CONTINUED $g$–FRACTIONS AND GEOMETRY OF BOUNDED ANALYTIC MAPS

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ABSTRACT. In this work we study qualitative properties of real analytic bounded maps. The main tool is approximation of real valued functions analytic in rectangular domains of the complex plane by continued $g$–fractions of Wall [8]. As an application, the Sundman-Poincaré method in the Newtonian three–body problem is revisited and applications to collision detection problem are considered.

1. THE CONTINUED $g$–FRACTION REPRESENTATION FOR REAL ANALYTIC BOUNDED FUNCTIONS

By $R_{T,B} \subset \mathbb{C}$ we denote the open domain which is the interior of the rectangle with vertices at the points $T + iB, T - iB, -T + iB, -T - iB, T, B > 0$ (see Fig. 1). Let $A_{M,T,B}$ be the set of all functions $f(z)$ analytic in $R_{T,B}$, real valued for $z \in I_T = (-T, T)$ and bounded in absolute value in $R_{T,B}$ by $M > 0$:

$$|f(z)| < M, \quad \forall z \in R_{T,B}.$$  

Let $\mathbb{H} = \mathbb{C}_- \cup \mathbb{C}_+ \cup (-1, +\infty)$ where $\mathbb{C}_+ = \{ z \in \mathbb{C} : Im(z) > 0 \}$, $\mathbb{C}_- = \{ z \in \mathbb{C} : Im(z) < 0 \}$. We shall construct a conformal map between the open connected sets $R_{T,B}$ and $\mathbb{H}$ using the Jacobi elliptic function $sn(z, k)$ and the theta functions $\theta_2(z, q), \theta_3(z, q)$ (see for definitions [1]).

Figure 1. Domain of analiticity $R_{T,B}$ of $f(z) \in A_{M,T,B}$.

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Let
\[ q = e^{-\pi B}, \] (1.2)
be the nome.

The corresponding real quarter-period \( K > 0 \) is defined as follows
\[ K = \frac{\pi}{2} \theta_3(0, q)^2. \] (1.3)
The elliptic modulus \( k \in (0, 1) \) is given by formula
\[ k = \frac{\theta_2(0, q)^2}{\theta_3(0, q)^2}, \] (1.4)
and defines the Jacobi elliptic function \( \text{sn}(z, k) \).

**Lemma 1.1.** Let
\[ \Phi(z) = 2\text{sn}(KzT, k) - \text{sn}(KzT, k). \] (1.5)
Then \( \Phi : R_{T,B} \to \mathbb{H} \) is conformal and maps bijectively \((-T, T)\) to \((-1, +\infty)\), \( \Phi(0) = 0 \).

The proof is straightforward and easily follows from properties of \( \text{sn}(z, k) \) described in [2], p. 119. Let
\[ \mathbb{D}_M = \{ z \in \mathbb{C} : |z| < M \}, \quad \mathbb{H}_+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}. \] (1.6)
One verifies that \( m : \mathbb{D}_M \to \mathbb{H}_+ \) defined by
\[ m(z) = \frac{M + z}{M - z}, \] (1.7)
is conformal.

The composition
\[ F = m \circ f \circ \Phi^{-1}, \quad \text{where} \quad f \in A_{M,T,B}, \] (1.8)
is then a holomorphic function in \( \mathbb{H} \) such that \( F(\mathbb{H}) \subseteq \mathbb{H}_+ \) and \( F : (-1, +\infty) \to (0, +\infty) \).

According to Theorem of Wall [3], p. 279 there exist \( \mu_0 > 0 \) and the sequence of real numbers
\[ g_i \in [0, 1], \quad i \geq 1, \] (1.9)
such that
\[ F(z) = \mu_0 \sqrt{1 + z} \{ g_1, g_2, \ldots |z \}, \quad z \in \mathbb{H}, \] (1.10)
where
\[ g(z) = \{ g_1, g_2, \ldots |z \} = \frac{1}{1} + \frac{g_1 z}{1} + \frac{(1 - g_1)g_2 z}{1} + \frac{(1 - g_2)g_3 z}{1} + \cdots, \] (1.11)
is a continued \( g \)-fraction converging uniformly on compact sets of \( \mathbb{H} \) to an analytic function \( g(z), \ z \in \mathbb{H} \) (for applications of \( g \)-fractions see [4]–[7]).
Remark 1.1. \(g(z)\) is a rational function of \(z\) if and only if \(g_k \in \{0, 1\}\), for some \(k \geq 1\).

As follows from (1.8): \(f = m^{-1} \circ F \circ \Phi\) and hence the following representation for \(f(z)\) holds
\[
f(z) = M \left(1 - \frac{2}{\mu_0 \sqrt{1 + \Phi(z)} \{g_1, g_2, \ldots |\Phi(z)\} + 1}\right), \quad z \in R_{T,B}.
\] (1.12)

To simplify (1.12) we make a rescaling and obtain the new function \(\phi\) given by
\[
\phi(z) = \frac{f(z/\alpha)}{M}, \quad \alpha = \frac{K}{T},
\] (1.13)

which is holomorphic in the rectangle \(R_{K,\alpha B}\).

We note that \(|\phi(z)| < 1, \forall z \in R_{K,\alpha B}\).

