SCHAUDER TYPE THEOREMS FOR MILD SOLUTIONS TO NON AUTONOMOUS ORNSTEIN-UHLENBECK EQUATIONS

Paolo De Fazio
University of Parma

Abstract

We prove smoothing properties along suitable directions of the Ornstein-Uhlenbeck evolution operator, namely the evolution operator formally associated to the non autonomous Ornstein-Uhlenbeck operator. Moreover we use the smoothing estimates to prove Schauder type theorems, again along suitable directions, for the mild solution of a class of evolution equations.

1 Introduction

Let \((X, \langle \cdot, \cdot \rangle_X, \| \cdot \|_X)\) be a separable Hilbert space, \(T > 0\) and \(\Delta = \{(s, t) \in [0, T]^2 \text{ s.t. } s < t\}\).

We consider the mild solution

\begin{equation}
(1.1) \quad u(s, x) = P_{s,t} \varphi(x) - \int_s^t \langle P_{s,\sigma} \psi(\sigma, \cdot) \rangle(x) d\sigma, \quad \varphi \in C_b(X), \psi \in C_b([0, T] \times X).
\end{equation}

to the following class of backward non autonomous initial value problems

\begin{equation}
(1.2) \quad \begin{cases}
\partial_s u(s, x) + L(s)u(s, x) = \psi(s, x), & (s, t) \in \Delta, \ x \in X, \\
u(t, x) = \varphi(x), & x \in X,
\end{cases}
\end{equation}

where the operators \(L(t)\) are of Ornstein-Uhlenbeck type

\begin{equation}
(1.3) \quad L(t)\varphi(x) = \frac{1}{2} \text{Tr} \left( Q(t)D^2 \varphi(x) \right) + (A(t)x + f(t), \nabla \varphi(x))_X,
\end{equation}

with \(Q(t) = B(t)B^*(t)\).

For any \(t \in [0, T]\) the evolution family \(\{P_{s,t}\}_{s \in [0, t]}\) is defined by

\begin{equation}
(1.4) \quad P_{t,t} = I \quad \text{for any } \ t \in [0, T],
\end{equation}

\begin{equation}
(1.5) \quad P_{s,t} \varphi(x) = \int_X \varphi(y) N_{m^* (t, s), Q(t, s)} (dy)
\end{equation}

\begin{equation}
= \int_X \varphi(y + m^* (t, s)) N_{0, Q(t, s)} (dy), \quad (s, t) \in \Delta, \ \varphi \in C_b(X)
\end{equation}
where $N_{m,Q}$ is the Gaussian measure in $X$ with mean $m$ and covariance $Q$.

For any $(s,t) \in \Sigma$

\begin{align}
(1.6) \quad Q(t,s) &= \int_s^t U(t, r) Q(r) U^*(t, r) \, dr, \\
(1.7) \quad m^\pi(t, s) &= U(t, s) x + g(t, s), \\
(1.8) \quad g(t, s) &= \int_s^t U(t, r) f(r) \, dr.
\end{align}

where $\{U(t, r)\}_{(t, r) \in \Sigma}$ is the strongly continuous evolution operator in $X$ associated to the family $\{A(t)\}_{t \in [0,T]}$. Under minimal assumptions, the mapping

\((s, \sigma, x) \in \Sigma \times X \mapsto (P_{s,\sigma} \psi(\sigma, \cdot))(x) \in \mathbb{R}\)

is measurable (e.g. [CL21, Lem.2.3]). So the integral in (1.1) is well defined. In this infinite dimensional setting, we need that the operators $Q(t, s)$ defined by (1.6) have finite trace.

If $\psi \equiv 0$, (1.2) is the Kolmogorov equation formally associated to the forward stochastic differential equation

\begin{equation}
(1.9) \quad \begin{cases}
  dX_t(s, x) = (A(t)X_t(s, x) + f(t))dt + B(t)dW_t, & 0 < s < t < T, \\
  X_s(s, x) = x \in X,
\end{cases}
\end{equation}

where $W_t$ is a $X$-valued cylindrical Wiener process. Namely it is the equation formally satisfied by

\[ u(s, x) = \mathbb{E}\varphi(X_t(s, x)), \quad 0 \leq s \leq t \leq T. \]

For a proof of this fact in the autonomous case see [DZ14].

We assume that for $0 \leq s < t \leq T$ there exists a normed space $(E, \| \cdot \|_E)$ such that $E \subseteq X$ with continuous embedding such that $U(t, s)(E)$ is contained in the Cameron-Martin space $\mathcal{H}_{t,s}$ of $Q(t, s)$ and $U(t, s)|_E \in \mathcal{L}(E, \mathcal{H}_{t,s})$. In this case we define the operators

\[ \Lambda(t, s) = Q^{-\frac{1}{2}}(t, s)U(t, s), \quad 0 \leq s < t \leq T, \]

where $Q^{-\frac{1}{2}}(t, s)$ is the pseudo-inverse of $Q^{\frac{1}{2}}(t, s)$. Under this hypothesis, we can prove that $P_{s,t}$ maps $C_b(X)$ (the space of continuous and bounded functions) into $C^k_{\mathcal{L}}(X)$ (the subspace of $k$-time Fréchet differentiable functions along $E$ having bounded Fréchet differentials along $E$). Moreover, we give an explicit formula for the Fréchet derivatives of $P_{s,t}$ of any order along $E$, that involves the operators $\Lambda(t, s)$. It turns out that $\Lambda(t, s) \in \mathcal{L}(E, X)$ for $0 \leq s < t \leq T$, but $\|\Lambda(t, s)\|_{\mathcal{L}(E, X)}$ blows up as $t - s \to 0^+$. If we assume that $\|\Lambda(t, s)\|_{\mathcal{L}(E, X)}$ has a powerlike behavior, namely that there exist $\theta, C > 0$ such that

\begin{equation}
(1.10) \quad \|\Lambda(t, s)\|_{\mathcal{L}(E, X)} \leq \frac{C}{(t-s)^{\theta}}
\end{equation}

and $E$ does not depend on any $s, t$, we can prove Hölder maximal regularity of (1.1) along directions of $E$.

In the autonomous case, Schauder estimates for Ornstein-Uhlenbeck type equations were proven in [DL95, Lun97] in finite dimension and in [CD96] in infinite dimension. Non autonomous equations in infinite dimensions were studied in [CL21] in the strong Feller case; namely they show
that $P_{s,t}$ maps $B_b(X)$ (the set of bounded Borel functions) into $C_b(X)$ under the assumption $U(t,s)(X) \subseteq Q^2(t,s)(X)$. Furthermore, it was proven that $P_{s,t}$ maps $B_b(X)$ into $C^k_b(X)$ for every $k \in \mathbb{N}$ and, under a suitable power like behavior of $\|A(t,s)\|_{C(X)}$, they proved Schauder type results. Techniques we adapted to our setting to prove Schauder estimates were developed in [LR21].

Others authors looked for regularizing results in the autonomous and perturbed case. For further readings we refer to [BF22a], [BF21], [BF22b] and [Mas07]. Moreover, for results on improvements of summability along suitable directions, we refer to [ABF23].

In section 2 we prove the smoothing properties of $P_{s,t}$. In particular we prove that $P_{s,t}$ maps $C_b(X)$ in $C_b^k(X)$ for every $(s,t) \in \Delta$ and $k \in \mathbb{N}$. Moreover there exist a constant $C_k > 0$ such that

\begin{equation}
\sup_{x \in X} \|D^k_{x}(P_{s,t}\varphi)(x)\|_{L^k(E)} \leq C_k \|\Lambda(t,s)\|_{L^k(E;X)} \|\varphi\|_{\infty}.
\end{equation}

Moreover, we extended to our non autonomous case a result of [GN03, Sct.3], giving sufficient conditions in order to take $E = Q^2(s)(X)$ for $s \in (0,t)$.

In section 3 we prove maximal Hölder regularity results. More precisely, if $\alpha \in (0,1)$, $\alpha + \frac{1}{\theta} \notin \mathbb{N}$, $\varphi \in C^{\alpha + \frac{1}{\theta}}(X)$ and $\psi \in C^{\alpha + \frac{1}{\theta}}([0,t] \times X)$, then $u(s,\cdot) \in C^{\alpha + \frac{1}{\theta}}([0,t] \times X)$. Moreover there exists $C = C(T,\alpha) > 0$, independent of $\varphi$ and $\psi$, such that

\begin{equation}
\|u\|_{C^{\alpha + \frac{1}{\theta}}([0,t] \times X)} \leq C \left(\|\varphi\|_{C^{\alpha + \frac{1}{\theta}}(X)} + \|\psi\|_{C^{\alpha + \frac{1}{\theta}}([0,t] \times X)}\right).
\end{equation}

If $\alpha + \frac{1}{\theta}$ is an integer, $u(s,\cdot)$ only belongs to the Zygmund space $Z^{\alpha + \frac{1}{\theta}}$ for every $s$. Zygmund regularity is not due to the infinite dimensional setting nor the time dependence of the data. Indeed, we have the same result even in finite dimension for the autonomous case for the Heat equation.

Last section concerns three genuinely non autonomous examples. In the first example $A(t)$ and $B(t)$ are diagonal operators with respect to the same Hilbert basis of $X$. In the second example we consider $A(t) = a(t)I$, where $a$ is a continuous function. We get a non autonomous version of the Ornstein-Uhlenbeck semigroup used in the Malliavin calculus and we extend to such non autonomous case the results of [CL19]. In the third example $A(t)$ are the realization of second order elliptic operators in $X = L^2(\Omega)$ with Dirichlet boundary conditions and smooth enough coefficients, $\Omega$ is a bounden open and smooth subset of $\mathbb{R}^d$ and $B(t) \in \mathcal{L}(L^2(\Omega);L^2(\Omega))$ for a suitable $q \geq 2$. As in the autonomous case, we can take $B(t) \equiv I$ if $d = 1$. In [CL21], the authors proved Schauder results on the whole space $X$ with $\theta = \frac{1}{q} + \frac{d(q-2)}{4q}$. In this present paper we show how we can achieve better regularity choosing a suitable subspace of $X$. Indeed choosing choosing $E = L^p(\Omega)$ with $p \in (2,q]$ we find $\theta = \frac{q}{2p}(1 - \frac{q}{q})$ and choosing $E = (X_q,D_q)_{\alpha,p}$ with $\alpha \in (0,\frac{1}{\theta})$ we get $\theta = \frac{1}{2} - \alpha$.

\section{Notations and assumptions}

If $X$ and $Y$ are real Banach spaces we denote by $\mathcal{L}(X;Y)$ the space of bounded linear operators from $X$ to $Y$. If $X = Y$, we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X;X)$ and if $Y = \mathbb{R}$ we simply write $X'$ instead of $\mathcal{L}(X;\mathbb{R})$. Moreover, we denote by $\mathcal{L}^+_1(X)$ the subset of $\mathcal{L}(X)$ consisting of all non-negative and
symmetric operators having finite trace. For \( k \geq 2, \mathcal{L}^k(X) \) is the space of the \( k \)-linear bounded operators \( T : X^k \rightarrow \mathbb{R} \) endowed with the norm

\[
\|T\|_{\mathcal{L}^k(X)} = \sup \left\{ \frac{|T(x_1, \ldots, x_k)|}{\|x_1\|_X \cdots \|x_k\|_X} : x_1, \ldots, x_k \in X \setminus \{0\} \right\}.
\]

By \( B_b(X; Y) \) and \( C_b(X; Y) \) we denote the space of bounded Borel functions from \( X \) to \( Y \) and the space of bounded and continuous functions from \( X \) to \( Y \), respectively. We endow them with the sup norm

\[
\|F\|_\infty = \sup_{x \in X} \|F(x)\|_Y.
\]

If \( Y = \mathbb{R} \), we simply write \( B_b(X) \) and \( C_b(X) \) instead of \( B_b(X; \mathbb{R}) \) and \( C_b(X; \mathbb{R}) \), respectively. Let \( (E, \| \cdot \|_E) \) be a normed space such that \( E \subseteq X \) with continuous embedding. For \( \alpha \in (0, 1) \) we define the Hölder spaces along \( E \) as

\[
C^\alpha_E(X; Y) = \left\{ F \in C_b(X; Y) : \|F\|_{C^\alpha_E(X; Y)} = \sup_{x \in X, \ h \in E \setminus \{0\}} \frac{\|F(x + h) - F(x)\|_Y}{\|h\|_E^\alpha} < +\infty \right\},
\]

\[
\|F\|_{C^\alpha_E(X; Y)} = \sup_{x \in X} \|F(x)\|_Y + |F|_{C^\alpha_E(X; Y)},
\]

and the Lipschitz space along \( E \) as

\[
\text{Lip}_E(X; Y) = \left\{ F \in C_b(X; Y) : \|F\|_{\text{Lip}_E(X; Y)} = \sup_{x \in X, \ h \in E \setminus \{0\}} \frac{\|F(x + h) - F(x)\|_Y}{\|h\|_E} < +\infty \right\},
\]

\[
\|F\|_{\text{Lip}_E(X; Y)} = \sup_{x \in X} \|F(x)\|_Y + |F|_{\text{Lip}_E(X; Y)}.
\]

Again, if \( Y = \mathbb{R} \) we write \( C^0_E(X) \) and \( \text{Lip}_E(X) \) instead of \( C^0_E(X; \mathbb{R}) \) and \( \text{Lip}_E(X; \mathbb{R}) \), respectively. Moreover we say that a map \( f : X \rightarrow \mathbb{R} \) is \( E \)-Gâteaux differentiable at \( x \in X \) if there exists a bounded linear operator \( l_x : E \rightarrow \mathbb{R} \) such that for any \( h \in E \), we have

\[
\lim_{t \rightarrow 0} \frac{f(x + th) - f(x) - l_x(h)}{t} = 0.
\]

\( l_x \) is the \( E \)-Gâteaux differential of \( f \) at \( x \) and we set \( l_x = D_E^G f(x) \). We say that a map \( f : X \rightarrow \mathbb{R} \) is \( E \)-Fréchet differentiable at \( x \in X \) if there exists a bounded linear operator \( t_x : E \rightarrow \mathbb{R} \) such that

\[
\lim_{\|h\|_E \rightarrow 0} \frac{f(x + h) - f(x) - t_x(h)}{\|h\|_E} = 0.
\]

\( t_x \) is the \( E \)-Fréchet differential of \( f \) at \( x \) and we set \( t_x = D_E f(x) \). Clearly, if \( f \) is Fréchet differentiable at \( x \) then \( f \) is \( E \)-Fréchet differentiable at \( x \) and this is due to the continuous embedding of \( E \) in \( X \). More generally, for any \( F : X \rightarrow Y \), we say that \( F \) is \( E \)-Gâteaux differentiable at \( x \) if there exists \( L_x \in \mathcal{L}(E, Y) \) such that for any \( h \in E \), we have

\[
Y - \lim_{t \rightarrow 0} \frac{F(x + th) - F(x) - L_x(h)}{t} = 0.
\]
$L_x$ is the $E$-Gâteaux differential of $F$ at $x$ and we denote it by $D_E^g F(x)$. Moreover we say that $F$ is $E$-Fréchet differentiable at $x$ if there exists $T_x \in \mathcal{L}(E,Y)$ such that

$$Y - \lim_{\|h\|_E \to 0} \frac{F(x + h) - F(x) - T_x(h)}{\|h\|_E} = 0.$$  \hspace{1cm} (2.4)

$T_x$ is the $E$-Fréchet differential of $F$ at $x$ and we denote it by $D_E F(x)$. Clearly, if $E = X$ the notions of $E$-Fréchet differentiability and $E$-Gâteaux differentiability coincide with the usual ones. In this case we omit the subindex $E$ in the notations above. Hence, for instance, we write $DF$ instead of $D_E F$ for the Fréchet derivative. If $F : X \to Y$ is $E$-Fréchet differentiable at $x$, we say that $F$ is twice $E$-Fréchet differentiable at $x$ if $D_E^2 F : X \to E'$ is $E$-Fréchet differentiable at $x$. Hence, we define the Hessian operator $D_E^2 F \in \mathcal{L}^2(E)$ by

$$D_E^2 F(x)(k,h) := (T_x k)(h),$$

where $T_x$ is the operator in the definition, with $F(x)$ replaced by $D_E F$ and $Y = E'$. If $F$ is $(k - 1)$-times $E$-Fréchet differentiable at $x$ with $k \geq 2$, we say that $F$ is $k$-times $E$-Fréchet differentiable at $x$ if $D_E^{k-1} F : X \to \mathcal{L}^{k-1}(E)$ is $E$-Fréchet differentiable at $x$. In this case $D_E^k \in \mathcal{L}^k(X)$ is defined as

$$D_E^k F(x)(h_1,\ldots,h_k) := (T_x h_1)(h_2,\ldots,h_k)$$

where $T_x$ is the operator in the definition with $F(x)$ replaced by $D_E^{k-1} F(x)$ and $Y = \mathcal{L}^{k-1}(E)$. For any $k \in \mathbb{N}$, $C^k_E(X;Y)$ is the subspace of $C_b(X)$ consisting of all functions $F : X \to Y$ $k$-times $E$-Fréchet differentiable at any point with $D_E^k F$ continuous and bounded in $\mathcal{L}^j(E)$ for $j \leq k$. $C^k_E(X;Y)$ is endowed with the norm

$$\|F\|_{C^k_E(X;Y)} := \|F\|_{\infty} + \sum_{j=1}^k \sup_{x \in X} \|D_E^j F(x)\|_{\mathcal{L}^j(E)}.$$  \hspace{1cm} (2.5)

If $Y = \mathbb{R}$ we write $C^k_E(X)$ instead of $C^k_E(X;\mathbb{R})$. $Z^1_E(X;Y)$ is the Zygmund space along $E$. It is defined by

$$Z^1_E(X;Y) = \left\{ F \in C_b(X;Y) : [F]_{Z^1_E(X;Y)} = \sup_{x \in X, h \in E \setminus \{0\}} \frac{\|F(x + 2h) - 2F(x + h) + F(x)\|_Y}{\|h\|_E} < +\infty \right\},$$

and it is endowed with the norm

$$\|F\|_{Z^1_E(X;Y)} = \|F\|_{\infty} + [F]_{Z^1_E(X;Y)}.$$  

Higher order Hölder and Zygmund spaces along $E$ are defined as follows. For $\alpha \in (0,1)$ and $n \in \mathbb{N}$, we set

$$C^\alpha_n(X;Y) := \left\{ F \in C^\alpha_n(X;Y) : D_E^g F \in C^\alpha_n(X;\mathcal{L}^n(E)) \right\},$$

$$\|F\|_{C^\alpha_n(X;Y)} := \|F\|_{C^\alpha_n(X;Y)} + \|D_E^g F\|_{C^\alpha_n(X;\mathcal{L}^n(E))}$$

and for $n \geq 2$,

$$Z^n_1(X;Y) = \left\{ F \in C^{n-1}_E(X;Y) : D_E^{n-1} F \in Z^1_1(X;\mathcal{L}^n(E)) \right\},$$
Hypothesis 2.1.

