SOLVABILITY AND SLIDING MODE CONTROL FOR THE VISCOUS CAHN–HILLIARD SYSTEM WITH A POSSIBLY SINGULAR POTENTIAL

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Abstract. In the present contribution we study a viscous Cahn–Hilliard system where a further leading term in the expression for the chemical potential $\mu$ is present. This term consists of a subdifferential operator $S$ in $L^2(\Omega)$ (where $\Omega$ is the domain where the evolution takes place) acting on the difference of the phase variable $\varphi$ and a given state $\varphi^*$, which is prescribed and may depend on space and time. We prove existence and continuous dependence results in case of both homogeneous Neumann and Dirichlet boundary conditions for the chemical potential $\mu$. Next, by assuming that $S = \rho \text{sign}$, a multiple of the sign operator, and for smoother data, we first show regularity results. Then, in the case of Dirichlet boundary conditions for $\mu$ and under suitable conditions on $\rho$ and $\Omega$, we also prove the sliding mode property, that is, that $\varphi$ is forced to join the evolution of $\varphi^*$ in some time $T^*$ lower than the given final time $T$. We point out that all our results hold true for a very general and possibly singular multi-well potential acting on $\varphi$.

1. Introduction. This paper deals with the viscous Cahn–Hilliard system, which is further generalized in order to admit an additional nonlinearity that plays as forcing term in order to reach a given evolution in the system. The resulting combination proposes an extension of the celebrated Cahn–Hilliard system, a phenomenological model that has its origin in the work of J.W. Cahn [6]. In fact, Cahn studied the effects of interfacial energy on the stability of spinodal states in solid binary solutions, and this took origin from a previous collaboration with J.W. Hilliard [7], where the functional

$$
\mathcal{F}(\varphi) = \int_{\Omega} \left( \frac{1}{2} |\nabla \varphi|^2 + f(\varphi) \right)
$$

was proposed as a model for the free energy of a non-uniform system whose composition is described by the scalar field $\varphi$. Actually, the viscous Cahn–Hilliard...
system considered here, which, up to our knowledge, was formulated by A. Novick-Cohen [30, 31], reads

$$\partial_t \varphi - \Delta \mu = 0 \quad (1.2)$$

$$\tau \partial_t \varphi - \Delta \varphi + f'(\varphi) = \mu + g \quad (1.3)$$

where the equations are understood to hold in a bounded domain $\Omega \subset \mathbb{R}^3$ and in some time interval $(0, T)$. Note that (1.3) contains the terms $-\Delta \varphi + f'(\varphi)$ that can be interpreted as the variational derivative of $F(\varphi)$. The viscous contribution $\tau \partial_t \varphi$ is completing the left-hand side of (1.3).

In the above system, the variables $\varphi$ and $\mu$ denote the order parameter and the associated chemical potential, respectively, with $\tau$ being the positive viscosity coefficient and $g$ standing for some given term. In general, $f$ represents a non-convex potential; typical and physically significant examples for $f$ are the so-called classical regular potential, the logarithmic double-well potential, and the double obstacle potential, which are given, in this order, by

$$f_{\text{reg}}(r) := \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R} \quad (1.4)$$

$$f_{\text{log}}(r) := ((1 + r) \ln(1 + r) + (1 - r) \ln(1 - r)) - c_1 r^2, \quad r \in (-1, 1) \quad (1.5)$$

$$f_{\text{2obs}}(r) := c_2 (1 - r^2) \quad \text{if} \quad |r| \leq 1 \quad \text{and} \quad f_{\text{2obs}}(r) := +\infty \quad \text{if} \quad |r| > 1. \quad (1.6)$$

Here, the constants $c_1$ in (1.5) and (1.6) satisfy $c_1 > 1$ and $c_2 > 0$, so that $f_{\text{log}}$ and $f_{\text{2obs}}$ are nonconvex. In cases like (1.6), one has to split $f$ into a nondifferentiable convex part $\hat{\beta}$ (the indicator function of $[-1, 1]$, in the present example) and a smooth perturbation $\hat{\pi}$. Accordingly, one has to replace the derivative of the convex part $\hat{\beta}$ by the subdifferential $\partial \hat{\beta}$ and interpret (1.3) as a differential inclusion. We will be more precise on that in the sequel.

On the other hand, we have to emphasize that a wide number of generalizations of the Cahn–Hilliard system have been proposed in the literature and it turns out that these contributions are so many that it would be difficult to make a list here. We prefer to refer to the recent review paper [28]. In this respect, it is also worth pointing out that a systematic approach to derive and generalize the C-H system has been proposed by M.E. Gurtin [21], by extending the thermodynamical framework of Continuum Mechanics, as reported in [27] as well. We aim to mention an alternative procedure due to P. Podio-Guidugli [38] that leads to another viscous Cahn–Hilliard system of nonstandard type [11, 10]. In addition, we report that in recent years Cahn–Hilliard and viscous Cahn–Hilliard systems have been employed successfully in many other branches of Science and Engineering, fields in which the segregation of a diffusant leads to pattern formation, such as population dynamics [26], image processing [4], dynamics for mixtures of fluids [18] and tumor modelling [9]. In the case of the variable $\varphi$ understood as concentration, the recent paper [5] faces with a doubly nonlinear Cahn-Hilliard system, where both an internal constraint on the time derivative of $\varphi$ and the potential $f$ for $\varphi$ are introduced, thus leading to an equation more general than (1.3).

Coming back to the system (1.2)–(1.3), we observe that proper supplementary conditions should complement it. As for $\varphi$, we consider the homogeneous Neumann boundary condition and the initial condition

$$\partial_\nu \varphi = 0 \quad \text{on} \quad \Gamma \times (0, T) \quad \text{and} \quad \varphi(0) = \varphi_0 \quad \text{in} \quad \Omega \quad (1.7)$$
where $\partial_n$ denotes the outward normal derivative on $\Gamma := \partial \Omega$ and $\varphi_0$ is a given initial datum. Regarding $\mu$, in view of (1.2) we have to add some boundary conditions. We impose

\begin{equation} \tag{1.8} \text{either } \partial_n \mu = 0 \text{ or } \mu = \mu_\Gamma \text{ on } \Gamma \times (0, T) \end{equation}

where $\mu_\Gamma$ is a given boundary datum. We remark that the first condition of no flux through the boundary for $\mu$ is quite natural in the framework of Cahn–Hilliard systems and entails the mean value conservation property for $\varphi$, as the reader can easily realize by integrating (1.2) over $\Omega \times (0, t)$, $t \in [0, T]$. Instead, the Dirichlet boundary condition for $\mu$ (as in [42] and [5]) is rather different and does not ensure any conservation, but it also looks reasonable from the modelling point of view and, as far as we know, classes of Dirichlet boundary data for $\mu$ are consistent with applications.

In this paper, we study a sliding mode control (SMC) problem. This consists in modifying the dynamics governed by the Cahn–Hilliard system by adding in equation (1.3) a further term that forces the solution of the new system to satisfy $\varphi(t) = \varphi^*(t)$ after some time $T^*$, where $\varphi^*$ is a given function. We stress that $\varphi^*$ is allowed to be time dependent, in contrast with the most part of the literature regarding sliding mode problems. In fact, SMC is considered a classic instrument for regulation of continuous or discrete systems in finite-dimensional settings (see e.g. the monographs [3, 15, 16, 17, 22, 40, 41, 44]), in order to reach some stable states. Here, we also want to allow the possibility that the variable $\varphi$ joins a prescribed evolution after the time $T^*$.

Then, the modified second equation is the following

\begin{equation} \tag{1.9} \tau \partial_t \varphi - \Delta \varphi + f'(\varphi) + S(\varphi - \varphi^*) \ni \mu + g \end{equation}

in place of (1.3), where now $S$ denotes some suitable maximal monotone graph in $L^2(\Omega) \times L^2(\Omega)$. In sliding mode problems, one generally has to choose operators that are singular at the origin. So, a typical choice for $S$ is given by the following rule

\begin{equation} \tag{1.10} \text{for } u, v \in L^2(\Omega), \quad v \in S(u) \text{ means that } v(x) \in \rho \text{sign } u(x) \text{ a.e. in } \Omega \end{equation}

where $\rho$ is a positive parameter and sign is the subdifferential of the modulus $|\cdot|$, i.e., sign $r := r/|r|$ if $r \neq 0$ and sign $0 := [-1, 1]$. However, it is clear that the assumption that $S$ is a graph in $L^2(\Omega) \times L^2(\Omega)$ (not necessarily induced in $L^2(\Omega)$ by a graph in $\mathbb{R} \times \mathbb{R}$) is much more general, and the first result of ours regards well-posedness for such a generalized problem. In proving it, we just have to reinforce the maximal monotonicity and subdifferential property by also assuming that $S$ grows at most linearly at infinity. Moreover, we can manage both the Neumann and the Dirichlet boundary conditions given in (1.8).

Next, under proper assumptions on the data of the problem that ensure further regularity for the solution, we study the existence of a sliding mode. This is done only for the Dirichlet boundary conditions for $\mu$ and in the particular case (1.10). More precisely, $\rho > 0$ has to be taken large enough and we also have to assume the domain $\Omega$ to be small enough. The result we prove is reminiscent of the one given in [9], but here we approach directly the viscous Cahn-Hilliard system and our key datum $\varphi^*$ is allowed to vary with time.

Now, we take the opportunity of reviewing some literature related to SMC, which offers a robust tool against abrupt variations, disturbances, time-delays, etc. in dynamics. The design procedure of a SMC scheme consists first in choosing a sliding set such that the original system restricted to it has a desired behavior, then
modifying the dynamics in order to force the involved variable to reach this set within a finite time. It is exactly for this aim that we add the term $\rho \text{sign}(\varphi - \varphi^*)$ in the Cahn–Hilliard evolution for $\varphi$ (cf. (1.9) and (1.10)), in order to force $\varphi$ to stay equal to a given desired function $\varphi^*$ in a finite time.

Sliding mode controls are pretty interesting in applications and in recent years the extension of well-developed methods for finite-dimensional systems (cf., e.g., [24, 33, 34, 36]) and the control of infinite-dimensional dynamical systems (see [36, 35, 32]) have been faced. The theoretical development for PDE systems is still in its early stages: one can see the contributions [8, 20, 23, 37, 43] dealing with semilinear PDE systems. We aim to quote [2], where a sliding mode approach has been applied to phase field systems of Caginalp type: these systems combine the evolution of a phase variable to the one of the relative temperature, and the chosen SMC laws force the system to reach within finite time a sliding manifold. In that case it was possible to have different choices for the manifold: in [2], and also in [12] which considers an extension of the Caginalp model, either one of the physical variables or a combination of them could reach a stable state. With reference to the results of [2, 12], we mention the analyses developed in [14, 13]: in particular, the second contribution is devoted to a conserved phase field system with a SMC feedback law for the internal energy in the temperature equation.

An outline of the present paper is as follows. In Section 2 we state precisely the problem, making clear the assumptions and presenting the different results we are going to prove. Section 3 brings the proof of the continuous dependence result, which ensures uniqueness at least for the component variable $\varphi$. The approximation of the problem, based on Yosida regularizations of graphs and a Faedo–Galerkin scheme, is discussed in Section 4. The existence of solutions is shown in Section 5 by proving some a priori estimates and passing to the limit with respect to the parameter of the Yosida regularizations. Finally, Section 6 is completely devoted to the proof of the sliding mode property, first dealing with the regularity of the solution, then proving the existence of a suitable time $T^*$, after which it occurs that $\varphi(t) = \varphi^*(t)$.

2. Statement of the problem. As in the Introduction, $\Omega$ is the domain where the evolution process takes place. We assume that $\Omega$ is a bounded and connected open set in $\mathbb{R}^3$ (more generally, one could take $\Omega \subset \mathbb{R}^d$ with $1 \leq d \leq 3$), which is supposed to have a smooth boundary $\Gamma := \partial \Omega$, and we write $|\Omega|$ and $\partial_\nu$ for the volume of $\Omega$ and the outward normal derivative on $\Gamma$, respectively. Given some final time $T > 0$, we set for convenience

$$Q_t \coloneqq \Omega \times (0, t) \quad \text{for} \quad t \in (0, T] \quad \text{and} \quad Q \coloneqq Q_T. \quad (2.1)$$

If $X$ is a Banach space, $\|\cdot\|_X$ denotes both its norm and the norm of $X^3$. Moreover, the dual space of $X$ and the dual pairing between $X^*$ and $X$ are denoted by $X^*$ and $\langle \cdot, \cdot \rangle$, the latter without indices since the choice of the space $X$ is clear every time from the context. The only exception from the convention for the norms is given by the the spaces $L^p$ constructed on $\Omega$ and $Q$ for $p \in [1, \infty]$, whose norms are denoted by $\|\cdot\|_p$. Furthermore, we put

$$H \coloneqq L^2(\Omega), \quad V \coloneqq H^1(\Omega) \quad \text{and} \quad W \coloneqq \{v \in H^2(\Omega) : \partial_\nu v = 0\} \quad (2.2)$$

$$V_0 \coloneqq H^1_0(\Omega) \quad \text{and} \quad W_0 \coloneqq H^2(\Omega) \cap H^1_0(\Omega). \quad (2.3)$$

We endow these spaces with their standard norms. Moreover, we identify $H$ with a subspace of $V^*$ in the usual way, i.e., in order that $\langle u, v \rangle = \int_\Omega uv$ for every $u \in H$.
and $v \in V$, and obtain the Hilbert triplet $(V, H, V^*)$. Analogously, we consider the Hilbert triplet $(V_0, H, V_0^*)$ when dealing with Dirichlet boundary conditions.

Now, we list our assumptions on the structure of the system at once. We assume that

\begin{equation}
\tau \text{ is positive real number} \tag{2.4}
\end{equation}
\begin{equation}
\beta := \partial \hat{\beta}, \quad S := \partial \hat{S} \quad \text{and} \quad \pi := \hat{\pi}' \quad \text{where} \tag{2.5}
\end{equation}
\begin{equation}
\hat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper and l.s.c. with} \quad \hat{\beta}(0) = 0 \tag{2.6}
\end{equation}
\begin{equation}
\hat{S} : H \to \mathbb{R} \quad \text{is convex, proper, l.s.c. and} \quad S \quad \text{satisfies} \quad \|v\|_H \leq C_S(\|u\|_H + 1) \tag{2.7}
\end{equation}
\begin{equation}
\hat{\pi} : \mathbb{R} \to \mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative}. \tag{2.8}
\end{equation}

Notice that all of the important examples (1.4)–(1.6) satisfy the above assumptions. We also remark that $\beta$ and $S$ are maximal monotone graphs. We denote by $D(\beta)$ the effective domain of $\beta$ and, for $r \in D(\beta)$, we use the notation $\beta^\circ(r)$ for the element of $\beta(r)$ having minimum modulus. The similar notation $S^\circ(u)$ for $u \in H$ refers to the minimum norm. For simplicity, we still write $\beta$ and $S$ for the graphs induced in $L^2(\Omega)$ and $L^2(Q)$ by $\beta$ and in $L^2(Q)$ by $S$, respectively.