Formula (1.12) then becomes
\[
\phi(z) = \left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} \{g_1, g_2, \ldots |\eta(z)\} + 1}\right), \quad z \in R_{K,\alpha B},
\] (1.14)

where
\[
\eta(z) = \frac{2\text{sn}(z, k)}{1 - \text{sn}(z, k)}.
\] (1.15)

The map \(z \mapsto \eta(z)\) is a bijection between the real intervals \((-K, K)\) and \((-1, +\infty)\), \(\eta(0) = 0\) what will be used later.

We define the truncated continued \(g\)-fraction as the \(n\)-order approximation of (1.11):
\[
\{g_1, g_2, \ldots, g_n|z\} = \frac{1}{1} + \frac{g_1 z}{1} + \frac{(1 - g_1)g_2 z}{1} \cdots \frac{(1 - g_{n-1})g_n z}{1},
\] (1.16)

which is a rational function of \(z\) analytic in \(\mathbb{H}\).

The next theorem gives the \textit{a priori} bounds for the \(g\)-fraction (1.11).

**Theorem 1.1.** (5)

\(a)\) Let \(k = 2n + 1, n = 0, 1, \ldots\), then
\[
A_k(z) \leq g(z) \leq B_k(z), \quad -1 < z < +\infty,
\] (1.17)

where
\[
A_k(z) = \{g_1, g_2, \ldots, g_k|z\}, \quad B_k(z) = \{g_1, g_2, \ldots, g_k, 1|z\}.
\] (1.18)

\(b)\) Let \(k = 2n, n = 1, 2, \ldots\), then
\[
A_k^+(z) \leq g(z) \leq B_k^+(z), \quad 0 \leq z < +\infty,
\] (1.19)

\[
A_k^-(z) \leq g(z) \leq B_k^-(z), \quad -1 < z < 0,
\] (1.20)

where
\[
A_k^+(z) = \{g_1, g_2, \ldots, g_k, 1|z\}, \quad B_k^+ = \{g_1, g_2, \ldots, g_k|z\},
\] (1.21)

\[
A_k^- = \{1, g_1, g_2, \ldots, g_k|z\}, \quad B_k^- = \{1, g_1, g_2, \ldots, g_k|z\}.
\] (1.22)
and $A_k^{-} = B_k^{+}$, $B_k^{-} = A_k^{+}$.

Using the above formulas we write below the rational a priori bounds for the $g$–fraction (1.11) corresponding to $k = 1, 2, 3$:

Case $k = 1$.

$$A_1(z) = \frac{1}{1 + g_1 z}, \quad B_1(z) = \frac{1 + (1 - g_1) z}{1 + z}.$$  
(1.22)

Case $k = 2$.

$$A_2^+(z) = \frac{(1 - g_1 g_2) z + 1}{(1 + z)(g_1(1 - g_2) z + 1)}, \quad B_2^+(z) = \frac{g_2(1 - g_1) z + 1}{(g_1 - g_1 g_2 + g_2) z + 1}.$$  
(1.23)

$$A_2^-, B_2^- = A_2^+.$$  
(1.24)

Case $k = 3$.

$$A_3(z) = \frac{(g_3 + g_2 - g_3 g_2 - g_2 g_1) z + 1}{g_1 g_3(1 - g_2) z^2 + (g_3 + g_2 + g_1 - g_3 g_2 - g_1 g_2) z + 1}.$$  
(1.25)

$$B_3(z) = \frac{g_2(1 - g_3)(1 - g_1) z^2 + (1 + g_2 - g_3 g_2 - g_1 g_2) z + 1}{(1 + z)((g_1 + g_2 - g_3 g_2 - g_1 g_2) z + 1)}.$$  
(1.26)

The coefficients $g_p$ in formula (1.14) are defined by

$$g_p = R_p(\phi(0), \phi'(0), \ldots, \phi^{(p)}(0)), \quad p \geq 1,$$  
(1.27)

with rational expressions $R_p$ which can be found by calculation of derivatives of both sides of (1.14) and evaluating them at $z = 0$. The recurrent formulas for $R_p$ can be derived from [8], p. 203.

In view of (1.13), $\phi_n = \phi^{(n)}(0)$ are functions of derivatives $f^{(n)}(0)$:

$$\phi_n = \frac{\alpha^{-n}}{M} f^{(n)}(0), \quad \alpha = K/T, \quad n \geq 0.$$  
(1.28)

Below we give explicit expressions for $\mu_0$, $g_1$, $g_2$ in terms of $\phi_n$, $n = 0, 1, 2$:

$$\mu_0 = \frac{1 + \phi_0}{1 - \phi_0},$$  
(1.29)

$$g_1 = \frac{1 - \phi_0^2 - 2\phi_1}{2 - \phi_0^2},$$  
(1.30)

$$g_2 = \frac{1}{2} \frac{(4\phi_1^2 - 2\phi_2 - \phi_0 + \phi_0^2 + \phi_0^3 - 2\phi_2 \phi_0 - 1)(1 - \phi_0)}{(2\phi_1 - \phi_0^2 + 1)(2\phi_1 + \phi_0^2 - 1)}.$$  
(1.31)

**Corollary 1.1.** As seen from (1.30), $f'(0) > 0$ is equivalent to $g_1 < 1/2$; $f'(0) < 0$ is equivalent to $g_1 > 1/2$ and $f'(0) = 0 \Leftrightarrow g_1 = 1/2$. 
2. Bounds on the time of the first return

Applying Theorem 1.1 to the $g$–fraction in (1.12) one can derive the a priori bounds for $f(z)$ holding inside the interval $(-T, T)$. Increasing the truncation order $n$ in (1.16) one obtains more and more precise information of qualitative character about $f(z)$ once the derivatives of $f(z)$ at $z = 0$ are known. In particular, if $f(z)$ is a solution of a system of analytic differential equations, one can find often recurrent formulas to calculate derivatives $f^{(n)}(0)$ of all orders $n \geq 0$ and write the $g$–fraction representation (1.12).

