The effective action and equations of motion of curved local and global vortices: Role of the field excitations.

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The effective actions for both local and global curved vortices are derived, based on the derivative expansion of the corresponding field theoretic actions of the nonrelativistic Abelian Higgs and Goldstone models. The role of excitations of the modulus and the phase of the scalar field and of the gauge field (the Bogolyubov-Anderson mode) emitted and reabsorbed by vortices is elucidated. In case of the local (gauge) magnetic vortex, they are necessary for cancellation of the long distance divergence when using the transverse form of the electric gauge field strength of the background field. In case of global vortex taking them into account results in the Greiter-Wilczek-Witten form of the effective action for the Goldstone mode. The expressions for transverse Magnus-like force and the vortex effective mass for both local and global vortices are found. The equations of motion of both type of vortices including the terms due to the field excitations are obtained and solved in cases of large and small contour displacements.

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I. INTRODUCTION

The active theoretical studies of the superfluid [1, 2] and superconducting vortices [3] take place over more than half a century [4]. Important issues concerning the dynamics of both types of vortices include, in particular, the problems of the transverse Magnus-like force acting on vortex [5, 6, 7], and the evaluation of the vortex effective mass [6, 8, 9]. The above incomplete list of references shows that different models are used in describing various aspects of vortex dynamics. This is due to complexity of the object under study put in real experimental conditions which include dissipation, finite-temperature effects etc. The common feature of both types of vortices in condensed matter physics is that they appear as topological defects in the models with spontaneously broken U(1) symmetry.

The purpose of the present paper is to study the dynamics of the curved line defects in the framework of a uniform approach based on the field theoretic models and to scrutinize the role of the excitations of the gauge and scalar fields exchanged between the different segments of the vortex. The local (gauge) vortex equations of motion are obtained from the effective action of the Abelian Higgs model (AHM) at zero temperature \( T = 0 \). The same task is performed for the global vortex by taking the limit of the vanishing gauge coupling constant. The earlier attempt to consider the role of the field excitations was undertaken in Ref. [9]. However, the background contribution which is absolutely essential for the static situation was not taken into account in [9]. Furthermore, the treatment in Ref. [9] was restricted to considering only quasi-two-dimensional situation (straight vortices), and no attempt was undertaken to derive the vortex equations of motion by means of the variation of effective action. In the present work we fill these gaps and treat the background fields and the excitations. We work in truly three-dimensional situation and allow for the curvature of the vortex contour by deriving expressions for the effective actions which systematically use the so called derivative expansion. The expressions for the transverse Magnus-like force and the effective masses of the local and global vortices are obtained.

II. FORMULATION OF THE MODEL AND BASIC NOTATIONS.

We will work in the London limit where the London penetration depth \( \lambda_L \) is much greater than the coherence length \( \xi \). [See Eq. (6.2) and (6.11) below for the definition of these quantities.] In this limit the gauge field configuration of
the magnetic vortex is generated by the singular phase $\chi_s$ of the order parameter $\psi$. The notion singular means that

$$[\nabla \times \nabla] \chi_s = 2\pi \sum_a n_a \int d\sigma_a X'_a \delta^{(3)}(x - X_a),$$

(2.1)

where sum goes over all contours given by the vector $X_a = X_a(t, \sigma_a)$, $n_a$ is the number of flux quanta trapped by the vortex. Hereafter, prime means the differentiation with respect to the corresponding contour parameter $\sigma_a$, and $[a \times b]$ stands for the vector product of two vectors $a$ and $b$. Since the length of the contour is $l = \int_{\sigma_a} |X'_a| d\sigma_a$, the natural choice of $\sigma_a$ is that it gives the length of the contour segment starting with initial value $\sigma_{a1}$ and ending with $\sigma_a$. Hence, the gauge condition

$$X'_a = 1$$

(2.2)

can be applied. In the rest of the paper the limits of integration over contour parameter are omitted for brevity. The action of the Abelian Higgs model is

$$S = \int d^4x \left\{ \frac{1}{8\pi} (E^2 - H^2) - \frac{g}{2} (|\psi|^2 - n_0)^2 + \frac{1}{2} [\psi^* (i\hbar \partial_t - q\varphi + qa_0) \psi + c.c.] - \frac{1}{2m} \left| -i\hbar \nabla - \frac{q}{c} A + \frac{q}{c} a \right|^2 \right\}.$$  

(2.3)

Here, $E = -\partial_t A/c - \nabla \varphi$ and $H = \nabla \times A$ are the electric and magnetic field strengths, with $\varphi$ standing for the time component of the gauge four-vector potential $A_\mu = (\varphi, A)$; $q = 2e$, $m = 2m_e$ are the charge, mass, of the scalar field in terms of the electronic ones, $c$ is the velocity of light. Further, $n_0$ and $\rho_0$ stand for the density of the scalar field condensate (the number density of the Cooper pairs) and for the homogeneous positive charge density introduced to provide the net neutrality of the system:

$$\rho_0 + qn_0 = 0.$$  

(2.4)

The coupling constant $g$ will be related below to the sound velocity $c_s$. The four-vector $a_\mu = -\frac{\hbar c}{q} \partial_\mu \chi_s$ represents the four-gradient of the singular phase. The field strength tensor corresponding to $a_\mu$ is $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \equiv (E_\mu, H_\mu$), where the mixed Fourier transforms of the singular parts of the electric and magnetic field strengths are

$$E_s(t, k) = -\frac{\Phi_0}{c} \sum_a n_a \int d\sigma_a [\dot{X}_a \times X'_a] e^{-i\mathbf{k} \cdot \mathbf{X}_a},$$

$$H_s(t, k) = \Phi_0 \sum_a n_a \int d\sigma_a X'_a e^{-i\mathbf{k} \cdot \mathbf{X}_a}.$$  

(2.5)

Hereafter dot means the differentiation with respect to time, and

$$\Phi_0 = \frac{2\pi \hbar c}{q}$$

is the magnetic flux quantum. Both the full Fourier representation in $k = (\omega, \mathbf{k})$ or the mixed one in $(t, \mathbf{k})$ turns out to be very suitable in what follows. The full representation is defined as usual:

$$b_k = b(\omega, k) = \int d^4x e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} b(t, x),$$

$$b(t, x) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} b_k.$$  

(2.6)

The mixed one is defined accordingly. One can find the necessary expressions for the singular part of the vector potential $a_k(t) \equiv \mathbf{a}(t, k)$ from the equation $i|\mathbf{k} \times a_k| = H_s(t, k)$:

$$a_k(t) = \frac{\Phi_0}{k^2} \sum_a n_a \int d\sigma_a [\mathbf{k} \times X'_a] e^{-i\mathbf{k} \cdot \mathbf{X}_a}.$$  

(2.7)
This vector is transverse, $k \mathbf{a}_k = 0$. The mixed Fourier component of the singular part of the Coulomb potential $a_0$ can be found from the expression $E_a(t, k) = -\dot{\mathbf{a}}_k/c - i k a_{0k}$:

$$\dot{a}_0(t) = -i \frac{\Phi_0}{c k^2} \sum_a n_a \int d\sigma_a \mathbf{k} [\dot{X}_a \times X'_a] e^{-i k X'_a}. \quad (2.8)$$

Note that both $E(t, k)$ and $a_{0k}$ look like the local Lorenz transforms of $\mathbf{H}(t, k)$ and $\mathbf{a}_k$, respectively. The end point contribution vanishes identically for the closed contour, while for the open one the end point term is proportional to the combination $X_f \cdot \nabla |x - X_f|^{-1} - X_i \cdot \nabla |x - X_i|^{-1}$ which can be made vanishing either by pinning the end points $X_{f,i}$ or pushing them to the spatial infinity. In what follows we will always assume the vanishing of the analogous end point contributions.

