A quantitative second order estimate for (weighted) \( p \)-harmonic functions in manifolds under curvature-dimension condition

Jiayin Liu, Shijin Zhang and Yuan Zhou

Abstract. We build up a quantitative second-order Sobolev estimate of \( \ln w \) for positive \( p \)-harmonic functions \( w \) in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted \( p \)-harmonic functions \( w \) in weighted manifolds under the Bakry-Émery curvature-dimension condition.

1 Introduction

Let \((M^n, g)\) be a complete non-compact Riemannian manifold with dimension \( n \geq 2 \). Suppose that the Ricci curvature is bounded from below, that is, \( \text{Ric}_g \geq -\kappa \) for some \( \kappa \geq 0 \). For any positive harmonic function \( w \) in a domain \( \Omega \subset M^n \), Cheng-Yau [2] established the following famous gradient estimate:

\[
|\nabla \ln w| = \frac{|\nabla w|}{w} \leq C(n) \frac{1 + \sqrt{\kappa r}}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega. \tag{1.1}
\]

Recall that a harmonic function \( w \) in \( \Omega \) is a weak solution to the Laplace equation

\[
\Delta w := \text{div}(\nabla w) = 0 \quad \text{in } \Omega.
\]

We also refer to [17, Theorem 1.3] for a quantitative \( W^{2,2}_{\text{loc}} \)-regularity of harmonic functions.

Motivated by the application in the inverse mean curvature flow (see [11, 15]), Cheng-Yau type gradient estimate was extended by [16, 11, 21, 15] to \( p \)-harmonic functions in \( \Omega \) for \( 1 < p < \infty \), that is, weak solutions to the \( p \)-Laplace equation

\[
\Delta_p w = \text{div}(|\nabla w|^{p-2} \nabla w) = 0 \quad \text{in } \Omega.
\]

Precisely, if \((M^n, g)\) is flat (that is, the Euclidean space \( \mathbb{R}^n \)) or its sectional curvature is bounded from below by \(-\kappa\), via Cheng-Yau’s approach Moser [16] and Kotschwar-Ni [11] showed that any positive \( p \)-harmonic function \( w \) in \( \Omega \) satisfies

\[
|\nabla \ln w| \leq C(n) \frac{1 + \sqrt{\kappa r}}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega, \tag{1.2}
\]

2020 Mathematics Subject Classification: 58J05 · 35J92

The first author is funded by the Academy of Finland (Grant No. 321896 and Grant No. 328846). The second author is funded by NSFC (Grant No. 12171018). The third author is funded by NSFC (Grant No. 12025102).

Data availability: No data was used for the research described in the article.

1
where the constant $C(n) > 0$ is independent of $p \in (1, \infty)$. Under the Ricci curvature lower bound $\text{Ric}_g \geq -\kappa$, it was asked in [11] whether (1.2) holds or not. Some progress was made as below. Based on Cheng-Yau’s argument, Wang-Zhang [21] proved that

$$|\nabla \ln w|^{\frac{p-\gamma}{p}} \in W^{1,2}_{\text{loc}}$$

with $\gamma < 0$ and also for positive weighted $p$ functions $w$ in a domain $\Omega \subset M^n$ is a weak solution to the weighted $p$-harmonic equation

$$\Delta_{p,h} w := e^h \text{div}(e^{-h}|\nabla w|^{p-2} \nabla w) = 0 \text{ in } \Omega.$$ 

Under the Bakry-Émery curvature-dimension condition $\text{Ric}^N_h \geq -\kappa$ for some $N \in [n, \infty)$ and $\kappa \geq 0$ (see Section 2 for details), Dung-Dat [5] showed that if $w > 0$, then $|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W^{1,2}_{\text{loc}}$ with $\gamma < 0$ and also

$$|\nabla \ln w| \leq C(n, N, p) \frac{1 + \sqrt{KR}}{r} \text{ in } B(z, r) \subset B(z, 2r) \subset \Omega.$$ 

The main aim of this paper is to build up a quantitative second-order Sobolev estimate of $\ln w$ for positive $p$-harmonic functions $w$ in Riemannian manifolds under the Ricci curvature bounded from below and also for positive weighted $p$-harmonic functions $w$ in weighted manifolds under the Bakry-Émery curvature-dimension condition. See Theorem 1.1 and Theorem 1.2 separately. These improve the corresponding second-order Sobolev regularity in [21] mentioned above.

To be precise, under the Ricci curvature lower bound, we have the following result. For convenience, below we write $\int_{E} f \, dm$ as the average of $f$ in the set $E$ with respect to the measure $m$, that is, $\int_{E} f \, dm = \frac{1}{m(E)} \int_{E} f \, dm$. We use $C(a_1, \ldots, a_m)$ to denote a positive constant depending on absolute constants $a_1, \ldots, a_m$.

**Theorem 1.1.** Suppose that $(M^n, g)$ satisfies $\text{Ric}_g \geq -\kappa$ for some $\kappa \geq 0$. Let $1 < p < \infty$ and $\gamma < 3 + \frac{n-1}{n-2}$. For any positive $p$-harmonic function $w$ in a domain $\Omega \subset M$, we have

$$|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W^{1,2}_{\text{loc}}(\Omega)$$

and

$$\int_{B(z,r)} \left| \nabla (|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w) \right|^2 \, dv_{g} \leq C(n, p, \gamma) \left[ \frac{1 + \sqrt{KR}}{r} \right]^{p-\gamma+4} e^{\sqrt{KR}}$$

whenever $B(z, 4r) \subset \Omega$.

In particular, if $1 < p < 3 + \frac{2}{n-2}$, then $\nabla^2 \ln w \in L^2_{\text{loc}}(\Omega)$ and

$$\int_{B(z,r)} |\nabla^2 \ln w|^2 \, dv_{g} \leq C(n, p) \left[ \frac{1 + \sqrt{KR}}{r} \right]^4 e^{\sqrt{KR}}$$

whenever $B(z, 4r) \subset \Omega$. 

2
Here and throughout the paper for domains $A$ and $B$, the notation $A \Subset B$ stands for that $A$ is a bounded subdomain of $B$ and its closure $A \subseteq B$.

Recall that if $(M^n, g)$ is flat, that is, the Euclidean space $\mathbb{R}^n$, $p$-harmonic functions $w$ in a domain $\Omega \subset \mathbb{R}^n$ are proved to satisfy $|\nabla w|^{2p} \nabla w \in W^{1,2}_{\text{loc}}(\Omega)$ with some quantitative bound whenever $\gamma < 3 + \frac{p-1}{n-1}$ see [13, 9, 4, 14] and also the references therein for some earlier partial results. In particular, if $1 < p < 3 + \frac{2}{n-2}$, noting $p < 3 + \frac{p-1}{n-1}$ and taking $\gamma = p$, one has $w \in W^{2,2}_{\text{loc}}(\Omega)$. When $n \geq 3$ and $p \geq 3 + \frac{2}{n-2}$, it is not clear whether $w \in W^{2,2}_{\text{loc}}(\Omega)$ or not. When $n = 2$, the range $\gamma < 3 + \frac{p-1}{n-1} = p + 2$ is optimal as witnessed by some construction in [9].

Moreover, we extend Theorem 1.1 to weighted manifolds satisfying Bakry-Émery curvature-dimension condition,

**Theorem 1.2.** Let $(M^n, g, e^{-h} \text{vol}_g)$ be a weighted manifold with $\text{Ric}_h^N \geq -\kappa$ for some $n \leq N < \infty$ and $\kappa \geq 0$. Let $1 < p < \infty$ and $\gamma < 3 + \frac{p-1}{n-1}$. For any positive weighted $p$-harmonic function $w$ in a domain $\Omega \subset M$, we have $|\nabla \ln w|^{\frac{p-2}{2}} \nabla \ln w \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$
\int_{B(z,r)} |\nabla \ln w|^{\frac{p-2}{2}} \nabla \ln w|^2 \, d\text{vol}_h \leq C(n, N, p, \gamma) \left[ \frac{1 + \sqrt{\kappa r}}{r} \right]^{p-\gamma + 4} e^{\sqrt{\kappa r}} \quad (1.8)
$$

whenever $B(z, 4r) \Subset \Omega$.

In particular, if $p \in (1, 3 + \frac{2}{n-2})$, then $\nabla^2 \ln w \in L^2_{\text{loc}}(\Omega)$ and

$$
\int_{B(z,r)} |\nabla^2 \ln w|^2 \, d\text{vol}_h \leq C(n, N, p) \left[ \frac{1 + \sqrt{\kappa r}}{r} \right]^4 e^{\sqrt{\kappa r}} \quad (1.9)
$$

whenever $B(z, 4r) \Subset \Omega$.

As a consequence of Theorem 1.1 and Theorem 1.2 one gets that $|\nabla \ln w|^{\frac{p-2}{2}} \nabla \ln w \in W^{1,2}_{\text{loc}}(\Omega)$ for $\gamma < 3 + \frac{p-1}{n-1}$ or $\gamma < 3 + \frac{p-1}{n-1}$, while in [21, 5], one has $|\nabla \ln w|^{\frac{p-2}{2}} \nabla \ln w \in W^{1,2}_{\text{loc}}(\Omega)$ for all $\gamma < 2$ (see (1.3) and the line above (1.5)). Thus our range for $\gamma$ obviously improves the one obtained in [21, 5] respectively.

