Diffusive limit of a spatially-extended kinetic FitzHugh-Nagumo model

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Abstract

We consider a spatially extended kinetic model of a FitzHugh-Nagumo neural network, with a rescaled interaction kernel. Our main purpose is to prove that its diffusive limit in the regime of strong local interactions converges towards a FitzHugh-Nagumo reaction-diffusion system, taking account for the average quantities of the network. Our approach is based on a relative entropy argument, to compare the macroscopic quantities computed from the solution of the kinetic equation, and the solution of the limiting system. The main difficulty, compared to the literature, lies in the need of regularity in space of the solutions of the limiting system and a careful control of an internal nonlocal kinetic dissipation.

Keywords: Diffusive limit, relative entropy, FitzHugh-Nagumo, neural network.

1 Introduction

The FitzHugh-Nagumo model [12, 22] focuses on the evolution of a couple \((v, w)\), where \(v \in \mathbb{R}\) is a fast excitable variable standing for the membrane electrical potential of a neural cell, and \(w \in \mathbb{R}\) is a slow refractory variable, which we shall call the adaptation variable throughout the sequel. It is usually written for a single neuron as follows:

\[
\begin{aligned}
\frac{dv}{dt} &= N(v) - w + I_{\text{ext}}, \\
\frac{dw}{dt} &= \tau (v - \gamma w),
\end{aligned}
\]

(1.1)

for \(t > 0\), where \(I_{\text{ext}}\) stands for the input current the neuron receives from its environment, \(N(v)\) is a bistable function modeling the cell excitability, \(\tau\) and \(\gamma\) are non-negative constants. Here, we consider small \(\tau > 0\) to account for the slow evolution of the adaptation variable. Furthermore, in the rest of this article, as in [4, 5, 14–16, 24], we consider the following cubic nonlinearity

\[
N(v) := v (1 - v) (v - a), \quad v \in \mathbb{R},
\]

(1.2)
where \( a \in (0,1) \) is fixed. Such a nonlinearity satisfies the following properties for four fixed positive constants \( \kappa_1, \kappa'_1, \kappa_2 \) and \( \kappa_3 \):

\[
\begin{align*}
  v N(v) &\leq \kappa_1 |v|^2 - \kappa'_1 |v|^4, & v \in \mathbb{R}, \\
  (v - u) (N(v) - N(u)) &\leq \kappa_2 |v - u|^2, & v, u \in \mathbb{R}, \\
  |N(v) - N(u)| &\leq \kappa_3 |v - u| \left(1 + |v|^2 + |u|^2\right), & v, u \in \mathbb{R}.
\end{align*}
\]

(1.3)

In a neural network, neurons interact through their synapses. Therefore, it can be modeled by a system of coupled equations of FitzHugh-Nagumo type, where the coupling is described through the input current \( I_{\text{ext}} \). In this article, we are interested in spatially structured coupling, motivated by all the complex dynamics it can produce, for example propagation of excitatory pulses through space, spiral waves in two-dimensions and so on. Thus, we consider a spatially structured network of \( d \) interacting neurons of FitzHugh-Nagumo type, where each neuron is indexed by \( i \in \{1,...,n\} \), and characterized by a couple potential-adaptation variable \((v_i, w_i) \in \mathbb{R}^2\) and a position \( x_i \in \mathbb{R}^d \) with \( d \in \{1,2,3\} \). Coupling the neurons through \( I_{\text{ext}} \) with Ohm’s law, the triplets \((x_i, v_i, w_i)_{1 \leq i \leq n}\) satisfy the following system:

\[
\begin{align*}
  \frac{d x_i}{dt} &= 0, \\
  \frac{d v_i}{dt} &= N(v_i) - w_i - \frac{1}{n} \sum_{j=1}^{n} \Phi_\varepsilon(\|x_i - x_j\|) (v_i - v_j), \\
  \frac{d w_i}{dt} &= \tau (v_i - \gamma w_i),
\end{align*}
\]

(1.4)

for \( t > 0 \). In (1.4), \( \Phi_\varepsilon : \mathbb{R} \to \mathbb{R} \) is a connectivity kernel modeling the influence of the distance between two neurons on their interactions, depending on a small scaling parameter \( \varepsilon > 0 \). In the rest of this article, we consider

\[
\Phi_\varepsilon(\|x\|) := \frac{1}{\varepsilon^d} \Psi_\varepsilon(\|x\|), \quad \Psi_\varepsilon(\|x\|) := \frac{1}{\varepsilon^d} \Psi \left(\frac{\|x\|}{\varepsilon}\right), \quad x \in \mathbb{R}^d, \quad \varepsilon > 0,
\]

(1.5)

where \( \Psi \) still denotes a function from \( \mathbb{R} \) to \( \mathbb{R}^+ \). Notice that in this model, the interactions between two neurons are only modulated by their distance and the electrical voltage between them.

As showed in [6], a mean-field description of this network when \( n \) goes to infinity is given by the following spatially-extended kinetic model for all \( \varepsilon > 0, t > 0, x \in \mathbb{R}^d \) and \( u = (v, w) \in \mathbb{R}^2 \), with the notation \( u' = (v', w') \in \mathbb{R}^2 \):

\[
\begin{align*}
  &\partial_t f^\varepsilon(t, x, u) + \nabla_x \left[ f^\varepsilon(t, x, u) \left( N(v) - w - \mathcal{K}_\varepsilon[f^\varepsilon](t, x, v) \right) \right] + \nabla_w \left[ f^\varepsilon(t, x, u) A(v, w) \right] = 0, \\
  &\mathcal{K}_\varepsilon[f^\varepsilon](t, x, v) := \frac{1}{\varepsilon^{d+2}} \int \int \Psi \left(\frac{\|x - x'\|}{\varepsilon}\right) (v - v') f^\varepsilon(t, x', u') \, dx' \, du', \\
  &A(v, w) := \tau (v - \gamma w), \\
  &f^\varepsilon|_{t=0} = f^\varepsilon_0,
\end{align*}
\]

(1.6)
where \( f^\varepsilon(t, x, u) \) is the density function of finding neurons at time \( t \) at position \( x \) and at the electrical state \( u = (v, w) \) within the cortex. Here, the interactions through the whole network are modeled by the nonlocal term \( \varepsilon \partial_v (f^\varepsilon K_\varepsilon[f^\varepsilon]) \) while all the other local terms model the excitability of neurons.

Our main purpose is to study the diffusive limit \( \varepsilon \to 0 \), in order to rigorously derive a macroscopic description of the system (1.4) in the limit \( n \to +\infty \). Our strategy is to introduce the kinetic equation (1.6) as an intermediary model, and then to pass to the limit \( \varepsilon \to 0 \) from this mesoscopic scale. Furthermore, the macroscopic description we derive is highly dependent of our choice of rescaling in \( \Phi_\varepsilon \). Indeed, in [7], we considered a family a connectivity kernels which can be decomposed into the sum of a long-range interaction kernel, and a Dirac mass centered in 0 with a mass \( \varepsilon^{-1} \). In the hydrodynamic limit \( \varepsilon \to 0 \), the kinetic equation (1.6) converges towards a nonlocal reaction-diffusion system of FitzHugh-Nagumo type. Despite the interest of this model, it is not quite realistic from a biological viewpoint, since it considers that two neurons even far away from each other can still interact. Since a neural cell only interacts with its closest neighbours through synapses, in this article, we rather consider the regime of strong local interactions, through the scaling given by (1.5).

A first idea to derive a macroscopic description from this kinetic model is to consider the average functions computed from \( f^\varepsilon \) a solution to the kinetic model (1.6), such as \( \rho^\varepsilon \) the average density of neurons, \( V^\varepsilon \) the average membrane potential and \( W^\varepsilon \) the average adaptation variable, defined through

\[
\begin{align*}
\rho^\varepsilon(t, x) &:= \int_{\mathbb{R}^2} f^\varepsilon(t, x, v, w) \, dv \, dw, \\
\rho^\varepsilon(t, x) V^\varepsilon(t, x) &:= \int_{\mathbb{R}^2} v f^\varepsilon(t, x, v, w) \, dv \, dw, \\
\rho^\varepsilon(t, x) W^\varepsilon(t, x) &:= \int_{\mathbb{R}^2} w f^\varepsilon(t, x, v, w) \, dv \, dw.
\end{align*}
\]

We can easily notice that the equation satisfied by \( \rho^\varepsilon V^\varepsilon \) is not closed, because of the nonlinearity \( N(v) \) which makes appear some moments of \( f^\varepsilon \) of higher order. Actually, the diffusive limit \( \varepsilon \to 0 \) in the kinetic equation (1.6) enables us to consider the regime of strong local interactions, in order to circumvent this issue. Indeed, in the spirit of [7], we expect the macroscopic functions computed from \( f^\varepsilon \) a solution to the kinetic equation (1.6) to converge in the limit \( \varepsilon \to 0 \) towards a solution to a reaction-diffusion system, which will therefore provide a macroscopic description of the FitzHugh-Nagumo model (1.4).

The problem tackled in this article lies within a rich literature of works aiming to establish a rigorous link between a mesoscopic model and a macroscopic characterization of large neural networks. For example, in [23], the authors derived the Integrate-and-Fire model as the macroscopic limit of a voltage-conductance kinetic system. Then, the article [3] studied the mean-field limit of a Hodgkin-Huxley neural network, and proved the existence of synchronized dynamics for the microscopical and macroscopic models. If we focus on the FitzHugh-Nagumo model, a recent example is the work of [24], which investigates the large coupling limit of a kinetic description of a noisy FitzHugh-Nagumo neural network, with uniform conductance, and highlight the emergence of clamping or synchronization between neurons in the limit. We also mention [19], in which the authors proved the mean-field limit of noisy spatially-extended FitzHugh-Nagumo network and propagation of chaos, so that in the limit, the electrical state of each neuron can be modeled by a stochastic differential equation. This last result has been extended in [20] to the mean-field limit of
spatially-extended FitzHugh-Nagumo neural networks on random graphs, to prove the convergence towards a nonlocal reaction-diffusion system.

Furthermore, there exists a large variety of works on the approximation of kinetic models with diffusion equations. We mention for instance the article [2] about the derivation of a diffusion-type equation as the limit of a rescaled collisional kinetic equation. In [9], the authors tackled the case in which the collisional term in the kinetic equation is heterogeneous in space. Then, in [21], this result has been extended to situations in which the equilibrium is a heavy-tailed distribution with finite variance, leading to a fractional diffusion in the limiting system.

Now, let us formally derive the behavior of a solution $f^\varepsilon$ of the kinetic equation (1.6) in the limit $\varepsilon \to 0$. Using the change of variable $y = (x - x')/\varepsilon$, we formally have for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $u = (v, w) \in \mathbb{R}^2$:

$$f^\varepsilon(t, x, u) K_\varepsilon[f^\varepsilon](t, x, v) = \varepsilon^{-(d+2)} \iint \Psi \left( \frac{\|x - x'|}{\varepsilon} \right) (v - v') f^\varepsilon(t, x', u') f^\varepsilon(t, x, u) \, dx' \, du'$$

Moreover, using a Taylor expansion, we get that for all $t \in [0, T]$, $u' \in \mathbb{R}^2$ and $(x, y) \in \mathbb{R}^{2d}$,

$$f^\varepsilon(t, x - \varepsilon y, u') = f^\varepsilon(t, x, u') - \varepsilon \nabla_x f^\varepsilon(t, x, u') \cdot y + \frac{\varepsilon^2}{2} y^T \nabla_x^2 f^\varepsilon(t, x, u') \cdot y + \ldots \quad (1.8)$$

In the following, for the sake of simplification, let us assume that the connectivity kernel $\Psi$ is in $L^1(\mathbb{R}^d)$ with $\|\Psi\|_{L^1(\mathbb{R}^d)} = 1$, is radially symmetric, and satisfies:

$$0 < \sigma := \int \Psi(\|y\|) \frac{\|y\|^2}{2} \, dy < \infty. \quad (1.9)$$

We resume the computation of $f^\varepsilon K_\varepsilon[f^\varepsilon]$ using the expansion (1.8), which gives:

$$f^\varepsilon(t, x, u) K_\varepsilon[f^\varepsilon](t, x, v) = \varepsilon^-2 \int (v - v') f^\varepsilon(t, x, u') f^\varepsilon(t, x, u) \, du'$$

$$+ \sigma \int (v - v') \Delta_x f^\varepsilon(t, x, u') f^\varepsilon(t, x, u) \, du' + R_\varepsilon(t, x, u), \quad (1.10)$$

where $R_\varepsilon(t, x, u)$ gathers all the remaining terms. Assume that the solution $f^\varepsilon$ converges towards a distribution $f$ in some weak sense, and that $R_\varepsilon$ tends towards 0 as $\varepsilon$ goes to 0. In the following, we define the triplet $(\rho, V, W)$ of macroscopic quantities associated to $f$ similarly as in (1.7). Then, we insert the expansion (1.10) into the kinetic equation (1.6), and we identify the orders $-2$ and 0 of $\varepsilon$ as $\varepsilon$ tends to 0, which formally leads to the following equations satisfied by $f$ in the sense of distributions for $t > 0, x \in \mathbb{R}^d$, $(v, w) \in \mathbb{R}^2$:

$$\mathcal{O}(\varepsilon^{-2}) : \quad \iint (v - v') f(t, x, v', u') f(t, x, v, w) \, dv' \, dw' = 0, \quad (1.11)$$

$$\mathcal{O}(1) : \quad \partial_t f + \partial_v [f (N(v) - w)] + \partial_w [f A(v, w)] - \sigma \partial_v [f (\Delta_x \rho v - \Delta_x (\rho V))] = 0. \quad (1.12)$$

