A FUNCTION CLASS OF STRICTLY POSITIVE DEFINITE AND LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE MODIFIED BESSEL FUNCTIONS

JAMEL EL KAMEL AND KHALED MEHREZ

Abstract. In this paper we give some conditions for a class of functions related to Bessel functions to be positive definite or strictly positive definite. We present some properties and relationships involving logarithmically completely monotonic functions and strictly positive definite functions. In particular, we are interested with the modified Bessel functions.

keywords: Bessel functions, Positive definite functions, Completely monotonic functions, logarithmically Completely monotonic functions.

1. Introduction

A complex valued continuous function $f$ is said positive definite (resp. strictly positive definite ) on $\mathbb{R}$ if for every reals numbers $x_1, x_2, ...x_n$ and every complex numbers $z_1, z_2, ...z_n$ not all zero, the inequality

$$\sum_{j=1}^{N} \sum_{k=1}^{N} z_j \bar{z}_k f(x_j - x_k) \geq 0 \quad \text{(resp. > 0)}$$

holds true ( see [6] ).

We denote by $\mathcal{P}$ (resp. $\mathcal{P}^s$) the class of such functions.

Bochner’s theorem [4] characterizes positive definite functions as Fourier transform of nonnegative finite Borel measure on the real line.

A function $f$ is said to be completely monotonic (CM) on an interval $I \subset \mathbb{R}$, if $f \in C(I)$ has derivatives of all orders on $I^0$ (the interior of $I$) and, for all $n \in \mathbb{N}$,

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^0; \quad n \in \mathbb{N}.$$

The class of all completely monotonic functions on $I$ is denoted by $CM(I)$.

A function $f$ is said to be logarithmically completely monotonic (LCM) on an interval $I \subset \mathbb{R}$, if

$$f > 0, \quad f \in C(I),$$

has derivatives of all orders on $I^0$ and, for all $n \in \mathbb{N} \setminus \{0\}$,

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, \quad x \in I^0; \quad n \geq 1.$$
The class of all logarithmically completely monotonic functions on $I$ is denoted by $LCM(I)$. We have $LCM(I) \subset CM(I)$.

Berstein’s theorem \cite{3} asserts that $f$ is completely monotonic function if and only if $f$ is the Laplace transform of nonnegative finite Borel measure on $[0, \infty[$.

In 1938, Schoenberg \cite{11} studied the completely monotonic functions and proved that these functions are closely related to positive definite functions.

H. Wendland \cite{14} was interested by strictly positive definite functions, and present a complete characterization of radial functions as being strictly positive definite on every $\mathbb{R}^d$.

Lévy Kinchin theorem asserts that a probability measure $d\mu$ supported on $[0, \infty[$ is infinitely divisible if and only if it’s Laplace transform of an logarithmically completely monotonic function.

In this paper, we consider a function class related to the modified Bessel functions. In the first part we prove that among these functions there are whose positive definite and strictly positive definite. In the second part we present some properties and relationships involving logarithmically completely monotonic functions and strictly positive definite functions. In particular, we are interested with the modified Bessel functions of the second kind.

Our paper is organized as follows: in Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we present some properties of the Bessel transform, the Bessel translation operator and the Bessel convolution product. All of these results can be found in \cite{7} and \cite{2}. In Section 4, we give some conditions for a class of functions related to Bessel functions to be positive definite or strictly positive definite. Much attention is devoted to the Bessel function, the modified Bessel function of second kind and the Bessel-Fourier transform. In Section 5, We present some properties and relationships involving logarithmically completely monotonic functions and strictly positive definite functions. As applications, in Section 6, using Schoenberg theorem \cite{11} and Wendland theorem \cite{14}, we prove logarithmically monotonicity for a class of functions related to the modified Bessel functions of second kind. In particular, it’s well known that the function $\frac{1}{x^{\frac{\alpha}{2}}K_{\alpha}(\sqrt{x})}$ is not completely monotonic, we will prove that the function $\frac{1}{x^{\frac{\alpha+1}{2}}K_{\alpha}(\sqrt{x})}$ is even logarithmically completely monotonic. We note that Ismail \cite{10} prove that the function $\frac{1}{e^{\sqrt{x}x^{\frac{\alpha}{2}}K_{\alpha}(\sqrt{x})}}$ is completely monotonic. We derive some new inequalities for the modified Bessel functions of second kind.
The normalized Bessel function of index $\alpha > -\frac{1}{2}$ is the even function defined by:

\[
\begin{align*}
\mathcal{J}_\alpha(x) &= \begin{cases} 
2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(x)}{x^\alpha}, & x > 0, \\
1, & x = 0
\end{cases} 
\end{align*}
\]

where $J_\alpha$ is the Bessel function of first kind and index $\alpha$. Thus

\[
\mathcal{J}_\alpha(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(\alpha + 1)}{n! \Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n},
\]

\[
|\mathcal{J}_\alpha(x)| \leq 1, \quad x \geq 0, \quad \alpha > -\frac{1}{2},
\]

\[
\mathcal{J}_\alpha(x) = \frac{1}{2^{\alpha-1} \sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - t^2)^{\alpha - \frac{1}{2}} \cos(xt) dt.
\]

