Third order Bose-Einstein correlations by means of Coulomb wave function revisited

Minoru Biyajima\textsuperscript{1*}
Takuya Mizoguchi\textsuperscript{2†}
Naomichi Suzuki\textsuperscript{3‡}

\textsuperscript{1}Department of Physics, Shinshu University, Matsumoto, 390-8621, Japan
\textsuperscript{2}Toba National College of Maritime Technology, Toba 517-8501, Japan
\textsuperscript{3}Department of Comprehensive Management, Matsumoto University, Matsumoto 390-1295, Japan

Abstract

In previous works, in order to include correction by the Coulomb wave function in Bose-Einstein correlations (BEC), the two-body Coulomb scattering wave functions have been utilized in the formulation of three-body BEC. However, the three-body Coulomb scattering wave function, which satisfies approximately the three-body Coulomb scattering Schrödinger equation, cannot be written by the product of the two-body scattering wave functions. Therefore, we reformulate the three-body BEC, and reanalyze the data. A set of reasonable parameters is obtained.

1 Introduction

Recently, in addition to the data on the two-body charged Bose-Einstein correlations (BEC), data on the three-body charged BEC have been reported \cite{1, 2, 3}. In some papers \cite{1, 3}, the Coulomb correction is done with fixed source radius, for example, 5 fm. On the other hand, the quasi-corrected data (raw data with acceptance correction) on the two-body $(2\pi^-)$ BEC \cite{4} and the three-body $(3\pi^-)$ BEC have been reported \cite{5}.

In Ref. \cite{6, 7}, authors proposed a theoretical formula for the $3\pi^-$BEC by the use of the two-body Coulomb wave functions, and outputted information on BEC with fixed source radii (5 fm and 10 fm). On the other hand, we have analyzed the $2\pi^-$ and $3\pi^-$BEC, using the CERN-MINUIT program with the two-body Coulomb wave functions and the source radius as a free parameter \cite{8, 9, 10}.

The formula for $2\pi^-$BEC reduces to that of plane wave formulation in the limit of plane wave approximation. However, the formula for $3\pi^-$BEC does not reduce to that of plane wave formulation \cite{11, 12}. Additional factor $(3/2)$ appears in the phase of plane wave \cite{8, 9, 10}. Therefore, we have re-interpreted the source radius estimated from the analysis of $3\pi^-$BEC.

\textsuperscript{*}E-mail: mbiyajima@azusa.shinshu-u.ac.jp
\textsuperscript{†}E-mail: mizoguti@toba-cmt.ac.jp
\textsuperscript{‡}E-mail: suzuki@matsu.ac.jp
In this paper, we would like to examine the relation between the two-body Coulomb wave function and the asymptotic solution of the three-body Coulomb wave function, which cannot be written by the product of two-body Coulomb wave functions. In addition, we would like to show that factor \( \frac{3}{2} \) disappears from the phase factors of plane wave in the formulation of the \( 3\pi^-\text{BEC} \), if the correct asymptotic three-body Coulomb wave function is used.

In the second section, an asymptotic solution for the three-body Coulomb scattering Schrödinger equation is shown. The formula for \( 3\pi^-\text{BEC} \) is derived from the analogy of the formula for the plane wave formulation in the third section. Analysis of \( 3\pi^-\text{BEC} \) is done in the fourth section. Final section is devoted to summary and discussions.

## 2 Approximate solution for Schrödinger equation of three-body Coulomb scattering

In order to describe the two-body charged BEC (for example, \( 2\pi^-\) system), we should solve the Shrödinger equation of Coulomb scattering. The solution, which is regular at the origin of the Coulomb potential, is given by

\[
\psi_{k_1k_2}^C(x_i, x_j) = e^{ik_1 \cdot r_{ij}} \phi_{k_1k_2}(r_{ij}),
\]

\[
\phi_{k_1k_2}(r_{ij}) = \Gamma(1 + i\eta_{ij}) e^{\pi\eta_{ij}/2} F\left[-i\eta_{ij}, 1; i(|k_{ij}| \cdot |r_{ij}| - k_{ij} \cdot r_{ij})\right],
\]

