The proof that the standard transformations of $E$ and $B$ are not the Lorentz transformations. Clifford algebra formalism

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In this paper it is exactly proved by using the Clifford algebra formalism that the standard transformations of the three-dimensional (3D) vectors of the electric and magnetic fields $E$ and $B$ are not the Lorentz transformations of well-defined quantities from the 4D spacetime but the 'apparent' transformations of the 3D quantities. Thence the usual Maxwell equations with the 3D $E$ and $B$ are not in agreement with special relativity. The 1-vectors $E$ and $B$, as well-defined 4D quantities, are introduced instead of ill-defined 3D $E$ and $B$.

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1. Introduction

It is generally accepted by physics community that there is an agreement between the classical electromagnetism and the special relativity (SR). Such opinion is prevailing in physics already from the Einstein first paper on SR [1]. The standard transformations of the 3D vectors of the electric and magnetic fields, $E$ and $B$ respectively, are first derived by Lorentz [2] and independently by Einstein [1], and subsequently quoted in almost every textbook and paper on relativistic electrodynamics. They are considered to be the Lorentz transformations (LT) of these vectors, see, e.g., [1-3]. The same opinion holds in all usual Clifford algebra formulations of the classical electromagnetism, e.g., the formulations with Clifford multivectors [4-6]. The usual Maxwell equations with the 3D vectors $E$ and $B$ are assumed to be physically equivalent to the field equations expressed in terms of the Faraday bivector field $F$ in the Clifford algebra formalism (or the electromagnetic field tensor $F_{ab}$ in the tensor formalism). In this paper it will be exactly proved that the above mentioned standard transformations of $E$ and $B$ (see eq. [12])
are not relativistically correct transformations in the 4D spacetime; they are not the LT of the 3D $E$ and $B$. Consequently the usual Maxwell equations with $E$ and $B$ and the field equations with the $F$ field are not physically equivalent. The correct LT (the active ones) of the electric and magnetic fields are given by the relations (8) and (9) (or (15) and (16)) below. In the Clifford algebra formalism (as in the tensor formalism) one deals either with 4D quantities that are defined without reference frames, e.g., Clifford multivector $F$ (the abstract tensor $F^{ab}$) or, when some basis has been introduced, with coordinate-based geometric quantity that comprises both components and a basis. The SR that exclusively deals with quantities defined without reference frames or, equivalently, with coordinate-based geometric quantities, can be called the invariant SR. The reason for this name is that upon the passive LT any coordinate-based geometric quantity remains unchanged. The invariance of some 4D coordinate-based geometric quantity upon the passive LT reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. It is taken in the invariant SR that such 4D geometric quantities are well-defined not only mathematically but also experimentally, as measurable quantities with real physical meaning. Thus they do have an independent physical reality. The invariant SR is discussed in [7] in the Clifford algebra formalism and in [8,9] in the tensor formalism. It is explicitly shown in [9] that the true agreement with experiments that test SR exists when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT. The usual standard transformations of the electric and magnetic fields, the transformations (10), (11) and (12) (or (17) and (18)) below are typical examples of the ‘apparent’ transformations that are first discussed in [10] and [11]. The ‘apparent’ transformations of the spatial distances (the Lorentz contraction) and the temporal distances (the dilatation of time) are elaborated in detail in [8] and [9] (see also [12]), and in [8] I have also discussed in the tensor formalism the ‘apparent’ transformations of the 3D vectors $E$ and $B$. The ‘apparent’ transformations relate, in fact, the quantities from ’3+1’ space and time (spatial and temporal distances and 3D vectors $E$ and $B$) and not well-defined 4D quantities. But, in contrast to the LT of well-defined 4D quantities, the ‘apparent’ transformations do not refer to the same physical object for relatively moving observers. In this paper it will be also shown that in the 4D spacetime the well-defined 4D quantities, the 1-vectors of the electric and magnetic fields $E$ and $B$ (see (21)) in the Clifford algebra formalism (as in [7]), have to be introduced instead of
ill-defined 3D vectors \( \mathbf{E} \) and \( \mathbf{B} \). The same proof is already presented in the tensor formalism in [13].

2. The \( \gamma_0 \) - split and the usual expressions for \( \mathbf{E} \) and \( \mathbf{B} \) in the \( \gamma_0 \) - frame

2.1. A brief summary of geometric algebra

First we provide a brief summary of Clifford algebra with multivectors (see, e.g., [4 − 6]). We write Clifford vectors in lower case (\( a \)) and general multivectors (Clifford aggregate) in upper case (\( A \)). The space of multivectors is graded and multivectors containing elements of a single grade, \( r \), are termed homogeneous and written \( A_r \). The geometric (Clifford) product is written by simply juxtaposing multivectors \( AB \). A basic operation on multivectors is the degree projection \( \langle A \rangle_r \) which selects from the multivector \( A \) its \( r \)− vector part (0 = scalar, 1 = vector, 2 = bivector ....). We write the scalar (grade-0) part simply as \( \langle A \rangle \).

