Strings on Curved Spacetimes: Black Holes, Torsion, and Duality

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We present a general discussion of strings propagating on noncompact coset spaces $G/H$ in terms of gauged WZW models, emphasizing the role played by isometries in the existence of target space duality. Fixed points of the gauged transformations induce metric singularities and, in the case of abelian subgroups $H$, become horizons in a dual geometry. We also give a classification of models with a single timelike coordinate together with an explicit list for dimensions $D \leq 10$. We study in detail the class of models described by the cosets $SL(2, \mathbb{R}) \otimes SO(1, 1)^{D-2}/SO(1, 1)$. For $D \geq 2$ each coset represents two different spacetime geometries: $(2D$ black hole)$\otimes \mathbb{R}^{D-2}$ and $(3D$ black string)$\otimes \mathbb{R}^{D-3}$ with nonvanishing torsion. They are shown to be dual in such a way that the singularity of the former geometry (which is not due to a fixed point) is mapped to a regular surface (i.e. not even a horizon) in the latter. These cosets also lead to the conformal field theory description of known and new cosmological string models.

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1. Introduction

Issues of principle in quantum gravity are among the most important questions that may eventually be addressed in the context of string theory. The study of curved spacetimes as string backgrounds could be used to investigate the string theoretical approach to physics at the Planck scale where classical methods are known to fail. The singularities in black hole and cosmological backgrounds are especially interesting.

The description of string backgrounds has been studied extensively by means of conformal field theory (CFT), but most of the effort thus far has been directed to the case where the noncompact part of the spacetime is flat, i.e. described by a trivial CFT, and only the internal space requires nontrivial CFT techniques. Coset models provide a rich class of explicit CFT’s for this case and lead to an understanding of the space of static tree-level vacua. Noncompact coset models provide a natural framework for nonstatic backgrounds and other nontrivial spacetimes which have recently received more attention, especially in the context of 2D gravity.

In this article we expand on previous approaches [1,2] to study noncompact cosets in the Wess-Zumino-Witten (WZW) formalism [3,4]. We present a general discussion of such spaces, classifying all those with a single timelike coordinate and any number of spacelike coordinates. These will provide the natural background for consistent string propagation. We also discuss the structure of singularities that occur in gauged WZW models, as well as the geometrical interpretation of the spaces obtained in this way. The metric obtained from the WZW model is singular and we show that there are singularities at fixed points of the gauge transformation. In cases where there is a dual gauging, these same group elements will provide horizons of the dual metric, generalizing the horizon/singularity duality.

We shall then discuss in detail a simple class of models that provides an example for any spacetime dimension: $SL(2, \mathbb{R}) \otimes SO(1,1)^{D-2}/SO(1,1)$. For $D = 2$, this is known to describe a single self-dual black hole geometry. We find that for $D > 2$, there are two geometries described by the two anomaly-free gaugings (vector and axial). Unlike the Lorentzian $D = 2$ case in which both gaugings give dual versions of the same black hole, the two geometries in this case are seemingly different. They are nonetheless dual to each other in a manner similar to mirror manifolds in string compactifications, where a single CFT can describe different target space geometries. For the axial gauging, we find the geometry $(2\text{D black hole})\otimes\mathbb{R}^{D-2}$, or a $D - 2$ black brane [5]. For the vector gauging, on the other hand, we find $(3\text{D black string})\otimes\mathbb{R}^{D-3}$ with nonvanishing torsion which is
a different black brane. We explicitly verify that the large $k$ (Kac-Moody level) limit of the gauged WZW model gives a solution of the field equations for dilaton, graviton and antisymmetric tensor field to lowest order in $\alpha'$. Unlike the 2D case, however, we are able to find a more general solution of those equations. The explicit form of the curvature scalar is used to clarify the nature of the singularities leading to the above black hole interpretation. Duality acts in a nontrivial way for these geometries, in particular the singularity for the axial gauging occurs for elements of the coset that are not fixed points, and its dual is at a regular surface in the asymptotically flat region of the vector gauged geometry. This illustrates the possibility that strings do not necessarily preclude spacetime singularities but may nonetheless be better behaved than expected on them.

In section 2 we present a general discussion of the geometry of noncompact coset spaces and their associated Kac-Moody algebras, and their description in terms of gauged WZW models. We discuss the conditions for anomaly-free gaugings and the conditions for the existence of isometries. The classification of all coset WZW models with a single timelike coordinate is a purely group theoretical question which we solve using the known properties of general coset spaces. Finally we discuss briefly the appearance of singularities in the large $k$ limit of gauged WZW models and their relation to fixed points of the corresponding gauge transformation.

In section 3 we construct explicitly the metric for the models $SL(2, \mathbb{R}) \otimes SO(1, 1)^{D-2}/SO(1, 1)$ for the two anomaly-free gaugings and obtain the geometries mentioned above. We explicitly verify that the background fields obtained from the large $k$ limit of the WZW model satisfy the string equations to lowest order in $\alpha'$. We also find the most general solution of those equations with the same number of isometries, and argue that it can be obtained from marginal perturbations of the coset CFT. Finally in section 4 we discuss duality of these solutions. We briefly review the duality of $\sigma$–models whenever there is an isometry (following reference [6]), and generalize to the case of several commuting isometries. This symmetry in particular relates spaces with torsion to spaces without torsion. We prove the relation between the two different geometries by identifying the vector transformation ($g \rightarrow hgh^{-1}$) as the isometry when gauging the axial transformation ($g \rightarrow hgh$), and vice-versa. We end with some final comments and future developments, and compare our work with other recent papers on the subject.
2. Noncompact coset CFT’s

The study of noncompact coset CFT’s was undertaken in [7] for SL(2,IR)/U(1) current algebra via the conventional GKO construction. The formalism was later generalized to any coset in [8]. Given a level \( k \) Kac-Moody algebra for a noncompact group \( G \),

\[
J_A(z) J_B(w) \sim -\frac{k \eta_{AB}}{(z-w)^2} + \frac{i f_{AB}^C J_C(w)}{(z-w)}
\]

(2.1)

(where \( \eta^{AB} = f_{AC}^D f_{BD}^C \) is the Cartan metric and \( g \) is the Coxeter number of \( G \)), the stress-energy tensor for a CFT based on \( G \) is given by the Sugawara from

\[
T_G(z) = \frac{\eta^{AB} : J_A(z) J_B(z):}{(-k + g)}.
\]

(2.2)

The corresponding central charge is \( c_G = k \dim G/(k - g) \). For the coset \( G/H \) with stress-energy tensor \( T_{G/H} = T_G - T_H \), the central charge is \( c_{G/H} = c_G - c_H \). The only changes from the compact case are the sign of \( k \) and of course the use of noncompact structure constants \( f_{AB}^C \). (The extension to supersymmetric coset models, discussed in [8], will not be considered here.) The spectrum and the corresponding elimination of negative norm states is not yet entirely understood for these models, and more progress is needed before we can properly treat the string vacua obtained from this approach.

In [2], it was shown that the SL(2,IR)/SO(1,1) current algebra could be interpreted as a two dimensional black hole spacetime. An implicit prescription for assembling the holomorphic and anti-holomorphic representations was given in terms of a gauged WZW model. In fig. 1, we reproduce the causal structure of this spacetime. In this section we shall provide some background on this construction, emphasizing the semi-classical limit in which various aspects of the geometry can be visualized, and which provides a geometric interpretation for the GKO current algebra construction. For an exact incorporation of all quantum corrections, however, we would need to return to the full conformal field theory / current algebra approach.

2.1. Anomalies

The WZW action in complex coordinates is written

\[
L(g) = \frac{k}{4\pi} \int d^2 z \text{tr}(g^{-1} \partial g g^{-1} \overline{\partial g}) - \frac{k}{12\pi} \int_B \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg),
\]

(2.3)
Fig. 1: The causal structure of the two dimensional black hole spacetime of [2]. Regions I,IV are asymptotic regions, regions II,III are inside the horizon, and regions V,VI are beyond the singularities.

where the boundary of $B$ is the 2D worldsheet. To promote the global $g \to h^{-1}_L g h_R$ invariance to a local $g \to h^{-1}_L(z) g h_R(\overline{z})$ invariance, we let $\partial g \to \partial g + A g$, and $\overline{\partial} g \to \overline{\partial} g - g \overline{A}$.

