Positive association and global connectivity in dependent percolation

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We study the effect of positive correlations on the critical threshold of site and bond percolation in a square lattice with $d = 2$. We propose two algorithms for generating dependent lattices with minimal correlation length and non-negative $n$-point correlations whose critical behavior is then compared with that of independent lattices. For site percolation, we show numerically that the introduction of this specific form of positive correlation results in a lower percolation threshold, i.e., higher connectivity. For bond percolation, the opposite is observed. In this case, however, we show that the dual lattice is also totally positively associated, demonstrating that positive association can result in either an increase or a decrease in global connectivity.

Keywords: percolation, FKG inequality, dependent percolation, renormalization

I. INTRODUCTION

Independent site and bond percolation have been the subject of extensive studies in the past few decades. The critical thresholds of several lattice geometries are known thanks to rigorous and non-rigorous arguments, and the critical behavior for general classes of lattices at or near criticality are well understood via scaling arguments [1, 2].

The problem of dependent percolation on the other hand, is relatively untouched due to the rapid growth of the mathematical complexity as correlations between sites or bonds are introduced. In a series of papers in the 1970s, Fortuin and Kasteleyn [3–5] proved the equivalence between various Potts models and dependent percolation processes obtained from independent ones by means of what they term an enhancement; a relationship that remains obscure.

A related question is whether monotonicity in the critical threshold (or lack thereof) can be deduced from the characterization of correlations in the lattice. In particular, what can be said about the critical threshold if all sites (bonds) are positively correlated?

Let us make this question precise. Let a finite lattice of sites (bonds) be defined as the edge set $E$ of a graph $G = (V, E)$ and let the sample space $\Omega = \{0, 1\}^E$ be the set of all realizations. Let $\mathcal{F} = 2^\Omega$, the power set of $\Omega$, be the set of all events and denote by $a(e_i) \in \mathcal{F}$ the event that the edge $e_i$ be open. Similarly, for a site percolation problem let $V = \{v_i\}$, $\Omega = \{0, 1\}^V$, $\mathcal{F} = 2^\Omega$ and $a(v_i) \in \mathcal{F}$ denote the analogous quantities. Most of our discussions are articulated in terms of the bond percolation problem, but the obvious analogues exist for site percolation as well.

Consider the following probability measures on the same lattice (graph):

1. $\Pr[a(e_i)] = p_i$ for all $i$, and all $a(e_i)$ are independent.

2. $\Pr[a(e_i)] = p_i$ for all $i$, and $\Pr[\cap_{e_i \in E_k} a(e_i)] \geq \prod_{e_i \in E_k} \Pr(a(e_i))$ for all subsets $E_k \subset E$. 

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In case (1), all edges are independent, whereas in case (2), all subsets of edges are positively correlated if not independent. In both cases though, the single-edge probabilities are given by the same set \( \{p_i\} \). This is a very particular construction, but as we will discuss below, it arises naturally in important problems involving coarse-graining of independent lattices.

We ask the following question: Is it true that if lattice (1) percolates for a particular set of edge probabilities \( \{p_i\} \), then lattice (2) also percolates at the same point in the parameter space? In other words, we set out to test the intuitive hypothesis that introducing positive association across the board while keeping everything else constant in fact increases global connectivity.

The rest of this paper is organized as follows: first, for each of the two percolation models (site and bond), we propose an algorithm by which a correlated lattice is derived from an un-correlated primal lattice. In both cases, each bond (site) is correlated only with its nearest neighbors. We believe this to be the simplest possible model of correlated percolation, with minimal additional control parameters necessary. Next, we present simulation results for both models, computing the critical point for each case. Finally, we demonstrate that common arguments fail to predict the results and discuss the difficulties in applying symmetry arguments to these problems.

II. DEPENDENT SITE PERCOLATION

Beginning with site percolation, we consider the problem where the only 2-point correlations exist between nearest neighbors. To be more precise, we ask what happens to the critical point of a lattice if uniform nearest-neighbor correlations are added while keeping the single site occupancies unchanged. Naturally, one has to ask first whether such setup is possible in the first place. Here, we present a simple algorithm that given an uncorrelated lattice with occupation probability \( p \), generates a second lattice with occupation probability also equal to \( p \), but in which there is a uniform nearest neighbor correlation between sites.

Consider a two-dimensional square lattice \( V \) of sites indexed by an ordered pair \((i,j) \in \mathbb{Z}^2\) and occupied with probability \( p \). The state of each site is represented by a random variable \( s_{i,j} \) where \( s_{i,j} = 1 \) if the site is occupied and 0 otherwise. One can think of the \( s_{i,j} \) as the indicator functions

\[
s_{i,j} = 1_{X_{i,j} \leq p}
\]  

where \( X_{i,j} \sim \text{Uniform}(0,1) \) are identical independent random variables. We shall henceforth refer to the \( X_{i,j} \) as the underlying random variables of \( V \).

