MAHARAM TRACES ON VON NEUMANN ALGEBRAS

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Abstract. Traces $\Phi$ on von Neumann algebras with values in complex order complete vector lattices are considered. The full description of these traces is given for the case when $\Phi$ is the Maharam trace. The version of Radon-Nikodym-type theorem for Maharam traces is established.

Mathematics Subject Classification (2000). 28B15, 46L50

Keywords: von Neumann algebra, measurable operator, vector-valued trace, order complete vector lattice, Radon-Nikodym-type theorem.

1. Introduction

The theory of integration for measures $\mu$ with values in order complete vector lattices has inspired the study of $(bo)$-complete lattice-normed spaces $L^p(\mu)$ (see, for example, [1], 6.1.8). The spaces $L^p(\mu)$ are the Banach-Kantorovich spaces if the measure $\mu$ possesses the Maharam property. In the proof of this fact, description of Maharam operators acting in order complete vector lattices plays an important role ([1], 3.4.3).

The existence of the center-valued traces in finite von Neumann algebras makes it natural to construct the theory of integration for traces with values in the complex order complete vector lattice $F_C = F \oplus iF$. If the von Neumann algebra is commutative, then construction of $F_C$-valued integration for it is the component part for the investigation of the properties of order continuous maps of vector lattices.

Let $M$ be a non-commutative von Neumann algebra, let $F_C$ be a von Neumann subalgebra in the center of $M$ and let $\Phi : M \to F_C$ be a trace with modularity property: $\Phi(zx) = z\Phi(x)$ for all $z \in F_C$, $x \in M$. It is known that the non-commutative $L^p$-space $L^p(M, \Phi)$ is a Banach-Kantorovich space [2], [3]. In addition, $\Phi$ possesses the Maharam property: if $0 \leq z \leq \Phi(x)$, $z \in F_C$, $0 \leq x \in M$, then there exists $0 \leq y \leq x$ such that $\Phi(y) = z$ (compare with [1], 3.4.1).
In the present article, we will study the faithful normal traces \( \Phi \) on a von Neumann algebra \( M \) with values in an arbitrary complex order complete vector lattice. We give the full description of such traces in the case when \( \Phi \) is a Maharam trace. With the help of the locally measure topology in the algebra \( S(M) \) of all measurable operators we construct the Banach-Kantorovich space \( L^1(M, \Phi) \subset S(M) \). We also state the version of Radon-Nikodym-type theorem for Maharam traces.

We use the terminology and results of the von Neumann algebras theory (see [4], [5]), measurable operators theory (see [6], [7]) and order complete vector lattices and Banach-Kantorovich spaces theory (see [1]).

2. Preliminaries

Let \( H \) be a Hilbert space, let \( B(H) \) be the \(*\)-algebra of all bounded linear operators on \( H \), and \( 1 \) be the identity operator on \( H \). Let \( M \) be a von Neumann algebra acting on \( H \), let \( Z(M) \) be the center of \( M \) and \( P(M) \) be the lattice of all projectors in \( M \). We denote by \( P_{fin}(M) \) the set of all finite projectors in \( M \).

A densely-defined closed linear operator \( x \) (possibly unbounded) affiliated with \( M \) is said to be measurable if there exists a sequence \( \{p_n\}_{n=1}^{\infty} \subset P(M) \) such that \( p_n \uparrow 1 \), \( p_n(H) \subset D(x) \) and \( p_n = 1 - p_n \in P_{fin}(M) \) for every \( n = 1, 2, \ldots \) (here \( D(x) \) is the domain of \( x \)). Let us denote by \( S(M) \) the set of all measurable operators.

Let \( x, y \) be measurable operators. Then \( x + y \), \( xy \) and \( x^* \) are densely-defined and preclosed. Moreover, the closures \( \overline{x+y} \) (strong sum), \( \overline{xy} \) (strong product) and \( \overline{x^*} \) are again measurable, and \( S(M) \) is a \(*\)-algebra with respect to the strong sum, strong product, and the adjoint operation (see [6]). It is clear that \( M \) is a \(*\)-subalgebra in \( S(M) \). For any subset \( A \subset S(M) \), let \( A_h = \{ x \in A : x = x^* \} \), \( A_+ = \{ x \in A : (x,\xi) \geq 0 \text{ for all } \xi \in D(x) \} \).

Let \( x \in S(M) \) and \( x = u|x| \) be the polar decomposition, where \( |x| = (x^*x)^{\frac{1}{2}} \), \( u \) is a partial isometry in \( B(H) \). Then \( u \in M \) and \( |x| \in S(M) \). If \( x \in S_h(M) \) and \( \{E_\lambda(x)\} \) are the spectral projections of \( x \), then \( \{E_\lambda(x)\} \subset P(M) \).

Let \( M \) be a commutative von Neumann algebra. Then \( M \) admits a faithful semi-finite normal trace \( \tau \), and \( M \) is \(*\)-isomorphic to the \(*\)-algebra \( L^\infty(\Omega, \Sigma, \mu) \) of all bounded complex measurable functions with the identification almost everywhere, where \( (\Omega, \Sigma, \mu) \) is a measurable space. In addition, \( \mu(A) = \tau(\chi_A) \), \( A \in \Sigma \). Moreover, \( S(M) \cong L^0(\Omega, \Sigma, \mu) \), where \( L^0(\Omega, \Sigma, \mu) \) is the \(*\)-algebra of all complex measurable functions with the identification almost everywhere [6].
The locally measure topology \( t(M) \) on \( L^0(\Omega, \Sigma, \mu) \) is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

\[
W(B, \varepsilon, \delta) = \{ f \in L^0(\Omega, \Sigma, \mu) : \text{there exists a set } E \in \Sigma, \text{ such that } E \subseteq B, \mu(B \setminus E) \leq \delta, f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \| f\chi_E \|_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon \}.
\]

Here \( \varepsilon, \delta \) run over all strictly positive numbers and \( B \in \Sigma, \mu(B) < \infty \).

It is known that \((S(M), t(M))\) is a complete topological *-algebra.

A net \( \{ f_\alpha \} \) converges to \( f \) locally in measure (notation: \( f_\alpha \xrightarrow{t(M)} f \)) if and only if \( f_\alpha \chi_B \) converges to \( f\chi_B \) in \( \mu \)-measure for each \( B \in \Sigma \) with \( \mu(B) < \infty \). Thus \( \{ f_\alpha \} \) remains convergent to \( f \) if \( \tau \) is replaced by another faithful semi-finite normal trace on \( M \). If \( M \) is \( \sigma \)-finite, i.e. any family of nonzero mutually orthogonal projectors from \( P(M) \) is at most countable, then there exists a faithful finite normal trace \( \tau \) on \( M \). In this case, the topology \( t(M) \) is metrizable, and convergence of a sequence \( f_n \xrightarrow{t(M)} f \) is equivalent to convergence of \( f_n \) to \( f \) in trace \( \tau \).

Let now \( M \) be an arbitrary finite von Neumann algebra, \( \Phi_M : M \rightarrow Z(M) \) be a center-valued trace on \( M \) ([4], 7.11). Let \( Z(M) \cong L^\infty(\Omega, \Sigma, \mu) \). The locally measure topology \( t(M) \) on \( S(M) \) is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

\[
V(B, \varepsilon, \delta) = \{ x \in S(M) : \text{there exists } p \in P(M), z \in P(Z(M)) \text{ such that } xp \in M, \| xp \|_M \leq \varepsilon, z^\perp \in W(B, \varepsilon, \delta), \Phi_M(zp^\perp) \leq \varepsilon \delta \},
\]

where \( \| \cdot \|_M \) is the \( C^* \)-norm in \( M \). It is known that, \((S(M), t(M))\) is a complete topological *-algebra [8].

