The Standard Model $\beta$-function and a matrix model renormalization of Yukawa interactions

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**Abstract**

We show that gauge-independent terms in the one-loop and multi-loops $\beta$-functions of the Standard Model can be equivalently computed from the Wetterich functional renormalization of a matrix model. Our framework is associated to the finite spectral triple underlying the computation of the Standard Model Lagrangian from the spectral action of Noncommutative Geometry. This matrix-Yukawa duality for the $\beta$-function provides a novel computational approach for multi-loop $\beta$-functions of particle physics models.

1 Introduction

The Standard Model is thus far the most accurate model for particle physics, successfully managing to formulate electromagnetism and the weak and strong nuclear forces in one unified framework. As it is formulated in the language of Quantum Field Theory, the parameters of its Lagrangian are subject to renormalization, and run with the energy scale.

The Standard Model is usually renormalized using perturbative quantum field theoretic techniques, in particular loop expansions and computations of Feynman diagrams, [2]. Those techniques, however, are computationally expensive and difficult to implement. In this text we present a new way to look at the renormalization of Yukawa interactions in the Standard Model extended with massive Dirac neutrinos, using the framework of noncommutative geometry of [8] and nonperturbative techniques from the theory of the functional renormalization group, [16] applied to the computationally simpler setting of a matrix model.

Noncommutative geometry provides a simple and elegant geometric framework, in terms of finite spectral triples and almost-commutative geometries, that makes it possible to compute the full Standard Model Lagrangian from simple geometric data (see [8], [9]), using the spectral action formalism of [7]. We show here that the gauge-independent Yukawa terms in the $\beta$-function of the Standard Model with massive Dirac neutrinos can be computed using a matrix model associated to the finite noncommutative spectral triple underlying the derivation of the Standard Model Lagrangian in [8]. The idea of considering finite spectral triples of noncommutative geometry as matrix models was developed in [4], [5] and in ongoing work [3]. Moreover, a general framework for the renormalization of almost-commutative geometries was developed in [14], [15].

2 Yukawa interactions and the Higgs mechanism

We consider here the minimal Standard Model extended with massive Dirac neutrinos. The Yukawa interactions couple the Higgs boson to the fermionic fields through a Lagrangian of the form

$$L_{\text{Yuk}} := -Y_{\nu}^{ij} \bar{\nu}_L^{ij} Q_L^{ii} \epsilon H - Y_{d}^{ij} \bar{d}_R^{ij} Q_L^{ii} \epsilon H^\dagger - Y_{u}^{ij} \bar{u}_R^{ij} Q_L^{ii} \epsilon H^\dagger - Y_{\bar{e}}^{ij} \bar{e}_R^{ij} L_L^{ii} \epsilon H^\dagger - Y_{\nu}^{ij} \bar{\nu}_R^{ij} L_L^{ii} \epsilon H^\dagger + \text{hc}. \quad (1)$$
Here $H$ is the Higgs boson field, the $Y$’s are the Yukawa matrices, $L$ and $Q$ are the lepton and quark doublets and $u, d, e, \nu$ the up, down, electron and neutrino singlets.

In what follows, we shall be primarily interested in the renormalization of the Yukawa matrices. The $\beta$-functions for the running of those couplings at one loop in the unbroken phase of the theory, following the results of [2], [1], take the form

$$16\pi^2 \partial_t Y_e = Y_e \left( \frac{3}{2} Y_e^d Y_e - \frac{3}{2} Y_e^u Y_e + \text{Tr}(Y_e^d Y_e + Y_e^u Y_e + 3Y_e^d Y_e + 3Y_e^u Y_e) + \text{gauge terms} \right),$$

$$16\pi^2 \partial_t Y_e = Y_e \left( \frac{3}{2} Y_e^d Y_e - \frac{3}{2} Y_e^u Y_e + \text{Tr}(Y_e^d Y_e + Y_e^d Y_e + 3Y_e^d Y_e + 3Y_e^u Y_e) + \text{gauge terms} \right),$$

$$16\pi^2 \partial_t Y_d = Y_d \left( \frac{3}{2} Y_d^d Y_d - \frac{3}{2} Y_d^u Y_e + \text{Tr}(Y_d^d Y_d + Y_d^d Y_d + 3Y_d^d Y_d + 3Y_d^u Y_e) + \text{gauge terms} \right),$$

