Complex-mass scheme and perturbative unitarity

T. Bauer, J. Gegelia, G. Japaridze, and S. Scherer

1Institut für Kernphysik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany
2Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany
3High Energy Physics Institute, Tbilisi State University, 0186 Tbilisi, Georgia
4Clark Atlanta University, Atlanta, GA 30314, USA

(Dated: October 9, 2012)

Abstract

We derive cutting rules for loop integrals containing propagators with complex masses. Using a field-theoretical model of a heavy vector boson interacting with a light fermion, we demonstrate that the complex-mass scheme respects unitarity order by order in a perturbative expansion provided that the renormalized coupling constant remains real.

PACS numbers: 11.10.Gh, 03.70.+k
I. INTRODUCTION

When describing resonances in perturbation theory, one needs to take their finite widths into account. A framework of handling this problem is provided by the complex-mass scheme (CMS) [1–4]. This approach has proven successful in various applications within the standard model [2–12]. Recently, the CMS has also been applied in chiral effective field theory [13–15].

In the framework of quantum field theory for unstable particles the usage of the CMS leads to complex-valued renormalized parameters. While the problems of unitarity and causality in field theories containing unstable particles were resolved long ago [16], the issue of perturbative unitarity of the $S$-matrix in the CMS is still open [4]. Using the CMS, one does not change the bare Lagrangian and, therefore, unitarity is not violated in the complete theory. On the other hand, perturbation theory is based on an order-by-order approximation to exact results. Therefore, it is not obvious that the approximate expressions for the $S$-matrix also satisfy the unitarity condition. In the present work we examine perturbative unitarity in a model of a heavy abelian vector field interacting with a light fermion. Keeping the renormalized coupling constant (the expansion parameter of the perturbation theory) as a real quantity, we derive the cutting rules for one-loop integrals involving propagators with complex masses and show that unitarity is satisfied up to higher-order corrections in the coupling constant. In full agreement with Ref. [16], the $S$-matrix connecting only stable states satisfies the unitarity condition.

II. IMAGINARY PARTS OF LOOP INTEGRALS

In this section we propose a method for deriving the imaginary parts of (one-loop) integrals involving propagators with complex masses. In the next section we use these results to demonstrate the perturbative unitarity at one-loop order in a model of a heavy vector boson interacting with a light fermion.

Before discussing the propagation of unstable particles, let us recall a few properties of the Feynman propagator of a stable particle. In momentum space the Feynman propagator of a scalar particle with mass $m$ and four-momentum $p$ is given by

\[
\Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} = \frac{p^2 - m^2 - i\epsilon}{(p^2 - m^2)^2 + \epsilon^2} = \frac{p^2 - m^2}{(p^2 - m^2)^2 + \epsilon^2} - \frac{i\epsilon}{(p^2 - m^2)^2 + \epsilon^2} = \frac{p^2 - m^2}{(p^2 - m^2)^2 + \epsilon^2} - i\pi\delta(p^2 - m^2).
\]

In Eq. (1), $\epsilon$ is a positively defined quantity and the limit $\epsilon \to 0^+$ is assumed. Defining $E_p = \sqrt{p^2 + m^2}$, the Feynman propagator has two simple poles at $p_0 = E_p - i\epsilon$ and $p_0 = -i\epsilon\delta(p^2 - m^2)$.

\footnote{Recall the representation of the Dirac delta function, $\pi\delta(x) = \frac{\epsilon}{x^2 + \epsilon^2}$, $\epsilon \to 0^+$.}
For a finite width $\Gamma$, the infinitesimal parameter $\epsilon$, respectively:

$$\Delta_F(p) = \frac{1}{2E_p} \left( \frac{1}{p_0 - E_p + i\epsilon} - \frac{1}{p_0 + E_p - i\epsilon} \right).$$

(2)

Besides the Feynman propagator, let us introduce the advanced and retarded propagators, respectively,

$$\Delta_A(p) = \frac{1}{2E_p} \left( \frac{1}{p_0 - E_p - i\epsilon} - \frac{1}{p_0 + E_p + i\epsilon} \right)$$

$$= \frac{p^2 - m^2}{(p^2 - m^2)^2 + \epsilon^2} + i\pi \delta(p^2 - m^2) \sigma_p,$$

(3)

$$\Delta_R(p) = \frac{1}{2E_p} \left( \frac{1}{p_0 - E_p + i\epsilon} - \frac{1}{p_0 + E_p - i\epsilon} \right)$$

$$= \frac{p^2 - m^2}{(p^2 - m^2)^2 + \epsilon^2} - i\pi \delta(p^2 - m^2) \sigma_p,$$

(4)

where $\sigma_p \equiv \text{sign}(p_0) = -\text{sign}(-p_0) = -\sigma_{-p}$. Note that $\Delta_A(p) = \Delta_R(-p)$. In terms of these propagators, the Feynman propagator can be written as

$$\Delta_F(p) = \Theta(p_0)\Delta_R(p) + \Theta(-p_0)\Delta_A(p),$$

(5)

with the Heaviside step function $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$.

