POSITIVE FUSS-CATALAN NUMBERS AND SIMPLE-MINDED SYSTEMS
IN NEGATIVE CALABI-YAU CATEGORIES

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Abstract. We establish a bijection between $d$-simple-minded systems ($d$-SMSs) of $(-d)$-Calabi-Yau cluster category $\mathcal{C}_{-d}(H)$ and silting objects of $D^b(H)$ contained in $D^{\leq 0} \cap D^{< -d}$ for hereditary algebra $H$ of Dynkin type and $d \geq 1$. We show that the number of $d$-SMSs in $\mathcal{C}_{-d}(H)$ is the positive Fuss-Catalan number $C_{d+1}^+(W)$ of the corresponding Weyl group $W$, by applying this bijection and Buan-Reiten-Thomas’ and Zhu’s results on Fomin-Reading’s generalized cluster complexes. Our results are based on a refined version of silting-$t$-structure correspondence.

1. Introduction

Fomin and Zelevinsky [FZ] showed that cluster algebras of finite type correspond bijectively with finite root systems $\Phi$. As a generalization of their combinatorial structure, Fomin and Reading [FR] introduced generalized cluster complex $\Delta^d(\Phi)$ for each positive integer $d$. It is a simplicial complex whose ground set is the disjoint union of $d$ copies of the set $\Phi^+$ of positive roots and the set of negative simple roots, and studied actively in combinatorics [Ar, STW]. It is known that $\Delta^d(\Phi)$ is categorified by $(d+1)$-Calabi-Yau ($\mathcal{C}_{d+1}(H)$ for the corresponding hereditary algebra $H$ of Dynkin type [KT]. The category $\mathcal{C}_{d+1}(H)$ has special objects called $(d+1)$-cluster tilting objects (see Definition 2.5), which correspond bijectively with maximal simplices in $\Delta^d(\Phi)$ [Z] and with silting objects (see Definition 2.3) contained in some subcategory of $D^b(H)$ [BRT1]. Cluster tilting objects also play a key role in Cohen-Macaulay representations [I].

Recently there is increasing interest in negative CY triangulated categories (Definition 2.4) (see [CS1, CS2, CS3, CSP, HLY, Ji1, Ji2, Jo, KYZ]), including $(-d)$-CY cluster categories $\mathcal{C}_{-d}(H)$ (see Section 1.1). These categories often contain special objects called $d$-simple-minded systems (or $d$-SMS) (see Definition 2.7). SMS plays a key role in the study of Cohen-Macaulay dg modules [Ji1].

In some important cases, cluster tilting objects and $d$-SMSs are shadows of more fundamental objects, namely, silting objects (Definition 2.3) and simple-minded collections (SMCs) (see Definition 2.6).

The aim of this paper is to show that there is a bijection between $d$-SMSs and maximal simplices in $\Delta^d(\Phi)$ without negative simple roots. In particular, the number of $d$-SMSs (Definition 2.7) in $\mathcal{C}_{-d}(H)$ is precisely the positive Fuss-Catalan number. Our method is based on a refined version...
of silting-t-structure correspondence, which is a bijection between silting objects in perfect derived categories of a finite-dimensional algebra $A$ and SMCs in bounded derived categories of $A$.

1.1. **Counting $d$-simple-minded systems.** Let $\Phi$ be a simply-laced finite root system, and $W$ the corresponding Weyl group. Let $\Phi_i$ be the set of positive roots and let $\alpha_i$ ($i \in I$) be the simple root. The **Fuss-Catalan number** is given by the formula

$$C_d(W) := \prod_{i=1}^{n} \frac{dh + e_i + 1}{e_i + 1},$$

where $n$ is the rank of $W$, $h$ is its Coxeter number, and $e_1, \ldots, e_n$ are its exponents (see Figure 1).

Recall that the generalized cluster complex $\Delta^d(\Phi)$ is a simplicial complex, whose ground set is

$$\Phi^d_{\geq -1} = (\Phi_+ \times \{1, \ldots, d\}) \cup \{(-\alpha_i, 1) | i \in I\},$$

and a simplex is a subset of $\Phi^d_{\geq -1}$ satisfying a compatibility condition (see [ER] Definition 3.1).

It is well-known that $C_d(W)$ equals the number of maximal simplices in $\Delta^d(\Phi)$ and also equals the number of $d$-noncrossing partitions for $W$ (see [AI, FR]). There is a variant of $C_d(W)$, called the **positive Fuss-Catalan number**, denoted by $C_d^+(W)$ and given by the formula

$$C_d^+(W) := \prod_{i=1}^{n} \frac{dh + e_i - 1}{e_i + 1},$$

see Figure 1 for the explicit value.

Let $k$ be a field and let $H$ be a hereditary $k$-algebra of Dynkin type. For an integer $d$, the $(-d)$-cluster category $\mathcal{C}_{-d}(H)$ is defined as the orbit category $\mathcal{C}_{-d}(H) := \mathcal{D}^{b}(H)/\nu[d]$, where $\nu := ? \otimes_H^{L} \mathcal{D}H$ is the Nakayama functor of $\mathcal{D}^{b}(H)$. This is a triangulated category by [K] and has AR quiver $Q/\nu[d]$ for the valued quiver $Q$ of $H$. We denote by $d$-SMS$\mathcal{C}_{-d}(H)$ the set of isomorphic classes of $d$-SMSs (see Definition 2.7) in $\mathcal{C}_{-d}(H)$, and by max-sim $\Delta^d(\Phi)$ (resp. max-sim$^+$ $\Delta^d(\Phi)$) the set of maximal simplices (resp. maximal simplices without negative simple roots) in $\Delta^d(\Phi)$. We will prove the following result.

**Theorem 1.1 (Theorem 3.1.2).** Let $H$ be a hereditary $k$-algebra of Dynkin type and let $d \geq 1$.

1. There is a bijection

$$d\text{-SMS} \mathcal{C}_{-d}(H) \leftrightarrow \text{max-sim}^+ \Delta^d(\Phi).$$

2. We have $\# d\text{-SM}\mathcal{C}_{-d}(H) = C_d^+(W)$, where $W$ is the Weyl group of $H$.

The result (2) is known for the case $d = 1$ by [CST] and for type $A_n$ by [Ji]. Figure 1 gives us concrete formulas for each Dynkin type.

To prove Theorem 1.1, we need to introduce some categorical notions. We define the following subcategories of $\mathcal{D}^{b}(H)$ for any $n, m \in \mathbb{Z}$.

$$\mathcal{D}^{\leq n} := \{X \in \mathcal{D}^{b}(H) | H^{\geq n}(X) = 0\}, \mathcal{D}^{\geq m} := \{X \in \mathcal{D}^{b}(H) | H^{\leq m}(X) = 0\}.$$ We have standard $t$-structures (see Definition 2.1) $\mathcal{D}^{b}(H) = \mathcal{D}^{\leq n} \perp \mathcal{D}^{\geq n+1}$ for any $n \in \mathbb{Z}$.

Let $m \leq n$. Since we have

$$\mathcal{D}^{\leq m+1} \subset \nu \mathcal{D}^{\leq m} \subset \mathcal{D}^{\leq m} \text{ and } \mathcal{D}^{\geq m+1} \subset \nu^{-1} \mathcal{D}^{\geq m} \subset \mathcal{D}^{\geq m},$$

we define three subcategories by

$$\mathcal{D}^{[m, n]} := \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m} \subset \mathcal{D}^{[m, n]} := \mathcal{D}^{\leq n} \cap \nu \mathcal{D}^{\geq m+1} \subset \mathcal{D}^{(m, n)} := \mathcal{D}^{\leq n} \cap \nu^{-1} \mathcal{D}^{\leq m-1}.$$ Then $\mathcal{D}^{[1-d, 0]}$ and $\mathcal{D}^{[-1, d]}$ are the fundamental domains of $\mathcal{C}_{d+1}(H)$ and $\mathcal{C}_{-d}(H)$ respectively. More precisely, the canonical functors $\mathcal{D}^{[1-d, 0]} \to \mathcal{C}_{d+1}(H)$ and $\mathcal{D}^{[-1, d]} \to \mathcal{C}_{-d}(H)$ induce bijections

$$\text{ind} \mathcal{D}^{[1-d, 0]} = \bigcup_{i=0}^{d-1} \text{ind} (\text{mod } H)[i] \bigcup \text{ind} (\text{proj } H)[d] \cong \text{ind} \mathcal{C}_{d+1}(H),$$

where $\text{ind} \mathcal{D}^{[1-d, 0]}$ and $\text{ind} \mathcal{C}_{d+1}(H)$ are the indecomposable objects of $\mathcal{D}^{[1-d, 0]}$ and $\mathcal{C}_{d+1}(H)$ respectively.
Theorem 1.2 (Theorem 3.11) Let \( H \) be a hereditary \( k \)-algebra of Dynkin type and \( d \geq 1 \). Then there are bijections

\[
\text{(silt} \mathcal{D}^b(H)\text{)} \cap \mathcal{D}^{[1-d,0]} \overset{1:1}{\sim} \text{(SMC} \mathcal{D}^b(H)\text{)} \cap \mathcal{D}^{[1-d,0]}, \tag{1.4}
\]

\[
\overset{1:1}{\sim} \text{d-SMS} \mathcal{C}_{-d}(H), \tag{1.5}
\]

where the map \( \text{1:1} \) is induced by the natural functor \( \mathcal{D}^b(H) \overset{\sim}{\rightarrow} \mathcal{D}^b(H)/\nu[d] \).