The equation (1.11) implies that $f$ is proportional to a Dirac mass in $v$, that is for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $(v, w) \in \mathbb{R}^2$,

$$f(t, x, v, w) = F(t, x, w) \otimes \delta_{V(t,x)}(v), \quad (1.13)$$
where we define

\[ F(t, x, w) := \int_{\mathbb{R}} f(t, x, v, w) dv, \quad \rho_0(x) W(t, x) := \int_{\mathbb{R}} w F(t, x, dw). \]

We also directly get the conservation of mass, that is for all \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \rho(t, x) = \rho(0, x) = \rho_0(x) \). Therefore, we deduce from (1.12)-(1.13) that the couple \((V, F)\) satisfies the following system for \( t > 0 \), \( x \in \mathbb{R}^d \) and \( w \in \mathbb{R} \):

\[
\begin{align*}
\partial_t F + \partial_w (A(V, w) F) &= 0, \\
\partial_t (\rho_0 V) - \sigma [\rho_0 \Delta_x (\rho_0 V) - (\Delta_x \rho_0) \rho_0 V] &= \rho_0 N(V) - \rho_0 W,
\end{align*}
\]

where \( \sigma \) is the constant defined with (1.9). Therefore, the limiting triplet \( Z := (\rho_0, \rho_0 V, \rho_0 W) \) is expected to satisfy the reaction-diffusion system

\[
\partial_t Z = \rho_0 \begin{pmatrix}
0 \\
N(V) - W + \sigma [\Delta_x (\rho_0 V) - \Delta_x \rho_0 V] \\
A(V, W)
\end{pmatrix}.
\]

Let us discuss the structure of the system (1.15). It is worth noticing that it is not well-defined for \( x \in \mathbb{R}^d \) if \( \rho_0(x) = 0 \). In the case \( \rho_0 > 0 \), the system (1.15) reduces to the FitzHugh-Nagumo reaction-diffusion system for \( t > 0 \) and \( x \in \mathbb{R}^d \):

\[
\begin{align*}
\partial_t V(t, x) - \sigma [\Delta_x (\rho_0 V)(t, x) - \Delta_x \rho_0(x) V(t, x)] &= N(V(t, x)) - W(t, x), \\
\partial_t W(t, x) &= A(V(t, x), W(t, x)).
\end{align*}
\]

Such a system has been extensively studied, namely regarding the formation and propagation of traveling fronts and pulses for example (see e.g. [4, 5, 16]). We also mention some recent works on the FitzHugh-Nagumo system in the discrete case [14, 15]. The contribution of our article is thus to prove that the system (1.14), which can be reduced under some assumptions to the FitzHugh-Nagumo reaction-diffusion system (1.16), is a macroscopic description of the FHN model (1.4) in the limit \( n \to +\infty \) and \( \varepsilon \to 0 \).

Our approach to prove the diffusive limit follows ideas from [7, 13, 17, 18], that is we use a relative entropy argument. The specificity of our problem is the absence of noise in the microscopic system (1.4), which implies the absence of a Laplace operator in \( v \) in the kinetic equation (1.6). Hence, without the regularizing effect of noise, the solution \( f^\varepsilon \) of the kinetic equation (1.6) converges towards a Dirac distribution in \( v \), which prevents us from using a more usual entropy of type \( f \log(f) \) as in [18]. As in [7, 13, 17], we rather focus on the evolution of moments of second order in \( v \) and \( w \) of \( f^\varepsilon \). Then, we encounter two main difficulties. The first one comes from the term \( \partial_v (f^\varepsilon N(v)) \) in (1.6), which introduces moments of \( f^\varepsilon \) in \( v \) of higher order. Similarly as in [7], it will be sufficient to control moments of \( f^\varepsilon \) of fourth order to circumvent this problem. Then, unlike [7], we have to precisely determine the regularity in space we need for the solutions of the limiting system (1.14) to estimate the relative entropy.
Outline of the paper. The rest of this paper is organised as follows. In Section 2, we present our hypotheses, and our results on the existence of solutions to the kinetic equation (1.6) and to the limit equation (1.14), and our main result about the diffusive limit from (1.6) to (1.14). Then, in Section 3, we prove some a priori estimates which will be key arguments for the proof of our main result. Then, Section 4 is devoted to the relative entropy estimate and the proof of our main result. Finally, in Section 5, we study the well-posedness of the limiting equation (1.14), constructing a solution from a couple $(V,W)$ satisfying the reaction-diffusion system (1.16) in a weak sense.

2 Main result

In this section, we state our main result on the diffusive limit of a weak solution $(f^\varepsilon)_{\varepsilon>0}$ of the kinetic equation (1.6) towards a solution $(V,F)$ of the reaction-diffusion equation (1.14). Before that, we have to precisely define our notion of solutions of the kinetic model (1.6) and of the system (1.14). But first, let us start the section setting the hypotheses we make on the connectivity kernel.

2.1 Preliminaries

Following the ideas from the formal analysis in the introduction, we consider a connectivity kernel $\Psi : \mathbb{R} \to \mathbb{R}^+$ satisfying the following assumptions:

$$
\Psi > 0 \text{ a.e., } \int \Psi(\|y\|) \, dy = 1, \quad 0 < \sigma = \int \Psi(\|y\|) \frac{\|y\|^2}{2} \, dy < \infty. \tag{2.1}
$$

Let us discuss the hypotheses on $\Psi$. First, the choice of a positive and integrable connectivity kernel implies that we only consider activatory interactions, and that neurons which are far away from each other have few interactions. Then, the assumptions of finite moments and symmetry are technical hypotheses, inspired by the formal analysis from the introduction.

2.2 Existence of a weak solution of the kinetic equation

In this subsection, we focus on the well-posedness of the kinetic model. First, let us specify our notion of weak solution of the kinetic equation (1.6).

**Definition 2.1.** We say that $f^\varepsilon$ is a weak solution of (1.6) with initial condition $f^\varepsilon_0 \geq 0$ if for any $T > 0$,

$$
f^\varepsilon \in \mathcal{C}^0\left([0,T], L^1(\mathbb{R}^{d+2}) \right) \cap L^\infty\left((0,T) \times \mathbb{R}^{d+2}\right),
$$

and for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^{d+2})$, the following weak formulation of (1.6) holds,

$$
\int_0^T \int f^\varepsilon [\partial_t \varphi + (N(v) - w - K_{\varepsilon}[f^\varepsilon]) \partial_v \varphi + A(v,w) \partial_w \varphi] \, dz \, dt + \int f^\varepsilon_0(\mathbf{z}) \varphi(\mathbf{0}, \mathbf{z}) \, dz = 0, \tag{2.2}
$$

where $\mathbf{z} = (\mathbf{x}, v, w) \in \mathbb{R}^{d+2}$. 
In the rest of this paper, if \( f^\varepsilon \) is a weak solution of (1.6), we note \( Z^\varepsilon := (\rho^e_0, \rho^e_0 V^e, \rho^e_0 W^e) \) the triplet of macroscopic quantities computed from \( f^\varepsilon \) as in (1.7). Then, let us quote our result of existence and uniqueness of a weak solution to the kinetic equation (1.6), whose proof can be found in the proposition 2.2 from [7].

**Proposition 2.2.** Let \( \varepsilon > 0 \). We consider a connectivity kernel \( \Psi \) satisfying (2.1), and an initial data \( f^e_0 \) such that

\[
f^e_0 \geq 0, \quad \| f^e_0 \|_{L^1(\mathbb{R}^{d+2})} = 1, \quad f^e_0, \nabla u f^e_0 \in L^\infty(\mathbb{R}^{d+2}),
\]

where \( u = (v, w) \), and there exists a positive constant \( R^e_0 > 0 \) such that for all \( x \in \mathbb{R}^d \),

\[
\text{Supp}(f^e_0(x, \cdot)) \subseteq B(0, R^e_0) \subseteq \mathbb{R}^2.
\]

Then, for any \( T > 0 \), there exists a unique non-negative weak solution \( f^\varepsilon \) of (1.6) in the sense of Definition 2.1, which is compactly supported in \( u = (v, w) \in \mathbb{R}^2 \).

**Remark 2.3.**

(i) In the following, since the conservation of the \( L^1 \) norm of a weak solution \( f^\varepsilon \) of (1.6) holds, we get that for all \( t \in [0, T] \), \( \| f^\varepsilon(t, \cdot) \|_{L^1(\mathbb{R}^{d+2})} = \| f^e_0 \|_{L^1(\mathbb{R}^{d+2})} = 1 \).

(ii) It is worth noticing that we do not need the weak solution \( f^\varepsilon \) of (1.6) to be differentiable in space.

### 2.3 Existence of a solution of the limit system

Our purpose is to prove the existence of a solution to the system (1.14). We proceed in two steps: first, we prove the existence and uniqueness of a solution to the FitzHugh-Nagumo reaction-diffusion system (1.16), and then we construct a solution to the sytsem (1.14) from the solution to the sytsem (1.16). Thus, we start by studying the following Cauchy problem for a given initial data \((V_0, W_0)\):

\[
\begin{cases}
\partial_t V(t, x) - \sigma \left[ \rho_0(x) \Delta x V(t, x) + 2 \nabla_x \rho_0(x) \cdot \nabla_x V(t, x) \right] = N(V(t, x)) - W(t, x), \\
\partial_t W(t, x) = A(V(t, x), W(t, x)), \\
V|_{t=0} = V_0, \quad W|_{t=0} = W_0.
\end{cases}
\]

(2.5)

Before stating our existence result, we have to precisely define the notion of weak solution to the reaction-diffusion system (2.5). Since all the mathematical difficulties come from the first equation from (2.5), we consider the second component \( W \) as a reaction term.

**Definition 2.4.** For any \( T > 0 \) and any given initial data \( V_0, W_0 \in H^2(\mathbb{R}^d) \), the couple \((V, W)\) is a weak solution of (2.5) on \([0, T]\) with initial data \((V_0, W_0)\) if \( V \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap W^{1,\infty}([0, T], L^2(\mathbb{R}^d)) \), and \((V, W)\) verifies for all \( \varphi \in H^1(\mathbb{R}^d) \), for all \( t \in [0, T] \) and almost every \( x \in \mathbb{R}^d \):

\[
\int \partial_t V \varphi \, dx = -\sigma \int \rho_0 \nabla_x V \cdot \nabla_x \varphi \, dx + \sigma \int \nabla_x \rho_0 \cdot \nabla_x \varphi \, dx + \int (N(V) - W) \varphi \, dx,
\]

\[
W(t, x) = e^{-\tau \gamma t} W_0(x) + \tau \int_0^t e^{-\tau \gamma (t-s)} V(s, x) \, ds,
\]

\[
V|_{t=0} = V_0.
\]

(2.6)
To conclude this subsection, we state our notion of solution to the limiting system (2.5).

**Proposition 2.5.** Let Ψ be a connectivity kernel satisfying (2.1). Consider an initial data ρ₀ such that

\[ ρ₀ ≥ 0, \quad ρ₀ ∈ C₀^∞(\mathbb{R}^d), \quad ∥ρ₀∥_{L^1} = 1, \]  

and also consider an initial data (V₀, W₀) such that

\[ V₀, W₀ ∈ H^2(\mathbb{R}^d). \]  

Then, for all T > 0, there exists a unique function (V, W) weak solution of the reaction-diffusion equation (2.5) on [0, T] in the sense of Definition 2.4 such that

\[ V, W ∈ L^∞ \left( [0, T], H^2(\mathbb{R}^d) \right) \cap C^0 \left( [0, T], H^1(\mathbb{R}^d) \right). \]

We postpone the proof of Proposition 2.5 to Section 5. Our strategy consists in approximating the reaction-diffusion system (2.5) with an uniformly parabolic system, and then to pass to the limit.

It remains to deduce from this proposition the existence of a solution to the limiting system (1.14). In order to work with a well-defined system on \( \mathbb{R}^d \), we make the convention that for all \( x ∈ \mathbb{R}^d \) such that \( ρ₀(x) = 0 \), \( V(⋅, x) = 0 \) and \( F(⋅, x, ⋅) = 0 \). From a modeling viewpoint, it means that there is no electrical activity wherever there is no neuron. Hence, we are led to study the following Cauchy problem for any given initial data \((V₀, F₀)\), for all \( t > 0, x ∈ \mathbb{R}^d \) and \( w ∈ \mathbb{R} \):

\[
\begin{cases}
\partial_t F + \partial_w (A(V, w) F) = 0, \\
\partial_t (ρ₀ V) - σ[ρ₀ Δ_x (ρ₀ V) - (Δ_x ρ₀) ρ₀ V] = ρ₀ N(V) - ρ₀ W, \\
V(t, x) = 0, \quad t > 0 \text{ and } x ∈ \mathbb{R}^d \setminus \text{Supp}_{ess}(ρ₀), \\
W(t, x) = \begin{cases}
\frac{1}{ρ₀(x)} \int_{\mathbb{R}} w F(t, x, dw) & \text{if } ρ₀(x) > 0, \\
0 & \text{else},
\end{cases} \\
V|_{t=0} = V₀, \quad F|_{t=0} = F₀, \quad ρ₀(x) = \int_{\mathbb{R}} F₀(x, w) dw.
\end{cases}
\]

To conclude this subsection, we state our notion of solution to the limiting system (2.9), and then our result of existence and uniqueness of a solution. In the rest of this article, we denote by \( \mathcal{M}(\mathbb{R}^{d+1}) \) the set of non-negative Radon measures on \( \mathbb{R}^{d+1} \).

