11. The Kernel $e^{θL}(y,x)$ of the semi-group $e^{θL}$ is given by

$$e^{θL}(y,x) = \frac{e^θ}{\sqrt{4π(1-e^{-θ})}} \exp\left[-\frac{(ye^{-θ/2} - x)^2}{4(1-e^{-θ})}\right],$$

(3.44)

for all $θ > 0$, and $e^{θL}$ is defined by

$$e^{θL}r(y) = \int_\mathbb{R} e^{θL}(y,x)r(x)dx.$$  

(3.45)

We have the following estimates:

1. If $r_1 \leq r_2$, then $e^{θL}r_1 \leq e^{θL}r_2$.

2. i) If $r \in W^{1,∞}(\mathbb{R})$, then $∥∇(e^{θL}r)∥_{L∞} \leq Ce^θ∥∇r∥_{L∞}$.

ii) If $r \in L^∞(\mathbb{R})$, then $∥∇(e^{θL}r)∥_{L∞} \leq \frac{Ce^θ}{\sqrt{1-e^{-θ}}} ∥r∥_{L∞}$.

3. For $m ≥ 0$, if $|r(x)| ≤ μ(1 + |x|^m)$, $∀x ∈ \mathbb{R}$, then

i) $|e^{θL}r(y)| ≤ Cμe^θ(1 + |y|^m)$, $∀y ∈ \mathbb{R}$.

ii) $|∇(e^{θL}r(y))| ≤ Cμ\frac{e^θ}{\sqrt{1-e^{-θ}}} (1 + |y|^m)$, $∀y ∈ \mathbb{R}$.

4. For $m ≥ 0$, if $|∇r(x)| ≤ μ(1 + |x|^m)$, $∀x ∈ \mathbb{R}$, then $|∇(e^{θL}r(y))| ≤ Cμe^θ(1 + |y|^m)$, $∀y ∈ \mathbb{R}$.

5. For $0 < m < 1$, we have

i) If $(1 + |y|^m)r \in W^{1,∞}(\mathbb{R})$, then

$$∥(1 + |y|^m)∇(e^{θL}r)(y)∥_{L∞} \leq Ce^θ∥(1 + |y|^m)∇r∥_{L∞}.$$  

ii) If $(1 + |y|^m)r \in L^∞(\mathbb{R})$, then

$$∥(1 + |y|^m)∇(e^{θL}r)(y)∥_{L∞} \leq C\frac{e^θ}{\sqrt{1-e^{-θ}}} ∥(1 + |y|^m)r∥_{L∞}.$$  

Proof. Because estimates 1) – 4) are not new, we refer the reader to Lemma 4.15 page 5926 in [25]. See also [3] page 554 – 555. Thus, we only prove 5). In order to avoid necessary technicalities here, we prove these in the Appendix A. □
We are now going to prove Proposition 3.5.

**Proof of Proposition 3.5**

Let $A \geq 1$, $s_0 \geq 1$ and consider $v(s)$ a solution of equation (2.10) defined on $[s_0, s_1]$, where $s_1 \geq s_0 \geq 1$ and $v(s_0) = \psi$ defined in (3.19) with $(d_0, d_1) \in D_s$ defined in Proposition 3.3. We assume in addition that $v(s) \in V_{A,p}(s)$, for all $s \in [s_0, s_1]$. We distinguish two cases.

**Case 1:** If $s \leq s_0 + 1$. Let $s' = \min(s_0 + 1, s_1)$ and take $s \in [s_0, s']$. Then we have for any $t \in [s_0, s]$

$$s_0 \leq t \leq s \leq s_0 + 1 \leq 2s_0,$$ hence $\frac{1}{s} \leq \frac{1}{t} \leq \frac{1}{s_0} \leq \frac{2}{s}$.

From equation (2.10), we write for any $s \in [s_0, s']$

$$v(s) = e^{(s-s_0)\xi} v(s_0) + \int_{s_0}^{s} e^{(s-t)\xi} F(t) dt,$$ (3.46)

where

$$F(y, t) = Vv(y, t) + B(v) + R(y, t) + N(y, t).$$

From Lemma 3.11, we see that for all $s \in [s_0, s']$

$$\| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s) \|_{L^{\infty}} \leq \left\| (1 + |y|^{\frac{2}{p-1}}) \nabla e^{(s-s_0)\xi} v(s_0) \right\|_{L^{\infty}}$$

$$+ \int_{s_0}^{s} \| (1 + |y|^{\frac{2}{p-1}}) \nabla e^{(s-t)\xi} F(t) \|_{L^{\infty}} dt,$$

$$\leq C \| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s_0) \|_{L^{\infty}}$$

$$+ C \int_{s_0}^{s} \left\| (1 + |y|^{\frac{2}{p-1}}) F(t) \right\|_{L^{\infty}} dt.$$ (3.49)

Using Proposition 3.3, we obtain

$$\| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s_0) \|_{L^{\infty}} \leq \frac{CA}{s_0^{2-p-1}}.$$ (3.48)

Since $s \leq 2s_0$, we deduce that

$$\| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s_0) \|_{L^{\infty}} \leq \frac{CA}{s^{2-p-1}}.$$ (3.47)

We now estimate $\| (1 + |y|^{\frac{2}{p-1}}) F(t) \|_{L^{\infty}}$: we write

$$(1 + |y|^{\frac{2}{p-1}}) F(y, t) = (1 + |y|^{\frac{2}{p-1}})(Vv(y, t) + B(v) + R(y, t)) + (1 + |y|^{\frac{2}{p-1}})N(y, t).$$
From Lemma 3.8, we see that $R, B(v) \in V_{C, p}(t)$, and $V v \in V_{C A, p}(t)$. Therefore by Proposition 3.7 we have

$$
\|(1 + |y|^{\frac{2}{p-1}})(V v + B(v) + R)\|_{L^\infty} \leq C \frac{A^2}{t^\frac{1}{2} - \frac{1}{p-1}}. \tag{3.50}
$$

Furthermore, using inequality (3.42) and (3.43), we see that

$$
\|(1 + |y|^{\frac{2}{p-1}})N\|_{L^\infty} \leq C e^{-\gamma t} \left( \frac{C}{t^\frac{1}{2} - \frac{1}{p-1}} + \|(1 + |y|^{\frac{2}{p-1}})\nabla v\|_{L^\infty} \right) \left( \frac{A^{2(q-1)}}{t^{\frac{1}{2} - \frac{1}{p-1}}} + \sqrt{t} \right).
$$

If we assume $s_0$ large enough, and consider $t \in [s_0, s]$, then we have

$$
\|(1 + |y|^{\frac{2}{p-1}})N\|_{L^\infty} \leq C e^{-\gamma t} \left( \frac{1}{t^\frac{1}{2} - \frac{1}{p-1}} + \|(1 + |y|^{\frac{2}{p-1}})\nabla v\|_{L^\infty} \right). \tag{3.51}
$$

Collecting all these bounds and using the fact that $t \leq s \leq 2t$, we obtain for all $t \in [s_0, s]$,

$$
\|(1 + |y|^{\frac{2}{p-1}})F(t)\|_{L^\infty} \leq C \frac{A^2}{s^\frac{1}{2} - \frac{1}{p-1}} + C e^{-\frac{\gamma s}{2}} \|(1 + |y|^{\frac{2}{p-1}})\nabla v\|_{L^\infty}. \tag{3.52}
$$

Therefore,

$$
\|(1 + |y|^{\frac{2}{p-1}})\nabla v(s)\|_{L^\infty} \leq C \frac{A}{s^{2 - \frac{1}{p-1}}} + C \frac{A^2}{s^\frac{1}{2} - \frac{1}{p-1}} + C e^{-\frac{\gamma s}{2}} \int_{s_0}^s \|(1 + |y|^{\frac{2}{p-1}})\nabla v\|_{L^\infty} dt.
$$

Using a Gronwall’s argument, we obtain that for $s_0$ large enough

$$
\|(1 + |y|^{\frac{2}{p-1}})\nabla v(s)\|_{L^\infty} \leq C \frac{A^2}{s^\frac{1}{2} - \frac{1}{p-1}}, \quad \forall s \in [s_0, s_1].
$$

**Case 2:** If $s > s_0 + 1$. Take $s \in [s_0 + 1, s_1]$ and rewrite equation (3.46) for any $s' \in (s - 1, s]$ (use the fact that $s \geq s_0 + 1 \geq 2$ and $s = s - 1 + 1 \leq 2(s - 1)$).