In the CMS, for an unstable scalar particle with mass $M$, width $\Gamma$, and four-momentum $p$, the Feynman propagator of Eq. (1) is replaced by

$$\Delta'(p) = \frac{1}{p^2 - M^2 + i(M\Gamma + \epsilon)}$$

$$= \frac{p^2 - M^2}{(p^2 - M^2)^2 + (M\Gamma + \epsilon)^2} - \frac{i(M\Gamma + \epsilon)}{(p^2 - M^2)^2 + (M\Gamma + \epsilon)^2}.$$  

(6)

For a finite width $\Gamma$, the infinitesimal parameter $\epsilon$ can be neglected. If one is also interested in the case of vanishing $\Gamma$, i.e. stable particles, one has to keep the infinitesimal parameter $\epsilon$. In the following, we drop $\epsilon$ with the understanding that it is easily reintroduced by replacing $M\Gamma \rightarrow M\Gamma + \epsilon$. Let us define auxiliary functions which we denote as "advanced" and "retarded" propagators $\Delta'_A$ and $\Delta'_R$ corresponding to $\Delta'$:

$$\Delta'_A(p) = \frac{1}{w(p) + w(p)^*} \left[ \frac{1}{p_0 - w(p)^*} - \frac{1}{p_0 + w(p)} \right]$$

$$= \frac{p^2 - M^2 - M^2\Gamma^2/(2x(p)^2)}{(p^2 - M^2)^2 + M^2\Gamma^2} + \frac{iM\Gamma}{(p^2 - M^2)^2 + M^2\Gamma^2} x(p),$$

(7)

$$\Delta'_R(p) = \frac{1}{w(p) + w(p)^*} \left[ \frac{1}{p_0 - w(p)} - \frac{1}{p_0 + w(p)^*} \right]$$

$$= \frac{p^2 - M^2 - M^2\Gamma^2/(2x(p)^2)}{(p^2 - M^2)^2 + M^2\Gamma^2} - \frac{iM\Gamma}{(p^2 - M^2)^2 + M^2\Gamma^2} x(p),$$

(8)

\footnote{We stress that in our notation the prime does not refer to the full, dressed propagator of the unstable particle.}
where

\[
    w(p) = x(p) - i y(p),
\]

\[
    x(p) = \frac{1}{\sqrt{2}} \sqrt{(\mathcal{E}_p^4 + M^2 \Gamma^2)^{1/2} + \mathcal{E}_p^2} = \mathcal{E}_p + \mathcal{O}(\Gamma^2),
\]

\[
    y(p) = \frac{1}{\sqrt{2}} \sqrt{(\mathcal{E}_p^4 + M^2 \Gamma^2)^{1/2} - \mathcal{E}_p^2} = M \Gamma / (2 \mathcal{E}_p) + \mathcal{O}(\Gamma^3),
\]

\[
    \mathcal{E}_p = \sqrt{p^2 + M^2}.
\]

Here and below it is understood that \( \Gamma \ll M \) and hence the small expansion parameter is \( \Gamma / M \). The ”advanced” (”retarded”) propagator has two simple poles in the upper (lower) complex half plane and approaches the advanced (retarded) propagator of a stable particle as \( \Gamma \to 0 \).

With the above definitions, let us consider a generic one-loop integral involving the propagation of both a stable and an unstable particle,

\[
    I_1 = i \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(k + q - p) \Delta_F(k) \Delta_R'(q).
\]

(10)

Using Eqs. (1) and (6), the imaginary part of the integral \( I_1 \) is given by

\[
    \text{Im}[I_1] = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(k + q - p) \left[ \frac{k^2 - m^2}{(k^2 - m^2)^2 + \epsilon^2} \frac{q^2 - M^2}{(q^2 - M^2)^2 + M^2 \Gamma^2} \right. \\
    \left. -\pi \delta \left( k^2 - m^2 \right) \frac{M \Gamma}{(q^2 - M^2)^2 + M^2 \Gamma^2} \right].
\]

(11)

The purpose of the subsequent manipulations is to bring the first term on the right-hand side of Eq. (11) into a more convenient form. To that end, we make use of the following observation. As functions of the complex variable \( q_0 \), both \( \Delta_R(q - p) \) and \( \Delta_R'(q) \) have simple poles in the lower half plane which is also true for their product. Closing the contour integration in the upper half plane including a vanishing contribution resulting from the semi circle at infinity, we find, using Cauchy’s theorem,

\[
    0 = i \int \frac{d^4q}{(2\pi)^4} \Delta_R(q - p) \Delta_R'(q) \\
    = i \int \frac{d^4q}{(2\pi)^4} \Delta_A(q - p) \Delta_R'(q) \\
    = i \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(k + q - p) \Delta_A(k) \Delta_R'(q).
\]

(12)

By substituting Eqs. (3) and (8) into Eq. (12) and taking the imaginary parts of both sides, we obtain

\[
    0 = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(k + q - p) \left[ \frac{k^2 - m^2}{(k^2 - m^2)^2 + \epsilon^2} \frac{q^2 - M^2 - M^2 \Gamma^2/(2 x(q)^2)}{(q^2 - M^2)^2 + M^2 \Gamma^2} \right. \\
    \left. +\pi \delta \left( k^2 - m^2 \right) \sigma_k \frac{M \Gamma}{(q^2 - M^2)^2 + M^2 \Gamma^2} \frac{q_0}{x(q)} \right].
\]

