Relation between the two geometric Satake equivalence via nearby cycle

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Abstract

Fargues and Scholze proved the geometric Satake equivalence over the Fargues–Fontaine curve. On the other hand, Zhu proved the geometric Satake equivalence using a Witt vector affine Grassmannian. In this paper, we explain the relation between the two version of the geometric Satake equivalence via nearby cycle.

1 Introduction

Let $F$ be a $p$-adic local field with a residue field $\mathbb{F}_q$. Write $O_F$ for its ring of integers. Put $k := \mathbb{F}_q$. Let $G$ be a reductive group scheme over $O_F$. Let $\Lambda$ be either of $\mathbb{Q}_\ell$, a finite extension $L$ of $\mathbb{Q}_\ell$, its ring of integers $O_L$, or its quotient ring. By [FS21, Theorem I.6.3], we have an equivalence of symmetric monoidal categories

$$\mathcal{S}_{FS}: \text{Sat}(\text{Hck}_{G,\text{Div}^1}, \Lambda) \xrightarrow{\sim} \text{Rep}(\hat{G}^{\text{tw}} \rtimes W_F, \Lambda)$$

where $\text{Sat}(\text{Hck}_{G,\text{Div}^1}, \Lambda)$ is the Satake category, which is the monoidal category of perverse sheaves with coefficients in $\Lambda$ on the Hecke stack $\text{Hck}_{G,\text{Div}^1}$ over the moduli space $\text{Div}^1 = \text{Spd} \hat{F}/\varphi\mathbb{Z}$ parametrizing degree 1 Cartier divisors on the Fargues–Fontaine curve for $F$. Let $\hat{G}$ be the Langlands dual group over $\Lambda$ and $W_F$ the Weil group of $F$. The group scheme $\hat{G}^{\text{tw}} \rtimes W_F$ is a semidirect product of $\hat{G}$ and $W_F$ with respect to the $W_F$-action on $\hat{G}$ obtained by twisting the usual action by an explicit cyclotomic

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action. Let $\text{Rep}(\hat{G}^{\text{tw}} \ltimes W_F, \Lambda)$ be the monoidal category of $\hat{G}^{\text{tw}} \ltimes W_F$-representations on finite projective $\Lambda$-modules.

Consider a diagram

$$
\begin{array}{c}
\text{Hck}_{G, \text{Spd}_k} \xrightarrow{i} \text{Hck}_{G, \text{Spd}_{\text{OP}}} \xrightarrow{j} \text{Hck}_{G, \text{Spd}_{\hat{F}}} \\
\downarrow p_F \\
\text{Hck}_{G, \text{Div}^1}.
\end{array}
$$

We have a nearby cycle functor

$$
\Psi := i^* \circ R_{j*} \circ (p_F)^*: \text{Sat}(\text{Hck}_{G, \text{Div}^1}, \Lambda) \to \text{Sat}(\text{Hck}_{G, \text{Spd}_k}, \Lambda).
$$

The main theorem is the following:

**Theorem 1.1.** There exists a symmetric monoidal structure on $(\text{Sat}(\text{Hck}_{G, \text{Spd}_k}, \Lambda), \star)$ and a symmetric monoidal equivalence

$$
\mathcal{J}_{YZ}: \text{Sat}(\text{Hck}_{G, \text{Spd}_k}, \Lambda) \xrightarrow{\sim} \text{Rep}(\hat{G}, \Lambda)
$$

such that for the diagram of functors

$$
\begin{array}{ccc}
\text{Sat}(\text{Hck}_{G, \text{Div}^1}, \Lambda) & \xrightarrow{\mathcal{J}_{FS}} & \text{Rep}(\hat{G}^{\text{tw}} \ltimes W_F, \Lambda) \\
\downarrow \Psi & & \downarrow \text{For} \\
\text{Sat}(\text{Hck}_{G, \text{Spd}_k}, \Lambda) & \xrightarrow{\mathcal{J}_{YZ}} & \text{Rep}(\hat{G}, \Lambda),
\end{array}
$$

there is a natural isomorphism

$$
\text{For} \circ \mathcal{J}_{FS} \cong \mathcal{J}_{YZ} \circ \Psi.
$$

The equivalence $\mathcal{J}_{YZ}$ with coefficients in $\overline{Q}_\ell, \mathbb{F}_\ell$ or $\mathbb{Z}_\ell$ has the same form as the geometric Satake equivalence in [Zhu17] or [Yu19]. We do not prove that the equivalence $\mathcal{J}_{YZ}$ is the same functor as the equivalence in [Zhu17] or [Yu19], which is too complicated to prove since the way of constructing a symmetric monoidal structure on Satake categories is quite different between [FS21] and [Zhu17]. However, we prove that the several properties of [Zhu17] also hold for $\mathcal{J}_{YZ}$.

The similar results to the above theorem are mentioned in [FS21, Remark I.2.14]. However, we show further that the nearby cycle functor, involving $R_{j*}$, can be used as the vertical functor.
While we are completing this work, a related paper [AGLR22] appeared. The results of Lemma 3.6, Theorem 3.10 and Corollary 3.11 overlap with [AGLR22] Lemma 6.11, Proposition 6.12, Corollary 6.14, respectively. Our proofs of these results are also similar to that of [AGLR22], but slightly different. We prove Lemma 3.6 by reducing to the scheme settings, showing some results on the comparison to schemes. Also, our proofs are simpler than [AGLR22] since the result of [AGLR22] is more general.

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2 Notations

Throughout this paper, $F$ is a $p$-adic local field with a residue field $\mathbb{F}_q$. Write $\mathcal{O}_F$ for its ring of integers. Put $k := \mathbb{F}_q$.

Let $\ell$ be a prime number not equal to $p$. Let $\Lambda$ be either of $\mathbb{Q}_\ell$, a finite extension $L$ of $\mathbb{Q}_\ell$, its ring of integers $\mathcal{O}_L$, or its quotient ring. The category of finite projective $\Lambda$-module is denoted by $\text{Proj}(\Lambda)$. If $\Lambda$ is a field, then we write $\text{Vec}(\Lambda)$ for $\text{Proj}(\Lambda)$.

Put

$$\text{Div}^1 := \text{Spd} \hat{F}/\phi \mathbb{Z}$$

where $\phi$ is the Frobenius. Let $G$ be a reductive group over $\text{Spec} \mathcal{O}_F$. For a small v-stack $S$ with a map $S \to \text{Div}^1$, or $S \to \text{Spd} \mathcal{O}_F$, let $\text{Gr}_{G,S}$ be the Beilison–Drinfeld affine Grassmannian of $G$ over $S$ and $\text{Hck}_{G,S}$ the local Hecke stack defined in [FS21, VI.1].

The Langlands dual group of $G$ over $\Lambda$ is denoted by $\hat{G}_\Lambda$ or simply $\hat{G}$. The scheme $\hat{G}$ has a natural $W_F$-action. We define the category $\text{Rep}(\hat{G}, \Lambda)$ as the monoidal category of algebraic $\hat{G}$-representations on finite projective $\Lambda$-modules. We write $\text{Rep}(\hat{G} \rtimes W_F, \Lambda)$ for the category of $\hat{G} \rtimes W_F$-representations on finite projective $\Lambda$-modules such that the $\hat{G}$-action is algebraic, and that the $W_F$-action is smooth. The category $\text{Rep}(\hat{G}^{tw} \rtimes W_F, \Lambda)$ is similarly defined with respect to the twisted $W_F$-action on $\hat{G}$ appearing in [FS21, Theorem I.6.3].

If $\Lambda$ is a torsion ring, then the derived category $D_{\text{et}}(X, \Lambda)$ for a small v-stack $X$ and the six functors with respect to this formalism are defined as in [Sch17, Definition 1.7].
3 Torsion coefficient case

Let $Λ$ be a proper quotient of $O_L$ where $L$ is a finite extension over $\mathbb{Q}_\ell$.

3.1 Generalities on étale cohomology of diamonds

Put $C := \widehat{F}$.

3.1.1 Excision distinguished triangle

Lemma 3.1. Let $X$ be a small $v$-stack. For $A \in D_{\text{ét}}(X, Λ)$, the functor

$$R \mathcal{H}om_{D_{\text{ét}}(X, Λ)}(-, A) : D_{\text{ét}}(X, Λ)^{\text{op}} \to D_{\text{ét}}(X, Λ)$$

preserves distinguished triangles.

