Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla \phi$ systems with non-convex potential

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April 28, 2010

Abstract

We consider a gradient interface model on the lattice with interaction potential which is a non-convex perturbation of a convex potential. Using a technique which decouples the neighboring vertices into even and odd vertices, we show for a class of non-convex potentials: the uniqueness of ergodic component for $\nabla \phi$-Gibbs measures, the decay of covariances, the scaling limit and the strict convexity of the surface tension.

AMS 2000 Subject Classification. 60K35, 82B24, 35J15

Key words and phrases. effective non-convex gradient interface models, uniqueness of ergodic component, decay of covariances, scaling limit, surface tension

1 Introduction

Phase separation in $\mathbb{R}^{d+1}$ can be described by effective interface models, where interfaces are sharp boundaries which separate the different regions of space occupied by different phases. In this class of models, the interface is modeled as the graph of a random function from $\mathbb{Z}^d$ to $\mathbb{Z}$ or $\mathbb{R}$ (discrete or continuous effective interface models). For more on interface models, see the reviews by Funaki [16] or Velenik [21]. In this setting we ignore overhangs and for $x \in \mathbb{Z}^d$, we denote by $\phi(x) \in \mathbb{R}$ the height of the interface above or below the site $x$. Let $\Lambda$ be a finite set in $\mathbb{Z}^d$ with boundary

$$\partial \Lambda := \{ x \notin \Lambda, \| x - y \| = 1 \text{ for some } y \in \Lambda \},$$

where $\| x - y \| = \sum_{i=1}^{d} |x_i - y_i|$ for $x, y \in \mathbb{Z}^d$ (1)

and with given boundary condition $\psi$ such that $\phi(x) = \psi(x)$ for $x \in \partial \Lambda$; a special case of boundary conditions are the tilted boundary conditions, with $\psi(x) = x \cdot u$ for all $x \in \partial \Lambda$, and where $u \in \mathbb{R}^d$ is fixed. Let $\Lambda := \Lambda \cup \partial \Lambda$ and let $d\phi_\Lambda = \prod_{x \in \Lambda} d\phi(x)$ be the Lebesgue measure over $\mathbb{R}^\Lambda$. For a finite region $\Lambda \subset \mathbb{Z}^d$, the finite volume Gibbs measure $\nu_{\Lambda, \psi}$ on $\mathbb{R}^{\mathbb{Z}^d}$ with boundary condition $\psi$ for the field of height variables $(\phi(x))_{x \in \mathbb{Z}^d}$ over $\Lambda$ is defined by

$$\nu_{\Lambda, \psi}(d\phi) = \frac{1}{Z_{\Lambda, \psi}} \exp\{-\beta H_{\Lambda, \psi}(\phi)\} \, d\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d \setminus \Lambda}),$$

with

$$Z_{\Lambda, \psi} = \int_{\mathbb{R}^{\mathbb{Z}^d}} \exp\{-\beta H_{\Lambda, \psi}(\phi)\} \, d\phi_\Lambda \delta_\psi(d\phi_{\mathbb{Z}^d \setminus \Lambda}),$$

$^*$Supported by the DFG-Forschergruppe 718 ‘Analysis and stochastics in complex physical systems’

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and where $\delta_\psi(d\phi \mid _\Lambda) = \prod_{x \in \mathbb{Z}^d \setminus \Lambda} \delta_\psi(x)(d\phi(x))$ and determines the boundary condition. Thus, $\nu_{\Lambda, \psi}$ is characterized by the inverse temperature $\beta > 0$ and the Hamiltonian $H_{\Lambda, \psi}$ on $\Lambda$, which we assume to be of gradient type:

$$H_{\Lambda, \psi}(\phi) = \sum_{i \in I} \sum_{x, x + e_i \in \Lambda} U(\nabla_i \phi(x)) + 2 \sum_{i \in I} \sum_{x \in \Lambda, x + e_i \in \partial \Lambda} U(\nabla_i \phi(x)),$$

where the sum inside $\Lambda$ is over ordered nearest neighbours pairs $(x, x + e_i)$. We denoted by

$$I = \{-d, -d + 1, \ldots, d\} \setminus \{0\}$$

and we introduced for each $x \in \mathbb{Z}^d$ and each $i \in I$, the discrete gradient

$$\nabla_i \phi(x) = \phi(x + e_i) - \phi(x),$$

that is, the interaction depends only on the differences of neighboring heights. Note that $e_i, i = 1, 2, \ldots, d$ denote the unit vectors and $e_{-i} = -e_i$. A model with such a Hamiltonian as defined in [3], is called a massless model with a continuous symmetry (see [16]). The potential $U \in C^2(\mathbb{R})$ is a symmetric function with quadratic growth at infinity:

$$U(\eta) \geq A|\eta|^2 - B, \quad \eta \in \mathbb{R} \quad (A0)$$

for some $A > 0, B \in \mathbb{R}$. Our state space $\mathbb{R}^{\mathbb{Z}^d}$ being unbounded, such models experience delocalization in lower dimensions $d = 1, 2$, and no infinite volume Gibbs state exists in these dimensions (see [14]). Instead of looking at the Gibbs measures of the $(\phi(x))_{x \in \mathbb{Z}^d}$, Funaki and Spohn proposed to consider the distribution of the gradients $(\nabla_i \phi(x))_{i \in I, x \in \mathbb{Z}^d}$ under $\nu$ (see Definition 2 in section 2.1 below) in the so-called gradient Gibbs measures $\mu$, which in view of the Hamiltonian [8], can also be given in terms of a Dobrushin-Lanford-Ruelle (DLR) description (see Definition 5 in section 2.2.2 below). Note that infinite volume gradient Gibbs measures exist in all dimensions, in particular for dimensions 1 and 2, which is one of the reasons that Funaki and Spohn introduced them. For a good background source on these models, see Funaki [16].

Assuming strict convexity of $U$:

$$0 < C_1 \leq U'' \leq C_2 < \infty, \quad (4)$$

Funaki and Spohn showed in [15] the existence and uniqueness of ergodic gradient Gibbs measures for every fixed tilt $u \in \mathbb{R}^d$, that is, if $E_\mu(\nabla_i \phi(x)) = u_i$ for all nearest-neighbour pairs $(x, x + e_i)$ (see also [20]). Moreover, they also proved that the corresponding free energy, or surface tension, $\sigma(u) \in C^1(\mathbb{R}^d)$ is convex in $u$; the surface tension, defined in section 8 of our paper, physically describes the macroscopic energy of a surface with tilt $u$, i.e., a $d$-dimensional hyperplane located in $\mathbb{R}^{d+1}$ with normal vector $(-u, 1) \in \mathbb{R}^{d+1}$. Both these results (ergodic component and convexity of surface tension) were used in [15] for the derivation of the hydrodynamical limit of the Ginzburg-Landau model.

In fact under the strict convexity assumption [11] of $U$, much more is known for the gradient field. At large scales it behaves much like the harmonic crystal or gradient free fields which is a Gaussian field with quadratic $U$. In particular, Brydges and Yao [7] (in the case of small analytic perturbations of quadratic potentials) and Naddaf and Spencer [19] (in the case of strictly convex potentials and tilt $u = 0$) showed that the rescaled gradient field converges weakly as $\epsilon \searrow 0$ to a continuous homogeneous Gaussian field, that is

$$S_\epsilon(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \sum_{i \in I} (\nabla_i \phi(x) - u_i) f_i(\epsilon x) \to N(0, \Sigma^2_\epsilon(f)) \quad \text{as} \quad \epsilon \to 0, \quad f \in C^\infty_0(\mathbb{R}^d, \mathbb{R}^d), \quad (5)$$

where the convergence takes place under ergodic $\mu$ with tilt $u$ (see Theorem 2.1 in Giacomin, Olla and Spohn [17] for an explicit expression of $\Sigma^2_\epsilon(f)$ in [5] and see Biskup and Spohn [3] for
similar results). This central limit theorem derived at standard scaling $e^{d/2}$, is far from trivial since as shown in Delmotte and Deuschel [9], the gradient field has slowly decaying, non-absolutely summable covariances of the algebraic order

$$|\text{Cov}_\nu(\nabla_i \phi(x), \nabla_j \phi(y))| \sim \frac{C}{1 + ||x - y||^2},$$  \hspace{1cm} (6)

All the above-mentioned results are proved under the essential assumption of strict convexity of the potential $U$, which assumption is necessary for the application of the Brascamp-Lieb inequality and of the Helffer-Sjostrand random walk representation (see [16] for a detailed review of these methods and results). While strict convexity is crucial for the proofs, one would expect some of these results to be valid under more general circumstances, in particular also for some classes of non-convex potentials. However, so far there have been very few results on non-convex potentials. This is where the focus of this paper comes in, which is to extend the results known for strictly convex potentials to some classes of non-convex potentials.

We will briefly summarize next the state of affairs regarding results for non-convex potentials, in the different regimes at inverse temperature $\beta$. At low temperature (i.e. large $\beta$) using the renormalization group techniques developed by Brydges [6], Adams et al. [1] show in on-going work for a class of non-convex potentials, the strict convexity of the surface tension for small tilt $u$. At moderate temperature ($\beta = 1$), Biskup and Kotecký [2] give an example of a non-convex potential $U$ for which uniqueness of the ergodic gradient Gibbs measures $\mu$ fails. The potential $U$ can be described as the mixture of two Gaussians with two different variances. For this particular case of $U$, [2] prove co-existence of two ergodic gradient Gibbs measures at tilt $u = 0$ (see also Figure 5 and example 4.2 (a) below). Similar results to [2] concerning discrete models have been proved for example by van Enter and Shlosman in [13]. For high temperature (i.e. small $\beta$), we have proved in a previous paper with S. Mueller [8] strict convexity of the surface tension in a regime similar to (A2) below. Our potentials are of the form

$$U(\nabla_i \phi(x)) = V(\nabla_i \phi(x)) + g(\nabla_i \phi(x))$$

where $V, g \in C^2(\mathbb{R})$ are such that

$$C_1 \leq V'' \leq C_2, \hspace{0.5cm} 0 < C_1 < C_2 \hspace{0.5cm} \text{and} \hspace{0.5cm} -C_0 \leq g'' \leq 0, \hspace{0.5cm} \text{with} \hspace{0.5cm} C_0 > C_2.$$  \hspace{1cm} (A1)

Specifically, we assumed in [8] that

$$\frac{4}{\pi} (12d\bar{C})^{1/2} \sqrt{\beta C_1} \frac{1}{C_1} ||g''||_{L^1(\mathbb{R})} \leq \frac{1}{2}, \hspace{0.5cm} \text{where} \hspace{0.5cm} \bar{C} = \max \left( \frac{C_0}{C_1}, \frac{C_2}{C_1} - 1, 1 \right).$$

The method used in [8], based on two scale decomposition of the free field, gives less sharp estimates for the temperature than our current paper as the estimates also depend on $C_0$. However, at this point it is not clear whether the method introduced in [8] could yield any other result of interest than the strict convexity of the surface tension.

The aim of our current paper is to use an alternative technique from the one we used in [8] and relax the strict convexity assumption [11] to obtain much more than just strict convexity of the surface tension; more precisely, we also prove uniqueness of the ergodic component at every tilt $u \in \mathbb{R}^d$, central limit theorem of form as given in [3] and decay of covariances as in [10]. As stated above, the hydrodynamical limit for the corresponding Ginzburg-Landau model should then essentially follow from our results. Our main results are proven under the assumption that the inverse temperature $\beta$ is sufficiently small, that is if:

$$\beta \frac{1}{2d} ||g''||_{L^2(\mathbb{R})} < \frac{(C_1)^{3/2}}{2C_2^{1/2} (2d)^{3/2}}, \hspace{0.5cm} \text{for some} \hspace{0.5cm} q \geq 1,$$  \hspace{1cm} (A2)
or if
\[ \beta^2 \|g'\|_{L^2(\mathbb{R})} \leq \frac{(C_1)^{\frac{3}{2}}}{2(C_2)^{\frac{3}{2}}(2d)^{\frac{3}{2}}} , \] (A3)

The condition (A1) with \( g'' \leq 0 \) may look a bit artificial, but as we elaborate in Remark 30 in section 4 below, any perturbation \( g \in C^2 \) with compact support can be substituted for the \( g'' \leq 0 \) assumption in (A1). Note that in contrast to the condition in our previous paper [8], \( \|g''\|_{L^\infty(\mathbb{R})} \) can be arbitrarily large as long as \( \|g''\|_{L^2(\mathbb{R})} \) is small. Note also that using an obvious rescaling argument (see Remark 24), we can always reduce our assumption (A1) to the case \( \beta = C_1 = 1 \); then (A2), respectively (A3), states that our condition is satisfied whenever the perturbation \( g'' \) is small in the \( L^2(\mathbb{R}) \), respectively \( g' \) is small in the \( L^2(\mathbb{R}) \) sense.

Due to the fact that some of these results are technical in nature, with a lot of notation to be defined precisely before the result can be stated, we will not formulate them formally in the introduction but defer this till later. To be more precise, the uniqueness of ergodic gradient Gibbs measure result is formulated precisely in Theorem 31 in section 5, the decay of covariance result in Theorem 37 in section 6, the central limit result in Theorem 39 in section 7 and the strict convexity of the surface tension result in Theorem 42 in section 8.

Even though our results are obtained for the high temperature case, nevertheless nothing was known apart from our previous result in [8], and the proofs of this paper require some crucial observations not made before. Moreover, in our main result Theorem 31, we prove uniqueness of ergodic gradient Gibbs measures \( \mu \) with a given arbitrary tilt \( u \in \mathbb{R}^d \) for the class of non-convex potentials satisfying (A0), (A1) and (A2). To the best of our knowledge, this is the first result where uniqueness of ergodic gradient Gibbs measures \( \mu \) is proved for a class of non-convex potentials \( U \).

For potentials that are mixtures of Gaussians as considered in Biskup and Kotecký [2], they prove non-uniqueness of ergodic gradient Gibbs measures for tilt \( u = 0 \) in the \( \beta = 1 \) regime. For the same example, we prove uniqueness of ergodic gradient Gibbs measures for given arbitrary tilt \( u \) in the high temperature regime. Therefore, our result also highlights the existence of phase transition for these models in different temperature regimes.

The basic idea relies on a one-step coarse graining procedure, in which we consider the marginal distribution of the gradient field restricted to the even sites, which is also a gradient Gibbs field. The corresponding Hamiltonian, although no longer a two-body Hamiltonian, is then obtained via integrating out the field at the odds sites. We can integrate out the field \( \phi \) at all odd sites, using the fact that they are independent conditional on the field \( \phi \) at even sites, which is a consequence of the bi-partiteness of the graph \( \mathbb{Z}^d \) with nearest-neighbor bonds. The crucial step, which is similar to the idea of our previous paper [8], is that strict convexity can be gained via integration at sufficiently high temperature (see also Brascamp et al. [5] for previous use of the even/odd representation). The essential observation is that we can formulate a condition for this multi-body potential, which we call the random walk representation condition, which allows us to obtain a strictly convex Hamiltonian, and implies the random walk representation, permitting us to apply the techniques of Helffer and Sjöstrand [15] or Deuschel [11]. The random walk representation condition, and implicitly the strict convexity of the new Hamiltonian, can be verified under our assumptions as in (A0), (A1) and (A2). Note that the method in [8] is more general and could be applied to non-bipartite graphs.

A natural question to ask is whether we can iterate the coarse graining procedure in our current paper and find a scheme which could possibly lower the temperature towards the critical \( \beta_c \), which \( \beta_c \) marks the transition from a unique gradient Gibbs measure \( \mu \) (as proved in Theorem 31 in our paper for arbitrary tilt \( u \)) to multiple gradient Gibbs measures \( \mu \) (as proved in [2] for tilt \( u = 0 \)). Note that iterating the coarse graining scheme is an interesting open problem. One of the main difficulties is that, after iteration, the bond structure on the even sites of \( \mathbb{Z}^d \) changes, and we no longer have a bi-partite graph. Currently, we could use our method as detailed in sections 3 and 4, to keep integrating out lattice points so that the new Hamiltonian at each step, always of gradient type, can be separated into a strictly convex part and a non-convex perturbation; however, at this point, our technique for estimation of covariances as given in section 4, is not robust enough to allow us to keep coarse graining the lattice points for more than a finite number of steps, before we stop...
being able to improve the assumptions on our initial perturbation $g$.