The bijection (1.4) holds in a more general setting, that is, it is true for any finite-dimensional Iwanaga-Gorenstein algebras (which will be proved in Corollary 3.6 and used in the proof of Theorem 1.2). Notice that, in the recent paper [CSPP], the bijection (1.5) was given for the path algebra \( kQ \) of any acyclic quiver \( Q \).

Our theorems and the results in [BRT1, Z] mentioned above are summarized as follows, where we denote by \((d+1)\text{-cilt}\mathcal{C}_{d+1}(H)\) the set of \((d+1)\text{-cluster tilting objects in } \mathcal{C}_{d+1}(H)\).

\[
\begin{array}{c}
\text{d-SMS} \mathcal{C}_{-d}(H) \overset{\text{Thm 3.11}}{\sim} \text{(SMC} \mathcal{D}^b(H)\text{)} \cap \mathcal{D}^{[1-d,0]} \overset{\text{Thm 3.11}}{\sim} \text{silt} \mathcal{D}^b(H) \cap \mathcal{D}^{[1-d,0]} \overset{\text{Thm 3.11}}{\sim} \text{max-sim} \Delta^d(\Phi) \overset{\text{Thm 3.11}}{\sim} \text{(d+1)-cilt} \mathcal{C}_{d+1}(H) \overset{\text{BRT1}}{\sim} \text{silt} \mathcal{D}^b(H) \cap \mathcal{D}^{[1-d,0]}
\end{array}
\]

We give an example of Theorem 3.11. Recall the cluster category \( \mathcal{C}_{-d}(H) \) has AR quiver \( Z/Q/\nu[d] \) for the valued quiver \( Q \) of \( H \).

**Example 1.3.** (1) Let \( H = kA_3 \) and \( d = 1 \), then the bijection between silting objects of \( \mathcal{D}^b(kA_3) \) contained in \( \text{mod } kA_3 = \mathcal{D}^{[0,0]} \) and 1-SMS of \( \mathcal{C}_{-1}(kA_3) \) is as follows.

| \( Q \) | \( h \) | \( e_1, \ldots, e_n \) | \( C^+_d(W) = \# \text{d-SMS} \mathcal{C}_{-d}(H) \) |
|---|---|---|---|
| \( A_n \) | \( n + 1 \) | \( 1, 2, \ldots, n \) | \( \frac{1}{n+1} \left( (n+1)^{n+d-1} \right) \) |
| \( B_n, C_n \) | \( 2n \) | \( 1, 3, \ldots, 2n-1 \) | \( \frac{(d+1)n-1}{n} \) |
| \( D_n \) | \( 2(n-1) \) | \( 1, 3, \ldots, 2n-3, n-1 \) | \( \frac{(2d+1)n-2d-2}{n} \) |
| \( E_6 \) | 12 | 1, 4, 5, 7, 8, 11 | \( \frac{d(2d+1)(3d+1)(4d+1)(6d+5)(12d+7)}{30} \) |
| \( E_7 \) | 18 | 1, 5, 7, 9, 11, 13, 17 | \( \frac{d(3d+1)(3d+2)(9d+2)(9d+4)(9d+8)}{280} \) |
| \( E_8 \) | 30 | 1, 7, 11, 13, 17, 19, 23, 29 | \( \frac{d(3d+1)(5d+1)(5d+2)(5d+3)(15d+8)(15d+11)(15d+14)}{144} \) |
| \( F_4 \) | 12 | 1, 5, 7, 11 | \( d(2d+1)(3d+1)(6d+5) \) |
| \( G_2 \) | 6 | 1, 5 | \( 3d^2 + 2d \) |
The five diagrams in left (resp. right) part are the AR quivers of $\mod kA_3$ (resp. $\mathcal{C}_{-1}(kA_3)$), where black vertices show all elements of $(\text{silt} \mathcal{D}^b(kA_3)) \cap \mod kA_3$ (resp. $1\text{-SMS} \mathcal{C}_{-1}(kA_3)$), and the arrows are given by mutation (see [BY]).

(2) Let $H = kA_2$ and $d = 2$. Then the bijection is as follows:

Let $H$ be a hereditary $k$-algebra. Recall that [BRT2] introduced the notion of Hom$_{\leq 0}$-configurations of $\mathcal{D}^b(H)$ (see Definition 2.8) and they gave a bijection between silting objects $s$ and Hom$_{\leq 0}$-configurations in [BRT2, Theorem 2.4]. This notion is similar to SMC, but quite different in non-hereditary case. At the end of this paper, we prove the following

**Theorem 1.4** (Theorem 3.13). Let $H$ be a hereditary $k$-algebra. Then Hom$_{\leq 0}$-configurations of $\mathcal{D}^b(H)$ are precisely SMCs of $\mathcal{D}^b(H)$.

One may also deduce the bijection (1.4) from Theorem 1.4 and [BRT2, Theorem 2.4].

1.2. Silting-structure correspondence. There is a useful structure in a triangulated category called $t$-structure (see Definition 2.1). In the derived category of a finite dimensional algebra, we have the following important bijection between silting objects and certain $t$-structures.

**Theorem 1.5.** [KYa] Let $A$ be a finite-dimensional $k$-algebra. Then there are bijections,

$$\text{silt per } A \overset{1:1}{\longleftrightarrow} \{ \text{bounded } t\text{-structures of } \mathcal{D}^b(A) \text{ with length hearts} \} \overset{1:1}{\longleftrightarrow} \text{SMC } \mathcal{D}^b(A),$$

where the first map sends $P \in \text{silt per } A$ to the $t$-structure $\mathcal{D}^b(A) = P[<0] \perp P[\geq 0]^\perp$, and the second map sends a $t$-structure to the simple objects in the heart.

In this subsection, we give two refined versions of Theorem 1.5 for triangulated categories, both of which imply the bijection (1.4) above. Our common assumption is the following, which is satisfied for $\mathcal{T} = \mathcal{D}^b(A)$ and the perfect derived category $\mathcal{U} = \text{per } A$ for a finite-dimensional $k$-algebra $A$.

**Assumption 1.6.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a thick subcategory of $\mathcal{T}$ (that is, $\mathcal{U}$ is a triangulated subcategory of $\mathcal{T}$ closed under direct summands). Assume that for any $P \in \text{silt } \mathcal{U}$, we have a bounded $t$-structure

$$\mathcal{T} = T_P^{-} \perp T_P^{+},$$

where $T_P^{-} := P[<0]^\perp$ and $T_P^{+} := P[\geq 0]^\perp$. (1.6)

See Section 2 for the definitions of $\perp$ and $(\cdot)^\perp$.

We call (1.6) the silting $t$-structure associated with $P$, and call its heart $\mathcal{H}_P := T_P^{-} \cap T_P^{+}$ the silting heart. Then $P$ can be recovered from the subcategory $\mathcal{H}_P$ (see Lemma 3.1). Denote by $\text{silt-heart } \mathcal{T}$ the set of silting hearts of $\mathcal{T}$. Notice that $\text{silt } \mathcal{U}$ and $\text{silt-heart } \mathcal{T}$ have canonical partial orders (see Section 2).
Theorem 1.7 (Theorem 3.3). Under Assumption 1.6 let $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^\perp = \mathcal{Y} \perp \mathcal{Y}^\perp$ be two silting $t$-structures. Then there is a poset isomorphism

$$\{P \in \text{silt} \mathcal{U} \mid P \in \mathcal{X} \cap \mathcal{Y}\} \xrightarrow{\sim} \{\mathcal{H} \in \text{silt-heart} \mathcal{T} \mid \mathcal{H} \subset \mathcal{X} \cap \mathcal{Y}^\perp\}.$$  

For a finite-dimensional $k$-algebra $A$ and $d \geq 1$, we call $P \in \text{silt per} A$ a $d$-term silting, if $\text{Hom}_{\mathcal{D}^b(A)}(A[<0], P) = 0 = \text{Hom}_{\mathcal{D}^b(A)}(P, A[\geq d])$. For $\mathcal{T} = \mathcal{D}^b(A)$, by applying Theorem 3.3 to $\mathcal{X} = \mathcal{D}^{\leq 0}$ and $\mathcal{Y} = \mathcal{D}^{\leq -d}$, we get the corollary below. It is well-known for the case $d = 2$ [BY] and plays an important role in cluster theory.

Corollary 1.8 (Corollary 3.4). There is a poset isomorphism

$$\{d\text{-term silting objects in per } A\} \xrightarrow{\sim} (\text{SMC } \mathcal{D}^b(A)) \cap \mathcal{D}^{[-d,0]}.$$  

When $\mathcal{T}$ has relative Serre functor in the following sense, we can improve Theorem 3.3 by dropping the assumption that two $t$-structures are silting.