**Definition 2.6.** For any \( T > 0 \), and any initial data \( V₀ ∈ H^2(\mathbb{R}^d) \) and \( F₀ ∈ \mathcal{M}(\mathbb{R}^{d+1}) \) satisfying

\[ \int_{\mathbb{R}^{d+1}} |w|^2 F₀(dx, dw) < +∞, \quad ρ₀ := \int_{\mathbb{R}} F₀(⋅, dw) ∈ L^1(\mathbb{R}^d), \]
we say that \((V, F)\) is a solution of \((2.9)\) if \(F\) is a measure solution of the first equation in \((1.14)\), that is for all \(\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^{d+1})\), for all \(t \in [0, T]\),

\[
\frac{d}{dt} \int_{\mathbb{R}^{d+1}} \varphi(x, w) F(t, dx, dw) - \int_{\mathbb{R}^{d+1}} A(V(t, x), w) \partial_w \varphi F(t, dx, dw) = 0, \tag{2.10}
\]

and \(V \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap W^{1,\infty}([0, T], L^2(\mathbb{R}^d))\) satisfies for all \(\varphi \in H^1(\mathbb{R}^d)\) and all \(t \in [0, T]\),

\[
\begin{aligned}
\int \partial_t (\rho_0 V) \varphi dx &= \sigma \int [(\rho_0 V) \nabla_x \rho_0 - \rho_0 \nabla_x V(\rho_0 V)] \cdot \nabla_x \varphi dx + \int \rho_0 (N(V) - W) \varphi dx, \\
W(t, x) &= \begin{cases} 
0 & \text{if } \rho_0(x) = 0, \\
\frac{1}{\rho_0(x)} \int w F(t, x, dw) & \text{else},
\end{cases} \tag{2.11}
\end{aligned}
\]

**Corollary 2.7.** Let \(\Psi\) be a connectivity kernel satisfying \((2.1)\). Consider an initial data \((V_0, F_0)\) such that \(F_0 \in \mathcal{M}(\mathbb{R}^{d+1})\) satisfies

\[
\int |w|^2 F_0(dx, dw) < +\infty, \tag{2.12}
\]

and define for all \(x \in \mathbb{R}^d\),

\[
\rho_0 := \int F_0(\cdot, dw), \quad W_0(x) := \begin{cases} 
\frac{1}{\rho_0(x)} \int w F_0(x, dw) & \text{if } \rho_0(x) > 0, \\
0 & \text{else.}
\end{cases} \tag{2.13}
\]

Let us assume that \(\rho_0\) satisfies \((2.7)\), \((V_0, W_0)\) satisfies \((2.8)\), and that

\[
V_0(x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus \text{Supp}_{ess}(\rho_0). \tag{2.14}
\]

Then, for all \(T > 0\), there exists a unique function \((V, F)\) solution of the reaction-diffusion equation \((2.9)\) on \([0, T]\) in the sense of Definition 2.6 such that

\[
\begin{cases} 
\rho_0 V \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap \mathcal{C}^0([0, T], H^1(\mathbb{R}^d)), \\
F \in L^\infty([0, T], \mathcal{M}(\mathbb{R}^{d+1})),
\end{cases}
\]

and such that there exists a constant \(C_T > 0\) such that for all \(t \in [0, T]\),

\[
\int |w|^2 F(t, dx, dw) \leq C_T.
\]

We postpone the proof of Corollary 2.7 to Section 5.

### 2.4 Main result

Now, we can state our main theorem about the diffusive limit.
Theorem 1 (Diffusive limit). Let \( T > 0 \), and let \( \Psi \) be a connectivity kernel satisfying (2.1). Consider a set of initial data \( (f_0^\varepsilon)_{\varepsilon > 0} \) satisfying the assumptions (2.3)-(2.4), and there exists a positive constant \( M > 0 \) such that
\[
\int (1 + |x|^4 + |v|^4 + |w|^4) f_0^\varepsilon(x,v,w) \, dx \, dv \, dw \leq M,
\]
\[
\|p_0^\varepsilon\|_{L^\infty({\mathbb{R}}^d)} \leq M.
\]
Also consider the initial data \((\rho_0, V_0, W_0)\) satisfying the assumptions (2.7)-(2.8) such that \( \rho_0 \in H^2({\mathbb{R}}^d) \), and verifying:
\[
\frac{1}{\varepsilon^2} \|p_0^\varepsilon - \rho_0\|_{L^2({\mathbb{R}}^d)} \to 0,
\]
\[
\int p_0^\varepsilon(x) \left[ |V_0^\varepsilon(x) - V_0(x)|^2 + |W_0^\varepsilon(x) - W_0(x)|^2 \right] \, dx \to 0
\]
as \( \varepsilon \to 0 \). Consider \((V,W)\) the weak solution of the reaction-diffusion equation (2.5) on \([0,T]\) provided by Proposition 2.5. For any \( \varepsilon > 0 \), let \( f^\varepsilon \) be the weak solution of the kinetic equation (1.6) on \([0,T]\) provided by Proposition 2.2. Then, for all \( \varepsilon > 0 \), the macroscopic functions \((\rho_0^\varepsilon, V^\varepsilon, W^\varepsilon)\) computed from \( f^\varepsilon \) satisfy
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \int p_0^\varepsilon(x) \left[ |V^\varepsilon(t,x) - V(t,x)|^2 + |W^\varepsilon(t,x) - W(t,x)|^2 \right] \, dx = 0.
\]
Moreover, assume that there exists a measure \( F_0 \in \mathcal{M}({\mathbb{R}}^{d+1}) \) satisfying (2.12)-(2.14). Further consider \((F_0^\varepsilon)_{\varepsilon > 0}\) the functions defined for all \( \varepsilon > 0 \), \((x,w) \in {\mathbb{R}}^{d+1}\), with
\[
F_0^\varepsilon(x,w) := \int_{\mathbb{R}} f_0^\varepsilon(x,v,w) \, dv.
\]
Therefore, if \( F_0^\varepsilon \to F_0 \) weakly-* in \( \mathcal{M}({\mathbb{R}}^{d+1}) \), then we have for all \( \varphi \in C_0^1({\mathbb{R}}^{d+2}) \),
\[
\iint \varphi(x,v,w) f^\varepsilon(t,x,v,w) \, dx \, dv \, dw \to \int \varphi(x,V(t,x),w) F(t,x,w) \, dx,
\]
strongly in \( L^1_{loc}(0,T) \) as \( \varepsilon \to 0 \), where \((V,F)\) is the solution of (1.14) provided by Corollary 2.7.

The proof is postponed to Section 4. Our approach is similar to the work done in [7], based on a relative entropy argument. To show that the relative entropy vanishes as \( \varepsilon \) goes to \( 0 \), we encounter some difficulties, coming from the reaction term \( \partial_v(f^\varepsilon N(v)) \), which makes appear some moments of \( f^\varepsilon \) of order higher than \( 2 \) that we need to control, and from the fact that \( V^\varepsilon \) and \( W^\varepsilon \) are not \textit{a priori} differentiable in space, and not uniformly bounded. We circumvent these issues with an \textit{a priori} kinetic dissipation estimate and an entropy estimate, detailed in Section 3.

Remark 2.8. We can further precise that the convergence of the estimate (2.19) is of order \( \varepsilon^{2/(d+6)} \) with further assumptions on the initial conditions. More precisely, we need to replace the assumption (2.17) with
\[
\|p_0^\varepsilon - \rho_0\|_{L^2({\mathbb{R}}^d)} \leq C\varepsilon^{2+1/(d+6)},
\]
and to assume that the convergence of the initial conditions in (2.18) is of rate \( \varepsilon^{2/(d+6)} \), and to suppose some additional regularity of \((\rho_0, V_0, W_0)\) so that
\[
\rho_0 V, \rho_0 W \in L^\infty([0,T], H^4({\mathbb{R}}^d)).
\]
3 A priori estimates for the kinetic equation

In this section, we prove some a priori estimates for the solution of the kinetic equation (1.6) that will be used later in the proof of Theorem 1. We start with an estimate of the moments of a solution of (1.6), which implies an estimate of a kinetic dissipation.

First of all, we define for all \(i \in \mathbb{N}\) and \(u \in \{v, w\}\) the moment of order \(i\) in \(u\) of \(f^\varepsilon\), denoted by \(\mu_i^u\), and the moment of order \(i\) in \(x\) of \(f^\varepsilon\), denoted by \(\mu_i^x\), with

\[
\mu_i^u(t) := \int_{\mathbb{R}^{d+1}} |u|^i f^\varepsilon(t, x, v, w) \, dx \, dv \, dw, \quad \mu_i^x(t) := \int_{\mathbb{R}^{d+1}} \|x\|^i f^\varepsilon(t, x, v, w) \, dx \, dv \, dw.
\]

We also define for all \(\varepsilon > 0\) and for all \(p \geq 1\) the kinetic dissipation

\[
D_p: t \mapsto \frac{1}{2} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+2}} \Psi \left(\frac{\|x - x'\|}{\varepsilon}\right) (v^{2p-1} - v'^{2p-1}) (v - v') f^\varepsilon(t, z') f^\varepsilon(t, z) \, dz' \, dz > 0, \quad (3.1)
\]

using the notation \(z = (x, v, w) \in \mathbb{R}^{d+2}\). In the following, let \(T > 0\), and \(\varepsilon > 0\) and suppose that there exists \(f^\varepsilon\) a well-defined solution of (1.6) on \([0, T]\).

**Proposition 3.1.** Consider \(f^\varepsilon\) a weak solution of the kinetic equation (1.6) on \([0, T]\) provided by Proposition 2.2. Assume that there exists \(p^* \in \mathbb{N}\) such that

\[
\mu_{2p^*}^v(0) + \mu_{2p^*}^w(0) < +\infty. \quad (3.2)
\]

Then, for all \(1 \leq p \leq p^*\), there exists a constant \(C_p\) which depends on \(p\) such that for all \(t \in [0, T]\), we have:

\[
\frac{1}{2p} \frac{d}{dt} \left( \mu_{2p}^v(t) + \mu_{2p}^w(t) \right) + \kappa'_1 \mu_{2(p+1)}^v(t) + \frac{1}{\varepsilon^2} D_p(t) \leq C_p \left( \mu_{2p}^v(t) + \mu_{2p}^w(t) \right), \quad (3.3)
\]

where \(\kappa'_1 > 0\) is the positive constant defined in (1.3).

**Proof.** Let \(t \in [0, T]\). In the rest of this proof, we use the notation \(u = (v, w) \in \mathbb{R}^2\) and \(z = (x, u) \in \mathbb{R}^{d+2}\).

Since \(f^\varepsilon\) is a solution of the kinetic equation (1.6), we have:

\[
\frac{1}{2p} \frac{d}{dt} \left( \mu_{2p}^v(t) + \mu_{2p}^w(t) \right) = I_1 + I_2,
\]

where

\[
I_1 := \int_{\mathbb{R}^{d+2}} v^{2p-1} (N(v) - w) f^\varepsilon(t, z) \, dz + \int_{\mathbb{R}^{d+2}} w^{2p-1} A(v, w) f^\varepsilon(t, z) \, dz,
\]

\[
I_2 := \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^{d+2}} \int_{\mathbb{R}^{d+2}} \Psi \left(\frac{\|x - x'\|}{\varepsilon}\right) v^{2p-1} (v' - v) f^\varepsilon(t, z') f^\varepsilon(t, z) \, dz' \, dz.
\]

First of all, using Young’s inequality and the properties of \(N\) given in (1.3), we treat the first term \(I_1\) as follows:

\[
I_1 \leq \int \left( \kappa_1 v^{2p} - \kappa'_1 v^{2p+2} + \frac{2p-1}{2p} v^{2p} + \frac{1}{2p} w^{2p} \right) f^\varepsilon(t, z) \, dz
\]

\[
+ \tau \int \left( \frac{2p-1}{2p} w^{2p} + \frac{1}{2p} v^{2p} - \gamma w^{2p} \right) f^\varepsilon(t, z) \, dz
\]

\[
= \frac{2p(1 + \kappa_1)}{2p} \mu_{2p}^v(t) + \frac{4\tau p - 2\tau + 1}{2p} \mu_{2p}^w(t) - \kappa'_1 \mu_{2(p+1)}^v(t).
\]
Then, to deal with the second term \( I_2 \), we reformulate it using the symmetry of \( \Psi \). Indeed, we have:

\[
I_2 = \frac{1}{2} \frac{1}{\varepsilon^{d+2}} \int \int \Psi \left( \frac{\|x - x'\|}{\varepsilon} \right) v^{2p-1} (v' - v) f^{\varepsilon}(t, z') f^{\varepsilon}(t, z) \, dz' \, dz \\
+ \frac{1}{2} \frac{1}{\varepsilon^{d+2}} \int \int \Psi \left( \frac{\|x - x'\|}{\varepsilon} \right) v^{2p-1} (v - v') f^{\varepsilon}(t, z') f^{\varepsilon}(t, z) \, dz' \, dz \\
= -\frac{1}{2} \frac{1}{\varepsilon^{d+2}} \int \int \Psi \left( \frac{\|x - x'\|}{\varepsilon} \right) (v^{2p-1} - v^{2p-1}) (v - v') f^{\varepsilon}(t, z') f^{\varepsilon}(t, z) \, dz' \, dz \\
= -\frac{1}{\varepsilon^2} D_p(t).
\]

This enables us to conclude that there exists a constant \( C_p > 0 \) such that for all \( t \in [0, T] \),

\[
\frac{1}{2p} \frac{d}{dt} (\mu_{2p}^v(t) + \mu_{2p}^w(t)) \leq C_p (\mu_{2p}^v(t) + \mu_{2p}^w(t)) - \mu_{2(p+1)}^v(t) - \frac{1}{\varepsilon^2} D_p(t).
\]

**Corollary 3.2.** Under the same assumptions than in Proposition 3.1 with \( p^* = 2 \), if we assume that for all \( \varepsilon > 0 \), (2.15) is satisfied, then there exists a constant \( C_T > 0 \) such that for all \( k \in [0, 4] \), for all \( \varepsilon > 0 \) and for all \( t \in [0, T] \),

\[
\begin{align*}
&\left\{ \mu_k^v(t) + \mu_k^w(t) + \mu_k^x(t) \leq C_T, \\
&\int_0^T \mu_{k+2}(t) \, dt \leq C_T.
\right.
\end{align*}
\] (3.4)

**Proof.** This result is a direct consequence of Proposition 3.1, integrating the inequality (3.3) between 0 and \( t \). \( \Box \)

Now, let us estimate the kinetic dissipation \( D_1 \) as defined in (3.1). This is definitely a crucial step to characterize the limit of the kinetic equation (1.6).

