Rapid Path Planning for Dubins Vehicles under Environmental Currents

Kushboo Mittal†*  Junnan Song‡†  Shalabh Gupta‡*  Thomas A. Wettergren‡

Abstract—This paper presents a rapid (real time) solution to the minimum-time path planning problem for Dubins vehicles under environmental currents (wind or ocean currents). Real-time solutions are essential in time-critical situations (such as rescheduling under dynamically changing environments or tracking fast moving targets). Typically, Dubins problem requires to solve for six path types; however, due to the presence of currents, four of these path types require to solve the root-finding problem involving transcendental functions. Thus, the existing methods result in high computation times and their applicability for real-time applications is limited. In this regard, in order to obtain a real-time solution, this paper proposes a novel approach where only a subset of two Dubins path types (LSL and RSR) are used which have direct analytical solutions in the presence of currents. However, these two path types do not provide full reachability. We show that by extending the feasible range of circular arcs in the LSL and RSR path types from $2\pi$ to $4\pi$: 1) full reachability of any goal pose is guaranteed, and 2) paths with lower time costs as compared to the corresponding $2\pi$-arc paths can be produced. Theoretical properties are rigorously established, supported by several examples, and evaluated in comparison to the Dubins solutions by extensive Monte-Carlo simulations.

Index Terms—Dubins paths, path planning, environmental currents, curvature-constrained vehicles.

LIST OF SYMBOLS

| Symbol | Description |
|--------|-------------|
| $\mathbf{v}$ | Vehicle velocity vector (m/s, m/s) |
| $\mathbf{v}_w$ | Current velocity vector (m/s, m/s) |
| $\mathbf{v}_{net}$ | Net velocity vector which is the vector sum of $\mathbf{v}$ and $\mathbf{v}_w$ (m/s, m/s) |
| $w_x$, $w_y$ | Components of $\mathbf{v}_w$ along the x and y axes (m/s) |
| $\theta$ | Vehicle heading (rad) |
| $\theta_w$ | Current heading (rad) |
| $u$ | Turn rate (rad/s) |
| $r$ | Turning radius (m) |
| $(x_0, y_0, \theta_0)$ | Start pose (m, m, rad) |
| $(x_f, y_f, \theta_f)$ | Goal pose (m, m, rad) |
| $k$ | Feasible range parameter for an LSL or RSR path (-) |
| $\alpha^k$, $\alpha_{\text{inf/sup}}^k$, $\beta^k$, $\gamma^k$, $\omega^k_{\text{LSL/RSR}}$, $\phi_{k_1,k_2}$ | Turning angle of the first arc of an LSL or RSR path (rad) | Minimum [supremum] of $\alpha$ for a given $k$ (rad) | Length of the straight line segment of an LSL or RSR path (m) | Turning angle of the last arc of an LSL or RSR path (rad) | Rotation of a reachability ray for a given $\alpha$ and $k$ for an LSL/RSR path (rad) | Rotation of $\omega^k_{\text{LSL/RSR}}(\alpha)$ by $\pi$ (rad) | Rotation of the line segment joining the centers of rotation of two path types with parameters $k_1$ and $k_2$, respectively (rad) | Total travel time (s) | Travel time of the $2\pi$-arc [4$\pi$-arc] path (s) | Travel time of the optimal Dubins path (s) |

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* These authors contributed equally to this work.
† Department of Electrical and Computer Engineering, University of Connecticut, Storrs, CT 06269, USA.
‡ Naval Undersea Warfare Center, Newport, RI 02841, USA.
* Corresponding Author (email id: shalabh.gupta@uconn.edu)

A fundamental problem in robotics is to find the minimum-time path from a start pose to a goal pose while considering several constraints on vehicles such as bounded curvature [1][2], bounded velocity [3][4] and bounded acceleration [5][6]. In particular, bounded curvature implies that the vehicle’s turning is subject to a non-zero minimum turning radius corresponding to its speed and maximum turn rate.

Dubins [7][8] used a geometrical approach to show that in absence of obstacles, the shortest path for a curvature-constrained vehicle between a pair of poses must be one of the following six path types (also known as the Dubins curves): LSL, RSR, LSR, RSL, LRL and RLR, where $L(R)$ refers to a left (right) turn with the maximum curvature, and $S$ indicates a straight line segment. Since each path type is composed of three segments, it is uniquely determined by three path parameters, which describe the angles of the circular arcs and the length of the straight line segment. Recently, the authors proposed the T* algorithm [9] which extended the Dubins approach to variable speed vehicles in obstacle-rich environments for time-optimal risk-aware motion planning. However, when environmental currents (e.g., wind or ocean currents) are present, the vehicle trajectory can be significantly distorted [10], resulting in a minimum-time trajectory which is different from the minimum-distance trajectory.

Along this line, the existing methods to compute the minimum-time trajectory for Dubins vehicles in the presence of environmental currents can be categorized into two types:
The Real-time Challenge

Although the above methods can produce the minimum-time trajectory for Dubins vehicles in the presence of static currents, their real-time application is limited due to their computational complexity. As shown in [11][13], the existing approaches require to solve for all six Dubins path types to find the minimum-time trajectory. Out of these six path types, only LSL and RSR paths have analytical solutions, while the remaining four path types require to solve a root-finding problem involving transcendental equations, which demand significant computational efforts. However, in dynamic situations (e.g., changing currents, adaptive exploration [16][17] and target tracking [18]), it is critical to obtain a real-time solution for fast replanning, which is the focus of this paper.

To motivate this further, we generated the computation time required to obtain the minimum-time path from all six Dubins path types, as shown in Fig. 2. Also, we compared this to the computation time required to get the minimum-time path from only the LSL and RSR path types. These computation times were obtained by averaging over 1000 randomly selected start and goal poses in an environment with steady currents. The simulations were run in MATLAB on a computer with a 2.4 GHz CPU and 8 GB RAM. It is seen that using only LSL and RSR paths takes $\sim 6.4 \times 10^{-4}$ s to get a solution. In contrast, using all six path types takes several orders of magnitude higher time to solve the transcendental equations. Furthermore, for practical applications, these numbers can become significantly larger for less powerful on-board processors. Moreover, these computation times depend on the non-linear solvers used. In addition, the implementation of these optimization solvers on on-board processors is challenging as compared to a system of equations with analytical solutions.

Example: The potential implications of computation times are shown with an example. Consider an underwater vehicle moving at 2.5 m/s in an environment with a time-varying current with a speed of 2 m/s. Now, suppose the current changes direction towards that of the vehicle motion, then a new path needs to be computed. Suppose that it takes $\sim 8.72$ s for the on-board processor to get a solution using all six path types. Then, the vehicle would drift by a distance of $8.72 \cdot (2 + 2.5) = 39.24$ m before it could compute a new path. In comparison, if it uses only LSL and RSR path types, then this drift would be as little as $6.4 \times 10^{-4} \cdot (2 + 2.5) = 0.0029$ m. Thus, computation time plays a crucial role in real-time path planning in dynamic environments.

Our Approach

Based on the above discussion, we propose a rapid (real-time) analytical solution as described below.

1) Proposed solution using $4\pi$-arc LSL and RSR paths: We propose a solution in the CF using only the LSL and RSR path types. However, the limitation of using only this subset of path types is the lack of full reachability, i.e., they cannot reach every goal pose in the presence of currents. To overcome the above limitation, we propose a simple yet powerful technique.
Instead of using the regular LSL and RSR paths where the arc angles are within a range of \([0, 2\pi]\), we propose to extend their arc range to \([0, 4\pi]\) [19]. Accordingly, we define the concepts of \(2\pi\)-arc and \(4\pi\)-arc paths below, where the parameters \(\alpha\) (\(\gamma\)) and \(\beta\) refer to the turning angle of the first (second) arc and the length of the straight line segment, respectively.

**Definition 1 (2\(\pi\)-arc Path).** An \(L^\alpha S^\beta L^\gamma\) or \(R^\alpha S^\beta R^\gamma\) path is called a 2\(\pi\)-arc path, if \(\alpha \in [0, 2\pi]\) and \(\gamma \in [0, 2\pi]\).

**Definition 2 (4\(\pi\)-arc Path).** An \(L^\alpha S^\beta L^\gamma\) or \(R^\alpha S^\beta R^\gamma\) path is called a 4\(\pi\)-arc path, if \(\alpha \in [0, 4\pi]\) and \(\gamma \in [0, 4\pi]\).

**Remark 1.** The six Dubins path types use the 2\(\pi\)-arcs.

**Remark 2.** It is shown that the 4\(\pi\)-arc LSL and RSR paths provide full reachability along with reduced total time costs as compared to the 2\(\pi\)-arc LSL and RSR paths.

*Example:* Figs. [1a] and [1b] show the minimum-time 2\(\pi\)-arc and 4\(\pi\)-arc paths, respectively, in both the IF and the CF. Fig. [1c] shows the optimal 2\(\pi\)-arc path, which is a RSR path with the total time cost of 20.91 s. In comparison, Fig. [1b] shows the optimal 4\(\pi\)-arc path, which is an LSL path with \(\gamma = 2.263\pi > 2\pi\) and the total time cost of 10.51 s. Intuitively, this happens because instead of traveling against the current, the vehicle spends more time on arcs which allows the current to help it to reach the goal in less time.

2) **Theoretical analysis of 4\(\pi\)-arc LSL and RSR paths:**

We present a rigorous theoretical analysis of the properties of 4\(\pi\)-arc LSL and RSR paths. First, we develop a comprehensive procedure for reachability analysis of the 2\(\pi\)-arc LSL and RSR paths. We present the conditions for full reachability using these two path types with the support from Lemmas [1-4]. The derivation of these conditions and the proofs of supporting lemmas are provided in Appendices [A] and [B], respectively. Next, it is numerically validated that the 2\(\pi\)-arc LSL and RSR paths fail to satisfy the reachability conditions under all goal poses and current velocities. Thus, we present Theorem [1] which provides a guarantee of full reachability using 4\(\pi\)-arc LSL and RSR paths. Further, it is established through Theorem [2] and Corollary [1] that the computational complexity of both 2\(\pi\)-arc and 4\(\pi\)-arc path solutions is the same. Along with providing full reachability, another important benefit of 4\(\pi\)-arc paths is their ability to generate faster, i.e., reduced time cost, paths in comparison to the 2\(\pi\)-arc paths, which is highlighted in Theorem [3]. Finally, Theorem [4] is presented to prove that \(\alpha, \gamma \in [0, 4\pi]\) is sufficient for optimality using LSL and RSR path types and thus further increasing of range is not needed. For validation of our approach, extensive Monte Carlo simulations are performed to compare the performance of Dubins solutions and the proposed 4\(\pi\)-arc path solutions.

3) **Comparison of 4\(\pi\)-arc LSL and RSR paths with Dubins:**

The solution obtained from the 4\(\pi\)-arc LSL and RSR paths might be sub-optimal for certain goal poses as compared to the one obtained from the six Dubins path types; however, the longer convergence time of the Dubins path solution might render it unsuitable for real-time applications.

For offline applications in static current environments, one can use the Dubins path types to compute the minimum-time path. In this regard, Section [6-A] provides a detailed comparison of the solution quality (i.e., travel time cost) obtained for the 4\(\pi\)-arc LSL and RSR solutions and the Dubins solutions. This analysis indicates that the advantage of the Dubins solutions over the 4\(\pi\)-arc LSL and RSR solutions in terms of travel time costs is not significant. Furthermore, upon adding the computation time costs, the advantage of Dubins solutions is further reduced. On the other hand, for time critical real-time applications (e.g., target tracking, planning under moving obstacles, and changing currents), 4\(\pi\)-arc paths provide rapid and reliable solutions without causing any vehicle drift. In contrast, the high computation times for Dubins solutions can cause vehicle drifts, thereby, resulting in longer sub-optimal trajectories which sometimes do not even converge to the goal pose. Section [6-E] presents a comparative analysis in the presence of dynamic currents, which highlights the benefits of the solutions obtained from the 4\(\pi\)-arc LSL and RSR paths over the ones obtained from the six Dubins paths.

