A TOPOLOGICALLY INDUCED 2-IN/2-OUT OPERATION ON LOOP COHOMOLOGY

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Abstract. We apply the Transfer Algorithm introduced in [7] to transfer an $A_\infty$-algebra structure that cannot be computed using the classical Basic Perturbation Lemma. We construct a space $X$ whose topology induces a nontrivial 2-in/2-out operation $\omega^2$ on loop cohomology $H^*(\Omega X; \mathbb{Z}_2)$.

1. Introduction

In [6] and [7], S. Saneblidze and this author defined the notions of a matrad and a relative matrad, and constructed the related families of polytopes known as biassociahedra $KK = \{KK_{n,m} = KK_{m,n}\}$ and bimultiplihedra $JJ = \{JJ_{n,m} = JJ_{m,n}\}$ of which $KK_{1,n}$ is the associahedron $K_n$ and $JJ_{1,n}$ is the multiplihedron $J_n$. Cells of $KK$ and $JJ$ are identified with certain fraction product monomials, for example,

In fact, $JJ_{m,n}$ is a subdivision of $KK_{m,n} \times I$ with $\partial JJ_{m,n}$ containing the cells $KK_{n,m} \times 0$ and $KK_{n,m} \times 1$.

Let $R$ be a commutative ring with unity. The free matrad $\mathcal{H}_\infty$ is represented by the DG $R$-module (DGM) of cellular chains $C_\ast (KK)$ by associating the top dimensional cell of $KK_{n,m}$ with the matrad generator $\theta_{n,m} \in \mathcal{H}_\infty$:

An $A_\infty$-bialgebra is a DGM $(A,d)$ together with a family of multilinear operations $\omega = \{\omega^m_n \in Hom^{m+n-3}(A^{\otimes m}, A^{\otimes n}) \mid mn \neq 1\}$ and a map of matrads $\mathcal{H}_\infty \to \mathcal{E}nd_{T A}$ such that $\theta_{n,m} \mapsto \omega^m_n$, i.e., $(A, \omega)$ is an algebra over $\mathcal{H}_\infty$. Note that we recover the operadic structure of $A_\infty$(co)algebras by setting $m = 1$ or $n = 1$.

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Similarly, the free relative matrad $\mathcal{G}$ is represented by the DGM of cellular chains $C_\ast (\mathcal{G})$ by associating the top dimensional cell of $\mathcal{G}_{m,n}$ with the relative matrad generator $f_{m,n} \in \mathcal{G}$:

\[ n \text{ outputs} \leftrightarrow f_{m,n}^n. \]

Let $(A, \omega_A)$ and $(B, \omega_B)$ be $A_\infty$-bialgebras. A morphism $G$ from $A$ to $B$, denoted by $G : A \Rightarrow B$, is a family of multilinear maps $G = \{ g_{m,n}^n \in \text{Hom}(A^m, B^n) \}$ together with a map of relative matrads $\mathcal{G} \rightarrow \text{Hom}(T_A, T_B)$ such that $f_{m,n}^n \mapsto g_{m,n}^n$, i.e., $G$ is an $A_\infty$-bimodule. The elements $\theta_{m,n}^n (\mathcal{G})^\otimes m$ and $(\mathcal{G})^\otimes n \theta_{m,n}$ of $\mathcal{G}$ are associated with the codimension 1 cells $KK_{n,m} \times 0$ and $KK_{n,m} \times 1$ of $\mathcal{G}_{m,n}$, respectively,

\[ n \leftrightarrow \theta_{m,n}^n (\mathcal{G})^\otimes m; \quad m \leftrightarrow (\mathcal{G})^\otimes n \theta_{m,n}^n, \]

and the aforementioned map of relative matrads sends $\theta_{m,n}^n (\mathcal{G})^\otimes m \mapsto \omega_{m,n}^n = \omega_{m,n}$. Again, we recover the structure of an $A_\infty$-(co)algebra morphism by setting $m = 1$ or $n = 1$. A morphism $\Phi = \{ \phi_{m,n}^n \}_{m,n \geq 1} : A \Rightarrow B$ is an isomorphism if $\phi_{1,1}^1$ is an isomorphism of underlying modules.

