The $q$-analog of higher order Hochschild homology and the Lie derivative

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Abstract

Let $A$ be a commutative algebra over $\mathbb{C}$. Given a pointed simplicial finite set $Y$ and $q \in \mathbb{C}$ a primitive $N$-th root of unity, we define the $q$-Hochschild homology groups $\{qHH^Y_n(A)\}_{n\geq 0}$ of $A$ of order $Y$. When $D$ is a derivation on $A$, we construct the corresponding Lie derivative on the groups $\{qHH^Y_n(A)\}_{n\geq 0}$. We also define the Lie derivative on $\{qHH^Y_n(A)\}_{n\geq 0}$ for a higher derivation $\{D_n\}_{n\geq 0}$ on $A$. Finally, we describe the morphisms induced on the bivariant $q$-Hochschild cohomology groups $\{qHH^Y_n(A, A)\}_{n \in \mathbb{Z}}$ of order $Y$ by a derivation $D$ on $A$.

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1 Introduction

Let $A$ be a commutative algebra over $\mathbb{C}$. Then, it is well known (see, for instance, [7 § 4.1]) that a derivation $D : A \rightarrow A$ induces morphisms

$$L^n_D : HH_n(A) \rightarrow HH_n(A) \quad \forall \ n \geq 0$$

(1.1)

on the Hochschild homology groups of the algebra $A$. The morphisms $L^n_D$, $n \geq 0$ play the role of the Lie derivative in noncommutative geometry. For more on these morphisms and for general properties of Hochschild homology, we refer the reader to [7]. Further, for any pointed simplicial finite set $Y$, Pirashvili [10] has introduced the Hochschild homology groups $\{HH^Y_n(A)\}_{n\geq 0}$ of $A$ of order $Y$ (see also Loday [6]). In particular, when $Y = S^1$ is the simplicial circle, the groups $\{HH^Y_n(A)\}_{n\geq 0}$ reduce to the usual Hochschild homology groups of the algebra $A$. Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. The purpose of this paper is to introduce the $q$-analogues $\{qHH^Y_n(A)\}_{n\geq 0}$ of these higher order Hochschild homology groups and study the morphisms induced on them by derivations on $A$.

More precisely, let $\Gamma$ denote the category whose objects are the finite sets $[n] = \{0, 1, 2, .., n\}$, $n \geq 0$ with basepoint $0 \in [n]$. Then, given the algebra $A$, we can define a functor $\mathcal{D}(A)$ from $\Gamma$ to the category $Vect$ of complex vector spaces that takes $[n]$ to $A \otimes A^\otimes n$ (see Section 2 for details). Then, we can
prolong $\mathcal{L}(A)$ by means of colimits to a functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$ from the category $\text{Fin}_*$ of all finite sets with basepoint. Given a pointed simplicial finite set $Y$, i.e., a functor $Y : \Delta^{op} \rightarrow \text{Fin}_*$ ($\Delta^{op}$ being the simplex category), we now have a simplicial vector space

$$\mathcal{L}^Y(A) : \Delta^{op} \rightarrow \text{Fin}_* \xrightarrow{\mathcal{L}(A)} \text{Vect}$$

(1.2)

Let $d^Y_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1}$, $0 \leq j \leq i$, $i \geq 0$ be the face maps of the simplicial vector space $\mathcal{L}^Y(A)$. We then construct the “$q$-Hochschild differentials”:

$$q^Y_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1} \quad q^Y_i := \sum_{j=0}^i q^j d^Y_i$$

(1.3)

Since $q \in \mathbb{C}$ is a primitive $N$-th root of unity, it follows that $q^n = 0$, i.e., $(\mathcal{L}^Y(A), q^Y)$ is an $N$-complex in the sense of Kapranov [5]. We now define the $q$-Hochschild homology groups $\{qHH_n^Y(A)\}_{n \geq 0}$ of $A$ of order $Y$ to be the homology objects of the $N$-complex $(\mathcal{L}^Y(A), q^Y)$ (see Definitions 2.1 and 2.2). When $q = -1$ (and hence $N = 2$), $q^Y$ reduces to the usual differential on the chain complex associated to the simplicial vector space $\mathcal{L}^Y(A)$ and we have $(-1)HH_n^Y(A) = HH_n^Y(A)$, $\forall n \geq 0$. Then, the main result of Section 2 is as follows.

**Theorem 1.1.** Let $D : A \rightarrow A$ be a derivation on $A$. Then, for each $n \geq 0$, the derivation $D$ induces a morphism $L^Y_n : qHH_n^Y(A) \rightarrow qHH_n^Y(A)$ of $q$-Hochschild homology groups of order $Y$. Additionally, if $\mathcal{H} = \mathcal{U}(\text{Der}(A))$ is the universal enveloping algebra of the Lie algebra $\text{Der}(A)$ of derivations on $A$, then $\{qHH_n^Y(A)\}_{n \geq 0}$ is a left $\mathcal{H}$-module, i.e., for any element $h \in \mathcal{H}$, there exist morphisms $L^Y_n : qHH_n^Y(A) \rightarrow qHH_n^Y(A)$ of $q$-Hochschild homology groups of order $Y$.

Thereafter, we consider a higher derivation $D = \{D_n\}_{n \geq 0}$ on $A$. We recall that a higher (or Hasse-Schmidt) derivation $D = \{D_n\}_{n \geq 0}$ on $A$ is a sequence of linear maps $D_n : A \rightarrow A$ satisfying the following relation (see, for example, [8]):

$$D_n(a \cdot a') = \sum_{i=0}^n D_i(a) \cdot D_{n-i}(a') \quad \forall a, a' \in A, n \geq 0$$

(1.4)

In this paper, we restrict ourselves to normalized higher derivations, i.e., higher derivations $D = \{D_n\}_{n \geq 0}$ such that $D_0 = 1$. Then, in Section 3, we construct the Lie derivative on the $q$-Hochschild homology groups of order $Y$ corresponding to a higher derivation $D = \{D_n\}_{n \geq 0}$.

