What does “ρ exchange” in πN scattering mean?

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We present an alternative method for calculating amplitudes for correlated ππ exchange in the “σ” and ρ channel in πN scattering. Starting from a fixed mass meson exchange potential, we introduce the width of the exchanged particles by integrating over a mass spectral function. The spectral functions are constructed from the pseudoempirical N\bar{N} → ππ data. Using this approach we develop a prescription for resolving ambiguities of the correlated ππ exchange in the ρ channel that occur when different dispersion theoretical formulations of ρ exchange are used to construct πN potentials.

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I. INTRODUCTION

The exchange of meson pairs plays an important role in hadron dynamics. The first meson exchange models of the two-nucleon interaction were based on the exchange of one meson only and had to introduce a scalar-isoscalar meson (the sigma) which is not seen as a resonance in the two-pion phase shifts and has remained a controversial issue \cite{sigma_controversy}. Models of the two-nucleon interaction based on dispersion relations show that the sigma meson can be understood as an effective degree of freedom which parameterizes the exchange of two pions
A natural extension of this approach leads to the inclusion of $\pi\pi$ correlations in the $\rho$-channel. In addition to the exchange of two pions, the exchange of a pion-rho pair is necessary to describe the phase shifts, in particular near the pion production threshold.

![Graph showing coupling constants and ratio calculated](image)

**FIG. 1.** Coupling constants $g_{\rho\pi\pi}g_v/4\pi$ (a), $g_{\rho\pi\pi}g_t/4\pi$ (b) and the ratio $\kappa = g_t/g_v$ (c) calculated by using different dispersion relations as in Ref. [7]. The solid line is for the $\Gamma$ amplitudes of eq. (12), the dashed line for the $f$ amplitudes of eq. (10) and the dotted line for the $h$ amplitude from eq. (15). The $g$ amplitude is proportional to the $\Gamma_2$ amplitude and therefore gives the same tensor coupling as the $\Gamma$ amplitudes. The amplitudes are defined in Sec. [11].

Correlated two-pion exchange has been incorporated in meson-theoretic models of both pion-nucleon and kaon-nucleon scattering. In these calculations ambiguities in the dispersion theoretic approach to the correlated two-meson exchange in the $\rho$ channel...
were encountered\textsuperscript{1}. The quantitative effect of these ambiguities is illustrated in Fig. 1, where we show the vector and tensor $\rho N N$ coupling as calculated using different dispersion relation formulations\textsuperscript{2}. Given the important role of correlated two-meson exchanges in many different hadronic reactions, one is compelled to address the question which of the various dispersion relation formulations is preferred. In order to answer this question, we calculate the correlated $\pi \pi$ exchange, not in the framework of dispersion relations, which suffer from the problem of being ambiguous, but in a formulation based on spectral functions.

We introduce this method in the next section using the $\pi N$ potential due to correlated $\pi \pi$ exchange in the $\sigma$ channel as an example, which is simpler than the exchange in the $\rho$ channel. In the third section we deal with the construction of the potential in the $\rho$ channel and discuss the problem of differing results in the dispersion relation approach. The last section summarizes our results.

**II. THE $\pi N$ POTENTIAL OF CORRELATED $\pi \pi$ EXCHANGE IN THE $\sigma$ CHANNEL**

In order to extract the dynamical information from the transfer matrix in both channels, $N \bar{N} \rightarrow \pi \pi$ (hereafter called the $t$-channel) and $\pi N \rightarrow \pi N$ (hereafter called the $s$-channel), one must first decompose the $T$-matrix into well-defined operators and amplitudes that contain the dynamics of the reaction. The most familiar amplitudes in this respect are the $f$ amplitudes (see Fig. 2) introduced by Frazer and Fulco\textsuperscript{3}, which we will use as input for our calculations. The details of this decomposition are given in the appendix A.

\textsuperscript{1}In the dispersion theoretic approach the $\rho$ exchange is defined by integrating right hand cuts only\textsuperscript{11}. By leaving out contributions from the left hand cuts, the mentioned ambiguities arise.