(13)
Expanding $x(q) = \mathcal{E}_q + \mathcal{O}(\Gamma^2)$, we can replace Eq. (13) by

$$0 = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(k + q - p) \left[ \frac{k^2 - m^2}{(k^2 - m^2)^2 + \epsilon^2} \frac{q^2 - M^2}{(q^2 - M^2)^2 + M^2 \Gamma^2} + \pi \delta(k^2 - m^2) \frac{M \Gamma}{(q^2 - M^2)^2 + M^2 \Gamma^2} \frac{\bar{q}_0 + \mathcal{O}(\Gamma^2)}{\mathcal{E}_q} \right].$$

Subtracting Eq. (14) from Eq. (11) and restoring $\epsilon$, we obtain

$$\text{Im}[I_1] = -\pi \int \frac{d^4k}{(2\pi)^4} \delta(k^2 - m^2) \left[ \frac{(M \Gamma + \epsilon) \left(1 + \sigma_k \frac{\bar{q}_0 - \bar{q}_0}{\bar{q}_0 - p} \right)}{[(p - k)^2 - M^2]^2 + (M \Gamma + \epsilon)^2} + \mathcal{O}(\Gamma^2) \right].$$

It is convenient to rewrite Eq. (15) in a form analogous to the standard cutting formula for loop integrals with real masses [18, 19]. The left-hand side of Eq. (15) is manifestly Lorentz invariant, and hence the right-hand side is Lorentz invariant order by order in $\Gamma$. Based on this fact we derive that

$$\text{Im}[I_1] = -\pi \int \frac{d^4k}{(2\pi)^4} \delta(k^2 - m^2) \left[ \frac{(M \Gamma + \epsilon) \left(1 + \sigma_k \frac{\bar{q}_0 - \bar{q}_0}{\bar{q}_0 - p} \right)}{[(p - k)^2 - M^2]^2 + (M \Gamma + \epsilon)^2} + \mathcal{O}(\Gamma^2) \right].$$

The symbol $\mathcal{O}(\Gamma^2)$ on the right-hand side of Eq. (16) indicates that the neglected terms contain at least one additional overall factor of $\Gamma$ as compared to the first term. Note that the integrand of Eq. (16) is not obtained by expanding the integrand of Eq. (15). The equivalence of expressions (15) and (16) can rather be seen by subtracting them from each other and considering, for $\epsilon = 0$, the limit $\Gamma \to 0$ of the integrated result. Both Eq. (16) and Eq. (15) turn into the expression for the stable particles in this limit. Hence the difference has to be suppressed by an additional factor of $\Gamma$.

The generalization of the above procedure to any one-loop integral containing propagators with complex masses is straightforward. In analogy to Eq. (16), cutting an unstable-particle line results in an overall factor of $\Gamma$, whereas cutting a stable-particle line generates a delta function. For integrals containing propagators of stable particles only, the usual cutting rules apply. A list of the imaginary parts of one-loop integrals needed for the calculations of the next sections is given in the appendix.

### III. THE MODEL

In the next section we demonstrate that using complex renormalized masses in unstable-particle propagators does not violate perturbative unitarity. To that end let us consider a model describing the interaction of an unstable vector boson ($B$) with a stable fermion ($\psi$),

$$\mathcal{L} = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \frac{M^2}{2} B_{0\mu} B_0^{\mu} + \bar{\psi}_0 \left(i\partial - m_0 \right) \psi_0 + g_0 \bar{\psi}_0 \gamma_\mu \psi_0 B_0^\mu,$$

where $F_{0\mu\nu} = \partial_\mu B_{0\nu} - \partial_\nu B_{0\mu}$. The subscript 0 indicates bare parameters and fields. The masses are chosen such that the vector boson can decay into a fermion-antifermion pair. Since the vector boson couples to a conserved vector current, the above model is on-mass-shell renormalizable [20], i.e. leads to finite physical quantities after renormalizing the masses and the coupling constant.
Our perturbative approach to the considered model is based on the path integral formalism. The integration over classical fields corresponding to particles (stable as well as unstable) is performed in the standard way, i.e., the Gaussian part is treated non-perturbatively and the rest perturbatively. For stable particles, perturbation theory based on the path integral formalism is equivalent to perturbation theory based on the operator formalism in the Dirac interaction representation. On the other hand, the functional integral also allows to incorporate the unstable degrees of freedom, while the application of the interaction picture using a “free” Hamiltonian for unstable states is conceptually problematic.