As for the data of the problem we assume that

\begin{equation}
g \in L^2(0, T; H) \tag{2.9}
\end{equation}
\begin{equation}
\varphi_0 \in V \quad \text{and} \quad \hat{\beta}(\varphi_0) \in L^1(\Omega) \tag{2.10}
\end{equation}
\begin{equation}
\varphi^* \in L^2(0, T; H) \tag{2.11}
\end{equation}

Moreover, in the case of the Neumann boundary conditions for $\mu$, we also assume that

\begin{equation}
m_0 := \text{mean } \varphi_0 \quad \text{belongs to the interior of } D(\beta) \tag{2.12}
\end{equation}

where the symbol mean $v$ denotes the mean value of the generic function $v \in L^1(\Omega)$. More generally (by denoting by 1 the function that is identically 1 on $\Omega$), we set

\begin{equation}
\text{mean } v := \frac{1}{|\Omega|} \langle v, 1 \rangle \quad \text{for every } v \in V^* \tag{2.13}
\end{equation}

and it is clear that mean $v$ is the usual mean value of $v$ if $v \in H$.

At this point, we can state the problem given by equations (1.2) and (1.9) and the boundary and initial conditions given in (1.7) and (1.8). We first distinguish between the two different boundary conditions for $\mu$. Then, we unify the two problems. We start with the case of the Neumann boundary conditions.

**The case of the Neumann boundary conditions.** We write a variational formulation. We set $V := V$ for convenience and look for a quadruple $(\varphi, \mu, \xi, \zeta)$ satisfying

\begin{equation}
\varphi \in H^1(0, T; H) \cap L^2(0, T; V) \tag{2.14}
\end{equation}
\begin{equation}
\mu \in L^2(0, T; V) \tag{2.15}
\end{equation}
\begin{equation}
\xi \in L^2(0, T; H) \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q \tag{2.16}
\end{equation}
\begin{equation}
\zeta \in L^2(0, T; H) \quad \text{and} \quad \zeta(t) \in S(\varphi(t) - \varphi^*(t)) \quad \text{for a.a. } t \in (0, T) \tag{2.17}
\end{equation}
and solving the following system
\[\int_{\Omega} \partial_t \varphi(t) v + \int_{\Omega} \nabla \mu(t) \cdot \nabla v = 0 \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V \quad (2.18)\]
\[\tau \int_{\Omega} \partial_t \varphi(t) v + \int_{\Omega} \nabla \varphi(t) \cdot \nabla v + \int_{\Omega} (\xi(t) + \pi(\varphi(t)) + \zeta(t)) v = \int_{\Omega} (\mu(t) + g(t)) v \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V \quad (2.19)\]
\[\varphi(0) = \varphi_0. \quad (2.20)\]

**Remark 1.** In fact, every solution enjoys some more regularity, namely
\[\varphi \in L^2(0, T; W) \quad \text{and} \quad \mu \in L^2(0, T; W) \quad (2.21)\]
so that equations (2.18) and (2.19) can be written in the strong form (1.2) and (1.9) complemented with homogeneous Neumann boundary conditions. Indeed, both (2.18) and (2.19) have the form (with \(u = \mu\) and \(u = \varphi\), respectively)
\[\int_{\Omega} \nabla u(t) \cdot \nabla v = \int_{\Omega} \psi(t)v \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V\]
with \(\psi \in L^2(0, T; H)\). This implies that
\[u \in L^2(0, T; W) \quad \text{and} \quad -\Delta u = \psi \quad \text{a.e. in } Q. \quad (2.22)\]
Moreover, we also point out that \(\varphi \in C^0([0, T]; V)\), which can be obtained for instance by interpolation between (2.14) and (2.21). However, in connection with the general result stated below, we just deal with the variational formulation and do not need such a further regularity of the solution. On the contrary, this remark is used in the last section.

**Remark 2.** It is worth noting that every solution also satisfies
\[\int_{\Omega} \partial_t \varphi = 0 \quad \text{a.e. in } (0, T), \ i.e., \ \text{mean } \varphi(t) = \text{mean } \varphi_0 \quad \text{for every } t \in [0, T] \quad (2.23)\]
as one immediately sees by choosing \(v = 1 \in V = V\) in (2.18).

**The case of the Dirichlet boundary conditions.** In the case of the Dirichlet boundary conditions for \(\mu\) given in (1.8), we first make the basic assumption on \(\mu_\Gamma\). Since we still look for \(\mu\) in \(L^2(0, T; H^1(\Omega))\), we require that
\[\mu_\Gamma \in L^2(0, T; H^{1/2}(\Gamma)) \quad (2.24)\]
As for the problem, we have to force \(\mu = \mu_\Gamma\), explicitly, and modify (2.18) as far as the test functions are concerned. Namely, (2.18) is required to hold just for \(v \in V_0\). However, it is convenient to reduce the boundary condition \(\mu = \mu_\Gamma\) to the homogeneous one. This can be done by introducing the harmonic extension \(\mu_H\) of \(\mu_\Gamma\), which is defined for a.a. \(t \in (0, T)\) by
\[\Delta \mu_H(t) = 0 \quad \text{in } \Omega \quad \text{and} \quad \mu_H(t) = \mu_\Gamma(t) \quad \text{on } \Gamma \quad (2.25)\]
and by considering the problem that the difference \(\mu - \mu_H\) has to solve. However, it is better to avoid a new notation (as we see in a moment) and still denote by \(\mu\) the above difference. Then, for the new \(\mu\), both the regularity requirement (2.15) and the first equation (2.18) remain unchanged provided that we set \(V = V_0\) now, while the forcing term \(g\) in (2.19) has to be replaced by the difference \(g_* := g - \mu_H\). Hence, (2.19) formally remains unchanged too provided that we still use the symbol
g for the difference $g_\ast$. Notice that the new $g$ belongs to $L^2(0,T;H)$ as the old one (see (2.9)) since (2.24) trivially implies $\mu_H \in L^2(0,T;H)$. Therefore the two problems corresponding to the two different boundary conditions for $\mu$ are unified and we just have different meanings of $\mu$ and $g$ in the two cases.

**Remark 3.** More regularity for the old chemical potential $\mu$ as in Remark 1 is ensured whenever $\mu_\Gamma \in L^2(0,T;H^{3/2}(\Gamma))$. Indeed, this implies that $\mu_H \in L^2(0,T;H^2(\Omega))$. On the contrary, the mass conservation stated in Remark 2 cannot be expected in the case of the Dirichlet boundary conditions for $\mu$. Indeed, the choice $v = 1$ in (2.18) is no longer allowed since $V = V_0$ now.

**Notation.** From now on, it is understood that $g$ has its new meaning in the case of the Dirichlet boundary conditions for $\mu$ and the Dirichlet datum $\mu_\Gamma$ and its harmonic $\mu_H$ are not mentioned any longer in the problem. The only exception to this rule will happen if we need more regularity for $g$. In that case, we come back to this point and give sufficient conditions on $\mu_\Gamma$ in order to satisfy the new requirements.

Furthermore, in the sequel of the paper, we simply write $V = V$ and $V = V_0$ to identify the boundary conditions for $\mu$ of the Neumann and Dirichlet type, respectively, in problem (2.18)-(2.20).

If $V = V_0$, the component $\mu$ of any solution is uniquely determined whenever uniqueness holds for the first component $\varphi$, since the first equation (2.18) is uniquely solvable for $\mu$ in this case. On the contrary, if $V = V$, it is clear that no uniqueness for $\mu$ can be expected unless both $\beta$ and $S$ are single-valued, and this is not the case in this paper. Hence, the best one can have is just existence of a solution $(\varphi, \mu, \xi, \zeta)$ and uniqueness and some continuous dependence for the first component of the solution and uniqueness for the second component if $V = V_0$. In the next sections, the following result is proved:

**Theorem 2.1.** Let the assumptions (2.4)-(2.8) on the structure and (2.9)-(2.11) on the data be satisfied. In addition, assume either $V = V$ and (2.12) or $V = V_0$. Then, there exists a quadruplet $(\varphi, \mu, \xi, \zeta)$ satisfying (2.14)-(2.17) and solving problem (2.18)-(2.20). Moreover, the component $\varphi$ of any solution is uniquely determined in any case and $\mu$ is uniquely determined if $V = V_0$. Furthermore, let $g_i$, $\varphi_{0,i}$ and $\varphi_1^i$, $i = 1,2$ be two choices of the data and assume that $\varphi_{0,1}$ and $\varphi_{0,2}$ have the same mean value if $V = V$. Then, for the first components of any corresponding solutions $(\varphi_i, \mu_i, \xi_i, \zeta_i)$, the continuous dependence inequality

$$
\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} 
\leq C_{cd} \left( \|g_1 - g_2\|_{L^2(0,T;H)} + \|\varphi_{0,1} - \varphi_{0,2}\|_H + \|\varphi_1^* - \varphi_2^*\|_{L^2(0,T;H)}^{1/2} \right)
$$

(2.26)

holds true for some constant $C_{cd}$ only depending on the structure of the system, $\Omega$, $T$ and an upper bound $M$ for the norms of $\varphi_i$ and $\varphi_i^*$ in $L^2(0,T;H)$.

The second result of the present paper, whose proof is given in Section 6, is the existence of a sliding mode. We cannot treat the Neumann boundary conditions for $\mu$, unfortunately, and just consider the case of the Dirichlet boundary conditions. Moreover, the convex function $\tilde{S}$ and its subdifferential $\tilde{S}$ have a particular shape, as said in the Introduction. Precisely, we assume that

$$
\tilde{S}(u) = \rho \int_\Omega |u| \text{ for } u \in H
$$

(2.27)
where $\rho$ is a positive real number. Then, $S$ is given by (1.10), i.e., it is the graph induced in $H$ by $\rho$ sign, where we recall that sign, the subdifferential of the modulus, is given by

$$\text{sign } r := \frac{r}{|r|} \text{ if } r \neq 0 \text{ and } \text{sign } 0 = [-1, 1].$$

(2.28)

Moreover, our hypotheses on the data have to be reinforced. Besides (2.9)-(2.11), we require that

$$g \in H^1(0, T; H) \cap L^\infty(Q)$$

(2.29)

$$\varphi_0 \in W \text{ and } \beta^0(\varphi_0) \in L^\infty(\Omega)$$

(2.30)

$$\varphi^* \in W^{2,1}(0, T; L^1(\Omega)) \cap L^2(0, T; W), \quad \varphi^*, \partial_t \varphi^*, \Delta \varphi^*, \beta^0(\varphi^*) \in L^\infty(Q).$$

(2.31)

**Remark 4.** According to the above Notation, the function $g$ appearing in (2.29) is the difference between the original forcing term and the harmonic extension $\mu_H$ of the inhomogeneous boundary datum $\mu_T$. In order to satisfy (2.29), we have to assume the same regularity for both the original forcing term and $\mu_H$. To obtain a sufficient condition for the latter, one can reinforce (2.24) by also assuming that

$$\mu_T \in L^\infty(\Gamma \times (0, T)) \quad \text{and} \quad \partial_t \mu_T \in L^2(\Gamma \times (0, T)).$$

(2.32)

**Remark 5.** Our new assumptions on $\varphi^*$ are just sufficient conditions for the existence of a sliding mode. However, some of them are necessary since they are satisfied by $\varphi$. Indeed, as the other data are more regular, more regularity for $\varphi$ is expected. Moreover, we observe that, in the case of an everywhere defined potential like (1.4), the boundedness condition on $\beta^0(\varphi^*)$ is satisfied whenever $\varphi^*$ is bounded. On the contrary, when dealing with potentials like (1.5) or (1.6), we also have to require the smallness condition $||\varphi^*||_\infty < 1$.

As announced in the Introduction, we also have to assume that $\rho$ is large enough and $\Omega$ is small enough. More precisely, once we fix the class $\mathcal{O}$ of the domains of $\mathbb{R}^3$ that have the same shape of $\Omega$, then $|\Omega|$ has to be small enough. To better explain this condition, we fix a class $\mathcal{O}$ as said above. Then, there exists a constant $C_{sh}$ realizing the inequalities

$$||v||_\infty \leq C_{sh}|\Omega|^{1/6}\|\Delta v\|_H \quad \text{for every } v \in W_0$$

(2.33)

$$||v||_\infty \leq C_{sh}(1|\Omega|^{-1/2}\|v\|_H + |\Omega|^{1/6}\|\Delta v\|_H) \quad \text{for every } v \in W$$

(2.34)

whenever $\Omega \in \mathcal{O}$. The smallness condition on $|\Omega|$ will involve the constant $C_{sh}$.

**Remark 6.** We summarize the argument of [9, Rem. 2.1] that shows that the constant $C_{sh}$ realizing (2.33)–(2.34) actually exists. The prototype for $\Omega$ in the given class $\mathcal{O}$ is an open set $\Omega_0 \subset \mathbb{R}^3$ (which is supposed to be bounded, connected and smooth) with $|\Omega_0| = 1$, and the general set $\Omega \in \mathcal{O}$ has the form

$$\Omega = x_0 + \lambda R \Omega_0$$

(2.35)

where $x_0$ is a point in $\mathbb{R}^3$, the real number $\lambda$ is positive and $R$ belongs to the rotation group $SO(3)$. We first notice that our assumptions on $\Omega_0$, the continuous embedding $W \subset L^\infty(\Omega)$ and elliptic regularity ensure that the inequalities (2.33) and (2.34) hold true for some constant $C_{sh}$ (depending only on $\Omega_0$) if $\Omega$ and $|\Omega|$ are replaced replaced by $\Omega_0$ and 1, respectively. Now, assume that the domain $\Omega$ we are dealing is given by (2.35) as said above. Then, it is easy to check that $|\Omega| = \lambda^3$, i.e., $\lambda = |\Omega|^{1/3}$, and that (2.33)–(2.34) are still satisfied for $\Omega$ with the same constant $C_{sh}$. 

