Twisted split category algebras as quasi-hereditary algebras

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Abstract

We show that if $C$ is a finite split category, $k$ is a field of characteristic 0 and $\alpha$ is a 2-cocycle of $C$ with values in $k^\times$ then the twisted category algebra $k\alpha C$ is quasi-hereditary.

1 Introduction

Throughout this paper we assume that $C$ is a finite category, that is, the objects of $C$ form a finite set, and for every $X, Y \in \text{Ob}(C)$, the morphism set $\text{Hom}_C(X, Y)$ is finite. The category $C$ is called split if, for each morphism $s \in \text{Hom}_C(X, Y)$, there is a (not necessarily unique) morphism $t \in \text{Hom}_C(Y, X)$ such that $s \circ t \circ s = s$. Note that $u := t \circ s \circ t$ then also satisfies $s \circ u \circ s = s$, and also $u \circ s \circ u = u$. In the special case where $C$ has only one object this leads to the notion of a regular monoid, see [10].

Let $k$ be a field, and let $\alpha$ be a 2-cocycle of $C$ with values in $k^\times$. That is, for every pair $s, t \in \text{Mor}(C)$ such that $t \circ s$ exists, one has an element $\alpha(t, s) \in k^\times$ such that the following holds: for any $s, t, u \in \text{Mor}(C)$ such that $t \circ s$ and $u \circ t$ exist, one has $\alpha(u \circ t, s)\alpha(u, t) = \alpha(u, t \circ s)\alpha(t, s)$. We will study the twisted category algebra $k\alpha C$, that is, the $k$-vector space with basis $\text{Mor}(C)$ and multiplication

$t \cdot s := \begin{cases} \alpha(t, s) \cdot t \circ s & \text{if } t \circ s \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$

The aim of this paper, see Theorem 3.5, is to show that if $C$ is a finite split category and if $k$ has characteristic 0 then $k\alpha C$ is a quasi-hereditary algebra. This generalizes a result of Putcha, see [16], who proved that regular monoid algebras are quasi-hereditary over $k = C$. In Theorem 4.2 we identify the standard modules, generalizing Putcha’s results in [16].

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Our main motivation for studying the quasi-hereditary structure of twisted category algebras comes from the theory of double Burnside rings and biset functors: by a result of Webb, see [19], the category of biset functors over a field of characteristic 0 is a highest weight category. In [1, Example 5.15(b)] we introduced an algebra $A$ with the property that the category of biset functors (on a finite set of groups) over a field of characteristic 0 is equivalent to the category of $eAe$-modules, where $e$ is an idempotent of $A$. Thus, by Webb’s result, $eAe$ is a quasi-hereditary algebra. It is natural to ask whether also $A$ is quasi-hereditary. In [1] it was also shown that $A$ is a twisted category algebra for a finite split category. Thus Theorem 3.5 of the present paper, in particular, implies that the algebra $A$ in [1] is indeed quasi-hereditary.

We further remark that Theorems 3.5 and 4.2 should be of independent interest, since, by work of Wilcox [21], they also cover various prominent classes of cellular algebras (for suitable parameters) such as Brauer algebras, cyclotomic Brauer algebras, Temperley–Lieb algebras, and partition algebras, so that the main result of this paper gives a unified proof for the known fact that these algebras are quasi-hereditary. Proofs of the quasi-heredity of the aforementioned diagram algebras can, for instance, be found in [9, 15, 17, 18, 20, 22], and also in work of König–Xi [12], who established necessary and sufficient criteria for a cellular algebra to be quasi-hereditary. For a more detailed discussion of the history of proofs that Brauer algebras, Temperley–Lieb algebras, and partition algebras are quasi-hereditary over coefficient fields of characteristic 0, we refer to [14].

We recently learnt that Linckelmann and Stolorz, see [14, Theorem 1.1], independently proved that, under certain conditions on the category, finite twisted category algebras are quasi-hereditary in characteristic 0. These conditions on the category are even weaker than being split, and therefore the results in [14] imply Theorem 3.5. However, the two approaches are slightly different; for instance, we explicitly determine the radical of the twisted category algebra as part of our proof. In addition, we construct the standard modules of $k\alpha C$. As has been pointed out by the referee, in the case where $k\alpha C$ is isomorphic to a Brauer algebra and $\text{char}(k) = 0$, standard modules have also been investigated by Cox–De Visscher–Martin in [4]; standard modules for cyclotomic Brauer algebras over fields of characteristic 0 have recently been studied by Bowman–Cox–De Visscher in [2].

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2 Notation and quoted results

Throughout this section we assume that $C$ is a finite split category. We begin by collecting some known facts concerning split categories that will be used repeatedly in this paper. For details and proofs of the results quoted here we refer the reader to [13] and [8].

In what follows, given subsets $S$ and $T$ of $\text{Mor}(C)$, we set $S \circ T := \{s \circ t \mid s \in S, t \in T \text{ such that } s \circ t \text{ exists}\}$. In the case where $S = \{s\}$ or $T = \{t\}$, we abbreviate $S \circ T$ by $s \circ T$ or $S \circ t$, respectively. Note that $S \circ T$ may be empty, even if neither $S$ nor $T$ is empty.

