Semi-Isolation and the Strict Order Property

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Abstract We study semi-isolation as a binary relation on the locus of a complete type and prove that—under some additional assumptions—it induces the strict order property.

0 Introduction

Throughout the paper $T$ is a fixed, complete, first-order theory in a countable language and $M$ is its (infinite) monster model. $T$ is an Ehrenfeucht theory if it has finitely many, but more than one, countable models. The class of Ehrenfeucht theories is quite interesting. There are numerous results and a large bibliography in this area (see Baizhanov, Sudoplatov, and Verbovskiy [1] and Sudoplatov [8] for references). The first example was found by Ehrenfeucht in Vaught [11, Section 6]: $T_E = \text{Th}(\mathbb{Q}, <, n)_{n \in \omega}$. It eliminates quantifiers and has three countable models: the prime model, the saturated model, and the model prime over a realization of a nonisolated type. $T_E$ is also a binary theory: every formula is equivalent modulo $T_E$ to a Boolean combination of formulas with at most two free variables. Not all Ehrenfeucht theories are binary: nonbinary examples can be found in Peretyat’kin [4] and Woodrow [13]. The motivating question for our work is the following.

Question 1 Is there a binary, Ehrenfeucht theory without the strict order property (SOP)? In particular, is there such a theory with three countable models?

An important relation in any Ehrenfeucht theory is semi-isolation as a binary relation on the locus of a powerful type $p \in S(\emptyset)$ in a model of $T$ (all these notions are defined in Section 1). There the semi-isolation relation is either empty (if $p$ is omitted) or a $\sqrt{\cdot}$-definable quasiorder with no maximal elements. If in addition $T$ has precisely three countable models, then the isomorphism type of any countable model $N$ can be described by combinatorial properties of the quasiorder:
1. \( N \) is prime if and only if \( p(N) = \emptyset \);
2. \( N \) is prime over a realization of \( p \) if and only if there is a minimal (with respect to semi-isolation) element in \( p(N) \); in this case, \( N \) is prime over any minimal element;
3. \( N \) is saturated if and only if \( p(N) \) has no minimal elements.

We note that in Ehrenfeucht’s example the type \( \{ n < x \mid n \in \omega \} \) determines a complete 1-type \( p \) on whose locus, in any countable model, the semi-isolation (defined precisely later and denoted by \( \text{SI}_p \)) coincides with \( \leq \). In particular, semi-isolation is a relatively definable relation on the locus of \( p \). The strict order property in this example is induced by the semi-isolation, and it is natural to examine whether this will happen in any binary Ehrenfeucht theory.

One result in this direction was obtained by Woodrow in [12]. He proved that if a theory in the language of Ehrenfeucht’s example eliminates quantifiers and has three countable models, then it is quite similar to the original one; in particular, semi-isolation is a relatively definable ordering on the locus of a powerful type. Ikeda, Pillay, and Tsuboi proved that the same happens in the case of an almost \( \mathcal{K}_0 \)-categorical theory with three countable models (see [3, Theorem 7]). Another result in this direction was obtained by Pillay in [5, Theorem 5], who proved that in any Ehrenfeucht theory with few links there exists a definable linear ordering. The ordering relation that he found, when restricted to the locus of a powerful type, is induced by the semi-isolation relation.

In this article we will investigate proper quasiorders of the form \( (p(M), \text{SI}_p) \), where \( p \in S(\emptyset) \) is a nonisolated type in an arbitrary first-order theory, and prove that, under some additional assumptions, a relatively definable suborder can be found. The additional assumptions have a topological flavor. That is not surprising because \( \text{SI}_p \) has a natural topological “definition” as a subspace of the compact space \( S_{p,p} \) consisting of all complete extensions of \( p(x) \cup p(y) \). The semi-isolation \( \text{SI}_p \) corresponds to the subspace \( S_{p,p}^\mathcal{P} \) of all types \( \text{tp}(a,b) \), where \( (a,b) \in \text{SI}_p \). We will decompose \( S_{p,p} \) into four parts, adequate for studying definability properties of \( \text{SI}_p \) (see Definition 1.1 and Remark 1.2). Then we will translate definability properties of semi-isolation into topological (complexity) properties of these parts.

In Section 2 we will prove that certain assumptions on the complexity imply the existence of a proper, relatively definable suborder of \( \text{SI}_p \). For example, we will prove in Theorem 2.7 that if the theory \( T \) has closed asymmetric links on \( p(M) \) (meaning that one of the parts, the set \( S_{p,p}^\mathcal{P} \), is nonempty and closed in \( S_{p,p} \)), then there exists a nontrivial, relatively definable suborder of \( \text{SI}_p \). This is one direction in which we generalize Pillay’s result: if \( p \) is a powerful type of an Ehrenfeucht theory with few links, then \( S_{p,p}^{\mathcal{P}} \) is finite (hence closed) and nonempty.

In Sections 3 and 4 we concentrate on the existence of antichains in \( \text{SI}_p \) in the case of the negation of the strict order property (NSOP), that is the case in which there is no formula \( \varphi(x, y) \) of given theory and tuples \( \vec{a}_i, i \in \omega \), such that the following equivalence holds:

\[
\vdash \varphi(\vec{a}_i, \vec{y}) \rightarrow \varphi(\vec{a}_j, \vec{y}) \iff i \leq j.
\]

We do not do much in this direction: assuming that the underlying theory is binary, NSOP, and has three countable models, with lots of effort we prove that there are at least two distinct types of \( \text{SI}_p \)-incomparable pairs of elements on the locus of a powerful type. This indicates that the answer to Question 1 may be affirmative.
In Section 5 we consider a powerful type $p$ in a binary theory for which SI$_p$ is downwards directed in a specific way (PGPIP; see Definition 5.1). We prove that in the NSOP case the Cantor–Bendixson rank of $S_{p,p}$ is finite, indicating that maybe there is no binary, Ehrenfeucht, NSOP theory with PGPIP at all. So the answer to Question 1 may be negative after all.

1 Preliminaries

Throughout the paper $S_n(A)$ denotes the set of all complete $n$-types with parameters from $A$. The topology on $S_n(A)$ is defined in the usual way. If $\phi(\bar{x})$ is a formula over $A$ in $n$ free variables, then by $[\phi]$ we will denote the set of all types from $S_n(A)$ containing $\phi(\bar{x})$. The set $S(A)$ denotes $\bigcup_n S_n(A)$. If $p, q \in S(\emptyset)$, then $S_{p,q}(\emptyset)$ is the subspace of all the extensions of $p(\bar{x}) \cup q(\bar{y})$ in $S_n(\emptyset)$ (where $\bar{x}$ and $\bar{y}$ are disjoint and $m = |\bar{x}| + |\bar{y}|$). Similarly, if $q \in S_n(\emptyset)$, then $S_q(A)$ denotes the set of all completions of $q(\bar{x})$ in $S_n(A)$. For any $\bar{c}$ realizing $p$ there is a canonical homeomorphism between $S_{p,q}(\emptyset)$ and $S_q(\bar{c})$: the one sending $r(\bar{x}, \bar{y})$ to $r(\bar{c}, \bar{y})$.

Next we recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space $X$ by induction: $\text{CB}_X(p) \geq 0$ for all $p \in X$; $\text{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, $p$ is an accumulation point of the points of $\text{CB}_X$-rank at least $\beta$. We have that $\text{CB}_X(p) = \alpha$ if and only if both $\text{CB}_X(p) \geq \alpha$ and $\text{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal $\alpha$ does not exist, then $\text{CB}_X(p) = \infty$. Isolated points of $X$ are precisely those having rank 0; points of rank 1 are those which are isolated in the subspace of all nonisolated points. For a nonempty $C \subseteq X$ we define $\text{CB}_X(C) = \sup\{\text{CB}_X(p) \mid p \in C\}$; in this way $\text{CB}_X(X)$ is defined and $\text{CB}_X(\{p\}) = \text{CB}_X(p)$ holds. If $X$ is compact and Hausdorff and $C$ is closed in $X$, then the sup is achieved: $\text{CB}_X(C)$ is the maximum value of $\text{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in $C$, and the number of such points is the $\text{CB}_X$-degree of $C$. If $X$ is countable and compact, then $\text{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite $\text{CB}_X$-degree.