$$16\pi^2 \partial_t Y_u = Y_u \left( \frac{3}{2} Y_u^d Y_u - \frac{3}{2} Y_u^d Y_u + \text{Tr}(Y_u^d Y_u + Y_u^d Y_u + 3Y_u^d Y_u + 3Y_u^d Y_u) + \text{gauge terms} \right),$$

in the variable $t = \log(\mu/\mu_0)$, with $\mu$ the energy scale. At the electroweak phase transition, the theory undergoes spontaneous symmetry breaking and $H$ acquires a vacuum expectation value for its second component. The charged component is then cancelled by the gauge freedom, and there remains only a neutral, scalar Higgs field.

The $\beta$-function of the Standard Model at one, two, and three loops is discussed in [2], [12] and [6], with explicit computations, while to our knowledge no full explicit computation for higher loop order is available in the literature. One of the purposes of this note is to show that the matrix model method presented here provides a simpler path to the explicit computation of some higher loop contributions.

3 Yukawa parameters and renormalization of the finite non-commutative geometry

It is shown in [8] (see also Chapter 1 of [8]) that the full Lagrangian of the Standard Model extended with right-handed neutrinos and Majorana mass terms can be computed from the spectral action principle of [2] applied to geometric fluctuations of an almost commutative space, which is the product of a four dimensional smooth manifold and a discrete noncommutative space (a finite spectral triple). Over this finite geometry the Dirac operator $D$ is a finite dimensional self-adjoint matrix. We recall here briefly the form of the finite spectral triple of [8] and then we focus on the case without Majorana masses but with non-trivial Dirac masses and the resulting Yukawa terms.

3.1 The finite spectral triple of the Standard Model

A finite spectral triple is a datum $(A, H, D)$ of a finite dimensional complex $C^*$-algebra $A$ acting on a finite dimensional Hilbert space $H$, together with a self-adjoint linear operator $D$ on $H$. A finite spectral triple $(A, H, D)$ is even if there is a $\mathbb{Z}/2\mathbb{Z}$-grading $\gamma$ on $H$ with $\gamma^2 = \gamma$, $\gamma^2 = 1$, $[\gamma, a] = 0$ for all $a \in A$ and $\gamma D + D\gamma = 0$. Moreover, a finite spectral triple $(A, H, D)$ has a real structure if there exists an anti-unitary operator $J : H \to H$ with the properties that $a^0 := Ja^*J$ defines a right action of $A$ on $H$ with $[a, b^0] = 0$, for all $a, b \in A$ and satisfying the “order-one condition” $[[D, a], b^0] = 0$ for all $a, b \in A$. Moreover, the anti-unitary $J : H \to H$ should satisfy

$$J^2 = \varepsilon, \quad JD = \varepsilon\epsilon'DJ, \quad \text{and} \quad J\gamma = \varepsilon\gamma'J$$
with \(c, \epsilon', \epsilon'' \in \{\pm 1\}\) where the third condition applies if the spectral triple is even.

The left-right symmetric spectral triple of [8] has a real algebra \(A_{LR,R} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})\), where \(\mathbb{H}\) is the real algebra of quaternions, and Hilbert space \(H\) is obtained by taking the sum \(M\) of all irreducible odd bimodule for the left-right symmetric algebra, where the odd condition means that the involution \(s = (1, -1, -1, 1)\) in the algebra acts by \(Ad(s) = -1\). (These are representations of the complex algebra \(B = \oplus^{4-times} M_2(\mathbb{C}) \oplus M_6(\mathbb{C})\).) The Hilbert space is then given by a direct sum of three (the number of generations) copies of this bimodule \(M\) of dimension \(\dim \mathbb{C} M = 32\). The bimodule \(M\) decomposes into two parts (matter/antimatter sectors) \(M = E \oplus E^o\) interchanged by the real structure \(J(\xi, \eta) = (\eta, \xi)\), while the grading is given by \(\gamma = c - JcJ\) with \(c = (0, 1, -1, 0) \in A_{LR,R}\), with \(J^2 = 1\) and \(J\gamma = -\gamma J\) (KO-dimension six). We refer the reader to [8] and to Chapter 1 of [9] for the details of this construction and the explicit identification of a basis of \(H\) with the fermion fields of the Standard Model. The main point that we need to recall here is the fact that the structure of the Dirac operator is determined by the self-adjointness and the order-one condition. In particular, it is shown in [8] that if one requires that the Dirac operator \(D\) intertwines the three copies of \(E\) with the three copies of \(E^o\) in the Hilbert space, then the largest subalgebra on which the order-one condition can be satisfied is \(A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\), with \(\mathbb{C}\) embedded diagonally in the complex numbers and one of the two copies of the quaternions in \(A_{LR,R}\), breaking the left-right symmetry. The term in the Dirac operator intertwining \(E\) and \(E^o\) sectors corresponds to the Majorana mass terms, see [8]. The general form of the Dirac operator is then shown to be given by

