New classes of analytic and bi-univalent functions

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Abstract: Using the (p,q)-derivative operator we introduce new subclasses of analytic and bi-univalent functions, we obtain estimates on coefficients and the Fekete-Szegö functional.

Keywords: Fekete-Szegö problem; (p,q)-derivative operator; univalent functions; bi-univalent functions; analytic functions; coefficient bounds and coefficient estimates

Mathematics Subject Classification: 30C45, 30C50

1. Introduction and Preliminary results

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

(1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$.

Let $S \subset A$ denote the class of all functions in $A$ which are univalent in $U$.

The Koebe One-Quarter Theorem [5] ensures that the image of the unit disk under every $f \in S$ functions contains a disk of radius $1/4$.

It is well known that every functions $f \in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

(1.2)
A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \).

Let \( \Sigma \) denote the class of all bi-univalent functions in \( U \) given by (1).

The class of bi-univalent functions was first introduced and studied by Lewin [12] and was showed that \( |a_2| < 1, 51 \).

The problem of maximizing the absolute value of the functional \( |a_3 - \mu a_2^2| \) is called the Fekete-Szegő problem. Many authors obtained Fekete-Szegő inequalities for different classes of functions [2], [6], [8], [11], [14]. Coefficient estimates of bi-univalent functions are studied in many papers, of which I mention: [1], [3], [7], [9], [10], [12], [17], [18], [19], [20].

For the new results posted in the last section we have to recall the necessary elements of the \((p, q)\)-calculus involving.

There is possibility of extension of the \( q \)-calculus to post quantum calculus denoted by the \((p, q)\)-calculus.

When the case \( p = 1 \) in \((p, q)\)-calculus, the \( q \)-calculus may be obtained.

In order to derive our main results, we need to following lemmas:

**Lemma 1.** [15] If \( p \in \mathcal{P} \), then \( |c_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( U \) for which

\[
\mathcal{R}(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \ldots,
\]

for \( z \in U \).

**Lemma 2.** [5] Let \( p \in \mathcal{P} \) be of the form \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \). Then

\[
|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2} \quad \text{and} \quad |c_k| \leq 2, \quad k \in \mathbb{N}.
\]

**Lemma 3.** [13] If \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \), \( z \in U \) is a function with positive real part in \( U \) and \( \mu \) is a complex number, then

\[
|c_2 - \mu c_1^2| \leq 2 \max\{1; 2\mu - 1\}.
\]

The result is sharp for the function given by

\[
p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}, \quad z \in U.
\]

**Definition 4.** Let \( f \in \mathcal{A} \) given by (1.1) and \( 0 < q < p \leq 1 \). Then the \((p, q)\)-derivative operator or \( p, q \)-difference operator for the function \( f \) of the form (1.1) is defined by

\[
D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad z \in U^* = U - \{0\}
\] (1.3)

and

\[
(D_{p,q}f)(0) = f'(0)
\] (1.4)

provided that the function \( f \) is differentiable at 0.

From the relation (1.2), we deduce that

\[
D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}
\] (1.5)
where the \((p, q)\)-bracket number or twin-basic is given by
\[
[k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + ... + pq^{k-2} + q^{k-1}, p \neq q
\]
which is a natural generalization of the \(q\)-number.

Also \(\lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}\)

See [4,16].

2. Main results

**Definition 5.** A function \(f\) given by (1.1) is said to be in the class \(H^{p,q,\alpha}_{\Sigma}(0 < q < p \leq 1, 0 < \alpha \leq 1)\), if the following conditions are satisfied:

\[
\begin{align*}
& f \in \Sigma \\
& \left| \arg(D_{p,q}f(z)) \right| < \frac{\alpha \pi}{2}, (z \in U)
\end{align*}
\]

and

\[
\left| \arg(D_{p,q}g(w)) \right| < \frac{\alpha \pi}{2}, (w \in U)
\]

where the function \(g\) is given by (1.2).

