H-STABILITY OF SYZYGY BUNDLES ON SOME REGULAR ALGEBRAIC SURFACES

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Abstract. Let $L$ be a globally generated line bundle over a smooth irreducible complex projective surface $X$. The syzygy bundle $M_L$ is the kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_X \to L$. We prove the $L$-stability of $M_L$ for Hirzebruch surfaces, del Pezzo surfaces and Enriques surfaces. The $(-\mathbb{K}_X)$-stability of syzygy bundles $M_L$ over del Pezzo surfaces is also obtained.

1. Introduction

Let $X$ be a smooth irreducible projective variety over $\mathbb{C}$ and let $L$ be a globally generated line bundle over $X$ (from now on simply a generated bundle). The kernel $M_L$ of the evaluation map $H^0(L) \otimes \mathcal{O}_X \to L$ fits into the following exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$  

The bundle $M_L$ is called a syzygy bundle. The rank of $M_L$ is $h^0(L) - 1$. The vector bundles $M_L$ have been extensively studied from different points of view.

When $X$ is a projective irreducible smooth curve of genus $g \geq 1$, Ein and R. Lazarsfeld showed in [9] that the syzygy bundle $M_L$ is stable for $d > 2g$ and it is semi-stable for $d = 2g$ (see also [3]). After this, the semi-stability of $M_L$ was proved for line bundles with $\text{deg}(L) \geq 2g - \text{Cliff}(C)$ (see [13, Corollary 5.4] and [5, Theorem 1.3]). In ([14]), Paranjape and Ramanan proved that $M_K_C$ is semi-stable and is stable if $C$ is non-hyperelliptic. In [15], Schneider showed that $M_L$ is semi-stable for a general curve $C$ (see also [4]). The semi-stability for incomplete linear series over general curves was proved in [1].

In [11], Flenner showed the stability of $M_L$ for projective spaces. The stability of syzygy bundles for incomplete linear series in projective spaces has been studied by several authors (see [8, 12, 2]). Recently, in [7], the authors proved that given an Abelian variety $A$ and any ample line bundle $L$ on $A$ the syzygy bundle $M_{L,d}$ is $L$-stable if $d \geq 2$.

In the case of a projective surface $X$, we must start by fixing a polarization $H$ and then ask for the $H$-stability of the syzygy bundle $M_L$. Recall that $H$-stability for a vector bundle $E$ on $X$ means that for any sub-bundle $F \subset E$

$$\mu_H(F) := \frac{c_1(F).H}{\text{rk}(F)} < \mu_H(E) := \frac{c_1(E).H}{\text{rk}(E)}.$$  

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In the pioneer work of Camere ([6]) it was proved that $M_L$ is $L$-stable for any ample and generated $L$ on a $K3$ surface and for any generated $L$ with $L^2 \geq 14$ on an abelian surface $X$. In [10] Ein, Lazarsfeld and Mustopa fixed an ample divisor $L$ and an arbitrary divisor $D$ over $X$, and setting $L_d = dL + D$ ($d \in \mathbb{N}$) showed that $M_{L_d}$ is $L$-stable for $d \gg 0$.

Most of our results derive from the following:

**Theorem 2.2** Let $X$ be a smooth projective surface. Let $L$ be an ample and generated line bundle over $X$ and $H$ be a divisor such that an irreducible and non-singular curve $C$ exists in $|H|$. Assume that

1. $h^1(L - H) = 0$;
2. $h^0(H) \geq h^0(L|_C)$;
3. $M_L|_C$ is semi-stable;

Then $M_L$ is $H$-stable.

Section 2 is devoted to the proof of Theorem 2.2.

If $L = H$ and $X$ is regular, then conditions (1) and (2) are automatically satisfied and in order to prove the stability of $M_L$ it is sufficient to prove the semi-stability of $M_L|_C$. In section 3, using this idea, we obtain the $L$-stability of the syzygy bundle $M_L$ in the following cases: if $|L|$ contains either a genus $\leq 1$ curve or a Brill–Noether general curve (Corollary 3.1), if $X$ is either a del Pezzo (Corollary 3.3) or a Hirzebruch surfaces (Corollary 3.4) and if $X$ is an Enriques surfaces under the condition that $\text{Cliff}(C) \geq 2$ (Corollary 3.6).

