Flow curvature in a self-coupled FitzHugh-Nagumo model

M Balabaev

1Mathematics and Computer Science Faculty, Nayanova Academy, Molodogvardeyskaya 196, Samara, Russia, 443001

e-mail: m.o.balabaev@gmail.com

Abstract. In this paper we consider a self-coupled FitzHugh-Nagumo model describes a prototype of an excitable system (e.g., a neuron). The aim is to construct the expansion of invariant manifold with variable stability and corresponding gluing function using flow curvature method. We show that a self-coupled FitzHugh-Nagumo model has sufficiently smooth invariant surface, called "black swan". Computer simulation and computer algebra methods are used for the quantitative analysis of the model.

1. Introduction

Let us consider an autonomous dynamical system:

\[
\begin{cases}
\dot{x} = f(x, y, \varepsilon), \\
\dot{y} = \frac{1}{\varepsilon}g(x, y, \varepsilon),
\end{cases}
\]

where \( x \) and \( y \) are functions of time, \( \varepsilon > 0 \) is a small parameter, and the dot refers to differentiation with respect to time \( t \). Assume that \( f : \mathbb{R}^{n+m} \to \mathbb{R}^n \) and \( g : \mathbb{R}^{n+m} \to \mathbb{R}^m \) are sufficiently smooth.

A slow surface of the system is the manifold described by equation

\[ g(x, y, 0) = 0. \]

The domain of the slow surface called stable if the spectrum of the Jacobian matrix

\[ \frac{\partial}{\partial y} g(x, y, 0) \]

is located in the left open complex half-plane. If there is at least one eigenvalue of the Jacobian matrix with a positive real part then the domain is called unstable. It is clear that the slow surface consists of stable and unstable domains.

A set of points of the slow surface is said to be the breakdown manifold if it separates stable and unstable domains. It follows in the standard way that breakdown manifold of the system is given by

\[
\begin{cases}
g(x, y, 0) = 0, \\
\frac{\partial}{\partial y} g(x, y, 0) = 0.
\end{cases}
\]
Let \( S \subseteq \mathbb{R}^{n+m} \) be a smooth surface; then the \( S \) is said to be the invariant manifold if any trajectory of the system that has at least one point in common with \( S \) lies on \( S \) entirely.

Now, suppose that we have an additional scalar parameter \( \mu \) in the differential system:

\[
\begin{align*}
\dot{x} &= f(x, y, \mu, \varepsilon), \\
\dot{y} &= \frac{1}{\varepsilon} g(x, y, \mu, \varepsilon),
\end{align*}
\]

(1)

Then, in some special cases [1, ch.8], we can glue the stable and unstable domains at one point of the breakdown manifold.

A trajectory of the singularly perturbed system (1) is called canard if at first moves along the stable slow invariant manifold and then continue for a while along the unstable slow invariant manifolds at one point of the breakdown surface. This approach was first proposed in [2, 3] and then applied in [1, 4–15]. The term ”canard” (or duck-trajectory) was originally introduced by French mathematicians [16–18].

Let \( \mu = \mu(x, y) \) be given as a control function. Then we can glue the stable and unstable domains of slow invariant manifold at all points of breakdown surface at the same time. As a result we obtain the continuous invariant surface. Such surfaces are said to be black swans [1, 19]. This approach was first proposed in [20] and was then applied in [7, 15, 21, 22].

In the framework of this paper, we consider the black swan as a multidimensional analog of a canard, called duck surface [6, 9–15, 17, 23]. In this case it is possible to regard the gluing function as a some special kind of partial feedback control function. This can guarantee the permanency of various processes, even with perturbations.

### 2. Flow curvature method

Consider an autonomous dynamical system:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, \varepsilon), \\
\dot{x}_2 &= f_2(x_1, \ldots, x_n, \varepsilon), \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, \varepsilon),
\end{align*}
\]

(2)

where \( x_1, \ldots, x_n \) are functions of time, \( \varepsilon \) is a small positive parameter, and the dot refers to differentiation with respect to time \( t \). Assume that functions \( f_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) are sufficiently smooth for all \( i \).

Let \( V \in \mathbb{R}^n \) be the velocity vector of the point \( P = (x_1, \ldots, x_n) \). According [24, p.168] the solution to autonomous system (2) would be associated with the corresponding coordinates of a point \( P \) at the instant \( t \).

So we have

\[
\dot{P} = V(P),
\]

where

\[
V(P) = \{f_1, \ldots, f_n\}.
\]

Let a smooth function \( \varphi(P) : \mathbb{R}^n \rightarrow \mathbb{R} \) be given by

\[
\varphi(P) = \left| V \wedge \dot{V} \wedge \cdots \wedge V^{(n-1)} \right|,
\]
where the wedge symbol inside discriminant refers to exterior product.

Then the manifold $\Omega \subset \mathbb{R}^n$ is called a flow curvature manifold associated with system (2) if we have $\varphi(P) = 0$ for all $P \in \Omega$. This term was applied in [15, 25, 26].