One calls $S$ a left ideal (respectively, right ideal) of $C$ if $\text{Mor}(C) \circ S \subseteq S$ (respectively, $S \circ \text{Mor}(C) \subseteq S$). Note that this is equivalent to $\text{Mor}(C) \circ S = S$ (respectively, $S \circ \text{Mor}(C) = S$),
since every object has an identity morphism. Analogously, one calls $S$ a (two-sided) ideal of $C$ if $\text{Mor}(C) \circ S \circ \text{Mor}(C) \subseteq S$, or equivalently, if $\text{Mor}(C) \circ S \circ \text{Mor}(C) = S$.

2.1 Idempotents and $\mathcal{J}$-classes. (a) For morphisms $s, t \in \text{Mor}(C)$ one defines

$$
 s \mathcal{J} t :\iff \text{Mor}(C) \circ s \circ \text{Mor}(C) = \text{Mor}(C) \circ t \circ \text{Mor}(C).
$$

This yields an equivalence relation $\mathcal{J}$ on the set $\text{Mor}(C)$, and the corresponding equivalence classes are called the $\mathcal{J}$-classes of $C$. We will denote the $\mathcal{J}$-class of a morphism $s \in \text{Mor}(C)$ by $\mathcal{J}(s)$.

(b) Let $I$ and $J$ be $\mathcal{J}$-classes of $C$. One sets

$$
 J \leq \mathcal{J} I :\iff \text{Mor}(C) \circ J \circ \text{Mor}(C) \subseteq \text{Mor}(C) \circ I \circ \text{Mor}(C).
$$

Note that this is also equivalent to $\text{Mor}(C) \circ s \circ \text{Mor}(C) \subseteq \text{Mor}(C) \circ u \circ \text{Mor}(C)$, where $s$ and $u$ are any representatives of $J$ and $I$, respectively. Note further that this defines a poset structure on the set of $\mathcal{J}$-classes of $C$.

(c) An idempotent of $C$ is an endomorphism $e \in \text{End}_C(X)$ such that $e \circ e = e$, where $X$ is an object of $C$. In this case we call $e$ an idempotent on $X$.

We say that idempotents $e$ on $X$ and $f$ on $Y$ are equivalent if there exist some $s \in e \circ \text{Hom}_C(Y, X) \circ f$ and some $t \in f \circ \text{Hom}_C(X, Y) \circ e$ such that $e = s \circ t$ and $f = t \circ s$. In this case we write $e \sim f$. It is straightforward to show that this defines an equivalence relation on the set of idempotents of $C$, we will denote the equivalence class of an idempotent $e$ by $[e]$.

(d) The next lemma shows that every $\mathcal{J}$-class of $C$ contains an idempotent. Furthermore, idempotents $e$ and $f$ of $C$ are equivalent if and only if $\mathcal{J}(e) = \mathcal{J}(f)$; a proof of this can be found in [13, Lemma 2.1]. Thus there is a bijection between the equivalence classes of idempotents of $C$ and the $\mathcal{J}$-classes of $C$.

2.2 Lemma Let $s \in \text{Mor}(C)$, and let $t, u \in \text{Mor}(C)$ be such that $s \circ t \circ s = s = s \circ u \circ s$. Then $s \circ t$ and $u \circ s$ are idempotents in $C$. Moreover,

$$
 \mathcal{J}(s \circ t) = \mathcal{J}(s) = \mathcal{J}(u \circ s);
$$

in particular, $s \circ t \sim u \circ s$.

Proof Clearly, $s \circ t$ and $u \circ s$ are idempotents in $C$ contained in the $\mathcal{J}$-class $\mathcal{J}(s)$. Thus, as already mentioned, [13, Lemma 2.1] implies $s \circ t \sim u \circ s$.

Suppose now that $k$ is a field, and let $\alpha$ be a 2-cocycle of $C$ with values in $k^\times$. The aim of the next section is to prove that, under suitable additional assumptions on $k$, the $k$-algebra $k_\alpha C$ is quasi-hereditary. To this end, we summarize and establish here some important facts concerning the algebra $k_\alpha C$ and, in particular, its simple modules and its Jacobson radical. For ease of notation, we will henceforth denote the twisted category algebra $k_\alpha C$ by $A$.

2.3 Idempotents of $C$ and simple $A$-modules. The isomorphism classes of simple $A$-modules have been parametrized by Linckelmann–Stolorz [13], generalizing previous work of Ganyushkin–Mazorchuk–Steinberg [8] concerning semigroup algebras.
(a) Given an idempotent $e$ on $X \in \text{Ob}(C)$, the group of invertible elements of the monoid $e \circ \text{End}_C(X) \circ e$ is denoted by $\Gamma_e$, and is called a maximal subgroup of $C$. Moreover, we set $J_e := e \circ \text{End}_C(X) \circ e \cdot \Gamma_e$. Restricting the 2-cocycle $\alpha$ to $\Gamma_e$, one can view the twisted group algebra $k \Gamma_e$ as (non-unitary) subalgebra of $A$.