Assumption 1.9. Assume that $\mathcal{T}$ is a $k$-linear triangulated category and we have a relative Serre functor $S$, that is, there is an auto-equivalence $S : \mathcal{T} \cong \mathcal{T}$ which restricts to an auto-equivalence $S : \mathcal{U} \cong \mathcal{U}$, such that there exists a functorial isomorphism

$$D \text{Hom}_{\mathcal{T}}(X, Y) \cong \text{Hom}_{\mathcal{T}}(Y, SX),$$  

for any $X \in \mathcal{U}$ and $Y \in \mathcal{T}$, where $D$ is the $k$-dual.

Then there is a poset isomorphism as follows.

Theorem 1.10 (Theorem 3.3). Under Assumption 1.6 let $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^\perp = \mathcal{Y} \perp \mathcal{Y}^\perp$ be any two $t$-structures. There is a poset isomorphism

$$\{P \in \text{silt} \mathcal{U} \mid P \in \mathcal{X} \cap \mathcal{Z}\} \xrightarrow{\sim} \{\mathcal{H} \in \text{silt-heart} \mathcal{T} \mid \mathcal{H} \subset \mathcal{X} \cap \mathcal{S} \mathcal{Z}\}.$$  

Now we consider a finite-dimensional $k$-algebra $A$, which is Iwanaga-Gorenstein (that is, the $A$-module $A$ has finite injective dimension both sides). For $d \geq 1$, we get the corollary below by applying Theorem 3.3 to $\mathcal{T} = \mathcal{D}^b(A)$, $\mathcal{U} = \text{per } A$, $\mathcal{X} = \mathcal{D}^{\leq 0}$ and $\mathcal{Z} = \mathcal{D}^{\geq 1-d}$. It plays a key role in proving Theorem 3.11.

Corollary 1.11 (Corollary 3.6). There is a poset isomorphism

$$(\text{silt per } A) \cap \mathcal{D}^{[-d,0]} \xrightarrow{\sim} (\text{SMC } \mathcal{D}^b(A)) \cap \mathcal{D}^{[-d,0]}.$$  

2. Preliminaries

Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{U}$ and $\mathcal{V}$ be two full subcategories of $\mathcal{T}$. We denote by $\text{add} \mathcal{U}$ the smallest subcategory containing $\mathcal{U}$, which is closed under direct summands and finite direct sums. We denote by $\text{thick} \mathcal{U}$ the thick subcategory generated by $\mathcal{U}$ (that is, the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{U}$ and closed under direct summands). We denote by $\text{Filt} \mathcal{U}$ the smallest extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{U}$. We define new subcategories

$$\mathcal{U}^\perp := \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(U, X) = 0 \text{ for any } U \in \mathcal{U}\},$$  

$$\perp \mathcal{U} := \{Y \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(Y, U) = 0 \text{ for any } U \in \mathcal{U}\},$$  

$$\mathcal{U} \ast \mathcal{V} := \{X \in \mathcal{T} \mid \text{there is a triangle } U \to X \to V \to U[1] \text{ with } U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}.$$  

If $\text{Hom}_{\mathcal{T}}(U, V) = 0$ for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$, we write $\mathcal{U} \ast \mathcal{V} = \mathcal{U} \perp \mathcal{V}$.

Definition 2.1. Let $\mathcal{T}, \mathcal{U}, \mathcal{V}$ be as above.

1. We call $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ a torsion pair of $\mathcal{T}$, if $\mathcal{T} = \mathcal{U} \perp \mathcal{V}, \mathcal{U}^\perp = \mathcal{V}$ and $\perp \mathcal{V} = \mathcal{U};$
2. We call a torsion pair $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ a $t$-structure of $\mathcal{T}$ if $\mathcal{U}[1] \subset \mathcal{U};$
3. We call a torsion pair $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ a co-$t$-structure of $\mathcal{T}$ if $\mathcal{V}[1] \subset \mathcal{V}.$
Let $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ be a $t$-structure (resp. co-$t$-structure) of $\mathcal{T}$. We denote by $\mathcal{H} = \mathcal{U} \cap \mathcal{V}[1]$ (resp. $\mathcal{P} = \mathcal{U} \cap \mathcal{V}[-1]$) the heart (resp. co-heart). We say a $t$-structure $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ is bounded, if $\bigcup_{i \in \mathbb{Z}} \mathcal{U}[i] = \mathcal{T} = \bigcup_{i \in \mathbb{Z}} \mathcal{V}[i]$. A bounded $t$-structure is determined by its heart.

**Definition 2.7.** Let $\mathcal{T} = \mathcal{U} \perp \mathcal{V}$ be a bounded $t$-structure with heart $\mathcal{H}$. Then $\mathcal{U} = \text{Filt}(\mathcal{H}[,0])$ and $\mathcal{V} = \text{Filt}(\mathcal{H}[,0])$.

On the set of $t$-structures on $\mathcal{T}$, there is a natural partial order defined by

$$(\mathcal{U}, \mathcal{V}) \geq (\mathcal{U}', \mathcal{V}') :\iff \mathcal{U} \supset \mathcal{U}' \iff \mathcal{V} \subset \mathcal{V}',$$

where $\mathcal{T} = \mathcal{U} \perp \mathcal{V} = \mathcal{U}' \perp \mathcal{V}'$ are $t$-structures with hearts $\mathcal{H}$ and $\mathcal{H}'$ respectively. It induces a partial order on the set of hearts of bounded $t$-structures by Lemma [2.2] that is

$$\mathcal{H} \geq \mathcal{H}' :\iff \mathcal{U} \supset \mathcal{U}' \iff \mathcal{V} \subset \mathcal{V}' \iff \text{Hom}_\mathcal{T}(\mathcal{H}', \mathcal{H}[,0]) = 0.$$  \hfill (2.1)

**Definition 2.3.** An object $P \in \mathcal{T}$ is called silting object if $\text{Hom}_\mathcal{T}(P, P[,0]) = 0$ and $\mathcal{T} = \text{thick } P$.

Two silting objects $P$ and $Q$ are equivalent if $\text{add } P = \text{add } Q$. We denote by $\text{silt } \mathcal{T}$ the set of equivalence classes of silting objects in $\mathcal{T}$. If $P \in \mathcal{T}$ is silting, then we have a natural co-$t$-structure

$$\mathcal{T} = \mathcal{T}_\geq P \perp \mathcal{T}_\leq P,$$

where $\mathcal{T}_\geq P := \text{Filt}(P[,0])$ and $\mathcal{T}_\leq P := \text{Filt}(P[,0])$. \hfill (2.2)

We have a partial order on $\text{silt } \mathcal{T}$, that is, for $P, Q \in \text{silt } \mathcal{T}$,

$$Q \geq P :\iff \mathcal{T}_\geq Q \supset \mathcal{T}_\leq P \iff \mathcal{T}_\leq Q \subset \mathcal{T}_\geq P \iff \text{Hom}_\mathcal{T}(Q, P[,0]) = 0.$$  \hfill (2.3)

Let $k$ be a field in the sequel.

**Definition 2.4.** Let $\mathcal{T}$ be a $k$-linear triangulated category. Let $S : \mathcal{T} \xrightarrow{\simeq} \mathcal{T}$ be an equivalence.

(1) We call $S$ a Serre functor of $\mathcal{T}$, if there exists a functorial isomorphism for any $X, Y \in \mathcal{T}$,

$$D \text{Hom}_\mathcal{T}(X, Y) \xrightarrow{\simeq} \text{Hom}_\mathcal{T}(Y, SX).$$

(2) Let $d \in \mathbb{Z}$. We call $\mathcal{T}$ a $d$-Calabi-Yau triangulated category, if $[d]$ gives a Serre functor.

**Definition 2.5.** Let $\mathcal{T}$ be a $k$-linear Hom-finite Krull-Schmidt triangulated category. For $d \geq 1$, one object $P \in \mathcal{T}$ is called cluster tilting, if the following conditions hold.

(1) $\text{Hom}_\mathcal{T}(P, P[j]) = 0$ for $1 \leq j \leq d - 1$.

(2) We have $\text{add } P = \{X \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(P, X[j]) = 0$ for $1 \leq j \leq d - 1\}$.

Next let us recall the notions of SMC and SMS, which are main study objects in our paper.

**Definition 2.6.** Let $\mathcal{T}$ be a $k$-linear Hom-finite Krull-Schmidt triangulated category and $S$ a set of objects of $\mathcal{T}$. We say $S$ is a simple-minded collection (or SMC), if the following conditions hold.

(1) $\text{End}_\mathcal{T}(X)$ is a division $k$-algebra for each $X \in \mathcal{T}$, and $\text{Hom}_\mathcal{T}(X, Y) = 0$ for $X \neq Y \in S$;

(2) $\text{Hom}_\mathcal{T}(X[,0], Y) = 0$ for $X, Y \in S$;

(3) $\mathcal{T} = \text{thick } S$.

We denote by $\text{SMC } \mathcal{T}$ the set of SMCs in $\mathcal{T}$. SMCs were first studied by [Ric] in the context of derived categories of symmetric algebras and they were also studied by [AI] under the name ‘cohomologically Schurian set of generators’. The name ‘SMC’ was introduced by [KYa] Definition 3.2) in general setting.

**Definition 2.7.** Let $\mathcal{T}$ be a $k$-linear Hom-finite Krull-Schmidt triangulated category and $S$ a set of objects of $\mathcal{T}$. For $d \geq 1$, we call $S$ a $d$-simple-minded system (or $d$-SMS for short), if the following conditions hold.

