Magnetic waves in a stratified medium: dispersion relations and exact solutions

Bindesh Tripathi1,2⋆ and Dhrubaditya Mitra2†

1Department of Physics, St. Xavier’s College, Tribhuvan University, Kathmandu, Nepal
2Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden

27 March 2019, Revision: 1.2

ABSTRACT
We solve for waves in an isothermal, stratified medium with a magnetic field that points along a direction perpendicular to that of gravity and varies exponentially in the direction of gravity. For waves propagating along the magnetic field, we calculate approximate dispersion relation as a function of the magnetic field strength. We also find exact solutions for two different cases: (a) waves propagating along the direction of the magnetic field and (b) waves propagating along the direction of gravity. In each of these cases, we find solutions in terms of either confluent hypergeometric functions or Gauss hypergeometric functions depending on whether the ratio of the scale height of the magnetic field over the density scale height is equal to two or not.

Key words: MHD - Sun: waves : Helioseismology

1 INTRODUCTION
Helioseismology has revolutionized solar physics. The technique has allowed us to probe sub-surface flows in the Sun. In near future it may be generalized to other stars too. But our knowledge of the subsurface magnetic field in the Sun remains limited. Several recent works (Schunker et al. 2005; Ilonidis et al. 2011; Singh et al. 2014, 2015, 2016, 2018) have attempted, in different ways, to infer, from surface signatures, the sub-surface magnetic field that is going to emerge at a certain location in the Sun. The problem of inferring the subsurface magnetic field from observations of the surface is an inverse problem. The corresponding direct problem is to calculate what the surface signature of a subsurface magnetic field would be. In this paper we do this calculation for one particular case. It is noteworthy to mention that almost all helioseismic calculations have been carried out without considering the effect of magnetic field on the solar atmospheric waves. Recently, there has been a growing concern to include the effects of magnetic field in helioseismic models (Cally 2007).

For simplicity, consider a Cartesian slab of plasma with constant gravity acting in the negative z direction as shown in Fig. 1. Assume the temperature to be a constant. Furthermore, assume periodic boundary conditions in the horizontal (x–y) directions. In the absence of any magnetic field, there is a static solution of the equations of magnetohydrodynamics (MHD); the density \( \rho = \rho_0(z) \) is a function of the vertical coordinate \( z \) alone and the velocity is zero everywhere. It is straightforward to apply the theory of linear waves to this problem. Due to the symmetry of the problem, the dynamical variables – velocity or Lagrangian displacement and density – can be written down as \( \exp(i(k_x x + k_y y - \omega t)) \), multiplied by a function of the vertical coordinate \( z \). In particular, the \( z \) component of the Lagrangian displacement satisfies a second order differential equation in \( z \), (see, e.g., Tolstoy 1963, section 4). By solving this equation with proper boundary conditions, we can find out the dispersion relation of the waves in this setup. How would this dispersion relation change in the presence of a horizontal magnetic field? For a constant magnetic field acting in the \( z \) direction, with a background density that varies exponentially as a function of the vertical coordinate, this problem was solved for waves propagating along the direction of the magnetic field (the \( z \) direction) by Nye & Thomas (1976); Adam (1977), (see also, Thomas 1983, for a more comprehensive treatment) and for waves propagating along the direction of gravity (the \( z \) direction) by Campos (1983). In both of these cases, it is possible to find an exact solution that uses the hypergeometric functions. Campos & Marta (2015) generalized the latter to the one where the magnetic field is also an exponential function of the vertical coordinate \( z \), with a characteristic length scale \( \ell_B \), but its direction remains fixed – along the \( x \) direction. For the particular case where \( \ell_B \) is twice the density scale height, \( \ell_\rho \), Campos & Marta (2015) found an exact solution for waves propagating along the gravity field in terms of confluent hypergeometric functions. The problem remains unsolved for any value of \( \ell_B/\ell_\rho \), not equal to two as well as for waves propagating along the \( x \) direction – the direction of the magnetic field and along the \( y \) direction – the direction orthogonal to both the direction of the magnetic field and the direction of gravity.

⋆ E-mail: bindeshtrip@gmail.com
† E-mail: dhruba.mitra@gmail.com

© 0000 The Authors
We do a straightforward calculation of an idealized problem: the properties of linear waves in a stratified isothermal plasma, permeated with magnetic field. The equations of motion of the plasma is formulated with magnetic field. The equations of motion of the plasma is given by the MHD equations (see, e.g., Choudhuri 1998) with uniform gravity, \( g \), acting in the negative \( z \) direction as:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho U) &= 0; \quad (1a) \\
\partial_t (\rho U_a) + \partial_j (\rho U_a U_j + P\delta_{a\beta}) &= -\rho g \delta_{a3} + \epsilon_{a\beta\gamma} J_\beta B_\gamma; \quad (1b) \\
\partial_t \mathbf{B} &= \nabla \times (\mathbf{U} \times \mathbf{B}). \quad (1c)
\end{align*}
\]

Here \( \rho, U, P, \) and \( \mathbf{B} \) are the density, the velocity, the pressure, and the magnetic field respectively. \( J = (1/\mu_0) \nabla \times \mathbf{B} \) is the current density. Equation (1a), (1b), and (1c) are respectively the continuity equation, the momentum equation, and the induction equation. As the equations of MHD are a consequence of Maxwell’s equations, we also have,

\[
\nabla \cdot \mathbf{B} = 0. \quad (2)
\]

To simplify our notation, we have written the momentum equation in tensor notation and the other two in vector notation. The second rank tensor \( \delta_{a\beta} \) is the Kroneker delta and the third rank tensor \( \epsilon_{a\beta\gamma} \) is the Levi-Civita tensor. We have used SI units where \( \mu_0 \) denotes the permeability of vacuum. We have also ignored all dissipative effects – viscosity and electrical resistance of the plasma. These equations must be supplemented by an equation of state, which we assume is the ideal gas equation of state,

\[
\mathcal{C}_s^2 = \frac{\gamma_{ad} P}{\rho}. \quad (3)
\]

where \( \mathcal{C}_s \) and \( \gamma_{ad} \) are the speed of sound and adiabatic index of the gas. We study this problem under isothermal condition, i.e., the speed of sound, \( \mathcal{C}_s \), is a constant.

