Tight Approximation for Unconstrained XOS Maximization

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Abstract

A set function is called XOS if it can be represented by the maximum of additive functions, where a set function $f$ is additive if $f(X) = \sum_{x \in X} f\{\{x\}\}$ for each $X$. When such a representation is fixed, the number of additive functions to define the XOS function is called the width.

In this paper, we study the problem of maximizing XOS functions in the value oracle model. The problem is trivial for the XOS functions of width 1 because they are just additive, but it is already nontrivial even when the width is restricted to 2. We show two types of tight bounds on polynomial-time approximability for this problem. First, in general, the approximation bound is between $O(n)$ and $\Omega(n^{1-\epsilon})$ for any $\epsilon > 0$, where $n$ is the ground set size. Second, when the width of the input XOS functions is bounded by a constant $k \geq 2$, the approximation bound is between $k - 1$ and $k - 1 - \epsilon$ for any $\epsilon > 0$. In particular, we give a linear-time algorithm to find an exact maximizer of a given XOS function of width 2, while we show that any exact algorithm requires an exponential number of value oracle calls even when the width is restricted to 3.

Keywords XOS functions, Value oracles, Approximation algorithms.
1 Introduction

Maximizing a set function is a fundamental task in combinatorial optimization as well as algorithmic game theory. For example, when an agent has a valuation \( v : 2^V \rightarrow \mathbb{R} \) on an item set \( V \), a demand of the agent under prices \( p \in \mathbb{R}^V \) is a bundle that maximizes her utility, and computing a demand is amount to maximizing a set function \( f(X) := v(X) - \sum_{x \in X} p_x \). We remark that, even if the valuation is submodular functions in the property \( k \) representation is fixed, an \( \alpha \)-maximizer of a given \( f \) is an algorithm that always returns an \( \alpha \)-maximizer of a given \( f \) in \( O(n^{1/\alpha}) \) time. Buchbinder et al. \[2\] gave a 2-approximation algorithm for maximizing nonnegative submodular functions in the value oracle model, in which we can access a set function only by asking the function value of each set, and this is tight in the sense that no polynomial-time algorithm can achieve (2 - \( \epsilon \))-approximation (in expectation) for any positive constant \( \epsilon \) \[4\].

In this paper, we study the maximization problem for another basic class of set functions called XOS functions\[^1\], which generalize submodular functions (see also Appendix A). A set function \( f : 2^V \rightarrow \mathbb{R} \) is called XOS if it can be represented by the maximum of additive functions, i.e., there are set functions \( f_i : 2^V \rightarrow \mathbb{R} \) (\( i \in [k] := \{1, 2, \ldots, k\} \)) with \( f(X) = \sum_{v \in X} f_i(v) \) for each \( X \subseteq V \) such that

\[
\forall X \subseteq V : f(X) = \max_{i \in [k]} f_i(X) = \max_{i \in [k]} \sum_{v \in X} f_i(v),
\]

where \( f_i(v) \) means \( f_i(\{v\}) \) (we often denote a singleton \( \{x\} \) by its element \( x \)). When such a representation is fixed, \( k \) is called the width of \( f \). An XOS function admitting a representation of width \( k \) is called \( k \)-XOS. A 1-XOS function is just an additive function. The width of an XOS function could be exponential in \( |V| \), and we may assume that it is at most \( 2^{|V|} \).

If we are given an XOS function \( f : 2^V \rightarrow \mathbb{R} \) explicitly as the maximum of additive functions \( f_i \) (\( i \in [k] \)), it is easy to maximize because we have

\[
\max_{X \subseteq V} f(X) = \max_{i \in [k]} \max_{X \subseteq V} f_i(X) = \max_{i \in [k]} \sum_{v \in V} \max\{f_i(v), 0\}.
\]

However, if an XOS function \( f \) is given by a value oracle similarly to submodular function maximization, maximization of \( f \) becomes nontrivial even when the width of \( f \) is restricted to 2. Our goal is to clarify what can and cannot be done in polynomial time for maximizing XOS functions given by value oracles.

Our contributions

The main contribution is to give two tight bounds on polynomial-time approximation for maximizing XOS functions \( f : 2^V \rightarrow \mathbb{R} \) given by value oracles. Throughout the paper, we denote by \( n \) the cardinality of the ground set \( V \).

First, for the general case, we prove that the optimal approximation ratio is almost linear in \( n \). More precisely, we show the following two theorems. Here, for \( \alpha \geq 1 \), a set \( \tilde{X} \subseteq V \) is called an \( \alpha \)-maximizer if \( f(\tilde{X}) \geq \frac{1}{\alpha} \cdot \max_{X \subseteq V} f(X) \), and an \( \alpha \)-approximation algorithm is an algorithm that always returns an \( \alpha \)-maximizer.

**Theorem 1.** For any \( \epsilon > 0 \), there exists an algorithm to find an \( (en) \)-maximizer of a given XOS function in \( O(n^{1/\epsilon}) \) time.

\[^1\]XOS stands for XOR-of-OR-of-Singletons, where XOR means max and OR means sum. While XOS functions are assumed to be monotone in most existing literature, we allow non-monotone XOS functions.
**Theorem 2.** Let $\varepsilon > 0$. Then, any $n^{1-\varepsilon}$-approximation algorithm for XOS maximization requires exponentially many value oracle calls.