$S_n(A)$ is compact, so $\text{CB}$-rank is defined there on points (complete types) and is well behaved on closed subsets (they correspond to partial types). So whenever $p$ is a partial type in $n$ free variables and parameters from $A$, then $\text{CB}_n^A(p)$ is the $\text{CB}$-rank of the compact space consisting of all completions of $p$ in $S_n(A)$; usually the meaning of $n$ and $A$ will be clear from the context, so we will simply write $\text{CB}(p)$. Similarly, the $\text{CB}$-degree is defined. Thus the $\text{CB}$-rank and degree are defined on all partial types and, in particular, they are defined on formulas. If $T$ is small (i.e., $|S(\emptyset)| = \aleph_0$), then the $\text{CB}$-rank of any partial type over a finite domain is an ordinal.

$\phi(M, \bar{a})$ denotes the solution set of $\phi(\bar{x}, \bar{a})$; if $p(\bar{x})$ is a (partial) type, then by $p(M)$ we denote the set of all its realizations. $D \subseteq M^n$ is definable if it is defined by a formula with parameters; it is $A$-definable (or definable over $A$) if the defining formula can be chosen to use only parameters from $A$. We have that $D$ is type-definable ($\bigvee$-definable) if it is the intersection (union) of $|M|$ definable sets; if all the sets in the intersection (union) are definable over a fixed set $A \subseteq M$, then we say that $D$ is type-definable ($\bigvee$-definable) over $A$. In this paper we will consider only countable intersections and unions of sets definable over a finite parameter set. Let $C \subseteq M^n$ be type-definable, and let $C_1 \subseteq C$. Then $C_1$ is relatively definable within $C$ if there is a definable $D \subseteq M$ such that $C_1 = C \cap D$; similarly, relative $\bigvee$-definability is defined.
Semi-isolation was introduced by Pillay in [5]; here we will sketch its basic properties (the reader may find more details in [1]). \( \bar{b} \) is semi-isolated over \( \bar{a} \) (or \( \bar{a} \) semi-isolates \( \bar{b} \)) if and only if there is a formula \( \varphi(\bar{a}, \bar{x}) \in \text{tp}(\bar{b}/\bar{a}) \) such that \( \varphi(\bar{a}, \bar{x}) \vdash \text{tp}(\bar{b}) \); we will denote that by \( \bar{b} \in \text{Sem}(\bar{a}) \) or by \( \bar{a} \rightarrow \bar{b} \). \( \varphi(\bar{x}, \bar{y}) \) is said to witness the semi-isolation; we will also write \( \bar{a} \xrightarrow{\varphi} \bar{b} \) (\( \bar{a} \varphi \)-arrows) Thus \[ \bar{a} \xrightarrow{\psi} \bar{b} \quad \text{if and only if} \quad \models \varphi(\bar{a}, \bar{b}) \text{ and } \varphi(\bar{a}, \bar{y}) \vdash \text{tp}_q(\bar{b}). \]

If \( \bar{a} \rightarrow \bar{b} \), then there are many formulas witnessing the semi-isolation: if \( \varphi(\bar{x}, \bar{y}) \) is a witness, then \( \varphi(\bar{x}, \bar{y}) \land \bar{x} = \bar{x} \) is a witness too. Therefore we can have many distinct named arrows between a fixed pair of tuples.

The reader may note that our definition of \( \bar{a} \rightarrow \bar{b} \) does not exclude the existence of an arrow in the opposite direction. If, in addition to \( \bar{a} \rightarrow \bar{b} \), we know that the opposite arrow does not exist (i.e., that \( \bar{a} \notin \text{Sem}(\bar{b}) \)), we will write \( \bar{a} \leftrightarrow \bar{b} \). Therefore \( \bar{a} \leftrightarrow \bar{b} \) means that both \( \bar{a} \rightarrow \bar{b} \) and \( \bar{a} \notin \text{Sem}(\bar{b}) \) hold; \( \bar{a} \rightarrow \bar{b} \) and \( \bar{a} \leftrightarrow \bar{b} \) may be consistent. \( \bar{a} \leftrightarrow \bar{b} \) means \( \bar{b} \rightarrow \bar{a} \). \( \bar{a} \leftrightarrow \bar{b} \) means that both \( \bar{a} \xrightarrow{\psi} \bar{b} \) and \( \bar{a} \rightarrow \bar{b} \) hold, while \( \bar{a} \leftrightarrow \bar{b} \) means that both \( \bar{a} \rightarrow \bar{b} \) and \( \bar{b} \rightarrow \bar{a} \) hold.

Consider semi-isolation as a binary relation on \( M^{<\omega} \). It is trivially reflexive and it is not hard to see that it is transitive:

\[ \bar{a} \xrightarrow{\varphi} \bar{b} \text{ and } \bar{b} \xrightarrow{\psi} \bar{c} \text{ together imply } \bar{a} \xrightarrow{\psi} \bar{c}, \]

where \( \varphi(\bar{x}, \bar{z}) = \exists \bar{y}(\varphi(\bar{x}, \bar{y}) \land \psi(\bar{y}, \bar{z})) \). Thus semi-isolation is a quasiorder on \( M^{<\omega} \).

We will be interested mainly in semi-isolation as a binary relation on the locus of a complete type \( p \in S(\emptyset) \). Then it is relatively \( \bigvee \)-definable within the locus: to simplify notation we will consider only 1-types; this is justified by passing to an appropriate sort in \( M^{eq} \). So fix for a while a \( p \in S_1(\emptyset) \). Define

\[ \text{SI}_p = \{(a, b) \in p(M)^2 \mid a \rightarrow b\}. \]

For any \( (a, b) \in \text{SI}_p \) there exists an \( L \)-formula \( \varphi(x, y) \) witnessing \( p \)-semi-isolation. This implies that \( \text{SI}_p \) is defined by \( \bigvee \varphi(x, y) \) within \( p(M)^2 \) (here the disjunction is taken over all such \( \varphi \)'s), so \( \text{SI}_p \) is a relatively \( \bigvee \)-definable subset of \( p(M)^2 \).

Define

\[ \overline{\text{SI}}_p = \{(a, b) \in p(M)^2 \mid a \rightarrow b \text{ or } b \rightarrow a \text{ holds}\}, \quad \perp_p = p(M)^2 \setminus \overline{\text{SI}}_p. \]

\((a, b) \in \perp_p \) means that \( a, b \) are incomparable in the quasiorder, in which case we will write \( a \perp_p b \). The semi-isolation \( \overline{\text{SI}}_p \) is relatively \( \bigvee \)-definable within \( p(M)^2 \), while \( \perp_p \) is type-definable.

We will use the following syntax: \( x \notin \text{Sem}_p(y) \) will denote the type consisting of all negated formulas witnessing that \( y \) \( p \)-semi-isolates \( x \); \( x \perp_p y \) will denote the type \( x \notin \text{Sem}_p(y) \cup y \notin \text{Sem}_p(x) \). Therefore the type \( p(x) \cup p(y) \cup x \perp_p y \) defines the set \( \{(a, b) \in p(M)^2 \mid a \perp_p b\} \) whose complement in \( p(M)^2 \) is \( \overline{\text{SI}}_p \).

Each \( \varphi(x, y) \) witnessing \( p \)-semi-isolation defines a binary relation on \( p(M) \), so the quasiorder \( \text{SI}_p \) may also be viewed as the union of a family of binary relations; this has already been suggested by the arrows notation. The relations defined by arrows correspond naturally to subsets of \( S_{p,p} \), and relative definability properties translate into topological properties of these subsets.
Definition 1.1  For a nonisolated \( p \in S(\emptyset) \) and \( \sigma \in \{\rightarrow, \leftarrow, \leftrightarrow, \iff, \bot\} \), define
\[
S^p_\sigma = \{ \text{tp}(ab) \in S_{p,p} \mid a \sigma b \}.
\]
The nonisolation of \( p \) in the definition is assumed in order to exclude the trivial case \( \text{SI}_p = p(M)^2 \), which is not interesting at all.

Remark 1.2  Let \( p \in S(\emptyset) \) be nonisolated. We list some observations related to the defined parts of \( S_{p,p} \).

