Finite generation of cohomology for Drinfeld doubles of finite group schemes

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Abstract
We prove that the Drinfeld double of an arbitrary finite group scheme has finitely generated cohomology. That is to say, for $G$ any finite group scheme, and $D(G)$ the Drinfeld double of the group ring $kG$, we show that the self-extension algebra of the trivial representation for $D(G)$ is a finitely generated algebra, and that for each $D(G)$-representation $V$ the extensions from the trivial representation to $V$ form a finitely generated module over the aforementioned algebra. As a corollary, we find that all categories $\text{rep}(G)^\ast$ dual to $\text{rep}(G)$ are also of finite type (i.e. have finitely generated cohomology), and we provide a uniform bound on their Krull dimensions. This paper completes earlier work of Friedlander and the author.

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1 Introduction

Fix $k$ an arbitrary field of finite characteristic. Let us recall some terminology [21]: A finite $k$-linear tensor category $\mathcal{C}$ is said to be of finite type (over $k$) if the self-extensions of the unit object $\text{Ext}_{\mathcal{C}}^\ast(1,1)$ form a finitely generated $k$-algebra, and for any object $V$ in $\mathcal{C}$ the extensions $\text{Ext}_{\mathcal{C}}^\ast(1,V)$ form a finitely generated module over this algebra. In this case, the Krull dimension $\text{Kdim} \mathcal{C}$ of $\mathcal{C}$ is the Krull dimension of the extension algebra of the unit. One is free to think of $\mathcal{C}$ here as the representation category $\text{rep}(A)$ of a finite-dimensional Hopf algebra $A$, with monoidal structure induced by the comultiplication, and unit $1 = k$ provided by the trivial representation.

It has been conjectured [10, Conjecture 2.18] [14] that any finite tensor category, over an arbitrary base field, is of finite type. Here we consider the category of representations for the Drinfeld double $D(G)$ of a finite group scheme $G$, which is identified with the so-called Drinfeld center $Z(\text{rep}(G))$ of the category of finite $G$-representations.

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The Drinfeld double $D(G)$ is the smash product $\mathcal{O}(G) \rtimes kG$ of the algebra of global functions on $G$ with the group ring $kG$, under the adjoint action. So, one can think of $Z(\text{rep}(G))$, alternatively, as the category of coherent $G$-equivariant sheaves on $G$ under the adjoint action

$$Z(\text{rep}(G)) = \text{rep}(D(G)) = \text{Coh}(G)^G.$$ 

In the present work we prove the following.

**Theorem (7.1)** For any finite group scheme $G$, the Drinfeld center $Z(\text{rep}(G))$ is of finite type and of Krull dimension

$$\text{Kdim } Z(\text{rep}(G)) \leq \text{Kdim } \text{rep}(G) + \text{embed. dim}(G).$$

Here embed. dim$(G)$ denotes the minimal dimension of a smooth (affine) algebraic group in which $G$ embeds as a closed subgroup. The above theorem was proved for $G = G(r)$ a Frobenius kernel in a smooth algebraic groups $G$ in work of Friedlander and the author [11]. Thus Theorem 7.1 completes, in a sense, the project of [11].

One can apply Theorem 7.1, and results of J. Plavnik and the author [21], to obtain an additional finite generation result for all dual tensor categories $\text{rep}(G)^\ast \mathcal{M}$ ($:= \text{End}_{\text{rep}(G)}(\mathcal{M})$), calculated relative to an exact $\text{rep}(G)$-module category $\mathcal{M}$ [10, Section 3.3].

**Corollary 1.1** Let $G$ be a finite group scheme, and $\mathcal{M}$ be an arbitrary exact $\text{rep}(G)$-module category. Then the dual category $\text{rep}(G)^\ast \mathcal{M}$ is of finite type and of uniformly bounded Krull dimension

$$\text{Kdim } \text{rep}(G)^\ast \mathcal{M} \leq \text{Kdim } \text{rep}(G) + \text{embed. dim}(G).$$

**Proof** Immediate from Theorem 7.1 and [21, Corollary 4.1].

We view Theorem 7.1, and Corollary 1.1, as occurring in a continuum of now very rich studies of cohomology for finite group schemes, e.g. [3,7,12–14,23,26].

