Boundary conditions for the spinor field in Rindler spacetime and the quantum field theoretical basis of the Unruh effect

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Abstract
We analyse the quantization procedure of the spinor field in the Rindler spacetime, showing the boundary conditions that should be imposed to the field, in order to have a well posed theory. Because of these boundary conditions we argue that this construction and the usual one in Minkowski spacetime are qualitatively different and cannot be compared and consequently the conventional interpretation of the Unruh effect, that is the thermal nature of the Minkowski vacuum state from the point of view of an accelerated observer, is questionable. We also analyse in detail the Unruh quantization scheme and we show that it is not valid in the whole Minkowski space but only in the double Rindler wedge, and it cannot be used as a basis for a quantum theoretical proof of the Unruh effect.

1 Introduction

The “Unruh effect” could be expressed by the following statements: 1) the Minkowski vacuum state, from the point of view of an accelerated observer, is a particle state described by a density matrix at the temperature

\[ T = \frac{a}{2 \pi k_B} \] (1)
called the Unruh-Davies temperature, where \( a \) is the (constant) acceleration of the observer and \( k_B \) is the Boltzmann constant; 2) an accelerated observer in the empty Minkowski space will detect a thermal bath of particles at the temperature \( T \).

As was stressed in \cite{1} \cite{2} \cite{3} \cite{4} (in particular \cite{4} is recommended for an especially clear account) the crucial point here is not that an accelerator detector (observer) would in someway react to the vacuum state of the field in Minkowski space, rather that this response is universal, i.e. independent from the structure of the detector itself, from the quantum field considered, and from the details of the interaction between them. This situation could remind the response of a probe massive body to the gravitational field, which is indeed universal. This universality is in this case determined by the pure geometrical nature of the gravitational field, so that the interaction between a body and the gravity is determined uniquely by the geometry of the spacetime in which the body moves, and not by its inner structure. If a similar universality would appear in the different context of accelerated observer in Minkowski space, this would mean that here only the quantum properties of Minkowski vacuum state matter.

Consequently, two problems are involved here, which in principle are different, but are claimed to be equivalent in the literature: the physical properties of a quantum field when restricted to a submanifold (the Rindler Space, (RS)) of Minkowski space (MS); the behaviour of a constantly accelerated particle detector in empty flat space.

The first one deals only with basic principles of quantum theory and appears to be more fundamental, whether the second one should in general involve also a description of structure and characteristics of the detector, and details of interaction with the quantum field. Again, it is the analysis of the first problem that made possible to claim for universality of the Unruh-like detector response.

We will treat here only the first problem above, which is a particular example of the analysis of the behaviour of a field in a submanifold of a maximally analytically extended manifold.

The procedure used by Unruh is based on a quantization scheme for a free field in MS, alternative but claimed to be equivalent to the standard one, which uses as the Hilbert space of solutions of the wave equation

\[
\mathcal{H}_U = \mathcal{H}_R \oplus \mathcal{H}_L
\]

where \( \mathcal{H}_R \) consists of solutions which are non-zero everywhere but in the L sector, which have positive frequency with respect to the “Rindler time” \( \eta \), and which reduce themselves to the well known Fulling modes \cite{5}, and \( \mathcal{H}_L \) is given by the solutions which are non-zero everywhere but in R and with negative frequency with respect to \( \eta \). Then it is obtained a representation of the Minkowski vacuum state as a state in \( \mathcal{F}_s(\mathcal{H}_U) \), i.e. the Fock space constructed on \( \mathcal{H}_U \). Finally it is derived the particle content of Minkowski vacuum and the expression for the density matrix associated to it, when expressed as a mixed state in the Fock space \( \mathcal{F}_s(\mathcal{H}_R) \) having traced out the degrees of freedom related to the L sector, which is unaccessible to the Rindler observer.
The usual explanation of the Unruh effect is based exactly on the presence, for a Rindler observer, which is confined inside the Rindler wedge, of an event horizon which prevent him from having part of the informations about the quantum field, so that he sees Minkowski vacuum state as a mixed state. But this explanation is (as was indicated in [1] [2] [3] [4] and also in our opinion) not entirely satisfying, for several reasons: on the one hand, the existence of horizons is due to overidealization of the problem, since for physical accelerations (which last finite amount of time) no horizon should be present, so that the response of an accelerated detector, if of the Unruh-Davies type, cannot be caused by it; on the other hand, in the purely quantum theoretical treatment of the problem, we would expect that the presence of event horizons affects deeply the fields, from the point of view of Rindler observer, in the form of some kind of boundary condition, which instead are totally absent in the Unruh scheme, and in the usual quantization in RS.

What we are going to do in this paper is to extend to the spinor field the results obtained in [1] [2] [3] [4] for the scalar field and to show the conditions to (and only to) which the quantum theory of the spinor field in the Rindler spacetime is well posed, to study the relationship between this construction and the usual one in MS, in order to find which role is played by these conditions in the derivation and interpretation of the Unruh effect in the spinor case, to analyse the Unruh quantization scheme and to understand finally what is its physical significance.

What we will find is that a correct quantization procedure for the spinor field in Rindler space requires the boundary condition

$$\lim_{\rho \to 0} \rho \Psi (0, \rho) = 0$$

i.e. the field should not grow up too rapidly at the origin of Rindler (and Minkowski) space. From a more general point of view, this means that the quantization on a background manifold which is not maximally extended requires a boundary condition which is absent in the quantization procedure over the extended manifold; in the literature this is not recognized clearly enough, and the usual procedure is to restrict the fields just considering a smaller domain of definition.

Moreover, we show that also for the spinor field the Unruh quantization scheme is not valid in the whole Minkowski space, but only in a sector of it, namely the double wedge $R \cup L$. Consequently, the Unruh quantization implies the same kind of boundary condition, and cannot be used as a proof of the Unruh effect.

These results, as we said, represent a generalization to the spinor field of similar ones obtained for the scalar field (and also recently extended also to the electromagnetic field [5]); consequently, they appear to be consequences of very general properties of Quantum Field Theory, and seem to be firmly established. It is worth to note here that the Unruh effect is often identified (mainly by mathematical physicists) with the so-called Bisognano-Wichmann Theorem, in the context of the algebraic approach to quantum field theory. We won’t deal
in this paper with the algebraic approach, but the interested reader could found the analog of our result for the scalar case extended to the algebraic framework in [3]. There it is shown that the physical interpretation of the Bisognano-Wichmann Theorem, in terms of accelerated observers in MS occurs in the same kind of problems encountered in the conventional approach, that we will now discuss.

In addition to the main results cited, we obtained some minor but original results, necessary to achieve the first, namely: the explicit expression of the Lorentz boost generator (or Lorentz Momentum) for the spinor field, its eigenfunctions and their analytical representation holding in the whole MS (except for the origin), which is in turn a generalization of the Gerlach’s Minkowski Bessel Modes [1].

2 Rindler Spacetime

Let’s consider a particle moving in MS with constant acceleration $a$ along the $x$-axis; it will follow the trajectory given by (parameter $\tau$):

$$
\begin{align*}
t & = a^{-1} \sinh(a\tau) \\
x & = a^{-1} \cosh(a\tau) \\
y & = y(0) \\
z & = z(0)
\end{align*}
$$

This is an hyperbola in the $(t,x)$ plane. The lines $t = \pm x$ represent asymptotes for it and event horizons for the moving particle. Varying $a$ we obtain different hyperbolas with the same characteristics. Let’s now perform, starting from the Minkowski metric, the change in coordinates given by:

$$
\begin{align*}
t & = \rho \sinh \eta \\
x & = \rho \cosh \eta \\
\rho & = \sqrt{x^2 - t^2} \\
\eta & = \arctgh \left( \frac{x}{\rho} \right)
\end{align*}
$$

with the other coordinates left unchanged. The metric assumes the form:

$$
ds^2 = \rho^2 d\eta^2 - d\rho^2 - dy^2 - dz^2
$$

Note that it describes a stationary spacetime. The worldlines $\rho = \text{const}, y = \text{const}, z = \text{const}$ correspond to uniformly accelerated observers, with $a = \rho^{-1}$ and proper time $\tau = \rho \eta$, as it can be seen comparing 4 and 5. We can think at the Rindler Space as the collection of these worldlines, and this is the reason why RS is generally regarded as the “natural” manifold in which to describe accelerated motion. The hypersurfaces $\eta = \text{const}$ describe events which are simultaneous from the point of view of a “Rindler (uniformly accelerated) observer”. The Rindler manifold cannot be extended to negative values of $\rho$ through $\rho = 0$. This hypersurface, in fact, represents, as we said, an event horizon for Rindler observers, which cannot see any event located beyond it. Nevertheless the horizon is a regular surface, of course, and the metric singularity is due only to the choice of the coordinates.