The gauge fields transform as $A \to h^{-1}_L(A + \partial) h_L$ and $\overline{A} \to h^{-1}_R(\overline{A} + \overline{\partial}) h_R$ (so that $D g \to h^{-1}_L D g h_R$ for $D$ equal to either holomorphic or anti-holomorphic covariant derivative). Vector gauge transformations correspond to $h_L = h_R$, and axial gauge transformations to $h_L = h^{-1}_R$. Substituting covariant derivatives in (2.3) gives the gauged action

$$L(g, A) = L(g) + \frac{k}{2\pi} \int d^2 z \, \text{tr}(A \overline{\partial} g^{-1} - \overline{A} g^{-1} \partial g - g^{-1} A g \overline{A}) .$$

(2.4)

Under the infinitesimal transformations $h_L \approx 1 + \alpha$, $h_R = 1 + \beta$, we have $\delta A = \partial \alpha + [A, \alpha]$ and $\delta \overline{A} = \overline{\partial} \beta + [\overline{A}, \beta]$. The anomalous variation of the effective action is (see e.g. [3] for a review)

$$\delta W = \frac{k}{2\pi} (\alpha \overline{\partial} A + \beta \partial \overline{A}) .$$

(2.5)

The variation of the (LR→VA) counterterm $\text{tr} A \overline{A}$, on the other hand, is

$$\delta (\text{tr} A \overline{A}) = \text{tr}\left(-\beta \overline{\partial} A - \alpha \partial \overline{A} + (\alpha - \beta) [\overline{A}, A]\right) .$$

(2.6)

For the abelian case, we see that (2.6) can compensate the variation (2.5) for either $\alpha = \pm \beta$ since the commutator term automatically vanishes. Thus both vector and axial-vector gauging are allowed. In the non-abelian case, only the vector gauging $\alpha = \beta$ is allowed. (An integrated form of this argument may be found in [10]: essentially the relevant $\pi_3$
obstruction is not captured by the trivial topology of $U(1)$. If we change sign $\bar{A} \rightarrow -\bar{A}$ for the axial gauged case (to give $A$ and $\bar{A}$ the same transformation properties), then the gauged action may be written

$$L(g, A) = L(g) + \frac{k}{2\pi} \int d^2z \text{tr}(A \partial_g g^{-1} \mp \bar{A} g^{-1} \partial g + A\bar{A} \mp g^{-1}Ag\bar{A}) , \quad (2.7)$$

where the upper and lower signs represent respectively vector ($g \rightarrow hgh^{-1}$) and axial-vector ($g \rightarrow hgh$) gauging.

It is intuitively reasonable that gauging $g \rightarrow hgh^{\mp 1}$ should result in a combination of holomorphic and antiholomorphic representations of $G/H$ algebras: since the equations of motion for the ungauged model result in $g(z, \bar{z}) = a(z) b(\bar{z})$, gauging left multiplication by $h(z)$ and right multiplication by $h^{\mp 1}(\bar{z})$ properly removes the $H$ degrees of freedom from both sides.

2.2. Semi-classical limit

We now consider some naive properties of the geometry described by (2.7) in the large $k$ (semi-classical) limit. Writing $A = A^a \sigma_a$ in terms of the generators $\sigma_a$ of $H$, and integrating out the components $A^a$ classically gives the effective action

$$L = L(g) \pm \frac{k}{2\pi} \int d^2z \text{tr}(\sigma_b g^{-1} \partial g) \text{tr}(\sigma_a \partial g g^{-1}) \Lambda_{ab}^{-1} , \quad (2.8)$$

with $\Lambda_{ab} \equiv \text{tr}(\sigma_a \sigma_b \mp \sigma_a g \sigma_b g^{-1})$. Notice that singularities of $\Lambda$ occur at least at fixed points of the gauge transformation $g \rightarrow hgh^{\mp 1}$. This is because for infinitesimal $h \approx 1 + \alpha^a \sigma_a$, we see that a fixed point $g$ satisfies $\sigma_a g \mp g \sigma_a = 0$. Multiplying by $g^{-1} \sigma_b$ and taking the trace, we see that $\Lambda = 0$ at a fixed point. (In the euclidean case it is easy to show the converse, i.e. that $\Lambda = 0$ implies a fixed point, whereas this is not true in the lorentzian case.)

From the transformation properties of the gauge fields and (2.6), we note that in the case of $H$ abelian the ungauged axial or vector symmetry remains a global symmetry, i.e. an isometry of the spacetime geometry. In the non-abelian case, not even a global vestige of the ungauged symmetry remains. In the abelian case, this implies that a fixed point of the ungauged symmetry corresponds to a point with vanishing Killing vector. For lorentzian signature, the surface carried into the fixed point by the isometry will be a null surface (the norm of the Killing vector is conserved), in general nonsingular and hence a horizon. We see that fixed points of symmetry transformations generically give
rise to metric singularities when the symmetry is gauged and to horizons when ungauged. This general property is the origin of the singularity/horizon duality of the 2D black hole of [2]. For the vector gauging, the metric can be written $ds^2 = -da db/(1 - ab)$, and the fixed point of the vector transformation (the gauged symmetry) corresponds to $ab = 1$, which is the singularity. The fixed point of the axial transformation (the isometry) is $a = b = 0$ indicating that the invariant surface $ab = 0$ is null, and provides the event horizon illustrated in fig. 1. For the axial gauging, the metric is identical (i.e. the geometry is self-dual) but the role of the fixed points is exchanged, implying the horizon/singularity duality pointed out in [11].

We now try to visualize in more detail the naive properties of the geometry described by (2.7) for $G = SL(2, \mathbb{R})$ in the large $k$ (semi-classical) limit. We take $SL(2, \mathbb{R})$ group elements parametrized as

$$g = w \mathbf{1} + x \sigma_1 + iy \sigma_2 + z \sigma_3 = \left( \begin{array}{cc} w + z & x + y \\ x - y & w - z \end{array} \right). \quad (2.9)$$

The condition that det $g = 1$ requires $w^2 + y^2 - x^2 - z^2 = 1$, so we see that $x, z$ parametrize the $\mathbb{R}^2$ and $w, y$ the $S^1$ of the $\mathbb{R}^2 \times S^1$ topology of $SL(2, \mathbb{R})$. We shall consider the actions $g \rightarrow hgh^{-1}$ for $h = h_{E,L}$, where

$$h_E \equiv e^{i\alpha \sigma_2}, \quad h_L \equiv e^{\alpha \sigma_3}. \quad (2.10)$$

Modding out by $h_E$ gives a Euclidean signature metric, and modding out by $h_L$ gives a Lorentzian signature metric.

For the euclidean case $h = h_E$, the action $g \rightarrow h_E gh_E^{-1}$ is easily seen to be rotation in the $1,3 = x, z$ plane. Thus we can always “gauge-fix” any point to, say, the positive $x$ axis (fig. 2). The origin is a fixed point of the transformation and results in a singularity of the metric. Crossing this half-line with endpoint singularity back together with the $w, y$ circle, we see that the spacetime takes the form of a “trumpet” (fig. 3).

The other action $g \rightarrow h_E gh_E$ is simply rotation in the $w, y$ circle. (This is easily seen by reparametrizing $g = \tilde{g} \sigma_1$, which exchanges $(x, z) \leftrightarrow (w, y)$, and noting that $g \rightarrow hgh \Rightarrow \tilde{g} \rightarrow h\tilde{g}h^{-1}$.) By rotation of the $w, y$ circle, we can always “gauge-fix” every point to, say, the point (1,0), thus eliminating the $S^1$ entirely. Since the action has no fixed points the

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1 The compact group $SU(2)$ has group elements $g = x_0 \mathbf{1} + i \vec{e} \cdot \vec{\sigma}$ so we identify $\sigma_1$ and $\sigma_3$ as the noncompact generators, giving signature $(- + -)$. 

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Fig. 2: Under rotation about the origin, any point in the plane can be gauge-fixed to the positive real axis. The origin is left fixed.

Fig. 3: The two dual versions of the Euclidean black hole. The upper “trumpet” version has a singularity at $r = 0$, reflecting the fixed point at the origin of fig. 2. The lower “cigar” version is free of singularities.

Metric on the remaining $\mathbb{R}^2$ can be entirely regular. For metrical reasons, we change to $r, \theta$ coordinates in which this $\mathbb{R}^2$ is naturally depicted as a “cigar”. Fig. 3 thus depicts the two dual semiclassical geometries of the Euclidean 2D black hole constructed in [2].

For the Lorentzian case $h = h_\mathbb{L}$, on the other hand, the action $g \rightarrow hgh^{-1}$ is a boost in the $(x, y)$ coordinates, and the action $g \rightarrow hgh$ is a boost in the $(w, z)$ coordinates. (The latter result is easily seen by the same reparametrization $g = \tilde{g}\sigma_1$ mentioned above.) Up to trivial reparametrization, these two actions are identical so we see that the Lorentzian version of the 2D black hole is self-dual. Instead of the compact action of fig. 2 and the gauge fixing to a single ray, for the Lorentz boost we have the action depicted in fig. 4,
which partitions the vicinity of the origin into four disjoint regions and leaves the origin fixed.

Modding out by the action \( g \rightarrow hgh \), we can gauge-fix the action of the \((w, z)\) boost to lines with \( z = 0 \) and \( w = 0 \) (which correspond respectively to \( a = \pm b \) in the parametrization \( g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \) used in [4]), as shown in fig. 4. In fig. 5, we transcribe this picture to the \( \text{SL}(2, \mathbb{R}) \) hyperboloid (with the \( x \) coordinate suppressed). The region \( z = 0 \) interpolates between the two fixed lines \( \pm (i\sigma_2 \cosh \alpha + \sigma_1 \sinh \alpha) \), passing along the \( x \) direction through the points \( g = \pm i\sigma_2 \) at \( y = \pm 1 \) in fig. 5. This \( z = 0 \) region encompasses two copies of regions I–IV of fig. 1. (The origins of the lightcones of fig. 1 appear at the points \( g = \pm 1 \), i.e. \( w = \pm 1 \), of fig. 5 with the \( x \) direction restored.) The region \( w = 0 \) encompasses two copies apiece of regions V and VI. As mentioned earlier, the (self-) duality in this case corresponds to interchanging \( w \leftrightarrow y, z \leftrightarrow x \), so we can see from fig. 5 how the duality exchanges region I (which is the asymptotic region moving upwards along the \( x \) direction from \( w = 1 \)) with region V, and similarly IV \( \leftrightarrow \) VI (see [1]). Region II (which lies between \( w = 1 \) and \( y = 1 \)) is mapped into itself, as is region III.

