Model-Order Reduction For Hyperbolic Relaxation Systems

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Abstract

We propose a novel framework for model-order reduction of hyperbolic differential equations. The approach combines a relaxation formulation of the hyperbolic equations with a discretization using shifted base functions. Model-order reduction techniques are then applied to the resulting system of coupled ordinary differential equations. On computational examples including in particular the case of shock waves we show the validity of the approach and the performance of the reduced system.

1. Introduction

Model-order reduction has been successfully applied to large-scale systems of ordinary differential equations as well as problems governed by elliptic or parabolic differential equations, see e.g. [1,2,12,14,20,28]. There is a large variety of methods out there for these problem classes and all of them base on the idea that the solution space as a subset of either a large finite dimensional space or possibly an infinite dimensional function space is well approximated by a finite dimensional linear subspace of relatively low dimension. There are several different methods to determine a suitable subspace and several methods to use it for a reduced order model. Some model order reduction methods only take the description of the system to create the projection onto that subspace, and some use data created from solving the full system. A crucial point in the interest and usefulness of a reduced model is that one is not interested in one single solution for one single equation but for a collection of solutions or equations. Sometimes this collection is created by a parameter in the differential equation, sometimes by a varying input function or by considering different starting values.

A way to quantify how reducible an equation is can be done by understanding how well the solution space is approximated by the best $n$-dimensional linear subspace. This concept is referred to as the Kolmogorov $n$-width in the literature. This is also studied for specific hyperbolic problems and the best approximation space in this setting is not satisfying. Therefore unfortunately we need to rethink the general strategy for nonlinear hyperbolic problem. So far a general method is not available. Several approaches have been proposed to provide a suitable finite dimensional approximation space. In particular, in the case of linear hyperbolic system the solution can be expressed as linear semigroup on suitable spaces and therefore an approximation by finite dimensional subspaces is feasible [10,15,18,21,30]. For linear hyperbolic problems

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the transport speed is constant and known a priori. This allows to exploit the idea of shifted base functions. Several different approaches exist and they have partially been extended to the nonlinear case \[10, 13, 21, 29\]. In the nonlinear case a major obstacle has been the loss of regularity of the solutions in the presence of shocks. Those also move at a speed determined through a possibly nonlinear relation out of the solution itself. This time-dependence in the approximate finite dimensional space has been dealt with by time dependent space transformation as part of the reduced model. There is a large body of literature addressing different solutions to this well-established problem \[4, 6, 11, 22, 27, 31, 32\]. They all use very different ways to deal with the creation of a non-linear subspace approximating the solution space.

We propose a method to treat loss of regularity due to shocks as well as the nonlinear transport speed. To that end we first lift the solution space and then find a linear subspace exploiting known techniques. The lifting is done in two steps, first a hyperbolic relaxation \[5, 16, 23\] and then a discretization using suitable spacetime Ansatz functions. The hyperbolic relaxation methods use a suitable reformulation of the nonlinear flux at the expense of an enlarged system. This in turn allows to keep possible discontinuous solutions but reduces the transport part to a linear transport. The linear part ensures further that the new system formally has fixed transport speeds. The latter system is therefore amendable for treatment within model order reduction as shown in this work. We propose to capture the movement of discontinuities by a suitable moving approximations. On those approximations we perform a suitable model order reduction. Based on the continuous formulation we discuss possible numerical discretizations and show computational results in the case of shocks.

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5. Summary

2. Reducability of Scalar non-linear hyperbolic equations

We consider a scalar nonlinear hyperbolic differential equation for the unknown \( U = U(t, x) \) on the torus \( \mathbb{T} = [-1, 1] \subset \mathbb{R} \) as solution to

\[
\partial_t U(t, x) + \partial_x f(U(t, x)) = 0
\]

subject to the initial conditions \( u_0 : \mathbb{T} \to \mathbb{R} \)

\[
 u(0, x) = u_0(x).
\]

The flux function \( f \in C^2(\mathbb{R}; \mathbb{R}) \) is assumed to be nonlinear. Even for smooth initial data \( u_0 \) the solution \( u \) may exhibit discontinuities in finite time \([9]\). Therefore, weak entropy solutions to \([1]\) have been introduced and we refer to \([9]\) for more details on well-posedness of weak solutions. This presentation is concerned with finding a reduced model to this system in the sense of approximating the solution on a lower dimensional manifold. For (linear) elliptic differential equations the lower dimensional manifold can be shown to be a linear subspace and the model-order reduction can be successfully applied. However, for nonlinear hyperbolic systems this approach is not straightforward as already mentioned in the introduction. For general nonlinear problems the typical way to create a reduced order model is to first solve the system at certain instances (in time) using a high dimensional solution technique. This information is used to define a linear subspace of the solution space which becomes the search space in which the equation is
then solved resulting in a so-called reduced system, which is then used to approximate the solution for different parameters or different input functions. For our setting we assume the flux function $f$ is given, however the initial condition given by $u_0$ could vary. Therefore, a suitable reduced modeling technique should allow to generate a reduced system which is able to approximate the solution to the original equation for different initial conditions.

In order to derive the discretization with the space-time ansatz function on the relaxation we consider as an example the linear case first where we already have a linear transport operator. Let

$$f(U) = \lambda U$$

with coefficient $\lambda \neq 0$. The explicit solution to \([1], [2]\) on $\mathbb{T}$ is given by

$$U(t,x) = u_0(x - \lambda t).$$

Classical model order reduction of partial differential equations is based on the idea that the numerical solution is computed as an approximation of the form

$$U(t,x) \approx \sum_{j=1}^{N} u_j(t)\phi_j(x),$$

for a set of basis functions $\phi_j$, like for example a finite element space. In general reduced solutions to the PDE are also described in a similar fashion but with different basis functions, which are picked in such a way that we do not need so many by exploiting the structure of the given equation. In other words model reduction tries to extract a lower dimensional space within the FE space which represents the solution of the given problem well. Assuming that we choose $u_0$ to be a compactly supported local finite element basis function, the solution $U(t,x)$ which is just the transported $u_0$ has a support that moves through the entire space over time. The collection of this functions evaluated at discrete time instances would fast span a large dimensional space within the finite element space leaving not too much hope that we can find a low dimensional subspace. This has been recognized as a problem for hyperbolic systems for a while [31] and a few techniques have been used to overcome that. The most promising approach being to use an Ansatz where the basis function contain a time dependent spatial shift. For linear problem as the speed is clear this can be done easily, and for nonlinear the spatial transformation is part of the hard work of finding the right reduced system and is still a work in progress, but with progress for certain problems [24].