1. \( S^p_\rightarrow \cup S^p_\leftarrow = S^p_\rightarrow \) and \( S^p_\leftarrow \cup S^p_\rightarrow = S^p_\rightarrow \). We have that \( S_{p,p} \) is the disjoint union
\[
S_{p,p} = S^p_\rightarrow \cup S^p_\leftarrow \cup S^p_\iff \cup S^p_\bot.
\]
2. The mapping taking \( \text{tp}(a, b) \) to \( \text{tp}(b, a) \) is a homeomorphism of \( S_{p,p} \). It fixes setwise \( S^p_\rightarrow \) and \( S^p_\leftarrow \), and maps \( S^p_\iff \) onto \( S^p_\leftarrow \) and \( S^p_\bot \) onto \( S^p_\rightarrow \). In particular, \( S^p_\bot \) and \( S^p_\leftarrow \), as well as \( S^p_\rightarrow \) and \( S^p_\iff \), are homeomorphic.
3. \( S^p_\bot \) has at least one member (containing \( x = y \)). We have that \( S^p_\bot \neq S_{p,p} \) holds; otherwise, there would be a formula \( \varphi(x, y) \) witnessing that each of \( x \) and \( y \) \( p \)-semi-isolates the other such that \( p(x) \cup p(y) \vdash \varphi(x, y) \). Then, by compactness, there would be a \( \theta(x) \in p \) such that \( \models (\theta(x) \land \theta(y)) \Rightarrow \varphi(x, y) \) and, if \( a \models p \) and \( b \in \theta(M) \sim p(M) \), we would get \( \models \varphi(a, b) \), which is not possible by our choice of \( \varphi(x, y) \).
4. Each of \( S^p_\bot \), \( S^p_\leftarrow \), and \( S^p_\rightarrow \) may be empty while their union is nonempty (because of \( S^p_\bot \neq S_{p,p} \)). By part (2), \( S^p_\bot \) and \( S^p_\leftarrow \) are homeomorphic, so they are either both empty or both nonempty.
   - Consider the theory of an infinite set with infinitely many elements named, and let \( p \in S_1(\emptyset) \) be the unique nonalgebraic type. Then \( S^p_\bot = S^p_\leftarrow = \emptyset \), while \( S^p_\bot \) is a singleton with a member containing \( x \neq y \).
   - Consider the type \( p \in S_1(\emptyset) \) containing \( \{n < x \mid n \in \omega\} \) in Ehrenfeucht’s theory \( T_E \). There \( S^p_\bot \) and \( S^p_\leftarrow \) have members containing \( x < y \) and \( y < x \), respectively, while \( S^p_\bot = \emptyset \) because any two elements are comparable.
5. \( S^p_\bot \), \( S^p_\leftarrow \), and \( S^p_\rightarrow \) are open in \( S_{p,p} \): \( S^p_\bot \) is open because \( S^p_\bot = \bigcup_\varphi [\varphi] \), where the union is taken over all formulas \( \varphi(x, y) \) witnessing \( p \)-semi-isolation; by homeomorphism, \( S^p_\leftarrow \) is open too. If \( \text{tp}(a, b) \in S^p_\bot \), then there is a formula \( \varphi(x, y) \in \text{tp}(a, b) \) witnessing \( a \leftrightarrow b \) and \( S^p_\bot \) is the union \( \bigcup_\varphi [\varphi] \) taken over all such \( \varphi(x, y) \). And so \( S^p_\bot \) is open in \( S_{p,p} \).
6. \( S^p_\bot \) is closed in \( S_{p,p} \) because it is the set of all completions of \( p(x) \cup p(y) \cup x \iff y \).
7. Since \( \text{SI}_p \) corresponds to \( S^p_\bot \), \( \text{SI}_p \) is relatively definable within \( p(M)^2 \) if and only if \( S^p_\bot \) is clopen in \( S_{p,p} \). But \( S^p_\bot \) is always open, so \( \text{SI}_p \) is relatively definable if and only if \( S^p_\bot \) is closed in \( S_{p,p} \).
8. \( \overline{\text{SI}}_p \) corresponds to \( S^p_\bot \cup S^p_\leftarrow \), which is open. Therefore relative definability of \( \overline{\text{SI}}_p \) within \( p(M)^2 \) is equivalent to any of the following conditions:
   - \( S^p_\bot \cup S^p_\leftarrow \) is clopen in \( S_{p,p} \);
   - \( S^p_\bot \cup S^p_\leftarrow \) is closed in \( S_{p,p} \);
   - \( S^p_\bot \) is clopen in \( S_{p,p} \) (because it is the relative complement of \( S^p_\bot \cup S^p_\leftarrow \)).
We have \( \text{cl}(S_p^p) \subseteq S_p^p \cup S_p^p \) (where \( \text{cl} \) denotes the topological closure in \( S_{p,p} \)). Since \( S_p^p \) is open and disjoint from \( S_p^p \), we have \( \text{cl}(S_p^p) \subseteq S_{p,p} \setminus S_p^p = S_p^p \cup S_p^p \). In particular, if \( S_p^p \) is not closed, then it has an accumulation point in \( S_p^p \) and \( S_p^p = \emptyset \).

**Definition 1.3** A nonisolated type \( p \in S(\emptyset) \) is symmetric if and only if \( S_I^p \) is a symmetric binary relation on \( p(M) \). Otherwise, \( p \) is asymmetric.

Since semi-isolation is transitive, it follows that \( p \) is asymmetric if and only if \((p(M), S_I^p)\) is a proper quasiorder (with infinite strictly increasing chains). Asymmetric types may exist even in an \( \omega \)-stable theory, so their existence, in general, does not imply the strict order property (examples of that kind can be found in Sudoplatov [7], [8] and Tanović [10]).

**Remark 1.4** It is well known that the symmetry of semi-isolation implies the symmetry of isolation. We will sketch the proof of this fact.

1. If \( \text{tp}(a/b) \) is isolated and \( b \in \text{Sem}(a) \), then \( \text{tp}(b/a) \) is isolated too. To prove this fact, choose \( \varphi(x,b) \in \text{tp}(a/b) \) witnessing the isolation and choose \( \psi(a,y) \in \text{tp}(b/a) \) witnessing the semi-isolation. Then \( \psi(a,y) \land \varphi(a,y) \models \text{tp}(b/a) \). If \( b' \) satisfies this formula, then \( \models \psi(a,b') \) implies \( \text{tp}(b') = \text{tp}(b) \). Combining with \( \models \varphi(a,b') \) and \( \varphi(x,b) \models \text{tp}(a/b) \), we derive \( \text{tp}(a'b') = \text{tp}(ab) \); \( \text{tp}(b/a) \) is isolated.

2. Suppose that \( \text{tp}(a/b) \) is isolated and that \( \text{tp}(b/a) \) is nonisolated. Then \( b \rightarrow a \) and, by part (1), \( b \notin \text{Sem}(a) \). This shows that the asymmetry of isolation on a pair of elements implies the asymmetry of semi-isolation on the same pair. In particular, if \( p \in S(\emptyset) \) and there are \( a, b \models p \) such that \( \text{tp}(a/b) \) is isolated and \( \text{tp}(b/a) \) is nonisolated, then \( p \) is asymmetric.

3. Suppose that \( \text{tp}(a/b) \) is isolated. By part (1) we have
\[
\text{tp}(b/a) \text{ is nonisolated iff } b \notin \text{Sem}(a) \text{ iff } b \mapsto a.
\]

We will use a version of Remark 1.4 localized to \( p \): if semi-isolation is symmetric on \( p(M) \), then isolation is symmetric on \( p(M) \) too. The following example shows that the converse is not true: symmetry of isolation on \( p(M) \) does not necessarily imply the symmetry of semi-isolation on \( p(M) \).

**Example 1.5** Let \( T = \text{Th}(\omega, <) \). Here there is a unique nonalgebraic 1-type \( p(x) \) over \( \emptyset \) (the type of an infinite element). Any infinite element has an immediate successor and a predecessor, so \( x \pm n \) are well-defined functions and
\[
S_I^p = \bigcup_{n \in \omega} \{(x,y) \in p(M)^2 \mid x = n < y \}
\]
(note that \( x + n < y \) implies \( x < y \)). We have that \( p \) is asymmetric. Take \( a, b \) realizing \( p \) such that \( a + n < b \) holds for all integers \( n \); then \( a \mapsto b \). On the other hand, isolation on \( p(M) \) is symmetric because it is witnessed by a formula of the form \( x = y \pm n \) for some \( n \).

Note that \( S_I^p \) is not relatively definable within \( p(M)^2 \) because the union is strictly increasing. On the other hand, \( \overline{S_I}^p = p(M)^2 \) is obviously relatively definable within \( p(M)^2 \). Therefore there are asymmetric types for which \( \overline{S_I}^p \) is relatively definable, while \( S_I^p \) is not relatively definable within the locus.
Recall that a nonisolated type \( p \in S(\emptyset) \) is called powerful if the model prime over a realization of \( p \) is weakly saturated (realizes all finitary types over \( \emptyset \)). Benda in [2] proved that powerful types exist in any Ehrenfeucht theory. Consider all the (isomorphism types of) countable models atomic over a finite subset, and order them by elementary embeddability. Then there is a maximal element (since there are finitely many isomorphism types); the maximal models are precisely those that are weakly saturated.