(Finally, for comparison with the semiclassical limit of compact cosets, we point out that the case of \( SU(2)/U(1) \) gives a disk with a singular boundary. Recall that \( g \in SU(2) \) can be parametrized as \( g = \cos \chi + i\hat{n} \cdot \vec{\sigma} \sin \chi \), where \( \chi \in [0, \pi] \) denotes an azimuthal angle on \( S^3 \) and \( \hat{n} \) parametrizes latitudinal \( S^2 \)’s with radius \( \sin \chi \). The action \( g \rightarrow hgh^{-1} \) with \( h = e^{i\alpha \sigma_3/2} \) simply rotates \( \hat{n} \) by angle \( \alpha \) about the 3-direction, i.e. for \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) the action is the translation \( \phi \rightarrow \phi + \alpha \). Modding out by this action simply removes the \( \phi \) coordinate and squashes each latitudinal \( S^2 \) parametrized
by \( \hat{n} \) to an interval \( \theta \in [0, \pi] \) with size still proportional to \( \sin \chi \). The result is a disk whose boundary, given by the circle \( e^{i \alpha \sigma_3/2} \), is a line fixed under the group action and consequently a singularity of the induced metric on the disk. By the argument used above in the noncompact case, modding out by the dual action \( g \rightarrow hgh \) results in an equivalent picture. (Write \( g = x_0 \mathbf{1} + i \vec{x} \cdot \vec{\sigma} \) as \( i \tilde{g} \sigma_1 \), which interchanges \( (x_0, x_3) \leftrightarrow (x_1, x_2) \), and has \( \tilde{g} \rightarrow h\tilde{g}h^{-1} \).) The \( U(1) \) gauged \( SU(2) \) WZW model thus gives another example of a theory that is self-dual in this sense.

2.3. Enumeration of possibilities

We have seen that the gauged "\( G/H \)" WZW models considered here are not the usual left or right \( G/H \) coset spaces with standard coset metric, as considered in the mathematical literature \([12]\) and in standard treatments of coset space nonlinear \( \sigma \)-models\([13]\). This is because we gauge \( g \rightarrow hgh^{\mp 1} \) type symmetries rather than \( g \rightarrow gh \) or \( g \rightarrow hg \), and as well we include a Wess-Zumino term which can add a torsion piece to the metric. Gauging the \( H \) subgroup nonetheless eliminates the \( H \) degrees of freedom, and it is easily verified that the signature of the resulting metric is the same as that of the standard coset metric. It is

\[ \text{Fig. 5:} \text{ The SL}(2,\mathbb{R}) \text{ hyperboloid with the } x \text{ coordinate suppressed. The two black dots represent the fixed points of fig. 4, and the gray lines represent the gauge-fixing. The two lightcones are the intersections of the hyperboloid with planes perpendicular to the } y \text{ axis at } y = \pm 1. \]
therefore straightforward to impose the phenomenological restriction to spaces with only a single timelike coordinate [1]. The only subtlety is that the level $k$ appears in front of the action. Positive $k$, in our sign conventions, results in a metric whose compact generators correspond to timelike directions and noncompact generators to spacelike directions. For negative $k$ (when allowed by unitarity), the roles of compact and noncompact generators are interchanged in the correspondence.

To classify all the coset CFT’s with a single timelike coordinate we consider first the case $k$ positive and examine the difference

$$N \equiv |G|_c - |H|_c,$$

for all possible cosets, where $|G, H|_c$ denote the number of compact generators. To this end we employ the known classification [12] of symmetric spaces $G/H$ (where $H$ is a maximal subgroup and $G$ is simple). From this list we eliminate all cases with $N > 1$, since for a given $G$ modding out by smaller (non-maximal) subgroups increases the value of $N$. For $N = 1$, this leaves only the case $SO(D - 1, 2)/SO(D - 1, 1)$ ([I]). For $N = 0$, which corresponds to maximal compact subgroup embeddings, the possibilities are listed in table 1. From this table we identify the cases for which $H$ has a $U(1)$ factor, $H = H' \times U(1)$ (hermitian symmetric spaces), so that $G/H'$ has an additional compact generator, hence one timelike coordinate. These latter cases are listed in table 2, and exhaust all possibilities in which $G$ is a simple group. For $k$ negative, we consider instead the difference $N = |G|_{nc} - |H|_{nc}$ of noncompact generators, and find that the only solution with $N = 1$ is $SO(D, 1)/SO(D - 1, 1)$.

For $G$ a product of simple groups and $U(1)$ factors, there are several possibilities to consider:

(i) $G = G_1 \otimes G_2 \otimes G_3$ and $H = H_1 \otimes H_2 \otimes H_3$ where $G_1/H_1$ is in table 2, $G_2/H_2$ is in table 1 (or products thereof) and $G_3/H_3$ is a (product of) compact coset(s).

(ii) $G = G'/\mathbb{R}$ where $G'/H$ has $N = 0$ (products of cases from table 1 and compact).

In this case $\mathbb{R}$ provides the timelike coordinate.

These are the most general cases. Possibilities such as products of cases in table 2 modded out by several $U(1)$’s, for example, are already included in case (i). In table 3, we list all such cases with coset dimension $\leq 10$ (and due to space limitations omit those with $G$ compact). In this table, it is implicitly understood that all possible embeddings $H \subset G$ are to be considered. The number of possible models so obtained is relatively small, particularly for lower dimensions.
Other possibilities may be obtained by enlarging consideration from semisimple groups \( G \) to non-semisimple groups of potential relevance, including of course the Poincaré group. In 3D, for example, ignoring the non-semisimple cases leaves only \( U(1)^3 \), \( SL(2,\mathbb{R}) \) and \( SU(2) \), whereas including them gives the nine groups corresponding to the Bianchi models considered in cosmology. We discuss briefly how to treat other potentially interesting cases involving non-semisimple groups.\(^2\) Let \( G \) be non-semisimple, then it is a semidirect product of \( S \) (a semisimple piece) and \( R \) (the radical, or maximal invariant subgroup) \(^1\). The algebra takes the form

\[
[S, S] = S, \quad [S, R] \in R, \quad [R, R] = R',
\]

where \( R' \subset N \subset R \). The Cartan matrix has zero eigenvalues, so the group manifold itself is not interesting, but modding out by subgroups may eliminate these zeroes to give a sensible space with a well-defined metric. The number of zeroes is the dimension of \( N \), the maximal nilpotent subalgebra of \( R \). The possible cosets are

1. \( G/N \). Since \( G/N = S \otimes R/N \) and \( R/N \) is an abelian invariant subalgebra, the zeroes are not eliminated but instead are equal in number to the dimension of \( R/N \).

2. \( G/R = S \). This case gives all the semisimple groups. Those with a single timelike coordinate are \( SL(2, R) \otimes C \) (we could also include \( C \otimes SO(1, 1)^n \)), where \( C \) is any compact semisimple group and one of the \( SO(1, 1)'s \) has negative level \( k \) to provide the timelike direction.

3. \( G/S = R \). In general \( R \) is an invariant subalgebra and will have abelian subalgebras with associated zero eigenvalues, so the zeroes are not eliminated. The only exception is when \( R \) itself is abelian, so the Cartan metric is not defined from the regular representation and will be nonsingular. This leads to interesting cases such as \( ISO(d-1, 1)/SO(d-1, 1) \), but the general classification is not known. Whenever \( R \) as a group has only a single timelike coordinate, the zeroes can be eliminated but since \( R \) is abelian the only choices are products of \( SO(1, 1)'s \) and \( U(1)'s \).

4. \( (G/S)/N = R/N \). As mentioned above this is an abelian subalgebra, so the Cartan metric is not defined from the regular representation and we are left with the situation of (3).

5. A more general situation would be to mod out by different subgroups not in the above decomposition, but this probably gives nothing new since \( R \) is a semidirect product

\(^2\) We thank V. Kaplunovsky for a question that prompted this discussion.
of abelian groups whose survival in the coset $G/H$ (for any $H$) would result in zero eigenvalues of the metric unless everything remaining is abelian as in (3) and (4) above. If they are all eliminated then we revert to case (2) above.

A general means of obtaining non-semisimple cosets is by group contractions, so it may well be possible to find a more systematic and complete procedure to generate all non-semisimple cosets using the properties of the semisimple cases.

3. $SL(2, \mathbb{R}) \otimes SO(1, 1)^{D-2}/SO(1, 1)$ Models

We now consider the simplest class of coset models with a single timelike coordinate and any number of spacelike coordinates. In order to find the metric in the large $k$ limit, we employ the standard procedure in nonlinear $\sigma$–models [13], as outlined in section 2: i.e. find a parametrization of the $G$ group elements, impose a unitary type gauge on the fields in the $\sigma$–model action and then solve for the (non-propagating) $H$-gauge fields to derive the $G/H$ worldsheet action. From that action we can read of the corresponding background fields. For the sake of generality, we write down the integrated action for a generic, not necessarily simple, group $G$ and a subgroup $H$, not necessarily abelian.

We first write the gauged WZW action (2.3),(2.7) as

$$L(g, A) = L(g) + \frac{1}{2\pi} \sum_i k_i \int d^2z \text{tr}(A \overline{dg}g^{-1} \mp \overline{A} g^{-1} \partial g + A \overline{A} \mp g^{-1} A g \overline{A})_i , \quad (3.1)$$

where $\mp$ represents respectively vector $(g \to hgh^{-1})$ and axial-vector $(g \to hgh)$ gauging.