As general XOS functions are too hard to maximize, we analyze the problem by restricting the width of the input XOS functions. When the width is bounded by $k \geq 2$, we prove that the optimal approximation ratio is $k - 1$. More precisely, we show the following two theorems.

**Theorem 3.** There exists an algorithm to find a $(k-1)$-maximizer of a given $k$-XOS function in $O(k^2n)$ time for any $k \geq 2$ (even if $k$ is unknown). In particular, when $k = 2$, it finds an exact maximizer in $O(n)$ time.

**Theorem 4.** Let $k \geq 3$ and $\varepsilon > 0$. Then, any $(k-1-\varepsilon)$-approximation algorithm for $k$-XOS maximization requires exponentially many value oracle calls.

We note that our algorithms are deterministic, and the hardness results (Theorems 2 and 4) can be extended to randomized settings as shown in Section 3.

As we will see in Theorem 17, the number of order-different $3$-XOS functions is only single exponential in $n$, whereas there are a doubly-exponential number of order-different set functions in general. In this sense, $3$-XOS functions look much more tractable than general set functions. Then, Theorem 4 is somewhat counterintuitive, because it shows that there is no polynomial-time algorithm just to find a maximizer of a $3$-XOS function if it is given by the value oracle.

In addition to the $2$-XOS functions, we show another special class of XOS functions that can be exactly maximized in polynomial time.

**Theorem 5.** There exists a deterministic algorithm to find an exact maximizer of a given $k$-XOS function $f$ with the condition

\[
(*) \text{ for every } v \in V \text{ and every } i \in [k], \text{ either } f_i(v) = f(v) \text{ or } f_i(v) \leq 0
\]

in $O(n^{k+1})$ time for any $k \geq 2$ (even if $k$ is unknown).

Finally, we show that the optimal approximation ratio is almost linear in $n$ for maximizing the maximum of only two set functions one of which is additive and the other is constant (see Theorem 20 in Appendix B).

**Related work**

The problem of maximizing monotone functions under cardinality constraint has also been paid much attention. For the submodular case (which is a special case of the XOS case), the greedy algorithm is the best possible and returns an $e/(e-1)$-approximation solution \cite{10, 11}. For the XOS case, no polynomial-time algorithm can achieve $n^{1/2-\varepsilon}$-approximation for any fixed $\varepsilon > 0$ in the value oracle model \cite{9, 13}. For the subadditive case, which includes the XOS case (see also Appendix A), Badanidiyuru et al. \cite{1} gave a tight $2$-approximation algorithm in the demand oracle model, in which we can access a set function only by asking a demand $S \in \arg \max_{X \subseteq V} \{f(X) - \sum_{v \in X} p_v\}$ under each price vector $p \in \mathbb{R}^V$. We remark that their algorithm does not imply a $2$-approximation algorithm in the value oracle model.

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Two set functions $f, g: 2^V \to \mathbb{R}$ is order-equivalent if $\{f(X) \leq f(Y) \text{ if and only if } g(X) \leq g(Y)\}$ for every $X, Y \subseteq V$, and order-different otherwise.
2 Algorithms

Let $V$ be a nonempty finite set of size $n$. Throughout this section, for the input XOS function $f : 2^V \to \mathbb{R}$, we may assume that $f(v) > 0$ for any $v \in V$, because any $v \in V$ with $f(v) \leq 0$ does not contribute to increasing the function values. This assumption can be tested in linear time, and when it is violated, one can modify the instance just by removing such unnecessary elements.

For each $X \subseteq V$, we define $I(X) := \{ i \mid f_i(X) = f(X) \}$. In addition, for each index $i$, we call $V_i := \{ v \in V \mid f_i(v) = f(v) \}$ a clique with respect to $i$. A subset of $V$ is called a clique if it is a clique with respect to some $i$.

2.1 $(\epsilon n)$-Approximation in general

In this section, we prove Theorem 1. In short, for any $\epsilon > 0$, an $(\epsilon n)$-maximizer is obtained just by taking the best one among all subsets of $V$ of size at most $\lceil 1/\epsilon \rceil$. A formal description is given in Algorithm 1 which clearly calls the value oracle $\sum_{i=0}^{\min\{\lceil 1/\epsilon \rceil, n\}} \binom{n}{i} = O(n^{\lceil 1/\epsilon \rceil})$ times, and we take the maximum in Line 2 while successively updating $X \in \mathcal{X}$ in constant time per once (cf. [7, § 7.2.1.3]).

\begin{algorithm}
\caption{($\epsilon n$)-approximation algorithm}
\begin{algorithmic}
  \STATE \textbf{Input:} An XOS function $f$ on $V$ (with $f(v) > 0$ for all $v \in V$)
  \STATE \textbf{Output:} An $(\epsilon n)$-maximizer $\tilde{X} \subseteq V$ of $f$
  \STATE 1 \hspace{1em} Let $\mathcal{X} \leftarrow \{ X \subseteq V \mid |X| \leq \lceil 1/\epsilon \rceil \}$; \hspace{1em} // Do not keep each $X$ explicitly. /*
  \STATE 2 \hspace{1em} \textbf{return} $\tilde{X} \in \arg \max_{X \in \mathcal{X}} f(X)$;
\end{algorithmic}
\end{algorithm}