**Remark 1.2** Indecomposable exact $\text{rep}(G)$-module categories have been classified by Gelaki [15], and correspond to pairs $(H, \psi)$ of a subgroup $H \subset G$ and certain 3-cocycle $\psi$ which introduces an associativity constraint for the action of $\text{rep}(G)$ on $\text{rep}(H)$.

**Remark 1.3** For an analysis of support theory for Drinfeld doubles of some solvable height 1 group schemes, one can see [19,20]. The problem of understanding support for general doubles $D(G)$ is, at this point, completely open.

### 1.1 Approach via equivariant deformation theory

In [11], where the Frobenius kernel $G(r)$ in a smooth algebraic group $G$ is considered, we basically use the fact that ambient group $G$ provides a smooth, equivariant,
deformation of $G_{(r)}$ parametrized by the quotient $G/G_{(r)} \cong G^{(r)}$ in order to gain a foothold in our analysis of cohomology. In particular, the adjoint action of $G_{(r)}$ on $G$ descends to a trivial action on the twist $G^{(r)}$, so that the Frobenius map $G \to G^{(r)}$ can be viewed as smoothly varying family of $G_{(r)}$-algebras which deforms the algebra of functions $O(G_{(r)})$. Such a deformation situation provides “deformation classes” in degree 2,

\[
\{\text{deformation classes}\} = T_1 G^{(r)} \subset \text{Ext}^2_{\text{Coh}(G_{(r)})} (G^{(r)} \cdot 1, 1) = \text{Ext}^2_{D(G^{(r)})} (1, 1).
\]

One uses these deformation classes, in conjunction with work of Friedlander and Suslin [14], to find a finite set of generators for extensions.

For a general finite group scheme $G$, we can try to pursue a similar deformation approach, where we embed $G$ into a smooth algebraic group $H$, and consider $H$ as a deformation of $G$ parametrized by the quotient $H/G$. However, a general finite group scheme may not admit any normal embedding into a smooth algebraic group. (This is the case for certain non-connected finite group schemes, and should also be the case for restricted enveloping algebras $kG = u^{res}(g)$ of Cartan type simple Lie algebras, for example). So, in general, one accepts that $G$ acts nontrivially on the parametrization space $H/G$, and that the fibers in the family $H$ are permuted by the action of $G$ here. Thus we do not obtain a smoothly varying family of $G$-algebras deforming $O(G)$ in this manner.

One can, however, consider a type of equivariant deformation theory where the group $G$ is allowed to act nontrivially on the parametrization space, and attempt to obtain higher deformation classes in this instance

\[
\{\text{higher deformation classes}\} \subset \text{Ext}^2_{\text{Coh}(G)} (G \cdot 1, 1) = \text{Ext}^2_{D(G)} (1, 1).
\]

We show in Sects. 3 and 5 that this equivariant deformation picture can indeed be formalized, and that–when considered in conjunction with work of Touzé and Van der Kallen [26]–it can be used to obtain the desired finite generation results for cohomology (see in particular Theorems 5.4 and 6.4).

**Remark 1.4** From a geometric perspective, one can interpret our main theorem as a finite generation result for the cohomology of non-tame stacky local complete intersections. (Formally speaking, we only deal with the maximal codimension case here, but the general situation is similar.) One can compare with works of Gulliksen [16], Eisenbud [8], and many others regarding the homological algebra of complete intersections.

### 2 Differential generalities

Throughout $k$ is a field of finite characteristic, which is not necessarily algebraically closed. Schemes and algebras are $k$-schemes and $k$-algebras, and $\otimes = \otimes_k$. All group schemes are affine group schemes which are of finite type over $k$, and throughout $G$ denotes an (affine) group scheme.
2.1 Commutative algebras and modules

A finite type commutative algebra over a field $k$ is a finitely generated $k$-algebra. A coherent module over a commutative Noetherian algebra is a finitely generated module. We adopt this language, at times, to distinguish clearly between these two notions of finite generation.