Note also that the element

$$e' := \alpha(e, e)^{-1} e$$

is an idempotent in the algebra $A$, and that $eAe = e'Ae' = k_{e}(e \circ \text{End}_C(X) \circ e)$. Furthermore, there is a $k$-vector space decomposition

$$e'Ae' = k_{e}\Gamma_e \oplus k J_e,$$

(1)

$k J_e$ is a two-sided ideal, and $k_{e}\Gamma_e$ is a unitary subalgebra of $e'Ae'$.

(b) Suppose that $e$ is an idempotent of $C$, and let again $e'$ denote the corresponding idempotent in $A$. Whenever $W$ is a $k_{e}\Gamma_e$-module, we obtain an $A$-module $Ae' \otimes_{e'Ae'} W$, where $W$ is the inflation of $W$ from $k_{e}\Gamma_e$ to $e'Ae'$ with respect to the decomposition (1). In the case where $W$ is a simple $k_{e}\Gamma_e$-module, the $A$-module $Ae' \otimes_{e'Ae'} W$ has a unique simple quotient module; see [11, Section 6.2].

(c) Let $e \in \text{End}_C(X)$ and $f \in \text{End}_C(Y)$ be equivalent idempotents of $C$ and let $s \in e \circ \text{Hom}_C(Y, X) \circ f$ and $t \in f \circ \text{Hom}_C(X, Y) \circ e$ be such that $e = s \circ t$ and $f = t \circ s$. Then the map $a \mapsto t \cdot a \cdot s$ defines a $k$-linear isomorphism between $e'Ae'$ and $f' Af'$, which takes $k_{e}\Gamma_e$ to $k J_f$. Similarly, one obtains an isomorphism of left $A$-modules between $Ae'$ and $Af'$. Altogether one obtains an isomorphism of left $A$-modules

$$Ae' \otimes_{e'Ae'} (e'Ae'/k J_e) \xrightarrow{\sim} Af' \otimes_{f' Af'} (f' Af'/k J_f).$$

(2)

2.4 Notation From now on, we denote by $e_1, \ldots, e_n$ representatives of the equivalence classes of idempotents of $C$, and for $i = 1, \ldots, n$ we fix representatives $T_{i1}, \ldots, T_{il_i}$ of the isomorphism classes of simple $k_{e_i}\Gamma_{e_i}$-modules. Moreover, for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$, we denote the inflation of the $k_{e_i}\Gamma_{e_i}$-module $T_{ij}$ to $e'_iAe'_i$ by $\tilde{T}_{ij}$, and the simple head of the $A$-module $Ae'_i \otimes_{e'_iAe'_i} \tilde{T}_{ij}$ by $D_{ij}$. With this, the following holds:

2.5 Theorem ([13], Theorem 1.2) The modules $D_{ij}$ ($i = 1, \ldots, n$, $j = 1, \ldots, l_i$) form a set of representatives of the isomorphism classes of simple $A$-modules.

Denoting the Jacobson radical of $A$ by $\text{J}(A)$, Theorem 2.5 now leads to the following description:

2.6 Proposition With the notation as in 2.4 one has

$$\text{J}(A) = \{u \in A \mid \forall i = 1, \ldots, n : e'_iAuAe'_i \subseteq k J_{e_i} + \text{J}(k_{e_i} \Gamma_{e_i})\}.$$

(3)

In particular, if in addition $|\Gamma_{e_i}| \in k^\times$, for all $i = 1, \ldots, n$, then

$$\text{J}(A) = \{u \in A \mid \forall i = 1, \ldots, n : e'_iAuAe'_i \subseteq k J_{e_i}\}.$$

(4)
Proof It suffices to prove that the set on the right-hand side of (3) is the common annihilator of the simple \( A \)-modules \( D_{ij} \) \((i = 1, \ldots, n, j = 1, \ldots, l_i)\). So let \( u \in A \), let \( i \in \{1, \ldots, n\} \), and let \( j \in \{1, \ldots, l_i\} \). By [1, Lemma 5.6], we know that \( uD_{ij} = 0 \) if and only if \( e_i'Ae_i' \subseteq \text{Ann}_{\mathbb{C}}(T_{ij}) = kJ_{e_i} + \text{Ann}_{k\alpha\Gamma_{e_i}}(T_{ij}) \). Therefore,

\[
 u \in J(A) \iff \forall i = 1, \ldots, n, j = 1, \ldots, l_i: uD_{ij} = 0
\]

\[
\iff \forall i = 1, \ldots, n: e_i'Ae_i' \subseteq \bigcap_{j=1}^{l_i} (kJ_{e_i} + \text{Ann}_{k\alpha\Gamma_{e_i}}(T_{ij}))
\]

\[
= kJ_{e_i} + \bigcap_{j=1}^{l_i} \text{Ann}_{k\alpha\Gamma_{e_i}}(T_{ij}) = kJ_{e_i} + J(k\alpha\Gamma_{e_i}),
\]

proving (3). If, moreover \(|\Gamma_{e_i}| \in k^\times\), for \( i = 1, \ldots, n \) then, by [5, Exercise 28.4], the twisted group algebras \( k\alpha\Gamma_{e_i} \) \((i = 1, \ldots, n)\) are semisimple, and we derive equation (4).

\[\square\]

3 A heredity chain for \( k\alpha\mathcal{C} \)

In this section we will prove the main result, Theorem 3.5. We start by recalling the definition of a quasi-heredity algebra.