Now we sketch the ideas to prove Theorem 1.1 and Theorem 1.2. Note that when $N = n$ and $h \equiv 1$, we have $\text{Ric}_h^N = \text{Ric}_g$, and hence Theorem 1.1 corresponds to the special case $N = n$ and $h \equiv 1$ in Theorem 1.2. We only need to prove Theorem 1.2. As usual, we approximate $u = -(p - 1) \ln w$ by smooth solution $u^\epsilon$ to the standard approximation/regularized equation (3.3), that is,

$$
e^h \text{div}(e^{-h}[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} \nabla u^\epsilon) = |\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla u^\epsilon|^2.
$$

(i) Using Bochner formula and the approximation equation (3.3), for $0 < \eta < 1/2$ we bound the integral of

$$(1 - \eta)|\nabla^2 u^\epsilon|^2 + (p - \gamma) \frac{|\nabla^2 u^\epsilon \cdot \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p - 2)\frac{(\Delta \nabla u^\epsilon)^2}{|\nabla u^\epsilon|^2 + \epsilon} (1.10)$$

from above by the integral of

$$\text{Ric}_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle$$

3
and other first order terms, where all integrals are taken against $\|\nabla u^\varepsilon\|^2 + \varepsilon \frac{m-2}{n} \phi^2 e^{-h} \mathrm{d} \operatorname{vol}_g$, where $\phi \in C^\infty_c(U)$ is a test function and $U \subseteq \Omega$; see Lemma 3.2. Here in (1.10) and in what follows, for any $C^2$ function $f$, $\Delta_\infty f := (\nabla^2 f, \nabla f)$.

(ii) If $\gamma < 3 + \frac{1}{n}$, via a fundamental inequality given in Lemma 2.4 and the approximation equation (5.3), for sufficiently small $\eta > 0$ we bound (1.10) as below

$$
\eta \|\nabla u^\varepsilon\|^2 - \frac{\langle \nabla h, \nabla u^\varepsilon \rangle^2}{N-n} - C \frac{1}{\eta} |\nabla u^\varepsilon|^4 \quad \text{everywhere;}
$$

see Lemma 3.4. This is crucial to get Theorem 1.2. Note that the approach in [21, 5] could not give Lemma 3.4; see Remark 3.8 for details.

(iii) Combining (i)&(ii) together, the integral of $\eta \|\nabla u^\varepsilon\|^2$ is bounded from above by the integral of $-\text{Ric}^N_h(\nabla u^\varepsilon, \nabla u^\varepsilon)$ and other first order terms, where all integrals are taken against $\|\nabla u^\varepsilon\|^2 + \varepsilon \frac{m-2}{n} \phi^2 e^{-h} \mathrm{d} \operatorname{vol}_g$; see Corollary 3.6.

Under the assumption $\text{Ric}^N_h \geq -\kappa$, in Lemma 3.7 we obtain an upper $L^2_{\text{loc}}$ bound for $\nabla \langle \nabla u^\varepsilon, \nabla u^\varepsilon \rangle \phi$ by the integral of some first order terms, where all integrals are against $e^{-h} \mathrm{d} \operatorname{vol}_g$. A standard argument then leads to the proof of Theorem 1.2.

Finally, we also notice that the Cheng-Yau gradient estimate (1.1) was generalized to positive harmonic functions $w$ in Alexandrov spaces with curvature bounded from below by Zhang-Zhu in [22], where the authors showed $|\nabla \ln w|^2 \in W^{1,2}_{\text{loc}}(\Omega)$ as a key step. Furthermore, one could study the regularity of $p$-harmonic functions in more general metric measure spaces. In these spaces, a natural generalization of the (weighted) Ricci curvature bound is the curvature-dimension condition $\text{RCD}(\kappa, N)$ in the sense of Bakry-Émery or Ambrosio-Gigli-Savaré. The two senses turned out to be equivalent by the work of Erbar-Kuwada-Sturm [6] (in the finite dimensional case) and Ambrosio-Gigli-Savaré [1] and the spaces satisfying one of the two equivalent conditions are known as $\text{RCD}(\kappa, N)$ spaces. Some progress was made in $\text{RCD}(\kappa, N)$ spaces. The Cheng-Yau gradient estimate was established by Jiang in [10] for positive harmonic functions $w$ in $\text{RCD}(\kappa, N)$ spaces; recently, Gigli-Violo in [7] established $|\nabla \ln w|^{\beta/2} \in W^{1,2}_{\text{loc}}(\Omega)$ under $\text{RCD}(0, N)$ spaces if $\beta > \frac{N-2}{N-1}$. However, when $p \neq 2$, it remains open to prove the Cheng-Yau type gradient estimates for positive $p$-harmonic functions in Alexandrov spaces and also $\text{RCD}(\kappa, N)$ spaces.

2 Preliminaries

Let $n \geq 2$ and $M^n$ be a Riemannian manifold, and $g$ be the Riemannian metric. By abuse of notation we also write $|\xi|^2 = g(\xi, \xi)$ and $\langle \xi, \eta \rangle = g(\xi, \eta)$ for all $\xi, \eta \in T_x M^n$. The corresponding Riemannian volume measure is written as $d\operatorname{vol}_g$, and the volume of a set $E$ is written as $\operatorname{vol}_g(E)$.

Denote by $\text{Ric}_g$ the Ricci curvature 2-tensor and write $\text{Ric}_g \geq -\kappa$ if $\text{Ric}_g(\xi, \xi) \geq -\kappa |\xi|^2$ for all $\xi \in T_x M^n$.

For $1 < p < \infty$, the $p$-Laplace operator $\Delta_p$ in $M^n$ is given by

$$
\Delta_p f = \text{div}(|\nabla f|^{p-2} \nabla f) \quad \forall f \in C^2(M^n).
$$
Obviously, $\Delta_2$ is exactly the Laplace-Beltrami operator $\Delta$ in $(M^n, g)$. A function $w$ defined in a domain $\Omega \subset M^n$ is called $p$-harmonic if $w \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to the $p$-Laplace equation $\Delta_p w = 0$ in $\Omega$, that is,

$$
\int_{\Omega} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle dvol_g = 0 \quad \forall \phi \in C^\infty_c(\Omega).
$$

Note that 2-harmonic functions are the well-known harmonic functions.

Next we recall some basic facts of weighted Riemannian manifolds $(M^n, g, e^{-h}dvol_g)$, where the weight $h$ is a positive smooth function in $M^n$. The weighted measure $dvol_h = e^{-h}dvol_g$ can be viewed as the volume form of a suitable conformal change of the metric $g$. Denote by $\text{vol}_h(E)$ the weighted volume of a set $E$. For $n \leq N < \infty$, the corresponding $N$-Bakry-Émery curvature tensor is

$$
\text{Ric}_h^N = \text{Ric}_g + \nabla^2 h - \frac{\nabla h \otimes \nabla h}{N - n},
$$

where when $N = n$, by convention, $h$ is a constant function and hence $\text{Ric}_h^N = \text{Ric}_g$. We say that $(M^n, g, e^{-h}dvol_g)$ satisfies the Bakry-Émery curvature-dimension condition $\text{Ric}_h^N \geq -\kappa$ if

$$
\text{Ric}_h^N(\xi, \xi) = \text{Ric}_g(\xi, \xi) + \langle \nabla^2 h \xi, \xi \rangle - \frac{\langle \nabla h, \xi \rangle^2}{N - n} \geq -\kappa \langle \xi, \xi \rangle \forall \xi \in T_x M^n
$$

By [13], under $\text{Ric}_h^N \geq -\kappa$, one has the following volume comparison result

$$
\text{vol}_h(B_{2r}(x)) \leq C(N)e^{\sqrt{-\kappa}r}\text{vol}_h(B_r(x)) \quad \forall x \in M, \ r > 0. \quad (2.1)
$$

For $1 < p < \infty$, the weighted $p$-Laplacian $\Delta_{p,h}$ is defined as

$$
\Delta_{p,h} f = e^h \text{div}(e^{-h}|\nabla f|^{p-2} \nabla f) = \Delta_p f - |\nabla f|^{p-2} \langle \nabla f, \nabla h \rangle \quad \forall f \in C^2(M^n).
$$

In the case $p = 2$, one writes $\Delta_{2,h}$ as $\Delta_h$, and hence

$$
\Delta_h f = \Delta f - \langle \nabla h, \nabla f \rangle.
$$

A function $w$ in a domain $\Omega \subset M^n$ is called a weighted $p$-harmonic function if $w \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to the weighted $p$-harmonic equation $\Delta_{p,h} w = 0$ in $\Omega$, that is,

$$
\int_{\Omega} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle e^{-h}dvol_g = 0 \quad \forall \phi \in C^\infty_c(\Omega). \quad (2.2)
$$

By a density argument, we can relax $\phi \in C^\infty_c(\Omega)$ to $\phi \in W^{1,p}_0(\Omega)$ in (2.2).

We also recall the following Bochner formula in $(M^n, g, e^{-h}dvol_g)$:

$$
\frac{1}{2} \Delta_h |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta_h f \rangle + \text{Ric}_g(\nabla f, \nabla f) + \langle \nabla^2 h \nabla f, \nabla f \rangle \quad \forall f \in C^3(M), \quad (2.3)
$$

which will be used in Section 3.

Finally, we recall the following fundamental inequality; see for example [21, 5, 14]. For the reader’s convenience we include it here. Recall that $\Delta_\infty f = \langle \nabla^2 f \nabla f, \nabla f \rangle$. 

5
Lemma 2.1. Let \( n \geq 2 \) and \( \Omega \) be a domain of \( M^n \). For any \( f \in C^2(\Omega) \), we have
\[
|\nabla f|^4|\nabla^2 f|^2 \geq 2|\nabla f|^2|\nabla^2 f|\nabla f|^2 + \frac{|\nabla f|^2\Delta f - \Delta_{\infty} f|^2}{n - 1} - (\Delta_{\infty} f)^2 \quad \text{in } \Omega, \tag{2.4}
\]
where when \( n = 2 \), \( \geq \) becomes \( = \).

