REES ALGEBRAS OF CLOSED DETERMINANTAL FACET IDEALS

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Abstract. Using SAGBI basis techniques, we find Gröbner bases for the presentation ideals of the Rees algebra and special fiber ring of a closed determinantal facet ideal. In particular, we show that closed determinantal facet ideals are of fiber type and their special fiber rings are Koszul. Moreover, their Rees algebras and special fiber rings are normal Cohen-Macaulay domains and have rational singularities.

1. Introduction

In this work, we study the blow-ups of certain determinantal varieties called determinantal facet ideals. To be more specific, we find the homogeneous coordinate rings of graphs and images of the blow-ups of a projective space along its subscheme defined by a certain class of determinantal varieties. Given an ideal $I$ in a polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$ over a field $\mathbb{K}$, the Rees algebra of $I$ is defined to be the graded algebra $\mathcal{R}(I) = \bigoplus_{i=1}^{\infty} I^i t^i \subset R[t]$, where $t$ is an indeterminate over $R$, and the special fiber ring $\mathcal{F}(I)$ is defined as $\mathcal{R}(I) \otimes \mathbb{K}$.

The Rees algebra is an important object in commutative algebra, algebraic geometry, elimination theory, intersection theory, geometric modeling, chemical reaction networks, and many more fields; see [6] and [7] for details on such applications. The Rees algebra has been the focus of many commutative algebra projects since the late 1950’s; see, for instance, [20]. If the ideal $I$ is minimally generated by $\mu$ elements, we find ideals $\mathcal{J}$ and $\mathcal{K}$ over polynomial rings $S = R[T_1, ..., T_\mu]$ and $\mathbb{K}[\mathcal{T}] = \mathbb{K}[T_1, ..., T_\mu]$ respectively, such that $\mathcal{R}(I) = S/\mathcal{J}$ and $\mathcal{F}(I) = \mathbb{K}[\mathcal{T}]/\mathcal{K}$. The defining equations of $\mathcal{J}$ and $\mathcal{K}$ are implicit equations of the varieties defined by the graph and image of a blow-up, respectively. Finding the implicit equations of the presentation ideals $\mathcal{J}$ and $\mathcal{K}$ of $\mathcal{R}(I)$ and $\mathcal{F}(I)$, respectively, is a challenging problem and is still open for many classes of ideals. In particular, the presentation ideals of the Rees algebra of determinantal ideals are only known in very special cases. Conca, Herzog, and Valla found the presentation ideal for the Rees algebra of the ideal of maximal minors of a generic matrix in [5]. The presentation ideal for the Rees algebra of the ideal of the rational normal scroll associated with a $2 \times n$ matrix was shown by Sammartano in [18]. Very recently, the case of two-minors of a generic $3 \times n$ matrix was resolved by Huang, Perlman, Polini, Raicu, and Sammartano in [15].

Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a ring $R = \mathbb{K}[X]$, and assume $m \leq n$. It is well-known that the ideal of maximal minors of $X$, denoted $I_m(X)$, is of fiber...
type; that is, $J$ is generated by linear relations with respect to the variables in $T$, and the generators of $K$ (see [5] and Subsection ??). The presentation ideal of the special fiber ring, $K$, is actually the ideal of a Grassmannian, and is defined by Plücker relations; see for example [17, Chapter 14]. Moreover, Eisenbud and Huneke proved in [9] that the Rees algebra of maximal minors is a normal Cohen-Macaulay domain. In the 1980s, Simis and his coauthors considered the Rees algebra and special fiber ring for the sub-ideal of maximal minors. More precisely, they considered the case where all minors share the first $k$ columns of $X$. Bruns and Simis found the symmetric algebra for this class of ideals in [2]. In later work with Trung, they used the Hodge algebra structure on these ideals to give the defining equations of the Rees algebra in [3] and concluded that they are of fiber type. To the best of the authors’ knowledge, not much more is known for Rees algebras of sub-ideals of $I_m(X)$. Even in the case when the ideal is generated by a subset of maximal minors of a $2 \times 5$ matrix, the ideal may not be of fiber type; see Example 2.8 Clearly, one needs to impose extra conditions in order to have hope of describing generators of presentation ideals of Rees algebras and their properties.

Determinantal facet ideals, which were introduced by Ene, Herzog, Hibi, and Mohammadi in [11], are generated by a subset of maximal minors of an $m \times n$ matrix indexed by the facets of a pure $(m-1)$-dimensional simplicial complex $\Delta$ on $n$ vertices. They are a natural generalization of binomial edge ideals, which were introduced by Herzog, et. al. in [13] due to their connections with algebraic statistics; see, for example, [8]. We restrict our attention to the case when a determinantal facet ideal is closed (see Definition 2.5). In this case, the generating set of the ideal corresponding to the set of facets of a simplicial complex with certain combinatorial properties forms a reduced Gröbner basis with respect to $>$, where $>$ denotes the lexicographic monomial order induced by $x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{mn}$. Ene, Herzog, and Hibi conjectured in [10] that the graded Betti numbers of a closed binomial edge ideal and its initial ideal with respect to $>$ coincide. This was confirmed in [1] for the case when $m \geq 3$. Thus, it is natural to study closed determinantal facet ideals via their initial ideals. For the Rees algebras and special fiber rings, the natural tool is the theory of SAGBI bases. It has been used successfully to find Gröbner bases for the presentation ideal of the Rees algebra for certain rational normal scrolls in [5], its secant varieties in [16], and sparse matrices in [4].

The paper is outlined as follows. We establish notation and recall some preliminaries in Section 2. Section 3 gives a novel proof of the presentation ideal of the Rees algebra of the initial ideal of maximal minors of a generic matrix. The techniques of Section 3 serve as a road map for the general case in Section 4. In Proposition 3.1, we show that a set of marked polynomials form a Gröbner basis for the presentation ideal of the Rees algebra with respect to some term order $\tau'$. To do this, we define the notion of “sorted” monomials and show that these sorted monomials are never leading terms of our set of marked polynomials with respect to $\tau'$. Moreover, we show that sorted monomials are linearly independent. See [19, Chapter 14] for more details on this technique. The more general case of a closed determinantal facet ideal is tackled in section 4. We define a further sorting, called “clique-sorted”, using a natural ordering of the maximal
cliques of $\Delta$ that is possible when $\Delta$ is closed, in Definition 4.2. An explicit Gröbner basis for the Rees algebra of the initial ideal of a closed determinantal facet ideal is provided in Theorem 4.1, extending a result of Ene, Herzog, Hibi, and Mohammadi in \cite[Corollary 1.4]{Ene}. Finally, we then show that these equations lift to a Gröbner basis for the Rees algebra of a closed determinantal facet ideal in Theorem 4.7. We conclude that any closed determinantal facet ideal is of fiber type and give necessary and sufficient conditions for it to be of linear type, recovering a theorem of Bruns, Simis, and Trung in \cite{Bruns}. In particular, the special fiber ring of any closed determinantal facet ideal is Koszul and its presentation ideal is generated by Plücker relations in Corollary 4.9. Finally, via the SAGBI basis deformation, we see that both the Rees algebra and special fiber ring of a closed determinantal facet ideal are normal Cohen-Macaulay domains and have rational singularities, extending a result of Eisenbud and Huneke in \cite{Eisenbud}.

