Quantum oscillator as 1D anyon

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It is shown that in one spatial dimension the quantum oscillator is dual to the charged particle situated in the field described by the superposition of Coulomb and Calogero–Sutherland potentials.

\textbf{Keywords:} 1D quantum oscillator, reduction, 1/x\textsuperscript{2} interaction, duality, 1D anyon.
I. INTRODUCTION

In one spatial dimension a particle moving in the Calogero–Sutherland potential $V_{cs} = -\hbar^2 \nu(1 - \nu)/2\mu x^2$ has a very unusual property. Unlike the potential $V_{cs}$, the wave function is not invariant under the replacement $\nu \to (1 - \nu)$. It describes a boson for even $\nu$ and a fermion for odd $\nu$. Statistics corresponding to the other values of $\nu$ is called the fractional statistics\(^1\), and the system influenced along with $V_{cs}$ by a potential binding the particle to the center is called the 1D anyon\(^2-4\). Nobody has observed a 1D anyon yet, but nevertheless it is of both theoretical\(^5\) and experimental\(^6\) interest. The purpose of the present note is to prove that such an extraordinary object can be constructed from a 1D quantum oscillator.

II. ANYON–OSCILLATOR DUALITY

Consider the Schrödinger equation

$$\frac{\partial^2}{\partial u^2} \Psi + \frac{2\mu}{\hbar^2} \left( E - \frac{\mu \omega^2 u^2}{2} \right) \Psi = 0,$$

(1)

which describes the 1D quantum oscillator. Introduce the quantum number $s = 0, 1/2$ and write $N = 2n + 2s$, with $N$ numerating the energy levels $E = \hbar \omega (N + 1/2)$ and $n$ being integer and nonnegative. Without loss of information we can assume $u$ to belong to the region $0 \leq u < \infty$. We interpret $s$ as a spin of the reduced oscillator. The corresponding wave function is denoted by $\Psi_n^{(s)}$.

Let us look for the function $\Psi_n^{(s)}$ in the form

$$\Psi_n^{(s)}(u) = C u^{2s} \overline{\Psi}_n,$$

(2)

where $\overline{\Psi}_n$ is subordinate to the condition $\overline{\Psi}_n(0) \neq 0$, and $C$ is a normalization constant. Eq. (1) is easily seen to take the form

$$\frac{\partial^2}{\partial x^2} \overline{\Psi}_n + \frac{4s}{u} \frac{\partial}{\partial u} \overline{\Psi}_n + \frac{2\mu}{\hbar^2} \left( E - \frac{\mu \omega^2 u^2}{2} \right) \overline{\Psi}_n = 0.$$

(3)

After change of the variable $x = u^2$, we arrive at the equation ($2\nu = 2s + 1/2$)

$$\frac{\partial^2}{\partial x^2} \overline{\Psi}_n + \frac{2\nu}{x} \frac{\partial}{\partial x} \overline{\Psi}_n + \frac{2\mu}{\hbar^2} \left( - \frac{\mu \omega^2}{8} + \frac{E}{4x} \right) \overline{\Psi}_n = 0.$$

(4)

Now we set

$$\overline{\Psi}_n = x^{-\nu} \Phi_n^{(\nu)},$$

(5)

then cancel the undesirable term with first derivative in (4) and obtain

$$\frac{\partial^2}{\partial x} \Phi_n^{(\nu)} + \frac{2\mu}{\hbar^2} (\varepsilon - V_c - V_{cs}) \Phi_n^{(\nu)} = 0,$$

(6)
where $V_c = -\alpha/x$, $V_{cs}$ is the Calogero–Sutherland potential with $\nu = 1/4$ or $3/4$ and

$$\varepsilon = -\frac{\mu \omega^2}{8}, \quad \alpha = \frac{E}{4}. \quad (7)$$

Eq. (3) describes a system which we call the 1D Coulomb anyon.

This equation realizes a special case of a more general equation that has a relation to $(2 + 1)$-dimensional anyons\(^7\).