The rest of the paper is organized as follows: In section 2 we define the model and recall the definition of gradient Gibbs measures. In section 3 we present the odd/even characterization of the gradient field. In section 4 we give the formulation of the random walk representation condition, which is verified in Theorem 22 under conditions (A0), (A1) and (A2). Section 4 also presents a few examples, in particular we show that our criteria gets close to the Biskup-Kotecký phase co-existence regime, both for the case of the zero and the non-zero tilt $u$ (see example 4.2 (a)). In section 5 we formulate and prove Theorem 31, our main result on uniqueness of ergodic gradient Gibbs measure with given tilt $u$, which is based on adaptations of [15], assuming the random walk representation condition. Section 6 deals with the decay of covariances and the proof is based on the random walk representation for the field at the even sites which allows us to use the result of [9]. Section 7 shows the central limit theorem, here again we focus on the field at even sites and apply the random walk representation idea of [17]. Section 8 proves the strict convexity of the surface tension, or free energy, which follows from the convexity of the Hamiltonian for the gradient field restricted to the even sites. Finally, the appendix provides explicit computations for our one-step coarse graining procedure in the special case of potentials considered by [2] (see also example 4.2 (a)).

2 General Definitions and Notations

2.1 $\phi$-Gibbs Measures

For $A \subset \mathbb{Z}^d$, we shall denote by $\mathcal{F}_A$ the $\sigma$-field generated by $\{\phi(x) : x \in A\}$.

**Definition 1 (\(\phi\)-Gibbs measure on \(\mathbb{Z}^d\))** The probability measure $\nu \in P(\mathbb{R}^{\mathbb{Z}^d})$ is called a Gibbs measure for the $\phi$-field with given Hamiltonian $H := (H_{\Lambda,\psi})_{\Lambda \subset \mathbb{Z}^d, \psi \in \mathbb{R}^{\mathbb{Z}^d}}$ ($\phi$-Gibbs measure for short), if its conditional probability of $\mathcal{F}_{\Lambda^c}$ satisfies the DLR equation

$$\nu(\cdot | \mathcal{F}_{\Lambda^c})(\psi) = \nu_{\Lambda,\psi}(\cdot), \ \nu - a.e. \ \psi,$$

for every finite $\Lambda \subset \mathbb{Z}^d$.

It is known that the $\phi$-Gibbs measures exist under condition (A0) when the dimension $d \geq 3$, but not for $d = 1, 2$, where the field “delocalizes” as $\Lambda \nearrow \mathbb{Z}^d$ (see [14]). An infinite volume limit (thermodynamic limit) for $\nu_{\Lambda,\psi}$ when $\Lambda \nearrow \mathbb{Z}^d$ exists only when $d \geq 3$.

2.2 $\nabla \phi$—Gibbs Measures

2.2.1 Notation for the Bond Variables on $\mathbb{Z}^d$

Let

$$(\mathbb{Z}^d)^* := \{b = (x_b, y_b) \mid x_b, y_b \in \mathbb{Z}^d, \|x_b - y_b\| = 1, b \text{ directed from } x_b \text{ to } y_b\};$$

note that each undirected bond appears twice in $(\mathbb{Z}^d)^*$. Let

$$\Lambda^* := (\mathbb{Z}^d)^* \cap (\Lambda \times \Lambda), \ \partial \Lambda^* := \{b = (x_b, y_b) \mid x_b \in \mathbb{Z}^d \setminus \Lambda, y_b \in \Lambda, \|x_b - y_b\| = 1\}$$

and

$$\overline{\Lambda^*} := \{b = (x_b, y_b) \in (\mathbb{Z}^d)^* \mid x_b \in \Lambda \text{ or } y_b \in \Lambda\}.$$
A sequence of bonds \( C = \{b^{(1)}, b^{(2)}, \ldots, b^{(n)}\} \) is called a chain connecting \( x \) and \( y \), \( x, y \in \mathbb{Z}^d \), if \( x_{b_1} = x, y_{b(i)} = x_{b(i+1)} \) for \( 1 \leq i \leq n-1 \) and \( y_{b(n)} = y \). The chain is called a closed loop if \( y_{b(1)} = x_{b(1)} \). A plaquette is a closed loop \( A = \{b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}\} \) such that \( \{x_{b(i)}, i = 1, \ldots, 4\} \) consists of 4 different points.

The field \( \eta = \{\eta(b)\} \in \mathbb{R}^{(\mathbb{Z}^d)^*}, b \in (\mathbb{Z}^d)^* \), is said to satisfy the plaquette condition if

\[
\eta(b) = -\eta(-b) \text{ for all } b \in (\mathbb{Z}^d)^* \text{ and } \sum_{b \in A} \eta(b) = 0 \text{ for all plaquettes } A \text{ in } \mathbb{Z}^d,
\]

where \(-b\) denotes the reversed bond of \( b \). Let

\[
\chi = \{\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*} \text{ which satisfy the plaquette condition}\}
\]

and let \( L_r^2, r > 0 \), be the set of all \( \eta \in \mathbb{R}^{(\mathbb{Z}^d)^*} \) such that

\[
|\eta|^2 := \sum_{b \in (\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r\|x_b\|} < \infty.
\]

We denote \( \chi_r = \chi \cap L_r^2 \) equipped with the norm \( | \cdot |_r \). For \( \phi = (\phi(x))_{x \in \mathbb{Z}^d} \) and \( b \in (\mathbb{Z}^d)^* \), we define \( \eta^{\phi}(b) := \nabla \phi(b) \). Then \( \nabla \phi = \{ \nabla \phi(b) \} \) satisfies the plaquette condition. Conversely, the heights \( \phi^{\eta, \phi(0)} \in \mathbb{R}^{\mathbb{Z}^d} \) can be constructed from height differences \( \eta \) and the height variable \( \phi(0) \) at \( x = 0 \) as

\[
\phi^{\eta, \phi(0)}(x) := \sum_{b \in C_{0,x}} \eta(b) + \phi(0),
\]

where \( C_{0,x} \) is an arbitrary chain connecting \( 0 \) and \( x \). Note that \( \phi^{\eta, \phi(0)} \) is well-defined if \( \eta = \{\eta(b)\} \in \chi \).

### 2.2.2 Definition of \( \nabla \phi \)-Gibbs measures

We next define the finite volume \( \nabla \phi \)-Gibbs measures. For every \( \xi \in \chi \) and finite \( \Lambda \subset \mathbb{Z}^d \) the space of all possible configurations of height differences on \( \overline{\chi} \) for given boundary condition \( \xi \) is defined as

\[
\chi_{\overline{\chi},\xi} = \{\eta = (\eta(b))_{b \in \overline{\chi}; \eta \vee \xi \in \chi}\},
\]

where \( \eta \vee \xi \in \chi \) is determined by \( (\eta \vee \xi)(b) = \eta(b) \) for \( b \in \overline{\chi} \) and \( = \xi(b) \) for \( b \not\in \overline{\chi} \).

**Remark 2** Note that when \( \mathbb{Z}^d \setminus \Lambda \) is connected, \( \chi_{\overline{\chi},\xi} \) is an affine space such that \( \dim \chi_{\overline{\chi},\xi} = |\Lambda| \).

Indeed, fixing a point \( x_0 \not\in \Lambda \), we consider the map \( \chi_{\overline{\chi},\xi} \to \mathbb{R}^\Lambda \), such that \( \eta \to \phi = \{\phi(x)\} \in \mathbb{R}^\Lambda \), with \( \phi(x) \) defined by

\[
\phi(x) = \sum_{b \in C_{x_0,x}} (\eta \vee \xi)(b)
\]

for a chain \( C_{x_0,x} \) connecting \( x_0 \) and \( x \in \Lambda \). This map then well-defined and an invertible linear transformation.

**Definition 3 (Finite Volume \( \nabla \phi \)-Gibbs measure)** The finite volume \( \nabla \phi \)-Gibbs measure in \( \Lambda \) (or more precisely, in \( \Lambda^* \)) with given Hamiltonian \( H := (H_{\Lambda,\xi})_{\xi \in \mathbb{Z}^d, \xi \in \chi} \) and with boundary condition \( \xi \) is defined by

\[
\mu_{\Lambda,\xi}(d\eta) = \frac{1}{Z_{\Lambda,\xi}} \exp \left\{ -\beta \sum_{b \in \Lambda} U(\eta(b)) \right\} d\eta_{\Lambda,\xi} \in P(\chi_{\overline{\chi},\xi}),
\]

where \( d\eta_{\Lambda,\xi} \) denotes the Lebesgue measure on the affine space \( \chi_{\overline{\chi},\xi} \) and \( Z_{\Lambda,\xi} \) is the normalization constant.

Let \( P(\chi) \) be the set of all probability measures on \( \chi \) and let \( P_2(\chi) \) be those \( \mu \in P(\chi) \) satisfying \( E^\mu[|\eta(b)|^2] < \infty \) for each \( b \in (\mathbb{Z}^d)^* \).
Remark 4 For every $\xi \in \chi$ and $a \in \mathbb{R}$, let $\psi = \phi^{a}_\xi$ be defined by (5) and consider the measure $\nu_{\Lambda, \psi}$. Then $\mu_{\Lambda, \xi}$ is the image measure of $\nu_{\lambda, \phi}$ under the map $\{\phi(x)\}_{x \in \Lambda} \rightarrow \{\eta(b) := \nabla(\phi \lor \psi)(b)\}_{b \in \Lambda^*}$ and where we defined $(\phi \lor \psi)(x) := \phi(x)$ for $x \in \Lambda$ and $(\phi \lor \psi)(x) := \psi(x)$ for $x \notin \Lambda$. Note that the image measure is determined only by $\xi$ and is independent of the choice of $a$. Let $K_{\Lambda}^{\psi} : \{\phi(x)\}_{x \in \mathbb{Z}^d} \rightarrow \{\eta(b)\}_{b \in (\mathbb{Z}^d)^*}$, with $\eta(b) := \nabla(\phi \lor \psi)(b)$.

Definition 5 ($\nabla \phi$-Gibbs measure on $(\mathbb{Z}^d)^*$) The probability measure $\mu \in P(\chi)$ is called a Gibbs measure for the height differences with given Hamiltonian $H := (H_{\Lambda, \xi})_{\Lambda \subset \mathbb{Z}^d, \xi \in \chi}$ ($\nabla \phi$-Gibbs measure for short), if it satisfies the DLR equation

$$
\mu(\cdot | F_{(\mathbb{Z}^d)^* \setminus \Lambda}^\phi)(\xi) = \mu_{\Lambda, \xi}(\cdot), \quad \mu - a.e. \varphi,
$$

for every finite $\Lambda \subset \mathbb{Z}^d$, where $F_{(\mathbb{Z}^d)^* \setminus \Lambda}$ stands for the $\sigma$-field of $\chi$ generated by $\{\eta(b), b \in (\mathbb{Z}^d)^* \setminus \Lambda\}$.

With the notations from (5) and Definition 3 let

$$
\mathcal{G}(H) := \{\mu \in P_2(\chi) : \mu \text{ is } \nabla \phi - \text{Gibbs measure on } (\mathbb{Z}^d)^* \text{ with given Hamiltonian } H\}.
$$

Remark 6 Throughout the rest of the paper, we will use the notation $\phi, \psi$ to denote height variables and $\eta, \xi$ to denote height differences.

3 Even/Odd Representation

There are two key results in this section. The first one is Lemma 16, where we are restricting the height differences to the even sites, which induces a $\nabla \phi$ measure on the even lattice with a different bond structure. The second main result of this section is Lemma 17, where we give a formula for the conditional of a $\nabla \phi$-Gibbs measure on the height differences between even sites. These two results will be essential for the proof for one of our main results, that is for the proof of the uniqueness of ergodic component of Theorem 31.

In Subsection 3.1 we introduce the notation for the bond variables on the even subset of $\mathbb{Z}^d$, in Subsection 3.2 we define the $\phi$-Gibbs measure and the $\nabla \phi$-Gibbs measure corresponding to the even subset of $\mathbb{Z}^d$ and in Subsection 3.3 we present the relationship between the $\nabla \phi$-Gibbs measures for the bonds on $\mathbb{Z}^d$ and the $\nabla \phi$ for the bonds on even subset of $\mathbb{Z}^d$, when their corresponding finite volume $\phi$-Gibbs measures are related by restriction.

3.1 Notation for the Bond Variables on the Even Subset of $\mathbb{Z}^d$

As $\mathbb{Z}^d$ is a bipartite graph, we will label the vertices of $\mathbb{Z}^d$ as even and odd vertices, such that every even vertex has only odd nearest neighbor vertices and vice-versa.

Let

$$
\mathbb{Z}_{ev}^d := \{a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d | \sum_{i=1}^{d} a_i = 2p, p \in \mathbb{Z}\}
$$

and

$$
\mathbb{Z}_{od}^d := \{a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d | \sum_{i=1}^{d} a_i = 2p + 1, p \in \mathbb{Z}\}.
$$

Let $\Lambda_{ev} \subset \mathbb{Z}_{ev}^d$ finite. We will next define the bonds in $\mathbb{Z}_{ev}^d$ in a similar fashion to the definitions for bonds on $\mathbb{Z}^d$. Let

$$
(\mathbb{Z}_{ev}^d)^* := \{b = (x_b, y_b) \mid x_b, y_b \in \mathbb{Z}_{ev}^d, \|x_b - y_b\| = 2, b \text{ directed from } x_b \text{ to } y_b\},
$$

$$
(\Lambda_{ev})^* := (\mathbb{Z}_{ev}^d)^* \cap (\Lambda_{ev} \times \Lambda_{ev}), \quad (\Lambda_{ev})^* := \{b = (x_b, y_b) \in (\mathbb{Z}_{ev}^d)^* \mid x_b \in \Lambda_{ev} \text{ or } y_b \in \Lambda_{ev}\},
$$
\[ \partial(\Lambda_{ev})^* := \{ b = (x_b, y_b) \mid x_b \in \mathbb{Z}_d^{ev}, y_b \in \Lambda_{ev}, \|x_b - y_b\| = 2 \} \]

and

\[ \partial\Lambda_{ev} := \{ y \in \mathbb{Z}_d^{ev} \setminus \Lambda_{ev} \mid \|y - x\| = 2 \text{ for some } x \in \Lambda_{ev} \}. \]

Note that throughout the rest of the paper, we will refer to the bonds on \((\mathbb{Z}_d^{ev})^*\) as the even bonds.