In this paper we use this idea of the spatial shift not to create the right reduced order model but to discretize the full model in order to get a large scale ordinary differential equation that no longer suffers from the transport phenomena. Our large dimensional ansatz space is given by a set of basis function $\phi_j$ but evaluated at a fixed spatial shift:

$$U(t,x) = \sum_{j=1}^{N} u_j(t)\phi_j(x - \lambda t).$$

And in fact if the initial condition $u_0$ is expanded in a truncated series of $N$ coefficients

$$u_0(x) = \sum_{j=1}^{N} u_{0,j}\phi_j(x),$$

for some functions $\{\phi_j\}_{j=1}^{N}$. The explicit solution \([6]\) then yields the exact solution for $u_j(t) = u_{0,j}$ on the linear transport problem.

If we choose a function $u_0(x)$ as the correct linear combination of our linear subspace as in equation \([7]\) and we consider the solution $u(t,x)$ within the one-dimensional manifold spanned by $u_0$

$$\{u(x,t) = u_0(x - \lambda t)\}$$

we obtain the exact solution. While this approach can be extended to linear transport equations with nonlinear right-hand side, as e.g.

$$\partial_t u(t,x) + \lambda \partial_x u(t,x) = g(u(t,x)),$$
this approach however does not extend to a non-linear equations. It is important to note that the previous approach only works because \( \lambda \) is constant. However, in the case of nonlinear flux \( U \rightarrow f(U) \) the characteristic \( \frac{dx}{dt} \) depends on the value of the initial datum \( u_0 \) at \( x_0 \):

\[
\frac{dx}{dt} = f'(U(t, x(t)), x(0) = x_0 \text{ and } U(t, x(t)) = u_0(x_0). \tag{10}
\]

In the nonlinear case it is challenging to determine the correct shift. There is a big e
erort to do so in the literature and for certain problems this has been applied successfully \([6,11,22,27,32]\). In the following we want to develop a general method allowing to have a fixed shift in the base functions.

### 3. Semi–Discretization Compatible to Model Order Reduction

Our approach is somewhat more robust with respect to the type of nonlinear function and the initial condition used as we do not have to track the speed or possibly more than one speed if the waves travel in different directions. The first ingredient is a stiff relaxation approximation \([11]\) considered e.g. in \([3,5,7,16,17,19,23,25,26,33]\). For the scalar problem \([1]\) the relaxation approximation reads

\[
\partial_t u(t, x) + \partial_x v(t, x) = 0 \tag{11}
\]

\[
\partial_t v(t, x) + \lambda^2 \partial_x u(t, x) = -\frac{1}{\varepsilon} (v(t, x) - f(u(t, x))). \tag{12}
\]

Here, \( \lambda > 0 \) is a positive fixed parameter that fulfills the subcharacteristic condition

\[
\lambda \geq \max_{x \in T} |f'(u_0(x))| \tag{13}
\]

and \( \varepsilon > 0 \) is the (small) relaxation parameter. At the expense of an additional variable \( v = v(t, x) \) the relaxation system \([11]\) introduces a linear, hyperbolic approximation to equation \([1]\). Using a Chapman–Enskog expansion in \( \varepsilon \) a formal computation shows that

\[
\partial_t u(t, x) + \partial_x f(u(t, x)) = \varepsilon \partial_x \left( \frac{(\lambda^2 - f'(u(t, x))^2)}{\varepsilon} \partial_x u(t, x) \right) + O(\varepsilon^2). \tag{14}
\]

Hence, \( u \) given by \([11]\) is a viscous approximation to the solution \( U \) of equation \([1]\). However, it needed to be pointed out that \([11]\) is linear hyperbolic and therefore a similar decomposition as shown above might be possible.

The eigenvalues of the linear part in equation \([11]\) are \( \lambda \) and \( -\lambda \), respectively. For small values of \( \varepsilon \) we expect \( v \approx f(u) \) and therefore we set the following initial conditions for \((u_0, v_0)\)

\[
u(0, x) = u_0(x) \text{ and } v(0, x) = f(u_0(x)). \tag{15}\]

Diagonalizing the system \([11]\) using the variables

\[
w^\pm(t, x) = v(t, x) \pm \lambda u(t, x) \tag{16}\]

and

\[
v(t, x) = \frac{1}{2} (w^+(t, x) + w^-(t, x)), \quad u(t, x) = \frac{1}{2\lambda} (w^+(t, x) - w^-(t, x)), \tag{17}\]

respectively, yields the following system

\[
\partial_t w^+ + \lambda \partial_x w^+ = -\frac{1}{\varepsilon} \left( w^+ + w^- \right) \frac{1}{2} - f \left( \frac{w^+ - w^-}{2\lambda} \right), \tag{18}\]

\[
\partial_t w^- - \lambda \partial_x w^- = -\frac{1}{\varepsilon} \left( w^+ + w^- \right) \frac{1}{2} - f \left( \frac{w^+ - w^-}{2\lambda} \right). \tag{19}\]

Their corresponding initial conditions are

\[
w^+(0, x) = f(u_0(x)) + \lambda u_0(x) \quad w^-(0, x) = f(u_0(x)) - \lambda u_0(x). \tag{20}\]
Following the procedure of the linear case we introduce \( \{ \phi_j(t) \}_{j=1}^{N} \) a set of \( N \) differentiable functions \( \phi_j : \mathbb{T} \rightarrow \mathbb{R} \) for \( j = 1, \ldots, N \). The initial data \( w_0^\pm \) is then expanded using the truncated series

\[
w_0^\pm(x) = \sum_{j=1}^{N} a_j^\pm(t) \phi_j(x),
\]

and the solution is expanded using the translated base functions

\[
w^+(t,x) \approx \sum_{j=1}^{N} a_j^+(t) \phi_j(x - \lambda t) \quad \text{and} \quad w^-(t,x) \approx \sum_{j=1}^{N} a_j^-(t) \phi_j(x + \lambda t),
\]

respectively. Note that in the case \( f(u) = au \) we in fact have that \( 22 \) is exact. However, due to the nonlinearity of the right-hand side of \( 18 \) and contrary to the linear case the previous ansatz \( 22 \) is in general not the exact solution to \( 18 \) and \( 20 \).