More precisely, we rewrite

$$
v(s') = e^{(s' - s_1)\xi} v(s - 1) + \int_{s - 1}^{s'} e^{(s' - t)\xi} F(t) dt.
$$

From Lemma 3.11 we see that for all $s \in [s_0, s']$

$$
\|(1 + |y|^{\frac{2}{p-1}})\nabla v(s')\|_{L^\infty} \leq \|(1 + |y|^{\frac{2}{p-1}}) e^{(s' - s_1)\xi} v(s - 1)\|_{L^\infty} + \int_{s - 1}^{s'} \|(1 + |y|^{\frac{2}{p-1}}) e^{(s' - t)\xi} F(t)\|_{L^\infty} dt
\leq C \frac{1}{\sqrt{1 - e^{-(s' - s_1)}}} \|(1 + |y|^{\frac{2}{p-1}} v(s - 1)\|_{L^\infty} + C \int_{s - 1}^{s'} \|(1 + |y|^{\frac{2}{p-1}}) F(t)\|_{L^\infty} dt. \tag{3.53}
$$
Since \( v \in V_{A,p} \) and \( \frac{s'}{2} < s - 1 < s' < s \), we have

\[
\| (1 + |y|^{\frac{2}{p-1}}) v(s-1) \|_{L^\infty} \leq \frac{CA^2}{(s-1)^{\frac{1}{2} - \frac{1}{p-1}}} \leq \frac{CA^2}{(s')^{\frac{1}{2} - \frac{1}{p-1}}}.
\]

\[
\| (1 + |y|^{\frac{2}{p-1}}) F(t) \|_{L^\infty} \leq C \frac{A^2}{t^{\frac{1}{2} - \frac{1}{p-1}}} + ce^{-\frac{s'}{2}} \| (1 + |y|^{\frac{2}{p-1}}) \nabla v(t) \|_{L^\infty} \leq C \frac{A^2}{(s')^{\frac{1}{2} - \frac{1}{p-1}}} + ce^{-\frac{s'}{2}} \| (1 + |y|^{\frac{2}{p-1}}) \nabla v(t) \|_{L^\infty}.
\]

Therefore,

\[
\| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s') \|_{L^\infty} \leq C \frac{A^2}{(s')^{\frac{1}{2} - \frac{1}{p-1}}} \sqrt{1 - e^{-(s'-s+1)}} + Ce^{-\frac{s'}{2}} \int_{s-1}^{s'} \frac{\| (1 + |y|^{\frac{2}{p-1}}) \nabla v \|_{L^\infty}}{\sqrt{1 - e^{-(s'-s+1)}}} \, dt.
\]

Using a Gronwall’s argument, we see that

\[
\| (1 + |y|^{\frac{2}{p-1}}) \nabla v(s') \|_{L^\infty} \leq 2C \frac{A^2}{(s')^{\frac{1}{2} - \frac{1}{p-1}}} \sqrt{1 - e^{-(s'-s+1)}}, \quad \forall s' \in [s - 1, s].
\]

Taking \( s' = s \), we conclude the proof of Proposition 3.5.

3.2.4 Reduction to a finite dimensional problem

This subsection is crucial in the proof of our result. It is dedicated to the proof of Proposition 3.4. In this subsection, we reduce the problem to a finite dimensional one. We prove through a priori estimates that the control of \( v(s) \) in \( V_{A,p}(s) \) is reduced to the control of \( (v_0, v_1)(s) \) in \( [-\frac{A}{s^2}, \frac{A}{s^2}]^2 \).

For this end, we project Equation (2.10) on the different components of the decomposition (2.14) and we get new bounds on all components of \( v \):

PROPOSITION 3.12. There exists \( A_6 \geq 1 \) such that for all \( A \geq A_6 \), there exists \( s_0, A_6(A) \) large enough such that the following holds for all \( s_0 \geq s_0, A \):

Assume that for some \( s_1 \geq \sigma \geq s_0 \), we have

\[
v(s) \in V_{A,p}(s), \quad \forall s \in [\sigma, s_1],
\]

and that \( \nabla v \) satisfies the estimate stated in Proposition 3.3. Then, the following holds for all \( s \in [\sigma, s_1] \),

i) (ODE satisfied by the positive mode) : For \( m \in \{0, 1\} \), we have

\[
|v_m'(s) - (1 - \frac{m}{2})v_m(s)| \leq \frac{C}{s^2}.
\]
ii) (ODE satisfied by the null mode) : We have

\[ |v'_2(s) + \frac{2}{s}v_2(s)| \leq \frac{C}{s^2}. \]

iii) (Control of the negative and outer modes): We have

\[
\|v-(s)\|_{L^\infty} \leq Ce^{-\frac{s-\sigma}{p}}\|v-(\sigma)\|_{L^\infty} + Ce^{\frac{(s-\sigma)^2}{s^2(\frac{1}{p}-1)}}\|v_e(\sigma)\|_{L^\infty} + C\frac{1 + (s-\sigma)e^{s-\sigma}}{\sqrt{s}}.
\]

iv) (Control of the term outside the blow-up area in the new functional space):

\[
\|(1 + |y|^{\frac{2}{p}-1})v_e(s)\|_{L^\infty} \leq Ce^{-\frac{s-\sigma}{p}}\|(1 + |y|^{\frac{2}{p}-1})v_e(\sigma)\|_{L^\infty} + Ce^{\frac{2\sigma}{s^2(\frac{1}{p}-1)}}\|v_e(\sigma)\|_{L^\infty} + C\frac{1 + (s-\sigma)e^{s-\sigma}}{s^{\frac{1}{2} - \frac{1}{p}-1}}.
\]

Proof. Note that estimates i) – iii) are stated in [25] for an equation lacking the new term \(N\). Since the new term \(N\) satisfies Proposition 3.9 and Lemma 3.10, the reader can adapt easily the proof of [25] to the new situation. For this reason, we only prove the estimate iv).

We write the integral form:

\[
v(s) = K(s,\sigma)v(\sigma) + \int_{\sigma}^{s} K(s,t)(B(v(t)) + R(t) + N(t))dt.
\]

The proof is given in two steps: In the first step, we need to understand the behavior of the Kernel \(K(s,\sigma)\), which plays an important role in estimating the new components of \(v\). In the second step, we use these estimates to give new bounds on different terms appearing in (3.54) and conclude the proof.

**Step 1:** It is clear that the kernel \(K(s,\sigma)\) has stronger influence in this formula. For this reason, it is convenient to give the following result of Bricmont and Kupiainen [3] (Note that estimate 3.ii) is new, crucial and ours):

**Lemma 3.13.** We have the following estimates for all \(1 \leq \sigma \leq s \leq 2\sigma\):

1. For all \(x, y \in \mathbb{R}\), \(K(s,\sigma,y,x) \leq Ce^{(s-\sigma)\xi}(y,x)\).

2. For all \(y \in \mathbb{R}\), we have \(|\int K(s,\sigma,y,x)1_{\{|x| \geq K\sqrt{\sigma}\}}dx| \leq Ce^{-\frac{s-\sigma}{p}}\).

3. For all \(m \geq 0\), \(y \in \mathbb{R}\), we have
LEMMA 3.14. For all ρ > 0, there exists σ₀ = σ₀(ρ) such that if σ ≥ σ₀ ≥ 1 and g(σ) satisfies

\[ \sum_{m=0}^{2} |g_m(σ)| + \frac{\|g_{-}(y, σ)\|}{1 + |y|^3} \|L_{∞} + \|g_e(σ)\|_{L_{∞}} < +∞, \]

then θ(s) = K(s, σ)g(σ) satisfies for all s ∈ [σ, σ + ρ],

1. \[ |\theta_2(s)| \leq \left( \frac{σ}{s} \right)^2 |g_2(σ)| + C \frac{s - α}{s} \left( \sum_{l=0}^{2} |g_l(σ)| + \frac{g_{-}(y, σ)}{1 + |y|^3} \right) \|L_{∞} \right) + C(s - σ)e^{-\frac{s}{2}} \|g_e(σ)\|_{L_{∞}}, \]

2. \[ \left\| \frac{θ_{-}(y, s)}{1 + |y|^3} \right\|_{L_{∞}} \leq C e^{s - σ} \left( (s - σ)^2 + 1 \right) \left( |g_0(σ)| + |g_1(σ)| + \sqrt{s} |g_2(σ)| \right) \]

+ Ce^{-\frac{s - σ}{2}} \|g_{-}(y, σ)\|_{1 + |y|^3} \|L_{∞} \right) + C \frac{e^{-(s - σ)^2}}{s^2} \|g_e(σ)\|_{L_{∞}}.