### 2.1 Static background state

To study linear waves, let us first consider the background state of the system. Assume that the fluid in the box is in a static state – velocity \( U = 0 \), with density \( \rho = \rho_0 \) and magnetic field \( \mathbf{B} = \mathbf{B}_0 \), which are constant in time. Here, in addition to density \( \rho_0 \) being a function of the vertical coordinate \( z \), the magnetic field is also a function of height. If the magnetic field had been uniform (or if the magnetic field were absent), it would have played no role in the stationary state. The background density \( \rho_0 \) would have been an exponential function of \( z \) with the scale height, \( \ell_p = \sqrt{c_s^2/(g\gamma_{ad})} \).

We consider the simplest possible case where the magnetic field is horizontal. The stationary state of such a system is a solution of

\[
\begin{align*}
\mathbf{B}_0(z) &= \hat{x} B_0 \exp(-z/\ell_B), \quad (5)
\end{align*}
\]

where \( \ell_B \) is the scale height of magnetic field. Rewriting this equation in terms of dimensionless variables we obtain

\[
B_0^\ast = \sqrt{e} \exp(-z^\ast/q) \quad (6)
\]

where \( q = \ell_B/\ell_p \) is the ratio of the characteristic length scale of magnetic field variation to the characteristic length scale of density variation. Equation (4a) rewritten in the non-dimensionalized form is

\[
D\rho_0^\ast + \rho_0^\ast + \frac{1}{2} \gamma_{ad} M_\Lambda (B_0^\ast)^2 = 0, \quad (7)
\]

where the differential operator \( D \equiv \ell_p(d/dz) \) and \( M_\Lambda \) is the dimensionless number defined in table 1. We have further assumed that the background magnetic field and the density are functions of only the vertical coordinate, \( z \). Integrating (7) we obtain a relationship between the background density and the background magnetic field,

\[
\rho_0^\ast(z^\ast) = -\left( \frac{\gamma_{ad} M_\Lambda}{2} \right) e^{-z^\ast} \int e^{z^\ast} D(B_0^\ast)^2 dz^\ast. \quad (8)
\]

This equation is true, in general, irrespective of whether the background magnetic field is an exponential function of the vertical coordinate or not. Substituting (6) in (8) and simplifying, we obtain:
\[ \rho_0^*(z^*) = \left(1 - \frac{\gamma_{\text{ad}} M_A}{q - 2} \right) e^{-z^*} + \frac{\gamma_{\text{ad}} M_A}{q - 2} e^{-2z^*/q} \quad \text{for} \quad q \neq 2; \tag{9a} \]
\[ \rho_0^*(z^*) = \left(1 + \frac{\gamma_{\text{ad}} M_A}{2} z^* \right) e^{-z^*} \quad \text{for} \quad q = 2. \tag{9b} \]

This means that the density drops exponentially with height for \( q \neq 2 \). However, when \( q = 2 \), it increases linearly with height for \( z^* \ll 1 \) and drops exponentially for \( z^* \gg 1 \).

For notational simplicity, in the rest of the paper, we shall drop the asterisks from the non-dimensional quantities.

### 2.2 Linearized MHD equations

The standard way of studying linear waves is to linearize the MHD equations about the static background state:
\[ (\rho, U, B) = (\rho_0, 0, B_0) + \varepsilon (\rho, U, B), \tag{10} \]
where \( \varepsilon \) is a small parameter. We have not included pressure as a separate variable as both the pressure and its perturbation are related to density by the equation of state. We substitute (10) in (1), expand it and keep terms up to first order in \( \varepsilon \). After some straightforward algebraic simplifications we reduce the problem to a second-order-in-time equation for the perturbed velocity alone:
\[ \frac{\partial^2 U}{\partial t^2} = \frac{1}{\rho_0} \nabla (\rho_0 \varepsilon^2 \nabla \cdot U) + \nabla (U \cdot g) - g (\nabla \cdot U) \]
\[ - \frac{1}{\rho_0} \nabla \left[ U \cdot \left( \nabla \left( \frac{B_0^2}{2\mu_0} \right) - \frac{1}{\mu_0} B_0 \cdot \nabla B_0 \right) \right], \tag{11} \]
\[ + \frac{1}{\mu_0 \rho_0} \left[ \nabla \times \left( \nabla \times (U \times B_0) \right) \right] \]
\[ + \frac{1}{\mu_0 \rho_0} \left[ \nabla \times (\nabla \times (U \times B_0)) \right] \]
This is a standard method of analysis, see, e.g., Chandrasekhar (1961).