3.1 Definition (Cline–Parshall–Scott [3], Section 3) Let \( k \) be any field. A finite-dimensional \( k \)-algebra \( A \) is called quasi-hereditary if there exists a chain

\[
\{0\} = J_0 \subset J_1 \subset \cdots \subset J_{n-1} \subset J_n = A
\]

of two-sided ideals in \( A \) such that, for every \( i = 1, \ldots, n \), when denoting by \( \tilde{\cdot}: A \rightarrow A/J_{i-1} = \tilde{A} \) the canonical epimorphism, the following conditions are satisfied:

(i) there is an idempotent \( \tilde{e}_i \in \tilde{A} \) with \( \tilde{J}_i = \tilde{A}\tilde{e}_i\tilde{A} \);
(ii) \( \tilde{J}_i \cdot \tilde{J}_i \cdot \tilde{J}_i = \{0\} \);
(iii) \( \tilde{J}_i \) is a projective right \( \tilde{A} \)-module.

In this case one calls the chain (5) a heredity chain for \( A \). Note that (iii) can be replaced by

(iii') \( \tilde{J}_i \) is a projective left \( \tilde{A} \)-module,

by [6, Statement 7].

For the remainder of this section assume again that \( \mathcal{C} \) is a finite split category. For ease of notation we denote from now on the morphism set \( \text{Mor}(\mathcal{C}) \) by \( S \). Recall from Section 2 that we also have the notions of left/right/two-sided ideals of the category \( \mathcal{C} \). So we will now use the term ‘ideal’ both in the context of categories and algebras.

We will next define a chain of two-sided ideals of \( \mathcal{C} \) that will then give rise to a heredity chain for the twisted category algebra in Theorem 3.5.

3.2 Definition Let \( e_1, \ldots, e_n \) be representatives of the equivalence classes of idempotents of \( \mathcal{C} \), ordered such that

\[
\mathcal{J}(e_i) \leq \mathcal{J}(e_j) \quad \text{implies} \quad i \leq j,
\]

\[
\mathcal{J}(e_i) \leq \mathcal{J}(e_j) \quad \text{implies} \quad i \leq j,
\]

\[
\mathcal{J}(e_i) \leq \mathcal{J}(e_j) \quad \text{implies} \quad i \leq j,
\]
in which case we also write \( i \leq j \). Moreover, for \( i = 1, \ldots, n \), we define

\[
S_i := \mathcal{J}(e_i), \quad S_{\leq j} := \bigcup_{j \leq i} S_j, \quad S_{\leq i} := \bigcup_{j \leq i} S_j.
\]

Then \( S_n = S \), and for convenience we also set \( S_0 := S_{\leq 0} := \emptyset \subseteq S \). Note that, by \cite[Lemma 2.6]{13}, one has

\[
\Gamma_e = (e \circ S \circ e) \cap S_i \quad \text{and} \quad J_e = (e \circ S \circ e) \cap S_{\leq i-1},
\]

for all \( i \in \{1, \ldots, n\} \) and any idempotent \( e \in S_i \).

### 3.3 Proposition

With the notation as in Definition \ref{def:12}, for \( i = 1, \ldots, n \), both \( S_{\leq j} \) and \( S_{\leq i} \) are ideals of \( C \).

**Proof** Let \( i \in \{1, \ldots, n\} \), let \( s \in S_i \), and let \( u, v \in S \) be such that \( s \circ u \) and \( v \circ s \) exist. Since \( S \circ s \circ u \circ S \subseteq S \circ s \circ S \) and \( S \circ v \circ s \circ O \subseteq S \circ v \circ S \), we immediately get \( \mathcal{J}(s \circ u) \leq \mathcal{J}(s) \) and \( \mathcal{J}(v \circ s) \leq \mathcal{J}(s) \). Thus \( S_{\leq j} \) is an ideal of \( C \), for \( i = 1, \ldots, n \), and since \( S_{\leq i} = \bigcup_{j \leq i} S_{\leq j} \), the latter is an ideal of \( C \) as well.

### 3.4 Proposition

(a) Let \( s, t \in S \) be such that \( s = s \circ t \circ s \). Then \( S \circ s = S \circ t \circ s \).

(b) Let \( i \in \{1, \ldots, n\} \), and let \( s, t \in S_i \) be such that \( S \circ s \subseteq S \circ t \). Then \( S \circ s = S \circ t \).

