Fermion production from preheating-amplified metric perturbations

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(December 11, 2018)

We study gravitational creation of light fermions in the presence of classical scalar metric perturbations about a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background. These perturbations can be large during preheating, breaking the conformal flatness of the background spacetime. We compute numerically the total number of particles generated by the modes of the metric perturbations which have grown sufficiently to become classical. In the absence of inhomogeneities massless fermions are not gravitationally produced, and then this effect may be relevant for abundance estimates of light gravitational relics.

I. INTRODUCTION

It is well known that particles are created in an expanding Universe\(^1\). Pioneering work by Parker\(^2\) highlighted the creation of nonconformally-invariant particles even in flat Friedmann-Lemaître-Robertson-Walker (FLRW) Universes. The extension to anisotropic Bianchi cosmologies by Zel’dovich and Starobinsky\(^3\), showed that massless, conformally coupled, scalar particles are created due to the breaking of conformal invariance which, in four dimensions, is signaled by a non-vanishing Weyl tensor. In the presence of small, inhomogeneous metric perturbations, Horowitz and Wald\(^4\) obtained the vacuum expectation value of the stress tensor for a conformally invariant field and showed that particle production only occurs if \(\langle T_{\mu\nu}(x) \rangle\) is non-local (i.e., has non-zero contributions with support on the past-null cone of the event \(x\)).

The number of particles created by inhomogeneities can be computed by a perturbative evaluation of the S-matrix\(^5–7\). In the inhomogeneous case, the occupation number of produced particles is composed by three parts, viz, the zeroth-order contribution due to the homogeneous expansion, a first-order (in the perturbation amplitude) part arising from the interference between 0- and 2-particle states, and a second-order contribution which comes from the interaction between nonzero particle states. In the case of massive or nonconformally coupled scalar fields, the last two terms (respectively linear and quadratic in the metric perturbations) typically give a contribution small compared with the first one. However, in the massless and conformally coupled cases, the first two terms vanish in rigid, exactly FLRW backgrounds, and only inhomogeneity contributes to the gravitational particle production.

Such theoretical studies were historically of interest both because they focus on the interplay between quantum field theory and gravity and because of their relevance to Misner’s “chaotic cosmology” program\(^8\) in which arbitrary initial inhomogeneity and anisotropy were to be damped to acceptable levels due to particle production. The non-renormalisability of standard quantum gravity, the successes of string theory and the dominance of cosmological inflation have since removed much of the motivation for studies of particle production from inhomogeneity, though research on non-equilibrium issues, generalized fluctuation-dissipation relations and effective Einstein-Langevin equations still continues\(^9\).

Inflationary cosmology seems to be less affected by the complex aspects of gravity beyond the semi-classical realm. For inflation to start, a homogeneous patch larger than the Hubble radius, \(H^{-1}\), is needed which may require some fine-tuning, especially for inflation at low energy scales. However, once inflation has begun, the no-hair conjecture\(^10\) should ensure that pre-existing inhomogeneities are driven to zero. The quantum metric perturbations generated during inflation can then be tuned to have small amplitudes \(\sim 10^{-5}\) and unless one wants to compute 4-point correlation functions which include graviton loops, comparison with the anisotropies in the Cosmic Microwave Background (CMB) is straightforward.

However, nonperturbative production during the coherent oscillation of the inflaton fields\(^11–15\), dubbed preheating\(^12\), can alter the simple picture of small amplitude metric perturbations. Preheating therefore provides a particularly useful arena for examining some aspects of the boundary between semi-classical and quantum gravity.
Preheating can lead to the growth of metric perturbations, which in turns is known to stimulate scalar particle creation \cite{16–18} and the production of seed magnetic fields \cite{19–23}. The amplification of metric perturbations may also yield runaway instabilities due to the negative specific heat of gravity, which leads to an interesting possibility of primordial black hole (PBH) formation \cite{24,25}. It can also amplify metric perturbations on very large scales \cite{12,26–30}. The growth of metric perturbations, in the case where a PBH does not form, is limited by backreaction effects which are explicitly nonlinear.

Hence on certain scales the Weyl tensor can actually be quite large. This means that there can be significant particle production even if the field is governed by a conformally invariant equation of motion, the classic examples being photons, massless fermions and conformally coupled scalars. In particular, if the effective mass of gravitational relics such as modulini and gravitini is small enough during reheating, then their main source of gravitational production will come from metric perturbations. Since stringent upper bounds hold for the abundances of these species, it is important to verify that creation from inhomogeneities (if sufficiently amplified at preheating) does not overcome this threshold. Our analysis suggests that this does not occur in the simplest models of chaotic inflation, at least when rescattering is neglected.

Particle creation via inhomogeneities may also be relevant for alternative scenarios such as the pre-big-bang model \cite{31}, where homogeneous gravitational production of moduli and gravitinos is known to be an issue \cite{32} and the Universe goes through a high-curvature phase where higher-order $\alpha'$ corrections are important.

The paper is organized as follows. In section II we review the formalism for computing the number of produced species, summarizing the calculations of \cite{5–7}. We consider perturbations around the a FLRW background. Sections IV and V present our numerical results and conclusions respectively.

II. FORMALISM FOR PRODUCTION OF FERMIONS

In this section we review the formalism for the production of (Dirac) fermionic quanta by cosmological inhomogeneities, summarizing the calculations of \cite{3,5}. We consider perturbations around the a FLRW background

$$g_{\mu \nu} = a^2(\eta) \left( \eta_{\mu \nu} + h_{\mu \nu} \right) ,$$

where $a(\eta)$ is the scale factor and $\eta$ the conformal time. We will also use proper time, $t$, related to conformal time by $d\eta = dt/a$ and choose to consider only scalar metric perturbations in the longitudinal gauge \cite{3,33,34}.

At linear order minimally coupled scalar fields do not induce an anisotropic stress \cite{17} and hence the metric perturbations are characterized by a single potential: $h_{\mu \nu} = 2\Phi \delta_{\mu \nu}$, viz:

$$ds^2 = a^2(1 + 2\Phi) d\eta^2 - a^2(1 - 2\Phi) \delta_{ij} dx^i dx^j .$$

The action of a fermionic field $\psi$ with mass $m$ in the background \cite{2,2} is, to first order in $\Phi$,

$$S = \int d^4x a^3 (1 - 2\Phi) \bar{\psi} \left\{ i \left[ (1 - \Phi) \gamma^0 \partial_\eta + \frac{3}{2a} \frac{da}{d\eta} (1 - \Phi) \gamma^0 - \frac{3}{2} \frac{d\Phi}{d\eta} \gamma^0 \right] + (1 + \Phi) \gamma^i \partial_i - \frac{3}{2} \gamma^i \partial_i \Phi \right\} \psi$$

$$= \int d^4x \bar{\psi} \left[ (\gamma^0 \partial_\eta + i \gamma^i \partial_i - m) - 3\Phi \gamma^0 \partial_\eta - \Phi \gamma^i \partial_i - \frac{3}{2} \frac{d\Phi}{d\eta} \gamma^0 - \frac{3}{2} \gamma^i \partial_i \Phi \right] \psi$$

$$= \int d^4x \left[ L_0 + L_I (\Phi) \right] ,$$

where we have defined $\bar{\psi} \equiv a^{3/2} \psi$ and $L_0, L_I$ denote the homogeneous and interaction Lagrangians.

