ON CARLITZ’S TYPE $q$-EULER NUMBERS ASSOCIATED WITH
THE FERMIONIC $p$-ADIC INTEGRAL ON $\mathbb{Z}_p$

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ABSTRACT. In this paper, we consider the following problem in [20]: “Find Witt’s formula for Carlitz’s type $q$-Euler numbers.” We give Witt’s formula for Carlitz’s type $q$-Euler numbers, which is an answer to the above problem. Moreover, we obtain a new $p$-adic $q$-l-function $l_{p,q}(s, \chi)$ for Dirichlet’s character $\chi$, with the property that

$$l_{p,q}(-n, \chi) = E_n,\chi_n,q \chi_n(p) [p]_q^n E_n,\chi_n,q^n, \ n = 0, 1, \ldots$$

using the fermionic $p$-adic integral on $\mathbb{Z}_p$.

1. Introduction

Throughout this paper, let $p$ be an odd prime number. The symbol $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ denote the rings of $p$-adic integers, the field of $p$-adic numbers and the field of $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p = p^{-1}$. Let $\mathbb{N}$ be the natural numbers and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$.

As the definition of $q$-number, we use the following notations:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Note that $\lim_{q \to 1} [x]_q = x$ for $x \in \mathbb{Z}_p$, where $q$ tends to 1 in the region $0 < |q-1|_p < 1$.

When one talks of $q$-analogue, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q = 1 + t \in \mathbb{C}_p$, one normally assumes $|t|_p < 1$. We shall further suppose that $\text{ord}_p(t) > 1/(p-1)$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$.

After Carlitz [3, 4] gave $q$-extensions of the classical Bernoulli numbers and polynomials, the $q$-extensions of Bernoulli and Euler numbers and polynomials have been studied by several authors (cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]). The Euler numbers and polynomials have been studied by researchers in the field of number theory, mathematical physics and so on (cf. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23]). Recently, various $q$-extensions of these numbers and polynomials have been studied by many mathematicians (cf. [3, 4, 10, 11, 12, 14, 19, 20, 22]). Also, some authors have studied in the several area of $q$-theory (cf. [1, 2, 6, 18, 21]).
It is known that the generating function of Euler numbers $F(t)$ is given by
\begin{equation}
F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.
\end{equation}
From (1.1), we known that the recurrence formula of Euler numbers is given by
\begin{equation}
E_0 = 1, \quad (E + 1)^n + E_n = 0 \text{ if } n > 0
\end{equation}
with the usual convention of replacing $E^n$ by $E_n$ (see [9, 20]).

In [19], the $q$-extension of Euler numbers $E_{n,q}$ are defined as
\begin{equation}
E_{0,q} = 1, \quad (qE + 1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\end{equation}
with the usual convention of replacing $(E^*)^n$ by $E_{n,q}$.

As the same motivation of the construction in [20], Carlitz’s type $q$-Euler numbers $E_{n,q}$ are defined as
\begin{equation}
E_{0,q} = 2 \frac{[2]_q}{[2]_q}, \quad q(qE + 1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\end{equation}
with the usual convention of replacing $E^n$ by $E_{n,q}$. It was shown that $\lim_{q \to 1} E_{n,q} = E_n$, where $E_n$ is the $n$th Euler number. In the complex case, the generating function of Carlitz’s type $q$-Euler numbers $F_q(t)$ is given by
\begin{equation}
F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} (-q)^n E_{n,q} t^n,
\end{equation}
where $q$ is complex number with $|q| < 1$ (see [20]). The remark point is that the series on the right-hand side of (1.5) is uniformly convergent in the wider sense.

In this paper, we obtain the generating function of Carlitz’s type $q$-Euler numbers in the $p$-adic case. Also, we give Witt’s formula for Carlitz’s type $q$-Euler numbers, which is a partial answer to the problem in [20]. Moreover, we obtain a new $p$-adic $q$-l-function $I_{p,q}(s, \chi)$ for Dirichlet’s character $\chi$, with the property that
\begin{equation}
l_{p,q}(-n, \chi) = E_{n,\chi_{-n,q}} - \chi_n(p) \mid_p^n \mu_q \chi_{n,q} n
\end{equation}
for $n \in \mathbb{Z}^+$ using the fermionic $p$-adic integral on $\mathbb{Z}_p$.