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Our starting point for the construction of the correlated $\pi\pi$ exchange is the Lagrangian

\[ \mathcal{L}_{\text{int}} = g_{NN\sigma} \bar{\Psi} \Psi \sigma + \frac{g_{\sigma\pi\pi}}{2m_\pi} \partial_\mu \vec{p} \cdot \partial^\mu \vec{\pi} \sigma. \]  

We use derivative coupling at the $\sigma\pi\pi$ vertex to ensure chiral symmetry. From this Lagrangian we calculate a $\pi N$ potential

\[ V_\sigma = \frac{g_{NN\sigma}g_{\sigma\pi\pi}}{2m_\pi} 2p_2 \cdot p_1 \sigma^\mu P_\sigma(t, m_\sigma^2) \bar{u}(\vec{p}_3, \lambda_3) u(\vec{p}_1, \lambda_1), \]  

where the propagator $P_\sigma(t, m_\sigma^2)$ depends on the theoretical framework in which one is working. In a covariant approach the propagator is

\[ P_\sigma(t, m_\sigma^2) = \frac{1}{t - m_\sigma^2}. \]

Our notation is shown in Fig. 5 in the appendix.

The potential $V_\sigma$ can be expressed in terms of the invariant amplitudes $A$ and $B$ defined in the appendix:
\[ A^{(+)}(\sigma) = -\frac{g_{\sigma N N_\sigma} g_{\pi \pi}}{2m_\pi} 2p_{2\pi} p_1^{\mu} P_\sigma(t, m_\sigma^2), \]
\[ B^{(+)}(\sigma) = 0. \]  

(3)

In order to obtain an idea of how to proceed, let us look again at the spectral functions in Fig. 2. The two spectral functions of the \( \rho \) channel suggest a Breit-Wigner parameterization of a resonance. The width of such a resonance is the decay width of the particle. If the particle is stable the width is zero, resulting in a delta function located at the mass squared of the particle. The spectral functions (\( Im(f) \)) can then be regarded as mass distributions of an unstable particle. We can then incorporate the width of the \( \sigma \) meson into our amplitudes by exchanging a continuum of \( \sigma \) mesons with different masses \( m_\sigma \), with amplitudes weighted according to the spectral function. This can be cast in spectral form [14–16]:

\[ A^{(+)}_{2\pi} = \frac{1}{\pi} \int dm_\sigma^2 \rho_\sigma(m_\sigma^2) A^{(+)}(m_\sigma^2), \]  

where the spectral function is normalized according to

\[ \frac{1}{\pi} \int dm_\sigma^2 \rho_\sigma(m_\sigma^2) = 1. \]

All that we need do is calculate the spectral function \( \rho_\sigma(m_\sigma) \). We do this by relating the spectral function to the dressed \( \sigma \) propagator, \( D_\sigma(m_\sigma^2) \), via

\[ \rho_\sigma(m_\sigma^2) = -Im(D_\sigma(m_\sigma^2)). \]  

(5)

In order to extract the \( \sigma \) propagator, we express the \( N\bar{N} \rightarrow \pi\pi \) amplitudes of eq. (A10) as a \( \sigma \) pole diagram with a dressed propagator, \( D_\sigma(t) \), which contains all the \( \pi\pi \) dynamics, as shown in Fig. 3. We thus obtain

\[ A^{(+)}(\sigma) = \frac{4\pi}{p_t^0} f_+^0(t) D_\sigma(t) = -\frac{4\pi}{p_t^0} f_+^0(t). \]  

(6)
FIG. 3. Expressing the $N\bar{N} \rightarrow \pi\pi$ $T$-matrix in term of a meson pole diagram with a dressed propagator, which contains all the dynamics.

The spectral function is then given by

$$\rho_{\sigma}(m_{\sigma}^2) = -\text{Im}(D_{\sigma}(m_{\sigma}^2)) = \frac{4\pi}{p_4^2} \text{Im}(f_0^0(t)) \frac{2m_{\pi}}{g_{\sigma\pi\pi}g_{\sigma NN}} \frac{1}{2q_3q_4},$$

(7)

from which we obtain the amplitude $A_{2\pi}^{(+)}$:

$$A_{2\pi}^{(+)} = -4(2p_2p_4^\mu) \int dt' \frac{\text{Im}(f_0^0(t'))}{p_4^2(t' - 2m_{\pi}^2)} P_{\sigma}(t, t')$$

$$= -16(2p_2p_4^\mu) \int dt' \frac{\text{Im}(f_0^0(t'))}{(t' - 4m_{\pi}^2)(t' - 2m_{\pi}^2)} P_{\sigma}(t, t'),$$

(8)

