Conformal metrics on $\mathbb{R}^{2m}$ with constant $Q$-curvature

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Abstract

We study the conformal metrics on $\mathbb{R}^{2m}$ with constant $Q$-curvature $Q \in \mathbb{R}$ having finite volume, particularly in the case $Q \leq 0$. We show that when $Q < 0$ such metrics exist in $\mathbb{R}^{2m}$ if and only if $m > 1$. Moreover we study their asymptotic behavior at infinity, in analogy with the case $Q > 0$, which we treated in a recent paper. When $Q = 0$, we show that such metrics have the form $e^{2p}g_{\mathbb{R}^{2m}}$, where $p$ is a polynomial such that $2 \leq \deg p \leq 2m - 2$ and $\sup_{\mathbb{R}^{2m}} p < +\infty$. In dimension 4, such metrics are exactly the polynomials $p$ of degree 2 with $\lim_{|x| \to +\infty} p(x) = -\infty$.

1 Introduction and statement of the main theorems

Given a constant $Q \in \mathbb{R}$, we consider the solutions to the equation

$$(-\Delta)^m u = Qe^{2mu} \text{ on } \mathbb{R}^{2m},$$

satisfying

$$\alpha := \frac{1}{|S^{2m}|} \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty.$$  \hfill (2)

Geometrically, if $u$ solves (1) and (2), then the conformal metric $g := e^{2u}g_{\mathbb{R}^{2m}}$ has $Q$-curvature $Q_{g}^{2m} \equiv Q$ and volume $\alpha|S^{2m}|$. For the definition of the $Q$-curvature and related remarks, we refer to [Mar1]. Notice that given a solution $u$ to (1) and $\lambda > 0$, the function $v := u - \frac{1}{2m} \log \lambda$ solves

$$(-\Delta)^m v = \lambda Qe^{2mv} \text{ in } \mathbb{R}^{2m},$$

hence what matters is just the sign of $Q$, and we can assume without loss of generality that $Q \in \{0, \pm(2m - 1)\}$.

Every solution to (1) is smooth. When $Q = 0$, that follows from standard elliptic estimates; when $Q \neq 0$ the proof is a bit more subtle, see [Mar1, Corollary 8].

For $Q \geq 0$, some explicit solutions to (1) are known. For instance every polynomial of degree at most $2m - 2$ satisfies (1) with $Q = 0$, and the function
\[ u(x) = \log \frac{2}{1 + |x|} \] satisfies (1) with \( Q = (2m - 1)! \) and \( \alpha = 1 \). This latter solution has the property that \( e^{2u} g_{2m} = (\pi^{-1})^* g_{S^{2m}} \), where \( \pi : S^{2m} \to \mathbb{R}^m \) is the stereographic projection.

For the negative case, we notice that the function \( w(x) = \log \frac{2}{1 + |x|} \) solves \((-\Delta)^m w = -(2m-1)! e^{2m w} \) on the unit ball \( B_1 \subset \mathbb{R}^m \) (in dimension 2 this corresponds to the Poincaré metric on the disk). However, no explicit entire solution to (1) with \( Q < 0 \) is known, hence one can ask whether such solutions actually exist. In dimension 2 \( (m = 1) \) it is easy to see that the answer is negative, but quite surprisingly the situation is different in dimension 4 and higher and we have:

**Theorem 1** Fix \( Q < 0 \). For \( m = 1 \) there is no solution to (1)-(2). For every \( m \geq 2 \), there exist (several) radially symmetric solutions to (1)-(2).

Having now an existence result, we turn to the study of the asymptotic behavior at infinity of solutions to (1)-(2) when \( m \geq 2, Q < 0 \), having in mind applications to concentration-compactness problems in conformal geometry. To this end, given a solution \( u \) to (1)-(2), we define the auxiliary function

\[ v(x) := -\frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^m} \log \left( \frac{|y|}{|x-y|} \right) e^{2mu(y)} dy, \tag{3} \]

where \( \gamma_m := \omega_{2m} 2^{2m-2} (m-1)!^2 \) is characterized by the following property:

\[ (-\Delta)^m \left( \frac{1}{\gamma_m} \log \frac{1}{|x|} \right) = \delta_0 \quad \text{in} \ \mathbb{R}^m. \]