2. CARLITZ’S TYPE $q$-EULER NUMBERS IN THE $p$-ADIC CASE

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. Then the $p$-adic $q$-integral of a function $f \in UD(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by
\begin{equation}
I_q(f) = \int_{\mathbb{Z}_p} f(a) d\mu_q(a) = \lim_{N \to \infty} \frac{1}{p^{N+1}_q} \sum_{a=0}^{p^N - 1} f(a) q^a
\end{equation}
(cf. [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22]). The bosonic $p$-adic integral on $\mathbb{Z}_p$ is considered as the limit $q \to 1$, i.e.,
\begin{equation}
I_1(f) = \int_{\mathbb{Z}_p} f(a) d\mu_1(a).
\end{equation}
From (2.1), we have the fermionic $p$-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q \to -1} I_q(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a).$$

Using formula (2.3), we can readily derive the classical Euler polynomials, $E_n(x)$, namely

$$2 \int_{\mathbb{Z}_p} e^{x+y} t d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$ 

In particular when $x = 0$, $E_n(0) = E_n$ is well known the Euler numbers (cf. [9, 13, 21]).

By definition of $I_{-1}(f)$, we show that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where $f_1(x) = f(x+1)$ (see [9]). By (2.5) and induction, we obtain the following fermionic $p$-adic integral equation

$$I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2 \sum_{i=0}^{n-1} (-1)^{n-i-1} f(i),$$

where $n = 1, 2, \ldots$ and $f_n(x) = f(x+n)$. From (2.6), we note that

$$I_{-1}(f_n) + I_{-1}(f) = 2 \sum_{i=0}^{n-1} (-1)^i f(i) \quad \text{if } n \text{ is odd;}$$

$$I_{-1}(f_n) - I_{-1}(f) = 2 \sum_{i=0}^{n-1} (-1)^{i+1} f(i) \quad \text{if } n \text{ is even.}$$

For $x \in \mathbb{Z}_p$ and any integer $i \geq 0$, we define

$$\left(\begin{array}{c} x \\ i \end{array}\right) = \begin{cases} \frac{x(x-1)\cdots(x-i+1)}{i!} & \text{if } i \geq 1, \\ 1 & \text{if } i = 0. \end{cases}$$

It is easy to see that $\binom{x}{i} \in \mathbb{Z}_p$ (see [23, p. 172]). We put $x \in \mathbb{C}_p$ with ord$_p(x) > 1/(p-1)$ and $|1-q|_p < 1$. We define $q^x$ for $x \in \mathbb{Z}_p$ by

$$q^x = \sum_{i=0}^{\infty} \binom{x}{i} (q-1)^i \quad \text{and} \quad [x]_q = \sum_{i=1}^{\infty} \binom{x}{i} (q-1)^{i-1}.$$ 

If we set $f(x) = q^x$ in (2.7) and (2.8), we have

$$I_{-1}(q^x) = \frac{2}{q^n+1} \sum_{i=0}^{n-1} (-1)^i q^i = \frac{2}{q+1} \quad \text{if } n \text{ is odd;}$$

$$I_{-1}(q^x) = \frac{2}{q^n-1} \sum_{i=0}^{n-1} (-1)^{i+1} q^i = \frac{2}{q+1} \quad \text{if } n \text{ is even.}$$
Thus for each \( l \in \mathbb{N} \) we obtain

\[
I_{-1}(q^lx) = \frac{2}{q^{l+1}}.
\]

Therefore we have

\[
(2.13) \quad I_{-1}(q^lx) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l} (-1)^li_{-1}(q^{l+1}x)
\]

\[
= \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l} (-1)^l \frac{2}{q^{l+1} + 1}
\]

Also, if \( f(x) = q^lx \) in (2.5), then

\[
(2.14) \quad I_{-1}(q^{l(x+1)}) + I_{-1}(q^lx) = 2f(0) = 2.
\]

