The universal R matrix for the Jordanian deformation of \( \text{sl}(2) \), and the contracted forms of \( \text{so}(4) \)

A. Shariati \(^{1,2,*} \), A. Aghamohammadi \(^{1,3} \), M. Khorrami \(^{1,2,4} \),

\(^{1} \) Institute for Studies in Theoretical Physics and Mathematics, P.O.Box 5746, Tehran 19395, Iran.
\(^{2} \) Institute for Advanced Studies in Basic Sciences, P.O.Box 159, Gava Zang, Zanjan 45195, Iran.
\(^{3} \) Department of Physics, Alzahra University, Tehran 19834, Iran.
\(^{4} \) Department of Physics, Tehran University, North-Kargar Ave. Tehran, Iran.
\(^* \) E-Mail: shahram@irearn.bitnet

Abstract

We introduce a universal R matrix for the Jordanian deformation of \( \text{U}(\text{sl}(2)) \). Using \( \text{U}_h(\text{so}(4)) = \text{U}_h(\text{sl}(2)) \oplus \text{U}^{-h}(\text{sl}(2)) \), we obtain the universal R matrix for \( \text{U}_h(\text{so}(4)) \). Applying the graded contractions on the universal R matrix of \( \text{U}_h(\text{so}(4)) \), we show that there exist three distinct R matrices for all of the contracted algebras. It is shown that \( \text{U}_h(\text{sl}(2)) \), \( \text{U}_h(\text{so}(4)) \), and all of these contracted algebras are triangular.

1 Introduction

The group \( \text{SL}(2) \) admits two distinct quantizations, one is the well known Drinfeld-Jimbo \((q-)\) deformation, and the other is the so called Jordanian \((h-)\) deformation \[^1,2\]. In fact the \( h\)-deformation itself can be obtained by a contraction procedure from the \( q\)- deformation \[^3\]. So far, however, there has been no expression for the Universal R matrix of \( \text{U}_h(\text{sl}(2)) \). It is worthy of mention that the R matrix which was introduced in \[^2\] does not satisfy the quantum Yang–Baxter equation \[^1\]. The universal R matrix for the positive Borel subalgebra of the Jordanian \( \text{sl}(2) \) was introduced in \[^4\].

Another interesting problem is the study of non semisimple quantum groups. One of the techniques for constructing inhomogeneous quantum groups is contraction. There are two distinct deformations of \( \text{U}(\text{e}(2)) \) both of which can be obtained by contraction of \( \text{U}_q(\text{su}(2)) \), neither of them has a universal R matrix \[^4,5,6\]. There is also a deformation of the two dimensional Poincaré algebra \( \text{U}_h(\text{p}(2)) \) which was obtained by a contraction of \( \text{U}_h(\text{sl}(2,\mathbb{R})) \) and has a universal R matrix \[^7\]. In \[^8\] two copies of the Jordanian deformation of \( \text{sl}(2) \) have been used to construct the deformed algebra of \( \text{so}(4) \). Then, the process of graded contraction \[^9\] has been used to construct a deformation for a fairly large class of non semisimple algebras.

In this article, we first introduce an expression for the universal R matrix of \( \text{U}_h(\text{sl}(2)) \), and show that this algebra is triangular \[^12\]. In fact we prove that the universal R matrix obtained in \[^1\] for the positive Borel subalgebra of the Jordanian \( \text{sl}(2) \), is the universal R matrix for the whole of \( \text{U}_h(\text{sl}(2)) \). Then we complete the study of \[^11\], that is, we list all possible contractions of \( \text{U}_h(\text{so}(4)) \) and, using the
R matrix of $U_h(\mathfrak{sl}(2))$, we obtain the R matrices of $U_h(\mathfrak{so}(4))$, and all its contracted algebras. As we will see, there are three distinct R matrices for all of the contracted algebras. It is also seen that $U_h(\mathfrak{so}(4))$, and all of these contracted algebras are triangular.

2 The Universal R matrix

A quasitriangular Hopf algebra is a Hopf algebra with a universal R matrix satisfying

$$(\Delta \otimes 1)R = R_{13}R_{23}, \quad (1 \otimes \Delta)R = R_{12}R_{13},$$

and

$$R\Delta(\cdot)R^{-1} = \Delta'(\cdot) := \sigma \circ \Delta(\cdot).$$

where $\sigma$ is the flip map: $\sigma(a \otimes b) = b \otimes a$. If in addition

$$\sigma(R^{-1}) = R,$$

the Hopf algebra is called triangular [12].