The number density of produced particles is easily computed in the interaction picture, where the evolution of the operators is just determined by the homogeneous expansion of the Universe, and the states evolve according to the small inhomogeneities in the metric. The field $\psi$ can be decomposed in the standard way

\footnote{Vector perturbations will induce magnetic fields via the induction equation \cite{23} but we do not consider them since inflation drives vector perturbations to zero. We also neglect tensor (gravitational wave) modes since they are not strongly amplified at preheating.}
\[
\tilde{\psi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ikx} \sum_r [u_r(k, \eta) a_r(k) + v_r(k, \eta)b_r^\dagger(-k)] ,
\]
where, as usual, the anti-commutation relations are
\[
\{a_r(k), a_r^\dagger(k')\} = \{b_r(k), b_r^\dagger(k')\} = \delta^{(3)}(k - k') \delta_{rs}.
\]
The Fock space is built at the initial time \(\eta_i\) starting from the vacuum state defined by
\[
a_r(k) |0\rangle = b_r(k) |0\rangle = 0 .
\]

For massless fermions (the conformally coupled case), the expansion of the Universe factors out of the free action \(\int d^4x \mathcal{L}_0\) and the spinors \(u_r\) and \(v_r\) evolve as in Minkowski spacetime. As a consequence, the operators \(a_r^{(1)}, b_r^{(1)}\) are the physical annihilation (creation) operators at all times. Introducing a mass term breaks the conformal symmetry, and the physical annihilation/creation operators are defined through the Bogolyubov transformations (see [36] for details)
\[
\hat{a}_r(k, \eta) \equiv \alpha_r(k, \eta) a_r(k) - \beta_r^* (k, \eta) b_r^\dagger(-k) ,
\]
\[
\hat{b}_r^\dagger(k, \eta) \equiv \beta_r(k, \eta) a_r(k) + \alpha_r^* (k, \eta) b_r^\dagger(-k) .
\]
The two Bogolyubov coefficients have initial conditions \(\alpha(\eta_i) = 1, \beta(\eta_i) = 0\) and evolve according to [37]
\[
\alpha' = -\frac{mka'}{2\omega_0^2} e^{2i \int^\eta \omega d\eta} \beta_r , \quad \beta' = \frac{mka'}{2\omega_0^2} e^{-2i \int^\eta \omega d\eta} \alpha_r ,
\]
with \(\omega \equiv \sqrt{k^2 + a^2 m^2}\). These equations preserve the normalization \(|\alpha|^2 + |\beta|^2 = 1\), which holds in the fermionic case.

In the interaction picture, the evolution of the initial vacuum (zero particle) state \(|0\rangle\) is determined by the interaction Lagrangian \(\mathcal{L}_I(\Phi)\). At first order in \(\Phi\) we have
\[
|\psi\rangle = |0\rangle + \frac{1}{2} \int d^3k d^3k' |k, r; k', s\rangle \langle k, r; k', s|S|0\rangle ,
\]
with the \(S\)-matrix element (\(T\) stands for time ordering)
\[
\langle k, r; k', s|S|0\rangle \equiv i T \langle k, r; k', s|\mathcal{L}_I(\Phi)|0\rangle .
\]

Rigorously speaking, the occupation number can be computed only in the asymptotic future, \(\eta_f\), with vanishing perturbations \(\Phi(\eta_f) = 0\). Identical results are obtained for particles and antiparticles, so we concentrate on the former. The expectation value of the number operator \(\hat{N} \equiv \langle 2\pi a \rangle^{-3} \int d^4p \hat{a}_r^\dagger(p, \eta_f) \hat{a}_r(p, \eta_f)\) in the state \(|\psi\rangle\) is given by the sum of three terms, \(N_0 + N_1 + N_2\), which are of zeroth, first, and second order in \(\Phi\), respectively [3 4],
\[
N_0 = \frac{V}{(2\pi a)^3} \int d^3k \langle 0| \hat{a}_r^\dagger(k) \hat{a}_r(k) |0\rangle = \frac{V}{(2\pi a)^3} \int d^3k |\beta|^2 ,
\]
\[
N_1 = -\frac{1}{(2\pi a)^3} \int d^3k \text{Re} \left[ \alpha_r(k) \beta_r^* (k) \langle k, r; -k, r |S|0\rangle \right] ,
\]
\[
N_2 = \frac{1}{4(2\pi a)^3} \int d^3k d^3k' \langle 0|S|k, r; k', s\rangle^2 \left[ |\alpha_r(k)|^2 + |\beta_r(-k')|^2 + 1 \right] .
\]
The zeroth-order term (2.11) is the well known expression arising from the homogeneous expansion of the Universe. \(V\) denotes the volume of the Universe at late times, when the perturbations can be neglected. The first-order contribution (2.12) comes from the combined effect of expansion and inhomogeneities. These two terms vanish for massless fermions, which, as we remarked, are conformally coupled to the FLRW background (more explicitly, we notice that \(\beta_r(k, \eta) = 0\) in this case, as can be seen from eqs. (2.8)).

In the massless case, only the last term (2.13) contributes to the particle production. Furthermore, in this limit \(N_2\) acquires the particularly simple form [3]
\[
N_2 = \frac{1}{160 \pi a^3} \int \frac{d^4p}{(2\pi)^4} \theta(p^0) \theta(p^2) |\tilde{C}^{abcd}(p)|^2 ,
\]
where \(\tilde{C}^{abcd}\) denotes the massless case.
in terms of the Fourier transform
\begin{equation}
\tilde{C}^{abcd}(p) \equiv \int d^4x \, e^{ip \cdot x} \, C^{abcd}(x)
\end{equation}
of the Weyl tensor $C^{abcd}$.

For the metric (2.2), one finds (see also 4 for useful intermediate steps)
\begin{equation}
\tilde{\Phi}(p, \eta) \equiv \int |p| \, d^3p \, \tilde{\Phi}(p, \eta) e^{i p \cdot x}.
\end{equation}

In these expressions, $\Phi$ should be regarded as a purely classical perturbation. The normalization as well as the classicality condition for $\tilde{\Phi}$ will be discussed in the subsequent sections.

III. EVOLUTION OF SCALAR METRIC PERTURBATIONS

A. Linearized equations and analytic solutions for metric perturbations

We now discuss the evolution of scalar perturbations $\Phi$ during inflation and reheating in simple models of chaotic inflation [3], denoting the minimally coupled inflaton field by $\phi$, with potential
\begin{equation}
V = \frac{1}{2} m^2 \phi^2 \quad \text{or} \quad V = \frac{1}{4} \lambda \phi^4.
\end{equation}

In subsection III.B.3 we will then discuss a model where two fields are present.

