A Black Hole inside Dark Matter and the Rotation Curves of Galaxies

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Abstract

In this article we find a four-dimensional metric for a large black hole immersed in dark matter. Specifically, we look for and find a static spherically symmetric black hole solution to the Einstein equations which gives, in the Newtonian limit, the rotation curves of galaxies, including the flat region and the Baryonic Tully-Fisher relation, and which has a regular horizon. We obtain as well the energy-momentum tensor of the dark matter sourcing this space-time and it turns, in special, to have a positive mass-energy density everywhere outside the horizon. This black-hole-dark-matter system represents a successful simplified model for galaxies and a new area for exploring the relativistic regime of dark matter.
Introduction

Since its discovery the rotation curve of galaxies has been a fuel to research in the fields of gravitation, astrophysics, cosmology, and to intense searches for new kinds of matter. The rotation curve problem can be stated as follows: On the one hand, according to Newtonian gravity the velocity of stars around the centers of galaxies must decrease with distance (as $v \propto 1/\sqrt{r}$), but observations, on the other hand, tell that the velocities at large distances are almost constant. The rotation curve problem split the physics community into two groups. The largest group took the opinion that the solution to this problem is that most of the mass inside galaxies is not of ordinary matter (i.e., baryonic matter) but of another kind of matter which we can not observe, as it does not interact with the baryonic matter and light except through the gravitational interaction and hence it is called "dark matter" (see for example the recent reviews [1, 2]). The second and smaller group took the opinion that it is the gravitational theories (Newtonian gravity or Einstein gravity) which must be modified in order to solve this problem (see for instance [3, 4, 5] and the references therein). However, this is not the whole story. After a lot of scientific research and observations another important piece of observational data, concerning the rotation curves of galaxies, showed up, the so-called (baryonic) Tully-Fisher relation [6, 7, 8, 9]. This is a universal relation in galaxies between the total baryonic mass and the constant velocity in the flat region. This important relation tells that the baryonic mass of a galaxy is proportional to the constant velocity to the power 4, namely, $M \propto v^4$ (see for example [10]). Thus, any candidate theory for solving the rotation curve problem must include the following main features: (1) a constant or almost constant velocity at large distances from the core, (2) the baryonic Tully-Fisher relation, and there are, in fact, two more natural requirements, (3) the Keplerian rotation curve must appear in some small region before the onset of the flat region, and (4) the theory must be contained in a relativistic framework.

In the first part of this work, we make an analysis in the non-relativistic regime and show that under the assumption that most of the baryonic mass in galaxies is concentrated in the cores, if the gravitational force inside galaxies gives both the baryonic Tully-Fisher relation and the flat part of the rotation curve, then the force must have the single form $F = -GM_b/m/r^2 - ma_0$ where $M_b$ is the baryonic mass of the galaxy and $a_0 > 0$ is a constant acceleration with a universal value. This conclusion, however, does not depend on whether we are adopting the dark matter or the modified-gravity theories. Afterwards, as an example, we show how this new force gives the main features of the observed rotation curve of the Milky Way to a very good precision.

In the second part of this work, we assume that the correct relativistic framework is the Einstein field equations and we adopt the dark matter solution to the rotation curve problem. Since very few information is known about the dark matter and its nature, we found it interesting to make a simplified model of the galaxy which could make us learn something new about dark matter. In this simplified model we assume that all the baryonic mass of the...
galaxy is enclosed inside a black hole in the centre of the galaxy. In more details, we look for
and find a black hole solution to the Einstein equations which gives in the Newtonian limit
(namely, the non-relativistic and weak field limit) the rotation curve of galaxies, including
the flat part and the baryonic Tully-Fisher relation, and which, furthermore, has a regular
horizon. The black hole solution that we find is a perturbative solution based on a derivative
expansion method, which is motivated by the fact that the gravitational effects of dark matter
in galaxies appear at large distances from the centers (several kilo-parsecs) and by the fact
that changes due to dark matter occurs on large scales (kilo-parsecs) as well. We compute
the dark matter energy-momentum tensor sourcing this black hole space-time, and find that
the mass-energy density is positive outside the horizon as expected from a classical system.
In addition, we find that the dark matter has a negligible pressure far away from the black
hole - which is also expected - but interestingly near the black hole we find that the pressure
is significant and of the same order as the mass-energy density, which signals a new "place"
for learning about dark matter in the relativistic regime.

We organise the article as follows. Sec. 1 contains the non-relativistic analysis. We write
down the force which gives both the baryonic Tully-fisher relation and the flat part of the
rotation curve, we determine the value of the universal constant acceleration, and we give
some plots showing how this force captures the main features of the rotation curve of the
Milky Way. Sec. 2 contains the relativistic analysis. We define the derivative expansion
method, we find the black hole metric up to first order in derivatives, and we calculate the
energy-momentum tensor of the dark matter. In Sec. 3 we summarise the results of this work
and discuss some important points.