The Jordanian deformation of $\mathfrak{sl}(2)$ is defined [2] through

$$[J^3, J^+] = 2\sinh(hJ^+) J^+ - \sinh(hJ^+) J^3$$

and

$$(\Delta J^+)^i = J^i \otimes e^{hJ^+} + e^{-hJ^+} \otimes J^i$$

$$\epsilon(X) = 0$$

$$\gamma(X) = -e^{hJ^+} X e^{-hJ^+}$$

where $i = -, 3$, and $X \in \{J^+, J^-, J^3\}$. Vladimirov [4] has considered the subalgebra of $U_h(\mathfrak{sl}(2))$ generated by the two generators $J^+$ and $J^3$ and has found the following universal R matrix for this subalgebra,

$$R = \exp \left\{ \frac{\Delta(hJ^+)}{\sinh(\Delta(hJ^+))} [J^3 \otimes \sinh(hJ^+) - \sinh(hJ^+) \otimes J^3] \right\}.$$  

This means that $R$ satisfies (1) and also,

$$R\Delta J^+ R^{-1} = \Delta' J^+, \quad R\Delta J^3 R^{-1} = \Delta' J^3.$$  

We are going to show that the above expression for $R$ is, in fact, the universal R matrix for the whole algebra $U_h(\mathfrak{sl}(2))$. To do so, we must show that

$$R\Delta J^- R^{-1} = \Delta' J^-.$$  

Now we define $E$ through

$$R\Delta J^- R^{-1} := \Delta' J^- + E.$$  

To prove (10), is equivalent to prove $E = 0$. From the commutation relations, it is seen that the power of $J^-$ in the right hand side of (10) does not exceed one. So, using Hausdorff identity it can be shown that

$$E = (J^- \otimes 1)C(J^+_1, J^+_2, J^3_1, J^3_2) + (1 \otimes J^-)D(J^+_1, J^+_2, J^3_1, J^3_2) + F(J^+_1, J^+_2, J^3_1, J^3_2).$$  

where $A_1 := A \otimes 1$, and $A_2 := 1 \otimes A$. The first step is to show that $C = D = 0$. To do so, we use the fact that the matrix (8) is a universal R matrix for the contracted form of $U_h(\mathfrak{sl}(2))$ [3]. The contraction procedure can be achieved through the definitions

$$P^- := \epsilon J^-, \quad P^+ := J^+, \quad J^3 := J^3.$$
Note that, in this basis it is not necessary to redefine both \( J^- \) and \( J^+ \). This appears slightly different from the procedure which we introduced in \( \text{(13)} \), but it is easy to see that the resulting Hopf algebra is the same.

Now we use the above redefinitions in \( \text{(14)} \). Note that, as \( R \) is a function of only \( J^+ \) and \( J^3 \), it does not change in this procedure. The result is that

\[
R \Delta P^- R^{-1} := \Delta' P^- + \lim_{\epsilon \to 0} \epsilon E. \tag{12}
\]

But in \( \text{(13)} \) we have shown that

\[
R \Delta P^- R^{-1} := \Delta' P^- . \tag{13}
\]

So, it is deduced that \( \lim_{\epsilon \to 0} \epsilon E = 0 \), in which we have to express the relation in terms of the new generators \( P^\pm \) and \( J^3 \). This results in

\[
(P^- \otimes 1)C + (1 \otimes P^-)D = 0 , \tag{14}
\]

which means that \( C = D = 0 \). We have ruled out the \( J^- \) dependence in \( E \). Now we can rewrite \( E \) as

\[
E = \sum_{m,n \geq 0} (J^3 \otimes 1)^m B^n g_{m,n}(J^+_1, J^+_2), \tag{15}
\]

where

\[
B := \frac{1}{2}[J^3 \otimes \sinh(hJ^+) - \sinh(hJ^+) \otimes J^3]. \tag{16}
\]

