Quantum Self-Correcting Stabilizer Codes

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In this paper, we explicitly construct (Abelian) anyonic excitations of arbitrary stabilizer Hamiltonians which are local on a 2D lattice of qubits. This leads directly to the conclusion that, in the presence of local thermal noise, such systems cannot be used for the fault-tolerant storage of quantum information by self-correction i.e. they are ruled out as candidates for a ‘quantum hard drive’. We suggest that in 3D, the same construction leads to an argument that self-correction is impossible.

I. INTRODUCTION

Our capacity for exploiting the properties of quantum systems for information processing tasks is critically dependent on the ability to protect this fragile information against the unwanted destructive effects of the environment. While theories of error correction \cite{1,2,3,4,5,6,7,8} and fault-tolerance \cite{9,10,11,12} have been proposed, these might be considered early steps towards proving what is possible, but that the resource requirements are prohibitive for useful implementation. A possible candidate for a ‘second generation’ architecture for fault-tolerant storage of information is known as self-correction. This concept has arisen from the study of the toric code, where information is encoded in the degenerate ground states of a Hamiltonian. If the energy cost for converting between the degenerate states using local operations grows with the number of qubits in the lattice, then logical errors can potentially be exponentially suppressed as the system size scales. This property is known as self-correction, and, if systems with such properties could be found, may reduce the energy cost of information storage. To date, no proof of self-correction exists, although candidate models in three \cite{10} and four \cite{11} spatial dimensions have been proposed.

In this paper, we investigate which Hamiltonians can possibly present self-correcting properties. Previous studies have restricted themselves to the toric code \cite{12,13,14,15,16,17}, except for one attempt at a more general no-go theorem \cite{12}. However, this latter result seems to address the question of whether one can store information in a thermal state rather than how long it takes for a system to thermalize.

Here we prove that stabilizer Hamiltonians on qubit lattices in two spatial dimensions are not self-correcting i.e. the time required for the system to develop an error via a local noise model is not exponentially increased by enlarging the lattice. Our proof involves explicitly constructing paths through which such noise destroys the stored information. We also argue the existence of such paths in 3D.

Let us begin by precisely defining what we are interested in, and the goal of this paper.

**Definition 1.** A topological quantum stabilizer code in two spatial dimensions is an instance from a family of qubit Hamiltonians parameterized by their size (N). The Hamiltonian is a sum of R terms $K_n$, where $[K_n, K_m] = 0$ and $K_n^2 = 1$. Each $K_n$ is a tensor product of operators confined to a local area on a 2D lattice, and whose size is independent of N.

Such codes can be designed such that there are degenerate ground states of the Hamiltonian in which quantum information can be stored. The classic example is the toric code \cite{11}. For concreteness, we restrict to the case of a square lattice with periodic boundary conditions, although the same arguments can be made for any planar lattice geometry and, with some modification, open boundary conditions.

There are many reasons why restricting to stabilizer codes is beneficial. The mutual commutation of terms makes calculations simpler than for general Hamiltonians, and thus provides a natural starting point. In particular, the ground state degeneracy is readily identified (and, usefully, the degenerate subspace persists at all energy levels). It also means that the implementation of error detection is easily described. Even in a self-correcting system, error detection and correction is important—at some stage it will be necessary to read out the stored information, and, while self correction will ensure that the number of errors is small and that consequently there will be no false read-out of the state, one nevertheless needs to detect the errors that have occurred.

Furthermore, one might hope that by examining the case of stabilizer codes, these turn out to be representative of a large class of Hamiltonians. For example, in the 1D scenario, there are some generic properties of cluster states (which are a specific example of a stabilizer state), such as correlation functions, which, when looked
at on a global scale, are very close to those for all local 1D gapped Hamiltonians. Specifically, two-body correlations of the ground states of these systems decay exponentially with distance \([18,19]\), and are hence negligibly small when we examine them over separations that scale with system size. In comparison, the correlation functions of the cluster state are identically zero \([21]\). Adding rigour to this possible connection is beyond the scope of the present work, but is certainly an interesting avenue for future studies. Nevertheless, this may be considered a difficult proposition. Properties of the ground states are certainly going to involve entanglement in some way, and yet our understanding of multipartite entanglement is quite poor. For instance, while we are able to discuss distillation of noisy stabilizer states \([21,22]\), including some optimality results \([23,24]\), results relating to non-stabilizer states are virtually non-existent \([25]\), and have only been realized by mapping the system into a stabilizer form (with the exception of some special cases developed in \([24]\).

The restriction to qubits is, again, a matter of convenience since it imposes a number of useful properties on the stabilizers. Furthermore, the literature on binary quantum codes, which have a direct relation to stabilizer states \([26,27]\), is vastly more developed than the non-binary case \([28,29]\). As a result, the generalization to qudits is not immediate.

Topological quantum memories are a useful step towards building fault-tolerant quantum computers, giving a degree of protection against Hamiltonian perturbations. However, as will be discussed in Sec. II, local thermal noise can destroy the stored information, in spite of the apparent protection due to the existence of an energy gap \([11,12,13,14,15]\). This means that active error correction is required, involving a supply of fresh ancillas, or a dissipative operation to reset them. Since the toric code is a stabilizer code, the error syndrome can be extracted by measuring the stabilizer operators, and hence the physical action of correction is relatively simple. However, it is more desirable to have self-correcting codes; those that exponentially suppress errors due to both Hamiltonian perturbations and thermal noise, without needing external interaction. One could simply leave such a memory in a ‘power off’ state and expect it to remain in the meta-stable state for a time that scales exponentially with the size of the system. Such a system could form a ‘quantum hard drive’ which would enable high fidelity storage of quantum information, as well as forming a building block for fault-tolerant information processing and Hamiltonian simulation.

**Definition 2.** A quantum self-correcting stabilizer code is a topological quantum stabilizer code, with the additional property that the survival time of the quantum information in the presence of local thermal noise, without active error correction except at the final, read-out, phase, scales exponentially with \(N\).