3. \[ |\theta_e(s)| \leq C e^{s - σ} \left( \sum_{l=0}^{2} s^\frac{l}{2} |g_l(σ)| + s^\frac{3}{2} \|g_{-}(y, σ)\|_{1 + |y|^3} \right) \|L_{∞} \right) + C e^{-\frac{s - σ}{ρ}} \|g_e(σ)\|_{L_{∞}}.

4. \[ \left\| \left( 1 + |y|^\frac{2}{ρ - 1} \right) θ_e(y, s) \right\|_{L_{∞}} \leq C e^{-\frac{s - σ}{ρ}} \left\| (1 + |y|^\frac{2}{ρ - 1}) g_e(y, σ) \right\|_{L_{∞}} \]

+ Ce^{-\frac{s - σ}{ρ}} s^\frac{2}{ρ - 1} \left( \sum_{l=0}^{2} s^\frac{l}{2} |g_l(σ)| + s^\frac{3}{2} \|g_{-}(y, σ)\|_{1 + |y|^3} \right) \|L_{∞}).

Proof. The proof of 1 – 3 is very close to that in [3] and [23]. We therefore give the sketch of the proof only for part 4). We write

\[ (1 + |y|^\frac{2}{ρ - 1}) θ_e(y, s) = (1 + |y|^\frac{2}{ρ - 1})(1 - χ(y, s))K(s, σ)(g_e(σ) + g_b(σ)) \]

For the first term, we remark that

\[ K(s, σ)g_e(σ) = \int \frac{K(s, σ, x)}{1 + |x|^\frac{2}{ρ - 1}} 1_{\{|x| ≥ K\sqrt{σ}\}}(1 + |x|^\frac{2}{ρ - 1}) g_e(x, σ) dx \]
Then, using part 3. ii) for Lemma 3.13 we obtain

\[ \| (1 + |y|^{\frac{2}{p+1}}) (1 - \chi(y, s)) K(s, \sigma) g_e(\sigma) \|_{L^\infty} \leq C e^{-\frac{\pi s}{p}} \| (1 + |y|^{\frac{2}{p+1}}) g_e(\sigma) \|_{L^\infty}. \]

For the second term, we use a Feynman-Kac representation for \( K \):

\[ K(s, \sigma, y, x) = e^{(s-\sigma)\xi}(y, x) \int d\nu^{s-\sigma}_{yx}(\omega) \exp \left( \int_0^{s-\sigma} V(\omega(t), \sigma + t) dt. \right) \]

We remark that

\[ \int d\nu^{s-\sigma}_{yx}(\omega) \exp \left( \int_0^{s-\sigma} V(\omega(t), \sigma + t) dt \leq C, \right. \]

and

\[ e^{(s-\sigma)\xi}(y, x) = \frac{e^{s-\sigma}}{\sqrt{4\pi(1 - e^{-(s-\sigma)})}} \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{4(1 - e^{-(s-\sigma)})}). \]

We distinguish two cases:

**Case 1:** If \( x \in A = \{ x; |ye^{-\frac{1}{2}x} - x| \geq \frac{|x|}{2} \} \), we first write

\[ \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{4(1 - e^{-(s-\sigma)})}) = \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{8(1 - e^{-(s-\sigma)})}) \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{8(1 - e^{-(s-\sigma)})}) \]

On the one hand, we remark that

\[ \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{8(1 - e^{-(s-\sigma)})}) \leq \exp(\frac{-ca^2}{1 - e^{-(s-\sigma)}}). \]

On the other hand, we write

\[ (ye^{-\frac{1}{2}x} - x)^2 = \left(\frac{4}{5} ye^{-\frac{1}{2}x} - x\right) \left(\frac{6}{5} ye^{-\frac{1}{2}x} - x\right) + \left(\frac{ye^{-\frac{1}{2}x}}{5}\right)^2, \]

we obtain easily that \( \left(\frac{4}{5} ye^{-\frac{1}{2}x} - x\right) \left(\frac{6}{5} ye^{-\frac{1}{2}x} - x\right) \geq 0 \), therefore

\[ \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{8(1 - e^{-(s-\sigma)})}) \leq \exp(\frac{-\left(\frac{1}{5} ye^{-\frac{1}{2}x}\right)^2}{8(1 - e^{-(s-\sigma)})}). \]

Since \( |y| \geq K\sqrt{s} \), we see that for \( s \) large enough

\[ \exp(\frac{-(ye^{-\frac{1}{2}x} - x)^2}{8(1 - e^{-(s-\sigma)})}) \leq \frac{C}{1 + |y|^{\frac{2}{p+1}}}. \]

Thus, for \( s \) large enough, we get

\[ (1 + |y|^{\frac{2}{p+1}}) (1 - \chi(y, s)) \int_A K(s, \sigma, y, x) 1_{\{|x| \leq 2K\sqrt{s}\}} dx \leq C. \]
Case 2: If \( x \in \mathbb{R} \setminus \mathbb{A} \), we obviously obtain
\[
1 + |y|^{\frac{2}{p-1}} \leq 1 + \left( \frac{5}{4} K \right)^{\frac{2}{p-1}} e^{\frac{s - \sigma}{2}} s^{\frac{1}{p-1}}.
\]

\[
(1 + |y|^{\frac{2}{p-1}})(1 - \chi(y, s)) \int_{\mathbb{R} \setminus \mathbb{A}} K(s, \sigma, y, x) 1_{\{|x| \leq 2K \sqrt{\sigma}\}} dx \leq C e^{\frac{s - \sigma}{2}} s^{\frac{1}{p-1}}.
\]

Collecting the above estimates, we obtain for \( s \) large enough
\[
\|(1 + |y|^{\frac{2}{p-1}})(1 - \chi(y, s)) K(s, \sigma) g_b(\sigma)\|_{L^\infty} \leq C e^{\frac{s - \sigma}{2}} s^{\frac{1}{p-1}} \left( \sum_{l=0}^{2} s^{\frac{l}{2}} |g_l(\sigma)| + s^{\frac{3}{2}} \| \frac{g_-(y, s)}{1 + |y|^{\frac{1}{3}}} \|_{L^\infty} \right).
\]

This concludes the proof of Lemma 3.14. \( \square \)

**Step 2:** Applying the above lemma, we get a new bound on all terms in the decomposition (3.54). More precisely, we have the following:

**Lemma 3.15.** There exists \( A_7 > 0 \) such that for all \( A \geq A_7 \) and \( \rho > 0 \), there exists \( s_0(A, \rho) > 0 \) with the following property: for all \( s_0 \geq s_0(A, \rho) \), assume that for all \( s \in [\sigma, \sigma + \rho] \), \( v(s) \) satisfies (2.10), \( v(s) \in \mathcal{V}_{A, p}(s) \) and \( \nabla v \) satisfies (3.36). Then, we have:

1. **Linear term:**

\[
\left\| \frac{A - (y, s)}{1 + |y|^3} \right\|_{L^\infty} \leq C e^{-\frac{s}{2}(s - \sigma)} \left\| \frac{v_-(y, \sigma)}{1 + |y|^3} \right\|_{L^\infty} + C e^{\frac{s - \sigma}{2}} \left\| v_e(\sigma) \right\|_{L^\infty} + \frac{C}{s^2},
\]

\[
\| A_e \|_{L^\infty} \leq C e^{-\frac{s}{p}} \left\| v_e(\sigma) \right\|_{L^\infty} + C e^{s - \sigma} s^{\frac{1}{2}} \left\| \frac{v_-(y, \sigma)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C}{\sqrt{s}}.
\]

\[
\|(1 + |y|^{\frac{2}{p-1}}) A_e(s)\|_{L^\infty} \leq C e^{-\frac{s}{p}} \left\| (1 + |y|^{\frac{2}{p-1}}) v_e(\sigma) \right\|_{L^\infty} + C e^{\frac{s - \sigma}{2}} s^{\frac{1}{2}} \left\| \frac{v_-(y, s)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C}{s^{\frac{1}{2}} - \frac{1}{p-1}}.
\]