### 3 GENERAL TREATMENT

We have chosen a domain that is periodic in the horizontal direction but not in the vertical direction. We now decompose the velocity perturbation as
\[ u(x, y, z) = \hat{u}(k_x, k_y, z, \omega) \exp(ixk_x + yk_y - \omega t) \tag{12} \]
where the amplitude, \( \hat{u}(k_x, k_y, z, \omega) \), depends on height, \( z \). Substituting in (11), we obtain three coupled ordinary differential equations (ODEs) for the three components of velocity \( \hat{u}_x, \hat{u}_y, \hat{u}_z \). We nondimensionalize them following the same prescription described in the section 2.1.1 to obtain:
\[ \left( \omega^2 - k_x^2 \right) \hat{u}_x - k_x k_y \hat{u}_y - ik_x \left( \frac{1}{\gamma_{\text{ad}}} - D \right) \hat{u}_z = 0, \tag{13a} \]
\[ -k_x k_y \hat{u}_x + \left( \omega^2 - k_y^2 - k^2 M_A^2 \frac{B_0^2}{\rho_0} \right) \hat{u}_y \]
\[ - ik_y \left[ \frac{1}{\gamma_{\text{ad}}} - \left( 1 + \frac{B_0^2}{\rho_0} M_A^2 \right) \right] \hat{u}_z = 0, \tag{13b} \]
and
\[ ik_x \left[ D + \frac{1}{\gamma_{\text{ad}}} - 1 - \frac{\gamma_{\text{ad}} M_A^2 (B_0^2)^2}{2 \rho_0} \right] \hat{u}_x \]
\[ + ik_y \left[ D + \frac{1}{\gamma_{\text{ad}}} - 1 + M_A^2 \frac{B_0^2}{\rho_0} D + \frac{2 - \gamma_{\text{ad}}^2}{2} \frac{M_A^2 (B_0^2)^2}{\rho_0} \right] \hat{u}_y \]
\[ + \left( \omega^2 + D^2 - D + \frac{2 - \gamma_{\text{ad}}^2}{2} \frac{M_A^2 (B_0^2)^2}{\rho_0} \right) \hat{u}_x \]
\[ + \frac{M_A^2 B_0^2 (D^2 - k_x^2)}{\rho_0} \hat{u}_z = 0. \tag{13c} \]

Equation (13a)-(13c) are completely general for the case where the background magnetic field everywhere points along the \( x \) direction and is a function of the \( z \) coordinate alone. These equations simplifies further if the magnetic field is a constant in space. Several such cases has already been studied in detail by Nye & Thomas (1976); Adam (1977); Thomas (1983); Campos (1983). In this paper we consider the more general case where \( B_0 \) is a function of the \( z \) coordinate.

In particular, we list the following cases:

(i) Waves propagating along the \( x \) direction (along the background magnetic field).
(ii) Waves propagating along the \( z \) direction (along the direction of gravity).
(iii) Waves propagating along the \( y \) direction (orthogonal to the direction of both the gravity and the background magnetic field).

For waves traveling along the gravity, Campos & Marta (2015) presented an exact solution for a very special case of \( q = 2 \). In this paper, we consider the first and the second cases and present exact solutions for arbitrary values of \( q \). For waves propagating along a general direction the problem becomes too cumbersome. We also point out the potential signatures of subsurface solar magnetic field by looking at the changes in dispersion relation for waves.

Clearly a general way to treat these problems is to eliminate two variables from the three equations, (13a)-(13c); then consider the ratio of the scale-heights, \( q \) to be a small-parameter and study these problems using perturbation theory. It turns out that in the two cases we consider, this route is not necessary as exact solutions in terms of hypergeometric functions exist. Furthermore, in each of these two cases, \( q = 2 \) is special and must be dealt with separately.

Let us first give a short summary of our calculations. In general, when we eliminate two of the three variables from the three equations we obtain a differential equation with coefficients that depend on the background density and magnetic field. As both of these quantities have exponential dependence on \( z \), we obtain a differential equation with exponential coefficients. Next, we look at dispersion relation for waves in Section 4 and obtain exact solutions in Section 5. For the latter, several change of variables is necessary to convert the differential equation with exponential coefficients to differential equations with coefficients that depend algebraically on the independent variable. This ultimately leads us to the hypergeometric differential equation.
4 DISPERSSION RELATION FOR WAVES PROPAGATING ALONG THE MAGNETIC FIELD

We set \( k_y = 0 \) and \( k_x = k \) in Equation (13a) - (13c) to obtain:

\[
\begin{align*}
\left( \omega^2 - k^2 \right) \hat{u}_x - i k \frac{1}{\gamma_m} \left( 1 - D \right) \hat{u}_x &= 0, \quad (14a) \\
\left( \omega^2 - k^2 \right) \frac{B_0^2}{\rho_0} \hat{u}_x &= 0, \quad (14b) \\
&+ \left( 1 + \gamma_m B_0^2 \right) D^2 \left( - \frac{1}{\gamma_m} \right) \frac{D}{B_0^2} \hat{u}_x + \left( \omega^2 - k^2 \right) \frac{B_0^2}{\rho_0} \hat{u}_x = 0. \quad (14c)
\end{align*}
\]

The \( \hat{u}_y \) mode in Equation (14b) is decoupled from the other two. We are primarily interested in the \( \hat{u}_z \) mode, so we eliminate \( \hat{u}_x \) between (14a) and (14c) to obtain

\[
\begin{align*}
\left( \omega^2 - k^2 \right) \frac{B_0^2}{\rho_0} + \omega^2 D^2 \hat{u}_z - \omega^2 D \hat{u}_z &+ \left( \omega^2 - k^2 \right) \frac{B_0^2}{\rho_0} \hat{u}_z + \frac{k^2}{\gamma_m} \left( \gamma_m - 1 \right) \hat{u}_z \\
&\left( \omega^2 - k^2 \right) \frac{B_0^2}{\rho_0} \hat{u}_z + \frac{k^2}{\gamma_m} \left( \gamma_m - 1 \right) \hat{u}_z = 0.
\end{align*}
\]

In the above equation, the expressions for \( \rho_0 \) and \( B_0 \) are given by \( \rho_0 \) and \( B_0^2 \) in (9) and (6). To simplify our notations we have now renamed our dimensionless variables. If the background magnetic field is a constant, these equations reduce to the ones considered by Nye & Thomas (1976).