Clearly if $E = X$ the functional spaces above coincide with the usual ones. We write $C^k_b(X;Y)$, $C^\alpha_b(X;Y)$, $\text{Lip}_b(X;Y)$, $Z^k_b(X;Y)$ instead of $C^k_b(X;Y)$, $C^\alpha_b(X;Y)$, $\text{Lip}_b(X;Y)$, $Z^k_b(X;Y)$, respectively.

Finally we introduce spaces of functions depending both on time and space variables. For every $a, b \in \mathbb{R}$, $a < b$ and $\alpha > 0$, we denote by $C^\alpha_b([a,b] \times X)$ the space of all bounded continuous functions $\psi : [a, b] \times X \rightarrow \mathbb{R}$ such that $\psi(s, \cdot) \in C^\alpha_b(X)$, for every $s \in [a, b]$, with

$$\|\psi\|_{C^\alpha_b([a,b] \times X)} = \sup_{s \in [a,b]} \|\psi(s, \cdot)\|_{C^\alpha_b(X)} < +\infty.$$  

If $\alpha \geq 1$ we also require that the mapping

$$(s, x) \mapsto \frac{\partial \psi}{\partial h_1 \ldots \partial h_k}(s, x)$$

are continuous in $[a, b] \times X$, for every $h_1, \ldots, h_k \in E$ with $k \leq [\alpha]$.

Now, for every $k \in \mathbb{N}$ we denote by $Z^k_b([a,b] \times X)$ the space of all bounded continuous functions $\psi : [a, b] \times X \rightarrow \mathbb{R}$ such that $\psi(s, \cdot) \in Z^k_b(X)$, for every $s \in [a, b]$, with

$$\|\psi\|_{Z^k_b([a,b] \times X)} = \sup_{s \in [a,b]} \|\psi(s, \cdot)\|_{Z^k_b(X)} < +\infty.$$  

If $k \geq 2$, we also require that the mapping

$$(s, x) \mapsto \frac{\partial \psi}{\partial h_1 \ldots \partial h_i}(s, x)$$

are continuous in $[a, b] \times X$, for every $h_1, \ldots, h_i \in E$ with $i \leq k - 1$.

**Hypothesis 2.1.**

1. $\{U(t, s)\}_{(s,t) \in \mathbb{R}} \subseteq \mathcal{L}(X)$ is a strongly continuous evolution operator, namely for every $x \in X$ the map

$$U(t, s) : x \mapsto U(t, s)x \in X,$$

is continuous and

(a) $U(t, t) = I$ for any $t \in [0, T]$,

(b) $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$.

2. The family of operators $\{B(t)\}_{t \in [0,T]} \subseteq \mathcal{L}(X)$ is bounded and strongly measurable, namely

(a) there exists $K > 0$ such that

$$\sup_{t \in [0,T]} \|B(t)\|_{\mathcal{L}(X)} \leq K,$$

(b) the map

$$B(t) : x \mapsto B(t)x \in X$$

is measurable for any $x \in X$. 
(3) The map \( f : [0, T] \rightarrow X \) is bounded and measurable.

(4) The trace of the operator \( Q(t, s) \) is finite for every \( 0 \leq s < t \leq T \).

By the Banach-Steinhaus theorem, there exists \( N > 0 \) such that
\[
\|U(t, s)\|_{L(X)} \leq N, \quad (s, t) \in \Delta.
\]

**Definition 2.2.** Let \( \mu = \mathcal{N}_{0, Q} \) be the Gaussian measure of mean \( m \) and covariance \( Q \), the consider the Cameron-Martin space \( \mathcal{H} \) of \( \mu \) is the space \( Q^{\frac{1}{2}}(X) \) endowed with the scalar product
\[
\langle h, k \rangle_{\mathcal{H}} = \langle Q^{-\frac{1}{2}}h, Q^{-\frac{1}{2}}k \rangle_X, \quad h, k \in Q^{\frac{1}{2}}(X),
\]
where \( Q^{-\frac{1}{2}} \) is the pseudo-inverse of \( Q \).

For all \( h \in \mathcal{H} \), there exists \( \hat{h} \in L^p(X, \mu) \), for every \( p \in [1, \infty) \) with
\[
\|\hat{h}\|_{L^p(X, \mu)} \leq \|h\|_{Q^{\frac{1}{2}}(X)}
\]
such that the Cameron-Martin formula
\[
\mathcal{N}_{z, Q}(dy) = \exp\left(-\frac{1}{2} \|h\|_{Q^{\frac{1}{2}}(X)}^2 + \hat{h}(y)\right) \mathcal{N}_{0, Q}(dy)
\]
holds. Let \( (e_k)_{k \in \mathbb{N}} \subseteq X \) be an orthonormal basis of \( X \) consisting of eigenvectors of \( Q \), \( Q e_k = \lambda_k e_k \) for all \( k \in \mathbb{N} \). We have
\[
\hat{h}(y) = \sum_{k \in \mathbb{N}, \lambda_k \neq 0} y_k \left(Q^{-\frac{1}{2}}h\right)_k \lambda_k^{-\frac{1}{2}} = \sum_{k \in \mathbb{N}, \lambda_k \neq 0} y_k h_k \lambda_k^{-1}
\]
where \( y_k = \langle y, e_k \rangle_X \) for any \( y \in X \). We remark that the series defined in (2.10) converges in \( L^p(X, \mathcal{N}_{0, Q}) \) for any \( p \in [1, \infty) \) but in general it does not converge pointwise.

Moreover for any \( p \in [1, \infty) \) there exists \( c_p > 0 \) such that
\[
\|y \mapsto \hat{h}(y)\|_{L^p(X, \mathcal{N}_{0, Q}(t, s))} \leq c_p \|h\|_X, \quad \text{for any } (s, t) \in \Delta.
\]

**Remark 2.3.** In our non-autonomous setting, for every \( (t, s) \in \Delta \) we denote by \( \mathcal{H}_{t, s} \) the Cameron-Martin space of the measure \( \mathcal{N}_{0, Q(t, s)} \). Moreover, for every \( t \in [0, T] \) we denote by \( H_t \) the space \( Q(t)^{\frac{1}{2}}(X) \) endowed with the scalar product
\[
\langle h, k \rangle_{H_t} = \langle Q^{-\frac{1}{2}}(t)h, Q^{-\frac{1}{2}}(t)k \rangle_X, \quad h, k \in Q^{\frac{1}{2}}(t)(X),
\]
where \( Q^{-\frac{1}{2}}(t) \) is the pseudo-inverse of \( Q(t) \).

**Hypothesis 2.4.** There exists a normed space \( (E, \| \cdot \|_E) \) such that \( E \subseteq X \) with continuous embedding such that \( U(t, s)(E) \subseteq E \), \( U(t, s)|_E \in L(E) \) and there exists \( M > 0 \) such that
\[
\|U(t, s)|_E\|_{L(E)} \leq M, \quad (s, t) \in \overline{\Delta}.
\]
Hypotesis 2.5. For a fixed \((s,t) \in \Delta\) there exists a normed space \((E, \| \cdot \|_E)\) such that \(E \subseteq X\) with continuous embedding such that \(U(t,s)(E) \subseteq \mathcal{H}_{t,s}\) and \(U(t,s)|_E \in \mathcal{L}(E, \mathcal{H}_{t,s})\).

Let \(Q^{-\frac{1}{2}}(t,s)\) be the pseudo-inverse of the operator \(Q^{\frac{1}{2}}(t,s)\). If Hypothesis 2.4 holds, we define the operator

\[
\Lambda(t, s) = Q^{-\frac{1}{2}}(t,s)U(t,s), \quad (s,t) \in \Delta.
\]

Remark 2.6. If \(E\) and \(F\) are a Banach spaces and \(U(t,s)(E) \subseteq F\), then \(U(t,s)|_E \in \mathcal{L}(E,F)\). Indeed, if \(x_n \xrightarrow{E} \rightarrow x\) and \(U(t,s)x_n \xrightarrow{F} \rightarrow y\), then \(x_n \xrightarrow{X} \rightarrow x\) and \(U(t,s)x_n \xrightarrow{X} \rightarrow y\), since the embeddings of \(E\) in \(X\) and of \(F\) in \(X\) are continuous. Since \(U(t,s) \in \mathcal{L}(X)\) then \(y = U(t,s)x\).

Remark 2.7. We remark that if \(E = X\) for all \((s,t) \in \Delta\), Hypothesis 2.4 implies that \(P_{s,t}\) is strong Feller as proved in [CL21].

Remark 2.8. It is convenient to rewrite Hypothesis 2.4, as in the autonomous case (e.g., [DZ02, Appendix B]). For \((s,t) \in \Delta\), we consider the operator \(L : L^2((s,t); X) \rightarrow X\) defined by

\[
Ly := \int_s^t U(t, \sigma)B(\sigma)y(\sigma) \, d\sigma, \quad y \in L^2((s,t); X).
\]

The adjoint operator \(L^* : X \rightarrow L^2((s,t); X)\) satisfies \(\langle Ly, x \rangle_X = \langle y, L^*x \rangle_{L^2((s,t); X)}\), which means

\[
\int_s^t \langle U(t, \sigma)B(\sigma)y(\sigma), x \rangle_X \, d\sigma = \int_s^t \langle y(\sigma), (L^*x)(\sigma) \rangle_X \, d\sigma, \quad x \in X, y \in L^2((s,t); X),
\]

so that

\[
(L^*x)(\sigma) = B^*(\sigma)U^*(t, \sigma)x, \quad x \in X, \text{ a.e \(\sigma \in (s,t)\)}
\]

and we get

\[
LL^*x = \int_s^t U(t, \sigma)B(\sigma)B^*(\sigma)U^*(t, \sigma)x \, d\sigma = Q(t,s)x, \quad x \in X.
\]

Therefore, by the general theory of linear operators in Hilbert spaces (e.g., [DZ14, Cor. B3]), we get

\[
\begin{cases}
\text{Range}(L) = \text{Range}(Q^{\frac{1}{2}}(t,s)) \\
\left\| Q^{-\frac{1}{2}}(t,s)x \right\|_X = \left\| L^{-1}x \right\|_{L^2((s,t); X)}, \quad x \in \mathcal{H}_{t,s}.
\end{cases}
\]

Since in general \(L\) is not invertible, we stress that \(L^{-1}\) is meant as the pseudo-inverse of \(L\). Hence

\[
\left\| L^{-1}x \right\|_{L^2((s,t); X)} = \min \left\{ \left\| y \right\|_{L^2((s,t); X)} : Ly = x \right\}, \quad x \in \mathcal{H}_{t,s}.
\]

Hence the range of \(L\) is the set of the traces at time \(t\) of the mild solutions of the evolution problems

\[
\begin{cases}
u'(r) = A(r)u(r) + B(r)y(r), \quad s < r < t, \\
u(s) = 0
\end{cases}
\]

where \(y\) varies in \(L^2((s,t); X)\). So Hypothesis 2.4 may be reformulated requiring that \(U(t,s)\) maps \(E\) in the trace space, for every \((s,t) \in \Delta\).
Finally we state also the last hypothesis that is essential to prove Schauder estimates.

**Hypothesis 2.9.** There exists a normed space \((E, \| \cdot \|_E)\) such that \(E \subseteq X\) with continuous embedding such that

1. \(U(t, s)(E) \subseteq E\), \(U(t, s)_E \in \mathcal{L}(E)\) and By the Banach-Steinhaus theorem, there exists \(M > 0\) such that

\[
\|U(t, s)_E\|_{\mathcal{L}(E)} \leq M, \quad (s, t) \in \Delta;
\]

2. \(U(t, s)(E) \subseteq \mathcal{H}_{t, s}\) for any \((s, t) \in \Delta\) and \(U(t, s)_E \in \mathcal{L}(E, \mathcal{H}_{t, s})\).

**Hypothesis 2.10.** There exist \(C, \theta > 0\) such that

\[
\|\Lambda(t, s)\|_{\mathcal{L}(E; X)} \leq \frac{C}{(t - s)\theta}, \quad (s, t) \in \Delta.
\]

## 3 Smoothing properties of \(P_{s, t}\)

If Hypothesis 2.1 hold, the evolution family

\[
P_{s, t}\varphi(x) = \int_X \varphi(y + m^s(t, s)) N_0 Q(t, s)(dy), \quad x \in X, \; (s, t) \in \Delta, \; \varphi \in C_b(X)
\]

is well defined. In [CL21] it was proved that \(P_{s, t}\) maps \(C^1_b(X)\) into itself, and

\[
\nabla(P_{s, t}\varphi)(x) = U^*(t, s) P_{s, t} \nabla \varphi(x), \quad (s, t) \in \Delta, \; x \in X, \; \varphi \in C^1_b(X)
\]

so that

\[
\sup_{x \in X} \|D(P_{s, t}\varphi)(x)\|_{X^*} \leq \|U(t, s)\|_{\mathcal{L}(X)} \|D\varphi\|_{\infty} \leq M \|\varphi\|_{C^1_b(X)}, \quad (s, t) \in \Delta, \; \varphi \in C^1_b(X);
\]

More generally it was proved that \(P_{s, t}\) maps \(C^k_b(X)\) into itself for every \(k \in \mathbb{N}\) Here we need similar properties for the spaces \(C^k_b(X)\).

**Lemma 3.1.** Under Hypotheses 2.1 and 2.4 for every \(k \in \mathbb{N}\) and \((s, t) \in \Delta\), \(P_{s, t}\) maps \(C^k_b(E)\) into itself, and

\[
D^k_E(P_{s, t}\varphi)(x)(h_1, ..., h_k) = P_{s, t}(D^k_E(\varphi)(U(t, s)h_1, ..., U(t, s)h_k))(x), \quad (s, t) \in \Delta,
\]

for any \(x \in X\) and \(h_1, ..., h_k \in E\). In particular it follows that for every \(\varphi \in C^k_b(X)\) and \((s, t) \in \Delta\)

\[
D^k_E P_{s, t} \varphi(h) \leq P_{s, t}\left(\|D^k_E \varphi(h)\|\right),
\]

\[
\|D^k_E P_{s, t} \varphi\|_{\mathcal{L}(E)} \leq \|U(t, s)\|_{\mathcal{L}(E)} \|D^k_E \varphi\|_{\infty} \leq M^k \|\varphi\|_{C^k_b(X)}.
\]

Moreover, for every fixed \(t \in [0, T]\) and \(h_1, ..., h_k \in X\), the mapping

\[
(s, x) \in [0, t] \times X \longmapsto D^k_E(P_{s, t}\varphi)(x)(h_1, ..., h_k) \in \mathbb{R},
\]

is continuous.
If in addition the mapping \((s,t) \in \Delta \mapsto U(t,s) \in \mathcal{L}(X)\) is continuous, then the function
\[
(s,t,x) \in \overline{\Delta} \times X \mapsto D^k_E(P_{s,t} \varphi)(x) (h_1, \ldots, h_k) \in \mathbb{R},
\]
is continuous.

If \(\varphi \in C^k_E(X)\), where \(\alpha = k + \sigma\), where \(k \in \mathbb{N} \cup \{0\}\) and \(\sigma \in (0,1)\), then \(P_{s,t} \varphi \in C^k_E(X)\) and
\[
[D^k_E(P_{s,t} \varphi)]_{C^k_E(X)} \leq \|U(t,s)\|_{L(E)}^k [D^k_E \varphi]_{C^k_E(X)} \leq M^k[D^k_E \varphi]_{C^k_E(X)}.
\]
If \(\varphi \in Z^k_E(X)\), where \(k \in \mathbb{N}\), then \(P_{s,t} \varphi \in Z^k_E(X)\) and
\[
[D^{k-1}_E(P_{s,t} \varphi)]_{Z^k_E(X)} \leq \|U(t,s)\|_{L(E)}^{k-1} [D^k_E \varphi]_{Z^k_E(X)} \leq M^{k-1}[D^k_E \varphi]_{Z^k_E(X)}.
\]

Proof. We start proving \((3.1)\) and \((3.3)\) by induction over \(k\). If \(k = 1\), let \(x \in X\), \(h \in E\) and \(\varepsilon > 0\), then for all \((s,t) \in \Delta\), we have
\[
P_{s,t} \varphi(x + \varepsilon h) - P_{s,t} \varphi(x) \leq \int_X \left| \varphi(y + \varepsilon h) - \varphi(y) \right| N_{0,Q(t,s)}(dy).
\]
Since
\[
\|\varphi(y + \varepsilon h) - \varphi(y)\| \leq \|D_E \varphi\| \|U(t,s)\|_E \leq \|D_E \varphi\| \|U(t,s)\|_E \|h\|_E,
\]
and by the Dominated Convergence Theorem, as \(\varepsilon \to 0^+\), we get
\[
D_E P_{s,t} \varphi(x)(h) = P_{s,t} D_E \varphi(\cdot)(U(t,s)h).
\]
Moreover
\[
|D_E P_{s,t} \varphi(x)(h)| \leq \int_X |D_E \varphi(y + \varepsilon h)| |U(t,s)h| N_{0,Q(t,s)}(dy) \leq \|D_E \varphi\| \|U(t,s)\|_{\mathcal{L}(E)} \|h\|_E,
\]
and
\[
\|D_E P_{s,t} \varphi(x)\|_{E'} \leq \|D_E \varphi\| \|U(t,s)\|_{\mathcal{L}(E)}.
\]
Hence
\[
\|D_E P_{s,t} \varphi\|_{\mathcal{L}(E)} \leq \|D_E \varphi\| \|U(t,s)\|_{\mathcal{L}(E)} \leq M \|\varphi\|_{C^k_E(X)}.
\]
We assume now \(\varphi \in C^k_E(X)\) and that \((3.1)\) and \((3.3)\) hold. Let \(x \in X\), \(h_1, \ldots, h_{k+1} \in E\) and \(\varepsilon > 0\), then for all \((s,t) \in \Delta\), we have
\[
P_{s,t}(D^k_E \varphi(\cdot)(U(t,s)h_1,\ldots,U(t,s)h_k)) (x + \varepsilon h_{k+1}) - P_{s,t}(D^k_E \varphi(\cdot)(U(t,s)h_1,\ldots,U(t,s)h_k))(x)
\]
\[
\leq \int_X \left| \frac{D^k_E \varphi(y + \varepsilon h_{k+1})}{\varepsilon} (U(t,s)h_1,\ldots,U(t,s)h_k) - D^k_E \varphi(y + \varepsilon h_{k+1}) (U(t,s)h_1,\ldots,U(t,s)h_k) \right| N_{0,Q(t,s)}(dy).
\]
Since
\[
\left| \frac{D^k_E \varphi(y + \varepsilon h_{k+1})}{\varepsilon} - D^k_E \varphi(y + \varepsilon h_{k+1}) \right| \leq \|D^k_E \varphi\| \|U(t,s)h_1,\ldots,U(t,s)h_k\|_{E'} \leq \|D^k_E \varphi\| \|U(t,s)h_1,\ldots,U(t,s)h_k\|_E
\]
\[
\leq \|D^k_E \varphi\| \prod_{i=1}^{k+1} \|U(t,s)h_{k+1}\|_{E'} \leq \|D^k_E \varphi\| \|U(t,s)\|_{E'}^k \prod_{i=1}^{k+1} \|h_i\|_E,
\]
and
by the Dominated Convergence Theorem, as \( \varepsilon \to 0^+ \), we get
\[
D_E^{k+1}(P_{s,t}\varphi)(x)(h_1, ..., h_{k+1}) = P_{s,t}(D_E^{k+1}\varphi(\cdot)(U(t,s)h_1, ..., U(t,s)h_{k+1}))(x).
\]
Moreover
\[
|D_E^{k+1}(P_{s,t}\varphi)(x)(h_1, ..., h_{k+1})| \leq \int_X |D_E^{k+1}\varphi(y + m^\varepsilon(t,s))(U(t,s)h_1, ..., U(t,s)h_{k+1})| \, N_0Q(t,s)(dy)
\]
\[
\leq \|D_E^{k+1}\varphi\|_\infty \|U(t,s)\| L(E) \prod_{i=1}^{k+1} \|h_i\|_E,
\]
and
\[
\|D_E^{k+1}(P_{s,t}\varphi)(x)\| L(E) \leq \|D_E^{k+1}\varphi\|_\infty \|U(t,s)\| L(E).
\]
Hence
\[
\|D_E^{k+1}(P_{s,t}\varphi)\|_\infty \leq \|U(t,s)\| L(E) \|D_E^{k+1}\varphi\|_\infty \leq M_k \|\varphi\| C_{E}^{k+1}(X).
\]

The proofs of the continuity properties are analogous to [CL21, Lem. 2.3] and (3.6), (3.7) are a consequence of (3.1) and of the definition of \( P_{s,t} \).

