Ordering properties of the smallest order statistic from Weibull-G random variables

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**ABSTRACT**  
In this article we compare the minimums of two heterogeneous samples each following Weibull-G distribution under three scenarios. In the first scenario, the units of the samples are assumed to be independently distributed and the comparisons are carried out through vector majorization. The minimums of the samples are compared in the second scenario when the independent units of the samples also experience random shocks. The last scenario describes the comparison when the units have a dependent structure sharing Archimedean copula.

**1. Introduction**  
The article by Alzaatreh, Lee, and Famoye (2013) proposed a new method for generating families of continuous distributions, known as $T-X$ family. The $T-X$ family is derived using a function $w(x)$, which links the support of the random variable (rv) $T$ with the range of the rv $X$. The family consists of a large number of new as well as existing distributions as special cases. Many of the new distributions derived from the family exhibit varying shapes including bimodal with both monotone and non-monotone failure rates. As in (Alzaatreh, Lee, and Famoye 2013), the $T-X$ family of distributions is generated as follows:

Let $X$ be a random variable with distribution function (survival function) $F(x)$ ($\bar{F}(x)$) and density function $f(x)$. Let $T$ be continuous random variable with pdf $h(t)$ defined on $[a,b]$. The cdf of $T-X$ family of distributions is defined as

$$G(x) = \int_{a}^{w(F(x))} h(t) dt, \quad x \in \mathbb{R}. \quad (1.1)$$

The scale model is a flexible family of distributions that have been used extensively in statistics. $X$ is said to belong to the scale model if $X_i \sim F(\gamma x)$, where $\gamma_i > 0$ for $i = 1,2,\ldots,n$, referred to as the scale parameter. In the scale model, $F(x)$ and $f(x)$ are said to be the baseline distribution function and the baseline density function, respectively.
When the rv $T$ in (1.1) follows Weibull distribution, and $w(F(x))$ is defined as $F(y|x) / (1 - F(y|x))$, Weibull-$G$ distribution is generated from the $T$-$X$ family with cumulative distribution function (cdf) given by

$$G(x) = 1 - e^{-x^{\gamma} \left( \frac{F(x)}{1 - F(x)} \right)^{\beta}}, \quad x > 0, \beta > 0, \gamma > 0,$$

where $\alpha$ and $\beta$ are the scale and shape parameters respectively, and $F(y|x)$ is a baseline cdf with scale parameter $\gamma$ and hazard function $r(x) = f(x) / F(x)$. For each cdf $F(y|x)$, a new Weibull-$G$ distribution can be defined with scale model link function bringing more flexibility in shapes and hazard rates. Exponential-$G$ distribution can be obtained as special case for $\beta = 1$. As in Cooray (2006), the Weibull-$G$ distribution, written as $W-G(\alpha, \beta, \gamma)$, can answer to the following questions found in survival analysis:

a. What are the odds that an individual will die prior to time $X$, if $X$ follows a life distribution with cdf $F$?

b. If these odds follow some other life distribution $T$, then what is the corrected distribution of $X$?

Suppose the odds ratio that an individual (or component) following a lifetime distribution with cdf $F(x)$ will die (failure) at time $x$ is $F(x)^{\gamma} / (1 - F(x))^{\beta}$. Also consider that the variability of these odds of death is represented by the random variable $T$ and assume that it follows Weibull distribution with parameters $\alpha$ and $\beta$. Then rv $X$ will have cdf as defined in Equation (1.2). A detail analysis of the distribution is found in Bourguignon, Silva, and Cordeiro (2014) with a data analysis which shows that the distribution performs better than the three parameter exponentiated Weibull distribution. The article aims to study the stochastic comparison of smallest order statistics from two heterogeneous samples (systems) with $W$-$G$ distributed units (components).

Order statistics have a prominent role in survival analysis, reliability theory, life testing, actuarial science, auction theory, hydrology and many other related and unrelated areas. If $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_n$, then the sample minimum corresponds to the smallest order statistics $X_{1:n}$. The results of stochastic comparisons of the order statistics (largely on the smallest and the largest order statistics) with independent sampling units can be seen in Dykstra, Kochar, and Rojo (1997), Zhao and Balakrishnan (2011), Torrado and Kochar (2015), Kundu et al. (2016), Kundu and Chowdhury (2016, 2018, 2020), Chowdhury and Kundu (2017), Kundu, Chowdhury, and Balakrishnan (2021), Kayal and Kundu (2021) and the references there in, for a variety of parametric models. Such comparisons are generally carried out with the assumption that the units of the sample die (fail) with certainty. In practice, the units may experience random shocks which eventually doesn’t guarantee its death (failure). Fang and Balakrishnan (2018) has compared two such systems with generalized Birnbaum-Saunders components. Chowdhury, Kundu, and Mishra (2021) and Kundu and Chowdhury (2021) also compared two such systems for different distributions. Such comparisons are also carried out by Barmalzan, Najafabadi, and Balakrishnan (2017), Balakrishnan, Zhang, and Zhao (2018) in the context of insurance. However, in many practical situations, the units of a sample may have a structural
dependence which result in a set of statistically dependent observations. Recently, the dependence structure of the components are investigated with the help of copulas by Navarro and Spizzichino (2010), Rezapour and Alamatsaz (2014), Li and Fang (2015), Fang, Li, and Li (2016), Kundu, Chowdhury, and Balakrishnan (2021), Chowdhury, Kundu, and Mishra (2021), and Kundu and Chowdhury (2020) among others.

In this article we intend to compare the minimums of two heterogeneous samples each following Weibull-\( G \) distribution under three scenarios. In the first scenario, the units of the samples are assumed to be independently distributed and the comparisons are carried out through vector majorization. The minimums of the samples are compared in the second scenario when the independent units of the samples also experience random shocks. The last scenario describes the comparison when the units have a dependent structure sharing Archimedean copula.

The organization of the article is as follows. In Section 2, we have given the required definitions and some useful lemmas which are used throughout the article. Results related to the comparison of two smallest order statistics from \( W-G \) distributions are derived in Section 3 under three scenarios as mentioned earlier. Finally, Section 4 concludes the article.