The modified Bessel function $I_\alpha$ of first kind and index $\alpha$ is defined by:

\[
I_\alpha(x) = \sum_{n=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2n+\alpha}}{n! \Gamma(n + \alpha + 1)}, \quad \alpha \neq -1, -2, \ldots; x \in \mathbb{R}.
\]

On can see easily that

\[
I_\alpha(x) > 0, \quad \forall \alpha > -1, \quad \forall x > 0.
\]

The modified Bessel function $K_\alpha$ of second kind (called sometimes Macdonald function) and index $\alpha$ is defined by:

\[
K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin \alpha \pi},
\]

where the right-hand side of this equation is replaced by its limiting value if $\alpha$ is an integer or zero.

By using the familiar integral representation

\[
K_\alpha(x) = \int_0^{+\infty} e^{-x \cosh(t)} \cosh(\alpha t) dt, \quad \alpha \in \mathbb{R}, \quad x > 0,
\]

we deduce that

\[
K_\alpha(x) > 0, \quad \forall \alpha \in \mathbb{R}, \quad \forall x > 0.
\]

Bell [3] showed that

\[
K_\alpha(x) = \frac{1}{\sqrt{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})}} \left(\frac{x}{2}\right)^\alpha \int_1^{+\infty} e^{-xt} (t^2 - 1)^{\alpha - \frac{3}{2}} dt, \quad \alpha > -\frac{1}{2}, \quad x > 0.
\]

We denote by $C$ the set of continuous functions and $C_0$ its subspace of functions vanishing at infinity, $S$ the Schwartz space of infinitely differentiable and rapidly
decreasing functions, \( D \) the space of infinitely differentiable even functions with compact support, \( L^p \) the set of \( p \)-power integrable functions with respect to the measure \( dx \) on \( \mathbb{R} \), \( L^p_\alpha \) the set of \( p \)-power integrable even functions with respect to the measure \( x^{2\alpha+1}dx \) on \([0, \infty[\). The symbol \( M^+ \) stands for the set of nonnegative finite Borel measure on \( \mathbb{R} \).

3. Harmonic analysis related to Bessel translation operator

In this section we present some properties of the Bessel transform, the Bessel translation operator and the Bessel convolution product \([7]\).

The Fourier-Bessel transform \( \mathcal{F}_\alpha \) is defined for \( f \in D(\mathbb{R}) \) by:

\[
\mathcal{F}_\alpha f(x) = c_\alpha \int_0^{+\infty} f(t) j_\alpha(xt) t^{2\alpha+1} dt, \quad x \geq 0,
\]

where

\[
c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha + 1)}
\]

**Theorem 1.** 1) For \( f \in L^2_\alpha \), we have \( \mathcal{F}_\alpha(f) \in L^2_\alpha \) and

\[
\| \mathcal{F}_\alpha(f) \|_2 = \| f \|_2.
\]

2) For \( f \in L^1_\alpha \), we have \( \mathcal{F}_\alpha(f) \in C_0 \) and

\[
\| \mathcal{F}_\alpha(f) \|_\infty \leq \| f \|_1.
\]

3) Let \( 1 < p \leq 2 \) and \( p' = \frac{p}{p-1} \). For \( f \in L^p_\alpha \), we have \( \mathcal{F}_\alpha(f) \in L^{p'}_\alpha \) and

\[
\| \mathcal{F}_\alpha(f) \|_{p'} \leq \left[ \frac{p}{p'} \right] \| f \|_p.
\]

The Bessel translation \( T^\alpha_x \) is defined for \( f \in L^p_\alpha, p \geq 1, \) a.e by:

\[
T^\alpha_x f(y) = c_\alpha \int_0^{+\infty} f(z) D(x, y, t) t^{2\alpha+1} dt, \quad x \neq 0.
\]

where

\[
T^\alpha_0 f(y) = f(y),
\]

\[
D(x, y, t) = \frac{2^{3\alpha-1} \Gamma^2(\alpha+1) \left[ \Delta(x, y, t) \right]^{2\alpha-1}}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2}) (xyt)^{2\alpha}},
\]