The net \( \{ x_\alpha \} \subset S(M) \) converges to \( x \in S(M) \) in trace \( \Phi_M \) (notation: \( x_\alpha \xrightarrow{\Phi_M} x \)) if \( \Phi_M(E^\perp_\chi(|x_\alpha - x|)) \xrightarrow{t(Z(M))} 0 \) for all \( \lambda > 0 \).

**Proposition 2.1.** (see [7], §3.5) Let \( M \) be a finite von Neumann algebra, \( x_\alpha, x \in S(M) \). The following conditions are equivalent:

(i) \( x_\alpha \xrightarrow{t(M)} x \);
(ii) \( x_\alpha \xrightarrow{\Phi_M} x \);
(iii) \( E^\perp_\chi(|x_\alpha - x|) \xrightarrow{t(M)} 0 \) for all \( \lambda > 0 \).

Let \( \tau \) be a faithful semi-finite normal trace on \( M \). An operator \( x \in S(M) \) is said to be \( \tau \)-measurable if \( \tau(E^\perp_\chi(|x|)) < \infty \) for some \( \lambda > 0 \).
The set $S(M, \tau)$ of all $\tau$-measurable operators is the $*$-subalgebra in $S(M)$, in addition $M \subset S(M, \tau)$. If $\tau(1) < \infty$, then $S(M, \tau) = S(M)$.

Denote by $t_\tau$ the locally measure topology in $S(M, \tau)$ generated by a trace $\tau$ (see, for example, [9]). If $x_\alpha, x \in S(M, \tau)$ and $x_\alpha$ converges to $x$ in topology $t_\tau$ (notation: $x_\alpha \longrightarrow^\tau x$), then $x_\alpha t(M) x$ ([7], §3.5). If $\tau$ is finite, then topologies $t(M)$ and $t_\tau$ coincide ([7], §3.5). It is known that $x_\alpha \longrightarrow^\tau x$ if and only if $\tau(E_\lambda^+(|x_\alpha - x|)) \rightarrow 0$ for all $\lambda > 0$ [10].

Denote by $T(M)$ the set of all nonzero finite normal traces on the finite von Neumann algebra $M$.

**Proposition 2.2.** Let $M$ be a finite von Neumann algebra, $x_\alpha, x \in S(M)$. Then

(i) if $x_\alpha \xrightarrow{t(M)} x$, then $|x_\alpha| \xrightarrow{t(M)} |x|$ and $\tau(E_\lambda^+(|x_\alpha - x|)) \rightarrow 0$ for all $\lambda > 0$ and $\tau \in T(M)$;

(ii) if $T_1(M)$ is a separating subset of $T(M)$ and $\tau(E_\lambda^+(|x_\alpha - x|)) \rightarrow 0$ for all $\lambda > 0$, $\tau \in T_1(M)$, then $x_\alpha \xrightarrow{t(M)} x$.

**Proof.** (i) Let $\tau \in T(M)$ and $s(\tau)$ be the support of a trace $\tau$. Then $s(\tau) \in P(Z(M))$ and $\tau(x) = \tau(xs(\tau))$ for all $x \in M$ ([4], 5.15, 7.13). Since $x_\alpha \xrightarrow{t(M)} x$, $x_\alpha s(\tau) \xrightarrow{t(M)} xs(\tau)$. The restriction of $\tau$ on $Ms(\tau)$ is a faithful finite normal trace. Therefore $\tau(E_\lambda^+(|x_\alpha - x|)) = \tau(E_\lambda^+(|x_\alpha s(\tau) - xs(\tau)|)) \rightarrow 0$ for all $\lambda > 0$.

If $|x_\alpha| \xrightarrow{t(M)} |x|$, then there are $\lambda_0 > 0$, $\tau \in T(M)$ such that $\tau(E_{\lambda_0}^+(|x_\alpha| - |x|)) \rightarrow 0$. The restriction $\tau_0$ of the trace $\tau$ on $Ms(\tau)$ is a faithful finite normal trace. Therefore convergence $x_\alpha s(\tau) \xrightarrow{t(M)} xs(\tau)$ implies $x_\alpha s(\tau) \xrightarrow{\tau_0} xs(\tau)$. Using continuity of the operator function $\sqrt{\tau}$, $y \in S_+(Ms(\tau))$ [11], we obtain

$$|x_\alpha|s(\tau) = \sqrt{(x_\alpha s(\tau))^*(x_\alpha s(\tau))} \xrightarrow{\tau_0} \sqrt{(xs(s(\tau)))s(\tau))} = |x|s(\tau).$$

Hence $\tau(E_{\lambda_0}^+(|x_\alpha| - |x|)) = \tau(E_{\lambda_0}^+(|x_\alpha s(\tau) - xs(\tau)|)) \rightarrow 0$, which is not the case.

(ii) Since $T_1(M)$ is the separating family traces on $M$, $\sup_{\tau \in T_1(M)} s(\tau) = 1$. Hence there is a family $\{z_i\}_{i \in I}$ of nonzero mutually orthogonal central projectors such that $\sup_{i \in I} z_i = 1$, and for any $i \in I$, there exists $\tau_i \in T_1(M)$ with $z_i \leq s(\tau_i)$ ([12], chapter III, §2). We defined the faithful semi-finite normal trace on $M$ as $\tau(x) = \sum_{i \in I} \tau_i(xz_i)$, $x \in M$.

It is clear that restrictions $\tau$ and $\tau_i$ coincide on $MZ_i$. In addition, $\tau_i(E_\lambda^+(|x_\alpha z_i - xz_i|)) = \tau_i(E_\lambda^+(|x_\alpha - x|)) \rightarrow 0$ for all $\lambda > 0$, $i \in I$. Hence, $E_\lambda^+(|x_\alpha - x|)z_i \xrightarrow{t(M)} 0$, and therefore $E_\lambda^+(|x_\alpha - x|)z_i \xrightarrow{t(M)} 0$. 

For any finite subset $\gamma \subseteq I$, let $u_\gamma = \sum_{i \in \gamma} z_i$. It is clear that $u_\gamma \uparrow 1$ and $\Phi_M(u_\gamma) \uparrow \Phi_M(1)$. Hence, $\Phi_M(u_\gamma) \xrightarrow{t(M)} 0$, i.e. $u_\gamma \xrightarrow{t(M)} 0$.

