Geometric properties of special orthogonal representations associated to exceptional Lie superalgebras

Philippe Meyer

Abstract

From an octonion algebra $\mathbb{O}$ over a field $k$ of characteristic not two or three, we show that the fundamental representation $\text{Im}(\mathbb{O})$ of the derivation algebra $\text{Der}(\mathbb{O})$ and the spinor representation $\mathbb{O}$ of $\mathfrak{so}(\text{Im}(\mathbb{O}))$ are special orthogonal representations. They have particular geometric properties coming from their similarities with binary cubics and we show that the covariants of these representations and their Mathews identities are related to the Fano plane and the affine space $(\mathbb{Z}_2)^3$. This also permits to give constructions of exceptional Lie superalgebras.

1 Introduction

The space of binary cubics, a symplectic representation of the Lie algebra $\mathfrak{sl}(2,k)$, has particular symplectic properties [Eis44], [SS12]. It admits three covariants, among the Hessien and the discriminant, satisfying remarkable geometric identities [Mat11]. This representation is an example of a larger class of representations sharing these properties: the special $\epsilon$-orthogonal representations of colour Lie algebras [SS15], [Mey19]. The terminology special comes from their role in symplectic geometry [CS09]. A special $\epsilon$-orthogonal representation $V$ of a colour Lie algebra $\mathfrak{g}$ can be extended to define a colour Lie algebra of the form

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{sl}(2,k) \otimes V \otimes k^2.$$ 

In this way, special symplectic representations of Lie algebras give rise to Lie algebras and special orthogonal representations of Lie algebras give rise to Lie superalgebras.

In this paper, from an octonion algebra $\mathbb{O}$ over $k$, we show that

- a one parameter family of 4-dimensional representations of $\mathfrak{sl}(2,k) \times \mathfrak{sl}(2,k)$;
- the 7-dimensional fundamental representation $\text{Im}(\mathbb{O})$ of the Lie algebra $\text{Der}(\mathbb{O})$;
- the 8-dimensional spinor representation $\mathbb{O}$ of the Lie algebra $\mathfrak{so}(\text{Im}(\mathbb{O}))$

are special orthogonal representations and give rise to exceptional Lie superalgebras of type $D(2,1;\alpha)$, $G_3$ and $F_4$ (in the Kac notation [Kac75]). This is similar to various constructions from Sudbery [Sud83], Kamiya and Okubo [KO03] and Elduque [Eld04].

We explicitly compute the covariants of these representations. In particular, we give formulae of the moment maps of $\text{Im}(\mathbb{O})$ and $\mathbb{O}$ and we show that the trilinear covariant of $\text{Im}(\mathbb{O})$ is, up to a constant, the associator. The quadrilinear covariant of $\text{Im}(\mathbb{O})$ admits a decomposition into a sum of 7 decomposable forms which naturally correspond to the 7 lines of the Fano plane and the two maps of the first Mathews identity are, up to constants, the Hodge duals of the cross-product on $\text{Im}(\mathbb{O})$. Then we give a decomposition of the quadrilinear covariant of $\mathbb{O}$ into a sum of 14 decomposable forms which naturally correspond to the 14 affine planes of the affine space $(\mathbb{Z}_2)^3$. The two maps of the first Mathews identity are, up to a constant, the Hodge duals of the trilinear covariant of $\mathbb{O}$ and the two maps of the second Mathews identity are, up to constants, the Hodge duals of the moment map of $\mathbb{O}$.

For special orthogonal representations associated to basic classical Lie superalgebras and interpretation of their covariants, see the appendix of [Mey19].

Key words: exceptional Lie superalgebra · covariant · spinor representation · octonions · Mathews identities.

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Notation

Let \( k \) be a field of characteristic not two or three.

For a finite-dimensional quadratic vector space \((V, q)\) and \( i \in \mathbb{N} \) such that \( i < \text{char}(k) \) if \( 0 < \text{char}(k) \), we denote by \( \eta : \Lambda^i(V) \to \Lambda^i(V)^* \) the canonical isomorphism given by the determinant and we consider the quadratic form \( q_\Lambda \) (resp. \( q_{\Lambda^*} \)) and the symmetric bilinear form \( B_\Lambda \) associated by polarisation (resp. \( B_{\Lambda^*} \)) on \( \Lambda^i(V) \) (resp. \( \Lambda^i(V)^* \)) given by \( \eta \).

If \( \{e_i\} \) is a basis of \( V \), we denote \( e_{i_1} \wedge \ldots \wedge e_{i_n} \) by \( e_{i_1 \ldots i_n} \).

2 Lie superalgebras from special orthogonal representations

In this section we explain how to construct a quadratic Lie superalgebra from an orthogonal representation of a quadratic Lie algebra, for details and proofs see [Mey19]. Let \((g, B_g)\) be a finite-dimensional quadratic Lie algebra, let \((V, (\ , \ ))\) be a finite-dimensional quadratic vector space and let \( \rho : g \to \mathfrak{so}(V, (\ , \ )) \) be an orthogonal representation of \( g \).

The moment map of the representation \( \rho : g \to \mathfrak{so}(V, (\ , \ )) \) is the \( g \)-equivariant alternating map \( \mu \in \text{Alt}_2(V, g) \) satisfying

\[
B_g(x, \mu(v, w)) = (\rho(x)(v), w) \quad \forall x \in g, \forall v, w \in V.
\]

The standard example is the moment map of the fundamental representation of \( \mathfrak{so}(V, (\ , \ )) \):

**Example 2.1.** Suppose that \( g = \mathfrak{so}(V, (\ , \ )) \) and \( B_g(f, g) = -\frac{1}{2} \text{Tr}(fg) \) for all \( f, g \in \mathfrak{so}(V, (\ , \ )) \). The corresponding moment map \( \mu_{\text{can}} \in \text{Alt}_2(V, \mathfrak{so}(V, (\ , \ ))) \) satisfies

\[
\mu_{\text{can}}(u, v)(w) = (u, w)v - (v, w)u \quad \forall u, v, w \in V,
\]

and is a \( g \)-equivariant isomorphism between \( \Lambda^2(V) \) and \( \mathfrak{so}(V, (\ , \ )) \).

We now define a particular class of orthogonal representations of quadratic Lie algebras:

**Definition 2.2.** The representation \( \rho : g \to \mathfrak{so}(V, (\ , \ )) \) is said to be special orthogonal if

\[
\mu(u, v)(w) + \mu(u, w)(v) = (u, v)w + (u, w)v - 2(v, w)u \quad \forall u, v, w \in V.
\]

Special orthogonal representations can be extended to define Lie superalgebras as follows:

**Theorem 2.3.** Let \( \rho : g \to \mathfrak{so}(V, (\ , \ )) \) be a finite-dimensional orthogonal representation of a finite-dimensional quadratic Lie algebra \((g, B_g)\) and let \( \mathfrak{sl}(2, k) \to \mathfrak{sp}(k^2, \omega) \) be the symplectic fundamental representation of the quadratic Lie algebra \((\mathfrak{sl}(2, k), B_2)\) where \( \omega \) is the canonical symplectic form on \( k^2 \) and where \( B_2(f, g) = \frac{1}{2} \text{Tr}(fg) \) for all \( f, g \in \mathfrak{sl}(2, k) \). Let \( \tilde{g} \) be the super vector space defined by

\[
\tilde{g} := g \oplus \mathfrak{sl}(2, k) \oplus V \otimes k^2,
\]

and let \( B_\tilde{g} := B_g \perp B_\omega \perp (\ , \ ) \otimes \omega. \) Then \((\tilde{g}, B_\tilde{g}, \{\ , \ }\)) is a quadratic Lie superalgebra extending the bracket of \( g \oplus \mathfrak{sl}(2, k) \) and the action of \( g \oplus \mathfrak{sl}(2, k) \) on \( V \otimes k^2 \) if and only if \( \rho : g \to \mathfrak{so}(V, (\ , \ )) \) is a special orthogonal representation.

In addition to the moment map, a trilinear and a quadrilinear alternating multilinear map can be naturally associated to a special orthogonal representation:
Definition 2.4. We define the multilinear alternating maps \( \psi \in \text{Alt}_3(V,V) \) and \( Q \in \text{Alt}_4(V,k) \) as follows:

\[
\psi(v_1, v_2, v_3) = \mu(v_1, v_2)(v_3) + \mu(v_3, v_1)(v_2) + \mu(v_2, v_3)(v_1), \\
Q(v_1, v_2, v_3, v_4) = (v_1, \psi(v_2, v_3, v_4)) - (v_4, \psi(v_1, v_2, v_3)) + (v_3, \psi(v_4, v_1, v_2)) - (v_2, \psi(v_3, v_4, v_1))
\]

for all \( v_1, v_2, v_3, v_4 \in V \). The maps \( \mu, \psi \) and \( Q \) are called the covariants of \( V \).