A series expansion of the original variables \( (u,v) \) is obtained applying the linear transformation \( 17 \). Hence, using ansatz \( 22 \) in equation \( 11 \) we obtain

\[
\begin{align*}
\partial_t u + \partial_x v &= \frac{1}{2\lambda} \left( \sum_{j=1}^{N} \alpha_j^+(t) \phi_j(x - \lambda t) - \sum_{j=1}^{N} \alpha_j^-(t) \phi_j(x + \lambda t) \right), \quad (23) \\
\partial_t v + \lambda^2 \partial_x u &= \frac{1}{2} \left( \sum_{j=1}^{N} \alpha_j^+(t) \phi_j(x - \lambda t) + \sum_{j=1}^{N} \alpha_j^-(t) \phi_j(x + \lambda t) \right) - \frac{1}{\epsilon}(v - f(u)), \quad (24)
\end{align*}
\]

where we did not expand \( v \) and \( u \) in terms of \( \phi_j \) in the right-hand side of equation \( 24 \) for the sake of readability. Define the family of matrices \( t \mapsto M(t) \in \mathbb{R}^{N,N} \) by

\[
M_{jk}(t) = \int_{\mathbb{T}} \phi_j(x) \phi_k(x + 2\lambda t) dx, \quad t \geq 0,
\]

and the projected initial data \( b_k^\pm \) for \( k = 1, \ldots, N \) as

\[
b_k^\pm = \int_{\mathbb{T}} \phi_k(x) (f(u_0(x)) \pm \lambda u_0(x)) dx.
\]

Then, the following system for the evolution of the coefficients \( a^\pm = (\alpha_j^\pm)_{j=1}^{N} \) is obtained

\[
\begin{align*}
M(0) a^+(t) - M(t) \dot{a}^-(t) &= 0 \quad (27) \\
M(0) \dot{a}^+(t) + M(t) a^-(t) &= -\frac{2}{\epsilon} \left( \frac{1}{2} (M(0) a^+(t) + M(t) a^-(t)) - \tilde{F}(t, a^\pm(t)) \right), \quad (28)
\end{align*}
\]

where \( \tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_N) \) and

\[
\begin{align*}
\tilde{F}_j(t, a^\pm(t)) &:= \int_{\mathbb{T}} \phi_j(x) f(\check{u}(t,x + \lambda t)) dx, \\
\check{u}(t,x) &:= \frac{1}{2\lambda} \left( \sum_{j=0}^{N} \alpha_j^+(t) \phi_j(x - \lambda t) - \alpha_j^-(t) \phi_j(x + \lambda t) \right).
\end{align*}
\]

This is a result of multiplying \( 24 \) and \( 23 \) by \( \phi_j(x - \lambda t) \) for all \( j \) and integrating it over \( x \) on \( \mathbb{T} \).

The initial data is given by

\[
M(0) a^+(0) = b^+ \quad \text{and} \quad M(0) a^-(0) = b^-,
\]

(31)
which also follows from multiplying by said basis function and integration. Summarizing, for fixed $N$ and $\epsilon > 0$, the stiff system (27)-(28) and (51) determine the coefficients $a^\pm(t)$ and $u$ given by equation (22) and (17), i.e.,

$$u^N(t,x) = \frac{1}{2\lambda} \left( \sum_{j=0}^{N} a^+_j(t) \phi_j(x - \lambda t) - a^-_j(t) \phi_j(x + \lambda t) \right).$$

(32)

Note that it is not clear a priori if $M(t)$ for $t \geq 0$ is invertible and therefore the governing equations are not necessarily an ordinary differential equation, but possibly a differential algebraic equation. This point will be discussed in more detail in the forthcoming section. For the further considerations assume

$$(Assumption) \forall t \geq 0: \ M(t) \text{ is invertible.}$$

(33)

Summarizing, under assumption (33) the system (27)-(28) with initial conditions (31) yield the approximation (32) to the solution $U = U(t,x)$ of the nonlinear conservation law (1) on $\mathbb{T}$. The proposed approximation (32) contains different approximation errors that have to be addressed in a numerical scheme. First, the solution is projected on the space spanned by the $N$ functions $\phi_j$. Since we expect discontinuities the choice of suitable functions $\phi_j$ is critical to the approximation error. Second, the derivation shows that $u^N$ given by (32) in fact approximates the relaxation solution $u$ to the system (11) for some fixed $\epsilon$. However, analytically, the sequence of weak solution $u^\epsilon$ to equation (11) converges weakly to the weak solution $U$ to equation (1) as $\epsilon \to 0$. The interplay of the obtained numerical errors with the choice of the parameters $\epsilon$ and $N$ will be investigated in the numerical results below.

3.1. Properties of the system (27)-(31)

Using the notation $\alpha = (a^+, a^-)$, we obtain

$$\begin{pmatrix} M(0) & -M(t) \\ M(0) & M(t) \end{pmatrix} \frac{d}{dt} \alpha(t) = -\frac{1}{\epsilon} \begin{pmatrix} 0 \\ [M(0), M(t)]\alpha(t) - 2\tilde{F}(t, \alpha(t)) \end{pmatrix}. $$

(34)

The left hand side of equation (34) consists of a $2 \times 2$ block matrix. This matrix is invertible provided that for all $t \geq 0$ $M(t)$ is invertible. In this case the inverse is explicitly given by

$$\frac{1}{2} \begin{pmatrix} M^{-1}(0) & M^{-1}(0) \\ M^{-1}(t) & M^{-1}(t) \end{pmatrix}$$

(35)

By suitable choice of $\{\phi_j(\cdot)\}_j$ we can guarantee that $M(0)$ is invertible. In fact, if for all $j, k = 1, \ldots, N$

$$\int_\mathbb{T} \phi_j(x) \phi_k(x) dx = \delta_{j,k}$$

(36)

holds true, then $M(0)$ is the identity matrix. Provided that $\phi_j$ is continuously differentiable we obtain under assumption (36) that $M(t)$ is invertible for $t > 0$ sufficiently small. Then, we obtain local existence and uniqueness of solutions $\alpha$. However, the following simple example shows that $M(t)$ is not necessarily invertible for all $t > 0$. Consider $\mathbb{T} = [-1, 1]$, $N = 2$, $2\lambda = 1$, $\phi_1(x) = \sin(x\pi)$ and $\phi_2(x) = \sin(2x\pi)$. Then, $M(0) = \text{Id}$ and $M(\frac{1}{2}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