Then \((-\Delta)^m v = -(2m-1)! e^{2mu} \). We prove

**Theorem 2** Let \( u \) be a solution of (1)-(2) with \( Q = -(2m-1)! \). Then

\[ u(x) = v(x) + p(x), \tag{4} \]

where \( p \) is a non-constant polynomial of even degree at most \( 2m - 2 \). Moreover there exist a constant \( a \neq 0 \), an integer \( 1 \leq j \leq m - 1 \) and a closed set \( Z \subset S^{2m-1} \) of Hausdorff dimension at most \( 2m - 2 \) such that for every compact subset \( K \subset S^{2m-1} \setminus Z \) we have

\[ \lim_{t \to +\infty} \Delta^\ell v(t\xi) = 0, \quad \ell = 1, \ldots, m - 1, \]

\[ v(t\xi) = 2\alpha \log t + o(\log t), \quad \text{as} \ t \to +\infty, \]

\[ \lim_{t \to +\infty} \Delta^j u(t\xi) = a, \tag{5} \]

for every \( \xi \in K \) uniformly in \( \xi \). If \( m = 2 \), then \( Z = 0 \) and \( \sup_{\mathbb{R}^2} u < +\infty \). Finally

\[ \lim_{|x| \to +\infty} R_{g_u}(x) = -\infty, \tag{6} \]

where \( R_{g_u} \) is the scalar curvature of \( g_u := e^{2u} g_{\mathbb{R}^2} \).

Following the proof of Theorem 1, it can be shown that the estimate on the degree of the polynomial is sharp. Recently J. Wei and D. Ye [WY] showed the existence of solutions to \( \Delta^2 u = 6 e^{4u} \) in \( \mathbb{R}^4 \) with \( \int_{\mathbb{R}^4} e^{4u} dx < +\infty \) which are not
radially symmetric. It is plausible that also in the negative case non-radially symmetric solutions exist.

For the case $Q = 0$ we have

**Theorem 3** When $Q = 0$, any solution to (1) is a polynomial $p$ with $2 \leq \deg p \leq 2m - 2$ and with

$$\sup_{\mathbb{R}^{2m}} p < +\infty.$$  

In particular in dimension 2 (case $m = 1$), there are no solutions. In dimension 4 the solutions are exactly the polynomials of degree 2 with $\lim_{|x| \to \infty} p(x) = -\infty$. Finally, there exist $1 \leq j \leq m - 1$ and $a < 0$ such that

$$\lim_{|x| \to \infty} \Delta^j p(x) = a. \quad (7)$$

The case when $Q > 0$, say $Q = (2m - 1)!$, has been exhaustively treated. The problem

$$(-\Delta)^m u = (2m - 1)! e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2mu} \, dx < +\infty \quad (8)$$

admits standard solutions, i.e. solutions of the form $u(x) := \log \frac{2}{1 + \lambda^2 |x-x_0|^2}$, $\lambda > 0$, $x_0 \in \mathbb{R}^{2m}$ that arise from the stereographic projection and the action of the Möbius group of conformal diffeomorphisms on $S^{2m}$. In dimension 2 W. Chen and C. Li [CL] showed that every solution to (8) is standard. Already in dimension 4, however, as shown by A. Chang and W. Chen [CC], (8) admits non-standard solutions. In dimension 4 C-S. Lin [Lin] classified all solutions $u$ to (8) and gave precise conditions in order for $u$ to be a standard solution in terms of its asymptotic behavior at infinity.

In arbitrary even dimension, A. Chang and P. Yang [CY] proved that solutions of the form

$$u(x) = \log \frac{2}{1 + |x|^2} + \xi(\pi^{-1}(x))$$

are standard, where $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection and $\xi$ is a smooth function on $S^{2m}$. J. Wei and X. Xu [WX] showed that any solution $u$ to (8) is standard under the weaker assumption that $u(x) = o(|x|^3)$ as $|x| \to \infty$, see also [Xu]. We recently treated the general case, see [Mar1], generalizing the work of C-S. Lin. In particular we proved a decomposition $u = p + v$ as in Theorem 2 and gave various analytic and geometric conditions which are equivalent to $u$ being standard.

The classification of the solutions to (8) has been applied in concentration-compactness problems, see e.g. [LS], [RS], [Mal], [MS], [DR], [Str1], [Str2], [Ndi]. There is an interesting geometric consequence of Theorems 2 and 3 with applications in concentration-compactness: In the case of a closed manifold, metrics of equibounded volumes and prescribed $Q$-curvatures of possibly varying sign cannot concentrate at points of negative or zero $Q$-curvature. For instance we shall prove in a forthcoming paper [Mar2]

**Theorem 4** Let $(M, g)$ be a $2m$-dimensional closed Riemannian manifold with Paneitz operator $P^m_g$ satisfying $\ker P^m_g = \{\text{const}\}$, and let $u_k : M \to \mathbb{R}$ be a sequence of solutions of

$$P^m_g u_k + Q^m_g = Q_k e^{2mu_k}, \quad (9)$$

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$$P^m_g u_k + Q^m_g = Q_k e^{2mu_k}, \quad (9)$$
where $Q_{2m}^2$ is the $Q$-curvature of $g$ (see e.g. [Cha]), and where the $Q_k$’s are given continuous functions with $Q_k \rightarrow Q_0$ in $C^0$. Assume also that there is a $\Lambda > 0$ such that

$$\int_M e^{2mu} \, d\text{vol}_g \leq \Lambda,$$

for all $k$. Then one of the following is true.