We perform the renormalization in two steps: first we get rid off the ultraviolet divergences by applying dimensional regularization in combination with the \( \overline{\text{MS}} \) scheme \([17]\). We refrain from showing the corresponding counter terms explicitly (including those leading to the wave function renormalization). Next we express the renormalized masses of the \( \overline{\text{MS}} \) scheme in terms of physical quantities—the poles of the dressed propagators—and substitute them back into the Lagrangian. This amounts to performing the following substitutions in Eq. (17),

\[
B_0^\mu \rightarrow B^\mu, \\
\psi_0 \rightarrow \psi, \\
m_0 \rightarrow m + \delta m, \\
M_0^2 \rightarrow M^2 - i M \Gamma + \delta z \equiv z + \delta z, \\
g_0 \rightarrow g, \tag{18}
\]

resulting in

\[
\mathcal{L} = \mathcal{L}_{\text{main}} + \mathcal{L}_{\text{ct}}, \\
\mathcal{L}_{\text{main}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{z}{2} B_\mu B^\mu + \bar{\psi} (i \partial - m) \psi + g \bar{\psi} \gamma_\mu \psi B^\mu, \\
\mathcal{L}_{\text{ct}} = \frac{\delta z}{2} B_\mu B^\mu - \delta m \bar{\psi} \psi. \tag{19}
\]

The main Lagrangian \( \mathcal{L}_{\text{main}} \) generates the propagators and the vector-boson fermion interaction vertex. The counter-term Lagrangian \( \mathcal{L}_{\text{ct}} \) is treated perturbatively in a loop expansion, i.e., we write the counter terms as \( \delta m = \sum_{k=1}^{\infty} \hbar^k \delta m_k \) and \( \delta z = \sum_{k=1}^{\infty} \hbar^k \delta z_k \) and include them order by order in perturbative calculations.

The undressed propagators of the fermion and the vector boson take the following form, respectively,

\[
S_F(p) = \frac{1}{p - m + i \epsilon}, \tag{20}
\]

\[
S_{\mu \nu}(p) = -\frac{g_{\mu \nu} - p_\mu p_\nu / z}{p^2 - z}, \tag{21}
\]

and, for later usage, we parameterize the self energy of the vector boson as

\[
i \Pi_{\mu \nu}(p) = i \left[ \Pi(p^2) g_{\mu \nu} + \Pi_p(p^2) p_\mu p_\nu \right]. \tag{22}
\]

IV. PERTURBATIVE UNITARITY OF THE S-MATRIX

Below we demonstrate that the unitarity condition for the forward-scattering amplitude,

\[
\text{Im} [\mathcal{T}_{ii}] = \frac{1}{2} \sum_{n} (2\pi)^4 \delta^4(P_n - P_i) \mathcal{T}_{ni}^* \mathcal{T}_{ni}, \tag{23}
\]
is satisfied at one-loop order in perturbation theory. In Eq. (23), the $T$-matrix element between initial and final four-momentum eigenstates is written as $\langle f|T|i \rangle = (2\pi)^4 \delta^4(P_f - P_i) \mathcal{T}_{fi}$. The imaginary parts of one-loop integrals needed in this section are given in the appendix.

We start with discussing the imaginary parts of diagrams c) and i) relevant for the calculation up to and including order $\Gamma$. Let us begin with Eq. (24)

$$i \mathcal{T}_a = -i g^2 \bar{u}(p_3, \sigma_3) \gamma^\mu v(p_4, \sigma_4) S_{\mu\nu}^\nu(p) \bar{v}(p_2, \sigma_2) \gamma^\nu u(p_1, \sigma_1) \equiv -i V_f^\mu S_{\mu\nu}^{\prime\nu}(p) V_i^\nu,$$

where $p = p_1 + p_2 = p_3 + p_4$ with $p^2 = s$ and our fermion states are normalized as $\langle \bar{v}(p, \sigma) | v(p, \sigma) \rangle = 2m = -\bar{v}(p, \sigma) v(p, \sigma)$. Because of current conservation, using Eq. (21), we obtain from Eq. (24)

$$\mathcal{T}_a = V_f^\mu \frac{1}{s - z} V_i^\mu. \quad \text{(25)}$$

From now on we consider forward scattering, i.e., $p_1 = p_3, p_2 = p_4, \sigma_1 = \sigma_3,$ and $\sigma_2 = \sigma_4$. Renaming $V_{i\mu} \rightarrow V_{\mu}$, and using $V_f^\mu = V^{\mu*}$, the imaginary part of the forward-scattering tree-order amplitude obtained from Eq. (25) reads

$$\text{Im} [\mathcal{T}_a] = \text{Im} \left[ V^{\mu*} \frac{1}{s - z} V_\mu \right] = \text{Im} \left[ V^{\mu*} \frac{s - z^*}{(s - z)(s - z^*)} V_\mu \right] = -V^{\mu*} \frac{M \Gamma}{(s - z)(s - z^*)} V_\mu. \quad \text{(26)}$$

We omit wave function renormalization constants for external fermion lines as they do not contribute to the obtained relations at the given accuracy.