Proof. Following [Sch17, Corollary 17.2], write

$$R_{X_{\text{ét}}} : D(X_v, Λ) \to D_{\text{ét}}(X, Λ)$$

for the right adjoint of the inclusion $D_{\text{ét}}(X, Λ) \hookrightarrow D(X_v, Λ)$. Then as in the remark just after Lemma 17.8 of [Sch17], we have a natural isomorphism

$$R \mathcal{H}om_{D_{\text{ét}}(X, Λ)}(-, A) \cong R_{X_{\text{ét}}} R \mathcal{H}om_{D(X_v, Λ)}(-, A) : D_{\text{ét}}(X, Λ)^{\text{op}} \to D_{\text{ét}}(X, Λ).$$

The functor $R \mathcal{H}om_{D(X_v, Λ)}(-, A)$ preserves distinguished triangles since it is the derived functor of the inner hom of $v$-sheaves. The functor $R_{X_{\text{ét}}}$ preserves distinguished triangles since its left adjoint $D_{\text{ét}}(X, Λ) \hookrightarrow D(X_v, Λ)$ preserves distinguished triangles.

Let $j : \text{Spd} C \to \text{Spd} O_C$ and $i : \text{Spd} k \to \text{Spd} O_C$ be inclusions.

Lemma 3.2. For $A \in D_{\text{ét}}(\text{Spd} O_C, Λ)^{bd}$,

$$j_! R j^* A \to A \to R i_* i^* A \to,$$  \hspace{1cm} (3.1)

and

$$R i_* R i^! A \to A \to R j_* j^* A \to$$  \hspace{1cm} (3.2)

are distinguished triangles.

Proof. The proof of (3.1) is reduced to the case that $A \in D_{\text{ét}}(\text{Spd} O_C, Λ)$ is concentrated in degree 0. The functors $j^*, i^*$ are exact (that is, commute with canonical truncations) by definition. The functor $R j_!$ is also exact (see [Sch17, Definition/Proposition 19.1]). Moreover, the functor $R i_*$ is exact. Hence it suffices to show that

$$0 \to j_! R j^* A \to A \to R i_* i^* A \to 0$$
is an exact sequence. This can be checked after taking $i^*$ and $j^*$, and that is easy.
For (3.2), by (3.1) and Lemma 3.1 we have a distinguished triangle

$$R \mathcal{H} \text{om}(Ri^* i^\ast \Lambda, A) \to R \mathcal{H} \text{om}(\Lambda, A) \to R \mathcal{H} \text{om}(Rj_! j^\ast \Lambda, A) \to.$$

By \cite{Sch17} Theorem 1.8(iv), (v)], we have

$$R \mathcal{H} \text{om}(Ri^* i^\ast \Lambda, A) \cong Ri_! Ri^! R \mathcal{H} \text{om}(i^\ast \Lambda, Ri^! A) \cong Ri_! Ri^! A$$

$$R \mathcal{H} \text{om}(\Lambda, A) \cong A$$

$$R \mathcal{H} \text{om}(j_! j^\ast \Lambda, A) \cong Rj_! Rj^! R \mathcal{H} \text{om}(j^\ast \Lambda, Rj^! A) \cong Rj_! Rj^! A \cong Rj_! j^\ast A,$$

hence the lemma follows. \hfill \Box

### 3.1.2 Comparison to scheme

Let $X$ be a scheme locally of finite type over $\text{Spec} \mathcal{O}_C$. Then we can define a diamond $X^{\diamond}$ over $\text{Spd} \mathcal{O}_C$ as follows:

$$X^{\diamond} : \text{Perf} \to \text{Sets}$$

$$S \mapsto \left\{ (S^\sharp, f : S^\sharp \to X) \mid \begin{array}{l}
S^\# \text{ is an untilt over } \text{Spa} \mathcal{O}_C = \text{Spa}(\mathcal{O}_C, \mathcal{O}_C) \\
f : S^\sharp \to X \text{ is a morphism of locally ringed space over } \text{Spec} \mathcal{O}_C.
\end{array} \right\}.$$

The functor $X \mapsto X^{\diamond}$ defines a morphism of site

$$c_X : (X^{\diamond})_v \to X_{\text{ét}}.$$

It defines a functor

$$c_X^* : D(X_{\text{ét}}, \Lambda) \to D_{\text{ét}}(X^{\diamond}, \Lambda).$$

It commute with colimits, and so admits a right adjoint $Rc_X^*$. We have the following proposition:

**Proposition 3.3.** Let $f : Y \to X$ be a separated map between schemes of finite type over $\mathcal{O}_C$. Then we have

$$(f^{\diamond})^* c_X^* \cong c_Y^* f^*,$$

$$Rf_! Rc_Y^* \cong Rc_X^* Rf_!^{\diamond}.$$

**Proof.** The first isomorphism follows from the definition. The second isomorphism is the adjunction of the first one. \hfill \Box
Proposition 3.4. Let $f : Y \rightarrow X$ be a separated map between schemes of finite type over $\mathcal{O}_C$. Then $f^\dagger : Y^\dagger \rightarrow X^\dagger$ is compactifiable and representable in locally spatial diamonds with $\dim \text{trg} f < \infty$, and
\[
Rf^\dagger c_Y^\ast \cong c_X Rf_!,
\]
\[
Rf_! Rc_{X^\ast} \cong Rc_{Y^\ast} R(f^\dagger)^!.
\]

Proof. The proof is the same as [Sch17, Proposition 27.5].

By Nagata’s compactification, we can assume $f$ is proper. Then $f^\dagger$ is also proper and representable in spatial diamonds with $\dim \text{trg} f < \infty$. We want to see that the natural transformation
\[
c_X Rf_! \rightarrow Rf_! c_Y^\ast
\]
is an equivalence. Let $T \rightarrow \text{Spa} \mathcal{O}_C$ be a smooth surjective map of adic spaces such that $T$ is analytic. Then $X \times_{\text{Spec} \mathcal{O}_C} T$ is an analytic adic space, and it holds that $(X \times_{\text{Spec} \mathcal{O}_C} T)^\dagger = X^\dagger \times_{\text{Spd} \mathcal{O}_C} T^\dagger$. Since $T^\dagger \rightarrow \text{Spd} \mathcal{O}_C$ is surjective and $\ell$-cohomologically smooth for all $\ell \neq p$ by [Sch17 Proposition 24.4], the pullback functor $D_{\text{et}}(X^\dagger, \Lambda) \rightarrow D_{\text{et}}(X^\dagger \times_{\text{Spd} \mathcal{O}_C} T^\dagger, \Lambda)$ is conservative. Thus, we only have to show the result after taking the fiber product with $T$. Then the proposition follows from [Hub96 Theorem 3.7.2] and some base change results.

Proposition 3.5. The three functors
\[
c_k := c_{\text{Spec} k}^\ast : D((\text{Spec} k)_{\text{et}}, \Lambda) \rightarrow D_{\text{et}}(\text{Spd} k, \Lambda),
\]
\[
c_{\mathcal{O}_C} := c_{\text{Spec} \mathcal{O}_C}^\ast : D((\text{Spec} \mathcal{O}_C)_{\text{et}}, \Lambda) \rightarrow D_{\text{et}}(\text{Spd} \mathcal{O}_C, \Lambda),
\]
\[
c_C := c_{\text{Spec} C}^\ast : D((\text{Spec} C)_{\text{et}}, \Lambda) \rightarrow D_{\text{et}}(\text{Spd} C, \Lambda)
\]
are isomorphisms.

