Off-diagonal Bethe Ansatz on the $so(5)$ spin chain

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Abstract

The $so(5)$ (i.e., $B_2$) quantum integrable spin chains with both periodic and non-diagonal boundaries are studied via the off-diagonal Bethe Ansatz method. By using the fusion technique, sufficient operator product identities (comparing to those in [2]) to determine the spectrum of the transfer matrices are derived. For the periodic case, we recover the results obtained in [1], while for the non-diagonal boundary case, a new inhomogeneous $T−Q$ relation is constructed. The present method can be directly generalized to deal with the $so(2n+1)$ (i.e., $B_n$) quantum integrable spin chains with general boundaries.

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1 Introduction

Both the algebraic and coordinate Bethe Ansatz are very powerful methods to construct exact solutions of quantum integrable models [2–7]. Nevertheless, those methods still have their restrictions for depending on existence of an obvious reference state. An important issue is that when $U(1)$-symmetry is broken, the systems may not have obvious reference states. In such cases, the problem becomes more frustrated and many interesting efforts have been made in this direction [8–25] in the past several decades. A generic method named off-diagonal Bethe ansatz (ODBA) for solving quantum integrable models either with or without $U(1)$-symmetry was proposed in [26]. By constructing the inhomogeneous $T − Q$ relations based on some operator identities, several typical models are solved exactly [27]. With the resulting eigenvalues, the corresponding Bethe-type eigenstates can also be retrieved [28,29].

The nested ODBA was initially proposed in studying the $su(n)$ (i.e., $A_n$) spin chain with generic boundaries [30,31]. However, ODBA to approach high-rank quantum integrable models associated with $B_n$, $C_n$ and $D_n$ Lie algebras is still missing. We note that such kind of models with obvious $U(1)$-symmetry has been studied extensively. For example, with some functional relations and algebraic Bethe Ansatz analysis (the analytic Bethe Ansatz method), Reshetikhin derived the energy spectrum of the periodic quantum spin chains associated with $B_n$, $C_n$, $D_n$ and other Lie algebras [1,32]. The algebraic Bethe Ansatz for those models with periodic boundary condition was constructed by Martins and Ramos [33], while the method for approaching such kind of models with diagonal open boundaries was developed by Li, Shi and Yue [34,35].

In this paper, we develop a nested ODBA method to approach the quantum integrable $so(5)$ (i.e., $B_2$) spin chain with either periodic or non-diagonal open boundary condition. This method can be generalized to $so(2n + 1)$ (i.e., $B_n$) case directly. The paper is organized as follows. In section 2, we study the $so(5)$ model with periodic boundary condition. Closed functional relations among the transfer matrices to determine the eigenvalues are constructed with fusion techniques. In section 3, we study the $so(5)$ model with an off-diagonal open boundary condition. By constructing some operator product identities, we derive the exact eigenvalues of the transfer matrix in terms of an inhomogeneous $T − Q$ relation. Section 4 is attributed to concluding remarks. Some detailed calculations are listed in Appendices A-C.
2 $so(5)$ spin chain with periodic boundary condition

2.1 The model

Let $V$ denote a 5-dimensional linear space with an orthonormal basis \{\{|i\rangle| i = 1, \cdots , 5\}\} which endows the fundamental representation of the $so(5)$ (or $B_2$) algebra. The quantum spin chain associated with the $B_2$ algebra is described by a $25 \times 25$ $R$-matrix $R_{12}^{vu}(u)$ defined in the $V \otimes V$ space with the matrix elements \[ R_{12}^{vu}(u)_{ij}^{kl} = u(u + \frac{3}{2})\delta_{ik}\delta_{jl} + (u + \frac{3}{2})\delta_{il}\delta_{jk} - u\delta_{ij}\delta_{kl}, \] (2.1)

where \{\{i, j, k, l\}\} = \{1, 2, 3, 4, 5\}, $i + \bar{i} = 6$. We introduce the notation for simplicity

\[ R_{12}^{vu}(u)_{ii}^{ii} = a_1(u) = (1 + u)(u + \frac{3}{2}), \quad i \neq 3, \]
\[ R_{12}^{vu}(u)_{ij}^{ij} = b_1(u) = u(u + \frac{3}{2}), \quad i \neq j, \bar{j}, \]
\[ R_{12}^{vu}(u)_{ii}^{i\bar{i}} = c_1(u) = \frac{3}{2}, \quad i \neq \bar{i}, \]
\[ R_{12}^{vu}(u)_{ij}^{j\bar{i}} = d_1(u) = -u, \quad i \neq j, \bar{j}, \]
\[ R_{12}^{vu}(u)_{ii}^{i\bar{i}} = e_1(u) = u(u + \frac{1}{2}), \quad i \neq \bar{i}, \]
\[ R_{12}^{vu}(u)_{ij}^{ji} = f_1(u) = a_1(u) + d_1(u), \quad i = 3, \]
\[ R_{12}^{vu}(u)_{ij}^{j\bar{i}} = g_1(u) = u + \frac{3}{2}, \quad i \neq j, \bar{j}. \] (2.2)

The $R$-matrix satisfies the properties

\[ R_{12}^{vu}(0) = \rho_1(0)^{\frac{1}{2}}\mathcal{P}_{12}, \]

regularity : \[ R_{12}^{vu}(u)R_{21}^{vu}(-u) = \rho_1(u) = a_1(u)a_1(-u), \]

unitarity : \[ R_{12}^{vu}(u) = \rho_1(u), \]

crossing – unitarity : \[ R_{12}^{vu}(u)^{t_1}R_{21}^{vu}(-u - 3)^{t_1} = \rho_1(u + \frac{3}{2}), \] (2.3)

where $\mathcal{P}_{12}$ is the permutation operator with the matrix elements $[\mathcal{P}_{12}]_{kl}^{ij} = \delta_{il}\delta_{jk}$, $t_i$ denotes the transposition in the $i$-th space, and $R_{21} = \mathcal{P}_{12}R_{12}\mathcal{P}_{12}$. Here and below we adopt the standard notation: for any matrix $A ∈ \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of $R$-matrix in the tensor space, which acts as an
identity on the factor spaces except for the \(i\)-th and \(j\)-th ones. The \(R\)-matrix satisfies the Yang-Baxter equation

\[
R_{12}^{uv}(u - v)R_{13}^{vw}(u)R_{23}^{vw}(v) = R_{23}^{vw}(v)R_{13}^{vw}(u)R_{12}^{uv}(u - v).
\]

For the periodic boundary condition, we introduce the monodromy matrix

\[
T_v^0(u) = R_{01}^{uv}(u - \theta_1)R_{02}^{uv}(u - \theta_2) \cdots R_{0N}^{uv}(u - \theta_N),
\]

where the index 0 indicates the auxiliary space and the other tensor space \(V^{\otimes N}\) is the physical or quantum space, \(N\) is the number of sites and \(\{\theta_j\}\) are the inhomogeneous parameters. The monodromy matrix satisfies the Yang-Baxter relation

\[
R_{12}^{uv}(u - v)T_1^v(u)T_2^v(v) = T_2^v(v)T_1^v(u)R_{12}^{uv}(u - v).
\]

The transfer matrix is the trace of monodromy matrix in the auxiliary space

\[
t(p)(u) \equiv t_1^p(u) = tr_0 T_0^v(u).
\]

From the Yang-Baxter relation, one can prove that the transfer matrices with different spectral parameters commute with each other, \([t(p)(u), t(q)(v)] = 0\). Therefore, \(t(p)(u)\) serves as the generating function of all the conserved quantities of the system. The Hamiltonian is given by

\[
H_p = \frac{\partial \ln t(p)(u)}{\partial u} \bigg|_{u=0, \{\theta_j\}=0}.
\]

### 2.2 Spinorial \(R\)-matrix and the fused ones

In order to obtain closed operator product identities (see (2.43)-(2.49) below) which allow one to completely determine the eigenvalues of the transfer matrix \(t(p)(u)\), we need further an \(R\)-matrix associated with the spinorial representation of the \(so(5)\) algebra. Let us denote the spinorial representation by \(V^{(s)}\) with an orthonormal basis \(\{|i\rangle^{(s)}| i = 1, \cdots, 4\}\). The spinorial \(16 \times 16\) \(R\)-matrix has the following non-zero matrix elements \[36\]

\[
R_{12}^{ss}(u)_{ii}^{ij} = a_2(u) = (u + \frac{1}{2})(u + \frac{3}{2}),
\]

\[
R_{12}^{ss}(u)_{ij}^{ij} = b_2(u) = u(u + \frac{3}{2}), \quad i \neq j, \bar{j},
\]

\[
R_{12}^{ss}(u)_{ii}^{ij} = c_2(u) = u + \frac{3}{4},
\]

\[
R_{12}^{ss}(u)_{ij}^{ij} = d_2(u) = (u + \frac{1}{2})(u + \frac{3}{2}).
\]
\[ \xi_i \xi_j R_{12}^{ss}(u)_{jj}^{ii} = d_2(u) = -\frac{u}{2}, \quad i \neq j, j, \]
\[ R_{12}^{ss}(u)_{ii}^{ii} = e_2(u) = u(u + 1), \]
\[ R_{12}^{ss}(u)_{ij}^{ij} = g_2(u) = \frac{u}{2} + \frac{3}{4}, \quad i \neq j, j, \]
(2.8)

where \( \{i, j\} = \{1, 2, 3, 4\}, i + \bar{i} = 5, \xi_i = 1 \) if \( i \in \{1, 2\} \) and \( \xi_i = -1 \) if \( i \in \{3, 4\} \). The spinorial \( R \)-matrix satisfies the properties

regularity : \( R_{12}^{ss}(0) = \rho_2(0)^{\frac{1}{2}} \mathcal{P}_{12}^{(s)} \),

unitarity : \( R_{12}^{ss}(u) R_{21}^{ss}(-u) = \rho_2(u) = a_2(u)a_2(-u), \)

crossing – unitarity : \( R_{12}^{ss}(u)^{t_{11}} R_{21}^{ss}(-u - 3)^{t_{11}} = \rho_2(u + \frac{3}{2}), \)
(2.9)

where \( \mathcal{P}_{12}^{(s)} \) is the permutation operator among the spinorial representation space (c.f., \( \mathcal{P}_{12} \) in \( 2.3 \)).

Following the fusion procedure \( 37, 43 \), we can construct another \( R \)-matrix \( R_{12}^{ss}(u) \) defined in \( V^{(s)} \otimes V \). It is easily to check that

\[ R_{12}^{ss}(-\frac{1}{2}) = P_{12}^{ss(5)} \times S, \]
(2.10)

where \( S \) is some non-degenerate constant matrix, and \( P_{12}^{ss(5)} \) is a 5-dimensional projector operator with the form

\[ P_{12}^{ss(5)} = \sum_{i=1}^{5} \langle \tilde{\psi}_i | \tilde{\psi}_i \rangle, \]
(2.11)

where the corresponding vectors are

\[ |\tilde{\psi}_1 \rangle = \frac{1}{\sqrt{2}}(|12\rangle^{(s)} - |21\rangle^{(s)}), \]
\[ |\tilde{\psi}_2 \rangle = \frac{1}{\sqrt{2}}(|31\rangle^{(s)} - |13\rangle^{(s)}), \]
\[ |\tilde{\psi}_3 \rangle = \frac{1}{2}(|14\rangle^{(s)} - |41\rangle^{(s)} + |23\rangle^{(s)} - |32\rangle^{(s)}), \]
\[ |\tilde{\psi}_4 \rangle = \frac{1}{\sqrt{2}}(|24\rangle^{(s)} - |42\rangle^{(s)}), \]
\[ |\tilde{\psi}_5 \rangle = \frac{1}{\sqrt{2}}(|34\rangle^{(s)} - |43\rangle^{(s)}). \]

Let \( V^{(ss)} \) denote the projected subspace of \( V^{(s)} \otimes V^{(s)} \) by the projector \( P_{12}^{ss(5)} \). Namely, \( V^{(ss)} \) is a 5-dimensional subspace and spanned by \( \{ |\tilde{\psi}_i \rangle | i = 1, \ldots, 5 \} \). Then we can construct a fused \( R_{12}^{ss(23)}(u) \) matrix

\[ R_{12}^{ss(23)}(u) = [(u - \frac{1}{4})(u + \frac{3}{4})(u + \frac{7}{4})]^{-1} P_{23}^{ss(5)} R_{12}^{ss}(u + \frac{1}{4}) R_{13}^{ss}(u - \frac{1}{4}) P_{23}^{ss(5)}. \]
(2.12)

\(^3\text{We used a temporal notation } |ij\rangle^{(s)} = |i\rangle^{(s)} \otimes |j\rangle^{(s)}.\)
Moreover, the fused $R$-matrix $R^{ss}_{12}(u)$ defined in $V^{(ss)} \otimes V^{(ss)}$ one has the identification: $V \equiv V^{(ss)}$, which leads to an $R$-matrix $R^{sv}_{12}(u)$ defined in $V^{(s)} \otimes V$. The non-vanishing matrix elements of the resulting $R$-matrix are

\[
R^{sv}(u)_{11}^{11} = R^{sv}(u)_{12}^{12} = R^{sv}(u)_{21}^{31} = R^{sv}(u)_{24}^{24} = R^{sv}(u)_{32}^{32} = R^{sv}(u)_{35}^{35} \\
= R^{sv}(u)_{44}^{44} = R^{sv}(u)_{45}^{45} = a_3(u) = u + \frac{5}{4}, \\
R^{sv}(u)_{14}^{14} = R^{sv}(u)_{15}^{15} = R^{sv}(u)_{22}^{22} = R^{sv}(u)_{25}^{25} = R^{sv}(u)_{31}^{31} = R^{sv}(u)_{34}^{34} \\
= R^{sv}(u)_{41}^{41} = R^{sv}(u)_{42}^{42} = b_3(u) = u + \frac{1}{4}, \\
R^{sv}(u)_{14}^{14} = -R^{sv}(u)_{15}^{15} = R^{sv}(u)_{22}^{22} = -R^{sv}(u)_{31}^{31} = R^{sv}(u)_{34}^{34} \\
= R^{sv}(u)_{41}^{41} = -R^{sv}(u)_{42}^{42} = c_3(u) = -1, \\
R^{sv}(u)_{13}^{13} = R^{sv}(u)_{23}^{23} = R^{sv}(u)_{33}^{33} = R^{sv}(u)_{43}^{43} = e_3(u) = u + \frac{3}{4},
\]

\[
R^{sv}(u)_{22}^{12} = -R^{sv}(u)_{23}^{13} = -R^{sv}(u)_{31}^{31} = R^{sv}(u)_{23}^{33} = -R^{sv}(u)_{13}^{33} = R^{sv}(u)_{32}^{32} = -R^{sv}(u)_{14}^{34} \\
= -R^{sv}(u)_{41}^{23} = R^{sv}(u)_{43}^{25} = R^{sv}(u)_{42}^{32} = -R^{sv}(u)_{43}^{43} = -g_3(u) = -\frac{1}{\sqrt{2}}.
\]

It is easily to check that the fused $R^{sv}_{12}$ matrix has the properties

\[
\text{unitarity} : R^{sv}_{12}(u) R^{sv}_{23}(u) = a_3(u) a_3(-u) \overset{\text{def}}{=} \rho_3(u),
\]

\[
\text{crossing} – \text{unitarity} : R^{sv}_{12}(u) R^{sv}_{23}(u) (-u - 3) \overset{\text{def}}{=} \rho_3(u + \frac{3}{2}).
\]

Moreover, the fused $R^{sv}_{12}$ also satisfy the Yang-Baxter equations

\[
R^{sv}_{12}(u_1 - u_2) R^{sv}_{13}(u_1 - u_3) R^{sv}_{23}(u_2 - u_3) = R^{sv}_{23}(u_2 - u_3) R^{sv}_{13}(u_1 - u_3) R^{sv}_{12}(u_1 - u_2),
\]

and

\[
R^{ss}_{12}(u_1 - u_2) R^{ss}_{13}(u_1 - u_3) R^{ss}_{23}(u_2 - u_3) = R^{ss}_{23}(u_2 - u_3) R^{ss}_{13}(u_1 - u_3) R^{ss}_{12}(u_1 - u_2).
\]