**D. Our Contributions**

The paper makes the following novel contributions:

- Provides an analytical solution of the path planning problem for Dubins vehicles under environmental currents, where the solution is based on a novel concept of 4\(\pi\)-arc LSL and RSR paths and can be computed in real-time. In this regard, the paper presents the following:
  - A detailed analytical method to construct the reachability graphs of LSL and RSR paths.
  - A detailed derivation of the conditions under which 2\(\pi\)-arc LSL and RSR paths provide full reachability.
  - A mathematical proof of full reachability of the 4\(\pi\)-arc LSL and RSR paths under all conditions unlike the corresponding 2\(\pi\)-arc paths (Theorem [1]).
  - A mathematical proof that a solution using 4\(\pi\)-arc LSL and RSR paths can be obtained with the same computational workload as that needed for 2\(\pi\)-arc paths (Theorem [2] and Corollary [1]).
  - A mathematical proof that 4\(\pi\)-arc LSL and RSR paths provide reduced travel time costs as compared to the corresponding 2\(\pi\)-arc paths. (Theorem [3]).

- Theoretical properties of 4\(\pi\)-arc LSL and RSR paths are rigorously established and evaluated in comparison to Dubins solutions by extensive Monte-Carlo simulations.

**E. Organization**

The rest of the paper is organized as follows. Section [2] reviews the existing literature. Section [3] presents the path planning problem and its analytical solution. Section [4] presents a detailed analytical procedure for the reachability analysis of the 2\(\pi\)-arc LSL and RSR paths. Section [5] presents the theoretical properties of 4\(\pi\)-arc paths and shows their advantages over the 2\(\pi\)-arc paths. Section [6] presents the comparative evaluation results. Finally, the paper is concluded in Section [7] with recommendations for future work. Appendices [A] and [B] provide proofs of reachability conditions and supporting lemmas.
2. Literature Review

Recently, several papers [20] have addressed the path planning problem in the presence of currents. Garau et al. [21][22] studied the minimum-time path planning problem in marine environments with spatial current variability, where the time cost was defined as the sum of step-wise costs that are specified by the traveling distance over the vehicle speed in the presence of ocean currents. However, the drawback in their design is that infeasible paths are penalized rather than being prohibited. Petres et al. [23] presented the FM* algorithm to find the minimum-time path for underwater vehicles, where the time cost is defined over the inner product of the distance function and the current field; however, their cost function still penalizes rather than restricts infeasible paths. In this regard, Soulignac et al. [24] proposed a time cost function that projects the speed vector to both axes as opposed to taking its norm as in [22]. Accordingly, their method is restricted to feasible paths. In addition, energy based cost functions [25][26] have also been used for planning in the presence of ocean currents.

However, the above-mentioned methods ignore any kinematic motion constraints for vehicles. Along this line, Techy and Woolsey [11] addressed the minimum-time path planning problem for a curvature-constrained vehicle in constant wind, based on the fact that the circular arcs are distorted by the wind into the trochoïdal curves in the inertial frame. They derived analytical solutions for LSL and RSR candidate paths, while for other paths of LSR, RSL, LRL and RLR, they must solve certain transcendental equations to obtain solutions. However, as we show in Fig. 2, the root finding problem for transcendental equations can be computationally expensive.

In contrast, McGee et al. [12] studied the minimum-time path planning problem in the current frame. They first used Pontryagin’s Minimum Principle to demonstrate that the optimal path is comprised of straight line segments and curves of maximum turn rate. Then, they introduced the concept of a "virtual target" which starts at the goal state but moves in the opposite direction as the wind. In this setup, the minimum-time problem is simplified into a target interception problem, where the objective is to find the earliest interception point in the current frame so that the Dubins path can meet with the virtual target in minimum time. However, one must repeatedly check for the validity of possible interception points, which can be arbitrarily heavy to compute if the actual interception point lies far from the beginning search point.

In this regard, Bakolas et al. [13] directly solved for the interception point in the current frame by introducing an extra parameter of interception time. They also showed that when the wind speed is less than the vehicle speed, the vehicle has full reachability, i.e., the optimal path always exists for any given goal pose. However, their solution methodology still involves solving for the roots of multiple transcendental equations, which could lead to heavy computational burden, thus prohibiting it from real-time applications.

Some researchers used the Nonlinear Trajectory Generation (NTG) algorithm [27] based on spline curves to obtain the optimal trajectory of a glider with kinematic constraints in presence of dynamically varying ocean currents. The proposed algorithm relies on Sequential Quadratic Programming (SQP) approach to solve the nonlinear programming problem which might lead to sub-optimal solutions and high computational time. In comparison, this paper proposes a novel method which provides a rapid analytical solution to the path planning problem under currents with guaranteed full reachability.

3. Problem Description and Solution

This section presents the minimum-time path planning problem for Dubins vehicles and its analytical solution.

A. Problem Description

Consider a vehicle moving at a velocity \( v = (v \cos \theta, v \sin \theta) \), where \( v \in \mathbb{R}^+ \) is its speed and \( \theta \in [0, 2\pi) \) is its heading. A steady current is assumed to be present in the environment with velocity \( \mathbf{v}_w = (v_w \cos \theta_w, v_w \sin \theta_w) = (w_x, w_y) \), where \( v_w \in \mathbb{R}^+ \) is its speed and \( \theta_w \in [0, 2\pi) \) is its direction. The current speed is assumed to be slower than the vehicle speed, i.e., \( v_w < v \). Then, the motion of the vehicle can be described as:

\[
\begin{align*}
\dot{x}(t) &= v \cdot \cos \theta(t) + w_x \\
\dot{y}(t) &= v \cdot \sin \theta(t) + w_y \\
\dot{\theta}(t) &= u(t)
\end{align*}
\]

where \( p = (x, y, \theta) \in SE(2) \) is the vehicle pose and \( u \) indicates its turn rate. By choosing a proper unit, the vehicle speed can be normalized to \( v = 1 \). The turn rate \( u \) is symmetric and bounded, i.e., \( u \in [-u_{\text{max}}, u_{\text{max}}] \), where \( u_{\text{max}} \in \mathbb{R}^+ \) is the maximum turn rate and the \( +/\) sign indicates a left/right turn. These constraints imply that the vehicle is subject to the minimum turning radius of \( r = 1/u_{\text{max}} \) (for \( v = 1 \)).

Then, for a vehicle operating in a current environment, as described in [1], the objective is to find the minimum-time path from a start pose \( p_{\text{start}} = (x_0, y_0, \theta_0) \) to a goal pose \( p_{\text{goal}} = (x_f, y_f, \theta_f) \). The state-of-the-art solutions [11][12][13] to this problem require to solve for all six Dubins path types to find the minimum-time path. However, as shown in (34) and (39) of [13], in order to obtain the path types of LSR, RSL, LRL and RLR, one must solve a root-finding problem involving transcendental equations for numerical solutions. This inevitably requires significant computation resources and thus can seriously restrict their usage in real-time applications.

In this regard, in order to achieve a real-time solution, we address the above problem using only two path types which have direct analytical solutions. These are \( L^2 \) and \( R^2 \), where \( \alpha \) and \( \gamma \) are the turning angles of the first and second arc segments, respectively; and \( \beta \geq 0 \) denotes the length of the straight line segment. Thus, the solution for each path type is uniquely determined by the 3-tuple \( \{\alpha, \beta, \gamma\} \) of path parameters. Since these parameters can be solved analytically, the solution is obtained very fast (in real-time).

However, due to using only a subset of the Dubins path types, there exist goal poses for which neither LSL nor RSR path can provide feasible solutions, i.e., LSL and RSR paths do not provide full reachability. To address this issue, we extend the feasible ranges of \( \alpha \) and \( \gamma \) from \([0, 2\pi)\) to \([0, 4\pi)\). It is shown later that the extended LSL and RSR path types guarantee full reachability, and can provide the solutions with even less time costs.
we get: velocity, and the start and goal poses. Then, using (2) and (3), which are constants that can be computed given the current equation  

\[ \beta = \pm \sqrt{(A^k w_x + B^k w_y)^2 + (A^k + B^k)(1 - v_w^2) - (A^k w_x + B^k w_y)} \]  

(5)

It is seen from (5) that when \( v_w < 1 \), \( \beta \) has valid solutions. Then, \( \alpha \) can be computed as  

\[ \alpha = \text{atan}2(B^k - \beta w_y, A^k - \beta w_x) \mod \kappa \]  

(6)

where \( \kappa = 2\pi \) for 2π-arc paths, and \( \kappa = 4\pi \) for 4π-arc paths. Thereafter, \( \gamma \) is computed as \( \gamma = 2k\pi + \theta_f - \alpha \mod \kappa \).

2) \( R^2 S^0 R^$ Path: As seen in Fig. 3b, the following boundary constraints must be satisfied for an RSR path:  

\[
\begin{align*}
A^k &= x_f - w_x T = -r \sin \theta_f + \beta \cos \alpha \\
B^k &= y_f - w_y T = -r(1 - \cos \theta_f) + \beta \sin \alpha \\
T &= (r(\alpha + \gamma) + \beta) / v \\
\alpha + \gamma &= 2k\pi + \theta_f \\
\end{align*}
\]

where \( v = 1 \) and \( T \in \mathbb{R}^+ \) is the total travel time.

In addition, we introduce \( k \in \mathbb{Z} \) to control the feasible ranges of \( \alpha \) and \( \gamma \). Specifically, for a 2π-arc LSL path, since \( \theta_f \in [0, 2\pi) \) and \( \alpha, \gamma \in [0, 2\pi) \), one has \( k \in \{0, 1\} \). In contrast, for a 4π-arc LSL path, since \( \alpha, \gamma \in [0, 4\pi) \), one has \( k \in \{0, 1, 2, 3\} \). Note: We show later that we need only \( k \in \{0, 1\} \) to find a feasible minimum-time 4π-arc LSL path.

Now, for a given \( k \), define \( A^k \) and \( B^k \) as follows:  

\[
\begin{align*}
A^k &= x_f - r \sin \theta_f - w_x r(2k\pi + \theta_f) \\
B^k &= y_f - r(1 - \cos \theta_f) - w_y r(2k\pi + \theta_f) \\
\end{align*}
\]

which are constants that can be computed given the current velocity, and the start and goal poses. Then, using (2) and (3), we get:  

\[
\begin{align*}
A^k &= \beta \cos \alpha + w_x \beta \\
B^k &= \beta \sin \alpha + w_y \beta \\
\end{align*}
\]

(4)

Based on (4), we can compute \( \beta \) by solving the quadratic equation \( (A^k - w_x \beta)^2 + (B^k - w_y \beta)^2 = \beta^2 \), such that

\[
\beta = \pm \sqrt{(A^k w_x + B^k w_y)^2 + (A^k + B^k)(1 - v_w^2) - (A^k w_x + B^k w_y)} \\
\]

C. Feasible Ranges of Path Parameters

According to Defn. 1 and Defn. 2, the parameters \( \alpha \) and \( \gamma \) are defined over \([0, 2\pi)\) and \([0, 4\pi)\) for 2π-arc paths and 4π-arc paths, respectively. Given the direction \( \theta_f \in [0, 2\pi) \) of the goal pose, we can obtain tighter feasible ranges for \( \alpha \) and \( \gamma \). Table II shows the feasible ranges of parameters for both 2π-arc and 4π-arc paths. An example is provided below.

**Example:** Consider a 4π-arc LSL path, where \( \alpha \in [0, 4\pi) \) and \( \gamma \in [0, 4\pi) \). There are four cases to study:

- \( k = 0 \) (i.e., \( \alpha + \gamma = \theta_f < 2\pi \)): Now, \( \gamma \geq 0 \implies \alpha \leq \theta_f \). Similarly, \( \alpha \geq 0 \implies \gamma \leq \theta_f \). Thus, the feasible range for both \( \alpha \) and \( \gamma \) is \([0, \theta_f]\).
- \( k = 1 \) (i.e., \( \alpha + \gamma = 2\pi + \theta_f < 4\pi \)): Again, \( \gamma \geq 0 \implies \alpha \leq 2\pi + \theta_f \). Similarly, \( \alpha \geq 0 \implies \gamma \leq 2\pi + \theta_f \). Thus, the feasible range for both \( \alpha \) and \( \gamma \) is \([0, 2\pi + \theta_f]\).  