Note that the commutation relations permit us to write \( E \) as \( \text{(15)} \). Using the commutation relations of \( J^+ \) and \( J^3 \) with \( J^- \), the fact that \( \Delta \) and \( \Delta' \) are homomorphisms of the algebra, and that \( \Delta' J^+ = \Delta J^+ \), one can see that

\[
[\Delta J^+, E] = 0, \tag{17}
\]

and

\[
[\Delta' J^3, E] = -\{ E \cosh(h \Delta J^+) + \cosh(h \Delta J^+) E \}. \tag{18}
\]

From \( \text{(17)} \) and \( [\Delta J^+, B] = 0 \), it is seen that \( g_{m,n} = 0 \) if \( m > 0 \). So,

\[
E = \sum_{n \geq 0} B^n g_{n}(J^+_1, J^+_2). \tag{19}
\]

Now, \( E \) is an analytic function of \( h \). Suppose that the smallest power of \( h \) in \( E \) is \( m \): \( E = h^m E_m + O(h^{m+1}) \). Then we can deduce from \( \text{(18)} \) that

\[
\lim_{h \to 0} [\Delta' J^3, h^{-m} E] = -\lim_{h \to 0} \{ h^{-m} E \cosh(h \Delta J^+) + \cosh(h \Delta J^+) h^{-m} E \}, \tag{20}
\]

or,

\[
\lim_{h \to 0} [1 \otimes J^3 + J^3 \otimes 1, E_m] = -2 E_m. \tag{21}
\]

From \( \text{(19)} \), it is seen that

\[
E_m = \sum_{n \geq 0} \tilde{B}^n \tilde{g}_{n}(J^+_1, J^+_2), \tag{22}
\]

where \( \tilde{B} := \frac{1}{2} J^3 \otimes J^+ - J^+ \otimes J^3 \), and \( \tilde{g}_{n} := \lim_{h \to 0} h^{n-m} g_{n} \). It is easy to see that

\[
\lim_{h \to 0} [1 \otimes J^3 + J^3 \otimes 1, \tilde{g}_{n}(J^+_1, J^+_2)] = 2(J^+_1 \frac{\partial}{\partial J^+_1} + J^+_2 \frac{\partial}{\partial J^+_2}) \tilde{g}_{n}(J^+_1, J^+_2). \tag{23}
\]

One can then write \( \text{(21)} \) as

\[
\sum_{n} \tilde{B}^n (n + 1 + J^+_1 \frac{\partial}{\partial J^+_1} + J^+_2 \frac{\partial}{\partial J^+_2}) \tilde{g}_{n} = 0, \tag{24}
\]
which gives
\[ (n + 1 + J^+_1 \frac{\partial}{\partial J^+_1} + J^+_2 \frac{\partial}{\partial J^+_2}) \tilde{g}_n = 0. \] (25)

This means that \( \tilde{g}_n \) is a homogeneous function of order \(-(n + 1)\) of \( J^+_1 \) and \( J^+_2 \). From this it is concluded that \( \tilde{B}^n \tilde{g}_n \) is a homogeneous function of order \(-1\) of \( J^+_1 \) and \( J^+_2 \), which means that \( E_m \) is such a function. But this is impossible for \( E_m \neq 0 \), because \( E \), and hence \( E_m \), is an analytic function of \( J^+_1 \) and \( J^+_2 \) (This can be deduced from the analyticity of \( R \) and the commutation relations), and there exist no analytic function which is homogeneous of a negative order. So, \( E_m \) should be zero. This means that \( E \) is zero, because we assumed that the lowest order term of \( E \) is \( h^m E_m \).

So (30) is correct. This completes the proof. It is worthy of mention that (31) can also be written in the form
\[ R = \exp \left\{ \frac{\Delta - \Delta'}{2} \left[ \frac{hJ^+}{\sinh hJ^+} \right] \right\}. \] (26)

As \( R \) is of the form \( R = \exp[(\Delta - \Delta')X] \), it is obvious that the algebra \( U_h(\text{sl}(2)) \) is triangular. In fact, as it will be seen, the \( R \) matrix of \( U_h(\text{so}(4)) \) and all its contractions which we consider, have this property. So, all of these algebras are triangular.

