Presentations of higher dimensional Thompson groups

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1. INTRODUCTION

In [2] we introduce groups nV and nV for integers n ≥ 1 and also ωV and ωV where ω represents the natural numbers. These are all subgroups of the homeomorphism group of the Cantor set. The group 1V is a group known as the Thompson group V which is infinite, simple and finitely presented (see [6]). In [2], we show that 2V is infinite, simple and finitely generated and that it is not isomorphic to V, and in fact not isomorphic to any member of an infinite collection of infinite, simple, finitely presented groups that are also known as Thompson groups.

In this paper, we put the group 2V on the same status as V and the other infinite, simple, finitely presented Thompson groups by calculating a finite presentation for 2V. The group 2V is a subgroup of the (non-simple) group 2V and we also calculate a finite presentation for 2V. This is done partly out of necessity since the group 2V is easier to work with than 2V and a fairly full analysis of 2V (tantamount to calculating a presentation) must be done before an analysis of 2V can be done.

The group 2V is a group of fractions of a particularly nice monoid Π that is also introduced in [2]. The monoid Π can be thought of as a countable, monoid approximation to the little cubes operads (in dimension 2) of [9] and [1]. The significance of this observation is unclear to the author. We gain the presentation and understanding of 2V by first calculating a presentation for Π.

In [2], we give generating sets for Π, 2V and 2V. In that paper, we also list relations that are satisfied by the elements, but do not show that the relations suffice to give presentations. In this paper, we perform the calculations that show that we have enough relations for a presentation.

The analysis of the presentation will proceed by using geometric representations of the elements that distinguish between different elements, and normal forms for words in the generators that are derived from the geometric representations. The work comes in showing that the relations suffice to reduce an arbitrary word to one of the normal forms. One step will make use of the analysis of 2V by mapping 2V into its subgroup 2V and using knowledge of 2V to perform part of the normalization.

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The first presentations that we obtain for \( \hat{2V} \) and \( 2V \) are infinite. These almost immediately turn into finite presentations by techniques that have never been formalized into a single machine, but probably should be. This machine contains much of the magical ability of the Thompson groups (with their strong finiteness properties) to imitate the behavior of much larger groups.

The monoid \( \Pi \) is not finitely presentable, and in fact is not finitely generated. We do not analyze the groups \( nV \) for \( n > 2 \). It seems reasonable to hope that these are also finitely presented. The group \( \omega V \) is the ascending union of the groups \( nV \) and is not finitely generated.

2. The monoid \( \Pi \)

The monoid \( \Pi \) is defined in [2] as a set of continuous functions from a topological space \( X \) to itself and also as a set of homeomorphisms from a topological space \( Y \) to itself. The invertibility of the homeomorphisms makes it easier to make the transition to the group of fractions, but the pictures in the setting of \( X \) are easier to draw. When discussing \( \Pi \), we will work with the continuous functions on \( X \) and will switch to the homeomorphisms on \( Y \) when we form the group of fractions. The shift will be painless.

2.1. Numbered patterns. The set \( X \) is the union of a countable set \( \{ S_0, S_1, \ldots \} \) of unit squares in the upper half plane. The intersection of each \( S_i \) with the \( x \)-axis is the closed interval \([2i, 2i+1]\).

Elements of \( \Pi \) are given by numbered patterns in \( X \). First we describe patterns and then we describe numbered patterns. Patterns are those derivable from a single trivial pattern by a finite number of simple increments. We now define these terms.

The trivial pattern in \( X \) is the set of rectangles \( S_i \). The trivial pattern has exactly one rectangle in each square \( S_i \). A simple increment to a pattern increases by one the number of rectangles in a single \( S_i \) by replacing one rectangle \( R \) in the pattern in \( S_i \) by two congruent rectangles obtained from \( R \) by dividing \( R \) exactly in half by either a horizontal line, or a vertical line. All other rectangles in the pattern are left alone by the simple increment. A pattern in \( X \) is a set of rectangles obtainable from the trivial pattern by a finite number of simple increments.

Note that a pattern in \( X \) (called a sequence of patterns in [2]) has more than one rectangle in only finitely many of the \( S_i \).

A numbered pattern is a bijection from \( \mathbb{N} = \{0, 1, 2, \ldots \} \) to the rectangles in a pattern for which there are \( j \) and \( k \) in \( \mathbb{N} \) so that if \( i > k \), then \( S_i \) has only one rectangle in the pattern and its number is \( i + j \). The element of \( \mathbb{N} \) associated to a rectangle by the bijection will be called the number of the rectangle. Below is a picture of a numbered pattern that is assumed to satisfy the defining requirement with \( k = 3 \) and \( j = 5 \).

2.2. A monoid of continuous functions. A numbered pattern determines a continuous function \( f \) from \( X \) to itself which we define separately on each \( S_i \). If \( R_i \) is the rectangle in the pattern with number \( i \), then \( f \) restricted to \( S_i \) is the
restriction to $S_i$ of the unique affine transformation of the plane of the form $(x, y) \mapsto (a + 2^p x, b + 2^q y)$ for integers $p$ and $q$ that carries $S_i$ onto the rectangle $R_i$. Note that this carries the lower left corner of $S_i$ to the lower left corner of $R_i$ and so forth.

It is an elementary exercise that the functions corresponding to numbered patterns in $X$ form a monoid $Π$ under composition of functions. We think of these functions as acting on the left and we compose from right to left.

Different numbered patterns lead to different functions, so when we list generators and relations, we will have a criterion for deciding when two words in the generators give the same element in $Π$.

2.3. Generators for the monoid. The following elements of $Π$ are introduced in [2]. For $i \geq 0$, let $v_i$ be as pictured below.

In the above picture, each square $S_j$ with $j \neq i$ has the one rectangle, each square $S_j$ with $j < i$ is numbered $j$ and each square $S_j$ with $j > i$ is numbered $j + 1$.

For $i \geq 0$, let $h_i$ be as pictured below.

In the above picture, each square $S_j$ with $j \neq i$ has the one rectangle, each square $S_j$ with $j < i$ is numbered $j$ and each square $S_j$ with $j > i$ is numbered $j + 1$.

For $i \geq 0$, let $σ_i$ be as pictured below.

In the above picture, every $S_j$ has the one rectangle and every $S_j$ with $j \notin \{i, i + 1\}$ is numbered $j$.

The following facts from [2] are clear. We do not distinguish between a numbered pattern and the element of $Π$ that it determines. If $P$ is a numbered pattern, then $P_i$ is the rectangle in $P$ numbered $i$.

Lemma 2.1. Let $P$ be a numbered pattern.

(a) The pattern for $Pv_i$ is gotten from $P$ by dividing $P_i$ vertically, giving the left half the number $i$, the right half the number $i + 1$, preserving the numbers of rectangles $P_j$ with $j < i$ and increasing by one the numbers of rectangles $P_j$ with $j > i$.

(b) The pattern for $Ph_i$ is gotten from $P$ by dividing $P_i$ horizontally, giving the bottom half the number $i$, the top half the number $i + 1$, preserving the numbers of rectangles $P_j$ with $j < i$ and increasing by one the numbers of rectangles $P_j$ with $j > i$. 
(c) The pattern for $P\sigma_i$ obtained from $P$ by exchanging the numbers of rectangles numbered $i$ and $i + 1$ and making no other changes.

(d) The set $\{v_i, h_i, \sigma_i \mid i \in \mathbb{N}\}$ is a generating set for the monoid $\Pi$.

2.4. Relations for $\Pi$. The following relations from [2] can be checked by hand. The easiest way is to draw pictures and use Lemma 2.1. In (7) below and in the rest of the paper, we will use the symbol $\bar{\sigma}_j$ to refer to the transposition on $\mathbb{N}$ that interchanges $j$ and $j + 1$.

Lemma 2.2. The following hold in $\Pi$. In the expressions below, the symbols $x$ and $y$ come from $\{h, v\}$.

\begin{equation}
\begin{align*}
(1) & \quad x_j y_i = y_i x_{j+1}, & i < j, \\
(2) & \quad \sigma_i^2 = 1, & i \geq 0, \\
(3) & \quad \sigma_i \sigma_j = \sigma_j \sigma_i, & |i - j| \geq 2, \\
(4) & \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & i \geq 0, \\
(5) & \quad \sigma_j x_i = (\sigma_j \cdot x_i) (\sigma_j)^{x_i}, & i \geq 0, j \geq 0, \\
(6) & \quad v_i h_{i+1} h_i = h_i v_{i+1} v_j \sigma_{i+1}, & i \geq 0,
\end{align*}
\end{equation}

where the right side of (5) is given by

\begin{equation}
\begin{align*}
(7) & \quad \sigma_j \cdot x_i = x_{\bar{\sigma}_j(i)}
\end{align*}
\end{equation}

and

\begin{equation}
(\sigma_j)^{x_i} = \begin{cases}
\sigma_{j+1}, & i < j, \\
\sigma_j \sigma_{j+1}, & i = j, \\
\sigma_{j+1} \sigma_j, & i = j + 1, \\
\sigma_j, & i > j + 1.
\end{cases}
\end{equation}

2.5. First normalization. We keep track of which relations we use as we simplify a word in the generators of $\Pi$.

Lemma 2.3. Using the relations (d), any word in the generators from Lemma 2.1(d) can be altered to a word in the form $pq$ where $p$ is a word in $\{v_i, h_i \mid i \in \mathbb{N}\}$ and $q$ is a word in $\{\sigma_i \mid i \in \mathbb{N}\}$.

Proof. If we break a word $w$ in the generators of $\Pi$ into a concatenation $w = p_a p_\sigma r$ where $p_a$ is the longest prefix of $w$ containing nothing but elements of $\{v_i, h_i \mid i \in \mathbb{N}\}$ and $p_\sigma$ is the longest prefix of $p_\sigma r$ containing nothing but elements of of $\{\sigma_i \mid i \in \mathbb{N}\}$, then we can form a complexity of $w$ out of the pair $(a, b)$ where $a$ is the length of $p_a$ and is the most significant part of $(a, b)$, and $b$ is the length of $p_\sigma$. Complexities are defined to decrease as $a$ increases and as $b$ decreases. The result follows from the observation that the relations (5) do not change the number of elements of $\{v_i, h_i \mid i \in \mathbb{N}\}$ that are in a word and the fact that applications of (5) lower the complexity.

We will need the following trivial strengthening of Lemma 2.3.

Lemma 2.4. Using the relations (d), any word in the generators from Lemma 2.1(d) of the form $p r$ with $p$ a word in $\{v_i, h_i \mid i \in \mathbb{N}\}$ can be altered to a word in the form $pq s$ where $q$ is a word in $\{v_i, h_i \mid i \in \mathbb{N}\}$ and $s$ is a word in $\{\sigma_i \mid i \in \mathbb{N}\}$.

Proof. Apply Lemma 2.3 to the subword $r$. \qed
2.6. **Labeled, numbered forests.** Elements of $\Pi$ are completely classified by numbered patterns, and many words in the generators of $\Pi$ lead to one element. We create a structure that is intermediate between numbered patterns and words in the generators from Lemma 2.1(d).

A forest will be a certain sequence of trees, so we start with trees. Our notation is fairly standard and we assume that the reader is familiar with trees, but we review the terms we will use.

A tree is a non-empty finite set of vertices with two relations **left child** and **right child**. Every vertex will either have one left child and one right child, or it will have no children. This makes a tree a binary tree, but all our trees will be binary and we will not use the adjective “binary” when discussing trees. A **child** is either a left or right child, the transitive closure of child is **descendent**, the inverse of child is **parent** and the transitive closure of parent is **ancestor**. Each tree has one vertex, the **root** that is the ancestor of all other vertices in the tree.

A vertex in a tree is called a **leaf** if it has no children, and it is called an **interior vertex** otherwise. The **trivial tree** has only one vertex which must therefore be both the root and a leaf. The trivial tree has no interior vertices. It is elementary that the number of leaves of a tree is one more than the number of interior vertices.

A **labeled tree** is a tree with a label on each interior vertex where the labels come from $\{v, h\}$. A **forest** is a sequence (indexed over $\mathbb{N}$) of trees of which all but finitely many are trivial. If we regard the trees of a forest as disjoint, then we have infinitely many vertices since every tree is non-empty, but we only have finitely many interior vertices. A **labeled** forest is a forest of labeled trees. The **leaves of the forest** are elements of the disjoint union of the leaves of the trees of the forest. A forest has infinitely many leaves. If $F$ is a forest then $F_i$ is its $i$-th tree.

A **labeled, numbered forest** is a labeled forest $F$ with a one-to-one correspondence between $\mathbb{N}$ and the leaves of the forest so that there are $j$ and $k$ in $\mathbb{N}$ so that $i > k$ implies that $F_i$ is trivial and its only leaf is numbered $i + j$. The **trivial numbered, labeled forest** is the sequence of trivial trees so that the sole leaf of the $i$-th tree is numbered $i$.

2.7. **Carets.** In a tree, a triple $(v, v_0, v_1)$ where $v$ is an interior vertex, $v_0$ is the left child of $v$ and $v_1$ is the right child of $v$ is called a **caret**. The **root** of the caret $(v, v_0, v_1)$ is $v$. Sending a caret to its root gives a one-to-one correspondence between the carets in a tree and the interior vertices of a tree. We can say that a tree is a union of a finite number of carets if we are sloppy and declare the trivial tree to be the union of zero carets. We talk about labeling interior vertices, but could just as easily talk about labeling (roots of) carets. We introduce carets since they are convenient when discussing modifications to a tree.

2.8. **Numbered patterns from numbered, labeled forests.** Each vertex of a labeled tree corresponds to a unique rectangle in a unit square. This can be said inductively. We start by declaring that the root corresponds to all of the unit square. If a vertex labeled $v$ corresponds to a rectangle $R$, then its left child corresponds to the left half of $R$ and the right child corresponds to the right half of $R$. If a vertex labeled $h$ corresponds to a rectangle $R$, then its left child corresponds to the bottom half of $R$ and the right child corresponds to the top half of $R$.

The rectangles corresponding to the leaves of a labeled tree form a pattern in a unit square. This is easy to see inductively on the size of the tree. A labeled forest
thus gives a pattern in \( X \). A numbered, labeled forest gives a numbered pattern in \( X \) by giving the rectangle corresponding to a leaf the number of the leaf. Different numbered, labeled forests can give the same numbered pattern. We can discuss examples more easily after the next topic.

2.9. **Numbered, labeled forests from words.** A word in the generators from Lemma 2.1(d) determines a numbered, labeled forest. We assign the trivial numbered, labeled forest to the empty word and we define the other assignments inductively on the length of the word. If \( w = pa \) with \( a \) from Lemma 2.1(d), then the forest \( F \) assigned to \( p \) is modified depending on \( a \).

If \( a = v_i \), then leaf \( i \) of \( F \) is given two children and the label \( v_i \). We can also describe this as attaching a new caret to \( F \) by attaching the root of the new caret to leaf \( i \) of \( F \). The new left child is numbered \( i \), the new right child is numbered \( i + 1 \), and each other leaf retains its old number if it was less than \( i \) and has its number increased by one if the old number was greater than \( i \).

If \( a = h_i \), then exactly the same thing happens as in the case \( a = v_i \) except that the new label is an \( h_i \).

If \( a = \sigma_i \), then the only change to \( F \) is to switch the numbers of the two leaves that have the numbers \( i \) and \( i + 1 \).

It is elementary that if \( w \) is a word in the generators from Lemma 2.1(d) and \( F \) is the numbered, labeled forest assigned to \( w \), then the numbered pattern corresponding to \( F \) is the numbered pattern that determines the same element of \( \Pi \) as the word \( w \). There are many words assigned to the same numbered, labeled forest.

