Skyrme Random-Phase-Approximation description of lowest $K^\pi = 2^+ \gamma$ states in axially deformed nuclei

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The lowest quadrupole $\gamma$-vibrational $K^\pi = 2^+ \gamma$ states in axially deformed rare-earth (Nd, Sm, Gd, Dy, Er, Yb, Hf, W) and actinide (U) nuclei are systematically investigated within the fully self-consistent separable random-phase-approximation (SRPA) based on the Skyrme functional. The energies $E_\gamma$, and reduced transition probabilities $B(E2)$ of $2^+ \gamma$-states are calculated with the Skyrme forces SV-mas10 and SkM$^*$. We demonstrate that the blocking effect in pairing plays an important role. It leads to a systematic downshift of $E_\gamma$ by 0.3-0.5 MeV and thus to a significant improvement of agreement with the experiment, especially in Sm, Gd, Dy, Hf, and W regions. For other isotopic chains, a noticeable overestimation of $E_\gamma$ and too weak collectivity of $2^+ \gamma$-states still persist. It is shown that domains of nuclei with a low and high $2^+ \gamma$-collectivity are related with the structure of the lowest 2-quasiparticle states and maintenance of the Nilsson selection rules. The description of $2^+ \gamma$ states with SV-mas10 and SkM$^*$ is generally similar in light rare-earth nuclei but deviates in heavier nuclei. As compared to SkM$^*$, SV-mas10 provides better description of the quadrupole deformation and energy of the isoscalar giant quadrupole resonance. The calculations suggest that coupling with complex configurations is needed for a satisfactory description of $2^+ \gamma$ states in all domains.

I. INTRODUCTION

During the last decades, remarkable progress was made in the description of the nuclear dynamics within self-consistent mean-field models (SC-MFM), see e.g. the reviews [1-4]. However, this progress mainly concerns giant resonances (GR) while the description of the lowest vibrational states (LVS) is still an open problem, see examples of recent studies in [5-6]. The main reason for the trouble is that LVS are very sensitive to various factors. Following early calculations within the schematic Quasiparticle-Phonon Model (QPM) [7,8], the description of LVS requires a proper treatment of the single-particle (s-p) spectra near the Fermi level, pairing with the blocking effect, residual interaction (with both particle-hole and particle-particle channels for $\beta$-vibrational states), coupling to complex configurations (with taking into account the Pauli principle), and exclusion of the spurious admixtures. The size of the configuration space can also affect the results. In deformed nuclei, a correct description of deformation is crucial.

Modern SC-MFM claim a universal description of nuclear matter and nuclear structure/dynamics in a multitude of atomic nuclei. However, the performance of SC-MFM concerning LVS is still an open question, especially in deformed nuclei. To the best of our knowledge, there exists so far only one systematic exploration of LVS for deformed nuclei using SC-MFM: random-phase-approximation (RPA) calculations using Skyrme forces for $\gamma$-vibrational $K^\pi = 2^+ \gamma$ and $\beta$-vibrational $K^\pi = 0^+_\beta$ states in rare-earth nuclei [5]. These calculations were performed with the Skyrme forces SkM$^*$ [10] and SLy4 [11]. It was shown that Skyrme RPA is a reasonable tool for investigation of LVS. The performance of SkM$^*$ was found to be noticeably better than that of SLy4.

In the present paper, we continue the systematic exploration of $2^+ \gamma$-states in axial deformed nuclei with self-consistent Skyrme RPA. The $2^+ \gamma$-states are chosen as the simplest LVS where we do not need the particle-particle interaction channel and do not meet the problem of the extraction of the spurious admixtures.

As compared to [5], our study has some important new aspects. First, it is desirable to use for the description of $2^+ \gamma$-states Skyrme forces which reproduce the energy of the isoscalar giant quadrupole resonance (ISGQR). Following [12], these forces should have a large isoscalar effective mass $m^*_0/m$. The forces from [5] have low effective masses, $m^*_0/m =0.70$ for SLy4 [11] and 0.79 for SkM$^*$ [10], and so overestimate the ISGQR, see [12] and discussion below. To make the description of ISGQR and $2^+ \gamma$-states consistent, we use in our calculations the recent SV-mas10 force [13] with $m^*_0/m=1$ (see the comment [14]). As shown below, SV-mas10 also manages to reproduce ground state deformations, a feature which is utterly crucial for a correct placing of LVS. For comparison, we also exploit the force SkM$^*$ which demonstrated a good performance in [5].

Second, we take into account the pairing blocking effect (PBE) which, following QPM studies [7,8], is of crucial importance for the description of LVS, especially in axially deformed nuclei. The PBE can significantly weaken pairing [6] and thus downshift the energies of low-
energy two-quasiparticle (2qp) states by a few hundreds keV \cite{7,8}. This in turn leads to lower RPA energies of 2\(^+\)-states. We implement PBE within BCS using volume pairing. The same volume pairing, though within the Hartree-Fock-Bogoliubov (HFB) approach and without the PBE, was used in \cite{5}.

Third, we provide a detailed analysis of the obtained results, both numerically and analytically (e.g. in terms of simplified models for schematic RPA). We determine domains of nuclei with low and high collectivity of 2\(^+\)-states and demonstrate that the lowest \(K^{\mp} = 2^+\) 2qp state plays a key role in formation of these domains. The study embraces 9 isotopic chains (Nd, Sm, Gd, Dy, Er, Yb, Hf, W, U) with 41 axial nuclei (as compared to 27 rare-earth nuclei in \cite{4}). Both rare-earth and actinide mass regions are covered.

The calculations are performed within the separable random-phase-approximation (SRPA) method based on the Skyrme functional \cite{15–17}. The method exists in one-dimensional (1D) version for spherical nuclei \cite{18} and in two-dimensional (2D) version \cite{19} for axial deformed nuclei. SRPA is fully self-consistent since i) both the mean field and residual interaction are obtained from the same Skyrme functional, ii) the residual interaction includes all terms of the functional, including Coulomb (direct and exchange) terms. The self-consistent factorization of the residual interaction dramatically reduces the computational effort for deformed nuclei while keeping high accuracy of the method. SRPA was successfully applied for the description of various excitations in spherical and deformed nuclei: (E1(T=1) and E2(T=0) GR \cite{12,13–22}, E1 toroidal/compression \cite{23,26} and pygmy \cite{22,26} resonances, M1 spin-flip and orbital GR \cite{27,29}).

The paper is organized as follows. In Sec. 2, the method and calculation details are outlined. It is shown that SV-mas10, unlike SkM\(^*\), reproduces nicely equilibrium quadrupole deformations and well describes, within SRPA, the ISQGR energy. Sec. 3 presents the main results for energies and reduced transition probabilities \(B(E2)\) of 2\(^+\)-states. In Sec. 4, these results are discussed and analyzed in detail. A summary is given in Sec. 5. In Appendix A, the expression for the pairing matrix element is derived. In Appendix B, the basic SRPA equations are outlined. In Appendix C, a simple two-pole RPA model is presented for application for explanation of the domains with low and high collectivity of 2\(^+\)-states. In Appendix D, SRPA strength constants of the residual interaction are compared with those of the QPM.