The (5) cover only a sector of the whole MS and the others are covered by the following charts:
\( L : \)
\[
    t = \rho \sinh \eta \quad x = \rho \cosh \eta
\]
\( \rho = -\sqrt{x^2 - t^2} \quad \eta = \text{arctgh}(\frac{t}{x}) \tag{8} \)

\( F : \)
\[
    t = \rho \cosh \eta \quad x = \rho \sinh \eta
\]
\( \rho = \sqrt{t^2 - x^2} \quad \eta = \text{arctgh}(\frac{x}{t}) \tag{10} \)

\( P : \)
\[
    t = \rho \cosh \eta \quad x = \rho \sinh \eta
\]
\( \rho = -\sqrt{t^2 - x^2} \quad \eta = \text{arctgh}(\frac{x}{t}) \tag{13} \)

3 Quantization in Rindler space

We now turn to the problem of quantization of the spinor field in the Rindler wedge. The procedure will be the standard one, so we will first solve the Dirac equation looking for hamiltonian eigenfunctions.

3.1 The Dirac equation and its solutions in RS

Using the tetrad formalism, the Dirac equation in a generic curved spacetime is given by:
\[
    (i \gamma^\mu \nabla_\mu - m) \Psi = 0 \tag{14}
\]
where: \( \gamma^\mu = \theta^\mu_{\bar{\mu}} \bar{\gamma}^\bar{\mu} \) are the analogous in curved spaces of the usual Dirac gamma matrices \( \bar{\gamma}^\bar{\mu} \), and they satisfy:
\[
    \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = g^{\mu\nu} \tag{15}
\]
la \( \theta^\mu_{\bar{\mu}} \) is the inverse of the tetrad vector \( \theta^\mu_{\bar{\mu}} \) with vectorial index \( \mu \) and tetradic index \( \bar{\mu} \); \( \nabla_\mu = \partial_\mu - \Gamma_\mu \) is the spinorial covariant derivative, defined in such a way that \( \nabla_\mu \Psi \) is a covariant vector;
\[
    \Gamma_\mu = \frac{1}{8} \theta^\bar{\mu}_{\bar{b}} \theta_{\bar{b} \bar{c} \mu} [\gamma^{\bar{c}}, \gamma^{\bar{b}}] \tag{16}
\]
is the spinorial connection.

Let’s consider the particular case of the Rindler metric:
\[
    ds^2 = \rho^2 d\eta^2 - d\rho^2 - dy^2 - dz^2 \tag{17}
\]
The Dirac equation assumes the form:

\[
\left( i \partial_\eta + i \rho \gamma^0 \gamma^1 \partial_\rho + i \rho \gamma^0 \gamma^2 \partial_y + i \rho \gamma^0 \gamma^3 \partial_z + \frac{i}{2} \gamma^0 \gamma^1 - m \rho \gamma^0 \right) \Psi = 0 \tag{18}
\]

\[
\implies i \partial_\eta \Psi = \left( -i \rho \alpha_i \partial_i - \frac{1}{2} i \alpha_1 + m \rho \beta \right) \Psi \tag{19}
\]

that is a Shroedinger-like form with an hamiltonian given by:

\[
H_R = -i \rho \alpha_i \partial_i - \frac{1}{2} i \alpha_1 + m \rho \beta = \tag{20}
\]

\[
= -i \rho \gamma^0 \gamma^1 \partial_\rho - i \rho \gamma^0 \gamma^2 \partial_y - i \rho \gamma^0 \gamma^3 \partial_z - \frac{1}{2} i \gamma^0 \gamma^1 + m \rho \gamma^0 \tag{21}
\]

We now look for solutions of the Dirac equation which are simultaneously eigenfunctions of the Rindler hamiltonian (21) and of the operators \( P_y \) and \( P_z \) (of course they should also have the correct behaviour at infinity). So we expect to find a degeneracy of these solutions, because we know that three operators are not sufficient to completely characterize the states of the spinor field. This degeneracy is however of no relevance for our purposes, so we will not deal with it.

The solutions are:

\[
\Psi_{1,M}^R = N_M \left( X_1^R K_{i,M-\frac{1}{2}}(\kappa \rho) + Y_1^R K_{i,M+\frac{1}{2}}(\kappa \rho) \right) e^{-i M \eta} e^{ik_2 y + ik_3 z} \tag{22}
\]

with:

\[
X_1^R = \begin{pmatrix} k_3 & \frac{k_3}{i (k_2 + i m)} & \frac{k_3}{i (k_2 + i m)} \\ \frac{k_3}{i (k_2 + i m)} & 0 & \frac{k_3}{i (k_2 + i m)} \end{pmatrix} \quad Y_1^R = \begin{pmatrix} 0 \\ i \kappa \\ -i \kappa \\ 0 \end{pmatrix} \tag{23}
\]

and

\[
\Psi_{2,M}^R = N_{M}' \left( X_2^R K_{i,M-\frac{1}{2}}(\kappa \rho) + Y_2^R K_{i,M+\frac{1}{2}}(\kappa \rho) \right) e^{-i M \eta} e^{ik_2 y + ik_3 z} \tag{24}
\]

with:

\[
X_2^R = \begin{pmatrix} 0 & \frac{k_3}{i (k_2 - i m)} & \frac{k_3}{i (k_2 - i m)} \\ i \kappa & 0 & \frac{k_3}{i (k_2 - i m)} \\ i \kappa & -i (k_2 - i m) & -k_3 \end{pmatrix} \tag{25}
\]

where \( k_2, k_3 \) are eigenvalues of \( P_2, P_3, \kappa = \sqrt{k_2 + k_3 + m^2}, M \) is the eigenvalue of the hamiltonian, and \( N_M, N_{M}' \) are normalization factors which are found to be:

\[
N_M = N_{M}' = \frac{1}{4 \pi^2 \sqrt{\kappa \cosh \pi M}} \tag{26}
\]
3.2 Physical characterization of the solutions

We now want to understand better the physical nature of the solutions (22)(24), and this means to characterize them in a clearer way than just saying they are eigenfunctions of the Rindler hamiltonian.

For this purpose we need to find the expressions for the solutions in minkowski coordinates.

In order to do it, it is necessary to consider carefully the way in which spinors transform under coordinate transformations.

It is well known that also in curved spacetimes spinors are characterized by their transformation properties under the action of the Lorentz group, but restricting the attention to the local minkowskian neighbour of the point in which the spinors have to be calculated, i.e., considering local Lorentz transformations on the tangent space of each point. Consequently, under a general coordinate transformation, the spinor will undergo a Lorentz transformation, but with a “velocity parameter” which will be a function of the coordinates; this local Lorentz transformation has to be determined linearizing the coordinate transformation in which we are interested. In our case (transformation between Minkowskian and Rindler coordinate systems) the result is that the spinor transformation is given by:

$$\Psi(t, x) = S(\eta) \Psi(\eta, \rho) = \exp \left( \frac{1}{2} \gamma^0 \gamma^1 \eta \right) \Psi(\eta, \rho)$$

Note that the operator we found, $S = \exp \left( \frac{1}{2} \gamma^0 \gamma^1 \eta \right)$, has the form of an operator resulting from a Lorentz coordinate transformation, with “velocity” parameter $\eta$.

The solutions (22)(24) in minkowskian coordinates take the form:

$$\Psi_{t,M}(t, x, y, z) = N_M \left( X_i^R K_{i,M-\frac{1}{2}}(\kappa \rho) e^{-(iM-\frac{1}{2})\eta} + Y_i^R K_{i,M+\frac{1}{2}}(\kappa \rho) e^{-(iM+\frac{1}{2})\eta} \right) e^{ik_2 y + ik_3 z}$$

It must be clear that these are again defined only in the RS, but expressed in minkowskian coordinates, and this is why the normalization factor is still $N$.

Now we can turn to the anticipated physical characterization of the solutions: they are found to be eigenfunctions of the Boost Generator Operator, or Lorentz Momentum. This could be aspected, because 1) $\Psi_{t,M}$ were eigenfunctions of the Rindler Hamiltonian $H_R$ and 2) $H_R$ is precisely the Lorentz Boost Generator written in Rindler coordinates, since (3) the time evolution of a Rindler (uniformly accelerated) observer is properly a infinite succession of infinitesimal boost transformations.

It is easy to verify this statement, once known the Lorentz Momentum operator, and this in turn can be obtained from the classical theory of fields.

In fact, given the conserved quantity:

$$M^{0i} = \int d^3x \left[ (x^0 T^{0i} - x^i T^{00}) + S^{0i} \right]$$

7
corresponding to the invariance of the Lagrangian under boost transformations along the \(i\)-axis, (which are isometries of the Rindler spacetime), the explicit calculation of \(T^{\mu \nu}\) and of \(S^{\mu \nu}\) gives:

\[
M^{0i} = \int d^3x \Psi^\dagger \left[ i t \partial^i + x^i \left( i \gamma^0 \gamma^j \partial_j - m \gamma^0 \right) + \frac{i}{2} \gamma^0 \gamma^i \right] \Psi
\]  

(30)

Interpreting it as a mean value of a quantum operator, we obtain for the Lorentz Momentum the expression (for contravariant and covariant components):

\[
M^{0i} = -i t \partial_i - x_i \left( -i \gamma^0 \gamma^j \partial_j + m \gamma^0 \right) + \frac{i}{2} \gamma^0 \gamma^i
\]  

(31)

\[
M_{0i} = +i t \partial_i + x_i \left( -i \gamma^0 \gamma^j \partial_j + m \gamma^0 \right) - \frac{i}{2} \gamma^0 \gamma^i
\]  

(32)

Given this expression it can be verified that our solutions are eigenfunctions of the operator \(M_{0i}\) with eigenvalue \(\mathcal{M}\), and this represent their physical characterization.