**Corollary 3.2.** Under Hypotheses 2.1 and 2.4, for every \( \alpha > 0 \) and for every \( \varphi \in C_E^\alpha(X) \), \( t \in (0,T] \), the function
\[
(s,x) \in [0,t] \times X \mapsto u_0(s,x) := P_{s,t}\varphi(x) \in \mathbb{R},
\]
belongs to \( C_E^{0,\alpha}([0,t] \times X) \), and there exists \( C = C(\alpha, T) > 0 \) such that
\[
\|u_0\| C_E^{0,\alpha}([0,t] \times X) \leq C \|\varphi\| C_E^\alpha(X).
\]
Similarly, if \( \varphi \in Z_E^k(X) \) for some \( k \in \mathbb{N} \), then the function \( u_0 \) belongs to \( Z_E^{0,k}([0,t] \times X) \), and there exists \( C = C(k, T) > 0 \) such that
\[
\|u_0\| Z_E^{0,k}([0,t] \times X) \leq C \|\varphi\| Z_E^k(X).
\]

**Theorem 3.3.** Under the hypotheses 2.1,2.5, \( P_{s,t}\varphi(x) \in \bigcap_{k \in \mathbb{N}} C_E^k(X) \) for every \( \varphi \in C_b(X) \) and all \( (s,t) \in \Delta \). In particular
\[
D_E(P_{s,t}\varphi)(x)(h) = \int_X \varphi(y + m^x(t,s)) U(t,s)h(y) \, N_0Q(t,s)(dy), \ h \in E
\]
and there exists \( C > 0 \) such that
\[
\sup_{x \in X} \|D_E(P_{s,t}\varphi)(x)\| E \leq C \|\varphi\| L(E;X) \|\varphi\|_\infty, \ \text{for any} \ (s,t) \in \Delta.
\]
Moreover for \( n \geq 2 \)
\[
D_E^n(P_{s,t}\varphi)(x)(h_1, ..., h_n) = \int_X \varphi(y + m^x(t,s)) I_n(t,s)(y)(h_1, ..., h_n) N_0Q(t,s)(dy), \ h_1, ..., h_n \in E,
\]
where
\[ I_n(t, s)(y)(h_1, ..., h_n) := \prod_{i=1}^{n} U(t, s) h_i(y) \]
\begin{equation}
(3.21) \quad + \sum_{s=1}^{r_n} \sum_{i_1 \ldots i_{2s} \in \mathcal{T}, \ i_{2k-1} < i_{2k}} \prod_{i=1}^{s} (\Lambda(t, s) h_{i_{2k-1}}, \Lambda(t, s) h_{i_{2k}}) X \prod_{i_{m}=1, \ i_{m} \neq i_{1} \ldots i_{2s}}^{n} U(t, s) h_{i_{m}}(y) \tag{3.21}
\end{equation}

and
\[ r_n = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even}, \\
\frac{n-1}{2} & \text{if } n \text{ is odd}.
\end{cases} \]

In particular, for every \( n \geq 2 \) there exists \( C_n > 0 \) such that
\begin{equation}
(3.22) \quad \sup_{x \in X} \|D_E^m(P_{s, t} \varphi)(x)\|_{L^n(E)} \leq C_n \|\Lambda(t, s)\|^{\frac{n}{2}}_{L^n(E; X)} \|\varphi\|_{\infty}, \text{ for any } (s, t) \in \Delta.
\end{equation}

Proof.

Step 1. We prove that \( P_{s, t} \varphi(x) \in C_b(X) \) for every \( \varphi \in C_b(X) \) and \( (s, t) \in \Delta \). For \( x, x_0 \in X \) we have
\begin{equation}
(3.23) \quad |P_{s, t} \varphi(x) - P_{s, t} \varphi(x_0)| \leq \int_X |\varphi(y + m^\tau(t, s)) - \varphi(y + m^\tau_0(t, s))| N_{0, Q_{1, s}}(dy).
\end{equation}
Since
\[ |\varphi(y + m^\tau(t, s)) - \varphi(y + m^\tau_0(t, s))| \leq 2 \|\varphi\|_{\infty}, \]
the statement follows as \( x \rightarrow x_0 \), by the Dominated Convergence theorem.

Step 2. We prove that \( P_{s, t} \varphi(x) \in C^1_b(X) \) for every \( \varphi \in C_b(X) \) and \( (s, t) \in \Delta \) and that (3.18) holds.
For every \( \varepsilon \in (0, 1) \), \( x \in X \) and \( h \in E \) we have
\[ \frac{P_{s, t} \varphi(x + \varepsilon h) - P_{s, t} \varphi(x)}{\varepsilon} = \frac{1}{\varepsilon} \left( \int_X \varphi(y + m^\tau(t, s)) N_{\varepsilon U(t, s) h, Q_{1, s}}(dy) \right. \]
\[ - \left. \int_X \varphi(y + m^\tau_0(t, s)) N_{0, Q_{1, s}}(dy) \right) . \]
Since \( \varepsilon U(t, s) h \in \mathcal{H}_{t, s} \), thanks to the Cameron-Martin formula we get
\[ N_{\varepsilon U(t, s) h, Q_{1, s}}(dy) = \exp\left( -\frac{1}{2} \|\varepsilon U(t, s) h\|^2_{\mathcal{H}_{t, s}} + \varepsilon \overline{U(t, s) h}(y) \right) N_{0, Q_{1, s}}(dy) \]
\[ = \exp\left( -\frac{1}{2} Q^{-\frac{1}{2}}(t, s) \varepsilon U(t, s) h, Q^{-\frac{1}{2}}(t, s) \varepsilon U(t, s) h \right)_X + \varepsilon \overline{U(t, s) h}(y) N_{0, Q_{1, s}}(dy) \]
\[ = \exp\left( -\frac{1}{2} \varepsilon^2 \|\Lambda(t, s) h\|^2_{X} + \varepsilon \overline{U(t, s) h}(y) \right) N_{0, Q_{1, s}}(dy). \]
Therefore, setting
\begin{equation}
(3.24) \quad f_\varepsilon(y) = -\frac{1}{2} \varepsilon^2 \|\Lambda(t, s) h\|^2_{X} + \varepsilon \overline{U(t, s) h}(y),
\end{equation}
we get
\[ \frac{P_{s,t}\varphi(x + \varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} = \int_X \frac{\exp(\varepsilon f_c(y)) - 1}{\varepsilon} \varphi(y + m^x(t, s)) N_{0,Q(t,s)}(dy). \]
Now
\[ \lim_{\varepsilon \to 0} f_c(y) = U(t,s)h(y) \text{ for a.e. } y \]
\[ \left| \frac{\exp(\varepsilon f_c(y)) - 1}{\varepsilon} \right| \leq C \left( \exp|f_c(y)| + 1 \right) \]
\[ y \mapsto U(t,s)h(y) \exp(U(t,s)h(y)) \in L^1(X, N_{0,Q(t,s)}). \]
Hence by the Dominated Convergence theorem we obtain
\[ (3.25) \lim_{\varepsilon \to 0} \frac{P_{s,t}\varphi(x + \varepsilon h) - P_{s,t}\varphi(x)}{\varepsilon} = \int_X \varphi(y + m^x(t, s)) U(t,s)h(y) N_{0,Q(t,s)}(dy). \]
Denoting by \( D_E^G(P_{s,t}\varphi)(x)(h) \) the right hand side of (3.25) we get by (2.11)
\[ |D_E^G(P_{s,t}\varphi)(x)(h)| \leq \|\varphi\|_{\infty} \left\| \frac{U(t,s)h(y)}{L^1(X, N_{0,Q(t,s)})} \right\| \leq \|\varphi\|_{\infty} c_1 \|\Lambda(t,s)\|_{L(E;X)} \|h\|_E \]
so that \( D_E^G(P_{s,t}\varphi)(x) \in E' \). This implies that \( P_{s,t}\varphi \) is \( E \)-Gâteaux differentiable at \( x \) and that \( D_E^G(P_{s,t}\varphi)(x) \) is its Gâteaux derivative.
To conclude we prove that \( D_E^G(P_{s,t}\varphi) : X \to E' \) is continuous. Indeed for \( x, x_0 \in X \) we have
\[ |D_E^G(P_{s,t}\varphi)(x)(h) - D_E^G(P_{s,t}\varphi)(x_0)(h)| \]
\[ \leq \int_X |\varphi(y + m^x(t, s)) - \varphi(y + m^{x_0}(t, s))| \left\| \frac{U(t,s)h(y)}{N(0,Q(t,s))} \right\|_{L^1(X, N_{0,Q(t,s)})} \|h\|_X \]
\[ \leq c_2 \|\varphi(y + m^x(t, s)) - \varphi(y + m^{x_0}(t, s))\|_{L^2(X, N_{0,Q(t,s)})} \|h\|_X \]
\[ \leq c_3 \|\varphi(y + m^x(t, s)) - \varphi(y + m^{x_0}(t, s))\|_{L^2(X, N_{0,Q(t,s)})} \|h\|_E. \]
Hence
\[ \|D_E^G(P_{s,t}\varphi)(x) - D_E^G(P_{s,t}\varphi)(x_0)\|_{E'} \leq c_3 \|\varphi(y + m^x(t, s)) - \varphi(y + m^{x_0}(t, s))\|_{L^2(X, N_{0,Q(t,s)})} \]
and since \( |\varphi(y + m^x(t, s)) - \varphi(y + m^{x_0}(t, s))| \leq 2 \|\varphi\|_{\infty} \), for \( x \to x_0 \) we have the statement by the Dominated Convergence theorem.
Step 3. We prove that \( P_{s,t}\varphi \in C_E^2(X) \). We show first that the for every \((s, t) \in \Delta \) and \( h \in E \), the mapping
\[ D_E(P_{s,t}\varphi)(\cdot)(h) : X \to \mathbb{R} \]
is \( E \)-Gâteau differentiable. Setting
\[ f_c(y) = -\frac{1}{2} \left\| \frac{\Lambda(t,s)h}{X} + \frac{U(t,s)h}{Y} \right\|_X^2, \]
due to (3.18) and using again the Cameron-Martin formula, for every \( \varepsilon > 0 \) and \( \tilde{h} \in E \) we have

\[
\frac{D_E(P_{s,t} \varphi)(x + \varepsilon \tilde{h}) - D_E(P_{s,t} \varphi)(x)}{\varepsilon} = \frac{1}{\varepsilon} \left[ \int_X \varphi(y + m^x(t,s)) U(t,s) \tilde{h}(y - \varepsilon U(t,s) \tilde{h}) N_{\varepsilon U(t,s) \tilde{h}, Q(t,s)}(dy) - \int_X \varphi(y + m^x(t,s)) U(t,s) \tilde{h}(y) N_{0, Q(t,s)}(dy) \right]
\]

\[
= \frac{1}{\varepsilon} \int_X \varphi(y + m^x(t,s)) U(t,s) \tilde{h}(y) \left( \exp(\varepsilon f_\varepsilon(x)) - 1 \right) N_{0, Q(t,s)}(dy)
\]

\[
- \int_X \varphi(y + m^x(t,s)) \langle \Lambda(t,s), \Lambda(t,s) \tilde{h} \rangle_X \exp(\varepsilon f_\varepsilon(y)) N_{0, Q(t,s)}(dy).
\]

Proceeding as in Step 2, we obtain

\[
\lim_{\varepsilon \to 0} \frac{D_E(P_{s,t} \varphi)(x + \varepsilon \tilde{h}) - D_E(P_{s,t} \varphi)(x)}{\varepsilon}
= \int_X \varphi(y + m^x(t,s)) U(t,s) \tilde{h}(y) U(t,s) \tilde{h}(y) N_{0, Q(t,s)}(dy)
\]

(3.26)

\[- \langle \Lambda(t,s), \Lambda(t,s) \tilde{h} \rangle_X P_{s,t} \varphi(x).\]

The right-hand side of (3.26) is the Gâteaux derivative of \( D_E P_{s,t} \varphi(\cdot) \tilde{h} \) at \( x \). In the same way we did in Step 2, we can show that the mapping \( D_E^0(D_E P_{s,t} \varphi(\cdot) \tilde{h}) : X \to E' \) is continuous, so that we conclude that \( P_{s,t} \varphi \in C_E^0(X) \).

Step 4. We prove that \( P_{s,t} \varphi \in C_E^0(X) \) and that (3.20) holds. We proceed by induction and we assume that \( P_{s,t} \varphi \in C_E^n(X) \) and that formula (3.20) holds for \( D_E^n(P_{s,t} \varphi) \).

We first show that \( D_E^{n-1}(P_{s,t} \varphi) \) is \( E \)-Gâteaux differentiable. Let \( x \in X, h_1, \ldots, h_{n-1}, h_n \in E \) and \( \varepsilon > 0 \), we set

\[
f_\varepsilon(y) = -\frac{1}{2\varepsilon} \left\| \Lambda(t,s) h_n \right\|_X^2 + \left\| U(t,s) h_n \right\|_X.
\]

and due to the Cameron-Martin Formula we have

\[
D_E^{n-1}(P_{s,t} \varphi)(x + \varepsilon h_n)(h_1, \ldots, h_{n-1})
= \int_X \varphi(m^x(t,s) + y) I_{n-1}(t,s) (y - \varepsilon U(t,s) h_n) \exp(\varepsilon f_\varepsilon(y)) N_{0, Q(t,s)}(dy).
\]
Moreover we have

\begin{equation*}
I_{n-1}(t, s)(y - \varepsilon U(t, s)h_n)(h_1, ..., h_{n-1}) = I_{n-1}(t, s)(y)(h_1, ..., h_{n-1})
- \varepsilon \sum_{i=1}^{n-1} \langle \Lambda(t, s)h_i, \Lambda(t, s)h_n \rangle_x \prod_{j=1, j \neq i}^{n-1} U(t, s)h_j(y)
- \varepsilon \sum_{s=1}^{r_{n-1}} (-1)^s \sum_{i_1, ..., i_{2s}, i_{2k-1} < i_{2k}, i_{2k-1} < i_{2k+1}} s \prod_{i=1}^{s} \langle \Lambda(t, s)h_{i_{2k-1}}, \Lambda(t, s)h_{i_{2k}} \rangle_x
\times \prod_{i_j=1, i_j \neq i_m, i_1, ..., i_{2s}}^{n-1} U(t, s)h_{i_j}(y) + O(\varepsilon^2).
\end{equation*}

In the same way we did in Step 2 and in Step 3, by the Dominated Convergence theorem we have

\begin{equation*}
\lim_{\varepsilon \to 0} \frac{D_{E}^{n-1}(P_{s,t}\varphi)(x + \varepsilon h_n)(h_1, ..., h_{n-1}) - D_{E}^{n-1}(P_{s,t}\varphi)(x)(h_1, ..., h_{n-1})}{\varepsilon} =
\int_X \varphi(m^x(t, s) + y)I_{n-1}(t, s)(y)(h_1, ..., h_{n-1}) U(t, s)h_n(y) N_0,Q(t,s)(dy) +
- \int_X \varphi(m^x(t, s) + y) \sum_{i=1}^{n-1} \langle \Lambda(t, s)h_i, \Lambda(t, s)h_n \rangle_x \prod_{j=1, j \neq i}^{n-1} U(t, s)h_j(y) +
+ \sum_{s=1}^{r_{n-1}} (-1)^s \sum_{i_1, ..., i_{2s}, i_{2k-1} < i_{2k}, i_{2k-1} < i_{2k+1}} s \prod_{i=1}^{s} \langle \Lambda(t, s)h_{i_{2k-1}}, \Lambda(t, s)h_{i_{2k}} \rangle_x \times \prod_{i_j=1, i_j \neq i_m, i_1, ..., i_{2s}}^{n-1} U(t, s)h_{i_j}(y) \right] N_0,Q(t,s)(dy).
\end{equation*}

It is easy to show that the right hand side of the equation above coincides with the expression of \( D_{E}^{k}(P_{s,t}\varphi)(h_1, ..., h_n) \) given in (3.20). The same arguments of Step 2 and Step 3 imply that \( P_{s,t}\varphi \in C^k_E(X) \).

Now we combine Lemma 3.1 and Theorem 3.3 to obtain the following result.