\( \Delta(x, y, t) \) is the area of the triangle \([x, y, t]\) if this triangle exists and 0 if not. We have

\[
c_\alpha \int_0^{+\infty} D(x, y, t) t^{2\alpha+1} dt = 1.
\]
If \( f \) is continuous on \([0, +\infty[\), we have

\[
T_\alpha^\alpha f(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^\pi f \left( \sqrt{x^2 + y^2 + 2xy\cos\theta} \right) (\sin\theta)^{2\alpha} d\theta.
\]

**Theorem 2.** 1) For \( \lambda, x, y \in [0, +\infty[ \) and \( f \in L^1_\alpha \), we have

\[
T_\alpha^\alpha j_\alpha(\lambda y) = j_\alpha(\lambda x)j_\alpha(\lambda y),
\]

(18)

(19)

2) For \( f \in L^p_\alpha \), we have \( T_\alpha^\alpha f(y) \in L^p_\alpha \) and

\[
\| T_\alpha^\alpha f \| \leq \| f \|_p.
\]

The Bessel convolution product is defined for \( f, g \in L^1_\alpha \) by:

\[
f *_\alpha g(x) = c_\alpha \int_0^{+\infty} T_\alpha^\alpha f(y)g(y)y^{2\alpha+1} dy.
\]

**Proposition 1.** Let \( f \) and \( g \) be in \( L^1_\alpha \), then

\[
f *_\alpha g \in L^1_\alpha
\]

and

\[
\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g).
\]

**Theorem 3.** 1) Let \( 1 \leq r \leq +\infty \), \( f \in L^r_\alpha \) and \( g \in L^1_\alpha \), then

\[
\| f *_\alpha g \|_r \leq \| f \|_r \| g \|_1,
\]

(24)

(25)

\[
\| f *_\alpha g \|_\infty \leq \| f \|_r \| g \|_{r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1.
\]

2) Let \( p, q, r \) be in \([1,2]\), such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \).

For \( f \in L^p_\alpha \) and \( g \in L^q_\alpha \), we have

\[
f *_\alpha g \in L^r_\alpha
\]

and

\[
\| f *_\alpha g \|_r \leq B_p B_q B_r \| f \|_p \| g \|_q, \quad \frac{1}{r} + \frac{1}{r'} = 1,
\]

(26)

(27)

where \( B_m = \left( \frac{m^\frac{1}{p} B_p B_r}{m^\frac{1}{p}} \right)^{\alpha+1} \).
4. **Strictly Positive Definite Functions Related to Bessel Functions**

A complex valued continuous function $f$ is said positive definite (resp. strictly positive definite) on $\mathbb{R}$ if for every real numbers $x_1, x_2, ... x_N$ and every complex numbers $z_1, z_2, ... z_N$ not all zero, the inequality

$$
\sum_{j=1}^{N} \sum_{k=1}^{N} z_j \bar{z}_k f(x_j - x_k) \geq 0 \quad \text{(resp.} \, > 0)$$

holds true.

In this section we give some conditions for a class of functions related to Bessel functions to be positive definite or strictly positive definite.

**Proposition 2.** For $\alpha > -\frac{1}{2}$, we have

1) 
$$j_\alpha \in \mathcal{P}.$$ 

2) Let $\mu \in M^+$ such that $\hat{\mu} \in \mathcal{P}^s$. Then the function

$$\int_{\mathbb{R}} j_\alpha(x\xi) d\mu(x) \in \mathcal{P}^s.$$ 

*Proof.* 1) It’s known that

$$j_\alpha(x) = \frac{1}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_{0}^{1} (1 - t^2)^{\alpha - \frac{3}{4}} \cos(xt) dt.$$ 

By Bochner’s theorem we conclude.

2) Since $j_\alpha \in \mathcal{P}$, hence the proof is done by corollary 6.6 in [5].

**Theorem 4.** For $\alpha > 0$, the even function $x^\alpha K_\alpha(x)$ is strictly positive definite on $\mathbb{R}$.

*Proof.* For $\alpha > 0$, the even function $x^\alpha K_\alpha(x)$ admit the Basset’s integral representation ([13], p 172):

$$x^\alpha K_\alpha(x) = \frac{2^{\alpha}\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{1}{2})} \int_{0}^{\infty} \frac{\cos(xt)}{(1 + t^2)^{\alpha + \frac{1}{2}}} dt, \quad x \in \mathbb{R}.$$ 