Proof. For $\text{Spec} C$, the claim follows from a canonical equivalence
\[
D_{\text{et}}(\text{Spd} C, \Lambda) \cong D(|\text{Spd} C|, \Lambda) = D(\Lambda-\text{mod})
\]
since $\text{Spd} C$ is strictly totally disconnected. For $\text{Spec} k$, it follows from the fact that the pullback functor
\[
D_{\text{et}}(\text{Spd} k, \Lambda) = D_{\text{et}}(\text{Spa} k, \Lambda)
\]
\[
\rightarrow D_{\text{et}}(\text{Spa} C^\diamond, \Lambda)
\]
\[
\simeq D(|\text{Spa} C^\diamond|, \Lambda)
\]
\[
\simeq D(\Lambda-\text{mod})
\]
is fully faithful by [FS21, Theorem 19.5], and essentially surjective considering constant complexes. Let us consider the case of Spec $\mathcal{O}_C$. First, the fully faithfulness is reduced to the fully faithfulness of 

$$c_{\mathcal{O}_C}^*: D((\text{Spec } \mathcal{O}_C)_{\text{et}}, \Lambda) \to D((\text{Spd } \mathcal{O}_C)_{\nu}, \Lambda),$$

that is, the fully faithfulness of the exact functor

$$c_{\mathcal{O}_C}^*: \text{Sh}((\text{Spec } \mathcal{O}_C)_{\text{et}}, \Lambda) \to \text{Sh}((\text{Spd } \mathcal{O}_C)_{\nu}, \Lambda).$$

where $\text{Sh}(\cdot, \Lambda)$ means the category of sheaves of $\Lambda$-modules. The site $(\text{Spec } \mathcal{O}_C)_{\text{et}}$ can be written explicitly, so one can check this directly.

Let $j: \text{Spd } C \to \text{Spd } \mathcal{O}_C$ and $i: \text{Spd } k \to \text{Spd } \mathcal{O}_C$ be inclusions. For $A \in D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)$, there exists a distinguished triangle

$$j_! j^* A \to A \to i_* i^* A \to j_! j^* A[1].$$

Hence the object of $D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)$ is determined by $B \in D_{\text{et}}(\text{Spd } k, \Lambda)$, $C \in D_{\text{et}}(\text{Spd } C, \Lambda)$ and a morphism $i_! B \to j_! C[1]$. Similarly, the object of $D(\text{Spec } \mathcal{O}_C, \Lambda)$ is determined by $B' \in D((\text{Spec } k)_{\text{et}}, \Lambda)$, $C' \in D((\text{Spec } C)_{\text{et}}, \Lambda)$ and a morphism $i_! B' \to j_! C'[1]$.

Since $i_!$ and $j_!$ are compatible with $c_X^*$ by Proposition 3.4, the equivalence of $c_{\mathcal{O}_C}^*$ follows from the equivalence of $c_k^*, c_C^*$ and the fully faithfulness of $c_{\mathcal{O}_C}^*$.

3.1.3 Pushforward of constant sheaves

Let $j: \text{Spd } C \to \text{Spd } \mathcal{O}_C$ and $i: \text{Spd } k \to \text{Spd } \mathcal{O}_C$ be inclusions.

**Lemma 3.6.** If $K \in D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)$ is constant with perfect fiber, then the canonical morphism

$$K \to Rj_* j^* K$$

is an isomorphism.

First, we show the scheme version of this lemma.

**Lemma 3.7.** Let $j_C: \text{Spec } C \to \text{Spec } \mathcal{O}_C$ be an inclusion. The canonical morphism

$$\Lambda \to Rj_* \Lambda$$

is an isomorphism.
Proof. Let \( j : \text{Spec} \, \widehat{F} \to \text{Spec} \, \mathcal{O}_\overline{F} \) and \( i : \text{Spec} \, k \to \text{Spec} \, \mathcal{O}_\overline{F} \) be inclusions. Apply \cite[Lemma 0EYM]{Sta18} to the inverse system of morphisms of schemes \( (j_E : \text{Spec} \, E \to \text{Spec} \, \mathcal{O}_E)_{E} \) and system of sheaves \( (\Lambda_{\text{Spec} \, E})_{E} \) where \( E \)'s are finite extensions of \( F \). Then we have

\[
R^p j_{\overline{F}*} \Lambda \cong \operatorname{colim} h_E^* R^p j_{E*} \Lambda
\]

where \( h_E : \text{Spec} \, \mathcal{O}_\overline{F} \to \text{Spec} \, \mathcal{O}_E \). By the calculation of Galois cohomologies, one has

\[
i_{\overline{F}}^* h_{E}^* R^p j_{E*} \Lambda \cong \begin{cases} \Lambda & (p = 0) \\ \operatorname{Hom}(I_E, \Lambda) & (p = 1) \\ 0 & \text{(otherwise)} \end{cases}
\]

where \( I_E \) is the inertia group of \( E \). Moreover, for \( E_1 \subset E_2 \), the map \( i_{E_2}^* h_{E_2}^* R^p j_{E_2*} \to i_{E_1}^* h_{E_1}^* R^p j_{E_1*} \) coincides with a canonical homomorphism induced by the inclusion \( I_{E_2} \subset I_{E_1} \). Therefore we have

\[
i_{\overline{F}}^* R^p j_{\overline{F}*} \Lambda \cong \begin{cases} \Lambda & (p = 0) \\ 0 & \text{(otherwise)} \end{cases}.
\]

That is, \( i_{\overline{F}}^* R j_{\overline{F}*} \Lambda \cong \Lambda \). This isomorphism is the inverse of the canonical morphism \( \Lambda = i_{\overline{F}}^* \Lambda \to i_{\overline{F}}^* R j_{\overline{F}*} j_{\overline{F}*} \Lambda = i_{\overline{F}}^* R j_{\overline{F}*} \Lambda \). It follows that the canonical morphism

\[
\Lambda \to R j_{\overline{F}*} \Lambda
\]

is an isomorphism.

Finally, let \( \alpha : \text{Spec} \, \widehat{F} \to \text{Spec} \, F \), and \( \beta : \text{Spec} \, \mathcal{O}_\widehat{F} \to \text{Spec} \, \mathcal{O}_F \) be the natural morphisms. One can show that the induced morphism of sites

\[
\begin{align*}
(Spec \, \widehat{F})_{\text{ét}} & \to (Spec \, F)_{\text{ét}}, \\
(Spec \, \mathcal{O}_\widehat{F})_{\text{ét}} & \to (Spec \, \mathcal{O}_F)_{\text{ét}}
\end{align*}
\]

are isomorphism since these four site can be written explicitly. Therefore, \( \alpha^* : D((\text{Spec} \, F)_{\text{ét}}, \Lambda) \to D((\text{Spec} \, \widehat{F})_{\text{ét}}, \Lambda) \) and \( \beta^* : D((\text{Spec} \, \mathcal{O}_F)_{\text{ét}}, \Lambda) \to D((\text{Spec} \, \mathcal{O}_\widehat{F})_{\text{ét}}, \Lambda) \) are equivalences. It follows that the diagram

\[
\begin{array}{ccc}
D((\text{Spec} \, F)_{\text{ét}}, \Lambda) & \xrightarrow{j_{\overline{F}*}} & D((\text{Spec} \, \mathcal{O}_F)_{\text{ét}}, \Lambda) \\
\downarrow{\alpha^*} & & \downarrow{\beta^*} \\
D((\text{Spec} \, \widehat{F})_{\text{ét}}, \Lambda) & \xrightarrow{j_*} & D((\text{Spec} \, \mathcal{O}_\widehat{F})_{\text{ét}}, \Lambda)
\end{array}
\]

is commutative up to natural isomorphism since \( \alpha^* \cong (\alpha_*)^{-1} \), \( \beta^* \cong (\beta_*)^{-1} \). By this diagram and the isomorphism \( (3.3) \), we get the lemma. \( \square \)
Corollary 3.8. If $K \in D((\text{Spec } \mathcal{O}_C)_{\text{et}}, \Lambda)$ is constant with perfect fiber, then the canonical morphism

$$K \to Rj_*j^*K$$

is an isomorphism.

Proof. Since the value of the constant complex $K$ is perfect, it can be represented by a bounded complex $(M^*)$ whose terms are finite projective over $\Lambda$. Since $\Lambda$ is a self-injective ring, each $M^i$ is injective, $Rj_*j^*K$ is represented by a complex $(Rj_*M^*)$, which is isomorphic to a constant complex $(M^*)$ by Lemma 3.7. This proves the claim.

Proof of Lemma 3.6 By Corollary 3.8 it suffices to show that the diagram

$$
\begin{array}{ccc}
D((\text{Spec } \mathcal{C})_{\text{et}}, \Lambda) & \xrightarrow{j_*} & D((\text{Spec } \mathcal{O}_C)_{\text{et}}, \Lambda) \\
\downarrow c^*_C & & \downarrow c^*_C \\
D_{\text{et}}(\text{Spd } \mathcal{C}, \Lambda) & \xrightarrow{j_*} & D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)
\end{array}
$$

is commutative up to a natural isomorphism. This follows from Proposition 3.5 since $c^*_C \cong (Rc^*_C)^{-1}$ and $c^*_O \cong (Rc^*_O)^{-1}$ holds.