Our aim is to estimate the time of return of $f(z)$ to the initial value $f(0)$ i.e to study the points $z_0 \in (-T, T), z_0 \neq 0$ such that $f(z_0) = f(0)$. To do this we will use the a priori bounds (1.22) applied to the $g$–fraction in formula (1.14). For $p = 2k + 1$ one obtains:

$$
\left( 1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)A_p(\eta(z)) + 1}} \right) \leq \phi(z) \leq \left( 1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)B_p(\eta(z)) + 1}} \right),
$$

(2.1)

for $z \in (-K, K)$.

If $p = 2k$ then

$$
\left( 1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)A_p^+(\eta(z)) + 1}} \right) \leq \phi(z) \leq \left( 1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)B_p^+(\eta(z)) + 1}} \right),
$$

(2.2)

for $z \in (0, K)$, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Bounds $r(x)A_1(x)$, $r(x)B_1(x)$ (bold line) and $r(x)A^{\pm}_1(x)$, $r(x)B^{\pm}_2(x)$ (dashed line) for $x \in (-0.9, 6)$, $g_1 = 0.7$, $g_2 = 0.3$, $r(x) = \sqrt{1 + x}$.}
\end{figure}
\[
\left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)A_p(\eta(z)) + 1}}\right) \leq \phi(z) \leq \left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)B_p(\eta(z)) + 1}}\right), \quad (2.3)
\]
for \(z \in (-K, 0]\).

**Definition 2.1.** We denote by \(A^{(k)}_{M,T,B} \subset A_{M,T,B}, k = 1, 2, \ldots\) the set of functions for which the \(g\)-fraction representation (1.12) satisfies the condition
\[
g_i \not\in \{0, 1\}, \quad \forall i = 1, \ldots, k. \quad (2.4)
\]

In the next theorem, for a given \(f \in A^{(1)}_{M,T,B}\), we will describe a neighborhood of origin in which \(z = 0\) is the only solution of \(f(z) = f(0)\).

**Theorem 2.1.** Let \(f(z) \in A^{(1)}_{M,T,B}\), \(f'(0) \neq 0\) where \(g_1\) is defined by (1.2), (1.3), (1.28) and (1.30) as a function of \(M, T, B, f(0), f'(0)\). Let \(\tau \in (-T, T), \tau \neq 0\) be the point such that \(f(\tau) = f(0)\). We define
\[
\tilde{\tau} = \frac{T}{K} |\text{sn}^{-1}(g, k)| = \frac{T}{K} \left| \int_0^{\delta} \frac{dt}{\sqrt{1 - t^2 \sqrt{1 - k^2 t^2}}} \right|, \quad (2.5)
\]
where
\[
g = \frac{1 - 2g_1}{g_1^2 + (1 - g_1)^2} \in (-1, 1), \quad (2.6)
\]
and \(k, K\) are given by (1.4) and (1.3).

Then \(0 < \tilde{\tau} < T\) and
\[
|\tau| \geq \tilde{\tau}. \quad (2.7)
\]

**Proof.** One considers (2.1) with \(p = 1\). We have \(A_1(0) = B_1(0) = 1\) and define \(t_1, t_2\) as non-zero solutions of the following algebraic equations
\[
\sqrt{1 + t_1}A_1(t_1) = 1, \quad \sqrt{1 + t_2}B_1(t_2) = 1, \quad t_1, t_2 \in (-1, +\infty). \quad (2.8)
\]

Simple algebraic calculations show that the only solutions satisfying (2.8) are given by
\[
t_1 = \frac{1 - 2g_1}{g_1^2}, \quad t_2 = \frac{2g_1 - 1}{(1 - g_1)^2}, \quad (2.9)
\]
which are related by
\[
\frac{1}{t_1} + \frac{1}{t_2} = -1. \quad (2.10)
\]
Since \(\eta(z)\) is a bijection between the intervals \((-K, K)\) and \((-1, +\infty)\) there exist unique real numbers \(T_1, T_2 \in (-K, K)\) satisfying the following equations:
\[
\eta(T_1) = t_1, \quad \eta(T_2) = t_2. \quad (2.11)
\]
As easily seen from (1.15): \( T_1 = -T_2 \) and
\[
\eta^{-1}(y) = \int_0^{\frac{2\pi}{T2}} \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}, \quad y \in (-1, 1).
\] (2.12)

We define
\[
\hat{\tau} = \frac{T}{K} |T_1| = \frac{T}{K} |\eta^{-1}(t_1)| \in (0, T).
\] (2.13)

The proof of Theorem 2.1 follows therefore directly from (2.1) and (2.8).