2. Preliminaries

2.1. Closed Determinantal Facet Ideals. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates where $m \leq n$, and let $R = \mathbb{K}[X]$ be the polynomial ring over a field $\mathbb{K}$ in the indeterminates $x_{ij}$. For indices $a = \{a_1, \ldots, a_m\}$ such that $1 \leq a_1 < \cdots < a_m \leq n$, set $[a] = [a_1, \ldots, a_m]$ to be the maximal minor of $X$ involving columns in $a$. The ideal generated by all $m$-minors of $X$ is denoted by $I_m(X)$.

**Definition 2.1.** Let $\Delta$ be a pure $(m-1)$-dimensional simplicial complex on the vertex set $V = [n]$. A determinantal facet ideal $J_\Delta \subseteq R$ is the ideal generated by determinants of the form $[a]$ where $a$ supports an $m-1$ face of $\Delta$; that is, the columns of $[a]$, $a_1, \ldots, a_m$, correspond to a facet $F = \{a_1, \ldots a_m\} \in \Delta$.

Notice that when $m = 2$, one may identify $\Delta$ with a graph $G$. In this case, $J_G$ is called a binomial edge ideal.

**Definition 2.2.** For a pure simplicial complex $\Delta$ and an integer $i$, the $i$-th skeleton $\Delta^{(i)}$ of $\Delta$ is the subcomplex of $\Delta$ whose faces are those faces of $\Delta$ whose dimension is at most $i$. Let $\mathcal{H}$ denote the set of simplices $\Gamma$ with $\dim(\Gamma) \geq m - 1$ and $\Gamma^{(m-1)} \subset \Delta$. Let $\Gamma_1, \ldots, \Gamma_r$ be the maximal elements in $\mathcal{H}$ with respect to inclusion, and let $\Delta_i := \Gamma_i^{(m-1)}$. Each $\Gamma_i$ is called a maximal clique, and any subset of the vertices of $\Gamma_i$ is called a clique. The simplicial complex $\Delta^{\text{clique}}$ whose facets are the cliques of $\Delta$ is called the clique complex associated to $\Delta$. The decomposition $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ is called the clique decomposition of $\Delta$.

**Remark 2.3.** Let $I$ be an ideal generated by an arbitrary subset of maximal minors of $X$. The simplicial complex $\Delta$ associated to a determinantal facet ideal can be viewed as a combinatorial tool to index generators of such an ideal, since the vertices of each facet correspond to the columns defining a minor in the generating set of $I$. For any $\Delta_i$ in the clique decomposition of $\Delta$, let $V_i$ denote the vertex set of $\Delta_i$. Then each $\Delta_i$ corresponds to a submatrix $X_{\Delta_i}$ of $X$ with columns in the set $V_i$ such that the ideal of maximal minors $I_m(X_{\Delta_i})$ is contained in $J_\Delta$.

**Notation 2.4.** Let $>$ denote the lexicographic monomial order induced by the natural order of indeterminates $x_{11} > x_{12} > \cdots > x_{21} > x_{22} > \cdots > x_{mn}$. Set $x_\mathbf{a} = \text{in}_>[\mathbf{a}] = \ldots$
$x_{1a_1}x_{2a_2} \cdots x_{ma_m}$. Frequently, we will drop $>$ and simply write $\text{in}(J_\Delta)$ for the initial ideal of $J_\Delta$.

In general, the initial ideal of a determinantal facet ideal with respect to an arbitrary term order is not very well understood. A combinatorial algorithm for obtaining the initial ideal for binomial edge ideals with respect to $>$ is observed in [13], but no such algorithm is known for the case when $m > 2$. However, the Gröbner basis of a determinantal facet ideal with respect to $>$ is well understood in the case when the corresponding simplicial complex is closed.

**Definition 2.5.** A simplicial complex $\Delta$ is said to be closed (with respect to a given labeling) if it satisfies the following equivalent conditions:

(a) Any two facets $F = \{a_1 < \cdots < a_m\}$ and $G = \{b_1 < \cdots < b_m\}$ with $a_i = b_i$ for some $i$ satisfy the property that the $(m - 1)$-skeleton of the simplex on the vertex set $F \cup G$ is contained in $\Delta$;

(b) All facets $F = \{a_1 < a_2 < \cdots < a_m\}$ and $G = \{b_1 < b_2 < \cdots < b_m\}$ such that $F$ and $G$ are not in the same clique of $\Delta$ satisfy $a_\ell \neq b_\ell$ for all $\ell$;

(c) All facets $F = \{a_1 < a_2 < \cdots < a_m\}$ and $G = \{b_1 < b_2 < \cdots < b_m\}$ such that $F$ and $G$ are not contained in the same clique of $\Delta$ satisfy that the monomials $\text{in}_> [a_1, \ldots, a_m]$ and $\text{in}_> [b_1, \ldots, b_m]$ are relatively prime.

We frequently refer to the ideal $J_\Delta$ as being closed when the simplicial complex $\Delta$ is closed.

**Proposition 2.6.** ([11, Theorem 1.1]) Let $\Delta$ be a pure and closed $(m - 1)$-dimensional simplicial complex. Then the generators of $J_\Delta$ form a Gröbner basis of $J_\Delta$ with respect to $>$. 

**Remark 2.7.** Theorem 1.1 in [11] is a necessary and sufficient condition for the minimal generating set of $J_\Delta$ to form a Gröbner basis with respect to $>$ in the case when no two maximal cliques of $\Delta$ intersect by more than $m - 1$ vertices. We do not impose this condition on $\Delta$, hence we only state the sufficient statement here.

### 2.2. Rees Algebras.

Given a pure $(m - 1)$-dimensional simplicial complex $\Delta$ and its determinantal facet ideal $J_\Delta$ in the polynomial ring $R = \mathbb{K}[x_{ij}]$, the *Rees algebra* of $J_\Delta$, denoted $\mathcal{R}(J_\Delta)$, is the graded subalgebra $R[J_\Delta t]$ of the polynomial ring $R[t]$. Set $T = \{T_a \mid a \text{ is an } (m - 1)\text{-face of } \Delta\}$. Define the following standard presentations of the symmetric algebra $S(J_\Delta)$, of Rees algebra $\mathcal{R}(J_\Delta)$ of $J_\Delta$, and of the special fiber ring $\mathcal{F}(J_\Delta)$:

$$
\begin{align*}
\rho : R[T] &\to S(J_\Delta), \\
\phi : R[T] &\to \mathcal{R}(J_\Delta), \\
\psi : \mathbb{K}[T] &\to \mathcal{F}(J_\Delta)
\end{align*}
$$

where for all $i,j$, $\rho(x_{ij}) = x_{ij} = \phi(x_{ij})$, $\rho(T_a) = [a] = \psi(T_a)$, and $\phi(T_a) = [a] \cdot t$.

Let $\mathcal{L} = \ker \rho$, $\mathcal{J} = \ker \phi$, and $\mathcal{K} = \ker \psi$. The ideals $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{K}$ are called the *presentation ideals* of $S(J_\Delta)$, $\mathcal{R}(J_\Delta)$, and $\mathcal{F}(J_\Delta)$, respectively. We sometimes refer to the ideals $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{K}$ as the symmetric ideal, the Rees ideal, and the special fiber.
ideal, respectively. When $\mathcal{L} = \mathcal{J}$, the ideal $J_\Delta$ is of linear type. If $\mathcal{J} = \mathcal{L} + \mathcal{K} \cdot R[T]$, then $J_\Delta$ is of fiber type.