The above equation, (15), can be rewritten as:

\[
\begin{align*}
D^2 \hat{u}_z + f(z) D \hat{u}_z + r(z) \hat{u}_z = 0 \quad (16)
\end{align*}
\]

where

\[
\begin{align*}
f(z) &= \frac{\int (B_0^2)}{\rho_0} M_A \left( \left. \frac{\omega^2 - k^2 - \frac{\omega^2 \gamma_m}{2} - \omega^2}{\omega^2 - k^2 \frac{B_0^2}{\rho_0} + \omega^2} \right|_{\gamma_m} \right) \quad (17a)
\end{align*}
\]

and

\[
\begin{align*}
r(z) &= \frac{\left( \omega^2 - k^2 \right) \left( \omega^2 - k^2 \frac{B_0^2}{\rho_0} + \frac{k^2}{\gamma_m} \left( \gamma_m - 1 \right) \right)}{\omega^2 - k^2 \frac{B_0^2}{\rho_0} + \omega^2} \\
&+ \frac{\int (B_0^2)}{\rho_0} M_A \frac{k^2}{\omega^2 - k^2} \frac{B_0^2}{\rho_0} + \omega^2).
\end{align*}
\]

Following Tolstoy (1963), we apply the transformation

\[
\hat{u}_z = \psi(z) \exp \left( - \frac{1}{2} \int^z f(z) dz \right) \quad (18)
\]

in equation (16) from which we obtain,

\[
D^2 \psi(z) + \Gamma^2(z) \psi(z) = 0, \quad (19)
\]

with

\[
\Gamma^2(z) = r(z) - \frac{f^2(z)}{4} - \frac{D f(z)}{2}. \quad (20)
\]

The equation (19) is the usual form of the wave equation. When \( \Gamma^2(z) > 0, \psi(z) \) locally exhibits oscillatory behaviour in \( z \) and when \( \Gamma^2(z) < 0, \) exponential or hyperbolic nature of \( \psi(z) \) is realized. Thus \( \Gamma^2(z) \) can be interpreted locally as the vertical component of the wavenumber (Tolstoy 1963).

We consider the boundary conditions: \( \hat{u}_z = 0 \) at \( z = 0 \) (bottom) and \( \partial_z \hat{u}_z = 0 \) at \( z = 1 \) (top). Our ansatz for discretising \( \Gamma \) is analogous to the organ pipes where we have:

\[
\Gamma = \left( n + \frac{1}{2} \right) \frac{\pi}{L_z}, \quad n = 0, 1, 2, 3, \ldots \quad (21)
\]

where \( n \) represents the number of nodes in the \( z \)-direction (radial nodes). This is valid only when \( \Gamma \) does not depend on \( z \). Since in this paper, we are interested in dealing with high-degree (i.e. high-wavenumber) \( p \)-modes, which are trapped in a very thin region near the solar surface, we consider the size of the box to be \( L_z = \ell_p. \) Under such circumstance, we find that \( \Gamma \) of equation (20) remains approximately constant for \( z \in [0, 1] \) and hence, the ansatz (21) is reasonable for obtaining a qualitative idea of how magnetic field could change the dispersion relation.

If \( \Gamma \) is not a function of \( z \), one should use the WKB approximation:

\[
\int^z \Gamma(z) dz = \left( n + \frac{1}{2} \right) \pi, \quad (22)
\]

which is a very good approximation for large \( n \). If accurate eigen-frequencies valid for all \( n \) are required, an exact quantization scheme for \( \Gamma \) is to be used:

\[
\int^z \Gamma(z) dz = \left( n + \frac{1}{2} \right) \pi + \delta_n. \quad (23)
\]

where \( \delta_n \) is the ‘quantum correction’, similar to the one proposed by Ma & Xu (2005). However, we do not employ such approach here.

It should be noted that the Duvall’s law, which was used in solar observations and which led to the beginning of inversion of sound speed with depth in the Sun, also had a similar functional form (Duvall 1982):

\[
\Gamma = \frac{(n_D + \alpha) \pi}{L_z}, \quad n_D = 1, 2, 3, \ldots \quad (24)
\]

Importantly, \( n_D = 0 \) was excluded in Duvall’s analysis as its frequencies did not fall along a curve where all frequencies corresponding to other values of \( n_D \) collapsed. On the other hand, the parameter \( \alpha \) was found to best fit the curve with its value nearly equal to 3/2. In fact, \( \alpha \) is shown to be a function of frequency (Gough 1987; Christensen-Dalsgaard 2003). We note that the equations (21) and (24) are identical if we include \( n_D = 0 \) and parametrize \( \alpha = 1/2 \).

Using equations (20) and (21), we find an approximate dispersion relation:

\[
\left( n + \frac{1}{2} \right)^2 \frac{\omega^2}{L_z^2} = r(z) - \frac{f^2(z)}{4} - \frac{D f(z)}{2}. \quad (25)
\]

which results into a sixth order polynomial equation in \( \omega \). They represent a pair of acoustic, Alfvén, and internal gravity waves, each modified by the other ones. This equation, (25), is the dispersion relation valid for waves that propagate along the magnetic field with the background horizontal magnetic field varying vertically in any arbitrary manner. We now choose the profile of the magnetic field

MNRA 000, 000-000 (0000)
in two different cases – uniform field and exponentially decaying field along the vertical direction.