We now prove that the output $\tilde{X} \subseteq V$ of Algorithm 1 is indeed an $(\epsilon n)$-maximizer. Let $X^* \subseteq V$ be a maximizer of $f$. If $|X^*| \leq \lceil 1/\epsilon \rceil$, then $X^* \in \mathcal{X}$ and hence $f(\tilde{X}) = f(X^*)$. Otherwise, $\mathcal{X}$ contains a subset $X \subseteq X^*$ such that $|X| = \lceil 1/\epsilon \rceil$ and $f_i(x) \geq f_i(x^*)$ for any $x \in X$ and any $x^* \in X^* \setminus X$, where $i \in I(X^*)$ is an index with $f_i(X^*) = f(X^*)$. Since $|X^*| \leq n$, this shows that

$$f(\tilde{X}) \geq f(X) \geq f_i(X) \geq \frac{|X|}{|X^*|} \cdot f_i(X^*) \geq \frac{1}{\epsilon n} \cdot f_i(X^*) = \frac{1}{\epsilon n} \cdot f(X^*).$$

2.2 $(k - 1)$-Approximation for $k$-XOS maximization

In this section, we prove Theorem 3. That is, we present a $(k - 1)$-approximation algorithm for maximizing $k$-XOS functions $f$ that runs in $O(k^2 n)$ time for any $k \geq 2$ (even if $k$ is unknown). In particular, when $k = 2$, it always finds an exact maximizer of $f$ in linear time. In addition, when $k = o(n)$, it achieves a better approximation ratio than Algorithm 1 in subcubic time.

Our algorithm is shown in Algorithm 2. It is worth remarking that the algorithm does not use the information of the width $k$. 
Algorithm 2: \((k - 1)\)-approximation algorithm

**Input:** A \(k\)-XOS function \(f\) on \(V\) (with \(f(v) > 0\) for all \(v \in V\))

**Output:** A \((k - 1)\)-maximizer \(\tilde{X} \subseteq V\) of \(f\)

1. Let \(R \leftarrow \emptyset\) and \(\ell \leftarrow 0\);
2. while \(R \neq V\) do
   3. Let \(\ell \leftarrow \ell + 1\);
   4. Pick \(v \in V \setminus R\) and let \(V_{\ell} \leftarrow \{v\}\) and \(R \leftarrow R + v\);
   5. foreach \(u \in V - v\) do
      6. if \(f(V_{\ell} + u) = f(V_{\ell}) + f(u)\) then let \(V_{\ell} \leftarrow V_{\ell} + u\) and \(R \leftarrow R + u\);
   7. For each \(i \in [\ell]\), let \(Y_i \leftarrow V_i \cup \{v \in V \mid f(V_i + v) > f(V_i)\}\);
   8. For each \((i, j) \in \binom{[\ell]}{2}\) = \(\{J \subseteq [\ell] \mid |J| = 2\}\) and \(v \in V\), let \(Z^v_{ij} \leftarrow V_i \cup V_j \cup \{v\}\);
   9. Let \(V \leftarrow \{V_1, \ldots, V_\ell\}, Y \leftarrow \{Y_1, \ldots, Y_\ell\}\), and \(Z \leftarrow \{Z^v_{ij} \mid (i, j) \in \binom{[\ell]}{2}, v \in V\}\);
10. return \(\tilde{X} \in \arg \max_{X \in \mathcal{W} \cup \mathcal{Y} \cup \mathcal{Z}} f(X)\);

In what follows, let \(\ell\) denote its value at the end of Algorithm 2.

**Lemma 6.** At the end of Algorithm 2, each \(V_i\) (\(i \in [\ell]\)) is a clique. In particular, \(\ell \leq k\) holds.

**Proof.** Each \(V_i\) is created as a singleton \(\{v\}\) in Line 1 and successively updated by adding \(u \in V - v\) if \(f(V_i + u) = f(V_i) + f(u)\) in Line 6. The condition is satisfied if and only if \(I(V_i) \cap I(u) \neq \emptyset\), and if satisfied, then \(I(V_i + u) = I(V_i) \cap I(u)\) holds. Hence, after the iteration, \(I(V_i) = \bigcap_{u \in V_i} I(u)\) and \(I(V_i) \cap I(u') = \emptyset\) for each \(u' \in V \setminus V_i\). This means that \(V_i = V_i^u\) for each \(i\) and \(\ell \leq k\).

Since \(V_i\) (\(i \in [\ell]\)) are pairwise distinct due to the choice of \(v\) in Line 4 and update of \(R\) in Lines 4 and 6, we conclude \(\ell \leq k\). \(\square\)

The following two lemmas complete the proof of Theorem 3.

**Lemma 7.** Algorithm 2 can be implemented to run in \(O(k^2 n)\) time.

**Proof.** For the while-loop (Lines 2–6), the number of iterations is \(\ell \leq k\) (Lemma 6). In each iteration step, the algorithm chooses an element \(v \in V\) and just checks whether \(f(X + u) = f(X) + f(u)\) or not for some \(X \subseteq V\) once for each element \(u \in V - v\). It requires \(O(n)\) time, and hence \(O(kn)\) time in total.

In Line 7, the algorithm computes \(\{v \in V \mid f(V_i + v) > f(V_i)\}\) for each \(i \in [\ell]\). It takes \(O(n)\) time for each \(i\), and hence \(O(kn)\) time in total.

In Line 8 (for \(Z\)), instead of keeping all \(Z^v_{ij}\) directly, we first construct \(V_i \cup V_j\) (\((i, j) \in \binom{[\ell]}{2}\)) in \(O(k^2 n)\) time. Then, each \(Z^v_{ij} \neq V_i \cup V_j\) can be successively constructed from \(V_i \cup V_j\) in constant time when taking the maximum in Line 10.