We have the following formulae:

Proposition 2.5. If \( \rho : g \rightarrow \mathfrak{so}(V,(\ ,\ )) \) is special orthogonal, then we have

\[
\psi(v_1, v_2, v_3) = 3(\mu(v_1, v_2)(v_3) - \mu_{\text{can}}(v_1, v_2)(v_3)), \\
Q(v_1, v_2, v_3, v_4) = 4(v_1, \psi(v_2, v_3, v_4)),
\]

for all \( v_1, v_2, v_3, v_4 \in V \).

For vector spaces \( E, F, G, H \), the exterior product \( f \wedge g \in \text{Alt}_{p+q}(E, H) \) of \( f \in \text{Alt}_p(E, F) \) and \( g \in \text{Alt}_q(E, G) \) relative to a bilinear map \( \phi : F \times G \rightarrow H \) is defined by

\[
f \wedge g(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in S([1,p],[p+1,p+q])} \text{sgn}(\sigma)\phi(f(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), g(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}))
\]

where the sum is over the \((p,q)\)-shuffle permutations in \( S_{p+q} \). If \( \phi \) is implicit, then we denote \( f \wedge g \) by \( f \wedge g \).

The composition \( f \circ g \in \text{Alt}_{pq}(G, F) \) of \( f \in \text{Alt}_p(E, F) \) and \( g \in \text{Alt}_q(G, E) \) is defined by

\[
f \circ g(v_1, \ldots, v_{pq}) = \sum_{\sigma \in S([1,q],\ldots,[p(q-1)+1,pq])} \text{sgn}(\sigma)f(g(v_{\sigma(1)}, \ldots, v_{\sigma(q)}), \ldots, g(v_{\sigma(p(q-1)+1)}, \ldots, v_{\sigma(pq)}))
\]

where the sum is over the \((q, \ldots, q)\)-shuffle permutations in \( S_{pq} \).

Covariants of special orthogonal representations satisfy to the following Mathews identities:

Theorem 2.6. Let \( \rho : g \rightarrow \mathfrak{so}(V,(\ ,\ )) \) be a finite-dimensional special orthogonal representation of a finite-dimensional quadratic Lie algebra and let \( \mu \in \text{Alt}_2(V,g) \), \( \psi \in \text{Alt}_3(V,V) \) and \( Q \in \text{Alt}_4(V,k) \) be its covariants. We have the following identities:

\[
a) \quad \mu \wedge \rho \psi = -\frac{3}{2}Q \wedge Id_V \quad \in \text{Alt}_5(V,V), \\
b) \quad \mu \circ \psi = 3Q \wedge \mu \quad \in \text{Alt}_6(V,g), \\
c) \quad \psi \circ \psi = -\frac{27}{2}Q \wedge Q \wedge Id_V \quad \in \text{Alt}_9(V,V), \\
d) \quad Q \circ \psi = -54Q \wedge Q \quad \in \text{Alt}_{12}(V,k).
\]

3 A one-parameter family of special orthogonal representations of \( \mathfrak{sl}(2, k) \times \mathfrak{sl}(2, k) \)

In this section we show that with respect to a one parameter family of invariant quadratic forms on \( \mathfrak{sl}(2, k) \times \mathfrak{sl}(2, k) \), the tensor product of the two fundamental representations is a special orthogonal representation.

Let \( (V, \omega_V) \) and \( (W, \omega_W) \) be two-dimensional symplectic vector spaces. The vector space \( V \otimes W \) is quadratic for the symmetric bilinear form \( \omega_V \otimes \omega_W \) given by

\[
\omega_V \otimes \omega_W(v_1 \otimes w_1, v_2 \otimes w_2) = -\omega_V(v_1, v_2)\omega_W(w_1, w_2) \quad \forall v_1, v_2 \in V, \forall w_1, w_2 \in W
\]
Consider the bilinear form $K_V$ (resp. $K_W$) on $\mathfrak{sp}(V, \omega_V)$ (resp. $\mathfrak{sp}(W, \omega_W)$) defined by $K_V(f, g) = \frac{1}{2} Tr(fg)$ (resp. $K_W(f, g) = \frac{1}{2} Tr(fg)$) for all $f, g \in \mathfrak{sp}(V, \omega_V)$ (resp. $\mathfrak{sp}(W, \omega_W)$). For $\alpha, \beta \in k^*$, we now consider the orthogonal representation

$$\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)$$

of the quadratic Lie algebra $(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W), \frac{\alpha}{\beta} K_V \perp \frac{\beta}{\alpha} K_W)$. Its moment map

$$\mu_{\alpha, \beta} : 	ext{Alt}_2(V \otimes W, \mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W))$$

satisfies

$$\mu_{\alpha, \beta}(v_1 \otimes w_1, v_2 \otimes w_2) = -(\alpha \mu_V(v_1, v_2) \omega_V(w_1, w_2) + \beta \mu_W(w_1, w_2) \omega_V(v_1, v_2)) \quad \forall v_1, v_2 \in V, \forall w_1, w_2 \in W,$$

where $\mu_i : S^2(V_i) \rightarrow \mathfrak{sp}(V_i, \omega_i)$ is the canonical symmetric moment map given by

$$\mu_i(v_1, v_2)(v_3) = -\omega_i(v_1, v_3)v_2 - \omega_i(v_2, v_3)v_1 \quad \forall v_1, v_2, v_3 \in V_i.
$$

**Proposition 3.1.** The orthogonal representation

$$\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)$$

of the quadratic Lie algebra $(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W), \frac{\alpha}{\beta} K_V \perp \frac{\beta}{\alpha} K_W)$ is a special orthogonal representation if and only if $\alpha + \beta = -1$.

**Proof.** Let $v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3 \in V \otimes W$. We want to know under what conditions on $\alpha$ and $\beta$ do we have

$$\mu_{\alpha, \beta}(v_1 \otimes w_1, v_2 \otimes w_2)(v_3 \otimes w_3) + \mu_{\alpha, \beta}(v_1 \otimes w_1, v_3 \otimes w_3)(v_2 \otimes w_2) = \omega_V \otimes \omega_W(v_1 \otimes w_1, v_2 \otimes w_2)v_3 \otimes w_3$$

$$+ \omega_V \otimes \omega_W(v_1 \otimes w_1, v_3 \otimes w_3)v_2 \otimes w_2 - 2\omega_V \otimes \omega_W(v_2 \otimes w_2, v_3 \otimes w_3)v_1 \otimes w_1. \quad (7)$$

Since $V$ and $W$ are two-dimensional (and after a permutation of $v_1, v_2, v_3$ or $w_1, w_2, w_3$ if necessary) we have $v_3 = av_1 + bv_2$ and $w_3 = cw_1 + dw_2$ where $a, b, c, d \in k$. Hence we have

$$\omega_V \otimes \omega_W(v_1 \otimes w_1, v_2 \otimes w_2)v_3 \otimes w_3 + \omega_V \otimes \omega_W(v_1 \otimes w_1, v_3 \otimes w_3)v_2 \otimes w_2 - 2\omega_V \otimes \omega_W(v_2 \otimes w_2, v_3 \otimes w_3)v_1 \otimes w_1$$

$$= \omega_V(v_1, v_2)\omega_W(v_1, w_2)\left(-av_1 \otimes dw_2 - bv_2 \otimes cw_1 - 2bv_2 \otimes dw_2 + av_1 \otimes cw_1\right).$$

On the other hand we have

$$\mu_{\alpha, \beta}(v_1 \otimes w_1, v_2 \otimes w_2)(v_3 \otimes w_3) + \mu_{\alpha, \beta}(v_1 \otimes w_1, v_3 \otimes w_3)(v_2 \otimes w_2)$$

$$= -(\alpha \mu_V(v_1, v_2)v_3 \otimes w_3) + \beta \omega_V(v_1, v_2)v_3 \otimes w_3 + \alpha \mu_V(v_1, v_2)v_3 \otimes w_3 + \alpha \mu_V(v_1, v_3)v_2 \otimes w_2$$

$$+ \beta \omega_V(v_1, v_3)v_2 \otimes w_2 + \beta \omega_V(v_1, v_3)v_2 \otimes w_2 + \alpha \mu_V(v_1, v_2)v_3 \otimes w_3 + \alpha \mu_V(v_1, v_3)v_2 \otimes w_2$$

$$+ \beta \omega_V(v_1, v_3)v_2 \otimes w_2 + \alpha \mu_V(v_1, v_2)v_3 \otimes w_3 + \alpha \mu_V(v_1, v_3)v_2 \otimes w_2$$

$$+ \beta \omega_V(v_1, v_3)v_2 \otimes w_2 + \alpha \mu_V(v_1, v_2)v_3 \otimes w_3 + \alpha \mu_V(v_1, v_3)v_2 \otimes w_2$$

Hence, Equation (7) is satisfied if and only if $\alpha + \beta = -1$ and so the representation $\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)$ is special orthogonal if and only if $\alpha + \beta = -1.$
Suppose that \(\alpha + \beta = -1\). By the previous proposition and Theorem 2.3 we have a Lie superalgebra \(\tilde{g}_\alpha\) of the form
\[
\tilde{g}_\alpha = \mathfrak{sp}(V, \omega_V) \oplus \mathfrak{sp}(W, \omega_W) \oplus \mathfrak{sl}(2, k) \oplus V \otimes W \otimes k^2.
\]
This a simple Lie superalgebra of type \(D(2,1;\alpha)\) which is an exceptional simple Lie superalgebra if \(\alpha\) is not equal to \(-\frac{1}{2}, -2\) or 1.