(1) $\text{End}_\mathcal{T}(X)$ is a division $k$-algebra for each $X \in \mathcal{T}$, and $\text{Hom}_\mathcal{T}(X, Y) = 0$ for $X \neq Y \in S$;

(2) $\text{Hom}_\mathcal{T}(X[,0], Y) = 0$ for any two objects $X, Y$ in $S$ and $1 \leq j \leq d - 1$;

(3) We have $\mathcal{T} = \text{Filt}(S[-j] \mid 0 \leq j \leq d - 1)$. 

We call $S$ a \textit{$(d)$-Calabi-Yau configuration} \textit{$(d)$-CY configuration} if it satisfies (1), (2) and
\[
\bigcap_{j=0}^{d-1} S[-j] = 0.
\]

The notion of SMS was introduced by \cite[Definition 2.1]{KYu}, and was generalized to $d$-SMS by \cite{CS2}. The term \textit{$(d)$-CY configuration} studied in \cite{BRT1} is also referenced to \textit{right $d$-Riedtmann configuration} in \cite{CS2} for the same concept.

If $H$ is a hereditary $k$-algebra of Dynkin type, then the notion \textit{$(d)$-CY configuration} of $C_{-d}(H)$ coincides with $d$-SMS by \cite[Proposition 2.13]{CSP}. Because in this case, $\text{Filt}(S)$ is functorially finite in $C_{-d}(H)$ for any \textit{$(d)$-CY configuration} $S$.

Finally, we recall the following definition.

\begin{definition} \cite{BRT1, Definition 2.2} \end{definition}

Let $H$ be a hereditary $k$-algebra. A basic object $X \in D^b(H)$ is a \textit{Hom$_{\leq 0}$-configuration} if
\begin{itemize}
  \item[(H1)] $X$ is the direct sum of $n$ exceptional indecomposable summands $X_1, \ldots, X_n$, where $n$ is the number of simple modules of $H$.
  \item[(H2)] $\text{Hom}_D(X_i, X_j) = 0$ for $i \neq j$.
  \item[(H3)] $\text{Hom}_D(X, X[t]) = 0$ for all $t < 0$.
  \item[(H4)] There is no subset $\{Y_1, \ldots, Y_r\}$ of the indecomposable summands of $X$ such that $\text{Hom}_D(Y_i, Y_{i+1}[1]) \neq 0$ and $\text{Hom}_D(Y_r, Y_1[1]) \neq 0$.
\end{itemize}

\section{Proof of main Theorems}

\subsection{Silting-$t$-structure correspondence} \begin{itemize}
  \item We first show Theorem \cite{14}. The following observation is useful.
  \end{itemize}

\begin{lemma} \label{lem:1} \end{lemma}

Under Assumption \cite{16}, there is a poset isomorphism $\text{silt} U \xrightarrow{\sim} \text{silt-heart} T$.

\begin{proof} The map $\text{silt} U \rightarrow \text{silt-heart} T$ is clearly surjective. For $P, Q \in \text{silt} U$, we have
\[
Q \geq P \Longleftrightarrow P[\geq 0] \subset Q[\geq 0] \xrightarrow{\text{1.1}} P[\geq 0] \subset T_Q^{<0} \xrightarrow{\text{1.1}} T_Q^{>0} \subset T_P^{>0} \xrightarrow{\text{1.1}} \mathcal{H}_Q \geq \mathcal{H}_P.
\]
Thus the map is a poset isomorphism. \end{proof}

\begin{proposition} \label{prop:2} \end{proposition}

Under Assumption \cite{16} let $Q, R \in \text{silt} U$. Then there is a poset isomorphism
\[
(\text{silt} U) \cap U_Q^{<0} \cap U_R^{\geq 0} \xrightarrow{\sim} \{ \mathcal{H} \in \text{silt-heart} T \mid \mathcal{H} \subset T_Q^{<0} \cap T_R^{>0} \}.
\]

\begin{proof} Let $P \in \text{silt} U$. Then
\[
P \in U_Q^{<0} \cap U_R^{\geq 0} \Longleftrightarrow Q \geq P \geq R \overset{\text{1.1}}{\Longrightarrow} \mathcal{H}_Q \geq \mathcal{H}_P \overset{\text{2.1}}{\Longrightarrow} \mathcal{H}_R \overset{\text{2.1}}{\Longrightarrow} \mathcal{H}_P \subset T_Q^{<0} \cap T_R^{>0}.
\]
Thus the assertion holds. \end{proof}

Now we are ready to prove Theorem \cite{17} which is stated as follows.

\begin{theorem} \cite{17}. \label{thm:3} \end{theorem}

Under Assumption \cite{16}, let $T = X \perp X' = Y \perp Y'$ be two silting $t$-structures. Then there is a poset isomorphism
\[
\{ P \in \text{silt} U \mid P \in X \cap Y \} \xrightarrow{\sim} \{ \mathcal{H} \in \text{silt-heart} T \mid \mathcal{H} \subset X \cap Y \}.
\]

\begin{proof} There exists $Q, R \in \text{silt} U$ such that $X = Q[\leq 0]$ and $Y = R[\leq 0]$. Since $X \cap Y = T_Q^{<0} \cap T_R^{>0}$ and $X \cap Y \cap U = U_Q^{<0} \cap U_R^{\geq 0}$ hold, the assertion follows from Proposition \ref{prop:2} \end{proof}

\begin{corollary} \cite{18}. \label{cor:4} \end{corollary}

There is a poset isomorphism
\[
\{ \text{$d$-term silting objects in per} A \} \xrightarrow{\sim} (\text{SMC} D^b(A)) \cap D^{[1-d,0]}.
\]
Proof. Let $\mathcal{T} = D^b(A)$ and $\mathcal{U} = \text{per} A$. Let $\mathcal{X} = D^{\leq 0}$ and $\mathcal{Y} = D^{\leq -d}$. Then $\mathcal{Y}^\perp = D^{\geq 1-d}$ and $\text{per} A \cap \mathcal{Y}^\perp = \text{Filt}(A[<d])$. By Theorem 3.5, we have a poset isomorphism $\{ P \in \text{per} A \mid P \in D^{\leq 0} \cap \text{Filt}(A[<d]) \} \xrightarrow{\sim} \{ H \in \text{silt-heart} D^b(A) \mid H \subset D^{\leq -d,0} \}$. Using the bijection silt-heart $D^b(A) \leftrightarrow \text{SMC} D^b(A)$ in Theorem 1.10, we obtain the assertion.

Next we prove Theorem 1.10.

**Theorem 3.5** (Theorem 1.10). Under Assumption 1.4, let $\mathcal{T} = \mathcal{X} \perp \mathcal{X}^\perp = \mathcal{Z} \perp \mathcal{Z}$ be any two $t$-structures. There is a poset isomorphism

$$\{ P \in \text{silt}\mathcal{U} \mid P \in \mathcal{X} \cap \mathcal{Z} \} \xrightarrow{\sim} \{ H \in \text{silt-heart} \mathcal{T} \mid H \subset \mathcal{X} \cap \mathcal{S}\mathcal{Z} \}.$$

**Proof.** Let $P \in \text{silt}\mathcal{U}$. Thanks to Lemma 3.1, it suffices to show that $P \in \mathcal{X}$ if and only if $H \subset \mathcal{X}$, and $P \in \mathcal{Z}$ if and only if $H \subset \mathcal{S}\mathcal{Z}$. Let $D_{\geq 0}$.

(a) By (1.6) and (2.2), we have $\displaystyle{\frac{1}{1}}(U_{\leq 0}^P) = \frac{1}{1}T^0_0 = T^0_0$. Thus

$$P \in \mathcal{X} \iff U_{\leq 0}^P \subset \mathcal{X} \iff U_{\leq 0}^P \subset \mathcal{X} \iff T^0_0 \subset \mathcal{X} \iff H \subset \mathcal{X}.$$

(b) By (1.6) and (2.2), we have $\displaystyle{\frac{1}{1}}(U_{\geq 0}^P)^\perp = T^0_0 = T^0_0$. Thus

$$P \in \mathcal{Z} \iff U_{\geq 0}^P \subset \mathcal{Z} \iff U_{\geq 0}^P \subset \mathcal{S}\mathcal{Z} \iff T^0_0 \subset \mathcal{S}\mathcal{Z} \iff H \subset \mathcal{S}\mathcal{Z}.$$ 

So the assertion is true. \hfill $\Box$

**Corollary 3.6** (Corollary 1.14). There is a poset isomorphism

$$(\text{silt per} A) \cap D^{[1-d,0]} \xrightarrow{\sim} (\text{SMC} D^b(A)) \cap D^{[-d,0]}.$$

**Proof.** Let $\mathcal{T} = D^b(A)$ and $\mathcal{U} = \text{per} A$. Let $\mathcal{X} = D^{\leq 0}$ and $\mathcal{Z} = D^{\geq 1-d}$. Since $A$ is Iwanaga-Gorenstein, then the Nakayama functor $\nu$ is the relative Serre functor. By Theorem 3.5, we have a poset isomorphism

$$\{ H \in \text{silt} D^b(A) \mid H \subset D_{[-d,0]} \}.$$ 

By Theorem 1.10, we obtain the assertion. \hfill $\Box$

### 3.2. Proof of results in Section 1.1

In this subsection, $H$ is a hereditary $k$-algebra. We will write $\mathcal{D}$ (resp. $\mathcal{C}$) for $D^b(H)$ (resp. $\text{C}_{\leq d}(H)$) for simplicity, and $H = \text{mod} H$. Let $S$ be an SMC of $\mathcal{D}$. Then $H_S := \text{Filt} S$ is the heart of a $t$-structure $(D_S^{\leq 0}, D_S^{\geq 0})$ given by

$$D_S^{\leq 0} := \text{Filt}(S[\geq 0]) \text{ and } D_S^{\geq 0} := \text{Filt}(S[\leq 0]).$$

We need the following observation.