For the potentials (3.1), the COBE normalization of the Cosmic Microwave Background Radiation [39] requires the coupling constants be $m \sim 10^{-6} M_p$ for a massive inflaton and $\lambda \sim 10^{-13}$ for the quartic case. Decomposing the inflaton as $\phi(t, x) \rightarrow \phi(t) + \delta \phi(t, x)$, the background equations are given by
\begin{align}
H^2 &= \frac{8 \pi}{3 M_p^2} \left( \frac{1}{2} \dot{\phi}^2 + V \right), \\
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} &= 0,
\end{align}
where dots denote derivatives with respect to physical time. In the numerical simulations which we present in the next section, we will implement the backreaction of field fluctuations within the Hartree approximation. This corresponds to adding the contributions
\begin{equation}
\langle \delta \phi^2 \rangle = \frac{1}{2} \int k^2 |\delta \phi_k|^2 dk, \quad \langle (\nabla \delta \phi)^2 \rangle = \frac{1}{2} \int k^4 |\nabla \delta \phi_k|^2 dk, \quad \langle \delta \dot{\phi} \rangle = \frac{1}{2} \int k^2 |\dot{\delta \phi}_k|^2 dk
\end{equation}
to Eqs. (3.2) and (3.3) (see Refs. 12,13 for details). Note that this approach neglects mode-mode coupling and rescattering effects [14], which are important around the end of preheating, and the backreaction effect of metric perturbations. The Fourier transformed, linearized Einstein equations for field and metric perturbations in the longitudinal gauge are
\begin{align}
\dot{\Phi}_k + H \Phi_k &= \frac{4 \pi}{M_p^2} \phi \delta \phi_k, \\
3H \dot{\Phi}_k + \left( \frac{k^2}{a^2} + 3H^2 - \frac{4 \pi}{M_p^2} \phi \right) \Phi_k &= \frac{4 \pi}{M_p^2} \left( \dot{\phi} \delta \phi_k + \frac{dV}{d\phi} \phi \delta \phi_k \right),
\end{align}
\[ \delta \ddot{\phi}_k + 3H \delta \dot{\phi}_k + \left( \frac{k^2}{a^2} + \frac{d^2V}{d\phi^2} \right) \delta \phi_k = 2(\dot{\phi} + 3H\dot{\phi})\Phi_k + 4\dot{\phi}\Phi_k. \] (3.7)

The analytic form of the solutions of the above equations are known in both the limits of \( k \to 0 \) and \( k \to \infty \). During inflation, metric perturbations exhibit adiabatic growth, \( \Phi_k \approx cH/H^2 \), after the first Hubble crossing \( (k \lesssim aH) \). During reheating, the super-Hubble modes \( (k \ll aH) \) are nearly constant in the single field case. In contrast, the solutions for the small-scale modes \( (k \gg aH) \) can be described by \( \Phi_k \approx \phi \left( c_1 e^{ik\eta} + c_2 e^{-ik\eta} \right) \), which shows adiabatic damping during reheating.

The system of scalar metric fluctuations \( \Phi_k \) and inflaton fluctuations \( \delta \phi_k \) can also be described in terms of the Mukhanov-Sasaki variable \( Q_k \), defined by

\[ Q_k \equiv \delta \phi_k + \frac{\dot{\phi}}{H}\Phi_k, \] (3.8)

which satisfies the equation

\[ \ddot{Q}_k + 3H \dot{Q}_k + \left[ \frac{k^2}{a^2} + \frac{d^2V}{d\phi^2} + \left( \frac{H}{\dot{\Phi}} + 3H \right) \right] Q_k = 0. \] (3.9)

The study of this equation is particularly convenient, since, contrary to the equation of motion for \( \Phi_k \) alone, it is nonsingular during the oscillations of the inflaton field. The gravitational potential \( \Phi_k \) is then related to \( Q_k \) by

\[ \frac{k^2}{a^2} \Phi_k = \frac{4\pi \phi^2}{M_p^2 H} \left( \frac{H}{\dot{\phi}} Q_k \right)^* \] (3.10).

During inflation, modes of cosmological interest initially (at \( t = t_0 \)) satisfy \( k \gg aH \), and their equation of motion is that of a free field in an expanding Universe. Quantization of this last quantity is thus straightforward. In the initial vacuum state only positive frequency waves are present (see [34] for more details), so that we take as initial conditions

\[ Q_k (t_0) = \frac{1}{a(t_0)} k^{1/2} e^{i\alpha_0}, \quad \dot{Q}_k (t_0) = -i k^{1/2} a(t_0) e^{i\alpha_0}, \] (3.11)

where \( \alpha_0 \) is an arbitrary phase.

As long as the modes remain much smaller than the Hubble scale they evolve as plane waves. When their physical lengths grow to of order the Hubble radius \( (k \sim aH) \), the variation of the frequency of \( Q_k \) becomes important. In fact, resonant amplification of some particular modes can occur during the coherent inflaton oscillations at reheating [10]. This phenomenon is mostly appreciated when the inflaton is non-gravitationally coupled to other fields [24] (see subsection [11B.3]). However, some resonant amplification occurs also in the single-field self coupled case with potential \( V = \lambda \phi^4/4 \) [18], as we will discuss in the next subsection.

**B. Evolution of metric perturbations during reheating**

1. \( V = \frac{1}{2} m^2 \phi^2 \)

Let us first summarize the behavior of metric fluctuations during reheating in the single field massive inflationary scenario. During reheating, from the time-averaged relation \( \langle \dot{\phi}^2 \rangle = \langle m^2 \dot{\phi}^2 \rangle \) one finds \( \dot{\phi} \approx M_p/(\sqrt{3\pi} mt) \sin mt \). Neglecting in eq. (3.3) the \( H^2 \) and \( \dot{H} \) terms, which decrease as \( \sim t^{-2} \) during reheating, the rescaled variable \( \tilde{Q}_k = \alpha^{3/2} Q_k \) evolves according to a Mathieu equation with time-dependent coefficients

\[ \frac{d^2}{dz^2} \tilde{Q}_k + (A_k - 2q \cos 2z) \tilde{Q}_k = 0, \] (3.12)

where \( z = mt/2 \), and

\[ A_k = 1 + \frac{k^2}{(ma)^2}, \quad q = \frac{1}{\sqrt{2z}}. \] (3.13)
The small $k$ modes ($k \lesssim ma$) lie in the resonance band around $A_k = 1$. However, the cosmic expansion makes $q$ smaller than unity already after one inflaton oscillation, and the resonance is not efficient. As a consequence, modes in the long wavelength limit ($k \to 0$) grow as $\tilde{Q}_k \propto a^{3/2}$ \cite{1,43}, which makes $Q_k$ and $\Phi_k$ nearly constant during reheating (for the mode $k \sim 0.1m$ one has $|k^{3/2}\Phi_k| \sim 10^{-6}$). On the contrary the sub-Hubble modes ($k \gtrsim m$) decrease due to adiabatic damping during reheating and are typically smaller than the modes which already crossed the Hubble scale by more than one order of magnitude. This behavior is shown for various wavelengths in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.pdf}
\caption{The evolution of metric perturbations for $\tilde{k} \equiv k/m = 0.1, 1, 5, 10$ in the model $V = m^2 \dot{\phi}^2/2$ with inflaton mass $m = 10^{-6} M_p$. We start integrating with an initial value, $\phi = 0.5 M_p$. The scale factor is normalized to unity at this time, and $aH$ is always of order $m$ in the time range plotted. Modes with $k \lesssim m$ exhibit adiabatic growth, and then (when reheating starts, i.e., at $mt \approx 1.5$ in the plot) $\Phi_k$ becomes nearly constant. Sub-Hubble modes ($k \gtrsim m$) instead decrease during reheating due to the adiabatic damping of the $\dot{\phi}$ term.}
\end{figure}

2. $V = \frac{1}{4} \lambda \phi^4$

Let us next consider the single field massless chaotic inflation model, with potential $V(\phi) = \lambda \phi^4/4$. As shown in \cite{13,44,45}, the evolution of $\phi$ is described (on dropping the term $\frac{1}{a} \frac{\partial a}{\partial \eta}$ from the equation of motion, eq. (3.3)) by

$$\phi \equiv \frac{\varphi}{a} = \frac{\dot{\phi}_0}{a} \operatorname{cn} \left( x - x_0, \frac{1}{\sqrt{2}} \right),$$

(3.14)

where $x \equiv \sqrt{\lambda} \varphi_0 \eta$ is the dimensionless conformal time, the suffix 0 indicates the value of the quantities at the beginning of reheating and $\operatorname{cn}(x - x_0, 1/\sqrt{2})$ is the elliptic cosine function.