The paper is organized as follows: In section 2 we review the Transfer Algorithm introduced in [7] and apply it to transfer an $A_\infty$-algebra structure that cannot be computed using the classical Basic Perturbation Lemma. In Section 3 we construct a space $X$ whose topology induces a nontrivial 2-in/2-out operation $\omega^2_2$ on loop cohomology $H^\ast (\Omega X; \mathbb{Z}_2)$.

2. Transfer of $A_\infty$-Structure

If $A$ is a free DGM, $B$ is an $A_\infty$-algebra, and $g : A \rightarrow B$ is a homology isomorphism with a right-homotopy inverse, the Basic Perturbation Lemma (BPL) transfers the $A_\infty$-algebra structure from $B$ to $A$ (see [2], [5], for example). When $B$ is an $A_\infty$-bialgebra, Theorem 1 generalizes the BPL in two directions:

1. The $A_\infty$-bialgebra structure on $B$ transfers to an $A_\infty$-bialgebra structure on $A$.

2. The transfer algorithm requires neither freeness in $A$ nor the existence of a right-homotopy inverse of $g$.

Given DGMs $(A, d_A)$ and $(B, d_B)$, let $\nabla$ be the induced differential on $U_{A,B} = \text{Hom}(T_A, T_B)$, i.e., for $f \in U_{A,B}$ define $\nabla f = d_B f - (-1)^{|f|} f d_A$, where $d_A$ and $d_B$ denote the free linear extensions of $d_A$ and $d_B$. A chain map $g : A \rightarrow B$ induces a cochain map $\tilde{g} : \text{End}_{T_A} \rightarrow U_{A,B}$ defined on $u \in \text{Hom}(A^m, A^n)$ by $\tilde{g} (u) = g^\otimes m u$. If $g$ is a homology isomorphism, so is $\tilde{g}$ provided condition (i) or (ii) in the following proposition is satisfied (the proof is left to the reader):
Proposition 1. Let \((A, d_A)\) and \((B, d_B)\) be DGMs, and let \(g : A \to B\) be a chain map that is also a homology isomorphism. Then \(\tilde{g} : \text{End}_{TA} \to U_{A,B}\) is a homology isomorphism if either of the following conditions holds:

(i) \(A\) is free as an \(R\)-module.
(ii) For each \(n \geq 1\), there is a DGM \(X(n)\) and a splitting \(B^\otimes n = A^\otimes n \oplus X(n)\) as a chain complex such that \(\text{H}^*\text{Hom}(A^\otimes k, X(n)) = 0\) for all \(k \geq 1\).

Thus there is the following generalization of the BPL:

Theorem 1 (The Transfer). Let \((A, d_A)\) be a DGM, let \((B, d_B, \omega_B)\) be an \(A\)-\(\infty\)-bialgebra, and let \(g : A \to B\) be a chain map and a homology isomorphism. If \(\tilde{g} : \text{End}_{TA} \to U_{A,B}\) is a homology isomorphism, then

(i) (Existence) \(g\) induces an \(A\)-\(\infty\)-bialgebra structure \(\omega_A = \{\omega_A^{m,n}\}\) on \(A\) and extends to a map \(G = \{g^n_A | g^1_1 = g\} : A \Rightarrow B\) of \(A\)-\(\infty\)-bialgebras.

(ii) (Uniqueness) \((\omega_A, G)\) is unique up to isomorphism, i.e., if \((\omega_A, G)\) and \((\tilde{\omega}_A, \tilde{G})\) are induced by chain homotopic maps \(g\) and \(\tilde{g}\), there is an isomorphism \(\Phi : (A, \tilde{\omega}_A) \Rightarrow (A, \omega_A)\) and a chain homotopy \(T : \tilde{G} \simeq G \circ \Phi\).