Throughout the article, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and \( \mathbb{R} \) denotes the set of real numbers \( \{x : -\infty < x < \infty\} \). We also write \( a \equiv b \) to mean that \( a \) and \( b \) have the same sign. For any differentiable function \( k(\cdot) \), we write \( k'(t) \) to denote the first derivative of \( k(t) \) with respect to \( t \).

2. Notations, definitions, and preliminaries

For two absolutely continuous random variables \( X \) and \( Y \) with distribution functions \( F(\cdot) \) and \( G(\cdot) \), survival functions \( \bar{F}(\cdot) \) and \( \bar{G}(\cdot) \), density functions \( f(\cdot) \) and \( g(\cdot) \) and hazard rate functions \( r(\cdot) \) and \( s(\cdot) \) respectively, \( X \) is said to be smaller than \( Y \) in

i. **likelihood ratio order** (denoted as \( X \leq_{lr} Y \)), if, for all \( t \), \( g(t) f(t) \) increases in \( t \),

ii. **hazard rate order** (denoted as \( X \leq_{hr} Y \)), if, for all \( t \), \( \frac{g(t)}{f(t)} \) increases in \( t \) or equivalently \( r(t) \geq s(t) g \), and

iii. **usual stochastic order** (denoted as \( X \leq_{st} Y \)), if \( F(t) \geq G(t) \) for all \( t \). For more on different stochastic orders, see Shaked and Shanthikumar (2007).

The notion of majorization (Marshall, Olkin, and Arnold 2011) is essential for the understanding of the stochastic inequalities for comparing order statistics. Let \( I^n \) be an \( n \)-dimensional Euclidean space where \( I \subseteq \mathbb{R} \). Further, for any two real vectors \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n \) and \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n \), write \( x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \) and \( y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)} \) as the increasing arrangements of the components of the vectors \( \mathbf{x} \) and \( \mathbf{y} \) respectively. The following definitions may be found in Marshall, Olkin, and Arnold (2011).

**Definition 2.1.**

i. The vector \( \mathbf{x} \) is said to majorize vector \( \mathbf{y} \) (written as \( \mathbf{x} \geq_{m} \mathbf{y} \)) if

\[
\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, j = 1, 2, \ldots, n - 1, \text{ and } \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)} ;
\]
ii. The vector \( \mathbf{x} \) is said to weakly submajorize the vector \( \mathbf{y} \) (written as \( \mathbf{x} \succeq_w \mathbf{y} \)) if
\[
\sum_{i=j}^{n} x(i) \geq \sum_{i=j}^{n} y(i) \quad \text{for } j = 1, 2, \ldots, n.
\]

**Definition 2.2.** A function \( \psi : I^n \to \mathbb{R} \) is said to be Schur-convex (resp. Schur-concave) on \( I^n \) if
\[
\mathbf{x} \succeq_w \mathbf{y} \quad \text{implies} \quad \psi(\mathbf{x}) \geq \psi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in I^n.
\]

**Definition 2.3.** For any positive integer \( r \), a function \( \psi : \mathbb{R} \to \mathbb{R} \) is said to be \( r \)-convex (resp. \( r \)-concave) on \( \mathbb{R} \) if \( \frac{d^r \psi(x)}{dx^r} \) is increasing (decreasing) for all \( x \in \mathbb{R} \).

Clearly any 1-convex (1-concave) function is a convex (concave) function. Again, while for any positive integer \( r \), \( g(x) = e^x \) is \( r \)-convex function; \( h(x) = e^{-x} \) is \( r \)-convex function for any odd positive integer \( r \) and \( r \)-concave function for any even positive integer \( r \).

Now, let us recall that a copula associated with a multivariate distribution function \( F \) is a function \( C : [0,1]^n \to [0,1] \) satisfying: \( F(x) = C(F_1(x_1), \ldots, F_n(x_n)) \), where the \( F_i \)'s, \( 1 \leq i \leq n \) are the univariate marginal distribution functions of \( X_i \)'s. Similarly, a survival copula associated with a multivariate survival function \( \bar{F} \) is a function \( \bar{C} : [0,1]^n \to [0,1] \) satisfying:
\[
\bar{F}(x) = P(X_1 > x_1, \ldots, X_n > x_n) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),
\]
where, for \( 1 \leq i \leq n \), \( \bar{F}_i(\cdot) = 1 - F_i(\cdot) \) are the univariate survival functions. In particular, a copula \( C \) is Archimedean if there exists a generator \( \psi : [0, \infty) \to [0,1] \) such that
\[
C(u) = \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)).
\]

For \( C \) to be Archimedean copula, it is sufficient and necessary that \( \psi \) satisfies i) \( \psi(0) = 1 \) and \( \psi(\infty) = 0 \) and ii) \( \psi \) is \( d \)-monotone, i.e., \( \frac{d^k \psi(s)}{ds^k} \geq 0 \) for \( k \in \{0, 1, \ldots, d - 2\} \) and \( \frac{d^k \psi(s)}{ds^k} \) is decreasing and convex. Archimedean copulas cover a wide range of dependence structures including the independence copula and the Clayton copula. For more detail on Archimedean copula, see, Nelsen (2006) and McNeil and NSlehov (2009). In this article, Archimedean copula is specifically employed to model on the dependence structure among random variables in a sample. The following important lemma is used in the next sections to prove some of the important theorems.

**Lemma 2.1.** (Li and Fang 2015) For two \( n \)-dimensional Archimedean copulas \( C_{\psi_1}(u) \) and \( C_{\psi_2}(u) \), with \( \phi_2 = \psi_2^{-1} = \sup \{ x \in \mathbb{R} : \psi_2(x) > u \} \), the right continuous inverse, if \( \phi_2 \circ \psi_1 \) is super-additive, then \( C_{\psi_1}(u) \leq C_{\psi_2}(u) \) for all \( u \in [0,1]^n \). Recall that a function \( f \) is said to be super-additive if \( f(x + y) \geq f(x) + f(y) \), for all \( x \) and \( y \) in the domain of \( f \).