While we cannot expect to store information for an arbitrarily long time, such a definition allows a beneficial scaling of the protection against errors with the size of the system. Our central thesis is that there are no quantum self-correcting stabilizer codes in two spatial dimensions. However, we have not proven this in the full generality, rather, our proof only applies to specified systems, where all the degeneracy is caused by products of stabilizers being identity. A trivial example where this is not the case would be defining an \(N\)-qubit Hamiltonian but considering \(N + 1\) qubits—this would be under-specified on the additional qubit.

The statement of our theorem is

**Theorem 1.** There are no quantum self-correcting stabilizer codes in two spatial dimensions in the case of specified Hamiltonians.

The proof of this will appear in Sec. IV D.

### II. PRELIMINARIES

In this section we illustrate the use of stabilizer systems for storing quantum information with two well-known examples, the 2D Ising model and the toric code. The 2D Ising model provides robust storage of classical information, while quantum information can be destroyed by a single local operation, whereas the toric code puts the logical \(X\) and \(Z\) rotations of a qubit on an equal footing, and we will discuss why the existence of these string-like operators implies that the model is not self correcting. This motivates our study, where we show that all stabilizer Hamiltonians in 2D behave like one of these two models, so there is no quantum self-correction.

#### A. Classical Self-Correction and the Ising Model

While we have defined self-correction and the Ising Model for storing quantum information, identical concepts exist in the classical case. In order to understand why systems such as the toric code are not self-correcting, it is instructive to examine the classical case, specifically the Ising model. In one dimension, the Ising model is a classical memory, but is unstable against thermal noise, whereas the Ising model in 2D is a self-correcting code for classical information. It is already known that the string-like properties of the toric code can be transformed into the 1D Ising model (in fact, two parallel copies) \([13,14]\).

The one-dimensional Ising model,

\[
H = -\frac{1}{2} \Delta \sum_i Z_i Z_{i+1}
\]

has degenerate ground states \(\left|00\ldots0\right\rangle\) and \(\left|11\ldots1\right\rangle\) which can encode a bit, and a gap to the first excited state of \(2\Delta\). A qubit cannot be reliably encoded because a single \(Z\) rotation has no energy cost, and performs a
logical $Z$-rotation on our bit, so there is no protection against this type of error. In contrast, to perform a bit-flip (logical $X$) one must apply the operator $X^\otimes N$. In the presence of noise, this can be realized with a sequence of local $X$ operations, the first of which costs an energy $2\Delta$ (which is recovered at the last step), with the remainder having no energy cost. Thus, while there is some protection afforded by the system against the first error, subsequent errors are not prevented, and one can arrive very quickly at a logical rotation.

In comparison, the two-dimensional Ising model on an $N \times N$ lattice has the Hamiltonian

$$H = -\frac{1}{2}\Delta \sum_{\langle i,j \rangle} Z_i Z_j,$$

where $\langle i,j \rangle$ denote neighbouring lattice sites. Like its one-dimensional counterpart, it has the degenerate ground states $|00\ldots0\rangle$ and $|11\ldots1\rangle$, so again only permits a classical code. However, in this case, a single bit flip, which costs $4\Delta$, is locked—to flip another neighbouring spin has a further energy cost of $2\Delta$. The energy cost of flipping a block grows with the surface area of the block, and so it becomes extremely difficult for an environment to perform a logical $X$ operation. These concepts have been made rigorous in a number of ways. For example, it was shown in [16] how perturbations on the Hamiltonian can propagate single errors into logical errors. For the 1D Ising model, the toric code, and indeed any stabilizer model with string-like logical operations, the time required is polynomial in the system size. A fixed density of errors can be converted into a logical gate operation in a time independent of the lattice size. On the other hand, it was also shown that such conversions in the case of the 2D Ising model require an exponentially long time, precisely because of the energy structure occurring due to the two-dimensional topology of the logical gate operation.

B. The Toric Code

The toric code, as depicted in Fig. 1, is described by a Hamiltonian

$$H = -\frac{1}{2}\Delta \left( \sum_{S} XXXX + \sum_{P} ZZZZ \right),$$

where the sums are taken over the plaquettes, $P$, (the sets of four qubits which surround a square) and sites, $S$, (the sets of four qubits which surround a corner). This model can be transformed into two parallel copies of the one-dimensional Ising model, and it is this transformation which indicates that the toric code is not self-correcting. Specifically, to perform operations within the ground state space of the Hamiltonian, strings of $X$ and $Z$ operations around the two topologically inequivalent loops of the torus are used. These are denoted $X_H$ and $X_V$ for the ‘horizontal’ and ‘vertical’ loops respectively.

One potentially useful property of the toric code is that if the Hamiltonian is affected by a sum of local perturbations, $V (\|V\| \ll 1)$, one has to consider $N$th order perturbation theory before the errors can possibly compose themselves into a string that affects ground state space. Since these are of strength $\|V\|^N$, a linear increase in the size of the lattice yields an exponential suppression in error.

However, if an environment is able to apply local rotations, the sequential flipping of the spins allows a logical rotation to be implemented. Such flips have the same energies as those of the bit-flips in the 1D Ising model, i.e. once the initial excitation has been created (with a single-qubit rotation), there is no further energy cost in extending the string.

For stabilizer codes, there is a concrete relation between the operations that convert between the degenerate ground states and the error pathways. Assuming that such an operation constitutes a string-like tensor product of Pauli operators around a loop, a truncation of this loop only has a finite energy because only stabilizers overlapping with the ends can possibly anti-commute with it. These string-like loops, if they exist, therefore describe a path through which noise destroys the stored information. This is exactly what is required for the application of the result in [16]. Other studies such as [17] reveal a similar result.
III. STABILIZER HAMILTONIANS AND DEGENERACIES

Consider a set of $R$ stabilizer operators $W_H = \{K_n\}$, $[K_n, K_m] = 0$, which are used to construct a Hamiltonian
\[ H = -\frac{1}{2} \Delta \sum_{K \in W_H} K \]
applied on an $N \times N$ square qubit lattice with periodic boundary conditions. The $\{K_n\}$ are tensor products of Pauli operators acting on systems of qubits, and hence have eigenvalues $\pm 1$. Since any tensor product of Pauli operators has trace 0 (except the trivial case of $\mathbb{I}$), exactly half of the eigenvalues are 1 and half are $-1$. We shall take each of the $K_n$ to be non-identity on at most a $k \times k$ block of spins on the 2D lattice, where $k$ is some fixed number, and $N \geq 2k$. In general, there may be values of $n$ for which $K_n$ acts on a smaller area.