for particles \( i \) and \( j \), where the coordinate and momentum of particle \( i \), are denoted by \( x_i \) and \( k_i \), respectively, and \( e_i \) in \( \eta_{ij} \) is the charge of particle \( i \). In Eq. (1), the relative coordinate and momentum of particles \( i \) and \( j \) are denoted by \( r_{ij} = x_i - x_j \), and \( k_{ij} = (m_j k_i - m_i k_j)/(m_i + m_j) \), respectively, \( \eta_{ij} = e_i e_j \mu_{ij}/|k_{ij}| \) where \( \mu_{ij} \) is reduced mass of \( m_i \) and \( m_j \), \( F[a, b; x] \) is the confluent hypergeometric function, and \( \Gamma(x) \) is the Gamma function.

In order to describe the three-body Coulomb scattering, the Jacobi coordinates [13] are introduced;

\[
\begin{align*}
\zeta_1 &= x_2 - x_1, \\
\zeta_2 &= x_3 - (m_1 x_1 + m_2 x_2)/M_2, \\
\zeta_3 &= (m_1 x_1 + m_2 x_2 + m_3 x_3)/M, \\
M_2 &= m_1 + m_2, \quad M = m_1 + m_2 + m_3.
\end{align*}
\]

The relative coordinates are written as,

\[
\begin{align*}
r_{21} &= x_2 - x_1 = \zeta_1, \\
r_{31} &= x_3 - x_1 = \alpha \zeta_1 + \zeta_2, \\
r_{32} &= x_3 - x_2 = -\beta \zeta_1 + \zeta_2, \\
\alpha &= m_2/M_2, \quad \beta = m_1/M_2.
\end{align*}
\]

The Schrödinger equation of the three-body Coulomb scattering is given by,

\[
\left[ -\frac{1}{2\mu_1} \nabla^2_{\zeta_1} - \frac{1}{2\mu_2} \nabla^2_{\zeta_2} + \frac{e_1 e_2}{r_{12}} + \frac{e_2 e_3}{r_{23}} + \frac{e_3 e_1}{r_{31}} - \frac{P_{1}^2}{2\mu_1} - \frac{P_{2}^2}{2\mu_2} \right] \Psi_f = 0,
\]

(3)
Figure 1: Jacobi coordinates of three-body system. The starting point of $\zeta_2$ is the center of mass of particles 1 and 2.

where

$$\mu_1 = \frac{m_1 m_2}{M_2}, \quad \mu_2 = \frac{M_2 m_3}{M},$$

$$P_1 = \mu_1 d\zeta_1/dt = \frac{(m_1 k_2 - m_2 k_1)}{M_2},$$

$$P_2 = \mu_2 d\zeta_2/dt = \frac{(M_2 k_3 - m_3 (k_1 + k_2))}{M}.$$  

Then, the approximate solution for the Schrödinger equation in $\Omega_0$, where $r_{12}, r_{23}, r_{31} >> 1$, is given by [14, 15],

$$\Psi_f = e^{i\left(P_1 \zeta_1 + P_2 \zeta_2\right)} \phi_{k_{12}}(r_{12}) \phi_{k_{23}}(r_{23}) \phi_{k_{31}}(r_{31}).$$  \hspace{1cm} (4)

The phase factor of the plane wave in Eq. (4) is rewritten as,

$$P_1 \zeta_1 + P_2 \zeta_2 = \frac{m_1 + m_2}{M} k_{12} \cdot r_{12} + \frac{m_2 + m_3}{M} k_{23} \cdot r_{23} + \frac{m_3 + m_1}{M} k_{31} \cdot r_{31},$$

$$= \frac{2}{3} (k_{12} \cdot r_{12} + k_{23} \cdot r_{23} + k_{31} \cdot r_{31}),$$

where $m_1 = m_2 = m_3$ is used.