Finding the presentation ideal $\mathcal{J}$ is not easy in general. Given the presenting matrix $M$ of $J_\Delta$, the generators of $\mathcal{L}$ are given by $[T_{a_1}, \ldots, T_{a_n}] \cdot M$ where $\mu$ denotes the number of $(m-1)$-faces of $\Delta$. In the best scenario when $J_\Delta$ is of linear type, this gives the Rees ideal $\mathcal{J}$. However, little is known about the resolutions of determinantal facet ideals, so even finding the symmetric algebra can be difficult. The next best case is when an ideal $J_\Delta$ is of fiber type. Although the ideal of maximal minors is known to be of fiber type, this is not true in general for a determinantal facet ideal.

**Example 2.8.** Let $J_G$ be the binomial edge ideal corresponding to the graph $G$ with edge set $\{(1,2), (1,4), (1,5), (2,3), (3,4), (3,5)\}$. Then

$$f = x_{12}T_{35}T_{14} - x_{14}T_{35}T_{12} - x_{12}T_{34}T_{15} + x_{15}T_{34}T_{12} - x_{14}T_{23}T_{15} + x_{15}T_{23}T_{14}$$

is a minimal generator of the Rees ideal, $\mathcal{J}$, of $J_G$, but is contained in neither the symmetric ideal, $\mathcal{L}$, nor the special fiber ideal, $\mathcal{K}$, of $J_G$.

### 2.3. SAGBI Bases.

Our goal is to use the theory of SAGBI basis deformations developed in [5] to find the presentation ideals of the Rees algebra and special fiber ring of closed determinantal facet ideals. In this way, one can use the Rees algebra of the initial ideal of $J_\Delta$ to understand the Rees algebra of $J_\Delta$. We recall the definition of a SAGBI basis below. For further reference on SAGBI bases, see [19] Chapter 11; for details about applications of SAGBI bases to Rees algebras, see [5].

**Definition 2.9.** Let $R$ be a polynomial ring over a field $\mathbb{K}$, and let $A \subset R$ be a finitely generated $\mathbb{K}$-subalgebra. Fix a term order $\tau$ on the monomials in $R$ and let $\text{in}_\tau(A)$ be the $\mathbb{K}$-subalgebra of $R$ generated by the initial monomials $\text{in}_\tau(a)$ where $a \in A$. We say that $\text{in}_\tau(A)$ is the initial algebra of $A$ with respect to $\tau$. A set of elements of $A \subseteq \mathcal{A}$ is called a **SAGBI basis** if $\text{in}_\tau(A) = \mathbb{K}[\text{in}_\tau(A)]$.

**Definition 2.10.** Let $> \tau$ be the lexicographic order on $R = \mathbb{K}[X]$ as in Notation [2,4]. Extend $> \tau$ to a monomial order $> \prime$ on $R[t]$ as follows: for monomials $m_1 \cdot t^i$ and $m_2 \cdot t^j$ of $\mathbb{K}[X][t]$, set $m_1 \cdot t^i > \prime m_2 \cdot t^j$ if $i > j$ or if $i = j$ and $m_1 > m_2$ in $R$.

The main goal of this paper is to use SAGBI basis deformation developed in [5] to study $\mathcal{R}(J_\Delta)$. In particular, we want to show $\text{in}_\tau(\mathcal{R}(J_\Delta)) = \mathcal{R}(\text{in}_\tau(J_\Delta))$. The first step is to understand $\mathcal{R}(\text{in}_\tau(J_\Delta))$. We define the presentations of the Rees algebra $\mathcal{R}(\text{in}(J_\Delta))$ of $\text{in}(J_\Delta)$, and of the special fiber ring $\mathcal{F}(\text{in}(J_\Delta))$, as follows:

$$\phi^* : R[T] \rightarrow \mathcal{R}(\text{in}(J_\Delta)),$$

$$\psi^* : \mathbb{K}[T] \rightarrow \mathcal{F}(\text{in}(J_\Delta))$$

where for all $i, j$, $\phi^*(x_{ij}) = x_{ij}$, $\phi^*(T_a) = x_a \cdot t$, and $\psi^*(T_a) = x_a$.

### 3. Rees Algebra of $\text{in}(I_m(X))$

The goal of this section is to determine the presentation ideal of $\mathcal{R}(\text{in}(J_\Delta))$ in the case when $J_\Delta$ corresponds to the ideal of maximal minors of a generic $m \times n$ matrix $X$. In this case, $\Delta$ has a unique clique that contains all $n$ vertices of $\Delta$; it is therefore the
simplest possible case of a closed determinantal facet ideal. Understanding this case will be crucial to finding the Rees algebra of more general closed determinantal facet ideals in the next section. The defining equations of \( \mathcal{R}(\text{in}(I_m(X))) \) are well-known and follow from the fact that \( \text{in}(I_m(X)) \) satisfies the so-called \( \ell \)-exchange property (see [14, Definition 4.1, Theorem 5.1]). Our proofs of the existence of monomial orders for which the generators of \( \mathcal{F} \) and \( \mathcal{K} \) form a Gröbner basis will serve a critical role in the proof for the existence of similar monomial orders for general closed determinantal facet ideals.

We first recall the Plücker poset from [17, Chapter 14]. Let \( \mathcal{P} = \{ a \mid a \text{ is an } m\text{-subset of } [n] \} \) be a poset. When \( a = \{a_1 < \cdots < a_m\} \) and \( b = \{b_1 < \cdots < b_m\} \) are two \( m \)-subsets of \( [n] \), set \( a \leq b \) if \( a_i \leq b_i \) for all \( i = 1, \ldots, m \). When \( a \) is an \( m \)-subset of columns of a generic \( m \times n \) matrix \( X \), the poset \( \mathcal{P} \) is called the Plücker poset.

**Proposition 3.1.** Let \( X \) be a generic \( m \times n \) matrix of indeterminates, and let \( I_m(X) \) denote the ideal of maximal minors of \( X \). Then there exists a monomial order \( \tau' \) on \( \mathbb{K}[T] \) such that the Gröbner basis of the presentation ideal of \( \mathcal{R}(\text{in}(J_\Delta)) \) is given by:

\[
x_{i_1}T_{c_{i_1}} - x_{i_{c_{i_1}+1}}T_{c_{i_{c_{i_1}+1}}}
\]

where \( c = \{c_1 < c_2 < \cdots < c_{m+1}\} \) is an \( m \)-face of \( \Delta_{\text{clique}} \) and \( 1 \leq i \leq m \),

\[
T_aT_b - T_cT_d
\]

where \( a \) and \( b \) are incomparable elements in the Plücker poset, and

\[
c = \{\min\{a_1, b_1\}, \ldots, \min\{a_m, b_m\}\},
\]

\[
d = \{\max\{a_1, b_1\}, \ldots, \max\{a_m, b_m\}\}.
\]

In addition, the presentation ideal of \( \mathcal{F}(\text{in}(J_\Delta)) \) has a Gröbner basis given by polynomials of type \( (2) \) under some monomial order \( \tau \) of \( \mathbb{K}[T] \) which is a restriction of \( \tau' \). In particular, \( \text{in}(I_m(X)) \) is of fiber type.

