Limit Theorems in Wasserstein Distance for Empirical Measures of Diffusion Processes on Riemannian Manifolds

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Abstract

Let \((M, \rho)\) be a connected compact Riemannian manifold without boundary or with a convex boundary \(\partial M\), let \(V \in C^2(M)\) such that \(\mu(dx) := e^{V(x)}dx\) is a probability measure, where \(dx\) is the volume measure. Let \(\{\lambda_i\}_{i \geq 1}\) be all non-trivial eigenvalues of \(-L\) with Neumann boundary condition if \(\partial M \neq \emptyset\), where \(L := \Delta + \nabla V\) for \(\Delta\) being the Laplace-Beltrami operator on \(M\). Then the empirical measures \(\{\mu_t\}_{t > 0}\) of the diffusion process generated by \(L\) (with reflecting boundary if \(\partial M \neq \emptyset\)) satisfy

\[
\lim_{t \to \infty} \left\{ t \mathbb{E}^x [W_2^2(\mu_t, \mu)]^2 \right\} = \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2} \quad \text{uniformly in } x \in M,
\]

where \(\mathbb{E}^x\) is the expectation for the diffusion process starting at point \(x\), and \(W_2\) is the \(L^2\)-Wasserstein distance induced by the Riemannian metric. The limit is finite if and only if \(d \leq 3\), and in this case we derive

\[
\lim_{t \to \infty} \sup_{x \in M} \left| \mathbb{P}^x \left( tW_2(\mu_t, \mu)^2 < a \right) - \mathbb{P} \left( \sum_{k=1}^{\infty} \frac{2\xi_k^2}{\lambda_k^2} < a \right) \right| = 0, \quad a \geq 0,
\]

where \(\mathbb{P}^x\) is the probability with respect to \(\mathbb{E}^x\), and \(\{\xi_k\}_{k \geq 1}\) are i.i.d. standard Gaussian random variables. Moreover, \(\mathbb{E}^x[W_2^2(\mu_t, \mu)] \sim t^{-\frac{d}{2}}\) for \(d \geq 5\), and when \(d = 4\) we have \(\mathbb{E}^x[W_2(\mu_t, \mu)] \leq c t^{-1} \log t\) for some constant \(c > 0\) and large \(t\) while the same type lower bound estimate holds for \(M = \mathbb{T}^d\). Finally, we establish the long-time large deviation principle for \(\{W_2(\mu_t, \mu)^2\}_{t \geq 0}\) with a good rate function given by the information with respect to \(\mu\).

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1 Introduction and Main results

The diffusion processes (for instance, the Brownian motion) on Riemannian manifolds have intrinsic link to properties (for instances, curvature, dimension, spectrum) of the infinitesimal generator, see, for instances, the monographs [7, 34] and references within. In this paper, we characterize the long time behaviour of empirical measures for diffusion processes by using eigenvalues of the generator.

Let $M$ be a $d$-dimensional connected complete Riemannian manifold possibly with a smooth boundary $\partial M$. Let $V \in C^2(M)$ such that $\mu(dx) = e^V \mu_M(dx)$ is a probability measure on $M$, where $\mu_M$ is the Riemannian volume measure on $M$. Then the (reflecting, if $\partial M \neq \emptyset$) diffusion process $X_t$ generated by $L := \Delta + \nabla V$ on $M$ is reversible; i.e. the associated diffusion semigroup $\{P_t\}_{t \geq 0}$ is symmetric in $L^2(\mu)$, where

$$P_t f(x) := \mathbb{E}^x f(X_t), \ t \geq 0, f \in \mathcal{B}_b(M).$$

Here, $\mathbb{E}^x$ is the expectation taken for the diffusion process $\{X_t\}_{t \geq 0}$ with $X_0 = x$, and we will use $\mathbb{P}^x$ to denote the associated probability measure. In general, for any probability measure $\nu$ on $M$, let $\mathbb{E}^\nu$ and $\mathbb{P}^\nu$ be the expectation and probability taken for the diffusion process with initial distribution $\nu$.

When the diffusion process generated by $L$ is exponentially ergodic, it is in particular the case when $M$ is compact, the empirical measure

$$\mu_t := \frac{1}{t} \int_0^t \delta_{X_s} ds, \ t > 0$$

converges weakly to $\mu$ as $t \to \infty$. More precisely, for any non-constant $f \in C_b(M)$, we have the law of large number

$$\lim_{t \to \infty} \mu_t(f) = \mu(f) \ a.s.$$ 

as well as the central limit theorem

$$\sqrt{t} \{\mu_t(f) - \mu(f)\} \to N(0, \delta(f)) \ \text{in law as} \ t \to \infty,$$

where $\delta(f) := \lim_{t \to \infty} t \mathbb{E}|\mu_t(f) - \mu(f)|^2 \in (0, \infty)$ exists, and $N(0, \delta(f))$ is the centered normal distribution with variance $\delta(f)$. Consequently, the average additive functional $\mu_t(f)$ converges to $\mu(f)$ in $L^2(\mathbb{P})$ with rate $t^{-\frac{1}{2}}$, which is universal and has nothing to do with specific properties of $M$ and $L$. See, for instance [21], for historical remarks and more results concerning limit theorems on additive functionals of Markov processes.
On the other hand, since the Wasserstein distance $W_2$ induced by the Riemannian distance $\rho$ on $M$ is associated with a natural Riemannian structure on the space of probability measures, see e.g. [24], the asymptotic behaviors of $W_2(\mu_t, \mu)$ should reflect intrinsic properties of $M$ and $L$. Indeed, as shown in Theorem 1.1 below, the long time behavior of $W_2(\mu_t, \mu)^2$ depends on the dimension of $M$ and all eigenvalues of $L$, this is essentially different from that of the additive functional $\mu_t(f)$ introduced above.

Let $\mathcal{P}$ be the set of all probability measures on $M$, and let $\rho$ be the Riemannian distance on $M$. For any $p \geq 1$, the $L^p$-Wasserstein distance $W_p$ is defined by

$$W_p(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{M \times M} \rho(x, y)^p \pi(dx, dy) \right)^{\frac{1}{p}}, \quad \mu_1, \mu_2 \in \mathcal{P},$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions $\mu_1$ and $\mu_2$. A measure $\pi \in \mathcal{C}(\mu_1, \mu_2)$ is called a coupling of $\mu_1$ and $\mu_2$.

In this paper, we aim to characterize the long time behavior of $W_2(\mu_t, \mu)^2$. When $M$ is compact, we will prove the large deviation principle with rate function

$$I(r) := \inf \{ I_\mu(\nu) : \nu \in \mathcal{P}, W_2(\nu, \mu) \geq r \}, \quad r \geq 0,$$

where $I_\mu$ is the information with respect to $\mu$; i.e.

$$I_\mu(\nu) := \begin{cases} \mu(|\nabla f|^2), & \text{if } \nu = f\mu, f \in W^{2,1}(\mu); \\ \infty, & \text{otherwise.} \end{cases}$$

Here, $W^{2,1}(\mu)$ is the closure of $C^\infty(M)$ under the Sobolev norm

$$\|h\|_{2,1} := \sqrt{\mu(h^2 + |\nabla h|^2)}.$$

By convention, we set $\inf \emptyset = \infty$, so that $I(r) = \infty$ for $r > r_0$, where since $\rho$ is bounded,

$$r_0 := \sup_{\nu} W_2(\nu, \mu)^2 = \sup_{x \in M} \mu(\rho(x, \cdot)^2) < \infty.$$

It is well known that when $M$ is compact, $L$ has purely discrete spectrum, and all non-trivial eigenvalues $\{\lambda_i\}_{i \geq 1}$ of $-L$ listed in the increasing order counting multiplicities satisfy (see for instance [12])

$$\kappa^{-1} i^{\frac{3}{2}} \leq \lambda_i \leq \kappa i^{\frac{3}{2}}, \quad i \geq 1$$

for some constant $\kappa > 1$. Our first result is the following.

**Theorem 1.1.** Let $M$ be compact.

1. If $\partial M$ is empty or convex, then the following limit formula holds uniformly in $x \in M$:

$$\lim_{t \to \infty} \left\{ t \mathbb{E}^x [W_2(\mu_t, \mu)^2] \right\} = \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2}.$$
In general, there exists a constant $c \in (0, 1]$ such that

\[
    c \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2} \leq \liminf_{t \to \infty} \inf_{x \in M} \left\{ t \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2]\right\}
\]

\[
    \leq \limsup_{t \to \infty} \sup_{x \in M} \left\{ t \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2]\right\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2},
\]

(1.3)

(2) $\{\mathbb{W}_2(\mu_t, \mu)^2\}_{t \geq 0}$ satisfies the uniform large deviation principle with good rate function $I$; that is, $\{I \leq \alpha\}$ is a compact subset of $[0, \infty)$ for any $\alpha \in [0, \infty)$, and

\[
    \inf_{r \in A^c} I(r) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in M} \mathbb{P}^x(\mathbb{W}_2(\mu_t, \mu)^2 \in A^c)
\]

\[
    \leq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in M} \mathbb{P}^x(\mathbb{W}_2(\mu_t, \mu)^2 \in \bar{A}) \leq - \inf_{r \in A} I(r), \quad A \subset [0, \infty),
\]

where $A^c$ and $\bar{A}$ denote the interior and the closure of $A$ respectively.

(3) If $d \leq 3$ and $\partial M$ is either empty or convex, then

\[
    \limsup_{t \to \infty} \left\{ t \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] - \nu_0((-\infty, a)) \right\} = 0, \quad a \in \mathbb{R},
\]

where $\nu_0$ is the distribution of $\Xi_0 := \sum_{k=1}^{\infty} \frac{2c^2}{\lambda_k}$ for a sequence of i.i.d. standard Gaussian random variables $\{\xi_k\}_{k \geq 1}$.

In Theorem 1.1(3) we only consider $d \leq 3$, since Theorem 1.1(1) and (1.1) yield

\[
    \liminf_{t \to \infty} \left\{ t \inf_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2]\right\} = \infty, \quad d \geq 4.
\]

(1.5)

So, for $d \geq 4$ the convergence of $\mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2]$ is slower than $t^{-1}$. In the next result we present two-sided estimates on the convergence rate of $\mathbb{E}[\mathbb{W}_2(\mu_t, \mu)^2]$ for $d \geq 4$.

**Theorem 1.2.** Let $M$ be compact with $d \geq 4$.

1. There exists a constant $c > 0$ such that for any $t \geq 1$,

\[
    \sup_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \leq \begin{cases} 
        ct^{-1} \log(1 + t), & \text{if } d = 4, \\
        ct^{-\frac{2}{d-2}}, & \text{if } d \geq 5.
    \end{cases}
\]

2. On the other hand, there exists a constant $c' > 0$ such that

\[
    \inf_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)]^2 \geq \inf_{x \in M} \{\mathbb{E}^x[\mathbb{W}_1(\mu_t, \mu)]^2\}^2 \geq c't^{-\frac{2}{d-2}}, \quad t \geq 1.
\]

Theorem 1.2 implies that when $d \geq 5$ we have $\mathbb{E}[\mathbb{W}_2(\mu_t, \mu)^2] \sim t^{-\frac{2}{d-2}}$ for large $t$. Due to (1.5), we hope that $\mathbb{E}[\mathbb{W}_2(\mu_t, \mu)^2] \sim t^{-1} \log t$ holds for $d = 4$, i.e. the order in the upper bound estimate is sharp. Although in general this is not yet proved in the paper, it is true for $M = \mathbb{T}^4(= \mathbb{R}^4 \setminus (2\pi \mathbb{Z}^4))$ and $V = 0$ according to the following result and $\{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]^2 \leq \mathbb{E}[\mathbb{W}_2(\mu_t, \mu)^2]$.
Theorem 1.3. Let $M = \mathbb{T}^4$ and $V = 0$. Then there exists a constant $c > 0$ such that

$$\inf_{x \in M} \{E^x W_1(\mu_t, \mu)\}^2 > ct^{-1} \log t, \quad t \geq 1$$

To conclude this section, we compare the convergence rate of $W_2(\mu_t, \mu)$ with that of $W_2(\bar{\mu}_n, \mu)$ investigated in [3, 5, 8, 9, 11, 15, 16], where

$$\bar{\mu}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \quad n \in \mathbb{N}$$

is the empirical measure of i.i.d. random variables $\{X_i\}_{i \geq 1}$ with common distribution $\mu$. In particular, for $\mu$ being the uniform distribution on a bounded domain in $\mathbb{R}^d$, we have

$$E[W_2(\bar{\mu}_n, \mu)]^2 \sim \begin{cases} n^{-2/d} & \text{if } d \geq 3, \\ n^{-1} \log n & \text{if } d = 2, \\ n^{-1} & \text{if } d = 1, \end{cases}$$

where the assertion for $d = 2$ is known as Ajtai-Komlós-Tusnády (AKT) optimal matching theorem [3], and $a_n \sim b_n$ means that $c_1 a_n \leq b_n \leq c_2 a_n$ holds for some constants $c_2 \geq c_1 > 0$ and large $n$. Moreover, the empirical measure for a discrete time Markov chain on a bounded domain has been investigated in [9].

Combining this Theorems 1.1-1.3 for $t = n$, we see that in the sense of empirical measures, diffusion processes converge faster than i.i.d. samples: the empirical measures of $d$-dimensional diffusion processes behave as those of $(d - 2) \lor 1$-dimensional i.i.d. samples. In particular, in the present setting the feature of AKT optimal matching theorem appears to dimension $d = 4$ rather than $d = 2$. However, unlike in the i.i.d. case for which Ambrosio-Stra-Trevisan [5] derived the exact value of $\lim_{n \to \infty} \frac{n}{\log n} E[W_2(\bar{\mu}_n, \mu)]^2$ for the uniform distribution $\mu$ on $\mathbb{T}^2$, in the present setting the exact value of $\lim_{t \to \infty} \frac{t}{\log t} E[W_2(\mu_t, \mu)]^2$ is unknown for the uniform distribution $\mu$ on $\mathbb{T}^4$. This could be a challenging problem.

Since $\mu_t$ is singular with respect to $\mu$, it is hard to estimate $W_2(\mu_t, \mu)$ using analytic methods. So, as in [4], we first investigate the modified empirical measures

$$\mu_{t,r} := \mu_t P_r = \frac{1}{t} \int_0^t \{\delta_{X_s} P_r\} ds, \quad t > 0, \quad r > 0,$$

where for a probability measure $\nu$ on $M$, $\nu P_r$ denotes the distribution of $X_r$ with $X_0$ having law $\nu$. Note that $\lim_{r \to 0} W_2(\mu_{t,r}, \mu_t) = 0$, see (3.3) below for an estimate of the convergence rate. The main results of the paper have been extended in [39] to subordinated diffusion processes on compact manifolds, see also [35, 36, 37, 38] for further development on the empirical measures for killed diffusion processes, SDEs and SPDEs.

The remainder of the paper is organized as follows. In Section 2, we investigate the long time behavior of the modified empirical measures $\mu_{t,r}$ for $r > 0$, where $M$ might be non-compact. We then prove Theorems 1.1, 1.2 and 1.3 in Sections 3, 4 and 5 respectively.
2 Asymptotics for modified empirical measures

In this part, we allow \( M \) to be non-compact, but assume that \( L \) satisfies the curvature condition

\[
Ric_V := \text{Ric} - \text{Hess}_V \geq -K
\]

for some constant \( K \geq 0 \), where \( \text{Ric} \) is the Ricci curvature on \( M \) and \( \text{Hess}_V \) is the Hessian tensor of \( V \). This condition means that \( \text{Ric}_V(X, X) \geq -K|X|^2 \) for all \( X \in TM \), the tangent bundle of \( M \).

When \( \partial M \neq \emptyset \), let \( N \) be the inward unit normal vector field of \( \partial M \). We call \( \partial M \) convex, if its second fundamental form \( I_{\partial M} \) is nonnegative; i.e.

\[
I_{\partial M}(X, X) := -\langle X, \nabla_X N \rangle \geq 0, \quad x \in \partial M, X \in T_x \partial M.
\]

We call \( \partial M \) convex on a set \( D \subset M \), if (2.2) holds for some function \( g \) which is non-negative on \( D \cap \partial M \).

For any \( q \geq p \geq 1 \), let \( \| \cdot \|_{p \to q} \) be the operator norm from \( L^p(\mu) \) to \( L^q(\mu) \). We will need the following assumptions.

(A1) \( P_t \) is ultracontractive, i.e. \( \| P_t \|_{1 \to \infty} := \sup_{\mu(\|f\|)} \| P_tf \|_{\infty} < \infty, \quad t > 0 \).

(A2) (2.1) holds for some constant \( K \geq 0 \), and there exists a compact set \( D \subset M \) such that either \( D^c \cap \partial M = \emptyset \) or \( \partial M \) is convex on \( D^c \).

Obviously, (A1) and (A2) hold if \( M \) is compact. When \( M \) is non-compact satisfying condition (A2), by [34, Theorem 3.5.5], (A1) holds if and only if \( \| P_te^{\lambda \rho_0(\cdot)} \|_{\infty} < \infty \) for any \( \lambda, t > 0 \), where \( \rho_0 := \rho(o, \cdot) \) is the distance function to a fixed point \( o \in M \), see [26, Corollary 2.5] for concrete examples with \( \| P_te^{\lambda \rho_0(\cdot)} \|_{\infty} < \infty \). See also [30, Proposition 4.1] for examples satisfying assumption (A1) when \( \text{Ric}_V \) is unbounded from below.

(A1) implies that the spectrum of \( L \) (with Neumann boundary condition if \( \partial M \neq \emptyset \)) is purely discrete. Since \( M \) is connected, in this case \( L \) has a spectral gap, i.e. \( 0 \) is a simple isolated eigenvalue of \( L \). Let \( \{\lambda_i\}_{i \geq 1} \) be all non-trivial eigenvalues of \( -L \) listed in the increasing order including multiplicities. By the concentration of \( \mu \) implied by the ultracontractivity condition (A1), we have

\[
\int_{M \times M} e^{\lambda \rho^2} \text{d}(\mu \times \mu) < \infty, \quad \lambda > 0.
\]

Indeed, according to [14, 18] (see for instance [26, Theorem 1.1]), (A1) implies that for some \( \beta : (0, \infty) \to (0, \infty) \),

\[
\mu(f^2 \log f^2) \leq r \mu(|\nabla f|^2) + \beta(r), \quad r > 0, f \in C^1_b(M), \mu(f^2) = 1,
\]
which then ensures (2.3) by [26, Corollary 6.3] or [2].

For any $r > 0$, let $\nu_r$ be the distribution of

$$
\Xi_r := \sum_{k=1}^{\infty} \frac{2\xi_k^2}{\lambda_k^2 e^{2\lambda_k r}},
$$

where $\{\xi_k\}_{k \geq 1}$ are i.i.d. standard Gaussian random variables.

**Theorem 2.1.** Assume (A1) and let $r > 0$. Then

$$
\limsup_{t \to \infty} \sup_{x \in M} \left\{ tE_x [W_2(\mu_{t,r}, \mu)^2] \right\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2\lambda_i r}} < \infty.
$$

If moreover (A2) holds, then

$$
\lim_{t \to \infty} \sup_{x \in M} \left| tE_x [W_2(\mu_{t,r}, \mu)^2] - \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2\lambda_i r}} \right| = 0,
$$

$$
\lim_{t \to \infty} \sup_{x \in M} \left| P_x \left( tW_2(\mu_{t,r}, \mu)^2 < a \right) - \nu_r ((-\infty, a)) \right| = 0, \quad a \in \mathbb{R}.
$$

**Remark 2.1.** Consider the measure

$$
\mu_{sp} := \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \delta_{\lambda_i},
$$

whose support consists of all non-trivial eigenvalues of $L$. Then (2.5) implies

$$
\int_0^\infty e^{-2rs} \mu_{sp} (ds) = \lim_{t \to \infty} \left\{ tE_{\nu} [W_2(\mu_{t,r}, \mu)^2] \right\}, \quad r > 0
$$

for any probability measure $\nu$ on $M$. This gives a probabilistic representation for the Laplace transform of $\mu_{sp}$, and hence determines all eigenvalues and multiplicities for $L$.

To investigate the long time behavior of $E[W_2(\mu_t, \mu)^2]$, i.e. $E[W_2(\mu_{t,r}, \mu)^2]$ with $r = 0$, one may consider the limit of formula (2.5) when $r \downarrow 0$.

