A rod-like piezoelectric controller for the improvement of the visco-elastic Beck’s beam linear stability

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Summary
The linear stability of a cantilever beam subject to a follower force is investigated. A system of distributed piezoelectric devices is attached to the beam with the purpose of improving the dynamic stability of the structure. They are connected to a second-order network and second-order analog electrical circuit. The assembly of the mechanical and piezoelectric subsystems constitutes a coupled Piezo-Electro-Mechanical (PEM) system, whose equations of motion, derived within a variational approach, are discretized via the Galerkin weighted residual method and the stability of the trivial equilibrium is addressed by numerically solving the associated eigenvalue problem. A sensitivity analysis, carried out on a numerical ground, is thus conducted on a wide range of the electrical parameters to investigate the effectiveness of the proposed controller.

KEYWORDS
Beck’s beam, Hopf bifurcation, linear stability, Piezo-Electro-Mechanical systems, Vibration Control

1 | INTRODUCTION

In the recent years, piezoelectric materials received increasing attention thanks to the great interest that they gained in several fields of engineering applications. Of particular interest is the structural vibration mitigation context, where several control approaches have been deeply investigated. Piezoelectric devices have been considered as structural vibration controllers by enforcing the “principle of similarity,” which is based on the concept that vibration mitigation can be successfully achieved when the controller resembles the behavior of the primary structure. The idea got its inspiration from the success obtained by more traditional control strategies, that is, the tuned mass damper (TMD) and the nonlinear energy sink (NES).

Most of the literature is focused on the control of nonautonomous systems; therefore, a challenging aspect is to analyze the behavior of autonomous systems, thus subject to nonconservative forces (e.g., follower forces), and investigate possible controlling strategies. Such kind of mechanical systems, whose applications are found in several engineering fields (e.g., aircrafts, wings, rocket motors, tall buildings, and flexible pipes), when subject to nonconservative forces of positional type, such as follower forces, may suffer the Ziegler’s paradox that is, a detrimental effect of damping that causes a finite reduction of the Hopf critical load.
Recent studies considered piezoelectric devices to enhance the response of autonomous systems subject to follower forces. However, the principle of similarity was not strictly adopted in Wang et al. since the nonconservative action of positional type was introduced only in the primary system. It was discussed in D'Annibale et al. that similar controllers are actually detrimental in the case of autonomous systems, since the gyroscopic coupling induces the pair of critical eigenvalue to split that in general triggers the instability. On the other hand, nonsimilar piezoelectric controllers were studied in D'Annibale et al. in the case of discrete autonomous systems via a perturbation approach based on an eigenvalue sensitivity analysis. This approach delivers closed-form expressions for the stability domains of the multi-parameter PEM system which are useful for the stability analysis, but as counterpart, these solutions are accurate in a restricted parameters range, that is, when the electrical quantities are small and the system is weakly coupled. It is shown that such controllers can significantly enhance the linear stability, but they are also effective in the improvement of the post-critical response of the Ziegler's column, also when nonlinear damping is considered. The effect of nonsimilar piezoelectric controllers on continuous autonomous systems was investigated in Casalotti and D'Annibale, where the zero-order network and zero-order dissipation controller, driven from Maurini et al., was adopted, and its capability to improve the linear stability of the system was shown.

In this work, a nonsimilar piezoelectric controller is considered to enhance the linear stability of the well-known visco-elastic Beck's beam, that is, a damped cantilever beam subject to a follower force at the free end, which induces the Hopf bifurcation. The beam is thus coupled to distributed piezoelectric devices (idealized as continuous layer), whose analogous circuit is the second-order network and second-order dissipation type \((S,S)\) derived in Maurini et al. The partial differential equations governing the coupled system are derived via a variational approach (see D'Annibale et al.), and the related eigenvalue problem is solved numerically after discretization through the Galerkin weighted residual approach. An extensive sensitivity analysis, grounded on a numerical approach, is carried out to investigate the regions of the parameter space for which an enhancement of the beam stability is achieved, also exploring ranges of the electrical quantities that are beyond the limits suggested by the perturbation analysis conducted in D'Annibale et al.

The paper is organized as follows. In Section 2, the Piezo-Electro-Mechanical (PEM) system equations are derived via a variational approach. In Section 3, the linear stability of the uncontrolled visco-elastic Beck's beam is discussed. In Section 4, suitable control strategies are introduced. In Section 5, the parametric investigations are presented, and the stability domains are investigated. In Section 6, the concluding remarks are summarized. Finally, in Appendix A, a comparison of the perturbation results obtained following D'Annibale et al. and those here derived numerically is discussed.

### 2 | THE PEM MODEL

The PEM system object of the present study is schematically illustrated in Figure 1. The primary mechanical system is a linear visco-elastic Euler–Bernoulli cantilever beam known in the literature as the visco-elastic Beck's beam (see, e.g., previous studies). The beam has length \(\ell\), cross-section inertia \(I\), and mass per unit-length \(\rho\). It is subject to a follower force \(F\) applied at the free end, inducing a compression state in the reference rectilinear configuration, while its direction remains aligned with the tangent to the beam deflected axis, in the current configuration. The material constituting the beam, obeying the Kelvin–Voigt visco-elastic law, has an elastic modulus \(E\), while the effect of

![Piezoelectric-controlled visco-elastic Beck's beam](image-url)
viscosity is introduced via the coefficient $\eta$ (internal damping); finally, the interaction with the surrounding air is introduced via the constant $c$ (external damping).