### 3.3 The second quantization in RS

We now possess all the necessary elements to perform the second quantization of the spinor field in RS, i.e. a set of normalized functions which are solutions of the equation of motion. We then expand the field in terms of them:

\[
\Psi^R(\eta, \rho, y, z) = \sum_{i=1,2} \int_{-\infty}^{+\infty} d\mathcal{M} \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3 a_{\mathcal{M}i}(k_2, k_3) \Psi_{i,\mathcal{M},k_2,k_3}(t, x, y, z) =
\]

\[
= \sum_{i=1,2} \int_{0}^{+\infty} d\mathcal{M} \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3 \times
\]

\[
\times \left( a_{\mathcal{M}i}(k_2, k_3) \Psi^R_{i,\mathcal{M},k_2,k_3}(\eta, \rho, y, z) + b^\dagger_{\mathcal{M}i}(k_2, k_3) \Psi^R_{i,-\mathcal{M},k_2,k_3}(\eta, \rho, y, z) \right)
\]  

(33)

The second quantization is now performed considering the coefficients \(a_{i,\mathcal{M}}\) and \(b_{i,\mathcal{M}}\) (and their hermitian conjugated) as operators, and requiring for their anticommutators:

\[
\{ a_{\mathcal{M},i}(k_2, k_3), a^\dagger_{\mathcal{M}',j}(k'_2, k'_3) \} = \delta_{ij} \delta(\mathcal{M} - \mathcal{M}') \delta(k_2 - k'_2) \delta(k_3 - k'_3)
\]  

(34)

\[
\{ b_{\mathcal{M},i}(k_2, k_3), b^\dagger_{\mathcal{M}',j}(k'_2, k'_3) \} = \delta_{ij} \delta(\mathcal{M} - \mathcal{M}') \delta(k_2 - k'_2) \delta(k_3 - k'_3)
\]  

(35)

\[
\{ a_{\mathcal{M},i}(k_2, k_3), b^\dagger_{\mathcal{M}',j}(k'_2, k'_3) \} = 0 \quad \forall i, j, \mathcal{M}, \mathcal{M}', k_2, k'_2, k_3, k'_3
\]  

(36)

\[
\{ a_{\mathcal{M},i}(k_2, k_3), b_{\mathcal{M}',j}(k'_2, k'_3) \} = 0 \quad \forall i, j, \mathcal{M}, \mathcal{M}', k_2, k'_2, k_3, k'_3
\]  

(37)

and the quantum states of the field are constructed from the Rindler vacuum state \(|0\rangle_R\), defined by:

\[
a_{\mathcal{M},i}(k_2, k_3) |0\rangle_R = 0 \quad \forall \ i, \mathcal{M}, k_2, k_3
\]  

(38)

Now we will study if this quantum construction is well posed and at which conditions.
3.4 Conditions for the quantization in RS

Let’s consider again the Rindler hamiltonian:

\[ H_R = -i \rho \alpha_1 \partial_\rho - \frac{1}{2} i \alpha_1 + m \rho \beta \]  \hspace{1cm} (39)

\[ = -i \rho \gamma^0 \gamma_1 \partial_\rho - i \rho \gamma^0 \gamma^2 \partial_y - i \rho \gamma^0 \gamma^3 \partial_z - \frac{1}{2} i \gamma^0 \gamma^1 + m \rho \gamma^0 \]  \hspace{1cm} (40)

and let’s check whether it represents an hermitian operator, that is a necessary and sufficient condition for the completeness and orthonormality of the modes used. We should verify the condition

\[ (H_R \Phi, \Psi) = (\Phi, H_R \Psi) \]  \hspace{1cm} (41)

with scalar product given by:

\[ (\Phi, \Psi) = \int d\Sigma_\mu \bar{\Phi} \gamma^\mu \Psi = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \int_{0}^{+\infty} d\rho \Phi^\dagger \Psi \]  \hspace{1cm} (42)

The explicit calculation shows that:

\[ (H_R \Phi, \Psi) = (\Phi, H_R \Psi) + \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \left[ i \Phi^\dagger \alpha_1 \rho \Psi \right]_{\rho = 0}^{\rho = +\infty} \]  \hspace{1cm} (43)

Then it is evident that the hermiticity of the hamiltonian is assured if and only if

\[ \lim_{\rho \to 0} \rho^{\frac{1}{2}} \Psi (\eta, \rho, y, z) = 0 \hspace{1cm} \forall \eta \]  \hspace{1cm} (44)

and of course with analogous condition at \( \rho \to +\infty \), i.e. the usual requirement of vanishing of the fields at spatial infinity.

We emphasize that, since the field \( \Psi \) is to be considered as an operator-valued distribution, this condition should be interpreted in the weak sense, this meaning that every matrix element of the quantity \( \rho^{\frac{1}{2}} \Psi \) calculated with respect to any pair of physical states has to go to zero as \( \rho \to 0 \).

We stress again that a similar condition was already found for the scalar field in RS \( [1] [2] [3] \), and for the vector field \( [9] \).

If the condition (44) is necessary for the hermiticity of the hamiltonian, it is expected to appear also in the analysis of the coefficients of the expansion (33). So we write the explicit expression of the coefficients \( a_{i,\mathcal{M}}(k_2, k_3) \):

\[ a_{i,\mathcal{M}}(k_2, k_3) = \left( \Psi_{i,\mathcal{M},k_2,k_3}, \Psi \right)_R = \]  \hspace{1cm} (45)

\[ = + \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \int_{0}^{+\infty} d\rho N^*_\mathcal{M} \left[ \left( X_i^\dagger K_{i,\mathcal{M}+\frac{1}{2}} + Y_i^\dagger K_{i,\mathcal{M}-\frac{1}{2}} \right) \Psi \right] \times \]  \hspace{1cm} (46)

\[ \times e^{(i,\mathcal{M} \eta + i k_2 y + i k_3 z)} \]

and consider the behaviour of the \( K_{i,\mathcal{M} \pm \frac{1}{2}}(\kappa \rho) \) for \( \rho \approx 0 \).
We have:

\[ K_{iM\pm\frac{1}{2}}(\kappa \rho) \simeq \frac{\pi}{2} \sin\left(\frac{\pi}{2} (iM \pm \frac{1}{2})\right) \times \]

\[ \times \left[ \frac{\kappa^{-iM+\frac{1}{2}}}{\Gamma\left(-\frac{1}{2} + iM + \frac{1}{2}\right)} \frac{\rho^{iM+\frac{1}{2}}}{\Gamma\left(-\frac{1}{2} + iM + \frac{1}{2}\right)} - \frac{\kappa^{iM+\frac{1}{2}}}{\Gamma\left(-\frac{1}{2} + iM + \frac{1}{2}\right)} \frac{\rho^{-iM-\frac{1}{2}}}{\Gamma\left(-\frac{1}{2} + iM + \frac{1}{2}\right)} \right] \]

(47)

We see that a divergence is present for \( \rho \to 0 \) (we note also that, for \( M = 0 \), we have the exact expression:

\[ K_{\frac{1}{2}}(\kappa \rho) = \frac{\pi}{2} \kappa \rho e^{-\kappa \rho} \]

(48)

again with the same type of divergence).

It is so evident that, in order the integral (46) to converge, is necessary to require the boundary condition (44) on the field. Otherwise, the coefficients \( a_{i,M}(k_2, k_3) \) are not defined and consequently the fundamental operators of quantum field theory like energy or particle number, which are built in terms of the annihilation and creation operators, are similarly not defined.

We want now to prove the necessity of the condition (44) in an even more apparent way, i.e. showing that the requirement of finiteness of the mean value of the energy in a generic state implies indeed (44). We work for simplicity in the plane \((\eta, \rho)\) Consider the state of the field \( |g\rangle \) given by

\[ |g\rangle = c^\dagger(g) |0\rangle_R = \sum_i \int_0^\infty \frac{dM}{M^2} g(M) c^\dagger_i,M |0\rangle_R \]

(49)

i.e. a generic sovrapposition of eigenstates of the hamiltonian with weight function \( g(M) \), and consider also the related one-particle amplitude for the field in RS given by

\[ \Psi^R_g = R(0 | \Psi_R | g) = e^{-iH_R \eta} \phi_g \]

(50)

where

\[ \phi_g = \sum_i \int_0^\infty \frac{dM}{M^2} g(M) \sqrt{\frac{\pi}{k}} \cosh \pi M \left( X^R_i K_{iM-\frac{1}{2}} + Y^R_{iM+\frac{1}{2}} \right) \]

(51)

Consequently we have the following translation of physical requirements (normalization of the states and finiteness of the mean value of the energy) into mathematical requirements on the weight function \( g(M) \):

\[ \langle g | g \rangle = \int_0^\infty \frac{dM}{M} | g(M) |^2 = 1 \]

(52)

\[ \langle g | H | g \rangle = \]

\[ = \int_0^\infty d\rho \phi^\dagger g H \phi_g = \int_0^\infty d\rho \phi^\dagger g \left\{ -i\alpha_1 \partial_\rho - \frac{i}{2} \alpha_1 + m \rho \beta \right\} \phi_g = \]

(53)

\[ = 2 \int_0^\infty dM | g(M) |^2 < \infty \]

(55)
From the equation (54) it is not immediately manifest what should be the behaviour of the functions $\phi_g$ with respect to $\rho$, and in order to understand it we have to analyse directly the expression (51), but taking into proper account the physical requirements (52) and (55).