Here is our result, which provides both further regularity for the solution and the existence of a sliding mode.

**Theorem 2.2.** In addition to the hypotheses of Theorem 2.1, assume $V = V_0$, (2.27) on the function $\bar{S}$ and (2.29)-(2.31) on the data. Then, every solution $(\varphi, \mu, \xi, \zeta)$ to problem (2.18)-(2.20) enjoys the further regularity

\[
\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \subset L^\infty(Q) \quad (2.36)
\]

\[
\mu \in L^\infty(0,T;W) \subset L^\infty(Q) \quad (2.37)
\]

\[
\xi \in L^\infty(0,T;H) \quad (2.38)
\]

Moreover, assume that $\Omega$ belongs to a class $\mathcal{O}$ of open sets for which (2.34) is guaranteed. Then, there exist $\rho^* > 0$ and $\delta^* > 0$ such that the following holds true: if $\rho > \rho^*$ and $|\Omega| < \delta^*$, then the component $\varphi$ of any solution satisfies for some $T^* \in (0,T)$ the sliding condition $\varphi(t) = \varphi^*(t)$ for every $t \in [T^*,T]$.

**Remark 7.** The properties specified in the above statement refer to any solution. As for the last sentence, we recall that the component $\varphi$ of any solution $(\varphi, \mu, \xi, \zeta)$ is uniquely determined (as well as $\mu$ since $V = V_0$). On the contrary, no uniqueness for the components $\xi$ and $\zeta$ is ensured since $S$ is multivalued. Nevertheless, the regularity property (2.38) holds for every solution. Indeed, (2.19) can be written as a PDE a.e. in $Q$ (apply Remark 1 to this equation), so that (2.38) follows by comparison, since all the other terms of the PDE belong to $L^\infty(0,T;H)$, due to (2.36)-(2.37) and the boundedness of the (possibly non unique) component $\zeta$, since $S$ is the graph induced in $H$ by $\rho$ sign.

The rest of the paper is organized as follows. In the next section, we prove the part of Theorem 2.1 concerning uniqueness and continuous dependence. The existence part is concluded in Section 5 and prepared in Section 4, where an approximating problem is introduced and solved. The proof of Theorem 2.2 is presented in the last Section 6.

In proving our results, we make a wide use of the Schwarz and Young inequalities. We recall the latter:

\[
ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0. \quad (2.39)
\]

Moreover, we often account for the Poincaré inequalities

\[
\|v\|_V \leq C_\Omega \left( \|\nabla v\|_H + |\text{mean } v| \right) \quad \text{and} \quad \|v\|_V \leq C_\Omega \|\nabla v\|_H
\]

for every $v \in V$ and every $v \in V_0$, respectively, (2.40)

with a constant $C_\Omega$ that only depends on $\Omega$. Furthermore, we take advantage of a tool that is rather common in the study of problems related to the Cahn–Hilliard equations. In order to introduce it, we recall that $\Omega$ is connected, define the subspaces

\[
V_{\text{mean}} := \{ v \in V : \text{mean } v = 0 \} \quad \text{and} \quad V_{\text{mean}}^* := \{ v \in V^* : \text{mean } v = 0 \} \quad (2.41)
\]

and consider, for $\psi \in V^*$, the problem of finding

\[
u \in V \quad \text{such that} \quad \int_\Omega \nabla u \cdot \nabla v = \langle \psi, v \rangle \quad \text{for every } v \in V. \quad (2.42)
\]

By the way, if $\psi \in H$, this is the usual Neumann problem

\[- \Delta u = \psi \quad \text{in } \Omega \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on } \Gamma.\]
Now, for \( \psi \in V^* \), (2.42) is solvable if and only if \( \psi \in V^*_{\text{mean}} \). Moreover, if \( \psi \in V^*_{\text{mean}} \), exactly one of the solutions belongs to \( V_{\text{mean}} \). This implies that the operator

\[
\mathcal{N} : V^*_{\text{mean}} \to V_{\text{mean}} \text{ is defined by the following rule: for } \psi \in V^*_{\text{mean}}, \]

\[
\mathcal{N}\psi \text{ is the unique solution } u \text{ to (2.42) belonging to } V_{\text{mean}} \quad (2.43)
\]
is well defined. It turns out that \( \mathcal{N} \) is an isomorphism and that the function

\[
V^* \ni \psi \mapsto \|\psi\|^2_* := \|\nabla\mathcal{N}(\psi - \text{mean } \psi)\|^2_H + |\text{mean } \psi|^2
\]

(in particular, \( \|\psi\|^2_* = \|\nabla\mathcal{N}\psi\|^2_H = \langle \psi, \mathcal{N}\psi \rangle \) if \( \psi \in V^*_{\text{mean}} \))

(2.44)
is the square of a norm on \( V^* \) that is equivalent to the standard one. In the sequel, we use (2.44) for the norm in \( V^* \). Similarly, we introduce the Dirichlet problem solver related to the Poisson equation

\[- \Delta u = \psi \quad \text{or} \quad \int_{\Omega} \nabla u \cdot \nabla v = \langle \psi, v \rangle \quad \text{for every } v \in V_0 \quad (2.45)
\]
with homogeneous Dirichlet boundary conditions. It is the operator

\[
\mathcal{D} : V^*_0 \to V_0 \text{ defined by the following rule: for } \psi \in V^*_0, \]

\[
\mathcal{D}\psi \text{ is the unique solution } u \text{ to (2.45) belonging to } V_0. \quad (2.46)
\]

We notice that the function

\[
V^*_0 \ni \psi \mapsto \|\psi\|_* := \|\nabla \mathcal{D}\psi\|_H \quad (2.47)
\]
is a norm on \( V^*_0 \) that is equivalent to the standard one since \( \mathcal{D} \) is an isomorphism. We remark that

\[
\langle \partial_t v(t), \mathcal{L}v(t) \rangle = \langle v(t), \mathcal{L}(\partial_t v(t)) \rangle = \frac{1}{2} \frac{d}{dt} \|v(t)\|^2_* \text{ for a.a. } t \in (0, T),
\]

for every \( v \in H^1(0, T; V^*) \), where \( \mathcal{L} = \mathcal{N} \) if \( V = V \) and \( \mathcal{L} = \mathcal{D} \) if \( V = V_0 \). \quad (2.48)

In (2.48), the notation \( \| \cdot \|_* \) means (2.44) if \( V = V \) and (2.47) if \( V = V_0 \), of course. Also in the sequel, the meaning of \( \| \cdot \|_* \) is clear from the context and no confusion can arise.

3. Partial uniqueness and continuous dependence. In this section, we prove the part of Theorem 2.1 regarding partial uniqueness and continuous dependence. Namely, we just prove the latter, since the former follows as a consequence. As for uniqueness of \( \mu \) in the case \( V = V_0 \), we have already noticed that the first equation (2.18) is uniquely solvable for \( \mu \) in this case, so that uniqueness of \( \mu \) follows from uniqueness for \( \varphi \). So, we pick two choices \( g_i, \varphi_{0,i} \) and \( \varphi^*_i, i = 1, 2, \) of the data and a constant \( M \) as in the statement. We assume that \( (\varphi_i, \mu_i, \xi_i, \zeta_i) \) are arbitrary corresponding solutions and we prove the continuous dependence inequality (2.26). For brevity, we use the same symbol \( e \) (even in the same line or a chain of inequalities) for different constants depending only on the structure of our system, \( \Omega, T \) and \( M \).

We set for convenience

\[
g := g_1 - g_2, \quad \varphi_0 := \varphi_{0,1} - \varphi_{0,2}, \quad \varphi^* := \varphi^*_1 - \varphi^*_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \mu := \mu_1 - \mu_2, \quad \xi := \xi_1 - \xi_2 \quad \text{and} \quad \zeta := \zeta_1 - \zeta_2.
\]

We recall that mean \( \varphi_{0,1} = \text{mean } \varphi_{0,2} \) if \( V = V \). In this case, by applying (2.23) to both \( \varphi_1 \) and \( \varphi_2 \), we infer that mean \( \varphi \) vanishes identically. Hence, we can write (2.18) at the time \( s \) for both solutions and take \( v = \mathcal{N}\varphi(s) \) as test function in the difference. In the case \( V = V_0 \) of the Dirichlet boundary conditions, we can take
\( v = D\varphi(s) \) without any trouble. In order to unify the arguments, we set \( \mathcal{L} = \mathcal{N} \) and \( \mathcal{L} = D \) according to \( V = V \) or \( V = V_0 \). We recall that the symbol \( \| \cdot \|_s \) means (2.44) or (2.47) according to \( V = V \) or \( V = V_0 \). Thus, we have in both cases

\[
\int_\Omega \partial_t \varphi(s) \mathcal{L}\varphi(s) + \int_\Omega \nabla \mu(s) \cdot \nabla \varphi(s) = 0.
\]

By then integrating over \( (0,t) \) with \( t \in (0,T) \) and recalling (2.48), we have

\[
\frac{1}{2} \| \varphi(t) \|^2 + \int_{Q_t} \nabla \mu \cdot \nabla \varphi = \frac{1}{2} \| \varphi_0 \|^2.
\]

Similarly, we write (2.19) for both solutions at the time \( s \). Moreover, we subtract the same quantity \( \int_{Q_t} \zeta \varphi^* \) to both sides and rearrange. We obtain

\[
\frac{\tau}{2} \| \varphi(t) \|^2_H + \int_{Q_t} |\nabla \varphi|^2 + \int_{Q_t} \xi \varphi + \int_{Q_t} \zeta (\varphi - \varphi^*)
= \frac{\tau}{2} \| \varphi_0 \|^2_H + \int_{Q_t} \mu \varphi - \int_{Q_t} \zeta (\pi(\varphi_1) - \pi(\varphi_2)) \varphi - \int_{Q_t} \zeta \varphi^*.
\]

At this point, we add the above equalities to each other. Then, the terms involving \( \mu \) cancel out by the definition of \( \mathcal{L} \) (see (2.43) if \( \mathcal{L} = \mathcal{N} \) and (2.46) if \( \mathcal{L} = D \)) and those containing \( \xi \) and \( \zeta \) on the left-hand side are nonnegative by monotonicity. Furthermore, we can owe to the Lipschitz continuity of \( \pi \) given by (2.8) and estimate the corresponding term. As for the last integral on the right-hand side, we can account for the linear growth of \( S \) given by (2.7) (since \( \zeta_i(s) \in S(\varphi_i(s) - \varphi_i^*(s)) \) for \( i = 1,2 \)) and use the upper bound \( M \) given in the statement. We have

\[
- \int_{Q_t} \zeta \varphi^* \leq \left( \| \zeta_1 \|_{L^2(0,T; H)} + \| \zeta_2 \|_{L^2(0,T; H)} \right) \| \varphi^* \|_{L^2(0,T; H)}
\leq c \| \varphi_1 - \varphi_1^* \|_{L^2(0,T; H)} + \| \varphi_2 - \varphi_2^* \|_{L^2(0,T; H)} + 1 \| \varphi^* \|_{L^2(0,T; H)}
\leq c \| \varphi^* \|_{L^2(0,T; H)}.
\]

Therefore, by also applying the Schwarz and Young inequalities, recalling the continuous embeddings \( H \subset V^* \) and \( H \subset V_0^* \) (in the two case of the boundary conditions) and rearranging, we deduce that

\[
\| \varphi(t) \|^2_H + \int_{Q_t} |\nabla \varphi|^2
\leq c \| \varphi_0 \|^2_H + c \| g \|^2_{L^2(0,T; H)} + c \int_{Q_t} |\varphi|^2 + c \| \varphi^* \|_{L^2(0,T; H)}.
\]

At this point, we can apply the Gronwall lemma and obtain the inequality (2.26). \( \square \)

**Remark 8.** We come back to the case of the Dirichlet boundary conditions for the chemical potential and make a remark concerning possibly different Dirichlet data. For clarity, we denote by \( \mu^*_H \), \( \mu^*_H \) and \( \mu^* \) the Dirichlet data, the corresponding harmonic extensions and the original chemical potentials associated to the given couple of data. Then, the meaning of \( \mu_i \) in the proof is \( \mu_i := \mu^* - \mu^*_H \) and the one of \( g_i \) is the difference between the original \( g \) and \( g^*_H \). Hence, the right-hand side of (2.26) contains both the norm in \( L^2(0,T; H) \) of the difference of the original \( g \) and \( g^*_H \) and the one of \( \mu^*_H \) and \( \mu^*_H \). The latter is then estimated by the norm in \( L^2(0,T; H^{1/2}(\Gamma)) \) (or by a weaker norm) of \( \mu^* \). This means that \( \varphi \) continuously depends also on the Dirichlet datum for the chemical potential.
4. Approximation. The method we use for the existence part of Theorem 2.1 consists in performing suitable a priori estimates on the solution to an approximating problem and using compactness and monotonicity arguments. This section is devoted to the approximating problem.

We introduce the Moreau-Yosida regularizations \( \beta_\varepsilon, S_\varepsilon, \hat{\beta}_\varepsilon \) and \( \hat{S}_\varepsilon \) of the graphs \( \beta \) and \( S \) and of their primitives \( \hat{\beta} \) and \( \hat{S} \) at the level \( \varepsilon \in (0,1) \), and replace \( \beta \) and \( S \) in (2.18)-(2.20) by \( \beta_\varepsilon \) and \( S_\varepsilon \), respectively. We obtain the problem of finding \((\varphi_\varepsilon, \mu_\varepsilon)\) satisfying (2.14)–(2.15) as well as

\[
\int_\Omega \partial_t \varphi_\varepsilon(t) v + \int_\Omega \nabla \mu_\varepsilon(t) \cdot \nabla v = 0 \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V \tag{4.1}
\]

\[
\tau \int_\Omega \partial_t \varphi_\varepsilon(t) v + \int_\Omega \nabla \varphi_\varepsilon(t) \cdot \nabla v + \int_\Omega (\beta_\varepsilon(\varphi_\varepsilon(t)) + \pi(\varphi_\varepsilon(t))) v
\]

\[
+ \int_\Omega S_\varepsilon(\varphi_\varepsilon(t) - \varphi^*(t)) v
\]

\[
= \int_\Omega (\mu_\varepsilon(t) + g(t)) v \quad \text{for a.a. } t \in (0,T) \text{ and every } v \in V \tag{4.2}
\]

\[
\varphi_\varepsilon(0) = \varphi_0 \tag{4.3}
\]

where \( V \) is either \( V \) or \( V_0 \) according to our Notation.