**Remark 3.2.**  
\(a)\) In [Ser83], Serganova shows that there are three families of simple real Lie superalgebras which are real forms of \(D(2,1;\alpha)\) (see also [Par86] for a discussion about the real forms of \(D(2,1;\alpha)\)). If \(k = \mathbb{R}\), the family \(\tilde{g}_\alpha\) defined above corresponds to one of these families.

\(b)\) There is a symmetry exchanging \(\alpha\) and \(\beta\). Hence, the special orthogonal representations \(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)\) of the quadratic Lie algebras \(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W), \frac{1}{\alpha}K_V \perp \frac{1}{\alpha}K_W\) and \((\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W), \frac{1}{\alpha}K_V \perp \frac{1}{\alpha}K_W)\) give rise to isomorphic Lie superalgebras \(\tilde{g}_\alpha\) and \(\tilde{g}_{-1-\alpha}\).

\(c)\) There is a singular case when \(\alpha = \beta = -\frac{1}{2}\). The Lie algebra \(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W)\) is isomorphic to \(\mathfrak{so}(W_0, (\ , \ ))\), where \((W_0, (\ , \ ))\) is a four-dimensional hyperbolic vector space, and under this isomorphism, the quadratic form \(\frac{1}{\alpha}K_V \perp \frac{1}{\alpha}K_W\) of \(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W)\) is isometric to the quadratic form \(-\frac{1}{\alpha}\text{Tr}(fg)\) for all \(f, g \in \mathfrak{so}(W_0, (\ , \ ))\). Hence, we have that \(\tilde{g}_{-\frac{1}{2}}\) is isomorphic to \(\mathfrak{osp}(W_0 \oplus W_1, (\ , \ ) \perp \omega)\) where \((W_1, \omega)\) is a two-dimensional symplectic vector space.

We now study the trilinear covariant and the quadrilinear covariant of the special orthogonal representation \(\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)\). Note that the Mathews identities of Theorem 2.6 vanish identically because \(V \otimes W\) is of dimension four.

**Proposition 3.3.** Suppose that \(\alpha + \beta = -1\). The trilinear covariant \(\psi \in \text{Alt}_3(V \otimes W, V \otimes W)\) and the quadrilinear covariant \(Q \in \text{Alt}_4(V \otimes W, k)\) of the special orthogonal representation
\[
\mathfrak{sp}(V, \omega_V) \times \mathfrak{sp}(W, \omega_W) \rightarrow \mathfrak{so}(V \otimes W, \omega_V \otimes \omega_W)
\]
satisfies:
\[
\psi(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3) = 3(2\alpha + 1)\left(\omega_V(v_1, v_3)\omega_V(v_2, w_1) + \omega_V(v_2, v_3)\omega_W(w_1, v_3)w_2 + \omega_V(v_2, v_3)\omega_V(v_1, v_3)\omega_W(w_1, v_3)\right),
\]
\[
Q(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3, v_4 \otimes w_4) = -12(2\alpha + 1)\left(\omega_V(v_1, v_4)\omega_V(v_2, v_3)\omega_W(w_4, w_3)\omega_W(w_1, w_2) + \omega_V(v_3, v_4)\omega_W(v_1, v_3)\omega_W(w_2, w_1)\right),
\]
for all \(v_1, v_2, v_3, v_4 \in V\), \(w_1, w_2, w_3, w_4 \in W\).

**Proof.** Let \(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3 \in V \otimes W\). Since \(V\) and \(W\) are two-dimensional (and after a permutation of \(v_1, v_2, v_3\) or \(w_1, w_2, w_3\) if necessary) we have \(v_3 = av_1 + bw_2\) and \(w_3 = cw_1 + dw_2\) where \(a, b, c, d \in k\). We have
\[
\mu_{\alpha,\beta}(v_1 \otimes w_1, v_2 \otimes w_2)(v_3 \otimes w_3) = (\alpha + \beta)\omega_V(v_1, v_3)\omega_W(w_3, w_2)w_1 + (\alpha + \beta)\omega_V(v_1, v_3)\omega_W(w_3, w_3)w_2 + (\alpha - \beta)\omega_W(v_2, v_3)\omega_W(w_1, w_3)w_2 + (\alpha - \beta)\omega_W(v_2, v_3)\omega_W(w_3, w_3)w_1,
\]
\[
\mu_{\alpha,\beta}(v_2 \otimes w_2, v_3 \otimes w_3)(v_1 \otimes w_1) = (\alpha + \beta)\omega_V(v_2, v_3)\omega_W(w_3, w_2)w_1 + 2\alpha\omega_V(v_3, v_1)\omega_W(w_2, w_3)w_1 + 2\beta\omega_V(v_2, v_3)\omega_W(w_1, w_3)w_2,
\]
\[
\mu_{\alpha,\beta}(v_3 \otimes w_3, v_1 \otimes w_1)(v_2 \otimes w_2) = (\alpha + \beta)\omega_V(v_1, v_3)\omega_W(w_3, w_2)w_1 + 2\alpha\omega_V(v_3, v_1)\omega_W(w_2, w_3)w_1 + 2\beta\omega_V(v_2, v_3)\omega_W(w_1, w_3)w_2.
\]
Hence, summing Equations (8), (9) and (10), we obtain
\[
\psi(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3) = 3(\alpha - \beta)\left(\omega_V(v_1, v_3)\omega_W(w_3, w_2)w_1 + \omega_V(v_2, v_3)\omega_W(w_1, w_3)w_2\right).
\]
The formula for \(Q\) follows by Proposition 2.5. \(\square\)
Remark 3.4. For the singular case $\alpha = -\frac{1}{2}$, we have that the covariants $\psi$ and $Q$ vanish identically. It means that the representation $sp(V,\omega_V) \times sp(W,\omega_W) \rightarrow so(V \otimes W,\omega_V \otimes \omega_W)$ is of $\mathbb{Z}_2$-Lie type in the sense of Kostant [Kos99] and then can be extended to define a Lie algebra structure on $sp(V,\omega_V) \oplus sp(W,\omega_W) \oplus V \otimes W$. This Lie algebra is isomorphic to the orthogonal Lie algebra $so(V \otimes W \oplus L,\omega_V \otimes \omega_W \perp \langle \ , \ \rangle_L)$ where $(L,\langle \ , \ \rangle_L)$ is a one-dimensional quadratic vector space.

4 The fundamental representation of $G_2$ is special orthogonal

In this section, we show that the irreducible 7-dimensional fundamental representation of an exceptional Lie algebra $g$ of type $G_2$ is special orthogonal. To do this we realise $g$ as the derivation algebra of an octonion algebra $\mathcal{O}$ and use octonionic calculations. We first recall some properties of the octonions, for details and proofs see [Sch95] and [SV06].

Let $\mathcal{O}$ be an octonion (or Cayley) algebra over $k$. This is a 8-dimensional unital composition algebra, the conjugation $\bar{}$ satisfies $\bar{q(u)} = u\bar{u}$ for all $u \in \mathcal{O}$, where $\bar{}$ is the norm of $\mathcal{O}$, and we have $\mathcal{O} = k \oplus \text{Im}(\mathcal{O})$, where $\text{Im}(\mathcal{O}) = \{ u \in \mathcal{O} \mid \bar{u} = -u \}$. Denote $B$ the symmetric bilinear form associated by polarisation to $q$. Let $e_1, e_2, e_4 \in \text{Im}(\mathcal{O})$ be such that $B = \{ e_1, e_2, e_1 e_2, e_1 e_4, e_2 e_4, (e_1 e_2) e_4 \}$ is an orthogonal and anisotropic basis of $\text{Im}(\mathcal{O})$ and set $e_3 := e_1 e_2$, $e_5 := e_1 e_4$, $e_6 := e_2 e_4$, $e_7 := (e_1 e_2) e_4$. This basis is related to the Fano plane:

![Fano Plane](image)

in the sense that, for $i \neq j$, the product between $e_i$ and $e_j$ is a multiple of $e_k$ where $k$ is the third point on the line going through $i$ and $j$.