The geometric product of a grade-\( r \) multivector \( A_r \) with a grade-\( s \) multivector \( B_s \) decomposes into

\[
A_r B_s = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} + \ldots + \langle AB \rangle_{|r-s|}.
\]

The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series. \( A_r \cdot B_s \equiv \langle AB \rangle_{|r-s|} \), and \( A_r \wedge B_s \equiv \langle AB \rangle_{r+s} \). For vectors \( a \) and \( b \) we have \( ab = a \cdot b + a \wedge b \), where \( a \cdot b \equiv (1/2)(ab + ba) \), and \( a \wedge b \equiv (1/2)(ab - ba) \). Reversion is an invariant kind of conjugation, which is defined by \( \overline{AB} = B \overline{A}, \overline{a} = a \), for any vector \( a \), and it reverses the order of vectors in any given expression. Any multivector \( A \) is a geometric 4D quantity defined without reference frame. When some basis has been introduced \( A \) can be written as a coordinate-based geometric quantity comprising both components and a basis. Usually, e.g., [4 − 6], one introduces the standard basis. The generators of the spacetime algebra are taken to be four basis vectors \( \{ \gamma_\mu \}, \mu = 0...3 \), satisfying \( \gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+−−−) \). This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime \( M^4 \) with \( \gamma_0 \) in the forward light cone. The \( \gamma_k \) (\( k = 1, 2, 3 \)) are spacelike vectors. This algebra is often called the Dirac algebra \( D \) and the elements of \( D \) are called \( d \)-numbers. The \( \gamma_\mu \) generate by multiplication a complete basis, the standard basis, for spacetime algebra: \( 1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5 \) (16 independent elements). \( \gamma_5 \) is the pseudoscalar for the frame \( \{ \gamma_\mu \} \).

We remark that the standard basis corresponds, in fact, to the specific system of coordinates, i.e., to Einstein’s system of coordinates. In the Einstein system of coordinates the Einstein synchronization [1] of distant clocks
and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference. However different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena. For example, in [8] two very different, but completely equivalent systems of coordinates, the Einstein system of coordinates and ”radio” (”r”) system of coordinates, are exposed and exploited throughout the paper. The coordinate-based geometric quantities representing some 4D physical quantity in different relatively moving inertial frames of reference, or in different systems of coordinates in the chosen inertial frame of reference, are all mathematically equal and thus they are the same quantity for different observers, or in different systems of coordinates. Then, e.g., the position 1-vector $x$ (a geometric quantity) can be decomposed in the $S$ and $S'$ frames and in the standard basis $\{\gamma_\mu\}$ as $x = x^\mu \gamma_\mu = x'^\mu \gamma'_\mu$. The primed quantities are the Lorentz transforms of the unprimed ones. In such interpretation the LT are considered as passive transformations; both the components and the basis vectors are transformed but the whole geometric quantity remains unchanged. Thus we see that under the passive LT a well-defined quantity on the 4D spacetime, i.e., a coordinate-based geometric quantity, is an invariant quantity. As already said in the Introduction the SR that exclusively deals with such quantities defined without reference frames or, equivalently, with coordinate-based geometric quantities, is called the invariant SR and it is considered in the tensor formalism in [8,9].

In the usual Clifford algebra formalism [4–6] instead of working only with such observer independent quantities one introduces a space-time split and the relative vectors. By singling out a particular time-like direction $\gamma_0$ we can get a unique mapping of spacetime into the even subalgebra of spacetime algebra. For each event $x$ this mapping is specified by $x \gamma_0 = ct + x, \quad ct = x \cdot \gamma_0, \quad x = x \wedge \gamma_0$. The set of all position vectors $x$ is the 3D position space of the observer $\gamma_0$ and it is designated by $P^3$. The elements of $P^3$ are called the relative vectors (relative to $\gamma_0$) and they will be designated in boldface. The explicit appearance of $\gamma_0$ implies that the space-time split is observer dependent. If we consider the position 1-vector $x$ in another relatively moving inertial frame of reference $S'$ (characterized by $\gamma'_0$) then the space-time split in $S'$ and in the Einstein system of coordinates is $x \gamma'_0 = ct' + x'$. This $x \gamma'_0$ is not obtained by the LT from $x \gamma_0$. (The hypersurface $t' = const.$ is not connected in any way with the hypersurface $t = const.$) Thence the spatial and the temporal components $(x, t)$ of some geometric 4D quantity ($x$) (and thus the relative vectors as well) are not physically well-defined quantities.
Only their union is physically well-defined quantity in the 4D spacetime from the invariant SR viewpoint.