An even plaquette is a closed loop \(A_{ev} = \{ b^{(1)}, b^{(2)}, \ldots, b^{(n)} \}\), where \(b^{(i)} \in (\mathbb{Z}_d^{ev})^*, \ n \in \{3, 4\}\), such that \(\{x_b^{(i)}, i = 1, \ldots, n\}\) consists of \(n\) different points in \(\mathbb{Z}_d^{ev}\). The field \(\eta = \{ \eta(b) \} \in \mathbb{R}^{(\mathbb{Z}_d^{ev})^*}\) is said to satisfy the even plaquette condition if

\[ \eta(b) = -\eta(-b) \text{ for all } b \in (\mathbb{Z}_d^{ev})^* \text{ and } \sum_{b \in A_{ev}} \eta(b) = 0 \text{ for all even plaquettes in } \mathbb{Z}_d^{ev}. \] (10)

Let \(\chi_{ev}\) be the set of all \(\eta \in \mathbb{R}^{(\mathbb{Z}_d^{ev})^*}\) which satisfy the even plaquette condition. For each \(b = (x_b, y_b) \in (\mathbb{Z}_d^{ev})^*\) we define the even height differences \(\eta_{ev}(b) := \nabla_{ev}\phi(b) = \phi(y_b) - \phi(x_b)\). The heights \(\phi_{\eta_{ev}, \phi(0)}\) can be constructed from the height differences \(\eta_{ev}\) and the height variable \(\phi(0)\) at \(x = 0\) as

\[ \phi_{\eta_{ev}, \phi(0)}(x) := \sum_{b \in \mathcal{E}_{0,x}} \eta_{ev}(b) + \phi(0), \]

where \(x \in \mathbb{Z}_d^d\) and \(\mathcal{E}_{0,x}\) is an arbitrary path in \(\mathbb{Z}_d^{ev}\) connecting 0 and \(x\). Note that \(\phi_{\eta_{ev}, \phi(0)}(x)\) is well-defined if \(\eta_{ev} = \{ \eta_{ev}(b) \} \in \chi_{ev}\). We also define \(\chi_{ev, r}\) similarly as we define \(\chi_r\). As on \(\mathbb{Z}_d\), let \(P(\chi_{ev})\) be the set of all probability measures on \(\chi_{ev}\) and let \(P_2(\chi_{ev})\) be those \(\mu \in P(\chi_{ev})\) satisfying \(\mathbb{E}[\|\eta_{ev}(b)\|^2] < \infty\) for each \(b \in (\mathbb{Z}_d^{ev})^*\). We denote \(\chi_{ev, r} = \chi \cap L^2_r\) equipped with the norm \(\| \cdot \|_r\).

**Remark 7** Let \(\eta \in \chi\). Using the plaquette condition property of \(\eta\), we will define \(\eta_{ev}\), the induced bond variables on the even lattice, from \(\eta\) thus: if \(b_1 = (x, x + e_i)\), \(b_2 = (x + e_j, x)\) and \(b_0 = (x + e_j, x + e_i)\), we define \(\eta_{ev}(b_0) = \eta(b_1) + \eta(b_2)\). Note that \(\eta_{ev} \in \chi_{ev}\).

**Remark 8** Throughout the rest of the paper, we will use the notation \(\phi_{ev}, \psi_{ev}\) either for a stand alone configuration on the even vertices, or in relation to the restriction of \(\phi\) to the even vertices. \(\eta_{ev}, \xi_{ev}\) will denote configurations on the even bonds. Similarly, \(\Lambda_{ev}\) will either be a stand alone subset of \(\mathbb{Z}_d^{ev}\) or will be used in relation to the restriction of a set \(\Lambda \subset \mathbb{Z}_d\) to \(\mathbb{Z}_d^{ev}\). For \(\Lambda \subset \mathbb{Z}_d\), we will denote \(\Lambda_{od} := \mathbb{Z}_d^{od} \cap \Lambda\).

### 3.2 Definition of \(\nabla\phi\)-Gibbs measure on \((\mathbb{Z}_d^{ev})^*\)

For every \(\xi_{ev} \in \chi_{ev}\) and finite \(\Lambda_{ev} \subset \mathbb{Z}_d^{ev}\), the space of all possible configurations of height differences on \((\Lambda_{ev})^*\) for given boundary condition \(\xi_{ev}\) is defined as

\[ \chi_{(\Lambda_{ev})^*, \xi_{ev}} = \{ \eta_{ev} = (\eta_{ev}(b))_{b \in (\Lambda_{ev})^*}, \eta_{ev} \vee \xi_{ev} \in \chi_{ev} \}, \]

where \(\eta_{ev} \vee \xi_{ev} \in \chi_{ev}\) is determined by \((\eta_{ev} \vee \xi_{ev})(b) = \eta_{ev}(b)\) for \(b \in (\Lambda_{ev})^*\) and \(= \xi_{ev}(b)\) for \(b \not\in (\Lambda_{ev})^*\).
The $\phi$-Gibbs measure $\nu^\phi$ on $\mathbb{Z}_d^d$ and the $\nabla \phi$-Gibbs measure $\mu^\phi$ on $(\mathbb{Z}_d^d)^*$ with given Hamiltonian $H^\phi$ can be defined similarly to the $\phi$-Gibbs measure and the $\nabla \phi$-Gibbs measure in Subsections 2.1 and 2.2.2. They are basically a $\phi$-Gibbs and $\nabla \phi$-Gibbs measure on a different graph, with vertex and edge sets $(\mathbb{Z}_d^d, (\mathbb{Z}_d^d)^*)$. They are defined via the corresponding Hamiltonian $H^\phi_{\Lambda_\phi,\psi\phi}$, assumed of even gradient type, via the finite volume Gibbs measure $\nu^\phi_{\Lambda_\phi,\psi\phi}$ on $\mathbb{Z}_d^d$ and the finite volume $\nabla$-Gibbs measure $\mu^\phi_{\Lambda_\phi,\psi\phi}$ on $(\mathbb{Z}_d^d)^*$.

Let

$$H^\phi := (H^\phi_{\Lambda_\phi,\psi\phi})_{\Lambda_\phi \subset \mathbb{Z}_d^d, \psi\phi \in \chi_d}$$

and let

$$G_{\psi\phi}(H^\phi) := \{\mu^\phi \in P_2(\chi_d) : \mu^\phi \text{ is } \nabla \phi - \text{Gibbs measure on } (\mathbb{Z}_d^d)^* \text{ with given Hamiltonian } H^\phi\}.$$

**Remark 9** Similar to Remark 2 when $\mathbb{Z}_d^d \setminus \Lambda_\phi$ is connected, $\chi_{(\Lambda_\phi)^*,\psi\phi}$ is an affine space such that

$$\dim \chi_{(\Lambda_\phi)^*,\psi\phi} = |\Lambda_\phi|. \text{ Fixing a point } x_0 \notin \Lambda_\phi, \text{ we consider the map } J^x_{\Lambda_\phi} : \chi_d \rightarrow \mathbb{R}^{\mathbb{Z}_d^d}, \text{ such that } \eta_\Lambda (x) = \{\phi(x)\}, \text{ with }$$

$$\phi(x) := \sum_{b \in C_{x_0,x}^\psi} (\eta_\Lambda \vee \psi\phi)(b), \quad x \in \Lambda_\phi$$

for a chain $C_{x_0,x}^\psi$ connecting $x$ and $x_0$ and for fixed $\phi(x_0)$,

$$\phi(x) := \psi\phi,\phi(x_0)(x) = \sum_{b \in C_{x_0,x}} \eta_\Lambda (b) + \phi(x_0), \quad x \notin \Lambda_\phi.$$

**Remark 10** For every $\xi_\phi \in \chi_d$ and $a \in \mathbb{R}$, let $\psi_\phi = \phi^{\xi_\phi,a}$ be defined by (11) and consider the measure $\nu_{\Lambda_\phi,\psi\phi}$. Then $\nu_{\Lambda_\phi,\psi\phi}$ is the image measure of $\nu_{\Lambda_\phi,\psi\phi}$ under the map $\phi(x) \in \Lambda_\phi \rightarrow \{\eta_\Lambda (b) := \nabla (\phi(x) \vee \psi_\phi)(b)\}_{b \in (\Lambda_\phi)^*}$. Note that the image measure is determined only by $\xi_\phi$ and is independent of the choice of $a$.

### 3.3 Induced $\nabla \phi$-Gibbs measure on $(\mathbb{Z}_d^d)^*$

Throughout this section, we will make the following notation conventions. For $\phi, \psi \in \mathbb{R}^{\mathbb{Z}_d^d}$, we define $\phi_\psi := (\phi(x))_{x \in \mathbb{Z}_d^d}, \psi_\phi := (\psi(x))_{x \in \mathbb{Z}_d^d}$. For $\eta, \xi \in \chi$, we define $\eta_\phi$ and $\xi_\psi$ according to Remark 7.

**Definition 11** Let $\Lambda_\phi$ be a finite set in $\mathbb{Z}_d^d$. We construct a finite set $\Lambda \subset \mathbb{Z}_d^d$ associated to $\Lambda_\phi$ as follows: if $x \in \Lambda_\phi$, then $x \in \Lambda$ and $x + e_i \in \Lambda$ for all $i \in I = \{-n, -n+1, \ldots, n\} \setminus \{0\}$. Note that by definition, $\partial \Lambda = \partial \Lambda_\phi$, where the boundary operations are performed in the graphs $(\mathbb{Z}_d^d, (\mathbb{Z}_d^d)^*)$ and $(\mathbb{Z}_d^d, (\mathbb{Z}_d^d)^*)$, respectively. (see Figures 2 and 3).

---

**Figure 2**: The graph of $\Lambda_\phi$  

**Figure 3**: The graph of $\Lambda$ associated to $\Lambda_\phi$
Lemma 12 (Induced finite volume \(\phi\)-Gibbs measure on \(\mathbb{Z}^d_{\text{ev}}\)) Let \(\Lambda_{\text{ev}} \subset \mathbb{Z}^d_{\text{ev}}\) and let \(\Lambda\) be the associated set in \(\mathbb{Z}^d\), as defined in Definition 11. Let \(\nu_{\Lambda,\psi}\) be the finite volume Gibbs measure on \(\Lambda\) with boundary condition \(\psi\) and with Hamiltonian \(H_{\Lambda,\psi}\) defined as in 13. We define the induced finite volume Gibbs measure on \(\mathbb{Z}^d_{\text{ev}}\) as \(\nu^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}} := \nu_{\Lambda,\psi} I_{\mathbb{Z}^d_{\text{ev}}}(\mathbb{Z}_{\text{ev}})\). Then \(\nu^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}}\) has Hamiltonian \(H^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}},\) where

\[
H^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}} (\phi_{\text{ev}}) := \sum_{x \in \Lambda_{\text{od}}} F_x ((\phi (x + e_i))_{i \in I}), \text{ with } F_x ((\phi (x + e_i))_{i \in I}) = -\log \int_{\mathbb{R}} e^{-2\beta \sum_{i \in I} U(\nabla_i \phi(x))} \, d\phi(x). \tag{12}
\]

Remark 13 Note that for any constant \(C \in \mathbb{R}\), by using the change of variables \(\phi(x) \to \phi(x) + C\) in the integral formula for \(F_x ((\phi (x + e_i))_{i \in I})\) in (12), we have

\[
F_x ((\phi (x + e_i))_{i \in I}) = F_x ((\phi (x + e_i) + C)_{i \in I}).
\]

In particular, this means that for any fixed \(k \in I\)

\[
F_x ((\phi (x + e_i))_{i \in I}) = F_x ((\phi (x + e_i) - \phi(x + e_k))_{i \in I}). \tag{13}
\]

Therefore we are still dealing with a gradient system. However, it is in general no longer a two-body gradient system. \(F_x ((\phi (x + e_i))_{i \in I})\), and consequently \(H^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}},\) are functions of the even gradients by (13) and (12).

Remark 14 We formulate next more explicitly the dependence of \(F_x\) and \(H^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}}\) on the even gradients. Let \(k \in I\) be arbitrarily fixed. For any \(x \in \mathbb{Z}^d\), let

\[
\mathcal{B}(x, k) = \{(x + e_k, x + e_i)\}_{i \in I}.
\]

For all \(\Lambda_{\text{ev}} \subset \mathbb{Z}^d_{\text{ev}}\), take the set \(\Lambda\) associated to \(\Lambda_{\text{ev}}\), as defined in Definition 11. We define here \(H^{\text{ev}} := (H^{\text{ev}}_{\Lambda_{\text{ev}},\xi_{\text{ev}}} : \Lambda_{\text{ev}} \subset \mathbb{Z}^d_{\text{ev}}, \xi_{\text{ev}} \in \mathcal{X}_{\text{ev}}\) as follows

\[
H^{\text{ev}}_{\Lambda_{\text{ev}},\xi_{\text{ev}}} (\eta) = \sum_{x \in \Lambda_{\text{od}}} F_x \left((\eta_{\text{ev}}(b))_{b \in \mathcal{B}(x,k)}\right). \tag{14}
\]

Note that, via Remark 9, one can easily obtain the equivalence between the corresponding finite volume \(\phi\)-Gibbs and \(\nabla \phi\)-Gibbs measures.

Remark 15 By definition, \(F_x ((\phi (x + e_i))_{i \in I})\) only depend on sites within distance 2 of \(x\). Note that the new Hamiltonian \(H^{\text{ev}}_{\Lambda_{\text{ev}},\psi_{\text{ev}}}\) depends on \(\beta\) through the functions \(F_x ((\phi (x + e_i))_{i \in I})\).

Proof of Lemma 12 The idea of this proof is just integrating out the height variables on the odd sites, conditioned on the even sites. The Gibbs property and specific graph structure imply that the odd height variables are independent conditional on the even sites.

Set

\[
H_x (\phi) = \sum_{i \in I} U(\nabla_i \phi(x)). \tag{15}
\]

Let \(\Lambda_{\text{ev}}\) be a finite set in \(\mathbb{Z}^d_{\text{ev}}\) and let \(\Lambda \subset \mathbb{Z}^d\) be the associated set as defined in Definition 11. Note now that due to the symmetry of the potential \(U\), to the specific boundary conditions on \(\Lambda\) and by 3, we have

\[
H_{\Lambda,\psi} (\phi) = \sum_{x \in \Lambda} H_x (\phi) = 2 \sum_{x \in \Lambda_{\text{od}}} H_x (\phi).
\]
Let $A \in \mathcal{F}_{Z_{ev}^d} \subset \mathcal{F}_{Z^d}$, $d\phi_{\Lambda_{ev}} = \prod_{x \in A_{ev}} d\phi(x)$ and $d\phi_{\Lambda_{od}} = \prod_{x \in A_{od}} d\phi(x)$. Then, by integrating out all the odd height variables conditional on the even height variables, due to the Gibbs property of $\nu_{\Lambda,\psi}$ (see Definition 11) and to the fact that $\partial \Lambda = \partial \Lambda_{ev}$, we have for every $\psi \in \mathbb{R}^{Z^d}$

$$
\nu_{\Lambda,\psi}(A) = \frac{1}{Z_{\Lambda,\psi}} \int_{\mathbb{R}^\Lambda} 1_A(\phi)e^{-\beta H_{\Lambda,\psi}(\phi)}d\phi_{\Lambda \setminus \Lambda_{ev}}(d\phi_{\Lambda_{ev}}),
$$

(16)

where for the last equality we also used the fact that $Z_{\Lambda,\psi} = Z_{\Lambda_{ev},\psi_{ev}}$ and where $\Lambda_{ev} = \Lambda \cap Z_{od}^d$ and $\Lambda_{od} = \Lambda \cap Z_{od}^d$.

**Lemma 16 (Induced $\nabla \phi$-Gibbs measure on $(Z_{ev}^d)^*$)** Let $\mu \in \mathcal{G}(H)$. We define the induced $\nabla \phi$-Gibbs measure on $(Z_{ev}^d)^*$ as $\mu_{ev} := \mu_{\mathcal{F}(Z_{ev}^d)^*}$. Then $\mu_{ev} \in \mathcal{G}_{ev}(H_{ev}^\Lambda_{ev},\xi_{ev})$ is defined as in Remark 14.

**Proof.** Let $\mathcal{F}_{(Z_{ev}^d)^*} := \sigma(\eta(b), b \in (Z_{ev}^d)^*)$ and $\mathcal{F}_{(Z_{ev}^d)^*} := \sigma(\eta_{ev}(b), b \in (Z_{ev}^d)^*)$.