\[
D = \begin{pmatrix} S & T^s \\ T & S^* \end{pmatrix}
\]

with \(T\) the Majorana masses term, and with matrices

\[
S_\ell = \begin{bmatrix} 0 & 0 & Y_{\nu}^\dagger & 0 \\ 0 & 0 & 0 & Y_e^\dagger \\ Y_{\nu} & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_q = \begin{bmatrix} 0 & 0 & Y_{u}^\dagger & 0 \\ 0 & 0 & 0 & Y_d^\dagger \\ Y_u & 0 & 0 & 0 \\ 0 & Y_d & 0 & 0 \end{bmatrix}, \tag{6}
\]

where \(Y_{\nu}, Y_e, Y_u, Y_d\) are, respectively, the Yukawa terms for the neutrinos, charged leptons, \(u/c/t\) and \(d/s/b\) quarks. These include the masses and the Cabibbo–Kobayashi–Maskawa and Pontecorvo–Maki–Nakagawa–Sakata mixing matrices, for the quarks and lepton sectors, respectively. In the following, we consider this same finite noncommutative geometry, in the absence of the intertwining term \(T\).

### 3.2 Renormalization of the finite geometry

We consider first the explicit form of the traces of even powers of the Dirac operator of the finite spectral triple. In the absence of Majorana masses for neutrinos, \(D\) is constituted of blocks of \(3 \times 3\) matrices, which are the Yukawa couplings of the theory, in the form

\[
D = \text{diag}(S_\ell, S_q, S_q, S_q, S_q, S_q, S_q, S_q, S_q).
\tag{7}
\]

with \(S_\ell\) and \(S_q\) as in (6). The traces of powers of the Dirac operator are therefore of the form

\[
\frac{1}{4} \text{Tr}(D^{2n}) = \text{Tr}((Y_{\nu}^\dagger Y_{\nu})^n) + \text{Tr}((Y_e^\dagger Y_e)^n) + 3\text{Tr}((Y_u^\dagger Y_u)^n) + 3\text{Tr}((Y_d^\dagger Y_d)^n). \tag{8}
\]

A natural question to ask is then how one can directly renormalize the Dirac operator. We consider the one-loop \(\beta\)-function of the Standard Model, as it is usually computed via perturbative quantum field theory, and we reinterpret it as an equation for the Dirac operator of the finite spectral triple.
At one loop, in the unbroken phase of the theory, the beta functions of its successive powers can be expressed as

\[ 16\pi^2 \partial_t \text{Tr}(D^{2n}) = 3n\text{Tr}(D^{2n+2}) + \frac{n}{2}\text{Tr}(D^2)\text{Tr}(D^{2n}) - 3n\text{Tr}(D^{2n}\bar{D}) + \text{gauge terms}, \tag{9} \]

where \(\bar{D}\) is the twisted Dirac operator, which is obtained by switching respectively the up and down, and electron and neutrino couplings.

The main idea here is that all gauge independent terms in the one loop \(\beta\)-functions of traces of even powers of the Dirac operator can be expressed in terms of traces of even powers of the Dirac operator, and possibly of \(\bar{D}\).

Notice that information about those successive traces is enough to know the eigenvalues of the square of the Dirac operator, and therefore the physical parameters of the Yukawa sector - as long as they are non-degenerate, as solving for them reduces to a Vandermonde determinant.