Figure 3: Geometric illustration for LSL and RSR paths.
Table 1: Feasible parameter ranges for 2π-arc and 4π-arc paths

| 2π-arc Paths (α, γ ranges are up to mod 2π) | RSR Path Type | LSL Path Type |
|---------------------------------------------|---------------|---------------|
| k α and γ β                                | k α and γ β  |
| 0 [0, θf] [0, \infty)                     | 0 [0, 2\pi - θf] [0, \infty) |
| 1 (θf, 2\pi) [0, \infty)                  | 1 (2\pi - θf, 2\pi) [0, \infty) |

| 4π-arc Paths (α, γ ranges are up to mod 4π) |
|---------------------------------------------|
| k α and γ β                                |
| 0 [0, θf] [0, \infty)                      |
| 1 (θf, 4\pi) [0, \infty)                  |
| 2 (2\pi - θf, 4\pi) [0, \infty)            |
| 3 (4\pi - θf, 4\pi) [0, \infty)            |

- **k = 2**: Here 2π - θf < 6π. Then, γ < 4π implies α > 0. Thus, the feasible range for both α and γ is (θf, 4π).
- **k = 3**: Similarly, α < 4π implies γ > 0. Thus, the feasible range for both α and γ is (2π + θf, 4π).

Similarly, we can obtain the feasible range of path parameters for 4π-arc RSR path and for 2π-arc LSL and RSR paths.

### 4. Reachability Analysis of 2π-Arc Paths

This section derives the analytical expressions for generating the reachability graphs of 2π-arc LSL and RSR path types and for finding the conditions of full reachability.

#### A. Construction of Reachability Graphs

First, we show that for a given α, the reachable goal points (xf, yf) lie on a ray. Then, we show that by varying α, this ray rotates to form the reachability graph.

**2π-arc LSL Paths:** Let us denote

\[ p_{\text{LSL}}^k \equiv r \sin \theta_f + w_x r (2k \pi + \theta_f), \]
\[ q_{\text{LSL}}^k \equiv r (1 - \cos \theta_f) + w_y r (2k \pi + \theta_f), \]

which are constants for \( k \in \{0, 1\} \) given \( \theta_f, w_x \) and \( w_y \). Further, let us denote

\[ a(\alpha) \equiv \sin \alpha + w_y, \]
\[ c(\alpha) \equiv \cos \alpha + w_x. \]

Then, using (3), (4), (11) and (12) we get:

\[ x_f = p_{\text{LSL}}^k + b(\alpha) c(\alpha), \]
\[ y_f = q_{\text{LSL}}^k + b(\alpha) a(\alpha). \]

By performing \( a(\alpha) \cdot [13a] - c(\alpha) \cdot [13b] \), [13] is equivalent to the following:

\[ a(\alpha) p_{\text{LSL}}^k - c(\alpha) q_{\text{LSL}}^k = 0, \]

\[ a(\alpha) > 0 \quad c(\alpha) < 0 \quad a(\alpha) > 0 \quad c(\alpha) > 0 \]

\[ y_f = \omega_{\text{LSL}}(\alpha_{\text{inf}}), \]

\[ \omega_{\text{LSL}}(\alpha_{\text{inf}}) = \arctan (a(\alpha), c(\alpha)) \mod 2\pi, \quad k \in \{0, 1\}. \]

#### 2π-arc RSR Paths:** Let us denote

\[ p_{\text{RSR}}^k \equiv -r \sin \theta_f - w_x r (2k \pi + \theta_f), \]
\[ q_{\text{RSR}}^k \equiv -r (1 - \cos \theta_f) - w_y r (2k \pi + \theta_f), \]

which are constants for \( k \in \{-1, -2\} \) given \( \theta_f, w_x \) and \( w_y \). Further, let us denote

\[ b(\alpha) \equiv \sin \alpha - w_y. \]

Then, using (8), (9), (16) and (17) we get:

\[ x_f = p_{\text{RSR}}^k + b(\alpha) c(\alpha), \]
\[ y_f = q_{\text{RSR}}^k - b(\alpha) a(\alpha). \]

By performing \( b(\alpha) \cdot [18a] + c(\alpha) \cdot [18b] \), [18] is equivalent to the following:

\[ a(\alpha) p_{\text{RSR}}^k - c(\alpha) q_{\text{RSR}}^k = 0, \]

\[ a(\alpha) > 0 \quad c(\alpha) < 0 \quad a(\alpha) > 0 \quad c(\alpha) > 0 \]

\[ y_f = \omega_{\text{RSR}}(\alpha_{\text{sup}}), \]

\[ \omega_{\text{RSR}}(\alpha_{\text{sup}}) = \arctan (a(\alpha), c(\alpha)) \mod 2\pi, \quad k \in \{0, 1\}. \]
Figure 5: An example showing the construction of reachability graph for $2\pi$-arc LSL and RSR path types. (a) MaRA for LSL with $k = 0$, (b) MiRA for LSL with $k = 1$, (c) union of MaRA and MiRA for LSL path, (d) MiRA for RSR with $k = -1$, (e) MaRA for RSR with $k = -2$, (f) union of MaRA and MiRA for RSR path, (g) complete reachability graph obtained by taking union of both LSL and RSR path types.

Lemma 1 implies that the reachable area for LSL paths is obtained by rotating (14) about the center $(p_{LSL}, q_{LSL})$, from $\omega_{LSL}(\alpha_{inf})$ to $\omega_{LSL}(\alpha_{sup})$, where $\alpha_{inf}$ to $\alpha_{sup}$ are the bounds of $\alpha$ (see Table I) for a given $k$. Fig. 4 shows the reachable area for LSL paths obtained by this rotation. Note that there are different reachable areas for each $k$. Similarly, the reachable region for RSR paths is obtained by rotating (19) from $\omega_{RSR}(\alpha_{inf})$ to $\omega_{RSR}(\alpha_{sup})$ for both its $k$ values.

Remark 3. Note that for simplicity of notation, we omit the superscript of $\alpha$ whenever it is used in the $\omega$ function, where it assumes the superscript of $\omega$.

For further explanation, we introduce the concepts of Major Reachable Area (MaRA) and Minor Reachable Area (MiRA).

Definition 3 (MaRA). For an LSL (RSR) path type, MaRA is the larger of the reachable areas spanned by $k = 0$ or $1$ ($k = -1$ or $-2$).

Definition 4 (MiRA). For an LSL (RSR) path type, MiRA is the smaller of the reachable areas spanned by $k = 0$ or $1$ ($k = -1$ or $-2$).

Example: Fig. 5 shows an example of the construction of the reachability graph for $2\pi$-arc LSL and RSR path types. Here, the environment has a current of speed $v_o = -0.5$ m/s and direction $\theta_o = \pi/3$. The goal pose has the heading angle $\theta_f = 7\pi/4$, while its position $(x_f, y_f)$ is varied within $[-10, 10]$. Figs. 5a and 5b show the MaRA ($k = 0$) and MiRA ($k = 1$) from $\omega_{inf}$ to $\omega_{sup}$ for both its $k$ values.

Now, we show a lemma that helps in constructing the reachability graphs using (14) and (19).

Lemma 1. As $\alpha$ increases from $\alpha_{inf}$ to $\alpha_{sup}$, then for:

- **LSL path type**: ray (14) rotates counterclockwise about the center $(p_{LSL}, q_{LSL})$, $\forall k \in \{0, 1\}$.
- **RSR path type**: ray (19) rotates clockwise about the center $(p_{RSR}, q_{RSR})$, $\forall k \in \{-1, -2\}$.

Proof. See Appendix B1.

The constraints in (19) are obtained by using the feasible range of $\beta \geq 0$ in (18a) and (18b). Again, these constraints define the quadrants of the coordinate frame with center at $(p_{RSR}, q_{RSR})$. For any given $\alpha$, (19) represents a reachable ray, and the goal $(x_f, y_f)$ is reachable if it lies on such ray. The rotation of (19) is given as

$$\omega_{RSR}(\alpha) = \text{atan2}(-b(\alpha), c(\alpha)) \mod 2\pi, \quad k \in \{-1, -2\}.$$  

(20)
of the LSL paths, respectively, which are obtained by rotating the ray (14) by varying \( \alpha \) from \( \alpha_{\text{inf}}^k \) to \( \alpha_{\text{sup}}^k \). The corresponding centers of rotation \( (p_{\text{LSL}}, q_{\text{LSL}}) = (0.67, 2.67) \) and \( (p_{\text{LSL}}, q_{\text{LSL}}) = (2.24, 5.39) \) are also shown. Fig. 5c shows the total reachable area of the LSL paths obtained by combining the MaRA and MiRA from Figs. 5a and 5b, respectively. Clearly, the LSL paths do not provide full reachability.

Similarly, Figs. 5d and 5e show the MiRA \((k = -1)\) and MaRA \((k = -2)\) of the RSR paths, respectively, which are obtained by rotating the ray (19) by varying \( \alpha \) from \( \alpha_{\text{inf}}^k \) to \( \alpha_{\text{sup}}^k \). The corresponding centers of rotation \( (p_{\text{RSR}}, q_{\text{RSR}}) = (0.90, 0.05) \) and \( (p_{\text{RSR}}, q_{\text{RSR}}) = (2.47, 2.77) \) are also shown. Again, Fig. 5f shows the total reachable area of the RSR path obtained by combining the MaRA and MiRA from Figs. 5d and 5e, respectively. As seen, the RSR paths also do not provide full reachability.

Finally, Fig. 5g shows the complete reachability graph using both LSL and RSR path types, which is obtained by combining Figs. 5c-5f. As seen in Fig. 5g, there is still some region that is unreachable, thus both LSL and RSR path types together also do not provide full reachability.

### B. Full Reachability Conditions for the 2\( \pi \)-arc Path Types

After acquiring the analytical expressions for generating the reachability graphs of the 2\( \pi \)-arc LSL and RSR path types, we now investigate the conditions under which these paths provide full reachability.

Note that full reachability is achieved if the entire space is covered by at least one of the following combinations:

1. Union of MaRA and MiRA of LSL, and/or
2. Union of MaRA and MiRA of RSR, and/or
3. Union of MaRA of LSL and MiRA of RSR, and/or
4. Union of MaRA of RSR and MiRA of LSL.

**Remark 4.** We show by Lemma 4 in Appendix A that these four cases are sufficient for reachability analysis.

For continuity of reading, the derivations of the full reachability conditions for the above four cases are presented in Appendix A and the results are summarized in Table III. If at some goal pose, all of the conditions in Table III are violated, then it is unreachable by 2\( \pi \)-arc paths. Next, we visually verify the unreachable regions using a numerical validation.

**Numerical Validation:** The reachability conditions for 2\( \pi \)-arc paths are shown in the last column of Table III in Appendix A. These reachability conditions only depend on parameters \( \theta_f, \theta_w \) and \( v_w \). Thus, we construct a 3D reachability graph by varying \( \theta_f \in [0, 2\pi] \) and \( \theta_w \in [0, 2\pi] \) in steps of \( \pi/100 \), and \( v_w \in (0, 1) \) in steps of 0.1. For any 3D parametric point, if at least one of the full reachability conditions is satisfied, then such point is colored, and the color varies with respect to \( v_w \), as shown in Fig. 6a. In contrast, the white area indicates the parametric space where all the reachability conditions are violated, i.e., providing no feasible solutions. This validation illustrates that full reachability is not achieved by 2\( \pi \)-arc LSL and RSR paths.

Figs. 6b and 6c show the cross sections of Fig. 6a at \( v_w = 0.25 \) m/s and \( v_w = 0.75 \) m/s, respectively. It is seen that a higher \( v_w \) leads to a smaller reachable space.

Fig. 7 shows a specific example where 2\( \pi \)-arc path does not exist, but 4\( \pi \)-arc path does. The start pose \((x_0, y_0, \theta_0) = (0, 0, 0)\), the goal pose \((x_f, y_f, \theta_f) = (6, 3.7\pi/4)\), and the current moves at speed \( v_w = 0.5 \) m/s in the direction of \( \theta_w = \pi/3 \). It is seen that the turning angle of the second turn in the optimal 4\( \pi \)-arc path is \( \gamma = 2.135\pi > 2\pi \), which drives the vehicle to circle around at the end so that it can meet with the exact goal heading with the help of external current.
include the $2\pi$-arc solutions, the above observation implies that there exist goal poses for which $4\pi$-arc paths can achieve even lower time costs as compared to the $2\pi$-arc paths.