**Theorem 4.1.** Problem (4.1)-(4.3) has a unique solution \((\varphi_\varepsilon, \mu_\varepsilon)\) satisfying the regularity requirements (2.14)–(2.15).

**Proof.** Uniqueness for \( \varphi_\varepsilon \) is still given by Section 3 since (4.1)-(4.3) is a particular case of problem (2.18)-(2.20). Indeed, \( \beta_\varepsilon \) and \( S_\varepsilon \) satisfy the assumptions we have required for \( \beta \) and \( S \). This is clear for \( \beta_\varepsilon \). As for \( S_\varepsilon \), we recall that \( \|S_\varepsilon(v)\|_H \leq \|S^0(v)\|_H \) and \( S^0(v) \in S(v) \) for every \( v \in H \), so that the linear growth condition (2.7) yields

\[
\|S_\varepsilon(v)\|_H \leq C_S(\|v\|_H + 1) \quad \text{for every } v \in H. \tag{4.4}
\]

Once uniqueness for \( \varphi_\varepsilon \) is established, uniqueness for \( \mu_\varepsilon \) trivially follows by comparison in (4.2) since both \( \beta_\varepsilon \) and \( S_\varepsilon \) are single-valued.

As for existence, we can follow the line of [9] only in the case of the Dirichlet boundary conditions for \( \mu_\varepsilon \). However, we have to take care on the dependence of \( \varphi^* \) on time (\( \varphi^* \) is constant in time in [9]) and on the general shape of \( S \). On the contrary, the treatment of the Neumann boundary conditions for \( \mu_\varepsilon \) needs a different argument. This is developed in the next lines by performing a discretization by means of a Faedo–Galerkin scheme corresponding to a Hilbert basis of eigenfunctions.

**The case of the Neumann boundary conditions: discretization.** We denote by \( \{\lambda_j\}_{j \geq 1} \) and \( \{e_j\}_{j \geq 1} \) the sequence of the eigenvalues and an orthonormal system of corresponding eigenfunctions of the Neumann problem for the Laplace equation, i.e.,

\[
0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \quad \text{and} \quad \lim_{j \to \infty} \lambda_j = +\infty
\]

\[
e_j \in V \quad \text{and} \quad \int_\Omega \nabla e_j \cdot \nabla e_j = \lambda_j \int_\Omega e_j v \quad \text{for every } v \in V \text{ and } j = 1,2,\ldots
\]

\[
\int_\Omega e_i e_j = \delta_{ij} \quad \text{for } i, j = 1,2,\ldots \quad \text{and} \quad \{e_j\}_{j \geq 1} \text{ is a complete system in } H.
\]
Notice that we can take $e_1 = |\Omega|^{-1/2}$. We set
\[ V_n := \text{span}\{e_1, \ldots, e_n\} \quad \text{for } n = 1, 2, \ldots \quad \text{and} \quad V_\infty := \bigcup_{n=1}^\infty V_n \quad (4.5) \]
and observe that $V_\infty$ is dense in both $V$ and $H$. The discretized problem consists in finding $\varphi^n \in H^1(0, T; V_n)$ and $\mu^n \in L^2(0, T; V_n)$ satisfying
\[ \int_\Omega \partial_t \varphi^n(t) v + \int_\Omega \nabla \mu^n(t) \cdot \nabla v = 0 \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V_n \quad (4.6) \]
\[ \tau \int_\Omega \partial_t \varphi^n(t) v + \int_\Omega \nabla \varphi^n(t) \cdot \nabla v + \int_\Omega (\beta_\varepsilon(\varphi^n(t)) + \pi(\varphi^n(t))) v \]
\[ + \int_\Omega S_\varepsilon(\varphi^n(t) - \varphi^*(t)) v \]
\[ = \int_\Omega (\mu^n(t) + g(t)) v \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V_n \quad (4.7) \]
\[ \int_\Omega \varphi^n(0)v = \int_\Omega \varphi_0 v \quad \text{for every } v \in V_n. \quad (4.8) \]
Thus, $\varphi^n$ and $\mu^n$ have to be expanded as
\[ \varphi^n(t) = \sum_{j=1}^n \phi_j(t)e_j \quad \text{and} \quad \mu^n(t) = \sum_{j=1}^n \eta_j(t)e_j \]
for some $\phi_j \in H^1(0, T)$ and $\eta_j \in L^2(0, T), \ j = 1, \ldots, n,$
and problem (4.6)-(4.8) written in terms of the column vectors $\phi := (\phi_1, \ldots, \phi_n)$ and $\eta := (\eta_1, \ldots, \eta_n)$ takes the form
\[ \phi'(t) + A\eta(t) = 0 \quad \text{for a.a. } t \in (0, T) \quad (4.9) \]
\[ \tau \phi'(t) + A\phi(t) + b^\varepsilon(\phi(t)) + \sigma^\varepsilon(t, \phi(t)) \]
\[ = \eta(t) + g^n(t) \quad \text{for a.a. } t \in (0, T) \quad (4.10) \]
with an addititional initial condition for $\phi$, where $A := (a_{ij}) \in \mathbb{R}^{n \times n}, b^\varepsilon := (b^\varepsilon_i) : \mathbb{R}^n \to \mathbb{R}^n, \sigma^\varepsilon := (\sigma^\varepsilon_i) : (0, T) \times \mathbb{R}^n \to \mathbb{R}^n \text{ and } g^n := (g^n_i) : (0, T) \to \mathbb{R}^n$ are given by
\[ a_{ij} := \int_\Omega \nabla e_j \cdot \nabla e_i, \quad b^\varepsilon_i(y_1, \ldots, y_n) := \int_\Omega (\beta_\varepsilon + \pi)(\sum_{j=1}^n y_j e_j)e_i \]
\[ \sigma^\varepsilon_i(t; y_1, \ldots, y_n) := \int_\Omega S_\varepsilon\left(\sum_{j=1}^n y_j e_j - \varphi^*(t)\right)e_i \quad \text{and} \quad g^n_i(t) := \int_\Omega g(t)e_i. \]
Since $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, $b^\varepsilon$ is Lipschitz continuous. On the other hand, $S_\varepsilon : H \to H$ is Lipschitz continuous too and the function $(y_1, \ldots, y_n) \mapsto \|\sum_{j=1}^n y_j e_j\|_H$ is a norm in $\mathbb{R}^n$. Hence, by denoting by $L_\varepsilon$ the Lipschitz constant of $S_\varepsilon$, we have for $i = 1, \ldots, n$ and every $y, z \in \mathbb{R}^n$
\[ |\sigma^\varepsilon_i(t, y) - \sigma^\varepsilon_i(t, z)| \]
\[ = \left| \int_\Omega S_\varepsilon(\sum_{j=1}^n y_j e_j - \varphi^*(t))e_i - \int_\Omega S_\varepsilon(\sum_{j=1}^n z_j e_j - \varphi^*(t))e_i \right| \]
\[ \leq \left\| S_\varepsilon(\sum_{j=1}^n y_j e_j - \varphi^*(t)) - S_\varepsilon(\sum_{j=1}^n z_j e_j - \varphi^*(t)) \right\|_H \]
\[ \leq L_\varepsilon \left\| \sum_{j=1}^n y_j e_j - \sum_{j=1}^n z_j e_j \right\|_H \approx |y - z| \]
that is, $\sigma^\varepsilon$ is a Charathéodory function that is Lipschitz continuous with respect to $y$ uniformly in $t$. Moreover, due to our assumptions (2.9) and (2.11) on $g$ and $\varphi^*$, we have that $t \mapsto \sigma^\varepsilon(t; y)$ and $g^n$ belong to $L^2(0, T; \mathbb{R}^n)$, the former for every $y \in \mathbb{R}^n$.

By replacing $\eta$ in (4.9) by the expression obtained by solving (4.10) for $\eta$, we obtain a Cauchy problem for the single equation (where $I \in \mathbb{R}^{n \times n}$ is the identity matrix)

$$(I + \tau A)\phi'(t) + A\left(A\phi(t) + b^* (\phi(t)) + \sigma^\varepsilon(t, \phi(t)) - g^n(t)\right) = 0$$

which can be put in its normal form since $I + \tau A$ is positive definite. Thus, we obtain a unique $\phi \in H^1(0, T; \mathbb{R}^n)$ and (4.10) provides the corresponding $\eta \in L^2(0, T; \mathbb{R}^n)$. This proves that the discrete problem (4.6)-(4.8) has a unique solution with the desired regularity. At this point, our aim is letting $n$ tend to infinity. To this end, we perform an a priori estimate. In the sequel, $c$ stands for different constants independent of $n$ (and possibly depending on the structure of our system, $\Omega$, $T$, the data and $\varepsilon$, which is fixed in the whole proof of Theorem 4.1).

**Estimate on the discrete solution.** We test (4.6) and (4.7) by the $V_n$-valued functions $\mu^n$ and $\partial_t \varphi^n$, respectively, sum up and integrate over $(0, t)$. Moreover, we add the same quantity $\frac{1}{2} \int_\Omega |\varphi^n(t)|^2 = \int_{Q_t} \varphi^n \partial_t \varphi^n + \frac{1}{2} \int_\Omega |\varphi^n(0)|^2$ to both sides. After an obvious cancellation and a rearrangement, we obtain

$$
\begin{align*}
\int_{Q_t} |\nabla \mu^n|^2 &+ \tau \int_{Q_t} |\partial_t \varphi^n|^2 + \frac{1}{2} \|\varphi^n(t)\|_V^2 + \int_{Q_t} \beta (\varphi^n(t)) \\
&= \frac{1}{2} \|\varphi^n(0)\|_V^2 + \int_{Q_t} \beta (\varphi^n(0)) \\
&\quad + \int_{Q_t} \left(g + \varphi^n - \pi(\varphi^n) - S_\varepsilon(\varphi^n - \varphi^*)\right) \partial_t \varphi^n. \\
\end{align*}
$$

The last integral can be treated by using the Lipschitz continuity of $\pi$ and $S_\varepsilon$ and applying the Schwarz and Young inequalities as follows

$$
\begin{align*}
\int_{Q_t} \left(g + \varphi^n - \pi(\varphi^n) - S_\varepsilon(\varphi^n - \varphi^*)\right) \partial_t \varphi^n \\
\leq \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi^n|^2 + c \int_{Q_t} \left(|\varphi^n|^2 + 1\right) + c \|g\|_{L^2(0,T;H)} + c \|\varphi^*\|_{L^2(0,T;H)}.
\end{align*}
$$

If we prove that $\varphi^n(0)$ is bounded in $V$, even the second term on the right-hand side of (4.11) remains bounded (since $\beta (r)$ grows at most as $r^2$) and we can apply the Gronwall lemma to conclude that

$$
\|\varphi^n\|_{H^1(0,T;V) \cap L^\infty(0,T;V)} + \|\nabla \mu^n\|_{L^2(0,T;H)} \leq c. \qquad (4.12)
$$

**Proof.** From (4.8) it follows that $\varphi^n(0)$ is the $H$-projection of $\varphi_0$ onto $V_n$, i.e.,

$$
\varphi^n(0) = \sum_{j=1}^n \alpha_j e_j \quad \text{where the sequence } \{\alpha_j\} \text{ satisfies } \varphi_0 = \sum_{j=1}^\infty \alpha_j e_j.
$$

Now, we observe that $W$ (see (2.2)) can be characterized as

$$
W = \{v \in H : -\Delta v \in H, \partial_n v = 0\} = \left\{\sum_{j=1}^\infty c_j e_j : \sum_{j=1}^\infty \lambda_j^2 |c_j|^2 < +\infty\right\}.
$$
Since $V$ is the interpolation space $(W, H)_{1/2}$, we also have that

$$V = \left\{ \sum_{j=1}^{\infty} c_j e_j : \sum_{j=1}^{\infty} \lambda_j |c_j|^2 < +\infty \right\}$$

with equivalence of Hilbert norms

$$\|v\|_V \approx \|v\|_\lambda := \left( \|v\|_H^2 + \sum_{j=1}^{\infty} \lambda_j |c_j|^2 \right)^{1/2}$$

if $v = \sum_{j=1}^{\infty} c_j e_j$.

Since $\varphi_0 \in V$, we deduce that

$$\|\varphi^n(0)\|^2 = \|\varphi^n(0)\|_H^2 + \sum_{j=1}^{n} \lambda_j |\alpha_j|^2 \leq \|\varphi_0\|_H^2 + \sum_{j=1}^{\infty} \lambda_j |\alpha_j|^2 = \|\varphi_0\|^2. \quad (4.13)$$

This concludes the proof of (4.12).

We immediately infer that $\beta_\varepsilon(\varphi^n)$, $\pi(\varphi^n)$ and $S_\varepsilon(\varphi^n - \varphi^*)$ are bounded in $L^2(0, T; H)$. At this point, we can test (4.6) and (4.7) by the Young inequality lead to

$$\|\varphi^n(t)\|_H^2 + \int_{Q_1} |\mu^n|^2 = \frac{1}{2} \|\varphi^n(0)\|_H^2$$

and

$$\leq c + \frac{1}{2} \int_{Q_1} |\mu^n|^2$$

whence a bound for $\mu^n$ in $L^2(0, T; H)$. Therefore, (4.12) is improved. Namely, we have

$$\|\varphi^n\|_{H^1(0, T; H) \cap L^\infty(0, T; V)} + \|\mu^n\|_{L^2(0, T; V)} \leq c. \quad (4.14)$$

**Conclusion.** From (4.14), the Aubin–Lions lemma (see, e.g., [25, Thm. 5.1, p. 58]) and the Lipschitz continuity of $\beta_\varepsilon$, $\pi$ and $S_\varepsilon$, we deduce that

$$\varphi^n \to \varphi_\varepsilon \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \quad (4.15)$$

$$\mu^n \to \mu_\varepsilon \text{ weakly in } L^2(0, T; V) \quad (4.16)$$

$$\varphi^n \to \varphi_\varepsilon, \quad (\beta_\varepsilon + \pi)(\varphi^n) \to (\beta_\varepsilon + \pi)(\varphi_\varepsilon) \quad (4.17)$$

and

$$S_\varepsilon(\varphi^n - \varphi^*) \to S_\varepsilon(\varphi_\varepsilon - \varphi^*) \text{ strongly in } L^2(0, T; H)$$

as $n$ tends to infinity, at least for a (not relabeled) subsequence. Actually, $\varphi^n$ converges to $\varphi_\varepsilon$ strongly in $C^0([0, T]; H)$ due to (4.15) and the generalized Ascoli theorem (see, e.g., [39, Sect. 8, Cor. 4]). We show that $(\varphi_\varepsilon, \mu_\varepsilon)$ is the solution to

(4.1)-(4.3) we are looking for. First, we have that $\varphi_\varepsilon(0) = \varphi_0$ since $\varphi^n(0)$ converges both to $\varphi_\varepsilon(0)$ and to $\varphi_0$ strongly in $H$, the latter since $V_\infty$ (see (4.5)) is dense in $H$ (in fact, (4.13) ensures that $\varphi^n(0) \to \varphi_0$ strongly in $V$). Now, fix $m \geq 1$, take any $v \in L^2(0, T; V_m)$ and assume that $n \geq m$. Then, $V_m \subset V_n$ so that both (4.6) and (4.7) can be tested by $v(t)$ and then integrated over $(0, T)$. We thus obtain

$$\int_Q \partial_t \varphi^n v + \int_Q \nabla \mu^n \cdot \nabla v = 0$$

$$\tau \int_Q \partial_t \varphi^n v + \int_Q \nabla \varphi^n \cdot \nabla v + \int_Q (\beta_\varepsilon(\varphi^n) + \pi(\varphi^n)) v + \int_Q S_\varepsilon(\varphi^n - \varphi^*) v$$

$$= \int_Q (\mu^n + g) v$$
for arbitrary choices of \( v \in L^2(0, T; V_m) \) in the above equations and every \( n \geq m \).

