Ambeswar Phukon

Hyponormality of Toeplitz operators with polynomial symbols on the weighted Bergman space

Received: 11 September 2016 / Accepted: 30 April 2017 / Published online: 16 May 2017
© The Author(s) 2017. This article is an open access publication

Abstract This paper gives the complete proof of the Conjecture given by Hazarika and this author jointly which deals with a necessary and sufficient condition for the hyponormality of Toeplitz operator, $T_\varphi$ on the weighted Bergman space with certain polynomial symbols under some assumptions about the Fourier coefficients of the symbol $\varphi$.

1 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. For $-1 < \alpha < \infty$, let $L^2(\mathbb{D}, dA_\alpha)$ denote the Hilbert space consisting of functions on $\mathbb{D}$ which are square integrable with respect to the measure $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)\alpha dA(z)$, where $dA$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The inner product on $L^2(\mathbb{D}, dA_\alpha)$ is defined as $\langle f, g \rangle_\alpha = \int_\mathbb{D} f(z)\overline{g(z)}dA_\alpha(z)$, $f, g \in L^2(\mathbb{D}, dA_\alpha)$. The weighted Bergman space $A^2_\alpha(\mathbb{D})$ is then defined as the closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of analytic functions on $\mathbb{D}$.

For a nonnegative integer $n$, if $e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)\Gamma(n+2)}}z^n$, $z \in \mathbb{D}$ with the usual Gamma function $\Gamma(s)$, then the set $\{e_n\}$ forms an orthonormal basis for $A^2_\alpha(\mathbb{D})$ [12,13]. The reproducing kernel in $A^2_\alpha(\mathbb{D})$ is defined as $K_z^{(\alpha)}(w) = \frac{1}{(1-\overline{z}w)^{\alpha+1}}$, for $z, w \in \mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D}, dA_\alpha)$, the multiplication operator $M_\varphi$ on $A^2_\alpha(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi.f$. The orthogonal projection $P_\alpha$ of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$ is given by $(P_\alpha f)(z) = \langle f, K_z^{(\alpha)} \rangle_\alpha = \int_\mathbb{D} \frac{f(w)}{(1-\overline{z}w)^{\alpha+1}}dA_\alpha(w)$, for $f \in L^2(\mathbb{D}, dA_\alpha)$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator $T_\varphi$ with symbol $\varphi$ is defined on $A^2_\alpha(\mathbb{D})$ by $T_\varphi f = P_\alpha(\varphi.f)$. We, thus, have $T_\varphi f(z) = \int_\mathbb{D} \frac{\varphi(w)f(w)}{(1-\overline{z}w)^{\alpha+1}}dA_\alpha(w)$, $f \in A^2_\alpha(\mathbb{D})$ and $w \in \mathbb{D}$. The Hankel operator on $A^2_\alpha(\mathbb{D})$ is defined by $H_\varphi f = (I - P_\alpha)(\varphi.f)$.
A bounded linear operator $T$ on a Hilbert space is said to be hyponormal if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive semi-definite. The hyponormality of Toeplitz operators $T_\varphi$ on the Hardy space $H^2(\mathbb{T})$ where $\mathbb{T} = \partial \mathbb{D}$ was first characterised by Cowen [2] which was simplified by Nakazi and Takahashi [9]. For $\varphi \in L^\infty(\mathbb{T})$, we write

$$\mathcal{E}(\varphi) = \{ k \in H^\infty : \| k \|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty(\mathbb{T}) \}.$$ 

Then $T_\varphi$ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty. The solution was given on the basis of a dilation theorem of Sarason [11]. As for the Bergman space, no such dilation theory exists [3]; so, the characterization for the hyponormality of Toeplitz operators on the Bergman space is still remained an open question. For $\varphi = f + \bar{g}$ where $f, g$ are bounded analytic functions, Sadraoui [10] gave a necessary and sufficient condition for the hyponormality of Toeplitz operators on the Bergman space which is equivalent true on the weighted Bergman space too. His theorem is stated below:

**Theorem 1.1** [10] Let $f, g$ be bounded and analytic in $L^2(\mathbb{D}, dA)$. The following statements are equivalent:

(i) $T_{f+\bar{g}}$ is hyponormal;
(ii) $H^2_f H_g \leq H^2_f H_{\bar{f}}$;
(iii) $H_g = CH_{\bar{f}}$, where $C$ is of norm less than or equal to one.