It is convenient to place the stabilizers in sets, such that the product of all stabilizers in each set is $\mathbb{I}$. The combination of two such sets generates a third set which also product to $\mathbb{I}$, where sets are combined under the operation $(G_i, G_j) \equiv G_i \cup G_j \setminus (G_i \cap G_j)$, with $(G_i, G_j, G_k) \equiv (G_i, (G_j, G_k))$ and so on. This motivates the following definitions.

**Definition 3.** Let $\{G_i\}$ be identity sets and denote $G$ as the set of all such sets. Then, for all $G_i \in G$,
\[ \prod_{K \in G_i} K = \mathbb{I}. \]  

**Definition 4.** Let $G'_i$ be a minimal identity set and denote $G'$ as the set of all such sets. Then, for all $G'_i \in G'$,
\[ \prod_{K \in G'_i} K = \mathbb{I}. \]
holds, but is not implied by $\{\prod_{K \in G'_{i,z}} K = \mathbb{I}\}$.

Note that the identity sets can be formed from the minimal set of identity sets by combining them under the operation $(G_i, G_j)$. The number of combinations is
\[ |G| = \sum_{i=1}^{|G'|} |G'| C_i = 2^{|G'|} - 1. \]

**Lemma 1.** Consider a Hamiltonian, $H$, composed of $R$ stabilizers defined on an $N \times N$ square lattice of qubits. Such a Hamiltonian has a ground state degeneracy of $2^M$ levels, where
\[ 2^M = 2^{N^2-R}(1 + |G|). \]

**Proof.** The ground state space of $H$ is given by projection onto the +1 eigenstate of each of the stabilizers,
\[ \rho = \frac{\prod_{i=1}^R (\mathbb{I} + K_n)}{\text{Tr} (\prod_{i=1}^R (\mathbb{I} + K_n))} \]
and the degeneracy is given by $2^M = \text{rank}(\rho)$. Alternatively, we can calculate the degeneracy with
\[ 2^M = \frac{1}{2^R} \text{Tr} \left( \prod_{n} (\mathbb{I} + K_n) \right) \]
\[ = \frac{1}{2^R} \text{Tr} \left( \mathbb{I} + \sum_{i} K_i + \sum_{i,j>i} K_i K_j + \ldots \right). \]
The expression within the trace is composed of ordered products of the stabilizers, and the trace is hence equal to the number of times the stabilizers product to $\mathbb{I}$, multiplied by a factor of $\text{Tr}(\mathbb{I}) = 2^N$. All sets that product to identity contribute to the sum, and there is an additional contribution from the first term in the trace in (4), hence
\[ 2^M = 2^{N^2-R}(1 + |G|). \]

Note that there are two possible causes of degeneracy: the first is that $R < N^2$, which means that there are insufficient stabilizers to break all the potential degeneracies. The second is that $|G| > 0$, i.e. products of stabilizers are identity. We show in this paper how to break degeneracies of the second type. A system is said to be specified if it only has degeneracies of this type.

**Definition 5.** Two operators, $S_1$ and $S_2$ are said to be independent with respect to a set of stabilizers $G_i$ if there is no set $W \subseteq G_i$ for which $S_1 S_2 = \prod_{K \in W} K$, i.e. if they are not related by a product of stabilizers in $G_i$. Otherwise, they are dependent with respect to $G_i$. If no set $G_i$ is indicated, the entire set of operators is intended.

For the sake of the general proof, it is helpful to define a subset of the minimal identity sets $G'$, which we refer to as the elementary sets, $\tilde{G}$. These are closely related to the concept of dependence of operators.

**Definition 6.** Consider a subset of the $R$ stabilizers, $W_R$, where $|W_R| = N^2 - M$, such that these stabilizers contain all the information of the Hamiltonian (i.e. $W_R$ does not contain any identity sets, and thus all stabilizers not in $W_R$ can be generated by products of members of $W_R$). The elementary sets $\tilde{G}_i$ are the minimal identity sets formed by adding back $|\tilde{G}_i| = M$ stabilizers such that each $\tilde{G}_i$ corresponds to a degeneracy.

**Definition 7.** We say an identity set $G$ is topologically trivial in the vertical direction if there exists a row, $t$, such that after removing all stabilizers which have support both above and in row $t$ from $G$, the remaining stabilizers still product to identity in and below row $t$. Otherwise, the set $G$ is topologically non-trivial \footnote{In the case of open boundary conditions, the definition of an}.
IV. CONSTRUCTING STRING OPERATORS FOR STABILIZER CODES

A. Proof Sketch

In order to show that a stabilizer Hamiltonian cannot store quantum bits in a self-correcting way, we show that there exists a product of local operators which converts between each of the ground states, the spatial patternings of which forms a one-dimensional structure. Such an operator is called a string operator. It is this one dimensional nature that prevents the exponential suppression of errors. Since the entire operator commutes with all the stabilizers, if one considers building it up by applying the local rotations sequentially, only the ends of the sequence anti-commute with terms in the Hamiltonian and hence the size of these bound the energy of the par-ticular string anti-commutes with terms in the Hamiltonian, and hence are local.

Once we have established that there are loop operators arising from each of the sets in $G$, we prove that they are independent i.e. that error pathways from two different loops have different effects on the degenerate space, so we are guaranteed that all the degeneracies are broken.