**Corollary 3.4.** Under Hypotheses 2.1 and 2.9, for every \( k, n \in \mathbb{N} \cup \{0\} \) such that \( k + n \geq 1 \), \( \varphi \in C^k_E(X) \) and \( (s, t) \in \Delta \), \( P_{s,t}\varphi \in C^{k+n}_E(X) \) and

\begin{equation*}
D_{E}^{k+n}(P_{s,t}\varphi)(x)(h_1, ..., h_{k+n}) = \int_X D_{E}^{k}(m^x(t, s) + y)(U(t, s)h_1, ..., U(t, s)h_k) I_n(t, s)(y)(h_{k+1}, ..., h_{k+n}) N_0,Q(t,s)(dy).
\end{equation*}
For every $\alpha_2 \geq \alpha_1 \geq 0$ there exists $C = C(\alpha_1, \alpha_2)$ such that
\begin{equation}
\|P_{s,t}\varphi\|_{c_{E}^{\alpha_2}(X)} \leq C(\|\Lambda(t,s)\|_{L(E;X)}^{\alpha_2} + 1) \|\varphi\|_{c_{E}^{\alpha_1}(X)}, \quad \varphi \in C_{E}^{\alpha_2}(X), \quad (s, t) \in \Delta.
\end{equation}

Moreover for every $j \in \mathbb{N}, h_1, \ldots, h_j \in E, \ t \in [0, T]$ and $\varphi \in C_{E}(X)$ the mapping
\[(s, x) \in [0, t) \times X \mapsto D_{E}^{j}P_{s,t}\varphi(x)(h_1, \ldots, h_j) \in \mathbb{R}\]
is continuous.

If in addition the mapping $(s, t) \in \Delta \mapsto U(t, s) \in L(X)$ is continuous, then the function
\begin{equation}
(s, t, x) \in \Delta \times X \mapsto D_{E}^{j}(P_{s,t}\varphi)(x)(h_1, \ldots, h_j) \in \mathbb{R},
\end{equation}
is continuous.

\textbf{Proof.} Formula (3.27) follows applying (3.1) and (3.20). Moreover, by (3.3) and (3.22) there exist $C = C(k, n) > 0$ such that
\begin{equation}
\sup_{x \in X} \|D_{E}^{n}(P_{s,t}\varphi)(x)\|_{L^{n+1}(E)} \leq C \|\Lambda(t, s)\|_{L(E;X)}^{n} \|\varphi\|_{c_{E}^{n}(X)}, \quad (s, t) \in \Delta, \ \varphi \in C_{E}^{k}(X),
\end{equation}
and for every $\alpha \in (0, 1)$
\begin{equation}
\|D_{E}^{k+n}(P_{s,t}\varphi)\|_{c_{E}^{n}(X;L^{n+k}(E))} \leq C \|\Lambda(t, s)\|_{P}^{n} \|\varphi\|_{c_{E}^{n}(X)}, \quad (s, t) \in \Delta, \ \varphi \in C_{E}^{k+n}(X).
\end{equation}

We point out that if $\alpha_1 = \alpha_2$, then (3.28) follows from Lemma 3.1; so we may assume $\alpha_2 > \alpha_1$. Now if $\alpha_2 - \alpha_1 = n \in \mathbb{N}$, we set $\alpha_1 = m + \sigma$, where $m \in \mathbb{N}$ and $\sigma \in (0, 1)$. We have
\begin{align*}
\|P_{s,t}\varphi\|_{c_{E}^{\alpha_2}(X)} & = \|P_{s,t}\varphi\|_{c_{E}^{\alpha_1}(X)} + \sup_{x \in X} \|D_{E}^{j}(P_{s,t}\varphi)(x)\|_{L^{n}(E)} + \|D_{E}^{m+n}(P_{s,t}\varphi)\|_{c_{E}^{n}(X)} \\
& \leq \|\varphi\|_{c_{E}^{\alpha_1}(X)} + \sum_{j=1}^{m+n} \sup_{x \in X} \|D_{E}^{j}(P_{s,t}\varphi)(x)\|_{L^{n}(E)} + \sup_{x \in X} \|\Lambda(t, s)\|_{L(E;X)}^{n} \|\varphi\|_{c_{E}^{n}(X)} + \\
& + \|\Lambda(t, s)\|_{L(E;X)}^{n+1} \|D_{E}^{n}(P_{s,t}\varphi)\|_{c_{E}^{n}(X)} \\
& \leq \|\varphi\|_{c_{E}^{\alpha_1}(X)} + \sum_{j=1}^{m+n} \sup_{x \in X} \|D_{E}^{j}(\varphi)(x)\|_{L^{n}(E)} + n(1 + \|\Lambda(t, s)\|_{L(E;X)}^{n}) \sup_{x \in X} \|D_{E}^{m+n}(\varphi)(x)\|_{L^{n}(E)} + \\
& + \|\Lambda(t, s)\|_{L(E;X)}^{n+1} \|D_{E}^{m+n}(\varphi)\|_{c_{E}^{n}(X)} \leq 2n(1 + \|\Lambda(t, s)\|_{L(E;X)}^{n}) \|\varphi\|_{c_{E}^{n}(X)}.
\end{align*}

If $\alpha_2 - \alpha_1 \notin \mathbb{N}$, we set $\alpha_2 = \alpha_1 + n + \sigma$ with $n \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0, 1)$. We apply the following interpolation inequality
\begin{equation}
\|\psi\|_{c_{E}^{\alpha_2}(X)} \leq \|\psi\|_{c_{E}^{\alpha_1+n}(X)}^{1-\sigma} \|\psi\|_{c_{E}^{\alpha_1+n+1}(X)}^{\sigma}, \quad \psi \in C_{E}^{\alpha_1+n+1}(X)
\end{equation}
to $\psi = P_{s,t}\varphi$. Then, due to (3.28) with $\alpha_1$ replaced by $\alpha_1 + n$ and $\alpha_2$ by $\alpha_1 + n + 1$, we have (3.28) in the general case.

The statement about the continuity of the derivatives is a consequence of Lemma 3.1 and Theorem 3.3. Indeed for a fixed $t > 0$ and $\varepsilon \in (0, t)$, we have
\[P_{s,t}\varphi = P_{s,t-\varepsilon}P_{t-\varepsilon,t}, \quad 0 \leq s - t - \varepsilon \leq T.\]
Since $\psi = P_{t-\varepsilon,t}\varphi \in C_E^k(X)$ by Theorem 3.3, the function
\[(s,x) \in [0,t-\varepsilon] \times X \mapsto D^k_E P_{s,t-\varepsilon}\psi(h_1,\ldots,h_j) = D^k_E P_{s,t}\varphi(h)(h_1,\ldots,h_j)\]
is continuous by Lemma 3.1. The proof of the last claim is similar. \(\square\)

**Remark 3.5.** The interpolation inequality (3.32) is proved in [CL21, Prop. 2.8] in the case $E = X$ with equivalent norms. Anyway the proof of the general case is analogous.

In the following propositions we give some examples of Hilbert spaces satisfying Hypothesis 2.4. We start with two preliminary results.

**Proposition 3.6.** Let $X, X_1, X_2$ be Hilbert spaces and $L_1 : X_1 \to X$, $L_2 : X_2 \to X$ be linear bounded operators. The following statements hold.

1. $\text{Range}(L_1) \subseteq \text{Range}(L_2)$ if and only if there exists a constant $C > 0$ such that
   \begin{equation}
   \|L_1^* x\|_{X_1} \leq C \|L_2^* x\|_{X_2}, \quad x \in X.
   \end{equation}
   In this case $\|L_2^{-1} L_1\|_{\mathcal{L}(X_1,X_2)} \leq C$; more precisely
   \begin{equation}
   \|L_2^{-1} L_1\|_{\mathcal{L}(X_1,X_2)} = \inf \{ C > 0 \text{ s.t. (3.33) holds} \}
   \end{equation}
2. If $\|L_1^* x\|_{X_1} = \|L_2^* x\|_{X_2}$ for every $x \in X$ then $\text{Range}(L_1) = \text{Range}(L_2)$ and $\|L_1^{-1} x\|_{X_1} = \|L_2^{-1} x\|_{X_2}$ for every $x \in X$.

**Proof.** See Proposition B.1 in Appendix B in [DZ14, pag. 429]. \(\square\)

**Lemma 3.7.** The following properties hold.

1. the mapping $s \mapsto Q(t,s)$ is continuous with values in $\mathcal{L}(X)$ and it is decreasing, namely
   $$(Q(t,s_1)x,x)_X \geq (Q(t,s_2)x,x)_X \quad \text{for any} \quad 0 \leq s_1 \leq s_2 < t < T.$$ 
2. $\mathcal{H}_{t,s_2} \subseteq \mathcal{H}_{t,s_1}$ for every $0 \leq s_1 < s_2 < t \leq T$ and the norm of the embedding is 1.
3. $\text{Ker}(Q(t,s_1)) \subseteq \text{Ker}(Q(t,s_2)) \subseteq \text{Ker}(Q(t))$ for any $0 \leq s_1 \leq s_2 < t < T$.

**Proof.**

1. We prove that the mapping $s \mapsto Q(t,s)$ is Lipschitz continuous. Let $0 \leq s_1 \leq s_2 < t < T$ and $x \in X$, we have
\[
\|Q(t,s_1)x - Q(t,s_2)x\|_X = \left\| \int_{s_1}^{s_2} U(t,r)Q(r) U^*(t,r)x dr \right\|_X \leq M^2 K^2 \|x\|_X |s_2 - s_1|
\]
and we obtain
\[
\|Q(t,s_1) - Q(t,s_2)\|_{\mathcal{L}(X)} \leq M^2 K^2 |s_2 - s_1|,
\]
where $M$ and $K$ are the constants defined in hypothesis 2.1 and in (2.12). Moreover

$$
\langle Q(t, s_2)x, x \rangle_X = \left\langle \int_{s_2}^t U(t, r)Q(r)U^*(t, r)x \, dr, x \right\rangle_X = \\
= \int_{s_2}^t \left\langle U(t, r)Q^\downarrow(r)Q^\downarrow(r)U^*(t, r)x, x \right\rangle_X \, dr = \\
= \int_{s_2}^t \left\| Q^\downarrow(r)U^*(t, r)x \right\|^2_X \, dr \leq \\
\leq \int_{s_2}^t \left\| Q^\downarrow(r)U^*(t, r)x \right\|^2_X \, dr = \langle Q(t, s_1)x, x \rangle_X.
$$

2. The continuous embedding $\mathcal{H}_{t, s_2} \subseteq \mathcal{H}_{t, s_1}$ is an immediate consequence of proposition 3.6 and statement 1, observing that $\left\| Q^\downarrow(t, s)x \right\|_X = \langle Q(t, s)x, x \rangle_X$.

3. Let $x \in \text{Ker}(Q(t, s_1))$. Since

$$
\langle Q(t, s_1)x, x \rangle_X = \int_{s_1}^t \left\| Q^\downarrow(r)U^*(t, r)x \right\|^2_X \, dr,
$$
we obtain $Q^\downarrow(r)U^*(t, r)x = 0$ for every $r \in (s_1, t)$ and in particular $x \in \text{Ker}(Q(t, s_2))$. Let $x \in \text{Ker}(Q(t, s_2))$, then $Q^\downarrow(r)U^*(t, r)x = 0$ for every $r \in (s_2, t)$. For $y \in X$ we have

$$
0 = \langle Q^\downarrow(r)U^*(t, r)x, y \rangle_X = \langle x, U(t, r)Q^\downarrow(r)y \rangle_X, \quad s_2 < r < t.
$$

For $r \to t^-$ we obtain

$$
(3.35) \quad 0 = \langle x, Q^\downarrow(t)y \rangle_X = \langle Q^\downarrow(t)x, y \rangle_X, \quad \text{for all } y \in X.
$$

Hence $Q^\downarrow(t)x = 0$ and since $\text{Ker}(Q(t)) = \text{Ker}(Q^\downarrow(t))$, the statement holds.

\[\square\]

**Proposition 3.8.** We assume that $U(t, s)(H_s) \subseteq H_t$ for any $(s, t) \in \Delta$ and that there exists $M > 0$ such that

$$
(3.36) \quad \left\| U(t, s) \right\|_{L(H_s, H_t)} \leq M, \quad (s, t) \in \Delta.
$$

Then the following statements hold.

1. For every $0 \leq s < t \leq T$, $U(t, s)(H_s) \subseteq \mathcal{H}_{t, s}$ and

$$
(3.37) \quad (t-s)^\frac{2}{p} \left\| U(t, s) \right\|_{L(H_s, H_t)} = (t-s)^\frac{2}{p} \left\| Q^\downarrow(t, s)U(t, s) \right\|_{L(H_s, H_t)} \leq M.
$$

2. For every $0 \leq s < t \leq T$, $U(t, s)^*(-\text{Ker}(Q(t))) \subseteq \text{Ker}(Q(s))$ and $\text{Ker}(Q(t, s)) = \text{Ker}(Q(t))$.

3. For every $0 \leq s_1 < s_2 < t \leq T$, $\mathcal{H}_{t, s_1} = \mathcal{H}_{t, s_2}$ and their norms are equivalent.

4. For every $0 \leq s < t \leq T$, $\mathcal{H}_{t, s}$ is continuously embedded in $H_t$. 
Proof.

1. We consider the operator \( L \) from remark 2.8. We know that \( \text{Range}(L) = \text{Range}(Q^\frac{1}{2}(t,s)) \). On the other hand, for every \( x \in H_s \) we have

\[
(t-s)U(t,s)x = \int_s^t U(t,s)x \, d\sigma = \int_s^t U(t,\sigma)U(\sigma,s)x \, d\sigma = \\
= \int_s^t U(t,\sigma)Q^\frac{1}{2}(\sigma)Q^{-\frac{1}{2}}(\sigma)U(\sigma,s)x \, d\sigma.
\]

where \( y(\sigma) = Q^{-\frac{1}{2}}(\sigma)U(\sigma,s)x \) belongs to \( L^\infty((s,t);X) \supseteq L^2((s,t);X) \). Hence \( U(t,s)x \) belongs to the range of \( L \), and by (2.16) we get

\[
\left\|Q^{-\frac{1}{2}}(t,s)((t-s)U(t,s)x)\right\|_X \leq \|y\|_{L^2((s,t);X)} \leq (t-s)^{\frac{1}{2}} \sup_{s<\sigma<t} \left\|Q^{-\frac{1}{2}}(\sigma)U(\sigma,s)x\right\|_X \leq (t-s)^{\frac{1}{2}}M \|x\|_{H_s}.
\]

2. We first remark some facts.

(a) Since \( U(t,s)(H_s) \subseteq H_s \), by continuity we get \( U(t,s)(\overline{H_s}) \subseteq \overline{H_s} \).

(b) \( \text{Ker}(Q^\frac{1}{2}(t)) = \text{Ker}(Q(t)) \), \( \text{Ker}(Q(t)) \perp = \overline{H_t} \) and consequently \( \overline{H_t} = (I-P_t)(X) \), where we denote by \( P_t \) the orthogonal projection on \( \text{Ker}(Q(t)) \).

(c) By statements (a) and (b) we have \( U(t,s)^*(\text{Ker}(Q(t))) \subseteq \text{Ker}(Q(s)) \). In fact

\[
U(t,s)(\overline{H_s}) \subseteq \overline{H_t} \iff U(t,s)(I-P_s)(X) \subseteq (I-P_t)(X) \iff P_tU(t,s)(I-P_s) = 0.
\]

Hence

\[
(I-P_s)^*U^*(t,s)P_t^* = (I-P_s)U^*(t,s)P_t = 0,
\]

namely \( U(t,s)^*(\text{Ker}(Q(t))) \subseteq \text{Ker}(Q(s)) \).

For every \( x \in \text{Ker}(Q(t)) \), since

\[
\left\|Q^\frac{1}{2}(t,s)x\right\|_X^2 = \langle Q(t,s)x,x \rangle_X = \int_s^t \left\|Q^\frac{1}{2}(r)U^*(t,r)x\right\|_X^2 \, dr,
\]

we obtain \( x \in \text{Ker}(Q(t,s)) \). Conversely, the embedding of \( \text{Ker}(Q(t,s)) \) in \( \text{Ker}(Q(t)) \) follows from statement 3 of lemma 3.7.

3. The continuous embedding \( \mathfrak{K}_{s_2} \subseteq \mathfrak{K}_{s_1} \) is statement 2 of lemma 3.7. Concerning the reverse embedding, we first point out that the adjoint of the operator \( U(t,s)|_{H_s} : H_s \rightarrow H_t \) is the operator \( (U(t,s)|_{H_s})^* : H_t \rightarrow H_s \) such that

\[
\langle x,(U(t,s)|_{H_s})^*y \rangle_{H_s} = \langle (U(t,s)|_{H_s}x,y \rangle_{H_s}, \quad \text{for all} \ x \in H_s, \ y \in H_t.
\]

Now we claim that

\[
(U(t,s)|_{H_s})^*Q(t)x = Q(s)U^*(t,s)x, \quad x \in X, \ (s,t) \in \Delta.
\]
where \( U^*(t, s) \) denotes the adjoint operator of \( U(t, s) \in \mathcal{L}(X) \). Indeed for all \( h \in H_x \) we have that \((I - P_s)h = h\) and

\[
\langle (U(t, s))_{|H_x})^*Q(t)x, h \rangle_{H_x} = \langle Q(t)x, U(t, s)|_{H_x}h \rangle_{H_x} = \langle x, U(t, s)h \rangle_X = \langle U^*(t, s)x, h \rangle_X = \langle Q(s)U^*(t, s)x, h \rangle_{H_x} = \langle Q(s)U^*(t, s)x, h \rangle_{H_x},
\]

where the last equality follows from statement 2. Hence we get for \( 0 \leq s_1 < s_2 < t \leq T \)

\[
\left\| Q^\frac{1}{2}(t, s_1)x \right\|_X^2 = \int_{s_1}^{t} \left\| Q^\frac{1}{2}(r)U^*(t, r)x \right\|_X^2 \, dr
\]

\[
= \int_{s_1}^{s_2} \left\| Q^\frac{1}{2}(r)U^*(t, r)x \right\|_X^2 \, dr + \int_{s_2}^{t} \left\| Q^\frac{1}{2}(r)U^*(t, r)x \right\|_X^2 \, dr
\]

\[
= \int_{s_2}^{2s_2 - s_1} \left\| Q^\frac{1}{2}(\sigma_0 + s_1 - s_2)U^*(t, \sigma_0 + s_1 - s_2)x \right\|_X^2 \, d\sigma_0 + \left\| Q^\frac{1}{2}(t, s_2)x \right\|_X^2,
\]

where \( \sigma_0 = r - s_1 + s_2 \) in the first integral.