By letting \( n \) tend to infinity, we deduce that

\[
\int_Q \partial_t \varphi_{\varepsilon} v + \int_Q \nabla \mu_{\varepsilon} \cdot \nabla v = 0 \quad (4.18)
\]

\[
\tau \int_Q \partial_t \varphi_{\varepsilon} v + \int_Q \nabla \varphi_{\varepsilon} \cdot \nabla v + \int_Q (\beta_{\varepsilon}(\varphi_{\varepsilon}) + \pi(\varphi_{\varepsilon}))v + \int_Q S_{\varepsilon}(\varphi_{\varepsilon} - \varphi^*)v = \int_Q (\mu_{\varepsilon} + g)v. \quad (4.19)
\]

In both equations, \( v \in L^2(0, T; V_m) \) is arbitrary and \( m \) is arbitrary too. Take now any \( V_\infty \)-valued step function. Then, \( v \in L^2(0, T; V_m) \) for some \( m \) so that (4.18) and (4.19) hold for \( v \). Since the set of such functions is dense in \( L^2(0, T; V) \), we conclude that both (4.18) and (4.19) hold for every \( v \in L^2(0, T; V) \). But this is equivalent to say that (4.1) and (4.2) hold true. This concludes the proof in the case of the Neumann conditions.

The case of the Dirichlet boundary conditions.

In this case, the argument used above does not work, since we should use two different systems of eigenfunctions corresponding to the different boundary conditions for \( \varphi_{\varepsilon} \) and \( \mu_{\varepsilon} \). Therefore, we follow the ideas of [9] and eliminate \( \mu_{\varepsilon} \). By recalling the definition (2.46) of \( D \), we write (4.1) in the equivalent form

\[
\mu_{\varepsilon} = -D(\partial_t \varphi_{\varepsilon}) \quad (4.20)
\]

and replace \( \mu_{\varepsilon} \) in (4.2) by this expression. We obtain

\[
\int_\Omega (\tau \partial_t \varphi_{\varepsilon}(t) + D(\partial_t \varphi_{\varepsilon}(t)))v + \int_\Omega \nabla \varphi_{\varepsilon}(t) \cdot \nabla v + \int_\Omega (\beta_{\varepsilon}(\varphi_{\varepsilon}(t)) + \pi(\varphi_{\varepsilon}(t)) + S_{\varepsilon}(\varphi_{\varepsilon}(t) - \varphi^*(t)))v = \int_\Omega g(t)v \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V. \quad (4.21)
\]

We present this variational equation as an abstract nonlinear equation of parabolic type. To this end, we introduce the bilinear form \( (\cdot, \cdot) \) in \( H \times H \) by setting

\[
(u, v) := \int_\Omega (Bu)v \quad \text{for } u, v \in H,
\]

\[
B := \tau I + D, \quad I \text{ being the identity map of } H
\]

and observe that, due to the definition (2.46) of \( D \), this form is continuous and symmetric and satisfies for every \( v \in H \)

\[
(u, v) = \tau \| u \|^2_H + \int_\Omega (Dv)v = \tau \| v \|^2_H + \int_\Omega |\nabla Dv|^2 \geq \tau \| v \|^2_H.
\]

Hence, it is an equivalent inner product in \( H \). This also shows that the operator \( B \) is an isomorphism from \( H \) into itself. Therefore, the variational equation (4.21) can be written as

\[
((\partial_t \varphi_{\varepsilon}, v)) + \int_\Omega \nabla \varphi_{\varepsilon} \cdot \nabla v + ((B^{-1}(\beta_{\varepsilon}(\varphi_{\varepsilon}) + \pi(\varphi_{\varepsilon}) + S_{\varepsilon}(\varphi_{\varepsilon} - \varphi^*)), v)) = ((B^{-1}g, v)) \quad \text{a.e. in } (0, T), \text{ for every } v \in V
\]
that is, as the abstract equation in $V^*$

$$\frac{d}{dt} \varphi_\varepsilon(t) + A\varphi_\varepsilon(t) + \mathcal{F}_\varepsilon(t, \varphi_\varepsilon(t)) = B^{-1}g(t) \quad \text{for a.a. } t \in (0, T)$$

(4.22)

provided that the Hilbert triplet $(V,H,V^*)$ is constructed starting from the new inner product $\langle \cdot, \cdot \rangle$ rather than the standard one (i.e., in order that $\langle u,v \rangle = \langle (u,v) \rangle$ for every $u \in H$ and $v \in V$) and the continuous linear operator $A : V \to V^*$ and the function $\mathcal{F}_\varepsilon : (0,T) \times H \to H \subset V^*$ are defined as follows

$$\langle Au, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v \quad \text{for } u, v \in V$$

$$\mathcal{F}_\varepsilon(t, v) := B^{-1}(\beta_\varepsilon(v) + \pi(v) + \lambda_\varepsilon(v - \varphi_\varepsilon^*(t))) \quad \text{for a.a. } t \in (0,T) \text{ and } v \in H.$$

So, it is sufficient to solve the Cauchy problem for (4.22) associated with the initial condition (4.3) and then recover $\mu_\varepsilon$ by means of (4.20). The Cauchy problem just mentioned has a unique solution $\varphi_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V)$ by the following reasons.

**Proof.** First, $A$ is linear and continuous and $A + I$ (where $I : V \to V^*$ is the embedding) is coercive. Next, since $\varphi^* \in L^2(0,T;H)$ and $\beta_\varepsilon$, $\pi$ and $\lambda_\varepsilon$ are Lipschitz continuous, $\mathcal{F}_\varepsilon$ is a Charatéodory function that is Lipschitz continuous with respect to $v$ uniformly in $t$. Finally, the right-hand side belongs to $L^2(0,T;H)$ and $\varphi_0 \in V$. For a more detailed proof, one could discretize (4.22) with a Faedo–Galerkin scheme, as we did before. Indeed, as $\mu_\varepsilon$ has been eliminated, one can use just one system of eigenfunction, namely, the same we have introduced to solve the problem in the case of the Neumann boundary conditions. This concludes the proof.

5. **Existence for the generalized problem.** We start from the approximating problem (4.1)-(4.3) and perform some a priori estimates on its solution $(\varphi_\varepsilon, \mu_\varepsilon)$. In the whole section, we use the same symbol $c$ to denote constants that only depend on the structure of the problem, the data, $\Omega$ and $T$, and can possibly be different from each other (even in the same chain of equalities or inequalities). We stress that the values of $c$ do not depend on $\varepsilon$.

**First a priori estimate.** Assume first that $\mathcal{V} = V$. Then, by taking $v = |\Omega|^{-1}$ in (4.1), we see that mean$(\partial_t \varphi_\varepsilon) = 0$ so that $\mathcal{N}(\partial_t \varphi_\varepsilon)$ is well defined. Moreover, it belongs to $L^2(0,T;V)$. If instead $\mathcal{V} = V_0$ one can consider $\mathcal{D}(\partial_t \varphi_\varepsilon)$, which is well defined and belongs to $L^2(0,T;V_0)$. Hence, in both cases, we can test (4.1) written at the time $s$ by $\mu_\varepsilon(s) + \mathcal{L}(\partial_t \varphi_\varepsilon(s))$ where $\mathcal{L} = \mathcal{N}$ if $\mathcal{V} = V$ and $\mathcal{L} = \mathcal{D}$ if $\mathcal{V} = V_0$. Then, we integrate the resulting equality over $(0,t)$ with respect to $s$. Similarly, we test (4.2) by $2\partial_t \varphi_\varepsilon$ and integrate in time. Then, we sum the equalities obtained this way to each other and notice that two cancellations occur: one of them is obvious and the other is due to the definitions of $\mathcal{N}$ and $\mathcal{D}$ given by (2.43) and (2.46). Finally, we add the same quantity $\int_\Omega |\varphi_\varepsilon(t)|^2 = \int_\Omega |\varphi_0|^2 + 2 \int_{Q_\varepsilon} \varphi_\varepsilon \partial_t \varphi_\varepsilon$ to both sides for convenience. Hence, by recalling (2.44) and (2.47), we obtain

$$\int_0^t \|\partial_t \varphi_\varepsilon(s)\|^2 ds + \int_{Q_\varepsilon} |\nabla \mu_\varepsilon|^2$$

$$+ 2\tau \int_{Q_\varepsilon} |\partial_t \varphi_\varepsilon|^2 + \int_{\Omega} |\varphi_\varepsilon(t)|^2$$

$$+ 2 \int_{\Omega} \hat{\beta}_\varepsilon(\varphi_\varepsilon(t)) + \int_{\Omega} |\varphi_\varepsilon(t)|^2.$$
Next, we use the Young inequality, the uniform linear growth condition (4.4) and inequalities. Hence, we can estimate it from above by the following expression
\[\int_{\Omega} |\nabla \varphi|^2 + 2 \int_{\Omega} \beta_{\varepsilon}(\varphi_0) + \int_{\Omega} |\varphi_0|^2 + 2 \int_{Q_t} (g - \pi(\varphi_\varepsilon) + \varphi_\varepsilon) \partial_t \varphi_\varepsilon - 2 \int_{Q_t} S_{\varepsilon}(\varphi_\varepsilon - \varphi^*) \partial_t \varphi_\varepsilon.\] (5.1)

All the terms on the left-hand side are nonnegative. As for the first line on the right-hand side, we account for the well-known inequality \(\beta_{\varepsilon}(r) \leq \hat{\beta}(r)\) (which holds for every \(r \in \mathbb{R}\)), the linear growth of \(\pi\) (see (2.8)) and the Schwarz and Young inequalities. Hence, we can estimate it from above by the following expression
\[\|\varphi_0\|^2 + 2 \int_{\Omega} \beta(\varphi_0) + \|\nabla \pi\|_{L^\infty(0,T;V)} + c \int_{Q_t} (|g|^2 + |\varphi|^2 + 1).\]

Next, we use the Young inequality, the uniform linear growth condition (4.4) and the assumption (2.11) on \(\varphi^*\). We obtain
\[- \int_{Q_t} S_{\varepsilon}(\varphi_\varepsilon - \varphi^*) \partial_t \varphi_\varepsilon \leq \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + c \int_{Q_t} (|\varphi_\varepsilon|^2 + |\varphi^*|^2)
\leq \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + c \int_{Q_t} |\varphi_\varepsilon|^2 + c.\] (5.2)

At this point, we come back to (5.1), combine with these estimates, rearrange and apply the Gronwall lemma. We conclude that
\[\|\varphi_\varepsilon\|_{H^1(0,T;H^1) \cap L^\infty(0,T;V)} + \|\nabla \mu_\varepsilon\|_{L^2(0,T;H)} \leq c.\] (5.3)

Hence, by accounting also for the Lipschitz continuity of \(\pi\), the inequality (4.4) and the \(L^2\) summability of \(\varphi^*\), we deduce that
\[\|\varphi_\varepsilon\|_{L^\infty(0,T;H)} + \|\pi(\varphi_\varepsilon)\|_{L^\infty(0,T;H)} + \|S_{\varepsilon}(\varphi_\varepsilon - \varphi^*)\|_{L^2(0,T;H)} \leq c.\] (5.4)

**Second a priori estimate.** Our aim is to improve a part of (5.3) by showing that
\[\|\mu_\varepsilon\|_{L^2(0,T;V)} \leq c.\] (5.5)

If \(V = V_0\) this trivially follows from (5.3) and the second Poincaré inequality (2.40). If instead \(V = V\), we can deduce (5.5) from the first Poincaré inequality provided we have an estimate of the mean value mean \(\mu_\varepsilon\) in \(L^2(0,T)\). Hence, we prepare a pointwise estimate in this direction.

