Heavy quark master equations in the Lindblad form at high temperatures

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(Dated: March 25, 2014)

We derive the quantum master equations for heavy quark systems in a high-temperature quark-gluon plasma in the Lindblad form. The master equations are derived in the influence functional formalism for open quantum systems in perturbation theory. These master equations have a wide range of applications, such as propagation of a heavy quarkonium in the quark-gluon plasma and quantum Brownian motion of a heavy quark. We also show the equivalence between the quarkonium master equations in the recoilless limit and the Schrödinger equations with stochastic potential.

I. INTRODUCTION

The fate of heavy quarkonium bound states in finite temperature QCD matter has long been considered as a sensitive probe of deconfined nature of such a matter. In the deconfined phase at finite temperature, the linear potential that confines a heavy quark-antiquark pair in the vacuum is screened by colored excitations (light quarks and gluons) in the medium and in such a short-ranged potential the quarkonium bound state levels will eventually disappear at high temperature. In relativistic heavy-ion collisions, suppression of the quarkonium yield, in particular Υ and J/Ψ states, is expected to serve as a signal for the formation of a quark-gluon plasma (QGP) \[ \Psi \]. The deconfined matter that once existed just after the Big Bang in the early universe. Indeed, experimental data from CMS collaboration at the LHC shows sequential suppression of Υ states (the ground (1S) and excited states (2S,3S)) \[ \Psi \], suggesting sequential melting of the Υ bound states in the QGP. To investigate the real-time dynamics of quarkonium quantum states and its suppression in the QGP, the appropriate theoretical framework that of open quantum systems \[ \Psi \].

There have been various studies on the quarkonium properties at finite temperature. Thermodynamic quantities such as free energy changes caused by putting two infinitely heavy quark and antiquark are calculated in lattice QCD simulations. The results clearly show that color charges are screened above the quark-hadron transition temperature \( T_c \). Spectral structures of Υ and J/Ψ are investigated by lattice QCD simulations and suggest that the ground states are fairly stable even at higher temperature \( T < 2T_c \). The stability of quarkonium ground states indicates a strongly coupled nature of the quark-gluon plasma \[ \Psi \]. It is not yet clear how these independent observations by numerical simulations can be understood in a unified viewpoint.

Recently, a real-time static potential, defined in terms of the real-time propagator of quarkonium operator at finite temperature, has been calculated in perturbation theory \[ \Psi \], non-perturbative lattice QCD simulations \[ \Psi \], and the potential non-relativistic QCD approach \[ \Psi \]. The same quantity has also been calculated for strong coupling plasmas using the conjectured gauge/gravity duality \[ \Psi \]. The real-time static potential is one of the crucial quantities to understand the quarkonium dynamics in the QGP. The potential is found to be complex valued with a negative imaginary part. Although the potential has an imaginary part, particle number conservation of the non-relativistic heavy quarks with infinite mass must not be violated. Quantum decoherence of quarkonium wave functions due to stochastic processes (stochastic potential) can give a physically natural explanation to the imaginary part \[ \Psi \]. Clearly, the complex potential and its stochastic potential interpretation indicates that the quantum mechanical properties of quarkonium in the QGP must be studied from the viewpoint of the open quantum systems \[ \Psi \], which we will summarize briefly.

In general, dynamics of open quantum systems is characterized by the reduced density matrix \( \hat{\rho}_S(t) \) defined as \( \hat{\rho}_S(t) \equiv \text{Tr}_E \hat{\rho}_{\text{tot}}(t) \). Here \( \hat{\rho}_{\text{tot}}(t) \) is the total density matrix for both system and environment degrees of freedom and \( \text{Tr}_E \) denotes the trace over the environment degrees of freedom. In the case of heavy quark systems in the QGP, the system consists of heavy quarks and the environment is composed of light quarks and gluons. Time-evolution of \( \hat{\rho}_S(t) \) in the Markov limit is given by the quantum master equation:

\[
\frac{d}{dt} \hat{\rho}_S(t) = \mathcal{L} \hat{\rho}_S(t). \tag{1}
\]

Note that the generator of the time evolution \( \mathcal{L} \) is a superoperator which acts on the operator \( \hat{\rho}_S(t) \).

Several studies have introduced descriptions of heavy quark systems as open quantum systems \[ \Psi \]. In Ref. \[ \Psi \], quantum master equations are first derived at leading-order in perturbation for non-relativistic heavy quark systems. In this derivation, the Feynman-Vernon’s influence functional formalism \[ \Psi \] is applied to the finite-temperature QCD at the level of quantum field theory. As a result, a quantum master equation is derived in the form of a functional differential equation of heavy quark fields, from which one can derive the quantum master equations for systems with arbitrary finite number of heavy quarks, such as a single heavy quark or a quarkonium, in the QGP.
One of the purposes of this paper is to extend the result of Ref. [15] and derive heavy quark master equations in the Lindblad form [17]. The Lindblad form is the form of the super-operator $\mathcal{L}$ which any Markovian master equation that preserves complete positivity must conform to. The Lindblad form is generally expressed with an hermitian Hamiltonian $\hat{H}$, Lindblad operators $\hat{L}_i$, and positive coefficients $\gamma_i > 0$ ($i = 1, 2, \cdots, N$):

$$\frac{d}{dt} \rho_S(t) = -i[\hat{H}, \rho_S]$$

$$+ \sum_{i=1}^{N} \gamma_i \left( \hat{L}_i \rho_S \hat{L}_i^\dagger - \frac{1}{2} \hat{L}_i^\dagger \hat{L}_i \rho_S - \frac{1}{2} \rho_S \hat{L}_i^\dagger \hat{L}_i \right).$$

In particular, we will derive explicit forms of the Lindblad-form master equations for single heavy quark systems and for heavy quark-antiquark systems. By deriving the master equations in the Lindblad form, we may be able to utilize several techniques, such as quantum state diffusion method [18] and quantum jump method [19], to numerically simulate the master equation in terms of wave function. Numerical calculation with wave function has substantial advantage over that of the master equation because the spatial dimension of a wave function is half of that of a density matrix.

The other purpose is to present a theoretical basis to the concept of stochastic potential, which was first introduced in Ref. [11] and has been recently simulated in Ref. [20]. This is partly because we find several confusing applications of the complex potential to the problem of quarkonium survival probability. 1 By definition, the master equation corresponding to the stochastic potential is of the Lindblad form because we derive it from the ensemble of wave functions with positive probability. Therefore the stochastic potential can be regarded as a method to calculate certain type of the Lindblad-form master equations in terms of wave function. The stochastic potential has two sources of quantum decoherence: One is decoherence among the wave functions in the ensemble at the same point $X = (\vec{x}_Q, \vec{x}_{\bar{Q}})$, and the other is decoherence in each wave function at different points $X$ and $Y$, where $\vec{x}_Q$ and $\vec{x}_{\bar{Q}}$ denote positions of heavy quark and antiquark [11]. With the information of the complex potential, we only know the former source for decoherence. Thus if we only know the complex potential, as is the case at present for non-perturbative lattice calculation for the complex potential [8], we can just guess the latter, for example, by referring to the perturbative results. In the perturbative analysis, we will see that the imaginary part of the complex potential has enough information to know the decoherence at different points.

This paper is organized as follows. In Sec. II, we begin with a review of the method developed in Ref. [15] and update it by including a new term necessary to obtain the master equations in the Lindblad form. In Sec. III we derive several master equations for a single heavy quark or a quarkonium in the QGP. We show that the master equations can be simplified for localized wave packets. We also show that wave function decoherence, if it proceeds fast enough, can be described by master equations in the recoilless limit. Each of them is shown to be in the Lindblad form. In Sec. IV we give the stochastic potential with color degrees of freedom, an extension of Ref. [11]. We also discuss the decoherence of quarkonium wave functions by comparing two scales, correlation length of thermal fluctuation and coherence length of a wave function. Section V is devoted to a summary. Throughout this paper, we adopt the natural units, $\hbar = c = k_B = 1$, and operators in Hilbert and Fock spaces are denoted by bold fonts.

II. INFLUENCE FUNCTIONAL OF HEAVY QUARKS

In this section, we review and also update the formalism developed in Ref. [15]. The formalism relies on three approximations for actual computations: (i) Expansion in terms of coupling constant $g \ll 1$, (ii) Expansion in terms of heavy quark velocity $v \ll 1$, and (iii) Coarse-graining in time. Through these approximations, we can obtain the influence functional and renormalized effective Hamiltonian, from which master equations for an arbitrary number of heavy quarks can be derived. The region of validity of these approximations is summarized in Table I.

| Approximation          | Heavy quark | Heavy quarkonium |
|------------------------|-------------|------------------|
| (i) Perturbation       | $g \ll 1$   | $g \ll 1$        |
| (ii) Velocity expansion| $\sqrt{T/M} \ll 1$ | $\alpha \ll 1$ |
| (iii) Coarse-graining  | $1/gT \ll 1/M\alpha^2$ |              |

TABLE I: Region of validity of the approximations

A. How small is heavy quark velocity?

Here we will explain why we only consider the case where heavy quark velocity is small. First of all, the heavy quark velocity in unbound states is estimated as

1 There is no conceptual problem in using the complex potential to calculate vector correlation function and its spectral function in the potential model. However, there is a conceptual problem if one calculates expectation values by using a wave function and its complex conjugate that are evolved by the Schrödinger equation with the complex potential. Also, using the complex potential, one can calculate the imaginary part of the self energy, or the width, which gives a rate of transition from one state to the other states in one scattering. However, since the width does not specify the final states in the scattering, it cannot tell anything about the occupation number other than the initial state. Also it does not contain enough information to study the case where there are repeated scatterings.
$v \sim \sqrt{T/M}$ in thermal equilibrium. Here the heavy quark mass is $M_h \approx 4.8$ GeV and $M_c \approx 1.5$ GeV for bottom and charm quarks and the typical temperature is $T \approx (1-3)T_{\text{crit}} \sim 200-500$ MeV in heavy-ion collision experiments. Therefore, close to thermal equilibrium, the heavy quark velocity is small for both bottom and charm quarks in unbound states.

In the case of quarkonium bound states, we need to take into account the relative velocity and acceleration of a heavy quark-antiquark pair. Let us now consider bound states in the Coulomb potential $V(r) = -\alpha/r$ since the fastest relative velocity can be estimated by the most deeply bound states. The Coulomb part of the phenomenological Cornell potential is typically chosen as $\alpha \approx 1/4$. The momentum of Coulomb bound states is estimated as $p \sim M\alpha$ and thus $v \sim \alpha \approx 1/4$. Therefore, in the temperature range of phenomenological interest, we can safely assume that the relative velocity of the heavy quark-antiquark pair is small.