Let $U$ be an arbitrary neighborhood of 0 in $(S(M), t(M))$. We choose $V(B, \varepsilon, \delta)$ such that $V(B, \varepsilon, \delta) + V(B, \varepsilon, \delta) \subseteq U$. Fix $\gamma_0$ with $(1 - u_{\gamma_0}) \in V(B, \frac{\varepsilon}{2}, \delta)$. Since $E^z_\lambda((|x_\alpha - x|)u_{\gamma_0}) \xrightarrow{t(M)} 0$, there is an $\alpha_0$ such that $E^z_\lambda((|x_\alpha - x|)u_{\gamma_0}) \in V(B, \varepsilon, \delta)$ as $\alpha \geq \alpha_0$. We have $aV(B, \frac{\varepsilon}{2}, \delta)b \subseteq V(B, \varepsilon, \delta)$, where $a, b \in M$, $\|a\|_M \leq 1$, $\|b\|_M \leq 1$ (see, for example, [7], §3.5). Hence

$$E^z_\lambda(|x_\alpha - x|) = E^z_\lambda(|x_\alpha - x|)u_{\gamma_0} + E^z_\lambda(|x_\alpha - x|)(1 - u_{\gamma_0}) \in V(B, \varepsilon, \delta) + V(B, \varepsilon, \delta) \subseteq U$$

for all $\alpha \geq \alpha_0$. Therefore $E^z_\lambda(|x_\alpha - x|) \xrightarrow{t(M)} 0$ for all $\lambda > 0$. Proposition 2.1 implies that $x_\alpha \xrightarrow{t(M)} x$. \hfill \Box

3. **Vector lattice-valued traces**

Throughout this section, let $M$ be a von Neumann algebra, let $F$ be an order complete vector lattice, and let $F_C = F \oplus iF$ be a complexification of $F$. If $z = \alpha + i\beta \in F_C$, $\alpha, \beta \in F$, then $\tau := \alpha - i\beta$, and $|z| := \sup\{Re(e^{\theta}z) : 0 \leq \theta < 2\pi\}$ (see [4], 1.3.13).

An $F_C$-valued trace on the von Neumann algebra $M$ is a linear mapping $\Phi : M \to F_C$ given $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_0) \subseteq F$, $\Phi(M_+) \subseteq F_+ = \{a \in F : a \geq 0\}$. A trace $\Phi$ is said to be faithful if the equality $\Phi(x^*x) = 0$ implies $x = 0$, normal if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_0$, $x_\alpha \uparrow x$.

If $M$ is a finite von Neumann algebra, then its center-valued trace $\Phi_M : M \to Z(M)$ is an example of a $Z(M)$-valued faithful normal trace.

Let $\Delta$ be a separating family of finite normal numerical traces on the von Neumann algebra $M$, $\mathbb{C}^\Delta = \prod_{\tau \in \Delta} \mathbb{C}_\tau$, where $\mathbb{C}_\tau = \mathbb{C}$ for all $\tau \in \Delta$.

Then $\Phi(x) = \{\tau(x)\}_{\tau \in \Delta}$ is also an example of an faithful normal $\mathbb{C}^\Delta$-valued trace on $M$.

Let us list some properties of the trace $\Phi : M \to F_C$.

**Proposition 3.1.** (i) Let $x, y, a, b \in M$. Then

$$\Phi(x^*) = \overline{\Phi(x)}$$

$$\Phi(xy) = \Phi(yx)$$

$$\Phi(|x^*|) = \Phi(|x|)$$

$$|\Phi(axb)| \leq \|a\|_M \|b\|_M \Phi(|x|)$$

(ii) If $\Phi$ is a faithful trace, then $M$ is finite;

(iii) If $x_\alpha, x \in M$ and $\|x_\alpha - x\|_M \to 0$, then $|\Phi(x_\alpha) - \Phi(x)|$ relative uniform converges to zero;
(iv) If $M$ is a finite von Neumann algebra, then $\Phi(\Phi_M(x)) = \Phi(x)$ for all $x \in M$;

(v) $\Phi(|x + y|) \leq \Phi(|x|) + \Phi(|y|)$ for all $x, y \in M$.

Proof. The proof of (i) and (ii) is the same as for numerical traces (see, for example, [5], chapter V, §2).

The proof of (iii) follows from the inequality $|\Phi(x_n) - \Phi(x)| \leq \|x_n - x\| M\Phi(1)$.

(iv) Let $U(M)$ be the set of all unitary operators in $M$. Then $\Phi_M(x)$ belongs to the closure of the convex hull $\text{co}\{uxu^* : u \in U(M)\}$ ([1], 7.11). Since $\Phi(uxu^*) = \Phi(u^*ux) = \Phi(x)$, we get $\Phi(y) = \Phi(x)$ for any $y \in \text{co}\{uxu^* : u \in U(M)\}$. Therefore, because of (iii), we have $\Phi(x) = \Phi(\Phi_M(x))$.

(v) Since $|x + y| \leq u|x|u^* + v|y|v^*$ for some partial isometries $u, v$ in $M$ (see [13]), we have, by virtue of (i)

\[
\Phi(|x + y|) \leq \Phi(u|x|u^*) + \Phi(v|y|v^*) = \Phi(u^*u|x|) + \Phi(v^*v|y|) \\
\leq \Phi(|x|) + \Phi(|y|).
\]

$\Box$

The trace $\Phi : M \to F_\mathbb{C}$ possesses the Maharam property if for any $x \in M_+$, $0 \leq f \leq \Phi(x)$, $f \in F$, there exists a positive $y \leq x$ such that $\Phi(y) = f$. A faithful normal $F_\mathbb{C}$-valued trace $\Phi$ with the Maharam property is called a Maharam trace (compare with [1], III, 3.4.1). Obviously, any faithful finite numerical trace on $M$ is a $\mathbb{C}$-valued Maharam trace.

Let us give another examples of Maharam traces. Let $M$ be a finite von Neumann algebra, let $A$ be a von Neumann subalgebra in $Z(M)$, and let $T : Z(M) \to A$ be an injective linear positive normal operator. If $f \in S(A)$ is a reversible positive element, then $\Phi(T,f)(x) = fT(\Phi_M(x))$ is an $S(A)$-valued faithful normal trace on $M$. In addition, if $T(ab) = aT(b)$ for all $a \in A, b \in Z(M)$, then $\Phi(T,f)$ is a Maharam trace on $M$.

Note that if $\tau$ is a faithful normal finite numerical trace on $M$ and $\dim(Z(M)) > 1$, then $\Phi(x) = \tau(x)1$ is a $Z(M)$-valued faithful normal trace. In addition, $\Phi$ does not possess the Maharam property. In fact, if $p \in Z(M)$, $0 \neq p \neq 1$, then for all $y \in M_+$, $y \leq 1$ the relation $\Phi(y) = \tau(y)1 \neq \tau(y)p \leq \Phi(1)$ is valid.

Let $F$ have an order unit $1_F$. Denote by $B(F)$ the complete Boolean algebra of unitary elements with respect to $1_F$, and by $s(a) := \sup\{1_F \wedge n\geq 1 n|a|\} \in B(F)$ the support of an element $a \in F$. Since $|\Phi(x)| \leq$




Theorem 3.2. Let $\Phi$ be an $F_\mathbb{C}$-valued Maharam trace on a von Neumann algebra $M$. Then there exists a von Neumann subalgebra $A$ in $Z(M)$, a $*$-isomorphism $\psi$ from $A$ onto the $*$-algebra $C(Q)_C$, an injective positive linear normal operator $E$ from $Z(M)$ onto $A$ with $E(1) = 1$, $E^2 = E$, such that

1. $\Phi(x) = \Phi(1)\psi(E(\Phi_M(x)))$ for all $x \in M$;
2. $\Phi(zy) = \Phi(zE(y))$ for all $z, y \in Z(M)$;
3. $\Phi(z) = \psi(z)\Phi(y)$ for all $z \in A$, $y \in M$.