1 The Gravitational Force inside Galaxies

In this section we are going to focus on the region outside the cores of galaxies where we are
assuming that all the baryonic mass is concentrated. Whether we are taking the dark matter
approach for the rotation curve problem [1, 2] or the other approaches of modified gravities
(see for example [3, 4]), in any case, in the non-relativistic regime one can write down the
gravitational force inside galaxies as

\[ F = -\frac{GM_b m}{r^2} - mg(r) \]  \hspace{1cm} (1.1)

where the first term is the Newtonian force due to the baryonic matter, \( M_b \) is the baryonic
mass of the galaxy, \( m \) is the mass of a test particle (e.g., a star), and the extra force \(-mg(r)\)
is the dark matter contribution to the gravitational force in theories of dark matter, or it is
the modification to the Newtonian force in modified-gravity theories. In what follows we are
going to determine the function \( g(r) \) from the requirement that the gravitational force gives
a flat, or almost flat, rotation curve at large distances and also the baryonic Tully-Fisher
relation.

\(^1\)Stars in galaxies are slowly moving bodies compared to light.
At small distances from the core the Newtonian force due to baryonic matter dominates
\[ F \approx -\frac{GM_b m}{r^2} \]  
(1.2)
and from the circular motion equation, \( a = \frac{v^2}{r} \), one obtains the famous Keplerian result that the circular velocity is
\[ v = \sqrt{\frac{GM_b}{r}} \]  
(1.3)

At large distances from the core the extra force dominates
\[ F \approx -mg(r) \]  
(1.4)
and since this force must be attractive (to cancel the centrifugal force) we conclude that \( g(r) \) must be positive. Here, from \( a = \frac{v^2}{r} \) the velocity of circular orbits is
\[ v = \sqrt{g(r)r} \]  
(1.5)

However, we are not going to rush and ask that \( "g(r)r = constant" \) since it might also be possible that \( g(r)r \) is only almost constant on galactic scales, that is, \( g(r)r \) might be a very slowly varying function of \( r \) on galactic scales. Therefore we find it instructive to impose first the baryonic Tully-Fisher relation and thereafter come back to this point.

1.1 Imposing the Baryonic Tully-Fisher Relation

The baryonic Tully-Fisher relation is a relation between the baryonic mass of galaxies and the constant velocity in the flat part of the rotation curve \([6, 7, 8, 9]\). The relation reads,
\[ M_b \propto v_f^4 \]  
(1.6)
where \( M_b \) is the baryonic mass of the galaxy and \( v_f \) is the velocity in the flat part. Now, it is easy to see that the flat part of the rotation curve starts approximately where the two forces in Eq.[1.1] are equal (or slightly after that), that is, where
\[ \frac{GM_b}{r^2} \approx g(r) \]  
(1.7)

since for larger distances the baryonic Newtonian force falls off rapidly while the second force \( -mg(r) \) (the force responsible for the flat part) is expected to dominate. Let us denote the radius where the flat region starts by \( r_f \) and so from the previous equation we see that this radius satisfies the following equation:
\[ g(r_f)r_f^2 \approx GM_b \]  
(1.8)

This equation, if solved, would give \( r_f \) as a function of \( M_b \), namely:\footnote{Note that the radius where the flat part approximately starts, \( r_f \), must indeed depend on the baryonic mass \( M_b \), because if, for example, the mass \( M_b \) were increased then it would take the force \( GM_b m/r^2 \) more distance to become comparable with \( -mg(r) \), which means that \( r_f \) would increase.}

3
\[ r_f = r_f(M_b) \]  
(1.9)

Now, as we increase the distance \( r \), moving deep into the flat region, Eq. [1.5] and Eq. [1.8] give

\[ v_f^4 = (g(r_f)r_f)^2 \approx GM_bg(r_f) \]  
(1.10)

where, of course, we are assuming that \( g(r)r \) is almost constant on galactic scales. Thus, in order to satisfy the baryonic Tully-Fisher relation, \( v_f \propto M_b \), we clearly, must have

\[ g(r) = \text{constant} \equiv a_0 \]  
(1.11)

because otherwise we will have \( g(r_f) = g(M_b) \) (since \( r_f = r_f(M_b) \)) and the baryonic Tully-Fisher relation will not be satisfied.

In summary, we have two interesting results. First, the baryonic Tully-Fisher relation reads

\[ M_b \approx \frac{v_f^4}{Ga_0} \]  
(1.12)

where we have determined the universal proportionality factor in terms of \( a_0 \); this implies, in particular, that \( a_0 \) itself is a universal constant. Second, we have reached the important conclusion that the gravitational force inside galaxies - and outside the cores - is given by

\[ F = -\frac{GM_bm}{r^2} - ma_0 \]  
(1.13)

where \( a_0 > 0 \) is a constant acceleration. In fact, this acceleration has already been noticed in the data and analysis of rotation curves [11, 12, 13]. According to the references just mentioned this is the acceleration below which the Newtonian force due to the baryonic mass must be replaced with the dark matter contribution or by a modified force. Note that this is exactly what happens here: The radius where the flat region starts occurs where

\[ \frac{GM_b}{r^2} \approx g(r) = a_0 \]  
(1.14)

and this splits the galaxy into two parts, one part with \( \frac{GM_b}{r^2} > a_0 \) where the baryonic force prevails, corresponding to small distances, and a second part with \( \frac{GM_b}{r^2} < a_0 \) where the baryonic force becomes subdominant, with respect to \( ma_0 \), corresponding to large distances.

