THE STANDARD MODEL WITHIN NON–ASSOCIATIVE GEOMETRY

RAIMAR WULKENHAAR

INSTITUT FÜR THEORETISCHE PHYSIK, UNIVERSITÄT LEIPZIG
AUGUSTUSPLATZ 10/11, D–04109 LEIPZIG, GERMANY
E-mail: wulkhaar@tph100.physik.uni-leipzig.de

ABSTRACT. We present the construction of the standard model within the framework of non–associative geometry. For the simplest scalar product we get the tree–level predictions $m_W = \frac{1}{2} m_t$, $m_H = \frac{3}{2} m_t$ and $\sin^2 \theta_W = \frac{3}{8}$. These relations differ slightly from predictions derived in non–commutative geometry.

1. Introduction

One of the most important applications of non–commutative geometry [1] to physics is a unified description of the standard model. The most elegant version rests upon a K–cycle [1, 2] with real structure [3], see [4, 5] for details and [6, 7] for an older version. There also exist numerous other formulations within non–commutative geometry (NCG), see for instance [8, 9]. The author of this paper has proposed in [10] a modification of non–commutative geometry. In that approach one uses unitary Lie algebras instead of unital associative $\ast$–algebras. Lie algebras are non–associative algebras – this is the motivation for the working title “non–associative geometry”. The only realistic physical model that one can construct within the most elegant NCG–prescription is the standard model [11]. The advantage of non–associative geometry is that a larger class of physical models can be constructed from the same amount of structures as in the most elegant NCG–formulation. That class includes the standard model, as we show in this paper.

We give in Section 2 a recipe how to construct classical gauge field theories within non–associative geometry. The arguments why this recipe works can be found in [10]. Section 3 contains the construction of the standard model. We derive the geometric structures and write down the bosonic action for the simplest scalar product. The fermionic action will not be displayed, because it is identical with the classical formulation.
2. The Recipe of Non–associative Geometry

The basic object in non–associative geometry is an L–cycle $(g, h, D, \pi, \Gamma)$, which consists of a *–representation $\pi$ of a unitary Lie algebra $g$ in bounded operators on a Hilbert space $h$, together with a selfadjoint operator $D$ on $h$ with compact resolvent and a selfadjoint operator $\Gamma$ on $h$, $\Gamma^2 = \text{id}_h$, which commutes with $\pi(a)$ and anticommutes with $D$. The operator $D$ may be unbounded, but such that $[D, \pi(g)]$ is bounded. L–cycles are naturally related to physical models if the following input data are given:

1) The (Lie) group of local gauge transformations $G$.
2) Chiral fermions $\psi$ transforming under a representation $\hat{\pi}$ of $G$.
3) The fermionic mass matrix $\widetilde{M}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4) Possibly the symmetry breaking pattern of $G$.

Take $g = C^\infty(X) \otimes a$ as the Lie algebra of $G$, where $a$ is a matrix Lie algebra and $C^\infty(X)$ the algebra of smooth functions on the (compact Euclidian) space–time manifold $X$. Take $h = L^2(X,S) \otimes \mathbb{C}^F$ as the completion of the Euclidian fermions, where $L^2(X,S)$ is the Hilbert space of square integrable bispinors. Take $\pi = 1 \otimes \hat{\pi}$ as the differential $\hat{\pi}_+$, where $\hat{\pi}$ is a representation of $a$ in $M_F\mathbb{C}$. Put $D = D \otimes 1_F + \gamma^5 \otimes M$, where $D$ is the Dirac operator on $X$ and $M \in M_F\mathbb{C}$ such that $\gamma^5 \otimes M$ coincides with $\widetilde{M}$ on chiral fermions. The chirality properties of the fermions are encoded in $\Gamma = \gamma^5 \otimes \hat{\Gamma}$.

The recipe towards the (classical) gauge field theory associated to the L–cycle is the following: Let $\Omega^1 a$ be the space of formal commutators

$$\omega^1 = \sum_{a,z \geq 0} [a_a^z, \ldots [a_a^1, da_a^0], \ldots] , \ a_a^i \in a .$$

Apply linear mappings $\hat{\pi} : \Omega^1 a \rightarrow M_F\mathbb{C}$ and $\hat{\sigma} : \Omega^1 a \rightarrow M_F\mathbb{C}$ defined by

$$\hat{\pi}(\omega^1) := \sum_{a,z \geq 0} [\hat{\pi}(a_a^z), \ldots [\hat{\pi}(a_a^1), [-iM, \hat{\pi}(a_a^0)], \ldots] , \ (1)$$

$$\hat{\sigma}(\omega^1) := \sum_{a,z \geq 0} [\hat{\sigma}(a_a^z), \ldots [\hat{\sigma}(a_a^1), [M^2, \hat{\sigma}(a_a^0)], \ldots] . \ (2)$$