**Theorem 1.2.** Let $D = \{D_n\}_{n \geq 0}$ be a normalized higher derivation on $A$. Then, for each $k \geq 0$, we have an induced morphism $L^Y_D : qHH^Y_k(A) = \bigoplus_{n=0}^{\infty} qHH_n^Y(A) \rightarrow qHH^Y_k(A) = \bigoplus_{n=0}^{\infty} qHH_n^Y(A)$ on the $q$-Hochschild homology groups of $A$ of order $Y$.

Further, in [9], Mirzavaziri has provided a characterization of normalized higher derivations on algebras over $\mathbb{C}$ from which it follows that if $D = \{D_k\}_{k \geq 0}$ is a higher derivation on $A$, each $D_k$ is an element of the Hopf algebra $\mathcal{H} = \mathcal{U}(\text{Der}(A))$ (see [E.7] for details). It follows therefore from Theorem 1.1 that for each $k$, the element $D_k \in \mathcal{H}$ induces a morphism $L^Y_D : qHH^Y_k(A) \rightarrow qHH^Y_k(A)$. Then, in Section 3, we prove the following result.
Theorem 1.3. Let $D = \{D_k\}_{k \geq 0}$ be a normalized higher derivation on $A$. Then, for each $k \geq 1$, we have $L_{D_k}^Y = L_{D_k}^{Y,k}$ as an endomorphism of $qHH_n^Y(A) = \bigoplus_{n=0}^{\infty} qHH_n^Y(A)$.

In Section 4, we start by defining bivariant $q$-Hochschild cohomology groups $\{qHH_n^Y(A,A)\}_{n \in \mathbb{Z}}$ of order $Y$. For this we consider the modules $\text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, $n \in \mathbb{Z}$ where an element $f \in \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$ is given by a family of morphisms $f = \{f_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+n}\}_{i \in \mathbb{Z}}$. Further, we define a differential $q\partial : \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n \rightarrow \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{n-1}$ (see Definition 4.1). Then, $(\text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), q\partial)$ is an $N$-complex and we let $qHH_n^Y(A,A)$ be the $(-n)$-th homology object of $(\text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), q\partial)$. We end with the following result.

Theorem 1.4. Let $D : A \rightarrow A$ be a derivation on $A$. Then, for each $n \in \mathbb{Z}$, the derivation $D$ induces a morphism $L_D^{Y,n} : qHH_n^Y(A,A) \rightarrow qHH_n^Y(A,A)$ on the bivariant $q$-Hochschild cohomology groups of order $Y$.

2 Lie Derivative on higher order $q$-Hochschild homology

Let $\text{Vec}$ denote the category of vector spaces over $\mathbb{C}$. Let $A$ be a commutative $\mathbb{C}$-algebra. We recall here the definition of higher order Hochschild homology groups of a commutative algebra $A$ as introduced by Pirashvili [10] (see also Loday [6]). Let $\Gamma$ denote the category whose objects are the pointed sets $[n] = \{0,1,2,\ldots,n\}$ with $0 \in [n]$ as base point for each $n \geq 0$. Then, a morphism $\phi : [m] \rightarrow [n]$ in $\Gamma$ is a map $\phi : \{0,1,2,\ldots,m\} \rightarrow \{0,1,2,\ldots,n\}$ of sets such that $\phi(0) = 0$. We now define a functor:

$$\mathcal{L}(A) : \Gamma \rightarrow \text{Vec} \quad [n] \mapsto A \otimes A^\otimes n \quad (2.1)$$

Given a morphism $\phi : [m] \rightarrow [n]$ in $\Gamma$, we have an induced map in $\text{Vec}$:

$$\mathcal{L}(A)(\phi) : A \otimes A^\otimes m \rightarrow A \otimes A^\otimes n \quad (2.2)$$

We now consider the category $\text{Fin}_*$ of finite pointed sets. There is a natural inclusion $\Gamma \hookrightarrow \text{Fin}_*$ of categories. Then, $\mathcal{L}(A) : \Gamma \rightarrow \text{Vec}$ can be extended to a functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vec}$ by setting:

$$\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vec} \quad T \mapsto \text{colim}_{\Gamma \ni T' \rightarrow T} \mathcal{L}(A)(T') \quad (2.3)$$

where the colimit in (2.3) is taken over all morphisms $T' \rightarrow T$ in $\text{Fin}_*$ such that $T' \in \Gamma$. Let $\Delta$ be the simplex category, i.e., the category whose objects are sets $[n] = \{0,1,2,\ldots,n\}$, $n \geq 0$ and whose morphisms are order preserving maps. Then, given a pointed simplicial finite set $Y$ corresponding to a functor $Y : \Delta^{op} \rightarrow \text{Fin}_*$, we have a simplicial vector space $\mathcal{L}^Y(A)$ determined by the composition of functors:

$$\mathcal{L}^Y(A) : \Delta^{op} \rightarrow \text{Fin}_* \xrightarrow{\mathcal{L}(A)} \text{Vec} \quad (2.4)$$
For any \( n \geq 0 \), let \( \HH^Y_n(A) \) denote the \( n \)-th homology group of the chain complex associated to the simplicial vector space \( \mathcal{L}^Y(A) \). Following Pirashvili [10], when \( Y = S^p \) (\( S^p \) being the sphere of dimension \( p \geq 1 \)), we say that the homology groups \( \{ \HH_n^S(A) \}_{n \geq 0} \) are the Hochschild homology groups of \( A \) of order \( p \). When \( p = 1 \), i.e., \( Y = S^1 \) is the simplicial circle, the Hochschild homology groups \( \{ \HH_n^{S^1}(A) \}_{n \geq 0} \) are identical to the usual Hochschild homology groups of \( A \).