Proof. Since $s(\Phi(1)) = 1_F$, we get that $\Phi_1(x) = \Phi(1)^{-1}\Phi(x)$ is a $(C(Q))_C$-valued Maharam trace on $M$. In addition, $\Phi_1(1) = 1_F$.

The set $Z_h(M)$ is an order complete vector lattice with a strong unit 1 with respect to algebraic operations, and the partial order induced from $M$. Moreover, the Boolean algebra of all unitary elements in $Z_h(M)$ with respect to 1 coincides with $P(Z(M))$. Let $T$ be a restriction of $\Phi_1$ on $Z_h(M)$. Since $|\Phi_1(x)| \leq \|x\|_M$, $T(Z_h(M)) \subset C(Q)$. It is clear that $T$ is an injective positive order continuous linear operator. If $x \in Z_+(M)$, $0 \leq a \leq Tx = \Phi_1(x)$, $a \in C(Q)$, then there exists $y \in M_+$ such that $y \leq x$ and $\Phi_1(y) = a$. By Proposition 3.1 (iv), we have $a = \Phi_1(y) = \Phi_1(\Phi_M(y)) = T(\Phi_M(y))$, moreover, $0 \leq \Phi_M(y) \leq \Phi_M(x) = x$. Hence, $T : Z_h(M) \to C(Q)$ is a Maharam operator ([1], 3.4.1). Theorem 3.4.3 from [1] guarantees the existence of a Boolean isomorphism $\varphi$ from $B(F)$ onto a regular Boolean subalgebra $B$ in $P(Z(M))$ such that $gT(x) = T(\varphi(g)x)$ for all $g \in B(F)$ and $x \in Z_h(M)$. We denote by $A$ a commutative von Neumann subalgebra in $Z(M)$ generated by $B$, i.e. $A$ coincides with the bicommutant of $B$. It is known that $A_h = \{x \in Z_h(M) : E_\lambda(x) \in B$ for all $\lambda\}$ where $\{E_\lambda(x)\}$ are the spectral projections of $x$. The Boolean isomorphism $\varphi$ is extended to the $*$-isomorphism $\tilde{\varphi}$ from the $*$-algebra $C(Q)_C$ onto the von Neumann algebra $A$. If $a = \sum_{i=1}^n \lambda_i e_i$ is a simple element, $\lambda_i \in \mathbb{R}$,
$e_i \in B(F), i = 1, \ldots, n$, then

$$T(\tilde{\varphi}(a)x) = \sum_{i=1}^{n} \lambda_i T(\varphi(e_i)x) = aT(x)$$

for all $x \in A_h$. Furthermore, we note $T(\tilde{\varphi}(a)x) = aT(x)$ for any $a \in C(Q), x \in A_h$. This is obtained by approximating the elements from $C(Q)$ by simple elements. Therefore, $\Phi_1(\tilde{\varphi}(a)x) = a\Phi_1(x)$ for all $a \in C(Q)_C, x \in A$, in particular,

$$\Phi_1(\tilde{\varphi}(a)) = a \quad (1)$$

Hence the restriction $T_0$ of the operator $T$ on $A_h$ is a lattice isomorphism from $A_h$ onto $C(Q)$. Therefore $T_0$ is a Maharam operator.

By Theorem 4.2.9 from [14], there exists an operator of conditionally mathematical expectation $E: Z_h(M) \to A_h$ satisfying the following conditions:

(E1) $E$ is an injective positive order continuous linear operator, $E^2 = E$ and $E(1) = 1$;

(E2) $T(xy) = T(xE(y))$ for all $x, y \in Z_h(M)$;

(E3) $E(zy) = zE(y)$ for all $z \in A_h, y \in Z_h(M)$.

The operator $E$ is extended to the operator $\tilde{E}: Z(M) \to A$. It is clear that the condition (E1) is satisfied for $\tilde{E}$, the condition (E2) has the form $\Phi_1(xy) = \Phi_1(x\tilde{E}(y))$ for all $x, y \in Z(M)$, and the condition (E3) is valid for all $z \in A, y \in Z(M)$. The condition (E2) implies that

$$\Phi_1(y) = \Phi_1(\tilde{E}(y)) \quad \text{for all } y \in Z(M). \quad (2)$$

Using equalities (1), (2) and Proposition 3.1 (iv), we get

$$\Phi_1(x) = \Phi_1(\Phi_M(x)) = \Phi_1(\tilde{E}(\Phi_M(x))) = \tilde{\varphi}^{-1}(\tilde{E}(\Phi_M(x))) \quad (3)$$

for any $x \in M$.

Taking in (3) $\psi = \tilde{\varphi}^{-1}$ and letting $\tilde{E}$ as $E$, we obtain the statement of Theorem 3.2. $\square$

Due to Theorem 3.2, the $*$-algebra $B = C(Q)_C$ is $*$-isomorphic to a von Neumann subalgebra in $Z(M)$. Therefore $B$ is a commutative von Neumann algebra, and $*$-algebra $C_\infty(Q)_C$ is identified with $*$-algebra $S(B)$. In particular, there exists a separating family of completely additive scalar-valued measures on $B(F)$, and therefore $F$ is a Kantorovich-Pinsker space ($[11], 1.4.10$).

We claim that a version of Radon-Nikodym-type theorem is valid for a Maharam trace $\Phi$. For this, we need the space $L^1(M, \Phi)$ of operators from $S(M)$ to be integrable with respect to $\Phi$.
Let $F$ be a Kantorovich-Pinsker space and let $\Phi$ be an $F_\mathbb{C}$-valued Maharam trace on the von Neumann algebra $M$. The net $\{x_n\} \subset S(M)$ converges to $x \in S(M)$ with respect to the trace $\Phi$ (notation: $x_n \xrightarrow{\Phi} x$) if $\Phi(E_\lambda^+(|x_n - x|)) \xrightarrow{t(B)} 0$ for all $\lambda > 0$.

**Proposition 3.3.** $x_n \xrightarrow{\Phi} x$ iff $x_n \xrightarrow{t(M)} x$.

**Proof.** Let $\nu$ be a faithful normal semi-finite numerical trace on $B$. Choose $\{e_i\}_{i \in I}$ to be a set of nonzero mutually orthogonal projections from $P(B)$ with $\sup_{i \in I} e_i = 1_F$ and $\nu(e_i) < \infty, i \in I$. Set $\tau_i(x) = \nu(\Phi(x)\Phi(1)^{-1}e_i), x \in M, i \in I$. It is clear that $\{\tau_i\}_{i \in I}$ is a separating family of finite traces on $M$. Due to Proposition 2.2, $x_n \xrightarrow{t(M)} x$ if and only if $\tau_i(E_\lambda^+(|x_n - x|)) \rightarrow 0$ for all $\lambda > 0, i \in I$. The last convergence is equivalent to convergence $\Phi(E_\lambda^+(|x_n - x|)) \xrightarrow{t(B)} 0$. □

For each $x \in M$, let $\|x\|_\Phi = \Phi(|x|)$. Proposition 3.1 implies that $\| \cdot \|_\Phi$ is an $F$-valued norm on $M$. In addition, $\|x\|_\Phi = \|x^*\|_\Phi = \| |x| \|_\Phi$ and $\|ab\|_\Phi \leq \|a\|_M \|b\|_M \|x\|_\Phi$ for all $x, a, b \in M$.