**Corollary 2.2.** If $M$ is compact, then:

1. For $d \leq 3$,

$$
\lim_{r \downarrow 0} \lim_{t \to \infty} \left\{ tE_x [W_2(\mu_{t,r}, \mu)^2] \right\} = \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2} < \infty \text{ uniformly in } x \in M.
$$

2. For $d = 4$,

$$
\lim_{r \downarrow 0} \lim_{t \to \infty} \frac{\log \log \left\{ tE_x [W_2(\mu_{t,r}, \mu)^2] \right\} \log \log r^{-1}}{\log \log r^{-1}} = 1 \text{ uniformly in } x \in M.
$$
For $d \geq 5$, 

$$
\lim_{r \downarrow 0} \lim_{t \to \infty} \frac{\log \{ t \mathbb{E} [ \mathbb{W}_2(\mu_{t,r}, \mu)^2] \}}{\log r^{-1}} = \frac{d - 4}{2} \text{ uniformly in } x \in M.
$$

In the following two subsections, we investigate the upper and lower bound estimates on $\mathbb{E} [ \mathbb{W}_2(\mu_{t,r}, \mu)^2]$ respectively, which then lead to proofs of Theorem 2.1 and Corollary 2.2 in the last subsection.

### 2.1 Upper bound estimate

We first estimate $\mathbb{W}_2(\mu_1, \mu_2)$ in terms of the energy for the difference of the density functions of $\mu_1$ and $\mu_2$ with respect to $\mu$. Let $\mathcal{D}(L)$ be the domain of the generator $L$ in $L^2(\mu)$, with Neumann boundary condition if $\partial M \neq \emptyset$. Then for any function $g \in L^2(\mu)_0 := \{ g \in L^2(\mu), \mu(g) = 0 \}$, we have

$$(2.7) \quad (-L)^{-1} g = \int_0^\infty P_s g \, ds \in \mathcal{D}(L), \quad L(-L)^{-1} g = g.$$  

Since $M$ is complete and $\mu$ is finite, we have $\mathcal{D}(L) \subset \mathcal{D}((-L)^{\frac{1}{2}}) = H^{1,2}(\mu) = W^{1,2}(\mu)$, where $H^{1,2}(\mu)$ is the completion of $C^\infty_0(M)$ under the Sobolev norm

$$\| f \|_{1,2} := \sqrt{\mu(f^2) + \mu(|\nabla f|^2)},$$

and $W^{1,2}(\mu)$ is the class of all weakly differentiable functions $f$ on $M$ such that $|f| + |\nabla f| \in L^2(\mu)$. In particular, $L^{-1} g \in W^{1,2}(\mu)$ for $g \in L^2(\mu)_0$. The following lemma is essentially due to [5, Proposition 2.3] where the case with compact $M$ and $V = 0$ is concerned, but its proof works also for the present setting.

**Lemma 2.3.** Let $f_0, f_1 \in L^2(\mu)$ be probability density functions with respect to $\mu$. Then

$$\mathbb{W}_2(f_0\mu, f_1\mu)^2 \leq \int_M \frac{\| \nabla L^{-1}(f_1 - f_0) \|^2}{\mathcal{M}(f_0, f_1)} \, d\mu,$$

where $\mathcal{M}(a, b) := \frac{a - b}{\log a - \log b}$ for $a, b > 0$, and $\mathcal{M}(a, b) := 0$ if one of $a$ and $b$ is zero.

**Proof.** Let $\text{Lip}_b(M)$ be the set of bounded Lipschitz continuous functions on $M$. Consider the Hamilton-Jacobi semigroup $(Q_t)_{t \geq 0}$ on $\text{Lip}_b(M)$:

$$Q_t \phi := \inf_{x \in M} \left\{ \phi(x) + \frac{1}{2t} \rho(x, \cdot)^2 \right\}, \quad t > 0, \phi \in \text{Lip}_b(M).$$

Then for any $\phi \in \text{Lip}_b(M)$, $Q_0 \phi := \lim_{t \downarrow 0} Q_t \phi = \phi$, $\| \nabla Q_t \phi \|_{\infty}$ is locally bounded in $t \geq 0$, and $Q_t \phi$ solves the Hamilton-Jacobi equation

$$d \frac{d}{dt} Q_t \phi = -\frac{1}{2} |\nabla Q_t \phi|^2, \quad t > 0.$$
In a more general setting of metric spaces, one has \( \frac{d}{dt}Q_t\phi \leq -\frac{1}{2}|\nabla Q_t\phi|^2 \) \( \mu \)-a.e., where the equality holds for length spaces which include the present framework, see e.g. [4, 5].

Letting \( \mu_i = \mu, i = 0, 1 \), the Kantorovich dual formula implies

\[
\frac{1}{2} \mathbb{W}_2(\mu_0, \mu_1)^2 = \sup_{\phi \in \text{Lip}_b(M)} \{ \mu_1(Q_1\phi) - \mu_0(\phi) \}.
\]

Let \( f_s = (1-s)f_0 + sf_1, s \in [0,1] \). By (2.3) and the boundedness of \( \|\nabla Q_t\phi\|_\infty \) in \( t \in [0,1] \), we deduce from (2.8) that

\[
\frac{d}{ds} \int_M f_sQ_s\phi d\mu = \int_M \left\{ -\frac{1}{2}|\nabla Q_s\phi|^2 f_s + (Q_s\phi)(f_1 - f_0) \right\} d\mu, \quad s \in (0,1].
\]

Moreover, (2.7) implies \( f := L^{-1}(f_0 - f_1) \in \mathcal{D}(L) \). Then by (2.10) and using the integration by parts formula, for any \( \phi \in \text{Lip}_b(M) \) we have

\[
\mu_1(Q_1\phi) - \mu_0(\phi) = \int_M \{ f_1Q_1\phi - f_0\phi \} d\mu = \int_0^1 \left( \frac{d}{ds} \int_M f_sQ_s\phi d\mu \right) ds
\]

\[
= \int_0^1 ds \int_M \left\{ -\frac{1}{2}|\nabla Q_s\phi|^2 f_s + (Q_s\phi)(f_1 - f_0) \right\} d\mu
\]

\[
= \int_0^1 ds \int_M \left\{ -\frac{1}{2}|\nabla Q_s\phi|^2 f_s - (Q_s\phi)Lf \right\} d\mu
\]

\[
= \int_0^1 ds \int_M \left\{ -\frac{1}{2}|\nabla Q_s\phi|^2 f_s + \langle \nabla f, \nabla Q_s\phi \rangle \right\} d\mu \leq \frac{1}{2} \int_0^1 ds \int_M \frac{\|\nabla f\|^2}{f_s} d\mu
\]

\[
= \frac{1}{2} \int_M |\nabla f|^2 d\mu \int_0^1 \frac{ds}{(1-s)f_0 + sf_1} = \frac{1}{2} \int_M \frac{\|\nabla f\|^2}{f_0} d\mu.
\]

Combining this with (2.9), we finish the proof. \( \square \)

To estimate \( \mathbb{W}_2(\mu_{t,r}, \mu)^2 \) using Lemma 2.3, we need to figure out the density function \( f_{t,r} \) of \( \mu_{t,r} \) with respect to \( \mu \), i.e. \( f_{t,r} \) is a nonnegative function such that \( \mu_{t,r}(A) = \int_A f_{t,r} d\mu \) for any measurable set \( A \subset M \). Obviously, letting \( p_t(x,y) \) be the heat kernel of \( P_t \) with respect to \( \mu \), i.e.

\[
P_tf(x) = \int_M p_t(x,y)f(y)\mu(dy), \quad t > 0, x \in M, f \in \mathcal{B}_b(M),
\]

we have

\[
f_{t,r} := \frac{1}{t} \int_0^t \mu_{t}(X_s) ds, \quad t > 0.
\]

On the other hand, the assumption (A1) implies

\[
\sup_{x,y \in M} p_t(x,y) = \|P_t\|_{1 \to \infty} < \infty, \quad t > 0,
\]

\[
p_t(x,y) = 1 + \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y), \quad t > 0, x, y \in M,
\]

}\]
where \( \{ \phi_i \}_{i \geq 1} \) are unit (Neumann if \( \partial M \neq \emptyset \)) eigenfunctions of \(-L\) with eigenvalues \( \{ \lambda_i \}_{i \geq 1} \). In particular, (2.13) implies

\[
(2.14) \quad f_{t,r}(y) - 1 = \frac{1}{\sqrt{t}} \sum_{i=1}^{\infty} e^{-\lambda r \psi_i(t)} \phi_i(y), \quad \psi_i(t) := \frac{1}{\sqrt{t}} \int_0^t \phi_i(X_s) ds, \quad i \in \mathbb{N}.
\]

The assumption \( \textbf{(A1)} \) also implies the following Poincaré inequality,

\[
(2.15) \quad \| P_t f \|_2 \leq e^{-\lambda_1 t} \| f \|_2, \quad t \geq 0, f \in L^2_0(\mu).
\]

Since \( P_t \) is contractive in \( L^p(\mu) \) for any \( p \in [1, \infty] \), (2.12) and (2.15) yield

\[
(2.16) \quad \| P_t f \|_p \leq c e^{-\lambda_1 t} \| f \|_p, \quad t \geq 0, p \in [1, \infty], f \in L^p_0(\mu)
\]

for some constant \( c > 0 \) independent of \( p \in [1, \infty] \).

By Lemma 2.3 with \( f_0 = 1 \) and \( f_1 = f_{t,r} \), where \( f_{t,r} \) is the density of \( \mu_{t,r} \) with respect to \( \mu \) given in (2.11), we have

\[
(2.17) \quad \mathbb{W}_2(\mu_{t,r}, \mu)^2 \leq \int_M \frac{\| \nabla L^{-1}(f_{t,r} - 1) \|^2}{\mathcal{M}(1, f_{t,r})} \, d\mu.
\]

Let

\[
(2.18) \quad \Xi_r(t) = t \int_M \| \nabla L^{-1}(f_{t,r} - 1) \|^2 \, d\mu, \quad t, r > 0.
\]

In the next two lemmas, we show that

\[
\lim_{t \to \infty} \left| \mathbb{E}^\nu \Xi_r(t) - \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} \right| = 0, \quad r > 0
\]

holds for \( \nu = h_\nu \mu \) with \( \| h_\nu \|_\infty < \infty \), and \( \mathcal{M}(1, f_{t,r}) \) is close to 1 for large \( t \), so that (2.17) implies the desired upper bound estimate (2.4) for \( \mathbb{E}^\nu \) replacing \( \mathbb{E}^x \).

**Lemma 2.4.** Assume \( \textbf{(A1)} \). There exists a constant \( c > 0 \) such that

\[
(2.19) \quad \left| \mathbb{E}^\nu \Xi_r(t) - \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} \right| \leq \frac{c \| h_\nu \|_\infty}{t} \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2 e^{2r \lambda_i}}, \quad t \geq 1, r > 0
\]

holds for any probability measure \( \nu = h_\nu \mu \), and

\[
(2.20) \quad \sup_{x \in M} \left| \mathbb{E}^x \Xi_r(t) - \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} \right| \leq \frac{c \| P_r/2 \|_{2 \to \infty}^2}{t} \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2 e^{r \lambda_i}}, \quad t \geq 1, r > 0.
\]
Proof. By (2.11) and (2.12), we have \(\mu(f_{t,r} - 1) = 0\) and \(\|f_{t,r}\|_\infty \leq \|P_r\|_{1 \to \infty} < \infty\). Consequently, (2.7) implies \((-L)^{-1}(f_{t,r} - 1) \in \mathcal{D}(L)\). Then the integration by parts formula and the symmetry of \(P_s\) in \(L^2(\mu)\) yield

\[
\int_M |\nabla L^{-1}(f_{t,r} - 1)|^2 d\mu = -\mathbb{E}^\nu \int_M \{L^{-1}(f_{t,r} - 1)\} \cdot L\{L^{-1}(f_{t,r} - 1)\} d\mu
\]

(2.21)

\[
= \int_M \{(-L)^{-1}(f_{t,r} - 1)\}(f_{t,r} - 1) d\mu = \int_0^\infty ds \int_M (f_{t,r} - 1) P_s(f_{t,r} - 1) d\mu
\]

\[
= \int_0^\infty ds \int_M |P_{\frac{s}{2}} f_{t,r} - 1|^2 d\mu.
\]

By (2.14), \(P_s \phi_i = e^{-\lambda_i s} \phi_i\) and \(\mu(\phi_i \phi_j) = 1_{i=j}\), we obtain

(2.22)

\[
t \int_M |P_{\frac{s}{2}} f_{t,r} - 1|^2 d\mu = \sum_{i=1}^\infty e^{-\lambda_i(2r+s)} |\psi_i(t)|^2.
\]

Combining this with (2.21) we get

(2.23)

\[
\Xi_r(t) = \sum_{i=1}^\infty \frac{|\psi_i(t)|^2}{\lambda_i e^{2\lambda_i r}}, \quad t, r > 0.
\]

Moreover, the Markov property and \(P_s \phi_i = e^{-\lambda_i s} \phi_i\) imply

\[
\mathbb{E}^\nu(\phi_i(X_{s_2})|\mathcal{F}_{s_1}) = P_{s_2-s_1} \phi_i(X_{s_1}) = e^{-\lambda_i(s_2-s_1)} \phi_i(X_{s_1}), \quad s_2 \geq s_1 \geq 0,
\]

so that

(2.24)

\[
\mathbb{E}^\nu|\psi_i(t)|^2 = \frac{1}{t} \mathbb{E}^\nu \left| \int_0^t \phi_i(X_s) ds \right|^2 = \frac{2}{t} \int_0^t ds_1 \int_{s_1}^t \mathbb{E}^\nu(\phi_i(X_{s_1})\phi_i(X_{s_2})) ds_2
\]

\[
= \frac{2}{t} \int_0^t \mathbb{E}^\nu|\phi_i(X_{s_1})|^2 ds_1 \int_{s_1}^t e^{-\lambda_i(s_2-s_1)} ds_2 = \frac{2}{\lambda_i} \int_0^t \nu(P_s \phi_i^2)(1 - e^{-\lambda_i(t-s)}) ds.
\]

Combining (2.23) and (2.24) gives

(2.25)

\[
\mathbb{E}^\nu \Xi_r(t) = \frac{2}{t} \sum_{i=1}^\infty \frac{e^{-2\lambda_i r}}{\lambda_i^2} \int_0^t \nu(P_s \phi_i^2)(1 - e^{-\lambda_i(t-s)}) ds = I_1 + I_2,
\]

where

(2.26)

\[
I_1 := \frac{2}{t} \sum_{i=1}^\infty \int_0^t \frac{1 - e^{-(t-s)\lambda_i}}{\lambda_i^2 e^{2\lambda_i r}} ds = \sum_{i=1}^\infty \frac{2}{\lambda_i^2 e^{2\lambda_i r}} - \frac{2}{t} \sum_{i=1}^\infty \frac{1 - e^{-\lambda_i t}}{\lambda_i^3 e^{2\lambda_i r}},
\]

and due to \(\nu(P_s \phi_i^2) = \mu(h_\nu P_s \phi_i^2) = \mu(\phi_i^2 P_s h_\nu)\),

(2.27)

\[
I_2 := \frac{2}{t} \sum_{i=1}^\infty \int_0^t \frac{1 - e^{-(t-s)\lambda_i}}{\lambda_i^2 e^{2\lambda_i r}} \mu(\phi_i^2 P_s h_\nu - 1) ds.
\]
Since $\mu(\phi_i^2) = 1$, by (2.16) we find a constant $c_1 > 0$ such that
\[
|\mu(\phi_i^2 P_s h_\nu - 1)| = |\mu((P_s h_\nu - 1)\phi_i^2)| \leq \|P_s (h_\nu - 1)\|_\infty \leq c_1 e^{-\lambda_1 s} \|h_\nu\|_\infty, \quad s \geq 0.
\]
Thus, there exists a constant $c_2 > 0$ such that
\[
|I_2| \leq \frac{c_2}{t} \|h_\nu\|_\infty \sum_{i=1}^\infty \frac{1}{\lambda_i^2 e^{2\lambda_i t}} < \infty.
\]
Combining this with (2.25) and (2.26), and noting that $t \geq 1$ and $\|h_\nu\|_\infty \geq 1$, we prove (2.19) for some constant $c > 0$.

Next, when $\nu = \delta_x$ (2.25) becomes
\[
(2.28) \quad \mathbb{E}^\nu_\Xi(t) \leq I_1 + I_2(x),
\]
where $I_1$ is in (2.26), and due to $\mu(\phi_i^2) = 1$ and $P_{r/2} \phi_i = e^{-r\lambda_i/2} \phi_i$,
\[
I_2(x) := \frac{2}{t} \sum_{i=1}^\infty \int_0^t \frac{1 - e^{-(t-s)\lambda_i}}{\lambda_i^2 e^{2\lambda_i t}} P_s \{\phi_i^2(x) - 1\} \, ds
\leq \frac{2}{t} \sum_{i=1}^\infty \int_0^t \frac{1}{\lambda_i^2 e^{\lambda_i t}} \|P_s (P_{r/2} \phi_i)^2(x) - \mu((P_{r/2} \phi_i)^2)\| \, ds.
\]
By (2.16) and noting that $\|P_s \phi_i\|_\infty \leq \|P_s\|_{2 \to \infty}$, we find a constant $c_3 > 0$ such that
\[
\sup_{x \in M} I_2(x) \leq \frac{c_3}{t} \sum_{i=1}^\infty \int_0^t \frac{1}{\lambda_i^2 e^{\lambda_i t}} \|P_{r/2} \phi_i\|_2^2 \|P_{r/2} \phi_i\|_\infty e^{-\lambda_1 s} \, ds
\leq \frac{c_3}{t} \|P_{r/2}\|_{2 \to \infty} \sum_{i=1}^\infty \frac{1}{\lambda_i^2 e^{\lambda_i t}} \int_0^t e^{-\lambda_1 s} \, ds
\leq \frac{c_3}{t} \|P_{r/2}\|_{2 \to \infty} \sum_{i=1}^\infty \frac{1}{\lambda_i^2 e^{\lambda_i t}}.
\]
Combining this with (2.28) and (2.26), we prove (2.20) for some constant $c > 0$. \hfill \square

The following lemma is similar to [27, Proposition 2.6], which ensures that $\mathcal{M}(1, f_{t,r}) \to 1$ as $t \to \infty$.

**Lemma 2.5.** Assume (A1). Let $\|f_{t,r} - 1\|_\infty = \sup_{y \in M} |f_{t,r}(y) - 1|$. Then there exists a function $c : \mathbb{N} \times (0, \infty) \to (0, \infty)$ such that
\[
\sup_{x \in M} \mathbb{E}^x \|f_{t,r} - 1\|_\infty^{2k} \leq c(k, r) t^{-k}, \quad t \geq 1, r > 0, k \in \mathbb{N}.
\]

**Proof.** For fixed $r > 0$ and $y \in M$, let $f = p_r(\cdot, y) - 1$. For any $k \in \mathbb{N}$, consider
\[
I_k(s) := \mathbb{E}^y \left| \int_0^s f(X_t) \, dt \right|^{2k} = (2k)! \mathbb{E}^y \int_{\Delta_k(s)} f(X_{s_1}) \cdots f(X_{s_{2k}}) ds_1 \cdots ds_{2k}, \quad s > 0,
\]
where $\Delta_k(s) := \{(s_1, \ldots, s_{2k}) \in [0, s] : 0 \leq s_1 \leq s_2 \leq \cdots \leq s_{2k} \leq s\}$. By the Markov property, we have

$$\mathbb{E}^\nu(f(X_{s_{2k}})|X_t, t \leq s_{2k-1}) = (P_{s_{2k}-s_{2k-1}}f)(X_{s_{2k-1}}).$$

So, letting $g(r_1, r_2) = (fP_{r_2-r_1}f)(X_{r_1})$ for $r_2 \geq r_1 \geq 0$, we obtain

$$I_k(s) = (2k)!\mathbb{E}^\nu\left[\int_0^s f(X_s)ds_1 \int_s^r f(X_{s_2})ds_2 \cdots \int_{s_{2k-2}}^r ds_{2k-1} \int_{s_{2k-1}}^r g(s_{2k-1}, s_{2k})ds_{2k}\right].$$

By the Fubini formula, we may rewrite $I_k(s)$ as

$$I_k(s) = (2k)!\mathbb{E}^\nu\left[\int_{\Delta_k(s)} g(r_1, r_2)dr_1dr_2 \int_{\Delta_{k-1}(r_1)} f(X_{s_1}) \cdots f(X_{s_{2k-2}})ds_1 \cdots ds_{2k-2}\right] = \frac{(2k)!}{(2k - 2)!} \int_{\Delta_k(s)} \mathbb{E}^\nu\left[g(r_1, r_2) \left| \int_0^{r_1} f(X_r)dr \right|^{2k-2}\right] dr_1dr_2.$$

Using Hölder’s inequality, we derive

$$I_k(s) \leq 2k(2k - 1) \int_{\Delta_k(s)} \left(\mathbb{E}^\nu|g(r_1, r_2)|^k\right)^{\frac{1}{k}} \left(\mathbb{E}^\nu\left|\int_0^{r_1} f(X_r)dr\right|^{2k}\right)^{\frac{k-1}{k}} dr_1dr_2 \leq 2k(2k - 1) \left(\sup_{u \in [0,s]} I_k(u)\right)^{\frac{k-1}{k}} \int_{\Delta_k(s)} \left(\mathbb{E}^\nu|g(r_1, r_2)|^k\right)^{\frac{1}{k}} dr_1dr_2.$$

Thus,

$$\sup_{s \in [0,t]} I_k(s) \leq 2k(2k - 1) \left(\sup_{s \in [0,t]} I_k(s)\right)^{\frac{k-1}{k}} \int_{\Delta_k(t)} \left(\mathbb{E}^\nu|g(r_1, r_2)|^k\right)^{\frac{1}{k}} dr_1dr_2, \quad t > 0.$$

Since $I_k(t) \leq (\|f\|_\infty t)^{2k} < \infty$, this implies

$$(2.29) \quad I_k(t) \leq \sup_{s \in [0,t]} I_k(s) \leq \left(2k(2k - 1)\right)^k \left(\int_{\Delta_k(t)} \left(\mathbb{E}^\nu|g(r_1, r_2)|^k\right)^{\frac{1}{k}} dr_1dr_2\right)^k.$$

Recalling that $g(r_1, r_2) = (fP_{r_2-r_1}f)(X_{r_1})$ and

$$\|f\|_\infty = \|p_x(\cdot, y) - 1\|_\infty \leq 2\|P_y\|_{1\rightarrow\infty} < \infty,$$

by (2.16) we obtain

$$|g(r_1, r_2)|^k \leq \|fP_{r_2-r_1}f\|^k \leq ce^{-\lambda_1(r_2-r_1)}k\|f\|_\infty^{2k} \leq c\|P_y\|_{1\rightarrow\infty}^{2k}e^{-\lambda_1(r_2-r_1)}k$$

for some constant $c > 0$. Thus,

$$\left(\int_{\Delta_k(t)} \left(\mathbb{E}^\nu|g(r_1, r_2)|^k\right)^{\frac{1}{k}} dr_1dr_2\right)^k.$$
So, we deduce from Lemma 2.3 and (2.20) that for some constant \( c \),

\[
\text{sup}_{x,y \in M} \mathbb{E}^x \left[ \| f_{t,r}(y) - 1 \|^{2k} \right] = t^{-2k} I_k(t) \leq c(k, r) \| P_r \|_{1 \to \infty}^{2k} t^{-k}, \quad t \geq 1, r > 0.
\]

for all \( k \in \mathbb{N} \) and some constant \( c(k) > 0 \).