In ref. [1], starting from the Jordanian deformation \( U_h(\text{sl}(2)) \), and using
\[ U_h(\text{so}(4)) = U_h(\text{sl}(2)) \oplus U_{-h}(\text{sl}(2)), \] (27)

\( U_h(\text{so}(4)) \) has been constructed. Consider \( U_h(\text{sl}(2)) \) with the generators \( \{J^0_1, J^1_1\} \), and \( U_{-h}(\text{sl}(2)) \) with the generators \( \{J^0_2, J^1_2\} \). The set of generators \( \{J^3, J^\pm, N^3, N^\pm\} \) defined by
\[ J^i := J^i_1 + J^i_2 \quad N^i := N^i_1 + N^i_2, \quad i = +, -, 3 \] (28)
closes the Hopf algebra \( U_h(\text{so}(4)) \), with the following Hopf structure.

\[
\begin{align*}
[J^3, J^\pm] &= \pm \frac{h}{4} \sinh(\frac{h}{4} J^+) \cosh(\frac{h}{4} N^+) \\
[J^3, J^-] &= -J^- \cosh(\frac{h}{4} J^+) \cosh(\frac{h}{4} N^+) - \cosh(\frac{h}{4} J^+) \cosh(\frac{h}{4} N^+) J^- \\
&\quad -N^- \sinh(\frac{h}{4} J^+) \sinh(\frac{h}{4} N^+) - \sinh(\frac{h}{4} J^+) \sinh(\frac{h}{4} N^+) N^- \\
[J^3, N^+] &= \pm \frac{h}{4} \sinh(\frac{h}{4} N^+) \cosh(\frac{h}{4} J^+) \\
[J^3, N^-] &= -N^- \cosh(\frac{h}{4} J^+) \cosh(\frac{h}{4} N^+) - \cosh(\frac{h}{4} J^+) \cosh(\frac{h}{4} N^+) N^- \\
&\quad -J^- \sinh(\frac{h}{4} J^+) \sinh(\frac{h}{4} N^+) - \sinh(\frac{h}{4} J^+) \sinh(\frac{h}{4} N^+) J^- \\
[N^3, J^\pm] &= [J^3, J^\pm], \\
[N^3, J^+] &= [J^3, N^+] \\
[J^+, J^-] &= [N^+, N^-] = J^3, \quad [J^+, N^\pm] = \pm N^3, \quad [J^i, N^i] = 0 \quad \text{where } i = +, -, 3
\end{align*}
\] (29)

and,
\[
\begin{align*}
\Delta J^+ &= 1 \otimes J^+ + J^+ \otimes 1 \\
\Delta N^+ &= 1 \otimes N^+ + N^+ \otimes 1 \\
\Delta J^i &= e^{-\frac{X}{2} N^+} \cos(h J^+ \otimes J^i + J^i \otimes \cos(h J^+) e^{\frac{X}{2} N^+} \\
&\quad - e^{-\frac{X}{2} N^+} \sinh(\frac{h}{2} J^+) \otimes N^i + N^i \otimes \sinh(\frac{h}{2} J^+) e^{\frac{X}{2} N^+} \\
\Delta N^i &= e^{-\frac{X}{2} N^+} \cosh(\frac{h}{2} J^+) \otimes N^i + N^i \otimes \cosh(\frac{h}{2} J^+) e^{\frac{X}{2} N^+} \\
&\quad - e^{-\frac{X}{2} N^+} \sinh(\frac{h}{2} J^+) \otimes J^i + J^i \otimes \sinh(\frac{h}{2} J^+) e^{\frac{X}{2} N^+} \quad \text{where } i = 3, \\
\epsilon(X) &= 0, \quad \gamma(X) = -e^{hN^+} X e^{-hN^+}, \quad \text{where } X \in \{J^3, J^\pm, N^3, N^\pm\}.
\end{align*}
\] (30)

This algebra has a universal \( R \) matrix which is simply the product of the two copies of (31). The resulting \( R \) matrix, in terms of the new generators, is
\[
R = \exp \left\{ \frac{\hbar}{2}(\Delta - \Delta') \left[ \frac{(J^3 J^+ + N^3 N^+) \sinh \frac{hJ^+}{2} \cosh \frac{hN^+}{2} - (J^3 N^+ + N^3 J^+) \sinh \frac{hN^+}{2} \cosh \frac{hJ^+}{2}}{\cosh hJ^+ - \cosh hN^+} \right] \right\}.
\] (31)
The definitions of the classical gradations and contractions have been extended to the quantum case, by assuming that they act on the algebra of the generators as in the classical case, and that $e^{-\frac{\mu}{\hbar}N^+}$ is invariant under the quantum gradations and contractions [10].