We have $\Phi(E_\lambda^+(|x_n - x|)) \leq \frac{1}{\lambda} \Phi(|x_n - x|)$, $\lambda > 0, x_n, x \in M$. Hence $\|x_n - x\|_\Phi \xrightarrow{t(B)} 0$ implies $x_n \xrightarrow{\Phi} x$, and therefore $x_n \xrightarrow{t(M)} x$ (Proposition 3.3).

An operator $x \in S(M)$ is said to be $\Phi$-integrable if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(B)} 0$ as $n, m \rightarrow \infty$. Denote by $L^1(M, \Phi)$ the set of all $\Phi$-integrable operators from $S(M)$. It is clear that $M \subset L^1(M, \Phi)$ and $L^1(M, \Phi)$ is a linear subset of $S(M)$. It follows from Proposition 3.1 and 3.3 that $ML^1(M, \Phi)M \subset L^1(M, \Phi)$ and $x^* \in L^1(M, \Phi)$ for all $x \in L^1(M, \Phi)$.

We now define an $S_h(B)$-valued $L^1$-norm on $L^1(M, \Phi)$.

**Proposition 3.4.** If $x_n \in M$, $x_n \xrightarrow{\Phi} 0$, $\|x_n - x_m\|_\Phi \xrightarrow{t(B)} 0$, then $\Phi(|x_n|) \xrightarrow{t(B)} 0$.

**Proof.** Since $\|x_n\|_\Phi - \|x_m\|_\Phi \leq \|x_n - x_m\|_\Phi, \Phi(|x_n|) = \|x_n\|_\Phi$ is a Cauchy sequence in $(S(B), t(B))$. Because of the completeness of $\ast$-algebra $(S(B), t(B))$, there exists $f \in S_+(B)$ such that $\Phi(|x_n|) \xrightarrow{t(B)} f$. We claim that $f = 0$. First, we assume that algebra $B$ is $\sigma$-finite. Then there exists a faithful normal finite numerical trace $\nu$ on $B$. We have $\Phi(|x_n|) \xrightarrow{\nu} f$ and the sequence $\{\Phi(|x_n|)\}$ has an $(\sigma)$-convergent subsequence. Therefore, as usual, we may and do assume that the sequence $\{\Phi(|x_n|)\}$ $(\sigma)$-converges to $f$ in $S_h(B)$ (notation: $\Phi(|x_n|) \xrightarrow{\sigma}$).
\( f \). Hence, there exists \( g = \sup_{n \geq 1} \Phi(|x_n|) \) in \( S_h(\mathcal{B}) \). It is clear that
\[
\tau(x) = \nu(\Phi(x)(1_F + g + \Phi(1))^{-1})
\]
is a faithful normal finite numerical trace on \( M \). Since topologies \( t_\nu \) and \( t(\mathcal{B}) \) coincide, \( \Phi(|x_n - x_m|) \xrightarrow{\nu} 0 \). Therefore inequalities \( 0 \leq \Phi(|x_n - x_m|) \leq 2g \), imply \( \tau(|x_n - x_m|) \xrightarrow{\nu} 0 \). It is known that \( (L^1(M, \tau), \| \cdot \|_{1,\tau}) \) is complete, where \( \| x \|_{1,\tau} = \tau(|x|) \). Hence there exists \( x \in L^1(M, \tau) \subset S(M) \) such that \( \| x - x_n \|_{1,\tau} \xrightarrow{\nu} 0 \) and therefore, \( x_n \xrightarrow{t} x \). Because of the equality of topologies \( t_\tau \) and \( t(M) \), we have \( x = 0 \). This means that \( \tau(|x_n|) \xrightarrow{\nu} 0 \), i.e. \( \Phi(|x_n|) \xrightarrow{\nu} 0 \).

Now let \( \mathcal{B} \) be a general (not necessarily \( \sigma \)-finite) von Neumann algebra. For each \( 0 \neq e \in P(\mathcal{B}) \), we set \( \Phi_e(x) = \Phi(x)e, \ x \in M \). It is clear that \( \Phi_e \) is a normal \( S_h(\mathcal{B}e) \)-valued trace on \( M \), which does not have, generally speaking, the faithfulness property. A projection \( s(\Phi_e) = 1 - \sup\{ p \in P(M) : \Phi_e(p) = 0 \} \) is called the support trace of \( \Phi_e \). As well as in the case of numerical traces ( see, for example, [4], 5.15, 7.13), one can establish that \( s(\Phi_e) \in P(Z(M)) \) and \( \Phi_e(x) = \Phi_e(xs(\Phi_e)) \) is a faithful normal \( S_h(e\mathcal{B}) \)-valued trace on \( Ms(\Phi_e) \).

If \( \Phi(|x_n|) \xrightarrow{t(\mathcal{B})} 0 \), then there is a nonzero \( \sigma \)-finite projection \( e \in P(\mathcal{B}) \) such that \( \Phi(|x_n|)e \not\xrightarrow{\nu} 0 \) where \( \nu \) is a faithful normal finite numerical trace on \( \mathcal{B}e \). The last contradicts to what we proved above. \( \square \)

Let \( x \in L^1(M, \Phi) \), \( x_n \in M \), \( x_n \xrightarrow{\Phi} x \) and \( \| x_n - x_m \|_\Phi \xrightarrow{t(\mathcal{B})} 0 \). The inequality \( \| \Phi(x_n) - \Phi(x_m) \| \leq \Phi(|x_n - x_m|) \) and completeness of the *-algebra \( (S(\mathcal{B}), t(\mathcal{B})) \) guarantees the existence of \( \Phi(x) \in S(\mathcal{B}) \) such that \( \Phi(x_n) \xrightarrow{t(\mathcal{B})} \Phi(x) \). Due to Proposition 3.4, \( \Phi(x) \) does not depend on the choice of a sequence \( \{ x_n \} \subset M \), for which \( x_n \xrightarrow{\Phi} x \) and \( \| x_n - x_m \|_\Phi \xrightarrow{t(\mathcal{B})} 0 \), in particular, \( \Phi(x) = \Phi(x) \) for all \( x \in M \). The element \( \Phi(x) \) is called an \( S(\mathcal{B}) \)-valued integral of \( x \in L^1(M, \Phi) \) by a trace \( \Phi \).

It follows immediately from the definition of \( \Phi \) and Proposition 3.1 that \( \Phi \) is a linear mapping from \( L^1(M, \Phi) \) into \( S(\mathcal{B}) \) and \( \Phi(xy) = \Phi(yx) \) for any \( x \in M, y \in L^1(M, \Phi) \). For each \( x \in L^1(M, \Phi) \), we set \( \| x \|_\Phi = \Phi(|x|) \).

**Theorem 3.5.** (i) The mapping \( \| \cdot \|_\Phi \) is an \( S_h(\mathcal{B}) \)-valued norm on \( L^1(M, \Phi) \).

(ii) \( (L^1(M, \Phi), \| \cdot \|_\Phi) \) is a Banach-Kantorovich space.
Proof. (i) Let $x \in L^1(M, \Phi)$, $x_n \in M$, $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_{\Phi} \xrightarrow{t(B)} 0$. It follows from Propositions 2.2(i) and 3.3 that $|x_n| \xrightarrow{\Phi} |x|$. We claim that $\|x_n| - |x_m|\|_{\Phi} \xrightarrow{t(B)} 0$.