Proof. It suffices to prove that for any symmetric \( n \times n \) matrix \( A \) one has
\[
|A|^2|\xi|^4 \geq \frac{1}{n - 1}(\text{tr} A|\xi|^2 - \langle A \xi, \xi \rangle^2 + 2|A \xi|^2|\xi|^2 - \langle A \xi, \xi \rangle^2) \quad \forall \xi \in \mathbb{R}^n. \tag{2.5}
\]
Note that if \( \xi = 0 \), (2.5) holds obviously. Below assume that \( \xi \neq 0 \). Up to a scaling we may assume \( |\xi| = 1 \). By a change of coordinates, we may further assume \( \xi = e_n = (0, \cdots, 0, 1) \); in this case, (2.5) reads as
\[
|A|^2 \geq \frac{1}{n - 1}(\text{tr} A - \langle A e_n, e_n \rangle)^2 + 2|A e_n|^2 - \langle A e_n, e_n \rangle^2.
\]
Denoting by \( A_{n-1} \) the \((n - 1)\) order principal submatrix of \( A \), one has
\[
|A|^2 = |A_{n-1}|^2 + 2|A e_n|^2 - \langle A e_n, e_n \rangle^2.
\]
Noting that
\[
|A_{n-1}|^2 \geq \frac{1}{n - 1}(\text{tr} A_{n-1})^2 = \frac{1}{n - 1}(\text{tr} A - \langle A e_n, e_n \rangle)^2,
\]
where when \( n = 2 \), one has \( |A_{n-1}|^2 = (\text{tr} A_{n-1})^2 \), one concludes (2.4). \( \square \)

3 Proof of Theorem 1.2

Let \( w \) be a positive weighted \( p \)-harmonic function in a domain \( \Omega \). Set \( u = -(p - 1) \ln w \). Then \( u \) is a weak solution to the equation
\[
\Delta_p u - |\nabla u|^{p-2} \langle \nabla u, \nabla h \rangle = |\nabla u|^p \quad \text{in } \Omega, \tag{3.1}
\]
that is,
\[- \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle e^{-h} d\text{vol}_g = \int_{\Omega} |\nabla u|^p \phi e^{-h} d\text{vol}_g \quad \forall \phi \in C^\infty_c(\Omega).\]

Given any smooth domain \( U \subset \Omega \) and \( \epsilon \in (0, 1] \), consider the approximation/regularized equation defined by
\[
e^h \text{div}(e^{-h}[|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} \nabla v) = [|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla v|^2 \quad \text{in } U; \ v = u \text{ on } \partial U. \tag{3.2}
\]
It is well known that if \( u \) is the solution to (3.1), then \( u \in C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \); see [3, 12, 19, 20]. Moreover, in the following lemma, we summarize some properties of the solution \( u \) to (3.1) and \( u^\epsilon \) to (3.3), which result from [3] as a special case. See also [19].
Lemma 3.1. For any \( \epsilon \in (0, 1] \), there exists a unique solution \( u^\epsilon \in C^\infty(U) \cap C^0(\overline{U}) \) to (3.3), and moreover, \( u^\epsilon \to u \in C^0(\overline{U}) \) and \( u^\epsilon \to u \in C^{1,\alpha}(V) \) uniformly in \( \epsilon > 0 \) as \( \epsilon \to 0 \) for all \( V \Subset U \) where \( u \) is the solution to (3.1).

To show Lemma 3.1 we just need to check that equations (3.1) and (3.3) are special cases of those considered in [3]. We put this verification in the appendix.

By Lemma 3.1, the solution \( u^\epsilon \) to (3.2) is \( C^\infty \), which implies that \( u^\epsilon \) satisfies (3.2) pointwise. Hence by a direct computation, (3.5) is equivalent to

\[
\int_U \left\{ (1 - \eta) \Delta_h u^\epsilon + (2 - \gamma) \frac{\Delta^2 u^\epsilon}{\nabla u^\epsilon^2} + (p - 2)(2 - \gamma) \frac{(\Delta^2 u^\epsilon)^2}{\nabla u^\epsilon^2 + \epsilon} \right\} \left[ \frac{\Delta^2 u^\epsilon}{\nabla u^\epsilon^2 + \epsilon} \right] d\sigma_g \\
\leq - \int_U \left[ Ric_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle \right] \frac{\Delta^2 u^\epsilon}{\nabla u^\epsilon^2 + \epsilon} d\sigma_g \\
+ C(p, \gamma) \frac{1}{\eta} \int_U \left[ \frac{\Delta^2 u^\epsilon}{\nabla u^\epsilon^2 + \epsilon} \right] d\sigma_g .
\] (3.4)

To prove this, we need the following identity.

Lemma 3.3. For any \( v \in C^3(U) \) and \( \psi \in C^\infty_c(U) \), one has

\[
\int_U |\nabla^2 v|^2 \psi e^{-h} d\sigma_g = - \int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\sigma_g + \int_U (\Delta_h v)^2 \psi e^{-h} d\sigma_g \\
- \int_U [Ric_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v \rangle] \psi e^{-h} d\sigma_g .
\] (3.5)

Proof. Applying the Bochner formula to \( v \), one has

\[
|\nabla^2 v|^2 + Ric_g(\nabla v, \nabla v) = \frac{1}{2} \Delta_h |\nabla v|^2 - \langle \nabla v, \nabla \Delta_h v \rangle - \langle \nabla^2 h \nabla v, \nabla v \rangle
\]

and hence

\[
|\nabla^2 v|^2 = \left[ \frac{1}{2} \Delta_h |\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla \Delta_h v \rangle \right] + (\Delta_h v)^2 \\
- [Ric_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v \rangle].
\]

By this, to get (3.5), it suffices to show the following identity

\[
\int_U \frac{1}{2} \Delta_h |\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla \Delta_h v \rangle \psi e^{-h} d\sigma_g
\]
\begin{equation}
\int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g.
\end{equation}

(3.6)

Note that

\begin{equation}
-[(\Delta_h v)^2 + \langle \nabla v, \nabla (\Delta_h v) \rangle] = -e^h \text{div}(e^{-h} \nabla v)(\Delta_h v) - e^h \langle e^{-h} \nabla v, \nabla (\Delta_h v) \rangle = -e^h \text{div}(e^{-h} \nabla v \Delta_h v).
\end{equation}

Via integration by parts, one has

\begin{equation}
-\int_U [(\Delta_h v)^2 + \langle \nabla v, \nabla (\Delta_h v) \rangle] \psi e^{-h} d\text{vol}_g = -\int_U \text{div}(e^{-h} \nabla v \Delta_h v) \psi d\text{vol}_g
= \int_U \langle \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g.
\end{equation}

Similarly, via integration by parts one also has

\begin{equation}
\frac{1}{2} \int_U \Delta_h |\nabla v|^2 \psi e^{-h} d\text{vol}_g = \int_U \frac{1}{2} \text{div}(e^{-h} \nabla |\nabla v|^2) \psi d\text{vol}_g
= -\int_U \frac{1}{2} \langle e^{-h} |\nabla v|^2, \nabla \psi \rangle d\text{vol}_g
= -\int_U \langle \nabla^2 v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g.
\end{equation}

Combining together we obtain (3.6) and hence, (3.5) as desired. \qed

We are ready prove Lemma 3.2 as below.

**Proof of Lemma 3.2** Taking \( v = u^\epsilon \) and \( \psi = |\nabla u^\epsilon|^2 + \epsilon \frac{p-1}{2} \phi^2 \) in (3.5) we get

\begin{equation}
\int_U |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^2 + \epsilon \frac{p-1}{2} \phi^2 e^{-h} d\text{vol}_g
= -\int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla [|\nabla u^\epsilon|^2 + \epsilon \frac{p-1}{2} \phi^2] \rangle e^{-h} d\text{vol}_g
+ \int_U (\Delta_h u^\epsilon)^2 |\nabla u^\epsilon|^2 + \epsilon \frac{p-1}{2} \phi^2 e^{-h} d\text{vol}_g
- \int_U [Ric(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle] |\nabla u^\epsilon|^2 + \epsilon \frac{p-1}{2} \phi^2 e^{-h} d\text{vol}_g.
\end{equation}

(3.7)

To bound the second term in the right-hand side in (3.7), recalling (3.3), that is,

\begin{equation}
\Delta_h u^\epsilon = |\nabla u^\epsilon|^2 - (p-2) \frac{\Delta_{\infty} u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon},
\end{equation}

(3.8)

by Cauchy-Schwarz’s inequality one has

\begin{equation}
(\Delta_h u^\epsilon)^2 \leq (p-2)^2 \frac{(\Delta_{\infty} u^\epsilon)^2}{|\nabla u^\epsilon|^2 + \epsilon} + \frac{\eta}{4} |\nabla^2 u^\epsilon|^2 + C(p) \frac{1}{\eta} |\nabla u^\epsilon|^4,
\end{equation}

8
where $0 < \eta < 1$ is any constant. Thus

$$
\int_U (\Delta_h u^\epsilon)^2 ||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2 e^{-h} \text{dvol}_g \leq (p - 2)^2 \int_U (\Delta_u u^\epsilon)^2 ||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2 e^{-h} \text{dvol}_g
$$
$$+ \frac{\eta}{4} \int_U ||\nabla^2 u^\epsilon||^2 ||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2 e^{-h} \text{dvol}_g
$$
$$+ \frac{C(p)}{\eta} \int_U ||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} + 2 \phi^2 e^{-h} \text{dvol}_g. \tag{3.9}
$$

