ON THE ALGEBRAIC HOLONOMY OF STABLE PRINCIPAL BUNDLES

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Abstract. Let $E_G$ be a stable principal $G$–bundle over a compact connected Kähler manifold, where $G$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. Let $H \subset G$ be a holomorphic reduction of structure group to $H$. We prove that $E_H$ is preserved by the Einstein–Hermitian connection on $E_G$. Using this we show that if $E_H$ is a minimal reductive reduction (which means that there is no complex reductive proper subgroup of $H$ to which $E_H$ admits a holomorphic reduction of structure group), then $E_H$ is unique in the following sense: For any other minimal reduction of structure group $(H', E_{H'})$ of $E_G$ to some reductive subgroup $H'$, there is some element $g \in G$ such that $H' = g^{-1} H g$ and $E_{H'} = E_{H} g$. As an application, we show the following:

Let $M$ be a simply connected, irreducible smooth complex projective variety of dimension $n$ such that the Picard number of $M$ is one. If the canonical line bundle $K_M$ is ample, then the algebraic holonomy of the holomorphic tangent bundle $T^{1,0} M$ is $\text{GL}(n, \mathbb{C})$. If $K_M^{-1}$ is ample, the rank of the Picard group of $M$ is one, the biholomorphic automorphism group of $M$ is finite, and $M$ admits a Kähler–Einstein metric, then the algebraic holonomy of $T^{1,0} M$ is $\text{GL}(n, \mathbb{C})$.

These answer some questions posed in [BK].

1. Introduction

In [BK], Balaji and Kollár introduced the notion of algebraic holonomy of polystable vector bundles on normal projective varieties defined over $\mathbb{C}$. Given a polystable vector bundle $E$ over $M$ together with a smooth point $x \in M$, the algebraic holonomy of $E$ is a canonically associated complex reductive subgroup of $\text{Aut}(E_x)$, the group of all linear automorphisms of the fiber $E_x$, which is constructed using the restrictions of $E$ to the general complete intersection curves of sufficiently large degrees that pass through $x$. Our aim here is to address some questions in [BK], which are recalled below, on the algebraic holonomy.

Let $M$ be a simply connected, irreducible smooth complex projective variety equipped with a Kähler–Einstein metric. In [BK], p. 187, Question 9 it is asked whether the algebraic holonomy of the holomorphic tangent bundle $T^{1,0} M$ coincides with the complexification of the differential geometric holonomy of $M$.

Let $M$ be a simply connected, irreducible smooth complex projective variety whose Néron–Severi group $\text{NS}(M)$ is of rank one. We note that $\text{rank}(\text{NS}(M)) = \text{rank}(\text{Pic}(M))$ because $M$ is simply connected. In [BK], p. 210, Question 51 it is asked whether the
condition that the canonical line bundle $K_M$ is ample implies that the algebraic holonomy of $T^{1,0}M$ is the full $GL(n,\mathbb{C})$. We note that from a theorem of Aubin and Yau it follows that the holomorphic tangent bundle $T^{1,0}M$ of $M$ is polystable provided $K_M$ is ample. If the anticanonical line bundle $K_{-1}M$ is ample, $\text{Aut}(X) < \infty$, and the holomorphic tangent bundle $T^{1,0}M$ is polystable, the Question 48 of [BK, p. 209] asks whether the algebraic holonomy of $T^{1,0}M$ is $GL(n,\mathbb{C})$.

We show that both Question 9 and Question 51 have an affirmative answer. We also show that Question 48 has an affirmative answer under the extra assumption that $M$ admits a Kähler–Einstein metric.

Let $M$ be a compact connected Kähler manifold equipped with a Kähler form. Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{C}$. It is known that any stable principal $G$–bundle over $M$ admits a unique Einstein–Hermitian connection. We prove the following (see Theorem 2.3):

**Theorem 1.1.** Let $E_G$ be a stable principal $G$–bundle over $M$. Let $H$ be a complex reductive subgroup of $G$ which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group of $E_G$ to $H$. Then the Einstein–Hermitian connection on $E_G$ is induced by a connection on $E_H$.

Let $H$ be a complex reductive subgroup of $G$ which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group to $H$ of a stable $G$–bundle $E_G$ defined over $M$. Assume that there is no complex reductive proper subgroup of $H$ to which $E_H$ admits a holomorphic reduction of structure group. Such reductions will be called the minimal reductive ones. Theorem 1.1 says that $E_H$ is preserved by the Einstein–Hermitian connection on $E_G$.

Fix a point $x_0 \in M$, and also choose a point in the fiber $z \in (E_G)_{x_0}$. Taking parallel translations, for the Einstein–Hermitian connection on $E_G$, of $z$ along piecewise smooth paths in $M$ based at $x_0$ we get a subset of $E_G$. The topological closure, in $E_G$, of this subset gives a smooth reduction of structure group of $E_G$ to a compact subgroup $\overline{K}_E \subset G$. The corresponding smooth reduction of structure group $E^{z_0}_{\overline{K}_E}$ of $E_G$ to the Zariski closure $\overline{K}_E^C$ of $\overline{K}_E$ in $G$ is actually holomorphic. We prove the following (see Theorem 3.1):

**Theorem 1.2.** There is a point $z_0 \in (E_G)_z$ such that the above minimal reductive reduction $E_H$ coincides with $E^{z_0}_{\overline{K}_E}$. In particular, $H$ coincides with $\overline{K}_E^C$ for such a base point $z_0$.