Define $\Omega^n a \supset \omega^n = \sum_{\alpha} [\omega_{n,a}^1, [\omega_{n-1,a}^1, \ldots [\omega_2^1, \omega_1^1], \ldots]]$, where $\omega_{1,a}^i \in \Omega^1 a$. Extend $\hat{\pi}$ and $\hat{\sigma}$ recursively to $\Omega^n a$ by

$$\hat{\pi}([\omega^1, \omega^k]) := \hat{\pi}(\omega^1)\hat{\pi}(\omega^k) - (-1)^k\hat{\pi}(\omega^k)\hat{\pi}(\omega^1) ,$$

$$\hat{\sigma}([\omega^1, \omega^k]) := \hat{\sigma}(\omega^1)\hat{\pi}(\omega^k) - \hat{\pi}(\omega^k)\hat{\sigma}(\omega^1) - \hat{\pi}(\omega^1)\hat{\sigma}(\omega^k) - (-1)^k\hat{\sigma}(\omega^k)\hat{\pi}(\omega^1) .$$

Define for $n \geq 2$

$$\hat{\pi}(\mathcal{J}^n a) := \{ \hat{\sigma}(\omega^{n-1}) , \ \omega^{n-1} \in \Omega^{n-1} a \cap \ker \hat{\pi} \} . \ (3)$$
Define spaces \( r^0 a \subset M_F \mathbb{C} \) and \( r^1 a \subset M_F \mathbb{C} \) elementwise by

\[
x^0 a = -(x^0)^* = \hat{\Gamma}(r^0 a) \hat{\Gamma}, \quad x^1 a = -(x^1)^* = -\hat{\Gamma}(r^1 a) \hat{\Gamma},
\]

\[
[x^0 a, \hat{\pi}(a)] \subset \hat{\pi}(a), \quad [x^0 a, \hat{\pi}(\Omega^1 a)] \subset \hat{\pi}(\Omega^1 a),
\]

\[
\{x^0 a, \hat{\pi}(a)\} \subset \{\hat{\pi}(a), \hat{\pi}(a)\} + \{\hat{\pi}(\Omega^2 a)\}, \quad \{x^0 a, \hat{\pi}(\Omega^1 a)\} \subset \{\hat{\pi}(a), \hat{\pi}(\Omega^1 a)\} + \{\hat{\pi}(\Omega^2 a)\},
\]

\[
[x^1 a, \hat{\pi}(a)] \subset \hat{\pi}(\Omega^1 a), \quad \{x^1 a, \hat{\pi}(\Omega^1 a)\} \subset \hat{\pi}(\Omega^2 a) + \{\hat{\pi}(a), \hat{\pi}(a)\}.
\]  

Define spaces \( j^0 a, j^1 a, j^2 a \subset M_F \mathbb{C} \) elementwise by

\[
j^0 a = -(j^0)^* = \hat{\Gamma}(j^0 a) \hat{\Gamma}, \quad j^1 a = -(j^1)^* = -\hat{\Gamma}(j^0 a) \hat{\Gamma}, \quad j^2 a = (j^0)^* = \hat{\Gamma}(j^0 a) \hat{\Gamma},
\]

\[
[j^0 a, \hat{\pi}(a)] = 0, \quad \{j^0 a, \hat{\pi}(a)\} \subset \hat{\pi}(J^2 a) + \{\hat{\pi}(a), \hat{\pi}(a)\},
\]

\[
[j^0 a, \hat{\pi}(\Omega^1 a)] = 0, \quad \{j^1 a, \hat{\pi}(\Omega^1 a)\} \subset \hat{\pi}(J^3 a) + \{\hat{\pi}(\Omega^1 a), \hat{\pi}(a)\},
\]

\[
[j^1 a, \hat{\pi}(a)] = 0, \quad \{j^1 a, \hat{\pi}(\Omega^1 a)\} \subset \hat{\pi}(J^2 a) + \{\hat{\pi}(a), \hat{\pi}(a)\},
\]

\[
\{j^1 a, \hat{\pi}(a)\} \subset \hat{\pi}(J^3 a) + \{\hat{\pi}(\Omega^1 a), \hat{\pi}(a)\} + \hat{\pi}(\Omega^2 a) + \{\hat{\pi}(a), \hat{\pi}(a)\},
\]

\[
\{j^2 a, \hat{\pi}(a)\} \subset \hat{\pi}(J^2 a) + \{\hat{\pi}(a), \hat{\pi}(a)\}, \quad \{j^2 a, \hat{\pi}(\Omega^1 a)\} \subset \hat{\pi}(J^3 a) + \{\hat{\pi}(\Omega^1 a), \hat{\pi}(a)\}.
\]  