### III. THE CONTRIBUTION OF BACKGROUND FIELD

Let us rewrite the action Eq. (2.3), first, in terms of the modulus and the phase of the scalar field $\psi = n^{1/2} e^{i \chi}$. Second, expand the field configuration in terms of the background fields and the fluctuations, $\varphi = \varphi + \delta \varphi$, $\mathbf{A} = \mathbf{A} + \delta \mathbf{A}$, $n \to n_0 + \delta n$, $\chi \to \delta \chi$. Note that the space-time derivatives of the singular background phase is already included in Eq. (2.3) as the four-vector $a_i$. The modulus of the scalar field is taken to be $n_0$ as it should in the London limit. One gets the total action Eq. (2.3) as the sum of the background action $S_{\text{bg}}$ and the action of the fluctuations $S_f$. The latter is discussed in Sec. IV. The background action is

$$S_{\text{bg}} = \int d^4x \left[ \frac{1}{8\pi} (E^2 - H^2) - \frac{1}{8\pi \lambda_L^2} (A - a)^2 + n_0 q a_0 \right]. \quad (3.1)$$

The term proportional to $\varphi$ drops drops in view of Eq. (2.4). Hereafter

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_0 q^2}} \quad (3.2)$$

is the the London penetration depth. The condition of the vanishing of terms linear in $\delta \mathbf{A}$ and $\delta \varphi$ gives the gauge field (Maxwell) equations for background gauge fields

$$[\nabla \times \mathbf{H}] + \frac{1}{\lambda_L^2} \mathbf{A} = \frac{1}{\lambda_L^2} \mathbf{a} + \frac{1}{c} \partial_i E, \quad (3.3)$$

$$\nabla E = 4\pi (n_0 q + \rho_0) = 0, \quad (3.4)$$

while the vanishing of terms linear in variation of the phase $\delta \chi$ gives the equation

$$\nabla (A - a) = 0. \quad (3.5)$$

The latter equation is satisfied automatically in the Coulomb gauge. Since the situation is non-relativistic, we will neglect the retardation effects in what follows. Hence, the displacement current in Eq. (3.3) will be neglected, too. One should not vary over density because we assume homogeneous density $n = n_0$ everywhere except the vortex core. The latter is ignored in the London limit. Using the relation $\mathbf{H}_k = i [k \times \mathbf{A}_k]$ one can find the solution of Eq. (3.3) in the mixed $(t, k)$ Fourier representation:

$$A_k = \frac{1/\lambda_L^2}{k^2 + 1/\lambda_L^2} a_k = \frac{i \Phi_0/\lambda_L^2}{k^2/(k^2 + 1/\lambda_L^2)} \sum_a n_a \int d\sigma_a [k \times X'_a] e^{-i k X'_a}. \quad (3.6)$$

Making the inverse Fourier transform to the coordinate space one can see that the vector potential $\mathbf{A}$ is decomposed into the long range and the short range terms, while the combination

$$A_k - a_k = -\frac{i \Phi_0}{k^2 + 1/\lambda_L^2} \sum_a n_a \int d\sigma_a [k \times X'_a] e^{-i k X'_a} \quad (3.7)$$
which appears in Eq. (3.11) contains only the short range exponentially damped contribution. The expression for the Fourier amplitude of the strength of the background magnetic field is

$$H_k = \frac{\Phi_0}{k^2 + 1/\lambda^2} \sum_n a_n \int d\sigma_a X_a' e^{-ikX_a}. \quad (3.8)$$

The Fourier amplitude of the strength of the electric field is

$$E_k = -\frac{1/(c\lambda^2)}{k^2 + 1/\lambda^2} a_k - ik\varphi_k. \quad (3.9)$$

The scalar potential $\varphi_k = 0$, as can be found from the above equation and the Gauss law Eq. (3.4). The resulting expression for $E_k$ looks like

$$E_k = -\frac{\Phi_0/(c\lambda^2)}{k^2 + 1/\lambda^2} \sum_n a_n \int d\sigma_a e^{-ikX_a} \left\{ [X_a \times X_a'] - \frac{k(k[X_a \times X_a'])}{k^2} \right\}. \quad (3.10)$$

Both $H_k$ and $E_k$ are transverse. As for $E_k$, it is evident from Eq. (3.11). In the case of $H_k$ the transverse character is proved by noting that $kH_k \propto \int d\sigma_a X_a e^{-ikX_a} = i \int d\sigma_a \partial_{\sigma_a} e^{-ikX_a} = 0$ for closed contours. For the open ones the transverse character is provided by pushing the end points to the spatial infinity.

With the help of the relation

$$\int d^3xf^2(t, x) = \int \frac{d^3k}{(2\pi)^3} |f_k(t)|^2$$

one can obtain the action of background fields in terms of the vortex contour variable $X_a \equiv X_a(t, \sigma_a)$. From now on we will restrict ourselves by the case of the single contour with the unit flux quantum. The notations adopted in what follows are $X_{1,2} \equiv X(\sigma_{1,2})$, $X_{12} \equiv X_1 - X_2$, where the points $\sigma_{1,2}$ belong to the same contour. Taking into account Eqs. (3.7), (3.8), (3.9), and (3.10) one obtains

$$S_{bg} = \frac{\Phi_0^2}{8\pi} \int dt \frac{d^3k}{(2\pi)^3} \left( \frac{1/\lambda^2}{k^2 + 1/\lambda^2} \right)^2 \int d\sigma_1 d\sigma_2 e^{-ikX_{12}} \times$$

$$\left\{ \frac{1}{c^2} [\dot{X}_1 \times X_1'][\dot{X}_2 \times X_2'] - \frac{(k[X_1 \times X_1')(k[X_2 \times X_2'])}{k^2} \right\} - X_1'X_2'(1 + \lambda_a^2) +$$

$$n_0g \int d^3x a_0. \quad (3.11)$$

In the braces, the terms with the time derivatives originate from $E^2$, the term $\propto \lambda^2$ does from the kinetic energy of the Higgs field $\propto (A - a)^2$, while the remaining one is due to $H^2$. Notice that the contribution of the term with $a_0$ in Eq. (3.11), at first sight, seems to be irrelevant because

$$\int d^3x a_0 = -\frac{\Phi_0}{4\pi c} \int d\sigma [\dot{X} \times X'] \left( \int d^3x \nabla^2 \frac{1}{|x - X|} \right) = 0.$$
can be used. When obtaining the bottom line in Eq. (3.12) the relations $X' \cdot X'' = -X''^2 = -\kappa^2$ and $X' \cdot X'' = 0$ should be taken into account. They follow from the Frenet-Serre equations

$$X'' = \kappa n,$$

$$n' = -\kappa X' + \tau b,$$

$$b' = -\tau n,$$  \hspace{1cm} (3.13)

where $\tau$ stands for the torsion of the contour, and $n, b$ are the vectors of normal and bi-normal, respectively. The vectors $(n, b, X')$ comprise the right triple of the unit orthogonal vectors, so that $X' = [n \times b]$ (and similar relations obtained by the cyclic permutation). Note also that $X'^2 = 1 + O(z^4)$ if $X'_L = 1$, hence the approximate expansions over $z$ do not break the gauge condition $X'^2 = 1$ within the adopted accuracy. The background action is represented as $S_{bg} = S_{bg}^{(0)} + \Delta S_{bg}$, where

$$S_{bg}^{(0)} = \frac{\Phi_0^2}{8\pi} \int dt \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\lambda^2_L} + 1/\lambda^2_L \right)^2 \int d\sigma_1 d\sigma_2 e^{-ikX_{12}} \left\{ [\hat{X}_1 \times X'_1] - X'_2 (1 + \lambda^2_L k^2) \right\} + l_0 q \int d^4x_0.$$  \hspace{1cm} (3.14)

It is finite at large distances while

$$\Delta S_{bg} = -\frac{\Phi_0^2}{8\pi^2} \int dt \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\lambda^2_L} + 1/\lambda^2_L \right)^2 \int d\sigma_1 d\sigma_2 e^{-ikX_{12}} (k [\hat{X}_1 \times X'_1]) (k [\hat{X}_2 \times X'_2])$$  \hspace{1cm} (3.15)

diverges at large distances in view of the factor $1/k^2$. As it will be shown below, the similar divergent contribution with the opposite sign arises from the term originating from the integrated out fluctuations. Hence, we postpone the evaluation of the terms $\propto 1/k^2$ to Sec. [V].

Performing the integration over momentum $k$ in Eq. (3.14) one gets

$$S_{bg}^{(0)} = \frac{\Phi_0^2}{32\pi^2\lambda^2_L} \int dt d\sigma_1 d\sigma_2 e^{-|X_{21}|/\lambda_L} \left\{ \frac{[\hat{X}_1 \times X'_1][\hat{X}_2 \times X'_2]}{2\epsilon^2 \lambda_L} - (X'_2)^2 \left[ \frac{\ln \lambda_L}{\xi} - C - \frac{13}{24} \lambda^2_L \kappa^2 \right] \right\} + l_0 q \int d^4x_0.$$  \hspace{1cm} (3.16)

The integrand in Eq. (3.16) is nonlocal in the contour parameter $\sigma$. This is due to the distribution of the gauge vortex profile over coordinate space. However, the contributions of the remote points are suppressed exponentially as $e^{-|X_{(s2)} - X_{(s2)}|/\lambda_L}$, and one can derive the approximate local expression. Indeed, using $\int d\sigma_1 \int d\sigma_2 \approx \int d\sigma_1 \int_{-\infty}^{\infty} dz$ and replacing $\sigma_1 \to \sigma$ one can integrate over $\sigma$ to obtain

$$S_{bg}^0 = \left( \frac{\Phi_0}{4\pi \lambda_L} \right)^2 \int dtdz \left\{ \frac{1}{2\epsilon^2} \left( (\hat{X} \times X')^2 - \lambda^2_L (\hat{X} \times X')^2 \right) - (X'^2) \left[ \frac{\ln \lambda_L}{\xi} - C - \frac{13}{24} \lambda^2_L \kappa^2 \right] \right\} + l_0 q \int d^4x_0.$$  \hspace{1cm} (3.17)

where $C = 0.577215...$ is the Euler constant. Notice that the term with $\propto \ln \lambda_L/\xi$ arises due to the usual short distance cutoff when the lower integration limit over $z$ is replaced by the coherence length $\xi$:

$$\int_{-\infty}^{\infty} dz e^{-|z|/\lambda_L} \rightarrow 2 \int_{-\infty}^{\infty} \frac{dz}{z} e^{-z/\lambda_L} \approx 2 \left( \ln \frac{\lambda_L}{\xi} - C \right).$$  \hspace{1cm} (3.18)