Recently, Hwang and Lee [5], Hwang, Lee and Park [6], Lu and Liu [7], Lu and Shi [8] and Hazarika along with this author in [4] gave some necessary and sufficient conditions for the hyponormality of Toeplitz operators on weighted Bergman space with the class of functions $\varphi = f + \bar{g}$ where $f, g \in L^2(\mathbb{D}, dA_\alpha)$. In [4], the authors gave a Conjecture in which it was made an attempt to give a complete criteria for the hyponormality of $T_\varphi$ for this class of functions in the weighted Bergman space. The Conjecture is:

**Conjecture 1.2** Let $\varphi(z) = g(z) + f(z)$, where $f(z) = a_mz^m + a_Nz^N, g(z) = a_{-m}z^m + a_{-N}z^N \ (1 \leq m < N)$. If $a_m a_N = a_{-m} a_{-N}$ and $\alpha$ is sufficiently large, then $T_\varphi$ on $A^2_\alpha(\mathbb{D})$ is hyponormal

$$\iff \quad \left\{ \begin{array}{l}
\prod_{j=m}^{N-1} \left( \alpha - j \right) \left| a_{-m} \right|^2 - \left| a_m \right|^2 \leq \prod_{j=m}^{N-1} \left( \alpha - j \right) \left| a_N \right|^2 - \left| a_{-N} \right|^2 
N^2 \left( \left| a_{-m} \right|^2 - \left| a_m \right|^2 \right) \leq m^2 \left( \left| a_m \right|^2 - \left| a_{-m} \right|^2 \right) \quad \text{if } \left| a_{-N} \right| \leq \left| a_N \right| 
\end{array} \right.$$ 

In this paper, we give the proof of this Conjecture for all $\alpha > -1$. Since, the hyponormality of operators is translation invariant, we may assume that $f(0) = g(0) = 0$. We have the following properties of Toeplitz operators: If $f, g \in L^\infty(\mathbb{D}, dA_\alpha)$, then

(i) $T_{f+\bar{g}} = T_f + T_{\bar{g}}$
(ii) $T_{f\bar{g}} = T_{\bar{f}}$
(iii) $T_f T_{\bar{g}} = T_{f\bar{g}}$ if $f$ or $g$ is analytic.

Also, for the projection $P_\alpha$ of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$, we have

$$P_\alpha(z^s \bar{z}^t) = \begin{cases} 
\frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(\alpha+1)\Gamma(s-t+1)}z^{s-t} & \text{if } s \geq t \\
0 & \text{if } s < t
\end{cases}$$

where $s$ and $t$ are nonnegative integers. Again, for $\gamma_k = \| z^k \|_\alpha$, we have

$$\gamma_k^2 = \| z^k \|_\alpha^2 = (z^k, z^k)_\alpha = \int_{\mathbb{D}} z^k \bar{z}^k dA_\alpha(z) = (\alpha + 1) \int_{\mathbb{D}} |z|^{2k} (1 - |z|^2)^\alpha dA(z)$$

$$= (\alpha + 1) \int_0^1 t^k (1 - t)^\alpha dt = (\alpha + 1) B(k+1, \alpha+1) = \frac{\Gamma(k+1)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)}$$

2 The proof of the Conjecture

To prove the Conjecture we need some specific lemmas.

**Lemma 2.1** [1, 8] Fix $m \geq 1$. Then for $\alpha \geq -1$, 

$$\gamma_k^2 = \| z^k \|_\alpha^2 = (z^k, z^k)_\alpha = \int_{\mathbb{D}} z^k \bar{z}^k dA_\alpha(z) = (\alpha + 1) \int_{\mathbb{D}} |z|^{2k} (1 - |z|^2)^\alpha dA(z)$$

$$= (\alpha + 1) \int_0^1 t^k (1 - t)^\alpha dt = (\alpha + 1) B(k+1, \alpha+1) = \frac{\Gamma(k+1)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)}$$
(i) \( H_{z^k}(\xi)(\xi) = \begin{cases} 
\frac{\xi^m}{\xi^{m-k}} k^k & \text{if } 0 \leq k < m \\
\frac{\xi^m}{\xi^{m-k}} - \frac{\gamma_k^2}{\gamma_{k-m}} \xi^{k-m} & \text{if } m \leq k; 
\end{cases} \)