The commutator and the associator are the alternating maps given by:

$$[u,v] = uv - vu,$$

$$(u,v,w) = (uv)w - u(vw)$$

for all $u,v,w \in \mathcal{O}$. The commutator doesn’t define a Lie algebra structure on $\mathcal{O}$ since the Jacobi tensor $J$ satisfies

$$J(u,v,w) = [u, [v,w]] + [v, [w,u]] + [w, [u,v]] = -6(u,v,w) \quad \forall u,v,w \in \mathcal{O}. \quad (11)$$

There is a cross-product on $\mathcal{O}$ defined by

$$u \times v = \frac{1}{2}(\bar{uv} - \bar{uv}) \quad \forall u,v \in \mathcal{O},$$

and we have

$$q(u \times v) = (\bar{u})q(v) - q(u)q(v) = q(u)q(v) - B(u,v)^2 \quad \forall u,v \in \mathcal{O}, \quad (12)$$

$$u \times v = \frac{1}{2}([u,v] = uv + B(u,v) \quad \forall u,v,w \in \text{Im}(\mathcal{O}), \quad (13)$$

$$u \times (v \times w) + v \times (u \times w) = B(v,w)u + B(u,w)v - 2B(u,v)w \quad \forall u,v,w \in \text{Im}(\mathcal{O}). \quad (14)$$

The associative form $\phi$ on $\text{Im}(\mathcal{O})$ is the trilinear alternating form defined by

$$\phi(u,v,w) = B(u,v \times w) \quad \forall u,v,w \in \text{Im}(\mathcal{O}).$$
and we have
\[ \eta^{-1}(\phi) = \frac{1}{q(e_1)q(e_2)} e_{123} + \frac{1}{q(e_1)q(e_2)q(e_4)} e_{167} + \frac{1}{q(e_1)q(e_3)q(e_4)} e_{257} - \frac{1}{q(e_1)q(e_2)q(e_4)} e_{356} \]
\[ + \frac{1}{q(e_1)q(e_4)} e_{145} + \frac{1}{q(e_2)q(e_4)} e_{246} + \frac{1}{q(e_1)q(e_2)q(e_4)} e_{347}. \]
(15)

Let \( \rho : \text{Im}(\mathfrak{O}) \to \text{End}(\mathfrak{O}) \) be the map defined by \( \rho(u)(x) = ux \) for \( u \in \text{Im}(\mathfrak{O}) \) and \( x \in \mathfrak{O} \). We have
\[ \rho(u)^2 = -q(u)\text{Id} \quad \forall u \in \text{Im}(\mathfrak{O}) \]
and so \( \rho \) extends to the Clifford algebra \( C(\text{Im}(\mathfrak{O}), -q) \). The quantisation map \( Q : \Lambda(\text{Im}(\mathfrak{O})) \to C(\text{Im}(\mathfrak{O}), -q) \) is an \( O(\text{Im}(\mathfrak{O}), B) \)-equivariant isomorphism of vector spaces and then we have \( C(\text{Im}(\mathfrak{O}), -q) = \bigoplus_i C^i(\text{Im}(\mathfrak{O}), -q) \)
where \( C^i(\text{Im}(\mathfrak{O}), -q) = Q(\Lambda^i(\text{Im}(\mathfrak{O}))) \). The map \( \mu_{\text{can}} \circ Q^{-1} : C^2(\text{Im}(\mathfrak{O}), -q) \to \mathfrak{so}(\text{Im}(\mathfrak{O}), q) \) is an isomorphism of Lie algebras. Let
\[ \mathfrak{g} := \{ x \in C^2(\text{Im}(\mathfrak{O}), -q) \mid \rho(x)(1) = 0 \}. \]
This is a Lie algebra of type \( G_2 \), the map \( \rho : \mathfrak{g} \to \mathfrak{so}(\text{Im}(\mathfrak{O}), q) \) is its 7-dimensional fundamental representation and \( \rho(\mathfrak{g}) \) is equal to the set of derivations of \( \mathfrak{O} \). Define the ad-invariant quadratic form \( B_\mathfrak{g} \) on \( \mathfrak{g} \) by
\[ B_\mathfrak{g}(x, y) = -\frac{1}{3} \text{Tr}(\rho(x)\rho(y)) \quad \forall x, y \in \mathfrak{g}. \]

**Proposition 4.1.** The moment map \( \mu_{\text{im}} : \Lambda^2(\text{Im}(\mathfrak{O})) \to \mathfrak{g} \) satisfies
\[ \mu_{\text{im}}(u, v)(w) = -\frac{1}{4} (w, [u, v]) + 3(u, v, w) \quad \forall u, v, w \in \text{Im}(\mathfrak{O}) \]

**Proof.** For \( u, v \in \text{Im}(\mathfrak{O}) \), let \( D(u, v) \in \mathfrak{g} \) be such that \( \rho(D(u, v))(x) = [x, [u, v]] + 3(u, v, x) \) for all \( x \in \text{Im}(\mathfrak{O}) \). Let \( D \) in \( \mathfrak{g} \). We want to show that
\[ \text{Tr}(\rho(D)\rho(D(u, v))) = 12B(D(u, v)). \]
Without loss of generality (changing \( \mathcal{B} \) if necessary) we can assume that \( u = e_1 \) and \( v = e_2 \). First of all
\[ \text{Tr}(\rho(D)\rho(D(u, v))) = \sum_{e_i \in \mathcal{B}} \frac{1}{q(e_i)} B(D(D(u, v))(e_i)), e_i). \]
We have
\[ \frac{1}{q(e_1)} B(D(D(e_1, e_2))(e_1), e_1) = 4B(D(e_1), e_2), \]
\[ \frac{1}{q(e_1)} B(D(D(e_1, e_2))(e_2), e_2) = 4B(D(e_1), e_2), \]
\[ \frac{1}{q(e_1 e_2)} B(D(D(e_1, e_2))(e_1 e_2), e_1 e_2) = 0, \]
\[ \frac{1}{q(e_3)} B(D(D(e_1, e_2))(e_3), e_3) = -\frac{2}{q(e_3)} B(D(e_3), (e_1 e_2)e_3), \]
\[ \frac{1}{q(e_1 e_3)} B(D(D(e_1, e_2)(e_1 e_3)), e_1 e_3) = 2B(D(e_1), e_2) + \frac{2}{q(e_3)} B(D(e_3), (e_1 e_2)e_3), \]
\[ \frac{1}{q(e_2 e_3)} B(D(D(e_1, e_2)(e_2 e_3)), e_2 e_3) = 2B(D(e_1), e_2) + \frac{2}{q(e_3)} B(D(e_3), (e_1 e_2)e_3), \]
\[ \frac{1}{q((e_1 e_2)e_3)} B(D(D(e_1, e_2)((e_1 e_2)e_3)), (e_1 e_2)e_3) = -\frac{2}{q(e_3)} B(D(e_3), (e_1 e_2)e_3), \]
and hence
\[ \text{Tr}(\rho(D)\rho(D(u, v))) = 12B(D(u, v)). \]
Corollary 4.2. For $u, v, w \in \text{Im}(\mathbb{O})$, we have

a) $\mu_{\text{Im}}(u, v \times w) + \mu_{\text{Im}}(w, u \times v) + \mu_{\text{Im}}(v, w \times u) = 0,$

b) $\mu_{\text{Im}}(u, v)(w) = \frac{3}{4} \mu_{\text{can}}(u, v)(w) + \frac{1}{8}[w, [u, v]].$

Proof. a) See (3.73) p.78 of [Sch95].

b) Let $u, v, w \in \text{Im}(\mathbb{O})$. We first show that

$$-\frac{1}{4}[w, [u, v]] - \frac{1}{2}(u, v, w) = \mu_{\text{can}}(u, v)(w).$$

(16)

Suppose that $u$ and $v$ are anisotropic and orthogonal.

- If $w = u$ then (16) follows from
  $$-\frac{1}{4}[w, [u, v]] - \frac{1}{2}(u, v, w) = -\frac{1}{2}[u, uv] = -u^2v = q(u)v = \mu_{\text{can}}(u, v)(u).$$

- If $w = uv$ then (16) is clear since $[w, [u, v]] = (u, v, w) = \mu_{\text{can}}(u, v)(w) = 0.$

- If $\{u, v, uv, w\}$ are orthogonal then we have $\mu_{\text{can}}(u, v)(w) = 0$ and
  $$-\frac{1}{4}[w, [u, v]] = -w(uv) = (uv)w = \frac{1}{2}(u, v, w).$$

Hence (16) is satisfied and this proves the corollary using Proposition 4.1.

We now give the main result of this section.

Theorem 4.3. The representation $\rho : g \to so(\text{Im}(\mathbb{O}), B)$ of the quadratic Lie algebra $(g, B_g)$ is a special orthogonal representation.

Proof. Let $u, v, w \in \text{Im}(\mathbb{O})$. Using Proposition 4.1 and (14) we have

$$\mu_{\text{Im}}(u, v)(w) + \mu_{\text{Im}}(u, w)(v) = -\frac{1}{4}([w, [u, v]] + [v, [u, w]]) = -w \times (u \times v) - v \times (u \times w)$$

$$= w \times (v \times u) + v \times (w \times u) = B(u, v)w + B(u, w)v - 2B(w, v)u.$$

By Theorems 2.3 and 4.3 we have a Lie superalgebra $\tilde{g}$ of the form

$$\tilde{g} = g \oplus sl(2, k) \oplus \text{Im}(\mathbb{O}) \otimes k^2.$$

This is an exceptional simple Lie superalgebra of type $G_3$.