As for the acceleration by the potential force, it can be estimated by $\dot{v} \sim \alpha/Mr^2 \sim M\alpha^3$. When we perform coarse-graining in time later, we need to assume that the effect of acceleration is small during a scattering event. Since typical correlation time of the medium is $\sim 1/gT$ or shorter ($\sim 1/T$), the medium is obtained as $M\alpha^2 \ll gT$. Using phenomenological values $\alpha \approx 1/4$ and $g \approx 2$, this condition is satisfied for charmonium but not very obvious for bottomonium at lower temperature. Nevertheless, we neglect the effect of acceleration in the bottomonium bound states because $M\alpha^2 \gg gT$ is also not at all obvious for these states.

### B. Influence functional

The influence functional can be defined using the closed-time path formalism of non-equilibrium field theory. In the closed-time path formalism, fields $\phi = (A, q, \psi)$ on forward (backward) time axis are denoted by $\phi_1 (\phi_2)$, where $A$ is gauge field, $q$ is light quark field, and $\psi$ is heavy quark field. The influence functional $S_{\text{IF}}$ is defined as a functional of the heavy quark color current $j^a_\mu = \bar{\psi}t^a \gamma_\mu \psi$:

\[
\exp[iS_{\text{IF}}[j_1, j_2]] = \int \mathcal{D}[q, A]_1\mathcal{D}[q, A]_2 |\rho^0_E[q_2, A_2]t_0 \times \exp[i \int_{t_0} d^4x \{ \mathcal{L}_{g+q} (q, A_1) - gj_1^a_\mu A_1^a_\mu \} ] \times \exp[-i \int_{t_0} d^4x \{ \mathcal{L}_{g+q} (q, A_2) - gj_2^a_\mu A_2^a_\mu \} ].
\]

Here $\rho^0_E$ is the equilibrium density matrix at temperature $T$ for interacting gluons and light quarks and $|A, q\rangle$ is the coherent state introduced to obtain path-integral formulation.

### 1. Perturbative expansion

In the perturbative expansion, assuming the medium temperature is very high but much lower than the heavy quark mass, the leading-order terms in $S_{\text{IF}}$ are given by

\[
iS_{\text{IF}}[j_1, j_2] = -\frac{g^2}{2} \int_{t_0} d^4x d^4y \left( j_1^a_\mu, j_2^a_\mu \right) (x) \times \left[ G^F_{ab,\mu
u} - G^<_{ab,\mu\nu} \right] (x-y) \left( j_1^b_\nu, j_2^b_\nu \right) (y) + O(g^3).
\]

The two-point functions of gluons are defined as

\[
G^F_{ab,\mu
u}(x-y) \equiv \langle T A^a_\mu(x) A^b_\nu(y) \rangle, \quad G^<_{ab,\mu\nu}(x-y) \equiv \langle A^a_\mu(x) A^b_\nu(y) \rangle, \quad G^S_{ab,\mu\nu}(x-y) \equiv \langle A^a_\mu(x), A^b_\nu(y) \rangle,
\]

where $\langle \mathcal{O} \rangle$ denotes thermal average in the gluon and light quark system. For completeness, let us also define following retarded and advanced propagators, symmetrized correlation function, and spectral function:

\[
G^R_{ab,\mu\nu}(x-y) \equiv i\theta(x^0-y^0) \langle [A^a_\mu(x), A^b_\nu(y)] \rangle, \quad G^A_{ab,\mu\nu}(x-y) \equiv -i\theta(y^0-x^0) \langle [A^a_\mu(x), A^b_\nu(y)] \rangle,
\]

\[
G^S_{ab,\mu\nu}(x-y) \equiv \langle [A^a_\mu(x), A^b_\nu(y)] \rangle, \quad \sigma_{ab,\mu\nu}(\omega, \vec{x} - \vec{y}) = \int dt e^{-i\omega(x^0-y^0)} \langle [A^a_\mu(x), A^b_\nu(y)] \rangle.
\]

### 2. Velocity expansion

When the heavy quark mass is much larger than the medium temperature, a non-relativistic approximation is justified. The heavy quark-antiquark pair contribution in the heavy quark color current is ignored since its effect is tiny so that only the non-relativistic velocity term remains to give an estimate $j^a \sim v^a$. This corresponds to the quenched approximation. In the leading order of the heavy quark velocity expansion, $S_{\text{IF}}$ is approximated as interaction terms of color densities ($\rho^a \equiv j^a$):

\[
iS_{\text{IF}}[j_1, j_2] = -\frac{g^2}{2} \int_{t_0} d^4x d^4y \left( \rho^0_1, \rho^0_2 \right) (x) \times \left[ G^F_{ab,00} - G^<_{ab,00} \right] (x-y) \left( \rho^b_1, \rho^b_2 \right) (y) + \cdots,
\]

where corrections are of the order of $O(g^3)$ or $O(g^2 v)$. For later purpose of coarse-graining in time, let us change the time variables from $(x^0, y^0)$ to $(t, s)$ with

\[
t = \max(x^0, y^0), \quad s = |x^0 - y^0|.
\]
The new time variable $t$ is taken to be always the later one of $x^0$ and $y^0$. This is essential in obtaining correct time-evolution equations. Based on the new time variables, the interaction terms can be schematically written as

$$\int_{t_0} d^4x d^4y \rho(x)G(x-y)\rho(y)$$

$$= \int_{t_0} dt \int_{t-t_0} dt' ds \int d^3x d^3y \rho(t,x)G(s,x-y)\rho(t-s,y)$$

$$\approx \int_{t_0} dt \int_0^\infty ds \int d^3x d^3y \rho(t,x)G(s,x-y)\rho(t-s,y)$$

$$\times \left( \rho(t,x)G(s,x-y)\rho(t-s,y) + \rho(t-s,x)G(s,x-y)\rho(t,y) + \rho(t-s,y)G(s,x-y)\rho(t,x) \right).$$

The final expression is obtained because the information of the initial time $t_0$ will become irrelevant after (a few times) the finite correlation time of gluons, which is much shorter than the dynamical time scales of the heavy quark systems, such as the relaxation time. The correlation time of gluons is approximately $1/g^2 T$ or shorter ($\sim 1/T$) while the shortest dynamical time scale of heavy quark systems is $1/g^2 T \gg 1/g T$ for decoherence and color diffusion. Using the symmetry of gluon two point functions, such as $G^F_{ab}(s,-\vec r) = G^-_{ab}(s,\vec r)$ and $G_0^F_{ab}(s,-\vec r) = G_0^-_{ab}(s,\vec r)$, we obtain

$$iS_{\Sigma} [j_1,j_2] \approx -g^2 \int_{t_0} dt \int d^3x d^3y \int_{t_0}^{\infty} ds \left( \rho_1^a, \rho_2^a \right)_{(t,x)}$$

$$\times \left[ G^F_{ab,00}(s,-\vec r) - G^-_{ab,00}(s,-\vec r) \right] \left( \rho_1^b, \rho_2^b \right)_{(t-s,y)}.$$

This influence functional is still non-local in time. By the Markov approximation as might be made shortly, the influence functional becomes local in time and Markovian master equations will be obtained.

3. **Coarse-graining in time**

When the intrinsic (not dynamical) scale of the heavy quark color density is long compared to the gluon correlation time, we can perform a coarse-graining in time in the form of the velocity expansion. The time scales of the heavy quark color density are $v/\bar v \sim \infty$ for single heavy quark kinetics and $v/\bar v \sim 1/\Lambda^2$ for relative motion in a quarkonium, while the gluon correlation time is $\sim 1/g T$ or shorter ($\sim 1/T$). Therefore, in order to perform the coarse-graining in time here, $\Lambda^2 \ll g T$ is required for a quarkonium.

Schematically, the coupling of the heavy quark color densities at different times is approximated by truncating the following expansions:

$$\int_0^\infty d\omega \rho(t) G(s) \approx \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \rho(t)(i\partial_\omega)^n G(s) \right] \left( \int_{-\infty}^\infty \frac{d\omega}{2\pi i} \frac{\partial^n G(\omega)}{\omega - i\epsilon} \right).$$

where $G(\omega) = \int dt e^{i\omega t} G(t,\vec r)$ and $\epsilon > 0$. The truncation corresponds to focusing on the long time behavior of the heavy quark color density $\rho(t)$. In our case, we keep the terms with $n \leq 2$, which corresponds to neglecting the effect of acceleration. Since $\dot\rho^a = -\nabla \cdot \vec v^a \sim \vec v \cdot \nabla \rho^a$, it formally takes a form of velocity expansion.

Using $G_{ab,00}^R(s,\vec r) = G_{ab,00}^S(s,\vec r)$ for $s > 0$, Eq. (19) can be expressed with $G_{ab,00}^R(s,\vec r)$ and $G_{ab,00}^S(s,\vec r)$. For the couplings with the retarded propagator, the analytic structure of $G_{ab,00}^R(\omega,\vec r)$ in the complex $\omega$ plane leads to

$$\int_0^\infty d\omega G_{ab,00}^R(s,\vec x - \vec y) \rho^a(t,\vec x) \rho^b(t-s,\vec y)$$

$$\approx \sum_{n=0,1,2} \frac{1}{n!} \left[ \rho^a(t,\vec x) (i\partial_\omega)^n \rho^b(t,\vec y) \partial^n G_{ab,00}^R(0,\vec x - \vec y) \right].$$

For the couplings with the symmetrized correlation function, we obtain

$$\int_0^\infty d\omega G_{ab,00}^S(s,\vec x - \vec y) \rho^a(t,\vec x) \rho^b(t-s,\vec y)$$

$$\approx \frac{1}{2} \sum_{n=0,1,2} \frac{1}{n!} \left[ \rho^a(t,\vec x) (i\partial_\omega)^n \rho^b(t,\vec y) \partial^n G_{ab,00}^S(0,\vec x - \vec y) \right]$$

$$+ \left[ \rho^a(t,\vec x) (i\partial_\omega)^n \rho^b(t,\vec y) \partial^n G_{ab,00}^S(0,\vec x - \vec y) \right]$$

$$\approx \frac{1}{2} \sum_{n=0,1,2} \frac{1}{n!} \left[ \rho^a(t,\vec x) (i\partial_\omega)^n \rho^b(t,\vec y) \partial^n G_{ab,00}^S(0,\vec x - \vec y) \right].$$

using the fact that $G_{ab,00}^S(\omega,\vec r) = \text{coth}(\omega/2T) \sigma_{ab,00}(\omega,\vec r)$ is an even function of $\omega$. Here the indices of the time contour in $\rho^a$ are dropped. By approximating the spectral function by an Ohmic one $\sigma_{ab,00}(\omega,\vec r) \sim \gamma_{ab,00}(\vec r)$ with a cutoff at $|\omega| = \Omega \ll g T$ or $T$ to ignore the memory effect of gluons, the integral in the third line of Eq. (19) turns out to be $\propto \Omega/T \ll 1$ and thus can be ignored.