2.2 $G$-equivariant dg algebras

Consider $G$ an affine group scheme. We let $\text{rep}(G)$ denote the category of finite-dimensional $G$-representations, $\text{Rep}(G)$ denote the category of integrable, i.e. locally finite, representations, and $\text{Ch}(\text{Rep}(G))$ denote the category of cochain complexes over $\text{Rep}(G)$. Each of these categories is considered along with its standard monoidal structure.

By a $G$-algebra we mean an algebra object in $\text{Rep}(G)$, and by a dg $G$-algebra we mean an algebra object in $\text{Ch}(\text{Rep}(G))$. For $T$ any commutative $G$-algebra, by a $G$-equivariant dg $T$-algebra $S$ we mean a $T$-algebra in $\text{Ch}(\text{Rep}(G))$. Note that, for such a dg algebra $S$, the associated sheaf $S^\sim$ on $\text{Spec}(T)$ is an equivariant sheaf of dg algebras, and vice versa. Note also that a dg $G$-algebra is the same thing as an equivariant dg algebra over $T = k$.

2.3 DG modules and resolutions

For $S$ a dg $G$-algebra, we let $S^{-\text{dgmod}}$ and $D(S)^G$ denote the category of $G$-equivariant dg modules over $S$ and its corresponding derived category $D(S)^G = (S^{-\text{dgmod}})[\text{quis}^{-1}]$. (Of course, by an equivariant dg module we mean an $S$-module in the category of cochains over $G$.) If we specify some commutative Noetherian graded $G$-algebra $T$, and equivariant $T$-algebra structure on cohomology $T \to H^*(S)$, then we take

$$D_{coh}(S)^G := \left\{ \text{The full subcategory in } D(S)^G\text{ consisting of dg modules } M \text{ with finitely generated cohomology over } T \right\}.$$ 

When $T = k$ we take $D_{fin}(S)^G = D_{coh}(S)^G$.

A (non-equivariant) free dg $S$-module is an $S$-module of the form $\bigoplus_{j \in J} \Sigma^n S$, where $J$ is some indexing set. A semi-projective resolution of a (non-equivariant) dg $S$-module $M$ is a quasi-isomorphism $F \to M$ from a dg module $F$ equipped with a filtration $F = \bigcup_{i \geq 0} F(i)$ by dg submodules such that each subquotient $F(i)/F(i-1)$ is a summand of a free $S$-module. An equivariant semi-projective resolution of an equivariant dg module $M$ is a $G$-linear quasi-isomorphism $F \to M$ from an equivariant dg module $F$ which is non-equivariantly semi-projective. The usual shenanigans, e.g. [6, Lemma 13.3], shows that equivariant semi-projective resolutions always exist.
2.4 Homotopy isomorphisms

Consider $S$ and $A$ dg $G$-algebras, over some given group scheme $G$. By an (equivariant) homotopy isomorphism $f : S \to A$ we mean a zig-zag of $G$-linear dg algebra quasi-isomorphism

$$S \sim S_1 \sim S_2 \cdots \sim S_{N-1} \sim A.$$  \hspace{1cm} (1)

We note that we use the term homotopy informally here, as we do not propose any particular model structure on the category of dg $G$-algebras (cf. [24,25]). Throughout the text, when we speak of homotopy isomorphisms between dg $G$-algebras we always mean equivariant homotopy isomorphisms.

A homotopy isomorphism $f : S \to A$ as in (1) specifies a triangulated equivalence between the corresponding derived categories of dg modules $f_* : D(S)^G \sim \to D(A)^G$, \hspace{1cm} (2)

via successive application of base change and restriction along the maps to/from the $S_i$. To elaborate, an equivariant quasi-isomorphism $f : S_1 \to S_2$ specifies mutually inverse equivalences $S_2 \otimes_{S_1}^L - : D(S_1)^G \to D(S_2)^G$ and $\text{res}_f : D(S_2)^G \to D(S_1)$. So for a homotopy isomorphism $f : S \to A$, compositions of restriction and base change produce the equivalence (2).

