UNITARY TRIDIAGONALISATION IN $M(4, \mathbb{C})$

VISHWAMBHAR PATI

Abstract. A question of interest in linear algebra is whether all $n \times n$ complex matrices can be unitarily tridiagonalised. The answer for all $n \neq 4$ (affirmative or negative) has been known for a while, whereas the case $n = 4$ seems to have remained open. In this paper we settle the $n = 4$ case in the affirmative. Some machinery from complex algebraic geometry needs to be used.

1. Main Theorem

Let $V = \mathbb{C}^n$, and $\langle \ , \ \rangle$ be the usual euclidean hermitian inner product on $V$. $U(V) = U(n)$ denotes the group of unitary automorphisms of $V$ with respect to $\langle \ , \ \rangle$. $\{e_i\}_{i=1}^n$ will denote the standard orthonormal basis of $V$. $A \in \text{End}_\mathbb{C}(V) = M(n, \mathbb{C})$ will always denote a $\mathbb{C}$-linear transformation of $V$.

A matrix $A = [a_{ij}]$ is said to be tridiagonal if $a_{ij} = 0$ for all $1 \leq i, j \leq n$ such that $|i - j| \geq 2$.

Then we have:

Theorem 1.1. For $n \leq 4$, and $A \in M(n, \mathbb{C})$, there exists a unitary $U \in U(n)$ such that $UAU^*$ is tridiagonal.

Remark 1.2. The case $n = 3$, and counterexamples for $n \geq 6$, are due to Longstaff, [4]. In the paper [3], Fong and Wu construct counterexamples for $n = 5$, and provide a proof in certain special cases for $n = 4$. The article §4 of [2] poses the $n = 4$ case in general as an open question. Our main theorem above answers this question in the affirmative. In passing, we also provide another elementary proof for the $n = 3$ case.

2. Some Lemmas

We need some preliminary lemmas, which we collect in this section. In the sequel, we will also use the letter $A$ to denote the unique linear transformation determined by $A$.

Lemma 2.1. Let $A \in M(n, \mathbb{C})$. For all $n$, the following are equivalent:

(i): There exists a unitary $U \in U(n)$ such that $UAU^*$ is tridiagonal.

(ii): There exists a flag (=ascending sequence of $\mathbb{C}$- subspaces) of $V = \mathbb{C}^n$:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that dim $W_i = i$, $AW_i \subset W_{i+1}$ and $A^*W_i \subset W_{i+1}$ for all $0 \leq i \leq n - 1$.

(iii): There exists a flag in $V$:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that dim $W_i = i$, $AW_i \subset W_{i+1}$ and $A(W_i^\perp) \subset W_{i+1}^\perp$ for all $0 \leq i \leq n - 1$.

Proof:

(i) $\Rightarrow$ (ii)

Set $W_i = \mathbb{C}\text{-span}(f_1, f_2, \ldots, f_i)$, where $f_i = U^*e_i$ and $e_i$ is the standard basis of $V = \mathbb{C}^n$. Since the matrix $[b_{ij}] := UAU^*$ is tridiagonal, we have:

$$Af_i = b_{i-1,i}f_{i-1} + b_{ii}f_i + b_{i+1,i}f_{i+1} \quad 1 \leq i \leq n$$
and by comparing the two equations (1), (2) above, it follows that

$$f^\perp_A W_i \subset W_{i+1}$$

which shows $A^*(W_i) \subset W_{i+1}$ for all $i$ as well, and (ii) follows.

$(iii) \Rightarrow (i)$

Inductively choose an orthonormal basis $f_i$ of $V = \mathbb{C}^n$ so that $W_i$ is the span of $\{f_1, ..., f_i\}$. Since $A(W_i) \subset W_{i+1}$, we have:

$$A f_i = a_{i1} f_1 + a_{i2} f_2 + ... + a_{i+i,i} f_{i+1}$$

(1)

Since $f_i \in (W_{i-1})^\perp$, and by hypothesis $A(W_{i-1}^\perp) \subset W_{i-2}^\perp$, and $W_{i-2}^\perp = \mathbb{C}\text{-span}(f_{i-1}, f_i, ..., f_n)$, we also have

$$A f_i = a_{i-1,i} f_{i-1} + a_{ii} f_i + ... + a_{ni} f_n$$

(2)

and by comparing the two equations (1), (2) above, it follows that

$$A f_i = a_{i-1,i} f_{i-1} + a_{ii} f_i + a_{i+1,i} f_{i+1}$$

for all $i$, and defining the unitary $U$ by $U^* e_i = f_i$ makes $U A U^*$ tridiagonal, so that (i) follows.

Lemma 2.2. Let $n \leq 4$. If there exists a 2-dimensional $\mathbb{C}$-subspace $W$ of $V = \mathbb{C}^n$ such that $A W \subset W$ and $A^* W \subset W$, then $A$ is unitarily tridiagonalisable.

Proof: If $n \leq 2$, there is nothing to prove. For $n = 3$ or 4, the hypothesis implies that $A$ maps $W^\perp$ into itself. Then, in an orthonormal basis $\{f_i\}_{i=1}^n$ of $W$ which satisfies $W = \mathbb{C}\text{-span}(f_1, f_2)$ and $W^\perp = \mathbb{C}\text{-span}(f_3, ..., f_n)$ the matrix of $A$ is in $(1, 2)$ (resp. $(2, 2)$) block-diagonal form for $n = 3$ (resp. $n = 4$), which is clearly tridiagonal.

Lemma 2.3. Every matrix $A \in M(3, \mathbb{C})$ is unitarily tridiagonalisable.

Proof:

For $A \in M(3, \mathbb{C})$, consider the homogeneous cubic polynomial in $v = (v_1, v_2, v_3)$ given by:

$$F(v_1, v_2, v_3) := \det(v, Av, A^* v)$$

Note $v \wedge Av \wedge A^* v = F(v_1, v_2, v_3) e_1 \wedge e_2 \wedge e_3$. By a standard result in dimension theory (see [1], p. 74, Theorem 5) each irreducible component of $V(F) \subset \mathbb{P}_3^2$ is of dimension $\geq 1$, and $V(F)$ is non-empty. Choose some $[v_1 : v_2 : v_3] \in V(F)$, and let $v = (v_1, v_2, v_3)$ which is non-zero. Then we have the two cases:

Case 1: $v$ is a common eigenvector for $A$ and $A^*$. Then the 2-dimensional subspace $W = (\mathbb{C} v)^\perp$ is an invariant subspace for both $A$ and $A^*$, and applying the lemma 2.2 to $W$ yields the result.

Case 2: $v$ is not a common eigenvector for $A$ and $A^*$. Say it is not an eigenvector for $A$ (otherwise interchange the roles of $A$ and $A^*$). Set $W_1 = \mathbb{C} v$, $W_2 = \mathbb{C}\text{-span}(v, Av), W_3 = \mathbb{C}^3$. Then $\dim W_i = i$, for $i = 1, 2, 3$, and the fact that $v \wedge Av \wedge A^* v = 0$ shows that $A^* W_1 \subset W_2$. Thus, by (ii) of lemma 2.1, we are done.

Let $V = \mathbb{C}^4$ from now on, and $A \in M(4, \mathbb{C})$.

Lemma 2.4. If $A$ and $A^*$ have a common eigenvector, then $A$ is unitarily tridiagonalisable.

Proof: If $v \neq 0$ is a common eigenvector for $A$ and $A^*$, the 2-dimensional subspace $W = (\mathbb{C} v)^\perp$ is invariant under both $A$ and $A^*$, and unitary tridiagonalisation of $A|_W$ exists from the $n = 3$ case of the lemma 2.3 by a $U_1 \in U(W) = U(3)$. The unitary $U = 1 \oplus U_1$ is the desired unitary in $U(4)$ tridiagonalising $A$. □
Lemma 2.5. If the main theorem holds for all \( A \in S \), where \( S \) is any dense (in the classical topology) subset of \( M(4, \mathbb{C}) \), then it holds for all \( A \in M(4, \mathbb{C}) \).