2.2. The $\gamma_0$ - split and the usual expressions for $E$ and $B$ in the $\gamma_0$ - frame

Let us now see how the space-time split is introduced in the usual Clifford algebra formalism [4,5] of the electromagnetism. The bivector field $F$ is expressed in terms of the sum of a relative vector $E$ and a relative bivector $\gamma_5 B$ by making a space-time split in the $\gamma_0$ - frame

$$F = E_H + c\gamma_5 B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0),$$
$$\gamma_5 B_H = (1/c)(F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F\gamma_0).$$

(1)

(The subscript 'H' is for - Hestenes.) Both $E_H$ and $B_H$ are, in fact, bivectors. Similarly in [6] $F$ is decomposed in terms of 1-vector $E_J$ and a bivector $B_J$ (the subscript 'J' is for - Jancewicz) as

$$F = \gamma_0 \wedge E_J - cB_J, \quad E_J = F \cdot \gamma_0, \quad B_J = -(1/c)(F \wedge \gamma_0)\gamma_0.$$

(2)

Instead of to use $E_H$, $B_H$ or $E_J$, $B_J$ we shall mainly deal (except in Sec. 3.3.) with simpler but completely equivalent expressions in the $\gamma_0$ - frame, i.e., with 1-vectors that will be denoted as $E_f$ and $B_f$. Then

$$F = E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0,$$
$$E_f = F \cdot \gamma_0, \quad B_f = -(1/c)(F \wedge \gamma_0).$$

(3)

All these quantities can be written as coordinate-based geometric quantities in the standard basis $\{\gamma_\mu\}$. Thus

$$F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = F^{0k}\gamma_0 \wedge \gamma_k + (1/2)F^{kl}\gamma_k \wedge \gamma_l, \quad k, l = 1, 2, 3.$$

(4)

$$E_f = E_f^\mu \gamma_\mu = 0\gamma_0 + F^{k0}\gamma_k,$$
$$B_f = B_f^\mu \gamma_\mu = 0\gamma_0 + (-1/2c)\epsilon^{0kli}F_{kl}\gamma_i.$$

(5)

We see from (4) and (5) that the components of $F$ in the $\{\gamma_\mu\}$ basis (i.e., in the Einstein system of coordinates) give rise to the tensor (components) $F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F$, which, written out as a matrix, has entries

$$E_f^\nu = F^{\nu0}, \quad B_f^\nu = (-1/2c)\epsilon^{0kl}F_{kl}.$$

(6)
The relation (6) is nothing else than the standard identification of the components $F_{\mu\nu}$ with the components of the 3D vectors $E$ and $B$, see, e.g., [3]. It is worth noting that all expressions with $\gamma_0$ actually refer to the 3D subspace orthogonal to the specific timelike direction $\gamma_0$. Really it can be easily checked that $E_f \cdot \gamma_0 = B_f \cdot \gamma_0 = 0$, which means that they are orthogonal to $\gamma_0$; $E_f$ and $B_f$ do not have the temporal components $E_f^0 = B_f^0 = 0$. These results are quoted in numerous textbooks and papers treating relativistic electrodynamics, see, e.g., [3]. Actually in the usual covariant approaches, e.g., [3], one forgets about temporal components $E_f^0$ and $B_f^0$ and simply makes the identification of six independent components of $F_{\mu\nu}$ with three components $E_i^f$ and three components $B_i^f$ according to (6). Since in SR we work with the 4D spacetime the mapping between some components of $F_{\mu\nu}$ and the components of the 3D vectors $E$ and $B$ is mathematically better founded by the relations (5) than by their simple identification. Note that the whole procedure is made in an inertial frame of reference with the Einstein system of coordinates. In another system of coordinates that is different than the Einstein system of coordinates, e.g., differing in the chosen synchronization (as it is the ‘r’ synchronization considered in [8]), the identification of $E_i^f$ with $F^{i0}$, as in (6) (and also for $B_i^f$), is impossible and meaningless.

3. The proof that the standard transformations of $E$ and $B$ are not the LT

3.1. The active LT of the electric and magnetic fields

Let us now explicitly show that the usual transformations of the 3D $E$ and $B$ are not relativistically correct, i.e., they are not the LT of quantities that are well-defined on the 4D spacetime. First we find the correct expressions for the LT (the active ones) of $E_f$ and $B_f$. In the usual Clifford algebra formalism, e.g., [4 – 6], the LT are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis $\{\gamma_\mu\}$) are transformed into the components of a new 1-vector relative to the same frame (the basis $\{\gamma_\mu\}$ is not changed). Furthermore the LT are described with rotors $R$, $R\tilde{R} = 1$, in the usual way as $p \rightarrow p' = Rp\tilde{R} = p'\mu\gamma^\mu$. But every rotor in spacetime can be written in terms of a bivector as $\tilde{R} = e^{\theta/2}$. For boosts in arbitrary direction

$$R = e^{\theta/2} = (1 + \gamma + \gamma\beta\gamma_0 n)/(2(1 + \gamma))^{1/2},$$