The choice of $t$ matters because if we took it to be $t = (x^0 + y^0)/2$, we would have integrals of the form $\int_{t_0}^{\infty} dt \int_{-\infty}^\infty dG(s) \rho(t)(t-s)$. Then $G_{ab,00}(s,\vec r)$ would also contribute in $G_{ab,00}^F(s,\vec r)$ for $s < 0$ and cancel the diagonal parts of Eq. (20). The reason why we have to take $t = \max(x^0, y^0)$ will become clear when we discuss how to obtain the functional master equation.
4. Influence functional in the Markov limit

Let us define following three functions to parametrize the influence functional:

\[ V(\vec{r})\delta_{ab} \equiv -g^2 \text{Re}G_{ab,00}^R(0, \vec{r}), \]
\[ D(\vec{r})\delta_{ab} \equiv -g^2 T \frac{\partial}{\partial \omega} \sigma_{ab,00}(0, \vec{r}), \]
\[ A(\vec{r})\delta_{ab} \equiv -g^2 \left( \frac{1}{6T} \frac{\partial}{\partial \omega} + \frac{T}{3} \frac{\partial^3}{\partial \omega^3} \right) \sigma_{ab,00}(0, \vec{r}), \]
\[ \simeq -g^2 \frac{\partial}{6T} \frac{\partial}{\partial \omega} \sigma_{ab,00}(0, \vec{r}), \]

where the Ohmic spectral function for \( \sigma_{ab,00}(\omega, \vec{r}) \) is assumed as before to obtain \( A(\vec{r}) \simeq D(\vec{r})/6T^2 \). Explicit forms of \( \text{Re}G_{ab,00}^R(0, \vec{r}) \) and \( \frac{\partial}{\partial \omega} \sigma_{ab,00}(0, \vec{r}) \) at typical distance \( r \sim 1/gT \) are given in Appendix A using the hard thermal loop resummed perturbation theory. Using these functions, the influence functional in the Markov limit is given by four terms: \(^2\)

\[ S_{\text{IF}} = S_{\text{pot}} + S_{\text{fluct}} + S_{\text{diss}} + S_{\text{new}} + \cdots, \]
\[ iS_{\text{pot}} = -\frac{i}{2} \int_{t_0}^t dt \int d^3x d^3y V(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{y})}, \]
\[ iS_{\text{fluct}} = -\frac{1}{2} \int_{t_0}^t dt \int d^3x d^3y D(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{y})}, \]
\[ iS_{\text{diss}} = -\frac{i}{4T} \int_{t_0}^t dt \int d^3x d^3y D(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} -1 & -1 \\ 1 & 1 \end{array} \right] \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{y})}, \]
\[ iS_{\text{new}} = \frac{1}{4} \int_{t_0}^t dt \int d^3x d^3y A(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{y})}. \]

Each term has physical meanings: \( S_{\text{pot}} \) gives a potential between two heavy quarks, \( S_{\text{fluct}} \) accounts for thermal fluctuations, \( S_{\text{diss}} \) gives rise to dissipative dynamics such as the drag force, and \( S_{\text{new}} \), which contains \( \rho_1^2, \rho_2^2 \), is a new term first introduced in this paper and makes an essential contribution to render the Lindblad-form master equations. This is analogous to the situation in the quantum Brownian motion \(^{23}\), where it is necessary to include \((\dot{x}_1 - \dot{x}_2)^2\) term in the influence functional in order to obtain the Lindblad-form master equation \(^{24}\).

The counting in the perturbative and velocity expansion is \( S_{\text{pot}}, S_{\text{fluct}} \sim g^2 v^0, S_{\text{diss}} \sim g^2 v, \) and \( S_{\text{new}} \sim g^2 v^2 \). In the counting of \( g \), the order is loosely counted as \( V(\vec{r}), D(\vec{r}), A(\vec{r}) \sim (g^2)^2 \) for all \( \vec{r} \) and similarly for their derivatives. We keep it loose unless it is worth making it more precise. In \( S_{\text{pot}} \), we ignore a term \( \propto (\rho_1 \dot{\rho}_1 - \rho_2 \dot{\rho}_2) \) because it would just give a correction of the order \( O(g^2 v^2) \) to the potential. In the Markov limit, terms ignored in Eq. (13) turn out to be \( O(g^3) \) or \( O(g^2 v^2) \).

It should also be remarked that we also implicitly rely on the perturbative expansion in the procedure of coarse-graining in time. The couplings in \( S_{\text{IF}} \) is originally non-local in time. Because of the coarse-graining, the couplings are approximated to be local. This approximation corresponds to the ladder approximation in the Bethe-Salpeter equation. The overlap of two interactions is thus neglected, which would yield cross-ladder contribution of higher order in \( g \).

Integrating by parts, \( S_{\text{diss}} \) and \( S_{\text{new}} \) become

\[ iS_{\text{diss}} = \frac{i}{4T} \int_{t_0}^t dt \int d^3x d^3y \nabla_{\vec{x}} D(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} -1 & -1 \\ 1 & 1 \end{array} \right] \left( j_1^a, j_2^a \right)_{(t,\vec{y})}, \]
\[ iS_{\text{new}} = \frac{1}{4} \int_{t_0}^t dt \int d^3x d^3y \nabla_{\vec{x}} A(\vec{x} - \vec{y}), \]
\[ \times \left( \rho_1^a, \rho_2^a \right)_{(t,\vec{x})} \left[ \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \left( j_1^a, j_2^a \right)_{(t,\vec{y})}. \]

Equations (24), (25), (26), and (27) constitute the influence functional in the leading orders in perturbative and velocity expansions up to the order of \( O(g^2 v^0, g^2 v^2) \) (and some terms of order \( O(g^2 v^3) \)) in the Markov limit.

C. Functional master equations

Here we review how to obtain the renormalized effective Hamiltonian described in Ref. 17. The total action \( S_{\text{CTP}} = \int_{t_0}^t dt L_{\text{CTP}} \) on the closed-time path is given by adding non-relativistic kinetic terms for
\(\psi_{1,2} = (Q, Q^\dagger)_{1,2}\), where \(Q_{(c)}\)s are Pauli spinors for heavy (anti)quark:

\[ S_{CTP}[\psi_1, \psi_2] = S_{kin}[\psi_1] - S_{kin}[\psi_2] + S_{FP}[j_1, j_2], \tag{30} \]

\[ S_{kin}[\psi] = \int_{t_0} d^4x Q^\dagger \left( \partial_0 - M + \frac{\nabla^2}{2M} \right) Q \tag{31} \]

\[ + \int_{t_0} d^4x Q e \left( \partial_0 + M - \frac{\nabla^2}{2M} \right) Q. \]

Since the partition functions is

\[ Z[\eta_1, \eta_2] = \int \mathcal{D}[\psi_1, \psi_2] \langle \psi_1^\dagger | \rho_S | \psi_2 \rangle e^{iS_{CTP}} \tag{32} \]

\[ \times e^{\int_{t_0} d^4x \left( \eta_1^\dagger \psi_1 + \eta_1 \psi_1^\dagger - \eta_2 \psi_2 - \eta_2^\dagger \psi_2^\dagger \right)}, \]

where \(\rho_S\) is the arbitrary initial density matrix in the heavy quark Fock space, the reduced density matrix at \(\langle \psi_1^\dagger | \rho_S | \psi_2 \rangle\) is

\[ \langle \psi_1^\dagger | \rho_S(t') | \psi_2 \rangle = \int \mathcal{D}[\psi_1, \psi_2] \langle \psi_1^\dagger | \rho_S | \psi_2 \rangle e^{\int_{t_0} d^4x \left( \eta_1^\dagger \psi_1 + \eta_1 \psi_1^\dagger - \eta_2 \psi_2 - \eta_2^\dagger \psi_2^\dagger \right)}, \tag{33} \]

with boundary conditions \(\psi_1^\dagger (t') = \psi_1^\dagger\) and \(\psi_2 (t') = \psi_2\). Note that time integration is limited to \(t < t'\). This is why we must choose \(t = \max (x^0, y^0)\) in the previous section.

The time-evolution equation for \(\rho_S \left[ t, \psi_1^\dagger, \psi_2 \right] = \langle \psi_1^\dagger | \rho_S(t) | \psi_2 \rangle\) is given by an analogy with the Schrödinger equation.

1. Derive the Hamiltonian \(H_{CTP} \left[ \psi_1^\dagger, \psi_1, \psi_2^\dagger, \psi_2 \right]\)

2. Obtain functional representation of \(H_{CTP}\) by the following replacement:

\[ H_{CTP} \left[ \psi_1^\dagger, \psi_1, \psi_2^\dagger, \psi_2 \right] \]

\[ \rightarrow H_{CTP} \left[ \psi_1^\dagger, \frac{\delta}{\delta \psi_1}, -\frac{\delta}{\delta \psi_2}, \psi_2 \right]. \tag{34} \]

3. The functional master equation is obtained as

\[ \frac{\partial}{\partial t} \rho_S \left[ t, \psi_1^\dagger, \psi_2 \right] \]

\[ = H_{CTP} \left[ \psi_1^\dagger, \frac{\delta}{\delta \psi_1}, -\frac{\delta}{\delta \psi_2}, \psi_2 \right] \rho_S \left[ t, \psi_1^\dagger, \psi_2 \right]. \tag{35} \]

In the first step, we must take care of the order of operators, which are ordered in time. For example, in the fermion bilinear in \(\rho_{Q^\dagger Q}^i(t, \vec{x})\) and in the kinetic terms, time is assigned as \(\psi_1^\dagger (t+\epsilon), \psi_1 (t-\epsilon)\) and \(\psi_2(t+\epsilon), \psi_2(t-\epsilon)\) with \(\epsilon > 0\). Also as is clear from Eq. (19), the time for \(\rho_{Q^\dagger Q}^i(t, \vec{x})\) is later than that for \(\rho_{Q^\dagger Q}^i(t, \vec{y})\) in Eq. (23). Use of new variables \(\tilde{\psi}_2 = \psi_1^\dagger, \tilde{\psi}_2^\dagger = \psi_2\) will make the fields on the 1 and 2 axes look symmetric.