(c) Let \( i \in \{1, \ldots, n\} \), and let \( s, t \in S_i \). Then the sets \( (S \circ s)_i := (S \circ s) \cap S_i \) and \( (S \circ t)_i := (S \circ t) \cap S_i \) are either equal or disjoint.

(d) Let \( i \in \{1, \ldots, n\} \). There is a subset \( \epsilon \) of \([e_i]\) such that the sets \( (S \circ e)_i := (S \circ e) \cap S_i \) \( (e \in \epsilon \) form a partition of \( S_i \).

**Proof** (a) This follows from \( S \circ s = S \circ s \circ t \circ s \subseteq S \circ t \circ s \subseteq S \circ s \).

(b) Let \( q, r \in S \) be such that \( s \circ q \circ s = s \) and \( t \circ r \circ t = t \), and set \( e := q \circ s \) and \( f := r \circ t \). Since \( s \in S \circ t \), the idempotents \( e \) and \( f \) are endomorphisms of the same object of \( C \), say \( X \). By Part (a), we have \( S \circ e \subseteq S \circ f \), and it suffices to show that \( S \circ f \subseteq S \circ e \). Recall that \( \mathcal{J}(s) = S_i = \mathcal{J}(t) \). Thus, by Lemma \ref{lem:22}, we have \( e \sim f \), so that there exist \( u \in e \circ S \circ f \) and \( v \in f \circ S \circ e \) with \( e = u \circ v \) and \( f = v \circ u \). Since \( S \circ e \subseteq S \circ f \), we also have \( e = e \circ f \). Note that \( u \) and \( v \) are endomorphisms of \( X \). Since \( \text{End}_C(X) \) is finite, there exist positive integers \( a \) and \( b \) such that \( v^{a+b} = v^b \). Composition with \( u^b \) from the right yields \( v^a = f \), since we have \( v \circ u = f \) and \( v \circ f = v \circ e \circ f = v \circ e = v \). Finally, we obtain \( f \circ e = v^a \circ e = v^a = f \), which implies that \( f \in S \circ e \) and \( S \circ f \subseteq S \circ e \).

(c) Assume that \( (S \circ s)_i \cap (S \circ t)_i \) is non-empty and that \( u \in (S \circ s)_i \cap (S \circ t)_i \). Then we obtain \( S \circ u \subseteq S \circ s \) and \( S \circ u \subseteq S \circ t \). Now Part (b) yields \( S \circ s = S \circ u = S \circ t \) and \( (S \circ s)_i = (S \circ t)_i \).

(d) Clearly, \( S_i \) is the union of its subsets \( (S \circ s)_i \), \( s \in S_i \), and by Part (a) also of the subsets \( (S \circ e)_i \), \( e \in [e_i] \). The condition \( (S \circ e)_i = (S \circ f)_i \) defines an equivalence relation on the set \([e_i] \). If \( \epsilon \) is a set of representatives of the corresponding equivalence classes then, by Part (c), \( S_i \) is the disjoint union of the subsets \( (S \circ e)_i \), \( e \in \epsilon \).
3.5 Theorem Let $C$ be a finite split category and let $\alpha$ be a 2-cocycle of $C$ with values in the multiplicative group $k^\times$ of a field $k$. Assume further that, for each idempotent $e$ of $C$, the order of $\Gamma_e$ is invertible in $k$. With the notation as in Definition 3.2 let $J_i := kS_{\leq i}$, for $i = 0, \ldots, n$. Then
\[
\{0\} = J_0 \subset J_1 \subset \cdots \subset J_n = k\alpha C
\]
is a heredity chain for $k\alpha C$. In particular, the twisted category algebra $k\alpha C$ is quasi-hereditary.

Proof We set $A := k\alpha C$. Since $S_{\leq i}$ is an ideal of $S$, $J_i$ is an ideal of $A$ for all $i = 0, \ldots, n$. We show that the chain (7) satisfies conditions (i), (ii) and (ii') in Definition 3.1. To this end, let $i \in \{1, \ldots, n\}$, and again let $\bar{\gamma} : A \to A/J_i$ denote the canonical epimorphism.

By definition, we have $S \circ o S = S \circ e_i \circ S$, for every $s \in S_i$, thus $S_i \subseteq S \circ e_i \circ S$. From this we get $\bar{J}_i = \bar{A}e_i \bar{A}$, and we have verified condition (i).