(i) For every $0 \leq \alpha < 1$, a subsequence is converging in $C^{2m-1,\alpha}(\mathcal{M})$.

(ii) There exists a finite set $S = \{ x^{(i)} : 1 \leq i \leq I \}$ such that $u_k \rightarrow -\infty$ in $L^\infty_{\text{loc}}(\mathcal{M} \setminus S)$. Moreover

$$\int_M Q_g \, d\text{vol}_g = I (2m-1)! |S^{2m}|,$$

and

$$Q_k e^{2mu_k} \, d\text{vol}_g \rightharpoonup \sum_{i=1}^{I} (2m-1)! |S^{2m}| \delta_{x^{(i)}},$$

in the sense of measures. Finally $Q_0(x^{(i)}) > 0$ for $1 \leq i \leq I$.

In sharp contrast with Theorem 4, on an open domain $\Omega \subset \mathbb{R}^{2m}$ (or a manifold with boundary), $m > 1$, concentration is possible at points of negative or zero curvature. Indeed, take any solution $u$ of (1)-(2) with $Q \leq 0$, whose existence is given by Theorem 1, and consider the sequence $u_k(x) = u(k(x-x_0)) + \log k$, for $x \in \Omega$ for some fixed $x_0 \in \Omega$. Then $(-\Delta)^m u_k = Q e^{2mu_k}$ and $u_k$ concentrates at $x_0$ in the sense that as $k \rightarrow \infty$ we have $u_k(x_0) \rightarrow +\infty$, $u_k \rightarrow -\infty$ a.e. in $\Omega$ and $e^{2mu_k} dx \rightharpoonup \alpha |S^{2m}| \delta_{x_0}$ in the sense of measures.

The 2 dimensional case ($m = 1$) is different and concentration at points of non-positive curvature can be ruled out on open domains too, because otherwise a standard blowing-up procedure would yield a solution to (1)-(2) with $Q \leq 0$, contradicting with Theorem 1.

An immediate consequence of Theorem 4 and the Gauss-Bonnet-Chern formula, is the following compactness result (see [Mar2]):

**Corollary 5** In the hypothesis of Theorem 4 assume that either

1. $\chi(M) \leq 0$ and $\dim M \in \{2, 4\}$, or

2. $\chi(M) \leq 0$, $\dim M \geq 6$ and $(M, g)$ is locally conformally flat,

where $\chi(M)$ is the Euler-Poincaré characteristic of $M$. Then only case (i) in Theorem 4 occurs.

The paper is organized as follows. The proof of Theorems 1, 2 and 3 is given in the following three sections; in the last section we collect some open questions. In the following, the letter $C$ denotes a generic constant, which may change from line to line and even within the same line.
2 Proof of Theorem

Theorem follows from Propositions 6 and 8 below.

**Proposition 6** For \( m = 1 \), \( Q < 0 \) there are no solutions to (1)–(2).

*Proof.* Assume that such a solution \( u \) exists. Then, by the maximum principle, and Jensen’s inequality,

\[
\int_{\partial B_R} u d\sigma \geq u(0), \quad \int_{\partial B_R} e^{2u} d\sigma \geq 2\pi Re^{2u(0)}.
\]

Integrating in \( R \) on \([1, +\infty)\), we get

\[
\int_{\mathbb{R}^2} e^{2u} dx = +\infty,
\]

contradiction. \( \square \)

**Lemma 7** Let \( u(r) \) be a smooth radial function on \( \mathbb{R}^n \), \( n \geq 1 \). Then there are positive constants \( b_m \) depending only on \( n \) such that

\[
\Delta^m u(0) = b_m u^{(2m)}(0), \quad (13)
\]

\( u^{(2m)} := \frac{d^{2m} u}{dr^{2m}} \). In particular \( \Delta^m u(0) \) has the sign of \( u^{(2m)}(0) \).

For a proof see [Mar1].

**Proposition 8** For \( m \geq 2 \), \( Q < 0 \) there exist radial solutions to (1)–(2).

*Proof.* We consider separately the cases when \( m \) is even and when \( m \) is odd. 

**Case 1: \( m \) even.** Let \( u = u(r) \) be the unique solution of the following ODE:

\[
\begin{cases}
\Delta^m u(r) = -(2m - 1)!e^{2mu(r)} \\
u^{(2j+1)}(0) = 0 & 0 \leq j \leq m - 1 \\
u^{(2j)}(0) = \alpha_j \leq 0 & 0 \leq j \leq m - 1,
\end{cases}
\]

where \( \alpha_0 = 0 \) and \( \alpha_1 < 0 \). We claim that the solution exists for all \( r \geq 0 \). To see that, we shall use barriers, compare [CC, Theorem 2]. Let us define

\[
w_+(r) = \frac{\alpha_1}{2} r^2, \quad g_+ := w_+ - u.
\]