II. MODEL AND CALCULATION SCHEME

The SRPA approach \cite{19} used in this paper is based on the Skyrme functional \cite{17}:

\[
\mathcal{E}(\rho, \tau, J, j, \sigma, T) = \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{Sk}} + \mathcal{E}_{\text{Coul}} + \mathcal{E}_{\text{pair}} \tag{1}
\]

where \(\mathcal{E}_{\text{kin}}\) is the kinetic energy, \(\mathcal{E}_{\text{Sk}}\) is the potential energy according to the Skyrme functional, \(\mathcal{E}_{\text{Coul}}\) is the Coulomb energy, and \(\mathcal{E}_{\text{pair}}\) is the pairing energy. The Coulomb exchange term is treated in Slater approximation. The volume pairing corresponds to a zero-range pairing interaction. The Skyrme part \(\mathcal{E}_{\text{Sk}}\) depends on the local densities and currents: density \(\rho(r)\), kinetic-energy density \(\tau(r)\), spin-orbit density \(J(r)\), current \(j(r)\), spin-density \(\sigma(r)\), and spin-kinetic-energy density \(T(r)\) \cite{17}. The mean-field Hamiltonian and SRPA residual interaction are self-consistently determined through the first and second functional derivatives of \(\mathcal{E}\), respectively \cite{19}. Further details of the model and calculation scheme are given below.

A. Mean field and quadrupole deformation

The stationary 2D mean-field calculations are performed with the SKYAX code \cite{30} in cylindrical coordinates using a mesh size of 0.5 fm and a calculation box of about three nuclear radii. The single-particle space is chosen to embrace the levels from the bottom of the potential well up to energy 15-20 MeV. For SV-mas10, the s-p schemes in \(^{150}\)Nd involve 304 proton and 375 neutron levels. In \(^{238}\)U, we deal with 379 proton and 485 neutron levels.

The ground state is obtained by solving the mean-field equations and resides at the minimum of the total energy \(\mathcal{E}\). Its axial quadrupole deformation is characterized by the dimensionless deformation parameter

\[
\beta_2 = \sqrt{\frac{\pi}{5}} \frac{Q_2}{Z \langle r^2 \rangle_p} \tag{2}
\]

where \(Q_2 = \int d\rho_p(r)r^2 Y_{20}\) is the quadrupole moment, \(\rho_p(r)\) is the proton density in the ground state, \(\langle r^2 \rangle_p = \int d\rho_p(r)r^2/Z\) is the r.m.s. proton radius.

Figure 1 shows deformation parameters calculated with SV-mas10 and compares them with available experimental data \cite{31}. Figure 2 shows the same for SkM\(^*\). We see very nice agreement for SV-mas10 while SkM\(^*\) systematically overestimates \(\beta_2\), especially in Yb, Hf, W, and U isotopes. Note that both SV-mas10 and SkM\(^*\) fail to describe specifically low values of experimental \(\beta_2\) in \(^{170}\)Yb and \(^{172,174}\)Hf.

B. Pairing and blocking effect

The volume pairing interaction reads

\[
V_{\text{pair}}(r, r') = V_q \delta(r - r'), \tag{3}
\]

where \(q\) stands for protons or neutrons and \(V_q\) are pairing strengths. The pairing is treated at the BCS level \cite{32}.

In the RPA description of the lowest vibrational states, the first (lowest energy) 2qp excitations play the crucial role. Their energies constitute poles of the SRPA equations. Usually the 2qp-energies are computed by using
2qp states is not ortho-normal any more. In the present treatment, we do not implement the blocking fully consistently. Instead, for a first estimate of the PBE, we simply use the 2qp energies from the blocking prescription and insert them into the standard SRPA treatment.

The PBE is particularly strong if the levels \( i \) and \( j \) lie near the Fermi level (or one of them coincides with it) because then their blocking can considerably decrease the density of pairing-active s-p levels in the Fermi region and thus essentially weaken the pairing \[7\]. Thus PBE should be taken into account at least for a few lowest 2qp states. The BCS-PBE formalism should be applied with care for a weak pairing because blocking may trigger its full break-down. In the worst cases, a more involved formalism (allowing the treatment of arbitrarily weak pairing) for calculation of the 2qp-energies should be used, e.g. the method with the projection to the particle number before variation \[32\]. The calculations with this method show BCS-PBE somewhat overestimates the actual blocking effect \[32\]. However, the projection method is involved and cannot be consistently applied in connection with the Skyrme energy functional \[34\]. So we use BCS-PBE, though with staying alert for suspect cases.

For the constant pairing gap, the BCS-PBE prescription was worked out previously \[7\]. Below we present the BCS-PBE formalism for the \( \delta \)-force volume pairing \[43\]. For each 2qp state \((ij)\in q\), one should solve the system of BCS+PBE equations

\[
\begin{align*}
\epsilon^2_{ij} &= \epsilon_i + \epsilon_j \quad (4) \\
\Delta^q_{ij} &= \sum_{k'=0, k' \in q} f_{k'} V^{(\text{pair})}_{kk'k} v_{k'}(ij) u_k(ij), \\
N_q &= 2+ \sum_{k' \neq i,j} f_{k'} \left( 1 - \frac{E_k - \lambda_q(ij)}{\sqrt{(E_k - \lambda_q(ij))^2 + \Delta^q_{ij}(ij)^2}} \right), \quad (9)
\end{align*}
\]

where \( u_k(ij), v_k(ij), \Delta^q_{ij}(ij), \lambda_q(ij) \) are Bogoliubov coefficients, pairing gaps and chemical potentials, calculated for the 2qp \((ij)\)-excitation. The sums in (4) and (9) include all s-p states \( k' \) with the momentum projection \( K' > 0 \) and isospin \( q \), but explicitly exclude \( k' = i \) and \( j \). Furthermore, \( N_p = Z \) and \( N_n = N \) are proton and neutron numbers. The smoothing energy-dependent cut-off weights \( f_{k'} \) are introduced to cure the well-known drawback of the zero-range pairing force to overestimate the coupling to the (continuum) high-energy states \[33\]. Expressions for weights \( f_{k'} \) and pairing matrix elements \( V^{(\text{pair})}_{kk'k} \) in axial nuclei are given in the Appendix A.
The PBE-corrected energy of the 2qp excitation reads
\[ E_{\text{bl}}(ij) = E^q(ij) - E_0^q \]
\[ E_{\text{bl}}(ij) \approx E_i + E_j + \sum_{K>0,k \in q} f_k [2E_k v_k(ij)]^2 - \Delta^q_k(ij) u_k(ij) v_k(ij), \]
is the energy of the system in the (ij)-state and
\[ E_0^q \approx 2 \sum_{k} f_k E_k v_k^2 - \sum_{k} f_k \Delta^q_k u_k v_k, \]
is the energy of the system in the ground state. The values \( u_k, v_k, \Delta^q_k, \lambda_q \) in (12) are those for the ground state. In both Eqs. (8), (9), and (11) we neglected small terms \( \propto v_k^2 \).