4.1 Uniform background magnetic field

From equation (7), we find that the background density varies exponentially along the direction of gravity when the background magnetic field is uniform in space. So, the non-dimensionalised Alfven speed becomes:

$$v_A^2(z) = \frac{B_0^2}{\rho_0} = e^z. \quad (26)$$

Hence, the expressions for $f(z)$ and $r(z)$, utilizing equations (17a) and (17b), are:

$$f(z) = \frac{-\omega^2}{[\omega^2 - k^2] \text{M}_A e^z + \omega^2} \quad (27a)$$

and

$$r(z) = \frac{\left(\omega^2 - k^2 \right) \left(\omega^2 - k^2 \text{M}_A e^z \right) + k^2 N^2}{[\omega^2 - k^2] \text{M}_A e^z + \omega^2} \quad (27b)$$

where the non-dimensionalised Brunt-Väisälä frequency, $N$, is defined as

$$N = \sqrt{\frac{1}{\gamma_{ad}} - \frac{1}{\gamma_{ad}^2}}. \quad (28)$$

This frequency, $N$, of course, changes in the presence of magnetic field. The first term inside the radical of $N$ accounts for density stratification and the other captures the effect of adiabatic expansion.

Substituting these expressions for $f(z)$ and $r(z)$ in equation (20), we get,

$$4\Gamma^2 \left[ \omega^2 - k^2 \right] \text{M}_A e^z + \omega^2 \right]^2 = 4 \left[ \omega^2 - k^2 \right] \left(\omega^2 - k^2 \text{M}_A e^z \right) + k^2 N^2 \right] \left(\omega^2 - k^2 \right) \text{M}_A e^z + \omega^2 \right] - 2\omega^2 \left(\omega^2 - k^2 \right) \text{M}_A e^z + \omega^2 \right) \omega^2 \quad (29)$$

This equation is cubic in $\omega^2$, representing three pairs of magneto-acoustic-gravity waves. We solve for the root of this equation that corresponds to the pressure mode.

The frequencies for pure acoustic and pure internal gravity waves in a stratified medium are respectively given as (Tolstoy 1963):

$$\omega_A^2 = k^2 + \Gamma^2 + 1/4, \quad (30a)$$

$$\omega^2 = \frac{k^2 N^2}{k^2 + \Gamma^2 + 1/4}, \quad (30b)$$

where $\Gamma$ is defined in the equation (21).

In a compressible fluid with gravity, these frequencies become:

$$\omega_A^2 = \frac{1}{2} \omega_A^2 \left[ 1 \pm \sqrt{1 - 4 \left(\omega_A^2 \omega_A^2 \right)} \right] \quad (31)$$

where $\omega_A$ and $\omega$ stand for the high-frequency acoustic wave and the low-frequency internal gravity wave, each modified by the other one. This gravity-modified acoustic wave frequency ($\omega_A$) is compared in Fig. 2 with its further modified wave frequency due to the presence of magnetic field, which is computed using the equation (29).

We find that the presence of a horizontal magnetic field increases the frequencies of $p$-modes. Fig. 2 demonstrates that the ridges in the $k_A - \omega$ diagram shift only when the Alfven speed is significantly close to (or more than) the sound speed. It can, therefore, be concluded that the ring-diagram analysis applied for the quite Sun is unaffected by the ubiquitous small-scale magnetic field. However, when the magnetic field becomes stronger, frequency shifts in the dispersion relation are observed.

4.2 Non-uniform background magnetic field

Now we need to consider two different cases, the more general $q \neq 2$ and the particular $q = 2$.

4.2.1 Ratio of scale-heights $q \neq 2$

For this case, the density profile is given by equation (9a).

Let’s calculate the non-dimensionalized Alfven speed as a function of height,

$$v_A^2(z) = \frac{B_0^2}{\rho_0} = \left[ \frac{\gamma_{ad} \text{M}_A}{q - 2} + \left( 1 - \frac{\gamma_{ad} \text{M}_A}{q - 2} \right) e^{-(1-2/q)z} \right]^{-1}. \quad (32)$$

Straightforward algebra show that, in equation (32), we can write

$$\gamma_{ad} \text{M}_A = 2/\beta \quad (33)$$

where $\beta$ is the plasma-beta, the ratio of the gas-pressure to the magnetic pressure at the bottom of our domain.
In order to obtain the dispersion relation for this case, we evaluate the expressions for \( f(z) \) and \( r(z) \) of equation (20) using the equations (17a), (17b), and (32). As in the previous section, 4.1, we get three pairs of waves, out of which the pressure-modes are shown in Fig. 3. We find that the frequency shift due to the presence of a horizontal magnetic field on the \( k_x - \omega \) diagram is less when the magnetic field drops with height, but is still significant for \( q = 10 \).

4.2.2 Ratio of scale-heights \( q = 2 \)

The Alfvén speed, in this case, is given as:

\[
v_A^2(z) = \frac{B_0^2}{\rho_0} = \left[ 1 + \frac{\gamma_{ad} M_A}{2} \right]^{-1}.
\]

The dispersion relation for acoustic wave for \( M_A = 1 \) is presented in the Fig. 3.

5 EXACT SOLUTIONS

5.1 Waves propagating along the magnetic field

We obtain an exact solution for waves that propagate along the magnetic field. The magnetic field with its scale height twice the density scale height turns out to be a special case and is, therefore, dealt with separately later.

5.1.1 Ratio of scale-heights \( q \neq 2 \)

We substitute (32) in (15), simplify it, and perform the following variable

\[
s = \left( 1 - \frac{2}{q} \right) z.
\]

Equation (15) can then be written in the following form:

\[
[A_2 e^s + B_2] \frac{d^2 \hat{u}_z}{ds^2} + [A_1 e^s + B_1] \frac{d\hat{u}_z}{ds} + [A_0 e^s + B_0] \hat{u}_z = 0
\]

where the expressions for \( A_0, A_1, A_2 \) and \( B_0, B_1, B_2 \) are given in appendix A.