**Remark 1.6** We note some well-known facts about powerful types. We sketch their proofs for the reader’s convenience.

1. Any powerful type is asymmetric. Let \( p(x) \) be powerful, and let \( a \models p \).
   Since \( p \) is nonisolated, we can find \( a_0 \) realizing a nonisolated extension of \( p \) in \( S(a) \). Further, because \( tp(aa') \) is realized in any maximal model, there is \( b \models p \) such that \( tp(aa'/b) \) is isolated. Note that \( tp(a'/ab) \) is isolated.
   If \( tp(b/a) \) were isolated, then by transitivity of isolation, \( tp(a'b/a) \) would be isolated too. The latter implies isolation of \( tp(a'/a) \), which is a contradiction.
   Therefore \( tp(b/a) \) is nonisolated while \( tp(a'b/a) \) is isolated, so isolation is asymmetric on \( p(M) \). By Remark 1.4(2), we conclude that \( p \) is asymmetric.

2. Let \( p \) be powerful. Then the proof of part (1) shows that for any \( a \models p \) there exists \( b \models p \) such that \( b \not\models a \).

3. Semi-isolation is a downwards-directed quasiorder on the locus of a powerful type. If \( a, b \models p \), then by maximality there is \( d \) realizing \( p \) such that \( tp(ab/d) \) is isolated. In particular, \( tp(a/d) \) and \( tp(b/d) \) are isolated, by \( \varphi(d, x) \) and \( \psi(d, y) \), say, and we have \( d \not\varphi \to a \) and \( d \not\psi \to b \). We have that \( d \) is a lower bound for \( a \) and \( b \).

By a \textit{p-principal formula} we mean an \( L \)-formula \( \varphi(x, y) \) such that for some (any) \( a \) realizing \( p \),

\[ \varphi(a, x) \text{ isolates an extension of } p \text{ in } S_1(a) \text{ and } a \not\varphi \to b \text{ holds for all } b \models \varphi(a, M). \]

By Remark 1.4(3), the condition \( a \not\varphi \to b \) can be replaced by “\( tp(a/b) \) is nonisolated.”

**Remark 1.7** Suppose that \( p \) is powerful. We strengthen the conclusion of Remark 1.6(3): for all \( a, b \in p(M) \) there is \( d \in p(M) \) and \( p \)-principal formulas \( \varphi \) and \( \psi \) such that both \( d \not\varphi \to a \) and \( d \not\psi \to b \) hold. To prove it, first choose \( c_a, c_b \models p \) satisfying \( c_a \not\varphi \to a \) and \( c_b \not\psi \to b \) (here we use Remark 1.6(2)). Then choose \( d \models p \) such that \( tp(c_a c_b ab/d) \) is isolated. Then \( tp(c_a/d) \) is isolated, by \( \varphi(d, x) \), say. Further, \( d \not\varphi \to c_a \) implies \( d \not\psi \to a \) and \( d \not\psi \to a \). Similarly, \( d \not\psi \to b \) for a suitably chosen \( \psi \).

Recall that a theory \( T \) is binary if every formula is equivalent modulo \( T \) to a Boolean combination of formulas with at most two free variables. Binary theories are a special case of \( \Delta \)-based theories (see Saffe, Palyutin, and Starchenko [6]). There \( \Delta \) is a fixed set of formulas (without parameters), and every formula without parameters is equivalent to a Boolean combination of formulas from \( \Delta \). As noted in [6], this means precisely that any complete type \( p \in S(\emptyset) \) is \( \Delta \)-based, that is, that \( p \) is forced by the set of formulas \( \varphi^\delta \in p \), where \( \varphi \in \Delta \) and \( \delta \in \{0, 1\} \). In particular, a theory is binary if and only if any complete type is forced by the union of its 2-subtypes.
2 Definability of Semi-Isolation

In this section we study definability properties of semi-isolation on the locus of an asymmetric type \( p \in S(\emptyset) \). We know that SI\(_p\) is \( \sqrt{\cdot} \)-definable within \( p(M)^2 \). We will prove that certain additional assumptions on the topological complexity of \( S_{p,p} \) imply the strict order property. The ordering relation found will always be a subset of SI\(_p\), as formalized in the next definition.

**Definition 2.1** Suppose that \( p \in S(\emptyset) \) and that \( (p(M), \leq) \) is a quasiorder with infinite strictly increasing chains. We will say that \( \leq \) is a \( p \)-order if

1. \( \leq \) is a relatively definable subset of \( p(M)^2 \), and
2. \( a \leq b \) implies \( (a, b) \in \text{SI}_p \).

The next proposition shows that a \( p \)-order is the restriction of a definable quasiorder to \( p(M) \); the domain of such a quasiorder can be chosen to be definable and unbounded (contains no maximal elements).

**Proposition 2.2** Suppose that \( p \in S(\emptyset) \), \( (p(M), \leq) \) is a \( p \)-order, and that \( \varphi(x, y) \) relatively defines \( \leq \) within \( p(M)^2 \). Then there exists \( \theta(x) \in p \) such that the formula \( \theta(x) \land \theta(y) \land \varphi(x, y) \) witnesses \( p \)-semi-isolation and defines an unbounded quasiorder on \( \theta(M) \).

**Proof** Denote by \( \tau(x, y, z) \) the formula \( \varphi(x, x) \land (\varphi(x, y) \land \varphi(y, z) \Rightarrow \varphi(x, z)) \).

The first condition from the definition of a \( p \)-order implies

\[
p(x) \cup p(y) \cup p(z) \models \tau(x, y, z). \tag{2.1}
\]

The second can be expressed by

\[
p(x) \cup p(y) \cup \{\varphi(x, y)\} \models \bigvee_{i \in I} \varphi_i(x, y), \tag{2.2}
\]

where the disjunction is taken over all formulas witnessing \( p \)-semi-isolation. By compactness there exists a finite \( I_0 \subset I \) such that \( (2.2) \) holds with \( I_0 \) in place of \( I \). Then

\[
p(x) \cup p(y) \cup \{\varphi(x, y)\} \models \varphi(x, y), \tag{2.3}
\]

where \( \varphi(x, y) \) is the formula \( \bigvee_{i \in I_0} \varphi_i(x, y) \). Note that \( \varphi(x, y) \) witnesses \( p \)-semi-isolation. Now we apply compactness simultaneously to (2.1) and (2.3): there exists a formula \( \theta_0(x) \) such that

\[
\theta_0(x) \land \theta_0(y) \land \theta_0(z) \models \tau(x, y, z) \quad \text{and} \quad \theta_0(x) \land \theta_0(y) \land \varphi(x, y) \models \varphi(x, y). \tag{2.4}
\]

The first relation here implies that \( \varphi(x, y) \) defines a quasi-order \( \leq_\varphi \) on \( \theta_0(M) \); its restriction to \( p(M) \) is \( \leq \). The second implies that \( \theta_0(x) \land \theta_0(y) \land \varphi(x, y) \) witnesses \( p \)-semi-isolation. Now we show that there is no \( \leq_\varphi \)-maximal element in \( \theta_0(M) \) above \( a \in p(M) \). We have that \( a \leq_\varphi b \) implies \( b \in p(M) \) and, because \( \leq \) is a \( p \)-order, there exists a strictly \( \leq \)-increasing chain above \( b \). Thus \( b \) is not \( \leq \)-maximal. But \( \leq \) is a restriction of \( \leq_\varphi \), so \( b \) is not \( \leq_\varphi \)-maximal.

Let \( \theta(x) \) be the conjunction of \( \theta_0(x) \) and the formula saying that there is no \( \leq_\varphi \)-maximal element above \( x \). Clearly, \( \theta(x) \land \theta(y) \land \varphi(x, y) \) witnesses \( p \)-semi-isolation and defines the restriction of \( \leq_\varphi \) on \( \theta(M) \). To finish the proof it remains to show that the restricted quasiorder is unbounded; this holds because \( \theta(M) \) is \( \leq_\varphi \)-closed upwards in \( \theta_0(M) \) and \( \theta_0(M) \) is unbounded. \( \square \)

As an immediate corollary we obtain the following.
Corollary 2.3 If \( p(x) \in S(\emptyset) \) is asymmetric and \( \text{SI}_p \) is a relatively definable subset of \( p(M)^2 \), then there is \( \theta(x) \in p \) and a definable, unbounded quasiorder on \( \theta(M) \) whose restriction to \( p(M) \) is \( \text{SI}_p \). In particular, \( T \) has the strict order property.