It is readily seen that

$$
\varphi(P) = \begin{vmatrix}
V_1 & \ldots & V_n \\
\dot{V}_1 & \ldots & \dot{V}_n \\
\vdots & \ddots & \vdots \\
(n-1) & \ldots & (n) \\
V_1 & \ldots & V_n
\end{vmatrix}
\begin{vmatrix}
\dot{P}_1 & \ldots & \dot{P}_n \\
\ddots & \ddots & \ddots \\
\dot{P}_1 & \ldots & \dot{P}_n
\end{vmatrix}
\begin{vmatrix}
\begin{vmatrix}
\dot{f}_1 & \ldots & \dot{f}_n \\
\ddots & \ddots & \ddots \\
\dot{f}_1 & \ldots & \dot{f}_n
\end{vmatrix}
\begin{vmatrix}
f_1 & \ldots & f_n \\
\ddots & \ddots & \ddots \\
f_1 & \ldots & f_n
\end{vmatrix}
\end{vmatrix}
$$

Therefore, yields

$$
\varphi(P) = \begin{vmatrix}
\dot{f}_1 & \ldots & \dot{f}_n \\
\ddots & \ddots & \ddots \\
\dot{f}_1 & \ldots & \dot{f}_n
\end{vmatrix}
= \begin{vmatrix}
\varphi(P)
\end{vmatrix}
$$

(3)

J. G. Darboux proved[27] that the flow curvature manifold is invariant manifold of the system (2). Since the Lie derivative $L_X\varphi = 0$, we have that $\varphi$ is the first integral of (2) [28]. According these we obtain that $\varphi(P)$ is globally invariant. Since the one, it follows that

$$
d\varphi \equiv 0,
$$

thus we have

$$
\frac{\partial\varphi}{\partial x_1} dx_1 + \frac{\partial\varphi}{\partial x_2} dx_2 + \ldots + \frac{\partial\varphi}{\partial x_n} dx_n + \frac{\partial\varphi}{\partial \varepsilon} d\varepsilon \equiv 0.
$$

(4)

It is the base equation of the flow curvature method. We will return to the one later.

3. FitzHugh-Nagumo model

The FitzHugh-Nagumo model describes a prototype of an excitable system. The model is a simplified version of the Hodgkin-Huxley model which models in a detailed manner activation and deactivation dynamics of a spiking neuron.

Following [29, 30], a self-coupled FitzHugh-Nagumo model has the form:

$$
\begin{cases}
\dot{v} = h - \frac{v^3 - v + 1}{2} - \gamma sv, \\
\dot{h} = -\varepsilon \left(2h + \frac{13}{5}v\right), \\
\dot{s} = -\varepsilon \delta s,
\end{cases}
$$

where $v, h, s \in \mathbb{R}$ and $0 < \varepsilon \ll 1$. The variable $v$ represent membrane potential, $h$ — inactivation of ionic channels, and $s$ is synaptic coupling. Parameter $\gamma$ is coupling strength, $\delta$ is inactivation of the synapse. In the framework of this paper we restrict the consideration to the domain $v > 1/\sqrt{3}$.

Changing variables, we obtain

$$
\begin{cases}
\dot{h} = -2h - \frac{13}{5}v, \\
\dot{s} = -\delta s, \\
\dot{v} = \frac{1}{\varepsilon} \left(h - \frac{v^3 - v + 1}{2} - sv\gamma\right),
\end{cases}
$$

(5)
It is obvious that a slow surface of system (5) is the manifold described by equation
\[ h = \frac{1 - v + v^3 + 2sv\gamma}{2}. \]

Let us remember that the slow surface consists of stable and unstable domains. By definition, a set of points of the slow surface is said to be the breakdown manifold if it separates stable and unstable domains. It follows in the standard way that breakdown manifold of the model is given by
\[
\begin{cases}
1 - 3v^2 - 2\gamma = 0, \\
h - v^3 - v + 1 - sv\gamma = 0,
\end{cases}
\]

therefore, we have breakdown curve of the model (5):
\[
\begin{cases}
h = \frac{1 - 2v^3}{2}, \\
s = \frac{1 - 3v^2}{2\gamma}.
\end{cases}
\]

Obviously, breakdown curve is embedded in slow surface (fig. 1).

Figure 1. Slow surface and breakdown curve of the model with \( \varepsilon = 0.05, \gamma = -1.05 \).

Let \( h = h(s, v, \varepsilon) \) be a solution of system (5). Suppose that expansion of the one has the form:
\[
h(s, v, \delta, \varepsilon) = h_0(s, v, \delta) + h_1(s, v, \delta) \varepsilon + O(\varepsilon^2). \quad (6)
\]

Let \( \delta = \delta(v) \) be given as a control function. Our goal is to find \( \delta(v) \) and expansion (6) such as all singularities are avoided.

Flow curvature method will be used. It follows from (3) that model (5) give us
\[
\varphi(h, s, v, \varepsilon) = \begin{vmatrix}
\dot{h} & \dot{s} & \dot{v} \\
\ddot{h} & \ddot{s} & \ddot{v} \\
\dddot{h} & \dddot{s} & \dddot{v}
\end{vmatrix}.
\]
We omit the expression of determinant for simplicity. All the calculations can be found in the Appendix.

