Abstract. We study a natural Dirac operator on a Lagrangian submanifold of a Kähler manifold. We first show that its square coincides with the Hodge-de Rham Laplacian provided the complex structure identifies the Spin structures of the tangent and normal bundles of the submanifold. We then give extrinsic estimates for the eigenvalues of that operator and discuss some examples.

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Key words: Dirac operators, Global Analysis, Spectral Geometry, Spin Geometry, Lagrangian submanifolds

Introduction

The main object of this paper is to initiate the study of the properties of a Dirac operator on Lagrangian submanifolds of Kähler manifolds.

Spin geometry has revealed as a powerful tool in intrinsic geometry for a long time (see e.g. [20]). It is however a recent and striking fact that spinors play a role in extrinsic geometry as well. Initiated by Witten [28], the use of Dirac operators on submanifolds has only been developed over the last years, especially about the following question: how can one relate analytical properties of some Dirac operators on a submanifold with extrinsic geometric quantities? For submanifolds of real space-forms, on which there exists particular spinor fields (mainly parallel spinor fields, up to a conformal change of the metric), a beautiful series of results has already appeared (see [11] for references). However, answering that question in presence of further geometric structures seems to have been little considered.

We propose in this paper to begin with the study of (immersed) submanifolds of Kählerian manifolds. The presence of a complex structure on the ambient manifold gives rise to a rich variety of submanifolds (totally real, Kählerian, real hypersurface,...). That is why we shall restrict our attention to a particular class of submanifolds, namely Lagrangian submanifolds. A submanifold of a Kählerian manifold is Lagrangian if and only if the (ambient) complex structure maps its tangent bundle onto its normal bundle. Like every submanifold, a Lagrangian submanifold carries a twisted-Dirac operator. We shall first prove that, if furthermore the complex structure identifies the Spin structures of the tangent and normal bundles, then this twisted-Dirac operator identifies with the Euler operator. This requires adapting some technical algebraic Lemmas (compare with [14, 2]), which we shall therefore recall in detail in the first part. Coming back to the original question, we then prove new eigenvalue estimates for the above twisted-Dirac operator, and show their sharpness through examples. The results obtained show analogies with [7, 26].

This work is partially based on the Ph.D.-thesis of the author [11].

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1 Spin structures and Dirac operators on a Lagrangian submanifold

We begin with collecting basic facts about Spin structures on Lagrangian submanifolds of Kählerian manifolds (see also [20, 6, 8] for general Spin geometry). We first describe the necessary algebraic material, then transport it to bundles with the help of a group-equivariance condition.

1.1 Clifford algebras and spinors

In this subsection, we recall some important isomorphisms between the complex Clifford algebra (see definition below) and other vector spaces. We point out that the isomorphism (9) below slightly differs from the equivalent one in [14] or [2], since we want here to keep track of the “Clifford action” in a more suitable way for our setting.

We fix a positive integer \( n \), and denote by “can” the standard Euclidean inner product of \( \mathbb{R}^n \). Throughout this paper, unless explicitly mentioned, all the isomorphisms will be denoted by the identity map.

Let \( \mathbb{C} \) be the complex (resp. real) algebra with unit generated by \( \mathbb{R} \) and all the isomorphisms will be denoted by the identity map.

Let \( \mathbb{C} \) (resp. \( \mathbb{C} \)) be the complex (resp. real) Clifford algebra of \( (\mathbb{R}^n, \text{can}) \), that is, the only associative complex (resp. real) algebra with unit generated by \( \mathbb{R}^n \) with the relation

\[
v \cdot w + w \cdot v = -2\text{can}(v, w)1,
\]

for all vectors \( v \) and \( w \) in \( \mathbb{R}^n \). The product “\( \cdot \)” is called the Clifford multiplication. We recall the properties of \( \mathbb{C} \) which will be important for the future:

- Let \( \Lambda \mathbb{C} \) be the complexified exterior algebra of \( \mathbb{R}^n \). Then there exists a canonical linear isomorphism \([20]\)

\[
\mathbb{C} \longrightarrow \Lambda \mathbb{C} \quad \text{(1)}
\]

which maps every element of the form \( v \cdot \varphi (v \in \mathbb{R}^n, \varphi \in \mathbb{C} \) onto \( v \wedge \varphi - v \cdot \varphi \), where “\( v \cdot \varphi \)” stands for \( v^\prime \cdot \varphi \) through the musical isomorphism \( v \longrightarrow v^\prime : = \text{can}(v, \cdot) \) between \( \mathbb{R}^n \) and \( (\mathbb{R}^n)^* \) . We hereby identify through \([11]\) the space \( \Lambda \mathbb{C} \) as a subspace of \( \mathbb{C} \).

- The algebra \( \mathbb{C} \) is either a matrix algebra or the copy of two such ones: there exists a complex algebra \( \Sigma \) of dimension \( 2^\frac{n}{2} \), called the space of spinors, and an isomorphism of complex algebras

\[
\mathbb{C} \cong \begin{cases} \text{End}_{\mathbb{C}}(\Sigma) & \text{if } n \text{ is even} \\ \text{End}_{\mathbb{C}}(\Sigma) \oplus \text{End}_{\mathbb{C}}(\Sigma) & \text{if } n \text{ is odd}. \end{cases} \quad \text{(2)}
\]

Without loss of generality (see e.g. [20]), we further assume that, when \( n \) is odd, the isomorphism \([2]\) maps the complex volume element of \( \mathbb{C} \) (see e.g. in [6] for its definition) onto \( 1 \text{Id}_{\Sigma} \oplus -1\text{Id}_{\Sigma} \). We define \( \delta_n \) as the isomorphism \([2]\) if \( n \) is even, and the composition of the projection onto the first subalgebra \( \text{End}_{\mathbb{C}}(\Sigma) \) with \([2]\) if \( n \) is odd. In particular, for \( n \) odd and for every \( v \) in \( \mathbb{R}^n \), the isomorphism \([2]\) reads \( v \longrightarrow \delta_n(v) \oplus -\delta_n(v) \), see [20].

- The space \( \Sigma \) carries a natural Hermitian inner product “\( \langle \cdot, \cdot \rangle \)” (which we assume to be complex-linear in the first argument) such that, for every \( v \) in \( \mathbb{R}^n \) and all \( \sigma, \sigma’ \) in \( \Sigma \),

\[
\langle \delta_n(v)\sigma, \sigma’ \rangle = -\langle \sigma, \delta_n(v)\sigma’ \rangle.
\]

The property \([3]\) determines the Hermitian inner product “\( \langle \cdot, \cdot \rangle \)” up to a positive scalar \([6]\).

Define now the Spin group \( \text{Spin}_n \) as

\[
\text{Spin}_n := \{ v_1 \cdot \ldots \cdot v_{2k} \mid k \geq 1, \ v_j \in \mathbb{R}^n, \ \text{can}(v_j, v_j) = 1 \}
\]

and the Spin representation to be the restriction of \( \delta_n \) to \( \text{Spin}_n \). The Spin group is a compact Lie-subgroup of the group of invertible elements in \( \mathbb{C} \) which has the following remarkable properties:

- There exists a two-fold covering Lie-group-homomorphism from \( \text{Spin}_n \) onto the special orthogonal group \( \text{SO}_n \), which we denote by “\( \text{Ad} \)”. 

2
Denoting also “Ad” the composition of the natural representation of $SO_n$ on $\mathbb{R}^n \otimes \mathbb{C}$ with Ad, the isomorphism (11) is Spin-$n$-equivariant, i.e., for every $u$ in Spin$_n$ and $\varphi$ in Cl$_n$,

$$u \cdot \varphi \cdot u^{-1} \simeq \text{Ad}(u)\varphi$$

through (11).

Every Hermitian inner product “($\cdot$, $\cdot$)” satisfying (3) is Spin$_n$-invariant, i.e., the Spin representation is unitary w.r.t. “($\cdot$, $\cdot$)”.

We now recall two lemmas and discuss their consequences.

**Lemma 1** There exists a complex-antilinear automorphism $\gamma$ of $\Sigma_n$ commuting with the Spin representation, i.e., for every $u$ in Spin$_n$, we have $\delta_n(u) \circ \gamma = \gamma \circ \delta_n(u)$.