Then \( \Delta^m g_+ \geq 0 \). By the divergence theorem,

\[
\int_{B_R} \Delta^j g_+ dx = \int_{\partial B_R} \frac{d^{j-1} g_+}{dr} d\sigma.
\]

Moreover, from Lemma 7 we infer

\[
\Delta^j g_+(0) \geq 0 \quad \text{for } 0 \leq j \leq m - 1,
\]

hence we see inductively that \( \Delta^j g_+ (r) \geq 0 \) for every \( r \) such that \( g_+(r) \) is defined and for \( 0 \leq j \leq m - 1 \). In particular \( g_+ \geq 0 \) as long as it exists.
Let us now define 
\[ w_-(r) := \sum_{i=0}^{m-1} \beta_i r^{2i} - A \log \frac{2}{1 + r^2}, \quad g_- := u - w_- , \]
where the \( \beta_i \)'s and \( A \) will be chosen later. Notice that 
\[ \Delta^m w_-(r) = \Delta^m \left( - A \log \frac{2}{1 + r^2} \right) = -(2m - 1)! A \left( \frac{2}{1 + r^2} \right)^{2m} . \]
Since \( \alpha_1 < 0 \), 
\[ \lim_{r \to +\infty} \frac{\left( \frac{2}{1 + r^2} \right)^{2m}}{e^{\alpha_1 r^2}} = +\infty , \]
and taking into account that \( u \leq w_+ \), we can choose \( A \) large enough, so that 
\[ \Delta^m g_-(r) = (2m - 1)! \left[ A \left( \frac{2}{1 + r^2} \right)^{2m} - e^{2m u(r)} \right] \geq (2m - 1)! \left[ A \left( \frac{2}{1 + r^2} \right)^{2m} - e^{m \alpha_1 r^2} \right] \geq 0 . \]
We now choose each \( \beta_i \) so that 
\[ \Delta^j g_-(0) \geq 0 , \quad 0 \leq j \leq m - 1 , \]
and proceed by induction as above to prove that \( g_+ \geq 0 \). Hence 
\[ w_-(r) \leq u(r) \leq w_+(r) \]
as long as \( u \) exists, and by standard ODE theory, that implies that \( u(r) \) exists for all \( r \geq 0 \). Finally 
\[ \int_{\mathbb{R}^{2m}} e^{2m u(|x|)} dx \leq \int_{\mathbb{R}^{2m}} e^{m \alpha_1 |x|^2} dx < +\infty . \]

Case 2: \( m \geq 3 \) odd. Let \( u = u(r) \) solve 
\[ \begin{cases} 
\Delta^m u(r) = (2m - 1)! e^{2m u(r)} \\
u^{(2j+1)}(0) = 0 & 0 \leq j \leq m - 1 \\
u^{(2j)}(0) = \alpha_j \leq 0 & 0 \leq j \leq m - 1 , \end{cases} \]
where the \( \alpha_i \)'s have to be chosen. Set 
\[ w_+(r) := \beta - r^2 - \log \frac{2}{1 + r^2} , \quad g_+ := w_+ - u , \]
where \( \beta < 0 \) is such that \( e^{-r^2+\beta} \leq \left( \frac{2}{1 + r^2} \right)^2 \), hence 
\[ \frac{2}{1 + r^2} - \frac{1 + r^2}{2} e^{-r^2+\beta} \geq 0 \quad \text{for all } r > 0 . \]
Then, as long as \( g_+ \geq 0 \), we have

\[
\Delta^m g_+(r) = (2m-1)! \left[ \left( \frac{2}{1+r^2} \right)^{2m} - e^{2mu(r)} \right]
\geq (2m-1)! \left[ \left( \frac{2}{1+r^2} \right)^{2m} - e^{2mw_+(r)} \right] \geq 0
\]

Choose now the \( \alpha_i \)'s so that, \( u^{(2i)}(0) < w^{(2i)}(0) \), for \( 0 \leq i \leq m-1 \). From Lemma \[\text{Lemma 7}\] we infer that

\[
\Delta^i g_+(0) \geq 0, \quad 0 \leq i \leq m-1,
\]

and we see by induction that \( g_+ \geq 0 \) as long as it is defined. As lower barrier, define

\[
w_-(r) = \sum_{i=0}^{m-1} \beta_i r^{2i}, \quad g_- := u - w_-,
\]

where the \( \beta_i \)'s are chosen so that \( \Delta^i g_-(0) \geq 0 \). Then, observing that

\[
\Delta^m g_-(r) = (2m-1)! e^{2mu(r)} > 0,
\]

as long as \( u \) is defined, we conclude as before that \( g_- \geq 0 \) as long as it is defined. Then \( u \) is defined for all times.