The connection form \( \rho \) has the structure

\[
\rho = \sum_\alpha (c^1_\alpha \otimes m^0_\alpha + c^0_\alpha \gamma^5 \otimes m^1_\alpha), \quad c^1_\alpha \in \Lambda^1, \quad c^0_\alpha \in \Lambda^0, \quad m^0_\alpha \in r^0 a, \quad m^1_\alpha \in r^1 a,
\]  

where \( \Lambda^k \) is the space of differential \( k \)-forms represented by gamma matrices.

The curvature \( \theta \) is computed from the connection form \( \rho \) by

\[
\theta = d\rho + \rho^2 - i\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}(\rho)\gamma^5 + J^2 g,
\]

\[
J^2 g = (\Lambda^2 \otimes j^0 a) + (\Lambda^1 \gamma^5 \otimes j^1 a) + (\Lambda^0 \otimes j^2 a),
\]

where \( d \) is the exterior differential and \( \hat{\sigma} \) the extension to elements of the form \( (\mathbb{C}) \). Select the representative \( c(\theta) \) orthogonal to \( J^2 g \), i.e. find \( j \in J^2 g \) such that

\[
c(\theta) = d\rho + \rho^2 - i\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}(\rho)\gamma^5 + j, \quad \int_X dx \: \text{tr}(c(\theta)j^2) = 0 \: \forall j^2 \in J^2 g.
\]

The trace includes the trace in \( M_F \mathbb{C} \) and over gamma matrices. Compute the bosonic and fermionic actions

\[
S_B = \int_X dx \: \frac{1}{g_0^2 F} \text{tr}(c(\theta)^2), \quad S_F = \int_X dx \: \psi^* (D + i\rho) \psi,
\]  

where \( g_0 \) is a coupling constant and \( \psi \in \mathfrak{h} \). Finally, perform a Wick rotation to Minkowski space.
3. The Construction

Our constructions requires that the mass matrices of all fermions of the same type, including the neutrinos, are different from zero and non-degenerated. In particular, the Kobayashi–Maskawa matrix in both the quark and the lepton sector must be non-trivial. This is necessary to avoid certain degeneracy effects. The matrix Lie algebra of the standard model is

\[ a = \text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1). \]

The Hilbert space is \( \mathbb{C}^{48} \), because we need right neutrinos. We label elements of \( \mathbb{C}^{48} \) in a suggestive way by the fermions of the first generation:

\[ (u_L, d_L, u_R, d_R, \nu_L, e_L, \nu_R, e_R)^T \in \mathbb{C}^{48}, \]

where \( u_L, d_L, u_R, d_R \in \mathbb{C}^3 \otimes \mathbb{C}^3 \) and \( \nu_L, e_L, \nu_R, e_R \in \mathbb{C}^3 \). The representation \( \hat{\pi} \) of \( a \) on \( \mathbb{C}^{48} \) is

\[
\hat{\pi}((a_1, a_2, a_3)) = \begin{pmatrix}
  i f_0 \text{diag}(\frac{1}{3}1_3 \otimes 1_3, \frac{1}{3}1_3 \otimes 1_3, \frac{4}{3}1_3 \otimes 1_3, -\frac{2}{3}1_3 \otimes 1_3, -1_3, -1_3, 0_3, -21_3) + \\
  (a_3 + i f_3 1_3) \otimes 1_3; (i(f_1 - i f_2) 1_3 \otimes 1_3) 0 0 \\
  i(f_1 + i f_2) 1_3 \otimes 1_3; (a_3 - i f_3 1_3) \otimes 1_3 0 0 \\
  0 0 a_3 \otimes 1_3 0 \\
  0 0 0 0 \\
  0 0 0 0 \\
  0 0 0 0 \\
  0 0 0 0 \\
  0 0 0 0 \\
  \end{pmatrix}.
\]