The beam is endowed with distributed piezoelectric devices that are here introduced as a couple of equivalent continuous layers bonded to the beam top and bottom faces, as depicted in Figure 1. They are shunted to the electrical circuit represented by the light blue E.C. box in Figure 1. The configuration of the adopted electrical circuit was derived in Maurini et al.\textsuperscript{21} to enhance the dynamic response of nonautonomous systems and is referred to as the Second-order network and Second-order dissipation, namely $(S, S)$, controller. Such nomenclature is driven from the order of the spatial derivative (i.e., the second) of the damping- and stiffness-like terms appearing in the flux-linkage equation: The $(S, S)$ controller reveals to be in analogy with the mechanical system known as the rod, that is, a linear elastic beam undergoing only axial displacements. According to the definition given in Maurini et al.\textsuperscript{21} the piezoelectric devices, shunted to the $(S, S)$ electrical circuit, are idealized as an array of infinite in-parallel RCL elements. In particular, the circuit is characterized by a linear density of piezoelectric capacitance $C$, of inductance $L$ and of resistances $R$ (proportional to the zero-order space derivative term), $r_R$ (proportional to the second-order space derivative term), while it has a piezoelectric coefficient $E_{em}$, that actually provides the coupling between the electrical and mechanical subsystems.

The assembly of the cantilever beam and the so-defined rod-like controller constitutes the PEM system, whose governing equations of motions are derived relying on the same variational approach, as previously mentioned (see D’Annibale et al.\textsuperscript{22} for a detailed derivation). To this end, the following notation and assumptions are introduced. The time is identified by $t$, and the material abscissa is introduced as $s \in [0, \ell]$: The overdot denotes derivation with respect to $t$ and the prime with respect to $s$. The hypothesis of linear kinematics is assumed, and the longitudinal displacement $u(s, t)$ is neglected, while the cross-section rotation is defined by $\theta(s, t) = \psi(s, t)$. The mass and stiffness of the piezoelectric devices are considered negligible, since they are small compared to their mechanical counterpart.

The PEM equations of motion are derived by enforcing the Extended Hamilton Principle for which the following condition has to be satisfied for any kinematically admissible field of variations:

$$\delta H + \delta W = 0,$$

(1)

being $\delta H$ the first variation of the action functional deriving from the conservative terms and $\delta W$ the work spent by the non-conservative actions. The first variation of the action functional is composed of the sum of a mechanical, electrical, and electro-mechanical contribution, namely, $\delta H_m$, $\delta H_e$, and $\delta H_{em}$, which are defined as

$$\delta H_m = \int_{t_1}^{t_2} \int_0^\ell (\rho \ddot{\delta \delta} - EI \dddot{\psi} \delta \delta) \, ds \, dt,$$

$$\delta H_e = \int_{t_1}^{t_2} \int_0^\ell \left( \frac{C \dddot{\psi} \delta \psi}{L} \right) \, ds \, dt,$$

(2)

$$\delta H_{em} = \int_{t_1}^{t_2} \int_0^\ell E_{em} (\psi' \dddot{\delta \psi}' - \dddot{\psi} \psi'') \, ds \, dt.$$

On the other hand, $\delta W$ is represented by the sum of the two contributions deriving from the nonconservative actions taking place in the mechanical and electrical subsystems, $\delta W_m$ and $\delta W_e$, respectively. According to the previous assumptions, they can be defined as

$$\delta W_m = -\int_{t_1}^{t_2} \int_0^\ell \left( c \dddot{\delta \psi} + \eta \dddot{\psi} \dddot{\psi} + F \dddot{\psi} \dddot{\psi} \right) \, ds \, dt,$$

(3)

$$\delta W_e = -\int_{t_1}^{t_2} \int_0^\ell \left( \frac{1}{R} \dddot{\psi} \dddot{\psi}' + \frac{1}{r_R} \dddot{\psi}' \dddot{\psi}'' \right) \, ds \, dt.$$

The variational principle (1) defined above, after substituting expressions (2) and (3) and integrating by parts, delivers the equations of motion governing the behavior of the PEM system. Such equations are suitably expressed in nondimensional form, by introducing the nondimensional time $t = \omega t$ (with $\omega^2 = EI/\rho \ell^4$), abscissa $\tilde{s} = s/\ell$, beam
The beam transverse displacement $\ddot{v}(s,t) = \ddot{v}/v_0$, and flux-linkage $\psi(s,t) = \psi/v_0$ (with $v_0 = v_0\sqrt{\rho/C_0}$) together with the following non-dimensional mechanical and electrical parameters

$$
\begin{align*}
\beta_m &= \frac{c}{\rho_0\omega}, \quad \alpha_m = \frac{\eta I}{\rho_0\omega^2}, \quad \gamma = \frac{\psi_0 E_{em}}{v_0\rho_0\omega^2}, \\
\nu_c &= \frac{C}{C_0}, \quad \beta_e = \frac{1}{RC_0\omega}, \quad \alpha_e = \frac{1}{\rho_0C_0\omega^2}, \quad \kappa_e = \frac{1}{LC_0\omega^2\ell^2}.
\end{align*}
$$

According to the latter assumptions, the non-dimensional PEM equations of motion read

$$
\ddot{v} + \beta_m \dot{v} + \alpha_m \ddot{v} + \nu_e \dot{v} + \kappa_e \psi' + \gamma \psi'' = 0,
$$

where the tilde has been removed for sake of notation. Accordingly, the dot denotes differentiation with respect to the nondimensional time $t$ and the prime with respect to the nondimensional abscissa $s$. Equations (5a–b) represent the nondimensional field equations, while Equations (5c–f) represent the corresponding geometrical and mechanical boundary conditions at $A$ and $B$, respectively. In the latter expressions, $\alpha_m$ and $\beta_m$ ($\theta = m, e$) represent the non-dimensional internal and external (mechanical and electrical) damping coefficients, respectively; $\nu_c$ and $\kappa_e$ are the, here referred to as, nondimensional electrical mass and stiffness, respectively. Finally, $\gamma$ is the coupling parameter, and $\mu$ is the magnitude of the nonconservative force.