From (4), we get the following:

\[
\frac{\partial \psi}{\partial h} \, dh + \frac{\partial \psi}{\partial s} \, ds + \frac{\partial \psi}{\partial s} \, ds + \frac{\partial \psi}{\partial \varepsilon} \, d\varepsilon \equiv 0.
\]

If we combine this with

\[
dh = \frac{\partial h}{\partial s} \, ds + \frac{\partial h}{\partial v} \, dv + \frac{\partial h}{\partial \varepsilon} \, d\varepsilon,
\]

we get

\[
\left( \frac{\partial \psi}{\partial h} \cdot \frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s} \right) \, ds + \left( \frac{\partial \psi}{\partial h} \cdot \frac{\partial h}{\partial v} + \frac{\partial \psi}{\partial v} \right) \, dv + \left( \frac{\partial \psi}{\partial \varepsilon} \cdot \frac{\partial \psi}{\partial \varepsilon} \right) \, d\varepsilon \equiv 0.
\]

It can be shown in the usual way [25, ch.11] that

\[
\begin{align*}
\frac{\partial \psi}{\partial h} \cdot \frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s} &= 0, \\
\frac{\partial \psi}{\partial h} \cdot \frac{\partial h}{\partial v} + \frac{\partial \psi}{\partial v} &= 0, \\
\frac{\partial \psi}{\partial h} \cdot \frac{\partial h}{\partial \varepsilon} + \frac{\partial \psi}{\partial \varepsilon} &= 0,
\end{align*}
\]

whence

\[
\frac{\partial h}{\partial \varepsilon} = -\frac{\partial \psi}{\partial \varepsilon} / \frac{\partial \psi}{\partial h}.
\]

Substituting (6) in both sides, we obtain

\[
h_1(s, v) + O(\varepsilon) = -\frac{\varphi_\varepsilon \left( h_0(s, v) + h_1(s, v) \varepsilon, s, v, \varepsilon \right)}{\varphi_h \left( h_0(s, v) + h_1(s, v) \varepsilon, s, v, \varepsilon \right)}.
\]

Evidently, setting \( \varepsilon \to 0 \) we obtain equation in \( h_1 \):

\[
h_1(s, v) = -\frac{\varphi_\varepsilon \left( h_0(s, v), s, v, 0 \right)}{\varphi_h \left( h_0(s, v), s, v, 0 \right)}.
\]  \hspace{1cm} (7)

Zero-order approximation is trivial and continuous:

\[
h_0(s, v) = \frac{1 - v + v^3 + 2sv\gamma}{2}.
\]  \hspace{1cm} (8)

If we replace \( h_0 \) by the one in (7), we get (Cf. Appendix for details)

\[
h_1(s, v) = \frac{15h_1(s, v) \left( 2s\gamma + 3v^2 - 1 \right) + 4 \left( 5 + 8v + 5v^3 + 10sv\gamma - 5sv\gamma\delta \right)}{5 \left( 2s\gamma + 3v^2 - 1 \right)}.
\]  \hspace{1cm} (9)
It follows easily that
\[ h_1(s,v) = -\frac{10 + 16v + 10v^3 + 20sv\gamma - 10sv\gamma\delta}{10\gamma s + 15v^2 - 5}. \]

It can trivially be checked that \( h_1(s,v) \) is continuous function whenever
\[ \delta(v) = \frac{20v^3 - 26v - 10}{15v^2 - 5v}. \] (10)

Finally, we obtain
\[ h_1(s,v) = \frac{10 + 16v + 10v^3}{5 - 15v^2}. \] (11)

Combining (8), (11) and (6) we get first-order smooth invariant manifold of the model (5).

Substituting (10) for \( \delta \) in (5), we obtain a new model
\[
\begin{align*}
\dot{h} &= -2h - \frac{13}{5}v, \\
\dot{s} &= \frac{20v^3s - 26vs - 10s}{5v - 15v^2}, \\
\dot{v} &= \frac{1}{\varepsilon} \left( h - \frac{v^3 - v + 1}{2} - sv\gamma \right).
\end{align*}
\]

There exists a nontrivial smooth solution of the new model. Projections of the one of it are demonstrated in (fig. 2).

\[ \text{Figure 2.} \text{ Breakdown curve (red) and projection of the canard (blue) with } \varepsilon = 0.05, \gamma = -1.05, \text{ and initial point } h = -0.1, s = 1.8, v = 1.6. \]

4. Conclusion

The result is the first-order approximation to the black swan surface (fig. 3) described by equation
\[ h(s,v) = \frac{1 - v + v^3 + 2sv\gamma}{2} + \frac{10 + 16v + 10v^3}{5 - 15v^2} \varepsilon, \]

and the corresponding gluing function is
\[ \delta(v) = \frac{20v^3 - 26v - 10}{15v^2 - 5v}. \]

Both of it are continuous functions in the domain \( v > 1/\sqrt{3} \).
Figure 3. Black swan (yellow), slow surface (green), breakdown curve (red), and canard of the model (blue) with $\varepsilon = 0.05$, $\gamma = -1.05$ and initial point $h = -0.1$, $s = 1.8$, $v = 1.6$.

