Ghost-free scalar-fermion interactions

Rampei Kimura,1,2,∗ Yuki Sakakihara,3,† and Masahide Yamaguchi2,‡

1 Waseda Institute for Advanced Study, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku, Tokyo 169-8050, Japan
2 Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
3 Department of Physics, Kwansei Gakuin University, Sanda, Hyogo 669-1337, Japan

We discuss a covariant extension of interactions between scalar fields and fermions in a flat spacetime. We show, in a covariant theory, how to evade fermionic ghosts appearing because of the extra degrees of freedom behind a fermionic nature even in Lagrangian with first derivatives. We will give a concrete example of a quadratic theory with up to the first derivative of multiple scalar fields and a Weyl fermion. We examine not only the maximally degenerate condition, which makes the number of degrees of freedom correct, but also a supplementary condition guaranteeing that the time evolution takes place properly. We also show that proposed derivative interaction terms between scalar fields and a Weyl fermion cannot be removed by field redefinitions.

I. INTRODUCTION

Construction of general theory without ghost degree of freedom has been discussed for a long time. According to the Ostrogradsky’s theorem, a ghost always appears in a higher (time) derivative theory as long as it is non-degenerate [1] (see also [2]). If a term with higher derivative plays an important role in the dynamics, the ghost degree of freedom associated with it must be removed because, otherwise, the dynamics is unstable. This is a different approach from effective field theory, which allows a ghost as long as it appears above the scale we are interested in, that is, a term with higher derivative can be treated as a perturbation.

Such an Ostrogradsky’s ghost could be circumvented by introducing the degeneracy of the kinetic matrix, which leads to the existence of a primary constraint and a series of subsequent constraints, which are responsible for eliminating extra ghost degrees of freedom associated with higher derivatives. After the recent rediscovery [3–5] of Horndeski theory [6] in the context of Galileon [7], the pursuit to find general theory without such ghost degrees of freedom has been revived, especially for bosonic degrees of freedom (DOFs), in the context of point particles [8–11], their field theoretical application [12], scalar-tensor theories [4–6, 13–20], vector-tensor theories [21–29], and form fields [30–32].

On the other hand, generic theory with fermionic degrees of freedom has not yet been investigated well. If we could find such fermionic theories which have not been explored, some applications of them will be expected. One will be fermionic dark matter, whose interaction could be constrained by the condition of the absence of ghost degrees of freedom. Another application is inflation and its subsequent reheating, which needs interactions between an inflaton and standard model particles, since their interactions are still unknown. The new interaction may affect the history during reheating era through particle production. The effect of the interactions can also appear in observables such as non-Gaussianity through the loop corrections of standard model particles, as pointed out in [33, 34]. The construction of a ghost free (higher derivative) supersymmetric theory is another direction though some attempts have already been made [35–38].

What is prominent in fermionic theory is that ghosts appear even with no higher derivatives in the usual meaning, i.e., no second derivatives and higher. As mentioned in [39] for purely fermion system, if N fermionic variables carry $2N$ degrees of freedom in the phase space, then, negative norm states inevitably appear. That is why we usually see that the canonical kinetic term of the fermionic field linearly depends on the derivative of the field like Dirac Lagrangian. Indeed, in the case of usual Weyl fermions, such unwanted states are evaded by the existence of an enough number of primary constraints. In our previous paper [40], as a starting point, we studied point particle theories, whose Lagrangian contains both of bosonic and fermionic variables with their first derivatives, and showed that the coexistence of fermionic degrees of freedom and bosonic ones allows us to have some extension of their kinetic terms. In the present paper, we apply the analysis done in [40] to the field system with scalar fields and Weyl fermions, which can be even as a preliminary step toward tensor-fermion or any other interesting theories including fermionic degrees of freedom.

∗ Email: rampei”at”aoni.waseda.jp
† Email: y.sakakihara”at”kwansei.ac.jp
‡ Email: gucci”at”phys.titech.ac.jp
This paper is organized as follows. In Sec. II, following the previous analysis for point particle theories, we give a general formulation of the construction of covariant theory with up to first derivatives of \( n \)-scalar fields and \( N \)-Weyl fermions. We then derive \( 4N \) primary constraints by introducing maximally degenerate conditions. In Sec. III, we concretely write down the quadratic theory of \( n \)-scalar and one Weyl fermion fields and apply the set of the maximally degenerate conditions, which we have proposed for removing the fermionic ghosts properly. In Sec. IV, we perform Hamiltonian analysis of the quadratic theory and derive a supplementary condition for the Lagrangian such that all the primary constraints are second class. In Sec. V, we obtain the explicit counterparts in Lagrangian formulation to the conditions in Hamiltonian formulation. In Sec. VI, we show that the obtained theories satisfying the maximally degenerate conditions cannot be mapped into theories whose Lagrangian linearly depend on the derivative of the fermionic fields. In Sec. VII, we give a summary of our work. In Appendix A, we summarize definitions and identities of the Pauli matrices. In Appendix B, we show the equivalence between the maximally degenerate conditions and the primary constraints obtained in Sec. II. In Appendix C, we extend our analysis in Sec. III to multiple Weyl fermions and derive the primary constraints.

II. GENERAL FORMALISM FOR CONSTRUCTING DEGENERATE LAGRANGIAN

In the present paper, we construct flat spacetime theories with \( n \) real scalar fields \( \phi^a(t,x) \) (\( a = 1,\ldots,n \)), \( N \) Weyl fermionic fields \( \psi^a_I(t,x) \) and their Hermitian conjugates \( \bar{\psi}^a_I(t,x) \) (\( a = 1,\ldots,N \)). Following to our previous work \[40\], we consider the Lagrangian whose fields carry up to the first derivative. The most general action is symbolized by

\[
S = \int d^4x \mathcal{L} \left[ \phi^\alpha, \partial_\mu \phi^\alpha, \psi^I, \partial_\mu \psi^I, \bar{\psi}^I, \partial_\mu \bar{\psi}^I \right].
\]

(1)

Here, the Lorentzian indices are raised and lowered by the Minkowski metric \( \eta_{\mu\nu} \), and the fermionic indices are raised and lowered by the anti-symmetric tensors \( \varepsilon_{\alpha\beta} \) and \( \varepsilon_{\dot{\alpha}\dot{\beta}} \). In addition, the capital Latin indices (\( I,J,K,\ldots \)) are contracted with the Kronecker delta \( \delta_{ij} \) such as \( \psi^I \psi_I, \bar{\psi}^I \bar{\psi}_I \). Throughout this paper, we use the metric signature \((+,−,−,−)\).

When the Lagrangian (1) consists only of scalar fields, the absence of Ostrogradsky’s ghosts is automatically ensured since the Euler-Lagrange equations contain no more than second derivatives with respect to time. On the other hand, second derivatives in fermionic equations of motion are generally dangerous because extra DOFs in the fermionic sector immediately lead to negative norm states (see \[39,40\] for the detail). In order to avoid such ghost DOFs, the system must contain the appropriate number of constraints, which originate from the degeneracy of the kinetic matrix. In this section, we derive degeneracy conditions and a supplementary condition for the Lagrangian (1). The basic treatment of Grassmann algebra and Hamiltonian formulation, which we are making use of, are summarized in e.g., \[39,40\], and we use left derivative throughout this paper.

We find degeneracy conditions of (1), which yield an appropriate number of primary constraints, eliminating half DOFs of fermions in phase space. For deriving the conditions, we first take a look at the canonical momenta defined as

\[
\pi^\mu_{\phi^a} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)}, \quad \pi^\mu_{\psi^I} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^I)}, \quad \pi^\mu_{\bar{\psi}^I} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}^I)} = -\left(\pi^\mu_{\psi^I}\right)^t.
\]

Then the variations of the 0-th component of canonical momenta with respect to all canonical variables give the following set of equations,

\[
\begin{pmatrix}
\delta \pi^\mu_{\phi^a} \\
\delta \pi^\mu_{\psi^I} \\
\delta \pi^\mu_{\bar{\psi}^I}
\end{pmatrix} = \mathbf{K}
\begin{pmatrix}
\delta \phi^a \\
\delta \psi^I \\
\delta \bar{\psi}^I
\end{pmatrix},
\]

(3)

where we have omitted the superscript of 0-th component, i.e., \( \pi_A \equiv \pi^0_A \), and defined the kinetic matrix,

\[
\mathbf{K} = \begin{pmatrix}
A_{ab} & B_{a\beta,J} \\
C_{ab,I} & D_{a\beta,J}
\end{pmatrix},
\]

(4)

and

\[
\begin{pmatrix}
\delta z^a \\
\delta \bar{z}^I \\
\delta \bar{z}_{\alpha,I}
\end{pmatrix} = \begin{pmatrix}
\delta (\bar{z}^a) \\
\delta \bar{z}^I \\
\delta \bar{z}_{\alpha,I}
\end{pmatrix} = \mathbf{K}^{-1}
\begin{pmatrix}
\delta \pi^\mu_{\phi^a} \\
\delta \pi^\mu_{\psi^I} \\
\delta \pi^\mu_{\bar{\psi}^I}
\end{pmatrix}.
\]

(5)
Here, we have introduced the shortcut notation,
\[ \mathcal{L}_{XY} = \frac{\partial^2 \mathcal{L}}{\partial Y \partial X} = \frac{\partial}{\partial Y} \left( \frac{\partial \mathcal{L}}{\partial X} \right) . \] 
(6)

Here, \( A_{ab} \) is a symmetric matrix, while \( D_{\alpha\beta,IJ}, D_{\alpha\beta,IJ}, D_{\alpha\beta,IJ} \), and \( D_{\alpha\beta,IJ} \) are anti-symmetric matrices under the exchange of the Greek indices as
\[ D_{\alpha\beta,IJ} = -D_{\beta\alpha,IJ}, \quad D_{\alpha\beta,IJ} = -D_{\beta\alpha,IJ}, \quad D_{\alpha\beta,IJ} = -D_{\beta\alpha,IJ} , \] 
(7)
\( B \) and \( C \) are Grassmann-odd matrices and related as \( C_{ab,I} = -B_{ab,I} \) and \( C_{\alpha I} = -B_{\alpha I} \). The Hermitian properties of \( \pi_{\alpha}^I \) and the anti-Hermitian properties of \( \pi_{\alpha}^I \), i.e., \( (\pi_{\alpha}^I)^I = \pi_{\alpha}^I \) and \( (\pi_{\alpha}^I)^I = -\pi_{\alpha}^I \), lead to
\[ A_{ab} = A_{ab}^\dagger, \quad B_{\alpha\beta,I} = -B_{\alpha\beta,I}^\dagger, \quad C_{ab,I} = -C_{ab,I}^\dagger, \quad D_{\alpha\beta,IJ} = -D_{\alpha\beta,IJ}^\dagger, \quad D_{\alpha\beta,IJ} = -D_{\alpha\beta,IJ}^\dagger . \] 
(8)

In the present paper, we assume that the scalar submatrix of the kinetic matrix \( A_{ab} \) is non-degenerate, i.e., invertible. This assumption is equivalent to requiring
\[ \det A_{ab}^{(0)} \neq 0 , \] 
(9)
where \( A_{ab}^{(0)} \) is defined by setting all fermionic variables and their derivatives to be zero. Then the first equation (3) can be solved for \( \delta \phi^b \),
\[ \delta \phi^b = A^{ba} \left( \delta \pi_{\alpha}^I - B_{ab,I} \delta \bar{\psi}_J^\dagger - B_{ab,I} \delta \bar{\psi}_J^\dagger - \delta z^a (\phi) \right) , \] 
(10)
where we have defined the inverse of the kinetic matrix in the scalar sector as \( A^{ab} = (A^{-1})^{ab} \). Plugging this into the infinitesimal momenta of \( \psi^a \) and \( \bar{\psi}^a \), we get
\[ \delta \pi_{I} = \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) \delta \bar{\psi}_J^\dagger + \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) \delta \bar{\psi}_J^\dagger + C_{ab,I} A^{ba} \left( \delta \pi_{\alpha}^I - \delta z^a (\phi) \right) + \delta z^a (\phi) , \] 
(11)
\[ \delta \bar{\psi}_J^\dagger = \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) \delta \bar{\psi}_J^\dagger + \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) \delta \bar{\psi}_J^\dagger + C_{ab,I} A^{ba} \left( \delta \pi_{\alpha}^I - \delta z^a (\phi) \right) + \delta z^a (\phi) . \] 
(12)

In order to obtain the sufficient number of constraints, we need conditions that the velocity terms \( \delta \dot{\psi}_I \) and \( \delta \dot{\bar{\psi}}_I \) cannot be expressed in terms of the canonical variables. In the present paper, we adopt the “maximally degenerate conditions” \(^1\), i.e.,
\[ D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} = 0, \quad D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} = 0 , \] 
(13)
\[ D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} = 0, \quad D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} = 0 . \] 
(14)

Two of these conditions are independent since they are related through Hermitian conjugates,
\[ \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right)^\dagger = -\left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) , \] 
(15)
\[ \left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right)^\dagger = -\left( D_{\alpha\beta,IJ} - C_{ab,I} A^{ba} B_{ab,J} \right) . \] 
(16)

The maximally degenerate conditions lead to the following 4N primary constraints,
\[ \Phi_{\psi_I} = \pi_{\psi_I} - F_{\alpha}, (\phi^a, \pi_{\phi^a}, \partial_1 \phi^a, \psi_J^\dagger, \bar{\psi}_J^\dagger, \bar{\psi}_J^\dagger, \partial_{\phi^a}) = 0 , \] 
(17)
\[ \Phi_{\bar{\psi}_I} = -\Phi_{\psi_I}^\dagger = \pi_{\bar{\psi}_I} - G_{\alpha}, (\phi^a, \pi_{\phi^a}, \partial_1 \phi^a, \psi_J^\dagger, \bar{\psi}_J^\dagger, \bar{\psi}_J^\dagger, \partial_{\phi^a}) = 0 , \] 
(18)
with \( G_{\alpha} = -(F_{\alpha})^\dagger \). The explicit proof of the equivalence between (17), (18) and (13), (14) is shown in the Appendix B. As discussed in [40], the existence of these primary constraints (17) and (18) based on the maximally degenerate conditions is enough to remove the extra DOFs in the fermionic sector. However, if these primary constraints yield secondary constraints, we have an even smaller number of physical DOFs. Here let us consider that we have the maximum number of physical DOFs in maximally degenerate Lagrangian, as we have with usual Weyl fields.