**Notation 2.1.** Let us define the following notations.

i. \( D_+ = \{ (x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \} \).

ii. \( E_+ = \{ (x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \} \).
3. Comparison of smallest order statistics

Suppose that \( U_i \sim W-G(\lambda_i, \beta, \gamma_i) \) and \( V_i \sim W-G(\lambda_i, \beta, \delta_i) \) \((i = 1, 2, \ldots, n)\) be two sets of \( n \) independent random variables. Also suppose that \( w(\gamma x) = \frac{f(\gamma x)}{F(\gamma x)} \), and the baseline distribution has failure rate \( r(x) = \frac{f(x)}{F(x)} \). If \( G_{1:n}(\cdot) \) and \( H_{1:n}(\cdot) \) be the survival functions of \( U_{1:n} \) and \( V_{1:n} \), respectively, then, for all \( x \geq 0 \),

\[
G_{1:n}(x) = e^{-\sum_{i=1}^{n} g_i(w(\gamma_i x))^\beta} \tag{3.1}
\]

and

\[
H_{1:n}(x) = e^{-\sum_{i=1}^{n} \lambda_i g_i(w(\delta_i x))^\beta} \tag{3.2}
\]

Again, if \( r_{1:n}(\cdot) \) and \( s_{1:n}(\cdot) \) are the hazard rate functions of \( U_{1:n} \) and \( V_{1:n} \), respectively, then

\[
r_{1:n}(x) = \sum_{i=1}^{n} \gamma_i \lambda_i \beta(w(\gamma_i x))^\beta - 1 w'(\gamma_i x), \tag{3.3}
\]

and

\[
s_{1:n}(x) = \sum_{i=1}^{n} \lambda_i \delta_i \beta(w(\delta_i x))^\beta - 1 w'(\delta_i x). \tag{3.4}
\]

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \), and \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \) \( \in I^n \).

3.1. Heterogeneous independent samples

Here, we compare two smallest order statistics with heterogeneous independent \( W-G \) distributed samples through vector majorization. The following two theorems show that under certain conditions on parameters, there exists hazard rate ordering between \( U_{1:n} \) and \( V_{1:n} \).

**Theorem 3.1.** For \( i = 1, 2, \ldots, n \), let \( U_i \) and \( V_i \) be two sets of mutually independent random variables with \( U_i \sim W-G(\lambda_i, \beta, \gamma_i) \) and \( V_i \sim W-G(\lambda_i, \beta, \gamma_i) \). If \( \beta \geq 1 \), and the baseline distribution has convex odds ratio, then \( \alpha \succeq_w \lambda \) implies \( X_{1:n} \leq_{hr} Y_{1:n} \) for \( \alpha, \lambda, \gamma \in D_+(E_+) \).

**Proof.** As in equation (3.3), let \( \Psi(x) = \sum_{i=1}^{n} \alpha_i \gamma_i (w(\gamma_i x))^\beta - 1 w'(\gamma_i x) \). Differentiating \( \Psi(x) \) with respect to \( \alpha_i \), we get \( \frac{\partial \Psi(x)}{\partial \alpha_i} = \gamma_i (w(\gamma_i x))^\beta - 1 w'(\gamma_i x) \geq 0 \) for all \( x \geq 0 \), giving that \( \Psi(x) \) is increasing in each \( \alpha_i \). Thus,

\[
\frac{\partial \Psi(x)}{\partial \alpha_i} - \frac{\partial \Psi(x)}{\partial \alpha_j} = \gamma_i (w(\gamma_i x))^\beta - 1 w'(\gamma_i x) - \gamma_j (w(\gamma_j x))^\beta - 1 w'(\gamma_j x). \tag{3.5}
\]

Now, as \( w(x) \) is increasing in \( x \), then for all \( x \geq 0 \), \( \beta \geq 1 \) and \( i \leq j \),

\[
\gamma_i \leq (\leq) \gamma_j \Rightarrow w(\gamma_i x) \geq (\leq) w(\gamma_j x) \Rightarrow \gamma_i (w(\gamma_i x))^\beta - 1 \geq (\leq) \gamma_j (w(\gamma_j x))^\beta - 1. \tag{3.6}
\]
Again, as \( w(x) \) is convex in \( x \), then for \( i \leq j, \gamma_i \geq (\leq) \gamma_j \) implies
\[
\frac{w' (\gamma_i x)}{\gamma_i} \geq (\leq) \frac{w' (\gamma_j x)}{\gamma_j}.
\]

Applying the results (3.6) and (3.7) in (3.5), and noticing the fact that for all \( x \geq 0 \) \( w'(x) \geq 0 \), it is clear that \( \frac{\partial w(x)}{\partial \gamma_i} \geq (\leq) \frac{\partial w(x)}{\partial \gamma_j} \). Thus by Lemma 3.1 (3.3) of Kundu et al. (2016) it can be written that \( \Psi(x) \) is Schur-convex in \( x \). So, using Lemma 2.2 of (Kundu and Chowdhury 2016) it can be proved that
\[
r_{1,n}(x) \geq s_{1,n}(x).
\]
This proves the result.

**Remark 3.1.** Let \( X \) and \( Y \) be two non negative random variables with odd ratios \( w_X(x) \) and \( w_Y(x) \), respectively. For \( i = 1, 2, \ldots, n \), let \( U_i \) and \( V_i \) be two sets of mutually independent random variables with \( U_i \sim W-G(\lambda_i, \beta, \delta_i) \) and \( V_i \sim W-G(\lambda_i, \beta, \delta_i) \) generated from baseline distributions of \( X \) and \( Y \), respectively. Then under same conditions as of Theorem 3.1, the result of the same will hold when \( X \leq_Y Y \).