This fact is well known and can be found in different forms in [1], [3], [5], and Tanović [9]. An example of an asymmetric type with relatively definable semi-isolation is the unique nonisolated 1-type in Ehrenfeucht’s example. A similar situation appears in any almost \( \aleph_0 \)-categorical theory: recall that \( T \) is almost \( \aleph_0 \)-categorical (see [3]) if \( p_1(x_1) \cup p_2(x_2) \cup \cdots \cup p_n(x_n) \) has only finitely many completions \( r(x_1, \ldots, x_n) \in S(\emptyset) \) for all \( n \) and all complete types \( p_i(x_i) \in S(\emptyset) \). For any \( p \) in such a theory, \( \text{SI}_p \) is relatively definable within \( p(M)^2 \): \( S_{p,p} \) is finite, so all its relevant parts are clopen, and by Remark 1.2(7), \( \text{SI}_p \) is relatively definable; alternatively, there are only finitely many inequivalent formulas witnessing \( p \)-semi-isolation, so their disjunction relatively defines \( \text{SI}_p \) within \( p(M)^2 \).

Corollary 2.4 If \( p(x) \in S(\emptyset) \) is asymmetric and \( S_{p,p} \) is finite, then there is \( \theta(x) \in p \) and a definable, unbounded quasiorder on \( \theta(M) \) whose restriction to \( p(M) \) is \( \text{SI}_p \). In particular, \( T \) has the strict order property.

Example 2.5 Let \( T = \left( \mathbb{Q}, <, c_n, d_n \right) \), where \( (c_n) \) is an increasing and \( (d_n) \) is a decreasing sequence such that both converge to \( \sqrt{2} \). We have that \( T \) is an Ehrenfeucht theory having six countable models. Let \( p \) be the 1-type representing \( \sqrt{2} \). Then the locus of \( p \) is convex and linearly ordered by \( < \). However, \( p \) is symmetric and \( \text{SI}_p \) is the identity relation. Thus there is no \( p \)-order there!

Therefore, the locus of a symmetric type may be properly ordered and the asymmetry of semi-isolation is not an exclusive reason for the presence of the strict order property. However, we believe that in this example the reason for the absence of \( p \)-orders lies in nonpowerfulness of \( p \).

Question 2 Suppose that \( p \) is a powerful type in an Ehrenfeucht theory. Does the existence of a nontrivial, relatively definable, partial order on \( p(M) \) always imply the existence of a \( p \)-order?

It is easy to realize that relative definability of \( \text{SI}_p \) implies relative definability of \( \overline{\text{SI}}_p \) within \( p(M)^2 \). The converse is, in general, not true as Example 1.5 shows. There the asymmetric type \( p \in S_1(\emptyset) \) is such that \( \overline{\text{SI}}_p \) is relatively definable within \( p(M)^2 \), while \( \text{SI}_p \) is not.

We will prove in Corollary 2.8 below that relative definability of \( \overline{\text{SI}}_p \) for asymmetric \( p \) implies the existence of a \( p \)-order. Actually, the order found in the proof will have an additional property which will witness that semi-isolation is partially definable on \( p(M) \). This notion was introduced in [10], and here we give an equivalent definition which relies on the notion of a \( p \)-order.

Definition 2.6 We say that semi-isolation is partially definable on \( p \) if there is a definable quasi-order \( \preceq \) such that for all \( a \in p(M) \),

(i) the restriction of \( \preceq \) to \( p(M) \) is a \( p \)-order, and

(ii) \( a \overset{\preceq}{\rightarrow} b \rightarrow b' \in p(M) \) imply \( a \overset{\preceq}{\rightarrow} b' \).

Clearly, partial definability of semi-isolation implies that \( T \) has the strict order property.
**Question 3** Does the existence of a $p$-order imply partial definability of semi-isolation on $p$?

**Theorem 2.7** Suppose that $p \in S(\emptyset)$ is asymmetric and that $S^p_\nearrow$ is closed in $S_{p,p}$. Then semi-isolation is partially definable on $p(M)$. In particular, $T$ has the strict order property.

**Proof** Suppose that $S^p_\nearrow$ is closed in $S_{p,p}$. Then it is compact. For each $q(x,y) \in S^p_\nearrow$, choose a formula $\varphi_q(x,y) \in q(x,y)$ witnessing $p$-semi-isolation. Then $S^p_\nearrow \subseteq \bigcup \{[\varphi_q] \mid q \in S^p_\nearrow\}$. Since $S^p_\nearrow$ is compact, there is a finite subcover. Let $\varphi(x,y)$ be the disjunction of all the $\varphi_q$’s from the subcover. Then $\varphi$ witnesses $p$-semi-isolation and $S^p_\nearrow \subseteq [\varphi] \subseteq S^p_\nearrow$ holds. Let $x \leq y$ be

$$x = y \lor (\varphi(x,y) \land (\forall t)(\varphi(y,t) \Rightarrow \varphi(x,t))).$$

Clearly, $\leq$ defines a quasiorder on $M$; it also witnesses $p$-semi-isolation.

**Claim 1** If $a \mapsto b$ realize $p$, then $\varphi(b,M) \not\subseteq \varphi(a,M)$ and $a < b$.

**Proof** Suppose that $d \in \varphi(b,M)$. Then $a \mapsto b \to d$ implies $a \mapsto d$ and $\text{tp}(ad) \in S^p_\nearrow \subseteq [\varphi]$. Thus $d \in \varphi(a,M)$ and $\varphi(b,M) \not\subseteq \varphi(a,M)$ holds. Similarly, $a \mapsto b$ implies $\text{tp}(ab) \in S^p_\nearrow \subseteq [\varphi]$, so $\models \varphi(a,b)$. Finally, $\models \varphi(a,b)$ and $\varphi(b,M) \not\subseteq \varphi(a,M)$ imply $a < b$. 

Since $p$ is asymmetric, no element of $p$ is maximal in the semi-isolation quasiorder. Then, by the claim, no realization of $p$ is $\leq$-maximal. We conclude that $\leq$ defines a $p$-order on $p(M)$, proving condition (i) from the definition of partial semi-isolation.

To prove (ii), suppose that $a \mapsto b \to c$ holds. Then $a \mapsto c$ and the claim implies $a < c$. Therefore $a \mapsto c$ holds, proving (ii). The symbol $\leq$ partially defines semi-isolation on $p$. 

**Corollary 2.8** Suppose that $p(x) \in S(\emptyset)$ is asymmetric and that $\overline{SI}_p$ is a relatively definable subset of $p(M)^2$. Then semi-isolation is partially definable on $p(M)$. In particular, $T$ has the strict order property.

**Proof** Suppose that $\overline{SI}_p$ is relatively definable within $p(M)^2$. We will show that $S^p_\nearrow$ is closed in $S_{p,p}$. By Remark 1.2(8) $S^p_\nearrow \cup S^p_\searrow$ is closed; clearly it contains $S^p_\nearrow$, so $\text{cl}(S^p_\nearrow) \subseteq S^p_\nearrow \cup S^p_\searrow$. On the other hand, by Remark 1.2(9) we have $\text{cl}(S^p_\searrow) \subseteq S^p_\searrow \cup S^p_\nearrow$. Therefore

$$\text{cl}(S^p_\searrow) \subseteq (S^p_\nearrow \cup S^p_\searrow) \cap (S^p_\searrow \cup S^p_\nearrow) = S^p_\nearrow.$$ 

Therefore $S^p_\nearrow$ is closed in $S_{p,p}$, and the conclusion follows by Theorem 2.7. 

**Corollary 2.9** ($T$ is NSOP) If $p \in S(\emptyset)$ is asymmetric, then $S^p_\searrow$ (is infinite and) has an accumulation point in $S^p_\nearrow$. In particular, $S^p_\nearrow \neq \emptyset$ and $p(x) \cup p(y) \cup x \perp^p y$ is consistent.