**Corollary 3.3.** Let \( \varepsilon > 0 \). Under the same assumptions as in Proposition 3.1 with \( p^* = 1 \), there exists a constant \( C_T \) such that:

\[
\int_0^T D_1(t) \, dt = \frac{1}{2} \frac{1}{\varepsilon^d} \int_0^T \left\| x - x' \right\| \, dz \, dz' \, \int |v - v'|^2 f^\varepsilon(t, z) f^\varepsilon(t, z') \, dz \, dz' \, dt \leq C_T \varepsilon^2.
\] (3.5)

**Proof.** This kinetic dissipation estimate comes from the inequality (3.3) with \( p = 1 \). Indeed, integrating between 0 and \( T \), we get that

\[
\frac{1}{\varepsilon^2} \int_0^T D_1(t) \, dt \leq C \int_0^T (\mu_2^v(t) + \mu_2^w(t)) \, dt + \mu_2^v(0) + \mu_2^w(0).
\]

We conclude using the moment estimate from Corollary 3.2. \( \Box \)

It turns out that this result is not enough to conclude the proof of Theorem 1 about the diffusive limit. From now, we note for all function \( U \in L^\infty(\mathbb{R}^d) \) and for all \( x \in \mathbb{R}^d \)

\[
\Psi_{\varepsilon \ast x} U(x) := \frac{1}{\varepsilon^d} \int \Psi \left( \frac{\|x - x'\|}{\varepsilon} \right) U(x') \, dx'.
\] (3.6)

Actually, we need to remove the weight \( \Psi_{\varepsilon \ast x} \rho_0^\varepsilon \) in the integrand of the previous estimate.
Corollary 3.4. Let $\varepsilon > 0$. We make the same assumptions than in Proposition 3.1 with $p^* = 2$, and we further assume that there exists a function $\rho_0 \in H^2(\mathbb{R}^d)$ such that for all $\varepsilon > 0$ small enough,

$$\|\rho_0^\varepsilon - \rho_0\|_{L^2(\mathbb{R}^d)} \leq C \varepsilon^2,$$

(3.7)

for some positive constant $C > 0$. Then, there exists a constant $C_T$ such that for all $t \in [0, T]$, we have:

$$\int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t, x, v, w) \left| v - V^\varepsilon(t, x) \right|^2 \, dx \, dv \, dw \leq C_T \varepsilon^{4/(d+6)}.$$  

(3.8)

Proof. Let $\varepsilon > 0$ and $T > 0$. We define the integral

$$I_\varepsilon := \int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t, x, v, w) \left| v - V^\varepsilon(t, x) \right|^2 \, dx \, dv \, dw \, dt.$$  

Using the definition of $V^\varepsilon$, let us notice that

$$I_\varepsilon = \int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t, x, v, w) |v|^2 \, dx \, dv \, dw \, dt - \int_0^T \int_{\mathbb{R}^d} \rho_0^\varepsilon(x) \left| V^\varepsilon(t, x) \right|^2 \, dx \, dt.$$  

In the rest of this proof, we use the notation $z = (x, v, w) \in \mathbb{R}^{d+2}$. Our strategy to estimate $I_\varepsilon$ consists in dividing the set of integration into subsets on which the integrand is easier to control. Let $\eta > 0$ be a constant depending on $\varepsilon$ to be determined later. We define:

$$\mathcal{A}_\eta^\varepsilon := \left\{ x \in \mathbb{R}^d, \; \Psi_x \ast x \rho_0^\varepsilon(x) \geq \eta \right\},$$  

$$\mathcal{B}_\eta^\varepsilon := \left\{ x \in \mathbb{R}^d, \; 0 < \Psi_x \ast x \rho_0^\varepsilon(x) < \eta \right\}.$$  

First of all, since $\Psi > 0$ almost everywhere and $\rho_0^\varepsilon \geq 0$, either we consider the nontrivial case $\rho_0^\varepsilon \equiv 0$, or $\rho_0^\varepsilon \not\equiv 0$ and we get that for all $x \in \mathbb{R}^d$,

$$\Psi_x \ast x \rho_0^\varepsilon(x) = \frac{1}{\varepsilon^d} \int \Psi \left( \frac{|x'|}{\varepsilon} \right) \rho_0^\varepsilon(x - x') \, dx' > 0,$$

and hence $\mathbb{R}^d = \mathcal{A}_\eta^\varepsilon \cup \mathcal{B}_\eta^\varepsilon$. Then, we have that

$$\int_0^T \int_{\mathcal{A}_\eta^\varepsilon} f^\varepsilon |v - V^\varepsilon|^2 \, dz \, dt \leq \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} f^\varepsilon |v - V^\varepsilon|^2 \Psi_x \ast x \rho_0^\varepsilon(x) \, dz \, dt$$

$$= \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} f^\varepsilon |v|^2 \Psi_x \ast x \rho_0^\varepsilon(x) \, dz \, dt - \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \rho_0^\varepsilon |V^\varepsilon|^2 \Psi_x \ast x \rho_0^\varepsilon(x) \, dx \, dt$$

$$= \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} f^\varepsilon |v|^2 \Psi_x \ast x \rho_0^\varepsilon(x) \, dz \, dt - \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \rho_0^\varepsilon V^\varepsilon \Psi_x \ast x \left[ \rho_0^\varepsilon V^\varepsilon \right](x) \, dx \, dt$$

$$+ \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \rho_0^\varepsilon V^\varepsilon \Psi_x \ast x \left[ \rho_0^\varepsilon V^\varepsilon \right](x) \, dz \, dt - \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \rho_0^\varepsilon |V^\varepsilon|^2 \Psi_x \ast x \rho_0^\varepsilon(x) \, dx \, dt$$

$$= \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \int \Psi \left( \frac{|x - x'|}{\varepsilon} \right) |v - v'|^2 f^\varepsilon(t, z) f^\varepsilon(t, z') \, dz' \, dz \, dt$$

$$- \frac{1}{\eta} \int_0^T \int_{\mathcal{A}_\eta^\varepsilon} \int \Psi \left( \frac{|x - x'|}{\varepsilon} \right) |V^\varepsilon(t, x) - V^\varepsilon(t, x')|^2 \rho_0^\varepsilon(x) \rho_0^\varepsilon(x') \, dx' \, dx \, dt$$

$$\leq \frac{1}{\eta} \int_0^T D_1(t) \, dt,$$
where \( D_1 \) is defined in (3.1). Consequently, using Corollary 3.3, we conclude that there exists a positive constant \( C_T > 0 \) such that
\[
\int_0^T \int_{B_R^0 \times \mathbb{R}^2} f^\varepsilon |v - V^\varepsilon|^2 \, dz \, dt \leq C_T \frac{\varepsilon^2}{\eta}.
\] (3.9)

Then, it remains to estimate
\[
\int_0^T \int_{B_R^1 \times \mathbb{R}^2} f^\varepsilon |v - V^\varepsilon|^2 \, dz \, dt \leq \int_0^T \int_{B_R^1 \times \mathbb{R}^2} f^\varepsilon |v|^2 \, dz \, dt = I_1 + I_2 + I_3,
\]
where
\[
\begin{align*}
I_1 & := \int_0^T \int_{B_R^1 \setminus \{|v| > R\}} f^\varepsilon |v|^2 \, dz \, dt, \\
I_2 & := \int_0^T \int_{B_R^1 \cap B^c(0,R)} \int_{\{|v| \leq R\}} f^\varepsilon |v|^2 \, dz \, dt, \\
I_3 & := \int_0^T \int_{B_R^1 \cap B(0,R)} \int_{\{|v| \leq R\}} f^\varepsilon |v|^2 \, dz \, dt,
\end{align*}
\]
where \( R > 0 \) is a constant depending on \( \eta \) and \( \varepsilon \) to be determined later. For \( k > 2 \), we get that
\[
I_1 \leq \frac{1}{R^{k-2}} \int_0^T \int_{B_R^1 \setminus \{|v| > R\}} f^\varepsilon |v|^k \, dz \, dt \leq \frac{1}{R^{k-2}} \int_0^T \mu_k^v(t) \, dt.
\]
Then, for \( q > 2 \), we also have that
\[
I_2 \leq \int_0^T \int_{B_R^1} f^\varepsilon R^2 \frac{|x|^q}{R^q} \, d\mathbf{z} \, dt \leq \frac{1}{R^{q-2}} \int_0^T \mu_q^X(t) \, dt.
\]
As for the last term, we compute:
\[
I_3 \leq \int_0^T \int_{B_R^1 \cap B(0,R)} \rho_0^\varepsilon(x) \, dx \, dt
\]
\[
\leq R^2 \int_0^T \int_{B_R^1 \cap B(0,R)} \Psi \ast \rho_0^\varepsilon(x) \, dx \, dt + R^2 \int_0^T \int_{B_R^1 \cap B(0,R)} |\rho_0^\varepsilon(x) - \Psi \ast \rho_0^\varepsilon(x)| \, dx \, dt
\]
\[
\leq C_T R^{d+2} \eta + R^2 T \left( \int_{B_R^1 \cap B(0,R)} 1 \, dx \right)^{1/2} \| \rho_0^\varepsilon - \Psi \ast \rho_0^\varepsilon \|_{L^2(\mathbb{R}^d)}
\]
\[
\leq C_T R^{d+2} \eta + C^{1/2} T R^{2+d/2} \| \rho_0^\varepsilon - \Psi \ast \rho_0^\varepsilon \|_{L^2(\mathbb{R}^d)}.
\]

Then, using Young’s inequality, we notice that
\[
\| \rho_0^\varepsilon - \Psi \ast \rho_0^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq \| \rho_0^\varepsilon - \rho_0 \|_{L^2(\mathbb{R}^d)} + \| \rho_0 - \Psi \ast \rho_0 \|_{L^2(\mathbb{R}^d)} + \| \Psi \ast (\rho_0^\varepsilon - \rho_0) \|_{L^2(\mathbb{R}^d)}
\]
\[
\leq (1 + \| \Psi \ast \|_{L^1(\mathbb{R}^d)}) \| \rho_0^\varepsilon - \rho_0 \|_{L^2(\mathbb{R}^d)} + \| \rho_0 - \Psi \ast \rho_0 \|_{L^2(\mathbb{R}^d)}
\]
\[
\leq 2 \| \rho_0^\varepsilon - \rho_0 \|_{L^2(\mathbb{R}^d)} + \| \rho_0 - \Psi \ast \rho_0 \|_{L^2(\mathbb{R}^d)}.
\]
On the one hand, we have assumed that the initial data satisfies the estimate (3.7). On the other hand, we can estimate \( \|\rho_0 - \Psi_\varepsilon * \rho_0\|_{L^2(\mathbb{R}^d)} \) with similar arguments as in [1]. Indeed, using the change of variable \( y = (x - x')/\varepsilon \) and a Taylor expansion, we get that

\[
\|\rho_0 - \Psi_\varepsilon * \rho_0\|_{L^2(\mathbb{R}^d)}^2 \leq \int \left| \frac{1}{\varepsilon^d} \int \Psi \left( \frac{x - x'}{\varepsilon} \right) (\rho_0(x') - \rho_0(x)) \, dx' \right|^2 \, dx \\
\leq \int \left| \int \Psi (y) (\rho_0(x - \varepsilon y) - \rho_0(x)) \, dy \right|^2 \, dx \\
\leq \varepsilon^4 \int \left| \int \Psi (y) \int_0^1 (1 - s) y^T \cdot \nabla_x^2 \rho_0(x - \varepsilon s y) \cdot y \, ds \, dy \right|^2 \, dx.
\]

Furthermore, using Cauchy-Schwarz inequality for the integral in \( s \) and then for the integral in \( y \), we get

\[
\|\rho_0 - \Psi_\varepsilon * \rho_0\|_{L^2(\mathbb{R}^d)}^2 \leq \varepsilon^4 \int \left| \int \Psi (y) \left( \int_0^1 |1 - s|/\|y\|^2 \, ds \right)^{1/2} \left( \int_0^1 |1 - s|/\|y\|^2 \|\nabla_x^2 \rho_0(x - \varepsilon s y)\|^2 \, ds \right)^{1/2} \, dy \right|^2 \, dx \\
\leq \varepsilon^4 \int \left( \int \Psi (\|y\|) \frac{\|y\|^2}{2} \, dy \right) \left( \int \Psi (\|y\|) \int_0^1 |1 - s|/\|y\|^2 \|\nabla_x^2 \rho_0(x - \varepsilon s y)\|^2 \, ds \, dy \right) \, dx.
\]

Consequently, \( \|\rho_0 - \Psi_\varepsilon * \rho_0\|_{L^2(\mathbb{R}^d)}^2 \leq \sigma \varepsilon^4 \int \Psi (\|y\|) \int_0^1 |1 - s|/\|y\|^2 \int \|\nabla_x^2 \rho_0(x - \varepsilon s y)\|^2 \, dx \, ds \, dy \),

\[
\leq \sigma^2 \|\rho_0\|_{H^2(\mathbb{R}^d)}^2 \varepsilon^4.
\]

Finally, we get that there exists a positive constant \( C > 0 \) such that

\[
I_3 \leq CT \left( R^{d+2} + R^{(d+4)/2} \varepsilon^2 \right).
\]

Finally, using the moment estimates, and the estimate (3.9), we get that there exists a positive constant \( C_T \) such that

\[
I_\varepsilon \leq C_T \left( \frac{\varepsilon^2}{\eta} + R^{d+2} \eta + R^{(d+4)/2} \varepsilon^2 + \frac{1}{R^{k-2}} + \frac{1}{R^{q-2}} \right).
\]

It remains to optimize the values of \( \eta \) and \( R \). For the sake of simplicity, we choose \( k = q = 4 \). We consider \( R = \eta^{-1/(d+4)} \), so that

\[
R^{d+2} \eta = \frac{1}{R^2}.
\]

Then, we take \( \eta = \varepsilon^{2(d+4)/(d+6)} \), so that

\[
R^{d+2} \eta = \frac{\varepsilon^2}{\eta}.
\]

This leads to

\[
I_\varepsilon \leq C_T \left( \varepsilon^{4/(d+6)} + \varepsilon^{(d+8)/(d+6)} \right) \leq \tilde{C}_t \varepsilon^{4/(d+6)},
\]

for a positive constant \( \tilde{C}_T > 0 \) and \( \varepsilon > 0 \) small enough. \( \blacksquare \)
4 Proof of Theorem 1

Our proof of Theorem 1 relies on a relative entropy argument, as developed in the works by Dafermos [8] and Di Perna [10] for conservation laws, and more recently used in [17] for the hydrodynamic limit of Vlasov-type equations. This leads to estimate the distance between the macroscopic functions derived from the solution of the kinetic equation (1.6), and the solution of the limit system (1.14). First, we introduce the notion of relative entropy we use in this article. Then, we write the variation of relative entropy in a more practical form, useful to get an estimate of relative entropy. Finally, we explain how this argument enables us to prove Theorem 1.

