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The quantum entropic uncertainty relation and entanglement witness in the two-atom system coupling with the non-Markovian environments are studied by the time-convolutionless master-equation approach. The influence of non-Markovian effect and detuning on the lower bound of the quantum entropic uncertainty relation and entanglement witness is discussed in detail. The results show that, only if the two non-Markovian reservoirs are identical, increasing detuning and non-Markovian effect can reduce the lower bound of the entropic uncertainty relation, lengthen the time region during which the entanglement can be witnessed, and effectively protect the entanglement region witnessed by the lower bound of the entropic uncertainty relation. The results can be applied in quantum measurement, quantum cryptography task and quantum information processing.

Keywords: entropic uncertainty relation, quantum memory, entanglement witness, non-Markovian environment

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1 Introduction

The entropy uncertainty relation and its application have received plenty of attention in quantum optics and quantum information processing, currently. The uncertainty principle is a core of quantum mechanics and remarkably illustrates the difference between classical and quantum mechanics. There are two different formulae for the uncertainty principle: one is the Heisenberg uncertainty relation[1], which measures the quantum fluctuations of two observables by standard deviations, i.e. \( \Delta R \cdot \Delta Q \geq \frac{1}{2} |\langle [R,Q] \rangle| \), for two incompatible observables \( R \) and \( Q \). Another is the entropic uncertainty relation[2, 3, 4] which quantifies the quantum fluctuations of two observables in terms of entropy. Suppose \( p(x) \) is a probability distributions of the measurement outcome \( x \) for a random variable \( X \), the Shannon entropy \( H(X) = -\sum_k p_k(x)\log_2 p_k(x) \) indicates the uncertainty of \( X \). Thus the entropic uncertainty relation[5, 6, 7, 8, 9] is expressed as \( H(Q) + H(R) \geq \log_2 \frac{1}{c} \) for two incompatible observables \( R \) and \( Q \), where \( \frac{1}{c} \) is defined as the complementarity of the two observables. For non-degenerate observables, \( c := \max_{j,k} |\langle \psi_j | \phi_k \rangle|^2 \) if the eigenvectors of \( Q \) and \( R \) are respectively \( |\psi_j \rangle \) and \( |\phi_k \rangle \). Recently, some important progress

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has been acquired about the entropic uncertainty relation and its application. Several uncer-
tainty relations have been proposed, one of them is the uncertainty principle in the presence of
quantum memory which is put forward by Renes group[10, 11]. This uncertainty principle has
important significance that the known quantum information stored in the quantum memory
can reduce or eliminate the uncertainty about measurement outcomes of another particle which
is entangled with the quantum memory, and is confirmed in recent experiments[12, 13]. The
entropic uncertainty relations have been widely used in quantum entanglement witness[11],
security analysis of quantum cryptographic protocols[14], locking of classical correlation in
quantum state[15], quantum phase transitions[16] and quantum information processing[17].

For real quantum systems, which unavoidably interact with their environments[18, 19], how
do environments affect the entropic uncertainty relation in the presence of quantum memory?
Recently, more and more attention has been paid to this topic, such as, M. Feng et al.[20]
explored the quantum-memory assisted entropic uncertainty relation under noises and found
that the unital noises only increase the uncertainty while the amplitude-damping nonunit
noises may reduce the uncertainty in the long-time limit. H. Fan et al.[17] studied the relations
between the quantum-memory-assisted entropic uncertainty principle, quantum teleportation
and entanglement witness. A. K. Pati et al.[21] researched the influence of quantum discord and
classical correlation on the entropic uncertainty relation in the presence of quantum memory
and showed that the quantum discord and the classical correlations can tighten the lower bound
of Berta et al.[11]. In this paper, we investigate the quantum entropic uncertainty relation and
entanglement witness in the two-atom system coupling with the non-Markovian environments
by the time-convolutionless master-equation approach. We propose a method to reduce the
lower bound of the entropic uncertainty relation and enhance entanglement witness in non-
Markovian environments.