**Proof:** Although it follows from representation theory (see e.g. [27], p. 21), we give here an elementary argument.

From the classification of real Clifford algebras (see [20]), we have:

$$\text{Cl}_n \cong \begin{cases} \mathbb{R} (2^{\lfloor n/2 \rfloor}) & \text{if } n \equiv 0 \text{ or } 6 \pmod{8} \\ \mathbb{R} (2^{\lfloor n/2 \rfloor}) \oplus \mathbb{R} (2^{\lfloor n/2 \rfloor}) & \text{if } n \equiv 7 \pmod{8} \\ \mathbb{H} (2^{\lfloor n/4 \rfloor}) & \text{if } n \equiv 2 \text{ or } 4 \pmod{8} \\ \mathbb{H} (2^{\lfloor n/4 \rfloor}) \oplus \mathbb{H} (2^{\lfloor n/4 \rfloor}) & \text{if } n \equiv 3 \pmod{8} \\ \mathbb{C} (2^{\lfloor n/2 \rfloor}) & \text{if } n \equiv 1 \pmod{8} \end{cases}$$

As $\text{Cl}_n \cong \text{Cl}_n \otimes \mathbb{C}$, we see that, if $n \equiv 6, 7$ or $8 \pmod{8}$, the complex representation $\delta_n : \text{Cl}_n \rightarrow \text{End}_\mathbb{C}(\Sigma_n)$ admits a real structure, i.e., there exists a $\mathbb{C}$-antilinear and involutive automorphism $\gamma$ of $\Sigma_n$ such that, for every vector $v$ in $\mathbb{R}^n$ (hence for every element in $\text{Cl}_n$),

$$\delta_n(v) \circ \gamma = \gamma \circ \delta_n(v).$$

If $n \equiv 2, 3$ or $4 \pmod{8}$, there exists a quaternionic structure on $\Sigma_n$, i.e. a $\mathbb{C}$-antilinear automorphism $\gamma$ of $\Sigma_n$ satisfying $\gamma^2 = -\text{Id}$ and the preceding relation. If $n \equiv 1 \pmod{4}$, the real representation of $\text{Cl}_n$ being already complex, there exists no $\mathbb{C}$-antilinear automorphism of $\Sigma_n$ commuting with the action of every vector of $\mathbb{R}^n$ as before. However, that relation needs only to hold on Spin$_n$ and not on Cl$_n$. We eliminate the case $n = 1$, for which this is obviously true. For $n > 1$, as Spin$_n$ is a subset of $\text{Cl}_n^o := \bigoplus_{p \text{ even}} \Lambda^p \mathbb{R}^n$, and $\text{Cl}_n^o$ identifies with $\text{Cl}_{n-1}$ through an isomorphism which provides the equivalence of $\delta_{n-1}$ (or “double copy” as in [2]) with $(\delta_n)$ on $\text{Cl}_n^o$ (see [20]), we just need to solve the problem for $\delta_{n-1}$. But from the preceding arguments, the representation $\delta_{n-1}$ admits such a structure, so that we again obtain a $\mathbb{C}$-antilinear automorphism $\gamma$ of $\Sigma_n$ which commutes with $(\delta_n)_{|_{\text{Spin}_n}}$.

To sum up, for every $n \geq 1$, there exists a $\mathbb{C}$-antilinear automorphism $\gamma$ of $\Sigma_n$ such that, for every $u$ in Spin$_n$,

$$\delta_n(u) \circ \gamma = \gamma \circ \delta_n(u),$$

which is the desired property.

\[\square\]

**Corollary 1** (10; 11, p. 244) There exists a complex-linear isomorphism

$$\text{Cl}_n \rightarrow \begin{cases} \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is even} \\ \Sigma_n \otimes \Sigma_n \oplus \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is odd} \end{cases}$$

(4)

satisfying:

- For every $v$ in $\mathbb{R}^n$ and every $\varphi$ in $\text{Cl}_n$, the element $v \cdot \varphi$ is mapped onto $\{\delta_n(v) \otimes \text{Id}\} \varphi$ when $n$ is even (resp. onto $\{\delta_n(v) \otimes \text{Id} \oplus -\delta_n(v) \otimes \text{Id}\} \varphi$ when $n$ is odd),
\[ \text{The isomorphism } \mathcal{J} \text{ is } \text{Spin}_n\text{-equivariant: for every } u \text{ in } \text{Spin}_n \text{ and every } \varphi \text{ in } \mathbb{C}l_n, \text{ the element } u \cdot \varphi \cdot u^{-1} \text{ is mapped onto } \{ \delta_n(u) \otimes \delta_n(u) \} \varphi \text{ when } n \text{ is even (resp. onto } \{ \delta_n(u) \otimes \delta_n(u) \otimes \delta_n(u) \otimes \delta_n(u) \} \varphi \text{ when } n \text{ is odd).} \]

**Proof:** The existence of \( \varphi \) allows to define a complex-linear isomorphism

\[
\begin{align*}
\Sigma_n & \rightarrow \Sigma_n^* \\
\sigma & \mapsto \langle \cdot , j(\sigma) \rangle,
\end{align*}
\]

which is Spin\(_n\)-equivariant: for every \( u \) in Spin\(_n\) and every \( \sigma \) in \( \Sigma_n \),

\[
\langle \cdot , j(\delta_n(u)\sigma) \rangle = \langle \cdot , \delta_n(u)j(\sigma) \rangle
\]

But, from (3), for every unit vector \( \sigma \) for the second factor of the right member does not depend on the choice of the o.n.b. \((\sigma)\) and we obtain \( \varphi \) and every \( \varphi \),

\[
\begin{align*}
\{ \delta_n(u)^{-1}, \cdot, \delta_n(u) \}& \\
& = \{ \delta_n(u)^{-1} \cdot, \cdot, \delta_n(u) \}
\end{align*}
\]

that is, the isomorphism \( \mathcal{J} \) maps \( \delta_n(u)\sigma \) onto \( \{ \delta_n(u)^{-1} \} \{ \cdot, j(\sigma) \} \). Composing then the canonical isomorphism \( \text{End}_{\mathbb{C}}(\Sigma_n) \cong \Sigma_n \otimes \Sigma_n^* \) with \( \mathcal{J} \) and using (3), we obtain the isomorphism \( \mathcal{J} \), which obviously satisfies the desired property w.r.t. the left Clifford action. Let now \( v \) in \( \mathbb{R}^n \) such that \( \text{can}(v,v) = 1 \) and \( \varphi \) in \( \mathbb{C}l_n \). Assume \( n \) even and fix an o.n.b. \( (\sigma_k)_{1 \leq k \leq n} \) of \( \Sigma_n \); then, noting that \( \mathcal{J} \) maps \( j^{-1}(\sigma_k) \) onto \( \sigma_k^\prime \),

\[
\sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi \cdot v)\sigma_k \otimes j^{-1}(\sigma_k) = \sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi)\delta_n(v)\sigma_k \otimes j^{-1}(\delta_n(-v)\delta_n(v)\sigma_k)
\]

\[
= -\sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi)\delta_n(v)\sigma_k \otimes \delta_n(v)j^{-1}(\delta_n(v)\sigma_k).
\]

But, from (3), for every unit vector \( v \) of \( \mathbb{R}^n \), the endomorphism \( \delta_n(v) \) of \( \Sigma_n \) is unitary, so that the vectors \( \sigma_k^\prime := \delta_n(v)\sigma_k \) form an o.n.b. of \( \Sigma_n \). Hence we have

\[
\sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi \cdot v)\sigma_k \otimes j^{-1}(\sigma_k) = -\sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi)\sigma_k^\prime \otimes \delta_n(v)j^{-1}(\sigma_k^\prime)
\]

\[
= -\{ \text{Id} \otimes \delta_n(v) \} \sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi)\sigma_k^\prime \otimes j^{-1}(\sigma_k^\prime)
\]

\[
= -\{ \text{Id} \otimes \delta_n(v) \} \sum_{k=1}^{2^{[n/2]}} \delta_n(\varphi)\sigma_k \otimes j^{-1}(\sigma_k),
\]

for the second factor of the right member does not depend on the choice of the o.n.b. \( (\sigma_k)_k \) of \( \Sigma_n \). Hence we have, through (3),

\[
\varphi \cdot v = -\{ \text{Id} \otimes \delta_n(v) \} \varphi,
\]

which then obviously holds for every \( v \) in \( \mathbb{R}^n \). For the case \( n \) odd, the computations go the same way (just beware of the sign!) and we obtain

\[
\varphi \cdot v = -\{ \text{Id} \otimes \delta_n(v) \} \varphi,
\]

so that, for every \( v \) in \( \mathbb{R}^n \) and every \( \varphi \) in \( \mathbb{C}l_n \),

\[
\varphi \cdot v = \begin{cases} -\{ \text{Id} \otimes \delta_n(v) \} \varphi & \text{if } n \text{ is even} \\ -\{ \text{Id} \otimes \delta_n(v) \} \varphi & \text{if } n \text{ is odd.} 
\end{cases}
\]