2. **Nonlinear source term:**

\[
\left\| \frac{B_e(y, s)}{1 + |y|^3} \right\|_{L^\infty} \leq C \frac{s^2}{s^2} (s - \sigma), \quad \| B_e(s) \|_{L^\infty} \leq C \frac{1 + e^{\frac{s}{p-1}}(s - \sigma)}{s^{\frac{1}{2}} - \frac{1}{p-1}}.
\]

3. **Corrective term:**

\[
\left\| \frac{C_e(y, s)}{1 + |y|^3} \right\|_{L^\infty} \leq C \frac{s^2}{s^2} (s - \sigma), \quad \| C_e(s) \|_{L^\infty} \leq C \frac{1 + e^{\frac{s}{p-1}}(s - \sigma)}{s^{\frac{1}{2}} - \frac{1}{p-1}}.
\]

\[
\|(1 + |y|^{\frac{2}{p-1}}) C_e(s)\|_{L^\infty} \leq C \frac{1 + e^{\frac{s}{p-1}}(s - \sigma)}{s^{\frac{1}{2}} - \frac{1}{p-1}}.
\]
4. New term:

\[
\|D_e^-(y,s)\|_{L^\infty} \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}, \quad \|D_e(s)\|_{L^\infty} \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}.
\]

\[
\|(1 + |y|^{\frac{2}{p-1}})D_e(s)\|_{L^\infty} \leq C(s - \sigma)e^{-\frac{s}{2}s}(1 + s^{\frac{1}{p-1}}e^{\frac{s-\sigma}{p-1}}).
\]

**Proof.** The proof of 1 – 3 follows by a simple modification of the argument in [25], Lemma 4.20. Therefore, we sketch only the proof of the new term \(D(s) = \int_\sigma^s K(s,t)N(t)dt\). Using Lemma [3.13] and Lemma [3.10] we deduce that

\[
|D(s)| \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}.
\]

In particular, for \(0 \leq i \leq 2\)

\[
|D_i(s)| \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}.
\]

We write

\[
|D^-_s(s)| = |D_b(s) - \sum_{i=0}^{2} D_i(s)k_i(y)| \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}(1 + |y| + |y|^2) \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}(1 + |y|^3).
\]

Also, we see that

\[
|D_e(s)| = |D(s) - D_b(s)| \leq 2|D(s)| \leq C(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s}.
\]

Finally, using Lemma [3.14] Part 4)

\[
\|(1 + |y|^{\frac{2}{p-1}})D_e(s)\|_{L^\infty} \leq \int_\sigma^s \|(1 + |y|^{\frac{2}{p-1}})(1 - \chi(y,s))K(s,t,y,x)N(t)\|_{L^\infty}dt \\
\leq C \int_\sigma^s e^{-\frac{s-\sigma}{p-1}}\|(1 + |y|^{\frac{2}{p-1}})N_e(y)\|_{L^\infty}dt \\
+ C \int_\sigma^s e^{\frac{s-\sigma}{p-1}t^{\frac{1}{p-1}}}(\sum_{l=0}^{2} t^\frac{1}{p-1} N_l(t))\|_{L^\infty} + t^\frac{1}{p-1} \|N_l(y)\|_{L^\infty} dt
\]

This yields the part 4 and concludes the proof of Lemma [3.15].

Now, we remark that if we choose \(s\) large enough such that

\[
(s - \sigma)e^{s-\sigma}e^{-\frac{s}{2}s} \leq \frac{1}{s^2}, \quad (s - \sigma)e^{-\frac{s}{2}s}(1 + s^{\frac{1}{p-1}}e^{\frac{s-\sigma}{p-1}}) \leq \frac{1}{s^2},
\]

we conclude the proof of Proposition [3.12].
Let us now give the proof of Proposition 3.4.

**Proof of Proposition 3.4**

We sketch only the proof of part i), since part ii) is follows exactly as in [25].

Let \( v \) be a solution of equation (2.10) with initial data \( \psi_{s_0, d_0, d_1} \) given by (3.19) with \( (d_0, d_1) \in D_{s_0} \) defined in Proposition 3.3, such that \( v(s) \in \mathcal{V}_{A,p}(s) \) for all \( s \in [s_0, s_1] \) with \( v(s_1) \in \partial \mathcal{V}_{A,p}(s_1) \). Our goal is to prove that

\[
|v_2(s_1)| < \frac{A^2 \log s_1}{s_1^2}, \quad \|v_-(y, s_1)\|_{L^\infty} < \frac{A}{s_1^2} \tag{3.55}
\]

\[
\|v_c(s_1)\|_{L^\infty} < \frac{A^2}{\sqrt{s_1}}, \quad \|1 + |y|^{\frac{2}{p-1}})v_c(s_1)\|_{L^\infty} < \frac{A^2}{s_1^{\frac{2}{p-1}}}. \tag{3.55}
\]

We prove only the last inequality, and refer the reader to [25] for the proof of the other estimates.

Let \( \rho = \log(A^\alpha), \alpha < 1 \) such that

\[
\rho \leq s_0 < \sigma < s = \sigma + \rho < 2\sigma. \tag{3.56}
\]

We distinguish two cases:

**Case 1:** \( s > s_0 + \rho \). Hence, \( \sigma = s - \rho > s_0 \) and from Proposition 3.12 part iv), we have

\[
\|1 + |y|^{\frac{2}{p-1}}v_c(s)\|_{L^\infty} \leq C e^{-\frac{\rho \sigma}{p}} \|1 + |y|^{\frac{2}{p-1}}v_c(\sigma)\|_{L^\infty} + Ce^{-\frac{\rho \sigma}{p}} s_0^{\frac{1}{2} - \frac{1}{p-1}} \|v_-(y, \sigma)\|_{L^\infty} \leq C e^{-\frac{\rho \sigma}{p}} s_0^{\frac{1}{2} - \frac{1}{p-1}} \]

\[
+ Ce^\frac{\rho (s - \rho) e^{\frac{\rho}{p-1} \sigma}}{s_0^{\frac{1}{2} - \frac{1}{p-1}}} \leq C A^2 e^{-\frac{s_0}{p}} A e^\frac{\rho \sigma}{p-1} + 1 + \rho e^\frac{\rho}{p-1} \leq C A^2 e^{-\frac{s_0}{p}} A e^\frac{\rho \sigma}{p-1} + 1 + \log(A^\alpha) A^\alpha e^{-\frac{s_0}{p}} A^\alpha e^\frac{\rho \sigma}{p-1}.
\]

Since \( \alpha < 1 \) and \( p > 3 \), taking \( A \) sufficiently large, we see that

\[
\|1 + |y|^{\frac{2}{p-1}}v_c(s)\|_{L^\infty} \leq \frac{1}{2} A^2 s_0^{\frac{1}{2} - \frac{1}{p-1}}.
\]

**Case 2:** \( s < s_0 + \rho \). Clearly, from this choice, we have \( s_0 < s < s_0 + \rho < 2s_0 \).

If we choose \( \sigma = s_0 \), then \( v_c(s_0) = \psi_c(s_0) = 0 \). Using Proposition 3.3 part ii) and Proposition 3.12 part iv), we have

\[
\|1 + |y|^{\frac{2}{p-1}}v_c(s)\|_{L^\infty} \leq C e^\frac{\rho}{p-1} + 1 + \rho e^\frac{\rho}{p-1} \leq C A^2 s_0^{\frac{1}{2} - \frac{1}{p-1}}
\]

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Taking $A$ sufficiently large, such that

$$C(A^{p-1} + 1 + \log(A^\alpha)) \leq \frac{A^2}{2},$$

we conclude that for all $s \in [s_0, s_1]$

$$\|(1 + |y|^{\frac{2}{p-1}})v_\epsilon(s)\|_{L^\infty} \leq \frac{A^2}{2} \frac{A^2}{s^{\frac{1}{2} - \frac{1}{p-1}}}. $$

In particular, in the two cases, we have

$$\|(1 + |y|^{\frac{2}{p-1}})v_\epsilon(s_1)\|_{L^\infty} < \frac{A^2}{s_1^{\frac{1}{2} - \frac{1}{p-1}}}. $$

Moreover, since $v(s_1) \in \partial V_{A,p}(s_1)$, we see that $(v_0(s_1), v_1(s_1)) \in \partial \left[ -\frac{A}{s_1^{\frac{1}{2}}}, A \right]$ and part $\iota$ of Proposition 3.4 is proved.