Finally, noting that \( f_{t,r} = P_{r/2} f_{t,r/2} \), we deduce from (2.30) that

\[
\text{sup}_{x \in M} \mathbb{E}^x \left[ \| f_{t,r} - 1 \|^{2k} \right] = \text{sup}_{x \in M} \mathbb{E}^x \left[ \| P_{r/2} f_{t,r/2} - 1 \|^{2k} \right] \leq \| P_{r/2} \|_{2k \to \infty} \text{sup}_{x \in M} \mathbb{E}^x \left[ \mu | f_{t,r/2} - 1 |^{2k} \right] \leq c(k) \| P_{r/2} \|_{1 \to \infty}^{4k} t^{-k}, \quad t \geq 1, r > 0.
\]

This finishes the proof. \( \square \)

We are now ready to prove the upper bound estimate (2.4) in Theorem 2.1.

**Proposition 2.6.** The assumption (A1) implies (2.4).

**Proof.** (a) Proof of (2.4). By (2.13), (2.12) and \( \mu(\phi_1^2) = 1 \), we have

\[
\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2 e^{2r \lambda_i}} \leq \frac{1}{\lambda_1^2} \sum_{i=1}^{\infty} e^{-2r \lambda_i} = \frac{1}{\lambda_1^2} \int_M (p_{2r}(x,x) - 1) \mu(dx) \leq \frac{\| P_{2r} \|_{1 \to \infty}}{\lambda_1^2} < \infty.
\]

So, it remains to prove the first inequality in (2.4).

For any \( \eta \in (0,1) \), consider the event

\[
A_\eta = \left\{ \| f_{t,r} - 1 \|_\infty \leq \eta \right\}.
\]

Noting that \( f_{t,r}(y) \geq 1 - \eta \) implies

\[
\mathcal{M}(1, f_{t,r}(y)) \geq \sqrt{f_{t,r}(y)} \geq \sqrt{1 - \eta},
\]

we deduce from Lemma 2.3 and (2.20) that for some constant \( c(r) > 0 \),

\[
t \sup_{x \in M} \mathbb{E}^x \left[ \mathbb{W}_2(\mu_{t,r}, \mu) \right] \leq \sup_{x \in M} \mathbb{E}^x \left\{ \mathbb{E}^x \left[ \frac{1}{\sqrt{1 - \eta}} \right] \right\} \\
\leq \frac{1}{\sqrt{1 - \eta}} \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} \left( 1 + \frac{c(r)}{t} \right), \quad t > 0, \eta \in (0,1).
\]

So,

\[
t \sup_{x \in M} \mathbb{E}^x \left[ \mathbb{W}_2(\mu_{t,r}, \mu) \right] \leq \frac{1}{\sqrt{1 - \eta}} \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} \left( 1 + \frac{c(r)}{t} \right) + t \sup_{x \in M} \mathbb{E}^x \left[ A_\eta \mathbb{W}_2(\mu_{t,r}, \mu)^2 \right] \\
\leq \frac{1 + c(r)t^{-1}}{\sqrt{1 - \eta}} \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}} + t \sup_{x \in M} \sqrt{\mathbb{P}(A_\eta) \mathbb{E}^x \left[ \mathbb{W}_2(\mu_{t,r}, \mu)^4 \right]}, \quad t, \eta \in (0,1).
\]
By Jensen’s inequality and (2.3), we obtain

\[
\begin{align*}
\mathbb{E}^x \mathbb{W}_2(\mu_{t,r}, \mu)^4 & \leq \mathbb{E}^x \left( \int_{M \times M} \rho(z, y)^2 \mu_{t,r}(dz) \mu(dy) \right)^2 \\
& \leq \mathbb{E}^x \int_{M \times M} \rho(z, y)^4 \mu_{t,r}(dz) \mu(dy) \leq \frac{1}{t} \int_0^t \mathbb{E}^x \mu(X_{r+s, \cdot})^4 ds \\
& \leq \frac{1}{t} \int_0^t \|P_{s+r}\|_{1 \rightarrow \infty}^4 (\mu \times \mu)(\rho^4) ds \leq \|P_r\|_{1 \rightarrow \infty}^4 (\mu \times \mu)(\rho^4) < \infty.
\end{align*}
\]

(2.33)

Moreover, Lemma 2.5 implies

\[
(2.34) \quad \text{sup}_{x \in M} \mathbb{E}^x(A_n^c) \leq \eta^{-2k} c(k, r) t^{-k}, \quad t \geq 1, k \in \mathbb{N}, \eta \in (0, 1)
\]

for some constant \( c(k, r) > 0 \). By taking \( k = 4 \) in (2.34) and applying (2.32) and (2.33), we conclude that

\[
\limsup_{t \to \infty} \left\{ t \sup_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}, \mu)^2] \right\} \leq \frac{1}{\sqrt{1 - \eta}} \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}}, \quad \eta \in (0, 1).
\]

By letting \( \eta \downarrow 0 \), we derive (2.4).

\[\square\]

2.2 Lower bound estimate

Due to (2.4), (2.5) follows from the lower bound estimate

\[
(2.35) \quad \liminf_{t \to \infty} \left\{ t \inf_{x \in M} \mathbb{E}^x [\mathbb{W}_2(\mu_{t,r}, \mu)^2] \right\} \geq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}}, \quad r > 0.
\]

To estimate \( \mathbb{W}_2(\mu_{t,r}, \mu) \) from below, we use the fact that

\[
(2.36) \quad \frac{1}{2} \mathbb{W}_2(\mu_{t,r}, \mu)^2 \geq \mu_{t,r}(\phi_1) - \mu(\phi_0), \quad (\phi_0, \phi_1) \in \mathcal{C},
\]

\[\mathcal{C} := \left\{ (\phi_0, \phi_1) : \phi_0, \phi_1 \in C_b(M), \phi_1(x) - \phi_0(y) \leq \frac{1}{2} \rho(x, y)^2 \text{ for } x, y \in M \right\}.
\]

We will construct the pair \((\phi_0, \phi_1)\) by using the idea of [5], where compact \( M \) without boundary has been considered. To realize the idea in the present more general setting, we need the following result on gradient estimate which is implied by [33, Corollary 1.2] for \( Z = \nabla V \).

Lemma 2.7 ([33]). If there exists \( \phi \in C^2_b(M) \) such that \( \inf \phi = 1, |\nabla \phi| \cdot |\nabla V| \) is bounded, \( \nabla \phi \parallel N \) (i.e. \( \nabla \phi \) is parallel to \( N \)) and \( I \geq -N \log \phi \) hold on \( \partial M \), and

\[
\text{Ric}_V - \frac{1}{2} \phi^2 L \phi^{-2} \geq -K_\phi
\]

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holds for some constant $K_{\phi} \geq 0$. Then

$$|\nabla P_t f|^2 \leq \frac{e^{2K_{\phi} t}}{\phi^2} P_t(\phi |\nabla f|)^2, \quad t \geq 0, f \in C^1_b(M),$$

$$|\nabla P_t f|^2 \leq \frac{\|\phi\|_{\infty} K_{\phi}}{e^{2K_{\phi} t} - 1} \{P_t(f^2) - (P_t f)^2\}, \quad t > 0, f \in \mathcal{B}_b(M),$$

$$P_t(f^2) \leq (P_t f)^2 + \frac{\|\phi\|_{\infty} (e^{2K_{\phi} t} - 1)}{K_{\phi}} P_t|\nabla f|^2, \quad t > 0, f \in C^1_b(M).$$

As a consequence of Lemma 2.7, we have the following result.

**Lemma 2.8.** Assume (A2). There exists a constant $\kappa > 0$ such that

$$|\nabla P_t f|^2 \leq (1 + \kappa \sqrt{t}) P_t|\nabla f|^2, \quad t \in [0, 1], f \in C^1_b(M),$$

$$|\nabla P_t f|^2 \leq \frac{K}{t} P_t(f^2), \quad t \in (0, 1], f \in \mathcal{B}_b(M),$$

$$P_t(f^2) \leq (P_t f)^2 + \kappa t P_t|\nabla f|^2, \quad t \in (0, 1], f \in C^1_b(M).$$

Consequently,

$$P_t e^{t f} \leq 8(P_t e^f)^4, \quad t \in (0, 1], \|\nabla f\|^2_\infty \leq \frac{1}{8\kappa t}.$$

**Proof.** Let $\text{Ric}_V \geq -K$ for some constant $K \geq 0$. If $\partial M$ is empty or convex, we may take $\phi = 1$ and $K_{\phi} = K$ in Lemma 2.7. Then (2.37)-(2.39) follow from estimates in Lemma 2.7.

If $\partial M \neq \emptyset$ and there exists a compact set $D$ such that $\partial M$ is convex outside $D$, we make use of Lemma 2.7. To this end, we construct a function $g \in C^\infty(\overline{M})$ such that $0 \leq g \leq 1$, $Ng|_{\partial M} = 0$, and $g = 1$ on the compact set $D$. Let $D'$ be the support of $g$. Since the distance $\rho_{\theta}$ to the boundary is smooth in a neighborhood of $\partial M$, we may take a constant $r_0 \in (0, 1)$ such that $\rho_{\theta}$ is smooth on $D' \cap \partial_{r_0}M$, where $\partial_{r_0}M := \{\rho_{\theta} \leq r_0\} \subset M$. Moreover, since $I_{\partial M}$ is nonnegative on $\partial M \setminus D$, there exists a constant $\kappa > 0$ such that $\|I_{\partial M}\| \geq -\kappa$. We choose $h \in C^\infty([0, \infty))$ such that $h$ is increasing, $h(r) = r$ for $r \in [0, \frac{r_0}{2}]$ and $h(r) = h(r_0)$ for $r \geq r_0$. For any $\varepsilon \in (0, 1)$, take

$$\phi = 1 + \kappa \varepsilon h(\varepsilon^{-1} \rho_{\theta}).$$

It is easy to see that $\inf \phi = 1, \nabla \phi \|N$ and $I \geq -N \log \phi$ hold on $\partial M$ as required by Lemma 2.7. Next, since $\phi \geq 1$ and $\nabla \phi = 0$ outside the compact set $D'$, there exists a constant $c_1 > 0$ such that

$$\frac{1}{2} \sup_M \{\phi^2 L\phi^{-2}\} = \sup_{D'} \{3\phi^{-2} |\nabla \phi|^2 - \phi^{-1} L\phi\} \leq c_1 \varepsilon^{-1}, \quad \varepsilon \in (0, 1).$$

Combining this with (2.1), we obtain

$$\text{Ric}_V - \frac{1}{2} \phi^2 L\phi^{-2} \geq -K - c_1 \varepsilon^{-1} \geq -c_2 \varepsilon^{-1}, \quad \varepsilon \in (0, 1).$$
for some constant $c_2 > 0$. Then the second and third estimates follow from (2.38) and (2.39), while (2.37) implies

$$
|\nabla P_t f|^2 \leq \frac{e^{2c_2 \varepsilon^{-1}t}}{\phi^2} P_t (\phi|\nabla f|)^2 \leq e^{2c_2 \varepsilon^{-1}t} \|\phi\|_\infty^2 P_t |\nabla f|^2 \leq e^{2c_2 \varepsilon^{-1}t} (1 + \kappa \|h\|_\infty \varepsilon)^2 P_t |\nabla f|^2, \quad t, \varepsilon \in (0, 1).
$$

Taking $\varepsilon = \sqrt{t}$, we prove the first estimate for some constant $c > 0$.

It remains to prove (2.40). By (2.39), we have

$$
P_t e^{2f} \leq (P_t e^{f})^2 + \kappa t P_t (|\nabla f|^2 e^{2f}) \leq (P_t e^{f})^2 + \kappa t \|\nabla f\|_\infty^2 P_t (e^{2f}).
$$

This implies

$$
P_t e^{2f} \leq 2(P_t e^{f})^2, \quad \text{if} \quad \|\nabla f\|_\infty^2 \leq \frac{1}{2\kappa t}.
$$

Using $2f$ replacing $f$ we obtain

$$
P_t e^{4f} \leq 2(P_t e^{2f})^2, \quad \text{if} \quad \|\nabla f\|_\infty^2 \leq \frac{1}{8\kappa t}.
$$

Therefore, (2.40) holds. \(\square\)

We are now ready to present the following key lemma for the lower bound estimate of $\mathcal{W}_2(\mu, t, \mu)$.

**Lemma 2.9.** Assume (A1) and (A2). For any $f \in C^2_b(M)$ with $\|\nabla f\|_\infty + \|L f\|_\infty < \infty$ and $N f|_{\partial M} = 0$ if $\partial M \neq 0$, let $\phi_t^\sigma = -\sigma \log P_t e^{-\sigma^{-1}f}$, $t \in [0, 1]$, $\sigma > 0$. Then $\phi_t^\sigma \in C^2(M)$ and

1. $\phi_0^\sigma = f$, $\|\phi_t^\sigma\|_\infty \leq \|f\|_\infty$, and $\partial_t \phi_t^\sigma = \frac{\sigma}{2} L \phi_t^\sigma - \frac{1}{2} |\nabla \phi_t^\sigma|^2$, $t > 0$;

2. There exist constants $c, \gamma > 0$ such that for any $\sigma, t \in (0, 1)$, when $\|\nabla f\|_\infty^2 \leq \gamma \sigma$ we have

$$
\phi_t^\sigma(y) - \phi_0^\sigma(x) \leq \frac{1}{2} \left\{ \rho(x,y)^2 + \sigma \|(L f)^+\|_\infty + c \sigma^{\frac{1}{2}} \|\nabla f\|_\infty^2 \right\},
$$

$$
\int_M (\phi_0^\sigma - \phi_t^\sigma) d\mu \leq \frac{1}{2} \int_M |\nabla f|^2 d\mu + c \sigma^{-1} \|\nabla f\|_\infty^4.
$$

**Proof.** (1) The first assertion follows from standard calculations. Indeed, by the chain rule and the heat equation $\partial_t P_t g = LP_t g$ for $t > 0$ and $g \in C_b(M)$, we have

$$
\partial_t \phi_t^\sigma = -\sigma^2 L P_t e^{-\sigma^{-1}f} \frac{\sigma^2}{2 P_t e^{-\sigma^{-1}f}} = \frac{\sigma}{2} L \phi_t^\sigma - \frac{1}{2} |\nabla \phi_t^\sigma|^2.
$$

(2) Let $\sigma, t \in (0, 1]$ and $\|\nabla f\|_\infty^2 \leq \gamma \sigma$ for $\gamma = \frac{1}{4\kappa}$, where $\kappa > 0$ is in Lemma 2.8. Then $\|\sigma^{-1} \nabla f\|_\infty^2 \leq \frac{1}{4 \kappa^2 t} \leq \frac{1}{8 \kappa^2 t}$ for $t' = \frac{t}{2}$, so that (2.40) holds for $(t', -\sigma^{-1} f)$ replacing $(t, f)$. This implies

$$
P_{t'} e^{-4\sigma^{-1}f} \leq 8(P_{t'} e^{-\sigma^{-1}f})^4 \quad (2.41)
$$
Moreover, by Lemma 2.2 in [31], there exists a constant $K > 0$ such that

$$\left| \nabla \phi^2 \right|^2 = \frac{\left| \nabla P_{t_x} e^{-\sigma^2 f} \right|^2}{(P_{t_x} e^{-\sigma^2 f})^2} \leq \frac{(1 + \kappa) P_{t_x} \left( \left| \nabla f \right|^2 e^{-2\sigma^2 f} \right)}{(P_{t_x} e^{-\sigma^2 f})^2} \leq c \left| \nabla f \right|^2_\infty$$

for some constant $c > 0$.

Next, by (2.1) in [29], for $g \in C_x^1(M)$,

$$\left| \nabla P_t g \right|(x) \leq E_x \left[ \left| \nabla g \right| (X_t) e^{Kt + \delta t} \right], \quad x \in M, \quad t > 0,$$

where $K \geq 0$ is the constant such that $\text{Ric}_{\mathcal{V}} \geq -K$ holds on $M$, $\delta > 0$ is the constant such that $\mathbb{I}_{\partial M} \geq -\delta$ and $l_t$ is the local time of the $L$-process on the boundary $\partial M$. Combining this with (2.41),

$$LP_{t_x} e^{-\sigma^2 f} = P_{t_x} e^{-\sigma^2 f} = \frac{1}{\sigma} P_{t_x} (e^{-\sigma^2 f} L f) + \frac{1}{\sigma^2} P_{t_x} \left( \left| \nabla f \right|^2 e^{-\sigma^2 f} \right),$$

and applying Hölder’s inequality, we find a constant $c_0 > 0$ such that

$$L \phi_t^\sigma = -\frac{\sigma LP_{t_x} e^{-\sigma^2 f}}{P_{t_x} e^{-\sigma^2 f}} + \frac{\sigma \left| \nabla P_{t_x} e^{-\sigma^2 f} \right|^2}{(P_{t_x} e^{-\sigma^2 f})^2} \leq \left| (L f)^+ \right|_\infty - \frac{P_{t_x} \left( \left| \nabla f \right|^2 e^{-\sigma^2 f} \right)}{\sigma P_{t_x} e^{-\sigma^2 f}} + \frac{E \left[ \left( e^{-\sigma^2 f} \left| \nabla f \right| (X_{t_x}) e^{Kt + \delta l_t^x} \right)^2 \right]}{\sigma (P_{t_x} e^{-\sigma^2 f})^2}$$

$$\leq \left| (L f)^+ \right|_\infty - \frac{P_{t_x} \left( \left| \nabla f \right|^2 e^{-\sigma^2 f} \right)}{\sigma P_{t_x} e^{-\sigma^2 f}} + \frac{P_{t_x} \left( \left| \nabla f \right|^2 e^{-\sigma^2 f} \right) E \left[ e^{-\sigma^2 f} (X_{t_x}) e^{Kt + 2\delta l_t^x} \right]}{\sigma (P_{t_x} e^{-\sigma^2 f})^2}$$

$$\leq \left| (L f)^+ \right|_\infty + \frac{P_{t_x} \left( \left| \nabla f \right|^2 e^{-\sigma^2 f} \right) E \left[ e^{-\sigma^2 f} (X_{t_x}) \left( e^{Kt + 2\delta l_t^x} - 1 \right) \right]}{\sigma (P_{t_x} e^{-\sigma^2 f})^2}$$

$$\leq \left| (L f)^+ \right|_\infty + c_0 \sigma^{-1} \left| \nabla f \right|^2_\infty \left( E \left[ \left( e^{Kt + 2\delta l_t^x} - 1 \right)^{\frac{3}{2}} \right] \right)^{\frac{1}{3}}.$$

By the proof of Lemma 2.1 in [29], for any $\lambda > 0$ there exists a constant $c > 0$ such that

$$E e^{\lambda l_t} \leq c(\lambda), \quad t \in (0, 1].$$

Moreover, by Lemma 2.2 in [31], there exists a constant $c_1 > 0$ such that

$$E l_t^2 \leq c_1 t, \quad t \in (0, 1].$$

Combining these facts, we find a constant $c = c(K, \delta) > 0$ such that

$$E \left[ \left( e^{Kt + 2\delta l_t^x} - 1 \right)^{\frac{3}{2}} \right] \leq E \left[ (Kt + 2\delta l_t^x)^{\frac{3}{2}} e^{\frac{3}{2} Kt + Kt + \delta l_t^x} \right]$$

$$\leq \left( E \left[ (Kt + 2\delta l_t^x)^2 \right] \right)^{\frac{3}{4}} \left( E e^{4Kt + 8\delta l_t^x} \right)^{\frac{1}{4}} \leq c \sigma^2, \quad \sigma, t \in (0, 1].$$
It then follows from (2.44) and (2.45) that
\[
L\phi^\sigma_t \leq \|(Lf)^+\|_\infty + c_2 \sigma^{-\frac 12} \|\nabla f\|_\infty^2, \quad \sigma, t \in (0, 1], \|\nabla f\|_\infty^2 \leq \gamma \sigma
\]
holds for some constant \(c_2 > 0\).