Let’s consider the quantity $\rho^{1/2} \phi_g$ for $\rho \to 0$, using the expression 51 and the formula 47 for the modified Bessel function for $\rho \to 0$. Simple manipulations lead to

$$\rho^{1/2} \phi_g \simeq_{\rho \to 0} \sum_i \int_0^\infty \frac{dM}{M^{1/2}} N_M g(M) \frac{\pi}{\sqrt{2\kappa}} \times$$

$$\times \left\{-X_i^R \frac{1}{\sin(\pi(iM - 1/2))} \Gamma \left( \frac{1}{2} + iM \right) \left( \frac{\kappa \rho}{2} \right)^{iM} +
+ Y_i^R \frac{1}{\sin(\pi(iM + 1/2))} \Gamma \left( \frac{1}{2} - iM \right) \left( \frac{\kappa \rho}{2} \right)^{-iM} \right\} = \sum_i \left\{-X_i^R G_A(\rho, \kappa) + Y_i^R G_B(\rho, \kappa) \right\}$$

(57)

Now we want to study in details these quantities $G_A$ and $G_B$; we will do an explicit calculation only for the first one, since the argument for the second one is analogous.

First let’s write $G_A$ as:

$$G_A = \int_0^\infty \frac{dM}{M^{1/2}} N_M g(M) \frac{\pi}{\sqrt{2\kappa}} \sin(\pi(iM - 1/2)) \Gamma \left( \frac{1}{2} + iM \right) \left( \frac{\kappa \rho}{2} \right)^{iM} = G_{A1} + G_{A2} + G_{A3}$$

(58)

where we have just splitted the integration domain into three parts, i.e. $(0, \infty) = (0, M_1) \cup (M_1, M_2) \cup (M_2, \infty)$, with $M_1 << 1$ and $M_2 >> 1$.

Consider the term $G_{A3}$. Using the explicit form of the normalization factor $N_M$, the asymptotic expression for the Gamma function (for $M \to \infty$), and basic formulas for hyperbolic functions, we obtain:

$$| G_{A3} |^2 \leq_{M_2 \to \infty} \frac{1}{32\pi^3 \kappa^2 M_2^3} \int_{M_2}^\infty dM \, |g(M)|^2$$

(59)

now, given the finiteness condition for the mean value of the energy (53), we have that $G_{A3}$ could be made as small as we want with sensible choices of $M_2 \to \infty$.

Consider the term $G_{A2}$. We can easily obtain the inequality:

$$| G_{A2} | \leq \int_{M_1}^{M_2} dM \, C(M) \, |g(M)|$$

(60)

where $C(M)$ is a non singular function in the interval of integration. Taking into account the inequality $|g(M)| \leq \frac{1}{2}(1 + |g(M)|^2)$, and the normalization
condition (52), it is easy to see that the integral above should converge. Now applying the Riemann-Lebesgue Lemma, we conclude that \( G_{A2} \) vanishes for \( \rho \to 0 \).

Coming to the term \( G_{A1} \), we have:

\[
G_{A1} = \int_{0}^{M_1} dM \frac{1}{\sqrt{\cosh \pi M g(M)}} \frac{1}{\sin(\pi(iM - 1/2))\Gamma(iM + 1/2)} \times \\
\left( \frac{\kappa \rho}{2} \right)^{iM} \lesssim -\frac{1}{2\pi \kappa} \int_{0}^{\infty} \frac{dM}{M^{1/2}} g(M) \left( \frac{\kappa \rho}{2} \right)^{iM} (61)
\]

being \( M_1 << 1 \). Let’s restrict the calculation to the case in which \( g(M) \) vanish with \( M \to 0 \) as a suitable high power of \( M \), i.e.

\[
g(M) \simeq a M^\alpha \quad \alpha \geq \frac{1}{2} \quad M \to 0 \quad (62)
\]

(the results could be generalised to other cases). Note that the vanishing of the weight function is required also by the condition (52).

We obtain the inequality:

\[
G_{A1} \leq \frac{a}{\frac{2\pi \kappa}{\ln \frac{\kappa \rho}{2}}} \quad (63)
\]

so that \( G_{A1} \to 0 \) as \( \rho \to 0 \).

Consequently we have proved that for the generic physical state (49) to be normalised and to have finite energy, the boundary condition (44) is necessary. In other words, since (52) and (55) imply \( \rho^2 \phi_\alpha \to 0 \) for \( \rho \to 0 \), we know that, if this condition is not satisfied, then (52) or (55) doesn’t hold, and consequently the state \( |g\rangle \) is not a physical state.

3.5 Discussion

We have found that the quantum theory of the spinor field in Rindler space is well defined if and only if we have the condition (44). This condition means that the field has to be quantized in a different way in MS and RS, because the horizons are not only of a causal but also of a physical significance for it. It also means that the usual procedure of quantize the field in RS, which just restrict its domain of definition, is not correct, because this kind of restriction is not enough to have a well posed theory. If one would like to study the spinor field in RS and work with physical states and modes, the two possible ways are to construct suitable wave packets made as combinations of the modes \( \Psi_{i,M} \), or to consider from the beginning a constrained hamiltonian, different from \( H_R \) which automatically assures that the condition (44) is satisfied. The crucial point is however that this condition prevents any relationship between the quantization in RS and that in MS, and it means that RS should be treated as an manifold on its own, and not as a submanifold of MS (the Rindler wedge), i.e. the two
quantizations define two different physical systems. This is because in the latter case we would not be free to choose any boundary condition for the field at the origin, because the state of the field would be determined from the beginning as a state in $\mathcal{H}_M$ (the Minkowski vacuum state $|0\rangle_M$). Consequently, it is not possible to describe any state of the spinor field quantized in Minkowski space, and then restricted to the Rindler wedge, as a state in $\mathcal{H}_R$, and we could expect that the necessity of boundary conditions on the field would manifest itself also in the analysis of the Unruh effect, which concerns exactly this relationship between RS and MS quantum constructions. To show this will be our next problem.

4 Analysis of the Unruh effect

In order to analyse this relationship, it is convenient to perform a quantization in Minkowski spacetime which is different from the standard one (that in plane waves) and which could be more easily compared with the Rindler one; namely, a quantization in terms of the Lorentz Momentum eigenfunctions defined in the whole Minkowski space, that represents analytical continuation of the $\Psi_{i,M}$ we found before, spinorial analogous of the Fulling modes for the scalar field, and that reduce themselves to these when restricted to the Rindler wedge. A physical motivation for this choice could be seen in the fact that the trajectories of a Rindler observer are the orbits of the Lorentz group, and that the R sector of MS is left invariant by the action of this group. The determination of these functions will require some preliminary steps.

4.1 Solutions in the other sectors and their relationship

First of all we will study the relations between the solutions of Dirac equation in the different sectors F, L, P (see \[\textcolor{red}{[3]}\]). Of course, we omit the passages and just write down the form of Dirac equation in the sector and its possible solutions. That is the following.