Our objective is to introduce a \( q \)-analog of the groups \( \HH^Y_n(A) \), where \( q \in \mathbb{C} \) is a primitive \( N \)-th root of unity. For this, we consider the face maps \( d_n^i : \mathcal{L}^Y(A)_n \to \mathcal{L}^Y(A)_{n-1}, 0 \leq i \leq n, n \geq 0 \), of the simplicial vector space \( \mathcal{L}^Y(A) \) defined in (2.4). We set:

\[
q b_n : \mathcal{L}^Y(A)_n \to \mathcal{L}^Y(A)_{n-1} \quad q b_n := \sum_{i=0}^{n} q^i d_n^i
\]  

(2.5)

For the sake of convenience, we will often write \( q b_n \) simply as \( q b \). Then, it is well known that the morphism \( q b \) satisfies \( q b^N = 0 \) (this is true in general for any simplicial vector space; see, for instance, Kapranov [5, Proposition 0.2]). In particular, if \( q = -1 \), i.e., \( N = 2 \), we have \( (-1)^2 b^2 = 0 \) and \( (-1)^1 b \) is the standard differential on the chain complex corresponding to the simplicial vector space \( \mathcal{L}^Y(A) \). In general, the pair \( (\mathcal{L}^Y(A), q b) \), i.e., the simplicial vector space \( \mathcal{L}^Y(A) \) equipped with the morphism \( q b \) is an “\( N \)-complex” in the sense defined below.

**Definition 2.1.** (see [3, § 2] and [5, Definition 0.1]) Let \( \mathcal{A} \) be an abelian category and \( N \geq 2 \) a positive integer. An \( N \)-complex in \( \mathcal{A} \) is a sequence of objects and morphisms of \( \mathcal{A} \)

\[
C_* = \{ \ldots \to C_1 \xrightarrow{b_1} C_0 \xrightarrow{b_0} C_{-1} \to \ldots \}
\]

(2.6)

such that the composition of any \( N \) consecutive morphisms in (2.6) is 0. For any \( n \in \mathbb{Z} \), the homology object \( H_{[n]}(C_*, b) \) of the \( N \)-complex \( (C_*, b) \) is defined as:

\[
H_{[n]}(C_*, b) := \bigoplus_{i=1}^{N-1} H_{[i,n]}(C_*, b) \quad H_{[i,n]}(C_*, b) := \frac{\text{Ker}(b^i : C_0 \to C_{n-i})}{\text{Im}(b^{N-i} : C_{N-i+n} \to C_n)}
\]  

(2.7)

**Definition 2.2.** Let \( A \) be a commutative algebra over \( \mathbb{C} \) and let \( Y \) be a pointed simplicial finite set. Let \( q \in \mathbb{C} \) be a primitive \( N \)-th root of unity. Then, the \( q \)-Hochschild homology groups \( q \HH^Y_n(A) \), \( n \geq 0 \) of \( A \) of order \( Y \) are defined to be the homology objects of the \( N \)-complex \( (\mathcal{L}^Y(A), q b) \) associated to the simplicial vector space \( \mathcal{L}^Y(A) \); in other words, we define:

\[
q \HH^Y_n(A) := H_{[n]}(\mathcal{L}^Y(A), q b)
\]  

(2.8)

As with the ordinary Hochschild homology of an algebra (see, for instance, [7, § 4.1]), given a derivation \( D : A \to A \), we want to construct the Lie derivative \( L^Y_D : q \HH^Y(A) \to q \HH^Y(A) \) on the Hochschild homology of order \( Y \). For this, we start with the following lemma.
Lemma 2.3. Let $A$ be a commutative $\mathbb{C}$-algebra and let $D : A \to A$ be a derivation on $A$. Then, the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \to \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \to \text{V}ect$.

Proof. We first consider the functor $\mathcal{L}(A)$ restricted to the subcategory $\Gamma$ of $\text{Fin}_*$, defined as in (2.1) and (2.2):

$$\mathcal{L}(A) : \Gamma \to \text{V}ect \quad [n] \to A \otimes A^\otimes n$$  

(2.9)

Given the derivation $D$ on $A$, we define morphisms (for all $n \geq 0$):

$$L_D([n]) : \mathcal{L}(A)([n]) \to \mathcal{L}(A)([n]) \quad (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes a_1 \otimes \ldots \otimes D(a_i) \otimes \ldots \otimes a_n)$$  

(2.10)

Further, for any morphism $\phi : [m] \to [n]$ in $\Gamma$, we have, for any $(a_0 \otimes a_1 \otimes \ldots \otimes a_m) \in A \otimes A^\otimes m$:

$$L_D([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes \ldots \otimes a_m) = L_D([n]) \left( \sum_{j=0}^{n} \bigotimes_{\phi(i)=j} a_i \right)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k-1} \bigotimes_{\phi(i)=j} a_i \otimes D \left( \bigotimes_{\phi(i)=k} a_i \right) \otimes \bigotimes_{j=k+1}^{n} \bigotimes_{\phi(i)=j} a_i$$

$$= \sum_{i=0}^{m} \mathcal{L}(A)(\phi)(a_0 \otimes \ldots \otimes D(a_i) \otimes \ldots \otimes a_m)$$

$$= \mathcal{L}(A)(\phi) \circ L_D([m])(a_0 \otimes a_1 \otimes \ldots \otimes a_m)$$

It follows that the derivation $D$ induces an endomorphism $L_D$ of the functor $\mathcal{L}(A) : \Gamma \to \text{V}ect$. More generally, for any object $T \in \text{Fin}_*$ and a morphism $T' \to T$ in $\text{Fin}_*$ such that $T' \in \Gamma$, we have a morphism $L_D(T') : \mathcal{L}(A)(T') \to \mathcal{L}(A)(T)$ as defined in (2.10). By definition, we know that $\mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \to T} \mathcal{L}(A)(T')$ and hence we have an induced morphism

$$L_D(T) : \mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \to T} \mathcal{L}(A)(T') \to \mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \to T} \mathcal{L}(A)(T')$$  