The one-loop diagrams contributing to the $f \bar{f} \rightarrow f \bar{f}$ amplitude are shown in Fig. 1; the results of these two diagrams read

$$i \mathcal{T}_i = i V^{\mu*} S_{\mu\alpha}^\nu(p) \Pi^{\alpha\beta}_1(p) S_{\beta\nu}^\nu(p) V_\mu, 
\quad i \mathcal{T}_c = i V^{\mu*} S_{\mu\alpha}^\nu(p) \delta z_1 g^{\alpha\beta} S_{\beta\nu}^\nu(p) V_\mu, \quad \text{(27)}$$

where $\Pi^{\alpha\beta}_1(p)$ and $\delta z_1$ denote the self energy and the counter term at first order in $\hbar$, respectively. Using Eq. (22), we obtain from Eq. (27)

$$\mathcal{T}_{i+c} = \mathcal{T}_i + \mathcal{T}_c = V^{\mu*} \frac{\Pi_1(s) + \delta z_1}{(s - z)^2} V_\mu, \quad \text{(28)}$$

with $\Pi_1(s)$ denoting the one-loop contribution to the function $\Pi(s)$ of Eq. (22). The imaginary part of the one-loop order amplitude for $s \neq M^2$ is obtained from Eq. (28) as

$$\text{Im} [\mathcal{T}_{i+c}] = V^{\mu*} \frac{\text{Im} \left[ (\Pi_1(s) + \delta z_1)(s - z)^2 \right]}{(s - z)^2} V_\mu = V^{\mu*} \frac{\text{Im} \left[ \Pi_1(s) + \delta z_1 \right]}{(s - z)(s - z^*)} V_\mu + \mathcal{O}(\Gamma^3). \quad \text{(29)}$$

In Eq. (29) we took into account that $g \sim \sqrt{\Gamma}$ and hence $V_\mu \sim \sqrt{\Gamma}$, $\Pi_1(s) \sim \Gamma$, and $\delta z_1 \sim \Gamma.$
FIG. 1: Tree and one-loop contributions to $f\bar{f}$ scattering. Solid and curved lines correspond to fermions and vector bosons, respectively. Crosses refer to counter-term contributions.

Let us now consider the contribution of the intermediate state consisting of one fermion ($f$) and one antifermion ($\bar{f}$) (”square” of the tree-order amplitude shown in Fig. 2) to the right-hand side of the unitarity condition of Eq. (23):

$$\frac{1}{2} \sum_{f,\bar{f}} (2\pi)^4 \delta^4(p_{ff} - p) T^*_{ff,i} T_{ff,i}$$

$$= \frac{1}{2} g^4 \sum_{\sigma,\bar{\sigma}} \int \frac{d^4 q}{(2\pi)^4} (2\pi) \delta \left[q^2 - m^2\right] \Theta(q_0) \Theta(p_0 - q_0)$$

$$ \times \bar{u}(p_1, \sigma_1) \gamma^\nu v(p_2, \sigma_2) \left[S^{'\beta\nu}(p)\right]^* \bar{v}(p - q, \bar{\sigma}) \gamma_\beta u(q, \sigma)$$

$$ \times \bar{u}(q, \sigma) \gamma_{\alpha} v(p - q, \bar{\sigma}) S^{\alpha \mu}(p) \bar{v}(p_2, \sigma_2) \gamma_\mu u(p_1, \sigma_1)$$

$$= 2\pi^2 g^2 \int \frac{d^4 q}{(2\pi)^4} \delta \left[q^2 - m^2\right] \Theta(q_0) \Theta(p_0 - q_0)$$

$$ \times \frac{1}{s - z} V^\nu_\mu \text{Tr} \left[\gamma^\nu (\not{q} + m) \gamma^\mu (\not{p} - \not{q} - m)\right] \frac{1}{s - z} V^\nu_\mu$$

$$= V^\nu_\mu \frac{\text{Im} \left[\Pi_1(s)\right]}{(s - z)(s - z^*)} V^\nu_\mu. \quad (30)$$

In the second-to-last step we made use of the projection operators over the positive and negative energy states. The last step is obtained by applying the cutting rules for stable particles to the one-loop self energy of the vector boson,

$$\Pi_1^{\nu\mu}(p) = ig^2 \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left(\frac{1}{\not{q} - \not{p} - m + i\epsilon} \gamma^\mu \frac{1}{\not{q} - m + i\epsilon} \gamma^\nu\right). \quad (31)$$

Using $1 + \sigma_{\rho\sigma} \rho_{\rho\rho} - 2[\Theta(q_0) \Theta(p_0 - q_0) + \Theta(-q_0) \Theta(q_0 - p_0)]$, the fact that $p_0 \geq 2m$ in Eq. (30), and Eq. (22), results in Eq. (30). Let us compare Eq. (30) with the sum of Eqs. (26) and
Taking into account that \( \text{Im}[\delta_{21}] = M \Gamma + \mathcal{O}(\hbar^2) \) [which follows from Eq. (18)], we see that the unitarity condition is satisfied up to and including order \( \Gamma^2 \).