The three terms in (9) have different origins in the renormalization process. The term involving \(\text{Tr}(D^2)\) comes from the fermion loop correction to the Higgs propagator, and the term involving the twist comes from charged Higgs interactions, which are suppressed by spontaneous symmetry breaking.

In [8], [9] it was shown that the possible choices of Dirac operators for the finite noncommutative geometry underlying the Standard Model with right-handed neutrinos and Majorana mass terms are parameterized by a moduli space given by a product \(\mathcal{C}_3 \times \mathcal{C}_1\), where \(\mathcal{C}_3\) consists of pairs \((Y_d, Y_u)\) modulo the equivalence \(Y_d' = W_1 Y_d W_3^*\) and \(Y_u' = W_2 Y_u W_3^*\) implemented by unitary matrices \(W_j\), so that

\[ \mathcal{C}_3 = (U(3) \times U(3))\backslash GL_2(\mathbb{C}) \times GL_2(\mathbb{C})/U(3). \tag{10} \]

This part of the moduli space of finite Dirac operators accounts for all the Yukawa parameters of the quark sector. The lepton part \(\mathcal{C}_1\) is a fibration over another copy of \(\mathcal{C}_3\), which accounts for the Yukawa terms for the leptons including Dirac neutrino masses and a Pontecorvo–Maki–Nakagawa–Sakata mixing matrix, and with fiber the set of symmetric matrices \(Y_R\) modulo the scaling equivalence \(Y_R \mapsto \lambda^2 Y_R\), which accounts for the Majorana terms. If we consider only the extension of the minimal standard model by Dirac neutrino masses without the Majorana terms, then the moduli space of the finite Dirac operators consists of a product of two copies of the quotient of (10).

The flow equation for the Dirac operator obtained in [9] from the one loop \(\beta\)-function of the standard model with Dirac neutrino masses can be interpreted as resulting from a flow on the moduli space of Dirac operators. Indeed, the fact that the flow is expressed in terms of traces of powers of the Yukawa matrices ensures that it is completely determined by invariants of the equivalence relation of (10).

In the next section we show that the equation (9) can in fact be obtained as a renormalization group flow for the finite noncommutative geometry, by constructing a matrix model associated to the finite Dirac operator and computing its renormalization group flow in terms of Wetterich’s functional renormalization.

4 Nonperturbative renormalization and the Functional Renormalization Group

We briefly review the setting of functional renormalization, adapted to our context. Since the results are about fixed size matrix models, we will present the theory in such a context, which has, to our knowledge, has never been done explicitly in the literature. For simplicity, we will assume that all matrices we use are Hermitian.
Let us consider a matrix field theory described by a generating functional

$$Z[J] = \int D[\tilde{A}] e^{-S[\tilde{A}]+\text{Tr}(J\tilde{A})},$$

(11)

where the integration variable $A$ is a finite size matrix of fixed dimensions and $J$ is a matrix of the same size. Note that this is the exact analogue of a quantum field theoretic path integral, for which Functional Renormalization is more frequently used, except here there is a finite number of degrees of freedom in the theory, because fields are matrices.

The idea is to introduce an additional energy scale $t$ dependent term in the exponential to account for the effects of renormalization and cancel the IR divergences of the model. This term has the form

$$\Delta S_t[\tilde{A}] = \frac{1}{2} \text{Tr}(\tilde{A}R_t\tilde{A}),$$

(12)

where $R_t$, called the regulator, has the size of the tensor product of $A$ with itself. It depends in general on the energy $q$, and it accounts for the effects of renormalization. It is chosen freely except for the three following properties:

$$\lim_{t\to 0} R_t(q) = 0.$$  

(13)

$$\lim_{t\to \infty} R_t(q) = \infty.$$  

(14)

$$\lim_{q\to 0} R_t(q) > 0.$$  

(15)

These properties have a precise meaning. Namely, (13) ensures that the theory has a classical limit, while (14) ensures that all modes are coupled in the UV. Finally, (15) shows that $R_t(q)$ behaves as an IR regulator.