Using the time-averaged relation $\langle \dot{\phi}^2 \rangle = \lambda \phi^4$, one finds $\langle H/H \rangle \propto \langle \dot{\phi}^2 \rangle \propto a^{-3}$ and $\dot{H} \propto a^{-4}$. Hence, the Hubble terms in Eq. (3.9) rapidly become negligible. Using $x$ and rescaling $\tilde{Q}_k = a Q_k$, Eq. (3.9) is reduced to

$$\frac{d^2}{dx^2} \tilde{Q}_k + \left[ \kappa^2 + 3 \operatorname{cn}^2 \left( x, \frac{1}{\sqrt{2}} \right) \right] \tilde{Q}_k = 0,$$

(3.15)

where we have defined $\kappa^2 \equiv k^2 / (\lambda \varphi_0^2)$, and set $x_0 = 0$ for simplicity. Eq. (3.15) is a Lamé equation. From the analytical study of \cite{45}, we deduce the presence of one resonance band in the interval

$$\frac{3}{2} < \kappa^2 < \sqrt{3},$$

(3.16)
characterized by an exponential growth $\tilde{Q}_k \propto e^{\mu x}$. The maximum value of the growth rate, $\mu_{\text{max}} \approx 0.03598$, occurs at $\kappa^2 \approx 1.615$. This growth dominates over the dilution from the expansion of the Universe even for the unrescaled variable $Q_k$, and hence also for $\Phi_k$. The growth of $\Phi_k$ continues until the backreaction of inflaton fluctuations shuts off the resonance. The maximal value of the metric perturbation is found numerically to be $|k^{3/2} \Phi_k| \approx 5 \times 10^{-5}$. Modes outside the interval (8.16) do not exhibit nonadiabatic growth, unless rescattering effects are taken into account [14,18].

3. $V = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g^2 \phi^2 \chi^2$

We now discuss the situation in which a massless inflaton $\phi$ is nongravitationally coupled to another scalar field $\chi$, with conformally invariant potential $V = \lambda \phi^4/4 + g^2 \phi^2 \chi^2/2$. This model was studied in detail, in the absence of metric perturbations, in Refs. [44,45], showing the presence of an infinite number of strong resonance bands for $\chi$ as a function of

$$R \equiv g^2/\lambda.$$ (3.17)

The conformal invariance allows exact Floquet theory to be used to prove analytically that metric fluctuations can exhibit parametric amplification [26]. This makes the model a very useful testing ground for a number of issues [27] including the possible formation of primordial black holes in some parameter space regions [28]. Here we briefly summarize those of the above results which are relevant for the calculations of the next section.

As before, it is convenient to introduce a Mukhanov-Sasaki variable for each field, $Q_k^{(i)} \equiv \delta \varphi^{(i)} + (\varphi^{(i)}/H) \Phi_k$, $i = \phi, \chi$ which satisfy a simple generalization of eq. (3.9) [26,27].

Using eq. (3.14) and rescaling all variables as $\tilde{F} \equiv a F$, the equation for $\tilde{Q}_k^\chi$ reads [26]

$$\frac{d^2 \tilde{Q}_k^\chi}{dx^2} + \left[ \kappa^2 + \frac{g^2}{\lambda} \chi^2 \left( x, \frac{1}{\sqrt{2}} \right) \right] \tilde{Q}_k^\chi = -2 \frac{g^2}{\lambda \phi_0} \chi^2 \left( x, \frac{1}{\sqrt{2}} \right) \tilde{\chi} \tilde{Q}_k^\phi + M_{\phi \chi} \tilde{Q}_k^\phi + M_{\chi \chi} \tilde{Q}_k^\chi,$$ (3.18)

where

$$M_{\phi \phi} = \frac{8 \pi}{a^2 M_p^2} \left[ \frac{1}{a H} \left( \varphi'_1 - \frac{a'}{a} \varphi_1 \right) \left( \varphi'_2 - \frac{a'}{a} \varphi_2 \right) \right].$$ (3.19)

Here prime denotes a derivative with respect to $\eta$. In the early stages of preheating, as long as $\chi$ fluctuations are small relative to $\phi$, one can neglect the right hand side of eq. (3.18), which then reduces to the generalized Lamé equation.
studied in [14,15] (in the following, we will apply the general analytical results for the Lamé equation given in [15]). Of particular cosmological interest are the longwave modes on the scales probed by the current CMB experiments. It can be shown [15] that the resonance bands extends to very longwave modes ($\kappa \to 0$) only when the ratio $R$ is in the range

$$n \left(2n - 1\right) < R < n \left(2n + 1\right),$$

where $n$ is a positive integer. In each of these intervals, the growth rate $\mu_k$ for $\hat{Q}_k^\chi$ can be analytically found as

$$\mu_k = \frac{2}{T} \ln \left( \sqrt{1 + e^{-\pi\epsilon}} + e^{-\pi\epsilon/2} \right),$$

where $\epsilon \equiv \sqrt{2/R\kappa^2}$ and $T \simeq 7.416$ is the period of inflaton oscillations. Then we have maximal growth when $R$ is exactly $2n^2$ and $\kappa = 0$, with large characteristic exponent $\mu_{\text{max}} = (2/T)\ln(\sqrt{2} + 1) \approx 0.2377$.

It is important to recognize that the exponential growth of $\hat{Q}_k^\chi$ just discussed does not necessarily translate into a parametric amplification of the gravitational potential $\Phi_k$. In the two-field case, Eq. (3.3) is modified to

$$\Phi_k + H\Phi_k = \frac{4\pi}{M_p^2} \left( \dot{\phi}\delta\phi_k + \dot{\chi}\delta\chi_k \right).$$

When $\chi$ and $\delta\chi_k$ are vanishingly small relative to $\phi$ and $\delta\phi_k$, the evolution of $\Phi_k$ is similar to the single field case. In fact, when $R \gg 1$ the homogeneous mode $\chi$ as well as the longwave $\delta\chi_k$ modes ($k \to 0$) are exponentially suppressed during inflation [24–29] due to the large effective mass $g\phi$ relative to the Hubble parameter (see also Ref. [18,19]). This can be explicitly seen by considering the equation of motion for $\delta\chi_k$, which for super-Hubble modes reads

$$\delta\chi_k + 3H\delta\chi_k + g^2\phi^2\delta\chi_k \simeq 0.$$  

Hence, during inflation, the amplitudes of the long-waves modes evolve as $20$ $|\delta\chi_k| \propto a^{-(3/2-\nu)}$, with $\nu = \text{Re}\left[9/4 - 3RM_p^2/\left(2\pi\phi^2\right)\right]^{1/2}$. This confirms that the inflationary suppression of the long-waves modes becomes more important as $R$ increases. In addition, considering backreaction in the Hartree approximation, one can show $29$ that for $R \gtrsim 8$ the growth of field perturbations shuts off the resonance before the super-Hubble metric perturbations ($k \to 0$) begin to be amplified. Hence, for large $R$, the evolution of the gravitational potential $\Phi_k$ on the scales probed by CMB experiments does not significantly differ from the “standard” one described in [14] unless rescattering is taken into account. For these modes, parametric amplification is expected to be effective in the first resonance band $\left(3.20\right)$ even when backreaction is considered $20,29$.