The next result shows that \( f(z) \in K_{M,T,B}^{(2)}, \) under some conditions on derivatives \( f^{(p)}(0), \) \( p = 0, 1, 2, \) always returns to the initial value \( f(0) \) in the interval \( (-T, T) \) i.e admits the oscillatory property.

**Theorem 2.2.** Let \( f(z) \in K_{M,T,B}^{(2)}, f'(0) \neq 0 \) where \( g_1, g_2 \) are defined by formulas (1.2), (1.3), (1.28) and (1.30), (1.31). We assume that one of the two following conditions (A) or (B) holds
\[
g_1 < \frac{1}{2}, \quad g_1^2 - 4(1 - g_1)^2(1 - g_2)g_2 \geq 0 \quad (A)
\] (2.14)
\[
g_1 > \frac{1}{2}, \quad (1 - g_1)^2 - 4g_1^2 g_2(1 - g_2) \geq 0 \quad (B)
\] (2.15)

Then there exists \( \tau \in (-T, T), \tau \neq 0 \) such that
\[
f(\tau) = f(0).
\] (2.16)

**Proof.** We consider (2.2) with \( p = 2 \) and define the following real algebraic equations
\[
\sqrt{1 + x} A_2^+(x) = 1, \quad x \in (0, +\infty), \quad (A_1),
\]
\[
\sqrt{1 + x} B_2^+(x) = 1, \quad x \in (0, +\infty), \quad (B_1),
\]
\[
\sqrt{1 + x} B_2^-(x) = 1, \quad x \in (-1, 0), \quad (\hat{A}_1),
\]
\[
\sqrt{1 + x} A_2^-(x) = 1, \quad x \in (-1, 0). \quad (\hat{B}_1),
\]

where \( A_2^- = B_2^+, B_2^- = A_2^+. \)

Making the change of variables
\[
x = -1 + t^2, \quad t \in \mathbb{R},
\] (2.17)

after some elementary transformations, it is easy to show that equations \( (A_1), (B_1) \) are equivalent respectively to quadratic equations \( (A_2) \) and \( (B_2) \) given below
\[
P_1(t) = g_1(1 - g_2)t^2 - (1 - g_1)t + g_1 g_2 = 0, \quad t \in \mathbb{R}, \quad (A_2)
\]
\[
P_2(t) = g_2(1 - g_1)t^2 - g_1 t + (1 - g_1)(1 - g_2) = 0, \quad t \in \mathbb{R}. \quad (B_2)
Remark 2.1. We notice that $P_2(t)$ is obtained from $P_1(t)$ by transformation

$$g_i \mapsto 1 - g_i, \quad i = 1, 2.$$  \hfill (2.18)

The polynomial $P_1(t)$ has two real roots $t_1^{(1)}, t_2^{(1)} \in \mathbb{R}$

$$t_1^{(1)} = \frac{1 - g_1 - \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_2^{(1)} = \frac{1 - g_1 + \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_1^{(1)} \leq t_2^{(1)},$$ \hfill (2.19)

if and only if the following condition holds

$$D_1 = (1 - g_1)^2 - 4g_1^2(1 - g_2) \geq 0.$$ \hfill (2.20)

$P_2(t) = 0$ has two real solutions $t_1^{(2)}, t_2^{(2)} \in \mathbb{R}$

$$t_1^{(2)} = \frac{g_1 - \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_2^{(2)} = \frac{g_1 + \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_1^{(2)} \leq t_2^{(2)},$$ \hfill (2.21)

if and only if

$$D_2 = g_1^2 - 4(1 - g_1)^2(1 - g_2) \geq 0.$$ \hfill (2.22)

Applying the Vieta’s formulas to polynomials $A_2$ and $B_2$, and taking into account that $g_i \in (0, 1), i = 1, 2$ one checks that:

$$t_j^{(i)} > 0, \quad i, j = 1, 2. \hfill (2.23)$$

Case A. Let $f'(0) > 0 (\Leftrightarrow g_1 < 1/2)$. Then $f(z)$ is increasing function in the interval $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$. We assume that inequality $D_2 \geq 0$ holds, so both roots $t_1^{(2)}$ and $t_2^{(2)}$ are real. One has $P_2(1) = 1 - 2g_1 > 0$, so, in view of (2.23), either $0 < t_1^{(2)} < t_2^{(2)} < 1$ (a) or $1 < t_1^{(2)} \leq t_2^{(2)}$ (b). One verifies with help of (2.21) that (a) is equivalent to $L_2 = g_1 - 2(1 - g_1)g_2 < 0$ and (b) to $L_2 > 0$. Thus, in view of (2.17), if (b) holds, the equation $(B_1)$ will have solution $x = -1 + t_1^{(2)} \in (0, +\infty)$ and if (a) holds, $(\tilde{B}_1)$ will have solution $x = -1 + t_2^{(2)} \in (-1, 0)$.