Remark 4.4. If $k = \mathbb{R}$, Serganova (see [Ser83]) showed that there are two real forms of $G_3$ whose even parts are isomorphic to the compact (resp. split) exceptional simple real Lie algebra of type $G_2$ in direct sum with $sl(2, \mathbb{R})$ and whose odd parts are isomorphic to the tensor product of the fundamental representations. In our construction, if $\mathbb{O}$ is the compact (resp. split) octonion algebra, the Lie algebra $g$ is the compact (resp. split) exceptional simple real Lie algebra of type $G_2$ and both real forms of $G_3$ are obtained by our construction.

Since the representation $g \to so(\text{Im}(\mathbb{O}), B)$ is special, we calculate its covariants and the Mathews identities they satisfy. Both-sides of Equations (5) and (6) vanish identically since $\text{Im}(\mathbb{O})$ is of dimension 7. It turns out, that both-sides of Equation (4) also vanish identically. However, both sides of Equation (3) do not vanish identically and, up to constants, $Q_{\text{Im}} \wedge Id$ and $\mu_{\text{Im}} \wedge \rho \psi_{\text{Im}} \in \text{Alt}_3(\text{Im}(\mathbb{O}), \text{Im}(\mathbb{O}))$ are the Hodge duals of the cross-product $\times \in \text{Alt}_2(\text{Im}(\mathbb{O}), \text{Im}(\mathbb{O})).$
Proposition 4.5. Let $\mu_{1m}, \psi_{1m}, Q_{1m}$ be the covariants of the special orthogonal representation $\rho: \mathfrak{g} \to \mathfrak{so}(\text{Im}(\mathbb{O})), B)$. We have

a) $\psi_{1m}(v_1, v_2, v_3) = -\frac{3}{4}(v_1, v_2, v_3)$ for all $v_1, v_2, v_3 \in \text{Im}(\mathbb{O})$, 
b) $Q_{1m}(v_1, v_2, v_3, v_4) = -3B(v_1, (v_2, v_3, v_4))$ for all $v_1, v_2, v_3, v_4 \in \text{Im}(\mathbb{O})$, 
c) 

\[
\eta^-(Q_{1m}) = \frac{6}{q(e_1)q(e_2)q(e_4)}e_{1247} - \frac{6}{q(e_1)q(e_2)q(e_4)}e_{1256} - \frac{6}{q(e_1)q(e_2)q(e_4)}e_{1346} - \frac{6}{q(e_1)q(e_2)q(e_4)}e_{1357} 
+ \frac{6}{q(e_1)q(e_2)q(e_4)}e_{2345} - \frac{6}{q(e_1)q(e_2)q(e_4)}e_{2365} - \frac{6}{q(e_1)q(e_2)q(e_4)}e_{4567},
\]  

(17) 
d) $\mu_{1m} \circ \psi_{1m} = 0$ and $Q_{1m} \wedge \mu_{1m} = 0$.

Proof. a) By Proposition 4.1 and by Equation (11) we obtain 

$\psi_{1m}(v_1, v_2, v_3) = -\frac{1}{4}(J(v_1, v_2, v_3) + 3(v_1, v_2, v_3) + 3(v_2, v_3, v_1) + 3(v_3, v_1, v_2))$ 

$= -\frac{1}{4}(J(v_1, v_2, v_3) + 9(v_1, v_2, v_3)) 

= -\frac{3}{4}(v_1, v_2, v_3)$.

b) Follows from Proposition 2.5.

c) The decomposition follows from b) and the fact that for $i_1 < i_2 < i_3 < i_4$, then $Q_{1m}(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is non-zero if and only if $(i_1, i_2, i_3, i_4) \in \{(1, 2, 4, 7), (1, 2, 5, 6), (1, 3, 4, 6), (1, 3, 5, 7), (2, 3, 4, 5), (2, 3, 6, 7), (4, 5, 6, 7)\}$.

d) Let $v_1, \ldots, v_6 \in \text{Im}(\mathbb{O})$. We have 

$\mu_{1m} \circ \psi_{1m}(v_1, \ldots, v_6) = \sum_{\sigma \in S\{1, 3, \ldots, 6\}} sgn(\sigma)\mu_{1m}(\psi_{1m}(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}), \psi_{1m}(v_{\sigma(4)}, v_{\sigma(5)}, v_{\sigma(6)})) 

= 2 \sum_{\sigma \in S'} sgn(\sigma)\mu_{1m}(\psi_{1m}(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}), \psi_{1m}(v_{\sigma(4)}, v_{\sigma(5)}, v_{\sigma(6)})) 

= \frac{9}{8} \sum_{\sigma \in S''} sgn(\sigma)\mu_{1m}(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}), (v_{\sigma(4)}, v_{\sigma(5)}, v_{\sigma(6)}))$

where $S' := \{\text{Id}, (14), (15), (16), (24), (25), (26), (34), (35), (36)\}$. 

Suppose that $v_i \in B$ for all $i \in [1, 6]$. Since there is no distinguished way to choose 5 different points on the Fano plane, then, without loss of generality, we can assume that $v_i = e_i$ for all $i \in [1, 6]$. Since 

$$(e_1, e_2, e_3) = (e_1, e_4, e_5) = (e_2, e_4, e_6) = (e_3, e_5, e_6) = 0$$

then we have 

$\mu_{1m} \circ \psi_{1m}(v_1, \ldots, v_6) = -\frac{9}{8} \sum_{\sigma \in S''} \mu_{1m}((e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}), (e_{\sigma(4)}, e_{\sigma(5)}, e_{\sigma(6)})) 

= -\frac{9}{2} \sum_{\sigma \in S''} \mu_{1m}((e_{\sigma(1)} , e_{\sigma(2)}}e_{\sigma(3)}) , (e_{\sigma(4)} , e_{\sigma(5)}, e_{\sigma(6)})$ 

where $S'' := \{(14), (15), (24), (26), (35), (36)\}$. Hence, we have that 

$\mu_{1m} \circ \psi_{1m}(v_1, \ldots, v_6) = -9q(e_1)q(e_2)q(e_4)\left(\mu_{1m}(e_1e_2, e_4) + \mu_{1m}(e_2e_4, e_1) + \mu_{1m}(e_4e_1, e_2)\right)$

and so, using a) of Corollary 4.2, we obtain $\mu_{1m} \circ \psi_{1m} = 0$ and by Theorem 2.6 we have $Q_{1m} \wedge \mu_{1m} = 0$. \qed
Remark 4.6. a) We have
\[ \phi \wedge Q_{\text{Im}}(e_1 \wedge \ldots \wedge e_7) = -42q(e_1)^2q(e_2)^2q(e_4)^2. \]

If \( \text{char}(k) \neq 7 \), then \( \phi \wedge Q_{\text{Im}} \) defines an orientation on \( \text{Im}(\mathcal{O}) \)

b) In the decomposition (15) (resp. (17)), the seven quadruples of indices \( \{i_1, i_2, i_3, i_4\} \) appearing are exactly (resp. the complements of) the seven lines of the Fano plane.

Suppose that \( \text{char}(k) = 0 \) or \( \text{char}(k) > 7 \). Define a quadratic form \( B_{\text{Alt}} \) on \( \text{Alt}_2(\text{Im}(\mathcal{O}), \text{Im}(\mathcal{O})) \) to be the tensor product of \( B_{\text{Alt}} \) and \( B \). For \( f \in \text{Alt}_4(\text{Im}(\mathcal{O}), \text{Im}(\mathcal{O})) \) define its Hodge dual \( *f \in \text{Alt}_{7-1}(\text{Im}(\mathcal{O}), \text{Im}(\mathcal{O})) \) to be the unique element which satisfies
\[ \alpha \wedge_B *f = B_{\text{Alt}}(\alpha, f)\phi \wedge Q_{\text{Im}} \quad \forall \alpha \in \text{Alt}_4(\text{Im}(\mathcal{O}), \text{Im}(\mathcal{O})). \]

Proposition 4.7. We have
\[ *\times = \frac{147}{8} Q_{\text{Im}} \wedge \text{Id} = -\frac{49}{4} \mu_{\text{Im}} \wedge \rho \psi_{\text{Im}}. \]

Proof. Let \( \alpha \in \text{Alt}_2(\text{Im}(\mathcal{O}), \text{Im}(\mathcal{O})) \). We have
\[ B_{\text{Alt}}(\alpha, x)\phi \wedge Q_{\text{Im}}(e_1 \wedge \ldots \wedge e_7) = -42 \sum_{i<j} \frac{1}{q(e_i)q(e_j)} B(\alpha(e_i, e_j), e_i \times e_j)q(e_1)^2q(e_2)^2q(e_4)^2. \quad (18) \]