### 2.1 The discrete PEM system

The weak form of the dynamic problem described by (1) represents the weak form of the PEM system equations and can be straightforwardly turned into a discretized form, by adopting a classical Galerkin projection approach. The beam transverse displacement $v$ and the flux-linkage $\psi$ are thus expressed according to the following series

$$
v(s,t) = \sum_{k=1}^{N^{(m)}} \phi_k^{(m)}(s) q_k^{(m)}(t) = \Phi_m^T(s) q_m(t) \quad \text{and} \quad \psi(s,t) = \sum_{k=1}^{N^{(e)}} \phi_k^{(e)}(s) q_k^{(e)}(t) = \Phi_e^T(s) q_e(t),
$$

where the superscripts $(m)$ and $(e)$ indicate the terms relative to the mechanical and electrical subsystems. The variables $v$ and $\psi$ are thus expressed as a linear combination of $N^{(m)}$ and $N^{(e)}$ terms, respectively, each representing the product of the $k$th shape function $\phi_k^{(\theta)}(s)$, which are collected in the vector $\Phi_{\theta}(s)$, and the corresponding time-depending amplitude $q_k^{(\theta)}(t)$, which are collected in the vector $q_{\theta}(t)$, with $(\theta = m, e)$.

By substituting expression (6) into Equation (1), the discretized form of the PEM system equations of motion is obtained as expressed below, directly in the nondimensional form:

$$
\begin{align*}
M_m \ddot{q}_m + (\alpha_m K_m + \beta_m M_m) \dot{q}_m + K_m q_m + 2\mu H q_m + \gamma G_m \dot{q}_e &= 0, \\
\nu_c M_e \ddot{q}_e + (\alpha_e K_e + \beta_e M_e) \dot{q}_e + \kappa_e K_e q_e + \gamma G_e q_m &= 0.
\end{align*}
$$

In the latter equations, the space-dependent terms have been integrated over the nondimensional domain and are defined as follows:

$$
\begin{align*}
M_m &= \int_1^0 \Phi_m \Phi_m^T ds, \quad K_m = \int_1^0 \Phi_m \Phi_m^{\nu T} ds, \quad H = \int_1^0 \Phi_m \Phi_m^{\nu T} ds, \quad G_m = \int_1^0 \Phi_m \Phi_m^{\nu T} ds, \\
M_e &= \int_1^0 \Phi_e \Phi_e^T ds, \quad K_e = \int_1^0 \Phi_e \Phi_e^{\nu T} ds, \quad G_e = \int_1^0 \Phi_e \Phi_e^{\nu T} ds, \quad G_e = \int_1^0 \Phi_e \Phi_e^{\nu T} ds.
\end{align*}
$$

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where \( \mathbf{M}_\theta, \mathbf{K}_\theta \) (with \( \theta = m, e \)) are the mass and stiffness matrices of the mechanical and electrical subsystem, respectively, \( \mathbf{H}_m \) is the external action matrix, and \( \mathbf{G}_m \) is the coupling matrix. It is remarked that, due to the gyroscope nature of the electro-mechanical coupling, it is obtained that \( \mathbf{G}_m = \mathbf{G}_e^T \).

In the following, the specific choice here adopted for the trial functions is introduced. The shape functions adopted for \( \nu \) represent the eigenfunctions of a Euler–Bernoulli beam undergoing flexural motion, while for \( \psi \), the analogy with the rod is recalled, thus

\[
\phi_k^{(m)}(s) = c_{k,1}^{(m)} \cos(\beta_k^{(m)} s) + c_{k,2}^{(m)} \cos(\beta_k^{(m)} s) + c_{k,3}^{(m)} \sin(\beta_k^{(m)} s) + c_{k,4}^{(m)} \sinh(\beta_k^{(m)} s),
\]

\[
\phi_k^{(e)}(s) = c_{k,1}^{(e)} \cos(\beta_k^{(e)} s) + c_{k,2}^{(e)} \sin(\beta_k^{(e)} s),
\]

where the coefficients \( c_{k,j}^{(\theta)} \) are found by enforcing the boundary conditions appearing in Equation (5) when \( \gamma = 0 \) and are normalized such that \( \int_0^1 \phi_k^{(\theta)} ds = 1 \), with \( \theta = m, e \), while the constants \( \beta_k^{(m)} \) and \( \beta_k^{(e)} \) assume the well-known expressions

\[
\beta_k^{(m)} = \frac{1}{2\pi} (2k - 1) \text{ with } k = 4, \ldots, N^{(m)},
\]

\[
\beta_k^{(e)} = \frac{\pi}{2} (2k - 1) \text{ with } k = 1, \ldots, N^{(e)}.
\]

