Abstract

We study the problem of online learning to re-rank, where users provide feedback to improve the quality of displayed lists. Learning to rank has been traditionally studied in two settings. In the offline setting, rankers are typically learned from relevance labels of judges. These approaches have become the industry standard. However, they lack exploration, and thus are limited by the information content of offline data. In the online setting, an algorithm can propose a list and learn from the feedback on it in a sequential fashion. Bandit algorithms developed for this setting actively experiment, and in this way overcome the biases of offline data. But they also tend to ignore offline data, which results in a high initial cost of exploration. We propose BubbleRank, a bandit algorithm for re-ranking that combines the strengths of both settings. The algorithm starts with an initial base list and improves it gradually by swapping higher-ranked less attractive items for lower-ranked more attractive items. We prove an upper bound on the $n$-step regret of BubbleRank that degrades gracefully with the quality of the initial base list. Our theoretical findings are supported by extensive numerical experiments on a large real-world click dataset.

1 Introduction

Learning to rank (LTR) is an important problem in many application domains, such as information retrieval, ad placement, and recommender systems [20]. More generally, LTR arises in any situation where multiple items, such as web pages, are presented to the user. It is particularly relevant when the diversity of users makes it hard to decide which item should be presented to a specific user.

A traditional approach to LTR is offline learning of rankers from either relevance labels of judges [23] or user interactions [10][21]. Recent experimental results [34] showed that such rankers, even in a highly-optimized search engine, can be improved by online LTR with exploration. Exploration is the key component in multi-armed bandit algorithms [3]. Many such algorithms have been proposed recently for online LTR in specific user-behavior models [14][12][16], the so-called click models [5]. Comparing to earlier online LTR algorithms [25], these model-based algorithms gain in statistical efficiency while giving up on generality. Empirical results indicate that the model-based algorithms are likely to be beneficial in practice.

∗This work was done while the author was at Adobe Research.
Yet the existing algorithms for online LTR in click models are impractical for at least three reasons. First, an actual model of user behavior is often unknown. This problem was recently addressed by Zoghi et al. [35]. They showed that the list of items in the descending order of relevance is optimal in several click models and proposed an online learning algorithm, BatchRank, for learning it. Second, the algorithms explore aggressively by placing potentially irrelevant items at high positions, which may significantly degrade user experience [22] [31]. A third and related problem is that the algorithms are not well suited for the so-called warming start, where the offline-trained production ranker already generates a good list, which only needs to be safely improved.

We make the following contributions. First, motivated by the exploration scheme of Radlinski and Joachims [24], we propose a bandit algorithm for online LTR that addresses all aforementioned issues. The algorithm gradually improves upon an initial base list by swapping higher-ranked less attractive items for lower-ranked more attractive items. The algorithm resembles bubble sort [7], and therefore we call it BubbleRank. Second, we prove an upper bound on the $n$-step regret of BubbleRank. The bound reflects the behavior of BubbleRank, that worse initial base lists lead to a higher regret. Finally, we evaluate BubbleRank extensively on a large real-world click dataset.

We denote $\{1, \ldots, n\}$ by $[n]$. For any sets $A$ and $B$, we denote by $AB$ the set of all vectors whose entries are indexed by $B$ and take values from $A$. We use boldface letters to denote random variables.

## 2 Background

In this section, we introduce our learning problem. We first review click models [5] and then introduce a stochastic click bandit [35], a learning to rank framework for multiple click models.

### 2.1 Click Models

A click model is a model of how a user clicks on a list of documents. We refer to the documents as items and denote the universe of all items by $D = [L]$. The user is presented a ranked list, an ordered list of $K$ documents out of $L$. We denote this list by $R \in \Pi_K(D)$, where $\Pi_K(D)$ is the set of all $K$-tuples with distinct items from $D$. We denote by $\mathcal{R}(k)$ the item at position $k$ in $\mathcal{R}$; and by $\mathcal{R}^{-1}(i)$ the position of item $i$ in $\mathcal{R}$, if that item is in $\mathcal{R}$.

Many click models are parameterized by item-dependent attraction probabilities $\alpha \in [0, 1]^L$, where $\alpha(i)$ is the attraction probability of item $i$. We discuss two most fundamental click models below.

In the cascade model (CM) [8], the user scans list $\mathcal{R}$ from the first item $\mathcal{R}(1)$ to the last $\mathcal{R}(K)$. If item $\mathcal{R}(k)$ is attractive, the user clicks on it and does not examine the remaining items. If item $\mathcal{R}(k)$ is not attractive, the user examines item $\mathcal{R}(k+1)$. The first item $\mathcal{R}(1)$ is examined with probability one. Therefore, the expected number of clicks is equal to the probability of clicking on any item, and is $r(\mathcal{R}) = \sum_{k=1}^{K} \chi(\mathcal{R}, k)\alpha(\mathcal{R}(k))$, where $\chi(\mathcal{R}, k) = \prod_{i=1}^{k-1} (1 - \alpha(\mathcal{R}(i)))$ is the examination probability of position $k$ is list $\mathcal{R}$.

In the position-based model (PBM) [26], the probability of clicking on an item depends on both its identity and position. Therefore, in addition to $\alpha$, the PBM is parameterized by $K$ position-dependent examination probabilities $\chi \in [0, 1]^K$, where $\chi(k)$ is the examination probability of position $k$. The user interacts with list $\mathcal{R}$ as follows. The user examines position $k \in [K]$ with probability $\chi(k)$ and then clicks on item $\mathcal{R}(k)$ at that position with probability $\alpha(\mathcal{R}(k))$. Thus, the expected number of clicks on list $\mathcal{R}$ is $r(\mathcal{R}) = \sum_{k=1}^{K} \chi(k)\alpha(\mathcal{R}(k))$.

Both above models are similar, because the probability of clicking factors into item and position dependent factors. Therefore, both in the CM and PBM, under the assumption that $\chi(1) \geq \cdots \geq \chi(K)$, the expected number of clicks is maximized by $K$ most attractive items in the descending order of their attraction. More precisely, the most clicked list is

$$\mathcal{R}^* = (1, \ldots, K)$$

when $\alpha(1) \geq \cdots \geq \alpha(L)$. Therefore, perhaps not surprisingly, the problem of learning the optimal list in both models can be viewed as the same problem, a stochastic click bandit [35].
2.2 Stochastic Click Bandit

An instance of a stochastic click bandit [35] is a tuple \((K, L, P_\alpha, P_\chi)\), where \(K \leq L\) is the number of positions, \(L\) is the number of items, \(P_\alpha\) is a distribution over binary attraction vectors \(\{0,1\}^L\), and \(P_\chi\) is a distribution over binary examination matrices \(\{0,1\}^{K(D) \times K}\).

