CORRIGENDUM: THE BASE CHANGE FUNDAMENTAL LEMMA FOR CENTRAL ELEMENTS IN PARAHORIC HECKE ALGEBRAS

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1. INTRODUCTION

In section 2.2 of [H09], there is a minor misstatement that this note will correct and clarify. It has no effect on the main results of [H09], but nevertheless this corrigendum seems necessary in order to avoid potential confusion. Also, I take this opportunity to point out a related typographical error in [BT2], section 5.2.4, and to address some matters of a similar nature.

I am very grateful to Brian Smithling and Tasho Kaletha, who informed me that something was amiss in section 2 of [H09].

2. NOTATION

All notation will be that of [H09], except for the correction in notation discussed below.

3. CORRECTION

In [H09], section 2.2, the “ambient” group scheme $G_{aJ}$ was incorrectly identified with the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of the facet $a_J$. In the notation of Bruhat-Tits [BT2], which I intended to follow in [H09], the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of $a_J$ is denoted $\hat{G}_{a_J}$. The group scheme $\hat{G}_{a_J}$ is defined and characterized in this way in [BT2], 4.6.26-28.

The group scheme denoted $G_{a_J}$ is defined in loc. cit. 4.6.26 (cf. also 4.6.3-6). In general, it can be a bit smaller than $\hat{G}_{a_J}$ (see below). In [H09], the symbol $G_{a_J}$ should be interpreted as this potentially proper subgroup of the full fixer $\hat{G}_{a_J}$.

We have, as stated in [H09], (2.3.2) and (2.3.3), the equalities\footnote{In light of the typographical error in [BT2], 5.2.4 explained in section 6 the reasoning used in [H09] to justify these equalities is correct.}

\begin{align}
(3.0.1) 
J(L) &= G_{a_J}^\circ(\mathcal{O}_L) = T(L)_1 \cdot U_{a_J}(\mathcal{O}_L) \\
(3.0.2) 
G_{a_J}(\mathcal{O}_L) &= T(L)_b \cdot U_{a_J}(\mathcal{O}_L).
\end{align}

In general,

$$
G_{a_J}(\mathcal{O}_L) = \hat{G}_{a_J}(\mathcal{O}_L) \subset G_{a_J}(\mathcal{O}_L) \subset \hat{G}_{a_J}(\mathcal{O}_L),
$$

and both inclusions can be strict.
4. Clarification of subsequent statements in [H09]

1. Theorem 2.3.1 of [H09] remains valid as stated, but can be slightly augmented: equation (2.3.1) can be replaced by

\[ J(L) = \text{Fix}(a_J^\gamma) \cap G(L)_1 = G_{a_J}(O_L) \cap G(L)_1 = \hat{G}_{a_J}(O_L) \cap G(L)_1. \]

Cf. [HRa], Remark 11.

2. Contrary to [H09], line above equation (2.3.2), our \( G_{a_J} \) should not now be identified with the scheme \( \hat{G}_{a_J} \) of [BT2].

3. Corollary 2.3.2 of [H09] remains valid, with the same proof. Indeed, when \( G_L \) is split we have \( T(L)_b = T(O_L) = T(L)_1 \) and then from (3.0.1) and (3.0.2) above we see that \( G_{a_J}(O_L) = G_{a_J}(O_L) \).

4. Lemma 2.9.1 of [H09] remains valid as stated, but in the proof (especially in equations (2.9.1) and (2.9.2)) the symbols \( G_{a_J}(O_L) \) and \( G_{a_M}(O_L) \) should be replaced by \( \hat{G}_{a_J}(O_L) \) and \( \hat{G}_{a_M}(O_L) \), respectively.