Appendix

First of all, recall that

$$
\varphi(h, s, v, \varepsilon) = \begin{vmatrix}
\dot{h} & \dot{s} & \dot{v} \\
\ddot{h} & \ddot{s} & \ddot{v} \\
\dddot{h} & \dddot{s} & \dddot{v}
\end{vmatrix} = -\begin{vmatrix}
2h + \frac{13\varepsilon}{5} & s\delta & -2h + 2s\varepsilon v^2 + s^2 - 1 \\
a_{21} & s\delta^2 & a_{23} \\
a_{31} & -s\delta^3 & a_{33}
\end{vmatrix},
$$

where

$$a_{21} = \frac{1}{10\varepsilon} \left( 40\varepsilon h - 26h + 26\gamma sv + 13v^3 + 52\varepsilon v - 13v + 13 \right),$$

$$a_{31} = \frac{1}{20\varepsilon^2} \left( -40\varepsilon h - 20\gamma hs - 30h v^2 + 10h + 20\gamma s^2 v + 10\gamma s + 40\gamma sv^3 + 20\gamma\varepsilon sv - 20\gamma sv + 15v^5 - 20v^3 + 15v^2 + 52\varepsilon v + 5v - 5 \right),$$

$$a_{23} = \frac{1}{100\varepsilon^2} \left( -260\varepsilon - 800\varepsilon^2 h + 1040\varepsilon h + 260\gamma hs + 390hv^2 - 130h - 260\gamma^2 s^2 v - 130\gamma s - 520\gamma sv^3 - 260\gamma \varepsilon sv - 520\gamma \varepsilon sv + 260\gamma lv - 195v^5 - 260\varepsilon v^3 + 260v^3 - 195v^2 - 1040\varepsilon^2 v + 936\varepsilon v - 65v + 65 \right).$$
\[ \alpha_{33} = \frac{1}{40 \varepsilon} \left( 52 \varepsilon - 120 h^2 v + 160 \varepsilon^2 h - 144 \varepsilon h + 40 \gamma^2 h s^2 + 80 \gamma \varepsilon h s + 
+ 80 \gamma \varepsilon h - 40 h s + 360 \gamma h s^2 v + 210 h v^4 + 120 \varepsilon^2 h^2 - 180 h v^2 + 120 h v + 
+ 10 h - 40 \gamma^3 \varepsilon^3 v - 20 \gamma^2 s^2 - 60 \gamma^2 s^2 v^2 - 120 \gamma^2 \varepsilon^2 s^2 v + 60 \gamma^2 \varepsilon^2 v - 40 \gamma \varepsilon s + 
+ 20 \gamma - 270 \gamma \varepsilon^5 - 100 \gamma \varepsilon s v^3 + 260 \gamma s v^3 - 180 \gamma s v^2 - 40 \gamma^2 \varepsilon^2 s v + 
+ 60 \gamma \varepsilon s v + 208 \gamma \varepsilon s v - 30 \gamma s v - 75 v^7 + 135 v^5 - 105 v^4 + 
+ 208 v^3 - 65 v^3 + 90 v^2 + 208 \varepsilon^2 v - 104 \varepsilon v - 25 v - 5 \right). \]