To prove the statement of the theorem, we need to prove that for all $A \in \mathcal{F}((Z_{ev}^d)^*)$, $\mu_{ev}$ satisfies

$$
\mu_{ev}(A|\mathcal{F}((Z_{ev}^d)^*)\setminus(\Lambda_{ev})^*) (\xi_{ev}) = \mu_{ev_{ev},\xi_{ev}}^\Lambda(A).
$$

(17)

In order to prove the above equality, we will first show that for all $A \in \mathcal{F}_{(Z_{ev}^d)^*}$ and for any $\Lambda_{ev}$ finite set in $Z_{ev}^d$ with associated set $\Lambda \subset Z^d$ as defined in Definition 11 we have

$$
\mu_{\Lambda,\xi}(A) = \mu_{\Lambda_{ev},\xi_{ev}}^\Lambda(A).
$$

(18)

Then using $\mathcal{F}_{(Z_{ev}^d)^*}\setminus(\Lambda_{ev})^* \subset \mathcal{F}_{(Z_{ev}^d)^*}\setminus(\Lambda_{ev})^*$, the definition of the $\nabla \phi$-Gibbs measure and (18), we have

$$
\mu(A|\mathcal{F}_{(Z_{ev}^d)^*}\setminus(\Lambda_{ev})^*) (\xi_{ev}) = E_{\mu} \left( E_{\mu} \left( 1_A|\mathcal{F}_{(Z_{ev}^d)^*}\setminus(\Lambda_{ev})^* \right) \right) |\mathcal{F}_{(Z_{ev}^d)^*}\setminus(\Lambda_{ev})^* (\xi_{ev}) = \mu_{ev_{ev},\xi_{ev}}^\Lambda(A).
$$

The key point in the above equation is that when we condition further, we get $\mu_{\Lambda,\xi'}$ where $\xi'$ is random and being integrated over, and $\xi'$ all have $\xi_{ev}$ as its restriction on the evens, and for all such $\xi'$, by (18) $\mu_{\Lambda,\xi'}$ all equal $\mu_{ev_{ev},\xi_{ev}}^\Lambda(A)$. To prove (18), first we start with the finite volume $\nabla \phi$-Gibbs measure $\mu_{\Lambda,\xi}$. Then we construct a finite volume $\phi$-Gibbs measure $\nu_{\Lambda,\psi}$ using the map $K_{\Lambda}$ defined in Remark 4. Next we restrict $\nu_{\Lambda,\psi}$ to the even vertices by means of Lemma 12, and then we pass to the finite volume $\nabla \phi$-Gibbs measure $\mu_{ev_{ev},\xi_{ev}}^\Lambda$ by applying the map $J_{ev_{ev},\xi_{ev}}$ defined in Remark 9.

The details in the derivation of (18) follow below.

Let $\xi \in \chi$. Fixing $\psi(0) \in \mathbb{R}$, for all $A \in \mathcal{F}((Z_{ev}^d)^*)$, we have by Remark 11 that

$$
\mu_{\Lambda,\xi}(A) = \mathbb{E}_{\nu_{\Lambda,\psi}}(1_A \circ K_{\Lambda}), \text{ with } \psi \text{ given as in (9) by } \psi(x) := \sum_{b \in C_{ev,x}} \xi(b) + \psi(0), \ x \in Z^d.
$$

(19)

For all $B \in \mathcal{F}_{Z_{ev}^d}$ and $\Lambda_{ev}$ finite sets in $Z_{ev}^d$ with $Z_{ev}^d \setminus \Lambda_{ev}$ connected, we have by Remark 11

$$
\mu_{ev_{ev},\xi_{ev}}^\Lambda(B) = \mathbb{E}_{\mu_{ev_{ev},\xi_{ev}}^\Lambda} \left( 1_B \circ J_{ev_{ev},\xi_{ev}} \right), \text{ with } \xi_{ev}(b) := \nabla \psi(b), \ b \in (Z_{ev}^d)^*.
$$

(20)
Let \( A \in \mathcal{F}(\mathbb{Z}_d)^* \subset \mathcal{F}(\mathbb{Z}_d)^* \); then by using Lemma 12, 19, and 20, we have for every \( \xi \in \chi \) such that \( \xi_{ev} \in \chi_{ev} \) (recall Remark 7)

\[
\mu_{\Lambda, \xi}(A) = E_{\nu_{\Lambda, \psi}}(1_{A} \circ K_{\Lambda}^{-1}) = \nu_{\Lambda, \psi}^* \left( ((K_{\Lambda}^{-1})^{-1}(A)) \right) = E_{\mu_{\Lambda, \xi, \psi}^*}(1_{(K_{\Lambda}^{-1})^{-1}(A)} \circ J_{\Lambda, \xi}) = \mu_{\Lambda, \xi, \psi}^*(A),
\]

where for the last equality we used the fact that \( 1_{(K_{\Lambda}^{-1})^{-1}(A)} \circ J_{\Lambda, \xi} = 1_A \).

The following statement is a consequence of the Markov property of the Gibbs measures.

**Lemma 17 (Conditional of \( \nabla \phi \)-Gibbs measure on \( (\mathbb{Z}_d)^* \))** Let \( G \) be a \( \mathcal{F}(\mathbb{Z}_d)^* \)-measurable and bounded function. Then for all \( \mu \in \mathcal{G}(H) \) and all \( \xi_{ev} \in \chi_{ev} \), we have

\[
E_{\mu} \left( G \mid \mathcal{F}(\mathbb{Z}_d)^* \right)(\xi_{ev}) = \int_{\mathbb{Z}_d^d} G(\nabla \phi) \prod_{x \in \mathbb{Z}_d^d} \nu_{x, \psi}(d\phi(x)) \delta(x, \psi_{\mathbb{Z}_d^d}(x)),
\]

(22)

where we use \( \nu_{x, \psi} \) to denote \( \nu_{\Lambda, \psi} \) with \( \Lambda = \{ x \} \) and \( \psi \) is given by \( \psi(x) := \sum_{b \in \mathcal{G}_{ev}} \xi_{ev}(b) + \psi(0), x \in \mathbb{Z}_d^d \), for a fixed \( \psi(0) \in \mathbb{R} \).

**Proof.** It is enough to prove (22) for functions \( G \) depending on finitely many coordinates. Define \( S^d_{n_0} = \{ x \in \mathbb{Z}_d^d : ||x|| \le n_0 \} \) such that \( \mathcal{F}(\mathbb{Z}_d)^* \)-measurable.

Note now that from the DLR conditions for \( \mu \), we have \( \mu(\cdot \mid \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ \xi \}) = \mu_{\Lambda, \xi}(\cdot) \) for all \( \Lambda = \{ x \} \in \mathbb{Z}_d^d \). In view of the definition of \( \chi \), we also have that for all \( y \in \mathbb{Z}_d^d \), \( \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ y \} \subset \cap_{x \in \mathbb{Z}_d^d} \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \} = \mathcal{F}(\mathbb{Z}_d)^* \). Take \( n \ge n_0 \) arbitrarily fixed. Then by repeated application, we have

\[
E_{\mu} \left( G \mid \cap_{x \in S^d_{n}} \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \} \right)(\xi_{ev}) = E_{\mu} \left( E_{\mu_{\psi}}(G) \mid \cap_{x \in S^d_{n}} \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \} \right)(\xi_{ev}) \ldots = E_{\mu} \left( E_{\mu_{\psi_{\mathbb{Z}_d^d}}}(G) \mid \cap_{x \in S^d_{n}} \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \} \right)(\xi_{ev}).
\]

The statement of the lemma follows now from the fact that

\[
\mu(\cdot \mid \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \})(\xi_{ev}) = \lim_{n \to \infty} \mu(\cdot \mid \cap_{x \in S^d_{n}} \mathcal{F}(\mathbb{Z}_d)^* \setminus \{ x \})(\xi_{ev}),
\]

and from Remark 2 as we switch from the finite \( \nabla \phi \)-Gibbs measures \( \mu_{\Lambda, \xi} \) to the finite \( \phi \)-Gibbs measures \( \nu_{x, \psi} \). □

In the next Corollary, we reformulate Lemma 17 to remove the dependence on the height field \( \psi \), and to make it more explicit that everything in the formula for \( E_{\mu} \left( G \mid \mathcal{F}(\mathbb{Z}_d)^* \right)(\xi_{ev}) \) depends only on the even gradients.

**Corollary 18** Let \( k \in I \) be an arbitrarily fixed element in \( I \) and let \( G \) be a \( \mathcal{F}(\mathbb{Z}_d)^* \)-measurable and bounded function. Then for all \( \mu \in \mathcal{G}(H) \) and all \( \xi_{ev} \in \chi_{ev} \), we have with the notations from Remark 14

\[
E_{\mu} \left( G \mid \mathcal{F}(\mathbb{Z}_d)^* \right)(\xi_{ev}) = \int G \left( (\xi_{ev}(b) - \phi(x))_{b \in \mathcal{B}(x,k)} \right) \prod_{x \in \mathbb{Z}_d^d} \mu_{x, \xi_{ev}}^k(d\phi(x)),
\]

(23)

where

\[
\mu_{x, \xi_{ev}}^k(d\phi(x)) = \frac{1}{Z_{x, \xi_{ev}}^k} \exp \left( -\beta \sum_{b \in \mathcal{B}(x,k)} U(\xi_{ev}(b) - \phi(x)) \right) d\phi(x),
\]

(24)

and \( Z_{x, \xi_{ev}}^k \) is the normalizing constant.

**Proof.** Note first that for all \( i \in I \) and all \( x \in \mathbb{Z}_d^d \), \( \nabla_i \phi(x) = \phi(x + e_i) - \phi(x + e_k) - \phi(x) + \phi(x + e_k) = \xi_{ev}(b) - \phi(x) + \phi(x + e_k) \), with \( b \in \mathcal{B}(x,k) \). The statement of the corollary follows now immediately, by making in (22) the change of variables \( \phi(x) \to \phi(x) + \phi(x + e_k) \) for all \( x \in \mathbb{Z}_d^d \). □
4 Random Walk Representation Condition

In this section, we prove that under suitable conditions on the perturbation $g$, the new Hamiltonian $H^\text{ev} = (H^\text{ev}_{\text{Ave}}, \psi_{\text{ev}})_{\lambda_{\text{ev}} \in \mathbb{Z}^d_{\text{ev}}, \psi_{\text{ev}} \in \mathbb{Z}^d_{\text{ev}}}$ induced on $\mathbb{Z}^d_{\text{ev}}$ and defined in [12], is strictly convex. More precisely, we will prove that $H^\text{ev}$ satisfies the so-called random walk representation condition (see Definition 19 below). This will allow us to adapt results known for strictly convex potentials, such as uniqueness of ergodic component and decay of covariance, to our non-convex setting.

Subsection 4.1 contains the main result of this section, Theorem 22, in which we prove that under assumption (A2) on $g$, the Hamiltonian $H^\text{ev}$ satisfies the random walk representation condition. Note that, in contrast to the condition in our previous paper [8], $\|g''\|_{L^\infty(\mathbb{R})}$ can be arbitrarily large as long as $\|g''\|_{L^\infty(\mathbb{R})}$ is small. In subsection 4.2, we present some examples of non-convex potentials which fulfill assumption (A2); our first example is the particular class of potentials treated both in [2] and in [3].

4.1 Definition and Main Result

For $i \in I$, let $D^i F_x(y_1, \ldots, y_d, y_{-1}, \ldots, y_d) := \frac{\partial}{\partial y_i} F_x(y_1, \ldots, y_d, y_{-1}, \ldots, y_d)$.

We will next formulate a condition on the multi-body potential, which we call the random walk representation condition, such that $F_x$ satisfies this condition, and we will adapt earlier results known for strictly convex two-body potentials to this setting.

Definition 19 We say that $H^\text{ev}$ satisfies the random walk representation condition if there exist $\underline{c}, \bar{c} > 0$ such that for all $x \in \mathbb{Z}^d_{\text{od}}$, for all $(\phi(x + e_k))_{x \in \mathbb{Z}^d_{\text{od}}, k \in I}$ and all $i, j \in I$

$$D^i-j F_x((\phi(x + e_k))_{k \in I}) = -\sum_{j \in I, j \neq i} D^i-j F_x((\phi(x + e_k))_{k \in I})$$

$$\underline{c} \leq -D^i-j F_x((\phi(x + e_k))_{k \in I}) \leq \bar{c} \text{ for } i \neq j.$$

Remark 20 Note that for each $x \in \mathbb{Z}^d_{\text{od}}$, $F_x$ is uniformly convex (with respect to the even heights). More precisely, for all $\alpha = (\alpha_1, \ldots, \alpha_{2d}) \in \mathbb{R}^{2d}$ we have

$$\underline{c} \sum_{i,j \in I, i \neq j} (\alpha_i - \alpha_j)^2 \leq \sum_{i,j \in I} \alpha_i \alpha_j D^i-j F_x((\phi(x + e_k))_{k \in I}) \leq \bar{c} \sum_{i,j \in I, i \neq j} (\alpha_i - \alpha_j)^2.$$

Remark 21 The random walk representation condition name comes from the fact that potentials satisfying this condition fulfill the random walk representation as explained for example in [12] or [16]; that is, for uniformly convex (with respect to heights) two-body gradient interactions, there is an extremely useful representation of the covariance matrix (with respect to the measure $\mu_{\Lambda, \xi}$) in terms of the Green function of a specific random walk.

The main result of this section is:

Theorem 22 (Random Walk Representation Condition) Let $U \in C^2(\mathbb{R})$ be such that it satisfies (A1). We also assume that $V, g \in C^2(\mathbb{R})$ satisfy (A1). Then, if for some $q \geq 1, g''$ satisfies (A2), more precisely, if

$$\beta \frac{1}{2q} ||g''||_{L^q(\mathbb{R})} < \frac{(C_1)^{\frac{3}{2}}}{2C_2^{\frac{q+1}{2}} (2d)^{\frac{1}{2q}}},$$

then there exist $\underline{c}, \bar{c} > 0$ such that $H^\text{ev}$ satisfies the random walk representation condition.

Remark 23 The main idea behind the proof of Theorem 22 is that one can gain convexity by one-step integration, which is possible if $||g''||_{L^q(\mathbb{R})}$ is sufficiently small compared to $V''$. 

What is crucial as regards the bounds $c, \bar{c}$, is that they are uniform in $x \in \mathbb{Z}_{od}^d$ and that they are independent of the possible values of $\phi_\nu \in \mathbb{Z}_{ov}^d$. This is necessary for us to adapt the arguments known for uniformly strictly convex potentials with two-body interaction to our setting of a generalized random walk representation condition for multi-body potentials.

Note that we only need $||g''||_{L^2(\mathbb{R})}$ to be small for the lower bound $c$, as the upper bound $\bar{c}$ only requires the perturbation to be finite, not small.

The first step in proving Theorem 22 is to prove the following lemma

**Lemma 24** Suppose $x \in \mathbb{Z}_{od}^d$. Then for all $j \in I$, we have

$$D^j F_x((\phi(x + e_k))_{k \in I}) = -\sum_{i \in I, i \neq j} D^i F_x((\phi(x + e_k))_{k \in I}),$$

and for all $i \in I, i \neq j$

$$D^{i,j} F_x((\phi(x + e_k))_{k \in I}) = -4\beta^2 \text{Cov}_{\nu,\psi} (U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))),$$

where $\nu, \psi$ are as defined in Lemma 17. $\psi = \phi'\nu$ and $E_{\nu,\psi}$ and $\text{Cov}_{\nu,\psi}$ are respectively the expectation and the covariance with respect to the measure $\nu_{x,\psi}$.