Proof: This is a consequence of the compactness of the unitary group \( U(4) \). Indeed, let \( T \) denote the closed subset of tridiagonal (with respect to the standard basis) matrices.

Let \( A \in M(4, \mathbb{C}) \) be any general element. By the density of \( S \), there exist \( A_n \in S \) such that \( A_n \to A \). By hypothesis, there are unitaries \( U_n \in U(4) \) such that \( U_nA_nU_n^* = T_n \), where \( T_n \) are tridiagonal. By the compactness of \( U(4) \), and by passing to a subsequence if necessary, we may assume that \( U_n \to U \in U(4) \). Then \( U_nA_nU_n^* \to UAU^* \). That is \( T_n \to UAU^* \). Since \( T \) is closed, and \( T_n \in T \), we have \( UAU^* \) is in \( T \), viz., is tridiagonal.

We shall now construct a suitable dense open subset \( S \subset M(4, \mathbb{C}) \), and prove tridiagonalisability for a general \( A \in S \) in the remainder of this paper. More precisely:

Lemma 2.6. There is a dense open subset \( S \subset M(4, \mathbb{C}) \) such that:

(i): \( A \) is nonsingular for all \( A \in S \).

(ii): \( A \) has distinct eigenvalues for all \( A \in S \).

(iii): For each \( A \in S \), the element \( (t_0I + t_1A + t_2A^*) \in M(4, \mathbb{C}) \) has rank \( \geq 3 \) for all \( (t_0, t_1, t_2) \neq (0, 0, 0) \) in \( \mathbb{C}^3 \).

Proof: The subset of singular matrices in \( M(4, \mathbb{C}) \) is the complex algebraic subvariety of complex codimension one defined by \( Z_1 = \{ A : \det A = 0 \} \). Let \( S_1 \), (which is just \( GL(4, \mathbb{C}) \)) be its complement. Clearly \( S_1 \) is open and dense in the classical topology (in fact, also in the Zariski topology).

A matrix \( A \) has distinct eigenvalues iff its characteristic polynomial \( \phi_A \) has distinct roots. This happens iff the discriminant polynomial of \( \phi_A \), which is a 4-th degree homogeneous polynomial \( \Delta(A) \) in the entries of \( A \), is not zero. The zero set \( Z_2 = V(\Delta) \) is again a codimension-1 subvariety in \( M(4, \mathbb{C}) \), so its complement \( S_2 = (V(\Delta))^c \) is open and dense in both the classical and Zariski topologies.

To enforce (iii), we claim that the set defined by

\[
Z_3 := \{ A \in M(4, \mathbb{C}) : \text{rank} (t_0I + t_1A + t_2A^*) \leq 2 \text { for some } (t_0, t_1, t_2) \neq (0, 0, 0) \text{ in } \mathbb{C}^3 \}
\]

is a proper real algebraic subset of \( M(4, \mathbb{C}) \). The proof hinges on the fact that three general cubic curves in \( \mathbb{P}^2_\mathbb{C} \) having a point in common imposes an algebraic condition on their coefficients.

Indeed, saying that rank \( (t_0I + t_1A + t_2A^*) \leq 2 \) for some \( (t_0, t_1, t_2) \neq (0, 0, 0) \) is equivalent to saying that the third exterior power \( \Lambda^3(t_0I + t_1A + t_2A^*) \) is the zero map, for some \( (t_0, t_1, t_2) \neq 0 \). This is equivalent to demanding that there exist a \( (t_0, t_1, t_2) \neq 0 \) such that the determinants of all the \( 3 \times 3 \)-minors of \( (t_0I + t_1A + t_2A^*) \) are zero.

Note that the (determinants of) the \( (3 \times 3) \)-minors of \( (t_0I + t_1A + t_2A^*) \), denoted as \( M_{ij}(A, t) \) (where the \( i \)-th row and \( j \)-th column are deleted) are complex valued, complex algebraic and \( \mathbb{C} \)-homogeneous of degree 3 in \( t = (t_0, t_1, t_2) \), with coefficients real algebraic of degree 3 in the variables \( A_{ij}, \bar{A}_{ij} \) (or, equivalently, in \( \text{Re} A_{ij}, \text{Im} A_{ij} \)), where \( A = [A_{ij}] \).

We know that the space of all homogeneous polynomials of degree 3 with complex coefficients in \( (t_0, t_1, t_2) \) (upto scaling) is parametrised by the projective space \( \mathbb{P}^9_\mathbb{C} \) (the Veronese variety, see [1], p.52). We first consider the complex algebraic variety:

\[
X = \{(P, Q, R, [t]) \in \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^2_\mathbb{C} : P(t) = Q(t) = R(t) = 0 \}
\]

where \([t] := [t_0 : t_1 : t_2] \), and \((P, Q, R)\) denotes a triple of homogeneous polynomials. This is just the subset of those \((P, Q, R, [t])\) in the product \( \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^9_\mathbb{C} \times \mathbb{P}^2_\mathbb{C} \) such that the point \([t]\) lies on all three of the plane
cubic curves \( V(P), V(Q), V(R) \). Since \( X \) is defined by bihomogenous degree \((1,1,1)\) equations, it is a complex algebraic subvariety of the quadruple product. Its image under the first projection \( Y := \pi_1(X) \subseteq \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \) is therefore an algebraic subvariety inside this triple product (see [1], p. 58, Theorem 3). \( Y \) is a proper subvariety because, for example, the cubic polynomials \( P = t_0^3, Q = t_1^3, R = t_2^3 \) have no common non-zero root.

Denote pairs \((i, j)\) with \(1 \leq i, j \leq 4\) by capital letters like \( I, J, K \) etc. From the minomial determinants \( M_I(A, t) \), we can define various real algebraic maps:

\[
\Theta_{IJK} : M(4, \mathbb{C}) \to \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}}
\]

\[
A \mapsto (M_I(A, t), M_J(A, t), M_K(A, t))
\]

for \(I, J, K\) distinct. Clearly, \( \bigwedge^3 (t_0 I + t_1 A + t_2 A^*) = 0 \) for some \( t = (t_0, t_1, t_2) \neq (0, 0, 0) \) iff \( \Theta_{IJK}(A) \) lies in in the complex algebraic subvariety \( Y \) of \( \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \), for all \( I, J, K \) distinct. Hence the subset \( Z_3 \subseteq M(4, \mathbb{C}) \) defined above is the intersection:

\[
Z_3 = \bigcap_{I, J, K} \Theta_{IJK}^{-1}(Y)
\]

where \( I, J, K \) runs over all distinct triples of pairs \((i, j), 1 \leq i, j \leq 4\).

We claim that \( Z_3 \) is a proper real algebraic subset of \( M(4, \mathbb{C}) \). Clearly, since each \( M_I(A, t) \) is real algebraic in the variables \( \text{Re} A_{ij}, \text{Im} A_{ij} \), the map \( \Theta_{IJK} \) is real algebraic. Since \( Y \) is complex and hence real algebraic, its inverse image \( \Theta_{IJK}^{-1}(Y) \), defined by the real algebraic equations obtained upon substitution of the components \( M_I(A, t), M_J(A, t), M_K(A, t) \) in the equations that define \( Y \), is also real algebraic. Hence the set \( Z_3 \) is a real algebraic subset of \( M(4, \mathbb{C}) \).

