Global existence of weak solution to the heat and moisture transport system in fibrous porous media

Buyang Li *, Weiwei Sun *, and Yi Wang †

Abstract

This paper is concerned with theoretical analysis of a heat and moisture transfer model arising from textile industries, which is described by a degenerate and strongly coupled parabolic system. We prove the global (in time) existence of weak solution by constructing an approximate solution with some standard smoothing. The proof is based on the physical nature of gas convection, in which the heat (energy) flux in convection is determined by the mass flux in convection.

Key words: Heat and moisture transfer, Porous media, Global weak solution.

1 Introduction

Mathematical modeling for heat and moisture transport with phase change in porous textile materials was studied by many authors, e.g., see [5, 6, 9, 12]. A typical application of these models is a clothing assembly, consisting of a thick porous fibrous batting sandwiched by two thin fabrics. The outside cover of the assembly is exposed to a cold environment with fixed temperature and relative humidity while the inside cover is exposed to a mixture of air and vapor at higher temperature and relative humidity. In general, the physical process can be viewed as a multiphase and single (or multi) component flow. In this process, the water vapor moves through the clothing assembly by convection which is induced by the pressure gradient. The heat is transferred by conduction in all phases (liquid, fiber and gas) and convection in gas. Phase changes occur in the form of evaporation/condensation and/or sublimation. Based on the conservation of mass and energy and the neglect of the water influence, the model can be described by

\[
\frac{\partial}{\partial t} (\epsilon C_v) + \frac{\partial}{\partial x} (\epsilon u C_v) = -\Gamma_{ce},
\]

\[
\frac{\partial}{\partial t} (\epsilon C_{vy} M C_v T + (1-\epsilon) C_{vs} T) + \frac{\partial}{\partial x} (\epsilon C_{vy} M u C_v T) = \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x} \right) + \lambda M \Gamma_{ce}.
\]

Here \( C_v \) is the vapor concentration (mol/m\(^3\)), \( T \) is the temperature (K), \( \epsilon \) the porosity of the fiber, \( M \) the molecular weight of water and \( \lambda \) the latent heat of evaporation/condensation in

*Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong (buyangli2@student.cityu.edu.hk, maweiw@math.cityu.edu.hk). The work of the authors was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 102005).

†Institute of Applied Mathematics, AMSS, CAS, Beijing 100190, China (wangyi@amss.ac.cn). The work of this author was supported in part by the NSFC grant (No. 10801128) and a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 102005).
the wet zone while in frozen zone, it represents the latent heat of sublimation. \( C_{vg} \) and \( C_{vs} \) are the heat capacities of the gas and mixture solid, respectively.

The evaporation/condensation (molar) rate of phase change per unit volume is defined by the Hertz-Knudsen equation \([10]\)

\[
\Gamma_{ce} = -\frac{E}{R_f} \sqrt{\frac{(1-\epsilon)(1-\epsilon')}{2\pi RM}} \left( \frac{P_{\text{sat}}(T) - P}{\sqrt{T}} \right)
\]  

(1.3)

where \( R \) is the universal gas constant, \( R_f \) is the radius of fibre and \( E \) is the nondimensional phase change coefficient. The vapor pressure is given by \( P = RC_v T \) because of the ideal gases’ assumption. The saturation pressure \( P_{\text{sat}} \) is determined from experimental measurements, see Figure ??.

The vapor velocity (volumetric discharge) is given by the Darcy’s law

\[
u = \frac{kk_{rg}}{\mu_g} \frac{\partial P}{\partial x}
\]  

(1.4)

where \( k \) is the permeability, \( k_{rg} \) and \( \mu_g \) are the relative permeability and the viscosity of the vapor, respectively.

Numerical methods and simulations for the heat and moisture transport in porous textile materials have been studied by many authors with various applications \([4, 8, 16, 17, 19]\). However, no theoretical analysis has been explored for the above system of nonlinear equations. A simple heat and moisture model was studied in \([18]\), where the model was described by a pure diffusion process (without convection and condensation) with a non-symmetric parabolic part. There are several related porous media flow problems from other physical applications. A popular one is a compressible (or incompressible) flow in porous media with applications in oil and underground water industries, which is described by an elliptic pressure equation coupled with a parabolic concentration equation for incompressible case and a system of parabolic equations for compressible case. The existence of weak solution for incompressible and compressible flows has been studied in \([3, 7]\) and \([1]\), respectively. However, in most of these works, the temperature is ignored and the phase change (condensation/evaporation) does not occur due to the nature of these applications, while both temperature and phase change play important roles in the textile model.

For the textile model, the water content in the batting area usually is relative small and one often assumes that all these physical parameters involved in the system (1.1)-(1.2) are positive constants. With nondimensionalization, the system (1.1)-(1.2) reduces to

\[
\begin{aligned}
\rho_t - ((\rho\theta)x\rho)_x &= -\Gamma, \\
(\rho\theta)_t + \sigma \theta_t - ((\rho\theta)x\rho\theta)_x - (\kappa\theta)_x &= \lambda \Gamma, \\
\end{aligned}
\]  

(1.5)

for \( x \in (0, 1), t > 0 \), where \( (\cdot)_\mu = \frac{\partial (\cdot)}{\partial \mu} \) for \( \mu = x, t \), \( \rho = \rho(x, t) \) and \( \theta = \theta(x, t) \) represent the density of vapor and the temperature, respectively,

\[
\Gamma = \rho \theta^{1/2} - p_s(\theta)
\]

and \( p_s(\theta) \sim P_{\text{sat}}(\theta)/\theta^{1/2} \). \( \sigma \) and \( \lambda \) are given positive constants and \( \kappa = \kappa_1 + \kappa_2 \rho^2 \) is the heat conductivity coefficient with \( \kappa_i (i = 1, 2) \) being positive constants. We consider a class of commonly used Robin type boundary conditions \([5, 6, 9, 20]\) defined by

\[
\begin{aligned}
(\rho\theta)x\rho|_{x=1} &= \alpha^1(\rho^1 - \rho(1, t)), \\
(\rho\theta)x\rho|_{x=0} &= \alpha^0(\rho(0, t) - \rho^0),
\end{aligned}
\]  

(1.6)
and
\[ \kappa \theta_x |_{x = 1} = \beta^1(\bar{\theta}^1 - \theta(1, t)), \quad \kappa \theta_x |_{x = 0} = \beta^0(\theta(0, t) - \bar{\theta}^0), \] (1.7)
and the initial condition is
\[ \rho(x, 0) = \rho_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in (0, 1) \] (1.8)
where \( \alpha^0, \alpha^1 \) represent the mass transfer coefficients, \( \rho^0, \rho^1 \) are the density of the gas in the inner background and outer background, respectively, \( \beta^0, \beta^1 \) the heat transfer coefficients, and \( \bar{\theta}^0, \bar{\theta}^1 \) the inner and outer background temperatures. Physically, all the parameters above are positive constants and \( \rho_0(x) > \rho, \theta_0(x) > \bar{\theta} \), where \( \rho \) and \( \bar{\theta} \) are positive constants. Based on the experimental data in Figure 2, we assume that \( \rho_0 \) is a smooth, increasing and nonnegative function defined on \( \mathbb{R}^+ \) which satisfies
\[ \lim_{\theta \to 0} \frac{\rho_s(\theta)}{\theta} = 0, \quad \lim_{\theta \to \infty} \frac{\rho_s(\theta)}{\theta^{1+\eta}} = \infty \] (1.9)
for some \( \eta > 0 \). For physical reasons, we set \( \rho_s(\theta) = 0 \) for \( \theta \leq 0 \).

The objective of this paper is to establish the global existence of weak solution to the initial-boundary value problem (1.5)-(1.8) under the general physical hypotheses (1.9) for the saturation pressure function \( \Gamma \). The difficulty lies on the strong nonlinearity and the coupling of equations. To the best of our knowledge, there are no theoretical results for the underlying model. More important is its significant applications in textile industries. Also analysis presented in this paper may provide a fundamental tool for theoretical analysis of existing numerical methods. Our proof is based on the equivalence of mass and heat transfer in convection.

2 The main result

Before we present our main result, we introduce some notations. Let \( T \) be a given positive number in the following sections. We define
\[ \Omega = (0, 1), \quad I = (0, T), \quad Q_t = \Omega \times (0, t), \quad Q_T = \Omega \times I, \]
\[ V_1(Q_T) = L^2(I; H^1(\Omega)), \quad V_2(Q_T) = \left\{ f \in L^2(Q_T) \mid \| f \|_{V_2(Q_T)} < +\infty \right\}, \]
\[ \| f \|_{V_2(Q_T)} = \text{ess sup}_{t \in [0, T]} \| f(\cdot, t) \|_{L^2(\Omega)} + \| f_x \|_{L^2(Q_T)}, \]
\[ W_2^{2,1}(Q_T) = \left\{ f \in L^2(Q_T) \mid f_t, f_x, f_{xx} \in L^2(Q_T) \right\}. \]
Let \( D(\Omega \times [0, T]) \) be the subspace of \( C^\infty(\mathbb{R}^2) \) consisting of functions which have compact support in \( \mathbb{R} \times [0, T] \), restricted to \( \Omega \times [0, T] \).

Now we give the definition of weak solution to the system (1.5)-(1.8) and then, state our main result.