**Proof** By Remark 1.2(9) we have $\text{cl}(S^p_\searrow) \subseteq S^p_\searrow \cup S^p_\nearrow$. The NSOP assumption combined with Theorem 2.7 implies that $S^p_\searrow$ is not closed in $S_{p,p}$, so there exists $q \in \text{cl}(S^p_\searrow) \setminus S^p_\nearrow$. Then $q$ is an accumulation point of $S^p_\nearrow$ and $q \in S^p_\nearrow$. In particular, $S^p_\nearrow \neq \emptyset$, so $p(x) \cup p(y) \cup x \perp^p y$ is consistent.
Theories with few links were introduced by Benda in [2]: \( T \) has few links if whenever \( p(\bar{x}) \) and \( q(\bar{y}) \) are complete types, then there are only finitely many complete types \( r(\bar{x}, \bar{y}) \supseteq p(\bar{x}) \cup q(\bar{y}) \) such that \( r(\bar{c}, \bar{y}) \) is nonisolated in \( S(\bar{c}) \) for all \( \bar{c} \) realizing \( p(\bar{x}) \). Pillay in [5, Theorem 5] proved that any Ehrenfeucht theory with few links has the strict order property. He noted that his proof uses only the assumption when \( p \| q \) is a powerful type. Indeed, it is not hard to realize that the few-links assumption implies that \( S_\rho^p \) is finite for any \( p \in S(\emptyset) \). If \( \bar{a}, \bar{b} \models p \) and \( \bar{a} \mapsto \bar{b} \), then \( tp(\bar{a}/\bar{b}) \) is nonisolated; there are only finitely many possibilities for \( tp(\bar{a}/\bar{b}) \), so \( S_\rho^p \) is finite. In particular, \( S_\rho^p \) is closed in \( S_{p,p} \), and we have the following.

**Corollary 2.10**  Any theory with few links and an asymmetric type has the strict order property.

In the same article, Pillay [5, Section 6] commented on the few-links assumption: “This condition is admittedly rather artificial, but it enables some proofs to go through.” An easy consequence of the few-links assumption is that \( CB(S_{p,p}) \leq 1 \) holds for all \( p \in S(\emptyset) \) (simply because \( S_{p,p} \) cannot have infinitely many accumulation points). So \( CB(S_{p,p}) = 1 \) seems to be a more natural condition. There are such Ehrenfeucht theories, the first example having been found by Woodrow in [13].

**Question 4**  Is there a powerful type \( p \) in an NSOP theory satisfying \( CB(S_{p,p}) = 1 \)?

In this article, we do not give much evidence towards answering this question.

**Corollary 2.11**  \((T \text{ is small, NSOP})\)  Suppose that \( p \in S(\emptyset) \) is asymmetric (not necessarily powerful) and that \( CB(S_{p,p}) = 1 \) holds. Then

1. \( |S_\rho^p| \geq \aleph_0 \) and \( |S^p_\rho| \geq 1 \), and
2. there are infinitely many pairwise inequivalent \( p \)-principal formulas.

**Proof**  Condition (1) follows from Corollary 2.9. To prove (2), note that \( CB(S_{p,p}) = 1 \) implies that there are infinitely many members of \( S_\rho^p \) isolated in \( S_{p,p} \). If \( tp(ab) \in S_\rho^p \) is such a type, then \( tp(b/a) \) is isolated and contains a \( p \)-principal formula.

\( \square \)

### 3 Incomparability

In this section, we start dealing with the \( SI_p \)-incomparability of realizations of an asymmetric type. By Corollary 2.9, it is an interesting relation especially in NSOP theories. The next theorem deals with the case when \( SI_p \) has relatively definable intersection with the product of two relatively definable subsets of \( p(M) \). We will prove that there is a pair of incomparable elements \( (a,b) \in D_1 \times D_2 \). The intended combinatorial description of this situation is formalized in Proposition 4.3: if we have two large, unbounded, relatively definable subsets of \( p(M) \), then some pair of their elements is incomparable.

**Theorem 3.1**  Suppose that \( p \in S_1(\emptyset) \) is nonisolated and that \( D_1, D_2 \subseteq M \) are \( \bar{e} \)-definable subsets of \( M \) such that the following conditions are satisfied.

1. \( SI_p \cap (D_1 \times D_2) \neq \emptyset \) is relatively \( \bar{e} \)-definable within \( D_1 \times D_2 \).
2. For all \( a \in D_1 \cap p(M) \) there is \( b \in D_2 \cap p(M) \) such that \( a \leftrightarrow b \).
3. For all \( b \in D_2 \cap p(M) \) there is \( a \in D_1 \cap p(M) \) such that \( b \rightarrow a \).
Then there is an $\bar{e}$-definable quasi-order on $M$ such that no element of $D_1 \cap p(M)$ is below a maximal one of $D_1$. In particular, $T$ has the strict order property.

**Proof** Suppose that $D_1$ is defined by $D_1(x, \bar{e})$ and that relative definability is witnessed by $\theta(x, y, \bar{e})$. So we have

$$p(x) \cup p(y) \cup \{D_1(x, \bar{e}), D_2(y, \bar{e}), \theta(x, y, \bar{e})\} \models y \in \text{Sem}_p(x) \lor x \in \text{Sem}_p(y).$$

The right-hand side is a long disjunction, so by compactness there is an $L$-formula $\varphi(x, y)$ witnessing $y \in \text{Sem}_p(x)$ and there is an $L$-formula $\psi^*(x, y)$ witnessing $x \in \text{Sem}_p(y)$ such that

$$p(x) \cup p(y) \cup \{D_1(x, \bar{e}), D_2(y, \bar{e}), \theta(x, y, \bar{e})\} \models \varphi(x, y) \lor \psi^*(y, x).$$

Let $\psi(x, y) := \psi^*(y, x)$. Then for any pair $(a, b) \in D_1 \times D_2$ of realizations of $p$, we have

either $\models \neg \theta(a, b, \bar{e})$ or: at least one of $a \not\psi b$ and $b \not\psi a$ holds. \hspace{1cm} (3.1)

(The first disjunction here is exclusive because $\theta(x, y, \bar{e})$ relatively defines $\overline{\text{ST}}_p \cap D_1 \times D_2$.) Further, we express assumption (3) by

$$p(x) \cup \{D_2(x, \bar{e})\} \models \exists y (D_1(y, \bar{e}) \land \psi'(x, y)), \hspace{1cm} (3.2)$$

where the disjunction is taken over all $\psi'(x, y)$ witnessing $p$-semi-isolation. By compactness, for some $\psi'(x, y)$ we have

for all $b \in D_2 \cap p(M)$ there is $c \in D_1 \cap p(M)$ such that $b \not\psi' c$ holds. \hspace{1cm} (3.3)

After replacing both $\psi$ and $\psi'$ by their disjunction, we may assume that $\psi = \psi'$.

Let $\varphi(x, y, \bar{e})$ be $\exists z (D_2(z, \bar{e}) \land \varphi(x, z) \land \varphi(z, y))$. Then $\varphi(a, y, \bar{e}) \models p(y)$ for any $a$ realizing $p$.

**Claim 1** For any $a \in D_1 \cap p(M)$, there exists $c \in D_1$ satisfying $a \leftrightarrow c$ and $\models \varphi(a, c, \bar{e})$.

**Proof** Let $a \in D_1 \cap p(M)$. By (3.2) there is $b \in D_2 \cap p(M)$, and by (3.3) there is $c \in D_1 \cap p(M)$ such that $a \iff b \not\psi c$ holds. Then $(a, b) \in \overline{\text{ST}}_p$ implies $\models \theta(a, b, \bar{e})$, and $a \notin \text{Sem}_p(b)$ implies that $b \not\psi a$ does not hold. By (3.1) we derive $a \not\psi b$. Thus $a \not\psi b \not\psi c$, and so $\models \varphi(a, c, \bar{e})$. \hfill $\Box$

Define $a' \leq b'$ iff $\varphi(b', M, \bar{e}) \cap D_1 \subseteq \varphi(a', M, \bar{e}) \cap D_1$. Clearly, $\leq$ is a definable quasi-order on $M$. We will show that no element of $D_1 \cap p(M)$ is below a maximal one of $D_1$.

