COMPLETION THEOREM FOR EQUIVARIANT $K$-THEORY

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Abstract. In this paper, we study the algebraic analogue of the topological Atiyah-Segal completion theorem for the equivariant $K$-theory. We show that this completion theorem for the equivariant algebraic $K$-theory holds for the action of connected groups on smooth projective schemes. In the opposite direction, we show that the completion theorem is false in general for non-projective smooth schemes.

1. Introduction

The equivariant $K$-theory for topological spaces with group action was invented by Atiyah much before Quillen discovered the algebraic $K$-theory of schemes. This theory had a significant impact on the subsequent works of Atiyah, Segal and others, including the celebrated work of Atiyah and Singer on the index theorem.

For a compact Lie group $G$, let $R(G)$ denote the ring of virtual representations of $G$ and let $I_G$ denote the augmentation ideal given by the kernel of the map $\epsilon : R(G) \rightarrow \mathbb{Z}$ that takes a virtual representation to its rank. In order to study the representations of $G$ in terms of the singular cohomology of its classifying space $B_G$, Atiyah [1] showed for a finite group $G$ that there is indeed a strong connection between $R(G)$ and the topological $K$-theory of $B_G$. More precisely, if $\mathcal{K}(B_G)$ denotes the inverse limit of the Grothendieck groups of complex vector bundles on the finite skeleta of the infinite $CW$-complex $B_G$, then there is a natural isomorphism

$$\widehat{R(G)}_{I_G} \cong \mathcal{K}(B_G).$$

This result was extended to the case of all compact Lie groups by Atiyah and Hirzebruch [2]. This was subsequently reinterpreted by Atiyah and Segal [3] in terms of the following very general statement about the equivariant $K$-theory of compact and Hausdorff topological spaces. Let $E_G \rightarrow B_G$ denote the universal $G$-bundle and for a compact $G$-space $X$ and let $X_G$ denote Borel space $X \times^G E_G$. Let $K^*_G(X)$ denote the equivariant $K$-theory of $G$-equivariant complex vector bundles on $X$.

Theorem 1.1 (Atiyah-Segal). Let $X$ be a compact $G$-space such that $K^*_G(X)$ is a finite $R(G)$-module. Then the map $K^*_G(X) \rightarrow K_*(X_G)$ induces an isomorphism

$$\widehat{K^*_G(X)}_{I_G} \cong K_*(X_G).$$

The equivariant algebraic $K$-theory of schemes under the action of group schemes was founded by Thomason [24] using the ideas of Quillen’s $K$-theory of exact categories and Waldhausen’s $K$-theory of categories with cofibrations and weak

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equivalences. In order to formulate and study the algebraic analogue of the Atiyah-Segal completion theorem for the algebraic $K$-theory, one needs to have algebraic objects which correspond to $X_G$. The problem is that $X_G$ is not a scheme and one does not know how to define $K$-theory of $X_G$. However, this problem now has a solution, thanks to the invention of $\mathbb{A}^1$-homotopy theory of schemes by Morel and Voevodsky [14]. These authors have constructed a category of motivic spaces which includes all smooth schemes over a base scheme as well as the colimits of such smooth schemes. They further show that there is a generalized cohomology theory on the stable homotopy category of motivic spaces which restricts to Thomason’s $K$-theory for smooth schemes.

The other problem is that the completion theorem of Atiyah and Segal is based on a strong assumption that $K^G_\ast(X)$ is a finite $R(G)$-module and this assumption is very crucial in their proofs. Thomason [23] studied the completion problem for the algebraic $K$-theory and was forced to work with the $K$-theory with finite coefficients and also had to invert the Bott element. He had to do this because one knows that the Bott inverted algebraic $K$-theory with finite coefficients has the above finiteness property and one could use the techniques of Atiyah-Segal. Such a finiteness assumption is almost never true (unless we work over finite fields) for the algebraic $K$-theory, even for the algebraic $K$-theory of a point.

The aim of this paper is to show using the $\mathbb{A}^1$-homotopy theory of schemes that the algebraic analogue of topological spaces like $X_G$ do exist as motivic spaces and this allows one to study the question of Atiyah-Segal completion theorem for algebraic varieties. We show that such a completion theorem does hold for smooth and projective schemes. We further show that this result is no longer true if we weaken the projectivity assumption. The main results of this paper roughly look as follows. We shall state these results in more precise form and explain the underlying terms in the body of the text.

**Theorem 1.2.** Let $G$ be a connected split reductive group over a field $k$ and let $G$ act on a smooth projective scheme $X$ over $k$. Then for every $p \geq 0$, there is an isomorphism

$$
\hat{K}^G_p(X)_{IG} \cong K_p(X_G).
$$

Since a connected linear algebraic group in characteristic zero has a Levi decomposition, the reductivity assumption is not necessary in this case.

For $p = 0$, the completion theorem holds in the most general situation without any condition on $G$ and $X$.

**Theorem 1.3.** Let $G$ be a linear algebraic group over $k$ acting on a smooth scheme $X$. Then there is an isomorphism

$$
\hat{K}^G_0(X)_{IG} \cong K_0(X_G).
$$

We remark that the equivariant Riemann-Roch theorem of Edidin and Graham [9] is an immediate consequence of Theorem 1.3 and its generalization for singular schemes (see Remark 9.9).

**Theorem 1.4.** Let $G$ be a one-dimensional torus over $\mathbb{C}$ and let $X$ be the quotient of $G$ by the subgroup $\mu_2$. Then

1. For $p > 0$ odd, the map $\hat{K}^G_p(X)_{IG} \rightarrow K_p(X_G)$ is an isomorphism.
(2) For $p > 0$ even, there is a short exact sequence

$$0 \to K_p^G(X)_{I_G} \to K_p(X_G) \to \mathbb{Z}_2 \to 0.$$ 

A brief outline of this paper is as follows. In § 2, we fix our notations and very briefly recall Morel and Voevodsky’s $\mathbb{A}^1$-homotopy theory of schemes. We introduce our motivic Borel spaces associated to group actions on smooth schemes and prove some of their properties. In § 3, we discuss the algebraic $K$-theory of motivic Borel spaces. We also introduce a variant, called $\mathcal{K}$-theory, for these spaces. This is an algebraic analogue of a similar topological object introduced by Atiyah [1].

In § 4 and § 5, we prove our basic results which yield decompositions of the equivariant $K$-theory of filtrable schemes and that of the $\mathcal{K}$-theory of the associated Borel spaces. These results allow us to prove the completion theorem for the torus action. The equivariant $K$-theory of smooth schemes for the action of connected reductive groups is studied in § 6. We define the Atiyah-Segal completion map and prove the completion theorem in § 8. The following section deals with the completion theorem for the Grothendieck group of equivariant bundles on all schemes. In § 10, we compute the $\mathcal{K}$-theory of the Borel spaces associated to some non-projective schemes. This allows us to show the failure of the completion theorem in such cases.

2. Motivic Borel spaces

In this section, we fix our notations and set up the machinery of $\mathbb{A}^1$-homotopy theory that we need in order to define our motivic Borel spaces. These motivic spaces are one of the main objects of study in this text.

2.1. Notations and conventions. A scheme in this paper will mean a quasi-projective scheme over a fixed field $k$. This ground field will be fixed throughout. Let $\text{Sch}_k$ denote the category of quasi-projective schemes over $k$. An object of this category will often be called a $k$-scheme. Let $\text{Sm}_k$ denote the subcategory of smooth schemes in $\text{Sch}_k$.

A linear algebraic group $G$ over $k$ will mean a smooth and affine group scheme over $k$. By a closed subgroup $H$ of an algebraic group $G$, we shall mean a morphism $H \to G$ of algebraic groups over $k$ which is a closed immersion of $k$-schemes. In particular, a closed subgroup of a linear algebraic group will be of the same type and hence smooth. Recall from [5, Proposition 1.10] that a linear algebraic group over $k$ is a closed subgroup of a general linear group, defined over $k$. Let $\text{Sch}^G_k$ (resp. $\text{Sm}^G_k$) denote the category of quasi-projective (resp. smooth) $k$-schemes with $G$-action and $G$-equivariant maps. An object of $\text{Sch}^G_k$ will often be called a $G$-scheme.

Recall that an action of a linear algebraic group $G$ on a $k$-scheme $X$ is said to be linear if $X$ admits a $G$-equivariant ample line bundle, a condition which is always satisfied if $X$ is normal (cf. [20, Theorem 2.5] for $G$ connected and [25, 5.7] for $G$ general). All $G$-actions in this paper will be assumed to be linear. We shall use the following other notations throughout this text.

(1) $\text{Nis}/k$: The Grothendieck site of smooth schemes over $k$ with Nisnevich topology.
(2) $\text{Shv(Nis}_k)$: The category of sheaves of sets on $\text{Nis}/k$. 
(3) $\Delta^{op}\text{Shv}(\text{Nis}_k)$: The category of sheaves of simplicial sets on Nis/$k$.

(4) $\mathcal{H}(k)$: The unstable $A^1$-homotopy category of simplicial sheaves on Nis/$k$, as defined in [14].

(5) $\mathcal{H}_*(k)$: The unstable $A^1$-homotopy category of pointed simplicial sheaves on Nis/$k$, as defined in [14].

(6) $\mathcal{S}\mathcal{H}(k)$: The stable $A^1$-homotopy category of pointed simplicial sheaves on Nis/$k$ as defined, for example, in [28].

Following the notations of [28], an object of $\Delta^{op}\text{Shv}(\text{Nis}_k)$ will be called a motivic space (or simply a space) and we shall often write this category of motivic spaces as $\text{Spc}$. The category of pointed motivic spaces over $k$ will be denoted by $\text{Spc}_\bullet$.

For any $X, Y \in \text{Spc}$, $S(X, Y)$ denotes the simplicial set of morphisms between spaces as in [14].

2.1.1. Quotients for free action. While dealing with the algebraic $K$-theory of schemes with group action, one often needs to know that certain kinds of quotient schemes for the group action exist. We shall usually only need to know that the following kind of quotient schemes exist (cf. [8, Proposition 23]).

**Lemma 2.1.** Let $H$ be a linear algebraic group acting freely and linearly on a $k$-scheme $U$ such that the quotient $U/H$ exists as a quasi-projective variety. Let $X$ be a $k$-scheme with a linear action of $H$. Then the mixed quotient $X^H \times U$ for the diagonal action on $X \times U$ exists as a scheme and is quasi-projective. Moreover, this quotient is smooth if both $U$ and $X$ are so. In particular, if $H$ is a closed subgroup of a linear algebraic group $G$ and $X$ is a $k$-scheme with a linear action of $H$, then the quotient $G^H \times X$ is a quasi-projective scheme.

**Proof.** It is already shown in [8, Proposition 23] using [10, Proposition 7.1] that the quotient $X^H \times U$ is a scheme. Moreover, as $U/H$ is quasi-projective, [10, Proposition 7.1] in fact shows that $X^H \times U$ is also quasi-projective. The similar conclusion about $G^H \times X$ follows from the first case by taking $U = G$ and by observing that $G/H$ is a smooth quasi-projective scheme (cf. [8, Theorem 6.8]). The assertion about the smoothness is clear since $X \times U \to X^H \times U$ is an $H$-torsor.

In this text, $\text{Sch}_{\text{free}/k}^G$ will denote the full subcategory of $\text{Sch}_k^G$ whose objects are those schemes $X$ on which $G$ acts freely such that the quotient $X/G$ exists and is quasi-projective over $k$. The full subcategory of $\text{Sch}_{\text{free}/k}^G$ consisting of smooth schemes will be denoted by $\text{Sm}_{\text{free}/k}^G$. The previous result shows that if $U \in \text{Sch}_{\text{free}/k}^G$, then $X \times U$ is also in $\text{Sch}_{\text{free}/k}^G$ for every $G$-scheme $X$.

2.2. Admissible gadgets. Let $G$ be a linear algebraic group over $k$. All representations of $G$ in this text will be assumed to be finite-dimensional. We shall say that a pair $(V, U)$ of smooth schemes over $k$ is a good pair for $G$ if $V$ is a $k$-rational representation of $G$ and $U \subseteq V$ is a $G$-invariant open subset which is an object of $\text{Sch}_{\text{free}/k}^G$. It is known (cf. [26, Remark 1.4]) that a good pair for $G$ always exists.
Definition 2.2. A sequence of pairs \( \rho = (V_i, U_i)_{i \geq 1} \) of smooth schemes over \( k \) is called an admissible gadget for \( G \), if there exists a good pair \((V, U)\) for \( G \) such that \( V_i = V^{\oplus i} \) and \( U_i \subseteq V_i \) is \( G \)-invariant open subset such that the following hold for each \( i \geq 1 \).

1. \( (U_i \oplus V) \cup (V \oplus U_i) \subseteq U_{i+1} \) as \( G \)-invariant open subsets.
2. \( \text{codim}_{U_{i+1}} (U_{i+1} \setminus (U_i \oplus V)) > \text{codim}_{U_{i+1}} (U_{i+1} \setminus (U_i \oplus V)) \).
3. \( \text{codim}_{V_i} (V_i \setminus U_i) > \text{codim}_{V_i} (V_i \setminus U_i) \).
4. \( U_i \in \text{Sm}^G_{\text{free}/k} \).

The above definition is a variant of the notion of admissible gadgets in [14, §4.2], where these terms are defined for vector bundles over a scheme. An example of an admissible gadget for \( G \) can be constructed as follows. Choose a faithful \( k \)-rational representation \( W \) of \( G \) of dimension \( n \). Then \( G \) acts freely on an open subset \( V \) of \( V = W^{\oplus n} \). Let \( Z = V \setminus U \). We now take \( V_i = V^{\oplus i}, U_1 = U \) and \( U_{i+1} = (U_i \oplus V) \cup (V \oplus U_i) \) for \( i \geq 1 \). Setting \( Z_1 = Z \) and \( Z_{i+1} = U_{i+1} \setminus (U_i \oplus V) \) for \( i \geq 1 \), one checks that \( V_i \setminus U_i = Z^i \) and \( Z_{i+1} = Z^i \oplus U \). In particular, \( \text{codim}_{V_i} (V_i \setminus U_i) = i(\text{codim}_{V}(Z)) \) and \( \text{codim}_{U_{i+1}} (Z_{i+1}) = (i+1)d - i(\text{dim}(Z)) - d = i(\text{codim}_{V}(Z)) \), where \( d = \text{dim}(V) \). Moreover, \( U_i \to U_i/G \) is a principal \( G \)-bundle.

2.3. The Borel spaces. Let \( X \in \text{Sm}^G_k \). For an admissible gadget \( \rho \), let \( X_G^i(\rho) \) denote the mixed quotient space \( X \times U_i \). If the admissible gadget \( \rho \) is clear from the given context, we shall write \( X_G^i(\rho) \) simply as \( X_G^i \).

We define the motivic Borel space \( X_G(\rho) \) to be the colimit \( \text{colim}_i X_G^i(\rho) \), where colimit is taken with respect the inclusions \( U_i \subset U_i \oplus V \subset U_{i+1} \) in the category of motivic spaces. We can think of \( X_G(\rho) \) as a smooth ind-scheme in \( \text{Spc} \). The finite-dimensional Borel spaces of the type \( X_G^i(\rho) \) were first considered by Totaro [26] in order to define the Chow ring of the classifying spaces of linear algebraic groups. For an admissible gadget \( \rho \), we shall denote the spaces \( \text{colim}_i U_i \) and \( \text{colim}_i (U_i/G) \) by \( E_G(\rho) \) and \( B_G(\rho) \) respectively. The definition of the motivic spaces \( X_G \) is based on the following observations.

Lemma 2.3. For any \( X \in \text{Sm}^G_k \), the natural map \( X_G(\rho) \overset{\sim}{\to} X \times E_G(\rho) \) is an isomorphism in \( \text{Spc} \).

Proof. We first observe that the map \( X \times U_i \to X \times U_{i+1} \) is a closed immersion of smooth schemes and \( \text{colim}_i (X \times U_i) \) is the union of its finite-dimensional sub-schemes \( (X \times U_i) \)'s. Moreover, \( G \) acts freely on \( \text{colim}_i (X \times U_i) \) such that each \( X \times U_i \) is \( G \)-invariant. Since any \( G \)-equivariant map \( f : \text{colim}_i (X \times U_i) \to Y \) with trivial \( G \)-action on \( Y \) factors through a unique map \( \text{colim}_i (X \times U_i) /G \to Y \), we see that the map \( X_G(\rho) \to (\text{colim}_i (X \times U_i))/G \) is an isomorphism. Thus we only need to show that the natural map \( \text{colim}_i (X \times U_i) \to X \times E_G(\rho) \) is an isomorphism.