3.2 Nearby cycle functor

Proposition 3.9. $K \in D_{\text{et}}(\mathcal{H}ck_G, \text{Spd } \mathcal{C}, \Lambda)^{\text{bd}}$ is universally locally acyclic (ULA) over $\text{Spd } \mathcal{C}$, then $Rj_*K \in D_{\text{et}}(\mathcal{H}ck_G, \text{Spd } \mathcal{O}_C, \Lambda)^{\text{bd}}$ is ULA over $\text{Spd } \mathcal{O}_C$.

Proof. Since we are working over $\text{Spd } \mathcal{O}_C$, we may assume that $G$ is split. Fix a maximal split torus $T \subset G$ and a Borel subgroup $B \subset G$ containing $T$. By [FS21 VI.6.4], for a small v-stack $S$ over $\text{Div}^1$ and $A \in D_{\text{et}}(\mathcal{H}ck_G, S, \Lambda)^{\text{bd}}$, the complex $A$ is ULA over $S$ if and only if

$$CT_B(A) \in D_{\text{et}}(\mathcal{H}ck_T, S, \Lambda)^{\text{bd}}$$

is ULA over $S$, where the functor $CT_B$ is the hyperbolic localization. Since $CT_B$ is compatible with $Rj_*$, we may assume $G = T$.

In this case, since $\mathcal{H}ck_T = \text{Gr}_{T,S} \cong X_*(T) \times S$ holds, we need to show that if $K \in D_{\text{et}}(\text{Spd } \mathcal{C}, \Lambda)^{\text{bd}}$ is ULA over $\text{Spd } \mathcal{C}$, then $Rj_*K \in D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)^{\text{bd}}$ is ULA over $\text{Spd } \mathcal{O}_C$. By [FS21 IV.2.9], we have to show that if $K \in D_{\text{et}}(\text{Spd } \mathcal{C}, \Lambda)^{\text{bd}}$ is locally constant with perfect fibers, then $Rj_*K \in D_{\text{et}}(\text{Spd } \mathcal{O}_C, \Lambda)^{\text{bd}}$ is locally constant with perfect fibers. This follows form Lemma 3.6.
Theorem 3.10. The pullback functor

\[ j^*: D^{ULA}_{\text{et}}(\mathcal{H}_{\text{ck}G, \text{Spd}O_C}, \Lambda)^{\text{bd}} \to D^{ULA}_{\text{et}}(\mathcal{H}_{\text{ck}G, \text{Spd}C}, \Lambda)^{\text{bd}} \]

is an equivalence of categories, and its quasi-inverse is \( Rj_* \).

Proof. The fact that \( j^* \) is an equivalence is proved in [FS21, Corollary VI.6.7]. By Proposition 3.9 and the identity \( j^* Rj_* = \text{id} \), its quasi-inverse is \( Rj_* \).

Corollary 3.11. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{H}_{\text{ck}G, \text{Spd}k} & \xrightarrow{i} & \mathcal{H}_{\text{ck}G, \text{Spd}O_C} \\
& j & \downarrow \rho_F \\
& & \mathcal{H}_{\text{ck}G, \text{Div}^1}.
\end{array}
\]

The functors \((p_F)^*, Rj_*, i^*\) preserves the object of the Satake categories. Moreover, \( Rj_* \) and \( i^* \) induce equivalences

\[ Rj_* : \text{Sat}(\mathcal{H}_{\text{ck}SpdC}, \Lambda) \xrightarrow{\sim} \text{Sat}(\mathcal{H}_{\text{ck}SpdO_C}, \Lambda), \]

\[ i^* : \text{Sat}(\mathcal{H}_{\text{ck}SpdO_C}, \Lambda) \xrightarrow{\sim} \text{Sat}(\mathcal{H}_{\text{ck}Spdk}, \Lambda). \]

Proof. Since we are considering the relative locally acyclicity and the relative perversity, \( i^*, j^*, (p_F)^* \) preserves ULA objects and flat perverse (i.e. for all \( \Lambda \)-modules \( M, A \otimes_{\mathbb{F}} M \) is perverse) objects.

Let us prove that \( Rj_* \) preserves the object of the Satake categories. Since we are working over \( \text{Spd}O_C \), we may assume that \( G \) is split. Fix a maximal split torus \( T \subset G \) and a Borel subgroup \( B \subset G \) containing \( T \). The pushforward \( Rj_* \) commutes with \( R\pi_{T,S}\alpha \text{CT}_{B}(-)[\text{deg}] \) where \( \pi_{T,S} : \text{Gr}_{T,S} \to S \). Hence by [FS21, Proposition VI.6.4, Proposition VI.7.7], we need to show that if \( A \in D_{\text{et}}(\text{Spd}C, \Lambda) \) is étale locally a finite projective \( \Lambda \)-module in degree 0, then so is \( Rj_* A \in D_{\text{et}}(\text{Spd}O_C, \Lambda) \). This follows from Lemma 3.6.

Moreover, \( Rj_* \) induces an equivalence since \( j^* \) gives the quasi-inverse. It remains to show that \( i^* \) induces an equivalence. As in [FS21, Corollary VI.6.7],

\[ i^* : D^{ULA}_{\text{et}}(\mathcal{H}_{\text{ck}SpdO_C}, \Lambda) \xrightarrow{\sim} D^{ULA}_{\text{et}}(\mathcal{H}_{\text{ck}Spdk}, \Lambda) \]

is an equivalence. Thus, as above, using the hyperbolic localization, the result follows from the fact that for \( A \in D^{ULA}_{\text{et}}(\text{Spd}O_C, \Lambda) \), the object \( i^* A \in D_{\text{et}}(\text{Spd}k, \Lambda) \) is étale locally a finite projective \( \Lambda \)-module in degree 0 if and only if so is \( A \in D_{\text{et}}(\text{Spd}O_C, \Lambda) \).
Therefore, we have a functor
\[ \Psi := i^* R j_* (p_F)^* : \text{Sat}(\mathcal{Hck}_{\text{Div}^1}, \Lambda) \to \text{Sat}(\mathcal{Hck}_{\text{Spd}^k}, \Lambda) \] (3.4)
called a nearby cycle functor.

### 3.3 Nearby cycle functor and convolution

In this subsection, we prove that

**Proposition 3.12.** Consider the nearby cycle functor
\[ \Psi := i^* R j_* (p_F)^* : \text{Sat}(\mathcal{Hck}_{G, \text{Div}^1}, \Lambda) \to \text{Sat}(\mathcal{Hck}_{G, \text{Spd}^k}, \Lambda). \]

There is a natural isomorphism
\[ \Psi(A \ast B) \cong \Psi(A) \ast \Psi(B) \]
for \( A, B \in \text{Sat}(\mathcal{Hck}_{\text{Div}^1}, \Lambda) \) where \( \ast \) denotes the convolution product defined in [ES21, VI.8].

Apply Theorem [3.10] to a reductive group \( G \times \mathcal{O}_F G \), noting that there is a canonical isomorphism
\[ \mathcal{Hck}_{G,S} \times_S \mathcal{Hck}_{G,S} \cong \mathcal{Hck}_{G \times \mathcal{O}_F G,S}. \]

We have the following proposition:

**Proposition 3.13.** Let \( j_G^2 : \mathcal{Hck}_{G, \text{Spd}^k} \times_{\text{Spd}^k} \mathcal{Hck}_{G, \text{Spd}^k} \to \mathcal{Hck}_{G, \text{Spd}^k \times \mathcal{O}_F G, \text{Spd}^k} \mathcal{Hck}_{G, \text{Spd}^k} \) be the natural inclusion. Then
\[ j_G^2^* : D^\text{ULA}_{\text{et}}(\mathcal{Hck}_{G, \text{Spd}^k \times \mathcal{O}_F G, \text{Spd}^k} \mathcal{Hck}_{G, \text{Spd}^k}, \Lambda)^{\text{bd}} \to D^\text{ULA}_{\text{et}}(\mathcal{Hck}_{G, \text{Spd}^k \times \mathcal{O}_F G, \text{Spd}^k} \mathcal{Hck}_{G, \text{Spd}^k}, \Lambda)^{\text{bd}} \]
is an equivalence of categories, and its quasi-inverse is \( R j_G^2^* \).