The first term in the right-hand side in (3.7) is further written as

$$
- \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] \rangle e^{-h} \text{dvol}_g
$$
$$= -(p - \gamma) \int_U \frac{||\nabla^2 u^\epsilon \nabla u^\epsilon||^2}{||\nabla u^\epsilon||^2 + \epsilon} [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$+ (p - \gamma) \int_U \Delta_h u^\epsilon \frac{\Delta_u u^\epsilon}{||\nabla u^\epsilon||^2 + \epsilon} [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$- \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$+ \int_U \langle \Delta_h u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g. \tag{3.10}
$$

Using (3.8) and Cauchy-Schwarz’s inequality, we obtain the following upper bound for the second term in (3.10):

$$(p - \gamma) \int_U \Delta_h u^\epsilon \frac{\Delta_u u^\epsilon}{||\nabla u^\epsilon||^2 + \epsilon} [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$= -(p - \gamma)(p - 2) \int_U (\Delta_u u^\epsilon)^2 [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} - 2 \phi^2] e^{-h} \text{dvol}_g
$$
$$+ (p - \gamma) \int_U \Delta_u u^\epsilon ||\nabla u^\epsilon||^2 [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$\leq -(p - \gamma)(p - 2) \int_U (\Delta_u u^\epsilon)^2 [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} - 2 \phi^2] e^{-h} \text{dvol}_g
$$
$$+ \frac{\eta}{4} \int_U ||\nabla^2 u^\epsilon||^2 [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g
$$
$$+ \frac{C(p)}{\eta} ||p - \gamma||^2 \int_U [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} + 2 \phi^2] e^{-h} \text{dvol}_g. \tag{3.11}
$$

For the third term in the right-hand side of (3.10), by Cauchy-Schwarz’s inequality, one has

$$\left| \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g \right|
$$
$$\leq \frac{\eta}{4} \int_U ||\nabla^2 u^\epsilon||^2 [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} \phi^2] e^{-h} \text{dvol}_g + \frac{C}{\eta} \int_U [||\nabla u^\epsilon||^2 + \epsilon \frac{p-\gamma}{2} + 1] ||\nabla \phi||^2 e^{-h} \text{dvol}_g. \tag{3.12}
$$
For the fourth term in the right-hand side of \((3.10)\), in a similar way, using \((3.8)\), one has
\[
\left| \int_U \left< \Delta h u^\epsilon, \nabla \phi^2 \right> \right| \leq \eta \int_U \left| \nabla^2 u^\epsilon \right|^2 e^{-h} dvol_g

\]
From \((3.14)\), \((3.9)\) and \((3.7)\) we conclude \((3.4)\).

If \(\gamma < 3 + \frac{p-1}{N-n}\), we get the following pointwise lower bound. Recall that when \(N = n\), we always assume that \(h\) is a constant function and \(\frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} = 0\).

**Lemma 3.4.** Let \(u^\epsilon\) be the solution to \((3.3)\). If \(\gamma < 3 + \frac{p-1}{N-n}\) for some \(N \geq n\), then for sufficiently small \(\eta > 0\) we have
\[
1 - \eta \left| \nabla^2 u^\epsilon \right|^2 + (p - \gamma) \frac{\left| \nabla^2 u^\epsilon \nabla u^\epsilon \right|^2}{\left| \nabla u^\epsilon \right|^2 + \epsilon} + (p - 2)(2 - \gamma) \frac{(\Delta u^\epsilon)^2}{\left| \nabla u^\epsilon \right|^2 + \epsilon} \\
\geq \eta \left| \nabla^2 u^\epsilon \right|^2 - \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} - C(n, N, p, \gamma) \frac{1}{\eta} \left| \nabla u^\epsilon \right|^4. \tag{3.15}
\]

To prove this, we need the following pointwise lower bound for \(|\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4\).

**Lemma 3.5.** Let \(u^\epsilon\) be the solution to \((3.3)\). If \(N \geq n\), then for \(0 < \eta < 1\) we have
\[
1 + \eta \left| \nabla^2 u^\epsilon \right|^2 |\nabla u^\epsilon|^4 \geq 2 |\nabla^2 u^\epsilon \nabla u^\epsilon|^2 |\nabla u^\epsilon|^2 + \left( \frac{1}{N-n} - 1 \right) \left( (p - 2) \frac{\left| \nabla u^\epsilon \right|^2}{\left| \nabla u^\epsilon \right|^2 + \epsilon} + 1 \right)^2 \frac{(\Delta u^\epsilon)^2}{\left| \nabla u^\epsilon \right|^2 + \epsilon} - \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta} |\nabla u^\epsilon|^8. \tag{3.16}
\]

\[\]
Proof. Applying (2.4) to \( u^\varepsilon \) one has

\[
|\nabla^2 u^\varepsilon| |\nabla u^\varepsilon|^4 \geq 2|\nabla u^\varepsilon|^2 |\nabla^2 u^\varepsilon| + \frac{||\nabla u^\varepsilon|^2 \Delta u^\varepsilon - \Delta_{\infty} u^\varepsilon|^2}{n-1} - (\Delta_{\infty} u^\varepsilon)^2 \tag{3.17}
\]

By (3.8) and \( \Delta u^\varepsilon = \Delta_h u^\varepsilon + \langle \nabla h, \nabla u^\varepsilon \rangle \), we have

\[
\Delta u^\varepsilon = |\nabla u^\varepsilon|^2 + \langle \nabla u^\varepsilon, \nabla h \rangle - (p-2) \frac{\Delta_{\infty} u^\varepsilon}{|\nabla u^\varepsilon|^2 + \varepsilon}.
\]

Thus

\[
|\nabla u^\varepsilon|^2 \Delta u^\varepsilon - \Delta_{\infty} u^\varepsilon = |\nabla u^\varepsilon|^2 (|\nabla u^\varepsilon|^2 + \langle \nabla u^\varepsilon, \nabla h \rangle) - \left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right) \Delta_{\infty} u^\varepsilon,
\]

and hence,

\[
\begin{align*}
|\nabla u^\varepsilon|^2 \Delta u^\varepsilon - \Delta_{\infty} u^\varepsilon &= \left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right)^2 (\Delta_{\infty} u^\varepsilon)^2 \\
& + |\nabla u^\varepsilon|^4 (|\nabla u^\varepsilon|^2 + \langle \nabla u^\varepsilon, \nabla h \rangle)^2 \\
& - 2 \left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right) |\nabla u^\varepsilon|^4 \Delta_{\infty} u^\varepsilon \\
& - 2 \left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right) |\nabla u^\varepsilon|^2 \Delta_{\infty} u^\varepsilon \langle \nabla u^\varepsilon, \nabla h \rangle \\
& =: I_1 + I_2 + I_3 + I_4. \tag{3.18}
\end{align*}
\]

Note that

\[
\left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right)^2 \leq 4p^2,
\]

which can be obtained by considering \( p > 2 \) and \( 1 < p < 2 \) separately. Using this, Cauchy-Schwarz inequality, for \( 0 < \eta < 1 \), we have

\[
I_3 \geq -\eta |\nabla^2 u^\varepsilon|^2 |\nabla u^\varepsilon|^4 - C(p) \frac{1}{\eta} |\nabla u^\varepsilon|^8. \tag{3.20}
\]

If \( h \) is a constant function and hence \( \nabla h = 0 \), \( I_2 \geq 0 \) and \( I_4 = 0 \), dividing by \( n-1 \) in both sides of (3.18), by (3.20) one has

\[
\frac{|\nabla u^\varepsilon|^2 \Delta u^\varepsilon - \Delta_{\infty} u^\varepsilon|^2}{n-1} \geq -\eta |\nabla^2 u^\varepsilon|^2 |\nabla u^\varepsilon|^4 + \left( (p-2) \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \varepsilon} + 1 \right)^2 \frac{\Delta_{\infty} u^\varepsilon)^2}{n-1} - \frac{C(p)}{\eta} |\nabla u^\varepsilon|^8.
\]

Plugging this in (3.17), noting \( N = n \), and adding \( \eta |\nabla^2 u^\varepsilon|^2 |\nabla u^\varepsilon|^4 \) in both side, one concludes (3.16).