On the other hand, by (2.14), we obtain that

\[
(2.15) \quad I_{-1}(q^{x+1}[x+1]^{n}_{q}) + I_{-1}(q^x[x]^{n}_{q}) = 0
\]

is equivalent to

\[
0 = I_{-1}(q^{x+1}[x+1]^{n}_{q}) + I_{-1}(q^x[x]^{n}_{q})
\]

\[
= qI_{-1}(q^{x+1}(1+q[x]^{n}_{q})) + I_{-1}(q^x[x]^{n}_{q})
\]

\[
= q \sum_{i=0}^{n} \binom{n}{l} q^l I_{-1}(q^x[x]^{i}_{q}) + I_{-1}(q^x[x]^{n}_{q})
\]

\[
(2.16) \quad I_{-1}(q^x[x]^{n}_{q}) = \lim_{N \to \infty} p^{-N-1} \sum_{a=0}^{p^{-N-1}} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l} (-1)^i q^a (-q)^a
\]

From the definition of fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) and (2.13), we can derive the following formula

\[
I_{-1}(q^x[x]^{n}_{q}) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]^{n}_{q} q^x d\mu_{-1}(x)
\]

\[
= \lim_{N \to \infty} p^{-N-1} \sum_{a=0}^{p^{-N-1}} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l} (-1)^i q^a (-q)^a
\]

\[
(2.17) \quad = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l} (-1)^l \frac{2}{q^{l+1} + 1}
\]
is equivalent to

\[
\sum_{n=0}^{\infty} I_1(q^n[x]_q)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{2}{1+q^{i+1}} \frac{t^n}{n!}
\]

(2.18)

\[
= 2 \sum_{n=0}^{\infty} (-q)^n e^{[n]_q} t^n.
\]

From (2.14), (2.15), (2.16), (2.17) and (2.18), it is easy to show that

\[
q \sum_{i=0}^{n} \binom{n}{i} q^i E_{i,q} + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}
\]

(2.19)

where \( E_{n,q} \) are Carlitz’s type \( q \)-Euler numbers defined by (see [20])

\[
F_q(t) = 2 \sum_{n=0}^{\infty} (-q)^n e^{[n]_q} t^n = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.
\]

Therefore, we obtain the recurrence formula for the Carlitz’s type \( q \)-Euler numbers as follows:

\[
q(qE + 1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\]

(2.21)

with the usual convention of replacing \( E_n \) by \( E_{n,q} \).

Therefore, by (2.18), (2.20) and (2.21), we obtain the following theorem.

**Theorem 2.1** (Witt’s formula for \( E_{n,q} \)). For \( n \in \mathbb{Z}^+ \),

\[
E_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{2}{1+q^{i+1}} = \int_{\mathbb{Z}_q} [x]_q^n q^x d\mu_{-1}(x),
\]

which is a partial answer to the problem in [20]. Carlitz’s type \( q \)-Euler numbers \( E_n = E_{n,q} \) can be determined inductively by

\[
q(qE + 1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\]

(2.22)

with the usual convention of replacing \( E_n \) by \( E_{n,q} \).

Carlitz type \( q \)-Euler polynomials \( E_{n,q}(x) \) are defined by means of the generating function \( F_q(x,t) \) as follows:

\[
F_q(x,t) = 2 \sum_{k=0}^{\infty} (-1)^k q^k e^{[k+x]_q} t^k = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]

(2.23)

In the cases \( x = 0 \), \( E_{n,q}(0) = E_{n,q} \) will be called Carlitz type \( q \)-Euler numbers (cf. [10], [21]). We also can see that the generating functions \( F_q(x,t) \) are determined as solutions of the following \( q \)-difference equation:

\[
F_q(x,t) = 2 e^{[x]_q t} - q e^t F_q(x,qt).
\]

From (2.23), we get the following:

**Lemma 2.2.** \( F_q(x,t) = 2 e^{[x]_q t} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^j q^x j^{j} \frac{t^j}{1+q^{j+1}} \).\]

(2) \( E_{n,q}(x) = 2 \sum_{k=0}^{\infty} (-1)^k q^k [k+x]_q^n. \)
It is clear from (1) and (2) of Lemma 2.2 that

\begin{equation}
E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^k
\end{equation}

and

\begin{equation}
\sum_{k=0}^{m-1} (-1)^k q^k [k+x]_q^n = \sum_{k=0}^{\infty} (-1)^k q^k [k+x]_q^n - \sum_{k=0}^{\infty} (-1)^k q^{k+m} [k+m+x]_q^n = \frac{1}{2} \left( E_{n,q}(x) + (-1)^{m+1} q^m E_{n,q}(x+m) \right).
\end{equation}