**Lemma 2.5.** Let \( w \) and \( w' \) be words in the generators from Lemma 2.1(d) that are related by relations (1)–(5). Then the numbered, labeled forests assigned to \( w \) and \( w' \) are identical.

**Proof.** Pictures can be drawn for each of the relations. □

The relations (1) do not preserve the forest. The simplest example built from a relation of the form (1) shows this and it also gives examples of different numbered, labeled forests that correspond to the same numbered pattern.

2.10. **Words in the \( v_i \) and \( h_i \).** We use the notions of “confluent” and “terminating” when applied to relations and rewriting systems. This material is covered in numerous places such as [8], [10] and [3]. If we change the relations in (1) to

\[
(9) \quad x_j y_i \rightarrow y_i x_{j+1} \text{ whenever } i < j,
\]

then we have a set of rewriting rules that can be applied to any word in \( \{v_i, h_i \mid i \in \mathbb{N} \} \) in that we are allowed to replace a subword like the left side of (9) by the right side of (9), but not the reverse. It is an elementary exercise that the rewriting rule (9) is terminating in that it cannot be applied an infinite number of times to a given word, and locally confluent in that two different single applications of (9) to a word \( w \) to give \( w_1 \) and \( w_2 \) can be “joined” by a fourth word \( z \) that can be obtained from each of \( w_1 \) and \( w_2 \) by zero or more applications of (9). In our situation, getting \( z \) from \( w_1 \) and \( w_2 \) will each take no more than two applications of (9).

The standard fact from such considerations is that the equivalence class of any word \( w \) in \( \{v_i, h_i \mid i \in \mathbb{N} \} \) under (1) contains a unique representative that admits no applications of (9). This unique representative is said to be irreducible under (9) and gives a convenient normal form for words in \( \{v_i, h_i \mid i \in \mathbb{N} \} \).
It is elementary that a word \( x_1 x_2 \cdots x_i \) where each \( x \) is chosen independently from \( \{v, h\} \) is irreducible under (9) if and only if \( i_1 \leq i_2 \leq \cdots \leq i_k \). It is now another easy exercise to show that if \( w \) and \( w' \) are two words in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) that are irreducible under (9) and are different, then they lead to different numbered, labeled forests. Combining this with Lemma 2.6 gives the following.

**Lemma 2.6.** Two words in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) lead to the same labeled, numbered forest if and only if they are related under (9).

The leaf numbering of a forest that comes from a word in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) is particularly simple. In the next lemma we use the standard fact that the left-right order on each child pair of a vertex in a tree leads to a linear, left-right order on the leaves of a tree.

**Lemma 2.7.** If the numbered, labeled forest \( F \) comes from a word in \( \{v_i, h_i \mid i \in \mathbb{N}\} \), then the leaves of \( F \) are numbered so that the leaves in \( F \) with numbers lower than those in \( F_{i'} \), whenever \( i < j \) and the leaves in each tree of \( F \) are numbered in increasing order under the natural left-right ordering of the leaves.

2.11. **Words leading to the same labeled, numbered forest.** If \( w \) and \( w' \) are words in the generators from Lemma 2.6(d) and they are associated to the same labeled, numbered forest, then we want to conclude that they are related in some known way. We know that we can use relations (4) to write \( w = pq \) and \( w' = p'q' \) where \( p \) and \( p' \) are words in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) and \( q \) and \( q' \) are words in \( \{\sigma | i \in \mathbb{N}\} \). Now words in \( \{\sigma | i \in \mathbb{N}\} \) can only affect the numbering of a forest, so \( p \) and \( p' \) must lead to the same labeled forest. By Lemma 2.7, the numbered, labeled forests corresponding to \( p \) and \( p' \) must be identical. Thus \( q \) and \( q' \) must have the same effects on the numbering. Since the effects of \( q \) and \( q' \) are permutations calculated from the transpositions (9), we know that the words \( q \) and \( q' \) represent the same permutation in the group of finitary permutations on \( \mathbb{N} \). It is well known that the group of finitary permutations on \( \mathbb{N} \) are presented by the transpositions (9), with the relations (4–9). Thus \( q \) and \( q' \) must be related by (4–9). Since the numbered, labeled forests corresponding to \( p \) and \( p' \) are identical, the words \( p \) and \( p' \) are related by (9) according to Lemma 2.6. We have proven the following converse to Lemma 2.6.

**Lemma 2.8.** If two words in the generators from Lemma 2.6(d) lead to the same numbered, labeled forest, then the words are related by (9)–(9).

2.12. **Ordering interior vertices.** We will characterize the words in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) that lead to a given labeled, numbered forest. Of course, by Lemma 2.4, the numbering of such a forest is restricted.

If \( w \) is a word in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) and \( F \) is the numbered, labeled forest derived from \( w \), then there is a one-to-one correspondence between the entries in \( w \) and the interior vertices in \( F \). The correspondence is easier to describe by referring to carets instead of interior vertices.

If \( p_{i-1} \) is the prefix of \( w \) of length \( i - 1 \) with \( i \geq 1 \), and \( F^{i-1} \) is the forest corresponding to \( p_{i-1} \), then every caret of \( F^{i-1} \) is a caret of \( F^{i} \) and every caret of \( F^{i} \) except one is a caret of \( F^{i-1} \). Thus it is seen that the set of carets of \( F \) is the ascending union of the sets of carets of the \( F^i \). The caret of \( F^i \) that is not in \( F^{i-1} \) is defined to correspond to the \( i \)-th entry in \( w \). Thus the order that the entries appear in \( w \) gives a linear order to the carets (and thus the interior vertices) in \( F \).
This linear order respects another order. It is clear that if \( w = a_1a_2 \cdots a_k \) is a word in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) with corresponding forest \( F \), then the linear order on the interior vertices of \( F \) given by the order of the entries in the word respects the ancestor relation in that the interior vertex for \( a_i \) is never the ancestor of the interior vertex for \( a_j \) when \( i > j \). In the proof of the next lemma we treat the ancestor relation as applied to carets using the one-to-one correspondence between interior vertices and carets.

**Lemma 2.9.** If \( F \) is a numbered, labeled forest with the numbering as in Lemma 2.7 and if a linear order is given on the interior vertices (and thus of the carets) of \( F \) that respects the ancestor relation, then there is a unique word \( w \) in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) leading to \( F \) so that the order on the interior vertices of \( F \) derived from the order on the entries in \( w \) is identical to the given linear order on the interior vertices.

**Proof.** Since the given linear order respects the ancestor relation, every ancestor of a given caret comes before that caret in the linear order. Therefore, there is a sequence of numbered, labeled forests starting with the trivial forest so that each term in the sequence is obtained from the previous by adding exactly one caret and so that the order of addition of carets in this sequence is exactly the given linear order. This sequence is unique in that it is completely determined by the given linear order. Now a word can be built up that adds these carets in exactly this order. Since each caret with label on its root can be added by exactly one generator from \( \{v_i, h_i \mid i \in \mathbb{N}\} \), this word is unique. \( \square \)

2.13. **Secondary labels and normalized forests.** We have seen that the labels in a labeled forest can be related to the arrangement of letters in a word. We will introduce extra labels to some of the interior vertices in a labeled forest that correspond to information gathered from the pattern in \( X \) that is associated to the forest.

Let \( F \) be a labeled forest, and let \( P \) be the pattern associated to (the leaves of) \( F \). Numbering will not be important here.

Let \( u \) be an interior vertex of \( F \). There is a rectangle \( R \) that corresponds to \( u \). Since \( u \) is an interior vertex, the rectangle \( R \) will be further subdivided by the pattern \( P \). If the left half of \( R \) is a union of rectangles of \( P \), then it is necessarily true that the right half of \( R \) is a union of rectangles of \( P \). In such case, we say that \( R \) is divided vertically by \( P \). Similarly, we say that \( R \) is divided horizontally by \( P \) if the bottom half of \( R \) is a union of rectangles of \( P \). In the case that \( R \) is divided vertically and also divided horizontally, we say that \( R \) is fully divided and add the secondary label “+” to \( u \). Note that not all interior vertices get secondary labels.

When an interior vertex has a secondary label, then the label from \( \{v, h\} \) is its primary label. We say that an interior vertex is normalized if it has no secondary label or if its primary label is \( v \). We say that a labeled forest \( F \) is normalized if every interior vertex is normalized.

2.14. **Uniqueness of normalized forests.** Uniqueness does not imply existence which will be covered later. The next lemma establishes uniqueness and is stated so as to be easy to prove.

**Lemma 2.10.** If two different forests correspond to the same pattern in \( X \), then at least one of the two forests is not normalized.
Proof. Since different trees in a forest correspond to patterns in different unit squares in X, we see that it suffices to look at different labeled trees and assume that they correspond to the same pattern P in the unit square.

Pick vertices closest to the root where the two trees differ (as labeled trees). A trivial check of cases shows that the vertices must be interior with different labels. Since the rectangle corresponding to a vertex depends only on the labeled path above it leading to the root and since our choice makes these two labeled paths the same in the two trees, our differing vertices correspond to the same rectangle R. The label must be v in one tree and h in the other, so R must be fully divided in P and the vertices in question must have a secondary label. Since one of the trees has h as the primary label, it is not normalized. \(\square\)

Existence of a normalized forest for a pattern is a triviality; one works directly from the pattern. However, it is not necessary to argue existence separately since it follows from the next (more difficult) proposition.

2.15. **Normalized forests from words.** We will prove the following proposition. If \(w\) is a word in the generators from Lemma 2.1(d), then we say the length of \(w\) is the number of appearances in \(w\) of elements of \(\{v_i, h_i | i \in N\}\). We note that length is preserved by the relations (1)–(6).

**Proposition 2.11.** Let \(w\) be a word in the generators from Lemma 2.1(d). Then \(w\) is related by (1)–(6) to a word corresponding to a normalized, labeled forest.

**Proof.** We will assume that the statement is false for some word \(w\) of length \(n\) and is true for all words of length less than \(n\). There are a number of immediate consequences of this assumption, not all of which are worth noting. Two that we need are that \(n > 2\) (since otherwise the corresponding forest has no secondary labels) and that the following lemma holds. We will complete the proof of the proposition after the lemma is stated and proven. \(\square\)

**Lemma 2.12.** Let \(w\) be a word in the generators from Lemma 2.1(d) of length \(n\) of the form \(w = as\) where \(a\) is an element of \(\{v_i, h_i | i \in N\}\) and \(s\) is of length \(n - 1\). Then we can alter \(w\) by applications of (1)–(6) to \(s\) alone to give a word whose corresponding forest has all non-root, interior vertices normalized.

**Proof.** By Lemma 2.13, we can assume that \(s\) is of the form \(pq\) with \(p\) a word in \(\{v_i, h_i | i \in N\}\) and \(q\) a word in the \(\sigma_i\) and that \(w = apq\). Since the order of the interior vertices of the forest for \(ap\) given by the order of the letters in \(ap\) must respect the ancestor relation, we know that the interior vertex corresponding to \(a\) is a root. By hypothesis, \(s\) is related by (1)–(6) to a word \(s'\) corresponding to a normalized forest. The pattern \(P\) for \(as'\) is obtained from the pattern \(P'\) for \(s'\) by applying the pattern of \(P'\) in unit square \(S_i\) to the rectangle numbered \(i\) in the pattern for \(a\). The forest \(F\) for \(as'\) is obtained from the forest \(F'\) for \(s'\) by attaching the \(i\)-th tree of \(F'\) to the \(i\)-th leaf of the forest for \(a\). Since \(F'\) is normalized, it is seen that \(F\) has all interior vertices normalized except possibly for the root vertex of one tree. \(\square\)

**Continuation of the proof of Proposition 2.7.1.** By Lemma 2.12 we can assume that the forest \(F\) for \(w\) has all non-root interior vertices normalized. Let \(P\) be the pattern corresponding to \(w\).
By Lemmas 2.3 and 2.5, we can assume that \( w \) is of the form \( w = pq \) with \( p \) a word in \( \{ v_i, h_i \mid i \in \mathbb{N} \} \) and \( q \) a word in the \( \sigma_i \). Let \( u \) be a root of \( F \) that is not normalized and let \( u_0 \) and \( u_1 \) be the left and right children, respectively, of \( u \). By Lemma 2.10 we can assume that the first three letters of \( p \) correspond, in order, to \( u_0, u_1 \) and \( u_0 \). This choice of order is deliberate.

Since \( u \) is not normalized, its corresponding rectangle \( R \) is fully divided and its label is \( h \). Thus the rectangles corresponding to \( u_0 \) and \( u_1 \) are the bottom and top rectangles, respectively, of \( R \) and must be both vertically divided since \( R \) is fully divided. Since \( u_0 \) and \( u_1 \) are normalized, a quick check of cases shows that they are both labeled \( v \). Thus the first three letters of \( p \) are \( h_i v_i v_{i+1} \). Using (6), these can be replaced by \( v_i h_{i+1} h_i \sigma_{i+1} \). This gives a word \( w' \) whose corresponding forest \( F' \) is different from \( F \) but whose corresponding pattern is still \( P \).

The root corresponding to the initial letter \( v_i \) is now normalized. Now a second application of Lemma 2.12 normalizes all other vertices. The result is a normalized forest. □

2.16. A presentation for \( \Pi \). We are ready for the following.

**Theorem 1.** The monoid \( \Pi \) is presented by using the generators from Lemma 2.1(d) and relations (1)–(6).

**Proof.** Let two words give the same element of \( \Pi \). Since elements correspond to numbered patterns, they give the same numbered pattern. From Proposition 2.11 and Lemma 2.10 we can assume that the two words correspond to the same labeled forest. Since the forests correspond to the same numbered pattern, the forests must have the same numbering. The result now follows from Lemma 2.8. □

3. The group \( \hat{2V} \)

The group \( \hat{2V} \) is the group of right fractions of \( \Pi \). However, as in [2] it is easier to change the representation of \( \Pi \) to make the elements invertible than it is to apply the usual theorem (Ore’s theorem, Theorem 1.23 of [7]).

3.1. New patterns. To invert the elements of \( \Pi \), we alter the meaning of vertical and horizontal divisions slightly. If \( R \) is a rectangle, than the new notion of vertical division replaces \( R \) by its left third and right third. The new horizontal division of \( R \) replaces \( R \) by its bottom third and top third. Patterns defined with these steps do not give collections of rectangles that cover all of the unit squares in \( X \). However, a pattern defined this way will cover copies of \( C \times C \) in \( X \) where \( C \) is the Cantor set defined in the usual way as the “deleted middle thirds” set in the unit interval and this inclusion of \( C \) in the unit interval \( I \) induces the natural inclusion of \( C \times C \) in the unit square \( I \times I \). The covering of \( C \times C \) will be by pairwise disjoint closed and open sets in \( C \times C \). Numberings will be handled in the same way in the new and old patterns and will be made to correspond.
The following is a picture from [2] that shows how an old numbered pattern in a unit square converts to a new numbered pattern covering $C \times C$ in that square.

\[ \begin{array}{c}
0 & 2 & 3 \\
4 & & \\
1 & & \\
\end{array} \quad \longleftrightarrow \quad \begin{array}{c}
1 & 3 & 3 \\
& & 1 \\
& & \\
\end{array} \]

(10)

3.2. Elements of the group. We put a copy of $C \times C$ in each unit square $S_i$ of $X$ and let $Y$ be the union of these copies of $C \times C$. Elements of $\hat{2V}$ are self homeomorphisms of $Y$ and are defined by pairs of numbered patterns $(P,Q)$. We think of $P$ as the range pattern and $Q$ as the domain pattern. We adopt this convention to make formulas for composition look nicer. For each $i$ in $\mathbb{N}$, the homeomorphism defined by $(P,Q)$ takes the intersection of $Y$ with the $i$-th rectangle under $Q$ onto the intersection $Y$ with the $i$-th rectangle of $P$ by the unique affine transformation $(x,y) \mapsto (a + 3^jx, b + 3^ky)$ with $j$ and $k$ integers that does so.