5. Example

It is sometimes but usually not the case that \( G_{a_J}(O_L) = \hat{G}_{a_J}(O_L) \). The following is perhaps the simplest example where this equality fails.\(^2\) Take \( G \) to be the split group \( \text{PSp}(4) \), and let \( a_J \) denote the non-special vertex in a base alcove. Then let \( \tau \) denote the element in the stabilizer \( \Omega \subset \hat{W}(L) \) of the base alcove, which interchanges the two special vertices and fixes \( a_J \). The element \( \tau \) does not belong to the group \( G_{a_J}(O_L) = G_{a_J}(O_L) \) (cf. 3 above), since \( \tau \) does not belong to \( G(L)_1 \). On the other hand \( \tau \in \hat{G}_{a_J}(O_L) \) since it fixes \( a_J \) and \( G(L)_1 = G(L) \) (cf. [BT2], 4.6.28).

6. Typographical error in [BT2], 5.2.4

Section 5.2.4 of [BT2] contains four displayed equations. In all of these equations, the “hats” should be removed. The fact that the final displayed equation

\[ \hat{G}_{01}^\gamma(O^\delta) = G_{01}^\gamma(O^\delta) \hat{3}(O^\delta) \]

is incorrect as stated is shown by the Example above (in light of the fact that for a \( K^2 \)-split group such as \( \text{PSp}(4) \) the group scheme \( \hat{3} \) is connected and the right hand side is simply \( G_{01}^\gamma(O^\delta) \)).

All of the displayed equations in [BT2], 5.2.4 become correct when the “hats” are removed.

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\(^2\) Brian Smithling and Tasho Kaletha provided me with another example for the split group \( \text{SO}(2n) \).
7. When is $\mathcal{G}_{\alpha_j}(\mathcal{O}_L) = \hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L)$?

Let us assume (for simplicity) that $G$ is split over $L$. Then the following give two cases where the equality $\mathcal{G}_{\alpha_j}(\mathcal{O}_L) = \hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L)$ holds. Since $G_L$ is split, by Corollary 2.3.2 of [H09] we automatically have $\mathcal{G}_{\alpha_j}(\mathcal{O}_L) = \hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L)$.

Lemma 7.0.1. If $G_{\text{der}} = G_{\text{sc}}$, then $\hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L) = \mathcal{G}_{\alpha_j}(\mathcal{O}_L)$.

Proof. Let $I = \text{Gal}(\overline{T}/L)$ denote the inertia group. Recall that $G(L)_1$ is the kernel of the Kottwitz homomorphism

$$G(L) \to X^*(Z(\hat{G}))$$

and $G(L)^1$ is the kernel of the map

$$G(L) \to X^*(Z(\hat{G}))/\text{torsion}$$
derived from the Kottwitz homomorphism. Our hypotheses imply that $X^*(Z(\hat{G}))$ is torsion-free, and hence $G(L)^1 = G(L)_1$. But then $\hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L)$, being by [BT2], 4.6.28 the fixer of $a^\alpha_j$ in $G(L)^1$, obviously coincides with $\mathcal{G}_{\alpha_j}(\mathcal{O}_L)$, the fixer of $a^\alpha_j$ in $G(L)_1$ (cf. [H09] above).

Lemma 7.0.2. If the closure of $a_j$ contains a special vertex $v$, then $\hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L) = \mathcal{G}_{\alpha_j}(\mathcal{O}_L)$.

Proof. By [BT2], 4.6.26, we have $\hat{\mathcal{G}}_{\alpha_j}(\mathcal{O}_L) = \hat{N}_{\alpha_j}^1 \mathcal{G}_{\alpha_j}(\mathcal{O}_L)$, where $\hat{N}_{\alpha_j}^1$ denotes the fixer in $N = N_G(T)(L)$ of $a_j$. Hence, it suffices to show that $\hat{N}_{\alpha_j}^1 \subset G(L)_1$. Let $K = K_v$ be the special maximal parahoric subgroup of $G(L)$ corresponding to $v$, and realize the finite Weyl group $W$ at $v$ as $W = (K \cap N_G(T))/T(\mathcal{O}_L)$, cf. [HRa]. As in loc. cit., the choice of the special vertex $v$ gives us a decomposition of the extended affine Weyl group as $X_*(T) \times W$. For $n \in \hat{N}_{\alpha_j}^1$ let $t_\lambda w \in X_*(T) \times W$ denote the corresponding element.