Moreover, we study the stability of syzygy bundle over del Pezzo surfaces with respect to the anti-canonical polarization:

**Theorem 3.7** Let $X$ be a del Pezzo surface and let $L$ be a generated line bundle on $X$. If $L$ contains an irreducible curve, then the vector bundle $M_L$ is $(-K_X)$-stable.

**Conventions:** We work over the field of complex numbers $\mathbb{C}$. Given a coherent sheaf $\mathcal{G}$ on a variety $X$ we write $h^i(\mathcal{G})$ to denote the dimension of the $i$-th cohomology group $H^i(X, \mathcal{G})$. The sheaf $K_X$ will denote the canonical sheaf on $X$. A surface always means a smooth (or non-singular) projective complex irreducible surface.

2. **Stability on some regular surfaces**

The aim of this section is to prove Theorem 2.2.

Let $X$ be a projective, irreducible and non-singular surface $X$. Let $H$ be an ample line bundle on $X$. The $H$-slope of a vector bundle $E$ is given by

$$\mu_H(E) := \frac{c_1(E)H}{\text{rk}(E)},$$

where $c_1(E)$ is the first Chern class of $E$ and $\text{rk}(E)$ is the rank of $E$. In particular, the $H$-slope of $M_L$ is given by

$$\mu_H(M_L) = \frac{c_1(M_L)H}{\text{rk}(M_L)} = \frac{-L.H}{h^0(L) - 1}.$$
the $H$-slope of a vector bundle $E$ as

$$
\mu_H(E) = \mu(E|_C) := \frac{\deg(E|_C)}{\text{rk}(E)},
$$

where $\deg(E|_C)$ is the degree of vector bundle $E$ restricted to the curve $C$.

Assume that there exists an irreducible and non-singular curve $C$ in the linear system $|H|$ and take a point $x \in C$. For the remainder of the argument $H, C \in |H|$ and $x \in C$ would be fixed. Note that $C \in |H \otimes m_x|$.

Given any sub-bundle $F \subset M_L$ we have, thanks to the condition $h^1(L - H) = 0$, a commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & F|_C & \rightarrow & N & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0(L - H) \otimes O_C & \rightarrow & (M_L)|_C & \rightarrow & M_L|_C & \rightarrow & 0,
\end{array}
\]

(see [10] page 76 and also [6], proof of Proposition 3).

**Lemma 2.1.** Let $X$ be a smooth projective surface. Let $L$ be an ample and generated line bundle over $X$ and $H$ be a divisor such that an irreducible and non-singular curve $C$ exists in $|H|$. Assume that the following statements are satisfied:

1. $h^1(L - H) = 0$.
2. $M_L|_C$ is semi-stable.

Let $F \subset M_L$ be a sub-bundle with $\mu_H(F) \geq \mu_H(M_L)$. Then,

$$\text{rk}(F) \geq h^0(H) - 1 + \text{rk}(K).$$

The proof of the Lemma is quite analogous to the one of Lemma 1.1 in [10]. We only need to replace $B$ by $L - H$ and $L_d - B$ by $H$ and to compute the dimension of a fiber. We include a proof for the reader convenience.

**Proof.** Note that for any vector bundle $E$ on $X$, $\mu_H(E) = \mu(E|_C)$. Therefore the condition (3) is equivalent to $\mu(F|_C) \geq \mu((M_L)|_C)$. If $K = 0$ in the exact sequence [2], then

$$\mu(N) = \mu(F|_C) \geq \mu((M_L)|_C) > \mu(M_L|_C)$$

gives a contradiction with the semi-stability of $M_L|_C$ and thus we have $K \neq 0$.