Eqs. (8), (9), and (11) show that PBE amounts to omit the states \( i \) and \( j \) from the pairing sums. These blocked states do not contribute to the pairing gap (8) and enter leads to a significant downshift of the energy of the first RPA solution.

In the present study, we block the five lowest \( K^\pi = 2^+ \) 2qp states (proton and neutron altogether). The calculations show that this number of blocked poles is optimal. More would involve poles with states too remote from the Fermi level and thus negligible PBE. Less is likely to miss some important PBE corrections.

Following [7], we expect that the major blocking effect comes from the shift of 2qp energies. Thus we use from BCS-PBE calculations only the corrected pole energies \( E_{\text{bl}}(ij) \) which are substituted to the SRPA equations instead of \( \epsilon_{ij} \). We do not utilize the modified Bogoliubov coefficients \( (u_k(ij), v_k(ij)) \), obtained for each \( (ij) \)-pair, but continue to use in RPA the g.s. sets \( (u_k, v_k) \). This leaves the 2qp basis ortho-normalized and so avoids problems with conservation of the proton and neutron numbers. Following our and previous [7] tests, the present prescription looks economical and, at the same time, quite accurate to take into account the major PBE impact in RPA calculations. A fully consistent RPA calculation with blocked pairs is still a demanding problem which goes beyond the present study.

C. SRPA scheme

The SRPA formalism for axial nuclei is described in detail elsewhere [19, 39]. Here we sketch only the points relevant for the present study. As mentioned above, the SRPA formalism starts from the functional [1]. The method is fully self-consistent. The residual interaction includes contributions from both time-even and time-odd densities and also takes care of the Coulomb interaction. The coupling between the quadrupole \( \lambda \mu = 22 \) and hexadecapole \( \lambda \mu = 42 \) modes, pertinent to deformed nuclei, is included. The basic SRPA equations and more calculation detail can be found in the Appendix B.

The structure and energies of the first RPA one-phonon 2+ states \( (\lambda \mu \nu = 221) \) are calculated in Nd, Sm, Gd, Dy, Er, Yb, Hf, W, and U isotopes. For each state, the reduced probability \( B(E2) = |\langle \nu = 1 | \sum_{k=1}^{\nu} Y^2_{22}(\theta_k)|0\rangle|^2 \) of the transition from the ground \( |0\rangle \) to the \( \gamma \)-vibrational state is computed.

The configuration space for \( \lambda \mu = 22 \) involves, depending on the isotope, 6600-9600 proton and 9400-14200 neutron 2qp-states with excitation energies up to 55-80 MeV. This basis is optimal for our aims. It results (together with the quadrupole components \( \lambda \mu = 20 \) and 21) in a reasonable exhaustion of the total energy-weighted sum rule \( \text{EWSR}(E2,T=0) = (R^2/\lambda^2)/(8\pi m_p)50A(r^2) \lambda \) by \( \sim 95 - 98\% \).

The calculations are performed for the Skyrme parametrizations SV-mas10 and SkM*. As mentioned in the Introduction, SV-mas10 is chosen because it provides good ground state deformations and ISGQR energy in connection with our 2D SRPA calculations. This is confirmed in Fig. 3, where the SRPA \( E_{2}(T=0) \) strength functions (defined in Appendix B) in nuclei \(^{164}\text{Dy}, ^{172}\text{Yb} \) and \(^{238}\text{U} \) are shown for SV-mas10 and SkM*. The results are compared with the empirical polynomial estimation for the ISGQR centroid [52]. It is seen that SV-mas10 well describes the resonance energy while SkM* systematically overestimates them. So SV-mas10 deliv-
ers a good performance in reproduction of both axial deformations (see the previous subsection) and ISGQR energies. This makes it a good candidate for the description of γ-vibrational states. For the comparison, we also use the force SkM∗ to establish a connection with the calculations of 2γ+ states in [3].

III. RESULTS AND DISCUSSION

A. Main results

Results of our calculations for the lowest 2qp states, RPA energies, and B(E2)-values of 2γ+ states are presented in Figs. 4-9. Cases without and with PBE are considered using for 2qp energies Eqs. (4) and (10), respectively. The results are compared with available experimental data [31].

Fig. 4 shows the results for Nd, Sm and Gd isotopes obtained with SV-mas10. Calculations without PBE (plots a-c) essentially overestimate the 2γ+-energies. The discrepancy decreases from Nd to Gd with the growth of the collective shift ΔE = E2qp − ESRPA (the difference between the 2qp and SRPA energies). Accounting for the PBE noticeably downshifts the poles and thus the RPA energies (plots d-f). The downshift reaches 0.1-0.6 MeV, depending on the isotope. As a result, the agreement with experimental energies improves, especially in heavy Gd isotopes. The trends of ESRPA with mass number A are approximately reproduced. The B(E2) values in Sm and Gd with and without blocking are about the same. In Nd isotopes, the calculated 2γ+ states demonstrate a weak collectivity, i.e. low B(E2) values. Here the PBE worsens the agreement. The SkM∗ results in Fig. 5 for the same isotopes provide a similar description.

Fig. 6 shows the results for Dy, Er, and Yb isotopes obtained with SV-mas10. The collectivity of calculated 2γ+ states reaches a maximum in Dy isotopes. Here we have the largest ΔE and B(E2). Furthermore, collectivity decreases in Er and almost vanishes in Yb. The PBE considerably decreases the 2qp pole and SRPA energies. In Dy isotopes, this leads to a nice agreement with the experimental energies and B(E2). In Er and Yb, the PBE noticeably improves the description of 2γ+ energies. However ESRPA still remain considerably higher than Eexp and calculated B(E2) are accordingly underestimated. The SkM∗ results for Dy-Er-Yb isotopes are given in Fig. 7. We again observe a decrease of collectivity of 2γ+ states from Dy to Yb isotopes. However, unlike the case of light rare-earth nuclei exhibited in Figs 4-5, the SkM∗ description considerably deviates here from the results of SV-mas10. First, as compared to SV-mas10 results and experimental data, the SkM∗ energies in Er and Yb isotopes strongly fluctuate with A, following variations of the poles (this feature of SkM∗ results was also mentioned in [3]). Such fluctuations point to a small collectivity of 2γ+ states, when the contribution of the first pole to the structure of 2γ+ state becomes significant. Furthermore, the 2qp energies are generally smaller for SkM∗ than for SV-mas10, which results in a better average description of Eexp in Er and Yb within SkM∗. The PBE gives here larger changes than for Nd-Sm-Gd isotopes. In particular, it leads to a huge decrease of 2γ+-energy in 164Dy (a similar effect was found in [3]). This state becomes extremely collective (see e.g. a huge overestimation of experimental B(E2)). It cannot be described with linear response methods like RPA and needs more involved prescriptions taking into account large-amplitude ground state correlations [10–32].