The ungauged action is

$$L(g) = \frac{1}{4\pi} \sum_i k_i \int d^2z \text{tr}(g^{-1} \partial g g^{-1} \partial g)_i - \frac{1}{12\pi} \sum_i \int_B \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)_i , \quad (3.2)$$

where $i$ runs over the simple group factors in $G = \otimes_i G_i$. Writing $A = A^a \sigma_a$ in terms of the generators $\sigma_a$ of $H$, and integrating out the components $A^a$ classically gives the effective action

$$L = L(g) \pm \frac{1}{2\pi} \sum_{i,j} k_i k_j \int d^2z \text{tr}(\sigma_b g^{-1} \partial g)_i \text{tr}(\overline{\sigma_a \overline{g} g^{-1}})_j \Lambda_{ab}^{-1} , \quad (3.3)$$

with

$$\Lambda_{ab} \equiv \sum_l k_l \text{tr}(\sigma_a \sigma_b \mp \sigma_a g \sigma_b g^{-1}) . \quad (3.4)$$
Notice that at the singular points of $\Lambda$ the classical integration of the gauge fields fails, and hence (according to the discussion following (2.8)) where singularities of the target space metric are expected.

For the $SL(2, \mathbb{R}) \otimes SO(1, 1)^{D-2}/SO(1, 1)$ models, we parametrize the group elements as

$$g = \begin{pmatrix} g_0 & 0 & \ldots & 0 \\ 0 & g_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_{D-2} \end{pmatrix},$$

where

$$g_0 = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}$$

(with $ab + uv = 1$) (3.5)

and

$$g_i = \begin{pmatrix} \cosh r_i & \sinh r_i \\ \sinh r_i & \cosh r_i \end{pmatrix}$$

with $i = 1, \ldots, D-2.$ (3.7)

We choose the embedding such that the generator of $H = SO(1,1)$ is

$$\sigma = \begin{pmatrix} s_0 & 0 & \ldots & 0 \\ 0 & s_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & s_{D-2} \end{pmatrix},$$

where

$$s_0 = q_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$s_i = q_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with coefficients normalized to $\sum_{i=0}^{D-2} q_i^2 = 1.$ (3.8)

3.1. Vector gauging

Under the infinitesimal vector gauge transformations $\delta g = \varepsilon(\sigma g - g\sigma)$, the parameters transform as $\delta a = \delta b = \delta r_i = 0$, and $\delta u = 2\varepsilon q_0 u$, $\delta v = -2\varepsilon q_0 v$. The choices $u = \pm v$ thus fix the gauge completely. From $\pm u^2 = 1 - ab$, we are left with the parameters $a, b, r_i$ as the $D$ spacetime coordinates. Substituting (3.5) and (3.8) into (3.3), we find the action (for both gauge choices $u = \pm v$):

$$L = \frac{k_0}{2\pi} \int d^2 z \left( -\frac{\partial a \partial b + \partial b \partial a}{2(1 - ab)} + \sum_i \kappa_i \left( \delta_{ij} + \frac{\kappa_j \eta_i \eta_j}{1 - ab} \right) \partial r_i \overline{\partial r}_j \\ + \frac{\kappa_i \eta_i}{2(1 - ab)} \left( (b \partial a - a \partial b) \overline{\partial r}_i + \partial r_i (b \overline{\partial a} - a \overline{\partial b}) \right) \right).$$

(3.10)
where \( \kappa_i \equiv k_i/k_0 \) and \( \eta_i \equiv q_i/g_0 \). (From (3.9) we see that the \( \eta_i \)'s parametrize the embedding of \( SO(1,1) \) into the factored \( SO(1,1)'s \) in \( G \).)

This action can be identified with a \( \sigma \)-model action of the form

\[
S = \int d^2z \left( G_{MN} + B_{MN} \right) \partial X^M \overline{\partial X^N}
\]

(3.11)
to read off the background metric and antisymmetric tensor field (torsion). We see that (3.10) gives for \( D = 2 \) the (dual) black hole metric of [2] \((ds^2 = - da db/(1 - ab))\). For \( \kappa_i \to 0 \), it reduces as expected to the 2D black hole and for \( \eta_i \to 0 \) gives the 2D black hole times \( D - 2 \) flat coordinates, again as expected since in this limit \( H = SO(1,1) \) is completely embedded in \( SL(2, \mathbb{R}) \). Note that for any \( D \) there is no torsion, in particular the WZ term can be seen to be a total derivative for our choice of gauge. Furthermore we can observe that there are at least \( D - 2 \) isometries since the metric does not depend explicitly on the coordinates \( r_i \). Finally, note that the metric blows up only at the fixed point \( ab = 1 \) which is the point where (3.4) vanishes and the classical integration is not justified. The fixed point of the isometry \( g \to hgh \) is at \( ab = 0 \), which we expect to lead to a horizon.

To further analyze this metric, we change to coordinates in which it is diagonal (such coordinates are to be expected due to the large number of isometries). We consider (as in the 2D case) the regions bounded by the horizon and singularity (fig. 1):

\[
(i) \ 0 < ab < 1, \quad (ii) \ ab < 0, \quad (iii) \ ab > 1.
\]

(3.12)

\( i \) corresponds to the interior regions II, III; \( ii \) to the asymptotic regions I, IV ; and \( iii \) to the additional regions V,VI.

In the interior regions \( i \), we can change to coordinates \( t, X_0, X_i \) by defining

\[
a = \sin t \ e^{(X_0 + m X_{D-2})} \\
b = \sin t \ e^{-(X_0 + m X_{D-2})} \\
r_i = N_{ij} X_j,
\]

(3.13)

with

\[
N_{ij} = \left\{ \begin{array}{ll}
-\frac{\rho_j}{\rho_i \sqrt{\kappa_i}} & i = j + 1 \\
\frac{\sqrt{\kappa_j + 1} \eta_i \eta_{j+1}}{\rho_j + 1 \rho_j} & j \geq i \\
\frac{\eta_i}{\rho_j (\rho_j^2 + 1)^{1/2}} & i \leq j = D - 2 \\
0 & \text{otherwise}
\end{array} \right.
\]

(3.14)
\[ m = -q_0 \rho^2 , \quad (3.15) \]

and

\[ \rho_i^2 \equiv \sum_{i=1}^{l} \kappa_i \eta_i^2 , \quad \text{and} \quad \rho \equiv \rho_{D-2} . \quad (3.16) \]

The matrix elements \( N_{ij} \) satisfy the relations

\[ \sum_{l} \kappa_l N_{li} N_{lj} = \delta_{ij} \quad i, j \neq D - 2 , \]
\[ \sum_{l} \kappa_l N_{lD-2}^2 = 1/(\rho^2 + 1) , \]
\[ \text{and} \quad \sum_{l} \kappa_l \eta_i N_{lj} = 0 \text{ for } j \neq D - 2 . \quad (3.17) \]

In these coordinates the metric takes the diagonal form

\[ ds^2 = \frac{k_0}{2\pi} \left( -dt^2 + \tan^2 t \, dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right) . \quad (3.18) \]

The remaining regions are described similarly. For the asymptotic regions \((ii)\), we use

\[ a = \sinh R e^{X_0 + m X_{D-2}} \]
\[ b = -\sinh R e^{-(X_0 + m X_{D-2})} \]
\[ r_i = N_{ij} X_j , \quad (3.19) \]

with the same \( m \) and \( N_{ij} \) as above. In these coordinates the metric takes the form

\[ ds^2 = \frac{k_0}{2\pi} \left( dR^2 - \tanh^2 R \, dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right) . \quad (3.20) \]

Finally, in the regions \((iii)\) beyond the singularity the new variables are defined by

\[ a = \cosh R e^{(X_0 + m X_{D-2})} \]
\[ b = \cosh R e^{-(X_0 + m X_{D-2})} \]
\[ r_i = N_{ij} X_j \quad (3.21) \]

with metric

\[ ds^2 = \frac{k_0}{2\pi} \left( dR^2 - \coth^2 R \, dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right) . \quad (3.22) \]

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Using the symmetry $a \to -a$, $b \to -b$, we identify the geometry (2D black hole)$\otimes \mathbb{R}^{D-2}$. In particular the isometry generated by $g \to hgh$ is now explicit (it is a linear combination of translation in $X_0$ and the $X_i$'s). We can see how the associated Killing vector changes signature on each boundary: it is timelike in (3.20) and (3.22) and spacelike in the region (3.18) in between. In (3.10), this was not explicit in the $a, b, r_i$ coordinates. Although we have chosen a general embedding of $H = SO(1, 1)$ in all of $G$, the resulting geometry nonetheless coincides with the case $\eta_i = 0$, where $SO(1, 1)$ was embedded only in $SL(2, \mathbb{R})$. This is as expected since the $SO(1, 1)$ factors in $G$ are abelian and therefore transform trivially under $g \to hgh^{-1}$. The spacetime diagram for the relevant 2D geometry was shown in fig. 1.