Given a realization of such a lattice, we define a second lattice \( V' \) indexed by \((2\mathbb{Z} - 1)^2\) and for every \((i,j) \in V'\),

\[
Y_{i,j} = \frac{1}{4} (X_{i+1,j} + X_{i-1,j} + X_{i,j+1} + X_{i,j-1})
\]

the state of the site is given by \( u_{i,j} = 1_{Y_{i,j} < q} \) where \( Y_{i,j} \) is a random variable defined as the arithmetic mean of the nearest neighbors of the \((i,j)\) site in \( V \) and our goal is to choose \( q \) such that \( \Pr(u_{i,j} = 1) = \Pr(s_{i,j} = 1) = p \).

We have

\[
Y_{i,j} = \frac{1}{4} (X_{i+1,j} + X_{i-1,j} + X_{i,j+1} + X_{i,j-1})
\]

Clearly, each site in \( V' \) is only correlated with its four nearest neighbors. The correlation between neighboring \( V' \) sites as well as the threshold probability \( q \) necessary to define the state of each site remain to be determined.

In order to compute \( q \), we first derive an expression for the probability density function of \( Y_{i,j} \). For the sake of readability, consider one site in \( V' \) with underlying random variable \( Y \) and denote its four \( V \) neighbors by \( X_1, X_2, X_3, X_4 \). Define \( \phi_X(t) \) to be the characteristic function of the \( X \) variables (which are identical):

\[
\phi_X(t) = \int_{-\infty}^{\infty} e^{-itx} f_X(x) \, dx
\]

where \( f_X(x) \) is the probability density function of \( X_i \):

\[
f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

It is an elementary result that if the \( X_i \) are independent, then \( \phi_{\sum X_i}(t) = \prod \phi_{X_i}(t) \). Furthermore, by the inversion formula for the characteristic function, the density function for \( \sum_{i=1}^{4} X_i \) is given by \( g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \phi_{\sum X_i}(t) \, dt \). Noting that the probability density function of \( Y = \frac{1}{4} \sum_{i=1}^{4} X_i \), \( f_Y(y) \) is equal to \( 4g(4y) \), a straightforward calculation yields:

\[
f_Y(y) = \frac{1}{3} \left[ 4y - 4|y|^3 - 4|y - 1|^3 + 64y^3 \right]
\]

for \( 0 \leq y \leq 1 \) and zero otherwise. Then, the cumulative
distribution function of $Y$ is $F_Y(y) = \int_0^y f_Y(z) dz$:

$$F_Y(y) = \begin{cases} \frac{32y^4}{4} & (192y^4 + 256y^3 - 96y^2 + 16y - 1) \\ \frac{1}{2} (192y^4 - 512y^3 + 480y^2 - 176y + 23) \\ \frac{1}{2} (-32y^4 + 128y^3 - 192y^2 + 28y - 29) \end{cases}$$

for $0 < y < \frac{1}{16}$, $\frac{1}{4} < y < \frac{1}{2}$, $\frac{1}{2} < y < \frac{3}{4}$, and $\frac{3}{4} < y < 1$ respectively.

Thus, with the sites in the $V'$ defined as $u_{i,j} = 1_{Y_{i,j} < q}$, we have

$$q = F_Y^{-1}(p) \implies Pr(u_{ij} = 1) = p. \quad (II.6)$$

That is, a dependent lattice populated with density $p$ must be derived from a primal lattice populated with density $q = F_Y^{-1}(p)$ which needs to be solved numerically. Next, we derive an expression for the nearest-neighbor conditional occupation probability in this model. For the sake of generality, we slightly modify the notation used above: we denote the cumulative distribution function of the average of $n$ independent uniformly distributed random variables, $Y = \frac{1}{n} \sum_{i=1}^{n} X_i$ by $F^{[n]}(y)$. Consider two such random variables: $Y_1$ and $Y_2$ defined by

$$Y_1 = \frac{1}{n} (Z + U_1 + U_2 + \cdots + U_{n-1}) \quad \text{and} \quad Y_2 = \frac{1}{n} (Z + W_1 + W_2 + \cdots + W_{n-1})$$

where $Z, U_i, W_i, i = 1, \cdots, n-1$ are independently drawn from Uniform(0,1). Then, $Pr(Y_1 < q | Y_2 < q) = Pr(Y_1 < q \cap Y_2 < q) / Pr(Y_2 < q)$. The numerator is

$$Pr(Y_1 < q \cap Y_2 < q) = \int_0^1 Pr(Y_1 < q \cap Y_2 < q | Z = s) f_Z(s) ds$$

$$= \int_0^1 \left[ \Pr \left( \sum_{i=1}^{n-1} U_i < nq - s \right) \Pr \left( \sum_{i=1}^{n-1} W_i < nq - s \right) f_Z(s) \right] ds$$

$$= \int_0^1 \left[ F^{[n-1]} \left( \frac{nq - s}{n-1} \right) \right]^{2} f_Z(s) ds. \quad (II.9)$$