4 Proof of Theorem 1.1 and Corollary 1.4

In this section, we prove our main result, using the previous subsections. We recall that, from Proposition 3.5 and Proposition 3.6, we obtain the existence of a solution $v$ of equation (2.10) defined for all $y \in \mathbb{R}$ and $s \geq s_0$, for some $s_0 > 1$ such that $v(s) \in \mathcal{V}_{A,p}(s)$. More precisely, we have

$$\|(1 + |y|^{\frac{2}{p-1}})v(s)\|_{L^\infty} + \|(1 + |y|^{\frac{2}{p-1}})\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{\frac{1}{2} - \frac{1}{p-1}}}. $$

Thus,

$$\|(1 + |y|^{\frac{2}{p-1}})(w(y, s) - \varphi(y, s))\|_{L^\infty} + \|(1 + |y|^{\frac{2}{p-1}})\nabla (w(y, s) - \varphi(y, s))\|_{L^\infty} \leq \frac{C}{s^{\frac{1}{2} - \frac{1}{p-1}}}, $$

where $\varphi$ is the profile introduced in (2.9). Let us first estimate this profile. We give the following Lemma:

**Lemma 4.1.** 1. For all $K_0 > 0$ and $y$ such that $|y| \geq 2K_0 \sqrt{s}$, we have

i) $|f\left(\frac{y}{\sqrt{s}}\right)| \sim \frac{s^{\frac{1}{p-1}}}{1 + |y|^{\frac{2}{p-1}}} \text{ as } \frac{y}{\sqrt{s}} \to +\infty \text{ and } s \to +\infty.$

ii) $\frac{1}{\sqrt{s}} f'(\frac{y}{\sqrt{s}}) \leq \frac{C}{(1 + |y|^{\frac{2}{p-1}})s^{\frac{1}{2} - \frac{1}{p-1}}}.$
2. For all $y \in \mathbb{R}$,

\[ i) \quad |\frac{\kappa}{2ps} \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right)| \leq \frac{C}{(1 + |y|^{\frac{2}{p-1}}) s^{\frac{1}{p-1}}}. \]

\[ ii) \quad |\frac{\kappa}{2ps} \frac{1}{g_\varepsilon(s)} \chi'_0 \left( \frac{y}{g_\varepsilon(s)} \right)| \leq \frac{C}{(1 + |y|^{\frac{2}{p-1}}) s^{\frac{1}{p-1}}}. \]

Proof. For the proof of 1), we recall that $f(z) = (p - 1 + bz^2)^{-\frac{1}{p-1}}$, where $z = \frac{y}{\sqrt{s}}$.

If $z$ is large enough, there exists a constant $C$ such that

\[ \frac{1}{C} z^{-\frac{2}{p-1}} \leq (p - 1 + bz^2)^{-\frac{1}{p-1}} \leq Cz^{-\frac{2}{p-1}}. \]

Thus,

\[ \frac{1}{C} \left| \frac{y}{z} \right|^{\frac{2}{p-1}} \leq (1 + |y|^{\frac{2}{p-1}})(p - 1 + bz^2)^{-\frac{1}{p-1}} \leq 2C \left| \frac{y}{z} \right|^{\frac{2}{p-1}}. \]

This concludes the proof of item i).

We now present the proof of item ii). We write

\[ \frac{1}{\sqrt{s}} f'(\frac{y}{\sqrt{s}}) = -\frac{2b}{p - 1} \left( p - 1 + b^2 \right)^{-\frac{1}{p-1}} \]

\[ = -\frac{2b}{p - 1} \frac{1}{\sqrt{s}} \left( 1 + \frac{p - 1}{b^2} \right)^{-\frac{1}{p-1}}. \]

Since $|z| \geq 2K_0$, we have

\[ \left| \frac{1}{\sqrt{s}} f'(\frac{y}{\sqrt{s}}) \right| \leq \frac{C}{K_0^3} \left| \frac{z}{\sqrt{s}} \right|^{-\frac{2}{p-1}}. \]

Thus, ii) is proved.

For the proof of 2), we see that

\[ (1 + |y|^{\frac{2}{p-1}}) \frac{1}{s} \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right) \leq \frac{C}{s} (1 + |y|^{\frac{2}{p-1}}) 1_{\{|y| \leq 2Kg_\varepsilon(s)\}} \leq \frac{C}{s} g_\varepsilon(s)^{\frac{2}{p-1}} \leq \frac{C}{s^{\frac{1}{p-1}}}. \]

Since $\varepsilon < \frac{p-1}{4}$, we get the conclusion of 2) i). The proof of ii) follows exactly as above. This concludes the proof of Lemma 4.1. □

Using the above Lemma and inequality (1.57), we deduce that

\[ \|(1 + |y|^{\frac{2}{p-1}})(w(y, s) - f(y, s))\|_{L^\infty} + \|(1 + |y|^{\frac{2}{p-1}}) \nabla (w(y, s) - f(y, s))\|_{L^\infty} \leq \frac{C}{s^{\frac{1}{p-1}}}. \]
In particular, the solution $u$ of equation (1.1) defined for all $x \in \mathbb{R}$ and $t \in [0, T)$ satisfies

$$|(T - t)^{\frac{1}{p-1}} u(x, t) - f(\frac{x}{\sqrt{(T - t)|\log(T - t)|}})| \leq \frac{C}{(1 + (\frac{|x|^2}{T - t})^{\frac{1}{p-1}})|\log(T - t)|^{\frac{1}{2} - \frac{1}{p-1}}},$$

hence $\lim_{t \to T} (T - t)^{\frac{1}{p-1}} u(0, t) = (p - 1)^{\frac{1}{p-1}}$ and $u$ blows up at time $t = T$ at the origin and

$$\|u\|_{L^\infty} \leq C \frac{(T - t)^{-\frac{1}{p-1}}}{|\log(T - t)|^{\frac{1}{2} - \frac{1}{p-1}}}.$$

On the other hand, we remark that

$$|(T - t)^{\frac{1}{p-1}} \nabla u(x, t) - \frac{1}{\sqrt{|\log(T - t)|}|\log(T - t)|}\nabla (\frac{x}{\sqrt{(T - t)|\log(T - t)|}})| \leq \frac{C}{(1 + (\frac{|x|^2}{T - t})^{\frac{1}{p-1}})|\log(T - t)|^{\frac{1}{2} - \frac{1}{p-1}}}.$$

Thus, $\lim_{t \to T} (T - t)^{\frac{1}{p-1}} \nabla u(0, t) = 0$ and $\nabla u$ blows up at time $T$ at the origin and

$$\|\nabla u\|_{L^\infty} \leq C \frac{(T - t)^{-\frac{1}{p-1}}}{|\log(T - t)|^{\frac{1}{2} - \frac{1}{p-1}}}.$$

Finally, we prove $ii)$ of Theorem 1.1. We assume $x_0 > 0$, for all $x \in B(x_0, \frac{\sqrt{2}}{2})$, we have

$$(T - t)^{\frac{1}{p-1}} |u(x, t)| \leq \frac{C}{|\log(T - t)|^{\frac{1}{2} - \frac{1}{p-1}}} + |f(\frac{x}{\sqrt{(T - t)|\log(T - t)|}})|.$$

From the definition of the profile, we have

$$|f(\frac{x}{\sqrt{(T - t)|\log(T - t)|}})| \leq c(\frac{x}{2\sqrt{(T - t)|\log(T - t)|}})^{-\frac{2}{p-1}}.$$

Thus, we can choose $t_0(x_0, \varepsilon_0)$ sufficiently large such that $0 < T - t_0(x_0, \varepsilon_0) = \delta << 1$ and

$$\frac{C}{|\log \delta|^{\frac{1}{2} - \frac{1}{p-1}}} + (\frac{2\sqrt{\delta} |\log \delta|}{x_0})^{\frac{2}{p-1}} \varepsilon_0.$$

This concludes the proof of Theorem 1.1.