Take any $g \in G$. If we replace the base point $z_0$ by $z_0g$, then the subgroup $\overline{K}_E \subset G$ gets replaced by $g^{-1}\overline{K}_E g$, and hence $\overline{K}_E^C$ gets replaced by $g^{-1}\overline{K}_E^C g$. Also,

$$E^{z_0g}_{\overline{K}_E} = E^{z_0}_{\overline{K}_E} g \subset E_G.$$ 

Therefore, if $E^{z_0}_{\overline{K}_E} = E^{z_0g}_{\overline{K}_E}$, then $g \in \overline{K}_E^C$. In particular, if $z_0$ satisfies the condition in Theorem 1.2 then $z_0g$ also satisfies the condition if and only if $g \in H$. 
The adjoint bundle of $E_G$ is the fiber bundle $\text{Ad}(E_G) \to M$ associated to $E_G$ for the adjoint action of $G$ on itself (see [KN, p. 55, Proposition 5.4] for the construction of associated bundles). Hence the fibers of $\text{Ad}(E_G)$ are groups isomorphic to $G$.

It follows from Theorem 1.2 and the above comments that if $(H, E_H)$ and $(H', E_{H'})$ are two minimal reductive reductions of $E_G$, then there is an element $g \in G$ such that $H' = g^{-1}Hg$ and $E_{H'} = E_Hg$. This gives us the following (see Corollary 3.2):

**Corollary 1.3.** Let $E_G$ be a stable principal $G$–bundle over $M$. Then there is a unique holomorphic sub–fiber bundle $G_{E_G}$ of the adjoint bundle $\text{Ad}(E_G)$, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction $E_H$ of $E_G$, the adjoint bundle $\text{Ad}(E_H)$, which is a sub–fiber bundle of $\text{Ad}(E_G)$, coincides with $G_{E_G}$.

The sub–fiber bundle $G_{E_G} \subset \text{Ad}(E_G)$ in Corollary 1.3 is the complexification of the holonomy of the Einstein–Hermitian connection on $E_G$. While the stability condition does not depend on the choice of a Kähler form in a given Kähler class, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. We note that the sub–fiber bundle $G_{E_G}$ does not depend either on the Kähler form or on the Kähler class as long as $E_G$ remains stable. When $M$ is a complex projective manifold, $G_{E_G}$ coincides with the algebraic holonomy of $E_G$. In particular, Theorem 1.2 gives an affirmative answer to Question 9 of [BK].

In Theorem 4.1 we prove the following:

**Theorem 1.4.** Let $M$ be a simply connected, irreducible smooth complex projective variety of dimension $n$ such that the canonical line bundle $K_M$ is ample, and $\text{rank}(\text{NS}(M)) = 1$. Then the algebraic holonomy of $T^{1,0}M$ is $\text{GL}(n, \mathbb{C})$.

In Theorem 4.2 we prove the following:

**Theorem 1.5.** Let $M$ be an irreducible smooth complex projective variety of dimension $n$ satisfying the following three conditions:

- the anticanonical line bundle $K_M^{-1}$ is ample, and $M$ admits a Kähler–Einstein metric,
- $\text{rank}(\text{NS}(M)) = 1$, and
- the biholomorphic automorphism group of $M$ is finite.

Then the algebraic holonomy of $T^{1,0}M$ is $\text{GL}(n, \mathbb{C})$.

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2. EINSTEIN–HERMITIAN CONNECTION AND REDUCTION OF STRUCTURE GROUP

Let $M$ be a compact connected Kähler manifold equipped with a Kähler form $\omega$. The degree of a torsion–free coherent analytic sheaf $V$ on $M$ is defined to be
\[
\text{degree}(V) := (c_1(V) \cup \omega^{\dim M-1}) \cap [M] \in \mathbb{R}.
\]

Let $G$ be a linear algebraic group defined over $\mathbb{C}$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. For any $g_0 \in G$, let $\text{Ad}(g_0) : G \rightarrow G$ be the inner automorphism defined by $g \mapsto g_0^{-1}gg_0$. The corresponding automorphism of $\mathfrak{g}$ will also be denoted by $\text{Ad}(g_0)$.

Let $E_G \rightarrow M$ be a holomorphic principal $G$–bundle. The adjoint vector bundle of $E_G$ will be denoted by $\text{ad}(E_G)$. We recall that $\text{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$, and two points $(z,v)$ and $(z',v')$ of $E_G \times \mathfrak{g}$ are identified in $\text{ad}(E_G)$ if and only if there is some $g_0 \in G$ such that $z' = zg_0$ and $v' = \text{Ad}(g_0)(v)$. Therefore, the fibers of $\text{ad}(E_G)$ are Lie algebras isomorphic to $\mathfrak{g}$. If $E$ is a vector bundle of rank $r$, and $E_{\text{GL}(r,\mathbb{C})}$ is the corresponding principal $\text{GL}(r,\mathbb{C})$–bundle, then $\text{ad}(E_{\text{GL}(r,\mathbb{C})}) = \mathcal{E}nd(E) = E \otimes E^*$.

Let $\text{Ad}(E_G)$ be the quotient of $E_G \times G$ where two points $(z,g)$ and $(z',g')$ are identified if and only if there is some $g_0 \in G$ such that $z' = zg_0$ and $g' = \text{Ad}(g_0)(g)$. Therefore, the adjoint bundle $\text{Ad}(E_G)$ is a fiber bundle over $M$, and the fibers are groups isomorphic to $G$. For any point $x \in M$, the fiber $\text{Ad}(E_G)_x$ is the group of all automorphisms of the fiber $(E_G)_x$ that commute with the action of $G$ on $(E_G)_x$. For the above principal $\text{GL}(r,\mathbb{C})$–bundle $E_{\text{GL}(r,\mathbb{C})}$ associated to $E$, the fiber of $\text{Ad}(E_{\text{GL}(r,\mathbb{C})})$ over any point $x \in M$ is the group of automorphisms of the fiber $E_x$.