From (2.25) and (2.26), we may state

Proposition 2.3. If \( m \in \mathbb{N} \) and \( n \in \mathbb{Z}^+ \), then

1. \( E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^k \).
2. \( \sum_{k=0}^{m-1} (-1)^k q^k [k+x]_q^n = \frac{1}{2} \left( E_{n,q}(x) + (-1)^{m+1} q^m E_{n,q}(x+m) \right) \).

Proposition 2.4. For \( n \in \mathbb{Z}^+ \), the value of \( \int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y) \) is \( n! \) times the coefficient of \( t^n \) in the formal expansion of \( 2 \sum_{k=0}^{\infty} (-1)^k q^k \mathcal{E}^{k+x}, \) \( \mathcal{E} \) in powers of \( t \). That is, \( E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y) \).

Proof. From (2.3), we have the relation

\[ \int_{\mathbb{Z}_p} q^{k(x+y)} q^y d\mu_{-1}(y) = q^x \lim_{N \to \infty} \sum_{a=0}^{p^N-1} (-q^{k+1})^a = \frac{2q^x}{1+q^{k+1}} \]

which leads to

\[ \int_{\mathbb{Z}_p} [x+y]_q^n q^y d\mu_{-1}(y) = 2 \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(1-q)^n} (-1)^k \int_{\mathbb{Z}_p} q^{k(x+y)} q^y d\mu_{-1}(y) \]

\[ = \frac{2}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{1+q^{k+1}} q^k. \]

The result now follows by using (1) of Proposition 2.3.

Corollary 2.5. If \( n \in \mathbb{Z}^+ \), then

\[ E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^k E_{k,q}. \]

Let \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \) and \( p \) be a fixed odd prime number. We set

\begin{equation}
X = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p,
\end{equation}

where \( a \in \mathbb{Z} \) with \( 0 \leq a \leq dp^N \) (cf. [9] [11]). Note that the natural map \( \mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z} \) induces

\begin{equation}
\pi : X \to \mathbb{Z}_p.
\end{equation}
Hereafter, if $f$ is a function on $\mathbb{Z}_p$, we denote by the same $f$ the function $f \circ \pi$ on $X$. Namely we consider $f$ as a function on $X$.

Let $\chi$ be the Dirichlet’s character with an odd conductor $d = d_\chi \in \mathbb{N}$. Then the generalized Carlitz type $q$-Euler polynomials attached to $\chi$ defined by

\[
E_{n, \chi, q}(x) = \int_X \chi(a)[x + y]^n q^y d\mu_1(y),
\]

where $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}_p$. Then we have the generating function of generalized Carlitz type $q$-Euler polynomials attached to $\chi$:

\[
F_{q, \chi}(t) = 2 \sum_{m=0}^\infty \chi(m)(-1)^m q^m e^{[m+x]t} = \sum_{n=0}^\infty E_{n, \chi, q}(x) \frac{t^n}{n!},
\]

Now fixed any $t \in \mathbb{C}_p$ with $\text{ord}_p(t) > 1/(p - 1)$ and $|1 - q|_p < 1$. From \((2.30)\), we have

\[
F_{q, \chi}(x, t) = 2 \sum_{m=0}^\infty \chi(m)(-q)^m \sum_{n=0}^\infty \frac{1}{(1 - q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{i(m+x)} \frac{t^n}{n!}
\]

\[
= 2 \sum_{n=0}^\infty \frac{1}{(1 - q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{ix} \sum_{j=0}^{d-1} \chi(j + dl)(-q)^{j + dl} q^{(j + dl)tl} \frac{t^n}{n!}
\]

\[
= 2 \sum_{n=0}^\infty \frac{1}{(1 - q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{ix} \sum_{j=0}^{d-1} \chi(j + dl)(-q)^{j + dl} q^{(j + dl)tl} \frac{t^n}{n!}
\]