Similarly, one can reconstruct the $R$-matrix $R^{sv}_{12}(u)$ from the fused one $R^{sv}_{12}(u)

\[
R^{sv}_{13}(u) \overset{\text{2.13}}{=} R^{ssv}_{12}(u) = P^{ssv}_{12}(u + \frac{1}{4}) R^{sv}_{13}(u - \frac{1}{4}) P^{ssv}_{12}.
\]

\[\]
We have checked that the $R$-matrices $R_{12}^{uv}(u)$ and $R_{12}^{uv}(u)$ enjoy the properties:

\[
R_{12}^{uv}(-\frac{3}{2}) = P_{12}^{uv(1)} \times S_1,
\]

\[
R_{12}^{uv}(-1) = P_{12} \times S_2,
\]

\[
R_{12}^{uv}(-1)R_{13}^{uv}(-2)R_{23}^{uv}(-1) = P_{123} \times S_3,
\]

\[
R_{12}^{uv}(-1)R_{13}^{uv}(-2)R_{14}^{uv}(-3)R_{23}^{uv}(-1)R_{24}^{uv}(-2)R_{34}^{uv}(-1) = P_{1234} \times S_4,
\]

\[
R_{12}^{uv}(-\frac{5}{4}) = P_{12}^{uv(4)} \times S_5,
\]

where $P_{12}^{uv(1)}$, $P_{12}$, $P_{123}$, $P_{1234}$ and $P_{12}^{uv(4)}$ are the projectors given by (A.14)-(A.19) in Appendix A and $S_i(i = 1, 2, \ldots, 5)$ are some irrelevant constant matrices. With the help of the above projectors and using the similar fusion procedure [37-43], we can construct the fused $R$-matrices

\[
R_{13}^{uv} \equiv P_{12}^{uv(3)} \equiv \tilde{\rho}_0^{-1}(u + \frac{1}{2})P_{21}R_{13}^{uv}(u + \frac{1}{2})R_{23}^{uv}(u - \frac{1}{2})P_{21},
\]

\[
R_{13}^{\tilde{v}u} \equiv P_{12}^{\tilde{v}u(4)} \equiv [\tilde{\rho}_0(u + 1)\tilde{\rho}_0(u)]^{-1}P_{321}R_{13}^{\tilde{v}u}(u+1)R_{24}^{\tilde{v}u}(u)R_{34}^{\tilde{v}u}(u-1)P_{321},
\]

\[
R_{(123)45}^{uv}(u) = \tilde{\rho}_1^{-1}(u)P_{4321}^{uv}R_{15}^{uv}(u)R_{25}^{uv}(u-1)R_{35}^{uv}(u-2)R_{45}^{uv}(u-3)P_{4321},
\]

where

\[
\tilde{\rho}_0(u) = (u - 1)(u + \frac{3}{2}), \quad \tilde{\rho}_1(u) = \tilde{\rho}_0(u)\tilde{\rho}_0(u-1)\tilde{\rho}_0(u-2).
\]

Here we have used $\tilde{V}$ (resp. $\tilde{V}$) to denote the projected subspace in $V \otimes V$ by $P_{21}$ (resp. the projected subspace in $V \otimes V \otimes V$ by $P_{321}$) and adopted the convention: $\bar{1} \equiv (12)$ and $\bar{1} \equiv (123)$.

Some remarks are in order. It is shown that each matrix elements of the above fused $R$-matrices, as a function of $u$, is a polynomial with degree up to two. Due to the fact that the 16-dimensional projected subspace in $V \otimes V \otimes V \otimes V$ by the projector $P_{4321}$ is equivalent to the tensor space $V^{(s)} \otimes V^{(s)}$ according to the correspondence (A.9) below, we have the equivalence

\[
R_{(123)45}^{uv}(u) \equiv S_{12}R_{15}^{uv}(u - \frac{1}{4})R_{25}^{uv}(u - \frac{11}{4})S_{12}^{-1},
\]

where the constant gauge transformation matrix $S_{12}$ is given by (A.11) below. Moreover, we have the identity

\[
P_{21}^{uv(1)}R_{13}^{uv}(u)R_{12}^{uv(u - \frac{3}{2})}P_{21}^{uv(1)} = a(u)e_1(u - \frac{3}{2})P_{21}^{uv(1)} \times \text{id}.
\]
2.3 Operator product identities

Besides the transfer matrix \( t_1^{(p)}(u) \) given by (2.30), let us introduce 3 fused transfer matrices:

\[
\tilde{t}_m^{(p)}(u) = tr_{(12...m)} \bar{T}_m^{v}(u), \quad m = 2, 3, 4, \tag{2.31}
\]

where \( \bar{T}_m^{v}(u) \) are the fused monodromy matrices

\[
\bar{T}_m^{v}(u) = P_{m-21} T_1^{v}(u) T_2^{v}(u-1) T_3^{v}(u-2) \cdots T_m^{v}(u-m+1) P_{m-21}, \tag{2.32}
\]

and the projectors \( \{ P_{m-21} | m = 2, 3, 4 \} \) are given by (A.14)-(A.18) below. Moreover, in order to have closed (or enough) operator product identities, we need to introduce an extra transfer matrix

\[
t_s^{(p)}(u) = tr_0 T_0^{s}(u), \quad T_0^{s}(u) = R_{01}^{sv}(u - \theta_1) R_{02}^{sv}(u - \theta_2) \cdots R_{0N}^{sv}(u - \theta_N), \tag{2.33}
\]

where the spinoral \( R \)-matrix \( R_{12}^{sv}(u) \) is given by (2.14). It is easily to show that all the transfer matrices constitute a commutative family, namely,

\[
[t_s^{(p)}(u), t_s^{(p)}(v)] = [\tilde{t}_m^{(p)}(u), \tilde{t}_m^{(p)}(v)] = [\tilde{t}_n^{(p)}(u), \tilde{t}_n^{(p)}(v)] = 0, \quad m, n = 1, 2, \cdots, 4. \tag{2.34}
\]

We have used the convention: \( \tilde{t}_1^{(p)}(u) = t_1^{(p)}(u) = t^{(p)}(u) \). Direct calculation shows that

\[
P_{21}^{v(1)} T_1^{v}(u) T_2^{v}(u - \frac{3}{2}) P_{21}^{v(1)} = \prod_{i=1}^{N} a_i(u - \theta_i) e_i(u - \theta_i - \frac{3}{2}) P_{21}^{v(1)} \times \text{id}, \tag{2.35}
\]

\[
\bar{T}_2^{v}(u) = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) T_1^{\tilde{v}}(u - \frac{1}{2}), \tag{2.36}
\]

\[
\bar{T}_3^{v}(u) = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) \tilde{\rho}_0(u - \theta_i - 1) T_1^{\tilde{v}}(u - 1), \tag{2.37}
\]

where we have introduced some normalized monodromy matrices:

\[
T_0^{\tilde{v}}(u) = R_{01}^{ev}(u - \theta_1) R_{02}^{ev}(u - \theta_2) \cdots R_{0N}^{ev}(u - \theta_N), \tag{2.38}
\]

\[
T_0^{\tilde{v}}(u) = R_{01}^{ev}(u - \theta_1) R_{02}^{ev}(u - \theta_2) \cdots R_{0N}^{ev}(u - \theta_N). \tag{2.39}
\]

It is remarked that the quantum spaces of the above monodromy matrices are the same (i.e., \( V^\otimes N \)) and that the corresponding auxiliary spaces are \( \bar{V} \) and \( \bar{V} \) with dimensions 11 and 15. Then the associated transfer matrices are given by

\[
t_2^{(p)}(u) = tr_0 T_0^{\tilde{v}}(u), \quad t_3^{(p)}(u) = tr_0 T_0^{\tilde{v}}(u). \tag{2.40}
\]
The equivalence (2.29) and the relations (2.35)-(2.37) imply that
\[
\bar{t}_2^{(p)}(u) = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) t_2^{(p)}(u - \frac{1}{2}),
\]
\[
\bar{t}_3^{(p)}(u) = \prod_{i=1}^{N} \tilde{\rho}_0(u - \theta_i) \tilde{\rho}_0(u - \theta_i - 1) t_3^{(p)}(u - 1),
\]
\[
\bar{t}_4^{(p)}(u) = \prod_{i=1}^{N} \tilde{\rho}_1(u - \theta_i) t_4^{(p)}(u - \frac{1}{4}) t_4^{(p)}(u - \frac{11}{4}).
\]

Following the method developed in [30], we obtain the identities
\[
T_1^w(\theta_j) T_2^w(\theta_j - \frac{3}{2}) = P_{21}^{w(1)} T_1^w(\theta_j) T_2^w(\theta_j - \frac{3}{2}),
\]
\[
T_1^w(\theta_j) T_2^w(\theta_j - 1) = P_{21} T_1^w(\theta_j) T_2^w(\theta_j - 1),
\]
\[
T_1^w(\theta_j) T_{(23)}^w(\theta_j - 1) = P_{321} T_1^w(\theta_j) T_{(23)}^w(\theta_j - 1),
\]
\[
T_1^w(\theta_j) T_{(234)}^w(\theta_j - 1) = P_{321} T_1^w(\theta_j) T_{(234)}^w(\theta_j - 1),
\]
\[
T_2^w(\theta_j) T_1^w(\theta_j - 1) = P_{12}^{w(5)} T_2^w(\theta_j) T_1^w(\theta_j - 1),
\]
\[
T_2^w(\theta_j) T_{(23)}^w(\theta_j - 1) = P_{12}^{w(11)} T_2^w(\theta_j) T_{(23)}^w(\theta_j - 1),
\]
\[
T_2^w(\theta_j) T_{(234)}^w(\theta_j - 1) = P_{12}^{w(4)} T_2^w(\theta_j) T_{(234)}^w(\theta_j - 1),
\]
where the explicit expressions of the projectors \( P_{21}^{w(1)} \), \( P_{21} \), \( P_{321} \), \( P_{321} \), \( P_{12}^{w(5)} \), \( P_{12}^{w(11)} \) and \( P_{12}^{w(4)} \) are given in Appendix A. With the help of the correspondences (2.29), (A.3), (A.8) and (A.13), we have that the transfer matrices satisfy the operator product identities:
\[
t^{(p)}(\theta_j) t^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} a_1(\theta_j - \theta_i) e_1(\theta_j - \theta_i - \frac{3}{2}) \times \text{id},
\]
\[
t^{(p)}(\theta_j) t^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t_2^{(p)}(\theta_j - \frac{1}{2}),
\]
\[
t^{(p)}(\theta_j) t_2^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t_3^{(p)}(\theta_j - 1),
\]
\[
t^{(p)}(\theta_j) t_3^{(p)}(\theta_j - 2) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t_s^{(p)}(\theta_j - \frac{11}{4}),
\]
\[
t^{(p)}(\theta_j) t_2^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t^{(p)}(\theta_j - \frac{1}{2}).
\]
Now, we consider the asymptotic behaviors of the fused transfer matrices. Direct calculation shows

\[
{t}^{(p)}(\theta_j) t_2^{(p)}(\theta_j - \frac{1}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t_2^{(p)}(\theta_j), \quad (2.48)
\]

\[
{t}^{(p)}(\theta_j) t_s^{(p)}(\theta_j - \frac{5}{4}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) t_s^{(p)}(\theta_j - \frac{1}{4}). \quad (2.49)
\]

Let us denote the eigenvalues of the transfer matrices \( t^{(p)}(u) \), \( t_2^{(p)}(u) \), \( t_3^{(p)}(u) \) and \( t_s^{(p)}(u) \) as \( \Lambda^{(p)}(u) \), \( \Lambda_2^{(p)}(u) \), \( \Lambda_3^{(p)}(u) \) and \( \Lambda_s^{(p)}(u) \), respectively. From the operator product identities \( (2.43)-(2.49) \), we have the functional relations among the eigenvalues\(^5\).

\[
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} a_1(\theta_j - \theta_i) c_1(\theta_j - \theta_i - \frac{3}{2}), \quad (2.51)
\]

\[
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda_2^{(p)}(\theta_j - \frac{1}{2}), \quad (2.52)
\]

\[
\Lambda^{(p)}(\theta_j) \Lambda_2^{(p)}(\theta_j - \frac{3}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda_3^{(p)}(\theta_j - 1), \quad (2.53)
\]

\[
\Lambda^{(p)}(\theta_j) \Lambda_3^{(p)}(\theta_j - 2) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda_s^{(p)}(\theta_j - \frac{1}{4}) \Lambda_2^{(p)}(\theta_j - \frac{11}{4}), \quad (2.54)
\]

\[
\Lambda^{(p)}(\theta_j) \Lambda_2^{(p)}(\theta_j - 1) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda^{(p)}(\theta_j - \frac{1}{2}), \quad (2.55)
\]

\[
\Lambda^{(p)}(\theta_j) \Lambda_3^{(p)}(\theta_j - \frac{1}{2}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \Lambda_2^{(p)}(\theta_j), \quad (2.56)
\]

\(^5\)It is remarked that only \( (2.51) \) and \( (2.57) \) were used to obtain \( \Lambda^{(p)}(u) \) and \( \Lambda_s^{(p)}(u) \) for the closed \( B_n \) chain \( [1132] \). Since that \( \Lambda^{(p)}(u) \) (resp. \( \Lambda_s^{(p)}(u) \)) is a polynomial of \( u \) with degree \( 2N \) (resp. degree \( N \)), one needs \( 3N + 2 \) conditions to determine them completely while \( (2.51) \) and \( (2.54) \), together with the asymptotic behaviors, only give \( 2N + 2 \) conditions. Therefore, in order to close the functional relations, \( (2.51)-(2.57) \) are necessary.
\[ \Lambda^{(p)}(\theta_j) \Lambda_s^{(p)}(\theta_j - \frac{5}{4}) = \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j - \theta_i) \lambda^{(p)}(\theta_j - \frac{1}{4}). \] 

(2.57)

The asymptotic behaviors (2.50) of the fused transfer matrices lead to the corresponding asymptotic behaviors of their eigenvalues:

\[
\Lambda^{(p)}(u)|_{u \to \pm \infty} = 5u^{2N} + \cdots, \quad (2.58)
\]

\[
\Lambda_2^{(p)}(u)|_{u \to \pm \infty} = 11u^{2N} + \cdots, \quad (2.59)
\]

\[
\Lambda_3^{(p)}(u)|_{u \to \pm \infty} = 15u^{2N} + \cdots, \quad (2.60)
\]

\[
\Lambda_s^{(p)}(u)|_{u \to \pm \infty} = 4u^{N} + \cdots. \quad (2.61)
\]

From the definitions (2.6), (2.33) and (2.40), we know that the eigenvalues \( \Lambda^{(p)}(u), \Lambda_2^{(p)}(u) \) and \( \Lambda_3^{(p)}(u) \) are polynomials of \( u \) with degree \( 2N \), while \( \Lambda_s^{(p)}(u) \) is a polynomial of \( u \) with degree \( N \). Hence the functional relations (2.58)-(2.61) could completely determine the eigenvalues, which allows us to express them in terms of some homogeneous \( T - Q \) relations in the next subsection.