Since \(V = V\), one of our assumption is (2.12). The following argument (which owes to [29, Appendix, Prop. A.1], see also [19, p. 908] for a detailed proof) is used in several papers. We repeat it here for the reader’s convenience. By (2.12), we have for some \(\delta_0 > 0\) depending only on \(\beta\) and \(m_0\)
\[\beta_\varepsilon(r)(r - m_0) \geq \delta_0 |\beta_\varepsilon(r)|^{-1} \quad \text{for every } r \in \mathbb{R} \text{ and every } \varepsilon \in (0,1).\] (5.6)

Now, we test (4.2) by \(\varphi_\varepsilon - m_0\) and avoid time integration. Thus, we argue for \(t\) fixed. However, for simplicity, we do not write the time \(t\) for a while. We have a.e. in \((0,T)\)
\[\delta_0 \int_{\Omega} |\beta_\varepsilon(\varphi_\varepsilon)| - \delta_0^{-1} |\Omega| \leq \int_{\Omega} \nabla \varphi_\varepsilon \cdot \nabla (\varphi_\varepsilon - m_0) + \int_{\Omega} \beta_\varepsilon(\varphi_\varepsilon)(\varphi_\varepsilon - m_0)
= \int_{\Omega} \mu_\varepsilon(\varphi_\varepsilon - m_0) + \int_{\Omega} (g - \tau \partial_t \varphi_\varepsilon - \pi(\varphi_\varepsilon) - S_{\varepsilon}(\varphi_\varepsilon - \varphi^*)) (\varphi_\varepsilon - m_0).\] (5.7)
We recall that \(\text{mean}(\varphi - m_0) = 0\) a.e. in \((0, T)\), thus we can take advantage of that and apply the first Poincaré inequality (2.40) to \(\mu - \text{mean} \mu\). In fact, using (5.4) as well, we have
\[
\int_{\Omega} \mu (\varphi - m_0) = \int_{\Omega} (\mu - \text{mean} \mu) (\varphi - m_0)
\leq \|\mu - \text{mean} \mu\|_H \|\varphi - m_0\|_H
\leq c \|\nabla \mu\|_H \|\varphi - m_0\|_H \. 
\]
As for the rest of the right-hand side of (5.7), we use the Schwarz inequality and the estimates in \(L^\infty(0, T; H)\) available from (5.4). Hence, we obtain
\[
\|\beta (\varphi)\|_1 \leq c \left( \|\nabla \mu\|_H + \|g\|_H + \|\partial_t \varphi\|_H + \|S \varphi - \varphi^*\|_H + 1 \right)
\]
a.e. in \((0, T)\). By taking \(v = \Omega^{-1}\) in (4.2) as before, using the inequality just obtained and estimating the other \(L^1\)-norms by the corresponding \(H\)-norms, we deduce that
\[
|\text{mean} \mu| \leq c \left( \|\nabla \mu\|_H + \|g\|_H + \|\partial_t \varphi\|_H + \|S \varphi - \varphi^*\|_H + 1 \right) \
\]
a.e. in \((0, T)\). This is the desired pointwise estimate.

By squaring (5.8), integrating over \((0, T)\) and taking the square root, we deduce an inequality. The left-hand side is the norm of \(\mu\) in \(L^2(0, T)\) and the right-hand side is what one obtains by replacing every \(H\)-norm in the right-hand side of (5.8) by the norm in \(L^2(0, T; H)\). Hence, by using (5.3) and (5.4), we deduce that
\[
|\text{mean} \mu|_{L^2(0, T)} \leq c .
\]
Hence, (5.5) follows.

**Third a priori estimate.** We test (4.2) by \(\beta (\varphi)\) and integrate over \((0, T)\). We obtain
\[
\int_Q \beta (\varphi) |\nabla \varphi|^2 + \int_Q |\beta (\varphi)|^2
= \int_Q (\mu + g - \tau \partial_t \varphi - \pi (\varphi - S \varphi^*) - S \varphi - \varphi^*) \beta (\varphi).
\]
From the previous estimates we immediately conclude that
\[
\|\beta (\varphi)\|_{L^2(0, T; H)} \leq c .
\]

**Conclusion.** By accounting for (5.3)–(5.5) and (5.10) and owing to well-known weak compactness results, we have in both cases \(\mathcal{V} = V\) and \(\mathcal{V} = V_0\) (along a subsequence)
\[
\varphi \to \varphi \quad \text{weakly in } H^1(0, T; H) \cap L^2(0, T; V)
\]
\[
\mu \to \mu \quad \text{weakly in } L^2(0, T; V)
\]
\[
\beta (\varphi) \to \xi \quad \text{weakly in } L^2(0, T; H)
\]
\[
S \varphi (\varphi - \varphi^*) \to \zeta \quad \text{weakly in } L^2(0, T; H)
\]
for some quadruplet \((\varphi, \mu, \xi, \zeta)\) with the regularity specified by the convergence properties. We claim that \((\varphi, \mu, \xi, \zeta)\) is the solution to problem (2.18)-(2.20) we are looking for. By letting \(\varepsilon\) tend to zero in (4.1), we clearly find that
\[
\int_Q \partial_t \varphi v + \int_Q \nabla \mu \cdot \nabla v = 0.
\]
for every \( v \in L^2(0,T; V) \), and this is equivalent to (2.18). Moreover, by (5.11), \( \varphi_\varepsilon \) converges to \( \varphi \) weakly in \( C^0([0,T]; H) \), whence we deduce that \( \varphi_\varepsilon(0) \) converge to \( \varphi(0) \) weakly in \( H \) and conclude that \( \varphi(0) = \varphi_0 \). Furthermore, we observe that the already mentioned Aubin–Lions lemma implies that
\[
\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^2(0,T; H).
\] (5.15)

By Lipschitz continuity, \( \pi(\varphi_\varepsilon) \) converges to \( \pi(\varphi) \) in the same topology. It follows that
\[
\tau \int_Q \partial_t \varphi v + \int_Q \nabla \varphi \cdot \nabla v + \int_Q (\xi + \pi(\varphi_\varepsilon) + \zeta)v = \int_Q (\mu_\varepsilon + g)v
\]
for every \( v \in L^2(0,T; V) \), and this is equivalent to (2.19). Finally, by accounting for the strong convergence (5.15) (which trivially implies the strong convergence of \( \varphi_\varepsilon - \varphi^* \) to \( \varphi - \varphi^* \) in the same topology) and the weak convergence given by (5.13)–(5.14), we can apply, e.g., [1, Lemma 2.3, p. 38] and conclude that \( \xi \in \beta(\varphi) \) and \( \zeta \in S(\varphi - \varphi^*) \) a.e. in \( Q \). Hence, the proof is complete.

6. Existence of a sliding mode. This section is devoted to the proof of Theorem 2.2. Hence, \( S \) is the graph induced in \( H \) by \( \rho \text{sign} \) (see (1.10) and (2.27)–(2.28)) and the data satisfy the further assumptions (2.29)–(2.31) as well. Moreover, we choose \( V = V_0 \) once and for all in both the original problem (2.18)–(2.20) and the approximating problem (4.1)–(4.3), since we only treat the Dirichlet boundary conditions for the chemical potential. The proof of our result relies on a number of careful a priori estimates on the solution \((\varphi_\varepsilon, \mu_\varepsilon)\) to the approximating problem and a comparison argument involving the solution of an ODE. In performing the a priori estimates, we have to take a particular care on the dependence of the constants on \( \rho \). Thus, from now on, the symbol \( c \) denotes (possibly different) constants that do not depend on \( \rho \) and \( \varepsilon \). We use capital letters (mainly with indices like \( C_i \)) to denote some constants we want to refer to. The quantities that such constants depend on are specified in introducing them. Our method is close to the ideas of [9] but it is different (in particular, we have to take care on the dependence of \( \varphi^* \) on time). For this reason, we present the whole argument with all the details that are necessary.

First of all, recalling the properties (2.31) on \( \varphi^* \), we can rewrite equations (4.1)–(4.2) in terms of both \( \varphi_\varepsilon \) and the auxiliary unknown
\[
\chi_\varepsilon := \varphi_\varepsilon - \varphi^*.
\] (6.1)
The new equations read
\[
\tau \int_\Omega \partial_t \chi_\varepsilon v + \int_\Omega \nabla \mu_\varepsilon \cdot \nabla v = -\int_\Omega \partial_t \varphi^* v \quad \text{for every } v \in V_0
\] (6.2)
\[
\tau \int_\Omega \partial_t \chi_\varepsilon v + \int_\Omega \nabla \chi_\varepsilon \cdot \nabla v + \int_\Omega (\beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon) + S_\varepsilon(\chi_\varepsilon))v = \int_\Omega (\mu_\varepsilon + g - \tau \partial_t \varphi^*)v - \int_\Omega \nabla \varphi^* \cdot \nabla v \quad \text{for every } v \in V
\] (6.3)
both a.e. in \((0,T)\), and the initial condition for \( \chi_\varepsilon \) is given by
\[
\chi_\varepsilon(0) = \chi_0 := \varphi_0 - \varphi^*(0).
\] (6.4)
In equation (6.3), \( S_\varepsilon : H \to H \) is the map associated to \( \sigma_\varepsilon := \rho \text{sign}_\varepsilon \), that is,
\[
\text{for } v \in H, \quad S_\varepsilon(v) \text{ is the function } \ x \mapsto \sigma_\varepsilon(v(x)) = \rho \text{sign}_\varepsilon v(x), \quad x \in \Omega,
\]
where sign$\epsilon$ is the Yosida approximation of sign. We point out that
\[ \text{sign}_\epsilon(0) = 0 \quad \text{and} \quad |\text{sign}_\epsilon(r)| \leq 1 \quad \text{for every } r \in \mathbb{R}. \] (6.5)

We also set for convenience
\[ \hat{\sigma}_\epsilon(r) := \int_0^r \sigma_\epsilon(r') \, dr' \quad \text{for } r \in \mathbb{R} \quad \text{and} \quad \hat{S}_\epsilon(v) := \int_\Omega \hat{\sigma}_\epsilon(v) \quad \text{for } v \in H \] (6.6)

and notice that $\hat{S}_\epsilon$ actually is the primitive of $S_\epsilon$. We also remark that
\[ 0 \leq \hat{\sigma}_\epsilon(r) \leq \rho|\epsilon| \quad \text{and} \quad 0 \leq \hat{S}_\epsilon(v) \leq \hat{S}(v) = \rho \int_\Omega |v| \] (6.7)
for every $r \in \mathbb{R}$ and every $v \in H$, respectively.

**Fourth a priori estimate.** We test (6.2) and (6.3) written at the time $s$ by $\mathcal{D}(X_\epsilon(s))$ and $X_\epsilon(s)$, respectively. Then, we sum up, integrate over $(0,t)$ with respect to $s$ and notice a cancellation due to the definition of $\mathcal{D}$ in the terms involving $\mu_\epsilon$.

By recalling (2.48), we obtain
\begin{align*}
&\frac{1}{2} \|X_\epsilon(t)\|_s^2 + \frac{\tau}{2} \int_\Omega |X_\epsilon(t)|^2 + \int_{Q_t} |\nabla X_\epsilon|^2 \\
&\quad + \int_{Q_t} \beta_\epsilon(\varphi_\epsilon)(\varphi_\epsilon - \varphi^*) + \rho \int_{Q_t} \text{sign}_\epsilon(X_\epsilon) X_\epsilon \\
&\quad = - \int_{Q_t} \partial_t \varphi^* \mathcal{D}(X_\epsilon) + \int_{Q_t} \left( g - \pi(X_\epsilon + \varphi^*) - \tau \partial_t \varphi^* \right) X_\epsilon - \int_{Q_t} \nabla \varphi^* \cdot \nabla X_\epsilon.
\end{align*}

The last term on the left-hand side is nonnegative since sign$\epsilon$ is monotone and vanishes on the origin. For the one involving $\beta_\epsilon$ we use the convexity inequality for $\widehat{\beta}_\epsilon$ as follows
\[ \beta_\epsilon(\varphi_\epsilon)(\varphi_\epsilon - \varphi^*) \geq \widehat{\beta}_\epsilon(\varphi_\epsilon) - \widehat{\beta}_\epsilon(\varphi^*) \]

and notice that
\[ \int_\Omega \widehat{\beta}_\epsilon(\varphi_\epsilon) \geq 0 \quad \text{and} \quad \int_{Q_t} \widehat{\beta}_\epsilon(\varphi^*) \leq \int_{Q_t} \widehat{\beta}(\varphi^*) \leq \int_{Q} \widehat{\beta}(\varphi^*) = c \]

where $c$ actually is finite as a consequence of the assumptions (2.31): indeed, it suffices that $\varphi^*, \beta^2(\varphi^*) \in L^2(Q)$. Finally, all the terms on right-hand side can be easily treated with the Young inequality since $\|\mathcal{D}v\|_H \leq c \|v\|_H$, as a trivial consequence of the continuity of $\mathcal{D}$ from $V^*_0$ into $V_0$. Hence, by applying the Gronwall lemma, we conclude that
\[ \|X_\epsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \] (6.8)

By recalling (2.31) on $\varphi^*$ and the Lipschitz continuity of $\pi$, we deduce that
\[ \|\varphi_\epsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c \quad \text{and} \quad \|\pi(\varphi_\epsilon)\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \] (6.9)

In the above estimates, $c$ does not depend on $\rho$, according to our general rule.

Now, we revisit the derivation of one of the estimates of Section 5 in order to explicit the dependence of the constant on $\rho$ under the new assumption on $S$.

**First a priori estimate revisited.** We come back to (5.1) and replace the treatment as in (5.2) by a different argument. Owing to (2.31) and (6.5)–(6.7), we can
write
\[-2\int_{Q_t} S_\varepsilon(\varphi_\varepsilon - \varphi^\ast) \partial_t \varphi_\varepsilon = -2\int_{Q_t} \sigma_\varepsilon(\chi_\varepsilon)(\partial_t \chi_\varepsilon + \partial_t \varphi^\ast)\]

\[= -2\int_\Omega \hat{\sigma}_\varepsilon(\chi_\varepsilon(t)) + 2\int_\Omega \hat{\sigma}_\varepsilon(\chi_0) - 2\int_{Q_t} \sigma_\varepsilon(\chi_\varepsilon) \partial_t \varphi^\ast\]

\[\leq 2p\int_\Omega |\chi_0| + 2p\int_Q |\partial_t \varphi^\ast| \leq c\rho.\]

Hence, by recalling the derivation of (5.3) and in view of (6.9), we have now the estimate
\[\|\varphi_\varepsilon\|_{L^\infty(0,T;V)} + \|\partial_t \varphi_\varepsilon\|_{L^2(0,T;H)} + \|\nabla \mu_\varepsilon\|_{L^2(0,T;H)} \leq c(1 + \rho^{1/2}).\] (6.10)

Thanks to the Poincaré inequality, we deduce that
\[\|\mu_\varepsilon\|_{L^2(0,T;V_0)} \leq c(1 + \rho^{1/2}).\] (6.11)

In particular, \(\mu_\varepsilon\) is bounded in \(L^2(0,T;H)\) by the same constant.

