A TALE OF TWO HECKE ALGEBRAS

GORDAN SAVIN

Abstract. We use Bernstein’s presentation of the Iwahori-Matsumoto Hecke algebra to obtain a simple proof of the Satake isomorphism and, in the same stroke, compute the center of the Iwahori-Matsumoto Hecke algebra.

1. Introduction

Let $G$ be a connected, split, reductive group over a non-archimedean local field $F$. Fix a maximal split torus $T$ in $G$. Then $T$ determines a root system $\Phi$. Let $W$ be the corresponding Weyl group. Let $K$ be a hyper-special maximal compact subgroup of $G$. More precisely, the torus $T$ preserves a unique apartment in the Bruhat-Tits building of $G$, and we pick $K$ to be the stabilizer of a hyper special vertex in the apartment. Then $T \cap K$ is a maximal compact subgroup of $T$, and the quotient $X = T/T_K$ is isomorphic to the co-character lattice of $T$. Let $H_K = C_c(K\backslash G/K)$ be the Hecke algebra of $K$-bi-invariant, compactly supported functions on $G$. Let $B = TN$ be a Borel subgroup containing $T$. Let $f \in C_c(G/K)$. Define $S(f)$, a function on $T/T_K$, by

$$S(f)(t) = \delta^{1/2}(t) \int_N f(tn) \, dn$$

where $\delta$ is the modular character. A famous theorem of Satake [Sa] states that the map $S$ is an isomorphism of $H_K$ and $\mathbb{C}[X]^W$.

Let $I \subset K$ be the Iwahori subgroup such that $I \cap B = K \cap B$. Let $H_I = C_c(I\backslash G/I)$ be the Hecke algebra of $I$-bi-invariant, compactly supported functions on $G$. Let $Z_I$ be the center of $H_I$. The space $C_c(I\backslash G/K)$ is naturally a left $H_I$-module and a right $H_K$-module. Using Bernstein’s description of $H_I$ we show, in Theorem 1, that the map $S$ gives an explicit isomorphism

$$S : C_c(I\backslash G/K) \to \mathbb{C}[X].$$

Then, as a simple consequence, we prove that the algebras $Z_I$, $H_K$ and $\mathbb{C}[X]^W$ are isomorphic.

2. Some preliminaries

The measure on $G$ is normalized so that the volume of $I$ is one. The space $C_c(G)$ of locally-constant, compactly supported functions is an algebra with respect to the convolution $*$ of functions. The unit of the algebra is $H_K$ is denoted by $1_K$. It is a function supported on $K$ such that $1_K(k) = \frac{1}{[K:F]}$ for all $k \in K$.

For every root $\alpha$ we fix a homomorphism $\varphi : SL_2(F) \to G$. The co-root $\alpha^\vee$ is an element of $X$ represented in $T$ by

$$\varphi_{\alpha} \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

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where \( v \in F \) has valuation 1. For every \( u \in F \), let
\[
x_\alpha(u) = \varphi_\alpha \begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}.
\]
We view the root \( \alpha \) as a homomorphism \( \alpha : X \to \mathbb{Z} \) such that, if \( x \in X \) and \( t_x \in T \) is a representative of \( x \), then
\[
t_x x_\alpha(u) t_x^{-1} = x_\alpha(vu)
\]
where the valuation of \( v \) is \( \alpha(x) \). We say that \( x \) is dominant if \( \alpha(x) \geq 0 \) for all positive roots \( \alpha \).

### 3. Iwahori Matsumoto Hecke algebra

Let \( q \) be the order of the residue field of \( F \). We summarize first some results of [IM].

The \( I \)-double co-sets in \( G \) are parameterized by \( \bar{W} = \mathbb{N}_G(T_K)/T_K \). This group is a semi-direct product of the lattice \( X \) and the Weyl group \( W \). The length function \( \ell : W \to \mathbb{Z} \) is defined by
\[
q^{\ell(w)} = [IwI : I].
\]
Let \( T_w \) denote the characteristic function of the double coset \( IwI \). Then \( T_w T_v = T_{wv} \) if and only if \( \ell(w) + \ell(v) = \ell(wv) \), and \( \ell(w) + \ell(v) = \ell(wv) \) if and only if \( IwIvI = IwvI \).