### 2.4 \( T - Q \) relations

Let us introduce some functions:

\[
Z^{(p)}_1(u) = \prod_{j=1}^{N} a_1(u - \theta_j) \frac{Q^{(1)}_p(u - 1)}{Q^{(1)}_p(u)},
\]

\[
Z^{(p)}_2(u) = \prod_{j=1}^{N} b_1(u - \theta_j) \frac{Q^{(1)}_p(u + 1)Q^{(2)}_p(u - 1)}{Q^{(1)}_p(u)Q^{(2)}_p(u)},
\]

\[
Z^{(p)}_3(u) = \prod_{j=1}^{N} b_1(u - \theta_j) \frac{Q^{(2)}_p(u - 1)Q^{(2)}_p(u + \frac{1}{2})}{Q^{(2)}_p(u)Q^{(2)}_p(u - \frac{1}{2})},
\]

\[
Z^{(p)}_4(u) = \prod_{j=1}^{N} b_1(u - \theta_j) \frac{Q^{(1)}_p(u - \frac{1}{2})Q^{(2)}_p(u + \frac{1}{2})}{Q^{(1)}_p(u + \frac{1}{2})Q^{(2)}_p(u - \frac{1}{2})},
\]

\[
Z^{(p)}_5(u) = \prod_{j=1}^{N} e_1(u - \theta_j) \frac{Q^{(1)}_p(u + \frac{3}{2})}{Q^{(1)}_p(u + \frac{1}{2})},
\]

\[
Q^{(m)}_p(u) = \prod_{k=1}^{L_m} (u - \mu_k^{(m)} + \frac{m}{2}), \quad m = 1, 2. \quad (2.62)
\]
The functional relations (2.58)-(2.61) enable us to parameterize the eigenvalues of the transfer matrices in terms of the $T - Q$ relations as follows:

\[
\Lambda^{(p)}(u) = Z_1^{(p)}(u) + Z_2^{(p)}(u) + Z_3^{(p)}(u) + Z_4^{(p)}(u) + Z_5^{(p)}(u), \tag{2.63}
\]

\[
\Lambda_2^{(p)}(u) = \prod_{i=1}^{N} \tilde{\rho}_0^{-1}(u - \theta_i + \frac{1}{2}) \left\{ Z_1^{(p)}(u + \frac{1}{2}) [Z_2^{(p)}(u - \frac{1}{2}) + Z_3^{(p)}(u - \frac{1}{2}) + Z_4^{(p)}(u - \frac{1}{2})] + Z_5^{(p)}(u - \frac{1}{2}) \right\} 
+ Z_2^{(p)}(u + \frac{1}{2}) [Z_3^{(p)}(u - \frac{1}{2}) + Z_4^{(p)}(u - \frac{1}{2}) + Z_5^{(p)}(u - \frac{1}{2})] 
+ Z_3^{(p)}(u + \frac{1}{2}) [Z_3^{(p)}(u - \frac{1}{2}) + Z_4^{(p)}(u - \frac{1}{2}) + Z_5^{(p)}(u - \frac{1}{2})] 
+ Z_4^{(p)}(u + \frac{1}{2}) Z_5^{(p)}(u - \frac{1}{2}) \}, \tag{2.64}
\]

\[
\Lambda_3^{(p)}(u) = \prod_{i=1}^{N} \left[ \tilde{\rho}_0(u - \theta_i + 1) \tilde{\rho}_0(u - \theta_i) \right]^{-1} \left\{ Z_1^{(p)}(u + 1) Z_2^{(p)}(u) [Z_3^{(p)}(u - 1) + Z_4^{(p)}(u - 1)] 
+ Z_2^{(p)}(u - 1) \right\} 
+ Z_1^{(p)}(u + 1) Z_3^{(p)}(u) [Z_3^{(p)}(u - 1) + Z_4^{(p)}(u - 1) + Z_5^{(p)}(u - 1)] 
+ Z_2^{(p)}(u + 1) Z_3^{(p)}(u) [Z_3^{(p)}(u - 1) + Z_4^{(p)}(u - 1) + Z_5^{(p)}(u - 1)] 
+ Z_3^{(p)}(u + 1) Z_4^{(p)}(u) [Z_3^{(p)}(u - 1) + Z_4^{(p)}(u - 1) + Z_5^{(p)}(u - 1)] 
+ (Z_1^{(p)}(u + 1) + Z_2^{(p)}(u + 1) + Z_3^{(p)}(u + 1)) Z_4^{(p)}(u) Z_5^{(p)}(u - 1) \}, \tag{2.65}
\]

\[
\Lambda_4^{(p)}(u) = \prod_{j=1}^{N} \left[ (u + \frac{7}{4} - \theta_j) b_1(u - \frac{3}{4} - \theta_j) \right]^{-1} \frac{Q^{(2)}_p(u + \frac{3}{4})}{Q^{(2)}_p(u + \frac{1}{4})} 
\times \left\{ Z_1^{(p)}(u + \frac{1}{4}) Z_2^{(p)}(u - \frac{3}{4}) + Z_1^{(p)}(u + \frac{1}{4}) Z_3^{(p)}(u - \frac{3}{4}) 
+ Z_2^{(p)}(u + \frac{1}{4}) Z_3^{(p)}(u - \frac{3}{4}) + Z_3^{(p)}(u + \frac{1}{4}) Z_3^{(p)}(u - \frac{3}{4}) \right\} 
= \left( \prod_{j=1}^{N} (u - \frac{1}{4} - \theta_j) b_1(u + \frac{3}{4} - \theta_j) \right)^{-1} \frac{Q^{(2)}_p(u + \frac{3}{4})}{Q^{(2)}_p(u + \frac{5}{4})} 
\times \left\{ Z_3^{(p)}(u + \frac{3}{4}) Z_3^{(p)}(u - \frac{1}{4}) + Z_3^{(p)}(u + \frac{3}{4}) Z_4^{(p)}(u - \frac{1}{4}) 
+ Z_3^{(p)}(u + \frac{3}{4}) Z_5^{(p)}(u - \frac{1}{4}) + Z_3^{(p)}(u + \frac{3}{4}) Z_5^{(p)}(u - \frac{1}{4}) \right\}, \tag{2.66}
\]
The regularity of the eigenvalue $\Lambda^{(p)}(u)$ given by (2.63) leads to that the parameters $\{\mu_k^{(m)}\}$ should satisfy the Bethe Ansatz equations (BAEs):

$$
\frac{Q_p^{(1)}(\mu_k^{(1)} + \frac{1}{2}) Q_p^{(2)}(\mu_k^{(1)} - \frac{3}{2})}{Q_p^{(1)}(\mu_k^{(1)} - \frac{3}{2}) Q_p^{(2)}(\mu_k^{(1)} - \frac{1}{2})} = -\prod_{j=1}^{N} \mu_k^{(1)} + \frac{1}{2} - \theta_j, \quad k = 1, \cdots, L_1,
$$

(2.67)

$$
\frac{Q_p^{(1)}(\mu_k^{(2)} + \frac{1}{2}) Q_p^{(2)}(\mu_k^{(2)} - \frac{3}{2})}{Q_p^{(1)}(\mu_k^{(2)} - \frac{3}{2}) Q_p^{(2)}(\mu_k^{(2)} - \frac{1}{2})} = -1, \quad l = 1, \cdots, L_2.
$$

(2.68)

We have verified that the above BAEs indeed guarantee all the $T-Q$ relations (2.63)-(2.66) are polynomials of $u$ with the required degrees. Moreover, these $T-Q$ relations also satisfy the function relations (2.51)-(2.57) and the asymptotic behaviors (2.58)-(2.61). Therefore, we conclude that $\Lambda^{(p)}(u)$, $\Lambda_2^{(p)}(u)$, $\Lambda_3^{(p)}(u)$ and $\Lambda_4^{(p)}(u)$ given by (2.63)-(2.66) are indeed the eigenvalues of the transfer matrices $t^{(p)}(u)$, $t_2^{(p)}(u)$, $t_3^{(p)}(u)$ and $t_4^{(p)}(u)$ provided that the $L_1+L_2$ parameters $\{\mu_k^{(m)}\}$ satisfy the associated BAEs (2.67)-(2.68). It is remarked that the $T-Q$ relations (2.63) and the associated BAEs (2.67)-(2.68) (after taking the homogeneous limit $\{\theta_j \rightarrow 0 | j = 1, 2, \cdots, N\}$) coincide with those obtained previously via conventional Bethe Ansatz methods [1][3][4].

3 Off-diagonal open boundary case

3.1 Open chain

Integrable open chain can be constructed as follows [6][7]. Let us introduce a pair of $K$-matrices $K^{v-}(u)$ and $K^{v+}(u)$. The former satisfies the reflection equation (RE)

$$
R_{12}^{uv}(u - v)K_1^{v-}(u)R_{21}^{uv}(u + v)K_2^{v-}(v) = K_2^{v-}(v)R_{12}^{uv}(u + v)K_1^{v-}(u)R_{21}^{uv}(u - v),
$$

(3.1)

and the latter satisfies the dual RE

$$
R_{12}^{uv}(-u + v)K_1^{v+}(u)R_{21}^{uv}(-u - v - 3)K_2^{v+}(v)
= K_2^{v+}(v)R_{12}^{uv}(-u - v - 3)K_1^{v+}(u)R_{21}^{uv}(-u + v).
$$

(3.2)

For open spin chains, instead of the “row-to-row” monodromy matrix $T^{uv}_0(u)$ (2.4), one needs to consider the “double-row” monodromy matrix as follows. Let us introduce another “row-to-row” monodromy matrix

$$
\hat{T}^{uv}_0(u) = R_{N0}^{uv}(u + \theta_N) \cdots R_{20}^{uv}(u + \theta_2)R_{10}^{uv}(u + \theta_1),
$$

(3.3)
which satisfies the Yang-Baxter relation

$$R_{12}^v(u - v)T_1^v(u)T_2^v(v) = T_2^v(v)T_1^v(u)R_{12}^v(u - v). \quad (3.4)$$

The transfer matrix $t(u)$ is defined as

$$t(u) \triangleq t_1(u) = \text{tr}_0 \{ K_0^v(u)T_0^v(u)K_0^v(u)\hat{T}_0^v(u) \}. \quad (3.5)$$

From the Yang-Baxter relation, reflection equation and its dual, one can prove that the transfer matrices with different spectral parameters commute with each other, $[t(u), t(v)] = 0$. Therefore, $t(u)$ serves as the generating function of all the conserved quantities of the system. The Hamiltonian of the open chain can be obtained by taking the derivative of the logarithm of the transfer matrix

$$H = \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0, \{ \theta_j \} = 0} = \sum_{k=1}^{N-1} H_{kk+1} + \frac{1}{2} K_1^v(0) + \frac{\text{tr}_0 \{ K_0^v(0)H_{N0} \}}{\text{tr}_0 K_0^v(0)} + \text{constant}. \quad (3.6)$$

In this paper, we consider an open chain associated with the off-diagonal $K$-matrix $K^v(u)$ given by

$$K^v(u) = \begin{pmatrix} K_{11}^v(u) & 0 & K_{13}^v(u) & 0 & K_{15}^v(u) \\ 0 & K_{22}^v(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{35}^v(u) & 0 \\ 0 & 0 & 0 & 0 & K_{44}^v(u) \\ K_{51}^v(u) & 0 & K_{53}^v(u) & 0 & K_{55}^v(u) \end{pmatrix}, \quad (3.7)$$

where the non-vanishing matrix elements are

$$K_{11}^v(u) = -1 - c_1 c_2 + 4u, \quad K_{13}^v(u) = 4\sqrt{2}c_1 u, \quad K_{15}^v(u) = -4c_1^2 u,$$

$$K_{22}^v(u) = -(1 + c_1 c_2)(4u + 1), \quad K_{33}^v(u) = 4\sqrt{2}c_2 u, \quad K_{35}^v(u) = 4\sqrt{2}c_1 u,$$

$$K_{33}^v(u) = -1 - 4u + c_1 c_2(4u - 1), \quad K_{44}^v(u) = -(1 + c_1 c_2)(4u + 1),$$

$$K_{51}^v(u) = -4c_2^2 u, \quad K_{53}^v(u) = 4\sqrt{2}c_2 u, \quad K_{55}^v(u) = -1 - c_1 c_2 + 4u. \quad (3.8)$$

Here $c_1$ and $c_2$ are arbitrary boundary parameters. The dual reflection matrix $K^{v\pm}(u)$ is also an off-diagonal one and given by

$$K^{v\pm}(u) = K^v(-u - \frac{3}{2})|_{c_1, c_2 \rightarrow \tilde{c}_1, \tilde{c}_2}, \quad (3.9)$$
where \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are the boundary parameters. For a generic choice of the four boundary parameters \( \{c_i, \tilde{c}_i | i = 1, 2\} \), it is easily to check that 
\[
[K_s^+(u), K_s^-(u)] \neq 0.
\]
This implies that the \( K_s^\pm(u) \) matrices cannot be diagonalized simultaneously. In this case, it is quite hard to derive solutions via conventional Bethe Ansatz methods due to the absence of a proper reference state. We will generalize the method developed in Section 2 to obtain eigenvalues of the transfer matrix \( t(u) \) \((3.5)\) specified by the \( K \)-matrices \((3.7)\) and \((3.9)\) in the following subsections.

### 3.2 Operator product relations

We define the dual fused monodromy matrices as
\[
\hat{T}_0^\rho(u) = R_{N0}^{\rho}(u + \theta_N) \cdots R_{20}^{\rho}(u + \theta_2)R_{10}^{\rho}(u + \theta_1),
\]
\[
\hat{T}_0^\alpha(u) = R_{N0}^{\alpha}(u + \theta_N) \cdots R_{20}^{\alpha}(u + \theta_2)R_{10}^{\alpha}(u + \theta_1),
\]
\[
\hat{T}_0^s(u) = R_{N0}^{us}(u + \theta_N) \cdots R_{20}^{us}(u + \theta_2)R_{10}^{us}(u + \theta_1),
\]
(3.10)

where the \( R \)-matrices \( R_{21}^{us}(u) \) is defined by \((2.15)\) and the others are defined by the relations
\[
R_{12}^{\rho}(u) R_{21}^{\rho}(-u) = (u + 1)(u - 1)(u + \frac{3}{2})(u - \frac{3}{2}) \times \text{id} = \rho_v(u) \times \text{id},
\]
(3.11)
\[
R_{12}^{\alpha}(u) R_{21}^{\alpha}(-u) = (u + 2)(u - 2)(u + \frac{1}{2})(u - \frac{1}{2}) \times \text{id} = \rho_v(u) \times \text{id}.
\]
(3.12)

We have checked that the \( R \)-matrices also enjoy the properties
\[
R_{12}^{\rho}(u)^t R_{21}^{\rho}(-u - 3)^t = (u + 3)(u + \frac{1}{2})(u + \frac{5}{2}) \times \text{id} = \tilde{\rho}_v(u) \times \text{id},
\]
(3.13)
\[
R_{12}^{\alpha}(u)^t R_{21}^{\alpha}(-u - 3)^t = (u + 1)(u + 2)(u + \frac{7}{2})(u + \frac{9}{2}) \times \text{id} = \tilde{\rho}_v(u) \times \text{id},
\]
(3.14)
which are used to derive the functional relations \((3.17)\) below.

Let us introduce the fused transfer matrices
\[
t_2(u) = tr_0\{K_0^{\rho+}(u)\hat{T}_0^\rho(u)K_0^{\rho-}(u)\hat{T}_0^\rho(u)\},
\]
\[
t_3(u) = tr_0\{K_0^{\rho+}(u)\hat{T}_0^\rho(u)K_0^{\rho-}(u)\hat{T}_0^\rho(u)\},
\]
\[
t_s(u) = tr_0\{K_0^{s+}(u)\hat{T}_0^s(u)K_0^{s-}(u)\hat{T}_0^s(u)\},
\]
(3.15)

where the associated \( K \)-matrices \( K^{s\pm}(u) \), \( K^{\rho\pm}(u) \) and \( K^{\rho\pm}(u) \) are given by \((3.3)-(3.4)\), \((3.11)-(3.12)\) and \((3.13)-(3.14)\) respectively. The Yang-Baxter relations, reflection equations and dual reflection equations allow one to show that all the transfer matrices constitute
a commutative family \([7]\), namely,

\[
[t_s(u), t_s(v)] = [t_s(u), t_m(v)] = [t_m(u), t_n(v)] = 0, \quad m, n = 1, 2, 3,
\]

where we have used the convention: \(t_1(u) = t(u)\). Moreover, from the construction of the transfer matrices we know that \(t_s(u)\), as a function of \(u\), is a polynomial with degree \(2N\), and that \(t_1(u)\) is a polynomial of \(u\) with degree \(4N + 2\), while \(t_2(u)\) and \(t_3(u)\) are polynomial of \(u\) with degree \(4N + 4\). Thus one need to look for \(14(N + 1)\) independent conditions to completely determine them.

