ENDOMORPHISM ALGEBRAS OF 2-TERM SILTING COMPLEXES

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Abstract. We study possible values of the global dimension of endomorphism algebras of 2-term silting complexes. We show that for any algebra $A$ whose global dimension $\text{gl.dim} A \leq 2$ and any 2-term silting complex $P$ in the bounded derived category $D^b(A)$ of $A$, the global dimension of $\text{End}_{D^b(A)}(P)$ is at most 7. We also show that for each $n > 2$, there is an algebra $A$ with $\text{gl.dim} A = n$ such that $D^b(A)$ admits a 2-term silting complex $P$ with $\text{gl.dim} \text{End}_{D^b(A)}(P)$ infinite.

Introduction

Let $A$ be a finite dimensional algebra over a field $k$. Let $T$ be a (classical) tilting module in the category $\text{mod} A$ of finite dimensional right $A$-modules; that is the projective dimension $\text{pd} T$ is at most 1, we have $\text{Ext}^1_A(T, T) = 0$ and there is an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with $T_1, T_2$ in $\text{add} T$, the additive closure of $T$. Let $B = \text{End}_A(T)$. Then, it is a well-known fact (see for example [8, III, Section 3.4] for a more general statement) that $\text{gl.dim} B \leq \text{gl.dim} A + 1$, where $\text{gl.dim} A$ denotes the global dimension of $A$.

In this paper we investigate to which extent this generalizes to the following setting. We now consider a 2-term silting complex $P$ in the bounded homotopy category of finitely generated projective $A$-modules, $K^b(\text{proj} A)$. This is just a map between projective $A$-modules, considered as a complex, with the property that $\text{Hom}_{K^b(\text{proj} A)}(P, P[1]) = 0$ where $[1]$ denotes the shift functor, and such that $P$ generates $K^b(\text{proj} A)$ as a triangulated category. Note that $K^b(\text{proj} A)$ can be considered to be a full triangulated subcategory of the derived category $D^b(A)$.

The concept of silting complexes originated from [11], and has more recently been studied by many authors, often motivated by combinatorial aspects related to mutations, as in [2]. Moreover, the case of 2-term silting is of particular interest, see e.g. [1], [4] and [12].

In the setting of 2-term silting, we have the following theorem:

Theorem 0.1. Let $B = \text{End}_{D^b(A)}(P)$, for a 2-term silting complex $P$ in $K^b(\text{proj} A)$. Then the following hold.

(a) If $\text{gl.dim} A = 1$, then $\text{gl.dim} B \leq 3$.
(b) If $\text{gl.dim} A = 2$, then $\text{gl.dim} B \leq 7$.

Moreover, for each $n > 2$, there is an algebra $A$, with $\text{gl.dim} A = n$, such that $K^b(\text{proj} A)$ admits a 2-term silting complex $P$ with $\text{gl.dim} \text{End}_{D^b(A)}(P) = \infty$.

Note that the projective presentation of a tilting $A$-module $T$ as defined above, gives rise to a 2-term silting complex $P_T$ in $K^b(\text{proj} A)$, and that we have an isomorphism of algebras $\text{End}_A(T) \cong \text{End}_{D^b(A)}(P_T)$.

The situation in part (a) was studied in [6]. In this case $B$ is a called a silted algebra, and it was proved that silted algebras are so-called shod algebras [7], in particular this implies that $\text{gl.dim} B \leq 3$, by [9].

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The main body of this paper is a proof of (b), an example that the global dimension of $B$ actually can be 7 in this case, and a class of examples that justifies the last statement of Theorem 0.1.

We also prove that with a stronger assumption on $P$, we actually get that $\text{gl.dim} B$ is bounded by $\text{gl.dim} A$. More precisely, we show the following.

**Theorem 0.2.** With the above notation, and assuming in addition that $\text{pd} H^0(P) \leq 1$, we have $\text{gl.dim} B \leq 2 (\text{gl.dim} A) + 2$.

In the first section, we recall some notation and background concerning 2-term silting complexes and their endomorphism algebras. In the second section, we prove some preliminary general results. Then, in Section 3 and 4, we prove respectively Theorem 0.2 and Theorem 0.1, while in the last section, we give some examples.

1. **Background and notation**

Let $A$ be a finite dimensional algebra with $\text{gl.dim} A = d$. Then $K^b(\text{proj} A) = D^b(A) := D$. Let $P$ be a 2-term silting complex in $D$ and let $B = \text{End}_D(P)$. We recall some classical notation (see e.g. [3]) and some results from [5], which will be used freely in the remaining of the paper.

Recall that a pair of subcategories $(X, Y)$ of $\text{mod} A$, is called a torsion pair, if the following hold:

- $\text{Hom}(X, Y) = 0$ if and only if $Y$ is in $Y$, and
- $\text{Hom}(X, Y) = 0$ if and only if $X$ is in $X$.

For a given torsion pair $(X, Y)$ and an object $M$ in $\text{mod} A$, there is a (unique) exact sequence

$$0 \to tM \to M \to M/tM \to 0$$

with $tM$ in $X$ and $M/tM$ in $Y$. This is called the canonical sequence of $M$. Furthermore, for an $A$-module $X$ we let $\text{add} X$ denote the additive closure of $X$ in $\text{mod} A$, and we let $\text{Fac} X$ denote the full subcategory of all quotients of modules in $\text{add} X$. The first notion is also used for a complex $X$ in $D$.

For a 2-term silting complex $P$, consider the full subcategories of $\text{mod} A$ given by

- $\mathcal{T}(P) = \{ X \in \text{mod} A \mid \text{Hom}_D(P, X[1]) = 0 \}$, and
- $\mathcal{F}(P) = \{ Y \in \text{mod} A \mid \text{Hom}_D(P, Y) = 0 \}$.

Furthermore, let $B = \text{End}_D(P)$. The following summarizes results from [5] which will be essential later in this paper.

**Proposition 1.1.** Let $P$ be a 2-term silting complex in $K^b(\text{proj} A)$. Then the following hold.

(a) The pair $(\mathcal{T}(P), \mathcal{F}(P))$ is a torsion pair in $\text{mod} A$.
(b) $\mathcal{T}(P) = \text{Fac} H^0(P)$.
(c) The category $C(P) = \{ X \in D \mid \text{Hom}(P, X[i]) = 0 \text{ for } i \neq 0 \}$ is an abelian category with short exact sequences coinciding with the triangles in $D$ whose vertices are in $C(P)$.
(d) Let $X$ be in $D$. Then we have that $X$ is in $C(P)$ if and only if $H^i(X)$ is in $\mathcal{T}(P)$, $H^{-1}(X) \notin \mathcal{T}(P)$ and $H^i(X) = 0$ for $i \neq -1, 0$.
(e) $\text{Hom}_D(P, -) : C(P) \to \text{mod} B$ is an equivalence of (abelian) categories.