Expanding this we obtain

\[ \varphi(h, s, \varepsilon) = -\frac{1000}{\varepsilon^6} \left( 975 s \delta^2 v^4 + 10 - 3900 s \delta^2 v^3 + 7800 s \delta v^8 - 2925 s \delta^2 v^8 + 
+ 5850 s^2 \delta^2 v^8 - 3750 s \delta^2 v^7 + 7500 h s \delta^2 v^7 - 5850 h s \delta^2 v^7 + 2925 s \delta^2 v^7 + 
+ 1300 s \delta^2 v^6 + 1300 s \delta^2 v^6 - 7800 s \delta^2 v^6 + 6500 s \delta^2 v^6 - 13650 s \delta^2 v^6 - 
- 9620 s \delta v^6 + 260000 s^2 \gamma \delta^2 v^6 + 11700 s^2 \gamma \delta^2 v^6 + 2925 s \delta v^6 + 11700 s^2 \gamma \delta^2 v^6 + 
+ 1500 h s \delta v^5 - 6000 h s \delta v^5 + 16500 h s \delta^2 v^5 - 4875 s \delta^2 v^5 - 13500 s \gamma \delta^2 v^5 - 
- 29100 h s \delta v^5 + 9750 s ^2 \delta^3 v^5 + 270000 s^2 \gamma \delta^3 v^5 + 11700 h s \delta v^5 - 5850 s \delta^3 v^5 - 
- 23400 h s^2 \gamma \delta^2 v^5 + 11700 s^2 \gamma \delta^2 v^5 + 26000 s^2 \gamma \delta^2 v^5 + 5200 s \delta^2 v^5 + 13000 s \delta^3 v^4 - 
- 13000 \gamma \delta^3 v^4 + 75400 s^2 \delta^3 v^4 + 15600 s^2 \gamma \delta^2 v^4 - 16640 s \delta^3 v^4 - 
- 208000 s^2 \gamma \delta^2 v^4 - 104000 s^2 \gamma \delta^2 v^4 + 10500 h s \delta^3 v^4 - 5250 h s \delta^3 v^4 - 
- 2600 s \delta^3 v^4 + 104000 s^2 \gamma \delta^2 v^4 + 208000 s^2 \gamma \delta^2 v^4 - 21000 h s \delta^3 v^4 + 
+ 10500 h s \delta^3 v^4 + 1820 s \delta^3 v^4 - 14040 s^2 \gamma \delta^3 v^4 + 78000 s^2 \gamma \delta^2 v^4 - 
- 117000 s^2 \gamma \delta^2 v^4 + 11700 h s \delta^2 v^4 + 11700 h s \delta^2 v^4 + 1950 s \delta^2 v^4 + 
+ 5850 s \delta^2 v^4 - 200000 h s \delta^3 v^4 + 400000 h s \delta^3 v^4 - 330000 h s \delta^3 v^4 + 
+ 65000 s^2 \delta^4 v^3 - 1000 h s \delta^3 v^4 + 100000 h s \delta^2 v^4 + 26000 s^2 \delta^2 v^4 + 
+ 100000 h s \delta^2 v^4 - 1280000 h s \delta^3 v^4 - 7800 s \delta^4 v^3 - 16000 h s \delta^4 v^3 - 
- 13000 h s^2 \gamma \delta^3 v^3 - 97500 h s \delta^2 v^3 + 32500 h s \delta^2 v^3 + 247000 h s \gamma \delta^3 v^3 - 
- 5850 s \gamma^2 \delta^2 v^3 + 260000 h s \gamma \delta^3 v^3 + 10920 h s \delta^3 v^3 - 4810 s \delta^3 v^3 - 
- 41600 h s^2 \gamma \delta^3 v^3 + 130000 s^2 \gamma \delta^3 v^3 - 23400 h s^2 \gamma \delta^3 v^3 + 11700 s^2 \gamma \delta^3 v^3 - 
- 58500 h s \delta^2 v^3 + 2925 s \delta^2 v^3 + 234000 h s \gamma \delta^2 v^3 + 11700 s^2 \gamma \delta^2 v^3 - 
- 41600 s \delta^2 v^4 + 83200 s^2 \delta^2 v^4 - 520000 s^2 \gamma \delta^3 v^2 - 
- 3000 h^2 s \delta^3 v^2 + 1500 h s \delta^3 v^2 + 1300 s^2 \gamma \delta^4 v^2 + 
\right). \]
\[
\begin{align*}