**Theorem 3.2.** For \( i = 1, 2, \ldots, n \), let \( U_i \) and \( V_i \) be two sets of mutually independent random variables with \( U_i \sim W-G(\lambda_i, \beta, \delta_i) \) and \( V_i \sim W-G(\lambda_i, \beta, \delta_i) \). If \( \beta \geq 2 \), and the odds ratio of the baseline distribution is convex and 2-convex, then \( \gamma \equiv_\alpha \delta \) implies \( X_{1:n} \leq_Y Y_{1:n} \) for \( \alpha, \gamma \in \mathcal{D}_+ \).

**Proof.** Let, \( \Psi_1(\gamma) = \sum_{i=1}^n x_i \gamma_i (w(\gamma_i x))^{\beta-1}w'(\gamma_i x) \). In view of Theorem 3.1 we need only to prove that \( \frac{\partial \Psi_1(\gamma)}{\partial \gamma_i} \geq (\leq) \frac{\partial \Psi_1(\gamma)}{\partial \gamma_j} \). Now,
\[
\frac{\partial \Psi_1(\gamma)}{\partial \gamma_i} = \frac{\partial \Psi_1(\gamma)}{\partial \gamma_i} = x_i (w(\gamma_i x))^{\beta-1}w'(\gamma_i x) + x_i \gamma_i (\beta - 1)(w(\gamma_i x))^{\beta-2} \left( w'(\gamma_i x) \right)^2
\]
\[
+ xx_i \gamma_i (w(\gamma_i x))^{\beta-1}w''(\gamma_i x) \geq 0,
\]
for all \( x \geq 0 \), giving that \( \Psi_1(\gamma) \) is increasing in each \( \gamma_i \). Now as \( w(x) \) is 2-convex in \( x \), \( w''(x) \) is increasing in \( x \). Therefore for \( i \leq j, \gamma_i \geq (\leq) \gamma_j \) implies that \( w''(\gamma_i x) \geq (\leq) w''(\gamma_j x) \). So, noticing the fact that \( w(x) \) is increasing and convex in \( x \) and \( \beta \geq 1 \), for all \( i \leq j, \gamma_i \geq (\leq) \gamma_j \) gives
\[
xx_i \gamma_i (w(\gamma_i x))^{\beta-1}w''(\gamma_i x) \geq (\leq) xx_i \gamma_i (w(\gamma_i x))^{\beta-1}w''(\gamma_j x),
\]
for all \( x \geq 0 \). Again, \( \beta \geq 2, x, \gamma \in \mathcal{D}_+ \), and \( w(x) \) is increasing and convex in \( x \) give, for all \( x \geq 0 \),
\[
xx_i \gamma_i (\beta - 1)(w(\gamma_i x))^{\beta-2} \left( w'(\gamma_i x) \right)^2 \geq (\leq) xx_i \gamma_i (\beta - 1)(w(\gamma_j x))^{\beta-2} \left( w'(\gamma_j x) \right)^2.
\]
Again, following the same logic as in Theorem 3.1, it can be proved that
\[
xx_i \gamma_i (\beta - 1)(w(\gamma_i x))^{\beta-2} \left( w'(\gamma_i x) \right)^2 \geq (\leq) xx_i \gamma_i (\beta - 1)(w(\gamma_j x))^{\beta-2} \left( w'(\gamma_j x) \right)^2.
\]

Applying the results in (3.8), it can be easily shown that \( \frac{\partial \Psi_1(\gamma)}{\partial \gamma_i} \geq (\leq) \frac{\partial \Psi_1(\gamma)}{\partial \gamma_j} \). Thus by Lemma 3.1 and 3.3 of Kundu et al. (2016) it can be written that \( \Psi(\gamma) \) is Schur-convex in \( \gamma \). Thus using Lemma 2.2 (Kundu and Chowdhury 2016) it can be proved that
\[
r_{1,n}(x) \geq s_{1,n}(x),
\]
which in turn proves that \( X_{1:n} \leq_Y Y_{1:n} \).
Remark 3.2. Let \( X \) and \( Y \) be two non-negative random variables with odd ratios \( w_X(x) \) and \( w_Y(x) \) respectively. For \( i = 1, 2, \ldots, n \), let \( U_i \) and \( V_i \) be two sets of mutually independent random variables with \( U_i \sim W-G(x_i, \beta, \gamma_i) \) and \( V_i \sim W-G(x_i, \beta, \gamma_i) \) generated from baseline distributions of \( X \) and \( Y \) respectively. Then under same conditions as of Theorem 3.2, the result of the same will hold when \( X \leq_{lr} Y \).

The following theorem shows that in case of multiple-outlier model, when \( x \geq w_\lambda \), \( \lambda \) ordering exists between \( X_{1:n} \) and \( Y_{1:n} \) for any positive integer \( n \).

**Theorem 3.3.** For \( i = 1, 2, \ldots, n \), let \( U_i \) and \( V_i \) be two sets of mutually independent random variables each following multiple-outlier \( W-G \) model such that \( U_i \sim W-G(x_1, \beta, \gamma_i) \) and \( V_i \sim W-G(x_2, \beta, \gamma_i) \) for \( i = 1, 2, \ldots, n \), \( U_i \sim W-G(x_2, \beta, \gamma_2) \) and \( V_i \sim W-G(x_2, \beta, \gamma_2) \) for \( i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2(= n) \). If \( \beta \geq 1 \), the baseline distribution has convex odds ratio, \( \frac{w_X'(x)}{w_X(x)} \) and \( \frac{w_Y'(x)}{w_Y(x)} \) are decreasing in \( x \), then

\[
(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2) \equiv \left( \lambda_1, \lambda_1, \ldots, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_2 \right)
\]

implies \( X_{1:n} \leq_{lr} Y_{1:n} \) for \( x, \lambda, \gamma \in D_+(E_+) \).