To see that \( Z_3 \) is a proper subset of \( M(4, \mathbb{C}) \), we simply consider the matrix (defined with respect to the standard orthonormal basis \( \{e_i\}_{i=1}^4 \) of \( \mathbb{C}^4 \)):

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For \( t = (t_0, t_1, t_2) \neq 0 \), we see that:

\[
t_0 I + t_1 A + t_2 A^* = \begin{bmatrix}
t_0 & t_1 & 0 & 0 \\
t_2 & t_0 & t_1 & 0 \\
0 & t_2 & t_0 & t_1 \\
0 & 0 & t_2 & t_0
\end{bmatrix}
\]

For this matrix above, the minomial determinant \( M_{ii}(A, t) = t_i^3 \), whereas \( M_{id}(A, t) = t_i^2 \). The only common zeros to these two minomial determinants are points \([t_0 : 0 : 0]\). Setting \( t_1 = t_2 = 0 \) in the matrix above gives \( M_{ii}(A, t) = t_0^3 \) for \(1 \leq i \leq 4\). Thus \( t_0 \) must also be 0 for all the minorial determinants to vanish. Hence the matrix \( A \) above lies outside the real algebraic set \( Z_3 \).

It is well known that a proper real algebraic subset in euclidean space cannot have a non-empty interior. Thus the complement \( Z_3^c \) is dense and open in the classical and real-Zariski topologies. Take \( S_3 = Z_3^c \).

Finally, set

\[
S := S_1 \cap S_2 \cap S_3 = \left( \bigcup_{i=1}^3 Z_i \right)^c
\]

which is also open and dense in the classical topology in \( M(4, \mathbb{C}) \). Hence the lemma. \( \square \)
Remark 2.7. One should note here that for each matrix $A \in M(4, \mathbb{C})$, there will be at least a curve of points $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}_1^2$ (defined by the vanishing of $\det (t_0 I + t_1 A + t_2 A^*)$), on which $(t_1 I + t_1 A + t_2 A^*)$ is singular. Similarly for each $A$ there is at least a curve of points on which the trace $\text{tr} \left( A^3(t_0 I + t_1 A + t_2 A^*) \right)$ vanishes, and so a non-empty (and generally a finite) set on which both these polynomials vanish, by dimension theory (9, Theorem 5, p. 74). Thus for each $A \in M(4, \mathbb{C})$, there is at least a non-empty finite set of points $[t]$ such that $(t_0 I + t_1 A + t_2 A^*)$ has 0 as a repeated eigenvalue. For example, for the matrix $A$ constructed at the end of the previous lemma, we see that the matrix $(t_0 I + t_1 A + t_2 A^*)$ is strictly upper-triangular and thus has 0 as an eigenvalue of multiplicity 4 for all $(0, t_1, 0) \neq 0$, but nevertheless has rank 3 for all $(t_0, t_1, t_2) \neq (0, 0, 0)$.

Indeed, as (iii) of the lemma above shows, for $A$ in the open dense subset $S$, the kernel $\ker (t_0 I + t_1 A + t_2 A^*)$ is at most 1-dimensional for all $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}_1^2$.

3. The varieties $C$, $\Gamma$, and $D$

Notation 3.1. In the light of the lemmas 2.3 and 2.6 above, we shall henceforth assume $A \in S$. As is easily verified, this implies $A^* \in S$ as well. We will also henceforth assume, in view of lemma 2.4 above, that $A$ and $A^*$ have no common eigenvectors. (For example, this rules out $A$ being normal, in which case we know that the main result for $A$ is true by the spectral theorem). Also, in view of lemma 2.4, we shall assume that $A$ and $A^*$ do not have a common 2-dimensional invariant subspace.

In $\mathbb{P}^3$, the complex projective space of $V = \mathbb{C}^4$, we denote the equivalence class of $v \in V \setminus 0$ by $[v]$. For a $[v] \in \mathbb{P}^3$, we define $W([v])$ (or simply $W(v)$ when no confusion is likely) by:

$$W([v]) := \mathbb{C} \text{-span}(v, Av, A^*v)$$

Since we are assuming that $A$ and $A^*$ have no common eigenvectors, we have $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3$.

Denote the four distinct points in $\mathbb{P}^3$ representing the four distinct eigenvectors of $A$ (resp. $A^*$) by $E$ (resp. $E^*$). By our assumption above, $E \cap E^* = \emptyset$.

Lemma 3.2. Let $A \in M(4, \mathbb{C})$ be as in 3.1 above. Then the closed subset:

$$C = \{[v] \in \mathbb{P}^3 : v \wedge Av \wedge A^*v = 0\}$$

is a closed projective variety. This variety $C$ is precisely the subset of $[v] \in \mathbb{P}^3$ for which the dimension $\dim W([v]) = \dim (\mathbb{C} \text{-span} \{v, Av, A^*v\})$ is exactly 2.

Proof: That $C$ is a closed projective variety is clear from the fact that it is defined as the set of common zeros of all the four $(3 \times 3)$-minor determinants of the $(3 \times 4)$-matrix

$$\Lambda := \begin{bmatrix} v \\ Av \\ A^*v \end{bmatrix}$$

(which are all degree-3 homogeneous polynomials in the components of $v$ with respect to some basis). Also $C$ is nonempty since it contains $E \cup E^*$.

Also, since $A$ and $A^*$ are nonsingular by the assumptions in 3.1, the wedge product $v \wedge Av \wedge A^*v$ of the three non-zero vectors $v, Av, A^*v$ vanishes precisely when the space $W([v]) = \mathbb{C} \text{-span} \{v, Av, A^*v\}$ is of dimension $\leq 2$. Since by 3.3, $A, A^*$ have no common eigenvectors, the dimension $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3$, so $C$ is precisely the locus of $[v] \in \mathbb{P}^3$ for which the space $W([v])$ is 2-dimensional.

Now we shall show that for $A$ as in 3.1, the variety $C$ defined above is of pure dimension one. For this, we need to define some more associated algebraic varieties and regular maps.
Lemma 3.4. Let us define the bilinear map: 
\[ B : \mathbb{C}^4 \times \mathbb{C}^3 \rightarrow \mathbb{C}^4 \]
\[ (v, t_0, t_1, t_2) \mapsto B(v, t) := (t_0I + t_1A + t_2A^*)v \]
We then have the linear maps \( K(t) := \ker(B(-, t) : \mathbb{C}^4 \rightarrow \mathbb{C}^4) \)

Denoting \([t_0 : t_1 : t_2]\) by \([t]\) and \([v_1 : v_2 : v_3 : v_4]\) by \([v]\) for brevity, we define:
\[ \Gamma := \{([v], [t]) \in \mathbb{P}_C^3 \times \mathbb{P}_C^2 : B(v, t) = 0\} \]

Finally, define the variety \( D \) by:
\[ D \subset \mathbb{P}_C^3 := \{[t] \in \mathbb{P}_C^2 : \det B(-, t) = \det (t_0I + t_1A + t_2A^*) = 0\} \]
Let
\[ \pi_1 : \mathbb{P}_C^3 \times \mathbb{P}_C^2 \rightarrow \mathbb{P}_C^3, \quad \pi_2 : \mathbb{P}_C^3 \times \mathbb{P}_C^2 \rightarrow \mathbb{P}_C^2 \]
denote the two projections.

Lemma 3.4. We have the following facts:

(i): \( \pi_1(\Gamma) = C \), and \( \pi_2(\Gamma) = D \).

(ii): \( \pi_1 : \Gamma \rightarrow C \) is 1-1, and the map \( g \) defined by
\[ g := \pi_2 \circ \pi_1^{-1} : C \rightarrow D \]
is a regular map so that \( \Gamma \) is the graph of \( g \) and isomorphic as a variety to \( C \).