In Lines 9–10, the algorithm just finds a maximizer of \(f\) over the family \(\mathcal{W} \cup \mathcal{Y} \cup \mathcal{Z}\), whose cardinality is at most \(\ell + \ell + \binom{\ell}{2} = O(k^2 n)\).

Thus the total computational time is bounded by \(O(k^2 n)\). \(\square\)

**Lemma 8.** Algorithm 2 returns a \((k - 1)\)-maximizer \(\tilde{X}\) of \(f\).

**Proof.** Let \(X^* \subseteq V\) be a maximizer of \(f\).
If $\ell < k$, then we have
\[
  f(\tilde{X}) \geq \max_{i \in [\ell]} f(V_i) \geq \frac{1}{\ell} \cdot \sum_{i \in [\ell]} f(V_i) = \frac{1}{\ell} \cdot \sum_{i \in [\ell]} \sum_{v \in V_i} f(v) \\
  \geq \frac{1}{\ell} \cdot \sum_{v \in V} f(v) \geq \frac{1}{\ell} \cdot f(\ell) \geq \frac{1}{k-1} \cdot f(X^*),
\]
where note that $f(V_i) = \sum_{v \in V_i} f(v)$ by Lemma 9, $f(v) > 0$ for each $v \in V$ by the assumption, and $\bigcup_{i \in [\ell]} V_i = V$ due to the condition of the while-loop (Line 2). Hence, $\tilde{X}$ is indeed a $(k-1)$-maximizer.

By Lemma 6 in what follows, we consider the case when $\ell = k$, and let us relabel the indices of $V_i$ ($i \in [k]$) so that $V_i = V^*_i$ for each $i \in [k]$. Without loss of generality, suppose that $X^* = \{ v \in V \mid f_p(v) > 0 \}$ for some $p \in [k]$. Then, we have $V_p \subseteq X^* \subseteq Y_p$.

**Case 1:** Suppose that $X^* = Y_p$. We then have $f(\tilde{X}) \geq f(Y_p) = f(X^*) \geq f(\tilde{X})$, and hence the output $\tilde{X}$ is also a maximizer of $f$.

**Case 2:** Suppose that $X^* \neq Y_p$. Fix any $v \in Y_p \setminus X^*$ and any $q \in I(V_p + v)$. Since $f_p(V_p + v) = f_p(V_p) + f_p(v) \leq f_p(V_p) = f(V_p) < f(V_p + v) = f_q(V_p + v)$, we have $q \neq p$. Then,
\[
  f(X^*) = f_p(X^*) = f_p(V_p) + f_p((X^* \setminus V_p) \cap V_q) + f_p((X^* \setminus (V_p \cup V_q)) \\
  < f_q(V_p + v) + f_p((X^* \setminus V_p) \cap V_q) + \sum_{i \in [k] \setminus \{p,q\}} f_i(V_i) \\
  = f_q(V_p \cup (X^* \cap V_q) \cup \{v\}) + \sum_{i \in [k] \setminus \{p,q\}} f_i(V_i) \\
  \leq f_q(Z^*_pq) + \sum_{i \in [k] \setminus \{p,q\}} f_i(V_i) \leq (k-1) \cdot f(\tilde{X}).
\]
Thus, $\tilde{X}$ is indeed a $(k-1)$-maximizer.

### 2.3 Finding all maximal cliques and its application

In this section, we prove Theorem 5. Fix $k \geq 2$ and let $f$ be a $k$-XOS function with the condition (*), i.e., for every $v \in V$ and every $i \in [k]$, either $f_i(v) = f(v) > 0$ or $f_i(v) \leq 0$.

**Lemma 9.** If an XOS function $f$ satisfies the condition (*), then there exists an inclusion-wise maximal clique that maximizes $f$.

**Proof.** Let $X^* \subseteq V$ be an inclusion-wise minimal maximizer of $f$. Then, for some $i \in [k]$, we have $X^* = \{ v \in V \mid f_i(v) > 0 \} = \{ v \in V \mid f_i(v) = f(v) \} = V^*_i$. Since $X^*$ maximizes $f$, such a clique $V^*_i$ must be inclusion-wise maximal.

By this lemma, it suffices to find all inclusion-wise maximal cliques. This can be done by Algorithm 8 which does not use the information of the width $k$.

**Lemma 10.** For any $k \geq 2$ and any $k$-XOS function $f$, Algorithm 8 returns the family $\mathbb{V}$ of all inclusion-wise maximal cliques in $O(n^{k+1})$ time.

**Proof.** After the first for-loop with $\ell = 1$, it is obvious that $|\mathbb{V}| \geq \ell$. Since we increase $\ell$ by one in each iteration, Algorithm 8 terminates in finite steps. In what follows, let $\ell$ denote its value when the algorithm terminates, i.e., $\ell = |\mathbb{V}|$. 