If \( t > 2s_2 - s_1 \) by (3.40), we have

\[
\int_{s_2}^{2s_2 - s_1} \left\| Q^\frac{1}{2}(\sigma_0 + s_1 - s_2)U^*(t, \sigma_0 + s_1 - s_2)x \right\|_X^2 \, d\sigma_0
\]

\[
\leq \int_{s_2}^{t} \left\| Q^\frac{1}{2}(\sigma_0 + s_1 - s_2)U^*(t, \sigma_0 + s_1 - s_2)x \right\|_X^2 \, d\sigma_0
\]

\[
= \int_{s_2}^{t} \left\| Q(\sigma_0 + s_1 - s_2)U^*(\sigma_0, \sigma_0 + s_1 - s_2; t, \sigma_0)x \right\|_X^2 \, d\sigma_0
\]

\[
= \int_{s_2}^{t} \int_{E_{\sigma_0 + s_1 - s_2}} \left\| Q(\sigma_0)U^*(t, \sigma_0)x \right\|_X^2 \, d\sigma_0
\]

\[
= M^2 \int_{s_2}^{t} \left\| Q(\sigma_0)U^*(t, \sigma_0)x \right\|_X^2 \, d\sigma_0 = M^2 \left\| Q^\frac{1}{2}(t, s_2)x \right\|_X^2.
\]

Hence

\[
(3.41) \quad \left\| Q^\frac{1}{2}(t, s_1)x \right\|_X^2 \leq (1 + M^2) \left\| Q^\frac{1}{2}(t, s_2)x \right\|_X^2.
\]

If \( s_2 < t \leq 2s_2 - s_1 \), by (3.41) we have

\[
\int_{s_2}^{2s_2 - s_1} \left\| Q^\frac{1}{2}(\sigma_0 + s_1 - s_2)U^*(t, \sigma_0 + s_1 - s_2)x \right\|_X^2 \, d\sigma_0 =
\]

\[
= \int_{s_2}^{t} \int_{s_2}^{2s_2 - s_1} \left\| Q^\frac{1}{2}(\sigma_0 + s_1 - s_2)U^*(t, \sigma_0 + s_1 - s_2)x \right\|_X^2 \, d\sigma_0 \leq
\]

\[
\leq M^2 \left\| Q^\frac{1}{2}(t, s_2)x \right\|_X^2 +
\]

\[
+ \int_{s_2}^{3s_2 - s_1 - t} \left\| Q^\frac{1}{2}((\sigma_1 + t + s_1 - 2s_2)U^*(t, \sigma_1 + t + s_1 - 2s_2)x \right\|_X^2 \, d\sigma_1,
\]
where \( \sigma_1 = \sigma_0 - t + s_2 \).

If \( t > 3s_2 - s_1 - t \iff t > \frac{3s_2 - s_1}{2} \), hence for any \( t \in \left( \frac{3s_2 - s_1}{2}, 2s_2 - s_1 \right] \) we have

\[
\int_{s_2}^{3s_2 - s_1} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_0 \leq \int_{s_2}^{t} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_1
\]

and by (3.40)

\[
\int_{s_2}^{t} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_1 = \\
= \int_{s_2}^{t} d\sigma_1 \] \( \sigma_1 \)

\[
(3.42) \quad \left\| Q^\star(t_1, s_1) \right\|_X^2 \leq (1 + 2M^2) \left\| Q^\star(t_2, s_2) \right\|_X^2.
\]

If \( s_2 < t \leq \frac{3s_2 - s_1}{2} \)

\[
\int_{s_2}^{3s_2 - s_1} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_1 = \\
= \int_{s_2}^{t} + \int_{t}^{3s_2 - s_1} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_1 \leq \\
\leq \int_{s_2}^{3s_2 - s_1} \left\| Q^\star(t_1 + t + s_1 - 2s_2)U^\star(t, \sigma_1 + t + s_1 - 2s_2) \right\|_X^2 d\sigma_1 + \\
(3.43) + M^2 \left\| Q^\star(t_2, s_2) \right\|_X^2,
\]

where \( \sigma_2 = \sigma_1 - t + s_2 \).

Let \( k \in \mathbb{N} \). Now we prove the following.

- If \( t \in \left( \frac{(k+2)s_2 - s_1}{k+1}, \frac{(k+1)s_2 - s_1}{k} \right] \)

\[
(3.44) \quad \left\| Q^\star(t_1, s_1) \right\|_X^2 \leq \left[ 1 + (k + 1)M^2 \right] \left\| Q^\star(t_2, s_2) \right\|_X^2.
\]

- If \( t \in \left( s_2, \frac{(k+2)s_2 - s_1}{k+1} \right] \)

\[
\int_{s_2}^{s_k} \left\| Q^\star(t_k)U^\star(t, t_k) \right\|_X^2 d\sigma_k \leq \int_{s_2}^{s_k+1} \left\| Q^\star(t_k+1)U^\star(t, t_k+1) \right\|_X^2 d\sigma_k+1
\]

\[
(3.45) + M^2 \left\| Q^\star(t_2, s_2) \right\|_X^2,
\]
where

\[ s_k = \frac{(k + 2)s_2 - s_1 - kt}{k+1}, \]
\[ t_k = \frac{\sigma_k + kt + s_1 - (k + 1)s_2}{k+1}, \]
\[ \sigma_{k+1} = \sigma_k - t + s_2. \]

We proceed by induction over \( k \).

For \( k = 1 \) the (3.44) and (3.45) have been already proven in (3.42) and (3.43).

We assume now (3.44) and (3.45) hold for a given \( k \in \mathbb{N} \) and we prove that (3.44) and (3.45) hold for \( k + 1 \).

Let \( t \in \left( s_2, \frac{(k+2)s_2-s_1}{k+1} \right) \). If \( t > s_{k+1} \iff t > \frac{(k+3)s_2-s_1}{k+2} \), for any \( t \in \left( \frac{(k+3)s_2-s_1}{k+2}, \frac{(k+2)s_2-s_1}{k+1} \right) \) we have

\[
\int_{s_2}^{s_{k+1}} \left\| Q^\frac{1}{2}(t_{k+1})U^*(t, t_{k+1}) \right\|_X^2 \, d\sigma_{k+1} \leq \int_{s_2}^{t} \left\| Q^\frac{1}{2}(t_{k+1})U^*(t, t_{k+1}) \right\|_X^2 \, d\sigma_{k+1}
\]

and by (3.40)

\[
\int_{s_2}^{t} \left\| Q^\frac{1}{2}(t_{k+1})U^*(t, t_{k+1}) \right\|_X^2 \, d\sigma_{k+1} = \int_{s_2}^{t} \left\| Q(t_{k+1})U^*(t, t_{k+1}) \right\|_E_{t_{k+1}}^2 \, d\sigma_{k+1} =
\]

\[
= \int_{s_2}^{t} \left\| U(\sigma_{k+1}, t, t_{k+1})^{*}Q(\sigma_{k+1})U^*(t, \sigma_{k+1})x \right\|_E_{t_{k+1}}^2 \, d\sigma_{k+1} =
\]

\[
\leq M^2 \left\| Q^\frac{1}{2}(t, s_2) \right\|_X^2.
\]

Finally

\[
\left\| Q^\frac{1}{2}(t, s_1)x \right\|_X^2 \leq \left[ 1 + (k + 2)M^2 \right] \left\| Q^\frac{1}{2}(t, s_2) \right\|_X^2.
\]

Now if \( t \in \left( s_2, \frac{(k+3)s_2-s_1}{k+2} \right) \), then

\[
\int_{s_2}^{s_{k+1}} \left\| Q^\frac{1}{2}(t_{k+1})U^*(t, t_{k+1}) \right\|_X^2 \, d\sigma_{k+1} = \int_{s_2}^{t} + \int_{t}^{s_{k+1}} \left\| Q^\frac{1}{2}(t_{k+1})U^*(t, t_{k+1}) \right\|_X^2 \, d\sigma_{k+1}
\]

\[
\leq \int_{s_2}^{s_{k+2}} \left\| Q^\frac{1}{2}(t_{k+2})U^*(t, t_{k+2}) \right\|_X^2 \, d\sigma_{k+2} + M^2 \left\| Q^\frac{1}{2}(t, s_2) \right\|_X^2.
\]

Therefore we have obtained that

\[
\left\| Q^\frac{1}{2}(t, s_1)x \right\|_X^2 \leq \left( 1 + M^2 \right) \left\| Q^\frac{1}{2}(t, s_2) \right\|_X^2, \quad t \in (2s_2 - s_1, +\infty)
\]

and if \( k \in \mathbb{N} \setminus \{0\} \) and \( t \in \left( \frac{(k+2)s_2-s_1}{k+1}, \frac{(k+1)s_2-s_1}{k} \right) \), we have

\[
\left\| Q^\frac{1}{2}(t, s_1)x \right\|_X^2 \leq \left[ 1 + (k + 1)M^2 \right] \left\| Q^\frac{1}{2}(t, s_2) \right\|_X^2.
\]

4. For \( (s, t) \in \Delta \) and \( x \in X \) we use again (3.40) and we have

\[
\left\| Q^\frac{1}{2}(t, s)x \right\|_X = \int_{s}^{t} \left\| Q^\frac{1}{2}(r)U^*(t, r)x \right\|_X^2 \, dr = \int_{s}^{t} \left\| Q(r)U^*(t, r)x \right\|_E_{r}^2 \, dr =
\]

\[
= \int_{s}^{t} \left\| (U(t, r)|_{E_{r}})^*Q(t)x \right\|_E_{r}^2 \, dr \leq M^2 (t - s) \left\| Q^\frac{1}{2}(t)x \right\|_X^2.
\]
By proposition 3.6 we the statement follows. □

4 Schauder type theorems

In this section we assume that Hypotheses 2.1, 2.9, 2.10 hold and we prove maximal Hölder regularity for the mild solution of problem (1.2), that is given by the formula

\[ u(s, x) = P_{s, t} \varphi(x) - \int_{s}^{t} (P_{s, \sigma} \psi(\sigma, \cdot))(x) \, d\sigma, \quad (s, t) \in \Delta, \ x \in X, \]

where \( \varphi \in C_b(X) \) and \( \psi \in C_b([0, t] \times X) \). We argue as in the papers [LR21] and [CL21]. More precisely, we rewrite for the reader the proofs of Theorems 3.12 and 3.13 of [LR21] in this setting.

We denote

\[ u_0(s, x) = P_{s, t} \varphi(x), \quad (s, t) \in \Delta, \ x \in X \]
\[ u_1(s, x) = - \int_{s}^{t} (P_{s, \sigma} \psi(\sigma, \cdot))(x) \, d\sigma, \quad (s, t) \in \Delta, \ x \in X. \]

We have immediately the following

**Corollary 4.1.** For every \( n \in \mathbb{N} \) there exists \( K_n > 0 \) such that

\[ \| D_E^k P_{s, t} \varphi(x) \|_{L^n(E)} \leq \frac{K_n}{(t - s)^{n \theta}} \| \varphi \|_{\infty}, \quad (s, t) \in \Delta, \ \varphi \in C_b(X). \]

For every \( \alpha \in (0, 1) \) and \( n \in \mathbb{N} \) there exists \( K_{n, \alpha} > 0 \) such that

\[ \| D_E^k P_{s, t} \varphi(x) \|_{L^n(E)} \leq \frac{K_{n, \alpha}}{(t - s)^{(n - \alpha)\theta}} \| \varphi \|_{C^\alpha_b(X)}, \quad (s, t) \in \Delta, \ \varphi \in C_b(X). \]

**Proof.** Estimates (4.2) and (4.3) follow from Theorem 3.3 and Corollary 3.4 taking into account Hypothesis 2.5. □

**Remark 4.2.** Corollary 4.1 allows us to extend continuously to \( \{t\} \times X \) the mapping

\[ (s, x) \in [0, t] \times X \mapsto D_E^k u_0(s, \cdot)(x)(h_1, \ldots, h_k) \in \mathbb{R}. \]

Indeed, for \( x_0, x \in X \) and \( s \in [0, t] \) we have

\[ |D_E^k u_0(s, \cdot)(x)(h_1, \ldots, h_k)| \leq \prod_{j=1}^{k} |h_j| \int_{s}^{t} \| D_E^k P_{s, \sigma} \varphi(\cdot)(x) \|_{L^k(E)} \, d\sigma \]
\[ \leq K_k \| \varphi \|_{\infty} \prod_{j=1}^{k} |h_j|_E (t - s)^{1 - k\sigma}, \]

where \( h_1, \ldots, h_k \in E \) and \( k \in 1, \ldots, n. \) As \( s \to t^- \) and \( x \to x_0 \) the right hand side in (4.5) vanishes and we get

\[ D_E^k u_0(t, \cdot)(x_0)(h_1, \ldots, h_k) = 0, \ \ x_0 \in X. \]
Proposition 4.3. For every $t \in [0,T]$ and $\psi \in C_b([0,t] \times X)$, $u_1$ is continuous and
\begin{equation}
\|u_1\|_\infty \leq t \|\psi\|_\infty.
\end{equation}

Moreover the following statements hold.

(1) Let $\theta < 1$. For every $n \in \mathbb{N}$ such that $n < \frac{1}{\theta}$, the function $u_1$ belongs to $C^{0,n}_E([0,t] \times X)$ and there exists $C > 0$, independent of $\psi$ and $t$, such that
\begin{equation}
\|u_1\|_{C^{0,n}_E([0,t] \times X)} \leq C \|\psi\|_\infty.
\end{equation}

(2) Let $\alpha \in (0,1)$ be such that $\alpha + \frac{1}{\theta} > 1$. For every $\psi \in C^\infty_0(X)$ and for every $n \in \mathbb{N}$ such that $n < \alpha + \frac{1}{\theta}$, the function $u_1$ belongs to $C^{0,n}_E([0,t] \times X)$ and there exists $C > 0$, independent of $\psi$ and $t$, such that
\begin{equation}
\|u_1\|_{C^{0,n}_E([0,t] \times X)} \leq C \|\psi\|_{C^{0,n}_E([0,t] \times X)}.
\end{equation}

Proof. Since estimate (4.6) is obvious, we prove that $u$ is continuous. Let $0 \leq s_0 \leq s < t \leq T$ and $x, x_0 \in X$. Then

$$
|u_1(s, x) - u_1(s_0, x_0)| \leq \int_s^t |P_{s, \sigma} \psi(\sigma, \cdot)(x) - P_{s_0, \sigma} \psi(\sigma, \cdot)(x_0)| \, d\sigma + \int_{s_0}^s |P_{s_0, \sigma} \psi(\sigma, \cdot)(x_0)| \, d\sigma.
$$

Since for every $\sigma \geq s_0$ the mapping

$$(s, x) \in [0,T] \times X \mapsto 1_{[s,t]}(\sigma)P_{s, \sigma} \psi(\sigma, \cdot)(x) \in \mathbb{R},$$

is continuous ([CL21, Lem. 2.3]) and

$$|P_{s_0, \sigma} \psi(\cdot)(x) - P_{s_0, \sigma} \psi(\cdot)(x_0)| \leq 2 \|\psi\|_\infty.$$

By the Dominated Convergence Theorem the first integral vanishes as $s \rightarrow s_0^+$ and $x \rightarrow x_0$. Moreover, since

$$
\int_{s_0}^s |P_{s_0, \sigma} \psi(\cdot)(x_0)| \, d\sigma \leq \|\psi\|_\infty (s - s_0),
$$

even the second integral vanishes as $s \rightarrow s_0^+$ and $x \rightarrow x_0$. If $s < s_0$ we split

$$
|u_1(s, x) - u_1(s_0, x_0)| \leq \int_{s_0}^t |P_{s, \sigma} \psi(\sigma, \cdot)(x) - P_{s_0, \sigma} \psi(\sigma, \cdot)(x_0)| \, d\sigma + \int_s^{s_0} |P_{s_0, \sigma} \psi(\sigma, \cdot)(x_0)| \, d\sigma.
$$

and the proof is analogous. So $u_1$ is continuous.

Concerning statements (1) and (2), due to Corollary 4.1 we can apply the Dominated Convergence Theorem to obtain
\begin{equation}
\frac{\partial u_1}{\partial h_1, \ldots, \partial h_k}(s, x) = \int_s^t D^k P_{s, \sigma} \psi(\cdot)(x)(h_1, \ldots, h_k) \, d\sigma, \quad s \in [0,t), \ x \in X,
\end{equation}
for every \( h_1, \ldots, h_k \in E \) and \( k \in \{1, \ldots, n\} \). The proof that the mapping

\[
(4.10) \quad (s, x) \in [0, t) \times X \rightarrow D^k_E u_1(s, \cdot)(x) (h_1, \ldots, h_k) \in \mathbb{R},
\]

for every \( h_1, \ldots, h_k \in E \) and \( k \in \{1, \ldots, n\} \), is similar to the proof of the continuity of \( u_1 \). Indeed, for \( 0 \leq s_0 \leq s < t \leq T \) and \( x, x_0 \in X \), we have

\[
\frac{\partial u_1}{\partial h_1, \ldots, \partial h_k}(s, x) - \frac{\partial u_1}{\partial h_1, \ldots, \partial h_k}(s_0, x_0) = \int_{s_0}^{s} D^k_E P_{s, \sigma} \psi(\sigma, \cdot)(x)(h_1, \ldots, h_k) \, d\sigma
\]

\[
+ \int_{s_0}^{t} \mathbb{1}_{[s, t]}(\sigma)(D^k_E P_{s, \sigma} \psi(\sigma, \cdot)(x)(h_1, \ldots, h_k) - D^k_E P_{s_0, \sigma} \psi(\sigma, \cdot)(x)(h_1, \ldots, h_k)) \, d\sigma,
\]

and we argue as before.

Corollary 4.1 allow us to extend continuously the mapping (4.10) to \( \{t\} \times X \). Indeed, for \( x_0, x \in X \) and \( s \in [0, t) \) we have

\[
|D^k_E u_1(s, \cdot)(x)(h_1, \ldots, h_k)| \leq \prod_{j=1}^{k} |h_j| \left| \int_{s}^{t} \|D^k_E P_{s, \sigma} \psi(\sigma, \cdot)(x)\|_{L^k(E)} \, d\sigma \right|
\]

\[
\leq K_k \|\psi\|_{\infty} \prod_{j=1}^{k} \|h_j\| E(t - s)^{1-k\sigma},
\]

where \( h_1, \ldots, h_k \in E \) and \( k \in \{1, \ldots, n\} \). As \( s \to t^- \) and \( x \to x_0 \) the right hand side in (4.11) vanishes and we get

\[
D^k_E u_1(t, \cdot)(x)(h_1, \ldots, h_k) = 0, \quad x_0 \in X.
\]

\[ \square \]

**Theorem 4.4.** Assume that Hypotheses 2.1, 2.9, 2.10 hold. Let \( \varphi \in C_b(X), \psi \in C_b([0, t] \times X) \) and let \( u \) be defined by (4.1).