The energy dependent generating functional $Z_t[J]$ then becomes

$$Z_t[J] = e^{-W_t[J]} = \int D[\tilde{A}] e^{-S[\tilde{A}]-\Delta S_t[\tilde{A}]+\text{Tr}(J\tilde{A})},$$

(16)

where $W_t[J]$ is the energy scale dependent connected generating function. It then follows that

$$\partial_t W_t = -\frac{1}{2} \text{Tr} \left( (\partial_t R_t) W_t^{(2)} \right) - \frac{1}{2} \text{Tr}(\tilde{A}(\partial_t R_t)\tilde{A}).$$

(17)

Then the background field $A$ is defined by

$$A_{ij} = \frac{\partial W_t[J]}{\partial J_{ij}},$$

(18)

and the energy scale dependent effective action as a modified Legendre transform of $W_t[J]$ is given by

$$\Gamma_t[A] := \text{Tr}(J\tilde{A}) - W_t[J] - \Delta S_t[A].$$

(19)

The identity

$$\delta_{ij} = \sum_k \left( \frac{\partial A}{\partial J} \right)_{ik} \left( \frac{\partial J}{\partial A} \right)_{kj}$$

(20)

implies that

$$W_t^{(2)} = (\Gamma_t^{(2)} + R_t)^{-1}.$$  

(21)

Deriving (19) and using identities (21) and (17) yields the Wetterich equation, which describes how the effective action of the theory flows with the energy scale. We have

$$\partial_t \Gamma_t = \frac{1}{2} \text{Tr} \left( \frac{\partial_t R_t}{R_t + \Gamma_t^{(2)}} \right).$$

(22)
The Wetterich equation is a powerful nonperturbative tool to renormalize a quantum field theory. In the next section we will see how to renormalize the Yukawa parameters by applying the Wetterich equation to a matrix model.

5 A matrix–Yukawa duality for $\beta$-functions

In this section we derive the $\beta$-functions for a matrix model using the Wetterich equation. We then comment on how they can be reduced to the ones of Yukawa couplings.

Let us start with the most general setting possible for clarity. Consider a theory with an effective action

$$ \Gamma[A] := Y + Z \text{Tr}(A^2) + \eta \sum_{n=2}^{\infty} M_{2n}(D, A), $$

where $D$ is the Dirac operator of the previously discussed discrete geometry, $A$ is a matrix of the same size which we consider as our field, and $M_{2n}$ is the trace of an even monomial in $D$ and $A$, of partial degree $2n$ in $A$.

The Wetterich equation applied to that effective action gives

$$ \partial_t \left( Y + Z \text{Tr}(A^2) + \eta \sum_{n=2}^{\infty} M_{2n}(D, A) \right) = \frac{1}{2} \text{Tr} \left( \frac{\partial_t R_t}{(R_t + Z) \mathbb{1} + \eta \sum_{n=2}^{\infty} M_{2n}(D, A)^{(2)}} \right). $$

We now choose our regulator $R_t \mathbb{1}$ in such a way that we have

$$ R_t + Z = C. $$

where $C$ is a scalar function which will be specified later. A perturbative expansion at first order in $\eta$ can then be performed, giving

$$ \partial_t \left( Y + Z \text{Tr}(A^2) + \eta \sum_{n=2}^{\infty} M_{2n}(D, A) \right) = \frac{1}{2C} (\partial_t R_t) \text{Tr} \left( \mathbb{1} - \frac{\eta}{C} \sum_{n=2}^{\infty} M_{2n}(D, A)^{(2)} \right). $$

In what follows we will use the following truncation ansatz: first set $A$ to $\epsilon \mathbb{1}$, and then identify the terms with the same power dependence in $\epsilon$.

We now apply this method to a specific theory. We define the effective action as

$$ \Gamma[A] := Y + Z \text{Tr}(A^2) + \eta \left( \sum_{n=2}^{\infty} a_{2n} \text{Tr}(D^{2n-2} A^{2n}) + b_{2n} \text{Tr}(D^2 A^2 D^{2n-4} A^{2n-2}) \right), $$

where the $a_{2n}$ and $b_{2n}$ do not vary with the energy scale. We will further specify those constants below.