Let \( R > 0 \) be such that, for every \( r \geq R \), \( w_+(r) \leq -\frac{2}{r} \). Then

\[
\int_{\mathbb{R}^{2m}} e^{2mu(|x|)} dx \leq \int_{B_R} e^{2mu(|x|)} dx + \int_{\mathbb{R}^{2m} \setminus B_R} e^{-m|x|^2} dx < +\infty.
\]

3 Proof of Theorem \[\text{Theorem 2}\]

The proof of Theorem \[\text{Theorem 2}\] is divided in several lemmas. The following Liouville-type theorem will prove very useful.

**Theorem 9** Consider \( h : \mathbb{R}^n \to \mathbb{R} \) with \( \Delta^m h = 0 \) and \( h \leq u - v \), where \( e^{mu} \in L^1(\mathbb{R}^n) \) for some \( p > 0 \), \((-v)^+ \in L^1(\mathbb{R}^n) \). Then \( h \) is a polynomial of degree at most \( 2m - 2 \).

**Proof.** As in \[\text{Mar1}\] Theorem 5, for any \( x \in \mathbb{R}^{2m} \) we have

\[
|D^{2m-1} h(x)| \leq \frac{C}{R^{2m-1}} \int_{B_R(x)} |h(y)| dy
\]

\[
= - \frac{C}{R^{2m-1}} \int_{B_R(x)} h(y) dy + 2C \left( \frac{2}{R^{2m-1}} \right) \int_{B_R(x)} h^+ dy \quad \text{(14)}
\]

and

\[
\int_{B_R(x)} h(y) dy = O(R^{2m-2}), \quad \text{as } R \to \infty.
\]

\[7\]
Then
\[
\int_{B_R(x)} h^+ dy \leq \int_{B_R(x)} u^+ dy + C \int_{B_R(x)} (-v)^+ dy \leq \frac{1}{p} \int_{B_R(x)} e^{\mu u} dy + \frac{C}{R^{2m}},
\]
and both terms in (14) divided by \(R^{2m-1}\) go to 0 as \(R \to \infty\). \(\square\)

**Lemma 10** Let \(u\) be a solution of (1)-(2). Then, for \(|x| \geq 4\)
\[
v(x) \leq 2\alpha \log |x| + C.
\]

**Proof.** As in [Mar1, Lemma 9], changing \(v\) with \(-v\). \(\square\)

**Lemma 11** For any \(\varepsilon > 0\), there is \(R > 0\) such that for \(|x| \geq R\),
\[
v(x) \geq \left(2\alpha - \frac{\varepsilon}{2}\right) \log |x| + \frac{(2m-1)!}{\gamma_m} \int_{B_{1}(x)} \log |x - y| e^{2\mu u(y)} dy.
\]
Moreover
\[
(-v)^+ \in L^1(\mathbb{R}^{2m}).
\]

**Proof.** To prove (16) we follow [Lin], Lemma 2.4. Choose \(R_0 > 0\) such that
\[
\frac{1}{|S^{2m}|} \int_{B_{R_0}} e^{2\mu u} dx \geq \alpha - \frac{\varepsilon}{16},
\]
and decompose
\[
\mathbb{R}^{2m} = B_{R_0} \cup A_1 \cup A_2,
\]
\[
A_1 := \{y \in \mathbb{R}^{2m} : 2|x - y| \leq |x|, |y| \geq R_0\},
\]
\[
A_2 := \{y \in \mathbb{R}^{2m} : 2|x - y| > |x|, |y| \geq R_0\}.
\]
Next choose \(R \geq 2\) such that for \(|x| > R\) and \(|y| \leq R_0\), we have \(\log \frac{|x - y|}{|y|} \geq \log |x| - \varepsilon\). Then, observing that \(\frac{(2m-1)!|S^{2m}|}{\gamma_m} = 2\), we have for \(|x| > R\)
\[
\frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} \log \frac{|x - y|}{|y|} e^{2\mu u(y)} dy \geq \left(\log |x| - \frac{\varepsilon}{16}\right) \frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} e^{2\mu u} dy \geq \left(2\alpha - \frac{\varepsilon}{8}\right) \log |x| - C\varepsilon.
\]
Observing that \(\log |x - y| \geq 0\) for \(y \notin B_1(x)\), \(\log |y| \leq \log(2|x|)\) for \(y \in A_1\),
\[
\int_{A_1} e^{2\mu u(y)} dy \leq \frac{\varepsilon|S^{2m}|}{16} \quad \text{and} \quad \log(2|x|) \leq 2 \log |x| \quad \text{for} \quad |x| \geq R, \quad \text{we infer}
\]
\[
\int_{A_1} \log \frac{|x - y|}{|y|} e^{2\mu u(y)} dy = \int_{A_1} \log |x - y| e^{2\mu u(y)} dy - \int_{A_1} \log |y| e^{2\mu u(y)} dy \geq \int_{B_1(x)} \log |x - y| e^{2\mu u(y)} dy - \log(2|x|) \int_{A_1} e^{2\mu u} dy \geq \int_{B_1(x)} \log |x - y| e^{2\mu u(y)} dy - \log |x| \frac{\varepsilon|S^{2m}|}{8},
\]
\[
(19)
\]
Finally, for $y \in A_2$, $|x| > R$ we have that $\frac{|x-y|}{|y|} \geq \frac{1}{4}$, hence
\[
\int_{A_2} \log \frac{|x-y|}{|y|} e^{2m u(y)} dy \geq -\log(4) \int_{A_2} e^{2m u} dy \geq -C\varepsilon. \tag{20}
\]
Putting together (18), (19) and (20), and possibly taking $R$ even larger, we obtain (16). From (16) and Fubini’s theorem
\[
\int_{\mathbb{R}^2} m \chi_{B_R(-v)} + dx \leq C \int_{\mathbb{R}^2} e^{2m u} \int_{B_1(y)} \log \frac{1}{|x-y|} dx dy \leq C \int_{\mathbb{R}^2} e^{2m u} dy < \infty.
\]
Since $v \in C^\infty(\mathbb{R}^2)$, we conclude that $\int_{B_R(-v)} + dx < \infty$ and (17) follows. □