(7)
θ = αγ₀n, β is the scalar velocity in units of c, γ = (1 − β²)⁻¹/², or in terms of an ‘angle’ α we have \( \tanh α = β \), \( \cosh α = γ \), \( \sinh α = βγ \), and \( n = \cosh α + γ₀n \sinh α \). One can also express the relationship between the two relatively moving frames \( S \) and \( S' \) in terms of the rotor as \( γ'_µ = Rγ_µ \tilde{R} \). For boosts in the direction \( γ₁ \) the rotor is given by the relation (7) with \( γ₁ \) replacing \( n \) (all in the standard basis \( \{γ_µ\} \)). Then using (5) the transformed \( E'_f \) can be written as

\[
E'_f = R(F \cdot γ₀)\tilde{R} = R(F^{k0}γ_k)\tilde{R} = E'\!^µ_µγ_µ =
\]

\[
= -βγE'_1γ₀ + γE'_1γ₁ + E'_2γ₂ + E'_3γ₃, \tag{8}
\]

what is the usual form for the active LT of the 1-vector \( E_f = E_f^µγ_µ \). Similarly we find for \( B'_f \)

\[
B'_f = R[−(1/c)γ₅(F \land γ₀)]\tilde{R} = R[(-1/2c)ε^{0kli}F_{kl}γ_i]\tilde{R} =
\]

\[
= B'\!^µ_µγ_µ = -βγB'₁γ₀ + γB'₁γ₁ + B'₂γ₂ + B'₃γ₃, \tag{9}
\]

what is the familiar form for the active LT of the 1-vector \( B_f = B_f^µγ_µ \). It is important to note that \( E'_f \) and \( B'_f \) are not orthogonal to \( γ₀ \), i.e., they do have the temporal components \( \neq 0 \). They do not belong to the same 3D subspace as \( E_f \) and \( B_f \), but they are in the 4D spacetime spanned by the whole standard basis \( \{γ_µ\} \). The relations (8) and (9) imply that the spacetime split in the \( γ₀ \)-system is not possible for the transformed \( F' = RF\tilde{R} \), i.e., \( F' \) cannot be decomposed into \( E'_f \) and \( B'_f \) as \( F \) is decomposed in the relation (3), \( F' \neq E'_f \land γ₀ + c(γ₅B'_f) \cdot γ₀ \). Notice, what is very important, that the components \( E'\!^µ_µ \) (\( B'_\!^µ_µ \)) from (5) transform upon the active LT again to the components \( E'\!^µ_µ \) (\( B'_\!^µ_µ \)) from (8) (9); there is no mixing of components. Thus by the active LT \( E_f \) transforms to \( E'_f \) and \( B_f \) to \( B'_f \). Actually, as we said, this is the way in which every 1-vector transforms upon the active LT.

3.2. The standard transformations of the electric and magnetic fields

However the standard transformations for \( E'_st \) and \( B'_st \) (the subscript - st - is for - standard) are derived wrongly assuming that the quantities obtained by the active LT of \( E_f \) and \( B_f \) are again in the 3D subspace of the \( γ₀ \)-observer. This means that it is wrongly assumed in all standard derivations, e.g., in the Clifford algebra formalism [4-6] (and in the tensor formalism [3] as well), that
one can again perform the same identification of the transformed components \( F'_{\mu\nu} \) with the components of the 3D \( \mathbf{E}' \) and \( \mathbf{B}' \). Thus it is taken in standard derivations that for the transformed \( E'_{st} \) and \( B'_{st} \) again hold \( E'_{st}^0 = B'_{st}^0 = 0 \), i.e., that \( E'_{st} \cdot \gamma_0 = B'_{st} \cdot \gamma_0 = 0 \) as for \( E_f \) and \( B_f \). Thence, in contrast to the correct LT of \( E_f \) and \( B_f \), formulas (8) and (9) respectively, it is taken in standard derivations that

\[
E'_{st}^0 = B'_{st}^0 = 0, \quad \text{i.e., that } \quad E'_{st} \cdot \gamma_0 = B'_{st} \cdot \gamma_0 = 0 \quad \text{as for } \quad E_f \text{ and } B_f.
\]

From the relativistically incorrect transformations (10) and (11) one simply finds the transformations of the spatial components \( E'_{st}^i \) and \( B'_{st}^i \):

\[
E'^i_{st} = F'^i_{00} \quad \text{and} \quad B'^i_{st} = (-1/2c)\varepsilon^{0ki}F'^k_{st}. \quad (12)
\]

As can be seen from (10), (11) and (12), the transformations for \( E'_{st}^i \) and \( B'_{st}^i \) are exactly the standard transformations of components of the 3D vectors \( \mathbf{E} \) and \( \mathbf{B} \) that are quoted in almost every textbook and paper on relativistic electrodynamics including [1] and [3]. These relations are explicitly derived and given in the Clifford algebra formalism, e.g., in [4], Space-Time Algebra (eq. (18.22)), New Foundations for Classical Mechanics (Ch. 9 eqs. (3.51a,b)) and in [6] (Ch. 7 eqs. (20a,b)). Notice that, in contrast to the active LT (8) and (9), according to the standard transformations (10) and (11) (i.e., (12)), the transformed components \( E'_{st}^i \) are expressed by the mixture of components \( E_f^i \) and \( B_f^i \), and the same holds for \( B'_{st}^i \). In all previous treatments of SR, e.g., [4-6] (and [1-3]) the transformations for \( E'_{st}^i \) and \( B'_{st}^i \) are considered to be the LT of the 3D electric and magnetic fields. However our analysis shows that the transformations for \( E_{st}^i \) and \( B_{st}^i \) (12) are derived from the relativistically incorrect transformations (10) and (11), which are not the LT; the LT are given by the relations (8) and (9).