The proof of Theorem 1 which appears in [7], suggests the following general Transfer Algorithm:

The Transfer Algorithm

**Initial data**

- A DGM \((A, d_A)\)
- An \(A\)-\(\infty\)-bialgebra \((B, d_B, \omega_B)\) and a map of matrads \(\alpha_B : C_s(KK) \to \text{End}_{TB}\) sending \(\theta^n_m \mapsto \omega_B^{m,n}\)
- A chain map/homology isomorphism \(g : A \to B\) such that \(\tilde{g}\) is a homology isomorphism

**Objectives**

- Define operations \(\omega_A^{m,n} : A^\otimes m \to A^\otimes n\) for all \(m, n, mn \neq 1\)
- Construct a map of matrads \(\alpha_A : C_s(KK) \to \text{End}_{TA}\) sending \(\theta^n_m \mapsto \omega_A^{m,n}\)
- Construct a map of \(A\)-\(\infty\)-bialgebras \(G = \{g^n_A | g^1_1 = g\} : A \Rightarrow B\)

**Initialization**

1. Define \(\beta : C_0(JJ_{1,1}) \to \text{Hom}(A, B)\) by \(f^1_1 \mapsto g\)
2. Define \(\beta\) on the vertex \(1\) of \(JJ_{1,2}\) by \(\theta^2_1 (f^1_1 \times f^1_1) \mapsto \omega_B^{1,2} (g \otimes g)\)
3. Define \(\beta\) on the vertex \(1\) of \(JJ_{2,1}\) by \(\theta^2_1 f^1_1 \mapsto \omega_B^{2,1} g\)
4. Consider the \(\nabla\)-cocycle \(\omega_B^{1,2} (g \otimes g)\)
   - Choose a cocycle \(\omega_A^{1,2} \in \text{End}_{TA}\) such that \(\tilde{g}_s[\omega_A^{1,2}] = [\omega_B^{1,2} (g \otimes g)]\)
   - Define \(\alpha_A : C_0(KK_{1,1}) \to \text{Hom}(A^\otimes 2, A)\) by \(\theta^2_2 \mapsto \omega_A^{1,2}\)
   - Define \(\alpha_A : C_0(\partial KK_{1,3}) \to \text{Hom}(A^\otimes 3, A)\) by
     \[ \mapsto \omega_A^{1,2} \left(\omega_A^{1,2} \otimes 1\right) \quad \text{and} \quad \mapsto \omega_A^{1,2} \left(1 \otimes \omega_A^{1,2}\right) \]
   - Extend \(\beta\) to the vertex \(1\) of \(JJ_{1,2}\) via \(f^1_1\theta^2_1 \mapsto g\omega_A^{1,2}\)
5. Dually, consider the \(\nabla\)-cocycle \(\omega_B^{2,1} g\)
   - Choose a cocycle \(\omega_A^{2,1} \in \text{End}_{TA}\) such that \(\tilde{g}_s[\omega_A^{2,1}] = [\omega_B^{2,1} g]\)
(4) Define $\alpha_A : C_0 (KK_{2,1}) \to \text{Hom} (A, A^{\otimes 2})$ by $\theta_1^2 = \begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto \omega_A^{2,1}$

5. Define $\alpha_A : C_0 (\partial KK_{3,1}) \to \text{Hom} (A, A^{\otimes 3})$ by $\begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto \left( \omega_A^{2,1} \otimes 1 \right) \omega_A^{2,1}$ and $\begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto \left( 1 \otimes \omega_A^{2,1} \right) \omega_A^{2,1}$

6. Extend $\beta$ to the vertex $\begin{array}{c} \text{Y} \\ \text{Y} \end{array} \subset JJ_{1,2}$ via $(f_1^1 \otimes f_1^1) \theta_1^2 \mapsto (g \otimes g) \omega_A^{2,1}$.