**Definition 2.1 (Weak solution)** We say that the measurable function pair \((\rho, \theta)\) defined on \( \Omega \times [0, T] \) is a global weak solution to (1.5)-(1.8) if \((\rho, \theta) \in (V_1(Q_T))^2\) and the density \( \rho \) and the temperature \( \theta \) are nonnegative functions satisfying
\[ \int_0^T \alpha^0(\rho(0, t) - \rho^0)\phi(0, t)dt + \int_0^T \alpha^1(\rho(1, t) - \rho^1)\phi(1, t)dt \]
\[ + \int_0^T \int_\Omega (-\rho \phi_t + (\rho \theta)_x \rho \phi_x + \Gamma \phi)dxdt = \int_\Omega \rho_0 \phi_0 dx \] (2.1)
and
\[
\int_0^T \left[ \alpha^0(\rho(0,t) - \rho^0)\theta(0,t) + \beta^0(\theta(0,t) - \bar{\theta}^0) \right] \psi(0,t) dt \\
+ \int_0^T \left[ \alpha^1(\rho(1,t) - \rho^1)\theta(1,t) + \beta^1(\theta(1,t) - \bar{\theta}^1) \right] \psi(1,t) dt \\
+ \int_0^\infty \int_\Omega \left[ - (\rho \theta + \sigma \theta) \psi_t + (\rho \theta)_{xx} \rho \theta \psi_x + \kappa \theta_x \psi_x - \lambda \Gamma \psi \right] dx dt \\
= \int_\Omega (\rho_0 \theta_0 + \sigma \theta_0) \psi_0 dx \tag{2.2}
\]
for any test functions \( \phi, \psi \in \mathcal{D}(\Omega \times [0,T]) \).

**Theorem 2.1** If the initial value \((\rho_0, \theta_0)\) satisfies \(\rho_0 \in L^{1+\gamma}(\Omega) \) (\(\gamma > 0\)), \(\theta_0 \in L^\infty(\Omega)\) and \(\rho_0 \geq 0, \theta_0 \geq \bar{\theta}\) for some positive constant \(\bar{\theta}\), then there exists a global weak solution \((\rho, \theta)\), in the sense of Definition 2.1, to the initial-boundary value problem (1.5)-(1.8) such that
\[
\rho \ln \rho \in L^\infty(0,T; L^1(\Omega)), \quad \rho \in L^4(Q_T), \quad \rho_x \in L^2(Q_T); \\
\theta, \theta^{-1} \in L^\infty(Q_T), \quad (1 + \rho) \theta_x \in L^2(Q_T).
\]

In the following sections, we denote by \(C_{p_1, p_2, \cdots, p_k}\) a generic positive constant, which depends solely upon \(p_1, p_2, \cdots, p_k\), the physical parameters \(\kappa_1, \kappa_2, \sigma\) and \(\lambda\) and the parameters involved in initial and boundary conditions. In addition, we denote by \(C(p_1, p_2, \cdots, p_k)\) a generic positive function, dependent upon the physical parameters \(\kappa_1, \kappa_2, \sigma\) and \(\lambda\) and the parameters involved in boundary conditions, which is bounded when \(p_1, p_2, \cdots, p_k\) are bounded.

### 3 Construction of approximate solutions

Throughout this section, we let \(\varepsilon\) be a fixed positive number which satisfies
\[
0 < \varepsilon \leq \min\{\rho^0, \rho^{-1}, \bar{\rho}^0, \bar{\rho}^{-1}, 1\},
\]
and \(0 < \nu < \varepsilon\). To prove the existence of global weak solutions to the system (1.5)-(1.8), we introduce a regularized approximate system as follows:
\[
\begin{align*}
\rho_t - ((\varepsilon + (\rho \theta)_{\nu}) \rho_x)_x - (\rho (\rho \theta_{\varepsilon})_x)_x & = - \rho \chi^\varepsilon(\sqrt{\theta}) + \chi^\varepsilon(p_s(\theta)), \\
(\rho \theta + \sigma \theta)_t - (\kappa^\varepsilon \theta_x)_x - ((\varepsilon + (\rho \theta)_{\nu}) \rho_x \theta)_x - (\rho (\rho \theta_{\varepsilon})_x \varepsilon \theta)_x & = \lambda \rho \chi^\varepsilon(\sqrt{\theta}) - \lambda \chi^\varepsilon(p_s(\theta)) + (\lambda + \theta) \left( \chi^\varepsilon(p_s(\theta)) - p_s(\theta) \right), \quad \text{in} \quad Q_T,
\end{align*}
\]
where \(\chi^\varepsilon\) is a cut-off function defined by
\[
\chi^\varepsilon(h) = \begin{cases} 
  h & \text{if } |h| \leq \varepsilon^{-1}, \\
  \varepsilon^{-1} & \text{if } |h| \geq \varepsilon^{-1},
\end{cases}
\]
and
\[
\kappa^\varepsilon = \kappa_1 + \kappa_2 |\rho_\varepsilon|^2,
\]
for any test functions \(\phi, \psi \in \mathcal{D}(\Omega \times [0,T])\).
and the subscriptions $\varepsilon, \nu$ define the smoothing operators in general by $f_\mu = \text{Ext}(f) * \eta_\mu$ with $\mu = \nu, \varepsilon$. Here $\eta_\mu$ is the standard mollifier and Ext($\cdot$) is the extension operator which extends any measurable functions defined on $\Omega_T$ to be zero on $\mathbb{R}^2 \setminus \Omega_T$.

The system (3.1) can be rewritten as

\[
\begin{cases}
\rho_t - ((\varepsilon + (\rho\theta)_\nu) \rho_x)_x - (\rho(\rho_x \theta_x)_x)_x + \rho \chi^\varepsilon(\sqrt{\theta}) = \chi^\varepsilon(p_s(\theta)), \\
(\rho + \sigma)\theta_t - (\kappa \theta_x)_x - [(\varepsilon + (\rho\theta)_\nu) \rho_x + \rho(\rho_x \theta_x)_x] \theta_x - \rho \chi^\varepsilon(\sqrt{\theta}) \theta + (\lambda + \theta) p_s(\theta) = \lambda \rho \chi^\varepsilon(\sqrt{\theta}).
\end{cases}
\] (3.2)

The corresponding initial and boundary conditions are given by

\[
\begin{align*}
(\varepsilon + (\rho\theta)_\nu) \rho_x + \rho(\rho_x \theta_x)_x |_{x=1} &= \alpha^1(\rho^1 - \rho(1, t)), \\
(\varepsilon + (\rho\theta)_\nu) \rho_x + \rho(\rho_x \theta_x)_x |_{x=0} &= \alpha^0(\rho(0, t) - \rho^0), \\
\rho(x, 0) &= \rho_0(x) := (\rho_0)_\varepsilon(x) + \varepsilon, \\
\kappa \theta_x |_{x=1} &= \beta^1(\theta^1 - \theta(1, t)), \\
\kappa \theta_x |_{x=0} &= \beta^0(\theta(0, t) - \theta^0), \\
\theta(x, 0) &= \theta_0(x) := (\theta_0)_\varepsilon(x).
\end{align*}
\] (3.3)

We prove the existence of solutions to the system (3.2)-(3.3) by using the Leray–Schauder fixed point theorem. The following lemma (see [13], [14]) is useful in our proof.

**Lemma 3.1 (Aubin–Lions)** Let $B_1 \hookrightarrow B_2 \hookrightarrow B_3$ be reflective and separable Banach spaces. Then

\[
\{u \in L^p(I; B_1) | u_t \in L^q(I; B_3)\} \hookrightarrow L^p(I; B_2), \quad 1 < p, q < \infty;
\]

\[
\{u \in L^q(I; B_2) \cap L^1(I; B_1) | u_t \in L^1(I; B_3)\} \hookrightarrow L^p(I; B_2), \quad 1 \leq p < q < \infty.
\]

### 3.1 Existence of approximate solutions

We define

\[
X = \{u \in L^2(I; H^1(\Omega)) | u \geq 0\}, \quad Y = \{u \in W^{2,1}_{2}(Q_T) | u \geq 0\}.
\]

By Aubin–Lions lemma, $Y \hookrightarrow X$. Let $\varepsilon$ and $\nu$ be given positive constants and the parameter $s \in [0, 1]$. For any given $(\rho^0, \theta^0) \in X^2$, we define $\rho$ to be the solution of the following linear parabolic equation

\[
\rho_t - ((\varepsilon + (\rho^0 \theta^0)_\nu) \rho_x)_x - (\rho(\rho_x \theta_x^0)_x)_x + s \rho \chi^\varepsilon(\sqrt{\theta^0}) = s \chi^\varepsilon(p_s(\theta^0)),
\] (3.4)

with the initial and boundary conditions

\[
\begin{align*}
(\varepsilon + (\rho^0 \theta^0)_\nu) \rho_x + \rho(\rho_x \theta_x^0)_x &= \alpha^1(s \rho^1 - \rho), \quad \text{at } x = 1, \\
(\varepsilon + (\rho^0 \theta^0)_\nu) \rho_x + \rho(\rho_x \theta_x^0)_x &= \alpha^0(\rho - s \rho^0), \quad \text{at } x = 0, \\
\rho(x, 0) &= s \rho_0(x), \quad \text{for } x \in \Omega.
\end{align*}
\] (3.5)

Now with $\rho$ in hand, we define $\theta$ to be the solution of the linear parabolic equation

\[
(\rho + \sigma)\theta_t - (\kappa \theta_x)_x - [(\varepsilon + (\rho^0 \theta^0)_\nu) \rho_x + \rho(\rho_x \theta_x^0)_x] \theta_x
+ s s \rho \chi^\varepsilon(\sqrt{\theta^0}) + s \chi^\varepsilon(p_s(\theta)) = s \alpha \rho \chi^\varepsilon(\sqrt{\theta^0}).
\] (3.6)
with the initial and boundary conditions
\[
\begin{aligned}
\kappa^c \theta_x &= \beta^1(s \theta^1 - \theta), \quad \text{at } x = 1, \\
\kappa^s \theta_x &= \beta^0(\theta - s \theta^0), \quad \text{at } x = 0, \\
\theta(x, 0) &= s \theta_0(x), \quad \text{for } x \in \Omega.
\end{aligned}
\]
(3.7)

Let \( M \) denote the mapping from \((\rho^0, \theta^0, s)\) to \((\rho, \theta)\). Then we have the following lemma.

