Matrix manipulations for properties of Fibonacci $p$-numbers and their generalizations

Ömür Deveci · Anthony G. Shannon

Abstract In this paper, we define the Fibonacci-Fibonacci $p$-sequence and then we discuss the connection of the Fibonacci-Fibonacci $p$-sequence with Fibonacci and Fibonacci $p$-sequences. We also provide a new Binet formula and a new combinatorial representation of Fibonacci $p$-numbers by the aid of the $n$th power of the generating matrix the Fibonacci-Fibonacci $p$-sequence. We furthermore develop relationships between the Fibonacci-Fibonacci $p$-numbers and their permanent, determinant and sums of certain matrices.

Keywords Fibonacci sequence · Fibonacci $p$-sequence · matrix · representation

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1 Introduction

The well-known Fibonacci sequence is defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ in which } F_0 = 0 \text{ and } F_1 = 1.$$  

There are many important generalizations of the Fibonacci sequence. The Fibonacci $p$-sequence [25, 26] is one of them, namely:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ for } p = 1, 2, 3, \ldots \text{ and } n > p$$

in which $F_p(0) = 0$, $F_p(1) = \cdots = F_p(p) = 1$. When $p = 1$, the Fibonacci $p$-sequence $\{F_p(n)\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$. There is a three-fold value in searching for elegant generalizations in number theory, namely, to investigate which identities are essential, to discover links with otherwise apparently unrelated results, and to formulate ideas for further research.
It is easy to see that the characteristic polynomials of the Fibonacci sequence and the Fibonacci $p$-sequence are $f_1(x) = x^2 - x - 1$ and $f_2(x) = x^{p+1} - x^p - 1$, respectively.

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding $k$ terms:

$$a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1}$$

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [14], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix $A$ be defined by

$$A = [a_{i,j}]_{k \times k} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1}
\end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied recently by many authors: see for example, [1, 2, 5, 9–13, 22–24, 27]. In [6–8, 15, 17–19, 25, 26, 28], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we discuss connections between Fibonacci and Fibonacci $p$-numbers. Firstly, we define the Fibonacci-Fibonacci $p$-sequence and then we give associated recurrence relations for Fibonacci and Fibonacci $p$-sequences. We also provide the relations between the generating matrix of the Fibonacci-Fibonacci $p$-numbers and the elements of the Fibonacci and the Fibonacci $p$-sequence. Furthermore, using the generating matrix for the Fibonacci-Fibonacci $p$-sequence, we obtain some new structural properties of the Fibonacci $p$-numbers such as the Binet formula and the combinatorial representations. Finally, we obtain relationships among the Fibonacci numbers, the Fibonacci-Fibonacci $p$-numbers and their sums, and permanents and, determinants of certain matrices.
2 The main results

We now define the Fibonacci-Fibonacci $p$-sequence by the following homogeneous linear recurrence relation for any given $p(3,4,5,\ldots)$ and $n \geq 0$

$$F_{n+p+3}^{p} = 2F_{n+p+2}^{p} - F_{n+p}^{p} + F_{n+p-2}^{p} - F_{n+1}^{p} + F_{n}^{p}$$ (2.1)

in which $F_{0}^{p} = 0$ and $F_{1}^{p} = \cdots = F_{p+2}^{p} = 1$.

First we consider a relationship between the Fibonacci-Fibonacci $p$-sequence, Fibonacci and Fibonacci $p$-sequences.

**Theorem 2.1** Let $p \geq 3$ and let $F_{n}$, $F_{p}(n)$ and $F_{n}^{p}$ be the $n$th Fibonacci number, the Fibonacci $p$-number and the Fibonacci-Fibonacci $p$-numbers, respectively, then

$$F_{n} = F_{n}^{p} - F_{n+p-1}^{p} + F_{p}(n + p - 1)$$

for $n \geq 0$.