First, we assume that algebra $B$ is $\sigma$-finite. Using the same trick as in the proof of Proposition 3.4, we can show that $\Phi(x_n) \xrightarrow{(o) \Phi} \hat{\Phi}(x)$ in $S_h(B)$. Therefore there exists $g = \sup_{n \geq 1} |\Phi(x_n)|$ in $S_h(B)$. Consider a faithful normal finite numerical trace $\tau$ on $M$ defined by (4). Since $\tau(|x_n - x_m|) \to 0$ as $n, m \to \infty$ (see the proof of Proposition 3.4), there exists $y \in L^1(M, \tau)$ such that $\|y - x_n\|_{1, \tau} \to 0$. Then $x_n \xrightarrow{\tau} y$, and therefore $x = y$. Moreover, $|x_n| \xrightarrow{\tau} |x|$ (Proposition 2.2(i)) and $\|x_n\|_{1, \tau} = \|x_n\|_{1, \tau} \to \|x\|_{1, \tau}$. It follows from (10), Theorem 3.7 that $\|x| - |x_n|\|_{1, \tau} \to 0$, in particular, $\tau(|x_n| - |x_m|) \to 0$ as $n, m \to \infty$.

Convergence $\hat{\Phi}(\|x_n| - |x_m|)(1_F + g + \Phi(1))^{-1} \xrightarrow{[]} 0$ implies $\|x_n| - |x_m|\|_{\Phi} \xrightarrow{t(B)} 0$.

Hence, $|x| \in L^1(M, \Phi)$ and $\Phi(|x_n|) \xrightarrow{t(B)} \hat{\Phi}(|x|)$. In particular, $\|x\|_{\Phi} = \hat{\Phi}(|x|) \geq 0$ for all $x \in L^1(M, \Phi)$. If $\hat{\Phi}(|x|) = 0$, then $0 \leq |x_n|_{\Phi} = \Phi(|x_n|) \xrightarrow{t(B)} 0$. Hence, $x_n \xrightarrow{\Phi} 0$, and therefore $x = 0$.

Let now $B$ be not a $\sigma$-finite algebra. Let $\{e_i\}_{i \in I}$ be a family of nonzero mutually orthogonal $\sigma$-finite projections in $B$ with $\sup_{i \in I} e_i = 1_F$. Since

$$\sup_{i \in I} s(\Phi_{e_i}) = 1$$ and $\hat{\Phi}(|x|)e_i = \hat{\Phi}_{e_i}(|x|s(\Phi_{e_i})) \geq 0$ for all $i \in I$, we get $\hat{\Phi}(|x|) \geq 0$. Similarly, the equality $\Phi(|x|) = 0$ implies $\hat{\Phi}_{e_i}(|x|s(\Phi_{e_i})) = 0$, and therefore $|x|s(\Phi_{e_i}) = 0$ for all $i \in I$. Hence, $x = 0$.

Finally, we have

$$\|x + y\|_{\Phi} \leq \|x\|_{\Phi} + \|y\|_{\Phi}, \ x, y \in L^1(M, \Phi),$$

due to the inequality $|x + y| \leq u|x|u^* + v|y|v^*$, $x, y \in S(M)$ (see [7], §2.4) and the trick in Proposition 3.1(v).

(ii) Let $x \in L^1(M, \Phi)$, $x_n \in M$, $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_{\Phi} \xrightarrow{t(B)} 0$. Fix $m$ and set $y_{nm} = x_n - x_m$ for $n \geq m$. We have $y_{nm} \xrightarrow{\Phi} x - x_m$ and $\|y_{nm} - y_{km}\|_{\Phi} \xrightarrow{t(B)} 0$ as $n, k \to \infty$. It follows from the proof of (i) that $\Phi(|y_{nm}|) \xrightarrow{t(B)} \hat{\Phi}(|x - x_m|) = \|x - x_m\|_{\Phi}$. Since $\Phi(|y_{nm}|) \xrightarrow{t(B)} 0$ as $n, m \to \infty$, $\|x - x_m\|_{\Phi} \xrightarrow{t(B)} 0$.

Let us now show that any $(bo)$-Cauchy sequence in $(L^1(M, \Phi), \|\cdot\|_{\Phi})$ $(bo)$-converges.
First, we assume that \( \mathcal{B} \) is a \( \sigma \)-finite von Neumann algebra. Let \( \{x_n\} \subset L^1(M, \Phi) \) and \( \|x_n - x_m\|_\Phi \overset{(o)}{\to} 0 \). Since \( \widehat{\Phi} \) is a positive mapping (see the proof of item (i)), the inequality \( \widehat{\Phi}(E^1_{\lambda}(|x_n - x_m|)) \leq \frac{1}{\lambda}\Phi(|x_n - x_m|), \lambda > 0 \) is valid. Hence, \( \{x_n\} \) is a Cauchy sequence in \((S(M), t(M))\) and therefore there exists \( x \in S(M) \) such that \( x_n \overset{t(M)}{\to} x \). Choose a system \( \{U_n\} \) of closed neighborhoods of 0 in \((S(\mathcal{B}), t(\mathcal{B}))\) with \( U_{n+1} + U_{n+1} \subset U_n \), \( n = 1, 2, \ldots \). Due to what we proved above, for any \( x_n \in L^1(M, \Phi) \), there exists \( y_n \in M \) such that \( \|x_n - y_n\|_\Phi \in U_n \). Since 
\[
\sum_{n=k+1}^\infty \|x_n - y_n\|_\Phi \in U_k \text{ for all } m \geq k + 1,
\]
the series \( \sum_{n=k+1}^\infty \|x_n - y_n\|_\Phi \) converges in \((S(\mathcal{B}), t(\mathcal{B}))\). Hence, \( \|x_n - y_n\|_\Phi \overset{(o)}{\to} 0 \), and therefore \( \|y_n - y_m\|_\Phi \overset{(o)}{\to} 0 \). Also, by Proposition 3.3, we get \( x_n - y_n \overset{t(\mathcal{B})}{\to} 0 \), and consequently \( y_n \overset{\Phi}{\to} x \). This means that \( x \in L^1(M, \Phi) \), in addition, 
\[
\|x - y_n\|_\Phi \overset{t(\mathcal{B})}{\to} 0 \text{ and } \|y_n - y_m\|_\Phi \overset{t(\mathcal{B})}{\to} \|x - y_m\|_\Phi \text{ as } n \to \infty.
\]
Since \( \|x - y_m\|_\Phi \leq \sup_{n \geq m} \|y_n - y_m\|_\Phi \downarrow 0 \), we get \( \|x - y_m\|_\Phi \overset{(o)}{\to} 0 \) and therefore \( \|x - x_n\|_\Phi \overset{(o)}{\to} 0 \).

