Implementation of lyapunov method to analyze the stability of pompano, cantang growth and nutrition dynamical systems

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Abstract. This study aims to determine the analysis stability of dynamical model of pompano, cantang growth, and nutrition on the Integrated Multi-Trophic Aquaculture (IMTA) systems by using the Lyapunov method. The analytical analysis was given to discuss the dynamic behavior of this model. Global stability analysis was performed based on the Lyapunov theory, i.e., construction the function $V(x)$ that is a definite positive scalar function and the derivative of $V(x)$ is definite negative. As a verification of the Lyapunov method, we conducted numerical simulations with data taken from IMTA systems in Sea Farming region, Kepulauan Seribu, Indonesia.

1. Introduction
IMTA system is a cultivation practice with the maintenance of more than one species of organism or a polyculture system, but the cultivated species are those that have a mutualistic relation ecologically as a food chain in the area. Polyculture cultivation is now widely researched and studied because it can improve water quality. The integration of seaweed Eucheuma and Gracilaria sp into the activities of polyculture of tiger shrimp (Penaeus monodon) and milkfish (Chanos Chanos) in an integrated manner to get maximum yields than monoculture cultivation and more efficient use of land [1].

Cultivation is also widely developed in an attempt to fulfill the nutritional needs of humans [2]. The development of polyculture cultivation Integrated Multi-Trophic Aquaculture (IMTA) can increase economic benefits and minimize the negative impact of aquaculture on the environment [3], the impact of sediment and water quality [4]. During the process of growing fish need good nutrition. Feeding too much of an impact on the environment such as the analysis of water quality due to floating net cages [5, 6].

In the culture system IMTA types of marine life that are often cultivated as pompano fish farming and cantang star because the value of consumption is very high. The market demand for this fish is good enough both of the pompano and cantang besides the rapid growth and relatively high resale value [7].

There are many studies on the management cultivation of monoculture and polyculture system. However, a most mathematical model developed to predict the growth is a dynamical model in monoculture. On the other hand, the mathematical model for the polyculture system is still minimal with the stability analysis of this mathematical model only in the analysis of local stability around the
equilibrium point [8]. So, we need a mathematical model to describe the growth of marine life more than one biota with a polyculture system and how to analyze the global stability of this model by using the Lyapunov method.

2. Lyapunov Method to Analyze the Dynamical Model Stability

In the modeling process, the component of growth in IMTA system studied was composed of three variables, i.e. pompano, cantang, and nutrition. Three component interaction model, with a population of Pompano \(B\), cantang \(M\) and nutrition \(P\). Process of growth in the IMTA system in this case the system is modeled in equations (1). The skeleton of the think to obtain the model can be found at Sulpiani, et al [8].

\[
\frac{dP}{dt} = \theta - \left( \frac{1}{\mu^2 \rho \eta} + \frac{1}{\gamma^2 \delta (1 - \eta)} \right) P \\
\frac{dB}{dt} = \frac{1}{\gamma^2 (1 - \eta)} \delta P B^2 - \varepsilon \\
+ \sigma M \\
\frac{dM}{dt} = \frac{1}{\mu^2 \eta \rho PM^2 - \tau M - \sigma M}
\]

\(P \geq 0, B \geq 0, M \geq 0, \mu, \rho, \gamma, \delta, \varepsilon, \sigma, \tau > 0, 0 < \eta < 1\),

where \(\eta\) is nutrition consumed by cantang, \(1 - \eta\) is nutrition consumed by pompano, \(\theta\) is nutrition into the cage, \(\sigma\) is the rest of cantang nutrition utilized pompano, \(\varepsilon\) is the nutrition necessary to maintain the viability of pompano, \(\tau\) is the nutrition necessary to maintain the viability of cantang, \(\gamma\) is constant stating the ratio of the pompano weight and length, \(\mu\) is constant stating the ratio of the cantang weight and length.

Next, the second Lyapunov method or often called by the Lyapunov direct method is used for globally stability analysis of the dynamical systems. This method commonly used to determine the stability of both linear and nonlinear systems and does not need to find a solution from the system.

The procedure of the Lyapunov method is given as follows,

i. Let \(x_e = 0\) be an equilibrium point of \(\dot{x} = f(x, t), f: D \rightarrow R^n\)

ii. Let \(V: D \rightarrow R\) be a continuously differentiable function, such that

a. \(V(0) = 0\)

b. \(V(x) > 0, \text{ in } D - \{0\}\)

c. \(V(x) < 0, \text{ in } D - \{0\}\)

iii. Then \(x_e = 0\) is globally asymptotically stable

Meanwhile, theorem corresponding to the global stability of the system with Lyapunov function can be represented in Theorem 2.1.[9, 10].