(6)

Since, for a (real) unit vector \( v \), we have \( v^{-1} = -v \) in \( \mathbb{C}l_n \setminus \{0\} \), we deduce from (3) that, for every \( u \) in \( \text{Spin}_n \),

\[
\varphi \cdot u = \begin{cases} \{ \text{Id} \otimes \delta_n(u^{-1}) \} \varphi & \text{if } n \text{ is even} \\ \{ \text{Id} \otimes \delta_n(u^{-1}) \} \varphi & \text{if } n \text{ is odd,}
\end{cases}
\]

from which follows the Spin\(_n\)-equivariance. \( \square \)
For the next lemma, we recall an explicit description of the space of spinors as a subspace of the complex Clifford algebra in even dimensions (compare with [6] [18] [19]). We consider $\mathbb{R}^{2n}$ endowed with its natural complex structure $J$, so that $\mathbb{R}^{2n} = \mathbb{R}^n \oplus J(\mathbb{R}^n)$. Let $p_\pm$ the two projectors of $\mathbb{R}^{2n} \otimes \mathbb{C}$ defined by

$$p_\pm = \frac{1}{2} (\mathrm{Id} \mp iJ),$$

where $J$ is extended as a complex-linear automorphism of $\mathbb{R}^{2n} \otimes \mathbb{C}$. The endomorphisms $p_+$ and $p_-$ satisfy the following: $p_+ \circ p_+ = p_+ \circ p_- = 0$, $p_+ \circ J = J \circ p_+ = \pm ip_\pm$, and $\mathrm{can}(p_+(Z), Z') = \mathrm{can}(Z, p_-(Z'))$ for all $Z, Z'$ in $\mathbb{R}^{2n} \otimes \mathbb{C}$. Let $(e_j)_{1 \leq j \leq n}$ be the canonical basis of $\mathbb{R}^n$. For $1 \leq j \leq n$, define $z_j := p_+(e_j)$ and $\overline{z}_j := p_-(e_j)$. The vectors $z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n$ form the so-called Witt-basis of $\mathbb{R}^{2n} \otimes \mathbb{C}$ associated to the basis $(e_j)_{1 \leq j \leq n}$ of $\mathbb{R}^n$. Set

$$\mathfrak{w} := \mathfrak{z} \cdot \mathfrak{z}_1 \cdot \ldots \cdot \mathfrak{z}_n.$$

From the above properties of $p_\pm$, that element of $\mathfrak{Cl}_{2n}$ is independent of the choice of the p.o.n.b. of $\mathbb{R}^{2n}$: replacing $(e_j)_{1 \leq j \leq n}$ by another p.o.n.b. of $\mathbb{R}^n$, and taking the associated Witt-basis of $\mathbb{R}^{2n} \otimes \mathbb{C}$, one obtains the same element $\mathfrak{w}$. For $1 \leq p \leq n$, set $\mathcal{L}^p := \mathrm{Span}_\mathbb{C} \{ z_{i_1} \cdot \ldots \cdot z_{i_p} \cdot \mathfrak{w}, \quad 1 \leq i_1 < \cdots < i_p \leq n \}$ and $\mathcal{L}^0 := \mathbb{C}\mathfrak{w}$. Those subspaces of $\mathfrak{Cl}_{2n}$ do not depend on the choice of a p.o.n.b. of $\mathbb{R}^n$, in the preceding sense. It can furthermore be shown that $\bigoplus_{p=0}^n \mathcal{L}^p$ is a left-ideal of dimension $2^n$ in $\mathfrak{Cl}_{2n}$, hence is isomorphic to $\Sigma_{2n}$ (see [3]). We can then set

$$\Sigma_{2n} := \bigoplus_{p=0}^n \mathcal{L}^p.$$ 

Through that identification, for each $\psi$ in $\mathfrak{Cl}_{2n}$, the endomorphism $\delta_{2n}(\psi)$ is given by the left-Clifford multiplication by $\psi$. For example (and this will be crucial for the future), the Clifford multiplication by the Kähler form $\tilde{\Omega}(\cdot, \cdot) = \mathrm{can}(J \cdot, \cdot)$ of $(\mathbb{R}^{2n}, J)$ is given by

$$\delta_{2n}(\tilde{\Omega}) = \bigoplus_{p=0}^n (2p-n) \mathrm{Id}_{\mathcal{L}^p},$$

that is, $\mathcal{L}^p$ is the eigenspace of $\delta_{2n}(\tilde{\Omega})$ for the eigenvalue $i(2p-n)$. Moreover, a Hermitian inner product satisfying (3) can be defined in the following way: for any $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq n$, set

$$(z_{i_1} \cdot \ldots \cdot z_{i_p} \cdot \mathfrak{w}, z_{j_1} \cdot \ldots \cdot z_{j_q} \cdot \mathfrak{w}) := \left\{ \begin{array}{ll} 0 & \text{if } \{i_1, \ldots, i_p\} \not= \{j_1, \ldots, j_q\} \\ 2^{[\frac{n+1}{2}]} & \text{otherwise.} \end{array} \right.$$ 

It can be shown that (3) holds and that this Hermitian inner product does not depend on the choice of a p.o.n.b. of $\mathbb{R}^n$.

We furthermore define $\mathrm{Spin}'_{n}$ to be the Spin group of $(\mathbb{J}\mathbb{R}^n, \mathrm{can})$, i.e.,

$$\mathrm{Spin}'_{n} := \{ w_1 \cdot \ldots \cdot w_{2k} \mid k \geq 1, \quad w_j \in \mathbb{J}\mathbb{R}^n, \quad \mathrm{can}(w_j, w_j) = 1 \} \subset \mathfrak{Cl}_{2n}.$$ 

From the universal property of Clifford algebras [20], the linear isometry $J : \mathbb{R}^n \rightarrow \mathbb{J}\mathbb{R}^n$ induces a Lie-group-isomorphism $\tilde{J} : \mathrm{Spin}_n \rightarrow \mathrm{Spin}'_{n}$. We then set $\delta'_{n} : = \delta_{n|_{\mathrm{Spin}_n}} \circ (\tilde{J})^{-1} : \mathrm{Spin}'_{n} \rightarrow \text{Aut}_{\mathbb{C}}(\Sigma_{n}).$

**Lemma 2** Consider $\mathfrak{Cl}_{n}$ as canonically embedded in $\mathfrak{Cl}_{2n}$. Then the map

$$\mathfrak{Cl}_{n} \rightarrow \Sigma_{2n}$$

$$\varphi \mapsto \varphi \cdot \mathfrak{w}$$

is a complex-linear isomorphism satisfying:

- For every $v$ in $\mathbb{R}^n$ and every $\varphi$ in $\mathfrak{Cl}_{n}$, the element $v \cdot \varphi$ is mapped onto $v \cdot \varphi \cdot \mathfrak{w}$.

- For every $w$ in $\mathbb{J}\mathbb{R}^n$ and every $\varphi$ in $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$, the element $(-1)^{p+1} i\varphi \cdot J(u)$ is mapped onto $w \cdot \varphi \cdot \mathfrak{w}$.

In particular, the isomorphism (7) is $\mathrm{Spin}_{n}$-equivariant w.r.t. the “diagonal immersion”

$$\mathrm{Spin}_{n} \rightarrow \mathrm{Spin}_{2n}$$

$$u \mapsto u \cdot \tilde{J}(u),$$

that is, for every $u$ in $\mathrm{Spin}_{n}$ and $\varphi$ in $\mathfrak{Cl}_{n}$, the element $u \cdot \varphi \cdot u^{-1}$ is mapped onto $u \cdot \tilde{J}(u) \cdot \varphi \cdot \mathfrak{w}$.
Proof: Since the linear map $\mathcal{L}$ is obviously surjective (for each $1 \leq p \leq n$ and each $1 \leq i_1 < \ldots < i_p \leq n$, the element $e_{i_1} \cdots e_{i_p}$ of $\mathbb{C}_n$ is mapped onto $z_{i_1} \cdots z_{i_p} \cdot \overline{w}$), and both spaces have the same dimension, it is a linear isomorphism. Furthermore, $\mathcal{L}$ maps the subspace $\Lambda^p \mathbb{R}^n \otimes \mathbb{C}$ onto $\mathcal{L}^p$.