1. If \( \frac{1}{q} \notin \mathbb{N}, \varphi \in C^+_E(X) \) and \( \psi \in C_b([0, t] \times X) \), then \( u \in C^0_E \left([0, t] \times X\right) \). Moreover there exists \( C = C(T) > 0 \), independent of \( \varphi \) and \( \psi \), such that

\[
\|u\|_{C^0_E([0, t] \times X)} \leq C \left(\|\varphi\|_{C^+_E(X)} + \|\psi\|_{\infty} \right).
\]

2. If \( \alpha \in (0, 1), \alpha + \frac{1}{q} \notin \mathbb{N}, \varphi \in C^{\alpha+\frac{1}{q}}_E(X) \) and \( \psi \in C^{\alpha}_E([0, t] \times X) \), then \( u \in C^{\alpha+\frac{1}{q}}_E([0, t] \times X) \). Moreover there exists \( C = C(T, \alpha) > 0 \), independent of \( \varphi \) and \( \psi \), such that

\[
\|u\|_{C^{\alpha+\frac{1}{q}}_E([0, t] \times X)} \leq C \left(\|\varphi\|_{C^{\alpha+\frac{1}{q}}_E(X)} + \|\psi\|_{C^{\alpha}_E([0, t] \times X)} \right).
\]

**Proof.** We can assume that \( \varphi \equiv 0 \) since, by Corollary 3.2, we know that for every non integer \( \gamma > 0 \) \( u_0 \) belongs to \( C^\gamma_E([0, t] \times X) \), if \( \varphi \in C^\gamma_E(X) \); moreover there exists \( C = C(\gamma, T) > 0 \) such that

\[
\|u_0\|_{C^{\gamma}_E([0, t] \times X)} \leq C \|\varphi\|_{C^\gamma_E(X)}.
\]

\[ \square \]
Let us prove statement (1). Setting \( n := \left\lfloor \frac{1}{\theta} \right\rfloor \), we have that
\[
 n\theta \in (0, 1), \quad (n+1)\theta > 1.
\]
we have to show that \( u_1(s, \cdot) \in C^0_b(X) \) for every \( s \in [0, t] \).

If \( n > 0 \), we know from Proposition 4.3 that \( u_1 \in C^{0,n}_b([0, t] \times X) \). Hence, we have to prove that \( D^n_E u_1(s, \cdot) \) is \( \left( \frac{1}{\theta} - n \right) \)-Hölder continuous with values in \( L^n(E) \), with Hölder constant independent of \( s \).

Let \( h, h_1, \ldots, h_n \in E \). We split every partial derivative \( D^n_E u_1(s, y)(h_1, \ldots, h_n) \) into \( a_h(s, y) + b_h(s, y) \) where
\[
\begin{align*}
a_h(s, y) &:= - \int_s^{(s+\|h\|^E_n)^{\frac{1}{\theta}}} D^n_E P_s,\sigma \psi(\sigma, \cdot)(h_1, \ldots, h_n) \, d\sigma, \quad s \in [0, t], \quad y \in X, \\
b_h(s, y) &:= - \int_t^{(s+\|h\|^E_n)^{\frac{1}{\theta}}} D^n_E P_s,\sigma \psi(\sigma, \cdot)(h_1, \ldots, h_n) \, d\sigma, \quad s \in [0, t], \quad y \in X.
\end{align*}
\]
Due to (4.2), we obtain
\[
|a_h(s, x+h) - a_h(s, x)| \leq |a_h(s, x+h)| + |a_h(s, x)|
\]
\[
\leq 2K_n \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E \int_s^{(s+\|h\|^E_n)^{\frac{1}{\theta}}} (\sigma - s)^{-n\sigma} \, d\sigma.
\]
\[
\leq \frac{2K_n}{1 - n\theta} \|h\|^E_n^{\frac{1}{\theta} - n} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E.
\]
Since if \( \|h\|^E_n \geq t - s \) \( b_h(s, \cdot) \) vanishes, we estimate \( |b_h(s, x+h) - b_h(s, x)| \) when \( \|h\|^E_n < t - s \). Again, by (4.2) we get
\[
\|D^n_E P_s,\sigma \psi(\sigma, \cdot)(x+h) - D^n_E P_s,\sigma \psi(\sigma, \cdot)(x)\|_{L^n(E)} \leq \sup_{y \in X} \|D^{n+1}_E P_s,\sigma \psi(\sigma, \cdot)(y)\|_{L^n(E)} \|h\|_E
\]
\[
\leq \frac{K_{n+1}}{(\sigma - s)^{(n+1)\theta}} \|\psi\|_\infty \|h\|_E \quad \sigma \in (s, t)
\]
and
\[
|b_h(s, x+h) - b_h(s, x)| \leq K_{n+1} \|\psi\|_\infty \|h\|_E \prod_{j=1}^n \|h_j\|_E \int_s^{(s+\|h\|^E_n)^{\frac{1}{\theta}}} (\sigma - s)^{-(n+1)\theta} \, d\sigma.
\]
\[
\leq \frac{K_{n+1}}{(n+1)\theta - 1} \|h\|^E_n^{\frac{1}{\theta} - n} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E.
\]
Summing up we get
\[
\left| (D^n_E u_1(s, x+h) - D^n_E u_1(s, x))(h_1, \ldots, h_n) \right| \leq C \|h\|^E_n^{\frac{1}{\theta} - n} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E
\]
4 SCHAUDER TYPE THEOREMS

with

\[ C = \frac{2K_n}{1 - n\theta} + K_n^{n+1} \frac{1}{(n + 1)\theta - 1}. \]

Therefore,

\[ [D^n_E u_1(s, \cdot)]_{C^\frac{1}{n} - n} (X; C^n(E)) \leq C \|\psi\|_\infty. \]

The case \( n = 0 \) is analogous to the previous one where instead of (4.2) we use (4.6).

Let us prove statement (2). Setting \( n := \left\lfloor \alpha + \frac{1}{\theta} \right\rfloor \), we have that

\[ (n - \alpha)\theta \in (0, 1), \quad (n + 1 - \alpha)\theta > 1. \]

We already know that \( u_1 \in C^{\alpha,n}_E (X) \) by Proposition 4.3 (2) and we have to show that \( u_1(s, \cdot) \in C^{\alpha+1,\frac{1}{n}}_E (X) \) for every \( s \in [0, t] \).

If \( n > 0 \), we have to prove that \( D^n_E u_1(s, \cdot) \) is \( (\alpha + 1 - n)\theta \)-Hölder continuous with values in \( L^n(E) \), with Hölder constant independent of \( s \).

Let \( h, h_1, \ldots, h_n \in E \). We split every partial derivative \( D^n_E u_1(s, y)(h_1, \ldots, h_n) \) in \( a_h(s, y) + b_h(s, y) \) in the same way as we did for the proof of statement (i). Due to (4.3), we obtain

\[ |a_h(s, x + h) - a_h(s, x)| \leq |a_h(s, x + h)| + |a_h(s, x)| \]
\[ \leq 2K_n,\alpha \|\psi\|_{C^\frac{h}{n}(0, t; X)} \prod_{j=1}^{n} \|h_j\|_E \int_s^{(s + \|h\|_E^\alpha)^\theta} (\sigma - s)^{-(n - \alpha)\sigma} \, d\sigma. \]
\[ \leq \frac{2K_n,\alpha}{1 - (n - \alpha)\theta} \|h\|_E^{1 - (n - \alpha)\theta} \|\psi\|_{C^\frac{h}{n}(0, t; X)} \prod_{j=1}^{n} \|h_j\|_E. \]

We observe that if \( \|h\|_E^\frac{1}{n} \geq t \), \( s \neq \cdot \) vanishes. Hence, we estimate now \( |b_h(s, x + h) - b_h(s, x)| \) when \( \|h\|_E^\frac{1}{n} < t \). Again, by (4.3) we get

\[ \|D^n_E P_s,\sigma \psi(\cdot)(x + h) - D^n_E P_s,\sigma \psi(\cdot)(x)\|_{L^n(E)} \leq \sup_{y \in X} \|D^{n+1}_E P_s,\sigma \psi(\cdot)(y)\|_{L^n(E)} \|h\|_E \]
\[ \leq \frac{K_{n+1,\alpha}}{(\sigma - s)^{(n + 1 - \alpha)\theta}} \|\psi\|_{C^\frac{h}{n}(0, t; X)} \|h\|_E, \sigma \in (s, t) \]

and

\[ |b_h(s, x + h) - b_h(s, x)| \leq K_{n+1,\alpha} \|\psi\|_{C^\frac{h}{n}(0, t; X)} \|h\|_E \prod_{j=1}^{n} \|h_j\|_E \int_s^{(s + \|h\|_E^\alpha)^\theta} (\sigma - s)^{-(n + 1 - \alpha)\theta} \, d\sigma. \]
\[ \leq \frac{K_{n+1,\alpha}}{(n + 1 - \alpha)\theta - 1} \|h\|_E^{\frac{1}{n} + \alpha - \eta} \|\psi\|_{C^\frac{h}{n}(0, t; X)} \prod_{j=1}^{n} \|h_j\|_E. \]

Summing up we get

\[ \left| (D^n_E u_1(s, x + h) - D^n_E u_1(s, x))(h_1, \ldots, h_n) \right| \leq C \|h\|_E^{\frac{1}{n} + \alpha - \eta} \|\psi\|_{C^\frac{h}{n}(0, t; X)} \prod_{j=1}^{n} \|h_j\|_E. \]
with
\[ C = \frac{2K_{n,\alpha}}{1 - (n - \alpha)\theta} + \frac{K_{n+1,\alpha}}{(n + 1 - \alpha)\theta - 1}. \]

Therefore,
\[ |D_E^n u_1(s, \cdot)|_{C_E^{\frac{1}{2} + \alpha - n}(X; L^n(E))} \leq C \|\psi\|_{C_E^n([0,t] \times X)}. \]

The case \( n = 0 \) is analogous to the previous one where instead of (4.3) we use (4.6).

\[ \square \]

Theorem 4.5. Assume that Hypotheses 2.1, 2.9, 2.10 hold. Let \( \varphi \in C_b(X) \), \( \psi \in C_b([0,t] \times X) \) and let \( u \) be defined by (4.1).

1. If \( \frac{1}{\theta^2} = k \in \mathbb{N} \) and if \( \varphi \in Z_E^k(X) \), then \( u \in Z_E^{0,k}([0,t] \times X) \). Moreover there exists \( C = C(T) > 0 \), independent of \( \varphi \) and \( \psi \), such that

\[ \|u\|_{Z_E^{0,k}([0,t] \times X)} \leq C (\|\varphi\|_{Z_E^k(X)} + \|\psi\|_{C_E^0}). \]

2. If \( \alpha \in (0,1) \) and \( \alpha + \frac{1}{\theta^2} = k \in \mathbb{N} \) and if \( \varphi \in Z_E^k(X) \) and \( \psi \in C_E^{0,\alpha}([0,t] \times X) \), then \( u \in Z_E^{0,k}([0,t] \times X) \). Moreover there exists \( C = C(T,\alpha) > 0 \), independent of \( \varphi \) and \( \psi \), such that

\[ \|u\|_{Z_E^{0,k}([0,t] \times X)} \leq C (\|\varphi\|_{Z_E^k(X)} + \|\psi\|_{C_E^{0,\alpha}}). \]

Proof. We can assume that \( \varphi \equiv 0 \) since, by Corollary 3.2, we know that for every \( k \in \mathbb{N} \) \( u_0 \) belongs to \( Z_E^k([0,t] \times X) \), if \( \varphi \in Z_E^k(X) \); moreover there exists \( C = C(k,T) > 0 \) such that

\[ \|u_0\|_{Z_E^{0,k}([0,t] \times X)} \leq C \|\varphi\|_{Z_E^k(X)}. \]

In the case of statement (1) we have \( \psi \in C_b([s,t] \times X) \) and \( k = \frac{1}{\theta^2} \). If \( k \geq 2 \), we know from Proposition 4.3 that \( u_1 \in Z_E^{0,k-1}([0,t] \times X) \). So we have to show that \( |D_E^{k-1}u_1(s,\cdot)|_{Z_E^{1,k-1}(X; L^{n-1}(E))} \) is bounded by a constant independent of \( s \).

Let \( h, h_1, \ldots, h_n \in E \). We split every partial derivative \( D_E^{k-1}u_1(s, y)(h_1, \ldots, h_n) \) into the sum \( a_h(s, y) + b_h(s, y) \) as we did in the Theorem 4.4. Due to (4.2), we obtain

\[ |a_h(s, x + 2h) - 2a_h(s, x + h) + a_h(s, x)| \leq |a_h(s, x + 2h)| + 2|a_h(s, x + h)| + |a_h(s, x)| \]

\[ \leq 4K_{k-1} \|\psi\|_\infty \prod_{j=1}^{k-1} \|h_j\|_E \int_s^{(s+||h||_E^2)\wedge t} (\sigma - s)^{-(k-1)\sigma} \, d\sigma. \]

\[ \leq \frac{4K_{k-1}}{1 - (k - 1)\theta} \|h\|_E^{\frac{t - (k-1)\theta}{k-1}} \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E \cdot \]

\[ = 4kK_{k-1} \|h\|_E \|\psi\|_\infty \prod_{j=1}^n \|h_j\|_E. \]
We observe that if \( \|h\|_{E}^{\frac{\beta}{\beta}} \geq t-s \), \( b_{h}(s,\cdot) \) vanishes. Hence we estimate

\[
|b_{h}(s,x+2h) - 2b_{h}(s,x+h) + b_{h}(s,x)|
\]

when \( \|h\|_{E}^{\frac{\beta}{\beta}} < t-s \). Again, by (4.2) we get

\[
|b_{h}(s,x+2h) - 2b_{h}(s,x+h) + b_{h}(s,x)|
\]

\[
\int_{s+\|h\|_{E}^{\frac{\beta}{\beta}}}^{t} \left| (D_{E}^{k-1} P_{\sigma} \psi(\sigma,\cdot)(x+2h) - 2D_{E}^{k-1} P_{\sigma} \psi(\sigma,\cdot)(x+h) + D_{E}^{k-1} P_{\sigma} \psi(\sigma,\cdot)(x)) (h_{1},\ldots,h_{k-1}) \right| d\sigma
\]

\[
\leq \|h\|_{E}^{2} \prod_{j=1}^{k-1} \|h_{j}\|_{E} \int_{s+\|h\|_{E}^{\frac{\beta}{\beta}}}^{t} \sup_{y \in X} \|D_{E}^{k+1} P_{\sigma} \psi(\sigma,\cdot)(y)\|_{L^{k+1}} d\sigma
\]

\[
\leq K_{k+1} \|\psi\|_{\infty} \|h\|_{E}^{2} \prod_{j=1}^{k-1} \|h_{j}\|_{E} \int_{s+\|h\|_{E}^{\frac{\beta}{\beta}}}^{t} (\sigma-s)^{-k(k+1)\theta} d\sigma \leq kK_{k+1} \|\psi\|_{\infty} \|h\|_{E} \prod_{j=1}^{k-1} \|h_{j}\|_{E}.
\]

Summing up we get

\[
[D_{E}^{k-1}u_{1}(s,\cdot)]_{Z_{k-1}^{E}(X;C^{k-1}(E))} \leq k(4K_{k-1} + K_{k+1}) \|\psi\|_{\infty}.
\]

The case \( k = \theta = 1 \) is analogous to the previous one where instead of (4.2) we use (4.6).

In the case of statement (2) we have \( \psi \in C_{E}^{\alpha,\alpha}[s,t] \times X \) and \( k = \alpha + \frac{1}{\theta} \). If \( k \geq 2 \), we know from Proposition 4.3 that \( u_{1} \in C_{E}^{0,k-1}[0,t] \times X \) and so we have to show that \( [D_{E}^{k-1}u_{1}(s,\cdot)]_{Z_{k-1}^{E}(X;C^{k-1}(E))} \) is bounded by a constant independent of \( s \).

Let \( h, h_{1}, \ldots, h_{n} \in E \). We split every partial derivative \( D_{E}^{k-1}u_{1}(s,y)(h_{1},...,h_{n}) \) into \( a_{h}(s,y) + b_{h}(s,y) \) as we did in the proof of Theorem 4.4. Due to (4.3), we obtain

\[
|a_{h}(s,x+2h) - 2a_{h}(s,x+h) + a_{h}(s,x)| \leq |a_{h}(s,x+2h)| + 2|a_{h}(s,x+h)| + |a_{h}(s,x)|
\]

\[
\leq 4K_{k-1,\alpha} \|\psi\|_{C_{E}^{\alpha,\alpha}([0,t] \times X)} \prod_{j=1}^{k-1} \|h_{j}\|_{E} \int_{s}^{(s+\|h\|_{E}^{\frac{\beta}{\beta}}) \wedge t} (\sigma-s)^{-k(k-1-\alpha)\sigma} d\sigma.
\]

\[
\leq \frac{4K_{k-1,\alpha}}{1 - (k-1-\alpha)\theta} \|h\|_{E}^{\frac{1-(k-1-\alpha)\theta}{\theta}} \|\psi\|_{C_{E}^{\alpha,\alpha}([0,t] \times X)} \prod_{j=1}^{n} \|h_{j}\|_{E}.
\]

We observe that if \( \|h\|_{E}^{\frac{\beta}{\beta}} \geq t-s \), \( b_{h}(s,\cdot) \) vanishes. Hence we estimate

\[
|b_{h}(s,x+2h) - 2b_{h}(s,x+h) + b_{h}(s,x)|
\]
when \( \|h\|_E^E < t - s \). Again, by (4.3) we get
\[
|b_h(s, x + 2h) - 2b_h(s, x + h) + b_h(s, x)|
\]
\[
\int_{s + \|h\|_E^E}^t \left| \left( D_{E}^{k-1} P_{s, \sigma} \psi(\sigma, \cdot)(x + 2h) - 2 D_{E}^{k-1} P_{s, \sigma} \psi(\sigma, \cdot)(x + h) + D_{E}^{k-1} P_{s, \sigma} \psi(\sigma, \cdot)(x) \right) (h_1, ..., h_{k-1}) \right| \, d\sigma
\]
\[
\leq \|h\|_E^E \sum_{j=1}^{k-1} \|h_j\|_E \int_{s + \|h\|_E^E}^t \sup_{y \in X} \|D_{E}^{k-1} P_{s, \sigma} \psi(\sigma, \cdot)(y)\|_{L_{k+1}} \, d\sigma
\]
\[
\leq K_{k+1, \alpha} \|\psi\|_{C^{\alpha}_{E}([0, t] \times X)} \|h\|_E^E \prod_{j=1}^{k-1} \|h_j\|_E \int_{s + \|h\|_E^E}^t (\sigma - s)^{-(k+1)\alpha} \, d\sigma
\]
\[
\leq (k - \alpha) K_{k+1, \alpha} \|\psi\|_{C^{\alpha}_{E}([0, t] \times X)} \|h\|_E \prod_{j=1}^{k-1} \|h_j\|_E.
\]
Summing up we get
\[
[D_{E}^{k-1} u_1(s, \cdot)]_{Z_k(X; L_{k-1}(E))} \leq (k - \alpha)(4 K_{k-1, \alpha} + K_{k+1, \alpha}) \|\psi\|_{H}.
\]
The case \( k = \theta = 1 \) is analogous to the previous one where instead of (4.3) we use (4.6).