Note that, on cohomology, such a homotopy isomorphism $f : S \to A$ induces an actual isomorphism of algebras $H^*(f) : H^*(S) \to H^*(A)$, and one can check that for a dg module $M$ over $S$ we have

$$H^*(f_* M) \cong H^*(A) \otimes_{H^*(S)} H^*(M) \cong \text{res}_{H^*(f)^{-1}} H^*(M).$$

So, in particular, if $H^*(S)$ and $H^*(A)$ are $T$-algebras, for some commutative Noetherian $T$, and $H^*(f)$ is $T$-linear, then the equivalence (2) restricts to an equivalence

$$f_* : D_{coh}(S)^G \sim \to D_{coh}(A)^G$$

between the corresponding equivariant, coherent, derived categories.

**Definition 2.1** We say a dg $G$-algebra $S$ is equivariantly formal if $S$ is equivariantly homotopy isomorphic to its cohomology $H^*(S)$.

2.5 Derived maps and derived endomorphisms

Fix $S$ a dg $G$-algebra, over a group scheme $G$. For such $S$, the dg Hom functor $\text{Hom}_S$ on $S$-dgmod$^G$ naturally takes values in Ch(Rep($G$)). Namely, for $x$ in the group ring $kG = \mathcal{O}(G)^*$, we act on functions $f \in \text{Hom}_S(M, N)$ according to the formula

$$(x \cdot f)(m) := x_1 f(S(x_2)m).$$
With these actions, each $\text{Hom}_S(M, N)$ is a dg $G$-representation, and composition

$$\text{Hom}_S(N, L) \otimes \text{Hom}_S(M, N) \to \text{Hom}_S(M, L)$$

is a map of dg $G$-representations. In particular, $\text{End}_S(M)$ is a dg $G$-algebra for any equivariant dg module $M$ over $S$.

**Remark 2.2** One needs to use cocommutativity of $kG$ here to see that $x \cdot f$ is in fact $S$-linear for $S$-linear $f$.

We derive the functor $\text{Hom}_S$ to $\text{Ch}(\text{Rep}(G))$ by taking

$$\text{RHom}_S(M, N) := \text{Hom}_S(M', N),$$

where $M' \to M$ is any equivariant semi-projective resolution of $M$. One can apply their favorite arguments to see that $\text{RHom}_S(M, N)$ is well-defined as an object in $D(\text{Rep}(G))$, or refer to the following lemma.

**Lemma 2.3** For any two equivariant resolutions $M_1 \to M$ and $M_2 \to M$ there is an equivariant semi-projective dg module $F$ which admits two surjective, equivariant, quasi-isomorphisms $F \to M_1$ and $F \to M_2$.

**Proof** By adding on acyclic semi-projective summands we may assume that the given maps $f_i : M_i \to M$ are surjective. For example, one can take a surjective resolution $N \to M$, consider the mapping cone $\text{cone}(\text{id}_N)$, then replaces the $M_i$ with $(\Sigma^{-1} \text{cone}(\text{id}_N)) \oplus M_i$. So, let us assume that the $f_i$ here are surjective.

We consider now the fiber product $F_0$ of the maps $f_1$ and $f_2$ to $M$. Note that the structure maps $F_0 \to M_i$ are surjective, since the $f_i$ are surjective. We have the exact sequence

$$0 \to F_0 \to M_1 \oplus M_2 \xrightarrow{[f_1 - f_2]^T} M \to 0$$

and by considering the long exact sequence on cohomology find that we have also an exact sequence

$$0 \to H^*(F_0) \to H^*(M_1) \oplus H^*(M_2) \to H^*(M) \to 0,$$

with the map from $H^*(M_1) \oplus H^*(M_2)$ the sum of isomorphisms $\pm H^*(f_i)$. It follows that the composites $H^*(F_0) \to H^*(M_1) \oplus H^*(M_2) \to H^*(M_1)$ are both isomorphisms, and hence that the maps $F_0 \to M_1$ and $F_0 \to M_2$ are quasi-isomorphisms. One considers $F \to F_0$ any surjective, equivariant, semi-projective resolution to obtain the claimed result.