The same results can be obtained with the passive LT, either by using the expression for the LT that is independent of the chosen system of coordinates (such one as in [7]), or by using the standard expressions for the LT in the Einstein system of coordinates from [3]. The passive LT transform always
the whole 4D quantity, basis and components, leaving the whole quantity unchanged. Thus under the passive LT the field bivector $F$ as well-defined 4D quantity remains unchanged, i.e., $F = (1/2)F^\nu\gamma_\mu \wedge \gamma_\nu = (1/2)F'^\nu\gamma'_\mu \wedge \gamma'_\nu$ (all primed quantities are the Lorentz transforms of the unprimed ones). In the same way it holds that, e.g., $E^\mu_\gamma \gamma_\mu = E'^\mu_\gamma' \gamma'_\mu$. The invariance of some 4D coordinate-based geometric quantity upon the passive LT is the crucial requirement that must be satisfied by any well-defined 4D quantity. It reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. The use of coordinate-based geometric quantities enables us to have clearly and correctly defined the concept of sameness of a physical system for different observers. Thus such quantity that does not change upon the passive LT does have an independent physical reality, both theoretically and experimentally.

However it can be easily shown that $E^\mu_\gamma \gamma_\mu \neq E'^\mu_\gamma' \gamma'_\mu$. This means that, e.g., $E^\mu_\gamma \gamma_\mu$ and $E'^\mu_\gamma' \gamma'_\mu$ are not the same quantity for observers in $S$ and $S'$. As far as relativity is concerned the quantities, e.g., $E^\mu_\gamma \gamma_\mu$ and $E'^\mu_\gamma' \gamma'_\mu$, are not related to one another. Their identification is the typical case of mistaken identity. The fact that they are measured by two observers ($\gamma_0$ - and $\gamma'_0$ - observers) does not mean that relativity has something to do with the problem. The reason is that observers in the $\gamma_0$ - system and in the $\gamma'_0$ - system are not looking at the same physical object but at two different objects. Every observer makes measurement on its own object and such measurements are not related by the LT. Thus from the point of view of the SR the transformations for $E^\mu_\gamma \gamma_\mu$ and $B^n_\iota$ (12) are not the LT of some well-defined 4D quantities. Therefore, contrary to the general belief, it is not true from SR viewpoint that, e.g., [3], Jackson’s Classical Electrodynamics, Sec. 11.10: “A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.”; or that [5], Handout 10 in Physical Applications of Geometric Algebra: ”Observers in relative motion see different fields.” This is also exactly proved in the tensor formalism in [13].

Both the transformations (10), (11) and the transformations (12) for $E^\mu_\gamma \gamma_\mu$ and $B^n_\iota$ (i.e., for the 3D vectors $E$ and $B$) are typical examples of the ‘apparent’ transformations that are first discussed in [10] and [11]. The ‘apparent’ transformations of the spatial distances (the Lorentz contraction) and the temporal distances (the dilatation of time) are elaborated in detail in [8] and [9] (see also [12]), and in [8] I have also discussed the ‘apparent’ transfor-
mations of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$. The ‘apparent’ transformations relate, in fact, the quantities from ‘3+1’ space and time (spatial and temporal distances and 3D vectors $\mathbf{E}$ and $\mathbf{B}$) and not well-defined 4D quantities. As shown in [8] two synchronously (for the observer) determined spatial lengths correspond to two different 4D quantities; two temporal distances connected by the relation for the dilatation of time also correspond to two different 4D quantities in two relatively moving 4D inertial frames of reference, see in [8] Figs. 3. and 4. that refer to the Lorentz contraction and the dilatation of time respectively and compare them with Figs. 1. and 2. that refer to the well-defined 4D quantities, the spacetime lengths for a moving rod and a moving clock respectively. Since the spatial length, the temporal distance and the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ are different for different observers in the 4D spacetime they do not have an independent physical reality. It is explicitly shown in [9] that the true agreement with experiments that test SR exists when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT; they do not change for different observers in the 4D spacetime.

These results (both with the active and the passive LT) entail that the standard transformations of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ are not mathematically correct in the 4D spacetime, which means that the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ themselves are not correctly defined quantities from the SR viewpoint. Consequently the usual Maxwell equations with 3D $\mathbf{E}$ and $\mathbf{B}$ are not in agreement with SR and they are not physically equivalent with the relativistically correct field equations with $F$ (e.g., eq. (8.1) in [4], Space-Time Algebra). The same conclusion is achieved in the tensor formalism in [13].

3.3. The LT and the standard transformations of $\mathbf{E}_H$, $\mathbf{B}_H$ and $\mathbf{E}_J$, $\mathbf{B}_J$

In this section, for the completeness, we shall repeat the proof from Secs. 3.1. and 3.2. but using $\mathbf{E}_H$, $\mathbf{B}_H$ from [4,5] and $\mathbf{E}_J$, $\mathbf{B}_J$ from [6]. In [4,5], as explained in Sec. 2.2., $F$ is decomposed in terms of bivectors $\mathbf{E}_H$ and $\mathbf{B}_H$, while in [6] $F$ is decomposed in terms of 1-vector $\mathbf{E}_J$ and a bivector $\mathbf{B}_J$. Our aim is to show that the relativistic incorrectness of the standard transformations for the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ will be obtained regardless of the used algebraic objects for the representation of the electric and magnetic parts in the decomposition of $F$. The correct transformations will be always, as in Sec. 3.1., simply obtained by the use of the LT of the considered 4D algebraic objects. Thus it is unimportant which algebraic objects represent the electric and magnetic fields. What is important is the way in which their
transformations are derived.