\(^1\) There should be other candidates for conditions eliminating ghost DOFs, but here, we adopt the simplest case, where all the constraints eliminating extra DOFs are primary constraints. See [40] for the other possibilities.
Then, we need to check that no secondary constraints appear by examining the consistency conditions of the primary constraints. To this end, we first define the total Hamiltonian density as

$$\mathcal{H}_T = \mathcal{H} + \Phi_{\psi}^{\dagger} \lambda_{I}^{\dagger} + \Phi_{\bar{\psi}}^{\dagger} \bar{\lambda}_{I}^{\dagger},$$

where the Hamiltonian and the total Hamiltonian are given by

$$H = \int d^3 x \mathcal{H}, \quad H_T = \int d^3 x \mathcal{H}_T.$$  

Now we would like to calculate the Poisson bracket, defined as

$$\{\mathcal{F}(t, x), \mathcal{G}(t, y)\} = \int d^3 z \left[ \frac{\delta \mathcal{F}(t, x)}{\delta \phi^\alpha(t, z)} \frac{\delta \mathcal{G}(t, y)}{\delta \phi^\beta(t, z)} - \frac{\delta \mathcal{F}(t, x)}{\delta \phi^\beta(t, z)} \frac{\delta \mathcal{G}(t, y)}{\delta \phi^\alpha(t, z)} \right]$$

$$\quad + (-)^{\alpha\beta} \left[ \frac{\delta \mathcal{F}(t, x)}{\delta \bar{\phi}^\alpha(t, z)} \frac{\delta \mathcal{G}(t, y)}{\delta \bar{\phi}^\beta(t, z)} - \frac{\delta \mathcal{F}(t, x)}{\delta \bar{\phi}^\beta(t, z)} \frac{\delta \mathcal{G}(t, y)}{\delta \bar{\phi}^\alpha(t, z)} \right].$$

The Poisson brackets between the canonical variables are given by

$$\{\phi^\alpha(t, x), \pi_{\phi^\alpha}(t, y)\} = \delta_\alpha^\beta \delta^3(x - y),$$

$$\{\psi^I(t, x), \pi_{\psi^I}(t, y)\} = -\delta_{IJ} \delta_{I\beta} \delta^3(x - y),$$

$$\{\bar{\psi}^I(t, x), \pi_{\bar{\psi}^I}(t, y)\} = -\delta_{IJ} \delta_{I\beta} \delta^3(x - y),$$

while other Poisson brackets are zero. Then, the Poisson brackets between the primary constraints are

$$\{\Phi_{\psi^I}(t, x), \Phi_{\psi^J}(t, y)\} = \frac{\delta F_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, y)} + \frac{\delta F_{\beta,J}(t, y)}{\delta \phi^\beta(t, x)} + \int d^3 z \left[ \frac{\delta F_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, z)} \frac{\delta F_{\beta,J}(t, y)}{\delta \phi^\beta(t, z)} - \frac{\delta F_{\alpha,I}(t, x)}{\delta \phi^\beta(t, z)} \frac{\delta F_{\beta,J}(t, y)}{\delta \phi^\alpha(t, z)} \right],$$

$$\{\Phi_{\psi^I}(t, x), \Phi_{\bar{\psi}^J}(t, y)\} = \frac{\delta G_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, y)} + \frac{\delta G_{\beta,J}(t, y)}{\delta \phi^\beta(t, x)} + \int d^3 z \left[ \frac{\delta G_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, z)} \frac{\delta G_{\beta,J}(t, y)}{\delta \phi^\beta(t, z)} - \frac{\delta G_{\alpha,I}(t, x)}{\delta \phi^\beta(t, z)} \frac{\delta G_{\beta,J}(t, y)}{\delta \phi^\alpha(t, z)} \right],$$

$$\{\Phi_{\bar{\psi}^I}(t, x), \Phi_{\bar{\psi}^J}(t, y)\} = \frac{\delta \bar{G}_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, y)} + \frac{\delta \bar{G}_{\beta,J}(t, y)}{\delta \phi^\beta(t, x)} + \int d^3 z \left[ \frac{\delta \bar{G}_{\alpha,I}(t, x)}{\delta \phi^\alpha(t, z)} \frac{\delta \bar{G}_{\beta,J}(t, y)}{\delta \phi^\beta(t, z)} - \frac{\delta \bar{G}_{\alpha,I}(t, x)}{\delta \phi^\beta(t, z)} \frac{\delta \bar{G}_{\beta,J}(t, y)}{\delta \phi^\alpha(t, z)} \right].$$

Then the time-evolution of the primary constraints are

$$\left( \dot{\Phi}_{\psi^I}(t, x), \dot{\Phi}_{\bar{\psi}^J}(t, x) \right) = \left( \{\Phi_{\psi^I}(t, x), H_T\}, \{\Phi_{\psi^J}(t, x), H_T\} \right) = \left( \{\Phi_{\psi^I}(t, x), H\}, \{\Phi_{\psi^J}(t, x), H\} \right) + \int d^3 y C_{IJ}(t, x, y) \left( \lambda_{J}^{\dagger}(t, y), \lambda_{J}^{\dagger}(t, y) \right) \approx 0,$$

where

$$C_{IJ}(t, x, y) = \left( \{\Phi_{\psi^I}(t, x), \Phi_{\psi^J}(t, y)\}, \{\Phi_{\bar{\psi}^I}(t, x), \Phi_{\bar{\psi}^J}(t, y)\}, \{\Phi_{\bar{\psi}^I}(t, x), \Phi_{\bar{\psi}^J}(t, y)\}, \{\Phi_{\psi^I}(t, x), \Phi_{\psi^J}(t, y)\} \right) .$$

In order not to have secondary constraints, all the Lagrange multipliers should be fixed by the equations (28). This can be realized if the coefficient matrix of the Lagrange multipliers (29) has the inverse after integrating over y, and then the primary constraints (17) and (18) are second class. In this case, the number of DOFs is

$$\text{DOFs} = \frac{2 \times n \text{ (boson)} + 2 \times 4N \text{ (fermions)} - 4N \text{ (constraints)}}{2} = n \text{ (boson)} + 2N \text{ (fermions)},$$

as desired. To find explicit expressions of these obtained conditions, one needs a concrete Lagrangian form. Hereafter, we will consider the most simplest case $N = 1$. The extension to multiple Weyl fields can be done in the same way although the analysis will be tedious. A brief introduction for constructing Lagrangians for arbitrary $N$ are given in the Appendix C.
III. DEGENERATE SCALAR-FERMION THEORIES

Since the fermionic fields obey the Grassmann algebra, one can dramatically simplify the analysis by restricting the number of fermionic fields. Hereafter, we confine the theory to \( n \) scalar fields and 1 Weyl field to simplify the analysis. In this section, we construct the most general scalar-fermion theory whose Lagrangian contains up to quadratic in first derivatives of scalar and fermionic fields. To this end, we first construct Lorentz scalars and vectors without any derivatives, which consist of the scalar fields \( \phi^a \) and the fermionic fields \( \psi^a, \bar{\psi}^a \). The fermionic indices can be contracted with the building block matrices, \( \varepsilon_{a\bar{b}}, \sigma^a_{\alpha\bar{a}}, (\sigma^{\mu\nu} \varepsilon)_{a\bar{b}}, \) and \((\varepsilon \sigma^{\mu\nu})_{a\bar{b}}\), and we have three possibilities

\[
\Psi = \psi^a \psi_\alpha, \quad \bar{\Psi} = \bar{\psi}_\bar{a} \bar{\psi}^\alpha, \quad J^\mu = \bar{\psi}^\alpha \sigma^a_{\alpha\bar{a}} \psi^a,
\]

where \( \Psi^\dagger = \bar{\Psi} \). Note that any contractions of \((\sigma^{\mu\nu} \varepsilon)_{a\bar{b}}\) and \((\varepsilon \sigma^{\mu\nu})_{a\bar{b}}\) with the fermionic fields always vanish due to the Grassmann properties\(^2\). One can also construct a scalar quantity from the square of \( J^\mu \) contracted with the Minkowski metric. However, it reduces to

\[
\eta_{\mu\nu} J^\mu J^\nu = -2 \Psi \bar{\Psi},
\]

where we have used (A4). Furthermore, the square of \( \Psi \) vanishes because of the fact that \( \Psi^2 \propto \psi^1 \psi^2 \bar{\psi}^2 \psi^1 \) and the Grassmann property, i.e., \( \Psi^2 = \bar{\Psi}^2 = 0 \), while \( \Psi \bar{\Psi} \) is non-zero. This implies that an arbitrary function \( A(\phi^a, \Psi, \bar{\Psi}) \) can be expanded in terms of \( \Psi \) and \( \bar{\Psi} \) as

\[
A(\phi^a, \Psi, \bar{\Psi}) = a_0(\phi^a) + a_1(\phi^a) \Psi + a_2(\phi^a) \bar{\Psi} + a_3(\phi^a) \Psi \bar{\Psi}.
\]

When the arbitrary function \( A \) is real i.e., \( A = A^\dagger \), then \( a_0 \) and \( a_3 \) are also real, and \( a_1^* = a_2 \). The inverse of \( A(\phi^a, \Psi, \bar{\Psi}) \) is given by

\[
A^{-1}(\phi^a, \Psi, \bar{\Psi}) = a_0^{-1}(\phi^a) \left( 1 - \frac{a_2(\phi^a)}{a_0(\phi^a)} \Psi - \frac{2a_1(\phi^a)a_2(\phi^a) - a_0(\phi^a)a_3(\phi^a)}{a_0(\phi^a)^2} \Psi \bar{\Psi} \right),
\]

and the condition that \( A(\phi^a, \Psi, \bar{\Psi}) \) has an inverse is simply \( a_0(\phi^a) \neq 0 \).

As we will show in the following subsection, the most general action up to quadratic in first derivatives of the scalar fields and the Weyl fermion can be written as

\[
S = \int d^4 x \left( \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \right), \quad (35)
\]

where

\[
\mathcal{L}_0 = P_0, \quad (36)
\]

\[
\mathcal{L}_1 = P^{(1)}_a \partial_\mu \phi^a J^\mu + P^{(2)}_a \left( \bar{\psi}^\alpha \sigma^a_{\alpha\bar{a}} \partial_\mu \psi^a - \partial_\mu \bar{\psi}^\alpha \sigma^a_{\alpha\bar{a}} \psi^a \right) + P^{(3)} \bar{\psi}^\alpha \partial_\mu \psi^a + P^{(3)} \bar{\psi}^\alpha \partial_\mu \bar{\psi}^\alpha, \quad (37)
\]

\[
\mathcal{L}_2 = \frac{1}{2} W^{\mu\nu}_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + S^{(1)}_{\alpha\bar{a}} \partial_\mu \psi^a \partial_\nu \psi^\alpha + \frac{1}{2} W^{\mu\nu}_{ab} \partial_\mu \bar{\psi}^\alpha \partial_\nu \bar{\psi}^\alpha + \bar{\psi}^\alpha (S^{(2)}_{\alpha\bar{a}}),
\]

\[
V^{\mu\nu}_{ab} = V_{ab} \eta^{\mu\nu}, \quad (38)
\]

\[
S^{(1)}_{\alpha\bar{a}} = S^{(1)}_{a\bar{a}} \psi_\alpha + S^{(2)}_{a\bar{a}} (\sigma^{\mu\nu} \varepsilon)_{a\bar{b}} \psi^\beta, \quad \varepsilon \sigma^{\mu\nu}, \quad (39)
\]

Here, \( P_0, P^{(i)}_a, V, S^{(i)}_a, W_1, W_3, Q_1, Q_3 \) are arbitrary functions of \( \phi^a, \Psi, \bar{\Psi} \). These arbitrary functions satisfy the following properties:

\[
V_{ab} = V_{ba}, \quad V_{ab} = V_{ab}^\dagger, \quad P_0 = P_0^\dagger, \quad P^{(1)}_a = P^{(1)}_a^\dagger, \quad P^{(2)} = -P^{(2)}^\dagger, \quad Q_1 = Q_1^\dagger, \quad Q_3 = Q_3^\dagger. \quad (40)
\]

\(^2\) See Appendix A for the detailed definition of each matrix.

\(^3\) For \( N \neq 1 \), \((\sigma^{\mu\nu} \varepsilon)_{a\bar{b}}\) and \((\varepsilon \sigma^{\mu\nu})_{a\bar{b}}\) can be contracted with different Weyl fields, and therefore rank-2 tensors exist. See Appendix C for the detail.
A. Construction of Lagrangian up to quadratic in the derivatives

In this subsection, we derive the most general Lagrangian \((35)-(40)\) one by one.

1. No derivative term

The Lagrangian without any derivatives should be the arbitrary function of the form in \((33)\), thus

\[
\mathcal{L}_0 = P_0(\phi^a, \Psi, \bar{\Psi}),
\]

where \(P_0\) is an arbitrary function of \(\phi^a, \Psi, \) and \(\bar{\Psi}\), and the Hermitian property of the Lagrangian requires \(P_0 = P_0^\dagger\).

2. Linear terms in the derivatives

The candidate for the Lagrangian including terms proportional to the first derivatives is

\[
\mathcal{L}_1 = P_1^{(1)} \frac{\partial}{\partial x} \phi^a J^\mu + P_1^{(2)} \frac{\partial}{\partial x} \bar{\psi}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_{\mu} \psi^\alpha + P_1^{(2)^\dagger} \frac{\partial}{\partial x} \bar{\psi}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_{\mu} \bar{\psi}^\dot{\alpha} + P_1^{(3)} J^\mu \psi^\alpha \bar{\psi}^\dot{\alpha} + P_1^{(3)^\dagger} J^\mu \partial_{\mu} \bar{\psi}^\dot{\alpha} \bar{\psi}^\dot{\alpha},
\]

where \(P_1^{(i)}\) are arbitrary functions of \(\phi^a, \Psi, \) and \(\bar{\Psi}\), and the Hermitian property of the Lagrangian requires \(P_1^{(1)} = P_1^{(1)^\dagger}\).

If we integrate the first term in the right hand side by parts, it becomes

\[
-\frac{1}{2} (P_1^{(2)} + P_1^{(2)^\dagger}) \partial_{\mu} \phi^a J^\mu + \frac{1}{2} (P_1^{(2)} - P_1^{(2)^\dagger}) \bar{\psi}^\alpha \partial_{\mu} \psi^\alpha + \frac{1}{2} (P_1^{(2)^\dagger} - P_1^{(2)}) \bar{\psi}^\alpha \partial_{\mu} \bar{\psi}^\dot{\alpha} \bar{\psi}^\dot{\alpha}
\]

where \(P_{1z} \equiv \partial P_{1}/\partial z\) and \(P_{1z}^\dagger \equiv \partial P_{1}^\dagger/\partial z\). Thus the real part of \(P_1^{(2)}\) can be absorbed into the other terms in \(\mathcal{L}_1\), and thus we can impose \(P_1^{(2)} = -P_1^{(2)^\dagger}\) without loss of generality.