The first property is trivial. On the other hand, for every $w$ in $\mathcal{L}^n$ and $\varphi$ in $\Lambda^p \mathbb{R}^n \otimes \mathbb{C}$,

$$(-1)^{p+1}i\varphi \cdot J(w) \cdot \overline{w} = (-1)^p \varphi \cdot (-ip_+ (J(w)) \cdot \overline{w})$$

$$= (-1)^p \varphi \cdot p_+(w) \cdot \overline{w}$$

$$= (-1)^p \varphi \cdot w \cdot \overline{w}$$

$$= w \cdot \varphi \cdot \overline{w},$$

hence the second point holds. As a consequence, for all $w_1, w_2$ in $\mathcal{L}^n$ with $\text{can}(w_1, w_1) = \text{can}(w_2, w_2) = 1$ and every $\varphi$ in $\mathcal{L}^p$, the preimage through $\mathcal{L}$ of $w_1 \cdot w_2 \cdot \varphi \cdot \overline{w}$ is given by

$$(-1)^p i\{w_2 \cdot \varphi \} \cdot J(w_1) \cdot \overline{w} = (-1)^p(-1)^{p+1}i\varphi \cdot J(w_2) \cdot J(w_1) \cdot \overline{w}$$

$$= \varphi \cdot J(w_2) \cdot J(w_1) \cdot \overline{w},$$

i.e., is equal to $\varphi \cdot J(w_2) \cdot J(w_1)$. Note that it does no longer depend on $p$. Since $J(w_2) \cdot J(w_1) = (\bar{J})^{-1}(w_2 \cdot w_1) = (\bar{J})^{-1}\{ (w_1 \cdot w_2)^{-1} \}$, we obtain

$$\varphi \cdot (\bar{J})^{-1}(u^{-1}) \sim u' \cdot \varphi \cdot \overline{w}$$

for every $u'$ in Spin'$_n$ and every $\varphi$ in $\mathbb{C}_n$, from which follows the last statement.

\[\square\]

Corollary 2 There exists a complex-linear isomorphism

$$\Sigma_{2n} \longrightarrow \begin{cases}
\Sigma_n \otimes \Sigma_n & \text{if } n \text{ is even} \\
\Sigma_n \otimes \Sigma_n \oplus \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is odd}
\end{cases}$$

(9)

satisfying:

- For every $v$ in $\mathbb{R}^n$ and $\varphi$ in $\Sigma_{2n}$, the element $\delta_{2n}(v)\varphi$ is mapped onto $\{\delta_n(v) \otimes \text{Id} \} \varphi$ if $n$ is even (resp. onto $\{\delta_n(v) \otimes -\delta_n(v) \otimes \text{Id} \} \varphi$ if $n$ is odd),

- For every $w$ in $\mathcal{L}^n$ and $\varphi$ in $\mathcal{L}^p$, the element $\delta_{2n}(w)\varphi$ is mapped onto $(-1)^pi\{\text{Id} \otimes \delta_n(J(w))\} \varphi$ if $n$ is even (resp. onto $(-1)^p\{\text{Id} \otimes \delta_n(J(w)) \otimes -\text{Id} \otimes \delta_n(J(w))\} \varphi$ if $n$ is odd),

- If $\Sigma_n \otimes \Sigma_n$ is endowed with the tensor product of a Hermitian inner product satisfying $\mathcal{L}$ with itself, then the isomorphism $\mathcal{L}$ is unitary.

In particular, the inverse of $\mathcal{L}$ is Spin$_n \times$ Spin'$_n$-equivariant w.r.t. the group-homomorphism

$$\text{Spin}_n \times \text{Spin}'_n \longrightarrow \text{Spin}_{2n}$$

$$(u, u') \longrightarrow u \cdot u',$$

that is: for every $(u, u')$ in Spin$_n \times$ Spin'$_n$ and every $\varphi$ in $\Sigma_{2n}$, the isomorphism $\mathcal{L}$ maps the element $\delta_{2n}(u \cdot u')\varphi$ onto $\{\delta_n(u) \otimes \delta'_n(u')\} \varphi$ if $n$ is even (resp. onto $\{\delta_n(u) \otimes \delta'_n(u') \otimes \delta_n(u) \otimes \delta'_n(u')\} \varphi$ if $n$ is odd).

Proof: The isomorphism $\mathcal{L}$ is obtained bringing together the isomorphisms $\mathcal{L}$ and $\mathcal{L}$, and straightforward satisfies the first property. The second one follows from Lemma 2 and from (7). The third one comes from the fact that the squared-norm of the image of $e_{i_1} \cdots e_{i_p} \cdot \overline{w}$ is $|\overline{w}|^2$ (remember that $(e_j)_{1 \leq j \leq n}$ stands here for the canonical basis of $\mathbb{R}^n$). The last statement follows from the two first ones, since for all vectors $w_1$ and $w_2$ in $\mathcal{L}^n$, the isomorphism $\mathcal{L}$ maps $\delta_{2n}(w_1)\delta_{2n}(w_2)$ onto $\text{Id} \otimes \{\delta_n(J(w_1))\delta_n(J(w_2))\}$ if $n$ is even (resp. onto $\text{Id} \otimes \{\delta_n(J(w_1))\delta_n(J(w_2))\}$) if $n$ is odd.

\[\square\]

Corollary 3 There exists a complex-linear isomorphism

$$\Sigma_{2n} \longrightarrow \Lambda^p \mathbb{R}^n \otimes \mathbb{C}$$

(10)

satisfying:
For every $v$ in $\mathbb{R}^n$ and $\varphi$ in $\Sigma_{2n}$, the element $\delta_{2n}(v)\varphi$ is mapped onto $v \wedge \varphi - v \cdot \varphi$.

For every $w$ in $J\mathbb{R}^n$ and $\varphi$ in $\Sigma_{2n}$, the element $\delta_{2n}(w)\varphi$ is mapped onto $-i\{J(w) \wedge \varphi + J(w) \cdot \varphi\}$.

In particular, the isomorphism (10) is $\text{Spin}_n$-equivariant w.r.t. (8), i.e., for every $u$ in $\text{Spin}_n$ and every $\varphi$ in $\Sigma_{2n}$, the isomorphism (10) maps $u \cdot J(u) \cdot \varphi$ onto $\text{Ad}(u)\varphi$.

Proof: As before, the isomorphism (10) is obtained from the isomorphisms (11) and (42), and satisfies the first property. The second statement comes from the fact that, for every vector $v$ in $\mathbb{R}^n$ and every $\varphi$ in $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$, the element $\varphi \cdot v$ corresponds through (11) to the form $(-1)^p \{v \wedge \varphi + v \cdot \varphi\}$ (see (24)). The last one straightforward follows from the preliminary remarks and Lemma 2.

\[\square\]

### 1.2 Spinor bundles on a Lagrangian submanifold

We now deduce from the first subsection isomorphisms between spinor bundles on a submanifold. Since we shall need those isomorphisms in the setting of Lagrangian submanifolds (see definition below), we restrict to the case of a Riemannian submanifold $(\tilde{M}^n, g)$ of dimension $n$ immersed in a Riemannian manifold $(M^{2n}, g)$ of (real) dimension $2n$. We shall always use the following notations: $II$ will be the bundle-valued second fundamental form of the immersion, $H$ its mean-curvature vector field (in our convention, $H := \frac{1}{n}\text{tr}(II)$), and $\nabla^M$ (resp. $\nabla$) will be the Levi-Civit\`a connection of $(M, g)$ (resp. of $(\tilde{M}, g)$). The induced covariant derivative on the exterior bundle will be denoted analogously.