**Lemma 12** Let $u$ be a solution of (11)–(2), with $m \geq 2$. Then $u = v + p$, where $p$ is a polynomial of degree at most $2m - 2$.

**Proof.** Let $p := u - v$. Then $\Delta^m p = 0$. Apply (17) and Theorem 9. □

**Lemma 13** Let $p$ be the polynomial of Lemma 12. Then if $m = 2$, there exists $\delta > 0$ such that
\[
p(x) \leq -\delta |x|^2 + C. \tag{21}
\]
In particular $\lim_{|x| \to \infty} p(x) = -\infty$ and $\deg p = 2$. For $m \geq 3$ there is a (possibly empty) closed set $Z \subset S^{2m-1}$ of Hausdorff dimension $\dim_H(Z) \leq 2m - 2$ such that for every $K \subset S^{2m-1} \setminus Z$ closed, there exists $\delta = \delta(K) > 0$ such that
\[
p(x) \leq -\delta |x|^2 + C \quad \text{for } x = \frac{x}{|x|} \in K. \tag{22}
\]
Consequently $\deg p$ is even.

**Proof.** From (17), we infer that there is a set $A_0$ of finite measure such that
\[
v(x) \geq -C \quad \text{in } \mathbb{R}^{2m} \setminus A_0. \tag{23}
\]

**Case** $m = 2$. Up to a rotation, we can write
\[
p(x) = \sum_{i=1}^4 (b_i x_i^2 + c_i x_i) + b_0.
\]
Assume that $b_{i_0} \geq 0$ for some $1 \leq i_0 \leq 4$. Then on the set
\[
A_1 := \{ x \in \mathbb{R}^4 : |x| \leq 1 \text{ for } i \neq i_0, \; c_{i_0} x_{i_0} \geq 0 \}
\]
we have $p(x) \geq -C$. Moreover $|A_1| = +\infty$. Then, from (23) we infer
\[
\int_{\mathbb{R}^4} e^{4u} dx \geq \int_{A_1 \setminus A_0} e^{4(v+p)} dx \geq C |A_1 \setminus A_0| = +\infty, \tag{24}
\]
contradicting \([2]\). Therefore \(b_i < 0\) for every \(i\) and \([21]\) follows at once.

**Case \(m \geq 3\).** From \([2]\) and \([23]\) we infer that \(p\) cannot be constant. Write

\[
p(t\xi) = \sum_{i=0}^{d} a_i(\xi)t^i, \quad d := \deg p,
\]

where for each \(0 \leq i \leq d\), \(a_i\) is a homogeneous polynomial of degree \(i\) or \(a_i \equiv 0\).

With a computation similar to \([24]\), \([2\phantom{1}]\) and \([23]\) imply that \(a_d(\xi) \leq 0\) for each \(\xi \in S^{2m-1}\). Moreover \(d\) is even, otherwise \(a_d(\xi) = -a_d(-\xi) \leq 0\) for every \(\xi \in S^{2m-1}\), which would imply \(a_d \equiv 0\). Set

\[
Z = \{ \xi \in S^{2m-1} : a_d(\xi) = 0 \}.
\]

We claim that \(\dim H(Z) \leq 2m - 2\). To see that, set

\[
V := \{ x \in \mathbb{R}^{2m} : a_d(x) = 0 \} = \{ t\xi : t \geq 0, \xi \in Z \}.
\]

Since \(V\) is a cone and \(Z = V \cap S^{2m-1}\), we only need to show that \(\dim H(V) \leq 2m - 1\). Set

\[
V_i := \{ x \in \mathbb{R}^{2m} : a_d(x) = \ldots = \nabla^i a_d(x) = 0, \ \nabla^{i+1} a_d(x) \neq 0 \}.
\]

Noticing that \(V_i = \emptyset\) for \(i > d\) (otherwise \(a_d \equiv 0\)), we find \(V = \cup_{i=0}^{d} V_i\). By the implicit function theorem, \(\dim H(V_i) \leq 2m - 1\) for every \(i \geq 0\) and the claim is proved.