**Remark 6.** When \(p = 1\), \(\lim_{q \to 1^-} H^{p,q,\alpha}_{\Sigma} = H^{\alpha}_{\Sigma}\), where \(H^{\alpha}_{\Sigma}\) is the class introduced in [19].

In the next theorem we obtain coefficient bounds for the functions class \(H^{p,q,\alpha}_{\Sigma}\).

**Theorem 7.** Let the function \(f\) given by (1.1) be in the function class \(H^{p,q,\alpha}_{\Sigma}(0 < q < p \leq 1, 0 < \alpha \leq 1)\). Then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{2[3]_{p,q}\alpha + (1-\alpha)[2]_{p,q}^3}}
\]

and

\[
|a_3| \leq \frac{4\alpha^2}{[2]_{p,q}^2} + \frac{2\alpha}{[3]_{p,q}}
\]

**Proof.** From the relations (2.1) and (2.2) it follows that

\[
D_{p,q}f(z) = [P(z)]^\alpha
\]

and

\[
D_{p,q}g(w) = [Q(w)]^\alpha, (z, w \in U)
\]

where \(P(z) = 1 + p_1z + p_2z^2 + ...\) and \(Q(z) = 1 + q_1w + q_2w^2 + ...\) in \(\mathcal{P}\).

From the relation (2.5), we obtain the next relations

\[
[2]_{p,q}a_2 = \frac{p^2 - q^2}{p - q} = \alpha p_1
\]
\[ [3]_{p,q}a_3 = a_3 \frac{p^3 - q^3}{p - q} = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 \]  
(2.7)

\[- [2]_{p,q}a_2 = -a_2 \frac{p^2 - q^2}{p - q} = \alpha q_1 \]  
(2.8)

and

\[ [3]_{p,q}(2a_2^2 - a_3) = (2a_2^2 - a_3) \frac{p^3 - q^3}{p - q} = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \]  
(2.9)

It follows that

\[ p_1 = -q_1 \]  
(2.10)

and

\[ 2[2]_{p,q}a_2^2 = \alpha^2(p_1^2 + q_1^2) \]  
(2.11)

The relations (2.10) and (2.11) are obtained from the relations (2.6) and (2.8). We obtain that

\[ 2[3]_{p,q}a_2^2 = 2a_2^2 \frac{p^3 - q^3}{p - q} = \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) = \]

\[ = \alpha (p_2 + q_2) + \frac{\alpha - 1}{\alpha} [2]_{p,q}a_2^2. \]  
(2.12)

This relation is obtained from (2.9) and (2.11). We get from (2.12)

\[ a_2^2 = \frac{\alpha^2}{2[3]_{p,q} \alpha + (1 - \alpha) [2]_{p,q}^2} (p_2 + q_2). \]  
(2.13)

From Lemma 1 for the above equality, we get the estimate on the coefficient \( |a_2| \) as asserted in the relation (2.3).

We subtract (2.9) from (2.7) and find the bound on the coefficient \( |a_3| \).

We get it that way

\[ 2[3]_{p,q}a_3 - 2[3]_{p,q}a_2^2 = 2a_3 \frac{p^3 - q^3}{p - q} - 2a_2^2 \frac{p^3 - q^3}{p - q} = \alpha (p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \]  
(2.14)

It follows that,

\[ a_3 = \frac{\alpha^2}{2[2]_{p,q}^2} (p_1^2 - q_1^2) + \frac{\alpha}{2[3]_{p,q}} (p_2 - q_2), \]

from the relations (2.10), (2.11) and (2.14).

And if we apply Lemma 1 for the above equality, we obtain the estimate on the coefficient \( |a_3| \) as asserted in (2.4). \( \square \)

When \( p = 1, \quad q \to 1^- \) in Theorem 7, we obtain the following result given in [19].