To show this, it suffices to prove that these two spaces coincide as representable functors on \( \text{Spc} \). Any object of \( \text{Spc} \) is a colimit of simplicial sheaves of the form \( Y \times \Delta[n] \), where \( Y \) is a smooth scheme. Since \( \text{Hom}_{\text{Spc}}(\text{colim} \, F, -) = \lim \text{Hom}_{\text{Spc}}(F, -) \), we only need to show that the map

\[
\text{Hom}_{\text{Spc}}(Y \times \Delta[n], \text{colim}_i (X \times U_i)) \to \text{Hom}_{\text{Spc}}(Y \times \Delta[n], X \times E_G(\rho))
\]

is bijective for all \( Y \in \text{Sm}^G_k \) and all \( n \geq 0 \).
For any \( F \in \Delta^{op}\text{Shv}(\text{Nis}_k) \), there are isomorphisms

\[
\text{Hom}_{\text{Spe}}(Y \times \Delta[n], F) \cong \mathcal{F}_n(Y) = \text{Hom}_{\text{Shv}(\text{Nis}_k)}(Y, F_n) = \text{Hom}_{\text{Spe}}(Y, F_n),
\]

where \( \mathcal{F}_n \) is the \( n \)-th level of the simplicial sheaf \( \mathcal{F} \). Since \( \text{colim}_i \) (\( X \times U_i \)) and \( X \times E_G(\rho) \) are constant simplicial sheaves, we are reduced to showing that the map

\[
\text{Hom}_{\text{Spe}}(Y, \text{colim}_i (X \times U_i)) \to \text{Hom}_{\text{Spe}}(Y, X \times E_G(\rho))
\]

is bijective.

On the other hand, it follows from [28, Proposition 2.4] that

\[
\text{Hom}_{\text{Spe}}(Y, \text{colim}_i (X \times U_i)) \cong \text{colim}_i \text{Hom}_{\text{Spe}}(Y, X \times U_i)
\]

\[
\cong \text{colim}_i [\text{Hom}_{\text{Spe}}(Y, X) \times \text{Hom}_{\text{Spe}}(Y, U_i)]
\]

\[
\cong \text{Hom}_{\text{Spe}}(Y, X) \times \text{Hom}_{\text{Spe}}(Y, \text{colim}_i U_i)
\]

\[
\cong \text{Hom}_{\text{Spe}}(Y, X \times E_G(\rho))
\]

which proves the lemma. \( \square \)

**Proposition 2.4.** For any two admissible gadgets \( \rho \) and \( \rho' \) for \( G \) and for any \( X \in \text{Sm}_k^G \), there is a canonical isomorphism \( X_G(\rho) \cong X_G(\rho') \) in \( \mathcal{H}(k) \).

**Proof.** This was proven by Morel-Voevodsky [14, Proposition 4.2.6] when \( X = \text{Spec}(k) \) and a similar argument works in the general case as well.

For \( i, j \geq 1 \), we consider the smooth scheme \( \mathcal{V}_{i,j} = (X \times U_i \times V_j')/G \) and the open subscheme \( \mathcal{U}_{i,j} = (X \times U_i \times V_j')/G \). For a fixed \( i \geq 1 \), this yields a sequence \( (\mathcal{V}_{i,j}, \mathcal{U}_{i,j}, f_{i,j})_{j \geq 1} \), where \( \mathcal{V}_{i,j} \xrightarrow{\pi_{i,j}} X_G(\rho) \) is a vector bundle, \( \mathcal{U}_{i,j} \subseteq \mathcal{V}_{i,j} \) is an open subscheme of this vector bundle and \( f_{i,j} : (\mathcal{V}_{i,j}, \mathcal{U}_{i,j}) \to (\mathcal{V}_{i,j+1}, \mathcal{U}_{i,j+1}) \) is the natural map of pairs of smooth schemes over \( X_G(\rho) \). Then \( (\mathcal{V}_{i,j}, \mathcal{U}_{i,j}, f_{i,j})_{j \geq 1} \) is an admissible gadget over \( X_G(\rho) \) in the sense of [14, Definition 4.2.1]. Setting \( \mathcal{U}_i = \text{colim}_j \mathcal{U}_{i,j} \) and \( \pi_{i} = \text{colim}_j \pi_{i,j} \), it follows from [loc. cit., Proposition 4.2.3] that the map \( \mathcal{U}_i \xrightarrow{\pi_{i}} X_G(\rho) \) is an \( A^1 \)-weak equivalence.

Taking the colimit of these maps as \( i \to \infty \) and using [loc. cit., Corollary 1.1.21], we conclude that the map \( \mathcal{U} \xrightarrow{\pi} X_G(\rho) \) is an \( A^1 \)-weak equivalence, where \( \mathcal{U} = \text{colim}_{i,j} \mathcal{U}_{i,j} \). Reversing the roles of \( \rho \) and \( \rho' \), we find that the obvious map \( \mathcal{U} \xrightarrow{\pi'} X_G(\rho') \) is also an \( A^1 \)-weak equivalence. This yields the canonical isomorphism \( \pi' \circ \pi^{-1} : X_G(\rho) \xrightarrow{\cong} X_G(\rho') \) in \( \mathcal{H}(k) \). \( \square \)

2.3.1. **Admissible gadgets associated to a given \( G \)-scheme.** A careful reader may have observed in the proof of Proposition 2.4 that we did not really use the fact that \( G \) acts freely on an open subset \( U_i \) (resp. \( U'_j \)) of the \( G \)-representation \( V_i \) (resp. \( V'_j \)). One only needs to know that for each \( i, j \geq 1 \), the quotients \( (X \times U_i)/G \) and \( (X \times U'_j)/G \) are smooth schemes and the maps \( (X \times U_i \times V_j')/G \to (X \times U_i)/G \) and \( (X \times V_i \times U'_j)/G \to (X \times U'_j)/G \) are vector bundles with appropriate properties. This observation leads us to the following variant of Proposition 2.4 which will sometimes be useful.

Let \( G \) be a linear algebraic group over \( k \) and let \( X \in \text{Sch}_k^G \). We shall say that a pair \((V, U)\) of smooth schemes over \( k \) is a good pair for the \( G \)-action on \( X \), if \( V \) is a \( k \)-rational representation of \( G \) and \( U \subseteq V \) is a \( G \)-invariant open subset such
that $X \times U$ is an object of $\text{Sch}_{\text{free/k}}^G$. We shall say that the sequence of pairs $\rho = (V_i, U_i)_{i \geq 1}$ of smooth schemes over $k$ is an admissible gadget for the $G$-action on $X$, if there exists a good pair $(V, U)$ for the $G$-action on $X$ such that $V_i = V_i^G$ and $U_i \subseteq V_i$ is $G$-invariant open subset such that the following hold for each $i \geq 1$.

1. $(U_i \oplus V) \cup (V \oplus U_i) \subseteq U_{i+1}$ as $G$-invariant open subsets.
2. $\text{codim}_{U_{i+1}}(U_{i+2} \setminus (U_i \oplus V)) > \text{codim}_{U_{i+1}}(U_{i+1} \setminus (U_i \oplus V))$.
3. $\text{codim}_{U_{i+1}}(V_{i+1} \setminus U_{i+1}) > \text{codim}_i(V \setminus U_i)$.
4. $X \times U_i \in \text{Sch}_{\text{free/k}}^G$.

Notice that an admissible gadget for $G$ as in Definition 2.2 is an admissible gadget for the $G$-action on every $G$-scheme $X$.

**Proposition 2.5.** Let $\rho_X$ and $\rho'_X$ be two admissible gadgets for the $G$-action on a smooth scheme $X$. Then there is a canonical isomorphism of motivic spaces

$$\text{colim}_i \left( X^G \times U_i \right) \cong \text{colim}_j \left( X^G \times U'_j \right).$$

In view of Proposition 2.4, we shall denote a motivic space $X_G(\rho)$ simply by $X_G$. The motivic space $B_G$ is called the classifying space of the linear algebraic group $G$ following the notations of [14]. It follows from [14, Proposition 4.2.3] that the space $E_G(\rho)$ is $\mathbb{A}^1$-contractible in $\mathcal{H}(k)$ and Lemma 2.3 implies that $B_G(\rho)$ is the quotient of $E_G(\rho)$ for the free $G$-action. Given $X \in \text{Sm}_{\text{k}}^G$, the motivic Borel space of $X$ will mean the motivic space $X_G \in \text{Spc}$.

**2.4. Morita equivalence for Borel spaces.** The motivic Borel spaces corresponding to different algebraic groups satisfy the following $\mathbb{A}^1$-homotopy version of the Morita equivalence.

**Proposition 2.6.** Let $H$ be a closed normal subgroup of a linear algebraic group $G$ and let $F = G/H$. Let $f : X \to Y$ be a morphism in $\text{Sm}_{\text{k}}^G$ which is an $H$-torsor for the restricted action. Then there is a canonical isomorphism $X_G \cong Y_F$ in $\mathcal{H}(k)$.

**Proof.** We first observe from [13, Corollary 12.2.2] that $F$ is also a linear algebraic group over the given ground field $k$. Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $F$. The natural morphism $G \to F$ shows that each $V_i$ is a $k$-rational representation of $G$ such that the open subset $U_i$ is $G$-invariant, even though $G$ may not act freely on $U_i$. In particular, $G$ acts on the product $X \times U_i$ via the diagonal action. Since $H$ acts freely on $X$ and $F$ acts freely on $U_i$, it follows that the map $X \times U_i \to X^G \times U_i$ is a $G$-torsor and hence $\rho = (V_i, U_i)_{i \geq 1}$ is an admissible gadget for the $G$-action on $X$.

Since the map $X^G \times U_i \to Y^F \times U_i$ is an isomorphism for every $i \geq 1$, we conclude from Proposition 2.5 that $X_G \cong \text{colim}_i \left( X^G \times U_i \right) \cong Y^F$ in $\mathcal{H}(k)$. \qed

**Corollary 2.7** (Morita isomorphism). Let $H$ be a closed subgroup of a linear algebraic group $G$ and let $X \in \text{Sm}_{\text{k}}^H$. Let $Y$ denote the space $X^H$ for the action $h \cdot (x, g) = (hx, gh^{-1})$. Then there is a canonical isomorphism $X_H \cong Y_G$ in $\mathcal{H}(k)$.

**Proof.** Define an action of $H \times G$ on $X \times G$ by

$$ (h, g) \cdot (x, g') = (hx, gg'h^{-1}) $$

(2.1)
and an action of $H \times G$ on $X$ by $(h, g) \cdot x = hx$. Then the projection map $X \times G \xrightarrow{p} X$ is $(H \times G)$-equivariant and a $G$-torsor. Hence there is canonical isomorphism $X_H \cong (X \times G)_{H \times G}$ in $\mathcal{H}(k)$ by Proposition 2.6.

On the other hand, the projection map $X \times G \to X \times G$ is $(H \times G)$-equivariant and an $H$-torsor. Hence there is a canonical isomorphism $(X \times G)_{H \times G} \cong Y_G$ in $\mathcal{H}(k)$ again by Proposition 2.6. Combining these two isomorphisms, we get $X_H \cong Y_G$ in $\mathcal{H}(k)$.

Recall that a unipotent group $U$ over $k$ is called split if it has a filtration $\{e\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = U$ by closed normal $k$-subgroups such that each quotient group $U_j/U_{j-1}$ is an elementary unipotent group (cf. [18, §3.4]).

**Proposition 2.8.** Let $G$ be a possibly non-reductive group over $k$. Assume that $G$ has a Levi decomposition $G = L \ltimes G^u$ such that $G^u$ is split over $k$ (e.g., if $k$ has characteristic zero). Then for any $X \in \text{Sm}_k^G$, the map $X_L \to X_G$ is an isomorphism in $\mathcal{H}(k)$.

*Proof.* Let us denote the unipotent radical $G^u$ by $U$ and let $\{e\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = U$ the filtration of $U$ as above. Set $G_i = LU_i$ for $0 \leq i \leq n$.

We have a sequence of morphisms $X_L \xrightarrow{p_0} X_{G_1} \xrightarrow{p_1} \cdots \xrightarrow{p_{n-1}} X_G$ in $\text{Spc}$ such that each $p_i$ is an $\mathbb{A}^1$-weak equivalence. Hence the map $X_L \to X_G$ is an $\mathbb{A}^1$-weak equivalence. \qed

### 3. $K$-theory of motivic Borel spaces

In this section, we define the algebraic versions of $K$-theory and $\mathcal{K}$-theory for motivic Borel spaces. We also study some properties of these two theories. The topological $K$-theory of such spaces already makes sense since this theory is defined for all topological spaces. The topological $\mathcal{K}$-theory was introduced by Atiyah [1] in order to ease the study the $K$-theory of infinite dimensional CW-complexes. We begin with the following result. Apart from being useful in defining $\mathcal{K}$-theory, it will also be used to prove the algebraic Atiyah-Segal completion theorem under the free action of a group.

#### 3.1. $K$-theory of quasi-bundles.

A quasi-bundle over a scheme $X$ is an open subset of a vector bundle over $X$. For any $X \in \text{Sch}_k$, let $G_*(X)$ (resp. $K_*(X)$) denote the Quillen $K$-theory of coherent sheaves (vector bundles) on $X$. One knows that the functor $X \mapsto G_*(X)$ is covariant for proper maps and contravariant for maps of finite Tor-dimension. In particular, for maps of finite Tor-dimension $X \xrightarrow{f} Y \xrightarrow{g} Z$, one has

$$ (g \circ f)^* = f^* \circ g^* : G_*(Z) \to G_*(X). $$

On the other hand, $X \mapsto K_*(X)$ is contravariant for all maps and covariant for proper maps of finite Tor-dimension (cf. [19, §5.10]). Moreover, $K$-theory and $G$-theory satisfy the appropriate projection formulas whenever defined.

**Proposition 3.1.** Let $X$ be a $k$-scheme and let $p : E \to X$ be a vector bundle of rank $r \geq 1$. Then there exists a positive integer $n$ such that for any closed subscheme $Y \subseteq E$ of codimension larger than $n$, the restriction map $G_i(E) \to G_i(E \setminus Y)$ is injective for all $i \geq 0$. 
Proof. Set $U = E \setminus Y$ and let $j : U \to Y$ be the inclusion map. We consider the commutative diagram

\[\begin{array}{ccc}
G_i(X) & \xrightarrow{j^* op^*} & G_i(E \setminus Y) \\
p^* \downarrow & & \downarrow f^* \\
G_i(E) & \xrightarrow{s} & G_i(U)
\end{array}\]

of algebraic $G$-groups. Since $p^*$ is an isomorphism by the homotopy invariance, the proposition is equivalent to showing that the composite map $j^* \circ p^*$ is injective.

We first assume that $E$ is the trivial bundle so that $p$ is the projection map $X \times \mathbb{A}^r \to X$. In this case, it is enough to show using (3.1) that the map $p \circ j : U \to X$ has a section $s : X \to U$ which is proper of finite Tor-dimension. Let $q : X \times \mathbb{A}^r \to \mathbb{A}^r$ be the other projection. Suppose that $\text{codim}_E(Y) > \dim(X)$. This will imply that $\dim(q(Y)) \leq \dim(Y) < r$ and hence there is a $k$-rational point $t \in \mathbb{A}^r$ such that $(X \times \{t\}) \cap Y = \emptyset$. Since the inclusion $X \times \{t\} \subseteq X \times \mathbb{A}^r$ is a regular closed immersion whose image is contained in $U$, we get a desired section of the map $p \circ j$.

Next suppose that $X$ is affine and that $\text{codim}_E(Y) > \dim(X)$. Then there is a smooth surjective map $q : X \times \mathbb{A}^s \xrightarrow{q} E$ of vector bundles over $X$. Setting $Z = q^{-1}(Y)$ and $V = q^{-1}(U)$, we see that $\dim(Z) \leq \dim(Y) + s - r < s$. Hence by the previous case, there is a section $X \to V$ of the projection map $V \to U \to X$.

The composite $X \to V \xrightarrow{q} U$ is then a closed immersion which yields a section of the map $U \to X$. This section is of finite Tor-dimension because $X \to V$ is such by the previous case and $q$ is smooth. Thus we have shown the injectivity of the map $j^* \circ p^*$ when $X$ is affine.