3. Quadratic terms in the derivatives

Next, the Lagrangian containing two derivatives is

\[
\mathcal{L}_2 = \frac{1}{2} V_{ab}^{\mu\nu} \partial_{\mu} \phi^a \partial_{\nu} \phi^b + S_{\alpha\beta}^{\mu\nu} \partial_{\mu} \phi^a \partial_{\nu} \psi^\alpha + \partial_{\mu} \phi^a \partial_{\nu} \bar{\psi}^\dot{\alpha} (S_{\alpha\dot{\alpha}}^{\mu\nu})^\dagger
\]

\[+ \frac{1}{2} W_{\alpha\beta}^{\mu\nu} \partial_{\mu} \psi^\alpha \partial_{\nu} \bar{\psi}^{\dot{\beta}} + \frac{1}{2} \partial_{\mu} \bar{\psi}^{\dot{\beta}} \partial_{\nu} \bar{\psi}^{\dot{\alpha}} (W_{\alpha\dot{\alpha}}^{\mu\nu})^\dagger + \partial_{\mu} \bar{\psi}^{\dot{\alpha}} Q_{\alpha\dot{\alpha}}^{\mu\nu} \partial_{\nu} \psi^\alpha,
\]

where \(V_{ab}^{\mu\nu}, S_{\alpha\beta}^{\mu\nu}, W_{\alpha\beta}^{\mu\nu}, \) and \(Q_{\alpha\dot{\alpha}}^{\mu\nu}\) consist of \(\phi^a, \psi^\alpha, \bar{\psi}^\dot{\alpha}\), and constant matrices. The coefficient \(V_{ab}^{\mu\nu}\) should take the form,

\[
V_{ab}^{\mu\nu} = V_{ab}^{(1)} \eta^{\mu\nu} + V_{ab}^{(2)} J^\mu J^\nu,
\]

where \(V_{ab}^{(i)}\) are arbitrary functions of \(\phi^a, \Psi, \) and \(\bar{\Psi}\). However, the identity \((A6)\) leads to

\[
J^\mu J^\nu = -\frac{1}{2} \eta^{\mu\nu} \bar{\Psi} \Psi.
\]

Therefore, \(V_{ab}^{(2)}\) term can be absorbed into \(V_{ab}^{(1)}\) term, and we can generally set \(V_{ab}^{(2)} = 0\). As a result, we have

\[
V_{ab}^{\mu\nu} = V_{ab}^{(1)} \eta^{\mu\nu},
\]

where \(V_{ab}\) is an arbitrary real function of \(\phi^a, \Psi, \) and \(\bar{\Psi}\), and symmetric under the exchange of \(a\) and \(b\).

Secondly, the general form of \(S_{\alpha\beta}^{\mu\nu}\) is

\[
S_{\alpha\beta}^{\mu\nu} = S_a^{(1)} \eta^{\mu\nu} \psi^\alpha + S_a^{(2)} J^\mu \bar{\psi}^{\dot{\alpha}} \sigma^\nu_{\alpha\dot{\alpha}} + S_a^{(3)} J^\nu \bar{\psi}^{\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} + S_a^{(4)} J^\mu J^\nu \psi^\alpha,
\]

where \(S_a\) is an arbitrary function of \(\phi^a, \Psi, \) and \(\bar{\Psi}\), and symmetric under the exchange of \(a\) and \(b\).
where $S_a^{(i)}$ are arbitrary functions of $\phi^\alpha, \Psi$, and $\bar{\Psi}$. By using (A5) and (A6), the second and third terms can be rewritten as

$$J^\mu \bar{\psi}^\alpha \sigma_\alpha^\nu = \frac{1}{2} \bar{\Psi} \eta^\mu \psi_\alpha + \bar{\Psi} (\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta,$$  

(50)

$$J^\nu \bar{\psi}^\beta \sigma_\alpha^\alpha = \frac{1}{2} \bar{\Psi} \eta^\mu \psi_\alpha - \bar{\Psi} (\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta.$$  

(51)

Therefore, the second and third terms can be compactly expressed as $S_a^{(2)}(\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta$, which is anti-symmetric under $\mu$ and $\nu$, by absorbing the symmetric remaining parts into $S_a^{(1)}$. The last term with $S_a^{(4)}$ is proportional to $\psi^1 \psi^1$ or $\psi^2 \psi^2$ in a component expression, thus this automatically vanishes. As a result, we can assign

$$S_a^{\mu\nu} = S_a^{(1)} \eta^\mu \psi_\alpha + S_a^{(2)}(\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta.$$  

(52)

Thirdly, the general form of $W_a^{\mu\nu}$ is

$$W_a^{\mu\nu} = W_a \eta^\mu \varepsilon_{\alpha\beta} + W_2 \eta^\mu \psi_\alpha \psi_\beta + \bar{W}_3 (\bar{\psi}^\alpha \sigma_\alpha^\mu) (\bar{\psi}^\beta \sigma_\beta^\nu) + \bar{W}_4 (\bar{\psi}^\alpha \sigma_\alpha^\mu) (\bar{\psi}^\beta \sigma_\beta^\nu) J + W_3 J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\alpha^\mu + W_4 J^\nu \psi_\alpha \bar{\psi}^\beta \sigma_\beta^\nu + W_7 J^\mu J^\nu \psi_\alpha \psi_\beta.$$  

(53)

We note that $J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\alpha^\mu \simeq -J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\beta^\mu$ and $J^\nu \psi_\beta \bar{\psi}^\alpha \sigma_\beta^\nu \simeq -J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\beta^\mu$, hold as long as they are contracted with $\partial_\mu \psi^\alpha \partial_\nu \psi^\beta$. Similarly to the case of $S^{(2)}$ and $S^{(3)}$, we can rewrite $\bar{W}_3, \bar{W}_4, W_5$, and $W_6$ as

$$\bar{W}_3^\mu = \frac{1}{2} \bar{\Psi} \eta^\mu \varepsilon_{\alpha\beta} - \bar{\Psi} (\sigma_{\mu\nu})_{\alpha\beta},$$  

(54)

$$\bar{W}_4^\mu = \frac{1}{2} \bar{\Psi} \eta^\mu \varepsilon_{\alpha\beta} + \bar{\Psi} (\sigma_{\mu\nu})_{\alpha\beta},$$  

(55)

$$J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\alpha^\mu = \frac{1}{2} \bar{\Psi} \eta^\mu \varepsilon_{\alpha\beta} \psi_\beta + \bar{\Psi} (\sigma_{\mu\nu})_{\beta\gamma} \psi_\gamma,$$  

(56)

$$J^\nu \psi_\alpha \bar{\psi}^\beta \sigma_\beta^\nu = - \frac{1}{2} \bar{\Psi} \eta^\mu \varepsilon_{\alpha\beta} \psi_\beta - \bar{\Psi} (\sigma_{\mu\nu})_{\beta\gamma} \psi_\gamma.$$  

(57)

Thus, the symmetric parts of $\bar{W}_3, \bar{W}_4$ and $W_5, W_6$ can be expressed in terms of $\eta^\mu \varepsilon_{\alpha\beta}$ and $\eta^\mu \psi_\alpha \psi_\beta$. On the other hand, anti-symmetric parts of them are taken into account by $\sigma_{\mu\nu}$ and $\psi_\alpha (\sigma_{\mu\nu})_{\beta\gamma} \psi_\gamma$. As before, the last term with $W_7$ is proportional to $\psi^1 \psi^1$ in a component expression, thus this automatically vanishes. As a result, we can express $W_a^{\mu\nu}$ as

$$W_a^{\mu\nu} = W_1 \eta^\mu \varepsilon_{\alpha\beta} + W_2 \eta^\mu \psi_\alpha \psi_\beta + W_3 (\sigma_{\mu\nu})_{\alpha\beta} + W_4 \left[ \psi_\alpha (\sigma_{\mu\nu})_{\beta\gamma} + \psi_\beta (\sigma_{\mu\nu})_{\alpha\gamma} \right] \psi_\gamma.$$  

(58)

$W_3$ term is manifestly symmetrized by making use of the fact that $\psi_\alpha (\sigma_{\mu\nu})_{\beta\gamma} \psi_\gamma \simeq \bar{\psi}_\beta (\sigma_{\mu\nu})_{\alpha\gamma} \psi_\gamma$ in the Lagrangian, which holds again since they are contracted with $\partial_\mu \psi^\alpha \partial_\nu \psi^\beta$.

Finally, let us take a look at $Q_a^{\mu\nu}$. The general form is given by

$$Q_a^{\mu\nu} = Q_1 \eta^\mu \psi_\alpha \bar{\psi}_\alpha + \bar{Q}_2 (\bar{\psi}^\beta \sigma_\beta^\mu) (\bar{\psi}^\alpha \sigma_\alpha^\nu) + \bar{Q}_3 (\bar{\psi}^\beta \sigma_\beta^\mu) (\bar{\psi}^\nu \sigma_\nu^\alpha) + Q_4 J^\mu \sigma_\alpha^\nu + Q_5 J^\nu \sigma_\alpha^\mu + Q_6 J^\mu J^\nu \psi_\alpha \psi_\beta + Q_7 J^\mu \psi_\alpha \bar{\psi}^\beta \sigma_\alpha^\nu + Q_8 J^\nu \psi_\alpha \bar{\psi}^\beta \sigma_\beta^\mu + Q_9 J^\mu \bar{\psi}_\alpha \bar{\psi}_\beta \sigma_\alpha^\nu + Q_{10} J^\nu \bar{\psi}_\alpha \bar{\psi}_\beta \sigma_\beta^\mu.$$  

(59)

The last five terms, $Q_{6,7,8,9,10}$, contains more than one $\psi^1$, $\psi^2$, $\bar{\psi}^1$ or $\bar{\psi}^2$, and therefore they automatically vanish. We can rewrite $\bar{Q}_2, \bar{Q}_3, Q_4$, and $Q_5$ as

$$\bar{Q}_2^\mu = - \frac{1}{2} \bar{\Psi} \eta^\mu \varepsilon_{\alpha\beta} - \bar{\Psi} (\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta - \psi_\alpha (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta - 2(\sigma_{\mu\nu})_{\alpha\beta} \psi_\beta (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta,$$  

(60)

$$\bar{Q}_3^\mu = - \frac{1}{2} \eta^\mu \psi_\alpha \bar{\psi}_\alpha -(\sigma_{\nu\mu})_{\alpha\beta} \bar{\psi}_\beta \psi_\alpha - \psi_\alpha (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta - 2(\sigma_{\nu\mu})_{\alpha\beta} \psi_\beta (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta,$$  

(61)

$$J^\mu \sigma_\alpha^\nu = \frac{1}{2} \eta^\mu \psi_\alpha \bar{\psi}_\alpha -(\sigma_{\nu\mu})_{\alpha\beta} \bar{\psi}_\beta \psi_\alpha + \psi_\alpha (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta - 2(\sigma_{\nu\mu})_{\alpha\beta} \psi_\beta (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta,$$  

(62)

$$J^\nu \sigma_\alpha^\mu = \frac{1}{2} \eta^\nu \psi_\alpha \bar{\psi}_\alpha -(\sigma_{\nu\mu})_{\alpha\beta} \bar{\psi}_\beta \psi_\alpha - \psi_\alpha (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta - 2(\sigma_{\nu\mu})_{\alpha\beta} \psi_\beta (\varepsilon \sigma_{\mu\nu})_{\alpha\beta} \bar{\psi}_\beta.$$  

(63)
Note that the last terms are symmetric under the replacement of $\mu$ and $\nu$, as shown in (A10). Thus, four functions from $Q_{1,4,5}$ and $Q_{2,3}$ are independent. We adopt $\eta^{\mu\nu}\psi_\alpha \tilde{\psi}_\beta$, $(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_\beta \tilde{\psi}_\alpha$, $\psi_\alpha (\varepsilon \tilde{\sigma}^{\mu\nu})_{\alpha\beta} \tilde{\psi}_\beta$, $(\epsilon^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_\beta \tilde{\psi}_\alpha$ as independent functions, and then we have
\[
Q^{\mu\nu}_{\alpha\alpha} = Q_1 \eta^{\mu\nu} \psi_\alpha \tilde{\psi}_\alpha + Q_2 (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_\beta \tilde{\psi}_\alpha - Q_4 \psi_\alpha \tilde{\psi}_\beta (\varepsilon \tilde{\sigma}^{\mu\nu})_{\beta\alpha} + Q_3 (\epsilon^{\mu\nu}\varepsilon)_{\alpha\beta}\psi_\beta \tilde{\psi}_\alpha .
\] (64)
Since the first and the fourth terms are Hermite, the coefficients are required to be real.

Collecting all the results, we obtain the most general Lagrangian (35)-(40).

### B. Degeneracy conditions

In the previous subsection, we have obtained the most general Lagrangian of $n$ scalar and one Weyl fields which contains up to quadratic in first derivatives. As explained earlier, the Euler-Lagrange equations, in general, contain second derivatives of the fermionic fields. Thus, one should choose the arbitrary functions appearing in the Lagrangian with care in order to have the correct number of DOFs. In this subsection, we derive the degeneracy conditions for the Lagrangian (38) with (39). (Note that linear Lagrangian in derivatives is irrelevant to the degeneracy conditions.)