Now, for any two points \(x, y \in M\), let \(\gamma : [0, 1] \to M\) be the minimal geodesic from \(x\) to \(y\), so that \(|\gamma_t| = \rho(x, y)\). By (1) and (2.46), we derive
\[
\frac{d}{dt}\phi^\sigma_t(\gamma_t) = (\partial_t\phi^\sigma_t)(\gamma_t) + \langle \nabla \phi^\sigma_t(\gamma_t), \dot{\gamma}_t \rangle \\
= -\frac{1}{2} |\nabla \phi^\sigma_t(\gamma_t)|^2 + \frac{\sigma}{2} L\phi^\sigma_t(\gamma_t) + \langle \nabla \phi^\sigma_t(\gamma_t), \dot{\gamma}_t \rangle \\
\leq \frac{1}{2} |\gamma_t|^2 + \frac{\sigma}{2} \|(Lf)^+\|_\infty + c\sqrt{\sigma} \|\nabla f\|_\infty^2 \\
= \frac{1}{2} \rho(x, y)^2 + \frac{\sigma}{2} \|(Lf)^+\|_\infty + c\sqrt{\sigma} \|\nabla f\|_\infty^2, \quad \sigma, t \in [0, 1], \|\nabla f\|_\infty^2 \leq \gamma \sigma
\]
for some constant \(c > 0\). Integrating over \(t \in [0, 1]\) and noting that \(\phi^\sigma_0(x) = f(x)\), we derive the first inequality in (2).

On the other hand, since \(\phi^\sigma_t \in C^2(M)\) with \(N\phi^\sigma_t|_{\partial M} = 0\) and bounded \(|\nabla \phi^\sigma_t| + |L\phi^\sigma_t|\), we have \(\mu(L\phi^\sigma_t) = 0\) so that assertion (1) yields
\[
\mu(f - \phi^\sigma_0) = \int_M (\phi^\sigma_0 - \phi^\sigma_t) d\mu = -\int_M d\mu \int_0^1 (\partial_t\phi^\sigma_t) dt \\
= \int_0^t dt \int_M \left\{ \frac{1}{2} |\nabla \phi^\sigma_t|^2 - \frac{\sigma}{2} L\phi^\sigma_t \right\} d\mu = \frac{1}{2} \int_0^1 \mu(|\nabla \phi^\sigma_t|^2) dt.
\]
Since \(\phi^\sigma \in C^2((0, \infty) \times M)\) with \(N\phi^\sigma_s|_{\partial M} = 0\) for \(s > 0\), we have
\[
N\partial_s \phi^\sigma_s|_{\partial M} = \partial_s N\phi^\sigma_s|_{\partial M} = 0.
\]
Combining this with assertion (1) and applying the integration by parts formula, we obtain
\[
\frac{d}{ds} \mu(|\nabla \phi^\sigma_s|^2) = -\frac{d}{ds} \int_M \phi^\sigma_s L\phi^\sigma_s d\mu = -\int_M (L\phi^\sigma_s) \partial_s \phi^\sigma_s d\mu - \int_M \phi^\sigma_s L(\partial_s \phi^\sigma_s) d\mu \\
= -2 \int_M (L\phi^\sigma_s) \partial_s \phi^\sigma_s d\mu = -2 \int_M (L\phi^\sigma_s) \left( \frac{\sigma}{2} \phi^\sigma_s - \frac{1}{2} |\nabla \phi^\sigma_s|^2 \right) d\mu, \quad s > 0.
\]
This and (2.42) imply
\[
\mu(|\nabla \phi^\sigma_t|^2) - \mu(|\nabla f|^2) = \int_0^t \left\{ \frac{d}{ds} \mu(|\nabla \phi^\sigma_s|^2) \right\} ds \\
\leq -2 \int_0^t ds \int_M (L\phi^\sigma_s) \left( \frac{\sigma}{2} \phi^\sigma_s - \frac{1}{2} |\nabla \phi^\sigma_s|^2 \right) d\mu \\
\leq \frac{1}{4\sigma} \int_0^t \mu(|\nabla \phi^\sigma_s|^4) ds \leq c\sigma^{-1} \|\nabla f\|_\infty^4, \quad \sigma, t \in [0, 1], \|\nabla f\|_\infty^2 \leq \gamma \sigma
\]
for some constant \(c > 0\). Substituting this into (2.47), we prove the second estimate in assertion (2).
We are now ready to prove the estimate (2.35).

**Proposition 2.10.** Assumptions (A1) and (A2) imply (2.35).

**Proof.** Let \( f = L^{-1}(f_{t,r} - 1) \), and denote

\[
C_1(f, \sigma) := \sigma^{-1} \| \nabla f \|_{\infty}^4,
\]

\[
C_2(f, \sigma) := \sigma \| f_{t,r} - 1 \|_{\infty} + c\sigma^\frac{3}{2} \| \nabla f \|_{\infty}^2,
\]

where \( c > 0 \) is the constant in Lemma 2.9(2). Then

\[
(2.49) \quad \| Lf \|_{\infty} = \| f_{t,r} - 1 \|_{\infty},
\]

and by (2.16) there exists a constant \( c_1 > 0 \) such that

\[
(2.50) \quad \| f \|_{\infty} \leq \int_0^\infty \| P_s(f_{t,r} - 1) \|_{\infty} ds \leq c_1 \| f_{t,r} - 1 \|_{\infty} \int_0^\infty e^{-\lambda_s} ds = \frac{c_1}{\lambda_1} \| f_{t,r} - 1 \|_{\infty}.
\]

Moreover, by Lemma 2.8, there exists a constant \( c_0 > 0 \) such that

\[
(2.51) \quad \| \nabla P_t g \|_{\infty} \leq c_0 (1 + t^{-\frac{1}{2}}) \| g \|_{\infty}, \quad t > 0, g \in \mathcal{B}_b(M).
\]

Combining this with (2.16) implied by (A1), we find constants \( c_2, c_3, c_4 > 0 \) such that

\[
\| \nabla f \|_{\infty} = \| \nabla L^{-1}(f_{t,r} - 1) \|_{\infty} \leq \int_0^\infty \| \nabla P_s(f_{t,r} - 1) \|_{\infty} ds,
\]

\[
(2.52) \quad \leq c_2 \int_0^\infty (1 + s^{-\frac{1}{2}}) \| P_s(f_{t,r} - 1) \|_{\infty} ds
\]

\[
\leq c_3 \| f_{t,r} - 1 \|_{\infty} \int_0^\infty (1 + s^{-\frac{1}{2}}) e^{-\lambda_s} ds \leq c_4 \| f_{t,r} - 1 \|_{\infty}.
\]

Let \( B_\sigma := \{ \| f_{t,r} - 1 \|_{\infty} \leq \sigma^\frac{3}{2} \} \). By (2.52), there exists a constant \( \sigma_0 \in (0, 1] \) such that

\( B_\sigma \subset \{ \| \nabla f \|_{\infty} \leq \gamma \sigma \} \) holds for \( \sigma \leq \sigma_0 \). So, we deduce from (2.49), (2.50) and (2.52) that

\[
(2.53) \quad C_1(f, \sigma) 1_{B_\sigma} \leq c_5 \sigma^\frac{3}{2}, \quad C_2(f, \sigma) 1_{B_\sigma} \leq c_5 \sigma^\frac{3}{2}, \quad \sigma \in (0, \sigma_0]
\]

holds for some constant \( c_5 > 0 \).

On the other hand, it is easy to see that \( f \) satisfies the Neumann boundary condition, so that by (2.49) and (2.52), Lemma 2.9 applies. By Lemma 2.9(2), the integration by parts formula and noting that \( f = L^{-1}(f_{t,r} - 1) \), we obtain that on the event \( B_\sigma \),

\[
C_2(f, \sigma) + \frac{1}{2} \mathbb{W}_2(\mu_{t,r}, \mu)^2 \geq \int_M \phi_1^\gamma d\mu - \int_M f d\mu_{t,r}
\]

\[
= \int_M (\phi_1^\gamma - f) d\mu - \int_M f(f_{t,r} - 1) d\mu
\]

\[
\geq -\frac{1}{2} \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2 d\mu - \int_M (f_{t,r} - 1) L^{-1}(f_{t,r} - 1) d\mu - C_1(f, \sigma)
\]

\[
= \frac{1}{2} \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2 d\mu - C_1(f, \sigma), \quad \sigma \in (0, \sigma_0].
\]
Since $\mathbb{W}_2(\mu_{t,r}, \mu)^2 \geq 0$, we deduce from this, (2.49) and (2.52) that on the event $B_\sigma$,

$$\frac{1}{2} \mathbb{W}_2(\mu_{t,r}, \mu)^2 \geq \frac{1}{2} \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2 d\mu - C_1(f, \sigma) - C_2(f, \sigma).$$

This and (2.53) yield

$$\frac{t}{2} \inf_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}, \mu)^2] \geq \frac{t}{2} \inf_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}, \mu)^2 1_{B_\sigma}]$$

$$\geq \frac{1}{2} \inf_{x \in M} \mathbb{E}^x \left[1_{B_\sigma} \xi(t)\right] - c_6 \sigma^{\frac{3}{4}} t$$

$$\geq \frac{1}{2} \inf_{x \in M} \mathbb{E}^x[\xi(t)] - I - c_6 \sigma^{\frac{3}{4}} t, \quad \sigma \in (0, \sigma_0]$$

for some constant $c_6 > 0$, where, by (2.52), Lemma 2.5 and noting that $\|f_{t,r} - 1\|_\infty \leq \|P_r\|_{1 \to \infty} < \infty$,

$$I := t \sup_{x \in M} \mathbb{E}^x [1_{B_\sigma} \mu(|\nabla L^{-1}(f_{t,r} - 1)|^2)]$$

$$\leq c_3^2 \|P_r\|_{1 \to \infty} t \sup_{x \in M} \mathbb{P}^x(B_\sigma^c) \leq \sigma^{-\frac{4k}{3}} c(k, r)t^{1-k}, \quad k \in \mathbb{N}, r > 0,$$

where $c(k, r) > 0$ is a constant depending on $k, r$. Now, let $\sigma = t^{-\alpha}$ for some $\alpha \in (\frac{3}{5}, \frac{3}{4})$ and take $k \geq 1$ such that $k(1 - \frac{4k}{3}) > 1$. Then we derive from (2.55) and (2.56) that

$$\frac{1}{2} \liminf_{t \to \infty} \left\{ t \inf_{x \in M} \mathbb{E}^x[\mathbb{W}_2(\mu_{t,r}, \mu)^2] \right\} \geq \frac{1}{2} \liminf_{t \to \infty} \inf_{x \in M} \mathbb{E}^x[\xi(t)].$$

Combining this with (2.20), we prove (2.35). \qed

### 2.3 Proofs of Theorem 2.1 and Corollary 2.2

Since (2.4) and (2.5) in Theorem 2.1 follow from Proposition 2.6 and Proposition 2.10, below we only prove (2.6) and Corollary 2.2. To this end, we first present the following two lemmas.

**Lemma 2.11.** Assume (A1). Then for any $r > 0$, $\xi(t)$ in (2.18) satisfies

$$\lim_{t \to \infty} \sup_{\|h_r\|_c \leq C} |\mathbb{P}^r(\xi(t) < a) - \nu_r((-\infty, a))| = 0, \quad a \in \mathbb{R}, C > 0.$$

If $M$ is compact and $d \leq 3$, then for any $r_t \downarrow 0$ as $t \uparrow \infty$,

$$\lim_{t \to \infty} \sup_{\|h_r\|_c \leq C} |\mathbb{P}^r(\xi(t) < a) - \nu_0((-\infty, a))| = 0, \quad a \in \mathbb{R}, C > 0.$$

**Proof.** By (2.23) and (2.24) we have

$$\xi(t) = \sum_{k=1}^\infty \int_0^\infty e^{-\lambda_k (2r+s)} |\psi_k(t)|^2 ds = \sum_{k=1}^\infty \frac{|\psi_k(t)|^2}{\lambda_k e^{2\lambda_k r}}, \quad t > 0.$$
For any \( n \geq 1 \), consider the \( n \)-dimensional process
\[
\Psi_n(t) := (\psi_1(t), \ldots, \psi_n(t)), \quad t > 0.
\]

For any \( \alpha \in \mathbb{R}^n \), we have
\[
\langle \Psi_n(t), \alpha \rangle = \frac{1}{\sqrt{t}} \int_0^t \left( \sum_{k=1}^n \alpha_k \phi_k(X_s) \right) ds.
\]

By [40, Theorem 2.4'], when \( t \to \infty \), the law of \( \langle \Psi_n(t), \alpha \rangle \) under \( P^\nu \) converges weakly to the Gaussian distribution \( N(0, \sigma_{n,\alpha}) \) uniformly in \( \nu \) with \( \|h_\nu\|_\infty \leq C \), where, due to (2.24) with \( \nu = \mu \) and \( \mu(P_s \phi^2_t) = \mu(\phi^2_t) = 1 \), the variance is given by
\[
\sigma_{n,\alpha} := \lim_{t \to \infty} E_P \langle \Psi_n(t), \alpha \rangle^2
= \lim_{t \to \infty} \frac{2}{t} \sum_{k=1}^n \alpha_k^2 \int_0^t \int_{s_1}^t e^{-\lambda_k(s_2-s_1)} ds_2 = \sum_{k=1}^n \frac{2\alpha_k^2}{\lambda_k}.
\]

Thus, uniformly in \( \nu \) with \( \|h_\nu\|_\infty \leq C \),
\[
\lim_{t \to \infty} E_P e^{i\langle \Psi_n(t), \alpha \rangle} = \int_{\mathbb{R}^n} e^{i\langle x, \alpha \rangle} \prod_{k=1}^n N(0, 2\lambda_k^{-1})(dx_k), \quad \alpha \in \mathbb{R}^n,
\]
so that the distribution of \( \Psi_n(t) \) under \( P^\nu \) converges weakly to \( \prod_{k=1}^n N(0, 2\lambda_k^{-1}) \). Therefore, letting
\[
(2.60) \quad \Xi_r^{(n)}(t) := \sum_{k=1}^n \frac{\psi_k(t)^2}{\lambda_k^2 e^{2\lambda_k r}}, \quad \Xi_r^{(n)} := \sum_{k=1}^n \frac{2\lambda_k^2}{\lambda_k^2 e^{2\lambda_k r}},
\]
we derive
\[
(2.61) \quad \lim_{t \to \infty} \sup_{\|h_\nu\|_\infty \leq C} \left| E_P \langle \Xi_r^{(n)}(t) < a \rangle - \mathbb{P}(\Xi_r^{(n)} < a) \right| = 0, \quad a \in \mathbb{R}.
\]

On the other hand, (2.23), (2.24) and (2.60) imply
\[
\sup_{\|h_\nu\|_\infty \leq C} \mathbb{P}^\nu|\Xi_r(t) - \Xi_r^{(n)}(t)|
= \frac{2}{t} \sup_{\|h_\nu\|_\infty \leq C} \sum_{k=n+1}^\infty \frac{e^{-2\lambda_k r}}{\lambda_k^2} \int_0^t \nu(P_s \phi^2_k)(1 - e^{-\lambda_k(t-s)}) ds \leq C\varepsilon_n,
\]
where \( \varepsilon_n := \sum_{k=n+1}^\infty \frac{2}{\lambda_k^2 e^{2\lambda_k r}} \to 0 \) as \( n \to \infty \). Combining this with (2.61) we see that for any \( a \in \mathbb{R} \) and \( \varepsilon > 0 \),
\[
\lim_{t \to \infty} \sup_{\|h_\nu\|_\infty \leq C} \left| E_P \langle \Xi_r(t) < a \rangle - \mathbb{P}(\Xi_r < a) \right|
\]
\[ \sum_{i=1}^{n}(t - a) \leq \limsup_{t \to \infty} \sup_{\|h_{r}\| \leq C} \left\{ |\mathbb{P}^{\mu}(\Xi_{r}^{(n)}(t) < a - \varepsilon) - \mathbb{P}(\Xi_{r}^{(n)} < a - \varepsilon)| + |\mathbb{P}^{\mu}(\Xi_{r}^{(n)}(t) < a - \varepsilon) - \mathbb{P}(\Xi_{r} < a)| \right\} \]

\[ \leq \limsup_{t \to \infty} \sup_{\|h_{r}\| \leq C} \left\{ |\mathbb{P}^{\mu}(\Xi_{r}^{(n)}(t) - \Xi_{r}^{(n)}(t) \geq \varepsilon) + \mathbb{P}^{\mu}(a - \varepsilon \leq \Xi_{r}^{(n)}(t) < a)| + |\mathbb{P}(\Xi_{r} - \Xi_{r}^{(n)}(t) \geq \varepsilon) + \mathbb{P}(a - \varepsilon \leq \Xi_{r}^{(n)}(t) < a)| \right\} \]

\[ \leq \frac{(1 + C)\varepsilon}{\varepsilon} + 2\mathbb{P}(a - \varepsilon \leq \Xi_{r}^{(n)} < a), \quad \varepsilon > 0, n \geq 1. \]

Letting first \( n \uparrow \infty \) then \( \varepsilon \downarrow 0 \), we prove (2.57).

(2) Next, let \( M \) be compact with \( d \leq 3 \). We have \( \sum_{k=1}^{\infty} \frac{2}{\lambda_{k}^{2}} < \infty \), so that the proof in (1) applies to \( r = 0 \) or \( r = r_{t} \) with \( r_{t} \downarrow 0 \) as \( t \uparrow \infty \), where \( \Xi_{0}(t) := \sum_{k=1}^{\infty} \lambda_{k}^{-1} |\psi_{k}(t)|^{2}, \Xi_{0} := 2 \sum_{k=1}^{\infty} \lambda_{k}^{-2} \varepsilon_{k}^{2} \). Then (2.58) holds. \( \square \)

**Lemma 2.12.** Assume (A1). For any \( 0 < \varepsilon < t \), let

\[ \mu_{t,r}^{\varepsilon} = \frac{1}{t - \varepsilon} \int_{\varepsilon}^{t} P_{r}(X_{s}, \cdot) ds, \quad r \geq 0, \]

where \( P_{r}(X_{s}, \cdot) := \delta_{X_{s}} \) for \( r = 0 \). Let \( D \) be the diameter of \( M \). Then

\[ |t\mathbb{W}_{2}(\mu_{t,r}, \mu) - (t - \varepsilon)\mathbb{W}_{2}(\mu_{t,r}^{\varepsilon}, \mu)| \leq 3c(r)\sqrt{\varepsilon} + \sqrt{\varepsilon}(t - \varepsilon)\mathbb{W}_{2}(\mu_{\varepsilon}^{\varepsilon}, \mu), \quad r \geq 0, t > \varepsilon, \varepsilon \in (0, 1) \]

holds for \( c(r) := \min \left\{ \|P_{r}\|_{\infty}^{2}(\mu \times \mu)(\rho^{2}), D^{2} \right\} \), which is finite if either \( r > 0 \) or \( D < \infty \).