**Proof.** Let $a = (a_1, a_2, \ldots, a_{2d}) \in \mathbb{R}^{2d}$. Since $F_x(a_1, \ldots, a_{2d}) = F_x(a_1 + t, \ldots, a_{2d} + t)$ for all $t > 0$, differentiating with respect to $t$ in it, gives the first identity in (25). The second assertion in (25) follows from the first, by differentiation. By differentiating now with respect to $\phi(x + e_i)$ and $\phi(x + e_j)$ in the formula for $F_x$, we have for all $i, j \in I, i \neq j$

$$D^{i,j} F_x((\phi(x + e_k))_{k \in I}) = -4\beta^2 \text{Cov}_{\nu,\psi} (U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))) .$$

□

The next lemma follows by Taylor expansion and will be needed for the proof of Theorem 22

**Lemma 25 (Representation of Covariances)** For all $L^2$-functions $F, G \in C^1(\mathbb{R}; \mathbb{R})$ with bounded derivatives and for all measures $\nu \in P(\mathbb{R})$, we have

$$\text{Cov}_\nu (F, G) = \frac{1}{2} \int \left[ F(\phi) - F(\psi) \right] \left[ G(\phi) - G(\psi) \right] \nu(d\phi)\nu(d\psi),$$

where we denote by

$$IF(\phi, \psi) := \int_0^1 F'(\psi + t(\phi - \psi)) \, dt, \quad IG(\phi, \psi) := \int_0^1 G'(\psi + s(\phi - \psi)) \, ds .$$

**Remark 26 (Scaling Argument)** A simple scaling argument shows that it suffices to prove Theorem 22 for

$$\beta = 1, C_1 = 1 .$$

Indeed, suppose that the result is true for $\beta = 1$ and $C_1 = 1$. Given $\beta, V$ and $g$ which satisfy (A1) and (A2), we define

$$\tilde{U}(s) = \tilde{V}(s) + \tilde{g}(s), \quad \tilde{V}(s) = \beta V \left( \frac{s}{\sqrt{\beta C_1}} \right), \quad \tilde{g}(s) = \beta g \left( \frac{s}{\sqrt{\beta C_1}} \right) .$$
Then 
\[ 1 \leq (\dot{V})'' \leq \frac{C_2}{C_1}, \quad -\frac{C_0}{C_1} \leq (\ddot{g})'' \leq 0, \quad ||\ddot{g})''||_{L^2(\mathbb{R})} = (\beta C_1)^{-\frac{1}{2}} ||g''||_{L^2(\mathbb{R})}, \quad ||\ddot{g})'||_{L^2(\mathbb{R})} = (\beta^3/C_1)^{\frac{1}{2}} ||g'||_{L^2(\mathbb{R})}. \]

Hence \( \dot{V}, \ddot{g} \) satisfy the assumptions of Theorem 22 with \( \beta = 1 \) and \( C_1 = 1 \). On the other hand, the change of variables \( \tilde{\phi}(x) = \sqrt{\beta C_1} \phi(x) \) yields \( \tilde{U}(\nabla \tilde{\phi}(x)) = \beta U(\nabla \phi(x)) \) and thus

\[
\tilde{F}_x((\tilde{\phi}(x + e_i))_{i \in I}) := -\log \int_{\mathbb{R}} e^{-2\sum_{i \in I} \tilde{U}(\nabla \tilde{\phi}(x))} d\tilde{\phi}(x) = -\log \beta C_1 - \log \int_{\mathbb{R}} e^{-2\sum_{i \in I} u(\nabla u(x))} d\phi(x) = -\frac{\log \beta C_1}{2} + F_x((\phi(x + e_i))_{i \in I}).
\]

**Proof of Theorem 22** From Definition 19 and Lemma 21 it follows that, in order to prove that the random walk representation condition holds for \( H^v \), all we need is to show that there exist \( c_l, c_u > 0 \) such that for all \( i, j \in I, i \neq j \), and uniformly in \( x \) and \( \psi \)

\[ c_l \leq \text{Cov}_{\nu,\psi}(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))) \leq c_u. \]

Recall that we have \( U = V + g \), where \( 1 \leq V'' \leq C_2 \) and therefore we can split the initial covariance term into four resulting covariance terms. We will next check which ones of the new covariance terms are positive and which are negative. Using Lemma 23 for \( V'(\nabla_i \phi(x)) \) and \( V'(\nabla_j \phi(x)) \), we see that

\[
\text{Cov}_{\nu,\psi}(V'(\nabla_i \phi(x)), V'(\nabla_j \phi(x))) = \frac{1}{2} \int_0^1 V''((1-t)\psi(x) - \phi(x + e_i) + t\phi(x)) dt 
\int_0^1 V''((1-s)\psi(x) - \phi(x + e_j) + s\phi(x)) d\nu_x(d\phi) \nu_x(d\psi).
\]

By combining the above equality with the similar one for \( \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_i \phi(x))) \) and with the bound \( C_1 \leq V'' \leq C_2 \), we have for all \( i, j \in I \)

\[ \text{Cov}_{\nu,\psi}(V'(\nabla_i \phi(x)), V'(\nabla_j \phi(x))) \geq \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x))) \geq \text{Var}_{\nu,\psi}(\phi(x)) \geq 0,
\]

\[ \text{Cov}_{\nu,\psi}(V'(\nabla_i \phi(x)), V'(\nabla_j \phi(x))) \leq C_2 \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x))). \] \hspace{1cm} (29)

Since \( -C_0 \leq g'' \leq 0 \), by similar reasoning

\[ 0 \leq \text{Cov}_{\nu,\psi}(g'(\nabla_i \phi(x)), g'(\nabla_j \phi(x))) \leq C_0^2 \text{Var}_{\nu,\psi}(\phi(x)), \] \hspace{1cm} (30)

and

\[ -C_0 \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x))) \leq \text{Cov}_{\nu,\psi}(V'(\nabla_j \phi(x)), g'(\nabla_i \phi(x))) < 0. \] \hspace{1cm} (31)

Given (29), (30) and (31), we have the following upper and lower bounds for \( \text{Cov}_{\nu,\psi}(U', U') \)

\[ \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x))) + \text{Cov}_{\nu,\psi}(g'(\nabla_j \phi(x)), V'(\nabla_i \phi(x))) + \text{Cov}_{\nu,\psi}(g'(\nabla_i \phi(x)), V'(\nabla_j \phi(x))) \leq \text{Cov}_{\nu,\psi}(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))) \leq (C_2 + C_0^2) \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x))). \] \hspace{1cm} (32)

Of more importance are the lower bound estimates, as they will determine the conditions on our perturbation \( g'' \) which give us convexity after the one-step integration. We will next get a lower bound for the \( \text{Cov}_{\nu,\psi}(g', V') \) terms in (32), which shows that the upper and lower bounds in (32) are all in terms of \( \text{Cov}_{\nu,\psi}(\phi, V') \). Using (31), the Cauchy-Schwarz inequality and (29), we have

\[
0 \leq -\text{Cov}_{\nu,\psi}(V'(\nabla_j \phi(x)), g'(\nabla_i \phi(x))) \leq \sqrt{\text{Var}_{\nu,\psi}(V'(\nabla_j \phi(x))) \text{Var}_{\nu,\psi}(g'(\nabla_i \phi(x)))} \\
\leq \sqrt{C_2 \text{Cov}_{\nu,\psi}(\phi(x), V'(\nabla_j \phi(x)))} \text{Var}_{\nu,\psi}(g'(\nabla_i \phi(x))). \] \hspace{1cm} (33)
Let now $q \geq 1$ be arbitrarily fixed. By Lemma 25 and Jensen’s inequality, we get

\[
\text{Var}_{\nu, \psi}(g'(\nabla_i \phi(x)))
\]

\[
= \frac{1}{2} \iint \left( \phi(x) - \psi(x) \right)^2 \left[ \int_0^1 g''(\psi(x) - \phi(x + e_i) + t(\phi(x) - \psi(x))) \, dt \right]^2 \nu_x(\, d\phi) \nu_x(\, d\psi)
\]

\[
\leq \frac{1}{2} \iint \left( \phi(x) - \psi(x) \right)^2 \left[ \int_0^1 |g''(\psi(x) - \phi(x + e_i) + t(\phi(x) - \psi(x)))|^q \, dt \right]^{\frac{2}{q}} \nu_x(\, d\phi) \nu_x(\, d\psi)
\]

\[
= \frac{1}{2} \iint |\phi(x) - \psi(x)|^{2-2/q} \left[ \int_0^1 |g''(s)|^q \, ds \right]^{\frac{2}{q}} \nu_x(\, d\phi) \nu_x(\, d\psi)
\]

\[
\leq \frac{1}{2} \|g''\|^2_{L^q(\mathbb{R})} \iint |\phi(x) - \psi(x)|^{2-2/q} \nu_x(\, d\phi) \nu_x(\, d\psi)
\]

\[
\leq \frac{1}{2^q} \|g''\|^2_{L^q(\mathbb{R})} \left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{2-q}{q}}.
\]

where for the second equality we made the change of variable $s = \psi(x) - \phi(x + e_i) + t(\phi(x) - \psi(x))$, in the penultimate inequality we used Lemma 25 and for the last inequality we used 24. The lower bound in 22 becomes by 31

\[
\text{Cov}_{\nu, \psi}(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x)))
\]

\[
\geq \left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{2q+1}{2q}} \left[ \left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{1}{2q}} - 2^{(2q-1)/2q} \sqrt{C_2} \|g''\|_{L^q(\mathbb{R})} \right].
\]

We now proceed to find upper and lower bounds for $\text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x)))$. From 24, we have by repeated application

\[
\text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \leq \frac{1}{2d} \text{Cov}_{\nu, \psi} \left( V'(\nabla_j \phi(x)), \sum_{i \in I} V'(\nabla_i \phi(x)) \right).
\]

Recall now that

\[
\text{Cov}_{\nu, \psi} \left( V'(\nabla_j \phi(x)), \sum_{i \in I} V'(\nabla_i \phi(x)) \right) = \frac{1}{Z_{\psi, \psi}} \int V'(\nabla_j \phi(x)) \left( \sum_{i \in I} V'(\nabla_i \phi(x)) \right) e^{-2H_{x, \psi}(\phi)} \, d\phi(x)
\]

\[
- \left[ \frac{1}{Z_{\psi, \psi}} \int V'(\nabla_j \phi(x)) e^{-2H_{x, \psi}(\phi)} \, d\phi(x) \right] \left[ \frac{1}{Z_{\psi, \psi}} \int \left( \sum_{i \in I} V'(\nabla_i \phi(x)) \right) e^{-2H_{x, \psi}(\phi)} \, d\phi(x) \right],
\]

where $Z_{\psi, \psi}$ is the normalizing constant and $H_{x, \psi}(\phi)$ has been defined in 15. Using integration by parts in the above, we have

\[
\text{Cov}_{\nu, \psi} \left( V'(\nabla_j \phi(x)), \sum_{i \in I} V'(\nabla_i \phi(x)) \right) = \frac{1}{2} \text{E}_{\nu, \psi}(V''(\nabla_j \phi(x)))
\]

\[
- \text{Cov}_{\nu, \psi} \left( V'(\nabla_j \phi(x)), \sum_{i \in I} g'(\nabla_i \phi(x)) \right) \leq \frac{C_2}{2} - \text{Cov}_{\nu, \psi} \left( V'(\nabla_j \phi(x)), \sum_{i \in I} g'(\nabla_i \phi(x)) \right).
\]

From 37, 37 and 34, we now get the upper bound

\[
\text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \leq \frac{C_2}{4d} + \frac{\sqrt{C_2}}{2^{(2q+1)/2q}} \|g''\|_{L^q(\mathbb{R})} \left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{2-q}{2q}}
\]

which is equivalent to

\[
\left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{2q-1}{2q}} \geq \left[ \text{Cov}_{\nu, \psi}(\phi(x), V'(\nabla_j \phi(x))) \right]^{\frac{1}{2q}} - b \leq a,
\]
Proof. Using integration by parts and Cauchy-Schwartz, we have

\[ \tau_{x,\psi}^2 := \text{Var}_{\nu_x,\psi}(\phi(x)) \leq \text{Cov}_{\nu_x,\psi}(\phi(x), V'(|j \phi(x)|)) \leq \max \left[ b^2 q, \left( \frac{a}{b^{2q-1}} + b \right)^{2q} \right] = \left( \frac{a}{b^{2q-1}} + b \right)^{2q}. \]

(39)

The upper bound on \( \text{Cov}_{\nu_x,\psi}(U'(\nabla_j \phi(x)), U'(|j \phi(x)|)) \) follows now from (32) and (39). To find a lower bound, note now that from (29) we get

\[ \text{Cov}_{\nu_x,\psi}(\phi(x), V'(|j \phi(x)|)) \geq \frac{1}{2dC_2} \text{Cov}_{\nu_x,\psi} \left( V'(|j \phi(x)|), \sum_{i \in I} V'(|j \phi(x)|) \right). \]

By using (37) and (31), we have

\[ \text{Cov}_{\nu_x,\psi}(\phi(x), V'(|j \phi(x)|)) \geq \frac{1}{4dC_2}. \]

(40)

From (40) and (35), the lower bound becomes

\[ \text{Cov}_{\nu_x,\psi}(U'(\nabla_i \phi(x)), U'(|j \phi(x)|)) \geq \frac{1}{(4dC_2)^{2q-1}} \left[ \frac{1}{(4dC_2)^{2q}} - \frac{2\sqrt{C_2} \|g''\|_{L^q(\mathbb{R})}}{2^{\frac{2q}{q}}} \right]. \]

To summarize, we obtain the following upper and lower bounds, uniform with respect to \( x \) and \( \psi \)

\[ c_l = \frac{1}{(4dC_2)^{2q-1}} \epsilon \leq \text{Cov}_{\nu_x,\psi}(U'(\nabla_i \phi(x)), U'(|j \phi(x)|)) \leq \left( C_2 + c_0^2 \right) \left( \frac{a}{b^{2q-1}} + b \right)^{2q} = c_u, \]

(41)

for \( \epsilon = \frac{1}{(4dC_2)^{2q}} - \frac{2\sqrt{C_2} \|g''\|_{L^q(\mathbb{R})}}{2^{\frac{2q}{q}}} > 0 \) by (A2).

\[ \Box \]

Remark 27 Another possible condition, (A3), is obtained if we use Lemma 28 below to replace (31) by

\[ \text{Var}_{\nu_x,\psi}(g'(|j \phi(x)|)) \leq \sqrt{2dC_2} \|g'\|_{L^2(\mathbb{R})}^2. \]

Lemma 28 If \( h \in L^1(\mathbb{R}) \), then we have

\[ |E_{\nu_x,\psi}(h)| \leq \sqrt{2dC_2} \|h\|_{L^1(\mathbb{R})}. \]

PROOF. Using integration by parts and Cauchy-Schwartz, we have

\[ |E_{\nu_x,\psi}(h)| = \left| E_{\nu_x,\psi} \left( \frac{\partial}{\partial y} \int_{-\infty}^y h(z) \, dz \right) \right| = 2\beta \left| E_{\nu_x,\psi} \left( H'_x(y) \left( \int_{-\infty}^y h(z) \, dz \right) \right) \right| \]

\[ \leq 2\beta E_{\nu_x,\psi}^{1/2} \left( (H_x')^2 \right) E_{\nu_x,\psi}^{1/2} \left( \left( \int_{-\infty}^y h(z) \, dz \right)^2 \right) = E_{\nu_x,\psi}^{1/2} \left( H_x'^2 \right) E_{\nu_x,\psi}^{1/2} \left( \int_{-\infty}^y h(z) \, dz \right)^2 \]

\[ \leq \sqrt{2dC_2} \|h\|_{L^1(\mathbb{R})}. \]

Note that we also used property (A0) in the above formula.

\[ \Box \]

Remark 29 Note that if we consider the case where \( U \) is strictly convex with \( C_1 \leq U'' \leq C_2 \) (that is \( U = V \) and \( g = 0 \)), in view of (29) and (38), the one step integration preserves the strict convexity of the induced Hamiltonian as

\[ \frac{C_1^2}{4dC_2} \leq \text{Cov}_{\nu_x,\psi}(U'(\nabla_i \phi(x)), U'(|j \phi(x)|)) \leq \frac{C_2^2}{4dC_1}. \]
Remark 30 (Perturbation with Compact Support) Note that we can extend the results from Theorem 22 to the case where we have a perturbation $g$ such that $g''$ has compact support (see also example (b) below and the graph in Figure 6). More precisely, assume that $U = Y + h$, where $U$ satisfies $\text{[A0]}$, $D_1 \leq Y'' \leq D_2$ and $-D_0 \leq h'' \leq 0$ on $[a, b]$ and $0 < h'' < D_3$ on $\mathbb{R} \setminus [a, b]$, with $a, b \in \mathbb{R}$ and $h''(a) = h''(b) = 0$. Then we just need to replace

$$C_1 := D_1, C_2 := D_1 + D_2, \text{ and } g'' := h''1_{\{h'' \leq 0\}}.$$ 

A sketch of the argument follows next. Set

$$g(s) = h(s)1_{\{s \in [a, b]\}} + [h(b) + h'(b)(s - b)]1_{\{s > b\}} + [h(a) + h'(a)(s - a)]1_{\{s < a\}}$$

and

$$V(s) = Y(s) + h(s)1_{\{s \notin [a, b]\}} - [h(b) + h'(b)(s - b)]1_{\{s > b\}} - [h(a) + h'(a)(s - a)]1_{\{s < a\}}.$$ 

Thus, we have $V, g \in C^2(\mathbb{R})$, with $-D_0 \leq h''(s) = g''(s) \leq 0$ for $s \in [a, b]$ and $g''(s) = 0$ for $s \in \mathbb{R} \setminus [a, b]$ and $D_1 \leq V''(s) = Y''(s) + h''(s)1_{\{s \notin [a, b]\}} \leq D_2 + D_3$. Note that this procedure can also be extended to the case where $h''$ changes sign more than once.