Thus,

$$Pr(Y_1 < q | Y_2 < q) = \int_0^1 \left[ F^{[n-1]} \left( \frac{nq - s}{n-1} \right) \right]^{2} ds$$

$$= \int_0^1 \left[ F^{[n]}(q) \right] \left[ F^{[n]}(q) \right] ds. \quad (II.10)$$

This relation may be used to compute—numerically or in closed form— the correlation between nearest neighbors in the model. Figure [II.2] shows $F^{[4]}(y)$ and the corresponding nearest-neighbor conditional probability. That $Y_1$ and $Y_2$ are positively correlated is evident.

### III. DEPENDENT BOND PERCOLATION

In order to illustrate a different application, we devise a different strategy for generating derived lattices with positive nearest-neighbor correlations in the case of bond percolation. Figure [II.1a] illustrates a renormalization scheme where a coarse-grained version of an independent square lattice is constructed as follows. First, the primal lattice (dashed lines) is populated with independent and identical bonds with occupation probability $p$. Given any such configuration, we then proceed to construct the corresponding configuration of the diagonal lattice (solid lines) by placing an open diagonal bond between diagonally opposite corners of any given squares of the primal lattice whenever there is an open path between the two corners through the primal edges on the boundary of the square. For instance, if we denote by $x, y, r, u$ respectively the events that the four boundary edges of a primal square be open and by $\alpha$ the event that the corners $A$ and $B$ are connected through those edges (figure [II.1a]), then $\alpha = (x \cap y) \cup (r \cap u)$, and $Pr(B \leftrightarrow C)$ is equal to

$$Pr(\alpha) = 2p^2 - p^4. \quad (III.1)$$

Percolation on the diagonal lattice, then, implies percolation on the primal lattice. Thus, the critical thresholds of the diagonal lattices of the primal lattice and its dual together yield upper and lower bounds on the critical threshold of the primal lattice.

For the purpose of simulating percolation on such a lattice, we require a square $N \times N$ diagonal lattice which may be generated easily as illustrated in figure [III.1b]. Here, the solid lines again represent the diagonal bonds, but they are rotated by 45 degrees. The independent bonds of the primal lattice are represented by dots which are populated independently, whereupon each diagonal bond is declared open depending on the state of the four dots surrounding it as discussed above.

In the diagonal lattice, then, any two nearest neighbors (more precisely, nearest neighbors at a right angle with respect to one another) are positively correlated, as they are both increasing functions of the primal bond they share. In figure [III.1a], the events $A \leftrightarrow B$ and $B \leftrightarrow C$ are positively correlated due to their mutual dependence on
the increasing event \( x \).

We may compute the correlation as follows:

\[
\Pr(B \leftrightarrow C | B \leftrightarrow A) = \Pr(\alpha | \beta) = \Pr(\alpha | \beta \cap x) \Pr(x | \beta) + \Pr(\alpha | \beta \cap \bar{x}) \Pr(\bar{x} | \beta) \quad (\text{III.2})
\]

Using \( \Pr(x | \beta) = \Pr(\beta | x) \Pr(x) / \Pr(\beta) \) and \( \Pr(\bar{x} | \beta) = \Pr(\beta | \bar{x}) \Pr(\bar{x}) / \Pr(\beta) \), a straightforward calculation leads to:

\[
\Pr(B \leftrightarrow C | B \leftrightarrow A) = \frac{p(1 + p - p^2)^2}{2 - p^2} + \frac{p^2(1 - p)}{2 - p^2} \quad (\text{III.3})
\]

On the other hand, the dual of the correlated lattice has an occupation probability \( \Pr(B \leftrightarrow A) = \Pr(\bar{\alpha}) = 1 - 2p^2 + p^4 \) and a nearest-neighbor conditional probability equal to

\[
\Pr(B \leftrightarrow C | B \leftrightarrow A) = \Pr(\bar{\alpha} | \bar{\beta}) = \frac{(1 - p)^3(1 + p)^2(1 + p - p^2)}{1 - 2p^2 + p^4}. \quad (\text{III.4})
\]

Interestingly, one can verify that for \( p \in [0, 1] \),

\[
\Pr(B \leftrightarrow C | B \leftrightarrow A) = \Pr(\bar{\alpha} | \bar{\beta}) = \frac{(1 - p)^3(1 + p)^2(1 + p - p^2)}{1 - 2p^2 + p^4}.
\]

For each of the two models introduced above, we perform a set of simulations to measure the critical occupation probability for various lattice sizes \( N \). For each \( N \), we measure the crossing probability \( \theta_N(p) \) for a large sample of occupation probabilities \( p \). Given the increasingly sharp transition in \( \theta_N(p) \) at the critical point, an adaptive algorithm must be employed in order to generate a sample of \( p \) values concentrated about the unknown critical point. To this end, we implement a Metropolis Monte-Carlo algorithm which stochastically samples the set of \( p \) values while attempting to minimize an “energy function” defined as an increasing function \( E(\theta_N(p) - 0.5) \) at a finite but low “temperature”.