We assume $\tilde{M}$ to be Spin, and fix a Spin structure $\text{Spin}(T\tilde{M}) \to \text{SO}(T\tilde{M})$. As it is in general impossible to induce a Spin structure from $\tilde{M}$ to $M$ (compare with the case of an oriented hypersurface [20, 2, 24]), we assume $M$ to be Spin as well and fix a Spin structure $\text{Spin}(TM) \to \text{SO}(TM)$ on $M$. Then the normal bundle $NM$ of $M$ in $\tilde{M}$ is Spin, and carries an induced Spin structure, $\text{Spin}(NM) \to \text{SO}(NM)$, for which there exists a principal-bundle-homomorphism $\text{Spin}(TM) \times_M \text{Spin}(NM) \to \text{Spin}(T\tilde{M})_{|M}$ making the following diagram commutative [24]:

Let $\Sigma M$ (resp. $\Sigma N$, $\Sigma \tilde{M}$) be the spinor bundle of $TM$ (resp. of $NM$, $T\tilde{M}$), i.e., the complex vector bundle associated to the Spin bundle through the Spin representation. There are three fundamental objects on the spinor bundle:

- The isomorphism (2) being obviously $\text{Spin}_n$-equivariant induces a bilinear map, called the Clifford multiplication

\[\text{TM} \times_M \Sigma M \to \Sigma M\]

\[(X, \varphi) \mapsto \gamma_M(X)\varphi\]

satisfying

\[\gamma_M(X)\gamma_M(Y) + \gamma_M(Y)\gamma_M(X) = -2g(X, Y)\text{Id}_{\Sigma M}\]

for all vectors $X$ and $Y$ in $TM$. The same holds for $\Sigma N$ and $\Sigma \tilde{M}$; we denote by $X \cdot \varphi := \gamma_M^{-1}(X)\varphi$ the Clifford multiplication by a vector $X$ on an element $\varphi$ of $\Sigma M$.

- The spinor bundle $\Sigma M$ also inherits from the space of spinors a Hermitian inner product \(\langle \cdot, \cdot \rangle_M\) satisfying

\[\langle \gamma_M(X)\varphi, \psi \rangle_M = -\langle \varphi, \gamma_M(X)\psi \rangle_M\]

for every $X$ in $TM$ and all $\varphi, \psi$ in $\Sigma M$. The same property holds for $\Sigma N$ and for $\Sigma \tilde{M}$, for which such a Hermitian inner product will be denoted by \(\langle \cdot, \cdot \rangle\).
• The Levi-Civitá connection of \((TM, g)\) induces a covariant derivative \(\nabla^{\Sigma M}\) on \(\Sigma M\) \([6, 20]\). This covariant derivative is metric w.r.t. \(\langle \cdot, \cdot \rangle_M\) and satisfies the Leibniz rule w.r.t. the Clifford multiplication. We denote by \(\nabla^{\Sigma N}\) (resp. \(\nabla\)) that covariant derivative on \(\Sigma N\) (resp. on \(\Sigma M\)).

We now compare the different spinor bundles on the submanifold \(M\). We need further notations in that purpose. For a tangent vector \(X\) to \(M\) and an element \(\phi\) of \(\Sigma M \otimes \Sigma N\) if \(n\) is even (resp. of \(\Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N\) if \(n\) is odd), we define “\(X \cdot \phi\)” to be

\[
\begin{cases}
\{\gamma_M(X) \otimes \text{Id}_{\Sigma N}\} \phi & \text{if } n \text{ is even} \\
\{\gamma_M(X) \otimes \text{Id}_{\Sigma N} \oplus -\gamma_M(X) \otimes \text{Id}_{\Sigma N}\} \phi & \text{if } n \text{ is odd}.
\end{cases}
\]

We furthermore set

\[
\nabla := \begin{cases}
\nabla^{\Sigma M \otimes \Sigma N} & \text{if } m \text{ or } n \text{ is even} \\
\nabla^{\Sigma M \otimes \Sigma N} \oplus \nabla^{\Sigma M \otimes \Sigma N} & \text{otherwise}.
\end{cases}
\]

Note that \(\nabla\) is not the natural covariant derivative of \(\Sigma M\), since from its definition it depends on the covariant derivative of the normal bundle.

From the above homomorphism between Spin bundles and the preceding subsection, we have the following:

**Lemma 3** There exists a complex-vector bundle isomorphism

\[
\Sigma M \to \begin{cases}
\Sigma M \otimes \Sigma N & \text{if } n \text{ is even} \\
\Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N & \text{if } n \text{ is odd}
\end{cases}
\]

satisfying:

- For every tangent vector field \(X\) on \(M\) and every section \(\phi\) of \(\Sigma M\), the isomorphism \(\Sigma M \to \Sigma M\) maps the section \(X \cdot \phi\) onto \(X \cdot \phi\),

- For every tangent vector field \(X\) on \(M\) and every section \(\phi\) of \(\Sigma M\),

\[
\nabla_X \phi = \nabla_X \phi + \frac{1}{2} \sum_{j=1}^n e_j \cdot II(X, e_j) \cdot \phi
\]

in any local o.n.b. \((e_j)_{1 \leq j \leq n}\) of \(TM\).

Furthermore, the isomorphism \(\Sigma M \to \Sigma M\) can be assumed to be unitary.

**Proof:** From its equivariance under the action of \(\text{Spin}_n \times \text{Spin}_n\), the isomorphism \(\Sigma M \to \Sigma M\) straightforwardly induces the isomorphism \(\Sigma M \to \Sigma M\) between the vector bundles. The first property is just the translation of that of \(\Sigma M \to \Sigma M\) on vector bundles. The second one is deduced in a quite analogous way as in \([11]\) from the three following points: use the local expressions of the covariant derivatives \(\nabla\) and \(\nabla\) \([6, 20]\) on \(\Sigma M\) and \(\Sigma M\), apply the Gauß-Weingarten formula on \(\Sigma M\) and \(\Sigma M\), and use the correspondence through \(\Sigma M \to \Sigma M\) between the Clifford multiplications by 2-forms, that is: for all vectors \(X_1\) and \(X_2\) in \(TM\),

\[
X_1 \cdot X_2 \equiv X_1 \cdot X_2,
\]

and for all vectors \(\nu_1, \nu_2\) in \(NM\),

\[
\nu_1 \cdot \nu_2 \equiv \text{Id} \otimes \gamma_N(\nu_1) \gamma_N(\nu_2) \oplus \text{Id} \otimes \gamma_N(\nu_1) \gamma_N(\nu_2),
\]

where the parentheses stand for the case “\(n\) odd” (see previous subsection). The last remark is also a direct consequence of Corollary \(2\).

\[\square\]

In this setup, the most natural Dirac operators that can be introduced on the manifold \(M\) are the so-called twisted-Dirac operator \(D^{\Sigma M}_M\) \([20]\) and the Dirac-Witten operator \(\hat{D}\) \([28]\), respectively defined in a local o.n.b \((e_j)_{1 \leq j \leq m}\) of \(TM\) by

\[
D^{\Sigma N}_M := \sum_{j=1}^m e_j \cdot \nabla e_j, \quad \hat{D} := \sum_{j=1}^m e_j \cdot \nabla e_j.
\]
Both operators, which act on the sections of $\Sigma_{M|\tilde{M}}$, are elliptic, and from Lemma 3 are related by

$$\hat{D} = D_{\Sigma N}^M - \frac{mH}{2}.$$ 

Furthermore, the operator $D_{\Sigma N}^M$ is formally self-adjoint (but $\hat{D}$ is not).

We now specialize to submanifolds with particular geometric structures. It is first important to point out that the spinor bundle $\Sigma_{N}$ is in general not isomorphic to $\Sigma_{M}$; this may hold even if there exists an isomorphism between $TM$ and $NM$, such as for Lagrangian submanifolds in Kählerian manifolds (see the examples in Notes 1). We therefore recall the notion of isomorphism between Spin structures:

**Definition 1** Let $E$ and $F$ be two Spin vector bundles on a manifold $M$, with fixed Spin structures $\text{Spin}(E) \to \text{SO}(E)$ and $\text{Spin}(F) \to \text{SO}(F)$. An isomorphism between the Spin structures of $E$ and $F$ is given by a pair of principal-bundle isomorphisms $\text{Spin}(E) \xrightarrow{\sim} \text{Spin}(F)$ and $\text{SO}(E) \xrightarrow{\sim} \text{SO}(F)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\text{Spin}(E) & \xrightarrow{f} & \text{Spin}(F) \\
\downarrow & & \downarrow \\
\text{SO}(E) & \xrightarrow{f} & \text{SO}(F)
\end{array}$$

If two vector bundles have isomorphic Spin structures, they obviously have isomorphic spinor bundles as well. Hence we give the following

**Corollary 4** Assume that there exists an orientation-preserving isometry $f$ from $TM$ to $NM$ which induces an isomorphism $(\tilde{f}, f)$ of the respective Spin structures. Then there exists a complex-vector bundle isomorphism

$$\Sigma_{M|\tilde{M}} \to \Lambda TM \otimes \mathbb{C}$$ (12)

satisfying:

- For every tangent vector $X$ to $M$ and every $\phi$ in $\Sigma_{M|\tilde{M}}$, the element $X \cdot \phi$ is mapped onto $X \wedge \phi - X \llcorner \phi$,
- For every vector $\nu$ in $NM$ and every $\phi$ in $\Sigma_{M|\tilde{M}}$, the element $\nu \cdot \phi$ is mapped onto $i\{f^{-1}(\nu) \wedge \phi + f^{-1}(\nu) \llcorner \phi\}$,
- If furthermore $f$ is parallel w.r.t. the respective connections on $TM$ and $NM$, then for every tangent vector field $X$ to $M$ and every section $\phi$ of $\Sigma_{M|\tilde{M}}$, the element $\nabla_X \phi$ is mapped onto $\nabla_X^M \phi$.