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We first confirm that each \( V_X \in V \) is indeed a maximal clique. In Line 6, we have \( V_X \subseteq V_i^* \) for \( i \in I(V_X) = \bigcap_{v \in V} I(v) \) because \( f(X) = \sum_{v \in X} f(v) \) holds. Hence, similarly to Lemma 6 in Line 7, the condition \( f(V_X + u) = f(V_X) + f(u) \) holds if and only if \( I(V_X) \cap I(u) \neq \emptyset \), and then \( I(V_X + u) = I(V_X) \cap I(u) \). Thus, after the innermost for-loop (Lines 6–7), \( I(V_X) \subseteq I(u) \) for each \( u \in V_X \) and \( f(V_X) \cap I(u') = \emptyset \) for each \( u' \in V \setminus V_X \). This means that \( V_X = V_i^* \) for each \( i \in I(V_X) \), i.e., \( V_X \) is a clique. Furthermore, \( V_X \) is a maximal clique because \( V_X \setminus V_j^* \neq \emptyset \) for each \( j \in [k] \setminus I(V_X) \) by the definition of \( I(V_X) \).

To show that the output contains all the maximal cliques, suppose to the contrary that some maximal clique \( V_i^* \) is not contained in the output \( V = \{ V_1, \ldots, V_t \} \). Since each \( V_j \in V \) is a maximal clique as shown above, we have \( V_i^* \setminus V_j \neq \emptyset \). Fix any \( x_{i,j} \in V_i^* \setminus V_j \) for each \( j \in [\ell] \), and let \( X_i := \{ x_{i,j} \mid j \in [\ell] \} \). Since \( X_i \subseteq V_i^* \), we have \( f(X_i) = f_i(X_i) = \sum_{v \in X_i} f_i(v) = \sum_{v \in X_i} f(v) \). This shows that \( X_i \in \mathcal{X}_\ell \) for some \( \ell \in [\ell] \), because \( |X_i| \leq \ell \). Then, we have \( V_X \subseteq \mathcal{V} \). However, \( V_X \neq V_j \) for each \( j \in [\ell] \) because \( x_{i,j} \notin V_j \), which is a contradiction.

The algorithm requires \( O(n) \) time to check whether \( f(X) = \sum_{v \in X} f(v) \) or not for each \( X \subseteq V \) with \( |X| \leq \ell \leq |V| \leq k \), and \( O(n) \) time in Lines 5–8 for each \( X \in \mathcal{X}_\ell \) (\( \ell \in [\ell] \)). The number of candidates of \( X \) is \( \sum_{i=1}^\ell \binom{n}{i} = O(n^k) \), and hence the total computational time is bounded by \( O(n^{k+1}) \).

By Lemmas 9 and 10, we obtain Theorem 3.

## 3 Hardness

In this section, we show two hardness results (Theorems 2 and 4), which claim that an exponential number of value oracle calls are required to beat the approximation ratios of Algorithms 1 and 2. Both hardness results are based on a probabilistic argument, and a key tool is the following lemma.

**Lemma 11.** Let \( \hat{V} = [\hat{n}] \) and let \( s, t, \delta \) be integers such that \( 1 \leq t \leq s \leq \hat{n} \). Suppose that we pick, uniformly at random, a set \( S \subseteq \hat{V} \) such that \( |S| = s \). Let \( \hat{f} : 2^\hat{V} \rightarrow \mathbb{R} \) be the function defined as

\[
\hat{f}(X) = \begin{cases} 
1 & \text{(if } X \subseteq S \text{ and } |X| \geq t), \\
0 & \text{(otherwise).}
\end{cases}
\]

Then, for any positive real \( \delta < 1 \), any algorithm (including a randomized one) to find \( X \subseteq \hat{V} \) with \( \hat{f}(X) = 1 \) calls the value oracle at least \( \delta \cdot (\hat{n}/s)^t \) times with probability at least \( 1 - \delta \).
Proof. Suppose to the contrary that there exists an algorithm to find \( X \subseteq \hat{V} \) with \( \hat{f}(X) = 1 \) that calls the value oracle less than \( \delta \cdot (\hat{n}/s)^t \) times with probability more than \( \delta \). Without loss of generality, we may assume that it is deterministic, and suppose that it calls the value oracle for \( X_1, X_2, \ldots \subseteq \hat{V} \) in this order. Note that \( \hat{f}(X_i) = 1 \) if and only if \( S \in \mathcal{X}_i := \{ X \mid X_i \subseteq X \subseteq \hat{V}, |X| = s \} \) and \( |X_i| \geq t \). Since \( |\mathcal{X}_i| \leq (\hat{n}/s)^{\ell} \) holds for any \( X_i \subseteq \hat{V} \) with \( |X_i| \geq t \), the probability that the algorithm finds \( X \subseteq \hat{V} \) with \( \hat{f}(X) = 1 \) before \( m \) oracle calls is at most

\[
\left| \bigcup_{i=1}^m \mathcal{X}_i \right| \leq \frac{m \cdot (\hat{n}/s)^{\ell}}{\binom{n}{s}} = m \cdot \frac{s}{n} \cdot \frac{s-1}{n-1} \cdots \frac{s-t+1}{n-t+1} \leq m \cdot (s/\hat{n})^t,
\]

which contradicts that the probability is larger than \( \delta \) when \( m = \delta \cdot (\hat{n}/s)^t \).

\( \square \)

3.1 Inapproximability within \( n^{1-\epsilon} \) in general

In this section, we prove Theorem 2 by showing the following stronger result.

**Theorem 12.** For any \( \epsilon > 0 \), there exist \( c > 0 \) and a distribution of instances of XOS maximization such that any (randomized) \( n^{1-\epsilon} \)-approximation algorithm calls the value oracle at least \( \delta \cdot 2^{\Omega(n^c)} \) times with probability at least \( 1 - \delta \) for any \( 0 < \delta < 1 \).