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. By \((2.30)\) and \((2.31)\), we can derive the following formula

\[
E_{n, \chi, q}(x) = \frac{1}{(1 - q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^n \binom{n}{i} (-1)^i q^{i(x + j)} \frac{2}{1 + q^{d(l + 1)}}
\]

\[
= \frac{1}{(1 - q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^n \binom{n}{i} (-1)^i q^{i(x + j)} 
\times \lim_{N \to \infty} \sum_{l=0}^{pN-1} (-1)^l q^{d(l + 1)l}
\]

\[
= \lim_{N \to \infty} \sum_{j=0}^{d-1} \sum_{l=0}^{pN-1} \chi(j + dl) \frac{1}{(1 - q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{i(j + dl + x)} 
\times (-1)^i q^{i(j + dl)}
\]

\[
= \lim_{N \to \infty} \sum_{a=0}^{dpN-1} \chi(a) \frac{1}{(1 - q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i q^{i(a + x)} (-q)^a
\]

\[
= \int_X \chi(y)[x + y]^n q^y d\mu_1(y),
\]

where $x \in \mathbb{Z}_p$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Therefore, we obtain the following
Theorem 2.6.

\[ E_{n,\chi,q}(x) = \frac{1}{(1 - q)^n} \sum_{j=0}^{d-1} \chi(j)(-q)^j \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{i(x+j)} \frac{2}{1 + q^{d(i+1)}}, \]

where \( n \in \mathbb{Z}^+ \) and \( x \in \mathbb{Z}_p \).

Let \( \omega \) denote the Teichmüller character mod \( p \). For \( x \in X^\ast \), we set

\[ \langle x \rangle = [x]_q \omega^{-1}(x) = \frac{[x]_q}{\omega(x)}. \]

(2.33) \[ \langle x \rangle \]

Note that since \(|\langle x \rangle - 1|_p < p^{-1/(p-1)}\), \( \langle x \rangle^s \) is defined by \( \exp(s \log \langle x \rangle) \) for \( |s|_p \leq 1 \) (cf. [12, 14, 24]). We note that \( \langle x \rangle^s \) is analytic for \( s \in \mathbb{Z}_p \).

We define an interpolation function for Carlitz type \( q \)-Euler numbers. For \( s \in \mathbb{Z}_p \),

\[ l_{p,q}(s, \chi) = \int_X \langle x \rangle^{-s} \chi(x) q^s d\mu_{-1}(x). \]

Then \( l_{p,q}(s, \chi) \) is analytic for \( s \in \mathbb{Z}_p \).

The proof of this function at non-positive integers are given by

Theorem 2.7. For integers \( n \geq 0 \),

\[ l_{p,q}(-n, \chi) = E_{n,\chi,q} - \chi_n(p)[p]_q^n E_{n,\chi,q}, \]

where \( \chi_n = \chi \omega^{-n} \). In particular, if \( \chi = \omega^n \), then \( l_{p,q}(-n, \omega^n) = E_{n,q} - [p]_q^n E_{n,q} \).

Proof.

\[ l_{p,q}(-n, \chi) = \int_X \langle x \rangle^{-n} \chi(x) q^s d\mu_{-1}(x) \]

\[ = \int_X [x]_q^n \chi_n(x) q^s d\mu_{-1}(x) - \int_X [px]_q^n \chi_n(px) q^s d\mu_{-1}(px) \]

\[ = \int_X [x]_q^n \chi_n(x) q^s d\mu_{-1}(x) - [p]_q^n \chi_n(p) \int_X [x]_q^n \chi_n(x) q^s d\mu_{-1}(x). \]

Therefore by (2.29), the theorem is proved. \( \square \)

Let \( \chi \) be the Dirichlet’s character with an odd conductor \( d = d_\chi \in \mathbb{N} \). Let \( F \) be a positive integer multiple of \( p \) and \( d \). Then by (2.23) and (2.30), we have

\[ F_{q,\chi}(x,t) = 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{[m+x]_q t} \]

\[ = 2 \sum_{a=0}^{F-1} \chi(a)(-q)^a \sum_{k=0}^{\infty} (-q)^{Fk} e^{[k + \frac{x}{F} + \frac{a}{F}]_q t} \]

\[ = \sum_{n=0}^{\infty} \left( [F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q} \left( \frac{x + a}{F} \right) \right) t^n \frac{n!}{n!}. \]