The simplified Wetterich equation applied to $A = \epsilon \mathbb{1}$ then reads

$$ \partial_t \left( Y + Z \epsilon^2 \text{Tr}(\mathbb{1}) + \eta \left( \sum_{n=2}^{\infty} \epsilon^{2n}(a_{2n} + b_{2n}) \text{Tr}(D^{2n-2}) \right) \right) = $$

$$ \frac{1}{2C} (\partial_t R_t) (\text{Tr}(\mathbb{1}) - \frac{\eta}{C} \sum_{n=2}^{\infty} \epsilon^{2n-2} (2n(2n-1)a_{2n} \text{Tr}(D^{2n-2}) + 8(n-1)b_{2n} \text{Tr}(D^2) \text{Tr}(D^{2n-4}) + (2n(2n-1) - 8(n-1))b_{2n} \text{Tr}(D^{2n-2})). $$
We then use our ansatz to project the equation on each power of $\epsilon$.

We obtain the set of equations

$$\partial_t Y = \frac{\text{Tr}(\mathbb{1})}{2C} \partial_t R_t.$$  

(29)

$$\text{Tr}(\mathbb{1}) \partial_t Z = -\frac{\eta \partial_t R_t}{2C^2} (12a_4 + 12b_4) \text{Tr}(D^2).$$  

(30)

and for $n \geq 2$ the equations

$$(a_2n + b_2n) \partial_t (\text{Tr}(D^{2n-2})) =$$

$$-\frac{1}{2C^2} (\partial_t R_t) \left( (2n+2)(2n+1)(a_{2n+2} + b_{2n+2}) - 8nb_{2n+2} \right) \text{Tr}(D^{2n}) + 8nb_{2n+2} \text{Tr}(D^2) \text{Tr}(D^{2n-2}).$$

(31)

The idea is now to choose $C$ and $R_\mu$ so that we have

$$\frac{\partial_t R_t}{C^2} = K,$$  

(32)

where $K$ is independent of the energy scale.

One can reasonably wonder whether satisfying (25), (30) and (32) simultaneously is possible. By substituting (30) and (32) into the logarithmic derivative of (25), we get an ODE for $C$ of the form

$$\text{Tr}(\mathbb{1}) \partial_t C = -\frac{\eta K}{2} (12a_4 + 12b_4) \text{Tr}(D^2) + KC^2.$$  

(33)

This is a particularly simple case of the standard Riccati equation

$$C' = \alpha_0(t) + \alpha_1(t)C + \alpha_2(t)C^2,$$  

(34)

where we have

$$\alpha_0 = -\frac{\eta K}{2 \text{Tr}(\mathbb{1})} (12a_4 + 12b_4) \text{Tr}(D^2), \quad \alpha_1 = 0, \quad \alpha_2 = K,$$  

(35)

where in our case $\alpha_2$ is independent of $t$. We use the standard Riccati substitution

$$V(t) = C(t)K \quad \text{and} \quad V = -\frac{U'}{U},$$

which gives the equation

$$U'' + K\alpha_0(t)U = 0.$$  

Note that the Cauchy-Lipschitz theorem guarantees that such a $U(t)$ is globally defined, as linear functions are globally Lipschitz. The solution of the equation is then given, as long as $U$ does not vanish, by

$$C(t) = -\frac{U'(t)}{\alpha_2 U(t)}.$$  

(36)

Then, we choose $a_4$ and $b_4$ such that $\alpha_0(t)$ is negative, and $K$ such that $\alpha_2$ is positive. We also impose our initial conditions by setting

$$U'(0) < 0$$  

(37)

and

$$U(0) < 0.$$  

(38)

These two conditions imply that $U$ decreases (therefore does not vanish) and goes to $-\infty$. We will also work under the additional assumption

$$t = O_+(-\alpha_0(t)).$$  

(39)
The form of equation (8) on the renormalization of the Dirac operator shows an explosive behavior, so at least where everything is defined, this assumption seems reasonable.