It turns out that many pairs will represent the same element of $\hat{2V}$. For example, the identity of $\hat{2V}$ is represented by all pairs of the form $(P,P)$.

In spite of the fact that the closed and open sets covering $C \times C$ look like the right side of (10), we will continue to think of patterns as drawn in the left side of (10). Thus we continue to talk about rectangles being divided in half and not thirds. When a pattern based on halves is used to create an element of $\hat{2V}$, it has to be converted first into a pattern based on thirds.

3.3. Group of right fractions. It is more useful to view elements as compositions of two homeomorphisms. If $E$ is the trivial pattern, then $(P,Q)$ is the composition of $(P,E)$ and $(E,Q)$.

With the homeomorphisms acting on the left, we write

$$(P,Q) = (P,E)(E,Q) = (P,E)(Q,E)^{-1}.$$

The elements of the form $(P,E)$ create a copy of the monoid $\Pi$. Specifically, sending $P$ interpreted as a pattern in $X$ to the pair $(P,E)$ in $\hat{2V}$ creates an isomorphic embedding of $\Pi$ into $\hat{2V}$. If we identify the element $P$ of $\Pi$ with $(P,E)$ in $\hat{2V}$, then this establishes $\Pi$ as a monoid in $\hat{2V}$ with the property that every element of $\hat{2V}$ is of the form $PQ^{-1}$ with both $P$ and $Q$ in $\Pi$. From [2] (Page 36 and Problem 3 of Page 37), this establishes $\hat{2V}$ as a group of right fractions of $\Pi$.

From this point, we will use both $(P,Q)$ and $PQ^{-1}$ to denote the same element of $\hat{2V}$. Note that if $M$ is any other element of $\Pi$, then $(PM,QM)$ represents the same element $(PM)(QM)^{-1} = PQ^{-1}$ as $(P,Q)$.

3.4. Presentation of $\hat{2V}$. By a well known extension of Ore’s theorem (see, for example, Proposition 2.4 of [3]), a monoid presentation for $\Pi$ is a group presentation for $\hat{2V}$. Thus we have the following.

**Theorem 2.** The group $\hat{2V}$ is presented by using the generators from Lemma 2.1(d) and relations (10)–(16).
3.5. A semi-normal form. There will be a natural homomorphism from \( \hat{2V} \) into \( 2V \) (actually and embedding, but the injective property will not be needed). We will use this homomorphism and the following lemma to prove consequences about the relations in \( 2V \). The lemma is stated in a way that will be easy to use.

**Lemma 3.1.** Let \( w \) be a word in the generators from Lemma 2.4(d) and their inverses. By applying the relations \( \{1, 2\} \) to \( w \), we can obtain a word of the form \( LMR \) where \( L \) and \( R^{-1} \) are words in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) and \( M \) is a word in \( \{\sigma_i \mid i \in \mathbb{N}\} \).

**Proof.** Since the relations \( \{1, 2\} \) are all of the presenting relations of \( \hat{2V} \), we only have to argue that every element of \( \hat{2V} \) has a representative in the desired form. But every element of \( \hat{2V} \) can be put in the form \( pq^{-1} \) with \( p \) and \( q \) a word in the generators from Lemma 2.4(d) and Lemma 2.6 puts each of \( p \) and \( q \) in the form \( ab \) with \( a \) a word in \( \{v_i, h_i \mid i \in \mathbb{N}\} \) and \( b \) a word in \( \{\sigma_i \mid i \in \mathbb{N}\} \). The fact that elements of \( \{\sigma_i \mid i \in \mathbb{N}\} \) are their own inverses completes the proof. \( \square \)

3.6. An interchange formula. Unfortunately we will need more detail than supplied by Lemma 3.1. The next lemma allows us to predict to some extent what we might see when putting a word in the generators from Lemma 2.4(d) into semi-normal form.

**Lemma 3.2.** Let \( w \) be a word in \( \{\sigma_i, v_i^{-1}, h_i^{-1} \mid i \in \mathbb{N}\} \). Then applications of \( \{1, 2\} \) can be used to put \( vw_i \) in the form \( pw' \) where \( p \) is a word in \( \{v_i \mid i \in \mathbb{N}\} \) and \( w' \) is a word in \( \{\sigma_i, v_i^{-1}, h_i^{-1} \mid i \in \mathbb{N}\} \).

**Proof.** Recall that elements of \( \hat{2V} \) of the form \( (P, E) \), where \( E \) is the trivial pattern, form a copy of the monoid \( \Pi \). The word \( w \) is the inverse of a word in this copy of \( \Pi \) and is of the form \( (E, P) \). The element \( v_i \) is of the form \( (V_i, E) \) where we use \( V_i \) to denote the first pattern shown in Section 2.3.

We can multiply the elements \( w = (E, P) \) and \( v_i = (V_i, E) \) if we can get a pairs representing \( w \) and \( v_i \) so that the second pattern for \( w \) is the same as the first pattern for \( v_i \). As pairs in a group of fractions, we can get new pairs from old by multiplying both entries on the right by the same thing. We will use Lemma 2.4 to understand this right multiplication.

We start to get a pattern that is a common right multiple of \( P \) and \( V_i \) by superposing the two patterns. Since \( V_i \) has only one non-trivial division consisting of a vertical line in the \( i \)-th square, we need only draw a vertical line in the \( i \)-th square of the pattern \( P \). This might have the consequence of vertically dividing several of the rectangles of \( P \). According to Lemma 2.4 this can be accomplished by multiplying the element of \( \Pi \) given by \( P \) on the right by a word \( p \) in the \( v_i \). Abusing notation somewhat, we now use the pair \( (Ep, Pp) \) to represent \( w \). Since \( E \) is the trivial pattern, we get \( w = (p, Pp) \).

Now \( V_i \) must be converted to \( Pp \). This is accomplished by subdividing various rectangles in \( V_i \) and applying a word in the \( \sigma_i \) to get the right numbering. According to Lemma 2.1 a word \( w' \) in \( \{\sigma_i, v_i, h_i \mid i \in \mathbb{N}\} \) must be applied to the right of \( V_i \). Thus we get \( V_i w' = Pp \) and \( v_i = (V_i w', Ew') = (Pp, w') \) and

\[ vw_i = (p, Pp)(Pp, w') = (p, w') = (p, E)(E, w'). \]

This is exactly what was wanted. \( \square \)
4. The group $2V$

4.1. The elements. Recall that $X$ is the disjoint union of the squares $S_i$, $i \in \mathbb{N}$, and that $Y$ is a subset of $X$. The group $2V$ is the subgroup of $\hat{2}V$ consisting of those elements that act as the identity off $Y \cap S_0$. It is easy to see that these are the elements that are representable by pairs $(P, Q)$ for which there is an $n \in \mathbb{N}$ so that each of $P$ and $Q$ satisfy the following: (1) the number of rectangles in $S_0$ is $n$, (2) the number of rectangles in each $S_i$, $i > 0$, is one, and (3) the number of the rectangle in $S_i$ is $i + n - 1$ for $i > 0$.

4.2. The generators. The following elements of $2V$ are shown in [2] to generate $2V$.

\begin{align*}
A_i &= (v_i^{i+1}v_1, v_i^{i+2}), \quad i \geq 0, \\
B_i &= (v_i^{i+1}h_1, v_i^{i+2}), \quad i \geq 0, \\
C_i &= (v_i^ih_0, v_i^{i+1}), \quad i \geq 0, \\
\pi_i &= (v_i^{i+2}\sigma_1, v_i^{i+2}), \quad i \geq 0, \\
\pi_i &= (v_i^{i+1}\sigma_0, v_i^{i+1}), \quad i \geq 0.
\end{align*}

We let

\[\Sigma = \{A_i, B_i, C_i, \pi_i, \pi_i | i \in \mathbb{N}\}.\]

The argument that $\Sigma$ is a generating set for $2V$ is not relevant to this paper.

4.3. The relations. The list of relations is longer. In [2] it is argued that the following relations hold in $2V$ where $X$ and $Y$ represent symbols from $\{A, B\}$.

\begin{align*}
(11) \quad X_qY_m &= Y_mX_{q+1}, \quad m < q, \\
(12) \quad \pi_qX_m &= X_m\pi_{q+1}, \quad m < q, \\
(13) \quad \pi_qX_q &= X_{q+1}\pi_{q+1}, \quad q \geq 0, \\
(14) \quad \pi_qX_m &= X_m\pi_q, \quad m > q + 1, \\
(15) \quad \pi_mA_m &= \pi_m\pi_{m+1}, \quad m \geq 0, \\
(16) \quad \pi_mB_m &= C_m+1\pi_m\pi_{m+1}, \quad m \geq 0, \\
(17) \quad C_qX_m &= X_mC_{q+1}, \quad m < q, \\
(18) \quad C_mA_m &= B_mC_{m+2}\pi_{m+1}, \quad m \geq 0, \\
(19) \quad \pi_mC_m &= C_m\pi_q, \quad m > q + 1, \\
(20) \quad A_mB_{m+1}B_m &= B_mA_m+1A_m\pi_{m+1}, \quad m \geq 0, \\
(21) \quad \pi_q\pi_m &= \pi_m\pi_q, \quad |m - q| \geq 2, \\
(22) \quad \pi_m\pi_{m+1}\pi_m &= \pi_{m+1}\pi_m\pi_{m+1}, \quad m \geq 0, \\
(23) \quad \pi_q\pi_m &= \pi_m\pi_q, \quad q \geq m + 2, \\
(24) \quad \pi_m\pi_{m+1}\pi_m &= \pi_{m+1}\pi_m\pi_{m+1}, \quad m \geq 0, \\
(25) \quad \pi^2_m &= 1, \quad m \geq 0, \\
(26) \quad \pi^2_m &= 1, \quad m \geq 0.
\end{align*}
It is our task to show that the generators in $\Sigma$ and the relations (11)–(27) present $2V$. We could eliminate the generators $C_i$ by using

$$C_m = (\tau_m B_m \pi_{m+1} \pi_m) (B_m \pi_{m+1} A_m^{-1}),$$

as is shown in [2]. However this does not seem to simplify the calculations below.

### 4.4. Strategy

We need only show that if a word represents the trivial element, then the word is reducible to the trivial word by the relations (11)–(27). However, it is hard to use the fact that the element represented is trivial until the word has been simplified significantly. Thus we reduce an arbitrary word to a particularly nice form first, and then take into account that the represented element is trivial.

### 4.5. Conventions

We let $G$ be the group with generators from Lemma 2.1(d) and relations (11)–(27). If two words $w$ and $w'$ in the generators of $G$ represent the same element of $G$, then we will write $w \sim w'$.

We will be giving different treatment to the positive and negative powers of the generators. Thus from now on we will work with the generating set

$$\Sigma_s = \{A_i, B_i, C_i, \pi_i, A_i^{-1}, B_i^{-1}, C_i^{-1} \mid i \in \mathbb{N}\}$$

and treat it as a group of semigroup generators of the group $G$. We will have no need to distinguish between $\pi_i$ and $\pi_i^{-1}$ or between $\pi_i$ and $\pi_i^{-1}$ because of the relations $\pi_i^2 = 1$ and $\pi_i^{-2} = 1$.

We will never have reason to discuss subsets of $\Sigma_s$ on the basis of the values of the subscripts. Thus we will often refer to subsets of $\Sigma_s$ by leaving out the subscripts and referring to words in these subsets in the following form. If $S$ is a subset of the symbols

$$\{A, B, C, \pi, \pi^{-1}, A^{-1}, B^{-1}, C^{-1}\},$$

then we will write $w(S)$ to indicate a word in the symbols from $S$, subscripted with values from $\mathbb{N}$. For example, $w(A, \pi, B^{-1})$ refers to a word in the subset $\{A_i, \pi_i, B_i^{-1} \mid i \in \mathbb{N}\}$ of $\Sigma_s$.

### 4.6. The LMR form, Part I

Let $w$ be a word in $\Sigma_s$. Our first task will be to show that $w \sim LMR$ where $L$ and $R^{-1}$ are words of the form $w(A, B, C)$ and $M$ is a word of the form $w(\pi, \tau)$. The calculations that do this are rather intricate and will be done in several steps. We will start with words in a specific subset of the generators. Then we will add generators one type at a time. The initial argument will be based on what we know about $2\hat{V}$. The remaining arguments will be detailed calculations based on the relations (11)–(27).

**Lemma 4.1.** (1) Let $w$ be of the form $w(A, B, \pi, A^{-1}, B^{-1})$. Then $w \sim LMR$ where $L$ and $R^{-1}$ are words of the form $w(A, B, C)$ and $M$ is of the form $w(\pi)$.

(2) Let $w$ be of the form $w(A, B, \pi)$. Then $w \sim LM$ where $L$ is a word of the form $w(A, B)$ and $M$ is of the form $w(\pi)$.

**Proof.** There is a homomorphism from $2\hat{V}$ to $G$ defined by $v_i \mapsto A_i$, $h_i \mapsto B_i$, $\sigma_i \mapsto \pi_i$. This is seen since the relations of $2\hat{V}$ correspond to the relations of $G$
according to the following table

\(\begin{align*}
1 & \rightarrow 11, \\
2 & \rightarrow 20, \\
3 & \rightarrow 22, \\
4 & \rightarrow 23, \\
5 & \rightarrow 12-14, \\
6 & \rightarrow 21.
\end{align*}\)

In the correspondence (5) with (12–14), we make use of the fact that

\[\pi_\varphi X_q = X_{q+1}\pi_\varphi \pi_{q+1}\] implies \(\pi_\varphi X_{q+1} = X_q\pi_{q+1}\pi_\varphi\) since the \(\pi_i\) are their own inverses in \(G\).

The statement (1) now follows from Lemma 3.1. The statement (2) follows from Lemma 2.3 and the corresponding monoid homomorphism from \(\Pi\) to \(G\). \(\square\)

We note that a direct proof of Lemma 4.1(1) from the relations seems rather complicated.

4.7. Subscript raising formulas. Because of the dependence of some relations on relative values of subscripts, it will be convenient to alter some subscripts. The next lemma allows a subscripted generator to be replaced by the same generator with a higher subscript at the expense of introducing words in the other generators.

**Lemma 4.2.** The following are consequences of the relations (11)–(27),

\[C_r \sim C_{r+1} B_r \pi_{r+1} A_r^{-1}\]
\[\pi_r \sim \pi_r \pi_{r+1} A_r^{-1}\]
\[\sim A_r \pi_{r+1} \pi_r\]

**Proof.** The first follows from \(C_m A_m \sim B_m C_{m+2} \pi_{m+1}\) and \(C_q B_m \sim B_m C_{q+1}\) when \(m < q\). The second and third follow from \(\pi_m A_m \sim \pi_m \pi_{m+1}\) and \(\pi^2 \sim 1\). \(\square\)

4.8. Interchanges. The basic tools for getting words into nicer form will be “reversals” of generators that are in the wrong order. If LMR form is desired, then the appearance of \(\pi_\varphi A_r\) in a word will an obstruction to getting this form. The resolution will depend on the relative values of \(q\) and \(r\). For example if \(r < q\), then we can replace the letters with \(A_r \pi_{q+1}\). However, if \(r = q\), then we get \(A_{q+1} \pi_q \pi_{q+1}\).