Here, the matrix \( a_3 \in \text{su}(3) \subset M_3 \mathbb{C} \) is written down in the standard matrix representation, \( a_2 = \begin{pmatrix} i f_3; & i(f_1 - i f_2) \\ i(f_1 + i f_2); & -i f_3 \end{pmatrix} \in \text{su}(2) \), for \( f_1, f_2, f_3 \in \mathbb{R} \), and \( a_1 = i f_0 \in \text{u}(1) \equiv i \mathbb{R} \). The generalized Dirac operator is

\[
M = \begin{pmatrix}
  0 & 0 & 1_3 \otimes M_\nu & 0 & 0 \\
  0 & 0 & 0 & 1_3 \otimes M_d \\
  1_3 \otimes M_\nu^* & 0 & 0 & 0 \\
  0 & 1_3 \otimes M_d^* & 0 & 0 \\
  & & & & \\
  0 & 0 & M_\nu & 0 \\
  0 & 0 & 0 & M_e \\
  M_\nu^* & 0 & 0 & 0 \\
  0 & M_e^* & 0 & 0 \\
  \end{pmatrix}.
\]
where \( M_u, M_d, M_\nu, M_e \in M_3 \mathbb{C} \) are the mass matrices of the fermions. It is easy to see that for \( a^i_\alpha = (a^i_{3,\alpha}, a^i_{2,\alpha}, a^i_{3,\alpha}) \in \mathfrak{a} \) one has

\[
\tau^1 := \sum_{\alpha, z \geq 0} [\hat{\pi}(a^2_\alpha) \ldots [\hat{\pi}(a^1_\alpha), [-iM, \hat{\pi}(a^0_\alpha)]] \ldots ] = \begin{pmatrix}
0 & 0 & 0 & b_2 \mathbb{1}_3 \otimes M_u & -b_1 \mathbb{1}_3 \otimes M_d & b_1 \mathbb{1}_3 \otimes M_d \\
0 & 0 & 0 & -b_1 \mathbb{1}_3 \otimes M_u & b_1 \mathbb{1}_3 \otimes M_d & 0 \\
(\mathbb{1}_3 \otimes M^*_u)^\dagger ; -b_1 \mathbb{1}_3 \otimes M^*_u & 0 & 0 & 0 & 0 & 0 \\
b_1 \mathbb{1}_3 \otimes M^*_d ; (b_2 \mathbb{1}_3 \otimes M^*_d) & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(10)

\[
(b_1) = \sum_{\alpha, z \geq 0} a^2_{2,\alpha} a^2_{1,\alpha} \ldots a^1_{2,\alpha} a^1_{1,\alpha} a^0_{2,\alpha} a^0_{1,\alpha} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.
\]

(11)

The matrix (10) is the general form of an element of \( \hat{\pi}(\Omega^1 \mathfrak{a}) \). The grading operator is

\[
\hat{\Gamma} = \text{diag}(-\mathbb{1}_3 \otimes \mathbb{1}_3, -\mathbb{1}_3 \otimes \mathbb{1}_3, \mathbb{1}_3 \otimes \mathbb{1}_3, \mathbb{1}_3 \otimes \mathbb{1}_3, -\mathbb{1}_3, -\mathbb{1}_3, \mathbb{1}_3, \mathbb{1}_3).
\]

One has \( \hat{\Gamma}^2 = \mathbb{1}_{48} \), \( [\hat{\Gamma}, \hat{\pi}(\mathfrak{a})] = 0 \), \( \{\hat{\Gamma}, \mathcal{M}\} = 0 \) and \( \{\hat{\Gamma}, \hat{\pi}(\Omega^1 \mathfrak{a})\} = 0 \). Let

\[
\begin{pmatrix}
if_3; i(f_1 - if_2) \\
i(f_1 + if_2); -if_3
\end{pmatrix} := \begin{pmatrix}
(i|b_2|^2 - |b_1|^2); -2i b_1 b_2 \\
-2i b_1 b_2; -i(|b_2|^2 - |b_1|^2)
\end{pmatrix} \in \text{su}(2),
\]

\[
\begin{align*}
M_{ud} &= M_u M^*_d - M_d M^*_u, \\
M_{\nu e} &= M_\nu M^*_\nu - M_\nu M^*_\nu, \\
M_{(ud)} &= M_u M^*_u + M_d M^*_d, \quad M_{(\nu e)} = M_\nu M^*_\nu + M_e M^*_e.
\end{align*}
\]

Then we have

\[
\{\tau^1, \tau^1\} = \begin{pmatrix}
if_3 \mathbb{1}_3 \otimes M_{ud}; & i(f_1 - if_2) \mathbb{1}_3 \otimes M_{ud} & 0 & 0 \\
i(f_1 + if_2) \mathbb{1}_3 \otimes M_{ud}; & -if_3 \mathbb{1}_3 \otimes M_{ud} & 0 & 0 \\
0 & 0 & 0_g & 0 \\
0 & 0 & 0 & 0_g
\end{pmatrix},
\]