Henceforth, $G$ will be a connected reductive group.

Since the degree has been defined, we have the notions of a stable $G$–bundle and a polystable $G$–bundle over $M$. See [RS], [AB], [Ra2] for definitions of stable and polystable principal $G$–bundles.

Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing $T$. We also fix a maximal compact subgroup $K$ of $G$. By a parabolic subgroup of $G$ we will mean one containing $B$. Therefore, the Levi quotient $L(Q)$ of any parabolic subgroup $Q \subset G$ is also a subgroup of $Q$. Let $Z_0(G)$ denote the connected component of the center of $G$ that contains the identity element.

Take any polystable principal $G$– bundle $E_G$ over $M$. From the definition of a polystable principal $G$–bundle (see [AB] p. 221, Definition 3.5), [RS] p. 23) it follows that there is a Levi subgroup $L(P)$ associated to some parabolic subgroup $P \subset G$ (the subgroup $P$ need not be proper) along with a holomorphic reduction of structure group $E_{L(P)} \subset E_G$ to $L(P)$, such that

- the principal $L(P)$–bundle $E_{L(P)}$ is stable, and
- the principal $P$–bundle $E_P := E_{L(P)}(P)$, obtained by extending the structure group of $E_{L(P)}$ using the inclusion $L(P)$ in $P$, is an admissible reduction of $E_G$ (the definition of an admissible reduction is recalled below).
Note that since $L(P) \subset P \subset G$, the $P$–bundle $E_P$ is a reduction of structure group of $E_G$ to $P$. The condition that $E_P$ is an admissible reduction of structure group of $E_G$ means that for each character $\chi$ of $P$, which is trivial on $Z_0(G)$, the associated line bundle $E_P(\chi)$ over $M$ is of degree zero. The $G$–bundle $E_G$ is stable if and only if $P = G$.

Since $E_G$ and $E_{L(P)}$ are polystable, they admit unique Einstein–Hermitian connections [AB] p. 208, Theorem 0.1. It should be clarified that the Einstein–Hermitian reduction of structure to a maximal compact subgroup depends on the choice of the maximal compact subgroup. Even for a fixed maximal compact subgroup, the Einstein–Hermitian reduction of structure group need not be unique. However, once the Kähler form on $M$ is fixed, the Einstein–Hermitian connection on a polystable principal bundle over $M$ is unique. Let $\nabla^{L(P)}$ be the Einstein–Hermitian connection on the stable $L(P)$–bundle $E_{L(P)}$. Let $\nabla^G$ be the connection on $E_G$ induced by $\nabla^{L(P)}$. Note that a connection on $L(P)$ induces a connection on any fiber bundle associated to $E_{L(P)}$. The principal $G$–bundle $E_G$ is associated to $E_{L(P)}$ for the left translation action of $L(P)$ on $G$, and $\nabla^G$ is the connection on it obtained from $\nabla^{L(P)}$.

**Proposition 2.1.** The induced connection $\nabla^G$ on $E_G$ coincides with the unique Einstein–Hermitian connection on $E_G$.

*Proof.* Since the inclusion map $L(P) \hookrightarrow G$ need not take the connected component, containing the identity element, of the center of $L(P)$ into $Z_0(G)$, the proposition does not follow immediately from [AB] p. 208, Theorem 0.1. The connection $\nabla^{L(P)}$ on $E_{L(P)}$ is induced by a connection on a smooth reduction of structure group of $E_{L(P)}$ to a maximal compact subgroup of $L(P)$ (this is a part of the definition of an Einstein–Hermitian connection). A maximal compact subgroup of $L(P)$ is contained in a maximal compact subgroup of $G$. Consequently, the connection $\nabla^G$ on $E_G$ is induced by a connection on a smooth reduction of structure group of $E_G$ to a maximal compact subgroup of $G$. Therefore, to prove that $\nabla^G$ is the Einstein–Hermitian connection on $E_G$ it suffices to show that $\nabla^G$ satisfies the Einstein–Hermitian equation.

Let $\mathfrak{z}(l(P))$ be the Lie algebra of the center of $L(P)$. For any $\theta \in \mathfrak{z}(l(P))$, the holomorphic section of the adjoint bundle $\text{ad}(E_{L(P)})$ given by $\theta$ will be denoted by $\hat{\theta}$; the vector bundle $\text{ad}(E_{L(P)})$ is associated to $E_{L(P)}$ for the adjoint action of $L(P)$ on its Lie algebra. Let $\Lambda_\omega$ be the adjoint of multiplication by $\omega$ of differential form on $M$. That the connection $\nabla^{L(P)}$ is Einstein–Hermitian means that there is an element $\theta \in \mathfrak{z}(l(P))$ such that the Einstein–Hermitian equation

(2.1) \[ \Lambda_\omega K_{\nabla^{L(P)}} = \hat{\theta} \]

holds.

If $\theta$ in (2.1) is in the Lie algebra $\mathfrak{z}(G)$ of $Z_0(G)$, then $\nabla^G$ is a Einstein–Hermitian connection on $E_G$. Therefore, in that case the proposition is proved. Assume that

(2.2) \[ \theta \notin \mathfrak{z}(G). \]
Fix a character $\chi$ of $P$ which is trivial on $Z_0(G)$ but satisfies the following condition: the homomorphism of Lie algebras
\begin{equation}
(2.3) \quad d\chi : \mathfrak{p} \rightarrow \mathbb{C}
\end{equation}
given by $\chi$, where $\mathfrak{p}$ is the Lie algebra of $P$, is nonzero on $\theta$. (It is easy to check that the group of characters of $P$ coincides with the group of characters of $L(P)$.)