**Proof.** It is easy to see that the measure

\[ \pi(dx, dy) := \left( \frac{1}{t} \int_{\varepsilon}^{t} P_{r}(X_{s}, dx) ds \right) \delta_{x}(dy) + \left( \frac{1}{t(t - \varepsilon)} \int_{\varepsilon}^{t} P_{r}(X_{s}, dx) ds \right) \int_{0}^{\varepsilon} P_{r}(X_{s}, dy) ds \]

is a coupling of \( \mu_{t,r}^{\varepsilon} \) and \( \mu_{t} \). So,

\[ t\mathbb{W}_{2}(\mu_{t,r}^{\varepsilon}, \mu_{t,r})^{2} \leq t \int_{M \times M} \rho(x, y)^{2} \pi(dx, dy) \]

\[ = \frac{1}{t - \varepsilon} \int_{\varepsilon}^{t} ds_{1} \int_{\varepsilon}^{\varepsilon} ds_{2} \int_{M \times M} \rho(x, y)^{2} \pi_{r}(X_{s_{1}}, x)p_{r}(X_{s_{2}}, y) \mu(dx) \mu(dy) \leq c(r)\varepsilon. \]

On the other hand,

\[ \mathbb{W}_{2}(\mu_{t,r}^{\varepsilon}, \mu)^{2} \leq \int_{M \times M} \rho(x, y)^{2} \mu_{t,r}^{\varepsilon}(dx) \mu(dy) \leq c(r), \quad r \geq 0. \]

Therefore,

\[ |t\mathbb{W}_{2}(\mu_{t,r}, \mu) - (t - \varepsilon)\mathbb{W}_{2}(\mu_{t,r}^{\varepsilon}, \mu)| \]

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\[ \leq \varepsilon \mathbb{W}_2(\mu_{t,r}, \mu)^2 + (t - \varepsilon) \left\{ |\mathbb{W}_2(\mu_{t,r}, \mu) - \mathbb{W}_2(\mu_{x,r}^\delta, \mu)|^2 + 2|\mathbb{W}_2(\mu_{t,r}, \mu) - \mathbb{W}_2(\mu_{x,r}^\delta, \mu)|\mathbb{W}_2(\mu_{x,r}^\delta, \mu) \right\} \]
\[ \leq \varepsilon \mathbb{W}_2(\mu_{t,r}, \mu)^2 + (1 + \varepsilon^{-\frac{1}{2}})(t - \varepsilon)|\mathbb{W}_2(\mu_{t,r}, \mu) - \mathbb{W}_2(\mu_{x,r}^\delta, \mu)|^2 + \varepsilon^{\frac{1}{2}}(t - \varepsilon)\mathbb{W}_2(\mu_{x,r}^\delta, \mu)^2 \]
\[ \leq 3c(r)\sqrt{\varepsilon} + \sqrt{\varepsilon}(t - \varepsilon)\mathbb{W}_2(\mu_{x,r}^\delta, \mu)^2, \quad t > \varepsilon, \varepsilon \in (0, 1). \]

\[ \square \]

Proof of (2.6). (a) We first prove for \( \nu \) with \( \|h_\nu\|_\infty \leq C \). Take \( \sigma = t^{-\frac{3}{2}} \). By (2.53) and (2.54), we find a constant \( c_1 > 0 \) such that for large enough \( t \geq 1 \), it holds on the event \( B_\sigma := \{ \|f_{t,r} - 1\|_\infty \leq \sigma^{\frac{3}{2}} \} \) that
\[ t\mathbb{W}_2(\mu_{t,r}, \mu)^2 \geq \Xi_r(t) - c_1t\sigma^{\frac{3}{2}} = \Xi_r(t) - c_1t^{-\frac{3}{2}}. \]
Moreover, Lemma 2.5 with \( k = 1 \) implies
\[ \limsup_{t \to \infty} \mathbb{P}^x(B_\sigma^c) \leq c(1, r) \lim_{t \to \infty} \sigma^{-\frac{3}{2}}t^{-1} = c(1, r) \lim_{t \to \infty} t^{-\frac{3}{2}} = 0. \]
This together with (2.64) and (2.19) yields
\[ \limsup_{t \to \infty} \mathbb{P}^x(|t\mathbb{W}_2(\mu_{t,r}, \mu)^2 - \Xi_r(t)| \geq \varepsilon) = 0, \quad \varepsilon > 0. \]
Combining this with (2.57) we prove
\[ \limsup_{t \to \infty} \sup_{\|h_\nu\|_\infty \leq C} \mathbb{P}^x(t\mathbb{W}_2(\mu_{t,r}, \mu)^2 < a) - \nu_r((-\infty, a)) = 0, \quad a \in \mathbb{R}. \]

(b) We now consider \( \nu = \delta_x \). By the Markov property, the law of \( \mu_{x,r}^\delta \) under \( \mathbb{P}^x \) coincides with that of \( \mu_{t,-r}^\delta \) under \( \mathbb{P}^\nu \) with \( \nu(dy) := p_r(x, y)\mu(dy) \). Moreover, since \( \sup_{x,y} p_r(x, y) = \|P_r\|_1 \to \infty =: c(\varepsilon) < \infty \), (2.66) implies
\[ \limsup_{t \to \infty} \mathbb{P}^x((t - \varepsilon)\mathbb{W}_2(\mu_{t,r}^\delta, \mu)^2 < a) - \nu_r((-\infty, a)) = 0, \quad a \in \mathbb{R}. \]
Taking $\delta = \frac{\varepsilon}{2}$ and letting $\varepsilon \to 0$ we finish the proof. \hfill \Box

Proof of Corollary 2.2. Obviously, when $d \leq 3$, (2.5) and (1.1) imply assertion (1). Next, for $d = 4$, (1.1) implies
\begin{equation}
\sum_{i=1}^{\infty} i^{-1} e^{-c_2 v^2 r^2} \leq 2 \int_{1}^{\infty} s^{-1} e^{-\frac{c_2 v^2}{4} r^2} ds \leq c_1 \sum_{i=1}^{\infty} i^{-1} e^{-c_2 v^2 r^2}, \quad r > 0
\end{equation}
for some constants $c_1, c_2, c_1', c_2' > 0$. Moreover, there exist constants $c_3, c_4 > 0$ such that
\begin{equation}
\sum_{i=1}^{\infty} i^{-1} e^{-c_2 v^2 r^2} \leq c_3 \int_{1}^{\infty} s^{-1} e^{-\frac{c_2 v^2}{4} r^2} ds \leq c_1 \sum_{i=1}^{\infty} i^{-1} e^{-c_2 v^2 r^2}, \quad r > 0
\end{equation}
while for some constants $c_3', c_4' > 0$,
\begin{equation}
\sum_{i=1}^{\infty} i^{-1} e^{-c_2 v^2 r^2} \geq c_3' \int_{1}^{\infty} s^{-1} e^{-\frac{c_2 v^2}{4} r^2} ds \leq c_4' \log r^{-1}, \quad r \in (0, 1/2),
\end{equation}
Combining this with (2.67), (2.68) and (2.5), we prove the second assertion.

Finally, when $d \geq 5$, (1.1) implies that for some constants $c_1, c_1', i = 1, 2, 3$ such that
\begin{equation}
\sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r\lambda_i}} \leq c_1 \sum_{i=1}^{\infty} i^{-\frac{4}{d} e^{-c_2 v^2 r^2}} \leq c_1 \int_{0}^{\infty} s^{-\frac{4}{d} e^{-c_2 v^2 r^2}} ds \leq c_2 r^{-\frac{d-4}{d}}, \quad r > 0,
\end{equation}
and
\begin{equation}
\sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r\lambda_i}} \geq c_1' \sum_{i=1}^{\infty} i^{-\frac{4}{d} e^{-c_2 v^2 r^2}} \geq c_1' \int_{1}^{\infty} s^{-\frac{4}{d} e^{-c_2 v^2 r^2}} ds \leq c_3' r^{-\frac{4-d}{d}}, \quad r \in (0, 1),
\end{equation}
Combining these with (2.5), we prove (3). \hfill \Box
3 Proof of Theorem 1.1

In this section we assume that $M$ is compact. We first present some lemmas which will be used in the proof.

3.1 Some lemmas

When $M$ is compact, there exist constants $\kappa, \lambda > 0$ such that

$$
\|P_t\|_{p \to q} \leq \kappa (1 \wedge t)^{-\frac{d}{2}(p^{-1} - q^{-1})}, \quad t > 0, q \geq p \geq 1.
$$

In particular, (A1) holds with $\|P_t\|_{1 \to \infty} \leq \kappa (1 \wedge t)^{-\frac{d}{2}}$ for some constant $\kappa > 0$ and all $t > 0$, so that (2.4) follows from Theorem 2.1.

To estimate $\mathbb{E}[W_2(\mu_t, \mu)]$ from (2.4), we use the triangle inequality to derive

$$
\mathbb{E}[W_2(\mu_t, \mu)]^2 \leq (1 + \varepsilon)\mathbb{E}[W_2(\mu_t, \mu)]^2 + (1 + \varepsilon^{-1})\mathbb{E}[W_2(\mu_t, \mu_t)^2], \quad \varepsilon > 0.
$$

We will show that $\mathbb{E}[W_2(\mu_t, \mu_t)^2] \leq cr$ holds for some constant $c > 0$ and all $r > 0$, which is known when $\partial M$ is either empty or convex, but is new when $\partial M$ is non-convex, see (3.3) below. If we could take $r_t > 0$ such that

$$
\lim_{t \to \infty} tr_t = 0, \quad \limsup_{t \to \infty} \{t\mathbb{E}[W_2(\mu_t, \mu)^2]\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2},
$$

we would deduce the desired estimate (3.30) from (3.2). Let us start with the following estimate of $\mathbb{E}[W_2(\mu_t, \mu_t)^2]$.

**Lemma 3.1.** Assume that $M$ is compact and let $\mu_{t,r,\varepsilon} = (1 - \varepsilon)\mu_{t,r} + \varepsilon \mu$, $\varepsilon \in [0, 1]$. Then there exists a constant $c > 0$ such that

$$
\mathbb{E}^\nu[W_2(\mu, \mu_{t,r,\varepsilon})^2] \leq c \|h^\nu\|_\infty r, \quad \nu = h^\nu \mu, r \geq 0,
$$

and for any initial value $X_0$ of the diffusion process,

$$
W_2(\mu_{t,r,\varepsilon}, \mu_{t,r})^2 \leq c \varepsilon, \quad t, r > 0, \varepsilon \in [0, 1].
$$

**Proof.** Since $M$ is compact, there exists $\hat{\rho} \in C_b^\infty(M \times M)$ and constants $\alpha_2 \geq \alpha_1 > 0$ such that

$$
\alpha_1 \hat{\rho} \leq \rho \leq \alpha_2 \hat{\rho}.
$$

By Itô’s formula, there exist constants $c_1, c_2 > 0$ such that

$$
d\hat{\rho}(X_0, X_r)^2 = \{L\hat{\rho}(X_0, \cdot)^2(X_r)\}dr + dM_r + \{N\hat{\rho}(X_0, \cdot)^2(X_r)\}dl_r
\leq c_1 dr + dM_r + c_2 dl_r,
$$

where $M_r$ is a martingale, when $\partial M$ exists $N$ is the inward unit normal vector field of $\partial M$ and $l_r$ is the local time of $X_r$ on $\partial M$, and $D$ is the diameter of $M$. If $\partial M = \emptyset$, then $l_r = 0$ so that

$$
\mathbb{E}^\nu[\rho(X_0, X_r)^2] \leq \alpha_2^2 \mathbb{E}^\nu[\hat{\rho}(X_0, X_r)^2] \leq c_1 \alpha_2^2 r \leq c_1 \alpha_2^2 \|h^\nu\|_\infty r, \quad r \geq 0.
$$
When $\partial M \neq \emptyset$, (3.5) implies

\begin{equation}
\mathbb{E}^\nu\left[\rho(X_0, X_t)^2\right] \leq c_1 \alpha_2^2 r + c_2 \alpha_2^2 \mathbb{E}^\nu l_r, \quad r > 0.
\end{equation}

Let $\tau = \inf\{t \geq 0 : X_t \in \partial M\}$. We have $l_r = 0$ for $r \leq \tau$, so that by the Markov property

\begin{equation}
\mathbb{E}^\nu l_r = \mathbb{E}^\nu[1_{\{\tau < r\}}] \mathbb{E}^X l_{r-\tau} \leq \mathbb{P}^{\nu}(\tau < r) \sup_{x \in \partial M} \mathbb{E}^x l_r.
\end{equation}

By [32, Proposition 4.1] and [6, Lemma 2.3], there exist constants $c_2, c_3, c_4 > 0$ such that

\[ \mathbb{E}^x l_r \leq c_2 \sqrt{r}, \quad x \in \partial M, \]

\[ \mathbb{P}^{\nu}(\tau < r) \leq \int_M e^{-c_2 \rho_0(x)^2/r} \nu(dx) \leq \|h_{\nu}\|_\infty \int_M e^{-c_3 \rho_0(x)^2/r} \mu(dx) \leq c_4 \|h_{\nu}\|_\infty \sqrt{r}. \]

Combining these with (3.8) we derive $\mathbb{E}^\nu l_r \leq c_2 c_4 \|h_{\nu}\|_\infty r$ for $r \geq 0$. Therefore, by (3.7) for $\partial M \neq \emptyset$ and (3.6) for $\partial M = \emptyset$, we find a constant $c > 0$ such that in any case

\begin{equation}
\mathbb{E}^\nu\left[\rho(X_0, X_t)^2\right] \leq c \|h_{\nu}\|_\infty r, \quad r \geq 0.
\end{equation}

It is easy to see that for any $t > 0$,

\[ \pi_t(dx, dy) := \left( \frac{1}{t} \int_0^t \left\{ p_r(x, y) \delta_{X_s} \right\}(dx) ds \right) \mu(dy) \in \mathcal{C}(\mu_t, \mu_{t,r}). \]

Then

\begin{equation}
\mathbb{W}_2^2(\mu_{t,r}, \mu_t) \leq \int_{M \times M} \rho(x, y)^2 \pi_t(dx, dy)
\end{equation}

\[ = \frac{1}{t} \int_0^t ds \int_M p_r(X_s, y) \rho(X_s, y)^2 \mu(dy), \quad r, t > 0. \]

Letting $\nu_s = (P_s h_{\nu})\mu$, which is the distribution of $X_s$ provided the law of $X_0$ is $\nu$, by the Markov property and (3.9), we obtain

\[ \mathbb{E}^\nu \int_M p_r(X_s, y) \rho(X_s, y)^2 \mu(dy) = \mathbb{E}^{\nu_s}\left[\rho(X_0, X_r)^2\right] \leq c \|P_s h_{\nu}\|_\infty r \leq c \|h_{\nu}\|_\infty r, \quad s, r > 0. \]

Substituting this into (3.10), we prove (3.3).

On the other hand, since $\mu_{t,r,\varepsilon} = (1 - \varepsilon) \mu_{t,r} + \varepsilon \mu$, we have

\[ \pi(dx, dy) := (1 - \varepsilon) \mu_{t,r}(dx) \delta_x(dy) + \varepsilon \mu(dx) \mu_{t,r}(dy) \in \mathcal{C}(\mu_{t,r,\varepsilon}, \mu_{t,r}), \]

so that

\[ \mathbb{W}_2^2(\mu_{t,r,\varepsilon}, \mu_{t,r}) \leq \int_{M \times M} \rho(x, y)^2 \pi(dx, dy) \leq \varepsilon D^2. \]

Therefore, (3.4) holds. \qed
The following lemmas is used for the estimates of \( \mathbb{E}[\mathcal{W}_2(\mu_t, \mu)^2] \). Let us start with the case \( d \leq 3 \).

**Lemma 3.2.** Assume that \( M \) is compact with \( d \leq 3 \). There exists a constant \( c > 0 \) such that for any probability measure \( \nu = h_\nu \mu \),

\[
\sup_{y \in M} \mathbb{E}^\nu\left[|f_{t,r}(y) - 1|^2\right] \leq \frac{c\|h_\nu\|_{\infty}}{t^{d/2}}, \quad t \geq 1, r \in (0, 1),
\]

(3.11)

**Proof.** We use the notation in the proof of Lemma 2.5. Noting that \( f = p_r(\cdot, y) - 1 \) and \( M \) is compact, by (2.15) and (3.1) there exists a constant \( c > 0 \) such that

\[
\mathbb{E}_\nu^\nu[g(r_1, r_2)] = \nu(P_{r_1}|f P_{r_2-r_1}f|) \leq \|h_\nu\|_{\infty}\nu(|f P_{r_2-r_1}f|)
\]

\[
\leq 2\|h_\nu\|_{\infty}\|P_{r_2-r_1}P_{r_2-r_1} (p_{r/2}(\cdot, y) - 1)\|_{\infty}
\]

\[
\leq c\|h_\nu\|_{\infty}(1 \wedge (r + r_2 - r_1))^{-\frac{d}{2}}e^{-\frac{\lambda}{2}(r_2-r_1)}.
\]

So,

\[
\int_{\Delta_t(t)} \mathbb{E}^\nu[g(r_1, r_2)]dr_1dr_2 \leq c\|h_\nu\|_{\infty}\int_0^t dr_1 \int_{r_1}^t (1 \wedge (r + r_2 - r_1))^{-\frac{d}{2}}e^{-\frac{\lambda}{2}(r_2-r_1)}dr_2
\]

(3.12)

\[
\leq c\|h_\nu\|_{\infty}\int_0^t dr_1 \int_{r_1}^t (1 + (r + r_2 - r_1))^{-\frac{d}{2}}e^{-\frac{\lambda}{2}(r_2-r_1)}dr_2.
\]

Noting that

\[
\int_{r_1}^t (r + r_2 - r_1)^{-\frac{d}{2}}e^{-\frac{\lambda}{2}(r_2-r_1)}dr_2 \leq \int_0^\infty (r + s)^{-\frac{d}{2}}e^{-\frac{\lambda}{2}s}ds \leq c' \begin{cases} 1, & \text{when } d = 1; \\ \log(1 + r^{-1}), & \text{when } d = 2; \\ r^{1-\frac{d}{2}}, & \text{when } d \geq 3, \end{cases}
\]

holds for some constant \( c' > 0 \), combining (2.29) with (3.12) for \( k = 1 \) and noticing that \( d \leq 3 \), we prove (3.11). \( \square \)

**Lemma 3.3.** Assume that \( M \) is compact with \( d \leq 3 \). For any \( \alpha \in (1, 2) \) and \( r_t := t^{-\alpha} \),

\[
\lim_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C, q \in M} \mathbb{E}^\nu\left[ |\mathcal{M}((1 - r_t)f_{t,r}(y) + r_t, 1)^{-1} - 1|^{q}\right] = 0, \quad C, q > 0.
\]

(3.13)

**Proof.** By [5, Lemma 3.12],

\[
\frac{\theta(ab)^\frac{\theta}{2}|a - b|}{|a^\theta - b^\theta|} \leq \mathcal{M}(a, b) \leq \frac{\theta(a^\theta + b^\theta)(a - b)}{2(a^\theta - b^\theta)}, \quad a, b, \theta > 0.
\]

(3.14)

Combining this with the simple inequality \(|a^\theta - 1| \leq |a - 1|\) for \( a \geq 0 \) and \( \theta \in [0, 1] \), we obtain

\[
\mathcal{M}((1 - r)f_{t,r}(y) + r, 1) \geq \frac{\theta((1 - r)f_{t,r}(y) + r)^\frac{\theta}{2}|(1 - r)f_{t,r}(y) + r - 1|}{|{(1 - r)f_{t,r}(y) + r}^\theta - 1|}
\]
Thus, we have

\[ (3.17) \]

\[ \text{Proof. Lemma 3.4.} \]

Then for any \( r \geq 1, \theta \in (0, 1), r > 0. \)

\[ (3.16) \]

Combining this with (3.11) and using (3.15), we obtain that for \( t \geq 1 \) and \( r \in (0, 1], \)

\[ \sup_{y \in M} \mathbb{E}^y \left[ |\mathcal{M}((1 - r)f_{t,r}(y) + r, 1)^{-1} - 1| \right] \leq \delta_y + (1 + \theta^{-1}r^{-\frac{q}{2}})^q \sup_{y \in M} \mathbb{P}^y(A_y) \]

\[ \leq \delta_y + C(\theta, \eta) \|h_\nu\|_\infty t^{-1}r^{-\frac{1}{2} - \frac{\alpha}{2}}. \]

Then for any \( \alpha \in (1, 2) \) and \( q > 0, \) we may take a small enough \( \theta \) such that \( \alpha(\frac{1}{2} + \frac{\alpha}{2}) < 1. \)

Then (3.13) follows from (3.16) with \( r = r_t = t^{-\alpha} \) and \( \eta \downarrow 0. \)

**Lemma 3.4.** Assume that \( M \) is compact. For any \( p \in [1, 2], \) there exists a constant \( c > 0 \) such that \( \psi_i(t) := \frac{1}{\sqrt{2\pi}} \int_0^t \phi_i(X_s) ds \) satisfies

\[ \mathbb{E}^\nu \left[ |\psi_i(t)|^{2p} \right] \leq c\|h_\nu\|_\infty \lambda_i^{-2+(p-1)(\frac{q}{2}-2)}, \quad t \geq 1, i \in \mathbb{N}, \nu = h_\nu \mu. \]

**Proof.** Let \( f = \phi_i. \) Then \( g(r_1, r_2) \) in (2.29) satisfies

\[ (3.17) \]

Since \( \mu(h_\nu P_1 \phi_i^2) \leq \|h_\nu\|_\infty \mu(\phi_i^2) = \|h_\nu\|_\infty < \infty, \) this and (2.29) with \( k = 1 \) imply

\[ (3.18) \]

for some constant \( c_1 > 0. \) On the other hand, taking \( k = 2 \) in (2.29) and using (3.17), we find a constant \( c_2 > 0 \) such that

\[ t^2 \mathbb{E}^\nu \left[ |\psi_i(t)|^4 \right] \leq c_2 \left( \int_0^t dr_1 \int_{r_1}^t \mathbb{E}^\nu \left[ |g(r_1, r_2)|^2 \right] \frac{1}{2} dr_2 \right)^2 \]