+15600s^3\gamma^2\epsilon^2v^2 + 6000h^2\delta^2\epsilon^2v^2 - 2080s\delta^2\epsilon^2v^2 + 9880s^2\gamma^2\epsilon^2v^2 - 10400s^3\gamma^2\epsilon^2v^2 - 6000hs\delta^2\epsilon^2v^2 - 6656s\delta^2\epsilon^2v^2 - 16640s^2\gamma^2\delta^2\epsilon^2v^2 + 2600s^4\gamma^2\delta^2\epsilon^2v^2 - 2600s^3\gamma^2\delta^2\epsilon^2v^2 - 12900h^2\delta^2\epsilon^2v^2 + 8400hs\delta^2\epsilon^2v^2 - 975s\delta^2\epsilon^2v^2 + 18000h^2\delta^2\epsilon^2v^2 - 9000hs^2\gamma^2\delta^2\epsilon^2v^2 + 650s^2\gamma^2\delta^2\epsilon^2v^2 + 18000h^2\delta^2\epsilon^2v^2 - 12900hs\delta^2\epsilon^2v^2 + 1950s\delta^2\epsilon^2v^2 - 36000h^2\gamma^2\delta^2\epsilon^2v^2 + 18000hs\gamma^2\delta^2\epsilon^2v^2 - 11700h^2\delta^2\epsilon^2v^2 + 11700hs\delta^2\epsilon^2v^2 - 2925s\delta^2\epsilon^2v^2 + 23400h^2\gamma^2\delta^2\epsilon^2v^2 - 23400h^2\gamma^2\delta^2\epsilon^2v^2 + 5850s^2\gamma^2\delta^2\epsilon^2v^2 - 3200hs\delta^3\epsilon^3v + 2600s\delta^3\epsilon^3v + 6400hs\delta^3\epsilon^3v + 5200s\delta^3\epsilon^3v - 4000hs^3\gamma^2\delta^3\epsilon^4v - 800hs\delta^3\epsilon^4v + 650s\delta^3\epsilon^4v + 3600hs^2\gamma^2\delta^3\epsilon^4v - 1300s^2\gamma^2\delta^3\epsilon^4v + 12000hs^3\gamma^2\delta^3\epsilon^4v - 800hs^2\gamma^2\delta^3\epsilon^4v + 7800s^2\gamma^2\delta^3\epsilon^4v - 8000hs^3\gamma^2\delta^3\epsilon^4v + 5120hs\delta^3\epsilon^4v - 4160s\delta^3\epsilon^4v - 1280hs^2\gamma^2\delta^3\epsilon^4v - 5200s^2\gamma^2\delta^3\epsilon^4v - 2000hs^3\gamma^2\delta^3\epsilon^4v - 7400hs^2\gamma^2\delta^3\epsilon^4v + 35200s^2\gamma^2\delta^3\epsilon^4v - 6000hs^3\gamma^2\delta^3\epsilon^4v + 6000hs^2\gamma^2\delta^3\epsilon^4v - 1900hs^3\gamma^2\delta^3\epsilon^4v + 325s^2\delta^2\epsilon^2v^3 + 5000hs^2\gamma^2\delta^3\epsilon^4v - 3250s^2\gamma^2\delta^3\epsilon^4v + 4400hs^3\gamma^2\delta^3\epsilon^4v + 4400hs^2\gamma^2\delta^3\epsilon^4v - 2600s^3\gamma^2\delta^3\epsilon^4v + 12000hs^3\gamma^2\delta^3\epsilon^4v - 12000hs^3\gamma^2\delta^3\epsilon^4v + 1720hs\delta^3\epsilon^4v + 1040s\delta^3\epsilon^4v - 640hs^2\gamma^2\delta^3\epsilon^4v - 780s^2\gamma^2\delta^3\epsilon^4v - 780hs^3\delta^3\epsilon^4v + 11700hs^2\delta^3\epsilon^4v - 5850hs\delta^3\epsilon^4v + 975s\delta^3\epsilon^4v - 2000hs\delta^3\epsilon^4v + 4000hs^2\gamma^2\delta^3\epsilon^4v + 2100hs\delta^3\epsilon^4v - 650s\delta^3\epsilon^4v + 2000h^2s^2\gamma^2\delta^3\epsilon^4v - 1000hs^2\gamma^2\delta^3\epsilon^4v + 3200h^2s^2\delta^2\epsilon^4v - 5200hs\delta^2\epsilon^4v + 1300s^2\delta^2\epsilon^4v - 4000h^2s^2\gamma^2\delta^2\epsilon^4v + 4000hs^2\gamma^2\delta^2\epsilon^4v - 3200hs\delta^2\epsilon^4v - 4000hs^2\gamma^2\delta^2\epsilon^4v + 2000h^2s^2\gamma^2\delta^2\epsilon^4v - 1000hs^2\gamma^2\delta^2\epsilon^4v - 800h^2s^2\delta^2\epsilon^4v + 1050hs\delta^2\epsilon^4v - 325s\delta^2\epsilon^4v + 5800h^2s^2\gamma^2\delta^2\epsilon^4v + 1950s^2\gamma^2\delta^2\epsilon^4v + 400h^2s^2\gamma^2\delta^2\epsilon^4v + 2000h^2s^2\gamma^2\delta^2\epsilon^4v - 2560h^2s^2\delta^2\epsilon^4v + 3360hs\delta^2\epsilon^4v - 1040s\delta^2\epsilon^4v - 640h^2s^2\gamma^2\delta^2\epsilon^4v + 5800h^2s^2\gamma^2\delta^2\epsilon^4v - 1300s^2\gamma^2\delta^2\epsilon^4v .
\end{align*}
\]