**Lemma 3.2** The mapping \( M : X^2 \times [0, 1] \to X^2 \) is well defined, continuous and compact.

**Proof.** By the \( L^2 \)-theory of linear parabolic equations [11], there exists a solution \( \rho \in W^{2,1}_2(Q_T) \) for the system (3.6)-(3.7) and integrating the resulting equation over \( E \) for the standard smoothing operator, we have
\[
\|\rho\|_{W^{2,1}_2(Q_T)} \leq C(\varepsilon^{-1}, \|\rho^0_\omega\|_{C^1(\partial \Omega)}, \|\rho^0\|_{C^1(\partial \Omega)}, \|\rho_\omega\|_{H^1(\Omega), T}).
\]
By noting the fact
\[
\|\rho^0_\omega\|_{H^1(\Omega)} \leq C\|\rho_\omega\|_{L^1(\Omega)}, \quad \|\rho^0\|_{C^1(\partial \Omega)} \leq C_{\nu,T}\|\rho^0\|_{L^2(Q_T)}\|\theta^0\|_{L^2(Q_T)},
\]
\[
\|\rho^0\|_{L^2(Q_T)} \leq C_{\nu,T}\|\rho^0\|_{L^2(Q_T)}\|\theta^0\|_{L^2(Q_T)},
\]
and therefore,
\[
\|\rho\|_{L^\infty(Q_T)} \leq \|\rho\|_{W^{2,1}_2(Q_T)} \leq C(\varepsilon^{-1}, \nu^{-1}, \|\rho^0\|_{X}, \|\theta^0\|_{X}, T).
\]
(3.8)

Let \( \rho^+ = \max\{\rho, 0\}, \rho^- = \max\{-\rho, 0\} \). Then \( \rho = \rho^+ - \rho^- \). By multiplying \( \rho^- \) on both sides of the equation (3.4) and integrating the resulting equation over \( Q_t \), we have
\[
\int_0^1 \frac{|\rho^-|^2}{2} dx + \int_0^t \int_0^1 (\varepsilon + (\rho^0_\omega)\nu)|\rho^-|^2 dx d\tau + \int_0^t \int_0^1 (s\chi^\varepsilon(\sqrt{\theta})|\rho^-|^2 + s\chi^\varepsilon(p_\varepsilon(\theta))\rho^-) dx d\tau
\]
\[
+ \int_0^t (\alpha^0|\rho^-|\theta^0 - 0(\tau)|^2 + \alpha^0 s\rho^0_\varepsilon \rho^- - 0(\tau)) d\tau + \int_0^t (\alpha^1|\rho^-|1(\tau)|^2 + \alpha^1 s\rho^1_\varepsilon \rho^- - 1(\tau)) d\tau
\]
\[
= -\int_0^t \int_0^1 \rho^- \rho^-_\varepsilon (\rho^0_\varepsilon \theta^0_\varepsilon) dx d\tau
\]
\[
\leq \int_0^t \int_0^1 \frac{(\rho^0_\varepsilon \theta^0_\varepsilon)_{L^\infty(Q_T)}|\rho^-|^2 + \frac{\varepsilon}{2}|\rho^-|^2}{2\varepsilon} dx d\tau.
\]
Notice that \( \rho^- \geq 0 \). Thus we have that
\[
\int_0^1 |\rho^-|^2 dx \leq \frac{(\rho^0_\varepsilon \theta^0_\varepsilon)_{L^\infty(Q_T)}|\rho^-|^2 + \varepsilon}{2\varepsilon} \int_0^1 |\rho^-|^2 dx d\tau.
\]
By Gronwall’s inequality, we can see that \( \rho^- \equiv 0 \). Thus \( \rho = \rho^+ \geq 0 \). This and (3.8) imply that \( \rho \in Y \hookrightarrow \hookrightarrow X \).

Similarly, by the \( L^2 \)-theory of quasi-linear parabolic equations [11], there exists a solution \( \theta \in W^{2,1}_2(Q_T) \) for the system (3.6)-(3.7) and
\[
\|\theta\|_{W^{2,1}_2(Q_T)} \leq C(\varepsilon^{-1}, \nu^{-1}, \|\rho^0\|_{X}, \|\theta^0\|_{X}, T).
\]
(3.9)

6
Let $\theta^+ = \max\{\theta, 0\}$, $\theta^- = \max\{-\theta, 0\}$. Then $\theta = \theta^+ - \theta^-$. Multiplying $\theta^-/(\rho + \sigma)$ on both sides of the equation (3.6) and integrating the resulting equation over $Q_t$, we can get

$$\int_0^1 |\theta^-|^2 \, dx + \int_0^t \int_0^1 \frac{\kappa_\varepsilon}{\rho + \sigma} |\theta_x^\varepsilon|^2 \, dx \, d\tau + \int_0^t \int_0^1 \frac{s(\lambda + \theta)p_s(\theta)}{(\rho + \sigma)} \theta^- \, dx \, d\tau$$

$$+ \int_0^t \int_0^1 s\lambda \rho \chi_\varepsilon(\sqrt{\theta^0}) \frac{\theta^-}{\rho + \sigma} \, dx \, d\tau + \int_0^t \frac{\theta^-}{\rho(1, \tau) + \sigma} \beta_1(s\theta^1 + \theta^-(1, \tau)) \, d\tau$$

$$+ \int_0^t \int_0^1 \rho(0, \tau) + \sigma \beta_1(s\theta^0 + \theta^-(0, \tau)) \, d\tau = \int_0^t \int_0^1 s\rho \chi_\varepsilon(\sqrt{\theta^0}) \frac{|\theta^-|^2}{\rho + \sigma} \, dx \, d\tau$$

$$+ \int_0^t \int_0^1 \kappa_\varepsilon \theta_x^\varepsilon \frac{\rho \rho_x}{\rho + \sigma} \, dx \, d\tau + \int_0^t \int_0^1 \left[ \varepsilon \left( \rho^0 \theta_0 \right) \rho_x + \rho(\rho_{\theta_x^0}) \right] \theta_x^\varepsilon \frac{\theta^-}{\rho + \sigma} \, dx \, d\tau.$$

Since $p_s(\theta) = 0$ for $\theta \leq 0$, we observe that $(\lambda + \theta)p_s(\theta)\theta^- = 0$ a.e in $\Omega_T$. By Cauchy inequality and the estimations (3.8)-(3.9), we can estimate the terms in the right hand side of the above equality. Thus we obtain

$$\int_0^1 |\theta^-|^2 \, dx \leq C(\varepsilon^{-1}, \nu^{-1}, ||\rho^0||_X, ||\theta^0||_X, T) \int_0^t \int_0^1 |\theta^-|^2 \, dx \, d\tau.$$

Gronwall’s inequality gives that $\theta^- \equiv 0$. Thus $\theta = \theta^+ \geq 0$. This and (3.9) imply that $\theta \in Y \hookrightarrow X$.

We conclude that the mapping $M : X^2 \times [0, 1] \rightarrow X^2$ is a compact mapping.

Now we prove the continuity of the mapping $M$. For any $(\hat{\rho}^0, \hat{\theta}^0, \hat{s}) \in X^2 \times [0, 1]$, let $(\hat{\rho}, \hat{\theta}) = M(\hat{\rho}^0, \hat{\theta}^0, \hat{s})$. Then

$$\hat{\rho}_t - [(\varepsilon + (\hat{\rho}^0 \hat{\theta}^0_0)_{\nu}) \hat{\rho}_x + \hat{\rho}(\hat{\rho}_{\theta_x^0})_{\varepsilon}]_x + \tilde{s} \hat{\rho} \chi_\varepsilon(\sqrt{\theta^0}) = \tilde{s} \chi_\varepsilon(p_s(\theta^0)),$$

$$(\hat{\rho} + \sigma) \hat{\theta}_t - (\kappa_\varepsilon \hat{\theta}_x)_x - [(\varepsilon + (\hat{\rho}^0 \hat{\theta}^0_0)_{\nu}) \hat{\rho}_x + \hat{\rho}(\hat{\rho}_{\theta_x^0})_{\varepsilon}] \hat{\theta}_x$$

$$- \tilde{s} \hat{\rho} \chi_\varepsilon(\sqrt{\theta^0}) + \tilde{s} \lambda \hat{\rho} \chi_\varepsilon(\sqrt{\theta^0}),$$

with the initial and boundary conditions

$$\begin{cases} 0 \\ \{ \varepsilon + (\rho^0 \theta^0_0)_{\nu} \} \rho_x + \hat{\rho}(\rho_{\theta_x^0})_{\varepsilon} = \alpha^0(\sqrt{(\rho - \hat{\rho})}) \\ \rho(x, 0) = \tilde{s} \rho_{0e}(x), \end{cases}$$

$$\{ \varepsilon + (\rho^0 \theta^0_0)_{\nu} \} \rho_x + \hat{\rho}(\rho_{\theta_x^0})_{\varepsilon} = \alpha^0(\sqrt{(\rho - \hat{\rho})})$$

$$\begin{cases} \varepsilon + (\rho^0 \theta^0_0)_{\nu} \rho_x + \hat{\rho}(\rho_{\theta_x^0})_{\varepsilon} = \alpha^0(\sqrt{(\rho - \hat{\rho})}) \end{cases},$$

$$\begin{cases} \rho(x, 0) = \tilde{s} \rho_{0e}(x), \end{cases}$$

for $x \in \Omega$.

and

$$\begin{cases} \hat{\rho}_x = \beta^3(s\theta^1 - \hat{\theta}) \\ \hat{\theta}_x = \beta^0(\theta - \hat{\theta}), \end{cases}$$

for $x \in \Omega$.