Before we move on to the next section we would like to highlight an immediate and important result which comes out form this analysis. From Eq. [1.8] it is immediately seen that

\[ r_f \approx \sqrt{\frac{GM_b}{a_0}} \]  
(1.15)

which tells that the radius where the flat curve starts increases monotonically with the baryonic mass of the galaxy according to a square-root relation. It is worth pointing out that it is natural and expected that \( r_f \) increases with \( M_b \) because if the mass \( M_b \) is increased then it will take the force \(-GM_bm/r^2\) more distance to become comparable to \(-ma_0\).

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3 Our \( a_0 \) is the constant called \( c^2\gamma/2 \) in [11] or \( c^2\gamma_0/2 \) in [12] and it is not the \( a_0 \) of MOND [8].
1.2 Determining the Order of Magnitude of $a_0$ from Observational Data

From Eq. [1.13], by assuming circular orbits, we can obtain the full rotation curve easily,

$$v^2 = \frac{GM_b}{r} + a_0r$$

and from this we can extract the constant (universal) acceleration $a_0$

$$a_0 = \left( \frac{v^2 - \frac{GM_b}{r}}{r} \right) / r$$

By plugging in correct observational values for the triple $v$, $M_b$, and $r$ in the previous equation one can obtain $a_0$. The value of $a_0$ is supposed to be universal for (spiral) galaxies. To determine the order of magnitude of $a_0$ take a typical spiral galaxy with velocity in the flat part of the curve between $v = 200 \text{ km/s}$ and $v = 250 \text{ km/s}$, with baryonic mass between $M_b \approx 0.5 \times 10^{11} M_{\text{sun}}$ and $M_b \approx 2 \times 10^{11} M_{\text{sun}}$ (see for example the review [14]), and take a point at a distance between $r = 30 \text{ kpc}$ and $r = 40 \text{ kpc}$; we have chosen points in the flat part to make sure that indeed all the baryonic mass $M_b$ lies inside. With the above "wide" ranges of the parameters $v$, $M_b$ and $r$ (we have taken wide ranges deliberately to show that the order magnitude of $a_0$ is insensitive) we get that $a_0$ must lie in the range:

$$a_0 \in [1.2, 5.9] \times 10^{-11} \text{ m/s}^2$$

Thus, we see that the order of magnitude of $a_0$ is definitely

$$a_0 \sim 10^{-11} \text{ m/s}^2$$

It is to be mentioned here that the value obtained in [12] ($a_0 = 1.4 \times 10^{-11} \text{ m/s}^2$) lies in the above range$^4$. In order to obtain the exact value of $a_0$ we must do a best fitting over a large number of spiral galaxies, a thing which we do not do in this paper and we leave for future work. In what follows we are going to make a rough estimation and determine $a_0$ from the Milky Way data.

1.3 The Constant Rotation Curve

Now it is time to show that the proposed force gives indeed the flat part of the rotation curve. As an example, we take the Milky Way galaxy, and we begin by obtaining a rough estimation of the acceleration $a_0$ as follows. For a point in the flat part, say at $r = 30 \text{ kpc}$$^5$ with reasonable values $v = 200 \text{ km/s}$ and $M_b = 1 \times 10^{11} M_{\text{sun}}$ (see, for example, the references [14] [15] $^8$) we obtain

$$a_0 = 2.8 \times 10^{-11} \text{ m/s}^2$$

$^4$In reference [12] $a_0$ is called $c^2\gamma_0/2$.

$^5$The radius $r = 30 \text{ kpc}$ (the radius of the galaxy’s disk) is considered as ideal since it encloses all the baryonic mass of the galaxy, $M_b$, whereas radii that are not very far from the core will enclose a mass that is slightly less than $M_b$.

$^8$See table 6 in the review [14].
If we plot [Eq. 1.16] using the above values \( a_0 = 2.8 \times 10^{-11} \text{ m/s}^2 \) and \( M_b = 10^{11} M_{\text{sun}} \) we obtain the following rotation curve:

![Rotation curve of the Milky Way](image)

**Figure 1:** The rotation curve of the Milky Way. The flat region is obvious in the outer half of the galaxy with a constant velocity of about 200 km/s. This graph must not be taken seriously in the core of the galaxy (say \( r < 5 \text{ kpc} \)) since our treatment in this section focuses only on the region outside the core.

The flat part is clear as the velocity is almost constant at about 200 km/s for several kilo-parsecs, and the Keplerian curve is also clear just before the beginning of the flat part. In the next plot we make a zoom-in on the flat region:

![Zoom-in on rotation curve](image)

**Figure 2:** The rotation curve of the Milky Way from 16-30 kpc (solid line) is almost constant at about 200 km/s. The two horizontal dashed lines \( v = 202 \text{ km/s} \) and \( v = 197 \text{ km/s} \) are the maximum and minimum velocities in this region.
We have chosen to start at 16 $kpc$ because according to equation Eq.[1.15] that is where the flat region starts. Clearly the curve is almost constant along a huge distance (along almost half of the galaxy size): from 16−30 $kpc$ the velocities lie in the small range [197 $km/s$, 202 $km/s$].