For $M$ in $D(S)^G$ we take $\text{REnd}_S(M) = \text{End}_S(M')$, for $M' \to M$ any equivariant semi-projective resolution. The following result should be known to experts. The proof we offer is due to Benjamin Briggs and Ragnar Buchweitz. I thank Briggs for communicating the proof to me, and allowing me to repeat it here.
Lemma 2.4  \( R\text{End}_S(M) \) is well-defined, as a dg \( G \)-algebra, up to homotopy isomorphism. Furthermore, if \( M \) and \( N \) are isomorphic in \( D(S)^G \), then \( R\text{End}_S(M) \) and \( R\text{End}_S(N) \) are homotopy isomorphic as well.

Given an explicit isomorphism \( \xi : M \to N \) in \( D(S)^G \), the homotopy isomorphism \( R\text{Hom}_S(M) \to R\text{Hom}_S(N) \) can in particular be chosen to lift the canonical isomorphism \( \text{Ad}_\xi : \text{Ext}^*_S(M, M) \to \text{Ext}^*_S(N, N) \) on cohomology.

**Proof**  Consider two equivariant semi-projective resolutions \( M_1 \to M \) and \( M_2 \to M \). By Lemma 2.3 we may assume that the map \( M_1 \to M \) lifts to a surjective, equivariant, quasi-isomorphism \( f : M_1 \to M_2 \). In this case we have the two quasi-isomorphisms \( f_* \) and \( f^* \) of Hom complexes, and consider the fiber product

\[
\begin{array}{ccc}
\text{End}_S(M_1) & \xrightarrow{f_*} & \text{Hom}_S(M_1, M_2) \\
\downarrow & & \downarrow \\
\text{End}_S(M_2) & \xleftarrow{f^*} & \\
\end{array}
\]

As \( f_* \) and \( f^* \) are maps of dg \( G \)-representations, \( B \) is a dg \( G \)-representation. Furthermore, one checks directly that \( B \) is a dg algebra, or more precisely a dg subalgebra in the product \( \text{End}(M_1) \times \text{End}(M_2) \). So the top portion of (3) is a diagram of maps of dg \( G \)-algebras.

As \( M_1 \) is projective, as a non-dg module, the map \( f_* \) is a surjective quasi-isomorphism. One can therefore argue as in the proof of Lemma 2.3 to see that the structure maps from \( B \) to the \( \text{End}_S(M_i) \) are quasi-isomorphisms. So we have the explicit homotopy isomorphism

\[
\text{End}_S(M_1) \sim \sim B \sim \sim \text{End}_S(M_2).
\]

Now, if \( M \) is isomorphic to \( N \) in \( D(S)^G \), then there is a third equivariant dg module \( \Omega \) with quasi-isomorphisms \( M \sim \sim \Omega \sim \sim N \). Any resolution \( F \sim \sim \Omega \) therefore provides a simultaneous resolution of \( M \) and \( N \), and we may take \( R\text{End}_S(M) = \text{End}_S(F) = R\text{End}_S(N) \).

3 Equivariant deformations and Koszul resolutions

In Sects. 3 and 5 we develop the basic homological algebra associated with equivariant deformations. Our main aim here is to provide equivariant versions of results of Bezrukavnikov and Ginzburg [4], and Pevtsova and the author [20, §4] (cf. [1,8]).

3.1 Equivariant deformations

We recall that a deformation of an algebra \( R \), parametrized by an augmented commutative algebra \( Z \), is a choice of flat \( Z \)-algebra \( Q \) along with an algebra map \( Q \to R \).
which reduces to an isomorphism \( k \otimes \mathbb{Z} \mathbb{Q} \cong \mathbb{R} \). We call such a deformation \( Q \to R \) an **equivariant deformation** if all of the algebras present are \( G \)-algebras, and all of the structure maps \( Z \to Q, Z \to k, \) and \( Q \to R \) are maps of \( G \)-algebras.

