Generalized Score Distribution

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2019
Model for Single PVS

\[ O_{ij} \sim N(\psi_o, \sigma_o) \]

We have two parameters, true quality \( \psi_o \) and standard deviation for particular PVS and Subject \( \sigma_o \). \( o \) stands for continuous model.
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After founding the new variable \( U \) has certain distribution with different parameters \( \psi_u \) and \( \sigma_u \).

Note that we estimate \( u \) not \( o \) parameters! Especially we estimate \( \psi_u \) not \( \psi_o \) which was entered to the simulator or MLE function.
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Note that we estimate \( u \) not \( o \) parameters! Especially we estimate \( \psi_u \) not \( \psi_o \) which was entered to the simulator or MLE function. So we can validate what is the relation between \( \psi_o \) and \( \psi_u \).
Theory

\[ O_{ij} \sim \mathcal{N}(\psi_o, \sigma_o) \]

\[
P(U_{ij} = k) = \begin{cases} 
\int_{-\infty}^{1.5} \frac{1}{\sqrt{2\pi}\sigma_o} e^{-\frac{(o-\psi_o)^2}{2\sigma_o}} & k = 1 \\
\int_{k-0.5}^{k+0.5} \frac{1}{\sqrt{2\pi}\sigma_o} e^{-\frac{(o-\psi_o)^2}{2\sigma_o}} & k \in \{2, 3, 4\} \\
\int_{k-0.5}^{k+0.5} \frac{1}{\sqrt{2\pi}\sigma_o} e^{-\frac{(o-\psi_o)^2}{2\sigma_o}} & k = 5 \\
\int_{4.5}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_o} e^{-\frac{(o-\psi_o)^2}{2\sigma_o}} & k = 5 
\end{cases}
\]

Knowing \( P(U_{ij} = k) \) we can calculate \( \psi_u \) and plot function \( \psi_u(\psi_o) \). It should be \( y = x \).
Let us assume that $\sigma_o = 0.1$. 

![Graph showing $\psi_u$ vs $\psi_o$ with a linear relationship and a wavy line representing the obtained results. The graph compares obtained and correct results.]
Let us assume that $\sigma_o = 1$. 

![Graph showing obtained and correct scores against $\psi_o$, with values ranging from 1 to 5 on both axes.](image-url)
Variance Limitation

\[ \mathbb{V}(U) \]

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Generalized Score Distribution
Instead of using continuous model we can use directly the discrete model.
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\[ P(X = 1) = F_1(\psi), \quad P(X = 2) = F_2(\psi) \]

Example: If \( \psi < 2 \): \( P(X = 1) = 2 - \psi, \quad P(X = 2) = \psi - 1 \)
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\[ U \sim GSD(\psi, \rho) \]

- \( \psi \) specify the mean
- \( \rho \) specify the answers spread
  - \( \rho = 1 \) is the minimum variance so only \( \lceil \psi \rceil \) and \( \lfloor \psi \rfloor \) are possible
  - \( \rho = 0 \) is the maximum variance so only the minimum and the maximum value is possible
### Existing Distributions

| $\psi$ | $V_{min}(\psi)$ | $V_{Bin}(\psi)$ | $V_{max}(\psi)$ |
|--------|-----------------|-----------------|-----------------|
| 3.0    | Bernoulli       | Binomial        | BetaBinomial    |
| 1      |                 |                 |                 |
| 0      |                 |                 |                 |
| 4      |                 |                 |                 |
Existing Distributions

\[ \psi = 3.0 \phi \]

Legend:
- \( V_{\min}(\psi) \) - Bernoulli distribution,
- \( V_{\max}(\psi) \) - Binomial distribution
- BetaBinomial distribution
- No distribution

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Generalized Score Distribution
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0

4

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Legend:

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Existing Distributions

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Our Proposal

1. For variances larger than Binomial distribution let us use BetaBinomial Distribution

2. For variances smaller we can mix Bernoulli with Binomial. Taking more Bernoulli we increase precision, taking more of Bernoulli we have more randomness
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\[ P(X = 1) = p\text{Ber}(X = 1) + (1 - p)\text{Bin}(X = 1) \]
Our Proposal