(ii) the functions \( \{H_{z^k}(\xi(\xi))\}_{k=0}^{\infty} \) are orthogonal in \( L^2(\mathbb{D}, \text{d}A) \);

(iii) \( H_{z^k}^*H_{z^m}(\xi(\xi)) = \omega_{mk}^2 \xi^k \); \( k = 0, 1, 2, \ldots \), where

\[ \omega_{mk}^2 = \begin{cases} 
\frac{\gamma_k^2 \xi^m}{\gamma_{m-k}} & \text{if } 0 \leq k < m \\
\frac{\gamma_k^2 \xi^m}{\gamma_{m-k}} - \frac{\gamma_k^2}{\gamma_{m-k}} & \text{if } m \leq k; 
\end{cases} \]

(iv) \( \|H_{z^k}(\xi(\xi))\|_a = \omega_{mk} \gamma_k \).

**Lemma 2.2** [5] For \( 0 \leq m \leq N \) and let \( K_i = \{k_i \in A_0^2(\mathbb{D}) : k_i(\xi) = \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i}\} \) where, \( i = 0, 1, 2, \ldots, N - 1 \), we have

\[ \|P_a(z^m)k_i(z)\|^2_a = \begin{cases} 
\sum_{k=0}^{\infty} \frac{(Nk+i)^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & \text{if } m \leq i \\
\sum_{k=1}^{\infty} \frac{(Nk+i)^2 \Gamma(Nk+i-m+\alpha+2) \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2 \Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & \text{if } m > i 
\end{cases} \]

**Lemma 2.3** Let \( k \geq 1 \) and \( 1 \leq m < N \). Let

\[ \xi(k) = \frac{(k+m)!}{\Gamma(k+m+\alpha+2)} - \frac{k^2 \Gamma(k-m+\alpha+2)}{\Gamma(k+\alpha+2)^2 \Gamma(k-m+1)} \]

\[ \zeta(k) = \frac{(k+N)!}{\Gamma(k+N+\alpha+2)} - \frac{k^2 \Gamma(k-N+\alpha+2)}{\Gamma(k+\alpha+2)^2 \Gamma(k-N+1)} \]

Then, \( \lim_{k \to \infty} \frac{\xi(k)}{\zeta(k)} = \frac{m^2}{N^2} \).

**Proof** By simplifying,

\[ \xi(k) = \frac{\prod_{j=0}^{m} (k+j)^{k+m}}{\prod_{j=0}^{m} (k+\alpha+2+j)^{k+m} - \prod_{j=0}^{m-1} (k+\alpha+2-j)^{k+m}} \]

\[ \zeta(k) = \frac{\prod_{j=0}^{m} (k+j)^{k+m}}{\prod_{j=0}^{m} (k+\alpha+2+j)^{k+m} - \prod_{j=0}^{m-1} (k+\alpha+2-j)^{k+m}} \]

If \( Q_n(k) \) is a polynomial of degree \( n \) in \( k \), then \( \prod_{j=1}^{m} (k+j) \prod_{j=1}^{m} (k+\alpha+2-j) - \prod_{j=0}^{m-1} (k-j) \prod_{j=0}^{m-1} (k+\alpha+2+j) \) is a polynomial \( Q_{2m-2}(k) \). Now, \( \prod_{j=1}^{m} (k+j) = k^m + \sum_{j=1}^{m} jk^{m-1} + Lk^{m-2} + \cdots + \prod_{j=1}^{m} j \)

where, \( L = 1.2 + (1+2) \cdot 3 + (1+2+3) \cdot 4 + \cdots + (1+2+3+\cdots+m-1) \cdot m \); and

\[ \prod_{j=1}^{m} (k+\alpha+2-j) = j(m+\alpha+2)^m - \sum_{j=1}^{m} j(j+\alpha+2)^{m-1} + L(k+\alpha+2)^{m-2} - \cdots \]

\[ = (k+\alpha+2)^m + \sum_{j=1}^{m-1} j(j+\alpha+2)^{k^{m-2} + \cdots} \]
If we denote the coefficients of $k^{(m)}$'s of the polynomial $Q_n(k)$ by $C_n$'s, then we have $C_{2m} = 0, C_{2m-1} = 0$.