It is worth to calculate as well the velocity of the solar system about the galaxy, which we found to be

$$v_0 = 243 \text{ km/s}$$

and, interestingly, this is very close to values obtained in recent works (see table 2 in reference [14]), where we have taken the distance between the solar system and the centre of the galaxy to be $R_0 = 8.3 \text{ kpc}$. This result shows that our model works very well not only for very far distances from the core but also for close distances down to 8 $kpc$ at the least.

As said in the beginning, our work in this section focuses only on the region outside the cores of galaxies, where most of the baryonic mass is concentrated. Therefore, the above proposed rotation curve of the Milky Way must not be taken seriously inside the core (say for $r < 5 \text{ kpc}$) since inside the core the baryonic mass is distributed according to some distance-dependent density, $\rho = \rho(r)$, which we are not going to study. However, for the sake of illustration of how our work may merge with other works treating the rotation curve inside the core, let us assume a constant density of the baryonic matter inside the core, $\rho = \text{constant}$. Then the velocity increases linearly with distance (like a rigid body), $v \propto r$. Below we give two figures showing how the two physics, inside and outside the core, can be approximately merged into one picture.

Figure 3: Left: Our plot of the rotation curve (solid line) and the plot of the rotation curve of a rigid body (dashed line), which we took for illustration only, are put together. Right: Since our plot is valid outside the core and the linear curve (straight line) is the model we took inside, the two plots can be merged (in an approximate way) together by simply taking off the additional parts after their intersection.

This combined plot resembles the observed rotation curve. It contains the main features,

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Our assumption that most of the baryonic mass in the galaxy lies within the first 5 $kpc$ has support in the literature [16, 17, 18, 19] where an exponential disk is assumed, with surface mass density $M(r) \propto \exp(-r/a)$, where the scale length $a$ determines how much mass there is in the core.
such as, the Keplerian region, the flat region, and the peak\(^8\). It is to be said that the way we have merged the two physics is a crude approximation; in the merging region one expects the physics to be rather complex since, for example, one can not tell exactly where the core region ends and how the density of baryonic matter changes behaviour there, a thing which could lead to a wide and unexpected merging profile as, in fact, is seen clearly in the observed rotation curve of the Milky Way.

1.4 The Corresponding Mass Density

For completeness and for later use we will write down here the gravitational potential which gives the force \(F = -GM_bm/r^2 - ma_0\) (see Eq.\([1.13]\)) and also the corresponding mass density in galaxies. It is clear that the potential is \(\phi = -GM_b/r + a_0r\). From the Poisson equation \(\nabla^2\phi = 4\pi G \rho\) one gets that the mass density inside galaxies is

\[\rho = \frac{1}{2\pi G} \frac{a_0}{r}\]

(1.22)

where as stated many times we are talking about the region outside the cores.

2 A Black Hole inside Dark Matter

Now we are going to look for a black hole solution to the Einstein equations which encloses all the baryonic matter of the galaxy and which gives in the Newtonian limit (the non-relativistic and weak-field limit) both, the flat rotation curve and the baryonic Tully-Fisher relation, and that, moreover, has a regular horizon. The general static spherically symmetric metric can be written as (see for example \([20, 21]\))

\[ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\]

(2.1)

We find it suitable for our purposes to write the functions \(f(r)\) and \(h(r)\) as

\[f(r) = 1 - \frac{2GM_b}{r} + V(r)\quad h(r) = \left(1 - \frac{2GM_b}{r} + H(r)\right)^{-1}\]

(2.2)

where \(M_b\) is the baryonic mass of the galaxy (enclosed inside the black hole), and where \(V(r)\) and \(H(r)\) are functions to be determined.

2.1 Derivative Expansion

Notice that the functions \(V(r)\) and \(H(r)\) are the contributions of the dark matter to the metric\(^9\) and therefore are expected to appear and to change significantly only on large scales.

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\(^8\)Many rotation curves display a peak between the core and the flat part.

\(^9\)The functions \(V(r)\) and \(H(r)\) are indeed the contributions of the dark matter to the metric since if the dark matter were absent then we would simply have the Schwarzschild black hole metric.
Therefore, the derivatives of \( V(r) \) and \( H(r) \) are expected to be very small and hence it is useful to Taylor expand them around the black hole radius \( r = r_0 \),

\[
V(r) = V_0 + V_1(r - r_0) + \frac{1}{2}V_2(r - r_0)^2 + \ldots \quad (2.3)
\]

\[
H(r) = H_0 + H_1(r - r_0) + \frac{1}{2}H_2(r - r_0)^2 + \ldots \quad (2.4)
\]

where \( V_0 = V(r_0) \), \( V_1 = dV(r_0)/dr \), \( V_2 = d^2V(r_0)/dr^2 \), etc., and similarly for \( H(r) \). Saying that the derivatives of \( V(r) \) and \( H(r) \) (namely, \( V_1, H_1, V_2, H_2, \text{etc.} \)) are very small means that

\[
1 \gg V_1(r - r_0) \gg \frac{1}{2}V_2(r - r_0)^2 \gg \ldots \quad (2.5)
\]

\[
1 \gg H_1(r - r_0) \gg \frac{1}{2}H_2(r - r_0)^2 \gg \ldots \quad (2.6)
\]

even for \( r \)'s of galactic scales (in fact, this is the key point in this derivative expansion).  