The interesting point here, and the point of deviation with other interpretations of equivariant deformation theory, is that we allow \( G \) to act nontrivially on the parametrization space \( \text{Spec}(Z) \) (or \( \text{Spf}(Z) \) in the formal setting).

### 3.2 An equivariant Koszul resolution

We fix a group scheme \( G \), and equivariant deformation \( Q \to R \) of a \( G \)-algebra \( R \) with formally smooth parametrization space space \( \text{Spf}(Z) \). We require specifically that \( Z \) is isomorphic to a power series \( \mathbb{K}[x_1, \ldots, x_n] \) in finitely many variables. As the distinguished point \( 1 \in \text{Spf}(Z) \) is a fixed point for the \( G \)-action, the cotangent space \( T_1 \text{Spf}(Z) = m_Z/m_Z^2 \) admits a natural \( G \)-action, and so does the graded algebra \( \text{Sym}(m_Z/m_Z^2) \), which we view as a dg \( G \)-algebra with vanishing differential.

**Lemma 3.1** (cf. [1, Lemma 5.1.4]) One can associate to the parametrization algebra \( Z \) a commutative equivariant dg \( Z \)-algebra \( K_Z \) such that

1. \( K_Z \) is finite and flat over \( Z \), and
2. \( K_Z \) admits quasi-isomorphisms \( K_Z \to k \) and \( k \otimes Z K_Z \to \text{Sym}(m_Z/m_Z^2) \) of equivariant dg algebras.

**Proof** (Construction) We first construct an unbounded dg resolution \( K' \) of \( k \), as in [5, Section 2.6], then truncate to obtain \( K \). We construct \( K' \) as a union \( K' = \lim_{\to i \geq 0} K(i) \) of dg subalgebras \( K(i) \) over \( Z \). We define the \( K(i) \) inductively as follows: Take \( K(0) = Z \) and, for \( V_1 \) a finite-dimensional \( G \)-subspace generating the maximal ideal \( m_Z \) in \( Z \), we take \( K(1) = \text{Sym}_Z(Z \otimes \Sigma V_1) \) with differential \( d(\Sigma v) = v, v \in V_1 \).

Suppose now that we have \( K(i) \) an equivariant dg algebra which is finite and flat over \( Z \) in each degree, and has (unique) augmentation \( K(i) \to k \) which is a quasi-isomorphism in degrees \( > -i \). Let \( V_i \) be an equivariant subspace of cocycles in \( K(i)^{-i} \) which generates \( H^{-i}(K(i)) \), as a \( Z \)-module. Define

\[
K(i + 1) = \text{Sym}_Z(Z \otimes \Sigma V_i) \otimes Z K(i), \quad \text{with extended differential } d(\Sigma v) = v \quad \text{for } v \in V_i.
\]

We then have the directed system of dg algebras \( K(0) \to K(1) \to \ldots \) with colimit \( K' = \lim_{\to i} K(i) \). By construction \( K' \) is finite and flat over \( Z \) in each degree, and has cohomology \( H^*(K') = k \).

Since \( Z \) is of finite flat dimension, say \( n \), the quotient

\[
(K_Z : = K = K' / ((K')_{< -n} + B^{-n}(K')))
\]
is finite and flat over \( Z \) in all degrees. Furthermore, \( K \) inherits a \( G \)-action so that the quotient map \( K' \rightarrow K \) is an equivariant quasi-isomorphism. So we have produced a finite flat dg \( Z \)-algebra \( K \) with equivariant quasi-isomorphism \( K \simto k \).