On the other hand
\[ \alpha \wedge_B Q_{\text{Im}} \wedge \text{Id}(e_1 \wedge \ldots \wedge e_7) = \frac{1}{126} \sum_{\sigma \in S_7} \text{sgn}(\sigma)B(\alpha(e_{\sigma(1)}, e_{\sigma(2)}), e_{\sigma(3)}Q_{\text{Im}}(e_{\sigma(4)}, e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}). \]

Since \( \alpha \) and \( Q_{\text{Im}} \) are alternating and using the decomposition of Equation (17), we have
\[ \alpha \wedge_B Q_{\text{Im}} \wedge \text{Id}(e_1 \wedge \ldots \wedge e_7) = \frac{8}{21} \sum_{\sigma \in \tilde{S}} \text{sgn}(\sigma)B(\alpha(e_{\sigma(1)}, e_{\sigma(2)}), e_{\sigma(3)}Q_{\text{Im}}(e_{\sigma(4)}, e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}) \quad (19) \]

where
\[ S = \{ \sigma \in S_7 \mid \sigma(1) < \sigma(2), \quad \sigma(4), \sigma(5), \sigma(6), \sigma(7) \} \in \{(1, 2, 4, 7), \ldots, (4, 5, 6, 7)\}. \]

We have \( |S| = 21 \), each summand in (19) correspond to one summand in (18) and so, a straightforward calculation gives
\[ \alpha \wedge_B Q_{\text{Im}} \wedge \text{Id}(e_1 \wedge \ldots \wedge e_7) = -\frac{16}{7} \sum_{i<j} \frac{1}{q(e_i)q(e_j)} B(\alpha(e_i, e_j), e_i \times e_j)q(e_1)^2q(e_2)^2q(e_4)^2 \]
and hence
\[ \alpha \wedge_B Q_{\text{Im}} \wedge \text{Id} = \frac{8}{147} B_{\text{Alt}}(\alpha, x)\phi \wedge Q_{\text{Im}}. \]

Remark 4.8. One can show similarly that, up to constants, the identity is the Hodge dual of \( \phi \wedge \psi_{\text{Im}} \), the covariant \( \mu_{\text{Im}} \) is the Hodge dual of \( \phi \wedge \mu_{\text{Im}} \) and the covariant \( \psi_{\text{Im}} \) is the Hodge dual of \( \phi \wedge \text{Id} \).

5 The spinor representation of a Lie algebra of type \( \mathfrak{so}(7) \) is special orthogonal

In this section, we show that the 8-dimensional spinor representation \( \mathcal{O} \) of \( C^2(\text{Im}(\mathcal{O}), -q) \) is special orthogonal. Let \( e_2, e_3, e_5 \in \text{Im}(\mathcal{O}) \) be such that \( \mathcal{B} = \{e_1, e_2, e_3, e_2e_3, e_5, e_2e_5, e_3e_5, (e_2e_3)e_5\} \) is an orthogonal and anisotropic basis of \( \mathcal{O} \) and set \( e_4 := e_2e_3, e_6 := e_2e_5, e_7 := e_3e_5, e_8 := (e_2e_3)e_5 \).
Proof. Let $\mathfrak{g} := C^2(\text{Im}(\mathbb{O}), -q)$ and define the ad-invariant quadratic form $B_\mathfrak{g}$ on $\mathfrak{g}$ by

$$B_\mathfrak{g}(x, y) = -\frac{3}{8} \text{Tr}(x y) \quad \forall x, y \in \mathfrak{g}.$$ 

**Definition 5.1.** Let $\Omega := Q(\eta^{-1}(\phi)) \in C(\text{Im}(\mathbb{O}), -q)$. For $u \in \text{Im}(\mathbb{O})$, define $c_u \in \mathfrak{g}$ by $c_u := \{u, \Omega\}$ and define the 7-dimensional subspace $W \subset \mathfrak{g}$ by $W := \text{span} \langle \{c_u \mid u \in \text{Im}(\mathbb{O})\} \rangle$.

The subspace $W$ acts on $\mathbb{O}$ as follows.

**Proposition 5.2.** Let $u, v \in \text{Im}(\mathbb{O})$. We have

$$\rho(c_u)(1) = -6u, \quad \rho(c_u)(v) = 2u \times v + 6B(u, v).$$

**Proof.** Using (15), we obtain

$$\rho(\Omega)(1) = -7, \quad \rho(\Omega)(u) = u,$$

and so

$$\rho(c_u)(1) = \rho(u)(\rho(\Omega)(1)) + \rho(\Omega)(\rho(u)(1)) = -7u + u = -6u.$$ 

Using (13), we have

$$\rho(c_u)(v) = \rho(u)(\rho(\Omega)(v)) + \rho(\Omega)(\rho(u)(v))$$

$$= uv + \rho(\Omega)(uv)$$

$$= u \times v - B(u, v) + \rho(\Omega)(u \times v) - B(u, v)\Omega(1)$$

$$= 2u \times v + 6B(u, v).$$

\[\square\]

Now, we can express the moment map of $\mathbb{O}$ in terms of the moment map of $\text{Im}(\mathbb{O})$ and $W$.

**Proposition 5.3.** The moment map $\mu_\mathbb{O} : \Lambda^2(\mathbb{O}) \to \mathfrak{g}$ satisfies to

$$\mu_\mathbb{O}(u, v) = \frac{8}{9} \mu_{\text{Im}(\mathbb{O})}(u, v) + \frac{1}{18} c_u \times v \quad \forall u, v \in \text{Im}(\mathbb{O}),$$

$$\mu_\mathbb{O}(u, 1) = \frac{1}{6} c_u \quad \forall u \in \text{Im}(\mathbb{O}).$$

**Proof.** We first need the following lemma:

**Lemma 5.4.** Let $u, v \in \text{Im}(\mathbb{O})$ and $D \in \mathfrak{g}$. We have

$$\text{Tr}(\rho(c_u)\rho(c_v)) = -96B(u, v), \quad \text{Tr}(\rho(D)\rho(c_u)) = 0.$$ 

**Proof.** Let $w \in \text{Im}(\mathbb{O})$. We have

$$B(\rho(c_u)(\rho(c_v)(w)), w) = 2B(\rho(c_u)(v \times w), w) + 6B(v, w)B(\rho(c_u)(1), w)$$

$$= 2B(2u \times (v \times w) + 6B(u, v \times w), w) - 36B(u, w)B(v, w)$$

$$= -4B(u \times w, v \times w) - 36B(u, w)B(v, w).$$

The linearisation of (12) gives

$$B(u \times w, v \times w) = B(u, v)q(w) - B(u, w)B(v, w)$$

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and so 
\[ B(\rho(c_u)(\rho(c_v)(w)), w) = -4B(u, v)q(w) - 32B(u, w)B(v, w). \]
We also have 
\[ B(\rho(c_u)(\rho(c_v)(1)), 1) = -6B(\rho(c_u)(v), 1) = -36B(u, v). \]
If \( u \) and \( v \) are orthogonal, without loss of generality (changing \( \mathcal{B} \) if necessary), we can assume that \( u = e_2 \) and \( v = e_3 \). Hence 
\[ Tr(\rho(c_u)\rho(c_v)) = \sum_{c_i \in \mathcal{B}} \frac{1}{q(e_i)} B(\rho(c_u)(\rho(c_v)(e_i)), e_i) = 0, \]
similarly we have 
\[ Tr(\rho(c_u)^2) = -96q(u) \]
and so 
\[ Tr(\rho(c_u)\rho(c_v)) = -96B(u, v). \]
A straightforward calculation shows that \( \mathfrak{g} \) and \( W \) are orthogonal. 

Let \( D \in \mathfrak{g} \). We have 
\[ B_h(D, \mu_\mathcal{O}(u, v)) = B(D(u), v) = B_\mathfrak{g}(D, \mu_\text{Im}(u, v)) = \frac{8}{9} B_h(D, \mu_\text{Im}(u, v)), \]
and, using the previous lemma, we also have 
\[ B_h(c_w, \mu_\mathcal{O}(u, v)) = B(c_w(u), v) = 2B(w, u \times v) = -\frac{1}{48} Tr(c_w c_u \times v) = \frac{1}{18} B_h(c_w, c_u \times v), \]
and so 
\[ \mu_\mathcal{O}(u, v) = \frac{8}{9} \mu_\text{Im}(u, v) + \frac{1}{18} c_u \times v. \]
Since \( \rho(D)(1) = 0 \), then we have \( B_h(D, \mu_\mathcal{O}(u, 1)) = 0 \). Moreover, 
\[ B_h(c_w, \mu(u, 1)) = B(c_w(u), 1) = 6B(u, w) = -\frac{1}{16} Tr(c_w c_u) = \frac{1}{6} B_h(c_w, c_u) \]
and so 
\[ \mu_\mathcal{O}(u, 1) = \frac{1}{6} c_u. \]

The counterpart of a) of Corollary 4.2 is the following property about the moment map of \( \mathfrak{h} \): 