5 Examples

- Example 1

Let \( A(t) \), \( B(t) \) be self-adjoint operators in diagonal form with respect to the same Hilbert basis \( \{e_k : k \in \mathbb{N}\} \), namely
\[
A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad 0 \leq t \leq T, \quad k \in \mathbb{N},
\]
with continuous coefficients \( a_k \), \( b_k \). We set
\[
\mu_k = \min_{t \in [0, T]} a_k(t), \quad \lambda_k = \max_{t \in [0, T]} a_k(t)
\]
and we assume that there exists \( \lambda_0 > 0 \) such that
\[
\lambda_k < \lambda_0, \quad \forall \ k \in \mathbb{N}.
\]
In this setting
\[
U(t, s)e_k = \exp \left( \int_s^t a_k(\tau) \, d\tau \right) e_k, \quad (s, t) \in \Delta, \quad k \in \mathbb{N}
\]
is the strongly continuous evolution operator formally associated to the family \( \{A(t)\}_{t \in [0, T]} \).
Moreover we assume that there exists \( K > 0 \) such that
\[
|b_k(t)| \leq K, \quad t \in [0, T], \quad k \in \mathbb{N}.
\]
Hence \( B(t) \in L(X) \) for all \( t \in [0, T] \) and
\[
\sup_{t \in [0, T]} \|B(t)\|_{L(X)} \leq K.
\]
The operators $Q(t, s)$ are given by

$$Q(t, s)e_k = \int_s^t \exp \left( 2 \int_\sigma^t a_k(\tau) \, d\tau \right) (b_k(\sigma))^2 \, d\sigma \, e_k =: q_k(t, s)e_k, \quad (s, t) \in \Delta, \ k \in \mathbb{N}.$$ 

Therefore, Hypothesis 2.1 is fullfilled if

$$\sum_{k=0}^{\infty} q_k(t, s) < +\infty, \quad (s, t) \in \Delta. \tag{5.1}$$

We give now a sufficient condition to have (5.1). We assume that $\lambda_k$ is eventually nonzero (say for $k \geq k_0$). Given $(s, t) \in \Delta$, we have

$$\left| \int_s^t \exp \left( 2 \int_\sigma^t a_k(\tau) \, d\tau \right) (b_k(\sigma))^2 \, d\sigma \right| \leq \|b_k\|_\infty^2 \int_s^t \exp(2\lambda_k(t-\sigma)) \, d\sigma$$

$$\frac{\|b_k\|_\infty^2}{2|\lambda_k|} \geq 1 - \exp(2\lambda_k(t-s)) \leq \frac{\|b_k\|_\infty^2}{2|\lambda_k|}(1 + \exp(2\lambda_0 T)). \tag{5.2}$$

Hence (5.1) holds provided

$$\sum_{k=0}^{\infty} \frac{\|b_k\|_\infty^2}{|\lambda_k|} < +\infty. \tag{5.3}$$

We look now for a sufficient condition such that

$$U(t, s)(X) \not\subset \mathcal{H}_{t, s}, \quad (s, t) \in \Delta. \tag{5.4}$$

Since $Q^+(t, s)e_k = (q_k(t, s))^2 e_k$ for every $(s, t) \in \Delta$ and $k \in \mathbb{N}$, (5.4) holds if $q_k(t, s)$ is eventually positive (say for $k \geq k_1 \geq k_0$) and

$$\sup_{k \geq k_1} \frac{\exp \left( \int_s^t 2a_k(\tau) \, d\tau \right)}{q_k(t, s)} = +\infty, \quad 0 \leq s < t \leq T. \tag{5.5}$$

We obtain a sufficient condition for (5.5) if we assume that $\lambda_k$ is eventually negative and $\|b_k\|_\infty$ is eventually nonzero (say for $k \geq k_2 \geq k_1$ for both), so for $(s, t) \in \Delta$ we have

$$\frac{\exp \left( \int_s^t 2a_k(\tau) \, d\tau \right)}{\int_s^t \exp \left( 2 \int_\sigma^t a_k(\tau) \, d\tau \right) (b_k(\sigma))^2 \, d\sigma} \geq \frac{\exp(2\mu_k(t-s))}{\|b_k\|_\infty^2}$$

$$= \frac{\exp(2\mu_k(t-s))}{\|b_k\|_\infty^2 (1 - \exp(2\lambda_k(t-s)))} = \frac{2|\lambda_k|}{\|b_k\|_\infty^2 (\exp(-2\mu_k(t-s)) - \exp(-2(\mu_k - \lambda_k)(t-s)))}. \tag{5.6}$$

So (5.18) is fulfilled if

$$\sup_{k \geq k_2} \frac{2|\lambda_k|}{\|b_k\|_\infty^2 (\exp(-2\mu_k(t-s)) - \exp(-2(\mu_k - \lambda_k)(t-s)))} = +\infty$$
Now we want to find some conditions such that the hypotheses of Proposition 3.8 are satisfied. Let \((s, t) \in \Delta\), we assume that \(b_k(s)\) and \(b_k(t)\) are eventually nonzero (say for \(k \geq k_3 \geq k_2\)). Hence we have that \(U(t, s)H_s \subseteq H_t\) if
\[
(5.7) \quad \sup_{k \geq k_3} \frac{b_k^2(s)}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau) \, d\tau\right) < +\infty.
\]
Since
\[
\frac{b_k^2(s)}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau) \right) \leq \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T), \quad k \geq k_3
\]
a sufficient condition to have (5.7) is given by
\[
(5.8) \quad \sup_{k \geq k_3} \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T) < +\infty.
\]
Now we investigate when (3.36) holds. We observe first that for \(y = Q^{\frac{1}{2}}(s)x\) with \(x \in H_s\), we have
\[
\|U(t, s)y\|_{H_t} = \left\|Q^{-\frac{1}{2}}(t)U(t, s)Q^{\frac{1}{2}}(s)x\right\|_{X} = \sum_{k \in \mathbb{N}, \ b_k(t) \neq 0} \left(\frac{|b_k(s)|}{|b_k(t)|}\right) \exp\left(\int_s^t a_k(\tau)\right) \langle x, e_k \rangle_{X}^2
\]
\[
= \sum_{k \in \mathbb{N}, \ b_k(t) \neq 0} \left(\frac{|b_k(s)|}{|b_k(t)|}\right) \exp\left(\int_s^t a_k(\tau)\right) \left(\frac{|y, e_k\rangle_{X}}{|b_k(s)|}\right)^2 = \sum_{k \in \mathbb{N}, \ b_k(t) \neq 0} \left(\frac{1}{|b_k(t)|}\right) \exp\left(\int_s^t a_k(\tau)\right) \left(\frac{\langle y, e_k \rangle_{H_s}}{|b_k(s)|}\right)^2.
\]
We assume that there exist \(L \geq 0\) and \(k_4 \in \mathbb{N}\) such that \(k_4 \geq k_3\) and \(|b_k(t)| \geq L\) for all \(k \geq k_4\). Hence for any \(k \geq k_4\), we have
\[
(5.9) \quad \frac{1}{b_k^2(t)} \exp\left(\int_s^t 2a_k(\tau) \, d\tau\right) \leq \frac{1}{L^2} \exp(2\lambda_0 T),
\]
and
\[
(5.10) \quad \|U(t, s)\|_{L(H_s, H_t)} \leq \frac{1}{L} \exp\lambda_0 T, \quad (t, s) \in \Delta.
\]
Moreover for all \(k \geq k_4\), by Proposition 3.8 we have that there exists \(M > 0\) such that
\[
(5.11) \quad \|U(t, s)\|_{L(H_s, X_t, s)} \leq \frac{M}{(t-s)^{\frac{1}{2}}}
\]
Moreover, for all \(k \geq k_4\), we have that \(H_{t_1} = H_{t_2}\) with equivalent norms for every \(t_1, t_2 \in [0, T]\). Indeed, let \(y \in H_{t_1}\), then there exists \(x \in X\) such that
\[
(5.12) \quad y = Q^{-\frac{1}{2}}(t_1)x = \sum_{k=1}^{\infty} b_k(t) \langle x, e_k \rangle_X = Q^{-\frac{1}{2}}(t_2) \left(\frac{b_k(t)}{b_k(s)}\right) x \in H_{t_2},
\]
and
\[
(5.13) \quad \|y\|_{H_{t_1}}^2 = \sum_{k=1}^{\infty} |b_k(t_1)|^2 \langle x, e_k \rangle_X^2 \leq \frac{K^2}{L^2} \sum_{k=1}^{\infty} |b_k(t_2)|^2 \langle x, e_k \rangle_X^2 = \frac{K^2}{L^2} \|y\|_{H_{t_2}}^2.
\]
Conversely, the other embedding is analogous.

So, identifying each \(H_t\) with one of them, Hypotheses 2.9 and 2.10 are satisfied too.
• Example 1 bis

Let $A(t), B(t)$ be self-adjoint operators in diagonal form with respect to the same Hilbert basis $\{e_k : k \in \mathbb{N}\}$, namely

$$A(t)e_k = a_k(t)e_k, \quad B(t)e_k = b_k(t)e_k \quad 0 \leq t \leq T, \quad k \in \mathbb{N},$$

with continuous coefficients $a_k, b_k$. We set

$$\mu_k = \min_{t \in [0,T]} a_k(t), \quad \lambda_k = \max_{t \in [0,T]} a_k(t)$$

and we assume that there exists $\lambda_0 > 0$ such that

$$\lambda_k < \lambda_0, \quad \forall \ k \in \mathbb{N}.$$ 

In this setting

$$U(t,s)e_k = \exp\left(\int_s^t a_k(\tau) \, d\tau\right)e_k, \quad (s,t) \in \Delta, \quad k \in \mathbb{N}$$

is the strongly continuous evolution operator formally associated to the family $\{A(t)\}_{t \in [0,T]}$. Moreover we assume that there exists $K > 0$ such that

$$|b_k(t)| \leq K, \quad t \in [0,T], \quad k \in \mathbb{N}.$$ 

Hence $B(t) \in \mathcal{L}(X)$ for all $t \in [0,T]$ and

$$\sup_{t \in [0,T]} \|B(t)\|_{\mathcal{L}(X)} \leq K.$$ 

The operators $Q(t,s)$ are given by

$$Q(t,s)e_k = \int_s^t \exp\left(2 \int_{\sigma}^t a_k(\tau) \, d\tau\right)(b_k(\sigma))^2 \, d\sigma e_k =: q_k(t,s)e_k, \quad (s,t) \in \Delta, \quad k \in \mathbb{N}.$$ 

We assume also that there exist some indexes $k$ (possibly, infinite many) such that $b_k$ is identically zero on the interval $(s,t)$.

Hypothesis 2.1 is fulfilled if

$$(5.14) \quad \sum_{k=0}^{\infty} q_k(t,s) < +\infty, \quad (s,t) \in \Delta.$$ 

We give now a sufficient condition to have (5.14). We assume that $\lambda_k$ is eventually nonzero (say for $k \geq k_0$). Given $(s,t) \in \Delta$, we have

$$\left|\int_s^t \exp\left(2 \int_{\sigma}^t a_k(\tau) \, d\tau\right)(b_k(\sigma))^2 \, d\sigma\right| \leq \|b_k\|_{\infty}^2 \left|\int_s^t \exp(2\lambda_k(t-\sigma)) \, d\sigma\right|$$

$$= \frac{\|b_k\|_{\infty}^2}{2|\lambda_k|} |1 - \exp(2\lambda_k(t-s))| \leq \frac{\|b_k\|_{\infty}^2}{2|\lambda_k|} (1 + \exp(2\lambda_0 T)).$$

Hence (5.14) holds if we require

$$(5.16) \quad \sum_{k=0}^{\infty} \frac{\|b_k\|_{\infty}^2}{|\lambda_k|} < +\infty.$$
We look now for a condition such that
\[ U(t, s)(X) \not\subseteq \mathcal{H}_{t,s}, \quad (s, t) \in \Delta. \]

Since \( Q^\frac{3}{2}(t, s)e_k = (q_k(t, s))^\frac{3}{2}e_k \) for every \( (s, t) \in \Delta \) and \( k \in \mathbb{N}, \) (5.17) holds if
\[
\sup_{k \geq k_0, q_k(t, s) \neq 0} \frac{\exp\left( \int_s^t 2a_k(\tau) d\tau \right)}{q_k(t, s)} = +\infty, \quad 0 \leq s < t \leq T.
\]

We obtain a sufficient condition for (5.18) if we assume \( \lambda_k \) eventually negative (say for \( k \geq k_1 \geq k_0 ). \) Let \( (s, t) \in \Delta, \) assuming \( b_k \neq 0 \) on \( (s, t), \) we have
\[
\frac{\exp\left( \int_s^t 2a_k(\tau) d\tau \right)}{\int_s^t \exp\left( 2 \int_\sigma^\tau a_k(\sigma) d\sigma \right)(b_k(\sigma))^2 d\sigma} \geq \frac{\exp(2\mu_k(t-s))}{\|b_k\|_\infty^2 \int_s^t \exp(2\lambda_k(t-\sigma)) d\sigma} \frac{2|\lambda_k|}{\|b_k\|_\infty^2 \left( \exp(-2\mu_k(t-s)) - \exp(-2(\mu_k - \lambda_k)(t-s)) \right)}.
\]

So (5.18) is fulfilled if
\[
\sup_{k \geq k_1, b_k \neq 0} \frac{2|\lambda_k|}{\|b_k\|_\infty^2 \left( \exp(-2\mu_k(t-s)) - \exp(-2(\mu_k - \lambda_k)(t-s)) \right)} = +\infty
\]

Now we want to find some conditions such that the hypotheses of Proposition 3.8 are satisfied. Let \( (s, t) \in \Delta, \) we assume that \( b_k(s) \) and \( b_k(t) \) are eventually nonzero (say for \( k \geq k_2 \geq k_1 ). \) Hence we have that \( U(t, s)H_s \subseteq H_t \) if
\[
\sup_{k \geq k_2} \frac{b_k^2(s)}{b_k^2(t)} \exp\left( \int_s^t 2a_k(\tau) d\tau \right) < +\infty.
\]

Since
\[
\frac{b_k^2(s)}{b_k^2(t)} \exp\left( \int_s^t 2a_k(\tau) d\tau \right) \leq \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T), \quad k \geq k_2
\]
a sufficient condition to have (5.20) is given by
\[
\sup_{k \geq k_2} \frac{b_k^2(s)}{b_k^2(t)} \exp(2\lambda_k T) < +\infty.
\]

Now we investigate when (3.36) holds. We observe first that for \( y = Q^\frac{3}{2}(s)x \) with \( x \in H_s, \) we have
\[
\|U(t, s)y\|_{H_t}^2 = \|Q^{-\frac{3}{2}}(t)U(t, s)Q^\frac{3}{2}(s)x\|_X^2 = \sum_{k \in \mathbb{N}, b_k(t) \neq 0} \left( \frac{|b_k(s)|}{|b_k(t)|} \exp\left( \int_s^t a_k(\tau) \langle x, e_k \rangle_X \right) \right)^2 \sum_{k \in \mathbb{N}, b_k(t) \neq 0} \left( \frac{1}{|b_k(t)|} \exp\left( \int_s^t a_k(\tau) \langle y, e_k \rangle_{H_s} \right) \right)^2.
\]
We assume that there exist \( L \geq 0 \) and \( k_3 \in \mathbb{N} \) such that \( k_3 \geq k_2 \) and \( |b_k(t)| \geq L \) for all \( k \geq k_3 \). Hence for any \( k \geq k_3 \), we have

\[
\frac{1}{b_k^2(t)} \exp \left( \int_s^t 2a_k(\tau) \, d\tau \right) \leq \frac{1}{L^2} \exp \left( 2\lambda_0 T \right),
\]

and

\[
\|U(t,s)\|_{\mathcal{L}(H_t,H_t)} \leq \frac{1}{L} \exp(\lambda_0 T), \quad 0 \leq s < t \leq T.
\]

**Example 2**

Let \( A(t) = a(t)I \), where \( a \) is a continuous real valued map on \( t \in [0,T] \) with \( \|a\|_\infty = a_0 \). Hence

\[
U(t,s) = \exp \left( \int_s^t a(\tau) \, d\tau \right) I, \quad (s,t) \in \Delta
\]

is the strongly continuous evolution operator formally associated to the family \( \{A(t)\}_{t \in [0,T]} \). This particular choice of \( A(t) \) is important because it is a non autonomous version of the Mallavin Operator. We refer to [CL19] for the autonomous case.

Let \( \{B(t)\}_{t \in [0,T]} \in \mathcal{L}(X) \) a family of continuous and strongly measurable operators such that there exists \( K > 0 \) such that

\[
\sup_{t \in [0,T]} \|B(t)\|_{\mathcal{L}(X)} \leq K.
\]

So Hypothesis 2.1 is fulfilled if \( Q(t,s) \in \mathcal{L}^1(X) \). Since \( U(t,s) \) is a multiple of the identity, \( U(t,s) \) cannot map \( X \) into \( H_t \), for all \( (s,t) \in \Delta \) and \( U(t,s)(H_s) = H_s \) for all \( (s,t) \in \Delta \). We look for sufficient conditions such that the hypotheses of Proposition 3.8 are satisfied. We require that for every \( (s,t) \in \Delta \) there exists \( C = C(t,s) > 0 \) such that \( \langle Q(s)x,x \rangle_X \leq C(\langle Q(t)x,x \rangle_X \) for all \( x \in X \) and we obtain

\[
\|Q^2(t)x\|_X^2 = \langle Q(s)x,x \rangle_X \leq C \langle Q(t)x,x \rangle_X = C \|Q^2(t)x\|_X^2, \quad x \in X.
\]

Then by Proposition 3.6 \( H_s \subset H_t \) with continuous imbedding. Denoting by \( \tilde{C} \) the norm of the embedding of \( H_s \) into \( H_t \), let \( x \in H_s \) we have

\[
\|U(t,s)x\|_{H_t} = \left\| \exp \left( \int_s^t a(\tau) \, d\tau \right) x \right\|_{H_t} \leq e^{aT \tilde{C}} \|x\|_{H_s},
\]

\[
\|U(t,s)\|_{\mathcal{L}(H_s,H_t)} \leq e^{aT \tilde{C}}.
\]

Hence, we can choose \( H_s \) as the set \( E \) of Theorem 3.3.