Such a replacement is justifiable because the modulus of the scalar field $|\psi|$, in fact, vanishes at the vortex core $r_{\perp} \leq \xi$. Thus, the only contribution to the action enhanced as $\ln \lambda_L/\xi$ is that of the kinetic energy of the scalar field given by the term $\propto k^2$ in Eq. (3.11). It is recognized as the contribution of the minus energy of the gauge vortex with the energy per unit length $\epsilon_v$, given, in the leading logarithmic approximation and by neglecting higher derivative term $\propto \kappa^2$, by the expression [10]

$$\epsilon_v \approx \left( \frac{\Phi_0}{4\pi \lambda_L} \right)^2 \ln \frac{\lambda_L}{\xi}.$$  \hspace{1cm} (3.19)

One should be cautious when dealing with the approximate local form Eq. (3.17) [together with the correction given by Eq. (5.10)]. Such local expressions are useful only when identifying various contributions, as, for example, the
energy per unit length above, or in deriving the expression for the effective mass of the vortex (see Sec. [VI, VII and Ref. [9]). They cannot be used for obtaining equations of the vortex motion because of the above mentioned finite size vortex gauge field distribution \( \propto e^{-|{x}(\sigma_2)-{x}(\sigma_2)|/\lambda_c} \). Indeed, the variation of the approximate local expression \( \propto \int d\sigma \Delta X^2 \) in Eq. (5.17) gives the result which is by the factor two greater than the correct expression obtained upon varying the corresponding nonlocal contribution \( \propto \int d\sigma_1 d\sigma_2 X_1^2 X_2^2 e^{-|{x}_{21}|/\lambda_c} \) in Eq. (5.16). The same is true for the dynamical terms containing the velocity \( \dot{X} \). See Sec. [VI for more detail. To summarize, only expressions Eq. (5.16) and (5.18) in Sec. [VI should be used to obtain the vortex equations of motion.

### IV. INTEGRATING OUT THE EXCITATIONS OF THE SCALAR AND GAUGE FIELDS.

The next step is to integrate out the propagating fluctuations of the modulus of the order parameter \( \delta n \), the phase of this parameter \( \delta \chi \), the Coulomb potential \( \delta \varphi \), and the vector potential \( \delta A \). This is necessary for obtaining the correction to the action of the background fields. To this end let us write the action \( S_{\ell} \) for fluctuations representing them for short as the row \( f^T = (\delta n, \delta \chi, \delta \varphi) \) and \( \delta A \). One obtains \( S_{\ell} = S_{\ell}^{(0)} + S_{\ell}^{(1)} \), where

\[
S_{\ell}^{(0)} = \int d^4x \left\{ \frac{1}{2} f^T M f - \frac{1}{8\pi} \left( \frac{1}{c^2} \nabla^2 - \frac{1}{\lambda_c^2} \right) \delta A + \delta n \left[ q_{a0} - \frac{q^2}{2mc^2} (A - a)^2 \right] \right\}
\]

and

\[
S_{\ell}^{(1)} = \frac{q_h}{mc} \int d^4x (A - a) \delta n \left( \nabla \delta \chi - \frac{q}{hc} \delta A \right).
\]

The Coulomb gauge \( \nabla \delta A = 0 \) is chosen. The matrix \( M \) is

\[
\begin{pmatrix}
\frac{h^2}{4\pi m_0} \nabla^2 - g & -h \partial_t & -q \\
-h \partial_t & \frac{h^2 n_0}{m} \nabla^2 & 0 \\
-q & 0 & -\frac{1}{\pi} \nabla^2
\end{pmatrix}.
\]

Formally, \( S_{\ell}^{(1)} \) in Eq. (4.12) is quadratic in fluctuations and should be attributed to the free action. However, because of highly inhomogeneous profile of the background vector potential \( A - a \), the path integrations with \( S_{\ell} \) cannot be performed in the closed form, so we prefer to treat \( S_{\ell}^{(1)} \) as the perturbation leaving \( S_{\ell}^{(0)} \) in Eq. (4.11) as the action of free fluctuations. The contribution of the term \( q^2 (A - a)^2 / 2mc^2 \) can be neglected. In fact, it would enter the effective action either quadratically giving the terms \( \propto \int \int \int d\sigma_1 d\sigma_2 d\sigma_3 X_1^2 X_2^2 X_3^2 \), or as the interference with \( q_{a0} \), giving the terms \( \propto \int \int \int d\sigma_1 d\sigma_2 d\sigma_3 \dot{X}_1 X_2 X'_3 \). Both type of the terms are attributed to higher derivatives and go beyond the scope of the present paper. One can show also that keeping \( S_{\ell}^{(1)} \) results in the higher derivative terms in the effective action. See the end of the present section.

As is known, one should find the generating functional \( Z[j] \) in order to integrate out the quantum excitations [11]. In the present case it looks as

\[
Z[j] = \int D[\delta n] D[\delta \chi] D[\delta \varphi] D[\delta A] \exp \left[ \frac{i}{\hbar} \tilde{S}_{\ell} \right]
\]

where the action \( \tilde{S}_{\ell} = S_{\ell}^{(0)} + \int d^4x (\int f^T j_f + j_A \delta A) \) is the functional of currents \( j_f = (j_n, j_\chi, j_\varphi) \), and \( j_A \). The gauge fixing and ghost terms are omitted because they are irrelevant in the context of the paper. As in the preceding section, it is suitable to work in the momentum space \( k = (\omega, k) \). Then the integration measure for the spacetime function \( f(x) \) is expressed in terms of its Fourier amplitudes as

\[
D[f] = \prod_k \frac{d\text{Re} f_k d\text{Im} f_k}{2\pi}.
\]

The path integral Eq. (4.4) is gaussian, so using the inverse of \( M \) in the momentum space,

\[
M^{-1} = \frac{1}{\left( \frac{\hbar^2}{k^2} + \frac{c^2}{\lambda_c^2} - \omega^2 - i\epsilon \right)} \begin{pmatrix}
-\frac{m}{\hbar^2} k^2 & \frac{\omega}{\hbar^2} & \frac{i\omega}{\hbar^2} & -\frac{4\pi n_0}{m} \\
-\hbar k^2 n_0 & -\frac{\hbar^2}{\lambda_c^2} k^2 + i\epsilon & 0 & 0 \\
-\frac{4\pi n_0 q}{m h} & \frac{4\pi n_0 q}{h k^2} & \frac{4\pi}{k^2} & \frac{\hbar^2}{k^2} - \omega^2
\end{pmatrix}.
\]


one finds the generating functional \( Z[j] = \exp \frac{i}{\hbar} \Delta \tilde{S}_t \), where the correction due to excitations is

\[
\Delta \tilde{S}_t = i\hbar \ln \det M + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \left\{ \left( \frac{c^2_B}{\hbar^2} + \frac{c_s^2}{\chi^2_L} - \omega^2 - i0 \right)^{-1} \times \right.
\]

\[
\left[ \frac{n_0 k^2}{m} |(j_n) + qa_0|^2 + \frac{m |(j_\chi) k^2}{\hbar^2 k^2 n_0} \left( \frac{c^2_B}{\hbar^2} + \frac{c_s^2}{\chi^2_L} \right) - \right.
\]

\[
\left. \frac{4\pi |(j_\varphi) k|^2}{k^2} \left( \frac{c^2_B}{\hbar^2} - \omega^2 \right) + i\frac{\omega}{\hbar} (j_n + qa_0)^k (j_\chi) k - c.c.] + \right.
\]

\[
\left. \frac{4\pi n_0 q}{m} (j_n + qa_0)^k (j_\varphi) k + c.c.] + \frac{4\pi i\omega}{\hbar k^2} [(j_\chi)^k (j_\varphi) k + c.c.] \right. \]

\[
\frac{4\pi |(j_\lambda) k|^2}{k^2 + \frac{1}{\chi_L^2} - \frac{c_s^2}{c^2_B} - i0} \right\}. \quad (4.6)
\]

Here,

\[
c_s^2 \equiv c_B^2(k) = \left( \frac{\hbar^2 k^2}{2m} \right)^2 + \hbar^2 c_s^2 k^2 \quad (4.7)
\]

is the square of the Bogolyubov spectrum \[12\] looking at small wave numbers as the spectrum of sound waves with the sound velocity

\[
c_s = \sqrt{\frac{n_0 g}{m}}. \quad (4.8)
\]

As is known, the regularization prescription \(-i0\) guarantees that the causal (Feynman) Green functions will result after taking the variational derivatives of \( Z[j] \) over \( j \). The correlation functions in the momentum space which are just the Fourier transforms of the Green functions are

\[
\langle f^{(1)}_{k_1} f^{(2)*}_{k_2} \cdots \rangle = Z^{-1}[0] \left( -2i\hbar \frac{\delta}{\delta j^{(1)}_{k_1}} \right) \left( -2i\hbar \frac{\delta}{\delta j^{(2)*}_{k_2}} \right) \cdots Z[j]_{|j=0}. \quad (4.9)
\]

The divergent constant \( \ln \det M \) drops from all expressions for the correlators Eq. \( (4.9) \).