**Proof.** We follow the method of Proposition 3.2 in [5]. It is clear that polynomials of types \( (1) \) and \( (2) \) sit inside \( \ker \phi^* \), and polynomials of types \( (2) \) sit inside \( \ker \psi^* \). A monomial order \( \tau' \) on \( \mathbb{K}[T] \) exists that selects the underlined monomials of polynomials of types \( (1) \) and \( (2) \) as leading terms; see Lemma 3.7. This monomial order \( \tau' \) restricts to a monomial order \( \tau \) on \( \mathbb{K}[T] \) selecting the underlined monomial of polynomials of type \( (2) \) as the leading term. Let \( L \) be the ideal generated by the underlined monomials. To see that polynomials of types \( (1) \) and \( (2) \) form a Gröbner basis with respect to \( \tau \) for the Rees ideal (or, that polynomials of type \( (2) \) form a Gröbner basis with respect to \( \tau \) for the special fiber ideal), it suffices to check that all monomials not contained in \( L \) are linearly independent; see Lemma 3.3. We show this in Lemma 3.8. \( \square \)

**Remark 3.2.** Observe that in the Plücker poset,

\[
\{a_1, \ldots, a_m\} \land \{b_1, \ldots, b_m\} = \{\min\{a_1, b_1\}, \ldots, \min\{a_m, b_m\}\}
\]

and

\[
\{a_1, \ldots, a_m\} \lor \{b_1, \ldots, b_m\} = \{\max\{a_1, b_1\}, \ldots, \max\{a_m, b_m\}\}.
\]
This is no coincidence: the special fiber ring of \( \text{in}(I_m(X)) \) is known to have a Hodge algebra structure induced by the Hibi ring structure on the Plücker poset. For more details on this perspective, see [12, Section 6.6].

The first step of the proof of Proposition 3.1 is to check that the marked polynomials of types (1) and (2) are marked coherently, i.e., the reduction relation modulo this set of polynomials is Noetherian (see [19, Theorem 3.12]). To this end, we define a class of monomials in \( R[T] \) called \textit{sorted} and show that there exist term orders \( \tau \) and \( \tau' \) for which they are precisely the standard monomials modulo the presentation ideals of \( \mathcal{F}(\text{in}(I_m(X))) \) and \( \mathcal{R}(\text{in}(I_m(X))) \). This class of sorted monomials will be exactly those not contained in the ideal generated by the underlined monomials in the polynomials of types (1) and (2). This fact will also allow us to employ the following lemma.

**Lemma 3.3.** [Lemma 3.1] Let \( K[Y] \) be a polynomial ring equipped with a term order \( \succeq \). Let \( J \) be an ideal of \( K[Y] \) and let \( f_1, \ldots, f_s \) be polynomials in \( J \). Assume that the monomials of the set \( \Omega = \{ m \mid m \notin (\text{in}_\succeq(f_1), \ldots, \text{in}_\succeq(f_s)) \} \) are linearly independent in \( K[Y]/J \). Then \( f_1, \ldots, f_s \) is a Gröbner basis of \( J \) with respect to \( \succeq \).

**Notation 3.4.** Let \( a \) be a face of \( \Delta \), and let \( j \in \{1, \ldots, n\} \) such that \( a_i < j < a_{i+1} \). Define \( (a \cup j) \) to be the ordered tuple
\[
(a_1, \ldots, a_i, j, a_{i+1}, \ldots, a_m).
\]
Observe that if \( a \) and \( j \) are in the same clique of \( \Delta \), then \( \{a \cup j\} \) forms an \( m \)-face of \( \Delta_{\text{clique}} \).

**Definition 3.5.** Adopt Notation 3.4. For any monomial \( m \in K[T] \), write it as \( T_{a_1} \cdots T_{a_k} \) where \( a_j < a_{j+1} \) for the first \( j \) where \( a^i \) and \( a^{i+1} \) differ. Consider the sequence
\[
a_1, a_2, \ldots, a_k
\]
for any \( 1 \leq j \leq m \). The monomial \( m \in K[T] \) is \textit{sorted} if for all \( 1 \leq j \leq m \) and all \( 1 \leq i < k \) we have \( a_i \leq a_{i+1} \).

Define an \textit{inversion} to be a pair \( (a_i, a_j) \) such that \( i < j \) but \( a_i > a_j \). Define \( \text{inv}(m) = (s_1, \ldots, s_m) \in \mathbb{Z}_{\geq 0}^m \) to be the \textit{inversion sequence} of \( m \), where \( s_j \) is the number of inversions in the sequence \( [3] \). In particular, if \( m \) is sorted in \( K[T] \), then \( \text{inv}(m) = (0, \ldots, 0) \).

For any monomial \( m \in R[T] \), define \( \text{sd}(m) = (r; s_1, \ldots, s_m) \in \mathbb{Z}_{\geq 0}^{m+1} \) to be the \textit{sorting distance} of \( m \) where
- \( r \) is the number of distinct pairs \( (x_{ij}, T_a) \) such that \( x_{ij} \) and \( T_a \) divide \( m \), \( j \notin a \), and \( (a \cup j), i = j \), and
- \( (s_1, \ldots, s_m) \) is the inversion sequence of the monomial defined by the \( T \) variables in \( m \).

If \( \text{sd}(m) = (0; 0, \ldots, 0) \), we say that the monomial \( m \) is \textit{sorted} in \( R[T] \).

**Example 3.6.** Let \( X \) be a \( 3 \times 5 \) matrix of indeterminates in \( R \), so \( I_3(X) = J_\Delta \) where \( \Delta \) is a pure 2-dimensional simplicial complex with facets corresponding to all possible 2-faces on five vertices. The monomial
\[
m_1 = x_{11}x_{22}T_{145}T_{234}
\]
in \( R[\mathbf{T}] \) has sorting distance \((2; 0, 1, 1)\).

- \( \{1, 4, 5\} \cup \{2\} = \{1, 2, 4, 5\} \) is a face of \( \Delta^{\text{clique}} \), and \((1, 2, 4, 5)_2 = 2. \) So the monomial \( x_{12}T_{145} \) is not sorted.
- \( \{2, 3, 4\} \cup \{1\} = \{1, 2, 3, 4\} \) is a face of \( \Delta^{\text{clique}} \), and \((1, 2, 3, 4)_1 = 1. \) So the monomial \( x_{11}T_{234} \) is not sorted.
- The minors \([145]\) and \([234]\) are incomparable in the Plücker poset, since we have that \( 1 < 2 \) but \( 4 > 3 \).

However, the monomial

\[
m_2 = x_{12}x_{24}T_{124}T_{135}
\]

is sorted. Observe that

\[
\phi^*(m_1) = \phi^*(m_2) = x_{11}x_{12}x_{22}x_{23}x_{24}x_{34}x_{35} \cdot t^2 \in R[\text{in}(I_m(X)) \cdot t].
\]

**Lemma 3.7.** There exists a term order \( \tau \) on \( \mathbb{K}[\mathbf{T}] \) such that the sorted monomials of Definition 3.5 are precisely the \( \tau \)-standard monomials modulo the ideal generated by polynomials of type (2). In addition, there exists a term order \( \tau' \) on \( R[\mathbf{T}] \) such that the sorted monomials in Definition 3.5 are precisely the \( \tau' \)-standard monomial modulo the ideal generated by polynomials of types (1) and (2).

**Proof.** Consider the reduction on \( R[\mathbf{T}] \) defined by polynomials of types (1) and (2). A monomial \( m \) is in normal form with respect to this reduction relation if and only if \( m \) is sorted.