The corollary 1.4 is obtained immediately from Theorem 1.1 and Lemma 4.1.
Appendix A

In this appendix, we give the proof of Lemma 3.11. As mentioned earlier, we will give the proof only of part 5), since the other estimates are proved in Lemma 4 of [3] and Lemma 4.15 of [25].

By definition \((3.44)\), we write

\[
\left( e^{\theta L} r \right)(y) = \int_{\mathbb{R}} \frac{e^{\theta}}{\sqrt{4\pi (1-e^{-\theta})}} e^{\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}} r(x) dx
\]

where \(z = ye^{-\theta/2} - x\), and

\[
\nabla \left( e^{\theta L} r \right)(y) = \int_{\mathbb{R}} \frac{e^{\theta}}{\sqrt{4\pi (1-e^{-\theta})}} e^{\frac{z^2}{4(1-e^{-\theta})}} \nabla r(ye^{-\theta/2} - z) dz.
\]

Since

\[
|y|^m \leq C (|ye^{-\theta/2} - z|^m + |z|^m),
\]

we have

\[
|(1 + |y|^m) \nabla (e^{\theta L} r)(y)| \leq \frac{Ce^\vartheta}{\sqrt{4\pi (1-e^{-\theta})}} \int_{\mathbb{R}} \frac{e^{\frac{z^2}{4(1-e^{-\theta})}}}{\sqrt{4\pi (1-e^{-\theta})}} \nabla r(ye^{-\theta/2} - z) dz
\]

\[
+ \int_{\mathbb{R}} |z|^m e^{\frac{z^2}{4(1-e^{-\theta})}} \nabla r(ye^{-\theta/2} - z) dz.
\]

Since

\[
\frac{C e^\vartheta}{\sqrt{4\pi (1-e^{-\theta})}} \int_{\mathbb{R}} \frac{e^{\frac{z^2}{4(1-e^{-\theta})}}}{\sqrt{4\pi (1-e^{-\theta})}} dz
\]

\[
+ \frac{C e^\vartheta}{\sqrt{4\pi (1-e^{-\theta})}} \int_{\mathbb{R}} |z|^m e^{\frac{z^2}{4(1-e^{-\theta})}} dz
\]

\[
\leq C e^\vartheta \frac{\| (1 + |y|^m) \nabla r(y) \|_{L^\infty}}{\sqrt{4\pi (1-e^{-\theta})}} \int_{\mathbb{R}} \frac{e^{\frac{z^2}{4(1-e^{-\theta})}}}{\sqrt{4\pi (1-e^{-\theta})}} dz
\]

\[
+ \frac{\| \nabla r(y) \|_{L^\infty}}{\sqrt{4\pi (1-e^{-\theta})}} \int_{\mathbb{R}} \frac{|z|^m e^{\frac{z^2}{4(1-e^{-\theta})}}}{\sqrt{4\pi (1-e^{-\theta})}} dz.
\]

We remark that, for \(\alpha \geq 0\), we get

\[
\int_{\mathbb{R}} \frac{|z|^\alpha e^{\frac{z^2}{4(1-e^{-\theta})}}}{\sqrt{4\pi (1-e^{-\theta})}} dz \leq C (1 - e^{-\theta})^\frac{\alpha}{2} \leq C.
\]

This yields part \(i\) of 5).

In order to prove the part \(ii\), we rewrite

\[
\nabla \left( e^{\theta L} r \right)(y) = \int_{\mathbb{R}} \frac{e^{\theta}}{\sqrt{4\pi (1-e^{-\theta})}} \frac{-e^{\frac{\vartheta}{2}}(ye^{-\frac{\vartheta}{2}} - x)}{2(1-e^{-\theta})} e^{-\frac{(ye^{-\frac{\vartheta}{2}} - x)^2}{4(1-e^{-\theta})}} r(x) dx.
\]
If we make the change of variable $z = ye^{-\frac{\theta}{2}} - x$, we obtain
\[
\nabla(e^{\theta L}r(y)) = 2\int_{\mathbb{R}} \frac{e^{\frac{\theta}{2}}}{\sqrt{\pi (4(1-e^{-\theta})^2)}} ze^{-\frac{z^2}{4(1-e^{-\theta})}} r(ye^{-\frac{\theta}{2}} - z)dz.
\]

In particular,
\[
(1 + |y|^m)\nabla(e^{\theta L}r(y)) \leq c \int_{\mathbb{R}} \frac{e^{\frac{\theta}{2}}}{\sqrt{\pi (4(1-e^{-\theta})^2)}} |z| e^{-\frac{z^2}{4(1-e^{-\theta})}} (1 + |ye^{-\frac{\theta}{2}} - z|^m) r(ye^{-\frac{\theta}{2}} - z)dz
\]
\[
+ c \int_{\mathbb{R}} \frac{e^{\frac{\theta}{2}}}{\sqrt{\pi (4(1-e^{-\theta})^2)}} |z|^{m+1} e^{-\frac{z^2}{4(1-e^{-\theta})}} r(ye^{-\frac{\theta}{2}} - z)dz.
\]

Using the fact that for all $\alpha \geq 0$, \[
\int_{\mathbb{R}} \frac{|z|}{\sqrt{1-e^{-\theta}}} |z| e^{-\frac{z^2}{4(1-e^{-\theta})}} dz \leq C,
\] we obtain
\[
\|(1 + |y|^m)\nabla(e^{\theta L}r(y))\|_{L^\infty} \leq C \left( \frac{e^{\frac{\theta}{2}}}{\sqrt{1-e^{-\theta}}} \|(1 + |y|^m) r(y)\|_{L^\infty} + \frac{e^{\frac{\theta}{2}}(1 - e^{-\theta})^m}{\sqrt{1-e^{-\theta}}} \|r(y)\|_{L^\infty} \right).
\]
Thus, part ii) of 5) is proved.

**Appendix B**

We prove now Lemma 3.13. Note that estimate 1, 2 and 3i) follow from Lemma 5 and Lemma 7 pages 555 – 559 in [3]. Thus, we only prove estimate 3 ii).

We recall the Feynman-Kac representation for $K$:
\[
K(s, \sigma, y, x) = e^{\theta c}(y, x) \int dv_{yx} \exp(\int_0^\theta V(w(t), \sigma + t)dt),
\]
where $\theta = s - \sigma$ and $e^{\theta c}(y, x) = e^{\frac{\theta}{\sqrt{4\pi(1-e^{-\theta})}}} \exp\left(-\frac{(ye^{-\theta} - 2 - x)^2}{4(1-e^{-\theta})}\right)$. We distinguish two cases: **Case 1:** If $x \in A = \{x, |ye^{-\frac{\theta}{2}} - x| \geq \frac{|y|}{4}\}$.

Arguing as in the proof of Lemma 3.14, we prove that
\[
\int dv_{yx} \exp(\int_0^\theta V(w(t), \sigma + t)dt) \leq C,
\]
and we decompose $\exp\left(-\frac{(ye^{-\theta} - 2 - x)^2}{4(1-e^{-\theta})}\right) = \exp\left(-\frac{(ye^{-\theta} - 2 - x)^2}{8(1-e^{-\theta})}\right) \exp\left(-\frac{(ye^{-\theta} - 2 - x)^2}{8(1-e^{-\theta})}\right)$. First, we remark that
\[
\exp\left(-\frac{(ye^{-\theta} - 2 - x)^2}{8(1-e^{-\theta})}\right) \leq Ce^{-\frac{|y|^2}{8}}.
\]
On the other hand, by a technical calculation, we prove that for $\sigma$ large enough

$$\exp\left(\frac{-(y e^{-\theta} - x)^2}{8(1 - e^{-\theta})}\right) \leq \frac{C}{1 + |y|^m}.$$  