**Proof.** Let \( \epsilon' := \epsilon/2 \), and \( V = [n] \). We pick, uniformly at random, a set \( S \subseteq V \) such that \( |S| = n/2 \). Suppose that \( f(X) = \max\{ f_i(X) \mid i \in [n+1] \} \), where

\[
f_i(v) = \begin{cases} 
n/2 & \text{(if } v = i), \\
0 & \text{(if } v \neq i), \end{cases} \quad \text{for } i \in [n], \quad \text{and } f_{n+1}(v) = \begin{cases} 
n^{1-\epsilon'} & \text{(if } v \in S), \\
-2^2 & \text{(if } v \notin S). \end{cases}
\]

We then have \( \max_{X \subseteq V} f(X) = f_{n+1}(S) = n^{2-\epsilon'}/2 \) and

\[
f(X) = \begin{cases} 
|X| \cdot n^{1-\epsilon'} & \text{(if } X \subseteq S \text{ and } |X| > n^{\epsilon'}/2), \\
n/2 & \text{(if } X = \emptyset), \\
0 & \text{(otherwise).}
\end{cases}
\]

Hence, by Lemma 11 (with \( \hat{n} = n \), \( s = n/2 \), and \( t = n^{\epsilon'}/2 \)), any algorithm to obtain an \( n^{1-\epsilon} \)-maximizer of \( f \) calls the value oracle at least \( \delta \cdot 2^{n^{\epsilon'}/2} \) times with probability at least \( 1 - \delta \).

This theorem shows that an exponential number of oracle calls are required with high probability, which implies Theorem 2. Furthermore, since Theorem 12 holds also for randomized algorithms, we obtain the following corollary.

**Corollary 13.** Let \( \epsilon > 0 \). Then, for any randomized \( n^{1-\epsilon} \)-approximation algorithm for XOS maximization, the expected number of value oracle calls is exponential in the ground set size \( n \) for the worst instance.

3.2 Inapproximability within \( k - 1 - \epsilon \) for \( k \)-XOS maximization

In this section, we prove Theorem 4 by showing the following stronger result.

\( ^3 \)We note that, for \( \alpha \geq 1 \), a randomized algorithm is said to be \( \alpha \)-approximation if the expected function value of its output is at least \( \frac{1}{\alpha} \cdot \max_{X \subseteq V} f(X) \).
**Theorem 14.** For any $k \geq 3$ and any $\epsilon > 0$, there exist $c > 0$ and a distribution of instances of XOS maximization such that any (randomized) $(k-1-\epsilon)$-approximation algorithm calls the value oracle at least $\delta \cdot 2^{\Theta(n^{1/(k-1)})}$ times with probability at least $1 - \delta$ for any $0 < \delta < 1$.

**Proof.** Let $\tilde{n}$ and $\gamma$ be a sufficiently large integer and a sufficiently small positive rational number, respectively, such that $(k-1)(1-\gamma)^2 > k-1 - \epsilon$ holds and $\gamma \tilde{n}$ is an integer. Suppose that $V$ is the union of $k-1$ disjoint sets $V_1, V_2, \ldots, V_{k-1}$ with $|V_i| = n_i$ for each $i \in [k-1]$. Then, $n = \Theta(n^{k-1})$ as well as $\tilde{n} = \Theta(n^{1/(k-1)})$. For each $i \in [k-1]$, we pick, uniformly at random, a set $S_i \subseteq V_i$ such that $|S_i| = (1-\gamma)|V_i| = (1-\gamma)n_i$.

Suppose that $f(X) = \max_{i \in [k]} f_i(X)$, where

$$f_i(v) = \begin{cases} \tilde{n}^{k-i} & \text{(if } v \in V_i), \\ 0 & \text{(otherwise),} \end{cases}$$

$$f_k(v) = \begin{cases} (1-\gamma)\tilde{n}^{k-i} & \text{(if } v \in S_i \text{ for some } i \in [k-1]), \\ -\tilde{n}^{k+1} & \text{(otherwise).} \end{cases}$$

Then, we have $\max_{X \subseteq V} f(X) = f_k(\bigcup_{i=1}^{k-1} S_i) = (k-1)(1-\gamma)^2\tilde{n}^k > (k-1-\epsilon)\tilde{n}^k$.

**Lemma 15.** If a nonempty subset $X \subseteq V$ satisfies that $f(X) = f_k(X)$, then there exists $i \in \{2, 3, \ldots, k-1\}$ such that $X \cap V_i \subseteq S_i$ and $|X \cap V_i| \geq \frac{\gamma}{k-2} \tilde{n}$.