By Bochner’s theorem we have

$$x^\alpha K_\alpha(x) \in \mathcal{P}.$$ 

Using the Bell integral representation [3], we obtain

$$x^\alpha K_\alpha(x) \in L_1.$$ 

we conclude by theorem 6.5 in [3].
Theorem 5. 1) Let $\varphi$ be a nonnegative function in $L^1_\alpha$, then

\begin{equation}
F_\alpha(\varphi) \in \mathcal{P}.
\end{equation}

2) Let $\varphi$ be a nonnegative continuous function in $L^1_\alpha$ and not identically zero, then

\begin{equation}
F_\alpha(\varphi) \in \mathcal{P}^s.
\end{equation}

Proof. 1) Let $x_1, x_2, \ldots, x_n$ are real numbers and $z_1, z_2, \ldots, z_n$ are complex numbers, we have

\begin{equation*}
\sum_{j=1}^n \sum_{k=1}^n z_j z_k F_\alpha(\varphi)(x_j - x_k) = \int_0^{+\infty} \sum_{j=1}^n \sum_{k=1}^n z_j \sqrt{\varphi(t)}(z_k \sqrt{\varphi(t)}) j_\alpha(x_j t - x_k t) d\mu_\alpha(t).
\end{equation*}

Since $j_\alpha \in \mathcal{P}$, we conclude.

2) By 1), we have $F_\alpha(\varphi) \in \mathcal{P}$. Bochner’s theorem asserts that there exists a nonnegative finite Borel measure $\mu$ such that $F_\alpha(\varphi) = \hat{\mu}$, with $\hat{\mu}(0) = F_\alpha(\varphi)(0) > 0$. By the Riemann-Lebesgue lemma, we have $\lim_{|\xi| \to \infty} F_\alpha(\varphi)(\xi) = 0$. Then the support of $\mu$ must contain an interior point and hence by theorem 6.8 in [14], we conclude.

Example 1: For $0 < \alpha < \beta$ and $a > 0$, the even function

\begin{equation}
\phi_{\alpha,\beta}(x) = \frac{a^{\alpha-\beta}}{2^\alpha \Gamma(\alpha + 1)} x^{\beta-\alpha} K_{\beta-\alpha}(ax) \in \mathcal{P}^s,
\end{equation}

where $K_{\alpha}$ is the modified Bessel function of second kind.

Proof: For $0 < \alpha < \beta$ and $a > 0$, we put

\begin{equation*}
\varphi_\beta(x) = \frac{1}{(x^2 + a^2)^{\beta+1}}.
\end{equation*}

Then, $\varphi_\beta \in L^1_\alpha$ is continuous positive function. By (11, p 254), we have:

\begin{equation*}
F_\alpha(\varphi_\beta)(x) = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \int_0^{+\infty} \frac{1}{(t^2 + a^2)^{\beta+1}} j_\alpha(x t) t^{2\alpha+1} dt = \frac{a^{\alpha-\beta}}{2^\alpha \Gamma(\alpha + 1)} x^{\beta-\alpha} K_{\beta-\alpha}(ax).
\end{equation*}

Thus, the proof is done by the last theorem.

Example 2: For $-1 < \alpha < \beta$ and $a \neq 0$, the function

\begin{equation}
\phi_{\alpha,\beta}(x) = a^{\beta+2} {}_1 F_1(1 + \frac{\beta}{2}; \alpha + 1; -\frac{1}{4} a^2 x^2) \in \mathcal{P}^s,
\end{equation}

where ${}_1 F_1$ is the hypergeometric function.

Proof: For $-1 < \alpha < \beta$ and $a \neq 0$, we consider the even function

\begin{equation*}
\varphi_{\alpha,\beta}(x) = x^{\beta-\alpha} e^{-\frac{x^2}{a^2}}, \quad x \geq 0.
\end{equation*}
Then, \( \varphi_{\alpha,\beta} \in L^1_{\alpha} \) is continuous, nonnegative function and 
\[
\phi_{\alpha,\beta}(x) = \mathcal{F}_\alpha(\varphi_{\alpha,\beta})(x),
\]
hence the proof is done by the last theorem.

**Corollary 1.** For \( \alpha \geq -\frac{1}{2} \), let \( \varphi \in L^1_{\alpha} \) be a nonnegative function, then
\[
(39) \quad \mathcal{F}_\alpha(T_x\varphi) \in \mathcal{P}.
\]
**Proof.** We have:
\[
\mathcal{F}_\alpha(T_x\varphi) = j_{\alpha}(x)\mathcal{F}_\alpha(\varphi).
\]
Since \( j_{\alpha} \in \mathcal{P} \), the last theorem complete the proof.

We consider the Wiener algebra
\[
(40) \quad \mathcal{A}_\alpha = \{ f \in L^1_{\alpha} / \mathcal{F}_\alpha(f) \in L^1_{\alpha} \}.
\]

**Theorem 6.** Let \( \varphi \in \mathcal{A}_\alpha \cap C \), \( \mathcal{F}_\alpha(\varphi) \geq 0 \) and \( \varphi \) non vanishing. Then \( \varphi \in \mathcal{P}^s \).
**Proof.** Let \( \varphi \in \mathcal{A}_\alpha \cap C \), by the inversion formula we have
\[
\varphi = \mathcal{F}_\alpha(\mathcal{F}_\alpha(\varphi)).
\]
Since \( \mathcal{F}_\alpha(\varphi) \geq 0 \), then \( \varphi \in \mathcal{P} \). Moreover \( \varphi \in L^1_{\alpha} \) and \( \varphi \) non vanishing, then \( \varphi \in \mathcal{P}^s \).