Hence,

$$\int_{A} K(s, \sigma, y, x) \frac{1}{1 + |x|^m} 1_{\{|x| \geq K\sqrt{\sigma}\}} dx \leq C \frac{e^\theta}{\sqrt{4\pi}(1 - e^{-\theta})} \frac{1}{1 + |y|^m} \int_{A} e^{-\frac{x^2}{1 - e^{-\theta}}} \frac{1}{1 + |x|^m} 1_{\{|x| \geq K\sqrt{\sigma}\}} dx.$$  

Moreover, we choose $\sigma$ large enough, such that

$$\int_{A} K(s, \sigma, y, x) \frac{1}{1 + |x|^m} 1_{\{|x| \geq K\sqrt{\sigma}\}} dx \leq C \frac{e^{-\frac{\theta}{p}}}{1 + |y|^m}.$$  

**Case 2:** If $x \in \mathbb{R} \setminus A$, we assume $x > 0$, the case $x < 0$ is exactly the same. First, we remark that $ye^{-\frac{\theta}{p}} \in \left[\frac{3}{4} x, \frac{5}{4} x\right]$, therefore

$$\frac{1}{1 + |x|^m} \leq \frac{(\frac{4}{5} e^{-\frac{\theta}{p}})^m}{1 + |y|^m} \leq C \frac{1}{1 + |y|^m}.$$  

Using part 2) of Lemma 3.13 we obtain

$$\int_{\mathbb{R} \setminus A} K(s, \sigma, y, x) \frac{1}{1 + |x|^m} 1_{\{|x| \geq K\sqrt{\sigma}\}} dx \leq C \frac{e^{-\frac{\theta}{p}}}{1 + |y|^m}.$$  

which is the desired conclusion of Lemma 3.13

**Appendix C**

In this appendix, we prove the existence and uniqueness of a solution of equation (1.1) in the functional space $W^{1,\infty}_{p}(\mathbb{R}^{N})$ by a fixed point argument. Let $S(t)$ be the heat semigroup and let us write the equation (1.1) in its Duhamel formuataion:

$$u(t) = S(t)u_0 + \int_{0}^{t} S(t - s)(g(u) + h(u))ds,$$

where $g(u) = |u|^{p-1}u$ and $h(u) = \mu |\nabla u| \int_{B(0,|x|)} |u|^{q-1}$.  

We introduce the functional:

$$F(u)(t) = S(t)u_0 + \int_{0}^{t} S(t - s)(g(u) + h(u))ds.$$  

The proof the existence of a solution $u$ of the Duhamel equation is reduced to the existence of a fixed point of $F$.  

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For the reader’s convenience, we recall the following well-known smoothing effect of the heat semigroup:

$$\| S(t)f \|_{L^\infty} \leq \| f \|_{L^\infty}, \quad \| \nabla S(t)f \|_{L^\infty} \leq \frac{C}{\sqrt{t}} \| f \|_{L^\infty}, \forall t > 0, \forall f \in L^\infty(\mathbb{R}^N). \quad (4.58)$$

Since we want to prove the existence in $W^{1,\infty}_p(\mathbb{R}^N)$, we need more information about the heat semigroup. We give our following result:

**LEMMA 4.2.** For all $0 < m < N$, the heat semigroup satisfies:

i) $\| (1 + |x|^m) S(t)f \|_{L^\infty} \leq C \| (1 + |x|^m) f \|_{L^\infty},$

ii) $\| (1 + |x|^m) \nabla S(t)f \|_{L^\infty} \leq \frac{C}{\sqrt{t}} \| (1 + |x|^m) f \|_{L^\infty},$

for all $t > 0$ and all $f$ such that $(1 + |x|^m) f \in L^\infty$.

**Proof.** We recall that the heat semigroup is defined explicitly by

$$S(t)f(x) = \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y) dy.$$

We see that

$$| |x|^m S(t)f(x) | \leq \int_{\mathbb{R}^N} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy \| (1 + |y|^m) f(y) \|_{L^\infty}.$$

Let $\mathcal{A} = \{ y \in \mathbb{R}^N, \text{ such that } |x| \leq 2|y| \}$. We decompose the previous integral as follows:

$$\int_{\mathbb{R}^N} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy = \int_{\mathcal{A}} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy + \int_{\mathbb{R}^N \setminus \mathcal{A}} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy.$$

It is easy to see that

$$\int_{\mathcal{A}} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy \leq 2^m \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} dy \leq C. \quad (4.59)$$

On the other hand, for $y \in \mathbb{R}^N \setminus \mathcal{A}$, we have

$$|x - y| \geq ||x| - |y|| \geq \frac{1}{2} |x|.$$

Hence,

$$\int_{\mathbb{R}^N \setminus \mathcal{A}} \frac{|x|^m e^{-|x-y|^2/4t}}{|y|^m (4\pi t)^{N/2}} dy \leq |x|^m \frac{e^{-|x|^2/16t}}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N \setminus \mathcal{A}} \frac{dy}{|y|^m}. $$

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Since $\mathbb{R}^N \setminus A = B(0, \frac{|x|}{2})$ and $m < N$, we get
\[
\int_{\mathbb{R}^N \setminus A} \frac{dy}{|y|^m} = C \int_0^{\frac{|x|}{2}} r^{N-m-1} dr = C (\frac{|x|}{2})^{N-m}. \tag{4.60}
\]

Therefore,
\[
\int_{\mathbb{R}^N \setminus A} \frac{|x|^m e^{-\frac{|x-y|^2}{4t}}}{|y|^m (4\pi t)^{\frac{N}{2}}} dy \leq C (\frac{|x|}{\sqrt{t}})^N e^{\frac{|y|^2}{16t}} (4\pi)^{-\frac{N}{2}}.
\]

Using the fact that $z \mapsto z^N e^{-\frac{z^2}{16(4\pi t)^{\frac{N}{2}}}}$ is a bounded function, we obtain
\[
\int_{\mathbb{R}^N \setminus A} |x|^m e^{-\frac{(x-y)^2}{4t}} |y|^m \frac{dy}{(4\pi)^{\frac{N}{2}}} \leq C. \tag{4.61}
\]

From (4.58), (4.59) and (4.61), we deduce the estimate $i$).

For simplicity, we give the proof of $ii$) in one dimension, $N = 1$.

By integration by part, we get
\[
S(t) \nabla f(x) = \int_{\mathbb{R}} \frac{x - y e^{-\frac{(x-y)^2}{4t}}}{2t \sqrt{4\pi t}} f(y) dy.
\]

Then, we write
\[
||x|^m S(t) \nabla f(x) | \leq \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \frac{|x|^m |x - y| e^{-\frac{(x-y)^2}{4t}}}{2 \sqrt{2\pi t}} \frac{dy}{\sqrt{4\pi t}} \|(1 + |y|^m) f(y)\|_{L^\infty}.
\]

As above, we introduce $A = \{ y \in \mathbb{R} \text{, such that } |x| \leq 2|y| \}$, and write
\[
\int_{A} \frac{|x|^m |x - y| e^{-\frac{(x-y)^2}{4t}}}{2 \sqrt{2\pi t}} \frac{dy}{\sqrt{4\pi t}} \leq 2^m \int_{\mathbb{R}} \frac{|x - y| e^{-\frac{(x-y)^2}{4t}}}{2 \sqrt{2\pi t}} \frac{dy}{\sqrt{4\pi t}}.
\]

Making the change of variable $z = \frac{x - y}{2\sqrt{t}}$ and using the boundness of $\int_{\mathbb{R}} |z| e^{-z^2} dz$, we obtain
\[
\int_{A} \frac{|x|^m |x - y| e^{-\frac{(x-y)^2}{4t}}}{2 \sqrt{2\pi t}} \frac{dy}{\sqrt{4\pi t}} \leq C.
\]

On the other hand, since $\frac{1}{2} |x| \leq |x - y| \leq \frac{3}{2} |x|$ whenever $y \notin A$, we obtain
\[
\int_{\mathbb{R} \setminus A} \frac{|x|^m |x - y| e^{-\frac{(x-y)^2}{4t}}}{2 \sqrt{2\pi t}} \frac{dy}{\sqrt{4\pi t}} \leq 3|x|^{m+1} e^{-\frac{x^2}{4t}} \frac{2}{4\sqrt{t}} \sqrt{4\pi} \int_{\mathbb{R} \setminus A} \frac{dy}{|y|^m}.
\]
Applying estimate (4.60), we obtain

$$\int_{\mathbb{R}^N} \frac{|x|^m |x-y|}{|y|^m} e^{-\frac{|x-y|^2}{4\pi t}} \sqrt{4\pi t} dy \leq C \frac{x^2}{t} e^{-\frac{t}{16}}.$$  

From the boundedness of the function $z \mapsto z^2 e^{-\frac{t}{16}}$, we deduce that

$$\int_{\mathbb{R}^N} \frac{|x|^m |x-y|}{|y|^m} e^{-\frac{|x-y|^2}{4\pi t}} \sqrt{4\pi t} dy \leq C.$$  

This concludes the proof of Lemma 4.2. \hfill \Box

Now, we start the application of a fixed-point argument to solve the Cauchy problem of equation (1.1), in the space $W^{1,p}_\infty(\mathbb{R}^N)$, locally in time.