Case B. Let $f'(0) < 0 (\Leftrightarrow g_1 > 1/2)$. Then $f(z)$ is decreasing function in the interval $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$. We assume that inequality $D_1 \geq 0$ holds, so both roots $t_1^{(1)}$ and $t_2^{(1)}$ are real. One has $P_1(1) = 2g_1 - 1 > 0$, so, in view of (2.23), either $0 < t_1^{(1)} < t_2^{(1)} < 1$ (c) or $1 < t_1^{(1)} \leq t_2^{(1)}$ (d). One verifies with help of (2.19) that (c) is equivalent to $L_1 = 1 - g_1 - 2g_1(1 - g_2) < 0$ and (d) to $L_2 = 1 - g_1 - 2g_1(1 - g_2) > 0$. Thus, in view of (2.17), if (d) holds, the equation $(A_1)$ will have solution $x = -1 + t_1^{(1)} \in (0, +\infty)$ and if (c) holds, $(\tilde{A}_1)$ will have solution $x = -1 + t_2^{(1)} \in (-1, 0)$ in view of (2.17).

As in proof of Theorem 2.1 since $\eta(z)$ is a bijection of $(-K, K)$ to $(-1, +\infty)$, there exists unique real number $\tilde{T} \in (-K, K)$ satisfying equation $\eta(\tilde{T}) = x$ with $x \in (-1, +\infty)$ defined above.
Let
\[ \zeta = \frac{T\hat{T}}{K} \in (-T, T) . \]  
(2.24)

Then, as follows from (2.2), (2.3), there exists \( \tau \) satisfying (2.16). One has \( \tau \in (0, \zeta) \) if \( \zeta > 0 \) and \( \tau \in (\zeta, 0) \) if \( \zeta < 0 \). That finishes the proof. \( \square \)

The next corollary contains explicit formulas and precises the intervals containing the point of return \( \tau \) defined in Theorem 2.2.

**Corollary 2.1.** We assume all conditions of Theorem 2.2 being satisfied and define four following subsets of \((0, 1)^2\) (see Fig. 3):

\[ E = \{(g_1, g_2) \in (0, 1)^2 : D_2 \geq 0, 0 < g_1 < 1/2, 0 < g_2 < 1/2\} , \]  
(2.25)
\[ F = \{(g_1, g_2) \in (0, 1)^2 : D_2 \geq 0, 0 < g_1 < 1/2, 1/2 < g_2 < 1\} , \]  
(2.26)
\[ G = \{(g_1, g_2) \in (0, 1)^2 : D_1 \geq 0, 1/2 < g_1 < 1, 0 < g_2 < 1/2\} , \]  
(2.27)
\[ H = \{(g_1, g_2) \in (0, 1)^2 : D_1 \geq 0, 1/2 < g_1 < 1, 1/2 < g_2 < 1\} . \]  
(2.28)

Let \( t_j^{(i)} \), \( i, j = 1, 2 \) be defined by (2.19), (2.21) and \( \eta^{-1}(z) \) by (2.12), then

\[ \zeta = \frac{T\hat{T}}{K} \eta^{-1}(-1 + t_1^{(2)}) \in (0, T) \text{ if } (g_1, g_2) \in E \]  
(2.29)
\[ \zeta = \frac{T\hat{T}}{K} \eta^{-1}(-1 + t_2^{(2)}) \in (-T, 0) \text{ if } (g_1, g_2) \in F \]  
(2.30)
\[ \zeta = \frac{T\hat{T}}{K} \eta^{-1}(-1 + t_1^{(1)}) \in (-T, 0) \text{ if } (g_1, g_2) \in G \]  
(2.31)
\[ \zeta = \frac{T\hat{T}}{K} \eta^{-1}(-1 + t_2^{(1)}) \in (0, T) \text{ if } (g_1, g_2) \in H \]  
(2.32)

For \( \tau \) defined by (2.16) one has

\[ \tau \in (0, \zeta) \text{ if } \zeta > 0 \text{ and } \tau \in (\zeta, 0) \text{ if } \zeta < 0 . \]  
(2.33)

One easy verifies using the above formulas that \( \zeta(g_1, g_2) \to 0 \) as \( g_1 \to 1/2 \).

### 3. Functions bounded in the complex strip

We denote \( A_{M, \infty, B} \) the set of functions \( f(z) \) satisfying the following conditions

1. \( f(z) \) is holomorphic in the infinite strip \( S_B = \{ z \in \mathbb{C} : |Im(z)| < B \} , B > 0 \).
2. \( f(\mathbb{R}) \subset \mathbb{R} \).
3. \( |f(z)| < M, M > 0, \forall z \in S_B \).

The next result gives a characterization of functions bounded in absolute value in \( S_B \) with the help of \( g \)-fractions and can be considered as the limit case of (1.12) as \( T \to +\infty \).
Figure 3. Four domains $E, F, G, H$ in the parameter space $(g_1, g_2) \in (0, 1)^2$ corresponding to the oscillatory behavior of $f(z) \in \mathbb{H}_{M,T,B}$.

Figure 4. Domain of analiticity $S_B$ of $f(z) \in \mathbb{H}_{M,\infty,B}$.