Next, following Campos & Marta (2015), we do two more change of variables,

\[
\xi = -\frac{B_2}{A_2} e^{-s},
\]

\[
W = \hat{u}_z e^{-\theta s},
\]

where \( \theta \) is yet undetermined constant. After some straightforward simplifications we obtain:

\[
\xi(1 - \xi) \frac{d^2 W}{d\xi^2} + \left( 2\theta + 1 - \frac{A_1}{A_2} \right) \left( 2\theta + 1 - \frac{B_1}{B_2} \right) \xi \frac{dW}{d\xi} - \left( \theta^2 - \frac{B_1}{B_2} \theta + \frac{B_0}{B_2} \right) W = 0
\]

Now, we choose \( \theta \) such that the last term in the above equation vanishes, i.e.

\[
\theta^2 - \frac{A_1}{A_2} \theta + \frac{A_0}{A_2} = 0.
\]

Consequently, Equation (38) turns into the hypergeometric equation:

\[
\xi(1 - \xi) \frac{d^2 W}{d\xi^2} + [C - (A + B + 1) \xi] \frac{dW}{d\xi} - AB W = 0. \tag{40}
\]

where the parameters \( A, B, \) and \( C \) are given by

\[
C = 2\theta + 1 - \frac{A_1}{A_2} \tag{41a}
\]

\[
A + B + 1 = 2\theta + 1 - \frac{B_1}{B_2}, \tag{41b}
\]

\[
AB = \theta^2 - \frac{B_1}{B_2} \theta + \frac{B_0}{B_2} \tag{41c}
\]

and \( \theta \) is given by the equation (39). This completes the first stage of our calculations.

The solution to the Gauss hypergeometric differential equation (40) may be expressed in terms of hypergeometric functions (for \( |\xi| < 1 \)) as:

\[
W(A, B; C; \xi) = \frac{\Gamma(C)}{\Gamma(A) \Gamma(B)} \sum_{n=0}^{\infty} \frac{\Gamma(A+n) \Gamma(B+n) \xi^n}{\Gamma(C+n)}
\]

Transforming back to the original variables, the solution can be

\[
MNRAS 000, 000-000 (0000)
\]
written down as

\[ \hat{u}_x(z) = D_1 e^{-\theta(1-2/q)} \mathcal{F}_1(A, B; C; -\frac{B_2}{A_2} e^{-z(1-2/q)}) + D_2 e^{-\theta(1-2/q)}(A_1/A_2)^{-\theta-1} \left[-\frac{B_2}{A_2}\right]^{1-C} \mathcal{F}_1(A - C + 1, \frac{B_2}{A_2} e^{-z(1-2/q)}) \]

\[ = 2D_1 e^{-\theta(1-2/q)}(A_1/A_2)^{\theta-1} \left[-\frac{B_2}{A_2}\right]^{1-C} \mathcal{F}_1(A + 1, \frac{B_2}{A_2} e^{-z(1-2/q)}) \]  

(43)

where \( D_1 \) and \( D_2 \) are two constants that must be determined by the boundary conditions and \( \mathcal{F}_1 \) is the Gauss hypergeometric function. Our result holds true for \( |\theta| < 1 \). All constants \(-A, B, C, A_0, A_1, A_2, B_0, B_1, B_2, \) and \( \theta \) are functions of \( \omega, k, \gamma_{ad}, \beta, \) and \( q \).

The eigenfunctions, \( \hat{u}_x \), for the first three \( p \)-modes of oscillations, using equation (43), are shown in Fig. 4. The values of eigenfrequencies, \( \omega \), were chosen after computing them for \( n = 0, 1, 2 \) using the equation (20) for \( k_x = 5 \). All eigenfunctions satisfy the boundary conditions of \( \hat{u}_x = 0 \) at \( z = 0 \) (bottom) and \( \partial_z \hat{u}_x = 0 \) at \( z = 1 \) (top).

5.1.2 Ratio of scale-heights \( q = 2 \)

In this case we proceed in a very similar fashion. We do not repeat the details of our calculations but merely write down the solution. The solution \( \hat{u}_x \) can, in general, be written down as

\[ \hat{u}_x(z, \omega, k) = A_1 F_1(m_+; 1; -\eta_1 (\eta_1 + z)) e^{\left(\frac{\omega - \nu}{\eta_1}\right)(\eta_1 + z)} + A_1 F_1(m_+; 1; \eta_1 (\eta_1 + z)) e^{\left(\frac{-1}{\eta_1}\right)(\eta_1 + z)} \]  

(44)

where the function \( F_1 \) is the confluent hypergeometric function of the first kind. Here the constants, \( m_+ \) and \( m_- \) are given by

\[ m_+ = \frac{1}{2} \left( 1 \pm \frac{1}{\eta_1} \right) \pm \frac{m_0}{\eta_0} (\eta_4 - \eta_1) \]  

(45)

All \( \eta \)'s are functions of \( \omega, k, \gamma_{ad}, \) and/or \( \beta \) as given in appendix B. The constants \( A_1 \) and \( A_2 \) must be determined by the choice of boundary conditions.

5.1.3 The special case of a constant \( B_0 \)

In the special case where \( B_0 \) is a constant in space we take the limit \( q \to \infty \) to obtain:

\[ A_2 = a_2 = (\omega^2 - k^2)M \]

\[ A_1 = a_1 = 0 \]

\[ A_0 = a_0 = -k^2 M(\omega^2 - k^2) \]

The equation (39) reduces to

\[ \theta = \pm k. \]  

(47)

We, thus, reproduce the results of Nye & Thomas (1976).