7. Define $\alpha_A : C_0 (\partial KK_{2,2}) \to \text{Hom} (A^{\otimes 2}, A^{\otimes 2})$ by $\begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto \left( \omega_A^{1,2} \otimes \omega_A^{1,2} \right) \sigma_{2,2} \left( \omega_A^{2,1} \otimes \omega_A^{2,1} \right)$ and $\begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto \omega_A^{2,1} \omega_A^{1,2}$, where $\sigma_{p,q} : (A^{\otimes p})^{\otimes q} \cong (A^{\otimes q})^{\otimes p}$ is the canonical permutation of tensor factors.

7. Note that $\left[ \omega_B^{1,2} (g \otimes g) - g \omega_A^{1,2} \right] = 0$

8. Choose a cochain $g_1^2$ such that $\nabla g_1^2 = \omega_B^{1,2} (g \otimes g) - g \omega_A^{1,2}$

9. Define $\beta : C_1 (JJ_{1,2}) \to \text{Hom} (A^{\otimes 2}, B)$ by $f_1^2 = \begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto g_1^2$

10. Define $\beta$ on a monomial in $C_\ast (\partial JJ_{1,3} \setminus \text{int} KK_{1,3} \times 1)$ to be the corresponding composition:

11. Choose a cochain $g_2^1 \in U_{A,B}$ such that $\nabla g_2^1 = \omega_B^{1,2} g - (g \otimes g) \omega_A^{2,1}$

12. Define $\beta : C_1 (JJ_{2,1}) \to \text{Hom} (A^{\otimes 2}, B^{\otimes 2})$ by $f_2^1 = \begin{array}{c} \text{Y} \\ \text{Y} \end{array} \mapsto g_2^1$

13. Define $\beta$ on a monomial in $C_\ast (\partial JJ_{3,1} \setminus \text{int} KK_{3,1} \times 1)$ to be the corresponding composition:

14. Define $\beta$ on a monomial in $C_\ast (\partial JJ_{2,2} \setminus \text{int} KK_{2,2} \times 1)$ to be the corresponding fraction product:
Induction hypothesis

Given $m + n \geq 4$, assume that for $i + j < m + n$, $ij \neq 1$, there exists a map

- $\alpha_A : C_*(KK_{i,j}) \to \text{Hom}(A^{\otimes i}, A^{\otimes j})$ of matrads sending $\theta^i_j \mapsto \omega^i_j A$
- $\beta : C_*(JJ_{i,j}) \to \text{Hom}(A^{\otimes i}, B^{\otimes j})$ of relative matrads sending $f^i_j \mapsto g^i_j$

Induction objectives

- Define $\alpha_A$ on the generator $\theta^m_n \in C_{m+n-3}(KK_{n,m})$
- Define $\beta$ on the monomial $(\theta^m_n)^{\otimes n} \in C_{m+n-3}(JJ_{n,m})$
- Define $\beta$ on the generator $\theta^m_n \in C_{m+n-2}(JJ_{n,m})$