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\[
= c_2 \left( \int_0^t dr_1 \int_{r_1}^t e^{-(r_2-r_1)\lambda_i} \sqrt{\mu(h_\nu P_{r_1} \phi_i^t) dr_2} \right)^2, \quad t \geq 1, i \in \mathbb{N}.
\]

By (3.1) and \( P_t \phi_i = e^{-\lambda_i t} \phi_i \), we obtain

\[
(3.19) \quad \| \phi_i \|_\infty = \inf_{t > 0} \left\{ e^{\lambda_i t} \| P_t \phi_i \|_\infty \right\} \leq \inf_{t > 0} \left\{ e^{\lambda_i t} \| P_t \|_{2 \to \infty} \right\} \leq c_3 \lambda_i^\frac{d}{4}, \quad i \geq 1
\]

for some constant \( c_3 > 0 \). Since \( h_\nu \) is bounded, (3.19) and \( \mu(\phi_i^t) = 1 \) imply

\[
\sqrt{\mu(h_\nu P_{r_1} \phi_i^t)} \leq \sqrt{\| h_\nu \|_\infty \mu(\phi_i)} \leq \sqrt{\| h_\nu \|_\infty \| \phi_i \|_\infty \| \phi_i \|_\infty \lambda_i^\frac{d}{4}}, \quad i \geq 1.
\]

Therefore, there exists a constant \( c_4 > 0 \) such that

\[
t^2 \mathbb{E}^\nu[|\psi_i(t)|^4] \leq c_4 \| h_\nu \|_\infty t^2 \lambda_i^\frac{d}{4} - 2, \quad t \geq 1, i \in \mathbb{N}.
\]

Combining this with (3.18) and Hölder’s inequality, we find a constant \( c > 0 \) such that for any \( p \in [1, 2], \)

\[
\begin{align*}
\mathbb{E}^\nu[|\psi_i(t)|^{2p}] &= \mathbb{E}^\nu[|\psi_i(t)|^{4-2p}|\psi_i(t)|^{4(p-1)}] \\
&\leq (\mathbb{E}^\nu|\psi_i(t)|^2) - p (\mathbb{E}^\nu|\psi_i(t)|^4)^{p-1} \leq c \| h_\nu \|_\infty \lambda_i^{p-2+(p-1)(\frac{d}{4}-2)}, \quad t \geq 1, i \in \mathbb{N}.
\end{align*}
\]

Lemma 3.5. Assume that \( M \) is compact with \( d \leq 3 \). There exists a constant \( p > 1 \) such that for any \( C > 1, \)

\[
\limsup_{t \to \infty} \sup_{r > 0, \| h_\nu \|_\infty \leq C} \left\{ t^p \mathbb{E}^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^{2p} d\mu \right\} < \infty.
\]

Proof. Let \( p > 1 \). By [29, (1.10)], the gradient estimate

\[
|\nabla P_t f| \leq \frac{c(p)}{\sqrt{t}} (P_t |f|^p)^{\frac{1}{p}}, \quad t > 0, f \in \mathcal{B}_b(M)
\]

holds for some constant \( c(p) > 0 \). Combining this with (2.7) and (2.11), we obtain

\[
\begin{align*}
\mathbb{E}^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^{2p} d\mu &\leq \mathbb{E}^\nu \int_M \left( \int_0^\infty |\nabla P_s(f_{t,r} - 1)| ds \right)^{2p} d\mu \\
&\leq c_1(p) \mathbb{E}^\nu \left( \int_0^\infty \frac{1}{s} \left\{ P_s |P_{\frac{1}{2}} f_{t,r} - 1|^{p} \right\}^{\frac{1}{p}} ds \right)^{2p} \\
&\leq c_1(p) \left( \int_0^\infty s^{-\frac{2p}{2p-1}} e^{-\frac{2p}{2p-1} s} ds \right)^{2p-1} \\
&\quad \times \mathbb{E}^\nu \int_0^\infty e^{\theta s} \mu \left( \{ P_{\frac{1}{2}} f_{t,r} - 1 \}^2 \right) ds, \quad t \geq 1, r > 0
\end{align*}
\]
for some constant $c_1(p) > 0$. Let $\theta \in \left(0, \frac{\lambda_1}{2}\right)$ and $p \in (1, 2)$. We have

$$\int_0^\infty s^{-\frac{2p}{p-1}}e^{-\frac{2\theta s}{2p-1}}ds < \infty.$$  

Combining (2.22), (3.1), (3.21), (3.22) and Hölder’s inequality, we arrive at

$$t^pE^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2p d\mu \leq c_2(p)t^pE^\nu \int_0^\infty e^{\theta s}\|P_2\|_2^{2p} \{\mu((P_2(f_{t,r} - 1))^2)\}^{p}ds$$

$$\leq c_3(p)E^\nu \int_0^\infty e^{\theta s}(1 + s)^{-\frac{d(p-1)}{2}} \left( \sum_{i=1}^\infty e^{-(2r+s)\lambda_i} |\psi_i(t)|^2 \right)^p ds$$

$$\leq c_3(p) \left( \sum_{i=1}^\infty i^{-\frac{\epsilon}{p-1}} \right)^{p-1} \int_0^\infty (1 + s)^{-\frac{d(p-1)}{2}} \sum_{i=1}^\infty i^\epsilon e^{-p(2r+s)\lambda_i + \theta s} E^\nu [|\psi_i(t)|^2p] ds, \quad t \geq 1, i \in \mathbb{N}$$

for some constants $c_2(p), c_3(p) > 0$. Since $-ps\lambda_i + \theta s \leq -\frac{\epsilon}{2}\lambda_i$, and noting that for any $c > 0$ and $\delta \in (0, 1)$ there exists a constant $c' > 0$ such that

$$\int_0^\infty (1 + s)^{-\delta}e^{-c\lambda_1^p}ds \leq c'\lambda_1^{p-1}, \quad i \geq 1,$$

this implies

$$t^pE^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2p d\mu \leq c_4(p) \left( \sum_{i=1}^\infty i^{-\frac{\epsilon}{p-1}} \right)^{p-1} \sum_{i=1}^\infty i^\epsilon \lambda_1^{d(p-1)+1}e^{-2r\lambda_i}E^\nu [|\psi_i(t)|^2p]$$

for some constant $c_4(p) > 0$. Therefore, for any $\epsilon > 0$ and $p > 1$ such that $\frac{\epsilon}{p-1} > 1$, there exists a constant $c(p, \epsilon) > 0$ such that this, (1.1) and Lemma 3.4 yield

$$E^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2p d\mu \leq c(p, \epsilon)t^{-p}\|h_\nu\|_\infty \sum_{i=1}^\infty i^\epsilon\lambda_i^{-2r/d}, \quad t \geq 1, r > 0,$$

$$(3.23) \quad \delta_{p,\epsilon} := \epsilon + \frac{2}{d}\{(p-1)(d-2) - 1 + p - 2\}.$$

When $d \leq 3$, by taking for instance $\epsilon = \frac{1}{12}$, and $p > 1$ close enough to 1 such that

$$\frac{d(p-1)}{2} \in (0, 1), \quad \frac{\epsilon}{p-1} > 1, \quad (p-1)(d-2) - 1 + p - 2 \leq -\frac{7}{4},$$

and noting $d \leq 3$ and (1.1) imply $\lambda_i \geq c''\epsilon^{\frac{2}{d}}$ for some constant $c'' > 0$, from (3.23) we find a constant $c > 0$ such that

$$E^\nu \int_M |\nabla L^{-1}(f_{t,r} - 1)|^2p d\mu \leq ct^{-p}\|h_\nu\|_\infty \sum_{i=1}^\infty i^{\epsilon - \frac{2}{d} - \frac{7}{4}}$$

$$= ct^{-p}\|h_\nu\|_\infty \sum_{i=1}^\infty i^{\epsilon - \frac{7}{4}} < \infty, \quad t \geq 1, r > 0, \nu = h_\nu \mu.$$

We finish the proof. \[\square\]
For $d \geq 4$ we will use the following lemma taken from [22]. Although this lemma is less sharper than Lemma 2.3, it is easier to apply since the term $\mathcal{M}(1, f)$ is dropped.

**Lemma 3.6.** Let $f$ be probability density functions with respect to $\mu$. Then

$$\mathbb{W}_2(f \mu, \mu)^2 \leq 4 \int_M |\nabla L^{-1}(f - 1)|^2 d\mu.$$ 

Finally, we have the following result on the large deviation of the empirical measures. Let $\mathcal{P}$ be the set of all probability measures on $M$.

**Lemma 3.7 ([41]).** Let $M$ be compact. Then for any open set $G \subset \mathcal{P}$ and closed set $F \subset \mathcal{P}$ under the weak topology,

$$- \inf_{\nu \in G} I_\mu(\nu) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in M} \mathbb{P}^x(\mu_t \in G),$$

$$- \inf_{\nu \in F} I_\mu(\nu) \geq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in M} \mathbb{P}^x(\mu_t \in F).$$

**Proof.** Since the $\tau$-topology induced by bounded measurable functions on $M$ is stronger than the weak topology, $G$ and $F$ are open and closed respectively under the $\tau$-topology. By the ultracontractivity and irreducibility of $P_t$, [41, Theorem 5.1(b) and Corollary B.11] imply

$$- \inf_{\nu \in G} I_\mu(\nu) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in M} \mathbb{P}^x(\mu_t \in G),$$

$$- \inf_{\nu \in F} I_\mu(\nu) \geq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in M} \mathbb{P}^x(\mu_t \in F).$$

Consequently, letting $\mathcal{P}_c$ be the set of all probability measures on $M$ which are absolutely continuous with respect to $\mu$, we have

$$- \inf_{\nu \in G} I_\mu(\nu) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{\nu \in \mathcal{P}_c} \mathbb{P}^\nu(\mu_t \in G),$$

$$- \inf_{\nu \in F} I_\mu(\nu) \geq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{\nu \in \mathcal{P}_c} \mathbb{P}^\nu(\mu_t \in F).$$

(3.25)

To replace $\mathcal{P}_c$ by $\mathcal{P}$, consider $\tilde{\mu}_t^\varepsilon := \frac{1}{t} \int_{\varepsilon}^{t+\varepsilon} \delta_{X_s} ds$ for $\varepsilon > 0$. By the Markov property, the law of $\tilde{\mu}_t^\varepsilon$ under $\mathbb{P}^\varepsilon$ coincides with that of $\mu_t$ under $\mathbb{P}^\nu$, where $\nu := p_{\varepsilon}(x, \cdot) \mu \in \mathcal{P}_c$. So, (3.25) implies

$$- \inf_{\nu \in G} I_\mu(\nu) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in M} \mathbb{P}^x(\tilde{\mu}_t^\varepsilon \in G)$$

(3.26)

$$- \inf_{\nu \in F} I_\mu(\nu) \geq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in M} \mathbb{P}^x(\tilde{\mu}_t^\varepsilon \in F), \quad \varepsilon > 0.$$

(a) Let $D$ be the diameter of $M$. By taking the Wasserstein coupling

$$\pi(dx, dy) := (\mu_t \land \tilde{\mu}_t^\varepsilon)(dx) \delta_x(dy) + \frac{(\mu_t - \tilde{\mu}_t^\varepsilon)^+(dx)(\mu_t - \tilde{\mu}_t^\varepsilon)^-(dy)}{(\mu_t - \tilde{\mu}_t^\varepsilon)^+(M)} \in \mathcal{C}(\mu_t, \tilde{\mu}_t^\varepsilon),$$

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we obtain

\[ (3.27) \quad W_2(\mu_t, \tilde{\mu}_t)^2 \leq \int_{M \times M} \rho^2 \, d\pi \leq D^2(\mu_t - \tilde{\mu}_t)^+(M) = \frac{D^2 \varepsilon}{t}. \]

So, when \( t > D^2 \), we have \( \{\tilde{\mu}_t \in G_\varepsilon\} \subset \{\mu_t \in D\} \), where

\[ G_\varepsilon := \left\{ \nu \in \mathcal{P} : W_2(\nu, G^c)^2 := \inf_{\nu' \in G^c} W_2(\nu, \nu')^2 \geq \varepsilon \right\} \]

is an open subset of \( \mathcal{P} \) under the weak topology, since for compact \( M \), \( W_2 \) is continuous under the weak topology. Combining this with (3.26) for \( G_\varepsilon \) replacing \( G \), we arrive at

\[- \inf_{\nu \in G_\varepsilon} I_\mu(\nu) \leq \liminf_{t \to \infty} \frac{1}{t} \log \inf_{x \in M} \mathbb{P}_x(\tilde{\mu}_t \in G), \quad \varepsilon > 0.\]

Noting that \( G_\varepsilon \uparrow G \) as \( \varepsilon \downarrow 0 \), we have \( \inf_{\nu \in G_\varepsilon} I_\mu(\nu) \downarrow \inf_{\nu \in G} I_\mu(\nu) \) as \( \varepsilon \downarrow 0 \). So, letting \( \varepsilon \downarrow 0 \) we prove the desired inequality for open \( G \).

(b) Similarly, let \( F_\varepsilon := \left\{ \nu \in \mathcal{P} : W_2(\nu, F)^2 := \inf_{\nu' \in F} W_2(\nu, \nu')^2 \leq \varepsilon \right\}, \) which is closed. When \( t > D^2 \), (3.26) and (3.27) imply

\[- \inf_{\nu \in F_\varepsilon} I_\mu(\nu) \geq \limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in M} \mathbb{P}_x(\mu_t \in F), \quad \varepsilon > 0.\]

So, it suffices to show that

\[ (3.28) \quad c := \liminf_{\varepsilon \downarrow 0} \inf_{F_\varepsilon} I_\mu = \inf_F I_\mu. \]

Since \( F_\varepsilon \downarrow F \) as \( \varepsilon \downarrow 0 \), we have \( c \leq \inf_F I_\mu \). On the other hand, if \( c < \infty \), then we may choose \( \varepsilon_n \downarrow 0 \) and \( \nu_n \in F_{\varepsilon_n} \) such that

\[ (3.29) \quad I_\mu(\nu_n) \leq \inf_{F_{\varepsilon_n}} I_\mu + \frac{1}{n} \leq \inf_{F_1} I_\mu + 1 < \infty, \quad n \geq 1. \]

So, \( \nu_n = f_n \mu \) with \( \sup_{n \geq 1} \mu(|\nabla f_n|^2) < \infty \). By the Sobolev embedding theorem, \( \{f_n^{\frac{1}{2}}\}_{n \geq 1} \) is relatively compact in \( L^2(\mu) \), so that up to a subsequence \( f_n^{\frac{1}{2}} \to f^{\frac{1}{2}} \) in \( L^2(\mu) \) for some probability density \( f \) with respect to \( \mu \). This and (3.29) yield \( f^{\frac{1}{2}} \in W^{2,1}(\mu) \) and

\[ I_\mu(f \mu) := \mu(|\nabla f^{\frac{1}{2}}|^2) \leq \liminf_{n \to \infty} I_\mu(\nu_n) \leq \liminf_{n \to \infty} \inf_{F_{\varepsilon_n}} I_\mu = c. \]

Since \( F \) is closed, \( \nu_n \in F_{\varepsilon_n} \downarrow F \) and \( \nu_n \to f \mu \) weakly as \( n \uparrow \infty \), we conclude that \( f \mu \in F \). Therefore, \( \inf_F I_\mu \leq c \) as desired.

\[ \square \]
3.2 Proof of Theorem 1.1

Proof of Theorem 1.1(1). Obviously, (1.2) can be reformulated as the following two estimates:

\[
\limsup_{t \to \infty} \sup_{x \in M} \left\{ t \mathbb{E}^x |\mathbb{W}_2(\mu_t, \mu)|^2 \right\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2}, \tag{3.30}
\]

\[
\liminf_{t \to \infty} \left\{ t \inf_{x \in M} \mathbb{E}^x |\mathbb{W}_2(\mu_t, \mu)|^2 \right\} \geq c \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2}, \tag{3.31}
\]

where \(c > 0\) is a constant which equals to 1 when \(\partial M\) is empty or convex. Below we prove these two estimates respectively.

(a) Let \(M\) be compact. Since \(\sum_{i=1}^{\infty} \frac{2}{\lambda_i^2} = \infty\) for \(d \geq 4\), we only consider \(d \leq 3\). As shown in (b) in the proof of (2.6), we only need to consider \(\nu = h_\nu \mu\) with \(\|h_\nu\|_{\infty} \leq C\) for some constant \(C > 0\). Let \(r_t = t^\alpha\) for \(t \geq 1\) and some \(\alpha \in (1, 2)\). By the triangle inequality of \(\mathbb{W}_2\), for any \(\varepsilon > 0\) we have

\[
\mathbb{W}_2(\mu_t, \mu)^2 \leq (1 + \varepsilon)\mathbb{W}_2(\mu_{t, r_t}, \mu)^2 + 2(1 + \varepsilon^{-1})\mathbb{W}_2(\mu_{t, r_t}, \mu_{t, r_t})^2 + \mathbb{W}_2(\mu_t, \mu_{t, r_t})^2. \tag{3.32}
\]

This and Lemma 3.1 yield

\[
\limsup_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C} \left\{ t \mathbb{E}^x |\mathbb{W}_2(\mu_t, \mu)|^2 \right\} \leq (1 + \varepsilon) \limsup_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C} \left\{ t \mathbb{E}^x |\mathbb{W}_2(\mu_{t, r_t}, \mu)|^2 \right\}, \quad \varepsilon > 0.
\]

Letting \(\varepsilon \downarrow 0\) implies

\[
\limsup_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C} \left\{ t \mathbb{E}^x |\mathbb{W}_2(\mu_t, \mu)|^2 \right\} \leq \limsup_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C} \left\{ t \mathbb{E}^x |\mathbb{W}_2(\mu_{t, r_t}, \mu)|^2 \right\}, \quad C > 0. \tag{3.33}
\]

Next, by Lemma 2.3 and noting that \(\frac{d\mu_{t, r_t}}{d\mu} = (1 - r_t)f_{t, r_t} + r_t\), for the \(p > 1\) in Lemma 3.5, we have

\[
\mathbb{E}'[\mathbb{W}_2(\mu_{t, r_t}, \mu)^2] \leq (1 - r_t)^2 \mathbb{E}' \int_M \frac{|\nabla L^{-1}(f_{t, r_t} - 1)|^2}{\mathcal{M}((1 - r_t)f_{t, r_t} + r_t, 1)} \, d\mu \leq \mathbb{E}' \int_M \left\{ |\nabla L^{-1}(f_{t, r_t} - 1)|^2 + |\nabla L^{-1}(f_{t, r_t} - 1)|^2 \right\} \mathcal{M}((1 - r_t)f_{t, r_t} + r_t, 1)^{-1} \, d\mu \leq \mathbb{E}' \int_M |\nabla L^{-1}(f_{t, r_t} - 1)|^2 \, d\mu + \left( \mathbb{E}' \int_M |\nabla L^{-1}(f_{t, r_t} - 1)|^2 \, d\mu \right)^{\frac{1}{p}} \times \left( \mathbb{E}' \int_M \mathcal{M}((1 - r_t)f_{t, r_t} + r_t, 1)^{-1} \, d\mu \right)^{\frac{p}{p-1}}. \tag{3.34}
\]

Combining this with Lemmas 2.4, 3.3 and 3.5, we arrive at

\[
\limsup_{t \to \infty} \sup_{\|h_\nu\|_{\infty} \leq C} \left\{ t \mathbb{E}'[\mathbb{W}_2(\mu_{t, r_t}, \mu)^2] \right\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2}, \quad C > 0,
\]

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which together with (3.33) yields

\begin{equation}
(3.35) \quad \lim_{t \to \infty} \sup_{\|h\|_\infty \leq C} \left\{ \frac{t E^t[\mathbb{W}_2(\mu_t, \mu)^2]}{\lambda_i^2} \right\} \leq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2} C > 0.
\end{equation}

Then (3.30) holds.