The maximally degenerate conditions (13) and (14) lead to
\[
D_{\alpha\beta} - C_{ab} A^{ba} B_{\alpha\beta} = W_1 \varepsilon_{\alpha\beta} + \left( W_2 + S_b^{(1)} V^{ba} S_a^{(1)} \right) \psi_\alpha \tilde{\psi}_\beta = 0 ,
\]
\[
D_{\alpha\beta} - C_{ab} A^{ba} B_{\alpha\beta} = - \left( Q_1 + S_b^{(1)} V^{ba} S_a^{(1)} \right) \psi_\alpha \tilde{\psi}_\beta - Q_3 (\sigma^{0\varepsilon})_{\alpha\beta} \psi_\gamma \tilde{\psi}_\gamma (\varepsilon \tilde{\sigma}^{0\varepsilon})_{\gamma\beta} = 0 ,
\]
where $V^{ab} = (V^{-1})^{ab}$. In general, they give us four conditions$^4$,
\[
W_1 = 0, \quad (W_2 + S_b^{(1)} V^{ba} S_a^{(1)}) \psi_\alpha \tilde{\psi}_\beta = 0, \quad (Q_1 + S_b^{(1)} V^{ba} S_a^{(1)} \right) \psi_\alpha \tilde{\psi}_\beta = 0, \quad Q_3 \psi_\gamma \tilde{\psi}_\gamma = 0 .
\] (67)

The functions included in $L_2$ are simplified after inserting these four conditions:
\[
V^{\mu\nu}_{\alpha\beta} = V_{ab} \eta^{\mu\nu} ,
\]
\[
S^{\mu\nu}_{\alpha\alpha} = S_a^{(1)} \eta^{\mu\nu} \psi_\alpha + S_a^{(2)} (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \tilde{\psi}_\beta ,
\]
\[
W^{\mu\nu}_{\alpha\beta} = - S_b^{(1)} V^{ba} S_a^{(1)} \eta^{\mu\nu} \psi_\alpha \tilde{\psi}_\beta + W_3 (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} + W_4 \left[ \psi_\alpha (\sigma^{\mu\nu}\varepsilon)_{\beta\gamma} + \psi_\beta (\sigma^{\mu\nu}\varepsilon)_{\alpha\gamma} \right] \tilde{\psi}_\gamma ,
\]
\[
Q^{\mu\nu}_{\alpha\alpha} = - S_b^{(1)} V^{ba} (S_a^{(1)})^\dagger \eta^{\mu\nu} \psi_\alpha \tilde{\psi}_\alpha + Q_2 (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \psi_\beta \tilde{\psi}_\alpha - Q_2 \psi_\alpha \tilde{\psi}_\beta (\varepsilon \tilde{\sigma}^{\mu\nu})_{\beta\alpha} .
\] (68)

### IV. HAMILTONIAN FORMULATION

In the previous section, we have derived the Lagrangian satisfying the degeneracy conditions which lead to 4 primary constraints, as explicitly shown in the following. They suggest that the extra DOFs associated with fermionic Ostrogradsky’s ghosts are removed; however, one need to make sure of the condition that no more (secondary) constraints arise in addition to have 4 physical DOFs in the phase space. Following Sec. II, we derive such the condition.

The momenta can be directly calculated from their definition,
\[
\pi_\phi = P_a^{(1)} J^0 + V_{ab} \phi^b + S_{ab}^{0\nu} \partial_\nu \psi^a + \partial_\nu \tilde{\psi}^a (S_{ab}^{0\nu})^\dagger ,
\]
\[
\pi_\psi = - P^{(2)} \tilde{\psi}^a (S_{ab}^{0\nu})^\dagger - P^{(3)} J^0 \psi_\alpha - S_{ab}^{0\nu} \partial_\nu \phi^a + W_{ab}^{0\nu} \partial_\nu \psi^\beta - \partial_\nu \tilde{\psi}^\beta Q_{ab}^{0\nu} ,
\]
\[
\pi_\tilde{\psi} = - (\pi_\psi)^\dagger .
\] (69)

Solving (69) for $\dot{\phi}^a$ and plugging it into (70) and (71), we indeed obtain primary constraints (17) and (18), where $F_\alpha$ and $G_\alpha$ are given by
\[
F_\alpha = - P^{(2)} \tilde{\psi}^a (S_{ab}^{0\nu})^\dagger - P^{(3)} J^0 \psi_\alpha - S_{ab}^{0\nu} V^{ab} \left[ \pi_\phi - P_{b}^{(1)} J^0 - S_{ab}^{0\nu} \partial_\nu \psi^a - \partial_\nu \tilde{\psi}^a (S_{ab}^{0\nu})^\dagger \right] ,
\]
\[
- S_{ab}^{0\nu} \partial_\nu \phi^a + W_{ab}^{0\nu} \partial_\nu \psi^\beta - \partial_\nu \tilde{\psi}^\beta Q_{ab}^{0\nu} ,
\]
\[
G_\alpha = -(F_\alpha)^\dagger .
\] (72)

$^4$ As far as we seek a healthy theory where we have no ghost not only in background but also in the perturbations, these four conditions are reasonable, e.g., even if the background evolution is like $\psi_\alpha \psi_\beta \propto \varepsilon_{\alpha\beta}$ as a result of the equations of motion, the perturbations on it will face serious instabilities.
Then, plugging these expression into (25) and (26) and picking up the bosonic parts, we obtain

\[
\{\Phi_\omega(t, x), \Phi_\phi(t, y)\}^{(0)} = 2(S_a(2))^{(0)} \partial_i \phi^a (\sigma^{0i} \varepsilon)_{\alpha\beta} \delta(x - y) + (\sigma^{0i} \varepsilon)_{\alpha\beta} \left[ W_3^{(0)}(x) \frac{\partial}{\partial x^i} \delta(x - y) + W_3^{(0)}(y) \frac{\partial}{\partial y^i} \delta(x - y) \right],
\]

(74)

\[
\{\Phi_\omega(t, x), \Phi_\phi(t, y)\}^{(0)} = -2(P^{(2)})^{(0)} \sigma^{0i} \delta(x - y),
\]

(75)

where (0) represents their bosonic parts. When we calculate the time derivative of the primary constraints, we integrate the product of the constraints matrix and the Lagrange multipliers over \(y\). Focusing on the second term in (74),

\[
\int d^3y (\sigma^{0i} \varepsilon)_{\alpha\beta} \left[ W_3^{(0)}(x) \frac{\partial}{\partial x^i} \delta(x - y) + W_3^{(0)}(y) \frac{\partial}{\partial y^i} \delta(x - y) \right] \lambda^\beta(t, y)
\]

\[
= - \int d^3y (\sigma^{0i} \varepsilon)_{\alpha\beta} \frac{\partial}{\partial y^i} \left( W_3^{(0)}(y) \right) \delta(x - y) \lambda^\beta(t, y)
\]

(76)

holds as far as we drop total derivative terms, where we have used the fact that \((\partial/\partial x^i)\delta(x - y) = -(\partial/\partial y^i)\delta(x - y)\). Using this result and the following identities,

\[
\{\Phi_\omega(t, x), \Phi_\phi(t, y)\} = -\{\Phi_\phi(t, x), \Phi_\omega(t, y)\}^\dagger, \quad \{\Phi_\phi(t, x), \Phi_\phi(t, y)\} = -\{\Phi_\omega(t, x), \Phi_\phi(t, y)\}^\dagger,
\]

(77)

we can write the bosonic part of the constraint matrix (29) as

\[
\mathbf{C}^{(0)}(t, x, y) = \begin{pmatrix} 2S_a(2) \partial_i \phi^a - \partial_i W_3 \delta^{0i} \varepsilon_{\alpha\beta} & -2P^{(2)} \sigma^{0i}_{\alpha\beta} \\ -2P^{(2)} \sigma^{0i}_{\alpha\beta} & -(2S_a(2))^{(0)} \partial_i \phi^a - \partial_i W_3 \delta^{0i} \varepsilon_{\alpha\beta} \end{pmatrix} \delta(x - y).
\]

(78)

After performing the integration in (28), we can define the matrix \(\mathbf{J}_H^{(0)}(t, x)\), thanks to the delta function, as

\[
\mathbf{J}_H^{(0)}(t, x) = \int d^3y \mathbf{C}^{(0)}(t, x, y).
\]

(79)

If this matrix \(\mathbf{J}_H^{(0)}(t, x)\) is invertible, i.e., the determinant is non-zero\(^5\),

\[
\det \mathbf{J}_H^{(0)}(t, x) \neq 0,
\]

(80)

all the Lagrange multipliers introduced in the total Hamiltonian (19) are determined by solving the simultaneous equations (28). In this case, the theory does not have secondary constraints, and all the (primary) constraints are second class. Therefore, the number of DOFs is \(n + 2\) as counted in (30), and the extra DOFs are properly removed.

V. EULER-LAGRANGE EQUATIONS

In this section, we show that the degeneracy conditions and the supplementary condition obtained in Hamiltonian formulation can be also derived in Lagrangian formulation. In addition, we show that if the supplementary condition is satisfied, the equations of motion for fermions can always be solved in terms of the first derivative of fermionic fields.

A. Equations of motion for fermions with the maximally degenerate conditions

We derive the equations of motion for the fields and see the EOMs for fermions become the first-order (non-linear) differential equations after applying the maximally degenerate conditions.

\(^5\) Allowing the theory to have solutions with \(\phi^a = \phi^a(t)\) requires non-vanishing \(P^{(2)}\).
The variations with respect to the scalar and Weyl fields yield a set of Euler-Lagrange equations,

\[
\left( \begin{array}{c}
\frac{V_{ab}^{00}}{S_{a\beta}^{00}} & \frac{S_{a\beta}^{00}}{Q_{\alpha\beta}^{00}} - \frac{S_{a\beta}^{00}}{Q_{\alpha\beta}^{00}} \\
-W_{a\beta}^{00} & \frac{W_{a\beta}^{00}}{Q_{\alpha\beta}^{00}} - \frac{Q_{\alpha\beta}^{00}}{Q_{\alpha\beta}^{00}} \\
(S_{ba}^{00})^\dagger & Q_{\beta\alpha}^{00} - (W_{\alpha\beta}^{00})^\dagger
\end{array} \right)
\left( \begin{array}{c}
\dot{\phi}^a \\
\dot{\bar{\psi}}^a \\
\dot{E}_a \\
\dot{\bar{E}}_a
\end{array} \right) = \left( \begin{array}{c}
E_a \\
\bar{E}_a
\end{array} \right) .
\]

(81)

where the coefficient matrix in the left hand side corresponds the kinetic matrix in (4), and we have defined

\[
E_a = \frac{\partial L}{\partial \phi^a} - \partial_\mu (P^{(1)} J^\mu) - \partial_\mu V^{\mu\nu} \partial_\nu \phi^a - \partial_\mu S_{a\alpha}^{\mu\nu} \partial_\nu \psi^a - \partial_\mu \bar{\psi}^a \partial_\mu (S_{a\alpha}^{\mu\nu})^\dagger \\
- V_{ab}^{ij} \partial_\mu \phi^b - S_{a\alpha}^{ij} \partial_\mu \psi^a - \partial_\mu \bar{\psi}^a \partial_\mu (S_{a\alpha}^{ij})^\dagger,
\]

(82)

\[
E_\alpha = \frac{\partial L}{\partial \alpha^a} + \partial_\mu (P^{(2)} \bar{\psi}^a) \sigma_{\alpha\alpha}^{a\beta} + \partial_\mu (P^{(3)} J^\mu \psi_\alpha) + \partial_\mu S_{\alpha\alpha}^{\mu\nu} \partial_\nu \phi^a - \partial_\mu W_{a\beta}^{\mu\nu} \partial_\nu \psi^a + \partial_\mu \bar{\psi}^a \partial_\mu Q_{\alpha\alpha}^{\mu\nu} \\
+ S_{\alpha\alpha}^{ij} \partial_\mu \phi^a - W_{a\beta}^{ij} \partial_\mu \psi^a + 2 \partial_\mu \bar{\psi}^a \partial_\mu (\bar{\psi}^a)^{(0)i) + \partial_\mu \bar{\psi}^a \partial_\mu Q_{\alpha\alpha}^{ij}.
\]

(83)

where \((0i)\) means symmetrized with respect to 0 and \(i\), and \(E_\alpha = -(E_\alpha)^\dagger\). Solving (82) for \(\ddot{\phi}^a\), we obtain

\[
\ddot{\phi}^a = V^{ab} \left[ E_b - S_{ba}^{00} \bar{\psi}^a - \bar{\psi}^a (S_{ba}^{00})^\dagger \right] .
\]

(84)

Eliminating \(\ddot{\phi}\) from the second lines of (81), we find the equations of motion for fermions,

\[
\left[ W_{a\beta}^{00} + S_{ba}^{00} V^{ba} S_{a\beta}^{00} \frac{\dot{\psi}}{\psi} \right] \bar{\psi}^a - \left[ Q_{a\alpha}^{00} + S_{ba}^{00} V^{ba} (S_{ba}^{00})^\dagger \right] \dot{\psi}^a = E_a + S_{ba}^{00} V^{ba} E_a .
\]

(85)

The condition that the dependence of the second derivative of fermions vanishes is exactly the same as the degeneracy conditions (65)-(66), and the third line of (81) is also reduced to the Hermitian conjugate. Imposing the maximally degenerate conditions, we obtain the first-order differential equations for \(\psi^a\) and \(\bar{\psi}^a\),

\[
\mathcal{Y}_a \equiv E_a + S_{a\alpha}^{00} V^{ab} E_b = 0,
\]

(86)

\[
\dot{\mathcal{Y}}_a \equiv -(\mathcal{Y}_a)^\dagger = (S_{a\alpha}^{00})^\dagger V^{ab} E_b = 0,
\]

(87)

and, by substituting (82) and (83), \(\mathcal{Y}_a\) is written as

\[
\mathcal{Y}_a = \frac{\partial L}{\partial \psi^a} + \partial_\mu (P^{(2)} \bar{\psi}^a) \sigma_{\alpha\alpha}^{a\beta} + \partial_\mu (P^{(3)} J^\mu \psi_\alpha) + \partial_\mu S_{\alpha\alpha}^{\mu\nu} \partial_\nu \phi^a - \partial_\mu W_{a\beta}^{\mu\nu} \partial_\nu \psi^a + \partial_\mu \bar{\psi}^a \partial_\mu Q_{\alpha\alpha}^{\mu\nu} \\
+ S_{\alpha\alpha}^{ij} \partial_\mu \phi^a - W_{a\beta}^{ij} \partial_\mu \psi^a + 2 \partial_\mu \bar{\psi}^a \partial_\mu (\bar{\psi}^a)^{(0)i) + \partial_\mu \bar{\psi}^a \partial_\mu Q_{\alpha\alpha}^{ij} .
\]

(88)

We note that the dependence on \(\partial_\mu \bar{\psi}^a\) and the second spatial derivatives of the fermions vanishes after applying (88), and they have become the first order differential equations for both time and spacial derivatives.

As we saw, thanks to the maximally degenerate conditions, one can remove the second derivative terms of the Weyl field in the Euler-Lagrange equations by combining the equations of motion for the scalar fields, suggesting that the number of initial conditions to solve the EOMs is appropriate. Thus we have confirmed even in Lagrangian formulation that the extra DOFs does not appear in the theory with the maximally degenerate conditions. Note that the second derivative terms of the Weyl field in the equations of motion for the scalar fields (84) can be removed by using the time derivative of (86) and (87), as discussed in [40].