$$(J^3, J^\pm, N^\pm, N^3, h) = (\hat{J}^3, \frac{\hat{J}^\pm}{\sqrt{\mu_2\mu_3}}, \frac{\hat{N}^\pm}{\sqrt{\mu_1\mu_3}}, \frac{\hat{N}^3}{\mu_1\mu_2\hbar}, \sqrt{\mu_1\mu_2\hbar}), \quad (32)$$

where $\mu_i$ fall into one of the values $\pm 1, 0$. (We use complex algebras, so $\mu_1$ is 1, or 0.) As it is shown in [10], for $\mu_3 \to 0$, the resulting Hopf algebra is not well defined. However, one can modify the above contraction such that it is well behaved for $\mu_3 \to 0$ too. The contraction is as follows.

$$(J^3, J^\pm, N^\pm, N^3, h) = (\hat{J}^3, \frac{\hat{J}^\pm}{\sqrt{\mu_2\mu_3}}, \frac{\hat{N}^\pm}{\sqrt{\mu_1\mu_3}}, \frac{\hat{N}^3}{\mu_3\mu_1\mu_2\hbar}, \mu_3\mu_1\mu_2\hbar), \quad (33)$$

Applying this contraction to (29,30), everything remains well behaved. The same is true for the universal R matrix. In this way one obtains three distinct R matrices which we present here. The full Hopf structure of the contracted algebras are given in the appendix.

1. For $\mu_3 = 0$; the algebras become classic (non deformed), although the coproducts remain deformed, and

$$R = \exp\left(\frac{\Delta - \Delta'}{2} \hat{J}^3\right). \quad (34)$$

2. For $\mu_3 = 1, \mu_1 = 0$,

$$R = \exp\left\{\frac{\hbar}{2}(\Delta - \Delta')\left\{\frac{\hbar^2 \hat{N}^3 \hat{N}^+}{2} - (\hat{J}^3 \hat{N}^+ + \hat{N}^3 \hat{J}^+) \sinh\left(\frac{\hbar \hat{N}^+}{2}\right)\right\}\right\} \quad (35)$$

3. For $\mu_3 = \mu_1 = 1, \mu_2 = 0$,

$$R = \exp\left\{\frac{\hbar}{2}(\Delta - \Delta')\left\{\frac{(\hat{J}^3 \hat{J}^+ + \hat{N}^3 \hat{N}^+)}{2} \cosh\left(\frac{\hbar \hat{J}^+}{2}\right) - (\hat{J}^3 \hat{N}^+ + \hat{N}^3 \hat{J}^+) \sinh\left(\frac{\hbar \hat{N}^+}{2}\right) \cosh\left(\frac{\hbar \hat{J}^+}{2}\right)\right\}\right\} \quad (36)$$

In fact the last R matrix is the R matrix for $U_\hbar(iso(2)) \oplus U_{-\hbar}(iso(2))$, and it is simply the product of two copies of the universal R matrix of the deformed $U(iso(2))$. Note that, by $\mu_i = 0$, it is meant that one must set $\mu_i = \epsilon$, and obtain the Hopf structure in the limit $\epsilon \to 0$.

3 Appendix

Here we present the full Hopf structure of the contracted algebras; that is the commutation relationships, and the nontrivial coproducts and antipodes. Throughout this appendix, $i = +,-,3$, $j = -,3$, and $X \in \{\hat{J}^3, \hat{J}^\pm, \hat{N}^3, \hat{N}^\pm\}$.

1. $(\mu_1, \mu_2, \mu_3) = (1,1,0)$

$$\begin{align*}
[\hat{J}^3, \hat{J}^\pm] &= \pm 2\hat{J}^\pm \\
[\hat{N}^3, \hat{N}^\pm] &= \pm 2\hat{N}^\pm \\
[\hat{J}^3, \hat{N}^+ + \hat{N}^3] &= 0 \\
[\hat{J}^3, \hat{N}^+ - \hat{N}^3] &= 0 \\
[\hat{J}^\pm, \hat{N}^\mp] &= \pm \hat{N}^3 \\
[\hat{J}^3, \hat{N}^3] &= 0 \\
\Delta \hat{J}^3 &= 1 \otimes \hat{J}^3 + \hat{J}^3 \otimes 1 + \frac{\hbar}{2}(\hat{N}^3 \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{N}^3) \\
\Delta \hat{N}^+ &= 1 \otimes \hat{N}^+ + \hat{N}^+ \otimes 1 + \frac{\hbar}{2}(\hat{J}^- \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{J}^-)
\end{align*}$$

This is a deformation of $U(iso(3))$. 