(b) Let \( \partial M \) be either convex or empty. In this case, we have

\begin{equation}
(3.36) \quad \mathbb{W}_p(\mu, \nu P_r)^2 \leq e^{2Kr} \mathbb{W}_p(\mu, \nu)^2, \quad r > 0, p \in [1, \infty),
\end{equation}

where \( K \geq 0 \) is such that \( \text{Ric}_V \geq -K \) holds on \( M \), and \( (\nu P_r)(f) := \mu(P_r f) \) for \( f \in \mathcal{B}_b(M) \), see [25] for empty \( \partial M \) and [34, Theorem 3.3.2] for convex \( \partial M \). Since \( \mu_{t,r} = \mu_{t,r} \), (3.36) and (2.35) imply

\[ e^{2Kr} \lim_{t \to \infty} \left\{ t \inf_{x \in M} E^t[\mathbb{W}_2(\mu_t, \mu)]^2 \right\} \geq \lim_{t \to \infty} \left\{ t \inf_{x \in M} E^t[\mathbb{W}_2(\mu_t, \mu)]^2 \right\} \geq \sum_{i=1}^{\infty} \frac{2}{\lambda_i^2 e^{2r \lambda_i}}, \quad r \in (0, 1], \]

which gives (3.31) for \( C = 1 \) by letting \( r \to 0 \). In general, by [13, Theorem 2.7], there exist constants \( C, \lambda > 0 \) such that

\begin{equation}
(3.37) \quad \mathbb{W}_2(\mu, \nu P_r)^2 \leq Ce^{3r} \mathbb{W}_2(\mu, \nu)^2, \quad r > 0.
\end{equation}

This together with (2.35) yields (3.31).

\[ \square \]

\textbf{Proof of Theorem 1.1(2).} The boundedness of \( \rho \) implies that the weak topology is induced by \( \mathbb{W}_2 \). So, \( G := \{ \nu \in \mathcal{P} : \mathbb{W}_2(\nu, \mu)^2 \in A^0 \} \) is open while \( F := \{ \nu \in \mathcal{P} : \mathbb{W}_2(\nu, \mu)^2 \in \bar{A} \} \) is closed in \( \mathcal{P} \) under the weak topology. Thus, by Lemma 3.7, it suffices to prove

(i) For any set \( A \subset [0, \infty), \inf\{I(\nu) : \mathbb{W}_2(\nu, \mu)^2 \in A\} = \inf_{r \in A} I(r) \).

(ii) For any \( \alpha \geq 0, \{I \leq \alpha\} \) is a compact subset of \([0, \infty)\).

Below we prove these two assertions respectively.

For (i). Let \( \hat{I}(r) := \inf_{\mathbb{W}_2(\nu, \mu)^2 = r} I(\nu) \). We have \( \inf\{I(\nu) : \mathbb{W}_2(\nu, \mu)^2 \in A\} = \inf_{r \in A} \hat{I}(r) \).

So, it suffices to show that \( \hat{I}(r) \) is increasing in \( r \geq 0 \), so that \( I(r) = \hat{I}(r) \). Let \( r_1 > r \geq 0 \) such that \( \hat{I}(r_1) < \infty \). For any \( \varepsilon > 0 \) there exists \( \nu = f \mu \) with \( \mathbb{W}_2(\nu, \mu)^2 = r_1 \) such that \( I_\nu(\nu) \leq I(r_1) + \varepsilon \). Consider \( \nu_s = s \mu + (1 - s)\nu \) for \( s \in [0, 1] \). Since \( \mathbb{W}_2(\nu_s, \mu) \) is continuous in \( s \) and \( \mathbb{W}_2(\nu_0, \mu)^2 = r_1 > r, \mathbb{W}_2(\nu_1, \mu)^2 = 0 \), there exists \( s \in [0, 1] \) such that \( \mathbb{W}_2(\nu_s, \mu)^2 = r \). We have \( \frac{d}{ds} \nu_s = s(1 - s)f \), so that

\[ \hat{I}(r) = I_\nu(\nu_s) = \mu\left( \frac{(1 - s)^2|\nabla f|^2}{4(s + (1 - s)f)} \right) \leq \mu(|\nabla f|^2) = I_\nu(\nu) \leq \hat{I}(r_1) + \varepsilon. \]

Letting \( \varepsilon \downarrow 0 \) we prove \( \hat{I}(r) \leq \hat{I}(r_1) \) as desired.
By Lemma 3.3 and Lemma 3.5, we have

\[
I_\mu(\nu_n) = \mu(\|\nabla f_n^\frac{1}{2}\|^2) \leq I(r_n) + \frac{1}{n} \leq \alpha + \frac{1}{n}.
\]

Then, up to a subsequence, \(f_n^\frac{1}{2} \rightarrow f^\frac{1}{2}\) in \(L^2(\mu)\) for some probability density \(f\) with respect to \(\mu\). Thus, \(\nu_n \rightarrow \nu := f \mu\) weakly such that

\[
\mathbb{W}_2(\nu, \mu)^2 = \lim_{n \rightarrow \infty} \mathbb{W}_2(\nu_n, \mu)^2 = \lim_{n \rightarrow \infty} r_n = r,
\]

and \(I_\mu(\nu) \leq \lim \inf_{n \rightarrow \infty} I_\mu(\nu_n) \leq \alpha\). Therefore, \(I(r) \leq \alpha\) as desired.

**Proof of Theorem 1.1(3).** As shown in step (b) in the proof of (2.6), we only need to prove for \(C > 0\) that

\[
(3.38) \quad \lim_{t \rightarrow \infty} \sup_{\|\nu\| \leq C} \left| \mathbb{P}^\nu(t\mathbb{W}_2(\mu, \mu)^2 < a) - \nu_0((-\infty, a)) \right| = 0, \quad a \in \mathbb{R}.
\]

Take \(r_t = t^{-\frac{3}{2}}\), and let

\[
\tilde{\Xi}(t) := t \int_M \frac{|\nabla L^{-1}(f_{t,r_t} - 1)|^2}{\mathcal{M}((1-r_t)f_{t,r_t} + r_t, 1)} d\mu, \quad t > 0.
\]

By Lemma 3.3 and Lemma 3.5, we have

\[
\begin{aligned}
\lim_{t \rightarrow \infty} \sup_{\|\nu\| \leq C} \mathbb{E}^\nu \left| \tilde{\Xi}(t) - \Xi_{r_t}(t) \right| &\leq \lim_{t \rightarrow \infty} \sup_{\|\nu\| \leq C} \left( t^p \mathbb{E}^\nu \int_M |\nabla L^{-1}(f_{t,r_t} - 1)|^{2p} d\mu \right)^\frac{1}{p} \\
&\quad \times \left( \mathbb{E}^\nu \int_M \mathcal{M}((1-r_t)f_{t,r_t} + r_t, 1)^{-1} - 1 \right)^{\frac{p-1}{p}} d\mu,
\end{aligned}
\]

\[
= 0.
\]

Combining this with (2.58), we obtain

\[
(3.39) \quad \lim_{t \rightarrow \infty} \sup_{\|\nu\| \leq C} \left| \mathbb{P}^\nu(\tilde{\Xi}(t) < a) - \nu_0((-\infty, a)) \right| = 0, \quad a \in \mathbb{R}.
\]

By Lemma 2.3 and (3.32) we obtain

\[
\begin{aligned}
\{ t^2\mathbb{W}_2(\mu, \mu)^2 - (1 + \varepsilon)\tilde{\Xi}(t) \}^+ &\leq \{ t^2\mathbb{W}_2(\mu, \mu)^2 - (1 + \varepsilon)\mathbb{W}_2(\mu, \mu)^2 \}^+ \\
&\leq 2(1 + \varepsilon^{-1}) \{ \mathbb{W}_2(\mu, \mu)^2 + \mathbb{W}_2(\mu, \mu)^2 \}.
\end{aligned}
\]

Combining this with (3.3), (3.4), (3.10) and \(r_t = t^{-\frac{3}{2}}\), we derive

\[
\begin{aligned}
\lim_{t \rightarrow \infty} \sup_{\|\nu\| \leq C} \mathbb{E}^\nu \left\{ t^2\mathbb{W}_2(\mu, \mu)^2 - (1 + \varepsilon)\tilde{\Xi}(t) \right\}^+ \\
&\leq 2(1 + \varepsilon^{-1}) \{ \mathbb{W}_2(\mu, \mu)^2 + \mathbb{W}_2(\mu, \mu)^2 \}.
\end{aligned}
\]
\[ \leq 2(1 + \varepsilon^{-1}) \lim_{t \to \infty} \sup_{\|h\|_\infty \leq C} \mathbb{E}^\nu \left[ \left( \mathbb{W}_2(\mu_{t,r}, \mu_{t,r})^2 + \mathbb{W}_2(\mu_t, \mu_{t,r})^2 \right) \right] = 0, \quad \varepsilon > 0. \]

Therefore,

\[ (3.40) \lim_{t \to \infty} \sup_{\|h\|_\infty \leq C} \mathbb{P}^\nu(t \mathbb{W}_2(\mu_t, \mu)^2 \geq (1 + \varepsilon)\overline{\Xi}(t) + \varepsilon) = 0, \quad \varepsilon > 0. \]

On the other hand, (3.36) and (2.65) yield

\[ \lim_{t \to \infty} \sup_{\|h\|_\infty \leq C} \mathbb{E}^\nu \left[ \left( \Xi_r(t)^2 + \Xi_{r'}(t)^2 \right) \right] \leq 2 \lim_{\varepsilon \to 0} \sup_{\|h\|_\infty \leq C} \mathbb{E}^\nu \left| \Xi_r(t) - \Xi_{r'}(t) \right|, \quad r > 0. \]

Since \( \sum_{k=1}^{\infty} \lambda_k^2 < \infty \), (2.24) implies

\[ \lim_{r, r' \to 0} \sup_{\|h\|_\infty \leq C} \mathbb{E}^\nu \left| \Xi_r(t) - \Xi_{r'}(t) \right| \leq \lim_{r, r' \to 0} \sum_{k=1}^{\infty} \frac{2C}{\lambda_k^2} \left| e^{-2\lambda_k r} - e^{-\lambda_k r'} \right| = 0. \]

So, letting \( r \to 0 \) in (3.41) we derive

\[ \lim_{t \to \infty} \sup_{\|h\|_\infty \leq C} \mathbb{P}^\nu(t \mathbb{W}_2(\mu_t, \mu)^2 \leq \Xi_{r_t}(t) - \varepsilon) = 0, \quad \varepsilon > 0. \]

Combining this with (2.58), (3.39) and (3.40), we prove (3.38).

\[
4 \quad \text{Proof of Theorem 1.2}
\]

Let \( d \geq 4 \). As explained in step (b) in the proof of (2.6), it suffices to prove that for any \( C > 0 \) there exist constant \( c > 0 \) such that

\[ (4.1) \sup_{\|h\|_\infty \leq C} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu)^2] \leq \begin{cases} ct^{-1} \log(1 + t), & \text{if } d = 4, \\ ct^{-\frac{2}{d-2}}, & \text{if } d \geq 5, \end{cases} \]

and

\[ (4.2) \lim_{t \to \infty} \left\{ t^{\frac{2}{d-2}} \inf_{\|h\|_\infty \leq C} \mathbb{E}^\nu[\mathbb{W}_1(\mu_t, \mu)]^2 \right\} > 0. \]

\[ 4.1 \quad \text{Proof of (4.1)} \]

By the triangle inequality, Lemma 2.4 and Lemma 3.1, we find a constant \( c > 0 \) such that for \( r_t \in (0, 1) \),

\[ \mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t}, \mu)^2] \leq 2\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t}, \mu_{t,r_t})^2] + 2\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t}, \mu)^2] \]

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Notice that for some constants \( c_i > 0, i = 1, \ldots, 5 \) we have
\[
\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2 e^{2\gamma_i}} \leq c_i \sum_{i=1}^{\infty} \frac{i^{-\frac{5}{2}} e^{-c_i r_i^2 \frac{3}{8} s}}{s} \leq c_3 \int_{1}^{\infty} \frac{s^{-\frac{5}{2}} e^{-c_i r_i^2 \frac{3}{8} s}}{s} ds
\]
\[
= c_3 r_i^{-\frac{5}{2}} \int_{r_i^2}^{\infty} u^{-\frac{5}{2}} e^{-c_i u^2 \frac{3}{8} du} \leq c_5 \{ \log (1 + r_i^{-1}) \} 1_{\{d=4\}} + r_i^{-\frac{d+4}{2}} 1_{\{d\geq 5\}}.
\]
Taking \( r_i = \frac{\log(t+1)}{t} 1_{\{d=4\}} + t^{-\frac{d}{2}} 1_{\{d\geq 5\}} \), we have
\[
\limsup_{t \to \infty} \left\{ \frac{t}{\log(t) \sup_{\|h\|_{\infty} \leq C} \mathbb{E}[\mathcal{W}_2(\mu_t, \mu)^2]} \right\} < \infty, \quad d = 4,
\]
and
\[
\lim_{t \to \infty} \left\{ \frac{t^{\frac{d}{2}}}{\sup_{\|h\|_{\infty} \leq C} \mathbb{E}[\mathcal{W}_2(\mu_t, \mu)^2]} \right\} < \infty, \quad d \geq 5.
\]
Therefore, (4.1) holds for some constant \( c > 0 \).

4.2 Proof of (4.2)

In this subsection, instead of \( \mu_{t,r} \), we use another method to approximate \( \mu_t \). For any \( t \geq 1 \) and \( N \in \mathbb{N} \), we consider \( \mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{t_i} \), where \( t_i := \frac{(i-1)t}{N}, 1 \leq i \leq N \). We write
\[
\mu_t = \frac{1}{N} \sum_{i=1}^{N} \frac{N}{t} \int_{t_i}^{t_{i+1}} \delta_{X_s} ds.
\]
By the convexity of \( \mathcal{W}_2 \), which follows from the Kantorovich dual formula (2.9), we have
\[
\mathcal{W}_2(\mu_N, \mu_t)^2 \leq \frac{1}{N} \sum_{i=1}^{N} \frac{N}{t} \int_{t_i}^{t_{i+1}} \mathcal{W}_2(\delta_{X_{t_i}}, \delta_{X_s})^2 ds = \frac{1}{t} \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \rho(X_{t_i}, X_s)^2 ds.
\]
Moreover, by (3.9) and the Markov property we have
\[
\mathbb{E}[\rho(X_{t_i}, X_s)] \leq c_1 \|P_t h\|_{\infty}(s - t_i) \leq c_1 C(s - t_i), \quad s \geq t_i, \|h\|_{\infty} \leq C.
\]
Therefore, there exists a constant \( c' > 0 \) such that
\[
\sup_{\|h\|_{\infty} \leq C} \mathbb{E}[\mathcal{W}_2(\mu_N, \mu_t)^2] \leq c' \frac{C}{t} \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} (s - t_i) ds \leq \frac{c't}{N}, \quad N \in \mathbb{N}, t \geq 1.
\]
On the other hand, since \( M \) is compact possibly with a smooth boundary, and \( \mu \) is comparable with the volume measure, we find a constant \( c > 0 \) such that for any \( r > 0 \),
\[
\sup_{x \in M} \mu(B(x, r)) \leq cr^d.
\]
So, [20, Proposition 4.2] (see also [19, Corollary 12.14]) implies
\[ \{\mathbb{W}_1(\mu_N, \mu)\}^2 \geq c_2 N^{-\frac{2}{d}}, \quad N \in \mathbb{N}, t \geq 1. \]

This together with (4.3) yields
\[
\inf_{\|h_\nu\|_\infty \leq C} \{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]\}^2 \geq \frac{c_2}{2N^\frac{2}{d}} - \frac{c't}{N}, \quad N \in \mathbb{N}, t \geq 1.
\]

Taking \( N = \sup\{i \in \mathbb{N} : i \leq \alpha t^{\frac{d}{2}}\} \) for some \( \alpha > 0 \), we derive
\[
t^\frac{2}{\alpha} \inf_{\|h_\nu\|_\infty \leq C} \{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]\}^2 \geq \frac{c_2}{2\alpha^\frac{2}{d}} - \frac{2c'}{\alpha} > 0.
\]

\section{Proof of Theorem 1.3}

Recently, a PDE proof of AKT theorem based on Lusin approximation has been developed in [5]. In our setting, we adapt a similar strategy. We first prove the following result.

\textbf{Proposition 5.1.} Let \( M \) be compact with \( d = 4 \) and \( \partial M \) either convex or empty. If for any \( C > 0 \) there exist constants \( \varepsilon, \gamma > 0 \) such that for large enough \( t \) we have
\[
\{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]\}^2 \geq \varepsilon \mathbb{E}[\mu(\nabla(-L)^{-1}(f_{t,t-\gamma} - 1)]^2, \quad \nu = h_\nu \mu, \|h_\nu\|_\infty \leq C,
\]
then there exists a constant \( c > 0 \) such that
\[
\liminf_{t \to \infty} t^{(\log t)^{-1}} \inf_{x \in M} \{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]\}^2 > 0.
\]

\textbf{Proof.} As shown in step (b) in the proof of (2.6), it suffices to prove that for any constant \( C > 0 \),
\[
\liminf_{t \to \infty} t^{(\log t)^{-1}} \inf_{\|h_\nu\|_\infty \leq C} \{\mathbb{E}[\mathbb{W}_1(\mu, \mu)]\}^2 > 0.
\]

By taking \( r = t^{-\gamma} \), we deduce from (2.19) and (5.1) that
\[
\frac{1}{t} \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2 e^{2t-\gamma} \lambda_i}.
\]

Since \( \lambda_i \sim \sqrt{i} \) for \( d = 4 \), combining this with (3.36) and (2.69), we prove (5.3). \(\square\)
To verify (5.1), we need the following fundamental lemma, where the first assertion is known as Sard’s lemma (see [17, p130, Excercise 5.5]), and the second is called Lusin’s approximation which is well-known for $M$ being a bounded open domain in $\mathbb{R}^d$ (see [1, 23]). For completeness we include a simple proof of the second assertion for the present Riemannian setting.

**Lemma 5.2.** Let $M$ be a compact Riemannian manifold, and let $p \in (1, \infty)$.

1. For any $f \in W^{1, p}(\mu_M)$ and $c \in \mathbb{R}$, we have $\mu_M(\{|\nabla f| > 0, f = c\}) = 0$.

2. There exists a constant $K > 0$ such that for any $\alpha > 0$,

\[
\inf_{\|\nabla u\|_{\infty} \leq \alpha} \mu_M(\{u \neq f\}) \leq \frac{K}{\alpha^p} \mu_M(\{|\nabla f|^p\}, f \in W^{1, p}(\mu_M).
\]

**Proof.** Let $M$ be isoperimetrically embedded into $\mathbb{R}^m$ for some $m > d$, where $d$ is the dimension of $M$. Let $N := \{N^1, \ldots, N^{m-d}\}$ be smooth vector fields on $\mathbb{R}^m$ such that for any $\theta \in M$,

1. $\langle N^i, N^j_\theta \rangle = 1_{i=j} \leq i, j \leq m - d$;
2. $N^i_\theta$ is orthogonal to the tangent space $T_M$ of $M$.

Let $B_s := \{r \in \mathbb{R}^{m-d} : |r| < s\}$ and $M_s := \{x \in \mathbb{R}^m : \text{dist}(x, M) < s\}, s > 0$. Since $M$ is compact, there exists $s_0 > 0$ such that the map

\[ M \times B_{s_0} \ni (\theta, r) \mapsto \theta + \langle r, N_\theta \rangle \in M_{s_0} \]

is diffeomorphic, where $\langle r, N_\theta \rangle := \sum_{i=1}^{m-d} r_i N^i_\theta, r = (r_1, \ldots, r_{m-d}) \in \mathbb{R}^{m-d}$. We simply denote $\theta + \langle r, N_\theta \rangle$ by its polar coordinate $(\theta, r)$. Let $\Lambda$ be the Lebesgue measure on the open set $M_{s_0} \subset \mathbb{R}^m$. Then there exist a constant $c_1 \geq 1$ such that under the polar coordinate $(\theta, r) \in M \times B_{s_0}$ we have

\[
c_1^{-1} \mu_M(d\theta)dr \leq \Lambda(d\theta, dr) \leq c_1 \mu_M(d\theta)dr,
\]

\[
|\nabla g|^2(\theta, r) \leq c_1 \left(|\nabla g(\cdot, r)(\theta)|^2 + |\nabla g(\theta, \cdot)|^2(r)\right),
\]

\[
c_1^{-1} \rho(\theta_1, \theta_2) \leq |\theta_1 - \theta_2| \leq c_1 \rho(\theta_1, \theta_2), \quad \theta_1, \theta_2 \in M.
\]

Now, for any $f \in W^{1, p}(\mu_M)$, we extend it to $M_{s_0}$ by letting

\[ \tilde{f}(x) = f(\theta), \quad \text{if } x = (\theta, r). \]

So, by the Lusin approximation result on the bounded open domain $M_{s_0}$ in $\mathbb{R}^m$, there exists a constant $c_2 > 0$ such that for any $\alpha > 0$, we find a $(c_1^{-1} \alpha)$-Lipschitz function $\tilde{u}$ on $M_{s_0}$ such that

\[
\Lambda(\{x \in M_{s_0} : \tilde{u}(x) \neq \tilde{f}(x)\}) \leq \frac{c_2}{\alpha^p} \int_{M_{s_0}} |\nabla \tilde{f}|^p d\Lambda.
\]
Proposition 5.3. Let following result, which together with Proposition 5.1 implies the assertion in Theorem 1.3. Fourier analytic technique is developed in [10]. We will follow the line of [10] to prove the constant $c$.