**B. Solvable condition for nonlinear equations including first derivatives**

So far, we have seen that the maximally degeneracy conditions lead to the first (second) order differential equations of motion for the Weyl (scalar) fields. As discussed in [40], it is not sure whether the equations of motion for fermions can be solved or not due to the Grassmann properties. Here we derive a condition, under which the equations of motion even nonlinear in the time derivatives can be explicitly solved. We start with the simplest example in a point particle system, and then we extend this analysis to the theory we focus on in the present paper.
1. Example: Two Grassmann-odd variables

Suppose that we have a system composed of bosons \( q^i \) and two fermionic variables \( \theta_1 \) and \( \theta_2 \). Due to the maximally degenerate conditions, the equations of motion for the fermionic variables should be the first-order differential equations after appropriately combining the bosonic equations of motion. Let us assume that we have already done this process, and we can generally write down the reduced equations of motion for fermionic variables as

\[
\begin{align*}
    a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2 + a_3 \dot{\theta}_1 \dot{\theta}_2 &= a_4, \\
    b_1 \dot{\theta}_1 + b_2 \dot{\theta}_2 + b_3 \dot{\theta}_1 \dot{\theta}_2 &= b_4,
\end{align*}
\]

where \( a_1, a_2, b_1 \) and \( b_2 \) \((a_3, a_4, b_3 \) and \( b_4 \)) are Grassmann-even (Grassmann-odd) functions depending on \( \theta_\alpha, q^i \) and \( \dot{q}^i \). We first assume at least one of \( a_1^{(0)}, a_2^{(0)}, b_1^{(0)} \) and \( b_2^{(0)} \) is non-zero. For instance, if \( a_1^{(0)} \neq 0 \), we can solve the first equation of (89) for \( \dot{\theta}_1 \):

\[
\dot{\theta}_1 = a_1^{-1} a_4 + a_1^{-1} (-a_2 + a_3 a_1^{-1} a_4) \dot{\theta}_2 =: A_1 + A_2 \dot{\theta}_2. 
\]

Then plugging this into the second equation of (89), we have

\[
(b_1 A_2 + b_2 + b_3 A_1) \dot{\theta}_2 = b_4 - b_1 A_1. 
\]

Thus, if the uniqueness condition,

\[
(b_1 A_2 + b_2 + b_3 A_1)^{(0)} = \left[ a_1^{-1} (-b_1 a_2 + a_1 b_2) \right]^{(0)} \neq 0,
\]

is satisfied, one can solve the set of nonlinear equations (89). In the other case, where the equations of motion have no linear terms in \( \dot{\theta}_\alpha \), there is no way to express each time derivative in terms of non-derivative variables, and the equations of motion are unsolvable. Therefore, the non-zero determinant of the coefficient matrix of the linear terms in \( \dot{\theta}_\alpha \) is necessary and sufficient condition for the equations to be uniquely solved for the time derivatives of the fermionic variables.

2. More general argument

In this subsection, we will extend the previous analysis to our degenerate theory, and show the equivalence between the condition (80) and the condition that the fermionic equations (86) and (87) are uniquely solved.

The equations of motion for fermions, (86) and (87), include up to first time derivatives, and, due to the Grassmann properties, they can be therefore generally written as (See also (88))

\[
\begin{align*}
    \mathcal{Y}_\alpha &= \sum_{n} \sum_{m} \chi_{\alpha \beta_1 \ldots \beta_n \gamma_1 \ldots \gamma_m} \dot{\psi}^{\beta_1} \ldots \dot{\psi}^{\beta_n} \dot{\bar{\psi}}^{\gamma_1} \ldots \dot{\bar{\psi}}^{\gamma_m} = 0, \quad (93) \\
    \mathcal{Y}_\bar{\alpha} &= -(\mathcal{Y}_\alpha)\dagger = -\sum_{n} \sum_{m} \dot{\bar{\psi}}^{\gamma_1} \ldots \dot{\bar{\psi}}^{\gamma_m} \dot{\psi}^{\beta_1} \ldots \dot{\psi}^{\beta_n} (\chi_{\alpha \beta_1 \ldots \beta_n \gamma_1 \ldots \gamma_m})\dagger = 0, \quad (94)
\end{align*}
\]

where \( \chi_{\alpha \beta_1 \ldots \beta_n \gamma_1 \ldots \gamma_m} \) consists of \( \psi^\alpha, \dot{\psi}^\alpha, \phi^\alpha, \dot{\psi}^\beta \), and their spatial derivatives. If these equations can be solved for \( \dot{\psi}^\alpha \) and \( \dot{\bar{\psi}}^{\dot{\alpha}} \), they should have the forms,

\[
\dot{\psi}^\alpha = (\dot{\bar{\psi}}^{\dot{\alpha}})\dagger = \sum_{k,l,m,n} C_{\beta_1 \ldots \beta_k \dot{\phi}^{\beta_1} \ldots \dot{\phi}^{\beta_k} \delta_1 \ldots \delta_m \dot{\psi}^{\beta_1} \ldots \dot{\psi}^{\beta_k} \partial_{1\delta_1} \psi^{\dot{\alpha}} \ldots \partial_{m\delta_m} \psi^{\dot{\alpha}} \partial_{j_1 \delta_1} \dot{\psi}^{\dot{\alpha}} \ldots \partial_{j_n \delta_n} \dot{\psi}^{\dot{\alpha}}, \quad (95)
\]

where the coefficients \( C_{\beta_1 \ldots \beta_k \dot{\phi}^{\beta_1} \ldots \dot{\phi}^{\beta_k} \delta_1 \ldots \delta_m} \) consist of \( \dot{\phi}^\alpha, \dot{\phi}^\alpha \) and \( \partial_{i} \phi^\alpha \) and has no dependence on \( \dot{\psi}^\alpha \) and \( \dot{\bar{\psi}}^{\dot{\alpha}} \). The nonlinear equations (93) and (94) can be solved for the first time derivatives of fermion if we could uniquely determine the coefficients \( C_{\beta_1 \ldots \beta_k} \) by substituting (95) into the them. After the substitution, the EOMs should become trivial at

\[\text{(k + l + m + n) is a finite odd number.}\]
each order of \((\psi^\gamma, \bar{\psi}^\gamma), (\partial_t \psi^\gamma, \partial_t \bar{\psi}^\gamma), (\psi^\gamma \psi^\delta \bar{\psi}^\gamma, \psi^\gamma \bar{\psi}^\gamma \bar{\psi}^\delta)\) and so on. At the lowest order, where the terms are linear in \(\psi^\gamma\) and \(\bar{\psi}^\gamma\), we have

\[
\left( \chi^{(0)}_{\alpha\beta} - \left( \chi^{(0)}_{\alpha\beta} \right)^* \right) \left( \left( C^{\beta \gamma}_{\gamma} \right)^* \left( \left( C^{\beta + \gamma}_{+ \gamma} \right)^* \right) \right) \left( \bar{\psi}^\gamma \right) = - \left( \left( \frac{\partial \chi_{\alpha}}{\partial \psi^\gamma} \right)^{(0)} - \left( \frac{\partial \chi_{\alpha}}{\partial \bar{\psi}^\gamma} \right)^{(0)} \right) \left( \psi^\gamma \right). \tag{96}
\]

Therefore, if the matrix,

\[
J^{(0)}_L(t, x) = \left( \chi^{(0)}_{\alpha\beta} - \left( \chi^{(0)}_{\alpha\beta} \right)^* \right),
\]

has non-zero determinant, we can uniquely determine \(C^{\beta \gamma}_{\gamma}\) and \(C^{\beta + \gamma}_{+ \gamma}\) from (96). Similarly, the coefficients associated with the spatial derivative terms \((\partial_t \psi^\gamma, \partial_t \bar{\psi}^\gamma)\) are determined by

\[
\left( - \left( \chi^{(0)}_{\alpha\beta} \right)^* \right) \left( \left( C^{\beta i}_{\gamma} \right) \left( C^{\beta i + \gamma}_{+ \gamma} \right)^* \right) \left( \partial_t \psi^\gamma \right) = - \left( \left( \frac{\partial \chi_{\alpha}}{\partial \psi^\gamma} \right)^{(0)} \right) \left( \partial_t \psi^\gamma \right). \tag{97}
\]

Once the coefficients at the linear order in the fermionic fields are determined, the coefficients at the cubic order can be determined by the equation,

\[
\left( \chi^{(0)}_{\alpha\beta} - \left( \chi^{(0)}_{\alpha\beta} \right)^* \right) \left( \left( C^{\beta \gamma}_{\gamma} \right) \left( C^{\beta i + \gamma}_{+ \gamma} \right)^* \right) \left( \psi^\gamma \bar{\psi}^i \bar{\psi}^\gamma \right) = - \left( \left( \frac{\partial \chi_{\alpha}}{\partial \psi^\gamma} \right)^{(0)} \right) \left( \psi^\gamma \bar{\psi}^i \bar{\psi}^\gamma \right). \tag{98}
\]

Here the coefficient matrices \(G_{\alpha\beta\gamma\delta}\) and \(G_{\alpha\gamma\delta\eta}\) are expressed in terms of \(\chi^{(0)}_{\alpha\beta\gamma\delta\eta}\) and \(\chi^{(0)}_{\alpha\beta\gamma\delta\eta}\) and their derivatives with respect to \(\psi^a, \bar{\psi}^\alpha\) and their spatial derivatives, as well as the linear coefficients \(C^{\beta \gamma}_{\gamma}, C^{\beta i}_{\gamma}, C^{\beta + \gamma}_{+ \gamma}\) and \(C^{\beta i + \gamma}_{+ \gamma}\). Again, we can solve the above equations for \(C^{\beta \gamma}_{\gamma}\) and \(C^{\beta + \gamma}_{+ \gamma}\) and successively determine all the coefficients appeared in (95) with the same procedure as long as the matrix \(J^{(0)}_L\) has an inverse.

Now it is straightforward to calculate the matrix \(J^{(0)}_L\) from the concrete expression of the equations of motion (86) and (87). The definition of \(J^{(0)}_L\) are rewritten with \(\mathcal{Y}_\alpha\) and \(\mathcal{Y}_\dot{\alpha}\) as

\[
J^{(0)}_L(t, x) = - \left( \frac{\partial \mathcal{Y}_\alpha}{\partial \psi^\beta} \right)^{(0)} \left( \frac{\partial \mathcal{Y}_\dot{\alpha}}{\partial \bar{\psi}^\beta} \right)^{(0)} , \tag{100}
\]

and the components are explicitly given by

\[
\left( \frac{\partial \mathcal{Y}_\alpha}{\partial \psi^\beta} \right)^{(0)} = - \left( \frac{\partial \mathcal{Y}_\dot{\alpha}}{\partial \bar{\psi}^\beta} \right)^{(0)} = - \left( \frac{1}{2} (S^{(2)}_{\alpha\beta})(0) \partial_t \phi^a - \partial_t W^{(0)}_\beta (\sigma^a \varepsilon)_{\alpha\beta} \right) , \tag{101}
\]

\[
\left( \frac{\partial \mathcal{Y}_\alpha}{\partial \bar{\psi}^\beta} \right)^{(0)} = - \left( \frac{\partial \mathcal{Y}_\dot{\alpha}}{\partial \psi^\beta} \right)^{(0)} = 2 (P^{(2)})(0) \sigma^0_{\alpha\beta} . \tag{102}
\]

It is manifest that \(J^{(0)}_L\) agrees with the matrix (79) with (78), obtained in Hamiltonian formulation. Therefore, the condition that the fermionic equations can be uniquely solved in Lagrange formulation is equivalent to the condition that all the primary constraints are second class.
VI. FIELD REDEFINITION

So far, we have successfully obtained most general quadratic Lagrangian for $n$ scalar and 1 Weyl fields satisfying the maximally degenerate conditions including up to first derivatives. However, one needs to carefully check whether the obtained theories can be mapped into a known theory, which trivially satisfy the degeneracy conditions. As an example, let us consider the Lagrangian containing a canonical scalar and a Weyl fields,

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{i}{2} (\bar{\psi}^\alpha \sigma^{\alpha\beta}_{\alpha\beta} \partial_\mu \psi_\alpha - \partial_\mu \bar{\psi}^\alpha \sigma^{\alpha\beta}_{\alpha\beta} \psi^\alpha).$$  \hfill (103)

Note that the kinetic matrix of $(103)$ is trivially degenerate since $B = C = D = 0$. Under the following field redefinition,

$$\phi = \varphi - \frac{i}{2} \bar{\Psi} + \frac{i}{2} \tilde{\Psi},$$  \hfill (104)

where $\varphi$ is a new scalar field, while keeping the Weyl field the same, the Lagrangian is transformed as

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - i \bar{\psi}^\alpha \varphi \partial_\mu \psi_\alpha - \bar{\bar{\psi}} \partial_\mu \bar{\psi}^\alpha + \frac{i}{2} (\bar{\psi}^\alpha \sigma^{\alpha\beta}_{\alpha\beta} \partial_\mu \psi_\alpha - \partial_\mu \bar{\psi}^\beta \sigma^{\beta\gamma}_{\beta\gamma} \psi^\gamma)
- \frac{1}{2} (\bar{\bar{\psi}} \partial_\mu \bar{\psi}^\alpha)^2 + (\psi^\alpha \partial_\mu \bar{\psi}^\beta)(\bar{\psi}^\alpha \partial_\mu \psi_\beta) - \frac{1}{2} (\bar{\psi} \partial_\mu \bar{\psi})^2.$$  \hfill (105)

As one can see, the submatrices $B, C$ and $D$ in the kinetic matrix become non-zero due to the transformation (104), and the mapped Lagrangian apparently yields the second-order differential equations for the fermionic fields. One can, however, easily check that this mapped Lagrangian satisfies the maximally degenerate conditions (65) and (66) as well as the supplementary condition (80); therefore the number of DOFs are the same as the original Lagrangian $(103)$. This is because the field redefinition (104) is an invertible transformation, keeping the number of DOFs under the transformation [41].