Denote $\hat{\rho} = \rho - \tilde{\rho}$ and $\tilde{\theta} = \theta - \hat{\theta}$. Then $\hat{\rho}$ satisfies the following equation,

$$\hat{\rho}_x - (F - F)_x + s\tilde{\rho} \chi_\varepsilon(\sqrt{\theta^0}) + (s - \tilde{s})\hat{\rho} \chi_\varepsilon(\sqrt{\theta^0}) + \tilde{s} \hat{\rho} \chi_\varepsilon(\sqrt{\theta^0}) - \chi_\varepsilon(\sqrt{\theta^0})$$

$$= (s - \tilde{s}) \chi_\varepsilon(p_s(\theta^0)) + \tilde{s} [\chi_\varepsilon(p_s(\theta^0)) - \chi_\varepsilon(p_s(\hat{\theta}^0))],$$

where

$$F = (\varepsilon + (\rho^0 \theta^0_0)_{\nu}) \rho_x + \rho(\rho_{\theta_x^0})_{\varepsilon},$$
\[
\hat{F} = (\varepsilon + (\rho^0 \theta^0)_\nu) \hat{\rho}_x + \hat{\rho}(\hat{\rho}^0 \theta^0)_\varepsilon.
\]

Multiplying the equation (3.14) by \(\hat{\rho}\) and integrating over \(Q_t\) gives
\[
\int_0^1 \hat{\rho}^2(x,t) dx + \int_0^1 \int_0^1 \hat{\rho}^2 dy dt \leq C \left[ \int_0^1 \int_0^1 \hat{\rho}^2 dy dt + (s - \hat{s})^2 + \|\rho^0 - \hat{\rho}^0\|^2_X + \|\theta^0 - \hat{\theta}^0\|^2_X \right]
\]
with \(C = C(\varepsilon^{-1}, \nu^{-1}, \|\rho_\varepsilon\|_{L^2(\Omega)}, \|\rho^0\|_X, \|\theta^0\|_X, \|\hat{\rho}^0\|_X, \|\hat{\theta}^0\|_X, T)\).

Thus Grönwall inequality implies that
\[
\|\hat{\rho}\|^2_X \leq C \left[ (s - \hat{s})^2 + \|\rho^0 - \hat{\rho}^0\|^2_X + \|\theta^0 - \hat{\theta}^0\|^2_X \right].
\]

Similarly, we can derive the equation for \(\hat{\theta}\) and get
\[
\|\hat{\theta}\|^2_X \leq C \left[ (s - \hat{s})^2 + \|\rho^0 - \hat{\rho}^0\|^2_X + \|\theta^0 - \hat{\theta}^0\|^2_X \right].
\]

Thus, the mapping \(M : X^2 \times [0,1] \rightarrow X^2\) is continuous. The proof of Lemma 3.2 is complete.

In addition, for \(s = 0\) we can see that \(M(\rho, \theta, 0) = 0\) for any \((\rho, \theta) \in X^2\). Thus, by the Leray–Schauder fixed point theorem, there exists a fixed point for the mapping \(M(\cdot, \cdot, 1) : X^2 \rightarrow X^2\) if all the functions \((\rho, \theta) \in X^2\) satisfying
\[
(\rho, \theta) = M(\rho, \theta, s)
\]
for some \(s \in [0,1]\) are uniformly bounded in \(X^2\). In fact, by the proof of Lemma 3.2, \(M\) maps \((\rho, \theta, s) \in X^2 \times [0,1]\) into \(Y^2\). Therefore, if \((\rho, \theta)\) is a fixed point of \(M(\cdot, \cdot, 1)\), then \((\rho, \theta) \in W^{2,1}_X(Q_T)\).

**Theorem 3.1** Under the assumptions of Theorem 2.1, the system (3.2)–(3.3) has a (strong) solution \((\rho, \theta) \in W^{2,1}_X(Q_T)\) which satisfies
\[
\rho \geq \rho_{\varepsilon,T} \quad \text{and} \quad \theta_T \leq \theta \leq \overline{\theta}_T \quad \text{for} \quad (x, t) \in Q_T.
\]
\[
\|\rho\|_{L^\infty(I_t; L^1(\Omega))}, \|\rho\|_{L^2(Q_T)}, \|\rho_x\|_{L^2(Q_T)} \leq C_{\varepsilon,T},
\]
\[
\|\theta\|_{L^\infty(Q_T)}, \|\rho\|_{L^\infty(I_t; L^1(\Omega))}, \|\theta_x\|_{L^2(Q_T)}, \|\rho_x\|_{L^2(Q_T)} \leq C_T.
\]
where \(\rho_{\varepsilon,T}\) and \(C_{\varepsilon,T}\) are positive constants which depend on \(\varepsilon\) and \(T\), independent of \(\nu; \dot{\theta}, \overline{\theta}_T\), and \(\dot{\theta}\) and \(\overline{\theta}_T\) are positive constants, dependent upon \(T\) and independent of \(\varepsilon\) and \(\nu\).

By the Leray-Schauder fixed point theorem, it suffices to prove the uniform boundedness of functions \((\rho, \theta) \in X^2\) satisfying the equation (3.15) and (3.16).

**3.2 Uniform estimates**

We assume that \((\rho, \theta) \in X^2\) and therefore, \((\rho, \theta) = M(\rho, \theta, s) \in Y^2\), for \(s \in [0,1]\), i.e., \((\rho, \theta)\) is a (strong) solution of the following system,
\[
\rho_t - ((\varepsilon + (\rho \theta)_\nu) \rho_x) \rho_x + s \rho \chi^\varepsilon(\sqrt{\theta}) = s \chi^\varepsilon(p_{s}(\theta)),
\]
\[
(\rho + \sigma) \theta_t - (\kappa \theta_x)_x - ((\varepsilon + (\rho \theta)_\nu) \rho_x + \rho (\rho \theta_x)_\varepsilon) \theta_x - \lambda \rho \chi^\varepsilon(\sqrt{\theta}) + s (\lambda + \theta) p_{s}(\theta) = s \lambda \rho \chi^\varepsilon(\sqrt{\theta}),
\]
with the initial and boundary conditions

\[
\begin{align*}
\begin{cases}
(\varepsilon + (\rho \theta)_{,\nu}) \rho_x + \rho(x_0 \theta_x)_{,\varepsilon} &= \alpha^1(s\rho^1 - \rho), & \text{at } x = 1, \\
(\varepsilon + (\rho \theta)_{,\nu}) \rho_x + \rho(x_0 \theta_x)_{,\varepsilon} &= \alpha^0(\rho - s\rho^0), & \text{at } x = 0, \\
\rho(x,0) &= s\rho_0(x), & \text{for } x \in \Omega,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\kappa^\varepsilon \theta_x &= \beta^1(s\theta^1 - \theta), & \text{at } x = 1, \\
\kappa^\varepsilon \theta_x &= \beta^0(\theta - s\theta^0), & \text{at } x = 0, \\
\theta(x,0) &= s\theta_0(x), & \text{for } x \in \Omega,
\end{cases}
\end{align*}
\]

In this subsection, we derive some uniform estimates for solutions to the above initial-boundary value problems.

Firstly we add the equation (3.18) multiplying by \((\lambda + \theta)\) into (3.19) and then, integrate the resulting equation over \(Q_t\). We arrive at

\[
\int_0^1 (\lambda \rho + \rho \theta + \sigma \theta)(x,t) \, dx - \int_0^t H_2(x,\tau) \bigg|_{x=0}^{x=1} d\tau \leq \int_0^1 (\lambda \rho_0 \varepsilon + \rho_0 \varepsilon \theta_0 + \sigma \theta_0)(x) \, dx
\]

where

\[
H_2(x,\tau) = [\varepsilon \rho_x + (\rho \theta)_{,\nu} \rho_x + \rho(x_0 \theta_x)_{,\varepsilon}]((\lambda + \theta) + \kappa^\varepsilon \theta_x).
\]

With boundary conditions in (3.20)-(3.21), we have

\[
-H_2(x,\tau) \bigg|_{x=0}^{x=1} = \alpha^1(\rho(1,\tau) - s\rho^1)(\lambda + \theta(1,\tau)) + \alpha^0(\rho(0,\tau) - s\rho^0)(\lambda + \theta(0,\tau)) + \beta^1(\theta(1,\tau) - s\theta^1) + \beta^0(\theta(0,\tau) - s\theta^0) \\
\geq -\alpha^1 s\rho^1 \theta(1,\tau) - \alpha^0 s\rho^0 \theta(0,\tau) - \lambda s(\alpha^1 \rho^1 + \alpha^0 \rho^0) - s(\beta^1 \theta^1 + \beta^0 \theta^0)
\]

and therefore,

\[
\int_0^1 (\lambda \rho + \rho \theta + \sigma \theta)(x,t) \, dx \leq C_T + C \int_0^t \|\theta(\cdot, \tau)\|_{C(\Omega)} \, d\tau,
\]

where

\[
C_T = (\lambda + \|\theta_0\|_{L^\infty})\|\rho_0\|_{L^1} + \sigma\|\theta_0\|_{L^\infty} + \left[\lambda(\alpha^1 \rho^1 + \alpha^0 \rho^0) + (\beta^1 \theta^1 + \beta^0 \theta^0)\right] T.
\]

