ON THE ISOMORPHISM PROBLEM FOR UNIT GROUPS OF MODULAR GROUP ALGEBRAS

A. KONOVALOV, A. KRIVOKHATA

Abstract. Using the computational algebra system GAP and the package LAGUNA, we checked that all 2-groups of order not greater than 32 are determined by normalized unit groups of their modular group algebras over the field of two elements.

1. Introduction

Let $KG$ be a modular group algebra of a finite $p$-group $G$ over the field $K$ of $p$ elements, and $V(KG)$ be the normalized unit group of $KG$.

The Modular Isomorphism Problem (MIP) asks whether a finite $p$-group $G$ is determined by its modular group algebra over the field of $p$ elements, i.e. is it true that

$$KG \cong KH \Rightarrow G \cong H ?$$

It remains open for more than 50 years, and up to now it is solved only for some classes of groups and for some groups of small orders.

In the context of MIP, S. D. Berman formulated the Modular Isomorphism Problem for Normalized Unit Groups (UMIP), asking whether a finite $p$-group $G$ is determined by the normalized unit group of its modular group algebra over the field of $p$ elements, i.e. is it true that

$$V(KG) \cong V(KH) \Rightarrow G \cong H ?$$

Obviously, UMIP is stronger than MIP, and from the solution of UMIP follows the solution of MIP. For a long time, the positive solution of UMIP was known only for abelian $p$-groups, and only recently it was solved for 2-groups of maximal class in [1], and for $p$-groups with the cyclic Frattini subgroup for $p > 2$ in [2].

Although UMIP is a stronger problem, it is well-suited for computer-aided tests. Since the order of the group $G$ is determined by $V(KG)$, for such a test it will be enough to check that all non-isomorphic groups of a given order have non-isomorphic normalized unit groups of their modular group algebras over the field of 2 elements. The structure of the normalized unit group $V(KG)$ can be investigated using the computational algebra system GAP [5] enhanced by the LAGUNA package [3] to calculate the power-commutator presentation of $V(KG)$. Having such a presentation, we can compare various invariants of normalized unit groups such as their exponent, sizes of certain subgroups, the number of involutions, etc. until we find a pair of different ones making evidence of their non-isomorphism. Note that here we have much more freedom in choosing invariants unlike in the similar

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approach to MIP tests when we can compare only properties of $G$ known to be determined by $KG$.

In the present paper we give a detailed report about checking that all 2-groups of order not greater than 32 are determined by the normalized unit group of their modular group algebras. Thus, our tests confirm UMIP for these groups.

2. Groups of orders up to 16

Since the solution of UMIP is known for abelian groups and also for 2-groups of maximal class [1], there is no need to consider groups of orders 2, 4, and 8, so we can start from order 16.

Among 14 groups of order 16, there are 5 abelian groups and 3 groups of maximal class, so it remains to consider 6 groups of this order.

In the following table for these groups we give the number of the group $G$ in the library of groups of order 16 from the Small Groups Library of the GAP system, the order of the Frattini subgroup of $V(KG)$, and the number of elements of order 2 in $V(KG)$:

| Catalogue number of $G$ | Order of the Frattini subgroup of $V(KG)$ | Number of elements of order 2 in $V(KG)$ |
|-------------------------|-------------------------------------------|----------------------------------------|
| 3                       | 256                                       | 5119                                   |
| 4                       | 256                                       | $3^2 \cdot 5 \cdot 7 \cdot 13$         |
| 6                       | 512                                       | $5 \cdot 307$                          |
| 11                      | 128                                       | 6143                                   |
| 12                      | 128                                       | $3^2 \cdot 5 \cdot 7 \cdot 13$         |
| 13                      | 128                                       | 3583                                   |

It is easy to see from the table that these two invariants split all groups.

Note that generators of $V(KG)$ for 2-groups of orders up to 16 were given by R. Sandling in [6], but, as it was confirmed by R. Sandling in private communication, he did not consider the isomorphism question.

3. Groups of order 32

Among 51 groups of order 32, there are 7 abelian groups and 3 groups of maximal class, so it remains to consider 41 groups of this order.

First we compute the order of the center of $V(KG)$ and the order of the Frattini subgroup of $V(KG)$ to split these groups into twelve families, and then we will show that unit groups within each family are pairwise non-isomorphic. In the next table we list catalogue numbers of groups and values of invariants for each family:
We see that families 11 and 12 contain only one group, so for these groups the test is already finished. Now we will consider each family 1–10 separately.