Induction

For each $i + j = m + n, ij \neq 1$

1. Define $\alpha_A$ on each monomial in $C_*(\partial KK_{n,m})$ to be its corresponding fraction product of operations in $\{\omega^i_j A\}$; let $z = \alpha_A(C_{m+n-4}(KK_{n,m}))$
2. Define $\beta$ on each monomial in $C_*(\partial JJ_{n,m} \setminus \text{int} KK_{n,m} \times 1)$ to be its corresponding fraction product of operations and maps in $\{\omega^i_j, g^i_j, \omega^j_i B\}$; let $\varphi = \beta(C_{m+n-3}(JJ_{n,m} \setminus \text{int} KK_{n,m} \times 1))$
3. Then $\tilde{g}(z) = \nabla \varphi$ implies $[z] = 0$; choose a cochain $b$ such that $\nabla b = z$
4. Note that $\nabla (\tilde{g}(b) - \varphi) = \nabla \tilde{g}(b) - \tilde{g}(z) = \tilde{g}(\nabla b - z) = 0$; choose a cocycle
   $$u \in \tilde{g}^{-1} \tilde{g}(b - \varphi)$$
5. Define $\alpha_A(\theta^m_n) = \omega^m_n A := b - u$
6. Define $\beta \left( (f^m_n)^{\otimes n} \right) = g^{\otimes n} \omega^m_n A$
7. Note that $[\tilde{g}(\omega^m_n A) - \varphi] = [\tilde{g}(b - u) - \varphi] = [\tilde{g}(b) - \varphi] - [\tilde{g}(u)] = 0$; choose a cochain $g^m_n$ such that
   $$\nabla g^m_n = \tilde{g}(\omega^m_n A) - \varphi = g^{\otimes n} \omega^m_n A - \varphi$$
8. Define $\beta(f^m_n) = g^m_n$

This completes the induction.

Let us apply the Transfer Algorithm to compute an induced $A_{\infty}$-algebra structure on the cohomology of a DGA, which cannot be computed using the classical BPL. The DGA $B$ considered here has no Hodge decomposition, its homology
we have $(a^2 + x, xc + cx, (ac + ca)^2, c^2, tb, bt), \ |t| > 0$. Although the DGA $B$ is not commutative, we do have $a (ac + ca) = (ac + ca) a$ and $(ac)^2 = (ca)^2$. Note that $B$ has no Hodge decomposition since $c$ is not a cocycle, $2c$ is a coboundary, and $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Furthermore,

$H^n (B) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2, 5, 7 \\ 0 & \text{otherwise} \end{cases}$

and $H = H^* (B)$ is not free. Define $g : H \rightarrow B$ by $g(1) = 1$ and

- $u = [a] \mapsto a$
- $v = [ac + ca] \mapsto ac + ca$
- $w = [a (ac + ca)] \mapsto a (ac + ca)$.

Then $g$ has no right-homotopy inverse $f$ since $gf(b) = \lambda a$ implies $b - \lambda a = (1 - gf)(b) = (sd + ds)(b) = sd(b) = s(2c) = 2s(c) = 0$, which is a contradiction. To compute the induced multiplication $\mu_H$, consider the following bases for $H$ and $H \otimes H$:

|   | $H$ | $u$ | $v$ | $w$ | $u|u|$ | $u|v$, $v|u|$ | $u|w$, $w|u|$ | $v|v$ | $v|w$, $w|v$ | $w|w$ |
|---|-----|-----|-----|-----|--------|----------------|----------------|-----|----------------|------|
| $H \otimes H$ | $u|u|$ | $u|v$, $v|u|$ | $u|w$, $w|u|$ | $v|v$ | $v|w$, $w|v$ | $w|w$ |

Ignoring the unit 1 and evaluating $\tilde{g}$ on the basis \{ $w \partial_{u|u}$, $w \partial_{v|u}$ \} for $Hom^0 (H^{\otimes 2}, H)$ we have

$\tilde{g} \left( w \left( \partial_{u|v} + \partial_{v|u} \right) \right) = g \left( w \right) \left( \partial_{u|v} + \partial_{v|u} \right) = \mu (g \otimes g)$.

Now thinking of $w \left( \partial_{u|v} + \partial_{v|u} \right)$ as a class in $H^* (Hom^* (H^{\otimes 2}, H))$ we have

$\tilde{g}^* \left[ w \left( \partial_{u|v} + \partial_{v|u} \right) \right] = \left[ g \left( w \right) \left( \partial_{u|v} + \partial_{v|u} \right) \right] = [\mu (g \otimes g)].$

Define $\mu_H = (\tilde{g}^*)^{-1} [\mu (g \otimes g)] = w \left( \partial_{u|v} + \partial_{v|u} \right)$; then $uv = vu = w$ and $\mu_H$ is associative. To extend $g$ to an $A(2)$-map, let $\mu$ denote the multiplication in $B$ and consider the expression

$z = \mu (g \otimes g) - g \mu_H = a^2 \partial_{u|u} + (a^3 c + c a^3) \left( \partial_{u|w} + \partial_{w|u} \right)$

$= da \partial_{u|u} + d (ca c ) \left( \partial_{u|w} + \partial_{w|u} \right)$.

