Path integration in relativistic quantum mechanics

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The simple physics of a free particle reveals important features of the path-integral formulation of relativistic quantum theories. The exact quantum-mechanical propagator is calculated here for a particle described by the simple relativistic action proportional to its proper time. This propagator is nonvanishing outside the light cone, implying that spacelike trajectories must be included in the path integral. The propagator matches the WKB approximation to the corresponding configuration-space path integral far from the light cone; outside the light cone that approximation consists of the contribution from a single spacelike geodesic. This propagator also has the unusual property that its short-time limit does not coincide with the WKB approximation, making the construction of a concrete skeletonized version of the path integral more complicated than in nonrelativistic theory.
1. Introduction

The path-integral formulation of relativistic quantum mechanics gives rise to problems not found in nonrelativistic theory. Some of these are similar to problems which arise in attempting to construct a quantum description of gravity. In this work we seek to elucidate some of these problems by considering the much simpler physics of a single, free, relativistic particle, a system offering the advantage of exact solubility.

The quantum mechanics of a particle can be completely described by a propagator, given by a functional integral

\[ G(x, t; x_0, t_0) = \int_C Dx e^{iS/\hbar}. \]  

Here \( C \) denotes the class of paths included in the integral, and \( S \) is the classical action associated with each path. In nonrelativistic theory, \( C \) properly includes all paths linking spacetime point \((x, t)\) to \((x_0, t_0)\) which move forward in time, in the sense that such a class \( C \) in (1.1) gives the same propagator as that in the canonical Hamiltonian quantization. In a relativistic theory, with Lorentz-invariant action \( S \), the light-cone structure of the spacetime comes into play: Should \( C \) include all paths that move forward in time, or only those inside the light cone, i.e., paths which are always timelike? If the latter, then the propagator (1.1) must vanish outside the light cone of \((x_0, t_0)\). If spacelike paths are included, this will not be so in general, and \( G \) will describe propagation outside the light cone.

Such propagation will be acausal—backward in time in some Lorentz frames. Whether such a feature is admissible, or perhaps necessary, is a question which
arises in quantum gravity as well as in particle mechanics. Indeed, Teitelboim [1] has argued that a quantum gravity theory cannot be both covariant and causal; he suggests [2] retaining causality, although Hartle [3] argues that covariance should be preserved instead. There too the problem can be framed as a choice of histories to be included in a path integral: It is the choice between the class of spacetime histories which includes those with negative lapse function, i.e., backward time displacement, and that which does not [2,3]. (In fact the action for general relativity can be converted into a form [4] like that for a relativistic particle; general relativity is analogous to parametrized theories for such particles [2,3,5].)

Here we evaluate the quantum-mechanical propagator for a free, relativistic point particle. We show the significance of spacelike paths by comparing the exact propagator with the WKB approximation to the formal, non-Gaussian, configuration-space path integral.

Comparison of the exact and WKB expressions for the propagator in the short-time-interval limit also yields the appropriate measure for the path integral. With this a concrete “skeletonized” version of the formal integral can be constructed. Here too the relativistic theory encounters complications not found in the nonrelativistic case.

For simplicity we treat a particle in 1 + 1-dimensional flat spacetime; generalization to higher dimensions is straightforward. Units with $\hbar = c = 1$ are used henceforth.
2. Propagator and path integral for a free relativistic particle

A single free particle can be described by the relativistic action

\[ S = - \int m \, d\tau = - \int m(1 - \dot{x}^2)^{1/2} \, dt , \quad (2.1) \]

where \( m \) is the particle’s mass, \( \tau \) its proper time, and \((x,t)\) its coordinates in some Lorentz frame, with \( \dot{x} \equiv dx/dt \). Hence this model has Lagrangian \( L = -m(1 - \dot{x}^2)^{1/2} \), momentum \( p = \partial L/\partial \dot{x} = m\dot{x}(1 - \dot{x}^2)^{-1/2} \), and Hamiltonian

\[ H = p\dot{x} - L = +\left(p^2 + m^2\right)^{1/2} . \quad (2.2) \]

Being nonpolynomial in \( p \), this corresponds to a nonlocal quantum operator; it is to be interpreted as acting on any wave function

\[ \psi(x,t) = \int dk \, e^{ikx} \, \phi(k,t) \quad (2.3a) \]

to give \([6,7]\)

\[ H\psi(x,t) = \int dk \, e^{ikx} \left(k^2 + m^2\right)^{1/2} \phi(k,t) . \quad (2.3b) \]