If a monomial \( f \in \mathbb{K}[\mathbf{T}] \) is reduced to another monomial \( g \in \mathbb{K}[\mathbf{T}] \) using polynomials of type (2), then at least one of the numbers in its inversion sequence must decrease. Therefore, the reduction relation in \( \mathbb{K}[\mathbf{T}] \) is Noetherian and \( \tau \) exists.

Take a non-sorted monomial \( m \) in \( R[\mathbf{T}] \) such that \( m = u \cdot f \), where \( u \) is a monomial in the \( x_{ij} \) and \( f \) is a monomial in \( \mathbf{T} \). By the above Noetherian reduction, we can reduce \( f \) to a standard monomial in \( \mathbb{K}[\mathbf{T}] \) modulo polynomials of type (2). Then \( m \) has been reduced to \( u = u \cdot T_{a_1} \cdots T_{a_d} \) where the \( a^1 \leq a^2 \leq \cdots \leq a^d \) in the Plücker poset.

Let \( a^\ell \) be the smallest \( a^i \) (with respect to the Plücker poset ordering) such that \( T_{a^\ell} \) divides \( u \) and some \( x_{ij} \) dividing \( u \) satisfies that \( (a^\ell \cup j)_i = j \); let \( x_{ij} \) be the smallest such variable with respect to lex monomial order \( > \) as in Definition ???. Set \( k = a^\ell \); then reduction modulo polynomials of type (1) gives a monomial \( n' = u'f' \) where \( u' = \frac{u}{x_{ij}} \) and \( f' = T_{a_1} \cdots T_{a^\ell} \cdots T_{a_d} \) and \( a^\ell = (a^\ell \cup j) \setminus k \). Observe that \( f' \) is still standard modulo polynomials of type (2) because \( j < k \), but \( j \geq a^\ell - 1 \) since \( \ell \) was the smallest possible \( \ell \) that satisfied \( (a^\ell \cup j)_i = j \). In particular, the inversion sequence associated to \( f' \) is still \((0, \ldots, 0)\) and the first index of the sorting distance has strictly decreased, so this reduction is Noetherian and \( \tau' \) exists.

\( \square \)

**Lemma 3.8.** Sorted monomials in \( R[\mathbf{T}] \) are linearly independent modulo \( \ker \phi^* \).

**Proof.** Since \( R[\mathbf{T}]/\ker \phi^* \cong R[\text{in}(I_m(X)) \cdot t] \), to show sorted monomials in \( R[\mathbf{T}] \) are linearly independent modulo \( \ker \phi^* \), it suffices to show that \( m \) corresponds to a unique sorted monomial in \( R[\mathbf{T}] \) modulo \( \ker \phi^* \) for any \( m \in R[\text{in}(I_m(X)) \cdot t] \). Let \( d \) be the power of \( t \) in \( m \). We show that there is a unique representation of \( m \) as a product of a monomial \( u \in R \) and a standard monomial corresponding to a product of lead terms of minors \( a^1, \ldots, a^d \) such that \( a^s \leq a^{s+1} \) in the Plücker poset.
Sort the $x_{1j}$ variables dividing $m$ by their values of $j$, so we have

$$x_{1j_1} \leq x_{1j_2} \leq \cdots x_{1j_d} \leq \cdots \leq x_{1j_1}$$

Then set $a_1^s = j_s$ for $1 \leq s \leq d$. Now, sort the $x_{2j}$ variables in the same way:

$$x_{2j_1} \leq \cdots x_{2j_d} \leq \cdots \leq x_{2j_2}$$

and set

$$a_2^s = \min\{j_{l} \mid j_{l} > a_1^s \text{ and } j_{l} \geq a_2^{s-1}\}$$

for all $1 \leq s \leq d$. Repeat this process, consecutively sorting all the $x_{ij}$ variables dividing $m$ for a fixed $i$ and setting

$$a_p^s = \min\{j_{l} \mid j_{l} > a_p^{s-1} \text{ and } j_{l} \geq a_p^{s-1}\}$$

for all $1 \leq s \leq d$ and $1 \leq p \leq m$. Clearly, for all $1 \leq p \leq m$ and all $1 \leq s < d$, we have $a_p^s \leq a_p^{s+1}$, i.e., $a^s \leq a^{s+1}$, and $a^s \in \Delta$. Finally, set

$$u = \frac{m}{x_{a^1} \cdots x_{a^d} \cdot t^d}.$$ 

Then $m = u \cdot (x_{a^1} \cdot t)(x_{a^2} \cdot t) \cdots (x_{a^d} \cdot t)$ corresponds to the sorted monomial $u \cdot T_{a^1} \cdots T_{a^d}$ in $R[T]$.

This presentation of $m$ is unique by construction, and it corresponds to a unique monomial in $R[T]$ modulo ker $\phi^s$. Therefore, every sorted monomial in $R[T]$ is distinct modulo ker $\phi^s$, so they are all linearly independent. □

Applying Lemma 3.8 completes the proof of Proposition 3.1.

Example 3.9. Again let $X$ be a $3 \times 5$ matrix of indeterminates in $R$. Consider the monomial $m = x_{11}x_{12}^2x_{22}x_{23}x_{24}x_{25}x_{35}t^2$ in $R[\det(I_m(X))]$. Apply the algorithm from Lemma 3.8 to fill the following $2 \times 3$ tableau column by column in a semi-standard way, i.e., strictly increasing along rows and weakly increasing along columns. Begin by sorting the $x_{1j}$ as $x_{11} < x_{13} \leq x_{13}$ and filling the tableau with the first two $j$’s. Next, sort the $x_{2j}$ variables as $x_{22} < x_{23} < x_{24}$, and then the $x_{3j}$ variables as $x_{34} < x_{35}$.

\[
\begin{array}{ccc}
\hline
1 & 2 & 3 \\
\hline
1 & 2 & 3 \\
\end{array}
\]

Finally,

\[
\begin{array}{ccc}
\hline
1 & 2 & 3 & 4 \\
\hline
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

and we conclude that $m$ can be written in the form $x_{13}x_{23}(x_{11}x_{22}x_{23}x_{24}x_{25}x_{35}t^2) \cdot (x_{13}x_{24}x_{25}x_{35}t^2)$, which corresponds to the unique sorted monomial $x_{13}x_{23}T_{124}T_{235}T_{345}$ in $R[T]$.

4. Rees Algebras of Closed Determinantal Facet Ideals

We turn our attention to the more general case where $\Delta$ is a pure and closed $(m-1)$-dimensional simplicial complex with possibly more than one clique in its clique decomposition. The special fiber ring of $\det(I_\Delta)$ and $J_\Delta$ are already known in the case when $\Delta$ is closed [14 Corollary 1.4] with the extra assumptions mentioned in Remark ??.

When $\Delta$ consists of more than one clique, $\det(I_\Delta)$ no longer satisfies the $\ell$-exchange property and it is not obvious that $\det(I_\Delta)$ is of fiber type. We find the defining equations of
\( R(\text{in}(J_\Delta)) \) and prove directly that they generate a Gröbner basis of the presentation ideal. We use the foundation built in Section 3, and a carefully crafted ordering defined later in Definition 4.2.

We open this section with one of the main theorems of this paper.