If \( h \) is not a constant function, set \( \eta_1 = \frac{N-n}{N-1} \). Then

\[
1 - \eta_1 = \frac{n-1}{N-1} > 0 \quad \text{and} \quad 1 - \frac{1}{\eta_1} = -\frac{n-1}{N-n} < 0. \tag{3.21}
\]
For any $0 < \eta < 1$ one has

$$I_2 \geq |\nabla u^\epsilon|^4 \langle \nabla u^\epsilon, \nabla h \rangle^2 + 2|\nabla u^\epsilon|^6 \langle \nabla u^\epsilon, \nabla h \rangle$$

$$\geq [1 + \eta(1 - \frac{1}{\hat{n}})]|\nabla u^\epsilon|^4 \langle \nabla u^\epsilon, \nabla h \rangle^2 - \frac{1}{\eta|1 - \frac{1}{\hat{n}}|}|\nabla u^\epsilon|^8. \tag{3.22}$$

Using Cauchy-Schwarz inequality, we have

$$I_4 \geq -\eta_1 \left[(p - 2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1\right]^2 (\Delta_{\infty} u^\epsilon)^2 - \frac{1}{\eta_1}(\nabla u^\epsilon, \nabla h)^2 |\nabla u^\epsilon|^4 \tag{3.23}$$

Dividing by $n - 1$ in both sides of (3.18), by (3.20), (3.22) and (3.23) one has

$$\frac{||\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_{\infty} u^\epsilon||^2}{n - 1} \geq -\eta|\nabla u^\epsilon|^2 |\nabla u^\epsilon|^4 + \frac{1 - \eta_1}{n - 1} \left[(p - 2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1\right]^2 (\Delta_{\infty} u^\epsilon)^2$$

$$+ (1 + \eta) \frac{1}{n - 1}(\nabla u^\epsilon, \nabla h)^2 |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta}|\nabla u^\epsilon|^8. \tag{3.21}$$

By (3.21),

$$\frac{||\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_{\infty} u^\epsilon||^2}{n - 1} \geq -\eta|\nabla u^\epsilon|^2 |\nabla u^\epsilon|^4 + \frac{1 - \eta}{n - 1} \left[(p - 2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1\right]^2 (\Delta_{\infty} u^\epsilon)^2$$

$$- (1 + \eta) \frac{1}{n - 1}(\nabla u^\epsilon, \nabla h)^2 |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta}|\nabla u^\epsilon|^8. \tag{3.23}$$

Plugging this in (3.17), and adding $\eta|\nabla u^\epsilon|^2 |\nabla u^\epsilon|^4$ in both side, we conclude (3.16) as desired.

We now prove Lemma 3.4 by using Lemma 3.5.

**Proof of Lemma 3.4.** Given any point $x \in U$, if $\nabla u^\epsilon(x) = 0$, then (3.15) holds trivially. Below we assume that $\nabla u^\epsilon(x) \neq 0$. At such point $x$, we already have (3.16) in Lemma 3.5. Dividing by $|\nabla u^\epsilon|^4$ in both sides of (3.16), for $0 < \eta < 1/2$ we obtain

$$(1 + \eta)|\nabla^2 u^\epsilon|^2 \geq 2 \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} + \left(\frac{1}{N - 1} \left[(p - 2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1\right]^2 - 1\right) \frac{(\Delta_{\infty} u^\epsilon)^2}{|\nabla u^\epsilon|^4}$$

$$- (1 + \eta) \frac{(\nabla u^\epsilon, \nabla h)^2}{N - n} - C(n, N, p) \frac{1}{\eta}|\nabla u^\epsilon|^4.$$

In both sides, multiplying by $\frac{1 - 2\eta}{1 + \eta} > 0$ and adding

$$\eta|\nabla u^\epsilon|^2 + (p - \gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p - 2)(2 - \gamma) \frac{(\Delta_{\infty} u^\epsilon)^2}{|\nabla u^\epsilon|^4},$$

we get

$$(1 - \eta)|\nabla^2 u^\epsilon|^2 + (p - \gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p - 2)(2 - \gamma) \frac{(\Delta_{\infty} u^\epsilon)^2}{|\nabla u^\epsilon|^4}.$$
≥ \eta|\nabla^2 u'|^2 + \left\{ \frac{1 - 2\eta}{1 + \eta} \left( \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 1 \right]^2 - 1 \right) + (p - 2)(2 - \gamma) \frac{|\nabla u'|^4}{|\nabla u'|^2 + \epsilon}^2 \right\} \frac{(\Delta_{\infty} u')^2}{|\nabla u'|^4}

+ \left\{ \frac{1 - 2\eta}{1 + \eta} \left( \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 1 \right]^2 - 1 \right) + (p - 2)(2 - \gamma) \frac{|\nabla u'|^4}{|\nabla u'|^2 + \epsilon}^2 \right\} \frac{(\Delta_{\infty} u')^2}{|\nabla u'|^4}

- (1 - 2\eta) \langle \nabla u', \nabla h \rangle^2 \frac{1}{N - n} - C(n, N, p) \frac{1}{\eta} |\nabla u'|^4

=: I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.24}

Recall that if \( N = n \) that is, \( h \) is a constant function, \( I_4 = 0 \) by our convention. If \( N > n \) that is, \( h \) is not a constant, then by \( 1 - 2\eta < 1 \), we have

\[ I_4 \geq - \frac{\langle \nabla u', \nabla h \rangle^2}{N - n}. \tag{3.25} \]

To bound \( I_2 + I_3 \) from below, since \( \gamma < 3 + \frac{p - 1}{N - 1} \) and \( N \geq 2 \) implies

\[ p + 2 - \gamma > p + 2 - 3 - \frac{p - 1}{N - 1} = (p - 1)(1 - \frac{1}{N - 1}) \geq 0, \]

we can find \( 0 < \hat{\eta}(p, \gamma) < 1/2 \) such that for \( 0 < \eta < \hat{\eta} \), one has \( p + 2\frac{1 - 2\eta}{1 + \eta} - \gamma > 0 \). Thus the coefficient of \( I_2 \) satisfies

\[ (p - \gamma) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 2 \frac{1 - 2\eta}{1 + \eta} \geq (p + 2 \frac{1 - 2\eta}{1 + \eta} - \gamma) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + \frac{1 - 2\eta}{1 + \eta} \frac{\epsilon}{|\nabla u'|^2 + \epsilon} > 0. \]

Using this and observing

\[ \frac{|\nabla^2 u' \nabla u'|^2}{|\nabla u'|^2} \geq \frac{|\Delta_{\infty} u'|^2}{|\nabla u'|^4}, \]

one has

\[ I_2 + I_3 \]

\[ \geq \left\{ (p - \gamma) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 2 \frac{1 - 2\eta}{1 + \eta} \right. \]

\[ + \left. \frac{1 - 2\eta}{1 + \eta} \left( \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 1 \right]^2 - 1 \right) + (p - 2)(2 - \gamma) \frac{|\nabla u'|^4}{|\nabla u'|^2 + \epsilon}^2 \right\} \frac{(\Delta_{\infty} u')^2}{|\nabla u'|^4}

\[ =: H(\eta) \frac{(\Delta_{\infty} u')^2}{|\nabla u'|^4}. \]

We claim that there exists \( 0 < \bar{\eta}(n, N, p, \gamma) < \hat{\eta} \) such that \( H(\eta) > 0 \) for all \( 0 < \eta < \bar{\eta} \). Assuming this claim holds for the moment, for any \( 0 < \eta < \bar{\eta} \), one has \( I_2 + I_3 > 0 \). From this, \( \text{(3.24)} \) and \( \text{(3.25)} \) we conclude \( \text{(3.15)} \) as desired.

Finally we prove the above claim. It suffices to show that

\[ H(0) := (p - \gamma) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 2 \left( \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 1 \right]^2 - 1 \right) \]

\[ \geq \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u'|^2}{|\nabla u'|^2 + \epsilon} + 1 \right]^2 - 1 \]
\[ + (p - 2)(2 - \gamma) \frac{|\nabla u'|^4}{||\nabla u'|^2 + \epsilon|^2} \]
\[ > \delta(N, p, \gamma), \quad (3.26) \]

where \( \delta(N, p, \gamma) > 0 \) is a constant. Indeed, by (3.19), one has
\[
H(\eta) \geq H(0) - 2\left[ 1 - \frac{1 - 2\eta}{1 + \eta} \right] - \left[ 1 - \frac{1 - 2\eta}{1 + \eta} \right] \frac{4p^2}{N - 1} - 1 \geq \delta(N, p, \gamma) - 15p^2\eta.
\]

If \( 0 < \eta < \bar{\eta} =: \min \{ \hat{\eta}, \delta(N, p, \gamma)/15p^2 \} \), one has \( H(\eta) > 0 \) and hence the claim holds as desired.

We prove (3.26) as below. Since
\[
(p - 2)^2 \frac{|\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} \geq \delta(N, p, \gamma) - 15p^2\eta,
\]
we rewrite
\[
H(0) = (p - 2)^2 \frac{|\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} + 2(p - 2) \frac{|\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} + 1,
\]
we further write
\[
H(0) = (p - 2)[2 - \gamma + \frac{p - 2}{N - 1}] \frac{|\nabla u'|^4}{||\nabla u'|^2 + \epsilon|^2} + \frac{\epsilon |\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} + \frac{N}{N - 1}.
\]

Observing
\[
\frac{|\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} = \frac{|\nabla u'|^4}{||\nabla u'|^2 + \epsilon|^2} + \frac{\epsilon |\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2}
\]
and
\[
1 = \frac{|\nabla u'|^4}{||\nabla u'|^2 + \epsilon|^2} + 2 \frac{\epsilon |\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2} + \frac{\epsilon^2}{||\nabla u'|^2 + \epsilon|^2},
\]
we further write
\[
H(0) = \left\{ (p - 2)[2 - \gamma + \frac{p - 2}{N - 1}] + \frac{N}{N - 1} \right\} \frac{|\nabla u'|^4}{||\nabla u'|^2 + \epsilon|^2}
\]
\[
+ \left\{ \frac{p - \gamma + \frac{2(p - 2)}{N - 1} + 2 \frac{N}{N - 1}}{N - 1} \right\} \frac{\epsilon |\nabla u'|^2}{||\nabla u'|^2 + \epsilon|^2}
\]
\[
+ \frac{N}{N - 1} \frac{\epsilon^2}{||\nabla u'|^2 + \epsilon|^2}.
\]