### 2.2 Regularity and Position of the Horizon

The regularity of the horizon is achieved only if (see Appendix A)

\[
h(r) = f(r)^{-1} \quad \text{as} \quad r \to r_0 \quad (2.7)
\]

and as can be easily seen this is satisfied only if

\[
H_0 = V_0 \quad (2.8)
\]

The position of the horizon, on the other hand, is obtained from the equation \( f(r_0) = 0 \), which gives

\[
r_0 = \frac{2GM_b}{1 + V_0} \quad (2.9)
\]

Since the dark matter is not expected to be significant in the cores of galaxies we will assume in this work that it is also insignificant inside our black hole, and thus we can neglect \( V_0 \) and set

\[
H_0 = V_0 = 0 \quad (2.10)
\]

though we expect this quantity to have interesting effects in general. Thus we have

\[
r_0 = 2GM_b \quad (2.11)
\]

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10 From the rotation curves of galaxies we know that the effects of dark matter appear after several kiloparsecs from the centre, and we also know that outside the cores things change significantly on the scale of kiloparsecs as well, which is much larger than the scale of a black hole containing all the baryonic mass of the galaxy, which has a radius (scale) that is much smaller than one parsec.

11 A similar derivative expansion method for solving the Einstein equations, but in a different context, was developed in [22].

12 It is clear that the quantity \( V_0 \) determines the amount of dark matter inside the black hole because it plays a role in determining its radius. Moreover, it is also clear that \( V_0 < 0 \) since the presence of dark matter is supposed to increase the mass inside the black hole and hence its radius too.

13 The cases with \( H_0 = V_0 \neq 0 \) will be studied in a future work.
2.3 Imposing the Rotation Curves of Galaxies in the Newtonian Limit

Here we will impose the boundary condition that in the Newtonian limit (the non-relativistic and weak-field limit) the metric Eq. 2.1 gives the rotation curve of galaxies, that is, it gives both the flat region of the rotation curve and also the baryonic Tully-Fisher relation. To do so let us assume as usual circular motions. Then, by consulting the geodesic equations one finds that (see Appendix B)

\[
\frac{v^2}{r} = \frac{1}{2} \frac{df}{dr}
\]  
(2.12)

Up to first order in derivatives one gets that

\[
v^2 = \frac{GM_b}{r} + \frac{V_1 r}{2}
\]  
(2.13)

Note that this expression for \( v^2 \) has the same form as the one obtained in the Newtonian treatment Eq. 1.16 and thus we make the identification

\[ V_1 = 2a_0 \]  
(2.14)

Interestingly, we have obtained the result that the rotation curve for this black hole is

\[
v^2 = \frac{GM_b}{r} + a_0 r
\]  
(2.15)

and it holds for any radius from the horizon to the far away region.

2.4 The Einstein Equations

Now we are going to impose (solve) the Einstein equations

\[
E_{\mu\nu} = 8\pi GT_{\mu\nu}
\]  
(2.16)

where \( E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \) is the Einstein tensor. We are going to assume the most general spherically symmetric energy-momentum tensor, which is that of an anisotropic fluid

\[ T_{\mu\nu} = \text{diagonal}(-\rho, P_r, P_\perp, P_\perp) \]  
(2.17)

where \( \rho \) is the mass-energy density, \( P_r \) is the radial pressure, and \( P_\perp \) is the transverse pressure (see references [23, 24]).

One can check that the equation \( E_{00} = 8\pi GT_{00} \) gives (see for example [21])

\[
\rho = \frac{-h + h^2 + rh'}{8\pi Gr^2h^2}
\]  
(2.18)

By inserting the expression of \( h(r) \) given in Eq. 2.2 in the above equation one obtains up to first order in derivatives that:

\[
\rho(r) = \frac{-H_1 (r - GM_b)}{4\pi Gr^2}
\]  
(2.19)
To fix $H_1$ let us take the large-radius limit (the weak field limit) of $\rho$,

$$\rho(r) = \frac{-H_1}{4\pi Gr} \quad (2.20)$$

and compare it to the expression for the mass density obtained in the Newtonian treatment of the first section (see Eq. [1.22])\textsuperscript{14} Upon comparison we see that we must fix:

$$H_1 = -2a_0 \quad (2.21)$$

Therefore, the mass-energy density in space up to first order in derivatives is:

$$\rho(r) = \frac{a_0 (r - GM_b)}{2\pi Gr^2} \quad (2.22)$$

where note that it is positive outside the horizon. Thus we have at hand now the metric Eq. [2.1] up to first order in derivatives

$$ds^2 = -\left(1 - \frac{2GM_b}{r} + 2a_0(r - 2GM_b)\right) dt^2 + \frac{dr^2}{1 - \frac{2GM_b}{r} - 2a_0(r - 2GM_b)} + r^2 d\Omega^2 \quad (2.23)$$

which is one of the main results of our paper\textsuperscript{15} Here, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