Next we verify condition (ii). Note that, since $A$ is a finite-dimensional algebra over a field, we have $\bar{J}(A) = \bar{J}(A)$. Hence it suffices to show that $s u t \in J_{i-1}$, for all $s, t \in S_{\leq i}$ and all $u \in J(A)$. If $s \in S_{\leq i-1}$ or $t \in S_{\leq i-1}$ then this is clearly true. So we may suppose that $s, t \in S_i$. Let $uJ, r \in S$ be such that $s o s \circ s = s$ and $t o r = t$. Then $S \circ o S = S \circ e_i \circ S = S \circ s \circ S$ and $|e_i| = |s \circ q| = |q \circ s| = |t \circ r| = |r \circ t|$, by Lemma 2.2. So there exist elements $x \in q \circ S \circ e_i$, $y \in e_i \circ S \circ q \circ s$, $v \in t \circ S \circ e_i$, and $w \in e_i \circ S \circ t \circ r$ such that
\[
q \circ s = x \circ y = x \circ e_i \circ y, \quad t \circ r = v \circ w = v \circ e_i \circ w, \quad e_i = y \circ x = w \circ v.
\]

Since $u \in J(A)$, Proposition 2.6 implies $(e_i \circ y)u(v \circ e_i) \in e_i' A e_i' \subseteq kJ_{e_i}$. Furthermore, we have $e_i \circ S \circ e_i \subseteq S_{\leq i}$, since $e_i \in S_i \subseteq S_{\leq i}$ and since, by Proposition 3.3, $S_{\leq i}$ is an ideal in $S$. By (6) we have $kJ_{e_i} \subseteq J_{i-1}$. This implies $(e_i \circ y)u(v \circ e_i) \in J_{i-1}$ and we obtain
\[
s u t = (s \circ o s)u(t \circ r \circ t) = (s \circ o e_i \circ o y)u(v \circ e_i \circ o w \circ t) \in J_{i-1},
\]
as required.

It remains to verify condition (iii'). By Proposition 3.4(d), we know that $\bar{J}_i$ is the direct sum of left ideals of the form $\bar{A}e_i$, for suitable idempotents $e \in A$. Since each such summand is a projective left $\bar{A}$-module, so is $\bar{J}_i$, and the proof of (iii') is complete.

4 Standard modules for $k\alpha C$

As before, we denote the twisted category algebra $k\alpha C$ by $A$. As in Theorem 3.5 we assume throughout this section that $|\Gamma_e| \in k^\times$ for each idempotent $e$ of $C$.

4.1 The modules $\Delta_{i, r}$. Recall from 2.4 that the heads of the $A$-modules
\[
\Delta_{i, r} := Ae_i' \otimes e_i' Ae_i' \overset{\bar{\delta}_{i, r}}{\to} T_{i, r} \quad (i \in \{1, \ldots, n\}, r \in \{1, \ldots, l_i\})
\]
yield a set of representatives of the isomorphism classes of simple $A$-modules. Here, $T_{i, 1}, \ldots, T_{i, l_i}$ again denote representatives of the isomorphism classes of simple $k\alpha \Gamma_{e_i}$-modules.

From now on we set $\Lambda := \{(i, r) \mid 1 \leq i \leq n, 1 \leq r \leq l_i\}$, and we define a partial order $\leq$ on $\Lambda$ via
\[
(i, r) < (j, s) :\iff S_j < \not\exists S_i.
\]

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The aim of this section is to prove the following theorem.

4.2 Theorem The modules $\Delta_{ir}$ $(i, r) \in \Lambda$ are the standard modules of the quasi-hereditary algebra $A$ with respect to the partial order $\leq$ on $\Lambda$.

4.3 Remark (a) In order to prove Theorem 4.2 we will have to show that, for each $(i, r) \in \Lambda$, the $A$-module $\Delta_{ir}$ is the unique maximal quotient module $M$ of the projective cover $P_{ir}$ of $D_{ir}$ such that all composition factors of $\text{Rad}(M)$ belong to the set $\{D_{js} \mid (j, s) < (i, r)\}$ (see the definition of a standard module with respect to the partial order $\leq$ in [7, A1]).

Note that it thus suffices to show that, for each $(i, r) \in \Lambda$, the module $\Delta_{ir}$ satisfies conditions (i) and (ii) below, and the projective cover $P_{ir}$ of $D_{ir}$ admits a filtration

$$0 = P_{ir}^{(0)} \subset \cdots \subset P_{ir}^{(m_{ir})} = P_{ir}$$

(9)
satisfying (iii) and (iv) below:

(i) $\text{Hd}(\Delta_{ir}) \cong D_{ir}$;

(ii) if $(j, s) \in \Lambda$ is such that $D_{js}$ occurs as a composition factor of $\text{Rad}(\Delta_{ir})$ then $(j, s) < (i, r)$;

(iii) $P_{ir}^{(m_{ir})}/P_{ir}^{(m_{ir}-1)} \cong \Delta_{ir}$;

(iv) for $q \in \{1, \ldots, m_{ir} - 1\}$, one has $P_{ir}^{(q)}/P_{ir}^{(q-1)} \cong \Delta_{jq,sq}$, for some $(jq, sq) \in \Lambda$ with $(i, r) < (jq, sq)$.

(b) Note also that the partial order on $\Lambda$ defined in [7, Proposition A3.7(ii)] from the heredity chain (7) in Theorem 3.5 is finer than the partial order defined in (8). Thus, the modules $\Delta_{ir}$ are also the standard modules associated with the heredity chain in (7).