Using the method we have used in previous sections and with the help of the relations \((C.2)\) and \((C.6)\), we can obtain the operator product identities among the fused transfer matrices as

\[
t(\pm \theta_j) t(\pm \theta_j) - \frac{3}{2} = 2^8 \frac{(\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{1}{4})^3(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{4})^3}{(\pm \theta_j - \frac{1}{4})(\pm \theta_j + \frac{1}{4})} \times H(\pm \theta_j) H(\pm \theta_j - \frac{1}{2}),
\]

\[
t(\pm \theta_j) t(\pm \theta_j - 1) = 2^2 \frac{(-\theta_j - 1)(\pm \theta_j + \frac{3}{2})}{(\pm \theta_j - \frac{1}{4})} \times H(\pm \theta_j) t_2(\pm \theta_j - \frac{1}{2}),
\]

\[
t(\pm \theta_j) t_2(\pm \theta_j) - \frac{3}{2} = -2^4 \frac{(\pm \theta_j - \frac{1}{2})(\pm \theta_j - \frac{3}{2})}{(\pm \theta_j - \frac{1}{4})} \times H(\pm \theta_j) t_2(\pm \theta_j - \frac{1}{2}),
\]

\[
t(\pm \theta_j) t_3(\pm \theta_j - 2) = -2^{10} (\pm \theta_j - 1)(\pm \theta_j - 2)(\pm \theta_j - \frac{3}{2}) \times \frac{(-\theta_j + \frac{1}{4})^2}{(\pm \theta_j + \frac{3}{2})} H(\pm \theta_j) t_4(\pm \theta_j - \frac{1}{4}) t_4(\pm \theta_j - \frac{11}{4}),
\]

\[
t(\pm \theta_j) t_2(\pm \theta_j) - \frac{1}{2} = 2^{10} (\pm \theta_j - 1)(\pm \theta_j - \frac{1}{4})^3(\pm \theta_j + \frac{3}{2})^3 \times \frac{(-\theta_j + \frac{1}{2})(\pm \theta_j + \frac{3}{4})^3}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{4})} H(\pm \theta_j) t_4(\pm \theta_j - \frac{1}{4}),
\]

\[
t(\pm \theta_j) t_3(\pm \theta_j - \frac{1}{2}) = -2^4 (\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{4})^2 \times \frac{(-\theta_j + \frac{1}{4})^2}{(\pm \theta_j + \frac{3}{4})} H(\pm \theta_j) t_2(\pm \theta_j),
\]

\[
t(\pm \theta_j) t_4(\pm \theta_j) - \frac{5}{4} = -2^4 (\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{4})^2 \times \frac{(-\theta_j + \frac{1}{2})(\pm \theta_j + \frac{3}{4})^2}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{4})} H(\pm \theta_j) t_4(\pm \theta_j - \frac{1}{4}).
\]
where

\[ H(u) = (1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \prod_{i=1}^{N} \bar{\rho}_0(u - \theta_i) \bar{\rho}_0(u + \theta_i). \] (3.18)

By direct calculation, we have

\[ t(0) = -\prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t\left(-\frac{3}{2}\right) = -\prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t_2(0) = -\frac{9}{4} \prod_{l=1}^{N} \bar{\rho}_l(-\theta_l)(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t_2\left(-\frac{3}{2}\right) = -\frac{9}{4} \prod_{l=1}^{N} \bar{\rho}_l(-\theta_l - \frac{3}{2})(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t_2\left(-\frac{1}{2}\right) = \frac{1}{4} t(-1), \quad t_2(-1) = \frac{1}{4} t\left(-\frac{1}{2}\right), \]

\[ t_3(0) = -\frac{45}{4} \prod_{l=1}^{N} \bar{\rho}_l(-\theta_l)(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t_3\left(-\frac{3}{2}\right) = -\frac{45}{4} \prod_{l=1}^{N} \bar{\rho}_l(-\theta_l - \frac{3}{2})(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ t_3\left(-\frac{1}{2}\right) = \frac{1}{3} t_2\left(-\frac{3}{2}\right), \quad t_3\left(-\frac{1}{2}\right) = \frac{3}{4} t\left(-\frac{3}{2}\right), \] (3.19)

where we have used the identities:

\[ tr\{K^v(0)\} = 1 + \bar{c}_1 \bar{c}_2, \quad K^v(0) = -(1 + c_1 c_2) \times \text{id}, \]

\[ tr\{K^v\left(-\frac{3}{2}\right)\} = 1 + c_1 c_2, \quad K^v\left(-\frac{3}{2}\right) = -(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ tr\{K^\bar{v}(0)\} = -\frac{3}{2}(1 + \bar{c}_1 \bar{c}_2), \quad K^\bar{v}(0) = \frac{3}{2}(1 + c_1 c_2) \times \text{id}, \]

\[ tr\{K^\bar{v}\left(-\frac{3}{2}\right)\} = -\frac{3}{2}(1 + c_1 c_2), \quad K^\bar{v}\left(-\frac{3}{2}\right) = \frac{3}{2}(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]

\[ tr\{K^\bar{v}(0)\} = \frac{15}{2}(1 + \bar{c}_1 \bar{c}_2), \quad K^\bar{v}(0) = -\frac{3}{2}(1 + c_1 c_2) \times \text{id}, \]

\[ tr\{K^\bar{v}\left(-\frac{3}{2}\right)\} = \frac{15}{2}(1 + c_1 c_2), \quad K^\bar{v}\left(-\frac{3}{2}\right) = -\frac{3}{2}(1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]
Moreover, we can derive the asymptotic behaviors of the transfer matrices

\[ t_1 \{ R_{12}^{uv}(-1) K_1^{u+}(0) R_{21}^{uv}(-2) \} = \frac{3}{2} (1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]
\[ t_2 \{ R_{21}^{uv}(-2) K_2^{v-}(-\frac{3}{2}) R_{12}^{uv}(-1) \}^{t_1 t_2} = \frac{3}{2} (1 + c_1 c_2) \times \text{id}, \]
\[ t_1 \{ R_{12}^{uv}(-1) R_{13}^{uv}(-2) K_1^{v+}(0) R_{31}^{uv}(-1) R_{21}^{uv}(-2) \} = \frac{9}{4} (1 + \bar{c}_1 \bar{c}_2) \times \text{id}, \]
\[ t_{23} \{ P_{123} R_{23}^{uv}(-1) R_{13}^{uv}(-2) K_2^{v+}(-\frac{1}{2}) R_{12}^{uv}(-3) K_1^{u+}(\frac{1}{2}) R_{32}^{uv}(-1) R_{31}^{uv}(-2) R_{21}^{uv}(0) \} = -\frac{35}{4} (1 + \bar{c}_1 \bar{c}_2)^2 \times \text{id}, \]
\[ K^{v-}(\frac{1}{2}) K^{v+}(\frac{1}{2}) = -3 (1 + c_1 c_2)^2 \times \text{id}. \]

Moreover, we can derive the asymptotic behaviors of the transfer matrices

\[
\begin{align*}
\left. t(u) \right|_{u \to \pm \infty} &= -4[(4 + 2c_1 \bar{c}_2 + 2c_2 \bar{c}_1)^2 + 4(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2)]u^{4N+2} \times \text{id} + \cdots, \\
\left. t_2(u) \right|_{u \to \pm \infty} &= 2^6 \left[ \frac{3}{4} (4 + 2c_1 \bar{c}_2 + 2c_2 \bar{c}_1)^2 - (1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2) \right]u^{4N+4} \times \text{id} + \cdots, \\
\left. t_3(u) \right|_{u \to \pm \infty} &= 2^6 [(4 + 2c_1 \bar{c}_2 + 2c_2 \bar{c}_1)^2 - (1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2)]u^{4N+4} \times \text{id} + \cdots, \\
\left. t_4(u) \right|_{u \to \pm \infty} &= (4 + 2c_1 \bar{c}_2 + 2c_2 \bar{c}_1)u^{2N} \times \text{id} + \cdots. \quad (3.20)
\end{align*}
\]

Let us denote the eigenvalues of the fused transfer matrices \( t(u) \), \( t_2(u) \), \( t_3(u) \) and \( t_4(u) \) as \( \Lambda(u) \), \( \Lambda_2(u) \), \( \Lambda_3(u) \) and \( \Lambda_4(u) \), respectively. The above 14\((N+1)\) operator production identities \((3.16)-(3.20)\) directly imply that

\[
\begin{align*}
\Lambda(\pm \theta_j) \Lambda(\pm \theta_j - \frac{3}{2}) &= 2^8 (\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{3}{4})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{4})^3 \\
&\times H(\pm \theta_j) H(\pm \theta_j - \frac{1}{2}), \quad (3.21) \\
\Lambda(\pm \theta_j) \Lambda(\pm \theta_j - 1) &= 2^2 (\pm \theta_j - 1)(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{4})^2 \\
&\times H(\pm \theta_j) \Lambda_2(\pm \theta_j - \frac{1}{2}), \quad (3.22) \\
\Lambda(\pm \theta_j) \Lambda_2(\pm \theta_j - \frac{3}{2}) &= -2^4 (\pm \theta_j - \frac{3}{4})(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{4})^2 \\
&\times H(\pm \theta_j) \Lambda_3(\pm \theta_j - 1), \quad (3.23) \\
\Lambda(\pm \theta_j) \Lambda_3(\pm \theta_j - 2) &= -2^{10} (\pm \theta_j - 1)(\pm \theta_j - 2)(\pm \theta_j - \frac{3}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j - \frac{1}{4})
\end{align*}
\]
\[
\Lambda(\pm \theta_j) \Lambda_2(\pm \theta_j - 1) = 2^6 \frac{(\pm \theta_j - 1)(\pm \theta_j - \frac{3}{4})^4(\pm \theta_j + \frac{1}{4})^4}{(\pm \theta_j - \frac{3}{4})^2(\pm \theta_j + 1)(\pm \theta_j + \frac{5}{4})} \times H(\pm \theta_j) \Lambda_0(\pm \theta_j - \frac{1}{2}),
\]

\[
\Lambda(\pm \theta_j) \Lambda_2(\pm \theta_j - \frac{1}{2}) = -2^4 \frac{(\pm \theta_j - \frac{1}{4})(\pm \theta_j + \frac{5}{4})^2}{(\pm \theta_j + \frac{3}{4})} H(\pm \theta_j) \Lambda_0(\pm \theta_j),
\]

\[
\Lambda(\pm \theta_j) \Lambda_3(\pm \theta_j - \frac{5}{4}) = -2^4 \frac{(\pm \theta_j - \frac{1}{4})(\pm \theta_j + \frac{3}{4})^2}{(\pm \theta_j - \frac{3}{4})^2(\pm \theta_j + \frac{3}{4})} \times H(\pm \theta_j) \Lambda_0(\pm \theta_j - \frac{1}{4}),
\]

\[
\Lambda(0) = -\prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda(-\frac{3}{2}) = -\prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda_2(0) = -\frac{9}{4} \prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda_2(-\frac{3}{2}) = -\frac{9}{4} \prod_{l=1}^{N} \tilde{\rho}_l(-\theta_l - \frac{3}{2})(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda_2(-\frac{1}{2}) = \frac{1}{4} \Lambda(-1), \quad \Lambda_2(-1) = \frac{1}{4} \Lambda(-\frac{1}{2}),
\]

\[
\Lambda_3(0) = -\frac{45}{4} \prod_{l=1}^{N} \rho_l(-\theta_l)(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda_3(-\frac{3}{2}) = -\frac{45}{4} \prod_{l=1}^{N} \tilde{\rho}_l(-\theta_l - \frac{3}{2})(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2),
\]

\[
\Lambda_3(-1) = \frac{1}{3} \Lambda_2(-\frac{3}{2}), \quad \Lambda_3(-\frac{1}{2}) = \frac{3}{4} \Lambda(-\frac{3}{2}),
\]

\[
\Lambda(u)|_{u \to \pm \infty} = -4[(4 + 2c_1 \tilde{c}_2 + 2c_2 \tilde{c}_1)^2 + 4(1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)] u^{4N+2} + \cdots,
\]

\[
\Lambda_2(u)|_{u \to \pm \infty} = 2^6 \frac{3}{4} [(4 + 2c_1 \tilde{c}_2 + 2c_2 \tilde{c}_1)^2 - (1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)] u^{4N+4} + \cdots,
\]

\[
\Lambda_3(u)|_{u \to \pm \infty} = 2^6 [(4 + 2c_1 \tilde{c}_2 + 2c_2 \tilde{c}_1)^2 - (1 + c_1 c_2)(1 + \tilde{c}_1 \tilde{c}_2)] u^{4N+4} + \cdots,
\]
\[
\Lambda_s(u)|_{u \to \pm \infty} = (4 + 2c_1\tilde{c}_2 + 2c_2\tilde{c}_1)u^{2N} + \cdots.
\]

The above 14 \((N + 1)\) conditions \([3.21]-[3.39]\) could completely determine the eigenvalues \(\Lambda(u), \Lambda_2(u), \Lambda_3(u)\) and \(\Lambda_s(u)\) in terms of some inhomogeneous \(T - Q\) relations.

### 3.3 Inhomogeneous \(T - Q\) relations

Define

\[
Z_1(u) = -2^4(u + \frac{1}{4})^3(u + \frac{3}{4})^2 \prod_{j=1}^{N} a_1(u - \theta_j)a_1(u + \theta_j) \frac{Q^{(1)}(u - 1)}{Q^{(1)}(u)}
\]

\[
\times (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
Z_2(u) = -2^4u(u + \frac{3}{2})(u + \frac{1}{4})(u + \frac{3}{4}) (u + 1)(u + \frac{1}{2}) \prod_{j=1}^{N} b_1(u - \theta_j)b_1(u + \theta_j) \frac{Q^{(1)}(u + 1)Q^{(2)}(u - 1)}{Q^{(1)}(u)Q^{(2)}(u)}
\]

\[
\times (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
Z_3(u) = -2^4u(u + \frac{3}{2})(u + \frac{1}{4})(u + \frac{3}{4}) (u + 1)(u + \frac{1}{2}) \prod_{j=1}^{N} b_1(u - \theta_j)b_1(u + \theta_j) \frac{Q^{(2)}(u - 1)Q^{(2)}(u + \frac{1}{2})}{Q^{(2)}(u)Q^{(2)}(u - \frac{1}{2})}
\]

\[
\times (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
Z_4(u) = -2^4u(u + \frac{3}{2})(u + \frac{3}{4})(u + \frac{3}{4}) (u + 1)(u + \frac{1}{2}) \prod_{j=1}^{N} b_1(u - \theta_j)b_1(u + \theta_j) \frac{Q^{(1)}(u - \frac{1}{2})Q^{(2)}(u + \frac{1}{2})}{Q^{(1)}(u)Q^{(2)}(u)}
\]

\[
\times (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
Z_5(u) = -2^4u(u + \frac{5}{4})^3(u + \frac{1}{4})(u + \frac{3}{4}) (u + 1)(u + \frac{1}{2}) \prod_{j=1}^{N} e_1(u - \theta_j)e_1(u + \theta_j) \frac{Q^{(1)}(u + \frac{3}{2})}{Q^{(1)}(u + \frac{1}{2})}
\]

\[
\times (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
Q^{(m)}(u) = \prod_{k=1}^{L_m}(u - \lambda_k^{(m)} + \frac{m}{2})(u + \lambda_k^{(m)} + \frac{m}{2}), \quad m = 1, 2,
\]

and

\[
f_1(u) = -2^4u(u + \frac{3}{2})(u + \frac{1}{4})(u + \frac{3}{4})(u + \frac{5}{4}) (u + \frac{1}{2}) \prod_{j=1}^{N} b_1(u - \theta_j)b_1(u + \theta_j)(u - \theta_j + 1)
\]

\[
\times (u + \theta_j + 1)] \frac{Q^{(1)}(u + 1)Q^{(1)}(u + \frac{1}{2})Q^{(2)}(u - 1)}{Q^{(2)}(u)Q^{(2)}(u - \frac{1}{2})}(1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2),
\]

\[
f_2(u) = -2^4u(u + \frac{3}{2})(u + \frac{3}{4})(u + \frac{5}{4})(u + \frac{1}{2}) \prod_{j=1}^{N} b_1(u - \theta_j)b_1(u + \theta_j)(u - \theta_j + 1)
\]
where $x$ is a parameter which will be determined later (see (3.46) below). The 14(N + 1)
functional relations (3.21)-(3.39) allow us to express the eigenvalues of the transfer matrices
in terms of some inhomogeneous $T - Q$ relations as follows:

$$\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + Z_5(u)$$
$$+ f_1(u) + f_2(u) + f_3(u) + f_4(u) + f_5(u),$$