**Corollary 8.** [19] Let \( f(z) \) given by (1.1) be in the class \( H^\alpha_\Sigma(0 < \alpha \leq 1) \). Then \( |a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 2}} \) and \( |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}. \)
Definition 9. A function $f$ given by (1.1) is said to be in the class $H^{p,q,\beta}_\Sigma (0 < q < p \leq 1, 0 \leq \beta < 1)$ if the following conditions are satisfied:

$$\begin{cases}
f \in \Sigma \\
\mathcal{R}(D_{p,q}f(z)) > \beta, z \in U \\
\mathcal{R}(D_{p,q}g(w)) > \beta, w \in U,
\end{cases}$$

where the function $g$ is defined by (1.2).

Remark 10. When $p = 1$ and $q \to 1^-$ we obtain the class $H^\Sigma_\Sigma(\beta)$ introduced in [19].

In the next theorem we obtain coefficient bounds for the functions class $H^{p,q,\beta}_\Sigma$.

Theorem 11. Let the function $f$ be given by (1.1) be in the function class $H^{p,q,\beta}_\Sigma (0 < q < p \leq 1)$. Then

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{[2]_p}, \sqrt{\frac{2(1-\beta)}{[3]_p}}\right\}$$

(2.17)

$$|a_3| \leq \frac{2(1-\beta)}{[3]_p}.$$

(2.18)

Proof. It follows from the conditions (2.15) and (2.16) that

$$D_{p,q}f(z) = \beta + (1-\beta)p(z) \quad \text{and} \quad D_{p,q}g(w) = \beta + (1-\beta)q(w), \quad z, w \in U$$

(2.19)

respectively, where

$$P(z) = 1 + p_1z + p_2z^2 + ...$$

and

$$Q(w) = 1 + q_1q + q_2w^2 + ... \quad \text{in} \quad \mathcal{P}.$$

We obtain

$$[2]_{p,q}a_2 = a_2 \frac{p^2 - q^2}{p - q} = (1-\beta)p_1$$

(2.20)

$$[3]_{p,q}a_3 = a_3 \frac{p^3 - q^3}{p - q} = (1-\beta)p_2$$

(2.21)

$$-[2]_{p,q}a_2 = -a_2 \frac{p^2 - q^2}{p - q} = (1-\beta)q_1$$

(2.22)

$$[3]_{p,q}(2a_2^2 - a_3) = (2a_2^2 - a_3) \frac{p^3 - q^3}{p - q} = (1-\beta)q_2,$$

(2.23)

upon equating the coefficients in (2.19). Now, from the relations (2.20) and (2.22), we obtain

$$p_1 = -q_1$$

(2.24)

and

$$2[2]_{p,q}a_2^2 = (1-\beta)^2(p_1^2 + q_1^2)$$

(2.25)
Now, from the relations (2.21) and (2.23) we have
\[ 2[3]_{p,q}a_2^2 = 2a_2^2 \frac{p^3 - q^3}{p - q} = (1 - \beta)(p_2 + q_2) \] (2.26)

From Lemma 1 for the relations (2.26) and (2.25), we obtain the estimate on the coefficient \(|a_2|\) as asserted in (2.17).

Now, we subtract (2.23) from (2.21) and we obtain
\[ 2[3]_{p,q}a_3 - 2[3]_{p,q}a_2^2 = 2a_3 \frac{p^3 - q^3}{p - q} - 2a_2^2 \frac{p^3 - q^3}{p - q} = (1 - \beta)(p_2 - q_2) \] (2.27)

From (2.25) and (2.26) we can obtain \(a_2^2\). From (2.27) we get
\[ a_3 = \frac{(1 - \beta)^2}{2[3]_{p,q}}(p_2^2 + q_2^2) + \frac{(1 - \beta)}{2[3]_{p,q}}(p_2 - q_2) \] (2.28)

By using the relation (2.26) into (2.27) it follows that
\[ a_3 = \frac{(1 - \beta)}{2[3]_{p,q}}(p_2 + q_2) + \frac{1 - \beta}{2[3]_{p,q}}(p_2 - q_2) = \frac{(1 - \beta)p_2}{p^3 - q^3} = \frac{(1 - \beta)p_2}{[3]_{p,q}} \] (2.29)

If we apply Lemma 1 for the relations (2.28) and (2.29) we get the estimate on the coefficient \(|a_3|\) as asserted in (2.18).