(2.11)

From (2.11) it follows that the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \to \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \to \text{V}ect$. This proves the claim.

\[ \square \]

Proposition 2.4. Let $A$ be a commutative $\mathbb{C}$-algebra and let $D : A \to A$ be a derivation on $A$. Let $Y$ be a pointed simplicial finite set. Then, for each $n \geq 0$, the derivation $D$ induces a morphism $L_D^Y : qHH_n^Y(A) \to qHH_n^Y(A)$ of $q$-Hochschild homology groups of order $Y$, where $q \in \mathbb{C}$ is a primitive $N$-th root of unity.

Proof. From Lemma 2.3 we know that the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \to \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \to \text{V}ect$. Given the pointed simplicial finite set $Y$, the endomorphism $L_D : \mathcal{L}(A) \to \mathcal{L}(A)$ of functors induces an endomorphism of the functor

$$\mathcal{L}^Y(A) : \Delta^\op \to \text{Fin}_* \xrightarrow{\mathcal{L}(A)} \text{V}ect$$  

(2.12)
From (2.12), it follows that we have an endomorphism $L_Y^D : \mathcal{L}^Y(A) \to \mathcal{L}^Y(A)$ of the simplicial vector space $\mathcal{L}^Y(A)$. Hence, we have induced morphisms $L_{Y,n}^D : qHH^Y_n(A) \to qHH^Y_n(A)$ on the homology objects of the $N$-complex $(\mathcal{L}^Y(A), q^n)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5).

We now let $Der(A)$ denote the vector space of all derivations on the commutative $\mathbb{C}$-algebra $A$. Then, $Der(A)$ is a Lie algebra, endowed with the Lie bracket $[D, D'] := D \circ D' - D' \circ D$, $\forall D, D' \in Der(A)$.

Let $\mathcal{H} := U(Der(A))$ denote the universal enveloping algebra of $Der(A)$. We will now show that for any pointed simplicial finite set $Y$, the operators $L_{Y,D}^n, D \in Der(A)$ on the $q$-Hochschild homology group of $A$ of order $Y$ make $qHH^Y_n(A)$ into a module over the Hopf algebra $\mathcal{H} = U(Der(A))$.

**Lemma 2.5.** Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative $\mathbb{C}$-algebra and let $D, D' \in Der(A)$ be derivations on $A$. Let $Y$ be a pointed simplicial finite set. Then, for each $n \geq 0$, the operators $L_{Y,D}^n, L_{Y,D'}^n : qHH^Y_n(A) \to qHH^Y_n(A)$ satisfy $[L_{Y,D}^n, L_{Y,D'}^n] = L_{D,D'}^n \circ L_{Y,D}^n - L_{Y,D'}^n \circ L_{Y,D}^n = L_{Y,D,D'}^n$.

**Proof.** For $D, D' \in Der(A)$, we consider the respective endomorphisms $L_D, L_{D'}$ of the functor $\mathcal{L}(A) : \Gamma \to Vect$. By definition, for any object $[n] \in \Gamma$, we have morphisms:

$$L_D([n]) : \mathcal{L}(A)([n]) \to \mathcal{L}(A)([n])$$

$$(a_0 \otimes \ldots \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes \ldots \otimes D(a_i) \otimes \ldots \otimes a_n)$$

$$L_{D'}([n]) : \mathcal{L}(A)([n]) \to \mathcal{L}(A)([n])$$

$$(a_0 \otimes \ldots \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes \ldots \otimes D'(a_i) \otimes \ldots \otimes a_n)$$

From (2.13), it may be verified easily that we have

$$(L_D \circ L_{D'} - L_{D'} \circ L_D)([n]) = (L_{D[D,D']})([n]) : \mathcal{L}(A)([n]) \to \mathcal{L}(A)([n]) \quad \forall n \geq 0$$

and it follows that $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{D[D,D']}$ as endomorphisms of the functor $\mathcal{L}(A) : \Gamma \to Vect$. More generally, for any object $T \in Fin_*$, we have $\mathcal{L}(A)(T) = \text{colim}_{T \to T'} \mathcal{L}(A)(T')$ and hence $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{D[D,D']}$ as endomorphisms of the functor $\mathcal{L}(A) : Fin_* \to Vect$. Finally, considering the composition of $\mathcal{L}(A) : Fin_* \to Vect$ with the functor $Y : \Delta^{op} \to Fin_*$ corresponding to the pointed simplicial finite set $Y$, it follows that $L_Y^D \circ L_{Y,D'} - L_{Y,D'} \circ L_Y^D = L_{D[D,D']}$ as endomorphisms of the functor $\mathcal{L}(A) : \Delta^{op} \to Vect$. Hence, we have $[L_{Y,D}^n, L_{Y,D'}^n] = L_{D,D'}^n \circ L_{Y,D}^n - L_{Y,D'}^n \circ L_{Y,D}^n = L_{Y,D,D'}^n$ on the homology objects $qHH^Y_n(A)$, $n \geq 0$ of the $N$-complex $(\mathcal{L}^Y(A), q^n)$ associated to the simplicial vector space $\mathcal{L}^Y(A) : \Delta^{op} \to Vect$ as in (2.5).

**Proposition 2.6.** Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Der(A)$ denote the Lie algebra of derivations on $A$. Let $\mathcal{H} = U(Der(A))$ denote the universal enveloping algebra of $Der(A)$. Then, for any pointed simplicial finite set $Y$ and any $n \geq 0$, the $q$-Hochschild homology group $qHH^Y_n(A)$ of order $Y$ is a left module over the Hopf algebra $\mathcal{H}$.