For each \( N \), then, an optimal linear regression yields the finite-size critical probability defined as \( p^*(N) = \theta_N^{-1}(0.5) \), and finally, the critical probability \( p_c = \lim_{N \to \infty} p^*(N) \) is estimated by fitting a power-law to the set of finite-size critical values:

\[
k | p^* - p_c |^{-k} = N. \quad (\text{IV.1})
\]

Figure [IV.1] shows our results for dependent site and bond percolation. Whereas the critical probability of independent percolation is roughly 0.5927, we find that the dependent lattice percolates at \( p_c \approx 0.5546 \), roughly 6% lower.
An extension of this argument may appear to apply to percolation models. Let $A_k$ be the event that all the edges in a subset $E_k \subset E$ be open, i.e., $A_k = \cap_i a(e_i)$. Any path between two points $x, y$ in the lattice (or any cluster in general) corresponds to one such event $A_k$. The event $A_k$ will occur in each of the two cases (1) and (2) with the following probabilities:

\[
\Pr(A_k) = \begin{cases} 
\prod_{e_i \in E_k} \Pr(a(e_i)) & \text{case 1} \\
\Pr(\bigcap_{e_i \in E_k} a(e_i)) & \text{case 2}
\end{cases}
\]

(V.1)

The FKG inequality states that for a set $\{x_i\}$ of increasing events (decreasing events), $\Pr(\bigcap x_i) \geq \prod \Pr(x_i)$. We now apply this inequality to $\{a(e_i)|e_i \in A_k\}$ for every $k$, noting that $a(e_i)$ are increasing events:

\[
\Pr(\bigcap_{e_i \in E_k} a(e_i)) \geq \prod_{e_i \in E_k} \Pr(a(e_i)).
\]

(V.2)

The result is that each path (or each cluster in general) is more likely to be open in the presence of correlations than without correlations. Note that the same inequality holds if we replace the increasing events $a(e_i)$ with the decreasing events $\overline{a}(e_i)$, meaning that each path or cluster of the dual lattice also occurs with higher probability once our correlations are introduced.

If we define percolation as the almost certain connectivity of opposite boundaries of $B(N)$ (box of size $2N$, centered at the origin), then as $N \to \infty$, this may seem to imply that If the uncorrelated lattice percolates at a point in the parameter space of single edge probabilities, then the lattice will still percolate even if we introduce positive correlations between edges, as long as we remain at
the same point in the single edge parameter space. This means that the supercritical zone of the parameter space with correlations is a superset of the supercritical zone of the parameter space without correlations. This is true for the lattice as well as its dual, but the supercritical zone of the lattice is the complement of the supercritical zone of the dual lattice, which leads to the conclusion that the border between the two—i.e., the critical surface—remains intact.

However, to arrive at that conclusion, one must justify one more proposition, namely that the union of all the paths considered above also increases in measure as a result of the introduction of the correlations. As it turns out, this does not follow from the above argument.

This is a very peculiar situation. Every single path is more likely to be open in the presence of correlations. However, the probability of at least one of them being open is not guaranteed to increase.

Similarly, one may attempt to show that the average cluster size does not decrease when positive correlations are introduced, by showing that each cluster $E_k \subset E$ appears with a higher probability in that case. We have the same inequality as before, but with the following caveat. While the probability of any cluster being open ($\Pr \left[ \bigcap_{e_i \in E_k} a(e_i) \right]$) increases by introducing positive correlations, the probability of occurrence of the lattice animal made up of the same edges does not necessarily increase, since the lattice animal consists of open interior edges and closed perimeter edges and the intersection of the two groups of events (open interior edges and closed perimeter edges) does not necessarily increase in measure. If we were able to prove an increase in the probability of occurrence of all lattice animals, then the result would follow trivially, but our situation is more complicated.

VI. CONCLUSIONS

Our results allow us to answer the question posed at the outset. Using our algorithm, we have simultaneously constructed two different dependent bond percolation problems, on the lattice and on its dual. In both cases, any subset of bonds is positively correlated. However, one of the two has a lower critical point than the independent lattice while (consequently), the other has a higher critical point. We see, then, that positive correlation is not sufficient for increased connectivity, nor is it sufficient for decreased connectivity.

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