By a direct calculation, \( \gamma < 3 + \frac{p - 1}{N - 1} \) implies that
\[
[\frac{p - \gamma + \frac{2(p - 2)}{N - 1} + 2 \frac{N}{N - 1}}{N - 1} + \frac{N}{N - 1}] > p + \frac{2(p - 2)}{N - 1} + 2 \frac{N}{N - 1} - 3 - \frac{p - 1}{N - 1}
\]
\[
= p - 1 + \frac{2(p - 2) + 2 - (p - 1)}{N - 1}
\]
\[
= (p - 1) \frac{N}{N - 1}
\]
\[
> 0.
\]
Moreover, \( \gamma < 3 + \frac{p-1}{N-1} \) also implies that
\[
(p - 2)[2 - \gamma + \frac{p - 2}{N - 1}] + [p - \gamma + \frac{2(p - 2)}{N - 1}] + \frac{N}{N - 1}
\]
\[
= 3(p - 1) + \frac{(p - 2)^2 + 2(p - 2) + 1}{N - 1} - (p - 1) \gamma
\]
\[
= 3(p - 1) + \frac{(p - 1)^2}{N - 1} - (p - 1) \gamma
\]
\[
= (p - 1)[3 + \frac{p - 1}{N - 1} - \gamma]
\]
\[
> 0.
\]
Thus
\[
H(0) > (p - 1)[3 + \frac{p - 1}{N - 1} - \gamma] \cdot \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \frac{N}{N - 1} \cdot \frac{e^2}{|\nabla u^\epsilon|^2 + \epsilon}^2
\]
\[
\geq \frac{1}{2} \min \left\{ (p - 1)[3 + \frac{p - 1}{N - 1} - \gamma], \frac{N}{N - 1} \right\}
\]
\[
= : \delta(N, p, \gamma)
\]
\[
> 0
\]
that is, (3.26) holds. 

Combining (3.15) and (3.4) we have the following. Recall that
\[
\text{Ric}_h(\nabla u^\epsilon, \nabla u^\epsilon) = \text{Ric}_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle - \frac{1}{N - n} \langle \nabla u^\epsilon, \nabla h \rangle^2.
\]

**Corollary 3.6.** Let \( u^\epsilon \) be the solution to (3.3). If \( \gamma < 3 + \frac{p-1}{N-1} \) for some \( N \geq n \), then for sufficiently small \( \eta > 0 \) one has
\[
\eta \int_U |\nabla^2 u^\epsilon|^2 \cdot \frac{|\nabla u^\epsilon|^2 + \epsilon}{2} \cdot \phi^2 e^{-h} \, dv \, g
\]
\[
\leq - \int_U \text{Ric}_h(\nabla u^\epsilon, \nabla u^\epsilon) \cdot \frac{|\nabla u^\epsilon|^2 + \epsilon}{2} \cdot \phi^2 e^{-h} \, dv \, g
\]
\[
+ C(n, N, p, \gamma, \eta) \int_U \left( \frac{|\nabla u^\epsilon|^2 + \epsilon}{2} \cdot \phi^2 + \frac{p - 2}{2} \cdot \phi^2 \right) \cdot e^{-h} \, dv \, g
\]
(3.27)

Under the Bakry-Émery curvature-dimension assumption, we have the following uniform upper bound.

**Lemma 3.7.** Let \( u^\epsilon \) be the solution to (3.3). If \( \gamma < 3 + \frac{p-1}{N-1} \) and \( \text{Ric}_h^N \geq -\kappa \), then one has
\[
\int_U \left| \nabla \right| \left[ \nabla \left( \frac{p - 2}{4} \cdot \nabla u^\epsilon \right) \right]^2 \cdot \phi^2 \cdot e^{-h} \, dv \, g
\]
\[
\leq C(n, N, p, \gamma) \int_U \kappa |\nabla u^\epsilon|^2 \cdot \frac{|\nabla u^\epsilon|^2 + \epsilon}{2} \cdot \phi^2 \cdot e^{-h} \, dv \, g
\]
\[
+ C(n, N, p, \gamma) \int_U \left( \frac{|\nabla u^\epsilon|^2 + \epsilon}{2} \cdot \phi^2 + \frac{p - 2}{2} \cdot \phi^2 \right) \cdot e^{-h} \, dv \, g.
\]
(3.28)
Proof. By $Ric_h^N \geq -\kappa$ we know that

$$-Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon) \leq \kappa |\nabla u^\epsilon|^2$$

Thus the first term in the right-hand side of (3.27) is bounded from above by

$$\kappa \int_U |\nabla u^\epsilon|^2(|\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \phi^2 e^{-h} dv g.$$

On the other hand, a direct calculation leads to

$$\begin{align*}
|\nabla(||\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u^\epsilon| |^2 \\
= |\nabla u^\epsilon|^2 + \epsilon |\nabla^2 u^\epsilon| + \frac{p-\gamma}{2} |\nabla u^\epsilon |^2 |\nabla u^\epsilon |^2 + \frac{(p-\gamma)^2}{4} |\nabla u^\epsilon|^2 |\nabla^2 u^\epsilon |^2 |\nabla u^\epsilon|^2 + \epsilon \\
&\leq C(n, p, \gamma) |\nabla u^\epsilon|^2 + \epsilon |\nabla u^\epsilon|^2 |\nabla u^\epsilon|^2 .
\end{align*}$$

Thus, up to a constant multiplier, the left-hand side of (3.27) is bounded by

$$\int_U |\nabla(||\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u^\epsilon| |^2 e^{-h} dv g.$$}

We therefore conclude (3.28) from (3.27).

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. Let $w \in W^{1,p}(\Omega)$ be any positive weighted $p$-harmonic function in the domain $\Omega$ and $u = -(p-1)\ln w$. Given any smooth domain $U \subseteq \Omega$, for each $\epsilon \in (0, 1)$, let $u^\epsilon \in C^\infty(U)$ be the solution to (3.3). By Lemma 3.1 we know that $u^\epsilon \to u \in C^{1,\alpha}(U)$, for some $\alpha \in (0, 1)$ uniformly in $\epsilon > 0$ as $\epsilon \to 0$. Using this and choosing suitable test functions $\phi \in C_c^\infty(U)$ in (3.28), one concludes $||\nabla u^\epsilon|^2 + \epsilon |\nabla u^\epsilon|^2 \nabla u^\epsilon | W^{1,2}_{loc}(U)$ uniformly in $\epsilon \in (0, 1)$.

Next, we claim that

$$|\nabla u|^{\frac{p-2}{2}} \nabla u \in W^{1,2}_{loc}(U),$$

and

$$\nabla(|\nabla u^\epsilon|^2 + \epsilon |\nabla u^\epsilon|^2) \to \nabla(|\nabla u|^{\frac{p-2}{2}} \nabla u) \text{ weakly in } L^2_{loc}(U, \mathbb{R}^{n \times n}) \text{ as } \epsilon \to 0. \ (3.30)$$

To see this, for any subdomain $V \subseteq U$, by Lemma 3.7 we already have

$$\sup_{\epsilon \in (0, 1]} \|\nabla(|\nabla u^\epsilon|^2 + \epsilon |\nabla u^\epsilon|^2)\|_{L^2(V, \mathbb{R}^{n \times n})} < C(\kappa, n, N, p, \gamma, V).$$

For any subsequence $\{\epsilon_j\}_{j \in \mathbb{N}}$ which converges to 0, by the weak compactness of $W^{2,2}(V)$, up to some subsequence one has $\nabla(|\nabla u^\epsilon|^2 + \epsilon_j |\nabla u^\epsilon|^2) \to z$ weakly in $L^2(V, \mathbb{R}^{n \times n})$ for some
function $z \in L^2(V, \mathbb{R}^{n \times n})$. Let $\{e_1, \cdots, e_n\} \subset T_x U$ be a local orthonormal frame at each $x \in U$. Notice that the $n \times n$ matrix
\[
\nabla (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) = \left( \nabla_{e_i} (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) \right)_{1 \leq k, l \leq n}.
\]
Recalling from Lemma 3.1 that $\nabla u^f \to \nabla u$ in $C^0(U)$ and $V \Subset U$, for any $\phi \in C_\infty^0(U)$ with $\phi|_V = 1$ and $1 \leq k, l \leq n$, we have
\[
\lim_{\epsilon \to 0} \int_U \nabla_{e_i} (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) \phi \, e^{-h} \, d\text{vol}_g
= - \lim_{\epsilon \to 0} \int_U (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) \nabla_{e_i} (\phi \, e^{-h}) \, d\text{vol}_g
= \int_U \nabla_{e_i} (|\nabla u|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u) \phi \, e^{-h} \, d\text{vol}_g.
\]
This shows that in the distributional sense
\[
\nabla (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) \to \nabla (|\nabla u|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u).
\]
Thus $z = \nabla (|\nabla u|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u) \in L^2(V, \mathbb{R}^{n \times n})$ in distributional sense. We therefore have $|\nabla u|^2 \nabla u|_V \in W^{1,2}(V)$, which gives (3.29).

Moreover, by the arbitrariness of subsequence $\{\epsilon_j\}$, we have
\[
\nabla (|\nabla u^f|^2 + \epsilon_j \frac{\partial^2}{\partial x^j} \nabla u^f) \to \nabla (|\nabla u|^2 - \gamma \frac{\partial^2}{\partial x^j} \nabla u)
\]
weakly in $L^2(V, \mathbb{R}^{n \times n})$ as $\epsilon \to 0$. Hence by the arbitrariness of $V \Subset U$, (3.30) holds.