**Proof.** In view of Theorem 3.1, we need only to prove that \( \frac{r_{1:n}(x)}{s_{1:n}(x)} \) is decreasing in \( x \). Now,

\[
\frac{d}{dx} \left( \frac{r_{1:n}(x)}{s_{1:n}(x)} \right) = \sum_{i=1}^{n} x_i \gamma_i^2 (w'/(\gamma_i x))^\beta - 2 \left[ (\beta - 1) \left( \frac{w'/(\gamma_i x)}{w'/(\gamma_i x)} \right)^2 + w(\gamma_i x)w''/(\gamma_i x) \right]
\]

\[
- \frac{\sum_{i=1}^{n} x_i \gamma_i^2 (w'/(\gamma_i x))^\beta - 2 \left[ (\beta - 1) \left( \frac{w'/(\gamma_i x)}{w'/(\gamma_i x)} \right)^2 + w(\gamma_i x)w''/(\gamma_i x) \right]}{\sum_{i=1}^{n} x_i \gamma_i^2 (w'/(\gamma_i x))^\beta - 1 w'/(\gamma_i x)}.
\]

Thus, to show that \( \frac{r_{1:n}(x)}{s_{1:n}(x)} \) is decreasing in \( x \), it is sufficient to show that

\[
\Psi_2(\alpha) = \sum_{i=1}^{n} x_i \gamma_i^2 (w(\gamma_i x))^\beta - 2 \left[ (\beta - 1) \left( \frac{w'/(\gamma_i x)}{w'/(\gamma_i x)} \right)^2 + w(\gamma_i x)w''/(\gamma_i x) \right]
\]

is Schur-concave in \( \alpha \). After simplifications, we get

\[
\frac{\partial \Psi_2(\alpha)}{\partial \alpha_1} = n_2 x_2 \gamma_2 w'/(\gamma_1 x)w'/(\gamma_2 x)(w(\gamma_1 x))^{\beta - 1} (w(\gamma_2 x))^{\beta - 1}
\]

\[
\left[ (\beta - 1) \left( \frac{\gamma_1 w'(\gamma_1 x)}{w'(\gamma_1 x)} - \gamma_2 w'(\gamma_2 x) \right) + \left( \frac{\gamma_1 w''(\gamma_1 x)}{w'(\gamma_1 x)} - \gamma_2 w''(\gamma_2 x) \right) \right],
\]

and
Now, three cases may arise:

Case(i) \(1 \leq i \leq j \leq n\). Here \(x_i = x_j = x_1\) and \(y_i = y_j = y_1\), so that

\[
\frac{\partial \Psi_2(x)}{\partial x_i} - \frac{\partial \Psi_2(x)}{\partial x_j} = \frac{\partial \Psi_2(x)}{\partial x_1} - \frac{\partial \Psi_2(x)}{\partial x_1} = 0.
\]

Case(ii) \(n_1 + 1 \leq i \leq j \leq n\). Here \(x_i = x_j = x_2\) and \(y_i = y_j = y_2\), so that

\[
\frac{\partial \Psi_2(x)}{\partial x_i} - \frac{\partial \Psi_2(x)}{\partial x_j} = \frac{\partial \Psi_2(x)}{\partial x_2} - \frac{\partial \Psi_2(x)}{\partial x_2} = 0.
\]

Case(iii) For, \(1 \leq i \leq n_1\) and \(n_1 + 1 \leq j \leq n\), then \(x_i = x_1, y_i = y_1\) and \(x_j = x_2, y_j = y_2\), giving \(x_1 \geq (\leq) x_2\) and \(y_1 \geq (\leq) y_2\). So, noticing the fact that, \(\frac{w'(x)}{w(x)}\) and \(\frac{w''(x)}{w(x)}\) are decreasing in \(x\), it can be shown that

\[
\frac{\partial \Psi_2(x)}{\partial x_i} - \frac{\partial \Psi_2(x)}{\partial x_j} = \frac{\partial \Psi_2(x)}{\partial x_1} - \frac{\partial \Psi_2(x)}{\partial x_2} \\
\left[\frac{\beta}{n_1x_1 + n_2x_2}\right] \left(\frac{\gamma_iw'(\gamma_1x)}{w'(\gamma_2x)} - \frac{\gamma_jw'(\gamma_1x)}{w'(\gamma_2x)}\right) + \left(\frac{\gamma_iw''(\gamma_1x)}{w''(\gamma_2x)} - \frac{\gamma_jw''(\gamma_1x)}{w''(\gamma_2x)}\right) \leq (\geq) 0.
\]

Thus, for all \(i \leq j\), \(\frac{\partial \Psi_2(x)}{\partial x_i} - \frac{\partial \Psi_2(x)}{\partial x_j} \leq (\geq) 0\), which by Lemma 3.1 (Lemma 3.3) of Kundu et al. (2016) gives \(\Psi_2(x)\) is Schur-concave in \(x\). This proves the result. \(\square\)

Theorem 3.3 shows that for multiple outlier model \(x \geq \lambda^{m}\) implies \(X_{1:n} \leq_{l} Y_{1:n}\). The counterexample given below shows that although the conditions of Theorem 3.3 are all satisfied but in general \(X_{1:n} \leq_{l} Y_{1:n}\) does not hold.

**Counterexample 3.1.** Let the baseline random variable follows Burr(3, 0.35) distribution. Then Figure 1a–c, respectively, show that \(w'(x)\) is increasing in \(x\), and \(\frac{w'(x)}{w(x)}, \frac{w''(x)}{w(x)}\) are decreasing in \(x\). Let, \(\gamma = (2, 1.5, 1.5)\) and \(\beta = 5\). Now, if \(\alpha = (4, 1, 1)\) and \(\gamma = (3, 1.5, 1.5)\) are taken, which are multiple outlier model, then Figure 2a shows that \(X_{1:n} \leq_{l} Y_{1:n}\). But, if \(\alpha = (0.95, 0.3, 0.1)\) and \(\gamma = (0.95, 0.25, 0.15)\) are taken, which are not multiple outlier model, then Figure 2b shows that \(X_{1:n} \not\leq_{l} Y_{1:n}\). Here the substitution \(x = -\ln y, 0 \leq y \leq 1\) is taken to plot the whole range of the curves.

The Counterexample given below shows that for \(\gamma \geq \delta^{m}\), even if for multiple-outlier model, \(X_{1:n} \leq_{l} Y_{1:n}\) does not hold.