The meaning of the poles in \( \omega \) in Eq. \( (4.5) \) is the following. Let us consider the freely propagating fluctuations decoupled from their source represented by the term in square brackets of Eq. \( (4.4) \). Their equations of motion obtained upon the condition of the vanishing variational derivatives look as

\[
0 = \hbar \partial_t \delta \chi + g \delta n + q \delta \varphi - \frac{\hbar^2}{4m n_0} \nabla^2 \delta n,
\]

\[
0 = \partial_t \delta n + \frac{\hbar n_0}{m} \nabla^2 \delta \chi,
\]

\[
0 = \nabla^2 \delta \varphi + 4\pi q \delta n. \quad (4.10)
\]

Applying \( \nabla^2 \) to the first line in Eq. \( (4.10) \) and using the second and third lines of the same equation one finds

\[
\left( -\partial_t^2 + c_s^2 \nabla^2 + \frac{c_s^2}{\chi^2_L} - \frac{\hbar^2}{4m^2} \nabla^4 \right) \delta n = 0.
\]

The propagating plane wave solution has the dispersion law with the gap

\[
\omega^2 \equiv \omega_k^2 = \frac{c_s^2}{\chi^2_L} + \frac{\varepsilon_B^2(k)}{\hbar^2},
\]

which is just the pole position of the matrix \( M^{-1} \). The same dispersion law is obtained for the phase fluctuations \( \delta \chi \).

The above dispersion law in the long wave limit acquires the form \( \omega^2 \approx \omega_p^2 + c_s^2 k^2 = \omega_p^2 + \frac{1}{2} v_F^2 k^2 \), where \( \omega_p, v_F \) stand for the plasma frequency and Fermi velocity, respectively. This is the dispersion law for the Bogolyubov-Anderson
mode in the charged Fermi system \[13, 14\]. The sound velocity \( c_s \) is related to \( v_F \) through the expression for the sound velocity \( c_s^2 = \frac{N}{m} \left( \frac{\partial v}{\partial N} \right)_s \) \[15\], by taking into account the expression for chemical potential of the weakly interacting Bose gas \( \mu = g n_0 / m \) \[12, 13\]. Recall that AHM mimics the Bose liquid of the Cooper pairs whose parameters are related to its electronic counterparts by the relations \( \mu = 2 \mu_c, n_0 = n_c / 2, m = 2m_c, q = 2e \). In the case of neutral superfluid where the gauge coupling constant is switched off, \( q \to 0 \), the spectrum is gapless, \( \omega^2 = \varepsilon_B^2 / h^2 \) \[12\]. On the other hand, taking the static limit and neglecting the gauge fields one finds the equation

\[ \nabla^2 \delta n - \frac{4mn_0 g}{\hbar^2} \delta n = 0, \]

which means that the space fluctuations of the modulus of the scalar field damp at the coherence (or healing) length

\[ \xi = \frac{\hbar}{2(mn_0 g)^{1/2}}. \] (4.11)

Taking into account Eq. (4.8) one obtains the relation

\[ \xi = \frac{\hbar}{2mc_s} \] (4.12)

which relates the sound velocity \( c_s \) with the coherence length \( \xi \). We dwell upon in this paragraph on the well known issues in order to show that the character of excitations and their dispersion laws can be established in the framework of effective lagrangians without references to the underlying microscopic picture.

The contribution of the term Eq. (4.2) can be evaluated with the help of Eq. (4.6). One finds the correction to the effective action due to the above term:

\[ \Delta S^{(1)}_{\text{eff}} = -\frac{g^3 n_0}{4m^2 \hbar} \int \frac{d^4k_1 d^4k_2}{(2\pi)^8} (A - a)_{k_1}^{*} a_{k_2} \left( \frac{\epsilon_B^2 k_1^2}{\hbar^2} + \frac{c^2}{X_L^2} - \omega^2 - i0 \right)^{-1} \times \left( \frac{\epsilon_B^2 k_2^2}{\hbar^2} + \frac{c^2}{X_L^2} - \omega^2 - i0 \right)^{-1}. \]

Using Eq. (2.8) and (3.7) one can see that because of the factor \( \omega_1 \) in the integrand the above expression starts with the terms of the type \( \int d\sigma_j d\sigma_j d\sigma_k X_1 X_2 X_3 X_4 \) etc. resulting in the third order time derivative terms in the vortex equations of motion. Such higher derivative terms can be neglected for sufficiently smooth evolution of the contours. By this reason the term Eq. (4.2) can be neglected.

V. DERIVATIVE EXPANSION AND THE EFFECTIVE ACTION FOR THE LOCAL VORTEX.

The correction due to fluctuations is obtained from Eq. (4.6) by setting \( j_n = 0, j_x = 0, j_\phi = 0 \) and neglecting the term \( \ln \det \Gamma \):

\[ \Delta S_l = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left( \frac{\epsilon_B^2 k^2}{\hbar^2} + \frac{c^2}{X_L^2} - \omega^2 - i0 \right)^{-1} \frac{m n_0 q^2}{k^2} a_{0k}^2 k^2, \] (5.1)

where \( a_{0k} = \int_{-\infty}^{\infty} dt e^{i\omega t} a_{0k}(t) \), with \( a_{0k}(t) \) given by Eq. (2.8), is the four-dimensional \( (\omega, k) \) Fourier transform of the time component of the singular part of the gauge field potential \( a_\mu \). The explicit expression will be obtained in the case of relatively slow dynamics of the vortex contour. In Sec. III the derivative expansion based on the smooth contour shape is used to obtain the explicit expression for the background part of the effective action. In the present section, the analogous derivative expansion based on the slow motion of the contour is the key point. Let us expose it in detail taking the typical integral:

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|f_k|^2}{\omega^2 - \Omega_k^2 + i0} = \int_{-\infty}^{\infty} dt \int \frac{d\omega}{2\pi} \frac{f_k(t) f_k^*(t + \tau) e^{i\omega \tau}}{\omega^2 - \Omega_k^2 + i0} = \int dt f_k(t) \sum_{l=0}^{\infty} \frac{(-i)^l}{l!} \frac{d^l f_k}{dt^l} \left[ \frac{1}{\partial^l \omega^2 - \Omega_k^2} \right]_{\omega=0} = \int dt \left[ -\frac{|f_k(t)|^2}{\Omega_k^2} - \frac{1}{2\Omega_k^4} \left( f_k \frac{\partial^2 f_k}{\partial t^2} + \text{c.c.} \right) + \cdots \right], \] (5.2)
where dots stand for the terms with the higher time derivatives. In the case of our interest, a typical $f_k$ contains $X$, hence $\partial^2 f_k/\partial t^2$, and, consequently, the equations of motion would contain the higher derivative term $\partial^3 X/\partial t^3$.