(iii): \( D \subset \mathbb{P}_C^3 \) is a plane curve, of pure dimension one. The map \( \pi_2 : \Gamma \rightarrow D \) is 1-1, and the map \( \pi_1 \circ \pi_2^{-1} : D \rightarrow C \) is the regular inverse of the regular map \( g \) defined above in (ii). Again \( \Gamma \) is also the graph of this regular inverse \( g^{-1} \), and \( D \) and \( \Gamma \) are isomorphic as varieties. In particular, \( C \) and \( D \) are isomorphic as varieties, and thus \( C \) is a curve in \( \mathbb{P}_C^3 \) of pure dimension one.

(iv): Inside \( \mathbb{P}_C^3 \times \mathbb{P}_C^2 \), each irreducible component of the intersection of the four divisors \( D_i := (B_i(v, t) = 0) \) for \( i = 1, 2, 3, 4 \) (where \( B_i(v, t) \) is the \( i \)-th component of \( B(v, t) \) with respect to a fixed basis of \( \mathbb{C}^4 \)) occurs with multiplicity 1. (Note that \( \Gamma \) is set-theoretically the intersection of these four divisors, by definition).

**Proof:**

It is clear that \( \pi_1(\Gamma) = C \), because \( B(v, t) = t_0v + t_1Av + t_2A^*v = 0 \) for some \([t_0 : t_1 : t_2] \in \mathbb{P}_C^2 \) iff \( \dim W(v) \leq 2 \), and since \( A \) and \( A^* \) have no common eigenvectors, this means \( \dim W(v) = 2 \). That is, \([v] \in C \).

Clearly \([t] \in \pi_2(\Gamma) \) iff there exists a \([v] \in \mathbb{P}_C^3 \) such that \( B(v, t) = 0 \). That is, iff \( \dim \ker B(-, t) \geq 1 \), that is, iff
\[ G(t_0, t_1, t_2) := \det B(-, t) = 0 \]
Thus \( D = \pi_2(\Gamma) \) and is defined by a single degree 4 homogeneous polynomial \( G \) inside \( \mathbb{P}_C^2 \). It is a curve of pure dimension 1 in \( \mathbb{P}_C^3 \) by standard dimension theory (see [1], p. 74, Theorem 5) because, for example \([1 : 0 : 0] \notin D \) so \( D \neq \mathbb{P}_C^2 \). So \( \pi_2(\Gamma) = D \), and this proves (i).

To see (ii), for a given \([v] \in C \), we claim there is exactly one \([t] \) such that \(([v], [t]) \in \Gamma \). Note that \(([v], [t]) \in \Gamma \) iff the linear map:
\[ B(v, -) : \mathbb{C}^3 \rightarrow \mathbb{C}^4 \]
\[ t \mapsto (t_0I + t_1A + t_2A^*)v \]
has a non-trivial kernel containing the line $Ct$. That is, $\dim \text{Im } B(v, -) \leq 2$. But the image $\text{Im } B(v, -) = W(v)$, which is of dimension 2 for all $v \in C$ by our assumptions. Thus its kernel must be exactly one dimensional, defined by $\ker B(v, -) = \mathbb{C}t$. Thus $([v], [t])$ is the unique point in $\Gamma$ lying in $\pi_1^{-1}([v])$, viz. for each $[v] \in C$, the vertical line $[v] \times \mathbb{P}_2^1$ intersects $\Gamma$ in a single point, call it $([v], g([v])$. So $\pi_1 : \Gamma \to C$ is 1-1, and $\Gamma$ is the graph of a map $g : C \to D$. Since $g([v]) = \pi_2\pi_1^{-1}([v])$ for $[v] \in C$, and $\Gamma$ is algebraic, $g$ is a regular map. This proves (ii).

To see (iii), note that for $[t] \in D$, by definition, the dimension $\dim \ker B(-, t) \geq 1$. By the fact that $A \in S$, and (iii) of the lemma [1], we know that $\dim \ker B(-, t) \leq 1$ for all $[t] \in \mathbb{P}_2^1$. Thus, denoting $K(t) := \ker B(-, t)$ for $[t] \in D$, we have:

$$\dim K(t) = 1 \quad \text{for all } t \in D$$

(3)

Hence we see that the unique projective line $[v]$ corresponding to $Cv = K(t)$ yields the unique element of $C$, such that $([v], [t]) \in \Gamma$. Thus $\pi_2 : \Gamma \to D$ is 1-1, and the regular map $\pi_1 \circ \pi_2^{-1} : D \to C$ is the regular inverse to the map $g$ of (ii) above. $\Gamma$ is thus also the graph of $g^{-1}$ and, in particular, is isomorphic to $D$. Since $g$ is an isomorphism of curves, and $D$ is of pure dimension 1, it follows that $C$ is of pure dimension one. This proves (iii).

To see (iv), we need some more notation.

Note that $D \subset \mathbb{P}_2^1 \setminus \{(1; 0; 0)\}$, (because there exists no $[v] \in \mathbb{P}_2^1$ such that $I.v = 0$). Thus there is a regular map:

$$\theta : D \to \mathbb{P}_2^1$$

$$[t_0 : t_1 : t_2] \mapsto [t_1 : t_2]$$

Let $\Delta(t_1, t_2)$ be the discriminant polynomial of the characteristic polynomial $\phi_{t_1A+t_2A^*}$ of $t_1A + t_2A^*$. Clearly $\Delta(t_1, t_2)$ is a homogeneous polynomial of degree 4 in $(t_1, t_2)$, and it is not the zero polynomial because, for example, $\Delta(1, 0) \neq 0$, for $\Delta(1, 0)$ is the discriminant of $\phi_A$, which has distinct roots (=the distinct eigenvalues of $A$) by the assumptions [2] on $A$. Let $\Sigma \subset \mathbb{P}_2^1$ be the zero locus of $\Delta$, which is a finite set of points. Note that the fibre $\theta^{-1}(1 : \mu)$ consists of all $[t : 1 : \mu] \in D$ such that $-t$ is an eigenvalue of $A + \mu A^*$, which are at most four in number. Similarly the fibres $\theta^{-1}((\lambda : 1))$ are also finite. Thus the subset of $D$ defined by:

$$F := \theta^{-1}(\Sigma)$$

is a finite subset of $D$. $F$ is precisely the set of points $[t] = [t_0 : t_1 : t_2]$ such that $B(-, t) = (t_0I + t_1A + t_2A^*)$ has 0 as a repeated eigenvalue.

Since $\pi_2 : \Gamma \to D$ is 1-1, the inverse image:

$$F_1 = \pi_2^{-1}(F) \subset \Gamma$$

is a finite subset of $\Gamma$.

We will now prove that for each irreducible component $\Gamma_\alpha$ of $\Gamma$, and each point $x = ([a], [b])$ in $\Gamma_\alpha \setminus F_1$, the four equations $\{B_i(v, t) = 0\}_{i=1}^4$ are the generators of the ideal of the variety $\Gamma_\alpha$ in an affine neighbourhood of $x$, where $B_i(v, t)$ are the components of $B(v, t)$ with respect to a fixed basis of $\mathbb{C}^4$. Since $F_1$ is a finite set, this will prove (iv), because the multiplicity of $\Gamma_\alpha$ in the intersection cycle of the four divisors $D_i = B_i(v, t) = 0$ in $\mathbb{P}_2^1 \times \mathbb{P}_2^1$ is determined by generic points on $\Gamma_\alpha$, for example all points of $\Gamma_\alpha \setminus F_1$. We will prove this by showing that for $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$, the four divisors $(B_i(v, t) = 0)$ intersect transversely at $x$.

So let $\Gamma_\alpha$ be some irreducible component of $\Gamma$, with $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$.