**Corollary 5.5.** We have 
\[ \mu_\mathcal{O}(u, v \times w) + \mu_\mathcal{O}(v, w \times u) + \mu_\mathcal{O}(w, u \times v) = -\frac{1}{2} \mu_\mathcal{O}((u, v, w), 1) \quad \forall u, v, w \in \text{Im}(\mathcal{O}). \]  

**Proof.** Using a) of Corollary 4.2, we have 
\[ \mu_\mathcal{O}(u, v \times w) + \mu_\mathcal{O}(v, w \times u) + \mu_\mathcal{O}(w, u \times v) = \frac{1}{18} c_u \times (v \times w) + v \times (w \times u) + w \times (u \times v). \]
Since, using (11), we have 
\[ u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = -\frac{3}{4} J(u, v, w) = -\frac{3}{2} (u, v, w) \]
then we obtain 
\[ \mu_\mathcal{O}(u, v \times w) + \mu_\mathcal{O}(v, w \times u) + \mu_\mathcal{O}(w, u \times v) = -\frac{1}{12} c_{(u, v, w)} = -\frac{1}{2} \mu_\mathcal{O}((u, v, w), 1). \]
Remark 5.6. By Proposition 5.3, the representation $\mathfrak{g} \rightarrow \mathfrak{so}(W, B_\mathfrak{h}|W)$ of the quadratic Lie algebra $(\mathfrak{g}, B_\mathfrak{h}|_\mathfrak{g})$ together with the non-trivial cubic term on $W$ given by a multiple of the cross-product is of Lie type in the sense of Kostant [Kos99).

We now give the main result of this section.

Theorem 5.7. The representation $\rho : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{O}, B)$ of the quadratic Lie algebra $(\mathfrak{h}, B_\mathfrak{h})$ is a special orthogonal representation.

Proof. We want to show Equation (2). Let $u, v, w \in \text{Im}(\mathfrak{O})$. We have

$$\mu_\mathfrak{O}(u, v)(w) + \mu_\mathfrak{O}(u, w)(v) = \frac{8}{9}(\mu_{1m}(u, v)(w) + \mu_{1m}(u, w)(v)) + \frac{1}{18}(c_{u,v}(w) + c_{u,w}(v))$$

$$= \frac{2}{9}([w, [u, v]] + [v, [u, w]]) + \frac{1}{9}((u \times v) \times w + (u \times w) \times v)$$

$$= \frac{8}{9}(w \times (u \times v) + v \times (u \times w)) + \frac{1}{9}((u \times v) \times w + (u \times w) \times v)$$

$$= w \times (v \times u) + v \times (w \times u)$$

and by Equation (14), we obtain

$$\mu_\mathfrak{O}(u, v)(w) + \mu_\mathfrak{O}(u, w)(v) = B(u, v)w + B(u, w)v - 2B(v, w)u$$

and so (2) is satisfied for $u, v, w \in \text{Im}(\mathfrak{O})$. We have

$$\mu_\mathfrak{O}(u, v)(1) + \mu_\mathfrak{O}(u, 1)(v) = \frac{1}{18}c_{u,v}(1) + \frac{1}{6}c_u(v) = B(u, v),$$

$$\mu_\mathfrak{O}(1, v)(w) + \mu_\mathfrak{O}(1, w)(v) = -\frac{1}{6}(c_w(v) + c_v(w)) = -2B(v, w),$$

and so (2) is satisfied whenever two elements $u, v$ or $w$ are in $\text{Im}(\mathfrak{O})$. Finally, since

$$2\mu_\mathfrak{O}(u, 1)(1) = -2u, \quad \mu_\mathfrak{O}(1, u)(1) = u,$$

then (2) is satisfied whenever $u, v$ or $w$ is in $\text{Im}(\mathfrak{O})$ and so (2) is satisfied for all $u, v, w \in \mathfrak{O}$. \hfill \Box

By Theorems 2.3 and 5.7 we have a Lie superalgebra $\mathfrak{f}$ of the form

$$\mathfrak{f} := \mathfrak{h} \oplus \mathfrak{sl}(2, k) \oplus \mathfrak{O} \otimes k^2.$$

This is an exceptional simple Lie superalgebra of type $F_4$ in the Kac notation.

Remark 5.8. If $k = \mathbb{R}$, Serganova (see [Serg83]) showed that there are four real forms of $F_4$. In particular, two of them have an even part isomorphic to $\mathfrak{so}(7) \oplus \mathfrak{sl}(2, \mathbb{R})$ (resp. $\mathfrak{so}(4, 3) \oplus \mathfrak{sl}(2, \mathbb{R})$) and an odd part isomorphic to the tensor product of the spinor representation of $\mathfrak{so}(7)$ (resp. $\mathfrak{so}(4, 3)$) and $\mathbb{R}^2$. In our construction, if $\mathfrak{O}$ is the compact or the split octonion algebra, both real forms of $F_4$ are obtained by our construction.

Since the representation $\mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{O}, B)$ is special, we calculate its covariants $\psi_\mathfrak{O}, Q_\mathfrak{O}$ and the Mathews identities they satisfy. Since $\mathfrak{O}$ is of dimension 8, both-sides of Equations (5) and (6) vanish identically. However, both sides of the identities (3) and (4) do no vanish identically. More precisely, up to constants, $\mu_\mathfrak{O} \wedge_\mathfrak{O} \psi_\mathfrak{O}$ and $Q_\mathfrak{O} \wedge_\mathfrak{O} \mathfrak{Id}_\mathfrak{O} \in \text{Alt}_3(\mathfrak{O}, \mathfrak{O})$ are the Hodge duals of the trilinear covariant $\psi_\mathfrak{O} \in \text{Alt}_3(\mathfrak{O}, \mathfrak{O})$ and $\mu_\mathfrak{O} \circ \psi_\mathfrak{O}$ and $Q_\mathfrak{O} \wedge_\mathfrak{O} \mu_\mathfrak{O} \in \text{Alt}_6(\mathfrak{O}, \mathfrak{h})$ are the Hodge duals of the moment map $\mu_\mathfrak{O} \in \text{Alt}_2(\mathfrak{O}, \mathfrak{h})$.

Proposition 5.9. Let $\mu_\mathfrak{O}, \psi_\mathfrak{O}, Q_\mathfrak{O}$ be the covariants of the special orthogonal representation $\rho : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{O}, B)$. We have

a) $\psi_\mathfrak{O}(v_1, v_2, v_3) = -\frac{1}{2}(v_1, v_2, v_3) + \phi(v_1, v_2, v_3)$ and $\psi_\mathfrak{O}(v_1, v_2, 1) = -v_1 \times v_2$ for all $v_1, v_2, v_3 \in \text{Im}(\mathfrak{O})$. 

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b) $Q_0(v_1, v_2, v_3, v_4) = \frac{4}{3} Q_{\text{Im}}(v_1, v_2, v_3, v_4)$ and $Q_0(v_1, v_2, v_3, 1) = -4\phi(v_1, v_2, v_3)$ for all $v_1, v_2, v_3, v_4 \in \text{Im}(\mathcal{O})$.

c) $\eta^{-1}(Q_0) = \frac{4}{q(e_2)q(e_3)} e_{1234} - \frac{4}{q(e_2)q(e_3)q(e_5)} e_{1278} + \frac{4}{q(e_2)q(e_3)q(e_5)} e_{1368} - \frac{4}{q(e_2)q(e_3)q(e_5)} e_{1467}
+ \frac{4}{q(e_2)q(e_3)} e_{1256} + \frac{4}{q(e_2)q(e_3)q(e_5)} e_{1357} + \frac{4}{q(e_2)q(e_3)q(e_5)} e_{1458} + \frac{4}{q(e_2)q(e_3)q(e_5)} e_{2358}
- \frac{4}{q(e_2)q(e_3)q(e_5)} e_{2367} - \frac{4}{q(e_2)q(e_3)q(e_5)} e_{2457} - \frac{4}{q(e_2)q(e_3)q(e_5)} e_{2468} + \frac{4}{q(e_2)q(e_3)q(e_5)} e_{3456}
- \frac{4}{q(e_2)q(e_3)^2q(e_5)} e_{3478} - \frac{4}{q(e_2)q(e_3)q(e_5)^2} e_{5678}.
(23)$

Proof. a) By Propositions 5.3 and 4.5, we have

$$\psi_0(v_1, v_2) = \frac{8}{9} \psi_{\text{Im}}(v_1, v_2, v_3) + \frac{1}{18} (c_{v_1 \times v_1}(v_3) + c_{v_2 \times v_2}(v_1) + c_{v_3 \times v_3}(v_1))$$

$$= -\frac{2}{3} (v_1 \times v_2) + \frac{1}{9} (v_1 \times v_2) \times v_3 + (v_2 \times v_3) \times v_1 + (v_3 \times v_1) \times v_2 + \phi(v_1, v_2, v_3).$$