Since we are interested in systems with a few heavy quarks in the QGP, coherent states \(\langle Q_1^c |\) and \(|Q_2^c \rangle\) defined as

\[ \langle Q_1^c | = \langle \Omega | e^{-\int d^4x (Q(\vec{x}) Q^c(\vec{x}) + Q^c(\vec{x}) Q(\vec{x}))} \]

\[ |Q_2^c \rangle = e^{-\int d^4x (Q_2^c(\vec{x}) Q^c(\vec{x}) + Q^c(\vec{x}) Q_2(\vec{x}))} \langle \Omega |, \tag{36} \]

are more convenient to express \(\rho_S(t)\). Here \(\Omega\) is the vacuum state that satisfies \(Q_1(\Omega) = 0\). This amounts to changing the variables for functional differentiation

\[ H_{CTP} \left[ Q_1^c, Q_1(\Omega), Q_2^c, Q_2(\Omega) \right] \]

\[ \rightarrow H_{CTP} \left[ Q_1^c, \frac{\delta}{\delta Q_1^c}, Q_2^c, -\frac{\delta}{\delta Q_2^c} \right]. \tag{38} \]

and the functional master equation is given by

\[ \frac{i}{\partial t} \rho_S \left[ t, Q_1^c, Q_2^c \right] \]

\[ = H_{CTP} \left[ Q_1^c, \frac{\delta}{\delta Q_1^c}, Q_2^c, -\frac{\delta}{\delta Q_2^c} \right] \rho_S \left[ t, Q_1^c, Q_2^c \right]. \tag{39} \]

In general, the time-ordered product does not give \(H_{CTP}\) such that all the differentiation is moved on the right. Therefore, in the course of doing so after deriving the time-ordered \(H_{CTP}\), we need to subtract divergent contributions in the vacuum, e.g. Coulomb potential at the origin \(V(0)\), by introducing counter terms.

### D. From fields to particles

The functional master equation can generate master equations for systems with arbitrary finite number of heavy quarks in the QGP. Since the coherent states act as generating functional for heavy quarks as in Eqs. (36) and (37), the reduced density matrices are given by functionally differentiating \(\rho_S \left[ t, Q_1^c, Q_2^c \right]\). For example, the reduced density matrix of a single heavy quark system in the QGP is obtained by

\[ \rho_Q^i(t, \vec{x}, y) = \langle \vec{x}, i | \rho_Q(t) | y, j \rangle \tag{40} \]

\[ = \langle \Omega | Q^i(\vec{x}) \rho_S(t) \langle \vec{x} | Q^j(\vec{y}) | \Omega \rangle \]

\[ = -\frac{\delta}{\delta Q^i(\vec{x})} \frac{\delta}{\delta Q^j(\vec{y})} \rho_S \left[ t, Q_1^c, Q_2^c \right] \bigg|_{Q^* = 0}. \]

Therefore in order to obtain the master equations for heavy quark reduced density matrices, we just need to perform appropriate functional differentiations on the
both sides of the functional master equation and switch off the source \( Q^*_{1(\ell)} = \hat{Q}^*_{2(\ell)} = 0 \).

Similarly, the forward propagators of heavy quarks are given by differentiating only with \( Q^*_{1(\ell)} \) fields, up to correction of order \( O(e^{-M/T}) \ll 1 \). For example, one-body forward propagator is obtained by

\[
G^*_{Q^j_{1(\ell)} Q^j_{2(\ell)}}(t, \vec{x}, \vec{y}) = \frac{\text{Tr} \left[ e^{-H_{QCD}/T} Q^j_{1(\ell)} Q^j_{2(\ell)}(t_0, \vec{y}) \right]}{\text{Tr} \left[ e^{-H_{QCD}/T} \right]} \mid_{t = 0},
\]

The time-evolution equations for forward propagators are also derived from the functional master equation by performing appropriate functional differentiations with \( Q^*_{1(\ell)} \) fields. In Appendix B we show the time-evolution equation for the forward propagator of a quarkonium, for which the leading correction in the velocity expansion is found to be \( O(v) \), not \( O(v^2) \) as in the vacuum. The \( O(v) \) term couples relative position and momentum of a heavy quark-antiquark pair in an intriguing way. It may be necessary to take into account the \( O(v) \) term when one computes vector current spectral function using the complex potential.

### III. Master Equations in the Lindblad Form

In this section, we derive master equations for a single heavy quark and a quarkonium in the QGP. We show that each master equation can be written in the Lindblad form

\[
\frac{d}{dt} \rho_s(t) = -i[H, \rho_s] + \sum_{i=1}^{N} \gamma_i \left( L_i \rho_s L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho_s - \frac{1}{2} \rho_s L_i^\dagger L_i \right),
\]

with \( H^\dagger = H \) and \( \gamma_i > 0 \) for \( S = Q \) and \( QQ \). This is equivalent to showing the master equations preserve complete positivity of the reduced density matrices.

#### A. Single heavy quark master equations

1. Full master equation

By following the procedures outlined in the previous section, the master equation for the reduced density matrix of a single heavy quark \( \hat{\rho}_Q(t, \vec{x}, \vec{y}) \) \((N_c \otimes N_c^c)\) representation of the color SU\((N_c)\) group is obtained as

\[
\frac{\partial}{\partial t} \hat{\rho}_Q(t, \vec{x}, \vec{y}) = i \left( \frac{\hat{\nabla}^2 - \hat{\nabla}^2}{2M} \right) \hat{\rho}_Q(t, \vec{x}, \vec{y}) + F_1(\vec{x} - \vec{y}) t^2 \hat{\rho}_Q(t, \vec{x}, \vec{y}) + C_F F_1(t) \hat{\rho}_Q(t, \vec{x}, \vec{y}) + \sum_{i,j=1}^{N_c} \left( \frac{\hat{\nabla}^2}_{x} \nabla^2_{y} t^2 \hat{\rho}_Q(t, \vec{x}, \vec{y}) + C_F F_3(t) \right) i \frac{\hat{\nabla}^2_{x} + \hat{\nabla}^2_{y}}{6} \hat{\rho}_Q(t, \vec{x}, \vec{y}),
\]

where \( C_F = \frac{(N_c^2 - 1)}{2N_c} \) and

\[
F_1(\vec{r}) = - \left( \frac{\nabla^2 D(\vec{r})}{4MT} + \frac{(\nabla^2)^2 A(\vec{r})}{8M^2} \right),
\]

\[
F_2(\vec{r}) = - \nabla \left( \frac{\nabla^2 D(\vec{r})}{4MT} \right),
\]

\[
F_3(\vec{r}) = \nabla^4 A(\vec{r}) \frac{(\nabla^2)^2}{2M^2}.
\]

In the master equation, by using \( \nabla \sim Mv \), the \( F_1, F_2, \) and \( F_3 \) terms are of order \( O(g^2v^0), O(g^2v), \) and \( O(g^2v^2) \) respectively. We also introduce a counting in the parameter \( \delta \equiv T/M \ll 1 \). Close to heavy quark kinetic equilibrium, we have \( v \sim \sqrt{T/M} = \delta^{1/2} \) that so that \( v \) and \( \delta \) are not independent. All the terms in the master equation can be evaluated by perturbative, velocity and \( T/M \) expansions, for example \( F_1(\vec{x} - \vec{y}) t^2 \hat{\rho}_Q(t, \vec{x}, \vec{y}) \) consists of \( O(g^2v^0 \delta), O(g^2v^0 \delta), \) and \( O(g^2v^0 \delta^2) \) terms. The velocity expansion \( O(v^n) \) in the influence functional \( S_{I\bar{y}} \) corresponds to an expansion \( O(v^n \delta^m) \) with \( n = l + m \) in the master equation as is recognized by the factors of \( 1/M \) in \( S_{I\bar{y}} \). For example, \( S_{\text{diss}} \sim g^2v \) produces \( O(g^2v^0 \delta) \) as well as \( O(g^2v^0 \delta^2) \) terms in the master equation. This observation is important in approximating the master equation in the recoilless limit later.

The master equation can be written in the Lindblad form Eq. (12) with \( H = \vec{p}^2/2M \). The label is \( i = (\vec{k}, a, \alpha) \), where \( \vec{k} \) is wave number in a box with volume \( L^3 \), \( a \) is the label for color matrix \( t^a \), and \( \alpha = 1, 2 \) is introduced for classification. The Lindblad operators \( L_{k\alpha} \) and coefficients \( \gamma_{k\alpha} \) are

\[
\begin{align*}
L_{k\alpha} &= e^{-i\vec{k} \cdot \vec{r}} / (2 \left( 1 - \frac{\vec{k} \cdot \vec{p}}{3MT} \right)) e^{i\vec{k} \cdot \vec{r} / 2T^a}, \\
\gamma_{k\alpha} &= \frac{\delta_{\alpha}}{L^3} \quad > 0,
\end{align*}
\]

\[
\begin{align*}
L_{k\alpha} &= e^{i\vec{k} \cdot \vec{r} / (2 \left( \frac{\vec{k} \cdot \vec{p}}{3MT} \right))} e^{i\vec{k} \cdot \vec{r} / 2T^a}, \\
\gamma_{k\alpha} &= \frac{\delta_{\alpha}}{T^3} \quad > 0.
\end{align*}
\]

Here \( \vec{D}(\vec{k}) = \int d^3y e^{-i\vec{k} \cdot \vec{y}} D(\vec{r}) \) and similarly for \( \vec{A}(\vec{k}) \). Without the term \( \vec{A}(\vec{k}) \), or \( S_{\text{new}} \) in the influence functional, the coefficient \( \gamma_{k\alpha} \) is negative and the master equation cannot be in the Lindblad form. Therefore,
keeping $S_{\text{new}}$ together with $S_{\text{disc}}$ in the influence functional is essential in obtaining the Lindblad-form master equation.

By tracing out the color space dynamics $\hat{\rho}_Q(t, \vec{x}, \vec{y}) = \text{Tr}_{\text{color}} \hat{\rho}_Q(t, \vec{x}, \vec{y}) = \rho^{\text{fluct}}_Q(t, \vec{x}, \vec{y})$, the master equation for $\hat{\rho}_Q(t, \vec{x}, \vec{y})$ reads

$$\frac{\partial}{\partial t} \hat{\rho}_Q(t, \vec{x}, \vec{y}) = \frac{i}{2M} \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$+ C_F \left( F_1(\vec{x} - \vec{y}) - F_1(0) \right) \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$+ C_F \hat{F}_2(\vec{x} - \vec{y}) \cdot \left( \nabla_\vec{x} - \nabla_\vec{y} \right) \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$+ C_F \left( \frac{\rho^{\text{fluct}}_Q(\vec{x} - \vec{y})}{\bar{D}_1^2} + \frac{\rho^{\text{fluct}}_Q(\vec{x})}{\bar{D}_1^2} \right) \frac{\partial}{\nabla_\vec{x}^2 + \nabla_\vec{y}^2} \hat{\rho}_Q(t, \vec{x}, \vec{y}),$$

and the Lindblad operators are obtained by replacing $\xi$s with $I$ in Eqs. (17) and (18) and the coefficients are $C_F$ times those in Eqs. (17) and (18).