**Proof.** Assume that $f(X) = f_k(X)$. Then, it is clear that $X \cap V_i \subseteq S_i$ for each $i \in [k-1]$. Let $j \in [k-1]$ be the minimum index such that $X \cap V_j \neq \emptyset$. Since $f_j(X) = \tilde{n}^{k-j} \cdot |X \cap V_j|$ and $f_k(X) = \sum_{i \geq j} (1-\gamma)\tilde{n}^{k-i} \cdot |X \cap V_i|$, we derive from $f_k(X) = f(X) \geq f_j(X)$ that

$$\sum_{i \geq j} (1-\gamma)\tilde{n}^{k-i} \cdot |X \cap V_i| \geq \gamma \tilde{n}^{k-j} \cdot |X \cap V_j|.$$ 

Since $|X \cap V_j| / (1-\gamma) \geq 1$, this shows that $\sum_{i>j} |X \cap V_i| \geq \gamma \tilde{n}$, which implies that $|X \cap V_i| \geq \frac{\gamma}{k-2} \tilde{n}$ for some $i \in \{j+1, j+2, \ldots, k-1\}$. \qed

This lemma shows that we cannot obtain a nonempty subset $X \subseteq V$ with $f(X) = f_k(X)$ unless we find a subset of $S_i$ of size $\frac{\gamma}{k-2} \tilde{n}$ for some $i$. By Lemma 11 (with $V = V_i$ as well as $\tilde{n} = n_i$, $S = S_i$ as well as $s = (1-\gamma)n_i$, and $t = \frac{\gamma}{k-2} \tilde{n}$), any algorithm to find such $X$ calls the value oracle at least

$$\delta \cdot \left( \frac{1}{1-\gamma} \right)^{\gamma \tilde{n}/(k-2)} = \delta \cdot 2^{\Theta(\tilde{n})} = \delta \cdot 2^{\Theta(n^{1/(k-1)})}$$

times with probability at least $1 - \delta$. The same number of oracle calls are required for obtaining $X \subseteq V$ such that $f(X) > \tilde{n}^k$, because $\max_{X \subseteq V} f_i(X) = \tilde{n}^k$ for $i \in [k-1]$. By combining this with $\max_{X \subseteq V} f(X) > (k-1-\epsilon)\tilde{n}^k$, we complete the proof of Theorem 13. \qed

This theorem shows that an exponential number of oracle calls are required with high probability, which implies Theorem 3. Furthermore, since Theorem 14 holds also for randomized algorithms, we obtain the following corollary.

**Corollary 16.** Let $k \geq 3$ and $\epsilon > 0$. Then, for any randomized $(k-1-\epsilon)$-approximation algorithm for $k$-XOS maximization, the expected number of value oracle calls is exponential in the ground set size $n$ for the worst instance.
4 The Number of XOS Functions

For set functions $f, g: 2^V \to \mathbb{R}$, we say that $f$ and $g$ are order-equivalent if $f(X) \leq f(Y)$ if and only if $g(X) \leq g(Y)$ for all $X, Y \subseteq V$. Also, we say that $f$ and $g$ are order-different if they are not order-equivalent.

We show that the number of order-different XOS functions with bounded width is single exponential in the ground set size $n$, whereas there are doubly-exponentially many order-different set functions on a fixed ground set.

**Theorem 17.** The number of order-different $k$-XOS functions on $V = [n]$ is $2^{O((kn)^2 \log n)}$.

To prove this theorem, it is sufficient to prove the following lemma.

**Lemma 18.** For any $k$-XOS function $f$ over $V = [n]$, there exist an order-equivalent XOS function $g$ and integer weights $|w_{i,v}| \leq (2n+1)^{kn/2}$ ($v \in V$) such that $g(X) = \max_{i \in [k]} \sum_{v \in X} w_{i,v}$ for all $X \subseteq V$.

**Proof.** Let $f_1, \ldots, f_k$ be additive functions such that $f(X) = \max_{i \in [k]} f_i(X)$ ($\forall X \subseteq V$). Fix a function $v: 2^V \to [k]$ that represents a maximizer index, i.e., $v(X) \in I(X)$ for each $X \subseteq V$. Let us consider the following polyhedron $P[f]$:

$$P[f] := \left\{ w \left| \begin{array}{l} \sum_{v \in X} w_{i,v}(X) - \sum_{u \in Y} w_{i,u}(Y) \geq 1 \ (X, Y \in 2^V \text{ with } f(X) > f(Y)), \\
\sum_{v \in X} w_{i,v}(X) - \sum_{u \in Y} w_{i,u}(Y) = 0 \ (X, Y \in 2^V \text{ with } f(X) = f(Y)), \\
\sum_{v \in X} w_{i,v} \geq 0 \ (X \in 2^V, i \in [k] \setminus \{v(X)\}) \end{array} \right. \right\}.$$

Here, we have $kn$ variables and $O(2^{kn})$ linear constraints.

The constraint matrix of polyhedron $P[f]$ is full-rank, because we have one of $w_{i,v} \geq 1$, $-w_{i,v} \geq 1$, and $w_{i,v} = 0$ for any $v \in V$ by the first and second inequalities, and $w_{i,v} - w_{i,v} \geq 0$ for any $v \in V$ and any $i \in [k] \setminus \{v\}$ by the third inequality. Also, a vector $w$ defined by $w_{i,v} := f_i(v)$ ($i \in [k], v \in V$) is in $P[f]$, and hence $P[f]$ is feasible (nonempty). Thus, $P[f]$ has a basic solution (vertex) (see, e.g., [12 § 8.5]).