$$C_{2m-2} = (\alpha + 2)^2 \sum_{j=1}^{m} j - (m-1)(\alpha + 2) \sum_{j=1}^{m} j + 2L + \sum_{j=1}^{m} j\left(m(\alpha + 2) - \sum_{j=1}^{m} j\right)$$

$$- (\alpha + 2)^2 \sum_{j=1}^{m-1} j - (m-1)(\alpha + 2) \sum_{j=1}^{m-1} j - 2L - \sum_{j=1}^{m-1} j\left(m(\alpha + 2) + \sum_{j=1}^{m-1} j\right)$$

$$= (\alpha + 2)\left(m + 2\sum_{j=1}^{m-1} j\right) + 2m \sum_{j=1}^{m-1} j - \left(\sum_{j=1}^{m-1} j\right)^2 + \left(\sum_{j=1}^{m-1} j\right)^2$$

$$= m^2(\alpha + 1)$$

Thus there exists a polynomial $Q_{2m-3}(k)$ such that $\xi(k) = m^2(\alpha + 1)k^{2m-2} + Q_{2m-3}(k)$. Similarly, $\xi(k) = N^2(\alpha + 1)k^{2N-2} + Q_{2N-3}(k)$.

Therefore,

$$\lim_{k \to \infty} \frac{\xi(k)}{\xi(k)} = \lim_{k \to \infty} \frac{m^2(\alpha + 1)k^{2m-2} + Q_{2m-3}(k)}{N^2(\alpha + 1)k^{2N-2} + Q_{2N-3}(k)} \prod_{j=-N}^{N-1} (k + \alpha + 2 + j) = \frac{m^2}{N^2}.$$ 

\[\square\]

Lemma 2.4 Let $f(z) = a_m z^m + a_N z^N$ and $g(z) = a_m z^m + a_N z^N$, with $1 \leq m < N$. Let $\alpha > -1$ and $a_m \tilde{a}_N = a_m \tilde{a}_N$. Then for $i \neq j$, we have $\langle H_j k_i(z), H_j k_j(z) \rangle_\alpha = \langle H_j k_i(z), H_j k_j(z) \rangle_\alpha$.

Proof

$$\langle H_j k_i(z), H_j k_j(z) \rangle_\alpha = \langle a_m H_{z^m} k_i(z), \tilde{a}_N H_{z^N} k_i(z), a_m H_{z^m} k_j(z), \tilde{a}_N H_{z^N} k_j(z) \rangle_\alpha$$

$$= |a_m|^2 \langle H_{z^m} k_i(z), H_{z^m} k_j(z) \rangle_\alpha + |\tilde{a}_N|^2 \langle H_{z^N} k_i(z), H_{z^N} k_j(z) \rangle_\alpha + a_m \tilde{a}_N \langle H_{z^N} k_i(z), H_{z^m} k_j(z) \rangle_\alpha + a_m \tilde{a}_N \langle H_{z^m} k_i(z), H_{z^N} k_j(z) \rangle_\alpha$$

$$= a_m \tilde{a}_N \langle H_{z^N} k_i(z), H_{z^m} k_j(z) \rangle_\alpha + a_m \tilde{a}_N \langle H_{z^m} k_i(z), H_{z^N} k_j(z) \rangle_\alpha.$$
Now we are ready to prove Conjecture 1.2.

Since, \( \langle H_{2^N} k_i(z), H_{2^N} k_j(z) \rangle_\alpha = 0 \) by Lemma 2.1.

Similarly, \( \langle H_{2^N} k_i(z), H_{2^N} k_j(z) \rangle_\alpha = a_{-m} \alpha_{-N}(H_{2^N} k_i(z), H_{2^N} k_j(z))_\alpha + \bar{a}_{-m} \alpha_{-N}(H_{2^N} k_i(z), H_{2^N} k_j(z))_\alpha \).

And, \( \langle H_{2^N} k_i(z), H_{2^N} k_j(z) \rangle_\alpha = 0 \).