Now, we would like to know whether our degenerate Lagrangian can be mapped into theories whose kinetic matrix is trivially degenerate by the covariant field redefinition $(\phi^a, \psi^\alpha, \bar{\psi}^\alpha) \rightarrow (f^A(\phi, \psi, \tilde{\psi}), \eta^A(\phi, \psi, \tilde{\psi}), \bar{\eta}^A(\phi, \psi, \tilde{\psi}))^8$, i.e., if it is possible that

$$K = \begin{pmatrix} A_{ab} & B_{a\beta,J} & B_{a\beta,J} \\ C_{ab,I} & D_{ab,J,I} & D_{ab,J,I} \\ C_{ab,I} & D_{ab,J,I} & D_{ab,J,I} \end{pmatrix} \text{ field redefinition } \rightarrow \begin{pmatrix} \tilde{A}_{ab} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$  \hfill (106)

where $\tilde{A}_{ab}$ is the kinetic matrix of the redefined scalar fields $f^A$. We note that, under the field redefinition, Lagrangian at each order (36), (37) and (38) are not mixed since the transformation do not include the derivatives of the fields. From the definition of the kinetic matrix, the quadratic terms in derivatives in Lagrangian, (38), can be responsible for the kinetic matrix, while linear or lower ones, (37) and (36), are obviously not. Furthermore, among the quadratic terms, anti-symmetrically contracted (with respect to Lorentzian indices of the derivatives) ones are irrelevant. The only contribution to the kinetic matrix is coming from the symmetrically contracted ones in $L_2$. When all the submatrices except $A_{ab}$ in the kinetic matrix vanish, the corresponding symmetrically contracted covariant terms in Lagrangian also should vanish simultaneously because of the covariance of the theory. As a result, we can just focus on the symmetrically contracted terms in $L_2$ in this section. Picking up these, we rewrite the relevant Lagrangian with the degeneracy conditions as

$$L_{\text{rel}} = \frac{1}{2} \left[ V_{ab} \partial_\mu \phi^a - \frac{1}{2} \left( S^{(1)}_a \partial_\mu \Psi + (S^{(1)}_a)^\dagger \partial_\mu \bar{\Psi} \right) \right] V^{ab} \left[ V_{bd} \partial^\mu \phi^d - \frac{1}{2} \left( S^{(1)}_b \partial^\mu \Psi + (S^{(1)}_b)^\dagger \partial^\mu \bar{\Psi} \right) \right].$$  \hfill (107)

If the coefficients $V_{ab}$ and $S^{(1)}_a$ are constants, the square brackets $[\cdots]$ can be redefined as the derivatives of new scalar fields. Then, the non-trivial terms in (107), which yield second time derivatives of the fermionic fields in Euler-Lagrange equations, can be removed just as a non-trivial Lagrangian (105) is reduced to a trivial Lagrangian (103) [40].

---

7 The Hamiltonian analysis of this Lagrangian has already examined in [40].

8 One can consider field redefinitions including derivatives of the scalar and fermionic fields, but, for instance, redefinitions like $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \partial_\mu \phi, \psi, \tilde{\psi}), \eta(\phi, \psi, \tilde{\psi}), \bar{\eta}(\phi, \psi, \tilde{\psi})), (\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi})), (\phi, \psi, \tilde{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi}))$ or $(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \tilde{\psi}), \eta(\phi, \psi, \tilde{\psi}), \bar{\eta}(\phi, \psi, \tilde{\psi})), \eta(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi})), \bar{\eta}(\phi, \psi, \bar{\psi}) \rightarrow (f(\phi, \psi, \bar{\psi}), \eta(\phi, \psi, \bar{\psi}), \bar{\eta}(\phi, \psi, \bar{\psi}))$ result in the appearance of the higher derivatives, implying that they might not keep the Lagrangian including just up to the first derivatives. To introduce such transformations, we also need to check the invertibility of them.
by the transformation (104). Under the field redefinition, \((\phi^a, \psi^a, \bar{\psi}^\alpha) \rightarrow (f^A(\phi, \psi, \bar{\psi}), \eta^\Lambda(\phi, \psi, \bar{\psi}), \bar{\eta}^\Lambda(\phi, \psi, \bar{\psi}))\), the above Lagrangian is rewritten in terms of the new variables as

\[
\mathcal{L}_{\text{rel}} = \frac{1}{2} G_A^{ab} V_{ab} \partial_{\mu} f^A \partial^\mu f^B - G_A^{ab} V_{ab} \partial_{\mu} f^A \partial^\mu \eta^\Lambda - G_A^{ab} V_{ab} \partial_{\mu} \bar{\eta}^\Lambda \partial^\mu \bar{\eta}^\Lambda \]

\[
- \frac{1}{2} G_A^{ab} V_{ab} \partial_{\mu} \bar{\eta}^\Lambda \partial^\mu \Sigma^\Lambda - G_A^{ab} V_{ab} \partial_{\mu} \eta^\Lambda \partial^\mu \bar{\eta}^\Lambda - \frac{1}{2} \bar{G}_A^{ab} \bar{V}_{ab} \partial_{\mu} \bar{\eta}^\Lambda \partial^\mu \Sigma^\Lambda ,
\]

where we have defined

\[
G_A^a = \phi^a f^A - \frac{1}{2} V^{ab} \left[ (S_b^{(1)})_a \right] \psi_a \]

\[
G_A^a = \phi^a \eta^\Lambda - \frac{1}{2} V^{ab} \left[ (S_b^{(1)})_a \right] \psi_a \]

\[
\bar{G}_A^a = \bar{G}_A^a = -(G_A^a)^\dagger, \]

and they satisfy \(G_A^a = (G_A^a)^\dagger\). Now let us seek the field redefinition such that the quadratic terms in the first derivative of \(\eta^\Lambda\) and \(\bar{\eta}^\Lambda\), the second line of (108), are removed. This becomes possible if we find a transformation realizing

\[
G_A^a = 0 .
\]

To find such a transformation, we first expand \(V^{ab}\) and \(S_a^{(1)}\) as

\[
V^{ab} = v^{ab(0)} + v^{ab(1)} \Psi + v^{ab(2)} \bar{\Psi} \Psi , \quad S_a^{(1)} = s_a^{(0)} + s_a^{(1)} \bar{\Psi} ,
\]

where \(v^{ab(0)}\) and \(s_a^{(i)}\) depend only on \(\phi^a\). Then, (111) can be rewritten as

\[
\phi^a_{,\eta^\Lambda} = \frac{1}{2} \left[ v^{ab(0)} s_b^{(0)} \Psi_{,\eta^\Lambda} + v^{ab(0)} \left( s_b^{(0)} \right) \bar{\Psi}_{,\eta^\Lambda} + \left( v^{ab(0)} s_b^{(1)} + \left( v^{ab(1)} s_b^{(0)} \right) \right) \Psi_{,\eta^\Lambda} \bar{\Psi} \right]
\]

\[
\phi^a_{,\bar{\eta}^\Lambda} = \frac{1}{2} \left[ v^{ab(0)} s_b^{(0)} \bar{\Psi}_{,\eta^\Lambda} + v^{ab(0)} \left( s_b^{(0)} \right) \bar{\Psi}_{,\eta^\Lambda} + \left( v^{ab(0)} s_b^{(1)} + \left( v^{ab(1)} s_b^{(0)} \right) \right) \Psi_{,\eta^\Lambda} \bar{\Psi} \right]
\]

\[
\phi^a_{,\eta^\Lambda} + \phi^a_{,\bar{\eta}^\Lambda} = \frac{1}{2} \left[ v^{ab(0)} s_b^{(1)} + \left( v^{ab(1)} s_b^{(0)} \right) \right] \Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi_{,\eta^\Lambda} \bar{\Psi} \]

It is obvious that the first three terms can have their integral forms, but the last term of (114), proportional to \(\Psi_{,\eta^\Lambda} \bar{\Psi} - \Psi_{,\eta^\Lambda} \bar{\Psi}\), cannot be integrated as we will see below. Let us explicitly see the transformation where \(\phi^a\) coincides with the integral of the right-hand side of (114) except the last term. Such the transformation is realized through

\[
f^A = f^A \left( \phi^b + \frac{1}{2} (A^b \Psi + (A^b)^* \bar{\Psi} + B^b \Psi \bar{\Psi}) \right) , \quad \eta^\Lambda = \eta^\Lambda (\phi^b, \psi^\beta, \bar{\psi}^\beta) , \quad \bar{\eta}^\Lambda = \bar{\eta}^\Lambda (\phi^b, \psi^\beta, \bar{\psi}^\beta) ,
\]

where \(A^b\) and \(B^b\) are functions of \(\phi^a\), determined properly in the following. For keeping the equivalence between the former and the latter Lagrangians, we assume the field redefinition invertible. The assumption of the invertible transformation requires the bosonic part of the whole Jacobian matrix,

\[
\text{Jacobian matrix} = \begin{pmatrix}
\frac{\partial f^A}{\partial \phi^b} & \frac{\partial f^A}{\partial \psi^\beta} & \frac{\partial f^A}{\partial \bar{\psi}^\beta} \\
\frac{\partial \eta^\Lambda}{\partial \phi^b} & \frac{\partial \eta^\Lambda}{\partial \psi^\beta} & \frac{\partial \eta^\Lambda}{\partial \bar{\psi}^\beta} \\
\frac{\partial \bar{\eta}^\Lambda}{\partial \phi^b} & \frac{\partial \bar{\eta}^\Lambda}{\partial \psi^\beta} & \frac{\partial \bar{\eta}^\Lambda}{\partial \bar{\psi}^\beta}
\end{pmatrix} ,
\]

has a non-zero determinant. The off-diagonal bosonic parts of the Jacobian matrix are inevitably zero because of the Grassmann-odd property. Therefore, the above requirement is equivalent to

\[
\det \left( \frac{\partial f^A}{\partial \phi^b} \right)^{(0)} \neq 0 , \quad \text{and} \quad \det \left( \frac{\partial \eta^\Lambda}{\partial \psi^\beta} \right)^{(0)} \neq 0 .
\]
Since we have assumed the transformation is invertible, we can invert the definition of $f^A$ in (115) for $\phi^a$,

$$\phi^a = \frac{1}{2} \left[ A^a \Psi + (A^a)^* \bar{\Psi} + B^a \Psi \bar{\Psi} \right] + \phi^{a(0)}(f) ,$$  \hspace{1cm} (118)

where $\phi^{a(0)}(f)$ is the inverse function of $f^A$ in (115). The derivative of (118) with respect to $\eta^A$ gives

$$\phi^{a(0)}_{,\eta^A} = \frac{1}{2} \left[ A^a \Psi,_{\eta^A} + (A^a)^* \bar{\Psi},_{\eta^A} + B^a \left( \Psi,_{\eta^A} \bar{\Psi} + \Psi \bar{\Psi},_{\eta^A} \right) \right] + \frac{1}{2} \phi^{b(0)}_{,\eta^B} \left[ A^a,_{\eta^B} \Psi + (A^a)^*_{,\eta^B} \bar{\Psi} + B^a,_{\eta^B} \Psi \bar{\Psi} \right]$$

$$= \frac{1}{2} \left[ A^a \Psi,_{\eta^A} + (A^a)^* \bar{\Psi},_{\eta^A} + \left( B^a + \frac{1}{2} A^b (A^b)^* \right) \Psi,_{\eta^A} \bar{\Psi} + \left( B^a + \frac{1}{2} (A^b)^* A^a,_{\eta^B} \right) \Psi \bar{\Psi},_{\eta^A} \right] ,$$  \hspace{1cm} (119)

where we have recursively solved the first line and have used $\Psi,_{\eta^A} \Psi = (1/2)(\Psi \bar{\Psi})_{,\eta^A} = 0$ and $\Psi,_{\eta^A} \bar{\Psi} = (1/2)(\bar{\Psi} \Psi)_{,\eta^A} = 0$ to find the second line without $\phi^{b(0)}_{,\eta^A}$. Now comparing (113) and (119), we can easily determine the coefficients $A^a$ and $B^a$ as

$$A^a = v^{ab(0)} s_b^{(0)} ,$$  \hspace{1cm} (120)

$$B^a = \Re[C^a] ,$$  \hspace{1cm} (121)

Introducing the transformation to the definitions (109) and (110), we then have

$$\mathcal{G}_A^a = -\frac{i}{2} \Im[C^a] (\Psi,_{\eta^A} \Psi - \Psi \bar{\Psi},_{\eta^A}) ,$$  \hspace{1cm} (122)

$$\mathcal{G}^a_{,\Sigma} = \left( \delta^a_b + \frac{1}{2}(A^b,_{\eta^A} \Psi + (A^b)^* \bar{\Psi}) + \frac{1}{2} (B^a,_{\eta^A} + \Re[C^a,_{\eta^A}] \bar{\Psi}) \right) \phi^{b(0)}_{,f^a} - \frac{i}{2} \Im[C^a] (\Psi,_{f^A} \Psi - \Psi \bar{\Psi},_{f^A}) .$$  \hspace{1cm} (123)

Since we cannot pick up the imaginary part of $C^a$ as one can see from (121), we still have non-vanishing $\mathcal{G}_A^a$. We cannot remove the dependence on $\Psi,_{\eta^A} \Psi - \Psi \bar{\Psi},_{\eta^A}$ in any way as we expected from the expression of (114). Nevertheless, the quadratic derivative interactions of fermions in (108) is the square of (122) (and its hermitian), and thus they accidentally vanish due to the Grassmann properties:

$$\mathcal{G}_A^a \mathcal{G}_a^{b(0)} \propto (\Psi,_{\eta^A} \Psi - \Psi \bar{\Psi},_{\eta^A}) (\Psi,_{\eta^A} \bar{\Psi} - \Psi,_{\eta^A} \bar{\Psi}) = 0 ,$$  \hspace{1cm} (124)

$$\mathcal{G}_A^a \mathcal{G}_a^{b(0)} \propto (\Psi,_{\eta^A} \bar{\Psi} - \Psi,_{\eta^A} \Psi) (\Psi,_{\eta^A} \bar{\Psi} - \Psi,_{\eta^A} \bar{\Psi}) = 0 ;$$  \hspace{1cm} (125)

whereas, the cross term of the redefined scalar field and fermionic field cannot be removed because

$$\mathcal{G}_A^a V_{ab} \mathcal{G}_A^b \partial_{\mu} f^{A} \partial^{\mu} \eta^A = -\frac{i}{2} v^{ab(0)} s_b^{(0)} \Im[C^b] (\Psi,_{\eta^A} \Psi - \Psi \bar{\Psi},_{\eta^A}) \partial_{\mu} f^{A} \partial^{\mu} \eta^A \neq 0 .$$  \hspace{1cm} (126)

As a result, the final Lagrangian with the field redefinition (115) with (120) and (121) is given by

$$L_{\text{rel}} = \frac{1}{2} \mathcal{G}_A^a V_{ab} \partial_{\mu} f^{A} \partial^{\mu} f^{B} - \mathcal{G}_A^a V_{ab} \mathcal{G}_A^b \partial_{\mu} f^{A} \partial^{\mu} \eta^A - \mathcal{G}_A^a V_{ab} \mathcal{G}_A^b \partial_{\mu} f^{A} \partial^{\mu} \bar{\eta}^A .$$  \hspace{1cm} (127)