The learning agent interacts with the stochastic click bandit as follows. At time \(t\), it chooses a list \(\mathcal{R}_t \in \Pi_K(D)\), which depends on its history up to time \(t\), and then observes clicks \(c_t \in \{0,1\}^K\) on all positions in \(\mathcal{R}_t\). The position is clicked if and only if it is examined and the item at that position is attractive. More specifically, for any \(k \in [K]\),

\[
e c_t(k) = X_t(\mathcal{R}_t, k)A_t(\mathcal{R}_t(k)),
\]

where \(X_t \in \{0,1\}^{K(D) \times K}\) and \(X_t(\mathcal{R}, k)\) is the examination indicator of position \(k\) in list \(\mathcal{R} \in \Pi_K(D)\) at time \(t\); and \(A_t \in \{0,1\}^L\) and \(A_t(i)\) is the attraction indicator of item \(i\) at time \(t\). Both \(A_t\) and \(X_t\) are stochastic and drawn i.i.d. from \(P_\alpha \otimes P_\chi\).

The key assumption that allows learning in this model is that the attraction of any item is independent of the examination of its position. In particular, for any list \(\mathcal{R} \in \Pi_K(D)\) and position \(k \in [K]\),

\[
\mathbb{E}[c_t(k) | \mathcal{R}_t = \mathcal{R}] = \chi(\mathcal{R}, k)\alpha(\mathcal{R}(k)),
\]

where \(\alpha = \mathbb{E}[A_t]\) and \(\alpha(i)\) represents the attraction probability of item \(i\); and \(\chi = \mathbb{E}[X_t]\) and \(\chi(\mathcal{R}, k)\) represents the examination probability of position \(k\) in \(\mathcal{R}\). We stress that this assumption does not require that the clicks are independent of the position or other displayed items.

The expected reward at time \(t\) is the expected number of clicks at time \(t\). Based on our independence assumption, \(\sum_{k=1}^K \mathbb{E}[c_t(k)] = r(\mathcal{R}_t, \alpha, \chi)\), where \(r(\mathcal{R}, A, X) = \sum_{k=1}^K X(\mathcal{R}, k)A(\mathcal{R}(k))\) for any \(\mathcal{R} \in \Pi_K(D)\), \(A \in \{0,1\}^L\), and \(X \in \{0,1\}^{K(D) \times K}\). The learning agent maximizes the expected number of clicks in \(n\) steps. This problem can be equivalently viewed as minimizing the expected cumulative regret in \(n\) steps, which we define as

\[
R(n) = \sum_{t=1}^n \mathbb{E}\left[ \max_{\mathcal{R} \in \Pi_K(D)} r(\mathcal{R}, \alpha, \chi) - r(\mathcal{R}_t, \alpha, \chi) \right].
\]

3 Learning to Re-rank

Multi-stage ranking is widely used in production ranking systems [11, 19, 29], with the re-ranking stage at the very end [4]. In the re-ranking stage, a relatively small number of items, typically 10–20, are re-ranked. One reason for re-ranking is that offline rankers are typically trained to minimize the average loss across a large number of queries. Naturally, such rankers perform well on very frequent queries and poorly on infrequent queries. On moderately frequent queries, known as torso queries, their performance varies. However, since torso queries are sufficiently frequent, an online algorithm can be used to correct for the bias of the offline ranker by re-ranking [33].

We propose an online algorithm that addresses the above problem and adaptively re-ranks a list of items generated by some production ranker with the goal of placing more attractive items at higher positions. We study a non-contextual variant of the problem, where we re-rank a small number of items in a single query. Generalization across queries and items is an interesting direction for future work. We follow the setting in Section 2.2 except that \(D = [K]\). Despite this, our problem remains a challenging bandit problem because the attraction of items is only observed through clicks in \(\mathcal{R}\), which are affected by other items in the list.

3.1 Algorithm BubbleRank

Our algorithm is presented in Algorithm 1. The algorithm gradually improves upon an initial base list \(\mathcal{R}_0\) by bubbling up more attractive items. Therefore, we refer to it as BubbleRank. BubbleRank determines more attractive items by randomly swapping neighboring items. If the lower-ranked item is found to be more attractive, the items are swapped and then never randomly swapped again. If the lower-ranked item is found to be less attractive, the items are never randomly swapped again. We describe BubbleRank in detail below.
Algorithm 1: BubbleRank