**Theorem 4.1.** Let \( \Delta \) be a pure and closed \((m-1)\)-dimensional simplicial complex with clique decomposition \( \Delta = \bigcup_{i=1}^r \Delta_i \). Under some monomial order \( \sigma' \) on the ring \( R[T] \), the presentation ideal of \( R(\text{in}(J_\Delta)) \) has a Gröbner basis given by polynomials of the following forms:

\[
\begin{align*}
\underline{x_a T_b} - \underline{x_b T_a} & \\
\underbrace{x_{i_1} T_{c_{i_1}}} - \underbrace{x_{i_{i+1}} T_{c_{i+1}}} & \\
\underbrace{T_a T_b} - \underbrace{T_c T_d}
\end{align*}
\]

where \( a, b \) are not contained in the same maximal clique of \( \Delta \) and the smallest clique containing \( a \) is strictly smaller than the smallest clique containing \( b \), \( c = \{c_1 < c_2 < \ldots < c_{m+1}\} \) is an \( m \)-face of \( \Delta \) clique and \( 1 \leq i \leq m \), and \( a, b \) are incomparable elements in the Plücker poset such that \( a \) and \( b \) are in the same maximal clique of \( \Delta \).

In addition, there exists some monomial order \( \sigma \) on \( K[T] \) such that the presentation ideal of \( F(\text{in}(J_\Delta)) \) has a Gröbner basis given by polynomials of type (6). In particular, \( \text{in}(J_\Delta) \) is of fiber type.

**Proof.** We apply the same strategy as in the proof of Proposition 3.1 and the proof of [5, Proposition 3.2]. We know that polynomials of types (4), (5), and (6) sit inside \( \ker \phi^* \), and polynomials of types (5) sit inside \( \ker \psi^* \). There exists a monomial order \( \sigma' \) on \( R[T] \) that selects the underlined monomials as leading terms, and it restricts to a monomial order \( \sigma \) on \( K[T] \) which selects the underlined monomials of polynomials of type (6) as leading terms; see Lemma 4.3. Let \( L \) be the ideal generated by the underlined monomials. To show that polynomials of types (4), (5), and (6) form a Gröbner basis for \( R(\text{in}(J_\Delta)) \), and that polynomials of type (6) form a Gröbner basis for \( F(\text{in}(J_\Delta)) \), it suffices to check that all monomials not contained in \( L \) are linearly independent by Lemma 3.3. We show this in Lemma 4.4. \( \square \)

Following the techniques of Section 3, we seek a term order for which polynomials of types (4), (5), and (6) form a Gröbner basis for the presentation ideal of \( R(\text{in}(J_\Delta)) \), so we must first check that these polynomials are marked coherently (see [19, Theorem 3.12]). Extending the notion of sorted (Definition 3.5) to arbitrary closed determinantal facet ideals, we define a class of monomials in \( R[T] \) called clique-sorted and show that there exist term orders \( \sigma \) and \( \sigma' \) for which they are precisely the standard monomials modulo the presentation ideals of \( F(\text{in}(J_\Delta)) \) and \( R(\text{in}(J_\Delta)) \), respectively. This class of clique-sorted monomials will be exactly those not contained in the ideal generated by the underlined monomials in the polynomials of types (4), (5), and (6). Once again, this allows us to employ Lemma 4.3 to show that these polynomials form a Gröbner basis for \( \ker \phi^* \).
Definition 4.2. Let \( \Delta \) be a pure and closed \((m-1)\)-dimensional simplicial complex on \( n \) vertices. Let \( \Delta = \bigcup_{i=1}^{r} \Delta_i \) be a clique decomposition of \( \Delta \) with the total order defined by \( \Delta_1 > \Delta_2 > \cdots > \Delta_r \) given by \( \min(V(\Delta_1)) < \min(V(\Delta_2)) < \cdots < \min(V(\Delta_r)) \). This is well-defined, since distinct cliques of a closed simplicial complex must have distinct minimal indexed vertices.

A monomial \( m \) in \( R[T] \) is **clique-sorted** if it can be written as a product of monomials \( u \cdot f_1 \cdots f_r \), where \( u \) is a monomial in the \( x_{ij} \) variables and \( f_i \) is a monomial in \( T \) satisfying the following properties:

(i) each \( f_i \) is a sorted monomial in \( K[T_a \mid a \in \Delta_i] \).
(ii) if \( T_a \) divides \( f_i \), then \( a \notin \Delta_j \) for any \( j < i \).
(iii) if \( T_a \) divides some \( f_i \), \( x_{ij} \) divides \( u \), and \( \{a \cup j\} \) is a face of \( \Delta^{\text{clique}} \), then \( (a \cup j)_i \neq j \).
(iv) for any \( x_a \) and \( T_b \) dividing \( m \) such that \( a \) and \( b \) are not in the same clique of \( \Delta \), the smallest clique containing \( b \) is less than the smallest clique containing \( a \).

A monomial in \( K[T] \) satisfying properties (i) and (ii) is **clique-sorted** in \( K[T] \).

Lemma 4.3. There exists a term order \( \sigma \) on \( K[T] \) such that the clique-sorted monomials of Definition 4.2 are precisely the \( \sigma \)-standard monomials modulo the ideal generated by polynomials of type (6). In addition, there exists a term order \( \sigma' \) on \( R[T] \) such that the clique-sorted monomials in Definition 4.2 are precisely the \( \sigma' \)-standard monomials modulo the ideal generated by polynomials of types (4), (5), and (6).

Proof. Consider the reduction on \( R[T] \) defined by the marked binomials of types (4), (5), and (6). Observe that a monomial \( m \) is in normal form with respect to this reduction relation if and only if \( m \) is clique-sorted.

First, we check that the reduction of \( m \) modulo polynomials of type (6) in \( K[T] \) is Noetherian. Take a monomial \( m = f_1 \cdots f_r \in K[T] \) such that if \( T_a \) divides \( f_i \), then \( a \in \Delta_i \) and \( a \notin \Delta_j \) for any \( j < i \). By Lemma 3.7, reduction using polynomials of type (6) within each clique is Noetherian. Now \( m \) has been reduced to a monomial \( m' = u \cdot g_1 \cdots g_r \), where each \( g_i \) is a sorted monomial such that each \( T_a \) dividing it corresponds to a facet \( a \) in the clique \( \Delta_i \). We claim that if some \( T_a \) divides some \( g_i \), then \( a \notin \Delta_j \) for any \( j < i \). To see this, suppose that after reduction modulo the Plücker relations, some \( T_a \) dividing some \( g_i \) satisfies \( a \in \Delta_j \) for some \( j < i \). By the definition of being closed (see Definition 2.3), this implies that there is some \( b \in \Delta_j \) such that \( b k = a k \) for some \( k \). But then the original \( T_c \) in \( f_i \) with \( c k = a k \) satisfies the condition that \( c \in \Delta_j \), contradicting the fact that each element of \( T \) was first placed in the earliest possible clique. This implies that the ordering \( \sigma \) exists.