**Claim 2** If $a, c \in D_1 \cap p(M)$ and $a \leftrightarrow c$, then $a \leq c$.

**Proof** Suppose that $d \in \varphi(c, M, \bar{e}) \cap D_1$, and let $b \in D_2$ be such that $c \not\psi b \not\psi d$. Then $a \not\psi b \not\psi d$ implies $a \not\psi b$, so $b \not\psi a$ does not hold; also, $(a, b) \in \overline{\text{ST}}_p$ implies $\models \theta(a, b, \bar{e})$. By (3.1) we conclude that $a \not\psi b$ holds, and then $a \not\psi b \not\psi d$ implies $\varphi(a, d, \bar{e})$. Thus $d \in \varphi(a, M, \bar{e})$. This shows that $\varphi(c, M, \bar{e}) \cap D_1 \subseteq \varphi(a, M, \bar{e}) \cap D_1$; that is, $a \leq c$. \hfill $\Box$
Now, let $a_1 \in D_1 \cap p(M)$. By Claim 1 there is $c_1 \in D_1$ such that $a_1 \mapsto c_1$ and $\models \varphi(a_1, c_1, \bar{e})$. By Claim 2 we have $a_1 \leq c_1$. Repeating the same procedure with $c_1$, we find $a_2 \in D_1$ satisfying $a_1 \mapsto a_2, \models \varphi(c_1, a_2, \bar{e})$, and $c_1 \leq a_2$. In particular, $a_1 \leq a_2$; that is, $\varphi(a_2, M, \bar{e}) \cap D_1 \subseteq \varphi(a_1, M, \bar{e}) \cap D_1$. Then $c_1 \notin \varphi(a_2, M, \bar{e})$; otherwise $\models \varphi(a_2, c_1, \bar{e})$ would witness $a_2 \mapsto c_1$, which is in contradiction with $c_1 \mapsto a_2$. Thus $c_1 \in \varphi(a_1, M, \bar{e}) \setminus \varphi(a_2, M, \bar{e})$ and $a_1 < a_2$. Continuing in this way we get an infinite strictly increasing chain of elements of $D_1 \cap p(M)$. 

4 Semi-Isolation on Minimal Powerful Types

Throughout this section we will assume that $T$ (is small and) has a powerful type. We will say that $p \in S(\emptyset)$ is a minimal powerful type if it is powerful and there is a formula $\theta(x) \in p$ such that $p$ is the unique powerful type containing $\theta$. Minimal powerful types exist in any Ehrenfeucht theory—take a powerful type of minimal CB-rank. To simplify notation, unless otherwise stated we will assume that $p \in S_1(\emptyset)$ is powerful.

We will be interested in sets definable over a single parameter; we do not a priori assume that the parameter realizes even a nonisolated type. We will say that $D = \varphi(d, M)$ is a $p$-set if $D \cap p(M) \neq \emptyset$ and there exists $b \in D \cap p(M)$ such that at least one of the following two conditions holds:

1. $b$ does not semi-isolate $d$;
2. $\text{tp}(d)$ is not powerful.

The intended intuitive description of a $p$-set is that $D \cap p(M)$ is large and unbounded; this is formalized in Lemma 4.2 below.

Remark 4.1 Suppose that $p$ is a powerful type.

1. If $\text{tp}(d)$ is not powerful, then the second condition from the definition of a $p$-set is satisfied, so $D = \varphi(d, M)$ is a $p$-set if and only if it contains a realization of $p$.
2. Suppose that $p$ is a minimal powerful type and that $\theta(x) \in p$ witnesses the minimality. Let $d \in \theta(M) \setminus p(M)$. Then, by part (1), $D = \varphi(d, M)$ is a $p$-set whenever it contains a realization of $p$.
3. Suppose that $d \models p$ and that $\varphi(x, y)$ witnesses the asymmetry of $p$-semi-isolation; there are $a, b \in p(M)$ such that $a \not\models \varphi \leftrightarrow b$. Then $b$ witnesses that the first condition from the definition holds for $D = \varphi(a, M)$, so $\varphi(a, M)$ is a $p$-set. In particular, $\psi(a, M)$ is a $p$-set for any $p$-principal formula $\psi(x, y)$ and $a \models p$.
4. Suppose that $p$ is a minimal powerful type and that the minimality is witnessed by $\theta(x) \in p(x)$. If $\varphi(x, y)$ is a $p$-principal formula, then for all $d \in \theta(M)$, $D = \varphi(d, M)$ is a $p$-set if and only if it contains a realization of $p$. For $d \in p(M)$ this follows from part (3), and for $d \notin p(M)$ from part (1).

Lemma 4.2 Suppose that $\theta(x) \in p(x)$ witnesses that $p \in S_1(\emptyset)$ is a minimal powerful type, $d \in \theta(M)$, and that $D = \varphi(d, M)$ is a $p$-set. Then $D \cap p(M)$ does not have an SI$_p$-upper bound.
Proof Suppose on the contrary that \( a \in p(M) \) is an upper bound for \( D \cap p(M) \). Then \( c \rightarrow a \) holds for all \( c \in D \cap p(M) \):

\[
p(x) \cup \{ \varphi(d, x) \} \models \bigvee \psi(x, a).
\]

By compactness there are \( \theta_0(x) \in p(x) \) (wlog implying \( \theta(x) \)) and \( \psi(x, y) \) witnessing \( p \)-semi-isolation such that \( \models (\theta_0(x) \land \varphi(d, x)) \Rightarrow \psi(x, a) \). Define

\[
\sigma(y, z) := \forall t ((\theta_0(t) \land \varphi(y, t)) \Rightarrow \psi(t, z)).
\]

Then \( \models \sigma(d, a) \) holds, and according to the definition we have two cases.

Case 1 There exists \( b \in D \cap p(M) \) such that \( b \) does not semi-isolate \( d \).

In this case, we have

\[
\models \varphi(d, b) \land \theta(d) \land \exists z \sigma(d, z). \tag{4.1}
\]

Since \( b \) does not semi-isolate \( d \), any formula from \( \text{tp}(d/b) \) is consistent with infinitely many types from \( S_1(\theta) \), so there exists \( d' \in M \) which does not realize \( p \) and satisfies (4.1) in place of \( d \). Note that \( \models \theta(d') \) and the minimality of \( p \) together imply that \( \text{tp}(d') \) is not powerful. Let \( a' \) be such that

\[
\models \varphi(d', b) \land \theta(d') \land \sigma(d', a').
\]

We claim that \( \sigma(d', z) \models p(z) \) holds. Assume \( \models \sigma(d', c) \). Then from \( b \in \theta_0(M) \cap \varphi(d', M) \) and the definition of \( \sigma \), we get \( \models \psi(b, c) \). Since \( \psi \) witnesses \( p \)-semi-isolation, the claim follows.

\( T \) is small, so there is an isolated type in \( S_1(d') \) containing \( \sigma(d', t) \) (it is an extension of \( p \)). Thus \( d' \) isolates an extension of \( p \), and because \( p \) is powerful, \( \text{tp}(d') \) has to be powerful too. This is a contradiction.

Case 2 \( \text{tp}(d) \) is not powerful.

Since \( D \) is a \( p \)-set, there exists \( b' \in \varphi(d, M) \cap p(M) \). Assuming \( \models \sigma(d, c') \) and arguing as in the first case, we derive \( b' \models c' \), so \( \sigma(d, z) \models p(z) \). Again, we can find an isolated extension of \( p \) in \( S_1(d) \) and conclude that \( \text{tp}(d) \) is powerful. This is a contradiction.

Next we show that \( \text{SI}_p \)-incomparability appears quite often on the locus of a minimal powerful type in an NSOP theory.

**Proposition 4.3 (T is NSOP)** Suppose that \( \theta(x) \in p(x) \) witnesses that \( p \) is a minimal powerful type, \( d_i \in \theta(M) \), and that each \( D_i = \varphi_i(d_i, M) \) is a \( p \)-set for \( i = 1, 2 \). Then there are \( a \in D_1, b \in D_2 \) realizing \( p \) such that \( a \perp_p b \).

**Proof** Otherwise, for all \( a \in D_1, b \in D_2 \) realizing \( p \) we have \( (a, b) \in \text{SI}_p \), so at least one of \( a \rightarrow b \) and \( b \rightarrow a \) holds.

\[
\text{at least one of } a \rightarrow b \text{ and } b \rightarrow a \text{ holds}. \tag{4.2}
\]

In particular, \( \text{SI}_p \cap (D_1 \times D_2) \) is relatively \( d_1d_2 \)-definable within \( p(M)^2 \), and the first assumption of Theorem 3.1 is satisfied. We will prove that the other two are satisfied too.

Suppose that the second condition fails, and witness the failure by \( a \in D_1 \cap p(M) \). Then, by (4.2), \( b \rightarrow a \) would hold for all \( b \in D_2 \cap p(M) \), so \( a \) would be an upper bound for \( D_2 \cap p(M) \); this is in contradiction with Lemma 4.2. Therefore the second and, similarly, the third condition are fulfilled. By Theorem 3.1, \( T \) has the strict order property. This is a contradiction.
Thus SI_p is in some sense a “wide” quasiorder. Because p is powerful, it is also directed downwards. It is interesting to know whether it has to be directed upwards.

**Question 5** Must SI_p be directed upwards on the locus of a minimal powerful type in an NSOP theory?

We have proved in Corollary 2.9 that S^P_\perp has at least one element, and here, under much stronger assumptions, we will prove that |S^P_\perp| ≥ 2.