We start out with the following lemma, which will be essential for verifying conditions (i)–(iv) above. As in Theorem 3.5 given $i \in \{1, \ldots, n\}$, we denote by $J_i$ the ideal $kS_{\leq i}$ of $A$, and we also set $J_0 := \{0\}$.

4.4 Lemma Let $i \in \{1, \ldots, n\}$, and let $\epsilon_i \subseteq [\epsilon_i] \subseteq S_i$ be a set of idempotents such that the sets $(S \circ e)_i := (S \circ e) \cap S_i$ form a partition of $S_i$ (cf. Proposition 3.4). Then one has a left $A$-module isomorphism

$$J_i/J_{i-1} \cong \bigoplus_{r=1}^{l_i} \Delta_{ir}^{[\epsilon_i]|n_{ir}} \quad \text{with} \quad n_{ir} := \frac{\dim_k(T_{ir})}{\dim_k(\text{End}_{kAe}T_{ir})}.$$ 

(10)

Proof We first show that, for each $e \in \epsilon_i$, we have

$$S_{\leq i-1} \cap S \circ e = S \circ J_e.$$ 

(11)

By (9), we know that $J_e \subseteq S_{\leq i-1}$, so that $S \circ J_e$ is contained in $S_{\leq i-1} \cap S \circ e$. Conversely, if $x \in S_{\leq i-1} \cap S \circ e$ then $x = x \circ e$, and there is some $y \in S$ such that $x = x \circ y \circ e$. Thus $x = x \circ e \circ y \circ x \circ e$, and $e \circ y \circ x \circ e \in e \circ S \circ e \cap S_{\leq i-1} = J_e$, by (9). Thus $x \in S \circ J_e$, as claimed.

Equation (11) implies $J_{i-1} \cap Ae^e = AJ_e$ and we obtain the following left $A$-module isomorphisms

$$J_i/J_{i-1} = k(S_{\leq i-1} \cup \bigcup_{e \in \epsilon_i} (S \circ e)_i)/J_{i-1} = \bigoplus_{e \in \epsilon_i} (J_{i-1} + Ae^e)/J_{i-1}$$

$$\cong \bigoplus_{e \in \epsilon_i} Ae^e/(Ae^e \cap J_{i-1}) = \bigoplus_{e \in \epsilon_i} Ae^e/AJ_e \cong \bigoplus_{e \in \epsilon_i} Ae^e \otimes e'Ae^e \ (e'Ae^e/kJ_e),$$

(10)
where the last isomorphism is given by the canonical isomorphisms

$$Ae' \otimes_{e' A e'} (e' A e' / k J) = Ae' \otimes_{e' A e'} (e' A e' / e' A e' J) \cong Ae' / Ae' J = Ae' / AJ.$$

By [2], $Ae' \otimes_{e' A e'} (e' A e' / k J) \cong Ae'_i \otimes_{e'_i A e'_i} (e'_i A e'_i / k J_{e'_i})$, for every $e \in e_i$. Moreover, recall that the twisted group algebra $k_0 \Gamma_{e_i}$ is semisimple, since we are assuming $|\Gamma_{e_i}| \in k^*$. Hence, since $T_{i_1}, \ldots, T_{i_l}$ are representatives of the isomorphism classes of simple $k_0 \Gamma_{e_i}$-modules, $k_0 \Gamma_{e_i} \cong \bigoplus_{r=1}^{l_i} T_{i_{r_1}}^n$ as left $k_0 \Gamma_{e_i}$-modules, where the multiplicity $n_{i_{r_1}}$ is as in (10). Then also

$$Ae'_i \otimes_{e'_i A e'_i} (e'_i A e'_i / k J_{e'_i}) \cong Ae'_i \otimes_{e'_i A e'_i} \bigoplus_{r=1}^{l_i} T_{i_{r_1}}^n \cong \bigoplus_{r=1}^{l_i} (Ae'_i \otimes_{e'_i A e'_i} T_{i_{r_1}})^n \cong \bigoplus_{r=1}^{l_i} \Delta_{i_{r_1}}$$

as left $A$-modules. Altogether this gives the desired left $A$-module isomorphism

$$J_{i_{r_1}} / J_{i_{r_1} - 1} \cong \bigoplus_{r=1}^{l_i} \Delta_{i_{r_1}}^{|e_i|}.$$

\[\Box\]

**Proof (of Theorem 4.2)** We follow the strategy outlined in Remark 4.3 and verify conditions (i)–(iv) listed there.

(i) This follows immediately from Green’s condensation theory and the definition of the modules $D_{i_{r_1}}$, see [2,3] and [2,4].

(ii) Suppose that $(i, r), (j, s) \in \Lambda$ are such that $D_{i_{r_1}}$ occurs as a composition factor of $\text{Rad}(\Delta_{i_{r_1}})$. We need to show that $(j, s) \neq (i, r)$, i.e., that $S_i \neq S_j$. By [13, Proposition 5.1], one has $S_j \cdot D_{i_{r_1}} \neq \{0\}$, and if $l \in \{1, \ldots, n\}$ is such that $S_l \cdot D_{i_{r_1}} \neq \{0\}$ then $S_j \subseteq S_l$. Assume that $S_j \subseteq S_l$. Then $S_j \cap S_{i_1} \subseteq S_{i_1 - 1}$, which, by Lemma 4.4, implies $S_j \cdot \Delta_{i_1} = \{0\}$. Thus, $S_j$ also annihilates every composition factor of $\Delta_{i_1}$ and we have $S_j \cdot D_{i_{r_1}} = \{0\}$, a contradiction.