Combining this with (5.5), (5.6) and noting that $\tilde{f}(\theta, r)$ does not depend on $r$, we find a constant $c_3 > 0$ such that

$$
\int_{B_{s_0}} \mu_M(\{f \neq \tilde{u}(\cdot, r)\}) dr \leq \frac{c_3}{\alpha^p} \int_M |\nabla f|^p d\mu_M.
$$

Thus, there exist $r \in B_{s_0}$ and a constant $c_4 > 0$ such that

$$
\mu_M(\{f \neq \tilde{u}(\cdot, r)\}) \leq \frac{c_4}{\alpha^p} \mu_M(|\nabla f|^p).
$$

Since $|\tilde{u}(x) - \tilde{u}(y)| \leq c_1^{-1} |x - y|$, (5.7) implies

$$
|\tilde{u}(\theta_1, r) - \tilde{u}(\theta_2, r)| \leq c_1^{-1} |\theta_1 - \theta_2| \leq \alpha \rho(\theta_1, \theta_2), \quad \theta_1, \theta_2 \in M.
$$

Therefore, (5.8) implies (5.4) for $K = c_4$. 

Besides the proof of AKT theorem in [5] using Riesz transform bounds, a simplified Fourier analytic technique is developed in [10]. We will follow the line of [10] to prove the following result, which together with Proposition 5.1 implies the assertion in Theorem 1.3.

**Proposition 5.3.** Let $M = \mathbb{T}^d$ and $V = 0$. Then for any $\gamma \in (0, \frac{2}{d})$, there exists a constant $c > 0$ such that

$$
\{\mathbb{E}^{\nu} \mathbb{W}_1(\mu_{t, t-\gamma}, \mu)\}^2 \geq c \mathbb{E}^{\nu} \mu(|\nabla (-\Delta)^{-1}(f_{t, t-\gamma} - 1)|^2), \quad t \geq 2
$$

holds for any probability measure $\nu$ on $M$.

**Proof.** (a) Let $f_t = (-\Delta)^{-1}(f_{t, t-\gamma} - 1)$. By Lemma 5.2, for any $\alpha > 0$ there exists an $\alpha$-Lipschitz function $u$ such that

$$
\mu(\{u - f_t \neq 0\}) \leq \frac{K}{\alpha^4} \mu(|\nabla f|^4)
$$

and

$$
\int_{\{u - f_t = 0\}} |\nabla (f_t - u)|^4 d\mu = 0.
$$

Then there exists a constant $K_1 > 0$ such that

$$
\begin{align*}
\alpha \mathbb{W}_1(\mu_{t, t-\gamma}, \mu) &\geq \mu_{t, t-\gamma}(u) - \mu(u) = \int_{\mathbb{T}^d} u(f_{t, t-\gamma} - 1) d\mu = \int_{\mathbb{T}^d} \langle \nabla u, \nabla f_t \rangle d\mu \\
&= \mu(|\nabla f_t|^2) - \int_{\mathbb{T}^d} \langle \nabla f_t, \nabla (f_t - u) \rangle d\mu = \mu(|\nabla f_t|^2) - \int_{\{f_t \neq u\}} \langle \nabla f_t, \nabla (f_t - u) \rangle d\mu \\
&\geq \mu(|\nabla f_t|^2) - \alpha \mu(1_{\{f_t \neq u\}} |\nabla f_t|^2) - \alpha \mu(1_{\{f_t \neq u\}} |\nabla f_t|) \\
&\geq \mu(|\nabla f_t|^2) - \sqrt{\mu(|\nabla f_t|^4) \mu(\{f_t \neq u\})} - \alpha \mu(\{f_t \neq u\})^{3/2} \mu(|\nabla f_t|^4)^{1/2} \\
&\geq \mu(|\nabla f_t|^2) - 2K_1 \alpha^{-2} \mu(|\nabla f_t|^4), \quad \alpha > 0.
\end{align*}
$$

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Thus,
\[ \mathbb{E}'[\mathbb{W}_1(\mu_{t,t^\gamma}, \mu)] \geq \alpha^{-1}\mathbb{E}'[\mu(|\nabla f_t|^2)] - 2K_1\alpha^{-3}\mathbb{E}'[\mu(|\nabla f_t|^4)], \quad \alpha > 0. \]

If we can find a constant \( K_2 > 0 \) such that
\[ \mathbb{E}'\mu(|\nabla f_t|^4) \leq K_2(\mathbb{E}'|\nabla f_t|^2)^2, \quad t \geq 2, \]
then we arrive at
\[ \mathbb{E}'\mathbb{W}_1(\mu_{t,t^\gamma}, \mu) \geq \alpha^{-1}\mu(|\nabla f_t|^2) - 2\alpha^{-3}\mathbb{E}'|\nabla f_t|^2, \quad \alpha > 0. \]

Taking \( \alpha = N\mu(|\nabla f_i|^2)^{-\frac{1}{2}} \) for large enough \( N > 1 \), we prove (5.9) for some constant \( c > 0 \).

(b) It remains to prove (5.10). To this end, we identify \( \mathbb{T} \) with \([0, 2\pi)\) by the one-to-one map
\[ [0, 2\pi) \ni s \mapsto e^{is}, \]
where \( i \) is the imaginary unit. In this way, a point in \( \mathbb{T}^4 \) is regarded as a point in \([0, 2\pi)^4\), so that \( \{e^{i(m,)}\}_{m \in \mathbb{Z}^4} \) consist of an eigenbasis of \( \Delta \) in the complex \( L^2 \)-space of \( \mu \), where \( \mu \) is the normalized volume measure on \( \mathbb{T}^4 \). We have
\[ f_t := (-\Delta)^{-1}(f_{t,t^\gamma} - 1) = \sum_{m \in \mathbb{Z}^4 \setminus \{0\}} b_m e^{-i(m,)}, \quad b_m := \frac{e^{-|m|^2t^{-\gamma}}}{|m|^2t^\gamma} \int_0^t e^{i(m,x)s}ds. \]
Then
\[ |\nabla f_t(x)|^2 = -\sum_{m_1,m_2 \in \mathbb{Z}^4 \setminus \{0\}} \langle m_1, m_2 \rangle b_{m_1}b_{m_2}e^{-i(m_1 + m_2,x)}, \]
\[ |\nabla f_t(x)|^4 = \sum_{m_1,\ldots,m_4 \in \mathbb{Z}^4 \setminus \{0\}} \langle m_1, m_2 \rangle \langle m_3, m_4 \rangle b_{m_1}b_{m_2}b_{m_3}b_{m_4}e^{-i(m_1 + m_2 + m_3 + m_4,x)}. \]
Noting that \( \mu(e^{-i(m,)}) = 0 \) for \( m \neq 0 \), we obtain
\[ \mathbb{E}'\mu(|\nabla f_t|^2) = \sum_{m \in \mathbb{Z}^4 \setminus \{0\}} |m|^2\mathbb{E}'[b_m b_{-m}], \]
\[ \mathbb{E}'\mu(|\nabla f_t|^4) = \sum_{(m_1,m_2,m_3,m_4) \in \mathbb{S}} \langle m_1, m_2 \rangle \langle m_3, m_4 \rangle \mathbb{E}'[b_{m_1}b_{m_2}b_{m_3}b_{m_4}], \]
where \( \mathbb{S} := \{(m_1,m_2,m_3,m_4) \in \mathbb{Z}^4 \setminus \{0\} : m_1 + m_2 + m_3 + m_4 = 0\} \).

By the definition of \( b_m \), we obtain
\[ \mathbb{E}'[b_m b_{-m}] = \frac{e^{-|m|^2t^{-\gamma}}}{|m|^2t^\gamma} \int_{[0,t]^2} \mathbb{E}'e^{i(m,x_2 - x_1)s}ds_1ds_2. \]
Since the Markov property and \( \mathbb{E}'e^{i(m,x)} = e^{-|m|^2s}e^{i(m,x)} \) imply
\[ \mathbb{E}'(e^{i(m,x_2 - x_1)}|\mathcal{F}_{s_1 \wedge s_2}) = e^{-|m|^2|s_1 - s_2|}, \]

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Similarly to (5.13), we have

$$
\mathbb{E}^\nu[b_m b_{-m}] = \frac{e^{-2|m|^2 t^{-\gamma}}}{|m|^2 t^2} \int_{[0,t]^2} e^{-|m|^2 |s_1-s_2|} ds_1 ds_2 \geq \frac{\kappa e^{-2|m|^2 t^{-\gamma}}}{|m|^6 t}, \quad t \geq 2.
$$

Combining this with (5.11) we find a constant $\kappa_1 > 0$ such that

$$
\mathbb{E}^\nu(\nu f_t)^2 \geq \sum_{m \in \mathbb{Z}^4 \setminus \{0\}} \frac{\kappa e^{-2|m|^2 t^{-\gamma}}}{|m|^4 t} \geq \frac{\kappa_1}{t} \int_1^\infty \frac{e^{-2s^2 t^{-\gamma}}}{s} ds
$$

(5.14)

$$
\geq \frac{\kappa_1}{2e^2} t^{-2} s^{-1} ds = \frac{\kappa_1 \gamma}{2e^2} (t^{-1} \log t), \quad t \geq 2.
$$

(c) By (5.12), to estimate $\mathbb{E}^\nu(\nu f_t)^4$, we calculate $\mathbb{E}^\nu[b_{m_1} b_{m_2} b_{m_3} b_{m_4}]$ for $(m_1, m_2, m_3, m_4) \in S$. Let $D(t) = \{(s_1, s_2, s_3, s_4) \in [0,t]^4 : 0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq t\}$, and let $S$ be the set of all the permutations of $\{1, 2, 3, 4\}$. We have

$$
\mathbb{E}^\nu[b_{m_1} b_{m_2} b_{m_3} b_{m_4}] = \frac{e^{-\sum_{j=1}^4 |m_j|^2 t^{-\gamma}}}{t^4 \prod_{p=1}^4 |m_p|^2} \int_{[0,t]^4} \mathbb{E}^\nu[e^{i|m_1|X_{s_1}} e^{i|m_2|X_{s_2}} e^{i|m_3|X_{s_3}} e^{i|m_4|X_{s_4}}] ds_1 ds_2 ds_3 ds_4
$$

(5.15)

$$
= \frac{e^{-\sum_{j=1}^4 |m_j|^2 t^{-\gamma}}}{t^4 \prod_{p=1}^4 |m_p|^2} \sum_{(i,j,k,l) \in S} \int_{D(t)} \mathbb{E}^\nu[e^{i|m_i|X_{s_1}} e^{i|m_j|X_{s_2}} e^{i|m_k|X_{s_3}} e^{i|m_l|X_{s_4}}] ds_1 ds_2 ds_3 ds_4.
$$

Similarly to (5.13), we have

$$
\mathbb{E}^\nu[e^{i|m_i|X_{s_1}} e^{i|m_j|X_{s_2}} e^{i|m_k|X_{s_3}} e^{i|m_l|X_{s_4}}]
$$

$$
= \mathbb{E}[e^{i(m_i X_{s_1}) e^{i(m_j X_{s_2})} e^{i(m_k X_{s_3})} e^{i(m_l X_{s_4})} | \mathcal{F}_{s_1}}]
$$

$$
= e^{-|m_i|^2 (s_4-s_3)} \mathbb{E}^\nu[e^{i|m_i|X_{s_1}} e^{i|m_j|X_{s_2}} e^{i|m_k X_{s_3}} e^{i|m_l+m_k|X_{s_4}} | \mathcal{F}_{s_1}] 
$$

$$
= e^{-|m_i|^2 (s_4-s_3)-|m_i+m_k|^2 (s_3-s_2)-|m_i|^2 (s_2-s_1)} 
$$

where in the last step we have used $m_i + m_j + m_k + m_l = 0$. Combining this with (5.15) we arrive at

$$
\mathbb{E}^\nu[b_{m_1} b_{m_2} b_{m_3} b_{m_4}]
$$

(5.16)

$$
= \frac{e^{-\sum_{j=1}^4 |m_j|^2 t^{-\gamma}}}{t^4 \prod_{p=1}^4 |m_p|^2} \sum_{(i,j,k,l) \in S} \int_{D(t)} e^{-|m_i|^2 (s_4-s_3)-|m_i+m_k|^2 (s_3-s_2)-|m_i|^2 (s_2-s_1)} ds_1 ds_2 ds_3 ds_4.
$$

When $m_l + m_k = 0$, we have

$$
\int_{D(t)} e^{-|m_i|^2 (s_4-s_3)} e^{-|m_i+m_k|^2 (s_3-s_2)} e^{-|m_i|^2 (s_2-s_1)} ds_1 ds_2 ds_3 ds_4
$$
\[ \int_{0}^{t} \int_{s_{1}}^{t} \int_{s_{2}}^{t} \int_{s_{3}}^{t} e^{-|m_{1}|^{2}(s_{4}-s_{3})} e^{-|m_{1}|^{2}(s_{2}-s_{1})} ds_{4} ds_{3} ds_{2} ds_{1} \leq \frac{t^{2}}{|m_{1}|^{2}|m_{l}|^{2}}. \]

When \( m_{l} + m_{k} \neq 0 \), we have

\[ \int_{D(t)} e^{-|m_{1}|^{2}(s_{4}-s_{3})} e^{-|m_{1}+m_{k}|^{2}(s_{3}-s_{2})} e^{-|m_{1}|^{2}(s_{2}-s_{1})} ds_{4} ds_{3} ds_{2} ds_{1} \]

\[ = \int_{0}^{t} \int_{s_{1}}^{t} \int_{s_{2}}^{t} \int_{s_{3}}^{t} e^{-|m_{1}|^{2}(s_{4}-s_{3})} e^{-|m_{1}+m_{k}|^{2}(s_{3}-s_{2})} e^{-|m_{1}|^{2}(s_{2}-s_{1})} ds_{4} ds_{3} ds_{2} ds_{1} \]

\[ \leq \frac{t^{2}|m_{l} + m_{k}|^{2}|m_{l}|^{2}}{|m_{l}|^{2}|m_{l} + m_{k}|^{2}|m_{l}|^{2}}. \]

Therefore, (5.16) implies

\[ \mathbb{E}^{\mu}[b_{m_{1}}b_{m_{2}}b_{m_{3}}b_{m_{4}}] \leq \frac{e^{-\sum_{p=1}^{4} |m_{p}|^{2}t^{-\gamma}}}{\prod_{p=1}^{4} |m_{p}|^{2}} \sum_{(i,j,k,l) \in S} \left\{ \frac{t^{-21}_{m_{l}+m_{k}=0}}{|m_{l}|^{2}|m_{l}|^{2}} + \frac{t^{-31}_{m_{l}+m_{k} \neq 0}}{|m_{l}|^{2}|m_{l} + m_{k}|^{2}|m_{l}|^{2}} \right\}, \]

so that (5.12) yields

\[ \mathbb{E}^{\mu}[|\nabla f_{t}|^{4}] \leq C(I_{1} + I_{2}), \quad t \geq 2 \]

for some constant \( C > 0 \), where

\[ I_{1} := \frac{1}{t^{2}} \sum_{a, b \in \mathbb{Z}^{4}\{0\}} \frac{1}{|a|^{4}|b|^{4}} e^{-2(|a|^{2}+|b|^{2})t^{-\gamma}}, \]

\[ I_{2} := \frac{1}{t^{3}} \sum_{m_{1}, m_{2}, m_{3}, m_{4} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-\sum_{p=1}^{4} |m_{p}|^{2}t^{-\gamma}}}{|m_{1}|^{3}|m_{2}|^{3}|m_{3} + m_{4}|^{2}|m_{4}|^{3}}. \]

Obviously, there exist constants \( c_{1}, c_{2} > 0 \) such that

\[ I_{1} \leq \frac{c_{1}}{t^{2}} \left( \int_{1}^{\infty} \frac{2s^{2}t^{-\gamma}}{s} ds \right)^{2} \leq c_{2}(t^{-1} \log t)^{2}, \quad t \geq 2. \]

Similarly, there exists a constant \( c_{3} > 0 \) such that

\[ \sum_{m_{1} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{1}|^{2}t^{-\gamma}}}{|m_{1}|^{3}} \leq c_{3}t^{\frac{3}{2}}, \quad \sum_{m_{2} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{2}|^{2}t^{-\gamma}}}{|m_{2}|} \leq c_{3}t^{\frac{3}{2}}, \quad t \geq 2. \]

So,

\[ I_{2} = \frac{1}{t^{3}} \left( \sum_{m_{1} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{1}|^{2}r}}{|m_{1}|^{3}} \right) \left( \sum_{m_{2} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{2}|^{2}r}}{|m_{2}|} \right) \sum_{m_{3}, m_{4} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{3}|^{2}+|m_{4}|^{2}r}}{|m_{3}||m_{3} + m_{4}|^{2}|m_{4}|^{3}} \]

\[ \leq c_{3}^{2}t^{2\gamma-3} \sum_{m_{4} \in \mathbb{Z}^{4}\{0\}} \frac{e^{-|m_{4}|^{2}r}}{|m_{4}|^{3}} \sum_{m_{3} \in \mathbb{Z}^{4}\{0\}, m_{3}-m_{4}} \frac{e^{-|m_{3}|^{2}r}}{|m_{3}||m_{3} + m_{4}|^{2}}. \]
Write
\[
\sum_{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\}} e^{-|m_3|^2 r} \frac{e^{-|m_4|^2 t - \gamma}}{|m_3||m_3 + m_4|^2} = S_1 + S_2 + S_3,
\]
where
\[
S_1 := \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\}} \frac{e^{-|m_3|^2 t - \gamma}}{|m_3||m_3 + m_4|^2},
\]
\[
S_2 := \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\}} \frac{e^{-|m_3|^2 t - \gamma}}{|m_3||m_3 + m_4|^2},
\]
\[
S_3 := \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\}} \frac{e^{-|m_3|^2 t - \gamma}}{|m_3||m_3 + m_4|^2}.
\]
Since on the region \(\{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\} : |m_3| \leq \frac{|m_4|}{2}\}\) we have \(|m_3 + m_4|^2 \sim |m_4|^2\), there exists a constant \(c_4 > 0\) such that
\[
S_1 \leq \frac{4}{|m_4|^2} \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0\}} \frac{e^{-|m_3|^2 t - \gamma}}{|m_3|^3} \leq \frac{c_4 t^{\frac{3\gamma}{2}}}{|m_4|^2}, \quad t \geq 2.
\]
Next, since on the region \(\{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\} : |m_3| > 2|m_4|\}\) it holds \(|m_3 + m_4|^2 \sim |m_3|^2\) and \(|m_3|^2 > \frac{|m_3|^2}{2} + 2|m_4|^2\), there exists a constant \(c_5 > 0\) such that
\[
S_3 \leq 4 \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0\}} \frac{e^{-|m_3|^2 t - \gamma}}{|m_3|^3} \leq 4e^{-2|m_4|^2 r} \sum_{m_3 \in \mathbb{Z}^4 \setminus \{0\}} \frac{e^{-|m_3|^2 r}}{|m_3|^3} \leq c_5 t^{2 - \frac{3\gamma}{2}} e^{-2|m_4|^2 t - \gamma}.
\]
Noting that \(e^{-s} \leq s^{-1}\) for \(s > 0\), this implies
\[
S_3 \leq \frac{c_5 t^{\frac{3\gamma}{2}}}{|m_4|^2}, \quad t \geq 2.
\]
Finally, on the region \(\{m_3 \in \mathbb{Z}^4 \setminus \{0, -m_4\} : \frac{|m_4|}{2} < |m_3| \leq 2|m_4|\}\) there holds \(|m_3| \sim |m_4|\) and \(1 \leq |m_3 + m_4| \leq 3|m_4|\), so that there exists a constant \(c_6 > 0\) such that
\[
S_2 \leq \frac{2e^{-|m_3|^2 t - \gamma}}{|m_4|} \sum_{1 \leq |m_3 + m_4| \leq 3|m_4|} \frac{1}{|m_3 + m_4|^2} \leq c_6 |m_4| e^{-\frac{|m_4|^2 t - \gamma}{8}}.
\]
Using \(e^{-s} \leq cs^{-\frac{3}{2}}\) for some constant \(c > 0\) and all \(s > 0\), we find a constant \(c_7 > 0\) such that
\[
S_2 \leq \frac{c_7 t^{\frac{3\gamma}{2}}}{|m_4|^2}, \quad t \geq 2.
\]
Combining this with (5.20), (5.21) and (5.22), we prove

\[ I_2 \leq c_8 t^{\frac{3\gamma}{2} + 2\gamma - 3} = c_8 t^{\frac{7\gamma}{2} - 3}, \quad t \geq 2 \]

for some constant \( c_8 > 0 \). Since \( \gamma \in (0, \frac{2}{7}) \), substituting this and (5.18) into (5.17) we derive

\[ \mathbb{E}^\nu \mu(|\nabla f_t|^4) \leq c_9 (t^{-1} \log t)^2, \quad t \geq 2 \]

for some constant \( c_9 > 0 \). This together with (5.14) implies (5.10) for some constant \( K_2 > 0 \), and hence finishes the proof.

\[ \Box \]

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