Therefore, the solution $\Psi_f$ is written as [16],

$$\Psi_f = \psi^C_{k_1 k_2, x_1, x_2} \psi^C_{k_2 k_3, x_2, x_3} \psi^C_{k_3 k_1, x_3, x_1},$$  \hspace{1cm} (5)

$$\psi^C_{k_i k_j, x_i, x_j} = e^{i(2/3)k_{ij}r_{ij}} \phi_{k_{ij}}(r_{ij}).$$  \hspace{1cm} (6)

The approximate solution for the three-body Coulomb scattering can be written by the product of $\psi^C_{k_i k_j, x_i, x_j}$, but not the product of the wave function of two-body scattering, $\psi^C_{k_i k_j, x_i, x_j}$. In the correct formula, factor $2/3$ is multiplied to the phase of plane wave.

In Ref.[7], the Coulomb correction for n-body scattering is discussed, where the n-body Coulomb scattering wave function is given by the product of two-body Coulomb scattering wave functions. However, the n-body Coulomb scattering wave function in $\Omega_0$ is approximately given by

$$\Psi_f = \prod_{i<j=1}^n \psi^C_{k_i k_j, x_i, x_j},$$

$$\psi^C_{k_i k_j, x_i, x_j} = e^{i(2/n)k_{ij}r_{ij}} \phi_{k_{ij}}(r_{ij}).$$
for \( n \geq 3 \).

## 3 formula for third order BEC

The wave function of identical Bose particles should be symmetrized. In Fig.2 or Fig.3 \( V_c \) denotes the interaction between two particles by the Coulomb potential, and cross (X) represents the exchange of particles.

As is shown in Fig.2 the two particle momentum density is given by,

\[
N^{(2\pi^-)} = \frac{1}{2} \prod_{i=1}^{2} \int \rho(x_i)d^3x_i|\psi_{k_1k_2}^C(x_1, x_2) + \psi_{k_1k_2}^C(x_2, x_1)|^2
\]

\[
= \prod_{i=1}^{2} \int \rho(x_i)d^3x_i(G_1 + G_2),
\]

\[
G_1 = \frac{1}{2} (|\psi_{k_1k_2}^C(x_1, x_2)|^2 + |\psi_{k_1k_2}^C(x_2, x_1)|^2),
\]

\[
G_2 = \text{Re} \left( \psi_{k_1k_2}^C(x_1, x_2)\psi_{k_1k_2}^{C*}(x_2, x_1) \right),
\]

where

\[
\rho(x) = \frac{1}{(2\pi R^2)^{3/2}} \exp\left[-\frac{x^2}{2R^2}\right].
\]

\[
N^{(3\pi^-)} = \frac{1}{6} \prod_{i=1}^{3} \int \rho(x_i)d^3x_i \sum_{j=1}^{6} |A(j)|^2,
\]

where

\[
A(1) = A_1 = \psi_{k_1k_2}^{C'}(x_1, x_2)\psi_{k_2k_3}^{C'}(x_2, x_3)\psi_{k_3k_1}^{C'}(x_3, x_1),
\]

\[
A(2) = A_{23} = \psi_{k_1k_2}^{C'}(x_1, x_3)\psi_{k_2k_3}^{C'}(x_3, x_2)\psi_{k_3k_1}^{C'}(x_2, x_1),
\]

\[
A(3) = A_{12} = \psi_{k_1k_2}^{C'}(x_2, x_1)\psi_{k_2k_3}^{C'}(x_1, x_3)\psi_{k_3k_1}^{C'}(x_3, x_2),
\]
\[ A(4) = A_{123} = \psi_{k_1k_2}^C(x_2, x_3)\psi_{k_3k_1}^C(x_3, x_1)\psi_{k_3k_1}^C(x_1, x_2), \]
\[ A(5) = A_{132} = \psi_{k_1k_3}^C(x_3, x_1)\psi_{k_2k_3}^C(x_1, x_2)\psi_{k_3k_1}^C(x_2, x_3), \]
\[ A(6) = A_{13} = \psi_{k_1k_2}^C(x_3, x_2)\psi_{k_2k_3}^C(x_2, x_1)\psi_{k_3k_1}^C(x_1, x_3). \]  

(9)

![Figure 3: Three-body BEC diagram](image)