**Proof of Proposition 3.12.** Put
\[ \mathcal{Hck}_{G, \text{Div}^1} := L^+_{\text{Div}^1} G \backslash L^+_{\text{Div}^1} G \times L^\text{et}_{\text{Div}^1} G \hookrightarrow L^+_{\text{Div}^1} G/L^+_{\text{Div}^1} G, \]
and put $\widehat{\text{Hck}}_{G,S} := \widehat{\text{Hck}}_{G,\text{Div}^1} \times_{\text{Div}^1} S$ for a small v-stack $S \to \text{Div}^1$. Consider the diagram

$$
\begin{array}{c}
\text{Hck}_{G,\text{Div}^1} \times_{\text{Div}^1} \text{Hck}_{G,\text{Div}^1} \\
\downarrow f_{F,G^2} \\
\text{Hck}_{G,\text{Spd}C} \times_{\text{Spd}C} \text{Hck}_{G,\text{Spd}C} \\
\downarrow j_{G^2} \\
\text{Hck}_{G,\text{Spd}O_C} \times_{\text{Spd}O_C} \text{Hck}_{G,\text{Spd}O_C} \\
\downarrow i_{G^2} \\
\text{Hck}_{G,\text{Spd}k} \times_{\text{Spd}k} \text{Hck}_{G,\text{Spd}k}
\end{array}
\quad
\begin{array}{c}
\widehat{\text{Hck}}_{G,\text{Div}^1} \\
\uparrow a_{\text{Div}^1} \\
\widehat{\text{Hck}}_{G,\text{Spd}C} \\
\uparrow a_C \\
\widehat{\text{Hck}}_{G,\text{Spd}O_C} \\
\uparrow a_{O_C} \\
\widehat{\text{Hck}}_{G,\text{Spd}k}
\end{array}
\quad
\begin{array}{c}
\widehat{\text{Hck}}_{G,\text{Div}^1} \\
\uparrow b_{\text{Div}^1} \\
\widehat{\text{Hck}}_{G,\text{Spd}C} \\
\uparrow b_C \\
\widehat{\text{Hck}}_{G,\text{Spd}O_C} \\
\uparrow b_{O_C} \\
\widehat{\text{Hck}}_{G,\text{Spd}k}
\end{array}
\quad
\begin{array}{c}
\text{Hck}_{G,\text{Div}^1} \\
\downarrow p_{F,G^2} \\
\text{Hck}_{G,\text{Spd}C} \times_{\text{Spd}C} \text{Hck}_{G,\text{Spd}C} \\
\downarrow p \\
\text{Hck}_{G,\text{Spd}O_C} \times_{\text{Spd}O_C} \text{Hck}_{G,\text{Spd}O_C} \\
\downarrow i \\
\text{Hck}_{G,\text{Spd}k} \times_{\text{Spd}k} \text{Hck}_{G,\text{Spd}k}
\end{array}
$$

Since $a_{O_C}$ is an $L^+G$-torsor and $b_{\text{Div}^1}$, $b_{O_C}$ is ind-proper, by using base changes, we have

$$R(b_k)^*(a_k)^*(i_{G^2})^*(j_{G^2})^*(A \boxtimes_{A,\text{Div}^1} B) \cong \Psi(A \ast B)$$

for $A, B \in D_\text{et}(\text{Hck}_{G,\text{Div}^1}, \Lambda)^{\text{bd}}$. It suffices to show that if $A, B \in D_\text{et}(\text{Hck}_{G,\text{Spd}C}, \Lambda)$ are ULA sheaves, then

$$R(j_{G^2})^*(A \boxtimes_{A,\text{Spd}C} B) \cong Rj_* A \boxtimes_{A,\text{Spd}O_C} Rj_* B.$$  \hspace{1cm} (3.5)

First, if $A, B \in D_\text{et}(\text{Hck}_{G,\text{Spd}C}, \Lambda)$ are ULA over $\text{Spd}C$, then

$$A \boxtimes_{A,\text{Spd}C} B \in D_\text{et}(\text{Hck}_{G,\text{Spd}C} \times_{\text{Spd}C} \text{Hck}_{G,\text{Spd}C}, \Lambda)^{\text{bd}}$$

is ULA over $\text{Spd}C$. In fact, by [FS21, Proposition VI.6.5], it suffices to show that if $A_1, A_2 \in D_\text{et}(\text{Spd}C, \Lambda)$ is locally constant with perfect fibers, then so is $A_1 \boxtimes_{A,\text{Spd}C} A_2 = A_1 \otimes_{\Lambda} A_2 \in D_\text{et}(\text{Spd}C, \Lambda)$. This is clear as $D_\text{et}(\text{Spd}C, \Lambda) \simeq D(\Lambda\text{-mod})$.

Therefore, by Proposition 3.13 we only have to show (3.5) after taking $j_{G^2}^*$. Since pullbacks are compatible with exterior tensor products, we have

$$j_{G^2}^* R(j_{G^2})^*(A \boxtimes_{A,\text{Spd}C} B) \cong A \boxtimes_{A,\text{Spd}C} B$$

$$\cong j^* Rj_* A \boxtimes_{A,\text{Spd}C} j^* Rj_* B$$

$$\cong j_{G^2}^* (Rj_* A \boxtimes_{A,\text{Spd}O_C} Rj_* B),$$

and the proposition follows. \qed
3.4 Relation between the two geometric Satake via nearby cycle

**Theorem 3.14.** Consider the diagram

\[
\begin{array}{c}
\text{Sat}(\mathcal{H}ck_{\text{Div}^1}, \Lambda) \xrightarrow{F_{\text{Div}^1}} \text{Rep}(W_F, \Lambda) \\
\downarrow (p_F)^* \downarrow \\
\text{Sat}(\mathcal{H}ck_{\text{Spd}C}, \Lambda) \xrightarrow{F_{\text{Spd}C}} \text{Proj}(\Lambda) \\
\downarrow R_j \downarrow \\
\text{Sat}(\mathcal{H}ck_{\text{Spd}O_C}, \Lambda) \xrightarrow{F_{\text{Spd}O_C}} \text{Proj}(\Lambda) \\
\downarrow i^* \downarrow \\
\text{Sat}(\mathcal{H}ck_{\text{Spd}k}, \Lambda) \xrightarrow{F_{\text{Spd}k}} \text{Proj}(\Lambda),
\end{array}
\]

where

\[F_S := F_{G,S}: D_\dR(\mathcal{H}ck_{G,S}, \Lambda) \rightarrow \text{LocSys}(S, \Lambda), \]

\[A \mapsto \bigoplus_i \mathcal{H}^i(R\pi_{G,S,*}A),\]

for a small \(v\)-stack \(S\) over \(\text{Div}^1\) or \(\text{Spd}O_E\). Here \(\text{LocSys}(S, \Lambda)\) is the category of \(\Lambda\)-local systems on \(S\). Then there are isomorphisms \(\text{For} \circ F_{\text{Div}^1} \cong F_{\text{Spd}C} \circ (p_F)^*\) and \(F_{\text{Spd}C} \cong F_{\text{Spd}k} \circ \Psi\).

**Proof.** We can prove this by applying the base change results to the cartesian squares

\[
\begin{array}{cccc}
\mathcal{H}ck_{\text{Div}^1} & \xrightarrow{} & \mathcal{H}ck_{\text{Spd}C} & \xrightarrow{} & \mathcal{H}ck_{\text{Spd}O_C} & \xrightarrow{} & \mathcal{H}ck_{\text{Spd}k} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Div}^1 & \xrightarrow{} & \text{Spd}C & \xrightarrow{} & \text{Spd}O_C & \xrightarrow{} & \text{Spd}k.
\end{array}
\]

\[\square\]

**Corollary 3.15.** There exists a symmetric monoidal structure on the category

\[\left(\text{Sat}(\mathcal{H}ck_{\text{Spd}k}, \Lambda), \ast\right)\]

and on the functors

\[i^* R_j*: \text{Sat}(\mathcal{H}ck_{\text{Spd}C}, \Lambda) \rightarrow \text{Sat}(\mathcal{H}ck_{\text{Spd}k}, \Lambda),\]

\[F_{\text{Spd}k}: \text{Sat}(\mathcal{H}ck_{\text{Spd}k}, \Lambda) \rightarrow \text{Proj}(\Lambda)\]
such that the isomorphism
\[ F_{\text{Spd}} \circ i^* R_j \ast \cong F_{\text{Spd}} C \]
in Theorem 3.14 is monoidal, with respect to the monoidal structure of \( F_{\text{Spd}} C \) using fusion products (see [FS21, VI.9]).

