ON CONTINUING CODES

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Abstract. We demonstrate that continuing block codes on general sofic shifts do not behave as well as in the case of shifts of finite type; a continuing block code on a sofic shift need not have a uniformly bounded retract, unlike one on a shift of finite type. A right resolving code on a sofic shift can display any behavior arbitrary block codes can have. We also show that a right continuing factor of a shift of finite type is necessarily a shift of finite type.

1. Introduction

Right continuing codes are a natural generalization of right resolving codes and right closing codes. A lot is known about right closing codes between shift spaces [1, 5]. Right closing codes are always finite-to-one (or bounded-to-one) factor maps. Continuing codes are infinite-to-one analog of closing codes.

Continuing codes between SFTs work nice with Markov measures: if \( \phi \) is a continuing code from an SFT \( X \) onto another SFT \( Y \), each Markov measure on \( Y \) lifts to uncountably many Markov measures on \( X \) (see [2]); if \( \phi \) is a fiber-mixing code from an SFT \( X \) onto another SFT \( Y \), i.e., given \( x, \bar{x} \in X \) with \( \phi(x) = \phi(\bar{x}) = y \in Y \), there is \( z \in \phi^{-1}(y) \) that is left asymptotic to \( x \) and right asymptotic to \( \bar{x} \), then \( \phi \) projects each Markov measure on \( X \) onto a Gibbs measure on \( Y \) with a Hölder continuous potential (see [3, 6]). Fiber-mixing condition implies that \( \phi \) is left continuing and right continuing.

We investigate how right continuing codes behave with respect to shift spaces that are not SFTs. We show that right continuing codes defined on general shift spaces don’t necessarily have a retract. We show that there is no right continuing codes from an SFT onto a non-SFT. We also learn that the property of right resolving codes on non-SFTs are not strong enough to tame block codes in any meaningful way, in particular, the property of right resolving doesn’t even imply the property of right continuing.

We hope that this work will be helpful in understanding right continuing codes and infinite-to-one codes on general shift spaces.

2. Background

An SFT or a shift of finite type is a shift space defined by a finite set of forbidden words. A non-SFT is a shift space that is not an SFT.

A block code is a homomorphism between two topological dynamical systems that are shift spaces. A block code \( \phi \) is a 1-block code if \( \phi(x)_0 \) is determined by \( x \). A block code from \( X \) onto \( Y \) is called a factor map between \( X \) and \( Y \), and \( Y \) is called a factor of \( X \). Factors of SFTs are called sofic shifts.

Definition 2.1. The language \( B(X) \) of a shift space \( X \) is the collection of words appearing in points of \( X \). The set \( B_n(X) \) is defined to be the collection of words of length \( n \) appearing in points of \( X \).

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Definition 2.2. Two points $x, x'$ in a shift space are left asymptotic (right asymptotic) to each other if there is $n \in \mathbb{Z}$ with $x_{[-\infty,n]} = x'_{[-\infty,n]}$ ($x_{[n,\infty)} = x'_{[n,\infty)}$).

Definition 2.3. A 1-block code $\phi$ from a shift space $X$ onto another shift space $Y$ is right resolving if for $a_0 \in \mathcal{B}_1(X)$ and $b_0h_1 \in \mathcal{B}_2(Y)$ with $\phi(a_0) = b_0$, there is at least one $a_1$ with $a_0a_1 \in \mathcal{B}_2(X)$ and $\phi(a_1) = b_1$.

Definition 2.4. A block code $\phi$ from a shift space $X$ onto another shift space $Y$ is right continuing or u-resolving if for each $x \in X$ and $y \in Y$ such that $\phi(x)$ is left-asymptotic to $y$, there is at least one $\bar{x} \in X$ such that $\bar{x}$ is left asymptotic to $x$ and $\phi(\bar{x}) = y$.

The term left resolving, left continuing and s-resolving are defined analogously.

Definition 2.5. A block code from a shift space onto another shift space is continuing if it is left continuing or right continuing.

Definition 2.6. A right continuing code without a retract $\phi$ from a shift space $X$ onto another shift space $Y$ is said to have a retract if for all $x \in X$ and $y \in Y$ with $(\phi x)_i = y_i$ for all $i \leq 0$, there is $\bar{x} \in X$ with $\phi \bar{x} = y$ and $\bar{x}_i = x_i$ for all $i \leq -n$.