Consider the holomorphic line bundle $L_\chi := E_P(\chi)$ over $M$ associated to $E_P$ for $\chi$. Let $\nabla^\chi$ be the connection on $L_\chi$ induced by the connection on $E_P$ given by $\nabla^{L(P)}$. Since $\nabla^{L(P)}$ is an Einstein–Hermitian connection, the connection $\nabla^\chi$ is also Einstein–Hermitian. Indeed, if $K(\nabla^\chi)$ is the curvature of $\nabla^\chi$, then
\begin{equation}
\Lambda_\omega K(\nabla^\chi) = d\chi(\theta),
\end{equation}
where $d\chi$ is the homomorphism in (2.3) and $\theta$ is the element in (2.1). Therefore,
\begin{equation}
(2.4) \quad \text{degree}(L_\chi) = \frac{d\chi(\theta) \sqrt{-1}}{2\pi} \int_M \omega^d,
\end{equation}
where $d = \dim_{\mathbb{C}} M$; see [Ko2, p. 103, Proposition 2.1].

The condition that $E_P \subset E_G$ is an admissible reduction of structure group says that
\begin{equation}
\text{degree}(L_\chi) = 0.
\end{equation}
This, in view of the assumption in (2.2) that $d\chi(\theta) \neq 0$, contradicts (2.4). Therefore, we conclude that $\theta \in \mathfrak{g}(G)$. This immediately implies that the connection $\nabla^G$ on $E_G$ is Einstein–Hermitian. This completes the proof of the proposition. \hfill \Box

Let $E_G$ be a holomorphic principal $G$–bundle over $M$ and $\nabla'$ a $C^\infty$ connection on $E_G$ compatible with the holomorphic structure of $E_G$. Such a connection is called a complex connection; see [AB, p. 230, Definition 3.1(1)] for the precise definition of a complex connection. A connection $\nabla'$ is complex if and only if the $(0,2)$–Hodge type component of the curvature of $\nabla'$ vanishes identically.

**Definition 2.2.** Let $H$ be a closed complex subgroup of $G$ and $E_H \subset E_G$ a $C^\infty$ reduction of structure group of $E_G$ to $H$. We will say that $E_H$ is preserved by the connection $\nabla'$ if $\nabla'$ induces a connection on $E_H$.

It follows immediately that $E_H$ is preserved by $\nabla'$ if and only if there is a smooth connection $\nabla''$ on $E_H$ such that the connection $\nabla'$ on $E_G$ coincides with the one given by $\nabla''$ using extension of structure group. It is easy to see that $E_H$ is preserved by $\nabla'$ if and only if for each point $z \in E_H$, the horizontal subspace in $T_z E_G$ for the connection $\nabla'$ is contained in $T_z E_H$. If $E_H$ is preserved by $\nabla'$, then $E_H$ is a holomorphic reduction of structure group of $E_G$.

**Theorem 2.3.** Let $E_G$ be a stable principal $G$–bundle over $M$ and $\nabla^G$ the Einstein–Hermitian connection on $E_G$. Let $H$ be a complex reductive subgroup of $G$ which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group of $E_G$ to $H$. Then $E_H$ is preserved by the connection $\nabla^G$. 
Proof. Let $H_0 \subset H$ be the connected component containing the identity element. Set
\[
X := E_H/H_0,
\]
which is finite étale Galois cover of $M$ with Galois group $H/H_0$. Let
\[
p : X \rightarrow M
\]
be the projection. We note that $p^*E_H$ has a canonical reduction of structure group to the subgroup $H_0 \subset H$. Set
\[
F_G := p^*E_G.
\]
Let $F_{H_0} \subset F_G$ be the reduction of structure group to $H_0$ obtained from the canonical reduction of structure group of $p^*E_H$ to $H_0$.

Equip $X$ with the Kähler form $p^*\omega$. The Einstein–Hermitian connection on $E_G$ pulls back to an Einstein–Hermitian connection on $F_G$ for the Kähler form $p^*\omega$. Therefore, $F_G$ is polystable with respect to $p^*\omega$. Let $\text{ad}(F_G)$ be the adjoint bundle over $X$. We recall that $\text{ad}(F_G)$ is associated to $F_G$ for the adjoint action of $G$ on its Lie algebra. The connection on the vector bundle $\text{ad}(F_G)$ induced by the Einstein–Hermitian connection of $F_G$ is clearly Einstein–Hermitian. Hence the adjoint vector bundle $\text{ad}(F_G)$ is polystable.

Let $Z(G)$ be the center of $G$. Set
\[
H' := H_0Z(G)/Z(G).
\]
Therefore, $H'$ is a complex reductive subgroup of the complex semisimple group $G' := G/Z(G)$. Let $\mathfrak{h}_0$ be the Lie algebra of $H_0$. As before, $\mathfrak{g}$ is the Lie algebra of $G$. Note that the adjoint action makes $\mathfrak{g}$ (respectively, $\mathfrak{h}_0$) a $G'$–module (respectively, $H'$–module). We will also consider $\mathfrak{g}$ as a $H'$–module using the inclusion of $H'$ in $G'$.