**Theorem 2.1**

Let a system in the form of \(\dot{x} = f(x, t)\) with \(f(0, t) = 0\) for all \(t\). If there is a scalar function \(V(x)\) which has the first derivative and meets the following conditions

1. \(V(x)\) is positive definite,

2. \(\dot{V}(x)\) is negative definite,

Then the equilibrium point is globally asymptotically stable. If it turns out that \(V(x) \rightarrow \infty \text{ for } \|x\| \rightarrow \infty\), then the equilibrium point is globally asymptotically stable.

Further, \(V(x)\) is called the Lyapunov function. Krasovskii method is one of the methods used to construct the Lyapunov function. From the dynamic system model (1) a Lyapunov function is constructed as follows,
Equations (1) can be written \( f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ M \end{bmatrix} \), so that \( \dot{\begin{bmatrix} B \\ M \end{bmatrix}} = \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \).

Krasovskii defines Lyapunov's function as follows

\[ V(x) = f^T R f, \]

where \( R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \)

is a positive definite symmetric matrix. Then obtain

\[ V(x) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \]

\[ V(x) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \]

Furthermore, calculate the derivative of the Lyapunov function as follows,

\[ \dot{V}(x) = f^T Q f + f^T R \dot{f}, \]

with \( \dot{f} = \frac{\partial f}{\partial x} \frac{dx}{dt} = J f, \) and \( Q = f^T R + R f, \)

Let \( \dot{f} = J f, \)

with the Jacobian matrix, \( J = \begin{bmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial B} & \frac{\partial f_1}{\partial M} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial B} & \frac{\partial f_2}{\partial M} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial B} & \frac{\partial f_3}{\partial M} \end{bmatrix} \)

then obtain,

\[ \dot{V}(x) = f^T Q f \]

\[ \dot{V}(x) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \]

\[ \dot{V}(x) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \]

\[ \dot{V}(x) = \begin{bmatrix} \theta - \left( \frac{1}{\mu^3} \rho \eta + \frac{1}{\gamma^3} \delta (1 - \eta) \right) P \\ \frac{1}{\gamma^3} (1 - \eta) \delta \bar{P} \bar{B} \bar{\sigma} - \varepsilon B + \sigma M \\ \frac{1}{\mu^3} \eta \rho P M^2 - \tau M - \sigma M \end{bmatrix} \]
3. Results and Discussions
In this section is given a numerical simulation from the model (1) and construct the Lyapunov function to analyze the stability of the model at the equilibrium point. Based on data from the IMTA system in Sea Farming region, Kepulauan Seribu, Indonesia [11], the mathematical model is a nonlinear differential equations system as follows,

\[
\begin{align*}
\frac{dP}{dt} &= 1 - 2.778 P \\
\frac{dB}{dt} &= 1.283 P B^2 - 0.13B + 0.10M \\
\frac{dM}{dt} &= 1.495 P M^2 - 0.21M
\end{align*}
\] (4)

The equilibrium point is obtained when fulfilling meet \( \frac{dP}{dt} = 0, \frac{dB}{dt} = 0, \frac{dM}{dt} = 0 \). Let \((P^*, B^*, M^*)\) stating the equilibrium point of the model (4) so that we have the model of equations at the point \((P^*, B^*, M^*)\) as follows, 

\[
\begin{align*}
\frac{dP}{dt} &= 1 - 2.778 P^* = 0 \\
\frac{dB}{dt} &= 1.283 P^* B^2 - 0.13B^* + 0.10M^* = 0 \\
\frac{dM}{dt} &= 1.495 P^* M^2 - 0.21M^* = 0
\end{align*}
\]

We found that the equilibrium point is \((P^*, B^*, M^*) = (0.35,0,0)\).

Further to analyze the global stability of a dynamic system model (4) at this equilibrium point, we apply the Lyapunov method. The lyapunov function is constructed as follows,

Let equations depending on the time

\[
\dot{x} = f(x,t), \quad x = \begin{bmatrix} \dot{P} \\ \dot{B} \\ M \end{bmatrix}
\]

thus obtained

\[
\begin{align*}
f_1 &= 1 - 2.778 P \\
f_2 &= 1.283 P B^2 - 0.13B + 0.10M \\
f_3 &= 1.495 P M^2 - 0.21M
\end{align*}
\]

moreover, can be written \( f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \dot{P} \\ \dot{B} \\ M \end{bmatrix} \), then 

\[
\begin{bmatrix} \dot{P} \\ \dot{B} \\ M \end{bmatrix} = \begin{bmatrix} 1 - 2.778 P \\ 1.283 P B^2 - 0.13B + 0.10M \\ 1.495 P M^2 - 0.21M \end{bmatrix}
\]