First we present this proof for \( \mathbf{E}_H, \mathbf{B}_H \). In [4,5], as already said in Sec.
2.2, the bivector field \( F \) is expressed in terms of the sum of a relative vector
\( \mathbf{E}_H \) and a relative bivector \( \gamma_0 \mathbf{B}_H \) by making a space-time split in the \( \gamma_0 - \) frame, eq. (1); \( \mathbf{E}_H = (F \cdot \gamma_0)\gamma_0 \) and \( \gamma_0 \mathbf{B}_H = (1/c)(F \wedge \gamma_0)\gamma_0 \). All these quantities can be written as coordinate-based geometric quantitie s in the standard basis \( \{ \gamma_{\mu} \} \). Thus

\[
\mathbf{E}_H = F^{i0}\gamma_i \wedge \gamma_0, \quad \mathbf{B}_H = (1/2c)\varepsilon^{kl0} F_{kl} \gamma_i \wedge \gamma_0. \tag{13}
\]

It is seen from (13) that both bivectors \( \mathbf{E}_H \) and \( \mathbf{B}_H \) are parallel to \( \gamma_0 \), that is, it holds that \( \mathbf{E}_H \wedge \gamma_0 = \mathbf{B}_H \wedge \gamma_0 = 0 \). Further we see from (13) that the components of \( \mathbf{E}_H, \mathbf{B}_H \) in the \( \{ \gamma_{\mu} \} \) basis (i.e., in the Einstein system of coordinates) give rise to the tensor (components) \( (\mathbf{E}_H)^{\mu \nu} = \gamma^\nu \cdot (\gamma^\mu \cdot \mathbf{E}_H) = (\gamma^\nu \wedge \gamma^\mu) \cdot \mathbf{E}_H \), (and the same for \( (\mathbf{B}_H)^{\mu \nu} \)) which, written out as a matrix, have entries

\[
(\mathbf{E}_H)^{i0} = F^{i0} = -(\mathbf{E}_H)^{0i} = E^i, \quad (\mathbf{E}_H)^{ij} = 0, \\
(\mathbf{B}_H)^{i0} = (1/2c)\varepsilon^{kl0} F_{kl} = -(\mathbf{B}_H)^{0i} = B^i, \quad (\mathbf{B}_H)^{ij} = 0. \tag{14}
\]

Using the results from Sec. 3.1. we now apply the active LT to \( \mathbf{E}_H \) and \( \mathbf{B}_H \) from (13). For simplicity, as in Secs. 3.1. and 3.2., we again consider boosts in the direction \( \gamma_1 \) for which the rotor \( \tilde{R} \) is given by the relation (7) with \( \gamma_1 \) replacing \( n \). Then using (13) the Lorentz transformed \( \mathbf{E}'_H \) can be written as

\[
\mathbf{E}'_H = R[(F \cdot \gamma_0)\gamma_0] \tilde{R} = E^1\gamma_1 \wedge \gamma_0 + \gamma(E^2\gamma_2 \wedge \gamma_0 + E^3\gamma_3 \wedge \gamma_0) - \beta \gamma(E^2\gamma_2 \wedge \gamma_1 + E^3\gamma_3 \wedge \gamma_1). \tag{15}
\]

The components \( (\mathbf{E}'_H)^{\mu \nu} \) that are different from zero are \( (\mathbf{E}'_H)^{01} = -E^1, (\mathbf{E}'_H)^{02} = -\gamma E^2, (\mathbf{E}'_H)^{03} = -\gamma E^3, (\mathbf{E}'_H)^{12} = \beta \gamma E^2, (\mathbf{E}'_H)^{13} = \beta \gamma E^2, \) \( (\mathbf{E}'_H)^{\mu \nu} \) is antisymmetric, i.e., \( (\mathbf{E}'_H)^{\nu \mu} = -(\mathbf{E}'_H)^{\mu \nu} \) and we denoted, as in (14), \( E^3 = F^{00} \). Similarly we find for \( \mathbf{B}'_H \)

\[
\mathbf{B}'_H = R[(-1/c)\gamma_5((F \wedge \gamma_0) \cdot \gamma_0)] \tilde{R} = B^1\gamma_1 \wedge \gamma_0 + \gamma(B^2\gamma_2 \wedge \gamma_0 + B^3\gamma_3 \wedge \gamma_0) - \beta \gamma(B^2\gamma_2 \wedge \gamma_1 + B^3\gamma_3 \wedge \gamma_1). \tag{16}
\]