**Proof.** From Lemma 2.5, it follows that $Der(A)$ has a Lie algebra action on each $qHH^Y_n(A)$, i.e., $[L_{Y,D}^n, L_{Y,D'}^n] = L_{D,D'}^n \circ L_{Y,D}^n - L_{Y,D'}^n \circ L_{Y,D}^n = L_{Y,D,D'}^n$ for any $D, D' \in Der(A)$. Since $\mathcal{H}$ is the universal enveloping algebra of $Der(A)$, it follows that this Lie algebra action of $Der(A)$ on $qHH^Y_n(A)$ makes $qHH^Y_n(A)$ into a left $\mathcal{H}$-module.
3 Higher derivations and the Lie derivative

As before, we work with a commutative algebra $A$ over $\mathbb{C}$, a pointed simplicial finite set $Y$ and $q \in \mathbb{C}$ a primitive $N$-th root of unity. In this section, we will describe the Lie derivative on the $q$-Hochschild homology groups $qHH^n_Y(A)$ corresponding to a higher derivation $D$ on $A$. Given an ordinary derivation $d$ on $A$, it is easy to verify that the sequence $\{D_n := d^n/n!\}_{n \geq 0}$ satisfies the following identity:

$$D_n(a \cdot a') = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(a') \quad \forall n \geq 0, \ a, a' \in A$$  (3.1)

More generally, we have the notion of a higher (or Hasse-Schmidt) derivation on $A$.

**Definition 3.1.** (see, for instance, [8]) Let $A$ be a commutative algebra over $\mathbb{C}$. A sequence $D = \{D_n\}_{n \geq 0}$ of $\mathbb{C}$-linear maps on $A$ is said to be a higher (or Hasse-Schmidt) derivation on $A$ if it satisfies:

$$D_n(a \cdot a') = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(a') \quad \forall n \geq 0, \ a, a' \in A$$  (3.2)

In this paper, we will only work with higher derivations $D = \{D_n\}_{n \geq 0}$ that are normalized, i.e., those higher derivations $D = \{D_n\}_{n \geq 0}$ which satisfy $D_0 = 1$. For a normalized higher derivation $D = \{D_n\}_{n \geq 0}$ it is easy to verify from relation (3.2) that $D_n(1) = 0$ for all $n > 0$. For more on the structure of higher derivations on an algebra, we refer the reader to [9], [11] and [12]. For a higher derivation on $A$, we have already described in [2] the corresponding Lie derivative on the ordinary Hochschild homology; we are now ready to introduce the action of a higher derivation on the $q$-Hochschild homology groups of order $Y$ of the algebra $A$.

**Lemma 3.2.** Let $A$ be a commutative algebra over $\mathbb{C}$ and let $D = \{D_n\}_{n \geq 0}$ be a (normalized) higher derivation on $A$. Then, for any given $k \geq 0$, the higher derivation $D$ induces an endomorphism $L^k_D : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$.

**Proof.** It suffices to prove that for each $k \geq 0$, we have an endomorphism $L^k_D : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$ restricted to the subcategory $\Gamma$ of $\text{Fin}_*$. Given the higher derivation $D = \{D_n\}_{n \geq 0}$ and the integer $k \geq 0$, we define morphisms ($\forall \ n \geq 0$)

$$L^k_D([n]) : \mathcal{L}(A)([n]) \rightarrow \mathcal{L}(A)([n]) \quad \sum_{(p_0, p_1, ..., p_n) \in \mathbb{N}^n \text{ such that } p_0 + p_1 + ... + p_n = k} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes ... \otimes D_{p_n}(a_n))$$  (3.3)

For the sake of convenience, we will often denote a sum as in (3.3) taken over all ordered tuples $(p_0, p_1, ..., p_n)$ of non-negative integers such that $p_0 + p_1 + ... + p_n = k$ simply as

$$\sum_{p_0 + p_1 + ... + p_n = k} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes ... \otimes D_{p_n}(a_n))$$  (3.4)
Let $\phi : [m] \rightarrow [n]$ be a morphism in $\Gamma$. We let $N(j)$ denote the cardinality of the set $\phi^{-1}(j) \subseteq [m]$ for any $0 \leq j \leq n$. Then, we have, for any $(a_0 \otimes a_1 \otimes \ldots \otimes a_m) \in A \otimes A^\otimes m$:

$$L^k_D([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes \ldots \otimes a_m) = L^k_D([n]) \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} a_i \right)$$

$$= \sum_{p_0+p_1+\ldots+p_n=k} \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} D_{p_j} \left( \prod_{\phi(i)=j} a_i \right) \right)$$

$$= \sum_{r_0+r_1+\ldots+r_m=k} \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} D_{r_i} (a_i) \right)$$

From (3.5), it follows that for each $k \geq 0$, $L^k_D : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ is an endomorphism of the functor $\mathcal{L}(A)$ restricted to $\Gamma$ and hence, taking colimits as in the proof of Lemma 2.3, $L^k_D$ induces an endomorphism of the functor $\mathcal{L}(A) : Fin_\ast \rightarrow Vect$.

Proposition 3.3. Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Y$ be a pointed simplicial finite set. Then, given a higher derivation $D = \{D_n\}_{n \geq 0}$ on $A$, for each $k \geq 0$, we have an induced morphism:

$$L^k_Y : qHH^Y_\ast (A) = \bigoplus_{n=0}^{\infty} qHH^Y_n (A) \rightarrow qHH^Y_\ast (A) = \bigoplus_{n=0}^{\infty} qHH^Y_n (A)$$

(3.6)

on the $q$-Hochschild homology groups of $A$ of order $Y$.

