World Nematic Crystal Model of Gravity
Explaining the Absence of Torsion

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Assuming that at small distances space-time is equivalent to an elastic medium which is isotropic in space and time directions, we demonstrate that the quantum nematic liquid arising from this crystal by spontaneous proliferation of dislocations corresponds with a medium which is merely carrying curvature rigidity. This medium is at large distances indistinguishable from Einstein’s spacetime of general relativity. It does not support torsion and possesses string-like curvature sources which in spacetime form world surfaces.

The 19-th century idea that space-time is filled with an elastic medium called the ether seemed once and for all defeated by Einstein’s relativity principle. However, it revives in a modern setting by the realization that the theory of crystals containing topological defects can be reformulated in the language of differential geometry [1]. In this language, the physics of the plastic medium has striking formal resemblances with the theory of general relativity. Dislocations and disclinations represent torus defect. First, in ordinary first gradient elasticity, curvature vanishes and torsion becomes indistinguishable from Einstein’s spacetime at distances large compared to lattice spacing which is assumed to be Planckian.

Our model will be formulated as before [2] in three euclidean dimensions, for simplicity. The generalization to four dimensions is straightforward. The elastic energy is expressed in terms of a material displacement field \( u_i(x) \) as

\[
E = \int d^3x \left[ \mu u_{ij}^2(x) + \frac{\lambda}{2} u_{ii}^2(x) \right],
\]

where

\[
u_{ij}(x) \equiv \frac{1}{2} \left[ \partial_i u_j(x) + \partial_j u_i(x) \right]
\]

is the strain tensor and \( \mu, \lambda \) are the shear modulus and the Lamé constant, respectively. The elastic energy goes to zero for infinite wave length since in this limit \( u_i(x) \) reduces to a pure translation under which the energy of the system is invariant. The crystallization process causes a spontaneous breakdown of the translational symmetry of the system. The elastic distortions describe the Nambu-Goldstone modes resulting from this symmetry breakdown. Note that so far the crystal has an extra longitudinal sound wave with a different velocity than the shear waves.

A crystalline material always contains defects. In their presence, the elastic energy is

\[
E = \int d^3x \left[ \mu (u_{ij} - u_{ij}^p)^2 + \frac{\lambda}{2} (u_{ii} - u_{ii}^p)^2 \right],
\]

where \( u_{ij}^p \) is the so-called plastic strain tensor describing the defects. It is composed of an ensemble of lines with a dislocation density

\[
\alpha_{il} = \epsilon_{ijk} \partial_j \partial_k u_l(x) = \delta_i(x; L) (b_l + \epsilon_iqr \Omega_q x_r).
\]

and a disclination density

\[
\theta_{il} = \epsilon_{ijk} \partial_j \phi_{ki} = \delta_i(x; L) \Omega_l,
\]

where \( b_l \) and \( \Omega_l \) are the so-called Burgers and Franck vectors of the defects. The densities satisfy the conservation laws

\[
\partial_t \alpha_{ik} = -\epsilon_{kmn} \alpha_{mn}, \quad \partial_t \theta_{il} = 0.
\]
Dislocation lines are either closed or they end in disclination lines, and disclination lines are closed. These are Bianchi identities of the defect system.

An important geometric quantity characterizing dislocation and disclination lines is the \textit{incompatibility or defect density}

\[ \eta_{ij}(x) = \epsilon_{ijklm} \partial_k \partial_m u^l_n(x). \]  

(7)

It can be decomposed into disclination and dislocation density as follows [3]:

\[ \eta_{ij}(x) = \theta_{ij}(x) + \frac{1}{2} \partial_m \left[ \epsilon_{min} \alpha_{jn}(x) + (i \leftrightarrow j) - \epsilon_{ijn} \alpha_{mn}(x) \right]. \]  

(8)

This tensor is symmetric and conserved

\[ \partial_i \eta_{ij}(x) = 0, \]  

(9)

again a Bianchi identity of the defect system.