Figs. 8-9 show the results for heavy rare-earth Hf-W and actinide U isotopes. For both forces, the collectivity of 2γ+ states increases from Hf to W and decreases in U. Moreover, both forces give rather similar trends of ESRPA with A, though deviating from the experimental ones. The PBE considerably downshifts the 2qp and RPA energies and thus in general improves the description. In average, SkM∗ energies are closer to Eexp than SV-mas10 ones, but give more fuzzy A-dependence, especially with PBE. In U isotopes, the description of the spectra with SkM∗ is much better than with SV-mas10, which again is explained by lower 2qp energies in SkM∗. The description of B(E2) is acceptable in heavy Hf isotopes with both SV-mas10 and SkM∗. Most often the PBE does not change description of B(E2) and sometimes even worsens it.

Altogether, the results allow to do the following conclusions: i) In rare-earth and actinide regions, there are pronounced isotopic domains with low and high collectivity of 2γ+ states. ii) The best agreement with the experimental data is obtained for Dy (except for 164Dy) and W isotopes, i.e. for the most collective 2γ+ states characterized by large ΔE and B(E2) values. iii) The PBE essentially downshifts 2qp and RPA energies, thus leading to a better agreement with experiment. At the same time, the blocking usually slightly affects the collectivity of the states, in particular B(E2) values. The above conclusions are supported by both SV-mas10 and SkM∗. In general, these two forces give similar results in light rare-earth nuclei but deviate in heavier nuclei. In SV-mas10, the trends of ESRPA are less fuzzy but usually overestimate Eexp. In SkM∗, trends of ESRPA are more fuzzy, especially with PBE. However, this force gives lower 2qp and SRPA energies, providing a better description of Eexp, e.g. in U isotopes.

B. Discussion

In this subsection, we analyze the above results and compare them with earlier studies [3, 36, 37].

First of all, it is worth to explore the origin of domains with low and high collectivity of 2γ+ states. The low-collectivity domains include most of Nd, Er, Yb, Hf, and U isotopes. High collectivity exists in Sm, Gd, Dy, and W isotopes. Table 1 shows that the appearance of such domains is determined by the structure of the first 2qp
state which, in turn, results in different absolute values of its matrix element $f_{ij}^{22}$ for the doorway operator $r^2 Y_{22}$. These 2qp states are built from the levels in the Fermi region. High collectivity (pertinent to $^{154}$Sm, $^{162,164}$Dy, $^{176}$Hf, and $^{182}$W) takes place if the state is characterized by a large value of $|f_{ij}^{22}|$. If $|f_{ij}^{22}|$ is small, then we get non-collective $2^+_z$ states ($^{172}$Yb and $^{174}$Hf). The magnitude of $|f_{ij}^{22}|$ is determined by Nilsson selection rules for E2($K=2$) transitions in axial nuclei [2,32]. The rules read

$$\Delta K = 2, \quad \Delta N = 0, \pm 2, \quad \Delta n_z = 0, \quad \Delta \Lambda = 2, \quad (13)$$

where $N$ is the principle quantum shell number, $n_z$ is the...
fraction of $N$ along the $z$-axis, $\Lambda$ is the orbital momentum projection onto $z$-axis. All the 2qp states in Table I fulfill the rules \([13]\) for $K$ and $N$ but not for $n_z$ and $\Lambda$. Table I shows that the rule $\Delta n_z = 0$ is decisive. The 2qp states which keep this rule ($^{154}$Sm, $^{162,164}$Dy, $^{176}$Hf, $^{182}$W) exhibit $|f^{22}_{ij}|$-values of one order of magnitude larger than states violating it ($^{172}$Yb and $^{174}$Hf). This effect is especially spectacular for neighboring isotopes $^{174}$Hf - $^{176}$Hf. The rule $\Delta \Lambda = 2$ is not so crucial. However, matrix elements increase further if it is obeyed ($^{176}$Hf, $^{182}$W).
Table I obviously suggests that just the strength $|f_{12}^{22}|$ of the first $2\nu$ state is decisive for the collectivity of the RPA $2^+_1$ state and formation of the domains with low and high collectivity. This finding can be understood within a simple two-pole model given in Appendix C. Following this model, the collectivity of the lowest RPA states is mainly determined by the ratio between the strengths of the first $\nu=1$ and second $\nu=2\nu$ poles where the second pole simulates a cumulative effect of all $2\nu$ states with $\nu>1$. Depending on this ratio, different scenarios can take place: high-collective limit, intermediate case and low-collective limit. In the last case, the first RPA energy can lie even a bit above the first $2\nu$ pole, which happens, e.g., in our calculations for Yb isotopes. Altogether, we
TABLE I: Features of the first $2^+_1$ (ij)-poles and $\lambda\mu\nu = 221$ RPA states in rare-earth nuclei, calculated with SV-mas10 and SkM* forces. The table includes: the pole notation $qq[NN_{\nu j}][NN_{\nu j}]_i$ in Nilsson quantum numbers; location of s-p levels $i$ and $j$ relative to the Fermi (F) level; the quadrupole 2qp matrix element $f_{ij}^{2^2} = (ij|^{2^2} Y_{22}|0)$; the 2qp energy $\epsilon_{ij}$ and collective shift $\Delta E_{ij} = \epsilon_{ij} - E_{221}$, calculated without the blocking; the 2qp energy $\epsilon_{bl}(ij)$ and collective shift $\Delta E_{bl} = \epsilon_{bl}(ij) - E_{221}$, calculated with the blocking; the blocking correction $\Delta \epsilon_{bl} = \epsilon_{bl} - \epsilon_{ij}$. See text for more details.