Proof. From Lemma 3.2 we know that for any $k \geq 0$, we have an endomorphism $L^k_D : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_\ast \rightarrow Vect$. Composing with the functor $Y : \Delta^{op} \rightarrow Fin_\ast$ corresponding to the pointed simplicial finite set $Y$, we have an induced endomorphism $L^k_Y : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y : \Delta^{op} \rightarrow Fin_\ast \rightarrow Vect$. Accordingly, $L^k_Y$ induces an endomorphism on the homology objects of the $N$-complex $(\mathcal{L}^Y(A), qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5). Hence, we have induced morphisms $L^k_Y : qHH^Y_\ast (A) \rightarrow qHH^Y_\ast (A)$ on the $q$-Hochschild homology groups of order $Y$.

We have already shown in the last section that $qHH^Y_\ast (A)$ is a left module over the universal enveloping algebra $\mathcal{H} = \mathcal{U}(Der(A))$ of the Lie algebra of derivations on $A$. Given a higher derivation $D = \{D_k\}_{k \geq 0}$
on a $\mathbb{C}$-algebra $A$, Mirzavaziri [9] has shown that the higher derivation $D$ may be expressed as follows: there exists a sequence of ordinary derivations $\{d_n\}_{n \geq 0}, d_n \in \text{Der}(A)$ such that:

$$D_k = \sum_{i=1}^{k} \left( \sum_{\sum_j r_j = k} \left( \prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) d_{r_1} \ldots d_{r_i} \right)$$  \hspace{1cm} (3.7)

From (3.7), it is clear that given a higher derivation $D = \{D_k\}_{k \geq 0}$ on $A$, each $D_k$ is an element of the Hopf algebra $H = U(Der(A))$. Hence, it follows from Proposition 2.6 that each operator $D_k \in H$ induces a morphism $L_{D_k}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ on the $q$-Hochschild homology groups of order $Y$. We will now show that the morphisms $L_{D_k}^{Y^k}$ are identical to the morphisms $L_{D}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ described in Proposition 3.3.

**Proposition 3.4.** Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Y$ be a pointed simplicial finite set. Let $D = \{D_k\}_{k \geq 0}$ denote a higher derivation on $A$. For any $k \geq 1$, let $L_{D_k}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ be the morphism induced by $D_k \in H$ as in Proposition 2.6 and let $L_{D}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ be the morphism induced by $D$ as in Proposition 3.3. Then, we have $L_{D_k}^{Y^k} = L_{D}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$.

**Proof.** From the proofs of Lemma 2.3 and Lemma 2.5, it follows that the element $D_k \in H = U(Der(A))$ of the universal enveloping algebra $H$ defines an endomorphism $L_{D_k} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_{\ast} \rightarrow \text{Vect}$. From the proofs of Proposition 2.4 and Proposition 2.6, it is clear that the morphism $L_{D_k}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ is obtained from the endomorphism $L_{D_k}^{Y} : \mathcal{L}^{Y}(A) \rightarrow \mathcal{L}^{Y}(A)$ of the functor $\mathcal{L}^{Y}(A) = \mathcal{L}(A) \circ Y$ induced by $L_{D_k} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$.

Similarly, from Lemma 3.2, it follows that the higher derivation $D$ induces an endomorphism $L_{D}^{Y^k} : \mathcal{L}^{Y}(A) \rightarrow \mathcal{L}^{Y}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_{\ast} \rightarrow \text{Vect}$. From the proof of Proposition 3.3, it follows that the morphism $L_{D}^{Y^k} : qHH_{*}^{Y}(A) \rightarrow qHH_{*}^{Y}(A)$ is obtained from the endomorphism $L_{D}^{Y} : \mathcal{L}^{Y}(A) \rightarrow \mathcal{L}^{Y}(A)$ of the functor $\mathcal{L}^{Y}(A) = \mathcal{L}(A) \circ Y$ induced by $L_{D} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$. Hence, in order to prove the result, we need to show that $L_{D_k}^{Y} = L_{D_k}^{Y} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ as endomorphisms of the functor $\mathcal{L}(A) : \text{Fin}_{\ast} \rightarrow \text{Vect}$. As before, it suffices to show that $L_{D_k}^{Y} = L_{D_k}^{Y}$ as endomorphisms of the functor $\mathcal{L}(A)$ restricted to the subcategory $\mathcal{L}(A)$.

Let $\Delta : H \rightarrow H \otimes H$ denote the coproduct on $H$. For any $h \in H$ and any $n \geq 0$, we write $\Delta^n(h) = \sum h_{(1)} \otimes h_{(2)} \otimes \ldots \otimes h_{(n+1)}$. Then, we have an induced endomorphism $L_{h} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_{\ast} \rightarrow \text{Vect}$. Further, we note that the equation

$$L_{h([n])}(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum (h_{(1)}(a_0) \otimes h_{(2)}(a_1) \otimes \ldots \otimes h_{(n+1)}(a_n)) \quad \forall (a_0 \otimes \ldots \otimes a_n) \in \mathcal{L}(A)([n]) \hspace{1cm} (3.8)$$

holds for all $h \in \text{Der}(A) \subseteq H$ and hence for all $h \in H = U(Der(A))$. From the definition of $L_{D_k}^{Y}$ in Lemma 3.2, we now see that in order to show that $L_{D_k}^{Y} = L_{D_k}^{Y}$, it suffices to show that

$$\Delta^n(D_k) = \sum_{\sum_{i=0}^{n} p_i = k} D_{p_0} \otimes D_{p_1} \otimes \ldots \otimes D_{p_n} \quad \forall n \geq 0 \hspace{1cm} (3.9)$$
We will prove (3.9) by induction on $k$. For any given $n \geq 0$, it is clear that the equation (3.9) holds for $k = 0$ and $k = 1$. We now suppose that its holds for any $0 \leq k \leq K$. From [9, Proposition 2.1], we know that

$$D_{M+1} = \frac{1}{M+1} \sum_{m=0}^{M} d_{m+1} D_{M-m} \quad \forall M \geq 0$$