We need to show that $t_\lambda w$ belongs to the affine Weyl group, since such an element will automatically belong to $G(L)_1$, and that would be enough to prove that $n \in G(L)_1$. We need to show $\lambda$ is in the coroot lattice $Q^\vee$. But $t_\lambda w$ fixes $v$, that is,

$$\lambda + w(v) = v.$$

On the other hand

$$v - w(v) \in Q^\vee,$$

since $v$ is a special vertex. Thus $\lambda \in Q^\vee$ and we are done.

8. Comparing Iwahori subgroups over $F$

The “naive” Iwahori subgroup that often appears in the literature (e.g. [C], [Mac]), can be identified with the group

$$\tilde{I} := G(F) \cap \text{Fix}(a^\alpha) = G(F)^1 \cap \text{Fix}(\langle a^\alpha \rangle^\sigma).$$

This contains the group

$$\tilde{\mathcal{G}}_\alpha(\mathcal{O}_F) = G(F)^1 \cap \text{Fix}(a),$$

(cf. [BT2], 4.6.28). The “true” Iwahori subgroup over $F$ is defined to be

$$I := G(F) \cap (G(L)_1 \cap \text{Fix}(a)) = \mathcal{G}_\alpha(\mathcal{O}_F)$$
(see [Hra]) which turns out to have the alternative description
\[ I = G(F)_1 \cap \text{Fix}(\mathfrak{a}^\sigma), \]
see [HRo], Remark 8.0.2. Thus, we always have the inclusions
\[ I \subseteq \hat{\mathcal{G}}_a(\mathcal{O}_F) \subseteq \bar{I}. \]

In general, we have \( \bar{I} \neq I \); for example, in the case of \( G = D^\times / F^\times \) we have \( \hat{\mathcal{G}}_a(\mathcal{O}_F) \neq \bar{I} \) (see Remark 8.0.2 of [HRo]).

**Lemma 8.0.3.** Suppose \( G \) is split over \( L \). Then \( I = \hat{\mathcal{G}}_a(\mathcal{O}_F) \).

**Proof.** Use Lemma 7.0.2. \( \square \)

**Proposition 8.0.4.** If \( G \) is unramified over \( F \), then \( I = \hat{\mathcal{G}}_a(\mathcal{O}_F) = \bar{I} \).

**Proof.** It is enough to prove \( I = \bar{I} \). Let \( v_F \) denote a hyperspecial vertex in the closure of \( (\mathfrak{a}^\sigma)^\sigma \), and let \( K = K_{v_F} \) denote the corresponding special maximal parahoric subgroup of \( G(F) \). Following [HRo], define \( \bar{K} = G(F)_1 \cap \text{Fix}(v_F) \); recall also that \( K = G(F)_1 \cap \text{Fix}(v_F) \). By loc. cit., it is clear that when \( G \) is unramified over \( F \) we have \( \bar{K} = K \). On the other hand, the inclusion \( \bar{I} \subset \bar{K} \) clearly induces an injection
\[ \bar{I}/I \hookrightarrow \bar{K}/K. \]

Thus \( \bar{I}/I \) is trivial. \( \square \)

**References**

[BT2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II*, Inst. Hautes Études Sci. Publ. Math. 60 (1984), 5-184.

[C] W. Casselman, *The unramified principal series of \( p \)-adic groups I. The spherical function*, Compositio Math. 40 (1980), 387-406.

[H09] T. Haines, *The base change fundamental lemma for central elements in parahoric Hecke algebras*, Duke Math. J, vol. 149, no. 3 (2009), 569-643.

[HRa] T. Haines, M. Rapoport, *Appendix: On parahoric subgroups*, Advances in Math. 219 (1), (2008), 188-198; appendix to: G. Pappas, M. Rapoport, *Twisted loop groups and their affine flag varieties*, Advances in Math. 219 (1), (2008), 118-198.

[HRo] T. Haines, S. Rostami, *The Satake isomorphism for special maximal parahoric Hecke algebras*, preprint 2009. Submitted. Available at www.math.umd.edu/~tjh.

[Mac] I. G. Macdonald, *Spherical functions on a group of \( p \)-adic type*, Ramanujan Institute, University of Madras Publ., 1971.

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