Similarly, subtracting the equation (3.19) multiplied by \(\theta^l/l\) from the equation (3.18) multiplied by \(\theta^{l+1}/(l+1)\) and integrating the resulting equation over \(Q_t\), we arrive at

\[
\begin{align*}
\int_0^1 (\rho + \sigma) \theta^{l+1}(x,t) \, dx - \int_0^t H_3(x,\tau) \bigg|_{x=0}^{x=1} d\tau + \int_0^t \int_0^1 \kappa^\varepsilon l(l+1)\theta^{l-1}|\theta_x|^2 \, dx \, d\tau \\
+ s(l+1) \int_0^t \int_0^1 (\lambda + \theta) p_s(\theta) \theta^l \, dx \, d\tau = \int_0^1 (\rho_0 \varepsilon + \sigma)(\theta_0 \varepsilon)^{l+1}(x) \, dx \\
+ s \int_0^t \int_0^1 l\theta^{l+1} + \lambda(l+1)\theta^l \rho \chi^\varepsilon(\sqrt{\theta}) \, dx \, d\tau + s \int_0^t \int_0^1 \chi^\varepsilon(p_s(\theta)) \theta^{l+1} \, dx \, d\tau,
\end{align*}
\]
where

\[-H_3(x, \tau) \bigg|_{x=0}^{x=1} = \alpha^1(\rho(1, \tau) - s\bar{\rho}^1)[\theta(1, \tau)]^{l+1} + \alpha^0(\rho(0, \tau) - s\bar{\rho}^0)[\theta(0, \tau)]^{l+1} + (l + 1) \beta^1(\theta(1, \tau) - s\bar{\theta}^1)[\theta(1, \tau)]^{l+1} + (l + 1) \beta^0(\theta(0, \tau) - s\bar{\theta}^0)[\theta(0, \tau)]^{l+1}\]

\[= [\alpha^1 \rho(1, \tau) + (l + 1) \beta^1 - \alpha^1 s\bar{\rho}^1][\theta(1, \tau)]^{l+1} - (l + 1) \beta^1 s\bar{\theta}^1[\theta(1, \tau)]^{l+1} + [\alpha^0 \rho(0, \tau) + (l + 1) \beta^0 - \alpha^0 s\bar{\rho}^0][\theta(0, \tau)]^{l+1} - (l + 1) \beta^0 s\bar{\theta}^0[\theta(0, \tau)]^{l+1},\]

\[\geq -2(l + 1)[\bar{\alpha}^1(s\bar{\rho}^1)^{l+1} + \bar{\beta}^0(s\bar{\theta}^0)^{l+1}]\]

when \(l\) is large enough. Since \(\theta^l \leq \theta^{1/2} + \theta^{l+1}\) for any \(\theta \geq 0\) and \(l \geq 1\), by (3.22)–(3.23),

\[
\int_0^1 (\rho + \sigma)\theta^{l+1}(x, t)dx + l(l + 1) \int_0^1 \int_0^1 \kappa^s \theta^{l-1}|\theta_x|^2 dxd\tau + sl \int_0^1 \int_0^1 (\lambda + \theta)p_s(\theta)\theta^l dxd\tau
\]

\[
\leq C_{l,T} + C_0s \int_0^1 \int_0^1 l(1 + \theta^{l+1/2})\rho\theta dxd\tau
\]

\[
\leq C'_{l,T} + C_0sl \int_0^t \|\theta(\cdot, \tau)\|_{L^{\infty}(\Omega)} d\tau
\]

where

\[C_{l,T} = \int_0^1 (\rho_0 + \sigma)\theta_0^{l+1}(x)dx + 2(l + 1)[\beta^1(s\bar{\rho}^1)^{l+1} + \beta^0(s\bar{\theta}^0)^{l+1}]\]

and \(C'_{l,T} = C_{l,T} + Cl\). Recall the Gagliardo–Nirenberg inequality

\[
\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)}^{1/2}\|fx\|_{L^2(\Omega)}^{1/2}, \quad \forall f \in H^1(\Omega).
\]

With \(f = \theta^{l+1/2}\) in the above inequality, we obtain

\[
\|\theta(\cdot, \tau)\|_{L^\infty(\Omega)} \leq \frac{C_2}{2} \int_0^1 \theta^{l+3/2}(x, \tau)dx + C_1\|\theta^{l+1}(\cdot, \tau)\|_{L^2(\Omega)} \|\theta^{l+1}x(\cdot, \tau)\|_{L^2(\Omega)}\]

and by Hölder’s inequality,

\[
\int_0^t \|\theta^{l+1}(\cdot, \tau)\|_{L^2(\Omega)} \|\theta^{l+1}x(\cdot, \tau)\|_{L^2(\Omega)} d\tau
\]

\[
\leq C_l \int_0^t \int_0^1 \left(\theta^{l+1}\right)^{\frac{2l+4}{l+1}} dx d\tau + \frac{1}{(l + 1)C_0C_1} \int_0^t \int_0^1 \kappa^s|\theta^{l+1}x|^2 dxd\tau
\]

\[
\leq C_l \int_0^t \int_0^1 \theta^\frac{(l+1)(2l+4)}{2l+1} dx d\tau + \frac{l + 1}{4C_0C_1} \int_0^t \int_0^1 \kappa^s\theta^{l-1}|\theta_x|^2 dxd\tau.
\]

It follows that

\[
\int_0^t \|\theta(\cdot, \tau)\|_{L^\infty(\Omega)} d\tau \leq \frac{C_2}{2} \int_0^t \int_0^1 \theta^{l+3/2} dx d\tau + Cl \int_0^t \int_0^1 \theta^\frac{(l+1)(2l+4)}{2l+1} dx d\tau + \frac{l + 1}{4C_0} \int_0^t \int_0^1 \kappa^s\theta^{l-1}|\theta_x|^2 dxd\tau.
\]
By the assumption (1.9), we observe that $C_0C_2\theta^{l+\frac{2}{p}} \leq p_\sigma(\theta)\theta^{l+1} + C$ for all $\theta \geq 0$. Substituting the last inequality into (3.24) gives

$$
\int_0^1 (\rho + \sigma)\theta^{l+1}(x,t)dx + \frac{l(l+1)}{2} \int_0^1 \int_0^1 \kappa^i \theta^{l+1} |\theta_x|^2 dxd\tau + \frac{sl}{2} \int_0^t \int_0^1 p_\sigma(\theta)\theta^{l+1} dxd\tau
\leq C_{l,T} + C_3 s^2 \int_0^t \int_0^1 \rho^{(l+1)(2l+3)\sigma} dxd\tau,
$$

(3.25)

for $l$ being large enough. Let $l_0$ be a positive integer satisfying

$$
\frac{(l_0 + 1)(2l_0 + 3)}{2l_0 + 1} = l_0 + 1 + \frac{2l_0 + 2}{2l_0 + 1} < l_0 + 1 + (1 + \eta)
$$

where $\eta$ is defined in (1.9). By noting the fact

$$
C_3^0 \theta^{(l_0+1)(2l_0+3)\sigma} \leq \frac{1}{4l_0} p_\sigma(\theta)\theta^{l_0+1} + (C_0)^{l_1}
$$

with $l_1 = 2(l_0 + 2 + \eta)/\eta$, we have

$$
\int_0^1 (\rho + \sigma)\theta^{l_0+1}(x,t)dx + \frac{l_0(l_0+1)}{2} \int_0^t \int_0^1 \kappa^i \theta^{l_0+1} |\theta_x|^2 dxd\tau + \frac{sl}{4} \int_0^t \int_0^1 p_\sigma(\theta)\theta^{l_0+1} dxd\tau \leq C_{l_0,T}^\prime
$$

where $C_{l_0,T}^\prime = C_{l_0,T} + C_T(C_0)^{l_1}$ for some constant $C_T$ independent of $l_0$. Furthermore,

$$
\sup_{0 \leq t \leq T} \int_0^1 \theta^{l_0+1}(x,t)dx + \int_0^T \int_0^1 |(\theta^{l_0+1})_x|^2 dxd\tau \leq C_{l_0,T}^\prime
$$

and by the Sobolev embedding inequality,

$$
\int_0^T \|\theta\|_{L_\infty(\Omega)}^{l_0+1} dxd\tau \leq C_T^{l_0+1} C_{l_0,T}^\prime
$$

Since $l_0$ is a fixed positive integer dependent solely upon $\eta$, we obtain the estimate

$$
\int_0^T \|\theta\|_{L_\infty(\Omega)} dxd\tau \leq C_T.
$$

(3.26)

From (3.22) and (3.24), we get

$$
\sup_{0 \leq t \leq T} \int_0^1 (\rho + \rho \theta)dx \leq C_T
$$

(3.27)

and

$$
\int_0^1 (\rho + \sigma)\theta^{l+1}(x,t)dx + l(l+1) \int_0^t \int_0^1 \kappa^i \theta^{l+1} |\theta_x|^2 dxd\tau + sl \int_0^t \int_0^1 (\lambda + \theta)p_\sigma(\theta)\theta^{l+1} dxd\tau
\leq C_{l,T} + Csl \int_0^t \int_0^1 \rho \theta dxd\tau + Csl \int_0^t \|\theta\|_{L_\infty(\Omega)}^{1/2} \int_0^1 \rho \theta^{l+1} dxd\tau
\leq (C_{l,T} + C_Tl) + Csl \int_0^t \|\theta\|_{L_\infty(\Omega)}^{1/2} \int_0^1 (\rho + \sigma)\theta^{l+1} dxd\tau.
$$
Moreover, by using Gronwall’s inequality,
\[ \int_0^1 (\rho + \sigma)\theta^l + 1(x, t)dx \leq (C_{l,T} + C_T l) + (C_{l,T} + C_T l)e^{C_T} \]
and
\[ \|\theta\|_{L^{l+1}(Q_T)} \leq [2(C_{l,T} + C_T l)]^{\frac{1}{l}} e^{C_T} . \]

On the other hand, by taking \( l \to \infty \), we have
\[ \|\theta\|_{L^\infty(Q_T)} \leq C_T \] (3.28)
where we have noted the fact
\[ C_{l,T}^{\frac{1}{l}} \leq C_T . \]