By introducing the additional variables \( \mathbf{p}_m = \dot{\mathbf{q}}_m \) and \( \mathbf{p}_e = \dot{\mathbf{q}}_e \), and collecting them into the state vector \( \mathbf{z} = \{\mathbf{q}_m, \mathbf{q}_e, \mathbf{p}_m, \mathbf{p}_e\}^T \), the system of ordinary differential Equations (7) can be expressed in the state space as

\[
\dot{\mathbf{z}} = \mathbf{A} \mathbf{z}
\]

where the matrix \( \mathbf{A} \) results defined by the following expression as

\[
\mathbf{A} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\mathbf{M}^{-1}(\mathbf{K} + 2\mu \mathbf{H}) & 0 & -\mathbf{M}^{-1}(\alpha_m \mathbf{K} + \beta_m \mathbf{M}) & \gamma \mathbf{M}^{-1} \mathbf{G}_m \\
0 & -\mathbf{M}_e^{-1} \mathbf{K}_e & -\gamma \mathbf{M}_e^{-1} \mathbf{G}_e & -\mathbf{M}_e^{-1}(\alpha_e \mathbf{K}_e + \beta_e \mathbf{M}_e)
\end{bmatrix},
\]

being \( \mathbf{I} \) the identity matrix and \( \mathbf{0} \) the square null matrix (\( N^{(m)} = N^{(e)} \) is considered). It turns out then that the PEM stability can be studied by solving the eigenvalue problem associated to the matrix \( \mathbf{A} \), that is, actually a function of the electro-mechanical parameters, that is, \( \mathbf{A} = \mathbf{A}(\mu, \alpha, \alpha_m, \alpha_e, \beta, \beta_m, \beta_e, \gamma, \theta) \). In what follows, the system stability is discussed first for the uncontrolled mechanical subsystem \( (\gamma = 0) \) to highlight important aspects that characterize the response of the primary system; subsequently, the PEM stability \( (\gamma \neq 0) \) is analyzed to investigate the capability of the rod-like controller to affect, and possibly enhance, the beam dynamic stability.

### 3 | LINEAR STABILITY ANALYSIS OF THE UNCONTROLLED SYSTEM

The linear stability of the uncontrolled visco-elastic Beck’s beam has been widely investigated in the literature, in particular the reader is referred to other works. Here, some of the major results are recalled to highlight the beam behavior and the phenomenon that has to be controlled. The linear stability diagram of the visco-elastic Beck’s beam is determined by solving the eigenvalue problem associated with system (5) in the square null matrix \( \mathbf{0} \) and seeking for the set of parameters values at which a Hopf bifurcation takes place. In particular, when \( \mu \) increases from zero, the first eigenvalue (with the lowest frequency) whose real part becomes zero is responsible for the loss of stability of the beam, which occurs at a critical load value \( \mu_c \) and at critical frequency \( \omega_c \), both depending on damping. The problem is generally solved via numerical asymptotic or numerical approaches (see, e.g., Kirillov and Seyranian and Luongo and D’Annibale), since no closed-form solutions can be found.
According to what observed in Luongo and D’Annibale, the effect of $\beta_m$ is always beneficial to the beam stability, while the effect of the small internal damping $\alpha_m$ reveals to be detrimental. In particular, small values of $\alpha_m$ cause a finite lowering of the critical load of the undamped beam, that is, $\mu_c$ (the subscript $c$ refers to the undamped beam, i.e. $\alpha_m = \beta_m = 0$), at which a circulatory Hopf bifurcation occurs. This phenomenon is known in the literature as the “Ziegler paradox” or the “destabilizing effect of damping.” An effective way to illustrate the beam behavior suffering the paradox is to express the (exact) linear stability diagram for the visco-elastic Beck’s beam in terms of $\beta$-isolines in the $(\alpha_m, \mu)$-plane. Those are represented in Figure 2, where the black lines identify the bifurcation loci, that is, the parameter combination $(\beta_m, \alpha_m, \mu_d, \omega_d)$ (the subscript $d$ refers to the damped beam, i.e., $\alpha_m \neq 0, \beta_m \neq 0$) at which the eigenvalue responsible for the system stability attains zero real part, thus giving rise to a Hopf bifurcation; the regions on the left are stable (S), while those on the right are unstable (U). It can be thus observed that, for small values of $\beta_m$, a small value of $\alpha_m$ can cause an important reduction of $\mu_d$ (up to $-50\%$), while if $\beta_m$ is sufficiently large, the presence of small $\alpha_m$ does not induce considerable detrimental effects; that is, the detrimental effect of the paradox is not recognizable. Of course, the case of small $\beta_m$ and $\alpha_m$ is of particular interest. The object of the present work is to investigate whether the presence of the proposed rod-like controller can improve the beam stability and possibly overcome the detrimental effects of damping. In what follows, $\beta_m = 1/10$ and $\alpha_m = 1/100$ are taken (see the black bullet in Figure 2), which correspond to a situation where the system suffers a considerable reduction of the critical load, that is, $\omega_d = 5.92$ and $\mu_d = 6.46$, that is a 30% reduction with respect to the undamped beam.

4 | CONTROL STRATEGY

The aim of this work is to find a suitable electrical parameter range for which the presence of the controller leads to an improvement of the beam stability in terms of critical load. According to what found in D’Annibale et al. and Casalotti and D’Annibale, the electro-mechanical coupling $\gamma$ is assumed to be small, that is, $\gamma < < 1$, to meet common practical applications requirements. Under this assumption, the working principle of the controller can be explained as follows.