**Proof.** The assertion may be proved by induction on $n$. It is clear that $F_{0} = F_{0}^{p} - F_{p}^{p} + F_{p}(p - 1) = 0$. Now we assume that the equation holds for $n > 0$. Then we show that the equation holds for $n + 1$. Since the characteristic polynomial of the Fibonacci-Fibonacci $p$-sequence $\{F_{n}^{p}\}$, is

$$p(x) = x^{p+3} - 2x^{p+2} + x^{p} - x^{2} + x + 1$$

and

$$p(x) = f_{1}(x)f_{2}(x)$$

where $f_{1}(x)$ and $f_{2}(x)$ are the characteristic polynomials of the Fibonacci sequence and the Fibonacci $p$-sequence, respectively, we obtain the following relations:

$$F_{n+p+3} = 2F_{n+p+2} - F_{n+p} + F_{n+2} - F_{n+1} - F_{n}$$

and

$$F_{p}(n + p + 3) = 2F_{p}(n + p + 2) - F_{p}(n + p) + F_{p}(n + 2) - F_{p}(n + 1) - F_{p}(n)$$

for $n > 0$. Thus, by a simple calculation, we have the conclusion. $\Box$

From the recurrence relation (2.1), we have

$$\begin{bmatrix}
F_{n}^{p} \\
F_{n+1}^{p} \\
F_{n+p}^{p} \\
F_{n+p+1}^{p} \\
\vdots \\
F_{n+p+3}^{p}
\end{bmatrix} =
\begin{bmatrix}
2 & 0 & -1 & 0 & \cdots & 0 & 0 & 1 & -1 & -1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
F_{n}^{p} \\
F_{n+p}^{p} \\
F_{n+p+1}^{p} \\
F_{n+p+2}^{p} \\
F_{n+p+3}^{p}
\end{bmatrix}$$
for the Fibonacci-Fibonacci $p$-sequence $\{F_n^{F,p}\}$. Now we define

$$M_p = \begin{bmatrix}
2 & 0 & -1 & 0 & \cdots & 0 & 0 & 1 & -1 & -1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$(p+3) \times (p+3)$. The companion matrix $M_p = [m_{ij}]_{(p+3) \times (p+3)}$ is said to be the Fibonacci-Fibonacci $p$-matrix. For more details on the companion type matrices, see [20,21]. From induction on $n$, we obtain

$$(M_p)^n = \begin{bmatrix}
F_{n+3}^{F,p} & F_p (n+1) + F_{n+1}^{F,p} & F_p (n+2) + F_{n+2}^{F,p} & F_p (n+3) \\
F_{n+2}^{F,p} & F_p (n) + F_{n}^{F,p} & F_p (n+1) + F_{n+1}^{F,p} & F_p (n+2) \\
F_{n+1}^{F,p} & F_p (n-1) + F_{n-1}^{F,p} & F_p (n) + F_{n}^{F,p} & F_p (n+1) \\
\vdots & \vdots & \vdots & \vdots \\
F_{n-p+1}^{F,p} & F_p (n-2p-1) + F_{n-2p-1}^{F,p} & F_p (n-2p) + F_{n-2p}^{F,p} & F_p (n-2p+1) \\
\end{bmatrix}$$

where

$$M_p = \begin{bmatrix}
F_p (n+4) & F_p (n+3) & F_p (n+2) & F_p (n+1) & F_p (n) \\
F_p (n+3) & F_p (n+2) & F_p (n+1) & F_p (n) & F_p (n-1) \\
F_p (n+2) & F_p (n+1) & F_p (n) & F_p (n-1) & F_p (n-2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
F_p (n-2p+2) & F_p (n-2p+1) & F_p (n-2p) & F_p (n-2p-1) & F_p (n-2p-2) \\
\end{bmatrix}$$

In [25], Stakhov defined the generalized Fibonacci $p$-matrix $Q_p$ and developed the $n$th power of the matrix $Q_p$. In [16], Kiliç gave a Binet formula for the Fibonacci $p$-numbers by the matrix method. Now we concentrate on finding another Binet formula for the Fibonacci $p$-numbers by the aid of the matrix $(M_p)^n$.