**Fifth a priori estimate.** We proceed formally for brevity (for the correct argument, one could follow the idea suggested at the end of the proof of Theorem 4.1, i.e., the discretization of the abstract equation (4.22) by a Faedo–Galerkin scheme, and then performing the proper estimates on the discrete solution). So, we formally differentiate (4.1)–(4.2) with respect to time and obtain the equations written below, which hold a.e. in \((0, T)\) and for every \(v \in V_0\) and every \(v \in V\), respectively.

\[\int_\Omega \partial_t^2 \varphi_\varepsilon v + \int_\Omega \nabla \partial_t \mu_\varepsilon \cdot \nabla v = 0\] (6.12)

\[\tau \int_\Omega \partial_t^2 \varphi_\varepsilon v + \int_\Omega \nabla \partial_t \varphi_\varepsilon \cdot \nabla v + \int_\Omega \beta'_\varepsilon(\varphi_\varepsilon) \partial_t \varphi_\varepsilon v + \int_\Omega \sigma_\varepsilon(\chi_\varepsilon) \partial_t \varphi_\varepsilon v
\]

\[= \int_\Omega (\partial_t \mu_\varepsilon + \partial_t g - \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) v.\] (6.13)

Then, we test the above equations written at the time \(s\) by \(\mathcal{D}(\partial_t \varphi_\varepsilon(s))\) and \(\partial_t \varphi_\varepsilon(s)\), respectively, we sum up and integrate with respect to \(s\) over \((0, t)\). Due to the definition (2.46) of \(\mathcal{D}\), a cancellation occurs. Hence, by recalling (2.48), we have

\[\frac{1}{2} \int_{Q_t} \|\partial_t \varphi_\varepsilon(t)\|^2 + \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon(t)|^2 + \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2
\]

\[+ \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon)|\partial_t \varphi_\varepsilon|^2 + \int_{Q_t} \sigma_\varepsilon(\chi_\varepsilon) |\partial_t \varphi_\varepsilon|^2
\]

\[= \frac{1}{2} \int_{Q_t} \|\partial_t \varphi_\varepsilon(0)\|^2 + \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon(0)|^2 + \int_{Q_t} (\partial_t g - \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \varphi_\varepsilon.\] (6.14)

The term involving \(\sigma_\varepsilon\) is treated this way

\[\int_{Q_t} \partial_t (\sigma_\varepsilon(\chi_\varepsilon)) \partial_t \varphi_\varepsilon = \int_{Q_t} \partial_t (\sigma_\varepsilon(\chi_\varepsilon)) \partial_t \chi_\varepsilon + \int_{Q_t} \partial_t (\sigma_\varepsilon(\chi_\varepsilon)) \partial_t \varphi^\ast
\]

\[= \int_{Q_t} \sigma'_\varepsilon(\chi_\varepsilon)|\partial_t \chi_\varepsilon|^2 - \int_{Q_t} \sigma_\varepsilon(\chi_\varepsilon) \partial_t^2 \varphi^\ast
\]

\[+ \int_{\Omega} \sigma_\varepsilon(\chi_\varepsilon(t)) \partial_t \varphi^\ast(t) - \int_{\Omega} \sigma_\varepsilon(\chi_0) \partial_t \varphi^\ast(0).\]
Hence, by substituting in (6.14) and then ignoring the nonnegative terms containing $\beta'_\varepsilon$ and $\sigma'_\varepsilon$, we obtain

$$
\frac{1}{2} \| \partial_t \varphi_\varepsilon (t) \|_2^2 + \frac{\tau}{2} \int_{Q_T} |\partial_t \varphi_\varepsilon (t)|^2 + \int_{Q_T} |\nabla \partial_t \varphi_\varepsilon|^2
\leq \frac{1}{2} \| \partial_t \varphi_\varepsilon (0) \|_2^2 + \frac{\tau}{2} \int_{Q_T} |\partial_t \varphi_\varepsilon (0)|^2 + \int_{Q_T} (\partial_t g - \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \varphi_\varepsilon
\leq \frac{1}{2} \| \partial_t \varphi_\varepsilon (0) \|_2^2 + \frac{\tau}{2} \int_{Q_T} |\partial_t \varphi_\varepsilon (0)|^2
+ \int_{Q_T} \sigma_\varepsilon (\varphi_\varepsilon) \partial_t \varphi_\varepsilon^2 - \int_{Q_T} \sigma_\varepsilon (\varphi_\varepsilon (t)) \partial_t \varphi_\varepsilon^2 (t) + \int_{Q_T} \sigma_\varepsilon (\varphi_\varepsilon (0)) \partial_t \varphi_\varepsilon (0).
$$

(6.15)

We just have to estimate the right-hand side. The initial value $\partial_t \varphi_\varepsilon (0)$ is read in equations (4.1)–(4.2) written at $t = 0$, i.e.,

$$
\int_{\Omega} \partial_t \varphi_\varepsilon (0) v + \int_{\Omega} \nabla \mu_\varepsilon (0) \cdot \nabla v = 0
$$

(6.16)

$$
\tau \int_{\Omega} \partial_t \varphi_\varepsilon (0) v + \int_{\Omega} \nabla \varphi_\varepsilon (0) \cdot \nabla v + \int_{\Omega} (\beta_\varepsilon (\varphi_\varepsilon (0)) + \pi (\varphi_\varepsilon (0))) v + \int_{\Omega} \sigma_\varepsilon (\varphi_\varepsilon (0)) v
= \int_{\Omega} (\mu_\varepsilon (0) + g (0)) v
$$

(6.17)

which hold for every $v \in V_0$ and every $v \in V$, respectively. To derive the needed estimate for $\varphi_\varepsilon (0)$, we test (6.16) and (6.17) by $D \partial_t \varphi_\varepsilon (0)$ and $\partial_t \varphi_\varepsilon (0)$, respectively. Then, we get rid of $\mu_\varepsilon (0)$ by adding the equalities we obtain to each other and owing to the cancellation that occurs due to the definition of $D$. We have

$$
\frac{1}{2} \| \partial_t \varphi_\varepsilon (0) \|_2^2 + \frac{\tau}{2} \int_{\Omega} |\partial_t \varphi_\varepsilon (0)|^2
= - \int_{\Omega} \nabla \varphi_\varepsilon (0) \cdot \nabla \partial_t \varphi_\varepsilon (0)
+ \int_{\Omega} (g (0) - \beta_\varepsilon (\varphi_\varepsilon (0)) - \pi (\varphi_\varepsilon (0)) - \rho \text{sign}_\varepsilon (\varphi_\varepsilon (0))) \partial_t \varphi_\varepsilon (0).
$$

As for the first term on the right-hand side, we recall that $\varphi_\varepsilon (0) \in W$ (see (2.30)) and have

$$
- \int_{\Omega} \nabla \varphi_\varepsilon (0) \cdot \nabla \partial_t \varphi_\varepsilon (0) = \int_{\Omega} \Delta \varphi_\varepsilon (0) \partial_t \varphi_\varepsilon (0)
$$

so that we can treat it with the Young inequality. The same inequality can be used for the second integral we have to estimate if we recall that

$$
0 \leq \beta_\varepsilon (\varphi_\varepsilon) \leq \beta^0 (\varphi_\varepsilon) \quad \text{and} \quad |\text{sign}_\varepsilon (\varphi_\varepsilon)| \leq 1 \quad \text{a.e. in} \ \Omega.
$$

Hence, the whole right-hand side is estimated from above by

$$
\frac{\tau}{4} \int_{\Omega} |\partial_t \varphi_\varepsilon (0)|^2 + \frac{1}{\tau} \left( 2 \| \Delta \varphi_\varepsilon | + | g (0) | + | \beta^0 (\varphi_\varepsilon) | + | \pi (\varphi_\varepsilon) | \|^2 + 2 \| \rho \text{sign}_\varepsilon (\varphi_\varepsilon) \|_H^2 \right)
\leq \frac{\tau}{4} \int_{\Omega} |\partial_t \varphi_\varepsilon (0)|^2 + c + \frac{2|\Omega|}{\tau} \rho^2
$$

and we conclude that

$$
\frac{1}{2} \| \partial_t \varphi_\varepsilon (0) \|_2^2 + \frac{\tau}{4} \| \partial_t \varphi_\varepsilon (0) \|_H^2 \leq c + \frac{2|\Omega|}{\tau} \rho^2.
$$

The next term on the right-hand side of (6.15) can be treated by means of (2.29) and (6.10), namely

$$
\int_{Q_T} \partial_t g \partial_t \varphi_\varepsilon - \int_{Q_T} \pi' (\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 \leq c \left( 1 + \| \partial_t \varphi_\varepsilon \|^2_{L^2 (0, T; H)} \right) \leq c (1 + \rho)
$$
and the last three terms of (6.15) can be trivially estimated by \( c \rho \) since \( |\sigma_r(r)| \leq \rho \) for every \( r \in \mathbb{R} \). Hence, by accounting for all the estimate we have obtained, we deduce from (6.15) that

\[
\|\partial_t \varphi_\varepsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C_1 |\Omega|^{1/2} \rho + c(\rho^{1/2} + 1) \leq C_2 |\Omega|^{1/2} \rho + c
\]

where \( C_1 \) is a constant that depends only on the structure and, e.g., \( C_2 := C_1 + 1 \).

**Sixth a priori estimate.** We can see (4.2) as a PDE (see (2.22) in Remark 1, which also applies to (4.2), of course). We write it at the time \( t \) (a.e. in \( (0,T) \)), multiply by \(-\Delta \varphi_\varepsilon(t)\) and integrate over \( \Omega \). However, for brevity, we avoid writing the time \( t \) for a while. We obtain

\[
\int_\Omega |\Delta \varphi_\varepsilon|^2 + \int_\Omega \beta'_\varepsilon(\varphi_\varepsilon)|\nabla \varphi_\varepsilon|^2 + \int_\Omega \sigma'_\varepsilon(\chi_\varepsilon) \nabla \chi_\varepsilon \cdot \nabla \varphi_\varepsilon
\]

\[
= \int_\Omega (\mu_\varepsilon + g - \tau \partial_t \varphi_\varepsilon - \pi(\varphi_\varepsilon))(-\Delta \varphi_\varepsilon).
\]

We treat the integral involving \( \sigma'_\varepsilon \) as follows

\[
\int_\Omega \sigma'_\varepsilon(\chi_\varepsilon) \nabla \chi_\varepsilon \cdot \nabla \varphi_\varepsilon = \int_\Omega \sigma'_\varepsilon(\chi_\varepsilon)|\nabla \chi_\varepsilon|^2 + \int_\Omega \nabla(\sigma'_\varepsilon(\chi_\varepsilon)) \cdot \nabla \varphi^*
\]

\[
\geq \int_\Omega \nabla(\sigma'_\varepsilon(\chi_\varepsilon)) \cdot \nabla \varphi^* = \int_\Omega \sigma'_\varepsilon(\chi_\varepsilon)(-\Delta \varphi^*) \geq -\rho \|\Delta \varphi^*\|_{L^\infty(0,T;L^1(\Omega))} = -c \rho
\]

the uniform summability of \( \Delta \varphi^* \) following from (2.31). We deduce for a.a. \( t \in (0,T) \)

\[
\frac{1}{2} \|\Delta \varphi_\varepsilon(t)\|^2_H \leq \frac{1}{2} \|\mu_\varepsilon(t) + g(t) - \tau \partial_t \varphi_\varepsilon(t) - \pi(\varphi_\varepsilon(t))\|^2_H + c \rho
\]

whence

\[
\|\Delta \varphi_\varepsilon(t)\|_H \leq \|\mu_\varepsilon(t) + g(t) - \tau \partial_t \varphi_\varepsilon(t) - \pi(\varphi_\varepsilon(t))\|_H + c \rho^{1/2}
\]

\[
\leq \|\mu_\varepsilon(t)\|_H + \|g(t)\|_H + \tau \|\partial_t \varphi_\varepsilon(t)\|_H + \|\pi(\varphi_\varepsilon(t))\|_H + c \rho^{1/2}.
\]

We aim to deduce two uniform bounds in \( L^\infty(Q) \). First, from (4.1), by adapting the argument of Remark 1, we have that \(-\Delta \mu_\varepsilon = \partial_t \varphi_\varepsilon \). Therefore, by (6.18) we infer that

\[
\|\Delta \mu_\varepsilon\|_{L^\infty(0,T;H)} \leq C_2 |\Omega|^{1/2} \rho + c
\]

so that the embedding inequality (2.33) yields

\[
\|\mu_\varepsilon\|_\infty \leq C_3 |\Omega|^{2/3} \rho + c
\]

with \( C_3 := C_\|C_2 \|. \) Now, by accounting for (6.19), (6.20), (2.29), (6.18) and the second estimate in (6.9), we infer that

\[
\|\Delta \varphi_\varepsilon\|_{L^\infty(0,T;H)} \leq |\Omega|^{1/2} \left[ C_5 |\Omega|^{2/3} + \tau C_2 \right] \rho + c(\rho^{1/2} + 1)
\]

\[
\leq C_4 |\Omega|^{1/2} \left( |\Omega|^{2/3} + 1 \right) \rho + c
\]

where, e.g., \( C_4 := \max\{C_3, \tau C_2 + 1\} \). Next, in the light of (6.21), the first condition in (6.9) and the embedding inequality (2.34), we conclude that

\[
\|\varphi_\varepsilon\|_\infty \leq C_5 |\Omega|^{2/3} \left( |\Omega|^{2/3} + 1 \right) \rho + c
\]

where, e.g., \( C_5 := C_\|C_4 \|. \) Since \( \pi \) is Lipschitz continuous, we deduce that

\[
\|\pi(\varphi_\varepsilon)\|_\infty \leq C_6 |\Omega|^{2/3} \left( |\Omega|^{2/3} + 1 \right) \rho + c
\]
with $C_6 := C_5 \sup |\pi'|$. At this point, if we set
\[ G_\varepsilon := \mu_\varepsilon + g - \pi(\phi_\varepsilon) - \tau \partial_t \phi^* - \Delta \phi^* \quad (6.23) \]
and recall the estimates \((6.20)\) and \((6.22)\) as well as our assumptions on \(g\) and \(\phi^*\), we conclude that
\[ ||G_\varepsilon||_\infty \leq C_{str}|\Omega|^{2/3} \left(|\Omega|^{2/3} + 1\right) \rho + \hat{C} \quad (6.24) \]
where \(C_{str} := C_3 + C_6\) only depends on the structure of the problem and the shape constant \(C_{sh}\) (see the construction of the previous \(C_i\)'s), while \(\hat{C}\) also depends on \(\Omega, T\) and the data \(g, \phi_0\) and \(\phi^*\).

As already announced, we show the existence of a sliding mode by a comparison argument. The function we use in this project is related to the solution to a Cauchy problem for an ordinary differential equation we study at once.