The paper is organized as follows. In Section 2, we present the time-convolutionless master-
equation of the two atoms in independent reservoirs. In Section 3, we introduce the entropic
uncertainty relation in the presence of quantum memory. Then we discuss in Section 4 the
influence of non-Markovian effect and detuning on the lower bound of the entropic uncertainty
relation and entanglement witness. Finally, a brief summary is given in Section 5.

2 The time-convolutionless master-equation of the two-
atom system in independent reservoirs

Suppose that two two-level atoms off-resonantly couple to two non-Markovian reservoirs
with zero temperature[19, 22]. The total Hamiltonian that describes such a system can be
written as($\hbar = 1$)

$$H = H_0 + \alpha H_1,$$

(1)
where

\[ H_0 = \omega_0 \sum_{j=A}^{B} S_j^z + \sum_{n} \omega_{n,A} b_{n,A}^{\dagger} b_{n,A} + \sum_{m} \omega_{m,B} b_{m,B}^{\dagger} b_{m,B} \]  

(2)

is the free Hamiltonian of the combined system. \( \omega_0 \) is the transition frequency of the two atoms, \( S_j^z \) is the inversion operators describing the atom \( j (j = A \text{ or } B) \), \( b_{n,A}^{\dagger} (b_{n,B}^{\dagger}) \) and \( b_{n,A} (b_{n,B}) \) are the creation and annihilation operators of the bosonic bath with the frequency \( \omega_{n,A} (\omega_{m,B}) \). The parameter \( \alpha \) is a dimensionless expansion parameter. The interaction Hamiltonian \( H_I \) is given by

\[ H_I = \sum_n g_{n,A} b_{n,A} S_A^+ + \sum_m g_{m,B} b_{m,B} S_B^+ + \text{h.c.}, \]  

(3)

where \( g_{n,A} (g_{m,B}) \) is the coupling constant between the atom and its corresponding reservoir, \( S_A^+ \) and \( S_A^- \) are the upward and downward operators of the atom, respectively. In the interaction picture, the Hamiltonian \( \alpha H_I \) reads as

\[ \alpha H_I(t) = \sum_{j=A}^{B} (S_j^+ \sum_n g_{n,j} b_{n,j} e^{i(\omega_0 - \omega_{n,j})t} + S_j^- \sum_n g_{n,j}^* b_{n,j}^* e^{-i(\omega_0 - \omega_{n,j})t}). \]  

(4)

In the second order approximation, the time-convolutionless (TCL) master equation [18], described by the density operator \( \rho_{AB}(t) \), has the following form

\[ \frac{d}{dt} \rho_{AB}(t) = -\alpha^2 \int_0^t d\tau Tr_E([H_I(t),[H_I(\tau),\rho_{AB}(t) \otimes \rho_E]]) \]  

(5)

with the environment state \( \rho_E \).

Here we have assumed that \( \rho(t) = \rho_{AB}(t) \otimes \rho_E \) and \( Tr_E([H_I(t),\rho_{AB}(0) \otimes \rho_E]) = 0 \), and Eq. (5) may be written as

\[ \frac{d}{dt} \rho_{AB}(t) = \mathcal{L}^{(A)} \rho_{AB}(t) + \mathcal{L}^{(B)} \rho_{AB}(t), \]  

(6)

where \( \mathcal{L}^{(j)} (j = A \text{ or } B) \) is the Liouville superoperator [23] associated to the Hamiltonian of Eq. (4) and it is defined by

\[ \mathcal{L}^{(j)} \rho_{AB}(t) = f_j(t)[S_{j-} \rho_{AB}(t),S_{j+}] + f_j^*(t)[S_{j+} \rho_{AB}(t),S_{j-}^+] + k_j(t)[S_{j+}^+ \rho_{AB}(t),S_{j-}] + k_j^*(t)[S_{j-}^+ \rho_{AB}(t),S_{j+}]. \]  