In the following, we qualitatively discuss the imaginary parts of the remaining diagrams of Fig. 1 using the formulae in the appendix. The sum of diagrams b) and d) may be expanded in powers of \( \Gamma \). The first term in this expansion is real, whereas at second order the respective imaginary parts of the two diagrams cancel each other. Therefore, the corresponding imaginary part of these two diagrams is of \( \mathcal{O}(\Gamma^3) \). Similarly, expanding the amplitude corresponding to diagram j), the first term in this expansion is of order \( g^4 \) and is real and the imaginary part starts contributing at \( \mathcal{O}(\Gamma^3) \). The imaginary parts of diagrams f), g), h), k), and l) are all of higher order. This is the case because all these diagrams are proportional to \( g^4 \) and applying vertical cuts (relevant to forward scattering) in each case means cutting at least one unstable-particle line producing an additional factor of \( \Gamma \).

Finally, cutting two stable fermion lines in diagrams e), m), and n) generates the imaginary parts corresponding, up to higher-order terms, to the “square” of the tree-order diagrams shown in Fig. 3. On the other hand, all remaining possible cuts involve at least one unstable line which leads at least to \( \mathcal{O}(\Gamma^3) \).

According to Ref. [16], only stable asymptotic states contribute to the unitarity condition of the \( S \)-matrix. In order to demonstrate that our cutting rules agree with this result, we consider the one-loop contribution to the forward-scattering amplitude of \( f \bar{f} f \bar{f} \to f \bar{f} f \bar{f} \) shown in Fig. 4 a). Using Eq. (16), it is easily seen that the imaginary part obtained by cutting the two lines of the loop in diagram a) is proportional to \( \Gamma \). In Fig. 4 b), it is schematically represented as the “square” of the tree-order diagram (modulo higher-order corrections). As the width \( \Gamma \) is generated by diagrams representing the decay of the vector boson into stable particles, it is clear that the imaginary part of the diagram in Fig. 4 a) corresponds to the “square” of diagrams with stable particles only in external legs [see diagram c) in Fig. 4]. Since the width \( \Gamma \) is calculated for an “on-mass-shell” vector boson, diagram c) contains also contributions corresponding to loop diagrams of higher order. Note that in the limit...
of vanishing $\Gamma$ ($M < 2m$) one is not allowed to drop $\epsilon$ in Eq. (16). Therefore, the result of diagram 4b) does not vanish. In fact, for $\Gamma = 0$ the limit $\epsilon \to 0$ leads to a delta function corresponding to the vector line, and we obtain the standard cutting rule for stable particles.

V. CONCLUSIONS

In this work we developed a procedure for deriving the imaginary parts of (one-loop) integrals involving propagators with complex masses. With the aid of this method, we demonstrated perturbative unitarity of the scattering amplitude within the complex-mass scheme at the one-loop level. This result was obtained under the assumption that the expansion parameter of perturbation theory (the renormalized coupling constant) remains real. Our results are in full agreement with the findings of Ref. [16] that unstable states do not appear as asymptotic states and are therefore excluded from the unitarity condition. A generalization of cutting rules for unstable particles to higher orders of the loop expansion is straightforward. However, because of the non-trivial dependence of imaginary parts on $\Gamma$, the analysis of perturbative unitarity in higher orders will become more involved.

Acknowledgments

J. G. acknowledges the support of the Deutsche Forschungsgemeinschaft (SFB 443) and Georgian National Foundation grant GNSF/ST08/4-400. T. B. was supported by the Deutsche Forschungsgemeinschaft (SCHE459/4-1) and the German Academic Exchange Service (DAAD). J. G. and T. B. would like to thank M. Paris for discussions and comments on the manuscript. T. B. would like to thank H. W. Grießhammer and M. R. Schindler for useful discussions and their hospitality during his stay at George Washington University.
Appendix A: Imaginary parts of one-loop integrals

In order to compactify the notation let us introduce the following abbreviations,
\[ 
d(k^2) = (k^2 - m^2)^2 + \epsilon^2, \]
\[ 
D(k^2) = (k^2 - M^2)^2 + M^2 \Gamma^2. \]

We consider the integral
\[ 
I_2 = i \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^4(k + l - p_1) (2\pi)^4 \delta^4(k + q - p_2) \Delta'(k) \Delta_F(q) \Delta_F(l) \]
\[ 
= i \int \frac{d^4 k}{(2\pi)^4} \Delta'(k) \Delta_F(p_2 - k) \Delta_F(p_1 - k). \quad \text{(A1)} \]

Its imaginary part reads
\[ 
\text{Im}[I_2] = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^4(k + l - p_1) (2\pi)^4 \delta^4(k + q - p_2) \]
\[ 
\times \left\{ \frac{k^2 - M^2}{D(k^2)} \frac{q^2 - m^2}{d(q^2)} \frac{l^2 - m^2}{d(l^2)} - \frac{k^2 - M^2}{D(k^2)} \frac{\pi^2 \delta(q^2 - m^2)}{d(l^2)} \delta(l^2 - m^2) \right. \]
\[ 
\left. - \frac{M \Gamma}{D(k^2)} \frac{q^2 - m^2}{d(q^2)} \pi \delta(l^2 - m^2) - \frac{M \Gamma}{D(k^2)} \frac{l^2 - m^2}{d(l^2)} \frac{\pi \delta(q^2 - m^2)}{d(q^2)} \right\}. \quad \text{(A2)} \]