For full subcategories $X$ and $Y$ of $D$, we let $X * Y$ denote the full subcategory of $D$ with objects $Z$ appearing in a triangle

$$X \to Z \to Y \to X[1]$$

with $X$ in $X$ and $Y$ in $Y$. It follows from the octahedral axiom that we have $(X * Y) * Z = X * (Y * Z)$, for three full subcategories $X$, $Y$ and $Z$. The subcategory $X$ is called extension
closed if $X \ast X = X$. We will need the following fact, which follows from [10, Propositions 2.1 and 2.4].

**Lemma 1.2.** Let $X_i$ be subcategories of $D$, with $\text{Hom}_D(X_i, X_j) = 0 = \text{Hom}_D(X_i, X_j)[1]$ for $i < j$. Then $X_1 \ast X_2 \ast \cdots \ast X_n$ is closed under extensions and direct summands.

## 2. Preliminaries

Now, fix a 2-term silting complex $P$ in $K^b(\text{proj} \, A)$, and let $\mathcal{P} = \text{add} \, P$. In this section we include some general observations on projective objects and projective dimensions in $C(P)$.

For each $P_0$ in $\mathcal{P}$, given by $P_0^{-1} \xrightarrow{p_0} P_0^0$, consider the canonical exact sequence of $H^{-1}(P_0)$ relative to the torsion pair $(\mathcal{T}(P), \mathcal{F}(P))$:

$$0 \to tH^{-1}(P_0) \to H^{-1}(P_0) \to H^{-1}(P_0)/tH^{-1}(P_0) \to 0.$$ 

So $tH^{-1}(P_0)$ is a submodule of $P_0^{-1}$ and we denote by $\pi: P_0^{-1} \to P_0^{-1}/tH^{-1}(P_0)$ the canonical epimorphism. Let $\tilde{P}_0$ be the complex $P_0^{-1}/tH^{-1}(P_0) \xrightarrow{\tilde{p}_0} P_0^0$, where $\tilde{p}_0$ is the unique homomorphism such that the diagram

$$
\begin{array}{ccc}
P_0^{-1}/tH^{-1}(P_0) & \xrightarrow{\pi} & P_0^0 \\
\downarrow \tilde{p}_0 & & \downarrow \\pi \\
P_0^{-1} & \xrightarrow{p_0} & P_0^0
\end{array}
$$

commutes.

Let $\mathcal{P}_C = \mathcal{P} \cap C(P)$.

**Lemma 2.1.** Let $P_0$ be in $\mathcal{P}$. Then $P_0$ is in $\mathcal{P}_C$ if and only if $P_0 \cong \tilde{P}_0$.

**Proof.** We have by definition that $P_0 \cong \tilde{P}_0$ if and only if $tH^{-1}(P_0) = 0$ if and only if $H^{-1}(P_0)$ is in $\mathcal{T}(P)$ if and only if $\text{Hom}(P, H^{-1}(P_0)) = 0$. It is straightforward to check that $\text{Hom}(P, H^{-1}(P_0)) = 0$ if and only if $\text{Hom}(P, P_0[-1]) = 0$. Moreover, we have that $\text{Hom}(P, P_0[-1]) = 0$ if and only if $P_0$ is in $C(P)$, and the statement follows from this. \qed

**Lemma 2.2.** With notation as above, the following hold.

(a) There is a triangle in $D$:

$$tH^{-1}(P)[1] \to P \to \tilde{P} \to tH^{-1}(P)[2].$$

(b) The object $\tilde{P}$ is a projective generator for $C(P)$.

**Proof.** The triangle in (a) exists by the construction of $\tilde{P}$.

Note that $H^0(\tilde{P}) = H^0(P)$ is in $\mathcal{T}(P)$ and $H^{-1}(\tilde{P}) = H^{-1}(P)/tH^{-1}(P)$ is in $\mathcal{F}(P)$. Then by Proposition 1.1(d), we have $\tilde{P} \in C(P)$. Applying the functor $\text{Hom}_D(P, -)$ to this triangle yields an isomorphism

$$\text{Hom}_D(P, P) \cong \text{Hom}_D(P, \tilde{P})$$

as $B$-modules. Now (b) follows from Proposition 1.1(e). \qed

For any integer $i$, we let $D^i(P) = \{ X \in D \mid \text{Hom}_D(P, X[j]) = 0 \text{ for } j > i \}$, and we let $D^{\geq i}(P) = \{ X \in D \mid \text{Hom}_D(P, X[j]) = 0 \text{ for } j < i \}$.

**Lemma 2.3.** With notation as above, we have: $C(P) \subset \mathcal{P} \ast \mathcal{P}[1] \ast \cdots \ast \mathcal{P}[d + 1]$. 
Proof. By \cite[Proposition 2.23]{2}, we have
\[ C(P) \subset D^{\leq 0}(P) \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l - 1] * \mathcal{P}[l] \]
for some \( l > 0 \). For any \( M \) in \( C(P) \), by Proposition \cite[(1.1)]{1} (d), we have \( H^i(M) = 0 \) for \( i \neq -1, 0 \). So there is a complex \( X \) of projective \( A \)-modules, which is equivalent to \( M \), and such that \( H^i(X) = 0 \) for \( i > 0 \) or \( i < -d - 1 \). So
\[ \text{Hom}_D(M, \mathcal{P}[i]) \equiv \text{Hom}_D(X, \mathcal{P}[i]) = 0, \ i \geq d + 2, \]
which implies that \( M \) is in \( \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d + 1] \). \( \square \)

Lemma 2.4. For a complex \( X \) in \( C(P) \cap (\mathcal{P}_C * \mathcal{P}_C[1] * \cdots * \mathcal{P}_C[m]) \) for some \( m \geq 0 \), we have \( \text{pd} \text{Hom}_D(P, X)_B \leq m \).

