REMARKS ON PARTIALLY ABELIAN EXACT CATEGORIES

THEO BÜHLER

Abstract. The purpose of this short and elementary note is to identify some classes of exact categories introduced in L. Previdi’s thesis. Among other things we show:

(i) An exact category is partially abelian exact if and only if it is abelian.
(ii) An exact category satisfies the axioms (AIC) and \((\text{AIC})^\circ\) if and only if it is quasi-abelian in the sense of J.-P. Schneiders.
(iii) An exact category satisfies (AIC) if and only if it is an additive category of the type considered by G. Laumon in his work on derived categories of filtered \(D\)-modules.

In all of the above classes all morphisms have kernels and cokernels and the exact structure must be given by all kernel-cokernel pairs.

1. Preliminary observations

For every morphism \(f: A \to B\) in an additive category there is an associated automorphism \([1 \, 0][f \, 1]\) of \(A \oplus B\). It yields an isomorphism of short sequences

\[
\begin{array}{cccc}
A & \xrightarrow{[1 \, 0]} & A \oplus B & \xrightarrow{[0 \, 1]} & B \\
\downarrow & & \downarrow & & \\
A & \xrightarrow{[f \, 1]} & A \oplus B & \xrightarrow{[0 \, 1]} & B.
\end{array}
\]

The morphism \([f \, 1]\) can be interpreted as the graph of \(f\). It has a left inverse given by the projection onto the first coordinate, while composition with the projection onto the second coordinate yields the morphism \(f = [0 \, 1][f \, 1]\). In particular, every morphism in an additive category is the composition of a split monic followed by a split epic.

The following lemma simply states that the intersection of the graph of \(f\) with the graph of the zero morphism coincides with the kernel of \(f\). The verification is straightforward.

Lemma. Let \(f: A \to B\) be a morphism in an additive category. Consider the diagram of solid arrows below

\[
\begin{array}{cccc}
K & \xrightarrow{h} & A & \xrightarrow{f} & B \\
\downarrow & & \downarrow & & \\
A & \xrightarrow{[f \, 1]} & A \oplus B & \xrightarrow{[0 \, 1]} & B.
\end{array}
\]

The pull-back square on the left exists if and only if \(f\) has a kernel \(k: K \to A\).

Corollary. An additive category admits pull-backs of all pairs of split monics if and only if it has kernels (if and only if it is finitely complete).

Date: February 25, 2013.

2010 Mathematics Subject Classification. 18E10, 18G25.
2. **Intrinsic Characterization of Exact Categories Satisfying (AIC)**

Previdi’s additional axiom (AIC) for exact categories [Pre10, 3.17] states that pull-backs of admissible monics along admissible monics exist and are admissible monics. Since split monics with cokernels are always admissible, we infer from (AIC) and the previous section that the underlying additive category must have kernels. Moreover, (AIC) and the lemma imply that all kernels are admissible monics.

Suppose \( f \) has a cokernel \( c \). Then \( \ker c \) is an admissible monic and \( c = \text{coker} \ker c \) implies that \( c \) is an admissible epic. Therefore the class of cokernels is equal to the class of admissible epics.

This discussion establishes the necessity part of the following characterization of exact categories satisfying (AIC):

**Proposition.** \((\mathcal{A}, \mathcal{E})\) is an exact category satisfying (AIC) if and only if:

(i) Every morphism in \( \mathcal{A} \) has a kernel.

(ii) The push-out of a kernel along an arbitrary morphism exists and is a kernel.

(iii) Cokernels are stable under pull-backs along arbitrary morphisms.

(iv) All kernels are admissible monics and all cokernels are admissible epics and \( \mathcal{E} \) is the class of all kernel-cokernel pairs in \( \mathcal{A} \).

In particular, if \( \mathcal{A} \) admits an exact structure satisfying (AIC), it is the unique maximal exact structure on \( \mathcal{A} \). Moreover, kernels are stable under pull-backs along arbitrary morphisms.

**Proof.** Suppose \((\mathcal{A}, \mathcal{E})\) satisfies (i), (ii), (iii) and (iv). Note that (ii) implies that every kernel has a cokernel and with (i) this shows that every morphism has a coimage (form the push-out of the kernel along zero).

In order to check that \( \mathcal{E} \) is an exact structure, Keller [Kel90, A.1] shows that it is enough to verify that the composition of kernels is again a kernel. To this end, let \((i, p)\) and \((j, q)\) be short exact such that \( ji \) is defined. We need to prove that \( j i \) is a kernel. Form the push-out under \( j \) and \( p \) to obtain the diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{i} & A & \xrightarrow{p} & A'' \\
\downarrow & & \downarrow^j & \text{PO} & \downarrow^{j''} \\
A' & \xrightarrow{j i} & B & \xrightarrow{r} & B'' \\
\downarrow & & \downarrow^q & & \downarrow^{q''} \\
C & \xrightarrow{} & C & & \\
\end{array}
\]

where \( q'' \) is uniquely determined by \( q'' r = q \) and \( q'' j'' = 0 \). Since \( q'' = \text{coker} j'' \) we have \((j'', q'') \in \mathcal{E}\). Using the push-out property one checks that \( r = \text{coker}(ji) \) and finally one checks that \( ji = \ker r \) so that \((ji, r) \in \mathcal{E}\).