| Nucleus    | Force   | $qq[NN_{\nu j}][NN_{\nu j}]_i$ | F-location | $f_{ij}^{2^2}$ [fm$^4$] | $\epsilon_{ij}$ [MeV] | $\Delta E_{ij}$ [MeV] | $\epsilon_{bl}(ij)$ [MeV] | $\Delta E_{bl}$ [MeV] | $\Delta \epsilon_{bl}$ [MeV] |
|------------|---------|---------------------------------|------------|-------------------------|-----------------------|------------------------|--------------------------|------------------------|--------------------------|
| $^{154}_{62}$Sm$^{92}$ | SV-mas10 | pp[411][411]↑ | F+3, F | 5.05 | 2.45 | 0.41 | 2.28 | 0.38 | 0.17 |
|            | SkM*    | pp[411][411]↑ | F+3, F-1 | 4.98 | 2.34 | 0.34 | 2.37 | 0.31 | 0.07 |
| $^{162}_{66}$Dy$^{96}$ | SV-mas10 | pp[411][411]↑ | F+1, F | 6.47 | 1.92 | 0.57 | 1.40 | 0.56 | 0.51 |
|            | SkM*    | pp[413][411]↓ | F, F-1 | -5.78 | 1.71 | 0.87 | 1.37 | 0.88 | 0.33 |
| $^{164}_{66}$Dy$^{98}$ | SV-mas10 | pp[411][411]↑ | F+1, F | 6.49 | 1.86 | 0.52 | 1.35 | 0.53 | 0.50 |
|            | SkM*    | nn[523][521]↓ | F, F-1 | 5.98 | 1.42 | 0.56 | 0.86 | 0.86 | 0.56 |
| $^{172}_{70}$Yb$^{102}$ | SV-mas10 | pp[402][411]↑ | F+3, F-1 | 0.22 | 2.20 | -0.03 | 1.98 | 0.004 | 0.23 |
|            | SkM*    | nn[512][521]↓ | F+1, F-1 | 0.086 | 1.63 | 0.06 | 1.30 | 0.06 | 0.33 |
| $^{174}_{72}$Hf$^{102}$ | SV-mas10 | nn[512][521]↓ | F+1, F-1 | 0.18 | 2.35 | 0.09 | 1.99 | -0.02 | 0.36 |
|            | SkM*    | nn[512][521]↓ | F+1, F-1 | 0.19 | 1.58 | 0.06 | 1.26 | 0.06 | 0.33 |
| $^{176}_{72}$Hf$^{104}$ | SV-mas10 | nn[512][510]↑ | F, F-2 | -8.17 | 2.48 | 0.47 | 2.14 | 0.34 | 0.33 |
|            | SkM*    | nn[512][510]↑ | F, F-2 | 8.61 | 2.07 | 0.40 | 1.92 | 0.38 | 0.15 |
| $^{184}_{72}$W$^{108}$ | SV-mas10 | nn[510][512]↓ | F+1, F+2 | 8.59 | 2.16 | 0.71 | 1.77 | 0.61 | 0.39 |
|            | SkM*    | nn[510][512]↓ | F+1, F+2 | 7.98 | 1.54 | 0.60 | 1.34 | 0.67 | 0.21 |

have a simple way for prediction of the collectivity (weak or large) of the first RPA state: it suffices to check for the Nilsson selection rules (13) for the lowest 2qp state, first of all the $\Delta N_z = 0$.

Following our analysis, SV-mas10 or SkM* can give other lowest 2qp states, as compared with those calculated with the Woods-Saxon potential [36]. Nonetheless, the correlation between $\Delta N_z = 0$ rule and collectivity of RPA $2^+_1$-states applies in any case.

In many respects, the forces SV-mas10 and SkM* provide similar descriptions. At least they predict the same domains of the low and high collectivity, determined by the $\Delta N_z = 0$ condition. In Table I, the matrix elements $|f_{ij}^{2^2}|$ for both forces are very similar though they may have different sign. At the same time, some results demonstrate a noticeable difference between SV-mas10 and SkM* which is, most probably caused by deviations in their s-p spectra near the Fermi surface. In some nuclei, these forces give different lowest 2qp states, see $^{162,164}$Dy and $^{172}$Yb in Table I. As mentioned above, SkM* has generally lower energies of the 2qp states than SV-mas10. The difference between $\epsilon_{ij}$ from SV-mas10 and SkM* increases with mass number $A$ and reaches 0.5 MeV in heavy rare-earth nuclei, see Table I. This seems to be the main agent for different performance of SV-mas10 and SkM* in description of $2^+_1$-states.

The nucleus $^{164}$Dy computed with SkM* shows a remarkable sequence of four strong ($|f_{ij}^{2^2}| = 5.4-6.5$ fm$^4$) 2qp states which are located with PBE at 0.86–1.96 MeV. The cumulative impact of these states delivers a dramatic effect: a break-down of RPA. Without PBE, these four poles lie at a higher energy 1.42–2.15 MeV and do not lead to such break-down. For comparison, SV-mas10 gives in $^{164}$Dy only three strong ($|f_{ij}^{2^2}| = 5.4-6.5$ fm$^4$) 2qp states and they are located at a higher energy 1.35–1.65 MeV. This gives a collective $2^+_1$-state still within RPA. Altogether, this discussion shows that RPA results for low lying states can be quite sensitive to the Skyrme force.

Table I shows that the collective shifts $\Delta E$ in most collective states reach 0.6-0.9 MeV. The PBE does not affect much $\Delta E$ and collectivity in general. However the PBE can significantly downshift the first 2qp state (see e.g. $\Delta \epsilon_{bl}$ in $^{162,164}$Dy) and thus the energy of $2^+_1$ state. This effect is indeed essential and should be taken into account in RPA calculations.

The results presented in Figs. 4–9 indicate that the present Skyrme RPA description is far from being perfect. Though we get rather good agreement with experimental data for collective $2^+_1$ states in Gd, Dy, and W isotopes, the $2^+_1$-collectivity is generally underestimated for other isotopic chains yielding often too high RPA energies and too low $B(E2)$-values. These cases require most probably a coupling to complex configurations (CCC) beyond RPA, which should additionally downshift the $2^+_1$-energies and affect $B(E2)$-values. In this respect, our RPA calculations indicate regions where a CCC is expected and should be taken into account in RPA calculations.
being in average $\sim$ 1 MeV, exceeds our $\Delta E$ values. The reason of this discrepancy is yet unclear. To clarify this point, a more involved exploration embracing BCS-PBE, full RPA and CCC is desirable. The dependence of the results on the size of the configuration space, including very large sizes, should be also scrutinized. In nuclei like $^{164}$Dy, the approach taking into account large ground state correlations [40, 41] is necessary.

Since SRPA represents the (self-consistent) residual interaction in separable form, it can be directly compared with (non-consistent) schematic RPA approaches, e.g. with QPM which is widely and successfully used in nuclear spectroscopy [42]. The QPM proposes some simple relations for the strength constants of the residual interaction which might be useful for a rough evaluation of the self-consistent SRPA constants. This analysis is done in the Appendix C. It is shown that the mixed isoscalar-isovector interaction might be essential in Skyrme RPA. If this interaction is not properly balanced, it can weaken a general isoscalar effect of the residual interaction and thus make $2^+_\gamma$ states less collective (which might be relevant for Nd, Yb, Hf, U isotopes).

IV. SUMMARY

We have performed a systematic study of the lowest $\gamma$-vibrational states in rare-earth and actinide axially deformed nuclei within a fully self-consistent separable random-phase-approximation (SRPA) [19]. Nine isotopic chains involving 41 nuclei were explored. The excitation energies and B(E2)-values of $2^+_\gamma$ states were computed and analyzed. The Skyrme forces SV-mas10 [10] and SkM* [10] were used. The force SV-mas10 was chosen since it provides good values for ground state deformations and a reasonable description of the isoscalar giant quadrupole resonance (ISGQR) in the SRPA context. SkM* was used as a force with the best performance in the previous study of $2^+_\gamma$ states [10] performed with a full (not factorized) Skyrme RPA.