In the general case, we can apply the Jouanolou’s trick to get a vector bundle torsor $f : X' \to X$ such that $X'$ is affine. Let $p' : E' \to X'$ be the pullback of the vector bundle $E$ to $X'$ and let $f' : E' \to E$ be the induced map between the vector bundles. Suppose $Y \subseteq E$ is such that $Y' = p'^{-1}(Y)$ and we have shown above that in this case, the map $G_i(X') \to G_i(E')$ is injective for all $i \geq 0$. The commutative diagram

\[\begin{array}{ccc}
G_i(X) & \xrightarrow{j^* op^*} & G_i(E \setminus Y) \\
p^* \downarrow & & \downarrow f^* \\
G_i(X') & \xrightarrow{s} & G_i(E' \setminus Y')
\end{array}\]

shows that $j^* \circ p^*$ must be injective since $f^*$ is an isomorphism by homotopy invariance. \qed

Remark 3.2. Under the hypothesis of Proposition 3.1, we can not claim that the map $G_i(E) \to G_i(E \setminus Y)$ is an isomorphism even if the codimension of $Y$ is arbitrarily large. To see this, we just take $X$ to be Spec($k$) and $Y$ to be the origin of an affine space $\mathbb{A}^r$. Then for any $i \geq 0$, we get a short exact sequence

\[0 \to K_i(\mathbb{A}^r) \to K_i(\mathbb{A}^r \setminus \{0\}) \to K_{i-1}(k) \to 0\]

and we know that $K_i(k)$ is not zero in general for any $i \geq 0$. \qed
3.2. **Isomorphism of two ind-objects in $\mathcal{SH}(k)$**. Recall that an ind-object $A = \{A_i, p_{i,j}\}_{i \in I}$ in a category $\mathcal{C}$ is a directed system of objects in $\mathcal{C}$. A morphism $f : A \to B$ between two ind-objects $A = \{A_i, p_{i,j}\}_{i \geq 1}$ and $B = \{B_i, q_{i,j}\}_{i \geq 1}$, directed by $\mathbb{N}^+$, is a function $\lambda : \mathbb{N}^+ \to \mathbb{N}^+$ and a morphism $f_i : A_i \to B_{\lambda(i)}$ in $\mathcal{C}$ for each $i \geq 1$ such that for any $j \geq i$, there is some $l \geq \lambda(i), l \geq \lambda(j)$ so that the diagram

$$
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_{\lambda(i)} \\
p_{i,j} & & \downarrow q_{\lambda(i),l} \\
A_j & \xrightarrow{f_j} & B_{\lambda(j)} \\
\end{array}
$$

commutes in $\mathcal{C}$. We shall say that the morphism $f : \{A_i\}_{i \geq 1} \to \{B_i\}_{i \geq 1}$ is strict if $\lambda$ is the identity map.

One has also the dual notion of pro-objects in $\mathcal{C}$, in which the morphisms $p_{i,j}$ and $f_i$ above are in the opposite direction. In particular, one notices that if $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then it induces a functor from the category of ind-objects in $\mathcal{C}$ to the category of pro-objects in $\mathcal{D}$. We shall use this obvious fact in the sequel. The notion of ind-objects and pro-objects in a category can be defined in a more general way where the index category could be any cofiltered or filtered category rather than a directed set. But we shall not have occasion to use this general notion.

Recall that $T$ denotes the pointed motivic space $(\mathbb{P}^1_k, \infty)$. Let $\Sigma^\infty_T : \mathcal{H}_s(k) \to \mathcal{SH}(k)$ be the functor which takes a motivic space to the corresponding $T$-spectrum by smashing repeatedly with $T$. For any $a \geq b \geq 0$, let $\Sigma^{a,b}_T$ denote the suspension functor $\Sigma^{a,b}_T = \Sigma^a_s \wedge \Sigma^b_t$, where $\Sigma_s$ and $\Sigma_t$ are the functors given by the suspension with the simplicial circle $S_s = S^1$ and the Tate circle $S_t = (\mathbb{G}_m, 1)$ respectively (cf. [14]). Our interest in the ind-objects and pro-objects is motivated by the following result in the stable homotopy category of smooth schemes over $k$. This result will form the main ingredient for our definition of the algebraic $K$-theory.

**Proposition 3.3.** Let $\rho = (V_i, U_i)_{i \geq 1}$ and $\rho' = (V'_i, U'_i)_{i \geq 1}$ be two admissible gadgets for a linear algebraic group $G$ over $k$. Then for any $X \in \text{Sm}_k^G$, there is a canonical isomorphism

$$\{\Sigma^\infty_T X^i_G(\rho)\}_{i \geq 1} \xrightarrow{\cong} \{\Sigma^\infty_T X^i_G(\rho')\}_{i \geq 1}$$

of ind-objects in $\mathcal{SH}(k)$.

**Proof.** For $i, i' \geq 1$, we set $X^i_{i'} = (X \times U_i \times U'_i)/G$. For any $j \geq i, j' \geq i'$, there is a closed immersion $\theta^i_{i',j'} : X^i_{i'} \to X^j_{j'}$ and this induces a natural map $\sigma^{i,j}_{i',j'} = \Sigma_T^i \theta^i_{i',j'} : \Sigma_T^i X^i_{i'} \to \Sigma_T^j X^j_{j'}$ in $\mathcal{SH}(k)$. Set $\mathcal{U} = \text{colim}_i \text{colim}_{i'} X^i_{i'}$ as a motivic space. Since for any $i, i' \geq 1$, the inclusion $X^i_{i'} \hookrightarrow \mathcal{U}$ factors through the inclusions $X^i_{i'} \hookrightarrow X^i_{i} \hookrightarrow \mathcal{U}$ where $l = \max(i, i')$, we see that $\mathcal{U}$ is also the colimit of the direct system of smooth schemes $\{U_i, \lambda_{i,j}\}$ if we let $U_i = X^i_{i}$ and $\lambda_{i,j} = \theta^i_{i,j}$.

There are natural projections $p_i : U_i \to X^i_G(\rho)$ and $q_i : U_i \to X^i_G(\rho')$. These maps combine together to yield a morphism of ind-schemes $p : \{U_i\} \to \{X^i_G(\rho)\}$ and $q : \{U_i\} \to \{X^i_G(\rho')\}$. In particular, there are strict morphisms of ind-objects
(3.4) \[ \{ \Sigma^\infty_T X^i_G(\rho) \} \xrightarrow{\hat{p}} \{ \Sigma^\infty_T U_i \} \xrightarrow{\hat{q}} \{ \Sigma^\infty_T X^i_G(\rho') \} \]

in \( \mathcal{SH}(k) \). It suffices to show that these two morphisms of ind-objects are isomorphisms. We shall show that \( \hat{p} \) is an isomorphism and exactly the same proof works for the isomorphism of \( \hat{q} \). Let \( \gamma_{i,j} : X^i_G(\rho) \to X^j_G(\rho) \) denote the structure maps of the direct system \( \{ X^i_G(\rho) \} \).

To show that \( \hat{p} \) is an isomorphism of ind-objects in \( \mathcal{SH}(k) \), we need to show that for every \( i \geq 1 \), there exists \( j \gg i \) and a morphism \( \beta_i : \Sigma^\infty_T X^i_G(\rho) \to \Sigma^\infty_T U_i \) in \( \mathcal{SH}(k) \) such that the diagram

\[
\begin{array}{ccc}
\Sigma^\infty_T U_i & \xrightarrow{\Sigma^\infty_T p_i} & \Sigma^\infty_T X^i_G(\rho) \\
\downarrow{\beta_i} & & \downarrow \\
\Sigma^\infty_T U_j & \xrightarrow{\Sigma^\infty_T p_j} & \Sigma^\infty_T X^j_G(\rho)
\end{array}
\]

commutes.

We have shown in Proposition 2.4 that the map \( p = \text{colim}_i p_i : U \to X_G(\rho) \) of colimits is an \( A^1 \)-weak equivalence. Since \( \Sigma^\infty_T \) preserves colimits, this implies in particular that the map \( \psi = \Sigma^\infty_T p : \text{colim}_i \Sigma^\infty_T U_i \to \text{colim}_i \Sigma^\infty_T X^i_G(\rho) \) is an isomorphism in \( \mathcal{SH}(k) \). Let \( \phi : \text{colim}_i \Sigma^\infty_T X^i_G(\rho) \to \text{colim}_i \Sigma^\infty_T U_i \) be the inverse of \( \psi \).

Since each \( \Sigma^\infty_T X^i_G(\rho) \) is a compact object of \( \mathcal{SH}(k) \) (cf. [28, Proposition 5.5]), the composite map \( \Sigma^\infty_T X^i_G(\rho) \to \text{colim}_i \Sigma^\infty_T X^i_G(\rho) \xrightarrow{\phi} \text{colim}_i \Sigma^\infty_T U_i \) factors through a map \( \Sigma^\infty_T X^i_G(\rho) \xrightarrow{\beta_i} \Sigma^\infty_T U_i \to \text{colim}_i \Sigma^\infty_T U_i \) for some \( i' \gg i \). For every \( j \geq i' \), let \( \beta_{i,j} \) denote the composite map

\[
\beta_{i,j} : \Sigma^\infty_T X^i_G(\rho) \xrightarrow{\beta_i} \Sigma^\infty_T U_i \xrightarrow{\Sigma^\infty_T \lambda_{i,j}} \Sigma^\infty_T U_j.
\]

We claim that the diagram

\[
\begin{array}{ccc}
\Sigma^\infty_T X^i_G(\rho) & \xrightarrow{\beta_{i,j}} & \Sigma^\infty_T X^j_G(\rho) \\
\downarrow{\Sigma^\infty_T \gamma_{i,j}} & & \downarrow{\Sigma^\infty_T \gamma_{i,j}} \\
\Sigma^\infty_T U_j & \xrightarrow{\Sigma^\infty_T p_j} & \Sigma^\infty_T X^j_G(\rho)
\end{array}
\]

commutes for all \( j \gg i \). To prove this claim, we consider the bigger diagram

\[
\begin{array}{ccc}
\Sigma^\infty_T X^i_G(\rho) & \xrightarrow{\beta_{i,j}} & \Sigma^\infty_T X^j_G(\rho) \\
\downarrow{\Sigma^\infty_T \gamma_{i,j}} & & \downarrow{\Sigma^\infty_T \gamma_{i,j}} \\
\Sigma^\infty_T U_j & \xrightarrow{\Sigma^\infty_T p_j} & \Sigma^\infty_T X^j_G(\rho)
\end{array}
\]

and a morphism \( \beta_{i,j} \) that for every \( i \geq 1 \), there exists \( j \gg i \) and a morphism \( \beta_i : \Sigma^\infty_T X^i_G(\rho) \to \Sigma^\infty_T U_i \) in \( \mathcal{SH}(k) \) such that the diagram

\[
\begin{array}{ccc}
\Sigma^\infty_T U_i & \xrightarrow{\Sigma^\infty_T p_i} & \Sigma^\infty_T X^i_G(\rho) \\
\downarrow{\beta_i} & & \downarrow \\
\Sigma^\infty_T U_j & \xrightarrow{\Sigma^\infty_T p_j} & \Sigma^\infty_T X^j_G(\rho)
\end{array}
\]

commutes.
For every $j \geq i'$, the lower square clearly commutes and the outer trapezium commutes by the construction of $\beta_{i,j}$ and the fact that $\phi^{-1} = \psi$. In particular, the maps $\Sigma^\infty_T p_j \circ \beta_{i,j}$ and $\Sigma^\infty_T \gamma_{i,j}$ become same when we go all the way to the colimit $\Sigma^\infty_T X_G(\rho)$. Since $\Sigma^\infty_T X^j_G(\rho)$ is compact, the map

$$\text{colim}_{j \geq i} \text{Hom}_{\mathcal{SH}(k)} \left( \Sigma^\infty_T X^i_G(\rho), \Sigma^\infty_T X^j_G(\rho) \right) \rightarrow \text{Hom}_{\mathcal{SH}(k)} \left( \Sigma^\infty_T X^i_G(\rho), \Sigma^\infty_T X_G(\rho) \right)$$

is an isomorphism of sets. We conclude that the maps $\Sigma^\infty_T p_j \circ \beta_{i,j}$ and $\Sigma^\infty_T \gamma_{i,j}$ must become same when $j \gg i'$. This proves the claim.

Our next claim is that the diagram

$$\Sigma^\infty_T U_i \xrightarrow{\Sigma^\infty_T p_i} \Sigma^\infty_T X^i_G(\rho)$$

$$\Sigma^\infty_T \lambda_{i,j} \downarrow \beta_{i,j}$$

$$\Sigma^\infty_T U_j$$

commutes for all $j \gg i$. To do this, we consider the diagram

$$\Sigma^\infty_T U_i \xrightarrow{\Sigma^\infty_T p_i} \Sigma^\infty_T X^i_G(\rho)$$

$$\Sigma^\infty_T \lambda_{i,j} \downarrow \beta_{i,j}$$

$$\Sigma^\infty_T U_j \rightarrow \Sigma^\infty_T X^j_G(\rho)$$

$$\Sigma^\infty_T \lambda_j \downarrow \Sigma^\infty_T \gamma_j$$

$$\Sigma^\infty_T U \xrightarrow{\phi} \Sigma^\infty_T X_G(\rho)$$

for every $j \geq i'$.

By the construction of $\beta_{i,j}$, we know that $\Sigma^\infty_T \lambda_j \circ \beta_{i,j} = \phi \circ \Sigma^\infty_T \gamma_{i,j}$. On the other hand, we also know that

$$\psi \circ \Sigma^\infty_T \lambda_j \circ \Sigma^\infty_T \lambda_{i,j} = \psi \circ \Sigma^\infty_T \lambda_j$$

$$= \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T p_i$$

$$= \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T p_{i}.$$ 

Equivalently, we get

$$\Sigma^\infty_T \lambda_j \circ \beta_{i,j} \circ \Sigma^\infty_T p_i = \phi \circ \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T p_{i}$$

$$= \phi \circ \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T \gamma_{i,j} \circ \Sigma^\infty_T p_{i}$$

$$= \Sigma^\infty_T \lambda_j \circ \Sigma^\infty_T \lambda_{i,j}.$$ 

Since $\Sigma^\infty_T U_i$ is a compact object of $\mathcal{SH}(k)$, the same argument as above shows that we must have $\beta_{i,j} \circ \Sigma^\infty_T p_{i} = \Sigma^\infty_T \lambda_{i,j}$ for all $j \gg i$. The two claims together prove (3.5) and hence the proposition. \qed

3.3. **Algebraic $K$-theory of motivic spaces.** Recall from [28, §6.2] that there is a $T$-spectrum $BGL \times \mathbb{Z}$ in $\mathcal{SH}(k)$ which represents the algebraic $K$-theory. For any $A \in \mathcal{SH}(k)$, one defines the algebraic $K$-theory of $A$ by

$$K^{a,b}(A) = \text{Hom}_{\mathcal{SH}(k)} \left( A, \Sigma^\infty_B (BGL \times \mathbb{Z}) \right).$$

For $X \in \mathsf{Spc}$, the algebraic $K$-theory $K^{a,b}(\Sigma^\infty_T X_+)$ is denoted by $K^{a,b}(X)$. Using the canonical isomorphism $BGL \cong T \wedge BGL$ (cf. [28, Theorem 6.9]), one finds
that for $X \in \text{Spc}$,

$$K^{a,b}(X) = \text{Hom}_{\text{SH}(k)} \left( \Sigma^\infty_T (S^{2b-a}_a \wedge X_+), (BGL \times \mathbb{Z}) \right)$$

$$= \text{Hom}_{\text{SH}(k)} \left( S^{2b-a}_a \wedge X_+, BGL \times \mathbb{Z} \right)$$

and it is shown in [14, Theorem 4.3.13] that the last term is same as the Quillen-Thomason $K$-theory $K_X^{2b-a}(X)$, if $X \in \text{Sm}_k$. This allows us to define the algebraic $K$-theory of a motivic space $X$ by

$$K^i(X) := \text{Hom}_{\text{SH}(k)} \left( S^i \wedge X_+, BGL \times \mathbb{Z} \right)$$

$$= \text{Hom}_{\text{SH}(k)} \left( \Sigma^\infty_T (S^i \wedge X_+), BGL \times \mathbb{Z} \right).$$

The following result summarizes some of the basic properties of the $K$-theory of motivic spaces.

**Proposition 3.4.** The assignment $X \mapsto K_*(X)$ is a contravariant functor on $\mathcal{H}(k)$. This coincides with the Quillen $K$-theory of algebraic vector bundles if $X \in \text{Sm}_k$. It has the following other properties.

1. If $H \subseteq G$ is a closed subgroup of a linear algebraic group $G$ over $k$ and if $X \in \text{Sm}^H_k$, then the map $K_*(X_H) \rightarrow K_*(Y_G)$ is an isomorphism, where $Y = X \times G$.

2. For $X \in \text{Sm}^G_{\text{free}/k}$, the map $K_*(X/G) \rightarrow K_*(X_G)$ is an isomorphism.

3. If $G$ has a Levi decomposition $G = L \times G^u$ such that $G^u$ is split over $k$, then $K_*(X_L) \cong K_*(X_G)$ for any $X \in \text{Sm}^G_k$.