[FIG. 3: The evolution of metric perturbations $\Phi_k$ and field variances $\langle \delta\chi^2 \rangle$ and $\langle \delta\phi^2 \rangle$ for $\kappa = 0.1, 3$ in the potential $V = \lambda\phi^4/4 + g^2\phi^2\chi^2/2$ with $g^2/\lambda = 2$ and $\lambda = 10^{-13}$. We start integrating with initial values $\phi = 0.1M_p$ and $\chi = 5 \times 10^{-6}M_p$, corresponding to $\chi = 10^{-3}M_p$ at $\phi = 4M_p$ (i.e., 55 e-folds before the end of inflation). The final results are weakly dependent on the choice of the initial $\chi$ as long as $10^{-6}M_p < \chi < M_p$ at $\phi = 4M_p$ 20,29].
In Fig. 3 we plot the evolution of $\Phi_k$ for the two modes $\kappa = 0.1$ and $\kappa = 3$ in the $R = 2$ case. We find that metric perturbations begin to grow after the $\delta\chi_k$ fluctuations are sufficiently amplified. The growth of $\Phi_k$ ends around $x = 90$ when the backreaction terminates the resonance, after which perturbations reach the plateau region found in Fig. 3. While only long-wave modes are necessary when comparing with CMB measurements, particle production is also induced by modes $\Phi_k$ with larger momentum. The sub-Hubble modes ($k > aH$) at the beginning of reheating are not severely suppressed during inflation, and therefore the modes $\delta\chi_k$ on those scales can be amplified to large values during preheating. However, from $m_\chi = g\langle \phi \rangle$ we see that large values of $R = g^2/\lambda$ results in a suppression of the homogeneous mode $\chi$ during inflation. This also suppresses amplification of $\Phi$ from the rhs of equation (3.22).

Indeed, for $R \gg 1$ the dominant effect for the source of metric perturbations is of second order in nonadiabatic (isocurvature) field perturbations [17]. To estimate this growth, we consider the curvature perturbation on uniform-density hypersurfaces [43, 50], denoted by $\zeta$. This quantity is related to the gravitational potential $\Phi$ via [50]

$$- \zeta = R + \frac{2\rho}{3(\rho + p)} \left( \frac{k}{aH} \right)^2 \Phi,$$

(3.24)

where $R \equiv \Phi - (H/\dot{H})(\dot{\Phi} + H\Phi)$ is the comoving curvature perturbation, and $\rho$ and $p$ are the energy and density, respectively. The variation of $\zeta$ occurs on large scales in the presence of a nonadiabatic contribution to the pressure perturbation, $\delta p_{\text{nad}} = \dot{p}(\delta p/\dot{\rho} - \delta\rho/\dot{\rho})$. One can estimate the second order contribution to $\zeta$, denoted $\zeta_{(2)}$, as [17]

$$\zeta_{(2)} = \frac{3H}{\dot{\rho}} \int \delta p_{\text{nad}} \dot{H} dt \simeq \frac{1}{\phi^2} \int \left( \frac{2\lambda\phi^3}{3H\phi} \right) g^2 \phi^2 \delta\chi^2 H dt,$$

(3.25)

where we used the time averaged relation, $\delta\chi^2 \simeq m_\chi^2 \delta\chi_0^2 \simeq g^2 \phi^2 \delta\chi^2$. We find that the second order term, $g^2 \phi^2 \delta\chi^2$, can induce a variation of $\zeta$, while the first order term including the homogeneous $\chi$ is vanishingly small for $R \gg 1$. From eq. (3.25) we find

$$\left| k^{3/2} \zeta_{(2)}(k) \right| \simeq \frac{2\pi}{\phi^2} \int \left( \frac{2\lambda\phi^3}{3H\phi} \right) g^2 \phi^2 \sqrt{P_{\delta\chi^2}} H dt,$$

(3.26)

where $P_{\delta\chi^2}$ is defined by $P_{\delta\chi^2} \equiv \frac{k^3}{2\pi^2} |\delta\chi_k|^2$. Following Ref. [17], the power spectrum of the $\delta\chi_k$ fluctuation is estimated as

$$P_{\delta\chi_k} \equiv \frac{k^3}{2\pi^2} |\delta\chi_k|^2 \simeq \frac{H_0}{g\phi_0} \left( \frac{H_0}{2\pi} \right)^2 \left( \frac{\kappa}{\kappa_0} \right)^3 e^{2\mu_\kappa^2} a^2.$$

(3.27)

Here $\kappa_0 \simeq 0.15$ is the mode corresponding to the horizon size at $\phi = 0.1M_p$. From eq. (3.27) one finds

$$P_{\delta\chi_k} \equiv \frac{k^3}{2\pi^2} |\delta\chi_k|^2 \simeq \frac{k^3}{2\pi} \int_0^{\kappa_0} |P_{\delta\chi}(|k'|)P_{\delta\chi}(|k-k'|)| d^3k' \simeq \frac{H_0}{g\phi_0} \left( \frac{H_0}{2\pi} \right)^2 \left( \frac{\kappa}{\kappa_0} \right)^3 \frac{M(\kappa, x)}{a^4},$$

(3.28)

where

$$M(\kappa, x) \equiv \left( \frac{\sqrt{\lambda\phi_0}}{\kappa_0} \right)^3 \int_0^{\epsilon_c} d\epsilon' \int_0^\pi d\theta e^{2(\mu_{\kappa'}+\mu_{\kappa-x})\kappa' x} \kappa'^2 \sin \theta \approx 2 \left( \frac{\sqrt{\lambda\phi_0}}{\kappa_0} \right)^3 \int_0^{\epsilon_c} d\kappa' e^{4\mu_x \kappa' x} \kappa'^2 + O(\kappa^2).$$

(3.29)

Note that we expanded the term $\mu_{\kappa-\kappa'}$ around $\mu_{\kappa'}$ in eq. (3.29). Due the $\kappa^3$ factor in eq. (3.28), the spectrum $P_{\delta\chi_k}$ vanishes in the large scale limit ($k \to 0$), which implies that also second order effects in field perturbations do not induce large metric perturbations on the scales probed by CMB measurements [17]. In contrast, on sub-Hubble scales at the beginning of preheating ($\kappa \sim \kappa_0$), $P_{\delta\chi_k}$ is nonvanishing. In this case parametric excitation of the $\chi$ fluctuation can lead to a growth of $\zeta$ and $\Phi$. 