**Proof.** This follows from Theorem 3.14 and the fact that \( i^* R_j : \text{Sat}(\mathcal{Hck}_{\text{Spd}} C, \Lambda) \to \text{Sat}(\mathcal{Hck}_{\text{Spd}} k, \Lambda) \) is an equivalence. \( \square \)

The main theorem follows from this corollary.

**Theorem 3.16.** There exists a symmetric monoidal structure on the category
\[(\text{Sat}(\mathcal{Hck}_{\text{Spd}} k, \Lambda), \ast)\]
and on the functor
\[ F_{\text{Spd}} k : \text{Sat}(\mathcal{Hck}_{\text{Spd}} k, \Lambda) \to \text{Proj}(\Lambda) \]
which induce a symmetric monoidal equivalence
\[ \mathcal{Y}_Z : \text{Sat}(\mathcal{Hck}_{G,\text{Spd}} k, \Lambda) \xrightarrow{\sim} \text{Rep}(\hat{G}, \Lambda) \]
such that the squares
\[
\begin{array}{c}
\text{Sat}(\mathcal{Hck}_{G,\text{Div}^1}, \Lambda) \xrightarrow{\mathcal{Y}_Z} \text{Rep}(\hat{G}^\text{tw} \rtimes W_F, \Lambda) \\
\downarrow (p_F)^* \quad \quad \quad \quad \downarrow \text{For} \\
\text{Sat}(\mathcal{Hck}_{G,\text{Spd}} C, \Lambda) \xrightarrow{\mathcal{Y}_Z} \text{Rep}(\hat{G}, \Lambda) \\
i^* R_j \downarrow \quad \quad \quad \quad \quad \downarrow \\
\text{Sat}(\mathcal{Hck}_{G,\text{Spd}} k, \Lambda) \xrightarrow{\mathcal{Y}_Z} \text{Rep}(\hat{G}, \Lambda)
\end{array}
\]
naturally commute.

**Proof.** Apply Tannakian theory to the monoidal natural isomorphism in Corollary 3.15. \( \square \)

### 3.5 Monoidal structure on hyperbolic localization functor

Let \( P \) be a parabolic subgroup of \( G \). Let \( M \) denote its Levi quotient, and \( \overline{M} \) its maximal torus quotient. Define a locally constant function \( \text{deg}_P : \text{Gr}_{M,\text{Spd}} k \to \mathbb{Z} \) as
the composition

\[
\begin{array}{c}
\text{Gr}_{M, \text{Spd}^k} \to \text{Gr}_{M, \text{Spd}^k} \\
\sim \to X_\star(\overline{M}) \\
i < 2\rho_G - 2\rho_M, \to \mathbb{Z}
\end{array}
\]

where \( \rho_G, \rho_M \) is a half sum of positive roots of \( G, M \), respectively.

**Theorem 3.17.** Consider the shifted hyperbolic localization functor

\[
\text{CT}_P[\deg_P] := R(p_k^+)(q_k^+)^*[\deg_P] : D_{\text{ét}}(\text{Hck}_{G, \text{Spd}^k}, \Lambda)_{\text{bd}} \to D_{\text{ét}}(\text{Hck}_{M, \text{Spd}^k}, \Lambda)_{\text{bd}}
\]

where \( p_k^+ : \text{Gr}_{P, \text{Spd}^k} \to \text{Gr}_{M, \text{Spd}^k}, q_k^+ : \text{Gr}_{P, \text{Spd}^k} \to \text{Gr}_{G, \text{Spd}^k} \) are natural maps. Then \( \text{CT}_P[\deg_P] \) induces the functor

\[
\text{CT}_P[\deg_P] : \text{Sat}(\text{Hck}_{G, \text{Spd}^k}, \Lambda) \to \text{Sat}(\text{Hck}_{P, \text{Spd}^k}, \Lambda)
\]

and there is a unique symmetric monoidal structure on \( \text{CT}_P[\deg_P] \) such that the natural isomorphism

\[
F_{G, \text{Spd}^k} \cong F_{M, \text{Spd}^k} \circ \text{CT}_P[\deg_P] : \text{Sat}(\text{Hck}_{G, \text{Spd}^k}, \Lambda) \to \text{Proj}(\Lambda)
\]  

(3.7)

is monoidal. Here the symmetric monoidal structures on the categories \( \text{Sat}(\text{Hck}_{G, \text{Spd}^k}, \Lambda) \), \( \text{Sat}(\text{Hck}_{M, \text{Spd}^k}, \Lambda) \) and the functors \( F_{G, \text{Spd}^k}, F_{M, \text{Spd}^k} \) are as in Corollary 3.15.

**Proof.** By the same argument as [FS21 Proposition VI.9.6], there is a natural symmetric monoidal structure on the functor

\[
\text{CT}_P[\deg_P] : \text{Sat}(\text{Hck}_{G, \text{Spd}^C}, \Lambda) \to \text{Sat}(\text{Hck}_{G, \text{Spd}^C}, \Lambda)
\]

such that such that the natural isomorphism

\[
F_{G, \text{Spd}^C} \cong F_{M, \text{Spd}^C} \circ \text{CT}_P[\deg_P] : \text{Sat}(\text{Hck}_{G, \text{Spd}^C}, \Lambda) \to \text{Proj}(\Lambda)
\]

is monoidal. The following square is naturally commutative

\[
\begin{array}{ccc}
D_{\text{ét}}(\text{Hck}_{G, \text{Spd}^C}, \Lambda) & \xrightarrow{\text{CT}_P[\deg_P]} & D_{\text{ét}}(\text{Hck}_{M, \text{Spd}^C}, \Lambda) \\
\downarrow i^* R_{j*} & & \downarrow i^* R_{j*} \\
D_{\text{ét}}(\text{Hck}_{G, \text{Spd}^k}, \Lambda) & \xrightarrow{\text{CT}_P[\deg_P]} & D_{\text{ét}}(\text{Hck}_{M, \text{Spd}^k}, \Lambda)
\end{array}
\]

and the vertical arrows \( i^* R_{j*} \) induce equivalences of the Satake categories. By this diagram, we can endow the functor \( \text{CT}_P[\deg_P] \) with a symmetric monoidal structure. The isomorphism (3.7) is monoidal by definition. The uniqueness of such monoidal structure follows from the fact that \( F_{G, \text{Spd}^k} \) is faithful. \( \square \)
4 Integral coefficient case

In this subsection, \( \Lambda \) is the ring of integers of a finite extension of \( \mathbb{Q}_\ell \). Let \( \lambda \in \Lambda \) be a uniformizer. First, recall the definition of \( D_{\text{ét}}(-, \Lambda) \).

**Definition 4.1.** ([Sch17, Definition 26.1]) For any small \( v \)-stack \( Y \), define

\[
D_{\text{ét}}(Y, \Lambda) \subset D(Y_v, \Lambda)
\]

as the full subcategory of all \( A \in D(Y_v, \Lambda) \) such that \( A \) is derived \((\lambda)\)-complete, and \( A \otimes^L_\Lambda \Lambda/\lambda \) lies in \( D_{\text{ét}}(Y/\Lambda/\lambda) \).