As can be seen, sometimes an interchange results in a word that is fairly complex. The above examples are quite simple and the interchanges get considerably worse. It is often more important to know the form that results from an interchange than the actual value of the word. Thus for example, we can write \(\pi_q C_r \sim C_w (A^{-1}, \pi, B)\) when \(r < q + 2\), rather than the more exact and complicated

\[\pi_q C_r \sim C_{q+2} \pi_q (B_{q+1} \pi_{q+2} A_{q+1}^{-1})(B_q \pi_{q+1} A_q^{-1}) \cdots (B_r \pi_{r+1} A_r^{-1})\]

The omission of the subscript of \(C\) on the right side of \(C\) in \(\pi_q C_r \sim C_w (A^{-1}, \pi, B)\) is deliberate since its exact value will not be important.

4.9. Interchange formulas. Below we give the formulas that we need to get a word into LMR form.

Our notation is best illustrated by example. In writing

\[B_q^{-1} A_r \sim \begin{cases} AB^{-1}, & r \neq q, \\
w(A)\pi w(B^{-1}), & r = q. \end{cases}\]

we say that the expression on the left can be replaced by the expressions on the right under the conditions stated. The subscripts are on the right are left unspecified as they will not be important.
We separate the formulas for moving the different generators to make them easier to refer to.

**Lemma 4.3.** The following are consequences of the relations (11)–(27) and are used to move positive powers of $A$ to the left. Their inverses can be used to move negative powers of $A$ to the right. In the last formula, the two words of form $w(\pi, A^{-1}, B^{-1})$ are not to be assumed identical.

$$\begin{align*}
A_q^{-1}A_r &\sim \begin{cases} 
AA^{-1}, & r \neq q, \\
1, & r = q.
\end{cases} \\
B_q^{-1}A_r &\sim \begin{cases} 
AB^{-1}, & r \neq q, \\
w(A)\pi w(B^{-1}), & r = q.
\end{cases} \\
C_q^{-1}A_r &\sim \begin{cases} 
AC^{-1}, & r < q, \\
w(A, \pi, B^{-1})C^{-1}, & r \geq q.
\end{cases} \\
\pi_qA_r &\sim Aw(\pi).
\end{align*}$$

$$\begin{align*}
\overline{\pi}_qA_r &\sim \begin{cases} 
A\overline{\pi}, & r < q, \\
\pi\overline{\pi}, & r = q, \\
w(A)\pi w(\pi), & r > q.
\end{cases}
\end{align*}$$

$$w(\pi, A^{-1}, B^{-1})A_r \sim w(A)w(\pi, A^{-1}, B^{-1}).$$

**Proof.** The last formula follows from Lemma 3.2 exactly as Lemma 4.1 follows from Lemma 3.1. For the rest, we will discuss the less simple instances and leave the others to the reader.

For $C_q^{-1}A_r$ with $r \geq q$, we use the inverse of the first line in Lemma 4.2 repeatedly to get

$$\begin{align*}
C_q^{-1}A_r &= (A_q\pi_{q+1}B_q^{-1})(A_{q+1}\pi_{q+2}B_{q+1}^{-1})\cdots (A_r\pi_{r+1}B_r^{-1})C_{r+1}^{-1}A_r \\
&= (A_q\pi_{q+1}B_q^{-1})(A_{q+1}\pi_{q+2}B_{q+1}^{-1})\cdots (A_r\pi_{r+1}B_r^{-1})A_rC_{r+2}^{-1}.
\end{align*}$$

For $\overline{\pi}_qA_r$ with $r > q$, we use the third line in Lemma 4.2 repeatedly to write

$$\begin{align*}
\overline{\pi}_qA_r &= A_qA_{q+1}A_{r-1}\overline{\pi}_r\pi_{r-1}\pi_{r-2}\cdots \pi_qA_r \\
&= A_qA_{q+1}A_{r-1}\overline{\pi}_r\pi_{r-1}A_r\pi_{r-2}\pi_{r-3}\cdots \pi_q \\
&= A_qA_{q+1}A_{r-1}\overline{\pi}_rA_{r-1}\pi_{r-1}\pi_{r-1}\pi_{r-2}\pi_{r-3}\cdots \pi_q \\
&= A_qA_{q+1}A_{r-1}A_{r-1}\overline{\pi}_r\pi_{r-1}\pi_{r-1}\pi_{r-2}\cdots \pi_q
\end{align*}$$

□

**Lemma 4.4.** The following are consequences of the relations (11)–(27) and are used to move positive powers of $B$ to the left. Their inverses can be used to move
negative powers of $B$ to the right.

\[
A_q^{-1} B_r \sim \begin{cases} 
& B A^{-1}, \\
& w(B) \pi w(A^{-1}), 
\end{cases} \quad r \neq q,
\]

\[
B_q^{-1} B_r \sim \begin{cases} 
& B B^{-1}, \\
& 1, 
\end{cases} \quad r = q.
\]

\[
C_q^{-1} B_r \sim \begin{cases} 
& B C^{-1}, \\
& w(A, \pi, B^{-1}) C^{-1}, 
\end{cases} \quad r < q.
\]

\[
\pi_q B_r \sim B w(\pi).
\]

\[
\pi_q B_r \sim \begin{cases} 
& B \pi, \\
& C \pi \pi, \\
& w(A) B \pi w(\pi), 
\end{cases} \quad r < q, \quad r = q, \quad r > q.
\]

**Proof.** The proof differs little from that of Lemma 4.3.

**Lemma 4.5.** The following are consequences of the relations (11)–(27) and are used to move positive powers of $C$ to the left. Their inverses can be used to move negative powers of $C$ to the right.

\[
A_q^{-1} C_r \sim \begin{cases} 
& C A^{-1}, \\
& w(A^{-1}, \pi, B), 
\end{cases} \quad q < r, \quad q \geq r.
\]

\[
B_q^{-1} C_r \sim \begin{cases} 
& C B^{-1}, \\
& w(A^{-1}, \pi, B), 
\end{cases} \quad q < r, \quad q \geq r.
\]

\[
C_q^{-1} C_r \sim \begin{cases} 
& 1 \\
& w(A, \pi, B^{-1}), 
\end{cases} \quad r = q, \quad r > q.
\]

\[
\pi_q C_r \sim \begin{cases} 
& C \pi, \\
& w(A^{-1}, \pi, B), 
\end{cases} \quad r > q + 1, \quad r \leq q + 1.
\]

\[
\pi_q C_r \sim \begin{cases} 
& B \pi \pi, \\
& w(A) B \pi w(\pi), \\
& w(B) C \pi \pi w(\pi, A^{-1}), 
\end{cases} \quad r = q + 1, \quad r > q + 1, \quad r < q + 1.
\]

**Proof.** The groups for $A_q^{-1} C_r$ and $B_q^{-1} C_r$ are inverses of cases covered in Lemmas 4.3 and 4.4. The first line for $C_q^{-1} C_r$ is handled much as in the proof of the case of $C_q^{-1} A_r$ in Lemma 4.3 and the third line for $C_q^{-1} C_r$ is the inverse of the first line. The second line of $\pi_q C_r$ is done by

\[
\pi_q C_r = \pi_q C_{q+2}(B_{q+1} \pi_{q+2} A^{-1}) (B_q \pi_q A^{-1}) \cdots (B_r \pi_r A^{-1})
\]

\[
= C_{q+2} \pi_q (B_{q+1} \pi_{q+2} A^{-1}) (B_q \pi_q A^{-1}) \cdots (B_r \pi_r A^{-1}).
\]

The second line for $\pi_q C_r$ is done by

\[
\pi_q C_r = A_q A_{q+1} \cdots A_{r-2} \pi_{r-1} \pi_{r-2} \pi_{r-3} \cdots \pi_q C_r
\]

\[
= A_q A_{q+1} \cdots A_{r-2} \pi_{r-1} \pi_{r-2} \pi_{r-3} \cdots \pi_q
\]

\[
= A_q A_{q+1} \cdots A_{r-2} B_{r-1} \pi_{r-1} \pi_{r-2} \pi_{r-3} \cdots \pi_q.
\]
The third line for $\pi_q C_r$ is the worst. As preparation, we write
\[
C_r = C_{q+1}(B_q \pi_{q+1} A_{q+1}^{-1})(B_{q-1} \pi_q A_{q-1}^{-1}) \cdots (B_r \pi_{r+1} A_{r+1}^{-1}) \\
= C_{q+1}(B_q B_{q-1} \cdots B_r)(\pi_{2q-r+1} A_{2q-r+1}^{-1})(\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1})
\]
which follows from the first line of Lemma 4.5 and from the relations 311-312.

Now we can write
\[
\pi_q C_r \\
= \pi_q C_{q+1}(B_q B_{q-1} \cdots B_r)(\pi_{2q-r+1} A_{2q-r+1}^{-1})(\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1}) \\
= B_q \pi_{q+1} B_q B_{q-1} \cdots B_r(\pi_{2q-r+1} A_{2q-r+1}^{-1})(\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1}) \\
= B_q \pi_{q+1} B_q B_{q-1} \cdots B_r \pi_{2q-r} A_{2q-r+1}^{-1} \\
(\pi_{2q-r+1} A_{2q-r+2}^{-1})(\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1}) \\
= B_q \pi_{q+1} B_q B_{q-1} \cdots B_r \pi_{2q-r} A_{2q-r+1}^{-1} \\
(\pi_{2q-r+1} A_{2q-r+2}^{-1})(\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1}) \\
= B_q B_{q-1} \cdots B_r C_{2q-r+2} \pi_{2q-r+1} \pi_{2q-r+2} \cdots (\pi_{2q-r+1} A_{2q-r+2}^{-1}) \cdots (\pi_{r+1} A_{r+1}^{-1}).
\]

4.10. The LMR form, Part II. We can now add $C$ and $C^{-1}$ to the list of generators that we can handle.

Lemma 4.6. Let $w$ be of the form $w(A, B, C, \pi, A^{-1}, B^{-1}, C^{-1})$. Then $w \sim LMR$ where $L$ and $R^{-1}$ are words of the form $w(A, B, C)$ and $M$ is of the form $w(\pi)$. Further the number of appearances of $C$ in $L$ will be no larger than the number of appearances of $C$ in $w$ and the number of appearances of $C^{-1}$ in $R$ will be no larger than the number of appearances of $C^{-1}$ in $w$.

Proof. Let $w$ be a word of form $w(A, B, C, \pi, A^{-1}, B^{-1}, C^{-1})$. We will deal in syllables of $w$. In this proof a syllable will be a maximal subword of $w$ of form $w(A, B, \pi, A^{-1}, B^{-1})$. Thus $w$ is an alternation of syllables and words of form $w(C, C^{-1})$. We will alter the word $w$ using the information in Lemma 4.5. At each stage, we can assume that each syllable is in the LMR form of Lemma 4.1.

All relations that we will use in this argument will not raise the number of appearances of $C$ and $C^{-1}$ and the last sentence of the lemma will follow from inspection the arguments.

Assume first that $w$ has a $C^{-1}$ that appears somewhere to the left of an appearance of $C$ in $w$. If these are adjacent, then the number of appearances of $C$ and $C^{-1}$ can be lowered by using the third group from Lemma 4.5. If there is no such adjacency, then there is a syllable with $C^{-1}$ on the left and $C$ on the right. Using the first, second and fourth groups from Lemma 4.5 the $C$ to the right can be moved over the maximal subword of form $w(\pi, A^{-1}, B^{-1})$ of the syllable at the expense of making the syllable to the right of the $C$ more complicated. Using the
inverses of the same groups from Lemma 4.6 we can move the $C^{-1}$ on the left over the remaining part of the syllable which now has the form $w(A, B)$ at the expense of making the syllable to the left of the $C^{-1}$ more complicated. Now the $C$ and $C^{-1}$ are adjacent and can be eliminated as before.

Thus we can assume that all appearances of $C$ in $w$ are to the left of all appearances of $C^{-1}$.

Let $p$ be the largest prefix of $w$ and let $s$ be the largest suffix of $w$ with $p$ and $s^{-1}$ both of form $w(A, B, C)$. We are done if we can get all appearances of $C$ in $p$ and all appearances of $C^{-1}$ in $s$.

Consider the leftmost appearances of $C$ that is not in $p$. It is separated from $p$ by a syllable. This syllable must be in LMR form as in Lemma 4.4. Further this syllable must also be of form $w(\pi, A^{-1}, B^{-1})$ since the $L$ part will be absorbed by $p$. As above, we use the inverses of groups one, two and four from Lemma 4.5 to move the $C$ past all letters in the syllable at the expense of making the syllable to the right of the $C$ more complex. Inductively we get all appearances of $C$ in $p$. The appearances of $C^{-1}$ are handled similarly.

\[ \square \]

4.11. **The LMR form, Part III.** We can now add $\pi$ to the list of generators that we can handle.

**Lemma 4.7.** Let $w$ be a word in $\Sigma_n$ of Section 4.6. Then $w \sim LMR$ where $L$ and $R^{-1}$ are words of the form $w(A, B, C)$ and $M$ is of the form $w(\pi, \bar{\pi})$.

**Proof.** We sketch the argument. We will only use relations that do not alter the number of appearances of the $\pi$ in a word. We will exploit the fact that the interchange rules of Lemma 4.3 are the least complex.

We are concerned with syllables that are maximal of the form $w(\pi, \bar{\pi})$. If there is more than one such syllable in a word $w$, then there are two $s_1$ and $s_2$ that are separated by a word in the LMR form of Lemma 4.6 giving a subword $s_1LMRs_2$. In $L$ we find generators $A$, $B$ and $C$.

Using the fourth and fifth groups from Lemmas 4.3 and 4.4 we can pass appearances of $A$ and $B$ from $L$ over a single appearance of $\pi$ in $s_1$ at the expense of introducing more complicated expressions to the left of the $\pi$ and copies of $\pi$ to the right of the $\pi$.

From the fourth and fifth groups from Lemma 4.6 we can move a copy of $C$ to the left at greater expense. Appearances of $A^{-1}$ and $B$ will be made to the right of the $\pi$ or $\bar{\pi}$ that is crossed over. When put in the LMR form of Lemma 4.1 we get copies of $A^{-1}$ that have to move to the right and copies of $B$ that have to move to the left. Using Lemma 4.6 and the inverses of the formulas in Lemma 4.3 we see that the copies of $A^{-1}$ can be migrated completely to the $R$ part of the altered $s_1LMRs_2$ making what is left of the $LM$ part more complicated, but without raising the number of appearances of $C$ that are left in the $L$ part.

Thus the appearances of $C$ in $L$ can be passed over each letter in $s_1$ as well as the (increasing number) of $A$ and $B$ generators between them. Eventually, $s_1LMRs_2$ is reduced to a word of the form $w(A, B, C)s_3R's_2$ where $s_3$ is the altered form of $s_1$.

Now we apply the inverses of what we have done to pass $R$ over $s_2$. This will result in the introduction of copies of $A$ which will have to pass over $s_3$. Eventually, $s_1LMRs_2$ is reduced to a word of the form $w(A, B, C)s_4w(A^{-1}, B^{-1}, C^{-1})$ where
s_4 is the combination of the altered form of s_3 and s_2. This reduces the number of syllables by one.

We now assume that our original word w is of the form psq where p and q are in the LMR form of Lemma 4.6. Thus w = LMRs'L'M'R' with the obvious comments. As before, we pass all of R over s and put the right side again in LMR form of Lemma 4.6. Now the new L' is passed to the left. All the while extra instances of A or A^{-1} that have to migrate “the other way” are handled as above. Eventually, we reach our goal. □

4.12. Improving L and R, Part I. We take the first step in getting the L and R parts of LMR in a more canonical form. In the following, we have to allow n = −1 since p might be the empty word. Similarly, we have to allow m = −1.