Here we have used the relation $2q_3q_4^\mu = t' - 2m_{\pi}^2$ and the definition of the on shell momentum $p_\nu^2 = \frac{t'}{4} - m_{\bar{N}}^2$ and changed the variable of integration from $m_{\sigma}^2$ to $t'$. This enables us to compare our expression directly with the results of the dispersion theoretical approach of Ref. [8]

$$\tilde{A}_{2\pi}^{(+)} = 16(t - 2m_{\pi}^2) \int dt' \frac{\text{Im}(f_0^0(t'))}{(t' - 4m_{\pi}^2)(t' - t')(t' - 2m_{\pi}^2)}.$$  

(9)

The tilde is used only as a reminder that this is a result from dispersion theory.
FIG. 4. The factors $(-2p_2 \cdot p_4')$ (solid line) and $(t - 2m_n^2)$ (dotted line) as a function of the off-shell momentum $|p_4|$ where $|p_2|=100$ MeV is fixed. The momenta $\vec{p}_2$ and $\vec{p}_4$ are chosen to be parallel.

In comparing the two results, eq. (8) derived by using the spectral function and eq. (9) calculated within the framework of dispersion relations, we find differences only in the factor in front of the integral: $-(2p_2 \cdot p_4')$ in eq. (8) and $(t - 2m_n^2)$ in eq. (9). For on-shell amplitudes, these two terms agree. Therefore the on-shell amplitudes of both methods are the same, as it must be. Off-shell, however, they are very different, as seen in Fig. 4. If one wishes to use this amplitude as a potential in a scattering equation or as an off-shell $\pi N$ amplitude in another reaction, care must be taken to use a potential with an off-shell behavior that is compatible with the off-shell behavior of other diagrams (e.g. nucleon exchange). The off-shell behavior can be implemented easily in eq. (8), because here only the momenta of the two pions appear and the off-shell prescription of these momenta is given by the formalism used to solve the scattering equation for which the correlated $\pi\pi$ exchange serves as a potential. The importance of a proper off-shell behavior of the $\pi N$ amplitude constructed in a dispersion theoretical way is also pointed out in the calculation of the $\pi NN$ vertex in Ref. [10]. There the authors prescribe the off-shell behavior of the
correlated two pion exchange by comparing the amplitude to the field theoretical amplitude for the exchange of a stable particle.

The fact that the on-shell amplitudes, which are the ones we are interested in at the moment, are the same with both methods shows that our method leads to correct results where dispersion theory is applicable. This observation encourages us to proceed in the calculation of the correlated $\pi\pi$ exchange in the $\rho$ channel.

III. THE CORRELATED $\pi\pi$ EXCHANGE IN THE $\rho$ CHANNEL

Let us start this section by taking a closer look at the source of the difference displayed in Fig. 1. The coupling constants shown there are calculated by using different dispersion relations [17]. The dashed lines were calculated by using the amplitudes

$$\tilde{A}_{\rho}^{(-)}(s, t) = 12 \frac{p_t q_x}{p_t^2} \left( \frac{m_N}{\sqrt{2}} \int dt' \frac{Im(f_1^1(t'))}{t' - t} - \int dt' \frac{Im(f_1^{-1}(t'))}{t' - t} \right)$$

$$\tilde{B}_{\rho}^{(-)}(s, t) = 6\sqrt{2} \int dt' \frac{Im(f_1^1(t'))}{t' - t},$$

which we get by writing dispersion relations for $f_1^1$ and $f_1^{-1}$ separately and inserting these into eq. (A11).

A different way was originally suggested by Frazer and Fulco [13] and later applied by Höhler and Pietarinen [11] to a calculation of the the $\rho NN$ couplings. They define the combinations

$$\Gamma_1(t) = -\frac{m_N}{p_t^2} \left( f_1^1(t) - \frac{t}{4\sqrt{2}m_N} f_1^{-1}(t) \right)$$

$$\Gamma_2(t) = \frac{m_N}{p_t^2} \left( f_1^1(t) - \frac{m_N}{\sqrt{2}} f_1^{-1}(t) \right),$$

for which dispersion relation are written. This leads to the amplitudes

$$\tilde{A}_{\rho}^{(-)}(s, t) = -12 \frac{p_t q_x}{m_N} \int dt' \frac{Im(\Gamma_2(t'))}{t' - t}$$

$$\tilde{B}_{\rho}^{(-)}(s, t) = 12 \left( \int dt' \frac{Im(\Gamma_1(t'))}{t' - t} + \int dt' \frac{Im(\Gamma_2(t'))}{t' - t} \right),$$