Before embarking on the complete proof, whose essence may be obscured by technicalities, we present the construction for the case of a translationally invariant system where the stabilizers act on $3 \times 3$ blocks of qubits.

B. An Example: $3 \times 3$ Translationally Invariant Codes

The simple case of $k = 3$, where all the stabilizers are the same (just displaced to different sites), removes many of the technicalities required for the general proof and, as such, provides a useful example that is entirely consistent with what will be presented later. Due to the translational invariance, it must be true that the product of all the stabilizers is equal to $I$, in order for there to be some degeneracy present. From now on, we assume this to be the case.

For each lattice site, there are 9 stabilizers which overlap with it, one for each term in the stabilizer. Since the product of stabilizers on this site is $I$, we conclude that the product of all operators in the stabilizer must be $I$. Let us represent the stabilizers as follows:

$$
\begin{align*}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{align*}
$$

where $\{C_{ij}\}$ are Pauli operators. We denote products of all the operators in a particular row of a stabilizer as $R_i := \prod_j C_{ij}$. Due to local unitary equivalence, without loss of generality, we take the first non-$I$ operator to be $X$. Since $R_1R_2R_3 = I$, there are only a limited set of possibilities that we have to consider:

$$(R_1, R_2, R_3) \in \{ (I, I, I), (X, I, X), (X, X, I), (X, Y, Z) \}.$$  

(7)

Necessary conditions for $[R_i, R_j]$ to be a stabilizer (i.e. for it to commute with itself in different positions on the lattice) are that

$$[R_1, R_3] = 0 \quad \text{and} \quad [R_1 \otimes R_2, R_2 \otimes R_3] = 0.$$  

(8)

We proceed to prove $[R_1, R_3] = 0$, since this is the only condition that we require. In order for the operator defined by (6) to be a stabilizer, it must satisfy the following commutation relations,

$$[C_{13} \otimes I \otimes I, C_{31} \otimes C_{32} \otimes C_{33}] = 0,$$

$$[C_{12} \otimes C_{13} \otimes I, C_{31} \otimes C_{32} \otimes C_{33}] = 0,$$

$$[C_{11} \otimes C_{12} \otimes C_{13}, C_{31} \otimes C_{32} \otimes C_{33}] = 0,$$

$$[I \otimes C_{11} \otimes C_{12}, C_{31} \otimes C_{32} \otimes C_{33}] = 0,$$

$$[I \otimes I \otimes C_{11}, C_{31} \otimes C_{32} \otimes C_{33}] = 0,$$

where we have retained unnecessary terms so that the following step is clearer. Now we use the fact that if

Identity set must be changed, specifically in the case of topolog-ically non-trivial identity sets. By assuming periodic boundary conditions, while allowing arbitrary spatial variation of the stabilizers, we are assuming that this situation does not arise.
case has many degeneracies since a single row of stabilizers only extend over 3 qubits, the height of the row, column operators constructed for the row of stabilizers deal with this case, one can first analyse the analogous condition for the operator to be a stabilizer.

For Pauli operators \( \{ P \} \) and \( P \),

\[
\prod_{i=1}^{m} P_i, P = 0 \iff [P_1 \otimes \cdots \otimes P_m, P = \cdots \otimes P] = 0. \tag{10}
\]

To see this, note that the LHS and RHS both imply that an even number of \( \{ P \} \) anticommute with \( P \). Hence, applying this to \([9]\) shows that \([R_1, R_3] = 0\) is a necessary condition for the operator to be a stabilizer.

This eliminates the \((X, Y, Z)\) possibility from Eqn. \([9]\). Now consider the cases \((X, X, \mathbb{1})\) and \((X, X, \mathbb{1})\) in Eqn. \([7]\). One finds that the string operators

\[
\cdots R_1 R_1 R_1 R_1 R_1 R_2 R_2 R_1 R_1 R_1 R_2 R_2 \cdots
\]

commute with the stabilizers, i.e., we get either two parallel rows of \(X\)s or a single row of \(X\)s. One can verify that in both cases, a single row of \(X\)s commutes with the stabilizers by using \([9]\). In fact, the case of \((R_1, R_2, R_3) = (X, \mathbb{1}, X)\) has degeneracy of at least \(2^M = 4\), since we can take the product of all stabilizers on every second row and get \(\mathbb{1}\). As will become clear in the general proof, this accounts for the difference between taking a single row of \(X\)s and two rows; a single row of \(X\)s on row 1 compared to a single row of \(X\)s on row 2 will implement a different logical operation on the logical qubits.

The remaining case is \((R_1, R_2, R_3) = (\mathbb{1}, \mathbb{1}, \mathbb{1})\). This case has many degeneracies since a single row of stabilizers products to \(\mathbb{1}\), and there are \(N\) such rows. To deal with this case, one can first analyse the analogous column operators constructed for the row of stabilizers that product to identity. An identical construction can be performed on these, except that the resulting operators only extend over 3 qubits, the height of the row, rather than taking the form of loops. Such operators are readily implemented by local noise. However, this construction also fails if the columns product to \((\mathbb{1}, \mathbb{1}, \mathbb{1})\) as well. In such a case, note that each column of the stabilizer must be either \((\mathbb{1}, \mathbb{1}, \mathbb{1})\), \((X, X, \mathbb{1})\) or \((X, Y, Z)\) (up to permutations). Any column operator \(X^{\otimes 3}, Y^{\otimes 3}\) or \(Z^{\otimes 3}\) commutes with all of these, and therefore commutes with all the stabilizers. Such an operator splits the ground state degeneracy, and is a fixed size, hence the system is again susceptible to local noise and cannot give any useful protection.

\[ A, C = 0 \text{ and } B, C = 0 \text{, then } [AB, C] = 0 \text{, combining all 5 equations to give } \]

\[ [R_1 \otimes R_1 \otimes R_1, C_{31} \otimes C_{32} \otimes C_{33}] = 0. \tag{9} \]

For Pauli operators \( \{ P \} \) and \( P \),

\[
\prod_{i=1}^{m} P_i, P = 0 \iff [P_1 \otimes \cdots \otimes P_m, P \otimes \cdots \otimes P] = 0. \tag{10}
\]

Lemma 2. An operator, \(S_i\), which satisfies \([S_i, K_n] = 0\) for all \(n\), performs a non-trivial action within the ground state space of \(H\) provided that \(S_i\) is independent of \(\mathbb{1}\).