3.1.1. Case of Compactly Supported Translated Base Functions

Consider a compactly supported function $\phi_0 : \mathbb{T} \to \mathbb{R}$. For fixed $\Delta x = \frac{1}{N}$ sufficiently small, define the family of base functions

$$\phi_j(x) := \phi_0(x - (j-1)\Delta x), \ j = 1, \ldots, N.$$  

(37)
By definition of $\phi_j$ the base functions fulfill $\phi_j(x) = \phi_k(x - (j - k)\Delta x)$. For $j = 1, \ldots, N$, $k = 2, \ldots, N$ we have
\[
M_{j,k}(t) = \int_{0}^{1} \phi_j(x)\phi_k(x + 2\lambda t)dx = \int_{0}^{1} \phi_j(x)\phi_{k-1}(x + 2\lambda t - \Delta x)dx = M_{j,k-1} \left( t - \frac{\Delta x}{2\lambda} \right)
\]
(38)
which implies that
\[
M(t) = P^k M \left( t - k \frac{\Delta x}{2\lambda} \right), \quad k = 1, \ldots, \text{ and } t \in \left[ k \frac{\Delta x}{2\lambda}, (k + 1) \frac{\Delta x}{2\lambda} \right].
\]
(39)

The explicit eigenvalues (42) determine possible singular. Hence, the family of matrices $M(t)$ for all $t \geq 0$ is uniquely defined by $t \to M(t)$ for $t \in [0, \frac{N}{2\lambda}]$. Since $\phi_0$ is defined on $T$ we obtain that $M(t)$ is a circulant matrix, i.e., for $i, j = 1, \ldots, N$,
\[
M_{i,j}(t) = M_{\text{mod}(i+1,N),\text{mod}(j+1,N)}(t).
\]
(41)
The family of matrices $M(t)$ therefore uniquely defined by a family of vectors $\bar{c} = c(t) \in \mathbb{R}^N$ with $c_j(t) = M_{j,j}(t)$ for $j = 1, \ldots, N$ and $t \geq 0$. For circulant matrices the eigenvalues $\Lambda_m$ and $m = 0, \ldots, N - 1$ are
\[
\Lambda_m(t) = \sum_{k=0}^{N-1} c_{k+1}(t) \exp \left( -2\pi i \frac{mk}{N} \right).
\]
(42)
The explicit eigenvalues (42) determine possible $t$ such that $M(t)$ is not invertible. We illustrate this on two examples. Let $\bar{c}_0(x) = \chi_{[-\frac{N}{2}, \frac{N}{2}]}(x)$ and let $\phi_j$ be defined by equation (37). Then, there exists $\Delta x > 0$ and $N$ such that the support $\Omega_j := \text{supp}_x \phi_j(x)$ fulfills
\[
\Omega_i \cap \Omega_j = \emptyset, \quad i \neq j, \quad \text{and} \quad \bigcup_{j=1}^{N} \Omega_j = T.
\]
(43)
For this choice of $\{\phi_j\}_{j=1}^{N}$ the vector $c(0) = (\Delta x, 0, \ldots, 0)^T$ and $c \left( \frac{N\Delta x}{2} \right) = \frac{\Delta x}{2}(1, 1, 0, \ldots, 0)^T$. Hence, if $N$ is even, then $\Lambda_m \left( \frac{N\Delta x}{2} \right) = 0$ for $m = \frac{N}{4}$ and hence $M \left( \frac{N\Delta x}{2} \right)$ is not invertible.

Similarly, if the support of $\phi_0$ is of size $\Delta x$, i.e., $\phi_0(x) = \chi_{[-\Delta x, \Delta x]}(x)$, then $c(0) = (c_0, c_1, c_2, 0, \ldots, 0)^T$ with $c_0 > c_j > 0$, $j = 2, 3$ and $M(0)$ is invertible. However, at time $t = \left( \frac{N\Delta x}{4} \right)$ and $N \geq 4$ even, we obtain $\Lambda_m = 0$ for $m = \frac{N}{4}$.

In the following we discuss properties of the matrix $M$ for the basis functions used in the numerical results later on. Hence, from now on we assume that $\phi_0$ is given by
\[
\phi_0(x) = \begin{cases} 
2x & x \in [0, \Delta x] \\
4\Delta x - 2x & x \in [\Delta x, 2\Delta x] \\
0 & x \not\in [0, 2\Delta x]
\end{cases}
\]
(44)
and $\phi_j$ for $j \geq 1$ are given by (37). There is an easy equivalence for when the circulant matrix is singular.

**Theorem 3.1** (48). A circulant matrix made from the vector $c = [c_0, c_1, \ldots, c_n]$ is singular if and only if $f(x) = \sum_{i=0}^{n-1} c_i x^i$ and $1 - x^n$ have a common zero.

The matrix $M(t)$ resulting from the given basis function is non-singular almost everywhere. It is only non-singular at discrete time instances and then there is only one zero eigenvalue:

**Theorem 3.2.** The matrix $M(t)$ given by (38) for $\phi$ given by (44) is non-singular on the interval $[0, \frac{1}{2\lambda}]$ as long as $t = t^* = \frac{1}{2\lambda}$ and the nullspace at $t^*$ is only one-dimensional.
Proof. In order to prove that the matrix is nonsingular we use Theorem 3.1. Our matrix is a circulant matrix composed of the vector \( c = [c_1, \ldots, c_N] \), where \( c_j = \int \phi_1(x)\phi_j(x + 2\lambda t)dx \). In the given interval we have \( c_0 = 0 \) except for \( c_1, c_2, c_3, c_N \). It is well known that the circulant matrix composed by \( c_1, \ldots, c_N \) has up to sign the same determinant as \( c_N, c_1, \ldots, c_{N-1} \). Therefore we can consider this matrix instead. Hence, the polynomial we are interested in is given by

\[
c_N + c_1 x + c_2 x^2 + c_3 x^3
\]

Next, we show that no root of unity is a zero of that polynomial except at time \( t = t^* \).

**Lemma 3.3.** For \( c_j(t) = \int \phi_1(x)\phi_j(x + 2\lambda t)dx \) the polynomial \( p(x, t) = c_N(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 \) has only a root of unity if \( t = t^* \).

**Proof.** Assume that \( \omega \) is a root of unity and also a root of \( p(x, t) \). Then \( \omega \) is either complex, equal to 1 or -1. However, \( \omega = 1 \) can not be a root of \( p \) as all \( c_j \) are positive. If \( \omega = -1 \) is a root we have that \( c_N(t) - c_1(t) + c_2(t) - c_3(t) = 0 \). It is straightforward by the definition of \( c_j \) to show that \( c_N(0) - c_1(0) + c_2(0) - c_3(0) < 0 \) and \( c_N(-\lambda N) - c_1(1) + c_2(1) - c_3(1) > 0 \). Further, the derivative is positive in the given interval and therefore it has exactly one zero in this interval. This is at \( t = t^* \). If \( \omega \) is complex then also \( \bar{\omega} \) has to be a root of \( p(x, t) \) and then we obtain

\[
p(x, t) = (x - \omega)(x - \bar{\omega})(\alpha + \beta x)
\]

for some \( \beta \) and \( \alpha \). Comparing the coefficients we get that

\[
\begin{align*}
\alpha &= \beta - 2\alpha \text{Re}(\omega) \\
\beta &= \alpha - 2\beta \text{Re}(\omega) \\
\alpha &= \beta
\end{align*}
\]

and from that we get that

\[
\text{Re}(\omega) = \frac{c_3 - c_1}{2c_N} = \frac{c_N - c_2}{2c_3}.
\]

This fraction is always less or equal to -1 and therefore \( \omega \) can only be -1 which has been treated before.