1: **Input:** initial list $R_0$ over $[K]$

2: $\forall i, j \in [K]: s_0(i, j) \leftarrow 0$, $n_0(i, j) \leftarrow 0$

3: $R_1 \leftarrow R_0$

4: for $t = 1, \ldots, n$ do

5: $h \leftarrow t \mod 2$

6: $R_t \leftarrow R_t$

7: for $k = 1, \ldots, \lfloor (K - h)/2 \rfloor$ do

8: $i \leftarrow R_t(2k - 1 + h)$, $j \leftarrow R_t(2k + h)$

9: if $s_{t-1}(i, j) \leq 2\sqrt{n_{t-1}(i, j)\log(1/\delta)}$ then

10: Randomly exchange items $R_t(2k - 1 + h)$ and $R_t(2k + h)$ in list $R_t$

11: Display list $R_t$ and observe clicks $c_t \in \{0, 1\}^K$

12: $s_t \leftarrow s_{t-1}$, $n_t \leftarrow n_{t-1}$

13: for $k = 1, \ldots, \lfloor (K - h)/2 \rfloor$ do

14: $i \leftarrow R_t(2k - 1 + h)$, $j \leftarrow R_t(2k + h)$

15: if $|c_t(2k - 1 + h) - c_t(2k + h)| = 1$ then

16: $s_t(i, j) \leftarrow s_t(i, j) + c_t(2k - 1 + h) - c_t(2k + h)$

17: $n_t(i, j) \leftarrow n_t(i, j) + 1$

18: $s_t(i, j) \leftarrow s_t(i, j) + c_t(2k + h) - c_t(2k - 1 + h)$

19: $n_t(i, j) \leftarrow n_t(i, j) + 1$

20: $R_{t+1} \leftarrow R_t$

21: for $k = 1, \ldots, K - 1$ do

22: $i \leftarrow R_{t+1}(k)$, $j \leftarrow R_{t+1}(k + 1)$

23: if $s_t(i, j) > 2\sqrt{n_t(i, j)\log(1/\delta)}$ then

24: Exchange items $R_{t+1}(k)$ and $R_{t+1}(k + 1)$ in list $R_{t+1}$

BubbleRank maintains a base list $\bar{R}_t$ at each time $t$. From the viewpoint of BubbleRank, this is the best list at time $t$. The list is initialized by the initial base list $\bar{R}_0$ (line 3). At time $t$, BubbleRank permutes $R_t$ into a displayed list $R_t$ (lines 6–10). Two kinds of permutations are employed. If $t$ is odd and so $h = 0$, the items at positions 1 and 2, 3 and 4, and so on are randomly swapped. If $t$ is even and so $h = 1$, the items at positions 2 and 3, 4 and 5, and so on are randomly swapped. The items are swapped only if BubbleRank is uncertain regarding which item is more attractive (line 9).

The list $R_t$ is displayed and BubbleRank collects feedback (line 11). Then it updates its statistics (lines 12–19). For any swapped items $i$ and $j$, if item $i$ is clicked and item $j$ is not, the belief that $i$ is more attractive than $j$, $s_t(i, j)$, increases; and the belief that $j$ is more attractive than $i$, $s_t(j, i)$, decreases. The number of observations, $n_t(i, j)$ and $n_t(j, i)$, increases. These statistics are updated only if one of the items is clicked (line 15), not both.

At the end of time $t$, the base list $\bar{R}_t$ can be improved as follows (lines 20–24). If any lower-ranked item $j$ is found to be more attractive than its higher-ranked neighbor $i$ (line 23), then the items are permanently swapped in the next base list $\bar{R}_{t+1}$.

A notable property of BubbleRank is that it explores “cautiously”, in that any item in the displayed list $R_t$ is at most one position away from its position in the base list $\bar{R}_t$. Any base list improves upon the initial base list $\bar{R}_0$, because it is obtained by bubbling up more attractive items with a high confidence. In fact, we can show that $r(\bar{R}_0, \alpha, \chi) \leq r(\bar{R}_1, \alpha, \chi) \leq \cdots \leq r(\bar{R}_n, \alpha, \chi)$ holds with probability of at least $1 - \delta^2 K^2 n$, by conditioning on the favorable event $E$ in Section 4.3.

4 Analysis

This section is organized as follows. In Section 4.1, we present an upper bound on the $n$-step regret of BubbleRank, together with our assumptions. In Section 4.2, we discuss the bound. The proof of bound is in Section 4.3. Our technical lemmas are stated and proved in Appendix A.
4.1 Regret Bound

Before we present our result, we introduce our assumptions and complexity metrics.

**Assumption 1.** For any lists $\mathcal{R}, \mathcal{R}' \in \Pi_K(D)$ and positions $k, \ell \in [K]$ such that $k < \ell$:

- A1. $r(\mathcal{R}, \alpha, \chi) \leq r(\mathcal{R}^*, \alpha, \chi)$, where $\mathcal{R}^*$ is defined in (1).
- A2. $\{\mathcal{R}(1), \ldots, \mathcal{R}(k-1)\} = \{\mathcal{R}'(1), \ldots, \mathcal{R}'(k-1)\} \implies \chi(\mathcal{R}, k) = \chi(\mathcal{R}', k)$
- A3. $\chi(\mathcal{R}, k) \geq \chi(\mathcal{R}, \ell)$
- A4. If $\mathcal{R}$ and $\mathcal{R}'$ differ only in that the items at positions $k$ and $\ell$ are exchanged, then $\alpha(\mathcal{R}(k)) \leq \alpha(\mathcal{R}(\ell)) \iff \chi(\mathcal{R}, k) \geq \chi(\mathcal{R}', k)$
- A5. $\chi(\mathcal{R}, k) \geq \chi(\mathcal{R}^*, k)$

The above assumptions hold in the CM. In the PBM, they hold when the examination probability decreases with the position. The assumptions can be interpreted as follows. Assumption [A1] says that the list of items in the descending order of attraction probabilities is optimal. Assumption [A2] says that the examination probability of any position depends only on the identities of higher-ranked items. Assumption [A3] says that a higher position is at least as examined as a lower position. Assumption [A4] says that a higher-ranked item is less attractive if and only if it increases the examination of a lower position. Assumption [A5] says that any position is examined the least in the optimal list.

For simplicity of exposition, let $\alpha(1) > \cdots > \alpha(K) > 0$. Let

$$
\chi_{\text{max}} = \chi(\mathcal{R}^*, 1), \quad \chi_{\text{min}} = \chi(\mathcal{R}^*, K), \quad \Delta_{\text{min}} = \min_{k \in [K-1]} \alpha(k) - \alpha(k+1)
$$

be the maximum examination probability, the minimum examination probability, and the minimum gap, respectively. Let

$$
V_0 = \{ (i, j) \in [K]^2 : i < j, \, R_0^{-1}(i) > R_0^{-1}(j) \} \tag{4}
$$

be the set of all incorrectly-ordered item pairs in the initial base list $R_0$. Then regret of BubbleRank can be bounded as follows.

**Theorem 1.** In any stochastic click bandit that satisfies Assumption [A1] the expected $n$-step regret of BubbleRank is bounded as

$$
R(n) \leq 180K \frac{\chi_{\text{max}} K - 1 + 2|V_0|}{\Delta_{\text{min}}} \log(1/\delta) + \delta^{\frac{3}{2}}K^3 n^2.
$$

4.2 Discussion

Our upper bound on the $n$-step regret of BubbleRank (Theorem [A1]) is $O(\Delta_{\text{min}}^{-1} \log n)$ for $\delta = n^{-4}$. This dependence is considered optimal in gap-dependent bounds. Our $\Delta_{\text{min}}$ gap is the minimum difference in the attraction probabilities of items, and reflects the hardness of sorting the items by their attraction probabilities. This is essentially the problem of learning $\mathcal{R}^*$. Therefore, a gap like $\Delta_{\text{min}}$ is expected, and in fact it is the same as that in Zoghi et al. [35]. In addition, our regret bound is notable because it reflects two key characteristics of BubbleRank.