**Theorem 3.1.** Let $f(z) \in \mathbb{H}_{M,\infty,B}$. Then for some $\mu_0 > 0$ and $g_k \in [0, 1]$, $k \geq 1$ one has

$$f(z) = M \left( 1 - \frac{2}{\mu_0 \exp \left( \frac{\pi z}{2B} \right) \{g_1, g_2, \ldots | \exp \left( \frac{\pi z^2}{2B} \right) - 1 \} + 1} \right). \tag{3.1}$$

**Proof.** Let $m : \mathbb{D}_M \to \mathbb{H}_+$ be defined by (1.7). We define the conformal map

$$l(z) = \frac{B}{\pi} \log(1 + z), \quad l : \mathbb{H} \to S_B. \tag{3.2}$$

One verifies that the composition $F = m \circ f \circ l$ is holomorphic in $\mathbb{H}$ and $F(\mathbb{H}) \subset \mathbb{H}_+$ with $F(z) \in \mathbb{R}$ for $z > -1$. Thus, according to theorem of Wall [8], p. 279 $F$ can be written as follows

$$F(z) = \mu_0 \sqrt{1 + z} \int_0^1 \frac{d\mu(u)}{1 + zu}, \tag{3.3}$$
for some nondecreasing real bounded function \( \mu(u) \), \( u \in (0, 1) \) and \( \mu_0 > 0 \).

For \( f = m^{-1} \circ F \circ t^{-1} \) one obtains the following formula

\[
f(z) = M \left( 1 - \frac{2}{\mu_0 \exp \left( \frac{\pi z}{2B} \right) \int_0^1 \frac{d\mu(u)}{1 + (\exp(\frac{\pi z}{B}) - 1)u} + 1} \right).
\]

The integral in (3.4) can be transformed to the continued \( g \)-fraction form \([8]\)

\[
\int_0^1 \frac{d\mu(u)}{1 + (\exp \left( \frac{\pi z}{B} \right) - 1)u} = \left\{ g_1, g_2, \ldots \left| \exp \left( \frac{\pi z}{B} \right) - 1 \right\} \right., \quad \text{for some} \quad g_k \in [0, 1], \quad (3.5)
\]

that together with (3.4) implies (3.1). The proof is finished. \( \square \)

Let

\[
\theta(z) = \frac{1}{M} f(Bz/\pi).
\]

To calculate the coefficients \( g_p \) in (3.1) one has formulas similar to (1.27):

\[
g_p = C_p(\theta(0), \theta'(0), \ldots, \theta^{(p)}(0)), \quad p \geq 1,
\]

with rational functions \( C_p \) determined by calculation of derivatives of both sides of (3.1) at \( z = 0 \).

Introducing

\[
\theta_n = \theta^{(n)}(0) = \frac{1}{M} \frac{B^n}{\pi^n} f^{(n)}(0), \quad n \geq 0,
\]

we provide below explicit formulas for \( \mu_0, g_1, g_2 \):

\[
\mu_0 = \frac{1 + \theta_0}{1 - \theta_0},
\]

\[
g_1 = \frac{1}{2} \frac{1 - 4\theta_1 - \theta_0^2}{1 - \theta_0^2},
\]

\[
g_2 = \frac{1}{2} \frac{(16\theta_1^2 - 8\theta_2 - \theta_0 + \theta_0^3 + \theta_0^3 - 8\theta_2\theta_0 - 1)(1 - \theta_0)}{(1 - \theta_0^2 + 4\theta_1)(4\theta_1 - 1 + \theta_0^2)}.
\]

**Definition 3.1.** We denote by \( \mathbb{A}^{(k)}_{M, \infty, B} \subset \mathbb{A}_{M, \infty, B}, \ k = 1, 2, \ldots \) the set of functions for which the \( g \)-fraction representation (1.12) satisfies the condition

\[
g_i \notin \{0, 1\}, \quad \forall i = 1, \ldots, k.
\]

The next result is analogous to Theorem 2.1 and is proved in the similar way.
Theorem 3.2. Let \( f(z) \in A_{M,\infty,B}^{(1)} \), \( f'(0) \neq 0 \) where \( g_1 \) is defined by (3.8) and (3.10) as function of \( M, B, f(0), f'(0) \). Let \( \tau \in \mathbb{R}, \tau \neq 0 \) be the point such that \( f(\tau) = f(0) \). We define

\[
\tilde{\tau} = \frac{2B}{\pi} \left| \log \left( \frac{1 - g_1}{g_1} \right) \right| > 0.
\] (3.13)

Then

\[
|\tau| \geq \tilde{\tau}.
\] (3.14)

The following theorem is equivalent to Theorem 2.2 for functions \( f \in A_{M,\infty,B}^{(2)} \) and has the similar proof.

Theorem 3.3. Let \( f(z) \in A_{M,\infty,B}^{(2)} \), \( f'(0) \neq 0 \) where \( g_1, g_2 \) are defined by formulas (3.8) and (3.10), (3.11). We assume that the point \((g_1, g_2) \in (0, 1)^2\) belongs to one of the four regions \( E, F, G, H \) defined by (2.25)-(2.28).

Let

\[
\zeta = \frac{2B}{\pi} \log(t^{(2)}_1) > 0 \quad \text{if} \quad (g_1, g_2) \in E
\] (3.15)

\[
\zeta = \frac{2B}{\pi} \log(t^{(2)}_2) < 0 \quad \text{if} \quad (g_1, g_2) \in F
\] (3.16)

\[
\zeta = \frac{2B}{\pi} \log(t^{(1)}_1) < 0 \quad \text{if} \quad (g_1, g_2) \in G
\] (3.17)

\[
\zeta = \frac{2B}{\pi} \log(t^{(1)}_2) > 0 \quad \text{if} \quad (g_1, g_2) \in H
\] (3.18)

where \( t^{(i)}_j, i, j = 1, 2 \) are defined as functions of \( g_1, g_2 \) by (2.19) and (2.21).