F sector:

\[
\left( i \gamma^0 \partial_\rho + \frac{i}{\rho} \gamma^1 \partial_\eta + \frac{i}{2} \frac{1}{\rho} \gamma^0 + i \gamma^2 \partial_\eta + i \gamma^3 \partial_\zeta - m \right) \Psi = 0 \tag{64}
\]

\[
\Psi_{i,M}^\pm = M_i \left( X_i^F K_{i,M-\frac{1}{2}}(\pm i \kappa \rho) + Y_i^F K_{i,M+\frac{1}{2}}(\pm i \kappa \rho) \right) e^{-iM \eta} e^{ik_2 \rho + ik_3 \zeta} \tag{65}
\]

with $i = 1, 2$ and

\[
X_1^F = \begin{pmatrix} k_3 \\ i (k_2 + i m) \\ i (k_2 + i m) \\ k_3 \end{pmatrix} \quad \quad Y_1^F = \begin{pmatrix} 0 \\ \mp i \kappa \\ \pm i \kappa \\ 0 \end{pmatrix} \tag{66}
\]
the passages through the horizons is given by the substitutions:

\[
X_2^F = \begin{pmatrix}
0 \\
\pm \kappa \\
\mp \kappa \\
0
\end{pmatrix} \quad Y_2^F = \begin{pmatrix}
k_3 \\
i (k_2 - i m) \\
-i (k_2 - i m) \\
-k_3
\end{pmatrix}
\]

(L sector):

\[
\left(\frac{1}{\rho} \gamma^0 \partial_\rho + i \gamma^1 \partial_\eta - i \gamma^2 \partial_y - i \gamma^3 \partial_z + \frac{1}{2} \frac{1}{\rho} \gamma^1 + m\right) \Psi = 0
\]

\[
\Psi_{i,M}^{L,\pm} = O_1^\pm \left(X^L_i K_{i,M-\frac{1}{2}}(\kappa \rho) + Y^L_i K_{i,M+\frac{1}{2}}(\kappa \rho)\right) e^{-i M \eta} e^{i k_2 y + i k_3 z}
\]

with

\[
X^L_1 = \begin{pmatrix}
k_3 \\
i (k_2 + i m) \\
i (k_2 + i m) \\
k_3
\end{pmatrix} \quad Y^L_1 = \begin{pmatrix}
0 \\
-i \kappa \\
i \kappa \\
0
\end{pmatrix}
\]

\[
X^L_2 = \begin{pmatrix}
0 \\
-i \kappa \\
-i \kappa \\
0
\end{pmatrix} \quad Y^L_2 = \begin{pmatrix}
k_3 \\
i (k_2 - i m) \\
-i (k_2 - i m) \\
-k_3
\end{pmatrix}
\]

(P sector):

\[
\left(i \gamma^0 \partial_\rho + \frac{i}{\rho} \gamma^1 \partial_\eta + \frac{i}{2} \frac{1}{\rho} \gamma^0 - i \gamma^2 \partial_y - i \gamma^3 \partial_z + m\right) \Psi = 0
\]

\[
\Psi_{i,M}^{P,\pm} = P_1^\pm \left(X^P_i K_{i,M-\frac{1}{2}}(\pm i \kappa \rho) + Y^P_i K_{i,M+\frac{1}{2}}(\pm i \kappa \rho)\right) e^{-i M \eta} e^{i k_2 y + i k_3 z}
\]

with

\[
X^P_1 = \begin{pmatrix}
k_3 \\
i (k_2 + i m) \\
i (k_2 + i m) \\
k_3
\end{pmatrix} \quad Y^P_1 = \begin{pmatrix}
0 \\
\pm \kappa \\
\mp \kappa \\
0
\end{pmatrix}
\]

\[
X^P_2 = \begin{pmatrix}
0 \\
\pm \kappa \\
\mp \kappa \\
0
\end{pmatrix} \quad Y^P_2 = \begin{pmatrix}
k_3 \\
i (k_2 - i m) \\
-i (k_2 - i m) \\
-k_3
\end{pmatrix}
\]

As regards to the relationship between these solutions, this could be found by analytically continuing the functions across the event horizons, which represent branch points for these functions. Using the variables

\[
x_+ = x + t \quad x_- = t - x
\]

the passages through the horizons is given by the substitutions: 

\(-x_- \to x_- e^{\pm i \pi}\) \(R \to F\) \(x_+ \to -x_+ e^{\pm i \pi}\) \(F \to L\), \(-x_- \to -x_- e^{\pm i \pi}\) \(L \to P\), \(-x_+ \to x_+ e^{\pm i \pi}\)
\((P \rightarrow R)\). The result is that the solutions of the Dirac equation are linked by two possible paths of analytical continuation, namely that one corresponding (we name it A) to the transformation \(-x_\pm \rightarrow x_\pm e^{\pm i\pi}\) and linking in succession the functions

\[
\Psi_{i,M}^R \rightarrow \Psi_{i,M}^{F,+} \rightarrow \Psi_{i,M}^{L,-} \rightarrow \Psi_{i,M}^{P,-} \rightarrow \Psi_{i,M}^R
\]

and that one corresponding to the transformation \(-x_\pm \rightarrow x_\pm e^{-\pm i\pi}\) (we name it B) and linking in succession the functions

\[
\Psi_{i,M}^R \rightarrow \Psi_{i,M}^{F,-} \rightarrow \Psi_{i,M}^{L,+} \rightarrow \Psi_{i,M}^{P,+} \rightarrow \Psi_{i,M}^R
\]

Moreover, it is possible to demonstrate that the normalization factors of the different solutions are related each one other in such a way that, once determined \(N\), all the others are determined consequently.

### 4.2 Lorentz Momentum eigenfunctions in MS

We now turn to the problem of finding a unified representation for the eigenfunctions of the Boost Generator. Let’s consider the integral representations of the Bessel functions \(K_{\nu}(\rho)\) given by:

\[
K_{\nu}(\rho) = \frac{1}{2} e^{-\pi \nu/2} \int_{-\infty}^{+\infty} e^{i \rho \sinh \vartheta} e^{\nu \vartheta} d\vartheta \quad (79)
\]

\[
K_{-\nu}(\rho) = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} e^{i \rho \sinh \vartheta} e^{-\nu \vartheta} d\vartheta \quad (80)
\]

Using the second one to express in an integral form and in minkowski coordinates the functions \(K_{\nu}(\rho(t,x))e^{-\nu \eta(t,x)}\) (using the coordinate transformation \([\,\,]\), we obtain:

\[
K_{\nu}(\kappa \rho) e^{-\nu \eta} = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} e^{i \kappa \rho \sinh \vartheta} e^{-\nu (\vartheta + \eta)} d\vartheta = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} e^{i \kappa \rho \sinh(\vartheta - \eta)} e^{-\nu \vartheta} d\vartheta = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} e^{i [\kappa \rho \sinh \vartheta - i \kappa \rho \cosh \theta \sinh \eta]} e^{-\nu \vartheta} d\vartheta = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} e^{i \kappa [\vartheta - t \cosh \theta]} e^{-\nu \vartheta} d\vartheta = \frac{1}{2} e^{i \pi \nu/2} \int_{-\infty}^{+\infty} P_{\vartheta}^-(t, x) e^{-\nu \vartheta} d\vartheta \quad (86)
\]

where \(P_{\vartheta}^-(t, x)\) represent (2-dim) positive frequency plane waves with \(\omega = \kappa \cosh \theta\) and \(k_x = \kappa \sinh \theta\). Using the first one, and with the same procedure,
we have:

\[
K_\nu(\kappa \rho) e^{-\nu \eta} = \frac{1}{2} e^{-\frac{\kappa}{2}} \int_{-\infty}^{+\infty} e^{i \nu [x \sinh \theta + t \cosh \theta]} e^{\nu \theta} d\theta = (87)
\]

\[
= \frac{1}{2} e^{-\frac{\kappa}{2}} \int_{-\infty}^{+\infty} P_\nu^+(t, x) e^{\nu \theta} d\theta \tag{88}
\]

where now \(P_\nu^+(t, x)\) are (2-dim) plane waves with negative frequency \(-\omega = -\kappa \cosh \theta\). What we found means that the functions \(K_\nu(\rho(t, x)) e^{-\nu \eta(t, x)}\) can be expressed equivalently as linear (integral) combination of positive or negative two dimensional plane waves.

Inserting these formulas into the expression for \(\Psi_{\bar{i}, \bar{\mathcal{M}}}(t, x, y, z)\), not taking care of the normalization factor, gives

\[
\Psi_{\bar{i}, \bar{\mathcal{M}}}(t, x, y, z) = \frac{1}{2} N^\mp \left[ X_i^R e^{\pm i\frac{\pi}{4}(i\mathcal{M}-\frac{1}{2})} \int_{-\infty}^{+\infty} e^{i \nu [x \sinh \theta + t \cosh \theta]} e^{\mp (i\mathcal{M}-\frac{1}{2}) \theta} d\theta +
+ Y_i^R e^{\pm i\frac{\pi}{4}(i\mathcal{M}+\frac{1}{2})} \int_{-\infty}^{+\infty} e^{i \nu [x \sinh \theta + t \cosh \theta]} e^{\mp (i\mathcal{M}+\frac{1}{2}) \theta} d\theta \right] \times
\times e^{i k_2 y + i k_3 z} \tag{89}
\]

These are the global functions we were looking for. For them, the following properties hold true:

- they are well behaved (analytical) on the entire Minkowski manifold, except for the origin;
- they are solutions of the Dirac equation;
- they are eigenfunctions of the boost generator operator \(M_{01}\), with eigenvalue \(\mathcal{M}\);
- they reduce themselves to the correct solutions of the Dirac equations in the different sectors;
- they correspond each to one of the two possible paths of analytical continuation we mentioned before, namely \(\Psi_{\bar{i}, \bar{\mathcal{M}}}\) corresponds to the path \(A\), and \(\Psi_{\bar{i}, \bar{\mathcal{M}}}^+\) corresponds to the path \(B\), so we could say that the reason for the existence of two different global representation is the existence of two different paths of analytical continuation across the horizons;
- they are orthonormalized with respect to the ordinary scalar product in MS, with normalization factors given by:

\[
N^\mp = \frac{e^{\pm \frac{1}{2} \pi \mathcal{M}}}{2\pi \sqrt{\kappa}} \tag{90}
\]
We note also that these functions represent the analogous in the spinor case of the Gerlach’s Minkowski Bessel modes for the scalar field \[10\]. Before considering the quantization of the field in terms of these modes, it is worth to notice that we could expect to have additional difficulties in using them instead of the standard plane wave basis. The reason for that is the divergence of these modes in the origin of Minkowski space, which doesn’t affect, of course, the quantization procedure in any deep way, as will be proved in the following, but requires additional care in the calculations.