The multiplication map of sections

$$\nu : \mathbb{P}(H^0(H \otimes m_x)) \times \mathbb{P}(H^0(L - H)) \rightarrow \mathbb{P}(H^0(L \otimes m_x))$$

is a finite morphism. After localizing at the given point $x$ we obtain a commutative diagram:

$\begin{array}{ccc}
(K)_x & \rightarrow & (F|_C)_x \\
\downarrow & & \downarrow \\
H^0(L - H)_x & \rightarrow & ((M_L)|_C)_x = H^0(L \otimes m_x).
\end{array}$
Let $Z = \nu^{-1}(\mathbb{P}(F|_C)_x)$, since $\nu$ is finite dim $\mathbb{P}((F|_C)_x) \geq \dim Z$. Given $s \in H^0(H \otimes m_x)$, we have that $s$ induces the injective morphism

$$H^0(L - H) \rightarrow H^0(L \otimes m_x) \quad \phi \mapsto s \otimes \phi.$$ 

Therefore for any $\phi$ in the image of the morphism $(K)_x \rightarrow H^0(L - H)$, we have that $(s, \phi) \in Z$ and $\pi_2(s, \phi) = s$. It follows that the projection $\pi_2 : Z \rightarrow \mathbb{P}(H^0(H \otimes m_x))$ is dominant and the dimension of the general fibre is greater or equal than $\text{rk}(K) - 1$. Hence

$$(3) \quad \text{rk}(F) \geq h^0(H) + \text{rk}(K) - 1.$$ 

\[\square\]

**Theorem 2.2.** Let $X$ be a smooth projective complex surface. Let $L$ be an ample and generated line bundle over $X$ and $H$ be a divisor such that an irreducible and non-singular curve $C$ exists in $|H|$. Assume that:

1. $h^1(L - H) = 0$,
2. $h^0(H) \geq h^0(L|_C)$,
3. $M_L|_C$ is semi-stable.

Then $M_L$ is $H$-stable.

**Proof.** Assume that $M_L$ is $H$-unstable, let $F \subseteq M_L$ be a sub-bundle such that $\mu_H(F) \geq \mu_H(M_L)$. By Lemma 2.1, we have

$$\text{rk}(F) \geq h^0(H) + \text{rk}(K) - 1.$$ 

We can repeat the proof of Lemma 1.2 in [10] replacing $B$ by $L - H$ and $L_d - B$ by $H$. More specifically, in Lemma 1.2 in [10] an inequality (inequality (*) at the beginning of page 78) is obtained that translated to our situation, just as we did with Lemma 2.1 yields:

$$\text{rk}(K) \geq h^0(L - H) \cdot \left( \frac{\text{rk}(F)}{\text{rk}(M_L)} \right).$$ 

Therefore, we have:

$$\begin{align*}
\text{rk}(F) &\geq h^0(H) + \text{rk}(K) - 1 \\
&\geq h^0(H) + h^0(L - H) \cdot \left( \frac{\text{rk}(F)}{\text{rk}(M_L)} \right) - 1.
\end{align*}$$

That is,

$$\text{rk}(F) \geq \frac{(h^0(H) - 1)\text{rk}(M_L)}{\text{rk}(M_L) - h^0(L - H)}.$$ 

Since $\text{rk}(M_L) - h^0(L - H) = \text{rk}(M_L|_C) = h^0(L|_C) - 1$, from the hypothesis (2), we obtain that $\text{rk}(F) \geq \text{rk}(M_L)$ which is impossible because $F$ is a sub-bundle of $M_L$. This proves the theorem. \[\square\]
Theorem 2.2 is especially useful when $X$ is a regular surface and $L = H$. In this case conditions (1) and (2) are automatically satisfied and only (3) remains to be verified. Also, as $H = L$ is ample and generated we can assume that $h^0(L) \geq 3$. Thus, a curve $C \in |L|$ exists which is non-singular and connected, and therefore irreducible (Bertini’s Theorem).

**Corollary 3.1.** Assume $X$ is regular and $L$ is an ample and generated line bundle on $X$. Let $C \in |L|$ be irreducible and non-singular and assume that either:

1. $g(C) \leq 1$
2. $C$ is Brill–Noether general,

then $ML$ is $L$-stable.

**Proof.** Assume that $g(C) = 0$. First note that $h^1(L|C) = 0$ because $L|C$ is a globally generated bundle over $C$. Taking cohomology in the exact sequence:

$$0 \to ML|C \to H^0(L|C) \otimes O_C \to L|C \to 0,$$

we see that $h^0(ML|C) = h^1(ML|C) = 0$. By Grothendieck’s Theorem, we get $ML|C \simeq \oplus O_C(-1)$. Therefore, $ML|C$ is semi-stable and the result follows at once from Theorem 2.2.