Proof. Let \( X_0 = X \). There are triangles
\[ X_{i+1} \rightarrow O_i \xrightarrow{g_i} X_i \rightarrow X_{i+1}[1], \ 0 \leq i \leq m - 1 \]
where \( O_i \) is in \( \mathcal{P}_C \) and \( X_i \) is in \( \mathcal{P}_C * \mathcal{P}_C[1] * \cdots * \mathcal{P}_C[m - i] \). Since \( \text{Hom}_D(P, \mathcal{P}[i]) = 0 \) for all \( i > 0 \), we have that \( g_i \) is a right \( \mathcal{P} \)-approximation of \( X_i \). By Lemma \cite[2.1]{2} and Lemma \cite[2.2]{2} (b), each \( O_i \) is projective in \( C(P) \). Assume that \( X_i \) is in \( C(P) \) for some \( 0 \leq i \leq m - 1 \). Then, since \( g_i \) is a right \( \mathcal{P} \)-approximation and \( O_i \) is projective in \( C(P) \), we have that \( g_i \) is an epimorphism in \( C(P) \). So \( X_{i+1} \) is the kernel of \( g_i \), by Proposition \cite[(1.1)]{1} (c). Note that \( X_0 \in C(P) \). Then by induction on \( i \), we have that \( X_i \in C(P) \) for all \( 0 \leq i \leq m \) and
\[ \text{pd} \text{Hom}_D(P, X)_B \leq \text{pd} \text{Hom}_D(P, X_{i+1})_B + 1. \]
Therefore \( \text{pd} \text{Hom}_D(P, X)_B \leq \text{pd} \text{Hom}_D(P, X_m)_B + m = m \) since \( X_m \in \mathcal{P}_C \) is projective in \( C(P) \). \( \square \)

We end this section by considering the following special case. Recall from \cite{13}, that a 2-term silting complex \( P \) in \( K^b(\text{proj} A) \) is a tilting complex if \( \text{Hom}_D(P, P[-1]) = 0 \).

Proposition 2.5. If the 2-term silting complex \( P \) is a tilting complex, then \( \text{gl. dim} \text{End}_D(P) \leq \text{gl. dim} A + 1 \).

Proof. If \( P \) is tilting, then \( P \) is in \( C(P) \). So we infer that \( \mathcal{P} = \mathcal{P}_C \). It follows from Lemma \cite[2.3]{2} and Lemma \cite[2.4]{2} that \( \text{gl. dim} \text{End}_D(P) \leq \text{gl. dim} A + 1 \). \( \square \)

Note that the classical situation (as in \cite[III, section 3.4]{8}) where \( P \) is the projective resolution of a classical tilting module, is covered by this result.

3. The partial tilting case

Throughout this section, we assume that \( \text{pd} H^0(P)_A \leq 1 \), that is: \( H^0(P) \) is a partial tilting \( A \)-module. Then we have that \( Q = H^{-1}(P) \) is projective as an \( A \)-module, and \( P \equiv H^0(P) \oplus Q[1] \). Consider the canonical exact sequence of \( Q \) relative to the torsion pair \( (\mathcal{T}(P), \mathcal{T}(P)) \):
\[ 0 \rightarrow tQ \rightarrow Q \rightarrow Q/tQ \rightarrow 0. \]
As before, we let \( d = \text{gl. dim} A \). We first prove two technical lemmas.

Lemma 3.1. With the above notation, we have
\[ tQ \in \text{add} H^0(P) * \text{add} H^0(P)[1] * \cdots * \text{add} H^0(P)[d - 1]. \]
Proof. We first note that \( tQ \in \mathcal{T}(P) \), so by definition \( \text{Hom}_{\mathcal{P}(A)}(P, tQ[i]) = 0 \) for \( i \neq 0 \). In particular, we have \( \text{Hom}_{\mathcal{P}(A)}(Q[1], tQ[i]) = 0 \) for \( i \neq 0 \). For \( i = 0 \), since both \( Q \) and \( tQ \) are \( A \)-modules, we also have that \( \text{Hom}_{\mathcal{P}(A)}(Q[1], tQ) = 0 \). It follows from \( \mathcal{T}(P) \in \mathcal{C}(P) \) that by Proposition 1.1 we have \( tQ \in \mathcal{P} \ast \mathcal{P}[1] \ast \cdots \ast \mathcal{P}[d + 1] \). Therefore, using \( P \cong H^0(P) \oplus Q[1] \), we get that \( tQ \) is in \( \text{add } H^0(P) \ast \text{add } H^0(P)[1] \ast \cdots \ast \text{add } H^0(P)[d + 1] \). By the canonical sequence of \( Q \), we have \( \text{pd}(tQ) \leq \text{pd}(Q) + 1 \leq d - 1 \). Hence, it follows that \( \text{Hom}(tQ, P[d]) = 0 = \text{Hom}(tQ, P[d + 1]) \). The claim of the lemma follows. \( \square \)

**Lemma 3.2.** With the above notation, we have \( \mathcal{C}(P) \subset \mathcal{P} \ast \mathcal{P}[1] \ast \cdots \ast \mathcal{P}[d] \ast \text{add } H^0(P)[d + 1] \).

**Proof.** Using that \( P \cong H^0(P) \oplus Q[1] \) in combination with Lemma 2.3 we only need to prove that \( \text{Hom}_2(X, Q[d + 1]) = 0 \) for \( X \in \mathcal{C}(P) \). This follows from \( \text{pd } H^1(X) \leq d \) for \( i = -1, 0 \) and \( H^i(X) = 0 \) for \( i \neq -1, 0 \). \( \square \)

We can now prove the main result of this section.

**Theorem 3.3.** If \( \text{pd}(H^0(P)) \leq 1 \), then \( \text{gl.dim } B \leq 2 \text{ gl.dim } A + 2 \).

**Proof.** Let \( X \) be an object in \( \mathcal{C}(P) \) with

\[
X \in \mathcal{P} \ast \cdots \ast \mathcal{P}[i] \ast \text{add } H^0(P)[i + 1] \ast \cdots \ast \text{add } H^0(P)[d + 1]
\]

for some \( 0 \leq i \leq d \). Then there is a triangle

\[
X_1 \to E \xrightarrow{g_X} X \to X_1[1]
\]

where \( g_X \) is a right \( \mathcal{P} \)-approximation of \( X \) and \( X_1 \) is in \( \mathcal{P} \ast \cdots \ast \mathcal{P}[i - 1] \ast \text{add } H^0(P)[i] \ast \cdots \ast \text{add } H^0(P)[d] \).

Then \( \text{Hom}_2(P, g_X) \) is an epimorphism and \( \text{Hom}_2(P, E) \) is projective in \( \text{mod } B \).