First, the bound is linear in the number of incorrectly-ordered item pairs in the initial base list $R_0$. This suggests that BubbleRank should have lower regret when initialized with a better list of items. We validate this dependence empirically in Section [A5]. In many domains, such lists already exist and are produced by existing ranking policies. They only need to be safely improved.

Second, the bound is $O(\chi_{\text{max}}^{-1} \chi_{\text{min}}^{-1})$, where $\chi_{\text{max}}$ and $\chi_{\text{min}}$ are the maximum and minimum examination probabilities, respectively. In Section [A5] we show that this dependence can be observed in problems where most attractive items are placed at infrequently examined positions. This limitation is intrinsic to BubbleRank, because attractive lower-ranked items cannot be placed at higher positions unless they are observed to be attractive at lower, potentially infrequently examined, positions.

Our assumptions are slightly weaker than those of Zoghi et al. [35]. In particular, Assumption [A2] is on the probability of examination. Zoghi et al. [35] make this assumption on the realization of examination.
4.3 Proof of Theorem 1

In Lemma 6, we establish that there exists a favorable event $E$ that holds with probability $1 - \delta^2 K^2 n$, when all beliefs $s_t(i,j)$ are at most $2\sqrt{n_t(i,j) \log(1/\delta)}$ from their respective means, uniformly for $i < j$ and $t \in [n]$. Since the maximum $n$-step regret is $Kn$, we get that

$$R(n) \leq \mathbb{E} \left[ \hat{R}(n) 1 \{ E \} \right] + \delta^2 K^3 n^2,$$

where $\hat{R}(n) = \sum_{t=1}^{n} r(R^*, \alpha, \chi) - r(R_t, \alpha, \chi)$. We bound $\hat{R}(n)$ next. For this, let

$$\mathcal{P}_t = \left\{ (i,j) \in [K]^2 : i < j, \left| \mathcal{R}_t^{-1}(i) - \mathcal{R}_t^{-1}(j) \right| = 1, s_{t-1}(i,j) \leq 2 \sqrt{n_{t-1}(i,j) \log(1/\delta)} \right\}$$

be the set of potentially randomized item pairs at time $t$. Then, by Lemma 4 on event $E$, which relates the regret of list $\mathcal{R}_t$ to the difference in the attraction probabilities of $(i,j) \in \mathcal{P}_t$, we have that

$$\hat{R}(n) \leq 3 K \chi_{\text{max}} \sum_{i=1}^{K} \sum_{j=i+1}^{K} \sum_{t=1}^{n} (\alpha(i) - \alpha(j)) 1 \{ (i,j) \in \mathcal{P}_t \}.$$

Now note that for any randomized $(i,j) \in \mathcal{P}_t$ at time $t$,

$$\chi_{\text{min}}(\alpha(i) - \alpha(j)) \leq \mathbb{E}_{t-1} \left[ c_t(\mathcal{R}_t^{-1}(i)) - c_t(\mathcal{R}_t^{-1}(j)) \right] = \mathbb{E}_{t-1} \left[ s_t(i,j) - s_{t-1}(i,j) \right],$$

where $\mathbb{E}_{t-1} [ \cdot ]$ is the conditional expectation given history $\mathcal{R}_1, c_1, \ldots, \mathcal{R}_{t-1}, c_{t-1}$ up to time $t$; and the inequality is from $\alpha(i) \geq \alpha(j)$, and Assumptions A2 and A4. The above two inequalities yield

$$\hat{R}(n) \leq 6 K \chi_{\text{max}} \sum_{i=1}^{K} \sum_{j=i+1}^{K} \sum_{t=1}^{n} \mathbb{E}_{t-1} \left[ s_t(i,j) - s_{t-1}(i,j) \right] 1 \{ (i,j) \in \mathcal{P}_t \}$$

$$\leq 6 K \chi_{\text{max}} \sum_{i=1}^{K} \sum_{j=i+1}^{K} 1 \{ \exists t \in [n] : (i,j) \in \mathcal{P}_t \} \sum_{t=1}^{n} \mathbb{E}_{t-1} \left[ s_t(i,j) - s_{t-1}(i,j) \right],$$

where the extra factor of 2 is because any given $(i,j) \in \mathcal{P}_t$ is randomized by BubbleRank at least once in any two consecutive times. Moreover, for any $i < j$ on event $E$,

$$\sum_{t=1}^{n} (s_t(i,j) - s_{t-1}(i,j)) = s_n(i,j) \leq 15 \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \log(1/\delta) \leq \frac{30}{\Delta_{\text{min}}} \log(1/\delta).$$

The first inequality is by Lemma 6, which establishes that the maximum difference in clicks of any randomized pair of items is bounded. After that, the better item is found and the pair of items is not randomized anymore. The second inequality is by $\alpha(i) + \alpha(j) \leq 2$ and $\alpha(i) - \alpha(j) \geq \Delta_{\text{min}}$. Now we chain the above two inequalities and get that

$$\hat{R}(n) \leq 180 K \chi_{\text{max}} \frac{1}{\Delta_{\text{min}}} \log(1/\delta) \sum_{i=1}^{K} \sum_{j=i+1}^{K} 1 \{ \exists t \in [n] : (i,j) \in \mathcal{P}_t \} + \delta^2 K^3 n^2.$$

Finally, let $\mathcal{P} = \bigcup_{t \in [n]} \mathcal{P}_t$. Then, on event $E$, $|\mathcal{P}| \leq K - 1 + 2 |\mathcal{V}_0|$. This follows from the design of BubbleRank (Lemma 5). This completes the proof.

5 Experiments

We evaluate BubbleRank on the Yandex click dataset\[35\] with more than 30 million search sessions. Each session contains at least one search query. We preprocess the queries as Zoghi et al.\[35\], who select 60 frequent search queries with 10 most attractive items in each query, and then learn their CMs and PBMs using PyClick\[35\]. These CMs and PBMs are used to mimic users and generate click feedback. The number of positions is equal to the number of items, $K = L = 10$. The objective of our re-ranking problem is to maximize the expected number of clicks at the 5 highest positions,

\[\text{https://academy.yandex.ru/events/data_analysis/relpred2011}\]

\[\text{https://github.com/markovi/PyClick}\]
The former is near optimal in the CM \cite{14}, but can have a linear regret in other click models. The CM and PBMs of all queries (Figure 1). The regret is reported from the highest positions.