(iii) and (iv): By Lemma 4.3, the left $A$-module $A$ has a filtration all of whose factors are isomorphic to modules of the form $\Delta_{i_{r_1}} ((i, r) \in \Lambda)$. Let $(i, r) \in \Lambda$, and let $f_{i_{r_1}} \in e'_i A e'_i$ be a primitive idempotent such that $e'_i A e'_i f_{i_{r_1}} = e'_i A f_{i_{r_1}}$ is a projective cover of the simple $e'_i A e'_i$-module $T_{i_{r_1}}$.

We claim that $P_{i_{r_1}} := A f_{i_{r_1}}$ is a projective cover of the $A$-module $D_{i_{r_1}}$. Since $f_{i_{r_1}}$ is primitive in $e'_i A e'_i$, it is primitive in $A$ as well, so that $P_{i_{r_1}}$ is an indecomposable projective $A$-module. We have an $e'_i A e'_i$-epimorphism $e'_i A f_{i_{r_1}} \rightarrow T_{i_{r_1}}$, and thus also an $A$-epimorphism

$$A f_{i_{r_1}} \cong A e'_i \otimes_{e'_i A e'_i} (e'_i A e'_i f_{i_{r_1}}) \rightarrow A e'_i \otimes_{e'_i A e'_i} T_{i_{r_1}} = \Delta_{i_{r_1}} \rightarrow D_{i_{r_1}},$$

since tensoring with $A e'_i$ is right exact. Hence $P_{i_{r_1}}$ must be a projective cover of $D_{i_{r_1}}$.

Since $A f_{i_{r_1}} = A e'_i f_{i_{r_1}} \subseteq k S_{i_{r_1}} f_{i_{r_1}} = J_{i_{r_1}} f_{i_{r_1}} \subseteq A f_{i_{r_1}}$, the filtration of the left $A$-module $A$ in (7) yields a filtration

$$\{0\} = J_0 \cdot f_{i_{r_1}} \subseteq J_1 \cdot f_{i_{r_1}} \subseteq \cdots \subseteq J_{i_{r_1} - 1} \cdot f_{i_{r_1}} \subseteq J_i \cdot f_{i_{r_1}} = A f_{i_{r_1}}. \quad (12)$$

If $j \in \{1, \ldots, i - 1\}$ is such that $S_j \not\subseteq S_{i_{r_1}}$ then we have

$$J_j f_{i_{r_1}} \subseteq J_j \cdot e'_i A e'_i \subseteq k S_{i_{r_1}} \cdot k S_{i_{r_1}} = k(S_{i_{r_1}} \cdot S_{i_{r_1}}) \subseteq k(S_{i_{r_1}} \cap S_{i_{r_1}}) \subseteq J_{i_{r_1} - 1},$$

where
since $S_j \cap S_{\leq j} = \emptyset$. So in this case we get $J_j f_{ir} = J_{j-1} f_{ir}$.

If $j \in \{1, \ldots, i-1\}$ is such that $S_j < S_i$, then, by Lemma 4.4, we deduce that, for $j = 1, \ldots, i-1$, the indecomposable direct summands of the quotient $J_j f_{ir} / J_{j-1} f_{ir}$ are again isomorphic to $\Delta_{js}$, for various $s \in \{1, \ldots, l_j\}$ such that $(i, r) < (j, s)$.

It thus suffices to show that $A f_{ir} / J_{i-1} f_{ir} \cong \Delta_{ir}$. For then the chain (12) gives rise to a filtration of $P_{ir}$ as desired.

Since $e_i' A e_i' f_{ir}$ is a projective cover of $T_{ir}$, we have $f_{ir} \cdot T_{ir} \neq \{0\}$. Since the ideal $k e_i \in e_i' A e_i'$ annihilates $T_{ir}$, and since $k e_i \in e_i' A e_i' \cap J_{i-1}$, this implies $f_{ir} \notin J_{i-1}$, hence $f_{ir} \in e_i' A e_i' \setminus J_{i-1} \subseteq J_i \setminus J_{i-1}$. Therefore, $A f_{ir} / J_{i-1} f_{ir} \neq \{0\}$. Since $A f_{ir}$ has a simple head isomorphic to $D_{ir}$, the same holds for $A f_{ir} / J_{i-1} f_{ir}$. Finally, since also $\Delta_{ir}$ has a simple head isomorphic to $D_{ir}$, Lemma 4.4 forces $A f_{ir} / J_{i-1} f_{ir} \cong \Delta_{ir}$, and the proof of the theorem is complete.

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