5.2 Waves propagating along the direction of gravity.

Next we consider the problem for waves propagating perpendicular to the magnetic field, along the direction of gravity. Again we need to consider two cases \( q \neq 2 \) and \( q = 2 \). The latter has already been solved exactly by Campos & Marta (2015). Below we present the solution for the former.

As we consider waves propagating along the \( z \) direction we substitute \( k_x = 0 = k_y \) in equations (13a)-(13c) to obtain:

\[ \omega^2 \hat{u}_x = 0, \]  

(48a)

\[ \omega^2 \hat{u}_y = 0, \]  

(48b)

\[ \left[ \omega^2 + D^2 \left( 1 + \frac{M_A B_0^2}{\rho_0} \right) - D \left( 2 - \gamma_{ad} \right) \frac{M_A D(B_0)^2}{\rho_0} D \right] \hat{u}_z = 0. \]  

(48c)

Clearly the solution to the differential equation (48c) can be obtained by substituting \( k_x = k = 0 \) in the solution presented in section 5.1.1:

\[ \hat{u}_z(z) = D_1 e^{-\theta(1-2/q)} \mathcal{F}_1(A; B; C; -\frac{B_2}{A_2} e^{-z(1-2/q)}) + D_2 e^{-\theta(1-2/q)}(A_1/A_2)^{-\theta-1} \left[-\frac{B_2}{A_2}\right]^{1-C} \mathcal{F}_1(A + 1, \frac{B_2}{A_2} e^{-z(1-2/q)}) \]  

(49)

where the parameters \( A, B, C, \theta, A_2, \) and \( B_2 \) are provided in the Appendix C, using \( k = 0 \).

The solution for \( q = 2 \), given by Campos & Marta (2015) can be reproduced from our results in section 5.1.2 if we substitute \( k = 0 \) in Equation (44).
6 CONCLUSIONS

To summarise, we investigate the detectable signatures of subsurface solar magnetic fields by looking at changes in the dispersion relation of waves in the presence of uniform and non-uniform magnetic fields. We find the changes are significant for a strong subsurface magnetic field.

We find exact solutions for waves in an isothermal, stratified medium with a magnetic field that points along a direction perpendicular to that of gravity and varies exponentially in the direction of gravity. Our exact solutions are for two different cases: (a) waves propagating along the direction of magnetic field, (b) waves propagating along the direction of gravity. In all of these cases we find solutions in terms of either Gauss hypergeometric or confluent hypergeometric functions. For waves propagating along a general direction in the horizontal plane - the plane perpendicular to gravity - no exact solution has been found so far. The case (a) above can also be considered as Alfvén waves in a non-uniform magnetic field modified by the presence of stratification. Two earlier results by Nye & Thomas (1976) and Campos & Marta (2015), emerges as special cases of our solutions.

We have considered the background magnetic field to be constant in time. This is physically justifiable as the time scale of emergence of magnetic flux and formation of active regions on the solar surface is very slow compared to characteristic time scales of helioseismic waves. Hence it is a good enough approximation to consider the background magnetic field to be static. Typical emergence phenomenon is the formation of a bipolar region, which is expected to form from an underlying magnetic field lines that are horizontal. Hence starting with a horizontal magnetic field can also be taken as a reasonable approximation. But, the assumptions that the gas is ideal and the atmosphere is isothermal cannot be used (Ulrich 1970). Hence our result cannot directly be compared with solar observations. However, certain qualitative results may still hold. For example, it is clear that a subsurface magnetic field will give rise to vertical oscillation propagating along horizontal directions on the surface. In other words, subsurface magnetic fields have surface signatures that may be detectable. Furthermore, we expect the dispersion relation to depend on the direction of propagation of the wave with respect to the direction of the sub-surface magnetic field. In addition, the exact solution that we provide can be used to verify the accuracy of numerical codes, which in turn may be more useful in studying the Sun.

ACKNOWLEDGMENTS

BT acknowledges Nordic Institute for Theoretical Physics (NORDITA) for providing a Visiting Fellowship. DM is supported by Swedish Research Council Grant no. 638-2013-9243. DM gratefully acknowledge Praphull Kumar for his help in earlier stages of these calculations.

REFERENCES

Adam J., 1977, Solar Physics, 52, 293
Campos L., 1983, Wave Motion, 5, 1
Campos L., Marta A., 2015, Gophysical & Astrophysical Fluid Dynamics, 109, 168
Cally P. S., 2007, Astronomische Nachrichten, 328, 286
Chandrasekhar S., 1961, Hydrodynamic and hydromagnetic stability. Oxford University Press
Christensen-Dalsgaard J., 2003, Lecture Notes on Stellar Oscillations. 5th edn
Choudhuri A. R., 1998, The physics of fluids and plasmas: an introduction for astrophysicists. Cambridge University Press
Duvall T. L. Jr., Nature, 1982, vol. 300 pg. 542
Ilonidis S., Zhao J., Kosovichev A., 2011, Science, 333, 993
Gough D. O., Zahn J.-P., Zinn-Justin J., Astrophysical Fluid Dynamics, 1987 Amsterdam North-Holland pg. 399
Ma Z.Q., Xu B. W., 2005, EPL (Europhysics Letters), 69(5), 685
Nye A. H., Thomas J. H., 1976, The Astrophysical Journal, 204, 573
Schunker H., Braun D. C., Cally P. S., Lindsey, C., 2005, The Astrophysical Journal Letters, 621, L149-L152
Singh, N. K., Brandenburg, A., Rheinhardt, M., 2014, The Astrophysical Journal Letters, 795, L8
Singh, N. K., Brandenburg, A., Chitre, S.M., Rheinhardt, M., 2015, Monthly Notices of the Royal Astronomical Society, 447(4), 3708-3722
Singh N. K., Raichur H., Brandenburg A., 2016, The Astrophysical Journal, 832, 120
Singh N. K., Raichur H., Kapyla M. J., Rheinhardt M., Brandenburg A., Kapyla P. J., 2018, arXiv preprint arXiv:1808.08904
Thomas J. H., 1983, Annual Review of Fluid Mechanics, 15, 321
Tolstoy I., 1963, Reviews of Modern Physics, 35, 207
Ulrich R. K., 1970, The Astrophysical Journal, 162, 993

