A Note on the Area Requirement of Euclidean Greedy Embeddings of Christmas Cactus Graphs

Roman Prutkin*

An Euclidean greedy embedding of a graph is a straight-line embedding in the plane, such that for every pair of vertices $s$ and $t$, the vertex $s$ has a neighbor $v$ with smaller distance to $t$ than $s$. This drawing style is motivated by greedy geometric routing in wireless sensor networks.

A Christmas cactus is a connected graph in which every two simple cycles have at most one vertex in common and in which every cutvertex is part of at most two biconnected blocks. It has been proved that Christmas cactus graphs have an Euclidean greedy embedding. This fact has played a crucial role in proving that every 3-connected planar graph has an Euclidean greedy embedding. The proofs construct greedy embeddings of Christmas cactuses of exponential size, and it has been an open question whether exponential area is necessary in the worst case for greedy embeddings of Christmas cactuses. We prove that this is indeed the case.

1 Introduction

Consider a graph $G = (V, E)$ and a straight-line embedding of $G$ in the Euclidean plane. For simplicity, we identify each vertex with the corresponding point in $\mathbb{R}^2$. An embedding of $G$ is greedy if for every pair $s, t \in V$, vertex $s$ has a neighbor $v$ in $G$, for which it is $|vt| < |st|$, where $|pq|$ denotes the Euclidean distance between points $p$ and $q$. Equivalently, every pair $s, t \in V$ is joined by a distance-decreasing, or greedy, path.

Greedy embeddings are motivated by geometric routing in wireless sensor networks. Given such an embedding, we can use vertex coordinates as addresses. To route a message to a destination, a vertex can simply forward the message to a neighbor that is closer to the destination, and a successful delivery is guaranteed.

The existence of greedy embeddings has been studied for various graph classes. Papadimitriou and Ratajczak [8] conjectured that every 3-connected planar graph has a greedy embedding in the Euclidean plane. This conjecture has been proved independently by Leighton and Moitra [5] and Angelini et al. [2]. Both proofs use the fact that 3-connected planar graphs have a spanning Christmas cactus subgraph. A Christmas cactus is a connected graph in which every two simple cycles have at most one vertex in common and in which every cutvertex is part of at most two biconnected blocks. The authors show that every Christmas cactus has a greedy embedding. However, both constructions produce embeddings of exponential size in the worst case.
We now present a family of Christmas cactuses that requires exponential aspect ratio of edge lengths. We prove Moitra’s conjecture that Euclidean greedy embeddings of Christmas cactuses require exponential size. Angelini et al. [1] proved that some trees require exponential aspect ratio of edge lengths in every greedy embedding. For an integer $k \geq 1$, consider the Christmas cactus $G_k$ with root $r_i$ in Fig. 1a. We then construct the cactus $F_k$ by attaching the roots of 30 copies of $G_k$ to a cycle of size 31; see Fig. 1b. We shall prove that the aspect ratio of edge lengths in every greedy embedding of $F_k$ is at least $2^k$. The following fact follows from Lemma 3 in [7].

**Fact 1.** Every greedy embedding of $F_k$ contains a greedy embedding of $G_k$, in which every pair of vectors from $\cup_i \{u_i u_{i+1}, u_i v_{i+1}, v_i w_{i+1}\}$ forms an angle of less than $120^{\circ}$.

From now on, we consider the embedding of $G_k$ from Fact 1.

**Lemma 1.** For $0 \leq i \leq k-1$, it holds: $|u_{i+1} u_{i+2}| < \frac{1}{2} |u_i u_{i+1}|$.

**Proof.** We rename the vertices for brevity: $a = u_i + 2$, $b = u_{i+1}$, $c = v_{i+1}$, $d = w_{i+2}$, $y = u_i$; see Fig. 2. Note that every greedy $a-d$ path as well as every greedy $d-a$ path must contain $b$ and $c$. Therefore, the path $abcd$ is greedy in both directions. Thus, the ray with origin $b$ and direction $\overrightarrow{ba}$ and the ray with origin $c$ and direction $\overrightarrow{cd}$ diverge [1]. The paths $abd$ and $acd$ are also greedy in both directions, therefore, $\alpha_1 = \angle abd > 60^{\circ}$ and $\alpha_4 = \angle acd > 60^{\circ}$.

Let $x$ be the intersection point of the lines through $ab$ and $cd$. Let $\varepsilon = 12^{\circ}$. Since $G_k$ has been chosen according to Fact 1, it is $\angle xby < \varepsilon$ and $\angle xcy < \varepsilon$.