**Lemma 2.2** The characteristic equation of all the Fibonacci-Fibonacci $p$-numbers $x^{p+3} - 2x^{p+2} + x^p - x^2 + x + 1 = 0$ does not have multiple roots for $p \geq 3$.

**Proof.** It is clear that

$$x^{p+3} - 2x^{p+2} + x^p - x^2 + x + 1 = (x^{p+1} - x^p - 1) (x^2 - x - 1).$$

In [16], it was shown that the equation $x^{p+1} - x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation
\(x^2 - x - 1 = 0\) are \(\frac{1 + \sqrt{5}}{2}\) and \(\frac{1 - \sqrt{5}}{2}\). Since \(\left(\frac{1 + \sqrt{5}}{2}\right)^p + 1 - \left(\frac{1 + \sqrt{5}}{2}\right)^p - 1 \neq 0\) and \(\left(\frac{1 - \sqrt{5}}{2}\right)^p + 1 - \left(\frac{1 - \sqrt{5}}{2}\right)^p - 1 \neq 0\) for \(p > 1\), the equation \(x^{p+3} - 2x^{p+2} + x^p - x^2 + x + 1 = 0\) does not have multiple roots for \(p \geq 3\).

\[\square\]

Let \(x_1, x_2, \ldots, x_{p+3}\) be the roots of the equation \(x^{p+3} - 2x^{p+2} + x^p - x^2 + x + 1 = 0\) and let \(V_p\) be a \((p + 3) \times (p + 3)\) Vandermonde matrix as follows:

\[
V_p = \begin{bmatrix}
(x_1)^{p+2} & (x_2)^{p+2} & \cdots & (x_{p+3})^{p+2} \\
(x_1)^{p+1} & (x_2)^{p+1} & \cdots & (x_{p+3})^{p+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & x_2 & \cdots & x_{p+3} \\
1 & 1 & \cdots & 1
\end{bmatrix}.
\]

Assume that \(V_p(i, j)\) is a \((p + 3) \times (p + 3)\) matrix derived from the Vandermonde matrix \(V_p\) by replacing the \(j^{th}\) column of \(V_p\) by \(W_p(i)\), where, \(W_p(i)\) is a \((p + 3) \times 1\) matrix as follows:

\[
W_p = \begin{bmatrix}
(x_1)^{n+p+3-i} \\
(x_2)^{n+p+3-i} \\
\vdots \\
(x_{p+3})^{n+p+3-i}
\end{bmatrix}.
\]

**Theorem 2.3** Let \(p\) be a positive integer such that \(p \geq 3\) and let \((M_p)^n = m_{i,j}^{(p,n)}\) for \(n \geq 1\), then

\[
m_{i,j}^{(p,n)} = \frac{\det V_{p(i,j)}}{\det V_p}.
\]

**Proof.** Since \(x^{p+3} - 2x^{p+2} + x^p - x^2 + x + 1 = 0\) does not have multiple roots for \(p \geq 3\), the eigenvalues of the Fibonacci-Fibonacci \(p\)-matrix \(M_p\) are distinct. Then, it is clear that \(M_p\) is diagonalizable. Let \(D_p = \text{diag}(x_1, x_2, \ldots, x_{p+3})\). Then we may write \(M_pV_p = V_pD_p\). Since the matrix \(V_p\) is invertible, we obtain the equation \((V_p)^{-1} M_p V_p = D_p\). Therefore, \(M_p\) is similar to \(D_p\); hence, \((M_p)^n V_p = V_p (D_p)^n\) for \(n \geq 1\). So we now have the following linear system of equations:

\[
\begin{align*}
m_{i,1}^{(p,n)} (x_1)^{p+2} + m_{i,2}^{(p,n)} (x_1)^{p+1} + \cdots + m_{i,p+3}^{(p,n)} = (x_1)^{n+p+3-i} \\
m_{i,1}^{(p,n)} (x_2)^{p+2} + m_{i,2}^{(p,n)} (x_2)^{p+1} + \cdots + m_{i,p+3}^{(p,n)} = (x_2)^{n+p+3-i} \\
\vdots \\
m_{i,1}^{(p,n)} (x_{p+3})^{p+2} + m_{i,2}^{(p,n)} (x_{p+3})^{p+1} + \cdots + m_{i,p+3}^{(p,n)} = (x_{p+3})^{n+p+3-i}.
\end{align*}
\]
Then we conclude that
\[ m_{i,j}^{(p,n)} = \frac{\det V_p^{(i,j)}}{\det V_p} \]
for each \( i, j = 1, 2, \ldots, p + 3 \).

Thus by Theorem 2.3 and the matrix \((M_p)^n\), we have the following useful result for the Fibonacci \( p \)-numbers.

**Corollary 2.4** Let \( p \) be a positive integer such that \( p \geq 3 \) and let \( F_p(n) \) be the \( n \)th element of the Fibonacci \( p \)-sequence, then
\[ F_p(n) = \frac{\det V_p(1, p + 1)}{\det V_p} \text{ for } n \geq 1. \]

Let \( K(k_1, k_2, \ldots, k_v) \) be a \( v \times v \) companion matrix as follows:
\[
K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.
\]

**Theorem 2.5** The \((i,j)\) entry \( k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) \) in the matrix \( K^n(k_1, k_2, \ldots, k_v) \) is given by the following formula [4]:
\[
k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} t_1 + t_2 + \cdots + t_v \times \left( \frac{t_1 + \cdots + t_v}{t_1 \cdot \cdot \cdot t_v} \right) k_1^{t_1} \cdots k_v^{t_v}
\]
where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + vt_v = n + i + j \), \((t_1 + \cdots + t_v) / (t_1 \cdot \cdot \cdot t_v) \) is a multinomial coefficient, and the coefficients in (2.2) are defined to be 1 if \( n = i - j \).

In [16], Kiliç also gave a combinatorial representation of the Fibonacci \( p \)-numbers using the \( n \)th power of the matrix \( Q_p \) in Theorem 2.5. Here we investigate a new combinatorial representation of the Fibonacci \( p \)-numbers by the matrix \((M_p)^n\).

**Corollary 2.6**
\[
F_p(n) = \sum_{(t_1, t_2, \ldots, t_{p+3})} \frac{t_{p+1}}{t_1 + t_2 + \cdots + t_{p+3}} \times \left( \frac{t_1 + t_2 + \cdots + t_{p+3}}{t_1, t_2, \ldots, t_{p+3}} \right) 2^{t_1} (-1)^{t_3 + t_{p+2} + t_{p+3}}, \quad (n \geq 1)
\]
where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + (p + 3) t_{p+3} = n + p \).
Proof. If we take $i = 1, j = p + 1$ in Theorem 2.5, then we can directly see
the conclusions from $(M_p)^n$.
\qed

We now focus on finding the permanental representations.

**Definition 2.7** A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix
in the $k^{th}$ column (resp. row.) if the $k^{th}$ column (resp. row.) contains exactly
two non-zero entries.

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix $M$. If $M$ is
contractible in the $k^{th}$ column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then
the $(u-1) \times (v-1)$ matrix $M_{ij}$ is obtained from $M$ by replacing the $i^{th}$
row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the $j^{th}$ row. The $k^{th}$ column is called
the contraction in the $k^{th}$ column relative to the $i^{th}$ row and the $j^{th}$ row.

In [3], Brualdi and Gibson obtained that $\text{per} (M) = \text{per} (N)$ if $M$ is a real
matrix of order $\alpha > 1$ and $N$ is a contraction of $M$.