Now take a non-clique-sorted monomial \( m \) in \( R[T] \) such that \( m = u \cdot f \), where \( u \) is a monomial in the \( x_{ij} \) and \( f \) is a monomial in the \( T \). Reduce \( f \) to a clique-sorted monomial in \( K[T] \); this reduction was shown above to be Noetherian. Then \( m \) has been reduced to \( m = u \cdot g_1 \cdots g_d \) such that \( g_1 \cdots g_d \) is a clique-sorted monomial. Rewrite \( u = u_1 \cdots u_i \), such that if \( x_{ij} \) divides \( u_k \), then there is some facet \( a \) of \( \Delta \) contained in the clique \( \Delta_k \) such that \( a_i = j \), and there is no facet in any earlier clique of \( \Delta \) with this property (allowing some of the \( u_i = 1 \)). Then by Lemma 3.7, each monomial \( u_i \cdot g_i \) can be reduced modulo polynomials of type (6) to a sorted monomial \( u'_i \cdot g'_i \) in \( R[T_a \mid a \in \Delta_i] \).
Lastly, reduce with respect to polynomials of type (4). Take the smallest $i$ and the smallest face $\mathbf{a} \in \Delta_i$ such that $\mathbf{x}_a$ divides $\mathbf{u}_i'$. Observe that for any $T_c$ in $g'_i$, $a_i \geq c_i$ for all $i$; otherwise, $\mathbf{u}_i' \cdot g'_i$ would not be a sorted monomial. Take the next $g'_k$ that is not equal to 1, and take the largest $b$ in the Plücker poset such that $T_b$ divides it. Reduce modulo polynomials of type (4) to obtain $u''_i = u''_k, u'''_k = \mathbf{x}_b u'_k, g''_k = g'_k T_a$, and $g''_k = \frac{g''_k}{T_b}$. Now, since $\mathbf{x}_b$ divides $u''_k$ and $k > i$, there is one fewer pair of facets $\mathbf{a}, \mathbf{b} \in \Delta$ violating condition (iv) of Definition 4.2 so this reduction will terminate, and the order $\sigma'$ exists.

Lemma 4.4. Clique-sorted monomials in $R[T]$ are linearly independent modulo $\ker \phi^*$.

Proof. Identify $R[T]/\ker \phi^*$ with $R[\text{in}(J_\Delta) \cdot t]$ via the natural isomorphism induced by $\phi^*$. We will show that every monomial in $R[\text{in}(J_\Delta) \cdot t]$ corresponds uniquely to a clique-sorted monomial in $R[T]$ modulo $\ker \phi^*$.

Take a monomial $\mathbf{m}$ in $R[\text{in}(J_\Delta) \cdot t]$. Let $d$ be the degree of $t$ in $\mathbf{m}$, and rewrite $\mathbf{m} = m_1 m_2 \cdots m_r \cdot t^d$ so that $m_k = \prod x_{ij}$ satisfying:

(a) $x_{ij}$ divides $m$.

(b) for some facet $a$ of $\Delta_k$, $a_i = j$.

(c) there is no facet $b \in \bigcup_{k \leq l} \Delta_l$ such that $b_i = j$.

Apply the algorithm in the proof of Lemma 3.8 to $\mathbf{m}_1$ to write it uniquely as a monomial $u_1 \cdot x_{a_1} x_{a_1} \cdots x_{a_1}$ where $d_1 \leq d$. For $k > 2$, as long as $\sum_{i=1}^{k-1} d_i < d$, set $m_k = v_{k-1} m_k$, where

$$v_{k-1} = \left\{ \prod x_{ij} \mid x_{ij} \text{ divides } u_{k-1} \text{ and } a_i = j \text{ for some } a \in \Delta_k \right\},$$

and

$$u_{k-1} = \frac{m_1 m_2 \cdots m_{k-1}}{\prod_{1 \leq p \leq k-1, 1 \leq q \leq d_p} x_a^{p,q}}.$$ 

In this way, one can consecutively write each $m_k = w_k \cdot x_{a_k} x_{a_k} \cdots x_{a_k}$ where $w_k = \frac{m_k}{x_{a_k} x_{a_k} \cdots x_{a_k}}$ using the algorithm in the proof of Lemma 3.8.

When $\sum_{1 \leq i \leq k} d_i = d$ for some $k \leq r$, set

$$\mathbf{u} = \frac{\mathbf{m}}{\left( \prod_{1 \leq i \leq k} x_{a_i} x_{a_i} x_{a_i} x_{a_i} \cdots x_{a_i} \cdot t^{d_i} \right)}$$

so $\mathbf{m}$ has the presentation

$$\mathbf{m} = \mathbf{u} \cdot \left( \prod_{1 \leq i \leq k} x_{a_i} x_{a_i} x_{a_i} x_{a_i} \cdots x_{a_i} \cdot t^{d_i} \right).$$

In this way, $\mathbf{m}$ corresponds to a unique monomial in $R[T]$ which is clique-sorted and is therefore not in $L$. Therefore, every clique-sorted monomial in $\mathbb{R}[T]$ is distinct modulo $\ker \phi^*$, so they are all linearly independent. \qed
Applying Lemma 3.3 completes the proof of Theorem 4.1. The following example illustrates the implicit equations listed in Theorem 4.1.

**Example 4.5.** Let $G$ be the closed graph on 6 vertices with two maximal cliques $G_1 = \{1, 2, 3, 4, 5\}$ and $G_2 = \{2, 3, 4, 5, 6\}$, so $J_G$ is generated by all two-minors $[a_1, a_2]$ of a generic $2 \times 6$ matrix such that either $(a_1, a_2) \neq (1, 6)$. Applying Theorem 4.1 we obtain the following defining equations for $R(\text{in}(J_G))$:

(i) Koszul relations between edges contained only in $G_1$ and edges only contained in $G_2$, e.g.,

$$x_{11}x_{22}T_{36} - x_{13}x_{26}T_{12},$$

(ii) all the linear relations from within $G_1$ and $G_2$, e.g.,

$$x_{11}T_{23} - x_{12}T_{13} \quad x_{23}T_{24} - x_{24}T_{23} \quad x_{14}T_{56} - x_{15}T_{46},$$

(iii) Plücker relations from the cliques of $\Delta_1$ and $\Delta_2$, e.g.,

$$T_{14}T_{23} - T_{13}T_{24} \quad T_{25}T_{34} - T_{24}T_{35} \quad T_{36}T_{45} - T_{35}T_{46}.$$

We are now ready to give a SAGBI basis of the Rees algebra for a closed determinantal facet ideal.

**Theorem 4.6.** Let $\Delta$ be a pure and closed $(m - 1)$-dimensional simplicial complex. The polynomials of the set $\{x_{ij}\} \cup \{[a] \cdot t \mid a \text{ is a facet of } \Delta\}$ form a SAGBI basis of the Rees algebra $R(J_\Delta)$ with respect to the monomial order $\succ'$ defined in Definition ???. In particular,

$$\text{in}_{\succ'}(R(J_\Delta)) = \mathbb{K}[X] \cdot \text{in}_{\succ'}(J_\Delta) \cdot t = R(\text{in}_{\succ'}(J_\Delta)).$$

Additionally, the polynomials of the set $\{[a] \mid a \text{ is a facet of } \Delta\}$ form a SAGBI basis of the $\mathbb{K}$-algebra $\mathbb{K}[J_\Delta]$ with respect to the lexicographic monomial order $\succ$ as in Notation [2.4]; in particular, $\text{in}_{\succ}(\mathbb{K}[J_\Delta]) = \mathbb{K}[\text{in}_{\succ}(J_\Delta)]$.

**Proof.** Polynomials of types [4], [5], and [6] form a Gröbner basis, and therefore a (not necessarily minimal) generating set, of ker $\phi^*$ by Theorem 4.1. It suffices to show that any $f$ in the generating set of ker $\phi^*$ lifts to a linear combination of elements of the form $\lambda u([a] \cdot t)^k$ with $\lambda \in \mathbb{K} \setminus \{0\}$, $k \in \mathbb{N}$, $u$ a monomial in the $x_{ij}$, and $\text{in}_{\succ'}(f) > \text{in}_{\succ'}(u([a] \cdot t)^k)$; see, for example, [12, Theorem 6.43].