(7)

The correlation functions \( k_j(t) \) and \( f_j(t) \) are given by

\[ k_j(t) = i \sum_n |g_{n,j}|^2 \langle b_{n,j} b_{n,j}^\dagger \rangle_{E_j} \frac{1 - e^{i(\omega_0 - \omega_{n,j})t}}{\omega_0 - \omega_{n,j}} \]  

(8)

and

\[ f_j(t) = i \sum_n |g_{n,j}|^2 \langle b_{n,j}^\dagger b_{n,j} \rangle_{E_j} \frac{1 - e^{i(\omega_0 - \omega_{n,j})t}}{\omega_0 - \omega_{n,j}}, \]  

(9)
where $\langle b_{n,j}^\dagger b_{n,j} \rangle_{E_j} = T \rho_{E_j} (b_{n,j}^\dagger b_{n,j})$ and $\langle b_{n,j} b_{n,j}^\dagger \rangle_{E_j} = T \rho_{E_j} (b_{n,j} b_{n,j}^\dagger)$.

Presuming that the two reservoirs are initially prepared in the thermal state with zero temperature, the correlation functions reduce to

$$k_j(t) = 0, \quad f_j(t) = i \sum_n |g_{n,j}|^2 \frac{1 - e^{i(\omega_0 - \omega_{n,j})t}}{\omega_0 - \omega_{n,j}},$$

(10)

For a sufficiently large environment, we can replace the sum over the discrete coupling constants with an integral over a continuous distribution of frequencies of the environmental modes, i.e. $\sum_n |g_{n,j}|^2 \rightarrow \int_0^\infty d\omega J(\omega_j)$.

We consider the $j$-th ($j = A, B$) reservoir with the Lorentzian spectral density

$$J(\omega_j) = \frac{1}{2\pi} \frac{\gamma_0 \lambda_j^2}{(\omega_0 - \omega_j - \delta_j)^2 + \lambda_j^2},$$

(11)

where $\delta_j$ is the detuning between $\omega_0$ and the center frequency $\omega_j$ of the $j$-th reservoir. And the parameter $\lambda_j$ defines the spectral width of the coupling, which is connected to the reservoir correlation time $\tau_{Rj}$ by $\tau_{Rj} \approx \lambda_j^{-1}$. On the other hand, the parameter $\gamma_0$ can be shown to be related to the decay of the excited state of the atom in the Markovian limit of flat spectrum. The relaxation time scale $\tau_S$ over which the state of the system changes is then related to $\gamma_0$ by $\tau_S \approx \gamma_0^{-1}$. Utilizing Eq. (10) and Eq. (11), we can obtain the correlation functions

$$k_j(t) = 0, \quad f_j(t) = \frac{\gamma_0 \lambda_j}{2(\lambda_j - i\delta_j)} [1 - e^{i(\delta_j - \lambda_j)t}].$$

(12)

In the subsequent analysis of dynamical evolution of the system, typically a weak and a strong coupling regimes can be distinguished. For a weak regime we mean the case $\lambda_j > 2\gamma_0$, that is, $\tau_S > 2\tau_{Rj}$. In this regime the relaxation time is greater than the reservoir correlation time and the behavior of dynamical evolution of the system is essentially a Markovian exponential decay controlled by $\gamma_0$. In the strong coupling regime, that is, for $\lambda_j < 2\gamma_0$, or $\tau_S < 2\tau_{Rj}$, the reservoir correlation time is greater than or of the same order as the relaxation time and non-Markovian effects become relevant. The dynamical evolution of the system will oscillate up and down due to the memory and feedback of non-Markovian environments. For this reason we are interested in this regime and we shall mainly limit our considerations to this case[18, 25, 26, 27, 28].