Then there exists \( \tau \in \mathbb{R}^* \) such that

\[
f(\tau) = f(0).
\] (3.19)

and \( \tau \in (0, \zeta) \) if \( \zeta > 0 \) and \( \tau \in (\zeta, 0) \) if \( \zeta < 0 \).

To conclude this section we will consider the case of functions analytic and bounded in a semi-infinite strip. Let \( B > 0, T > 0 \). We denote by \( S_{B,T} \subset \mathbb{C} \) the complex domain which is the interior of the semi-infinite strip formed by segments \( t + iB, t - iB, t \in [-T, +\infty) \) and \( -T + it, t \in [-B, B] \) (see Fig. 5).

Let \( B_{M,T,B} \) be the set of functions \( f(z) \) satisfying the following conditions:

1. \( f(z) \) is holomorphic in \( S_{B,T} \).
2. \( f(\mathbb{R}) \subset \mathbb{R} \).
3. \( |f(z)| < M, M > 0, \forall z \in S_{B,T} \).
It is straightforward to verify that the following function
\[
L(z) = \frac{\cosh \left( \frac{\pi (z + T)}{B} \right) - \cosh \left( \frac{\pi T}{B} \right)}{\cosh \left( \frac{\pi T}{B} \right) - 1},
\]
defines a conformal map \( L : S_{B,T} \to \mathbb{H} \) and maps bijectively \((-T, +\infty)\) to \((-1, +\infty)\), \(L(0) = 0\). One can formulate now the result analogous to (1.12) and (3.1):

**Theorem 3.4.** Let \( f(z) \in \mathbb{B}_{M,T,B} \). Then for some \( \mu_0 > 0 \) and \( g_k \in [0, 1], k \geq 1 \) one has
\[
f(z) = M \left( 1 - \frac{2}{\mu_0 \sqrt{1 + L(z) \{g_1, g_2, \ldots |L(z)|\}}} + 1 \right).
\]

**4. Applications**

We will apply the results from previous sections to the Newtonian three–body problem, whose solutions in many situations are analytic functions in the strip along the real axis of the complex time plane.

We consider three mass points \( P_1, P_2, P_3 \) in \( \mathbb{R}^3 \) which attract each other according to the Newtonian law with finite positive masses \( m_1, m_2, m_3 \). Let \( R_i = (x_i, y_i, z_i) \) be the position vector of \( P_i \) and \( r_{ij} \) the distance between it and mass \( j \). One writes equations of motion as follows:
\[
m_i \frac{dR'_i}{dz} = - \sum_{j \neq i} m_i m_j \frac{R_i - R_j}{r_{ij}^3},
\]
\[
R'_i = \frac{dR_i}{dz} = (x'_i, y'_i, z'_i), \quad i = 1, 2, 3,
\]
which have the integral of energy:
\[
H = T + U = h = -\frac{m_1 m_2 m_3}{2 \Gamma} K, \quad h, K = \text{const},
\]
\[
T = \sum_{i=1}^{3} \frac{m_i (x_i^2 + y_i^2 + z_i^2)}{2},
\]
the first integrals of the impulse of the system:

\[
\sum_{i=1}^{3} m_i x_i = 0, \quad \sum_{i=1}^{3} m_i z_i = 0, \quad \sum_{i=1}^{3} m_i z_i = 0, \quad (4.7)
\]

\[
\sum_{i=1}^{3} m_i x_i' = 0, \quad \sum_{i=1}^{3} m_i y_i' = 0, \quad \sum_{i=1}^{3} m_i z_i' = 0, \quad (4.8)
\]

and the first integrals of the angular momentum:

\[
\sum_{i=1}^{3} m_i (x_i y_i' - x_i' y_i) = c_1, \quad \sum_{i=1}^{3} m_i (y_i z_i' - y_i' z_i) = c_2, \quad \sum_{i=1}^{3} m_i (z_i x_i' - z_i' x_i) = c_3, \quad (4.9)
\]

\[c_1, c_2, c_3 = \text{const.}\]

**Figure 6.** The planar three–planet problem.

We shall need the following result due to Sundman [3] which we write in a slightly different form.

**Theorem 4.1.** Let \( J \subset \mathbb{R} \) be a connected open interval and \( x_i(z), y_i(z), z_i(z), i = 1, 2, 3 \) be a solution of the three-body problem (4.1) defined for \( z \in J \) and satisfying the following inequalities

\[
r_{32}(z) > 2 \chi, \quad r_{13}(z) > 2 \chi, \quad r_{21}(z) > 2 \chi, \quad \forall z \in J. \quad (4.10)
\]

Then \( \forall z_0 \in J \) the positions \( R_i(z) \), \( i = 1, 2, 3 \) are holomorphic functions in the complex disk

\[
\Delta_\chi = \{ z \in \mathbb{C} : |z - z_0| < B_\chi \}, \quad (4.11)
\]
where
\[
B_{\chi} = \frac{1}{14} \frac{2\chi}{\sqrt{\frac{28}{21} \frac{\tau}{m\chi} + \Gamma K}}, \quad m = \min\{m_1, m_2, m_3\},
\] (4.12)
and satisfy \( \forall z \in \Delta_{\chi} \) the inequalities
\[
|x_i(z_0) - x_i(z)| < \chi/7, \quad |y_i(z_0) - y_i(z)| < \chi/7, \quad |z_i(z_0) - z_i(z)| < \chi/7, \quad i = 1, 2, 3. \] (4.13)

Such a solution can be interpreted as a collision free motion for \( z \in J \) of three planets each of radius \( \chi > 0 \) (see Fig. 6). The collision between bodies \( P_i \) and \( P_j \) at the moment of time \( z \in \mathbb{R} \) happens if and only if \( r_{ij}(z) = 2\chi \).