**Proof.** The contravariance follows from the fact that $A \mapsto K_*(A)$ is a generalized cohomology theory on $\text{SH}(k)$. The isomorphism with Quillen $K$-theory for smooth schemes is proven in [14, Theorem 4.3.13]. The property (1) follows from Corollary 2.7. The property (2) follows from Lemma 2.3 and [14, Lemma 4.2.9]. The last property follows from Proposition 2.8.

3.4. **Algebraic $K$-theory of Borel spaces.** Having defined the algebraic $K$-theory of motivic spaces and $T$-spectra as in (3.10), we obtain the following immediate consequence of Proposition 3.3.

**Corollary 3.5.** Let $\rho = (V_i, U_i)_{i \geq 1}$ and $\rho' = (V'_i, U'_i)_{i \geq 1}$ be two admissible gadgets for a linear algebraic group $G$ over $k$. Then for any $X \in \text{Sm}^G_k$ and $p \geq 0$, there is a canonical isomorphism

$$\left\{ K_p \left( X^i_G(\rho) \right) \right\}_{i \geq 1} \xrightarrow{\cong} \left\{ K_p \left( X^i_G(\rho') \right) \right\}_{i \geq 1}$$

of pro-abelian groups. In particular,

1. $\lim^m_i K_p \left( X^i_G(\rho) \right) \cong \lim^m_i K_p \left( X^i_G(\rho') \right)$ for $m \geq 0$.

2. $\left\{ K_p \left( X^i_G(\rho) \right) \right\}_{i \geq 1}$ satisfies the Mittag-Leffler condition if and only if so does $\left\{ K_p \left( X^i_G(\rho') \right) \right\}_{i \geq 1}$.

**Proof.** Since a contravariant functor on $\text{SH}(k)$ takes an isomorphism of ind-objects to an isomorphism of pro-objects, the isomorphism (3.13) follows directly from Propositions 3.3 and 3.4. The second assertion follows from the isomorphism (3.13) and the elementary fact that the derived limits of two isomorphic pro-abelian groups are isomorphic (cf. [17, Corollary 7.3.7]). Also, a pro-abelian group $\{A_i\}$
satisfies the Mittag-Leffler condition if and only if it is isomorphic to a pro-abelian group \( \{ B_i \} \) such that the map \( B_j \to B_i \) is surjective for all \( j \geq i \).

Recall that in order to study the connection between the representation ring and the cohomology of the classifying spaces of compact Lie groups, Atiyah [1] (see also [2, §4.6]) introduced the \( K \)-theory for infinite CW-complexes. This is defined in terms of the projective limit of the usual topological \( K \)-theory of the various skeleta of the given CW-complex and it plays a very important role in understanding the topological \( K \)-theory of such complexes. The above corollary allows us to define an algebraic analogue of such objects for motivic spaces \( X_G \). This will be one of three main objects of study in the proof of the algebraic analogue of the Atiyah-Segal theorem.

**Definition 3.6.** Let \( G \) be a linear algebraic group over \( k \) and let \( X \in \text{Sm}_k^G \). We define

\[
\mathcal{K}_p(X_G) := \lim_{\leftarrow i} K_p \left( X^G \times U_i \right)
\]

where \( \rho = (V_i, U_i) \) is any admissible gadget for \( G \).

It follows from Corollary 3.5 that \( \mathcal{K}_p(X_G) \) is well-defined for every \( p \geq 0 \). Since \( \mathcal{K}_p(X_G) \) is the limit of an inverse system of \( R(G) \)-linear maps, we see that it is an \( R(G) \)-module. Since the structure maps of the inverse system \( \left\{ K_p \left( X^G \times U_i \right) \right\} \) are defined in terms of the pull-back maps, one checks easily from the various properties of the ordinary algebraic \( K \)-theory that the functor \( X \mapsto \mathcal{K}_*(X_G) \) on \( \text{Sm}_k^G \) satisfies all the properties of an oriented cohomology theory except possibly the localization sequence. However, it is true that this theory satisfies the localization sequence as well. A proof of this will appear elsewhere.

For any \( X \in \text{Sm}_k^G \), the inclusions \( X_G^i(\rho) \hookrightarrow X_G(\rho) \) induce a natural map

\[
\tau^G_X : \mathcal{K}_*(X_G) \to \mathcal{K}_* (X_G)
\]

which is surjective. This can be seen from the Milnor exact sequence (cf. [12, Proposition 7.3.2])

\[
0 \to \lim_{\leftarrow i} K_{p+1} (X_G^i(\rho)) \to K_p (X_G(\rho)) \to \lim_{\leftarrow i} K_p (X_G^i(\rho)) \to 0.
\]

Following the terminology of Atiyah [1], we shall say that \( \mathcal{K}_*(X_G) \) has no element of finite order if it contains no non-zero element which dies in all of \( K_*(X_G^i) \). This is equivalent to saying that the map \( \tau^G_X \) is injective. One of the key steps in the proof of our main results of this paper is to show in the cases of interest that \( \mathcal{K}_*(X_G) \) has no element of finite order.

4. **Equivariant \( K \)-theory of filtrable schemes**

Our strategy for the algebraic Atiyah-Segal theorem is to reduce the general problem to the case when the underlying group is a split torus. Recall that a torus \( T \) over \( k \) is called split if it is isomorphic to a finite product of the multiplicative group \( \mathbb{G}_m \) as a \( k \)-group scheme. The number of copies of \( \mathbb{G}_m \) in \( T \) is called the rank of \( T \). We shall prove some general structural results for the equivariant \( K \)-theory of a class of schemes. It will turn out that this class contains a large number of schemes with group action.
4.0.1. Equivariant $K$-theory. We recall the equivariant $K$-theory of group scheme actions from \cite{22}, \cite{24} and \cite{25}. Let $G$ be a linear algebraic group over $k$. For $X \in \text{Sch}_k^G$, $G^G(X)$ (resp. $K^G(X)$) is the $K$-theory spectrum of the $G$-equivariant coherent sheaves (vector bundles) on $X$. $G^G(X)$ is a contravariant functor in the group $G$, it is contravariant with respect to equivariant flat maps, and is covariant with respect to equivariant proper maps. $K^G(X)$ is contravariant in $G$ as well as in $X$ and it is a ring spectrum with respect to the tensor product of equivariant vector bundles. The spectrum $G^G(X)$ is a module over the ring spectrum $K^G(X)$. The projection formula holds for equivariant proper maps. That is, for an equivariant $X$ and it is a ring spectrum with respect to the tensor product of equivariant vector bundles. The spectrum $G^G(X)$ is a module over the ring spectrum $K^G(X)$. The projection formula holds for equivariant proper maps. That is, for an equivariant proper map $f : X \to Y$, $f_* : G^G(X) \to G^G(Y)$ is a morphism of $K^G(Y)$-modules.

For $p \geq 0$, $G^G_p(X)$ and $K^G_p(X)$ denote the stable homotopy groups of the corresponding spectra. For $X \in \text{Sm}_k^G$, the map $K^G(X) \to G^G(X)$ is a homotopy equivalence of spectra. Furthermore, for $X \in \text{Sch}_{f\text{ree}/k}^G$, the natural map $G(X/G) \to G^G(X)$ is a homotopy equivalence of spectra (cf. \cite{9} §3.2).

The ring $K^G_0(k)$ is also denoted by $R(G)$. It is isomorphic to the Grothendieck ring of finite-dimensional representations of $G$. For any $G$-equivariant map $f : X \to Y$, the maps $f_*$ and $f^*$ are morphisms of $R(G)$-modules. Let $\epsilon : R(G) \to \mathbb{Z}$ denote the augmentation map which takes any virtual representation to its rank. The kernel $I_G$ of this map is called the augmentation ideal of $G$. It is known that $R(G)$ is a commutative noetherian ring (cf. \cite{13} Lemma 14.1). The $I_G$-adic completion of the ring $R(G)$ will be denoted by $\hat{R}(G)$.

The equivariant $K$-theory also satisfies the following Morita equivalence by a result of Thomason.

**Theorem 4.1** (\cite{25} Theorem 1.13]). Let $G$ be a linear algebraic group over $k$ and let $H$ be a closed subgroup of $G$. For any $X \in \text{Sch}_k^H$, there is a canonical isomorphism $G^G_*(X \times^H G) \cong G^H_*(X)$ of $R(G)$-modules which is natural in $X$. If $G$ has a Levi decomposition $G = L \ltimes G^u$ such that $G^u$ is split over $k$, then $G^G_*(X) \cong G^L_*(X)$ for any $X \in \text{Sch}_k^G$.

4.1. Filtrable schemes. Let $G$ be a linear algebraic group over $k$ and let $X \in \text{Sch}_k^G$. Following \cite{9} Section 3, we shall say that $X$ is $G$-filtrable if the fixed point locus $X^G$ is smooth and projective, and there is an ordering $X^G = \bigcup_{m=0}^n Z_m$ of the connected components of the fixed point locus, a filtration of $X$ by $G$-invariant closed subschemes

$$ (4.1) \quad \emptyset = X_{-1} \subsetneq X_0 \subsetneq \cdots \subsetneq X_n = X $$

and maps $\phi_m : W_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \leq m \leq n$ which are all $G$-equivariant vector bundles such that the inclusions $Z_m \to W_m$ are the 0-section embeddings. It is important to note that the closed subschemes $X_m$’s may not be smooth even if $X$ is so. Observe also that if $X$ is $G$-filtrable, then each closed subscheme $X_m$ is also $G$-filtrable. If $X$ is a smooth $G$-filtrable scheme, the associated motivic Borel space $X_G$ will be called filtrable.

We shall say that a $k$-scheme $X$ is filtrable if there are closed subschemes $\{Z_0, \cdots, Z_m\}$ of $X$ which are connected, smooth and projective, a filtration of
$X$ by closed subschemes

$$\emptyset = X_{-1} \subsetneq X_0 \subsetneq \cdots \subsetneq X_n = X$$

and maps $\phi_m : W_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \leq m \leq n$ which are all vector bundles such that the inclusions $Z_m \hookrightarrow W_m$ are the 0-section embeddings. It is clear that a $G$-filtrable scheme is also filtrable. We prove the following structural result for the equivariant and ordinary $K$-theory of filtrable schemes.

**Theorem 4.2.** Let $G$ be a linear algebraic group over $k$ and let $X \in \text{Sch}_k^G$ be $G$-filtrable with $G$-invariant filtration as in (4.2). Then there is a canonical isomorphism of $R(G)$-modules

$$\bigoplus_{m=0}^n G_s^G(Z_m) \xrightarrow{\cong} G_s^G(X).$$

For a filtrable scheme $X$, there is a canonical isomorphism

$$\bigoplus_{m=0}^n G_s(Z_m) \xrightarrow{\cong} G_s(X).$$

**Proof.** We prove the first isomorphism and the second isomorphism is proven in identical way. We prove this by induction on the length of the filtration. For $n = 0$, the map $X = X_0 \xrightarrow{\phi_0} Z_0$ is a $G$-equivariant vector bundle and hence the proposition follows from the homotopy invariance.

We now assume by induction that $1 \leq m \leq n$ and that there is a canonical isomorphism of $R(G)$-modules

$$\bigoplus_{j=0}^{m-1} G_s^G(Z_j) \xrightarrow{\cong} G_s^G(X_{m-1}).$$

The localization exact sequence for the inclusions $i_{m-1} : X_{m-1} \hookrightarrow X_m$ and $j_m : W_m = X_m \setminus X_{m-1}$ of the $G$-invariant closed and open subschemes yields a long exact sequence of $R(G)$-linear maps

$$\cdots \to G_p^G(X_{m-1}) \xrightarrow{i_{m-1,*}} G_p^G(X_m) \xrightarrow{j_m^*} G_p^G(W_m) \xrightarrow{\partial} G_{p-1}^G(X_{m-1}) \to \cdots.$$\[4.5\]

Using (4.5), it suffices now to construct a canonical $R(G)$-linear splitting of the pull-back $j_m^*$ in order to prove the theorem.

Let $V_m \subset W_m \times Z_m$ be the graph of the projection $W_m \xrightarrow{\phi_m} Z_m$ and let $Y_m$ denote the closure of $V_m$ in $X_m \times Z_m$. Then $Y_m$ is a $G$-invariant closed subset of $X_m \times Z_m$ and $V_m$ is $G$-invariant and open in $Y_m$. We consider the composite maps

$$p_m : V_m \hookrightarrow W_m \times Z_m \to W_m, \quad q_m : V_m \hookrightarrow W_m \times Z_m \to Z_m$$

in $\text{Sch}_k^G$. Note that $p_m$ is a projective morphism since $Z_m$ is projective. The map $q_m$ is smooth and $p_m$ is an isomorphism.
We consider the diagram

\[
\begin{array}{ccc}
G^*_s(Z_m) & \xrightarrow{\overline{q}_m} & G^*_s(Y_m) \\
\phi_m \cong & & \overline{p}_m \\
G^*_s(W_m) & \xrightarrow{j_m^*} & G^*_s(X_m).
\end{array}
\]

The map $\overline{q}_m$ is the composite $K^G_*(Z_m) \rightarrow K^G_*(Y_m) \rightarrow G^*_s(Y_m)$, where one identifies $K^G_*(Z_m)$ and $G^*_s(Z_m)$ since $Z_m$ is smooth. The map $\phi_m$ is an isomorphism by the homotopy invariance. It suffices to show that this diagram commutes. For, the map $s_m := \overline{p}_m \circ \overline{q}_m \circ \phi_m^{-1}$ will then give the desired splitting of the map $j_m^*$. Note that $s_m$ is $R(G)$-linear since so are all the maps in (4.8).

We now consider the commutative diagram

\[
\begin{array}{ccc}
X_m & \xrightarrow{j_m} & W_m \\
\overline{p}_m \downarrow & & \downarrow \overline{p}_m \\
Y_m & \xrightarrow{j_m} & W_m \\
\overline{q}_m \downarrow & & \downarrow \overline{q}_m \\
Z_m & \xrightarrow{i_m \circ \phi_m} & W_m \\
\end{array}
\]

Since the top left square is Cartesian with $\overline{p}_m$ projective and $j_m$ an open immersion, it follows that $j_m^* \circ \overline{p}_m = p_m^* \circ j_m^*$. Now, using the fact that $(id, \phi_m)$ is an isomorphism, we get

\[
j_m^* \circ \overline{p}_m \circ \overline{q}_m = p_m^* \circ \overline{q}_m \circ \phi_m^* = p_m^* \circ q_m^* = id_m \circ \phi_m^* = \phi_m^*.
\]

This proves the commutativity of (4.8) and the proof of the theorem is complete.

\[\square\]

5. $K$-theory of filtrable Borel spaces

Let $G$ be any linear algebraic group over $k$ and let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$ given by a good pair $(V, U)$. Then for any $X \in \textbf{Sch}_k^G$ and $i \geq 1$, there are natural maps $X \times U_i \hookrightarrow X \times (U_i \oplus V) \xrightarrow{t_i} X \times U_{i+1}$, where the first map is the 0-section of a vector bundle and the second map is an open immersion. The homotopy invariance of the $G$-theory implies that $s_i$ induces a natural pull-back map $s_i^* : G_* \left( X \times (U_i \oplus V) \right) \rightarrow G_* \left( X \times U_i \right)$ which is an isomorphism.

In particular, we get a natural pull-back map $f_i^* = s_i^* \circ t_i^* : G_* \left( X \times U_{i+1} \right) \rightarrow G_* \left( X \times U_i \right)$ for each $i \geq 1$. Note that the maps $s_i$ and $t_i$ are induced by the $G$-equivariant maps $X \times U_i \rightarrow X \times (U_i \oplus V) \rightarrow X \times U_{i+1} \rightarrow X$. In particular, $s_i^*$ and $t_i^*$ are morphisms of $R(G)$-modules. Thus for any $X \in \textbf{Sch}_k^G$, we get an inverse system $\{G_* (X^i_G), f_i^* \}_{i \geq 1}$ of $R(G)$-modules. This inverse system is covariant for proper
maps and contravariant for flat maps in \( \text{Sch}_k^G \). The inverse system \( \{ K_\ast(X_G^i), f^\ast_i \}_{i \geq 1} \) of \( R(G) \)-modules is contravariant for all maps in \( \text{Sch}_k^G \) and covariant for smooth and proper maps.

We can define a local versions of \( K \)-theory and \( G \)-theory by

\[
G^\ast_p(X_G) := \lim_{\leftarrow i} G_p \left( X^i_G(\rho) \right); \quad K^\ast_p(X_G) := \lim_{\leftarrow i} K_p \left( X^i_G(\rho) \right).
\]

We call these local versions and specify the admissible pair \( \rho \) because we do not know if these are independent of \( \rho \) if \( X \) is not smooth. We see that the assignment \( X \mapsto G^\ast(X_G) \) is covariant for proper maps and contravariant for flat maps in \( \text{Sch}_k^G \).