Therefore, we conclude that the fermionic derivative interactions like $\partial_{\mu} \eta^A \partial^{\mu} \eta^C$ can be eliminated by the field redefinition, but the cross terms between scalar fields and fermions like $\partial_{\mu} f^{A} \partial^{\mu} \eta^A$ cannot be removed. Such the cross terms indicate that the Euler-Lagrange equations apparently contain the second derivatives of the fermionic field, while they are reduced to first order equations because the maximally degenerate conditions are satisfied,

$$D_{A\Sigma} - C_{AB} A^{BA} B_{A\Sigma} = \mathcal{G}_B^a V_{ab} \mathcal{G}_A^b A^{BA} C_{A\Sigma} \mathcal{G}_A^{b(0)} \propto \mathcal{G}_A^a \mathcal{G}_a^{b(0)} = 0 ,$$  \hspace{1cm} (128)

$$D_{A\Sigma} - C_{AB} A^{BA} B_{A\Sigma} = \mathcal{G}_B^a V_{ab} \mathcal{G}_A^b A^{BA} C_{A\Sigma} \mathcal{G}_A^{b(0)} \propto \mathcal{G}_A^a \mathcal{G}_a^{b(0)} = 0 .$$  \hspace{1cm} (129)

To summarize, the kinetic matrix cannot be transformed to the form in (106) with any field redefinition but it can be reduced to

$$K = \begin{pmatrix} A_{ab} & B_{a\beta,J} & E_{a\beta,J} \\ C_{ab,I} & D_{a\beta,I} & 0 \\ C_{ab,I} & D_{a\beta,I} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{A}_{ab} & \tilde{B}_{a\beta,J} & \tilde{E}_{a\beta,J} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$  \hspace{1cm} (130)

This fact indicates that the cross terms are really newly found derivative interactions.
The crucial difference between purely bosonic degenerate system (such as Horndeski theory [4–6] and DHOST theories [17, 20]) and our scalar-fermion degenerate system is the highest derivatives appearing in Lagrangian. This is because the Lagrangian containing first derivatives of the fermionic fields in general yields second order differential equations of motion, which is a signal of fermionic (Ostrogradsky’s) ghosts. Therefore, this situation corresponds to the bosonic Lagrangian containing up to second derivatives. On the analogy of the conformal and disformal transformation in the scalar-tensor theory, the fermionic version of conformal (disformal) transformations does not involve any derivatives of the fermionic field. Thus, the conformal transformation, which mix the metric and the Weyl field, takes the following form\(^9\),

\[
\bar{g}_{\mu\nu} = A(\Psi, \Psi)g_{\mu\nu}.
\]

Such a transformation will be significantly useful to find new theories of “tensor-fermion theories”, and we will leave this interesting issue in a future work.

\[\]

VII. SUMMARY AND DISCUSSION

As usually discussed in Lagrangian composed of bosonic degrees of freedom, we have to avoid the appearance of ghosts even in boson-fermion coexisting Lagrangian. Fermions easily suffer from the fermionic ghost as pointed out in [39], i.e., fermionic degrees of freedom should be constrained by the same number of constraints as the (physical) DOFs. That is true even when we additionally have bosonic degrees of freedom.

Our purpose is the construction of covariant derivative interactions between fermions or between scalar fields and fermions, free from fermionic ghosts. We have given the prescription to find new interactions in Lagrangian with up to the first derivative of fields, made up of a set of conditions imposed on Lagrangian. One of the conditions is the maximally degenerate condition which produces the enough number of primary constraints. In this paper, all of the constraints are simply assumed to be second class. If we have gauge invariance, that is, first class constraints, the number of primary constraints can be smaller. However, it should be notice that, after the gauge fixing, all of the primary constraints can be turned into second class. In Lagrangian formulation, the equations of motion for fermions can be reduced to the first order differential equations by imposing the maximally degenerate condition, which should be solved for the first derivative of each fermionic variable. Whether they are solvable is not a priori because of the Grassmann-odd nature, and the requirement is exactly the same with the invertibility of the constraint matrix.

The coefficients of the derivative of fields in Lagrangian are complicated but classified thanks to the covariance, and they become much simpler in case of one Weyl field with multiple scalar fields. One of the prominent features which we can learn from the concrete analysis is that the first condition, that is, the maximally degenerate condition, affects the symmetric part of the Lagrangian with respect to the spacetime indices of the derivatives, but the supplementary second condition the anti-symmetric part.

The proposed Lagrangian are possible to be used as a model for the interaction between scalar fields and a Weyl field, but some may ask what happens when we perform field redefinitions. Transformation including derivatives seems to introduce cubic or higher nonlinear derivative terms as well as second or higher derivatives in the Lagrangian. In a simple setup where the whole Lagrangian is expressed by the proposed quadratic Lagrangian, the quadratic terms in the derivative of fermions are absorbed by certain invertible field redefinitions, but the derivative interaction terms between the derivative of the scalar fields and the fermions remain, which suggest that they are strictly new terms. As shown in [10], in a bosonic case, any healthy theory even with arbitrary higher derivative terms can be reduced to a non-degenerate Lagrangian written in terms of (unconstrained) variables with at most the first-order time derivatives through canonical transformation. On the other hand, in a fermionic case, any healthy theories require constraints to reduce extra degrees of freedom of fermionic variables. Then, it would be interesting to investigate what kind of a simpler Lagrangian can be generally found through canonical transformation from healthy fermionic theories even with arbitrary higher derivative terms.

There are several directions applying the method we have developed: inclusion of nonlinear derivative interactions, second or higher derivatives, gauge invariance and/or gravity, and so on. Some of these issues will be discussed in future publications.

\[9\] A candidate for the disformal transformation including the Weyl field is

\[
g_{\mu\nu} = A(\Psi, \Psi)g_{\mu\nu} + B(\Psi, \Psi)J_{\mu}J_{\nu},
\]

where \(J_{\mu}\) is now appropriately mapped with tetrad fields \(e_{\mu}^{a}\), defined by \(g_{\mu\nu} = e_{\mu}^{a}e_{\nu}^{b}g_{ab}\), from a flat tangent space. However, one can easily show that the arbitrary function \(B\) can be absorbed into the conformal factor \(A\) by using (47). On the other hand, the identity (C3) tells us that the disformal transformation with multiple Weyl fields is independent from the conformal transformation.
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Appendix A: Identities of Pauli matrices

In this Appendix, we summarize notations and identities of Pauli matrices [42]. We have used the following matrices defined as

\[ \tilde{\sigma}^{\mu\nu\alpha} = \epsilon^{\alpha\beta\gamma}\sigma^{\mu\beta}\sigma^{\nu\gamma}, \tag{A1} \]

\[ (\sigma^{\mu\nu})_{\alpha}^{\beta} = \frac{1}{4} \left( \sigma_{\alpha\alpha}^{\mu} \sigma_{\nu\beta}^{\alpha} - \sigma_{\alpha\alpha}^{\nu} \sigma_{\mu\beta}^{\alpha} \right), \quad (\tilde{\sigma}^{\mu\nu})_{\alpha}^{\beta} = \frac{1}{4} \left( \sigma^{\mu\alpha\nu\alpha} - \sigma^{\nu\alpha\mu\alpha} \right), \tag{A2} \]

\[ (\sigma^{\mu\nu\epsilon})_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha}^{\gamma} \epsilon_{\gamma\beta}, \quad (\epsilon^{\sigma^{\mu\nu}})_{\alpha\beta} = \epsilon_{\alpha\gamma}(\sigma^{\mu\nu})_{\gamma\beta}. \tag{A3} \]

Note that the last two matrices are symmetric under the exchange of the fermionic indices, i.e., \((\sigma^{\mu\nu\epsilon})_{\alpha\beta} = (\sigma^{\mu\nu\epsilon})_{\beta\alpha}\) and \((\epsilon^{\sigma^{\mu\nu}})_{\alpha\beta} = (\epsilon^{\sigma^{\mu\nu}})_{\beta\alpha}\). They satisfy the following useful properties:

\[ \sigma_{\alpha\alpha}^{\mu} \tilde{\sigma}_{\beta}^{\alpha} = -2 \delta_{\alpha}^{\beta} \delta_{\alpha}^{\beta}, \quad \sigma_{\alpha\alpha}^{\mu} \sigma_{\beta\beta}^{\alpha} = -2 \epsilon_{\alpha\beta}^{\epsilon} \epsilon_{\alpha\beta}^{\epsilon}, \quad \tilde{\sigma}_{\alpha\alpha}^{\mu} \tilde{\sigma}_{\beta}^{\alpha} = -2 \epsilon_{\alpha\beta}^{\epsilon} \epsilon_{\alpha\beta}^{\epsilon}, \tag{A4} \]

\[ \sigma_{\alpha\alpha}^{\mu} \sigma_{\beta\beta}^{\nu} - \sigma_{\alpha\alpha}^{\nu} \sigma_{\beta\beta}^{\mu} = 2(\sigma^{\mu\nu})_{\alpha\beta} \epsilon_{\epsilon\beta}^{\epsilon} + (\epsilon^{\sigma^{\mu\nu}})_{\alpha\beta} \epsilon_{\epsilon\beta}^{\epsilon}, \tag{A5} \]

\[ \sigma_{\alpha\alpha}^{\mu} \sigma_{\beta\beta}^{\nu} + \sigma_{\alpha\alpha}^{\nu} \sigma_{\beta\beta}^{\mu} = -\eta^{\mu\nu} \epsilon_{\alpha\beta}^{\epsilon} \epsilon_{\alpha\beta}^{\epsilon} + 4(\sigma^{\mu\nu})_{\alpha\beta}(\epsilon^{\sigma^{\mu\nu}})_{\alpha\beta}. \tag{A6} \]

In addition, one can easily derive the following useful identities:

\[ (\sigma^{\mu\nu\epsilon})_{\alpha\beta} \sigma_{\epsilon\delta}^{\mu} = \frac{1}{2}(\delta_{\alpha}^{\epsilon} \sigma_{\beta}^{\mu} + \epsilon_{\beta}^{\epsilon} \sigma_{\alpha}^{\mu}), \tag{A7} \]

\[ \epsilon_{\alpha\beta}^{\epsilon} \sigma_{\alpha\alpha}^{\mu} \tilde{\sigma}_{\gamma\beta}^{\epsilon} = -2(\sigma^{\mu\lambda})_{\alpha\beta} - \eta^{\epsilon\mu} \epsilon_{\alpha\beta}^{\epsilon}, \tag{A8} \]

\[ \epsilon_{\alpha\beta}^{\epsilon} \sigma_{\alpha\alpha}^{\mu} \tilde{\sigma}_{\epsilon\beta}^{\lambda} = -2(\sigma^{\mu\lambda})_{\alpha\beta} + \eta^{\epsilon\mu} \epsilon_{\alpha\beta}^{\epsilon}, \tag{A9} \]

\[ (\sigma^{\mu\nu\epsilon})_{\alpha\beta}(\epsilon^{\sigma^{\mu\nu}})_{\gamma\delta} = \frac{1}{8}(\sigma_{\alpha\gamma}^{\mu} \sigma_{\beta\gamma}^{\nu} + \sigma_{\alpha\gamma}^{\nu} \sigma_{\beta\gamma}^{\mu} + \sigma_{\alpha\beta}^{\mu} \sigma_{\gamma\beta}^{\nu} + \sigma_{\alpha\beta}^{\nu} \sigma_{\gamma\beta}^{\mu}), \tag{A10} \]

\[ (\sigma^{\mu\nu\epsilon})_{\alpha\beta}(\sigma^{\mu\nu\epsilon})_{\gamma\delta} = \frac{1}{4} \left[ -\eta^{\epsilon\mu} (\epsilon_{\alpha\gamma}^{\epsilon} \epsilon_{\beta\delta}^{\epsilon} + \epsilon_{\beta\gamma}^{\epsilon} \epsilon_{\alpha\delta}^{\epsilon}) + (\sigma^{\mu\lambda})_{\alpha\gamma} \epsilon_{\beta\gamma}^{\epsilon} \epsilon_{\alpha\delta} + (\sigma^{\mu\lambda})_{\beta\gamma} \epsilon_{\alpha\gamma}^{\epsilon} \epsilon_{\alpha\delta} + (\sigma^{\mu\lambda})_{\beta\delta} \epsilon_{\alpha\gamma}^{\epsilon} \epsilon_{\alpha\gamma} \right]. \tag{A11} \]

Therefore, any contraction of the building block matrices, \(\epsilon_{\alpha\beta}, \epsilon_{\alpha\beta}, \sigma_{\alpha\alpha}^{\mu}, (\sigma^{\mu\nu\epsilon})_{\alpha\beta},\) and \((\epsilon^{\sigma^{\mu\nu}})_{\alpha\beta}\), with respect to the Lorentzian or fermionic indices reduces to the un-contracted combination of them. Here, \((\sigma^{\mu\nu\epsilon})_{\alpha\beta}(\epsilon^{\sigma^{\mu\nu}})_{\gamma\delta}\) is symmetric under the exchange of \(\mu\) and \(\nu\), and traceless, \(\sigma^{\mu\nu\epsilon})_{\alpha\beta}(\epsilon^{\sigma^{\mu\nu}})_{\gamma\delta} = 0\), through \((A4)\). The matrices and their complex conjugates are related through

\[ (\sigma_{\alpha\beta}^{\mu})^* = \sigma_{\beta\alpha}^{\mu}, \quad ((\sigma^{\mu\nu\epsilon})_{\alpha\beta})^* = (\epsilon^{\sigma^{\mu\nu}})_{\beta\alpha}. \tag{A13} \]

Appendix B: Maximally degenerate conditions and primary constraints

As discussed in the the section II, we need \(4N\) constraints making the number of the DOFs of fermions half to avoid negative norm states in a n-scalar and N-fermion Lagrangian with up to first derivatives. For the purpose, we have adopted the maximally degenerate conditions, \((13)\) and \((14)\), having \(4N\) primary constraints for the fermions, \((17)\) and \((18)\). Here we show that, if we have \(4N\) primary constraints for the fermions, the maximally degenerate conditions are automatically satisfied.