Let $\tilde{w}$ be a basic solution of $P[f]$. Then, by considering the corresponding inequalities, we have $A\tilde{w} = b$ for a nonsingular matrix $A \in \{-1, 0, 1\}^{kn \times kn}$ and a vector $b \in \{0, 1\}^{kn}$. By Cramer’s rule, we obtain $\tilde{w}_{i,v} = \det A_{(i,v)}/\det A$, where $A_{(i,v)}$ is the matrix obtained from $A$ by replacing the $(i, v)$th column with the column vector $b$. Let us define

$$\hat{w}_{i,v} := |\det A| \cdot \tilde{w}_{i,v} = \text{sgn}(\det A) \cdot \det A_{(i,v)} \ (i \in [k], v \in V).$$

Now we observe that $\hat{w}$ is desired weights. Since $A_{(i,v)}$ is an integer matrix, $\hat{w}_{i,v}$ is an integer. Moreover, as $A_{(i,v)} \in \{-1, 0, 1\}^{kn \times kn}$ and each row vector has at most $2n+1$ nonzero entries, we have $|\hat{w}_{i,v}| \leq (2n+1)^{kn/2}$ by Hadamard’s inequality [5]. Hence, the claim holds.

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A Classes of Set Functions

The following classes are well-studied in combinatorial optimization and recently also in algorithmic game theory as valuations of agents.

Definition 19. A set function \( f : 2^V \to \mathbb{R} \) is

- normalized, if \( f(\emptyset) = 0 \).
- monotone, if \( f(X) \leq f(Y) \) for every \( X \subseteq Y \subseteq V \).
- additive, if \( f(X) = \sum_{v \in X} f(v) \) for every \( X \subseteq V \).
- gross-substitute, if for every \( p, q \in \mathbb{R}^V \) with \( p \leq q \) and every \( X \in \arg\max \{ f(S) - \sum_{v \in S} p_v \mid S \subseteq V \} \), there exists \( Y \in \arg\max \{ f(S) - \sum_{v \in S} q_v \mid S \subseteq V \} \) such that \( \{ v \in X \mid p_v = q_v \} \subseteq Y \).
- submodular, if \( f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \) for every \( X, Y \subseteq V \).
- fractionally subadditive: \( f(T) \leq \sum_i \alpha_i f(S_i) \) for every \( T, S_i \subseteq V \) whenever \( \alpha_i \geq 0 \) and \( \sum_i: v \in S, \alpha_i \geq 1 \) (\( \forall v \in T \)).
- subadditive if \( f(X \cup Y) \leq f(X) + f(Y) \) for every \( X, Y \subseteq V \).

Let us denote by Add, GS, SubM, FSubA, SubA, and XOS the sets of (normalized) additive functions, of normalized gross-substitute functions, of normalized submodular functions, of normalized fractionally subadditive functions, of normalized subadditive functions, and of (normalized) XOS functions, respectively. Also, we add * to each of them to assume the monotonicity in addition to each property. We then have the following relations (3 1 3):

\[
\text{Add} \subseteq \text{GS} \subseteq \text{SubM} \subseteq \text{XOS},
\text{Add}^* \subseteq \text{GS}^* \subseteq \text{SubM}^* \subseteq \text{XOS}^* = \text{FSubA}^* (= \text{FSubA}) \subseteq \text{SubA}^*.
\]

B Maximum of an Additive Function and a Constant

In this section, we consider the maximization problem for a variant class of set functions named LTA. We say that a set function \( f : 2^V \to \mathbb{R} \) is LTA\(^4\) if it is the maximum of an additive function and a constant, i.e., \( f(X) = \max \{ g(X), a \} \) \( \forall X \subseteq V \) for an additive function \( g : 2^V \to \mathbb{R} \) and a constant \( a \in \mathbb{R} \). We assume that the function \( f \) is given by a value oracle. By the same argument as in Section 2.1, one can obtain \((cn)\)-maximizer for any \( \epsilon > 0 \) by Algorithm 1.

We show an inapproximability result for the LTA maximization problem, which is based on a similar idea used to show the general hardness of XOS maximization (Theorem 12).

Theorem 20. For any \( \epsilon > 0 \), there exist \( c > 0 \) and a distribution of instances of LTA maximization such that any (randomized) \( n^{1-\epsilon} \)-approximation algorithm calls the value oracle at least \( \delta \cdot 2^{\Omega(n^c)} \) times with probability at least \( 1 - \delta \) for any \( 0 < \delta < 1 \).

Proof. Let \( \epsilon' := \epsilon/2 \), and \( V = [n] \). We pick, uniformly at random, a set \( S \subseteq V \) such that \( |S| = n/2 \). Suppose that \( f(X) = \max \{ g(X), n^{\epsilon'}/2 \} \), where \( g \) is an additive function such that \( g(v) = 1 \) if \( v \in S \) and \( g(v) = -n \) if \( v \notin S \).

\(^4\)LTA stands for Lower-Truncated Additive.
Then, we have $\max_{X \subseteq V} f(X) = g(S) = n/2$ and

$$f(X) = \begin{cases} |X| & \text{(if } X \subseteq S \text{ and } |X| > n^{\epsilon^*}/2), \\ n^{\epsilon^*}/2 & \text{(otherwise).} \end{cases}$$

Hence, by Lemma 11 (with $\hat{n} = n$, $s = n/2$, and $t = n^{\epsilon^*}/2$), any algorithm to obtain an $n^{1-\epsilon}$-maximizer of $f$ calls the value oracle at least $\delta \cdot 2^{n^{\epsilon^*}/2}$ times with probability at least $1 - \delta$.

This theorem shows that an exponential number of oracle calls are required with high probability. Furthermore, since Theorem 20 holds also for randomized algorithms, we obtain the following corollary.

**Corollary 21.** Let $\epsilon > 0$. Then, for any randomized $n^{1-\epsilon}$-approximation algorithm for LTA maximization, the expected number of value oracle calls is exponential in the ground set size $n$ for the worst instance.