Recall that \( Q = H^{-1}(P) \). Then, by Lemma 2.2 there is a triangle

\[
F[1] \to E \to \widetilde{E} \to F[2]
\]

where \( F \) is in \( \text{add } tQ \subset \mathcal{T}(P) \subset \mathcal{C}(P) \) and \( \widetilde{E} \) is projective in \( \mathcal{C}(P) \). So \( \text{Hom}_2(F[1], X) = 0 \) since \( X \in \mathcal{C}(P) \). It follows that the map \( g_X \) factors through the map \( E \to \widetilde{E} \). Then, by the octahedral axiom, we have the following commutative diagram of triangles:

\[
\begin{array}{c}
\text{X[1]} \\
\text{F[1]} \downarrow \text{X} \downarrow \text{F[2]} \\
\text{F[1]} \downarrow \text{E} \downarrow \text{F[2]} \\
\text{X} \downarrow \text{gX} \downarrow \text{gX} \\
\end{array}
\]

Then we have that

\[
\begin{align*}
X' & \in \text{add } X_1 \ast \text{add } F[2] \\
& \subset (\mathcal{P} \ast \cdots \ast \mathcal{P}[i - 1] \ast \text{add } H^0(P)[i] \ast \cdots \ast \text{add } H^0(P)[d]) \\
& \ast (\text{add } H^0(P) \ast \cdots \ast \text{add } H^0(P)[d - 1])[2] \\
& = (\mathcal{P} \ast \cdots \ast \mathcal{P}[i - 1] \ast \text{add } H^0(P)[i] \ast \cdots \ast \text{add } H^0(P)[d]) \ast \text{add } H^0(P)[d + 1]
\end{align*}
\]
where the inclusion is due to Lemma 3.1 and the equality follows from
\[ P \ast \cdots \ast P[i-1] \ast \text{add} \, H^0(P)[i] \ast \cdots \ast \text{add} \, H^0(P)[d] \]
being closed under extensions by Lemma 1.2. Applying \( \text{Hom}_D(P, -) \) to the above diagram, we obtain a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_D(P, E) & \xrightarrow{\sim} & \text{Hom}_D(P, \tilde{E}) \\
\text{Hom}_D(P, DX) & & \downarrow & & \text{Hom}_D(P, DX) \\
\text{Hom}_D(P, X) & & & & \text{Hom}_D(P, X)
\end{array}
\]
Using that the map \( \text{Hom}_D(P, g_X) \) is an epimorphism in \( \text{mod} \, B \), it follows that the map \( g_X \) is an epimorphism in \( C(P) \). Then \( X' \) is the kernel of \( g_X \) in \( C(P) \). Hence
\[ \text{pd} \, \text{Hom}_D(P, X)_B \leq \text{pd} \, \text{Hom}_D(P, X')_B + 1. \]
Using induction on \( i \) and Lemma 3.2, we have that for \( X \in C(P) \), there is \( X' \) such that
\[ X' \in C(P) \cap (\text{add} \, H^0(P) \ast \text{add} \, H^0(P)[1] \ast \cdots \ast \text{add} \, H^0(P)[d+1]) \]
and \( \text{pd} \, \text{Hom}_D(P, X)_B \leq \text{pd} \, \text{Hom}_D(P, X')_B + d + 1 \). By Lemma 2.4 and equation (1), we have \( \text{pd} \, \text{Hom}_D(P, X')_B \leq d + 1 \). It then follows that \( \text{pd} \, \text{Hom}_D(P, X)_B \leq 2d + 2 \), and hence \( \text{gl.dim} \, B \leq 2d + 2 \).

4. The case of global dimension 2.

In this section, we consider the case when \( \text{gl.dim} \, A \leq 2 \). Our aim is to prove part (b) of Theorem 0.1 stating that in this case we have that the global dimension is at most 7 for the endomorphism algebra of any 2-term silting complex.

We prepare by showing four technical lemmas. Let \( \mathcal{P}_C^{[0,1]} = (\mathcal{P} \ast \mathcal{P}[1]) \cap C(P) \).

**Lemma 4.1.** If \( X \) is in \( \mathcal{P}_C^{[0,1]} \), then \( \text{pd} \, \text{Hom}_D(P, X)_B \leq 1 \).

**Proof.** Since \( X \) is in \( \mathcal{P} \ast \mathcal{P}[1] \), there is a triangle \( O_1 \to O_0 \to X \to O_1[1] \) with \( O_0, O_1 \in \mathcal{P} \). Applying the functor \( \text{Hom}_D(P, -) \) to this triangle, we get a long exact sequence
\[
\text{Hom}_D(P, X)[-1] \to \text{Hom}_D(P, O_1) \to \text{Hom}_D(P, O_0) \to \text{Hom}_D(P, X) \to \text{Hom}_D(P, O_1[1])
\]
where the first term is zero since \( X \) is in \( C(P) \), and the last term is zero since \( \text{Hom}_D(P, P[1]) = 0 \). Therefore, \( \text{pd} \, \text{Hom}_D(P, X)_B \leq 1 \).

**Lemma 4.2.** If \( X \) is in \( C(P) \cap (\mathcal{P}_C^{[0,1]} \ast \mathcal{P}_C^{[0,1]}[1] \ast \mathcal{P}_C^{[0,1]}[2]) \), then \( \text{pd} \, \text{Hom}_D(P, X)_B \leq 3 \).

**Proof.** By \( X \in \mathcal{P}_C^{[0,1]} \ast \mathcal{P}_C^{[0,1]}[1] \ast \mathcal{P}_C^{[0,1]}[2] \), there are triangles
\[ L \to D_1 \to X \to L[1] \]
and
\[ D_1 \to D_2 \to L \to D_3[1] \]
with \( D_1, D_2, D_3 \in \mathcal{P}_C^{[0,1]} \) and \( L \in \mathcal{P}_C^{[0,1]} \ast \mathcal{P}_C^{[0,1]}[1] \subset \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2] \). Applying \( \text{Hom}_D(P, -) \) to triangle (2), we obtain a long exact sequence
\[
\text{Hom}_D(P, X)[-2] \to \text{Hom}_D(P, L)[-1] \to \text{Hom}_D(P, D_1[-1]) \to \text{Hom}_D(P, X[-1]) \\
\to \text{Hom}_D(P, L) \to \text{Hom}_D(P, D_1) \to \text{Hom}_D(P, X) \to \text{Hom}_D(P, L[1]).
\]
We have $\text{Hom}_D(P, X[i]) = 0$ for $i = -1$ or $i = -2$, since $X$ is in $C(P)$. Furthermore, we have $\text{Hom}_D(D_1[1][i]) = 0$, by $D_1 \in C(P)$ and $\text{Hom}_D(L[1][i]) = 0$ by $L \in \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2]$. From this it follows that we have a short exact sequence

$$0 \to \text{Hom}_D(P, L) \to \text{Hom}_D(P, D_1) \to \text{Hom}_D(P, X) \to 0$$

and that $\text{Hom}_D(L[1][i]) = 0$. Using this short exact sequence, it follows that

$$\text{pd Hom}_D(P, X)[i] \leq \text{max}[\text{pd Hom}_D(P, D_1)[i], \text{pd Hom}_D(P, L)[i] + 1].$$