3.2. Axial gauging

We now consider the axial gauging for which things are less trivial. Under the infinitesimal gauge transformation $\delta g = \varepsilon (\sigma g + gg \sigma)$, we see that $\delta u = \delta v = 0$ and $\delta a = 2\varepsilon q_0 a$, $\delta b = -2\varepsilon q_0 b$, $\delta r_i = 2\varepsilon q_i$. A simple choice that fixes the gauge completely is $a = \pm b$. Using $\pm a^2 = 1 - uv$ leaves $u$, $v$, and $r_i$ as the spacetime coordinates. The gauged WZW action for the axial gauging is (3.1) with the lower (+) signs, and integration over the gauge fields gives again (3.3) but with $\Lambda$ defined by the lower (+) sign in (3.4). Substituting (3.5) and (3.8) into (3.3), and using the above gauge fixing gives the effective action

$$L = \frac{k_0}{2\pi} \int d^2 z \left( \left( \kappa_i \delta_{ij} - \frac{\kappa_i \kappa_j \eta_i \eta_j}{1 - uv + \rho} \right) \partial r_i \partial r_j + \frac{(u \partial v - v \partial u)(u \partial v' - v \partial u')}{4(1 - uv + \rho)} \right)$$

$$- \frac{1}{2}(\partial u \partial v + \partial v \partial u) - \frac{(u \partial v + v \partial u)(u \partial v' + v \partial u')}{4(1 - uv)}$$

$$- \frac{\kappa_i \eta_i}{2(1 - uv + \rho)} \left[ (u \partial v - v \partial u) \partial r_i - \partial r_i (u \partial v - v \partial u) \right] \right). \quad (3.23)$$

From this expression we can make the following observations. First, unlike the vector gauging, there is nonvanishing torsion given by the term in square brackets in (3.23), even though the WZ term vanishes for the gauge choice made. We also can see that the metric has singularities at $uv = 1$, which in 2D is the fixed point of the axial transformation, and also at $uv = 1 + \rho$, which is not a fixed point. Again the lines $uv = 0$ represent horizons, and the metric and torsion do not depend on the $r_i$ variables so there are also the $D - 2$ isometries $r_i \to r_i + \text{constant}$. As in the vector case, the $D = 2$ ($\kappa_i = 0$) limit reproduces the 2D black hole of [2]. Furthermore the $\eta_i = 0$ limit gives the geometry (2D black
hole)⊗R^{D−2} (with vanishing torsion) as in the vector case, recovering the self-duality of those solutions.

The general case is more conveniently studied via variables that diagonalize the metric in different regions. We will consider the analog of the three regions (i), (ii), (iii) of the vector case (3.12), but with $a, b$ exchanged for $u, v$. In principle we could add an additional region due to the extra metric singularity which forms the inner horizon (fig. 6), but we shall find it is already included as part of region (iii).

It is straightforward to see that the same changes of variables (3.13), (3.19), (3.21) made for the vector case also diagonalize this metric, but now for $m = 0$ instead of (3.15). For (i) $0 < uv < 1$, we find

$$ds^2 = \frac{k_0}{2\pi} \left( -dt^2 + \frac{1}{(\rho^2 + 1)\tan^2 t + \rho^2} (dX_0^2 + \tan^2 t \, dX_{D-2}^2) + \sum_{l=1}^{D-3} dX_l^2 \right), \quad (3.24)$$

and the antisymmetric tensor is

$$B_{X_0 X_{D-2}} = \left( (\rho^2 + 1)\tan^2 t + \rho^2 \right)^{-1}. \quad (3.25)$$

In the region (ii) $uv < 0$, we have

$$ds^2 = \frac{k_0}{2\pi} \left( dR^2 + \frac{1}{(\rho^2 + 1)\coth^2 R - \rho^2} (dX_0^2 + \coth^2 R \, dX_{D-2}^2) + \sum_{l=1}^{D-3} dX_l^2 \right), \quad (3.26)$$
with torsion
\[ B_{X_0 X_{D-2}} = ((\rho^2 + 1) \coth^2 R - \rho^2)^{-1} . \quad (3.27) \]

Finally in the region (iii) \( uv > 1 \) the metric is
\[ ds^2 = \frac{k_0}{2\pi} \left( dR^2 + \frac{1}{(\rho^2 + 1) \tanh^2 R - \rho^2} (-dX_0^2 + \tanh^2 R dX_{D-2}^2) + \sum_{l=1}^{D-3} dX_l^2 \right) , \quad (3.28) \]

with torsion
\[ B_{X_0 X_{D-2}} = ((\rho^2 + 1) \tanh^2 R - \rho^2)^{-1} . \quad (3.29) \]

From these metrics we can compute the corresponding curvature scalar in each of the regions and find
\[ R = B_{X_0 X_{D-2}} + \text{constant} . \quad (3.30) \]

We see that \( R \) blows up only in region \( uv > 1 \) at the hyperbola \( uv = 1 + \rho^2 \) which is the real singularity, whereas the surface \( uv = 1 \) is only a metric singularity where the signature of the metric changes. The latter is another horizon, in addition to \( uv = 0 \). The geometry is thus \((3\text{D black string}) \otimes \mathbb{R}^{D-3}\) with nonvanishing torsion and an inner horizon. The 2D representation \((uv\) diagram\) with the eight different regions separated by the horizons and singularity, is presented in fig. 6. It is not surprising that there is a trivial \( \mathbb{R}^{D-3} \) crossed on since the high action of \( SO(1,1) \) only acts on one nontrivial linear combination of \( SO(1,1) \) generators of \( G \).

We have thus far given the expressions for the metric and antisymmetric tensor field for both gaugings, but not the expression for the dilaton. This can be found in principle by considering the correct measure in the path integral, but it is technically simpler to find it by solving the background field equations to lowest order in \( \alpha' \). This procedure will also be useful to verify that the expressions we have given are solutions of those equations. This is to be expected since they are valid for large values of \( k_0 \) (equivalent to \( 1/\alpha' \) in the sigma model expansion).

Let us consider the string background equations [14]
\[ R_{MN} + D_M D_N \Phi - \frac{1}{4} H_{MNP}^L H_{NLP}^L = 0 \quad (3.31) \]
\[ D_L H_{MN}^L - (D_L \Phi) H_{MN}^L = 0 \quad (3.32) \]
\[ R - 2\Lambda - (D \Phi)^2 + 2D_M D^M \Phi - \frac{1}{12} H_{MNP} H^{MNP} = 0 \, , \quad (3.33) \]
where \( \Lambda \equiv (D - 26)/3 \) is the cosmological constant in the effective string action and as usual \( H_{MNP} \equiv \partial_{[M}B_{NP]} \). To check whether the expressions obtained above for the metric and antisymmetric fields satisfy these equations, we can restrict to one of the regions. We choose the “cosmological” region \( 0 < uv, ab < 1 \) and assume an ansatz

\[
ds^2 = -dt^2 + \sum_{i=1}^{D-1} r_i^2(t) \, dX_i^2 ,
\]

(3.34)

\( \Phi = \Phi(t) \), and \( H_{MNP} = H_{MNP}(t) \). The two cases above are particular cases of this ansatz. For the vector gauging, \( r_i = \) constant for \( i \geq 2 \) and \( H_{MNP} = 0 \). For the axial gauging \( r_i = \) constant for \( i \geq 3 \) and \( H_{MNP} \) is nonvanishing only for \( M, N, P = 0, 1, 2 \). The case without torsion was solved in general in \([15]\) and has solutions

\[
r_i(t) = \alpha_i \tan^{p_i} \gamma t , \quad \sum_i p_i^2 = 1
\]

(3.35)

\[
e^\Phi = \beta \tan^{2p} \gamma t \sec^2 \gamma t
\]

(3.36)

with \( 2p = 1 + \sum p_i, \alpha_i \) and \( \beta \) arbitrary constants and \( \gamma^2 \equiv (26 - D)/6 \). It is easy to see that our solution for the vector gauging is a particular case of this class of solutions with \( p_i = 0, i > 1 \), as long as we make the shift \( k_0 \rightarrow k_0 - 2 \). This is suggested by the relation

\[
c = \frac{3k_0}{k_0 - 2} - 1 + (D - 2) = 26
\]

(3.37)

which implies \( (k_0 - 2)^{-1} = 26 - D/6 = \gamma^2 \). This representation makes it straightforward to discuss the limit \( D \rightarrow 26 \ (\gamma \rightarrow 0) \) of our solutions. Expanding \( \tan \gamma t \) and rescaling the variables we see that the metric depends on powers of \( t \) \([15]\) in the cosmological region and similarly for the other regions. The curvature scalar behaves similarly so there is no singularity and the black hole picture disappears. For \( D > 26 \ (k_0 > 2) \), if allowed by unitarity constraints (which have not yet been entirely clarified for noncompact cosets)m we can analytically continue the coordinates and get back the same black hole picture as for \( D < 26 \) with the interchanges \( \Pi \leftrightarrow \Pi(V) \) and \( \Pi \leftrightarrow \Pi(VI) \).

For the axial case the ansatz \((3.34)\) substituted into \((3.31)-(3.33)\) gives the equations

\[
- \sum_i \ddot{r}_i r_i + \ddot{\Phi} - \sum_{i<j} \frac{\dot{B}_{ij}^2}{2r_i^2 r_j^2} = 0
\]

(3.38)

\[
\frac{\ddot{r}_i}{r_i} + \sum_{j \neq i} \frac{\dot{r}_i \dot{r}_j}{r_i r_j} - \frac{\dot{r}_i \dot{\Phi}}{r_i} + \sum_{j \neq i} \frac{\dot{B}_{ij}^2}{2r_i^2 r_j^2} = 0
\]

(3.39)
\[
\Phi = \frac{\ddot{B}_{ij}}{B_{ij}} - \sum_i \frac{\dot{r}_i}{r_i} \tag{3.40}
\]

\[
2 \sum_{i<j} \dot{r}_i \dot{r}_j + \Phi^2 - \sum_{i<j} \frac{\dot{B}_{ij}^2}{2r_i^2 r_j^2} = 2\Lambda \tag{3.41}
\]

To make contact with (3.24)–(3.29) where the non-vanishing torsion \(B_{X_0 X_{D-2}}\) depends only on \(t\), we need consider only the values \(i, j = 0, 1, 2\). The most general solution of the equations for the case of interest is thus

\[
\begin{align*}
    r_1^2 &= \left( \sum_i \alpha_i^2 \tan^{2p_i} \gamma t \right)^{-1} \\
    r_2^2 &= r_1^2 A \tan^{2p_1 + 2p_2} \gamma t \\
    B_{12} &= r_1^2 \sum_i B_i \tan^{2p_i} \gamma t \\
    e^\Phi &= e^{\Phi_0} \frac{\dot{B}_{12}}{r_1 r_2} = \beta r_1^2 \tan^{p_1 + p_2 - 1} \gamma t \sec^2 \gamma t,
\end{align*}
\]

where \(B_i, \Phi_0, \beta, \) and \(\alpha_i\) are arbitrary, the \(p_i\)'s are constrained as above, \(A\) is given by \(A\alpha_1\alpha_2 = \alpha_1^2 B_2 - \alpha_2^2 B_1\), and the constant \(\gamma\) remains as above.