2. Master equation for wave packets

Now let us assume that the heavy quark is kinetically thermalized and its wave function is localized compared to the length scale of functions $D(\vec{r})$ and $A(\vec{r})$. Close to heavy quark kinetic equilibrium, the heavy quark wave function extends over the thermal de Broglie wavelength $\lambda_{\text{dB}} \sim 1/\sqrt{M T}$. The size of the wave function is characterized by the “correlation length” $|\vec{x} - \vec{y}|$ of $\hat{\rho}_Q(t, \vec{x}, \vec{y})$. Therefore, in the master equation (49), we can approximate $D(\vec{r})$ and $A(\vec{r})$ by

$$D(\vec{r}) \approx D(\vec{0}) + \frac{\rho^{\text{fluct}}_Q(\vec{0})}{6} \nabla^2 D(\vec{0}) = D_0 + \frac{D_2}{6} r^2,$$

$$A(\vec{r}) \approx A(\vec{0}) + \frac{\rho^{\text{fluct}}_Q(\vec{0})}{6} \nabla^2 A(\vec{0}) = A_0 + \frac{A_2}{6} r^2,$$

which yields

$$\frac{\partial}{\partial t} \hat{\rho}_Q(t, \vec{x}, \vec{y}) = \frac{i}{2M} \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$- \frac{C_F D_2}{6} \left( (\vec{x} - \vec{y})^2 + (\vec{x} - \vec{y}) \cdot \nabla_\vec{x} - \nabla_\vec{y} \right) \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$+ \frac{C_F A_2}{12M^2} \left( \nabla_\vec{x}^2 + \nabla_\vec{y}^2 \right) \hat{\rho}_Q(t, \vec{x}, \vec{y}).$$

By means of the counting in $g$, $\delta = T/M$, and $\nu = \sqrt{T/M} = \delta^{1/2} / 2$, we can make the above argument more precise. The thermal de Broglie wave length of a heavy quark $\lambda_{\text{dB}} \sim 1/\sqrt{M T} = \delta^{1/2} / 2$ is much smaller than the length scale $\lambda_{\text{fluct}} \sim 1/g T$ of $D(\vec{r})$ and $A(\vec{r})$. The latter is defined so that for $|\vec{r}| < \lambda_{\text{fluct}}$, $D(\vec{r})$, $A(\vec{r}) \approx 0$ holds. Then, Eqs. (49) and (52) are evaluated as expansions up to $(1/\lambda_{\text{dB}}^2) \sim g^2 \delta$. The master equation is also expanded in terms of $g^2 \delta$ and keeping the terms up to $O(g^2 \delta)$ in this expansion yields Eq. (52). Using $D_2 \sim D_0/\lambda_{\text{fluct}}^2 \sim g^4 T^3$ and $A_2 \sim A_0/\lambda_{\text{fluct}}^2 \sim (D_0/T^2)/\lambda_{\text{fluct}}^2$, time scale of the quantum Brownian motion of Eq. (52) is estimated to be $\sim M/g^4 T^2$.

The Lindblad operators and coefficients are labeled with $i = (l, \alpha)$, where $l = x, y, z$,

$$\left\{ \begin{array}{l} L^L_l = \frac{(\vec{x} + i\vec{p})}{4MT}, \\ \gamma^L_l = \frac{C_F D_2}{3} > 0, \\ L^H_l = \frac{(\vec{p})^2}{4MT}, \\ \gamma^H_l = \frac{C_F A_2}{12MT} (8T^2 A_2 - D_2) > 0, \end{array} \right.$$ 

and the Hamiltonian is

$$H = \frac{\vec{p}^2}{2M} + \frac{C_F D_0}{12MT} \langle \vec{x}, \vec{p} \rangle.$$

Here the number of the Lindblad operators is reduced to only 6. If we neglect $A_2$, the master equation (52) is the same with that of the Caldeira-Leggett model of quantum Brownian motion. Note that, without $A_2$, the second coefficient becomes $\gamma^{H}_l = 0$ and the master equation is no longer in the Lindblad form. Thus again, we find that $S_{\text{new}}$ makes an essential contribution in obtaining the Lindblad-form master equation.

3. Master equation in the recoilless limit

If one is interested in decoherence of a wave function at distant points, which takes place in a much shorter time scale than the momentum dissipation, one can approximate the full master equation (43) by just keeping the kinetic term and terms from $\lambda_{\text{fluct}} \sim O(g^2 v^0)$ in the influence functional:

$$\frac{\partial}{\partial t} \hat{\rho}_Q(t, \vec{x}, \vec{y}) = \frac{i}{2M} \hat{\rho}_Q(t, \vec{x}, \vec{y})$$

$$- D(\vec{x} - \vec{y})^a \rho^{\text{fluct}}_Q(t, \vec{x}, \vec{y})^a + C_F D(\vec{0}) \hat{\rho}_Q(t, \vec{x}, \vec{y}).$$

This is called recoilless limit of the full master equation.

Let us examine in more detail under which conditions the decoherence takes place rapidly compared to the momentum dissipation. The condition for the distance $\Delta x$ is $|F_1(\Delta \vec{x}) - F_1(0)| > |\hat{F}_2(\Delta \vec{x})| M v$ or $|D(\Delta \vec{x}) - D(0)| > |\sqrt{D(\Delta \vec{x})} v / 4T|$. At large enough distance $\Delta x > \lambda_{\text{fluct}}$ where $D(\Delta \vec{x}) \approx 0$ holds, the condition is satisfied. At short distance $\Delta x < \lambda_{\text{fluct}}$, we can derive a condition $\Delta x > v/T$, that is $\Delta x > l_{\text{fB}}$. Therefore for $\Delta x > l_{\text{fB}}$, the decoherence takes place very rapidly and the full master equation (43) can be approximated by taking the recoilless limit. The time scale depends on $\Delta x$: For $\Delta x > l_{\text{fB}}$, the time scale is $\sim 1/D(0) \sim 1/g^2 T$ and for $\Delta x < l_{\text{fB}}$, the time scale is $\sim (1/D(0)) \sim (1/g^2 T^3)/(\Delta x)^2$. Even if an initial wave function is coherent over $\Delta x \gg l_{\text{fB}}$, its coherence is lost ($\hat{\rho}(t, \vec{x}, \vec{y}) \approx 0$ for $|\vec{x} - \vec{y}| \gg \Delta x$) through a few scatterings with medium particles. Note that for heavy quarks to be kinetically
thermalized, it requires many scatterings ($\propto M/T$) and thus takes much longer time than decoherence. Close to heavy quark kinetic equilibrium, the typical wave function is coherent only over $\Delta x \sim l_{4B}$ and thus the master equation is not applicable there.

The master equation in the recoilless limit can be written in the Lindblad form. The Lindblad operator is $L_{\delta a} = \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{L^3}}$ and the coefficient is $\gamma_{\delta a} = -\frac{D(k)}{L^3} > 0$. As mentioned before, the master equation in the Lindblad form but cannot describe heavy quark kinetic equilibration.

The same approximation can be made to the color-traced master equation. Or equivalently one can trace out the color space dynamics in the master equation. The form of the master equation is different only in $D(\vec{x} - \vec{y})^a \rho(t, \vec{x}, \vec{y}) t^a \rightarrow C_F D(\vec{x} - \vec{y}) \rho(t, \vec{x}, \vec{y})$. The Lindblad operator is $L_{\delta a} = \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{L^3}}$ and the coefficient is $\gamma_{\delta a} = -C_F D(k)/L^3 > 0$.

B. Heavy quarkonium master equations

1. Full master equation

In the case of quarkonium, the reduced density matrix $\hat{\rho}_{QQ}(t, \vec{x}_Q, \vec{y}_Q, \vec{y}_Q)$ is in the $(N_c \otimes N^*_c) \otimes (N^*_c \otimes N_c)$ representation. The master equation has the following structure: $\frac{\partial}{\partial t} \hat{\rho}_{QQ}(t, \vec{x}_Q, \vec{y}_Q, \vec{y}_Q) = \mathcal{L}_{QQ} \hat{\rho}_{QQ}(t, \vec{x}_Q, \vec{Q}_c, \vec{y}_Q, \vec{y}_Q)$, $\mathcal{L}_{QQ} = \mathcal{L}_Q + \mathcal{L}_{Qc} + \mathcal{L}_{QQ}^{(2)}$. Here $\mathcal{L}_Q$ denotes the super-operator in the right hand side of Eq. and $\mathcal{L}_{Qc}$ is obtained by substituting $-t^a$ for $t^a$ in $\mathcal{L}_Q$. $\mathcal{L}_Q$ acts on variables of heavy quark while $\mathcal{L}_{Qc}$ acts on those of heavy antiquark. The “interaction” between the heavy quark and antiquark is given by $\mathcal{L}_{QQ}^{(2)}$, whose explicit form is shown in Appendix C.

The structure of the master equation is quite complicated but the Lindblad operators and coefficients turn out to be remarkably simple. We just need to add a contribution from a heavy antiquark with appropriate color representation in Eqs. and 43:

$$\begin{cases} L_{\delta a}^{\alpha=1} = \frac{e^{i\vec{k} \cdot \vec{x}_Q}}{\sqrt{L^3}} \left(1 - \frac{\hat{\rho}_{QQ}}{\frac{4M}{t^a}}\right) e^{i\vec{k} \cdot \vec{x}_Q}/(t^a \otimes 1) \\ -\frac{e^{i\vec{k} \cdot \vec{x}_Q}}{\sqrt{L^3}} \left(1 - \frac{\hat{\rho}_{QQ}}{\frac{4M}{t^a}}\right) e^{i\vec{k} \cdot \vec{x}_Q}/(1 \otimes t^a), \end{cases} \quad (59)$$

$$\gamma_{\delta a}^{\alpha=1} = -\frac{D(k)}{L^3} > 0.$$  

$$\begin{cases} L_{\delta a}^{\alpha=2} = \frac{e^{i\vec{k} \cdot \vec{x}_Q}}{\sqrt{L^3}} \left(1 - \frac{\hat{\rho}_{QQ}}{\frac{4M}{t^a}}\right) e^{i\vec{k} \cdot \vec{x}_Q}/(t^a \otimes 1) \\ -\frac{e^{i\vec{k} \cdot \vec{x}_Q}}{\sqrt{L^3}} \left(1 - \frac{\hat{\rho}_{QQ}}{\frac{4M}{t^a}}\right) e^{i\vec{k} \cdot \vec{x}_Q}/(1 \otimes t^a), \end{cases} \quad (60)$$

$$\gamma_{\delta a}^{\alpha=2} = -\frac{1}{L^3} \left(8T^2 \hat{A}(k) - \hat{D}(k)\right) > 0.$$  

Here $\vec{x}_Q, \vec{p}_Q$ are position and momentum operators for the heavy quark and $\vec{x}_{Qc}, \vec{p}_{Qc}$ are those for the heavy antiquark. The Hamiltonian in the Lindblad form has two contributions in the potential: one is the screened potential from $S_{pot}$ and the other is from $S_{diss}$ in the influence functional.

$$\begin{align} H &= \frac{\hat{p}_Q^2 + \hat{p}_{Qc}^2}{2M} - V(\vec{x}_Q - \vec{y}_Q)(t^a \otimes 1) \\ &+ \frac{1}{8MT} \left\{ (\vec{p}_Q - \vec{p}_{Qc}), \vec{\nabla} D(\vec{x}_Q - \vec{y}_Q) \right\} (t^a \otimes 1). \end{align} \quad (61)$$

Physical meaning of the second line of Eq. is remarkable. In the classical Hamiltonian, the anti-commutator part is positive (negative) when $(\vec{x}_Q - \vec{y}_Q) \cdot (\vec{p}_Q - \vec{p}_{Qc})$ is positive (negative), because $D(\vec{r})$ is an increasing function of $r$. Therefore when a heavy quark-antiquark pair in the singlet state is moving apart from (approaching) each other, the term makes a positive (negative) contribution to the Hamiltonian, while the sign is opposite for a heavy quark-antiquark pair in the octet states.