We want to prove an analogue of Theorem 4.2 for these groups.

Let \( \rho = (V_i, U_i) \) be an admissible gadget for \( G \) such that each \( U_i/G \) is projective. Let \( X \in \text{Sch}_k^G \) be a \( G \)-filtrable scheme with the filtration given by (4.2). We set

\[
X^i = X \times U_i, \quad X^i_m = X^i_m \times U_i, \quad W^i_m = W^i_m \times U_i \quad \text{and} \quad Z^i_m = Z^i_m \times U_i.
\]

Given the \( G \)-equivariant filtration of \( X \) as in (4.2), it is easy to see that for each \( i \geq 1 \), there is an associated system of filtrations

\[
\emptyset \subseteq X^i_0 \subseteq \cdots \subseteq X^i_n = X^i
\]

and maps \( \phi^i_m : W^i_m = (X^i_m \setminus X^i_{m-1}) \to Z^i_m \) for \( 0 \leq m \leq n \) which are all vector bundles. Moreover, as \( G \) acts trivially on each \( Z^i_m \), we have that \( Z^i_m \cong Z^i_m \times (U_i/T) \) is smooth and projective since \( Z^i_m \) is smooth and projective. We conclude that the filtration (5.2) of each \( X^i \) satisfies all the conditions of Theorem 4.2. In other words, each \( X^i \) is filtrable. In particular, there are split exact sequences

\[
0 \to G_\ast \left( X^i_{m-1} \right) \to G_\ast \left( X^i_m \right) \to G_\ast \left( W^i_m \right) \to 0
\]

for all \( 0 \leq m \leq n \) and \( i \geq 1 \) by Theorem 4.2.

**Lemma 5.1.** Let \( X \in \text{Sch}_k^G \) be a \( G \)-filtrable scheme. Then for all \( 0 \leq m \leq n \) and \( i \geq 1 \), the diagram

\[
\begin{array}{ccc}
G_\ast(Z^i_{m+1}) & \xrightarrow{f^\ast_{Zm,i}} & G_\ast(Y^i_{m+1}) \\
\phi^i_m \downarrow & & \phi^i_m \downarrow \\
G_\ast(W^i_{m+1}) & \xrightarrow{f^\ast_{Wm,i}} & G_\ast(X^i_{m+1})
\end{array}
\]

\[
\begin{array}{ccc}
G_\ast(Z^i_{m}) & \xrightarrow{f^\ast_{Zm,i}} & G_\ast(Y^i_{m}) \\
\phi^i_m \downarrow & & \phi^i_m \downarrow \\
G_\ast(W^i_{m}) & \xrightarrow{f^\ast_{Wm,i}} & G_\ast(X^i_{m})
\end{array}
\]

commutes.

**Proof.** We have shown in the proof of Theorem 4.2 (cf. (4.8)) that the front and the back squares commute. The right, left and the bottom squares commute by
the covariant and contravariant functoriality of the inverse systems \( \{ G_s(X^i_m) \} \). The top square commutes by the contravariant property of the inverse systems \( \{ K_s(X^i_m) \} \) and the functorial isomorphism \( K_s(Z^i_m) \cong G_s(Z^i_m) \). \( \square \)

Let \( X \in \text{Sch}_k^G \) be as above. We consider the commutative diagram

\[
\begin{array}{c}
0 \rightarrow G_s(X^{i+1}_m) \xrightarrow{(i^{i+1}_m)^*} G_s(X^i_m) \xrightarrow{(j^i_m)^*} G_s(W^i_m) \rightarrow 0
\end{array}
\]

of split short exact sequences (5.3), where the map \( (j^i_m)^* \) is split by \( s^i_m := (p^i_m)^* \circ (q^i_m)^* \circ ((\phi^i_m)^*)^{-1} \) for each \( i \geq 1 \) (cf. diagram 5.4).

We now show that

\[
(5.6) \quad s^i_m \circ f^*_W = f^*_X \circ s^{i+1}_m.
\]

To show this, it is equivalent to showing that

\[
(5.7) \quad f^*_X \circ (p^i_m)^* \circ (q^i_m)^* = (p^i_m)^* \circ (q^i_m)^* \circ ((\phi^i_m)^*)^{-1} \circ f^*_W \circ (\phi^{i+1}_m)^*.
\]

On the other hand, it follows from Lemma 5.1 that

\[
((\phi^i_m)^*)^{-1} \circ f^*_W \circ (\phi^{i+1}_m)^* = ((\phi^i_m)^*)^{-1} \circ (\phi^i_m)^* \circ f^*_Z.
\]

Applying Lemma 5.1 once again, we get

\[
(p^i_m)^* \circ (q^i_m)^* \circ ((\phi^i_m)^*)^{-1} \circ f^*_W \circ (\phi^{i+1}_m)^* = (p^i_m)^* \circ (q^i_m)^* \circ f^*_Z.
\]

This shows (5.7) and hence (5.6).

The following result is an analogue of Theorem 4.2 for the \( K \)-theory for the filtrable Borel spaces.

**Theorem 5.2.** Let \( G \) be a linear algebraic group and let \( \rho = (V_i, U_i)_{i \geq 1} \) be an admissible gadget for \( G \) such that each \( U_i/G \) is projective. Let \( X \in \text{Sch}_k^G \) be a \( G \)-filtrable scheme with the filtration given by (1.2). Then for every \( 0 \leq m \leq n \), there is a canonical split exact sequence of \( R(G) \)-modules

\[
(5.8) \quad 0 \rightarrow \mathcal{G}^0_s(\mathbb{A}_{m-1}) \xrightarrow{i^m_{m-1}} \mathcal{G}^0_s(\mathbb{A}_m) \xrightarrow{j^m_n} \mathcal{G}^0_s(\mathbb{W}_m) \rightarrow 0.
\]

In particular, there is a canonical isomorphism of \( R(G) \)-modules

\[
(5.9) \quad \bigoplus_{m=0}^n \mathcal{G}^0_s(\mathbb{A}_m) \cong \mathcal{G}^0_s(X_G).
\]
Proof. It follows from (5.5) and the left exactness of the inverse limit that there is a sequence as above which is exact except possibly at the right end. But (5.6) shows that \( j_m^* \circ s_m^* \) is identity on \( G^e_\kappa ((W_m)_G) \). This yields the split short exact sequence (5.8).

The isomorphism of (5.10) follows from (5.8) and using induction on the length of the filtration. For \( n = 0 \), the map \( X_0 \to Z_0 \) is a \( G \)-equivariant vector bundle and hence each map \( X_0^i \to Z_0^i \) is a vector bundle and this induces isomorphism on the \( G \)-theory. We can now argue as in the proof of Theorem 4.2 to complete the proof. \( \square \)

5.1. Filtrable schemes for torus action. The above structural results become important for us when we deal with the equivariant and the ordinary \( K \)-theory under the action of a split torus. This is due to the following fundamental result, proven by Bialynicki-Birula [4] when \( k \) is algebraically closed and by Hesselink [11] when \( k \) is any field.

**Theorem 5.3** (Bialynicki-Birula, Hesselink). Let \( X \) be a smooth projective scheme with an action of a split torus \( T \). Then \( X \) is \( T \)-filtrable.

5.1.1. Good admissible gadgets. Let \( T \) be a split torus of rank \( r \). For a character \( \chi \) of \( T \), let \( L_\chi \) denote the one-dimensional representation of \( T \) on which it acts via \( \chi \). Given a basis \( \{ \chi_1, \cdots, \chi_r \} \) of the character group \( \hat{T} \) of \( T \) and given \( i \geq 1 \), we set \( V_i = \prod_{j=1}^{r} L_{\chi_j}^{\otimes i} \) and \( U_i = \prod_{j=1}^{r} \left( L_{\chi_j}^{\otimes i} \setminus \{0\} \right) \). Then \( T \) acts on \( V_i \) by \( (t_1, \cdots, t_r)(x_1, \cdots, x_r) = (\chi_1(t_1)(x_1), \cdots, \chi_r(t_r)(x_r)) \). It is then easy to see that \( \rho = (V_i, U_i)_{i \geq 1} \) is an admissible gadget for \( T \) such that \( U_i/T \cong (\mathbb{P}^{i-1})^r \). Moreover, the line bundle \( L_{\chi_j}^{\otimes i} \times (\mathbb{P}^{i-1})^r \to \mathbb{P}^{i-1} \) is the line bundle \( \mathcal{O}(\pm 1) \) for each \( 1 \leq j \leq r \). An admissible gadget for \( T \) of this form will be called a good admissible gadget.

Let \( X \in \text{Sch}_k^T \) be a filtrable scheme with the filtration given by (4.2). Given a good admissible gadget \( \rho = (V_i, U_i)_{i \geq 1} \) for \( T \), we see that each \( X^i \) has a filtration as in (5.2) such that \( U_i/T \cong (\mathbb{P}^{i-1})^r \) for each \( i \geq 1 \).

If \( T \) is a split torus over \( k \), a good admissible pair satisfies the hypothesis of Theorem 5.2. Hence we have the following.

**Corollary 5.4.** Let \( T \) be a split torus over \( k \) and let \( X \in \text{Sm}_k^T \) be a \( T \)-filtrable scheme with the filtration given by (4.2). Then there is a canonical isomorphism of \( R(T) \)-modules

\[
(5.10) \quad \bigoplus_{m=0}^{n} \mathcal{K}_s((Z_m)_T) \xrightarrow{\cong} \mathcal{K}_s(X_T).
\]

We end this section with the following result which compares the \( K \)-theory and \( \mathcal{K} \)-theory of motivic Borel spaces with torus action. This will be generalized to the case of all connected split reductive groups in Section 6.

**Proposition 5.5.** Let \( T \) be a split torus of rank \( r \). Let \( X \in \text{Sm}_k^T \) be \( T \)-filtrable and let \( p \geq 0 \). Then for any admissible gadget \( \rho = (V_i, U_i)_{i \geq 1} \) for \( T \), the inverse
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system $\{K_p(X^i_T(\rho))\}_{i \geq 1}$ satisfies the Mittag-Leffler condition. In particular, the map

$$\tau_X^T : K_p(X_T) \to K_p(X_T)$$

is an isomorphism.

**Proof.** In view of Corollary 3.5, it is enough to prove the proposition when $\rho$ is a good admissible gadget for $T$. We shall prove in this case the stronger assertion that for any $T$-filtrable scheme $X \in \text{Sch}^T$, each map $f_i^*$ in the inverse system $\{G_p(X^i_T(\rho)) : f_i^*\}_{i \geq 1}$ is surjective.

Let us consider the $T$-equivariant filtration of $X$ as in (4.2) and the associated system of filtrations (5.2) for each $X_i = X^i_T(\rho)$. We prove our surjectivity assertion by induction on the length of the filtration.

For every $0 \leq m \leq n$, there is a commutative diagram

$$\begin{array}{ccc}
G_*(Z_{i+1}^m) & \xrightarrow{f_{Z_{m,i}}^*} & G_*(Z_m^i) \\
\cong & & \cong \\
G_*(W_{m,i}^i) & \xrightarrow{f_{W_{m,i}}^*} & G_*(W_i^i)
\end{array}$$

of the $G$-theory of smooth schemes where all the vertical arrows are isomorphisms by the homotopy invariance.

Next, we observe that $T$ acts trivially on each $Z_m$ and hence there is an isomorphism $Z_m^i \cong Z_m \times (U_i/T) \cong Z_m \times (\mathbb{P}^{i-1})^\ast$. Hence the projective bundle formula for the $G$-theory of smooth schemes implies that the map $G_*(Z_{m+1}^i) \xrightarrow{f_{Z_{m,i}}^*} G_*(Z_m^i)$ is surjective. We conclude from (5.11) that each $f_{W_{m,i}}^*$ is surjective. Taking $m = 0$, we see in particular that the map $G_*(X_{0+1}^i) \to G_*(X_0^i)$ is surjective for all $i \geq 1$.

Assume now that $m \geq 1$ and that the surjectivity assertion holds for all $j \leq m - 1$. We consider the diagram

$$\begin{array}{ccc}
0 & \to & G_*(X_{m-1}^i) \\
\downarrow f_{X_{m-1,i}} & & \downarrow f_{X_{m,i}} \\
0 & \to & G_*(X_m^i)
\end{array}$$

$$\begin{array}{ccc}
& & \downarrow f_{W_{m,i}} \\
G_*(W_{m,i}^i) & \to & G_*(W_i^i)
\end{array}$$

which commutes by the functorial properties of proper push-forward and pull-back maps in $G$-theory and where the two rows are exact by (5.3). We have shown above that the right vertical arrow is surjective. The left vertical arrow is surjective by induction. We conclude that the middle vertical arrow is also surjective. This completes the proof of the Mittag-Leffler condition. The second assertion of the proposition follows from the Mittag-Leffler condition and the Milnor exact sequence (3.15). \hfill \square

6. $K$-theory of Borel spaces for reductive groups

Our aim in this section is to prove the analogue of Proposition 5.5 for any connected split reductive group. This is achieved by reducing the problem to the case of torus using the push-pull operators which we now discuss.
6.1. **Push-pull operators in Equivariant $K$-theory.** Let $G$ be a connected reductive group with a (not necessarily split) maximal torus $T$. Let $B$ be a Borel subgroup of $G$ containing $T$. For any $X \in \text{Sch}_k^G$, we know that there is a functorial restriction map $r_G^{T,X}: G_*(X) \to G_*(X)$ and it is known (cf. [25, Theorem 1.13]) that there is an isomorphism

$$\text{(6.1)} \quad r_B^{T,X}: G^B_*(X) \cong G^T_*(X).$$

Since $X$ is a $G$-scheme, the map $X \times G \to X \times G/B$ is an isomorphism of $G$-schemes. In particular, there is a $G$-equivariant smooth and projective map $p_X : X \times G \to X$. The projective push-forward yields a natural induction map $s_G^{T,X}: G^T_*(X) \to G^B_*(X)$. Note that the restriction map $r_G^{T,X}$ is same as the flat pull-back $G^T_*(X) \to G^B_*(X \times G) \cong G^B_*(X)$. In particular, $r_G^{T,X}$ and $s_G^{T,X}$ satisfy the usual projection formula and hence they are $K^*_G(X)$-linear. The following result was proven by Thomason.

**Theorem 6.1** ([25, Theorem 1.13]). For $X \in \text{Sch}_k^G$, the composite map

$$G^G_*(X) \xrightarrow{\eta_G} G^T_*(X) \xrightarrow{s_G^{T,X}} G^B_*(X)$$

is identity.

6.2. **Push-pull operators in $K$-theory.** We want to construct the above restriction and induction maps for the $K$-theory of the Borel spaces. We need the following intermediate results to construct these operators. Recall from [15, Definition 2.1] that a commutative square of smooth schemes

$$\begin{array}{ccc}
Y' & \xrightarrow{g} & Y \\
\downarrow{\iota'} & & \downarrow{\iota} \\
X' & \xrightarrow{f} & X
\end{array}$$

is called transverse if it is Cartesian, the map $\iota$ is a closed immersion and the map $N_{X'}(Y') \to g^* (N_X(Y))$ of normal bundles is an isomorphism.

**Lemma 6.2.** Let $f : X \to Y$ be a morphism in $\text{Sm}_{\text{free}/k}^G$. Then the diagram of quotients

$$\begin{array}{ccc}
X/B & \xrightarrow{\pi} & X/G \\
\downarrow & & \downarrow \\
Y/B & \xrightarrow{\pi} & Y/G
\end{array}$$

is Cartesian such that the horizontal maps are smooth and projective. If $f$ is a closed immersion, then this diagram is transverse.

**Proof.** The top horizontal map is an étale locally trivial smooth fibration with fiber $G/B$. Hence this map is proper by the descent property of properness. Since this map is also quasi-projective, it must be projective. The same holds for the bottom
horizontal map. Proving the other properties is an elementary exercise and can be shown using the commutative diagram

(6.3) \[
\begin{array}{ccc}
X & \rightarrow & X/B \\
\downarrow f & & \downarrow \\
Y & \rightarrow & Y/B
\end{array}
\rightarrow
\begin{array}{ccc}
X/G & \rightarrow & X/G \\
\downarrow f & & \downarrow \\
Y & \rightarrow & Y/G.
\end{array}
\]

One easily checks that the left and the big outer squares are Cartesian and transverse if \( f \) is a closed immersion. Since all the horizontal maps are smooth and surjective, the right square must have the similar property. \( \square \)

**Lemma 6.3.** Consider the Cartesian square

(6.4) \[
\begin{array}{ccc}
Y' & \rightarrow & Y \\
\downarrow g' & & \downarrow \\
X' & \rightarrow & X
\end{array}
\]

in \( \text{Sm}_k \) such that \( f \) is projective. Suppose that either

1. \( g \) is a closed immersion and \((6.4)\) is transverse, or,
2. \( g \) is smooth.