In order to have the equations of motion with the time derivative not higher that two one should keep only the first term in the expansion Eq. (5.2). Hence, in the lowest order in the number of the derivatives over time, one finds from Eq. (5.1) the correction to the effective action due to the fluctuations:

$$\Delta S_l = \frac{\Phi_0^2}{8\pi\lambda^2} \int dt d\sigma_1 d\sigma_2 \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot X_{21}}}{k^2} \left(k[X_1 \times X_{1j}](k[X_2 \times X_{2j}])\right).$$  (5.3)

One can see that due to the factor $1/k^2$ this expression is logarithmically divergent at large $\sigma_2 - \sigma_1$. It is the same large distance divergence as is observed in the background action Eq. (3.11), (3.15), but with the opposite sign. Let us examine the divergent and finite contributions in Eq. (3.15) and (5.3). Denoting the their sum as $\Delta S_{\text{eff}} = \Delta S_{\text{bg}} + \Delta S_l$ and taking into account the relation $4m^2c^2/h^2 = 1/\xi^2$, let us represent $\Delta S_{\text{eff}}$ in the form suitable for integration over momentum:

$$\Delta S_{\text{eff}} = \frac{\Phi_0^2}{8\pi c^2} \int dt d\sigma_2 d\sigma_1 [X_1 \times X_{1j}](X_2 \times X_{2j})\nabla_{21i} \nabla_{21j} I(\sigma_1, \sigma_2),$$

$$I(\sigma_1, \sigma_2) = \left\{ \frac{\partial}{\partial \lambda L} \right\}^2 \left[ \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2} - \frac{1}{k^2 + 1/\lambda^2} \right) \right] e^{ik \cdot X_{21}} - \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot X_{21}} \left[ \frac{1}{k^2} - \frac{k^2 + 1/\xi^2}{k^2 + k^2/\xi^2 + 1/\xi^2\lambda^2} \right],$$  (5.4)

where $\nabla_{21i} = \partial/\partial X_{21i}$. The above equation displays clearly the cancellation of the divergent contributions arising at $|k| \to 0$. Notice the appearance of additional length scale $\xi$.

$$\lambda_s = \lambda_L \cdot \frac{c_s}{c} \ll \lambda_L,$$  (5.5)

see Eq. (3.2), (4.8), and (4.12). This scale is absent in the static case, but arises in the dynamical situation. After convincing oneself that the divergences cancel one can write the resulting expression for the quantity $I(\sigma_1, \sigma_2)$ in Eq. (5.4):

$$I(\sigma_1, \sigma_2) = -\frac{e^{-|X_{21}|/\lambda_L}}{4\pi|X_{21}|} \left( 1 + \frac{|X_{21}|}{2\lambda_L} \right) + \frac{1}{8\pi|X_{21}|} \times$$

$$\left[ e^{-|X_{21}|(1-\sqrt{1-4\xi^2/\lambda^2})^{1/2}}/\xi \sqrt{1+1/\sqrt{1-4\xi^2/\lambda^2}} \right] +$$

$$e^{-|X_{21}|(1+\sqrt{1-4\xi^2/\lambda^2})^{1/2}}/\xi \sqrt{1-1/\sqrt{1-4\xi^2/\lambda^2}} \right]$$  (5.6)

at $\xi \leq \lambda_s/2$, and

$$I(\sigma_1, \sigma_2) = -\frac{e^{-|X_{21}|/\lambda_L}}{4\pi|X_{21}|} \left( 1 + \frac{|X_{21}|}{2\lambda_L} \right) \frac{1}{4\pi|X_{21}|} e^{-|X_{21}|/\sqrt{1/2}\lambda_s + 1/4\xi^2} \times$$

$$\left[ \cos \left( \frac{|X_{21}|}{\sqrt{1/2}\lambda_s + 1/4\xi^2} \right) + \frac{1}{\sqrt{1/2}\lambda_s - 1} \times$$

$$\sin \left( \frac{|X_{21}|}{\sqrt{1/2}\lambda_s + 1/4\xi^2} \right) \right]$$  (5.7)

at $\xi > \lambda_s/2$. Both above expressions are interrelated by the analytical continuation. The scale $\lambda_s$ divides the London part of the parameter space $\xi \ll \lambda_L$ into two pieces, $\lambda_s < \xi < \lambda_L$ and $\xi < \lambda_s < \lambda_L$. Let us consider the latter one and, in addition, choose a special case $\xi \ll \lambda_s/2 < \lambda_L$, leaving the case $\xi > \lambda_s/2$ for a future work. Expanding square root in Eq. (5.5), neglecting the terms exponentially small at $|X_{21}| > \xi$, and applying to the resulting expression $\nabla_{21i} \nabla_{21j}$ one finds:

$$\Delta S_{\text{eff}} = \frac{\Phi_0^2}{32\pi^2 c^2} \int dt d\sigma_1 d\sigma_2 [X_1 \times X_{1j}][X_2 \times X_{2j}] \left\{ \frac{\delta_{ij}}{|X_{21}|} \left( 1 + \frac{|X_{21}|}{\lambda_s} \right) \times$$

$$(1 + \frac{|X_{21}|}{\lambda_s}) \right\}.$$
\[ e^{-|X_{21}|/\lambda} - \left( 1 + \frac{|X_{21}|}{\lambda L} \right) e^{-|X_{21}|/\lambda L} + \frac{\delta_{ij}}{2\lambda^2 |X_{21}|} e^{-|X_{21}|/\lambda L} + \]
\[
\frac{X_{21} X_{21j}}{|X_{21}|^5} \left[ \left( 1 + 3 \frac{|X_{21}|}{\lambda L} + \frac{X_{21}^2}{\lambda^2} \right) e^{-|X_{21}|/\lambda L} - \right.
\]
\[
\left( 1 + 3 \frac{|X_{21}|}{\lambda L} + \frac{X_{21}^2}{\lambda^2} \right) e^{-|X_{21}|/\lambda L} + \frac{X_{21}^2}{2\lambda L} \left( 1 + \frac{|X_{21}|}{\lambda L} \right) e^{-|X_{21}|/\lambda L} \right] \}
\] (5.8)

Note that \( \delta_{ij}\delta^3(X_{21}) \) arising due to \( \nabla_{21} \nabla_{21j} \left( 1/|X_{21}| \right) \) is multiplied by the factor vanishing at \( X_{21} = 0 \) and hence does not contribute to the effective action. The derivative expansion adopted in the present paper, results in setting \( \sigma_2 = \sigma_1 + z \) and expanding all quantities depending on \( \sigma_2 \) in series over \( z \), see Eq. (3.12). In view of the exponential damping of the integrands at large \( |X_{21}| \) one can replace in Eq. (5.8) approximately \( \int d\sigma_1 \int d\sigma_2 = \int d\sigma_1 \int_{-\infty}^{\infty} dz \). The contributions proportional to \( \delta_{ij} \) possess the logarithmic divergence at short distances. This is an artifact due to the ignorance of the vortex core \( |X_{21}| \leq \xi \). One can handle this divergence in the same manner as in Eq. (3.13). Taking into account another regularized expression
\[
\int_{\xi}^{\infty} \frac{dz}{z^3} \left( 1 + \frac{z}{\lambda} \right) e^{-z/\lambda} = \frac{1}{2} \left( 1 + \frac{1}{\xi^2} + C - \frac{\ln \lambda/\xi}{\lambda^2} \right)
\] (5.9)

where \( C = 0.577215... \) is the Euler constant, one can see that the terms \( \propto \ln(\lambda L/\xi) \) cancel, the term \( \propto \ln(\lambda_s/\xi) \) survives. Gathering all the integral together and replacing \( \sigma_1 \to \sigma \) one obtains the correction to the effective action due to the combined contributions of the excitations integrated out, and of the part of the background action Eq. (3.15):
\[
\Delta S_{\text{eff}} = \frac{\Phi_0^2}{32\pi^2 c^2} \int dt d\sigma \left\{ \left[ \frac{1}{\lambda^2} \left( \ln \frac{\lambda_s}{\xi} - C \right) + \kappa^2 \left( \ln \frac{\lambda L}{\lambda s} - \frac{1}{24} \right) \right] |\hat{X} \times X'|^2 - \right.
\]
\[
\left. \left[ \ln \frac{\lambda L}{\lambda s} + \frac{1}{2} \right] \left( \frac{\partial}{\partial \sigma} |\hat{X} \times X'| \right)^2 + \frac{1}{2} \left( \ln \frac{\lambda L}{\lambda s} + \frac{3}{2} \right) (\hat{X} |\hat{X} \times X'|)^2 \right\}. \] (5.10)

The higher derivative terms do not have the logarithmic enhancement factors \( \ln \lambda_L/\xi \) nor \( \ln \lambda_s/\xi \). Hence, we neglect them to obtain
\[
S_{\text{eff}} \approx \frac{\Phi_0}{4\pi \lambda_L}^2 \int dt d\sigma \left\{ \frac{1}{2 \xi^2} |\hat{X} \times X'|^2 \ln \frac{\lambda_s}{\xi} - X'^2 \ln \frac{\lambda L}{\xi} \right\} + n_0 q \int d^4 x a_0. \] (5.11)

The contribution of the short range part of the electric field has the same form \( |\hat{X} \times X'|^2 \), see Eq. (3.17). But it is not enhanced as logarithm and is suppressed by the factor \( c^2/e^2 \). Hence, it can be neglected. Since \( |\hat{X} \times X'|^2 = \hat{X}^2 \) in the gauge \( \hat{X} \cdot X' = 0, X'^2 = 1 \), one can see that the first term in the braces of Eq. (5.11) can be interpreted as the kinetic energy of the vortex motion with the effective mass per length \( L \)
\[
\frac{m_{\text{eff}}}{L} = \left( \frac{\Phi_0}{4\pi \lambda_c c_s} \right)^2 \ln \frac{\lambda_s}{\xi} = \frac{\pi \hbar^2}{g} \ln \frac{\lambda_s}{\xi} = m_e n_c \xi^2 \times 4\pi \ln \frac{\lambda_s}{\xi}. \] (5.12)

The first equality in Eq. (5.12) coincides with the expression obtained earlier for the straight vortex. The last equality shows that the effective mass of the gauge vortex equals to the mass of superconducting electrons expelled from the region with the transverse dimension \( \xi \), multiplied by the dynamical enhancement factor.