**Proof:** The existence of (12) is a direct consequence of Corollary 5 and the fact that the Spin structure of $T\tilde{M}_{|\tilde{M}}$ reduces via $f$ to the Spin structure of $TM$. The first property comes straightforward. For the second one, it is to be noted that the automorphism

$$J := \begin{pmatrix} 0 & -f^{-1} \\ f & 0 \end{pmatrix}$$

of $T\tilde{M}_{|\tilde{M}}$ is described through the Spin$_n$-reduction as

$$\text{Spin}(TM) \times_{\text{Ad}} \mathbb{R}^{2n} \to \text{Spin}(TM) \times_{\text{Ad}} \mathbb{R}^{2n}$$

$$[\bar{s}, v] \mapsto [\bar{s}, J(v)].$$

The last statement follows from a short computation using the properties of compatibility between $\nabla$ and the other objects on $\Sigma M \otimes \Sigma M$ ($\oplus \Sigma M \otimes \Sigma M$). \qed
Remark that, from the preceding proof, the existence of an orientation-preserving isometry \( f : TM \to NM \) is equivalent to the existence of an almost-Hermitian structure \( J \) on \( T\tilde{M}_{|M} \) mapping \( TM \) onto \( NM \). Let \( \Omega \) then denote the Kähler form of \( (T\tilde{M}_{|M}, g, J) \), i.e., \( \Omega(X, Y) := g(J(X), Y) \) for all \( X \) and \( Y \) in \( T\tilde{M}_{|M} \). Under the hypotheses of Corollary \( \ref{corollary5} \) the following holds: for every \( 0 \leq p \leq n \) and \( \phi \) in \( \Lambda^p TM \otimes \mathbb{C} \),

\[
\tilde{\Omega} \cdot \phi = i(2p - n)\phi
\]

through \( \ref{eq:7} \). This also follows from the properties of \( \ref{eq:7} \), see previous subsection.

The existence of an almost-complex structure on \( T\tilde{M}_{|M} \) is precisely the case we shall be interested in, since we shall consider submanifolds of Kählerian manifolds. We now recall the following

**Definition 2** A submanifold \( M^n \) of a Kählerian manifold \( (\tilde{M}^{2n}, g, J) \) is called Lagrangian if and only if

\[
J(TM) = NM,
\]

i.e., the complex structure identifies the tangent and normal bundles of the submanifold.

For a Lagrangian submanifold in a Kählerian manifold, the complex structure \( J \) obviously preserves the metric and the orientation of \( T\tilde{M}_{|M} \), and is parallel.

**Corollary 5** Let \( (M^n, g) \) be a Spin Lagrangian submanifold immersed in a Kählerian Spin manifold \( (\tilde{M}^{2n}, g, J) \). Let the normal bundle \( NM \) carry the induced Spin structure. Assume that the complex structure \( J \) induces an isomorphism between the Spin structures of \( TM \) and \( NM \). Then there exists a complex-vector bundle isomorphism

\[
\Sigma\tilde{M}_{|M} \to \Lambda TM \otimes \mathbb{C}
\]

satisfying:

- For every tangent vector \( X \) to \( M \) and every \( \phi \) in \( \Sigma\tilde{M}_{|M} \), the element \( X \cdot \phi \) is mapped onto \( X \wedge \phi - X \cdot \phi \).
- For every vector \( \nu \) in \( NM \) and every \( \phi \) in \( \Sigma\tilde{M}_{|M} \), the element \( \nu \cdot \phi \) is mapped onto \( -i\{ J(\nu) \wedge \phi + J(\nu) \cdot \phi \} \).
- For every tangent vector field \( X \) to \( M \) and every section \( \phi \) of \( \Sigma\tilde{M}_{|M} \), the element \( \nabla_X \phi \) is mapped onto \( \nabla_X^M \phi \).

In particular, for each \( 0 \leq p \leq n \), the subspace \( \Lambda^p TM \otimes \mathbb{C} \) is the eigenspace associated to the eigenvalue \( i(2p - n) \) of the action of the Kähler form \( \Omega \) of \( (\tilde{M}^{2n}, g, J) \). Furthermore, for every section \( \phi \) of \( \Sigma\tilde{M}_{|M} \),

\[
D^\Sigma_{M} \phi = (d + \delta)\phi \quad \text{and} \quad \hat{D}\phi = (d + \delta)\phi + \frac{iM}{2}\{ J(H) \wedge \phi + J(H) \cdot \phi \},
\]

where \( d \) (resp. \( \delta \)) denotes the exterior differential (resp. codifferential).

**Proof:** The only statement to be proved is the last one, for which it suffices to know that, for any local o.n.b. \( (e_j)_{1 \leq j \leq n} \) of \( TM \),

\[
d = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j}^M \quad \text{and} \quad \delta = -\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}^M.
\]

\[ \square \]

**Notes 1**

1) In the same way as above, one can give a “bundle-version” of Corollary \( \ref{corollary1} \) let \( E \) be any arbitrary Riemannian Spin vector bundle on a Spin manifold \( M \) such that there exists an isomorphism from \( TM \) to \( E \), preserving the metric, the orientation and the Spin structure. Then there exists a complex-vector bundle isomorphism between the Clifford bundle and the tensor product \( \Sigma M \otimes \Sigma E \) (or double copy), mapping \( X \cdot \phi \) onto \( X \cdot \phi \) for every \( X \) in \( TM \) and \( \phi \) in the Clifford bundle; if furthermore the isomorphism from
2 An upper eigenvalue bound for the twisted Dirac operator on a Lagrangian submanifold

In this section, we consider a compact Lagrangian Spin submanifold \((M^n, g)\) in a Kählerian Spin manifold \((\tilde{M}^{2n}, g, J)\). Since the operator \(D^N_{\tilde{M}}\) is elliptic and formally self-adjoint, it has a discrete spectrum; we then denote by \(\lambda_k\) \((k \in \mathbb{N} \setminus \{0\})\) its eigenvalues, counted with their multiplicities, assuming that \(|\lambda_{k+1}| \geq |\lambda_k|\) for every \(k \geq 1\).