3 The entropic uncertainty relation in the presence of quantum memory

We consider the entropic uncertainty game model illustrated in Ref.[11]. Before the game commences, Alice and Bob agree on the two measurements, the atomic polarization components $S_x$ and $S_y$. The game proceeds as follows: Bob sends an atom $A$, initially entangled with another atom $B$ (quantum memory), to Alice. Then, Alice measures $S_x$ and $S_y$, and announces
her measurement choice to Bob. The entropic uncertainty relation about Alice’s measurement outcomes in the presence of quantum memory is written as

$$H(S_x|B) + H(S_y|B) \geq \log_2 \frac{1}{c} + H(A|B),$$

where $H(A|B) = H(\rho_{AB}) - H(\rho_B)$, $H(S_x|B) = H(\rho_{S_xB}) - H(\rho_B)$ and $H(S_y|B) = H(\rho_{S_yB}) - H(\rho_B)$ are respectively the conditional von Neumann entropies of the states $\rho_{AB}$, $\rho_{S_xB}$ and $\rho_{S_yB}$, where $H(\rho) = -\text{tr}(\rho \log_2 \rho) = -\sum \lambda_j \log_2 \lambda_j$ is the von Neumann entropy of the state $\rho$.

The significance of this entropic uncertainty relation is that the known quantum information about $A$, stored in the quantum memory $B$, can reduce or eliminate the entropic uncertainty about Alice’s measurement outcomes. Thus the bigger the entanglement between $A$ and $B$ is, the smaller the lower bound of the entropic uncertainty relation is. The term $\frac{1}{c}$ quantifies the complementarity of the two observables. If $|\psi_j\rangle$ and $|\phi_k\rangle$ are respectively the eigenvectors of $S_x$ and $S_y$, $c := \max_{j,k} |\langle \psi_j|\phi_k\rangle|^2 = \frac{1}{2}$. The results in Refs.\[30, 31, 32\] show that a negative conditional entropy is a signature of entanglement, i.e. $\rho_{AB}$ is entangled when $H(A|B) < 0$. Hence the entanglement can be witnessed by the right-hand side of the inequality in Eq. (13). That is, $A$ is entangled with $B$ if $\log_2 \frac{1}{c} + H(A|B) < 1$.

Let that the initial state of the two atoms is

$$|\Psi(0)\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB}. \tag{14}$$

Solving Eq. (6), the density matrix of the two-atom system at time $t$ has the following form

$$\rho_{AB}(t) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{15}$$

After Alice measures $S_x$ or $S_y$, the system state is

$$\rho_{S_xB} = \sum_j (|\psi_j\rangle \langle \psi_j| \otimes I_B) \rho_{AB} (|\psi_j\rangle \langle \psi_j| \otimes I_B) \tag{16}$$

or

$$\rho_{S_yB} = \sum_k (|\phi_k\rangle \langle \phi_k| \otimes I_B) \rho_{AB} (|\phi_k\rangle \langle \phi_k| \otimes I_B). \tag{17}$$

In the following, we analyze numerically the lower bound of the entropic uncertainty relation and witness entanglement according to this lower bound.

4 Results and Discussions

Based on the formulae introduced in section 3, we can obtain numerically the lower bound of the entropic uncertainty relation and witness entanglement according to this lower bound.
We employ Wootter’s concurrence quantifying the entanglement between the two atoms, which is defined as
\[
C_{AB} = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})
\]  
(18)

where \(\lambda_i\) are the eigenvalues, organized in a descending order, of the matrix \(\tilde{\rho} = \rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)\).

For simplicity, we introduce three abbreviations. The minimum uncertainty \((MU)\) represents the lower bound of the entropic uncertainty relation in Eq. (13), i.e. \(MU = \log_2 \frac{1}{c} + H(A|B)\). The time of entanglement witnessed \((T_{EW})\) expresses the time region during which the entanglement can be witnessed by \(MU < 1\). The witnessed concurrence \((C_{EW})\) indicates the entanglement region witnessed by \(MU < 1\).