**Lemma 3.7.** Let $S, T$ be two SMCs of $\mathcal{D}$. Then the following are equivalent.

1. $H_S \subset D_T^{\leq 0} \cap \nu D_T^{\geq 1-d}$;
2. $H_T \subset \nu^{-1} D_T^{\leq d-1} \cap D_T^{\geq 0}$;
3. $D_S^{\leq 0} \subset D_T^0$ and $D_S^{\geq 0} \subset \nu D_T^{\geq 1-d}$;
4. $D_T^{\leq 0} \subset \nu^{-1} D_T^{\leq d-1}$ and $D_T^{\geq 0} \subset D_T^{\geq 0}$.

**Proof.** (1) $\iff$ (3) follows from

$$(H_S \subset D_T^{0} \iff D_S^{\leq 0} \subset D_T^{\leq 0}) \text{ and } (H_S \subset \nu D_T^{\geq 1-d} \iff D_S^{\geq 0} \subset \nu D_T^{\geq 1-d}).$$

The similar argument shows (2) $\iff$ (4). Finally, (3) $\iff$ (4) follows from

$$(D_S^{\leq 0} \subset D_T^{\leq 0} \iff D_T^{\leq 0} \subset D_T^{\leq 0}) \text{ and } (D_S^{\geq 0} \subset \nu D_T^{\geq 1-d} \iff D_T^{\geq 0} \subset \nu^{-1} D_T^{\leq d-1}).$$ \hfill $\Box$
Recall that \( \pi : \mathcal{D} \to \mathcal{C} \) is the natural functor and by (1.3), there is a bijection
\[
\text{ind } \mathcal{D}_{[-d,0]} \cong \text{ind } \mathcal{C}.
\] (3.1)

In the rest, we write \( \pi(X) \) as \( X \) for any \( X \in \mathcal{D} \). We give a lemma which plays an important role in the sequel.

**Lemma 3.8.** Let \( X, Y \in \mathcal{D}_{-d,0} \) and \( 0 \leq i \leq d \). Then we have
\[
\Hom_{\mathcal{C}}(X,Y[-i]) = \Hom_{\mathcal{D}}(X,Y[-i]) \oplus D \Hom_{\mathcal{D}}(Y,X[i-d]).
\]

**Proof.** By the definition of \( \mathcal{C} \), we have
\[
\Hom_{\mathcal{C}}(X,Y[-i]) = \bigoplus_{n \in \mathbb{Z}} \Hom_{\mathcal{D}}(X,\nu^nY[nd-i]).
\]

If \( n < 0 \), then \( nd-i \leq -d \) and \( \nu^nY[nd-i] \in \nu^{n+1}\mathcal{D}^{\leq 1} \subset \mathcal{D}^{\geq 1} \) by (1.1). So \( \Hom_{\mathcal{D}}(X,\nu^nY[nd-i]) = 0 \). If \( n > 1 \), then \( m := 1-n < 0 \) and
\[
\Hom_{\mathcal{D}}(X,\nu^nY[nd-i]) = D \Hom_{\mathcal{D}}(\nu^{n-1}Y[nd-i],X) = D \Hom_{\mathcal{D}}(Y,\nu^{m}X[md-(d-i)]) = 0,
\]
by the first case. Thus the assertion follows. \( \square \)

For \( m \geq n \), we denote by \( \mathcal{D}_S^{[m,n]} \) the intersection \( \mathcal{D}_S^{\leq n} \cap \mathcal{D}_S^{\geq m} \). The following lemma is useful.

**Lemma 3.9.** Let \( H \) be a hereditary \( k \)-algebra, \( S \in \text{SMC } \mathcal{D} \cap \mathcal{D}_S^{\leq 0} \cap \nu \mathcal{D}_S^{\geq 1-d} \) and \( N \in \mathcal{D}_S^{\leq 0} \cap \nu \mathcal{D}_S^{\geq 1-d} \). For \( 1 \leq i \leq d \), if \( \Hom_{\mathcal{D}}(S[i],N) = 0 = \Hom_{\mathcal{D}}(N,S[i-d]) \), then \( N = 0 \).

**Proof.** By (1.1) and Lemma 3.7, we have
\[
\mathcal{H} = \text{mod } A \subset \nu^D \subset \mathcal{D}_S^{\leq 1} \cap \mathcal{D}_S^{\leq 0} \subset \mathcal{D}_S^{\leq d} \cap \mathcal{D}_S^{\geq 0}.
\] (3.2)

Since \( DA \in \nu \mathcal{D}_S^{\leq 0} \subset \mathcal{D}_S^{\leq d-1} \) by Lemma 3.7, \( DA \in \mathcal{D}_S^{[0,d-1]} \) by (3.2). Note that \( \Hom_{\mathcal{D}}(N,S[i-d]) = 0 \) for \( 1 \leq i \leq d \), then \( \Hom_{\mathcal{D}}(N,X) = 0 \) for any \( X \in \mathcal{D}_S^{[0,d-1]} \). In particular, we have \( \Hom_{\mathcal{D}}(N,DA) = 0 \) (that is \( H_0(N) = 0 \)), and moreover, \( N \in \mathcal{D}_S^{\leq 1-d} \cap \nu \mathcal{D}_S^{\geq 1-d} \subset \mathcal{D}_S^{[d-1,d-1]} \). Since \( \mathcal{D}_S^{[d-1,d-1]} = \text{Filt}(\mathcal{H}[i] \mid 1 \leq i \leq d) \), then (3.2) implies that \( N \in \mathcal{D}_S^{[-d-1]} \subset \mathcal{D}_S^{[d-1,d-1]} \). Recall that \( \mathcal{D}_S^{[d-1]} = \frac{1}{d}(1[S]_{\leq -d]} \) and \( \mathcal{D}_S^{[-d]} = \frac{1}{d}(S[d]) \), thus \( N \in \mathcal{D}_S^{[-1]} \cap \mathcal{D}_S^{[0]} = 0 \) by the orthogonality conditions on \( N \). \( \square \)

We denote by \((d)-\text{CY-conf } \mathcal{C}\) the set of \((d)-\text{Calabi-Yau configurations of } \mathcal{C}\).

**Proposition 3.10.** Let \( H \) be a hereditary \( k \)-algebra. Then the map
\[
(SMC \mathcal{D}) \cap \mathcal{D}_{[-d,0]} \xrightarrow{\pi} (d)-\text{CY-conf } \mathcal{C}.
\] (3.3)
is well-defined.

**Proof.** Let \( S \) be an SMC contained in \( \mathcal{D}_{[-d,0]} \). We show \( S \) is a \((d)-\text{CY configuration in } \mathcal{C} \). Let \( X,Y \in S \) and \( 0 \leq i < d \). By Lemma 3.8, we have
\[
\Hom_{\mathcal{C}}(X,Y[-i]) = \Hom_{\mathcal{D}}(X,Y[-i]) \oplus D \Hom_{\mathcal{D}}(Y,X[i-d]).
\]

Immediately \( S \) satisfies the conditions (1) and (2) in Definition 2.4.

It remains to check that \( \bigcap_{j=0}^{d-1} \frac{1}{j}S[-j] = 0 \). Let \( M \in \mathcal{C} \) be an indecomposable object satisfying \( \Hom_{\mathcal{C}}(M,S[-j]) = 0 \) for any \( 0 \leq j \leq d-1 \). By (3.1), there exists an indecomposable object \( N \in \mathcal{D}_{[-d,0]} \), such that \( \pi(N) = M \). By Lemma 3.8, we have \( \Hom_{\mathcal{D}}(N,S[i-d]) = 0 \) and \( \Hom_{\mathcal{D}}(S[i],N) = 0 \) for any \( 1 \leq i \leq d \). Then \( N = 0 \) by Lemma 3.7. \( \square \)

We are ready to prove Theorem 3.2 which is stated as follows.
Theorem 3.11. Let $H$ be a hereditary $k$-algebra of Dynkin type and $d \geq 1$. Then there are bijections
\[
(silt \mathcal{D}^b(H)) \cap \mathcal{D}^{[1-d,0]} \overset{1:1}{\leftrightarrow} (\text{SMC } \mathcal{D}^b(H)) \cap \mathcal{D}^{[-d,0]},
\]
(3.4)
\[
\overset{1:1}{\leftrightarrow} \text{d-SMC } C_{-d}(H),
\]
(3.5)
where the map (3.5) is induced by the natural functor $\mathcal{D}^b(H) \xrightarrow{\pi} \mathcal{D}^b(H)/\nu[d]$.