Since $\mathfrak{g}$ is a faithful $G'$–module, and $H'$ is a reductive subgroup of $G'$, there is a positive integer $N$ and nonnegative integers $a_i, b_i, i \in [1, N]$, such that the $H'$–module $\mathfrak{h}_0$ is a direct summand of the $H'$–module
\[
\bigoplus_{i=1}^N \mathfrak{g}^{\otimes a_i} \otimes (\mathfrak{g}^*)^{\otimes b_i} = \bigoplus_{i=1}^N \mathfrak{g}^{\otimes a_i} \otimes \mathfrak{g}^{\otimes b_i} \tag{2.6}
\]
[De, p. 40, Proposition 3.1]; since $H'$ is complex reductive, any exact sequence of $H'$–modules splits; also, $\mathfrak{g} = \mathfrak{g}^*$ as $G$ is reductive. Therefore, the adjoint vector bundle $\text{ad}(F_{H_0})$ is a direct summand of the vector bundle
\[
\bigoplus_{i=1}^N \text{ad}(F_G)^{\otimes a_i} \otimes \text{ad}(F_G)^{\otimes b_i}. \tag{2.7}
\]
We note that the vector bundle in (2.7) is associated to $F_{H_0}$ for the $H_0$–module in (2.6).

Since the adjoint vector bundle $\text{ad}(F_G)$ is polystable of degree zero (recall that $\text{ad}(F_G) = \text{ad}(F_G)^*$), the vector bundle
\[
\text{ad}(F_G)^{\otimes a} \otimes \text{ad}(F_G)^{\otimes b}
\]

is polystable of degree zero. Therefore, the adjoint vector bundle $\text{ad}(F_{H_0})$ is a direct summand of the vector bundle
\[
\bigoplus_{i=1}^N \text{ad}(F_G)^{\otimes a_i} \otimes \text{ad}(F_G)^{\otimes b_i}.
\]
Lemma 5 for a similar argument.)

The principal $H_0$–bundle $F_{H_0}$ over $X$ is polystable because the vector bundle $\text{ad}(F_{H_0})$ is polystable [AB p. 224, Corollary 3.8]. Hence $F_{H_0}$ admits a unique Einstein–Hermitian connection [AB p. 208, Theorem 0.1]. Let $\nabla^{H_0}$ be the Einstein–Hermitian connection on $F_{H_0}$. So, there is an element $\nu \in \mathfrak{h}$ such that

$$(2.8) \quad \Lambda_{p^*\omega(\nabla^{H_0})} = \hat{\nu},$$

where $\mathcal{K}(\nabla^{H_0})$ is the curvature of $\nabla^{H_0}$, and $\hat{\nu}$ is the holomorphic section of $\text{ad}(p^*E_H) = \text{ad}(F_{H_0})$ given by $\nu$.

Since $H/H_0$ is a finite group, giving a connection on a principal $H_0$–bundle is equivalent to giving a connection on the principal $H$–bundle obtained from it by extension of structure group. From the uniqueness of the Einstein–Hermitian connection on $F_{H_0}$ it follows that the corresponding connection on $p^*E_H$ is left invariant by the action of the Galois group $H/H_0$. Consequently, the connection on $p^*E_H$ given by $\nabla^{H_0}$ descends to a connection on $E_H$. Let $\nabla^H$ denote the connection on $E_H$ obtained this way. It is clear that $\nabla^H$ is a complex connection.

Let $\nabla$ be the complex connection on $E_G$ induced by the above connection $\nabla^H$ on $E_H$. We will first show that $\nabla$ is unitary, which means that $\nabla$ is induced by connection on a smooth reduction of structure group of $E_G$ to a maximal compact subgroup of $G$. Then we will show that $\nabla$ satisfies the Einstein–Hermitian equation.

To prove that $\nabla$ is unitary, fix a point $x_0 \in M$, and also fix a point $z_0 \in (E_G)_{x_0}$ in the fiber of $E_G$ over $x_0$. Taking parallel translations of $z_0$, with respect to the connection $\nabla$, along piecewise smooth paths in $M$ based at $x_0$ we get a subset $\mathcal{S}$ of $E_G$. Sending any $g \in G$ to the point $z_0g \in (E_G)_{x_0}$ we get an isomorphism $G \to (E_G)_{x_0}$. Using this isomorphism, the intersection $(E_G)_{x_0} \cap \mathcal{S}$ is a subgroup $K_E$ of $G$. The condition that the connection $\nabla$ is unitary is equivalent to the condition that $K_E$ is contained in some compact subgroup of $G$.

Fix a point $x \in p^{-1}(x_0)$, and also fix a point $z \in (p^*E_G)_x$, where $p$ is the covering map in (2.5). Consider parallel translations of $z$, with respect to the connection $p^*\nabla$ on $p^*E_G =: F_G$, along piecewise smooth paths in $X$ based at $x$. As before, we get a subgroup $K_F \subset G$ from the resulting subset of $F_G$. It is easy to see that $K_F$ is a finite index subgroup of the group $K_E$ constructed above.

We note that the connection $p^*\nabla$ on $F_G$ is induced by the unitary connection $\nabla^{H_0}$ on the reduction $F_{H_0}$ of $F_G$. Therefore, the connection $p^*\nabla$ is unitary. Consequently, the subgroup $K_F \subset G$ is contained in a compact subgroup of $G$. Using this together with the observation that $K_F$ is a finite index subgroup of the subgroup $K_E \subset G$ we conclude that $K_E$ is also contained in a compact subgroup of $G$. Thus the connection $\nabla$ is unitary.