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and to the solid curves in Fig. 1. Note that the factor $\frac{1}{p_\perp}$ does not generate a pole at $t = 4m_N^2$, since the $f$ amplitudes obey the relation

$$f_+^1(4m_N^2) - \frac{m_N}{\sqrt{2}} f_-^1(4m_N^2) = 0. \quad (13)$$

Yet another possibility was suggested by J.M. Richard [18]. He defines two amplitudes

$$g(t) = f_+^1(t) - \frac{m_N}{\sqrt{2}} f_-^1(t)$$
$$h(t) = f_+^1(t) + \frac{m_N}{\sqrt{2}} f_-^1(t), \quad (14)$$

and formulates a subtracted dispersion relation for $g$ and an unsubtracted dispersion relation for $h$

$$\tilde{g}(t) = \frac{t - 4m_N^2}{\pi} \int dt' \frac{\text{Im}(g(t'))}{(t' - t)(t' - 4m_N^2)}$$
$$\tilde{h}(t) = \frac{1}{\pi} \int dt' \frac{\text{Im}(h(t'))}{t' - t}. \quad (15)$$

Since $g(t) \propto \Gamma_2$ the amplitude $A_{2\pi}^{(-)}$ (and therefore the tensor coupling) is the same as in eq. (12). The amplitude $B_{2\pi}^{(-)}$, however, reads now

$$\tilde{B}_{2\pi}^{(-)} = 12\pi m_N (\tilde{h}(t) - \tilde{g}(t)) \quad (16)$$

and leads to a different vector coupling, shown in Fig. 1.

The fact that these three alternatives lead to very different amplitudes can be seen directly by comparing the eqs. (10), (12), and (15). One can see for example, that the factor $\frac{1}{p_\perp}$ appears in front of the dispersion relation integral over the $f$ amplitudes in eq. (10), whereas it is absorbed in the definition of the $\Gamma$ amplitudes in eq. (12). Of course this influences the convergence of the dispersion integrals which causes the differences between the couplings displayed in Fig. 1. These are the ambiguities mentioned above.

Let us now calculate the $\rho$ exchange by using the spectral function approach we have introduced in section II. The calculation of the amplitudes in the $\rho$ channel proceeds along the same lines as outlined in the previous section. The Lagrangian we start with is
\[ \mathcal{L}_{\text{int}} = g_v \bar{\psi} (\gamma_\mu \tilde{\rho}^\mu - \frac{\kappa}{2m_N} \sigma_{\mu\nu} \tilde{\rho}^\nu \tilde{\rho}^\nu) \frac{i}{2} \bar{\psi} + g_{\rho\pi\pi} (\bar{\pi} \times \partial_\mu \pi) \rho^\mu \]  \quad (17)

\( (\kappa = \frac{g_t}{g_v}) \). From this we calculate the invariant amplitudes

\[
A_{\rho}^{-} = g_{\rho\pi\pi} \frac{g_t}{m_N} \frac{1}{2} Q^\mu (p_1 + p_3)_\mu P_\rho(t, m_\rho^2) \\
B_{\rho}^{-} = -g_{\rho\pi\pi} (g_v + g_t) P_\rho(t, m_\rho^2),
\]

which we have to weight with a spectral function to get the correlated \( \pi\pi \) exchange.

The two coupling schemes of the \( \rho \) to the nucleon as given in eq. (17), manifest themselves in two spectral functions which must be calculated from the two amplitudes \( f_+^1 \) and \( f_+^1 \) by using eq. (A11), but how to assign the propagators (spectral functions) is not yet clear. We could either define (a) separate spectral functions for the amplitudes \( A \) and \( B \) or (b) spectral functions for the vector and tensor part of the coupling.