Observe the following elementary facts: $\sum_{p \in \mathcal{S}_m} \text{sgn}(p)x_{1p(a_1)} \cdots x_{mp(a_m)} - [a] = \sum_{p \in \mathcal{S}_m} \text{sgn}(p)x_{1p(a_1)} \cdots x_{mp(a_m)}$ and $[a][b] - [b][a] = 0$. Then the linear relation [11] lifts to

$$x_a[b] \cdot t - x_b[a] \cdot t = \left(\sum_{p \in \mathcal{S}_m} \text{sgn}(p)x_{1p(a_1)} \cdots x_{mp(a_m)} \right)[b] \cdot t - \left(\sum_{p \in \mathcal{S}_m} \text{sgn}(p)x_{1p(b_1)} \cdots x_{mp(b_m)} \right)[a] \cdot t$$

where $\mathcal{S}_m$ denotes the symmetric group on $m$ letters. Observe that every monomial in

$$\sum_{p \in \mathcal{S}_m} x_{1p(a_1)} \cdots x_{mp(a_m)}$$

is less than $x_a$ with respect to $\succ$. 


The linear relation \( (5) \) lifts to
\[
x_{ic_1}[c \setminus c_1] \cdot t - x_{ic_{i+1}}[c \setminus c_{i+1}] \cdot t = \sum_{j \in \{1, \ldots, m+1\}} (-1)^j x_{ic_j} [c \setminus c_j] \cdot t
\]
where \( c = \{c_1 < c_2 < \ldots < c_{m+1}\} \) is an \( m \)-face of \( \Delta^{\text{clique}} \). If \( j < i \), then the lead monomial of any \( x_{ic_j} [c \setminus c_j] \) on the right hand side of the equation differs from the lead monomial of the lefthand side at \( x_{j} \). If \( j > i + 1 \), then the leading monomial of \( x_{ic_j} [c \setminus c_j] \) first differs from the leading monomial of the lefthand side at \( x_{ic_j} < x_{ic_{i+1}} \).

The Plücker relation \( (6) \) lifts to the standard Plücker relation
\[
[a][b] \cdot t^2 - [c][d] \cdot t^2 = \sum_{[e], [f] \neq [a], [b], [c], [d]} c_{e,f} \cdot [e_1, \ldots, e_m] \cdot [f_1, \ldots, f_m] \cdot t^2
\]
where \( c_{e,f} \in \mathbb{K} \) and \( [e] \leq [a], [b] \) for all terms with \( c_{e,f} \neq 0 \) in the Plücker poset. It is well-known that \( \text{in}_>([e][f]) < \text{in}_>([a][b]) \); see, for example, [12, Theorem 6.46].

Now, applying [19, Corollary 11.6], we obtain our main result.

**Theorem 4.7.** Let \( \Delta \) be a closed and pure \((m-1)\)-dimensional simplicial complex. Under some monomial order \( \omega' \) on the ring \( R[T] \), the presentation ideal of \( R(J_\Delta) \) has a Gröbner basis consisting of
\[
[a] \cdot T_b - [b] \cdot T_a
\]
where \( a \) and \( b \) are contained in distinct cliques of \( \Delta \),
\[
\sum_{j \in \{1, \ldots, m+1\}} (-1)^j x_{ic_j} T_{c \setminus c_j}
\]
where \( c = \{c_1, \ldots, c_{m+1}\} \) is an \( m \)-face of \( \Delta^{\text{clique}} \), and
\[
\sum_{i_1 < \ldots < i_{m-k}} \sum_{i_{m-k+1} < \ldots < i_{m+1}} \sum_{\left\{i_1, \ldots, i_{m+1}\right\} = \{1, \ldots, m+1\}} \text{sgn}(i_{\bullet}) \cdot T_{c_1 \ldots c_k a_{i_1} \ldots a_{i_{m-k}}} \cdot T_{a_{i_{m-k+1}} \ldots a_{i_{m+1}} d_{k+2} \ldots d_m}
\]
for a fixed \( k \in \{1, \ldots, m-1\} \) and elements \( c_1, \ldots, c_k, d_{k+2}, \ldots, d_m, a_1, \ldots, a_{m+1} \in \{1, \ldots, n\} \) such that \( \{c_1, \ldots, c_k, a_{i_1}, \ldots, a_{i_{m-k}}\} = \{a_{i_{m-k+1}}, \ldots, a_{i_{m+1}}, d_{k+2}, \ldots, d_m\} \) are \((m-1)\)-faces of \( \Delta^{\text{clique}} \). Here we regard \( i_{\bullet} \) as the permutation \( \mathbf{p} \in \mathfrak{S}_{m+1} \) given by \( \mathbf{p}(j) = i_j \) and define \( \text{sgn}(i_{\bullet}) = \text{sgn}(\mathbf{p}) \).

In addition, the presentation ideal of \( \mathcal{F}(J_\Delta) \) has a Gröbner basis given by polynomials of type \( \mathcal{A} \) with respect to some monomial order \( \omega \) on \( \mathbb{K}[T] \). In particular, \( J_\Delta \) is of fiber type.

**Example 4.8.** Consider again the graph \( G \) from Example 4.5. Lifting the defining equations of the Rees ideal of \( \text{in}(J_G) \), we obtain the defining equations of \( R(J_G) \):

(i) Koszul relations between edges in \( G_1 \) and edges in \( G_2 \),
(ii) all the linear “Eagon-Northcott” relations from within cliques of \( G_1 \) and \( G_2 \),
(iii) Plücker relations from cliques of \( G_1 \) and \( G_2 \), e.g., \( T_{26}T_{34} - T_{24}T_{35} + T_{25}T_{45} \) (which corresponds to the second equation in Example 4.5).
Corollary 4.9. Let $\Delta$ be a closed and pure $(m - 1)$-dimensional simplicial complex with clique decomposition $\Delta = \bigcup_{i=1}^{r} \Delta_i$ and $J_\Delta$ be its corresponding determinantal facet ideal, and let $n_i$ be the size of the vertex set of each $\Delta_i$ in the clique decomposition. Then we have the following properties:

(a) $F(J_\Delta)$ is Koszul.
(b) $R(J_\Delta)$ is Koszul if $\Delta$ is a clique.
(c) $J_\Delta$ is of linear type if and only if $n_i < m + 2$ for all $i$.
(d) $R(J_\Delta)$ and $F(J_\Delta)$ are normal Cohen-Macaulay domains. In particular, $R(J_\Delta)$ and $F(J_\Delta)$ have rational singularities if $\text{char}\ K = 0$, and they are F-rational if $\text{char}\ K > 0$.

Proof. It is well-known that if the presentation ideal for an algebra has a quadratic Gröbner basis, then the algebra is Koszul; see, for instance, [12, Theorem 6.7]. This gives (a) and (b). To see (c), observe that all relations of type (9) come from faces of $\Delta$ clique which are dimension $m + 1$ or larger. By [19, Proposition 13.15], the semigroup rings $R(\text{in}_\omega(J_\Delta))$ and $F(\text{in}_\omega(J_\Delta))$ are normal because their presentation ideals have square-free initial ideals by Theorem 4.1. Applying [5, Corollary 2.3] and Theorem 4.6, we obtain (d). □

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