**Counterexample 3.2.** Let the baseline random variable follows Weibull(0.02,2) distribution. Then, Figure 3(a) and (b) show that \(w(x)\) is convex and 2-convex. Again, for \(\beta =\)
3.4, and multiple outlier model having $\mathbf{z} = (3, 3, 1), \, \gamma = (3, 3, 1)$ and $\mathbf{d} = (2.5, 2.5, 2)$ although $\gamma \geq m \delta$ but Figure 3c shows that there exists no lr ordering between $X_{1:n}$ and $Y_{1:n}$. Here the substitution $x = -\ln y$, $0 \leq y \leq 1$ is taken to plot the whole range of the curves.

### 3.2. Heterogeneous independent samples under random shocks

The assumption in the previous section lies in the fact that each of the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ occurs with certainty and the comparison is carried out on the minimums of the order statistics. Now, it may so happen that the order statistics experience random shocks which may or may not result in its occurrence and it is of interest to compare two such systems stochastically. The model could arise in the context of reliability and actuarial sciences as described next.
Let us assume a series system consists of \( n \) independent components in working conditions. Each component of the system receives a shock which may cause the component to fail. Let the random variable (rv) \( T_i \) denote lifetime of the \( i^{th} \) component in the system which experiences a random shock at binging. Also suppose that \( I_i \) denote independent Bernoulli rvs, independent of the \( T_i \)'s, with \( E(I_i) = p_i \), will be called shock parameter hereafter. Then, the random shock impacts the \( i^{th} \) component \((I_i = 1)\) with probability \( p_i \) or doesn’t impact the \( i^{th} \) component \((I_i = 0)\) with probability \( 1 - p_i \). Hence, the rv \( X_i = I_i T_i \) corresponds to the lifetime of the \( i^{th} \) component in a system under shock. In this section, we compare two smallest order statistics with heterogeneous independent \( W-G \) distributed samples under random shocks through matrix majorization. For \( i = 1, 2, \ldots, n \), let \( U_i \) (resp. \( V_i \)) be \( n \) independent nonnegative rvs following \( W-G \) distribution as given in (1.2). Under random shock, let us assume \( W_i = U_i I_i \) and \( Y_i = V_i I_i^* \). Thus, for \( x > 0 \), the cdf of \( W_i \) and \( Y_i \) are given by

\[
F_i^W(x) = P(U_i I_i \geq x) = P(U_i I_i \geq x | I_i = 1)P(I_i = 1) = p_i e^{-\lambda_i (w(x))} , \quad x \geq 0,
\]

and

\[
F_i^Y(x) = P(V_i I_i^* \geq x) = P(V_i I_i^* \geq x | I_i^* = 1)P(I_i^* = 1) = p_i^* e^{-\lambda_i^* (w(x))} , \quad x \geq 0
\]

respectively. where \( E(I_i) = p_i \) and \( E(I_i^*) = p_i^* \).

If \( F_{1:n}^W(\cdot) \) and \( F_{1:n}^Y(\cdot) \) be the cdf of \( W_{1:n} \) and \( Y_{1:n} \) respectively, then from (1.2) it can be written that, for \( x > 0 \),

\[
\bar{F}_{1:n}^W(x) = \left( \prod_{i=1}^n p_i \right) e^{-\sum_{i=1}^n \lambda_i (w(x))} \tag{3.9}
\]

and

\[
\bar{F}_{1:n}^Y(x) = \left( \prod_{i=1}^n p_i^* \right) e^{-\sum_{i=1}^n \lambda_i^* (w(x))} , \tag{3.10}
\]

with \( \bar{F}_{1:n}(0) = \prod_{i=1}^n p_i \) and \( \bar{F}_{1:n}(0) = \prod_{i=1}^n p_i^* \).
The following two theorems show that under certain conditions on parameters, there exists hazard rate and likelihood ratio ordering between $W_{1:n}$ and $Y_{1:n}$ when the units of the sample experience random shocks.

**Theorem 3.4.** For $i = 1, 2, \ldots, n$, let $U_i$ and $V_i$ be two sets of mutually independent random variables with $U_i \sim W-G(x_i, \beta, \gamma_i)$ and $V_i \sim W-G(\lambda_i, \beta, \gamma_i)$. Further, suppose that $I_i (I_i^+)$ be a set of independent Bernoulli rv, independent of $U_i$'s (or $V_i$'s) with $E(I_i) = p_i$, $E(I_i^+) = p_i^+$, $i = 1, 2, \ldots, n$. If $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^+$, the baseline distribution has convex odds ratio, $\alpha, \lambda, \gamma, \in D_+(E_+)$, and $\alpha \succeq W \lambda$, then $W_{1:n} \succeq hr Y_{1:n}$.

**Proof.** Using (3.9-3.10) and (3.1-3.2), we can write for $x > 0$,

$$
\frac{F_{1:n}^W(x)}{F_{1:n}^Y(x)} = \prod_{i=1}^{n} p_i e^{-\sum_{i=1}^{n} x_i(\gamma_i, x)} = \prod_{i=1}^{n} p_i^+ \frac{\bar{G}_{1:n}(x)}{\bar{H}_{1:n}(x)},
$$

which is decreasing in $x$ by **Theorem 3.4**.