We will now show that $\nabla$ satisfies the Einstein–Hermitian equation.
Let $\mathcal{K}(\nabla)$ be the curvature of the connection $\nabla$. From (2.8) it follows immediately that $\Lambda_p^* \omega \mathcal{K}(\nabla^{H_0})$ is a holomorphic section of $\text{ad}(p^*E_H)$. Consequently, $\Lambda_\omega \mathcal{K}(\nabla)$ is a holomorphic section of the adjoint vector bundle $\text{ad}(E_G)$.

Since $E_G$ is stable, it can be shown that all holomorphic sections of $\text{ad}(E_G)$ are given by the Lie algebra $\mathfrak{g}(G)$ of $Z_0(G)$. To prove this consider the Einstein–Hermitian connection on $\text{ad}(E_G)$ induced by the Einstein–Hermitian connection on $E_G$. We have $\text{ad}(E_G)^* = \text{ad}(E_G)$ because $G$ is reductive. Hence the constant in the Einstein–Hermitian equation for $\text{ad}(E_G)$ vanishes (see [Ko2, p. 99, Proposition (1.4)(2)]). Therefore, the mean curvature for the Einstein–Hermitian connection on $\text{ad}(E_G)$ vanishes (see [Ko2, p. 51, (1.7)] for the definition of mean curvature, and [Ko2, p. 99] for the expression of the Einstein–Hermitian equation in terms of mean curvature). Since the mean curvature of the Einstein–Hermitian connection on $\text{ad}(E_G)$ vanishes, from [Ko2, p. 52, Theorem (1.9)] it follows that any holomorphic section of $\text{ad}(E_G)$ is flat with respect to the Einstein–Hermitian connection on it. Now the fact that $E_G$ is stable implies that any flat section of $\text{ad}(E_G)$ for the Einstein–Hermitian connection on it is given by some element of $\mathfrak{g}(G)$; see the proof of [Ra1, p. 136, Proposition 3.2].

In other words, there is an element $\theta \in \mathfrak{g}(G)$ such that $\Lambda_\omega \mathcal{K}(\nabla)$ coincides with the section of $\text{ad}(E_G)$ given by $\theta$. Thus, we conclude that the connection $\nabla$ is the unique Einstein–Hermitian connection on $E_G$. Since $\nabla$ is induced by a connection on $E_H$, the proof of the theorem is complete. \[\square\]

Proposition 2.1 and Theorem 2.3 together have the following:

**Corollary 2.4.** Let $E_G$ be a polystable $G$–bundle over $M$. Take $E_{L(P)}$ as in Proposition 2.7. Let $H$ be a complex reductive subgroup of $L(P)$ (not necessarily connected), and let $E_H \subset E_{L(P)}$ be a holomorphic reduction of structure group of $E_{L(P)}$ to $H$. Then $E_H$ is preserved by the Einstein–Hermitian connection on $E_G$.

3. Properties of a minimal reduction

Let $E_G$ be a stable principal $G$–bundle over $M$. Fix a point $x_0 \in M$, and also fix a point $z_0 \in (E_G)_{x_0}$ in the fiber over $x_0$. Let $\nabla^G$ be the Einstein–Hermitian connection on $E_G$.

Taking parallel translations of $z_0$, with respect to $\nabla^G$, along piecewise smooth paths in $M$ based at $x_0$ we get a subset $\mathcal{S}$ of $E_G$. Let $\overline{\mathcal{S}}$ be the topological closure of $\mathcal{S}$ in $E_G$. The map $G \rightarrow (E_G)_{x_0}$ defined by $g \rightarrow z_0g$ is an isomorphism. Using this isomorphism, the intersection $(E_G)_{x_0} \cap \overline{\mathcal{S}}$ gives a compact subgroup $\overline{K}_E \subset G$. The subset $\overline{\mathcal{S}} \subset E_G$ is a $C^\infty$ reduction of structure group of $E_G$ to the subgroup $\overline{K}_E$. Let $K^{\mathbb{C}}_E$ be the complex reductive subgroup of $G$ obtained by taking the Zariski closure of $\overline{K}_E$ in $G$. The subset

$$E_{z_0}^{\overline{K}_E} := \overline{\mathcal{S}}K^{\mathbb{C}}_E \subset E_G$$
is a holomorphic reduction of structure group of $E_G$ to $\overline{K}_E$; see [Bi, Section 3] for the details. It is easy to see that the principal $\overline{K}_E$-bundle $E^{\overline{z}_0}_{\overline{K}_E}$ is the extension of structure group of the principal $\overline{K}_E$-bundle $\mathcal{S}$.

The subgroup $\overline{K}_E \subset G$ is called the (differential geometric) holonomy of the connection $\nabla^G$ with reference point $z_0$ (see [KN, p. 72]). The subset $\mathcal{S}$ is called the (differential geometric) holonomy bundle through $z_0$ (see [KN, p. 85]).

We note that in [Bi], the point $z_0$ is taken to be in the subset of $\overline{E}_G$ given by an Einstein–Hermitian reduction of structure group (see [Bi, p. 71, (3.20)]). This was done only to make $K_E$ lie inside a fixed maximal compact subgroup of $G$. If we replace the base point $z_0$ by $z_0g$, where $g$ is any point of $G$, then it is easy to see that the subset $\mathcal{S}$ constructed above using $z_0$ gets replaced by $\mathcal{S}g$. Therefore, the subgroup $K_C \subset \overline{K}_E$ in (3.1) gets replaced by $g^{-1}K_C g$, and the reduction $E^{\overline{z}_0}_{\overline{K}_E} \subset E_G$ in (3.1) gets replaced by $E^{\overline{z}_0}_{\overline{K}_E} g$.