APPENDIX A: EXPRESSIONS FOR A AND B

We present here the expressions for constants introduced in equation (36) of section 5.1.1.

\[ A_2 = \left( \omega^2 - k^2 \right) M_A + \frac{\omega^2 \gamma_{ad} M_A}{q - 2} \left( 1 - \frac{2}{q} \right)^2 \]
\[ B_2 = \omega^2 \left( 1 - \frac{\gamma_{ad} M_A}{q - 2} \right) \left( 1 - \frac{2}{q} \right)^2 \]
\[ A_1 = \left( \frac{2 M_A}{q} \right) \left( \omega^2 - k^2 - \frac{\omega^2 \gamma_{ad}}{2} - \omega^2 \frac{\gamma_{ad} M_A}{q - 2} \right) \left( 1 - \frac{2}{q} \right) \]
\[ B_1 = -\omega^2 \left( 1 - \frac{\gamma_{ad} M_A}{q - 2} \right) \left( 1 - \frac{2}{q} \right) \]
\[ A_0 = \left( \omega^2 - k^2 \right) \left( \omega^2 - \frac{\gamma_{ad} M_A}{q - 2} - k^2 M_A \right) \]
\[ + \frac{k^2}{\gamma_{ad} (\gamma_{ad} - 1)} \left( \frac{\gamma_{ad} M_A}{q - 2} - \frac{M_A k^2}{q} \right) \]
\[ B_0 = \omega^2 \left( \omega^2 - k^2 \right) \left( 1 - \frac{\gamma_{ad} M_A}{q - 2} \right) \]
\[ + \frac{k^2}{\gamma_{ad} (\gamma_{ad} - 1)} \left( 1 - \frac{\gamma_{ad} M_A}{q - 2} \right) \]

MNRAS 000, 000-000 (0000)
APPENDIX B: EXPRESSIONS FOR $\eta$

The expressions for $\eta$’s used in equation (45) of section 5.1.2 are listed here:

$$
\eta_1 = \frac{2}{\gamma_{ad}} \left( 1 - \frac{k^2}{\omega^2} \right) + \beta
$$

$$
\eta_2 = -\frac{2}{\gamma_{ad}} \left( 1 - \frac{k^2}{\omega^2} - \frac{\gamma_{ad}}{2} \right) - \beta
$$

$$
\eta_3 = \frac{k^2 (\gamma_{ad} - 1)}{\omega^2 \gamma_{ad}^2} + (\omega^2 - k^2)
$$

$$
\eta_4 = \frac{\left( 1 - \frac{k^2}{\omega^2} \right) \left( \omega^2 \beta - \frac{2 k^2}{\omega^2} \right) + \frac{k^2 (\gamma_{ad} - 1)}{\omega^2 \gamma_{ad}^2} \beta - \frac{k^2}{\omega^2} \gamma_{ad}}{(\omega^2 - k^2)}
$$

$$
\eta_5 = (1 - 4 \eta_3)^{1/2}
$$

APPENDIX C: EXPRESSIONS FOR $A$, $B$, AND $C$

We provide here the parameters utilized in equation (49) of section 5.2

$$
A_2 = \omega^2 M \left( 1 + \frac{\gamma_{ad}}{q - 2} \right) \left( 1 - \frac{2}{q} \right)^2
$$

$$
B_2 = \omega^2 \left( 1 - \frac{\gamma_{ad} M}{q - 2} \right) \left( 1 - \frac{2}{q} \right)^2
$$

$$
A_1 = -\omega^2 M \left( \frac{2 - \gamma_{ad}}{q - 2} \right) \left( 1 - \frac{2}{q} \right)
$$

$$
B_1 = -\omega^2 \left( 1 - \frac{\gamma_{ad} M}{q - 2} \right) \left( 1 - \frac{2}{q} \right)
$$

$$
A_0 = \omega^4 \frac{\gamma_{ad} M}{q - 2}
$$

$$
B_0 = \omega^4 \left( 1 - \frac{\gamma_{ad} M}{q - 2} \right).
$$

Now, $\theta$ can be calculated using the equation (39) using the following set of equations:

$$
\theta = \frac{1}{2} \left( \frac{A_1}{A_2} \pm \sqrt{\left( \frac{A_1}{A_2} \right)^2 - \frac{4 A_0}{A_2}} \right).
$$

Lastly, the parameters $A$, $B$, and $C$ in equation (49) are evaluated as:

$$
C = 2 \theta + 1 - \frac{A_1}{A_2}
$$

$$
A + B + 1 = 2 \theta + 1 - \frac{B_1}{B_2},
$$

$$
AB = \theta^2 - \frac{B_1}{B_2} \theta + \frac{B_0}{B_2}.
$$

This completes the specification for obtaining the Gauss hypergeometric function as the solution for vertical propagation of magneto-acoustic-gravity waves.