It is useful to separate from the dislocation density the contribution from the disclinations which causes the nonzero right-hand side of (8). Thus we define a pure dislocation density

\[ \alpha^b_{ij}(x) \equiv \alpha_{ij}(x) - \alpha^\Omega_{ij}(x) \]  

(10)

which satisfies \( \partial_i \alpha^b_{ij} = 0 \). Accordingly, we split

\[ \eta_{ij}(x) = \eta^b_{ij}(x) + \eta^\Omega_{ij}(x), \]  

(11)

where

\[ \eta^b_{ij}(x) = \frac{1}{2} \left[ \epsilon_{min} \alpha^b_{jn}(x) + (i \leftrightarrow j) - \epsilon_{ijn} \alpha^b_{mn}(x) \right], \]  

(12)

and the pure disclination part of the defect tensor looks like [5], but with superscripts \( \Omega \) on \( \eta^\Omega \) and \( \alpha^\Omega \).

The tensors \( \alpha_{ij} \), \( \theta_{ij} \), and \( \eta_{ij} \) are linearized versions of important geometric tensors in the \textit{Riemann-Cartan space of defects}, a noneuclidean space with curvature and torsion. Such a space can be generated from a flat space by a plastic distortion, which is mathematically represented by a \textit{nonholonomic mapping} \( x_i \rightarrow x_i + u_i(x) \). Such a mapping is nonintegrable. The displacement fields and their first derivatives fail to satisfy the Schwarz integrability criterion:

\[ (\partial_i \partial_j - \partial_j \partial_i) u(x) \neq 0, \quad (\partial_i \partial_j - \partial_j \partial_i) \partial_k u_l(x) \neq 0. \]  

(13)

The metric and the affine connection of the geometry in the plastically distorted space are \( g_{ij} = \delta_{ij} + \partial_i u_j + \partial_j u_i \) and \( \Gamma_{ij} = \partial_i u_j - \partial_j u_i \), respectively. The noncommutativity of the derivatives in front of \( u_l(x) \) implies a nonzero torsion, the torsion tensor being \( S_{ijk} = (\Gamma_{ijk} - \Gamma_{ikj})/2 \). The dislocation density \( \alpha_{ij} \) is equal to \( \alpha_{ij} = \epsilon_{ijk} S_{kij} \).

The noncommutativity of the derivatives in front of \( \partial_k u_l(x) \) implies a nonzero curvature. The disclination density \( \theta_{ij} \) is the Einstein tensor \( \theta_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \) of this Einstein-Cartan defect geometry. The tensor \( \eta_{ij} \), finally, is the Belinfante symmetric energy momentum tensor, which is defined in terms of the canonical energy-momentum tensor and the spin density by a relation just like [3]. For more details on the geometric aspects see Part IV in Vol. II of [6], where the full one-to-one correspondence between defect systems and Riemann-Cartan geometry is developed as well as a gravitational theory based on this analogy.

Let us now show how linearized gravity emerges from the energy [3]. For this we eliminate the jumping surfaces in the defect gauge fields from the partition function by introducing conjugate variables and associated stress gauge fields. This is done by rewriting the elastic action of defect lines as

\[ E = \int d^3x \left[ \frac{1}{4\mu} \left( \sigma^2_{ij} - \frac{\nu}{1 + \nu} \sigma^2_{ii} \right) + i \sigma_{ij} (u_{ij} - u^b_{ij}) \right], \]  

(14)

where \( \nu \equiv \lambda/2(\lambda + \mu) \) is Poisson’s ratio, and forming the partition function, integrating the Boltzmann factor \( e^{-E/k_BT} \) over \( \sigma_{ij}, u_i \) and summing over all jumping surfaces \( S \) in the plastic fields. The integrals over \( u_i \) yield the conservation law \( \partial_i \sigma_{ij} = 0 \). This can be enforced as a Bianchi identity by introducing a stress gauge field \( h_{ij} \) and writing \( \sigma_{ij} = G_{ij} \equiv \epsilon_{ijklm} \partial_k \partial_m h_{lm} \). The double curl on the right-hand side is recognized as the Einstein tensor in the geometric description of stresses, expressed in terms of a small deviation \( h_{ij} \equiv g_{ij} - \delta_{ij} \) of the metric from the flat-space form. Inserting \( G_{ij} \) into (14) and using (7), we can replace the energy in the partition function by \( E = E^{\text{stress}} + E^{\text{def}} \) where

\[ E^{\text{stress}} + E^{\text{def}} \equiv \int d^3x \left[ \frac{1}{4\mu} \left( G^2_{ij} - \frac{\nu}{1 + \nu} G^2_{ii} \right) + ih_{ij} \eta_{ij} \right], \]  