One has then \( g^* \circ f_* = f'_* \circ g'^* : K_*(X') \rightarrow K_*(Y) \).

**Proof.** We can write \( f = p \circ i \), where \( X' \hookrightarrow \mathbb{P}^n \times X \) is a closed immersion and \( p : \mathbb{P}^n \times X \rightarrow X \) is the projection. This yields a commutative diagram

\[
\begin{array}{ccc}
Y' & \rightarrow & \mathbb{P}^n \times Y \\
\downarrow g' & & \downarrow \\
X' & \rightarrow & \mathbb{P}^n \times X
\end{array}
\rightarrow
\begin{array}{ccc}
Y & \rightarrow & Y \\
\downarrow h & & \downarrow \\
X & \rightarrow & X
\end{array}
\]

where both squares are Cartesian.

First suppose that \( g \) is a closed immersion and \((6.4)\) is transverse. Since the right square is transverse and so is the big outer square, it follows that the left square is also transverse. In particular, we have \( h^* \circ i_* = i'_* \circ g'^* \) by [15, Definition 2.2-(2)]. On the other hand, we have \( g^* \circ p_* = p'_* \circ h^* \) by [15, Definition 2.2-(3)]. Combining these two, we get

\[
g^* \circ f_* = g^* \circ p_* \circ i_* = p'_* \circ h^* \circ i_* = p'_* \circ i'_* \circ g'^* = f'_* \circ g'^*
\]

where the first and the last equalities follow from the functoriality property of the push-forward (cf. [15, Definition 2.2-(1)]).

Now suppose that \( g \) is smooth. For the above proof to go through, only thing we need to know is that the left square in the above diagram is still transverse. But this is an elementary exercise using the fact that \( g, h, g' \) are all smooth and \( T(g') = i'^* (T(h)) \), where \( T(f) \) denotes the relative tangent bundle of a smooth map \( f \). This proves the lemma. \( \square \)
Proposition 6.4. Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$. For any $X \in \text{Sm}_k^G$, let $X^i_G$ denote the mixed space $X^G \times U_i$. Then for any $p \geq 0$, there are strict morphisms of inverse systems

$$\{ K_p(X^i_G) \} \xrightarrow{\tilde{s}^G_{T,X}} \{ K_p(X^i_T) \} \xrightarrow{\tilde{s}^G_{T,X}} \{ K_p(X^i_G) \}$$

such that the composite $\tilde{s}^G_{T,X} \circ \tilde{r}^G_{T,X}$ is identity. Furthermore, these morphisms are contravariant for all maps and covariant for proper maps in $\text{Sm}_k^G$. These maps satisfy the projection formula $\tilde{s}^G_{T,X}(x \cdot \tilde{r}^G_{T,X}(y)) = \tilde{s}^G_{T,X} \cdot y$ for $x \in \{ K_p(X^i_T) \}$ and $y \in \{ K_p(X^i_G) \}$.

Proof. We first observe that for any $i \geq 1$, the map $K_*(X^i_B) \to K_*(X^i_G)$ is an isomorphism by the homotopy invariance. Hence, we can replace the torus $T$ with the Borel subgroup $B$ everywhere in the proof.

For every $i \geq 1$, there are smooth and projective maps $\pi^i_X : X^i_B \to X^i_G$. These maps induce the corresponding push-forward map $(\pi^i_X)^*$ on the equivariant and ordinary $K$-groups. These maps on the equivariant and ordinary $K$-groups are connected by the commutative diagram

$$(6.5) \quad K_*(X^i_G) \xrightarrow{(\pi^i_X)^*} K_*(X^i_B) \xrightarrow{(\pi^i_X)^*} K_*(X^i_G)$$

The functorial properties of the pull-back map on the $K$-theory yield a strict morphism of the projective systems $\{ K_*(X^i_G) \} \xrightarrow{\tilde{s}^G_{T,X}} \{ K_*(X^i_B) \}$.

For any $j \geq i \geq 1$, Lemma 6.2 yields a transverse square

$$(6.6) \quad X^i_B \xrightarrow{\pi^i_X} X^i_G$$

where the vertical maps are closed immersions and the horizontal maps are smooth and projective. It follows from Lemma 6.3 that push-forward maps $(\pi^i_X)^*$ give rise to a strict morphism of projective systems $\{ K_*(X^i_B) \} \xrightarrow{\tilde{s}^G_{T,X}} \{ K_*(X^i_G) \}$. The assertion that $\tilde{s}^G_{T,X} \circ \tilde{r}^G_{T,X}$ is identity follows from the diagram (6.5) because the composite map on the bottom row is identity by Theorem 6.1.

The covariant and contravariant functoriality of $\tilde{r}^G_{T,X}$ and $\tilde{s}^G_{T,X}$ follow directly from Lemmas 6.2 and 6.3. The projection formula for $\tilde{s}^G_{T,X}$ and $\tilde{r}^G_{T,X}$ follows from the projection formula for the maps $X^i_B \xrightarrow{\pi^i_X} X^i_G$. \hfill $\Box$

As an immediate consequence of Proposition 6.4, we obtain the following analogue of Theorem 6.1 for the $K$-theory of the motivic Borel spaces.
For a commutative ring completion theorem. We shall use the following notations in what follows.

Lemma 7.1. The following fact.

Corollary 6.5. For any $X \in \text{Sm}_k^G$ and $p \geq 0$, there are restriction and induction maps

$$\tilde{r}_{T,X}^G : K_p(X_G) \to K_p(X_T); \quad \tilde{s}_{T,X}^G : K_p(X_T) \to K_p(X_G)$$

which are contravariant for all maps and covariant for proper maps in $\text{Sm}_k^G$. These maps satisfy the projection formula $\tilde{s}_{T,X}^G (x \cdot \tilde{r}_{T,X}^G (y)) = \tilde{s}_{T,X}^G \cdot y$ for $x \in K_p(X_T)$ and $y \in K_p(X_G)$. Moreover, the composite map $\tilde{s}_{T,X}^G \circ \tilde{r}_{T,X}^G$ is identity.

As another consequence of Proposition 6.4, we obtain the following main result of this section.

Theorem 6.6. Let $G$ be a connected reductive group over $k$ with a split maximal torus $T$. Let $X \in \text{Sm}_k^G$ be such that it is $T$-filtrable. Then for any admissible gadget $\rho = (V_i, U_i)_{i \geq 1}$ for $G$ and any $p \geq 0$, the inverse system $\{K_p(X_G^i(\rho))\}_{i \geq 1}$ satisfies the Mittag-Leffler condition. In particular, the map

$$\tau^G_X : K_p(X_G) \to K_p(X_G)$$

is an isomorphism and hence $K_p(X_G)$ is naturally an $R(G)$-module.

Proof. We only need to show the Mittag-Leffler condition. Setting $X_G^i = X_G^i(\rho)$, we conclude from Proposition 6.4 that there is a strict $R(G)$-linear morphism

$$\{K_p(X_G^i)\} \xrightarrow{\tilde{s}_{T,X}^G} \{K_p(X_G^i)\}$$

of inverse systems which is surjective. On the other hand, the inverse system $\{K_p(X_G^i)\}$ satisfies the Mittag-Leffler condition by Proposition 5.5. Hence, this condition must hold for $\{K_p(X_G^i)\}$ as well.

7. The Atiyah-Segal theorem: two easy cases

As an starter, we dispose of the two easy cases of the algebraic Atiyah-Segal completion theorem. We shall use the following notations in what follows.

Notation: For a commutative ring $A$ and an $A$-module $M$, the symbol $M[[t_1, \ldots, t_r]]$ (resp. $M[t_1, \ldots, t_r]$) will denote the set of all formal ‘power series’ (resp. polynomials) in variables $\{t_1, \ldots, t_r\}$ with coefficients in $M$. Notice that $M[[t_1, \ldots, t_r]]$ is a module over the formal power series ring $A[[t_1, \ldots, t_r]]$, but it is not necessarily same as $M \otimes_A A[[t_1, \ldots, t_r]].$

7.1. Case of free action. To deal with the case of free action, we first observe the following fact.

Lemma 7.1. Let $G$ be a connected linear algebraic group over $k$. Then for any $X \in \text{Sch}_{\text{free}/k}^G$ and $p \geq 0$, $I_G^m G_p^G(X) = 0$ for $m \gg 0$. In particular, $G_p^G(X)$ is $I_G$-adically complete.

Proof. Since $G$ is connected, it keeps each connected component of $X$ invariant and hence it suffices to consider the case when $X$ is connected. Setting $Y = X/G$ and letting $m_Y$ denote the ideal of $K_0(Y)$ consisting of virtual bundles of rank zero, one has that $I_G^m G_p^G(X) \subseteq m_Y^m G_p^G(Y)$ once we identify $G_p^G(X)$ with $G_p(Y)$. Thus it suffices to show that $m_Y^m G_p^G(Y) = 0$ for $m \gg 0$.

If we consider the filtration of $G_p(Y)$ by the subgroups

$$F_q G_p(Y) = \bigcup \limits_{\dim(Z) \leq q} \text{Ker } (G_p(Y) \to G_p(Y \setminus Z)),$$
then it follows from [7, Theorem 4] that there is a pairing

\[ F_\gamma^m K_0(Y) \otimes G_\rho(Y) \to F_{d-m} G_\rho(Y), \]

where \( d = \dim(Y) \) and \( F_\gamma^m K_0(Y) \) is the gamma filtration on \( K_0(Y) \). Since \( m^p \subseteq F_\gamma^m K_0(Y) \), we get the desired vanishing for \( m \geq d + 1 \).

Remark 7.2. Using the ideas of [3, Proposition 4.3], one can in fact show that the converse of Lemma 7.1 is also true. That is, if \( G^*_s(X) \) is \( I_G \)-adically complete, then \( G \) must act freely on \( X \).

Suppose now that \( G \) is a connected linear algebraic group over \( k \) and \( X \in \text{Sm}^G_{\text{free}/k} \). In this case, we identify \( K^*_s(X) \) with \( K_*(X/G) \). It follows from Lemma 7.1 that \( K^*_s(X) \cong K^*_s(X)_I \). Moreover, it follows from Lemma 2.3 and [14, Lemma 4.2.9] that the map \( K_*(X/G) \to K_*(X_G) \) is an isomorphism. Hence, for any \( \rho \geq 0 \), we have isomorphisms

\[ K^*_p(X) \cong K^*_p(X)_I \cong K_*(X_G). \]  

(7.1)

It follows from (3.13) that the map \( K^*_p(X_G) \to K_*(X_G) \) is surjective. To show that it is injective, we use Proposition 3.1. It suffices to show using Proposition 2.4 that for any admissible gadget \( \rho = (V_i, U_i) \) for \( G \), the map \( K^*_p(X/G) \to K_*(X_G) \) is injective for all large \( i \). However, as \( X \times V_i \to X/G \) is a vector bundle and \( X^*_G(\rho) \subseteq X \times V_i \) is open, this injectivity follows from the definition of admissible gadgets and Proposition 3.1. We have thus shown for \( X \in \text{Sm}^G_{\text{free}/k} \) that the maps

\[ K^*_p(X) \cong K^*_p(X)_I \cong K_*(X_G) \cong K_*(X_G) \]

(7.2)

are all isomorphisms.

7.2. The case of trivial action. The case of trivial action of a split torus can be handled using some results of Thomason. We now show how this works. So let \( T \) be a split torus over \( k \) of rank \( r \) acting trivially on a smooth scheme \( X \). Given an admissible gadget \( \rho = (V_i, U_i) \) for \( T \), there is a natural map of inverse systems \( \{K^*_p(X)\} \to \{K_*(X_T)\} \) of \( R(T) \)-modules and this induces a map \( K^*_p(X) \to K_*(X_T) \).

Lemma 7.3. The maps \( K^*_p(X) \to K_*(X_T) \to K_*(X_T)_I \) induce isomorphisms of \( R(T) \)-modules

\[ K^*_p(X) \to K_*(X_T) \cong K_*(X_T)_I \cong K_*(X_T). \]

(7.3)

Proof. Since \( T \) acts trivially on \( X \), it follows from [22, Lemma 5.6] (which is true for all diagonalizable groups) that the exact functor \( \text{Vec}(X) \times \text{Rep}(T) \to \text{Vec}^T(X) \) induces an isomorphism

\[ K_*(X) \otimes \mathbb{Z} R(T) \cong K_*(X). \]

(7.4)

To show the first two isomorphisms of (7.3), let \( \rho = (V_i, U_i) \) be a good admissible gadget for \( T \). For \( i \geq 1 \), let \( J^i_T \) denote the ideal \((\rho^i_1, \cdots, \rho^i_r)\) of \( R(T) = \)
Z[t_1, \cdots, t_r, (t_1 \cdots t_r)^{-1}]$, where $\rho_j = 1 - t_j$. Notice that $J^T_I = I_T$ and $J^T_I \subseteq I^T_I$ for each $i \geq 1$. We claim for each $i \geq 1$ that there is a short exact sequence

\begin{equation}
(7.5) \quad 0 \to J^T_I (K^T_p (X)) \to K^T_p (X) \to K^T_p (X^T_I) \to 0.
\end{equation}

Using (7.3) and the isomorphism $K^T_p (X^T_I) \cong K^T_p (X) \otimes \kappa_0 ((\mathbb{P}^{i-1})^r)$ (by the projective bundle formula), it is enough to show the exactness of the sequence

\begin{equation}
(7.6) \quad 0 \to J^T_I R(T) \to R(T) \to K_0 ((\mathbb{P}^{i-1})^r) \to 0.
\end{equation}

Since $R(T) = R(G_m) \otimes \cdots \otimes R(G_m)$ and $K_0 ((\mathbb{P}^{i-1})^r) = K_0 (\mathbb{P}^{i-1}) \otimes \cdots \otimes K_0 (\mathbb{P}^{i-1})$, we can further assume that $T$ has rank one.

In this case, we have the localization sequence

\begin{equation}
K_0^T (k) \to K_0^T (V_i) \to K_0^T (\mathbb{P}^{i-1}) \to 0.
\end{equation}

Moreover, one knows by the self intersection formula (cf. [27, Theorem 2.1]) that under the isomorphism $R(T) \cong K^T_p (V_i)$, the first map in this exact sequence is multiplication by the top $T$-equivariant Chern class of vector bundle $V_i$. Since $V_i = V^i$ and since the first Chern class of $V$ in $R(T) = \mathbb{Z}[t, t^{-1}]$ is $1 - t$, the Whitney sum formula shows that the above exact sequence is same as (7.6). This proves the claim.

The lemma follows immediately from the claim. The $R(T)$-module structure on each $K_p (X^T_I)$ is given by taking the inverse limit of the $R(T)$-modules $K_p (X^T_I)$. It follows from (7.3) and (7.5) that there is a natural strict isomorphism of inverse systems of $R(T)$-modules

\begin{equation}
(7.7) \quad \left\{ \frac{K^T_p (X)}{J^T_I (K^T_p (X))} \right\} \cong \left\{ K^T_p (X^T_I) \right\}.
\end{equation}

Since $J^T_I \subseteq I^T_I$ for $i \geq 1$ and since for any $i \geq 1$, one has $I^T_I \subseteq J^T_I$ for $j \gg i$, we see that the map of inverse systems of $R(T)$-modules \[\left\{ \frac{K^T_p (X)}{J^T_I (K^T_p (X))} \right\} \to \left\{ \frac{K^T_p (X)}{J^T_I (K^T_p (X))} \right\}\] is an isomorphism. We conclude that $K^T_p (X)_{I_T} \cong K^T_p (X^T_I)$ in $I_T$-adically complete for each $p \geq 0$.

The isomorphism $K^T_p (X^T_I) \cong K^T_p (X) \otimes \kappa_0 ((\mathbb{P}^{i-1})^r)$ also shows that the inverse system $\{ K^T_p (X^T_I) \}$ is surjective. We conclude from (3.13) that the map $K^T_p (X) \to K^T_p (X^T_I)$ is an isomorphism.

In fact, what the above shows is that the map

\begin{equation}
(7.8) \quad \gamma \!: K^T_p (X)[[t_1, \cdots, t_r]] \to K^T_p (X^T_I); \quad t_j \mapsto 1 - \xi_j
\end{equation}

is an isomorphism of $R(T)$-modules, where $\xi \in \kappa_0 ((\mathbb{P}^{\infty}_k)^r)$ is the class of the line bundle $p^*_j (\mathcal{O}(-1))$ under the projection $p_j : (\mathbb{P}^{\infty}_k)^r \to \mathbb{P}^{\infty}_k$.

\[ \Box \]

8. The Atiyah-Segal completion theorem

In this section, we formulate and prove the algebraic analogue of the Atiyah-Segal completion theorem and prove it for filtrable schemes. Let $G$ be a linear algebraic group over $k$ and let $X \in \text{Sm}_k^G$. We have seen before that $K^T_p (X^G)$ is an $R(G)$-module and so, there is a natural map $i^G_X : K^T_p (X^G) \to K^T_p (X^G)_G$, where
the latter is the $I_G$-adic completion of $\mathcal{K}_p(X_G)$. Notice that $\iota_X^G$ is contravariant functorial in $X$ and $G$. We begin with the following.