Consider the re-ranking problem in Section 3 where a production ranker generates a list of items to re-rank. Neither CascadeKL-UCB nor BatchRank can take advantage of this initial list, and would learn to order its items from scratch. To show the effectiveness of BubbleRank in this setting, we study its two variants. BubbleRank0 is initialized with a randomly ordered list. BubbleRankKL is initialized with a list generated by CascadeKL-UCB. In particular, we run CascadeKL-UCB for 20k steps and then order the items in the initial base list $R_0$ in the descending order of their estimated click probabilities. This initialization is rather naive, and existing production rankers may produce better lists in practice. We choose 20k steps because we observe empirically that CascadeKL-UCB learns good lists in most queries in about this many steps. This allows us to study whether BubbleRank is able to exploit a good initialization.

In the first experiment, we compare BubbleRank to CascadeKL-UCB and BatchRank in the CM and PBM of a single query (Figure 1). The regret is reported from 1k steps because we aggregate the per-step regret in batches of 1k steps. We observe that CascadeKL-UCB learns $R^*$ quickly in the CM, but may have a linear regret in the PBM. BatchRank learns $R^*$ in both click models, but has a much higher regret than CascadeKL-UCB in the CM. BubbleRank0 can also learn $R^*$, but has a higher regret than BatchRank. This is expected since BubbleRank is more conservative, because it can only learn better lists by swapping neighboring items in the base list. BubbleRankKL has a lower regret than BatchRank; and can learn $R^*$ in both the CM and PBM, unlike CascadeKL-UCB. This supports our hypothesis that BatchRank can exploit a good initialization very efficiently.

In the second experiment, we compare BubbleRank to CascadeKL-UCB and BatchRank in the CMs and PBMs of all 60 queries (Figure 2). We observe similar trends to those in Figure 1. Note that

Figure 1: The $n$-step regret of BubbleRank0 (green), BubbleRankKL (red), CascadeKL-UCB (cyan) and BatchRank (blue) in the CM and PBM of query 82523 in up to 10 million steps. The shaded regions represent standard errors of our estimates.

Figure 2: The $n$-step regret of BubbleRank0 (green), BubbleRankKL (red), CascadeKL-UCB (cyan) and BatchRank (blue) in the CM and PBM in up to 10 million steps. The results are averaged over all 60 queries. The shaded regions represent standard errors of our estimates.
We observe that the regret of BubbleRank another related topic are conservative bandits [32, 13], where the objective is to improve upon an warm start [28]. But they are limited to small action sets, and thus unsuitable for ranking. work are contextual bandits [17, 1], which deal with a broader class of models and can address the parameterized online LTR methods try to learn the best ranker in a parameterized class of rankers [33, 9]. These methods are not guaranteed to learn optimal lists in all queries. In addition, the class of rankers is limited to linear rankers, which tend to perform poorly in practice. Related to this line of work are contextual bandits [17, 11], which deal with a broader class of models and can address the warm start [28]. But they are limited to small action sets, and thus unsuitable for ranking.

Another related topic are conservative bandits [32, 13], where the objective is to improve upon an existing policy without suffering a much higher loss up to any time step. These approaches are not practical for our setting because they require the action space to be small. Actions in our problem are ranked lists, and their number is exponential in $K$.

Figure 3: Regret of BubbleRank as a function of the number of swaps, $|V_0|$, and $\chi_{\min}$. In the right plot, the purple, red, green, orange and blue colors represent $\chi_{\min}$ equals $0.5, 0.5^2, 0.5^3, 0.5^4$, and $0.5^5$, respectively.

the regret in Figure 2 is reported on a log scale, and that the improvements of BubbleRankKL over BatchRank are quite significant. In the CM at 10 million steps, the regret of BubbleRankKL and BatchRank are $541 \pm 33$ and $1475 \pm 39$, respectively. So BubbleRankKL has $63\%$ lower regret. In the PBM at 10 million steps, the regret of BubbleRankKL and BatchRank are $4409 \pm 218$ and $6833 \pm 179$, respectively. So BubbleRankKL has $35\%$ lower regret.

In the third experiment, we study how the number of incorrectly-ordered item pairs in the initial list $R_0, |V_0|$, impacts the regret of BubbleRank. We choose 10 random initial lists in each of our 60 queries and plot the regret of BubbleRank as a function of $|V_0|$. Our results are shown in Figure 3. We observe that the regret of BubbleRank is linear in $|V_0|$ in both the CM and PBM, which is the same dependence as in our regret bound (Theorem 1).

In the last experiment, we study the impact of the minimum examination probability $\chi_{\min}$ on the regret of BubbleRank. We experiment with a synthetic PBM with 10 items, which is parameterized by $\alpha = (0.9, 0.5, \ldots, 0.5)$ and $\chi = (0.9, \ldots, 0.9, 0.5^i, 0.5^i)$ for $i \geq 1$. The most attractive item is placed at the last position in $R_0$, $R_0 = (2, \ldots, K - 1, 1)$. Since this position is examined with probability $0.5^i$, we expect the regret to double when $i$ increases by one. We experiment with $i \in [5]$ in Figure 3 and observe this trend in one million steps. This confirms that the dependence on $1/\chi_{\min}$ in Theorem 1 is generally unavoidable.

6 Related Work

Ranked bandits [25, 27] are one of the earliest approaches to online learning to rank. The key idea in ranked bandits is to model each position in the recommended list as an individual bandit problem, which is solved by an adversarial algorithm [2], because clicks at lower positions are affected by the choices at higher positions. Therefore, their regret bounds are $O(\sqrt{n})$ and instance-independent.

Online LTR in specific click models was recently studied in multiple papers [14, 6, 15, 12, 36, 18, 16]. In all of these papers, the attraction of items is estimated from clicks based on the structure of the click model. The algorithms do not have guarantees beyond their specific click model.

A complementary approach to the methods discussed in this paper involves de-biasing training data for an offline LTR model either through the use of randomization [30] or by training a click model [31]. These methods lack guarantees of convergence to the optimal list, but they can provide a good starting list for methods like BubbleRank.