**Corollary 2.** Let \( \varphi \in \mathcal{A}_\alpha \cap C \), \( \mathcal{F}_\alpha(\varphi) \geq 0 \) and \( \varphi \) non vanishing. Then
\[
(41) \quad T^t\varphi = E_t *_{\alpha} \varphi \in \mathcal{P}^s
\]
and
\[
(42) \quad P^t\varphi = p_t *_{\alpha} \varphi \in \mathcal{P}^s,
\]
where \( E_t \) and \( p_t \) are respectively the Gauss and the Poisson kernels associated to the Bessel operator given by:
\[
(43) \quad E_t(x) = \frac{e^{-\frac{x^2}{4t}}}{(2t)^{\alpha+1}},
\]
\[
(44) \quad p_t(x) = \frac{2^{\alpha+1}\Gamma(\alpha + \frac{3}{2})}{\sqrt{\pi}} \frac{t}{(t^2 + x^2)^{\Gamma(\alpha + \frac{3}{2})}}.
\]
5. Relation between strictly positive definite function and logarithmically completely monotonic functions

5.1. Definitions and properties:

**Definition 1.** A function $f$ is said to be completely monotonic (CM) on an interval $I \subset \mathbb{R}$ if $f \in C(I)$ has derivatives of all orders on $I^0$ and, for all $n \in \mathbb{N}$,

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^0; \quad n \in \mathbb{N}. \quad (45)$$

The class of all completely monotonic functions on $I$ is denoted by $CM(I)$.

**Lemma 1.** The sum and the product of completely monotonic functions are also.

**Definition 2.** A function $f$ is said to be logarithmically completely monotonic (LCM) on an interval $I \subset \mathbb{R}$ if $f > 0$, $f \in C(I)$, has derivatives of all orders on $I^0$ and, for all $n \in \mathbb{N} \setminus \{0\}$,

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, \quad x \in I^0; \quad n \geq 1. \quad (46)$$

The class of all logarithmically completely monotonic on $I$ is denoted by $LCM(I)$.

**Proposition 3.**

$$LCM(I) \subset CM(I). \quad (47)$$

**Definition 3.** A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is said to be positive definite on $\mathbb{R}^d$ if the corresponding multivariate function $\phi := \varphi(\|\cdot\|^2_2)$ is positive definite on every $\mathbb{R}^d$.

In [11] Schoenberg establish the connexion between positive definite radial and completely monotone functions.

**Theorem 7.** (Schoenberg) A function $\varphi$ is completely monotone on $[0, \infty)$ if and only if $\phi := \varphi(\|\cdot\|^2_2)$ is positive definite on every $\mathbb{R}^d$.

In [14] Wendland was interested by strictly positive definite functions and present a complete characterization of radial functions as being strictly positive definite on every $\mathbb{R}^d$.

**Theorem 8.** (Wendland) For a function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ the following three properties are equivalent

1. $\varphi$ is strictly positive definite on every $\mathbb{R}^d$;
2. $\varphi(\sqrt{\cdot})$ is completely monotone on $[0, \infty)$ and non constant;
3. there exists a finite nonnegative Borel measure $\nu$ on $[0, \infty)$ that is non concentrated at zero, such that

$$\varphi(r) = \int_0^\infty e^{-r^2t}d\nu(t).$$
Proposition 4. For $\alpha > 0$, the even function
\[ x^{2\alpha} K_\alpha(\sqrt{x}) \in CM([0, \infty[), \]
where $K_\alpha$ is the modified Bessel function of second kind.

Proof. By Theorem 4, if $\alpha > 0$, then
\[ x^\alpha K_\alpha(x) \in \mathcal{P}^s. \]
Since this function is $C^\infty(\mathbb{R})$ and positive on $]0, \infty[$, we conclude. 

The next result exist in [8]. We give an elementary proof.