Moreover, by taking \( l = 1 \) in the equation (3.24), we obtain
\[ \int_0^1 (\rho + \sigma)\theta^2(x, t)dx + \frac{1}{2} \int_0^t \int_0^1 ((\kappa_1 + \kappa_2|\rho|_2^2)|\theta_x|^2 + s\theta^2 p_s(\theta))dxdt \leq C_T , \]
which implies that
\[ \|\theta_x\|_{L^2(Q_T)}, \|\rho \varepsilon \theta_x\|_{L^2(Q_T)} \leq C_T . \] (3.29)

Secondly we present some estimates for \( \rho \). By multiplying \( \rho \) on both sides of the equation (3.18) and integrating the resulting equation over \( Q_T \), with Gronwall’s inequality we get
\[ \sup_{0 \leq t \leq T} \int_0^1 \rho^2 dx + \int_0^T \int_0^1 |\rho|^2 dxdt \leq C_{\varepsilon,T} + C(\varepsilon, \|\rho \varepsilon \theta_x\|_{L^\infty(Q_T)}) \leq C_{\varepsilon,T} , \] (3.30)
which together with the Sobolev embedding inequality gives
\[ \int_0^T \int_0^1 \rho^6 dxdt \leq C_{\varepsilon,T} . \]

Once again, multiplying \( \rho^3 \) on both sides of the equation (3.18) and integrating the resulting equation over \( Q_T \) lead to
\[ \sup_{0 \leq t \leq T} \int_0^1 \rho^4 dx + \int_0^T \int_0^1 \rho^2 |\rho|^2 dxdt \leq C_{\varepsilon,T} . \] (3.31)

From (3.28), (3.29) and (3.30), we conclude that \( (\rho, \theta) \) is uniformly bounded in \( X^2 \). Thus, by the Leray–Schauder fixed point theorem, there exists a fixed point \( (\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu}) \) for the mapping \( M(\cdot, \cdot, 1) : X^2 \to X^2 \) and \( (\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu}) \) is a solution of the system (3.2)-(3.3).

### 3.3 Positivity of the approximate solutions

Finally we prove the positivity of the approximate solutions \( (\rho^{\varepsilon,\nu}, \theta^{\varepsilon,\nu}) \). Let \( \tilde{\theta}^\delta = \theta e^t - \delta \). Then \( \tilde{\theta}^\delta \) is the solution of the following problem,
\[ (\rho + \sigma)\tilde{\theta}^\delta_t - (\kappa^\varepsilon \tilde{\theta}^\delta_x)_x - [(\varepsilon + (\rho \theta)_{\nu})\rho_x + \rho(\rho \varepsilon \theta_x)_\varepsilon] \tilde{\theta}^\delta_x - (\rho + \sigma)\tilde{\theta}^\delta - \rho \chi^\varepsilon(\sqrt{\theta})\tilde{\theta}^\delta + q(\theta e^t, \delta)\tilde{\theta}^\delta = \rho \chi^\varepsilon(\sqrt{\theta})\theta e^t + \lambda \rho \chi^\varepsilon(\sqrt{\theta})e^t + (\rho + \sigma)\delta - (\lambda + e^{-t} \delta)p_s(e^{-t} \delta)e^t , \] (3.32)
with the initial and boundary conditions
\[
\begin{cases}
\kappa^e \tilde{\theta}^\delta_x + \beta^1 \tilde{\theta}^\delta = \beta^1 (\tilde{\theta}^l e^t - \delta), & \text{at } x = 1, \\
-\kappa^e \tilde{\theta}^\delta_x + \beta^0 \tilde{\theta}^\delta = \beta^1 (\tilde{\theta}^l e^t - \delta), & \text{at } x = 0, \\
\tilde{\theta}^\delta(x, 0) = \theta_{0e}(x) - \delta, & \text{for } x \in \Omega,
\end{cases}
\]
where
\[
\tilde{q}(\tilde{\theta}, \delta) = \frac{(\lambda + e^{-t}\tilde{\theta})p_s(e^{-t}\tilde{\theta}) - (\lambda + e^{-t}\delta)p_s(e^{-t}\delta)}{\theta - \delta} e^t \geq 0.
\]
By the assumption (1.9), the right hand side of the equations (3.32)-(3.33) are nonnegative if \( \delta \) is small enough (independent of \( \varepsilon \) and \( \nu \)). Multiplying (\( \tilde{\theta}^\delta \))^-/(\( \rho + \sigma \)) on both sides of the equation (3.32) and integrating the resulting equation over \( Q_t \), we derive \( \tilde{\theta}^\delta \geq 0 \), i.e. \( \theta \geq e^{-T}\delta \), which together with (3.28) implies that
\[
\bar{\theta}_T \leq \theta(x, t) \leq \tilde{\theta}_T \text{ for } (x, t) \in Q_T.
\]
where \( \bar{\theta}_T \) and \( \tilde{\theta}_T \) are positive constants independent of \( \varepsilon \) and \( \nu \).

For \( \rho \), we define \( \rho^\delta = \rho - \delta \). Then \( \rho^\delta \) is the solution of the following equation
\[
\rho^\delta_x - ((\varepsilon + (\rho\theta)_{\nu})\rho^\delta_x) - (\rho^\delta(\rho e\theta_x)_{\varepsilon}) x + \rho^\delta \chi^\varepsilon(\sqrt{\theta}) = \chi^\varepsilon(p_s(\theta)) + \delta[(\rho e\theta_x)_{\varepsilon}]_x - \delta \chi^\varepsilon(\sqrt{\theta}),
\]
with the initial and boundary conditions
\[
\begin{cases}
(\varepsilon + (\rho\theta)_{\nu})\rho^\delta_x + \rho^\delta(\rho e\theta_x)_{\varepsilon} + \alpha^1 \rho^\delta = \alpha^1(\rho^l - \delta) - \delta(\rho e\theta_x)_{\varepsilon}, & \text{at } x = 1, \\
-((\varepsilon + (\rho\theta)_{\nu})\rho^\delta_x - \rho^\delta(\rho e\theta_x)_{\varepsilon} + \alpha^0 \rho^\delta = \alpha^0(\rho^0 - \delta) + \delta(\rho e\theta_x)_{\varepsilon}, & \text{at } x = 0, \\
\rho^\delta(x, 0) = \rho_{0e}(x) - \delta, & \text{for } x \in \Omega.
\end{cases}
\]
Since \( \chi^\varepsilon(p_s(\theta)) \geq \varepsilon \), the right hand side of the equations (3.35)-(3.36) are nonnegative if
\[
\delta = \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{1 + \|p_s\|_{L^1(Q_T)}}\right\},
\]
in which case \( \rho^\delta \geq 0 \), or equivalently \( \rho \geq \delta \). On the other hand, from (3.29) we have
\[
\|p_s\|_{L^1(Q_T)} \leq \frac{1}{\varepsilon^2} \leq \frac{1}{\varepsilon^2} \|\rho^l\|_{L^1(Q_T)} \leq C \varepsilon.
\]
Thus, there exists a positive constant \( \rho_{\varepsilon,T} \) such that
\[
\rho \geq \rho_{\varepsilon,T} \text{ for } (x, t) \in Q_T.
\]
\[\text{(3.37)}\]

### 4 Global existence

We have constructed an approximate solution \( (\rho^{e,\nu}, \theta^{e,\nu}) \) to the system (3.1) and (3.3) (or equivalently (3.2) \( , \) (3.3)) in the last section. In this section, we prove the global existence of weak solutions for the system (1.5)-(1.8). Firstly we fix \( \varepsilon > 0 \) and study the convergence as \( \nu \to 0 \).