Fix an $a \in \mathbb{C}^4$ representing $[a] \in C_\alpha := \pi_1(\Gamma_\alpha)$, and also fix $b \in \mathbb{C}^3$ representing $[b] = g([a]) \in g(C_\alpha)$. Also fix a 3-dimensional linear complement $V_1 := T_{[a]}(\mathbb{P}_2^1) \subset \mathbb{C}^4$ to $a$ and similarly, fix a 2-dimensional linear complement $V_2 = T_{[b]}(\mathbb{P}_2^1) \subset \mathbb{C}^3$ to $b$. (The notation comes from the fact that $T_{[v]}(\mathbb{P}_2^1) \simeq \mathbb{C}^{n+1}/\mathbb{C}v$, which we are identifying non-canonically with these respective complements $V_i$). These complements also provide local coordinates in the respective projective spaces as follows. Set coordinate charts $\phi$ around $[a] \in \mathbb{P}_2^1$ by $[v] = \phi(u) := [a + u]$, and $\psi$ around $[b] \in \mathbb{P}_2^1$ by $[t] = \psi(s) := [b + s]$, where $u \in V_1 \simeq \mathbb{C}^3$, and $s \in V_2 \simeq \mathbb{C}^2$. The images $\phi(V_1)$ and $\psi(V_2)$ are affine neighbourhoods of $[a]$ and $[b]$ respectively. These charts are like `stereographic
projection’ onto the tangent space and depend on the initial choice of \(a\) (resp. \(b\)) representing \([a]\) (resp. \([b]\)), and are not the standard coordinate systems on projective space, but more convenient for our purposes.

Then the local affine representation of \(B(v, t)\) on the affine open \(V_1 \times V_2 = \mathbb{C}^3 \times \mathbb{C}^2\), which we denote by \(\beta\), is given by:

\[
\beta(u, s) := B(a + u, b + s)
\]

Note that \( \ker B(a, -) = \mathbb{C}b \), where \([b] = g([a])\), so that \(B(a, -)\) passes to the quotient as an isomorphism:

\[
B(a, -) : V_2 \longrightarrow W([a])
\]  

(5)

where \(W([a])\) is 2-dimensional.

Similarly, since \(B(-, b)\) has one dimensional kernel \(\mathbb{C}a = K(b) \subset \mathbb{C}^4\), by (3) above, we also have the other isomorphism:

\[
B(-, b) : V_1 \longrightarrow \text{Im } B(-, b)
\]  

(6)

where \(\text{Im } B(-, b)\) is 3-dimensional, therefore.

Now one can easily calculate the derivative \(D\beta(0, 0)\) of \(\beta\) at \((u, s) = (0, 0)\). Let \((X, Y) \in V_1 \times V_2\). Then, by bilinearity of \(B\), we have:

\[
\beta(X, Y) - \beta(0, 0) = B(a + X, b + Y) - B(a, b) = B(X, b) + B(a, Y) + B(X, Y)
\]

Now since \(B(X, Y)\) is quadratic, it follows that:

\[
D\beta(0, 0) : V_1 \times V_2 \rightarrow \mathbb{C}^4
\]  

(7)

\[ (X, Y) \mapsto B(X, b) + B(a, Y) \]

By the equations (5) and (6) above, we see that the image of \(D\beta(0, 0)\) is precisely \(\text{Im } B(-, b) + W([a])\).

Claim: For \(([a], [b]) \in \Gamma_\alpha \setminus F_1\), the space \(\text{Im } B(-, b) + W([a])\) is all of \(\mathbb{C}^4\).

Proof of Claim: Denote \(T := B(-, b)\) for brevity. Clearly \(a \in \text{W([a])}\) by definition of \(W([a])\). Also, \(a \in \ker T = K(b)\). Then it is enough to show that 0 \(\neq a\) is not in the image of \(T\). For, if \(a \in \text{Im } T\), we would have \(a = Tw\) for some \(w \notin K(b) = \ker T\) and \(w \neq 0\). In fact \(w\) is not a multiple of \(a\) since \(Tw = a \neq 0\) whereas \(a \in \ker T\). Thus we would have \(T^2w = 0\), and completing \(e_1 = a = Tw, e_2 = w\) to a basis \(\{e_i\}_{i=1}^4\) of \(\mathbb{C}^4\), the matrix of \(T\) with respect to such a basis would be of the form:

\[
\begin{bmatrix}
0 & 1 & * & * \\
0 & 0 & * & * \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{bmatrix}
\]

Thus \(T = B(-, b)\) would have 0 as a repeated eigenvalue. But we have stipulated that \(([a], [b]) \notin F_1\), so that \([b] \notin F\), and hence \(B(-, b)\) does not have 0 as a repeated eigenvalue. Hence the non-zero vector \(a \in \text{W([a])}\) is not in \(\text{Im } T\). Since \(\text{Im } T\) is 3-dimensional, we have \(\mathbb{C}^4 = \text{Im } T + \text{W([a])}\), and this proves the claim. 

In conclusion, all the points of \(\Gamma_\alpha \setminus F_1\) are in fact smooth points of \(\Gamma_\alpha\), and the local equations for \(\Gamma_\alpha\) in a small neighbourhood of such a point are precisely the four equations \(\beta_i(u, s) = 0, 1 \leq i \leq 4\). This proves (iv), and the lemma. 

\[\Box\]
4. Some algebraic bundles

We construct an algebraic line bundle with a (regular) global section over $C$. By showing that this line bundle has positive degree, we will conclude that the section has zeroes in $C$. Any zero of this section will yield a flag of the kind required by Lemma 2.1. One of the technical complications is that none of the bundles we define below are allowed to use the hermitian metric on $V$, orthogonal complements, orthonormal bases etc., because we wish to remain in the $C$-algebraic category. As a general reference for this section and the next, the reader may consult [3].

**Definition 4.1.** For $0 \neq v \in V = \mathbb{C}^4$, we will denote the point $[v] \in \mathbb{P}_C^3$, by $v$, whenever no confusion is likely, to simplify notation. We have already denoted the vector subspace $\mathbb{C}$-span$\{v, Av, A^*v\} \subset \mathbb{C}^4$ as $W(v)$. Further define $W_3(v) := W(v) + AW(v)$, and $\tilde{W}_3(v) := W(v) + A^*W(v)$. Clearly both $W_3(v)$ and $\tilde{W}_3(v)$ contain $W(v)$.

Since $A$ and $A^*$ have no common eigenvectors, we have $\dim W(v) \geq 2$ for all $v \in \mathbb{P}_C^3$, and $\dim W(v) = 2$ for all $v \in C$, because of the defining equation $v \wedge Av \wedge A^*v = 0$ of $C$. Also, since $\dim W(v) = 2 = \dim AW(v)$ for $v \in C$, and since $0 \neq Av \in W(v) \cap AW(v)$, we have $\dim W_3(v) \leq 3$ for all $v \in C$. Similarly $\dim \tilde{W}_3(v) \leq 3$ for all $v \in C$.

If there exists a $v \in C$ such that $\dim W_3(v) = 2$, then we are done. For, in this case $W_3(v)$ must equal $W(v)$ since it contains $W(v)$. Then the dimension $\dim \tilde{W}_3(v) = 2$ or $= 3$. If it is $2$, $W_3(v)$ will be a 2-dimensional invariant space for both $A$ and $A^*$, and the main theorem will follow by Lemma 2.2. If $\dim \tilde{W}_3(v) = 3$, then the flag:

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = \tilde{W}_3(v) \subset W_4 = V$$

satisfies the requirements of (ii) in the Lemma 2.1, and we are done. Similarly, if there exists a $v \in C$ with $\dim \tilde{W}_3(v) = 2$, we are again done. Hence we may assume that:

$$\dim W_3(v) = \dim \tilde{W}_3(v) = 3 \quad \text{for all} \quad v \in C$$  \hspace{1cm} (8)

In the light of the above:

**Remark 4.2.** We are reduced to the situation where the following condition holds:

For each $v \in C$, $\dim W(v) = 2$, $\dim W_3(v) = \dim \tilde{W}_3(v) = 3$.