(12)

\[
-((|b_1|^2 + |b_2|^2)) \begin{pmatrix}
\text{diag}(\mathbb{1}_3 \otimes M_{(ud)}), \mathbb{1}_3 \otimes M_{(ud)}; & 1_3 \otimes 2M^*_u M_u, & 1_3 \otimes 2M^*_d M_d, \\
M_{(\nu e)}, & M_{(\nu e)}, & 2M^*_\nu M_\nu, & 2M^*_e M_e
\end{pmatrix}.
\]

(13)
Next, for \( \tau^1 = \hat{\pi}(\omega^1) \) given by \((10)\) we obtain with \((2)\)

\[
\hat{\sigma}(\omega^1) = \sum_{\alpha, z \geq 0} [\hat{\pi}(a_{\alpha}^z), \ldots, [\hat{\pi}(a_{\alpha}^1), [\mathcal{M}^2, \hat{\pi}(\omega_0^0)] \ldots] =
\frac{1}{2} \left( \begin{array}{cccc}
if_3 1_3 \otimes M_{ud}; & i(f_1 - i f_2) 1_3 \otimes M_{ud} & 0 & 0 \\
if(f_1 + i f_2) 1_3 \otimes M_{ud}; & -i f_3 1_3 \otimes M_{ud} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right)
\]

Choosing

\[
\omega_0^1 = d a_0^0 + [a_2^0, [a_2^0, d a_0^0]] , \quad a_0^1 = \left( \begin{array}{c} 0 \\ i \\ 0 \end{array} \right) \in \text{su}(2) , \quad a_0^1 = \left( \begin{array}{c} 1 \\ 0 \\ -i \end{array} \right) \in \text{su}(2) ,
\]

we have \( \omega_0^1 \in \ker \hat{\pi} \) due to \((a_0^0 + a_2^0 a_0^0)(0,0) = (0,0) \), see \((11)\). On the other hand, \( \hat{\sigma}(\omega^1) \neq 0 \) is the matrix \((14)\), with \( f_1 = 0, f_2 = -3, f_3 = 0 \). Obviously, each matrix of the form \((14)\) can be represented as \( \hat{\sigma}(\omega^1) \), for \( \omega^1 = \sum_{\alpha, z \geq 0} [a_{\alpha}^z, \ldots, [a_0^0, \omega_0^1] \ldots] \in \ker \hat{\pi} \). Therefore, each element of \( \hat{\pi}(J^2 a) \) is precisely of the form \((14)\), see \((3)\):

\[
\hat{\sigma}(\Omega^1 a) \equiv \hat{\pi}(J^2 a) .
\]

Comparing the results \((15)\) and \((14)\) with \((12)\) and \((13)\) we get

\[
\{\tau^1, \tau^1\} = -\|b_1\|^2 + \|b_2\|^2 \text{diag}(1_3 \otimes M_{ud}, 1_3 \otimes M_{ud}, 1_3 \otimes 2M^*_{ud} M_u, 1_3 \otimes 2M^*_{ud} M_d, M_{ud}, M_{ud}, 2M^*_{ud} M_u, 2M^*_{ud} M_d) \mod \hat{\pi}(J^2 a) .
\]

It is clear that \((16)\) is orthogonal to \( \hat{\pi}(J^2 a) \).

Next, we need the structure of the space \( \{\hat{\pi}(a), \hat{\pi}(a)\} \). A simple calculation yields for elements of \( \{\hat{\pi}(a), \hat{\pi}(a)\} \) the form

\[
\{\hat{\pi}(a), \hat{\pi}(a)\} \ni \text{diag}(A_q + \Delta_q, A_{\ell} + \Delta_{\ell}) ,
\]

\[
A_q = \sum_{\alpha} i \frac{1}{3} \left( \begin{array}{cccc}
[\frac{1}{2} \bar{\lambda}_\alpha^0 + \lambda_\alpha^1 + \bar{\lambda}_\alpha^0 a_{3,0}] & [\lambda_\alpha^0 - i \bar{\lambda}_\alpha^0 a_{3,0}] & 0 & 0 \\
\frac{1}{2} \bar{\lambda}_\alpha^1 + \lambda_\alpha^1 a_{3,0} & 0 & 0 & 0 \\
\frac{1}{2} \bar{\lambda}_\alpha^0 + \bar{\lambda}_\alpha^1 a_{3,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \otimes 1_3 ,
\]