In general, the quality of obtained results is similar to that in the previous full RPA calculations [10]. The description looks reasonable though yet far from being perfect. The main drawback is the lack of collectivity of $2^+_\gamma$ states in Nd, Er, Yb, Hf and U isotopes. In this sense, the problem of the Skyrme RPA description of $2^+_\gamma$ still leaves open questions. At the same time, as outlined below, our study has taken some important steps which were not realized earlier [10]. Besides, some points of the physical interpretation of the problem were clarified.

The most important novel aspect of our study consists in taking into account the pairing blocking effect (PBE). The blocking was applied to a few lowest two-quasiparticle (2qp) states whose corrected energies were then used in SRPA calculations. The PBE significantly downshifts the RPA energies of $2^+_\gamma$ states and thus in general improves the agreement with experiment. At the same time, PBE usually does not influence much collectivity of the states, characterized by collective shifts and transition probabilities $B(E2)$. Note that importance of PBE was also indicated in the Skyrme-BCS calculations for odd nuclei [43].

Next, the domains of nuclei with low and high collectivity of $2^+_\gamma$ states were figured out. It was shown that the collectivity is mostly determined by the structure of the lowest 2qp state constituting the first SRPA pole. The effect was explained in terms of the Nilsson selection rule $\Delta n_z = 0$ which delivers a simple recipe to predict the $2^+_\gamma$-collectivity without performing RPA calculations. Some results and SRPA characteristics were compared with those from the schematic Quasiparticle-Phonon Model (QPM) [7] which was successfully used for a long time in nuclear spectroscopy.

Note that, unlike the SkM*, the force SV-mas10 combines the description of $2^+_\gamma$ states with the simultaneous reproduction of quadrupole equilibrium deformations and ISGQR energy (while Skyrme forces in the study [5] have too low effective mass to describe the ISGQR).

Both forces SV-mas10 and SkM* perform similar in the description of $2^+_\gamma$ for light rare-earth nuclei but start to deviate in heavier nuclei. The latter is mainly explained by the fact that SkM* produces lower 2qp energies than SV-mas10. SV-mas10 delivers less fuzzy trends of energies and B(E2) values well describes Dy isotopes but fails in U isotopes. SkM* is better in U isotopes but its results fluctuate more with the mass number.

In accordance to [5], the present calculations show that RPA is probably not sufficient for a detailed description of $2^+_\gamma$ states and thus the coupling to complex configurations (CCC) is desirable. This calls for a more involved description embracing the PBE and CCC. The question of a proper choice of the Skyrme parametrization for low-energy states is also to be checked. The stability of the results with using larger configuration spaces should be further analyzed. Some of these points will be a subject of our next studies.

Acknowledgments

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Appendix A: Pairing cut-off weight and pairing matrix elements

To simulate the effect of a finite range pairing force, the pairing-active space is limited by using a smooth energy-
dependent cut-off \[ f_k = \frac{1}{1 + \exp\left(\frac{E_k - \lambda_0 - \Delta E_0}{\eta_q}\right)} \quad (A1) \]

in the sums in Eqs. (8), (9), (11), and (12). The cut-off parameters \( \Delta E_q \) and \( \eta_q = \Delta E_q/10 \) are chosen self-adjusting to the actual level density in the vicinity of the Fermi energy [33].

For the \( \delta \)-force pairing interaction [33], the antisymmetrized pairing matrix elements read

\[
V_{ij}^{(\text{pair})} = \langle ij | V_{\text{pair}}(r,r') | jj \rangle
\]

\[
\begin{align*}
&= \int d^3r \int d^3r' \Phi_+^i(r) \Phi_+^j(r') \left( V_0^{(\tau)} \delta(r-r') ight) \\
&\quad \cdot \left( \Phi_+^i(r) \Phi_+^j(r') - \Phi_-^i(r) \Phi_-^j(r') \right) \\
&= V_0^{(\tau)} \int d^3r \left( \Phi_+^i(r) \cdot \Phi_+^j(r) \right) \\
&\quad - \left( \Phi_+^i(r) \cdot \Phi_-^j(r) \right) \left( \Phi_+^j(r) \cdot \Phi_-^i(r) \right)
\end{align*}
\]

where

\[
\Phi_i(r) = \left( \begin{array}{c}
R_i^+(\rho, z) e^{i(K_i - \frac{\pi}{4})} \\
R_i^-(\rho, z) e^{i(K_i + \frac{\pi}{4})}
\end{array} \right)
\]

\[
\Phi_i(r) = \left( \begin{array}{c}
-R_i^+(\rho, z) e^{-i(K_i + \frac{\pi}{4})} \\
-R_i^-(\rho, z) e^{-i(K_i - \frac{\pi}{4})}
\end{array} \right)
\]

are spinor s-p. wave functions in cylindrical coordinates \((\rho, z, \vartheta)\) and \(\langle \Phi_+^i(r) \cdot \Phi_+(r) \rangle\) are scalar products. Denoting the first (Hartree) and second (exchange) terms in the last line of (A2) as \(V_{ij}^{(\text{pair})-\text{H}}\) and \(V_{ij}^{(\text{pair})-\text{ex}}\), we obtain

\[
\begin{align*}
V_{ij}^{(\text{pair})-\text{H}} &= 2\pi V_0^{(\tau)} \int_0^\infty d\rho \int_0^{\infty} dz \rho \\
&\quad \times \left[ 2R_i^+(r) R_i^-(r) R_j^+(r) R_j^-(r) + (R_i^-(r) R_j^+(r))^2 + (R_i^+(r) R_j^-(r))^2 \right] \\
V_{ij}^{(\text{pair})-\text{ex}} &= 2\pi V_0^{(\tau)} \int_0^\infty d\rho \int_0^{\infty} dz \rho \\
&\quad \times \left[ -2R_i^+(r) R_i^-(r) R_j^+(r) R_j^-(r) + (R_i^-(r) R_j^+(r))^2 + (R_i^+(r) R_j^-(r))^2 \right]
\end{align*}
\]

and finally

\[
V_{ij}^{(\text{pair})} = V_{ij}^{(\text{pair})-\text{H}} + V_{ij}^{(\text{pair})-\text{ex}}
\]