We obtain the next corollary.

**Corollary 12.** Let \( f(z) \) given by (1.1) be in the function class \( H_2(\beta)(0 \leq \beta < 1) \). Then
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{3}} \]
and
\[ |a_3| < \frac{(1 - \beta)(5 - 3\beta)}{3}. \]

**Definition 13.** Let \( b, t : U \rightarrow \mathbb{C} \) be analytic functions and
\[ \min\{R(b(z)), R(t(z))\} > 0, \forall z \in U, \]
\[ b(0) = t(0) = 1. \]

A function \( f \) given by (1.1) is said to be in the class \( H^{p,q,b,t}_\Sigma \) if the following conditions are satisfied:
\[ D_{p,q}f(z) \in b(U) \] (2.30)
and
\[ D_{p,q}g(w) \in t(U), \] (2.31)
where \( z, w \in U \) and the function \( g \) is given by (1.2).

In the next theorem we obtain coefficient bounds for the functions class \( H^{p,q,b,t}_\Sigma \).
Theorem 14. Let $f$ given by (1.1) be in the class $H^{p,q,b,t}_\Sigma$. Then

$$|a_2| \leq \min\{\sqrt{\frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}}}, \sqrt{\frac{|b''(0)| + |t''(0)|}{2[3]_{p,q}}}\}$$  \hspace{1cm} (2.32)

$$|a_3| \leq \min\{\sqrt{\frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}}} + \frac{|b''(0)| + |t''(0)|}{4[3]_{p,q}} + \frac{|b''(0)|}{2[3]_{p,q}}\}$$  \hspace{1cm} (2.33)

Proof. We will write the equivalent forms of the argument inequalities in the relations (2.32) and (2.33).

$$D_{p,q}f(z) = b(z)$$  \hspace{1cm} (2.34)

and

$$D_{p,q}g(w) = t(w),$$  \hspace{1cm} (2.35)

where $b(z)$ and $t(w)$ satisfy the conditions from Definition 13 and it have the following Taylor-Maclaurin series expansions:

$$b(z) = 1 + b_1 z + b_2 z^2 + ...$$  \hspace{1cm} (2.36)

$$t(w) = 1 + t_1 w + t_2 w^2 + ....$$  \hspace{1cm} (2.37)

We find that

$$[2]_{p,q} a_2 = b_1$$  \hspace{1cm} (2.38)

$$[3]_{p,q} a_3 = b_2$$  \hspace{1cm} (2.39)

$$- [2]_{p,q} a_2 = t_1$$  \hspace{1cm} (2.40)

$$[3]_{p,q}(2a_2^2 - a_3) = t_2,$$  \hspace{1cm} (2.41)

subtracting from (2.36) and (2.37) into (2.34) and (2.35) and equating the coefficients.

We obtain that

$$b_1 = -t_1$$  \hspace{1cm} (2.42)

and

$$2[2]_{p,q} a_2^2 = b_1^2 + t_1^2$$  \hspace{1cm} (2.43)

from the relations (2.38) and (2.40). We obtain that

$$[3]_{p,q} a_3 + [3]_{p,q}(2a_2^2 - a_3) = b_2 + t_2$$  \hspace{1cm} (2.44)

Adding the relation (2.2) and (2.3). From the relations (2.44) and (2.45), we can find

$$a_2^2 = \frac{b_1^2 + t_1^2}{2[2]_{p,q}}$$  \hspace{1cm} (2.45)

and

$$a_2^2 = \frac{t_2 + b_2}{2[3]_{p,q}}.$$  \hspace{1cm} (2.46)
We can calculate from the relations (2.45) and (2.46) that
\[ |a_2|^2 \leq \frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}} \]
and
\[ |a_2|^2 \leq \frac{|b''(0)| + |t''(0)|}{2[3]_{p,q}}. \]

So, we obtain the desired estimate on the coefficient \(|a_2|^2\) as asserted in the relation (2.32).