**3.1.2. Differential algebraic nature of the system**

As discussed in the previous section \( M(t) \) could be singular for base functions fulfilling \( f(t) \). For the choices discussed above \( M(t) \) is singular only at a single point in time \( t^* \) within the interval \( [0, \frac{N}{2\lambda}] \), i.e. for the last example \( t^* = \frac{N}{2\lambda} \). Furthermore, it exist a vector \( e \) such that \( M(t^*)e = 0 \) and for all vectors \( v \) orthogonal to that we have \( M(t^*)v \neq 0 \) unless \( v = 0 \).

Let \( V, W \) be the \( N \times (N-1) \) dimensional orthogonal matrices and \( f \) the vector orthogonal to \( W \) such that \( W^TM(t)V \) is invertible and \( f^TM(t)V = 0 \). Then, decompose \( \alpha^- \) into

\[
\alpha^-(t) = \alpha_0^-(t)e + V\alpha^-(t).
\]

For \( \tilde{\beta} = (\alpha^+, \alpha^-) \) problem reads

\[
\begin{bmatrix}
M(0) & -M(t)V \\
W^TM(0) & W^TM(t)V \end{bmatrix}
\begin{bmatrix}
M(0) & -M(t)e \\
W^TM(0) & W^TM(t)e \end{bmatrix}
\begin{bmatrix}
dM(t) \\
fM(t)e \end{bmatrix}
= \frac{\rho}{\epsilon} \begin{bmatrix}
0 \\
0 \end{bmatrix}
- \frac{\rho}{\epsilon} \begin{bmatrix}
0 \\
0 \end{bmatrix}
\int \bar{\beta}(t) + \bar{\beta}(t) \end{bmatrix}.
\]

This system is not an ordinary differential equation at \( t = t^* \), since \( M(t^*)e = 0 \). The resulting system is a semi-explicit differential algebraic equation. We introduce a small parameter \( \rho > 0 \) and regularize equation 50 by
We rewrite (52) as
\[
\begin{bmatrix}
M(0) & -M(t)V \\
W^T M(0) & W^T M(t) V \\
\dot{f}^T M(0) & \dot{f}^T M(t) e + \rho
\end{bmatrix}
\begin{bmatrix}
\Delta t \\
\frac{d}{dt} \hat{\beta}
\end{bmatrix}
= \frac{1}{\epsilon} \left( [M(0), M(t)] \hat{\beta}(t) - 2 \hat{F}(t, \hat{\beta}(t)) \right),
\]
(51)
or in terms of \(\alpha\) we have
\[
\begin{bmatrix}
M(0) & -M(t) \\
M(0) & M(t) + \rho f e^T
\end{bmatrix}
\frac{d}{dt} \begin{bmatrix}
\alpha \\
\alpha^+ - \alpha^-\end{bmatrix}
= \frac{1}{\epsilon} \left( [M(0), M(t)] \alpha(t) - 2 \hat{F}(t, \alpha(t)) \right).
\]
(52)

For \(\rho > 0\) the matrix is invertible and it inverse is given by
\[
\begin{bmatrix}
M(0)^{-1}(1 + M(t)(2M(t) + \rho f e^T)^{-1}) & M(0)^{-1}M(t)(2M(t) + \rho f e^T)^{-1} \\
-(2M(t) + \rho f e^T)^{-1} & (2M(t) + \rho f e^T)^{-1}
\end{bmatrix}
\]
(53)

### 3.2. Temporal Discretization and Model Order Reduction

Fix a positive parameter \(\rho > 0\) and consider system (52) subject to initial conditions (31). Consider a temporal grid \(t^n = \Delta t n\) for \(n = 0, \ldots\), where for simplicity we consider an equi-distant grid in time. Denote by \(a^+_n = \alpha^+(t^n)\). Furthermore, denote by
\[N(t) := M(t) + \rho f e^T.\]

We rewrite (52) as
\[
\frac{d}{dt}(M(0)\alpha^+ - M(t)\alpha^-) = -\dot{M}(t)\alpha^-,
\]
(54)
\[
\frac{d}{dt}(M(0)\alpha^+ + N(t)\alpha^-) = \dot{M}(t)\alpha^- - \frac{1}{\epsilon} \left( \frac{1}{2} [M(0)\alpha^+ + M(t)\alpha^-] - \hat{F}(t, \alpha^+(t)) \right).
\]
(55)

where \(\hat{F}\) is given by equation (29) and \(M(t) \int \phi_j(x)\phi'_j(x + 2\lambda t) 2\lambda dx\). An implicit discretization of (55) is preferable to resolve small scales of \(\epsilon\). Since the term \(\hat{F}\) is an integral term in both \(\alpha^+\) and \(\alpha^-\), a fully implicit discretization is computationally too costly. We therefore proceed using a semi-implicit discretization, i.e.,
\[
(M_0 \alpha^+_{n+1} - M_{n+1} \alpha^-_{n+1}) - (M_0 \alpha^+_n - M_n \alpha^-_n) = -\Delta t \dot{M}_n \alpha^-_n,
\]
(56)
\[
(M_0 \alpha^+_n + N_{n+1} \alpha^-_{n+1}) - (M_0 \alpha^+_n + N_n \alpha^-_n) = \Delta t \dot{M}_n \alpha^-_n - \frac{\Delta t}{\epsilon} \left( M_0 \alpha^+_n + M_{n+1} \alpha^-_{n+1} - \hat{F}_n \right),
\]
(57)
\[
\hat{F}_n = 2\hat{F}(t^n, \alpha^+_n),
\]
(58)

leading to the following system
\[
\begin{bmatrix}
M_0 & -M_{n+1} \\
M_0 & N_{n+1}
\end{bmatrix}
\begin{bmatrix}
\alpha^+_{n+1} \\
\alpha^-_{n+1}
\end{bmatrix}
= \frac{\epsilon}{\Delta t} \left( M_0 \alpha^+_n + N_n \alpha^-_n + \Delta t \dot{M}_n \alpha^-_n \right) + \frac{\Delta t}{\epsilon - \Delta t} \hat{F}_n,
\]
(59)