Theorem 9. Suppose that
\[ f \in C(I), \quad f > 0 \quad \text{and} \quad f' \in CM(I). \]
Then
\[ \frac{1}{f} \in LCM(I). \]

Proof. Proving by induction that:
\[ (-1)^n \left( \ln \left( \frac{1}{f} \right) \right)^{(n)} \geq 0, \quad \forall n \geq 1. \]

For $n = 1$, we have
\[ (-1) \left( \ln \left( \frac{1}{f} \right) \right)^{(1)} = \frac{f'}{f} \geq 0. \]

Suppose that
\[ (-1)^k \left( \ln \left( \frac{1}{f} \right) \right)^{(k)} \geq 0, \quad \forall 1 \leq k \leq n. \]

Put $f' = \frac{f'}{f}$. By the Leibnitz formula, we get
\[ (f')^{(n)} = - \left[ f \left( \ln \left( \frac{1}{f} \right) \right)^{(1)} \right]^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} \left( \ln \left( \frac{1}{f} \right) \right)^{(n-k+1)}. \]

Then
\[ (-1) f \left( \ln \left( \frac{1}{f} \right) \right)^{(n+1)} = (f')^{(n)} + \sum_{k=1}^{n} C_n^k f^{(k)} \left( \ln \left( \frac{1}{f} \right) \right)^{(n-k+1)}, \]
which readily yields
\[ (-1)^{n+1} f \left( \ln \left( \frac{1}{f} \right) \right)^{(n+1)} = (-1)^n (f')^{(n)} + \]
\[ \sum_{k=1}^{n} C_n^k \left[ (-1)^{k-1} (f')^{(k-1)} \right] \left[ (-1)^{n-k+1} \left( \ln \left( \frac{1}{f} \right) \right)^{(n-k+1)} \right]. \]
Since $f > 0$, $f' \in \text{CM}(I)$ and $(-1)^k \left( \frac{\ln(1)}{f} \right)^{(k)} \geq 0$, \quad \forall 1 \leq k \leq n$, we obtain

$$(-1)^{n+1} f \left( \frac{\ln(1)}{f} \right)^{(n+1)} \geq 0,$$

which complete the proof.

**Theorem 10.** Let $\varphi : [0, \infty[ \rightarrow \mathbb{R}$ and $\varphi'$ non vanishing.

Suppose that $\varphi = \hat{\mu} \in \mathcal{P}^s \cap C^\infty$.

Then

$$-\frac{1}{\varphi'(\sqrt{x})} \in \text{LCM}(0, \infty).$$

**Proof.** Since $\varphi \in \mathcal{P}$, by Bochner's theorem there exist $\mu \in M^+$ such that $\varphi = \hat{\mu}$. Since $\varphi \in C^\infty$, By corollary 6.3 in [5], we have $x^2 \mu \in M^+$ and $-\varphi'' \in \mathcal{P}^s$

Let

$$f = -\varphi'(\sqrt{x}),$$

then

$$f'(x) = -\frac{1}{2\sqrt{x}} \varphi''(\sqrt{x}).$$

Since $-\varphi'' \in \mathcal{P}^s$, then $-\varphi''(\sqrt{x}) \in \text{CM}(0, \infty)$.

Moreover $\frac{1}{\sqrt{x}} \in \text{CM}(0, \infty)$, hence $f' \in \text{CM}(0, \infty)$. Since $f > 0$, we conclude by the last theorem.

**Corollary 3.** Let $\varphi : [0, \infty[ \rightarrow \mathbb{R}$ and $\varphi^{(n)}$ non vanishing for all $n \geq 1$.

Suppose that $\varphi = \hat{\mu} \in \mathcal{P}^s \cap C^\infty$.

Then

$$-\frac{1}{\varphi^{(n)}(\sqrt{x})} \in \mathcal{P}^s \quad \forall n \geq 1.$$  

**Proof.** By induction:

Since $\varphi = \hat{\mu} \in \mathcal{P}^s \cap C^\infty$ and $\varphi'$ non vanishing, then

$$-\frac{1}{\varphi'(\sqrt{x})} \in \text{CM}.$$

Thus by Wendland theorem
\[-\frac{1}{\phi'} \in \mathcal{P}^s.\]

Suppose that
\[\phi = -\frac{1}{\phi^{(n)}} \in \mathcal{P}^s, \quad \forall n \geq 1.\]

Then
\[-\frac{1}{\phi'} \in \mathcal{P}^s.\]

Hence
\[-\frac{1}{\phi^{(n+1)}} = \left[-\frac{1}{\phi^{(n)}}\right]^2 \in \mathcal{P}^s,\]

which complete the proof. ■

6. Applications

6.1. A class of logarithmically completely monotonic functions related to the modified Bessel functions of second kind.

**Theorem 11.** For \(\alpha > 0\), the function
\[
(49) \quad \frac{1}{x^{\frac{\alpha+1}{2}}K_{\alpha}(\sqrt{x})} \in LCM(0, \infty).
\]

**Proof.** By theorem 6.4 in [3], we have
\[
(50) \quad \frac{d}{dx} \left[x^{\alpha+1}K_{\alpha+1}(x)\right] = -x^{\alpha+1}K_{\alpha}(x).
\]