• Family 1. Since the number of involutions in $V(KG)$ for both groups are the same, we need to employ other invariants. Using the GAP4 Package AutPGrp by B. Eick and E. O’Brien [4], we computed the automorphism groups $Aut(V(KG))$ for both cases. It appears that their orders, given in the next table, are different:

| Number of the family | Catalogue numbers of groups of order 32 | Order of the center of $V(KG)$ | Order of the Frattini subgroup of $V(KG)$ |
|----------------------|----------------------------------------|-------------------------------|------------------------------------------|
| 1                    | 43, 44                                 | $2^{19}$                      | $2^{19}$                                 |
| 2                    | 6, 7, 8                                | $2^{19}$                      | $2^{19}$                                 |
| 3                    | 28, 29, 39, 40, 41, 42                 | $2^{13}$                      | $2^{13}$                                 |
| 4                    | 9, 10, 13, 14, 27, 30, 31, 32, 33, 34, 35 | $2^{13}$                      | $2^{13}$                                 |
| 5                    | 11, 15                                 | $2^{19}$                      | $2^{19}$                                 |
| 6                    | 49, 50                                 | $2^{19}$                      | $2^{19}$                                 |
| 7                    | 46, 47, 48                             | $2^{19}$                      | $2^{19}$                                 |
| 8                    | 22, 23                                 | $2^{19}$                      | $2^{19}$                                 |
| 9                    | 2, 24, 25, 26, 37, 38                  | $2^{19}$                      | $2^{19}$                                 |
| 10                   | 5, 12                                  | $2^{19}$                      | $2^{19}$                                 |
| 11                   | 4                                      | $2^{19}$                      | $2^{19}$                                 |
| 12                   | 17                                     | $2^{19}$                      | $2^{19}$                                 |

• Family 2. The number of involutions in $V(KG_n)$ is equal to $2^{18} \cdot 19$, $2^{19} \cdot 7$ and $2^{20} \cdot 3$ for $n = 6, 7$ and 8, respectively.

• Family 3. First we note that the exponent of the center of $V(KG)$ is equal to 2 for $n \in \{28, 29\}$ and to 4 for $n \in \{39, 40, 41, 42\}$. In case of exponent 2, automorphism groups of $V(KG)$ have different orders, given in the following table:

| Catalogue number of $G$ | Order of the automorphism group of $V(KG)$ |
|-------------------------|------------------------------------------|
| 28                      | $2^{173}$                                 |
| 29                      | $2^{173}$                                 |

In the case of exponent 4, the number of involutions in the center of $V(KG)$ is equal to 4095 for $n \in \{39, 40, 41\}$ and to 2047 for $n = 42$, so the latter group is determined. Now it remains to check that the number of involutions in $V(KG_n)$ is equal to $2^{23}$, $2^{18} \cdot 31$ and $2^{19} \cdot 3 \cdot 5$ for $n = 39, 40$ and 41, respectively.
Family 4. First we give a table containing the number of involutions in $V(KG)$:

| Catalogue number of $G$ | Number of elements of order 2 in $V(KG)$ |
|-------------------------|------------------------------------------|
| 9                       | $2^{18} \cdot 29$                       |
| 10                      | $2^{20} \cdot 7$                       |
| 13                      | $2^{19} \cdot 13$                       |
| 14                      | $2^{19} \cdot 13$                       |
| 27                      | $2^{18} \cdot 97$                       |
| 30                      | $2^{18} \cdot 3^4$                      |
| 31                      | $2^{19} \cdot 3 \cdot 11$              |
| 32                      | $2^{24}$                                |
| 33                      | $2^{18} \cdot 73$                       |
| 34                      | $2^{20} \cdot 17$                       |
| 35                      | $2^{24}$                                |

We see that this parameter splits almost all groups except two pairs $n = 13, 14$ and $n = 32, 35$. To split these pairs, first we compute the order of the automorphism group of $V(KG)$:

| Catalogue number of $G$ | Order of the automorphism group of $V(KG)$ |
|-------------------------|-------------------------------------------|
| 13                      | $2^{49}$                                  |
| 14                      | $2^{149}$                                 |
| 32                      | $2^{158}$                                 |
| 35                      | $2^{159}$                                 |

Now to split the remaining pair for $n = 13$ and $14$, we need one more step. We were trying to compare a number of invariants but they were the same. Finally, using the development version of the AutPGrp package [4] we checked that the minimal number of generators of $\text{Aut}(V(KG))$ is equal to 15 for $n = 13$ and to 16 for $n = 14$. Note that these groups, given by relations

\[ G_{13} = \langle a, b \mid a^8 = 1, b^4 = 1, b^{-1}ab = a^3 \rangle; \]
\[ G_{14} = \langle a, b \mid a^8 = 1, b^4 = 1, b^{-1}ab = a^{-1} \rangle, \]

also appeared in the preprint by Newman, Michler and O’Brien [7], where this pair also was the most complicated case in confirming (MIP) for groups of order 32.