Applying $\text{Hom}_D(P, -)$ to the triangle (3), we obtain an exact sequence

$$0 = \text{Hom}_D(P, L[1][i]) \to \text{Hom}_D(P, D_3) \to \text{Hom}_D(P, D_2) \to \text{Hom}_D(P, L) \to \text{Hom}_D(P, D_3[1][i])$$

where the last term is zero due to $D_3 \in \mathcal{P}_C^{[0,1]}$. As above, we obtain that

$$\text{pd Hom}_D(P, L)[i] \leq \text{max}[\text{pd Hom}_D(P, D_2)[i], \text{pd Hom}_D(P, D_3)[i] + 1].$$

Now, combining the inequalities (4) and (5) with Lemma 4.1, we obtain $\text{pd Hom}_D(P, X)[i] \leq 3$.

**Lemma 4.3.** If $N$ is in $D^{[0]}(P) \cap \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2]$, then there is an object $\tilde{N} \in C(P)$ such that $\text{Hom}_D(P, N) \cong \text{Hom}_D(P, \tilde{N})$ as $B$-modules and $\tilde{N} \in \text{add } N \ast \mathcal{P}_C^{[0,1]}$.

**Proof.** Since $(D^{[0]}(P), D^{[0]}(P))$ is a $t$-structure (see [12, Lemma 5.10]), there is a triangle

$$M \to N \to \tilde{N} \to M[1]$$

with $M \in D^{[0]}(P)[1]$ and $\tilde{N} \in D^{[0]}(P)$. Then $M \in \mathcal{P}[1] \ast \cdots \ast \mathcal{P}[l]$ for some $l$ by [2, Proposition 2.23]. Applying the functor $\text{Hom}_D(P, -)$ to the triangle (6), we have a long exact sequence

$$\cdots \to \text{Hom}_D(P, M[i]) \to \text{Hom}_D(P, N[i]) \to \text{Hom}_D(P, \tilde{N}[i]) \to \text{Hom}_D(P, M[i + 1]) \to \cdots$$

Since $\text{Hom}_D(P, M[i]) = 0$ for $i \geq 0$ by $M \in \mathcal{P}[1] \ast \cdots \ast \mathcal{P}[l]$ and $\text{Hom}_D(P, \tilde{N}[i]) = 0$ for $i < 0$ by $\tilde{N} \in D^{[0]}(P)$, and also $\text{Hom}_D(P, N[i]) = 0$ for $i \neq -1, 0$ by the assumption $N \in D^{[0]}(P) \cap \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2]$, we have that

$$\text{Hom}_D(P, N) \cong \text{Hom}_D(P, \tilde{N})$$

as $B$-modules,

$$\text{Hom}_D(P, \tilde{N}[i]) = 0, \text{ for } i > 0,$$

and

$$\text{Hom}_D(P, M[i]) = 0, \text{ for } i < -1.$$

Thus, we obtain $\tilde{N} \in D^{[0]}(P) \cap D^{[0]}(P) = C(P)$ and $M \in D^{[0]}(P)[1] \cap D^{[0]}(P)[1] = C(P)[1]$. Then by Lemma 2.3, we have $\tilde{N} \in \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2] \ast \mathcal{P}[3]$. Applying the functor $\text{Hom}_D(P, -)$ to the triangle (6), we obtain a long exact sequence

$$\cdots \to \text{Hom}_D(\tilde{N}, P[i]) \to \text{Hom}_D(N, P[i]) \to \text{Hom}_D(M, P[i]) \to \text{Hom}_D(\tilde{N}, P[i + 1]) \to \cdots$$

We have $\text{Hom}_D(N, P[i]) = 0$ for $i > 2$ by $N \in \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2]$, and we have $\text{Hom}_D(\tilde{N}, P[i]) = 0$ for $i > 3$ by $\tilde{N} \in \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}[2] \ast \mathcal{P}[3]$. From this it follows that $\text{Hom}_D(M, P[i]) = 0$ for $i > 2$, and hence $M \in (\mathcal{P}[1] \ast \mathcal{P}[2]) \cap C(P) = \mathcal{P}_C^{[0,1]}[1]$. Therefore we have that

$$\tilde{N} \in \text{add } N \ast \text{add } M[1] \subset \text{add } N \ast \mathcal{P}_C^{[0,1]}[2].$$
Lemma 4.4. Let $X \in C(P) \cap (P \ast P[1] \ast \cdots \ast P[t] \ast \mathcal{H}[t+1])$ for some $t$ with $0 \leq t \leq 3$, where $\mathcal{H} \subset P \ast P[1] \ast \cdots \ast P[2-t]$ (and $\mathcal{H} = 0$ for $t = 3$). Then for each $r$ with $0 \leq r \leq \min(t+1, 3)$, there is an object $\tilde{X}_r \in C(P)$ such that

$$\text{pd } \text{Hom}_D(P, X)_B \leq \text{pd } \text{Hom}_D(P, \tilde{X}_r)_B + r$$

and

$$\tilde{X}_r \in P \ast P[1] \ast \cdots \ast P[t-r] \ast \mathcal{H}[t+1-r] \ast P_C^{(1)}[3-r] \ast \cdots \ast P_C^{(1)}[2]$$

where $P \ast P[1] \ast \cdots \ast P[t-r]$ is taken to be 0 when $r = t + 1$ and $P_C^{(1)}[3-r] \ast \cdots \ast P_C^{(1)}[2]$ is taken to be 0 when $r = 0$.

Proof. Let $\tilde{X}_0 = X$. Then $\tilde{X}_0$ satisfies the conditions in the lemma. Assume that $\tilde{X}_{r-1}$ satisfying the conditions.