As in equation (54) the left-hand side of equation (59) consists of a 2 x 2 block matrix
\[
R_{n+1} := \begin{bmatrix}
M_0 & -M_{n+1} \\
M_0 & N_{n+1}
\end{bmatrix}
\]
which is invertible provided that \(M(0)\) is invertible and \(\rho\) is non-negative. In this case its inverse is given by equation (53) evaluated at \(t = t^n\). Furthermore, \(\hat{F}_n\) and \(\dot{M}_n = M(t^n)\) needs to be discretized using a numerical quadrature formula of sufficient high–order. Note that the previous formulation can be formally evaluated for all values of \(\epsilon\) (even \(\epsilon = 0\)). However, since \(\hat{F}_n\) is \(\hat{F}(t^n, \alpha^+_n, \alpha^-_n)\) the previous scheme requires a time step restriction of the type
\[
\Delta t \leq C \epsilon
\]
(60)
for some constant $C$ to be stable. Clearly, this leads to small steps for sufficiently small $\epsilon$. The only way to circumvent this restriction is to discretize $\tilde{F}$ implicit. Since our focus is on the model order reduction for the system (59) we leave the efficient computation of the fully implicit scheme for future investigation. For sake of completeness we also state the alternative fully explicit discretization as

\begin{align}
(M_0\alpha_{n+1}^+ - M_n\alpha_{n+1}^-) - (M_0\alpha_n^+ - M_n\alpha_n^-) & = - \Delta t M_n \alpha_n^- \\
(M_0\alpha_{n+1}^+ + N_n\alpha_{n+1}^-) - (M_0\alpha_n^+ + N_n\alpha_n^-) & = \Delta t M_n \alpha_n^- - \frac{2\Delta t}{\epsilon}(M_0\alpha_n^+ + M_n\alpha_n^-) - \tilde{F}_n \tag{62}
\end{align}

\begin{equation}
\tilde{F}_n = \tilde{F}(t^n, \alpha_n^+) \tag{63}
\end{equation}

The same restriction (60) applies for this discretization.

Note that the original scheme [16] does not require a time step restriction of the order of $\epsilon$. The reason being that in the case of a finite volume scheme the implicit discretization of the source vectors of the full size. This is for arbitrary nonlinear flux function and arbitrary basis functions $\phi$ we can do so. However we first solve for $V_\pm \dot{\alpha}$ and then multiply the equation by the transpose of the projections matrices $V_\pm$:

\begin{align}
\dot{\alpha}^+(t) & = -\frac{1}{2} V_+^T M^{-1}(0) \frac{2}{\epsilon} \left( \frac{1}{2} (M(0) V_+ \dot{\alpha}^+(t) + M(t) V_- \dot{\alpha}^-(t)) - \tilde{F}(t, V_+ \dot{\alpha}^+(t)) \right) \\
\dot{\alpha}^-(t) & = -V_-^T M^{-1}(t) \frac{1}{2} \left( \frac{1}{2} (M(0) V_+ \dot{\alpha}^+(t) + M(t) V_- \dot{\alpha}^-(t)) - \tilde{F}(t, V_- \dot{\alpha}^-(t)) \right) \tag{66, 67}
\end{align}

As above if $M(t)$ is not invertible we replace it by $N(t)$.

In order to gain computation speed solving equation (66 and 67) over the full system (52) we need to make sure that the right hand sides are evaluated fast and do not need the computation of vectors of the full size. This is for arbitrary nonlinear flux function and arbitrary basis functions $\phi$ not a trivial problem. However this paper is concerned with the proof of concept of the general method, namely the fact that the solution of $\alpha$ in $R^{2N}$ lives in a low-dimensional space and this fact can be exploited to create a reduced model with standard methods for $\alpha$.

### 3.3. Projection based Model Order Reduction for system (27) - (28)

The previous formulation (59) is amendable for model order reduction. Hence, we approximate $\alpha^+(t) \in R^N$ within a lower dimensional linear subspace of $R^N$, meaning there exist $V_+$ and $V_-$ such that $\alpha_+(t) \approx V_+ V_-^T \alpha^+(t)$ and therefore an $\hat{\alpha}^+(t)$ exist such that $\alpha^+(t) \approx V_+ \hat{\alpha}^+(t)$.

Using this approximation in the ODE we get the following

\begin{align}
M(0)V_+ \dot{\hat{\alpha}}^+ + M(t)V_- \dot{\hat{\alpha}}^- = 0 \\
M(0)V_+ \dot{\hat{\alpha}}^+ + M(t)V_- \dot{\hat{\alpha}}^- = -\frac{2}{\epsilon} \left( \frac{1}{2} (M(0) V_+ \dot{\alpha}^+(t) + M(t) V_- \dot{\alpha}^-(t)) - \tilde{F}(t, V_+ \dot{\alpha}^+(t)) \right) \tag{64, 65}
\end{align}

which is then projected to get a system of ordinary differential equation in a lower dimension. We use a Galerkin projection for simplicity. However we first solve for $V_+ \dot{\alpha}^+$ and then multiply the equation by the transpose of the projections matrices $V_+$:

\begin{align}
\dot{\alpha}^+(t) & = -\frac{1}{2} V_+^T M^{-1}(0) \frac{2}{\epsilon} \left( \frac{1}{2} (M(0) V_+ \dot{\alpha}^+(t) + M(t) V_- \dot{\alpha}^-(t)) - \tilde{F}(t, V_+ \dot{\alpha}^+(t)) \right) \\
\dot{\alpha}^-(t) & = -V_-^T M^{-1}(t) \frac{1}{2} \left( \frac{1}{2} (M(0) V_+ \dot{\alpha}^+(t) + M(t) V_- \dot{\alpha}^-(t)) - \tilde{F}(t, V_- \dot{\alpha}^-(t)) \right) \tag{66, 67}
\end{align}

As above if $M(t)$ is not invertible we replace it by $N(t)$.