2. Master equation in the recoilless limit

Suppose we have a quarkonium initial state where the wave function for the relative motion is in a bound state which is coherent over some length $l_{coh}$. The decoherence of the wave function takes place rapidly and the momentum dissipation can be neglected if $l_{coh} > v_{rel}/T$. When the total momentum of the quarkonium is zero, we have $v_{Qc} \approx v_{rel}/2$, where $v_{rel}$ is the relative velocity of the heavy quark and anti-quark in the quarkonium. Since $v_{rel} \approx 1/Ml_{coh}$, the condition $l_{coh} \gg l_{4B} \sim 1/\sqrt{MT}$ must be satisfied for each bound state of interest. Note that $l_{coh} \sim 1/M \gg l_{4B}$ is satisfied by all the bound states if the condition for the coarse-graining in time $Ml_{coh}^2 \ll gT$ is satisfied.

When studying decoherence of bound states with $l_{coh} \gg l_{4B}$, the master equation can be approximated by keeping the kinetic term and terms from $S_{pot}, S_{fluct} \sim O(g^2 l_{coh}^2)$ in the influence functional. The super-operator $\mathcal{L}_{QQ}$ in the recoilless limit is

$$\begin{align} \mathcal{L}_{QQ} \hat{\rho}_{QQ} &= \frac{\hat{V}^2_{QQ}}{2M} \hat{\rho}_{QQ} \\ &- D(\vec{x}_Q - \vec{y}_Q)(t^a \otimes 1) \hat{\rho}_{QQ}(t^a \otimes 1) \\ &+ C_F D(\vec{0}) \hat{\rho}_{QQ}, \end{align} \quad (62)$$

$$\begin{align} \mathcal{L}_{Qc} \hat{\rho}_{QQ} &= \frac{\hat{V}^2_{QQ}}{2M} \hat{\rho}_{QQ} \\ &- D(\vec{x}_Q - \vec{y}_Q)(1 \otimes t^a) \hat{\rho}_{QQ}(1 \otimes t^a) \\ &+ C_F D(\vec{0}) \hat{\rho}_{QQ}, \end{align} \quad (63)$$

3 To be precise, another condition $v_{rel} \gg v_{tot}$ is necessary, where $v_{tot}$ is velocity of global motion of a quarkonium. When $l_{coh} \gg l_{4B}$ is satisfied, it indicates that the heavy quark and antiquark in the quarkonium have not yet receive many scatterings and $v_{rel} \gg v_{tot}$ is still expected to be satisfied as well.
and

\[
\mathcal{L}_{\text{QQ}}^{(2)} \hat{\rho}_{\text{QQ}} =
\begin{align*}
(iV(\vec{x}_Q - \vec{x}_{Q_*}) - D(\vec{x}_Q - \vec{x}_{Q_*}))(t^a \otimes t^{a*}) \hat{\rho}_{\text{QQ}} \\
- (iV(\vec{y}_Q - \vec{y}_{Q_*}) + D(\vec{y}_Q - \vec{y}_{Q_*})) \hat{\rho}_{\text{QQ}}(t^a \otimes t^{a*}) \\
+ D(\vec{x}_Q - \vec{y}_{Q_*})(t^a \otimes 1) \hat{\rho}_{\text{QQ}}(1 \otimes t^{a*}) \\
+ D(\vec{y}_Q - \vec{x}_{Q_*})(1 \otimes t^{a*}) \hat{\rho}_{\text{QQ}}(t^a \otimes 1).
\end{align*}
\] (64)

This master equation is in the Lindblad form with

\[
\begin{align*}
\gamma_{ka} &= e^{-\vec{k}\cdot\vec{r}_Q}(t^a \otimes 1) - e^{-\vec{k}\cdot\vec{x}_{Q_*}}(1 \otimes t^{a*}), \\
\gamma_{ka} = - \frac{D(\vec{k})}{L} > 0,
\end{align*}
\] (65)

and with the Hamiltonian

\[
H = \frac{\vec{P}_Q^2 + \vec{P}_{Q_*}^2}{2M} - V(\vec{x}_Q - \vec{x}_{Q_*})(t^a \otimes t^{a*}).
\] (66)

In the master equation given by the super-operators \[62\], \[63\], and \[64\], the relative motion and center-of-mass motion decouple. Note that these motions decouple only after taking the recoilless limit. Let us define the reduced density matrix for the relative motion \(\hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})\):

\[
\begin{align*}
\hat{R} &= \frac{\vec{x}_Q + \vec{x}_{Q_*}}{2}, \\
\hat{S} &= \frac{\vec{y}_Q + \vec{y}_{Q_*}}{2}, \\
\hat{c} &= \frac{\vec{y}_Q - \vec{y}_{Q_*}},
\end{align*}
\] (67)

\[
\hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s}) = \int d^3R d^3S \delta(\hat{R} - \hat{S}) \times \hat{\rho}_{\text{QQ}}(t, \vec{x}_Q, \vec{x}_{Q_*}, \vec{y}_Q, \vec{y}_{Q_*}),
\] (68)

and derive a master equation for it. The result is

\[
\frac{\partial}{\partial t} \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s}) = \left(\frac{i\hat{\nabla}_Q^2 - \hat{\nabla}_Q^2}{2M} + 2C_T D(\hat{\vec{0}})\right) \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})
+ (iV(\vec{r}) - D(\vec{r}))(t^a \otimes t^{a*}) \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})
- (iV(\vec{s}) + D(\vec{s})) \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})(t^a \otimes t^{a*})
- D\left(\frac{\vec{r} - \vec{s}}{2}\right) \left(t^a \otimes 1\right) \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})(1 \otimes t^{a*})
+ D\left(\frac{\vec{r} + \vec{s}}{2}\right) \left(t^a \otimes 1\right) \hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s})(1 \otimes t^{a*})
\] (69)

The Lindblad operator is obtained by substituting \(\vec{x}_Q \rightarrow \vec{r}/2\) and \(\vec{x}_{Q_*} \rightarrow -\vec{r}/2\) in Eq. \[63\] and the coefficient is the same with Eq. \[64\]. The Hamiltonian is obtained by just expressing Eq. \[66\] in the relative coordinate.

Since the potential and thermal fluctuation depends on the color states of quarkonium, one cannot trace out the color space dynamics in the master equation \[64\]. We can obtain coupled master equations for the color singlet occupation \(\rho_1(t, \vec{r}, \vec{s}) \equiv \text{Tr}_{\text{color}}[\hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s}) P_1]\) and for the color octet (or \((N_c^2 - 1)\) representation) occupation \(\rho_8(t, \vec{r}, \vec{s}) \equiv \text{Tr}_{\text{color}}[\hat{\rho}_{\text{QQ}}(t, \vec{r}, \vec{s}) P_8]\), where \(P_1\) and \(P_8\) are projection operators onto color singlet and octet states.

**IV. QUANTUM DECOHERENCE OF HEAVY QUARKONIUM**

In this section, we show that the master equations in the recoilless limit are equivalent to stochastic Schrödinger equations. The stochastic Schrödinger equations describe the effects of thermal fluctuation on the quantum states of heavy quarks and the wave function at distant points becomes decoherent. Decoherence is essential for quarkonium dissociation so that we here concentrate on quarkonium in the QGP. Since the stochastic Schrödinger equations can be understood as Hamiltonian dynamics with time-dependent random potential, they cannot describe irreversible processes such as momentum dissipation and thus checking the criterion for the recoilless limit is very important for applications. We also discuss decoherence of a wave function and its relevance to the bound state dissociation.

**A. Stochastic potential**

The basics of the stochastic potential are given in \[11\]. The wave function of a quarkonium \(\psi(t, \vec{x}_Q, \vec{x}_{Q_*})\) is in the \(N_c \otimes N_c^*\) representation. The stochastic and unitary time-evolution of \(\psi(t, \vec{x}_Q, \vec{x}_{Q_*})\) is

\[
\psi(t + dt, \vec{x}_Q, \vec{x}_{Q_*}) = e^{-i\theta t H}(t) \psi(t, \vec{x}_Q, \vec{x}_{Q_*}),
\] (70)

with the following stochastic Hamiltonian

\[
H_\theta(t) = H + \theta^a(t, \vec{x}_Q)(t^a \otimes 1) - \theta^a(t, \vec{x}_{Q_*})(1 \otimes t^{a*}),
\]

\[
H = - \frac{\hat{\nabla}_Q^2 + \hat{\nabla}_{Q_*}^2}{2M} - V(\vec{x}_Q - \vec{x}_{Q_*})(t^a \otimes t^{a*}),
\]

\[
[\theta^a(t, \vec{x} \theta^b(s, \vec{y}) = -D(\vec{x} - \vec{y})(\delta_{ts}/dt)\delta^{ab}.
\] (71)

Note that \(-D(\vec{r})\) is positive definite. In the limit \(dt \to 0\), the stochastic Schrödinger equation becomes (in the Itô discretization)

\[
\frac{\partial}{\partial t} \psi(t, \vec{x}_Q, \vec{x}_{Q_*}) = H_\xi(t) \psi(t, \vec{x}_Q, \vec{x}_{Q_*}),
\] (72)

\[
H_\xi(t) = H_\theta(t) + iC_T D(\hat{\vec{0}}) - iD(\vec{x}_Q - \vec{x}_{Q_*})(t^a \otimes t^{a*}).
\] (73)

In the stochastic Hamiltonian \(H_\xi(t)\), we omit terms of the form \(dt(\theta^2 - \langle \theta^2 \rangle) \sim dt \hat{\vec{0}}\) because they do not contribute in the master equation. In the stochastic Schrödinger equation, the reduced density matrix is defined as \(\hat{\rho}_{\text{QQ}}(t, \vec{x}_Q, \vec{x}_{Q_*}, \vec{y}_Q, \vec{y}_{Q_*}) \equiv \langle \psi(t, \vec{x}_Q, \vec{x}_{Q_*}) \psi^*(t, \vec{y}_Q, \vec{y}_{Q_*}) \rangle_\theta\) and its time evolution is governed by the master equation obtained previously.