Comparing Eq. (1) with Eqs. (6) and (7), we summarize that there are two alternative possibilities connected with Eq. (1) – explicit and hidden. In the first case, the parameter $\omega$ is fixed ($\omega = \text{fix.} > 0$) and plays a role of coupling constant, the parameter $E$ is quantized and has a meaning of energy, and the system is a 1D quantum oscillator. For the hidden possibility, the parameter $E$ is fixed ($E = \text{fix.} > 0$), the coupling constant is equal to $E/4$, $\omega$ is quantized, the meaning of energy takes the quantity $\varepsilon = -\mu \omega^2/8$, and the system is the 1D Coulomb anyon. In the above-mentioned sense, the 1D quantum oscillator is dual to the 1D Coulomb anyon.

### III. ENERGY LEVELS AND WAVE FUNCTIONS

Let us return to Eq. (3) and make the substitution

$$\Phi^{(\nu)}_n = y^{\nu} e^{-y/2} Q(y), \quad (8)$$

where $y = x(-8 \mu \varepsilon/\hbar^2)^{1/2}$ and $Q(0) \neq 0$ and is finite. The function $Q(y)$ can diverge at infinity but not higher than the finite power of $y$. Using (8) and (3) we come to the equation

$$y \partial^2_y Q + (2 \nu - y) \partial_y Q - (\nu - \lambda) Q = 0, \quad (9)$$

with $\lambda = (-\mu \alpha^2/2 \hbar^2 \varepsilon)^{1/2}$. Eq. (9) is the equation for a confluent hypergeometric function. It has a general solution\(^8\)

$$Q(y) = C_1 F(\nu - \lambda, 2 \nu, y) + C_2 y^{1-2\nu} F(1 - \lambda - \nu, 2 - 2\nu, y), \quad (10)$$

where $F(a, b, y)$ is given by the series

$$F(a, b, y) = 1 + \frac{a}{b} \frac{y}{1!} + \frac{a(a+1)}{b(b+1)} \frac{y^2}{2!} + \ldots$$

convergent for all finite $y$. For large $y$ the asymptotic formula\(^8\) is valid

$$F(a, b, y) \sim \frac{\Gamma(b)}{\Gamma(b-a)} (-y)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^y (y)^{a-b}. \quad (11)$$

The second term in (11) for $\nu = 3/4$ is singular at $y = 0$, and hence $C_2$ has to be taken zero. The first term in (11), as it is evident from (11), is “well-behaved” at infinity under the condition $3/4 - \lambda = -n$, where $n$ is an integer number greater or equal to zero. For $\nu = 1/4$ both the terms in (11) are regular at $y = 0$, but the satisfactory behavior at
infinity needs the simultaneous requirements $1/4 - \lambda = -n$, $3/4 - \lambda = -m$, or $n - m = 1/2$, which is impossible. Hence, either $C_1 = 0$ or $C_2 = 0$. But for $C_1 = 0$ the function $Q(y)$ will become zero at $y = 0$. This contradicts the condition $Q(0) \neq 0$, and, therefore, we put $C_2 = 0$ and $1/4 - \lambda = -n$. Thus, we conclude that $\nu - \lambda = -n$, i.e.,

$$\varepsilon_{n}^{(\nu)} = -\frac{\mu \omega^2}{2\hbar^2(n + \nu)^2}, \quad n = 0, 1, 2, \ldots$$

(12)

Returning to the corresponding eigenfunctions, we put

$$\Phi_n^{(\nu)} = C_n^{(\nu)} y^{\nu} e^{-y/2} F(-n, 2\nu, y).$$

(13)

It is known that

$$F(-n, 2\nu, y) = \frac{n! \Gamma(2\nu)}{[\Gamma(n + 2\nu)]^2} \Lambda_n^{2\nu-1}(y),$$

and

$$\int_0^\infty e^{-y} y^{2\nu} [\Lambda_n^{2\nu-1}(y)]^2 \, dy = 2(n + \nu) \frac{[\Gamma(n + 2\nu)]^3}{n!},$$

where $\Lambda_n^{2\nu-1}(y)$ is an associated Laguerre polynomial. Using this results and taking into account the relation

$$\left(-\frac{8\mu \varepsilon}{\hbar^2}\right)^{1/4} = \frac{1}{\hbar} \left(\frac{2\mu \alpha}{n + \nu}\right)^{1/2},$$

we find

$$C^{(\nu)} = \frac{\sqrt{\mu \alpha}}{\hbar} \frac{1}{n + \nu} \frac{1}{\Gamma(2\nu)} \sqrt{\frac{\Gamma(n + 2\nu)}{n!}}.$$

Summarizing, we write

$$\Phi_n^{(\nu)} = \frac{\sqrt{\mu \alpha}}{\hbar} \frac{1}{n + \nu} \frac{1}{\Gamma(2\nu)} \sqrt{\frac{\Gamma(n + 2\nu)}{n!}} y^{\nu} e^{-y/2} F(-n, 2\nu, y).$$

(14)

So, we have two types of the 1D Coulomb anyons with $\nu = 1/4$ and $\nu = 3/4$. They are dual to reduced oscillators with $s = 0$ and $s = 1/2$, respectively.