(3.42)

$$\Lambda_2(u) = 2^{-6}[(u - \frac{1}{2})(u + 2)(u + \frac{3}{4})]^2 H(u + \frac{1}{2})^{-1} \tilde{p}_v(2u)$$
$$\times \{Z_1(u + \frac{1}{2})[Z_2(u - \frac{1}{2}) + Z_3(u - \frac{1}{2}) + Z_4(u - \frac{1}{2}) + f_1(u - \frac{1}{2}) + f_2(u - \frac{1}{2}) + f_3(u - \frac{1}{2}) + f_4(u - \frac{1}{2}) + f_5(u - \frac{1}{2})] + [Z_2(u + \frac{1}{2}) + Z_3(u + \frac{1}{2}) + Z_4(u + \frac{1}{2}) + f_1(u + \frac{1}{2}) + f_2(u + \frac{1}{2}) + f_3(u + \frac{1}{2}) + f_4(u + \frac{1}{2}) + f_5(u + \frac{1}{2})]Z_5(u - \frac{1}{2})$$
$$+ [Z_2(u + \frac{1}{2}) + Z_3(u + \frac{1}{2}) + f_1(u + \frac{1}{2})][Z_3(u - \frac{1}{2})$$
$$+ Z_4(u + \frac{1}{2}) + f_5(u + \frac{1}{2})] + Z_1(u + \frac{1}{2})Z_5(u - \frac{1}{2})\},$$

$$\Lambda_3(u) = -2^{-18}[(u + \frac{5}{4})^3(u + \frac{3}{4})^2(u - \frac{1}{2})u(u - 1)(u + \frac{1}{4})^3(u + \frac{3}{2})$$

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\[ \times (u + \frac{5}{2})(u + 2)H(u + 1)H(u) - \tilde{\rho}_v(2u + 1)\tilde{\rho}_v(2u - 1) \]
\[ \times \left\{ [Z_1(u + 1)Z_2(u) + Z_1(u + 1)Z_3(u) + Z_2(u + 1)Z_3(u) + Z_3(u + 1)Z_3(u) + Z_1(u + 1)f_1(u) + f_1(u + 1)Z_3(u)]Z_3(u - 1) + Z_1(u - 1) \right\] 
\[ + Z_3(u + 1)Z_3(u) + Z_1(u + 1)f_1(u) + f_1(u + 1)Z_3(u) + Z_4(u) + f_1(u) + f_2(u) \]
\[ + f_3(u) + f_4(u) + f_5(u)]Z_5(u - 1) + [Z_2(u + 1) + Z_3(u + 1) \right\} \]
\[ \Lambda_4(u) = \frac{1}{2^8} \frac{(u - \frac{7}{4})(u + \frac{1}{4})}{(u + \frac{1}{2})\left(\frac{u - \frac{3}{4}}{u - \frac{1}{2}}\right)\left(\frac{u + \frac{3}{4}}{u + \frac{1}{2}}\right)Q(2)(u - \frac{3}{4})}{(u - \frac{3}{4})Q(2)(u - \frac{1}{4})} \]
\[ \times \prod_{j=1}^{N} [(u + \frac{7}{4} - \theta_j)(u + \frac{7}{4} + \theta_j)b_1(u - \frac{3}{4} - \theta_j)b_1(u - \frac{3}{4} + \theta_j)]^{-1} \]
\[ \times \{ Z_1(u + \frac{1}{4})Z_2(u - \frac{3}{4}) + Z_1(u + \frac{1}{4})Z_3(u - \frac{3}{4}) + Z_2(u + \frac{1}{4})Z_3(u - \frac{3}{4}) \]
\[ + Z_3(u + \frac{1}{4})Z_3(u - \frac{3}{4}) + Z_1(u + \frac{1}{4})f_1(u - \frac{3}{4}) + f_1(u + \frac{1}{4})Z_3(u - \frac{3}{4}) \} \]
\[ = \frac{1}{2^8} \frac{(u + \frac{1}{2})(u + \frac{3}{4})Q(2)(u + \frac{3}{4})}{(u - \frac{1}{4})(u + \frac{1}{4})Q(2)(u - \frac{3}{4})} \]
\[ \times \prod_{j=1}^{N} [(u - \frac{1}{4} - \theta_j)(u - \frac{1}{4} + \theta_j)b_1(u + \frac{3}{4} - \theta_j)b_1(u + \frac{3}{4} + \theta_j)]^{-1} \]
\[ \times \{ Z_3(u + \frac{3}{4})Z_3(u - \frac{1}{4}) + Z_3(u + \frac{3}{4})Z_4(u - \frac{1}{4}) + Z_3(u + \frac{3}{4})Z_5(u - \frac{1}{4}) \]
\[ + Z_4(u + \frac{3}{4})Z_5(u - \frac{1}{4}) + Z_3(u + \frac{3}{4})f_5(u - \frac{1}{4}) + f_5(u + \frac{3}{4})Z_5(u - \frac{1}{4}) \}. \quad (3.43) \]

All the eigenvalues are polynomials of \( u \), the residues of right hand sides of Eq. (3.42) should be zero, which gives rise to the BAEs

\[ \frac{Q^{(1)}(\lambda_k^{(1)} + \frac{1}{2})Q^{(2)}(\lambda_k^{(1)} - \frac{3}{2})}{Q^{(1)}(\lambda_k^{(1)} - \frac{3}{2})Q^{(2)}(\lambda_k^{(1)} - \frac{1}{2})} = -\frac{\left(\lambda_k^{(1)} + \frac{1}{2}\right)^2}{\left(\lambda_k^{(1)} - \frac{1}{2}\right)^2} \]
\[ \times \prod_{j=1}^{N} \frac{(\lambda_k^{(1)} - \theta_j + \frac{1}{2})(\lambda_k^{(1)} + \theta_j + \frac{1}{2})}{(\lambda_k^{(1)} - \theta_j - \frac{1}{2})(\lambda_k^{(1)} + \theta_j - \frac{1}{2})}, \quad k = 1, 2, \cdots, L_1, \quad (3.44) \]
\[ = -x\lambda_l^{(2)}(\lambda_l^{(2)} - \frac{1}{4}) \prod_{j=1}^{N}(\lambda_l^{(2)} - \theta_j)(\lambda_l^{(2)} + \theta_j), \quad l = 1, 2, \cdots, L. \]  

(3.45)

Considering the asymptotic behaviors of \( \Lambda(u) \), \( \Lambda_2(u) \), \( \Lambda_3(u) \) and \( \Lambda_s(u) \), we obtain a constraint \( L_2 = 2L_1 + N + 1 \) and the value of parameter \( x \) in the functions \( \{f_m(u)\} \) given in (3.41) as

\[
x = \frac{4 + 2c_1\tilde{c}_2 + 2c_2\tilde{c}_1}{2\sqrt{(1 + c_1\tilde{c}_2)(1 + \tilde{c}_1\tilde{c}_2)}} - 2. \]  

(3.46)

Due to the choice of reflection matrices (3.7) and (3.9), one conserved charge is survived, which means that the number of Bethe roots \( L_1 \) or \( L_2 \) could be any non-negative integer but the constraint \( L_2 = 2L_1 + N + 1 \) must be hold. We have checked that the BAEs (3.44)-(3.45) indeed ensure that all the \( T - Q \) relations (3.42)-(3.43) are polynomials of \( u \) with the required degrees and also satisfy the required 14\((N+1)\) function relations (3.21)-(3.39).

Therefore, we conclude that \( \Lambda(u) \), \( \Lambda_2(u) \), \( \Lambda_3(u) \) and \( \Lambda_s(u) \) given by the inhomogeneous \( T - Q \) relations (3.42)-(3.43) are indeed the eigenvalues of the transfer matrices \( t(u) \), \( t_2(u) \), \( t_3(u) \) and \( t_s(u) \) provided that the \( L_1 + L_2 \) parameters \( \{\lambda_k^{(m)}\} \) satisfy the associated BAEs (3.44)-(3.45). Moreover, when \( c_1 = c_2 = \tilde{c}_1 = \tilde{c}_2 = 0 \), the boundary reflection matrices degenerate to diagonal ones and our results recover those obtained by the algebraic Bethe method [34].

4 Discussion

In this paper, we study the \( so(5) \) quantum spin chains with integrable boundary conditions. By using the fusion technique, we obtain the closed operator product identities of the fused transfer matrices (2.43)-(2.49) for the periodic boundary condition and (3.16)-(3.17) for the off-diagonal boundary case. Based on them and the asymptotic behaviors as well as the special values at certain points, we obtain the exact solutions of the systems. For the periodic case, the eigenvalues of the transfer matrices are given in terms of the homogeneous \( T - Q \) relations (2.63)-(2.66) and the associated BAEs are (2.67)-(2.68), which recover those obtained previously via conventional Bethe Ansatz methods [1,32,33]. For the open boundary case specified by the off-diagonal boundary \( K \)-matrices (3.7)-(3.9), (B.3)-(B.4), (B.11)-(B.12) and (B.13)-(B.14), the eigenvalues of the transfer matrices are given in terms of the inhomogeneous \( T - Q \) relations (3.42)-(3.43) and the associated BAEs (3.44)-(3.45).

The method and the results in this paper can be generalized to the \( so(2n+1) \) (i.e., \( B_n \)) case directly.

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Appendix A: Projectors and the fusion of the $R$-matrices

A.1: Fused $R$-matrices

From the definition (2.25) of the fused $R$-matrix $R_{12}^{\tilde{v}v}(u)$, we have

$$R_{12}^{\tilde{v}v}(-1) = P_{12}^{v(5)} \times \tilde{S},$$

where $P_{12}^{v(5)}$ is a 5-dimensional projector (see (A.23) below). The new projector allows us to construct a new fused $R$-matrix,

$$R_{(12)3}^{v\tilde{v}}(u) = \tilde{\rho}_0^{-1}(u + \frac{1}{2})P_{12}^{v(5)} R_{23}^{v\tilde{u}}(u + \frac{1}{2}) R_{13}^{\tilde{v}v}(u - \frac{1}{2}) P_{12}^{\tilde{v}v}. \quad (A.1)$$

Taking the correspondence

$$|\phi_i^{(5)}\rangle \longrightarrow |i\rangle, \quad i = 1, \ldots, 5,$$

where the basis $\{|\phi_i^{(5)}\rangle\}$ is given by (A.24) below, we have

$$R_{(12)3}^{v\tilde{v}}(u) \equiv R_{13}^{uv}(u). \quad (A.3)$$

The definition (2.25) of the fused $R$-matrix $R_{12}^{\tilde{v}v}(u)$ implies that

$$R_{12}^{\tilde{v}v}(-\frac{1}{2}) = P_{12}^{\tilde{v}v(11)} \times \tilde{S}, \quad (A.4)$$

where $P_{12}^{\tilde{v}v(11)}$ is a 11-dimensional projector (see below (A.26)) and $\tilde{S}$ is an irrelevant constant matrix. The projector allows us to construct the fused $R$-matrix

$$R_{(12)3}^{v\tilde{v}}(u) = \tilde{\rho}_0^{-1}(u) S^{-1}_{\tilde{v}} P_{12}^{\tilde{v}v(11)} R_{23}^{v\tilde{u}}(u) R_{13}^{\tilde{v}v}(u - \frac{1}{2}) P_{12}^{\tilde{v}v(11)} S_{\tilde{v}}, \quad (A.5)$$

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where $S_v$ is a $11 \times 11$ diagonal constant matrix with the elements

$$S_v = \text{diag} \left[ 1, 1, 1, 1, 1, 1, 1, 1, \frac{\sqrt{13}}{61} \right].$$  \hfill (A.6)

After taking the correspondence

$$| \phi_i^{(11)} \rangle \longrightarrow | \varphi_i^{(11)} \rangle, \quad i = 1, \ldots, 11,$$

where the bases $\{ | \phi_i^{(11)} \rangle \}$ and $\{ | \varphi_i^{(11)} \rangle \}$ are given by (A.16) and (A.27) below, we have

$$R_{(12)3}^{sv}(u) \equiv R_{13}^{sv}(u).$$  \hfill (A.8)

Taking the correspondence

$$| \phi_1^{(16)} \rangle \longrightarrow | 11 \rangle^{(s)}, \ldots, | \phi_4^{(16)} \rangle \longrightarrow | 14 \rangle^{(s)}, | \phi_5^{(16)} \rangle \longrightarrow | 21 \rangle^{(s)}, \ldots, | \phi_{16}^{(16)} \rangle \longrightarrow | 44 \rangle^{(s)},$$

and after some calculations, we arrive at (2.29), namely,

$$R_{(1234)5}(u) \equiv S_{12} R_{15}^{sv}(u - \frac{1}{4}) R_{25}^{sv}(u - \frac{11}{4}) S_{12}^{-1},$$  \hfill (A.10)

where $S_{12}$ is the $16 \times 16$ gauge transformation on the tensor space $V^{(s)} \otimes V^{(s)}$ with the non-zero matrix elements

$$S_{12}^{11} = S_{12}^{12} = -S_{12}^{14} = -S_{12}^{21} = S_{12}^{24} = S_{12}^{34} = -S_{12}^{35} = -a_4 = -\sqrt{\frac{6}{11}},$$

$$S_{12}^{13} = S_{12}^{14} = -S_{12}^{32} = -S_{12}^{42} = S_{12}^{43} = \frac{1}{\sqrt{2}},$$

$$S_{12}^{22} = S_{12}^{23} = S_{12}^{33} = S_{12}^{44} = c_4 = \frac{1}{\sqrt{2}},$$

$$S_{12}^{23} = S_{12}^{31} = -S_{12}^{32} = -S_{12}^{33} = S_{12}^{42} = S_{12}^{43} = S_{12}^{44} = \frac{1}{\sqrt{2}},$$

$$S_{12}^{31} = S_{12}^{34} = -S_{12}^{34} = -S_{12}^{34} = S_{12}^{41} = S_{12}^{41} = e_4 = \frac{1}{\sqrt{2}},$$

$$S_{12}^{41} = S_{12}^{44} = S_{12}^{44} = S_{12}^{44} = g_4 = \frac{1}{\sqrt{2}}. \hfill (A.11)$$

With the help of the 4-dimensional projector $P_{21}^{sv(4)}$ given by (A.19) and after taking the correspondence

$$| \psi_i^{(s)} \rangle \longrightarrow | i \rangle^{(s)}, \quad i = 1, \ldots, 4,$$

where the basis $\{ | \psi_i^{(s)} \rangle \}$ is given by (A.22) below, we have

$$R_{13}^{sv}(u) \equiv R_{(12)3}^{sv}(u) = \tilde{P}_0^{-1}(u + \frac{1}{4}) R_{25}^{sv}(u - \frac{11}{4}) P_{21}^{sv(4)} R_{(12)3}^{sv}(u + \frac{1}{4}) R_{25}^{sv}(u - 1) P_{21}^{sv(4)}. \hfill (A.13)$$

25
A.2: Projectors

Here we list the projectors used in this paper. $P_{12}^{\nu(1)}$ is the one-dimensional projector

$$P_{12}^{\nu(1)} = |\psi_0\rangle\langle \psi_0|, \quad P_{21}^{\nu(1)} = P_{12}^{\nu(1)},$$

(A.14)

where

$$|\psi_0\rangle = \frac{1}{\sqrt{5}}(|15\rangle + |24\rangle + |33\rangle + |42\rangle + |51\rangle).$$

$P_{12}$ is a 11-dimensional projector

$$P_{12} = \sum_{i=1}^{11} |\phi_i^{(11)}\rangle\langle \phi_i^{(11)}|, \quad P_{21} = P_{12},$$