- At first, the controller is inactive, and the beam is subject to its critical load $\mu_d$: Its response is thus a large amplitude oscillation at the critical frequency $\omega_d$.
- When the controller is activated, due to the gyroscopic coupling, it results subject to a forcing term having frequency equal to $\omega_d$, and it starts oscillating as well.
- The response of the controller then grows and returns back to the beam, again because of the presence of the gyroscopic coupling, and allows a modification of the mechanical response.

![Figure 2](image-url)  
**Figure 2** Linear stability diagram of the uncontrolled visco-elastic Beck's beam in terms of $\beta$-isolines in the $(\alpha_m, \mu)$-plane. S stable region, U unstable region
Depending on the other electrical parameters values, the electrical response may grow significantly, and thus, its effect on the mechanical subsystem may become considerable. In particular, for controllers with moderate electrical damping, a larger response is expected when \( \omega_d \) is close to the electrical frequency, giving rise to the so-called resonant controller. In particular, according to the control strategies presented in D’Annibale et al. and Casalotti and D’Annibale,22,59 if the electrical mass- and stiffness-like terms, that is, \( \nu_e, k_e \), are comparable to those of the mechanical subsystem, the controller is referred to as the Non-Singular (resonant) Controller (NSC); on the other hand, if \( \nu_e, k_e \) are small, the controller is referred to as the Singular Resonant Controller (SRC). However, as already discussed in D’Annibale et al. and Casalotti and D’Annibale,22,59 this strategy reveals to be mainly detrimental to the beam stability, even though small increments of the Hopf critical load may be found with the SRC, while those with the NSC are negligible. It has been observed, in Casalotti and D’Annibale,59 that the best performance is obtained when the controller has small \( \nu_e, k_e \) and is far from resonance, thus giving rise to the Singular Non-Resonant Controller (SNRC), and in particular when the electrical frequency is smaller than \( \omega_d \) (sub-resonant SNRC), while if it is larger than \( \omega_d \) the controller is always detrimental (super-resonant SNRC).

It is worth to note that a perturbation approach may allow to derive close-form solutions to study the stability of the PEM system. In particular, the discretized equations of motion (7) are formally identical to those describing the naturally discrete PEM system studied in D’Annibale et al.52; thus, the same analytical procedure may be conducted to perform a sensitivity analysis and obtain the stability domains. However, the perturbation approach relies on a specific scaling of the parameters that actually limits the accuracy of such analysis to a restricted parameters space; see Appendix A for further details. Besides, in this work, the interest is to investigate the capability of the proposed controller to improve the system stability on a wider range of parameters, thus entailing that the numerical approach is here preferred.

In what follows, the results of the conducted parametric analyses are presented to discuss the influence of the controller resonance and investigate the effect of the electrical parameters in improving the beam stability.

### 5 NUMERICAL RESULTS

In this section, numerical simulations are carried out to perform a sensitivity analysis on the PEM stability to variations of the nondimensional electrical parameters, with the purpose of evaluating the capability of the rod-like controller to enhance the beam response. As discussed in the previous section, the beam parameters, namely, \( \alpha_m \) and \( \beta_m \), are fixed in order to analyze a situation in which the beam suffers a considerable detrimental effect due to the “Ziegler paradox.” Therefore, an extensive parametric analysis is conducted to find suitable parameter ranges that can improve the PEM stability. To this end, the eigenvalue problem associated to the PEM system is solved by adopting \( N_m^{(m)} = N_m^{(e)} = 9 \), and the stability diagrams are derived by varying the load multiplier \( \mu \) around \( \mu_d \) (the critical Hopf load of the uncontrolled beam) together with selected electrical parameters. The eigenvalue problem is solved for each set of parameters, and the stable part of the domain is found by identifying the eigenvalues having negative real part.

#### 5.1 Sensitivity to the controller frequency

According to what previously discussed, the system stability is strongly influenced by the frequency of the piezoelectric controller, that is, how close it is to the critical frequency of the uncontrolled system \( \omega_d \). In the case here analyzed, the behavior of the proposed rod-like controller is governed by a multimodal response: thanks to the rod analogy, the expression defining the natural frequencies of the undamped electrical subsystem can be defined as

\[
\omega_{\text{e}k} = \frac{\pi}{2}(2k-1)\sqrt{\frac{k_e}{\nu_e}} \quad \text{with} \quad k = 1, 2, ...
\]

where the electrical damping has been neglected. It is thus of interest to investigate the effect of detuning: \( k_e \) is varied across a wide range such that resonance can take place across the first nine electrical frequencies, that is, \( \omega_{\text{e}k} = \omega_d \) with \( k = 1, ..., 9 \).

This aspect is first investigated in absence of electrical damping, that is, \( \alpha_e = \beta_e = 0 \), and the results are illustrated in Figure 3 where the stability diagrams of the PEM system are represented in the plane \( (k_e, \Delta \mu) \) with
The effect of the external electrical damping $\beta_e$ is now analyzed together with the effect of the electro-mechanical coupling $\gamma$. In this case, the stability diagrams of the PEM system are represented in the plane $(\Delta \mu, \beta_e)$ in Figure 5 with $\alpha_e = 0.001$, $\beta_e = 0$. Significant stable regions above the horizontal axis appear at specific values of $\kappa_e$, and the obtained critical load increment is of the same magnitude. In particular, when resonance takes place close to the first electrical mode, that is, $\omega_{e1} \simeq \omega_d$, the stable region is limited to negative increments of the Hopf load $(\Delta \mu < 0)$; that is, the controller is detrimental. On the contrary, when $\kappa_e$ decreases, considerable positive increments of the Hopf load $(\Delta \mu > 0)$ are encountered. In both cases, it is remarked that the largest load increments are encountered when $\kappa_e$ is comprised between two adjacent resonances, except the first electrical mode. This implies that, quite counterintuitively, the controller is effective only when it is far from resonance; however, the positive increment of $\Delta \mu$ becomes weaker as $\kappa_e$ reaches smaller values. Therefore, in the following, the electrical mass is set to $\nu_e = 0.1$, and two selected values of $\kappa_e$ will be considered, namely, $\kappa_{e,1} = 0.075$ and $\kappa_{e,2} = 0.21$, that lead to the largest load increments in both cases.