(15)

where the defect tensor [3] has the decomposition

\[ \eta_{ij} = \eta^\Omega_{ij} + \partial_m \epsilon_{min} \alpha^b_{jm}. \]  

(16)

The defects have also core energies which has been ignored so far. Adding these for the dislocations and ignoring, for a moment, the disclination part of the defect density in (16), we obtain

\[ E^{\text{def}} \equiv i \int d^3x \left( \epsilon_{ijn} \partial_m h_{ij} \alpha^b_{mn} + \frac{\epsilon}{2} \alpha^b_{jn} \right). \]  

(17)

We now assume that the world crystal has undergone a transition to a condensed phase in which dislocations are condensed. To reach such a state, whose existence was conjectured for two-dimensional crystals in Ref. [14], the model requires a modification by an additional rotational energy, as shown in (16) and verified by Monte Carlo simulations in [17]. The three-dimensional extension of the model is described in [14], and analyzed in great detail for the non-relativistic 2+1D quantum-fluid in [15].

The condensed phase is described by a partition function in which the discrete sum over the pure dislocation densities in \( \alpha^b_{jn} \) is approximated by an ordinary functional integral. This has been shown in Ref. [3]. The rule
for summing over closed-line ensembles \( l_i(x) = \delta_i(x; L) \) in a phase where the lines have proliferated is

\[
\int d^3l \delta(\partial_l l_i) e^{-\epsilon_i \nu^2 l_i^2/2 + il_i a_i} = e^{-a_i^2/2\epsilon_e},
\]

where \( a_{Ti} = -i\epsilon_{ijk}\partial_j a_k / \sqrt{-\partial^2} \). The Boltzmann factor resulting in this way from \( E_{\text{stress}} \) plus \( \Omega \) has now the energy

\[
E' = \int d^3x \left[ \frac{1}{4\mu} \left( G_{ij}^2 - \frac{\nu}{1 + \nu} G_{ij}^2 \right) + \frac{1}{2\epsilon_e} G_{ij} \left( \frac{1}{-\partial^2} G_{ij} + i h_{ij} \eta_{ij}^2 \right) \right].
\]

The second term implies a Meissner-like screening of the initially confining gravitational forces between the disclination part of the defect tensor to Newton-like forces. For distances longer than the Planck scale, we may ignore the stress term and find the effective gravitational action for the disclination part of the defect tensor:

\[
E \approx \int d^3x \left( \frac{1}{2\epsilon_e} G_{ij} \left( \frac{1}{-\partial^2} G_{ij} + i h_{ij} \eta_{ij}^2 \right) \right).
\]

A path integral over \( h_{ij} \) and a sum over all line ensembles applied to the Boltzmann factor \( e^{-E/k} \) is a simple Euclidean model of pure quantum gravity. The line fluctuations of \( \eta_{ij}^2 \) describe a fluctuating Riemann geometry perforated by a grand-canonical ensemble arbitrarily shaped lines of curvature. As long as the loops are small they merely renormalize the first term in the energy \( \Omega \). Such effects were calculated in closely related theories in great detail in Ref. \cite{24}. They also give rise to post-Newtonian terms in the above linearized description of the Riemann space.

We may now add matter to this gravitational environment \cite{5}. It is coupled by the usual Einstein interaction

\[
E'_{\text{int}} \approx \int d^3x h_{ij} T^{ij},
\]

where \( T^{ij} \) is the symmetric Belinfante energy momentum tensor of matter. Inserting for \( G_{ij} \) the double-curl of \( h_{ij} \) we see that the energy \cite{20} produces the correct Newton law if the core energy is \( \epsilon_c = 8\pi G \), where \( G \) is Newton’s constant.

Note that the condensation process of dislocations has led to a pure Riemann space without torsion. Just as a molten crystal shows residues of the original crystal structure only at molecular distances, remnants of the initial torsion could be observed only near the Planck scale. This explains why present-day general relativity requires only a Riemann space, not a Riemann-Cartan space.