Letting $\epsilon \to 0$ in (3.28) and using the convergence in the above verified claim, we obtain
\[
\int_U \nabla |\nabla u|^2 \frac{\partial^2}{\partial x^j} \nabla u \phi \, e^{-h} \, d\text{vol}_g
\leq C(n, N, p, \gamma) \int_U |\nabla u|^{p-\gamma+2} \phi \, e^{-h} \, d\text{vol}_g
+ C(n, N, p, \gamma) \int_U (|\nabla u|^{p-\gamma+2} |\nabla \phi|^2 + |\nabla u|^{p-\gamma+4} \phi^2) \, e^{-h} \, d\text{vol}_g.
\]  
(3.31)

Let $\phi \in C_\infty^0(B_{2r})$, where $B_{4r} \subset U$, such that $\phi = 1$ in $B_r$ and $\nabla \phi \leq \frac{C}{r}$. Then (3.31) becomes
\[
\int_{B_r} \nabla |\nabla u|^2 \frac{\partial^2}{\partial x^j} \nabla u \, e^{-h} \, d\text{vol}_g \leq C(n, N, p, \gamma) \int_{B_{2r}} \left( \frac{1}{r^2} + \frac{C}{r} \right) |\nabla u|^{p-\gamma+2} + |\nabla u|^{p-\gamma+4} \, e^{-h} \, d\text{vol}_g.
\]
Recalling from (3.5) the Cheng-Yau type gradient estimate that $|\nabla u| \leq C(n, N, p, \gamma) \frac{1 + \sqrt{Kr}}{r}$ and noting that $\gamma < 3 + \frac{2}{p-1}$ guarantees $p-\gamma+2 > 0$, we deduce
\[
|\nabla u|^{p-\gamma+2} \leq C(n, N, p, \gamma) \left( \frac{1 + \sqrt{Kr}}{r} \right)^{p-\gamma+2}.
\]
Together with \( \frac{1}{r^p} + \kappa \leq \left( \frac{1 + \sqrt{\kappa r}}{r} \right)^2 \), we conclude

\[
\int_{B_r} |\nabla |(\nabla u|^2 + \nabla u)|^2 e^{-b} \text{dvol}_g \leq C(n, N, p, \gamma) \text{vol}_h(B_{2r}) \left[ \frac{1 + \sqrt{\kappa r}}{r} \right]^{p - \gamma + 4}.
\]

Dividing both sides by \( \text{vol}_h(B_r) \), noting \( \text{vol}_h(B_{2r}) \leq e^{\sqrt{\kappa r} \text{vol}_h(B_r)} \) from the volume comparison (2.1), and recalling \( u = -(p - 1) \ln w \), we conclude (1.8).

Note that (1.9) is just the special case \( \gamma = \frac{18}{11} \), where \( p < 3 + \frac{2}{N - 2} \) guarantees \( p < 3 + \frac{p - 1}{N - 1} \) and hence one can take \( \gamma = p \) in (1.8).

Finally, we compare our proof with [21, 5], in particular, the crucial pointwise lower bound given in Lemma 3.4 and Lemma 3.5.

**Remark 3.8.** (i) It was well known that a positive (weighted) \( p \)-harmonic function \( w \), and hence \( \ln w \), is always smooth outside of the null set \( E_w \) of \( \nabla \ln w \). In \( \Omega \setminus E_w \), the proof of Lemma 3.5 works for \( \ln w \) so to get (3.16) with \( u^\epsilon \) replaced by \( \ln w \) and \( \epsilon = 0 \), dividing both sides of which by \( |\nabla \ln w|^4 \), for \( 0 < \eta < 1/2 \) one gets

\[
(1 + \eta)|\nabla^2 \ln w|^2 \geq 2 \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} + \left( \frac{(p - 1)^2}{N - 1} - 1 \right) \frac{(\Delta \infty \ln w)^2}{|\nabla \ln w|^4} - (1 + \eta) \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N - n} - C(n, N, p, \gamma, \frac{1}{\eta}) |\nabla \ln w|^4.
\]

(3.32)

If \( \gamma < 3 + \frac{p - 1}{N - 1} \), using (3.32) and noting that the proof of Lemma 3.4 works for \( \ln w \) we get (3.15) with \( u^\epsilon \) replaced by \( \ln w \) and \( \epsilon = 0 \), that is, for \( \eta > 0 \) sufficiently small,

\[
(1 - \eta)|\nabla^2 \ln w|^2 + (p - \gamma) \frac{\langle \nabla \ln w, \nabla \ln w \rangle^2}{|\nabla \ln w|^2} + (p - 2)(2 - \gamma) \frac{(\Delta \infty \ln w)^2}{|\nabla \ln w|^4} \geq \eta|\nabla^2 \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N - n} - C(n, N, p, \gamma, \frac{1}{\eta}) |\nabla \ln w|^4.
\]

(3.33)

From the proof, we see that both of the coefficient 2 of \( \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} \) and the coefficient \( \frac{(p - 1)^2}{N - 1} - 1 \) of \( (\Delta \infty \ln w)^2 \) in (3.32) are critical to guarantee the existence of sufficiently small \( \eta > 0 \) in (3.33) when \( \gamma < 3 + \frac{p - 1}{N - 1} \).

On the other hand, instead of (3.32), recall the following lower bound obtained in [5] by using Lemma 2.1 and the equation (3.1):

\[
|\nabla^2 \ln w|^2 \geq \left| \frac{\nabla^2 \ln w \nabla \ln w}{|\nabla \ln w|^2} \right| - \frac{2}{n - 1} \Delta \infty \ln w + \frac{1}{N - 1} |\nabla \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N - n}.
\]

(3.34)

and also, when \( N = n \) and \( h \equiv 1 \), recall the following lower bound derived in [21] via Lemma 2.1 and (3.1):

\[
|\nabla^2 \ln w|^2 \geq \left[ 1 + \min \{ \frac{(p - 1)^2}{n - 1}, 1 \} \right] \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} - \frac{2}{n - 1} \Delta \infty \ln w + \frac{1}{n - 1} |\nabla \ln w|^2.
\]

(3.35)
From (3.34) and (3.35), via a direct check one can conclude $|\nabla \ln w|^\frac{\mu-2}{2-\mu} \in W^{1,2}_{\text{loc}}$ for $\gamma < 2$, but NOT for all $\gamma < 3 + \frac{p-1}{N-1}$.

(ii) Moreover, unlike [21, 5] where the authors differentiate the equation (3.1) for $\ln w$, we directly derive an upper bound from Bochner formula for the left-hand side of (3.33) with respect to $|\nabla u'|^2 + \epsilon |\nabla u|^2 e^{-h} \text{dvol}_g$.

Appendix A  Proof of Lemma 3.1

In the appendix, we show Lemma 3.1 by checking equations (3.1) and (3.3) are special cases considered in [3]. To this end, we recall the result in [3].

Let $\Omega$ be a domain of $M^n$. Consider the equation

$$-\text{div} \tilde{a}(x, \nabla u) + b(x, \nabla u) = 0 \quad \text{in } \Omega$$  \hspace{1cm} (A.1)

where $\tilde{a}$ is a map from $\Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $b$ maps $\Omega \times \mathbb{R}^n$ to $\mathbb{R}$. Let $\{e_1, \cdots, e_n\} \subset T_x \Omega$ be a local orthonormal frame at each $x \in \Omega$. By a weak solution of (A.1) we mean a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} [\tilde{a}(x, \nabla u), \nabla \phi] + b(x, \nabla u) \phi \text{dvol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega).$$  \hspace{1cm} (A.2)

Assume the following holds for $\tilde{a} = (a_1, \cdots, a_n)$ and $b$.

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j \geq \gamma_0 |\eta|^p |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, p > 1,$$  \hspace{1cm} (A1)

$$\left| \frac{\partial a_i}{\partial \eta_j} \right| \leq \gamma_1 |\eta|^{p-2}, \quad 1 \leq i, j \leq n,$$  \hspace{1cm} (A2)

$$|\nabla e_i a_j(x, \eta)| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i, j \leq n,$$  \hspace{1cm} (A3)

$$|b(x, \eta)| \leq \gamma_1 |\eta|^p,$$  \hspace{1cm} (A4)

and

$$|\nabla e_i b(x, \eta)| \leq \gamma_1 |\eta|^p, \quad \left| \frac{\partial b}{\partial \eta_i}(x, \eta) \right| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i \leq n,$$  \hspace{1cm} (B)

for all $\eta \in \mathbb{R}^n$, where $\gamma_i$ are positive constants, $i = 0, 1$.

For any smooth domain $U \Subset \Omega$ and $\epsilon \in (0, 1]$, consider the regularized equation

$$-\text{div} a^\epsilon(x, \nabla u^\epsilon) + b^\epsilon(x, \nabla u^\epsilon) = 0 \quad \text{in } U \text{ and } u^\epsilon = u \text{ on } \partial U$$  \hspace{1cm} (A.3)

where $a^\epsilon$ is a map from $U \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $b^\epsilon$ maps $U \times \mathbb{R}^n$ to $\mathbb{R}$ such that

$$\lim_{\epsilon \to 0} a^\epsilon(x, \eta) = \tilde{a}(x, \eta) \quad \text{and} \quad \lim_{\epsilon \to 0} b^\epsilon(x, \eta) = b(x, \eta) \quad \forall (x, \eta) \in \Omega \times \mathbb{R}^n.$$

The weak solution of (A.3) is defined similarly as (A.2). Assume the following holds for $a^\epsilon = (a_1^\epsilon, \cdots, a_n^\epsilon)$ and $b^\epsilon$.

$$\sum_{i,j=1}^{n} \frac{\partial a^\epsilon_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j \geq \gamma_0 (\epsilon + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2, \quad \xi \in \mathbb{R}^n, p > 1,$$  \hspace{1cm} (A1,\epsilon)
\[ |\partial a^e_{ij}(x,\eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-2}{2}}, \quad 1 \leq i, j \leq n, \quad (A_{2,e}) \]

\[ |\nabla e \circ a^e_{ij}(x,\eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-1}{2}}, \quad 1 \leq i, j \leq n, \quad (A_{3,e}) \]

\[ |b^e(x,\eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p}{2}}, \quad (A_{4,e}) \]

for all \( \eta \in \mathbb{R}^n \setminus \{0\} \).