Parameterized online LTR methods try to learn the best ranker in a parameterized class of rankers [33, 9]. These methods are not guaranteed to learn optimal lists in all queries. In addition, the class of rankers is limited to linear rankers, which tend to perform poorly in practice. Related to this line of work are contextual bandits [17, 11], which deal with a broader class of models and can address the warm start [28]. But they are limited to small action sets, and thus unsuitable for ranking.

Another related topic are conservative bandits [32, 13], where the objective is to improve upon an existing policy without suffering a much higher loss up to any time step. These approaches are not practical for our setting because they require the action space to be small. Actions in our problem are ranked lists, and their number is exponential in $K$. 

8
7 Conclusions

In this paper, we fill a gap in the LTR literature by proposing BubbleRank, a re-ranking algorithm that gradually improves an initial base list, which is provided by some offline LTR method. The improvements are learned from small perturbations of base lists, which are unlikely to degrade user experience greatly. We prove a gap-dependent upper bound on the regret of BubbleRank and evaluate it on a large-scale click dataset from a commercial search engine.

We leave open several questions of interest. For instance, our paper studies BubbleRank only in the setting of re-ranking. We believe that BubbleRank can be extended to the setting where the number of items is larger than the number of positions. Our general topic of interest are exploration schemes that are more conservative than those of the existing online LTR methods. These methods are not very practical because they can explore highly irrelevant items at frequently examined positions.

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A Technical Lemmas

Lemma 2. Let $\mathcal{R}$ be any list over $[K]$. Let

$$\Delta(\mathcal{R}) = \sum_{k=1}^{K-1} \mathbb{I}\{\alpha(\mathcal{R}(k+1)) - \alpha(\mathcal{R}(k)) > 0\} (\alpha(\mathcal{R}(k+1)) - \alpha(\mathcal{R}(k)))$$

be the attraction gap of list $\mathcal{R}$. Then the expected regret of $\mathcal{R}$ is bounded as

$$\sum_{k=1}^{K} (\chi(\mathcal{R}^*, k)\alpha(k) - \chi(\mathcal{R}, k)\alpha(\mathcal{R}(k))) \leq K\chi_{\max}\Delta(\mathcal{R}) .$$

Proof. Fix position $k \in [K]$. Then

$$\chi(\mathcal{R}^*, k)\alpha(k) - \chi(\mathcal{R}, k)\alpha(\mathcal{R}(k)) \leq \chi(\mathcal{R}^*, k)(\alpha(k) - \alpha(\mathcal{R}(k)))$$

$$\leq \chi_{\max}(\alpha(k) - \alpha(\mathcal{R}(k))) ,$$

where the first inequality follows from the fact that the examination probability of any position is the lowest in the optimal list (Assumption A5) and the second inequality follows from the definition of $\chi_{\max}$. In the rest of the proof, we bound $\alpha(k) - \alpha(\mathcal{R}(k))$. We consider three cases.

First, let $\alpha(\mathcal{R}(k)) \geq \alpha(k)$. Then $\alpha(k) - \alpha(\mathcal{R}(k)) \leq 0$ and bounded by $\Delta(\mathcal{R})$.

Second, let $\alpha(\mathcal{R}(k)) < \alpha(k)$ and $\pi(k) > k$, where $\pi(k)$ is the position of item $k$ in list $\mathcal{R}$. Then

$$\alpha(k) - \alpha(\mathcal{R}(k)) = \alpha(\mathcal{R}(\pi(k))) - \alpha(\mathcal{R}(k))$$

$$\leq \sum_{i=k}^{\pi(k)-1} \mathbb{I}\{\alpha(\mathcal{R}(i+1)) - \alpha(\mathcal{R}(i)) > 0\} (\alpha(\mathcal{R}(i+1)) - \alpha(\mathcal{R}(i))) .$$

From the definition of $\Delta(\mathcal{R})$, this quantity is bounded by $\Delta(\mathcal{R})$.

Finally, let $\alpha(\mathcal{R}(k)) < \alpha(k)$ and $\pi(k) < k$. This implies that there exists an item at a lower position than $k$, $j > k$, such that $\alpha(\mathcal{R}(j)) \geq \alpha(k)$. Then

$$\alpha(k) - \alpha(\mathcal{R}(k)) \leq \alpha(\mathcal{R}(j)) - \alpha(\mathcal{R}(k))$$

$$\leq \sum_{i=k}^{j-1} \mathbb{I}\{\alpha(\mathcal{R}(i+1)) - \alpha(\mathcal{R}(i)) > 0\} (\alpha(\mathcal{R}(i+1)) - \alpha(\mathcal{R}(i))) .$$

From the definition of $\Delta(\mathcal{R})$, this quantity is bounded by $\Delta(\mathcal{R})$. This concludes the proof.

Lemma 3. Let

$$\mathcal{P}_t = \{(i, j) \in [K]^2 : i < j, \left| \mathcal{R}_{t-1}^{-1}(i) - \mathcal{R}_{t-1}^{-1}(j) \right| = 1, s_{t-1}(i, j) \leq 2\sqrt{m_{t-1}(i, j) \log(1/\delta)} \}$$

be the set of potentially randomized item pairs at time $t$ and $\Delta_t = \max_{\mathcal{R}_t} \Delta(\mathcal{R}_t)$ be the maximum attraction gap of any list $\mathcal{R}_t$, where $\Delta(\mathcal{R}_t)$ is defined in (5). Then on event $\mathcal{E}$ in Lemma 8

$$\Delta_t \leq 3 \sum_{i=1}^{K} \sum_{j=i+1}^{K} \mathbb{I}\{(i, j) \in \mathcal{P}_t\} (\alpha(i) - \alpha(j))$$

holds at any time $t \in [n]$.

Proof. Fix list $\mathcal{R}_t$ and position $k \in [K-1]$. Let $i', i, j, j'$ be items at positions $k-1, \ldots, k+2$ in $\mathcal{R}_t$. If $k = 1$, let $i' = i$; and if $k = K-1$, let $j' = j$. We consider two cases.