Then $\nabla \left( c \partial_{u|u} + cac \left( \partial_{u|w} + \partial_{w|u} \right) \right) = z$ and we define $g_2 = c \partial_{u|u} + cac \left( \partial_{u|w} + \partial_{w|u} \right)$ so that

$\nabla g_2 = \mu (g \otimes g) - g \mu_H$.

Thus $g$ is homotopy multiplicative. To compute the induced associator $\mu_H^3$, consider the following bases for $H$ and $H^{\otimes 3}$:

|   | $2$ | $5$ | $6$ | $7$ | $\ldots$ |
|---|-----|-----|-----|-----|---------|
| $H$ | $u$ | $v$ |     |     |         |
| $H \otimes H \otimes H$ | $u|u|u$ |     |     |     |         |
Since $B$ has trivial higher order structure, we consider the cochain

$$\varphi = \mu (g_2 \otimes g - g \otimes g_2) + g_2 (\mu_H \otimes 1 - 1 \otimes \mu_H),$$

which vanishes on $H^{\otimes 3}$ except $\varphi(u|u|u) = ac + ca = g(v)$; thus $\varphi = g(v) \partial_{u|u|u}$. Since $z = \mu_H (\mu_H \otimes 1 - 1 \otimes \mu_H) = 0$, every cocycle $b \in Hom^{-1} (H^{\otimes 3}, H)$ satisfies $\nabla b = z$. Since $v \partial_{u|u|u}$ is the only candidate, we set $b = v \partial_{u|u|u}$; then

$$[\tilde{g}(b) - \varphi] = [g(v) \partial_{u|u|u} - \varphi] = [0].$$

Choose $u = 0 \in \tilde{g}_s^{-1} [\tilde{g}(b) - \varphi]$ and define $\mu_3^H = v \partial_{u|u|u}$; then $\mu_3^H (u|u|u) = v$. Finally, since $\varphi - g(\mu_3^H) \equiv 0$; we may set $g_n = 0$ and $\mu_3^H = 0$ for all $n \geq 4$ to obtain an induced $A_\infty$-algebra structure $(H, \mu_H, \mu_3^H)$ and a map $G = g + g_2$ of $A_\infty$-algebras.

### 3. A Topological Example

Let $k$ be a field. Given a space $X$, let $S_* (\Omega X; k)$ denote the singular chains on the space of (base pointed) Moore loops on $X$, and choose a homology isomorphism $g : H_*(\Omega X; k) \to S_* (\Omega X; k)$. Since $H = H_*(\Omega X; k)$ is free and $S = S_*(\Omega X; k)$ is a Hopf algebra, the induced map $\tilde{g} : \mathcal{E}nd_{TH} \to U_{TH}$ is a homology isomorphism by Proposition 1, and the Transfer Algorithm induces an $A_\infty$-bialgebra structure on $H$. Let us apply this fact to a particular space $X$ and identify a non-trivial operation $\omega^2 : H \otimes H \to H \otimes H$.