4.2 Examples

(a) Let $p \in (0, 1)$ and $0 < k_2 < k_1$. Let

$$U(s) = -\log \left( p e^{-k_1 \frac{s^2}{2}} + (1 - p) e^{-k_2 \frac{s^2}{2}} \right).$$

Take $\frac{p}{1 - p} > \frac{k_2}{k_1}$ in order that the potential $U$ is non-convex. Let $\beta = 1, d = 2$ and $k_1 \gg k_2$. In this particular case, as Christof Külske pointed out to us, we are dealing entirely with sums of Gaussian integrals, so we can compute $\text{Cov}_{\nu_{2,\phi}}(U'(\nabla \phi(x)), U'(\nabla \phi(x)))$ directly, which explicit computation is not possible in general; the random walk representation condition holds then if $p < O \left( \left( \frac{k_2}{k_1} \right)^{1/2} \right)$ (see the Appendix for a sketch of the explicit computations).

This particular example is of independent interest and has been the focus of two other papers in the area (see [2] and [3]). For the case $d = 2$ and $\beta = 1$, it was proved in [2] that at the critical point $p := p_c$, such that $\frac{p}{1 - p} = \left( \frac{k_2}{k_1} \right)^{1/4}$, uniqueness of ergodic states is violated for this example of potential $U$ and there are multiple ergodic, invariant $\nabla \phi$-Gibbs measures with zero tilt; the same example is also treated in [3], where they prove CLT for the this particular class of potentials in the case of $\nabla \phi$-Gibbs measures with zero tilt.

Note that we can use (A3) to show that the random walk representation condition holds if $p < O \left( \left( \frac{k_2}{k_1} \right)^{2/3} \right)$. To show this, take $V$ and $g$ such that

$$V''(s) = \frac{pk_1 e^{-k_1 \frac{s^2}{2}} + (1 - p)k_2 e^{-k_2 \frac{s^2}{2}}}{p e^{-k_1 \frac{s^2}{2}} + (1 - p) e^{-k_2 \frac{s^2}{2}}}, \quad g''(s) = \frac{p(1 - p)(k_1 - k_2)^2 s^2}{p^2 e^{-(k_1 - k_2)^2 \frac{s^2}{2}} + 2p(1 - p) + (1 - p)^2 e^{(k_1 - k_2)^2 \frac{s^2}{2}}}.$$ 

Then

$$k_2 \leq V''(s) \leq pk_1 + (1 - p)k_2, \quad \|g''(s)\|_{L^2(\mathbb{R})} \leq O \left( \frac{p}{1 - p} (k_1 - k_2)^{1/4} \right),$$

$$\frac{p}{1 - p} (k_1 - k_2)^{1/4} \leq O \left( \frac{(k_2)^{3/2}}{(pk_1 + (1 - p)k_2)^{5/4}} \right) = O \left( \frac{(k_2)^{3/2}}{(pk_1)^{5/4}} \right).$$
(b) $U(s) = s^2 + a - \log(s^2 + a)$, where $0 < a < 1$. Let $0 < \beta < \frac{a}{4\sqrt{d}(2 + \frac{2}{2\sqrt{a}})}$. This example is interesting, as it has two global minimums.

Then, using the notation from Remark 30 take $Y(s) = s^2$ and $h(s) = -\log(s^2 + a)$. We have $Y''(s) = 2$, so $D_1 = D_2 = 2$; also $h''(s) = 2 - \frac{2a}{(s^2 + a)^2}$, with $-\frac{2}{a} \leq h''(s) \leq 0$ for $s \in [-\sqrt{a}, \sqrt{a}]$ and $0 < h''(s) \leq \frac{2}{2\sqrt{a}}$ otherwise. Then $C_0 = \frac{2}{a}$, $C_1 = 2, C_2 = 2 + \frac{2}{2\sqrt{a}}$ and $\|g''(s)\|_{L^1(\mathbb{R})} = \frac{2}{\sqrt{a}}$.

By using condition (A2) with $q = 1$, the random walk representation condition holds.

5 Uniqueness of ergodic component

In this section, we extend the uniqueness of ergodic component result, proved for strictly convex potentials in \([15]\), to the class of non-convex potentials $U = V + g$ which satisfy (A0) such $V$ and $g$ satisfy (A1) and (A2).

For $x \in \mathbb{Z}^d$, we define the shift operators: $\sigma_x : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ for the heights by $\sigma_x \phi(y) = \phi(y - x)$ for $y \in \mathbb{Z}^d$ and $\phi \in \mathbb{R}^{2d}$, and $\sigma_x : \mathbb{R}^{(2d)^*} \to \mathbb{R}^{(2d)^*}$ for the bonds by $(\sigma_x \eta)(b) = \eta(b - x)$, for $b \in (\mathbb{Z}^d)^*$ and $\eta \in \chi$. Then shift-invariance and ergodicity for $\mu$ (with respect to $\sigma_x$ for all $x \in \mathbb{Z}^d$) is defined in the usual way (see for example page 122 in \([16]\)). We say that the shift-invariant $\mu \in \mathbb{P}_2(\chi)$ has a given tilt $u \in \mathbb{R}^d$ if $E_\mu(\eta(b)) = \langle u, y_b - x_b \rangle$ for all bonds $b = (x_b, y_b) \in (\mathbb{Z}^d)^*$.

We will prove that:

**Theorem 31 (Uniqueness of an ergodic $\mu_u$)** Let $U = V + g$, where $U$ satisfy (A0) and $V$ and $g$ satisfy (A1) and (A2). Then for every $u \in \mathbb{R}^d$, there exists at most one ergodic, shift-invariant $\mu_u \in \mathcal{G}(H)$ with a given tilt $u \in \mathbb{R}^d$.

Note that existence of an ergodic $\mu_u$ is guaranteed for our class of non-convex potentials by Remark 30 below.

The proof of Theorem 31 will be done in two steps. First, in subsection 5.1 we will prove the uniqueness of ergodic, shift-invariant $\mu^\text{ev}_u \in \mathcal{G}_\text{ev}(H^\text{ev})$ with a given tilt $u \in \mathbb{R}^d$, when the potentials $F_x$ are of form as defined in \([14]\) and therefore $H^\text{ev}$ satisfies the random walk representation condition. For that, we will be adapting earlier results for two-body potentials under uniformly strictly convex condition, to multi-body potentials satisfying the random walk representation condition. Then we will use this result combined with Lemma 17 in subsection 5.2, to extend the result to $\mu_u \in \mathcal{G}(H)$.

5.1 Step 1: Uniqueness of ergodic component for $(\mathbb{Z}^d_{\text{ev}})^*$

For $x \in \mathbb{Z}^d_{\text{ev}}$, we define the even shift operators: $\sigma_x : \mathbb{R}^{2d}_{\text{ev}} \to \mathbb{R}^{2d}_{\text{ev}}$ and $\sigma_x : (\mathbb{Z}^d_{\text{ev}})^* \to (\mathbb{Z}^d_{\text{ev}})^*$ similarly as for $x \in \mathbb{Z}^d$. Then shift-invariance and ergodicity for $\mu^\text{ev}$ (with respect to $\sigma_x$ for all $x \in \mathbb{Z}^d_{\text{ev}}$) are defined similarly as for $\mu$. The main result in this section is:

**Theorem 32** For every $u \in \mathbb{R}^d$, there exists at most one $\mu^\text{ev}_u \in \mathcal{G}_\text{ev}(H^\text{ev})$, shift-invariant and ergodic with tilt $u$. 
We will prove Theorem 32 by coupling techniques. We will follow the same line of argument as in [15], by introducing dynamics on the gradient field which keeps the measure in $G_{ev}(H^{ev})$ invariant. Suppose the dynamics of the even height variables $\phi_t = \{\phi_t(y)\}_{y \in \mathbb{Z}^d_{ev}}$ are generated by the family of SDEs

$$d\phi_t(y) = -\sum_{x \in \mathbb{Z}^d_{ev} \setminus \{x-y\}=1} \frac{\partial}{\partial \phi(y)} F_x((\phi_t(x + e_i))_{i \in I})\,dt + \sqrt{2}dW_t(y), \quad y \in \mathbb{Z}^d_{ev},$$

where for all $x \in \mathbb{Z}^d_{od}$, $F_x$ are the functions defined in Lemma 12 satisfying the properties in Definition 19 and $\{W_t(y), y \in \mathbb{Z}^d_{ev}\}$ is a family of independent Brownian motions. Using standard SDE methods, one can show that equation (42) has a unique solution.

We denote by $S_{ev}$ the class of all shift invariant $\mu \in P_2(\chi_{ev})$ which are stationary for the SDE (42) and by $\text{ext} S_{ev}$ those $\mu_{ev} \in S_{ev}$ which are ergodic. For each $u \in \mathbb{R}^d$, we denote by $(\text{ext} S_{ev})_u$ the family of all $\mu_{ev} \in \text{ext} S_{ev}$ such that $E_{\mu_{ev}}(\eta_{ev}(b)) = \langle u, y_b - x_b \rangle$ for all bonds $b = (x_b, y_b) \in (\mathbb{Z}^d_{ev})^*$. Note that all translation invariant measures in $G_{ev}(H^{ev})$ are stationary under the dynamics (see Proposition 3.1 in [15]).

The next theorem is a key result in the proof of Theorem 32.

**Theorem 33** For every $u \in \mathbb{R}^d$, there exists at most one $\mu_{ev}^u \in (\text{ext} S_{ev})_u$.

Theorem 32 now follows from Theorem 33 and Proposition 3.1 in [15], which shows that if $\mu_{ev}^u \in G_{ev}(H^{ev})$ is shift-invariant and ergodic, then $\mu_{ev}^u \in \text{ext} S_{ev}$.

The proof of Theorem 33 is based on a coupling lemma, Lemma 34 below; a key ingredient for the coupling lemma is a bound on the distance between two measures evolving under the same dynamics. The main ingredients needed to prove it are Lemma 33 below and a special ergodic theorem (see (41) below). The deduction of Theorem 33 from the coupling lemma follows the same arguments as the proof of Theorem 2.1 in [15] and will be omitted.

**Dynamics** We will first derive a differential inequality for the difference of two solutions evolving under the same dynamics, which will be a key ingredient in the proof of the coupling Lemma 35 below.

**Lemma 34** Let $\phi_t$ and $\tilde{\phi}_t$ be two solutions for (42), coupled via the same Brownian motion in (42), and set $\tilde{\phi}_t(y) := \phi_t(y) - \tilde{\phi}_t(y)$, where $y \in \mathbb{Z}^d_{ev}$. Then for every finite $\Lambda_{ev} \subset \mathbb{Z}^d_{ev}$, we have

$$\frac{\partial}{\partial t} \sum_{y \in \Lambda_{ev}} (\tilde{\phi}_t(y))^2 \leq -c \sum_{b \in (\Lambda_{ev})^*} \left[ \nabla \tilde{\phi}_t(b) \right]^2 + 2c \sum_{b \in (\Lambda_{ev})^*} |\phi_t(y_b)||\nabla \tilde{\phi}_t(b)|. \quad (43)$$

**Proof.** The proof of Lemma 34 is an adaptation of an earlier result by [15], where we replace the uniform strictly convex condition on the two-body potential $V$ with the random walk representation condition on a multi-body potential of gradient type.

Let $y \in \Lambda_{ev}$. Then from (42), we have

$$\frac{\partial}{\partial t}(\tilde{\phi}_t(y))^2 = -2 \sum_{x \in \Lambda_{od} \cap \{x-y\}=1} \left[ \frac{\partial}{\partial \phi(y)} F_x((\phi_t(x + e_i))_{i \in I}) - \frac{\partial}{\partial \phi(y)} F_x((\tilde{\phi}_t(x + e_i))_{i \in I}) \right] \tilde{\phi}_t(y). \quad (44)$$

By summing now in (41) over all $y \in \Lambda_{ev}$ in (44), we get

$$\frac{\partial}{\partial t} \sum_{y \in \Lambda_{ev}} (\tilde{\phi}_t(y))^2 = -2 \sum_{x \in \Lambda_{od} \cap \{x+y\}=1} \left[ D^j F_x((\phi_t(x + e_i))_{i \in I}) - D^j F_x((\tilde{\phi}_t(x + e_i))_{i \in I}) \right] \tilde{\phi}_t(x + e_j), \quad (45)$$

where $\Lambda_{od} = \Lambda \cap \mathbb{Z}^d_{od}$ and $\Lambda$ is the associated set to $\Lambda_{ev}$, as defined in Definition 11. To prove (43), we expand now $D^j F_x((\phi_t(x + e_i))_{i \in I})$ in Taylor series around $(\tilde{\phi}_t(x + e_i))_{i \in I}$ to get

$$D^j F_x((\phi_t(x + e_i))_{i \in I}) - D^j F_x((\tilde{\phi}_t(x + e_i))_{i \in I})$$

$$= \sum_{k \in I} \tilde{\phi}_t(x + e_k) \int_0^1 D^{j,k} F_x ((s\phi_t(x + e_i) + (1-s)\tilde{\phi}_t(x + e_i))_{i \in I}) \,ds. \quad (46)$$
Plugging (46) in (45), we have
\[
\frac{\partial}{\partial t} \sum_{y \in \Lambda_{\text{ev}}} (\tilde{\phi}_t(y))^2
\]
\[
= -2 \sum_{x \in \Lambda_{\text{od}}} \sum_{\{j \in \ell, \text{ } x+e_j \in \Lambda_{\text{ev}}\}} \sum_{k \in I} \tilde{\phi}_t(x+e_k) \tilde{\phi}_t(x+e_j) \int_0^1 D^{j,k} F_x ((s \phi_t(x+e_i) + (1-s) \phi_t(x+e_i))_{i \in I}) \, ds
\]
\[
= 2 \sum_{x \in \Lambda_{\text{od}}} \sum_{\{j \in \ell, \text{ } x+e_j \in \Lambda_{\text{ev}}\}} \sum_{k \in I, k \neq j} \left[ \tilde{\phi}_t^2(x+e_j) - \tilde{\phi}_t(x+e_k) \tilde{\phi}_t(x+e_j) \right]
\]
\[
\int_0^1 D^{j,k} F_x ((s \phi_t(x+e_i) + (1-s) \phi_t(x+e_i))_{i \in I}) \, ds
\]
\[
= \sum_{x \in \Lambda_{\text{od}}} \sum_{\{j \in \ell, \text{ } x+e_j \in \Lambda_{\text{ev}}\}} \left[ \tilde{\phi}_t^2(x+e_j) - \tilde{\phi}_t(x+e_k) \tilde{\phi}_t(x+e_j) \right]
\]
\[
\int_0^1 D^{j,k} F_x ((s \phi_t(x+e_i) + (1-s) \phi_t(x+e_i))_{i \in I}) \, ds
\]
\[
\leq -c \sum_{b \in (\Lambda_{\text{ev}})^*} \left| \nabla \tilde{\phi}_t(b) \right|^2 + 2\bar{c} \sum_{b \in \partial(\Lambda_{\text{ev}})^*} \left| \phi_t(b) \right| \left| \nabla \tilde{\phi}_t(b) \right|,
\]
where we used (25) in the second equality, symmetry of the equation and the fact that $D^{j,k} F_x = D^{k,j} F_x$ in the third equality and Definition (19) in the inequality on the last line.