**VI. THE LOCAL VORTEX EQUATIONS OF MOTION.**

First of all, let us show that the variation of the term \( n_0 q \int d^4 x a_0 \) in Eq. (3.16) gives nonzero result despite the fact that the contribution of the above term to the effective action vanishes. Using the Fourier amplitude of the singular part of the time component of the vector potential \( a_0 \) [see Eq. (2.38)] one obtains
\[
\delta \int d^4 x a_0 = -\frac{\Phi_0}{c} \int dt d\sigma (\delta \hat{X} |\hat{X} \times X'|). \] (6.1)

Here, a number of integrations by parts over both \( t \) and \( \sigma \) should be performed, and the known vector relation \( [A \times [B \times C]] = B(AC) - C(AB) \) should be taken into account. The equations of motion of the local (gauge) vortex
in the approximation of large logarithms can be obtained upon varying Eq. (3.16) and (5.8) over the contour variable $X$ and with the neglect of the higher derivative terms. To do so, one should first vary over $X$ including the factor $e^{-|X_{21}|/\lambda_L}$ then, second, use the expansions Eq. (5.12) keeping the terms with the lowest non-vanishing order in the expansion variable $\delta = \sigma_2 - \sigma_1$:

$$\delta \int d\sigma_1 d\sigma_2 (X_1' X_2') e^{-|X_{21}|/\lambda_L} = 2 \int d\sigma_1 d\sigma_2 (\delta X_1' \cdot [X_2' \times [X_2' \times X_2']]) \frac{1 + |X_{21}|/\lambda_L}{|X_{21}|} e^{-|X_{21}|/\lambda_L} =$$

$$2 \int d\sigma \int_{-\infty}^{\infty} e^{-|z|/\lambda_L} \frac{\delta X \cdot [X' \times [X' \times X'']]}{2|z|} = -2 \ln \frac{\lambda_L}{\xi} \int d\sigma \delta X \cdot X''.$$

(Eq. 3.12)

Eq. (3.12) should be kept in mind when deriving the above chain of calculations. As far as $\Delta S_{\text{eff}}$ are concerned, one can see that the terms in Eq. (5.5) arising upon contracting $[X_1' \times [X_2' \times X_2']_j$ with $X_{21i}, X_{21j}$ are just the higher derivative terms which are not enhanced as $\ln \lambda_\nu/\xi$. They are displayed in Eq. (5.10). Using Eq. (3.13) and (5.9) for the short distance regularization and performing the manipulations analogous to the above, one can obtain the variations of the terms containing velocity $X$. Collecting all variations together and keeping only the terms corresponding to the lowest derivative ones in Eq. (5.10) one gets

$$\delta S_{\text{eff}} = - \left(\frac{\Phi_0}{4 \pi \lambda_L}\right)^2 \int dt d\sigma \left\{ \frac{1}{c_s^2} [X' \cdot \partial \delta X \times [X' \times X']] \ln \frac{\lambda_L}{\xi} - X'' \ln \frac{\lambda_L}{\xi} \right\} - \frac{n_0 q \Phi_0}{c} \int dt d\sigma (\delta X [X' \times X']) ,$$

which results in the equations of motion

$$[X' \times X'] = \frac{\hbar}{2m} \left( X'' \ln \frac{\lambda_L}{\xi} + \frac{1}{c_s^2} \partial \partial [X' \times [X' \times X']] \ln \frac{\lambda_L}{\xi} \right).$$

(Eq. 3.13)

Let us denote for the sake of brevity

$$\gamma = \frac{\hbar}{2m} \ln \frac{\lambda_L}{\xi}$$

and consider the large amplitude motions, when the nonlinearity in Eq. (6.4) could be essential. The zeroth order approximate solution obtained upon neglecting the terms $\propto 1/c_s^2$ is

$$[X' \times X']^{(0)} = \gamma \kappa n, \quad X^{(0)} = \gamma \kappa b + V^{(0)} X',$$

where the first line in Eq. (3.13) and the relation $X' = [n \times b]$ are used. The longitudinal component of the velocity $V^{(0)}$ cannot be determined from the first relation in Eq. (6.6). However, $V^{(0)}$ is locally unobservable, because of physical homogeneity of the vortex in the (local) longitudinal direction. It can be excluded by additional gauge choice $\dot{X} X' = 0$. So, the zeroth order intrinsic velocity of the curved gauge vortex looks as

$$X^{(0)} = b \frac{\hbar}{2m} [X'' \ln \frac{\lambda_L}{\xi}] .$$

(Eq. 6.4)

It is proportional to the curvature of the contour and is directed along the bi-normal vector. Its magnitude relative to the sound velocity $c_s$ is

$$\frac{[X^{(0)}]}{c_s} = \frac{\hbar}{2mc_s} |X''| \ln \frac{\lambda_L}{\xi} = \frac{\xi}{R} \ln \frac{\lambda_L}{\xi} \ll 1,$$

where $R = |X''|^{-1}$ stands for the radius of curvature of the contour. The large scale nonlinear motion of the contour is rather slow. One can see that the straight vortex does not possess an intrinsic motion as it should.

To find the correction $O(1/c_s^2)$ to the zeroth order solution it is necessary to obtain the time derivatives of the curvature $\kappa$ and torsion $\tau$ as well as the triple of basic vectors of normal $n$, bi-normal $b$, and the tangent $X'$. Using Eq. (3.13), (6.6), the definitions $\kappa = (n X'')$, $\tau = (b \cdot n)$ one obtains the equations

$$\frac{\partial X'}{\partial t} = \gamma (\kappa' b - \kappa \tau n),$$

$$\frac{\partial n}{\partial t} = \gamma \left[ \kappa \tau X' + \left( \frac{\kappa''}{\kappa} - \tau^2 \right) b \right],$$

$$\frac{\partial b}{\partial t} = -\gamma \left[ \kappa' X' + \left( \frac{\kappa''}{\kappa} - \tau^2 \right) n \right].$$

(Eq. 6.8)
and

\[ \frac{\partial \kappa}{\partial t} = -\gamma (2\kappa' + \kappa''), \]

\[ \frac{\partial \tau}{\partial t} = \gamma \left[ \kappa \kappa' + \left( \frac{\kappa''}{\kappa} - \tau^2 \right) \right]. \]  

(6.9)

Equations (6.9) (however, without the factor \( \gamma \)) are known in hydrodynamics as Da Rios equations [16]. Making use of Eq. (6.13), (6.4), (6.3), (6.8), and (6.9) one can obtain the relation for finding the correction \( \hat{X}^{(1)} \) due to the term \( O(1/\epsilon^2) \):

\[ [\hat{X}^{(1)} \times \hat{X}'] = \frac{\hbar \gamma^2}{2mc^2} \ln \frac{\lambda_\alpha}{\xi} \left[ n(\kappa'' - \kappa\tau^2) + b(2\kappa' + \kappa'' \tau') \right]. \]  

(6.10)

Using the above relation one gets

\[ \hat{X}^{(1)} = \frac{\hbar \gamma^2}{2mc^2} \ln \frac{\lambda_\alpha}{\xi} \left[ b(\kappa'' - \kappa\tau^2) - n(2\kappa' + \kappa'' \tau') \right] = |\hat{X}^{(0)}| \xi^2 \ln \frac{\lambda_\alpha}{\xi} \ln \frac{\lambda_\xi}{\xi} \times \]

\[ \left[ b(\kappa'' - \kappa\tau^2) - n(2\kappa' + \kappa'' \tau') \right] \frac{1}{\kappa}. \]  

(6.11)

The ratio of correction to the zeroth order velocity

\[ \frac{|\hat{X}^{(1)}|}{|X^{(0)}|} = \xi^2 \kappa^2 \ln \frac{\lambda_\alpha}{\xi} \ln \frac{\lambda_\xi}{\xi} < 1 \]

is small but non-negligible, because the truly small ratio of the coherence length to the curvature radius squared is enhanced by the product of two large logarithms.