First, suppose that the permutation at time $t$ is such that $i$ and $j$ could be swapped. Then

$$\alpha(\mathcal{R}_t^{-1}(k+1)) - \alpha(\mathcal{R}_t^{-1}(k)) \leq$$

$$\mathbb{I}\{\{\min\{i, j\}, \max\{i, j\}\} \in \mathcal{P}_t\} (\alpha(\min\{i, j\}) - \alpha(\max\{i, j\}))$$
The proof has two parts. First, suppose that

This implies that

Therefore, for any position

Let

Lemma 4. Let \( P_t \) be defined as in Lemma 2. Then on event \( E \) in Lemma 8

holds at any time \( t \in [n] \).

Proof. The claim follows directly from chaining Lemmas 2 and 3.

Lemma 5. Let \( P_t \) be defined as in Lemma 2. \( P = \bigcup_{t=1}^{n} P_t \), and \( V_0 \) be defined as in 4. Then on event \( E \) in Lemma 8

\[ |P| \leq K - 1 + 2|V_0|. \]

Proof. From the design of BubbleRank, \( |P_t| = K - 1 \). The set of randomized item pairs grows only if the base list in BubbleRank changes. When this happens, the number of incorrectly-ordered item pairs decreases by one, on event \( E \), and the set of randomized item pairs increases by at most two pairs. This event occurs at most \( |V_0| \) times. This concludes our proof.

Lemma 6. For any items \( i \) and \( j \) such that \( i < j \), \( s_n(i, j) \leq 15 \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \log(1/\delta) \) on event \( E \) in Lemma 8.

Proof. Let \( s_t = s_t(i, j) \) and \( n_t = n_t(i, j) \) for any \( t \in [n] \).

The proof has two parts. First, suppose that \( s_t \leq 2\sqrt{n_t \log(1/\delta)} \) holds at all times \( t \in [n] \). Then from this assumption and on event \( E \) in Lemma 8

This implies that

\[ n_t \leq \left( \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \right)^2 \log(1/\delta) \]
at any time \( t \), and in turn that
\[
s_t \leq 2\sqrt{n_t \log(1/\delta)} \leq 8 \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \log(1/\delta)
\]
at any time \( t \). Our claim follows from setting \( t = n \).

On the other hand, suppose that \( s_t \leq 2\sqrt{n_t \log(1/\delta)} \) does not hold at all times \( t \in [n] \). Let \( \tau \) be the first time when \( s_\tau > 2\sqrt{n_\tau \log(1/\delta)} \). Then from the definition of \( \tau \) and on event \( F \) in Lemma 8,
\[
\frac{\alpha(i) - \alpha(j)}{\alpha(i) + \alpha(j)} n_\tau - 2\sqrt{n_\tau \log(1/\delta)} \leq s_\tau \leq s_{\tau - 1} + 1 \leq 2\sqrt{n_\tau \log(1/\delta)} + 1 \leq 3\sqrt{n_\tau \log(1/\delta)},
\]
where the last inequality holds for any \( \delta \leq 1/e \). This implies that
\[
n_\tau \leq \left\lfloor 5 \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \right\rfloor^2 \log(1/\delta),
\]
and in turn that
\[
s_\tau \leq 3\sqrt{n_\tau \log(1/\delta)} \leq 15 \frac{\alpha(i) + \alpha(j)}{\alpha(i) - \alpha(j)} \log(1/\delta).
\]

Now note that \( s_t = s_\tau \) for any \( t > \tau \), from the design of BubblesRank. This concludes our proof. \( \Box \)

For some \( F_t = \sigma(\mathcal{R}_1, c_1, \ldots, \mathcal{R}_t, c_t) \)-measurable event \( A \), let \( \mathbb{P}_t(A) = \mathbb{P}(A \mid F_t) \) be the conditional probability of \( A \) given history \( \mathcal{R}_1, c_1, \ldots, \mathcal{R}_t, c_t \). Let the corresponding conditional expectation operator be \( \mathbb{E}_t[\cdot] \). Note that \( \mathcal{R}_t \) is \( F_{t-1} \)-measurable.

**Lemma 7.** Let \( i, j \in [K] \) be any items at consecutive positions in \( \mathcal{R}_\tau \) and
\[
z = c_i(\mathcal{R}_\tau^{-1}(i)) - c_i(\mathcal{R}_\tau^{-1}(j)).
\]
Then, on the event that \( i \) and \( j \) are subject to randomization at time \( t \),
\[
\mathbb{E}_{t-1}[z \mid z \neq 0] \geq \frac{\alpha(i) - \alpha(j)}{\alpha(i) + \alpha(j)}
\]
when \( \alpha(i) > \alpha(j) \), and \( \mathbb{E}_{t-1}[-z \mid z \neq 0] \leq 0 \) when \( \alpha(i) < \alpha(j) \).

**Proof.** The first claim is proved as follows. From the definition of expectation and \( z \in \{-1, 0, 1\} \),
\[
\mathbb{E}_{t-1}[z \mid z \neq 0] = \mathbb{P}_{t-1}(z = 1 \mid z \neq 0) - \mathbb{P}_{t-1}(z = -1 \mid z \neq 0) = \frac{\mathbb{P}_{t-1}(z = 1, z \neq 0) - \mathbb{P}_{t-1}(z = -1, z \neq 0)}{\mathbb{P}_{t-1}(z \neq 0)} = \frac{\mathbb{P}_{t-1}(z = 1) - \mathbb{P}_{t-1}(z = -1)}{\mathbb{P}_{t-1}(z \neq 0)} = \frac{\mathbb{E}_{t-1}[z]}{\mathbb{P}_{t-1}(z \neq 0)},
\]
where the third equality follows from \( z = 1 \implies z \neq 0 \) and \( z = -1 \implies z \neq 0 \).