In the relative coordinate, the stochastic Schrödinger equation for the wave function \(\psi^r(t, \vec{r})\) is also obtained
similarly:
\[
\frac{\partial}{\partial t}\psi^\alpha(t, \vec{x}_Q, \vec{x}_{Q_c}) = H^\alpha_I(t)\psi^\alpha(t, \vec{x}_Q, \vec{x}_{Q_c}), \tag{74}
\]
\[
H^\alpha_I(t) = -\frac{\nabla^2}{M} + iC_F D(\vec{0}) + (V(\vec{r}) - iD(\vec{r})) (t^n \otimes t^{n*}) + \theta^\alpha(t, \vec{r}/2)(t^n \otimes 1) - \theta^\alpha(t, -\vec{r}/2)(1 \otimes t^{n*}), \tag{75}
\]
and the master equation (69) is obtained by defining the reduced density matrix \(\hat{\rho}^{\alpha Q_c}_{Q_c}(t, \vec{r}, \vec{s}) \equiv \langle \psi^\alpha(t, \vec{r}) \psi^{*\alpha}(t, \vec{s}) \rangle_\phi\).

In a numerical simulation, solving the stochastic Schrödinger equation has substantial advantage over solving the master equation because the dimension of the former is half of the latter.

### B. Heavy quarkonium dissociation

In the stochastic Schrödinger equations (72) and (74), the noise and imaginary part describes how the color density fluctuation in the medium affects quantum dynamics while the potential describes how the color charges of heavy quarks interact with each other in the medium. The important scales here are correlation length \(l_{\text{fluct}}\) of the color density fluctuation \(-D(\vec{r})\) and the range of the screened potential \(V(\vec{r})\) (or more precisely coherence lengths \(l_{\text{coh}}\) of the eigenfunctions). If the former is much longer than the latter \(l_{\text{fluct}} \gg l_{\text{coh}} \gg l_{\text{IB}}\), the wave function remains almost unchanged by a scattering except for receiving a nearly uniform but random phase factor. In the opposite case \(l_{\text{fluct}} \ll l_{\text{coh}}\), the wave function easily becomes decoherent by a scattering.

To see these features explicitly, let us write down the coupled master equations for density matrices projected onto color singlet and octet states \((\rho_1(t, \vec{r}, \vec{s})\) and \(\rho_8(t, \vec{r}, \vec{s})\) defined previously):

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho_1 \\ \rho_8 \end{pmatrix}_{(t, \vec{r}, \vec{s})} = \begin{pmatrix} \nabla^2/2 - \nabla^2/2 \\ i( -2C_F) \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_8 \end{pmatrix}_{(t, \vec{r}, \vec{s})} + i\left(\frac{V(\vec{r}) - V(\vec{s})}{2}\right) \begin{pmatrix} C_F & 0 \\ 0 & -1/2N_c \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_8 \end{pmatrix}_{(t, \vec{r}, \vec{s})} + D(\vec{r}, \vec{s}) \begin{pmatrix} \rho_1 \\ \rho_8 \end{pmatrix}_{(t, \vec{r}, \vec{s})}, \tag{76}
\]

where \(D(\vec{r}, \vec{s})\), which describes decoherence, is defined as

\[
D(\vec{r}, \vec{s}) = 2C_F D(\vec{0}) - (D(\vec{r}) + D(\vec{s})) \begin{pmatrix} C_F & 0 \\ 0 & -1/2N_c \end{pmatrix} - 2D\left(\frac{\vec{r} - \vec{s}}{2}\right) \begin{pmatrix} 0 & 1/2N_c \\ C_F & (N_c^2 - 2)/2N_c \end{pmatrix} + 2D\left(\frac{\vec{r} + \vec{s}}{2}\right) \begin{pmatrix} 0 & 1/2N_c \\ C_F & -1/2N_c \end{pmatrix}. \tag{77}
\]

Before discussing decoherence, let us start from a simpler case with \(\vec{r} = \vec{s}\) as a warm up. Since \(\rho_1(t, \vec{r}, \vec{r})\) and \(\rho_8(t, \vec{r}, \vec{r})\) represent probability densities to find a quarkonium with separation \(\vec{r}\) in the singlet and octet states, \(D(\vec{r}, \vec{s})\) at the same points \(\vec{r} = \vec{s}\) gives the rate of color singlet-octet transitions there:

\[
D(\vec{r}, \vec{r}) = 2(D(\vec{0}) - D(\vec{r})) \begin{pmatrix} C_F & -1/2N_c \\ -C_F & 1/2N_c \end{pmatrix}. \tag{78}
\]

Because \(D(\vec{0}) - D(\vec{r}) < 0\), \(D(\vec{r}, \vec{r})\) has zero and negative eigenvalues for eigenvectors \(\{1, N_c^2 - 1\}\) and \(\{-1, -1\}\). The eigenvector \(\{1, N_c^2 - 1\}\) represents the equal occupation in the color singlet and octet states. If we ignore the kinetic and potential terms, a color-space configuration would approach this state, within shorter time scale at larger \(|\vec{r}|\). It is also important to observe that \(\rho_1(t, \vec{r}, \vec{r}) + \rho_8(t, \vec{r}, \vec{r})\) is conserved by \(D(\vec{r}, \vec{r})\). In the recoilless limit, the scatterings are equivalently described by a stochastic potential, which randomly gives phase and color rotations to a wave function. Therefore the probability density to find a quarkonium with a given separation \(\vec{r}\) either in the color singlet or octet states must be conserved in each scattering in the recoilless limit.

Now let us discuss the decoherence of a wave function. If the coherence length of a wave function is small \(l_{\text{IB}} \ll l_{\text{coh}} \ll l_{\text{fluct}}\), we have \(D(\vec{r}), D(\vec{s}) \sim D(\vec{0})\) for \(\vec{r}, \vec{s} < l_{\text{coh}}\) in the domain of the wave function. In this case, the decoherence is not effective \(D(\vec{r}, \vec{s}) \sim 0\). Note that this holds both for \(\rho_1(t, \vec{r}, \vec{s})\) and \(\rho_8(t, \vec{r}, \vec{s})\) even though the color singlet and octet states are quite different in their interaction with medium particles: Since the wave function is localized, the singlet state is almost invisible to them while the octet states clearly interact with them. An octet state does interact with the medium but it remains as one of the octet states. This is why \(D(\vec{r}, \vec{s}) \approx 0\) also for the octet states. This kind of information cannot be gained just from the imaginary part of the potential. By taking \(\vec{s} \sim -\vec{r}\) and expanding \(D(\vec{r}, \vec{s})\) in terms of \(\vec{r}^2/2\) \(\sim l_{\text{coh}}/l_{\text{fluct}} \ll 1\) to second order, the time scale of decoherence at the opposite edges of the wave function is estimated as \(\sim 1/(D_{2\text{coh}})^2 \sim 1/g^4T^3/l_{\text{coh}}^2\).

If the coherence length of a wave function is large \(l_{\text{coh}} \gg l_{\text{fluct}}\), we have \(D(\vec{r}), D(\vec{s}) \approx 0\) for \(l_{\text{fluct}} \ll l_{\text{coh}}\) and thus the decoherence at the edges of the wave function \(\vec{s} \sim -\vec{r}\) is given by

\[
D(\vec{r}, \vec{s}) \approx 2D(\vec{0}) \begin{pmatrix} C_F & 1/2N_c \\ C_F & C_F - 1/N_c \end{pmatrix}. \tag{79}
\]

Because \(D(\vec{0}) < 0\), \(D(\vec{r}, \vec{s})\) has only negative eigenvalues so that it makes the wave function decoherent by scatterings. The time scale for the decoherence is \(\sim 1/D(0) \sim 1/g^2T\).

In summary, we have shown that the bound states with larger size dissociate more easily by scattering with medium particles, as one can imagine quite intuitively.
V. SUMMARY

In this paper, we have derived the Lindblad-form master equations for heavy quark systems in the quark-gluon plasma. In order to obtain the master equations in the Lindblad form, we derive the influence functional $S_{\text{IF}}$ by perturbative and velocity expansions and by coarse-graining in time. The influence functional consists of $S_{\text{pot}}, S_{\text{fluct}} \sim \mathcal{O}(g^2v^0), S_{\text{diss}} \sim \mathcal{O}(g^2v)$, and $S_{\text{new}} \sim \mathcal{O}(g^2v^2)$. Here $S_{\text{new}}$ plays an essential role in deriving the master equations in the Lindblad form. In the velocity expansion, the velocity is estimated to be $v \sim \sqrt{T/M} \ll 1$ for a single heavy quark and $v \sim \alpha \ll 1$ for the relative motion in the quarkonium bound states with Coulomb potential $-\alpha/r$. In the coarse-graining in time, we need a condition $\alpha \lesssim gT$ in order to neglect the effect of acceleration in the quarkonium bound states. When $\alpha \lesssim gT$ is not satisfied, which indicates that quantum optical description works better for a quarkonium, the master equations for a single heavy quark and those for a quarkonium are not derived from a common influence functional $S_{\text{IF}}$.

The master equation in the Lindblad form ensures the complete positivity of the reduced density matrix as it evolves in real time. Therefore deriving the master equations in the Lindblad form is an important theoretical advance in the formulation of quantum dynamics of heavy quarks. Also the Lindblad-form master equation can be simulated by quantum state diffusion method [18] or quantum jump method [19], which evolves the wave functions, not the reduced density matrices.

After deriving the master equations in the Lindblad form, we have made approximations to obtain more effective master equations appropriate to the physical conditions of the problems. One is for quantum Brownian motion of localized wave packets and the other is for the decoherence of extended wave functions. Both approximations yield master equations in the Lindblad form.