Finally, for every compact set \(K \subset S^{2m-1} \setminus Z\), there is a constant \(\delta > 0\) such that \(a_d(\xi) \leq -\frac{\delta}{2}\), and since \(d \geq 2\), \([22]\) follows.

**Corollary 14** Any solution \(u\) of \([1] - [2]\) with \(m = 2\), \(Q < 0\) is bounded from above.

**Proof.** Indeed \(u = v + p\) and, for some \(\delta > 0\),

\[
v(x) \leq 2\alpha \log |x| + C, \quad p(x) \leq -\delta |x|^2 + C.
\]

\[\square\]

**Lemma 15** Let \(v : \mathbb{R}^{2m} \to \mathbb{R}\) be defined as in \([3]\) and \(Z\) as in Lemma \([13]\). Then for every \(K \subset S^{2m-1} \setminus Z\) compact we have

\[
\lim_{t \to +\infty} |x|^j \Delta^{m-j} v(t\xi) = 0, \quad j = 1, \ldots, m-1
\]

for every \(\xi \in K\) uniformly in \(\xi\); for every \(\varepsilon > 0\) there is \(R = R(\varepsilon, K) > 0\) such that, for \(t > R\), \(\xi \in K\),

\[
v(t\xi) \geq (2\alpha - \varepsilon) \log t
\]

**Proof.** Fix \(K \subset S^{2m-1} \setminus Z\) compact and set \(C_K := \{ t\xi : t \geq 0, \xi \in K \}\). For any \(\sigma > 0\), \(1 \leq j \leq 2m-1\),

\[
\int_{\mathbb{R}^{2m} \setminus B_{\varepsilon}(x)} e^{2m\alpha(y)} |x - y|^{-2j} dy \to 0 \quad \text{as } |x| \to \infty
\]

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by dominated convergence. Choose a compact set $\tilde{K} \subset S^{2m-1 \setminus Z}$ such that $K \subset \text{int}(\tilde{K}) \subset S^{2m-1}$. Since $u \leq C(\tilde{K})$ on $C_{\tilde{K}}$ we can choose $\sigma = \sigma(\varepsilon) > 0$ so small that

$$\int_{B_\sigma(x)} \frac{e^{2mu}}{|x-y|^{2j}} dy \leq C(\tilde{K}) \int_{B_\sigma(x)} \frac{1}{|x-y|^{2j}} dy \leq C(\tilde{K}) \varepsilon, \quad \text{for } x \in C_K, \ |x| \text{ large},$$

where $|x|$ is so large that $B_\sigma(x) \subset C_{\tilde{K}}$. Therefore

$$(−1)^{j+1} \Delta^j v(x) = C \int_{\mathbb{R}^{2m}} \frac{e^{2mu}}{|x-y|^{2j}} dy \to 0, \quad \text{for } x \in C_K, \ \text{as } |x| \to \infty,$$

We have seen in Lemma [11] that for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \geq R$

$$v(x) \geq \left(2\alpha - \frac{\varepsilon}{2}\right) \log|x| + \frac{(2m-1)!}{\gamma_m} \int_{B_1(x)} \log|x-y|e^{2mu(y)} dy,$$

and [28] follows easily by choosing $\tilde{K}$ as above and observing that $u \leq C(\tilde{K})$ on $C_{\tilde{K}}$, hence on $B_1(x)$ for $x \in C_K$ with $|x|$ large enough. □

Proof of Theorem 2. The decomposition $u = v + p$ and the properties of $v$ and $p$ follow at once from Lemmas [10] [12] [13] and [15] [0] follow as in [Mar1, Theorem 2]. As for [13], let $j$ be the largest integer such that $\Delta^j p \neq 0$. Then $\Delta^{j+1} p \equiv 0$ and from Theorem 9 we infer that $\deg p = 2j$, hence $\Delta^{j+1} p = a \neq 0$. □

4 The case $Q = 0$

Proof of Theorem 3. From Theorem 9 with $v \equiv 0$, we have that $u$ is a polynomial of degree at most $2m - 2$. Then, as in [Mar1, Lemma 11], we have

$$\sup_{\mathbb{R}^{2m}} u < +\infty,$$

and, since $u$ cannot be constant, we infer that $\deg u \geq 2$ is even. The proof of [7] is analogous to the case $Q < 0$, as long as we do not care about the sign of $a$. To show that $a < 0$, one proceeds as in [Mar1, Theorem 2]. For the case $m = 2$ one proceeds as in Lemma [13] setting $v \equiv 0$ and $A_0 = 0$. □