Now, noticing the fact that $\lim_{x \to 0} \frac{F_{1:n}^W(x)}{F_{1:n}^Y(x)} = 1$, it can be written that

$$
1 \geq \prod_{i=1}^{n} \frac{p_i}{p_i^+} \Rightarrow \lim_{x \to 0} \frac{F_{1:n}^W(x)}{F_{1:n}^Y(x)} \geq \prod_{i=1}^{n} p_i \frac{\bar{G}_{1:n}(0)}{\bar{H}_{1:n}(0)} = \frac{F_{1:n}^W(0)}{F_{1:n}^Y(0)},
$$

proving that $\frac{F_{1:n}^W(x)}{F_{1:n}^Y(x)}$ is decreasing at $x = 0$. This proves the result.
Theorem 3.5. Let \( U_i \) and \( V_i \) be two sets of mutually independent random variables each following multiple-outlier \( W-G \) model such that with \( U_i \sim W-G(\alpha, \beta, \gamma_i) \) and \( V_i \sim W-G(\lambda, \beta, \gamma_i) \) for \( i = 1, 2, \ldots, n \). Then it is obvious that \( \nu_{11}^{U_i} \leq \nu_{11}^{V_i} \) if \( \nu_1^{U_i} = \nu_1^{V_i} \), \( \nu_2^{U_i} = \nu_2^{V_i} \), and \( \gamma_i = \gamma \). Further, suppose that \( I_i (I_i^*) \) be a set of independent Bernoulli rv, independent of \( U_i \)’s (or \( V_i \)’s) with \( E(I_i) = p_i \) (or \( E(I_i^*) = p_i^* \), \( i = 1, 2, \ldots, n \). If \( \prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^* \), the baseline distribution has convex odds ratio, \( \frac{\nu_{11}^{U_i}(x)}{\nu_{11}^{V_i}(x)} \) and \( \frac{\nu_{11}^{U_i}(x)}{\nu_{11}^{V_i}(x)} \) are decreasing in \( x \), then

\[
\left( x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots, x_{2n} \right) \leq_{hr} \left( \lambda_{11}, \lambda_{12}, \ldots, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n} \right),
\]

implies \( W_{1:n} \leq_{hr} Y_{1:n} \) for \( \alpha, \lambda, \gamma \in D_+ (E+) \).

Proof. If \( r_{1:n}^W(x) \) and \( s_{1:n}^Y(x) \) are the hazard rate functions of \( W_{1:n} \) and \( Y_{1:n} \), respectively, then it is obvious that \( r_{1:n}^W(x) = r_{1:n}^W(x) \) and \( s_{1:n}^Y(x) = s_{1:n}^Y(x) \). Then, by Theorem 3.4, \( W_{1:n} \leq_{hr} Y_{1:n} \) and by Theorem 3.3, \( \frac{r_{1:n}^W(x)}{r_{1:n}^Y(x)} \) is decreasing in \( x \). Thus by Theorem 1.C.4 of Shaked and Shanthikumar (2007), the result is proved.

3.3. Heterogeneous dependent samples

Here, we compare two smallest order statistics with heterogeneous dependent \( W-G \) distributed samples. The first two theorems show that usual stochastic ordering exists between \( U_{1:n} \) and \( V_{1:n} \) under majorization order of the scale parameters.

Theorem 3.6. Let \( U_1, U_2, \ldots, U_n \) be a set of dependent random variables sharing Archimedean copula having generator \( \psi_1 (\phi_1 = \psi_1^{-1}) \) such that \( U_i \sim W-G(\alpha, \beta, \gamma_i) \), \( i = 1, 2, \ldots, n \). Let \( V_1, V_2, \ldots, V_n \) be another set of dependent random variables sharing Archimedean copula having generator \( \psi_2 (\phi_2 = \psi_2^{-1}) \) such that \( V_i \sim W-G(\lambda, \beta, \gamma_i) \), \( i = 1, 2, \ldots, n \). Assume that \( \alpha, \lambda, \gamma \in D_+ \) (or \( E_+ \)), \( \phi_2 \circ \psi_1 \) is super-additive, \( \psi_1 \) or \( \psi_2 \) is log-convex. Then \( \alpha \geq_{w} \lambda \) implies \( X_{1:n} \leq_{st} Y_{1:n} \).

Proof.

\[
\tilde{G}_{1:n}(x) = \psi_1 \left[ \sum_{k=1}^n \phi_1 \left\{ e^{-2\alpha (w(\gamma_k x))^\beta} \right\} \right],
\]

and

\[
\tilde{H}_{1:n}(t) = \psi_2 \left[ \sum_{k=1}^n \phi_2 \left\{ e^{-2\lambda (w(\gamma_k x))^\beta} \right\} \right].
\]

By Lemma 2.1, super-additivity of \( \phi_2 \circ \psi_1 \) implies that

\[
\psi_1 \left[ \sum_{k=1}^n \phi_1 \left\{ e^{-2\alpha (w(\gamma_k x))^\beta} \right\} \right] \leq \psi_2 \left[ \sum_{k=1}^n \phi_2 \left\{ e^{-2\lambda (w(\gamma_k x))^\beta} \right\} \right]. \tag{3.11}
\]
Thus, by Lemma 2.2 of Kundu and Chowdhury (2016) it can be written that
\[ E^+ \text{ is log-convex,} \]
which combining with (3.11) proves that
\[ e^{-\lambda_3(w(\gamma_j x))^\beta} \leq (\geq) e^{-\lambda_3(w(\gamma_j x))^\beta} \Rightarrow \phi_2 (e^{-\lambda_3(w(\gamma_j x))^\beta}) \geq (\leq) \phi_2 (e^{-\lambda_3(w(\gamma_j x))^\beta}). \]

As \( \psi_2 \) is log-convex, \( \frac{\partial \psi_2}{\partial x} \) is decreasing in \( x \), implying that
\[ \frac{\partial \psi_2}{\partial \lambda_i} \leq \frac{\partial \psi_2}{\partial \lambda_i} \]
which in turn implies that
\[ \frac{\partial \psi_2}{\partial \lambda_i} \leq \frac{\partial \psi_2}{\partial \lambda_i} \]
proving that \( \frac{\partial \psi_2}{\partial \lambda_i} \leq \frac{\partial \psi_2}{\partial \lambda_i} \).

