Complex geometry three-dimensional curvilinear grids construction for numerical flow calculations

M M Gorokhov¹,², A V Korepanov¹,³, V A Tenenev¹ and G A Blagodatskiy¹

¹ Kalashnikov Izhevsk State Technical University, 7, Studencheskaya street, Izhevsk, 426069, Russia
² Research Institute of Federal Penitentiary Service, 15a, Narvskaya street, Moscow, 125130, Russia
³ Email: kor-and@yandex.ru

Abstract. The urgency of constructing spatial curvilinear grids in areas with complex geometry is formulated. An overview of the construction of two-dimensional curvilinear computational grids is given. The article describes the rationale for the fact that the methods described in the review cannot fully calculate the flow parameters in regions with complex geometry. A method for constructing three-dimensional grids for bodies with arbitrary surface geometry is described. The results of constructing a three-dimensional grid are presented using a hemisphere as an example. An estimate of the measures of non-orthogonality for the test case is given.

1. Introduction
In the numerical solution of equations describing multiphase gas dynamics and heat-mass transfer, one has to face a number of difficulties associated with the complexity of approximating the initial differential equations and boundary conditions in regions with complex geometry. The complex shape of the boundaries of the study area requires the use of computational grids adapted to the flow conditions. The use of rectangular grids in the approximation of boundaries leads to a significant complication of computational algorithms for the implementation of boundary conditions. In addition, such grids do not allow the refinement of the grid spacing near curvilinear solid boundaries, which is necessary when calculating flows with large Reynolds numbers, and lead to the emergence and strong influence of the scheme viscosity.

2. Creation of two-dimensional grids
The most obvious way to transform coordinates is based on the properties of harmonic functions that satisfy the Laplace’s equation. This method is described in detail in the works of S.K. Godunov and co-authors [1]. According to this method, the lines of the curvilinear grid are found as the level lines of functions \( \xi(x, y) \), \( \eta(x, y) \) that satisfy the Laplace’s equations

\[
\xi_{xx} + \xi_{yy} = 0, \quad \eta_{xx} + \eta_{yy} = 0
\]

where \( x, y \) – coordinates in the physical plane.

To find the coordinates of the grid nodes, equations (1) turn into equations for the functions \( x(\xi, \eta) \), \( y(\xi, \eta) \).
\begin{equation}
\alpha_1 x_{\xi\xi} + \alpha_2 x_{\xi\eta} + \alpha_3 x_{\eta\eta} = P_1, \quad \alpha_1 y_{\xi\xi} + \alpha_2 y_{\xi\eta} + \alpha_3 y_{\eta\eta} = P_2,
\end{equation}

where $\alpha_1, \alpha_2, \alpha_3, P_1, P_2$ – functions of $x_\xi, y_\eta, x_\eta, y_\xi$ [2]. In the case of orthogonal grids, it is necessary to take $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, P_1 = P_2 = 0$.

In [1], the class of quasiconformal mappings is considered, that is, not requiring the fulfillment of the grid orthogonality conditions (Cauchy-Riemann equations) $\nabla x \cdot \nabla y = 0$.

One of the features which arise when solving equations (2) in a complex domain is a favorable location of the boundary nodes. With an arbitrary arrangement of nodes on the border, this cannot be achieved. Therefore, as the calculations have shown, when using the methods described in [1], for strongly curved regions, it is impossible to construct a grid without overlaps of coordinate levels.

The problem of constructing computational grids, including orthogonal ones, in areas of complex shape can be solved using the complex method of boundary elements [3]. The solution to the problem is found by using the boundary element method for harmonic functions $\xi, \eta$ satisfying equations (1).

It is known that for a potential flow of an incompressible fluid, the potential and the stream function satisfy the Laplace’s equations under certain boundary conditions. The isolines $\xi$ and $\eta$ drawn in the investigated area are the equipotential lines and streamlines of the potential plane flow of an incompressible fluid. Consequently, the grid tracks the main flow direction in the area for hydro-gasdynamic problems. Streamlines can also be interpreted as stationary isotherms for two-dimensional heat transfer problems; therefore, the use of such grids greatly simplifies the setting of boundary conditions and reduces the effect of circuit dissipation. The field of application of the method described is limited to the two-dimensional case, for which the theory of conformal mappings exists. When calculating three-dimensional flows, a different approach is required.