Let us consider the small amplitude oscillations around the locally straight vortex. Then, locally, one may choose \( \hat{X}' = e_z, \hat{X}(\sigma, t) = u(\sigma, t) + e_z \sigma \), where \( u \) is orthogonal to the contour direction \( e_z \): \( e_z \cdot u(\sigma, t) = 0 \). Introducing the characteristic velocity

\[ c_0^2 = c_s^2 \times \frac{\ln(\lambda L/\xi)}{\ln(\lambda_\alpha/\xi)}, \]  

(6.12)

and restricting oneself in Eq. (6.4) by the linear approximation, one obtains the equation

\[ \frac{\dot{u}}{c_0^2} - u'' + \frac{1}{\gamma} [u \times e_z] = 0 \]  

(6.13)

for the two-dimensional vector \( u = (u_x, u_y) \). The plain wave solution \( u_{x,y} \propto e^{i(k \sigma - \omega t)} \) corresponds to the oscillation spectrum

\[ \omega = \frac{c_0^2}{2 \gamma} \left( \pm 1 \pm \sqrt{1 + \frac{4 \gamma^2 k^2}{c_0^2}} \right), \]

and the relations among the amplitudes \( u_y = \pm i u_x \). In our case, \( \gamma^2 k^2/c_0^2 = 2k^2 \xi^2 \ln(\lambda L/\xi) \ln(\lambda_\alpha/\xi) \ll 1 \), because the typical wave numbers \( k \ll 1/\xi \), so that the square root can be expanded. Then the general solution of Eq. (6.13), assuming as usual the periodic boundary conditions, can be represented in the form of the superposition of the left and right circularly polarized waves:

\[ u(\sigma, t) = \frac{e_x + i e_y}{\sqrt{2}} \sum_{l=-\infty}^{\infty} \left( \alpha_l e^{-i \omega_1 t} + \beta_l e^{-i \omega_2 t} \right) e^{i k_l \sigma} + \text{c.c.} \]  

(6.14)

Here, \( k_l = 2\pi l/L, L \) stands for the periodicity length, \( l = 0, 1, 2... \),

\[ \omega_1 = \gamma k_l^2 = \frac{\hbar k_l^2}{2m} \ln \frac{\lambda L}{\xi}, \]

\[ \omega_2 = \frac{c_0^2}{\gamma} + \gamma k_l^2 = \frac{2n_0 g}{\hbar \ln(\lambda_\alpha/\xi)} + \frac{\hbar k_l^2}{2m} \ln \frac{\lambda L}{\xi}, \]  

(6.15)
and $\alpha_l, \beta_l$ being the arbitrary complex constants. The first branch $\omega_1$ coincides with the Friedel-De Gennes-Matricon waves \cite{17,18}, if one allows for the fact that here the scalar field mimics the condensate of the Cooper pairs with the mass $m = 2m_c$ and charge $q = 2e$. It originates from the fact that the vortex segment being accelerated by the tension force $\propto X''$ acquires velocity $\Delta v \propto n$ in the normal vector direction. The Magnus-like force $\propto [e_z \times \dot{u}]$ results in the circular motion of the segment in the plane perpendicular to $e_z$ and $n$, that is in the direction of bi-normal vector $\theta = [X' \times n]$. The high frequency branch $\omega_2$ possesses the gap

$$\omega_{\text{min}} = \frac{2n_0 g}{\hbar \ln(\lambda_n/\xi)}.$$  

It can be interpreted as follows. Since $\omega_{\text{min}} = \omega_2(k_l = 0)$, this mode corresponds to the neglect of the term $\propto X''$ in the vortex equations of motion Eq. (6.13). Using Eq. (5.12) for the effective vortex mass together with Eq. (4.8), (6.5), and (6.12) one can rewrite the equations of motion in this limiting case as

$$\frac{m_{\text{eff}}}{L} \ddot{\theta} = 2\pi \hbar n_0 [e_z \times \dot{u}],$$  

(6.16)

which is the dissipation- and pinning-free part of the vortex equation of motion including the transverse force \cite{3].

The characteristic velocity $c_0$ in the case of gauge vortex exceeds the sound velocity $c_s$ [see Eq. (6.12)], hence both branches of small transverse oscillations are expected to be damped due to the emission of the Bogolyubov-Anderson excitations (sound waves in the large wave length limit). The calculation of the damping rate goes beyond the scope of the present paper.

VII. THE CASE OF THE GLOBAL VORTEX.

The global vortex can be treated in the same manner. The corresponding field theoretic model can be obtained from AHM one upon switching off the gauge field degrees of freedom and is called the Goldstone model \cite{19} in particle physics and the Gross-Pitaevskii one \cite{1,2} in condensed matter physics. Omitting the chain of derivations completely analogous to those presented in proceeding sections let us write the effective action $S_{\text{eff}} = S_{\text{bg}} + \Delta S_1$ of the global model as the sum of the action of the background field $S_{\text{bg}}$ and that of the integrated out fluctuations of the modulus and the phase of the scalar field $\Delta S_1$. Here $S_{\text{bg}}$ looks like

$$S_{\text{bg}} = \hbar n_0 \int d^4x \left[ a_0 - \partial_t \chi_b - \frac{\hbar}{2m} (a + \nabla \chi_b)^2 \right],$$  

(7.1)

and $\chi_b$ stands for the smooth background phase representing the possible potential flow whose velocity is $v_b = \frac{\hbar}{2m} \nabla \chi_b$, while $a_0 = -\partial_t \chi_s$ and $a = \nabla \chi_s$ represent the time and space derivatives of the singular phase $\chi_s$ responsible for the vortex. Their space Fourier components are given by the equations analogous to Eq. (2.8) and (2.9), respectively:

$$a_0(k,t) = -\frac{2\pi i}{k^2} \sum_n n_a \int d\sigma |k| [X_a \times X'_a] e^{-i k X_a},$$

$$a_k(t) = \frac{2\pi i}{k^2} \sum_n n_a \int d\sigma |k| X'_a e^{-i k X_a}.$$  

(7.2)

From now on let us restrict ourselves by the single vortex with the unit quantum of circulation $n_a = 1$. Then the coordinate space expressions found from Eq. (7.2) by the inverse Fourier transform are

$$a_0(t, x) = \frac{1}{2} \int d\sigma \frac{(x - X) \cdot [X \times X']}{|x - X|^3},$$

$$a(t, x) = \frac{1}{2} \int d\sigma |X' \times (x - X)|}{|x - X|^3}.$$  

(7.3)

$X = X(t, \sigma)$. The term originating from the integrated out fluctuations can be obtained from Eq. (5.1) by taking the limit of vanishing gauge coupling constant $g$ and restoring the term $\propto a^2/2m$ omitted when discussing the gauge vortex:

$$\Delta S_1 = \frac{\hbar^2 n_0}{2m} \int \frac{d^3k}{(2\pi)^3} \frac{|p_k|^2}{\hbar^2 k^2/4m^2 + c_s^2},$$  

(7.4)
where, for the sake of brevity, we introduce the quantity \( \rho_k = \int d^3 x e^{-i k \cdot \tau} \), which is the Fourier amplitude of the quantity

\[
\rho = a_0 - \partial_t \chi_b - \frac{\hbar}{2m} (a + \nabla \chi_b)^2. \tag{7.5}
\]

Notice that this \( \rho \) (taken at \( a_0 = 0, a = 0 \), that is, in the case of no vortex present) is just that combination of the derivatives of the Goldstone field \( \chi_b \) which was introduced in Ref. [21, 22] on the basis of the Galilean invariance of the effective action. Since the typical momenta are subjected to the condition \(|k| \ll 1/\xi\), where the healing length \( \xi \) is given by Eq. (1.12), the correction Eq. (7.4) to the background action can be represented as the series in the number of derivatives

\[
\Delta S_t = \frac{\hbar^2 n_0}{2mc_s^2} \int dtd^3 x \left[ \rho^2 - \xi^2 \sum_{l=0}^{\infty} (\xi^2 \nabla^2)^l (\nabla \rho)^2 \right]. \tag{7.6}
\]

The leading order effective action in the case of no vortex present is, omitting the total derivative term,

\[
S_{\text{eff}} = \frac{\hbar^2 n_0}{2m} \int d^4 x \left[ \frac{1}{c_s} (\partial_t \chi_b)^2 - (\nabla \chi_b)^2 + \ldots \right], \tag{7.7}
\]

which is just the action of the Goldstone mode in neutral superfluid. This agrees with the general analysis [21], where the coefficient in the effective action was fixed on the basis of the equations of motion of the Goldstone field, about which it is known that this field is the sound wave.