By $X_r \in P \ast P[1] \ast \cdots \ast P[t-(r-1)] \ast \mathcal{H}[t+1-(r-1)] \ast P_C^{(1)}[3-(r-1)] \ast \cdots \ast P_C^{(1)}[2]$, there is a triangle

$$X_r \rightarrow P_0 \rightarrow \tilde{X}_{r-1} \rightarrow X_r[1]$$

with $P_0 \in P, X_r \in P \ast \cdots \ast P[t-r] \ast \mathcal{H}[t+1-r] \ast P_C^{(1)}[3-r] \ast \cdots \ast P_C^{(1)}[1] \subset P \ast P[1] \ast P[2]$. The inclusion follows from Lemma 1.2. Applying $\text{Hom}_D(P, -)$ to this triangle, we have a long exact sequence

$$\cdots \rightarrow \text{Hom}_D(P, X_r[i]) \rightarrow \text{Hom}_D(P, P_0[i]) \rightarrow \text{Hom}_D(P, \tilde{X}_{r-1}[i]) \rightarrow \text{Hom}_D(P, X_r[i+1]) \rightarrow \cdots$$

Since $\text{Hom}_D(P, \tilde{X}_{r-1}[i]) = 0$ for $i \neq 0$ by $\tilde{X}_{r-1} \in C(P)$, $\text{Hom}_D(P, X_r[1]) = 0$ by $X_r \in P \ast P[1] \ast P[2]$, and also $\text{Hom}_D(P, P_0[i]) = 0$ for $i < -1$ by $P$ being 2-term, we have a short exact sequence

$$0 \rightarrow \text{Hom}_D(P, X_r) \rightarrow \text{Hom}_D(P, P_0) \rightarrow \text{Hom}_D(P, \tilde{X}_{r-1}) \rightarrow 0$$

and

$$\text{Hom}_D(P, X_r[i]) = 0 \text{ for } i < -1.$$}

Then $\text{pd } \text{Hom}_D(P, \tilde{X}_{r-1})_B \leq \text{pd } \text{Hom}_D(P, X_r)_B + 1$ and by Lemma 4.3, there is an object $\tilde{X}_r \in C(P)$ such that $\text{Hom}_D(P, \tilde{X}_r)_B \cong \text{Hom}_D(P, X_r)_B$ and

$$\tilde{X}_r \in \text{add } X_r \ast P_C^{(1)}[2] \subset P \ast P[1] \ast \cdots \ast P[t-r] \ast \mathcal{H}[t+1-r] \ast P_C^{(1)}[3-r] \ast \cdots \ast P_C^{(1)}[1] \ast P_C^{(1)}[2].$$

Now we prove the main result in this section.

Theorem 4.5. If $\text{gl.dim } A \leq 2$, then $\text{gl.dim } \text{End}_D(P) \leq 7$ for any 2-term silting complex $P$ in $K^b(\text{proj } A)$.

Proof. Let $X \in C(P)$. Then by Lemma 2.3, we have that $X \in P \ast P[1] \ast P[2] \ast P[3]$. By Lemma 4.4, (taking $t = 3$, $r = 2$ and hence $\mathcal{H} = 0$), there is an $\tilde{X} \in C(P)$ such that $\tilde{X} \in P \ast P[1] \ast P_C^{(1)}[1] \ast P_C^{(1)}[2]$, and

$$(7) \quad \text{pd } \text{Hom}_D(P, X)_B \leq \text{pd } \text{Hom}_D(P, \tilde{X})_B + 2.$$

Then there is a triangle

$$Z \rightarrow Y \rightarrow \tilde{X} \rightarrow Z[1]$$

with $Y \in P \ast P[1]$ and $Z \in P_C^{(1)} \ast P_C^{(1)}[1]$. Applying the functor $\text{Hom}_D(P, -)$ to this triangle, we have a long exact sequence

$$\cdots \rightarrow \text{Hom}_D(P, Z[i]) \rightarrow \text{Hom}_D(P, Y[i]) \rightarrow \text{Hom}_D(P, \tilde{X}[i]) \rightarrow \text{Hom}_D(P, Z[i+1]) \rightarrow \cdots$$
Since $\text{Hom}_D(P, \tilde{X}[i]) = 0$ for $i \neq 0$ by $\tilde{X} \in C(P)$, and $\text{Hom}_D(P, Z[i]) = 0$ for $i \neq -1, 0$ by $Z \in C(P) + C(P)[1]$, we have a short exact sequence

$$0 \to \text{Hom}_D(P, Z) \to \text{Hom}_D(P, Y) \to \text{Hom}_D(P, \tilde{X}) \to 0,$$

and

$$\text{Hom}_D(P, Y[i]) = 0 \text{ for } i < -1.$$

Then we have that

$$(8) \quad \text{pd} \text{ Hom}_D(P, \tilde{X})_B \leq \max \{\text{pd} \text{ Hom}_D(P, Y)_B, \text{pd} \text{ Hom}_D(P, Z)_B + 1\}$$

and $Y, Z \in \mathcal{D}^{\geq -1}(P)$. By Lemma 4.3, there are objects $\tilde{Y}, \tilde{Z} \in C(P)$ such that:

$$\tilde{Y} \in \mathcal{P} \ast \mathcal{P}[1] \ast \mathcal{P}^{[0,1]}[2] \quad \tilde{Z} \in \mathcal{P}^{[0,1]}[1] \ast \mathcal{P}^{[0,1]}[2]$$

$$\text{Hom}_D(P, \tilde{Y})_B \cong \text{Hom}_D(P, Y)_B \quad \text{Hom}_D(P, \tilde{Z})_B \cong \text{Hom}_D(P, Z)_B$$

By Lemma 4.4 (taking $t = 1, r = 2$ and $\mathcal{H} = \mathcal{P}^{[0,1]}[2]$), there is an object $\tilde{Y}' \in C(P)$ such that

$$\tilde{Y}' \in \mathcal{P}^{[0,1]}[1] \ast \mathcal{P}^{[0,1]}[2]$$

and

$$(9) \quad \text{pd} \text{ Hom}_D(P, \tilde{Y})_B \leq \text{pd} \text{ Hom}_D(P, \tilde{Y}')_B + 2.$$ 

By Lemma 4.2 we have $\text{pd} \text{ Hom}_D(P, \tilde{Z})_B \leq 3$ and $\text{pd} \text{ Hom}_D(P, \tilde{Y}')_B \leq 3$. Hence, combining (7), (8) and (9), we obtain

$$\text{pd} \text{ Hom}_D(P, X)_B \leq \text{pd} \text{ Hom}_D(P, \tilde{X})_B + 2 \leq \max \{\text{pd} \text{ Hom}_D(P, Y)_B, \text{pd} \text{ Hom}_D(P, Z)_B + 1\} + 2 \leq \max \{\text{pd} \text{ Hom}_D(P, \tilde{Y})_B, \text{pd} \text{ Hom}_D(P, \tilde{Z})_B + 1\} + 2 \leq 7.$$