4. Computational Results

The theoretical findings are exemplified on a series of linear and nonlinear numerical examples. All computational results are obtained on torus $T = [-1, 1]$. The matrix $M(t)$ defined by equation (25) and the $j$ the component of the right-hand side $\tilde{F}$ are given by

\begin{align}
M_{j,k}(t) = \int_{-1+\lambda t}^{1+\lambda t} \phi_j(x - \lambda t) \phi_k(x + \lambda t) dx, \quad \tilde{F}_j(t, \alpha) = \int_{-1+\lambda t}^{1+\lambda t} \phi_j(x - \lambda t) f(\tilde{u}(t,x)) dx, \tag{68}
\end{align}
where \( \bar{u} \) is given by equation \((50)\). As base function we choose compactly supported, piecewise linear functions fulfilling property \((43)\). We divide the torus in cells \([j-1,j]\Delta x\) where for fixed \(N\) we set \(\Delta x = 2/N\) and \(j = 1,\ldots,N\). Then, the set of base functions \(\{\phi_j : j = 1,\ldots,N\}\) are defined by

\[
\phi_j(x) = \begin{cases} 
  x - (j-1)\Delta x & x \in [j-1,j]\Delta x \\
  (j+1)\Delta x - x & x \in [j-1,j]\Delta x \\
  0 & \text{else} 
\end{cases} \tag{69}
\]

As shown above the matrix \(M(0)\) is invertible for the previous choice of \(\phi_j\). Further, \(M(t)\) is invertible for \(t > 0\), except at the discrete time \(t_n = \frac{n\epsilon_1}{2N}\). The number of time steps is denoted by \(N_{\Delta t}\).

The further parameters are set as follows:

\[
\rho = \epsilon_1; \Delta t = \frac{1}{2} \epsilon_1 \text{ and } T = N_{\Delta t}. \tag{70}
\]

Since the basis function is only nonzero on a small interval we use this in the numerical implementation with MATLAB® built in function \texttt{integral}. When we compute the matrix \(M\) at any time \(t = t_n\) where we use the fact that our basis functions \(\phi\) are shifted. This implies that we have to compute only a single row of the matrix as the matrix is a circulant matrix. To be even more precise, as only four of the values are potentially nonzero, we only have to compute those. Besides \(M\) and \(N\) we have to also compute \(\hat{F}\) at time \(t_n\). As \(a_n^\pm\) are given we can define the function \(\bar{u}(t_n, x)\) and \(\hat{F}\) compute via a quadrature rule which we do by using the build in MATLAB® function \texttt{integral}. Once we have the initial values for \(a_n^\pm\) and the possibility to evaluate \(M\) and \(\hat{F}\) we use \((59)\) to compute further timesteps of \(a_n^\pm\). Since we are only interested in the qualitative behaviour, we do not discuss the possibilities for improving this numerical computation. In the numerical results we will first show that the approach using translated base functions yields qualitative and quantitative correct solutions in the case of linear transport with and without nonlinear source terms, showing that this discretization produces feasible solution. We then show that a linear subspace in the solution space of \(a_n^\pm\) produces correct results and with that the reducability of the ordinary differential equation in \(a\). Secondly, we show that also for nonlinear transport the proposed method yields a good qualitative and quantitative agreement with standard results by finite-volume methods. The later however are not amendable for model order reduction. In the case of strong shocks the reduction in dimension of the reduced model order system is however not as significant as in the linear case. However, the computed reduced order system is able to correctly reproduce solutions to different initial data. This example shows that the chosen formulation is amendable for model order reduction even in the nonlinear case and in the case of discontinuous solutions.

### 4.1. Linear Transport with Nonlinear Source

In order to validate the Ansatz \((22)\) we present numerical results for linear transport equation with nonlinear source term:

\[
\begin{align*}
\partial_t w(t,x) + \lambda \partial_x w(t,x) &= \gamma w^2(t,x) + \delta w \\
w(0,x) &= w_0(x) = 0
\end{align*} \tag{71}
\]

The equation \((71)\) contains three parameters \(\lambda \neq 0\) and \(\gamma, \delta \in \mathbb{R}\). The case \(\gamma = \delta = 0\) corresponds to a linear transport equation. On the full space \(x \in \mathbb{R}\) the explicit solution to equation \((71)\) and \((72)\) is given by

\[
w(t,x) = \frac{1}{w_0(x-\lambda t)} \cdot \frac{\gamma}{\delta} \frac{1}{1 - e^{-\delta t}}. \tag{73}\]

for \(t\) sufficiently small such that \((73)\) is well-defined. Due to the finite speed of propagation a numerical comparison of approximation errors with the exact solution is possible provided that \(\text{supp} w_0(x) \subset \mathbb{T}\). In this case the exact solution \(w(t,x)\) is given by equation \((73)\) for \(x \in \mathbb{S}(t)\) where
\( S(t) := \{ x \in \mathbb{R}^d : x - \lambda t \in \text{suppw}_0 \} \) and \( w = 0 \) zero else. The exact solution \( w(t,x) \) is defined for any \( t \) such that \( S(t) \subset \mathbb{T} \). In the case of the linear equation our Ansatz reduces to

\[
  w(t,x) = \sum_{i=1}^{N} \alpha_i(t) \phi_i(x - \lambda t)
\]

(74)

In the numerical test shown in Figure 1 we choose \( \gamma = 2, \delta = 1 \) and simulate until \( T = 0.5 \).

![Figure 1: Simulation result for a linear transport equation with a nonlinear right hand side with a solution ansatz as in equation (74) for two different values of \( N \) and also a simulated reduced system arising from the larger system and a reduced order equivalent to the size of the smaller one.](image)

We compute the solution for \( N = 100 \) and a reduced model projected on a \( r = 30 \) dimensional subspace. We compare the analytical solution, the solution on the subspace with \( N = 100 \) and the solution with \( N = 30 \) modes. The initial value is given by \( w_0(x) = e^{-4x^2} - e^{-1} \) on \((-1/2, 1/2)\) and zero otherwise. Its approximation on the subspace is also shown. As seen in Figure 1 we observe very good agreement between the reduced basis approximation and the analytical solution. Clearly, higher-dimensional subspaces provide better agreement than lower dimensional ones. This example indicates that the use of translate base functions leads to qualitative and quantitative correct results in the linear case.

### 4.2. Relaxation Approach for a Linear Problems

Consider the same linear flux as in the previous example. Here, we apply the relaxation formulation with \( \lambda = a \) and \( \epsilon = 10^{-3} \) to the linear problem. Clearly, this is not necessary in order to solve the linear problem but the numerical result following illustrates that no additional numerical approximation error appears. The initial condition is

\[
  u_0 = \sin(\pi x)
\]

and the analytical solution is given by \( u(t,x) = u_0(x - t) \). In Figure 2 we show initial condition and analytical solution at final time \( T = 0 \). Figure 3 shows that using \( N = 40 \) basis functions the numerical solution is indistinguishable from the analytical solution. Further, we observe that in
this particular example a reduced system of dimension two can already capture the complete behavior due to the fact that there is only linear transport. For sake of completeness we also show the result with only a single base function that is equal to zero. The test case only contains smooth data and solution and as expected the a low dimensional reduced base formulation recovers the behavior well.