In the rest of this article, for any given \( \varepsilon > 0 \) and for \( \rho : \mathbb{R}^d \to \mathbb{R} \) and \( V : (0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) regular enough, we define the following local and nonlocal differential operators:

\[
\begin{align*}
L_\rho(V) &:= \sigma [\Delta_x (\rho V) - \Delta_x \rho V] = \sigma [\rho \Delta_x V + 2 \nabla_x \rho \cdot \nabla_x V], \\
\mathcal{L}_\rho(V) &= \Psi \ast_x [\rho V] - [\Psi \ast_x \rho] \; V = \frac{1}{\varepsilon^{d+2}} \iint \psi \left( \frac{\|x - x'\|}{\varepsilon} \right) (V(t,x') - V(t,x)) \; \rho(x') \, dx',
\end{align*}
\]

respectively defined on \( H^2(\mathbb{R}^d) \) and \( L^\infty(\mathbb{R}^d) \).

4.1 Definition of relative entropy

Here, motivated by the form of the nonlocal interaction term and by the absence of Laplace operator in the kinetic equation (1.6), we use the same entropy as in [7, 13, 17], given in the following definition.

Definition 4.1. For all functions \( V \) and \( W : \mathbb{R}^d \to \mathbb{R} \), and for any non-negative function \( \rho : \mathbb{R}^d \to \mathbb{R} \), we define for \( Z = (\rho, \rho V, \rho W) \) the entropy \( \eta(Z) \) by

\[
\eta(Z) := \frac{\rho |V|^2 + |W|^2}{2}.
\]

From this notion of entropy, we define the relative entropy as follows.

Definition 4.2. For all functions \( V_1, W_1, V_2 \) and \( W_2 : \mathbb{R}^d \to \mathbb{R} \), and for any non-negative function \( \rho_1 \) and \( \rho_2 : \mathbb{R}^d \to \mathbb{R} \), we define for \( Z_i = (\rho_i, \rho_i V_i, \rho_i W_i) \), \( i \in \{1, 2\} \) the entropy \( \eta(Z_1|Z_2) \) by

\[
\eta(Z_1|Z_2) := \eta(Z_1) - \eta(Z_2) - D\eta(Z_2) \cdot (Z_1 - Z_2),
\]

where

\[
D\eta(Z_2) = \begin{pmatrix}
\frac{|V_2|^2 + |W_2|^2}{2} \\
V_2 \\
W_2
\end{pmatrix},
\]

which yields after computation,

\[
\eta(Z_1|Z_2) := \rho_1 \frac{|V_2 - V_1|^2 + |W_2 - W_1|^2}{2}.
\]
4.2 Relative entropy estimate

This subsection is devoted to the proof of the relative entropy estimate (2.19) under the same assumptions as in Theorem 1. The classical algorithm, as explained in previous works as [7, 13, 17], consists in splitting the relative entropy dissipation into one part due to the solution of the limiting system (2.5), and another part which estimates the difference between the solution to (2.5) and the macroscopic functions computed from the solution to the kinetic equation (1.6). Here, we cannot use exactly the same argument, since for all \( \varepsilon > 0 \), the macroscopic function \( V^\varepsilon \) does not have enough regularity in space, so we cannot apply the local operator \( \mathcal{L}_{\rho_0} \) on it. Thus, we proceed with a direct computation of the variation of relative entropy.

For all \( \varepsilon > 0 \), let \( f^\varepsilon \) be the solution of the kinetic equation (1.6). According to Corollary 3.2, we know that for all \( t \in [0, T] \), the moment of order 4 of \( f^\varepsilon(t) \) is uniformly bounded with respect to \( \varepsilon > 0 \). Therefore, using Hölder’s inequality, we obtain that for all \( x \in \mathbb{R}^d \) such that \( \rho_0(x) > 0 \) and for all \( t \in [0, T] \),

\[
\rho_0^\varepsilon(x) |V^\varepsilon(t, x)|^4 = \frac{1}{\rho_0^\varepsilon(x)^3} \left( \int v f^\varepsilon(t, x, v, w) \, dv \, dw \right)^4 \leq \int |v|^4 f^\varepsilon(t, x, v, w) \, dv \, dw. \tag{4.4}
\]

This last inequality (4.4) remains true where \( \rho_0^\varepsilon(x) = 0 \) and with \( W^\varepsilon \) instead of \( V^\varepsilon \). Consequently, since \( \|\rho_0^\varepsilon\|_{L^1(\mathbb{R}^d)} = 1 \), we get that for any \( 0 \leq p \leq 4 \) and for all \( t \in [0, T] \),

\[
\rho_0^\varepsilon(|V^\varepsilon(t)|^p + |W^\varepsilon(t)|^p) \in L^1(\mathbb{R}^d).
\]

Let us note for all \( \varepsilon > 0 \) the macroscopic functions \( Z^\varepsilon := (\rho_0^\varepsilon, \rho_0^\varepsilon V^\varepsilon, \rho_0^\varepsilon W^\varepsilon) \) computed from the solution \( f^\varepsilon \) of the kinetic equation. Moreover, let \((V, W)\) be the weak solution of the reaction-diffusion system (2.5) provided by Proposition 2.5. Consequently, we note the triplet \( Z := (\rho_0, \rho_0 V, \rho_0 W) \), solution of the reaction-diffusion equation (1.15). Since \( V \) and \( W \) are in \( W^{1,\infty}([0, T], L^2(\mathbb{R}^d)) \) by definition, for all \( t \in [0, T] \), we can compute:

\[
\int \eta(Z^\varepsilon | Z)(t, x) \, dx = \int \eta(Z^\varepsilon | Z)(0, x) \, dx \\
+ \int_0^t \left[ \int (V^\varepsilon - V) (\partial_t (\rho_0^\varepsilon V^\varepsilon) - \rho_0^\varepsilon \partial_t V) \, dx + \int (W^\varepsilon - W) (\partial_t (\rho_0^\varepsilon W^\varepsilon) - \rho_0^\varepsilon \partial_t W) \, dx \right](s) \, ds \\
= \int \eta(Z^\varepsilon | Z)(0, x) \, dx + \int_0^t [T_1(s) + T_2(s) + T_3(s)] \, ds,
\]

where for all \( s \in [0, T] \), we define

\[
\begin{align*}
T_1(s) &:= \int \rho_0^\varepsilon (W^\varepsilon - W) (A(V^\varepsilon, W^\varepsilon) - A(V, W)) \, dx - \int \rho_0^\varepsilon (V^\varepsilon - V) (W^\varepsilon - W) \, dx, \\
T_2(s) &:= \int_{\mathbb{R}^d} (V^\varepsilon - V) \int_{\mathbb{R}^2} (N(v) - N(V)) \, f^\varepsilon(s, x, u) \, du \, dx, \\
T_3(s) &:= \int \rho_0^\varepsilon (V^\varepsilon - V) (\mathcal{L}_{\rho_0^\varepsilon} V^\varepsilon - \mathcal{L}_{\rho_0} V) \, dx,
\end{align*}
\]

which respectively stand for the difference between the linear reaction terms, the nonlinear reaction terms, and the diffusion terms.
Estimate of the linear reaction terms. First of all, we can directly treat the first term $T_1$ with Young’s inequality, which yields that,

\[
\int_0^T T_1(t) \, dt \leqslant (1 + \tau) \int_0^T \eta(|Z| \, |Z|) \, dx \, dt. \tag{4.5}
\]

Estimate of the nonlinear reaction terms. Then, we deal with the second term $T_2$ using the assumptions (1.3) satisfied by $N$, as follows for all $t \in [0, T]$:

\[
T_2(t) = \int_{\mathbb{R}^d} (V^\varepsilon - V) \int_{\mathbb{R}^d} (N(v) - N(V^\varepsilon)) f^\varepsilon(t, x, u) \, du \, dx + \int_{\mathbb{R}^d} (V^\varepsilon - V) (N(V^\varepsilon) - N(V)) \rho_0(x) \, dx
\]
\[
\leq \kappa_3 \int_{\mathbb{R}^d+2} |V^\varepsilon - V| |V^\varepsilon - v| \left[ 1 + v^2 + (V^\varepsilon)^2 \right] f^\varepsilon(t, x, u) \, dx \, du + 2 \kappa_2 \int_{\mathbb{R}^d} \eta(|Z| \, |Z|) \, dx,
\]

where the constants $\kappa_2$ and $\kappa_3$ are given in (1.3). Then, in order to estimate $T_2$ using the kinetic dissipation estimate from Corollary 3.4, Cauchy-Schwarz inequality yields that

\[
T_2(t) \leq \alpha(t) \left( \int |V^\varepsilon(t) - v|^2 f^\varepsilon(t, x, u) \, dx \, du \right)^{1/2} + 2 \kappa_2 \int \eta(|Z| \, |Z|) \, dx,
\]

where

\[
\alpha(t) := \kappa_3 \left( \int [1 + (V^\varepsilon(t))^2 + v^2]^2 (|V^\varepsilon(t) - V(t)|^2 f^\varepsilon(t, x, u) \, dx \, du \right)^{1/2}.
\]

We recall that $V \in L^\infty([0, T], H^2(\mathbb{R}^d))$, and $H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ since $d \leq 3$. Hence, using the moment estimate from Corollary 3.2, and the fact that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$,

\[
\rho_0(x) |V^\varepsilon(t, x)|^6 \leq \int |v|^6 f^\varepsilon(t, x, u) \, dx \, du,
\]

we can conclude that there exists a positive constant $C_T > 0$ such that

\[
\int_0^T \alpha(t)^2 \, dt \leq C_T.
\]

Consequently, according to the estimate from Corollary 3.4,

\[
\int_0^T T_2(t) \, dt \leq C_T \varepsilon^{2/(d+6)} + 2 \kappa_2 \int \eta(|Z| \, |Z|)(t) \, dx \, dt. \tag{4.6}
\]

Estimate of the diffusion terms. Finally, it remains to estimate the third term $T_3$, involving the difference between the nonlocal diffusion term $\mathcal{L}_{\rho_0}^\varepsilon(V^\varepsilon)$ and the local diffusion term $\mathcal{L}_{\rho}(V)$. On the one hand, since $V^\varepsilon$ is not regular enough in space, we cannot apply the operator $\mathcal{L}_{\rho_0}$ to it. On the other hand, we can apply the nonlocal operator $\mathcal{L}_{\rho_0}^\varepsilon$ to both $V^\varepsilon$ and $V$. This leads to rewrite $T_3$ as follows:

\[
T_3 = T_{3,1} + T_{3,2},
\]

where for all $t \in [0, T]$

\[
\begin{aligned}
T_{3,1}(t) &:= \int \rho_0^\varepsilon (V^\varepsilon - V) \left( \mathcal{L}_{\rho_0}^\varepsilon(V^\varepsilon) - \mathcal{L}_{\rho_0}(V) \right) \, dx, \\
T_{3,2}(t) &:= \int \rho_0^\varepsilon (V^\varepsilon - V) \left( \mathcal{L}_{\rho_0}(V) - \mathcal{L}_{\rho_0}(V) \right) \, dx.
\end{aligned}
\]
To estimate the first term $\mathcal{T}_{3,1}$, using the shorthand notations $V := V(t, x)$, $V' := V(t, x')$, and the same for $V^\varepsilon$ and $V^\varepsilon'$, we compute:

$$
\mathcal{T}_{3,1}(t) = \frac{1}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) \rho_0^\varepsilon(x) \left( V^\varepsilon - V \right) \left[ \rho_0^\varepsilon(x') \left( V'^\varepsilon - V' \right) - \rho_0(x') \left( V' - V \right) \right] \, dx \, dx'
$$

$$
= \frac{1}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) \left( V^\varepsilon - V \right) \rho_0^\varepsilon(x) \left[ (V'^\varepsilon - V') \rho_0^\varepsilon(x') - (V'^\varepsilon - V') \rho_0(x') \right] \, dx \, dx'
$$

$$
+ \frac{1}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) \left( V^\varepsilon - V \right) \rho_0^\varepsilon(x) \left( \rho_0^\varepsilon(x') - \rho_0(x') \right) (V' - V) \, dx \, dx'
$$

$$
\leq \frac{1}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) \rho_0^\varepsilon(x) \left( |V^\varepsilon - V| \cdot | \rho_0^\varepsilon - \rho_0 | \cdot | (x') | \right) \, dx \, dx'
$$

$$
\leq \frac{2 \| V \|_{L^\infty}}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) \rho_0^\varepsilon(x) \left( |V^\varepsilon - V| \cdot | \rho_0^\varepsilon - \rho_0 | \cdot | (x') | \right) \, dx \, dx',
$$

and then, using Young’s inequality, we have

$$
\mathcal{T}_{3,1}(t) \leq \| V \|_{L^\infty} \int \rho_0^\varepsilon \left( |V^\varepsilon - V|^2 \right) \, dx + \| V \|_{L^\infty} \int \left[ \frac{1}{\varepsilon^{d+2}} \int \Psi \left( \frac{\| x - x' \|}{\varepsilon} \right) | \rho_0^\varepsilon - \rho_0 | (x') \, dx \right]^2 \rho_0(x) \, dx
$$

$$
\leq 2 \| V \|_{L^\infty} \int \eta(\mathcal{Z} \mid \mathcal{Z}) \, dx + \frac{1}{\varepsilon^d} \| V \|_{L^\infty} \| \rho_0^\varepsilon \|_{L^\infty} \| \eta \|_{L^1} \| \rho_0 - \rho_0^\varepsilon \|_{L^2}^2.
$$