\[
A_{\ell} = \sum_{\alpha} i \frac{1}{3} \left( \begin{array}{cccc}
[\frac{1}{2} \bar{\lambda}_\alpha^0 + \lambda_\alpha^1 + \bar{\lambda}_\alpha^0 a_{3,0}] & [\lambda_\alpha^0 - i \bar{\lambda}_\alpha^0 a_{3,0}] & 0 & 0 \\
\frac{1}{2} \bar{\lambda}_\alpha^1 + \lambda_\alpha^1 a_{3,0} & 0 & 0 & 0 \\
\frac{1}{2} \bar{\lambda}_\alpha^0 + \bar{\lambda}_\alpha^1 a_{3,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \otimes 1_3 ,
\]
\[
A_\ell = \sum_{\alpha} i \begin{pmatrix}
-\hat{i}\lambda_\alpha^0 \mathbb{1}_3; & -i(\hat{\lambda}_\alpha^0 - i\hat{i}\lambda_\alpha^0) \mathbb{1}_3 \\
-\hat{i}(\hat{\lambda}_\alpha^1 + i\lambda_\alpha^0) \mathbb{1}_3 & \hat{i}\lambda_\alpha^3 \mathbb{1}_3
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\]

\[\Delta_q = \text{diag} \left( (\lambda + \tilde{\lambda} + \frac{1}{3} \tilde{\lambda}) \mathbb{1}_3, (\lambda + \tilde{\lambda} + \frac{1}{9} \tilde{\lambda}) \mathbb{1}_3, (\lambda + \frac{16}{9} \tilde{\lambda}) \mathbb{1}_3, (\lambda + \frac{4}{9} \tilde{\lambda}) \mathbb{1}_3 \right) \otimes \mathbb{1}_3,
\]

\[\Delta_\ell = \text{diag} \left( (\hat{\lambda} + \hat{\lambda}) \mathbb{1}_3, (\hat{\lambda} + \hat{\lambda}) \mathbb{1}_3, 0_3, 4\mathbb{1}_3 \right),
\]

where \(a_{3,\alpha} \in \text{su}(3)\) and \(\lambda_0^0, \lambda_0^1, \lambda_0^2, \tilde{\lambda}_0^0, \tilde{\lambda}_0^1, \tilde{\lambda}_0^2, \lambda, \tilde{\lambda}, \tilde{\lambda} \in \mathbb{R}\).

In order to write down the structure of the connection form we must find the simple result

\[r^0 a = \hat{\pi}(a), \quad r^1 a = \hat{\pi}(\Omega^1 a).
\]

For generic mass matrices, equations (18) have the solution \(j^0 a = 0, j^1 a = 0\) and

\[j^2 a = \hat{\pi}(J^2 a) \oplus \{\hat{\pi}(a), \hat{\pi}(a)\} \oplus \text{diag}(R \mathbb{1}_6, \mathbb{R} \mathbb{1}_6, \mathbb{R} \mathbb{1}_6, R \mathbb{1}_6)
\]

\[J_q = \text{diag} \left( (\lambda_1 + \frac{1}{9} \lambda_0) \mathbb{1}_3, (\lambda_1 + \frac{1}{9} \lambda_0) \mathbb{1}_3, (\lambda_2 + \frac{16}{9} \lambda_0) \mathbb{1}_3, (\lambda_2 + \frac{4}{9} \lambda_0) \mathbb{1}_3 \right) \otimes \mathbb{1}_3,
\]

\[J_\ell = \text{diag} \left( (\lambda_3 + \lambda_0) \mathbb{1}_3, (\lambda_3 + \lambda_0) \mathbb{1}_3, (\lambda_4 + 4 \lambda_0) \mathbb{1}_3 \right),
\]

for \(J_2 \in \hat{\pi}(J^2 a)\) and \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}\).

In order to write down the bosonic action it is necessary to select the representative \(\mathbf{e}(\{\tau^1, \tau^1\})\) of \(\{\tau^1, \tau^1\}\) + \(j^2 a\) orthogonal to \(j^2 a\). This problem is easy to solve. Let

\[\tilde{M}_{\text{uu}} := M_u M_u^* + M_d M_d^* + \frac{1}{4} \text{tr}(M_u M_u^* + M_d M_d^*) \mathbb{1}_3,
\]

\[\tilde{M}_{\text{dd}} := M_u M_u^* + M_d M_d^* - \frac{1}{4} \text{tr}(M_u M_u^* + M_d M_d^*) \mathbb{1}_3,
\]

\[\tilde{M}_{\text{uu}} := M_u M_u^* + M_d M_d^* + \frac{1}{4} \text{tr}(5M_u M_u^* + 3M_d M_d^* - M_u M_u^* + M_d M_d^*) \mathbb{1}_3,
\]