4.3 Alternative quantization in MS

Having obtained these global functions, we can perform the quantization of the spinor field in terms of them. We remember that the usual plane wave expansion is given by:

\[\Psi = \sum_{r=1,2} \int d^3k \left[ a_r(k) \Psi^+_r(k) + b_r^\dagger(k) \Psi^-_r(k) \right] \tag{91}\]

where the \(\Psi^+_r(k)\) are positive frequency plane waves and \(\Psi^-_r(k)\) are negative frequency ones, and that the quantum vacuum state is defined by the relation:

\[a_r(k) \ket{0}_M = b_r(k) \ket{0}_M = 0 \quad \forall r, \vec{k} \tag{92}\]

But we know that our \(\Psi^-_{i,M}\) are linear combinations of positive frequency plane waves and \(\Psi^+_{i,M}\) of negative frequency ones, so we can have this kind of expansion:

\[\Psi(t,x,y,z) = \sum_{i=1,2} \int_{-\infty}^{+\infty} dM \int dk_2 \int dk_3 \left[ c_{i,M}(k_2,k_3) \Psi^-_{i,M} + d_{i,M}(k_2,k_3) \Psi^+_{i,M} \right] \tag{93}\]

then imposing the usual anticommutations rules

\[\{ c_{i,M}(k_2,k_3), c^\dagger_{j,M'}(k'_2,k'_3) \} = \delta_{ij} \delta(M-M') \delta(k_2-k'_2) \delta(k_3-k'_3) \ldots \ldots \tag{94}\]

so defining a vacuum state \(\ket{0}\) by means of:

\[c_{i,M}(k_2,k_3) \ket{0} = d_{i,M}(k_2,k_3) \ket{0} = 0 \quad \forall i, M, k_2, k_3 \tag{95}\]

It is easy now to show that this quantization in equivalent to the usual one and so that the state \(\ket{0}\) is the usual Minkowski vacuum state \(\ket{0}_M\). In fact we have the following relations:

\[c_{i,M}(k_2,k_3) = (\Psi^-_{i,M}, \Psi)_M = \int_M d^3x \Psi^-_{i,M} \Psi = \]

\[= \sum_r \int_0^{+\infty} dk'_3 \frac{2\pi^2}{\omega'} N_{-N_{k'}} e^{i\frac{\pi}{4}(iM+\frac{1}{2})} e^{(iM+\frac{1}{2})\omega'} X_1^r u_r(k') + \]

17
with \( \vartheta' = \frac{1}{2} \ln \left( \frac{\omega' + k'_1}{\omega' - k'_1} \right) \)
and, for \( d^i_{\lambda M}(k_2, k_3) \):

\[
d^i_{\lambda M}(k_2, k_3) = (\Psi^+_{iM} : \Psi) = \int_M d^3x \Psi^+_{iM} \Psi =
\]

\[
\sum_r \int_0^{+\infty} dk'_1 \frac{2\pi^2}{\omega'} N^+_k \left[ e^{i \frac{\pi}{2} (iM + \frac{1}{2})} e^{i \frac{\vartheta'}{2} Y^+_{iM}(k')} \right] \times b^r(k'_1, k_2, k_3) = (98)
\]

\[
\sum_r \int_0^{+\infty} dk'_1 G^i_{hr}(k'_1, M) b^r(k'_1, k_2, k_3)
\]

So it is demonstrated that the vacuum states defined by the two quantization procedures are the same. Moreover, by explicit calculation it is possible to show that

\[
\{ a^r(k'_1, k_2, k_3) , a^r(k'_1, k'_2, k'_3) \} = \delta_{rr} \delta(k'_1 - k''_1) \delta(k_2 - k'_2) \delta(k_3 - k'_3) \quad \Leftrightarrow \quad \{ \epsilon_i{}_{\lambda M}(k_2, k_3), \epsilon_j{}_{\lambda M'}(k'_2, k'_3) \} = C_{\text{ost}} \times \delta_{ij} \delta(M - M') \delta(k_2 - k'_2) \delta(k_3 - k'_3)
\]

so the two quantum constructions are totally equivalent.

### 4.4 The Unruh construction and the Unruh effect

We will now derive the Unruh effect for the spinor field following the standard procedure first used by Unruh himself.

Let’s first recall that an attempt in finding the relationship between the quantum construction in RS and that in MS was made by Fulling \[5\] (for the scalar field), who simply identified the RS with the R-sector of the MS, consequently considered the Rindler vacuum state as a state in the Minkowski Hilbert space, and tried to express the Rindler annihilation (and creation) operators in terms of the usual plane waves ones. He then argued that Minkowski vacuum state could be considered as a particle state with respect to the Rindler vacuum state. Of course, because of the boundary condition we found necessary for the quantization in RS, this procedure is meaningless, since, as we already stressed, RS cannot be identified with R-sector of MS, but should be considered as a manifold on its own. Apart from this, however, there is another reason why the Fulling scheme is not valid. This scheme, in fact, implies to consider field modes in MS which corresponds to the Fulling ones in R and are zero everywhere else. This is equivalent to use a representation of the boost modes \[39\] given by
\[ \Psi_M = \theta(x_+) \theta(-x_-) \Psi^R_M + \theta(x_+) \theta(x_-) \Psi^F_M + \theta(-x_+) \theta(x_-) \Psi^L_M + \theta(-x_+) \theta(-x_-) \Psi^P_M \] (100)

with \( x_\pm = t \pm x \), but then using for the quantization only the first term of this expression, so to be restricted in the R-wedge. This procedure cannot be valid, since physically these modes correspond to solutions of the field equation when infinite sources of energy are placed at the horizon, because of the presence of the theta function.

The procedure used by Unruh to compare the Minkowski quantization to the Rindler one is to construct a new quantization scheme, which should be valid in the whole MS, but should also reproduce the Fulling quantization in RS when restricted to to the R-sector. The idea is to build the Hilbert space of Minkowski states \( \mathcal{H}_M \) out of the Hilbert space of solutions of the wave equation of the form \( \mathcal{H}_R \oplus \mathcal{H}_L \), i.e. sum of solution of Minkowski equation of motion which are the same as the Fulling modes in the R (L) sector, but vanish identically in the L (R) sector.

Let’s perform the Unruh construction for quantization in MS, using our globally defined functions \( \Psi_{\pm iM} \). Consider the functions:

\[
R_{i,M} = \frac{1}{\sqrt{2 \cosh \pi M}} \left( e^{\frac{\pi M}{2}} \Psi_{iM} - e^{-\frac{\pi M}{2}} \Psi_{iM}^+ \right) \tag{101}
\]

\[
L_{i,M} = \frac{1}{\sqrt{2 \cosh \pi M}} \left( e^{-\frac{\pi M}{2}} \Psi_{iM} - e^{\frac{\pi M}{2}} \Psi_{iM}^+ \right) \tag{102}
\]

which are solutions of Dirac equations, are eigenfunctions of \( M_{01} \), are analytical in the whole Minkowski space, except for the origin, and orthonormalized in MS. Moreover, it happens that the \( R_{i,M} \) are defined everywhere but in L sector and reduce themselves to the \( \Psi^R_{iM} \) in the R one, while the \( L_{i,M} \) manifest the inverse behaviour. Inverting these relations, and inserting into the expansion \( 93 \), we have:

\[
\Psi(t, x, y, z) = \sum_{i = 1, 2} \int_{-\infty}^{+\infty} dM \int dk_2 \int dk_3 \left[ c_{i,M}(k_2, k_3) \Psi_{iM} - d_{i,M}(k_2, k_3) \Psi_{iM}^+ \right] = \sum_i \int_{-\infty}^{+\infty} dM \left[ r_{i,M} R_{i,M} + l_{i,M}^\dagger L_{i,M} \right] = \sum_i \int_0^{+\infty} dM \left[ r_{i,M} R_{i,M} + l_{i,M} L_{i,M} + r_{i,-M}^\dagger R_{i,-M} + l_{i,-M}^\dagger L_{i,-M} \right] \tag{103}
\]

having introduced the coefficients:

\[
r_{i,M} = \frac{c_{i,M} e^{\frac{\pi M}{2}} + d_{i,M}^\dagger e^{-\frac{\pi M}{2}}}{\sqrt{2 \cosh \pi M}} \tag{105}
\]

\[
l_{i,M}^\dagger = \frac{c_{i,M} e^{-\frac{\pi M}{2}} - d_{i,M} e^{\frac{\pi M}{2}}}{\sqrt{2 \cosh \pi M}} \tag{106}
\]
These relations represent the Bogolubov transformation between the quantum constructions (13) and (104).