Proof. Consider the projection of \(\rho\) (as defined by \([9]\)) onto the +1 eigenspace of \(S_i\),

\[
\rho' = \frac{1}{2} (\mathbb{1} + S_i)\rho.
\]

If there exists a set \(W\) such that \(S_i = \prod_{K \in W} K\), then we already know that \(S_i\) must have eigenvalue +1 when acting on \(\rho\), hence \(S_i\) has trivial action. Conversely, if there is no set \(W\), then \(\text{Tr}(\rho') = \frac{1}{2} \text{Tr}(\prod_{j}(\mathbb{1} + K_j)) = 2^{M-1}\), so the ground state space is split in half, i.e. within the space defined by \(\rho\), there are states \(\langle \psi_{\pm} \rangle \) which satisfy \(S_i|\psi_{\pm}\rangle = \pm|\psi_{\pm}\rangle\) so \(S_i\) acts like a logical phase gate on this space. (Alternatively, we could rewrite it to appear as a logical \(X\) rotation, using basis states \((|\psi_{(+)}\rangle \pm |\psi_{(-)}\rangle)/\sqrt{2}\).)

Lemma 3. Two dependent operators \(S_i\) and \(S_j\) have the same action on the ground states.

Proof. The state \(\rho'\) has eigenvalues +1 for all \(K\) and \(S_i\). Given that \(S_j\) is dependent on these, there exists a \(W\) such that \(S_j = S_i \prod_{K \in W} K\), and hence the value of \(S_j\) when acting on \(\rho'\) is +1, and the action of \(S_j\) is trivial i.e. it had the same effect as \(S_i\).

Suppose that we find a set of \(\tilde{G}\) independent string operators, \(\{ S_i \} \), obeying \([S_i, K_n] = 0\) for all \(i, n\) and \([S_i, S_j] = 0\) for all \(i, j\). We can repeat the argument in the lemma for each member of the set, so, the rank of

\[
\rho'' = \prod_{i} (\mathbb{1} + S_i) \prod_{j} (\mathbb{1} + K_j) \tag{11}
\]

is 1, and hence no quantum bits can be stored in the degenerate ground space of \(H\). (This argument does not rule out storing up to \(\tilde{G}\) classical bits.)
D. General Construction of String Operators

In this section, we will describe how to construct two operators $S_i^H$ and $S_i^V$, for each identity set of stabilizers, $G_i \in G$. We show that, for each $G_i$, either

1. at least one of the operators performs a non-trivial operation on the ground states, or
2. the set of stabilizers $G_i$ is defined over an area of fixed size, and cannot provide any protection against noise.

We first introduce some notation that will be used throughout this section. Let us use $\bigwedge_A K$ to denote the restriction of the operator $K$ to a particular area of the lattice $A$, replacing all terms outside this area with $\mathbb{I}$. So, for example, if the area $A$ is just a single site, $\bigwedge_A K$ is just the Pauli operator of $K$ that acts on that site.

**Definition 9.** A set of stabilizers, $G_i^A$, is a subset of $G_i$ for which the elements $K_j$ satisfy

$$\bigwedge_A K_j = \mathbb{I},$$

where $\bar{A}$ is the entire lattice not contained within the area $A$. In other words, the action of operators in $G_i^A$ is entirely within $A$.

We use $L_i^l$ to denote the area corresponding to a horizontal band of height $j$, whose top edge coincides with row $l$ (according to a numbering system of the rows with numbers increasing going down, and counting is performed modulo $N$ to account for the periodic boundary conditions). Our constructions will involve horizontal strips of spins of height $k - 1$, across the whole width of the lattice i.e. $L_{k-1}$. To simplify the notation, we denote $L_{k-1} = L, L_{2k-2} = L_1$ and $L_{2k-2} = L_1$. $\bigwedge_{L_1}$ selects all operators that are either entirely contained within $L$, or extend below $L$, still overlapping with it. By taking the height to be $k - 1$, there is never a stabilizer which extends both above and below $L$ (assuming a suitably large lattice size). There are equivalent terms involving vertical bands, but since horizontal and vertical are arbitrary labels, constructions for both work identically, although, importantly, they can give rise to topologically inequivalent loops.

If we apply $\bigwedge_{L_i^m}$ to Eqn. (1), we obtain the identity

$$\bigwedge_{L_i^m} \left( \prod_{K \in G_i^l_m} K \right) \left( \prod_{K \in G_i^{l+k-1}_m} K \right) \bigwedge_{L_i^m} \left( \prod_{K \in G_i^{l+k-1}_m} K \right) = \mathbb{I}, \quad (12)$$

which simply states that if we take the product of all stabilizers that are not $\mathbb{I}$ on a particular area, then the overall product must be $\mathbb{I}$ so that Eqn. (1) is satisfied. We have chosen to state this for the area $L_i^m$, as this will be most useful to us. There is a more general form of the identity,

$$\bigwedge_{L_i^m} \left( \prod_{K \in G_i^{l+n+k-1}_m} K \right) \bigwedge_{L_i^m} \left( \prod_{K \in G_i^{l+n+k-1}_m} K \right) \bigwedge_{L_i^m} \left( \prod_{K \in G_i^{l+n+k-1}_m} K \right) = \mathbb{I}, \quad (13)$$

which holds for any integers $n < l + k + m - 1, r > n-l+k-1$. Identity (12) is the case $n = l, r = m+k-1$. It is worth noting that the strips $L_i$ for the first two terms have the same bottom edge, and that the last two terms have the same top edge. Also, the top edge of the first term and the bottom edge of the last term are such that all possible operators that hit the strip $L_i^m$ are contained within it.