\[\tilde{M}_{\text{dd}} := M_u M_u^* + M_d M_d^* - \frac{1}{4} \text{tr}(3M_u M_u^* + 5M_d M_d^* + M_u M_u^* - M_d M_d^*) \mathbb{1}_3,
\]

\[\tilde{M}_{\text{uu}} := M_u M_u^* + M_d M_d^* + \frac{1}{4} \text{tr}(M_u M_u^* + M_d M_d^* + 7M_u M_u^* + M_d M_d^*) \mathbb{1}_3,
\]

\[\tilde{M}_{\text{ee}} := M_e^* M_e^* - \frac{1}{21} \text{tr}(3M_u M_u^* - 3M_d M_d^* + M_u M_u^* + 7M_e M_e^*) \mathbb{1}_3.
\]

Then, the canonical embedding \(\mathbf{e}(\{\tau^1, \tau^1\})\) of \(\{\tau^1, \tau^1\}\) into \(M_{49} \mathbb{C}\) is given by

\[\mathbf{e}(\{\tau^1, \tau^1\}) = -((b_1)^2 + (b_2)^2) \text{diag}(\mathbb{1}_3 \otimes \tilde{M}_{\text{uu}}, \mathbb{1}_3 \otimes \tilde{M}_{\text{dd}}, 2M_{uu}, 2M_{dd}, \tilde{M}_{\text{uu}}, \tilde{M}_{\text{dd}}, 2\tilde{M}_{\text{uu}}, 2\tilde{M}_{\text{dd}}).
\]

Now we include the four dimensional Riemannian spin manifold \(X\) and choose a self-adjoint local basis \(\{\gamma^\mu\}_{\mu=1,2,3,4}\) of \(\Lambda^1\). The connection form \(\rho\) has due to (19), (20) and (22) the structure

\[\rho = \begin{pmatrix}
\rho_q & 0 \\
0 & \rho_e
\end{pmatrix},
\]
\[ \rho_q = \begin{pmatrix} \mathbf{A} + i \frac{1}{3} \mathbf{A}^0 + i \mathbf{A}^3 & \mathbf{1}_3 \otimes \mathbf{1}_3 \otimes \mathbf{1}_3 & -i \gamma^5 \Phi_2 \mathbf{1}_3 \otimes M_u & -i \gamma^5 \Phi_1 \mathbf{1}_3 \otimes M_d \\ i \mathbf{A}^0 - i \mathbf{A}^2 & \mathbf{1}_3 \otimes \mathbf{1}_3 \otimes \mathbf{1}_3 & -i \gamma^5 \Phi_1 \mathbf{1}_3 \otimes M_u & -i \gamma^5 \Phi_2 \mathbf{1}_3 \otimes M_d \\ -i \gamma^5 \Phi_2 \mathbf{1}_3 \otimes M^*_{u} & i \gamma^5 \Phi_1 \mathbf{1}_3 \otimes M^*_{u} & (\mathbf{A} + \frac{2}{3} i \mathbf{A}^0) \mathbf{1}_3 \otimes \mathbf{1}_3 & 0 \\ -i \gamma^5 \Phi_1 \mathbf{1}_3 \otimes M^*_{d} & -i \gamma^5 \Phi_2 \mathbf{1}_3 \otimes M^*_{d} & 0 & (\mathbf{A} - \frac{2}{3} i \mathbf{A}^0) \mathbf{1}_3 \otimes \mathbf{1}_3 \end{pmatrix}, \]

\[ \rho_e = \begin{pmatrix} i \mathbf{A}^0 & i \mathbf{A}^1 & i \mathbf{A}^2 & i \mathbf{A}^3 \\ i \mathbf{A}^1 & i \mathbf{A}^2 & i \mathbf{A}^3 & i \mathbf{A}^0 \\ i \mathbf{A}^2 & i \mathbf{A}^3 & i \mathbf{A}^0 & i \mathbf{A}^1 \\ i \mathbf{A}^3 & i \mathbf{A}^0 & i \mathbf{A}^1 & i \mathbf{A}^2 \end{pmatrix}, \]

where \( \mathbf{A} \in \Lambda^1 \otimes \text{su}(3), \quad \bar{\mathbf{A}} := \begin{pmatrix} i \mathbf{A}^0; & i \mathbf{A}^1; & i \mathbf{A}^2; & i \mathbf{A}^3 \end{pmatrix} \in \Lambda^1 \otimes \text{su}(2), \quad A^0 \in \Lambda^1, \quad \Phi_1, \Phi_2 \in \Lambda^0 \otimes \mathbb{C}. \) In formula (7) for the curvature note that \( \sigma(\omega^1) = 0 \mod \pi(J^2 a) \). Inserting (20) into (7) we obtain for the bosonic action given in [8]