Now our main task is to prove that there exists a $v \in C$ such that the two 3-dimensional subspaces $W_3(v)$ and $\tilde{W}_3(v)$ are the same. In that event, the flag

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = W(v) + AW(v) \subset W_4 = V$$

will meet the requirements of (ii) of the Lemma 2.1. The remainder of this discussion is aimed at proving this.

**Definition 4.3.** Denote the trivial rank 4 algebraic bundle on $\mathbb{P}_C^3$ by $O^4_{\mathbb{P}_C^3}$, with fibre $V = \mathbb{C}^4$ at each point (following standard algebraic geometry notation). Similarly, $O^4_C$ is the trivial bundle on $C$. In $O^4_{\mathbb{P}_C^3}$, there is the tautological line-subbundle $O_{\mathbb{P}_C^3}(-1)$, whose fibre at $v$ is $\mathbb{C}v$. Its restriction to the curve $C$ is denoted as $W_1 := O_{C}(-1)$.

There are also the line subbundles $A^*O_{\mathbb{P}_C^3}(-1)$ (respectively $A^*O_{\mathbb{P}_C^3}(-1)$) of $O^4_{\mathbb{P}_C^3}$, whose fibre at $v$ is $Av$ (respectively $A^*v$). Both are isomorphic to $O_{\mathbb{P}_C^3}(-1)$ (via the global linear automorphisms $A$ (resp. $A^*$) of $V$). Similarly, their restrictions $A^*O_{C}(-1)$, $A^*O_{C}(-1)$, both isomorphic to $O_{C}(-1)$. Note that throughout what follows, bundle isomorphism over any variety $X$ will mean algebraic isomorphism, i.e. isomorphism of the corresponding sheaves of algebraic sections as $O_X$-modules.

Denote the rank 2 algebraic bundle with fibre $W(v) \subset V$ at $v \in C$ as $W_2$. It is an algebraic subbundle of $O^4_{\mathbb{P}_C^3}$, for its sheaf of sections is the restriction of the subsheaf $O^4_{\mathbb{P}_C^3} + A^*O_{\mathbb{P}_C^3} + A^*O_{\mathbb{P}_C^3}(-1) \subset O^4_{\mathbb{P}_C^3}$.
to the curve $C$, which is precisely the subvariety of $\mathbb{P}^3_C$ on which the sheaf above is locally free of rank 2 (=rank 2 algebraic bundle).

Denote the rank 3 algebraic subbundle of $\mathcal{O}_C^4$ with fibre $W_3(v) = W(v) + AW(v)$ (respectively $\tilde{W}_3(v) = W(v) + A^*W(v)$) by $\mathcal{W}_3$ (respectively $\tilde{\mathcal{W}}_3$). Both $\mathcal{W}_3$ and $\tilde{\mathcal{W}}_3$ are of rank 3 on $C$ because of (4.2) above, and both contain $\mathcal{W}_2$ as a subbundle. We denote the line bundles $\wedge^2 \mathcal{W}_2$ by $L_2$, and $\wedge^3 \mathcal{W}_3$ (resp. $\wedge^3 \tilde{\mathcal{W}}_3$) by $L_3$ (resp. $\tilde{L}_3$). Then $L_2$ is a line subbundle of $\wedge^2 \mathcal{O}_C^4$, and $L_3$, $\tilde{L}_3$ are line subbundles of $\wedge^3 \mathcal{O}_C^4$.

Finally, for $X$ any variety, with a bundle $E$ on $X$ which is a subbundle of a trivial bundle $\mathcal{O}_X^m$, the annihilator of $E$ is defined as:

$$\text{Ann}E = \{ \phi \in \text{hom}_X(\mathcal{O}_X^m, \mathcal{O}_X) : \phi(E) = 0 \}$$

Clearly, by taking $\text{hom}_X(-, \mathcal{O}_X)$ of the exact sequence

$$0 \to E \to \mathcal{O}_X^m \to \mathcal{O}_X^m / E \to 0$$

the bundle

$$\text{Ann}E \cong \text{hom}_X(\mathcal{O}_X^m / E, \mathcal{O}_X) = (\mathcal{O}_X^m / E)^*$$

where * always denotes the (complex) dual bundle.

**Lemma 4.4.** Denote the bundle $\mathcal{W}_3 / \mathcal{W}_2$ (resp. $\tilde{\mathcal{W}}_3 / \mathcal{W}_2$) by $\Lambda$ (resp. $\tilde{\Lambda}$). Then we have the following identities of bundles on $C$:

(i):

$$0 \to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0$$

(ii):

$$\mathcal{L}_3 \cong \mathcal{L}_2 \otimes \Lambda \quad \text{and} \quad \tilde{\mathcal{L}}_3 \cong \mathcal{L}_2 \otimes \tilde{\Lambda}$$

(iii):

$$\wedge^2 \text{Ann} \mathcal{W}_2 \cong \wedge^2 \mathcal{W}_2$$

(iv):

$$\Lambda \cong \tilde{\Lambda}$$

(v):

$$\mathcal{L}_2 \cong \Lambda \otimes \mathcal{O}_C(-1) \cong \tilde{\Lambda} \otimes \mathcal{O}_C(-1)$$

(vi):

$$\text{hom}_C(\mathcal{L}_3, \tilde{\Lambda}^*) \cong \mathcal{L}_2 \otimes \tilde{\Lambda}^* \cong \mathcal{L}_2 \otimes \mathcal{O}_C(-2)$$

**Proof:** From the definition of $\Lambda$, we have the exact sequence:

$$0 \to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0$$

from which it follows that:

$$0 \to \Lambda \to \mathcal{O}_C^4 / \mathcal{W}_2 \to \mathcal{O}_C^4 / \mathcal{W}_3 \to 0$$

is exact. Taking $\text{hom}_C(-, \mathcal{O}_C)$ of this exact sequence yields the exact sequence:

$$0 \to \text{Ann} \mathcal{W}_3 \to \text{Ann} \mathcal{W}_2 \to \Lambda^* \to 0$$

Now, via the canonical isomorphism $\wedge^3 V \to V^*$ which arises from the non-degenerate pairing
\[3 \bigwedge V \otimes V \to 4 \bigwedge V \simeq \mathbb{C}\]

it is clear that \(\text{Ann} W_3 \simeq \bigwedge^3 W_3 = L_3\).

Thus the first and third exact sequences of (i) follow. The proofs of the second and fourth are similar. From the first exact sequence in (i), it follows that \(\bigwedge^3 W_3 \simeq \bigwedge^2 W_2 \otimes \Lambda\). This implies the first identity of (ii). Similarly the second exact sequence of (i) implies the other identity of (ii).

Since for every line bundle \(\gamma, \gamma \otimes \gamma^*\) is trivial, we get from the first identity of (ii) that \(L_2 \simeq L_3 \otimes \Lambda^*\). From third exact sequence in (i) it follows that \(\bigwedge^2 \text{Ann} W_2 \simeq L_3 \otimes \Lambda^*,\) and this implies (iii).

To see (iv), note that
\[
\Lambda \simeq \frac{W_2 + AW_2}{W_2} \simeq \frac{AW_2}{W_2} \otimes \bigwedge W_2
\]
The automorphism \(A^{-1}\) of \(V\) makes the last bundle on the right isomorphic to the line bundle \(W_2/(W_2 \cap A^{-1} W_2)\) (note all these operations are happening inside the rank 4 trivial bundle \(O_C^4\)). Similarly, \(\Lambda\) is isomorphic (via the global isomorphism \(A^{-1}\) of \(V\)) to the line bundle \(W_2/(W_2 \cap A^{-1} W_2)\). But for each \(v \in C, W(v) \cap A^{-1} W(v) = C(v) = W(v) \cap A^{-1} W(v)\), from which it follows that the line subbundles \(W_2 \cap A^{-1} W_2\) and \(W_2 \cap A^{-1} W_2\) of \(W_2\) are the same (= \(W_1 \simeq O_C(-1)\)). Thus \(\Lambda \simeq \tilde{\Lambda}\), proving (iv).