**Roadmap of this Section:** In the following subsections, we present four theorems to highlight the theoretical properties of $4\pi$-arc LSL and RSR paths and compare them with the corresponding $2\pi$-arc paths. First, Theorem 1 proves that both the LSL and RSR $4\pi$-arc paths provide full reachability unlike the $2\pi$-arc paths. Then, Theorem 2 and Corollary 4 show that the computation workload required to get a solution using the $4\pi$-arc paths is the same as that using the $2\pi$-arc paths. Next, Theorem 3 compares the optimality of $4\pi$-arc and $2\pi$-arc path solutions and shows that the optimal trajectory provided by $4\pi$-arc paths is either of shorter time or same as that provided by $2\pi$-arc paths. Finally, Theorem 4 proves that $\alpha, \gamma \in [0, 4\pi]$ is sufficient for optimality and increasing the range of these arc segments beyond $4\pi$ does not lead to a shorter time path.

### A. Full Reachability of $4\pi$-arc Paths

The following theorem relates to the reachability of the $4\pi$-arc solutions for the LSL and RSR path types.

**Theorem 1 (Full reachability of $4\pi$-arc paths).** The $4\pi$-arc LSL and RSR paths individually provide full reachability.

**Proof.** Full reachability implies the existence of solution for any goal pose. We prove for LSL and RSR paths below.

- **$4\pi$-arc LSL paths:** Consider $k = 1$. From Table 1, $\alpha_{inf} = 0$ and $\alpha_{sup} = 2\pi + \theta_f$. Using Lemma 2, we construct the reachable space for $k = 1$ by rotating the ray (14) around $(p_{LSL}^{1}, q_{LSL}^{1})$ by varying $\alpha$ from 0 to $2\pi + \theta_f$. In this process, the ray (14) sweeps in the anticycloidal direction from $\omega_{LSL}^{1}(0)$ to $\omega_{LSL}^{1}(2\pi + \theta_f)$. However, when $\alpha = 2\pi < 2\pi + \theta_f$, the rotation of ray (14) becomes $\omega_{LSL}^{1}(2\pi) = \omega_{LSL}^{1}(0) = \tan(2\pi, 1 + w_s) (\mod 2\pi)$, which implies that the ray comes back to the start again and continues sweeping thereafter. This means that for $k = 1$, the whole area is covered and full reachability is obtained. Now consider $k = 2$. From Table 1, $\alpha_{inf} = \theta_f$ and $\alpha_{sup} = 4\pi$. Following the same process as for the $k = 1$ case, one can see that the swept area for $k = 3$ also covers the whole area and full reachability is obtained. In summary, $4\pi$-arc LSL paths guarantee full reachability. (Note: for $k = 0$ and 3, the swept area does not cover the whole space, hence they do not provide full reachability.)

- **$4\pi$-arc RSR paths:** Consider $k = 2$. From Table 1, $\alpha_{inf} = 0$ and $\alpha_{sup} = 4\pi - \theta_f$. Using Lemma 4, as $\alpha$ grows, the ray (19) rotates around $(p_{RSR}^{2}, q_{RSR}^{2})$ in the clockwise direction from $\omega_{RSR}^{2}(0)$ to $\omega_{RSR}^{2}(4\pi - \theta_f)$. During this process, when $\alpha = 2\pi < 4\pi - \theta_f$, the rotation of ray (19) becomes $\omega_{RSR}^{2}(2\pi) = \omega_{RSR}^{2}(0) = \tan(2\pi, 1 + w_s) (\mod 2\pi)$, which implies that it comes back to the start again and continues sweeping thereafter. This means that for $k = 2$, the whole space is covered and full reachability is obtained. Now consider $k = -3$. From Table 1, $\alpha_{inf} = 2\pi - \theta_f$ and $\alpha_{sup} = 4\pi$. Following the same process as for the $k = -2$ case, one can see that...
the swiped area for \( k = -3 \) also covers the whole space and full reachability is obtained. In summary, 4\( \pi \)-arc RSR paths guarantee full reachability. (Note: for \( k = -1 \) and \(-4\), the swiped area does not cover the whole space, hence they do not provide full reachability.)

Hence proved.

**B. Time Costs of 4\( \pi \)-arc LSL and RSR Paths**

Now, we analyse the time costs of 4\( \pi \)-arc LSL and RSR paths and compare them to the corresponding 2\( \pi \)-arc paths.

Based on (22) and substituting \( v = 1 \), the time cost for an LSL path type is given as
\[
T = r(\alpha + \gamma) + \beta = 2k\pi r + r\theta_f + \beta.
\] (21)

Similarly, based on (7), the time cost for an RSR path type is given as
\[
T = r(\alpha + \gamma) + \beta = -2k\pi r - r\theta_f + \beta.
\] (22)

From this point on, let us denote \( T_k \) and \( \beta_k \) as the values of \( T \) and \( \beta \) for a given \( k \), i.e., \( T_k = 2k\pi r + r\theta_f + \beta_k \) for an LSL path and \( T_k = -2k\pi r - r\theta_f + \beta_k \) for an RSR path.

**Theorem 2.** The following are true:

- \( T_0 < T_1 < T_2 < T_3 \), for 4\( \pi \)-arc LSL paths.
- \( T_{-1} < T_{-2} < T_{-3} < T_{-4} \), for 4\( \pi \)-arc RSR paths.

**Proof.** Let us denote \( \Delta T_k \) as the difference in time cost \( T_k \) between two consecutive \( k \) values, i.e., for LSL path type,
\[
\Delta T_k \triangleq T_{k+1} - T_k = 2\pi r + \beta_{k+1} - \beta_k, \quad k = 0, 1, 2, \quad (23)
\]
and for RSR path type,
\[
\Delta T_k \triangleq T_{k+1} - T_k = 2\pi r + \beta_{k+1} - \beta_k, \quad k = -1, -2, -3. \quad (24)
\]

Consider 4\( \pi \)-arc LSL paths. To prove the theorem, we show that \( \Delta T_k > 0, \forall k = 0, 1, 2 \). Fig. 9a shows the feasible 4\( \pi \)-arc LSL paths in the CF, corresponding to \( k \) (shown in solid blue) and \( k + 1 \) (shown in solid red), to reach the goal pose \((x_f,y_f,\theta_f)\). These paths have the time costs \( T_k \) and \( T_{k+1} \), respectively. While these two paths share the same start pose, due to different travel times, the corresponding goal poses in the CF become \( G_k = (x_f - w_k T_k, y_f - w_k T_k, \theta_f) \) and \( G_{k+1} = (x_f - w_k T_{k+1}, y_f - w_k T_{k+1}, \theta_f) \), where \( \|G_{k+1} - G_k\| = \sqrt{w_k^2\Delta T_{k+1}^2 + w_k^2\Delta T_k^2} = v_w |\Delta T_k| \).

Since an LSL path is comprised of an \( \alpha \) arc, a straight line and a \( \gamma \) arc, one can equivalently combine the two arcs followed by the straight line to reach the same goal pose, as shown by the dotted line paths in Fig. 9a corresponding to \( k \) (shown in dotted blue) and \( k + 1 \) (shown in dotted red). According to (22), \( \alpha + \gamma = 2\pi + \theta_f \), so if \( k \) is increased by 1, it adds a full \( 2\pi \) rotation to this combined \( \alpha \) and \( \gamma \) arc. This implies that after combining these arcs, the red and blue dotted straight lines share the same start point \( O_k \in \mathbb{R}^2 \). Note that the solid straight lines are parallel to the corresponding dotted straight lines, with lengths \( \beta_k \) and \( \beta_{k+1} \), respectively.

Now consider the triangle formed by \( O_k, G_k \) and \( G_{k+1} \), shown by the shaded region in Fig. 9b, where \( \|O_k - G_k\| = \beta_k \) and \( \|O_k - G_{k+1}\| = \beta_k+1 \). Next, we consider three cases:

1) \( \Delta T_k > 0 \): In this case, \( \|G_{k+1} - G_k\| = v_w \Delta T_k \). Using the triangle inequalities, we get \( \|G_{k+1} - G_k\| < v_w \Delta T_k \). By (23), \( \beta_{k+1} - \beta_k = \Delta T_k - 2\pi r \). Hence, \( \|G_{k+1} - G_k\| < v_w \Delta T_k \). Note that if \( O_k, G_k \) and \( G_{k+1} \) fall on one line, then \( \|G_{k+1} - G_k\| = v_w \Delta T_k \). Therefore, the feasible range of \( \Delta T_k \) is
\[
\Delta T_k \in \left[ \frac{2\pi r - 2\pi r}{1 + v_w}, \frac{2\pi r}{1 - v_w} \right].
\] (25)

2) \( \Delta T_k < 0 \): In this case, \( \|G_{k+1} - G_k\| = -v_w \Delta T_k \). Then, based on the triangle inequalities, \( \|G_{k+1} - G_k\| < v_w \Delta T_k \). Again substituting \( \|G_{k+1} - G_k\| = \Delta T_k - 2\pi r \) from (23), we get \( \frac{2\pi r}{1 - v_w} < \Delta T_k < \frac{2\pi r}{1 + v_w} \). However, since \( 0 < v_w < 1 \), this inequality is invalid. Thus, \( \Delta T_k < 0 \) is impossible.

3) \( \Delta T_k = 0 \): In this case, \( \|G_{k+1} - G_k\| = 0 \). Then, \( \|G_k + G_{k+1}\| = 0 \), which is a contradiction, hence \( \Delta T_k = 0 \) is impossible.

Thus, \( \Delta T_k > 0, \forall k \), and its bounds are given in (25). Similarly, for 4\( \pi \)-arc RSR paths, the bounds of \( \Delta T_k \) can be derived using Fig. 9b leading to the same bounds and the derivation is omitted here. Hence proved.

The following corollary shows that in order to obtain the minimum-time solutions using 4\( \pi \)-arc paths, it is sufficient to use \( k = \{0, 1\} \) for LSL path type and \( k = \{-1, -2\} \) for RSR path type and the remaining \( k \) values are not needed.

**Corollary 1.** A minimum-time solution for the 4\( \pi \)-arc paths can be obtained by using

- \( k \in \{0, 1\} \) for LSL paths and
- \( k \in \{-1, -2\} \) for RSR paths.

**Proof.** Theorem 2 implies that based on time costs, the preferred solutions follow the order \( k = 0, 1, 2, 3 \) for LSL paths and \( k = -1, -2, -3, -4 \) for RSR paths. Theorem 1 suggests that
for LSL paths, \( k = 0 \) solutions do not provide full reachability; however full reachability can be achieved by \( k = 1 \) solutions. Similarly, for RSR paths, \( k = -1 \) solutions do not provide full reachability; however full reachability can be achieved by \( k = -2 \) solutions. Thus, in order to get full reachability and to obtain minimum-time paths, one must solve only for \( k \in \{0, 1\} \) for LSL paths, and \( k \in \{-1, -2\} \) for RSR paths. Hence proved.

**Remark 5.** Corollary 1 implies that the computation workload required to get a solution using the 4\( \pi \)-arc paths is the same as that using the 2\( \pi \)-arc paths.

**Corollary 2.** A minimum-time 4\( \pi \)-arc LSL or RSR solution must satisfy \( \alpha + \gamma < 4\pi \).

**Proof.** Using Corollary 1 and that \( \theta_f < 2\pi \), substitute \( k = 1 \) into 2 and \( k = -2 \) into 7, one can easily get the result. Hence proved.

**Remark 6.** As seen from Table 1, the feasible ranges of parameters \( \alpha \) and \( \gamma \) for the 4\( \pi \)-arc LSL (RSR) paths for \( k = 0 \) \((k = -1)\) are the same as those of the corresponding 2\( \pi \)-arc paths. However, for \( k = 1 \) \((k = -2)\), the parameter ranges for 4\( \pi \)-arc LSL (RSR) paths form supersets of the corresponding ranges of the 2\( \pi \)-arc paths.

**Theorem 3.** The time costs of 4\( \pi \)-arc path solutions are lower than or same as those of the 2\( \pi \)-arc path solutions.

**Proof.** First, consider the case when both 2\( \pi \)-arc LSL and RSR solutions exist for a given goal pose. Remark 6 indicates that any valid 2\( \pi \)-arc path solution is also a valid 4\( \pi \)-arc path solution. Hence, in this case the time cost of 4\( \pi \)-arc path solution is the same as that of the 2\( \pi \)-arc path solution.

Second, consider the case when neither of the 2\( \pi \)-arc LSL and RSR solutions exist for a given goal pose. In this case, Theorem 1 guarantees that 4\( \pi \)-arc LSL and RSR solutions exist for that goal pose.