We proceed to prove several important properties of the string operators, that will be needed in the proof of Theorem 1. The following identities are also useful:

$$K \left( \bigwedge_A K \right) = \bigwedge_A K \quad (14)$$

and

$$\bigwedge_A \left( K_1, K_2 \right) = \bigwedge_A \left[ K_1, K_2 \right] \bigwedge_A K_2, \quad (15)$$

where the latter identity follows from

$$\bigwedge_A \left( K_1 K_2 \right) = \bigwedge_A K_1 \bigwedge_A K_2,$$

and the definition of the commutator.

**Lemma 4.** The operator $S_i^H$, defined by

$$S_i^H := \bigwedge_L \left( \prod_{K \in G_i^l} K \right), \quad (16)$$
satisﬁes \([S_i^H, K_n] = 0\) for all \(i, n\).

Proof. Identity (15) gives

\[
[S_i^H, K_n] = \bigwedge_L \left[ \prod_{K \in G_i^L} K, K_n \right] \bigwedge K_n,
\]

which equals zero because all the stabilizers commute.

Thus, for each set, \(G_i\), we have found an operator that commutes with all the stabilizers. As we will show later, it is also independent of \(I\) and hence has non-trivial action on the ground states (cf. Lemma 2). Since \(L\) is defined with a speciﬁc upper row, \(l\), one might apply this construction for each \(l\) (with ﬁxed \(G_i\)) to generate further operators. However, such operators will not be independent as the following lemma shows.

Lemma 5. Consider the operator \(S_i^l\), a string operator \(S_i^H\) where the corresponding area \(L\) has its top edge on row \(l\). Then, for any \(l, l' \neq l\), \(S_i^l\) and \(S_i^{l'}\) are dependent. Further, the set under which they are dependent with respect to is \(G_i\).

Proof. It sufﬁces to show that \(S_i^l\) and \(S_i^{l-1}\) are dependent; the general relation then follows by induction. We have

\[
S_i^l S_i^{l-1} = \bigwedge_{L_{k-1}^{l-1}} \left( \prod_{K \in G_i^{L_{k-1}^{l-1}}} K \right) \bigwedge \left( \prod_{K \in G_i^{L_{k+1}^{l-1}}} K \right) \bigwedge \left( \prod_{K \in G_i^{L_k^{l-1}}} K \right). 
\]

Using (14) for \(K \in G_i^L\) gives \(\bigwedge_{L_{k-1}^{l-1}} K = K \bigwedge_{L_{k+1}^{l-2}} K\) etc., hence

\[
S_i^l S_i^{l-1} = \left( \prod_{K \in G_i^{L_{k-1}^{l-1}}} K \right) \left( \prod_{K \in G_i^{L_{k+1}^{l-1}}} K \right) \bigwedge \left( \prod_{K \in G_i^{L_k^{l-1}}} K \right)
\]

where the ﬁnal line follows from (13).

Thus, the operators \(S_i^l\) and \(S_i^{l'}\) \(l' \neq l\) are dependent and hence Lemma 5 implies that they have the same action on the ground states. Similarly, all column operators for a particular \(G_i\) are related by products of stabilizers. Provided the row and column operators implement closed loops around the boundaries of the lattice (i.e. around inequivalent loops of a torus), they are topologically inequivalent, and therefore cannot be related by products of stabilizers.

Lemma 6. If the operator \(S_i^H\) formed from a set \(G_i\) is dependent on \(I\), then it can be written as a product of stabilizers entirely contained within \(L\).

Proof. Suppose that \(S_i^H\) is dependent on \(I\), i.e. that it can be written in the form \(\prod_{K \in W} K\) for some set \(W\). In order that \(S_i^H\) be \(I\) on \(L\), \(W\) must take all the members of sets \(G_k \setminus G_k^L\), for some \(k\) or none of them. To see that this is a necessary condition, note that \(\prod_{K_j \in U} K_j = I\), with \(U^L = \emptyset\) implies \(U = G_k\) for some \(k\). Furthermore, since \(\prod_{K \in G_k^L} K = \prod_{K \in G_k \setminus G_k^L} K\) for all \(k\), we can always redefine \(W\) as \(W'\) such that \(W' = W''\).

Lemma 7. An elementary set \(\tilde{G}_i\) that is topologically trivial horizontally (vertically) can be replaced with an elementary set consisting of stabilizers deﬁned within a region of width \(k\) (height \(k\)).

Proof. Lemma 5 tells us that the operators formed by using our construction on the same identity set, but starting in different places are dependent. Since the set \(\tilde{G}_i\) is topologically trivial, we can deﬁne a top row, the \(t\)th row, the highest row which contains a stabilizer. Consider forming the horizontal string operator starting at the top (i.e. forming \(S_i^t\) in the notation of Lemma 3). We have \(S_i^t = \bigwedge_{L_{k-1}^{t-1}} (\prod_{K \in \tilde{G}_i} K) = I\), since when choosing this top row, we can replace \(\tilde{G}_i^L\) by \(\tilde{G}_i\). Hence \(S_i^H\) must be dependent on \(I\), for an arbitrary starting row.

Consider then constructing the horizontal string operator \(S_i^{t-1}\). This operator is dependent on identity, so is
FIG. 2: To construct $S_i$ from $k \times k$ stabilizers, we consider a strip $L$ of height $k - 1$ (the solid black lines). In the depicted example, $k = 5$. Within this, we take all stabilizers that extend below the bottom line (these are the members of $L_1$), but truncate them to only include the parts in the area (so we remove the hatched components, acting $\bigwedge$ on the stabilizers). In this figure, we only depict one column of stabilizers, and offset them horizontally for clarity. If we were to construct the $S_i$ on one row higher, then by taking their product, we can see that they are related by the product of stabilizers. The final equality only holds when we consider all stabilizers along the strip, not just a single column.

a product of stabilizers. Furthermore, it is entirely contained in a strip of height $k - 1$. From the argument in Lemma 8, we can define it in terms of a set of stabilizers contained in this strip. We write $S_{t+1}^i = \prod_{K \in W} K$, where $W$ is a set of stabilizers entirely within the strip.