In the plane wave approximation, each amplitude \( A(i) \) approaches to the following form,

\[ A(1) = A_1 \text{ PW} e^{i(2/3)(k_{12} \cdot r_{12} + k_{23} \cdot r_{23} + k_{31} \cdot r_{31})} = e^{i(k_1 \cdot x_1 + k_2 \cdot x_2 + k_3 \cdot x_3)}, \]
\[ A(2) = A_{23} \text{ PW} e^{i(2/3)(k_{12} \cdot r_{13} + k_{23} \cdot r_{12} + k_{13} \cdot r_{21})} = e^{i(k_1 \cdot x_1 + k_2 \cdot x_3 + k_3 \cdot x_2)}, \]
\[ A(3) = A_{12} \text{ PW} e^{i(2/3)(k_{12} \cdot r_{21} + k_{23} \cdot r_{13} + k_{31} \cdot r_{32})} = e^{i(k_1 \cdot x_2 + k_2 \cdot x_1 + k_3 \cdot x_3)}, \]
\[ A(4) = A_{132} \text{ PW} e^{i(2/3)(k_{12} \cdot r_{23} + k_{23} \cdot r_{31} + k_{31} \cdot r_{12})} = e^{i(k_1 \cdot x_2 + k_2 \cdot x_3 + k_3 \cdot x_1)}, \]
\[ A(5) = A_{13} \text{ PW} e^{i(2/3)(k_{12} \cdot r_{31} + k_{23} \cdot r_{12} + k_{13} \cdot r_{23})} = e^{i(k_1 \cdot x_3 + k_2 \cdot x_1 + k_3 \cdot x_2)}, \]
\[ A(6) = A_{13} \text{ PW} e^{i(2/3)(k_{12} \cdot r_{32} + k_{23} \cdot r_{21} + k_{31} \cdot r_{13})} = e^{i(k_1 \cdot x_1 + k_2 \cdot x_2 + k_3 \cdot x_3)}. \]  

(10)

In Eq. (10), \( \text{PW} \) means the plane wave approximation of the amplitude, and the condition in the center of mass system, \( \exp[-i(k_1 + k_2 + k_3) \cdot \xi] = 1 \) is used.

The amplitudes squared in Eq. (9) can be classified into the following groups,

\[ F_1 = \frac{1}{6}[A_1 A_1^* + A_{12} A_{12}^* + A_{23} A_{23}^* + A_{13} A_{13}^* + A_{123} A_{123}^* + A_{132} A_{132}^*], \]
\[ F_{12} = \frac{1}{6}[A_1 A_{12}^* + A_{23} A_{123}^* + A_{13} A_{132}^* + c.c.], \]
\[ F_{23} = \frac{1}{6}[A_1 A_{23}^* + A_{12} A_{132}^* + A_{13} A_{123}^* + c.c.], \]
\[ F_{31} = \frac{1}{6} \{ A_1 A_{13} + A_{23} A_{123} + A_{12} A_{123}^* + \text{c.c.} \}, \]
\[ F_{123} = \frac{1}{6} \{ A_1 A_{123} + A_{23} A_{12} + A_{12} A_{23} + A_{23} A_{13} + A_{13} A_{12} \}, \]
\[ F_{132} = \frac{1}{6} \{ A_1 A_{123} + A_{23} A_{12} + A_{12} A_{23}^* + A_{23} A_{13} + A_{13} A_{12}^* \}, \]
\[ \text{where, c.c. denotes the complex conjugate, and } F_{132} \text{ is the complex conjugate of } F_{123}. \]

In the plane wave approximation, \( F_1 \) reduces to 1, \( F_{ij} \) corresponds to exchange between \( i \) and \( j \) charged particles, and \( F_{123} \) correspond to exchange among three charged particles.

Phenomenologically, the coherence parameter \( \lambda \) is introduced into the formula for \( 2\pi^- \text{BEC} \) as,
\[ \frac{N^{2\pi^-}}{N_{BG}} = C \prod_{i=1}^{2} \int \rho(x_i) d^3 x_i (G_1 + \lambda G_2), \]
where \( C \) is the normalization factor.