To see (v), we need another exact sequence. For each \(v \in C\), we noted in the proof of (iv) above that \(C(v) = W(v) \cap A^{-1} W(v)\). Thus the sequence of bundles:
\[
0 \to O_C(1) \to W_2 \to \frac{W_2}{W_2 \cap A^{-1} W_2} \to 0
\]
is exact. But, as we noted in the proof of (iv) above, the bundle on the right is isomorphic to \(\Lambda\), so that
\[
0 \to O_C(1) \to W_2 \to \Lambda \to 0
\]
is exact. Hence \(L_2 = \bigwedge^2 W_2 \simeq \Lambda \otimes O_C(-1)\). The other identity follows from (iv), thus proving (v).

To see (vi) note that we have by (ii) \(L_3^* \simeq L_2^* \otimes \Lambda^*\). Thus
\[
\text{hom}_C(L_3, \Lambda^*) \simeq L_3^* \otimes \Lambda^* \simeq L_2^* \otimes \Lambda^* \otimes \tilde{\Lambda}^*
\]
However, since by (iv), \(\Lambda \simeq \tilde{\Lambda}\), we have \(\text{hom}_C(L_3, \Lambda^*) \simeq L_2^* \otimes \Lambda^* 2\). Now, substituting \(\Lambda^* = L_2^* \otimes O_C(-1)\) from (v), we have the rest of (vi). Hence the lemma.

We need one more bundle identity:

**Lemma 4.5.** There is a bundle isomorphism:
\[
L_2 \simeq g^* O_D(1) \otimes O_C(-2)
\]

**Proof:** When \([t] = [t_0 : t_1 : t_2] = g([v])\), we saw in (3) that the linear map \(B(v, -) : \mathbb{C}^3 \to \mathbb{C}^4\) acquires a 1-dimensional kernel, which is precisely the line \(Ct\), which is the fibre of \(O_D(-1)\) at \([t]\). The image of \(B(v, -)\) was the 2-dimensional span \(W(v)\) of \(v, Av, A^* v\), as noted there. Thus for \(v \in C\), \(B(-, -)\) induces a canonical isomorphism of vector spaces:
\[
O_C(-1) \otimes (\mathbb{C}^3/O_D(-1))_{g(v)} \to W_{2,v}
\]
which, being defined by the global map \(B(-, -)\), gives an isomorphism of bundles:
\[
O_C(-1) \otimes g^* (O_D^3/O_D(-1)) \simeq W_2
\]
From the short exact sequence:
\[
0 \to O_D(-1) \to O_D^3 \to O_D^3/O_D(-1) \to 0
\]
it follows that \( \bigwedge^2(\mathcal{O}_D^3/\mathcal{O}_D(-1)) \simeq \mathcal{O}_D(1) \). Thus:
\[
\mathcal{L}_2 = \bigwedge^2 W_2 \simeq g^* \left( \bigwedge^2 (\mathcal{O}_D^3/\mathcal{O}_D(-1)) \right) \otimes \mathcal{O}_C(-2) \\
\simeq g^* \mathcal{O}_D(1) \otimes \mathcal{O}_C(-2)
\]
This proves the lemma. \( \square \)

5. Degree computations

In this section, we compute the degrees of the various line bundles introduced in the previous section.

**Definition 5.1.** Note that an irreducible complex projective curve \( C \), as a topological space, is a canonically oriented pseudomanifold of real dimension 2, and has a canonical generator \( \mu_C \in H_2(C, \mathbb{Z}) = \mathbb{Z} \). Indeed, it is the image \( \pi_* \mu_C \), where \( \pi : \tilde{C} \to C \) is the normalisation map, and \( \mu_C \in H_2(\tilde{C}, \mathbb{Z}) = \mathbb{Z} \) is the canonical orientation class for the smooth connected compact complex manifold \( \tilde{C} \), where \( \pi_* : H_2(\tilde{C}, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \) is an isomorphism for elementary topological reasons.

If \( C = \bigcup_{\alpha=1}^n C_{\alpha} \) is a projective curve of pure dimension 1, with the curves \( C_{\alpha} \) as irreducible components, then since the intersections \( C_{\alpha} \cap C_{\beta} \) are finite sets of points (or empty), \( H_2(C, \mathbb{Z}) = \bigoplus \alpha H_2(C_{\alpha}, \mathbb{Z}) \). Letting \( \mu_\alpha \) denote the canonical orientation classes of \( C_{\alpha} \) as above, there is a unique class \( \mu_C = \sum \alpha \mu_\alpha \in H_2(C, \mathbb{Z}) \).

Thinking of \( C \) as an oriented 2-pseudomanifold, \( \mu_C \) is just the sum of all the oriented 2-simplices of \( C \).

If \( \mathcal{F} \) is a complex line bundle on \( C \), it has a first Chern class \( c_1(\mathcal{F}) \in H^2(X, \mathbb{Z}) \), and the degree of \( \mathcal{F} \) is defined by:
\[
\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_C \rangle \in \mathbb{Z}
\]

It is known that a complex line bundle on a pseudomanifold is topologically trivial if and only if its first Chern class is zero. In particular, if an algebraic line bundle on a projective variety has non-zero degree, then it is topologically (and hence algebraically) non-trivial.

Finally, if \( i : C \hookrightarrow \mathbb{P}^n \) is an (algebraic) embedding of a curve in some projective space, we define the degree of the bundle \( \mathcal{O}_C(1) = i^* \mathcal{O}_{\mathbb{P}^n}(1) \) as the degree of the curve \( C \) (in \( \mathbb{P}^n \)). We note that \( [C] := i_* (\mu_C) \in H_2(\mathbb{P}^n, \mathbb{Z}) \) is called the fundamental class of \( C \) in \( \mathbb{P}^n \), and by definition \( \deg C = \langle c_1(\mathcal{O}_C(1)), \mu_C \rangle = \langle c_1(\mathcal{O}_{\mathbb{P}^n}(1)), [C] \rangle \).

Geometrically, one intersects \( C \) with a generic hyperplane, which intersects \( C \) away from its singular locus in a finite set of points, and then counts these points of intersection with their multiplicity.

More generally, a complex projective variety \( X \subset \mathbb{P}^n \) of complex dimension \( m \) has a unique orientation class \( \mu_X \in H_2m(X, \mathbb{Z}) \). Its image in \( H_2m(\mathbb{P}^n, \mathbb{Z}) \) is denoted \( [X] \), and the degree \( \deg X \) of \( X \) is defined as \( \langle (c_1(\mathcal{O}_{\mathbb{P}^n}(1)))^m, [X] \rangle \). It is known that if \( X = V(F) \) for a homogeneous polynomial \( F \) of degree \( d \), then \( \deg X = d \).