Let \( \rho \) be the sum of all positive roots. Then \( \ell(x) = \rho(x) \) for a dominant \( x \in X \). It follows that \( T_x \cdot T_y = T_{x+y} \) for any two dominant \( x \) and \( y \). Any \( x \in X \) can be written as \( x = y - z \) where \( y \) and \( z \) are two dominant elements in \( X \). Following Bernstein, let
\[
\theta_x = q^{(\ell(x) - \ell(y))/2} \cdot T_y T_x^{-1}.
\]

**Proposition 1.** Let \( x \in X \), and \( s \in W \) a reflection corresponding to a simple root \( \alpha \). Then
\[
T_s \theta_x - \theta_{s(x)} T_s = (1 - q) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha^\vee}}.
\]

Lusztig [Lu] derives this proposition from [IM]. It can be also verified by a direct calculation in \( \varphi_\alpha(\text{SL}_2(F)) \), see [S2].

**Corollary 1.** Let \( x \in X \), and \( s \in W \) a simple reflection, as in Proposition 1. Then
\[
T_s (\theta_x + \theta_{s(x)}) = (\theta_x + \theta_{s(x)}) T_s.
\]

**Proposition 2.** (Bernstein’s basis) Elements \( \theta_x T_w \), where \( x \in X \) and \( w \in W \), form a basis of \( H_I \).

**Proof.** Since \( T_w, w \in W \) and \( T_x \), with \( x \) dominant generate \( H_I \), Proposition 1 implies that \( \theta_x T_w \) span \( H_I \). Thus it remains to prove the linear independence. We follow an argument from [S1]. Assume that
\[
\sum_{i,j} c_{i,j} \theta_{x_i} T_{w_j} = 0.
\]
Let \( x_0 \in X \) be dominant such that \( x_0 + x_i \) is dominant for all \( x_i \) appearing in the sum. Then, after multiplying by \( \theta_{x_0} \) from the left,
\[
\sum_{i,j} c_{i,j} \theta_{x_0 + x_i} T_{w_j} = 0.
\]
However, if \( x \) is dominant then \( T_x \cdot T_w = T_{x+w} \). In particular, \( \theta_{x_0+x_i} T_{w_j} \) are linearly independent. Thus \( c_{i,j} = 0 \).  \( \square \)
Let $A$ be the sub algebra of $H_I$ generated by $\theta_x$. Then $A \cong \mathbb{C}[X]$ via the isomorphism $\theta_x \mapsto [x]$. (We shall write an element in the group algebra $\mathbb{C}[X]$ as $\sum_{x \in X} c_x [x]$, where $c_x \in \mathbb{C}$, in order to distinguish $[x - y]$ from $[x] - [y]$.)

**Proposition 3.** The centralizer of $A$ in $H_I$ is $A$.

**Proof.** Let $z \in H_I$. Express $z$ in the Bernstein’s basis, and let $\theta_x T_w$ be a term in the expression such that $\ell(w)$ is maximal. If $w = 1$, then $z \in A$. Otherwise, there exists $y \in X$ such that $w(y) \neq y$. Now notice that $\theta_y \cdot \theta_x T_w = \theta_{y+x}$, while

$$\theta_x T_w \cdot \theta_y = \theta_{x+w(y)} T_w + \sum_{z,v} c_{z,v} \theta_z T_v$$

where $\ell(v) < \ell(w)$. As $y - w(y)$ can be made arbitrarily large, $z$ does not commute with all elements in $A$. \hfill \Box

4. **Satake Map**

We fix the measure on $N$ so that the volume of $(N \cap K)$ is $[K : I]$. We identify $C_c(T/T_K)$ with $\mathbb{C}[X]$ by $f \mapsto \sum_{x \in X} f(x)[x]$. The Satake map $S : C_c(G/K) \to C_c(T/T_K) = \mathbb{C}[X]$ is defined by

$$S(f)(t) = \delta(t)^{1/2} \int_N f(tn) \, dn.$$  

It is a formal check (see [Ca]) that $S$, when restricted to $H_K = C_c(K \setminus G/K)$, is a homomorphism and the image of $H_K$ is contained in $\mathbb{C}[X]^W$.

**Proposition 4.** Let $1_K$ be the identity element of $H_K$. Then $\theta_x * 1_K$, $x \in X$, form a basis of $C_c(I \setminus G/K)$.

**Proof.** Note that $C_c(I/G/K) = C_c(I \setminus G/I) * 1_K$. Since $1_K = \frac{1}{[K:I]} \sum_{w \in W} T_w$, the proposition follows from Proposition 2. \hfill \Box

**Lemma 1.** Let $(\pi, V)$ be a smooth $G$-module and $(\pi', V')$ a smooth $B$-module with the trivial action of $N$. Let $S : V \to V'$ be a map such that $S(\pi(b)v) = \delta^{-1/2}(b) \pi'(b) S(v)$ for every $b \in B$. Then, for every $x \in X$ and $v \in V'$, $S(\pi(\theta_x)v) = \pi'(t_x)v$.