Now, subtracting the relation (2.41) from the relation (2.39), we obtain
\[ 2[3]_{p,q}a_3 - 2a_2^2[3]_{p,q} = b_2 - t_2 \tag{2.47} \]
Substituting \(a_2^2\) from (2.45) into (2.47) it follows that
\[ a_3 = \frac{b_2 - t_2}{2[3]_{p,q}} + \frac{b_1^2 + t_1^2}{2[2]_{p,q}}. \]

It follows that
\[ |a_3| \leq \frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}} + \frac{|b''(0)| + |t''(0)|}{4[3]_{p,q}}. \]

Substituting the value of \(a_2^2\) from the relation (2.46) into the relation (2.47), we get
\[ a_3 = \frac{b_2}{[3]_{p,q}}. \]

It follows that
\[ |a_3| \leq \frac{|b''(0)|}{2[3]_{p,q}}. \]

Now, we compute the Fekete-Szegö functional for the class \(H^{p,q,\alpha}_\Sigma\).

**Theorem 15.** Let \(f\) of the form (1.1) be in the class \(H^{p,q,\alpha}_\Sigma\). Then
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha}{|\alpha|_{p,q}}, |r(\mu)| \leq \frac{1}{|\alpha|_{p,q}} \\ 2\alpha|r(\mu)|, |r(\mu)| \geq \frac{1}{|\alpha|_{p,q}} \end{cases} \tag{2.48} \]

**Proof.** From Theorem 7 we can use the value of the coefficients \(a_2^2\) and \(a_3\) to compute \(a_3 - \mu a_2^2\).

\[ a_3 - \mu a_2^2 = \alpha[p_2(\frac{1}{2[3]_{p,q}} + \frac{(1 - \mu)\alpha}{2[3]_{p,q}\alpha + (1 - \alpha)[2]_{p,q}^2}) + \]
\[ + q_2(\frac{(1 - \mu)\alpha}{2[3]_{p,q}\alpha + (1 - \alpha)[2]_{p,q}^2} - \frac{1}{2[3]_{p,q}})]. \]

If we denote by
\[ r(\mu) = (1 - \mu)\alpha \frac{\alpha}{2[3]_{p,q}\alpha + (1 - \alpha)[2]_{p,q}^2}, \]

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It follows that
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha}{3[3]_{p,q}}, |r(\mu)| \leq \frac{1}{2[3]_{p,q}} \\ 2\alpha |r(\mu)|, |r(\mu)| \geq \frac{1}{3[3]_{p,q}} \end{cases} \]

Now, we compute the Fekete-Szegö functional for the class \( H^{\mu}_{\Sigma} \).

**Theorem 16.** Let \( f \) of the form (1.1) be in the class \( H^{\mu}_{\Sigma} \). Then
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{3[3]_{p,q}}, r(\mu) \leq \frac{1}{3[3]_{p,q}} \\ 2(1-\beta) |r(\mu)|, |r(\mu)| \geq \frac{1}{3[3]_{p,q}} \end{cases} \] (2.49)

**Proof.** From Theorem 11 we can use the value of the coefficients \( a_2^2 \) and \( a_3 \) to compute \( a_3 - \mu a_2^2 \).

\[
a_3 - \mu a_2^2 = (1 - \beta) [p_2(\frac{1}{2[3]_{p,q}} + \frac{1-\mu}{2[3]_{p,q}}) + q_2(\frac{1-\mu}{2[3]_{p,q}} - \frac{1}{2[3]_{p,q}})].
\]

If we denote by 
\[ r(\mu) = (1 - \mu) \frac{1}{2[3]_{p,q}} \]

It follows that
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{3[3]_{p,q}}, r(\mu) \leq \frac{1}{3[3]_{p,q}} \\ 2(1-\beta) |r(\mu)|, |r(\mu)| \geq \frac{1}{3[3]_{p,q}} \end{cases} \]

**Conflict of interest**

The author declares no conflicts of interest in this paper.

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