Third, consider the case when only one of the 2\( \pi \)-arc LSL or RSR path solution exists for a given goal pose, i.e., the other path type does not provide a solution. Thus, the dominant solution is the only existing path type. However, from Theorem 1 for 4\( \pi \)-arc paths both LSL and RSR paths exist and the dominant solution is selected from these two path types with the minimum time cost. Thus, due to the existence of an extra solution provided by the 4\( \pi \)-arc paths, the time cost of the dominant path could be better than or same as that of the single solution provided by the 2\( \pi \)-arc paths. The examples below validate this case. Hence proved.

**Example:** We show an example where the 4\( \pi \)-arc paths provide faster (i.e., lower time cost) solutions as compared to the 2\( \pi \)-arc paths. We first construct the time cost map for a fixed set of \( \theta_f \), \( v_w \) and \( \theta_w \), where each \((x_f, y_f)\) is assigned the time cost of the dominant path between LSL and RSR paths.

Fig. 10 shows the example generated for an environment with current of \( v_w = 0.5 \) m/s and \( \theta_w = \pi \). For constructing the time cost map, the goal poses are varied within \( x_f, y_f \in [-10, 10] \) m with a fixed heading angle \( \theta_f = \pi/4 \). Figs. 10a and 10b show the time cost maps for 2\( \pi \)-arc paths and 4\( \pi \)-arc paths, respectively. The color code indicates the value of the time cost. Clearly, there exist many goal poses where 4\( \pi \)-arc paths provide significantly lower time costs.

Next, we pick a goal pose where 4\( \pi \)-arc paths provide a lower time cost, say \((x_f, y_f, \theta_f) = (-1, 4, \pi/4)\). Then, we draw the optimal 2\( \pi \)-arc and 4\( \pi \)-arc paths in the IF and the CF, as shown in Fig. 10c. The 2\( \pi \)-arc path follows the RSR path type, and requires a total time cost of 24.47 s. In comparison, the 4\( \pi \)-arc path follows the LSL path type and the total time cost is reduced to 13.21 s. This is because on the 2\( \pi \)-arc path, the vehicle has to travel a longer straight-line segment that is almost in an opposite direction to the current, hence its actual speed in the inertial frame becomes slower. On the other hand, the 4\( \pi \)-arc path first makes a small left turn, followed by a much shorter straight-line segment; then, it starts circling for over 2\( \pi \) while letting the current help it to reach the goal.

**Theorem 4.** The time cost \( T \) cannot be reduced further by extending the ranges of arc segments (\( \alpha \) and \( \gamma \)) over \( 4\pi \).

**Proof.** Suppose the ranges of \( \alpha \) and \( \gamma \) are defined over \([0, 2n\pi)\), where \( n > 2 \) and \( n \in \mathbb{N}^+ \). Then, using the same procedure as described in Section 3-B, we get a larger set of feasible values of \( k \), s.t. for LSL paths, \( k \in \{0, 1, \ldots, 2n - 1\} \), and for RSR paths, \( k \in \{-1, -2, \ldots, -2n\} \).

Then, one can derive the feasible ranges for \( \alpha \) and \( \gamma \). Con-
sider a $2n\pi$-arc LSL path, where $\alpha \in [0, 2n\pi)$ and $\gamma \in [0, 2n\pi)$. We examine only $k = 0, 1$ cases as necessary.

- $k = 0$ (i.e., $\alpha + \gamma = \theta_f < 2\pi$): Now, $\gamma \geq 0$ $\implies \alpha \leq \theta_f$. Similarly, $\alpha \geq 0$ $\implies \gamma \leq \theta_f$. Thus, the feasible range for both $\alpha$ and $\gamma$ is $[0, \theta_f]$.
- $k = 1$ (i.e., $\alpha + \gamma = 2\pi + \theta_f < 4\pi$): Again, $\gamma \geq 0$ $\implies \alpha \leq 2\pi + \theta_f$. Similarly, $\alpha \geq 0$ $\implies \gamma \leq 2\pi + \theta_f$. Thus, the feasible range for both $\alpha$ and $\gamma$ is $[0, 2\pi + \theta_f]$.

The above analysis indicates that for $2n\pi$-arc LSL paths, if $n > 2$, the feasible ranges of $\alpha$ and $\gamma$ for $k = 0, 1$ are the same as those for $4\pi$-arc LSL paths, as presented in Table 1. Similarly, one can verify that for $2n\pi$-arc RSR paths, if $n > 2$, the feasible ranges of $\alpha$ and $\gamma$ for $k = -1, -2$ are also the same as those for $4\pi$-arc RSR paths.

Since the feasible ranges of $\alpha$ and $\gamma$ for $2n\pi$-arcs are the same as those for $4\pi$-arc paths, by Theorem 1, full reachability is achieved using $k = 0, 1$ for LSL paths and $k = -1, -2$ for RSR paths. Further, by Theorem 2, $\Delta T_{ik} > 0, \forall k$. Therefore, for $n > 2$, we only need to search over $k = 0, 1$ for LSL paths and $k = -1, -2$ for RSR paths to get the minimum-time path. This implies that the time cost $T$ is not reduced by extending the feasible ranges of $\alpha$ and $\gamma$ over $4\pi$. Hence proved.

6. RESULTS AND DISCUSSION

This section presents the results of the proposed approach, which uses the $4\pi$-arc LSL and RSR paths, in comparison to the Dubins approach, which uses the six $2\pi$-arc paths. We discuss the performance of these two approaches first in an environment with static current and then in an environment with dynamically changing current. We conduct Monte Carlo simulations as needed for statistical performance evaluation. The simulations were done on a computer with 2.4 GHz and 8 GB RAM. In order to obtain a solution using the Dubins approach, the transcendental functions are solved using the function ffsolve in MATLAB. On average, the Dubins approach took $\sim 8.72$ s to get a solution with 100 initial guesses, while the $4\pi$-arc paths approach took only $\sim 0.64$ ms which is orders of magnitude faster than that of the Dubins computation.

A. Comparison of $4\pi$-arc LSL and RSR solutions with Dubins solutions in a static current environment

First, we considered an environment with a static current where the planning is done offline. This comparative study is presented using two metrics: a) the solution quality (i.e., the travel time cost) and b) the total time cost (i.e., the offline computation time cost plus the travel time cost).

**Simulation Setup:** The start pose is fixed at $(x_0, y_0, \theta_0) = (0, 0, 0)$. Then, 80 different goal positions are distributed uniformly on the boundaries of concentric squares at a distance of $R = \{5, 10, 50, 100, 200\}$ m around the origin. For each goal position, 6 different heading angles $\theta_f \in \{ \frac{m\pi}{5}, m = 0, \ldots, 5 \}$ are considered. This leads to a total of 480 goal poses. The vehicle and current speeds are taken to be $v = 1$ m/s and $v_x = 0.5$ m/s, respectively, where 6 different current heading angles $\theta_0 \in \{ \frac{m\pi}{5}, m = 0, \ldots, 5 \}$ are considered, thus leading to a total number of 2880 runs.

For each run, the travel time cost and computation time cost are obtained for the two approaches. Fig 11 shows the savings obtained with the proposed $4\pi$-arc path solutions as compared to the Dubins solutions. Fig 11a shows the savings in travel time, computed as $T_{Dubins} - T_{4\pi}$, where $T_{Dubins}$ and $T_{4\pi}$ refer to the travel time costs of Dubins paths and $4\pi$-arc paths, respectively. As seen in the figure, in more than 50% of the cases, the travel time costs of the $4\pi$-arc path solutions match those of the Dubins solutions. Although the performance of Dubins paths is better than the $4\pi$-arc paths for the remaining cases, the travel time cost difference is not that significant.

Fig. 11b shows the total time cost obtained by adding the computation time costs taken by the two approaches to their respective travel time costs. It is seen that in more than 90% of the cases the total time of the $4\pi$-arc solutions is lower than that of the Dubins solutions; thus, $4\pi$-arc solutions yield a superior performance upon considering the computation times.

Based on these trends, it is observed that although Dubins solutions are suitable for applications requiring offline planning, they do not provide significant advantage over the $4\pi$-arc LSL and RSR solutions in terms of travel time costs. Furthermore, when computation times are added then Dubins solutions provide worse total time costs in a significant majority of cases. Moreover, as discussed in Section 6-E, for applications requiring online planning in dynamic current environments, the high computation times of Dubins solutions cause significant vehicle drifts, thus, resulting in longer sub-optimal trajectories which sometimes do not even converge to the goal pose. In such situations, $4\pi$-arc paths lead to faster and reliable solutions with negligible drifts allowing the vehicle to reach the goal pose precisely in shorter times.

B. Effect of a Change in Current

During path execution, a change in the current’s speed or heading could deviate the vehicle from its original path if left unattended. Hence, it is necessary to replan online upon detection of a change in current. However, as explained in Section 1 using Dubins solution to regenerate the path to reach the goal pose requires considerable amount of computation time to solve the transcendental functions, during which the vehicle can drift noticeably. In particular, the vehicle drift would be along the direction of the net velocity of the vehicle

![Figure 11: Time savings of the $4\pi$-arc solutions w.r.t. the Dubins solutions over 2880 different simulation runs in a static current environment.](image-url)
(a) An example of path replanning under changing current. Start pose \((x_0, y_0, \theta_0) = (0, 0, 0)\) and goal pose \((x_f, y_f, \theta_f) = (5, 8.5, 3\pi/4)\). Initially, the current has \(v_w = 0.5\) m/s and \(\theta_w = \pi\), which changed at time 3.2 s to a new current with \(v_w = 0.75\) m/s and \(\theta_w = 3\pi/2\). The radius of precision circle is 1 m.

(b) An example to show the effect of the net velocity of the vehicle drift. Start pose \((x_0, y_0, \theta_0) = (0, 0, 0)\) and goal pose \((x_f, y_f, \theta_f) = (5, 8.5, 3\pi/4)\). Initially, the current has \(v_w = 0.5\) m/s and \(\theta_w = 3\pi/2\), which changed at time 3.72 s to a new current.

Fig. 12: Illustrative examples of replanning under changing current, and the effect of \(v_{net}\) on the vehicle drift.
end-point lying inside the precision circle with an acceptable heading error. The total time taken by the vehicle to reach the goal is obtained by adding the initial execution time of \( \sim 3.2 \) s before the change of current, the replanning time of \( \sim 8 \) s, and the execution time of \( \sim 40.08 \) s along the replanned path, which leads to the total travel time of \( \sim 51.28 \) s.

In comparison, Fig. 12a(3) shows the replanning process using the 4π-arc LSL and RSR paths approach. Due to the negligible computation time, the points \( A, B \) and \( \hat{B} \) coincided, thus resulting in a much faster total travel time of \( \sim 33.57 \) s. Also, the goal pose was achieved more accurately as compared to the Dubins solution. This example clearly highlights the benefits of the proposed rapid solution using the 4π-arc paths over the Dubins approach.

C. Effect of \( v_{net} \)

During replanning, the vehicle is drifted along the direction of \( v_{net} \), with a magnitude of \( v_{net} \in \mathbb{R}^+ \) times the computation time. To examine the effect of \( v_{net} \) over the vehicle drift, we tested three scenarios over a range of \( v_{net} \) and the results are shown in Fig. 12b(1)–(3). The start pose, the goal pose and the initial environmental current are set to be the same as those shown in Fig. 12b(1), where the drift occurring due to a change of current after 3.2 s, when the vehicle has reached point \( A \).

As seen in Fig. 12b(1)–(3), the 4π-arc path solution generates trajectories with negligible drifts, while the Dubins solution results in significant vehicle drifts of lengths 0.875 m for low \( v_{net} = 0.112 \) m/s, 3.56 m for medium \( v_{net} = 0.432 \) m/s and 7.39 m for high \( v_{net} = 0.924 \) m/s. In all cases, since the Dubins solution incurs high computation time, it leads to a higher overall execution time. In particular, even for the scenario with low \( v_{net} \) as shown in Fig. 12b(1), where the drift is very close to the vehicle’s initial state and within its turning radius, 4π-arc paths provide a faster solution than the Dubins solution because of the high computation time of the latter.

