A note on the combination of equilibrium problems

Nguyen Thi Thanh Ha, Tran Thi Huyen Thanh, Nguyen Ngoc Hai, Hy Duc Manh, Bui Van Dinh

Department of Mathematics, Le Quy Don Technical University, Hanoi, Vietnam

1,2,4,5Department of Scientific Fundamentals, Trade Union University, Hanoi, Vietnam

Abstract. In this short paper, we show that the solution set of a combination of equilibrium problems is not necessary contained in the intersection of a finite family of solution sets of equilibrium problems. As a corollary, we deduce that statements in recent papers given by S. Suwannaut, A. Kangtunyakarn (Fixed Point Theory Appl. 2013, 2014; Thai Journal of Maths. 2016), W. Khuangsatung, A. Kangtunyakarn (Fixed Point Theory Appl. 2014), and A.A. Khan, W. Cholamjiak, and K.R. Kazmi (Comput. Appl. Maths. 2018) are not correct.

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1 Introduction

Let $C$ be a nonempty closed convex subset in the Euclidean space $\mathbb{R}^n$ and $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem (shortly EP($C, f$)), in the sense of Blum, Muu and Oettli [1, 6] (see also [3]), consists of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C.$$ 

We denote the solution set of EP($C, f$) by $\text{Sol}(C, f)$. Solution methods for EP($C, f$) can be found in [10, 2].

Let $f_i : C \times C \rightarrow \mathbb{R}, i = 1, 2, ..., N$, be bifunctions defined on $C$. Recently, many researchers are interested in finding a common solution of a finite family of equilibrium problems [7, 8, 9, 4] (CSEP for short).

Find $x^* \in C$ such that $f_i(x^*, y) \geq 0, \quad \forall y \in C$ and $i = 1, 2, ..., N$. $\text{CSEP}(C, f_i)$
Or, equivalently,
\[ \text{find } x^* \in \Omega := \cap_{i=1}^{N} \text{Sol}(C, f_i). \]

Given bifunctions \( f_i, i = 1, \ldots, N \) defined on \( C \). Let \( \alpha_i \in (0,1), i = 1, \ldots, N \) such that \( \sum_{i=1}^{N} \alpha_i = 1 \). Set
\[ f(x, y) = \sum_{i=1}^{N} \alpha_i f_i(x, y). \]

The combination of equilibrium problems (shortly, \( \text{CEP}(C, \sum_{i=1}^{N} \alpha_i f_i) \)) consists of finding \( x^* \in C \) such that
\[ f(x^*, y) = \sum_{i=1}^{N} \alpha_i f_i(x^*, y) \geq 0, \forall y \in C. \]

By \( \text{Sol}(C, \sum_{i=1}^{N} \alpha_i f_i) \), we denote the solution set of the combination of equilibrium problems.

In 2013, S. Suwannaut and A. Kangtunyakarn \[7\] said that under certain conditions
\[ \Omega := \cap_{i=1}^{N} \text{Sol}(C, f_i) = \text{Sol}(C, \sum_{i=1}^{N} \alpha_i f_i). \]

Therefore, to find a common solution of a finite family of equilibrium problems leads to find a solution of a combination of equilibrium problems \( \text{CEP}(C, \sum_{i=1}^{N} \alpha_i f_i) \). Based on this relation, S. Suwannaut and Kangtunyakarn \[7 8 9\], W. Khuangsatung and A. Kangtunyakarn \[5\], S.A. Khan, W. Cholamjiak, and K.R. Kazmi \[4\] gave algorithms for finding a common element of the fixed point sets of a family of mappings and the solution sets of equilibrium problems and/or the zero point sets of a family of mappings.

In this short paper, we show that, under the same conditions given in \[7\], the relation
\[ \text{Sol}(C, \sum_{i=1}^{N} \alpha_i f_i) \subset \cap_{i=1}^{N} \text{Sol}(C, f_i), \]
does not hold true. Therefore, presenting of recent papers \[7 8 9 5 4\] using this formula are not correct.