In the non-relativistic context, a dislocation condensate is characteristic for a nematic liquid crystal, whose order is translationally invariant, but breaks rotational symmetry \cite{3,12} in two dimensions and \cite{17} in the 2+1-dimensional quantum theory). The Burgers vector of a dislocation is a vectorial topological charge, and nematic order may be viewed as an ordering of the Burgers vectors in the dislocation condensate. Initially, a nematic order would break the low energy Lorentz-invariance of space-time. We may, however, imagine the stiffness of the directional field of Burgers vectors to be so low that, by the criterion of Ref. \cite{18}, they undergo a Heisenberg-type of phase transition into a directionally disordered phase in an environment with only a few disclinations \cite{15}.

In three dimensions, dislocations and disclinations are line-like. This has the pleasant consequence, that they can be described by the disorder field theories developed in Ref. \cite{19} in which the proliferation of dislocations follows the typical Ginzburg-Landau pattern of the field expectation acquiring a nonzero expectation value. A cubic interaction becomes isotropic in the continuum limit \cite{20}(this is the famous fluctuation-induced symmetry restoration of the Heisenberg fixed-point in a \( \phi^4 \)-theory with O(3)-symmetric plus cubic interactions \cite{21}). The disordered, isotropic phase is just a physical realization of what has been called a topological form of nematic order by Lammert et al. \cite{22} in their O(3)/Z_2 gauge theory of nematic order: disclinations are massive although rotational symmetry is unbroken.

The above description of defects was formulated in what has been named tangential approximation to the Euclidean group \cite{4}, in which the discrete rotations are treated as if they took place in the tangential place with arbitrary real Franck vectors \( \Omega \) \cite{20}. In a more accurate formulation, the nonabelian nature of the rotations and the quantization of \( \Omega \) must be taken into account. Their discreteness is certainly remembered in the nematic phase, even if the directions of the Burgers vectors become disordered \cite{18} as also the discussion in Ref. \cite{24}). This implies that there are elements of quantized curvature fluctuating in spacetime. Fortunately, this does not introduce any observable consequence at presently accessible length scales since these fluctuations mainly renormalize the basic curvature energy in \cite{20}, as discussed before.

In one regard, our relativistic nematic fluid departs radically from the nematics formed from non-relativistic matter. In the energy \cite{15}, all components of the stress tensor \( G_{ij} \) have acquired the same mass which makes both shear- and compression stresses short-ranged. This is in contrast to non-relativistic crystalline matter, where shear rigidity is associated with translational symmetry breaking, and the dislocation condensate gives a Meissner-Higgs mass only to shear modes. Accordingly, the non-relativistic quantum nematic supports a massless compressional mode which is just the phase mode of the superfluid \cite{15}. The absence of compressional rigidity in the relativistic nematic is in fact quite natural. In order to decouple compression from the dislocation condensate an independent dynamical constraint is required: the glide condition, implying that dislocations only propagate in the directions of their (spatial) Burgers vector. Orthogonal directions would involve an excessive quantum migration of atoms in the crystal. Glide can only
meaningfully be defined in a non-relativistic spacetime, with the consequence that compressional rigidity has to disappear in the relativistic fluid.

In conclusion, we have demonstrated that the spacetime of general relativity can in principle be interpreted in terms of a material medium similar to a relativistic quantum-nematic fluid. Our construction is explicit in three Euclidean space-time dimensions. The generalization to four dimensions changes mainly the geometry of the defects. These become world sheets, and a second-quantized disorder field description of surfaces has not yet been found. But the approximation of representing a sum over dislocation surfaces in the proliferated phase as an integral as in Eq. (18) does remain valid, so that the above line of arguments will survive, this being a natural generalization of the Meissner-Higgs mechanism.

One might want to view our finding as a metaphor, offering an alternative perspective on the nature of spacetime, which might be of use in the greater context of the quantum gravity enigma. However, despite its simplicity, it cannot be a-priori excluded that the idea has a more literal meaning. It implies that Lorentz invariance is emergent and this alone cures the singularities of quantum gravity. Next, our model has automatically a vanishing cosmological constant. Since the atoms in the crystal are in equilibrium, the pressure is zero. This explanation is similar to that given by Volovik with his helium droplet analogies. Finally, the theory predicts that at the Planck scale, the disclination sources of curvature should become visible. As an attractive feature for string theorists, these will be world sheets. However, the high energy properties will be completely different from the common strings, because these surfaces behave non-relativistically as the energies approach the Planck scale. Accordingly, they will neither exhibit the recurrent particle spectra at arbitrary multiples of the Planck mass, nor the deep conceptual problems associated with explaining the low-energy universe. The deviations from relativity at high energies or short distances associated with a literal interpretation of the ‘world nematic’ may come into experimentalists reach in the not too distant future.