We mention a result [2] on a relationship between right resolving codes and right continuing codes on SFTs. (The term right resolving in the paper [2] corresponds to the term right resolving in this paper)

Proposition 2.1. For a factor map $\phi$ between SFTs, the followings are equivalent.

1. $\phi$ is topologically equivalent to a right resolving 1-block factor map between 1-step SFTs.
2. $\phi$ is right continuing.
3. $\phi$ is right continuing and has a retract.

After we prove that the right continuing factor of an SFT is always an SFT, the requirement of the codomain being an SFT from the above proposition can be safely dropped.

3. A RIGHT CONTINUING CODE WITHOUT A RETRACT

Example 3.1. Let $X$ be a shift space with the alphabet $\{1, 2, 3\}$ defined by forbidden blocks $\{12^n3 : n \geq 0\}$ and let $Y$ be the full shift $\{1, 2, 3\}^\mathbb{Z}$. Then $X$ is an irreducible sofic shift. Let $\phi : X \to Y$ be the 1-block code defined by $\phi(1) = \phi(1) = 1, \phi(2) = 2, \phi(3) = 3$. Then the map $\phi$ is a right-continuing factor without a retract.

Proof. The map $\phi$ is clearly onto $Y$.

Next, we need to check that it is right continuing. Suppose $x \in X$ and $y \in Y$ with $\phi(x_i) = y_i$ for all $i \leq 0$. If $x^{(1)} := x_{[-\infty,-1]}x_0y_{[1,\infty)}$ is in $X$, then we are done because $x^{(1)}$ is left-asymptotic to $x$ and $\phi(x^{(1)}) = y$. Otherwise, the word $12^n3$ for some $n \geq 0$ occurs exactly once in $x^{(1)}$. Let $x^{(2)}$ be the point obtained from $x^{(1)}$ by replacing the occurrence of $12^n3$ with $12^n3$. Then the point $x^{(2)}$ is in $X$, is left-asymptotic to $x$ and satisfies $\phi(x^{(2)}) = y$. This completes the proof of right continuing property.

Finally, we check that the map $\phi$ has no retract. This follows from considering, for each $n \geq 0$, the points $x = 1^{\infty}2^n.22^{\infty} \in X$ and $y = 1^{\infty}2^n.23^{\infty} \in Y$.

This example and the proof are also included in [3] by personal communication.
4. Right eresolving codes on non-SFTs

Consider an arbitrary 1-block code $\phi$ from a shift space $X$ onto another shift space $Y$. One can transform $\phi : X \to Y$ into a right eresolving code $\sqrt{\phi} : \sqrt{X} \to \sqrt{Y}$ without losing too much of the original code’s property.

First, we choose a symbol $a$ disjoint from the alphabets of $X$ and $Y$. Then define shift spaces $\sqrt{X} = \{ \eta(x), \sigma(\eta(x)) : x \in X \}$ and $\sqrt{Y} = \{ \eta(y), \sigma(\eta(y)) : y \in Y \}$ where $\eta$ is defined by $\eta(x)_{2i} = x_i$ and $\eta(x)_{2i+1} = a$ for all $i \in \mathbb{Z}$. Then define a 1-block code $\sqrt{\phi}$ from $\sqrt{X}$ onto $\sqrt{Y}$ by

$$
\sqrt{\phi}(c) = \begin{cases} 
a & \text{if } c = a, \\ \phi(a) & \text{if } c \neq a
\end{cases}
$$

for $c \in B_1(\sqrt{X})$.

The map $\sqrt{\phi}$ is easily seen to be right eresolving and shares many properties with the original $\phi$. For example, if $\phi$ is not right continuing, then neither is $\sqrt{\phi}$. Thus right eresolving codes are not necessarily right continuing. This is in contrast with Proposition 2.4 in the case of SFTs.