Now let \( \{x_\alpha\}_{\alpha \in A} \) be an arbitrary \((bo)\)-Cauchy net in \( L^1(M, \Phi) \), i.e. \( \sup_{\alpha, \beta \geq \gamma} \|x_\alpha - x_\beta\|_\Phi \downarrow 0 \). We choose a sequence of indices \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \ldots \) in \( A \) such that \( \sup_{n, m \geq k} \|x_\alpha - x_\beta\|_\Phi \in U_k \). Then \( \sup_{\beta \geq \alpha_n} \|x_\alpha - x_\beta\|_\Phi \in U_k \), and therefore \( \{x_\alpha\} \) is a \((bo)\)-Cauchy sequence in \( L^1(M, \Phi) \). It follows from what we proved above that there exists \( x \in L^1(M, \Phi) \) such that \( \|x - x_\alpha\|_\Phi \overset{(o)}{\to} 0 \). Let us claim that \( \|x - x_\alpha\|_\Phi \overset{(o)}{\to} 0 \), i.e. \( \sup_{\alpha \geq \beta} \|x - x_\alpha\|_\Phi \downarrow 0 \). Fix \( \beta \in A \) and consider the net \( \{x_\alpha\}_{\alpha \geq \beta} \). We construct a sequence of indices \( \beta \leq \alpha_1 \leq \alpha_2 \leq \ldots \) such that \( \alpha_n \leq \beta_n \). Then \( \|x_\beta_n - x_\alpha_n\|_\Phi \in U_n \), and therefore \( \|x_\beta_n - x_\alpha_n\|_\Phi \overset{(o)}{\to} 0 \). Hence, 
\[
\|x - x_\beta_n\|_\Phi \overset{(o)}{\to} 0 \text{ and } \|x_\beta_n - x_\beta\|_\Phi \overset{(o)}{\to} \|x - x_\beta\|_\Phi \text{ as } n \to \infty.
\]
Thus, 
\[
\|x - x_\beta\|_\Phi \leq \sup_{n \geq 1} \|x_\beta_n - x_\beta\|_\Phi \leq \sup_{\alpha \geq \beta} \|x_\alpha - x_\beta\|_\Phi \|x - x_\beta\|_\Phi \overset{(o)}{\to} 0.
\]

Let now \( \mathcal{B} \) be not a \( \sigma \)-finite algebra and let \( \{x_\alpha\} \) be a \((bo)\)-Cauchy net in \( L^1(M, \Phi) \). Due to the completeness of \((S(M), t(M))\), there is \( x \in S(M) \) such that \( x_\alpha \overset{\Phi}{\to} x \). Let \( \{e_i\}_{i \in I} \) be the same family of projections in \( \mathcal{B} \), as in the proof of (i). It is clear that \( \{x_\alpha s(\Phi_{e_i})\} \) is a \((bo)\)-Cauchy net in \( L^1(M s(\Phi_{e_i}), \Phi_{e_i}) \), and therefore, by virtue of what we proved above, there exists \( x_i \in L^1(M s(\Phi_{e_i}), \Phi_{e_i}) \) such that \( \|x_i -
$x_{α}s(Φ_{ε_i})∥Φ_{ε_i} \xrightarrow{(o)} 0$. Convergence $x_{α}s(Φ_{ε_i}) \xrightarrow{Φ} x_{α}s(Φ_{ε_i})$ implies $x_i = x_{α}s(Φ_{ε_i})$ for all $i ∈ I$. Thus, $\hat{Φ}|x - x_{α}|e_i = \hat{Φ}|x - x_{α}|s(Φ_{ε_i})e_i \xrightarrow{(o)} 0$ and $∥x - x_{α}∥_Φ \xrightarrow{(o)} 0$.

Hence, $(L^1(M, Φ), ∥ · ∥_Φ)$ is a (bo)-complete lattice-normed space.

Now let us show that $(L^1(M, Φ), ∥ · ∥_Φ)$ is a Banach-Kantorovich space, i.e. for any element $x ∈ L^1(M, Φ)$ and any decomposition $∥x∥_Φ = f_1 + f_2$, $f_1, f_2 ∈ S_+(B)$, $f_1 ∧ f_2 = 0$, there exist $x_1, x_2 ∈ L^1(M, Φ)$ such that $x = x_1 + x_2$ and $∥x_i∥_Φ = f_i$, $i = 1, 2$.

Set $e_i = s(f_i)$. It is clear that $e_i ∈ P(B)$, $e_1 + e_2 = 0$, $e_1 + e_2 = s(∥x∥_Φ)$. Since $Φ$ is a Maharam trace, we have $Φ\left(\left| \sum_{i} s(Φ_{ε_i}) e_i \right| \right) = Φ\left(\left| \sum_{i} s(Φ_{ε_i}) e_i \right| \right)$ (see Theorem 3.2).

Let $p_i = Ψ^{-1}(e_i)$, $x_i = xp_i$. Since $p_i ∈ P(\mathcal{A}) ⊂ P(Z(M))$, $∥x_i∥ = |x|p_i ∈ L^1(M, Φ)$. We choose $y_n ∈ M$ such that $y_n \xrightarrow{Φ} x$ and $∥y_n - y_m∥_Φ \xrightarrow{t(B)} 0$. Then $|y_n| \xrightarrow{Φ} |x|$, $∥y_n - |y_m∥_Φ \xrightarrow{t(B)} 0$ and $Φ(|y_n|) \xrightarrow{t(B)} Φ(|x|)$ (see the proof of (i)). Set $y_n^{(i)} = y_np_i$, $i = 1, 2$. We have $|y_n^{(i)}| \xrightarrow{Φ} |x_i|$ and $∥|y_n^{(i)}| - |y_m^{(i)}∥_Φ \leq ∥|y_n| - |y_m∥_Φ$. Hence, $Φ(|y_n^{(i)}|) \xrightarrow{t(B)} Φ(|x_i|)$. Due to the property 3) from Theorem 3.2, we have $Φ(|y_n^{(i)}|) = Φ(p_i)Φ(|y_n|) = e_iΦ(|y_n|)$.

Thus, $∥x_i∥_Φ = Φ(|x_i|) = e_iΦ(|x|) = f_i$, in addition $x_1 + x_2 = x(p_1 + p_2) = xψ^{-1}(s(∥x∥_Φ))$. As well as above, one can establish that $qΦ(|x|) = Φ(|x|ψ^{-1}(q))$ for all $q ∈ P(B)$. Taking $q = 1_F - s(∥x∥_Φ)$, we get $Φ(|x|)(1 - ψ^{-1}(s(∥x∥_Φ))) = 0$. Hence, $|x| = |x|ψ^{-1}(s(∥x∥_Φ))$. Using the polar decomposition $x = u|x|$, we obtain $x = xψ^{-1}(s(∥x∥_Φ)) = x_1 + x_2$.

Note another useful properties of mapping $\hat{Φ}$.

Let $Φ, M, Q, Φ_M, A, Ψ$ be the same as in Theorem 3.2. $B = C(Q)_C$. It is clear that the $*$-isomorphism $Ψ$ from $A$ onto $B$ can be extended to the $*$-isomorphism from $S(A)$ onto $S(B)$. We denote this mapping also by $Ψ$.

**Proposition 3.6.** $S(A)L^1(M, Φ) ⊂ L^1(M, Φ)$, in particular, $S(A) ⊂ L^1(M, Φ)$, in addition, $\hat{Φ}(zx) = ψ(z)\hat{Φ}(x)$ and $\hat{Φ}(Φ_M(zx)) = Φ(zx)$ for all $z ∈ S(A), x ∈ L^1(M, Φ)$.

**Proof.** It is sufficient to show that $x ∈ L^1_+(M, Φ), z ∈ S_+(A)$ implies $zx ∈ L^1_+(M, Φ)$ and $\hat{Φ}(zx) = ψ(z)\hat{Φ}(x), \hat{Φ}(Φ_M(zx)) = Φ(zx)$.