5
2. \((\mu_1, \mu_2, \mu_3) = (1, 0, 0)\)
\[
\begin{align*}
[\hat{J}^3, \hat{J}^\pm] &= \pm \hat{J}^\pm \\
[\hat{N}^3, \hat{N}^\pm] &= \pm \hat{J}^\pm \\
[\hat{J}^+, \hat{J}^-] &= [\hat{N}^+, \hat{N}^-] = [\hat{J}^\pm, \hat{N}^\mp] = 0 \\
\end{align*}
\]
\[
\Delta \hat{J}^3 = 1 \otimes \hat{J}^3 + \hat{J}^3 \otimes 1 + \frac{\hbar}{2} \left( \hat{N}^3 \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{N}^3 \right)
\]
\[
\Delta \hat{N}^- = 1 \otimes \hat{N}^- + \hat{N}^- \otimes 1 + \frac{\hbar}{2} \left( \hat{J}^- \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{J}^- \right)
\]
This is a deformation of \(U(\text{iso}(2))\).

3. \((\mu_1, \mu_2, \mu_3) = (0, 1, 0)\)
\[
\begin{align*}
[\hat{J}^3, \hat{J}^\pm] &= \pm 2\hat{J}^\pm \\
[\hat{N}^3, \hat{N}^\pm] &= [\hat{N}^3, \hat{J}^\pm] = 0 \\
[\hat{J}^\pm, \hat{N}^\mp] &= \hat{N}^3 \\
[\hat{J}^+, \hat{J}^-] &= [\hat{N}^+, \hat{N}^-] = 0 \\
\end{align*}
\]
\[
\Delta \hat{J}^3 = 1 \otimes \hat{J}^3 + \hat{J}^3 \otimes 1 + \frac{\hbar}{2} \left( \hat{N}^3 \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{N}^3 \right)
\]
This is a deformation of \(U(\text{i'}\text{iso}(2))\).

4. \((\mu_1, \mu_2, \mu_3) = (0, 0, 0)\)
\[
\begin{align*}
[\hat{J}^3, \hat{J}^\pm] &= \pm 2\hat{J}^\pm \\
[\hat{N}^3, \hat{N}^\pm] &= [\hat{N}^3, \hat{J}^\pm] = 0 \\
[\hat{J}^\pm, \hat{N}^\mp] &= \hat{N}^3 \\
[\hat{J}^+, \hat{J}^-] &= [\hat{N}^+, \hat{N}^-] = 0 \\
\end{align*}
\]
\[
\Delta \hat{J}^3 = 1 \otimes \hat{J}^3 + \hat{J}^3 \otimes 1 + \frac{\hbar}{2} \left( \hat{N}^3 \otimes \hat{J}^+ - \hat{J}^+ \otimes \hat{N}^3 \right)
\]
This is a deformation of \(U(\text{R} \oplus (\text{R}^4 \otimes \text{so}(2)))\).