Family 5. The number of involutions in $V(KG_n)$ is equal to $2^{20} \cdot 3$ for $n = 11$ and $2^{18} \cdot 3^2$ for $n = 15$.

Family 6. The number of involutions in $V(KG_n)$ is equal to $2^{21} \cdot 7$ for $n = 49$ and $2^{22} \cdot 3$ for $n = 50$.

Family 7. First we note that the exponent of the center of $V(KG)$ is equal to 4 for $n = 48$, while it is equal to 2 for $n = 46, 47$. Then, the number of involutions in $V(KG_n)$ is equal to $2^{25} \cdot 3$ for $n = 46$ and $2^{23} \cdot 11$ for $n = 47$.

Family 8. The number of involutions in $V(KG_n)$ is equal to $2^{22} \cdot 23$ for $n = 22$ and $2^{23} \cdot 11$ for $n = 23$.

Family 9. First we note that the exponent of the center of $V(KG)$ is equal to 2 for $n = 2$ and to 8 for $n = 38$, while it is equal to 4 for $n \in \{24, 25, 26, 37\}$. Furthermore, the $p$-class of $V(KG)$ is equal to 2 for $n \in \{2, 24, 25, 26\}$ and to 3 for $n \in \{37, 38\}$. Combining these two invariants, we split groups with $n \in \{2, 37, 38\}$.
For the three remaining groups the number of involutions in \( V(KG_n) \) is equal to 2\(^{21} \cdot 3 \cdot 5\), 2\(^{22} \cdot 7\) and 2\(^{24}\) for \( n = 24, 25 \) and 26, respectively.

- **Family 10.** The number of involutions in \( V(KG_n) \) is equal to 2\(^{23}\) for \( n = 5 \) and 2\(^{24}\) for \( n = 12 \).

Thus, we confirmed UMIP for groups of order 32, because each group \( G \) of order 32 can be determined from the unique set of invariants of the normalized unit group \( V(KG) \) of its modular group algebra over the field of two elements.

### 4. Computational issues

Computations of \( V(KG) \) and its invariants were performed using the computational algebra system GAP [5] enhanced with the LAGUNA package [3]. Note that without the LAGUNA package it would not be possible to compute the unit group of the modular group algebra for groups of order 16 and 32.

First we tried to employ some "cheap" invariants such as the order and the exponent of the center of \( V(KG) \), and the Frattini subgroup of \( V(KG) \). Computations of such kind could be performed very fast, especially on modern computers.

For those groups, which cannot be split using "cheap" invariants, we decided to compare the number of involutions in \( V(KG) \). This parameter proved to be useful to deal with (UMIP) for 2-groups of maximal class [1], where it was computed by theoretical means. We used the results of [1] to check that our program in GAP returns correct output for 2-groups of maximal class. We hope that results of our computations may motivate further attempts for evaluating the number of involutions in unit groups and provide a number of examples for such research.

For groups of order 16 it takes a very short time to count involutions in \( V(KG) \), but this is already not the case for groups of order 32. This is why computations of the number of involutions for groups of order 32 were performed on the Computational Cluster of the Kiev National Taras Shevchenko University, created with the support of Intel corporation (http://www.cluster.kiev.ua/), where it takes about 24 hours to enumerate elements of one unit group. Also, because of some internal GAP limitations, we were forced first to enumerate cosets of \( V(KG) \) by one of its proper subgroup, and then to enumerate elements of these cosets.

In that cases when the number of involutions was not useful, we tried to compare the number of elements of each order, but this did not give any new information. It was the AutPGrp package [4] that brings new information useful for splitting the most difficult pairs. We especially acknowledge the help of B. Eick in splitting the pair \( (G_{13}, G_{14}) \) using the development version of AutPGrp, and for her helpful advises in AutPGrp usage. Note that some other pairs of groups, which were split by the number of involutions, could be also splitted in shorter time by the order of \( Aut(V(KG)) \). Since this is easier for a GAP user, we did not give these parameters here, being more interested in the number of involutions by the above mentioned motivation.

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Alexander Konovalov
School of Computer Science
University of St Andrews
Jack Cole Building, North Haugh,
St Andrews, Fife, KY16 9SX, Scotland
e-mail: konovalov@member.ams.org

Anastasiya Krivokhata
Department of Mathematics
Zaporozhye National University
ul.Zhukovskogo, 66, Zaporozhye
69063, Ukraine
e-mail: k-algebra@zsu.zp.ua