Therefore we obtain the following

\[ E_{n,\chi,q}(x) = [F]_q^n \sum_{a=0}^{F-1} \chi(a)(-q)^a E_{n,q} \left( \frac{x + a}{F} \right). \]
If \( \chi_n(p) \neq 0 \), then \((p, d_{\chi_n}) = 1\), so that \( F/p \) is a multiple of \( d_{\chi_n} \). From (2.36), we derive
\[
\chi_n(p)[p]_q^n E_{n, \chi_n, q^n} = \chi_n(p)[p]_q^n [F/p]_q^{n-1} \sum_{a=0}^{F/p-1} \chi_n(a)(-q)^a E_{n,(q^n)^{F/p}} \left( \frac{a}{F/p} \right)
\]
(2.37)
\[
= [F]_q^n \sum_{a=0 \atop p \nmid a}^F \chi_n(a)(-q)^a E_{n,q^n} \left( \frac{a}{F} \right).
\]
Thus we have
\[
E_{n, \chi_n, q} - \chi_n(p)[p]_q^n E_{n, \chi_n, q^n} = [F]_q^n \sum_{a=0 \atop p \nmid a}^{F-1} \chi_n(a)(-q)^a E_{n,q^n} \left( \frac{a}{F} \right).
\]
(2.38)
By Corollary 2.5, we easily see that
\[
E_{n,q^n} \left( \frac{a}{F} \right) = \sum_{k=0}^n \binom{n}{k} \left[ \frac{F}{a} \right]_{q^n}^{n-k} q^{ka} E_{k,q^n}
\]
(2.39)
\[
= [F]_q^{-n} [a]_q^n \sum_{k=0}^n \binom{n}{k} \left[ \frac{F}{a} \right]_{q^n}^k q^{ka} E_{k,q^n}.
\]
From (2.38) and (2.39), we have
\[
E_{n, \chi_n, q} - \chi_n(p)[p]_q^n E_{n, \chi_n, q^n}
\]
\[
= [F]_q^n \sum_{a=0 \atop p \nmid a}^{F-1} \chi_n(a)(-q)^a E_{n,q^n} \left( \frac{a}{F} \right)
\]
(2.40)
\[
= \sum_{a=0 \atop p \nmid a}^{F-1} \chi(a)(a)^{n-\infty} \sum_{k=0}^n \binom{n}{k} \left[ \frac{F}{a} \right]_{q^n}^k q^{ka} E_{k,q^n},
\]
\[
\text{since } \chi_n(a) = \chi(a) \omega^{-n}(a). \text{ From Theorem 2.7 and (2.40),}
\]
\[
l_{p,q}(-n, \chi) = \sum_{a=0 \atop p \nmid a}^{F-1} \chi(a)(a)^{n-\infty} \sum_{k=0}^n \binom{n}{k} \left[ \frac{F}{a} \right]_{q^n}^k q^{ka} E_{k,q^n}
\]
(2.41)
for \( n \in \mathbb{Z}^+ \). Therefore we have the following theorem.

**Theorem 2.8.** Let \( F \) be a positive integer multiple of \( p \) and \( d = d_{\chi_n} \), and let
\[
l_{p,q}(s, \chi) = \int_{X_+} (x)^{-s} \chi(x) q^x d\mu_{-1}(x), \quad s \in \mathbb{Z}_p.
\]
Then \( l_{p,q}(s, \chi) \) is analytic for \( s \in \mathbb{Z}_p \) and
\[
l_{p,q}(s, \chi) = \sum_{a=0 \atop p \nmid a}^{F-1} \chi(a)(a)^{-s} (-q)^a \sum_{k=0}^\infty \binom{-s}{k} \left[ \frac{F}{a} \right]_{q^n}^k q^{ka} E_{k,q^n}.
\]
Furthermore, for \( n \in \mathbb{Z}^+ \)
\[
l_{p,q}(-n, \chi) = E_{n, \chi_n, q} - \chi_n(p)[p]_q^n E_{n, \chi_n, q^n}.
\]
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