In Fig.1, we display the influence of detuning and non-Markovian effect on \(MU\) and \(C_{AB}\) versus \(\gamma_0 t\) in the two identical reservoirs \((\lambda_A = \lambda_B = \lambda, \delta_A = \delta_B = \delta)\). \(MU\)(blue line), \(C_{AB}\)(red line). (a)the influence of \(\delta\) on the entanglement witness in the non-Markovian regime \((\lambda = 0.1\gamma_0)\): \(\delta = 0\)(dotted-dashed line), \(\delta = 1.2\gamma_0\)(dashed line), \(\delta = 1.6\gamma_0\)(solid line); (b)the influence of \(\lambda\) on the entanglement witness with detuning \((\delta = \gamma_0)\): \(\lambda = 5\gamma_0\)(dotted-dashed line), \(\lambda = 0.1\gamma_0\)(dashed line), \(\lambda = 0.08\gamma_0\)(solid line).

In Fig.1, we display the influence of detuning and non-Markovian effect on \(MU\) and \(C_{AB}\) versus \(\gamma_0 t\) in the two identical reservoirs \((\lambda_A = \lambda_B = \lambda, \delta_A = \delta_B = \delta)\). From Fig.1(a), we can see that, in the non-Markovian regime \((\lambda = 0.1\gamma_0)\), if \(\delta = 0\), \(C_{AB}\)(red dotted-dashed line) quickly decays to 0 and \(MU\)(blue dotted-dashed line) becomes larger than 1 in a very short time. That is, when \(0 \leq \gamma_0 t < 2.5\), \(MU < 1\) and \(1 \geq C_{AB} > 0.568\), the entanglement between \(A\) and \(B\) can be witnessed by \(MU\), while \(2.5 \leq \gamma_0 t \leq 8.6\), \(MU \geq 1\) and \(0.568 \geq C_{AB} \geq 0\), thus the entanglement between \(A\) and \(B\) cannot be witnessed by \(MU\) though \(A\) is still entangled with \(B\). In this case, \(C_{EW}\) is from 0.568 to 1.0 and \(T_{EW}\) is from 0 to 2.5. A short \(T_{EW}\) will restrict the application of entanglement witness in quantum information. When \(\delta = 1.2\gamma_0\), \(C_{AB}\)(red dashed line) oscillates damply and disappears in a long time, but \(MU\)(blue dashed line) will be larger than 1 after finite timescales, i.e. when \(0 \leq \gamma_0 t < 31.2\), \(MU < 1\) and \(1 \geq C_{AB} > 0.568\), the entanglement between \(A\) and \(B\) can be witnessed by \(MU\), while \(\gamma_0 t \geq 31.2\), the entanglement between \(A\) and \(B\) cannot be witnessed due to \(MU \geq 1\) though \(A\) will be still entangled with \(B\) for a long time, so that \(T_{EW} \in [0, 31.2]\), but it is very interesting that there is still
$C_{EW} \in (0.568, 1]$. When $\delta = 1.6\gamma_0$, $C_{AB}$ (red solid line) is obviously protected and $MU$ (blue solid line) slow rises, $T_{EW} \in [0, 0.619]$ and $C_{EW} \in (0.568, 1]$. Thus $MU$ can be reduced, $T_{EW}$ can be lengthened but $C_{EW}$ can be effectively protected by increasing the detuning in the non-Markovian regime. Fig.1(b) shows that, in the detuning ($\delta = \gamma_0$), when $\lambda = 5\gamma_0$ (i.e. in the Markovian regime), $C_{AB}$ (red dotted-dashed line) fast decays and $MU$ (blue dotted-dashed line) rapidly increases, $T_{EW} \in [0, 0.46]$ and $C_{EW} \in (0.568, 1]$. When $\lambda = 0.1\gamma_0$ (i.e. with the small non-Markovian effect), $C_{AB}$ (red dashed line) reduces and $MU$ (blue dashed line) increases, $T_{EW} \in [0, 19.8]$ and $C_{EW} \in (0.568, 1]$. When $\lambda = 0.08\gamma_0$ (i.e. with the strong non-Markovian effect), $C_{AB}$ (red solid line) is effectively protected and $MU$ (blue solid line) slow rises, $T_{EW} \in [0, 32.5]$ and $C_{EW} \in (0.568, 1]$. Hence, in the detuning ($\delta = \gamma_0$), with $\lambda$ reducing, $MU$ will become small and $T_{EW}$ will extend, but $C_{EW}$ is invariant. Therefore, when both non-Markovian effect and detuning are present simultaneously, increasing detuning and non-Markovian effect can reduce the minimum uncertainty ($MU$), lengthen the time of entanglement witnessed ($T_{EW}$), and effectively protect the witnessed concurrence ($C_{EW}$).