a) By using the ansatz

\[
A_{\rho}^{(-)} = g_{\rho\pi\pi} \frac{g_t}{m_N} p_t q_t x D_A^{\rho}(t) \\
B_{\rho}^{(-)} = -g_{\rho\pi\pi} (g_v + g_t) D_B^{\rho}(t),
\]

the first method leads us to the spectral functions

\[
\rho_{\rho}^{A}(t) = -\frac{12\pi}{p_t^2} \frac{m_N}{g_{\rho\pi\pi} g_t} \left( \frac{m_N}{\sqrt{2}} \text{Im}(f_-^1(t)) - \text{Im}(f_+^1(t)) \right) \\
\rho_{\rho}^{B}(t) = \frac{12\pi}{\sqrt{2}} \frac{\text{Im}(f_-^1(t))}{g_{\rho\pi\pi} (g_v + g_t)},
\]

which we now use to construct the amplitude for correlated \( \pi\pi \) exchange via

\[
A_{2\pi}^{(-)} = \frac{1}{\pi} \int dm_\rho^2 \rho_{\rho}^{A}(m_\rho^2) A_{\rho}^{(-)} \\
B_{2\pi}^{(-)} = \frac{1}{\pi} \int dm_\rho^2 \rho_{\rho}^{B}(m_\rho^2) B_{\rho}^{(-)}.
\]

We finally obtain for the amplitudes

\[
A_{2\pi}^{(-)} = -6 \left( \frac{1}{2} Q^\mu (p_1 + p_3)_\mu \right) \int dt' \frac{1}{p_{t'}^2} \left( \sqrt{2} m_N \text{Im}(f_-^1(t')) - 2 \text{Im}(f_+^1(t')) \right) P_\rho(t, t') \\
B_{2\pi}^{(-)} = -6 \sqrt{2} \int dt' \text{Im}(f_-^1(t')) P_\rho(t, t').
\]

(22)
b) The different momentum dependence of the vector and tensor coupling as given in eq. (17) may lead to different dressing of these couplings in expressing the amplitude by a dressed propagator (as displayed in Fig. 3). The second possibility of defining spectral functions for the vector and tensor coupling takes this into account and leads us to a system of coupled equations

\[- g_{\rho\pi\pi}(g_v D^\rho_v(t) + g_t D^\rho_t(t)) = \frac{12\pi}{\sqrt{2}} f^1(t)\]

\[g_{\rho\pi\pi} \frac{g_t}{m_N} p_t q_t x D^\rho_t(t) = \frac{12\pi}{p_t^2} p_t q_t x \left( \frac{m_N}{\sqrt{2}} f^1(t) - f^1_+(t) \right), \tag{23}\]

Solving this system of propagators and using the definition of the spectral function given in eq. (5) as well as the relations

\[A^{(-)}_2 = \frac{1}{\pi} \int dm^2_\rho \rho^i_\rho(m^2_\rho) A^{(-)}_\rho\]

\[B^{(-)}_2 = -\frac{g_{\rho\pi\pi}}{\pi} \int dm^2_\rho (\rho^i_\rho(m^2_\rho) g_v + \rho^i_\rho(m^2_\rho) g_t) P_\rho(t, m^2_\rho). \tag{24}\]

for integrating over the \( \rho \) mass distribution we get

\[A^{(-)}_2 = 12 \frac{Q^\mu(p_1 + p_3)_\mu}{m_N} \int dt' \frac{m_N}{p_{t'}^2} \left( \text{Im}(f^1_+(t')) - \frac{m_N}{\sqrt{2}} \text{Im}(f^1_-(t')) \right) P_\rho(t, t')\]

\[= 12 \frac{Q^\mu(p_1 + p_3)_\mu}{m_N} \int dt' \text{Im}(\Gamma_2(t')) P_\rho(t, t')\]

\[B^{(-)}_2 = 12 \int dt' \frac{m_N}{p_{t'}^2} \left( \text{Im}(f^1_+(t')) - \frac{t'}{4\sqrt{2}m_N} \text{Im}(f^1_-(t')) \right) P_\rho(t, t')\]

\[-12 \int dt' \frac{m_N}{p_{t'}^2} \left( \text{Im}(f^1_+(t')) - \frac{m_N}{\sqrt{2}} \text{Im}(f^1_-(t')) \right) P_\rho(t, t')\]

\[= -12 \int dt' \text{Im}(\Gamma_2(t')) P_\rho(t, t'), \tag{25}\]

where we have written our result also in terms of the \( \Gamma \) amplitudes defined in eq. (11)

The amplitude \( B^{(-)}_2 \) can be reduced further by using \( p_{t'}^2 = t'/4 - m^2_N \) to give

\[B^{(-)}_2 = -12 \int dt' \frac{\text{Im}(f^1_+(t'))}{\sqrt{2}} P_\rho(t, t'). \tag{26}\]
As can be seen by comparing $A_{2\pi}^{(-)}$ from eq. (25) and $B_{2\pi}^{(-)}$ from eq. (26) with the amplitudes in eq. (22) both methods lead to the same amplitudes. Therefore it does not matter how we assign the propagators to the different amplitudes or coupling schemes; they all lead to the same result. This makes our prescription for calculating the correlated $\pi\pi$ exchange in the $\rho$ channel unambiguous.