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\[ P(X = 1) = p\text{Ber}(X = 1) + (1 - p)\text{Bin}(X = 1) \]

\[ P(X = 1) = p(2 - \psi) + (1 - p) \binom{4}{0} \left(\frac{\psi - 1}{4}\right)^4 \]

\[ H_\rho = G_\rho \ I(\rho < C(\psi)) + F_\rho \ I(\rho \geq C(\psi)) \]
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Equations

\[ H_\rho = G_\rho \ I(\rho < C(\psi)) + F_\rho \ I(\rho \geq C(\psi)) \]

\[ P_{F_\rho}(\epsilon = k - \psi) = \]
\[ \frac{\rho - C(\psi)}{1 - C(\psi)} \left[ 1 - |k - \psi| \right]_+ + \]
\[ \frac{1 - \rho}{1 - C(\psi)} \left( \frac{4}{k - 1} \right) \left( \frac{\psi - 1}{4} \right)^{k-1} \left( \frac{5 - \psi}{4} \right)^{5-k} \]
\[ H_\rho = G_\rho \, I(\rho < C(\psi)) + F_\rho \, I(\rho \geq C(\psi)) \]

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\[ \frac{\rho - C(\psi)}{1 - C(\psi)} \left[ 1 - |k - \psi| \right]_+ + \]
\[ \frac{1 - \rho}{1 - C(\psi)} \left( \begin{array}{c} 4 \\ k - 1 \end{array} \right) \left( \frac{\psi - 1}{4} \right)^{k-1} \left( \frac{5 - \psi}{4} \right)^{5-k} \]

\[ P_{G_\rho}(\epsilon = k - \psi) = \left( \begin{array}{c} 4 \\ k - 1 \end{array} \right) \times \]
\[ B \left( \frac{\rho(\psi - 1)}{4(C(\psi) - \rho)} + k - 1, \frac{(5 - \psi)\rho}{4(C(\psi) - \rho)} + 5 - k \right) \]

\[ B \left( \frac{(\psi - 1)\rho}{4(C(\psi) - \rho)}, \frac{(5 - \psi)\rho}{4(C(\psi) - \rho)} \right) \]
GSD Examples

\[ \psi = 1.30 \]

\[ P(U_{ij} = s) \]

Score \( s \):

- \( 0.95 \)
- \( 0.88 \)
- \( 0.81 \)
- \( 0.72 \)
- \( 0.61 \)
- \( 0.38 \)
GSD Examples

\[ \psi = 2.10 \]

Score distribution examples:

- \( P(U_{ij} = s) \) for various values of \( \psi \):
  - \( \psi = 0.95 \)
  - \( \psi = 0.88 \)
  - \( \psi = 0.81 \)
  - \( \psi = 0.72 \)
  - \( \psi = 0.61 \)
  - \( \psi = 0.38 \)
GSD Examples

$\psi = 2.85$

$P(U_{ij} = s)$

Score $s$

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Generalized Score Distribution
GSD Examples

Score $s$

$P(U_{ij} = s)$

$\psi = 3.90$

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Generalized Score Distribution
Parameters Estimation

Simulation:

- a number of subjects $N = \{6, 12, 24, 48\}$,
- $\psi$ from 1.05 to 4.95 (with 23 different values),
- $\rho$ from 0.01 to 0.99 (with 23 different values).
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Subjective data $u_i$
Testing Distribution

Subjective data

GSD or Normal model

Parameters $(\psi, \rho)$ or $(\psi, \sigma)$

Theoretical probabilities $p_s$

Observed frequencies $O_s$

$\chi^2$ test, goodness of fit $p$-value

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- Observed frequencies $O_s$
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Subjective data

\[ u_i \]

Observed frequencies \( O_s \)

GSD or Normal model

Parameters \((\psi, \rho)\) or \((\psi, \sigma)\)

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\( \chi^2 \) test, goodness of fit

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Generalized Score Distribution
Testing Distribution

Subjective data $u_i$

Observed frequencies $O_s$

GSD or Normal model
Parameters $(\psi, \rho)$ or $(\psi, \sigma)$
Theoretical probabilities $p_s$

$\chi^2$ test, goodness of fit

$p$-value
Results for 1874 Sequences

Generalized Score Distribution

p-value

GSD

QNormal
Prior Distribution $\psi$

Generalized Score Distribution
Prior Distribution $\rho$

![Graph showing a bar chart with density on the y-axis and $\hat{\rho}$ on the x-axis. The bars represent the density values at different $\hat{\rho}$ values. The red line is a smoothing curve that fits the data points.]
The full paper describing the GSD can be found here: https://arxiv.org/pdf/1909.04369.
Further Steps

- Use GSD for bootstrap
- Advance the GSD to model taking into account subject bias or error coming from subjects and work on better estimation method like Bayes
- Use GSD for different data like not pixel quality but whole movie (like imdb) or product quality
- Create a correct test with GSD so we can test influencing factors like if gender influences the quality score
- Test which scale is correct by analyzing the answer spread