By proposition [4], we have
\[x^{\alpha+1}K_{\alpha+1}(x) \in \mathcal{P}^s \cap C^\infty\]
and is non vanishing. Theorem 10 complete the proof. ■

**Proposition 5.** For \(\alpha > 0\) and \(\beta > 0\),
\[
(51) \quad g_{\alpha,\beta}(x) = \frac{K_{\alpha+1}(\sqrt{x})}{x^\beta K_{\alpha}(\sqrt{x})} \in CM(0, \infty).
\]

**Proof.** For \(\alpha > 0\) and \(\beta > 0\), using proposition 4 and the last proposition we have
\[\frac{1}{x^\beta} \in CM(0, \infty),\]
\[x^{\frac{\alpha+1}{2}}K_{\alpha+1}(\sqrt{x}) \in CM(0, \infty),\]

and
\[\frac{1}{x^{\frac{\alpha+1}{2}}K_{\alpha}(\sqrt{x})} \in LCM(0, \infty) \subset CM(0, \infty).\]
Proposition 6. For $\alpha > 0$,

$$K_\alpha(\sqrt{x}) \in LCM([0, \infty]).$$

Proof. In [9], Ismail gave the integral representation

$$K_{\alpha-1}(\sqrt{x}) = \frac{4}{\pi^2} \int_0^\infty \frac{t^{-1}dt}{(x + t^2) \left[ J_\alpha^2(t) + Y_\alpha^2(t) \right]} \quad x > 0, \; \alpha \geq 0.$$  

Using the relationship, Watson ([13], p: 79)

$$K'_{\alpha}(x) = -\frac{1}{2} \{K_{\alpha-1}(x) + K_{\alpha+1}(x)\} \quad x > 0, \; \alpha \geq 0,$$

we find

$$[\ln (K_\alpha(\sqrt{x})')]' = -\frac{K'_{\alpha}(\sqrt{x})}{2 \sqrt{x} K_\alpha(\sqrt{x})} = -\frac{1}{4} \left\{ \frac{K_{\alpha-1}(\sqrt{x})}{\sqrt{x} K_\alpha(\sqrt{x})} + \frac{K_{\alpha+1}(\sqrt{x})}{\sqrt{x} K_\alpha(\sqrt{x})} \right\}.$$  

By the last integral representation, it’s clear that

$$\frac{K_{\alpha-1}(\sqrt{x})}{\sqrt{x} K_\alpha(\sqrt{x})} \in CM([0, \infty]).$$

Since

$$\frac{K_{\alpha+1}(\sqrt{x})}{\sqrt{x} K_\alpha(\sqrt{x})} = g_{\alpha, \frac{1}{2}}(x) \in CM([0, \infty]).$$

Thus

$$- [\ln (K_\alpha(\sqrt{x})')]' \in CM([0, \infty]).$$

and

$$K_\alpha(\sqrt{x}) \in LCM([0, \infty]).$$

Proposition 7. Let $\alpha > 0$, $x_0 > 0$ and $K_\alpha(\sqrt{x_0}) = 1$, then

$$\Delta_\alpha(x) = \frac{-1}{[\ln (K_\alpha(\sqrt{x}))']} \in LCM([x_0, \infty]).$$

Proof. Put

$$y_\alpha(x) = -\ln (K_\alpha(\sqrt{x})), \quad x > x_0.$$  

Since $K_\alpha$ is decreasing on $[x_0, +\infty[$, we have

$$0 < K_\alpha(\sqrt{x}) < K_\alpha(\sqrt{x_0}) = 1, \quad x > x_0.$$  

We have prove in the last proposition that

$$y_\alpha'(x) = - [\ln (K_\alpha(\sqrt{x}))]' \in CM([x_0, \infty]).$$

Theorem 9 completes the proof.
6.2. Inequalities for the modified Bessel function of the second kind.

**Definition 4. (logarithmically-convex)** A positive function defined on an interval \( I \) is said to be log-convex if \( \ln(f) \) is convex, i.e for all \( x, y \in I \) and \( \lambda \in [0, 1] \), we have
\[
(57) \quad f(\lambda x + (1 - \lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}.
\]

We have immediately the following result:

**Lemma 2.** If \( f \) is LCM function on an interval \( I \subset \mathbb{R} \), then \( f \) is log-convex on \( I \).