The algorithm presented in [4] for constructing finite-difference grids adapted to the flow parameters, based on the use of the complex boundary element method [3], was successfully applied to calculate the flow parameters near axisymmetric surfaces [5, 6] and was used to calculate the parameters of the spatial flow near a curved surface in [6]. In the case of an axisymmetric surface, the method [4] allows one to construct a spatial orthogonal curvilinear finite-difference grid. If the body does not have axial symmetry, then the method [4] does not allow constructing an orthogonal spatial grid. Thus, the equations of hydromechanics and their discrete analogs acquire a rather complicated form for numerical implementation [6]. In addition, the non-orthogonality of the finite-difference grid enhances the influence of "circuit viscosity" on the numerical solution.

3. Creation of three-dimensional grids

The starting point for creating a curvilinear orthogonal grid around a spatial body with arbitrary geometry (aircraft, car body, etc.) is an orthogonal grid applied to the surface of this body.

The geometry of a regular piece of surface defined by equations in three-dimensional Euclidean space

\begin{equation}
x = x(u,v), \quad y = y(u,v), \quad z = z(u,v),
\end{equation}

where $u, v$ - some arbitrary coordinates. The orthogonal grid on the surface is given by the equations $\xi = \xi(u,v) = \text{const}, \eta = \eta(u,v) = \text{const}$. For orthogonal lines on the surface, the following relation is valid

\begin{equation}
\frac{\partial \eta}{\partial n} = \frac{\partial \xi}{\partial \tau},
\end{equation}

where $n, \tau$ – normal and tangent to the line.

Functions $\xi(u,v), \eta(u,v)$ must satisfy the system of equations

\begin{equation}
\frac{G\eta_u - F\eta_u}{W} = \xi_v, \quad \frac{E\eta_u - F\eta_v}{W} = \xi_v,
\end{equation}

where $\alpha, \beta, \gamma, \delta$ are the coefficients of the second form. The criteria is that the grid is orthogonal and the grid lines are tangential to the equipotential and streamline lines.
which is a generalization of the Cauchy-Riemann equations.

To solve system (5), the first equation is differentiated with respect to \( \nu \), the second - with respect to \( \mu \) and are added. The result is the second Beltrami differential operator

\[
\nabla^2 \eta = \frac{\partial}{\partial \nu} \left( \frac{G \eta - F \eta_u}{W} \right) + \frac{\partial}{\partial \mu} \left( \frac{E \eta_u - F \eta_n}{W} \right) = 0,
\]

where \( E(u,\nu), \ G(u,\nu), \ F(u,\nu) \) – the coefficients of the basic quadratic surface shape:

\[
E(u,\nu) = \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial y}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial u} \right)^2, \ G(u,\nu) = \left( \frac{\partial x}{\partial \nu} \right)^2 + \left( \frac{\partial y}{\partial \nu} \right)^2 + \left( \frac{\partial z}{\partial \nu} \right)^2, \ F(u,\nu) = \frac{\partial x}{\partial u} \frac{\partial x}{\partial \nu} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial \nu} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial \nu}.
\]

The algorithm for constructing an orthogonal mesh consists of a sequential solution of equation (6) and the equation \( \nabla^2 \xi = 0 \) under the appropriate boundary conditions. For the variable \( \eta \) on one boundary, a value was set \( \eta = 0 \), on the opposite one \( \eta = 1 \), on the other boundaries \( \partial \eta / \partial n = 0, \) where \( n \) – the normal to the boundary. From equation (4) the boundary conditions for the variable \( \xi \) are determined and the equation is solved \( \nabla^2 \xi = 0 \). Equations (3) and \( \xi = \xi(u,\nu) = const, \ \eta = \eta(u,\nu) = const \) determine the position of the nodes of the curved grid on the surface.

Equation (6) was approximated by a system of difference equations on a nine-point stencil. The system of difference equations was solved by the iterative conjugate gradient method with regularization. For the plane corresponding to the condition \( z = 0 \), the results obtained using this method are completely identical to the results corresponding to the complex boundary element method.