The rest of paper is organized as follows. The next section contains some preliminaries on equilibrium problems and some statements in papers \[7 8 9 5 4\] related with combination of equilibrium problems. The last section is devoted to show that the common points of a finite family of equilibrium problems is truly contained in a solution set of a combination of equilibrium problems and its corollaries.
2 Preliminaries

In this section, we present some statements presented in recent papers related to combination of equilibrium problems. Let \( \varphi : C \times C \to \mathbb{R} \) be a bifunction defined on \( C \). In the sequel, we need the following blanket assumptions:

**Assumptions** \( \mathcal{A} \).

\((\mathcal{A}_1)\) \( \varphi(x, x) = 0 \) for every \( x \in C \);

\((\mathcal{A}_2)\) \( \varphi \) is monotone on \( C \);

\((\mathcal{A}_3)\) \( \varphi \) is upper hemicontinuous, i.e., for each \( x, y, z \in C \) we have

\[
\limsup_{t \to 0^+} \varphi(tz + (1 - t)x, y) \leq \varphi(x, y);
\]

\((\mathcal{A}_4)\) for each \( x \in C \), \( \varphi(x, \cdot) \) is lower semicontinous and convex on \( C \);

\((\mathcal{A}_5)\) for fixed \( r > 0 \) and \( z \in C \), there exists a nonempty compact convex subset \( B \) of \( \mathbb{R}^n \) and \( x \in C \cap B \), such that

\[
\varphi(y, x) + \frac{1}{r} \langle y - z, z - x \rangle < 0, \forall y \in C \setminus B.
\]

The following statement is in [7].

**Statement 2.1** (See [7, Lemma 2.7]). Let \( f_i, i = 1, 2, \ldots, N \) be bifunctions satisfying \( \mathcal{A}_1 - \mathcal{A}_4 \) with \( \cap_{i=1}^N \text{Sol}(C, f_i) \neq \emptyset \). Then

\[
\cap_{i=1}^N \text{Sol}(C, f_i) = \text{Sol}(C, f),
\]

where \( f(x, y) = \sum_{i=1}^N \alpha_i f_i(x, y), \alpha_i > 0, \forall i = 1, 2, \ldots, N \) and \( \sum_{i=1}^N \alpha_i = 1 \).

If Statement 2.1 holds true then it allows us to find common solutions of \( N \) equilibrium problems by solving a combination of equilibrium problems.

The following statement is in [8].

**Statement 2.2** (See [8, Theorem 3.1]). Let \( F \) be an \( \tau \)-contractive mapping on \( \mathbb{R}^n \) and let \( A \) be a strongly positive linear bounded operator on \( \mathbb{R}^n \) with coefficient \( \bar{\gamma} \) and \( 0 < \gamma < \frac{\bar{\gamma}}{\tau} \). For every \( i = 1, 2, \ldots, N \) let \( f_i : C \times C \to \mathbb{R} \) be a bifunction satisfying \( \mathcal{A}_1 - \mathcal{A}_4 \) with \( \Omega = \cap_{i=1}^N \text{Sol}(C, f_i) \neq \emptyset \). Let \( \{x^n\}, \{y^n\}, \{z^n\} \) be sequences generated by \( x^1 \in \mathbb{R}^n \) and

\[
\begin{cases}
\sum_{i=1}^N \alpha_i f_i(z^k, y) + \frac{1}{\rho_k} \langle y - z^k, x^k - z^k \rangle \geq 0, \forall y \in C, \\
y^k = \theta_k P_C(x^k) + (1 - \theta_k)z^k, \\
x^{k+1} = \delta_k F(x^k) + (I - \delta_k A)y^k,
\end{cases}
\]

where \( \{\delta_k\}, \{\theta_k\}, \{\rho_k\} \subset (0, 1), 0 < \alpha_i < 1, \forall i = 1, \ldots, N \). Suppose the conditions \((i) - (vi)\) hold.
(i) \( \lim_{k \to \infty} \delta_k = 0 \) and \( \sum_{k=0}^{\infty} \delta_k = \infty \);