(A.15)

where

$$|\phi_1^{(11)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |\phi_2^{(11)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle),$$

$$|\phi_3^{(11)}\rangle = \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle), \quad |\phi_4^{(11)}\rangle = \frac{1}{\sqrt{2}}(|15\rangle - |51\rangle),$$

$$|\phi_5^{(11)}\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \quad |\phi_6^{(11)}\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle),$$

$$|\phi_7^{(11)}\rangle = \frac{1}{\sqrt{2}}(|25\rangle - |52\rangle), \quad |\phi_8^{(11)}\rangle = \frac{1}{\sqrt{2}}(|43\rangle - |34\rangle),$$

$$|\phi_9^{(11)}\rangle = \frac{1}{\sqrt{2}}(|53\rangle - |35\rangle), \quad |\phi_{10}^{(11)}\rangle = \frac{1}{\sqrt{2}}(|45\rangle - |54\rangle),$$

$$|\phi_{11}^{(11)}\rangle = \frac{1}{\sqrt{5}}(|15\rangle + |24\rangle + |33\rangle + |42\rangle + |51\rangle).$$

(A.16)

$P_{123}$ is a 15-dimensional projector

$$P = \sum_{i=1}^{15} |\phi_i^{(15)}\rangle\langle \phi_i^{(15)}|, \quad P_{321} = P_{123},$$

(A.17)

where

$$|\phi_1^{(15)}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle - |213\rangle + |231\rangle + |312\rangle - |321\rangle),$$

$$|\phi_2^{(15)}\rangle = \frac{1}{\sqrt{6}}(|124\rangle - |142\rangle - |214\rangle + |241\rangle + |412\rangle - |421\rangle),$$

$$|\phi_3^{(15)}\rangle = \frac{1}{\sqrt{6}}(|125\rangle - |152\rangle - |215\rangle + |251\rangle + |512\rangle - |521\rangle),$$

26
$|\phi_{4}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|134\rangle - |143\rangle - |314\rangle + |341\rangle + |413\rangle - |431\rangle)$,$

$|\phi_{5}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|135\rangle - |153\rangle - |315\rangle + |351\rangle + |513\rangle - |531\rangle)$,$

$|\phi_{6}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|145\rangle - |154\rangle - |415\rangle + |451\rangle + |514\rangle - |541\rangle)$,$

$|\phi_{7}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|234\rangle - |243\rangle - |324\rangle + |342\rangle + |423\rangle - |432\rangle)$,$

$|\phi_{8}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|235\rangle - |253\rangle - |325\rangle + |352\rangle + |523\rangle - |532\rangle)$,$

$|\phi_{9}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|245\rangle - |254\rangle - |425\rangle + |452\rangle + |524\rangle - |542\rangle)$,$

$|\phi_{10}^{(15)}\rangle = \frac{1}{\sqrt{6}}(|345\rangle - |354\rangle - |435\rangle + |453\rangle + |534\rangle - |543\rangle)$,$

$|\phi_{11}^{(15)}\rangle = \frac{1}{\sqrt{13}}(2|151\rangle + |124\rangle + |133\rangle + |142\rangle - |214\rangle + |241\rangle

-|313\rangle + |331\rangle - |412\rangle + |421\rangle)$,$

$|\phi_{12}^{(15)}\rangle = \frac{1}{\sqrt{13}}(|215\rangle + 2|242\rangle + |233\rangle + |251\rangle - |125\rangle + |152\rangle

-|323\rangle + |332\rangle + |512\rangle - |521\rangle)$,$

$|\phi_{13}^{(15)}\rangle = \frac{1}{\sqrt{13}}(|345\rangle - |354\rangle - |435\rangle + |453\rangle + |534\rangle - |543\rangle)$,$

$|\phi_{14}^{(15)}\rangle = \frac{1}{\sqrt{13}}(|145\rangle + 2|424\rangle + |433\rangle + |451\rangle - |145\rangle + |154\rangle

-|343\rangle + |334\rangle + |514\rangle - |541\rangle)$,$

$|\phi_{15}^{(15)}\rangle = \frac{1}{\sqrt{13}}(|251\rangle + |524\rangle + |533\rangle + |542\rangle - |254\rangle + |245\rangle

-|353\rangle + |335\rangle - |452\rangle + |425\rangle)$.

$P_{1234}$ is a 16-dimensional projector

$$P_{1234} = \sum_{i=1}^{16} |\phi_{i}^{(16)}\rangle\langle \phi_{i}^{(16)}|, \quad P_{4321} = P_{1234},$$

where

$$|\phi_{1}^{(16)}\rangle = \frac{1}{2\sqrt{6}}(|4123\rangle - |4132\rangle - |4213\rangle + |4231\rangle + |4312\rangle - |4321\rangle$$
\[
φ_2^{(16)} = \frac{1}{2\sqrt{6}} \{[5123] - [5132] - [5213] + [5231] + [5312] - [5321] \\
- [1523] + [1532] + [2513] - [2531] - [3512] + [3521] \\
+ [1253] - [1352] - [2153] + [2351] + [3152] - [3251] \\
- [1235] + [1325] + [2135] - [2315] - [3125] + [3215]\},
\]

\[
φ_3^{(16)} = \frac{1}{2\sqrt{6}} \{[5124] - [5142] - [5214] + [5241] + [5412] - [5421] \\
- [1524] + [1542] + [2514] - [2541] - [4512] + [4521] \\
+ [1254] - [1452] - [2154] + [2451] + [4152] - [4251] \\
- [1245] + [1425] + [2145] - [2415] - [4125] + [4215]\},
\]

\[
φ_4^{(16)} = \frac{1}{2\sqrt{6}} \{[5134] - [5143] - [5314] + [5341] + [5413] - [5431] \\
- [1534] + [1543] + [3514] - [3541] - [4513] + [4531] \\
+ [1354] - [1453] - [3154] + [3451] + [4153] - [4351] \\
- [1345] + [1435] + [3145] - [3415] - [4135] + [4315]\},
\]

\[
φ_5^{(16)} = \frac{1}{2\sqrt{6}} \{[5234] - [5243] - [5324] + [5342] + [5423] - [5432] \\
- [2534] + [2543] + [3524] - [3542] - [4523] + [4532] \\
+ [2354] - [2453] - [3254] + [3452] + [4253] - [4352] \\
- [2345] + [2435] + [3245] - [3425] - [4235] + [4325]\},
\]

\[
φ_6^{(16)} = \frac{1}{2\sqrt{11}} \{[2][1251] + [2][1242] + [1233] - [1323] + [1332] \\
- [2][2151] - [2][2142] - [2][133] + [3123] - [3132] \\
+ [2][1512] + [2][2412] + [2][313] - [3213] + [3312] \\
- [2][1521] - [2][2421] - [2][3231] + [3231] - [3321]\},
\]

\[
φ_7^{(16)} = \frac{1}{2\sqrt{11}} \{[2][3151] + [3142] + [3133] + [3124] - [2][1351] - [1342] \\
- [3][1333] - [1324] + [2][1531] + [1432] - [3][3133] + [1234] \\
- [2][1513] - [1423] + [3][331] - [1243] - [3][2143] + [3][3241] \\
- [3][412] + [3][421] + [2][314] - [2][341] + [4][312] - [4][321] \\
- [2][134] + [2][431] - [4][132] + [4][231]\},
\]
\[ |\phi_{10}^{(16)}\rangle = \frac{1}{2\sqrt{11}} [2|2342\rangle + |3233\rangle + |3215\rangle + |3251\rangle - 2|2333\rangle - |2315\rangle - |2513\rangle - |2531\rangle + 2|2432\rangle + |3332\rangle + |2135\rangle + |2531\rangle - 2|2423\rangle - |3323\rangle - |2153\rangle - |2513\rangle - |3521\rangle + |3512\rangle - |3125\rangle + |3152\rangle + |5321\rangle - |5312\rangle + |1325\rangle - |1352\rangle - |5231\rangle + |5132\rangle - |1235\rangle + |1532\rangle + |5213\rangle - |5123\rangle + |1253\rangle - |1523\rangle], \]

\[ |\phi_{11}^{(16)}\rangle = \frac{1}{2\sqrt{11}} [|4215\rangle + |4233\rangle + |4251\rangle - |4323\rangle + |4332\rangle - |2415\rangle - |2433\rangle - |2451\rangle + |3423\rangle - |3432\rangle + |2145\rangle + |2343\rangle + |2541\rangle - |3243\rangle + |3342\rangle - |2154\rangle - |2334\rangle - |2514\rangle + |3234\rangle - |3324\rangle - |4521\rangle + |4512\rangle - |4125\rangle + |4152\rangle + 2|4242\rangle + |5421\rangle - |5412\rangle + |1425\rangle - |1452\rangle - 2|2424\rangle + |5241\rangle + |5142\rangle - |1245\rangle + |1542\rangle + |5214\rangle - |5124\rangle + |1254\rangle - |1524\rangle], \]

\[ |\phi_{12}^{(16)}\rangle = \frac{1}{2\sqrt{11}} [2|5215\rangle + 2|5242\rangle + |5233\rangle - |5323\rangle + |5332\rangle - 2|2515\rangle - 2|2542\rangle - |2533\rangle + |3523\rangle - |3532\rangle + 2|5152\rangle + 2|2452\rangle + |2353\rangle - |3253\rangle + |3352\rangle]. \]
\[
|\phi_{13}^{(16)}\rangle = \frac{1}{\sqrt{65}} \left[ 2|5151\rangle + |5142\rangle + |5133\rangle + |5124\rangle - |5214\rangle + |5241\rangle - |5313\rangle + |5331\rangle - |5412\rangle + |5421\rangle + |4215\rangle + |4233\rangle + 2|4242\rangle + |4251\rangle - |4323\rangle + |4332\rangle + |4512\rangle - |4521\rangle + |4152\rangle - |4125\rangle + |3315\rangle + |3324\rangle + |3333\rangle + |3342\rangle - |3351\rangle - |3135\rangle + |3153\rangle - |3234\rangle + |3243\rangle + |3423\rangle - |3432\rangle - |3531\rangle + |3513\rangle + |2415\rangle + |2451\rangle + |2242\rangle + |2433\rangle - |2541\rangle + |2514\rangle - |2343\rangle + |2334\rangle + |2154\rangle - |2145\rangle + 2|1515\rangle + |1524\rangle + |1533\rangle + |1542\rangle + |1425\rangle - |1452\rangle - |1353\rangle + |1335\rangle - |1254\rangle + |1245\rangle, \\
|\phi_{14}^{(16)}\rangle = \frac{1}{2\sqrt{11}} \left[ 2|4324\rangle + |4333\rangle + |4315\rangle + |4351\rangle - 2|3424\rangle - |3433\rangle - |3415\rangle - |3451\rangle + 2|4243\rangle + |3343\rangle + |3145\rangle + |3541\rangle - 2|4234\rangle - |3334\rangle - |3154\rangle - |3514\rangle - |4135\rangle + |4153\rangle - |4531\rangle + |4513\rangle + |1435\rangle - |1453\rangle + |5431\rangle - |5413\rangle - |1345\rangle + |1543\rangle - |5341\rangle + |5143\rangle + |1354\rangle - |1534\rangle + |5314\rangle - |5134\rangle, \\
|\phi_{15}^{(16)}\rangle = \frac{1}{2\sqrt{11}} \left[ 2|5315\rangle + |5333\rangle + |5324\rangle + |5342\rangle - 2|3515\rangle - |3533\rangle - |3524\rangle - |3542\rangle + 2|5153\rangle + |3353\rangle + |3254\rangle + |3452\rangle - 2|5135\rangle - |3335\rangle - |3245\rangle - |3425\rangle - |5234\rangle + |5243\rangle - |5432\rangle + |5423\rangle + |2534\rangle - |2543\rangle + |4532\rangle - |4523\rangle - |2354\rangle + |2453\rangle - |4352\rangle + |4253\rangle + |2345\rangle - |2435\rangle + |4325\rangle - |4235\rangle, \\
|\phi_{16}^{(16)}\rangle = \frac{1}{2\sqrt{11}} \left[ 2|5415\rangle + 2|5424\rangle + |5433\rangle - |5343\rangle + |5334\rangle - 2|4515\rangle - 2|4524\rangle - |4533\rangle + |3543\rangle - |3534\rangle + 2|5154\rangle + 2|4254\rangle + |4353\rangle - |3453\rangle + |3354\rangle - 2|5145\rangle - 2|4245\rangle - |4335\rangle + |3435\rangle - |3345\rangle].
\]
\( P_{12}^{sv(4)} \) is a 4-dimensional projector

\[
P_{12}^{sv(4)} = \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|, \tag{A.19}
\]

where

\[
|\psi_1\rangle = \frac{1}{\sqrt{5}}(|1, 3\rangle + \sqrt{2}|2, 2\rangle + \sqrt{2}|3, 1\rangle),
\]
\[
|\psi_2\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|1, 4\rangle - |2, 3\rangle + \sqrt{2}|4, 1\rangle),
\]
\[
|\psi_3\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|1, 5\rangle - |3, 3\rangle - \sqrt{2}|4, 2\rangle),
\]
\[
|\psi_4\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|2, 5\rangle - \sqrt{2}|3, 4\rangle + |4, 3\rangle). \tag{A.20}
\]

Similarly, we can construct the projector \( P_{21}^{vs(4)} \)

\[
P_{21}^{vs(4)} = \sum_{i=1}^{4} |\psi'_i\rangle\langle\psi'_i|, \tag{A.21}
\]

where

\[
|\psi'_1\rangle = \frac{1}{\sqrt{5}}(|3, 1\rangle + \sqrt{2}|2, 2\rangle + \sqrt{2}|1, 3\rangle),
\]
\[
|\psi'_2\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|1, 5\rangle - |3, 2\rangle + \sqrt{2}|1, 4\rangle),
\]
\[
|\psi'_3\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|5, 1\rangle - |3, 3\rangle - \sqrt{2}|2, 4\rangle),
\]
\[
|\psi'_4\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|5, 2\rangle - \sqrt{2}|4, 3\rangle + |3, 4\rangle). \tag{A.22}
\]

\( P_{12}^{sv(5)} \) is the 5-dimensional projector

\[
P_{12}^{sv(5)} = \sum_{i=1}^{5} |\varphi_i^{(5)}\rangle\langle\varphi_i^{(5)}|, \tag{A.23}
\]

where

\[
|\varphi_1^{(5)}\rangle = \frac{1}{\sqrt{14}}(-|1, 4\rangle - |2, 3\rangle - |3, 2\rangle - |4, 1\rangle - \sqrt{10}|11, 1\rangle),
\]
\[
|\varphi_2^{(5)}\rangle = \frac{1}{\sqrt{14}}(|1, 5\rangle - |5, 3\rangle - |6, 2\rangle - |7, 1\rangle - \sqrt{10}|11, 2\rangle),
\]
\[
|\varphi_3^{(5)}\rangle = \frac{1}{\sqrt{14}}(|2, 5\rangle + |5, 4\rangle - |8, 2\rangle - |91\rangle - \sqrt{10}|11, 3\rangle),
\]
\[
|\varphi_4^{(5)}\rangle = \frac{1}{\sqrt{14}}(-|1, 4\rangle - |2, 3\rangle - |3, 2\rangle - |4, 1\rangle - \sqrt{10}|11, 1\rangle),
\]
\[
|\varphi_5^{(5)}\rangle = \frac{1}{\sqrt{14}}(|1, 5\rangle - |5, 3\rangle - |6, 2\rangle - |7, 1\rangle - \sqrt{10}|11, 2\rangle),
\]
\[
|\varphi_6^{(5)}\rangle = \frac{1}{\sqrt{14}}(|2, 5\rangle + |5, 4\rangle - |8, 2\rangle - |91\rangle - \sqrt{10}|11, 3\rangle),
\]
\[
|\varphi_7^{(5)}\rangle = \frac{1}{\sqrt{14}}(-|1, 4\rangle - |2, 3\rangle - |3, 2\rangle - |4, 1\rangle - \sqrt{10}|11, 1\rangle),
\]
\[
|\varphi_8^{(5)}\rangle = \frac{1}{\sqrt{14}}(|1, 5\rangle - |5, 3\rangle - |6, 2\rangle - |7, 1\rangle - \sqrt{10}|11, 2\rangle),
\]
\[
|\varphi_9^{(5)}\rangle = \frac{1}{\sqrt{14}}(|2, 5\rangle + |5, 4\rangle - |8, 2\rangle - |91\rangle - \sqrt{10}|11, 3\rangle),
\]
\[
|\varphi_{10}^{(5)}\rangle = \frac{1}{\sqrt{14}}(-|1, 4\rangle - |2, 3\rangle - |3, 2\rangle - |4, 1\rangle - \sqrt{10}|11, 1\rangle). \tag{A.24}
\]
\begin{align*}
|\varphi_4^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|3, 5\rangle + |6, 4\rangle + |8, 3\rangle - |10, 1\rangle - \sqrt{10}|11, 4\rangle), \\
|\varphi_5^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|4, 5\rangle + |7, 4\rangle + |9, 3\rangle + |10, 2\rangle - \sqrt{10}|11, 5\rangle). 
\end{align*}