Proof. The bijection (3.4) follows directly from Corollary 3.6. The map (3.5) is well-defined by Proposition 3.10. Since this is injective by (3.1), it suffices to show that (3.5) is surjective.

Let $S$ be any d-SMS of $C$. We also denote by $S$ the preimage $\pi^{-1}(S)$ of $S$ via the bijection (3.1). We claim $S$ is an SMC of $D$. For $X \in S$, by Lemma 3.8 and $\text{Hom}_D(X, X[\leq 0]) = 0$, we have $\text{End}_C(X) = \text{End}_D(X)$, hence $\text{End}_D(X)$ is a division ring. Let $X, Y \in S$. We have
\[
\text{Hom}_D(X, Y[\leq i]) \supset \text{Hom}_D(X, Y[\leq i]),
\]
for any $i \in \mathbb{Z}$. For $0 \leq i \leq d-1$ and $X \neq Y$, the left hand side is 0, so is the right hand side. Now we show $\text{Hom}_D(X, Y[-d]) = 0$. This is clear if $X = Y$. If $X \neq Y$, then by Lemma 3.8
\[
0 = \text{Hom}_C(Y, X) = \text{Hom}_D(Y, X) \oplus D \text{Hom}_D(X, Y[-d]),
\]
and hence $\text{Hom}_D(X, Y[-d]) = 0$. For $i > d$, we have
\[
Y[-i] \in \nu \mathcal{D}^{\geq 1-d+i} \subseteq \nu \mathcal{D}^{> 1} \subseteq \mathcal{D}^{\geq 1}
\]
by (1.1). Thus $\text{Hom}_D(X, Y[-d]) = 0$.

It remains to show $D = \text{thick } S$. Since $D$ is locally finite, $S = \text{thick } S$ is functorially finite in $D$. Thus we have a torsion pair $D = \perp S \perp \mathcal{S}$ by [LY] Proposition 2.3. Thus it suffices to show $\perp \mathcal{S} = 0$. Let $X \in \perp \mathcal{S}$ be an indecomposable object. If $H$ is hereditary, $D = \text{add}(\mathcal{H}[i] || i \in \mathbb{Z})$. We may assume $X \in \mathcal{H}$. Then $\text{Hom}_D(X, S[-i]) = 0$ for all $i \in \mathbb{Z}$. Moreover, for any $0 \leq i < d$, we have $X[i-d] \in \mathcal{D}^{d+1}$, and hence $\text{Hom}_D(S, X[i-d]) = 0$. Since $X, S \in \mathcal{D}^{[1-d,0]}$, we have $\text{Hom}_C(X, S[-i]) = 0$ by Lemma 3.8. Since $S$ is a d-SMS, $X = 0$. Thus $\perp \mathcal{S} = 0$ as desired. $\Box$

Theorem 1.1 is clear now, which is stated as follows.

Theorem 3.12. Let $H$ be a hereditary $k$-algebra of Dynkin type and let $d \geq 1$.
(1) There is a bijection
\[
\text{d-SMC } C_{-d}(H) \overset{1:1}{\leftrightarrow} \text{max-sim } \Delta^d(\Phi).
\]
(2) We have $\# \text{d-SMC } C_{-d}(H) = C^+_d(W)$, where $W$ is the Weyl group of $H$.

Proof. (1) By [Z] Theorem 5.7 and [BRT2] Proposition 2.4, we have bijections
\[
\text{max-sim } \Delta^d(\Phi) \overset{\sim}{\rightarrow} (d+1)-\text{ctilt } C_{d+1}(H) \overset{\sim}{\rightarrow} (\text{silt } \mathcal{D}^b(H)) \cap \mathcal{D}^{[1-d,0]},
\]
which restrict to a bijection
\[
\text{max-sim } \Delta^d(\Phi) \overset{\sim}{\rightarrow} (d+1)-\text{ctilt } C_{d+1}(H) \overset{\sim}{\rightarrow} (\text{silt } \mathcal{D}^b(H)) \cap \mathcal{D}^{[1-d,0]},
\]
where $(d+1)$-ctilt $C_{d+1}(H)$ consists of $P \in (d+1)$-ctilt $C_{d+1}(H)$, which does not have a non-zero common direct summand with $(\text{proj } H)[d]$. Combine with Theorem 3.11 we get a bijection $\text{max-sim } \Delta^d(\Phi) \overset{\sim}{\rightarrow} \text{d-SMC } C_{-d}(H)$. Then (1) is true.

(2) By [FR] Proposition 12.4, $\# \text{max-sim } \Delta^d(\Phi) = C^+_d(W)$. Then (2) follows by (1). $\Box$

Let $H$ be a hereditary $k$-algebra. Recall that [BRT2] introduced the notion of $\text{Hom}_{<0}$-configurations of $\mathcal{D}^b(H)$ (see Definition 2.8), and they gave a bijection between silting objects and $\text{Hom}_{<0}$-configurations in [BRT2] Theorem 2.4. Combining their result and a general result Theorem 3.1 on SMC-exceptional sequence, we prove the following result.

Theorem 3.13. Let $H$ be a hereditary $k$-algebra. Then $\text{Hom}_{<0}$-configurations of $\mathcal{D}^b(H)$ are precisely SMCs of $\mathcal{D}^b(H)$. 


To prove this theorem, we need the following observation.

**Proposition 3.14.** Let $H$ be a hereditary algebra. Then

1. For each $X_1 \oplus \cdots \oplus X_n \in \text{silt} \, \mathcal{D}^b(H)$, the sequence $(X_1, \ldots, X_n)$ can be ordered into a silting-exceptional sequence.\[ \text{Proof.} \]
2. For each $\{S_1, \ldots, S_n\} \in \text{SMC} \, \mathcal{D}^b(H)$, the sequence $(S_1, \ldots, S_n)$ can be ordered into a SMC-exceptional sequence.\[ \text{Proof.} \]

Notice that $n \bigoplus (1)$ See [AI, Proposition 3.11].

Let $\mu^\leftrightarrow$ be a D-structure\[ \text{Proof.} \]

For each $X$, $Y$ let $(X, Y)$ be an exceptional sequence of $D$.\[ \text{Proof.} \]

Let $(X, Y)$ be an exceptional sequence of $D$.\[ \text{Proof.} \]

Then $X = Y$.\[ \text{Proof.} \]

If $X = Y$, then $X = Y$.\[ \text{Proof.} \]

Then $X = Y$.\[ \text{Proof.} \]

Notice that $\mu^{-1}$ induces bijections
\[ A := \{\text{SMC-exceptional sequences of } \mathcal{D}^b(H)\} \]
\[ \mu^{-1} \Rightarrow \text{Theorem A.1} \]
\[ \{\text{silting-exceptional sequences of } \mathcal{D}^b(H)\} \]
\[ \mu^{-1} \Rightarrow \text{BRT2 Theorem 2.4} \]

Then $A = B$. Thus, by Proposition 8.14 (2) (resp. BRT2 Lemma 2.3), any SMC (resp. Hom-configuration) of $D$ is given by an object in $A$ (resp. $B$) by forgetting the order. So the assertion is true. \[ \square \]

**Appendix A. Silting-$t$-structure correspondence via exceptional mutation**

In this appendix, we study the correspondence between silting objects and SMCs by exceptional mutation [CG, GR]. Throughout this section, $k$ is a field. Let $\mathcal{T}$ be a $k$-linear triangulated category such that $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X, Y[i])$ is finite-dimensional for any $X, Y \in \mathcal{T}$. An object $X \in \mathcal{T}$ is exceptional if $\text{End}_\mathcal{T}(X)$ is a division ring and $\text{Hom}_\mathcal{T}(X, X[i]) = 0$ for any $i \neq 0$. We call a sequence $(X_1, \ldots, X_n)$ of exceptional objects of $\mathcal{T}$ an exceptional sequence if $\text{Hom}_\mathcal{T}(X_i, X_j[0]) = 0$ for any $1 \leq j < i \leq n$ and any $l \in \mathbb{Z}$. An exceptional sequence $(X_1, \ldots, X_n)$ is full if $\text{thick}_\mathcal{T}(X_1 \oplus \cdots \oplus X_n) = \mathcal{T}$. If $\bigoplus_{i=1}^n X_i$ is a silting object (resp. SMC) of $\mathcal{T}$, we say the exceptional sequence $(X_1, \ldots, X_n)$ is silting-exceptional (resp. SMC-exceptional).