If the two reservoirs have the same spectral width but different detunings, $MU$ (blue line) and $C_{AB}$ (red line) in Fig.2 exhibit different behaviors from Fig.1. From Fig.2(a), it is found that, in the non-Markovian regime ($\lambda_A = \lambda_B = 0.1\gamma_0$), when $\delta_A = 0$ and $\delta_B \geq 0$, $C_{AB}$ quickly reduces to 0 and $MU$ quickly rises to 2, $T_{EW}$ is very short. With $\delta_B$ increasing, $T_{EW}$ has only a little change but $C_{EW}$ becomes clearly narrow. For instance, when $\delta_B = 0$, $T_{EW} \in [0, 2.5]$ and $C_{EW} \in (0.568, 1]$, but when $\delta_B = 4\gamma_0$, $T_{EW} \in [0, 3.8]$ and $C_{EW} \in (0.663, 1]$. Fig.2 (b) indicates that, when $\delta_A = 0$ and $\delta_B = 2\gamma_0$, the value of $\lambda$ affects $MU$ and $C_{AB}$. With $\lambda$ decreasing, $T_{EW}$ has a little change but $C_{EW}$ becomes also narrow. For example, when $\lambda_A = \lambda_B = 5\gamma_0$, $T_{EW} \in [0, 0.38]$ and $C_{EW} \in (0.568, 1]$, but when $\lambda_A = \lambda_B = 0.05\gamma_0$, $T_{EW} \in [0, 5.3]$ and $C_{EW} \in (0.652, 1]$.  

![Fig2. MU and C_{AB} versus \gamma_0t in the different reservoirs which have the same spectral width but different detunings. MU (blue line), C_{AB} (red line). (a) the influence of \delta_B on the entanglement witness when \lambda_A = \lambda_B = 0.1\gamma_0 and \delta_A = 0: \delta_B = 0 (dotted-dashed line), \delta_B = 2\gamma_0 (dashed line), \delta_B = 4\gamma_0 (solid line); (b) the influence of \lambda on the entanglement witness when \delta_A = 0 and \delta_B = 2\gamma_0: \lambda_A = \lambda_B = 5\gamma_0 (dotted-dashed line), \lambda_A = \lambda_B = 0.1\gamma_0 (dashed line), \lambda_A = \lambda_B = 0.05\gamma_0 (solid line).](image)
Fig. 3 depicts \( MU \) (blue line) and \( C_{AB} \) (red line) versus \( \gamma_0 t \) when a reservoir is non-Markovian \( (\lambda_A = 0.1 \gamma_0) \) and another is Markovian \( (\lambda_B = 5 \gamma_0) \). The results show that the detunings \( \delta_A \) and \( \delta_B \) hardly affect entanglement witness. \( T_{EW} \in [0, 0.855] \) and \( C_{EW} \in (0, 0.645, 1] \) in Fig.3(a), \( T_{EW} \in [0, 0.838] \) and \( C_{EW} \in (0.641, 1] \) in Fig.3(b). As a result, \( T_{EW} \) and \( C_{EW} \) remain unchanged in this case.