**Lemma 4.2.** ([Sch17, Remark 26.3])

(i) For any map \( f : X \to Y \) of small \( v \)-stacks, the following squares are commutative:

\[
\begin{align*}
D_{\text{ét}}(Y, \Lambda) & \xrightarrow{f^*} D_{\text{ét}}(X, \Lambda) & D_{\text{ét}}(X, \Lambda) & \xrightarrow{Rf_*} D_{\text{ét}}(Y, \Lambda) \\
-\otimes^L_\Lambda \Lambda/\lambda^n & \downarrow & -\otimes^L_\Lambda \Lambda/\lambda^n & \downarrow \\
D_{\text{ét}}(Y, \Lambda/\lambda^n) & \xrightarrow{f^*} D_{\text{ét}}(X, \Lambda/\lambda^n), & D_{\text{ét}}(X, \Lambda/\lambda^n) & \xrightarrow{Rf_*} D_{\text{ét}}(Y, \Lambda/\lambda^n).
\end{align*}
\]

(ii) Let \( f : X \to Y \) be a map of small \( v \)-stacks which is compactifiable, representable in locally spatial diamonds and with \( \dim \text{trg} f < \infty \). The following squares are commutative:

\[
\begin{align*}
D_{\text{ét}}(X, \Lambda) & \xrightarrow{Rf_*} D_{\text{ét}}(Y, \Lambda) & D_{\text{ét}}(Y, \Lambda) & \xrightarrow{Rf^*} D_{\text{ét}}(X, \Lambda) \\
-\otimes^L_\Lambda \Lambda/\lambda^n & \downarrow & -\otimes^L_\Lambda \Lambda/\lambda^n & \downarrow \\
D_{\text{ét}}(X, \Lambda/\lambda^n) & \xrightarrow{Rf_*} D_{\text{ét}}(Y, \Lambda/\lambda^n), & D_{\text{ét}}(Y, \Lambda/\lambda^n) & \xrightarrow{Rf^*} D_{\text{ét}}(X, \Lambda/\lambda^n).
\end{align*}
\]

**Proof.** See [Sch17, Remark 26.3]. \( \square \)

Also, the definition of universal locally acyclicity is as follows:

**Definition 4.3.** ([PF21, VII.5]) Let \( f : X \to S \) be a compactifiable map of small \( v \)-stacks representable by locally spatial diamonds with locally \( \dim \text{trg} f < \infty \). Then we define the category of ULA complexes \( D_{\text{ét}}^{\text{ULA}}(X/S, \Lambda) \) by \( \lim D_{\text{ét}}^{\text{ULA}}(X/S, \Lambda/\lambda^n) \).

**Definition 4.4.** We say that \( A \in D_{\text{ét}}(\text{Hck}_G, S, \Lambda) \) is ULA if its pullback to \( \text{Gr}_G, S \) is ULA. Write \( D_{\text{ét}}^{\text{ULA}}(\text{Hck}_G, S, \Lambda) \) for the full subcategory of \( D_{\text{ét}}(\text{Hck}_G, S, \Lambda) \) consisting of ULA objects.
Lemma 4.5. Let $S$ be a small $v$-stack over $\text{Div}^1$. For an object $A \in D_{\text{et}}(\text{Hck}_{G,S},\Lambda)$, the following conditions are equivalent:

(i) $A \in D^\text{ULA}_{\text{et}}(\text{Hck}_{G,S},\Lambda)$.

(ii) $A \otimes^L_A \Lambda/\lambda^n \in D^\text{ULA}_{\text{et}}(\text{Hck}_{G,S},\Lambda/\lambda^n)$ for all $n$.

Proof. The square

$$
\begin{array}{ccc}
D_{\text{et}}(\text{Hck}_{G,S},\Lambda) & \xrightarrow{(-)^*} & D_{\text{et}}(\text{Gr}_{G,S},\Lambda) \\
- \otimes^L_A \Lambda/\lambda^n & \downarrow & - \otimes^L_A \Lambda/\lambda^n \\
D_{\text{et}}(\text{Hck}_{G,S},\Lambda/\lambda^n) & \xrightarrow{(-)^*} & D_{\text{et}}(\text{Gr}_{G,S},\Lambda/\lambda^n)
\end{array}
$$

is commutative. For $A \in D_{\text{et}}(\text{Hck}_{G,S},\Lambda)$, the object $\pi^*A$ is ULA if and only if $(\pi^*A) \otimes^L \Lambda/\lambda^n \cong \pi^*(A \otimes^L \Lambda/\lambda^n)$ is ULA for all $n$. Hence $A$ is ULA if and only if $A \otimes^L \Lambda/\lambda^n$ is ULA for all $n$. \hfill \Box

This equivalence implies that the essential image of $D^\text{ULA}_{\text{et}}(\text{Hck}_{G,S},\Lambda)$ under the categorical equivalence

$$
D_{\text{et}}(\text{Hck}_{G,S},\Lambda) \cong \lim_n D_{\text{et}}(\text{Hck}_{G,S},\Lambda/\lambda^n)
$$

in [Sch17, Proposition 26.2] is $\lim_n D^\text{ULA/S}_{\text{et}}(\text{Hck}_{G,S},\Lambda/\lambda^n)$. Also, we define the flat perversity as follows.

**Definition 4.6.** For $A \in D^\text{ULA}_{\text{et}}(\text{Hck}_{G,S},\Lambda)$, we say that $A$ is flat perverse over $\Lambda$ if and only if $A \otimes^L_A \Lambda/\lambda^n$ is flat perverse over $\Lambda/\lambda^n$ for all $n$.

From this definition, we have

$$
\text{Sat}(\text{Hck}_{G,S},\Lambda) = \lim_n \text{Sat}(\text{Hck}_{G,S},\Lambda/\lambda^n).
$$

By passing to the limit of the results in §3.2, 3.3, 3.4, 3.5 and using Lemma 4.2, we get the following theorem:

**Theorem 4.7.** Theorem 3.10, Corollary 3.11, Proposition 3.12, Theorem 3.16 and Theorem 3.17 hold even if $\Lambda$ is the ring of integers in a finite extension over $\mathbb{Q}_\ell$. 

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5 Fractional coefficient case

Moreover, we can prove a similar result for $\Lambda[\ell^{-1}]$-coefficient.

**Definition 5.1.** Let $X$ be a small v-stack. The category $D_{\text{et}}(X, \Lambda[\ell^{-1}])$ is defined as the category

\[ D_{\text{et}}(X, \Lambda)[\ell^{-1}] \]

obtained by inverting $\ell$ in the Hom-sets of $D_{\text{et}}(X, \Lambda)$. There is a natural functor

\[- \otimes_{\Lambda} \Lambda[\ell^{-1}]: D_{\text{et}}(X, \Lambda) \to D_{\text{et}}(X, \Lambda[\ell^{-1}]).\]

For any map $f: X \to Y$, the adjoint pair $(f^*, Rf_*)$ with coefficients in $\Lambda$ induces the adjoint pair

\[ f^*: D_{\text{et}}(Y, \Lambda[\ell^{-1}]) \to D_{\text{et}}(X, \Lambda[\ell^{-1}]), \]
\[ Rf_*: D_{\text{et}}(X, \Lambda[\ell^{-1}]) \to D_{\text{et}}(Y, \Lambda[\ell^{-1}]). \]

Let $f: X \to Y$ be a map of small v-stacks which is compactifiable, representable in locally spatial diamonds and with $\text{dim.\,trg}\,f < \infty$. The the adjoint pair $(Rf_1, Rf^1)$ with coefficients in $\Lambda$ induces the adjoint pair

\[ Rf_1: D_{\text{et}}(X, \Lambda[\ell^{-1}]) \to D_{\text{et}}(Y, \Lambda[\ell^{-1}]), \]
\[ Rf^1: D_{\text{et}}(Y, \Lambda[\ell^{-1}]) \to D_{\text{et}}(X, \Lambda[\ell^{-1}]). \]

**Definition 5.2.** We say that $A \in D_{\text{et}}(\text{Hck}_{G,S}, \Lambda[\ell^{-1}])$ is universal locally acyclic (ULA) if $A$ is in the essential image of $D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,S}, \Lambda)$ under the functor

\[- \otimes_{\Lambda} \Lambda[\ell^{-1}]: D_{\text{et}}(\text{Hck}_{G,S}, \Lambda) \to D_{\text{et}}(\text{Hck}_{G,S}, \Lambda[\ell^{-1}]).\]

Write $D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,S}, \Lambda[\ell^{-1}])$ for this essential image.

By definition, $D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,\text{Spd} C}, \Lambda[\ell^{-1}])$ is equivalent to the category

\[ D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,S}, \Lambda)[\ell^{-1}] \]

Also, we define the (flat) perversity as follows.

**Definition 5.3.** For $A \in D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,S}, \Lambda[\ell^{-1}])$, we say that $A$ is perverse over $\Lambda[\ell^{-1}]$ if and only if $A$ is isomorphic to an image of a flat perverse sheaf in $D_{\text{et}}^{\text{ULA}}(\text{Hck}_{G,S}, \Lambda)$ under the functor $- \otimes_{\Lambda}^{L} \Lambda[\ell^{-1}]$. 