5. \((\mu_1, \mu_2, \mu_3) = (0, 1, 1)\)
\[
\begin{align*}
[\hat{J}^3, \hat{J}^+] &= 2 \hat{J}^+ \cosh(\frac{\hbar}{2} \hat{N}^+) \\
[\hat{J}^3, \hat{J}^-] &= - \hat{J}^- \cosh(\frac{\hbar}{2} \hat{N}^+) - \cosh(\frac{\hbar}{2} \hat{N}^+ \hat{J}^- - \frac{\hbar}{2} \hat{N}^- \hat{J}^+ \sinh(\frac{\hbar}{2} \hat{N}^+ )) + \hat{J}^+ \sinh(\frac{\hbar}{2} \hat{N}^+ \hat{N}^- ) \\
[\hat{J}^3, \hat{N}^+] &= \frac{\hbar}{2} \sinh(\frac{\hbar}{2} \hat{N}^+ ) \\
[\hat{J}^3, \hat{N}^-] &= - \hat{N}^- \cosh(\frac{\hbar}{2} \hat{N}^+ ) - \cosh(\frac{\hbar}{2} \hat{N}^+ \hat{N}^-) \hat{N}^- \\
[\hat{N}^3, \hat{J}^\pm] &= 0 \\
[\hat{N}^3, \hat{J}^\pm] &= [\hat{J}^3, \hat{N}^\pm] \\
[\hat{J}^+, \hat{J}^-] &= \hat{J}^3, \\
[\hat{N}^+, \hat{N}^-] &= 0 \\
[\hat{J}^\pm, \hat{N}^\mp] &= \pm \hat{N}^3, \\
\end{align*}
\]
\[
\begin{align*}
\Delta \hat{J}^3 &= e^{-\frac{\hbar}{2} \hat{N}^+} \hat{J}^+ \otimes \hat{J}^3 + \hat{J}^3 \otimes e^{\frac{\hbar}{2} \hat{N}^+} - \frac{\hbar}{2} (e^{-\frac{\hbar}{2} \hat{N}^+} \hat{J}^+ \otimes \hat{N}^- - \hat{N}^+ \otimes \hat{J}^3 e^{\frac{\hbar}{2} \hat{N}^+}) \\
\Delta \hat{N}^j &= e^{-\frac{\hbar}{2} \hat{N}^+} \hat{J}^j \otimes \hat{N}^j + \hat{N}^j \otimes e^{\frac{\hbar}{2} \hat{N}^+} \\
\gamma(X) &= - e^{\hbar \hat{N}^+} X e^{-\hbar \hat{N}^+} \quad \text{where} \quad j = -, 3
\end{align*}
\]
This is a deformation of \(U(\text{iso}(3))\).
6. \((\mu_1, \mu_2, \mu_3) = (0, 0, 1)\)

\[
\begin{align*}
[\hat{J}^3, \hat{J}^+] &= \frac{4}{h} \sinh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) \\
[\hat{J}^3, \hat{J}^-] &= -\hat{J}^- \cosh(\frac{h}{2} \hat{N}^+) - \cosh(\frac{h}{2} \hat{N}^+) \hat{J}^- - \frac{h}{2} \hat{\hat{N}}^- \sinh(\frac{h}{2} \hat{N}^+) \hat{\hat{J}}^+ + \hat{J}^+ \sinh(\frac{h}{2} \hat{N}^+) \hat{\hat{N}}^- \\
[\hat{J}^3, \hat{N}^+] &= \frac{4}{h} \sinh(\frac{h}{2} \hat{N}^+) \cosh(\frac{h}{2} \hat{J}^+) \\
[\hat{J}^3, \hat{N}^-] &= -\hat{\hat{N}}^- \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) - \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) \hat{\hat{N}}^- \\
\hat{\hat{N}}^3, \hat{\hat{N}}^\pm &= [\hat{J}^3, \hat{J}^\pm], \quad [\hat{\hat{N}}^3, \hat{\hat{J}}^\pm] = [\hat{J}^3, \hat{\hat{N}}^\pm] \\
[\hat{J}^+, \hat{J}^-] &= [\hat{\hat{N}}^+, \hat{\hat{N}}^-] = [\hat{J}^+, \hat{\hat{N}}^-] = 0
\end{align*}
\]

\[\Delta \hat{J}^j = e^{-\frac{h}{2} \hat{N}^+} \hat{J}^j + \hat{J}^j \otimes e^{-\frac{h}{2} \hat{N}^+} \hat{J}^j - \frac{h}{2} (e^{-\frac{h}{2} \hat{N}^+} \hat{J}^j \otimes \hat{\hat{N}}^j - \hat{\hat{N}}^j \otimes \hat{J}^j + e^{\frac{h}{2} \hat{N}^+} \hat{J}^j)\]
\[\Delta \hat{\hat{N}}^j = e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j + \hat{\hat{N}}^j \otimes e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j - \frac{h}{2} (e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j \otimes \hat{J}^j + \hat{J}^j \otimes e^{\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j)\quad \text{where } j = -3\]
\[\gamma(X) = -e^{h \hat{N}^+} X e^{-h \hat{N}^+}.\]

This is a deformation of \(U(\text{iso}(2))\).