We note that $(\overline{K}_E, E^{\overline{z}_0}_{\overline{K}_E})$ in (3.1) is a minimal complex reduction of $E_G$ in the following sense: There is no complex proper subgroup of $\overline{K}_E$ to which $E^{\overline{z}_0}_{\overline{K}_E}$ admits a holomorphic reduction of structure group which is preserved by the connection $\nabla^G$. Indeed, this follows immediately from the construction of $E^{\overline{z}_0}_{\overline{K}_E}$. Furthermore, if $E_{H'} \subset E_G$ is a holomorphic reduction of structure group of $E_G$, to a complex subgroup $H' \subset G$, satisfying the two conditions:

- $E_{H'}$ is preserved by $\nabla^G$, and
- there is no complex proper subgroup of $H'$ to which $E_{H'}$ admits a holomorphic reduction of structure group which is also preserved by $\nabla^G$,

then it follows immediately that there is a point $z_0 \in (E_G)_{x_0}$ such that $E_{H'} = E^{\overline{z}_0}_{\overline{K}_E}$. Indeed, $z_0$ can be taken to be any point of the fiber $(E_{H'})_{x_0}$.

Let $H$ be a complex reductive subgroup of $G$ which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group to $H$ of the stable $G$–bundle $E_G$. This reduction $E_H$ will be called a minimal reductive reduction of $E_G$ if there is no complex reductive proper subgroup of $H$ to which $E_H$ admits a holomorphic reduction of structure group.

In view of the above observations, using Theorem 2.3 we get the following:

**Theorem 3.1.** Let $E_G$ be a stable principal $G$–bundle over a compact connected Kähler manifold $M$, where $G$ is a connected reductive complex linear algebraic group. Let $E_H \subset E_G$ be a minimal reductive reduction of $E_G$. Fix a point $x_0 \in M$. Then there is a point $z_0$ in the fiber $(E_G)_{x_0}$ such that $E^{\overline{z}_0}_{\overline{K}_E}$ (defined in (3.1)) coincides with $E_H$. In particular, the subgroup $H$ coincides with $\overline{K}_E$ for such a base point $z_0$.

Let $\text{Ad}(E_G)$ be the adjoint bundle for $E_G$. So $\text{Ad}(E_G)$ is the fiber bundle over $M$ associated to $E_G$ for the adjoint action of $G$ on itself. The fibers of $\text{Ad}(E_G)$ are groups
isomorphic to $G$. Let $\text{Ad}(E_{K_E}^{z_0})$ be the adjoint bundle of the principal $K_E^C$–bundle $E_{K_E}^{z_0}$ defined in (3.1). Since $E_{K_E}^{z_0}$ is a holomorphic reduction of structure group of $E_G$, the fiber bundle $\text{Ad}(E_{K_E}^{z_0})$ is a holomorphic sub–fiber bundle of $\text{Ad}(E_G)$ with the fibers of $\text{Ad}(E_{K_E}^{z_0})$ being subgroups of the fibers of $\text{Ad}(E_G)$.

We noted earlier that if $z_0$ is replaced by $z_0g$, where $g \in G$, then the subgroup $K_E$ gets replaced by $g^{-1}K_Eg$, and $E_{K_E}^{z_0}$ gets replaced by $E_{K_E}^{z_0}g$. From this it follows immediately that the subbundle

$\text{Ad}(E_{K_E}^{z_0}) \subset \text{Ad}(E_G)$

is independent of the choice of the base point $z_0$.

Therefore, from Theorem 3.1 we have the following:

**Corollary 3.2.** Let $E_G$ be a stable principal $G$–bundle over a compact connected Kähler manifold $M$, where $G$ is a connected reductive complex linear algebraic group. Then there is a unique holomorphic sub–fiber bundle $G_{E_G}$ of the adjoint bundle $\text{Ad}(E_G)$, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction $E_H$ of $E_G$, the adjoint bundle $\text{Ad}(E_H)$, which is a sub–fiber bundle of $\text{Ad}(E_G)$, coincides with $G_{E_G}$.

From Theorem 3.1 it follows that the sub–fiber bundle $G_{E_G} \subset \text{Ad}(E_G)$ in Corollary 3.2 is the complexification of the holonomy of the Einstein–Hermitian connection on $E_G$. We note that for defining stable bundles we need only the Kähler class $[\omega] \in H^2(M, \mathbb{R})$. In other words, the stability condition does not depend on the choice of the Kähler form in a given Kähler class. On the other hand, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. From Corollary 3.2 it follows that the sub–fiber bundle $G_{E_G}$ does not depend on the choice of the Kähler form in a given Kähler class. In fact, it does not even depend on the Kähler class as long as $E_G$ remains stable.

Assume that $M$ is a complex projective manifold. The algebraic holonomy $H_{x_0}(E_G)$ of $E_G$ constructed in [BK] is an algebraic subgroup of $\text{Ad}(E_G)_{x_0}$, recall that $\text{Ad}(E_G)_{x_0}$ is the group of automorphisms of $(E_G)_{x_0}$ that commute with the action of $G$. Hence $H_{x_0}(E_G)$ gives a conjugacy class of algebraic subgroups of $G$. Take any subgroup $H_{x_0} \subset G$ in this conjugacy class. The principal $G$–bundle admits a minimal reductive reduction of structure group to this subgroup $H_{x_0}$ [BK, p. 193, Theorem 20(3)]. Hence from Corollary 3.2 it follows that the algebraic holonomy of $E_G$ coincides with the fiber $(G_{E_G})_{x_0}$ when $M$ is a complex projective manifold.