5.2  Sensitivity to the electrical external-like damping

The effect of the external electrical damping $\beta_e$ is now analyzed together with the effect of the electro-mechanical coupling $\gamma$. In this case, the stability diagrams of the PEM system are represented in the plane $(\Delta \mu, \beta_e)$ in Figure 5 with $\alpha_e = 0.001$, $\beta_e = 0.075$ (Figure 4b). The light blue regions in the plot identify the stability domain, also denoted by $S$; on the contrary, the white regions indicate the unstable part ($U$). The horizontal axis represents the stability boundary of the uncontrolled Beck’s beam, that is, $\Delta \mu = 0$. The results clearly illustrate that when the controller is undamped, that is, $\alpha_e = \beta_e = 0$, its effect is detrimental to the beam stability since the stable region of the PEM system is entirely located below the horizontal axis, resulting only in negative increments of the Hopf load, $\Delta \mu < 0$. The same behavior is observed independently from the adopted control strategy; that is, both the NSC ($\nu_e = 1$, Figure 3a) and the SRC ($\nu_e = 1$, Figure 3b) are detrimental if the electrical damping is not considered. This is in agreement with the results from the literature; however, when the electrical damping is taken into account, the increments of critical load that may be reached when $\nu_e$ is large are considerably smaller than those obtained when $\nu_e$ is small (results not shown here). Therefore, the subsequent analyses are focused on the case $\nu_e = 0.1$.

The effect of electrical damping, both of internal and external type, is now considered. The stability diagrams of the PEM system are illustrated in Figure 4, again in the plane $(\kappa_e, \Delta \mu)$, having set $\gamma = 0.1$, $\nu_e = 0.1$, $\kappa_e \in [0, 5]$ and $\alpha_e = 0.001$, $\beta_e = 0$ (Figure 4a); $\alpha_e = 0$, $\beta_e = 0.075$ (Figure 4b). The choice of these specific values will be clarified next. The same color legend adopted before is maintained, while the dashed gray lines indicate the values of $\kappa_e$ at which a certain electrical mode is tuned to the beam critical frequency $\omega_d$; they thus identify the resonance condition $\omega_{ek} = \omega_d$.

A similar behavior is obtained in both cases that are $\alpha_e = 0.001$, $\beta_e = 0$ (see Figure 4a); $\alpha_e = 0$, $\beta_e = 0.075$ (see Figure 4b): significant stable regions above the horizontal axis appear at specific values of $\kappa_e$, and the obtained critical load increment is of the same magnitude. In particular, when resonance takes place close to the first electrical mode, that is, $\omega_{e1} \simeq \omega_d$, the stable region is limited to negative increments of the Hopf load $(\Delta \mu < 0)$; that is, the controller is detrimental. On the contrary, when $\kappa_e$ decreases, considerable positive increments of the Hopf load $(\Delta \mu > 0)$ are encountered. In both cases, it is remarked that the largest load increments are encountered when $\kappa_e$ is comprised between two adjacent resonances, except the first electrical mode. This implies that, quite counterintuitively, the controller is effective only when it is far from resonance; however, the positive increment of $\Delta \mu$ becomes weaker as $\kappa_e$ reaches smaller values. Therefore, in the following, the electrical mass is set to $\nu_e = 0.1$, and two selected values of $\kappa_e$ will be considered, namely, $\kappa_{e,1} = 0.075$ and $\kappa_{e,2} = 0.21$, that lead to the largest load increments in both cases.
Stability domains of the discretized PEM system in the $(\kappa_e, \Delta \mu)$-plane, when $\gamma = 0.1$, $\kappa_e \in [0, 5]$ and (a) $\alpha_e = 0.001$, $\beta_e = 0$; (b) $\alpha_e = 0$, $\beta_e = 0.075$. Stable regions in light blue, denoted by $S$. Unstable regions in white, denoted by $U$. The dashed gray lines indicate the values of $\kappa_e$ at which the $k$th electrical frequency $\omega_{e,k}$ equals $\omega_d$.

$\nu_e = 0.1$, $\alpha_e = 0$, while $\kappa_e$ and $\gamma$ are set equal to $\kappa_e = \kappa_{e,1}$, $\gamma = 0.1$ (Figure 5a); $\kappa_e = \kappa_{e,2}$, $\gamma = 0.1$ (Figure 5b); $\kappa_e = \kappa_{e,1}$, $\gamma = 0.15$ (Figure 5c); $\kappa_e = \kappa_{e,2}$, $\gamma = 0.15$ (Figure 5d).