Lemma 4.8. Let w be a word in Σ_n of Section 4.7 and in addition, L = pq where p = C_{i_0}C_{i_1}\cdots C_{i_n} with n ≥ −1, with i_0 < i_1 < \cdots < i_n and q is a word of form w(A, B), and R^{-1} = p'q' where p' = C_{j_0}C_{j_1}\cdots C_{j_m} with m ≥ −1, with j_0 < j_1 < \cdots < j_m and q' is a word of form w(A, B).

Proof. Let L be as given by Lemma 4.7. Let p be the longest (possibly empty) prefix of L of the form L = C_{i_0}C_{i_1}\cdots C_{i_n} with i_0 < i_1 < \cdots < i_n and let r be the remainder of L in that L = pr. Let C_j be the leftmost appearance of the generator C in s. We have L in the correct form if there is none.

We write s = uC_jv. Using Lemma 4.2 we can raise the subscript of C_j as far as we like at the expense of introducing a word of form w(B, π, A^{-1}) before v. As in the proof of Lemma 4.7 we can move the appearances of A^{-1} past vM without raising the number of appearances of C in v.

Using the previous paragraph, we raise the subscript of C_j so that it is higher than i_n plus the maximum of all the subscripts in u plus the number of letters in u. Since u is a word of form w(A, B), we can use (13) to pass the altered C_j to the left of each letter in u, lowering the subscript of C by one with each application of (13). Our elevation of the subscript guarantees that (13) applies at each step of this passage and that the ending subscript will be higher than i_n. We have L in the right form by induction.

We now look at (LMR)^{-1} = R^{-1}M^{-1}L^{-1} and apply what we have done to R^{-1}. We get the right form for R^{-1} at the expense of adding a word of form w(A^{-1}) to the left of L^{-1}. This keeps the correct form for L. □

4.13. Structure from L. We will extract structure from L (and R^{-1}) assumed to be in the form from Lemma 4.8. We will do so inductively, so we will have to describe the structure before we prove it exists.

We will show that as an element in 2\hat{V}, the pair representing such an L will be in the form (t, v_k^t) where k is the length of t and t is of form w(v, h). Further, the word t will correspond to a forest whose only non-trivial tree T is the 0-th tree. Note that v_k^t also corresponds to a forest whose only non-trivial tree is the 0-th tree. We call the tree T, the tree corresponding to t.

With L, t, T and k as in the previous paragraph, L = (tv_0^j, v_0^{j+k}) also holds for any j ≥ 0. From our methods of building forests from an element of Π described in Section 2.9 and from the fact given in Lemma 2.7 that the leaf numbering of a forest corresponding to a word of form w(v, h) is the standard left-right numbering,
we know that appending \(v_0^j\) to the right of \(t\) just adds a caret to the leftmost leaf of the forest for \(t\) repeatedly \(j\) times. This is pictured below.

\[
\begin{array}{c}
\text{\(v_0^j\)}
\end{array} = \begin{array}{c}
\text{\(v_0\)}
\end{array}
\]

Thus the tree corresponding to \(tv_0^j\) is obtained from \(T\) by adding a caret with label \(v\) to the leftmost leaf of \(T\) exactly \(j\) times. We refer to this as an extension of \(T\) to the left. Of course, the same thing happens in the passage from \(v_0^k\) to \(v_0^{k+1}\).

4.14. The right-left leaf order. We saw in Section 2.8 how the leaf numbering and the letter subscripts cooperated in telling where the next caret is to be attached. We also saw in Lemma 2.7 that the leaf numbering of a forest for a word of form \(w(v, h)\) is the standard left-right numbering of the leaves.

We will discover that the left-right leaf numbering will not cooperate well with the subscripts of the letters in a word of form \(w(A, B, C)\). However, a right-left numbering does. Since all the information from a word of form \(w(A, B, C)\) is concentrated in a single tree, we will only discuss trees here. Later, we will extend the discussion to forests.

Given a tree \(T\), we will refer to two numberings of the leaves. If the tree has \(k\) cares (\(k\) internal vertices), it will have \(k+1\) leaves which can be numbered from 0 through \(k\). The left-right numbering and right-left numbering should be self descriptive, but we make sure by pointing out that the following is true of each vertex in \(T\) in the left-right numbering: all leaves below the left child are numbered less than all the leaves below the right child. For the right-left numbering, the phrase “less than” is replaced by “greater than.” Note that for each leaf of the tree, the numbers from the two numberings will add up to \(k\).

4.15. Extending the right-left leaf order. Let \(T\) be a tree corresponding to a word \(t\) of form \(w(v, h)\) that arises from an \(L\) from Lemma 1.8. An extension \(T'\) of \(T\) to the left corresponds to \(tv_0^j\) for some \(j \geq 0\). The extra cares that make \(T'\) from \(T\) are constantly added to the leftmost leaf. Thus the right-left leaf order on \(T\) carries over to those leaves that \(T\) and \(T'\) have in common. On \(T\), this consists of all leaves of \(T\) except the leftmost. This observation will be used repeatedly in what follows.

4.16. Building a tree from \(L\). From Lemma 2.8 we are motivated to study words such as \(C_{i_0}C_{i_1} \cdots C_{i_n}w(A, B)\) with \(i_0 < i_1 < \cdots < i_n\). Later we will be obliged to apply relations \((18)\) and \((21)\) to such words which will bring in elements of \(\{\pi_i \mid i \in \mathbb{N}\}\). With a little work and Lemma 2.11 we will be able to move appearances of \(\{\pi_i \mid i \in \mathbb{N}\}\) to the end of the words. This briefly justifies our concentration on the words that appear in the next few lemmas.

We start without any appearances of \(\pi_i\). Let \(L = C_{i_0}C_{i_1} \cdots C_{i_n}w(A, B)\) with \(i_0 < i_1 < \cdots < i_n\). Let \(l\) be the length of \(L\). For each \(j\) with \(0 \leq j \leq l\), let \(p_j\) be the prefix of \(L\) of length \(j\). We will show that \(L\) corresponds to an element of the form \((t, v_0^j)\) in \(2V\) and we want to describe \(t\) and the tree \(T\) corresponding to \(t\). We will do so inductively by describing these items for each \(p_j\) and how they are obtained from the corresponding items for \(p_{j-1}\). The prefix \(p_0\) is the empty string and its element of \(2V\) is \((v_0^0, v_0^0)\) and its tree is the trivial tree. We write \(p_j = (t_j, v_0^{k_j})\) and
the tree for \( t_j \) is \( T_j \). Note that for \( 1 \leq j \leq n+1 \), we have \( p_j = C_{i_0} \cdots C_{i_{j-1}} \). We now give the inductive lemmas.

**Lemma 4.9.** If we take the notation and assumptions of the previous paragraph and restrict \( j \) so that \( 1 \leq j \leq n+1 \), then we have

(a) \( k_j = i_{j-1} + 1 \),

(b) \( t_j = t_{j-1}v_0^dh_0 \) where \( d = i_{j-1} - i_{j-2} - 1 \), and

(c) \( T_j \) is obtained from \( T_{j-1} \) by attaching a caret labeled \( h \) to the leaf numbered \( i_{j-1} \) in the right-left leaf order in the smallest left extension of \( T_{j-1} \) that has a leaf numbered \( i_{j-1} \) in the right-left leaf order.

**Proof.** Item (a) follows from (b) by induction and item (c) follows from (b) directly. Thus we must show (b).

Let \( m = i_{j-1} \) and \( n = k_{j-1} \) for typographical reasons. We have \( m > i_{j-2} \) by assumption and \( i_{j-2} = n - 1 \) by induction, so \( m \geq n \). We set

\[ d = m - n = i_{j-1} - i_{j-2} - 1. \]

From Section 4.2 we have \( C_m = (v_0^n h_0, v_0^{m+1}) \). Now

\[
p_j = p_{j-1}C_{i_{j-1}} = p_{j-1}C_m = (t_{j-1}, v_0^n)(v_0^m h_0, v_0^{m+1})
\]

\[ = (t_{j-1}v_0^d h_0, v_0^m d h_0)(v_0^m h_0, v_0^{m+1}) \]

\[ = (t_{j-1}v_0^d h_0, v_0^m h_0)(v_0^m h_0, v_0^{m+1}) = (t_{j-1}v_0^d h_0, v_0^{m+1}) \]

which is what we needed to show. \( \square \)

**Lemma 4.10.** Take the notation and assumptions of the paragraph before Lemma 4.9 and restrict \( j \) so that \( n + 1 < j \leq l \). Let \( X_i \) be such that \( p_j = p_{j-1}X_i \) with \( X \) one of \( \{A, B\} \). Let \( x = v \) if \( X = A \) and \( x = h \) if \( X = B \). Let \( n = k_{j-1} \). Then

\[
p_j = p_{j-1}X_i = \begin{cases} 
(t_{j-1}v_0^{i+1-n}x_1, v_0^{i+1}), & n \leq i + 1, \\
(t_{j-1}x_{n-i}, v_0^{n+1}), & n > i + 1.
\end{cases}
\]

and \( T_j \) is obtained from \( T_{j-1} \) by attaching a caret labeled \( x \) to the leaf numbered \( i \) in the right-left leaf order in the smallest left extension of \( T_{j-1} \) that has at least \( i + 2 \) leaves.

**Proof.** From Section 4.2 we have \( X_i = (v_0^{i+1}x_1, v_0^{i+2}) \). If \( n \leq i + 1 \), then we have

\[
p_{j-1}X_i = (t_{j-1}, v_0^n)(v_0^{i+1}x_1, v_0^{i+2}) = (t_{j-1}v_0^{i+1-n}x_1, v_0^{i+1}x_1)(v_0^{i+1}x_1, v_0^{i+2}) = (t_{j-1}v_0^{i+1-n}x_1, v_0^{i+2}).
\]

If \( n > i + 1 \), then we have

\[
p_{j-1}X_i = (t_{j-1}, v_0^n)(v_0^{i+1}x_1, v_0^{i+2}) = (t_{j-1}, v_0^n)(v_0^{i+1}x_1v_0^{n-i-1}, v_0^{n+1}) = (t_{j-1}, v_0^n)(v_0^nx_{n-i}, v_0^{n+1}) = (t_{j-1}x_{n-i}, v_0^{n+1}).
\]
When \( n \leq i + 1 \), we are adding a caret with label \( x \) at leaf \( 1 \) in the left-right order to a tree with \( i + 1 \) carets and thus \( i + 2 \) leaves numbered from 0 through \( i + 1 \). Thus the addition is at leaf \( i \) in the right-left order and the tree is the smallest left extension of \( T_{j-1} \) that has at least \( i + 2 \) leaves. When \( n > i + 1 \), we are adding a caret with label \( x \) directly to \( T_{j-1} \) at the leaf numbered \( k_{j-1} - i \) in the left-right order, or the leaf numbered \( i \) in the right-left order. Note that in this case, the tree \( T_{j-1} \) already has at least \( i + 2 \) leaves.

We now add appearances of the \( \pi_i \), but don’t worry about the tree structure.

**Lemma 4.11.** Let \( p = (\pi_i) \) where \( t \) is a word of form \( w(v, h, \sigma) \) and \( n \) is the number of appearances of \( v \) and \( h \) in \( t \). Then \( \pi_i = (tv_0^{i+2}\sigma_{(n+1)\ldots-i},v_0^{i+2}) \) where \( j \) is the smallest value in \( \mathbb{N} \) so that \( (n+j-1)-i > 0 \). In particular, \( j = 0 \) if \( (n-1)-i > 0 \) (equivalently, \( n \geq i + 2 \)), and \( (n+j-1)-i = 1 \) if \( n < i + 2 \).

**Proof.** A calculation similar to that in Lemma 4.10 using \( \pi_i = (v_0^{i+2}\sigma_1,v_0^{i+2}) \) gives

\[
\pi_i = \begin{cases} 
(tv_0^{i+2-n}\sigma_1,v_0^{i+2}), & n < i + 2, \\
(t\sigma_{(n-1)\ldots-i},v_0^{i+2}), & n \geq i + 2.
\end{cases}
\]

The rest is straightforward.

**Lemma 4.12.** Let \( L = C_{i_0}C_{i_1}\ldots C_{i_n}w(A, B) \) with \( i_0 < i_1 < \ldots < i_n \), and let \( p = Lv \) with \( v \) a word in \( \{\pi_i \mid 0 \leq i \leq k\} \). Let \( r \in \mathbb{N} \) be such that \( n + r \geq k + 2 \). Then \( L = (t, v_0^i) \) where \( t \) is a word of form \( w(v, h) \) and \( n \) is the length of \( t \), and \( p = (tv_0^r,v_0^{i+n+r}) \) where \( s \) is a word of form \( w(\sigma) \).

**Proof.** The claim about \( L \) follows from Lemmas 4.9 and 4.10. The claim about \( p \) follows from Lemma 4.11. This is seen by noting that if we set \( L = (tv_0^r,v_0^{i+n+r}) \), then the only case that arises in applying Lemma 4.11 to each letter in \( v \) is the case in which the \( j \) of that lemma is equal to 0.

4.17. **The primary tree from \( L \).** We show how much flexibility there is in representations of the form \( (t, v_0^i) \) in \( \hat{2V} \).

**Lemma 4.13.** Let \( (t, v_0^i) = (s, v_0^j) \) in \( \hat{2V} \) where \( k \leq j \). Then \( s = tv_0^{j-k} \).

**Proof.** From the structure of \( \hat{2V} \) as a group of right fractions of \( \Pi \), there are \( p \) and \( q \) in \( \Pi \) so that \( tp, v_0^i \) is a pair of \( \{\pi_i \mid 0 \leq i \leq k\} \) as pairs, giving \( p = v_0^{j-k}q \) from the cancellativity of \( \Pi \). The claim follows from \( tv_0^{j-k}q = tp = sq \).

This immediately gives the following.

**Lemma 4.14.** Let \( L = C_{i_0}C_{i_1}\ldots C_{i_n}w(A, B) \) with \( i_0 < i_1 < \ldots < i_n \). Let \( k \) be the smallest in \( \mathbb{N} \) so that \( L = (t, v_0^i) \) as an element in \( \hat{2V} \) where \( t \) is of form \( w(v, h) \) and \( k \) is the length of \( t \). Let \( L = (s, v_0^j) \) where \( s \) is of form \( w(v, h) \) and \( j \) is the length of \( s \). Then \( s = tv_0^{j-k} \).

With \( L, k \) and \( t \) as in Lemma 4.14, we call the tree \( T \) corresponding to \( t \), the **primary tree for the word \( L \)**. If \( L = (s, v_0^i) \) is any other representation of \( L \) with \( s \) a word of form \( w(v, h) \) and \( j \) equal to the length of \( s \), then we know that the tree corresponding to \( s \) is an extension of \( T \) to the left. Since \( L = (tv_0^n,v_0^{i+n}) \) is a valid representation of \( L \) of the correct form for any \( n \in \mathbb{N} \), we know that all extensions
of $T$ to the left can show up in this way. We call such an extension with a secondary tree for $L$ even when $n = 0$. Thus the primary tree for $L$ is also a secondary tree for $L$.

4.18. The structure of the primary tree for $L$. We consider a word

$$L = C_{i_0}C_{i_1} \cdots C_{i_n}X_{i_{n+1}} \cdots X_{i_{l-1}}$$

where $i_0 < i_1 < \cdots i_n$ and where each $X$ comes separately from $\{A, B\}$. The primary tree for $L$ will be described as a particularly simple tree with a finite forest attached. The right-left leaf order will be used throughout. We must state how this order extends to forests, since we use one of two obvious choices and have to be explicit as to which.