To see that this is the most general solution consistent with the ansatz is to count the number of independent parameters. Equations (3.38)–(3.40) provide four second order equations for the variables \(r_1, r_2, \Phi, B_{12}\). This allows eight free parameters given by the initial conditions on the variables and their first derivatives. Equation (3.41) gives a non-trivial relation among them, which reduces the number to seven. These seven parameters can be extracted from the expressions above, taking into account a seventh parameter \(t_0\) which results from the freedom \(t \rightarrow t + t_0\) to choose the origin of time (not an isometry). Notice that the symmetry \(r_1 \leftrightarrow r_2\) of the field equations (3.38)–(3.41) is realized by the symmetries \(p_i \rightarrow -p_i\) and \(p_1 \leftrightarrow p_2\) of the circle described by the parameters \(p_i\). Again we can see that the expressions (3.24), (3.25) obtained for the axial gauging are particular solutions of the above equations for \(p_1 = 1, p_2 = 0\), verifying that the WZW approach provides solutions of the field equations to lowest order in \(\alpha'\) if we shift \(k_0\) as before. Note also that we now have as well a solution for the dilaton field.

This solution was only valid for one of the regions of the black hole geometry \((0 < uv < 1)\). It is easy to treat the other regions. For \(uv < 0\) we can make the rotations \(it \rightarrow R, r_2 \rightarrow ir_1\) and \(r_1 \rightarrow r_2\). A similar rotation, including the shift \(t \rightarrow t + \pi/2\) gives the results for region \(uv > 1\). There we see that the dilaton field, which gives the string coupling
constant, blows up as expected only at the singularity $uv = 1 + \rho^2$. In the $D \to 26$ limit of these solutions, the torsion vanishes and the solution collapses to the vector gauging case (making it self-dual!). For $D > 26$ analytic continuation gives the same picture as for $D < 26$ (as in the vector case), so we see that the critical dimension plays an interesting role in our solutions. Note that we are unable to obtain solutions for all allowed values of the $p_i$ parameters via a WZW construction, but expect that exactly marginal deformations of the present CFT’s will access those parameters to complete the class of solutions.

4. Duality

The two different spacetime geometries, corresponding to the vector and axial gaugings of the $G/H$ WZW model, can be viewed as different modular invariant combinations of representations of the same holomorphic and anti-holomorphic chiral algebras. There are general arguments \[10\] that show that the vector and axial gaugings are dual, in the sense of having equal partition functions. The duality is similar to the familiar $r \to 1/r$ duality in $c = 1$ conformal field theory where two seemingly different theories are as well related by a changing the sign of the left (or holomorphic) currents $J = J_L$ with respect to the right (or anti-holomorphic) currents $\overline{J} = J_R$. The lorentzian $D = 2$ case is special since the same geometry (2D black hole) is obtained by either the vector or axial gauging, so we say that the model is self-dual. We now point out the sense in which geometries for $D \geq 2$ are dual, placing the vector/axial duality in a more generalized context.

In ref. \[6\], following previous developments in supergravity, the $r \to 1/r$ duality of compactified string theories was generalized to any string background for which the worldsheet action has at least one isometry. For completeness, we review this analysis and treat explicitly the case of $N$ commuting isometries in bosonic string theory. The worldsheet action for the bosonic string is

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left( (G_{MN}(X) + B_{MN}(X))\partial X^M \overline{\partial} X^N + \alpha'R^{(2)} \Phi(X) \right), \quad (4.1)$$

where $M, N = 1, \ldots D$; $G_{MN}, B_{MN}$ and $\Phi$ are the metric, antisymmetric tensor and dilaton backgrounds respectively, and $R^{(2)}$ is the 2D curvature. For a background with $N$ commuting isometries, we write the action in the form

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left( Q_{\mu\nu}(X_\alpha) \partial X^\mu \overline{\partial} X^\nu + Q_{\mu n}(X_\alpha) \partial X^\mu \overline{\partial} X^n + Q_{n\mu}(X_\alpha) \partial X^n \overline{\partial} X^\mu + Q_{mn}(X_\alpha) \overline{\partial} X^m \overline{\partial} X^n + \alpha'R^{(2)} \Phi(X_\alpha) \right), \quad (4.2)$$

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where \( Q_{MN} \equiv G_{MN} + B_{MN} \) and lower case latin indices \( m, n \) label the isometry directions. Since the Lagrangian (4.2) depends on \( X_m \) only through their derivatives, we can describe it in terms of the first order variables \( V^m = \partial X^m \),

\[
S = \frac{1}{4\pi\alpha'} \int d^2 z \left( Q_{\mu\nu}(X_\alpha) \partial X^\mu \overline{\partial} X^\nu + Q_{\mu n}(X_\alpha) \partial X^\mu \nabla^m + Q_{n \mu}(X_\alpha) V^n \overline{\partial} X^\mu + Q_{mn}(X_\alpha) V^m \overline{\partial} V^n + \alpha' R^{(2)} \Phi(X_\alpha) \right) .
\]

This can be alternatively interpreted as gauging the isometry with the constraint of vanishing gauge field strength [16]. Integrating the Lagrange multipliers \( \hat{X}_m \) in the above gives back (4.2). After partial integration and solving for \( V^m \) and \( V^m \), we find the dual action

\[
S' = \frac{1}{4\pi\alpha'} \int d^2 z \left( Q'_{\mu\nu}(X_\alpha) \partial X^\mu \overline{\partial} X^\nu + Q'_{\mu n}(X_\alpha) \partial X^\mu \overline{\partial} \hat{X}^n + Q'_{n \mu}(X_\alpha) \partial \hat{X}^n \overline{\partial} \hat{X}^\mu + Q'_{mn}(X_\alpha) \partial \hat{X}^m \overline{\partial} \hat{X}^n + \alpha' R^{(2)} \Phi'(X_\alpha) \right) .
\]

The dual backgrounds are given in terms of the original ones by

\[
\begin{align*}
Q'_{mn} & = Q_{mn}^{-1} \\
Q'_{\mu\nu} & = Q_{\mu\nu} - Q_{mn}^{-1} Q_{n\nu} Q_{\mu m} \\
Q'_{n \mu} & = Q_{nm}^{-1} Q_{m \mu} \\
Q'_{\mu n} & = -Q_{mn}^{-1} Q_{\mu m} .
\end{align*}
\]

To preserve conformal invariance, it can be seen by a careful consideration of the path integral [3] (and from other approaches [17]) that

\[
\Phi' = \Phi - \log \sqrt{\det G_{mn}}
\]

Notice that equations (4.3) reduce to the usual duality transformations for the toroidal compactifications of [18] in the limit \( Q_{m\mu} = Q_{\mu m} = 0 \). For the case of a single isometry \( (m = n = 0) \), we recover the explicit expressions of [3]. This is to our knowledge the most general statement of duality in string theory. In particular we see that a space with no torsion \( (Q_{m\mu} = Q_{\mu m}) \) can be dual to a space with torsion \( (Q'_{m\mu} = -Q'_{\mu m}) \), as found in the previous section. To prove the duality we should identify the particular isometry (of the \( D + 1 \) total) that relates them. Notice that for every isometry we do not have to go to the first order formalism, i.e. we can integrate the Lagrange multipliers \( \hat{X}_m \) for some of the fields and instead the \( V^m \) for the remaining fields with isometries. This is the most
general form of these duality transformations, and eqs. (4.3) and (4.6) should be read with indices \( m, n \) running over only the variables with isometries that have been dualized.

Before applying this formalism to the solutions we found in the previous section, we compare this approach to duality with others in the literature. In string theory, duality symmetry was originally discovered in toroidal compactifications and found to interchange winding states with momentum (Kaluza–Klein) states in the compactified theory. We have seen that the toroidal compactification is a particular case of a \( \sigma \)–model with isometries and thus has this symmetry manifest. The interchange of winding and momenta states realizes the duality symmetry in this particular background but is not necessarily a generic feature of duality, so we might expect duality even in backgrounds where winding modes are not present. A particular example is given by the 2D Lorentzian black hole reviewed here in section 2.

The existence of duality as well implies a continuous noncompact global symmetry (3.6) at the classical level that relates the field equations of the theory (4.2) with the Bianchi identities of the dual theory (4.4). For the case of two dimensional \( \sigma \)–models in the context of string theory, this symmetry was found in [13] to be \( SO(N, N) \). Duality is of course a discrete subgroup of this continuous symmetry. The noncompact continuous symmetries have been very useful to identify the moduli space in certain string compactifications and more recently have been used to find new nonstatic solutions from known ones [20,21]. We wish to emphasize that they are not true string symmetries since, just as in static backgrounds, they are broken by nonperturbative effects on the worldsheet. The surviving symmetry is a discrete symmetry which includes duality and integer shifts of the antisymmetric tensor, and as shown in [22] generalizes to \( SO(N, N, \mathbb{Z}) \).