Because the dimension of \(V_1\) and that of \(V_2\) are not equal, we need introduce another 5-dimensional projector

\[ P_{21}^{ev(5)} =\sum_{i=1}^{5} |\varphi_i^{(5)}\rangle \langle \varphi_i^{(5)}|, \tag{A.25} \]

where

\begin{align*}
|\varphi_1^{(5)}\rangle &= \frac{1}{\sqrt{14}}(-|4, 1\rangle - |3, 2\rangle - |2, 3\rangle - |1, 4\rangle + \sqrt{10}|11, 1\rangle), \\
|\varphi_2^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|5, 1\rangle - |3, 5\rangle - |2, 6\rangle - |1, 7\rangle + \sqrt{10}|2, 11\rangle), \\
|\varphi_3^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|5, 2\rangle + |4, 5\rangle - |2, 8\rangle - |1, 9\rangle + \sqrt{10}|3, 11\rangle), \\
|\varphi_4^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|5, 3\rangle + |4, 6\rangle + |3, 8\rangle - |1, 10\rangle + \sqrt{10}|4, 11\rangle), \\
|\varphi_5^{(5)}\rangle &= \frac{1}{\sqrt{14}}(|5, 4\rangle + |4, 7\rangle + |3, 9\rangle + |2, 10\rangle + \sqrt{10}|5, 11\rangle).
\end{align*}

\(P_{12}^{ev(11)}\) is the 11-dimensional projector

\[ P_{12}^{ev(11)} = \sum_{i=1}^{11} |\varphi_i^{(11)}\rangle \langle \varphi_i^{(11)}|, \tag{A.26} \]

where

\begin{align*}
|\varphi_1^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|1, 3\rangle - \sqrt{3}|2, 2\rangle - \sqrt{3}|3, 1\rangle + \sqrt{26}|11, 2\rangle - \sqrt{26}|12, 1\rangle), \\
|\varphi_2^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|1, 4\rangle - \sqrt{3}|4, 2\rangle - \sqrt{3}|5, 1\rangle + \sqrt{26}|11, 3\rangle - \sqrt{26}|13, 1\rangle), \\
|\varphi_3^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|2, 4\rangle + \sqrt{3}|4, 3\rangle - \sqrt{3}|6, 1\rangle + \sqrt{26}|11, 4\rangle - \sqrt{26}|14, 1\rangle), \\
|\varphi_4^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|3, 4\rangle + \sqrt{3}|5, 3\rangle + \sqrt{3}|6, 2\rangle + \sqrt{26}|11, 5\rangle - \sqrt{26}|15, 1\rangle), \\
|\varphi_5^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|1, 5\rangle - \sqrt{3}|7, 2\rangle - \sqrt{3}|8, 1\rangle + \sqrt{26}|12, 3\rangle - \sqrt{26}|13, 2\rangle), \\
|\varphi_6^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|2, 5\rangle + \sqrt{3}|7, 3\rangle - \sqrt{3}|9, 1\rangle + \sqrt{26}|12, 4\rangle - \sqrt{26}|14, 2\rangle), \\
|\varphi_7^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|3, 5\rangle + \sqrt{3}|8, 3\rangle + \sqrt{3}|9, 2\rangle + \sqrt{26}|12, 5\rangle - \sqrt{26}|15, 2\rangle),
\end{align*}
\begin{align}
|\varphi_{8}^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|4, 5\rangle - \sqrt{3}|7, 4\rangle - \sqrt{3}|10, 1\rangle + \sqrt{26}|13, 4\rangle - \sqrt{26}|14, 3\rangle), \\
|\varphi_{9}^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 5\rangle - \sqrt{3}|8, 4\rangle + \sqrt{3}|10, 2\rangle + \sqrt{26}|13, 5\rangle - \sqrt{26}|15, 3\rangle), \\
|\varphi_{10}^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|6, 5\rangle - \sqrt{3}|9, 4\rangle - \sqrt{3}|10, 3\rangle + \sqrt{26}|14, 5\rangle - \sqrt{26}|15, 4\rangle), \\
|\varphi_{11}^{(11)}\rangle &= \frac{1}{\sqrt{5}}(|11, 5\rangle + |12, 4\rangle + |13, 3\rangle + |14, 2\rangle + |15, 1\rangle). \quad (A.27)
\end{align}

Because the dimensions of spaces \(V_1\) and \(V_2\) in the operator \(P_{12}^{\phi(11)}\) are not equal, we should introduce the related projector

\[ P_{21}^{\phi(11)} = \sum_{i=1}^{11} |\varphi_i^{(11)}\rangle \langle \varphi_i^{(11)}|, \quad (A.28) \]

where

\begin{align}
|\varphi_1^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|3, 1\rangle - \sqrt{3}|2, 2\rangle - \sqrt{3}|1, 3\rangle - \sqrt{26}|2, 11\rangle + \sqrt{26}|1, 12\rangle), \\
|\varphi_2^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|4, 1\rangle - \sqrt{3}|2, 4\rangle - \sqrt{3}|1, 5\rangle - \sqrt{26}|3, 11\rangle + \sqrt{26}|1, 13\rangle), \\
|\varphi_3^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|4, 2\rangle + \sqrt{3}|3, 4\rangle - \sqrt{3}|1, 6\rangle - \sqrt{26}|4, 11\rangle + \sqrt{26}|1, 14\rangle), \\
|\varphi_4^{(11)}\rangle &= \frac{1}{\sqrt{61}}(\sqrt{3}|4, 3\rangle + \sqrt{3}|3, 5\rangle + \sqrt{3}|2, 6\rangle - \sqrt{26}|5, 11\rangle + \sqrt{26}|1, 15\rangle), \\
|\varphi_5^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 1\rangle - \sqrt{3}|2, 7\rangle - \sqrt{3}|1, 8\rangle - \sqrt{26}|3, 12\rangle + \sqrt{26}|2, 13\rangle), \\
|\varphi_6^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 2\rangle + \sqrt{3}|3, 7\rangle - \sqrt{3}|1, 9\rangle - \sqrt{26}|4, 12\rangle + \sqrt{26}|2, 14\rangle), \\
|\varphi_7^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 3\rangle + \sqrt{3}|3, 8\rangle + \sqrt{3}|2, 9\rangle - \sqrt{26}|5, 12\rangle + \sqrt{26}|2, 15\rangle), \\
|\varphi_8^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 4\rangle - \sqrt{3}|4, 7\rangle - \sqrt{3}|1, 10\rangle - \sqrt{26}|4, 13\rangle + \sqrt{26}|3, 14\rangle), \\
|\varphi_9^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 5\rangle - \sqrt{3}|4, 8\rangle + \sqrt{3}|2, 10\rangle - \sqrt{26}|5, 13\rangle + \sqrt{26}|3, 15\rangle), \\
|\varphi_{10}^{(11)}\rangle &= \frac{1}{\sqrt{61}}(-\sqrt{3}|5, 6\rangle - \sqrt{3}|4, 9\rangle - \sqrt{3}|3, 10\rangle - \sqrt{26}|5, 14\rangle + \sqrt{26}|4, 15\rangle), \\
|\varphi_{11}^{(11)}\rangle &= \frac{1}{\sqrt{5}}(|5, 11\rangle + |4, 12\rangle + |3, 13\rangle + |2, 14\rangle + |1, 15\rangle).
\end{align}
Appendix B: Fusion of the $K$-matrices

B.1: Associated spinorial reflection matrices

We introduce the corresponding spinorial $K$-matrix $K^s-(u)$ and the dual reflecting matrix $K^s+(u)$, which satisfy the reflection equation

$$R_{12}^{ss}(u-v)K_1^{s-}(u)R_{21}^{ss}(u+v)K_2^{s-}(v) = K_2^{s-}(v)R_{12}^{ss}(u+v)K_1^{s-}(u)R_{21}^{ss}(u-v),$$  \(\text{(B.1)}\)

and its dual

$$R_{12}^{ss}(-u+v)K_1^{s+}(u)R_{21}^{ss}(-u-v-3)K_2^{s+}(v) = K_2^{s+}(v)R_{12}^{ss}(-u-v-3)K_1^{s+}(u)R_{21}^{ss}(-u+v),$$  \(\text{(B.2)}\)

respectively. The matrix forms of $K^{s\pm}(u)$ are

$$K^{s-}(u) = M^s, \quad M^s = \begin{pmatrix} 1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & c_1 \\ -c_2 & 0 & -1 & 0 \\ 0 & c_2 & 0 & -1 \end{pmatrix},$$  \(\text{(B.3)}\)

$$K^{s+}(u) = \tilde{M}^s, \quad \tilde{M}^s = \begin{pmatrix} 1 & 0 & -\tilde{c}_1 & 0 \\ 0 & 1 & 0 & \tilde{c}_1 \\ -\tilde{c}_2 & 0 & -1 & 0 \\ 0 & \tilde{c}_2 & 0 & -1 \end{pmatrix}.$$  \(\text{(B.4)}\)

With the help of the projector $P^{ss(5)}_{12}$ given by \(\text{(2.11)}\), one can construct the fused $K$-matrices \(\text{(29)}\) as follows:

$$K^{v-}_{(12)}(u) = (u + \frac{3}{4})^{-1}P^{ss(5)}_{12}K_2^{s-}(u + \frac{1}{4})R_{12}^{ss}(2u)K_1^{s-}(u - \frac{1}{4})P^{ss(5)}_{21},$$  \(\text{(B.5)}\)

$$K^{v+}_{(12)}(u) = -(u + \frac{3}{4})^{-1}P^{ss(5)}_{12}K_2^{s+}(u - \frac{1}{4})R_{12}^{ss}(-2u-3)K_1^{s+}(u + \frac{1}{4})P^{ss(5)}_{21}. $$  \(\text{(B.6)}\)

We remark that $P^{ss(5)}_{21} = P^{ss(5)}_{12}$. With the help of the equivalence \(\text{(2.13)}\), we have

$$K^{v\pm}_1(u) \equiv K^{v\pm}_{(12)}(u).$$  \(\text{(B.7)}\)

B.2: Other fused $K$-matrices

The one-dimensional projector $P^{rv(1)}_{21}$ given by \(\text{(2.20)}\) allows us to compute the quantum determinants $\text{Det}_q(K^{v\pm}(u))$

$$P^{rv(1)}_{21} K_1^{v-}(u)R_{21}^{rv}(2u - \frac{3}{2})K_2^{v-}(u - \frac{3}{2})P^{rv(1)}_{12} \equiv \text{Det}_q(K^{v-}(u)) P^{rv(1)}_{12},$$  \(\text{(B.8)}\)
where

\[
\text{Det}_q(K^{\nu-}(u)) = -2^2(u - \frac{3}{2})(u - \frac{1}{4})h(u)h(-u), \quad h(u) = (1 + c_1 c_2)(4u + 1),
\]

\[
\text{Det}_q(K^{\nu+}(u)) = -2^2(u + \frac{3}{2})(u + \frac{1}{4})\tilde{h}(u)\tilde{h}(-u), \quad \tilde{h}(u) = (1 + \tilde{c}_1 \tilde{c}_2)(4u + 1). \tag{B.10}
\]

Using the the 11-dimensional projector \( P_{21}^{\nu} \) given by (A.15), we can construct the 11 \times 11 \( K \)-matrices \( K^{\nu\pm}(u) \)

\[
K_1^{\nu-}(u) = \frac{1}{2(u - \frac{1}{2})h(u + \frac{1}{2})} P_{21} K_1^{\nu-}(u + \frac{1}{2}) R_{21}^{\nu}(2u) K_2^{\nu-}(u - \frac{1}{2}) P_{12}, \tag{B.11}
\]

\[
K_1^{\nu+}(u) = \frac{1}{2(u + \frac{1}{2})h(u + \frac{1}{2})} P_{12} K_2^{\nu+}(u - \frac{1}{2}) R_{12}^{\nu}(-2u - 3) K_1^{\nu+}(u + \frac{1}{2}) P_{21}. \tag{B.12}
\]

The 15-dimensional projector \( P_{321} \) given by (A.17) allows us to construct the 15 \times 15 \( K \)-matrices \( K^{\nu\pm}(u) \)

\[
K^{\nu-}(u) \equiv K_{(123)}^{\nu-}(u + 1) = \frac{1}{2^5(u + \frac{1}{2})h(u + \frac{1}{2})} P_{321} K_1^{\nu-}(u + 1) R_{21}^{\nu}(2u + 1) R_{31}^{\nu}(2u)
\]

\[
\times K_2^{\nu-}(u - 1) R_{32}^{\nu}(2u - 1) K_3^{\nu-}(u - 1) P_{123}, \tag{B.13}
\]

\[
K^{\nu+}(u) \equiv K_{(123)}^{\nu+}(u + 1) = \frac{1}{2^5(u + \frac{1}{2})h(u + \frac{1}{2})} \left( -P_{123} K_3^{\nu+}(u - 1) R_{21}^{\nu}(2u - 2) R_{13}^{\nu}(2u - 3) \right)
\]

\[
\times K_2^{\nu+}(u) R_{12}^{\nu}(-2u - 4) K_1^{\nu+}(u + 1) P_{321}. \tag{B.14}
\]

All these \( K \)-matrices satisfy the associate reflection equations or the dual reflection equations.