It is observed that, when $\kappa_e = \kappa_{e,1}$, $\gamma = 0.1$, the stable region at $\Delta \mu > 0$ first increases with $\beta_e$ up to a maximum of $+30\%$ then decreases tending to negative $\Delta \mu$, as illustrated in Figure 5a, when the electrical damping assumes larger values, that is, $\beta_e > 1$, the stability domain tends to the value $\Delta \mu = 0$ (not shown in the figure). When $\kappa_e = \kappa_{e,2}$, $\gamma = 0.1$, an analogous behavior is observed, see Figure 5b, and an optimal value for $\beta_e$ can be detected around 0.075 leading to an increase in the critical load of $\Delta \mu \simeq +33\%$; however, it is noted that for low values of $\beta_e$, the Hopf load drastically reduces, thus indicating that in this case, an external electrical damping $\beta_e < 0.05$ should not be considered. On the other hand, when $\gamma$ increases, as shown in Figure 5c,d, the maximum $\Delta \mu$ increases up to $40\%$ when $\kappa_e = \kappa_{e,1}$ and up to $\sim 50\%$ when $\kappa_e = \kappa_{e,2}$, but the optimal value of $\beta_e$ moderately changes. Moreover, the results illustrated in Figure 5b,d clearly highlight that for $\beta_e < 0.15$, the PEM system suffers a considerable reduction of stable region. In particular, a sharp cusp is observed on the stability boundary which derives from a change in the eigenvalue governing the system stability. This latter aspect will be further discussed in the follow. It is then remarked that, in the design of the controller, careful attention should be payed to the choice of the external electrical damping to avoid an anticipated Hopf bifurcation: if $\beta_e$ is taken too low or too high the PEM may encounter the instability at moderate increments of $\Delta \mu$, while if it is taken in the optimal range, significant positive load increments may be achieved. The choice of the electromechanical coupling should be also well assessed: in general, a larger $\gamma$ is expected to produce larger load increments; however, the multimodal nature of the PEM system response may lead to premature instability if the choice of $\beta_e$ is not derived accordingly. Finally, it is remarked that, even though small values of $\beta_e$ can lead to positive $\Delta \mu$, the undamped controller, that is $\beta_e \equiv 0$, reveals to be always detrimental (remember Figure 3b).

### 5.3 Sensitivity to the electrical internal-like damping

The effect of the internal damping is finally investigated. In this case, the stability diagrams are derived in the plane $(\Delta \mu, \alpha_e)$ and are illustrated in Figure 6 by adopting $\nu_e = 0.1$, $\gamma = 0.1$ while $\kappa_e = \kappa_{e,1}$, $\beta_e = 0$ (Figure 6a); $\kappa_e = \kappa_{e,2}$, $\beta_e = 0$ (Figure 6b); $\kappa_e = \kappa_{e,1}$, $\beta_e = 0.075$ (Figure 6c); $\kappa_e = \kappa_{e,2}$, $\beta_e = 0.075$ (Figure 6d).

According to the choice of the parameters, as shown in Figure 6a,b, a significant load increment is observed when $\alpha_e$ is small; however, the optimal value is not 0 (remember Figure 3a), but very small, that is, $\alpha_e = 0.001$. If $\alpha_e$ assumes larger values, $\Delta \mu$ decreases considerably until it becomes negative for relatively low values of $\alpha_e$. 

[Figure 4 and Figure 6 images for visual reference are included]
On the other hand, when also $\beta_e$ is introduced, as shown in Figure 6c,d, the detrimental effect at smaller $\alpha_e$ ($<0.001$) disappears, and the largest positive $\Delta \mu$ is achieved at $\alpha_e = 0$, thus corresponding to the maximum $\Delta \mu$ observed in Figure 5c,d, respectively.

It can be then remarked that the electrical internal damping may lead to load increments similar to those obtained with the external damping, but the extension of the stability domain at $\Delta \mu > 0$ is limited to a very small range.

5.4 | Critical mode shape

As already discussed about the results of Figure 5b,d, the stability boundary is characterized by a sharp cusp when the electrical damping overcomes a certain threshold around the value $\beta_e = 0.075$; in particular, this is caused by a change in the mode experiencing the instability. It is thus of interest to analyze the shape of the mode that encounters the Hopf bifurcation in the state of incipient instability (i.e., on the stability boundary) evaluated before and after the cusp (see the gray dots in Figure 5b). The results are illustrated in Figure 7, in terms of $v$ versus $s$ (Figure 7a) and $\psi$ versus $s$ (Figure 7b), when $\nu_e = 0.1$, $\kappa_e = \kappa_{e,1}$, $\gamma = 0.1$. Two selected values are considered for the electrical damping, namely, $\beta_e = 0.05, 0.1$, that are represented by the gray bullets in Figure 5b, so that the system encounters the instability at $\Delta \mu \simeq +33\%$ before the cusp as well as after the cusp. The eigenvectors have been normalized with respect the beam tip displacement, that is set as $v(s = 1) = 1$.