We now want the regulator to match the asymptotic requirements of the previous section. Checking that it is the case reduces to showing that \( C^2 = \left( \frac{\beta}{2U(t)} \right)^2 \) does not have a convergent integral. To show this we use an energy estimate. For \( t_0 > 0 \), the linear ODE on \( U(t) \) gives

\[
\int_{t_0}^{t} U''(t)U''(t)\,dt + \alpha_2 \int_{t_0}^{t} \alpha_0(t)U(t)U''(t)\,dt = 0.
\]

Then we obtain

\[
\frac{U''(t)^2}{2} - \frac{U(t_0)^2}{2} = \alpha_2 \int_{t_0}^{t} (-\alpha_0(s))U(s)U'(s)\,ds.
\]

Our hypothesis on the asymptotic behavior of \(-\alpha_0(t)\) then allows to write, for some \( \beta > 0\),

\[
\frac{U''(t)^2}{2} - \frac{U(t_0)^2}{2} \geq \beta \int_{t_0}^{t} sU(s)U'(s)\,ds = \beta t \frac{U(t)^2}{2} - \beta t_0 \frac{U(t_0)^2}{2} - \beta \int_{t_0}^{t} \frac{U(s)^2}{2}\,ds.
\]

Dividing everything by \( U(t)^2 \) gives

\[
\frac{U''(t)^2}{2U(t)^2} \geq \frac{U(t_0)^2}{2U(t)^2} - \frac{U(t_0)^2}{2U(t)^2} + \frac{\beta t}{2} \frac{U(t)^2}{2U(t)^2} \int_{t_0}^{t} \frac{U(s)^2}{2}\,ds.
\]

The two first terms go to 0 at infinity, therefore for \( t \) large enough they are greater than \( -\frac{\beta t_0}{4} \). As \( U^2 \) is increasing, the last term is greater than \( -\frac{\beta t_0}{2} \). We therefore finally get

\[
\left( \frac{U'(t)}{U(t)} \right)^2 \geq \frac{\beta t_0}{2}, \tag{41}
\]

for large \( t \), which shows that \( C^2 \) does not have a convergent integral, and therefore that \( R_t \) blows up at infinity. Choosing the integration constant wisely while integrating \( KC(t)^2 \) therefore allows one to set the limit at 0 of the regulator to be 0, and the third condition being obviously satisfied (since the regulator has no momentum dependence). We have thus proved that our choice of \( R_t \) is suitable.

Now, choosing the constants \( a_{2n} \) and \( b_{2n} \) successively to solve the two equations

\[
 \frac{(2n + 1)(2n + 2)(a_{2n+2} + b_{2n+2}) - 8nb_{2n+2}}{a_{2n} + b_{2n}} = \frac{3(n - 2)}{16\pi^2}, \tag{42}
\]

\[
 \frac{8nb_{2n+2}}{a_{2n} + b_{2n}} = \frac{n}{32\pi^2}, \tag{43}
\]

the last set of equations yields for \( n \geq 1 \) the expression

\[
16\pi^2 \partial_t \text{Tr}(D^{2n}) = 3n\text{Tr}(D^{2n+2}) + \frac{n}{2} \text{Tr}(D^2)\text{Tr}(D^{2n}). \tag{44}
\]

We have exactly recovered the terms of the renormalization equation of the Dirac operator which do not involve either the twisted operator \( \bar{D} \) or gauge terms.

This shows that our matrix model can be regarded as being dual to a simplified model of Yukawa interactions, in the sense that the Yukawa part of the \( \beta \)-function can be fully computed in terms of the matrix model.
6 Adding more loops: traces of the finite Dirac operator and the topology of Yukawa Feynman diagrams

In order to find a $\beta$-function for the Dirac operator that only involves traces of its even powers, it is quite straightforward to see that it is sufficient to consider equations for the Yukawa matrices $Y_i$ that are of the form

$$\partial_t Y_i = Y_i f \left( Y_i^\dagger Y_i, \{ \text{Tr}(D^{2n}) \}_{n \in \mathbb{N}} \right).$$

In [12] and [6], the equations of the two and three-loop $\beta$-functions are of this form, up to the presence of the Higgs self-coupling. In this section, we give some arguments to show that a dual matrix model can be constructed for the renormalization of Yukawa interactions at an arbitrary number of loops, up to diagrams which involve a four Higgs vertex.

The counterterms to the Yukawa matrices come from the renormalization of the fermion-antifermion-Higgs vertex, which can be computed at a fixed loop order once the fermionic and bosonic two-point functions are already renormalized. A diagram will yield a counterterm of the right form if it contains an odd number of vertices involving $Y_i$, where $i$ is the type of the incoming fermion, and an even number of vertices involving other fermions.