Hence, by Lemma 3.1 and 3.3 of Kundu et al. (2016), \( \Psi_3(\lambda) \) is Schur-concave in \( \lambda \). Thus by Lemma 2.2 of Kundu and Chowdhury (2016) it can be written that
\[ \mathbf{X} \equiv W \lambda \Rightarrow \psi_2 \left[ \sum_{k=1}^{n} \phi_2 (e^{-\lambda_3(w(\gamma_j x))^\beta}) \right] \leq \psi_2 \left[ \sum_{k=1}^{n} \phi_2 (e^{-\lambda_3(w(\gamma_j x))^\beta}) \right], \]
which combining with (3.11) proves that \( G_{1:n}(x) \leq H_{1:n}(x) \) which gives \( X_{1:n} \leq s_t Y_{1:n} \).
**Example 3.1.** Let the baseline random variable follows Weibull(0.02, 1.02) distribution. Also let the two generators $\psi_1 = (1 + x)^{-1}$ and $\psi_2 = (1 + 2x)^{-1/2}$ be the generators of Clayton copula, a well-known Archimedean copula. Then $\phi_1(x) = \frac{1-x}{x}$ and $\phi_2(x) = \frac{1-x^2}{2x^2}$. Again for all $x, y \geq 0$,

$$
\phi_2 \circ \psi_1(x + y) = \frac{1}{2} [(x + y)^2 + 2(x + y)] = \phi_2(x) + \phi_2(y),
$$

showing that the function $\phi_2 \circ \psi_1$ is super-additive. Clearly, $\psi_1$ and $\psi_2$ are log convex functions in $x$. Again, let, $\alpha = (3, 0.03, 0.02)$, $\lambda = (2.9, 0.11, 0.04)$. Thus, $\alpha, \lambda \in D_+$ and $\alpha \succeq_x \lambda$. Again, if $\beta = 0.25$ and $\gamma = (0.1, 0.03, 0.003) \in D_+$ are taken, then, Figure 4 shows that $H_{1,3}(x) - \tilde{G}_{1,3}(x) \geq 0$, demonstrating the result of the Theorem 3.6. Here the substitution $x = -\ln y$, $0 \leq y \leq 1$ is taken to plot the whole range of the curves.

**Remark 3.3.** Comparing Theorem 3.1 and Theorem 3.6, it can be observed that for independent case, when $\alpha \succeq_x \lambda$, stochastic ordering exits between $X_{1:n}$ and $Y_{1:n}$ under less restrictive condition than hazard rate ordering, which is expected.

**Theorem 3.7.** Let $U_1, U_2, \ldots, U_n$ be a set of dependent random variables sharing Archimedean copula having generator $\psi_1$ ($\phi_1 = \psi_1^{-1}$) such that $U_i \sim W-G(\alpha_i, \beta, \gamma_i), i = 1, 2, \ldots, n.$ Let $V_1, V_2, \ldots, V_n$ be another set of dependent random variables sharing Archimedean copula having generator $\psi_2$ ($\phi_2 = \psi_2^{-1}$) such that $V_i \sim W-G(\alpha_i, \beta, \delta_i), i = 1, 2, \ldots, n.$ Assume that $\alpha, \gamma, \delta, \in D_+$ (or $E_+$), $\beta \geq 1, \phi_2 \circ \psi_1$ is super-additive, and $\psi_1$ or $\psi_2$ is log-convex and the odd function of the baseline distribution is convex. Then $\gamma \succeq_x \delta$ implies $X_{1:n} \leq_{H} Y_{1:n}$.

**Proof.** Let,

$$
\tilde{G}_{1:n}(x) = \psi_1 \left[ \sum_{k=1}^{n} \phi_1 \left\{ e^{-\alpha_k(x)} \right\} \right],
$$

and

$$
\tilde{H}_{1:n}(t) = \psi_2 \left[ \sum_{k=1}^{n} \phi_2 \left\{ e^{-\alpha_k(t)} \right\} \right].
$$

As $\phi_2 \circ \psi_1$ is super-additive, by **Lemma 2.1** it can be written that

$$
\psi_1 \left[ \sum_{k=1}^{n} \phi_1 \left\{ e^{-\alpha_k(x)} \right\} \right] \leq \psi_2 \left[ \sum_{k=1}^{n} \phi_2 \left\{ e^{-\alpha_k(x)} \right\} \right]. \quad (3.12)
$$

**Now, let**

$$
\phi_2 \left[ \sum_{k=1}^{n} \phi_2 \left\{ e^{-\alpha_k(x)} \right\} \right] = \Psi\delta.
$$
So, \( \partial \Psi_4(\delta) \)/\( \partial \delta_i \) = \(-\psi'_2 \left[ \sum_{k=1}^n \phi_2 \left\{ e^{-2k(w(\delta_k)x)^p} \right\} \right] (w(\delta_ix))^b-1 \)

\[
\begin{align*}
\psi'_2 \left[ \phi_2 \left\{ e^{-2k(w(\delta_k)x)^p} \right\} \right] - \psi'_2 \left[ \phi_2 \left\{ e^{-2k(w(\delta_k)x)^p} \right\} \right] = x_i \delta_i \beta w'(\delta_ix) \leq 0,
\end{align*}
\]

for all \( x \geq 0 \), proving that \( \Psi_4(\delta) \) is decreasing in each \( \delta_i \). Proceeding in the similar manner as of the previous theorem it can be proved that \( \partial \Psi_4(\delta) \)/\( \partial \delta_j \) is Schur-concave in \( \delta \). Thus, using Lemma 2.2 of Kundu and Chowdhury (2016) it can be written that,

\[
\begin{align*}
\gamma \succeq_w \delta \Rightarrow \psi'_2 \left[ \sum_{k=1}^n \phi_2 \left\{ e^{-2k(w(\gamma)x)^p} \right\} \right] \leq \psi'_2 \left[ \sum_{k=1}^n \phi_2 \left\{ e^{-2k(w(\delta_k)x)^p} \right\} \right].
\end{align*}
\]

So, combining the above inequality with (3.12) it can be written that \( G_{1:n}(x) \leq \bar{H}_{1:n}(x) \), which gives \( X_{1:n} \preceq_{st} Y_{1:n} \).

**Remark 3.4.** Comparing Theorem 3.2 and Theorem 3.7, it can be observed that for independent case, when \( \gamma \succeq_w \delta \), stochastic ordering exits between \( X_{1:n} \) and \( Y_{1:n} \) under less restrictive condition than hazard rate ordering, which is expected.

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