Example. One might believe that every polynomial $p$ on $\mathbb{R}^{2m}$ of degree at most $2m - 2$ with $\int_{\mathbb{R}^{2m}} e^{2mp} dx < \infty$ satisfies $\lim_{|x| \to \infty} p(x) = -\infty$, as in the case $m = 2$. Consider on $\mathbb{R}^{2m}$, $m \geq 3$ the polynomial $u(x) = -(1 + x_1^2)|\tilde{x}|^2$, where $\tilde{x} = (x_2, \ldots, x_{2m})$. Then $\Delta^m u \equiv 0$ and

$$\int_{\mathbb{R}^{2m}} e^{2mu} dx = \int_{\mathbb{R}} \int_{\mathbb{R}^{2m-1}} e^{-2m(1+x_1^2)|\tilde{x}|^2} d\tilde{x} dx_1$$

$$= \int_{\mathbb{R}} \frac{dx_1}{(1 + x_1^2)^{m-2}} \cdot \int_{\mathbb{R}^{2m-1}} e^{-2m|\tilde{y}|^2} d\tilde{y} < +\infty.$$
5 Open questions

Open Question 1 Does the claim of Corollary [14] hold for \( m > 2 \)? In other words, is any solution \( u \) to (1)-(2) with \( Q < 0 \) bounded from above?

This is an important regularity issue, in particular with regard to the behavior at infinity of the function \( v \) defined in [3]. If \( \sup_{B^2m} u < +\infty \), then one can take \( Z = \emptyset \) in Theorem [2] as in the case \( Q > 0 \), see [Mar1, Theorem 1].

Definition 16 Let \( P_{0m}^2 \) be the set of polynomials \( p \) of degree at most \( 2m - 2 \) on \( \mathbb{R}^{2m} \) such that \( e^{2mp} \in L^1(\mathbb{R}^{2m}) \). Let \( P_+^2m \) be the set of polynomials \( p \) of degree at most \( 2m - 2 \) on \( \mathbb{R}^{2m} \) such that there exists a solution \( u = v + p \) to (1)-(2) with \( Q > 0 \). Similarly for \( P_{0m}^2 \) with \( Q < 0 \).

Related to the first question is the following

Open Question 2 What are the sets \( P_{0m}^2 \), \( P_+^2m \)? Is it true that \( P_{0m}^2 \subset P_+^2m \) and \( P_0^{2m} \subset P_+^{2m} \)?

J. Wei and D. Ye [WY] proved that \( P_0^4 \subset P_+^4 \) (and actually more). Consider now on \( \mathbb{R}^{2m} \), \( m \geq 3 \), the polynomial
\[
p(x) = -(1 + x_1^2)|x|^2, \quad x = (x_2, \ldots, x_{2m}).
\]

As seen above, \( e^{2mp} \in L^1(\mathbb{R}^{2m}) \), hence \( p \in P_{0m}^2 \). Assume that \( p \in P_{0m}^2 \) as well, i.e. there is a function \( u = v + p \) satisfying (1)-(2) and \( Q < 0 \). Then we claim that \( \sup_{B^2m} u = \infty \). Assume by contradiction that \( u \) is bounded from above. Then (15) and (16) imply that
\[
v(x) = 2\alpha \log |x| + o(\log |x|), \quad \text{as } |x| \to \infty.
\]
Therefore,
\[
\lim_{x_1 \to \infty} u(x_1, 0, \ldots, 0) = \lim_{x_1 \to \infty} 2\alpha \log x_1 = \infty,
\]
contradiction.

Open Question 3 Even in the case that \( u \) is not bounded from above, is it true that one can take \( Z = \emptyset \) in Theorem [2] for \( m \geq 3 \) also?

For instance, in order to show that \( v(x) = 2\alpha \log |x| + o(\log |x|) \) as \( |x| \to +\infty \), thanks to (16), it is enough to show that
\[
\int_{B_1(x)} \log |x - y|e^{2mu(y)}dy = o(\log |x|), \quad \text{as } |x| \to +\infty,
\]
which is true if \( \sup_{B^2m} u < \infty \), but it might also be true if \( \sup_{B^2m} u = \infty \).

Open Question 4 What values can the \( \alpha \) given by (1)-(2) assume for a fixed \( Q \)?

As usual, it is enough to consider \( Q \in \{0, \pm(2m - 1)!\} \). When \( m = 1, Q = 1 \), then \( \alpha = 1 \), see [CL]. When \( m = 2, Q = 6 \), then \( \alpha \) can take any value in \( (0, 1] \), as shown in [CC]. Moreover \( \alpha \) cannot be greater than 1 and the case \( \alpha = 1 \) corresponds to standard solutions, as proved in [Lin]. For the trivial case \( Q = 0 \), \( \alpha \) can take any positive value, and for the other cases we have no answer.
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