(3.10)

where the $d_{m+1}$ are the derivations corresponding to the higher derivation $D = \{D_n\}_{n \geq 0}$ as described in (3.7). From (3.10), it follows that $\Delta^n(D_{K+1}) = \frac{1}{K+1} \sum_{m=0}^{K} \Delta^n(d_{m+1}) \Delta^n(D_{K-m})$ and hence

$$\Delta^n(D_{K+1}) = \frac{1}{K+1} \sum_{m=0}^{K} \left( \sum_{j=0}^{n} d_{m+1} \right) \left( \sum_{i=0}^{n} D_{p_0} \otimes D_{p_1} \otimes \ldots \otimes D_{p_n} \right)$$

(3.11)

where $d_{m+1}$ denotes the term $1 \otimes 1 \otimes \ldots \otimes d_{m+1} \otimes \ldots \otimes 1$ (i.e., $d_{m+1}$ at the $j$-th position) appearing in the expression for $\Delta^n(d_{m+1})$. We now consider ordered tuples $(p'_0, p'_1, \ldots, p'_n)$ of non-negative integers such that $p'_0 + p'_1 + \ldots + p'_n = K + 1$. Then, we can write:

$$\sum_{m=0}^{K} \left( \sum_{j=0}^{n} d_{m+1} \right) \left( \sum_{i=0}^{n} D_{p_0} \otimes D_{p_1} \otimes \ldots \otimes D_{p_n} \right)$$

(3.12)

From (3.10), it follows that $\sum_{m=0}^{K} \sum_{j=0}^{n} d_{m+1} = p'_j \cdot D_{p'_j}$ and hence:

$$\sum_{m=0}^{p'_j-1} (D_{p'_0} \otimes \ldots \otimes d_{m+1} D_{p'_j-m-1} \otimes \ldots \otimes D_{p'_n}) = p'_j \cdot (D_{p'_0} \otimes \ldots \otimes D_{p'_j} \otimes \ldots \otimes D_{p'_n})$$

(3.13)

Combining (3.11), (3.12) and (3.13), it follows that:

$$\Delta^n(D_{K+1}) = \frac{1}{K+1} \left( \sum_{i=0}^{n} \sum_{j=0}^{n} p'_j \cdot (D_{p'_0} \otimes \ldots \otimes D_{p'_j} \otimes \ldots \otimes D_{p'_n}) \right)$$

(3.14)

This proves the result of (3.9) for $K + 1$. \qed
4 Action on bivariant $q$-Hochschild cohomology groups

Let $A$ be a commutative algebra over $\mathbb{C}$ and let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $Y$ be a pointed simplicial finite set. In this section, we will define the bivariant $q$-Hochschild cohomology groups $\{HH^n_Y(A, A)\}_{n \in \mathbb{Z}}$ of $A$ of order $Y$ and show that a derivation $D$ on $A$ induces a morphism $L^D_Y^n(A, A) : qHH^n_Y(A, A) \rightarrow qHH^n_Y(A, A)$. For the ordinary bivariant Hochschild cohomology groups $\{HH^n(A, A)\}_{n \in \mathbb{Z}}$, we have already studied this morphism in [1]. For the definition and properties of ordinary bivariant Hochschild cohomology, we refer the reader to [71, § 5.1] (see also the original paper of Jones and Kassel [4]). We start by defining the bivariant $q$-Hochschild cohomology groups of order $Y$.

**Definition 4.1.** Let $(\mathcal{L}^Y(A), q)$ be the $N$-complex corresponding to the simplicial vector space $\mathcal{L}^Y(A)$ as defined in (2.5). We consider the $q$-Hom complex $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of these $N$-complexes which is defined as follows:

$$\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n := \prod_{i \in \mathbb{Z}} Hom_{Vect}(\mathcal{L}^Y(A)_i, \mathcal{L}^Y(A)_{i+n})$$

(4.1)

Further, if the family $f = \{ f_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+n}\}_{i \in \mathbb{Z}}$ is an element of $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, then the differential $q\partial_n : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n \rightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{n-1}$ is defined by setting:

$$q\partial_n(f) := \{ q\partial_n(f)_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+n-1}\}_{i \in \mathbb{Z}}$$

(4.2)

For any given $n \in \mathbb{Z}$, we define the bivariant $q$-Hochschild cohomology group $qHH^n_Y(A, A)$ of $A$ of order $Y$ to be the homology object

$$qHH^n_Y(A, A) := H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), q\partial)$$

(4.3)

of the $N$-complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), q\partial)$.

We mention that it follows from [5, Proposition 1.8] that the $q$-Hom complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), q\partial)$ as defined in (2.5) and (4.2) is also an $N$-complex. We now make the convention that if $M = \oplus_{i \in \mathbb{Z}} M_i$ is a graded vector space and $f = \{ f_i : M_i \rightarrow M_{i+n}\}_{i \in \mathbb{Z}}$ and $g = \{ g_i : M_i \rightarrow M_{i+n}\}_{i \in \mathbb{Z}}$ are two morphisms of homogenous degree $m$ and $n$ respectively, we will write $[f, g] := f \circ g - q^m g \circ f$ for their graded $q$-commutator.