The remaining of the Einstein equations ($E_{rr} = 8\pi GT_{rr}$ and $E_{\theta\theta} = 8\pi GT_{\theta\theta}$) will be used to obtain $P_r$ and $P_\perp$ as we show next. From the equation $E_{rr} = 8\pi GT_{rr}$ we obtain

$$P_r = \frac{f(1 - h) + rf'}{8\pi Gr^2 fh} \quad (2.24)$$

which up to first order in derivatives gives

$$P_r = -\frac{a_0 M_b}{2\pi r^2} \quad (2.25)$$

From the equation $E_{\theta\theta} = 8\pi GT_{\theta\theta}$ we obtain

$$P_\perp = \frac{-rfh^2 - 2f^2 h' - rf f'h' + 2fh(f' + rf'')}{32\pi Gr f^2 h^2} \quad (2.26)$$

and to first order in derivatives

$$P_\perp = \frac{a_0 M_b}{4\pi r^2} \quad (2.27)$$

For completeness, the Ricci scalar for our black hole metric is

$$R = \frac{4a_0 (r - GM_b)}{r^2} \quad (2.28)$$

and so we see that it is positive outside the black hole and we see again that the space-time is regular on the horizon as the Ricci scalar is finite there.

\textsuperscript{14} Notice that this is another boundary condition.

\textsuperscript{15} This black hole is different from the black hole given in [11] by the crucial minus sign multiplying $a_0$ in the $g_{rr}$ component. In fact, the black hole in [11] gives a negative mass-energy density if plugged into the Einstein equations.
2.5 Validity of the Derivative Expansion

In the large-radius limit \((r >> 2GM_b/c^2)\) our metric \([2.23]\) reads

\[
d s^2 = - \left(1 - \frac{2GM_b}{c^2 r} + \frac{2a_0 r}{c^2} \right) c^2 d t^2 + \left(1 - \frac{2GM_b}{c^2 r} - \frac{2a_0 r}{c^2} \right)^{-1} d r^2 + r^2 d \Omega^2 \tag{2.29}
\]

where we have returned the speed of light \(c\) to the metric. Upon recalling that \(a_0 \sim 10^{-11} \text{ m/s}^2\) we see that indeed

\[
- \frac{2GM_b}{c^2 r} \pm \frac{2a_0 r}{c^2} << 1 \tag{2.30}
\]

since the constant \(a_0/c^2\) is very small in galactic scales,

\[
a_0/c^2 \sim 10^{-28} \text{ m}^{-1} \tag{2.31}
\]

and thus even for a large distance, say \(r = 100 \text{ kpc}\), we have

\[
\frac{2a_0 r}{c^2} \sim 10^{-6} \tag{2.32}
\]

3 Summary and Discussion

In the first part of the article we have made a Newtonian analysis of the rotation curves of galaxies and concluded that if both the flat curve and the baryonic Tully-Fisher relation are to be obtained then the gravitational force must be \(F = -GM_b m/r^2 - ma_0\), where \(a_0 \sim 10^{-11} \text{ m/s}^2\) is a universal constant. We have also fixed the proportionality factor in the baryonic Tully-Fisher relation in terms of \(a_0\), namely, \(M_b \approx v_f^4/Ga_0\).

In the second part of the article we have proposed a black-hole-dark-matter model for the galaxy, with the purpose to learn new information about dark matter in the relativistic regime. The black hole metric was obtained based on a derivative expansion method that is motivated and justified by the fact that the length scale of the black hole is much smaller than the galactic scale. The black hole metric which we have found up to first order in the derivative expansion is

\[
d s^2 = -f(r) dt^2 + h(r) dr^2 + r^2 (d \theta^2 + \sin^2 \theta d \phi^2) \tag{3.1}
\]

with

\[
f(r) = 1 - \frac{2GM_b}{r} + 2a_0 (r - 2GM_b) \quad h(r) = \left(1 - \frac{2GM_b}{r} - 2a_0 (r - 2GM_b) \right)^{-1} \tag{3.2}
\]

where \(M_b\) is the baryonic mass in space and it is enclosed inside the black hole horizon and \(a_0 \sim 10^{-11} \text{ m/s}^2\) is the universal constant. This metric describes a static spherically symmetric black hole immersed in dark matter. This black hole gives the galactic rotation curve with many of its essential features, such as the small Keplerian region, the large flat
region, and the baryonic Tully-Fisher relation. Therefore this black hole space-time can be viewed as a simplified model for galaxies.