The vertices involving other types of fermions only appear in fermion loops, and exactly yield traces of powers of the Dirac operator, as Yukawa matrices are traced over and summed with the right color factors. Therefore, the only diagrams which could pose a problem to generalize our matrix duality are diagrams containing fermion loops with an odd number of propagators. However, such diagrams cannot exist because of the chiral nature of Yukawa interactions in the Standard Model: a Yukawa vertex always couples a right-handed fermion to a left-handed fermion. As a result, this model can be extended to an arbitrary loop level.

However, notice that this scheme does not encompass the renormalization of the Higgs four-point function, for which a quartic Higgs self-coupling needs to be introduced. At one loop, the
Figure 3: This divergence in the Higgs four-point function needs a quartic self-coupling to be cancelled.

Figure 4: At two loops and more, the new vertex makes diagrams which are not taken into account appear in the matrix model.

Yukawa matrices corrections do not depend on it, but that result is no longer true for more loops. Therefore, a renormalization scheme for the Yukawa matrices which also encompasses the Higgs four-point function should have a more complicated multiloop structure.

7 Conclusion

We have exhibited an intriguing duality between the renormalization of Yukawa interactions at one loop in the standard model extended with right handed neutrinos and the functional renormalization of a matrix model. The $\beta$-function of the matrix model looks exactly like the one of Yukawa interactions, without the contribution of the charged Higgs and the gauge bosons. This suggests that for any Yukawa Lagrangian with neutral Higgs and no gauge freedom, it should be possible to write a dual matrix model at one loop.

In particular, more terms seem to be possible to recover in the broken phase of the Standard Model, as charged Higgs interactions are suppressed by gauge freedom through the Higgs mechanism. Nonetheless, whether or not charged Higgs interactions and gauge couplings can have a nice interpretation in this setting remains an open question.

This matrix-Yukawa duality can have multiple interesting potential applications. The first one is the renormalization of the spectral action in Noncommutative Geometry. In noncommutative field theories like the standard model, the action of the geometry is

\[ S := \text{Tr} f \left( \frac{D + A + JAJ^{-1}}{\Lambda} \right), \]

(46)

where $D$ is the unperturbed Dirac operator, $A + JAJ^{-1}$ is a perturbation and $\Lambda$ is a cutoff, $J$ is the real structure, and $f$ is an even test function. In the Noncommutative Standard Model,
the renormalization of the spectral action is done in two steps: it is first expanded to get back the usual form of the Standard Model action, and then renormalized using the usual Feynman diagram techniques. Therefore, it is not intrinsic at all. Our matrix model renormalization theory is conceptually much closer to the spectral action, and our duality hints towards the possibility of a more conceptual understanding of the renormalization of spectral actions, which could include nonperturbative effects. Another possible application of this result would be to find a way of renormalizing field theories without having to resort to Feynman diagrams, which could turn out to be computationally easier.

Another interesting question is whether or not an expression like (44) could still work for more loops. The answer is yes as long as one does not care about the renormalization of the Higgs four-point function. However, that four-point function needs a new Higgs vertex to be renormalized, and the corresponding self-coupling is not intrinsically linked to the Dirac operator anymore. In fact, in the Noncommutative Standard Model the Higgs self-coupling comes from a term of the form

$$\lambda = \frac{f_0}{2\pi^2} \text{Tr}(D^4).$$

(47)

This term contains a factor $f_0$ which comes from the test function of the spectral action, and also appears in the gauge couplings [1]. The presence of this extra factor seems inevitable: without it, in the expansion of the spectral action [2], the Higgs self-coupling would only depend on the trace of an even power of the Dirac operator, whose renormalization equation is already determined by equation (44). Therefore there would not be enough freedom in the model to be able to renormalize the Higgs self-coupling to cancel the divergences of the new vertex. Our method then seems to require an extension on a more fundamental level to take this new vertex contribution into account.

It is finally worth noting that the spectral action contains all geometric information needed to formulate electromagnetism, weak and strong interactions, and gravity altogether in the framework of Noncommutative Geometry. Consequently, a more conceptual understanding of its renormalization could lead to interesting new perspectives for several physically relevant geometric models, especially matrix models for quantum gravity which could share some features with those explored in [10] and [1].

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