Next, let $C$ be an elliptic non-singular curve and let $L$ be a line bundle on $C$ of degree $d \geq 2$, observe that $ML$ is stable. Indeed, the rank of $ML^\vee$ is $d - 1$, therefore the slope of $ML^\vee$ is given by

$$\mu(ML^\vee) = \frac{d}{d - 1}.$$

Let $F$ be a quotient of $ML^\vee$, we want to prove that $\mu(ML^\vee) < \mu(F)$. Since $F^\vee$ is a sub-bundle of $ML$, we get that $H^0(F^\vee) = 0$ and thus $H^1(F) = 0$, by Serre duality. By Riemann Roch Theorem, we get

$$h^0(F) = \deg(F) + \rk(F)(1 - g(C)) + h^1(F) = \deg(F).$$

Since $F$ is globally generated over $C$, it follows that $h^0(F) \geq \rk(F)$. Assume that $h^0(F) = \rk(F)$, then the evaluation map

$$H^0(F) \otimes O_C \to F$$

is an isomorphism, which is impossible because $H^1(F) = 0$. Therefore $\deg(F) \geq \rk(F) + 1$ and

$$\mu(F) \geq \frac{\rk(F) + 1}{\rk(F)} > \frac{d}{d - 1} = \mu(ML^\vee),$$

the last inequality following from $\rk(F) < d - 1$.

Finally, assume that $C$ is Brill–Noether general. By [15] we have that $ML|C$ is semi-stable.

Another general case that can be treated is when the anticanonical divisor $-K_X$ is nef.

**Proposition 3.2.** Let $X$ be a regular surface and let $L$ be an ample and generated line bundle on $X$. Let $C \in |L|$ be irreducible and non-singular. Then:

1. If $-L.K_X \geq 2$, then $ML$ is $L$-stable.
2. If $-L.K_X = 1$ and $\Cliff(C) \geq 1$; then $ML$ is $L$-stable.
(3) if \( L.K_X = 0 \) and \( \text{Cliff}(C) \geq 2 \); then \( M_L \) is \( L \)-stable.

Proof. If \( g(C) \leq 1 \), then apply Corollary 3.1. If \( -L.K_X \geq 2 \), then by the Adjunction Formula we have \( \text{deg}(L|_C) = L^2 \geq 2g(C) \). Therefore \( M_{L|_C} \) is semi-stable and \( M_L \) is \( L \)-stable by Theorem 2.2. This proves (1). The proofs of (2) and (3) are similar: using Adjunction Formula we get \( \text{deg}(L|_C) = L^2 \geq 2g(C) - \text{Cliff}(C) \), therefore \( M_{L|_C} \) is semi-stable. □

Some particular cases of this situation are:

Corollary 3.3. Let \( L \) be an ample and generated line bundle over a del Pezzo surface, then \( M_L \) is \( L \)-stable.

Proof. If \( C \in |L| \) is irreducible and non-singular and \( g(C) \leq 1 \), then the result follows from Corollary 3.1. Otherwise, from \( h^0(−K_X) \geq 2 \) we have \( -L.K_X \geq 2 \) and the result follows from Proposition 3.2. □

Corollary 3.4. Let \( \mathbb{F}_n \) be a non-singular Hirzebruch surface and let \( L \) be an ample and generated line bundle. Then \( M_L \) is \( L \)-stable.

Proof. The canonical line bundle is given by \( K_{\mathbb{F}_n} = -2C_n - (n + 2)F \), where \( C_n \) and \( F \) are respectively the section and the fiber of the structural fibration \( \mathbb{F}_n \to \mathbb{P}^1 \). If \( L \) is ample, then \( c_1(L) = aC_n + bF \) with \( a > 0 \) and \( b > na \). Since \( -L.K_{\mathbb{F}_n} = 2b - an + 2a > 3 \), it follows that \( M_L \) is \( L \)-stable by Proposition 3.2. □

The \( L \)-stability of \( M_L \) for \( X \) a K3 surface was studied in ([6], Theorem 1). As a byproduct of our method we can recover the quoted result:

Corollary 3.5. Let \( X \) be a smooth projective K3 surface and \( L \) an ample and generated line bundle over \( X \). Then \( M_L \) is \( L \)-stable.