Let $z_n = E_n(z)n$. It is clear that $z_n ∈ A_+, z_n ↑ z, z_n x ∈ L^1_+(M, Φ)$. Since $z_n x = \sqrt{x}z_n \sqrt{x} ↑ \sqrt{x}z \sqrt{x} = zx$, we get $ψ(z_n)\hat{Φ}(x) = \hat{Φ}(z_n x) ≤ \hat{Φ}(z_{n+1} x) = ψ(z_{n+1})\hat{Φ}(x) ↑ ψ(z)\hat{Φ}(x)$.
In addition, \( \psi \) all

Hence,

\[
\sup_{n \geq m} \| z_n x - z_m x \|_\Phi = \sup_{n \geq m} | \hat{\Phi}(z_n x) - \hat{\Phi}(z_m x) | \downarrow 0,
\]

i.e. \( \{ z_n x \} \) is a \((bo)\)-Cauchy sequence. By Theorem 3.5 there exists \( y \in L^1(M, \Phi) \) such that \( \| z_n x - y \|_\Phi \xrightarrow{\omega} 0 \). The inequality \( \Phi(E^+_{\lambda}(|z_n x - y|)) \leq \frac{1}{\Delta} \Phi(|z_n x - y|) \) implies \( z_n x \xrightarrow{\Phi} y \). Therefore \( y = zx \), i.e. \( zx \in L^1(M, \Phi) \).

In addition, \( \psi(z_n) \hat{\Phi}(x) = \hat{\Phi}(z_n x) = \| z_n x \|_\Phi \xrightarrow{\text{loc}} \| zx \| = \hat{\Phi}(zx) \). Hence, \( \hat{\Phi}(zx) = \psi(z) \hat{\Phi}(x) \).

Set \( x_k = E_k(x)x \). Then \( 0 \leq x_k \uparrow x \), \( x_k \in M \). By virtue of Proposition 3.1(ii), \( \Phi(z_n x_k) = \Phi(\Phi_M(z_n x_k)) = \Phi(z_n \Phi_M(x_k)) \). Since \( (z_n x_k) \uparrow (z_n x) \) as \( k \to \infty \), we have \( \Phi(z_n x_k) \uparrow \hat{\Phi}(z_n x) \) and \( \Phi(\Phi_M(z_n x_k)) \uparrow \hat{\Phi}(\Phi_M(z_n x)) \).

Therefore \( \hat{\Phi}(z_n x) = \hat{\Phi}(\Phi_M(z_n x)) \) for all \( n = 1, 2, \ldots \). After switching to the limit as \( n \to \infty \), we obtain \( \hat{\Phi}(zx) = \hat{\Phi}(\Phi_M(zx)) \).

Let \( \Phi \) be an \( F_C \)-valued Maharam trace on \( M \) and let \( \Psi \) be a normal \( F_C \)-valued trace on \( M \). A trace \( \Psi \) is called absolutely continuous with respect to \( \Phi \) (notation \( \Psi \ll \Phi \)) if \( s(\Psi(p)) \leq s(\Phi(p)) \) for all \( p \in P(M) \). The last condition is equivalent to inclusion \( \Psi(p) \in \{ \Phi(p) \}^{\downarrow \downarrow} = s(\Phi(p))S_h(B), p \in P(M) \) where \( B^\perp := \{ x \in S_h(B) : (\forall y \in B)|x| \wedge |y| = 0 \} \) for a nonempty subset \( B \subset S_h(B) \) (compare with [1], 6.1.11).

The next theorem is a non-commutative version of the Radon-Nikodym-type theorem for Maharam traces.

**Theorem 3.7.** Let \( \Phi \) be an \( F_C \)-valued Maharam trace on the von Neumann algebra \( M \). If \( \Psi \) is a normal \( F_C \)-valued trace on \( M \) absolutely continuous with respect to \( \Phi \), then there exists an operator \( y \in L^1_{+}(M, \Phi) \cap S(Z(M)) \) such that

\[
\Psi(x) = \hat{\Phi}(yx)
\]

for all \( x \in M \).

**Proof.** Let \( l \) be the restriction of \( \Psi \) on the complete Boolean algebra \( P(Z(M)) \), and let \( m \) be the restriction of \( \Phi \) on \( P(Z(M)) \). Obviously, \( l \) and \( m \) are \( S_h(B) \)-valued completely additive measures on \( P(Z(M)) \). In addition, \( m(ze) = \psi(z)m(e) \) for all \( z \in P(A), e \in P(Z(M)) \) (see Theorem 3.2). Hence, \( m \) is a \( \psi \)-modular measure on \( P(Z(M)) \) (see [1], 6.1.9). Since the measure \( l \) is absolutely continuous with respect to \( m \), by the Radon-Nikodym-type theorem from (1], 6.1.11), there exists \( y \in L^1_{+}(Z(M), m) = L^1_{+}(Z(M), \Phi) \) such that \( l(e) = \hat{\Phi}(ye) \) for all \( e \in P(Z(M)) \).
If \( a = \sum_{i=1}^{n} \lambda_i e_i \) is a simple element from \( Z(M) \), where \( \lambda_i \in \mathbb{C} \), \( e_i \in P(Z(M)) \), \( i = 1, \ldots, n \), then \( \Psi(a) = \sum_{i=1}^{n} \lambda_i \Psi(e_i) = \sum_{i=1}^{n} \lambda_i \Phi(ye_i) = \Phi(ya) \). Let \( a \in Z_+(M) \) and \( \{a_n\} \) be a sequence of simple elements from \( Z_+(M) \) with \( a_n \uparrow a \). Then \( \Psi(a_n) \uparrow \Psi(a) \), \( ya_n \uparrow ya \), and \( \Phi(ya_n) \uparrow \Phi(ya) \) (see the proof of Proposition \( \ref{prop:Maharam property} \)). Hence, \( \Psi(a) = \Phi(ya) \) for all \( a \in Z_+(M) \). Now using the linearity of traces \( \Psi \) and \( \Phi \), we obtain \( \Psi(a) = \Phi(ya) \) for all \( a \in M \).

Furthermore, due to Propositions \( \ref{prop:Properties of traces} \)(iv) and \( \ref{prop:Maharam property} \) we get
\[
\Psi(x) = \Psi(\Phi_M(x)) = \Phi(y\Phi_M(x)) = \Phi(\Phi_M(yx)) = \Phi(yx)
\]
for all \( x \in M \).

\( \square \)

**Remark 3.8.** If \( \Psi \) is a normal \( F_\mathbb{C} \)-valued trace on \( M \) and \( \Psi \ll \Phi \), then \( \Psi \) possesses the Maharam property.

In fact, by Theorem \( \ref{thm:Maharam property} \), \( \Psi(x) = \Phi(yx) \) for all \( x \in M \) where \( y \in L_1^+(M, \Phi) \cap S(Z(M)) \). Let \( 0 \neq x \in M_+ \), \( f \leq \Psi(x) \), \( f \in S_+(\mathcal{B}) \), \( g \in S_+(\mathcal{B}) \), \( g\Psi(x) = s(\Psi(x)) \). Set \( h = gf \), \( z = \psi^{-1}(h) \), \( a = zx \). Then
\[
0 \leq h \leq g\Psi(x) = s(\Psi(x)) \leq 1_F, \quad 0 \leq z \leq 1, \quad 0 \leq a \leq x
\]
and
\[
\Psi(a) = \Phi(ya) = \Phi(zyx) = \psi(z)\Phi(yx) = h\Psi(x) = fs(\Psi(x)) = f.
\]

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