**An ordinary differential equation.** Given two real numbers \(M \in [0, \rho]\) and \(w_0 \geq 0\), we consider the problem of finding \(w_\varepsilon \in W^{1,\infty}(0, T)\) such that
\[ \tau w_\varepsilon'(t) + \rho \text{sign}_\varepsilon w_\varepsilon(t) = M \quad \text{for a.a. } t \in (0, T) \quad \text{and} \quad w_\varepsilon(0) = w_0. \quad (6.25) \]

First of all, since \(\text{sign}_\varepsilon\) is Lipschitz continuous, such a problem has a unique solution. In the (less interesting) case \(w_0 = 0\), the solution is given by
\[ w_\varepsilon(t) = \frac{\varepsilon M}{\rho} \left(1 - \exp \frac{-\rho t}{\varepsilon \tau}\right) \quad \text{for } t \in [0, T]. \quad (6.26) \]

Indeed, such a formula provides \(w_\varepsilon\) satisfying \(w_\varepsilon(0) = 0\) and \(\tau w_\varepsilon' + \rho w_\varepsilon/\varepsilon = M\).

Since \(0 \leq w_\varepsilon \leq \varepsilon M/\rho < \varepsilon\), we also have that \(\text{sign}_\varepsilon w_\varepsilon = w_\varepsilon/\varepsilon\) so that \((6.25)\) is fulfilled. Suppose now that \(w_0 > 0\). Then we can assume \(\varepsilon \in (0, w_0)\). We prove that
\[ 0 \leq w_\varepsilon \leq w_0 \quad \text{a.e. in } (0, T). \quad (6.27) \]

In order to derive the first inequality, we multiply the equation \((6.25)\) by \(-w_\varepsilon^-,\) where here and in the sequel \((\cdot)^-\) denotes the negative part (later on, we also use the symbol \((\cdot)^+\) for the positive part). Then, we integrate over \((0, t)\) and rearrange. Since \(w_0 \geq 0\), we obtain
\[ \frac{\tau}{2} |w_\varepsilon^-(t)|^2 - \rho \int_0^t (\text{sign}_\varepsilon w_\varepsilon(s)) w_\varepsilon^-(s) ds = -M \int_0^t w_\varepsilon^-(s) ds \leq 0. \quad (6.28) \]

On the other hand, \(\text{sign}_\varepsilon r \leq 0\) for \(r \leq 0\) so that the second term on the left-hand side is nonnegative. Thus \(w_\varepsilon^- = 0\), whence \(w_\varepsilon \geq 0\). In order to show the second inequality in \((6.27)\) we consider the open set \(P\) of points \(t \in (0, T)\) such that \(w_\varepsilon(t) > \varepsilon\). We have \(\text{sign}_\varepsilon w_\varepsilon = 1\) in \(P\) whence also \(\tau w_\varepsilon^+ = M - \rho < 0\). Hence, by recalling that \(P\) is a (finite or countably infinite) union of open intervals \(I_n\), we infer that the restriction of \(w_\varepsilon\) to each of them is a strictly decreasing affine function. On the other hand, one of these intervals, say \(I_1\), has 0 as an end-point since \(w_\varepsilon(0) = w_0 > \varepsilon\). Therefore, one can easily derive that \(P = I_1\), so that there are two possibilities. It might happen that \(w_\varepsilon \geq \varepsilon\) in the whole of \([0, T]\). In this case, \(w_\varepsilon\) is a strictly decreasing affine function. In the opposite case, \(w_\varepsilon\) is strictly decreasing till it reaches the value \(\varepsilon\) at some \(T_\varepsilon < T\) and then it remains under such a level. Thus, the second inequality of \((6.27)\) is established in any case. We notice that \(w_\varepsilon\) could be explicitly computed, but the calculation is not necessary.

Finally, we prove that \(w_\varepsilon\) converges as \(\varepsilon \searrow 0\) to the (unique) solution \(w\) to the following problem
\[ \tau w'(t) + \rho \text{sign} w(t) \ni M \quad \text{for a.a. } t \in (0, T) \quad \text{and} \quad w(0) = w_0. \quad (6.29) \]
If \( w_0 = 0 \), then \( 0 \leq w_\varepsilon \leq \varepsilon \), whence \( w_\varepsilon \) tends to zero uniformly. On the other hand, \( w = 0 \) solves (6.29) since \( M \in [0, \rho) \subset \rho \text{sign} 0 \). Suppose now that \( w_0 > 0 \). Then we can assume \( \varepsilon \in (0, w_0) \). We trivially have \( \tau |w'_\varepsilon| \leq M + \rho \). Moreover, \( w_\varepsilon(0) \) is independent of \( \varepsilon \). By also applying the Ascoli–Arzelà theorem, we deduce that

\[
\text{for some function } w \in W^{1,\infty}(0, T), \text{ in principle for a subsequence. Since } w_\varepsilon(0) \text{ converges to } w(0), \text{ we obtain } w(0) = w_0. \text{ Next, we recall that sign } \varepsilon \text{ is bounded, so that sign } \varepsilon w_\varepsilon \rightarrow \sigma \text{ weakly star in } L^\infty(0, T) \text{ for some } \sigma \in L^\infty(0, T) \text{ (once more for a subsequence, in principle), whence we immediately infer that } \tau w' + \rho \sigma = M \text{ a.e. in } (0, T). \text{ By applying, e.g., [1, Lemma 2.3, p. 38], we deduce that } \sigma \in \text{sign } w \text{ so that } w \text{ solves problem (6.29).}
\]

**Remark 9.** The function \( w \) can be explicitly computed. Namely, we have

\[
w(t) = \left(w_0 - \frac{\rho - M}{\tau} t\right)^+ \text{ for } t \in [0, T]. \tag{6.30}
\]

In particular, if we define the nonnegative number

\[
T^* := \frac{\tau w_0}{\rho - M} \tag{6.31}
\]

and we reinforce our assumption on \( \rho \) by assuming that

\[
\rho > M + \frac{\tau w_0}{T} \tag{6.32}
\]

then \( T^* < T \) and \( w(t) = 0 \) for every \( t \in [T^*, T] \).

**The comparison argument.** The function we use in our argument is the space independent function \((x, t) \mapsto w_\varepsilon(t)\) (still termed \( w_\varepsilon \) for simplicity) where \( w_\varepsilon \) is the solution to (6.25) with a proper choice of \( w_0 \) and \( M \). We recall that \( C_{str} \) and \( \tilde{C} \) are the constants that appear in (6.24). We stress once more that \( C_{str} \) only depends on the structure of the problem and the shape constant \( C_{sh} \). We assume that

\[
|\Omega| < \delta^* := \left| \sqrt{1 + 4/C_{str}} - 1 \right|^{3/2} \tag{6.33}
\]

so that \( C_{str} |\Omega|^{2/3} \left( |\Omega|^{2/3} + 1 \right) < 1 \). By also recalling (2.31), we see that the real number

\[
\rho^* := \frac{\tilde{C} + \beta^* + (\tau w_0)/T}{1 - C_{str} |\Omega|^{2/3} \left( |\Omega|^{2/3} + 1 \right)} \text{ where } \beta^* := \|\beta^o(\varphi^*)\|_\infty \tag{6.34}
\]

is well defined. At this point, we fix \( \rho \) by assuming that

\[
\rho > \rho^* \tag{6.35}
\]

and choose

\[
w_0 := \|\chi_0\|_\infty \text{ and } M := C_{str} |\Omega|^{2/3} \left( |\Omega|^{2/3} + 1 \right) \rho + \tilde{C} + \beta^*. \tag{6.36}
\]

Our assumptions and choices are made in order that \( \rho > M + (\tau w_0)/T \), whence in particular \( \rho > M \). Hence, the conditions assumed in the study of the solution \( w_\varepsilon \) performed above and in Remark 9 are fulfilled, and the time \( T^* \) given by (6.31) is well-defined and belongs to \((0, T)\). At this point, we can start our comparison
argument. We recall that \( \varphi^*(t) \in W \) for a.a. \( t \in (0, T) \) by (2.31) so that we can integrate by parts in the right-hand side of (6.3) and write the equation as

\[
\tau \int_\Omega \partial_t \chi \varepsilon v + \int_\Omega \nabla \chi \varepsilon \cdot \nabla v + \int_\Omega \beta(\chi \varepsilon + \varphi^*) v + \rho \int_\Omega \text{sign}_\varepsilon(\chi \varepsilon) v = \int_\Omega G \varepsilon v \\
\text{a.e. in } (0, T) \text{ and for every } v \in V
\] (6.37)

where \( G \varepsilon \) is given by (6.23). We recall (6.24), which provides a uniform bound for \( G \varepsilon \). On the other hand, by reading \( w \varepsilon \) as a space independent function defined in \( Q \) as said before, we can write the ordinary differential equation (6.25) as a partial differential equation with homogeneous Neumann boundary conditions. It is convenient to choose the following two forms

\[
\tau \int_\Omega \partial_t w \varepsilon v + \int_\Omega \nabla w \varepsilon \cdot \nabla v + \int_\Omega (\beta(\chi \varepsilon + \varphi^*) + \rho \text{sign}_\varepsilon w \varepsilon) v \\
= \int_\Omega (M + \beta(\chi \varepsilon + \varphi^*)) v \quad \text{a.e. in } (0, T) \text{ and for every } v \in V
\] (6.38)

\[
\tau \int_\Omega \partial_t w \varepsilon v + \int_\Omega \nabla w \varepsilon \cdot \nabla v + \int_\Omega (-\beta(-\chi \varepsilon + \varphi^*) - \rho \text{sign}_\varepsilon(-\chi \varepsilon)) v \\
= \int_\Omega (M - \beta(-\chi \varepsilon + \varphi^*)) v \quad \text{a.e. in } (0, T) \text{ and for every } v \in V.
\] (6.39)

We prove that \( |\chi \varepsilon| \leq w \varepsilon \) a.e. in \( Q \) by showing that

\[
\chi \varepsilon \leq w \varepsilon \quad \text{and} \quad -\chi \varepsilon \leq w \varepsilon \quad \text{a.e. in } Q.
\] (6.40)

To obtain the first inequality, we take the difference between (6.37) and (6.38) and choose \( v = (\chi \varepsilon - w \varepsilon)^+ \). Then, we integrate over \((0, t)\). We have

\[
\frac{\tau}{2} \int_\Omega (\chi \varepsilon - w \varepsilon)^+(t)^2 + \int_{Q_t} (\chi \varepsilon - w \varepsilon)^+^2 \\
+ \int_{Q_t} \{\beta(\chi \varepsilon + \varphi^*) - \beta(\chi \varepsilon + \varphi^*) + \rho \text{sign}_\varepsilon \chi \varepsilon - \rho \text{sign}_\varepsilon w \varepsilon\}(\chi \varepsilon - w \varepsilon)^+ \\
= \int_{Q_t} \{G \varepsilon - M - \beta(\chi \varepsilon + \varphi^*)\}(\chi \varepsilon - w \varepsilon)^+.
\]

Clearly, the expression between braces on the left-hand side is nonnegative in the set where \( \varepsilon \chi > \varepsilon \varepsilon \) so that the corresponding integral is nonnegative. On the other hand, since \( w \varepsilon \) is nonnegative and recalling the estimate (6.24) and the definitions of \( \beta^* \) and \( M \) (see (6.34) and (6.36)), we have

\[
G \varepsilon - M - \beta(\chi \varepsilon + \varphi^*) \leq \|G \varepsilon\|_\infty - M - \beta(\varphi^*) \\
\leq \|G \varepsilon\|_\infty - M + |\beta^*(\varphi^*)| \leq 0.
\]

Hence, \( (\chi \varepsilon - w \varepsilon)^+ = 0 \) and the desired inequality is proved. To obtain the other one, we add equations (6.37) and (6.39) to each other and test the equality we get...
by $-(\chi_\varepsilon + w_\varepsilon)^-$. Then, we integrate over $(0,t)$. We have

$$\frac{T}{2} \int_\Omega \vert (\chi_\varepsilon + w_\varepsilon)^-(t) \vert^2 + \int_{Q_t} \vert \nabla (\chi_\varepsilon + w_\varepsilon)^- \vert^2$$

$$- \int_{Q_t} \{ \beta_\varepsilon (\chi_\varepsilon + \varphi^*) - \beta_\varepsilon (-w_\varepsilon + \varphi^*) + \rho \text{sign}_\varepsilon \chi_\varepsilon - \rho \text{sign}_\varepsilon (-w_\varepsilon) \} (\chi_\varepsilon + w_\varepsilon)^-$$

$$= - \int_{Q_t} \{ G_\varepsilon + M - \beta_\varepsilon (-w_\varepsilon + \varphi^*) \} (\chi_\varepsilon + w_\varepsilon)^-. $$

In the set where $\chi_\varepsilon + w_\varepsilon$ is negative, we have $\chi_\varepsilon < -w_\varepsilon$ so that the expression between braces on the left-hand side is nonpositive and the corresponding integral is nonpositive. On the other hand, we have

$G_\varepsilon + M - \beta_\varepsilon (-w_\varepsilon + \varphi^*) \geq -\|G_\varepsilon\|_\infty + M - \beta_\varepsilon (\varphi^*) \geq -\|G_\varepsilon\|_\infty + M - \|\beta_\circ (\varphi^*)\| \geq 0.$

Hence $(\chi_\varepsilon + w_\varepsilon)^- = 0$ and (6.40) is completely proved. Since $\chi_\varepsilon$ and $w_\varepsilon$ converge to $\chi$ and $w$, respectively, where $w$ is the solution to (6.29), we deduce that

$$|\chi| \leq w \quad \text{a.e. in } Q.$$ 

As already noticed, we can apply Remark 9. Hence, $T^* < T$ and $\chi(t) = 0$ for every $t \in [T^*, T]$, i.e., $\varphi(t) = \varphi^*(t)$ for every $t \in [T^*, T]$.

**Remark 10.** In terms of the original physical variables, i.e., the order parameter $\varphi$ and the associated chemical potential $\mu$ satisfying the inhomogeneous Dirichlet boundary condition, the behavior of the solution after the time $T^*$ is the following:

$$\varphi(t) = \varphi^*(t) \quad \text{for every } t \in [T^*, T]$$

$$-\Delta \mu(t) = -\partial_t \varphi^*(t) \quad \text{and} \quad \mu(t)|_\Gamma = \mu_\Gamma(t) \quad \text{for a.a. } t \in (T^*, T).$$

In particular, even $\mu$ can be explicitly computed on $(T^*, T)$.

**Remark 11.** Due to our choice (6.36) of $M$, the difference $\rho - M$ is almost proportional to $\rho$ for large values of it. It follows that the time $T^*$ given by (6.31) tends to zero as $\rho$ tends to infinity. Therefore, the sliding mode can be imposed to occur in an arbitrarily short time by assuming that $\rho$ is large enough.

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