To rewrite the expression of Eq. (A2) in a more convenient form we consider the following integral,
\[ 
0 = i \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^4(k + l - p_1) (2\pi)^4 \delta^4(k + q - p_2) \Delta_A'(k) \Delta_R(q) \Delta_R(l), \quad \text{(A3)} \]

and take its imaginary part and obtain
\[ 
0 = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^4(k + l - p_1) (2\pi)^4 \delta^4(k + q - p_2) \]
\[ 
\times \left\{ \frac{k^2 - M^2}{D(k^2)} \frac{q^2 - m^2}{d(q^2)} \frac{l^2 - m^2}{d(l^2)} - \frac{k^2 - M^2}{D(k^2)} \frac{\pi^2 \delta(q^2 - m^2)}{d(l^2)} \delta(l^2 - m^2) \sigma_q \sigma_l \right. \]
\[ 
\left. + \frac{M \Gamma}{D(k^2)} \frac{q^2 - m^2}{d(q^2)} \pi \delta(l^2 - m^2) \sigma_q \frac{k_0}{\Sigma_k} + \frac{M \Gamma}{D(k^2)} \frac{l^2 - m^2}{d(l^2)} \pi \delta(q^2 - m^2) \sigma_q \right\} \]
\[ 
+ O(l^2). \quad \text{(A4)} \]

By subtracting Eq. (A4) from Eq. (A2), we obtain
\[ 
\text{Im}[I_2] = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^4(k + l - p_1) (2\pi)^4 \delta^4(k + q - p_2) \]
\[ 
\times \left\{ - \frac{k^2 - M^2}{D(k^2)} \pi^2 \delta(q^2 - m^2) \delta(l^2 - m^2) (1 - \sigma_q \sigma_l) \right. \]
\[ 
\left. - \frac{M \Gamma}{D(k^2)} \frac{(1 + \sigma_q \frac{k_0}{\Sigma_k})}{d(q^2)} q^2 - m^2 - \frac{M \Gamma}{D(k^2)} \frac{(1 + \sigma_q \frac{k_0}{\Sigma_k})}{d(l^2)} l^2 - m^2 \pi \delta(q^2 - m^2) + O(\Gamma^2) \right\}. \quad \text{(A5)} \]
The first term in Eq. (A5) corresponds to cutting both stable-particle lines [18, 19]. The other two terms correspond to cutting one of the two stable-particle lines together with the unstable-particle line. These two terms are proportional to $\Gamma$.

Next, let us consider an integral

$$
I_3 = i \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(k + p_1 - p_3 - q) (2\pi)^4 \delta^4(k + p_1 - l) \\
\times (2\pi)^4 \delta^4(k - p_2 + p) \Delta'(k) \Delta'(q) \Delta_F(l) \Delta_F(p)
$$

$$
= i \int \frac{d^4k}{(2\pi)^4} \Delta'(k) \Delta'(p_1 - p_3 + k) \Delta_F(p_1 + k) \Delta_F(p_2 - k). \quad (A6)
$$

Its imaginary part reads

$$
\text{Im}[I_3] = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(k + p_1 - p_3 - q) (2\pi)^4 \delta^4(k + p_1 - l) \\
\times (2\pi)^4 \delta^4(k - p_2 + p) \left\{ \frac{k^2 - M^2}{D(k^2)} - \frac{q^2 - M^2}{D(q^2)} - \frac{l^2 - m^2}{D(l^2)} \right\} \\
\times \left\{ \frac{p^2 - m^2}{D(p^2)} - \frac{M \Gamma}{D(k^2)} \right\} \\
\times \left\{ \frac{\pi \delta(l^2 - m^2)}{D(q^2)} \delta(p^2 - m^2) - \frac{\pi \delta(p^2 - m^2)}{D(q^2)} \delta(l^2 - m^2) \right\} \\
+ \frac{\pi \delta(l^2 - m^2)}{D(k^2)} \right\}. \quad (A7)
$$

To rewrite the expression of Eq. (A7) in a more convenient form we consider the following integral,

$$
0 = i \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(k + p_1 - p_3 - q) (2\pi)^4 \delta^4(k + p_1 - l) \\
\times (2\pi)^4 \delta^4(k - p_2 + p) \Delta'(k) \Delta'(q) \Delta_A(l) \Delta_F(p)
$$

$$
= i \int \frac{d^4k}{(2\pi)^4} \Delta'(k) \Delta'(p_1 - p_3 + k) \Delta_A(p_1 + k) \Delta_F(p_2 - k), \quad (A8)
$$

take its imaginary part and obtain

$$
0 = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(k + p_1 - p_3 - q) (2\pi)^4 \delta^4(k + p_1 - l) \\
\times (2\pi)^4 \delta^4(k - p_2 + p) \left\{ \frac{k^2 - M^2}{D(k^2)} - \frac{q^2 - M^2}{D(q^2)} - \frac{l^2 - m^2}{D(l^2)} \right\} \\
\times \left\{ \frac{p^2 - m^2}{D(p^2)} - \frac{M \Gamma}{D(k^2)} \right\} \\
\times \left\{ \frac{\pi \delta(l^2 - m^2)}{D(q^2)} \delta(p^2 - m^2) - \frac{\pi \delta(p^2 - m^2)}{D(q^2)} \delta(l^2 - m^2) \right\} \\
- \frac{M \Gamma}{D(k^2)} \right\}. \quad (A7)
$$
Subtracting Eq. (A9) from Eq. (A7), we obtain