In the third order \( \text{BEC} \), factor \( \lambda^{n/2} \) is multiplied to the amplitudes squared according to the number \( n \) of exchange particles. After \( \zeta_3 \) integration, the \( 3\pi^- \text{BEC} \) is given by
\[ \frac{N^{3\pi^-}}{N_{BG}} = C \prod_{i=1}^{3} \int \rho(x_i) d^3 x_i \left[ F_1 + 3\lambda F_{12} + 2\lambda^{3/2} \text{Re}(F_{123}) \right] \]
\[ = \frac{C}{(2\sqrt{3\pi R^2})^3} \int d^3 \zeta_1 d^3 \zeta_2 \exp \left[ -\frac{1}{2R^2} \left( \frac{1}{2} \zeta_1^2 + \frac{2}{3} \zeta_2^2 \right) \right] \left[ F_1 + 3\lambda F_{12} + \lambda^{3/2} \text{Re}(F_{123}) \right]. \]

(13)

The set of following variables is used in the concrete calculations of Eq.(13),
\[ \begin{align*}
  k_{12} &= -P_1, \\
  k_{23} &= \frac{1}{2} P_1 - \frac{3}{4} P_2, \\
  k_{31} &= \frac{1}{2} P_1 + \frac{3}{4} P_2, \\
  Q_3 &= \sqrt{4(k_{12}^2 + k_{23}^2 + k_{31}^2)} = \sqrt{6P_1^2 + 2P_2^2}.
\end{align*} \]

(14)

4 Analysis of \( 3\pi^- \text{BEC} \)

The formula (13) is applied to the analysis of quasi-corrected data on \( 3\pi^- \text{BEC} \) by STAR Collaboration [5]. The results are shown in Table 1 and Fig. 4. For comparison, the result of previous work [9] is also shown in the lower part of Table 1. The result for \( 2\pi^- \text{BEC} \) by STAR Collaboration [4] is shown in Table 2.

The source radius \( R_{3rd} \) estimated from the data on \( 3\pi^- \text{BEC} \) with Eq.(13) is comparable with that \( R_{2nd} \) from \( 2\pi^- \text{BEC} \). However, the source radius \( R_{3rd}^{\text{pre}} \) estimated in the previous work [9], namely with the two-body Coulomb wave functions is much smaller than \( R_{3rd} \) with Eq.(13). The re-interpreted radius becomes \( (3/2)R_{3rd}^{\text{pre}} = 8.01 \text{ [fm]} \), which is nearly equal to \( R_{3rd} \). The coherence parameter \( \lambda_{3rd} \) estimated from \( 3\pi^- \text{BEC} \) with Eq.(13) is somewhat smaller than that \( \lambda_{2nd} \) from \( 2\pi^- \text{BEC} \).
The problem on the phase factors appearing in the two-body BEC among three identical particles \[11, 12\] is proposed in \[17\]. If these factors are taken into account, the formula for \(3\pi\) BEC is given by,

\[
\frac{N_{3\pi}^\text{BG}}{N_{BG}} = C \prod_{i=1}^{3} \int \rho(x_i)d^3x_i \left[ F_1 + 3\lambda F_{12} + 2\lambda^{3/2} Re[F_{123}] \times W \right],
\]

where \(W = \cos(\phi_{12} + \phi_{23} + \phi_{31})\), which is parameterized as \(W = \cos(g \times Q_3)\) in the simplest form with parameter \(g\) and \(Q_3^2 = (k_1 - k_2)^2 + (k_2 - k_3)^2 + (k_3 - k_1)^2\). The result is shown in Table 3 and Fig. 4. Estimated source radius shown in Table 3 is smaller than \(R_{3rd}\) with Eq. (13), and is not consistent with \(R_{2nd}\).