We recall the results in [3] as follows.

**Theorem A.1.** Let \( \epsilon \in (0,1] \) and \( U \subset \Omega \). Assume \((A_1)\) - \((A_4)\), \((B)\) and \((A_{1,e})\) - \((A_{4,e})\) hold. Then there exists a unique solution \( u^e \in C^\infty(U) \cap C^0(\overline{U}) \) to \((A.3)\), and moreover, \( u^e \to u \) in \( C^0(\overline{U}) \) and \( u^e \to u \) in \( C^{1,\alpha}(V) \) uniformly in \( \epsilon > 0 \) as \( \epsilon \to 0 \) for all \( V \subset U \) where \( u \) is the solution to \((A.1)\). As a consequence, \( u \in C^{1,\alpha}(\Omega) \).

Theorem A.1 is a combination of Theorem 1 and Theorem 2 in [3] and several intermediate results in the proof of these two theorems in [3]. Indeed, the existence, uniqueness and \( C^{\infty}\) regularity of \( u^e \) is by elliptic theory in PDE; see for example [3]. Based on these facts, in [3], the author first showed that under \((A_1)\) - \((A_4)\), \((B)\) and \((A_{1,e})\) - \((A_{4,e})\), \( u^e \to u \) in \( W^{1,p}(U) \) uniformly in \( \epsilon > 0 \) in section 2. Moreover, \( ||u^e||_{L^\infty(V)} \leq \max_{x \in \partial V} \{ |u(x)| \} \). Thus recalling that \( u^e|_{\partial U} = u|_{\partial U} \), we know \( u^e \to u \) in \( C^0(\overline{U}) \). See the discussion around (2.7) in [3]. Then the author showed that \( ||u^e||_{C^{1,\alpha}(V)} \) is uniformly bounded independently of \( \epsilon \in (0,1] \) and finally showed that \( u^e \to u \) in \( C^{1,\alpha}(V) \) and \( u \in C^{1,\alpha}(\Omega) \) for all \( V \subset U \). By the arbitrariness of \( U \subset \Omega \), one has \( u \in C^{1,\alpha}(\Omega) \).

**Proof of Lemma 3.1.** It suffices to check equations \((3.1)\) and \((3.2)\) are special ones of \((A.1)\) and \((A.3)\) respectively. To this end, let \( \bar{a}(x,\eta) = e^{-h(x)}|\eta|^{p-2}\eta, b(x,\eta) = -e^{-h(x)}|\eta|^p, \bar{a}(x,\eta) = e^{-h(x)}(|\eta|^2 + \epsilon)\frac{p-2}{2}\eta, \) and \( b^e(x,\eta) = -e^{-h(x)}(|\eta|^2 + \epsilon)\frac{p-2}{2}|\eta|^2 \) for all \( x \in U \) and \( \eta \in \mathbb{R}^n \). Then in the weak sense, the equations

\[
\int_{\Omega} [(\bar{a}(x,\nabla u), \nabla \phi) + b(x, \nabla u)\phi] \, d\text{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)
\]

and

\[
\int_{\Omega} [(\bar{a}^e(x,\nabla u), \nabla \phi) + b^e(x, \nabla u)\phi] \, d\text{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)
\]

are exactly \((3.1)\) and \((3.2)\) respectively.

We show \( \bar{a} \) satisfies \((A_1)\). Noting that \( a_j(x,\eta) = e^{-h(x)}|\eta|^{p-2}\eta_j \), we compute

\[
\frac{\partial a^e_{ij}}{\partial \eta_i}(x,\eta) = e^{-h(x)}[(p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2}], \quad \forall 1 \leq i, j \leq n
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \). Thus

\[
\sum_{i,j=1}^n \frac{\partial a^e_{ij}}{\partial \eta_i}(x,\eta)\xi_i\xi_j = e^{-h(x)}\sum_{i,j=1}^n [(p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2})\xi_i\xi_j]
\]

\[
= e^{-h(x)}|\eta|^{p-4}[(p-2)(\sum_{i=1}^n \eta_i\xi_i)^2 + |\eta|^2|\xi|^2], \quad \forall \xi \in \mathbb{R}^n.
\]
If $1 < p < 2$, we have
\[
\sum_{i,j=1}^{n} \frac{\partial a_j}{\partial \eta_i} (x, \eta) \xi_i \xi_j \geq e^{-h(x)} (p-1)|\eta|^{p-2} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.
\]
And if $p \geq 2$, we have
\[
\sum_{i,j=1}^{n} \frac{\partial a_j}{\partial \eta_i} (x, \eta) \xi_i \xi_j \geq e^{-h(x)} |\eta|^{p-2} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.
\]
By taking $\gamma_0 := \min_{x \in U} \{e^{-h(x)}\}$, we conclude that $a$ satisfies (A_1). By direct computations, one can also check $\tilde{a}, \tilde{a}^2 \in C^\infty(U \times \mathbb{R}^n, \mathbb{R}^n)$, $b, b' \in C^\infty(U \times \mathbb{R}^n)$ satisfy (A_3)-(A_4), (B) and (A_{1,4})-(A_{4,4}) respectively. We omit the details. Thus by Theorem A.1, we get the desired result.

**Acknowledgement.** The authors would like to thank the anonymous referees for the careful reading, valuable comments, and many detailed corrections which improve the final presentation of the paper significantly. In particular, we are grateful to the referees for pointing out references [1, 6] that enhance the precision of the literature review in the introduction.

**References**

[1] L. Ambrosio, N. Gigli and G. Savaré, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*. Ann. Probab. 43(1), 339-404 (2015)

[2] S. Y. Cheng and S. T. Yau. *Differential equations on Riemannian manifolds and their geometric applications*. Comm. Pure Appl. Math. 28(1975), no. 3, 333-354.

[3] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*. Nonlinear Anal. 7 (1983), 827-850.

[4] H. Dong, F. Peng, Y. Zhang and Y. Zhou, *Hessian estimates for elliptic and parabolic equations involving $p$-Laplacian via a fundamental inequality*. Adv. Math. 370(2020), Article ID 107212.

[5] N. T. Dung and N. D. Dat, *Weighted $p$–harmonic functions and rigidity of smooth metric measure spaces*. J. Math. Anal. Appl. 443 (2016), no. 2, 959-980.

[6] M. Erbar, K. Kuwada, and K.T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces*. Invent. math. 201, 993-1071, (2015).

[7] N. Gigli and I. Violo *Monotonicity formulas for harmonic functions in RCD(0, N) spaces*. J. Geom. Anal., 33, 100 (2023).

[8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, (1977).

[9] T. Iwaniec and J. J. Manfredi, *Regularity of p-harmonic functions on the plane*. Rev. Mat. Ibero. 5 (1989), no. 1-2, 1-19.

[10] R. Jiang, *Cheeger-harmonic functions in metric measure spaces revisited*. J. Funct. Anal. 266 (2014), 1373-1394.
[11] B. Kotschwar and L. Ni, *Local gradient estimates of p-harmonic functions, 1/H flow, and an entropy formula*. Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 1, 1-36.

[12] J. Lewis, *Regularity of the derivatives of solutions to certain elliptic equations*. Indiana Univ. Math. J. 32 (1983), 849-858.

[13] J. J. Manfredi and A. Weitsman, *On the Fatou theorem for p-harmonic functions*, Commun. Part. Diff. Equ. 13 (1988), 651-668.

[14] S. Sarsa, *Note on an elementary inequality and its application to the regularity of p-harmonic functions*. Annales Fennici Mathematici, vol. 47, no. 1, pp. 139-153, 2021.

[15] L. Mari, M. Rigoli and A. Setti, *On the 1/H-flow by p-Laplace approximation: new estimates via fake distances under Ricci lower bounds*. American Journal of Mathematics, Volume 144, Number 3, 2022.

[16] R. Moser, *The inverse mean curvature flow and p-harmonic functions*. J. Eur. Math. Soc. 9 (2007), 77-83.

[17] L. Ni, Y. Shi and L. Tam, *Poisson Equation, Poincaré-Lelong Equation and Curvature Decay on Complete Kähler Manifolds*. J. Differential Geometry 57 (2001), 339-388.

[18] Z. Qian, *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford Ser. (2) 48 (1997), 235-242.

[19] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*. J. Differential Equations 51 (1984), 126-150.

[20] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*. Acta Math. 138 (1977), 219–240.

[21] X. Wang and L. Zhang, *Local gradient estimate for p-harmonic functions on Riemannian Manifolds*. Comm. Anal. Geom. 19 (2011), no. 4, 759-771.

[22] H. Zhang and X. Zhu, *Yau’s Gradient Estimates on Alexandrov Spaces*. Journal of Differential Geometry, 91 (2012), 445-522.

---

**Jiayin Liu**  
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, Jyväskylä, Finland  
*E-mail*: jiayin.mat.liu@jyu.fi

**Shijin Zhang**  
School of Mathematical Science, Beihang University, Changping District Shahe Higher Education Park South Third Street No. 9, Beijing 102206, P. R. China  
*E-mail*: shijinzhang@buaa.edu.cn

**Yuan Zhou**  
School of Mathematical Science, Beijing Normal University, Haidian District Xinjieku Waidajie No.19, Beijing 10875, P. R. China  
*E-mail*: yuan.zhou@bnu.edu.cn