$\square$

5. Examples

5.1. First example. We first give an example to show that the bound in Theorem 3.3 is possible. Let $n \geq 2$ and $A = \mathbb{k}Q/I$ where $Q$ is the following quiver

$$\begin{array}{ccccccccc}
2n+3 & \overset{a_1}{\longrightarrow} & \cdots & \overset{a_2}{\longrightarrow} & 7 & \overset{a_3}{\longrightarrow} & 5 & \overset{a_4}{\longrightarrow} & 3 & \overset{e_1}{\longrightarrow} & 1 \\
2n+4 & \overset{b_1}{\longrightarrow} & \cdots & \overset{b_2}{\longrightarrow} & 8 & \overset{b_3}{\longrightarrow} & 6 & \overset{b_4}{\longrightarrow} & 4 & \overset{d_1}{\longrightarrow} \end{array}$$

and the ideal $I$ is generated by $c_1d_1 - c_2d_2, a_i + a_i; b_jb_{j+1}, 1 \leq i \leq n - 1$. Then $\text{gl.dim} A = n$. Let $P$ be the direct sum of the following complexes in $K^b(\text{proj} A)$:

$$\begin{array}{ll}
0 & \rightarrow \bigoplus_{1 \leq i \leq n+2, i \neq 2} P_{2i} \ , \\
P_4 & \rightarrow P_2 \ , \\
P_1 & \rightarrow P_3 \ , \\
\bigoplus_{1 \leq i \leq n+2, i \neq 2} P_{2i-1} & \rightarrow 0 \ .
\end{array}$$
It is easily verified that $P$ is a 2-term silting complex. The quiver of the endomorphism ring $\End \mathcal{D}(P)$ is the Dynkin quiver of type $A_{2n+4}$:

$$
\begin{align*}
2n + 3 & \overset{a_n}{\longrightarrow} \cdots \overset{a_3}{\longrightarrow} 7 & \overset{a_2}{\longrightarrow} 5 & \overset{a_1}{\longrightarrow} 3 & \overset{a_0}{\longrightarrow} 1 \\
2n + 4 & \overset{b_n}{\longrightarrow} \cdots \overset{b_3}{\longrightarrow} 8 & \overset{b_2}{\longrightarrow} 6 & \overset{b_1}{\longrightarrow} 4 & \overset{b_0}{\longrightarrow} 2
\end{align*}
$$

with the relations $a_{i+1}a_i = 0$, $b_ib_{i+1} = 0$, $1 \leq i \leq n-1$ and $a_0cb_0 = 0$. Hence the global dimension of $\End \mathcal{D}(P)$ is $2n + 2$.

5.2. **Second example.** The next example shows that 7 is a possible value for the global dimension of the endomorphism algebra of a 2-term silting complex over an algebra of global dimension two. Let $A = kQ/I$ with $Q$ the following quiver

$$
\begin{align*}
1 & \overset{a_1}{\longrightarrow} 2 & \overset{a_2}{\longrightarrow} 3 & \overset{a_3}{\longrightarrow} 4 & \overset{a_4}{\longrightarrow} 5 & \overset{a_5}{\longrightarrow} 6 & \overset{a_6}{\longrightarrow} 7 & \overset{a_7}{\longrightarrow} 8
\end{align*}
$$

and $I$ the ideal generated by $a_1a_2$, $a_3a_4a_6$ and $a_7a_8$. Then $A$ has global dimension two, and the complex $P$ given by the direct sum of the complexes

$$
\begin{align*}
0 & \rightarrow \bigoplus_{i=5,7,8} P_i \\
P_6 & \rightarrow P_5 \\
P_4 & \rightarrow P_3 \\
\bigoplus_{i=1,2,4} P_i & \rightarrow 0
\end{align*}
$$

is a 2-term silting complex. It is easily verified, that the quiver of $\End \mathcal{D}(P)$ is a linearly oriented Dynkin quiver of type $A_8$ and the ideal of relation equals the square of the Jacobson radical. Hence the global dimension of $\End \mathcal{D}(P)$ is 7.

5.3. **Third example.** The last example shows that there is no bound on the global dimension of the endomorphism algebra of a 2-silting object over an algebra with global dimension $d \geq 3$. This example then completes the proof of Theorem 0.1.

Let first $A = kQ/I$ where the quiver $Q$ is

$$
\begin{align*}
3 & \overset{d}{\longrightarrow} 2 & \overset{c}{\longrightarrow} 4 \\
\downarrow & & \\
1 & \overset{b}{\longrightarrow} & \overset{a}{\longrightarrow}
\end{align*}
$$

and $I = \langle ba, bd, abc, de \rangle$. The indecomposable projective $A$–modules are

$$
P_1 = 1, \quad P_2 = 1 \frac{1}{2} 4, \quad P_3 = \frac{1}{2}, \quad P_4 = \frac{4}{1} \frac{1}{2} 4.
$$

The integers here denote the corresponding simples, and the notation indicates the radical filtration. The global dimension of $A$ is 3. Let $P$ be the direct sum of $P_i$ (concentrated in degree -1) and

$$
P_i = \cdots \rightarrow 0 \rightarrow P_i \rightarrow 0 \rightarrow 0 \rightarrow \cdots, \quad i = 1, 3, 4,$$

where $p$ is a projective presentation of $S_2$. Then it is easily verified that $P$ is a 2-term silting complex.
By Proposition 1.1(b) we have that $T(P) = \text{Fac} H^0(P)$, and hence $T(P) = \text{add} S_2$. We will show the projective dimension of $S_2$ in $C(P)$ is infinite, by proving that its third syzygy equals $S_2$. This implies that a minimal projective resolution of $S_2$ is periodic and hence infinite.

Using the notation in Section 2, we have that $\tilde{P}_1 = P_1$ and $\tilde{P}_3 = P_3$. Moreover $\tilde{P}_4 = (P_4/S_2)[1]$, and $\tilde{P}_2$ is given by the complex

$$\cdots \rightarrow 0 \rightarrow P_1 \oplus P_3 \oplus (P_4/S_2) \xrightarrow{\bar{\pi}} P_2 \rightarrow 0 \rightarrow \cdots$$

Consider now the triangle

$$\text{cone}(\pi)[-1] \rightarrow \tilde{P}_2 \xrightarrow{\pi} S_2 \rightarrow \text{cone}(\pi)$$

where $\pi$ is

$$\cdots \rightarrow 0 \rightarrow P_1 \oplus P_3 \oplus (P_4/S_2) \xrightarrow{\bar{\pi}} P_2 \rightarrow 0 \rightarrow \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow S_2 \rightarrow 0 \rightarrow \cdots$$

with $\pi^0$ being a projective cover of $S_2$ in mod $A$.