**Proposition 4.4** Suppose that T is a binary NSOP theory with three countable models and that p ∈ S_1(\emptyset) has CB-rank 1. Then q(x, y) = p(x) ∪ p(y) ∪ x_\perp y has at least two completions in S_2(\emptyset).

**Proof** In a theory with three countable models there is a unique isomorphism type of a “middle model,” that is, a countable model prime over a realization of a nonisolated type. The middle model is weakly saturated because every finitary type is realized in some finitely generated model. Thus any nonisolated type is powerful and, in particular, p is powerful. Let \theta(x) ∈ p be a formula of CB-rank 1 and CB-degree 1. Then p is the unique nonisolated type containing \theta(x), and p is a minimal powerful type.

p is asymmetric, so by Corollary 2.9, q(x, y) is consistent. Now suppose that the conclusion of the proposition fails: q(x, y) has a unique completion q'(x, y) ∈ S_2(\emptyset). Choose a b ⊨ q'; then a _\perp b holds. By Corollary 2.9, q' is an accumulation point of S^P_\perp, so each of tp(ab), tp(a/b), and tp(b/a) is nonisolated. By the three model assumption, we know that the model prime over ab is also prime over a realization d of p (because any two models prime over a realization of a nonisolated type are isomorphic). Note that both tp(ab/d) and tp(d/ab) are isolated. Hence there is a formula τ(x, y, z) ∈ tp(abd) such that τ(d, y, z) isolates tp_\perp(ab/d) and τ(x, a, b) isolates tp_x(d/ab). Now we use the assumption that T is binary: there are formulas \varphi', \psi', \sigma such that

\[ \models (\varphi'(x, y) \land \psi'(x, z) \land \sigma(y, z)) \iff \tau(x, y, z). \]

The assumed isolation properties of τ imply

\[ \varphi'(x, a) \land \psi'(x, b) \land \sigma(a, b) \vdash p(x); \quad (4.3) \]
\[ \varphi'(d, y) \land \psi'(d, z) \land \sigma(y, z) \vdash tp(ab/d). \quad (4.4) \]

Let tp(ab/d) be isolated by \varphi(d, y) ∈ tp(ab/d), and let tp(b/d) be isolated by \psi(d, z) ∈ tp(b/d). Without loss of generality, assume that they are chosen so that

\[ \models (\varphi(x, y) \Rightarrow \varphi'(x, y)) \land (\psi(x, y) \Rightarrow \psi'(x, y)). \]

Then by (4.3) and (4.4):

\[ \varphi(x, a) \land \psi(x, b) \land \sigma(a, b) \vdash p(x); \quad (4.5) \]
\[ \varphi(d, y) \land \psi(d, z) \land \sigma(y, z) \vdash tp(ab/d). \quad (4.6) \]

Now consider the formula (\exists x)(\theta(x) \land \varphi(x, y) \land \psi(x, z) \land \sigma(y, z)) which is in tp_\perp(ab) = q'(y, z). Since S^P_\perp = \{q'\}, by Corollary 2.9, q' is an accumulation point of S^P_\perp. Hence (a', b') ∈ SI_p. Then for some d' we have

\[ \models \theta(d') \land \varphi(d', a') \land \psi(d', b') \land \sigma(a', b'). \quad (4.7) \]

d' does not realize p; otherwise (4.6) would imply a'b' ⊨ q', which is in contradiction with (a', b') ∈ SI_p. Thus d' ∈ θ(M) ∪ p(M), so by Remark 4.1(2),
$D_1 = \psi(d', M)$ and $D_2 = \psi(d', M)$ are $p$-sets. By Proposition 4.3, there are $a'' \in D_1$ and $b'' \in D_2$ realizing $p$ such that $a'' \perp_p b''$ holds. The uniqueness of $q'$ implies $a''b'' \models q'$ and $\models \sigma(a'', b'')$. Thus

$$\models \varphi(d', a'') \land \psi(d', b'') \land \sigma(a'', b'').$$

By (4.5) and $\text{tp}(ab) = \text{tp}(a''b'') = q'$, we get $d' \models p$. This is a contradiction.  

5 PGPIP for Binary Theories

Throughout this section we will assume that $T$ is a small, binary theory and that $p$ is a powerful 1-type. We have already noted in Remark 1.6 that $\text{SI}_p$ is directed downwards. In Remark 1.7 we noted a stronger form: for any pair of elements $a, b \in p(M)$ there exists $d \in p(M)$ and $p$-principal formulas $\varphi, \psi$ such that both $d \not\models \varphi a$ and $d \models \psi b$ hold. In all the basic examples $\varphi$ and $\psi$ can be chosen from a finite (fixed in advance) set. This property is labeled in [8] as the global pairwise intersection property (GPIP) for $p$. Precisely, it means that there is a formula $'.x; y/'$ which is a disjunction of $p$-principal formulas and such that $(p(M); '.x; y'/2)$ is an acyclic digraph satisfying

$$\forall a, b \in p(M) \exists d \models p \text{ such that } \models '.d; a/ \land '.d; b/.$$

Here we introduce a somewhat stronger property.

Definition 5.1 $p$ has PGPIP if there is a formula $'.x; y/'$ which is a disjunction of $p$-principal formulas and is such that $(p(M); '.x; y'/2)$ is an acyclic digraph, and for all $a, b \in p(M)$ there exists $d \models p$ satisfying

$$\text{tp}(ab/d) \text{ is isolated and } \models '.d; a/ \land '.d; b/.$$

We leave it to the reader to check that nonisolated 1-types from Ehrenfeucht’s and Peretyatkin’s (see [4]) examples have PGPIP.

Theorem 5.2 ($T$ is binary, NSOP) Suppose that $\varphi(x, y) = \bigvee_{i=1}^n \varphi_i(x, y)$, where each $\varphi_i(x, y)$ is $p$-principal, witnesses PGPIP for $p$. Then $n \geq 2$ and $\text{CB}(S_{p, p}(\emptyset)) < n^2$.

Proof Fix $d$ realizing $p$. For each pair $i, j \leq n$, define

$$D_{(i, j)} = \{(a, b) \in p(M)^2 \mid \text{tp}(ab/d) \text{ is isolated and } \models \varphi_i(d, a) \land \varphi_j(d, b)\}.$$ $$C_{(i, j)} = \{\text{tp}(ab/d) \mid (a, b) \in D_{(i, j)}\}.$$ $$S_{(i, j)} = \{\text{tp}(ab) \mid (a, b) \in D_{(i, j)}\}.$$ Note that PGPIP implies that $\bigcup_{(i, j)} S_{(i, j)} = S_{p, p}(\emptyset)$ holds; in particular, if $n = 1$, then $S_{(1, 1)} = S_{p, p}(\emptyset)$.

Claim 1 For every $q(x, y) \in S_{(i, j)}$ there is $\theta_q(x, y) \in q$ which has a unique extension in $C_{(i, j)}$.

Proof Let $(a, b) \in D_{(i, j)}$ realize $q$. Then $\text{tp}(ab/d)$ is isolated and, because $T$ is binary and $\varphi_i$’s are $p$-principal, there is a formula $\theta_q(x, y) \in q(x, y)$ such that

$$\varphi_i(d, x) \land \varphi_j(d, y) \land \theta_q(x, y) \models \text{tp}(ab/d).$$

Since any extension of $\theta_q(x, y)$ in $C_{(i, j)}$ contains the formula on the left-hand side, we conclude that the extension is unique.
Now, we claim that each $S_{i,j}$ is a discrete subset of $S_{p,p}(\emptyset)$. Suppose on the contrary that $q(x, y) \in S_{i,j}$ is an accumulation point of $S_{i,j}$. Then $\theta_q$ is contained in some $q' \in S_{i,j}$ which is distinct from $q$. Thus $\theta_q$ has at least two extensions in $C_{i,j}$: the one extending $q$ and the one extending $q'$. This is a contradiction.

The first part of our theorem follows: if $n = 1$, then $S_{1,1} = S_{p,p}(\emptyset)$ is discrete and, because it is compact, it has to be finite. Then by Corollary 2.4, $T$ has the strict order property. This is a contradiction. Therefore $n \geq 2$.

The second part follows from the following topological fact: a compact space which is a union of $m$ discrete subsets has CB-rank smaller than $m$ (easily proved by induction). In our situation $S_{p,p}(\emptyset) = \bigcup_{i,j} S_{i,j}$ is a union of $n^2$ discrete subsets, so $\text{CB}(S_{p,p}(\emptyset)) < n^2$.

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