D. Effect of the Size of Precision Circle

Next, we study the effect of the size of precision circle, centered at the goal, on the total travel time using the two approaches. The vehicle is assumed to keep replanning until it converges inside the precision circle with an acceptable heading error. Fig. 13 shows the results obtained by varying the radii of the precision circle as: 1 m, 1 m and 0.5 m. The start pose is \((x_0, y_0, \theta_0) = (0, 0, 0)\) and the goal pose is \((x_f, y_f, \theta_f) = (2, 8, \pi/2)\). Initially, the current has \(v_w = 0.75 \) m/s and \(\theta_w = 0\), which changed at time \(3.72\) s to a new current with \(v_w = 0.65 \) m/s and \(\theta_w = \pi\).

E. Comparison of 4π-arc LSL and RSR solutions with Dubins solutions in a dynamic current environment

Now, we present a comparative evaluation of the 4π-arc LSL and RSR solutions with Dubins solutions in a dynamic current environment. The performance of the two approaches is evaluated statistically using Monte Carlo simulations which cover a wide range of environmental conditions, considering realistic vehicle properties and sensing capabilities. The simulation setup is described as follows.
Sampled Goal Poses: The start pose is fixed at $\left(x_0, y_0, \theta_0\right) = (0,0,0)$. Then, six different goal positions are chosen located at a distance of $R = 100$ m from the origin. For each goal position, six different heading angles $\theta_m \in \left\{\frac{m\pi}{6}, m = 0, \ldots, 5\right\}$ are considered, which leads to a total number of 36 start and goal pose pairs. Due to noise (discussed later), 10 Monte Carlo simulation runs were conducted for each goal pose, thus leading to a total number of 360 runs.

Changing Environment: To validate the effectiveness of the proposed method, the current with speed $v_w$ is set to change its direction with a random heading angle $\theta_w \in \left\{\frac{m\pi}{6}, m = 0, \ldots, 11\right\}$. This change happens after a random time interval $T_0 \in \{30, 45, 60\}$ s. Specifically, for each simulation run, the current heading $\theta_w$ and its time period $T_0$ are randomly generated from their corresponding sets. Then, after $T_0$, the updated current heading $\theta_w$ and its time period $T_0$ are randomly chosen again and the process is repeated. Thus, the vehicle has to replan its path based on the updated $\theta_w$ every time the current changes. Since the measurements of $\theta_w$ include noise (discussed later), the vehicle estimates its value using a Maximum Likelihood Estimator (MLE) [28], which utilizes measurements of $\theta_w$ within a period of $T_1 = 12$ s.

Termination Conditions: The vehicle is assumed to successfully reach the goal pose if: (1) it arrives within a precision circle of radius $1.5$ m centered at the goal, and (2) its heading falls between $\theta_f \pm 5^\circ$. However, if the vehicle cannot converge to the goal pose in $T_{\text{max}} = 1000$ s, then the solution is considered to be not convergent.

Performance Metric: The performance of the proposed $4\pi$-arc solution is evaluated in comparison to the Dubins solution based on the percentage of savings in the total travel time:

$$\text{Savings}(\%) = \frac{T_{\text{Dubins}} - T_{4\pi}}{T_{\text{Dubins}}} \cdot 100, \quad (26)$$

where $T_{\text{Dubins}}$ and $T_{4\pi}$ denote the total time cost using Dubins solution and the proposed $4\pi$-arc solution, respectively.

Applications: Since sensing capabilities can vary significantly for different vehicles and in different operation environments, we evaluated the performance for two different applications: 1) naval (unmanned underwater vehicles (UUVs)) and 2) aerial (unmanned aerial vehicles (UAVs)).

1) Naval Application: Consider a typical UUV that travels at a speed of $v = 2.5$ m/s. The ocean environment is assumed to have currents that move at a speed of $v_w = 2$ m/s with an initial heading of $\theta_w = 0$. Regarding the sensing systems, the ocean current speed and heading are usually measured using an Acoustic Doppler Current Profiler (ADCP) [29] with a sampling rate of 1 Hz. On the other hand, the location and heading of UUV can be measured using Long Baseline (LBL) localization system [30] and compass, respectively. The sensor uncertainties are modeled using Additive White Gaussian Noise (AWGN) with parameters listed in Table II.

Fig. 14a shows the distribution of percentage savings in time for the $4\pi$-arc path solutions in comparison to the corresponding Dubins solutions over all Monte Carlo runs. While $4\pi$-arc path solutions always converged, Dubins solutions could not converge within the precision circle in $T_{\text{max}}$ time for 6.11% of the runs. As explained in Section 6-B this happens mainly due to their significantly high computation times during replanning which makes them keep replanning due to errors caused by the vehicle drift. For the remaining runs where both methods converged, the proposed $4\pi$-arc path solutions achieved an average of 57.62% time savings, thus showing their superiority over Dubins solutions in a dynamic naval environment. This

### Table II: The specifics in Monte Carlo simulations

| Application   | Naval                  | Aerial                  |
|---------------|------------------------|-------------------------|
| Vehicle speed | $v = 2.5$ m/s          | $v = 10$ m/s            |
| External current | Ocean currents  | Wind                  |
| $v_w = 2$ m/s     | $v_w = 8$ m/s         |
| Noise in vehicle state measurement | $\sigma_{\text{GPS}} = 0.3$ m | $\sigma_{\text{GPS}} = 0.01$ m |
| $\sigma_{\text{compass}} = 0.5^\circ$ | $\sigma_{\text{compass}} = 0.5^\circ$ |
| Noise in current state measurement | $\sigma_{\theta_w} = 0.75\% \cdot v_w$ | $\sigma_{\theta_w} = 1.25\% \cdot v_w$ |
| $\sigma_{\theta_v} = 0.67^\circ$ | $\sigma_{\theta_v} = 4^\circ$ |
implies that the $4\pi$-arc path solutions can guide the UUV to successfully reach the goal pose in significantly less time cost as compared to the Dubins solutions. Furthermore, we note that only a very small fraction of all test cases result in negative time savings, which could be perhaps when the vehicle drift directly took the vehicle to the goal.

2) **Aerial Application:** Consider a typical UAV that travels at a speed of $v = 10$ m/s. The environment is assumed to have wind that moves at a speed of $v_w = 8$ m/s with an initial heading $\theta_w = 0$. As for the sensing systems, the wind profile can be measured using the Acoustic Resonance Wind Sensor system of FT 205 [31], which has a sampling rate of 10 Hz. For localization of the UAV, a Real-Time Kinematic (RTK) GPS is used [32]. The sensor uncertainties are modeled using AWGN, with parameters listed in Table II.

AWS is used [32]. The sensor uncertainties are modeled using AWGN, with parameters listed in Table II. For localization of the UAV, a Real-Time Kinematic (RTK) GPS is used [32]. The sensor uncertainties are modeled using AWGN, with parameters listed in Table II.

For this purpose, two applications were considered: i) naval and ii) aerial, where extensive Monte Carlo simulations were conducted for statistical analysis under stochastic uncertainties in dynamically changing environments. The results showed that the $4\pi$-arc solutions converged to the goal pose in all runs as opposed to the Dubins solutions which failed to converge in a significant portion of runs. For the cases where Dubins solutions converged, the $4\pi$-arc solutions yielded superior performance and achieved significantly lower time costs to reach the goal poses with high precision.

### B. Future Work

Future research will consider the following challenging problems for Dubins vehicles: 1) minimum-time path planning under spatio-temporally varying currents, 2) complete coverage in unknown environments [33] [34], and 3) Dubins orienteering problem in dynamic environments [35].

### APPENDIX

#### A. Derivation of conditions under which $2\pi$-arc LSL and RSR path types provide full reachability

From (15) and (20), we note that the boundaries of the reachable areas have the following rotations:

- $\omega^k_{LSL}(\alpha_{inf})$ and $\omega^k_{LSL}(\alpha_{sup})$, for $k = 0, 1,
- $\omega^k_{RSR}(\alpha_{inf})$ and $\omega^k_{RSR}(\alpha_{sup})$, for $k = -1, -2$.

Now, we present a lemma related to these boundary rotations, which helps us in deriving the reachability conditions.

**Lemma 2.** The following are true:

- $\omega^0_{LSL}(\alpha_{inf}) = \omega^1_{LSL}(\alpha_{sup}) = \omega^1_{RSR}(\alpha_{inf}) = \omega^2_{RSR}(\alpha_{sup})$
- $\omega^0_{LSL}(\alpha_{sup}) = \omega^1_{LSL}(\alpha_{inf}) = \omega^1_{RSR}(\alpha_{sup}) = \omega^2_{RSR}(\alpha_{inf})$

**Proof.** See Appendix [B2]

By Lemma 2, the boundary lines of certain reachability regions of LSL and RSR path types are parallel to each other. This fact is explored to derive the full reachability conditions. Before we start with the detailed analysis of the reachability conditions, we present a useful notation. Let $\delta \in [0, 2\pi)$ be the rotation of a ray, then denote

$$\delta = \bar{\delta} + \pi \mod 2\pi.$$  

(27)

to be the rotation of the ray in its opposite direction.

As discussed in Section [H-B], full reachability is achieved by $2\pi$-arc LSL and RSR paths, if the entire space is covered by at least one of the following cases:

- 1) Union of MaRA and MiRA of LSL, and/or
- 2) Union of MaRA and MiRA of RSR, and/or
- 3) Union of MaRA of LSL and MiRA of RSR, and/or
- 4) Union of MaRA of RSR and MiRA of LSL.
Now, we derive the full reachability conditions for Case 1, while the derivation of the rest of the cases are similar.

Case 1: Conditions under which the union of LSL MaRA and LSL MiRA provide full reachability

Consider the centers \((p_{LSL}^0, q_{LSL}^0)\) and \((p_{LSL}^1, q_{LSL}^1)\), as described in \([11]\), for \(k = 0\) and \(k = 1\), respectively. There are two subcases:

1.1 \(k = 0\) forms LSL MaRA and \(k = 1\) forms LSL MiRA:

An illustrative example is shown in Fig. 15a. Note that the boundaries of LSL MaRA are formed by rays with rotations \(\omega_{LSL}^0(\alpha_{inf})\) and \(\omega_{LSL}^1(\alpha_{inf})\). Similarly, the boundaries of LSL MiRA are formed by rays with rotations \(\omega_{LSL}^0(\alpha_{sup})\) and \(\omega_{LSL}^1(\alpha_{sup})\). Now, using the notation in \([27]\), we define \(\overline{\omega}_{LSL}^0(\alpha_{sup})\) and \(\overline{\omega}_{LSL}^1(\alpha_{inf})\) to denote the rotations of the boundaries of LSL MaRA by \(\pi\) about the center \((p_{LSL}^0, q_{LSL}^0)\).

Further, since \(\overline{\omega}_{LSL}^0(\alpha_{sup}) \neq \overline{\omega}_{LSL}^0(\alpha_{inf})\), we can have:

1) \(\omega_{LSL}^0(\alpha_{sup}) < \omega_{LSL}^1(\alpha_{inf})\), as shown in Fig. 15a (1)

2) \(\omega_{LSL}^1(\alpha_{inf}) < \omega_{LSL}^1(\alpha_{sup})\), as shown in Fig. 15a (2)

The region enclosed within the \(\pi\) rotations of LSL MaRA boundaries is shown as the shaded area in Fig. 15a.

For full reachability, LSL MiRA should cover the unreachable area of LSL MaRA. From Lemma 2, we know that \(\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^1(\alpha_{sup})\) and \(\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^1(\alpha_{sup})\), thus the respective boundaries of LSL MaRA and LSL MiRA are parallel. This fact implies that, to achieve full reachability, the center of rotation \((p_{LSL}^1, q_{LSL}^1)\) of LSL MiRA should lie within the shaded area of LSL MaRA (see Fig. 15a).

To implement this full reachability condition, we find the rotation of the line segment joining the centers \((p_{LSL}^0, q_{LSL}^0)\) and \((p_{LSL}^1, q_{LSL}^1)\) as

\[\phi_{0,1} = \text{atan2}(q_{LSL}^1 - q_{LSL}^0, p_{LSL}^1 - p_{LSL}^0) \mod 2\pi\] (28a)

\[= \text{atan2}(w_y, w_x) \mod 2\pi,\] (28b)

where \(28b\) is obtained using \([11]\).