**Proposition 8.1.** Let $G$ be a connected reductive group over $k$ with a split maximal torus $T$ and let $X \in \text{Sm}^G_k$ be $T$-filtrable. Then for any $p \geq 0$, the map

$$\iota_X^G : \mathcal{K}_p(X_G) \to \mathcal{K}_p(X_G)_{I_G}$$

is an isomorphism.

**Proof.** We consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}_p(X_G) & \xrightarrow{\iota_X^G} & \mathcal{K}_p(X_T) \\
\downarrow & & \downarrow \\
\mathcal{K}_p(X_G)_{I_G} & \xrightarrow{\iota_X^G} & \mathcal{K}_p(X_T)_{I_G}
\end{array}$$

It follows from Corollary [6.5] that the composite horizontal maps on both rows are identity. Hence, it suffices to show that the middle vertical arrow is an isomorphism.

It follows from [9, Corollary 6.1] that the $I_G$-adic and the $I_T$-adic topologies on $R(T)$ coincide. Hence for any $R(T)$-module $M$, the $I_G$-adic and $I_T$-adic topologies on $M$ coincide. Applying this to $\mathcal{K}_p(X_T)$, we see that the map $\mathcal{K}_p(X_T)_{I_G} \to \mathcal{K}_p(X_T)_{I_T}$ is an isomorphism. Thus we have reduced the problem to the case of a split torus.

Let $r$ denote the rank of $T$ and let $\rho = (V_i, U_i)$ be a good admissible gadget for $T$ so that $U_i/T \cong (\mathbb{P}^{i-1}_k)_r$. Let $X \in \text{Sm}^G_k$ be a $T$-filtrable scheme with the filtration given by (4.2). By Corollary [5.4] it suffices to prove the result when $T$ acts trivially on $X$. But this case follows from Lemma [7.3].

### 8.1. The Atiyah-Segal map.

Let $G$ be a linear algebraic group over $k$ and let $\rho = (V_i, U_i)$ be an admissible gadget for $G$. Recall for any $X \in \text{Sm}^G_k$ that $X^G_i$ denotes the mixed quotient $X \times U_i$. The projection map $X \times U_i \xrightarrow{\pi} X$ is $G$-equivariant and hence induces the pull-back $R(G)$-linear map $p_i^* : K^G_*(X) \to K^G_*(X \times U_i) = K_* (X^G_i)$. This map is clearly compatible with the maps $f_i^* : K_* (X^G_{i+1}) \to K_* (X^G_i)$. Hence we get a natural map from $K^G_*(X)$ to the limit of the inverse system $\{K_* (X^G_i)\}$. In other words, we conclude that there is a natural map

$$\beta^G_X : K^G_p(X) \to \mathcal{K}_p(X_G)$$

for every $p \geq 0$. This map is contravariant functorial in $X$ and $G$ and is $R(G)$-linear. In fact, $\mathcal{K}_* (X_G)$ has a natural structure of $K^G_*(X)$-module and $\beta^G_X$ is then $K^G_*(X)$-linear. We shall call $\beta^G_X$, the Atiyah-Segal map in tribute to Atiyah and Segal, who first studied this map in the topological context in [3]. Since $\beta^G_X$ is a morphism of $R(G)$-modules, it induces a natural morphism of $I_G$-adic completions:

$$\widehat{\beta}^G_X : \mathcal{K}^G_p(X)_{I_G} \to \mathcal{K}_p(X_G)_{I_G}.$$
Suppose now that $G$ is a connected reductive group over $k$ with a split maximal torus $T$ and suppose that $X \in \text{Sm}^G_k$ is $T$-filtrable. It follows then from Proposition 8.1 that the map $\hat{\beta}_X^G$ actually lifts canonically to a map of $\widehat{R}(G)$-modules $K_p^G(X)_{IG} \to K_p(X_T)$. We wish to prove the following.

**Proposition 8.2.** Let $G$ be a connected reductive group over $k$ with a split maximal torus $T$. Let $X \in \text{Sm}^G_k$ be $T$-filtrable. Then for every $p \geq 0$, the morphism of $\widehat{R}(G)$-modules

\[(8.4) \quad \hat{\beta}_X^G : K_p^G(X)_{IG} \to K_p(X_T)\]

is an isomorphism.

**Proof.** We consider the commutative diagram of $\widehat{R}(G)$-modules

\[(8.5) \quad \begin{array}{ccc}
K_p^G(X)_{IG} & \xrightarrow{\hat{\alpha}_X^G} & K_p^T(X)_{IG} \xrightarrow{\hat{\beta}_X^T} K_p^T(X_G)_{IG} \\
\downarrow{\hat{\beta}_X^G} & & \downarrow{\hat{\beta}_X^T} \\
K_p(X_G) & \xrightarrow{\hat{\alpha}_X^G} & K_p(X_T) \xrightarrow{\hat{\beta}_X^T} K_p(X_G).
\end{array}\]

It follows from Theorem 6.1 and Corollary 6.5 that the composite horizontal maps on both rows are identity. Hence, it suffices to show that the middle vertical arrow is an isomorphism. The arguments in the proof of Proposition 8.1 show that the map $K_p^T(X)_{IG} \to K_p^T(X_G)_{IG}$ is an isomorphism. This reduces the problem to the case of a split torus. Using Theorems 4.2 and 5.2 we further reduce to the case of trivial action of a split torus. But this case follows from Lemma 7.3. □

As an immediate consequence of Theorem 6.6 and Proposition 8.2, we get the following final algebraic Atiyah-Segal completion theorem.

**Theorem 8.3.** Let $G$ be a connected and reductive group over $k$ with a split maximal torus $T$. Let $X \in \text{Sm}^G_k$ be $T$-filtrable. Then for every $p \geq 0$, the morphisms of $\widehat{R}(G)$-modules

\[(8.6) \quad \hat{\beta}_X^G : K_p^G(X)_{IG} \to K_p(X_G)\]

are isomorphisms.

Recall that a connected linear algebraic group $G$ over $k$ is called split, if it contains a split maximal torus. Since every smooth projective $T$-scheme is $T$-filtrable by Theorem 5.3, we get the following.

**Corollary 8.4.** Let $G$ be a connected and split reductive group over $k$ and let $X \in \text{Sm}^G_k$ be projective. Then for every $p \geq 0$, the morphisms of $\widehat{R}(G)$-modules

\[(8.7) \quad \hat{\beta}_X^G : K_p^G(X)_{IG} \to K_p(X_G)\]

are isomorphisms.
It follows from [14, Theorem 4.3.13] that a morphism \( f : X \to Y \) of smooth \( k \)-schemes induces an isomorphism on the ordinary algebraic \( K \)-theory, if \( f \) is an \( \mathbb{A}^1 \)-weak equivalence. However, if \( f : X \to Y \) is a \( G \)-equivariant map of smooth \( G \)-schemes, one does not yet know as to what kind of weak equivalence will induce isomorphism on the equivariant \( K \)-theory. The following algebraic analogue of [3, Proposition 5.1] gives a partial solution to this problem.

**Corollary 8.5.** Let \( G \) be a connected and split reductive group over \( k \) and let \( f : X \to Y \) be a \( G \)-equivariant morphism of smooth and projective \( G \)-schemes. Assume that \( f \) is an \( \mathbb{A}^1 \)-homotopy equivalence of schemes (after forgetting the \( G \)-action). Then for every \( p \geq 0 \), the map \( f^* : K_p^G(Y) \to K_p^G(X) \) induces an isomorphism of the \( I_G \)-adic completions.

**Proof.** By Corollary 8.4, it is equivalent to showing that the map \( f^* : K_p(Y_G) \to K_p(X_G) \) is an isomorphism. On the other hand, we have a commutative diagram of cofibration sequences

\[
\begin{array}{ccc}
X & \longrightarrow & X_G \\
\downarrow{f} & & \downarrow{f_G} \\
Y & \longrightarrow & Y_G \\
\end{array}
\]

in \( \mathcal{H}_*(k) \). This gives a commutative diagram of long exact sequences (cf. [12, Proposition 6.5.3])

\[
\begin{array}{cccc}
K_{p+1}(B_G) & \longrightarrow & K_p(Y) & \longrightarrow & K_p(Y_G) & \longrightarrow & K_p(B_G) & \longrightarrow & K_{p-1}(Y) \\
\downarrow{f^*} & & \downarrow{f_G^*} & & \downarrow{f^*} & & \downarrow{f^*} & & \\
K_{p+1}(B_G) & \longrightarrow & K_p(X) & \longrightarrow & K_p(X_G) & \longrightarrow & K_p(B_G) & \longrightarrow & K_{p-1}(X).
\end{array}
\]

Our assumption implies that \( f^* : K_*(Y) \to K_*(X) \) is an isomorphism. We conclude from the 5-lemma that \( K_*(Y_G) \to K_*(X_G) \) is an isomorphism. \( \square \)

**Remark 8.6.** One can use Proposition 2.8 and Theorem 4.1 to conclude that Theorem 8.3, Corollary 8.4 and Corollary 8.5 are true for the action of any connected and split (not necessarily reductive) linear algebraic group in characteristic zero.

9. **Completion theorem for non-projective schemes**

In this section, we show that the algebraic Atiyah-Segal completion theorem holds for \( G^G_0(-) \) for all \( G \)-schemes and for all linear algebraic groups \( G \). We begin with the following application of Proposition 3.1

**Lemma 9.1.** Let \( G \) be a linear algebraic group over \( k \) and let \( X \in \text{Sch}_k^G \). Let \( \rho = (V_i, U_i)_{i \geq 1} \) and \( \rho' = (V'_i, U'_i)_{i \geq 1} \) be two admissible gadgets for the \( G \)-action on \( X \) (cf. §2.3.1). Then there is a canonical isomorphism

\[
\lim_{\leftarrow i} G_0 \left( X_G^i(\rho) \right) \cong \lim_{\leftarrow i} G_0 \left( X_G^i(\rho') \right).
\]

**Proof.** Since \( \rho \) and \( \rho' \) are the admissible gadgets for the \( G \)-action on \( X \), \( G \) acts freely on each \( X \times U_i \) and on \( X \times U'_i \). Furthermore, the map \( X \times (U_i \oplus V'_j) \to X \times U_i \)
is a vector bundle and hence the map $G_0\left( X \times U_i \right) \to G_0\left( X \times (U_i \oplus U'_j) \right)$ is surjective. It follows from Proposition 3.1 that this map is in fact an isomorphism for all $j \gg 0$. Taking the limit, we get

$$\lim_{i} G_0\left( X \times U_i \right) \xrightarrow{\cong} \lim_{i} \lim_{j} G_0\left( X \times (U_i \oplus U'_j) \right).$$

The same argument shows that

$$\lim_{i} G_0\left( X \times U'_i \right) \xrightarrow{\cong} \lim_{i} \lim_{j} G_0\left( X \times (U_i \oplus U'_j) \right).$$

These two isomorphisms prove the lemma.

As a consequence of Lemma 9.1, we can define the $G_0$-theory for schemes with group actions as follows.

**Definition 9.2.** For any linear algebraic group $G$ over $k$ and any $X \in \text{Sch}_k^G$, we define

$$G_0(X_G) := \lim_{i} G_0\left( X \times U_i \right)$$

where $\rho = (V_i, U_i)$ is any admissible gadget for the $G$-action on $X$.

The functor $X \mapsto G_0(X_G)$ is covariant for proper maps and contravariant for flat maps in $\text{Sch}_k^G$.

9.1. **Case of split reductive groups.** We first consider the case of the action of split reductive groups over $k$. Let $T$ be a split torus of rank $r$ over $k$.

**Lemma 9.3.** For any $X \in \text{Sch}_k^T$, the map $G_0^T(X) \to G_0(X_T)$ induces an isomorphism

$$\widehat{G_0^T(X)} \cong G_0(X_T).$$

**Proof.** Let $\rho = (V_i, U_i)$ be a good admissible gadget for $T$. It suffices to show that the map $G_0^T(X) \xrightarrow{\cong} G_0^T(X)_T \cong G_0(X_T)$ is an isomorphism.

For any $i \geq 0$, let $J^i(X) = \text{Ker} \left( G_0^T(X) \to G_0(X_T^i) \right)$. The surjections

$$G_0^T(X) \xrightarrow{\cong} G_0^G(X \times V_i) \to G_0(X_T^i).$$

imply that there is an isomorphism of the inverse systems $\left\{ \frac{G_0^G(Y)}{J(Y)} \right\} \to \left\{ G_0(X_T^i) \right\}$. Hence, it suffices to show that the pro-$R(T)$-modules $\left\{ \frac{G_0^G(Y)}{J(Y)} \right\}$ and $\left\{ \frac{G_0^T(X)}{J(Y)} \right\}$ are isomorphic. But this follows from [9, Theorem 2.1].

**Lemma 9.4.** Let $G$ be a connected split reductive group over $k$ and let $X \in \text{Sm}_k^G$. Then for any admissible gadget $(V_i, U_i)$ for $G$, the inverse system $\left\{ K_0(X_G^i) \right\}$ satisfies the Mittag-Leffler condition. In particular, the map $K_0(X_G) \to K_0(X_G)$ is an isomorphism.
Proof. We first consider the case of a split torus $T$. By Corollary 3.3, we can assume that $(V_i, U_i)$ is a good admissible gadget. In that case, we have shown in Lemma 9.3 that there is an isomorphism of the inverse systems $\left\{ K_0^T(X_i) \right\} \cong \left\{ K_0(X_i^+) \right\}$ of $R(T)$-modules. Since the first inverse system is Mittag-Leffler, so should be $\left\{ K_0(X_i^+) \right\}$. The general case of connected split reductive groups now follows from Proposition 6.4. □

**Proposition 9.5.** Let $G$ be a connected split reductive group over $k$ and let $X \in \text{Sm}^G_k$. Then the morphisms of $\widehat{R}(G)$-modules

\begin{equation}
K_0^G(X)_{I_G} \xrightarrow{\widehat{\beta}_G} K_0(X_G) \xleftarrow{r_G} K_0(X_G)
\end{equation}

are isomorphisms. For $X \in \text{Sch}^G_k$, the map

\begin{equation}
G_0^G(X)_{I_G} \xrightarrow{\widehat{\beta}_G} \mathcal{G}_0(X_G)
\end{equation}

is an isomorphism of $\widehat{R}(G)$-modules.

Proof. The second isomorphism of (9.3) follows from Lemma 9.4. To prove the first isomorphism, we can use Theorem 6.1 and Corollary 6.5 and argue as in the proof of Theorem 8.3 to reduce to the case of a split torus.

For the singular case, one has to observe that the variant of Corollary 6.5 also holds for $G_0$. All the arguments in the proof of Proposition 6.4 go through in the singular case as well except the diagram (6.6). But using the definition of the inverse system $\{ X_i^G \}$, we can replace pull-back $G_*(X_i^{G+1}) \to G_*(X_i^G)$ with the pull-back via the open inclusion $G_*(X_i^{G+1}) \to G_*\left( X \times (U_i \oplus V) \right)$ and this pull-back always commutes with projective push-forward.

Finally, the case of split torus follows from Lemma 9.3. □

9.2. The general case. We now deduce the completion theorem for all groups $G$ and all $G$-schemes from the case of split reductive groups using the Morita equivalence as follows. Recall from Definition 3.6 that $\mathcal{K}$-theory is defined for all linear algebraic groups $G$ and all $X \in \text{Sm}^G_k$. We have also seen above that $\mathcal{G}_0$-theory is defined for all $X \in \text{Sch}^G_k$.

**Lemma 9.6.** Let $H$ be a closed normal subgroup of a linear algebraic group $G$ and let $F = G/H$. Let $f : X \to Y$ be a morphism in $\text{Sch}^G_k$ which is an $H$-torsor. Then there is a canonical isomorphism $\mathcal{G}_0(Y_F) \cong \mathcal{G}_0(X_G)$.