Now, consider the stabilizers from $G_i$ whose top row is $t$. Denote this set by $T$. We can write $S_{t+1}^i = \bigwedge_{L_{t+1}} \left( \prod_{K \in \bar{G}_i \setminus T} K \right)$ (recall that $\bigwedge_{L_{t+1}}$ simply removes the area below row $t + k$). We have $\prod_{K \in \bar{G}_i \setminus T} K \prod_{K \in T} K = \mathbb{1}$. Thus,

$$
\bigwedge_{L_{t+1}} \left( \prod_{K \in \bar{G}_i \setminus T} K \right) \bigwedge_{L_{t+1}} \left( \prod_{K \in T} K \right) = \mathbb{1}.
$$

However, $\bigwedge_{L_{t+1}} \left( \prod_{K \in T} K \right) = \prod_{K \in T} K$, hence $\prod_{K \in W} K \prod_{K \in T} K = \mathbb{1}$. We have thus found a new identity set contained entirely within a strip of height $k$.

**Corollary 1.** An elementary set $\bar{G}_i$ that is topologically trivial in both directions can be replaced with an elementary set consisting of stabilizers defined within a $k \times k$ area.

We have hence shown that the elementary sets can be chosen such that each is either local or topologically non-trivial. It remains to prove that for the subsets $\bar{G}_i$, the operators break the corresponding degeneracy and are independent of each other.

First we argue that for a local elementary set, $\bar{G}_i$, there exists a local operator that will break the degeneracy. To see this, consider a $k \times k$ lattice with a Hamiltonian containing only the stabilizers in $\bar{G}_i$. There are $M_k = 2^{k^2 - |\bar{G}_i| + 1}$ groundstates, hence $M_k$ commuting operators on this $k \times k$ region breaking these degeneracies. If one then expands the lattice to be $N \times N$ (without adding any new stabilizers), then this same set of localized operators splits the degeneracy. Now consider adding back the other stabilizers. They break many of the $M_k$ degeneracies, but by assumption, one remains. However, all of the operators breaking such a degeneracy are local.

It is therefore impossible to get self-correction by encoding information in ground states whose degeneracy is caused by a local elementary set.

**Lemma 8.** The operator $S_i^H$ formed from a set $G_i$ that is composed of only topologically non-trivial elementary sets in the vertical direction is independent of $\mathbb{1}$.

**Proof.** Consider a set of stabilizers $W_0$ that is the union of $W_R$ and $G_i$, so that the only identity sets that are present are due to $G_i$. Now assume that, contrary to the lemma, $S_i^H = \prod_{K \in W'} K$. From Lemma 8, we can take $W_1$ such that $W_1 = W_0'$ This may require the enlargement of set $W_0$, which could introduce new identity sets. However, any identity sets that are introduced cannot be restricted to the area $L$ (otherwise it would not be necessary to introduce the set), and hence must be topologically non-trivial in the vertical direction. Now, recall from Lemma 5 that $S_i^{l-1} S_i^l = \prod_{K \in W'} K$ where $W' \subseteq G_i$. We have

$$
\prod_{K \in W_1} K \prod_{K \in W_{l-1}} K \prod_{K \in W'} K = \mathbb{1}.
$$

So, if $(W_i, W_{l-1}) \not\subseteq G_i$, we have formed a new identity set from members of $W_i, W_{l-1}$ and $W'$, in contradiction with the assumption.

The remaining possibility is that $(W_i, W_{l-1}) \subseteq G_i$. One way to satisfy this is if $W'$ has members not in $(W_i, W_{l-1})$. Then we have formed an identity set which is topologically trivial vertically, a contradiction. Alternatively, all members of $W'$ are also members of $(W_i, W_{l-1})$, so do not have height more than $k - 1$, instead of the $k$ that we were assuming. Thus, one reapplies the construction of $S_i^H$ such that it has height $k - 2$. Either one of the previous cases occurs, and thus the lemma holds, or
we conclude that the stabilizers must have height \(k - 2\). Hence, we can continue making this same argument until we either conclude that the lemma holds, or the height of the stabilizers is 0, i.e. \(G_i = \emptyset\).

We state a corollary of this lemma which follows because the product \(S_i^H S_j^H\) is equal to the corresponding operator of the combined set \((\tilde{G}_i, \tilde{G}_j)\).

**Corollary 2.** For any pair of elementary sets, \(\tilde{G}_i\) and \(\tilde{G}_j\), which are topologically non-trivial in the vertical direction, the corresponding operators, \(S_i^H\) and \(S_j^H\) are independent.

Note that if \(\tilde{G}_i\) is topologically non-trivial vertically, but topologically trivial horizontally, then (cf. Lemma 8) one could redefine \(\tilde{G}_i\) such that it contained stabilizers in a strip of width \(k\). The \(S_i^H\) formed from this set will therefore be local. It follows that we only generate a loop operator for elementary sets which are topologically non-trivial in both directions.

**Lemma 9.** Let \(\tilde{G}_i\) be an elementary set which is topologically non-trivial in the vertical direction, but topologically trivial horizontally, and \(\tilde{G}_j\) be an elementary set which is topologically non-trivial in the horizontal direction, but topologically trivial vertically. The corresponding operators, \(S_i^H\) and \(S_j^V\), formed from these sets are independent.

**Proof.** Since \(\tilde{G}_i\) and \(\tilde{G}_j\) are topologically trivial in one direction, we can use Lemma 7 to redefine them as having a strip of width, respectively height, \(k\). The sets then have overlap in a \(k \times k\) area. We choose this area to form \(S_i^H\) and \(S_j^V\). Moreover, the operators \(S_i^H S_j^V\) and \(S_i^H S_j^V\) are also restricted to this \(k \times k\) block (with upper-left corner \((m-1, l-1)\)). We can now make an argument that closely parallels that of Lemma 8.