Using the 5-dimensional projectors \( P_{12}^{\nu(5)} \) and \( P_{21}^{\nu(5)} \) given by (A.25)-(A.26) and the correspondence (A.2), we have

\[
K_1^{\nu-}(u) = K_{(12)}^{\nu-}(u + \frac{1}{2}) = -\frac{P_{12}^{\nu(5)} K_2^{\nu-}(u + \frac{1}{2}) P_{12}^{\nu(5)}(2u) K_1^{\nu-}(u - \frac{1}{2}) P_{21}^{\nu(5)}}{2^3(u - \frac{1}{2})(u + \frac{1}{2})h(u + \frac{1}{2})}, \tag{B.15}
\]

\[
K_1^{\nu+}(u) = K_{(12)}^{\nu+}(u + \frac{1}{2}) = -\frac{P_{21}^{\nu(5)} K_1^{\nu+}(u - \frac{1}{2}) P_{21}^{\nu(5)}(-2u - 3) K_2^{\nu+}(u + \frac{1}{2}) P_{12}^{\nu(5)}}{2^3(u + 2)(u + \frac{1}{2})h(u + \frac{1}{2})}. \tag{B.16}
\]

Similarly with the help of the 11-dimensional projectors \( P_{12}^{\nu(11)} \) and \( P_{21}^{\nu(11)} \) given by (A.26)-(A.28) and the correspondence (A.7), we have

\[
K_1^{\nu-}(u) = S_\nu^{-1} K_{(12)}^{\nu-}(u) S_\nu = -\frac{S_\nu^{-1} P_{12}^{\nu(11)} K_2^{\nu-}(u) P_{12}^{\nu(11)}(2u - \frac{1}{2}) K_1^{\nu-}(u - \frac{1}{2}) P_{21}^{\nu(11)} S_\nu}{2^3(u - \frac{1}{2})h(u)}, \tag{B.17}
\]
\[ K^{\tilde{\nu}+}(u) \equiv S^{-1}_\nu K^{\tilde{\nu}+}_{\nu}(u) P^{\tilde{\nu}+}_{\nu} = -\frac{S^{-1}_\nu P^{\tilde{\nu}+}_{\nu}}{2^2(u+\frac{1}{4})(u+\frac{3}{4})h(u)} (2.32), \]

where the 11×11 diagonal matrix \( S_\nu \) is given by (A.6). Moreover the 16-dimensional projector \( P_{4321} \) and the correspondence (A.9) allow us to have the identifications:

\[ \tilde{K}^{-}_{(1234)}(u) = P_{4321} K^{-}_{1}(u) R^{uv}_{21}(2u-1) R^{uv}_{31}(2u-2) P^{uv}_{41}(2u-3) K^{\nu-}_{2}(u-1) \]

\[ \equiv \tilde{\rho}^-(u) S_{12} K^{\nu-}_{1}(u-\frac{1}{4}) R^{ss}_{21}(2u-3) K^{\nu-}_{2}(u-\frac{11}{4}) S^{-1}_{12}, \]

\[ \tilde{K}^{+}_{(1234)}(u) = P_{1234} K^{\nu+}_{4}(u-3) R^{uv}_{34}(-2u+2) R^{uv}_{23}(-2u+1) R^{uv}_{14}(-2u) K^{\nu+}_{3}(u-2) \]

\[ \equiv \tilde{\rho}^+(u) S_{12} K^{\nu+}_{2}(u-\frac{11}{4}) R^{ss}_{12}(-2u) K^{\nu+}_{1}(u-\frac{1}{4}) S^{-1}_{12}, \]

where the 16 × 16 constant matrix is given by (A.11) and

\[ \tilde{\rho}^-(u) = -2^{12}(u-1)(u-2)^2(u-3)(u-\frac{5}{2})(u-\frac{3}{2})(u-\frac{1}{4})^2 \]

\[ \times(u-\frac{5}{4})(u+\frac{1}{4})h(u)h(u-1)h(u-2), \]

\[ \tilde{\rho}^+(u) = -2^{12}u(u+1)(u+\frac{1}{2})^2(u-\frac{1}{2})(u+\frac{3}{2})(u-\frac{3}{4})(u-\frac{7}{4}) \]

\[ \times(u-\frac{5}{4})^2(u+\frac{1}{4})h(u)h(u-1)h(u-2). \]

In order to complete the whole fusion of the \( K \)-matrices, we need the fusion between vectorial \( K^{\nu-}_{1}(u) \) and spinorial \( K^{\nu-}_{2}(u) \). The 4-dimensional projectors \( P^{uv}_{12} \) and \( P^{uv}_{21} \) given by (A.19)-(A.21) and the correspondence (A.12) enable us to have the identifications:

\[ K^{s-}(u) \equiv K^{s-}_{(12)}(u+\frac{1}{4}) = -\frac{P^{uv}_{21} K^{-}_{1}(u+\frac{1}{4}) P^{uv}_{12}(2u-\frac{3}{4}) K^{s-}_{2}(u-1)}{2u h(u+\frac{1}{4})}, \]

\[ K^{s-}(u) \equiv K^{s+}_{(12)}(u+\frac{1}{4}) = -\frac{P^{uv}_{12} K^{s+}_{2}(u-1) P^{uv}_{12}(-2u-\frac{9}{4}) K^{s+}_{1}(u+\frac{1}{4})}{2(u+\frac{3}{2}) h(u+\frac{1}{4})}. \]

**Appendix C: Operator identities for the open case**

Similar to the periodic case (see (2.32)-(2.32)), let us introduce some fused transfer matrices as follows:

\[ \tilde{t}_m(u) = tr_{12...m} \{ \tilde{K}^{\nu+}_{(12...m)}(u) \tilde{\bar{T}}^{uv}_{(12...m)}(u) \tilde{K}^{\nu-}_{(12...m)}(u) \tilde{T}^{uv}_{(12...m)}(u) \}, \quad m = 2, 3, 4, \]
where
\[
\bar{K}_{(12\ldots m)}^{-}(u) = P_{m-1} K_{12\ldots m}^{-}(u) R_{21}^{u v} (2u - 1) R_{31}^{u v} (2u - 2) \cdots \\
x R_{m-1}^{u v} (2u - m + 1) K_{(2\ldots m)}^{-}(u - 1) P_{m-1},
\]
\[
\bar{K}_{(12\ldots m)}^{+}(u) = P_{12\ldots m} \bar{K}_{(2\ldots m)}^{+}(u - 1) R_{1m}^{u v} (-2u + (m - 1) - 3) \cdots \\
x R_{12}^{u v} (-2u + 1 - 3) K_{1}^{u v}(u) P_{12\ldots m},
\]
\[
T_{(12\ldots m)}^{v}(u) = P_{m-21} T_{1}^{v}(u) T_{2}^{v}(u - 1) T_{3}^{v}(u - 2) \cdots T_{m}^{v}(u - m + 1) P_{m-21},
\]
\[
\hat{T}_{(12\ldots m)}^{v}(u) = P_{12\ldots m} \hat{T}_{1}^{v}(u) \hat{T}_{2}^{v}(u - 1) \hat{T}_{3}^{v}(u - 2) \cdots \hat{T}_{m}^{v}(u - m + 1) P_{12\ldots m}.
\]

The fused reflecting monodromy matrices satisfy the fusion relations (c.f., [2.33]-[2.34] for the periodic case)
\[
P_{12}^{v(w)(1)} \hat{T}_{1}^{v}(u) \hat{T}_{2}^{v}(u - \frac{3}{2}) P_{12}^{v(w)(1)} = \prod_{i=1}^{N} a_{1}(u + \theta_{i}) e_{1}(u + \theta_{i} - \frac{3}{2}) P_{12}^{v(w)(1)} \times \text{id},
\]
\[
\hat{T}_{12}^{v}(u) = P_{12} \hat{T}_{1}^{v}(u) \hat{T}_{2}^{v}(u - 1) P_{12} = \prod_{i=1}^{N} \rho_{0}(u + \theta_{i}) \hat{T}_{1}^{v}(u - \frac{1}{2}),
\]
\[
\hat{T}_{(123)}^{v}(u) = P_{123} \hat{T}_{1}^{v}(u) \hat{T}_{2}^{v}(u - 1) \hat{T}_{3}^{v}(u - 2) P_{123}
\]
\[
\quad = \prod_{i=1}^{N} \rho_{0}(u + \theta_{i}) \rho_{0}(u + \theta_{i} - 1) \hat{T}_{1}^{v}(u - 1),
\]
\[
\hat{T}_{(1234)}^{v}(u) = P_{1234} \hat{T}_{1}^{v}(u) \hat{T}_{2}^{v}(u - 1) \hat{T}_{3}^{v}(u - 2) \hat{T}_{4}^{v}(u - 3) P_{1234}
\]
\[
\quad = \prod_{i=1}^{N} \rho_{1}(u + \theta_{i}) S_{12} \hat{T}_{1}^{s}(u - \frac{1}{4}) \hat{T}_{2}^{s}(u - \frac{11}{4}) S_{12}^{-1},
\]
\[
P_{21}^{v(v)(5)} \hat{T}_{2}^{v}(u) \hat{T}_{1}^{v}(u - 1) P_{21}^{v(v)(5)} = \prod_{i=1}^{N} \rho_{0}(u + \theta_{i}) \hat{T}_{(12)}^{v}(u - \frac{1}{2}),
\]
\[
P_{21}^{v(v)(11)} \hat{T}_{2}^{v}(u) \hat{T}_{1}^{v}(u - \frac{1}{2}) P_{21}^{v(v)(11)} = \prod_{i=1}^{N} \rho_{0}(u + \theta_{i}) S_{1} \hat{T}_{(12)}^{v}(u) S_{1}^{-1},
\]
\[
P_{21}^{v(v)(4)} \hat{T}_{2}^{v}(u) \hat{T}_{1}^{v}(u - \frac{5}{4}) P_{21}^{v(v)(4)} = \prod_{i=1}^{N} \rho_{0}(u + \theta_{i}) \hat{T}_{(12)}^{v}(u - \frac{1}{4}).
\]

The above relations imply that the fused transfer matrices are indeed proportional to those
given by (3.15) by some polynomials (c.f., (2.41) for the periodic case)

\[ \tilde{t}_2(u) = 2^6(u-1)(u+\frac{3}{2})(u+\frac{1}{4})^2H(u)t_2(u-\frac{1}{2}), \]  
(C.3)

\[ \tilde{t}_3(u) = -2^{18}(u+\frac{1}{4})^3(u-\frac{1}{4})^2(u-\frac{3}{2})^2(u-1)(u-2)(u-\frac{3}{4})^3 \times (u+\frac{1}{2})(u+\frac{3}{2})(u+1)H(u)H(u-1)t_3(u-1), \]  
(C.4)

\[ \tilde{t}_4(u) = 2^{26}(u-1)(u-2)^2(u-3)(u-\frac{3}{2})(u-\frac{5}{2})(u-\frac{3}{4})^4(u-\frac{5}{4})^3 \times (u-\frac{1}{4})^3(u+\frac{1}{4})^3u(u+1)(u+\frac{1}{2})^2(u-\frac{1}{2})(u+\frac{3}{2})(u-\frac{7}{4})^3 \times \rho_2(2u-\frac{3}{2}) \times H(u)H(u-1)H(u-2)t_s(u-\frac{1}{4})t_s(u-\frac{11}{4}). \]  
(C.5)

Similar to (2.42) for the periodic case, we can derive the relations among the reflecting monodromy matrices given by (3.10)

\[ \hat{T}_1^u(-\theta_j)\hat{T}_2^u(-\theta_j-\frac{3}{2}) = P_{12}^{uv(1)}\hat{T}_1^u(-\theta_j)\hat{T}_2^u(-\theta_j-\frac{3}{2}), \]
\[ \hat{T}_1^u(-\theta_j)\hat{T}_2^u(-\theta_j-1) = P_{12}^{uv}(-\theta_j)\hat{T}_2^u(-\theta_j-1), \]
\[ \hat{T}_1^u(-\theta_j)\hat{T}_{(23)}^u(-\theta_j-1) = P_{123}^{uv}(-\theta_j)\hat{T}_{(23)}^u(-\theta_j-1), \]
\[ \hat{T}_1^u(-\theta_j)\hat{T}_{(234)}^u(-\theta_j-1) = P_{1234}^{uv}(-\theta_j)\hat{T}_{(234)}^u(-\theta_j-1), \]
\[ \hat{T}_2^u(-\theta_j)\hat{T}_{(23)}^u(-\theta_j-1) = P_{21}^{uv(5)}\hat{T}_2^u(-\theta_j)\hat{T}_1^u(-\theta_j-1), \]
\[ \hat{T}_2^u(-\theta_j)\hat{T}_1^u(-\theta_j-\frac{1}{2}) = P_{21}^{uv(11)}\hat{T}_2^u(-\theta_j)\hat{T}_1^u(-\theta_j-\frac{1}{2}), \]
\[ \hat{T}_2^u(-\theta_j)\hat{T}_1^u(-\theta_j-\frac{5}{4}) = P_{21}^{uv(4)}\hat{T}_2^u(-\theta_j)\hat{T}_1^u(-\theta_j-\frac{5}{4}). \]  
(C.6)

References

[1] N. Yu. Reshetikhin, Sov. Phys. JETP 57 (1983), 691.
[2] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, 1982.
[3] C. N. Yang, Phys. Rev. Lett. 19 (1967), 1312.
[4] L. A. Takhtadzhan and L. D. Faddeev, Rush. Math. Surveys 34 (1979), 11.
[5] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Function, Cambridge University Press, 1993.
[6] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys.* A 20 (1987), 6397.

[7] E. K. Sklyanin, *J. Phys.* A 21 (1988), 2375.

[8] H. Fan, B. -Y. Hou, K. -J. Shi and Z. -X. Yang, *Nucl. Phys.* B 478 (1996), 723.

[9] R. I. Nepomechie, *Nucl. Phys.* B 622 (2002), 615; *J. Stat. Phys.* 111 (2003), 1363; *J. Phys.* A 37 (2004), 433.

[10] J. Cao, H. -Q. Lin, K. -J. Shi and Y. Wang, *Nucl. Phys.* B 663 (2003), 487.

[11] J. de Gier and P. Pyatov, *J. Stat. Mech.* (2004), P03002.

[12] W. -L. Yang, Y. -Z. Zhang and M. Gould, *Nucl. Phys.* B 698 (2004), 503.

[13] J. de Gier and F. H. L. Essler, *Phys. Rev. Lett.* 95 (2005), 240601; *J. Stat. Mech.* (2006), P12011.

[14] P. Baseilhac, *Nucl. Phys.* B 754 (2006), 309.

[15] P. Baseilhac and K. Koizumi, *J. Stat. Mech.* (2007), P09006.

[16] W. Galleas, *Nucl. Phys.* B 790 (2008), 524.

[17] H. Frahm, A. Seel and T. Wirth, *Nucl. Phys.* B 802 (2008), 351.

[18] P. Baseilhac and S. Belliard, *Lett. Math. Phys.* 93 (2010), 213; *Nucl. Phys.* B 873 (2013), 550.

[19] G. Niccoli, *J. Stat. Mech.* (2012), P10025; *Nucl. Phys.* B 870 (2013), 397; *J. Phys.* A 46 (2013), 075003.

[20] J. Cao, W. -L. Yang, K. Shi and Y. Wang, *Nucl. Phys.* B 875 (2013), 152; *Nucl. Phys.* B 877 (2013), 152.

[21] R. I. Nepomechie, *J. Phys.* A 46 (2013), 442002.

[22] S. Belliard and N. Crampé, *SIGMA* 9 (2013), 072.

[23] S. Belliard, *Nucl. Phys.* B 892 (2015), 1.
[24] S. Belliard and R. A. Pimenta, *Nucl. Phys.* B 894 (2015), 527.

[25] J. Avan, S. Belliard, N. Grosjean and R. A. Pimenta, *Nucl. Phys.* B 899 (2015), 229.

[26] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Phys. Rev. Lett.* 111 (2013), 137201.

[27] Y. Wang, W.-L. Yang, J. Cao and K. Shi, *Off-Diagonal Bethe Ansatz for Exactly Solvable Models*, Springer Press, 2015.

[28] X. Zhang, J. Cao, W.-L. Yang, K. Shi and Y. Wang, *J. Stat. Mech.* (2014), P04031.

[29] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 06 (2014), 128.

[30] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 04 (2014), 143.

[31] J. Cao, S. Cui, W.-L. Yang, K. Shi and Y. Wang, *JHEP* 02 (2015), 036.

[32] N. Yu. Reshetikhin, *Lett. Math. Phys.* 14 (1987), 235.

[33] M. J. Martins and P. B. Ramos, *Nucl. Phys.* B 500 (1997), 579.

[34] G.-L. Li, K. J. Shi and R. H. Yue, *Nucl. Phys.* B 696 (2004), 381.

[35] G. -L. Li, K. J. Shi and R. H. Yue, *Commun. Theor. Phys.* 44 (2005), 8; G.-L. Li and K. J. Shi, *J. Stat. Mech.* (2007), P01018.

[36] D. Chicherin, S. Derkachov and A. P. Isaev, *J. Phys.* A 46 (2013), 485201.

[37] M. Karowski, *Nucl. Phys.* B 153 (1979), 244.

[38] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Lett. Math. Phys.* 5 (1981), 393.

[39] P. P. Kulish and E. K. Sklyanin, *Lecture Notes in Physics* 151 (1982), 61.

[40] A. N. Kirillov and N. Yu. Reshetikhin, *J. Sov. Math.* 35 (1986), 2627.

[41] A. N. Kirillov and N. Yu. Reshetikhin, *J. Phys.* A 20 (1987), 1565.

[42] L. Mezincescu and R. I. Nepomechie, *Nucl. Phys.* B 372 (1992), 597.

[43] Y.-K. Zhou, *Nucl. Phys.* B 458 (1996), 504.

[44] H. J. de Vega and A. González-Ruiz, *Nucl. Phys.* B 417 (1994), 553.
[45] H. J. de Vega and A. González-Ruiz, *Mod. Phys. Lett.* A **09** (1994), 2207.

[46] G. -L. Li, R. H. Yue and B. Y. Hou, *Nucl. Phys.* B **586** (2000), 711.