This leads to the estimate

$$
\mathcal{T}_{3,1}(t) \leq C_T \left( \frac{1}{\varepsilon^d} \| \rho_0^\varepsilon - \rho_0 \|_{L^2(\mathbb{R}^d)}^2 + \int \eta(\mathcal{Z} \mid \mathcal{Z}) \, dx \right), \tag{4.7}
$$

where $C_T > 0$ is a positive constant independent of $\varepsilon$. It remains to control the final term $\mathcal{T}_{3,2}$. We start by separating the diffusions on $\rho_0 V$ and on $\rho_0$ alone, as follows:

$$
\mathcal{T}_{3,2} = \mathcal{T}_{3,2,1} + \mathcal{T}_{3,2,2},
$$

where for all $t \in [0, T]$

$$
\begin{align*}
\mathcal{T}_{3,2,1}(t) &:= \int \rho_0^\varepsilon \left( V^\varepsilon - V \right) \left[ \frac{1}{\varepsilon^2} (\Psi^\varepsilon \ast_x [\rho_0 V](t, x) - \rho_0 V(t, x)) - \sigma \Delta_x(\rho_0 V)(t, x) \right] \, dx, \\
\mathcal{T}_{3,2,2}(t) &:= - \int \rho_0^\varepsilon \left( V^\varepsilon - V \right) V(t, x) \left[ \frac{1}{\varepsilon^2} (\Psi^\varepsilon \ast_x \rho_0(x) - \rho_0(x)) - \sigma \Delta_x \rho_0(x) \right] \, dx.
\end{align*}
$$

Our strategy to estimate both $\mathcal{T}_{3,2,1}$ and $\mathcal{T}_{3,2,2}$ follows the idea from [1] with a Taylor expansion. Using Young’s inequality, we get that

$$
\begin{align*}
\mathcal{T}_{3,2,1}(t) &\leq \eta(\mathcal{Z} \mid \mathcal{Z}) + \frac{1}{2} \| \rho_0^\varepsilon \|_{L^\infty} \int \left[ \frac{1}{\varepsilon^2} (\Psi^\varepsilon \ast_x [\rho_0 V](t, x) - \rho_0 V(t, x)) - \sigma \Delta_x(\rho_0 V)(t, x) \right]^2 \, dx, \\
\mathcal{T}_{3,2,2}(t) &\leq \eta(\mathcal{Z} \mid \mathcal{Z}) + \frac{1}{2} \| \rho_0^\varepsilon \|_{L^\infty} \| V \|_{L^\infty}^2 \int \left[ \frac{1}{\varepsilon^2} (\Psi^\varepsilon \ast_x \rho_0(x) - \rho_0(x)) - \sigma \Delta_x \rho_0(x) \right]^2 \, dx.
\end{align*}
$$
Then, for all $x \in \mathbb{R}^d$, we apply in the convolution products as defined in (3.6) the change of variable $y = (x - x')/\varepsilon$, so that using a Taylor expansion, we get:

$$\left| \frac{1}{\varepsilon^2} (\Psi *_{x} [\rho_0 V](t, x) - \rho_0 V(t, x)) - \sigma \Delta_x(\rho_0 V)(t, x) \right|^2$$

$$= \left| \frac{1}{\varepsilon^2} \int \Psi(\|y\|) (\rho_0 V)(t, x - \varepsilon y) \, dy - \frac{1}{\varepsilon^2} \rho_0 V(t, x) - \sigma \Delta_x(\rho_0 V)(t, x) \right|^2$$

$$= \left| \int \Psi(\|y\|) \int_0^1 (1 - s) y^T \cdot (\nabla_x^2(\rho_0 V)(t, x - \varepsilon s y) - \nabla_x^2(\rho_0 V)(t, x)) \cdot y \, ds \, dy \right|^2.$$ 

Besides, we consecutively use the Cauchy-Schwarz inequality in the integrals in $s$ and then in $y$, which gives:

$$\left| \frac{1}{\varepsilon^2} (\Psi *_{x} [\rho_0 V](t, x) - \rho_0 V(t, x)) - \sigma \Delta_x(\rho_0 V)(t, x) \right|^2$$

$$\leq \left| \int \Psi(\|y\|) \left( \int_0^1 |1 - s| \|y\|^2 \, ds \right)^{1/2} \left( \int_0^1 |1 - s| \|y\|^2 \|\nabla_x^2(\rho_0 V)(t, x - \varepsilon s y) - \nabla_x^2(\rho_0 V)(t, x)\|^2 \, ds \right)^{1/2} \, dy \right|^2$$

$$\leq \sigma \int \Psi(\|y\|) \left( \int_0^1 |1 - s| \|y\|^2 \|\nabla_x^2(\rho_0 V)(t, x - \varepsilon s y) - \nabla_x^2(\rho_0 V)(t, x)\|^2 \, ds \right) \, dy \, dx \right. \leq 2\sigma^2 \|\rho_0 V\|^2_{L^\infty([0,T],H^2(\mathbb{R}^d))}.$$ 

Then, using these two last inequalities and the Lebesgue’s Dominated Convergence Theorem, and the fact that $\|\rho_0\|_{L^\infty}$ is uniformly bounded, we get that as $\varepsilon$ goes to 0, for all $t \in [0,T]$

$$T_{3.2.1}(t) \leq \int \eta(Z^\varepsilon | Z)(t, x) \, dx + \alpha_{\varepsilon \rightarrow 0}(1), \quad (4.8)$$

where $\alpha_{\varepsilon \rightarrow 0}(1)$ denotes a function which converges towards 0 as $\varepsilon$ goes to 0, uniformly in $t$. Using similar arguments, and the fact that $\rho_0 \in H^2(\mathbb{R}^d)$ and $V \in L^\infty([0,T],L^\infty(\mathbb{R}^d))$, we also get that

$$T_{3.2.2}(t) \leq \int \eta(Z^\varepsilon | Z)(t, x) \, dx + \alpha_{\varepsilon \rightarrow 0}(1). \quad (4.9)$$

**Relative entropy estimate.** Finally, putting together the estimates (4.5)–(4.9), we get that there exists a positive constant $C_T > 0$ such that for all $t \in [0,T]$

$$\int \eta(Z^\varepsilon | Z)(t, x) \, dx \leq \int \eta(Z^\varepsilon | Z)(0, x) \, dx$$

$$+ C_T \left( \frac{1}{\varepsilon^4} \|\rho_0 - \rho_0\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{2/(d+6)} + \alpha_{\varepsilon \rightarrow 0}(1) \right) + \int_0^t \int \eta(Z^\varepsilon | Z)(s, x) \, dx \, ds.$$

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According to the assumptions (2.17) and (2.18) satisfied by the initial conditions, we get
\[
\int \eta(Z^\varepsilon | Z)(t, x) \, dx \leq o_{\varepsilon \to 0}(1) + C_T \int_0^t \int \eta(Z^\varepsilon | Z)(s, x) \, dx \, ds.
\]
Therefore, Grönwall’s inequality yields that
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} \int \eta(Z^\varepsilon | Z)(t) \, dx = 0. \tag{4.10}
\]

### 4.3 Conclusion

Finally, let us conclude the proof of Theorem 1 using the relative entropy estimate (4.10) established in the previous subsection. Let $T > 0$. We want to prove that the weak solution of the kinetic equation (1.6) converges towards a monokinetic distribution as $\varepsilon$ vanishes. First, we set
\[
F^\varepsilon(t, x, w) := \int f^\varepsilon(t, x, v, w) \, dv, \quad F^\varepsilon(0, x, v) = F_0(x, w) := \int f_0^\varepsilon(x, v, w) \, dv.
\]
Let us notice that since $f^\varepsilon$ is compactly supported in $v$ for any $\varepsilon > 0$, we can choose a test function in (2.2) independent of $v \in \mathbb{R}$, so that the distribution $F^\varepsilon$ satisfies the following equation for all $\varphi \in C^\infty_c((0, T) \times \mathbb{R}^{d+1})$:
\[
\int_0^T \int_{\mathbb{R}^{d+1}} \left( F^\varepsilon \partial_t \varphi + \tau \left( \int_{\mathbb{R}} v f^\varepsilon \, dv - \gamma \, w \, F^\varepsilon \right) \partial_w \varphi \right) \, dx \, dw \, dt + \int_{\mathbb{R}^{d+1}} F_0^\varepsilon \varphi(0) \, dx \, dw = 0,
\]
which is equivalent to satisfying for all $\varphi \in C^1_c((0, T) \times \mathbb{R}^{d+1})$ the equation
\[
\int_0^T \int_{\mathbb{R}^{d+1}} F^\varepsilon \left[ \partial_t \varphi + A(V(t, x), w) \, \partial_w \varphi \right] \, dx \, dw \, dt + \int_{\mathbb{R}^{d+1}} F_0^\varepsilon \varphi(0) \, dx \, dw
\]
\[
= \tau \int_0^T \int_{\mathbb{R}^{d+2}} (V(t, x) - v) f^\varepsilon \partial_w \varphi \, dv \, dx \, dt, \tag{4.11}
\]
where $V$ is solution to the second equation in (2.9). On the one hand, since $(F^\varepsilon)_{\varepsilon > 0}$ in uniformly bounded by 1 in $L^\infty([0, T), L^1(\mathbb{R}^{d+1}))$, we get that it converges weakly-* in $\mathcal{M}((0, T) \times \mathbb{R}^{d+1})$ towards a limit $F \in \mathcal{M}((0, T) \times \mathbb{R}^{d+1})$. Thus, we can pass to the limit on the left hand side of (4.11) by linearity. On the other hand, from the kinetic dissipation estimate in Corollary 3.4 and the relative entropy estimate (4.10), we get that
\[
\int_0^T \int f^\varepsilon |v - V(t, x)|^2 \, dx \, dv \, dw \, dt \quad \leq \quad 2 \int_0^T \int f^\varepsilon (|v - V^\varepsilon(t, x)|^2 + |V^\varepsilon(t, x) - V(t, x)|^2) \, dx \, dv \, dw \, dt
\]
\[
\to \quad 0, \quad \text{as } \varepsilon \to 0.
\]
(4.12)
Consequently, since $\|\rho_0^0\|_{L^1} = 1$, it yields with Cauchy-Schwarz inequality that:
\[
\left| \int_0^T \int_{\mathbb{R}^{d+2}} (V(t, x) - v) f^\varepsilon \partial_w \varphi \, dv \, dx \, dt \right| \leq T^{1/2} \|\partial_w \varphi\|_{\infty} \int_0^T \int |V(t, x) - v|^2 f^\varepsilon \, dv \, dx \, dt \to 0,
\]
as $\varepsilon \to 0$. Therefore, passing to the limit $\varepsilon \to 0$ in (4.11), it proves that $(V, F)$ is a solution of the system (1.14). Furthermore, by uniqueness of the solution of (1.14), we get the convergence of the sequence $(F^\varepsilon)_{\varepsilon > 0}$.
Now, let us prove that for any \( \varphi \in C_c^0(\mathbb{R}^{d+2}) \),
\[
\int \varphi(x, v, w) f^\varepsilon(t, x, v, w) \, dx \, dv \, dw \longrightarrow \int \varphi(x, V(t, x), w) F(t, dx, dw),
\]
strongly in \( L^1_{\text{loc}}(0, T) \) as \( \varepsilon \to 0 \). We start with proving that for all \( 0 < t < t' \leq T \), for all \( \varphi \in C_c^1(\mathbb{R}^{d+2}) \),
\[
\int_t^{t'} \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) \varphi(x, v, w) \, dv \, dw \, dx \, ds \longrightarrow \int_t^{t'} \int \varphi(x, V(s, x), w) F(s, dx, dw) \, ds,
\]
(4.13)
where \((V, F)\) is the solution on \([0, T]\) of the reaction-diffusion system (1.14) provided by Proposition 2.5, and we conclude using a density argument. Let \( 0 < t < t' \leq T \). We can compute:
\[
\mathcal{I} := \left| \int_t^{t'} \left( \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) \varphi(x, v, w) \, dv \, dw \, dx \right) \, ds \right| \leq \mathcal{I}_1 + \mathcal{I}_2,
\]
where
\[
\begin{align*}
\mathcal{I}_1 &:= \int_t^{t'} \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) |\varphi(x, v, w) - \varphi(x, V(s, x), w)| \, dv \, dw \, dx \, ds, \\
\mathcal{I}_2 &:= \int_t^{t'} \int |\varphi(x, V(s, x), w)| |F^\varepsilon(s, dx, dw) - F(s, dx, dw)| \, ds.
\end{align*}
\]
On the one hand, using Cauchy-Schwarz inequality, we have
\[
\mathcal{I}_1 \leq \|\partial_v \varphi\|_\infty \int_t^{t'} \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) |v - V(s, x)| \, dv \, dw \, dx \, ds
\]
\[
\leq \|\partial_v \varphi\|_\infty \left( \int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) \, dv \, dw \, dx \, ds \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^{d+2}} |v - V(s, x)|^2 \, dv \, dw \, dx \, ds \right)^{1/2}
\]
\[
= \|\partial_v \varphi\|_\infty T^{1/2} \left( \int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(s, x, v, w) |v - V(s, x)|^2 \, dv \, dw \, dx \, ds \right)^{1/2}.
\]
Consequently, using the convergence from (4.12), we get
\[
\lim_{\varepsilon \to 0} \mathcal{I}_1 = 0.
\]
(4.14)
On the other hand, the second term \( \mathcal{I}_2 \) converges to zero as \( \varepsilon \) goes to zero since \((F^\varepsilon)_{\varepsilon \geq 0}\) converges weakly-* towards \( F \) in \( M((0, T) \times \mathbb{R}^{d+1}) \). Consequently, we can conclude that
\[
\lim_{\varepsilon \to 0} \mathcal{I} = 0.
\]
(4.15)
Using a density argument, this shows the convergence of \( f^\varepsilon \) in \( L^1_{\text{loc}}((0, T), M(\mathbb{R}^{d+2})) \) towards a monokinetic distribution, which concludes the proof of Theorem 1.