It remains to verify (AIC). Consider the following diagram in which \((b, b') \in \mathcal{E}\) and \( f \) is arbitrary:

\[
\begin{array}{ccc}
A' & \xrightarrow{a} & A & \xrightarrow{a'} & A'' \\
\downarrow^f & & \downarrow & & \downarrow^f'' \\
B' & \xrightarrow{b} & B & \xrightarrow{b'} & B'' \\
\end{array}
\]

Define \((a, a') = (\ker(b' f), \text{coim}(b' f))\). Since \( b' f a = 0 \), there are unique \( f' \) and \( f'' \) making the diagram commutative. If \( \alpha : X \rightarrow A \) and \( \beta : X \rightarrow B' \) are such that \( f \alpha = b \beta' \) then \( b' f \alpha = b' b \beta = 0 \) and since \( a = \ker(b' f) \), there is a unique morphism \( \alpha' : X \rightarrow A' \) such that \( a \alpha' = \alpha \). Because \( b \) is monic we conclude \( \beta' = f' \alpha' \). Thus, the left hand square is a pull-back and (AIC) follows. \( \square \)
Remark. Exact categories satisfying (AIC) are studied in [Lau83, (1.3.0), p.160ff].

Remark. Since admissible monics have cokernels, every morphism has in addition a coimage.

\[
A \xrightarrow{f} B \\
\downarrow p = \text{coim } f \\
\downarrow \text{Ker } f \\
\downarrow \text{Coim } f
\]

The morphism \( m : \text{Coim } f \to B \) is monic: if \( mx = 0 \), form the pull-back over \( x \) and \( p \), notice that \( fy = 0 \) so that \( y \) factors through \( \text{Ker } f \). Therefore \( 0 = py = xq \), so \( x = 0 \) since \( q \) is epic. [A variant of this argument can be used to give a direct proof of [Pre10, Lemma 3.18].]

3. Exact categories satisfying \((\text{AIC}) \) and \((\text{AIC})^\circ\)

The characterization of exact categories satisfying \((\text{AIC}) \) in the previous section is very nearly self-dual. The only missing piece is the existence of cokernels. Thus:

**Proposition.** \((\mathcal{A}, \mathcal{E})\) is an exact category with \((\text{AIC}) \) and \((\text{AIC})^\circ\) if and only if:

(i) Every morphism in \( \mathcal{A} \) has a kernel and a cokernel.
(ii) Kernels are stable under push-outs along arbitrary morphisms.
(iii) Cokernels are stable under pull-backs along arbitrary morphisms.
(iv) \( \mathcal{E} \) is the class of all kernel-cokernel pairs in \( \mathcal{A} \).

These are precisely the quasi-abelian categories of Schneider [Sch99] and the almost abelian categories of Rump [Rum01]. \( \square \)

4. Partially abelian exact categories are abelian

An exact category is called **partially abelian exact** if every morphism which is the composition of an admissible monic followed by an admissible epic has a factorization as an admissible epic followed by an admissible monic:

\[
A \xrightarrow{f} B \\
\downarrow \exists p \\
\downarrow \exists i
\]

that is, \( f = em \) with \( m \) admissible monic and \( e \) admissible epic implies \( f = ip \) with \( p \) admissible epic and \( i \) admissible monic. This factorization shows that \( \ker f = \ker p \) and \( \coker f = \coker i \) and thus \( I \) is both coimage and image of \( f \).

From the first section we know that every morphism \( f : A \to B \) in an additive category can be written as the composition of a split monic followed by a split epic \( f = \left[ \begin{array}{r} 0 & 1 \\ \frac{i}{j} \end{array} \right] \). Thus, in a partially abelian exact category every morphism \( f \) factors as \( f = ip \) via admissible epic \( p \) followed by an admissible monic \( i \) and hence every \( f \) has a kernel and a cokernel.

Consider the factorization \( f = ip \) of a monic \( f \) into an admissible epic followed by an admissible monic. Then \( p \) is monic and since monic cokernels in an additive category are isomorphisms, \( p \) is an isomorphism from \( f \) to the admissible monic \( i \), so \( f \) is an admissible monic, and hence it is a kernel. Dually, every epic is an admissible epic and hence it is a cokernel.

**Proposition.** A category is partially abelian exact if and only if it is abelian. \( \square \)
5. Discussion of Previdi’s Theorem 3.24 and Proposition 3.22

The facts that for an exact category we have

\[ \text{partially abelian exact } \iff \text{abelian} \]

\[ (\text{AIC}) \iff (\text{AIC})^0 \iff \text{quasi-abelian} \]

contradict Theorem 3.24 in Previdi [Pre10]. The reason is that the proof of the theorem is based on the incorrect Proposition 3.22.