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As an example, we plot in Fig. 4 the evolution of $|k^{3/2}\Phi_{(2)}|$ and $|k^{3/2}\zeta_{(2)}|$ for $\kappa = 0.1$ in the case of $g^2/\lambda = 5000$. $|k^{3/2}\Phi_{(2)}|$ is evaluated by making use of eqs. (3.24) and (3.26). Note that the final field variances are smaller than in the $g^2/\lambda = 2$ case, due to the stronger backreaction effect (c.f. Fig. 3). The growth of nonadiabatic perturbations continues until backreaction ends the resonance. Although the final value of $|k^{3/2}\Phi_{k}|$ is somewhat smaller than that of $|k^{3/2}\Phi_{k}|$ in the $g^2/\lambda = 2$ case, the fermion number density (2.16) can be larger due to the contribution of higher momentum modes, as we will see in the next section.

IV. NUMERICAL RESULTS FOR THE PRODUCTION OF FERMIONS

A. Quantum to classical transition

The results of section II refer to particle production by a classical external source. On the contrary, the relation (3.10) of the previous section relates the modes of the decompositions

$$\Phi(\eta, x) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \Phi_k(\eta) e^{ik\cdot x} a(k) + \Phi^*_k(\eta) e^{-ik\cdot x} a^\dagger(k) \right],$$

$$Q^{(i)}(\eta, x) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{3/2}} \left[ Q_k^{(i)}(\eta) e^{ik\cdot x} a(k) + Q^{*(i)}_k(\eta) e^{-ik\cdot x} a^\dagger(k) \right],$$

to the annihilation and creation operators $a(k)$ and $a^\dagger(k)$. As is well known, metric perturbations arise from quantum fluctuations during inflation and some of them eventually undergo a transition to classicality. A consistent treatment of fermion production requires the integral (2.16) to be limited to those modes of $\Phi$ that have grown enough to become classical.

A simple criterion for classicality is given in Ref. [51]. Classicality is expected to be a good approximation when the following condition is satisfied:

$$\Delta_k^{(i)} \equiv \left| Q_k^{(i)} a \right| / \left| Q_k^{(i)} a^\dagger \right| \gg 1 \quad i = \phi, \chi$$

(4.3)
The quantity $\Delta_k^{(i)}$ is related to the uncertainty in the determination of $\langle \Delta Q^{(i)} \Delta \Pi^{(i)} \rangle$, where $\Pi^{(i)}$ is the conjugate momentum of $Q^{(i)}$. In the case of pure vacuum fluctuations, the equality $\Delta_k^{(i)} = 1$ holds, in accordance with the rules of quantum mechanics. Notice that this is the case for the initial states (3.11). A much bigger uncertainty (4.3) is associated to large fluctuations of classical nature.

**FIG. 5:** The quantum to classical transition as signaled by the evolution of the quantity $\Delta_k^\phi$ during inflation and subsequent reheating stage in the single field model, $V = \lambda \phi^4/4$. The mode $k = a_0 H_0$ leaves the Hubble scale at $\phi = 1.5 M_p$ (i.e., at the initial time plotted). $\Delta_k^\phi$ grows significantly from unity after Hubble radius crossing. In this figure the end of inflation corresponds to $\bar{t} \equiv \sqrt{\lambda \phi_0} t \approx 10$.

**FIG. 6:** The spectra of $\Delta_k^\chi$ when the variance $\langle \delta \chi^2 \rangle$ reaches its maximum value in the two-field model of preheating, $V = \lambda \phi^4/4 + g^2 \phi^2 \chi^2 / 2$, for different values of $R = g^2 / \lambda$. The small $k$ modes ($\kappa < 0.5$) are sufficiently enhanced by parametric resonance for $R = 2 n^2$, which makes the perturbations highly classical. In the case of $R < 20$ we find that $\Delta_k^\chi$ is very close to unity for the modes, $\kappa \gtrsim 3$. Therefore the cut-off $\kappa_c = 3$ is justified in evaluating the occupation number of fermions for $R < 20$. 
In Fig. 5 the time evolution of $\Delta_k^\phi$ is shown in the model $V = \lambda \phi^4/4$, for some particular values of $k$. We notice that $\Delta_k^\phi$ begins to increase around the region where a mode leaves the Hubble scale during inflation. This is expected to occur quite generically, irrespective of the specific form of the inflaton potential. On the contrary, modes which are always inside the Hubble scale typically conserve their quantum nature ($\Delta_k^\phi \simeq 1$), unless some physical (model dependent) process drives them classical. Among these processes, nonperturbative particle creation during preheating can drive to a classical level the sub-Hubble modes in the resonance bands discussed in the previous section [14,15,18,30].

In Fig. 6 we show instead the spectrum $\Delta_k^\chi$ in the two-field model of preheating (III B 3) for some values of $R$ (the largest time plotted corresponds to the moment at which Hartree approximation breaks down). The case $R = 3$ reproduces the results of the one massless field case upon identification of $\chi$ with the fluctuations of the field $\phi$. For this value, Fig. 6 shows the enhancement of the modes in the resonance band (4.16). In the two-field case, strong resonance occurs in the parameter ranges described by Eq. (3.20). In the center of the resonance bands, $R = 2n^2$, small $k$ modes ($\kappa \lesssim 0.5$) are efficiently excited, while the modes with $\kappa \gtrsim 3$ are regarded as quantum fluctuations ($\Delta_k^\chi \simeq 1$) as long as $R$ is not much larger than unity (this evolution does not include rescattering effects which become important in the nonlinear regime).

In the next section we numerically evaluate the formula for the occupation numbers reported in Sec. II. In the calculation we do not include those modes which are not excited to a classical level ($\Delta_k^\phi \simeq 1$). For modes which become classical, from the two decompositions (2.17) and (4.1) we get

$$\chi \rightarrow \eta \times \eta,$$

(4.4)

\[ \Phi(k) = \frac{V^{1/2}}{\sqrt{2} (2 \pi)^{3/2}} \Phi_k. \]

B. Particle production

We can combine eqs. (2.16) and (4.4) to evaluate the number density of produced fermions in terms of the modes $\Phi_k$ computed in the previous section. Moreover, we assume the metric perturbations to be statistically isotropic and homogeneous, so that $\Phi_k = \Phi_{|k|}$. In this case, the final particle density is given by

$$N_\phi \equiv \frac{N_2}{V} = \frac{1}{15 (2 \pi)^4 a^3} \int_0^\infty dp_0 \int_0^{p_0} dp \int_{\eta_f}^{\infty} d\eta \left| e^{i p_0 \eta} \Phi_p(\eta) \right|^2.$$

(4.5)

As discussed in the previous subsection, the integral over $p \equiv |p|$ must be restricted to the region where the perturbations are classical. We have seen in section III that this typically occurs for $p$ smaller than a cut-off value $p_c$ which approximately corresponds to modes which never crossed the Hubble scale (or, in the massless inflaton case, to the highest resonance band excited). In the numerical computation of $N_2$ we also integrated $p_0$ up to $p_c$. Higher frequencies are not expected to give a sizable contribution to eq. (4.5), since the rapidly oscillating phase $e^{i p_0 \eta}$ averages to nearly zero the time integral.