Now, consider carefully the nature of the operators $r_{i,M}$ and $l_{i,M}$ (and $r_{i,M}^\dagger$ and $l_{i,M}^\dagger$).

First of all suppose that the (104) represents an expansion of the spinor field, that leads to a correct quantization of it in MS, providing we impose the conditions

$$\left\{ r_{i,M}, r_{j,M}^\dagger \right\} = \delta_{ij} \delta(M - M') \quad \left\{ r_{i,M}, r_{j,M} \right\} = 0 \quad \left\{ r_{i,M}, l_{j,M} \right\} = 0$$

and the analogous for $l_{i,M}$ and $l_{i,M}^\dagger$.

Suppose also that the operators above could be considered as annihilation (and creation) operators for R-particles and L-particles.

We stress that these hypothesis are crucial for the following derivation being physically meaningful.

In fact, if these both hold, the operators $r_{i,M}$, defined as scalar products in MS, could be also expressed as integrals over the surface ($t = 0, x > 0$) which is a Cauchy (hyper)surface for the R wedge.

Moreover, this, together with the particular functional behaviour of the R-function (that reduce to the $\Psi^R_{i,M}$ in the R sector), would mean that we can identify the operators $r_{i,M}$ ($r_{i,M}^\dagger$) with the Rindler annihilation (creation) operators $a_{i,M}$ ($a_{i,M}^\dagger$). Consequently, the R-particles would be identified with the Rindler particles, constructed in terms of the $a_{i,M}^\dagger$, that an accelerated observer detects.

Let's now consider the operator $N_{i,j,M,M'} = r_{i,M}^\dagger r_{j,M'}$ (we can of course not consider the quantum numbers $k_2$ and $k_3$, because they don’t play any significant role here). We remind that $M$ is the eigenvalue of $M_0$ but also the energy of a Rindler observer. It is easy to calculate the mean value of this operator in the Minkowski vacuum state, having:

$$M\langle 0 \left| N_{i,j,M,M'} \right| 0 \rangle_M = M\langle 0 \left| r_{j,M}^\dagger r_{i,M} \right| 0 \rangle_M = \frac{1}{e^{2\pi M} + 1} \delta_{ij} \delta(M - M') \delta(k_2 - k_2') \delta(k_3 - k_3')$$

If we now identify the operators $r_{i,M}$ with the operators $a_{i,M}$, then we can interpret $N_{i,j,M,M'}$ as a Rindler particle number operator. This is exactly what is usually done in literature. The reasons for this identification were explained before and appear to be quite convincing. Nevertheless this passage is not trivial at all, as we will show.

Anyway, once identified the operators $r_{i,M}$ with the operators $a_{i,M}$, the result (109) can be interpreted as meaning that an inertial observer and a Rindler observer don’t share the same vacuum state, and that Minkowski vacuum state is a particle state for a Rindler observer.
We can also calculate the total number of particles that a Rindler observer will perceive if the field is in Minkowski vacuum state, for any quantum number and for unity of proper time. The result is:

\[
dN = \sum_{i,j} \int_0^{+\infty} dM \int_0^{+\infty} dM' \int dk_2 \int dk_2' \int dk_3 \int dk_3' N_{i,j,M,M'} \ d\eta = \tag{110}
\]

\[
= \sum_{i,j} \int_0^{+\infty} dM \int_0^{+\infty} dM' \int dk_2 \int dk_2' \int dk_3 \int dk_3' M \langle 0 \mid r^\dagger_{j,M'} r_{i,M} \mid 0 \rangle_M \ d\eta = \tag{111}
\]

\[
= \sum_{i,j} \int_0^{+\infty} dM \int_0^{+\infty} dM' \int dk_2 \int dk_2' \int dk_3 \int dk_3' \frac{1}{e^{2\pi M'} + 1} \times \delta_{ij} \delta(M - M') \delta(k_2 - k_2') \delta(k_3 - k_3') \ d\eta = \tag{112}
\]

\[
= \sum_i \int_0^{+\infty} dM \int dk_2 \int dk_3 \frac{1}{e^{2\pi M} + 1} \ d\eta = \tag{113}
\]

\[
= \sum_i \int_0^{+\infty} dh_R \int dk_2 \int dk_3 \frac{1}{e^{2\pi h_R} + 1} \ d\tau = \tag{114}
\]

\[
= 2 \int_0^{+\infty} dh_R \int dk_2 \int dk_3 \frac{1}{e^{\frac{2\pi h_R}{a}} + 1} \ d\tau \tag{115}
\]

where we have used the quantities: \( h_R = aM \), Rindler energy, and \( \tau = \frac{a}{\beta} \) proper time, with \( a \) acceleration of the Rindler observer. This result could be stated saying that the particle distribution of the Minkowski vacuum state, with respect to the quantization performed using the \( R_{i,M} \) modes, is given by a thermal spectrum, according to Fermi-Dirac statistics with temperature:

\[
T = \frac{1}{\beta k_B} = \frac{a}{2\pi k_B} \tag{116}
\]

We saw that the identification between the operators \( r_{i,M} \) and \( a_{i,M} \) is crucial in its interpretation. We also pointed out that this identification is based on two strong hypotheses: that the Unruh construction is valid in the whole MS, and that the operators \( r_{i,M} \) (\( r^\dagger_{i,M} \)), and \( l_{i,M} \) (\( l^\dagger_{i,M} \)) can be considered as annihilation (creation) operators.

Consequently, we will now analyse in detail these crucial points.

### 4.5 Analysis of the Unruh construction

We are going now to show that the same arguments against the validity of the Unruh construction in the whole Minkowski Space, that were indicated in 1, 2, 3, 4 for the case of the scalar field, are preserved also for the spinor field and support the conclusion that the Unruh scheme is valid instead in the disjoint union of the R and L wedge. This conclusion will be also proved by an explicit calculation.

First of all we argue that the Unruh quantization scheme is not suitable for the quantization of the spinor field in MS. This can be easily seen by looking
at the initial expansion of the field (104). This is based on a separation of
the integration interval into two parts corresponding to positive and negative
values of the eigenvalue \( \mathcal{M} \), and this separation is necessary in order to have
an expansion in terms of R and L modes. But we should recall the divergence
of the functions \( \Psi_{i,\mathcal{M}} \) (hence of the Unruh R and L modes) at the origin of
MS, as we already noticed. This, we said, doesn’t affect the quantization in MS
in terms of the boost modes, but nevertheless implies that we cannot perform
the separation of the integration interval as in the Unruh procedure. In fact,
discarding the eigenfunction correspondent to the eigenvalue \( \mathcal{M} = 0 \) (as any
other eigenfunction, because the divergence at the origin is a common property
of all of them) it would mean to discard an infinite number of degrees of freedom
of the field, if the origin is in the domain of definition of our field, so commuting
from our initial system to a physically different one. Consequently we could say
that the Unruh construction cannot be valid in the whole MS, which of course
include the origin.

Moreover, the Unruh operators \( r \) and \( l \) (and their conjugates) cannot be
considered as annihilation (and creation) operators for the field in MS, even
if they satisfy the anticommutation relations (107). For this being possible,
it is necessary the existence of a stationary ground state for the field in MS,
which is defined with respect to a global timelike variable in the whole space,
and which is annihilated by the \( r \) and \( l \) operators. But such a ground state is
definitely missing in MS. In fact, there is no timelike variable with respect to
which the Unruh modes (or their adjoints) are positive frequency solutions of the
Dirac equation in MS (remember that the Killing vector \( \frac{\partial}{\partial \eta} \) is spacelike in the
\( F \) and \( P \) sectors) and consequently the Unruh operators will always be linear
combinations of operators which create or annihilate particles with opposite
frequency signs, i.e. there is not a stationary vacuum state in MS with respect
to \( r \)-particles or \( l \)-particles.

These problems disappear if we consider the field only defined in the double
Rindler wedge, i.e. in the disjoint union of the R and L sectors. Here the
origin of MS is not considered, and so the divergence of the R and L modes
has no physical consequences for the separation of the integration domain in
the expansion (104). In this manifold there exists a timelike killing vector with
respect to which the R and L modes are positive frequency solutions of the Dirac
equation, consequently there exists a stationary ground state for the system
defined by the Unruh expansion, and the \( r \) and \( l \) operators could be interpreted
as annihilation operators for the field in this double wedge. The result is that
the Unruh construction is well defined in this case but, we emphasize, it refers
to a field restricted to the union of the R and L sectors of MS and so physically
different from the field defined in MS. In addition, since the R and L sectors
are totally independent of each other, because they are separated by a spacelike
interval, the quantization in these two wedges should be carried on separately.