Since the system (3.13)-(3.19) reduces to (3.2)-(3.3) when \( s = 1 \), the uniform estimates (3.28), (3.29), (3.30) and (3.34) given in the last section still hold for the approximate solution \( (\rho^{e,\nu}, \theta^{e,\nu}) \). We rewrite the first equation in (3.2) by
\[
\rho_t = -f_x + g
\]
with \( g \) uniformly bounded in \( L^2(Q_T) \) and
\[
f = (\varepsilon + (\rho\theta)_x)\rho + \rho(\rho\theta_x)_\varepsilon.
\]
Since \( \rho \) is uniformly bounded in \( L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)) \), we derive that
\[
\|\rho(\rho\theta_x)_\varepsilon\|_{L^2(Q_T)}, \quad \|((\rho\theta)_x)\|_{L^6(Q_T)}, \quad \|((\rho\theta)_x)\|_{L^{1/2}(Q_T)} \leq C_{\varepsilon,T}
\]
and
\[
\|\rho_t\|_{L^{5/4}(I; W^{-1,5/4}_0(\Omega))} \leq C_{\varepsilon,T}.
\]
From the first equation in (3.1) we derive that
\[
\|((\rho\theta + \sigma\theta)_t\|_{L^{5/4}(I; W^{-1,5/4}_0(\Omega))} \leq C_{\varepsilon,T}
\]
where we have noted (3.28), and moreover, from (3.31), we observe that \( \rho^{\varepsilon,\nu} \) is uniformly bounded in \( L^6(I; L^6(\Omega)) \cap L^2(I; H^1(\Omega)) \) and \( \rho^{\varepsilon,\nu}_t \) is uniformly bounded in \( L^{5/4}(I; W^{-1,5/4}_0(\Omega)) \). Using Aubin–Lions lemma, we conclude that there exists a sequence \( \nu_j \to 0 \) such that
\[
\rho^{\varepsilon,\nu_j} \to \rho^{\varepsilon} \quad \text{strongly in} \quad L^p(Q_T) \quad (\forall 1 \leq p < 6),
\]
\[
\rho^{\varepsilon,\nu_j} \to \rho^{\varepsilon} \quad \text{strongly in} \quad L^2(I; C(\Omega)),
\]
\[
\rho^{\varepsilon,\nu_j} \to \rho^{\varepsilon} \quad \text{weakly in} \quad L^2(I, H^1(\Omega)),
\]
\[
\rho^{\varepsilon,\nu_j}_t \to \rho^{\varepsilon}_t \quad \text{weakly in} \quad L^{5/4}(I; W^{-1,5/4}_0(\Omega))
\]
and
\[
\rho^{\varepsilon,\nu_j}(0, \cdot) \to \rho^{\varepsilon}(0, \cdot) \quad \text{and} \quad \rho^{\varepsilon,\nu_j}(1, \cdot) \to \rho^{\varepsilon}(1, \cdot) \quad \text{strongly in} \quad L^2(0, T).
\]
Similarly, by noting the uniform estimates (3.28), (3.29) and (3.31), we conclude that there exists a subsequence of \( \theta^{\varepsilon,\nu_j} \) (also denoted by \( \theta^{\varepsilon,\nu_j} \)) such that
\[
\theta^{\varepsilon,\nu_j} \to \theta^{\varepsilon} \quad \text{strongly in} \quad L^p(Q_T) \quad (\forall 1 \leq p < \infty),
\]
\[
\theta^{\varepsilon,\nu_j} \to \theta^{\varepsilon} \quad \text{strongly in} \quad L^2(I; C(\Omega)),
\]
\[
\theta^{\varepsilon,\nu_j} \to \theta^{\varepsilon} \quad \text{weakly in} \quad L^2(I, H^1(\Omega)),
\]
\[
(\rho^{\varepsilon,\nu_j} \theta^{\varepsilon,\nu_j} + \sigma \theta^{\varepsilon,\nu_j})_t \to (\rho^{\varepsilon} \theta^{\varepsilon} + \sigma \theta^{\varepsilon})_t \quad \text{weakly in} \quad L^{5/4}(I; W^{-1,5/4}_0(\Omega))
\]
and
\[
\theta^{\varepsilon,\nu_j}(0, \cdot) \to \theta^{\varepsilon}(0, \cdot) \quad \text{and} \quad \theta^{\varepsilon,\nu_j}(1, \cdot) \to \theta^{\varepsilon}(1, \cdot) \quad \text{strongly in} \quad L^p(0, T), \ 1 \leq p < \infty.
\]
Since \( (\rho^{\varepsilon,\nu_j}, \theta^{\varepsilon,\nu_j}) \) is a strong solution of the system (3.1) and (3.3), it satisfies
\[
\int_0^T \alpha^0(\rho^{\varepsilon,\nu_j}(0, t) - \rho^0)\phi(0, t)dt + \int_0^T \alpha^1(\rho^{\varepsilon,\nu_j}(1, t) - \rho^1)\phi(1, t)dt
\]
\[
+ \int_0^T \int_{\Omega} \rho^{\varepsilon,\nu_j}_t \phi dx dt + \int_0^T \left[ (\varepsilon + (\rho^{\varepsilon,\nu_j} \theta^{\varepsilon,\nu_j})_x) \rho^{\varepsilon,\nu_j} + \rho^{\varepsilon,\nu_j}(\rho^{\varepsilon,\nu_j} \theta^{\varepsilon,\nu_j})_x \right] \phi_x dx dt
\]
\[
= \int_0^T \int_{\Omega} \chi^\varepsilon(p_\mu(\theta^{\varepsilon,\nu_j})) dx dt - \int_0^T \int_{\Omega} \rho^{\varepsilon,\nu_j} \chi^\varepsilon(\sqrt{\theta^{\varepsilon,\nu_j}}) \phi dx dt
\]
\[
= \int_0^T \int_{\Omega} \chi^\varepsilon(p_\mu(\theta^{\varepsilon,\nu_j})) dx dt - \int_0^T \int_{\Omega} \rho^{\varepsilon,\nu_j} \chi^\varepsilon(\sqrt{\theta^{\varepsilon,\nu_j}}) \phi dx dt
\]

14
and
\[
\int_0^T \int_0^1 [(\rho^{\varepsilon, \nu}_s + \sigma) \theta^{\varepsilon, \nu}_s] \psi dx dt + \int_0^T \beta_0 (\theta^{\varepsilon, \nu}(0, t) - \bar{\theta}_0) \psi(0, t) dt + \int_0^T \beta_1 (\theta^{\varepsilon, \nu}(1, t) - \bar{\theta}_1) \psi(1, t) dt \\
+ \int_0^T \alpha_0 (\rho^{\varepsilon, \nu}_s(0, t) - \bar{\rho}_0) \theta^{\varepsilon, \nu}_s(0, t) \psi(0, t) dt + \int_0^T \alpha_1 (\rho^{\varepsilon, \nu}_s(1, t) - \bar{\rho}_1) \theta^{\varepsilon, \nu}_s(1, t) \psi(1, t) dt \\
+ \int_0^T \int_0^1 \kappa \theta^{\varepsilon, \nu}_x \psi_x dx dt + \int_0^T \int_0^1 [(\varepsilon + (\rho^{\varepsilon, \nu}_s \theta^{\varepsilon, \nu}_s)_\nu) \rho^{\varepsilon, \nu}_s \theta^{\varepsilon, \nu}_s + \rho^{\varepsilon, \nu}_s (\rho^{\varepsilon, \nu}_s \theta^{\varepsilon, \nu}_s)_\varepsilon \theta^{\varepsilon, \nu}_s] \psi_x dx dt \\
+ \int_0^T \int_0^1 (\lambda + \theta^{\varepsilon, \nu}_s)p_s (\theta^{\varepsilon, \nu}_s) \psi dx dt \\
= \lambda \int_0^T \int_0^1 \rho^{\varepsilon, \nu}_s \chi^{\varepsilon} (\sqrt{\theta^{\varepsilon, \nu}_s}) \psi dx dt + \lambda \int_0^T \int_0^1 \theta^{\varepsilon, \nu}_s \chi^{\varepsilon} (p_s (\theta^{\varepsilon, \nu}_s)) \psi dx dt,
\]
for any \( \phi, \psi \in L^5(I; W^{1,5}(\Omega)) \). By taking the limit \( j \to \infty \), we obtain a global weak solution \((\rho^\varepsilon, \theta^\varepsilon)\) to the approximate system
\[
\rho_t - ((\varepsilon + \rho \theta) \rho_x)_x - (\rho_x \theta_x)_{\varepsilon x} = -\Gamma_\varepsilon,
\]
\[
(\rho_t + \sigma \theta)_x - ((\varepsilon + \rho \theta)) \rho_x (\theta_x)_{\varepsilon x} - ((\varepsilon + \rho \theta)) \rho_x \theta_x - (\rho \theta_x)_{\varepsilon x} = (\varepsilon + \rho \theta)(\rho_x \theta_x)_{\varepsilon x} - (\rho \theta_x)_{\varepsilon x}
\]
\[
= \lambda \Gamma_\varepsilon + (\lambda + \theta)(\chi^{\varepsilon} (p_s (\theta)) - p_s (\theta)),
\]
with the boundary and initial conditions
\[
(\varepsilon + \rho \theta) \rho_x |_{x=1} + \rho (\rho_x \theta_x)_{\varepsilon} |_{x=1} = \alpha_1 \bar{\rho}^1 - \rho (1, t)),
\]
\[
(\varepsilon + \rho \theta) \rho_x |_{x=0} + \rho (\rho_x \theta_x)_{\varepsilon} |_{x=0} = \sigma (\rho(0, t) - \bar{\rho}^0),
\]
\[
\rho(x, 0) = \rho_0 (x) := \rho_0 \ast \eta_\varepsilon (x) + \varepsilon,
\]
\[
\kappa \theta_x |_{x=1} = \beta_1 (\bar{\theta}^1 - \theta (1, t)),
\]
\[
\kappa \theta_x |_{x=0} = \beta_0 (\theta (0, t) - \bar{\theta}^0),
\]
\[
\theta(x, 0) = \theta_0 (x) := \theta_0 \ast \eta_\varepsilon (x).
\]

Secondly, we study the convergence as \( \varepsilon \to 0 \). To take the limit \( \varepsilon \to 0 \), we need more uniform estimates for \( \rho \) with respect to \( \varepsilon \).