(ii) \( 0 < \theta \leq \theta_k \leq \bar{\theta} < 1 \), for some \( \bar{\theta}, \bar{\theta} \in (0, 1) \);

(iii) \( 0 < \underline{\alpha} \leq \alpha_k \leq \bar{\alpha} < 1 \), for some \( \underline{\alpha}, \bar{\alpha} \in (0, 1) \);

(iv) \( \sum_{i=1}^{N} \alpha_i = 1 \);

(v) \( \sum_{i=1}^{N} |\delta_{k+1} - \delta_k| < \infty \), \( \sum_{i=1}^{\infty} |\theta_{k+1} - \theta_k| < \infty \), \( \sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty \).

Then the sequences \( \{x^k\}, \{y^k\} \), and \( \{z^k\} \) converge to \( q = P_\Omega(I - A + \gamma F)q \).

From Theorem 3.1 in [5], we get the following statement.

**Statement 2.3** (See [5] Theorem 3.1). Let \( f_i, i = 1, 2, ..., N \) satisfy assumption \( A_1 - A_4 \). Assume that \( \Omega = \cap_{i=1}^{N} \text{Sol}(C, f_i) \neq \emptyset \). Let the sequence \( \{x^k\} \) and \( \{y^k\} \) be generated by \( u, x^1 \in \mathbb{R}^n \) and

\[
\begin{align*}
\sum_{i=1}^{N} \alpha_i f_i(y^k; y) + \frac{1}{\mu_k} (y - y^k, y^k - x^k) & \geq 0, \forall y \in C, \\
x^{k+1} = \lambda_k u + \mu_k x^k + \delta_k y^k
\end{align*}
\]

where \( \{\lambda_k\}, \{\mu_k\}, \{\delta_k\} \subset (0, 1) \) and \( \mu_k + \lambda_k + \delta_k = 1 \); \( \{\rho_k\} \subset (\rho, \bar{\rho}) \subset (0, 1) \), \( 0 < \alpha_i < 1, \forall i = 1, ..., N \). Suppose the conditions (i) - (iii) hold.

(i) \( \lim_{k \to \infty} \lambda_k = 0 \) and \( \sum_{k=0}^{\infty} \lambda_k = \infty \);

(ii) \( \sum_{i=1}^{N} \alpha_i = 1 \);

(iii) \( \sum_{i=1}^{N} |\delta_{k+1} - \delta_k| < \infty \).

Then the sequences \( \{x^k\}, \{y^k\} \) converge to \( q = P_\Omega(u) \).

The next statement is deduced from Theorem 3.1 in [9].

**Statement 2.4** [9] Theorem 3.1. Let \( F \) be an \( \tau \)-contractive mapping on \( \mathbb{R}^n \) and let \( f_i, i = 1, 2, ..., N \) satisfy assumption \( A_1 - A_4 \). Assume that \( \Omega = \cap_{i=1}^{N} \text{Sol}(C, f_i) \neq \emptyset \). Let the sequence \( \{x^k\} \) and \( \{y^k\} \) be generated by \( x^1 \in C \) and

\[
\begin{align*}
\sum_{i=1}^{N} \alpha_i f_i(y^k; y) + \frac{1}{\mu_k} (y - y^k, y^k - x^k) & \geq 0, \forall y \in C, \\
x^{k+1} = \lambda_k F(x^k) + \mu_k x^k + \delta_k y^k
\end{align*}
\]

where \( \{\lambda_k\}, \{\mu_k\}, \{\delta_k\} \subset (0, 1) \); \( \{\rho_k\} \subset (\rho, \bar{\rho}) \subset (0, 1) \), \( 0 < \alpha_i < 1, \forall i = 1, ..., N \). Suppose the conditions (i) - (iii) hold.