\begin{align}
\text{Im}[I_3] &= \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta^4(k + p_1 - p_3 - q) (2\pi)^4 \delta^4(k + p_1 - l) \\
&\quad \times (2\pi)^4 \delta^4(k - p_2 + p) \left\{ -\frac{k^2 - M^2}{D(k^2)} \frac{q^2 - M^2}{D(q^2)} \frac{l^2 - m^2}{d(l^2)} \pi \delta(l^2 - m^2) \delta(p^2 - m^2) (1 + \sigma_l \sigma_p) \\
&\quad - \frac{M \Gamma}{D(k^2)} \left( 1 - \frac{q_0}{\xi_\pi} \sigma_l \right) \frac{q^2 - M^2}{D(q^2)} \frac{l^2 - m^2}{d(l^2)} \pi \delta(l^2 - m^2) \\
&\quad - \frac{M \Gamma}{D(k^2)} \left( 1 + \frac{q_0}{\xi_\pi} \sigma_p \right) \frac{q^2 - M^2}{D(q^2)} \frac{l^2 - m^2}{d(l^2)} \pi \delta(p^2 - m^2) \\
&\quad - \frac{k^2 - M^2}{D(k^2)} \frac{M \Gamma}{D(q^2)} \left( 1 - \frac{q_0}{\xi_\pi} \sigma_l \right) \frac{p^2 - m^2}{d(p^2)} \pi \delta(l^2 - m^2) \\
&\quad - \frac{k^2 - M^2}{D(k^2)} \frac{M \Gamma}{D(q^2)} \left( 1 + \frac{q_0}{\xi_\pi} \sigma_p \right) \frac{l^2 - m^2}{d(l^2)} \pi \delta(p^2 - m^2) + O(\Gamma^2) \right\}.
\end{align}

(A10)

Note that we have kept the second line in the brackets for completeness although it is of $O(\Gamma^2)$. In analogy to the step from Eq. (15) to Eq. (16), one can replace expressions like $\frac{q_0}{\xi_\pi}$ by the corresponding functions $\sigma_q$ in Eqs. (A5) and (A10). Finally, the case $\Gamma = 0$ is obtained by first replacing $M \Gamma \to M \Gamma + \epsilon$ in all integrals above. Setting $\Gamma = 0$ and then taking the limit $\epsilon \to 0^+$, we exactly reproduce the standard cutting formulas for loop integrals with real masses \[18, 19\].

[1] R. G. Stuart, in $Z^0$ Physics, ed. J. Tran Thanh Van (Editions Frontiers, Gif-sur-Yvette, 1990), p. 41.
[2] A. Denner, S. Dittmaier, M. Roth, and D. Wackeroth, Nucl. Phys. B560, 33 (1999).
[3] A. Denner, S. Dittmaier, M. Roth, and L. H. Wieders, Nucl. Phys. B724, 247 (2005).
[4] A. Denner and S. Dittmaier, Nucl. Phys. Proc. Suppl. 160, 22 (2006).
[5] A. Denner, S. Dittmaier, M. Roth, and L. H. Wieders, Nucl. Phys. Proc. Suppl. 157, 68 (2006).
[6] A. Bredenstein, A. Denner, S. Dittmaier, and M. M. Weber, Nucl. Phys. Proc. Suppl. 160, 131 (2006).
[7] A. Bredenstein, A. Denner, S. Dittmaier, and M. M. Weber, JHEP 0702, 080 (2007).
[8] S. Actis and G. Passarino, Nucl. Phys. B777, 100 (2007).
[9] S. Actis, G. Passarino, C. Sturm, and S. Uccirati, Phys. Lett. B 669, 62 (2008).
[10] S. Actis, G. Passarino, C. Sturm, and S. Uccirati, Phys. Lett. B 670, 12 (2008).
[11] A. Denner, S. Dittmaier, T. Kasprzik, and A. Muck, JHEP 0908, 075 (2009).
[12] A. Denner, S. Dittmaier, T. Gehrmann, and C. Kurz, Nucl. Phys. B836, 37 (2010).
[13] D. Djukanovic, J. Gegelia, A. Keller, and S. Scherer, Phys. Lett. B 680, 235 (2009).
[14] D. Djukanovic, J. Gegelia, and S. Scherer, Phys. Lett. B 690, 123 (2010).
[15] T. Bauer, J. Gegelia, and S. Scherer, Phys. Lett. B 715, 234 (2012).
[16] M. J. G. Veltman, Physica 29, 186 (1963).
[17] J. C. Collins, Renormalization (Cambridge University Press, Cambridge, England, 1984).
[18] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
[19] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, Reading, USA, 1995).
[20] D. G. Boulware, Annals Phys. 56, 140 (1970).