The energy-momentum tensor of dark matter in this space-time has the form of anisotropic fluid

$$T_{\mu}^{\nu} = \text{diagonal}(-\rho, P_r, P_\perp, P_\perp)$$ (3.3)

with

$$\rho = \frac{a_0}{2\pi G r} - \frac{a_0 M_b}{2\pi r^2}, \quad P_r = -\frac{a_0 M_b}{2\pi r^2}, \quad P_\perp = \frac{a_0 M_b}{4\pi r^2}$$ (3.4)

Note that the mass-energy density is positive outside the black hole horizon as expected in classical physics, and note that the radial pressure is negative. Notice, furthermore, that the pressure in the far region is negligible compared to the density; if one takes the expressions for $P_r$ and $P_\perp$ given above and compute them at, say $10 \text{ kpc}$, one gets that $P_r \sim P_\perp \sim 10^{-12}\text{Pa}$ which is even smaller than the pressure coming from the cosmological constant ($\sim 10^{-10}\text{Pa}$). Hence, this conforms with the literature that the dark matter is pressure-less and non-relativistic in the outer regions of galaxies. Nevertheless, in the near region, close to the black hole, the mass-energy density and the pressures are of the same order of magnitude which tells that the dark matter is relativistic there. Therefore, the region close to the horizon is a place where we can learn something new about dark matter; it is a place to learn about its relativistic nature.

It is important to highlight the result that the rotation curve for this black hole is $v^2 = \frac{GM_b}{r} + a_0 r$ for any radius from the horizon tell the far region. This result allows us to uncover the relativistic regime of stars; stars in orbits close to the black hole move at relativistic speeds.

We find it important to mention also that the derivative expansion method we have developed here will bring out the cosmological constant term at second order in derivatives, which, of course, is expected and consistent with our picture and analysis since the cosmological constant effects must enter on the cosmological scale - the scale next to the galactic scale.

Finally, it is worth mentioning that the black hole metric we have found is interesting in itself, regardless of the galactic context we have followed in this paper. That is, this derivative expansion approach for constructing the black hole metric can be used whenever a black hole is immersed in a background which changes on a large scale compared with the black hole scale.

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A Eddington-Finkelstein Coordinates and Regularity

Since we have spherical symmetry, to find the location of the horizon we look for the null \( r = \text{constant} \) surface,

\[
g^{\mu\nu} \partial_{\mu} r \partial_{\nu} r = 0 \tag{A.1}
\]

which for our metric Eq.[2.1] gives

\[
g^{rr} = h^{-1} = 0 \tag{A.2}
\]

To make the horizon regularity manifest we move to the new coordinate \( v = t + r_* \) where \( dr_* / dr = \sqrt{h/f} \), upon which our metric Eq.[2.1] becomes

\[
ds^2 = -f(r)dv^2 + 2\sqrt{f(r)h(r)}dvdr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{A.3}
\]

and so in order to prevent the singularity of \( g_{rv} \) at the horizon (as \( h \to \infty \)) clearly we must have

\[
f(r) = h(r)^{-1} \quad \text{as} \quad r \to r_0 \tag{A.4}
\]

B Geodesic Equation and Circular Motion

For the general spherically symmetric static metric

\[
ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{B.1}
\]

the geodesic equations are

\[
\frac{d^2 x^{\mu}}{dp^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{dp} \frac{dx^{\lambda}}{dp} = 0 \tag{B.2}
\]

where \( p \) is a parameter along the trajectory. Because of spherical symmetry the motion will take place in a plane, and without loss of generality we will take it to be in the \( \theta = \pi/2 \) plane. The radial equation of motion will be (here we are going to follow reference [20])

\[
\frac{d^2 r}{dp^2} + \frac{h'}{2h} \left( \frac{dr}{dp} \right)^2 - \frac{J^2}{r^3 h} + \frac{f'}{2hf^2} = 0 \tag{B.3}
\]

where the prime denotes derivative with respect to \( r \). There are also the two familiar relations

\[
\frac{dt}{dp} = \frac{1}{f(r)} \quad \text{and} \quad r^2 \frac{d\phi}{dp} = J \tag{B.4}
\]

where the constant \( J \) is the angular momentum per unit mass.

On the one hand, for a circular motion, \( r = \text{constant} \), the equation [B.3] becomes

\[
\frac{J^2}{r^3} = \frac{f'}{2f^2} \tag{B.5}
\]
On the other hand, upon combining the two equations in B.4 one gets that

\[ J = r^2 \frac{d\phi}{dt} f^{-1} \quad (B.6) \]

Finally, if we insert equation B.6 into equation B.5 we get

\[ \left( \frac{v_\phi}{r} \right)^2 = \frac{1}{2} f' \quad (B.7) \]

where \( v_\phi = rd\phi/dt \) is the angular velocity.

References

[1] P. J. E. Peebles, “Growth of the nonbaryonic dark matter theory,” Nat. Astron. 1 (2017) no.3, 0057 doi:10.1038/s41550-017-0057 [arXiv:1701.05837 [astro-ph.CO]].