The sign of \( H \) in Eq. (2.2) emerges unambiguously from the canonical formalism, given the action (2.1). It implies that the quantum-mechanical description is entirely in terms of positive-frequency functions. Such a description is adequate for a noninteracting particle \([6]\). [It is formally the same as the positive-frequency branch of a Klein-Gordon theory. In the Klein-Gordon picture interactions could “scatter the particle into the negative-frequency branch.” Teitelboim \([2]\), however, has constructed a theory of \textit{interacting} relativistic point particles based on an action of form (2.1), plus interaction terms.]
The propagator for the wave equation $i\dot{\psi} = H\psi$ is given by the integral

$$G(x, t; x_0, t_0) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik\Delta x} e^{-i(k^2+m^2)^{1/2}\Delta t},$$

(2.4)

with $\Delta x \equiv x - x_0$ and $\Delta t \equiv t - t_0$; clearly this is a solution of the equation, with “initial value” $\delta(\Delta x)$ at $\Delta t = 0$. The same integral expression can be obtained using the localized states of Newton and Wigner [7], i.e., as the projection of the state localized at $x_0$ at time $t_0$ on that localized at $x$ at time $t$. Hartle and Kuchar [5] derive this same “Newton-Wigner propagator” via a phase-space path integral.

A very simple calculation yields the propagator $G$ in closed form: The integral (2.4) can be evaluated by adding a small negative imaginary part to $\Delta t$ for convergence. The result is

$$G = \lim_{\epsilon \to 0^+} \frac{m(i\Delta t + \epsilon)}{\pi \lambda_\epsilon^{1/2}} K_1(m\lambda_\epsilon^{1/2}),$$

(2.5)

with $\lambda_\epsilon \equiv (\Delta x)^2 + (i\Delta t + \epsilon)^2$ and $K_1$ the familiar modified Bessel function. This propagator contains the complete description of the behavior of a free particle in this formulation of relativistic quantum mechanics.

The corresponding configuration-space path integral (1.1) for the propagator is, formally,

$$G = \int \mathcal{D}x \exp \left[ -im \int_{(x_0,t_0)}^{(x,t)} (1 - \dot{x}^2)^{1/2} \, dt \right].$$

(2.6)

The treatment of spacelike paths in this functional integral is clearly problematic [2]. If such paths are to be excluded from the integral by fiat then $G$ must vanish for all $(x, t)$ outside the light cone of $(x_0, t_0)$, i.e., for $|\Delta x| > |\Delta t|$. But expression (2.5) does not do this: Outside the light cone $m\lambda_\epsilon^{1/2}$ is (nearly) real, and the Bessel
function is nonvanishing—decreasing exponentially for large argument. Obviously
the path integral for a relativistic particle must include the contributions of spacelike
paths in general.

Such contributions are manifest in the WKB approximation to the propagator. In such circumstances as the integral (2.6) is dominated by paths near the classical
trajectory between \((x_0, t_0)\) and \((x, t)\), it is approximated by the WKB expression [8]

\[
\int \mathcal{D}x \, e^{iS} \sim \left( \frac{i}{2\pi} \frac{\partial^2 S_{cl}}{\partial x \partial x_0} \right)^{1/2} e^{iS_{cl}},
\]

with \(S_{cl}\) the action evaluated along that classical path. Here that path simply has
constant speed \(\dot{x} = \Delta x / \Delta t\). Hence the classical action is \(S_{cl} = +im\lambda^{1/2} (\epsilon \to 0^+)\),
and the resulting approximation is

\[
G \sim \lim_{\epsilon \to 0^+} \left( \frac{m(i\Delta t + \epsilon)^2}{2\pi \lambda^{3/2}} \right)^{1/2} e^{-m\lambda^{1/2}}. \tag{2.8}
\]

(Here the \(\epsilon\) term serves to specify the phases of the square roots.) The exact propa-
gator (2.5) approaches just this form in the regime \(m|\lambda^{1/2}| \gg 1\), i.e., many Compton
wavelengths from the light cone; in other words, as expected, the WKB approxima-
tion to the path integral (2.6) is accurate when the magnitude of the classical action
is large compared to \(\hbar\). For \((x, t)\) well outside the light cone of \((x_0, t_0)\), the dominant
trajectory giving rise to form (2.8) of the propagator is the spacelike geodesic (line)
between the two points.

These features of the propagator are illustrated in Fig. 1, which shows the
evolution of a simple Gaussian initial wave function via \(G\). The extension of the
propagator outside the light cone and the coincidence of the exact and WKB forms away from the light cone are evident.

Unlike familiar nonrelativistic propagators, the relativistic-particle propagator does not coincide with the WKB form in the limit $\Delta t \to 0$. In the regime $|\Delta x|, |\Delta t| \ll m^{-1}$, the exact propagator (2.5) takes the form

$$G(\text{exact}) \sim \frac{1}{\pi} \lim_{\epsilon \to 0^+} i\Delta t + \epsilon \left\{ 1 + O[m^2|\lambda_\epsilon\ln(m\lambda_\epsilon^{1/2})]| \right\}$$

(2.9a)

while the WKB approximation (2.8) takes the form

$$G(\text{WKB}) \sim \left( \frac{m}{2\pi} \right)^{1/2} \lim_{\epsilon \to 0^+} i\Delta t + \epsilon \left[ 1 + O(m|\lambda_\epsilon^{1/2}|) \right].$$

(2.9b)

At $\Delta t = 0$ the former becomes $\delta(\Delta x)$, as required; the latter does not. In this disagreement between the $\Delta t \to 0$ and WKB limits, the relativistic-particle propagator resembles certain curved-space propagators [9].