Finally we have examined the decoherence of a quarkonium wave function. The decoherence is described by the master equation in the recoilless limit, which is equivalent to the Schrödinger equation with a stochastic potential. In terms of a stochastic potential, quarkonium dissociation can be understood as an interplay of two length scales, the coherence length of a state $l_{\text{coh}}$ and the correlation length of the thermal fluctuation $l_{\text{fluct}}$. For $l_{\text{coh}} \ll l_{\text{fluct}}$, the decoherence of the wave function is not effective and quarkonium dissociation requires a longer time of the order $\sim 1/g^2T^3l_{\text{coh}}^2$. For $l_{\text{coh}} \gg l_{\text{fluct}}$, the decoherence is so efficient that the quarkonium dissociates quickly with typical time scale $\sim 1/g^2T$.

By the same procedure and approximation, one can derive the master equations for an open quantum system with electromagnetic interaction, in which muons are treated as the system and a plasma of electrons, positrons, and photons constitutes the environment. In an open quantum system, where electrons or muons are treated as the system in a photon gas environment, the real photon emission and absorption are the only interaction processes between the system and the environment. Therefore one has to continue the velocity expansion to the next-to-leading order in order to include the finite temperature effect.

Before closing, let us mention the elementary but important limitations of other approaches using the complex potential. If a wave function is evolved by the Schrödinger equation with the complex potential [21, 22], such a wave function does not have enough information to construct either the density matrix or occupation number of a state. However, this approach can still calculate a vector correlation function and its spectral function in the potential model description and hence its application should be limited to these observables. If a complex potential is used to calculate the width of some initial state [27], it gives the characteristic frequency of the interaction between the quarkonium state and the medium. However, the width cannot tell us the occupation number of a state other than the initial state after one scattering. Moreover, since the width does not specify the final state, it does not contain enough information to study the case where there are many repeated scatterings. Therefore, although these approaches and quantities contain useful information of the in-medium dynamics of quarkonium, we have to keep these limitations in mind.

As future prospects, the calculation of the $\Upsilon$ spectrum at the LHC is one of the important applications of our approach. For this application, we need to model the dynamics of the open quantum system in the non-perturbative region by referring to and extending the perturbative results. It is also necessary to continue the velocity expansion to higher order in order to describe real gluon processes, such as excitation of quarkonium by absorbing a real gluon (glu-dissociation).

Acknowledgement

The author would like to thank François Gelis and Derek Teaney for enlightening discussions during their stay at the Kobayashi-Maskawa Institute as KMI visitors. He would also like to thank Jean-Paul Blaizot, Tetsuo Hatsuda, and Alexander Rothkopf for valuable comments on the manuscript.

Appendix A: Two-point functions of gluons

The influence functional $S_{\text{IF}}$ up to the order of $\mathcal{O}(g^2v^0, g^4v)$ and some of $\mathcal{O}(g^2v^2)$ is given by two-point functions of gluons. Since we are interested in the dis-
tance scale of $r \approx 1/gT$, where the Debye screening of the color charges becomes important, we need to include hard thermalloop (HTL) resummations to obtain the two-point functions at the leading order $O(g^3)$. The two-point functions $V(\vec{r})$, $D(\vec{r})$, and $A(\vec{r})$ are defined by using the retarded propagator $\tilde{C}_{ab,00}^R(\omega, \vec{r})$ and the spectral function $\sigma_{ab,00}(\omega, \vec{r})$, as shown in Eqs. (20), (21), and (22). The explicit forms of the retarded propagator and the spectral function are

$$\tilde{C}_{ab,00}^R(0, \vec{r}) = -\frac{e^{-\omega_D r}}{4\pi r}, \quad (A1)$$

and

$$\frac{\partial}{\partial \omega} \sigma_{ab,00}(0, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} \frac{\pi \omega_D^3 e^{i \vec{k} \cdot \vec{r}}}{k^2 + \omega_D^2}, \quad (A2)$$

with the Debye screening mass $\omega_D^2 = (g^2T^2/3)(N_c + N_f/2)$ for QCD with $N_f$ light flavors. With these, $V(\vec{r})$, $D(\vec{r})$, and $A(\vec{r})$ are determined to leading order in $g$.

It should be emphasized that the HTL-resummed calculation gives the leading-order result for $r \approx 1/gT$ but does not give a correct extrapolation from $r \approx 1/gT$ to $r \approx 1/T$. For example, if we calculate the heavy quark momentum diffusion constant, which is given by $(C_F/3)\bar{\nu}^2 D(\vec{r})|_{r=1/T}$, the scattering processes with exchanged momentum $k \approx T$ as well as $k \approx gT$ become relevant. In this case, we need to split the momentum integral at some intermediate scale $gT \ll \Lambda \ll T$ in Fourier space and add the two contributions to obtain the heavy quark diffusion constant. In $k < \Lambda$ the HTL-resummed result is reliable, while in $k > \Lambda$ scattering processes with exchanged momentum $k$ need to be considered separately. The contributions from different momentum regions are logarithmically sensitive to the scale $\Lambda$ but these dependences are canceled in the sum, yielding a finite heavy quark momentum diffusion constant $28$.

Appendix B: Time-evolution equation for forward propagator

By the method explained in Sec. [11], we can derive the time-evolution equation for the forward propagator $G_{QQ}(t, \vec{x}_Q, \vec{x}_Q^*), \quad \text{in the } N_c \otimes N_c^* \text{ representation. The time-evolution equation is often called "the Schrödinger equation," causing a lot of confusion by its name. Using the influence functional up to } O(g^2v^0, g^2v^2) \text{ (thus we do not consider } S_{\text{new}} \sim O(g^2v^2) \text{ here), the time evolution of the forward propagator is given by an operator $K(\vec{x}_Q, \vec{x}_Q^*)$}:

$$\begin{align*}
&i \frac{\partial}{\partial t} G_{QQ}(t, \vec{x}_Q, \vec{x}_Q^*) = K(\vec{x}_Q, \vec{x}_Q^*)G_{QQ}(t, \vec{x}_Q, \vec{x}_Q^*), \\
&K(\vec{x}_Q, \vec{x}_Q^*) = \left\{ \begin{array}{l}
2M - \frac{\vec{\pi} \omega}{15} + iD(0) \\
+ \frac{C_F}{2M} \left\{ \begin{array}{l}
-\frac{V(\vec{x}_Q - \vec{x}_Q^*) - iD(\vec{x}_Q) - iD(\vec{x}_Q^*)}{M} \\
\left( \frac{\bar{\nu}^2 D(\vec{x}_Q) - \bar{\nu}^2 D(\vec{x}_Q^*)}{M} \right)
\end{array} \right. \\
\times (t^a \otimes t^{a*})
\end{array} \right.
\end{align*}$$

We find that there are terms not only from $S_{\text{pot}}, S_{\text{luct}} \sim O(g^2v^0)$ but also from $S_{\text{diss}} \sim O(g^2v)$ in the operator $K(\vec{x}_Q, \vec{x}_Q^*)$. This shows that the leading correction to the operator $K(\vec{x}_Q, \vec{x}_Q^*)$ in the velocity expansion is $O(v)$, not $O(v^2)$ as in the vacuum. The $O(v)$ term comes from the diagonal parts of $S_{\text{diss}}$. This is correctly obtained by choosing $t = \max(x^0, y^0)$.

In Eq. (B2), the term $-\frac{i}{4MT}(\cdots) (t^a \otimes t^{a*})$ in $K(\vec{x}_Q, \vec{x}_Q^*)$ is hermitian and identical to the second line of Eq. (61). Therefore, by projecting Eqs. (B1) and (B2) onto the singlet channel, we can see that the term makes a positive (negative) contribution to $K(\vec{x}_Q, \vec{x}_Q^*)$ when a heavy quark-antiquark pair is moving apart from (approaching) each other. The sign is opposite if we project onto the octet channel. This coupling of the relative position and momentum may be relevant to the stability of the ground states of charmonium and bottomonium even at $T < 2T_c$ found in Refs. 24, 25.

Appendix C: Explicit form of $\mathcal{L}^{(2)}_{QQ}$

The explicit form of $\mathcal{L}^{(2)}_{QQ}$ consists of 4 terms:

$$\begin{align*}
\mathcal{L}^{(2)}_{QQ} &= \mathcal{L}^{(2)}_{1Q}(\vec{x}_Q, \vec{x}_Q^*) (t^a \otimes t^{a*}) \hat{\rho}_{QQ} \\
&+ \mathcal{L}^{(2)}_{2Q}(\vec{x}_Q, \vec{y}_Q) (t^a \otimes 1) \hat{\rho}_{QQ} (1 \otimes t^{a*}) \\
&+ \mathcal{L}^{(2)}_{3Q}(\vec{y}_Q, \vec{x}_Q^*) (1 \otimes t^{a*}) \hat{\rho}_{QQ} (t^a \otimes 1) \\
&+ \mathcal{L}^{(2)}_{4Q}(\vec{y}_Q, \vec{y}_Q^*) \hat{\rho}_{QQ} (t^a \otimes t^{a*}).
\end{align*}$$

With $\vec{r}_1 = \vec{x}_Q - \vec{x}_Q^*$ and $\vec{r}_2 = \vec{x}_Q - \vec{y}_Q^*$, each of them is given by

$$\begin{align*}
\mathcal{L}^{(1)}_{QQ}(\vec{x}_Q, \vec{x}_Q^*) &= iV(\vec{r}_1) - D(\vec{r}_1) - \frac{\bar{\nu}^2 D(\vec{r}_1)}{4MT} + \frac{(\bar{\nu}^2)^2 A(\vec{r}_1)}{8M^2} \\
&- \bar{\nu} \left( \frac{D(\vec{r}_1)}{4MT} - \frac{\bar{\nu}^2 A(\vec{r}_1)}{4M^2} \right) (\bar{\nu}_x - \bar{\nu}_{x_Q}) \\
&- \frac{\vec{\nabla}^i \vec{\nabla}^j A(\vec{r}_1)}{2M^2} \vec{\nabla}_x^i \vec{\nabla}_x^j \bar{\nu}_{x_Q},
\end{align*}$$

\[13\]
\[
\mathcal{L}^{12}_{QQ}(\vec{x}_Q, \vec{y}_Q) = \frac{D(\vec{r}_{12})}{4M} + \frac{\nabla^2 A(\vec{r}_{12})}{8M^2} + \nabla \left( \frac{D(\vec{r}_{12})}{4M} + \frac{\nabla^2 A(\vec{r}_{12})}{4M^2} \right) \cdot (\nabla \vec{x}_Q - \nabla \vec{y}_Q), \quad (C3)
\]

\[
\mathcal{L}^{21}_{QQ}(\vec{y}_Q, \vec{x}_Q) = \mathcal{L}^{12}_{QQ}(\vec{y}_Q, \vec{x}_Q), \quad (C4)
\]

\[
\mathcal{L}^{22}_{QQ}(\vec{y}_Q, \vec{y}_Q) = (\mathcal{L}^{11}_{QQ}(\vec{y}_Q, \vec{y}_Q))^*, \quad (C5)
\]