We consider a linear section \( m_Z/m_Z^2 \rightarrow V_1 \) of the projection \( V_1 \rightarrow m_Z/m_Z^2 \), and let \( S_1 \subset V_1 \) denote the image of this section. Take \( S = \mathrm{Sym}_Z(Z \otimes S_1) \) with differential specified by \( d(\Sigma v) = v \) for \( v \in S_1 \). Then \( S \) is the standard Koszul resolution for \( k \), and the inclusion \( S \rightarrow K \) is a (non-equivariant) dg algebra quasi-isomorphism. Since \( K \) and \( S \) are bounded above and flat over \( Z \) in each degree, the reduction \( k \otimes_Z S \rightarrow k \otimes_Z K \) remains a quasi-isomorphism and we have an isomorphism of algebras

\[
\mathrm{Sym}(\Sigma m_Z/m_Z^2) \cong H^*(k \otimes_Z S) \xrightarrow{\sim} H^*(k \otimes_Z K).
\]

Note that the dg subalgebra \( \mathrm{Sym}(\Sigma V_1) \subset k \otimes_Z K \) consists entirely of cocycles, and furthermore \( Z^{-1}(k \otimes_Z K) = \Sigma V_1 \). We see also that the intersection \( V_1 \cap m_Z^2 \) consists entirely of coboundaries, as such vectors \( v \) lift to cocycles in the acyclic complex \( K \) which lie in the respective spaces \( v + m_Z \otimes V_1 \). A dimension count now implies that the projection

\[
V_1 = Z^{-1}(k \otimes_Z K) \rightarrow H^{-1}(k \otimes_Z K)
\]

reduces to an isomorphism \( V_1/(m_Z^2 \cap V_1) = H^1(k \otimes_Z K) \). Hence, for the degree \(-1\) coboundaries in \( k \otimes_Z K \), we have \( B^{-1} = V_1 \cap m_Z^2 \). One now consults the diagram

\[
\begin{array}{ccc}
\mathrm{Sym}(\Sigma m_Z/m_Z^2) & \xrightarrow{\text{incl}} & \mathrm{Sym}(\Sigma V_1) \\
& \searrow & \downarrow \text{proj} \\
H^*(k \otimes_Z S) & \xrightarrow{\sim} & H^*(k \otimes_Z K),
\end{array}
\]

\[
\mathrm{Sym}(\Sigma m_Z/m_Z^2)/(B^{-1}) \cong \mathrm{Sym}(\Sigma m_Z/m_Z^2)
\]

to see that the intersection \( B^*(k \otimes_Z K) \cap \mathrm{Sym}(\Sigma V_1) \) is necessarily the ideal \( (B^{-1}) \) generated by the degree \(-1\) coboundaries. So we find that the projection

\[
f : k \otimes_Z K \rightarrow \mathrm{Sym}(\Sigma V_1)/(B^{-1}) \cong \mathrm{Sym}(\Sigma m_Z/m_Z^2)
\]

which annihilates (the images of) all cells \( \Sigma V_i \) with \( i > 1 \) is an equivariant dg algebra map, and furthermore an equivariant dg algebra quasi-isomorphism.

In the following \( Z \) is a commutative \( G \)-algebra which is isomorphic to a power series in finitely many variables, as above.

**Definition 3.2** An equivariant Koszul resolution of \( k \) over \( Z \) is a \( G \)-equivariant dg \( Z \)-algebra \( K_Z \) which is finite and flat over \( Z \), comes equipped with an equivariant dg algebra quasi-isomorphism \( \epsilon : K_Z \rightarrow k \), and also comes equipped with an equivariant dg map \( \pi : K_Z \rightarrow \mathrm{Sym}(\Sigma m_Z/m_Z^2) \) which reduces to a quasi-isomorphism \( k \otimes_Z K_Z \rightarrow \mathrm{Sym}(\Sigma m_Z/m_Z^2) \) along the augmentation \( Z \rightarrow k \).