It is observed that the critical mode shape in terms of beam displacement $v$, as shown in Figure 7a, is proportional to the first cantilever mode shape when $\beta = 0.1$ (after the cusp), thus resembling the critical mode shape of the uncontrolled visco-elastic Beck’s beam; however, when $\beta_e = 0.05$ (before the cusp), the mode shape resembles the
second mode of the uncontrolled visco-elastic Beck’s beam. On the other hand, in terms of flux-linkage \( \psi \), see Figure 7b, the critical mode shape turns out to be a combination of the first and second rod modes: The electrical stiffness is here set to \( \kappa_e = \kappa_{e,2} \); and this implies that \( \omega_d \) is comprised between the first and second controller frequencies (remember Figure 4b); thus, it is expected that the controller response will resemble a combination of those modes. It is remarked that there are actually two modes concurring to the PEM stability: depending on the selection of the electrical damping, the mode shape of the mechanical part changes from the first beam mode (\( \beta_e = 0.1 \)) to the second (\( \beta_e = 0.05 \)).
This behavior can be explained by recalling the previously described working principle of the controller. When the load $\mu$ grows and reaches the critical condition $\mu = \mu_d$, the beam experiences the instability: it starts oscillating with a frequency $\omega_d$, and its response $v$ is large. The controller starts oscillating as well due to the electro-mechanical coupling $\gamma v''$ (remember Equation 5b) that represents a forcing term having frequency equal to $\omega_d$. The electrical response will thus be governed by the mode with frequency close to $\omega_d$, and it will grow depending on the values of the other electrical parameters. The electrical contribution returns then back to the mechanical subsystem via the $\gamma \psi''$ term appearing in Equation (5a), thus providing a correction ($\gamma$ is small) for the beam damping and stiffness terms, which result in a modification of the critical Hopf load and the corresponding mode shape.

6 | CONCLUSIONS

A piezoelectric controller for the visco-elastic Beck’s beam subject to a follower force has been proposed. The beam suffers the detrimental effects due to the so-called “Ziegler paradox” and the capability of the controller to overcome them has been investigated. The rod-like controller is composed of a set of distributed piezoelectric patches that are shunted to the so-called $(S,S)$ electrical circuit (from which the analogy with the rod is derived). The assembly of the beam and the controller constitutes the PEM system, whose governing equations of motion have been derived through a variational approach. The eigenvalue problem has been solved numerically via a Galerkin projection approach, and the system stability diagrams have been derived in order to investigate the sensitivity to the controller parameters and evaluate its effectiveness to enhance the beam stability. The major outcomes are summarized below.

1. The controller can be tuned to the beam critical frequency. If resonance is enforced close to the first electrical mode, the controller reveals to be detrimental, while a significant enhancement of the stability domain can be obtained when the electrical stiffness reduces but is kept far from resonance on higher electrical modes.
2. If the controller is not provided with suitable electrical internal or external damping, it can be detrimental.
3. The presence of external electrical damping allows an interesting enhancement of the Hopf critical load, though it reduces if excessive values are adopted. Thus, optimal values can be found, depending on the other electrical parameters, with particular attention to the coupling coefficient.
4. The electro-mechanical coupling allows enhancing the Hopf load of the PEM system, however induces significant detrimental effects if the corresponding external electrical damping is too low.
5. The PEM stability is mainly governed by one mode; however, specific parameter ranges are found to activate instability of multiple modes. This may cause a premature Hopf bifurcation.
6. The internal electrical damping may lead to load increments similar to those obtained with the external damping, but the extension of the improved stability domain is limited to a very small range.

Future developments of the proposed controller will be devoted to the derivation of an asymptotic procedure. This will lead to a more detailed understanding of the PEM response and will allow extensive parametric investigations that may lead to a more accurate definition of the parameter space in which the proposed controller succeeds in enhancing the system stability. The study may also be extended to evaluate the response of the PEM in the post-critical regime and to investigate the capability of the controller to reduce the limit-cycle oscillations arising when the Hopf bifurcation is overcome.

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AUTHOR CONTRIBUTIONS
All authors contributed equally to this work. Francesco D’Annibale conceived the scientific idea of this paper. All authors carried out the numerical simulations. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.
DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A: A Perturbation approach

The discretized system (7) is formally analogous to the naturally discrete PEM system analyzed in D’Annibale et al.; thus, a similar sensitivity analysis can be conducted, provided that the electrical parameters are sufficiently small (i.e., they satisfy the adopted scaling). The approach consists of addressing the eigenvalue problem associated to Equation (7) and finding the closed-form solutions for the critical eigenvalue of the PEM system (depending on the system parameters) as a perturbation of the critical eigenvalue of the uncontrolled beam, namely, $\lambda_0 = \omega_d$. The descending sensitivity analysis must be carried out distinguishing two cases: (a) for the SRC, the critical eigenvalue is defective which requires a fractional power series expansion of the variables; (b) for the SNRC, the critical eigenvalue is simple, and the variables can be expanded following a MacLaurin series. For the complete derivation of the different perturbation schemes, the reader is referred to D’Annibale et al.

Here, the stability domains numerically derived are compared with those obtained via the analytical approach of D’Annibale et al. The adopted electrical parameters are chosen in such a way that (i) they match the scaling of D’Annibale et al.; (ii) they are moderately larger. The aim of this analysis is to show that the analytical approach is accurate in the former case, while in the latter, more complex behaviors arise, and a loss of accuracy is recognized.

Indeed, as illustrated in Figure A1a, when the electrical parameters are sufficiently small, the boundary of the stability domain predicted by the sensitivity analysis (blue line) well resembles the numerical solution, though quantitatively a moderate difference can be observed. However, when the electrical parameters are moderately larger, the relevant stability domain is shown in Figure A1b. In this case, qualitative differences between the numerical and asymptotic results are found since, due to the chosen parameters, a stronger interaction between the primary (beam) and secondary (controller) subsystems occurs. This leads to richer behaviors, also involving complex modal interactions, that cannot be well captured by the perturbation approach.

![Figure A1](image-url)