Turning back to deriving the global vortex equations of motion let us set the smooth phase \( \chi_b \) to zero and neglect in Eq. (7.8) all terms \( \propto \xi^2 \). When finding the variation of the background effective action Eq. (7.11) over \( X \) the following expressions derived from Eq. (7.2) are necessary:

\[
\begin{align*}
\delta a_0(x) &= -\int \frac{d^3 k}{(2\pi)^2} e^{i k \cdot x} \int d\sigma e^{-i k \cdot x} \left\{ \delta X \cdot [\dot{X} \times X'] + \frac{\delta X \cdot [k \times X']}{k^2} \right\} i \partial_{\tau} \\
\delta a(x) &= \int \frac{d^3 k}{(2\pi)^2} e^{i k \cdot x} \int d\sigma e^{-i k \cdot x} \left\{ \delta X \times X' - \frac{k (k \cdot [\delta X \times X'])}{k^2} \right\}.
\end{align*} \tag{7.8}
\]

In fact, the time derivative term in \( \delta a_0 \) in Eq. (7.8) vanishes when applied to the background action. The zeroth order equations of motion resulting from the variation of the background action are

\[
[\dot{X} \times X'] - \frac{\hbar}{m} [a(X) \times X'] = 0. \tag{7.9}
\]

One can see from Eq. (7.9) that upon neglecting the integrated out fluctuations of the phase and the modulus of the scalar field the classic law of the vortex motion

\[
\dot{X} = \frac{\hbar}{m} a(X) = V \tag{7.10}
\]

is recovered, because \( \frac{\hbar}{m} a(X) \) is the velocity induced at the given vortex location \( x = X(\sigma) \) by other elements of the vortex. Inserting here Eq. (7.3), using Eq. (3.12), and taking the low and upper integration limits to be, respectively, \( \xi \) and \( R \) (see below), one obtains in the zeroth order

\[
\dot{X}^{(0)}(\sigma_1) = \frac{\hbar}{m} a(X) = \frac{\hbar}{2m} \int d\sigma_2 \frac{[X_{21} \times X'_2]}{|X_{21}|^3} \approx \frac{\hbar}{2m} |X'_1 \times X'_0| \ln \frac{R}{\xi} \equiv \gamma_2 \kappa b, \tag{7.11}
\]

where \( b \) is the bi-normal vector, and

\[
\gamma_2 = \frac{\hbar}{2m} \ln \frac{R}{\xi}. \tag{7.12}
\]

Recall that \( X_{1,2} \equiv X(\sigma_{1,2}) \), and \( \kappa = |X''| \). The result (7.11) coincides with the velocity of the vortex ring in superfluid HeII [112]. The large-scale motions of the contour in the case of the global vortex is slow, because

\[
\left| \frac{\dot{X}^{(0)}}{c_s} \right| = \frac{\hbar \kappa}{2mc_s} \ln \frac{R}{\xi} \sim \frac{\xi}{R} \ln \frac{R}{\xi} \ll 1.
\]
Here the curvature of the contour $\kappa$ is approximated by the inverse cutoff $1/R$.

As far as $\Delta S_I$ are concerned, we evaluate its contribution under the same assumption $|k| \ll 1/\xi$ as made when discussing the local vortex case. However, in addition to the short distance divergence at $r \leq \xi$ the energy per unit length of global vortex has the pertinent divergence at large distances due to the real gapless Goldstone field. As usual, one regularizes it by means of the integration cutoff at the distances $r \sim R$, where $R$ is of the order of the curvature radius of the contour. We make here the large distance regularization by means of the large distance cutoff $k^{-2} \to (k^2 + 1/R^2)^{-1}$, $R \to \infty$. Performing the integration over momentum and making the same approximation as in the case of the local vortex one finds

$$\Delta S_I \approx \frac{\pi \hbar^2 n_0}{2mc^2} \int dt d\sigma \left\{ [\dot{X} \times X']^2 \ln \frac{R}{\xi} + \frac{3R^2}{4} (\ddot{X} \cdot [X' \times X''])^2 \right\}. \quad (7.13)$$

Contrary to the case of magnetic flux vortex, here the higher derivative term is multiplied by the large factor $R^2$. The ratio of the second higher derivative term in the braces to the first lowest order derivative term is estimated with the help of Eq. (7.11) to be

$$\frac{R^2\kappa^2}{\ln R/\xi} \sim \frac{1}{\ln R/\xi} \ll 1,$$

where we take the cutoff $R \sim 1/\kappa$. So, the higher derivative here in the case of the global vortex is also suppressed, although without additional small factor $\lambda_0^2 \kappa^2$ pertinent to the gauge vortex case. As in Sec. [X] when obtaining the correction to the equations of motions, the higher derivative term is neglected. Analogously to the gauge vortex case, here the lowest derivative term in the action can be interpreted as the contribution of the kinetic energy of the vortex segment with the effective mass per length

$$\frac{m_{\text{eff}}}{L} = \frac{\pi \hbar^2 n_0}{2mc^2} \ln \frac{R}{\xi} = \frac{\pi \hbar^2}{g} \ln \frac{R}{\xi}. \quad (7.14)$$

This coincides with the expressions obtained in Ref. [6, 9], allowing for apparent typo (extra mass in the denominator) in Ref. [6].

The correction to the zeroth approximation should be derived upon varying the nonlocal form of the action Eq. (7.13). Taking into account the zeroth order equations of motion, one finds

$$[\dot{X} \times X'] = \frac{\hbar}{m} [a(X) \times X'] + \frac{\hbar}{2mc^2} \ln \frac{R}{\xi} \left[\partial_t [\dot{X} \times X'] \times X'\right]. \quad (7.15)$$

Recalling that $a(X) \approx \frac{\hbar}{m} \ln \frac{R}{\xi} b$ [see Eq. (6.11)] Eq. (7.15) looks similar to Eq. (6.4). Hence the correction due to excitations of the modulus and the phase of the scalar field can obtained from Eq. (6.11) by means of the replacements $\gamma \to \gamma_g$, $\lambda_a, \lambda_L \to R$:

$$\dot{X}^{(1)} = \frac{\hbar \gamma_g^2}{2mc^2} \ln \frac{R}{\xi} \left[b(\kappa'' - \kappa \tau^2) - n(2\kappa' \tau + \kappa \tau')\right] = \frac{[\dot{X}^{(0)}] \xi^2 \ln^2 \frac{R}{\xi} \left[b(\kappa'' - \kappa \tau^2) - n(2\kappa' \tau + \kappa \tau')\right] 1}{\kappa}. \quad (7.16)$$

The ratio of the correction to the zeroth order approximation is small,

$$\frac{|X^{(1)}|}{|X^{(0)}|} = \xi^2 \kappa^2 \ln^2 \frac{R}{\xi} < 1,$$

but non-negligible, because of the large logarithm squared. As in the case of the magnetic vortex, the large scale nonlinear motion is slow.

The linear small amplitude motion of the global vortex is analyzed by taking the limit of locally straight contour, see preceding section for more detail. The corresponding equations of motion looks similar to Eq. (6.13), in which one should make the replacements $c_0 \to c_a$, because the ratio of logarithms in Eq. (6.12) drops in the global vortex limit, and $\gamma \to \gamma_g$. Hence, the small transverse motions of the global vortex are characterized by the velocity of sound which means that the kinematics is exactly on the threshold of the emission of sound waves, and the phase space of the emitted waves shrinks to zero. By this reason the small amplitude motions are not expected to be damped. The
oscillation spectrum is given by Eq. (6.15) with the proper replacements just mentioned, together with \(\lambda_L, \lambda_s \to R\). The lower branch of the spectrum is well-known as the Kelvin waves \([1, 15, 22]\). The branch with the gap

\[
\omega_{\text{min}} = \frac{2n_0 g}{\hbar \ln(R/\xi)}
\]

is interpreted similarly to that of the local case as being due to the acceleration of the vortex segment with the mass per length Eq. (7.14) subjected to the Magnus force. The equations of motion look the same as Eq. (6.10), with the effective mass Eq. (5.12) replaced by one in Eq. (7.14).

VIII. DISCUSSION.

The usual attitude to the effective lagrangians is that they do not refer to underlying basic theory and its fundamental constituents but are formulated in terms of effective degrees of freedom. In the case of condensed matter physics they are the modulus and phase of the order parameter together with the screened electromagnetic field in case of charged superfluid, and the same except the electromagnetic fields in case of neutral superfluids. Guided by analogous approach based on the field theoretic models possessing the property of spontaneous symmetry breaking we obtain here the effective actions and the equations of motion to the lowest number of derivatives over contour parameter and time for the local (gauge) and global vortices. Of course, the models used in the present paper are idealized in that they ignore the friction and pinning effects, but the inertial properties of the vortex state are most conveniently treated in the framework of such models. The key feature of the present consideration is the usage of the explicitly transverse electric gauge field strength in the study of the gauge vortex dynamics. If the field excitations were not included, the resulting effective action of the background field configuration would contain divergence at large distance. The crucial role of the excitations is that they cancel this divergence so that the gauge vortex dynamics becomes finite at large distances as it should, because all gauge fields in spontaneously broken gauge model are screened at the distances larger than the penetration depth. As a consequence, the higher derivative terms in the effective action for curved gauge vortex are small. Taking into account the scalar field excitations in the global models like the Gross-Pitaevskii or the Goldstone one results in the correct Galilean-invariant form of effective action for the Goldstone mode suggested in Ref. \([21]\) on the symmetry grounds.

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