In comparing the amplitudes from the dispersion relation approach and the amplitudes from our approach using the spectral functions, we see, that the amplitudes of eq. (12) generated by the dispersion over the $\Gamma$ amplitudes are in complete agreement with our amplitudes from eq. (25). Furthermore we recognize, that the only difference between our amplitudes from eq. (22) and the amplitudes from dispersing the $f$ amplitudes of eq. (10) is the position of the term $\frac{1}{p^2_t}$. In our case this term appears inside the integration whereas in eq. (10) it appears in front of the integral. A dispersion relation over the amplitudes $\frac{f}{p^2_t}$ would therefore lead also to the eq. (12). The dispersion relation over $\frac{f}{p^2_t}$ was done for the $\sigma$ exchange right from the beginning, since the known (model dependent) part of the $f_0^\sigma$ amplitude does not have the necessary asymptotic behavior as $t \to \infty$ [13].

**IV. SUMMARY**

To summarize, we have introduced an alternative method for the construction of amplitudes for the correlated $\pi\pi$ exchange that does not suffer from ambiguities of the $\rho$ exchange in approaches through dispersion relations. In the $\sigma$ channel our result agrees completely with the on-shell results from dispersion theory, however by starting from a field theoretically constructed potential, our approach removes ambiguities in the construction of the off shell amplitudes.

In the $\rho$ channel the source of ambiguity in the coupling constants is traced to the choice whether to perform a dispersion relation over the amplitudes $f$ or the ratio $\frac{f}{p^2_t}$. The latter choice agrees completely with dispersion relations over the $\Gamma$ amplitudes used by Höhler and Pietarinen. Our approach solves the problem of non-uniqueness and yields amplitudes that
are in complete agreement with the results from dispersion relations over the $\Gamma$ amplitudes for on-shell processes.

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APPENDIX A: DECOMPOSITION OF THE TRANSFER MATRIX

FIG. 5. Notation in the reaction channels $\pi N \to \pi N$ a) and $N\bar{N} \to \pi\pi$ b)
The transfer matrix $T$ is related to the standard $S$-matrix by
\[ S_{fi} = \delta_{fi} - i(2\pi)^{-2}\delta^{(4)}(P_f - P_i) \left( \frac{m_N m_N}{E_{p_1} E_{p_3}} \right)^{\frac{1}{2}} (2\omega_{p_2}2\omega_{p_4})^{-\frac{1}{2}} T_{fi}. \]

In the $s$-channel the transfer matrix $T$ can be represented as
\[ T_s(p_3, p_4, p_1, p_2) = \bar{u}(\vec{p}_3, \lambda_3)\xi(\mu_3)\zeta(\alpha_4)\hat{T}(s, t)u(\vec{p}_1, \lambda_1)\xi(\mu_1)\zeta(\alpha_2), \quad (A1) \]
where $u(p, \lambda)$ is the Dirac spinor of the nucleon with momentum $\vec{p}$ and helicity $\lambda$. The isospin wave functions -- $\xi$ for the nucleon and $\zeta$ for the pion -- explicitly depend on the third components as given in Fig. 5.

The operator $\hat{T}$ acts in spin-momentum and isospin space and can be decomposed into
\[ \hat{T}(s, t) = \hat{T}^{(+)}(s, t)1 - \hat{T}^{(-)}(s, t)\vec{r} \cdot \vec{t}, \quad (A2) \]
where $\vec{r}$ and $\vec{t}$ are the isospin operators of nucleon and pion, respectively. The index $(+)$ = symmetric and $(−)$ = antisymmetric tells us how the amplitudes behave under exchange of the two pions. These amplitudes can be represented by
\[ \hat{T}^{(±)} = -(A^{(±)}(s, t)I_4 + \gamma^{\mu}Q_{\mu}B^{(±)}(s, t)), \quad (A3) \]
where $Q = \frac{1}{2}(p_2 + p_4)$.