The components \( (\mathbf{B}'_H)^{\mu \nu} \) that are different from zero are \( (\mathbf{B}'_H)^{01} = -B^1, (\mathbf{B}'_H)^{02} = -\gamma B^2, (\mathbf{B}'_H)^{03} = -\gamma B^3, (\mathbf{B}'_H)^{12} = \beta \gamma B^2, (\mathbf{B}'_H)^{13} = \beta \gamma B^3. (\mathbf{B}'_H)^{\mu \nu} \)
is antisymmetric, i.e., \((B'_H)^\mu_{\nu} = -(B'_H)^{\nu}_{\mu}\) and we denoted, as in (12), \(B^i = (1/2c)\epsilon^{kli0}F_{kl}\). Both (15) and (16) are the familiar forms for the active LT of the bivectors, here \(E_H\) and \(B_H\). It is important to note that \(E'_H\) and \(B'_H\), in contrast to \(E_H\) and \(B_H\), are not parallel to \(\gamma_0\), i.e., it does not hold that \(E'_H \wedge \gamma_0 = B'_H \wedge \gamma_0 = 0\) and thus there are \((E'_H)^{ij}_{} \neq 0\) and \((B'_H)^{ij}_{} \neq 0\). Further, as in Sec. 3.1., the components \((E_H)^{\mu\nu}_{} \ (B_H)^{\mu\nu}_{}\) transform upon the active LT again to the components \((E'_H)^{\mu\nu}_{} \ (B'_H)^{\mu\nu}_{}\); there is no mixing of components. Thus by the active LT \(E_H\) transforms to \(E'_H\) and \(B_H\) to \(B'_H\). Actually, as we said, this is the way in which every bivector transforms upon the active LT.

However the standard transformations for \(E'_{H,st}\) and \(B'_{H,st}\) are derived wrongly assuming that the quantities obtained by the active LT of \(E_H\) and \(B_H\) are again parallel to \(\gamma_0\), i.e., that again holds \(E'_H \wedge \gamma_0 = B'_H \wedge \gamma_0 = 0\) and consequently that \((E'_{H,st})^{ij}_{} = (B'_{H,st})^{ij}_{} = 0\). Thence, in contrast to the correct LT of \(E_H\) and \(B_H\), (15) and (16) respectively, it is taken in standard derivations ([4], Space-Time Algebra (eq. (18.22)), New Foundations for Classical Mechanics (Ch. 9 eqs. (3.51a,b))) that

\[
E'_{H,st} = [(RF\tilde{R}) \cdot \gamma_0] \gamma_0 = (F' \cdot \gamma_0) \gamma_0 = E^1 \gamma_1 \wedge \gamma_0 + (\gamma E^2 - \beta \gamma c B^3) \gamma_2 \wedge \gamma_0 + (\gamma E^3 + \beta \gamma c B^2) \gamma_3 \wedge \gamma_0, \tag{17}
\]

where \(F' = RF\tilde{R}\). Similarly we find for \(B'_{H,st}\)

\[
B'_{H,st} = (-1/c)\gamma_5 [(F' \wedge \gamma_0) \cdot \gamma_0] = B^1 \gamma_1 \wedge \gamma_0 + (\gamma B^2 + \beta \gamma E^3/c) \gamma_2 \wedge \gamma_0 + (\gamma B^3 - \beta \gamma E^2/c) \gamma_3 \wedge \gamma_0. \tag{18}
\]

The relations (17) and (18) give the familiar expressions for the standard transformations of the 3D vectors \(E\) and \(B\). Now, in contrast to the correct LT of \(E_H\) and \(B_H\), (15) and (16) respectively, the components of the transformed \(E'_{H,st}\) are expressed by the mixture of components \(E^1\) and \(B^i\), and the same holds for \(B'_{H,st}\).

The same procedure can be easily applied to the transformations of \(E_J\), \(B_J\) from [6] and it will lead to the same fundamental difference between the standard transformations of \(E_J\), \(B_J\) obtained in [6] and their correct LT. Again the active LT of \(E_J\), \(B_J\) will be given by

\[
E'_J = R(F \cdot \gamma_0)\tilde{R}, \quad B'_J = R[(-1/c)\gamma_5(F \wedge \gamma_0)]\tilde{R}, \tag{19}
\]
while the standard transformations from [6] will follow from

\[ \mathbf{E}'_{J,st} = (RF\bar{R}) \cdot \gamma_0, \quad \mathbf{B}'_{J,st} = -(1/c)[\gamma_5((RF\bar{R}) \wedge \gamma_0)]. \quad (20) \]

For brevity the whole discussion will not be done here. Of course the discussion from Sec. 3.2. regarding the passive LT and the ‘apparent’ transformations applies in the same measure to the results of this section.