Let $(X, Y)$ be an exceptional sequence of $\mathcal{T}$. Consider the following two triangles

\[ Y'[1] \to \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X[i], Y) \otimes_k X[i] \to Y \to Y', \]
\[ X' \to X \to \bigoplus_{j \in \mathbb{Z}} D \text{Hom}_\mathcal{T}(X, Y[j]) \otimes_k Y[j] \to X'[1]. \]

We denote by $\mu^+(X, Y) := (Y', X)$ and $\mu^-(X, Y) := (Y, X')$ the left and right mutations of $(X, Y)$ respectively. For an exceptional sequence $(X_1, \ldots, X_n)$ and $1 \leq i \leq n-1$, we define

\[ \mu^+_i(X_1, \ldots, X_n) := (X_1, \ldots, X_{i-1}, \mu^+(X_i, X_{i+1}), X_{i+2}, \ldots, X_n), \]
\[ \mu^-_i(X_1, \ldots, X_n) := (X_1, \ldots, X_{i-1}, \mu^-(X_i, X_{i+1}), X_{i+2}, \ldots, X_n). \]

Then $\mu^+_i(X_1, \ldots, X_n)$ and $\mu^-_i(X_1, \ldots, X_n)$ are also exceptional sequences. Moreover, $\mu^+_i$ and $\mu^-_i$ are mutual inverses, and they satisfy the braid relations. Let

\[ \mu^{-1}_{\text{rev}} := \mu^+_2 \mu^+_1 \cdots (\mu^+_n \cdots \mu^+_1) \quad \text{and} \quad \mu^{-1}_{\text{rev}} := \mu^-_2 \mu^-_1 \cdots (\mu^-_{n-1} \cdots \mu^-_1). \]
Let full-exc $\mathcal{T}$ (resp. silt-exc $\mathcal{T}$, SMC-exc $\mathcal{T}$) be the set of isomorphism classes of full exceptional (resp. silting-exceptional, SMC-exceptional) sequences in $\mathcal{T}$.

**Theorem A.1.** $\mu_{\text{rev}}^{\pm}$ gives bijections

\[
\text{silt-exc } \mathcal{T} \xrightarrow{1:1} \text{SMC-exc } \mathcal{T} \quad \text{and} \quad \text{SMC-exc } \mathcal{T} \xrightarrow{1:1} \text{silt-exc } \mathcal{T}.
\]

**Remark A.2.** Theorem A.1 is given in [BRT2, Theorem 4.6] for the derived category $D^b(H)$ of a hereditary algebra $H$, where they considered Hom$_{\geq 0}$-configurations instead of SMCs. In Proposition 5.13 we showed Hom$_{\geq 0}$-configurations are precisely SMCs in the derived category of hereditary algebras by using Theorem A.1.

To prove Theorem A.1 the following observation is useful.

**Lemma A.3.** Let $(X_1, \ldots, X_n)$ be an exceptional sequence and $(X_i^\vee) := \mu_{\text{rev}}^+(X_1, \ldots, X_n)$. Then

1. $X_i^\vee \in X_i \ast \text{add}(X_i^\vee[0]) \ast \cdots \ast \text{add}(X_i^\vee[1][Z]) \subset X_i \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0])$.
2. For any $1 \leq s, t \leq n$ and $m \in \mathbb{Z}$, we have

\[
\text{Hom}_\mathcal{T}(X_s, X_t^\vee[m]) = \begin{cases} \text{division ring,} & \text{if } s = t \text{ and } m = 0. \\ 0, & \text{otherwise.} \end{cases}
\]

(A.1)

3. If $(X_1, \ldots, X_n)$ is a silting-exceptional sequence, then for each $1 \leq i \leq n$, we have

\[
X_i^\vee \in X_i \ast \text{add}(X_i^\vee[0]) \ast \cdots \ast \text{add}(X_i^\vee[1][Z]) \subset X_i \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0]).
\]

**Proof.** (1) Let $\mu^{(i)} := \mu_{n-i}^+ \ast \cdots \ast \mu_1^+$. Then $\mu_{\text{rev}} = \mu^{(n-1)} \cdots \mu^{(1)}$. Let $X_i^{(0)} := X_i$ for any $1 \leq i \leq n$, and $(X_i^{(i)}, X_i^{(i-1)}, \ldots, X_i^{(1)}) := \mu^{(i)}(X_1, \ldots, X_n)$. Since

\[
(X_i^{(i)}, \ldots, X_i^{(1)}) = \mu^{(i)}(X_i^{(i-1)}, \ldots, X_n^{(i-1)}, X_{i-1}^{(i-1)}, \ldots, X_1^{(i-1)}),
\]

then we have $X_j^{(i)} = X_j^{(i-1)}$ for $1 \leq j \leq i$, and moreover, for any $i + 1 \leq t \leq n$, we have

\[
X_i^{(i)} \in X_i^{(i-1)} \ast \text{add}(X_i^{(i-1)}[Z]).
\]

(A.2)

Therefore,

\[
X_i^\vee = X_i^{(n-1)} = \cdots = X_i^{(i-1)} \in X_i \ast \text{add}(X_i^{(i-1)}[Z]) \subset X_i \ast \text{add}(X_i^{(i-2)}[Z]) \ast \text{add}(X_i^{(i-3)}[Z]) \ast \cdots \ast \text{add}(X_i^{(0)}[Z]) \subset X_i \ast \text{add}(X_i^{\vee}[Z]) \ast \cdots \ast \text{add}(X_i^{\vee-1}[Z]).
\]

Inductively, we have $X_i \ast \text{add}(X_i^{\vee}[Z]) \ast \cdots \ast \text{add}(X_i^{\vee-1}[Z]) \subset X_i \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0])$. So (1) holds.

(2) Assume $s \geq t$. Since $X_i^\vee \in X_{\leq t} \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0])$ by (1) and $\text{Hom}_\mathcal{T}(X_s, X_i^\vee[m]) = 0$, we obtain

\[
\text{Hom}_\mathcal{T}(X_s, X_t^\vee[m]) = \text{Hom}_\mathcal{T}(X_s, X_t[m]) = \begin{cases} \text{division ring,} & \text{if } s = t \text{ and } m = 0. \\ 0, & \text{otherwise.} \end{cases}
\]

Assume $s < t$. Then $X_s \in \text{add}(X_i^\vee[0]) \ast \cdots \ast \text{add}(X_i^{\vee-1}[0])$ by (1). Since $(X_1^\vee, \ldots, X_n^\vee)$ is an exceptional sequence, $\text{Hom}_\mathcal{T}(X_s^\vee, X_i^\vee[m]) = 0$ holds, thus $\text{Hom}_\mathcal{T}(X_s, X_t^\vee[m]) = 0$. So the assertion is true.

(3) As in the proof of (1), it is enough to show that for any $1 \leq i \leq n$, we have

\[
X_i^{(i)} \in X_i^{(i-1)} \ast \text{add}(X_i^{(i-1)}[0]) \ast \cdots \ast \text{add}(X_i^{(i-1)}[t]) \rightarrow X_i^{(i-1)} \rightarrow X_i^{(i)}.
\]

(A.3)

In fact, $X_i^{(i)}$ is given by the triangle

\[
X_i^{(i)} \rightarrow \bigoplus_{t \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X_i^{(i-1)}[t], X_i^{(i-1)} \ast_k X_i^{(i-1)}[t]) \rightarrow X_i^{(i-1)} \rightarrow X_i^{(i)}.
\]
To prove (A.3), it suffices to show that
\[ \text{Hom}_\mathcal{T}(X_i^{(i-1)}[l], X_i^{(i-1)}) = 0 \text{ for } l \leq 0. \]  
(A.4)

This is clearly for \( i = 1 \). Now we assume (A.3) is true for \( 1, 2, \ldots, i-1 \). Since \( X_i^{(i-1)} = X_i^\vee \in X_i \ast \text{add}(X_i^\vee[2]) \ast \cdots \ast \text{add}(X_i^\vee[-1]) \) by (1), and \( (X_i^{(i-1)}, \ldots, X_i^{(i-1)}, X_i^{(i-1)}, \ldots, X_i^{(i-1)}) \) is an exceptional sequence, then \( \text{Hom}_\mathcal{T}(X_i^\vee[l], X_i^{(i-1)}) = \text{Hom}_\mathcal{T}(X_i^\vee[l], X_i^{(i-1)}) = 0 \) for any \( 1 \leq j \leq i-1 \). Therefore, we have
\[ \text{Hom}_\mathcal{T}(X_i^{(i-1)}[l], X_i^{(i-1)}) = \text{Hom}_\mathcal{T}(X_i[l], X_i^{(i-1)}). \]

It is zero for \( l < 0 \), because we know \( X_i^{(i-1)} \in X_i \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0]) \) by our assumption that (A.3) is true for \( 1, 2, \ldots, i-1 \). Thus (A.3) holds. \( \square \)

Now we prove Theorem A.1.

**Proof of Theorem A.1** For any full exceptional sequence \( (X_1, \ldots, X_n) \), we know \( (X_n^\vee, \ldots, X_1^\vee) := \mu(X_1, \ldots, X_n) \) is also a full exceptional sequence. Therefore we have \( \text{thick}(X_1^\vee, \ldots, X_n^\vee) = \mathcal{T} \) and \( \text{Hom}_\mathcal{T}(X_1^\vee, \ldots, X_n^\vee \neq 0) = 0 \).