In the numerical calculations we take as the upper limit $\eta_f$ of the time integral the moment in which backreaction effects start to shut off the resonance. In the single self-coupling case this occurs at $x_f \equiv \sqrt{\lambda \phi(t_{co}) \eta_f} \simeq 500$, while in the two-field case the precise value of $\eta_f$ is a function of the ratio $R$. Note that this is just an approximate calculation, since the above integral is defined rigorously only for the asymptotic future, $\eta_f \to \infty$, with vanishing perturbations $\Phi_k \to 0$. That is, our numerical calculation is approximately

$$n_\phi \equiv a^3 N_\phi \simeq \frac{1}{15 (2 \pi)^4 a} \int_0^{p_c} dp_0 \int_0^{p_0} dp \int_{\eta_f}^{\infty} d\eta \left| e^{i p_0 \eta} \Phi_p(\eta) \right|^2.$$

(4.6)

Before presenting the numerical results, we briefly consider the occupation number of massive fermions produced by the homogeneous FLRW expansion, eq. (2.11). For sufficiently small masses, the final occupation number is still (approximately) given by (4.6), since the Bogolyubov coefficient $\beta$ is very close to zero in this limit (see eq. (2.8) and (2.13)). The term (2.11), although quadratic in $\beta$, is not suppressed by the small perturbations. Thus, by solving the equation

$$N_0 (m_\phi) = N_2 (m_\phi = 0)$$

we can estimate up to which mass $m_\phi$ the productions by the inhomogeneities is comparable with that from the homogeneous expansion.
In this expression, \( H \) (shown in Fig. 2), one can see that the main contribution to particle production is given by the resonance band \( (3.16) \). From simple estimates (for example combining eq. (4.6) with the values for the perturbations \( \Delta \phi^2 \)), the parameter \( \alpha \) is instead the exponent appearing in the expansion law \( a(t) \propto t^\alpha \) when \( H(t) = m_\psi \) (as is well known, the coherent inflaton oscillations give effective matter domination, \( \alpha = 2/3 \), in the \( V = m^2 \phi^2/2 \) case, and effective radiation domination, \( \alpha = 1/2 \), for \( V = \lambda \phi^4/4 \)). For the first coefficient of eq. (4.8), the two values \( C_{2/3} \approx 3 \times 10^{-3} \) and \( C_{1/2} \approx 10^{-2} \) have been numerically found [52].

\[
\frac{H_0}{m_\psi} \equiv n_\psi \text{hom} \equiv a^3 N_\psi / V \simeq C_\alpha m_\psi^3 \left( \frac{H_0}{m_\psi} \right)^{3\alpha}.
\]

In this expression, \( H_0 \) denotes the value of the Hubble rate at the initial time \( t_0 \), where the scale factor \( a(t_0) \) is normalized to unity. The parameter \( \alpha \) is instead the exponent appearing in the expansion law \( a(t) \propto t^\alpha \) when \( H(t) = m_\psi \).

The calculation of \( N_\psi \) has been performed numerically in [52] (see also references therein). For low fermionic masses, the gravitational production from the homogeneous expansion mainly occurs when the Hubble expansion rate equals \( m_\psi \). After this time, the comoving fermion number density is given by [52]

\[
n_\psi \text{hom} \equiv a^3 N_\psi / V \simeq C_\alpha m_\psi^3 \left( \frac{H_0}{m_\psi} \right)^{3\alpha}.
\]

1. \( V = \frac{1}{2} m^2 \phi^2 \)

Let us first present the numerical results for the massive inflaton case. The precise value deduced from the integral (4.6) is sensitive to the cut-off \( p_c \). To discriminate which cut-off should be taken, we consider the quantity \( \Delta \phi^2 \). We found that \( \Delta \phi^2 \) approaches unity around \( k = 10 m \), so we chose \( p_c = 10 m \). In this case, for the comoving occupation number (normalizing \( a = 1 \) at \( \phi = 0.5 M_p \)) we have numerically found

\[
n_\psi \equiv a^3 N_\psi \simeq 3 \times 10^{-14} m^3
\]

(just to give an example on how the final result is sensitive to \( p_c \), we found \( n_\psi \simeq 1 \times 10^{-14} m^3 \) for \( p_c = 5 m \) and \( n_\psi \simeq 1 \times 10^{-13} m^3 \) for \( p_c = 20 m \)).

Equating (4.6) with (4.8), the mass \( m_\psi \) below which the production from inhomogeneities dominates over the one from the homogeneous expansion is estimated as

\[
m_\psi = \frac{n_\psi}{C_{2/3} H_0} \simeq 10^2 \text{ GeV}.
\]

Of more physical interest is the fermionic abundance deduced from (4.9). By assuming an instantaneous inflaton decay into a thermal bath characterized by the reheat temperature \( T_{rh} \), the particle density divided by the entropy density, \( s \), is given by

\[
Y_\psi (T_{rh}) \equiv \frac{N_{\psi}}{s} (T_{rh}) \simeq 10^{-29} \frac{T_{rh}}{10^3 \text{ GeV}}.
\]

In the case of other light gravitational relics, such as gravitini or modulini, the primordial abundance is severely constrained by the successful predictions of Big Bang nucleosynthesis. For masses of order the electroweak breaking scale, the limit is about \( Y \lesssim 10^{-13} \) [53]. In the present case, eq. (4.11) gives a much smaller abundance.

2. \( V = \frac{1}{4} \lambda \phi^4 \)

In the quartic case, both super-Hubble modes and the modes in the interval (3.16) contribute to the particle production, eq. (4.6). From simple estimates (for example combining eq. (4.6) with the values for the perturbations shown in Fig. 2), one can see that the main contribution to particle production is given by the resonance band (3.16).

In calculating the total number densities of produced fermions due to inhomogeneities, we take the cut-off value in momentum space at \( p_c = 3 \), over which \( \Delta \phi^2 \) approaches unity. Then the final particle density reached at the end of preheating is numerically found to be

\[
n_\psi \simeq 1 \times 10^{-13} (\sqrt{\lambda \phi_0})^3.
\]

which is larger than in the massive inflaton case.

Proceeding as in the previous subsection we find that the production from metric perturbations dominates over that from the homogeneous expansion as long as the fermions have masses smaller than
\[ m_\psi = \frac{1}{H_0} \left( \frac{n_\psi}{C_{1/2}} \right)^{2/3} \simeq 10^5 \text{ GeV} . \] (4.13)

The final abundance is also greater than in the previous case (thus confirming that the integral (4.6) is dominated by the modes in the resonance band),

\[ Y_\psi (T_{rh}) \simeq 10^{-22} . \] (4.14)

Nevertheless it is still too small to have any cosmological effect.

\[ \mathcal{V} = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g^2 \phi^2 \chi^2 \]

Since this potential leads to an infinite number of resonance bands it is the case of most interest. In Fig. 7, our numerical results for fermionic production are presented as a function of \( R \equiv g^2 / \lambda \), for \( R \leq 10 \). This is obtained by solving the linearized equation (3.22) using the Hartree approximation for field perturbations. By comparison with the single field case \( (g = 0, \text{ here reproduced by the choice } R = 3) \) the two equations (4.13) and (4.14) are generalized to

\[ m_\psi \simeq 5 \times 10^{13} \tilde{n}_\psi^{2/3} \text{ GeV} , \quad Y_\psi (T_{rh}) \simeq 10^{-9} \tilde{n}_\psi , \] (4.15)

where \( \tilde{n}_\psi \equiv n_\psi / (\sqrt{\lambda} \phi_0)^3 \).