**Proposition.** Consider the additive category \( \text{Ban} \) of Banach spaces and bounded linear maps. Equip it with the usual maximal exact structure consisting of the short sequences whose underlying sequence of vector spaces is exact. Then \( \text{Ban} \) satisfies axioms (AIC) and (AIC)^0. On the other hand, \( \text{Ban} \) is not partially abelian exact.

**Proof.** One could appeal to the known fact that \( \text{Ban} \) is quasi-abelian and the proposition in section 3, but it is just as easy to verify (AIC) and (AIC)^0 directly:

Admissible monics are precisely the injective maps with closed range. Every admissible monic is isomorphic to the inclusion of a closed subspace. Given two closed subspaces \( U \) and \( V \) of a Banach space \( E \), their pull-back is the intersection \( U \cap V \) which is clearly closed in both \( U \) and \( V \), and (AIC) follows.

Admissible epics are precisely the surjective maps. The push-out of two morphisms in \( \text{Ban} \) is a quotient of their push-out in the category of vector spaces, so (AIC)^0 follows from the facts that \( \text{Vect} \) satisfies (AIC)^0 and that the composition of two surjective maps is surjective.

Since there are morphisms in \( \text{Ban} \) which do not have closed range, it cannot be partially abelian exact. \( \square \)

To see what goes wrong in the proof of Proposition 3.22, consider a bimorphism (monic-epic) \( f: A \to B \) which is not an isomorphism, e.g. the inclusion \( \ell^1 \subset \ell^2 \) in \( \text{Ban} \). In the following diagram the first two rows and columns are split exact:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow 1 \\
PB & \rightarrow & A \\
\downarrow & & \downarrow f \\
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & B \\
\end{array}
\]

Since \( f \) is monic we have \( 0 = \text{Ker} f \). It follows from the lemma in the first section that the upper left corner is a pull-back. Dually, the push-out of the two morphisms in the lower right corner is 0.

The Quillen embedding preserves pull-backs, so the morphism \( f \) is still monic in the abelian envelope \( F \): the left square of a morphism of short exact sequences in an abelian category is a pull-back if and only if the rightmost arrow is monic.

Of course, \( f \) cannot be the kernel of \( B \rightarrow 0 \) (neither in the abelian envelope \( F \) nor in \( A \)) since it is not an isomorphism. This gives an explicit example showing that the Quillen embedding preserves neither epics (\( f \) is epic in \( A \) but it is not epic in \( F \)), nor cokernels (the cokernel of \( f \) is no longer \( B \rightarrow 0 \)), nor push-outs (not even those involving only admissible epics), as seems to be assumed in the proof of Previdi’s Proposition 3.22. To reiterate: it does not follow that \( t \in A \), contrary to what is claimed in the proof.

Note that the above diagram can be substituted as diagram (3.21) by adding the cokernel \( e'' : B \rightarrow t \in F \) of \( f \) in the abelian envelope and the corresponding factorization \( j'' \) of \( [o e''] : A \oplus B \rightarrow t \) over \([ -f 1] \).
PARTIALLY ABELIAN EXACT CATEGORIES

Postscriptum

This is an unmodified version of a note sent to Braunling, Groechenig and Wilson. It was written late 2012 or early 2013 when I answered some questions that arose in their work on Tate objects in exact categories. At that point I also informed Previdi of these results. While others have since rediscovered variants of these ideas, there is still no citable reference for this material, so I made this publicly available on Braunling’s request.

Brüstle pointed out to the author that Hassoun and Roy [HR19] independently introduced AI-categories (pre-abelian exact categories satisfying (AIC)). Brüstle, Hassoun, Shah, Tatar and Wegner showed that AI-categories and quasi-abelian categories are the same, see [BHT20] Theorem 1.3. This is a variant of the results in sections 2 and 3 of this note. Readers who found this simple note interesting will find more in [BHT20] and the references therein.

References

[BHT20] Thomas Brüstle, Souheila Hassoun, and Aran Tattar, Intersections, sums, and the Jordan-Hölder property for exact categories, preprint arXiv:2006.03505 (2020), 1–34.

[HR19] Souheila Hassoun and Sunny Roy, Admissible intersection and sum property, preprint arXiv:1906.03246 (2019), 1–14.

[Ke90] Bernhard Keller, Chain complexes and stable categories, Manuscripta Math. 67 (1990), no. 4, 379–417. MR1052551 (91h:18006)

[Lau83] G. Laumon, Sur la catégorie dérivée des D-modules filtrés, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 151–237. MR726427 (85d:32022)

[Pre10] Luigi Previdi, Sato Grassmannians for generalized Tate spaces, preprint arXiv:1002.4863 (2010), 1–57.

[Rum01] Wolfgang Rump, Almost abelian categories, Cahiers Topologie Géom. Differentielle Catég. 42 (2001), no. 3, 163–225. MR1856638 (2002m:18008)

[Sch99] Jean-Pierre Schneiders, Quasi-abelian categories and sheaves, Mém. Soc. Math. Fr. (N.S.) (1999), no. 76, vi+134. MR1779315 (2001i:18023)

Email address: math@theobuehler.org