**IV. DUALITY FOR SOLUTIONS**

Now we will calculate the energy levels $\varepsilon_{n}$ and wave functions $\Phi_n^{(\nu)}$ in another, more straightforward, way. For energy levels we have

$$\varepsilon = -\frac{\mu \omega^2}{8} = -\mu \left[ \frac{E}{2\hbar(n + \nu)} \right]^2 = -\mu \left[ \frac{2\alpha}{\hbar(n + \nu)} \right]^2 = -\frac{\mu \alpha^2}{2\hbar^2(n + \nu)^2}.$$ 

It follows from Eqs. (2) and (3) that

$$\Phi_n^{(\nu)} = \frac{1}{C} x^{1/4} \Phi_n^{(\nu)}$$
and, therefore,
\[ \int_0^\infty |\Phi_n(\nu)|^2 \, dx = \frac{1}{|C|^2} \int_0^\infty x^{1/2} |\Psi_n(s)|^2 \, dx. \]

The integral in the left-hand side is equal to 1, from which it follows that
\[ |C|^2 = 2 \int_{-\infty}^\infty u^2 |\Psi_N(u)|^2 \, du = 2 \bar{u}^2 = \frac{4(n + \nu) \hbar}{\mu \omega}, \]

where \( \Psi_N \) is the normalized wave function of a 1D quantum oscillator. Thus,
\[ \Phi_n(\nu) = \frac{(-1)^n}{2} \sqrt{\frac{\mu \omega}{\hbar (n + \nu)}} x^{1/4} \Psi_n(s). \]  (15)

Remind that according to the theory of quantum oscillator\(^9\),
\[ \Psi_n(s) = \sqrt{2} \left( \frac{\mu \omega}{\pi \hbar} \right)^{1/4} \frac{1}{2^n N!} e^{-\mu \omega n^2 / \hbar} H_N \left( u \sqrt{\frac{\mu \omega}{\hbar}} \right). \]  (16)

Further, it is known\(^10\) that Hermite polynomials could be expressed in terms of confluent hypergeometric functions. For our case
\[ H_{2n+2s}(\sqrt{y}) = (-1)^n \frac{(2n + 2s)!}{n!} (2 \sqrt{y})^{2s} F(-n, 2s + 1/2, y). \]  (17)

Using the identification \( y = x \mu \omega / \hbar \) and the relations \( 2s + 1/2 = 2\nu \) and \( \mu \omega / \hbar = 2 \mu \alpha / \hbar^2 (n + \nu) \), and taking into account Eqs. (15)-(17) we get
\[ \Phi_n(\nu) = \tilde{C}_n^{(\nu)} y^{\nu} e^{-y^{1/2}} F(-n, 2\nu, y), \]  (18)

where
\[ \tilde{C}_n^{(\nu)} = \sqrt{\frac{\mu \alpha}{\hbar^2}} \frac{1}{2^{n-1/2} \pi^{1/4} n!(n + \nu)}. \]  (19)

or more explicitly
\[ \tilde{C}_n^{(1/4)} = \sqrt{\frac{\mu \alpha}{\hbar}} \frac{1}{2^n \pi^{1/4} n!(n + 1/4)}, \]
\[ \tilde{C}_n^{(3/4)} = \sqrt{\frac{\mu \alpha}{\hbar}} \frac{1}{2^{n-1/2} \pi^{1/4} n!(n + 3/4)}. \]

From the duplication formula for a gamma-function
\[ \Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2) \]

and taking into account that \( \Gamma(1/2) = \pi^{1/2}, \Gamma(3/2) = \frac{1}{2} \pi^{1/2} \), we conclude that \( \tilde{C}_n^{(\nu)} = C_n^{(\nu)} \) and, consequently, Eqs. (13) and (14) are identical.