5. A right continuing factor of an SFT is an SFT

A right eresolving factor of 1-step SFT is easily seen to be a 1-step SFT. Since a right continuing codes between SFTs can be assumed right eresolving by recording (see Proposition 2.4), one might think that this would easily imply that right continuing factor of an SFT is an SFT. But the proof of Proposition 2.4 uses the assumption of $Y$ being an SFT, which in turn is what we need to prove. Nevertheless, we manage to prove the following result.

Theorem 5.1. If $X$ is an SFT and $\phi$ a right continuing code onto a shift space $Y$. Then $Y$ is an SFT.

In this section from now on, we assume that $\phi$ is a right continuing 1-block code from a 1-step SFT $X$ onto a shift space $Y$ unless stated otherwise. Note that we cannot assume without loss of generality that $\phi$ is eresolving to make things easier, because use of Proposition 2.4 requires the condition that $Y$ is an SFT, which in turn is what we are trying to prove. We will not assume the retract to be 0 without loss of generality because we want to see how the memory of $Y$ grows depending on the size of the retract. We establish our theorem through a series of lemmas.

Lemma 5.2. There is $R \in \mathbb{N}$ such that for all $x \in X$ and $y \in Y$ with $\phi(x_i) = y_i$, $i \leq 0$, there exist $x'_{[-R,1]} \in B(X)$ with $x'_{-R} = x_{-R}$ and $\phi(x'_1) = y_1$, $i \in [-R,1]$.

Proof. Suppose it is false. Then for all $n \in \mathbb{N}$, there exist $x^{(n)} \in X$ and $y^{(n)} \in Y$ such that $\phi(x^{(n)}_i) = y^{(n)}_i$, $i \leq 0$ and that there does not exist a word $x'_{[-n,1]} \in B(X)$ satisfying

$$(*) \quad x^{(n)} = x'_{-n}, \quad \phi(x'_{[-n,1]}) = y^{(n)}_{[-n,1]}.$$

By compactness argument, there is a limit point $(x,y) \in X \times Y$ of the sequence of pairs $(x^{(n)},y^{(n)})$. We have $\phi(x_i) = y_i$, $i \leq 0$ and since $\phi$ is right continuing, there are $\bar{n} \in \mathbb{N}$ and $\bar{x} \in X$ such that $\phi(\bar{x}) = y$ and $\bar{x}_i = x_i$, $i \leq -\bar{n}$. We can choose $n \in \mathbb{N}$ such that $n > \bar{n}$ and $x_i = x^{(n)}_i$, $y_i = y^{(n)}_i$, $i \in [-\bar{n},1]$. But the word $x'_{[-n,1]} := x^{(n)}_{-n-\bar{n}-1}x'_{[-n,1]}$, which is in $B(X)$ because $X$ is a 1-step SFT, satisfies $(*)$ and that contradicts our initial assumption.

Since $X$ is assumed to be a 1-step SFT, it is easily shown by repeated application of the previous lemma that the number $R$ is a retract of $\phi$. 
Lemma 5.3. There is $K \in \mathbb{N}$ such that for all $\pi \in \mathcal{B}_K(X)$ and $u \in \mathcal{B}_{K+1}(Y)$ with $\phi(\pi)$ being a prefix of $u$, there exist $\pi' \in \mathcal{B}_{K+1}(X)$ with $\pi_1 = \pi'_1$ and $\phi(\pi') = u$.

Proof. Let $d := (|\mathcal{B}_1(X)|)^2 + 1$ and $K := d + R + 1$ and consider two words $x_{[-R-d,0]} \in \mathcal{B}_K(X)$ and $y_{[-R-d,1]} \in \mathcal{B}_{K+1}(Y)$ with $\phi(x_{[-R-d,0]}) = y_{[-R-d,0]}$.