\[ S_B = \frac{1}{4 g_0^2} \int_X dx \, tr(\epsilon(\theta)^2) = \int_X dx \, (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0), \]

\[ \mathcal{L}_2 = \frac{1}{4 g_0^2} \text{tr}((dA + \frac{1}{2} [A, A], A^3)^2) + \frac{1}{4 g_0^2} \text{tr}((d A + \frac{1}{2} \{A, A\})^2) + \frac{5}{4 g_0^2} \text{tr}((dA^0)^2), \]

\[ \mathcal{L}_1 = \frac{1}{2 g_0^2} \text{tr}(|d \Phi_1 + i (A^0 + A^3) \Phi_1 + i (A^1 - i A^2) (\Phi_2 + 1) |^2 + d \Phi_2 + i (A^0 - A^3) (\Phi_2 + 1) + i (A^1 + i A^2) (\Phi_2 + 1)), \]

\[ \mathcal{L}_0 = \frac{1}{192 g_0^8} |(\Phi_1)^2 + |\Phi_2 + 1|^2 - 1|^2 tr(1) \times \text{tr}(6 \tilde{M}_{uu}^2 + 12 \tilde{M}_{uu}^2 + 12 \tilde{M}_{ud}^2 + 2 \tilde{M}_{ud}^2 + 4 \tilde{M}_{uu}^2 + 4 \tilde{M}_{ud}^2). \]

We perform the reparameterizations

\[ \mathbf{A} = \sum_{a=1}^{8} \frac{i g_0}{2} W^{a} \mathbf{\gamma}^{a} \otimes \mathbf{\lambda}^{a}, \quad \bar{\mathbf{A}} = \sum_{a=1}^{3} \frac{i g_0}{2} W^{a} \mathbf{\gamma}^{a} \otimes \mathbf{\sigma}^{a}, \quad A^0 = \frac{i g_0}{2} \sqrt{3} W^{0} \mathbf{\gamma}^{0}, \]

\[ \Phi_i = g_0 \phi_i / \sqrt{\text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{1}{3} M_{e} M_{e}^{*})}, \quad i = 1, 2, \]

where \( \{\sigma^a\} \) are the Pauli matrices and \( \{\lambda^a\} \) the Gell-Mann matrices. Using

\[ \text{tr}(\mathbf{\gamma}^{a} \otimes \mathbf{\gamma}^{a})(\mathbf{\gamma}^{b} \otimes \mathbf{\gamma}^{b}) = 4 (\delta^{ab} \delta^{\nu} - \delta^{a\nu} \delta^{b\nu}), \quad \text{tr}(\mathbf{\gamma}^{a} \otimes \mathbf{\gamma}^{b}) = 4 \delta^{ab}, \quad \text{tr}(1) = 4 \]

and performing a Wick rotation to Minkowski space we obtain for (21) precisely the bosonic action of the standard model, see [12]. Here, the Weinberg angle \( \theta_W \) and the masses \( m_W, m_Z \) and \( m_H \) of the \( W, Z \) and Higgs bosons are given by

\[ m_W = \frac{1}{2} \sqrt{\text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{1}{3} M_{e} M_{e}^{*})} = \frac{1}{2} m_t, \]

\[ m_Z = m_W / \cos \theta_W, \quad \sin^2 \theta_W = \frac{3}{8}, \]

\[ m_H = \frac{\sqrt{\text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{2}{3} M_{e} M_{e}^{*} + \frac{2}{3} M_{e} M_{e}^{*})}}{\text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{1}{3} M_{e} M_{e}^{*})} = \frac{3}{2} m_t, \]

\[ \text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{1}{3} M_{e} M_{e}^{*} + \frac{2}{3} M_{e} M_{e}^{*}) = \text{tr}(M_u M_u + M_d M_d^{*} + \frac{1}{3} M_{\nu} M_{\nu} + \frac{1}{3} M_{e} M_{e}^{*}). \]
where \( m_t \) is the mass of the top quark. Here we have neglected the other fermion masses against \( m_t \). The analogous relations in non-commutative geometry read for the simplest scalar product \[6\]

\[
m_W = \frac{1}{2} m_t, \quad m_H = \sqrt{\frac{69}{28}} m_t, \quad \sin^2 \theta_W = \frac{12}{29}.
\]

(24)

Inserting (20) and (22) into the fermionic action in (8) we arrive after a Wick rotation to Minkowski space and imposing the chirality condition \( \Gamma h = h \) at the usual fermionic action of the standard model [12].

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