(i) \( \lim_{k \to \infty} \lambda_k = 0 \) and \( \sum_{k=0}^{\infty} \lambda_k = \infty \);

(ii) \( \sum_{i=1}^{N} \alpha_i = 1 \);
(iii) \( \sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty. \)

Then the sequences \( \{x^k\}, \{y^k\} \) converge to \( q = P_\Omega(u). \)

From Theorem 4.2 in [4] we get the following statement.

**Statement 2.5** [4, Theorem 3.1]. Let \( f_i, i = 1, 2, ..., N \) satisfy assumption \( A. \) Assume that \( \Omega = \cap_{i=1}^{N} \text{Sol}(C, f_i) \neq \emptyset. \) For given \( x^0, x^1 \in \mathbb{R}^n, \) let the sequence \( \{x^k\}, \{y^k\} \) and \( z^k \) be generated by

\[
\begin{align*}
    x^k &= \sum_{i=1}^{N} \alpha_i f_i(z^k, y) + \frac{1}{\rho_k} \langle y - z^k, z^k - y^k \rangle \\
    y^k &= x^k + \theta_k (x^k - x^{k-1}) \\
    x^{k+1} &= \lambda_k x^k + \mu_k z^k
\end{align*}
\]

where \( \{\theta_k\} \subset [0, \theta], \theta \in [0; 1], \{\alpha_i\}, \{\lambda_k\}, \{\mu_k\} \subset (0, 1) \) and \( \lambda_k + \mu_k = 1 \) for all \( k; \) \( \rho_k \subset (\rho, \bar{\rho}) \subset (0, 1), \) \( 0 < \alpha_i < 1, \forall i = 1, ..., N. \) Suppose that the following conditions hold

(i) \( \theta_k \|x^k - x_{k-1}\| < \infty; \)

(ii) \( \sum_{i=1}^{\infty} \alpha_i < \infty \) and \( \lim_{i \to \infty} \alpha_i = 0; \)

(iii) \( \sum_{i=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty, \sum_{i=1}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty. \)

Then the sequence \( \{x^k\} \) converges to \( q = P_\Omega(u). \)

### 3 Main Results

Now, given natural number \( N \geq 2 \) and a nonempty, closed convex set \( C \) and bifunctions \( f_i (i = 1...N) \) defined on \( C \) such that

\[
\cap_{i=1}^{N} \text{Sol}(C, f_i) \neq \emptyset.
\]

For \( \alpha_i \in (0, 1), i = 1, ..., N \) and \( \sum_{i=1}^{N} \alpha_i = 1. \) We define

\[
f(x, y) = \sum_{i=1}^{N} \alpha_i f_i(x, y).
\]

It is clear that if \( x^* \in \cap_{i=1}^{N} \text{Sol}(C, f_i) \) then \( f_i(x^*, y) \geq 0, \forall y \in C, i = 1, 2, ..., N. \) Therefore \( f(x^*, y) = \sum_{i=1}^{N} \alpha_i f_i(x^*, y) \geq 0, \forall y \in C. \) So \( x^* \in \text{Sol}(C, f). \)

Hence

\[
\cap_{i=1}^{N} \text{Sol}(C, f_i) \subset \text{Sol}(C, f).
\]

The following theorem show that under assumptions \( A_1 - A_4, \) the inversion is not true.
Theorem 3.1 For any integer number $N \geq 2$, there exist a nonempty, closed convex set $C$ and bifunctions $f_1, f_2, \ldots, f_N$ defined on $C$ satisfy assumptions $A_1 - A_4$ and $\alpha_i \in (0, 1), i = 1, 2, \ldots, N, \sum_{i=1}^{N} \alpha_i = 1$ such that

$$\text{Sol}(C, \sum_{i=1}^{N} \alpha_i f_i) \not\subset \cap_{i=1}^{N} \text{Sol}(C, f_i).$$

Proof. It is clear that, we only need prove for the case $n = 2$ and $N = 2$. Indeed, for $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Consider the set $C$ and bifunctions are given as follow

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\},$$

$$f_1(x, y) = x_2 y_1 - x_1 y_2,$$

$$f_2(x, y) = x_1 y_2 - x_2 y_1.$$

Then we have: $f_1(x, x) = 0, \forall x \in C$. For all $x, y \in C$, we have

$$f_1(x, y) + f_1(y, x) = x_2 y_1 - x_1 y_2 + y_2 x_1 - y_1 x_2 = 0.$$

Hence, $f_1$ is monotone on $C$.