In other words we have the expansion:

\[
\Psi_{RL}(x) = \sum_i \int_0^{\infty} d\mathcal{M} \left\{ r_{i,\mathcal{M}} R_{i,\mathcal{M}}(x) + r_{i,\mathcal{M}}^\dagger R_{i,-\mathcal{M}}(x) \right\} +
\]
\[ + \sum_i \int_0^\infty dM \left\{ l_{i,M} L_{i,M}(x) + l_{i,M}^\dagger L_{i,M}(x) \right\} \quad x \in R \cup L \quad (117) \]

We already showed that the quantization of the spinor field in the R sector requires the boundary condition (14), so we expect it to be manifest also for the field \( \Psi_{RL} \). In particular, if the Unruh expansion is valid (only) for the field in the double Rindler wedge, the Unruh operators \( r \) should coincide with the operators \( a_{i,M} \) in terms of which the quantization of the field in RS is performed, provided that the field satisfy the boundary condition (14). Let’s now prove with an explicit calculation this statement.

First of all, we are going to find the explicit expression of the coefficients \( r_{i,M} \) as functions of the values of the field \( \Psi_{RL} \). We recall that these are defined as:

\[ r_{i,M} = \frac{c_{i,M} e^{\frac{z_{i,M}}{2}} + \tilde{d}_{i,M}^\dagger e^{-\frac{z_{i,M}}{2}}}{\sqrt{2 \cosh \pi M}} \quad (118) \]

so we first need to find the expression for the coefficients \( c_{i,M} \) and \( \tilde{d}_{i,M}^\dagger \), in terms of the field and of its spatial derivative. This task implies the proper treatment of integral whose hypersurface of integration is the \( t = 0 \) hypersurface, which pass across the origin of Minkowski space, that is the intersection of the branch points where the functions we used are not well defined; this requires great attention.

By performing this calculation it is also possible to see that, as it should be, the coefficients \( c_{i,M} \) and \( \tilde{d}_{i,M}^\dagger \) are well defined without the need of any additional boundary condition other than the vanishing of the field at spatial infinity, in contrast to the “Rindler coefficients” in (33).

Inserting the expressions so found for \( c_{i,M} \) and \( \tilde{d}_{i,M}^\dagger \) into the (118), we obtain:

\[
\begin{align*}
\frac{r_{i,M}}{r_{i,M}} &= \frac{c_{i,M} e^{\frac{z_{i,M}}{2}} + \tilde{d}_{i,M}^\dagger e^{-\frac{z_{i,M}}{2}}}{\sqrt{2 \cosh \pi M}} \\
&= \frac{1}{\sqrt{2 \cosh \pi M}} \int_0^\infty dy e^{-i k_2 y} \int_{-\infty}^{+\infty} dz e^{-i k_3 z} \times \\
&\quad \times \left\{ X^\dagger \int_0^{+\infty} dx \left[ K_{i,M+\frac{1}{2}}(k x) \Psi_{RL}(0, x) - \\
- \frac{1}{2} \Gamma \left( \frac{1}{2} + i M \right) \left( \frac{k x}{2} \right)^{-iM-\frac{1}{2}} \Psi_{RL}(0, x) - \\
- \frac{1}{\kappa} \frac{1}{\frac{1}{2} - i M} \Gamma \left( \frac{1}{2} + i M \right) \left( \frac{k x}{2} \right)^{-iM+\frac{1}{2}} \frac{d}{dx} \Psi(0, x) \right] + \\
+ \frac{1}{\sqrt{2 \cosh \pi M}} \int_0^{+\infty} dx \left[ K_{i,M-\frac{1}{2}}(k x) \Psi_{RL}(0, x) - \\
- \frac{1}{2} \Gamma \left( \frac{1}{2} - i M \right) \left( \frac{k x}{2} \right)^{+iM-\frac{1}{2}} \Psi_{RL}(0, x) - \\
- \frac{1}{\kappa} \frac{1}{\frac{1}{2} + i M} \Gamma \left( \frac{1}{2} - i M \right) \left( \frac{k x}{2} \right)^{+iM+\frac{1}{2}} \frac{d}{dx} \Psi_{RL}(0, x) \right] \right\} \quad (119)
\end{align*}
\]
Changing coordinates to the Rindler ones (R sector), we have:

\[ r_{i,M} = \sqrt{2} a_{i,M} + \frac{1}{2 \pi \sqrt{\kappa}} \sqrt{2} \pi \int_{-\infty}^{+\infty} dy e^{-ik_2y} \int_{-\infty}^{+\infty}dz e^{-ik_3z} \times \]

\[ \left\{ \lim_{\rho \to 0} \frac{1}{2} \frac{1}{1 - i\mathcal{M}} \Gamma \left( \frac{1}{2} + i\mathcal{M} \right) \left( \frac{\kappa \rho}{2} \right)^{-i\mathcal{M} + \frac{1}{2}} X^\dagger_i \Psi_R (0, \rho) + \right. \]

\[ \left. + \lim_{\rho \to 0} \frac{1}{2} \frac{1}{1 + i\mathcal{M}} \Gamma \left( \frac{1}{2} - i\mathcal{M} \right) \left( \frac{\kappa \rho}{2} \right)^{+i\mathcal{M} + \frac{1}{2}} Y^\dagger_i \Psi_R (0, \rho) \right\} \quad (120) \]

It is so clear that the coefficients \( r_{i,M} \) and \( a_{i,M} \) cannot be identified unless

\[ \lim_{\rho \to 0} \rho^{\frac{1}{2}} \Psi_R (0, \rho) = 0 \quad (121) \]

which is just the boundary condition we found in sec. 4.

Let’s now discuss this result. We saw that it is not possible to identify the operators \( r_{i,M} \) and \( a_{i,M} \), unless (121); but there are no physical reasons to impose this boundary condition on the spinor field in MS; we have seen indeed that we obtain a meaningful quantum theory in terms of the Unruh modes only if we perform the “Unruh construction” just in the R and L sectors of MS, which are completely disjoint, from the causal and physical point of view; moreover, for an observer living in the Rindler wedge it is not possible, because of condition (121), to perform measurement in the whole MS, in order to put the field in the Minkowski vacuum state; in other words, the Unruh construction outlined in 4.1 does not represent a valid quantization scheme for the whole MS; consequently the operator \( r^\dagger r \) cannot be interpreted as a particle number operator in MS and therefore, the relation (115) cannot be interpreted in any sense as a proof of the “Unruh effect”.

5 Conclusions

Here we come to our conclusions. We can say that our analysis of the spinor field confirms the conclusion made in (1), (2), (3), (4) that the basic principles of quantum field theory imply that the Unruh procedure does not represent a correct derivation of the Unruh effect.

We saw that the reason for this conclusion is the existence of boundary conditions for the quantization of the spinor field in Rindler spacetime, preventing any relationship between this quantum construction and that one in Minkowski spacetime. The role played by this boundary conditions was analytically showed in our analysis of the Unruh procedure. We already noted that the existence of such boundary conditions could be expected since a Rindler observer is confined inside the Rindler wedge by event horizons, so he would see these like spatial infinity of an inertial observer. For the same reason a Rindler observer has no relationship with MS and he cannot in any way prepare the quantum field in the Minkowski vacuum state.
We have also shown that the Unruh quantization scheme is valid only in the double Rindler wedge $R \cup L$, and so it is also in the spinor case, because of the boundary condition (44); consequently the Unruh construction cannot be used to analyse the relationship between Rindler and Minkowski quantization schemes.

It seems to us that there are enough reasons to assert that this is a general feature of the analysed problem and that it holds true for any quantum field, and we are supported in this conclusion by the previously mentioned and similar results obtained for the scalar field and for the electromagnetic field.

On the other hand, the appearance of the fermion factor in the distribution (115) is entirely due to the particular form of the Bogolubov transformation (105), and there is no real need to interpretate it as an proof of a thermal nature of the spectrum; a similar situation is encountered in other physical problems where the concept of temperature doesn’t arise at all, as it is explained in details in [3].

Of course, there are many different approaches to the Unruh problem and many derivations of the effect have been proposed. It is then worth to study these carefully in order to see if the difficulties found here for the Unruh procedure survive also in these cases, or they can be considered as correct derivations of the Unruh effect and of its consequences. Particularly interesting are the results in [11] and [12], which deserve further study and attention.

As regards to the other aspect of the Unruh problem, that we mentioned in the beginning, namely the behaviour of an accelerated detector in Minkowski space, we could only say that it remains an open question. This should be clear also just considering that the Unruh effect is generally explained using the key role of the event horizons, but there are no horizon at all for a non-ideal accelerating detector, whose acceleration lasts a finite amount of time.

Our results show, however, that there are no reasons to expect that it will be of the type predicted by Unruh, at least as far as only the conventional derivation of it is considered, and that it should not be expected to be universal and independent from the nature and characteristics of the detector itself. It is worth to note that, as a partial confirmation of what we are saying, it was showed in great details in [13] that elementary particles accelerated by a constant electric field don’t follow, in general, the Unruh behaviour, i.e. the thermal response with temperature (1).

We are sure, of course, that an accelerated detector behaves, in general, in a different way from an inertial one, and we admit that in some cases it could follow the Unruh behaviour, but we think that there are no quantum theoretical reasons to believe that this is the universal one.

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