As in Lemma 8 we form a set \(W_0\), here as the union of \(W_R, \tilde{G}_i\) and \(\tilde{G}_j\). Assume that \(S_i^H\) and \(S_j^V\) are dependent, i.e. \(S_i^H S_j^V\) = \(\prod_{K \in W_{1-m}} K\), where each of the members of \(W_{l,m}\) must have either height or width \(k - 1\). As before, we must have

\[
\prod_{K \in W_{l,m}} K \prod_{K \in W_{l-1,m}} K \prod_{K \in W_{l,m-1}} K \prod_{K \in W'} K = 1. \tag{18}
\]

where \(W' \subseteq (\tilde{G}_i, \tilde{G}_j)\). If \((W_{l,m}, W_{l-1,m}, W_{l,m-1})\) contains elements not in \((\tilde{G}_i, \tilde{G}_j)\), then there exists a topologically trivial identity set, a contradiction. The same holds if \(W'\) contains elements not in \((W_{l,m}, W_{l-1,m}, W_{l,m-1})\), which means that no members of \(G_i\) can have height \(k\), and no members if \(G_j\) can have width \(k\), unless \(\tilde{G}_i\) and \(\tilde{G}_j\) contain the same terms of size \(k \times k\). It is not possible for \(\tilde{G}_i\) and \(\tilde{G}_j\) to contain the same \(k \times k\) term, since, were we to remove it, it would break both identity sets, which is in contradiction with the definition of elementary sets. Thus we can repeat this construction for strips of dimension \(k - 2\), and the argument recurses (in the same way as in Lemma 8) until we conclude that the lemma holds, or the sets are null.

We are now ready to prove our main theorem.

**Proof of Theorem.** For every elementary set \(\tilde{G}_i\), there are two possible forms for the logical operations. If \(\tilde{G}_i\) is topologically trivial, then there is a local operator breaking the relevant degeneracy. Any data encoded in ground states that are degenerate due to such a set is not protected against thermal noise or Hamiltonian perturbations (due to its fixed size).

If \(\tilde{G}_i\) is topologically non-trivial vertically, then we can generate a string operator \(S_i^H\) breaking the degeneracy, and likewise if it is topologically non-trivial horizontally, then we can generate \(S_i^V\) breaking the degeneracy. It follows from Corollary 2 and Lemma 9 that for different elementary sets, these are independent. We have therefore found a way to break all the degeneracies caused by products of stabilizers being identity.

To complete the proof of Theorem, it remains to show that any string operators \(S_i^H\) and \(S_i^V\) provide error paths for thermal noise. If the operators are not point-like, then they take the form of a loop of fixed width around the torus. If we truncate such a loop, then only a finite number \((O(k^2))\) of stabilizers do not commute with it, so there is an approximately constant energy cost as the amount of truncation is varied. The ends of the truncated string thus behave precisely like the strings in the toric code. Hence, the survival time against thermal noise is not exponential, and these codes are not self-correcting.

It may also serve as a useful side observation that the truncations of our string operators give the excited states of the Hamiltonian, describing pairs of anyons. Since the strings are tensor products of Pauli operators, two such strings either commute or anti-commute, and Abelian braiding properties of the anyons can be realised, which are identical to those of the toric code, except with up to \(|G|\) different particle types, and subsequent composite particles.

**V. HIGHER SPATIAL DIMENSIONS**

While our constructions have been formed specifically for 2D square lattices, they work equally well on arbitrary 2D geometries. We can also choose to apply an identical construction in \(d\) dimensional systems to find the structure of one logical operation per qubit. The proof proceeds as before, replacing the loop \(L\) with a \((d - 1)\)-dimensional object with height \(k - 1\) in the other dimension.

With \(d = 3\), for example, we arrive at an area operator, which provides protection for classical data. For each elementary set, we take one such operator, all of which act in the same plane, to define the \(Z\)-basis of our
qubits. What happens for the other logical gate operation? Clearly, to generate the $SU(2)$ algebra, we would need an operation that anticommutes with the plane (we can shift the plane to arbitrary positions). Thus, the only logical structure is a string-like object, although we have no explicit construction for its form. This argument suggests that self-correction is also impossible for 3D stabilizer Hamiltonians.

On the other hand, in $d = 4$, the same argument does not apply. This is because although our construction can give a 3D object, if the stabilizers in an identity set only span 3 dimensions themselves, the logical operation is two-dimensional, and an anti-commuting 2-dimensional term can also be found, giving protection to a whole qubit, which is identical to the classical protection afforded by the 2D Ising model. This is the case in the 4D toric code.

VI. CONCLUSIONS

In this paper, we have shown that for all qubit stabilizer Hamiltonians in 2D with periodic boundary conditions (subject to the condition that the Hamiltonian is specified), there are always string operators that loop around the torus, and perform non-trivial actions on the ground state space. There are always enough of these to ensure that there is no non-trivial subspace of the ground state space which is not affected, and hence there is no possibility of storing quantum data that cannot be affected by such an operator. Furthermore, we give an explicit method for constructing the operators that give the paths for noise. Given the existence of these, we are led to conclude that none of these systems are self-correcting, in the same way that the toric code is not self-correcting. In addition, we suggest that a similar construction might work for stabilizer Hamiltonians in 3D.

There are several potential routes for further investigation. It would be interesting to understand if this proof for stabilizer Hamiltonians can be applied to non-stabilizer Hamiltonians. In particular, many Hamiltonians are in some sense ‘close’ to a stabilizer Hamiltonian, so it might seem surprising if these could exhibit self-correction. However, there are radically different Hamiltonians with, for example, chiral terms [30], which stabilizer codes cannot encapsulate, and clearly it would be interesting to investigate if these can be self-correcting.

Additional Note: We recently became aware of independent work of a similar nature [31].

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