Then, based on the above discussion, we obtain the condition for full reachability as
Table III: Full reachability conditions using 2\(\pi\)-arc LSL and RSR paths.

| Case | MaRA | MiRA | Rotation of the line segment joining the centers of MaRA and MiRA | Full Reachability Conditions |
|------|------|------|-------------------------------------------------------------|--------------------------------|
|      | Path Type | Path Type | \(\phi_{0,1} = \text{atan2} \left( w_y, w_x \right) \pmod{2\pi} \) | If \( \alpha \) \begin{align*}
\omega_1^{\text{LSL}}(\alpha_{\text{sup}}) &< \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{0,1} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \\
\omega_1^{\text{LSL}}(\alpha_{\text{sup}}) &> \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{0,1} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{0,1} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}})
\end{align*} |
| 1   | LSL   | LSL   | \(\phi_{1,0} = \text{atan2} \left( -w_y, -w_x \right) \pmod{2\pi} \) | \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) < \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \\
\omega_1^{\text{LSL}}(\alpha_{\text{sup}}) > \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{1,0} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) 
\end{align*} |
| 2   | RSR   | RSR   | \(\phi_{1,0} = \text{atan2} \left( w_y, w_x \right) \pmod{2\pi} \) | \( \omega_1^{\text{RSR}}(\alpha_{\text{sup}}) < \omega_1^{\text{RSR}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \leq \phi_{1,0} \leq \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \\
\omega_1^{\text{RSR}}(\alpha_{\text{sup}}) > \omega_1^{\text{RSR}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \leq \phi_{1,0} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{1,0} \leq \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) 
\end{align*} |
| 3   | LSL   | RSR   | \(\phi_{1,0} = \text{atan2} \left( \cos \Theta - 1 + w_y (1 - \Theta), \sin \Theta + w_x (1 - \Theta) \right) \pmod{2\pi} \) | \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) < \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \\
\omega_1^{\text{LSL}}(\alpha_{\text{sup}}) > \omega_1^{\text{LSL}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \leq \phi_{1,0} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) 
\end{align*} |
| 4   | RSR   | LSL   | \(\phi_{1,0} = \text{atan2} \left( 1 - \cos \Theta - w_y (1 - \Theta), \sin \Theta - w_x (1 - \Theta) \right) \pmod{2\pi} \) | \( \omega_1^{\text{RSR}}(\alpha_{\text{sup}}) < \omega_1^{\text{RSR}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \leq \phi_{1,0} \leq \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \\
\omega_1^{\text{RSR}}(\alpha_{\text{sup}}) > \omega_1^{\text{RSR}}(\alpha_{\text{inf}}), \quad \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) \leq \phi_{1,0} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{1,0} \leq \omega_1^{\text{RSR}}(\alpha_{\text{inf}}) 
\end{align*} |

- If \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) < \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \), then:
  \[
  \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) \leq \phi_{0,1} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}). \tag{29}
  \]

- If \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) > \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \), then:
  \[
  \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) \leq \phi_{0,1} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{0,1} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}). \tag{30}
  \]

\subsection{1.2} \( k = 1 \) forms LSL MaRA and \( k = 0 \) forms LSL MiRA:
Since this subcase is similar to the first subcase of Case 1, we do not show the corresponding figure here. Using the same logic as for the first subcase, we find the rotation of the line segment joining the above two centers as
\[
\phi_{1,0} = \text{atan2} \left( q_1^{\text{LSL}} - q_1^{\text{LSL}}, p_1^{\text{LSL}} - p_1^{\text{LSL}} \right) \pmod{2\pi}, \tag{31a}
\]
\[
\phi_{1,0} = \text{atan2} \left( -w_y, -w_x \right) \pmod{2\pi}, \tag{31b}
\]
where (31b) is obtained using (11).
Then, we obtain the condition for full reachability as

- If \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) < \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \), then:
  \[
  \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}). \tag{32}
  \]

- If \( \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) > \omega_1^{\text{LSL}}(\alpha_{\text{inf}}) \), then:
  \[
  \omega_1^{\text{LSL}}(\alpha_{\text{sup}}) \leq \phi_{1,0} < 2\pi, \quad \text{or} \quad 0 \leq \phi_{1,0} \leq \omega_1^{\text{LSL}}(\alpha_{\text{inf}}). \tag{33}
  \]
The reachability conditions for Cases 2 – 4 can be derived in a similar fashion as Case 1, and their illustrative examples are shown in Figs. 15b, 15c and 15d respectively. However, for Cases 3 and 4, the union of different path types is used. Therefore, to obtain reachability conditions for Cases 3 and 4, we need Lemma 3 which connects the $k$ values associated with the MaRA and MiRA regions across different path types.

**Lemma 3.** The following are true:

1. If $k = 0$ forms LSL MaRA (MiRA), then $k = -1$ forms RSR MaRA (MaRA).
2. If $k = 1$ forms LSL MaRA (MiRA), then $k = -2$ forms RSR MiRA (MaRA).

**Proof.** See Appendix B3.

Table III presents the reachability conditions for all cases.

**Remark 7.** Besides Cases 1 – 4, there are other cases that can be considered for reachability analysis. However, Lemma 2 below negates those cases and shows that Cases 1 – 4 are sufficient for full reachability analysis.

**Lemma 4.** The following are true:

1. LSL (RSR) MaRA alone cannot provide full reachability.
2. Union of LSL MaRA and RSR MaRA cannot provide full reachability.
3. If Cases 1-4 do not provide full reachability, then the union of LSL MaRA, LSL MiRA, RSR MaRA and RSR MiRA cannot provide full reachability.

**Proof.** See Appendix B4.

**Corollary 3.** Cases 1 – 4 and the conditions therein are sufficient for full reachability analysis.

**Proof.** Lemma 3 discards all cases for full reachability analysis beyond Cases 1 – 4. Hence proved.

**B. Lemma proofs**

1) **Proof of Lemma 7**

**Proof.** Lemma 1 is proved in two steps. First, we show that as $\alpha$ varies within its feasible range as shown in Table I, the rays (14) (corresponding to the LSL path type) and (19) (corresponding to the RSR path type) rotate, where the points $(p_{LSL}^k, q_{LSL}^k)$ and $(p_{RSR}^k, q_{RSR}^k)$ form their centers of rotation, respectively. Second, we show that as $\alpha$ increases, (14) rotates anticlockwise, while (19) rotates clockwise.

For LSL path type, (14) can be re-written as

$$a(\alpha) \cdot (x_f - p_{LSL}^k) - c(\alpha) \cdot (y_f - q_{LSL}^k) = 0.$$  \hfill (34)

Thus, the slope of (34) varies when $\alpha$ changes, while the point $(p_{LSL}^k, q_{LSL}^k)$ always lies on (34) for all rotations. This indicates that $(p_{LSL}^k, q_{LSL}^k)$ is the center of rotation of (14). Moreover, for any given $\alpha$, one can determine the signs of $a(\alpha)$ and $c(\alpha)$, and the corresponding inequality constraint in (14), which in turn determines the quadrant of the coordinate system with center at $(p_{LSL}^k, q_{LSL}^k)$, within which (14) falls in. Thus (14) represents a ray starting from the center $(p_{LSL}^k, q_{LSL}^k)$.

Now, we show that as $\alpha$ increases from $\alpha_{inf}$ to $\alpha_{sup}$, (14) rotates in the anticlockwise manner. To see this, denote the slope of (14) as $S_{LSL}(\alpha) = \frac{a(\alpha)}{c(\alpha)}$, $\alpha \neq 0$. Note that $S_{LSL}(\alpha)$ is a continuous function of $\alpha$.

Taking the first-order derivative of $S_{LSL}(\alpha)$, we get

$$\frac{\partial S_{LSL}(\alpha)}{\partial \alpha} = \frac{1 + v_w \cos(\alpha - \theta_w)}{(\cos \alpha + v_w \cos \theta_w)^2}.$$  \hfill (35)

Since, $v_w < 1$ and $\cos(\alpha - \theta_w) \in [-1, 1]$, we get $\frac{\partial S_{LSL}(\alpha)}{\partial \alpha} > 0$.

Thus, as $\alpha$ grows, (14) rotates in the clockwise manner.

For RSR path type, (19) can be re-written as

$$b(\alpha) \cdot (x_f - p_{RSR}^k) + c(\alpha) \cdot (y_f - q_{RSR}^k) = 0.$$  \hfill (36)

Thus, the point $(p_{RSR}^k, q_{RSR}^k)$ always lies on (36) for all rotations. This indicates that $(p_{RSR}^k, q_{RSR}^k)$ is the center of rotation of (19). Moreover, for any given $\alpha$, one can determine the signs of $b(\alpha)$ and $c(\alpha)$, and the corresponding inequality constraint in (19), which in turn determines the quadrant of the coordinate system with center at $(p_{RSR}^k, q_{RSR}^k)$, within which (19) falls in. This implies that (19) represents a ray starting from the center $(p_{RSR}^k, q_{RSR}^k)$.

Now, we show that as $\alpha$ increases from $\alpha_{inf}$ to $\alpha_{sup}$, (19) rotates in the clockwise manner. To see this, denote the slope of (19) as $S_{RSR}(\alpha) = -\frac{b(\alpha)}{c(\alpha)}$, $\alpha \neq 0$. Note that $S_{RSR}(\alpha)$ is a continuous function of $\alpha$.

Taking the first-order derivative of $S_{RSR}(\alpha)$, we get

$$\frac{\partial S_{RSR}(\alpha)}{\partial \alpha} = -\frac{1 + v_w \cos(\alpha + \theta_w)}{(\cos \alpha + v_w \cos \theta_w)^2}.$$  \hfill (37)

Since $v_w < 1$ and $\cos(\alpha + \theta_w) \in [-1, 1]$, we get $\frac{\partial S_{RSR}(\alpha)}{\partial \alpha} < 0$.

Thus, as $\alpha$ grows, (19) rotates in the clockwise manner.

2) **Proof of Lemma 2**

**Proof.** First, consider $2\pi$-arc LSL paths. From Table I, for $k = 0$: $\alpha_{inf} = 0$ and $\alpha_{sup} = \theta_f$; while for $k = 1$: $\alpha_{inf} = \theta_f$ and $\alpha_{sup} = 2\pi$. Then, using (15) we get

$$\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^0(\alpha_{sup}) = \text{atan}(w_x, 1 + w_x) \mod 2\pi,$$  \hfill (38a)

$$\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^0(\alpha_{sup}) = \text{atan}(2\sin \theta_f + w_x, \cos \theta_f + w_x) \mod 2\pi.$$  \hfill (38b)

Now, consider $2\pi$-arc RSR paths. From Table I, for $k = -1$: $\alpha_{inf} = 0$ and $\alpha_{sup} = 2\pi - \theta_f$; while for $k = -2$: $\alpha_{inf} = 2\pi - \theta_f$ and $\alpha_{sup} = 2\pi$. Then, using (20) and we get

$$\omega_{RSR}^1(\alpha_{inf}) = \omega_{RSR}^1(\alpha_{sup}) = \text{atan}(w_x, 1 + w_x) \mod 2\pi,$$  \hfill (39a)

$$\omega_{RSR}^1(\alpha_{inf}) = \omega_{RSR}^1(\alpha_{sup}) = \text{atan}(2\sin \theta_f + w_x, \cos \theta_f + w_x) \mod 2\pi.$$  \hfill (39b)

Therefore, from (38a) and (39a) we get:

$$\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^0(\alpha_{sup}) = \omega_{RSR}^1(\alpha_{inf}) = \omega_{RSR}^1(\alpha_{sup}).$$

And from (38b) and (39b) we get:

$$\omega_{LSL}^0(\alpha_{inf}) = \omega_{LSL}^0(\alpha_{sup}) = \omega_{RSR}^1(\alpha_{inf}) = \omega_{RSR}^1(\alpha_{sup}).$$

\hfill \Box
3) Proof of Lemma 3

Proof. From Lemma 2 we get:
- $\omega^{0}_{\text{LSL}}(\alpha_{nf}) = \omega^{0}_{\text{RSR}}(\alpha_{nf})$
- $\omega^{0}_{\text{LSL}}(\alpha_{sup}) = \omega^{0}_{\text{RSR}}(\alpha_{sup})$.