[2] G. Bertone and D. Hooper, “History of dark matter,” Rev. Mod. Phys. 90 (2018) no.4, 045002 doi:10.1103/RevModPhys.90.045002 [arXiv:1605.04909 [astro-ph.CO]].

[3] P. D. Mannheim and J. G. O’Brien, “Galactic rotation curves in conformal gravity,” J. Phys. Conf. Ser. 437 (2013) 012002 doi:10.1088/1742-6596/437/1/012002 [arXiv:1211.0188 [astro-ph.CO]].

[4] J. W. Moffat, “Scalar-tensor-vector gravity theory,” JCAP 0603 (2006) 004 doi:10.1088/1475-7516/2006/03/004 [gr-qc/0506021].

[5] M. Milgrom, “MOND vs. dark matter in light of historical parallels,” arXiv:1910.04368 [astro-ph.GA].

[6] R. B. Tully and J. R. Fisher, “A New method of determining distances to galaxies,” Astron. Astrophys. 54 (1977) 661.

[7] S. S. McGaugh, J. M. Schombert, G. D. Bothun and W. J. G. de Blok, “The Baryonic Tully-Fisher relation,” Astrophys. J. 533 (2000) L99 doi:10.1086/312628 [astro-ph/0003001].

[8] M. Milgrom, “MOND: A pedagogical review,” Acta Phys. Polon. B 32 (2001) 3613 [astro-ph/0112069].

[9] M. Milgrom, “A modification of the Newtonian dynamics: implications for galaxy systems,” Astrophys. J. 270 (1983) 384. doi:10.1086/161132

[10] S. McGaugh, “The Baryonic Tully-Fisher Relation of Gas Rich Galaxies as a Test of LCDM and MOND,” Astron. J. 143 (2012) 40 doi:10.1088/0004-6256/143/2/40 [arXiv:1107.2934 [astro-ph.CO]].

[11] P. D. Mannheim and D. Kazanas, “Exact Vacuum Solution to Conformal Weyl Gravity and Galactic Rotation Curves,” Astrophys. J. 342 (1989) 635. doi:10.1086/167623
[12] J. G. O’Brien, T. L. Chiarelli and P. D. Mannheim, “Universal properties of galactic rotation curves and a first principles derivation of the Tully-Fisher relation,” Phys. Lett. B 782 (2018) 433 doi:10.1016/j.physletb.2018.05.060 [arXiv:1704.03921 [astro-ph.GA]].

[13] S. McGaugh, F. Lelli and J. Schombert, “Radial Acceleration Relation in Rotationally Supported Galaxies,” Phys. Rev. Lett. 117 (2016) no.20, 201101 doi:10.1103/PhysRevLett.117.201101 [arXiv:1609.05917 [astro-ph.GA]].

[14] Y. Sufue, “Rotation and mass in the Milky Way and spiral galaxies”, Publications of the Astronomical Society of Japan, Volume 69, Issue 1, February 2017, R1, https://doi.org/10.1093/pasj/psw103

[15] Y. Sufue, ”Dark halos of M31 and the Milky Way”, Publications of the Astronomical Society of Japan, Volume 67, Issue 4, August 2015, 75, https://doi.org/10.1093/pasj/psv042

[16] K. C. Freeman, “On the disks of spiral and SO Galaxies,” Astrophys. J. 160 (1970) 811. doi:10.1086/150474

[17] P. D. Sackett, “Does the Milky Way have a maximal disk?,” Astrophys. J. 483 (1997) 103 doi:10.1086/304223 [astro-ph/9608164].

[18] J. Bovy and H. W. Rix, “A Direct Dynamical Measurement of the Milky Way’s Disk Surface Density Profile, Disk Scale Length, and Dark Matter Profile at 4 kpc ∼ R ∼ 9 kpc,” Astrophys. J. 779 (2013) 115 doi:10.1088/0004-637X/779/2/115 [arXiv:1309.0809 [astro-ph.GA]].

[19] C. Porcel, F. Garzon, J. Jimenez-Vicente and E. Battaner, “The radial scale length of the milky way,” Astron. Astrophys. 330 (1998) 136 [astro-ph/9710197].

[20] S. Weinberg, “Gravitation and Cosmology : Principles and Applications of the General Theory of Relativity,”

[21] R. M. Wald, “General Relativity,” doi:10.7208/chicago/9780226870373.001.0001

[22] N. Haddad, “Black Strings Ending on Horizons,” Class. Quant. Grav. 29 (2012) 245001 doi:10.1088/0264-9381/29/24/245001 [arXiv:1207.2305 [hep-th]].

[23] R. L. Bowers and E. P. T. Liang, “Anisotropic Spheres in General Relativity,” Astrophys. J. 188 (1974) 657. doi:10.1086/152760

[24] G. Raposo, P. Pani, M. Bezares, C. Palenzuela and V. Cardoso, “Anisotropic stars as ultracompact objects in General Relativity,” Phys. Rev. D 99 (2019) no.10, 104072 doi:10.1103/PhysRevD.99.104072 [arXiv:1811.07917 [gr-qc]].