We need the following remark later on:

**Remark 5.2.** If \( f : C \to D \) is a regular isomorphism of complex projective curves \( C \) and \( D \), both of pure dimension 1, and if \( \mathcal{F} \) is a complex line bundle on \( D \), then \( \deg f^* \mathcal{F} = \deg \mathcal{F} \). This is because \( f_* (\mu_C) = \mu_D \), so that
\[
\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_D \rangle = \langle c_1(\mathcal{F}), f_* \mu_C \rangle = \langle f^* c_1(\mathcal{F}), \mu_C \rangle = \langle c_1(f^* \mathcal{F}), \mu_C \rangle = \deg f^* \mathcal{F}
\]

Now we can compute the degrees of all the line bundles introduced.
**Lemma 5.3.** The degrees of the various line bundles above are as follows:

(i): \( \deg O_C(1) = \deg C = 6 \)

(ii): \( \deg O_D(1) = \deg D = 4 \)

(iii): \( \deg L_2 = 8 \)

(iv): \( \deg \text{hom}_C(L_3, \tilde\Lambda^*) = \deg (L_2^3 \otimes O_C(-2)) = 12 \)

**Proof:**

We denote the image of orientation class \( \mu_\Gamma \) of the curve \( \Gamma \) (see definition 3.3 for the definition of \( \Gamma \)) in \( H_2(\mathbb{P}^1_C \times \mathbb{P}^2_C, \mathbb{Z}) \) by \( [\Gamma] \). By the part (iv) of the Lemma 3.4, we have that the homology class \( [\Gamma] \) is the same as the homology class of the intersection cycle defined by the four divisors \( D_i := (B_i(v, t) = 0) \) inside \( H_2(\mathbb{P}^1_C \times \mathbb{P}^2_C, \mathbb{Z}) \).

By the generalised Bezout theorem in \( \mathbb{P}^1_C \times \mathbb{P}^2_C \), the homology class of the last-mentioned intersection cycle is the homology class Poincare-dual to the cup product

\[ d := d_1 \cup d_2 \cup d_3 \cup d_4 \]

where \( d_i \) is the first Chern class of the the line bundle \( L_i \) corresponding to \( D_i \), for \( i = 1, 2, 3, 4 \). (See \[1\], p. 237, Ex.2).

Since each \( B_i(v, t) \) is separately linear in \( v, t \), the line bundle defined by the divisor \( D_i \) is the bundle \( \pi_1^*O_{\mathbb{P}^1_C}(1) \otimes \pi_2^*O_{\mathbb{P}^2_C}(1) \), where \( \pi_1, \pi_2 \) are the projections to \( \mathbb{P}^1_C \) and \( \mathbb{P}^2_C \) respectively. If we denote the hyperplane classes which are the generators of the cohomologies \( H^2(\mathbb{P}^1_C, \mathbb{Z}) \) and \( H^2(\mathbb{P}^2_C, \mathbb{Z}) \) by \( x \) and \( y \) respectively, we have:

\[ d_i = c_1(L_i) = \pi_1^*(x) + \pi_2^*(y) \]

Then we have, from the cohomology ring structures of \( \mathbb{P}^1_C \) and \( \mathbb{P}^2_C \) that \( x \cup x \cup x \cup x = y \cup y \cup y = 0 \). Hence the cohomology class in \( H^8(\mathbb{P}^1_C \times \mathbb{P}^2_C, \mathbb{Z}) \) given by the cup-product of \( d_i \) is:

\[ d := d_1 \cup d_2 \cup d_3 \cup d_4 = (\pi_1^*(x) + \pi_2^*(y))^4 = 4\pi_1^*(x^4)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2) \]

where \( x^3 = x \cup x \cup x \) etc. By part (ii) of the lemma 3.4, the map \( \pi_1 : \Gamma \to C \) is an isomorphism, so applying the remark 5.2 to it, we have:

\[
\deg O_C(1) = \deg \pi_1^*O_C(1) \\
= \left\langle c_1(\pi_1^*(O_{\mathbb{P}^1_C}(1))), [\Gamma] \right\rangle \\
= \left\langle c_1(\pi_1^*(O_{\mathbb{P}^1_C}(1))), \mathbb{P}^3_C \times \mathbb{P}^2_C \right\rangle \\
= \left\langle \pi_1^*(x) \cup (4\pi_1^*(x^3)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2)), \mathbb{P}^3_C \times \mathbb{P}^2_C \right\rangle \\
= \left\langle 6\pi_1^*(x^3) \cup \pi_2^*(y^2), \mathbb{P}^3_C \times \mathbb{P}^2_C \right\rangle \\
= 6
\]

where we have used the Poincare duality cap-product relation \( [\Gamma] = [\mathbb{P}^3_C \times \mathbb{P}^2_C] \cap d \) mentioned above, and that \( \pi_1^*(x^3) \cup \pi_2^*(y^2) \) is the generator of \( H^{10}(\mathbb{P}^3_C \times \mathbb{P}^2_C, \mathbb{Z}) \), so evaluates to 1 on the orientation class \( [\mathbb{P}^3_C \times \mathbb{P}^2_C] \), and \( x^4 = 0 \). This proves (i).

The proof of (ii) is similar, we just replace \( C \) by \( D \), and \( \pi_1 \) by \( \pi_2 \), and \( \pi_1^*(x) \) by \( \pi_2^*(y) \) in the equalities of (i) above, and get 4 (as one should expect, since \( D \) is defined by a degree 4 homogeneous polynomial in \( \mathbb{P}^2_C \)). This proves (ii).

For (iii), we use the identity of lemma 4.3 that \( L_2 = g^*O_D(1) \otimes O_C(-2) \), and the remark 5.2 applied to the isomorphism of curves \( g : C \to D \) (part (iii) of the lemma 3.4) to conclude that \( \deg L_2 = \deg D - 2\deg C = 4 - 12 = -8 \), by (i) and (ii) above, so that \( \deg L_2 = 8 \).

For (iv), we have by (vi) of the lemma 4.4 that \( \text{hom}_C(L_3, \tilde\Lambda^*) \simeq L_2^3 \otimes O_C(-2) \), so that its degree is \( 3\deg L_2 - 2\deg C = 24 - 12 = 12 \) by (i) and (iii) above.
This proves the lemma.

From (iv) of the lemma above, we have the:

**Corollary 5.4.** The line bundle $\text{hom}_C(L_3, \tilde{\Lambda}^*)$ is a non-trivial line bundle.

### 6. Proof of the Main Theorem

**Proof of Theorem 1.1:** By the third exact sequence in (i) of the lemma 4.4, we have a bundle morphism $s$ of line bundles on $C$ defined as the composite:

$$\text{Ann}W_3 = L_3 \rightarrow \text{Ann}W_2 \rightarrow \tilde{\Lambda}^* = \text{Ann}W_2/\text{Ann}\tilde{W}_3$$

which vanishes at $v \in C$ if and only if the fibre $\text{Ann}W_3,v$ is equal to the fibre $\text{Ann}\tilde{W}_3,v$ inside $\text{Ann}W_2,v$. At such a point $v \in C$, we have $\text{Ann}W_3,v = \text{Ann}\tilde{W}_3,v$, so that $W_3,v = W(v) + AW(v) = \tilde{W}_3,v = W(v) + A^*W(v)$. But this morphism $s$ is a global section of the bundle $\text{hom}_C(L_3, \tilde{\Lambda}^*)$, which is not a trivial bundle by the corollary 5.4 of the last section.

Thus there exists a $v \in C$, satisfying $s(v) = 0$, and consequently the flag

$$0 \subset W_1 = W_1,v \subset W_2 = W_2,v = W(v) = \mathbb{C}\text{-span}\{v, Av, A^*v\} \subset W_3 = W_3,v = W(v) + AW(v) = W(v) + A^*W(v) = \tilde{W}_3,v \subset W_4 = V = \mathbb{C}^4$$

satisfies the requirements of (ii) of lemma 2.1, and the main theorem 1.1 follows.

**Remark 6.1.** Note that since $\dim C = 1$, then the set of points $v \in C$ such $s(v) = 0$, where $s$ is the section above, will be a finite set. Then the set of flags that satisfy (ii) of lemma 2.1 which tridiagonalise $A$ of the kind considered above (viz. $A$ satisfying the assumptions of 3.1), will only be finitely many (at most 12 in number!).

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Stat-Math Unit, Indian Statistical Institute, RVCE P.O., Bangalore 560 059, India