**4. Algebraic holonomy and Kähler–Einstein metric**

Let $(M, g)$ be a simply connected compact Kähler manifold. The de Rham decomposition says that there is a biholomorphic isometry

\begin{equation}
(M, g) \sim \prod_{i=1}^\ell (M_i, g_i),
\end{equation}

\[4.1\]
where each \((M_i, g_i)\) is a compact Kähler manifold whose holonomy group is irreducible (see [Ko1, § 4, pp. 327–328], [Jo, p. 49, Theorem 3.2.7]). Furthermore, for \(i \in [1, \ell]\), the Kähler manifold \((M_i, g_i)\) is either an irreducible Hermitian symmetric space or its holonomy is one of the following three:

\[
(4.2) \quad \text{U}(m_i), \text{SU}(m_i), \text{and Sp}(m_i/2),
\]

where \(m_i = \dim_C M_i\) [Jo, p. 58, § 3.4.2] (this also follows from the combination of [Ko1 p. 327, Corollary 4] and [Be, p. 294, § 10.66]).

**Theorem 4.1.** Let \(M\) be a simply connected, irreducible smooth complex projective variety, of complex dimension \(n\), such that the canonical line bundle \(K_M\) is ample, and \(\text{rank}(\text{NS}(M)) = 1\). Then the algebraic holonomy of the holomorphic tangent bundle \(T^{1,0}M\) is \(\text{GL}(n, \mathbb{C})\).

**Proof.** Fix a Kähler–Einstein metric \(g\) on \(M\), which exists by [Au], [Ya]. We know that the algebraic holonomy of \(M\) is a complexification of the holonomy of the Kähler–Einstein metric \(g\) (see Theorem 3.1). Therefore, it suffices to show that the holonomy of \(g\) is full \(U(n)\).

Consider the decomposition in (4.1). Since \(\text{rank}(\text{NS}(M)) = 1\), we have \(\ell = 1\). A simply connected compact Hermitian symmetric space is a Fano manifold. Hence the holonomy of \(g\) is one of the three groups in (4.2). Since the canonical line bundle \(K_M\) is ample, only \(U(n)\) in (4.2) can be the holonomy. This completes the proof of the theorem. \(\square\)

**Theorem 4.2.** Let \(M\) be an irreducible smooth complex projective variety of complex dimension \(n\) satisfying the following four conditions:

- the anticanonical line bundle \(K_M^{-1}\) is ample,
- \(\text{rank}(\text{NS}(M)) = 1\),
- the biholomorphic automorphism group of \(M\) is finite, and
- \(M\) admits a Kähler–Einstein metric.

Then the algebraic holonomy of the holomorphic tangent bundle \(T^{1,0}M\) is \(\text{GL}(n, \mathbb{C})\).

**Proof.** Fix a Kähler–Einstein metric \(g\) on \(M\). As in the proof of Theorem 4.1, we need to show that the holonomy of \(g\) is \(U(n)\).

Consider the decomposition in (4.1). As before, \(\ell = 1\) because \(\text{rank}(\text{NS}(M)) = 1\). Also, \(M\) is not a Hermitian symmetric space because the biholomorphic automorphism group of \(M\) is finite. Since \(K_M^{-1}\) is ample, except \(U(n)\) none of the other groups in (4.2) can be the holonomy of \(g\). This completes the proof of the theorem. \(\square\)

**References**

[Au] T. Aubin: Équations du type Monge–Ampère sur les variétés kähleriennes compactes, *Comp. Ren. Acad. Sci. Paris Sér. A* 283 (1976), 119–121.
B. Anchouche and I. Biswas: Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, *Amer. Jour. Math.* **123** (2001), 207–228.

B. Anchouche H. Azad and I. Biswas: Harder–Narasimhan reduction for principal bundles over a compact Kähler manifold, *Math. Ann.* **323** (2002), 693–712.

V. Balaji and J. Kollár: Holonomy groups of stable vector bundles, *Publ. Res. Inst. Math. Sci.* **44** (2008), 183–211.

A. Besse: *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Band 10, Springer–Verlag, Berlin, 1987.

I. Biswas: Stable bundles and extension of structure group, *Diff. Geom. Appl.* **23** (2005), 67–78.

P. Deligne: Hodge cycles on abelian varieties, (in: *Hodge cycles, motives, and Shimura varieties*, by P. Deligne, J. S. Milne, A. Ogus and K.-Y. Shih), Lecture Notes in Mathematics, 900, Springer-Verlag, Berlin-New York, 1982.

D. D. Joyce: *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

S. Kobayashi: The first Chern class and holomorphic symmetric tensor fields, *Jour. Math. Soc. Japan* **32** (1980), 325–329.

S. Kobayashi: *Differential Geometry of Complex Vector Bundles*, Publications of the Math. Society of Japan 15, Iwanami Shoten Publishers and Princeton University Press, 1987.

S. Kobayashi and K. Nomizu: *Foundations of differential geometry. Vol. I*, Interscience Publishers, New York-London, 1963.

A. Ramanathan and S. Subramanian: Einstein–Hermitian connections on principal bundles and stability, *Jour. Reine Angew. Math.* **390** (1988), 21–31.

A. Ramanathan: Stable principal bundles on a compact Riemann surface, *Math. Ann.* **213** (1975), 129–152.

A. Ramanathan: Moduli for principal bundles over algebraic curves: I, *Proc. Ind. Acad. Sci. (Math. Sci.)* **106** (1996), 301–328.

S.-T. Yau: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

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