5 Proof of Proposition 2.5

This subsection is devoted to the proofs of Proposition 2.5 and its Corollary 2.7, that is to the construction of a solution to the system (2.9). The main difficulty lies in the fact that the function \( \rho_0 \) can reach 0, so
the second equation in (2.9) is not well-defined on $\mathbb{R}^d$. A solution to overcome this problem is to construct a solution to (2.9) from a weak solution of the reaction-diffusion FitzHugh-Nagumo system (2.5) in the sense of Definition 2.4.

The difficulty to prove the existence and uniqueness of a weak solution to the reaction-diffusion system (2.5) is that it is not a parabolic system. A way to circumvent this issue is to consider for all $\delta \geq 0$ the linear operator:

$$L_{\rho_0+\delta} : V \mapsto \sigma[(\rho_0 + \delta) \Delta x V + 2 \nabla x \rho_0 \cdot \nabla x V],$$

(5.1)

in order to study a parabolic approximation of the system.

Our approach to prove the existence and uniqueness of the solution to the reaction-diffusion system (2.5) follows several steps. First of all, for any given initial data $V_0 \in L^2(\mathbb{R}^d)$ and for all $\delta > 0$, we prove the existence and uniqueness of $V_\delta$ a weak solution of the approximated parabolic system

$$\partial_t V_\delta + L_{\rho_0+\delta}(V_\delta) = N(V_\delta) - W[V_\delta],$$

(5.2)

where for all $V : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, we define

$$W[V] : (t, x) \mapsto e^{-\tau \gamma t}W_0(x) + \tau \int_0^t e^{-\tau \gamma (t-s)}V(s, x) ds,$$

(5.3)

where $W_0 : \mathbb{R}^d \to \mathbb{R}$ is a given initial data in $H^2(\mathbb{R}^d)$. Let us notice that if $V$ is a weak solution to the equation (5.2) with $\delta = 0$, then the couple $(V, W[V])$ is a weak solution to the reaction-diffusion system (2.5) in the sense of Definition 2.4.

Then, adding some regularity to the initial data, we get some estimates of $V_\delta$ which are uniform in $\delta$. Thus, we will be able to pass to the limit $\delta \to 0$. Finally, we will conclude this section with the construction of a solution to the system (2.9). But before giving the details of the proof, we begin with some a priori estimates.

In the rest of this article, for all $k \in \{0, 1, 2\}$, we note $\langle \cdot, \cdot \rangle_{H^k(\mathbb{R}^d)}$ the scalar product of of $H^k(\mathbb{R}^d)$ defined as follows:

$$\langle U, V \rangle_{H^k(\mathbb{R}^d)} := \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} \int \partial^\alpha U \partial^\alpha V \, dx,$$

for all $U, V \in H^k(\mathbb{R}^d)$, where for all $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, $\partial^\alpha = \partial^\alpha_{x_1} ... \partial^\alpha_{x_d}$.

5.1 A priori estimates

In this subsection, we prove some a priori estimates of the solution of the reaction-diffusion equation (5.2) with $\delta \geq 0$. We start with estimating the $H^2$ norm of a weak solution (5.2) with a general reaction term. Then, we will get an estimate of the $H^2$ norm of a weak solution to (5.2) uniform in $\delta$, which will be crucial to pass to the limit $\delta \to 0$.

Lemma 5.1. Let $T > 0$. Consider an initial data $\rho_0$ satisfying (2.7) and $V_0 \in H^2(\mathbb{R}^d)$ satisfying (2.8). Let $\delta \geq 0$. Assume that there exists

$$V_\delta \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap C^0([0, T], L^2(\mathbb{R}^d))$$
a weak solution of the reaction-diffusion equation for $t > 0$ and $x \in \mathbb{R}^d$:

\[
\partial_t V_\delta - \mathcal{L}_{\rho_0 + \delta} V_\delta = S,
\]

where $S \in L^\infty([0, T], H^2(\mathbb{R}^d))$ are two general source terms. Then, there exists a positive constant $C > 0$ independent of $\delta$ such that for all $t \in [0, T]$

\[
\|V_\delta(t)\|_{H^2(\mathbb{R}^d)}^2 + \delta \int_0^t \|V_\delta(s)\|_{H^1(\mathbb{R}^d)}^2 ds \lesssim \|V_0\|_{H^2(\mathbb{R}^d)}^2 + C \int_0^t \left( \|V_\delta(s)\|_{H^2(\mathbb{R}^d)}^2 + \langle S(s), V_\delta(s) \rangle_{H^2(\mathbb{R}^d)} \right) ds.
\]

(5.5)

**Proof.** We postpone the proof to Appendix A. \hfill \blacksquare

Then, let us deduce from the estimate (5.5) an upper bound of the $H^2$ norm of a weak solution to the FitzHugh-Nagumo reaction-diffusion system (5.2) uniform in $\delta$, which will be crucial to get global existence of the solution, and to pass to the limit $\delta \to 0$.

**Corollary 5.2.** Let $\delta \geq 0$. Consider an initial data $\rho_0$ satisfying (2.7) and $(V_0, W_0)$ satisfying (2.8). Let $T > 0$ such that there exists

\[
V_\delta \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap \mathcal{C}^0([0, T], L^2(\mathbb{R}^d))
\]

a weak solution of the reaction-diffusion system (5.2). Then, there exists a finite constant $C_T > 0$ independent of $\delta$ such that for all $t \in [0, T],$

\[
\|V_\delta(t)\|_{H^2(\mathbb{R}^d)} \leq C_T.
\]

(5.6)

**Proof.** We postpone the proof to Appendix B. \hfill \blacksquare

### 5.2 Case of a positive $\delta$

Let $\delta > 0$. Hence, the operator $\partial_t - \mathcal{L}_{\rho_0 + \delta}$ is uniformly parabolic if $\rho_0 + \delta$ and $\nabla_x \rho_0$ are bounded, since $\rho_0 + \delta \geq \delta > 0$. Consequently, we can study the well-posedness of the reaction-diffusion equation (5.2) with classical arguments.

**Lemma 5.3.** Consider an initial data $\rho_0$ satisfying (2.7) and $V_0 \in H^2(\mathbb{R}^d)$. Then, for all $T > 0$ and for all $\delta > 0$, there exists a unique weak solution $V_\delta$ of the diffusion equation

\[
\partial_t V_\delta - \mathcal{L}_{\rho_0 + \delta} V_\delta = N(V_\delta) - W[V_\delta],
\]

(5.7)

such that

\[
V_\delta \in L^\infty([0, T], H^2(\mathbb{R}^d)) \cap L^2([0, T], H^3(\mathbb{R}^d)) \cap \mathcal{C}^0([0, T], L^2(\mathbb{R}^d)),
\]

and $V_\delta$ satisfies the energy estimate (5.5) with $S = N(V_\delta) - W[V_\delta]$.

**Proof.** The proof relies on classical methods explained in [11]. According to Lemma 5.1, $V_\delta$ satisfies the energy estimate (5.5) with $S = N(V_\delta) - W[V_\delta]$. \hfill \blacksquare
5.3 Proof of Proposition 2.5

Now, let us pass to the limit \( \delta \to 0 \) in the approximated equation (5.2), to prove Proposition 2.5. Let \( V_0 \) and \( W_0 \in H^2(\mathbb{R}^d) \). For all \( \delta > 0 \), Lemma 5.3 yields the existence of a weak solution \( V_\delta \) to the reaction-diffusion equation (5.2). According to Corollary 5.2, there exists a positive constant \( K_1 > 0 \) independent of \( \delta \) such that for all \( \delta > 0 \),

\[
\|V_\delta\|_{L^\infty([0,T],H^2(\mathbb{R}^d))} \leq K_1.
\]

Let \((\delta_n)_{n \in \mathbb{N}}\) be a sequence of positive reals such that \( \delta_n \to 0 \) as \( n \to \infty \). Therefore, there exists a function \( V \in L^\infty([0,T],H^2(\mathbb{R}^d)) \) such that up to extraction,

\[
V_\delta \rightharpoonup V,
\]

weakly-\( * \) in \( L^\infty([0,T],H^2(\mathbb{R}^d)) \) as \( \delta \to 0 \). Furthermore, since for all \( n \in \mathbb{N} \), \( V_\delta \) is a weak solution of (5.7), then there exists a constant \( K_2 > 0 \) independent of \( \delta \) such that for all \( n \in \mathbb{N} \),

\[
\|\partial_t V_\delta\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \leq K_2.
\]

Consequently, using Arzelà-Ascoli theorem, we get that there exists \( \widetilde{V} \in \mathcal{C}^0([0,T],L^2(\mathbb{R}^d)) \) such that up to extraction, \((V_\delta)_{n \in \mathbb{N}}\) converges strongly respectively towards \( \widetilde{V} \) in the space \( \mathcal{C}^0([0,T],L^2(\mathbb{R}^d)) \). Therefore, \( V = \widetilde{V} \), and thus,

\[
V \in L^\infty\left([0,T],H^2(\mathbb{R}^d)\right) \cap \mathcal{C}^0\left([0,T],L^2(\mathbb{R}^d)\right).
\]

Since \((V_\delta)_{n \in \mathbb{N}}\) converges strongly towards \( V \) up to extraction, we can pass to the limit in the weak formulation of the equation (5.2). Consequently, \( V \) is a weak solution of (2.5) with \( \delta = 0 \).

Finally, it remains to obtain additional regularity to conclude the proof of Corollary 2.7. We just have to notice that \( V \in L^\infty([0,T],H^2(\mathbb{R}^d)) \cap W^{1,\infty}(0,T,L^2(\mathbb{R}^d)) \), so using classical arguments detailed in the paragraph 5.9.2 from [11], we get:

\[
V \in \mathcal{C}^0([0,T],H^1(\mathbb{R}^d)).
\]

5.4 Conclusion: proof of Corollary 2.7

This subsection is devoted to the construction of a solution to the system (1.14) from a weak solution to the FitzHugh-Nagumo reaction-diffusion system.

Let \( T > 0 \). Let us denote with \((\widetilde{V},\widetilde{W})\) the weak solution of the equation (2.5) provided by Proposition 2.5 with initial condition \((\rho_0,\rho_0,\rho_0)\). Our proof is organised in two steps. First of all, we claim that for all solution \((V,F)\) of the system (1.14), the two functions \( V \) and \( \widetilde{V} \) coincide almost everywhere on \([0,T] \times \mathbb{R}^d\).

Then, we prove the existence of a measure solution \( F \) so that \((\widetilde{V},F)\) is a solution of the system (1.14).

Before starting the proof, notice that for all \( x \in \mathbb{R}^d \) such that \( \rho_0(x) = 0 \), the system (2.5) reduces to the following system of ODEs

\[
\begin{cases}
\partial_t \widetilde{V} = N(\widetilde{V}) - \widetilde{W}, \\
\partial_t \widetilde{W} = A(\widetilde{V},\widetilde{W}).
\end{cases} \tag{5.8}
\]

Since \( V_0(x) = W_0(x) = 0 \) of \( \rho_0(x) = 0 \), one can directly conclude that for all \( t \in [0,T] \) and for almost every \( x \in \mathbb{R}^d \) such that \( \rho_0(x) = 0 \),

\[
\widetilde{V}(t,x) = \widetilde{W}(t,x) = 0.
\]
Step 1: Uniqueness. Now, let us prove that for any solution \((V, F)\) on \([0,T]\) of (1.14) in the sense of Definition 2.6, the function \(V\) coincides with \(\widetilde{V}\) on \([0,T] \times \mathbb{R}^d\). Suppose that \((V, F)\) is a solution of (1.14) such that \(F\) has a finite second moment in \(w\). If we define for all \((t, x) \in [0,T] \times \mathbb{R}^d\)
\[
\rho_0(x) W(t, x) := \int w F(t, x, dw),
\]
then the triplet \((\rho_0, \rho_0 V, \rho_0 W)\) satisfies the reaction-diffusion equation (1.15). On the one hand, by definition, for almost every \(x \in \mathbb{R}^d\) such that \(\rho_0(x) = 0\), for all \(t \in [0,T]\),
\[
V(t, x) = \widetilde{V}(t, x) = 0, \quad W(t, x) = \widetilde{W}(t, x) = 0.
\]
According to the notion of solution of (1.14) from Definition 2.6, \(V \in L^\infty([0,T], H^2(\mathbb{R}^d))\), so \(\Delta_x V(t, x)\) is defined for almost every \(x \in \mathbb{R}^d\). Consequently, the couple \((V, W)\) satisfies (2.5) pointwise for all \(t \in [0,T]\) and almost every \(x \in \mathbb{R}^d\) such that \(\rho_0(x) = 0\).