The $T$ matrix in the $t$-channel can be decomposed in the same way
\[ T_t(q_3, q_4, q_1, q_2) = \bar{v}(\vec{q}_2, \lambda_2)\xi(\mu_2)\zeta(\alpha_3)\hat{T}(s, t)u(\vec{q}_1, \lambda_1)\xi(\mu_1)\zeta(\alpha_4), \quad (A4) \]
where the amplitudes $\hat{T}$ and therefore also $A^{(±)}$ and $B^{(±)}$ are the same functions as in eq. (A1) and eq. (A3), however, in a different kinematic domain.

In order to isolate the contributions of the $\sigma$ and $\rho$ we have to perform a partial wave decomposition of the amplitudes $A^{(±)}$ and $B^{(±)}$,
\[ A^{(±)}(s, t) = \sum_J \frac{1}{2}(2J + 1)P_J(x)A^{(±)}_J(t), \quad (A5) \]
and the same for $B^{(±)}$. The argument $x$ of the Legendre polynomials $P_J(x)$ is the cosine of the scattering angle in the $t$-channel. Due to crossing symmetry, the amplitudes $A^{(−)}_J$ and
$B_J^{(+)}$ have to vanish for even $J$ and $A_J^{(+)}$ and $B_J^{(-)}$ have to vanish for odd $J$, so that the index $J$ determines also the symmetry $(\pm)$, which can then be dropped from the partial wave amplitudes.

The $f$ amplitudes introduced by Frazer and Fulco [13] are free of kinematical singularities. They are connected to the partial wave amplitudes in the following way:

$$f_J^+(t) = \frac{1}{8\pi} \left( -\frac{p_t^2}{(p_t q_t) J} A_J + \frac{m_N((J+1)B_{J+1} + JB_{J-1})}{(2J+1)(p_t q_t)^{J-1}} \right)$$

$$f_J^-(t) = \frac{1}{8\pi} \frac{\sqrt{J(J+1)}}{2J+1} \frac{1}{(p_t q_t)^{J-1}} (B_{J-1} - B_{J+1}).$$ \hspace{1cm} (A6)

The momenta

$$p_t = \sqrt{\frac{t}{4} - m_N^2} \quad \text{and} \quad (A7)$$

$$q_t = \sqrt{\frac{t}{4} - m^2} \quad (A8)$$

are the $t$-channel momenta of the $NN$ and $\pi \pi$ system in their c.m., respectively. The index $\pm$ gives the helicity of the nucleon $\pm \frac{1}{2}$. The helicity of the antinucleon is fixed to $+\frac{1}{2}$. Using these amplitudes, the invariant amplitudes can be written as

$$A^{(\pm)}(s, t) = \frac{8\pi}{p_t^2} \sum_J \frac{1}{2} (2J+1)(p_t q_t)^J x \times$$

$$\times \left( \frac{m_N}{\sqrt{J(J+1)}} x P'_J(x)f_J^+(t) - P_J(x)f_J^+(t) \right)$$

$$B^{(\pm)}(s, t) = 8\pi \sum_J \frac{1}{2} (2J+1) \frac{(p_t q_t)^{J-1}}{\sqrt{J(J+1)}} P'_J(x)f_J^-(t), \quad (A9)$$

where $P'_J(x) = \frac{d}{dx} P_J(x)$. The symmetry $(\pm)$ of the amplitudes is determined by summing only even (odd) $J$. The $\sigma$ and $\rho$ contribution is contained in the $J = 0$ and $J = 1$ term in eq. (A9), respectively. Explicitly, the amplitudes for the $\sigma$ channel read

$$A^{(+)}_{\sigma}(s, t) = -\frac{4\pi}{p_t^2} f_0^+(t)$$

$$B^{(+)}_{\sigma}(s, t) = 0, \quad \text{(A10)}$$

and for the $\rho$ channel
\[ A_{p}^{(+)}(s, t) = 12\pi \frac{p_{t} q_{t} x}{P_{t}^{2}} \left( \frac{m_{N}}{\sqrt{2}} f_{-}^{1}(t) - f_{+}^{1}(t) \right) \]
\[ B_{p}^{(+)}(s, t) = 6\sqrt{2}\pi f_{-}^{1}(t). \] (A11)
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