Let \( \chi_i = \mathbb{E}_{t-1} \left[ \chi(\mathcal{R}_t, \mathcal{R}_\tau^{-1}(i)) \right] \) and \( \chi_j = \mathbb{E}_{t-1} \left[ \chi(\mathcal{R}_t, \mathcal{R}_\tau^{-1}(j)) \right] \) denote the average examination probabilities of the positions with items \( i \) and \( j \), respectively, in \( \mathcal{R}_\tau \); and consider the event that \( i \) and \( j \) are subject to randomization at time \( t \). By Assumption \( A2 \) the values of \( \chi_i \) and \( \chi_j \) do not depend on the randomization of other parts of \( \mathcal{R}_t \), only on the positions where \( i \) and \( j \) are. Then \( \chi_i \geq \chi_j \); from \( \alpha(i) > \alpha(j) \) and Assumption \( A4 \). Based on this fact, \( \mathbb{E}_{t-1}[z] \) is bounded from below as
\[
\mathbb{E}_{t-1}[z] = \chi_i \alpha(i) - \chi_j \alpha(j) \geq \chi_i (\alpha(i) - \alpha(j)),
\]
where the inequality is from \( \chi_i \geq \chi_j \). Moreover, \( \mathbb{P}_{t-1}(z \neq 0) \) is bounded from above as
\[
\mathbb{P}_{t-1}(z \neq 0) = \mathbb{P}_{t-1}(z = 1) + \mathbb{P}_{t-1}(z = -1) \leq \chi_i \alpha(i) + \chi_j \alpha(j) \leq \chi_i (\alpha(i) + \alpha(j)),
\]
where the first inequality is from \( \mathbb{P}_{t-1}(z = 1) \leq \chi_i \alpha(i) \) and \( \mathbb{P}_{t-1}(z = -1) \leq \chi_j \alpha(j) \), and the second inequality is from \( \chi_i \geq \chi_j \).

Finally, we chain all above inequalities and get our first claim. The second claim follows from the observation that \( \mathbb{E}_{t-1}[-z \mid z \neq 0] = -\mathbb{E}_{t-1}[z \mid z \neq 0] \). \( \Box \)
Lemma 8. Let
\[ S_1 = \{(i, j) \in [K]^2 : i < j\} , \quad S_2 = \{(i, j) \in [K]^2 : i > j\} . \]

Let
\[ E_{t,1} = \left\{ \forall (i, j) \in S_1 : \frac{\alpha(i) - \alpha(j)}{\alpha(i) + \alpha(j)} n_t(i, j) - 2 \sqrt{n_t(i, j) \log(1/\delta)} \leq s_t(i, j) \right\} , \]
\[ E_{t,2} = \left\{ \forall (i, j) \in S_2 : s_t(i, j) \leq 2 \sqrt{n_t(i, j) \log(1/\delta)} \right\} . \]

Let \( E = \bigcap_{t \in [n]} (E_{t,1} \cap E_{t,2}) \) and \( \bar{E} \) be the complement of \( E \). Then \( P(\bar{E}) \leq \delta^{1/2} K^2 n \).

Proof. First, we bound \( P(\bar{E}_{t,1}) \). Fix \((i, j) \in S_1, t \in [n] \), and \((n_t(i, j))_{t=1}^n\). Let
\[ \tau(m) = \min \{ \ell \in [t] : n_t(i, j) = m \} \]
for \( m \in [n_t(i, j)] \). In plain English, \( \tau(m) \) is the time of observing item pair \((i, j)\) for the \( m \)-th time.

Let \( z_\ell = c_\ell(\mathcal{R}_1^{-1}(i)) - c_\ell(\mathcal{R}_1^{-1}(j)) \). Since \((n_t(i, j))_{t=1}^n\) is fixed, note that \( z_\ell \neq 0 \) if \( \ell = \tau(m) \) for some \( m \in [n_t(i, j)] \).

Let
\[ X_\ell = \sum_{\ell' = 1}^\ell E_{t'(\ell' - 1) \mid z_{t'(\ell')} \neq 0} - s_{t'(\ell')}(i, j) \]
for \( \ell \in [n_t(i, j)] \) and \( X_0 = 0 \). Then \((X_\ell)_{\ell=1}^{n_t(i, j)}\) is a martingale, because

\[
X_\ell - X_{\ell-1} = E_{t'(\ell-1) \mid z_{t'(\ell')} \neq 0} \left[ z_{t'(\ell')} - (s_{t'(\ell')}(i, j) - s_{t'(\ell-1)}(i, j)) \right] = E_{t'(\ell-1) \mid z_{t'(\ell')} \neq 0} \left[ z_{t'(\ell')} - s_{t'(\ell')} \right],
\]

where the last equality follows from the definition of \( s_{t'(\ell')}(i, j) - s_{t'(\ell-1)}(i, j) \). Now we apply the Azuma-Hoeffding inequality and get that
\[
P \left( X_{n_t(i, j)} - X_0 \geq 2 \sqrt{n_t(i, j) \log(1/\delta)} \right) \leq \delta^{1/2}.
\]

Moreover, from the definitions of \( X_0 \) and \( X_{n_t(i, j)} \), and by Lemma 7, we have that
\[
\delta^{1/2} \geq P \left( X_{n_t(i, j)} - X_0 \geq 2 \sqrt{n_t(i, j) \log(1/\delta)} \right)
\]
\[
= P \left( \sum_{\ell = 1}^{n_t(i, j)} E_{t'(\ell-1) \mid z_{t'(\ell')} \neq 0} \left[ z_{t'(\ell')} \right] - s_t(i, j) \geq 2 \sqrt{n_t(i, j) \log(1/\delta)} \right)
\]
\[
\geq P \left( \frac{\alpha(i) - \alpha(j)}{\alpha(i) + \alpha(j)} n_t(i, j) - s_t(i, j) \geq 2 \sqrt{n_t(i, j) \log(1/\delta)} \right)
\]
\[
= P \left( \frac{\alpha(i) - \alpha(j)}{\alpha(i) + \alpha(j)} n_t(i, j) - 2 \sqrt{n_t(i, j) \log(1/\delta)} \geq s_t(i, j) \right). \]

The above inequality holds for any \((n_t(i, j))_{t=1}^n\), and thus in expectation over \((n_t(i, j))_{t=1}^n\). From the definition of \( E_{t,1} \) and the union bound, we have that
\[
P(\bar{E}_{t,1}) \leq \frac{1}{2} \delta^{1/2} K(K - 1).
\]

The claim that
\[
P(\bar{E}_{t,2}) \leq \frac{1}{2} \delta^{1/2} K(K - 1).
\]

is proved similarly, except that we use \( E_{t'(\ell-1)} \mid z_{t'(\ell')} \neq 0 \) \leq 0. From the definition of \( \bar{E} \) and the union bound,
\[
P(\bar{E}) \leq \sum_{t=1}^{n} P(\bar{E}_{t,1}) + \sum_{t=1}^{n} P(\bar{E}_{t,2}) \leq \delta^{1/2} K^2 n.
\]

This completes our proof. \( \square \)