It is $\angle cbx = 180^{\circ} - \angle abc < 120^{\circ}$. Similarly, $\angle bxc < 120^{\circ}$. Also, $\angle bxc < \varepsilon$. Thus, by considering the triangle $bxc$ it follows: $\angle cbx > 60^{\circ} - \varepsilon$ and $\angle bxc > 60^{\circ} - \varepsilon$. Since it is $60^{\circ} - \varepsilon < \angle cbx < 120^{\circ}$, it is $60^{\circ} - 2\varepsilon < \angle cbx < 120^{\circ} + \varepsilon$. Analogously, it is $60^{\circ} - 2\varepsilon < \angle bcy < 120^{\circ} + \varepsilon$. It follows:

$$\frac{|bc|}{|by|} = \frac{\sin \angle byc}{\sin \angle bcy} < \frac{\sin \varepsilon}{\sin(60^{\circ} - 2\varepsilon)} < 0.36.$$
Therefore, it is $|bc| < 0.36|by|$ and, analogously, $|bc| < 0.36|cy|$.

Next, recall that it is $\angle bxc = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 180^\circ < \varepsilon$, for $\alpha_2 = \angle dbc$ and $\alpha_3 = \angle acb$. Therefore, $\angle bac = 180^\circ - \alpha_1 - \alpha_2 - \alpha_3 > \alpha_4 - \varepsilon > 60^\circ - \varepsilon$. Also, since the path $abc$ is greedy in both directions, it is $\angle bac < 90^\circ$. Now consider $\angle acb = \alpha_3$. Since $\angle bcx > 60^\circ - \varepsilon$, it is $\alpha_3 + \alpha_4 < 120^\circ + \varepsilon$, and $\alpha_3 < 60^\circ + \varepsilon$. Therefore,

$$\frac{|ab|}{|bc|} = \frac{\sin \angle acb}{\sin \angle bac} = \frac{\sin \alpha_3}{\sin(180^\circ - \alpha_1 - \alpha_2 - \alpha_3)} < \frac{\sin(60^\circ + \varepsilon)}{\sin(60^\circ - \varepsilon)} < 1.28.$$

Thus, $|ab| < 1.28|bc|$. It follows: $|ab| < 1.28|bc| < 1.28 \cdot 0.36|by| < 0.461|by|$. Therefore, we have $|u_{i+1}u_{i+2}| < \frac{1}{2}|u_iu_{i+1}|$.

**Theorem 1.** In every greedy embedding of cactus $F_k$, the ratio of the longest and the shortest edge is in $\Omega(2^n/90)$, where $n$ is the number of vertices of $F_k$.

**Proof.** Cactus $G_k$ has $3k+2$ vertices. Thus, cactus $F_k$ has $n = 90k + 61$ vertices. By Lemma 1 every greedy embedding of $F_k$ contains an embedding of $G_k$, such that it is $|u_ku_{k+1}| < \frac{1}{2}|u_0u_1|$. Therefore, the ratio of the longest and shortest edge in every greedy embedding of $F_k$ is at least $2^k = \Omega(2^n/90)$.

**Acknowledgements**

The author thanks Martin Nöllenburg for valuable discussions and comments.

**References**

[1] P. Angelini, G. Di Battista, and F. Frati. Succinct greedy drawings do not always exist. *Networks*, 59(3):267–274, 2012.

[2] P. Angelini, F. Frati, and L. Grilli. An algorithm to construct greedy drawings of triangulations. *J. Graph Algorithms Appl.*, 14(1):19–51, 2010.
[3] D. Eppstein and M. T. Goodrich. Succinct greedy geometric routing using hyperbolic geometry. *IEEE Transactions on Computers*, 60(11):1571–1580, 2011. 2

[4] M. T. Goodrich and D. Strash. Succinct greedy geometric routing in the Euclidean plane. In Y. Dong, D.-Z. Du, and O. Ibarra, editors, *Algorithms and Computation (ISAAC’09)*, volume 5878 of *LNCS*, pages 781–791. Springer, 2009. 2

[5] T. Leighton and A. Moitra. Some results on greedy embeddings in metric spaces. *Discrete Comput. Geom.*, 44(3):686–705, 2009. 1 2

[6] A. Moitra and T. Leighton. Some results on greedy embeddings in metric spaces. In *Foundations of Computer Science (FOCS’08)*, pages 337–346, 2008. 1

[7] M. Nöllenburg, R. Prutkin, and I. Rutter. On self-approaching and increasing-chord drawings of 3-connected planar graphs. *J. Comput. Geom.*, 7(1):47–69, 2016. 2

[8] C. H. Papadimitriou and D. Ratajczak. On a conjecture related to geometric routing. *Theoret. Comput. Sci.*, 344(1):3–14, 2005. 1