On the other hand, for all \(x \in \mathbb{R}^d\) such that \(\rho_0(x) > 0\), the equation (1.15) reduces to the reaction-diffusion system (2.5). Therefore, \((V, W)\) satisfies the equation (2.5) for all \(t \in [0,T]\) and almost every \(x \in \mathbb{R}^d\), with initial condition \((V_0, W_0)\). Consequently, the couple \((V, W)\) satisfies the reaction-diffusion system (2.5) in the sense of Definition 2.4. Thus, by uniqueness of the solution of the equation (2.5), we can conclude that
\[
(V, W) = (\widetilde{V}, \widetilde{W}).
\]

Step 2: Existence. Then, let us to prove the existence of a measure solution of the first equation in (1.14). With \(V\) the first component of the solution of the system (2.5), let us consider the transport equation for \(t > 0, x \in \mathbb{R}^d\) and \(w \in \mathbb{R}\):
\[
\begin{cases}
\partial_t F(t, x, w) + \partial_w (A(V(t, x), w) F(t, x, w)) = 0, \\
F|_{t=0} = F_0.
\end{cases}
\]
(5.9)

In order to solve (5.9), we introduce the associated system of characteristic curves for all \((s, t, x, w) \in [0,T]^2 \times \mathbb{R}^{d+1}\):
\[
\begin{cases}
\frac{d}{ds} W(s) = A(V(s, x), W(s)), \\
W(t) = w.
\end{cases}
\]
(5.10)

Since the function \(A\) grows linearly with respect to \(w\), and the function \(V\) is regular enough, we get the global existence and uniqueness of a solution of the characteristic equation (5.10), denoted by \(W\), such that for all \((s, t, x, w) \in [0,T]^2 \times \mathbb{R}^{d+1}\),
\[
W(s, t, x, w) := e^{-\tau \gamma (s-t)}w + \int_s^t e^{-\tau \gamma (s-t')}V(t', x) \, dt'.
\]

Then, using the theory of characteristics, we can conclude the existence of a unique solution of the transport equation (5.9) given by
\[
F(t, x, w) := F_0(x, W(0, t, x, w)) e^{\tau \gamma t}.
\]
The expression above of the solution $F$ directly yields that there exists a constant $C_T > 0$ such that for all $t \in [0, T],
\int |w|^2 F(t, dx, dw) \leq C_T.
Moreover, we can conclude from the uniqueness of the solutions of (2.5) that for all $t \in [0, T]$ and all $x \in \mathbb{R}^d$ such that $\rho_0(x) > 0$,
\[
\frac{1}{\rho_0(x)} \int w F(t, x, dw) = W(t, x).
\]
Consequently, the unique solution of the system (1.14) is $(V, F)$ where $(V, W)$ is the weak solution of (2.5) and $F$ is the unique solution of the transport equation (5.9).

A Proof of Lemma 5.1

Our approach consists in studying the variations of $\|V_\delta\|_{L^2(\mathbb{R}^d)}$ and $\|\Delta_x V_\delta\|_{L^2(\mathbb{R}^d)}$, in order to conclude with interpolations. Let $t \in [0, T]$. For the sake of simplicity, in the rest of this proof, we note $V$ instead of $V_\delta$. First of all, we get:

\[
\frac{1}{2} \frac{d}{dt} \|V\|_{L^2(\mathbb{R}^d)}^2 = -\sigma \left( \int (\rho_0 + \delta) |\nabla_x V|^2 \, dx - \int (\nabla_x \rho_0 \cdot \nabla_x V) \, dx \right) + \int S \, V \, dx
= -\sigma \left( \int (\rho_0 + \delta) |\nabla_x V|^2 \, dx - \frac{1}{2} \int \nabla_x \rho_0 \cdot |\nabla_x V|^2 \, dx \right) + \int S \, V \, dx.
\]

Thus, using Young’s inequality, we can conclude that there exists a constant $C_0 > 0$ such that

\[
\frac{1}{2} \frac{d}{dt} \|V\|_{L^2(\mathbb{R}^d)}^2 + \sigma \int (\rho_0 + \delta) |\nabla_x V|^2 \, dx \leq C_0 \|V\|_{L^2(\mathbb{R}^d)}^2 + \int S \, V \, dx. \tag{A.1}
\]

Then, we also get that

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_x V\|_{L^2(\mathbb{R}^d)}^2
= -\int \Delta_x ((\rho_0 + \delta) \nabla_x V) \cdot \nabla_x \Delta_x V \, dx + \int \Delta_x (\nabla_x \rho_0 \cdot \nabla_x V) \, dx + \int \Delta_x S \Delta_x V \, dx
\]

\[
= -\int [\Delta_x \rho_0 \nabla_x V + 2 \Delta_x V \nabla_x \rho_0 + (\rho_0 + \delta) \nabla_x \Delta_x V] \cdot \nabla_x \Delta_x V \, dx
+ \int [\nabla_x \Delta_x \rho_0 \cdot \nabla_x V + 2 \Delta_x \rho_0 \Delta_x V + \nabla_x \rho_0 \cdot \nabla_x \Delta_x V] \, dx
+ \int \Delta_x S \Delta_x V \, dx.
\]

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Using Green’s formula on the term \( \int \Delta_x \rho_0 \nabla_x V \cdot \nabla_x \Delta_x V \, dx \), we compute that

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_x V \|_{L^2(\mathbb{R}^d)}^2 = - \int (\rho_0 + \delta) |\nabla_x \Delta_x V|^2 \, dx + 3 \int \Delta_x \rho_0 |\Delta_x V|^2 \, dx + \int \nabla_x \Delta_x \rho_0 \cdot \nabla_x V \Delta_x V \, dx \\
+ \int \Delta_x S \Delta_x V \, dx \\
\leq - \int (\rho_0 + \delta) |\nabla_x \Delta_x V|^2 \, dx + 3 \| \rho_0 \|_{L^2(\mathbb{R}^d)}^2 \| \Delta_x V \|_{L^2(\mathbb{R}^d)}^2 + \| \rho_0 \|_{L^3(\mathbb{R}^d)} \| \nabla_x V \|_{L^2(\mathbb{R}^d)}^2 \| \Delta_x V \|_{L^2(\mathbb{R}^d)} \\
+ \int \Delta_x S \Delta_x V \, dx \\
\leq - \int (\rho_0 + \delta) |\nabla_x \Delta_x V|^2 \, dx + C \left( \| V \|_{L^2(\mathbb{R}^d)}^2 + \| \Delta_x V \|_{L^2(\mathbb{R}^d)}^2 \right) + \int \Delta_x S \Delta_x V \, dx,
\]

where \( C > 0 \) is a positive constant. Consequently, we get that for all \( t \in [0, T] \),

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_x (t) \|_{L^2(\mathbb{R}^d)}^2 \leq C \left( \| \Delta_x V(t) \|_{L^2(\mathbb{R}^d)}^2 + \| V(t) \|_{L^2(\mathbb{R}^d)}^2 \right) + \int \Delta_x (t) \Delta_x V(t) \, dx. \tag{A.2}
\]

Finally, integrating the estimates (A.1) and (A.2) between 0 and \( t \) for \( t \in [0, T] \), we get the estimate (5.5) with interpolations.

**B Proof of Corollary 5.2**

For the sake of simplicity, in the rest of this section, we note \( V \) instead of \( V_\delta \). According to Lemma 5.1, the estimate (5.5) holds with \( S = N(V) - W[V] \). To obtain Corollary 5.2 from the energy estimate (5.5), we need to estimate the scalar product

\[
\langle V, N(V) \rangle_{H^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq 2} \int \partial^\alpha V \partial^\alpha N(V) \, dx.
\]

First of all, for all \( t \in [0, T] \), since \( N \) satisfies the property (1.3), we have:

\[
\int V N(V) \, dx \leq \kappa_1 \| V \|_{L^2(\mathbb{R}^d)}^2 - \kappa_1 \| V \|_{L^4(\mathbb{R}^d)}^4 \leq \kappa_1 \| V \|_{L^2(\mathbb{R}^d)}^2.
\]

Then, for all \( t \in [0, T] \), we also have:

\[
\int \nabla_x V \nabla_x N(V) \, dx \leq \int |\nabla_x V|^2 ( -3 V^2 + 2 (1 + a) V - a ) \, dx \\
\leq \int |\nabla_x V|^2 \, dx.
\]
Finally, as for the order 2, using Young’s inequality for some small parameter $\theta > 0$, we also obtain:

$$\int \Delta_x V \Delta_x N(V) \, dx = \int |\Delta_x V|^2 \left[-3|V|^2 + 2(1+a)V - a\right] \, dx + \int \Delta_x V |\nabla_x V|^2 \left[-6V + 2(1+a)\right] \, dx$$

$$\leq -3 \int |\Delta_x V|^2 |V|^2 \, dx - a \int |\Delta_x V|^2 \, dx$$

$$+ \frac{(1+a)}{\theta} \int |\Delta_x V|^2 \, dx + (1+a) \theta \int |\Delta_x V|^2 |V|^2 \, dx$$

$$+ \frac{3}{\theta} \int |\nabla_x V|^4 \, dx + 3 \theta \int |V|^2 |\Delta_x V|^2 \, dx$$

$$+ (1+a) \int |\Delta_x V|^2 \, dx + (1+a) \int |\nabla_x V|^4 \, dx.$$

Consequently, if we consider $\theta$ small enough so that $(1+a) \theta + 3 \theta \leq 3$, we obtain:

$$\int \Delta_x V \Delta_x N(V) \, dx \leq \left(1 + \frac{1}{\theta}\right) (1+a) \int |\Delta_x V|^2 \, dx + \left(1 + a + \frac{3}{\theta}\right) \int |\nabla_x V|^4 \, dx.$$  (B.1)

Therefore, to conclude, we only need to find a uniform bound of the $L^4$ norm of $\nabla_x V(t)$. We apply the Gagliardo-Nirenberg inequality on $\|\nabla_x V_0\|_{L^4(\mathbb{R}^d)}$, which yields that there exists a positive constant $C > 0$ such that:

$$\|\nabla_x V_0\|_{L^4(\mathbb{R}^d)} \leq C \|V_0\|_{H^2(\mathbb{R}^d)}^{\frac{4}{3}} \|\nabla_x V_0\|_{L^4(\mathbb{R}^d)}^{1 - \frac{4}{3}} < +\infty,$$

Hence, since $V_0 \in H^2(\mathbb{R}^d)$, we have $\nabla_x V_0 \in L^4(\mathbb{R}^d)$, and it still holds if we replace $V_0$ with $W_0$. Furthermore, $\nabla_x V$ satisfies in the weak sense the following equation on $(0, T) \times \mathbb{R}^d$

$$\frac{\partial}{\partial t} (\nabla_x V) = \sigma \left[\nabla_x (\rho_\theta \Delta_x V) + 2 \nabla_x (\nabla_x \rho_\theta \cdot \nabla_x V)\right] + \nabla_x V \cdot N'(V) - \nabla_x W[V].$$

Consequently, using Green’s formula, we get for all $t \in [0, T]$,

$$\frac{1}{4} \frac{d}{dt} \|\nabla_x V\|_{L^4(\mathbb{R}^d)}^4$$

$$= \sigma \int |\nabla_x V|^2 \nabla_x V \cdot \nabla_x ((\rho_\theta + \delta) \Delta_x V) \, dx + 2 \sigma \int |\nabla_x V|^2 \nabla_x V \cdot \nabla_x (\nabla_x \rho_\theta \cdot \nabla_x V) \, dx$$

$$+ \int |\nabla_x V|^4 N'(V) \, dx - \int |\nabla_x V|^2 \nabla_x V \cdot \nabla_x W[V] \, dx$$

$$= -3 \sigma \int |\nabla_x V|^2 |\Delta_x V|^2 (\rho_\theta + \delta) \, dx + \frac{3}{2} \sigma \int |\nabla_x V|^4 \Delta_x \rho_\theta \, dx$$

$$+ \int |\nabla_x V|^4 N'(V) \, dx - \int |\nabla_x V|^2 \nabla_x V \cdot \left(e^{-\gamma(t-s)} \nabla_x W_0 + \int_0^t e^{-\gamma(t-s)} \nabla_x W(s, x) \, ds\right) \, dx.$$
Thus, Grönwall’s inequality yields that there exists a positive constant $K_T$ independent of $\|V\|_{L^\infty([0,T],H^2(\mathbb{R}^d))}$ such that for all $t \in [0,T]$,
\[
\|\nabla_x V(t)\|_{L^4(\mathbb{R}^d)}^4 \leq K_T.
\]
Therefore, we conclude from (B.1)-(B.2) that there exists a positive constant $C > 0$ such that
\[
\int \Delta_x V \Delta_x N(V) \, dx \leq C \|V\|_{H^2(\mathbb{R}^d)}^2 + K_T.
\]

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