Therefore, from the above analysis, we find that, only if the two environments are identically non-Markovian reservoirs, increasing detuning and non-Markovian effect can reduce the minimum uncertainty \( (MU) \), lengthen the time of entanglement witnessed \( (T_{EW}) \), and effectively protect the witnessed concurrence \( (C_{EW}) \).

We can explain the above results using the correlation function \( f(t) \) in Eq. (12). In Fig.4, we describe \( f(t) \) as a function of \( \gamma_0 t \). In the non-Markovian regime \( (\lambda = 0.1 \gamma_0) \), the effect of detuning on \( f(t) \) is shown in Fig.4(a). When \( \delta = 0 \), \( f(t) \) is always positive and quickly reach a bigger value. The quantum information will very speedily outflow from the atom so that \( C_{AB} \) rapidly declines, \( MU \) quickly increases and \( T_{EW} \) is very short. Nevertheless, when \( \delta > 0 \), \( f(t) \) oscillates and its amplitude reduces with \( \delta \) increasing. Negative value of \( f(t) \) can be understood as the feedback of information from the environment into the atom due to the memory effect of the environment. The amplitude decrease means the atomic decay rate reducing. Only if the two environments are identically non-Markovian reservoirs, the decay of two atoms can slows synchronously and the information is synchronously returned to the two atoms from the environments. In this case, increasing detuning can reduce \( MU \), lengthen \( T_{EW} \), and effectively protect \( C_{EW} \). The influence of the spectral width \( \lambda \) on \( f(t) \) is shown in Fig.4(b) with detuning \( (\delta = \gamma_0) \). When \( \lambda = 5 \gamma_0 \), \( f(t) \) is also always positive and immediately attains a stationary value, which leads to \( C_{AB} \) decreasing and \( MU \) increasing rapidly. However, when
\( \lambda < 2\gamma_0 \), \( f(t) \) oscillates and its amplitude reduces with \( \lambda \) decreasing. At the same time, only if the two non-Markovian reservoirs are identical, \( MU \) can be reduced, \( T_{EW} \) can be lengthened and \( C_{EW} \) can be effectively protected by enhancing the non-Markovian effect.

![Fig4](image-url)

**Fig4.** \( f(t) \) as a function of \( \gamma_0 t \). (a)the influence of \( \delta \) on \( f(t) \) in the non-Markovian regime(\( \lambda = 0.1\gamma_0 \)): \( \delta = 0 \) (blue dotted-dashed line), \( \delta = 1.2\gamma_0 \) (red dashed line), \( \delta = 1.6\gamma_0 \) (black solid line); (b)the influence of \( \lambda \) on \( f(t) \) with detuning(\( \delta = \gamma_0 \)): \( \lambda = 5\gamma_0 \) (blue dotted-dashed line), \( \lambda = 0.1\gamma_0 \) (red dashed line), \( \lambda = 0.05\gamma_0 \) (black solid line).

## 5 Conclusion

In conclusion, we have investigated the quantum entropic uncertainty relation and entanglement witness in the two-atom system coupling with the non-Markovian environments by the time-convolutionless master-equation approach. The influence of non-Markovian effect and detuning on the lower bound of the quantum entropic uncertainty relation and entanglement witness in the presence of quantum memory is discussed in detail. The results show that, only if the two non-Markovian reservoirs are identical, increasing detuning and non-Markovian effect can reduce the lower bound of the entropic uncertainty relation, lengthen the time region during which the entanglement can be witnessed, and effectively protect the entanglement region witnessed by the lower bound of the entropic uncertainty relation. The results can be applied in quantum measurement, entanglement detecting, quantum cryptography task and quantum information processing.

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