Given a $1$-connected DGA $(A, d_A)$ over $Z_2$, the bar construction of $A$, denoted by $BA$, is the cofree DGC $T^c (\frac{1}{X})$ with differential $d$ and coproduct $\Delta$ defined as follows: Let $[x_1] \cdots [x_n]$ denote the element $\downarrow x_1 \otimes \cdots \downarrow x_n \in BA$; then

$$d [x_1] \cdots [x_n] = \sum_{i=1}^{n} [x_1] \cdots [dx_i] \cdots [x_n] + \sum_{i=1}^{n-1} [x_1] \cdots [x_i, x_{i+1}] \cdots [x_n];$$

$$\Delta [x_1] \cdots [x_n] = [\ ] \otimes [x_1] \cdots [x_n] + [x_1] \cdots [x_n] \otimes [\ ] + \sum_{i=1}^{n-1} [x_1] \cdots [x_i] \otimes [x_i] \cdots [x_n].$$

Consider the space $Y = (S^2 \times S^3) \vee \Sigma CP^2$, multiplicative generators $\tilde{a}_i \in H^*(S^3; Z_2)$, $\tilde{b} \in H^3 (\Sigma CP^2; Z_2)$ and $Sq^2 \tilde{b} \in H^5 (\Sigma CP^2; Z_2)$, a map $f : Y \to K(Z_2, 5)$ such that $f^* (i_5) = \tilde{a}_2 \tilde{a}_3 + Sq^2 \tilde{b}$, and the pullback $p : X \to Y$ of the following path fibration:

$$K(Z_2, 4) \longrightarrow X \longrightarrow \mathcal{L} K(Z_2, 5)$$

$$p \downarrow \quad \downarrow$$

$$Y \quad \quad \quad f \quad \quad \quad f^* \quad \quad \quad K(Z_2, 5).$$

Let $a_i = p^* (\tilde{a}_i)$ and $b = p^* (\tilde{b})$; then $A = H^*(X; Z_2) = \{1, a_2, a_3, b, a_2 a_3 = Sq^2 b, \ldots \}.$

Form the bar construction $BA$; since $H = H^* (BA) \approx H^* (\Omega X; Z_2)$ as coalgebras, $(BA, d, \Delta)$ is a DG coalgebra model for cochains on $\Omega X$. In [1], H.-J. Baues identified a compatible multiplication $\mu : BA \otimes BA \to BA$ and a DG Hopf algebra model $(BA, d, \Delta, \mu)$ in the following way: The twisting in $X$ induces Steenrod's
\( \sim_1: A \otimes A \to A \), which acts non-trivially via \( b \sim_1 b = a_2a_3 \) and the induced map \( \phi: BA \otimes BA \to A \) acts non-trivially via

\[
\phi([x] \otimes [1]) = \phi([1] \otimes [x]) = x \quad \text{and} \quad \phi([b] \otimes [b]) = b \sim_1 b = a_2a_3.
\]

(cf. [2], [3]). Consider the tensor product of coalgebras \( BA \otimes BA \) with coproduct \( \psi = \sigma_{2,2}(\Delta \otimes \Delta) \) and define

\[
\mu := \sum_{k \geq 0} (\underbrace{\phi \cdots \phi}_{k+1 \text{ factors}}) \bar{\psi}^{(k)},
\]

where \( \bar{\psi}^{(0)} = 1, \bar{\psi}^{(k)} = (\bar{\psi} \otimes 1^{\otimes k-1}) \cdots (\bar{\psi} \otimes 1) \bar{\psi} \) for \( k > 0 \), and \( \bar{\psi} \) is the reduced coproduct. Then for example, \( \mu([b] \otimes [b]) = [a_2a_3] \).

Let \( \mu_H \) be the multiplication on \( H \) induced by \( \mu \) and consider the classes \( \alpha_i = \text{cls}([a_i], \beta = \text{cls}([b]) \in H \). Choose a cocycle-selecting map \( g: H \to BA \) such that \( g(\text{cls}[x_1| \cdots| x_n]) = [x_1| \cdots| x_n] \). Then \( \mu([b] \otimes [b]) = [a_2a_3] = d([a_2a_3]) \) implies \( \mu_H(\beta \otimes \beta) = 0 \) and \( (g \mu_H + \mu(g \otimes g))(\beta \otimes \beta) = [a_2a_3] \). Nevertheless, by the Transfer Theorem, there is a cochain homotopy \( g^2_1: H \otimes H \to BA \) satisfying the relation \( \nabla g^2_1 = g \mu_H + \mu(g \otimes g) \) such that \( g^2_1(\beta \otimes \beta) = [a_i|a_{i-1}] \) for some \( i \in \{2,3\} \); and in particular, we may choose