The attached forest is finite in that it has finitely many trees. The right-left leaf order of a finite forest $F$ with trees $F_i$ with $0 \leq i \leq q$ has its leaves numbered consecutively starting from 0 with all leaves in $F_i$ numbered above those in $F_j$ whenever $i > j$ and the numbering of the leaves in any one $F_i$ following the right-left leaf order. This is best pictured if the forest is drawn with $F_0$ the rightmost tree and $F_q$ the leftmost. This is the reverse of the usual picture.

The particularly simple tree that our forest is attached to will correspond to the subword $C_{i_0}C_{i_1} \cdots C_{i_n}$ and will have the form pictured below.

(28)

The tree in (28) is the result of any word of the form $a_0a_1 \cdots a_n$ where each $a_i$ is from $\{v_0, h_0\}$. Note that the subscript $i$ of $a_i$ is not part of the symbol that $a_i$ represents. The tree in (28) will be referred to as a trunk.

If $\Lambda$ is a trunk with $m$ caret$e$s and $m + 1$ leaves with the leaves numbered from 0 through $m$ in the right-left order, and $F$ is a finite forest with $m$ trees, then we combine $\Lambda$ with $F$ to produce a tree $T$ by attaching each $F_i$ with $0 \leq i < m$ to the leaf in $\Lambda$ numbered $i$ in the right-left order. It is deliberate that we never use the leaf in $\Lambda$ that is numbered $m$ (the leftmost leaf).

The forest $F$ will be built from $X_{i_{n+1}} \cdots X_{i_{l-1}}$ much as forests are built from words in Section 2.9 with a few differences. As in Section 2.9, we build the forest caret by caret as we build the word from left to right letter by letter. The forests we build here will end up with the right-left ordering on the leaves. We start with the trivial forest and then for each $A_i$, we add a caret to leaf $i$ with label $v$, we keep the numbering on all leaves with number less than $i$, we increase by 1 the numbers of all leaves with number greater than $i$, we number the new left leaf $i + 1$, and we number the new right leaf $i$. The numbering of the new leaves is different from the scheme in Section 2.9. For each $B_i$, we do exactly the same thing, except the label of the new caret is $h$.

Lemma 4.15. Let $L = C_{i_0}C_{i_1} \cdots C_{i_n}X_{i_{n+1}} \cdots X_{i_{l-1}}$ where $i_0 < i_1 < \cdots i_n$ and where each $X$ comes separately from $\{A, B\}$. Let $m$ equal the maximum of

$$\{i_j + n + 2 - j \mid n + 1 \leq j \leq l - 1\} \cup \{i_n + 1\}.$$

Then $L$ can be represented as $L = (t, v^k)$ where $t$ is of form $w(v, h)$ and $k$ is the length of $t$, so that $k = m + l - n$, and so that the tree $T$ for $t$ is the primary tree for
of carets in $F$ case, the bottom caret in the trunk $\Lambda$ has a caret attached to one of its leaves. In Section 4.17, and the definition of extension to the left as found in Section 4.13.

**Proof.** We can discuss the tree $T$ as built from $L$ letter by letter because of Lemmas 4.14 and 4.15. The subtree $T'$ coming from the prefix $C_{i_0}C_{i_1}\ldots C_{i_n}$ that we get from Lemma 4.13 is as $\Lambda$ is described (including the labeling) in the statement above, except that the number of carets in $T'$ will only by $i_n + 1$. The trunk $\Lambda$ is an extension of $T'$ to the left so as to have $m$ carets and highest leaf number $m$.

To add $X_i$ to an existing tree, we need to create a left extension of the tree if the tree has no leaf numbered $i + 1$. Let us assume that we can build the tree from $\Lambda$ as described in the statement with no extra left extensions needed through $X_{i_n-1}$. The tree for $C_{i_0}C_{i_1}\ldots C_{i_n}X_{i_{n+1}}\ldots X_{i_{l-1}}$ will have highest leaf number equal to $m + j - (n + 1)$. The next letter to be treated will be $X_i$, and our hypothesis dictates that $m \geq i_j + n + 2 - j$ or $m + j - (n + 1) \geq i_j + 1$. Thus the caret for $X_i$, can be added without further extension. The forest $F$ is simply the tree $T$ for $L$ as built from Lemmas 4.14 and 4.15 with the trunk $\Lambda$ removed where the trunk $\Lambda$ consists of all carets reachable from the root by repeatedly going to the left child.

The number of carets in $T$ is $m$ plus the number of carets in $F$, and the number of carets in $F$ is $l - n$. The number of carets in $T$ must be the length of the word $t$, so $k = m + l - n$.

Note that either $m + j - (n + 1) = i_j + 1$ for some $j$ or $m = i_n + 1$. In the first case, the bottom caret in the trunk $\Lambda$ has a caret attached to one of its leaves. In the second case, the bottom caret of $\Lambda$ has label $h$. If the tree $T$ is not the primary tree for $L$, then it is an extension to the left of another tree by Lemma 4.15. This is impossible by the remarks we have just made.

Note that $m$ must come out to be at least 1 in Lemma 4.16. This applies even if $n = -1$ which happens if there is no appearance of any $C_i$ in $L$. Thus the trunk $\Lambda$ is never empty.

**Lemma 4.16.** Let $L = C_{i_0}C_{i_1}\ldots C_{i_n}w(A,B)$ with $i_0 < i_1 < \cdots < i_n$ and let $L = (s,v^k_b)$ where $s$ is a word of form $w(v,h)$ and $k$ is the length of $s$. Then the tree $T'$ for $s$ is a secondary tree for $L$. Further, $T'$ is an extension to the left of the primary tree $T$ for $L$ and has the same description as $T$ as given in Lemma 4.19 except that the trunk for $T'$ is an extension to the left of the trunk for $\Lambda$ in Lemma 4.19.

**Proof.** This follows immediately from Lemmas 4.14 and 4.16 the definitions in Section 4.17 and the definition of extension to the left as found in Section 4.18. □

4.19. **Improving $M$.** We now take care of the $M$ part of the $LMR$ form.

**Lemma 4.17.** Let $w$ be a word in $\Sigma_n$ of Section 4.3. Then $w \sim LMR$ as in Lemma 4.8 with $L = (s, v^m_w)$ and $R^{-1} = (t,v^n_w)$ so that $s$ and $t$ are words of form $w(v,h)$, $m$ and $n$ are, respectively, the lengths of $s$ and $t$, and so that there is a $p \geq \max\{m,n\}$ so that $M$ is a word in $\{\pi_{p-1}, \pi_i | i \leq p - 2\}$.

**Proof.** This is a direct consequence of Lemmas 4.7 and 4.11 of [5] and the definitions made in [5] just before those lemmas. The cited lemmas of [5] apply since they are
about a group called $BV$ in \[5\] presented by a generating set \( \{v_i, \pi_i, \pi_i | i \in \mathbb{N} \} \) and a set of relations given in Lemma 4.2 of \[5\] that are all seen to hold in our setting when the generators of \[5\] are mapped to generators of \( \Sigma \) under the mapping \( v_i \mapsto A_i, \pi_i \mapsto \pi_i \) and \( \pi_i \mapsto \pi_i \). The relations \( \pi_i^2 = 1 \) and \( \pi_i^2 = 1 \) of \( V \) are not relations of \( BV \) and as a consequence the relations of \( BV \) in \[5\] mention both positive and negative powers of the \( \pi_i \) and \( \pi_i \). However, these all reduce to relations in (11)–(27) because \( \pi_i^2 = 1 \) and \( \pi_i^2 = 1 \) are assumed here. The improvements of Lemmas 4.7 and 14.11 of \[5\] are gained at the expense of introducing (after the translation \( v \mapsto A \)) a word of form \( w(A) \) to the left of \( M \) and a word of form \( w(A^{-1}) \) to the right of \( M \). The control of the powers of \( v_0 \) (corresponding to \( \lambda_0 \) in \[5\]) carries over to our setting and does not disturb the forms of \( L \) and \( R^{-1} \) guaranteed by Lemma 4.8.

In the next two lemmas, it will be more convenient to express elements of \( \hat{2V} \) as \( PQ^{-1} \) with \( P \) and \( Q \) from \( \Pi \) rather than \( (P, Q) \).

The next lemma will be used in analyzing not only \( M \), but also \( L \) and \( R \). It is lifted from the proof of Proposition 4.13 of \[5\]. It gives properties about a certain translation function and its inverse. We need some definitions. Let \( M \) be a word in \( \{\pi_{p-1}, \pi_i | i \leq p-2\} \) written as

\[(29) \quad M = Y_1, Y_2 \cdots Y_q\]

where \( 0 \leq i_j \leq p-1 \) and

\[(30) \quad Y_{i_j} = \begin{cases} \pi_{i_j}, & i_j < p-1, \\ \pi_{p-1}, & i_j = p-1. \end{cases}\]

We let

\[(31) \quad \Psi_p(M) = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_q}\]

where \( k_j = (p-1) - i_j \) for \( 1 \leq j \leq q \). In the other direction, if \( u \) is a word in \( \{\sigma_i | i \leq p-1\} \) written as in the right side of (31), then \( \Psi_p^{-1}(u) \) is taken to be \( M \) as in (29) with letters interpreted by (30) with each \( i_j = (p-1) - k_j \) for \( 1 \leq j \leq q \). Note that \( \Psi_p \) and \( \Psi_p^{-1} \) are truly inverse to each other as transformations on words.

**Lemma 4.18.** Let \( M \) be a word in \( \{\pi_{p-1}, \pi_i | i \leq p-2\} \) and let \( u = \Psi_p(M) \). Then \( M = v_0^p w_0^{-p} \). Further if \( u \) can be taken to a word \( u' \) in \( \{\sigma_j | 0 \leq j \leq p-1\} \) by relations (23)–(4) from Lemma 4.4, then \( M \) can be taken to a word \( M' \) in \( \{\pi_{p-1}, \pi_i | i \leq p-2\} \) by relations (23)–(27) so that \( u' = \Psi_p(M') \).

**Proof.** That \( M = v_0^p w_0^{-p} \) follows from

\[(p-1) = (v_0^p \sigma_0, v_0^p) = v_0^p \sigma_0 v_0^{-p} = v_0^p \sigma_{(p-1) - (p-1)} v_0^{-p}\]

and

\[\pi_i = (v_0^{i+2} \sigma_1, v_0^{i+2}) = (v_0^{i+2} \sigma_1 v_0^{-i-2}, v_0^p) = (v_0^p \sigma_{(p-1) - i}, v_0^p) = v_0^p \sigma_{(p-1) - i} v_0^{-p}.\]

The last sentence of the lemma follows by noting that an application of (22)–(27) to \( M \) results in an application of (2)–(4) according to the following associatation

\[(26) \quad \text{or} \quad (27) \quad \leftrightarrow \quad (2)\]

\[(22) \quad \text{or} \quad (24) \quad \leftrightarrow \quad (3)\]

\[(23) \quad \text{or} \quad (26) \quad \leftrightarrow \quad (1).\]

and conversely. \( \square \)
Lemma 4.19. Let \( w \) be a word in \( \Sigma_n \) of Section 4.4. Then \( w \sim LMR \) as in Lemma 4.8 and further when \( L \), \( M \) and \( R^{-1} \) are expressed as elements of \( \hat{2V} \), they are expressible as \( L = sv_0 p \), \( R^{-1} = tv_o p \) and \( M = v_0 p w v_0 p \) where \( s \) and \( t \) are words of form \( w(v, h) \), \( u \) is a word in \( \{ \sigma_j \mid 0 \leq j \leq p - 1 \} \), and the lengths of \( s \) and \( t \) are both \( p \). Further, if \( u \) can be reduced to the trivial word using relations (3.3) from Lemma 2.4 then \( M \) can be reduced to the trivial word using relations (28)–(27).

Proof. If we take \( w \sim LMR \) as given by Lemma 4.17 then we let \( L \) be represented by \( (sv_o p n, v_0 p) \) and \( R^{-1} \) be represented by \( (tv_o p n, v_0 p) \). The rest follows from Lemma 4.18.

\( \square \)

4.20. Normalizing the trees from \( L \) and \( R^{-1} \). As pointed out in the proof of Lemma 4.14 the assignments \( v_i \mapsto A_i \), \( h_i \mapsto B_i \), \( \sigma_i \mapsto \pi_i \) extend to a group homomorphism from \( \hat{2V} \) to \( G \) and a monoid homomorphism from \( \Pi \) to \( G \). The forest \( F \) built in Section 4.18 from a word of form \( w(A, B) \) is the mirror image of the forest that would have been built from the corresponding word of form \( w(u, h) \) built from \( w(A, B) \) by replacing each \( A \) by \( v \) and each \( B \) by \( h \). We will use these facts to improve the word \( L \) even more.

Let \( L = C_{i_0} C_{i_1} \cdots C_{i_n} X_{i_{n+1}} \cdots X_{i_{n+7}} \) where \( i_0 < i_1 < \cdots < i_n \) and where \( X \) comes separately from \( \{A, B\} \). Let \( T \) be a secondary tree for the word \( L \) as defined in Section 4.17. The labeling on the tree lets us build a numbered pattern in \( S_0 \) from \( T \) and we can use this pattern to add secondary labels to \( T \) as in Section 2.4.8.

We can then define when \( T \) is normalized exactly as is done in that section.

The purpose of this section is to prove the following.

Proposition 4.20. Given the notation, hypotheses and conclusion as expressed in Lemma 4.12, we can further assume that the trees for \( s \) and \( t \) are normalized.

We will build a reductive proof of Proposition 4.20 from two lemmas. The lemmas will be applied alternately to reduce the pattern of non-normalized vertices. Thus the hypotheses of each lemma will be designed to match the conclusions of the other. This partly explains their rather strange statements. In the next lemma, the double appearance of \( n \) is deliberate.

Lemma 4.21. Let \( L = C_{i_0} C_{i_1} \cdots C_{i_n} u \) and \( L' = C_{k_0} C_{k_1} \cdots C_{k_n} u' \) where \( i_0 < i_1 < \cdots < i_n \), where \( k_0 < k_1 < \cdots < k_n \), where \( u \) is a word of form \( w(A, B) \), and where \( u' \) is a word of form \( w(A, B, \pi) \). Assume that \( L \) is expressible as \((t, v_0)\) as an element of \( \hat{2V} \) with \( t \) a word of form \( w(v, h) \) and \( p \) is the length of \( t \). Let \( m \) be the number of carets of the trunk of \( T \) and assume that \( m \geq k_n + 1 \).

If \( L \sim L' \), then there is a word \( u'' \) of form \( w(A, B) \), and there is a word \( z \) in \( \{\pi_i \mid i \leq p - 2\} \) so that setting \( L_1 = C_{k_0} C_{k_1} \cdots C_{k_n} u'' \) and \( L_2 = L_1 z' \) gives that \( L \sim L_2 \) and \( L_1 \) is expressible as \((t', v_0')\) with \( y \) a word of form \( w(A, B) \) of length \( p \) so that the tree \( T' \) for \( t' \) is normalized except possibly at interior vertices in the trunk of the tree, and so that the trunk of \( T' \) has \( m \) carets.