The qualitative behavior of Fig. 7 is readily understood from the structure of resonance (3.20). In the range of \( R \) plotted, we have maximal production when \( R \) is in the center of the first resonance band, in which case \( n_\psi \) is found to be \( \tilde{n}_\psi \simeq 5 \times 10^{-10} \), which is about 5000 times larger than in the single-field case. In the second resonance band, particle production is slightly smaller than in the \( R = 2 \) case.

\[ \text{FIG. 7: The comoving number density, } \tilde{n}_\psi = n_\psi / (\sqrt{\lambda} \phi_0)^3 , \text{ vs the ratio } g^2 / \lambda , \text{ for } g^2 / \lambda \leq 10. \text{ Note that this is based on the linear calculation using eq. (3.22), which does not include the second order effect in field perturbations. We take } \chi = 10^{-3} M_p \text{ at } \phi = 4 M_p. \text{ We see that particle creation is enhanced in the super-Hubble resonance bands (3.20).} \]

As \( R \) increases, the inflationary suppression of \( \chi \) becomes more and more relevant, and particle production from first order perturbations is consequently reduced. Nevertheless, the second order effect of field perturbations described in subsection III B 3 can lead to the excitation of metric perturbations on sub-Hubble scales. Just for indicative purposes, we report a couple of our numerical results in this range. For large \( R \), the resonance bands cover only modes up to \( 10^{15} \)

\[ \kappa \lesssim \left( \frac{R}{2 \pi^2} \right)^{1/4} . \] (4.16)
In the center of the 10-th instability band, $R = 200$, one finds resonant amplification up to $\kappa \lesssim 1.78$, as can be appreciated in Fig. 1. By implementing the analysis reported in subsection III B 3 we find $\bar{n}_\psi \approx 6 \times 10^{-11}$. As $R$ increases, higher momentum modes contribute to the growth of metric perturbations. For example, eq. (4.16) gives $\kappa \lesssim 4$ in the $R = 5000$ case. As a consequence, this enhances fermionic production. The comoving occupation number in this case is found to be $\bar{n}_\psi \approx 1 \times 10^{-7}$, $Y_\psi (T_{\text{rh}}) \approx 10^{-16}$.

It is however important to mention that for $R \gg 1$ the whole analysis becomes very delicate. For large coupling $g$, the backreaction effect of produced particles ends resonant amplification of field perturbations earlier. In addition, it was found in Ref. [14] that the final field variances get smaller if rescattering of the $\delta \chi$ fluctuations is taken into account in the rigid FLRW spacetime ($\Phi = 0$). On the contrary, the effect of rescattering can lead to the excitation of inflaton fluctuations through the amplification of $\delta \chi$ fluctuations. Since (contrary to the field $\chi$) the homogeneous inflaton component is not suppressed during inflation, the gravitational potential can then acquire a potentially large additional source (see eq. (3.22)). Finally, the anisotropic stress is expected to be important in the nonlinear regime, in which case the relation $\Phi = \Psi$ no longer holds [3]. The study of these effects is certainly worth separate detailed investigation.

V. CONCLUSIONS

In the present work we have discussed gravitational creation of fermions by inhomogeneous scalar perturbations of a Friedmann-Lemaître-Robertson-Walker (FLRW) Universe. Massless fermions are conformally coupled to this background, and thus are not generated by the homogeneous gravitational field. As a consequence, gravitational production of light particles is solely induced by metric perturbations which break the conformal flatness of the background. Although the small size of perturbations on very large scales indicated by CMB experiments seems to suggest a small production, a precise computation is still worth while for light gravitational relics.

This is particularly true in scenarios in which metric perturbations undergo parametric amplification during preheating at the end of inflation. To examine this issue more quantitatively, we have studied some particularly simple models of chaotic inflation. The simplest example is a single massive inflaton field ($V = m^2 \phi^2/2$), where parametric resonance is actually absent. In this model, metric perturbations are nearly constant during reheating for modes which left the Hubble scale during inflation, whereas the modes deep inside the Hubble scale exhibit adiabatic damping. As may be expected, particle production is negligibly small in this situation.

We then studied the massless self-coupled inflaton field ($V = \lambda \phi^4/4$) case, in which parametric resonance occurs for modes in a small band in momentum space near the Hubble scale. As a consequence, particle production is enhanced. Moreover, for a massless inflaton the background energy density decreases more quickly with respect to the massive case, and this results in a larger abundance at reheating of the particles generated by the inhomogeneities. Despite these two effects, the final result for the production is however still well below the limit from nucleosynthesis.

More efficient production is expected when more fields are present, such as the model characterized by the potential $V = \lambda \phi^4/4 + g^2 \phi^2 \chi^2/2$. In this case, the crucial parameter which determines the resonance bands is $R \equiv g^2/\lambda$. In particular, the longwavelength modes of the second field $\chi$ are parametrically amplified whenever $R \gtrsim 2 n^2$ with integer $n$. At linear order in field perturbations, these modes are coupled to metric perturbations through a term proportional to $\dot{\chi} \delta \chi$. For relatively small $R$, metric perturbations are amplified by this term, and particle production is strongly enhanced.

For $R > 10$ the large $\chi$ effective mass ($m_\chi = g \langle \phi \rangle \propto R^{1/2}$) causes the $\chi$ field to be exponentially suppressed during inflation. As a consequence, for large $R$ metric perturbations mainly increase due to second order field fluctuations [17]. Particle production may be expected to become more significant at very large $R$. This is indeed suggested by our numerical results, although in the cases considered the final abundance is still below the nucleosynthesis bounds.

Our numerical simulations include the backreaction of the fluctuations on the background evolution in the Hartree approximation. Rescattering effects – mode-mode coupling between fields – are instead neglected. This may considerably affect the final abundances, particularly in the large $R$ limit where the mode coupling becomes particularly important. Strong rescattering effects imply an earlier end to preheating, which may decrease the final values of field and metric perturbations. On the other hand, rescattering leads to amplification of modes outside the resonance bands. In particular, inflaton field fluctuations can be significantly amplified. Since (contrary to the field $\chi$) the homogeneous inflaton component is not suppressed during inflation, metric fluctuations may be strongly enhanced by the source term proportional to $\dot{\phi} \delta \phi_k$ (analogous to the $\dot{\chi} \delta \chi_k$ term considered above). This would be reflected in more particle creation.

Stronger production from inhomogeneities may also occur in different contexts. In the multi-field inflationary scenario, the gravitational potential can exhibit nonadiabatic growth during inflation [21,23], especially when the second field $\chi$ has a negative nonminimal coupling to the Ricci scalar [36,37]. In addition, it was shown that the
growth of metric perturbations can be present even in the single field case in the context of non-slow roll inflation with potential gap \[58\], and during graceful exit in string cosmology \[59\]. It may be interesting to study gravitational particle production in these contexts along the lines presented in this work.

ACKNOWLEDGEMENTS

The authors thank Jürgen Baacke and Fermin Viniegra for contributions at early stages of this project and Robert H. Brandenberger for comments on the draft. ST also thanks Alexei A. Starobinsky and Jun’ichi Yokoyama for useful discussions. ST is thankful for financial support from the JSPS (No. 04942). The work of MP is supported by the European Commission RTN programmes HPRN-CT-2000-00148 and 00152. MP thanks Lev Kofman for some useful discussions. He also thanks the Canadian Institute for Theoretical Astrophysics of Toronto and the Astrophysics group of Fermilab for their friendly hospitality during the early stages of this work.

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