We can choose a $\phi$-preimage $\bar{x}_{[-R-d,1]} \in \mathcal{B}(X)$ of $y_{[-R-d,1]}$. By pigeon hole principle, there exist $-R - d \leq I < J \leq -R - 1$ such that $x_I = x_J$ and $\bar{x}_I = \bar{x}_J$. Build a sequence $x_{1',x_{2},\cdots}$ that extends $x_{[-R-d,0]}$ and $\bar{x}_{1',\bar{x}_2,\cdots}$ that extends $\bar{x}_{[-R-d,1]}$. Now $x_{[-R-d,\infty)}$ and $\bar{x}_{[-R-d,\infty]}$ are right-infinite sequences allowed in $X$. The point $\hat{x} := (x_{[I,J-1]}^\infty,x_{[J-1],x_{[0,\infty)}}$ is in $X$. The point $(\bar{x}_{[I,J-1]}^\infty,\bar{x}_{[J-1],\bar{x}_{[0,\infty)}}$ is also in $X$ and its image $\bar{y} := (y_{[I,J-1]}^\infty,y_{[J-1],y_0y_1\phi(\bar{x}_2)\phi(\bar{x}_3)\cdots}$ is in $Y$ and we have $\phi(\hat{x}_i) = \bar{y}_i$, $i \leq 0$. By applying the previous lemma to $\hat{x}$ and $\bar{y}$, we can see that there exist $x_{[-R,1]}' \in \mathcal{B}(X)$ satisfying $\hat{x}'_{-R} = \hat{x}_{-R} = x_{-R}$ and $\phi(\hat{x}_{[-R,1]}) = \bar{y}_{[-R,1]} = y_{[-R,1]}$.

Define $x_{[-R-d,-R-1]}' := x_{[-R-d,-R-1]}$, then we have $x_{[-R-d,1]}' \in \mathcal{B}(X)$ and $\phi(x_{[-R-d,1]}) = y_{[-R,1]}$.

Remark 5.1. In order to prove above lemma, one might be tempted to just set $K := R$ and start by extending the word $\pi$ to a point $x \in X$ such that $x_{[-R,0]} = \pi$ and then somehow extend $y$ to apply Lemma 5.2. But choosing an appropriate $y \in Y$ requires showing that $x_{[-\infty,-R-1]}u \in \mathcal{B}(Y)$, which you cannot show because $Y$ is not yet known to be an SFT, let alone an $R$-step SFT.

Lemma 5.4. For words $u,v \in \mathcal{B}(Y)$ and a symbol $a \in \mathcal{B}_1(Y)$ such that $uv \in \mathcal{B}(Y)$, $va \in \mathcal{B}(Y)$ and $|v| = K$, we have $uva \in \mathcal{B}(Y)$.

Proof. There exist $\pi,\tau \in \mathcal{B}(X)$ such that $\pi \tau \in \mathcal{B}(X)$, $\phi(\pi \tau) = uv$ and $|\tau| = K$. Applying the previous lemma to $\tau$ and $va$, we see that there is $\tau' \in \mathcal{B}(X)$ with $\tau_1 = \tau'_1$ and $\phi(\tau') = va$. Clearly $\pi \tau' \in \mathcal{B}(X)$ and $\phi(\pi \tau') = uva$.

Proposition 5.5. $Y$ is $K$-step SFT.

Proof. It follows easily from the previous lemma.

The proof of Theorem 5.1 is now complete.

Remark 5.2. Since $Y$ is now shown to be an SFT, Proposition 2.1 applies and therefore $\phi$ is topologically equivalent to a right-eresolving code from a 1-step SFT onto another 1-step SFT.

Remark 5.3. This result implies in particular that there is no right continuing code from an SFT onto a strictly sofic shift. This contrasts with the fact that every sofic shift is a right closing factor of an SFT.

Question 5.6. We showed that $Y$ is a $K$-step SFT where $K = R + 2 + (|\mathcal{B}_1(X)|)^2$. Is there a better bound on $K$?

6. Decreasing the Retract of a Right Continuing Map

Theorem 6.1. Given $\phi$ a right continuing factor map with a retract from a shift space $X$ onto a shift space $Y$, there exist a topological conjugacy $\psi$ from $X$ to a shift space $\hat{X}$ such that $\phi \circ \psi^{-1} : \hat{X} \to Y$ is a right continuing 1-block factor map with retract $\psi$.

Proof. We may assume that $\phi$ is a 1-block code and its retract is $R$.

Define a block code $\psi$ on $X$ by

$$(\psi x)_i = (x_{i-R}, (\phi x)_{[i-R,i]}).$$
for all \( x \in X \) and \( i \in \mathbb{Z} \). This code is injective, therefore a conjugacy onto its image \( X \).