The absence of a factor $e^{iS_{\text{cl}}} = 1 - m\lambda_\epsilon^{1/2} + O(m^2|\lambda_\epsilon|)$ in form (2.9a) might lead one to conclude that a Lagrangian, i.e., configuration-space, path integral for the relativistic particle cannot be constructed, as indicated by Hartle and Kuchar [5]. This can be done, however, by including appropriate factors in the path-integral measure. Thus a skeletonized version of integral (2.6) can be given as the composition

$$G(x, t; x_0, t_0) = \lim_{N \to \infty} \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_N \lim_{\epsilon \to 0^+} \frac{m(i\Delta t + \epsilon)}{\pi \lambda_\epsilon^{1/2}} K_1(m\lambda_\epsilon^{1/2}) \exp(m\lambda_\epsilon^{1/2}) \right] e^{iS_{\text{cl}}},$$

(2.10a)

with the infinitesimal-interval propagators

$$G(x_k, t_k; x_{k-1}, t_{k-1}) = \lim_{\epsilon \to 0^+} \left[ \frac{m(i\Delta t + \epsilon)}{\pi \lambda_\epsilon^{1/2}} K_1(m\lambda_\epsilon^{1/2}) \exp(m\lambda_\epsilon^{1/2}) \right] e^{iS_{\text{cl}}},$$

(2.10b)
where $\Delta t$, $\lambda$, and $S_{\text{cl}}$ are taken between $(x_{k-1}, t_{k-1})$ and $(x_k, t_k)$. This form follows directly from the exact result (2.5); since $\Delta x$, hence $\lambda$, need not be small compared to $m^{-1}$ even when $\Delta t$ is, no expansion of that result is suitable. The measure factors, those in square brackets preceding $e^{iS_{\text{cl}}}$ in Eq. (2.10b), are considerably more complicated than their nonrelativistic counterparts. This is to be expected since the kinetic term in the relativistic action is more complicated than the nonrelativistic term, which is simply quadratic in the velocity.

3. Conclusions

The free, relativistic point particle provides a simple, exactly soluble example of some remarkable features of path integrals for relativistic quantum theories. The Newton-Wigner [7] propagator for such a particle can be expressed in the closed form (2.5). Although the corresponding configuration-space path integral (2.6) cannot be evaluated exactly [except in the sense that Eq. (2.5) is the evaluation of integral (2.6)], it can be approximated by the formal WKB expression (2.8). The WKB approximation can be calculated without knowing the precise form of the path integral measure. It agrees with the exact result far from the light cone, which substantiates the suitability of the path-integral formulation of this theory. The exact and approximate forms are seen to differ only near the light cone, i.e., where the classical action is not large compared to $\hbar$, as expected.

The propagator is manifestly nonvanishing outside the light cone. Hence the path integral must include the contributions of spacelike trajectories; in particular
spacelike classical paths give the dominant contribution in the construction of the WKB approximation far outside the light cone, where that approximation is valid.

The exact and WKB forms of the propagator show a difference between the short-time and WKB limits, i.e., the limits $\Delta t \to 0$ and $\hbar \to 0$, of the relativistic-particle propagator, unlike its nonrelativistic counterpart. This discrepancy introduces additional complexity into the relativistic theory, i.e., into the path-integral measure. This does not appear to invalidate the configuration-space path-integral formulation [5], though it does diminish its intuitive appeal.

The contribution of spacelike paths to the relativistic-particle path integral, i.e., the fact that the propagator is nonvanishing outside the light cone, shows that acausality is a feature of at least this simple relativistic quantum theory. Strict causality is recovered as a classical limit: The propagator falls off exponentially outside the light cone, on a scale of the particle’s Compton wavelength. It might be argued that a first-quantized theory is inappropriate, that quantum field theory is called for. Nonetheless the same questions of acausality and the proper domain of histories in the path integral are known to arise in quantum gravity [1–3,10]. The example treated here lends support to the conjecture [3] that path integrals in that more complex case should also include the contributions of acausal histories.

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FIG. 1. Real (a) and imaginary (b) parts of a wave function propagated via the relativistic-particle propagator $G$. The wave function with initial form $\psi(x,0) = \exp(-m^2x^2)$ is shown at time $t = 10m^{-1}$. The solid curves are the exact wave function, the dotted that obtained using the WKB approximation to the propagator. The values of $\psi$ are in units of $m$. 