Using Equations (13) and (11), we have

$$(v_1 \times v_2) \times v_3 + (v_2 \times v_3) \times v_1 + (v_3 \times v_1) \times v_2 = -\frac{1}{4} J(v_1, v_2, v_3) = \frac{3}{2} (v_1, v_2, v_3)$$

and so

$$\psi_0(v_1, v_2, v_3) = \frac{1}{2} (v_1, v_2, v_3) + \phi(v_1, v_2, v_3).$$

We also have

$$\psi_0(v_1, v_2, 1) = \frac{1}{18} c_{v_1 \times v_2}(1) + \frac{1}{6} c_{v_2}(v_1) - \frac{1}{6} c_{v_3}(v_2) = -\frac{1}{3} v_1 \times v_2 + \frac{1}{3} v_2 \times v_1 - \frac{1}{3} v_1 \times v_2 = -v_1 \times v_2.$$ b) Follows from a) and Propositions 2.5 and 4.5.

c) Using b), the decomposition of $Q_0$ follows from the decompositions (17) and (15). \hfill \Box

Remark 5.10. a) We have

$$Q_0 \wedge Q_0(e_1 \wedge \ldots \wedge e_8) = -224 q(e_2)^2 q(e_3)^2 q(e_5)^2.$$

If $\text{char}(k) \neq 7$, then $Q_0 \wedge Q_0$ defines an orientation on $\mathcal{O}$.

b) In the decomposition (23), there are fourteen 4-vectors of the form $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$. The fourteen quadruples of indices $\{i_1, i_2, i_3, i_4\}$ appearing are not arbitrary. There is a numbering of the eight points of the affine space $(\mathbb{Z}_2)^3$ such that each quadruple corresponds to one of the fourteen affine planes.
Suppose that $\text{char}(k) = 0$ or $\text{char}(k) > 7$. Define a quadratic form $B_{\text{Alt}(\mathfrak{O}, \mathfrak{O})}$ (resp. $B_{\text{Alt}(\mathfrak{O}, \mathfrak{h})}$) on $\text{Alt}_1(\mathfrak{O}, \mathfrak{O}) \cong \Lambda^i(\mathfrak{O})^* \otimes \mathfrak{O}$ (resp. $\text{Alt}_1(\mathfrak{O}, \mathfrak{h}) \cong \Lambda^i(\mathfrak{O})^* \otimes \mathfrak{h}$) to be the tensor product of $B_{\Lambda^i}$ and $B$ (resp. $B_{\mathfrak{h}}$). For $f \in \text{Alt}_i(\mathfrak{O}, \mathfrak{O})$ define its Hodge dual $*f \in \text{Alt}_{n-i}(\mathfrak{O}, \mathfrak{O})$ to be the unique element which satisfies

\[ \alpha \wedge *f = B_{\text{Alt}(\mathfrak{O}, \mathfrak{O})}(\alpha, f)Q_0 \wedge Q_0 \quad \forall \alpha \in \text{Alt}_i(\mathfrak{O}, \mathfrak{O}) \]

and for $f \in \text{Alt}_i(\mathfrak{O}, \mathfrak{h})$ define its Hodge dual $*f \in \text{Alt}_{n-i}(\mathfrak{O}, \mathfrak{h})$ to be the unique element which satisfies

\[ \alpha \wedge _B *f = B_{\text{Alt}(\mathfrak{O}, \mathfrak{h})}(\alpha, f)Q_0 \wedge Q_0 \quad \forall \alpha \in \text{Alt}_i(\mathfrak{O}, \mathfrak{h}). \]

**Proposition 5.11.** We have

\begin{enumerate}
\item $*\psi_0 = -56Q_0 \wedge \text{Id} = \frac{112}{3} \mu_\mathfrak{O} \wedge_\rho \psi_0$,
\item $*\mu_\mathfrak{O} = -56Q_0 \wedge \mu_\mathfrak{O} = -\frac{56}{3} \mu_\mathfrak{O} \circ \psi_0$.
\end{enumerate}

**Proof.**

\begin{enumerate}
\item Let $\alpha \in \text{Alt}_3(\mathfrak{O}, \mathfrak{O})$. We have

\[ B_{\text{Alt}(\mathfrak{O}, \mathfrak{O})}(\alpha, \psi_0)Q_0 \wedge Q_0(e_1 \wedge \ldots \wedge e_8) = -224 \sum_{i < j < k} \frac{1}{q(e_i)q(e_j)q(e_k)} B(\alpha(e_i, e_j, e_k), \psi_0(e_i, e_j, e_k))q(e_2)^2q(e_3)^2q(e_5)^2. \]

On the other hand

\[ \alpha \wedge_B Q_0 \wedge \text{Id}(e_1 \wedge \ldots \wedge e_8) = \frac{1}{144} \sum_{\sigma \in S_8} \text{sgn}(\sigma)B(\alpha(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}), e_{\sigma(4)}Q_0(e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}, e_{\sigma(8)}). \]

Since $\alpha$ and $Q_0$ are alternating and using the decomposition of Equation (23), we have

\[ \alpha \wedge_B Q_0 \wedge \text{Id}(e_1 \wedge \ldots \wedge e_8) = \sum_{\sigma \in S} \text{sgn}(\sigma)B(\alpha(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}), e_{\sigma(4)}Q_0(e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}, e_{\sigma(8)}) \quad (25) \]

where

\[ S = \{ \sigma \in S_8 \mid \sigma(1) < \sigma(2) < \sigma(3), \quad (\sigma(5), \sigma(6), \sigma(7), \sigma(8)) \in \{ (1, 2, 3, 4), \ldots, (5, 6, 7, 8) \} \}. \]

We have $|S| = 56$ and each summand in (24) correspond to one summand in (25). A straightforward calculation gives

\[ \alpha \wedge_B Q_0 \wedge \text{Id}(e_1 \wedge \ldots \wedge e_8) = 4 \sum_{i < j < k} \frac{1}{q(e_i)q(e_j)q(e_k)} B(\alpha(e_i, e_j, e_k), \psi_0(e_i, e_j, e_k))q(e_2)^2q(e_3)^2q(e_5)^2 \]

and so

\[ \alpha \wedge_B Q_0 \wedge \text{Id} = -\frac{1}{56} B_{\text{Alt}(\mathfrak{O}, \mathfrak{O})}(\alpha, \psi_0)Q_0 \wedge Q_0. \]

\item Let $\alpha \in \text{Alt}_2(\mathfrak{O}, \mathfrak{h})$. We have

\[ B_{\text{Alt}(\mathfrak{O}, \mathfrak{h})}(\alpha, \mu_\mathfrak{O})Q_0 \wedge Q_0(e_1 \wedge \ldots \wedge e_8) = -224 \sum_{i < j} \frac{1}{q(e_i)q(e_j)} B_h(\alpha(e_i, e_j), \mu_\mathfrak{O}(e_i, e_j))q(e_2)^2q(e_3)^2q(e_5)^2. \]

On the other hand

\[ \alpha \wedge_B Q_0 \wedge \mu_\mathfrak{O}(e_1 \wedge \ldots \wedge e_8) = \frac{1}{96} \sum_{\sigma \in S_8} \text{sgn}(\sigma)B_h(\alpha(e_{\sigma(1)}, e_{\sigma(2)}), \mu_\mathfrak{O}(e_{\sigma(3)}, e_{\sigma(4)}))Q_0(e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}, e_{\sigma(8)}). \]

Since $\alpha, \mu_\mathfrak{O}$ and $Q_0$ are alternating and using the decomposition of Equation (23), we have

\[ \alpha \wedge_B Q_0 \wedge \mu_\mathfrak{O}(e_1 \wedge \ldots \wedge e_8) = \sum_{\sigma \in S} \text{sgn}(\sigma)B_h(\alpha(e_{\sigma(1)}, e_{\sigma(2)}), \mu_\mathfrak{O}(e_{\sigma(3)}, e_{\sigma(4)}))Q_0(e_{\sigma(5)}, e_{\sigma(6)}, e_{\sigma(7)}, e_{\sigma(8)}). \]
where
\[ S = \{ \sigma \in S_8 \mid \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), (\sigma(5), \sigma(6), \sigma(7), \sigma(8)) \in \{(1, 2, 3, 4), \ldots, (5, 6, 7, 8)\} \}. \]

We have \(|S| = 84\) and, using Equation (22), a straightforward calculation gives
\[
\alpha \wedge B O \mu O \wedge \psi O (e_1 \wedge \ldots \wedge e_8) = 4 \sum_{i<j} \frac{1}{q(e_i)q(e_j)} B_0(\alpha(e_i, e_j), \mu_0(e_i, e_j))q(e_2)^2q(e_3)^2q(e_5)^2, 
\]
and so
\[
\alpha \wedge B O \mu O (e_1 \wedge \ldots \wedge e_8) = -\frac{1}{56} B_{Alt(O,H)}(\alpha, \mu_0)Q O \wedge Q O. 
\]

\(\square\)

**Remark 5.12.** One can show similarly that, up to a constant, the identity is the Hodge dual of \(Q O \wedge \psi O\).

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