In order to make a connection between duality in this formulation and the vector–axial duality in \( G/H \) WZW models, we recall the discussion of duality for the latter [10]. If \( H \) is abelian, group elements \( g \in G \) can be parametrized as \( g = e^{i\sigma \phi} \) where \( \sigma \) is the generator of \( H \) (considered to be \( U(1) \) for concreteness). Substituting into the vector gauged WZW action, it turns out that the action depends only on \( \partial \phi \), and proceeding with the standard duality transformations (4.5) the axial gauged action is obtained. The isometry in this case is then \( \phi \rightarrow \phi + \delta \phi \) which is generated by the right transformation \( g \rightarrow gh \). After gauging the \( g \rightarrow hgh^{-1} \) transformation, we see that there will always be a remaining isometry generated by the other independent transformation \( g \rightarrow hg \), or more symmetrically by the axial transformation \( g \rightarrow hgh \). The same occurs for the axial gauging, where the remaining isometry is given by the vector action \( g \rightarrow hgh^{-1} \). (In the
case of gauged non-abelian symmetries, it followed from the analysis of section 2 that due to quantum effects the ungauged symmetry does not remain even as a global symmetry.)

Now we are ready to analyze duality in the models of the previous section. Starting with the vector gauging, we have to see how the gauged fixed parameters transform under $g \rightarrow hg$, and go to a basis where only one of the coordinates transforms. In that basis the metric is not necessarily diagonal so the duality transformations (4.5) will be non-trivial. Let us consider the $0 < ab < 1$ region for concreteness. In that case we see that under $g \rightarrow hg$ the original parameters transform as

$$
\delta a = \varepsilon q_0 a, \quad \delta b = -\varepsilon q_0 b, \quad \delta r_i = \varepsilon q_i. \tag{4.7}
$$

We can see that the coordinates which diagonalize the metric transform as

$$
\begin{align*}
\delta t & = 0 \\
\delta X_0 & = \varepsilon (\rho^2 + 1) \\
\delta X_i & = 0, \quad i = 1, \ldots, D - 3 \\
\delta X_{D-2} & = \varepsilon,
\end{align*}
$$

where we have used (3.17). Note that if we exchange $X_{D-2}$ for $Y \equiv X_0 - \Omega X_{D-2}$, with $\Omega \equiv \rho^2 + 1$, the independent system of coordinates $t, X_0, Y$ and $X_i$ ($i = 1, \ldots, D - 3$) is such that only $X_0$ transforms, so this defines the isometry.

In these coordinates, the metric takes the form

$$
\begin{align*}
\mathrm{d}s^2 & = -\mathrm{d}t^2 + \left(\frac{\tan^2 t + 1}{\Omega^2}\right)\mathrm{d}X_0^2 + \frac{1}{\Omega^2} \mathrm{d}Y^2 - \frac{1}{\Omega^2} \mathrm{d}X_0 \mathrm{d}Y + \sum_{i=1}^{D-3} \mathrm{d}X_i^2. \tag{4.9}
\end{align*}
$$

From equations (4.5), we can find the dual metric with the single isometry $X_0 \rightarrow X_0 + \delta X_0$. It is straightforward to verify that it coincides with the one given in equation (3.24) coming from the $hgh$ gauging. This proves that regions III of both geometries are mapped to each other under duality. An identical analysis can be carried out for the other regions obtained

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3 Duality for the diagonal metric was explicitly proved in [23] to lowest order in $\alpha'$, and extended to next order in [24] where either a field redefinition or equivalently a change in the dilaton transformation was required. This had the interesting consequence that large to small radius duality could be explicitly realized in cosmology, as treated in [25], but requiring only weak coupling information from string theory, as recently analyzed in [26,27]. Duality for nondiagonal metrics relates not only large to small radius, but as well relates different cosmological models.
by analytic continuation of this region. It is straightforward to see that region V of fig. 1
is mapped to region I of fig. 6 and in particular the singularity of the first is mapped to
one horizon in the second. Also, region I of the vector gauging black hole gets mapped
to regions V and VII together of the axial gauging black hole. This has the interesting
implication that a surface which in one geometry is perfectly regular \((ab = \rho^2)\) is mapped
to the singularity in the other geometry \((uv = 1 + \rho^2)\). This goes even further than
the black hole singularity/horizon duality of the 2D black holes \([11]\), since in that case
the horizon is a better behaved region than the singularity but there remains nontrivial
behavior such as the exchange of spacelike and timelike coordinates. In the present case it
can be seen explicitly that string theory can deal with spacetimes that have singularities
at the classical level, in the sense that there still exists a description of interactions, etc.
for that region of spacetime. The situation is not so different from situations have been
encountered in compactified cases where singular spaces (orbifolds) are dual to nonsingular
ones (tori). (For a review in the simplest \(c = 1\) case, see \([28]\); in a more general context that
has recently arisen, see e.g. \([29]\).) It would be interesting to study the present geometries
at the singularity in more detail to start probing string theory in those regimes.

We have therefore established the duality between the \((2D \text{ black hole}) \otimes \mathbb{R}^{D-2}\) and
\((3D \text{ black string}) \otimes \mathbb{R}^{D-3}\) geometries. A more general analysis may be performed for the
combination of all the isometries using \((4.5)\) and for the more general solutions discussed
earlier. Also we can easily check that the solutions \((3.42)\) are related to those of \((3.35)\)
for three dimensions by duality — we rotate the spacelike coordinates in \((3.35)\) to get a
nondiagonal metric and apply \((4.5)\).

Similarly, starting from \((3.35)\) for any number of dimensions we can find new (cosmo-
logical) string solutions with torsion by applying \((4.5)\) after going to a nondiagonal basis.

For the boosted variables,
\[
X_M \equiv \Lambda_{MN} Y_N ,
\]
we see that dualizing equations \((3.35)\) gives the metric (using the same index convention
mentioned after \((4.5)\))

\[
G_{mn} = \left( \sum_M \alpha_M^2 \tan^{2p_M} \gamma t \Lambda_{Mm} \Lambda_{Mn} \right)^{-1}
\]

\[
G_{\mu\nu} = G_{mn} \sum_{M,N} \alpha_M^2 \alpha_N^2 \tan^{2p_M+2p_N} \gamma t \cdot \Lambda_{N\mu} \Lambda_{Mn} (\Lambda_{N\nu} \Lambda_{Mm} - \Lambda_{M\nu} \Lambda_{Nm}) ,
\]

\[
(4.11)
\]

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and torsion

\[ B_{\mu} = G_{\mu m} \sum_M \alpha_M \tan^{2p_M} \gamma t \Lambda_M \Lambda_{\mu} , \quad (4.12) \]

with a corresponding expression for the dilaton determined from (4.6). These expressions provide new explicit cosmological solutions with torsion for any value of \( m, n \leq N \) (and \( N \) varying from one to the total number of isometries). Notice that they depend on \( D^2 - D + 1 \) arbitrary parameters \( (\Gamma_{MN} = \alpha_M \Lambda_{MN}, p_M, \text{ and } t_0) \) which equals the number of boundary conditions allowed for \( G_{MN}, B_{MN} \) and \( \Phi \) and their first derivatives (minus the constraint provided by equation (4.6)). Eq. (3.42) is the \( D = 3 \) case of these general solutions. It is certainly interesting to explore the possible cosmological consequences of these solutions as well as the implications of their duality to the known solutions of [15]. Similar remarks apply for the generalization of the other black hole regions (by analytic continuation) and it would be interesting to determine if together they could lead to a geodesically complete system of coordinates for a black hole type of geometry with (4.11) as the cosmological region. This is certainly true for the cases we constructed in section 3 (\( p_1 = 1, \text{ all other } p_i = 0 \)), where (4.11) and (4.12) generate new dual black branes (although they are not directly obtained from the WZW construction).

5. Conclusions

We have presented a general discussion of the class of geometries that can be obtained in string theory from noncompact coset conformal field theories in the large \( k \) limit. The WZW approach plays a crucial role in allowing identification the background fields, in particular the metric of the target spacetime. This is not necessary for the CFT describing the internal degrees of freedom in superstring compactifications, since in that case a geometrical interpretation for that sector of the string vacua is unnecessary (and does not even necessarily exist in general).

We have found that the number of possible geometries obtained in this way is very restricted due to the constraint of having a single timelike coordinate.\footnote{Euclidean cosets, corresponding to the cases listed in table 1, remain useful for describing instanton-like configurations such as euclidean black holes, and for describing compactified dimensions.} In the case of superstring compactifications, however, the coset \( G/H \) describing the internal degrees of freedom actually provides many string vacua. These arise both from the different possible...
choices of boundary conditions (orbifoldizing) and also because they are only special points in a degenerate space of vacua parametrized by the exactly marginal deformations of the CFT. In the case of (2,2) compactifications, for example, the coset models describe particular points of a Calabi-Yau manifold.