[1] K. Kondo, Proceedings of the II Japan National Congress on Applied Mechanics (Tokyo, 1952); E. Kröner in Physics of Defects, Les Houches lectures, Session XXXV, (ed.’s R. Balian et al., North Holland, Amsterdam, 1981).
[2] H. Kleinert,  Ann. d. Physik, 44, 117 (1987) (http://www.physik.fu-berlin.de/~kleinert/172)
[3] H. Kleinert, Proc. NATO Advanced Study Institute on Formation and Interaction of Topological Defects at the University of Cambridge, Plenum Press, New York, 1995, S. 201–232 (cond-mat/9503030)
[4] H. Kleinert, Gen. Rel. Grav. 32, 769 (2000) (http://www.physik.fu-berlin.de/~kleinert/258/258j.pdf)
[5] A. de Padua, F. Parisio-Filho, and F. Moraes, Phys. Lett. A 238, 153 (1998).
[6] H. Kleinert, Gauge fields in Condensed Matter, Vol. I: Superflow and Vortex Lines, Disorder Fields, Phase Transitions, Vol. II: Stresses and Defects, Differential Geometry, Crystal Defects, World Scientific, Singapore, 1989. (http://www.physik.fu-berlin.de/~kleinert/b1)
[7] G.E. Volovik, Phys. Rep. 351, 195-348 (2001) (gr-qc/0005091): Phenomenology of effective gravity, (gr-qc/0304061)
[8] G. Chapline, E. Hohlfeld, R. B. Laughlin, D. I. Santiago, Quantum Phase Transitions and the Breakdown of Classical General Relativity, (gr-qc/0012094)
[9] S.C. Zhang, To see a world in a grain of sand, hep-th/0210162
[10] H. Kleinert, Gen. Rel. Grav. 32, 1271 (2000) (gr-qc/9807021)
[11] S.A. Kivelson, E. Fradkin, and V.J. Emery, Nature 393, 550 (1998).
[12] E. Fradkin, S.A. Kivelson, E. Manousakis, and K. Nho, Phys. Rev. Lett. 84, 1982 (2000).
[13] J. Zaanen, Science 286, 251 (1999) and references therein.
[14] B.I. Halperin and D.R. Nelson, Phys. Rev. Lett. 41, 121 (1978).
[15] J. Zaanen, S.I. Mukhin, and Z. Nussinov, Duality in 2+1D quantum elasticity: superconductivity and quantum nematic order, in preparation.
[16] H. Kleinert, Phys. Lett. A 130, 443 (1988) (http://www.physik.fu-berlin.de/~kleinert/174).
[17] W. Janke and H. Kleinert, Phys. Rev. Lett. 61(20), 2344(1988) (http://www.physik.fu-berlin.de/~kleinert/179).
[18] H. Kleinert, Phys. Rev. Lett. 84, 286 (2000) (cond-mat/9908239).
[19] For the derivation of this connection see pp. 133–164 in Vol. I of the textbook (http://www.physik.fu-berlin.de/~kleinert/179).
[20] See the discussion on pp. 938–952 in Vol. I of the textbook (http://www.physik.fu-berlin.de/~kleinert/179).
[21] See Chapter 18 in H. Kleinert and V. Schulte-Frohlinde, Critical Properties of Φ⁴-Theories, World Scientific, Singapore 2001, pp. 1–487 (http://www.physik.fu-berlin.de/~kleinert/b8).
[22] P.E. Lammert, D.S. Rokhsar, and J. Toner, Phys. Rev. E 52, 1778 (1995).
[23] H. Kleinert and W. Miller, Phys. Rev. Lett. 56, 11 (1986); Phys. Rev. D 38, 1239(1988).
[24] N.M. Muller, Thesis, University of Amsterdam (1998); N.M. Muller and F.A. Bais (unpublished).
[25] For a summary see T. Banks, A Critique of Pure String Theory: Heterodox Opinions of Diverse Dimensions, hep-th/0306074.