Figure 2: Relaxation formulation applied to a linear flux \( f(u) = u \) and smooth initial data. Shown are the numerical solution at time \( T = 1 \) with \( N = 40 \) piecewise defined basis function and \( r = 2 \) and \( r = 1 \) reduced basis functions, respectively.

4.3. Burgers Equation and Approximation of Shock Solution

We consider the relaxation formulation for Burgers equation, i.e., the flux is given by \( f(u) = \frac{1}{2}u^2 \). Smooth periodic initial data

\[
\begin{align*}
    u_0(x) &= \frac{1}{2} + \sin(\pi x)
\end{align*}
\]
	on \( \mathbb{T} \) is considered. It is known that at time \( T = 1 \) a shock is formed due to the nonlinear transport. In Figure 3 we show the quality of the proposed approximation for different numbers of base functions \( N \). We choose \( \lambda \) larger than the norm of the initial data, i.e.,

\[ \lambda = 2 \]

and

\[ \epsilon = 10^{-3} \]

for this test. In the subfigures of Figure 3 initial data and the solution at terminal time \( T = 1 \) is shown. We observe that for \( N \) sufficiently large the expected shock is recovered in detail. For small \( N \) we observe a Gibb’s phenomena due to the strong discontinuity of the underlying solution. To compare the solution we also included a figure showing the result of a second–order finite volume scheme applied to the same relaxation formulation. In particular, we observe that the size of the jump discontinuity is the same for the proposed approximation and the finite–volume scheme. The later is taken from reference [16] and the spatial discretization is given by \( \Delta x = 1/320 \). Since \( \epsilon > 0 \) we observe in all simulations a slight decay of the maxima and minima over time. For smaller values of \( \epsilon \) the decay of the extreme values is expected to be smaller. However, the time step of the proposed method scales with \( \epsilon \) and this leads to inefficiencies in the numerical scheme. Compared to the method [16] we can not resolve in the regime \( \Delta t > \epsilon \).
Figure 3: Relaxation approximation to a solution to Burgers equation with smooth initial data. At time $T = 1$ a shock develops that is captured by the proposed approximation for $N$ sufficiently large. In red a comparison with a first–order finite volume scheme with $N = 320$ discretization points in space.

In Figure 5 the red curve we observe decay of singular values in the solution such that we can derive efficient model order reduction formulations in the classical sense. As expected the decay is not as significant as in elliptic or parabolic problems.

4.4. Model Order Reduction Strong Shock

As a second example with non–smooth data we consider Burgers’ equations and initial data of the type

$$u_0(x) = a(\chi_{0,1/2}(x) - 1),$$

for a parameter $a > 0$. The value of $a$ controls the size of the jump discontinuity. The solution $u$ to Burgers equation and the given initial data consists of a shock wave followed by a rarefaction. The later wave is a linear function. The parameter $a$ also controls the speed of propagation of the shock wave due to the Rankine-Hugenoit condition: the speed is $s = -a^{1/2}$. For the numerical test we set $\epsilon = 10^{-3}$ and $\Delta t = \frac{\epsilon}{\Delta x}$. In Figure 4 the initial condition and its approximation with a discretization of $N = 160$ are shown in the left part of the figure. Small oscillations due to the strong discontinuity are visible. On the right we show the solution for two reduced model order
approximations as well as the full model and a reference solution. The later is computed as in the previous section using a second–order finite volume scheme with $\Delta x = \frac{1}{160}$.

In order to investigate the oscillatory behavior we show the normalized singular values of reduced solutions for smooth and non–smooth initial data in Figure 5. The decay of the singular values in the beginning is similar but then, as expected, the smooth initial condition shows a significant decrease in the singular values compared with the non–smooth case. This also validates the observation that a low number of base functions might not necessary be sufficient to resolve the strong discontinuities. However, as seen in Figure 4 a model order reduction is still possible and we will investigate the found reduced basis for different initial conditions in the forthcoming section. The singular value decay of the matrix created by both solution is also shown in Figure 5. The dominant modes are also the basis of the reduced model used in the next section.

4.5. Model Order Reduction for Different Initial Conditions

We use a reduced model obtained from a combination of the above initial conditions to predict model output for different initial conditions. We consider the solutions to the two different initial conditions given in the previous section. A reduced model from the dominant basis functions of the first two problems is obtained. The solution to this reduced system for initial data given by equation (75) is compared with classical finite–volume integration. The initial condition is chosen as a linear combination of the two previous initial conditions.

$$u_0 = \sin(\pi x) + a (x_{0,1/2}(x) - 1)$$  \hspace{1cm} (75)$$

with $a = 0.2$ Note that the solution $u(T,x)$ is not a linear combination of the two previous solutions due to the nonlinear nature of the problem. Hence, we generate computational efficiency by reducing the size of the Ansatz space needed to solve for a given initial datum. Results are shown in Figure 6. The initial condition is sinusoidal with an additional discontinuities. The reduced
Figure 5: Singular values for the $N = 160$ solution. Compared with singular values of the solution to parabolic or elliptic problems the decay is not as strong. We show the singular values for two examples: the solution to a smooth profile developing a shock (red) and for a discontinuous solution. Further, the decay for the combination both initial conditions is shown (green).

system solution at $T = 0.3$ as well as the finite volume comparison showing good qualitative and quantitative agreement. In this example we set the number of base functions as $N = 160$, the dimension of the reduced space $r = 80$. The further parameters are $\epsilon = 10^{-3}$ as above and $\Delta t = \frac{\epsilon}{10}$. The results confirm that the chosen approach allows to efficiently apply model order reduction to hyperbolic problems.

5. Summary

We proposed a relaxation formulation of hyperbolic conservation laws that allows to use shifted base functions for a formulation that is amendable for model-order reduction. The resulting discretized scheme is reduced using snapshots in time and shows qualitative good approximation properties even in the case of shock waves. The approach has been tested on linear hyperbolic problems with nonlinear source terms but known exact solution as well as nonlinear hyperbolic problems with strong shocks. A numerical investigation of the approximation quality, the singular value decay as well as comparisons with classical finite-volume schemes have been conducted.

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Figure 6: Left: Initial condition on the full space and reduced space with $r = 80$ out of $N = 160$ base functions. Right: Solution at time $T = 0.3$ on full and reduced space.

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