Thus, using Lemma 2.4 in (2), we have that \( T_\psi \) is hyponormal if and only if

\[
\sum_{i=0}^{N-1} \langle H_j k_i(z), H_j k_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle H_{g_j} k_i(z), H_{g_j} k_i(z) \rangle_\alpha \geq 0.
\]

That is, if and only if

\[
\left( H_j \sum_{i=0}^{N-1} k_i(z), H_j \sum_{i=0}^{N-1} k_i(z) \right)_\alpha - \left( H_{g_j} \sum_{i=0}^{N-1} k_i(z), H_{g_j} \sum_{i=0}^{N-1} k_i(z) \right)_\alpha \geq 0.
\]

Similarly,

\[
\sum_{i=0}^{N-1} \langle M_j k_i(z), M_j k_i(z) \rangle_\alpha = \sum_{i=0}^{N-1} \langle (a m \bar{z}^m + a_N \bar{z}^N) k_i(z), (a m \bar{z}^m + a_N \bar{z}^N) k_i(z) \rangle_\alpha = |a_m|^2 \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^m k_i(z) \rangle_\alpha + |a_N|^2 \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_m a_N \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_m a_N \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha.
\]

Thus,

\[
\sum_{i=0}^{N-1} \langle M_{g_j} k_i(z), M_{g_j} k_i(z) \rangle_\alpha = |a_{-m}|^2 \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^m k_i(z) \rangle_\alpha + |a_{-N}|^2 \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_{-m} a_{-N} \sum_{i=0}^{N-1} \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_{-m} a_{-N} \sum_{i=0}^{N-1} \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha.
\]
Therefore, using the assumption \(a_m\bar{a}_N = a_{-m}\bar{a}_{-N}\), we have

\[
\sum_{i=0}^{N-1} \langle M_j k_i(z), M_j k_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle \tilde{M}_j \bar{k}_i(z), \tilde{M}_j \bar{k}_i(z) \rangle_\alpha
= \left( |a_m|^2 - |a_{-m}|^2 \right) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} |C_{Nk+i}|^2 |z^{Nk+i} \tilde{z}^m + z^m \tilde{z}^{Nk+i}|_\alpha
+ \left( |a_N|^2 - |a_{-N}|^2 \right) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} |C_{Nk+i}|^2 |z^{Nk+i} \tilde{z}^m + z^m \tilde{z}^{Nk+i}|_\alpha
= \left( |a_m|^2 - |a_{-m}|^2 \right) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} \left( Nk + i + m \right)! \Gamma(\alpha + 2) |C_{Nk+i}|^2 \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} \left( Nk + i + m + \alpha + 2 \right)! |C_{Nk+i}|^2
+ \left( |a_N|^2 - |a_{-N}|^2 \right) \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} \left( N(k+1) + i \right)! \Gamma(\alpha + 2) |C_{Nk+i}|^2.
\]

(4)

Also, we have

\[
\langle \tilde{M}_j \bar{k}_i(z), \tilde{M}_j \bar{k}_i(z) \rangle_\alpha = \langle \bar{a}_{-m} T_{\bar{z}^m} k_i(z), \bar{a}_{-N} T_{\bar{z}^N} k_i(z), \bar{a}_{-m} T_{\bar{z}^m} k_i(z) + \bar{a}_{-N} T_{\bar{z}^N} k_i(z) \rangle_\alpha
+ \left( |a_N|^2 \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha \right) + \bar{a}_{-m} a_N \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha
+ a_m \bar{a}_N \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha.
\]

And,

\[
\langle T_j k_i(z), T_j k_i(z) \rangle_\alpha = \left( |a_m|^2 \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha \right) + \bar{a}_{-m} a_N \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha
+ a_m \bar{a}_N \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha.
\]

Therefore, using the assumption \(a_m\bar{a}_N = a_{-m}\bar{a}_{-N}\) and Lemma 2.2, we have

\[
\sum_{i=0}^{N-1} \langle \tilde{M}_j \bar{k}_i(z), \tilde{M}_j \bar{k}_i(z) \rangle_\alpha - \sum_{i=0}^{N-1} \langle T_j k_i(z), T_j k_i(z) \rangle_\alpha
= \left( |a_m|^2 - |a_{-m}|^2 \right) \sum_{i=0}^{N-1} \sum_{m=0}^{\infty} \| P_\alpha (\bar{z}^m k_i(z)) \|_\alpha^2
+ \left( |a_N|^2 - |a_{-N}|^2 \right) \sum_{i=0}^{N-1} \sum_{m=0}^{\infty} \| P_\alpha (\bar{z}^N k_i(z)) \|_\alpha^2
= \left( |a_m|^2 - |a_{-m}|^2 \right) \sum_{i=0}^{N-1} \sum_{m=0}^{\infty} \left( Nk + i + m + \alpha + 2 \right)! \Gamma(\alpha + 2) \| C_{Nk+i} \|_\alpha^2
+ \left( |a_N|^2 - |a_{-N}|^2 \right) \sum_{i=0}^{N-1} \sum_{m=0}^{\infty} \left( Nk + i - N + \alpha + 2 \right)! \Gamma(\alpha + 2) \| C_{Nk+i} \|_\alpha^2
+ \left( |a_N|^2 - |a_{-N}|^2 \right) \sum_{i=0}^{N-1} \sum_{m=0}^{\infty} \left( Nk + i - N + \alpha + 2 \right)! \Gamma(\alpha + 2) \| C_{Nk+i} \|_\alpha^2.
\]