The study of the spectrum of the noncompact coset models, especially the marginal deformations, will thus generate new classes of geometries which would be interesting to investigate and still represent exact CFT’s. A motivation for considering the models of section 3 was to find a conformal field theory realization of the cosmological solutions found in [15]. We have only partially succeeded since in the vector gauging our solutions in the cosmological region correspond to those of [15] only for fixed values of the parameters $p_i$ of equations (3.35). A natural expectation is that exactly marginal deformations of our models will turn on those parameters $p_i$ and generate the whole class of cosmological models of reference [15], as well as the new class of models with torsion we give in (3.42) for the axial gauging. This could be an interesting extension of the results here. Another possibility is that by marginal deformations, the black hole–like singularities of the noncompact cosets could be “blown-up”, reminiscent of the way orbifold singularities can be blown-up to obtain smooth string compactifications. This would illustrate the ameliorating control that string theory seems to have over singularities, already exemplified in section 4 by the duality between singular and regular regions of the different black hole geometries of section 3.

Our discussion of duality in section 4, following [6], is based entirely on the existence of isometries of the $\sigma$–model. Although it is the most general statement of that symmetry to date (including the known toroidal compactifications as particular cases), it cannot be the final statement because we know that there are geometries, such as Calabi-Yau spaces, that have no continuous isometries and still are known to have duality-like symmetries, for example the mirror symmetry of [30]. It would be very interesting to have a unified understanding of these dualities from a $\sigma$–model point of view.

Finally, since the subject of the present article has been evolving faster than our ability to write it up, we briefly discuss some of those recent results that have partial overlap with our work. First, the list of single-time cosets for the case of simple groups (table 2) was independently given in [31] using the known list of supersymmetric (Kähler) cosets (with the exception of the $SO(D – 1, 2)/SO(D – 1, 1)$ given earlier in [11]) but with no claim to completeness. The particular case $D = 3$ of the axial gauging geometry of the models of section 3, was discussed in [32] where a complete discussion of the black string geometry.

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can be found. The expressions for the background fields given therein are in agreement with those given here after a simple change of variables. Duality of that geometry was discussed in [33] and the relation with the vector gauging was briefly mentioned, although the proof of the duality of both geometries was not explicitly presented. More recently, duality for several commuting symmetries was discussed in [34] and an $SO(N, N, \mathbb{Z})$ was identified as the modular group in agreement with our comments about the general results of [19] and [22]. The periodic coordinates and the winding mode / momentum duality insisted upon by these authors, however, is not considered essential here. The solution of 3D cosmological backgrounds discussed in [35] are particular cases of our solutions (3.42) (up to analytic continuations). Other models have been recently explored [36] and together with the models considered here, most of the possible cases of our table 3 for $D \leq 4$ have been investigated.

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5 In [33], duality is found to map the singularity of the axial gauged geometry to a region inside the singularity of the vector gauged geometry, instead of to the asymptotically flat region as found here. The discrepancy is due to the presence of two isometries, and hence more than one possible duality. The two results are easily reconciled by applying a second duality transformation.
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| $G$          | $H$                | #H Generators | Dim($G/H$) |
|--------------|--------------------|---------------|------------|
| $SL(p, \mathbb{C})$ | $SU(p)$            | $p^2 - 1$     | $p^2 - 1$  |
| $SL(p, \mathbb{R})$ | $SO(p)$            | $\frac{1}{2}p(p - 1)$ | $\frac{1}{2}p(p + 1) - 1$ |
| $SU^*(2p)$   | $USp(2p)$          | $p(2p + 1)$   | $(p - 1)(2p + 1)$ |
| $SU(p, q)$   | $SU(p) \times SU(q) \times U(1)$ | $p^2 + q^2 - 1$ | $2pq$ |
| $SO(p, \mathbb{C})$ | $SO(p, \mathbb{R})$ | $\frac{1}{2}p(p - 1)$ | $\frac{1}{2}p(p - 1)$ |
| $SO(p, q)$   | $SO(p) \times SO(q)$ | $\frac{1}{2}(p(p - 1) + q(q - 1))$ | $pq$ |
| $SO^*(2p)$   | $SU(p) \times U(1)$  | $p^2$         | $p(p - 1)$  |
| $Sp(2p, \mathbb{C})$ | $USp(2p)$          | $p(2p + 1)$   | $p(2p + 1)$ |
| $Sp(2p, \mathbb{R})$ | $SU(p) \times U(1)$  | $p^2$         | $p(p + 1)$  |
| $USp(2p, 2q)$ | $USp(2p) \times USp(2q)$ | $p(2p + 1) + q(2q + 1)$ | $4pq$ |
| $G^5_2$      | $G_2(-14)$         | 14            | 14         |
| $G_2(+2)$    | $SU(2) \times SU(2)$ | 6             | 8          |
| $F^c_4$      | $F_4(-52)$         | 52            | 52         |
| $F_4(+4)$    | $USp(6) \times SU(2)$ | 24           | 28         |
| $F_4(-26)$   | $SO(9)$            | 36            | 16         |
| $E^c_6$      | $E_6(-78)$         | 78            | 78         |
| $E_6(+6)$    | $USp(8)$           | 36            | 42         |
| $E_6(+2)$    | $SU(6) \times SU(2)$ | 38           | 40         |
| $E_6(-14)$   | $SO(10) \times SO(2)$ | 46           | 32         |
| $E_6(-26)$   | $F_4(-52)$         | 52            | 26         |
| $E^c_7$      | $E_7(-133)$        | 133           | 133        |
| $E_7(+7)$    | $SU(8)$            | 63            | 70         |
| $E_7(-5)$    | $SO(12) \times SO(3)$ | 69           | 64         |
| $E_7(-25)$   | $E_6(-78) \times SO(2)$ | 79           | 54         |
| $E^c_8$      | $E_8(-248)$        | 248           | 248        |
| $E_8(+8)$    | $SO(16)$           | 120           | 128        |
| $E_8(-24)$   | $E_7(-133)$        | 136           | 112        |

Table 1: Noncompact coset spaces $G/H$ with no timelike coordinates. $G$ is simple and $H$ is the maximal compact subgroup.
| $G$       | $H$       | # $G$ Generators | # $H$ Generators | Signature |
|---------|---------|------------------|------------------|-----------|
|         |         | compact | non compact | compact | non compact |
| $SU(p, q)$ | $SU(p) \times SU(q)$ | $p^2 + q^2 - 1$ | $2pq$ | $p^2 + q^2 - 2$ | $0$ | $(1, 2pq)$ |
| $SO(p, 2)$ | $SO(p, 1)$ | $\frac{1}{2}p(p - 1) + 1$ | $2p$ | $\frac{1}{2}p(p - 1)$ | $p$ | $(1, p)$ |
| $SO(p, 2)$ | $SO(p)$ | $\frac{1}{2}p(p - 1) + 1$ | $2p$ | $\frac{1}{2}p(p - 1)$ | $0$ | $(1, 2p)$ |
| $Sp(2p, \mathbb{R})$ | $SU(p)$ | $p^2$ | $p(p + 1)$ | $p^2 - 1$ | $0$ | $(1, p(p + 1))$ |
| $SO^*(2p)$ | $SU(p)$ | $p^2$ | $p(p - 1)$ | $p^2 - 1$ | $0$ | $(1, p(p - 1))$ |
| $E_6(-14)$ | $SO(10)$ | $46$ | $32$ | $45$ | $0$ | $(1, 32)$ |
| $E_7(-25)$ | $E_6(-78)$ | $79$ | $54$ | $78$ | $0$ | $(1, 54)$ |

Table 2: Coset spaces $G/H$ with only one time coordinate (for simple groups $G$)
| $D$ | G/H |
|-----|-----|
| 2   | $\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(1,1)}$ |
| 3   | $\frac{\text{SO}(2,2)}{\text{SO}(2,1)}$; $\{(D = 2 \text{ case}); \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)}\} \times \mathbb{R}$ |
| 4   | $\frac{\text{SO}(3,2)}{\text{SO}(3,1)}$; $\frac{\text{SO}(2,2)}{\text{SO}(1,1) \times \text{SO}(2)}$; $\frac{\text{SO}(3)}{\text{SO}(3)}$; $\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(3)}$; $\{(D = 3)\} \times \mathbb{R}$ |
| 5   | $\frac{\text{SU}(2,1)}{\text{SU}(1,1)}$; $\frac{\text{SO}(2,2)}{\text{SO}(2)}$; $\frac{\text{SU}(2,2)}{\text{SO}(1,1) \times \text{SO}(2)}$. $\{(D = 4)\} \times \mathbb{R}$ |
| 6   | $\frac{\text{SO}(5,2)}{\text{SO}(5,1)}$; $\{(\text{SL}(3, \mathbb{R}) = 3)\}; \frac{\text{SO}(5,1)}{\text{SO}(5)}$; $\{(D = 5)\} \times \mathbb{R}$; $\{(D = 4) \times \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}\}$ |
| 7   | $\{(\text{SU}(3,1) = 6)\} \times \mathbb{R}$. $\{(D = 6)\} \times \mathbb{R}$ |
| 8   | $\frac{\text{SO}(7,2)}{\text{SO}(7,1)}$; $\{(\text{SU}(3,1) = 6)\}; \frac{\text{SO}(7)}{\text{SO}(7)}$. $\{(D = 7)\} \times \mathbb{R}$; $\{(D = 5) \times \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}\}$ |
| 9   | $\{(\text{SU}(2,2) = 6)\} \times \mathbb{R}$. $\{(D = 6) \times \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}\}$ |
| 10  | $\{(\text{SU}(2,2) = 6)\} \times \mathbb{R}$. $\{(D = 6) \times \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}\}$ |

Table 3: Noncompact coset spaces $G/H$ with one time coordinate with $\dim(G/H) \leq 10$, where $G$ is a product of simple noncompact groups.