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V. CONCLUSIONS

a) The 1D oscillator has only a discrete energy spectrum and, therefore, is a model provided by the property which is known in QCD as confinement. A particle situated in the confinement potential cannot be removed from the center and transferred to infinity. On the other hand, the 1D Coulomb anyon is a system possessing both the discrete and continuous part in the energy spectrum. At the same time, it includes $1/x^2$ interaction and, therefore, pretends to be a magnetic monopole in one spatial dimension. All these ideas confirm that our result can be interpreted in the spirit of the Seiberg–Witten duality$^{11}$: The theories with strong coupling (i.e., including confinement) are equivalent to the theories with weak coupling (i.e., without confinement) accompanied by magnetic monopoles. We conclude that the Seiberg–Witten duality has its prototype in 1D quantum mechanics.

b) The anyon–oscillator duality is a simple example of a more complicated dyon–oscillator duality$^{12-23}$. The latter connects the isotropic oscillator with charge–dyon bound system (dyon is a hypothetical object which has both the electric and magnetic charge$^{24}$). The passage from an oscillator to a charge–dyon system is realized by non-bijective bilinear transformations$^{25}$ (for the mapping of the 1D Coulomb system into the oscillator refer to$^{26}$).

c) The wave function (13) of 1D Coulomb anyon can formally be extended to the region $-\infty < y < 0$. Such a continuation is an arbitrary-rule operation and we choose the following one. First, still being in the region $0 < y < \infty$, we change $y$ in the exponent and confluent hypergeometric function by $|y|$ and remain unchanged the factor $y^\nu$. Then, we extend the expression to the region $-\infty < y < 0$. These steps allow us to get rid of divergence in the exponent for large negative values of $y$ and conserve the normalization condition in $-\infty < y < \infty$ by multiplying the function $\Phi_n^{(\nu)}$ by the factor $1/\sqrt{2}$. The obtained wave function $\Phi_n^{(\nu)}(y)$ satisfies Eq. (13) in the region $-\infty < y < \infty$ and has the parity $(-1)^\nu$, i.e. describes the 1D anyon$^4$.

d) Eq. (13) for $-\infty < x < \infty$ and $\nu = 0$ corresponds to the so-called 1D hydrogen atom$^{27}$ (for later references see$^{28}$) which has some mysterious properties. For example, the ground state corresponds to an infinite negative value of the energy and the exited levels are double degenerated. The reason is that the potential $(-1/|x|)$ is singular in 1D space and the system is provided by hidden symmetry$^{29-31}$ and supersymmetry$^{32,33}$. As it is follows from (13) and (12), the Calogero–Sutherland potential transforms the 1D hydrogen atom into two modified atoms with the statistical parameter $\nu = 1/4$ and $\nu = 3/4$. This transformation leads to the formation of the ground states with a finite energy level and remove the problem of degeneracy (replacement $n \rightarrow n + \nu$).

e) It is easily to be convinced that Eq. (4) is identical to the Schrödinger equation with the Hamiltonian

$$\hat{H} = \frac{1}{2\mu} \left( -i\hbar \frac{\partial}{\partial x} - \frac{e}{c} A \right)^2 - \frac{\alpha}{x} - \frac{\hbar^2 \nu(1 - \nu)}{2\mu x^2}$$

(20)

where $\alpha = e^2$, $A = g/x$, $g = iv\hbar c/e$. So, we deal with a charged particle moving in the field created by the 1D Coulomb dyon of the electric charge $e$ and purely imaginary magnetic
charge $g$. The Calogero–Sutherland potential gains the meaning of the Goldhaber term typical of theory of magnetic monopoles\cite{34,35}.

Note that the Hamiltonian in (20) is not Hermitian, but it could be transformed into the Hermitian one if we do the following: 1) consider instead of the semiaxis $x \in (0, \infty)$ the axis $x \in (-\infty, \infty)$; 2) substitute $\alpha/x$ by $\alpha/|x|$; 3) introduce the Yang–Dunkl operator\cite{36}

$$\hat{D} = -i\hbar \partial_x - eA\hat{R}/c$$

for the Calogero model, where $\hat{R}$ is the reflection operator.

Exactly the same “covariant derivative” appears in the Calogero-like models with exchange interaction\cite{37,38}.

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