From the definition of the canonical momenta of fermions, we generally have

\[ \pi_{\psi_{\alpha}} = \mathcal{F}_{\alpha\beta}(\phi^{\alpha}, \pi_{\phi^{\alpha}}, \partial_{t}\phi^{\alpha}, \psi_{\alpha}^{\beta}, \partial_{t}\psi_{\alpha}^{\beta}, \psi_{\alpha}^{\beta}, \partial_{t}\psi_{\alpha}^{\beta}, \psi_{\alpha}^{\beta}, \tilde{\psi}_{\alpha}^{\beta}, \tilde{\psi}_{\alpha}^{\beta}), \tag{B1} \]
after we locally solve the canonical momenta of the scalar fields for \( \dot{\phi}^a \), which can be justified by our assumption that \( \partial \pi_a / \partial \phi^b = A_{ab} \) has an inverse. Taking the derivative of (B1) with respect to \( \dot{\psi}_j^\alpha \) and \( \dot{\bar{\psi}}_j^\alpha \), we have

\[
\mathcal{L}_{\dot{\psi}_j^\alpha \dot{\bar{\psi}}_j^\alpha} = \mathcal{L}_{\phi^a \phi^a} \frac{\partial F_{a,l}}{\partial \dot{\phi}^a} + \frac{\partial F_{a,l}}{\partial \dot{\psi}_j^\alpha} \quad \leftrightarrow \quad \frac{\partial F_{a,l}}{\partial \dot{\phi}^a} = D_{\alpha \beta, J} + B_{\alpha \beta, J} \frac{\partial F_{a,l}}{\partial \pi^\alpha} = D_{\alpha \beta, J} - C_{\alpha b, J} A_{ab} B_{\alpha \beta, J} ,
\]

(B2)

\[
\mathcal{L}_{\dot{\psi}_j^\alpha \dot{\bar{\psi}}_j^\alpha} = \mathcal{L}_{\phi^a \phi^a} \frac{\partial F_{a,l}}{\partial \dot{\phi}^a} + \frac{\partial F_{a,l}}{\partial \dot{\psi}_j^\alpha} \quad \leftrightarrow \quad \frac{\partial F_{a,l}}{\partial \dot{\phi}^a} = D_{\alpha \beta, J} + B_{\alpha \beta, J} \frac{\partial F_{a,l}}{\partial \pi^\alpha} = D_{\alpha \beta, J} - C_{\alpha b, J} A_{ab} B_{\alpha \beta, J} ,
\]

(B3)

where we have used the relations derived from the derivative of (B1) with respect to \( \dot{\phi}^a \),

\[
\mathcal{L}_{\dot{\psi}_j^\alpha \dot{\bar{\psi}}_j^\alpha} = \mathcal{L}_{\phi^a \phi^a} \frac{\partial F_{a,l}}{\partial \pi^\alpha} \quad \leftrightarrow \quad C_{\alpha a, l} = A_{ba} \frac{\partial F_{a,l}}{\partial \pi^\alpha} ,
\]

(B4)

in the last equalities. Thus (B2) and (B3) tell us that no dependence of \( F_{a,l} \) on the time derivative of the fermions means that we have \( 4N \) primary constraints for the fermions. Thus (B2) and (B3) suggest that the maximally degeneracy conditions, (13) and (14), are not only the sufficient condition for the existence of \( 4N \) primary constraints for fermions, but also the necessary condition for that.

**Appendix C: Scalar-fermion theories quadratic in first derivatives of multiple Weyl fields**

In this Appendix, we extend our analysis in Sec. III to the case of multiple Weyl fields and construct the most general scalar-fermion theory whose Lagrangian contains up to quadratic in first derivatives of scalar and fermionic fields. As in Sec. III, we first construct Lorentz-invariant scalar quantities with no derivatives, which only contain scalar fields \( \phi^a \) and fermionic fields \( \psi^\alpha, \bar{\psi}^\alpha \). Since the fermionic indices can be contracted with \( \varepsilon_{\alpha \beta}, \varepsilon_{\alpha \beta}, \sigma^{\mu \nu \varepsilon}_{\alpha \beta}, (\sigma^{\mu \nu \varepsilon})_{\alpha \beta} \), and \( (\varepsilon \sigma^{\mu \nu})_{\alpha \beta} \), we have five possibilities

\[
\Psi_{IJ} = \psi_I^\alpha \psi_{J, \alpha}, \quad \bar{\Psi}_{IJ} = \bar{\psi}_I^\alpha \bar{\psi}_{J, \alpha}, \quad J_{IJ}^\mu = \psi_I^\alpha \sigma^{\mu \nu \varepsilon}_{\alpha \beta} \bar{\psi}_{J, \beta}, \quad K_{IJ}^{\mu \nu} = \psi_I^\alpha (\varepsilon \sigma^{\mu \nu \varepsilon})_{\alpha \beta} \bar{\psi}_{J, \beta},
\]

(C1)

which satisfy the following properties:

\[
(\Psi_{IJ})^\dagger = \bar{\Psi}_{IJ}, \quad (J_{IJ}^\mu)^\dagger = J_{IJ}^\mu, \quad (K_{IJ}^{\mu \nu})^\dagger = K_{IJ}^{\mu \nu},
\]

(C2)

and

\[
J_{IJ}^\mu J_{KL}^{\nu} = -\frac{1}{2} \bar{\Psi}_{IK} \Psi_{JL} - 2 \bar{K}_{IK}^{\nu} K_{JL}^{\mu} + \bar{\Psi}_{IK} K_{JL}^{\mu \nu} - \Psi_{JL} K_{IK}^{\mu \nu} ,
\]

(C3)

\[
J_{IJ}^\mu J_{KL}^{\nu} = \eta_{\mu \nu} J_{IJ}^{\nu} J_{KL}^\mu = -2 \bar{\Psi}_{IK} \Psi_{JL} ,
\]

(C4)

\[
K_{IJ}^{\mu \nu} J_{KL}^{\nu} = \frac{1}{2} (J_{IK}^{\mu} \Psi_{JL} - J_{IK}^{\nu} \Psi_{JL}^{\mu} ) ,
\]

(C5)

\[
K_{IJ}^{\mu \nu} J_{KL}^{\nu} J_{JM}^{\mu} = -\bar{\Psi}_{KM} (\Psi_{IJ} J_{KL} - \Psi_{JL} J_{IJ} ) ,
\]

(C6)

\[
\bar{K}_{IJ}^{\mu \nu} J_{KL}^{\nu} J_{JM}^{\mu} = \frac{1}{2} (-J_{IKJ}^{\mu \nu} + J_{IKJ}^{\mu} J_{IJ}^{\nu} + J_{IJ}^{\nu} J_{IKJ}^{\mu} ) ,
\]

(C7)

\[
\bar{K}_{IJ}^{\mu \nu} J_{KL}^{\nu} J_{JM}^{\mu} = -\bar{\Psi}_{NJ} (\Psi_{IM} J_{IJ} - \Psi_{IJ} J_{IM} ) ,
\]

(C8)

\[
K_{IJ}^{\mu \nu} K_{KL}^{\nu} K_{JM}^{\mu} = \frac{1}{4} \bar{\Psi}_{KM} (\bar{\Psi}_{IKJ} - \bar{\Psi}_{IMJ} ) - \bar{\Psi}_{IMJ} \bar{\Psi}_{IKJ} ,
\]

(C9)

\[
K_{IJ}^{\mu \nu} \bar{K}_{KL}^{\nu} K_{JM}^{\mu} = \frac{1}{8} (J_{IKJ}^{\mu \nu} - J_{IKJ}^{\mu} J_{IJ}^{\nu} - J_{IKJ}^{\nu} J_{IJ}^{\mu} ) ,
\]

(C10)

\[
K_{IJ}^{\mu \nu} \bar{K}_{KL}^{\nu} K_{JM}^{\mu} = \frac{1}{4} (J_{IKJ}^{\mu \nu} - J_{IKJ}^{\mu} J_{IJ}^{\nu} - J_{IKJ}^{\nu} J_{IJ}^{\mu} ) = 0 ,
\]

(C11)

where we have used (A4)-(A11). In general, any quantities without derivatives and Weyl indices (Weyl indices are appropriately contracted.) are written as

\[
A^{\mu_1 \mu_2 \cdots \mu_n} = A^{\mu_1 \mu_2 \cdots \mu_n} (\eta_{\mu \nu}, \phi^a, \Psi_{IJ}, \bar{\Psi}_{IJ}, J_{IJ}^\mu, K_{IJ}^{\mu \nu}, \bar{K}_{IJ}^{\mu \nu}) ,
\]

(C12)
where \( n \) is zero or a natural number. In addition, from the above identities, we know that all scalar quantities without derivatives can be expressed only by \( \phi^a \), \( \Psi_{IJ} \) and \( \bar{\Psi}_{IJ} \). We also see that the spacetime index of vector quantities are described by \( J^I_{IJ} \), and those of 2nd-rank tensors by \( \eta_{\mu\nu}, K_{IJ}^{\mu\nu}, \bar{K}_{IJ}^{\mu\nu}, \) and \( K_{IJ}^{\mu\lambda}K_{KL}^{\nu}, \) i.e.,

\[
F = \mathcal{F}(\phi^a, \Psi_{IJ}, \bar{\Psi}_{IJ}), \quad G^\mu = G_{IJ}^I J^J_{IJ}, \quad H^\mu = H(1)_{IJ} \eta_{\mu\nu} + H(2)_{IJ} K_{IJ}^{\mu\nu} + H(3)_{IJ} \bar{K}_{IJ}^{\mu\nu} + H(4)_{IJ} K_{IJ}^{\mu\lambda}K_{KL}^{\nu},
\]

where \( F, G^\mu \) and \( H^\mu \) are arbitrary scalar, vector and 2nd-rank tensor, respectively. \( G_{IJ}, H(1), H(2) \) and so on are scalar quantities just like \( F \). Though further investigation is needed for the construction of arbitrary functions with Weyl indices, we can formally write down the most general action with arbitrary functions, and this is given by

\[
S = \int d^4x \left( L_0 + L_1 + L_2 \right)
\]

where

\[
L_0 = P_0, \\
L_1 = P_1^{(\mu} \partial_{\nu} \phi^a + S_{\mu\alpha}^{\nu} \partial_{\nu} \phi^a + \partial_{\mu} \phi^a \left( P^{(2)(\mu} \right), \\
L_2 = \frac{1}{2} V_{ab}^{\mu\nu} \partial_{\nu} \phi^a \partial_{\nu} \phi^b + S_{\alpha\mu\nu}^{\beta} \partial_{\mu} \phi^a \partial_{\nu} \phi^a + \partial_{\mu} \phi^a \partial_{\nu} \phi^a \left( S_{\alpha\mu\nu}^{\beta} \right),
\]

\[
+ \frac{1}{2} \partial_{\mu} \phi^a \partial_{\nu} \phi^a \left( W_{ab}^{\mu\nu,JI} \right) + \partial_{\mu} \phi^a \left( G_{ab}^{\mu\nu,JI} \right),
\]

and \( P_0, P_1^{(\mu} \partial_{\nu} \phi^a + S_{\mu\alpha}^{\nu} \partial_{\nu} \phi^a + \partial_{\mu} \phi^a \left( P^{(2)(\mu} \right), V_{ab}^{\mu\nu} = V_{ba}^{\mu\nu}, W_{ab}^{\mu\nu,JI}, \) and \( G_{ab}^{\mu\nu,JI} \) are appropriately contracted arbitary functions of Lorentz scalar quantities, Pauli matrices, and fermionic fields. The properties of these coefficients are

\[
(P_0) \dagger = P_0, \quad (P_1^{(\mu}) \dagger = P_1^{(\mu)}, \quad (V_{ab}^{\mu\nu}) \dagger = V_{ab}^{\mu\nu}, \quad (Q_{ab}^{\mu\nu,JI}) \dagger = Q_{ab}^{\mu\nu,JI},
\]

\[
V_{ab}^{\mu\nu} = V_{ba}^{\mu\nu}, \quad W_{ab}^{\mu\nu,JI} = -W_{ba}^{\mu\nu,JI}.
\]

The maximally degenerate conditions, (13) and (14), are

\[
D_{a\alpha,JI} - C_{ab,JI} A_{ba},J = W_{a\alpha}^{00,JI} + S_{ba}^{00,I} (V_{\alpha\beta}^{00})^{-1} ba_{a\alpha}^{00,J} = 0, \\
D_{a\alpha,JI} - C_{ab,JI} A_{ba},J = -Q_{a\alpha}^{00,JI} - S_{ba}^{00,I} (V_{\alpha\beta}^{00})^{-1} ba_{a\alpha}^{00,J} = 0.
\]

We can write down the momenta as

\[
\pi_{\phi^a} = P^{(1)(\mu} A_{ab}^{\nu} \partial_{\nu} \phi^b + S_{\alpha\mu\nu}^{\beta} \partial_{\nu} \phi^a + \partial_{\mu} \phi^a \left( S_{\alpha\mu\nu}^{\beta} \right), \\
\pi^{(\nu)}_{\phi^a} = -P^{(2)(\mu} A_{ab}^{\nu} \partial_{\nu} \phi^b + W_{a\alpha}^{\mu\nu,JI} \partial_{\mu} \phi^a + W_{a\alpha}^{\mu\nu,JI} \partial_{\nu} \phi^a - \partial_{\mu} \phi^a \left( Q_{a\alpha}^{\mu\nu,JI} \right),
\]

\[
\pi^{(\nu)}_{\phi^a} = -\left( \pi^{(\nu)}_{\phi^a} \right)^\dagger.
\]

As in Sec. III, we assume det \( (V_{ab}^{(\mu)})^{(0)} \neq 0 \). Substituting (C19) into (C20) for eliminating \( \dot{\phi}^a \), we obtain the 4N primary constraints,

\[
\Phi_{\psi^a} = \pi^{(\nu)}_{\psi^a} + P^{(2)(\mu} A_{ab}^{\nu} \partial_{\nu} \phi^b + S_{\alpha\mu\nu}^{\beta} \partial_{\nu} \phi^a + \partial_{\mu} \phi^a \left( S_{\alpha\mu\nu}^{\beta} \right) - [S_{\alpha\mu}^{00,I} (V_{\alpha\beta}^{00})^{-1} ab_{b\alpha}^{00,J} \partial_{b} \phi^c - S_{\alpha\mu}^{00,J} (V_{\alpha\beta}^{00})^{-1} ab_{b\alpha}^{00,J} \partial_{b} \phi^c] \partial_{a} \phi^c - \partial_{\mu} \phi^a \left( S_{\alpha\mu\nu}^{\beta} \right) + \partial_{\mu} \phi^a \left( Q_{a\alpha}^{\mu\nu,JI} \right).
\]

\[
\Phi_{\psi^a} = -\left( \Phi_{\psi^a} \right)^\dagger.
\]

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