Then $H^0(\text{cone}(\pi)[-1]) = 0$, $H^{-1}(\text{cone}(\pi)[-1]) \cong H^{-1}(\tilde{P}_2) \in \mathcal{T}(P)$ and $H^i(\text{cone}(\pi)[-1]) = 0$ for $i \neq -1, 0$. So cone($\pi$)[-1] is in $C(P)$, using Proposition 1.1(d). Hence $\pi$ is a projective cover of $S_2$ in $C(P)$ and cone($\pi$)[-1] is its kernel in $C(P)$.

Note that $\text{cone}(\pi)[-1] \cong H^{-1}(\text{cone}(\pi)[-1]) \cong H^{-1}(\tilde{P}_2) \cong P_1 \oplus M$, where $M = \frac{2}{134} \in \mathcal{T}(P)$. Consider the triangle

$$\text{cone}(\pi_1)[-1] \rightarrow \tilde{P}_1 \oplus \tilde{P}_3 \oplus \tilde{P}_4 \xrightarrow{\pi_1} M[1] \rightarrow \text{cone}(\pi_1)$$

where $\pi_1$ is

$$\cdots \rightarrow 0 \rightarrow P_1 \oplus P_3 \oplus (P_4/S_2) \rightarrow 0 \rightarrow 0 \rightarrow \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow \pi^{-1}_1 \rightarrow M \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

with $\pi^{-1}_1$ being the unique (up to a scalar) right minimal homomorphism from $P_1 \oplus P_3 \oplus (P_4/S_2)$ to $M$. Then $H^0(\text{cone}(\pi_1)[-1]) \cong S_2 \in \mathcal{T}(P)$, $H^{-1}(\text{cone}(\pi_1)[-1]) \cong \frac{2}{134} \oplus M \in \mathcal{T}(P)$ (since $\text{Hom}(S_2, \frac{2}{134}) = 0$) and $H^i(\text{cone}(\pi_1)[-1]) = 0$ for $i \neq -1, 0$. So cone($\pi_1$)[-1] is in $C(P)$, hence $\pi_1$ is a projective cover of $M[1]$ in $C(P)$ and cone($\pi_1$)[-1] is its kernel.

Now consider the triangle

$$\text{cone}(\pi_2)[-1] \rightarrow \tilde{P}_2 \xrightarrow{\pi_2} \text{cone}(\pi_1)[-1] \rightarrow \text{cone}(\pi_2)$$

where $\pi_2$ is

$$\cdots \rightarrow 0 \rightarrow P_1 \oplus P_3 \oplus (P_4/S_2) \rightarrow 0 \rightarrow 0 \rightarrow \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow \pi^{-1}_2 \rightarrow M \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

with $\pi^{-1}_2$ being the identity map and $\pi^0_2$ being a projective cover of $M$ in mod $A$. Then we have $H^0(\text{cone}(\pi_2)[-1]) \cong S_2 \in \mathcal{T}(P)$ and $H^i(\text{cone}(\pi_2)[-1]) = 0$ for $i \neq 0$. Hence cone($\pi_2$)[-1] $\cong S_2$ in $C(P)$ and we have a short exact sequence in $C(P)$:

$$0 \rightarrow S_2 \xrightarrow{\pi_2} \text{cone}(\pi_1)[-1] \rightarrow 0.$$

Thus, the projective resolution of $S_2$ in $C(P)$ is periodic and hence the projective dimension is infinite. Therefore, also the global dimension of $B$ is infinite, by Proposition 1.1(e).
Now, for any \( n \) consider the quiver \( Q_n \) given by

\[
\begin{array}{c}
3 \overset{a}{\longrightarrow} 2 \overset{e}{\longrightarrow} 4 \\
\downarrow \quad \downarrow \quad \downarrow \\
1_0 \overset{c_0}{\longrightarrow} 1_1 \overset{c_1}{\longrightarrow} 1_2 \overset{c_2}{\longrightarrow} \cdots \overset{c_n}{\longrightarrow} 1_n
\end{array}
\]

with relations \( I_n = \langle ba, bd, abc_0, de, c_0c_1, c_1c_2, \ldots, c_{n-1}c_n \rangle \). Consider the algebra \( A(n) = kQ_n/I_n \). We leave it as an exercise to check that \( A(n) \) has global dimension \( n + 3 \), and to find a 2-term silting complex \( P' \), such that \( \text{End}_{D^b(A(n))}(P') \) has infinite global dimension.

References

[1] T. Adachi, O. Iyama and I. Reiten, \( \tau \)-tilting theory, Compos. Math. 150 (2015), no. 3, 415–452.
[2] T. Aihara and O. Iyama, Silting mutation in triangulated categories, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668.
[3] I. Assem, D. Simson and A. Skowronski, Elements of the representation theory of associative algebras. Vol. 1. Techniques of Representation Theory, London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.
[4] T. Brüstle and D. Yang, Ordered Exchange Graphs, Advances in representation theory of algebras, 135–193, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.
[5] A. B. Buan and Y. Zhou, A silting theorem, J. Pure Appl. Algebra 220 (2016), no. 7, 2748–2770.
[6] A. B. Buan and Y. Zhou, Silted algebras, preprint arXiv:1506.03649.
[7] F. U. Coelho and M. A. Lanzilotta, Algebras with small homological dimensions, Manuscripta Mathematica 100 (1999) 1–11.
[8] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Mathematical Society Lecture Note Series, 119, Cambridge University Press, Cambridge, 1988.
[9] D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebra, Mem. Amer. Math. Soc. 120 (1996), no. 575.
[10] O. Iyama and Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, Invent. Math. 172 (2008), no. 1, 117–168.
[11] B. Keller and D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. Sr. A 40 (1988), no. 2, 239–253.
[12] S. Koenig and D. Yang, Silting objects, simple-minded collections, \( t \)-structures and \( \text{co-}t \)-structures for finite-dimensional algebras, Doc. Math. 19 (2014), 403–438.
[13] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989), no. 3, 436–456.

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