**Lemma 4.1.** Let all conditions of Theorem 4.1 hold and \( z_0 \in J \). Then the inverse mutual distances \( r_{ij}^{-1}(z) \) are holomorphic functions in \( \Delta_{\chi} \) and bounded in the absolute value:
\[
|r_{ij}(z)^{-1}| < M_{\chi} = \frac{7}{4\chi}, \quad \forall z \in \Delta_{\chi}. \] (4.14)

**Proof.** Let \( \langle , \rangle \) denotes the Euclidean scalar product in \( \mathbb{R}^3 \). For some fixed \( i < j \) we introduce \( X(z) = (X_1(z), X_2(z), X_3(z)) = r_i(z), \ Y(z) = (Y_1(z), Y_2(z), Y_3(z)) = r_j(z) \). Let \( X(z) = X(z_0) + \tilde{X}(z), \ Y(z) = Y(z_0) + \tilde{Y}(z), \ z \in \Delta_{\chi} \) where \( X(z) = (\tilde{X}_1(z), \tilde{X}_2(z), \tilde{X}_3(z)), \ Y(z) = (\tilde{Y}_1(z), \tilde{Y}_2(z), \tilde{Y}_3(z)) \). Then
\[
|\tilde{X}_i(z)| < \chi/7, \quad |\tilde{Y}_i(z)| < \chi/7, \quad \forall z \in \Delta_{\chi}, \quad i = 1, 2, 3, \] (4.15)

as follows from Sundman’s Theorem 4.1.

Let \( E = X(z_0) - Y(z_0), \ R^2 = \langle E, E \rangle, \ K = \tilde{X}(z) - \tilde{Y}(z) \). Then \( |r_{ij}^2(z)| = | \langle E + K, E + K \rangle | \). Applying the triangular inequality one obtains for \( z \in \Delta_{\chi} \):
\[
|r_{ij}^2(z)| \geq R^2 - 2|E, K| - |K, K| > R^2 - 12R\frac{\chi}{7} - 12 \left( \frac{\chi}{7} \right)^2 = \frac{4\chi^2}{7}, \] (4.16)
where we have used the inequalities \( R^2 > 4\chi^2 \) and (4.15). That implies (4.14) and finishes the proof.

We shall consider the case that \( J = \mathbb{R} \) which corresponds to the collision–free motion of three rigid spherical bodies, each of radius \( \chi > 0 \), for \(-\infty < z < +\infty \). In this case, according to Sundman’s Theorem 4.1 all inverse mutual distances \( r_{ij}^{-1}(z) \) are analytic functions in the complex infinite strip \( S_{B_{\chi}} \). As follows from Lemma 4.1 \( r_{ij}^{-1}(z) \) are bounded in \( S_{B_{\chi}} \) in absolute value by \( M_{\chi} \). Moreover, \( r_{ij}^{-1}(z) \in \mathbb{A}_{M,\infty,B}^{(k)}; \ k = 1, 2 \). Indeed, in the opposite case the \( g \)-fraction in (3.1) would be rational one of the form \( \frac{a + b}{cz + d} \). \( a, b, c, d \in \mathbb{R} \). It is straightforward to verify that this case is eliminated with help of the equations of motion (1.1).
Thus, the Theorems 3.2 and 3.3 can be applied in this case. Below we describe one possible application to the collision problem:

Let us consider the motion of three spherical rigid bodies (planets) in 3–dimensional Euclidean space each of radius $\chi > 0$ with masses $m_1, m_2, m_3$ interacting according to Newtonian low. The motion is assumed to be collision free inside some small interval $z \in (-\epsilon, \epsilon)$, $\epsilon > 0$. Let $r_{ij}(z) \in \{r_{32}(z), r_{13}(z), r_{21}(z)\}$ be one of three mutual distances. We put $f(z) = r_{ij}^{-1}(z)$. One calculates $f(0), f'(0), f''(0)$ and finds the upper and lower bounds on the time of the first return of $f(z)$ to its initial value $f(0)$ using Theorems 3.2 and 3.3 with $B = B_\chi, M = M_\chi$ defined by (4.12), (4.14). Then, if these bounds are not fulfilled for the observed motion, then there has to be a collision between two of bodies $P_1, P_2, P_3$ for some negative or positive value of time $z_0 \in (-\infty, +\infty)$ $i.e$ $r_{kl}(z_0) = 2\chi$ for some $(kl) \in \{(13), (21), (32)\}$.

Indeed, let us assume that the bounds on the time of the first return given by Theorems 3.2, 3.3 and applied to $f(z) = r_{ij}^{-1}(z)$ with $B = B_\chi, M = M_\chi$ are not fulfilled. Then the condition (4.11) is not satisfied for some $z_0 \in (-\infty, +\infty)$ and thus at least two bodies collide at the the moment of time $z = z_0$.

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