A hemisphere on a plane is considered as an example of a surface. The initial position of the grid nodes on the surface is shown in figure 1a. It can be seen that the mesh is not orthogonal. The optimized mesh is shown in figure 1b.

![Figure 1. Position of grid nodes.](image)

The visual distribution of the measure of non-orthogonality \( F(u,\nu) \) is shown on the surface for the original mesh (Figure 2a) and for the optimized one (Figure 2b). It can be seen that the optimized mesh has a slight deviation of the measure \( F(u,\nu) \) from zero, compared to the non-orthogonal grid.

The developed algorithm forms the basis of the method for constructing spatial curvilinear difference grids close to orthogonal. The method consists in building a three-dimensional grid layer by layer, starting from the surface of the streamlined body. The nodes of the subsequent layer are located on the normals to the previous surface. If a step along the third coordinate \( \Delta \zeta \) is chosen, then the equations for the nodes of the new layer have the form:
When the conditions $F = 0$ are fulfilled on each layer, the other orthogonality conditions are also satisfied:

$$P = \left( \frac{\partial x}{\partial \xi} \right)_{ij}^k \left( \frac{\partial x}{\partial \eta} \right)_{ij}^k + \left( \frac{\partial y}{\partial \xi} \right)_{ij}^k \left( \frac{\partial y}{\partial \eta} \right)_{ij}^k + \left( \frac{\partial z}{\partial \xi} \right)_{ij}^k \left( \frac{\partial z}{\partial \eta} \right)_{ij}^k = 0, \quad Q = \left( \frac{\partial x}{\partial \xi} \right)_{ij}^k \left( \frac{\partial x}{\partial \eta} \right)_{ij}^k + \left( \frac{\partial y}{\partial \xi} \right)_{ij}^k \left( \frac{\partial y}{\partial \eta} \right)_{ij}^k + \left( \frac{\partial z}{\partial \xi} \right)_{ij}^k \left( \frac{\partial z}{\partial \eta} \right)_{ij}^k.$$
For the grid shown in Figure 3, the values of the measures of non-orthogonality are given in Table 1.

Table 1. Non-orthogonality measure values.

|          | average | Maximum |
|----------|---------|---------|
| $F^m$   | $1.3 \cdot 10^{-4}$ | $F^{\text{max}} = 5.1 \cdot 10^{-2}$ |
| $P^m$   | $2.4 \cdot 10^{-4}$ | $P^{\text{max}} = 9.1 \cdot 10^{-2}$ |
| $Q^m$   | $3.3 \cdot 10^{-4}$ | $Q^{\text{max}} = 8.4 \cdot 10^{-2}$ |

The outlined methods allow constructing grids with a high degree of orthogonality for two-dimensional regions of arbitrary complexity. For three-dimensional calculations, the method makes it possible to construct grids close to orthogonal, but only for external flows.

Acknowledgements
The article was written with the support of the BGA / 20-28-09 grant from the Kalashnikov Izhrevsk State Technical University.

References
[1] Godunov S K, Zabrodin A V and Ivanov M Ya 1976 Numerical Solution of Multidimensional Problems of Gas Dynamics (Moscow: Nauka) p 400
[2] Godunov S K and Prokopov G P 1967 Computational Mathematics and Mathematical Physics 7(5) 1031-59
[3] Gromadka T and Ley Ch 1990 Complex Boundary Element Method in Engineering Problems (Moscow: Mir) p 303
[4] Tenenev V A, Rusyak I G and Gorokhov M M 1997 Math Modeling 9(5) 87-96
[5] Gorokhov M M, Rusyak I G and Tenenev V A 1996 Izvestiya RAN. Seriya Fluid and Gas Mechanics 4 162
[6] Gorokhov M M, Korepanov A V and Mikryukov A V 2006 Proc.of the Institute of Mathematics and Informatics at Udmurt State University 2 155-8
[7] Gorokhov M M, Korepanov A V and Tenenev V A 2013 Vestnik of Kalashnikov Izhrevsk State Technical University 3 155-60