7. \((\mu_1, \mu_2, \mu_3) = (1, 0, 1)\)

\[
\begin{align*}
[\hat{J}^3, \hat{J}^+] &= \frac{4}{h} \sinh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) \\
[\hat{J}^3, \hat{J}^-] &= -\hat{J}^- \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) - \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) \hat{J}^- \\
&\quad - \hat{\hat{N}}^- \sinh(\frac{h}{2} \hat{J}^+) \sinh(\frac{h}{2} \hat{N}^+) - \sinh(\frac{h}{2} \hat{J}^+) \sinh(\frac{h}{2} \hat{N}^+) \hat{\hat{N}}^- \\
[\hat{J}^3, \hat{N}^+] &= \frac{4}{h} \sinh(\frac{h}{2} \hat{N}^+) \cosh(\frac{h}{2} \hat{J}^+) \\
[\hat{J}^3, \hat{N}^-] &= -\hat{\hat{N}}^- \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) - \cosh(\frac{h}{2} \hat{J}^+) \cosh(\frac{h}{2} \hat{N}^+) \hat{\hat{N}}^- \\
&\quad - \hat{J}^- \sinh(\frac{h}{2} \hat{J}^+) \sinh(\frac{h}{2} \hat{N}^+) - \sinh(\frac{h}{2} \hat{J}^+) \sinh(\frac{h}{2} \hat{N}^+) \hat{\hat{J}}^+ \\
\hat{\hat{N}}^3, \hat{\hat{N}}^\pm &= [\hat{J}^3, \hat{J}^\pm], \quad [\hat{\hat{N}}^3, \hat{\hat{J}}^\pm] = [\hat{J}^3, \hat{\hat{N}}^\pm] \\
[\hat{J}^+, \hat{J}^-] &= [\hat{\hat{N}}^+, \hat{\hat{N}}^-] = [\hat{J}^+, \hat{\hat{N}}^-] = 0
\end{align*}
\]

\[\Delta \hat{J}^j = e^{-\frac{h}{2} \hat{N}^+} \cosh(\frac{h}{2} \hat{J}^+) \hat{J}^j + \hat{J}^j \otimes e^{\frac{h}{2} \hat{N}^+} \hat{\hat{J}}^j - \frac{h}{2} (e^{\frac{h}{2} \hat{N}^+} \hat{J}^j \otimes \hat{\hat{J}}^j + \hat{\hat{J}}^j \otimes e^{\frac{h}{2} \hat{N}^+} \hat{J}^j + e^{-\frac{h}{2} \hat{N}^+} \hat{J}^j \otimes e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{J}}^j)
\]
\[\Delta \hat{\hat{N}}^j = e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j + \hat{\hat{N}}^j \otimes e^{\frac{h}{2} \hat{N}^+} \hat{\hat{J}}^j - \frac{h}{2} (e^{\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j \otimes \hat{\hat{J}}^j + \hat{\hat{J}}^j \otimes e^{\frac{h}{2} \hat{N}^+} \hat{\hat{N}}^j + e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{J}}^j \otimes e^{-\frac{h}{2} \hat{N}^+} \hat{\hat{J}}^j)
\quad \text{where } j = -3\]
\[\gamma(X) = -e^{h \hat{N}^+} X e^{-h \hat{N}^+}.
\]

This is a deformation of \(U(\text{iso}(2) \oplus \text{iso}(2))\).

References

[1] E. E. Demidov, Yu. I. Manin, E. E. Mukhin, and D. V. Zhdanovich; Prog. Theor. Phys. Suppl. 102, 203 (1990)

[2] Ch. Ohn; Lett. Math Phys. 25, 85 (1992)

[3] A. Aghamohammadi, M. Khorrami, and A. Shariati; J. Phys. A28, L225 (1995)

[4] A. A. Vladimirov; Mod. Phys. Lett. A8, 2573 (1993)

[5] L. L. Vaksman, and L. I. Korogodski; Sov. Math. Dokl. 39, 173 (1989)

[6] S. L. Woronowicz; Lett. Math. Phys. 23, 251 (1991)

[7] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini; J. Math. Phys. 32, 1159 (1991)

[8] A. Ballesteros, E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini; J. Phys. A26, 7495 (1993)
[9] M. Khorrami, A. Shariati, M. R. Abolhassani and A. Aghamohammadi; Mod. Phys. Lett. A10, 873 (1995)

[10] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander, J. Phys. A28, 941 (1995)

[11] R. V. Moody, J. Patera; J. Phys. A24, 2227 (1991)

[12] S. Majid; Int. J. Mod. Phys. A5, 1 (1990)