Now we show the first part. Fix \( (P_1, \ldots, P_n) \in \text{silt-} \mathcal{T} \). To show \( (P_n^\vee, \ldots, P_1^\vee) \in \text{SMC-} \mathcal{T} \), it remains to show \( \text{Hom}_\mathcal{T}(P_s^\vee, P_t^\vee[\leq 0] = 0 \) for each \( s \neq t \). By Lemma (A.3), we have \( P_s^\vee \in X_s \ast \text{Filt}(X_1[0], \ldots, X_{s-1}[0]). \) By Lemma (A.3), \( \text{Hom}_\mathcal{T}(P_s^\vee, P_t^\vee[\leq 0] = 0 \) and \( \text{Hom}_\mathcal{T}(P_s^\vee, P_t^\vee[\geq 0] = 0 \) for any \( s \neq t \). Then \( \text{Hom}_\mathcal{T}(X_s, X_t[0]) = 0 \) and \( (X_1, \ldots, X_n) \in \text{silt-} \mathcal{T} \). Thus the assertion follows.

We have a bijection \( \iota : \text{full-} \mathcal{T} \xrightarrow{1:1} \text{full-} \mathcal{T} \) given by \( \iota(X_1, \ldots, X_n) = (X_n, \ldots, X_1) \), which gives bijections \( \iota : \text{full-} \mathcal{T} \xrightarrow{1:1} \text{silt-} \mathcal{T} \) and \( \iota : \text{SMC-} \mathcal{T} \xrightarrow{1:1} \text{SMC-} \mathcal{T} \). Since \( \iota \circ \mu^\vee \circ \iota = \mu^\vee \) and hence, \( \iota \circ \mu^\vee \circ \iota = \mu^\vee \) holds, then the second claim follows from the first one. \( \square \)

**Remark A.4**. Similarly to Lemma (A.3), we can show that, if \( (X_1, \ldots, X_n) \) is a SMC-exceptional sequence, then for each \( 1 \leq i \leq n \), we have
\[ X_i^\vee \in X_i \ast \text{add}(X_i^\vee[\leq 0]) \ast \cdots \ast \text{add}(X_i^\vee[\leq 0]) \subset X_i \ast \text{Filt}(X_1[0], \ldots, X_{i-1}[0]). \]

The second part of Theorem A.1 can be shown directly by this result.

We give an application of Theorem A.1.

**Corollary A.5**. Let \( A \) be a proper \( dg \) \( k \)-algebra. If \( \text{per} \ A \) has a full exceptional sequence, then \( \text{per} \ A \) has sitting-exceptional and SMC-exceptional sequences. Moreover, we have \( \text{per} \ A = \mathcal{D}^b(A) \).

**Proof**. Let \( (X_1, \ldots, X_n) \) be a full exceptional sequence in \( \text{per} \ A \). Then by [A1] Proposition 3.5, we have a sitting-exceptional sequence \( (X_1[l_1], \ldots, X_n[l_n]) \) by shifting the original one with proper integers \( l_1, \ldots, l_n \). By Theorem A.1, we get a SMC-exceptional sequence \( \mu^\vee(X_1[l_1], \ldots, X_n[l_n]). \) Let \( X = \bigoplus_{i=1}^n X_i[l_i]. \) Then \( B := \delta\text{ind}_A(X) \) is a non-positive proper \( dg \) algebra. Moreover, we have a triangle equivalence \( \text{per} \ B \simeq \text{per} \ A \) given by \( R\text{Hom}_A(X, ?) \), and hence \( \mathcal{D}^b(B) \simeq \mathcal{D}^b(A) \).

Recall that \( \mathcal{D}^b(B) \) has a standard \( t \)-structure \( \mathcal{D}^b(B) = \mathcal{D}^{\leq 0} \perp \mathcal{D}^{> 0} \), which is bounded and the heart is a length category. By Lemma (A.3), the simple objects in the heart are given by \( \mu_\text{rev}(X_1[l_1], \ldots, X_n[l_n]) \in \text{per} \ B \). Therefore \( \mathcal{D}^b(B) = \text{thick}(\mu_\text{rev}(X_1[l_1], X_n[l_n])) \in \text{per} \ B \). Since \( B \) is proper, we have \( \mathcal{D}^b(B) = \text{per} \ B \) as desired. \( \square \)

**Acknowledgements** We thank the referees for their useful comments and suggestions, which improved the writing of our paper.
References

[Al] Salah Al-Nofayee, Simple objects in the heart of a t-structure, J. Pure Appl. Algebra 213 (2009), no. 1, 54–59.
[Ar] Drew Armstrong, Generalized noncrossing partitions and combinatorics of Coxeter groups, Mem. Amer. Math. Soc. 202 (2009), no. 949.
[AI] Takuma Aihara, Osamu Iyama, Silting mutation in triangulated categories, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 635–668.
[BMRRT] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, Gordana Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006), no. 2, 572–618.
[BRT1] Aslak Bakke Buan, Idun Reiten, Hugh Thomas, Three kinds of mutation, J. Algebra 339 (2011), 97–113.
[BRT2] Aslak Bakke Buan, Idun Reiten, Hugh Thomas, From m-clusters to m-noncrossing partitions via exceptional sequences, Math. Z. 271 (2012), no. 3-4, 1117–1139.
[BY] Thomas Brüstle, Dong Yang, Ordered exchange graphs, Advances in representation theory of algebras, 135–193, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.
[C] William Crawley-Boevey, Exceptional sequences of representations of quivers, Representations of algebras (Ottawa, ON, 1992), 117–124, CMS Conf. Proc., 14, Amer. Math. Soc., Providence, RI, 1993.
[CS1] Raquel Coelho Simões, Hom-configurations and noncrossing partitions, J. Algebraic Combin. 35 (2012), no. 2, 313–343.
[CS2] Raquel Coelho Simões, Hom-configurations in triangulated categories generated by spherical objects, J. Pure Appl. Algebra 219 (2015), no. 8, 3322–3336.
[CS3] Raquel Coelho Simões, Mutations of simple-minded systems in Calabi-Yau categories generated by a spherical object, Forum Math. 29 (2017), no. 5, 1065–1081.
[CSPP] Raquel Coelho Simões, David Pauksztello, Simple-minded systems and reduction for negative Calabi-Yau triangulated categories, to appear in Trans. Amer. Math. Soc., arXiv:1808.02519
[CSPP] Raquel Coelho Simões, David Pauksztello, David Ploog, Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions, arXiv:2001.06693.
[D] Alex Dugas, Torsion pairs and simple-minded systems in triangulated categories, Appl. Categ. Structures 23 (2015), no. 3, 507–526.
[FR] Sergey Fomin, Nathan Reading, Generalized cluster complexes and Coxeter combinatorics, Int. Math. Res. Not. 2005, no. 44, 2709–2757.
[FZ] Sergey Fomin, André Zelevinsky, Cluster algebras. II. Finite type classification, Invent. Math. 154 (2003), no. 1, 63–121.
[GR] A. L. Gorodentsev, A.N. Rudakov, Exceptional vector bundles on projective spaces, Duke Math. J. 54 (1987), no. 1, 115–130.
[HJY] Thorsten Holm, Peter Jørgensen, Dong Yang, Sparseness of t-structures and negative Calabi-Yau dimension in triangulated categories generated by a spherical object, Bull. Lond. Math. Soc. 45 (2013), no. 1, 120–130.
[I] Osamu Iyama, Tilting Cohen-Macaulay representations, Proceedings of the International Congress of Mathematicians–Rio de Janeiro 2018. Vol. II. Invited lectures, 125–162, World Sci. Publ., Hackensack, NJ, 2018.
[IYa] Osamu Iyama, Dong Yang, Silting reduction and Calabi-Yau reduction of triangulated categories, Trans. Amer. Math. Soc. 370 (2018), no. 11, 7861–7898.
[IYo] Osamu Iyama, Yuji Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, Invent. Math. 172 (2008), no. 1, 117–168.
[Ji] Haibo Jin, Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations, arXiv:1812.09737
[Ji2] Haibo Jin, Reductions of triangulated categories and simple-minded collections, arXiv:1907.05114
[Joj] Peter Jørgensen, Auslander-Reiten theory over topological spaces, Comment. Math. Helv. 79 (2004), no. 1, 160–182.
[K] Bernhard Keller, On triangulated orbit categories, Doc. Math. 10 (2005), 551–581.
[KN] Bernhard Keller, Pedro Nicolas, Cluster hearts and cluster tilting objects, work in preparation.
[KV] Bernhard Keller, Dieter Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 239–253.
[KYZ] Bernhard Keller, Dong Yang, Guodong Zhou, The Hall algebra of a spherical object, J. Lond. Math. Soc. (2) 80 (2009), no. 3, 771–784.
[KYa] Steffen Koenig, Dong Yang, Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras, Doc. Math. 19 (2014), 403–438.
[KYu] Steffen Koenig, Yuming Liu, Simple-minded systems in stable module categories Q. J. Math. 63 (2012), no. 3, 653–674.
[Ric] Jeremy Rickard, Equivalences of derived categories for symmetric algebras, J. Algebra 257 (2002), no. 2, 460–481.
[Rie] Christine Riedtmann, Representation-finite self-injective algebras of class $A_n$, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 449–520, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[STW] Christian Stump, Hugh Thomas, Nathan Williams, Cataland: Why the Fuss?, arXiv:1503.00710.

[T] Hugh Thomas, Defining an m-cluster category, J. Algebra 318 (2007), no. 1, 37–46.

[Z] Bin, Zhu, Generalized cluster complexes via quiver representations, J. Algebraic Combin. 27 (2008), no. 1, 35–54.

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