The self-consistent derivation [19, 39] yields the SRPA Hamiltonian

\[
\hat{H} = \sum_q \hat{h}_{\text{HF+BCS}}^q + \hat{V}_{\text{res}}
\]

where

\[
\hat{h}_{\text{HF+BCS}}^q = \int dr \sum_{\alpha,\alpha'} \left[ \frac{\delta E}{\delta J_{\alpha}^q(r)} \hat{J}_{\alpha}^q(r) \right]
\]

is the mean field and pairing contribution and

\[
\hat{V}_{\text{res}} = \frac{1}{2} \sum_{qq'} \sum_{m,m'=1}^M \left[ \kappa_{qq',mm'} \hat{X}_{mm'} \hat{X}_{mm'}^\dagger \right]
\]

\[
+ \eta_{qq',mm'} \hat{Y}_{mm'} \hat{Y}_{mm'}^\dagger
\]

is the separable residual interaction with one-body operators

\[
\hat{X}_{mm'} = \sum_{qq'} \hat{X}_{mm'}^q = i \sum_{q' \alpha, \alpha'} \int dr \left[ \delta^2 E \right] \left[ \delta J_{\alpha}^q(r) \delta J_{\alpha'}^{q'}(r') \right] \langle \hat{P}_{mm'}, \hat{J}_{\alpha}^q(r) \rangle \langle \hat{J}_{\alpha'}^{q'}(r') \rangle
\]

\[
\hat{Y}_{mm'} = \sum_{qq'} \hat{Y}_{mm'}^q = i \sum_{q' \alpha, \alpha'} \int dr \left[ \delta^2 E \right] \left[ \delta J_{\alpha}^q(r) \delta J_{\alpha'}^{q'}(r) \right] \langle \hat{Q}_{mm'}, \hat{J}_{\alpha}^q(r) \rangle \langle \hat{J}_{\alpha'}^{q'}(r') \rangle
\]

and inverse strength matrices

\[
\kappa_{qq',mm'}^{-1} = -i \langle \hat{P}_{mm'}, \hat{X}_{mm'} \rangle \]

\[
\eta_{qq',mm'}^{-1} = -i \langle \hat{Q}_{mm'}, \hat{Y}_{mm'} \rangle
\]

Here \(\alpha = \rho, \tau, J, H, \chi, j, s, t, T\) enumerates densities \(J_{\alpha}^q\) and their operators \(\hat{J}_{\alpha}^q\) while \(m\) marks time-even \(\hat{Q}_{mm}\) and time-odd \(\hat{P}_{mm}\) Hermitian input (doorway) operators. The number \(M\) of separable terms in (B3) is determined by the number of the input operators \(Q_{mm}\) chosen from physical arguments [13]. Usually we have \(M = 3-5\).

The operators \(\hat{Q}_{mm}\) constitute the key input for SRPA [13]. They are chosen from physical arguments, namely to produce doorway states for particular excitations. In present calculations, four operators are used. The first one, \(Q_{11}(r) = r^2 Y_{22}(\theta) + \text{h.c.}\), generates the quadrupole

\[
\kappa_{qq',mm'} = -i \langle \hat{P}_{mm'}, \hat{X}_{mm'} \rangle
\]

\[
\eta_{qq',mm'} = -i \langle \hat{Q}_{mm'}, \hat{Y}_{mm'} \rangle
\]
mode of interest in the long-wave approximation ($Y_{22}(\theta)$ is the spherical harmonic). Usually, already one such operator (generator) is enough for a rough description of the spectrum. However the corresponding Tassie mode is mainly of the surface character. So, to improve accuracy of the description, two other generators, $Q_{B3}(r) = r^2 Y_{22}(\theta) + \text{h.c.}$ and $Q_{B4}(r) = j_2(0.6r) Y_{22}(\theta) + \text{h.c.}$ (with $j_2(0.6r)$ being the spherical Bessel function), are added. These generators result in $\hat{X}_{B3}(r)$ operators peaked more in the nuclear interior \[19\]. Finally, the generator $Q_{B4}(r) = r^2 Y_{22}(\theta) + \text{h.c.}$ is added to take into account the coupling between quadrupole and hexadecapole excitations in axially deformed nuclei. Note that these input operators do not form directly the separable residual interaction \[\text{B3}\] but generate its operators $\hat{X}_{\text{B3}}(r)$, $\hat{Y}_{\text{B3}}(r)$ and strength constants $\kappa_{\text{B3},q,m'}$, $\eta_{\text{B3},q,m'}$, based on the initial Skyrme functional. The number $M$ of input operators determines the number of the separable terms in \[\text{B3}\]. Larger $M$ brings the separable interaction closer to the true (not factorized) one, but makes SRPA calculations more time consuming. The four operators which we are using here constitute a good compromise between reliability and expense.

SRPA allows to calculate the energies $\omega_\nu$ and wave function (with forward $\psi^\nu_{i,j}$ and backward $\phi^\nu_{i,j}$ 2qp amplitudes) of one-phonon $\nu$-states. Besides, various strength functions can be directly computed (without calculation of $\nu$-states). In this study, we use for description of ISGQR the strength function

$$S_\gamma(E_{22}, E) = \sum_\nu | \langle \nu | r^2 Y_{22} | 0 \rangle |^2 \xi_\Delta(E - E_\nu) \quad \text{(B8)}$$

where $\xi_\Delta(E - E_\nu) = \Delta/[2\pi((E - E_\nu)^2 + (\Delta/2)^2)]$ is the Lorentz weight with the averaging parameter $\Delta = 1 \text{ MeV}$.

**Appendix C: Simple two-pole RPA model**

Let’s consider SRPA with one input (doorway) operator and without time-odd contributions. Then the SRPA secular equation is reduced to the familiar equation for the schematic separable RPA \[\text{B3}\]:

$$\kappa^{-1} = \sum_{ij} \frac{f^2_{ij}}{e^2_{ij} - E^2_\nu} \quad \text{(C1)}$$

where $\kappa$ is the strength constant, $f_{ij}$ is the matrix element of the residual interaction (including the pairing factors) between the states $i$ and $j$, $e_{ij}$ is the 2qp energy, and $E_\nu$ is the energy of the $\nu$-th RPA states. This equation may be simplified to the case of two 2qp states, yielding two poles in the schematic RPA equation:

$$1 = \kappa f^2 \left[ \frac{k^2}{e^2_1 - E^2} + \frac{1}{e^2_2 - E^2} \right]. \quad \text{(C2)}$$

Here the first pole is characterized by the 2qp energy $e_1$ and matrix element $kf$. The second pole (with the 2qp energy $e_2 > e_1$ and matrix element $f$) is assumed to simulate the effect of all the poles above the lowest one. The coefficient $k$ determines the ratio between the matrix elements of the first and second poles. We suppose $\kappa > 0$, which is common for low-energy isoscalar excitations \[7\].

Equation \text{(C2)} is reduced to a standard quadratic equation

$$E^4 + bE^2 + c = 0 \quad \text{(C3)}$$

with

$$b = -(e_1^2 + e_2^2 + \kappa f^2(1 + k^2)), \quad \text{(C4)}$$

$$c = e_1^2 e_2^2 - \kappa f^2(e_1^2 + k^2 e_2^2). \quad \text{(C5)}$$

This equation allows to get useful analytical estimations for three important cases: i) $k \gg 1$ (strong first pole, typical for Gd, Dy, and W isotopes), ii) $k \ll 1$ (weak first pole, typical for Nd, Yb, Hf, and U isotopes), iii) $k = 1$ (intermediate case with equal strengths of the first and second poles).