4. The 1-vectors of the electric and magnetic fields \( E \) and \( B \)

In order to have the electric and magnetic fields defined without reference frames, i.e., \textit{independent of the chosen reference frame and of the chosen system of coordinates in it}, one has to replace \( \gamma_0 \) (the velocity in units of \( c \) of an observer at rest in the \( \gamma_0 \)-system) in the relation (3) (and (1), (2) as well) with \( v \). The velocity \( v \) and all other quantities entering into the relations (3) (and (1), (2) as well), but with \( v \) replacing \( \gamma_0 \), are then defined without reference frames. \( v \) characterizes some general observer. We can say, as in tensor formalism \[14\], that \( v \) is the velocity (1-vector) of a family of observers who measures \( E \) and \( B \) fields. With such replacement the relation (3) becomes

\[ F = (1/c)E \wedge v + (e_5B) \cdot v, \]
\[ E = (1/c)F \cdot v, \quad B = -(1/c^2)e_5(F \wedge v), \quad (21) \]

and it holds that \( E \cdot v = B \cdot v = 0 \). Of course \textit{the relations for \( E \) and \( B \) are independent of the chosen observer; i.e., they hold for any observer}. When some reference frame is chosen with the Einstein system of coordinates in it and when \( v \) is specified to be in the time direction in that frame, i.e., \( v = c\gamma_0 \), then all results of the classical electromagnetism are recovered in that frame. Namely we can always select a particular - but otherwise arbitrary - inertial frame of reference \( S \), the frame of our ‘fiducial’ observers in which \( v = c\gamma_0 \) and consequently the temporal components of \( E^\mu_f \) and \( B^\mu_f \) are zero (the subscript ‘\( f \)’ is for ‘fiducial’). Then in that frame the usual Maxwell equations for the spatial components \( E^i_f \) and \( B^i_f \) (of \( E^\mu_f \) and \( B^\mu_f \)) will be fulfilled. As a consequence the usual Maxwell equations can explain all experiments that are performed in one reference frame. Thus the correspondence principle is simply and naturally satisfied. However as shown above the temporal components of \( E^\mu_f \) and \( B^\mu_f \) are not zero; (8) and (9) are
relativistically correct, but it is not the case with (10) and (11). This means that the usual Maxwell equations cannot be used for the explanation of any experiment that test SR, i.e., in which relatively moving observers have to compare their data *obtained by measurements on the same physical object.* However, in contrast to the description of the electromagnetism with the 3D \( E \) and \( B \), the description with \( E \) and \( B \) is correct not only in that frame but in all other relatively moving frames and it holds for any permissible choice of coordinates. It is worth noting that the relations (21) are not the definitions of \( E \) and \( B \) but they are the relations that connect two equivalent formulations of electrodynamics, the standard formulations with the \( F \) field and the new one with the \( E \) and \( B \) fields. Every of these formulations is an independent, complete and consistent formulation. For more detail see [7] where four equivalent formulations are presented, the \( F \) and \( E, B \) - formulations and two new additional formulations with real and complex combinations of \( E \) and \( B \) fields. All four formulations are given in terms of quantities that are defined without reference frames. In the recent work [15] I have presented the formulation of relativistic electrodynamics (independent of the reference frame and of the chosen system of coordinates in it) that uses the bivector field \( F \). This formulation with \( F \) field is, as already said, a self-contained, complete and consistent formulation that dispenses with either electric and magnetic fields or the electromagnetic potentials. Note however that in the \( E, B \) - formulation of electrodynamics in [7] the expression for the stress-energy vector \( T(v) \) and all quantities derived from \( T(v) \) are written for the special case when \( v \), the velocity of observers who measure \( E \) and \( B \) fields is \( v = cn \), where \( n \) is the unit normal to a hypersurface through which the flow of energy-momentum \( (T(n)) \) is calculated. The more general case with \( v \neq n \) will be reported elsewhere.

In addition, as we have already said, the replacement of \( \gamma_0 \) with \( v \) in the relations (1) and (2) also yields the electric and magnetic fields defined without reference frames. However, it is much simpler and, in fact, closer to the classical formulation of the electromagnetism with the 3D \( E \) and \( B \) to work with 1-vectors \( E \) and \( B \) instead of to use the bivectors \( E_H, B_H \) and \( B_J \) (but all with \( v \) replacing \( \gamma_0 \)).

We have not mentioned some other references that refer to the Clifford algebra formalism and its application to the electrodynamics as are, e.g., [16]. The reason is that they use the Clifford algebra formalism with spinors but, of course, they also erroneously consider that the standard transformations of the 3D \( E \) and \( B \) (12) are the LT of the electric and magnetic fields.
5. Conclusions

The whole consideration explicitly shows that the 3D quantities $E$ and $B$, their transformations and the equations with them are ill-defined in the 4D spacetime. More generally, the 3D quantities do not have an independent physical reality in the 4D spacetime. Contrary to the general belief we find that it is not true from the SR viewpoint that observers in relative motion see different fields; the transformations (10), (11) and (12) (or (17) and (18)) are not relativistically correct. According to the relativistically correct transformations, the LT (8) and (9), (or (15) and (16)) the electric field transforms only to the electric field and the same holds for the magnetic field. Thence the relativistically correct physics must be formulated with 4D quantities that are defined without reference frames, or by the 4D coordinate-based geometric quantities, e.g., as in [7] in the Clifford algebra formalism with multivectors, or [8,9] in the tensor formalism. The principle of relativity is automatically included in such theory with well-defined 4D quantities, while in the standard approach to SR [1] it is postulated outside the mathematical formulation of the theory. The comparison with experiments from [9] (and [7]) reveals that the true agreement with experiments that test SR can be achieved when such well-defined 4D quantities are considered.

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