\begin{proof}
Coupling Argument Suppose that there exist $\mu^{\text{ev}} \in (\text{ext } S_{\text{ev}})_u$ and $\mu^{\text{ev}} \in (\text{ext } S_{\text{ev}})_v$ for $u, v \in \mathbb{R}^d$. For $r > 0$, recall the definition of $\chi_{\text{ev}, r}$ as given in subsection 3.1. Let us construct two independent $\chi_{\text{ev}, r}$-valued random variables $\eta_{\text{ev}} = \{\eta_{\text{ev}}(b)\}_{b \in (\mathbb{Z}_d^2)^*}$ and $\bar{\eta}_{\text{ev}} = \{\bar{\eta}_{\text{ev}}(b)\}_{b \in (\mathbb{Z}_d^2)^*}$ on a common probability space $(\Omega, F, P)$ in such a manner that $\eta_{\text{ev}}$ and $\bar{\eta}_{\text{ev}}$ are distributed by $\mu^{\text{ev}}$ and $\mu^{\text{ev}}$ respectively. We define $\phi_0 = \phi^{\text{ev}, 0}$ and $\tilde{\phi}_0 = \phi^{\text{ev}, 0}$ using the notation in (11). Let $\phi_t$ and $\tilde{\phi}_t$ be two solutions of the SDE (12) with common Brownian motions having initial data $\phi_0$ and $\tilde{\phi}_0$. Let $\eta_{\text{ev}, t}$ and $\bar{\eta}_{\text{ev}, t}$ be defined by $\eta_{\text{ev}, t}(b) := \nabla \phi_t(b)$ and $\bar{\eta}_{\text{ev}, t}(b) := \nabla \tilde{\phi}_t(b)$, for all $b \in (\mathbb{Z}_d^2)^*$. Since $\mu^{\text{ev}}, \mu^{\text{ev}} \in S_{\text{ev}}$, we conclude that $\eta_{\text{ev}, t}$ and $\bar{\eta}_{\text{ev}, t}$ are distributed by $\mu^{\text{ev}}$ and $\mu^{\text{ev}}$ respectively, for all $t \geq 0$.

Change of Basis To adapt the coupling argument from Lemma 2.1 in [15] to the even bonds, we will use the generator set in $\mathbb{Z}_{\text{ev}}^d$ outlined below:

$e_{\text{ev}, i} = e_i + e_{i+1}$, $i = 1, 2, \ldots, d - 1$ and $e_{\text{ev}, d} = \begin{cases} e_d - e_1 & \text{d even,} \\ e_d + e_1 & \text{d odd}. \end{cases}$

Once we have defined this generator set, we can proceed with our arguments. We claim that:

\begin{lemma}
There exists a constant $C > 0$ independent of $u, v \in \mathbb{R}^d$ such that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{i=1}^T \mathbb{E}^P \left[ (\eta_{\text{ev}, t}(e_{\text{ev}, i}) - \bar{\eta}_{\text{ev}, t}(e_{\text{ev}, i}))^2 \right] \, dt \leq C ||u - v||^2.
\]
\end{lemma}

\begin{proof}
To prove (48), we apply Lemma 34 to the differences $\{\phi_t(x) := \phi_t(x) - \tilde{\phi}_t(x)\}$ to bound, with the choice $\Lambda_N = [-N, N]^d$, the term
\[
\int_0^T \sum_{x \in \Lambda_N} \mathbb{E}^P |\phi_t(x)|^2 dt.
\]
By using shift-invariance in the resulting inequality, we will obtain a bound for the term on the left of (48). We will next use a special ergodic theorem for co-cycles (see for example Theorem 4 in [4]), which we can use in our case because \( Z_{\text{ev}}^d \) is a sub-algebra; we apply it to \( \mu^{\text{ev}} \in (\text{ext} S_{\text{ev}})_u \) to obtain

\[
\lim_{\|x\| \to \infty} \frac{1}{\|x\|} \| \phi^{\text{ev},0}(x) - x \cdot u \|_{L^2(\mu^{\text{ev}})} = 0.
\]

This ergodic theorem will allow us to further estimate the bound and to obtain the statement of the lemma. The details of the proof, following the same arguments as Lemma 2.1 from [15], will be omitted and are left to an interested reader.

\[ \Box \]

5.2 Step 2: Uniqueness of ergodic component for \((\mathbb{Z}^d)^*\)

Proof of Theorem 31 Let \( u \in \mathbb{R}^d \). Suppose now that there exist \( \mu, \bar{\mu} \in G(H) \) ergodic and shift-invariant such that \( E_{\mu}(\eta(b)) = E_{\bar{\mu}}(\eta(b)) = \langle u, y_b - x_b \rangle \) for all bonds \( b = (x_b, y_b) \in (\mathbb{Z}^d)^* \). Note now that \( E_{\mu^{\text{ev}}}(\eta_{\text{ev}}(b)) = E_{\bar{\mu}^{\text{ev}}}(\eta_{\text{ev}}(b)) = \langle u, y_b - x_b \rangle \) for all bonds \( b = (x_b, y_b) \in (\mathbb{Z}_{\text{ev}}^d)^* \).

From Lemma 16 and with the same notation as there, we get that \( \mu^{\text{ev}}, \bar{\mu}^{\text{ev}} \in G_{\text{ev}}(H^{\text{ev}}) \). As for all \( \eta_{\text{ev}} \in \chi_{\text{ev}} \), with \( \eta_{\text{ev}}(b) = \phi(y_b) - \phi(x_b), b = (x_b, y_b) \in (\mathbb{Z}_{\text{ev}}^d)^* \), we can write \( \eta_{\text{ev}}(b) = \eta(b_1) + \eta(b_2), b_1, b_2 \in (\mathbb{Z}^d)^*, \) shift-invariance and ergodicity under the even shifts for \( \mu^{\text{ev}}, \bar{\mu}^{\text{ev}} \) follow immediately from the similar properties for \( \mu, \bar{\mu} \). Therefore \( \mu^{\text{ev}}, \bar{\mu}^{\text{ev}} \in (\text{ext} S_{\text{ev}})_u \), so we can apply Theorem 32 to get \( \mu^{\text{ev}} = \bar{\mu}^{\text{ev}} \). Then for any \( A \in F(\mathbb{Z}^d)^* \), we have from Lemma 17 that \( E_{\mu}(1_A|F(\mathbb{Z}_{\text{ev}}^d)^*) = E_{\bar{\mu}}(1_A|F(\mathbb{Z}_{\text{ev}}^d)^*) \) and we have

\[
\mu(A) = E_{\mu}(1_A) = E_{\mu}(E_{\mu}(1_A|F(\mathbb{Z}_{\text{ev}}^d)^*)) = E_{\bar{\mu}}(E_{\mu}(1_A|F(\mathbb{Z}_{\text{ev}}^d)^*)) = E_{\bar{\mu}}(E_{\mu}(1_A|F(\mathbb{Z}_{\text{ev}}^d)^*)) = E_{\bar{\mu}}(A) = \bar{\mu}(A).
\]

\[ \Box \]

Remark 36 (Existence of ergodic component on \((\mathbb{Z}^d)^*\)) Tightness of the family \( \{\mu_{A, \chi}\}_{A \subset \mathbb{Z}^d, \chi \in \chi} \) is known for non-convex potentials with quadratic growth at \( \infty \) (see also Remark 4.4 page 152 in [16]). Therefore a limiting measure exists by taking \( |A| \to \infty \) along a suitable sub-sequence. Thus existence of shift-invariant \( \mu \in P_2(\chi) \) with given tilt \( u \in \mathbb{R}^d \) is assured; nevertheless, existence of an ergodic and shift-invariant \( \mu_u \in P_2(\chi) \) with given tilt \( u \in \mathbb{R}^d \), is not assured for non-convex potentials. However, due to the strict convexity of the \( F_u \) potentials, we can use the Brascamp-Lieb inequality and a similar reasoning to the one of Theorem 3.2 in [15], to easily show the existence, for every \( u \in \mathbb{R}^d \), of at least one \( \mu_u \in G(H) \) ergodic and shift-invariant and with tilt \( u \in \mathbb{R}^d \).

6 Decay of Covariances

In this section, we extend the covariance estimates of [9] to the class of non-convex potentials \( U = V + g \) which satisfy (A0) such \( V \) and \( g \) satisfy (A1) and (A2).

Let \( F \in C^1_b(\chi_\tau) \), where \( C^1_b(\chi_\tau) \) denotes the set of differentiable functions depending on finitely many coordinates with bounded derivatives and where \( \chi_\tau \) was defined in subsection 2.2.1. For \( \eta, \eta' \in \chi_\tau \), let

\[
\lim_{\epsilon \to 0} \frac{F(\eta + \epsilon \eta') - F(\eta)}{\epsilon} = \langle DF(\eta), \eta' \rangle = \sum_{b \in (\mathbb{Z}^d)^*} \alpha(b) \eta'(b).
\]

We denote by

\[
\partial_b F(\eta) := \alpha(b) \text{ and } ||\partial_b F||_{\infty} = \sup_{\eta \in \chi} ||\partial_b F(\eta)||.
\]

Using now \( \eta, \eta' \in \chi_{\text{ev}} \) in the above, we define \( \partial_{b_{\text{ev}}} F \) and \( ||\partial_{b_{\text{ev}}} F||_{\infty} \) similarly for \( b_{\text{ev}} \in (\mathbb{Z}_{\text{ev}}^d)^* \). The main result of this section is
Theorem 37 (Decay of Covariances) Let \( u \in \mathbb{R}^d \). Assume \( U = V + g \), where \( U \) satisfies (A1) and \( V \) and \( g \) satisfy (A2). Let \( F, G \in C_b^1 (\mathcal{X}) \). Then there exists \( C > 0 \) such that

\[
|\text{Cov}_{\mu_u}(F(\eta), G(\eta))| \leq C \sum_{b, b' \in \mathbb{Z}^d} \frac{||\partial_b F||_{\infty} ||\partial_b G||_{\infty}}{1 + ||x_b - x_{b'}||^2},
\]

(50)

where \( b = (x_b, y_b) \) and \( b' = (x_{b'}, y_{b'}) \).

Before proving Theorem 37, we make a remark which we will use in our proof.

Remark 38 Take \( b_{ev} = (x + e_i, x + e_j) \in (\mathbb{Z}^d)^* \). In view of the definition, we have

\[
||\partial_{b_{ev}} F||_{\infty} = \sup_{\eta \in \mathcal{X}_{ev}} |\partial_{b_{ev}} F(\eta)| \leq \sum_{b \in \mathbb{Z}^d} \sup_{\eta \in \mathcal{X}} |\partial_b F(\eta)| = \sum_{b \sim b_{ev}} ||\partial_b F||_{\infty},
\]

(51)

where \( b \sim b_{ev} \) are those \( b = (x, x + e_s) \in (\mathbb{Z}^d)^* \), \( x \in \mathbb{Z}^d_{od} \), such that \( s \in \{i, j\} \).

Proof of Theorem 37 We have

\[
\text{Cov}_{\mu_u}(F(\eta), G(\eta)) = E_{\mu_u} \left[ \text{Cov}_{\mu_u}(F(\eta), G(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*) \right] + \text{Cov}_{\mu_u} \left( E_{\mu_u}[F(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*], E_{\mu_u}[G(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*] \right),
\]

where by Corollary 24 and with the same notations, we have

\[
E_{\mu} \left( F | \mathcal{F}(\mathbb{Z}^d_{ev})^* \right) (\eta_{ev}) = \int F \left( (\eta_{ev}(b) - \phi(x))_{b \in B(x, k), x \in \mathbb{Z}^d_{od}} \right) \prod_{x \in \mathbb{Z}^d_{od}} \mu_{x, \eta_{ev}} (d\phi(x));
\]

a similar formula holds for \( G \). Note that under \( \mu_u(\cdot \mathcal{F}(\mathbb{Z}^d_{ev})^*) \), the gradients \( (\nabla \phi_i(x), x \in \mathbb{Z}^d_{od}, i \in I) \) are independent. Thus, there exists \( c > 0 \) such that

\[
|\text{Cov}_{\mu_u}(F(\eta), G(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*)| \leq c \sum_{b \in \mathbb{Z}^d} ||\partial_b F||_{\infty} ||\partial_b G||_{\infty} \text{Var}_{\mu_u}(\nabla \phi(b) | \mathcal{F}(\mathbb{Z}^d_{ev})^*)
\]

\[
\leq c' \tau^2 \sum_{b \in \mathbb{Z}^d} ||\partial_b F||_{\infty} ||\partial_b G||_{\infty},
\]

(52)

where the first inequality is an application of Lemma 3.1 in [10], and for the second inequality we used (39). Note that, due to the fact that the random walk representation holds, Theorem 6.2 in [9] can be adapted to the case of the infinite even lattice with strictly convex potential; thus, a decay of covariance statement, similar to the one in Theorem 37, holds for the even setting. In view of Lemma 16 there exists \( c'' > 0 \) such that

\[
|\text{Cov}_{\mu_u}(\hat{F}, \hat{G})| \leq c'' \sum_{b_{ev}, b'_{ev} \in \mathbb{Z}^d_{ev}^*} \frac{||\partial_{b_{ev}} \hat{F}||_{\infty} ||\partial_{b'_{ev}} \hat{G}||_{\infty}}{1 + ||x_{b_{ev}} - x_{b'_{ev}}||^2},
\]

(53)

where \( \hat{F} = E_{\mu_u}[F(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*] \) and \( \hat{G} = E_{\mu_u}[G(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*] \). We need to estimate now \( \partial_{b_{ev}} \hat{F} \) and \( \partial_{b_{ev}} \hat{G} \).

But

\[
\partial_{b_{ev}} \hat{F} = \partial_{b_{ev}} E_{\mu_u}[F(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*] = E_{\mu_u}[\partial_{b_{ev}} F(\eta) | \mathcal{F}(\mathbb{Z}^d_{ev})^*]
\]

\[
- \text{Cov}_{\mu_u} \left( F(\eta), \partial_{b_{ev}} \left( \sum_{x \in \mathbb{Z}^d_{od}} \sum_{b \in B(x, k)} U(\eta_{ev}(b) - \phi(x)) \right) | \mathcal{F}(\mathbb{Z}^d_{ev})^* \right),
\]

(54)
Remark 21. Also, since by Theorem 37 uses the same arguments as in [17] and is based on the random walk representation, as explained in Section 6.

Let \( S = \sum_{x \in \mathbb{Z}^d} f(x) \) where \( \Sigma \) is the covariance function. We will extend next the scaling limit results from [17] to our class of potentials.

7 Central Limit Theorem

The statement of the theorem follows now from (55), (56), (52), (53) and (51).

Applying (52) to the covariance in (55) and using \( |U''| \leq C_0 + C_2 \) and (39), we get for some \( \epsilon'' > 0 \)

\[
|\text{Cov}_{\mu_u} \left( F(\nabla \phi), \partial_{b_{\text{ev}}} \left( \sum_{x \in \mathbb{Z}^d} \sum_{b \in B(x)} U(\eta_{\text{ev}}(b) - \phi(x)) \right) \right) \left| F_{(\Sigma_{\text{ev}}^*)} \right| \leq 2d\epsilon''(C_0 + C_2)||\partial_{b_{\text{ev}}} F||_{\infty} \text{Var}_{\mu_u}(\eta(b)|F_{(\Sigma_{\text{ev}}^*)}) \leq \epsilon||\partial_{b_{\text{ev}}} F||_{\infty}. \]

The statement of the theorem follows now from (55), (56), (52), (53) and (51).

\[ \square \]

Theorem 39 (Central Limit Theorem) Let \( u \in \mathbb{R}^d \). Assume \( U = V + g \), where \( U \) satisfies (A1) and \( V \) and \( g \) satisfy (A2). Set

\[ S_{\epsilon}(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \sum_{i \in I}(\nabla_i \phi(x) - u_i)f_i(cx), \]

where \( f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) \). Then

\[ S_{\epsilon}(f) \Rightarrow N(0, \Sigma_{\text{ev}}^2(f)) \quad \text{as} \quad \epsilon \to 0, \]

where \( \Sigma_{\text{ev}}^2(f) \) can be identified explicitly as in Theorem 2.1 in [17] and \( \Rightarrow \) signifies convergence in distribution.