This lemma appears in the literature in a special case when $V' = V_N$, the normalized Jacquet functor. The proof is the same and therefore omitted.

**Theorem 1.** The map $S$ induces an isomorphism of left $A \cong \mathbb{C}[X]$-modules

$$C_c(I \setminus G/K) \rightarrow \mathbb{C}[X]$$

which sends the basis elements $\theta_x * 1_K$ to the basis elements $[x]$.

**Proof.** We apply Lemma 1 to $V = C_c(G/K)$, $V' = C_c(T/T_K)$ (considered as left $G$ and $T$-modules) and $S$ the Satake map. Then, for every $f \in C_c(I \setminus G/K)$, $S(\theta_x * f)(t) = S(f)(t_x^{-1} t)$. Thus $S(\theta_x * f) = [x] \cdot S(f)$. In particular, $S(\theta_x * 1_K) = [x] \cdot S(1_K) = [x]$, and the theorem follows. \hfill \Box

Let $Z_I$ be the center of $H_I$. Let $A^W$ be the span of $\sum_{w \in W} \theta_{w(x)}$ for $x \in X$. Corollary 1 implies that $A^W \subseteq Z_I$. Let $Z : Z_I \to H_K$ be a homomorphism defined by $Z(z) = z * 1_K$. 

**Theorem 2.** The maps $Z$ and $S$ induce isomorphisms of algebras

$$A^W \cong Z_I \cong H_K \cong \mathbb{C}[X]^W.$$ 

**Proof.** Theorem 1 implies that $S$, restricted to $H_K$, is injective. Proposition 3 implies that $Z_I \subseteq A$. This and Theorem 1 imply that the map $S \circ Z$ is injective. Thus, we have the injections

$$A^W \subseteq Z_I \subseteq H_K \subseteq \mathbb{C}[X]^W.$$ 

Since $(S \circ Z)(\sum_{w \in W} \theta_w(x)) = S(\sum_{w \in W} \theta_{w(x)} \ast 1_K) = \sum_{w \in W} [w(x)]$, the above injections are isomorphisms. □

**Final Remarks.** A proof of the isomorphism $Z_I \cong [X]^W$ can be found in [Da] and [HKP]. Both approaches are based on the explicit description of the Bernstein component of the category of smooth $G$-modules containing the trivial representation. Dat also shows that the map $Z$ gives an isomorphism of $Z_I$ and $H_K$. On the other hand, Lusztig [Lu] considers a version of the algebra $H_I$ over the ring $\mathbb{Z}[q^{\pm1/2}]$ where $q$ is considered a formal variable. He shows that the center is isomorphic to $\mathbb{Z}[q^{\pm1/2}][X]^W$ by specializing $q^{1/2} = 1$. No claim is made as to what the center is when $q$ is specialized to a power of a prime number.

**References**

[Ca] P. Cartier, *Representations of $p$-adic groups: a survey*. Automorphic Forms, Representations and $L$-functions, Proc. Symp. Pure Math., vol 33, part 1, AMS, Providence, RI, 1979, 111-155.

[Da] J.-F. Dat, *Caractères à valeurs dans le centre de Bernstein*. J. reine angew. Math. 508 (1999), 61-83.

[HKP] T. Haines, R. Kottwitz, A. Prasad, *Iwahori-Hecke algebras*. J. Ramanujan Math. Soc. 25, No 2 (2010), 113-145.

[Lu] G. Lusztig, *Singularities, character formulas, and a $q$-analog of weight multiplicities*. Astérisque 101-102 Soc. Math. France, Paris, 1983, 208-229.

[IM] Iwahori, H. Matsumoto, *On some Bruhat decompositions and the structure of the Hecke ring of a $p$-adic Chevalley group*. Publ. Math. IHES 25 (1965), 5-48.

[Sa] I. Satake, *Theory of spherical functions on reductive algebraic groups over $p$-adic fields*. Publ. Math. IHES 18 (1963), 1-69.

[S1] G. Savin, *Local Shimura correspondence*. Math. Ann. 280 (1988), 185-190.

[S2] G. Savin, *On unramified representations of covering groups*. J. reine angew. Math. 566 (2004), 111-134.

Department of Mathematics, University of Utah, Salt Lake City, UT 84112

E-mail address: savin@math.utah.edu