Further,

\[
\lim_{\varepsilon \to 0} \frac{\varphi_s(h(s,v), s, v, \varepsilon)}{\varphi_h(h(s,v), s, v, \varepsilon)} = -\frac{1}{10\varepsilon} \cdot \frac{b_1}{b_2} .
\]
where

\[ b_1 = -3900v^{10} - 23400s^2\gamma v^6 + 11700\delta v^8 - 23400v^8 - 11700h^8 + 23400hv^7 - \\
-22500h\delta v^7 - 11250h\delta v^7 - 11700v^7 - 46800s^2\gamma v^6 - 2600\delta v^8 - \\
-2600\delta v^8 + 15600s^2\delta v^6 + 46800s^2\gamma v^6 - 78000s\gamma v^6 + 40950s\gamma v^6 - \\
-19500h\delta v^6 + 28860v^6 - 11700v^6 - 3000h\delta v^5 + 12000h^2v^5 - \\
-46800hv^5 + 93600hs\gamma v^5 - 46800s\gamma v^5 + 87300h\delta v^5 - 81000hs\gamma v^5 - \\
-49500h\delta v^5 + 40500h^2\gamma v^5 + 14625\delta v^5 - 29250v^5 + 23400h^5 - \\
-31200s^3\gamma v^4 + 2600h^2\delta v^4 - 5200\delta v^4 - 46800h^2v^4 + 46800s^2\gamma v^4 + \\
+2600s\gamma \delta v^2 + 2600d^2v^2 + 41600s^2\gamma v^4 - 31200s\gamma \delta v^2 - 15800h^2v^4 + \\
+33280s^2v^4 + 46800hv^4 - 23400s\gamma v^4 + 63000h^2v^4 - 62400s^2\gamma v^4 - \\
-31500hv^4 + 42120s^2\gamma v^4 - 31500h^2\delta v^4 + 31200s^2\gamma \delta v^2 + \\
+15750h\delta v^4 + 31200s^2\gamma \delta v^4 + 7800v^4 - 5460v^4 - 7800v^4 + 2000h^2\delta v^3 + \\
-4000h\delta v^3 - 39600h^2s^2\gamma v^3 - 46800s^2\gamma v^3 + 6600h^2\delta v^3 + 2000hs\gamma \delta v^3 - 1300h\delta v^3 + \\
+25600h^2\delta v^3 + 32000hs\gamma \delta v^3 - 20800h\delta v^2 - 2000hs\gamma \delta v^2 - 5200h\delta v^2 + 15600h^2v^2 + \\
+23400hv^3 - 93600hs\gamma v^3 + 46800s\gamma v^3 - 78000hs^2\gamma v^3 - 32760h\delta v^3 + 124800hs\gamma v^3 - \\
-39000s\gamma v^3 + 39000h^2s^2\gamma \delta v^3 + 29250h\delta v^3 - 74100hs\gamma \delta v^3 + 17550s\gamma \delta v^3 - \\
-9750h\delta v^3 + 14430s^2\gamma v^3 - 11700v^3 + 41600s^2\gamma \delta v^2 - 8320\delta v^3 + 5200h^2v^2 + \\
+14600s^2\gamma v^2 + 13312s^2v^2 - 46800hv^2 - 93600h^2s^2\gamma v^2 + 93600hs\gamma v^2 - 23400s^2\gamma v^2 - 54000h^2v^2 + \\
+38700hv^2 + 108000h^2s\gamma v^2 - 54000hs\gamma v^2 - 7800s^3\gamma v^2 - 38700s^2\delta v^2 + \\
+7800s^2\gamma \delta v^2 + 25200h\delta v^2 - 54000h^2s\gamma v^2 - 27000s\gamma \delta v^2 - 19500s\gamma v^2 + \\
+2925\delta v^2 - 5850v^2 + 11700v^2 + 31200h^3v + 3200h^2\delta v^3 + 2600h^2\delta v^3 + 6400h\delta v^3 - \\
-5200h^3v - 46800h^2v + 16000hs^2\gamma v^2 + 8000s^2\gamma \delta v^2 + 1600h^2\delta v^2 - 7200hs\gamma \delta v^2 + \\
+2600s\gamma \delta v^2 + 1300h\delta v^2 + 10240h\delta v^2 + 25600hs\gamma v^2 + 10400s\gamma \delta v^2 - 24000h^2\gamma \delta v^2 + \\
+1600hs\gamma \delta v^2 - 15600s\gamma \delta v^2 + 5200h\delta v^2 + 3820v^2 + 23400hv - 36000h^3v - 12000hs^3\gamma v^2 + \\
+3600h^2v^2 - 13200h^2\gamma v^2 - 7800s^2\gamma v^2 - 5160hv + 1920hs\gamma v^2 + 2340s\gamma v^2 + 1800h^3v + \\
+6000h^3s^3\gamma v - 1800h^2\delta v - 22200hs^2\gamma \delta v - 15600s^2\gamma \delta v + 5700h\delta v - \\
-15000hs\gamma \delta v - 9750s\gamma \delta v - 9750v^2 - 3120v^2 - 3900v^2 + 2000h^2\delta v^3 - 4000h^2\delta v^3 + \\
+3200h^2\delta v^2 - 4200h^2\delta v^2 - 4000h^2s^2\gamma v^2 + 2000hs\gamma \delta v^2 + 1300h^2\gamma v^2 + 6400h\delta v^2 + \\
+8000h^2\gamma v^2 - 4600h\delta v^2 + 10400h\delta v^2 + 8000s^2\gamma v^2 - 8000h\gamma \delta v^2 - 2600h\delta v^2 + 7680h^2v + \\
+12000h^2s^3\gamma v - 6000hs^2\gamma v^2 - 1080hv - 1920hs\gamma v^2 - 1740h\gamma v^2 + 3900s\gamma v + 2400h^2\gamma v - \\
-6000h^2s^3\gamma v^2 + 3000hs^2\gamma v^2 - 3150hv - 17400h^2\gamma v^2 + 20400hs\gamma v^2 - 5850s\gamma v + 9750\delta v + 3120\delta v,
and
\[ b_2 = 375 \delta e v^2 - 750 \delta e v^2 + 585 \delta e v^2 + 150 \delta e^2 v^2 + 600 \delta e^2 v^2 + 2340 s \gamma v^2 - 2700 s \gamma v^2 + \\
+1350 s \gamma v^2 - 1650 s \gamma v^2 + 2910 \delta e v^2 + 2340 \delta e v^2 + 4200 \delta e v^2 - 2100 h \delta e v^2 + \\
+525 \delta e v^2 - 1050 \delta e v^2 + 1170 \delta e v^2 + 200 \delta e v^2 - 400 \delta e v^2 + 2340 \delta e v^2 + 100 \delta e v^2 + \\
+330 \delta e v^2 - 1600 \delta e v^2 - 1000 s \gamma \delta e v^2 - 1040 \delta e v^2 - 1280 \delta e v^2 - 2340 s \gamma v^2 - 2600 s \gamma v^2 + \\
+4160 s \gamma v^3 + 1300 s \gamma^2 \delta e v^2 - 2470 s \gamma \delta e v^2 + 750 \delta e v^3 - 1092 v^3 + 585 v^3 + 600 h \delta e^2 v^2 - \\
-150 \delta e v^2 - 1200 h \delta e v^2 + 200 \delta e v^2 + 2340 h \delta e v^2 - 4680 h s \gamma v^2 + 2340 s \gamma v^2 - 3600 h s \gamma v^2 + 7200 h s \gamma v^2 - \\
-1800 s \gamma v^2 + 2580 h \delta e v^2 - 3600 h s \gamma \delta e v^2 + 900 s \gamma \delta e v^2 - 840 \delta e v^2 + 1290 \delta e v^2 + \\
+320 \delta e v^2 - 640 \delta e v^2 + 2340 h \delta e v^2 + 800 s \gamma \delta e v^2 + 400 s \gamma \delta e v^2 - 360 s \gamma^2 v^2 + \\
+80 \delta e v^2 + 1280 \delta e v^2 - 1200 s \gamma \delta e v^2 + 80 s \gamma \delta e v^2 + 512 \delta e v^2 - 2340 h \delta e v^2 + 400 s \gamma \delta e v^2 - \\
-3600 h^2 \delta e v^2 - 440 s \gamma \delta e v^2 + 2400 h \delta e v + 64 s \gamma e v^2 + 200 s \gamma^2 \delta e v + 1800 h \delta e v^2 + \\
+740 s \gamma \delta e v^2 - 1200 h \delta e v + 500 s \gamma \delta e v + 190 \delta e v^2 + 720 s \gamma e v + 585 \delta e v + 200 \delta e v^2 - 400 \delta e^3 + \\
+320 h \delta e v^2 - 400 h s \gamma \delta e v^2 + 100 s \gamma \delta e v^2 - 210 \delta e v^2 + 400 s \gamma^2 v^2 - 640 h \delta e^2 v^2 + \\
+800 h s \gamma \delta e v^2 - 400 s \gamma \delta e v^2 + 520 \delta e^2 v^2 + 320 \delta e v^2 + 800 h s \gamma \gamma \delta e v^2 - 200 s \gamma^2 \delta e v + 512 \delta e v + \\
+1280 h s \gamma e v - 580 s \gamma e v - 400 s \gamma \gamma \delta e v + 100 s \gamma \delta e v + 160 h \delta e v^2 - 1160 h \delta e v^2 + \\
+680 s \gamma \delta e - 105 \delta e - 336 \delta e.
\]