Proof. It suffices to prove that for all \( i \in I \)

\[ S_{\epsilon,i}(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + e_i) - u_i \Rightarrow N(0, \sigma_{u,i}^2(f)) \quad \text{as} \quad \epsilon \to 0, \]

where \( \sigma_{u,i}^2 > 0 \). Note that

\[
S_{\epsilon,i}(f) = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + e_i) - \phi(x) - u_i = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + 2e_i) - \phi(x) - 2u_i
\]

\[- \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + 2e_i) - \phi(x + e_i) - u_i + \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + e_i) - \phi(x) - u_i
\]

\[ = \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f(x)\phi(x + 2e_i) - \phi(x) - 2u_i
\]

\[ + \epsilon^{d/2} \sum_{x \in \mathbb{Z}^d} f((x + e_i)\epsilon - f(x))\phi(x + 2e_i) - \phi(x + e_i) - u_i = S_{\epsilon,i}(f) + R_\epsilon(f). \]

We can show the CLT for \( S_{\epsilon,i}(f) \) since the summation is concentrated on the even sites; the proof uses the same arguments as in [17] and is based on the random walk representation, as explained in Remark 21. Also, since by Theorem 37

\[ |\text{Cov}_{\mu_u}(\nabla_i \phi(x), \nabla_j \phi(y))| \leq \frac{C}{(\|x - y\| + 1)^d}, \]
we have
\[
\text{Var}_{\mu_u}(R_{\epsilon,i}(f)) \leq \epsilon^d \sum_{x, y \in \mathbb{Z}^d_v} |\nabla_i f(x)| |\nabla_i f(y)| \text{Cov}_{\mu_u}(\phi(x + \epsilon_i) - \phi(x), \phi(y + \epsilon_i) - \phi(y))|
\]
\[
\leq \epsilon^d \sum_{x, y \in \mathbb{Z}^d_v} |\nabla_i f(x)| |\nabla_i f(y)| \frac{C}{(\|x - y\| + 1)^d},
\]
where \(\nabla_i f(x) = f((x + \epsilon_i)e) - f(x)\). Expanding \(f((x + \epsilon_i)e)\) in Taylor expansion around \(x\), we have \(\nabla_i f(x) = D_i^1 f(a)\epsilon\), for some \(a \in \mathbb{R}^d\). As \(f \in C_0^\infty(\mathbb{R}^d)\), there exist \(M, N > 0\) such that for all \(x \in \mathbb{R}^d\) with \(|\epsilon x| \leq N\) we have \(f(\epsilon x) \leq M, |D^1 f(\epsilon x)| \leq M\) and both functions equal to 0 for \(|\epsilon x| > N\). Therefore
\[
\text{Var}_{\mu_u}(R_{\epsilon,i}(f)) \leq \sum_{x, y \in \mathbb{Z}^d_v, |\epsilon x| \leq N} \frac{\epsilon^{d+2} M^2 C}{(\|x - y\| + 1)^d} \leq \epsilon^{d+2} M^2 C \sum_{y \in \mathbb{Z}^d_v, |\epsilon y| \leq N} \int_{-N \epsilon}^N \cdots \int_{-N \epsilon}^N \frac{dx_1 dx_2 \cdots dx_d}{(\sum_{i=1}^d |x_i - y_i| + 1)^d}
\]
\[
\leq \epsilon^2 C(d, N, M) \log (1 + 2dN/\epsilon) \leq 2dNC(d, N, M)\epsilon,
\]
where \(C(d, N, M)\) is a positive constant depending on \(d, M\) and \(N\). It follows that \(R_{\epsilon,i}(f) \to 0\) in probability as \(\epsilon \to 0\).

8 Surface tension

We will extend here the surface tension strict convexity results from [15] and [12] to the family of non-convex potentials satisfying (A0), (A1) and (A2).

Take \(N \in \mathbb{N}\) and let \(T^d_N = (\mathbb{Z}/N\mathbb{Z})^d\) be the lattice torus in \(\mathbb{R}^d\) and let \(u \in \mathbb{R}^d\). Then, we define the surface tension on the torus \(T^d_N\) as
\[
\sigma^\beta_{T^d_N}(u) = -\frac{1}{|T^d_N|} \log \frac{Z^\beta_{T^d_N}(u)}{Z^\beta_{T^d_N}(0)}, \text{ with } Z^\beta_{T^d_N}(u) = \int_{T^d_N} \exp(-\beta H_{T^d_N}(\phi, u)) \prod_{x \in T^d_N \setminus \{0\}} d\phi(x)
\]
and where \(H_{T^d_N}\) is given by
\[
H_{T^d_N}(\phi, u) = \sum_{x \in T^d_N} \sum_{i=1}^d U(\nabla_i \phi(x) + u_i) = \sum_{x \in T^d_N} \sum_{i=1}^d [V(\nabla_i \phi(x) + u_i) + g(\nabla_i \phi(x) + u_i)].
\]
We define \(u_{-i} = -u_i\) for \(i = 1, 2, \ldots, d\). Take now \(N\) to be even. Just as in the previous sections, let us label the vertices of the torus as odd and even; let the set of odd vertices on the torus be \(T^d_{N,od}\) and the set of even vertices be \(T^d_{N,ev}\). Then we can of course first integrate all the odd coordinates and:
\[
Z^\beta_{T^d_N}(u) = \int_{\mathbb{R}^d_N} \left( \int_{T^d_{N,od}} \exp(-\beta H_{T^d_N}(\phi, u)) \prod_{x \in T^d_{N,od}} d\phi(x) \right) \prod_{x \in T^d_{N,ev} \setminus \{0\}} d\phi(x)
\]
\[
= \int_{\mathbb{R}^d_N} \exp(-\beta H_{T^d_{N,ev}}(\phi, u)) \prod_{x \in T^d_{N,ev} \setminus \{0\}} d\phi(x),
\]
where, similarly to (12)
\[
H_{T^d_{N,ev}}(\phi, u) = \sum_{x \in T^d_{N,od}} F_x((\phi(x + e_i))_{i \in I}, u), \text{ I = \{-d, \ldots, d\} \setminus \{0\},}
\]
with
\[ F_x((\phi(x + e_i))_{i \in I}, u) = -\log \int_{\mathbb{R}} e^{-\beta \sum_{i \in I} U(\nabla_i \phi(x) + u_i)} \, d\phi(x). \]

Then, defining the **even** surface tension on \( \mathbb{T}^d_{N, ev} \)
\[
\sigma^\beta_{\mathbb{T}^d_{N, ev}}(u) = -\frac{1}{|\mathbb{T}^d_{N, ev}|} \log \frac{Z^\beta_{\mathbb{T}^d_{N, ev}}(u)}{Z^\beta_{\mathbb{T}^d_{N, ev}}(0)}, \quad \text{with } Z^\beta_{\mathbb{T}^d_{N, ev}}(0) = \int_{\mathbb{R}^d_{N, ev}} \exp(-\beta H^\text{ev}_{\mathbb{T}^d_{N, ev}}(\phi, u)) \prod_{x \in \mathbb{T}^d_{N, ev}} d\phi(x),
\]
we obtain the following result by integrating out the odds

**Lemma 40**
\[
\sigma^\beta_{\mathbb{T}^d_{N, ev}}(u) = \frac{1}{2} \sigma^\beta_{\mathbb{T}^d_{N}}(u).
\]

We will next prove strict convexity for the even surface tension, uniformly in \( N \) even.

**Theorem 41 (Strict convexity of the even surface tension)** Suppose that \( V, g \in C^2(\mathbb{R}) \) such that they satisfy (A0), (A1) and (A2). Then, for all \( N = 2k \), we have
\[
D^2 \sigma^\beta_{\mathbb{T}^d_{N, ev}}(u) = 2D^2 \sigma^\beta_{\mathbb{T}^d_{N, ev}}(u) \geq 4d \beta^2 c_1 \text{Id}, \quad \forall u \in \mathbb{R}^d,
\]
where \( c_1 \) is given in (41). That is, the even surface tension is uniformly strictly convex in \( u \in \mathbb{R}^d \), uniformly in all \( N \) even.

**Proof.** Since \( H^\text{ev} \) fulfills the random walk representation condition by Theorem 22, \( F_x \) are uniformly convex and we can apply Lemma 3.2 in [8] to \( \sigma^\beta_{\mathbb{T}^d_{N, ev}}(u) \), to get the statement of our theorem. \( \square \)

Note now that by the same reasoning as in [15], we can prove the existence of
\[
\sigma^\beta(u) = \lim_{|\mathbb{T}^d_{N}| \to \infty} \sigma^\beta_{\mathbb{T}^d_{N}}(u).
\]
Together with Theorem 41, this gives

**Theorem 42 (Strict convexity of the surface tension)** Suppose that \( V, g \in C^2(\mathbb{R}) \) such that they satisfy (A0), (A1) and (A2). Then the surface tension \( \sigma^\beta(u) \) is strictly convex in \( u \in \mathbb{R}^d \).

**9 Appendix**

Due to the fact that Example 4.2 (a) has been the subject of two other papers in the area (see [2] and [3]), we will provide here a sketch of the explicit computations for this example, which provide us with the \( p < O \left( \left( \frac{k_1}{k_2} \right)^{1/2} \right) \) order. The explicit computations are worth separate consideration, as they don’t follow from Theorem 22. As before, it is sufficient to estimate \( \text{Cov}_{\nu_z, \phi} (U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))) \).
Denote by \( \theta_k := \phi(x + e_k), k = 1, \ldots, 4 \). By standard Gaussian computations, we have

\[
\text{Cov}_{\nu, \psi}(U'(\nabla_i \phi(x)), U'(\nabla_j \phi(x))) = \sqrt{2\pi} \cdot \frac{1}{Z} \sum_{\alpha_k, \beta_k \in \{0, 1, \ldots, 4\}, \alpha_k + \beta_k = 1, \ldots, 4} \frac{1}{(k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k)^{3/2}} \cdot \exp \left( - \frac{1}{2} \left[ k_1 \sum_{k=1}^4 \alpha_k \theta_k^2 + k_2 \sum_{k=1}^4 \beta_k \theta_k^2 - \frac{(k_1 \sum_{k=1}^4 \alpha_k \theta_k + k_2 \sum_{k=1}^4 \beta_k \theta_k)^2}{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k} \right] \right),
\]

where

\[
Z = \sum_{\alpha_k, \beta_k \in \{0, 1, \ldots, 4\}, \alpha_k + \beta_k = 1, \ldots, 4} \frac{1}{(k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k)^{1/2}} \cdot \exp \left( - \frac{1}{2} \left[ k_1 \sum_{k=1}^4 \alpha_k \theta_k^2 + k_2 \sum_{k=1}^4 \beta_k \theta_k^2 - \frac{(k_1 \sum_{k=1}^4 \alpha_k \theta_k + k_2 \sum_{k=1}^4 \beta_k \theta_k)^2}{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k} \right] \right).
\]

By simple arithmetic, one can easily prove that

\[
\frac{1}{Z} \sum_{\alpha_k, \beta_k \in \{0, 1, \ldots, 4\}, \alpha_k + \beta_k = 1, \ldots, 4} \frac{1}{(k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k)^{3/2}} \cdot \exp \left( - \frac{1}{2} \left[ k_1 \sum_{k=1}^4 \alpha_k \theta_k^2 + k_2 \sum_{k=1}^4 \beta_k \theta_k^2 - \frac{(k_1 \sum_{k=1}^4 \alpha_k \theta_k + k_2 \sum_{k=1}^4 \beta_k \theta_k)^2}{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k} \right] \right) \geq \frac{k_2 \sqrt{2\pi}}{4}.
\]

Using (58), we have

\[
\text{Cov}_{\nu, \psi}(V'(\nabla_i \phi(x)), V'(\nabla_j \phi(x))) \geq \frac{k_2 \sqrt{2\pi}}{4},
\]

(59)

Note now that, by a reasoning similar to (57)

\[
\text{Cov}_{\nu, \psi}(g'(\nabla_j \phi(x)), V'(\nabla_i \phi(x))) \geq \text{Cov}_{\nu, \psi} \left( g'(\nabla_j \phi(x)), \sum_{k \in I} V'(\nabla_k \phi(x)) \right)
\]

\[
= \frac{1}{2} \mathbb{E}_{\nu, \psi} \left( g''(\nabla_j \phi(x)) \right) - \text{Cov}_{\nu, \psi} \left( g'(\nabla_j \phi(x)), \sum_{k \in I} g'(\nabla_k \phi(x)) \right),
\]

We next estimate \( \mathbb{E}_{\nu, \psi}(-g''(\nabla_j \phi(x))) \); by the change of variables

\[
\phi(x) = \frac{1}{\sqrt{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k}} \left[ t + k_1 \sum_{k=1}^4 \alpha_k \theta_k + k_2 \sum_{k=1}^4 \beta_k \theta_k + (k_1 - k_2) \theta_1 \right],
\]

(60)
in each of the ensuing Gaussian integrals, we obtain

\[
\mathbb{E}_{\nu_{x,\psi}} \left( -g''(\nabla_j \phi(x)) \right) \leq p(1-p) \sqrt{2\pi k_1 k_2} + \sum_{\alpha_k, \beta_k \in \{0,1\}, \alpha_k + \beta_k = 1, \ldots, 4} p(1-p)^{-3}(k_1 - k_2)^2 p^{\sum_{k=1}^4 \alpha_k} (1-p)^{\sum_{k=1}^4 \beta_k} (k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k + k_1 - k_2)^{1/2}
\]

\[
\left( \frac{k_1 \sum_{k=1}^4 \alpha_k \theta_k - \theta_1 + k_2 \sum_{k=1}^4 \alpha_k \theta_k - \theta_1}{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k + k_1 - k_2} \right)^2 e^{(k_1-k_2) \left( \frac{k_1 \sum_{k=1}^4 \alpha_k \theta_k - \theta_1 + k_2 \sum_{k=1}^4 \alpha_k \theta_k - \theta_1}{k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k + k_1 - k_2} \right)^2}
\]

\[
\leq p(1-p) \sqrt{2\pi k_1 k_2} + \sum_{\alpha_k, \beta_k \in \{0,1\}, \alpha_k + \beta_k = 1, \ldots, 4} p(1-p)^{-3}(k_1 - k_2) p^{\sum_{k=1}^4 \alpha_k} (1-p)^{\sum_{k=1}^4 \beta_k} (k_1 \sum_{k=1}^4 \alpha_k + k_2 \sum_{k=1}^4 \beta_k + k_1 - k_2)^{1/2}.
\]

By a similar reasoning, we get

\[
\text{Cov}_{\nu_{x,\psi}} \left( g'(\nabla_j \phi(x)), \sum_{k \in \mathcal{I}} g'((\nabla_k \phi(x)) \right) \leq \left( \frac{p}{1-p} \right)^2 k_1.
\]

Combining (60), (61) and (62), the conclusion follows.

**Acknowledgment**

We thank Elliot Lieb for suggesting to us the use of the even/odd representation and Christof Külkse for pointing out to us that we can explicitly compute the 1-step iteration in example 4.2 (a). We also thank Nicolas Petrelis and Rongfeng Sun for many useful comments and suggestions, which greatly improved the presentation of the manuscript.

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