Now we concentrate on finding relationships among the Fibonacci numbers, the Fibonacci-Fibonacci $p$-numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Fibonacci-Fibonacci $p$-numbers. Let $F_{m,p} = \begin{bmatrix} f_{i,j}^{(p)} \end{bmatrix}$ be the $m \times m$ super-diagonal matrix,
defined by

$$f_{i,j}^{(p)} = \begin{cases} 2 & \text{if } i = r \text{ and } j = r \text{ for } 1 \leq r \leq m, \\ 1 & \text{if } i = r + 1 \text{ and } j = r \text{ for } 1 \leq r \leq m - 1 \\ & \text{and} \\ 1 & \text{if } i = r \text{ and } j = r + p \text{ for } 1 \leq r \leq m - p, \\ -1 & \text{if } i = r \text{ and } j = r + p + 1 \text{ for } 1 \leq r \leq m - p - 1 \\ & \text{and} \\ 0 & \text{otherwise.} \\ \end{cases}$$

Then we have the following Theorem.

**Theorem 2.8** For $m \geq p + 3$,

$$\text{per} F_{m,p}^F = F_{m+3}^F - F_{m+3}^{F,p}.$$

Proof. Let $x_m^p = F_m - F_{m+1}^F$ and let the equation be hold for $m \geq p + 3$.
Then we show that the equation holds for $m + 1$. If we expand the $\text{per} F_{m,p}^F$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per} F_{m+1,p}^F = 2\text{per} F_{m,p}^F - \text{per} F_{m-2,p}^F + \text{per} F_{m-p,p}^F - \text{per} F_{m-p-1,p}^F - \text{per} F_{m-p-2,p}^F.$$
Since $\text{per}_{F m,p} = x_{m+3}^p$, $\text{per}_{F m-2,p} = x_{m+1}^p$, $\text{per}_{F m-p+3} = x_{m-p+3}^p$, $\text{per}_{F m-p} = x_{m-p+1}^p$, and $\text{per}_{F m-p-1} = x_{m-p+5}^p$, we easily obtain that $\text{per}_{F m+1,p} = x_{m+4}^p = F_{m+4} - F_{m-2, p}$. 

Then we have the following results:

Let $G_{m,p}^F = \begin{bmatrix} g_{i,j}^{(p)} \end{bmatrix}$ be the $m \times m$ matrix, defined by

$$
g_{i,j}^{(p)} = \begin{cases} 2 & \text{if } i = r \text{ and } j = r \text{ for } 1 \leq r \leq m - p + 1, \\
2 & \text{if } i = r + 1 \text{ and } j = r \text{ for } 1 \leq r \leq m - p, \\
1 & \text{if } i = r \text{ and } j = r + p \text{ for } 1 \leq r \leq m - p, \\
1 & \text{if } i = r + 1 \text{ and } j = r + p \text{ for } 1 \leq r \leq m - p - 1, \\
-1 & \text{if } i = r \text{ and } j = r + p + 1 \text{ for } 1 \leq r \leq m - p - 1, \\
-1 & \text{if } i = r + 1 \text{ and } j = r + p + 1 \text{ for } 1 \leq r \leq m - p - 2, \\
0 & \text{otherwise.} \\
\end{cases}$$

Then we have the following Theorem.

**Theorem 2.9** For $m \geq p + 3$,

$$
\text{per}_{G_{m,p}^F} = F_{m-p+4} - F_{m-2, p}.
$$

**Proof.** Let $x_{m}^p = F_{m} - F_{m}^{p}$ and let the equation hold for $m \geq p + 3$. Then we show that the equation holds for $m + 1$. If we expand $\text{per}_{G_{m,p}^F}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$
\text{per}_{G_{m+1,p}^F} = 2 \text{per}_{G_{m,p}^F} - \text{per}_{G_{m-2,p}^F} + \text{per}_{G_{m-p+2,p}^F} - \text{per}_{G_{m-p-1,p}^F} - \text{per}_{G_{m-p-2,p}^F}.
$$

Also, since $\text{per}_{G_{m,p}^F} = x_{m-p+4}^p$, $\text{per}_{G_{m-2,p}^F} = x_{m-p+2}^p$, $\text{per}_{G_{m-p+2,p}^F} = x_{m-2p+4}^p$, $\text{per}_{G_{m-p-1,p}^F} = x_{m-2p+3}^p$, and $\text{per}_{G_{m-p-2,p}^F} = x_{m-2p+2}^p$, it is clear that $\text{per}_{G_{m+1,p}^F} = x_{m-p+5}^p$. So the proof is complete. 