We are interested in the following question: how can one control the smallest eigenvalues of the twisted Dirac operator in terms of extrinsic geometric invariants? For submanifolds of certain real space-forms, it was proved by C. Bär in [2] and the author in [11] that the ambient curvature together with either the \(L^2\) or the \(L^\infty\) norm of the mean curvature appear as the best candidates in that purpose. Those results were obtained considering restrictions to the submanifold of particular spinor fields on the ambient manifold, called Killing spinors (see [4] about those). As non Ricci-flat Kählerian Spin manifolds of (real) dimension greater than 2 do not admit such spinor fields [10] [21] [22], it comes as a natural question whether such kind of estimates could still hold in our context. We give an affirmative and sharp answer to that problem, using the notion of Kählerian Killing spinors introduced by K.-D. Kirchberg in [19] and O. Hijazi in [17]. Remember that, for a complex constant \(\alpha\), an \(\alpha\)-Kählerian Killing spinor on the Kählerian Spin manifold \((\tilde{M}^{2n}, g, J)\) is a couple of sections \((\psi, \phi)\) of \(\Sigma \tilde{M}\) satisfying, for every tangent vector field \(Z\) on \(\tilde{M}\),

\[
\begin{cases}
\bar{\nabla}_Z \psi + \alpha p_-(Z) \cdot \phi &= 0 \\
\bar{\nabla}_Z \phi + \alpha p_+(Z) \cdot \psi &= 0,
\end{cases}
\]

where \(p_{\pm}(Z) := \frac{1}{2}(Z \mp iJ(Z))\). When \(\alpha = 0\), an \(\alpha\)-Kählerian Killing spinor is just a pair of parallel spinor fields. As for Killing spinors, the presence of non-zero Kählerian Killing spinors yields strong conditions on the geometry of \(\tilde{M}\) (see [19] [17]): if \(\alpha \neq 0\), the complex dimension \(n\) of \(\tilde{M}\) has to be odd, the manifold \((\tilde{M}, g, J)\) has to be Einstein with scalar curvature \(n(n+1)\alpha^2\) (therefore \(\alpha\) must be either real or purely imaginary), and the sections \(\psi\) and \(\phi\) have to lie in particular eigenspaces of the Clifford action of the Kähler form \(\Omega\) of \((\tilde{M}, g, J)\):

\[
\begin{align*}
\tilde{\Omega} \cdot \psi &= -i \psi \\
\tilde{\Omega} \cdot \phi &= i \phi
\end{align*}
\]

(remember that, in our convention, \(\tilde{\Omega}(X, Y) := g(J(X), Y)\) for all vectors \(X\) and \(Y\) in \(T \tilde{M}\)).

2.1 Main result

From here on, we denote by \(K_\alpha\) the space of \(\alpha\)-Kählerian Killing spinors on \((\tilde{M}, g, J)\) (note that, if \(\alpha \neq 0\), then \(K_\alpha \cap K_{-\alpha} = \{0\}\)). Manifolds carrying a non-zero \(K_\alpha\) have been completely characterized by A. Moroianu in [24] when \(\alpha\) is a non-zero real number, and partially by K.-D. Kirchberg in [19] and M.
Let the normal bundle of $M$ quotient $D$ and apply the Min-Max principle. To obtain $\psi$, we deduce from that identity that $H$ is the mean curvature vector field of $M$ in $\tilde{M}$.

**Proof:** Let $(\psi, \phi)$ be a non-zero $\alpha$-Kählerian Killing spinor as above on $\tilde{M}$. We compute the Rayleigh-quotient

$$Q((D^N_M)^2, \psi + \phi) := \frac{\int_M \langle (D^N_M)^2(\psi + \phi), \psi + \phi \rangle v_g}{\int_M \langle \psi + \phi, \psi + \phi \rangle v_g}$$

and apply the Min-Max principle. To obtain $(D^N_M)^2(\psi + \phi)$, we first evaluate $\hat{D}^2$ on $\psi + \phi$, then use the following relation ([11], Lemme 4.1): for every section $\varphi$ of $\tilde{M}|_M$ and in every local o.n.b. $(e_j)_{1 \leq j \leq n}$ of $TM$,

$$(D^N_M)^2 \varphi = \hat{D}^2 \varphi + \frac{n^2|H|^2}{4} \varphi + \sum_{j=1}^n e_j \cdot \nabla^N_{e_j} H \cdot \varphi,$$  \(13\)

where $\nabla^N H$ denotes the normal covariant derivative of $H$.

Let us fix a local o.n.b. $(e_j)_{1 \leq j \leq n}$ of $TM$. From the hypotheses,

$$\hat{D} \psi = \sum_{j=1}^n e_j \cdot \tilde{\nabla}_{e_j} \psi = \alpha \sum_{j=1}^n e_j \cdot p_-(e_j) \cdot \phi.$$

For every vector $X$ on $T\tilde{M}$, we have $g(p_-(X), p_-(X)) = 0$ and therefore $p_-(X) \cdot p_-(X) \cdot \varphi = 0$ for every $\varphi$ in $\tilde{M}|_M$. Hence

$$\hat{D} \psi = -\alpha \sum_{j=1}^n p_+(e_j) \cdot p_-(e_j) \cdot \phi.$$

But, since $M$ is Lagrangian in $\tilde{M}$, the complex vectors $Z_j := p_+(e_j)$ and $\overline{Z}_j := p_-(e_j)$ $(1 \leq j \leq n)$ form a Witt-basis for $TM \otimes \mathbb{C}$. Now remember the expression of the Kähler form $\Omega$ of $(\tilde{M}, g, J)$ in that basis:

$$\Omega = -2i \sum_{j=1}^n Z_j \wedge \overline{Z}_j.$$

We deduce from that identity that

$$\hat{D} \psi = -\alpha \sum_{j=1}^n (Z_j \wedge \overline{Z}_j) \cdot \phi + \alpha \sum_{j=1}^n g(Z_j, \overline{Z}_j) \phi = -\frac{i\alpha}{2} \Omega \cdot \phi + \frac{n\alpha}{2} \phi = \frac{(n+1)\alpha}{2} \phi,$$
since $\tilde{\Omega} \cdot \phi = i\phi$. A similar computation gives
\[
\tilde{D}\phi = -\alpha \sum_{j=1}^{n} p_-(e_j) \cdot p_+(e_j) \cdot \psi
\]
\[
= -\alpha \sum_{j=1}^{n} (Z_j \wedge Z_j) \cdot \psi + \alpha \sum_{j=1}^{n} g(Z_j, Z_j) \psi
\]
\[
= \frac{i\alpha}{2} \tilde{\Omega} \cdot \psi + \frac{n\alpha}{2} \psi
\]
\[
= \left(\frac{n+1}{2}\right) \Omega \cdot \psi,
\]
\[
\text{since } \tilde{\Omega} \cdot \psi = -i\psi. \text{ We therefore obtain: } \tilde{D}^2\psi = \left(\frac{n+1}{2}\right)^2 \psi \text{ and } \tilde{D}^2\phi = \left(\frac{n+1}{2}\right)^2 \phi, \text{ i.e.,}
\]
\[
(D_{\mathcal{M}}^\Sigma \mathcal{N})^2 \varphi = \frac{(n+1)^2 \alpha^2 4}{4} \varphi + \frac{n^2 |H|^2 4}{4} \varphi + \frac{n}{2} \sum_{j=1}^{n} e_j \cdot \nabla e_j H \cdot \varphi
\]
\[
(14)
\]
for $\varphi := \psi$ or $\phi$. Taking the Hermitian inner product of the sum of the two identities for $\psi$ and $\phi$ with $\psi + \phi$ and integrating lead to
\[
Q \left( (D_{\mathcal{M}}^\Sigma \mathcal{N})^2, \psi + \phi \right) = \frac{(n+1)^2 \alpha^2 4}{4} + \frac{n^2 \int_M |H|^2 4}{4} \varphi + \frac{n}{2} \sum_{j=1}^{n} e_j \cdot \nabla e_j H \cdot \varphi
\]
\[
(15)
\]
Here we recall that, as the operator $D_{\mathcal{M}}^\Sigma \mathcal{N}$ is self-adjoint, we only keep the real parts when taking the Hermitian inner product of both members of (14) with $\psi + \phi$. That is why the term involving $\nabla H \cdot \varphi$ doesn’t give any contribution to (15).