Proof. Let $\rho = (V_i, U_i)$ be an admissible gadget for $F$. Then $G$ acts freely on $X \times U_i$ and the map $X \times U_i \to Y \times U_i$ is $G$-equivariant which is an $H$-torsor. This in turn implies that the map $X \times U_i \to Y \times U_i$ is an isomorphism. We conclude that the map of inverse systems $\left\{ G_*\left( X \times U_i \right) \right\} \to \left\{ G_*\left( Y_i^F(\rho) \right) \right\}$ is an isomorphism. To prove the lemma, we only have to show that there is a canonical isomorphism

\begin{equation}
\lim_{\leftarrow} G_0\left( X \times U_i \right) \xrightarrow{\cong} \mathcal{G}_0(X_G).
\end{equation}
To show this, we first observe that if $\rho = (V_i, U_i)$ is the admissible gadget for $F$ as chosen above, then each $V_i$ is a $k$-rational representation of $G$ with $G$-invariant open subset $U_i$. Since $H$ acts freely on $X$ and $F$ acts freely on each $U_i$, we see that $G$ acts freely on each $X \times U_i$. In particular, $\rho$ is an admissible gadget for the $G$-action on $X$. The isomorphism (9.5) now follows at once from Lemma 9.1. □

Corollary 9.7. Let $H$ be a closed subgroup of a linear algebraic group $G$ over $k$ and let $X \in \text{Sch}_k^H$. Then there is a canonical isomorphism $G_0(X_H) \cong G_0(Y_G)$, where $Y = X \times G$.

Proof. The proof is exactly same as the proof of Corollary 2.7 once we have Lemma 9.6. We omit the details. □

Theorem 9.8. Let $G$ be a linear algebraic group over $k$ and let $X \in \text{Sm}_k^G$. Then the morphisms of $\hat{R}(G)$-modules

\[ (9.6) \quad K_0^G(X)_{IG} \xrightarrow{\beta^G} K_0(X) \xleftarrow{\alpha^G} K_0(X_G) \]

are isomorphisms. For $X \in \text{Sch}_k^G$, the map

\[ (9.7) \quad G_0^G(X)_{IG} \xrightarrow{\beta^G} G_0(X_G) \]

is an isomorphism of $\hat{R}(G)$-modules.

Proof. We embed $G$ as a closed subgroup of some general linear group $GL_n$ over $k$ and set $Y = X \times GL_n$. Then $Y \in \text{Sch}_k^{GL_n}$ and is smooth if $X$ is so. Moreover, it follows from Theorem 4.1 that the map $G_*^G(X) \to G_*^{GL_n}(Y)$ is an isomorphism of $R(GL_n)$-modules. Hence the map $G_*^G(X)_{IG} \to G_*^{GL_n}(Y)_{IGL_n}$ is an isomorphism. On the other hand, we have already observed before that the $IGL_n$-adic and $IG$-adic topologies on $G_*^G(X)$ coincide. We conclude that the map $G_*^G(X)_{IG} \to G_*^{GL_n}(Y)_{IGL_n}$ is an isomorphism.

It follows from Corollary 2.7 that $K_*(X_G) \cong K_*(Y_{GL_n})$ if $X \in \text{Sm}_k^G$. It follows from Corollary 9.7 that $G_0(X_G) \cong G_0(Y_{GL_n})$. Thus we have reduced the proof of the theorem to case of $GL_n$. But this case is covered by Proposition 9.5 since $GL_n$ is connected and split reductive. □

Remark 9.9. It follows immediately from the non-equivariant Riemann-Roch theorem that there is an isomorphism $G_0(X_G) \cong \prod_p \text{CH}_p^G(X)$, if we are working with the rational coefficients. Here, $\text{CH}_p^G(X)$ is the equivariant Chow group of Totaro and Edidin-Graham [8]. We see at once that the equivariant Riemann-Roch theorem of Edidin and Graham [9] is an immediate consequence of Theorem 9.8. In this sense, Theorem 9.8 may be called the integral version of the equivariant Riemann-Roch theorem.

As a consequence of Theorem 9.8 we get the following extension of Corollary 8.5 for $p = 0$. 
Corollary 9.10. Let $G$ be a linear algebraic group over $k$ and let $f : X \to Y$ be a $G$-equivariant morphism of smooth $G$-schemes. Assume that $f$ is an $\mathbb{A}^1$-homotopy equivalence of schemes (after forgetting the $G$-action). Then the map $f^* : K_0^G(Y) \to K_0^G(X)$ induces an isomorphism of the $I_G$-adic completions.

10. Failure of Atiyah-Segal completion theorem

In this section, we show by an example that the completion theorem is in general false for smooth schemes which are not filtrable. To show this, we take our ground field to be the field of complex numbers $\mathbb{C}$ and $G = \mathbb{G}_m$. Consider the closed subgroup $H = \{1, -1\} \subseteq G$ and set $X = G/H$. Let $\mathbb{Z}_2$ denote the 2-adic completion of $\mathbb{Z}$. For $m \geq 1$, let $\mu_m$ denote the subgroup of $m$-th roots of unity in $\mathbb{C}^*$. Given a commutative ring $A$, an element $a \in A$ and an $A$-module $M$, let $a^M$ denote the submodule of $M$ consisting of those elements which are annihilated by $a$. We wish to prove the following. This will produce counter examples to the completion theorem if we weaken the filtrability condition.

Theorem 10.1. Let $X = G/H$ be the homogeneous space as above. Then

1. For $p \geq 0$, the map $K_p(X_G) \to K_p(X_G)$ is an isomorphism.
2. $K_p^G(X)_{I_G} \xrightarrow{\cong} K_0(X_G)$.
3. For $p > 0$ odd, the map $K_p^G(X)_{I_G} \to K_p(X_G)$ is an isomorphism.
4. For $p > 0$ even, there is a short exact sequence

$$0 \to K_p^G(X)_{I_G} \to K_p(X_G) \to \mathbb{Z}_2 \to 0.$$ 

We shall prove this theorem in several steps. Let us consider the inverse system of rings $\left\{ R_n = \frac{\mathbb{Z}[u]}{(u^n, u(2-u))} \right\}_{n \geq 1}$ with the obvious quotient homomorphisms. We begin with the following elementary lemma.

Lemma 10.2. For any abelian group $M$, the inverse system $\left\{ \text{Tor}^1_\mathbb{Z}(R_n, M) \right\}$ is isomorphic to the inverse system $\left\{ 2^{n-1}M \right\}$ whose structure maps are given by multiplication by 2.

Proof. For a fixed $n \geq 1$, there is an exact sequence

$$0 \to \left( \frac{u^n, 2-u}{(u^n, u(2-u))} \right) \to R_n \to \frac{\mathbb{Z}[u]}{(u)} \times \frac{\mathbb{Z}[u]}{(u^n, 2-u)} \to \mathbb{Z}/2 \to 0.$$ 

The exactness of this sequence is clear except possibly at the third term. However, we notice that $\frac{\mathbb{Z}[u]}{(u^n, 2-u)} \cong \frac{\mathbb{Z}/2u}{(u^n, 2-u)} \cong \mathbb{Z}/2$. So we only have to show that if $f(u)$ and $g(u)$ are two polynomials such that $f(u) - g(u) \in (u, 2-u)$, then there is a polynomial $h(u)$ such that $h(u) - f(u) \in (u)$ and $h(u) - g(u) \in (u^n, 2-u)$. But this is an elementary exercise. We write $f(u) - g(u) = h_1(u) - h_2(u)$, where $h_1(u) \in (u)$ and $h_2(u) \in (u^n, 2-u)$. Equivalently, we have that $f(u) - h_1(u) = g(u) - h_2(u)$. Set $h(u) = f(u) - h_1(u) = g(u) - h_2(u)$. Then, we see that $h(u) - f(u) \in (u)$ and $h(u) - g(u) \in (u^n, 2-u)$. This proves the exactness of the above sequence.

We next claim that the first term of the above exact sequence is zero. To see this, let $f(u) \in (u^n, 2-u) \cap (u)$. Then we can write
\[ f(u) = u g(u) \] and \[ f(u) = u^n h_1(u) + (2 - u) h_2(u) \]
\[ \Rightarrow u(g(u) - u^{n-1} h_1(u)) = (2 - u) h_2(u). \]
\[ \Rightarrow u(2 - u) h_2(u). \]
\[ \Rightarrow f(u) = u^n h_1(u) + (2 - u) u h_3(u). \]
\[ \Rightarrow f(u) \in (u^n, u(2 - u)). \]

This proves the claim. Using this claim, the above exact sequence is simplified as

\[ (10.1) \quad 0 \to R_n \to \mathbb{Z} \times \mathbb{Z}/2^n \to \mathbb{Z}/2 \to 0, \]

where \( \mathbb{Z} \to \mathbb{Z}/2 \) and \( \mathbb{Z}/2^n \to \mathbb{Z}/2 \) are the obvious quotient maps.

Since \( \text{Tor}_2^\mathbb{Z}(\mathbb{Z}/2, M) = 0 \), the exact sequence (10.1) yields an exact sequence

\[ (10.2) \quad 0 \to \text{Tor}_1^\mathbb{Z}(R_n, M) \to 2^a M \to 2M. \]

Moreover, the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \mathbb{Z}^{2^{n+1}} & \mathbb{Z} \to \mathbb{Z}/2^{n+1} & 0 \\
& & 2 & \downarrow & \Downarrow & \downarrow \\
0 & \to & \mathbb{Z} & \mathbb{Z}^{2^n} & \mathbb{Z} \to \mathbb{Z}/2^n & 0 \\
& & 2^{n-1} & \downarrow & \Downarrow & \downarrow \\
0 & \to & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}/2 & 0
\end{array}
\]

gives rise to the commutative diagram

\[
\begin{array}{cccc}
0 & \to & 2^a M & \to M & 2^n M & \to M \\
& & \downarrow & \Downarrow & \downarrow & \Downarrow \\
0 & \to & 2M & \to M & 2M &
\end{array}
\]

and this shows that the last map in (10.2) is multiplication by \( 2^{n-1} \). We conclude that the map \( \text{Tor}_1^\mathbb{Z}(R_n, M) \to 2^{n-1} M \) is an isomorphism and the map \( \text{Tor}_2^\mathbb{Z}(R_{n+1}, M) \to \text{Tor}_2^\mathbb{Z}(R_n, M) \) is multiplication by 2. This completes the proof of the lemma. \( \square \)

**Corollary 10.3.** Let \( \{S_n\} \) denote the inverse system of rings \( S_n = \frac{\mathbb{Z}[t]}{((1-t)^{n}, t^2-1)} \) \( n \geq 1 \) with the obvious quotient homomorphisms. Then

1. For \( p \geq 0 \) even, \( \text{Tor}_1^\mathbb{Z}(S_n, K_p(\mathbb{C})) = 0 \) for each \( n \geq 1 \).
2. For \( p \geq 0 \) odd, \( \lim_{\frac{n}{n}} \text{Tor}_1^\mathbb{Z}(S_n, K_p(\mathbb{C})) \cong \mathbb{Z}_2 \) and \( \lim_{\frac{n}{n}} \text{Tor}_1^\mathbb{Z}(S_n, K_p(\mathbb{C})) = 0 \).

**Proof.** There is a strict isomorphism of the inverse systems of rings \( \{S_n\} \xrightarrow{\sim} \{R_n\} \) via the transformation \( t \mapsto 1 - u \). The corollary is now a consequence of Lemma 10.2 and the solution of the Lichtenbaum’s conjecture (about the \( K \)-theory of algebraically closed fields) by Suslin [21].

Suslin has shown that for \( p = 2q > 0 \), \( K_p(\mathbb{C}) \) is uniquely divisible. He has also shown that for \( p = 2q - 1 > 0 \), there is an isomorphism \( 2^a K_p(\mathbb{C}) \cong \mu_{2^a} \).
and the map $\rho^{m+1}K_p(C) \to \rho^mK_p(C)$ is multiplication by 2 (cf. [29, Proof of Theorem IV.1.6]). In particular, this map is surjective. The corollary now follows from this and Lemma [10.2].

**Proposition 10.4.** Let $X = G/H$ be as chosen in the beginning of this section and let $\rho = 1 - t \in R(G) = \mathbb{Z}[t, t^{-1}]$. Then

1. For $p \geq 0$ even, $\lim_{\rightarrow}^{n} \rho^nK_p^G(X) = 0$ for $m = 0, 1$.
2. For $p \geq 0$ odd, $\lim_{\rightarrow}^{n} \rho^nK_p^G(X) = 0$ and $\lim_{\rightarrow}^{n} \rho^nK_p^G(X) \cong \mathbb{Z}/2$.

**Proof.** The Morita equivalence and (7.4) imply that $K_p^G(X) \cong K_*(C) \otimes Z R(H)$. Moreover, we know that $R(H) \cong \mathbb{Z}[t]/(t^2 - 1)$. We have the exact sequence

$$0 \to \rho^nR(H) \to R(H) \overset{\rho^n}{\to} R(H) \to S_n \to 0,$$

where $S_n$ is as in Corollary [10.3]. This yields a commutative diagram of exact sequences

$$\begin{align*}
\rho^nR(H) \otimes_{Z} K_*(C) &\to R(H) \otimes_{Z} K_*(C) \to \rho^nR(H) \otimes_{Z} K_*(C) \to 0 \\
\rho^nR(H) \otimes_{Z} K_*(C) &\to \rho^nR(H) \otimes_{Z} K_*(C) \to \text{Tor}^1_Z(S_n, K_*(C)) \to 0
\end{align*}$$

where the map $\text{Tor}^1_Z(S_n, K_*(C)) \to \rho^nR(H) \otimes_{Z} K_*(C)$ is injective because the possible kernel of this map comes from $\text{Tor}^1_Z(R(H), K_*(C))$, and this is zero since $R(H)$ is a torsion-free abelian group. A diagram chase gives us an exact sequence

$$(10.4) \quad \rho^nR(H) \otimes_{Z} K_*(C) \to \rho^nK_p^G(X) \to \text{Tor}^1_Z(S_n, K_*(C)) \to 0.$$

Let $\rho^nR(H) \otimes_{Z} K_*(C)$ denote the image of the first map in (10.4). We claim that the inverse system $\left\{ \rho^nR(H) \otimes_{Z} K_*(C) \right\}$ is zero as a pro-abelian group. In particular, it satisfies the Mittag-Leffler condition. For this, it is enough to show that the inverse system $\left\{ \rho^nR(H) \otimes_{Z} K_*(C) \right\}$ is zero as a pro-abelian group.

Since $R(H)$ is a noetherian ring, the chain $\rho^nR(H) \subseteq \rho^{n+1}R(H) \subseteq \cdots$ of ideals in $R(H)$ must stabilize. In other words, there exists $m \gg 0$ such that $\rho^mR(H) = \rho^{m+1}R(H) = \cdots$. This implies that $\rho^m$ annihilates $\rho^nR(H)$ for every $n \geq 1$. That is, the map $\rho^{n+m}R(H) \overset{\rho^m}{\to} \rho^nR(H)$ is zero. Hence, the map

$$\rho^{n+m}R(H) \otimes_{Z} K_*(C) \overset{\rho^m}{\to} \rho^mR(H) \otimes_{Z} K_*(C)$$

is zero for every $n \geq 1$. This proves the claim. We conclude from this claim that there is a short exact sequence of pro-abelian groups

$$0 \to \rho^nR(H) \otimes_{Z} K_*(C) \to \rho^nK_p^G(X) \to \text{Tor}^1_Z(S_n, K_*(C)) \to 0$$

in which the first pro-abelian group is zero. In particular, we get an isomorphism of pro-abelian groups

$$(10.5) \quad \left\{ \rho^nK_p^G(X) \right\} \cong \left\{ \text{Tor}^1_Z(S_n, K_*(C)) \right\}.$$

We now apply Corollary [10.3] to conclude the proof of the proposition.
Proof of Theorem 10.1: Recall that $\rho = 1 - t \in R(G) = \mathbb{Z}[t, t^{-1}]$. It is then clear that $I_G = (\rho)$. Let $(V_i, U_i)$ be a good admissible gadget for $G$. It follows from [27, Theorem 2.1] that for every $p \geq 0$, there is a short exact sequence of the inverse systems of $R(G)$-modules

$$0 \to K^G_p(X) \xrightarrow{(\rho^i)} K_p(X_G^i) \to \rho^i K^G_{p-1}(X) \to 0.$$  

Since the inverse system on the left is surjective, taking the limits and using Corollary 3.5, we conclude that

$$0 \to K^G_p(X)_{I_G} \to K_p(X_G) \to \lim_{i} \rho^i K^G_{p-1}(X) \to 0$$

is exact and $\lim_{i} \rho^i K^G_{p-1}(X)$ for each $p \geq 0$. It follows from Proposition 10.4 and the Milnor exact sequence (3.15) that the map $K_p(X_G) \to K_p(X_G)$ is an isomorphism for all $p \geq 0$, $K^G_p(X)_{I_G} \xrightarrow{\cong} K_p(X_G)$ for $p > 0$ odd, and there is a short exact sequence

$$0 \to K^G_p(X)_{I_G} \to K_p(X_G) \to \mathbb{Z}_2 \to 0$$

for $p > 0$ even. This finishes the proof of Theorem 10.1. ∎

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