For each $x \in C$, $f(x, y)$ is linear in $y$, so $f(x, \cdot)$ is convex. It is trivial that $f_1$ is continuous on $C \times C$.

Therefore bifunction $f_1$ satisfies assumptions $A_1, A_2, A_3,$ and $A_4$.

Similarly, $f_2$ satisfies assumptions $A_1, A_2, A_3,$ and $A_4$. In addition, It can be seen that

$$\text{Sol}(C, f_1) = \{0\} \times [0, +\infty),$$

$$\text{Sol}(C, f_2) = [0, +\infty) \times \{0\}.$$  

So,

$$\text{Sol}(C, f_1) \cap \text{Sol}(C, f_2) = \{(0, 0)\}.$$  

Now, we consider a combination of $f_1, f_2$ given as follows

$$f(x, y) = \frac{1}{2} f_1(x, y) + \frac{1}{2} f_2(x, y)$$

$$= \frac{1}{2} \left[ f_1(x, y) + f_2(x, y) \right]$$

$$= 0, \forall x, y \in C.$$  

It is obvious that $f$ satisfies assumptions $A_1, A_2, A_3,$ and $A_4$. Moreover

$$\text{Sol}(C, f) = C = [0, +\infty) \times [0, +\infty).$$  

Therefore

$$\text{Sol}(C, f) \not\subset \text{Sol}(C, f_1) \cap \text{Sol}(C, f_2).$$  

From this theorem, we have the following corollary
**Corollary 3.1** Statement 2.1 - Statement 2.5 are not correct.

**Proof.** We take $N = 2$, the set $C$, bifunctions $f_1$ and $f_2$ defined as in Theorem 3.1. The combination of $f_1$ and $f_2$ is given by $f(x, y) = \frac{1}{2}f_1(x, y) + \frac{1}{2}f_2(x, y) = 0, \forall x, y \in C$. Hence, $\Omega = \text{Sol}(C, f_1) \cap \text{Sol}(C, f_2) = \{(0, 0)\}$, $\text{Sol}(C, f) = C = [0, +\infty) \times [0, +\infty)$. Then we have the followings:

(a) Statement 2.1 is false.

(b) Take $x^1 \in C$ such that $x^1 \neq (0, 0)$ and set $F(x) = x^1, Ax = x, \forall x \in \mathbb{R}^n$. Choose $\gamma = 1$, then the sequence $\{x^k\}$ generated by Statement 2.2 takes the form

$$x^{k+1} = \delta_k x^1 + (1 - \delta_k)x^k = x^1, \forall k.$$

Therefore, it converges to $x^1 \notin \Omega$. It means that Statement 2 is false.

(c) By taking any $u = x^1 \in C$ such that $x^1 \neq (0, 0)$. Then the sequence $\{x^k\}$ generated by the scheme in Statement 2.3 becomes

$$x^{k+1} = \lambda_k x^1 + (1 - \lambda_k)x^k = x^1, \forall k.$$

It leads to $x^k \to x^1 \notin \Omega$. Hence Statement 2.3 is not correct.

(d) Similar to the case (b), we have Statement 2.4 is false.

(e) By taking any $x^1 = x^0 \in C$, then the sequence $\{x^k\}$ generated by Statement 2.5 takes the form

$$x^k = x^1, \forall k.$$

So, Statement 2.5 does not true.

\[\square\]
Conclusion. We have proved that there exist a finite family of monotone equilibrium problems such that the common solution set of them does not contain the solution set of a combination of those equilibrium problems. Based on this fact, we imply that recent papers [4, 5, 7, 8, 9] are not correct.

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