**Lemma 4.2.** Let $L^m \{ L^m_i \}_{i \in \mathbb{Z}}$ denote a collection of maps $L^m_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+m}$. Given an element $f = \{ f_i \}_{i \in \mathbb{Z}}$ in $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, we define $L^m(f) \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{m+n}$ by setting:

$$L^m(f)_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+m+n}$$

$$L^m_i(f) := L^m_{i+n} \circ f_i - q^m f_{i+m} \circ L^m_i$$

(4.4)

Then, if $q^{2m} = 1$, the endomorphism $L^m : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \rightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of homogenous degree $m$ satisfies the following relation:

$$[q\partial, L^m](f) = [q, L^m]f + q^{m+n}f[L^m, q]b \quad \forall \ f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n, \ n \in \mathbb{Z}$$

(4.5)
Proof. We consider:

\[
\begin{align*}
(q \partial \circ L^m)(f) &= q^{b_{i+m+n}} \circ L^m(f) - q^{m+n} \circ L^m(f)_i - q^{m} \circ b_i \\
&= q^{b_{i+m+n}} \circ L^m_i - q^{m} \circ b_i + q^{mn+n+m} \circ L^m_i - q^{m} \circ b_i \\
&= L^m_i \circ q \partial(f)_i - q^{m(n-1)} \circ q \partial(f)_i \circ L^m_i \\
&= L^m_{i+n-1} \circ q \partial(f)_i - q^{m(n-1)} \circ q \partial(f)_i \circ L^m_i \\
&= L^m_{i+n-1} \circ q \partial(f)_i - q^{m(n-1)} \circ q \partial(f)_i \circ L^m_i \\
&= L^m_{i+n-1} \circ q \partial(f)_i - q^{m(n-1)} \circ q \partial(f)_i \circ L^m_i \\
&= L^m_{i+n-1} \circ q \partial(f)_i - q^{m(n-1)} \circ q \partial(f)_i \circ L^m_i.
\end{align*}
\]

(4.6)

From (4.6), it follows that:

\[
\begin{align*}
(q \partial, L^m)(f)_i &= (q \partial \circ L^m)(f)_i - q^{-m}(L^m \circ q \partial)(f)_i \\
&= (q^{b_{i+m+n}} \circ L^m_i - q^{-m}L^m_{i+n-1} \circ q^{b_i+n}) \circ f_i - q^{m+n} \circ q^{mn+m+n}(L^m_i - q^{m} \circ b_i - q^{-2m}(q^{-m} \circ b_i \circ L^m_i)) \\
&= (1 - q^{-2m})q^{b_{i+m+n}} \circ f_i - q^{m+n} \circ q^{mn+m+n}(1 - q^{-2m})L^m_{i+n-1} \circ f_i \\
&= q^{m+n} \circ q^{mn+m+n} \circ f_i \circ L^m_i - q^{m+n} \circ q^{mn+m+n} \circ q^{m} \circ b_i \\
&= q^{m+n} \circ q^{mn+m+n} \circ f_i \circ L^m_i.
\end{align*}
\]

From (4.6), it follows from the above expression that:

\[
[q \partial, L^m](f) = [q \partial, L^m]f + q^{m+n} \circ q^{mn+m+n} \circ f_i \circ L^m_i.
\]

(4.7)

Proposition 4.3. Let \( q \in \mathbb{C} \) be a primitive \( N \)-th root of unity. Let \( A \) be a commutative algebra over \( \mathbb{C} \) and let \( D : A \rightarrow A \) be a derivation on \( A \). Let \( Y \) be a pointed simplicial finite set. Then, for each \( n \in \mathbb{Z} \), the derivation \( D \) on \( A \) induces a morphism

\[
L^Y_D : qHH^Y_n(A, A) \rightarrow qHH^Y_n(A, A)
\]

on the bivariant \( q \)-Hochschild cohomology groups of order \( Y \).

Proof. From the proof of Proposition 2.6, we know that the derivation \( D \) induces an endomorphism

\[
L^Y_D : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)
\]

of the simplicial vector space \( \mathcal{L}^Y(A) \). Accordingly, we have a collection of maps \( L^Y_D = \{L^Y_{D,i} : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_i\}_{i \in \mathbb{Z}} \) determined by the endomorphism \( L^Y_D \). Applying Lemma 4.2 with \( m = 0 \) (and hence \( q^{2m} = 1 \)), it follows that \( L^Y_D \) determines a morphism

\[
L^Y_D : \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \rightarrow \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))
\]

(4.9)

of homogeneous degree \( m = 0 \) satisfying:

\[
[q \partial, L^Y_D](f) = [q \partial, L^Y_D]f + q^n f[L^Y_D, q] \quad \forall f \in \text{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_.
\]

(4.10)

Again, since \( L^Y_D : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A) \) is a morphism of simplicial vector spaces, the morphisms \( \{L^Y_{D,i} : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_i\}_{i \in \mathbb{Z}} \) commute with the face maps \( d^j_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1}, 0 \leq j \leq i, i \geq 0 \) of the simplicial vector space \( \mathcal{L}^Y(A) \). By definition, \( q^d_{i,j} := \sum_{j=0}^i q^j d^j_i \) and hence we have:

\[
[q, L^Y_D] = [L^Y_D, q] = 0
\]

(4.11)
Applying this to (4.10), it follows that:
\[
[q \partial, L_D^Y] = q \partial \circ L_D^Y - q^{-m} L_D^Y \circ q \partial = q \partial \circ L_D^Y - L_D^Y \circ q \partial = 0 \quad (4.12)
\]
From (4.12), it follows that the endomorphism \( L_D^Y \): \( \text{Hom}(L^Y(A), L^Y(A)) \rightarrow \text{Hom}(L^Y(A), L^Y(A)) \) of degree zero commutes with the differential \( q \partial \) on the \( N \)-complex \( \text{Hom}(L^Y(A), L^Y(A)) \). This induces morphisms (\( \forall n \in \mathbb{Z} \)):
\[
qHH^n_Y(A, A) = H_{\{-n\}}(\text{Hom}(L^Y(A), L^Y(A)), q \partial)
\]
\[
L_D^{Y,n} \downarrow
\]
\[
qHH^n_Y(A, A) = H_{\{-n\}}(\text{Hom}(L^Y(A), L^Y(A)), q \partial)
\]
(4.13)
on the bivariant \( q \)-Hochschild cohomology groups of order \( Y \).

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