**Theorem 12.** For \( \alpha > 0 \),

1) if \( x, y > 0 \), then
\[
(58) \quad K'_\alpha\left(\sqrt{\frac{x+y}{2}}\right) \leq \sqrt{K'_\alpha(\sqrt{x})K'_\alpha(\sqrt{y})} \leq (\frac{x+y}{2\sqrt{xy}})^{\frac{\alpha+1}{2}} K'_\alpha\left(\sqrt{\frac{x+y}{2}}\right).
\]

2) if \( x, y > x_0 > 0 \), where \( K'_\alpha(\sqrt{x_0}) = 1 \), then
\[
(59) \quad \sqrt{-\ln(K'_\alpha(\sqrt{x}))}\sqrt{-\ln(K'_\alpha(\sqrt{y}))} \leq -\ln(K'_\alpha\left(\sqrt{\frac{x+y}{2}}\right)).
\]

In each of the above inequalities equality hold if and only if \( x = y \).

**Proof.** 1) By Proposition 6, we know that for \( \alpha > 0 \), the function \( K'_\alpha(\sqrt{x}) \in LCM([0, \infty]) \).

Thus it is log-convex on \( ]0, \infty[ \). We get
\[
K'_\alpha\left(\sqrt{\frac{x+y}{2}}\right) \leq \sqrt{K'_\alpha(\sqrt{x})K'_\alpha(\sqrt{y})}.
\]

By Theorem 11, we know that for \( \alpha > 0 \), the function
\[
\frac{1}{x^{\frac{\alpha+1}{2}}K'_\alpha(\sqrt{x})} \in LCM([0, \infty]).
\]

Thus, it is log-convex on \( ]0, \infty[ \). We get
\[
\sqrt{K'_\alpha(\sqrt{x})K'_\alpha(\sqrt{y})} \leq \left(\frac{x+y}{2\sqrt{xy}}\right)^{\frac{\alpha+1}{2}} K'_\alpha\left(\sqrt{\frac{x+y}{2}}\right).
\]

2) By Proposition 7, we have, for \( x, y > x_0 > 0 \), where \( K'_\alpha(\sqrt{x_0}) = 1 \)
\[
\Delta'_\alpha(x) = \frac{-1}{\ln(K'_\alpha(\sqrt{x}))} \in LCM([x_0, \infty]).
\]

Thus, it is log-convex on \( ]x_0, \infty[ \). We get:
\[
\Delta'_\alpha\left(\frac{x+y}{2}\right) \leq \Delta'_\alpha(x)\Delta'_\alpha(y),
\]
which completes the proof.

REFERENCES

[1] G. Andrews, R. Askey and Q. Ranjan, special function, Cambridge University Press, (1999).
[2] W. Bekner, Inequalities in Fourier analysis, Ann. Math, (2), 102 (1975), 159-182.
[3] W.W. Bell, Special functions for scientists and engineers, London 1967. Encyclopedia of Mathematics and its application, Vol 35 Cambridge Univ. Press, Cambridge, UK, 1990.
[4] S. Bochner, Integral transforms and their applications. Applied Math. Sciences 25. Springer-Verlag. New York Berlin Heidelberg Tokyo.
[5] F. Derrien, Strictly positive definite functions on the real line, hal-00519325,version 1-20sep 2010.
[6] M. Ky Fan, Les fonctions définies positives et les fonctions complètement monotones, Memorial Sciences Mathématiques Paris, (1950).
[7] A. Fitouhi, Inégalité de Babenko et inégalité logarithmique de Sobolev pour l’opérateur de Bessel, C.R. Acad. Sci. Paris, 305(1) (1987) 877-880.
[8] S. Guo and H.M. Srivastava, A certain function class related to the class of logarithmically completely monotonic functions, Mathematical and Computer Modelling 49 (2009), 2073-2079.
[9] M.E.H. Ismail, Bessel functions and the infinite divisibility of the student t-distribution, Ann. Prob. 5 (1977), 582-585.
[10] M.E.H. Ismail, Complete monotonicity of the modified Bessel functions, Proceeding of the American Mathematical society, Vol. 108, No 2(1990), 353-361.
[11] I.J. Schoenberg, Metric spaces and completely monotonic functions. Ann. Math. 39, (1938), 811-841.
[12] E.C. Titchmarsh. Introduction to The Theory of Fourier Integrals, Oxford University Press, second ed. 1937.
[13] G.N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press.
[14] H. Wendland, Scattered data approximations, Cambridge University Press, Cambridge, 2005.
[15] D.V. Widder, The Laplace transform, Princeton Univ Press, Princeton, NJ, 1941.

JAMEL EL KAMEL. DÉPARTEMENT DE MATHEMATIQUES FSM. MONASTIR 5000, TUNISIA. 
E-mail address: jamel.elkamel@fsm.rnu.tn

KHALED MEHREZ. DÉPARTEMENT DE MATHEMATIQUES IPEIM. MONASTIR 5000, TUNISIA. 
E-mail address: k.mehrez@yahoo.fr