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From this definition, we have
\[ \text{Sat}(Hck_G, S, \Lambda)[\ell^{-1}] \simeq \text{Sat}(Hck_G, S)[\ell^{-1}]. \]

By inverting $\ell$ in Theorem 4.7, we get the following theorem:

**Theorem 5.4.** Theorem 3.10, Corollary 3.11, Proposition 3.12, Theorem 3.16 and Theorem 3.17 hold even if $\Lambda$ is a finite extension over $\mathbb{Q}_\ell$.

### 6 $\overline{\mathbb{Q}}_\ell$-coefficient case

We can prove the result for the $\overline{\mathbb{Q}}_\ell$-coefficient case.

**Definition 6.1.** Let $X$ be a small $v$-stack. $D_{\text{et}}(X, \overline{\mathbb{Q}}_\ell)$ is defined by
\[ D_{\text{et}}(X, \overline{\mathbb{Q}}_\ell) = \lim_{\overset{\longrightarrow}{L}} D_{\text{et}}(X, L) \]
where $L$ is a finite extension over $\mathbb{Q}_\ell$, and if $L/L'/\mathbb{Q}_\ell$ is a tower of finite extension, the transition functor is the functor
\[ - \otimes_{L'} L : D_{\text{et}}(X, L') \to D_{\text{et}}(X, L) \]
induced by
\[ - \otimes_{O_{L'}} O_L : D_{\text{et}}(X, O_{L'}) \to D_{\text{et}}(X, O_L). \]

There is a natural functor
\[ - \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell : D_{\text{et}}(X, L) \to D_{\text{et}}(X, \overline{\mathbb{Q}}_\ell). \]

For any map $X \to Y$ of small $v$-stacks, the adjoint pairs $(f^*, Rf_*)$ with coefficient in a finite extension $L$ over $\mathbb{Q}_\ell$ induce the adjoint pair
\[ f^* : D_{\text{et}}(Y, \overline{\mathbb{Q}}_\ell) \to D_{\text{et}}(X, \overline{\mathbb{Q}}_\ell), \]
\[ Rf_* : D_{\text{et}}(X, \overline{\mathbb{Q}}_\ell) \to D_{\text{et}}(Y, \overline{\mathbb{Q}}_\ell). \]

In fact, the following lemma holds:

**Lemma 6.2.** Let $L/L'/\mathbb{Q}_\ell$ be a tower of finite extension.

(i) For any map $f : X \to Y$ of small $v$-stacks, the following squares are commutative:
\[
\begin{array}{ccc}
D_{\text{et}}(Y, L') & \xrightarrow{f^*} & D_{\text{et}}(X, L') \\
\downarrow - \otimes_{L'} L & & \downarrow - \otimes_{L'} L \\
D_{\text{et}}(Y, L) & \xrightarrow{f^*} & D_{\text{et}}(X, L)
\end{array}
\]
\[
\begin{array}{ccc}
D_{\text{et}}(Y, L') & \xrightarrow{Rf_*} & D_{\text{et}}(Y, L') \\
\downarrow - \otimes_{L'} L & & \downarrow - \otimes_{L'} L \\
D_{\text{et}}(X, L) & \xrightarrow{Rf_*} & D_{\text{et}}(Y, L)
\end{array}
\]

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(ii) Let \( f : X \to Y \) be a map of small \( v \)-stacks which is compactifiable, representable in locally spatial diamonds and with \( \dim \text{trgf} < \infty \). The following squares are commutative:

\[
\begin{array}{cccc}
D_{\text{ét}}(X, L') & \xrightarrow{Rf} & D_{\text{ét}}(Y, L') \\
\downarrow \otimes L & & \downarrow \otimes L \\
D_{\text{ét}}(X, L) & \xrightarrow{Rf} & D_{\text{ét}}(Y, L),
\end{array}
\]

Proof. For (i), we need to show that the squares

\[
\begin{array}{cccc}
D_{\text{ét}}(Y, \mathcal{O}_{L'}) & \xrightarrow{j^*} & D_{\text{ét}}(X, \mathcal{O}_{L'}) \\
\downarrow \otimes L_{\mathcal{O}_{L}} & & \downarrow \otimes L_{\mathcal{O}_{L}} \\
D_{\text{ét}}(Y, \mathcal{O}_{L}) & \xrightarrow{j^*} & D_{\text{ét}}(X, \mathcal{O}_{L}),
\end{array}
\]

are naturally commutative. By Lemma 3.7, it is enough to prove that the squares

\[
\begin{array}{cccc}
D_{\text{ét}}(Y, \mathcal{O}_{L}/\ell^n) & \xrightarrow{j^*} & D_{\text{ét}}(X, \mathcal{O}_{L}/\ell^n) \\
\downarrow \otimes L_{\mathcal{O}_{L}/\ell^n} & & \downarrow \otimes L_{\mathcal{O}_{L}/\ell^n} \\
D_{\text{ét}}(Y, \mathcal{O}_{L}/\ell^n) & \xrightarrow{j^*} & D_{\text{ét}}(X, \mathcal{O}_{L}/\ell^n),
\end{array}
\]

are naturally commutative. The commutativity of the first square follows from the definition. For the second square, since \( \mathcal{O}_{L}/\ell^n \) is a finite free module over \( \mathcal{O}_{L'}/\ell^n \), the outer square in the diagram

\[
\begin{array}{cccc}
D_{\text{ét}}(X, \mathcal{O}_{L'}/\ell^n) & \xrightarrow{Rj^*} & D_{\text{ét}}(Y, \mathcal{O}_{L'/\ell^n}) \\
\downarrow \otimes L_{\mathcal{O}_{L'/\ell^n}} & & \downarrow \otimes L_{\mathcal{O}_{L'/\ell^n}} \\
D_{\text{ét}}(X, \mathcal{O}_{L'}/\ell^n) & \xrightarrow{Rj^*} & D_{\text{ét}}(Y, \mathcal{O}_{L'/\ell^n}),
\end{array}
\]

is commutative, where the vertical functors For are forgetful functors. As the functor

\[
\text{For}: D_{\text{ét}}(Y, \mathcal{O}_{L}/\ell^n) \to D_{\text{ét}}(Y, \mathcal{O}_{L'/\ell^n})
\]

is conservative, it suffices to show that the following square is naturally commutative:

\[
\begin{array}{cccc}
D_{\text{ét}}(X, \mathcal{O}_{L}/\ell^n) & \xrightarrow{Rj^*} & D_{\text{ét}}(Y, \mathcal{O}_{L}/\ell^n) \\
\downarrow \text{For} & & \downarrow \text{For} \\
D_{\text{ét}}(X, \mathcal{O}_{L'}/\ell^n) & \xrightarrow{Rj^*} & D_{\text{ét}}(Y, \mathcal{O}_{L'/\ell^n}).
\end{array}
\]
This square is the right adjoint of the first square. The proof of (ii) is similar.

**Definition 6.3.** We say that $A \in D_{\text{ét}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell})$ is universal locally acyclic (ULA) if there exists a finite extension $L$ over $\mathbb{Q}_\ell$ such that $A$ is in the essential image of $D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,L)$ under the functor

$$- \otimes_L \overline{\mathbb{Q}_\ell} : D_{\text{ét}}(\mathcal{H}ck_G,S,L) \rightarrow D_{\text{ét}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell}).$$

Write $D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell})$ for the full subcategory of ULA sheaves in $D_{\text{ét}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell})$.

By definition, it holds that

$$D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell}) = \lim_{L} D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,L).$$

We define the perversity as follows.

**Definition 6.4.** We say that $A \in D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell})$ is perverse if there exists a finite extension $L$ over $\mathbb{Q}_\ell$ such that $A$ is in the essential image of $\text{Sat}(\mathcal{H}ck_G,S,L)$ under the functor

$$- \otimes_L \overline{\mathbb{Q}_\ell} : D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,L) \rightarrow D_{\text{ét}}^{\text{ULA}}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell}).$$

By definition, it holds that

$$\text{Sat}(\mathcal{H}ck_G,S,\overline{\mathbb{Q}_\ell}) = \lim_{L} \text{Sat}(\mathcal{H}ck_G,S,L).$$

By passing the limit in Theorem 5.4 and using Lemma 6.2, we get the following theorem:

**Theorem 6.5.** Theorem 3.10, Corollary 3.11, Proposition 3.12, Theorem 3.16 and Theorem 3.17 hold even if $\Lambda = \overline{\mathbb{Q}_\ell}$.

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