Krasovskii defines the following Lyapunov function:

\[
V(x) = f^T R f
\]

\[
V(x) = \begin{bmatrix} 1 - 2.778 P \\ 1.283 P B^2 - 0.13B + 0.10M \\ 1.495 P M^2 - 0.21M \end{bmatrix}^T \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 1 - 2.778 P \\ 1.283 P B^2 - 0.13B + 0.10M \\ 1.495 P M^2 - 0.21M \end{bmatrix}
\]
moreover, the derivative of the Lyapunov function,
\[
\dot{V}(x, t) = f^T(J^T R + R) f
\]
\[
= \begin{bmatrix}
1 - 2.778 P \\
1.283 P B^2 - 0.13B + 0.10M \\
1.494 P M^2 - 0.21M
\end{bmatrix} \begin{bmatrix}
-2.778 \\
0.1855P B^2 - 0.13 \\
0.1 \end{bmatrix} \begin{bmatrix}
1 - 2.778 P \\
1.283 P B^2 - 0.13B + 0.10M \\
1.494 P M^2 - 0.21M
\end{bmatrix}
\]
Note that, by algebra manipulation, we found that \( V (x) \) is a positive definite scalar function and \( \dot{V} (x) \) is the definite negative. In addition, based on Theorem 2.1. we have \( V (x) \to \infty \) when \( \| x \| \to \infty \), so that the system is globally asymptotically stable. In this case we obtained globally asymptotically stable system which means that the cultivation practice of IMTA systems can be assumed the growth of pompano and cantang fish to normally run well.

4. Conclusion
In this paper, the model discussed is a dynamical system model of the pompano and cantang growth with nutrition as a significant factor in the cultivation of IMTA systems. Globally stability analysis of dynamical systems obtained by the Lyapunov method, i.e., we construct the Lyapunov function which is definite positive, and the derivative of the Lyapunov function is definite negative. From the numerical simulations we obtained that dynamical systems of the pompano, cantang growth and nutrition at the equilibrium point is globally asymptotically stable. This indicates that pompano and cantang fish with sufficient nutrition can grow well in the IMTA system in Sea Farming region, Kepulauan Seribu, Indonesia.

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References
[1] Sudaryono, Putro S, Suminto S, dan Malina A 2014 Pengembangan Teknik Budidaya, Diversifikasi Produk, dan Pengelolaan Lingkungan Sektor Perikanan Budidaya untuk Mendukung Ketahanan Pangan Nasional. [Development of Cultivation Technique, Product Diversification, and Environmental Management of Cultivation Fishery Sector to Support National Food Security], Final Report Year 2 of the National Priorities Research Master Plan for the Acceleration and Expansion of Indonesian Economic Development 2011-2025 (Penprinas MP3EI 2011-2025), (Semarang: Diponegoro University)
[2] Corilee, Watters, Iwamura S, Ako H, and Deng D 2012 Nutrition Considerations in Aquaculture: The Importance of Omega-3 Fatty Acids in Fish Development and Human Health, College of Tropical Agriculture and Human Resources, University of Hawai’i at Manoa.
[3] Putro S, Widowati, Suhartana, and Muhammad F 2015 Internat. J. Sci. Eng. 9 85.
[4] Putro S, Febria I, Muhammad F 2014 Jurnal Sains dan Matematika 22 79
[5] Parrello S, Tomassetti P, Mansueto L, Finoia M, Persia E, Mercatali I, and Stipa P 2005 Aquaculture 249 145
[6] McGhie T, Crawford C, Mitchell I, and O’Brien D 2000 Aquaculture 187 351
[7] Langkosono 2007 *Neptunus* 14 61.

[8] Sulpiani R, Widowati, Putro S, and Sunarsih 2016 *Stability Analysis and Mathematical Modeling of Pompano and Cantang Growth on Integrated Multi-Trophic Aquaculture Systems*, presented in the Asian Mathematics Conference, Juli 2016, Bali.

[9] Wang Q 1983 *Using Liapunov Principles in the Analysis of System Dynamics Models*. Shanghai Institute of Mechanical Engineering, China.

[10] Panikhom S and Sujitjorn S 2010 *Latest Trends on Circuits, Systems and Signals* 58

[11] Putro S, Widowati, Muhammad F and Suhartana 2016 *Peningkatan kapasitas perikanan budidaya berkelanjutan melalui aplikasi stratified double floating net cages (SDFNC) dengan pendekatan integrated multi-trophic aquaculture (IMTA)* [Capacity building for sustainable aquaculture through net applications stratified double floating cages (SDNFC) with integrated multi-trophic aquaculture (IMTA)] (Pusnas Kemenristek Dikti), Diponegoro University, [Bahasa Indonesia].