Let $T > 0$ and consider $C([0,T], W^{1,p}_\infty(\mathbb{R}^N))$ the space of all continuous functions from $[0,T]$ into $W^{1,p}_\infty(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_{L^p_c(W^{1,p}_\infty)} = \sup_{t \in [0,T]} \left(\|\left(1 + |x|^{\frac{2}{p-1}}\right)u\|_L^2 + \|\left(1 + |x|^{\frac{2}{p-1}}\right)\nabla u\|_L^2\right)^{\frac{1}{2}}.$$  

Our next goal is to find a positive constant $r$ such that the function $F$ is a strict contraction in $B(0, r)$, where $B(0, r)$ is the ball in $C([0,T], W^{1,p}_\infty(\mathbb{R}^N))$ of center 0 and radius $r$.

In a first step, we prove that $F$ is locally lipschitz continuous.

Let $r > 0$, for any $u_1, u_2 \in B(0, r)$, we write

$$(F(u_1) - F(u_2))(t) = \int_0^t S(t-s)(g(u_1) - g(u_2) + h(u_1) - h(u_2)) ds.$$  

Applying the above Lemma, we obtain

$$\|(1 + |x|^{\frac{2}{p-1}})(F(u_1) - F(u_2))\|_L \leq C \left[ \int_0^t \|(1 + |x|^{\frac{2}{p-1}})(g(u_1) - g(u_2))\|_L \right.$$ 

$$+ \left. \int_0^t \|(1 + |x|^{\frac{2}{p-1}})(h(u_1) - h(u_2))\|_L \right] \quad (4.62)$$  

and

$$\|(1 + |x|^{\frac{2}{p-1}})\nabla (F(u_1) - F(u_2))\|_L \leq \int_0^t \frac{C}{\sqrt{t-s}} \|(1 + |x|^{\frac{2}{p-1}})(g(u_1) - g(u_2))\|_L + \int_0^t \frac{C}{\sqrt{t-s}} \|(1 + |x|^{\frac{2}{p-1}})(h(u_1) - h(u_2))\|_L. \quad (4.63)$$  

It is easy to prove for any $u_1, u_2 \in B(0, r)$,

$$\|(1 + |x|^{\frac{2}{p-1}})(g(u_1) - g(u_2))\|_L \leq C r^{p-1} \|(1 + |x|^{\frac{2}{p-1}})(u_1 - u_2)\|_L. \quad (4.64)$$  

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Now, we estimate \( \| (1 + |x|^{\frac{2}{p-1}})(h(u_1) - h(u_2)) \|_{L^\infty} \). We write

\[
|h(u_1) - h(u_2)| \leq \mu \left( |\nabla u_1 - \nabla u_2| \int_{B(0,|x|)} |u_1|^{q-1} + |\nabla u_2| \int_{B(0,|x|)} |u_1|^{q-1} - |u_2|^{q-1}| \right)
\]

On the one hand, since \( \| (1 + |y|^{\frac{2}{p-1}})u_1 \|_{L^\infty} \leq r \) and \( \frac{2(q-1)}{p-1} > N \) by (1.2), we have

\[
\int_{B(0,|x|)} |u_1|^{q-1} \leq \| (1 + |y|^{\frac{2}{p-1}})u_1 \|_{L^\infty}^{\frac{q-1}{2}} \frac{dy}{(1 + |y|^{\frac{2}{p-1}})^{q-1}} \leq Cr^{q-1}.
\]

On the other hand, we write

\[
\int_{B(0,|x|)} \left| |u_1|^{q-1} - |u_2|^{q-1} \right| \leq \int_{B(0,|x|)} \frac{(1 + |y|^{\frac{2}{p-1}})^{q-1} |u_1|^{q-1} - |u_2|^{q-1}|}{(1 + |y|^{\frac{2}{p-1}})^{q-1}} dy.
\]

Since \( q > 2 \) by (1.2), we see that

\[
(1 + |y|^{\frac{2}{p-1}})^{q-1} \left| |u_1|^{q-1} - |u_2|^{q-1} \right| \leq Cr^{q-2}(1 + |y|^{\frac{2}{p-1}})|u_1 - u_2|.
\]

Thus,

\[
\int_{B(0,|x|)} \left| |u_1|^{q-1} - |u_2|^{q-1} \right| \leq Cr^{q-2} \| (1 + |x|^{\frac{2}{p-1}})(u_1 - u_2) \|_{L^\infty}.
\]

Since \( \| \nabla u_2 \|_{L^\infty} \leq \| (1 + |x|^{\frac{2}{p-1}}) \nabla u_2 \|_{L^\infty} \leq r \), we deduce that

\[
\| (1 + |x|^{\frac{2}{p-1}})(h(u_1) - h(u_2)) \|_{L^\infty} \leq Cr^{q-1} \left( \| (1 + |x|^{\frac{2}{p-1}})(u_1 - u_2) \|_{L^\infty} + \| (1 + |x|^{\frac{2}{p-1}}) \nabla (u_1 - u_2) \|_{L^\infty} \right) \tag{4.65}
\]

Collecting estimates (4.62), (4.63), (4.64) and (4.65), we obtain

\[
\| F(u_1) - F(u_2) \|_{L^\infty_{t}(W^{1,p}_{\infty})} \leq C(r^{p-1} + r^{q-1})(T^2 + T)^{\frac{1}{2}} \| u_1 - u_2 \|_{L^\infty_{t}(W^{1,p}_{\infty})}.
\]

If we assume \( r \) small enough such that \( C(r^{p-1} + r^{q-1})(T^2 + T)^{\frac{1}{2}} \leq \frac{1}{2} \), we obtain

\[
\| F(u_1) - F(u_2) \|_{L^\infty_{t}(W^{1,p}_{\infty})} \leq \frac{1}{2} \| u_1 - u_2 \|_{L^\infty_{t}(W^{1,p}_{\infty})}.
\]

In the next step, we prove that \( F(B(0,r)) \subset B(0,r) \). We rewrite, for \( u \in B(0,r) \)

\[
\| F(u) \|_{L^\infty_{t}(W^{1,p}_{\infty})} = \| F(u) - F(0) + F(0) \|_{L^\infty_{t}(W^{1,p}_{\infty})} \leq \frac{r}{2} + \| F(0) \|_{L^\infty_{t}(W^{1,p}_{\infty})}.
\]

According to Lemma 4.2, we have

\[
\| (1 + |x|^{\frac{2}{p-1}})S(t)u_0 \|_{L^\infty} \leq C(1 + |x|^{\frac{2}{p-1}})u_0 \|_{L^\infty},
\]

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and
\[ \|(1 + |x|^\frac{2}{p}) S(t) \nabla u_0 \|_{L^\infty} \leq C \|(1 + |x|^\frac{2}{p}) \nabla u_0 \|_{L^\infty}. \]

It follows then that
\[ \|F(0)\|_{L^\infty(W^{1,p}_\infty)} \leq C \|u_0\|_{L^\infty(W^{1,p}_\infty)}. \]

If we assume that
\[ C \|u_0\|_{L^\infty(W^{1,p}_\infty)} \leq \frac{r}{4}, \]

then
\[ \|F(u)\|_{L^\infty(W^{1,p}_\infty)} \leq \frac{3}{4} r. \]

We conclude that, there exist \( r > 0 \) such that the function \( F : B(0, r) \to B(0, r) \) is a strict contraction and that \( F \) admits a unique fixed point \( u \in B(0, r) \).

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