(5)
Thus by applying (4), (5) in (3) and simplifying, we have that $T_\psi$ is hyponormal if and only if
\[
(|a_m|^2 - |a_{-m}|^2) \left( \sum_{i=0}^{m-1} \frac{(i+m)! \Gamma(\alpha+2)}{\Gamma(i+m+\alpha+2)} |c_i|^2 + \sum_{k=1}^{N} \frac{(Nk+i+m)! \Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)} |c_{Nk+i}|^2 \right) + \sum_{k=1}^{N-1} \sum_{i=0}^{\infty} \left( \frac{(Nk+i+m)! \Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)} \right) |c_{Nk+i}|^2
\]
\[
- \left( \frac{(Nk+i)! \Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)} \right) |c_{Nk+i}|^2 \right) + \left( |a_N|^2 - |a_{-N}|^2 \right) \left( \sum_{k=0}^{N-1} \frac{(k+N)! \Gamma(\alpha+2)}{\Gamma(k+N+\alpha+2)} |c_k|^2 \right)
\]
\[
+ \sum_{k=N}^{\infty} \left( \frac{(k+N)! \Gamma(\alpha+2)}{\Gamma(k+N+\alpha+2)} \right) k^2 |c_k|^2 \geq 0.
\]

Let for all $k \geq 1$
\[
\xi(k) = \frac{(k+m)!}{\Gamma(k+m+\alpha+2)} - \frac{k^2 \Gamma(k-m+\alpha+2)}{\Gamma(k-m+\alpha+2) \Gamma(k-m+\alpha+1)}
\]
3 Conclusion

This theorem may give a clue to the readers to think about the generalised form of the hyponormality of Toeplitz operators with a class of polynomial symbols relaxing some of the restrictions to the Fourier coefficients.

Open Access  This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Arazy, J.; Fisher, S.D.; Peetre, J.: Hankel operators on weighted Bergman spaces. Am. J. Math. 110, 989–1053 (1988)
2. Cowen, C.C.: Hyponormality of Toeplitz operators. Proc. Am. Math. Soc. 103, 809–812 (1988)
3. Hwang, I.S.: Hyponormal Toeplitz operators on the Bergman space. J. Korean Math. Soc. 42, 387–403 (2005)
4. Hazarika, M.; Phukon, A.: On hyponormality of Toeplitz operators on the weighted Bergman space. Commun. Math. Appl. 3, 147–157 (2012)
5. Hwang, I.S.; Lee, J.: Hyponormal Toeplitz operators on the weighted Bergman spaces. Math. Inequal. Appl. 15, 323–330 (2012)
6. Hwang, I.S.; Lee, J.; Park, S.W.: Hyponormal Toeplitz operators with polynomial symbols on weighted Bergman spaces. J. Inequal. Appl. 2014, 335 (2014)
7. Lu, Y.; Liu, C.: Commutativity and hyponormality of Toeplitz operators on the weighted Bergman space. J. Korean Math. Soc. 46, 621–642 (2009)
8. Lu, Y.; Shi, Y.: Hyponormal Toeplitz operators on the weighted Bergman space. Integr. Equ. Oper. Theory 65, 115–129 (2009)
9. Nakazi, T.; Takahashi, K.: Hyponormal Toeplitz operators and extremal problems of Hardy spaces. Trans. Am. Math. Soc. 338, 753–769 (1993)
10. Sadraoui, H.: Hyponormality of Toeplitz operators and composition operators. Thesis, Purdue University (1992)
11. Sarason, D.: Generalized interpolation on $H^\infty$. Trans. Am. Math. Soc. 127, 179–203 (1967)
12. Zhu, K.: Theory of Bergman Spaces. Springer, New York (2000)
13. Zhu, K.: Operator Theory in Function Spaces. Marcel Dekker, INC., New York (2007)