Let \( \tilde{\phi} := \phi \circ \psi^{-1} \). This is a 1-block code because for \( \bar{x} \in \bar{X} \) and \( x := \psi^{-1}(\bar{x}) \) we have

\[
(\tilde{\phi} x)_0 = (\phi x)_0 \\
= F((\psi x)_0) \\
= F(x_0)
\]

where \( F \) is a function that maps \((a, b_0 b_1 \ldots b_R)\) to \( b_R \).

It remains to show that \( \tilde{\phi} \) is right continuing with retract 0. Suppose we are given \( \bar{x} \in \bar{X} \), \( y \in Y \) with \((\phi \bar{x})_{[-\infty,0]} = y_{[-\infty,0]} \). Then the point \( x := \psi^{-1}\bar{x} \) satisfies \((\phi x)_{[-\infty,0]} = y_{[-\infty,0]} \). But since \( \phi \) has retract \( R \), there is \( z \in X \) satisfying \( x_{[-\infty, -R]} = z_{[-\infty, -R]} \) and \( \phi z = y \). The point \( \bar{z} := \psi z \) satisfies the desired properties: for \( i \leq 0 \)

\[
\bar{z}_i = (z_{i-R}, (\phi z)_{i-R, i+R}) \\
= (z_{i-R}, y_{i-R, i+R}) \\
= (x_{i-R}, (\phi x)_{i-R, i+R}) \\
= \bar{x}_i
\]

and \( \tilde{\phi} \bar{z} = y \).

**Theorem 6.2.** Given a bi-continuous factor map \( \phi \) with retracts from a shift space \( X \) to a shift space \( Y \), there are conjugacies \( \psi : X \to \bar{X} \) and \( \theta : Y \to \bar{Y} \) such that \( \theta \circ \phi \circ \psi^{-1} : X \to \bar{Y} \) is a bi-continuous 1-block factor map with retracts \( \theta \).

**Proof.** We may assume \( \phi \) is a 1-block code with retracts \( R \) (same \( R \) in both directions). Let \( \bar{X} \) and \( \bar{Y} \) be images of block codes \( \psi \) and \( \theta \) defined by

\[
(\psi x)_i = (x_i, (\phi x)_{i-R, i+R}) \\
(\theta y)_i = y_{i-R, i+R}
\]

for \( i \in \mathbb{Z}, x \in X \) and \( y \in Y \). They are easily checked to be injective, hence they are conjugacies.

Let \( \tilde{\phi} := \theta \circ \phi \circ \psi^{-1} : \bar{X} \to \bar{Y} \), then it is a 1-block code since for all \( \bar{x} \in \bar{X} \) and \( x := \psi^{-1}\bar{x} \) we have

\[
(\tilde{\phi} x)_0 = (\theta \phi x)_0 \\
= (\phi x)_{i-R, i+R} \\
= F((\psi x)_0) \\
= F(x_0)
\]

where the function \( F \) maps \((a, b)\) to \( b \).

To show that \( \tilde{\phi} \) is right continuing with retract 0, suppose we are given \( \bar{x} \in \bar{X} \) and \( \bar{y} \in \bar{Y} \) with \((\tilde{\phi} \bar{x})_{[-\infty,0]} = \bar{y}_{[-\infty,0]} \) and let \( x = \psi^{-1}\bar{x} \) and \( y = \theta^{-1}\bar{y} \). Then we have \( y_{i-R, i+R} = (\phi x)_{i-R, i+R} \) and since \( \phi \) is right continuing with retract \( R \), there is \( z \in X \) such that \( x_{[-\infty,0]} = z_{[-\infty,0]} \) and \( \phi z = y \). Let \( \bar{z} = \psi z \), then we have for \( i \leq 0 \),

\[
\bar{x}_i = (x_i, (\phi x)_{i-R, i+R}) \\
= (x_i, y_{i-R, i+R}) \\
= (z_i, (\phi z)_{i-R, i+R}) \\
= \bar{z}_i
\]

and that \( \tilde{\phi} \bar{z} = \bar{y} \).

Similarly, \( \tilde{\phi} \) is also left continuing with retract 0.

\( \square \)
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