To conclude, it remains to note that, if $\alpha$ is real, then the length-function of $\psi + \phi$ is constant on $\tilde{\mathcal{M}}$ (hence on $\mathcal{M}$), whereas it cannot be constant when $\alpha$ is imaginary (see (13)). From the Min-Max principle, we straightforward obtain the result.

\[
\square
\]

**Note 1** If $\alpha \neq 0$, one can actually bound a greater number of eigenvalues for $D_{\mathcal{M}}^\Sigma \mathcal{N}$ in Theorem 1: indeed, let $\mathcal{K}_{-1}$ (resp. $\mathcal{K}_{+1}$) be the (pointwise) orthogonal projection of $\mathcal{K}_{\alpha}$ onto the $-i$- (resp. $i$-) eigenspace of the Clifford action of $\tilde{\Omega}$, that is,
\[
\mathcal{K}_{-1} := \left\{ \psi \in \Gamma(\Sigma \tilde{\mathcal{M}}) / \tilde{\Omega} \cdot \psi = -i\psi \text{ and } \exists \phi \in \Gamma(\Sigma \tilde{\mathcal{M}}) / (\psi, \phi) \in \mathcal{K}_{\alpha} \right\}
\]
\[
\mathcal{K}_{+1} := \left\{ \phi \in \Gamma(\Sigma \tilde{\mathcal{M}}) / \tilde{\Omega} \cdot \phi = i\phi \text{ and } \exists \psi \in \Gamma(\Sigma \tilde{\mathcal{M}}) / (\psi, \phi) \in \mathcal{K}_{\alpha} \right\}.
\]
From the obvious injectivity of the orthogonal projections $\mathcal{K}_{\alpha} \rightarrow \mathcal{K}_{\pm 1}$, we deduce that $\text{dim}_\mathbb{C} \left( \mathcal{K}_{\pm 1} \right) = \dim_\mathbb{C} \left( \mathcal{K}_{\pm 1} \right) = N$. The identity (14) holding then on $\mathcal{K}_{\pm 1} \oplus \mathcal{K}_{\pm 1}$, it follows that, for every $\varphi$ in $\mathcal{K}_{\pm 1} \oplus \mathcal{K}_{\pm 1}$,
\[
Q \left( (D_{\mathcal{M}}^\Sigma \mathcal{N})^2, \varphi \right) = \frac{(n+1)^2 \alpha^2 4}{4} + \frac{n^2 \int_M |H|^2 4}{4} \varphi + \frac{n}{2} \sum_{j=1}^{n} e_j \cdot \nabla e_j H \cdot \varphi
\]
in particular
\[
Q \left( (D_{\mathcal{M}}^\Sigma \mathcal{N})^2, \varphi \right) \leq \frac{(n+1)^2 \alpha^2 4}{4} + \frac{n^2 |H|^2 8}{4},
\]
from which the following estimate holds:
\[
\lambda_{2N}^2 \leq \frac{(n+1)^2 \alpha^2 4}{4} + \frac{n^2 |H|^2 8}{4},
\]
(16)
We can therefore estimate $2N$ eigenvalues with the right member of (16); when $\alpha$ is a non-zero real number, we do not obtain more information on the $N$ first eigenvalues, who remain sharply bounded from Theorem 1. However, when $\alpha$ is purely imaginary, we obtain an estimate for a larger number of eigenvalues.
**Corollary 6** Under the hypotheses of Theorem 11 assume furthermore that the complex structure \( J \) induces an isomorphism between the Spin structures of \( TM \) and \( NM \). Then the \( N \) smallest eigenvalues (counted with their multiplicities) \( \lambda \) of the Hodge-de Rham Laplacian on \( \Omega^{\frac{n+1}{2}}(M) \oplus \Omega^{\frac{n+1}{2}}(M) \) satisfy

\[
\lambda \leq \frac{(n+1)^2 \alpha^2}{4} + \begin{cases} 
\frac{n^2}{4 \text{Vol}(M)} \int_M |H|^2 \nu_g & \text{if } \alpha \text{ is real} \\
\frac{n^2 \|H\|^2}{4} & \text{if } \alpha \text{ is purely imaginary},
\end{cases}
\]

where \( H \) is the mean curvature vector field of \( M \) in \( \tilde{M} \). Moreover, if \( M \) is minimal in \( \tilde{M} \), then the same result holds for the \( \lfloor \frac{N+1}{2} \rfloor \) smallest eigenvalues of the Hodge-de Rham Laplacian on the space of closed \( \frac{n+1}{2} \)-forms.

**Proof:** From Corollary 5 if \( J \) identifies the Spin structures of \( TM \) and \( NM \), then \( (D_M^N)^2 = d\delta + \delta d \). Furthermore, the isomorphism 12 identifies the eigenspace associated to the eigenvalue \( i(2p - n) \) of the Clifford action of \( \tilde{\Omega} \) with \( \Lambda^p TM \otimes \mathbb{C} \); since, under that action, the spinor field \( \phi \) (resp. \( \psi \)) is eigen for the eigenvalue \( i \) (resp. \(-i\)), it is a \( \frac{n+1}{2} \)-form (resp. a \( \frac{n-1}{2} \)-form) on \( M \). Hence the first statement holds.

If moreover \( H = 0 \), then \( \tilde{D} = D_M^N = d + \delta \). From the equalities \( \tilde{D} \psi = (\frac{n+1}{2}) \phi \) and \( \tilde{D} \phi = (\frac{n+1}{2}) \psi \), we then deduce that

\[
\begin{align*}
d\phi &= 0 \\
\delta \phi &= \frac{n+1}{2} \psi,
\end{align*}
\]

i.e., \( \psi \) is coclosed and \( \phi \) is closed. As the spectrum of the \( \frac{n+1}{2} \)-Laplacian on coclosed forms coincides with that of the \( \frac{n+1}{2} \)-Laplacian on closed forms (use the Hodge star operator), we obtain the second property. \( \square \)

### 2.2 Examples

For an odd integer \( n \geq 3 \), consider the round sphere \( S^n \) (of constant sectional curvature 1) of dimension \( n \) as canonically embedded in the \( 2n + 1 \)-dimensional round sphere \( S^{2n+1} \). That embedding is isometric, totally geodesic, and the canonical complex structure of \( \mathbb{R}^{2n+2} \) maps the tangent bundle of \( S^n \) into the horizontal space \( \mathcal{H} \) defined, for each \( z \) in \( S^{2n+1} \) as

\[
\mathcal{H}_z := \{ \mathbb{R}z \oplus \mathbb{R}Jz \}^\perp \subset T_z S^{2n+1},
\]

with the following property:

\[
\mathcal{H}_{|S^n} = TS^n \oplus J(TS^n).
\]

Let then \( \mathbb{CP}^n \) be the complex projective space of complex dimension \( n \). Composing the Hopf fibration \( S^{2n+1} \longrightarrow \mathbb{CP}^n \) with the above embedding yields an immersion

\[
S^n \rightarrow \mathbb{CP}^n
\]

(17)

satisfying the following: it is isometric (the Hopf fibration induces an isometry from \( \mathcal{H} \) onto \( T\mathbb{CP}^n \)), totally geodesic (the Hopf fibration maps horizontal geodesics onto geodesics) and Lagrangian (the Hopf fibration is “holomorphic” w.r.t. the complex structures of \( \mathcal{H} \) and \( \mathbb{CP}^n \)). Furthermore, if \( n \) is odd, the manifold \( \mathbb{CP}^n \) is Spin, has a unique Spin structure since it is simply-connected [20], and carries a \( 2C_n \frac{n+1}{2} \)-dimensional space of 1-K"ahlerian Killing spinors [18]. The round sphere \( S^n \) is also Spin, and for the same reason has a unique Spin structure; more generally, every Spin vector bundle on \( S^n \) has a unique Spin structure, that is, two Spin structures on a vector bundle on \( S^n \) will always be isomorphic.

Consider then the (canonical) Spin structure of \( TS^n \) and the induced one on the normal bundle of \( S^n \) in \( \mathbb{CP}^n \) w.r.t. (17); then the complex structure of \( \mathbb{CP}^n \) will necessarily induce an isomorphism between the Spin structures of the tangent and normal bundles of \( S^n \). Hence we obtain from Corollary 5 and Note 11 the existence of the following upper bound for the \( 2C_n \frac{n+1}{2} \)-smallest eigenvalues \( \lambda \) of the Hodge-de Rham Laplacian on the closed \( \frac{n+1}{2} \)-forms:

\[
\lambda \leq \frac{(n+1)^2}{4}.
\]
That estimate is sharp: indeed, for \(1 \leq p \leq n-1\), the spectrum of the Hodge-de Rham Laplacian on the closed \(p\)-forms on \(S^n\) is\footnote{\((k+p)(n-p+k+1) / k \in \mathbb{N}\),}

and the multiplicity of the first eigenvalue \((k=0)\) is \(C_{n+1}^p\). But, for \(p := \frac{n+1}{2}\), we have \(2C_{n+1}^{n+1} = C_n^{n+1} + C_n^{n+1} = C_{n+1}^{n+1}\), which is precisely the multiplicity of the first eigenvalue of the Hodge-de Rham Laplacian on the closed \(^{n+1}\)-forms.

A further interesting example would be to consider the real \(n\)-dimensional projective space (with \(n = 4k + 3\)) in the complex projective space \(\mathbb{C}P^n\). That question, which is linked to determining all the Lagrangian submanifolds which satisfy the equality in Theorem \(\text{II}\) will be considered in a forthcoming work.

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