We go through these three cases step by step:

i) For the strong first pole ($k \gg 1$), we get $(1 \pm k^2) \rightarrow \pm k^2$ and so

$$E^2 \approx \frac{1}{2}[e_1^2 + e_2^2 - \kappa(fk)^2] \quad \text{(C6)}$$

with two solutions

$$E^2_+ \approx e_1^2 - \kappa(fk)^2, \quad E^2_- \approx e_2^2 - \kappa f^2. \quad \text{(C7)}$$

The solution $E_+$ gives the energy of the 1st RPA state below the first pole which is a common case in phenomenological QPM \[8, 36, 37\]. In our calculations, this case is met in Gd, Dy, and W isotopes.

ii) For the weak first pole ($k \ll 1$), we get $(1 \pm k^2) \rightarrow 1$ and so

$$E^2 \approx \frac{1}{2}[e_1^2 + e_2^2 - \kappa f^2] \quad \text{(C8)}$$

$$E^2_+ \approx e_1^2, \quad E^2_- \approx e_2^2 - \kappa f^2. \quad \text{(C9)}$$

The solution $E_+$ is the energy of the 1st RPA state close to the first pole. This energy can be both a bit smaller or larger than $e_1$. We have this case for Nd, Yb and Hf isotopes.

iii) If the pole strengths are equal ($k = 1$), then $(1 - k^2) \rightarrow 0$, $(1 + k^2) \rightarrow 2$ and

$$E^2 \approx \frac{1}{2}[e_1^2 + e_2^2 - 2\kappa f^2 \pm (e_1^2 - e_2^2 + 4\kappa^2 f^2)^{1/2}] \quad \text{(C10)}$$

Supposing that $(e_1^2 - e_2^2)^2 \gg 4\kappa^2 f^4$, we get

$$E^2 \approx \frac{1}{2}[e_1^2 + e_2^2 - 2\kappa f^2 \pm (e_1^2 - e_2^2 + \kappa f^2)], \quad \text{(C11)}$$

$$E^2_+ \approx e_1^2 - \frac{1}{2}\kappa f^2, \quad E^2_- \approx e_2^2 - \frac{3}{2}\kappa f^2. \quad \text{(C12)}$$
This simple model indicates that collectivity (collective shift $\Delta E = E_+ - e_1$) of the first RPA state is determined to a large extent by the relative strength of the first pole. This conclusion is confirmed by our numerical results, see discussion of Table I. Thus we have found a simple way for the prediction of the collectivity (weak or large) of the first RPA state. In practice, it is enough to compare the matrix elements of the first and next poles. Or, which is even easier, one should check if the first pole fulfills the $\Delta n_z = 0$ Nilsson selection rule.

Appendix D: Comparison with QPM

Since SRPA deals with a (self-consistent) separable residual interaction, this method can be directly compared with the schematic separable RPA exploited in QPM \cite{7}. The QPM is not self-consistent: it uses the Woods-Saxon s-p basis and its isoscalar $\kappa_{00}$ and isovector $\kappa_{11}$ strength constants of the residual interaction are adjusted to reproduce the experimental energies of lowest vibrational states and giant resonances. However, just because of the successful combination of the microscopic and phenomenological aspects, the QPM is known to be quite accurate in description of low-energy states. Thus it is instructive to compare the characteristics of self-consistent models, like Skyrme RPA, with the relevant QPM parameters.

In this connection, let’s briefly discuss the QPM strength constants of the residual interaction and compare them with the SRPA ones. The strength constants in the proton-neutron domain (nn, pp, np) can be related to their counterparts in the isoscalar-isovector domain (00,11, 01) as

$$\kappa_{00} = \frac{1}{2} (\kappa_{pp} + \kappa_{pn} + \kappa_{np} + \kappa_{nn}),$$

$$\kappa_{11} = \frac{1}{2} (\kappa_{pp} - \kappa_{pn} - \kappa_{np} + \kappa_{nn}),$$

$$\kappa_{10} = \frac{1}{2} (\kappa_{pp} - \kappa_{pn} + \kappa_{np} - \kappa_{nn}) = \kappa_{10}. \quad (D3)$$

The constants $\kappa_{01} = \kappa_{10}$ represent the mixing between isoscalar (00) and isovector (11) excitations. This mixing can be motivated by both physical (Coulomb interaction, etc) and technical (different sizes of neutron and proton s-p basis, etc) reasons. Since nuclei roughly keep the isospin symmetry, then

$$|\kappa_{00}|, |\kappa_{11}| \gg |\kappa_{01} = \kappa_{10}|. \quad (D4)$$

If to assume $\kappa_{01} = \kappa_{10} = 0$ and $\kappa_{np} = \kappa_{pn}$, then we get

$$\kappa_{pp} = \kappa_{nn} \quad (D5)$$

and the familiar QPM relations \cite{7}

$$\kappa_{00} = \kappa_{pp} + \kappa_{pn}, \quad \kappa_{11} = \kappa_{pp} - \kappa_{pn}. \quad (D6)$$

From (D6) one gets

$$\kappa_{pp} = \kappa_{nn} = \frac{1}{2} (\kappa_{00} + \kappa_{11}), \quad (D7)$$

$$\kappa_{pn} = \kappa_{np} = \frac{1}{2} (\kappa_{00} - \kappa_{11}) \quad (D8)$$

where $\kappa_{11} = \alpha \kappa_{00} = 0$. Usually $\alpha = -1.5$ is used \cite{7}, which results in a dominance of the np-interaction, $\kappa_{pn}/\kappa_{pp} = -2.5$ with $\kappa_{pp} = \kappa_{np} > 0$ and $\kappa_{np} = \kappa_{nn} < 0$.

For the comparison, the self-consistent SRPA calculations give somewhat different picture. As a relevant example, the strength constants $\kappa_{1l,q} = \kappa_{qq'}$ for the dominant first input operator $r^2Y_{22}$ in $^{162}$Dy are considered. Note that in SRPA the relation $\kappa_{pp} = \kappa_{np}$ is kept. SV-mas10 gives strength constants $\kappa_{pp}, \kappa_{nn} > 0$ with the relations $\kappa_{pp}/\kappa_{nn} = 2.7$, $\kappa_{pn}/\kappa_{pp} = 7.7$ and $\kappa_{np}/\kappa_{nn} = 2.9$. Similar results are obtained in other nuclei. SkM* gives $\kappa_{nn}, \kappa_{np} > 0$, $\kappa_{pp} < 0$ and relations $\kappa_{pp}/\kappa_{nn} = -2.0$, $\kappa_{pn}/\kappa_{pp} = -4.4$, and $\kappa_{np}/\kappa_{nn} = 2.2$. In agreement with QPM, both forces provide a dominant np-interaction with the proper sign. However, in contrast to (D6), the weak SRPA constants $\kappa_{pp}$ and $\kappa_{nn}$ noticeably deviate from each other, which might be a signature of an large mixing of the isoscalar and isovector interaction. Perhaps just this mixing, if not be properly balanced with other parts of the interaction, partly leads to the troubles of Skyrme RPA with the description of $2^+_1$ states. A difference in sign of SV-mas10 and SkM* constants $\kappa_{pp}$ should be also mentioned as demonstration of the noticeable dependence of the residual interaction on the Skyrme force.

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