This completes the right-hand side of the expression (9).

5. References
[1] Shchepakina E A, Sobolev V A and Mortell M P 2014 Singular Perturbations: Introduction to System Order Reduction Methods with Applications vol 2114 (Cham: Springer Lecture Notes in Math)
[2] Gorelov G N and Sobolev V A 1991 Combust. Flame 87 203-210
[3] Gorelov G N and Sobolev V A 1992 Appl. Math. Lett 5 3-6
[4] Gol’dshtein V, Zinoviev A, Sobolev V and Shchepakina E 1996 Proc. London Roy. Soc. Ser. A. 452 2103-2119
[5] Gorelov G N, Shchepakina E A and Sobolev V A 2006 J. Eng. Math 56 143-160
[6] Shchepakina E and Korotkova O 2013 Discrete and Continuous Dynamical Systems - Series B. 18 495-512
[7] Shchepakina E A 2003 Nonlinear Anal.: Real World Appl 4 45-50
[8] Sobolev V A and Shchepakina E A 1993 J. Combustion, Explosion and Shock Waves 29 378-381
[9] Gavin C, Pokrovskii A, Prentice M and Sobolev V 2006 Journal of Physics: Conference Series 55 80-93
[10] Shchepakina E and Korotkova O 2011 Journal of the Optical Society of America B: Optical Physics 28 1988-1993
[11] Pokrovskii A, Shchepakina E and Sobolev V 2008 Journal of Physics: Conference Series 138
[12] Pokrovskii A, Rachinskii D, Sobolev V and Zhezherun A 2011 Applicable Analysis 90 1123-1139
[13] Sobolev V A and Shchepakina E A 1993 Explosion and Shock Waves 29 378-381
[14] Sobolev VA and Shchepakina EA 1996 Differential Equations 32 1177-1186
[15] Balabaev M O 2017 Procedia Engineering 201 561-566
[16] Benoit E 1983 Société Mathématique De France 109-110 159-191
[17] Benoit E, Calot J L and Diener M 1981 Collectanea Mathematica 31-32 37119

gapore: World Scientific)
[18] Diener M 1979 Nessie et les Canards (Strasbourg: Publication IRMA)
[19] Shchepakina E and Sobolev V 2001 *Nonlinear Analysis. Ser. A: Theory Methods* **44** 897-908
[20] Shchepakina E 2002 *Differential Equations* **38** 1146-1452
[21] Shchepakina E and Sobolev V 2016 *Journal of Physics: Conference Series* 727
[22] Shchepakina E 2005 *Journal of Physics: Conference Series* **22** 194-207
[23] Sobolev V 2013 *Discrete and Continuous Dynamical Systems - Series B* **18** 513-521
[24] Poincare´ H 1886 *J. De Math. Pures et Appl. S´erie IV* **2** 15-17
[25] Ginoux J M 2009 *Differential Geometry Applied to Dynamical Systems* vol 3
[26] Rossetto B 1986 *Lecture Notes in Physics* **278** 12-14
[27] Darboux J G 1878 *Bull. Sci. Math. S´er 2* 2 60-96, 123-143, 151-200
[28] Demazure M 1989 Catastrophes et Bifurcations (Paris: Ellipses)
[29] FitzHugh R 1969 *Biological Engineering* 1-85
[30] Desroches M, Krauskopf B and M Osinga H 2008 *Chaos* (Woodbury, N Y) **18** 015107