Proof. The homomorphism from \( \hat{2V} \) to \( G \) defined by \( v_i \mapsto A_i \), \( h_i \mapsto B_i \), \( \sigma_i \mapsto \pi_i \) allows us to write \( u' \sim u''z' \) where \( u'' \) is a word of form \( w(A, B) \), where \( z' \) is a word in \( \{\pi_i \mid i \in N\} \), and where the forest \( F \) for \( u'' \) as built in Section 4.18 is normalized. Since the primary tree for \( L_1 = C_{k_0} C_{k_1} \cdots C_{k_n} u'' \) is a trunk with \( F \) attached, we have satisfied the normalization requirements. The rest of the argument is devoted to improving \( z' \) and understanding the structure of trees associated to \( L_1 \).
From Lemma 4.12 we know that \( L_1 = (\hat{t}, v_0^q) \) where \( \hat{t} \) is a word of form \( w(v, h) \) and \( q \) is the length of \( \hat{t} \), and that \( L_2 = (\hat{t}v_0^r s, v_0^{q+r}) \) where \( r \geq 0 \) and where \( s \) is a word of form \( w(\sigma) \). It is seen from Lemma 4.11 that \( s \) is equal to \( \Psi_{q+r}(z') \). Since Lemma 4.12 allows \( r \) to be any sufficiently large value, we cover all cases by saying that there is a \( k \in \mathbb{N} \) so that \( p + k = q + r \) and

\[
L = (tv_0^k, v_0^{p+k}) = (\hat{t}v_0^r s, v_0^{q+r}) = L_2
\]
as elements of \( 2\hat{V} \). Thus the numbered patterns in the unit square represented by \( tv_0^k \) and \( \hat{t}v_0^r s \) are identical.

Since \( s \) is a word of form \( w(\sigma) \), the only difference between the numbered pattern for \( \hat{t}v_0^r s \) and \( \hat{t}v_0^r s \) is in the numbering. The unnumbered patterns for \( \hat{t}v_0^r s \) and \( \hat{t}v_0^r s \) are identical. Thus the unnumbered patterns for \( tv_0^k \) and \( \hat{t}v_0^r s \) are identical.

We consider the vertices of a tree that are reachable from the root by always going to the left. We call these the “left edge vertices.” In the tree for \( tv_0^k \) let the left edge vertices be \( a_0, a_1, \ldots, a_b \) reading from the top. This makes \( a_0 \) the root of the tree and \( a_b \) the only non-interior vertex among them. Thus all \( a_i \) with \( 0 \leq i < b \) have labels. Since the trunk for \( T \) the tree for \( t \) has \( m \) carets, we know that \( b = m + k \). We also know that for \( m \leq i < b \) the label for \( a_i \) is \( v \) and that the right child of \( a_i \) is a leaf.

In the tree for \( \hat{t}v_0^r s \), let the left edge vertices be \( a'_0, a'_1, \ldots, a'_c \) reading from the top. Since \( m \geq k_n + 1 \), we know from Lemma 4.15 that the label for \( a'_i \) is \( v \) for \( m \leq i < c \).

The following fact is elementary from the description in Section 2.8 of how vertices in a labeled tree correspond to rectangles in the unit square:

(*) The rectangle corresponding to a left edge vertex depends only on two numbers; the number of left edge vertices above it and the number of left edge vertices above it with label \( h \).

From (*), from Lemma 4.15 and from the fact that there are \( n + 1 \) appearances of a \( C \) in both \( L \) and \( L' \), we know that the rectangle \( R \) corresponding to \( a_m \) is identical to the rectangle corresponding to \( a'_m \). Since \( R \) is divided \( k \) times vertically according to the carets below \( a_m \) in the tree for \( tv_0^k \), it must be divided in exactly the same way by the tree for \( \hat{t}v_0^r s \). Thus the tree below \( a'_m \) in the tree for \( \hat{t}v_0^r s \) must consist of an extension to the left by \( k \) carets all labeled \( v \). From this we know that \( r \geq k \).

If we give the left-right leaf numbering to the trees for \( tv_0^k \) and \( \hat{t}v_0^r s \) (this is the leaf numbering that works with the subscripts of the letters in \( \{v, h\} \)), then the carets below \( a_m \) and \( a'_m \) appear as follows in both of these trees.

\[
(32)
\]

The last \( k \) carets in \( v_0^{p+k} = v_0^{q+r} \) with the left-right ordering are also as pictured in (32). Since the numbered patterns for \( tv_0^k \) and \( \hat{t}v_0^r s \) are identical, the permutation
given by \( s \) must be trivial on \( \{0, 1, \ldots, k\} \). Thus there is a way to modify \( s \) using the relations (20)–(24) from Lemma 2.22 to give an \( s' \) that is a word in \( \{\sigma_i \mid i \geq k+1\} \).

Since there are \( p + k = q + r \) carets in the relevant trees, there are \( p + k + 1 = q + r + 1 \) leaves numbered from 0 through \( p + k = q + r \). From Lemma 4.18 there is a way to modify \( z' \) using the relations (22)–(27) to give a word \( z \) so that \( \Psi_{q+r}(z) = s' \).

Since there are no appearances of \( \pi \) in \( z' \), there will be none in \( z \).

Now we use the fact that the only subscripts of the \( \sigma \) in \( s' \) are above \( k \) to conclude that all the subscripts of the \( \pi \) in \( z \) are below \( (q+r-1)-k = (p+k-1)-k = p-1 \). Thus \( z \) is a word in \( \{\pi_i \mid 0 \leq i \leq p-2\} \). This is what was wanted.

We now turn our attention to trees for \( L_1 \). We know that the tree for \( \hat{v}_0^r \) below \( a_m' \) is pictured as in (22). From Lemma 2.22 we know that \( \hat{v}_0^r \) can be rewritten using the relations of \( \Pi \) to end in \( k \) appearances of \( v_0 \). Thus we may assume that \( j \geq k \). In the word \( \hat{v}_0^r s' \) that corresponds to \( L_1 z \), we know that \( s' \) is a word in \( \{\pi_i \mid i \geq k+1\} \). Thus the last \( k \) appearances of \( v_0 \) in \( \hat{v}_0^r \) can be moved to the right of \( s' \) using relations (3) at the expense of lowering each subscript in \( s' \) by \( k \). This changes \( s' \) to a word \( s'' \) and changes \( \hat{v}_0^r s' \) to \( \hat{v}_0^{r-k} s'' v_0^p \). Now we have

\[
L_1 z = (\hat{v}_0^r s, v_0^{q+r}) = (\hat{v}_0^r s', v_0^{p+k}) = (\hat{v}_0^{r-k} s'' v_0^p, v_0^p)
\]

and \( L_1 \) can be represented by \((\hat{v}_0^{r-k}, v_0^p)\). Since \( \hat{v} \) has \( q \) letters, and \( p + k = q + r \), the word \( \hat{v}_0^{r-k} \) has \( p \) letters. The trunk of the tree for \( \hat{v}_0^{r-k} \) ends at vertex \( a_m' \) and so has \( m \) carets. Thus setting \( t' = \hat{v}_0^{r-k} \) completes the proof.

The next lemma attacks non-normalized vertices in the trunk of a tree. We need a notion of complexity to measure progress. In the intermediate stages of the argument, it is extra work to define the non-normalized vertices, so we use a different measure of progress. The lemma will change the locations of labels, so we will focus on the labels. It turns out that only finitely many such changes can be done, so this will be sufficient.

If \( T \) is a labeled tree, then we let \( a_0, a_1, \ldots, a_n \) be the interior, left edge vertices of \( T \) reading from top to bottom so that \( a_0 \) is the root of \( T \). We let \( b_0 b_1 \cdots b_n \) be a word in \( \{0, 1\} \) defined so that \( b_0 = 0 \) if \( a_i \) is labeled \( v \) and \( b_1 = 1 \) if \( a_i \) is labeled \( h \). We call \( b_0 b_1 \cdots b_n \) the complexity of \( T \). If \( w_1 \) and \( w_2 \) are two such words, then we say \( w_1 < w_2 \) if \( w_1 \) is shorter than \( w_2 \) or if \( w_1 \) and \( w_2 \) are the same length and \( w_1 \) represents a smaller binary number than \( w_2 \). Note that this gives the label of the root the most significant position. The complexity will not be mentioned directly in the lemma, but will be mentioned in its application. However, the statement is easier to understand if the complexity is kept in mind.

**Lemma 4.22.** Let \( L = C_{i_0} C_{i_1} \cdots C_{i_n} u \) where \( i_0 < i_1 < \cdots < i_n \) and \( u \) is a word of form \( w(A, B) \). Assume that the primary tree \( T \) for \( L \) is normalized except at one or more vertices in the trunk of \( T \). Let \( m \) be the number of carets in the trunk of \( T \). Then \( L \sim L' = C_{k_0} C_{k_1} \cdots C_{k_n} u' \) where \( k_0 < k_1 < \cdots < k_n \), and \( u' \) is a word of form \( w(A, B, \pi) \), so that \( m \geq k_n + 1 \), and so that the smallest \( j \) so that \( i_j \neq k_j \) has \( i_j < k_j \).

**Proof.** Let \( \Lambda \) be the trunk of \( T \). The interior vertices of \( \Lambda \) are the interior, left edge vertices of \( T \) and let these be \( a_0, a_1, \ldots, a_{m-1} \). Let \( r \) be the highest value with \( 0 \leq r < m \) for which \( a_r \) is not normalized. Note that this is the lowest non-normalized interior vertex of \( \Lambda \).
It follows that $a_r$ has label $h$ and that its children are both interior vertices of $T$ with label $v$. Because the left child of $a_r$ is an interior vertex, we must have $r < m - 1$. The vertex $a_r$ must correspond to some $C_{i_j}$ in $L$ and from Lemma 4.15 we have $i_j = r$.

Represent $T$ as $\Lambda$ with a forest $F$ attached. Since the left child of $a_r$ is in $\Lambda$ and has label $v$, we know that if $j < n$, then $i_{j+1} > r + 1 = i_j + 1$. Since the right child of $a_r$ is an interior vertex, there is a letter in $u$ corresponding to it. This letter must be some $A_q$. Also, the right child of $a_r$ is a root of $F$. By Lemma 2.9 we can assume that $A_q$ occurs as the first letter of $u$. Thus we are looking at a word

$$C_{i_0}C_{i_1} \cdots C_{i_j}C_{i_{j+1}} \cdots C_{i_n}A_qu''$$

where $u''$ is the remainder of $u$ after $A_q$. The subword $C_{i_0}C_{i_1} \cdots C_{i_j}C_{i_{j+1}} \cdots C_{i_n}A_q$ of (33) is a trunk with a single caret labeled $v$ attached at caret $i_j$ of the trunk on its right child. From the details of the right-left numbering, this implies that $q = i_j$. Thus (33) reads as

$$C_{i_0}C_{i_1} \cdots C_{i_j}C_{i_{j+1}} \cdots C_{i_n}A_ju''.$$

Since $i_0 < i_1 < \cdots < i_n$ and $i_{j+1} > i_j + 1$, we can use relations (15) to rewrite (34) as

$$C_{i_0}C_{i_1} \cdots C_{i_j}A_i C_{i_{j+1}} \cdots C_{i_n}u''.$$

Combining relations (14) and (18) we get $C_mA_m = C_{m+1}B_m\pi_{m+1}$ so (35) becomes

$$C_{i_0}C_{i_1} \cdots C_{i_j+1}B_i\pi_{i_j+1}C_{i_{j+1}} \cdots C_{i_n}u''.$$

Now $i_{j+1} + 1 > i_j + 2$, so relations (20) and (18) allow us to rewrite (36) as

$$C_{i_0}C_{i_1} \cdots C_{i_j+1}B_i\pi_{i_j+1}C_{i_{j+1}} \cdots C_{i_n}u''.$$

If we now set $B_i\pi_{i_j+1}u'' = u'$ in (37), then (37) clearly satisfies all provisions of the lemma except possibly $m \geq k_n + 1$. However, we observed above that $i_j = r < m - 1$ so $i_j + 2 \leq m$. This gives what we want in the case that $j = n$. If $j < n$, then $k_n = i_n$ and $m \geq i_n + 1$ by Lemma 4.13.

**Proof of Proposition 4.20.** We start with a word $w$ in $\Sigma_\pi$ of Section 4.5 and assume $w = LMR$ as specified in both Lemmas 4.19 and 4.21. We work first on $L$. We have $L$ expressible as $(t, v_0^p)$ with $t$ a word in $w(v, h)$ of length $p$ and with the trunk of the tree $T$ for $t$ having $m$ carets.

We apply Lemma 4.21 by letting $L' = L$ in the hypothesis of that lemma. This gives $L_1$ and $L_2 = L_1z$ with $L \sim L_2$, with $z$ a word in $\{\pi_i | i \leq p - 2\}$. Also $L_1$ is expressible as $(t', v_0^p)$ with $t'$ a word of form $w(v, h)$ of length $p$, with the trunk of the tree $T'$ for $t'$ of length $m$ and with $T'$ normalized off the trunk. Since we set $L' = L$, we see that the trunks of $T$ and $T'$ are identical. Since the word $z$ is in $\{\pi_i | i \leq p - 2\}$, it can be absorbed into $M$ without disrupting our assumptions on $M$. We now replace $L$ with $L_1$ and proceed.

We now apply Lemma 4.22 to get $L \sim L'$ as specified in that lemma and then apply Lemma 4.21 to $L$ and $L'$. This gives $L \sim L_2z$ as above. The word $z$ can be added to $M$. The complexity of the tree $T'$ as given in Lemma 4.21 is dictated by the subword $C_{i_0}C_{i_1} \cdots C_{i_n}$ mentioned in that lemma. But this is the sequence with the same notation from Lemma 4.22 which gives a complexity that is strictly less than that for the tree $T$ associated to $L$. Since there are finitely many complexities and
they are linearly ordered, the process of repeatedly applying Lemma 4.22 followed by Lemma 4.21 must stop. At that point, the associated tree will be normalized.

To normalize the tree for $R$, we apply this process to the inverse of $LMR$. □

4.21. An assumption of triviality. We now look at our simplification of a word in the generators of $2V$ under the assumption that the word represents the trivial element. We will use the fact that this is also the trivial element in $2\tilde{V}$.

Lemma 4.23. Let $w$ be a word in $\Sigma_s$ of Section 4.5 that represents the trivial element of $2V$. Then $w \sim 1$.

Proof. Express $w \sim LMR$ as in Lemma 4.19 so that

$$LMR = (sv_0^{-p})(w_0^{-p})(tv_0^{-p})^{-1} = sut^{-1}$$

where $s$ and $t$ are of form $w(v, h)$ and $u$ is of form $w(\sigma)$. By Proposition 4.20, we can assume that the trees for $t$ and $s$ are normalized.

Since $sut^{-1} = (su, t)$ is the trivial element of $2\tilde{V}$, we know that the elements $su$ and $t$ are the same elements of $\Pi$ and represent the same numbered patterns. Since $s$ and $t$ are words of form $w(v, h)$ and $u$ is a word in $\{\sigma \mid 0 \leq j \leq p - 1\}$, the words $s$ and $t$ must give the same unnumbered pattern, and $u$ must simply renumber the numbered pattern for $s$ to give the numbered pattern $su$.

The forests for $s$ and $t$ are trivial after the first trees and so are normalized since the first trees are normalized. Since the forests for $s$ and $t$ are normalized and lead to the same patterns, the forests are identical by Lemma 2.10. Since $s$ and $t$ are words of form $w(v, h)$, the numbering on the leaves is the left-right order and so $s$ and $t$ represent the same numbered patterns. Thus $u$ affects the trivial permutation on the numbering. By Lemma 4.19, we know $M \sim 1$.

It remains to show that $L \sim R^{-1}$.