Clearly the system (3.22)-(3.33) reduces to the system (4.3)-(4.4) when \( \nu = 0 \). Then the uniform estimates (3.16) and (3.17) hold for the obtained solution \((\rho^\varepsilon, \theta^\varepsilon)\). From (3.31) we see that
\[
\|\rho^\varepsilon \rho_x\|_{L^2(Q_T)} \leq C_{\varepsilon, T}
\]
and from the first equation of (4.3) we deduce that \( \rho_t \in L^2(I; H^{-1}_0 (\Omega)) \). Note that \( \ln \rho \in L^2(I; H^1(\Omega)) \). By multiplying the first equation of (4.3) by \( \ln \rho \) and integrating the equation over \( Q_T \), we arrive at
\[
\int_0^1 \rho \ln \rho (x, t) dx - \int_0^1 \rho (x, t) dx + \int_0^t \int_0^1 \left[ \varepsilon \rho_x + \rho \theta \rho_x + \rho (\rho_x \theta_x)_{\varepsilon} \right] \ln \rho |_{x=1} \rho_x dx dt + \int_0^t \int_0^1 \rho_x^2 dx dt \\
\leq \int_0^1 \rho_0 \varepsilon \ln \rho_0 (x) dx - \int_0^1 \rho_0 (x) dx - \int_0^t \int_0^1 \rho_x (\rho_x)_{\varepsilon x} \rho_x dx dt - \int_0^t \left( \rho \sqrt{\theta} - p_s (\theta) \right) \ln \rho dx dt.
\]
Since
\[
\int_0^t \int_0^1 |\rho \varepsilon \partial_x \rho_x| dx d\tau \leq \frac{1}{2} \int_0^t \int_0^1 \theta \rho_x^2 dx d\tau + \frac{1}{2} \int_0^t \int_0^1 \frac{|\rho \varepsilon \partial_x \rho_x|^2}{\theta} dx d\tau
\]
\[
\leq \frac{1}{2} \int_0^t \int_0^1 \theta \rho_x^2 dx d\tau + C_T \|\rho \varepsilon \partial_x \rho_x\|_{L^2(Q_T)}^2
\]
\[
\leq \frac{1}{2} \int_0^t \int_0^1 \theta \rho_x^2 dx d\tau + C_T,
\]
we get
\[
\int_{[0,1] \cap \{\rho \geq 1\}} \rho \ln \rho(x,t) dx + \int_0^t \int_{[0,1] \cap \{\rho \leq 1\}} \rho \ln \rho dx dt
\]
\[
\leq \int_0^1 \rho_0 \ln \rho_0 \rho(x) dx + \int_{[0,1] \cap \{\rho \leq 1\}} \rho \ln \rho dx dt
\]
\[
+ \int_0^t \int_{[0,1] \cap \{\rho \leq 1\}} \rho \ln \rho dx dt + \int_0^t \int_{[0,1] \cap \{\rho \geq 1\}} p_\varepsilon(\theta) \ln \rho dx dt + C_T
\]
\[
\leq C_T,
\]
which, together with (3.34), leads to
\[
\|\rho \ln \rho\|_{L^\infty(0,T;L^1(\Omega))}, \quad \|\rho_x\|_{L^2(Q_T)} \leq C_T.
\] (4.5)

From the inequalities (3.27) and (4.5) we derive
\[
\|\rho\|_{L^2(0,T;H^1(\Omega))} \leq C_T
\] (4.6)
and
\[
\|\rho\|_{L^\infty(\Omega)}^2 \leq \|\rho\|_{L^1(\Omega)}^3 + \|\rho\|_{L^2(\Omega)}^{3/2} \|\rho_x\|_{L^2(\Omega)}^{3/2}
\]
\[
\leq C_T + C\|\rho\|_{L^1(\Omega)}^{3/4} \|\rho\|_{L^\infty(\Omega)}^{3/4} \|\rho_x\|_{L^2(\Omega)}^{3/2}
\]
\[
\leq C_T + \frac{1}{2} \|\rho\|_{L^\infty(\Omega)}^2 + C_T \|\rho_x\|_{L^2(\Omega)}^2,
\]
which results in
\[
\int_0^T \|\rho\|_{L^\infty(\Omega)}^2 dt \leq C_T + C_T \int_0^T \|\rho_x\|_{L^2(\Omega)}^2 dt \leq C_T.
\]

Moreover, we have
\[
\int_0^T \int_0^1 \rho^4 dx dt \leq \left( \int_0^T \|\rho\|_{L^\infty(\Omega)}^3 dt \right) \left( \sup_{0 \leq t \leq T} \int_0^1 \rho dx \right) \leq C_T,
\] (4.7)
i.e. \(\rho\) is uniformly bounded in \(L^4(Q_T)\).

Finally, we let
\[
B_1 = H^1(\Omega), \quad B_2 = L^4(\Omega), \quad B_3 = W_0^{-1,6/5}(\Omega).
\]

Then \(B_1 \hookrightarrow B_2 \hookrightarrow B_3\) and \(\{\rho^\varepsilon\}\) is uniformly bounded in \(L^4(I;B_2) \cap L^2(I;B_1)\). From the first equation in (3.2), i.e.
\[
\rho_t = \left[ \varepsilon \rho_x + \rho \theta_x + \rho (\rho \theta_x)_x \right]_x - \rho \chi^\varepsilon(\sqrt{\theta}) + \chi^\varepsilon(p_\varepsilon(\theta)),
\]

i.e. \(\rho\) is uniformly bounded in \(L^4(Q_T)\).
we observe that $\{\rho_\varepsilon^j\}$ is uniformly bounded in $L^{6/5}(I;B_3)$. By Aubin–Lions lemma, $\{\rho_\varepsilon\}$ is relatively compact in $L^p(I;L^4(\Omega))$ for $(1 \leq p < 4)$. Thus, there exists a sequence $\rho_\varepsilon^j$ such that

$$\lim_{j \to \infty} \varepsilon_j = 0 \quad \text{and} \quad \rho_\varepsilon^j \to \rho \quad \text{strongly in} \quad L^p(I,L^4(\Omega)) \quad (\forall 1 \leq p < 4),$$

$$\rho_\varepsilon^j \to \rho \quad \text{strongly in} \quad L^2(I,C(\Omega)),$$

$$\rho_\varepsilon^j \rightharpoonup \rho \quad \text{weakly in} \quad L^2(I,H^1(\Omega)),$$

$$\rho_\varepsilon^j \to \rho_t \quad \text{weakly in} \quad L^{6/5}(I;W_0^{-1,6/5}(\Omega)).$$

Similarly, by (3.16) and (3.17), there exists a subsequence of $\theta_\varepsilon^j$ (also denoted by $\theta_\varepsilon^j$) such that

$$\theta_\varepsilon^j \to \theta \quad \text{strongly in} \quad L^p(Q_T) \quad (\forall 1 \leq p < \infty),$$

$$\theta_\varepsilon^j \to \theta \quad \text{strongly in} \quad L^2(I,C(\Omega)),$$

$$\theta_\varepsilon^j \rightharpoonup \theta \quad \text{weakly in} \quad L^2(I,H^1(\Omega)),$$

$$(\rho_\varepsilon^j \theta_\varepsilon^j + \sigma \theta_\varepsilon^j)_t \rightharpoonup (\rho \theta + \sigma \theta)_t \quad \text{weakly in} \quad L^{6/5}(I;W_0^{-1,6/5}(\Omega)).$$

Now we take the limit $j \to \infty$ and by (4.8) and (4.9), we obtain a weak solution $(\rho, \theta)$ which satisfies (2.1) and (2.2).

**Acknowledgements** The authors wish to thank Professors P. Lei, T. Yang, G. Yuan and X. Xu for helpful discussions.

**References**

[1] Y. Amirat and V. Shelukhin, *Global weak solutions to equations of compressible miscible flow in porous media*, SIAM J. Math. Anal., 38 (2007), 1825-1846.

[2] Y. Z. Chen, *Parabolic Partial Differential Equations of Second Order*, Peking University Press, 2003, (in Chinese).

[3] Z. Chen and R. Ewing, Mathematical analysis for reservoir models, *SIAM J. Math. Anal.*, 30 (1999), pp. 431–453.

[4] A. Cheng and H. Wang, An error estimate on a Galerkin method for modeling heat and moisture transfer in fibrous insulation, *Numer. Methods Partial Differential Equations*, 24(2008), pp. 504–517.

[5] J. Fan, Z. Luo and Y. Li, Heat and moisture transfer with sorption and condensation in porous clothing assemblies and numerical simulation, *Int. J. Heat Mass Transfer*, 43 (2000), pp. 2989-3000.

[6] J. Fan, X. Cheng, X. Wen and W. Sun, An improved model of heat and moisture transfer with phase change and mobile condensates in fibrous insulation and comparison with experimental results, *Int. J. Heat Mass Transfer*, 47 (2004), pp. 2343-2352.

[7] X. Feng, On existence and uniqueness results for a coupled system modeling miscible displacement in porous media, *J. Math. Anal. Appl.*, 194 (1995), pp. 883–910.

[8] X. Hang, W. Sun and C. Ye, Finite volume solution of heat and moisture transfer in three-dimensional textile materials, submitted.
[9] H. Huang, C. Ye and W. Sun, *Moisture transport in fibrous clothing assemblies*, J. Engrg. Math., 61 (2008), pp. 35–54.

[10] F.E. Jones, *Evaporation of Water*, Lewis Publishers Inc., Michigan, 1992, pp. 25-43.

[11] O. A. Ladyzenskaja, V. Solomnikov and N. N. Uralceva, *Linear and nonlinear equations of parabolic type*, Translations of Mathematical Monographs, 23, 1968.

[12] Y. Li and Q. Zhu, Simultaneous heat and moisture transfer with moisture sorption, condensation, and capillary liquid diffusion in porous textiles, *Textile Res. J.*, 73 (2003), pp. 515-524.

[13] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris (in French).

[14] P. L. Lions, *Mathematical topics in fluid mechanics*, Vol. 2 of Oxford Lecture series in Mathematics and its applications. Clarendon Press, Oxford 1998.

[15] A. Novotny, I. Straskraba, *Introduction to the mathematical theory of compressible flow*, Oxford University Press 2004.

[16] Y. Ogniewicz and C.L. Tien, Analysis of condensation in porous insulation, *J. Heat Mass Transfer*, 24 (1981), pp. 421-429.

[17] P. Smith and E.T. Twizell, A transient model of thermoregulation in a clothed human, *Applied Math. Modeling*, 8 (1984), pp. 211-216.

[18] J. Vala, On a system of equations of evolution with a non-symmetrical parabolic part occuring in the analysis of moisture and heat transfer in porous media, *Applications of Math.*, 47 (2002), pp.187–214.

[19] J.A. Wehner, B. Miller, and L. Rebenfeld Moisture Dynamics of water vapor transmission through fabric barriers, *Textile Research Journal* 58 (1988) 581-592.

[20] C. Ye, H. Huang, J. Fan and W. Sun, Numerical study of heat and moisture transfer in textile materials by a finite volume method, *Communications in computational Physics*, 4 (2008), pp. 929-948.