Assume next that $H_{m,p}^F = \begin{bmatrix} h_{i,j}^{(p)} \end{bmatrix}$ be the $m \times m$ matrix, defined by

$$
H_{m,p}^F = \begin{bmatrix} (m - 2) \text{ th} \\
1 & 1 \\
1 & 0 \\
\vdots & G_{m-1,p} \\
0 & 0 \\
\end{bmatrix}, \text{ for } m > p + 3,
$$

then we have the following results:
Properties of Fibonacci-Fibonacci \( p \)-sequence

Theorem 2.10 For \( m > p + 3 \),

\[
\text{per} H^F_{m,p} = \sum_{i=0}^{m-p+3} x_i^p.
\]

Proof. If we extend \( \text{per} H^E_{m,p} \) with respect to the first row, we write

\[
\text{per} H^F_{m,p} = \text{per} H^F_{m-1,p} + \text{per} G^F_{m-1,p}.
\]

Thus, by the results and an inductive argument, the proof is easily seen. \( \square \)

A matrix \( M \) is called convertible if there is an \( n \times n \) \((1, -1)\)-matrix \( K \) such that \( \text{per} M = \det (M \circ K) \), where \( M \circ K \) denotes the Hadamard product of \( M \) and \( K \).

Then we can develop relationships among the Fibonacci numbers, the Fibonacci-Fibonacci \( p \)-numbers and the determinants of certain matrices which are obtained by using the matrix \( F^F_{m,p}, G^F_{m,p} \) and \( H^F_{m,p} \). Let \( m > p + 3 \) and let \( R \) be the \( m \times m \) matrix, defined by

\[
R = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & -1 & 1 & 1 \\
1 & \cdots & 1 & 1 & -1 & 1
\end{bmatrix}
\]

Corollary 2.11 For \( m > p + 3 \),

\[
\det (F^F_{m,p} \circ R) = x_{m+3}^p,
\]

\[
\det (G^F_{m,p} \circ R) = x_{m-p+4}^p,
\]

and

\[
\det (H^F_{m,p} \circ R) = \sum_{i=0}^{m-p+3} x_i^p.
\]

Proof. Since \( \text{per} F^F_{m,p} = \det (F^F_{m,p} \circ R) \), \( \text{per} G^F_{m,p} = \det (G^F_{m,p} \circ R) \) and \( \text{per} H^F_{m,p} = \det (H^F_{m,p} \circ R) \) for \( m > p + 3 \), by Theorem 2.8, Theorem 2.9 and Theorem 2.10, we have the conclusion. \( \square \)
Finally we consider the sums of the Fibonacci-Fibonacci \( p \)-numbers. Let 
\[
S_n = \sum_{i=0}^{n} x_i^F
\]
for \( n \geq p \) and let \( T_p^F \) and \((T_p^F)^n\) be the \((p + 4) \times (p + 4)\) matrix such that
\[
T_p^F = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & & & & & \\
: & & M_p & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & 
\end{bmatrix}
\]
If we use induction on \( n \), then we obtain
\[
(T_p^F)^n = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
S_{n+2} & S_{n+1} & S_n & (M_p)^n \\
: & & & & & & \\
S_{n-p+1} & S_{n-p}
\end{bmatrix}
\]

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Authors

Ömür Deveci (Corresponding author),
Department of Mathematics,
Faculty of Science and Letters,
Kafkas University 36100, Turkey,
E-mail: odeveci30@hotmail.com

Anthony G. Shannon,
Warrane College,
University of New South Wales,
2033, Australia,
E-mail: t.shannon@warrane.unsw.edu.au

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