The trunks of the trees for $L$ and $R$ are identical. We have

$$L = C_{i_0}C_{i_1} \cdots C_{i_n} w(A, B)$$

and

$$R^{-1} = C_{i_0}C_{i_1} \cdots C_{i_m} w'(A, B)$$

where $i_0 < i_1 < \cdots < i_n$ and $k_0 < k_1 < \cdots k_m$. Since the sequences $(i_0, i_1, \ldots, i_n)$ and $(k_0, k_1, \ldots, k_m)$ are determined by the labeling of the trunks of the trees for $L$ and $R^{-1}$, they are identical as sequences. What remains to be shown is that $w(A, B) \sim w'(A, B)$.

The numbered, labeled forests $F$ and $F'$ for $w(A, B)$ and $w'(A, B)$, respectively, that obtained by removing the trunks of the trees for $s$ and $t$ are mirror images of the numbered, labeled forests built from $w(A, B)$ and $w'(A, B)$ by replacing each $A$ by $v$ and each $B$ by $h$. Let these words be, respectively, $w(v, h)$ and $w'(v, h)$. Since the labeled forests $F$ and $F'$ are equal, we know from Lemma 2.8 that $w(v, h)$ and $w'(v, h)$ are related by relations (11–15). However, these relations are preserved under the homomorphism from $\Pi$ to $G$ under the assignment $v \mapsto A$ and $h \mapsto B$. Thus $w(A, B) \sim w'(A, B)$ and the proof is done. □

4.22. The presentation. Lemma 4.23 immediately gives the following.

Theorem 3. The group $2V$ is presented with the generators of Section 4.3 and the relations (11–24).
5. Finite presentations

We give finite presentations for $2\hat{V}$ and $2V$. We proceed by showing that the relations that we have established for $2\hat{V}$ and $2V$ are consequences of finitely many of those relations. The techniques for doing this form a sort of machine that would take about as long to describe as to use. Thus we do not make a theory out of it. It is not clear that such a theory is needed.

5.1. A finite presentation for $2\hat{V}$. We have an infinite presentation from Theorem 2. First we cut down the generating set $\{v_i, h_i, \sigma_i \mid i \in \mathbb{N}\}$. The relations (1) and (5) when $i < j$ give $v_i^{-1}x_jv_i = x_{j+1}$ when $x$ is any of $v$, $h$ or $\sigma$. This allows us to use

\begin{align*}
v_i &= v_0^{1-i}v_1v_0^{i-1}, \\
h_i &= v_0^{1-i}h_1v_0^{i-1}, \\
\sigma_i &= v_0^{1-i}\sigma_1v_0^{i-1},
\end{align*}

as definitions for all $i \geq 2$. Thus $2\hat{V}$ is generated by $\{v_i, h_i, \sigma_i \mid i \in \{0, 1\}\}$.

The relations (1)--(6) break into two classes: the relations (1), (3), and (5) whose subscripts incorporate two parameters $i$ and $j$, and the relations (2), (4), and (6) whose subscripts incorporate only the one parameter $i$. We treat these two classes separately.

Conjugating the relations (2), (4), and (6) for $i = 1$ by powers of $v_0$ shows that the relations (2), (4), and (6) for $i \geq 2$ follow from (2), (4), and (6) for $i = 1$. Thus (2), (4), and (6) are all consequences of the six relations obtained when $i$ is set to 0 or 1 in (2), (4), and (6).

We consider (1). If we rewrite (1) when $x$ and $y$ are both $v$ as

\begin{equation}
\tag{38}
v_i^{-1}v_{i+k}v_i = v_{i+k+1}, \quad \text{for all } k > 0,
\end{equation}

then this is known to be true for $i = 0$ by definition. If (38) is known for $i = 1$ and a set of values of $k > 0$, then it is known for all $i$ and that same set of $k$ values by conjugating the known expressions with $i = 1$ by powers of $v_0$. If (38) is known for $i = 1$ and $1 \leq k \leq j$ with $j \geq 2$, then the calculation

\begin{align*}
v_1^{-1}v_{1+j+1}v_1 &= v_1^{-1}v_{j+2}v_1 = v_1^{-1}v_2^{-1}v_{j+1}v_2v_1 = v_3^{-1}v_{j+2}v_3 = v_{j+3}
\end{align*}

is the inductive step that lets us conclude (38) for $i = 1$ and all $k > 0$ if we know (38) for $i = 1$ and $k \in \{1, 2\}$. Thus (38) for all needed $i$ and $k$ follow from

\begin{align*}
v_1^{-1}v_2v_1 &= v_3, \quad \text{and} \quad v_1^{-1}v_3v_1 = v_4.
\end{align*}

If we now look at (1) when $x = v$ and $y = h$, then we still have $v_i^{-1}h_{i+k}v_i = v_{i+k+1}$ by definition when $i = 0$. Arguments similar to those in the previous paragraph show that all of (1) for $x = v$ and $y = h$ follow from

\begin{align*}
v_1^{-1}h_2v_1 &= h_3, \quad \text{and} \quad v_1^{-1}h_3v_1 = h_4.
\end{align*}

When $x = h$ in (1), we no longer have definitions to help with $i = 0$. Repetitions of the arguments above show that the remaining cases of (1) follow from the following eight relations:

\begin{align*}
h_0^{-1}v_1h_0 &= v_2, \quad h_1^{-1}v_2h_1 = v_3, \quad h_0^{-1}h_1h_0 = h_2, \quad h_1^{-1}h_2h_1 = h_3, \\
h_0^{-1}v_2h_0 &= v_3, \quad h_1^{-1}v_3h_1 = v_4, \quad h_0^{-1}h_2h_0 = h_3, \quad h_1^{-1}h_3h_1 = h_4.
\end{align*}
The next set to consider is (15). We give the discussion and leave the results to be summarized after. The relations (15) break into four smaller sets depending on the relative sizes of \( i \) and \( j \). For \( i = j \) and \( i = j + 1 \), we are dealing with a single parameter, and we are in the same situation as (2), (4), and (6). Further, the fact that the \( \sigma_i \) are their own inverses immediately shows that the cases \( i = j \) and \( i = j + 1 \) give equivalent relations. When \( i < j \), we are in a situation “isomorphic” to some of the cases in (11). When \( i > j + 1 \), we are to prove that \( \sigma_i \) or \( h_i \) commutes with \( \sigma_j \). When \( i \geq j + 4 \), the calculation

\[
\sigma_j^{-1} \sigma_i \sigma_j = \sigma_j^{-1} \sigma_j = v_{i-2} v_{i-1} v_i = v_i
\]

gives the required inductive step.

Lastly, when \( i \geq j + 4 \), the inductive step

\[
\sigma_j^{-1} \sigma_i \sigma_j = \sigma_j^{-1} \sigma_j = v_{i-2} \sigma_{i-1} v_i = v_{i-2} = \sigma_i
\]

does the job for (15).

Summarizing all of the above gives the following.

**Theorem 4.** The group \( \overline{2V} \) is presented by the generating set \( \{ \sigma_i, h_i, \sigma_j | i \in \{0, 1 \} \} \) and the following set of 40 relations:

- \( v_2 v_1 = v_1 v_3, \ v_3 v_1 = v_1 v_4, \ h_2 v_1 = v_1 h_3, \ h_3 v_1 = v_1 h_4, \)
- \( v_1 h_0 = h_0 v_2, \ v_2 h_0 = h_0 v_3, \ v_2 h_1 = h_1 v_3, \ v_3 h_1 = h_1 v_4, \)
- \( h_1 h_0 = h_0 h_2, \ h_2 h_0 = h_0 h_3, \ h_2 h_1 = h_1 h_3, \ h_3 h_1 = h_1 h_4, \)
- \( \sigma_0 v_2 = v_2 \sigma_0, \ \sigma_0 v_3 = v_3 \sigma_0, \ \sigma_1 v_3 = v_3 \sigma_1, \ \sigma_0 v_4 = v_4 \sigma_0, \)
- \( \sigma_0 h_2 = h_2 \sigma_0, \ \sigma_0 h_3 = h_3 \sigma_0, \ \sigma_1 h_3 = h_3 \sigma_1, \ \sigma_0 h_4 = h_4 \sigma_0, \)
- \( \sigma_0 v_0 = v_1 \sigma_0 \sigma_1, \ \sigma_1 v_1 = v_2 \sigma_1 \sigma_2, \ \sigma_0 h_0 = h_1 \sigma_0 \sigma_1, \ \sigma_1 h_1 = h_2 \sigma_1 \sigma_2, \)
- \( \sigma_0 h_0 = h_0 \sigma_2, \ \sigma_2 h_0 = h_0 \sigma_3, \ \sigma_2 h_1 = h_1 \sigma_3, \ \sigma_3 h_1 = h_1 \sigma_4, \)
- \( \sigma_2 v_1 = v_1 \sigma_3, \ \sigma_3 v_1 = v_1 \sigma_4, \ \sigma_4 = 1, \ \sigma_5 = 1, \)
- \( \sigma_1 \sigma_2 = \sigma_2 \sigma_0, \ \sigma_0 \sigma_3 = \sigma_3 \sigma_0, \ \sigma_1 \sigma_3 = \sigma_3 \sigma_1, \ \sigma_1 \sigma_4 = \sigma_4 \sigma_1, \)
- \( \sigma_0 \sigma_1 \sigma_0 = \sigma_1 \sigma_0 \sigma_1, \ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \ \sigma_0 h_0 v_0 = h_0 v_1 \sigma_0, \ \sigma_1 v h_1 h_0 = h_0 v_1 v_0 \sigma_0, \)

5.2. A finite presentation for \( 2V \). As in \( \overline{2V} \) we get the following as definitions

- \( A_i = A_0^{1-i} A_1 A_0^{i-1} \),
- \( B_i = A_0^{1-i} B_1 A_0^{i-1} \),
- \( \pi_i = A_0^{1-i} \pi_1 A_0^{i-1} \),
- \( \pi_i = A_0^{1-i} \pi A_1 A_0^{i-1} \),
- \( C_m = (\pi_m B_m \pi_m + 1) (B_m \pi_m + 1 A_m^{-1}) \)

for \( i \geq 2 \) and \( m \geq 0 \). The last line was mentioned at the end of Section 3.3 and is easily shown.

Because of the homomorphism from \( \overline{2V} \to 2V \) determined by \( v \mapsto A, h \mapsto B, \sigma \mapsto \pi \), all of the facts that we know about relations (11)–(27) apply to the relations (11)–(15), (21)–(25), and (26) as mentioned in Lemma 4.1. Thus these relations reduce to the image under the homomorphism \( \overline{2V} \to 2V \) of the relations in Theorem 4. If the remaining relations in the list (11)–(27) are treated in a manner similar to that of Theorem 4, the following is proven.
Theorem 5. The group $2V$ is presented by the generating set \{ $A_i, B_i, \pi_i, \bar{\pi}_i \mid i \in \{0,1\}$\} the 40 relations obtained from the relations in Theorem 4 under the transformation $v \mapsto A$, $h \mapsto B$, $\sigma \mapsto \pi$ and the 30 relations below:

\[
\begin{align*}
\pi_2 A_1 &= A_1 \pi_3, & \pi_3 A_1 &= A_1 \pi_4, & \pi_1 B_0 &= B_0 \pi_2, & \pi_2 B_0 &= B_0 \pi_3, \\
\pi_2 B_1 &= B_1 \pi_3, & \pi_3 B_1 &= B_1 \pi_4, & \pi_0 A_0 &= \pi_0 \pi_1, & \pi_1 A_1 &= \pi_1 \pi_2, \\
\pi_0 B_0 &= C_1 \pi_0 \pi_1, & \pi_1 B_1 &= C_2 \pi_1 \pi_2, & C_2 A_1 &= A_1 C_3, & C_1 A_1 &= A_1 C_4, \\
C_1 B_0 &= B_0 C_2, & C_2 B_0 &= B_0 C_3, & C_2 B_1 &= B_1 C_3, & C_3 B_1 &= B_1 C_4, \\
C_0 A_0 &= B_0 C_2 \pi_1, & C_1 A_1 &= B_1 C_3 \pi_2, & \pi_0^{2} &= 1, & \pi_1^{2} &= 1, \\
\pi_0 C_2 &= C_2 \pi_0, & \pi_0 C_3 &= C_3 \pi_0, & \pi_1 C_3 &= C_3 \pi_1, & \pi_1 C_4 &= C_4 \pi_1, \\
\pi_0 \pi_2 &= \pi_2 \pi_0, & \pi_0 \pi_3 &= \pi_3 \pi_0, & \pi_1 \pi_3 &= \pi_3 \pi_1, & \pi_1 \pi_4 &= \pi_4 \pi_1, \\
\pi_0 \pi_1 \pi_0 &= \pi_1 \pi_0 \pi_1, & \pi_1 \pi_2 \pi_1 &= \pi_2 \pi_1 \pi_2.
\end{align*}
\]

6. Normal forms

The group $2V$ and $V$ resemble each other greatly, but differ in one important aspect. Both $2V$ and $V$ live in larger groups of fractions, respectively $\hat{2V}$ and $\hat{V}$, which in some ways are better behaved than $2V$ and $V$. However, $\hat{V}$ has a property not possessed by $2V$.

Elements of $\hat{V}$ have a nice normal form when regarded as pairs. The positive monoid $P$ for $\hat{V}$ has a nice length function, and a given element of $\hat{V}$ has a nice class of pairs $(Y, Z)$ that represent it that is characterized by the fact that they are minimal in length in the monoid $P$. In particular, if $(Y, Z)$ and $(Y', Z')$ are in this class, then $Y$ and $Y'$ (and also $Z$ and $Z'$) differ by an invertible element (a permutation) in $P$. This gives $\hat{V}$ a nice semi-normal form that can be turned into a normal form by an easy shift of point of view. See Theorem 2 in Section 6.3 of [4].

The remarks above about $\hat{V}$ follow from a property of $P$ called least common left multiples in [4]. A common left multiple of two elements $Y$ and $Z$ in $P$ is an element $L = AY = BZ$ with $A$ and $B$ in $P$. If $L$ and $L'$ are two left multiples of $Y$ and $Z$, then we write $L \leq L'$ if there is a $C$ in $P$ so that $L' = CL$. A least common left multiple of $Y$ and $Z$ is a common left multiple of $Y$ and $Z$ that is least in this order. A monoid had least common left multiples if every pair in the monoid with a common left multiple has a least common left multiple. The definition is worded so that a monoid can have least common left multiples even if not every pair has a common left multiple.

The semi-normal form discussed above for elements of $\hat{V}$ comes directly from the fact that the positive monoid $P$ of $\hat{V}$ has least common left multiples. The monoid $\Pi$ of this paper does not. The culprit is relation (6). We leave it as an exercise to show that while $v_0$ and $h_0 \sigma_1$ have common left multiples

\[(h_0 v_1 \sigma_2) v_0 = h_0 v_1 v_0 \sigma_3 = v_0 h_1 h_0 \sigma_1 \sigma_3 = (v_0 h_1 \sigma_2)(h_0 \sigma_1)\]

and

\[(h_0 v_1) v_0 = (v_0 h_1)(h_0 \sigma_1),\]

there is no least common left multiple for $v_0$ and $h_0 \sigma_1$. From this it follows that the element

\[(h_0 v_1 \sigma_2, h_0 v_1) = (v_0 h_1 \sigma_2, v_0 h_1)\]
of $\hat{2V}$ has no unique “minimal” representative. This accounts for the arbitrary choice made (a normalized vertex with a secondary label must have label $v$) in the definition of a normalized forest.

It would be nice to know if $\hat{2V}$ and $2V$ are of type $F_\infty$ (have classifying spaces that are finite in each dimension). It seems at the moment that the absence of the least common left multiples property will make the question harder.

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