Thus, the rotations of the two boundaries of the region spanned by $k = 0$ (i.e., $(\omega^{0}_{\text{LSL}}(\alpha_{nf})$ and $\omega^{0}_{\text{LSL}}(\alpha_{sup})$) are the same as the rotations of the corresponding boundaries of the region spanned by $k = -1$ (i.e., $\omega_{\text{RGR}}(\alpha_{nf})$ and $\omega_{\text{RGR}}(\alpha_{sup})$), respectively. Therefore, the acute angles between the boundaries corresponding to $k = 0$ and $k = 1$ are the same. Note that the centres of these two regions could be different. However, from Lemma 1 the swiping direction for $k = 0$ and $k = -1$ are opposite. Thus, if $k = 0$ forms LSL MaRA (MiRA), then $k = -1$ forms RSR MiRA (MaRA). This proves a). The proof of b) follows similar logic and is omitted here.

4) Proof of Lemma 4

Proof. a) Consider LSL MaRA formed by $k = 0$. From Table 1 we have the feasible range of $\alpha$ as $[0, \theta_{f}]$, where $\theta_{f} \in [0, 2\pi)$. Thus, if $\omega^{0}_{\text{LSL}}(0) = \omega^{0}_{\text{LSL}}(2\pi) = \text{atan2}(w_{y}, 1 + w_{x})$, then given any $\theta_{f} < 2\pi$, $\omega^{0}_{\text{LSL}}(\alpha)$ cannot make a full rotation as $\alpha$ varies from 0 to $\theta_{f}$. Thus, for $k = 0$, LSL MaRA cannot provide full reachability. Similarly, we can show that the MaRAs formed by $k = 1$ and $-2$ cannot provide full reachability.

b) First, we show that either LSL MaRA completely covers the RSR MaRA (i.e., RSR MaRA is a subset of LSL MaRA), or RSR MaRA completely covers the LSL MaRA (i.e., LSL MaRA is a subset of RSR MaRA).

Suppose LSL MaRA is formed by $k = 0$ (hence LSL MiRA is formed by $k = 1$). By Lemma 3, RSR MaRA is formed by $k = -2$. According to Lemma 2, $\omega^{0}_{\text{LSL}}(\alpha_{nf}) = \omega^{0}_{\text{RSR}}(\alpha_{sup})$ and $\omega^{0}_{\text{LSL}}(\alpha_{sup}) = \omega^{0}_{\text{RSR}}(\alpha_{nf})$. This implies that the boundaries of LSL MaRA and RSR MaRA are parallel to each other, and that they form the same acute angle, as shown in Fig. 16a. Thus, LSL MaRA can completely cover RSR MaRA if the center $(\rho^{0}_{\text{RGR}}, \phi^{0}_{\text{RGR}})$ falls inside the shadow region in Fig. 16a, which is formed by the boundaries with angles $\omega^{0}_{\text{LSL}}(\alpha_{sup})$ and $\omega^{0}_{\text{LSL}}(\alpha_{nf})$. Similarly, we can also determine the other condition when LSL MaRA is formed by $k = 1$ and RSR MaRA is formed by $k = -1$. Subsequently, we checked the truthness of both conditions for the full range of $\theta_{f}$ and $\theta_{w}$ from 0 to $2\pi$, and the results are presented in Fig. 16b. It is seen that for any given pair of $\theta_{f}$ and $\theta_{w}$, one of the above conditions is always true. Thus, either LSL MaRA completely covers the RSR MaRA or RSR MaRA completely covers the LSL MaRA. This indicates that the union of both MaRAs equals to the larger MaRA, then following part a) above, this in turn implies that their union cannot provide full reachability.

c) According to part b) above, either LSL MaRA completely covers the RSR MaRA or RSR MaRA completely covers the LSL MaRA. First, suppose that LSL MaRA is the larger of the two and covers the RSR MaRA. If the condition of Case 1 is not satisfied, then the union of LSL MaRA and LSL MiRA cannot provide full reachability and there exists some region that is unreachable, say $\mathcal{R}_{1}$ (e.g., see the white region in Fig. 15a). Thus, the center of LSL MiRA is not in the shadow region formed by the $\pi$ rotations of LSL MaRA boundaries.

Also, if the condition of Case 3 is not satisfied, then the union of LSL MaRA and RSR MiRA cannot provide full reachability and there exists some region that is unreachable, say $\mathcal{R}_{2}$ (e.g., see the white region in Fig. 15c). Thus, the center of RSR MiRA is not in the shadow region formed by the $\pi$ rotations of LSL MaRA boundaries. Since by Lemma 3, the boundaries of LSL MaRA, LSL MiRA and RSR MiRA are parallel to each other, as long as the centers of LSL MiRA and RSR MiRA are outside the shadow region of LSL MaRA, there is no way they can together cover the reachability gaps $\mathcal{R}_{1}$ and $\mathcal{R}_{2}$ completely. Thus, in this case, because RSR MaRA is a subset of LSL MaRA, the union of LSL MaRA, LSL MiRA, RSR MaRA and RSR MiRA cannot provide full reachability.

Using a similar logic, when RSR MaRA is the larger MaRA, one can show that if the conditions of Case 2 and Case 4 are not satisfied, then the union of LSL MaRA, LSL MiRA, RSR MaRA and RSR MiRA cannot provide full reachability.

REFERENCES

[1] I. S. Dolinskaya and A. Maggiar, “Time-optimal trajectories with bounded curvature in anisotropic media,” The International Journal of Robotics Research, vol. 31, no. 14, pp. 1761–1793, 2012.

[2] T. Fraichard and A. Scheuer, “From reeds and shepp’s to continuous-curvature paths,” IEEE Transactions on Robotics, vol. 20, no. 6, pp. 1025–1035, 2004.

[3] D. J. Balkcom and M. T. Mason, “Time optimal trajectories for bounded velocity differential drive vehicles,” The International Journal of Robotics Research, vol. 21, no. 3, pp. 199–217, 2002.

[4] S. G. Loizou and K. J. Kyriakopoulos, “Navigation of multiple kinematically constrained robots,” IEEE Transactions on Robotics, vol. 24, no. 1, pp. 221–231, 2008.

[5] D. B. Reister and F. G. Pin, “Time-optimal trajectories for mobile robots with two independently driven wheels,” The International Journal of Robotics Research, vol. 13, no. 1, pp. 38–54, 1994.

[6] Y. Bestaoui, “On line motion generation with velocity and acceleration constraints,” Robotics and Autonomous Systems, vol. 5, no. 3, pp. 279–288, 1989.

[7] L. Dubins, “On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents,” American Journal of Mathematics, vol. 79, no. 3, pp. 497–516, 1957.

[8] A. M. Shkel and V. Lumelsky, “Classification of the dubins set,” Robotics and Autonomous Systems, vol. 34, no. 4, pp. 179–202, 2001.

[9] J. Song, S. Gupta, and T.A. Wettergren, “T*: Time-optimal risk-aware motion planning for curvature-constrained vehicles,” IEEE Robotics and Automation Letters, vol. 4, no. 1, pp. 33–40, 2019.
[10] K. Mittal and S. Gupta, “Minimum-time motion-planning of auvs under spatially varying ocean currents,” in Proceedings of the IEEE/MTS OCEANS 2019, Seattle, Washington, 2019, pp. 1–5.

[11] L. Techy and C. Woolsey, “Minimum-time path planning for unmanned aerial vehicles in steady uniform winds,” Journal of Guidance, Control, and Dynamics, vol. 32, no. 6, p. 1736, 2009.

[12] T. McGee, S. Spry, and J.K. Hedrick, “Optimal path planning in a constant wind with a bounded turning rate,” in Proceedings of the AIAA Guidance, Navigation, and Control Conference and Exhibit, Reston, VA, 2005, pp. 1–11.

[13] E. Bakolas and P. Tsiotras, “Optimal synthesis of the zermelo–markov–dubins problem in a constant drift field,” Journal of Optimization Theory and Applications, vol. 156, no. 2, pp. 469–492, 2013.

[14] Y. Meyer, P. Isaiah, and T. Shima, “On dubins paths to intercept a moving target,” Automatica, vol. 53, pp. 256–263, 2015.

[15] Y. Ding, B. Xin, and J. Chen, “Curvature-constrained path elongation with expected length for dubins vehicle,” Automatica, vol. 108, p. 108495, 2019.

[16] J. Song, S. Gupta, J. Hare, and S. Zhou, “Adaptive cleaning of oil spills by autonomous vehicles under partial information,” in OCEANS’13 MTS/IEEE, San Diego, CA, September 2013, doi: 10.23919/OCEANS.2013.6741246.

[17] S. Gupta, A. Ray, and S. Phoha, “Generalized ising model for dynamic adaptation in autonomous systems,” EPL (Europhysics Letters), vol. 87, no. 1, p. 10009, 2009.

[18] J. Z. Hare, S. Gupta, and T. A. Wettergren, “Pose. 3c: Prediction-based opportunistic sensing using distributed classification, clustering, and control in heterogeneous sensor networks,” IEEE Transactions on Control of Network Systems, vol. 6, no. 4, pp. 1438–1450, 2019.

[19] K. Mittal, J. Song, and S. Gupta, “Real-time motion-planning of curvature-constrained auvs under steady ocean currents,” in Proceedings of the IEEE/MTS OCEANS 2019, Seattle, Washington, 2019, pp. 1–6.

[20] Z. Zeng, K. Sammut, E. Lian, F. He, A. Lammus, and Y. Tang, “A comparison of optimization techniques for auv path planning in environments with ocean currents,” Robotics and Autonomous Systems, vol. 82, pp. 61–72, 2016.

[21] B. Garau, M. Bonet, A. Alvarez, S. Ruiz, and A. Pascual, “Path planning for autonomous underwater vehicles in realistic oceanic current fields: Application to gliders in the western mediterranean sea,” Journal of Maritime Research, vol. 6, no. 2, pp. 5–22, 2009.

[22] B. Garau, A. Alvarez, and G. Oliver, “Path planning of autonomous underwater vehicles in current fields with complex spatial variability: an A* approach,” in Proceedings of the IEEE International Conference on Robotics and Automation, Barcelona, Spain, 2005, pp. 194–198.

[23] C. Petres, Y. Pailhais, P. Patron, Y. Pettot, J. Evans, and D. Lane, “Path planning for autonomous underwater vehicles,” IEEE Transactions on Robotics, vol. 23, no. 2, pp. 331–341, 2007.

[24] M. Soulignac, P. Taillibert, and M. Rueher, “Time-minimal path planning in dynamic current fields,” in Proceedings of the IEEE International Conference on Robotics and Automation, Kobe, Japan, 2009, pp. 2473–2479.

[25] A. Alvarez, A. Caiti, and R. Onken, “Evolutionary path planning for autonomous underwater vehicles in a variable ocean,” IEEE Journal of Oceanic Engineering, vol. 29, no. 2, pp. 418–429, 2004.

[26] W. Zhang, T. Inanc, S. Ober-Blobaum, and J. E. Marsden, “Optimal trajectory generation for a glider in time-varying 2D ocean flows B-spline model,” in Proceedings of the IEEE International Conference on Robotics and Automation, Pasadena, CA, 2008, pp. 1083–1088.

[27] T. Inanc, S. C. Shadden, and J. E. Marsden, “Optimal trajectory generation in ocean flows,” in Proceedings of the American Control Conference, Portland, OR, 2005, pp. 674–679.

[28] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, Estimation with applications to tracking and navigation: theory algorithms and software. John Wiley & Sons, 2004.

[29] Teledyne RD Instruments, Workhorse Mariner ADCP, http://www.teledynemarine.com/workhorse-mariner-adcp.

[30] L. Paull, S. Saeedi, M. Seto, and H. Li, “AUV navigation and localization: A review,” IEEE Journal of Oceanic Engineering, vol. 39, no. 1, pp. 131–149, 2014.

[31] FT Technologies Ltd., FT 205 Lightweight Acoustic Resonance Wind Sensor, https://fttechnologies.com.

[32] Inertial Sense, GNSS-INS Sensors (RTK), https://inertialsense.com/products/gnss-ins-rtk-sensors/.

[33] J. Song and S. Gupta, “ε*: An online coverage path planning algorithm,” IEEE Transactions on Robotics, vol. 34, pp. 526 – 533, 2018.

[34] ——, Care: Cooperative autonomy for resilience and efficiency of robot teams for complete coverage of unknown environments under robot failures,” Autonomous Robots, vol. 44, pp. 647–671, 2020.

[35] R. Pěnička, J. Faigl, P. Váňa, and M.aska, “Dubins orienteering problem,” IEEE Robotics and Automation Letters, vol. 2, no. 2, pp. 1210–1217, 2017.