Proof. Let \( C \in |L| \) be irreducible and non-singular. By Adjunction Formula, \( L|_C = K_C \) is the canonical line bundle. In ([14]), Paranjape and Ramanan proved that \( M_{K_C} \) is semi-stable. The Corollary follows from Theorem 2.2. □

Similar results can be obtained for regular surfaces with numerically trivial canonical divisor:

Corollary 3.6. Let \( X \) be an Enriques surface. Let \( L \) be an ample and generated line bundle on \( X \). Assume that an irreducible and non-singular curve \( C \) in the linear system \( |L| \) exists such that \( \text{Cliff}(C) \geq 2 \). Then \( M_L \) is \( L \)-stable.

Proof. Note that \( \text{deg}(L|_C) = L^2 = 2g - 2 \geq 2g - \text{Cliff}(C) \), thus \( M_{L|_C} \) is semi-stable. □

Finally we address the case of \( (−K_X) \)-stability for del Pezzo surfaces.

Theorem 3.7. Let \( X \) be a del Pezzo surface and let \( L \) be a globally generated line bundle on \( X \). If \( |L| \) contains an irreducible curve, then the vector bundle \( M_L \) is \( (−K_X) \)-stable.

Proof. Note that \( h^1(L) = h^2(L) = 0 \). From Riemann-Roch Theorem we get
\[
h^0(L) = 1 + \frac{L^2 - L.K_X}{2}.
\]
Therefore the rank of $M_L$ is equal to $\frac{L^2 - L.K_X}{L^2}$. We want to prove that $M_L^\vee$ is stable with respect to the polarization $-K_X$. Let $C$ be a non-singular projective curve in the linear system $| - K_X|$. By Adjunction Formula $C$ is an elliptic curve. The slope of $M_L^\vee$ with respect to $-K_X$ is given by

$$\mu(M_L^\vee) = -\frac{2L.K_X}{L^2 - L.K_X}.$$  

Let $F$ be a torsion-free quotient sheaf of $M_L^\vee$ of rank $0 < \text{rk } F < r$; then $F|_C$ is a quotient of $(M_L^\vee)|_C$. We want to prove that

$$\mu(M_L^\vee) < \mu(F).$$

First we assume that $F|_C$ is a vector bundle on $C$. The following properties are satisfied:

(i) $H^1(C, F|_C) = 0$.

(ii) $\deg(F|_C) \geq \text{rk}(F) + 1$.

Indeed, since $F$ is globally generated, we have $F|_C$ is globally generated, and therefore $h^0(F^\vee|_C) = 0$ and $h^1(F|_C) = 0$ by Serre duality; this proves (i). To prove (ii), by Riemann Roch Theorem for curves, we get

$$h^0(F|_C) = \deg(F|_C) + \text{rk}(F|_C)(1 - g(C)) + h^1(F|_C) = \deg(F|_C).$$

Since $F|_C$ is globally generated over $C$, it follows that $h^0(F|_C) \geq \text{rk}(F|_C)$. Assume that $h^0(F|_C) = \text{rk}(F|_C)$, then the evaluation map

$$H^0(F|_C) \otimes \mathcal{O}_C \to F|_C$$

is an isomorphism but this is impossible because $H^1(F|_C) = 0$. Therefore $\deg(F|_C) \geq \text{rk}(F) + 1$ and we obtain (ii).

Hence, we have

$$\mu(F) = \mu(F|_C) \geq 1 + \frac{1}{\text{rk}(F|_C)} > 1 + \frac{2}{L^2 - L.K_X},$$

therefore

$$\mu(M_L^\vee) = -\frac{2L.K_X}{L^2 - L.K_X} \leq 1 + \frac{2}{L^2 - L.K_X} < \mu(F),$$

the first inequality follows by Adjunction Formula and the fact that $|L|$ contains an irreducible curve. Hence $\mu(M_L^\vee) < \mu(F)$.

Now, if $F|_C$ is not a vector bundle, then $F|_C = E \oplus \tau$, where $E$ is a vector bundle and $\tau$ is a torsion sheaf over $C$. The above proof can be repeated to obtain $\mu(M_L^\vee) < \mu(E)$. Then

$$\mu(M_L^\vee) < \mu(E) \leq \mu(E \oplus \tau) = \mu(F)$$

Therefore $M_L^\vee$ is stable with respect to polarization $-K_X$. 

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$\square$
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