Table 3: Analyses of \(3\pi\) BEC by STAR Collaboration with Eq. (15).

| \(R\) [fm] | \(\lambda\) | \(g\) | \(\chi^2/\text{n.d.f.}\) |
|------------|------------|------|------------------|
| Eq. (15)   | 7.70±0.57  | 0.55±0.05 | 31.13±10.87     | 0.51/34        |
5 Summary and discussions

We reformulate the formula for $3\pi^-\text{BEC}$, using the asymptotic three-body Coulomb wave function. We apply the formula to the analysis of data on $3\pi^-\text{BEC}$ by STAR Collaboration. The source radius $R_{3rd}$ estimated from $3\pi^-\text{BEC}$ is consistent with that $R_{2nd}$ from $2\pi^-\text{BEC}$. The coherence parameter $\lambda_{3rd}$ estimated from $3\pi^-\text{BEC}$ with Eq.(6) is almost the same with $\lambda_{2nd}$ from $2\pi^-\text{BEC}$. Whether a set of preferable parameters can be estimated from the analyses of $2\pi^-\text{BEC}$ and $3\pi^-\text{BEC}$ or not in other approaches will be reported elsewhere.

By the use of our formula, we can estimate source radius with Coulomb correction, without the re-interpretation due to the factor $(3/2)$.

Acknowledgments

One of the authors (N.S.) would like to thank J.R.Glauber for variable comments on the behaviors of charged particles in the electric field at Kromeritz, August, 2005. They would like to thank E.O.Alt, T.Csörgő, and members of WA8 Collaboration (Y.Miake, L.Rosselet) for discussing on this subject. They also would like to thank participants at a meeting of RCNP at Osaka University, Faculty of Science, Shinshu University, Toba National College of Maritime Technology and Matsumoto University for financial support.

References

[1] H. Bøggild et al., NA44 Collaboration, Phys. Lett. B455(1999)77.
[2] M.M.Aggarwal et al., WA98 Collaboration, Phys. Rev. C67(2003)014906
[3] J.Adams et al., STAR Collaboration, Phys. Rev. Lett. 91(2003)262301
[4] C.Adler, et al. STAR Collaboration, Rhys. Rev. Lett. 87(2001)082301
[5] R. Willson, Dr. Thesis at Ohio University (2002)

[6] E. O. Alt, T. Cs"org"o, B. L"orstad and J. Schmidt-Sørensen, Phys. Lett. B458 (1999) 407

[7] E. O. Alt, T. Cs"org"o, B. L"orstad and J. Schmidt-Sørensen, Eur. Phys. J. C13 (2000) 663

[8] T. Mizoguchi and M. Biyajima, Phys. Lett. B499 (2001) 245

[9] M. Biyajima, M. Kaneyama and T. Mizoguchi, Phys. Lett. B601 (2004) 41

[10] M. Biyajima, T. Mizoguchi and N. Suzuki, [hep-ph/0510015] to be published in the proceedings of the Workshop on Particle Correlations and Femtoscopy, Kromeritz, Czech Republic, August 15-17, 2005

[11] M. Biyajima, A. Bartl, et al., Prog. Theor. Phys. 84 (1990) 931; [addenda] Prog. Theor. Phys. 88 (1992) 157

[12] N. Suzuki and M. Biyajima, Prog. Theor. Phys. 88 (1992) 609; Phys Rev. C 60 (1999) 034903

[13] M. Reed and B. Simon, Methods of modern mathematical physics V: Scattering theory, Academic press, 1979

[14] E. O. Alt, Few-Body Systems Suppl., 10 (1999) 65-74; [nucl-th/9809046]

[15] M. Brauner, J. S. Briggs and H. Klar, J. Phys. B22 (1989) 2265

[16] N. Suzuki, K. Ide, M. Biyajima and T. Mizoguchi, Soryushiron Kenkyu (Kyoto), in Japanese, 112-4 (2006)

[17] U. Heinz and Q. H. Zhang, Phys. Rev. C56 (1997) 426

[18] Previous result $R_{3rd}^{pre} = 1.53 \pm 0.20$ fm for S+Pb collision obtained in [3] becomes $4.40 \pm 0.58$ fm in re-analyses by the present formula Eq. [13]. This is almost in agreement with the result $R_{2nd} = 4.69 \pm 0.46$ fm obtained therein [3].

[19] M. Biyajima, T. Mizoguchi and N. Suzuki, in preparation