Finally we study the special case \( B(t) = b(t)B \), where \( b \) is a continuous real valued map on \( [0,T] \) and \( B \in \mathcal{L}(X) \). We set

\[
\max_{t \in [0,T]} b(t) = b_0.
\]

The operators \( Q(t,s) \) are given by Let \( \{e_k : k \in \mathbb{N}\} \) be a Hilbert basis that diagonalizes \( Q(t,s) \), we have

\[
\langle Q(t,s)e_k,e_k \rangle_X = \int_s^t b^2(\sigma) \|U^*(t,\sigma)B^*e_k\|_X^2 \, d\sigma.
\]
and

\[(5.27) \quad \langle Q(t,s)e_k,e_k \rangle \leq T N^2 b_0^2 \langle BB^*e_k,e_k \rangle x \]

and \( Q(t,s) \in \mathcal{L}_+^1(X) \) if and only if \( Q := BB^* \in \mathcal{L}_+^1(X) \).

Since \( U(t,s) \) is invertible, \( U(t,s) \) cannot map \( X \) into \( \mathcal{H}_{t,s} \) for all \((s,t) \in \Delta \) and we look for sufficient conditions such that the hypotheses of Proposition 3.8 are satisfied. For all \((s,t) \in \Delta \) such that \( b(s) \) and \( b(t) \) are nonzero, we have \( U(t,s)H_s = H_t \) and let \( y \in H_s \) we obtain

\[ \|U(t,s)y\|_{H_t} = \left\| \exp \left( \int_s^t a(\tau) d\tau \right) y \right\|_{H_t} \leq e^{a_0(t-s)} \|y\|_{H_t} \leq \frac{|b(s)|}{|b(t)|} e^{a_0T} \|y\|_{H_s}. \]

Hence

\[ \|U(t,s)\|_{\mathcal{L}(H_s,H_t)} \leq \frac{|b(s)|}{|b(t)|} e^{a_0T}. \]

If \( b(t) \neq 0 \) for all \( t \in [0,T] \), then \( H_{t_1} = H_{t_2} \) with equivalent norms. Indeed, setting \( B_1 = \min_{t \in [0,T]} |b(t)| \) and \( B_2 = \sup_{t \in [0,T]} |b(t)| \) for \( x \in H_{t_2} \) we have

\[(5.28) \quad \|x\|^2_{H_{t_2}} = \frac{1}{|b(t_2)|} (Q^{-\frac{T}{2}}x,Q^{-\frac{T}{2}})X = \frac{B_2}{B_1} \|x\|^2_{H_{t_1}}. \]

The other inequality is analogous.

**Example 3**

We consider now Example 2 of [CL21] and for the readers we write again all preliminary results. Let \( \mathcal{O} \subset \mathbb{R}^d \) be an open and with \( C^2 \) boundary, we consider the evolution operator \( U(t,s) \) in \( X = L^2(\mathcal{O}) \) associated to an evolution equation of parabolic type,

\[(5.29) \quad \begin{cases} u_t(t,x) = A(t)u(t, \cdot)(x), \quad (t,x) \in (s,T) \times \mathcal{O}, \\ u(t,x) = 0, \quad (t,x) \in (s,T) \times \partial \mathcal{O}, \\ u(s,x) = u_0(x), \quad x \in \mathcal{O}. \end{cases} \]

The differential operators \( A(t) \) are defined by

\[(5.30) \quad A(t)\varphi(x) = \sum_{i,j=1}^d a_{i,j}(t,x)D_{ij}\varphi(x) = \sum_{i=1}^d a_i(t, \cdot)D_i\varphi(x) + a_0(t,x)\varphi(x), \quad t \in [0,T], \quad x \in \mathcal{O}, \]

and we make the following assumptions on the coefficients.

**Hypothesis 5.1.** For some \( \rho > 0 \), \( a_{ij} \in C^{0,1+\rho}([0,T] \times \overline{\mathcal{O}}) \) and there exists \( \nu > 0 \) such that for all \( \xi \in \mathbb{R}^d \)

\[(5.31) \quad \sum_{i,j=1}^d a_{i,j}(t,x)\xi_i\xi_j \geq \nu |\xi|^2, \quad t \in [0,T], \quad x \in \mathcal{O}. \]

On the operators \( B(t) \) we make the following assumptions.
Hypothesis 5.2. There exists $q \geq 2$, $q > d$ such that for a.e. $t \in [0, T]$, $B(t) \in \mathcal{L}(L^2(\Omega); L^q(\Omega))$ has bounded inverse and

$$\text{ess sup}_{0 < t < T} (\| B(t) \|_{\mathcal{L}(L^2(\Omega); L^q(\Omega))} + \| B(t)^{-1} \|_{\mathcal{L}(L^q(\Omega); L^2(\Omega))}) < +\infty.$$  

Moreover for all $f \in L^2(\Omega)$ the mapping $t \in [0, T] \mapsto B(t)f \in L^q(\Omega)$ is measurable.

Since $q \geq 2$, $L^q(\Omega) \subseteq L^2(\Omega)$ and $B(t) \in \mathcal{L}(X)$ a.e. $t \in (0, T)$. Hence Hypothesis 2.4 (2) is fulfilled. Moreover, with abuse of notation, we denote by $(\text{still denoted by})$

where $q$ satisfies

Since Hypothesis 5.1 holds, results of the paper [PS01, Sect. 4] can be applied. Hence for $q \in (1, +\infty)$ there exists a strongly continuous evolution operator $U_q(t, s)$ on $X_q = L^q(\Omega)$ such that, setting $D_q = W^2,q(\Omega) \cap W^{1,2}(\Omega)$, for every $\varphi \in L^p((s, t); X_q)$ with $p \in (1, +\infty)$ and $u_0 \in (X_q, D_q)_{1-p, p}$, the function $U_q(t, s)u_0$ is the unique strong solution to (5.29), namely it is the unique function that belongs to $L^p((s, T); D_q) \cap W^{1,p}((s, T); X_q) \cap C([s, T]; (X_q, D_q)_{1-p, p})$ and that satisfies

$$\begin{cases}
u'(\tau) = A_q(\tau)\nu(\tau) + \varphi(\tau), \quad \text{a.e. } \tau \in (s, t), \\ \nu(s) = u_0 \end{cases}$$

Using this tool in [Dan00, Thm 4.2] it is proved that for all $u_0 \in X$ there exists a unique weak solution of (5.29), namely there exists a unique $u \in W := L^2((s, T); H^1_0(\Omega)) \cap W^{1,2}((s, T); H^{-1}(\Omega))$ such that for every $v \in W$ satisfying $v(T) = 0$ we have

$$\begin{align}
(5.32) \quad & - \int_s^T \langle u(t), v'(t) \rangle_{L^2(\Omega)} dt + \int_s^T a(t, u(t), v(t)) dt = \langle u_0, v(0) \rangle_{L^2(\Omega)},
\end{align}$$

where $a(t, \cdot, \cdot)$ is the quadratic form associated to the operator $A(t)$ in $H^1_0(\Omega)$, namely

$$\begin{align}
a(t, \varphi, \psi) = & \int_\Omega \sum_{i,j=1}^d a_{ij}(t, x) D_j \varphi(x) D_i \psi(x) dx \\
& - \int_\Omega \left( \sum_{j=1}^d \left( \sum_{i=1}^d D_i a_{ij}(t, x) \right) D_j \varphi(x) \right) \psi(x) dx.
\end{align}$$

Setting $U(t, s)u_0 := u(t)$, $U(t, s)$ turns to be an evolution operator in $L^2(\Omega)$. Moreover in [Dan00] it is shown that $U(t, s)$ can be extended to the whole space $L^1(\Omega)$ and the extension (still denoted by $U(t, s)$) belongs to $\mathcal{L}(L^1(\Omega); L^\infty(\Omega))$ and it is represented by

$$\begin{align}
(5.33) \quad & U(t, s)\varphi(x) = \int_\Omega k(x, y, t, s) \varphi(y) dy, \quad \varphi \in L^1(\Omega)
\end{align}$$

where for every $(s, t) \in \Delta$, $k(\cdot, \cdot, t, s) \in L^\infty(\Omega \times \Omega)$. Moreover there exist $M, m > 0$ such that

$$\begin{align}
(5.34) \quad & |k(x, y, t, s)| \leq \frac{M}{(t-s)^p} e^{-\frac{(t-s)^2}{m(t-s)}}, \quad x, y \in \Omega, \quad (s, t) \in \Delta.
\end{align}$$

Using this tools in [CL21, Lem 4.3], it is proven that the operator $Q(t, s)$ has finite trace for all $(s, t) \in \Delta$. Since Hypothesis 5.1 holds, results of the paper [PS01, Sect. 4] can be applied. Hence for $q \in (1, +\infty)$ there exists a strongly continuous evolution operator $U_q(t, s)$ on $X_q = L^q(\Omega)$ such that, setting $D_q = W^2,q(\Omega) \cap W^{1,2}(\Omega)$, for every $\varphi \in L^p((s, t); X_q)$ with $p \in (1, +\infty)$ and $u_0 \in (X_q, D_q)_{1-p, p}$, the function $U_q(t, s)u_0$ is the unique strong solution to (5.29), namely it is the unique function that belongs to $L^p((s, T); D_q) \cap W^{1,p}((s, T); X_q) \cap C([s, T]; (X_q, D_q)_{1-p, p})$ and that satisfies

$$\begin{cases}
u'(\tau) = A_q(\tau)\nu(\tau) + \varphi(\tau), \quad \text{a.e. } \tau \in (s, t), \\ \nu(s) = u_0 \end{cases}$$
where

\[ A_q(\tau) : D_q \longrightarrow X_q, \quad A_q(\tau)\varphi = A(\tau)\varphi \]

is the realization of \( A(\tau) \) in \( X_q \). Taking \( p = q = 2 \), we obtain that \( U_2(t, s) \) coincides with \( U(t, s) \), since for every \( u_0 \in W^{1,2}_0(\Omega) = (X_2, D_2)_{1/2,2} \), the function \( U_2(t, s)u_0 \) is a strong solution to (5.29), hence it is a weak solution. By uniqueness of the weak solution, the bounded operators \( U(t, s) \) and \( U_2(t, s) \) coincide on a dense subset of \( X \), and therefore they coincide on the whole \( X \). Morover, again by uniqueness, for \( q > 2 \) the operators \( U_q(t, s) \) are the parts of \( U(t, s) \) in \( X_q \). Therefore Hypotheses 2.1(1) and (2) are fulfilled.

Let us check that Hypothesis 2.9 is satisfied by the spaces \( X_q \), \( E_{\alpha, p} = (X_q, D_q)_{\alpha, p} \) with \( \alpha \in (0, 1/2) \) and \( X_p \) with \( p \in (2, q) \).

Theorem 2.5 of [PS01], for every \( (s, t) \in \Delta \) and \( \varphi \in L^2((s, t); X_q) \), the problem

\[
\begin{align*}
\phi'(\tau) &= A_q(\tau)\phi(\tau) + \varphi(\tau), \quad \text{a.e. } \tau \in (s, t), \\
\phi(s) &= 0
\end{align*}
\]

has an unique solution \( \phi \in L^2((s, t); D_q) \cap W^{1,2}((s, t); D_q) \), given by the variation of constants formula

\[
\phi(\tau) = \int_s^t U(\tau, \sigma)\varphi(\sigma)\,d\sigma \quad \tau \in (s, t),
\]

and there exists a constant \( C > 0 \) independent of \( s, t \), and \( \varphi \) such that

\[
\|\phi\|_{L^2((s, t); D_q)} + \|\phi\|_{W^{1,2}((s, t); X_q)} \leq C \|\varphi\|_{L^2((s, t); X_q)}.
\]

Therefore the mapping

\[
\{ \phi \in L^2((s, t); D_q) \cap W^{1,2}((s, t); X_q) : \phi(s) = 0 \} \longrightarrow L^2((s, t); X_q), \quad \phi \longmapsto \Phi(\phi) := \phi' - A_q(\cdot)\phi
\]

is an isomorphism. We recall now that for every couples of Banach spaces \( X, D \) such that \( D \subseteq X \) with continuous embedding, the space \( L^2((s, t); D) \cap W^{1,2}((s, t); X) \) is continuously embedded in \( C((s, t]; (X, D)_{1/2,2}) \) and the range of the trace operator \( \phi \longmapsto T\phi := \phi(t) \) is precisely \( (X, D)_{1/2,2} \). It follows that the range of the mapping

\[
L^2((s, t); X_q) \longrightarrow X_q, \quad \phi \longmapsto \int_s^t U(\tau, \sigma)\varphi(\sigma)\,d\sigma = T\Phi^{-1}\phi
\]

is equal to \( (X_q, D_q)_{1/2,2} \). By Hypothesis 5.2 the operator

\[
L^2((s, t); X_2) \longrightarrow L^2((s, t); X_q), \quad \phi \longmapsto B(\cdot)\phi
\]

is bounded and onto. Hence, the range of the operator \( L \) defined in (2.14) is still \( (X_q, D_q)_{1/2,2} \). In [PS01], it is proven that \( U(t, s) \) maps \( X_q \) into \( (X_q, D_q)_{1/2,2} \) and \( U(t, s) \) maps \( X_q \) into itself. Moreover there exist \( C, C_q > 0 \) such that

\[
\begin{align*}
\|U(t, s)x\|_{(X_q, D_q)_{1/2,2}} &\leq \frac{C}{(t-s)^{1/2}} \|x\|_{X_q}, \quad (s, t) \in \Delta, \quad x \in X_q \\
\|U(t, s)x\|_{X_q} &\leq C_q \|x\|_{X_q}, \quad (s, t) \in \Delta, \quad x \in X_q.
\end{align*}
\]
Hence Hypotheses 2.9 and 2.10 are satisfied by $X_q$. Moreover by (5.34), we have that for every $x \in \mathcal{O}$ and $p > 1$

$$ (5.41) \quad \|k(x, \cdot, t, s)\|_{L^p(\mathcal{O})} \leq \frac{M}{(t-s)^{\frac{d-p}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2(t-s)}} \, dy =: M_p(t-s)^{\frac{d-p}{2}} $$

and for every $\varphi \in X_p$ and $x \in \mathcal{O}$

$$ (5.42) \quad |U(t, s)\varphi(x)| \leq \int_{\mathcal{O}} |k(x, y, t, s)||\varphi(y)| \, dy \leq \frac{M_p'}{(t-s)^{\frac{d}{p'}}} \|\varphi\|_{X_p} $$

and

$$ (5.43) \quad \|U(t, s)\varphi\|_{\infty} \leq \int_{\mathcal{O}} |k(x, y, t, s)||\varphi(y)| \, dy \leq \frac{M_p'}{(t-s)^{\frac{d}{p'}}} \|\varphi\|_{X_p} $$

Again, as proven in [PS01], there exist $C_p > 0$ such that

$$ (5.44) \quad \|U(t, s)x\|_{X_p} \leq C_q \|x\|_{X_p}, \quad (s, t) \in \Delta, \quad x \in X_p. $$

Hence $U(t, s) \in \mathcal{L}(X_p; X_p) \cap \mathcal{L}(X_p; X_{\infty})$ and, by interpolation, $U(t, s) \in \mathcal{L}(X_p; X_q)$ since we know that $(X_p, X_{\infty})_{\theta,q} = X_q$ with the choice $\theta = 1 - \frac{2}{q}$. Moreover we have that there exists $C_{p,q} > 0$

$$ (5.45) \quad \|U(t, s)\|_{\mathcal{L}(X_p; X_q)} \leq \frac{C_{p,q}}{(t-s)^{\frac{d}{p}}(1-\frac{2}{q})}, \quad (s, t) \in \Delta. $$

For $(s, t) \in \Delta$ we split $U(t, s) = U(t, \frac{t+s}{2})U(t, \frac{t+s}{2}, s)$ and by (5.40) and (5.45) we have that there exist a constant $C > 0$ independent of $s$ and $t$ such that

$$ (5.46) \quad \|U(t, s)\varphi\|_{\mathcal{L}(X_p; (X_q, D_q)_{\frac{1}{2}, 2})} \leq \frac{C}{(t-s)^{\frac{d}{2}+\frac{d}{p}}(1-\frac{2}{q})}, \quad (s, t) \in \Delta. $$

Hence Hypotheses 2.9 and 2.10 are satisfied by $X_p$ with $p \in (2, q)$. Finally, we know that $U(t, s) \in \mathcal{L}(X_q; (X_q, D_q)_{\frac{1}{2}, 2})$ and by [PS01] $U(t, s) \in \mathcal{L}((X_q, D_q)_{\frac{1}{2}, 2}; (X_q, D_q)_{\frac{1}{2}, 2})$. Since $(X_q, (X_q, D_q)_{\frac{1}{2}, 2})_{2\alpha,p} = (X_q, D_q)_{\alpha,p}$ with $\alpha \in (0, \frac{1}{2})$, then by interpolation we have $U(t, s) \in \mathcal{L}(E_{\alpha,p}; (X_q, D_q)_{\frac{1}{2}, 2})$ and there exists a constant $C > 0$ independent of $s$ and $t$ such that

$$ (5.47) \quad \|U(t, s)\|_{\mathcal{L}(E_{\alpha,p}; (X_q, D_q)_{\frac{1}{2}, 2})} \leq \frac{C}{(t-s)^{\frac{d}{2}-\alpha}}. $$

Since the family $\{U(t, s)\}_{0,T}$ is a strongly continuous evolution operator on $E_{\alpha,p}$ (see again [PS01]), Hypotheses 2.9 and 2.10 are satisfied by $E_{\alpha,p}$.

**References**

[ABF23] Luciana Angiuli, Davide A. Bignamini, and Simone Ferrari. “Harnack inequalities with power $p \in (0, +\infty)$ for transition semigroups in Hilbert spaces”. In: *Nonlinear Differential Equations and Applications* 30.1 (2023). DOI: 10.1007/s00030-022-00812-0.
[Mas07] Federica Masiero. “Regularizing properties for transition semigroups and semilinear parabolic equations in Banach spaces”. In: Electron. J. Probab. 12 (2007), no. 13, 387–419. issn: 1083-6489. doi: 10.1214/EJP.v12-401.

[PS01] Jan Prüss and Roland Schnaubelt. “Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time”. In: J. Math. Anal. Appl. 256.2 (2001), pp. 405–430. issn: 0022-247X. doi: 10.1006/jmaa.2000.7247.