\[
g^2_1(\beta \otimes \beta) = [a_2|a_3]
\]

since either choice gives rise to isomorphic structures. Let \( \Delta_H \) be the coproduct induced by \( \Delta \); then \( \{\Delta g + (g \otimes g) \Delta_H\}(\beta) = 0 \) since \( \beta \) is primitive. By the Transfer Theorem, there is a cochain homotopy \( g^2_1: H \to BA \otimes BA \) satisfying the relation \( \nabla g^2_1 = \Delta g + (g \otimes g) \Delta_H \) such that \( \nabla g^2_1(\beta) = 0 \). Thus \( g^2_1(\beta) = \lambda \otimes [a_2] + [a_2] \otimes \rho \) for some \( \lambda, \rho \in \mathbb{Z}_2 \); and in particular, we may choose

\[
g^2_1(\beta) = 0.
\]

Again by the Transfer Theorem, there is a cochain homotopy \( g_2^2: H \to BA \otimes BA \) satisfying the following relation on \( JJ_{2,2} \):

\[
\nabla g^2_1 = (\mu \otimes \mu) \sigma_{2,2}(\Delta g \otimes g^2_1 + g^2_1 \otimes (g \otimes g) \Delta_H) + (\mu(g \otimes g) \otimes g^2_1 + g^2_1 \otimes g_H \mu) \sigma_{2,2}(\Delta_H \otimes \Delta_H) + \omega_{2,2}^{BA}(g \otimes g) + (g \otimes g) \omega_{2,2}^{BA} + \Delta g^2_1 + g^2_1 \mu_H.
\]

The component \( \omega_{2,2}^{BA}(g \otimes g) \) vanishes since \( BA \) has trivial higher order structure; the non-triviality of \( (g \otimes g) \omega_{2,2}^{BA} \) is to be determined.

Let us evaluate relation \((\text{3.1})\) at \( \beta \otimes \beta \). First, \( g^2_1 \mu_H(\beta \otimes \beta) = 0 \) by the observation above, and \( (\mu \otimes \mu) \sigma_{2,2}(\Delta g \otimes g^2_1 + g^2_1 \otimes (g \otimes g) \Delta_H)(\beta \otimes \beta) = 0 \) by our choice of \( g^2_1 \). Second, \( (\mu(g \otimes g) \otimes g^2_1 + g^2_1 \otimes g_H \mu) \sigma_{2,2}(\Delta_H \otimes \Delta_H)(\beta \otimes \beta) = [\ ] \otimes g^2_1(\beta \otimes \beta) + g^2_1(\beta \otimes \beta) \otimes [\ ] = (\Delta + \bar{\Delta}) g^2_1(\beta \otimes \beta) \). Thus relation \((\text{3.1})\) reduces to

\[
\nabla g^2_1(\beta \otimes \beta) = (g \otimes g) \omega_{2,2}^{BA}(\beta \otimes \beta) + \Delta g^2_1(\beta \otimes \beta) = (g \otimes g) \omega_{2,2}^{BA}(\beta \otimes \beta) + [a_2] \otimes [a_3],
\]

and we conclude that

\[
\omega_{2,2}^{BA}(\beta \otimes \beta) = a_2 \otimes a_3.
\]

Thus the topology of the total space \( X \) in the fibration \( p: X \to (S^2 \times S^3) \vee \Sigma CP^2 \) above induces a nontrivial 2-in/2-out operation \( \omega_{2,2}^{BA} \) on \( H = H^*(\Omega X; \mathbb{Z}_2) \). A variation of this example, with a nontrivial topologically induced 2-in/n-out operation on loop cohomology for each \( n \geq 2 \), appears in [7].
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