Lemma 3.1 says that equivariant Koszul resolutions of \( k \), over such \( Z \), always exists.
3.3 The Koszul resolution associated to an equivariant deformation

Consider $Q \rightarrow R$ an equivariant deformation, parameterized by a formally smooth space $\text{Spf}(Z)$, as in Sect. 3.2. For any equivariant Koszul resolution $K_Z \sim k$ over $Z$, the product

$$K_Q := Q \otimes_Z K_Z$$

(4)

is naturally a dg $G$-algebra which is a finite and flat extension of $Q$. Since finite flat modules over $Z$ are in fact free, $K_Q$ is more specifically free over $Q$ in each degree. Flatness of $Q$ over $Z$ implies that the projection

$$\text{id}_Q \otimes_Z \epsilon : K_Q \sim Q \otimes_Z k = R$$

is a quasi-isomorphism of dg $G$-algebras (cf. [1, Section 5.2], [4, Section 3], [2, Section 2]). We call the dg algebra (4), deduced from a particular choice of equivariant Koszul resolution for $Z$, the (or a) Koszul resolution of $R$ associated to the equivariant deformation $Q \rightarrow R$.

4 Deformations associated to group embeddings

Consider now $G$ a finite group scheme, and a closed embedding of $G$ into a smooth affine algebraic group $H$. (We mean specifically a map of group schemes $G \rightarrow H$ which is, in addition, a closed embedding.) We explain in this section how such an embedding $G \rightarrow H$ determines an equivariant deformation $\mathcal{O} \rightarrow \mathcal{O}(G)$ which fits into the general framework of Sect. 3.

Note that such closed embeddings $G \rightarrow H$ always exists for finite $G$. For example, if we choose a faithful $G$-representation $V$ then the corresponding action map $G \rightarrow \text{GL}(V)$ is a closed embedding of $G$ into the associated general linear group.

4.1 The quotient space

For any embedding $G \rightarrow H$ of $G$ into smooth $H$ we consider the quotient space $H/G$. The associated quotient map $H \rightarrow H/G$ is $G$-equivariant, where we act on $H$ via the adjoint action and on $H/G$ via translation.

This is all clear geometrically, but let us consider this situation algebraically. Functions on the quotient $\mathcal{O}(H/G)$ are the right $G$-invariants $\mathcal{O}(H)^G$ in $\mathcal{O}(H)$, or rather the left $\mathcal{O}(G)$-coinvariants. Then $\mathcal{O}(H/G)$ is a right $\mathcal{O}(H)$-coideal subalgebra in $\mathcal{O}(H)$, in the sense that the comultiplication on $\mathcal{O}(H)$ restricts to a coaction

$$\rho : \mathcal{O}(H/G) \rightarrow \mathcal{O}(H/G) \otimes \mathcal{O}(H).$$
We project along $\mathcal{O}(\mathcal{H}) \to \mathcal{O}(G)$ to obtain the translation coaction of $\mathcal{O}(G)$ on $\mathcal{O}(\mathcal{H}/G)$. The left translation coaction of $\mathcal{O}(G)$ on $\mathcal{O}(\mathcal{H})$ restricts to a trivial coaction on $\mathcal{O}(\mathcal{H}/G)$. So, $\mathcal{O}(\mathcal{H}/G)$ is a sub $\mathcal{O}(G)$-bicomodule in $\mathcal{O}(\mathcal{H})$.

We consider the dual action of the group ring $kG = \mathcal{O}(G)^*$ on $\mathcal{O}(\mathcal{H})$, and find that the inclusion $\mathcal{O}(\mathcal{H}/G) \to \mathcal{O}(\mathcal{H})$ is an inclusion of $G$-algebras, where we act on $\mathcal{O}(\mathcal{H})$ via the adjoint action and on $\mathcal{O}(\mathcal{H}/G)$ by translation. We have the following classical result, which can be found in [17, Proposition 5.25 and Corollary 5.26].

**Theorem 4.1** Consider a closed embedding $G \to \mathcal{H}$ of a finite group scheme into a smooth algebraic group $\mathcal{H}$. The algebra of functions $\mathcal{O}(\mathcal{H})$ is finite and flat over $\mathcal{O}(\mathcal{H}/G)$, and $\mathcal{O}(\mathcal{H}/G)$ is a smooth $k$-algebra.

### 4.2 The associated equivariant deformation sequence

Consider $G \to \mathcal{H}$ as above and let $1 \in \mathcal{H}/G$ denote the image of the identity in $\mathcal{H}$, by abuse of notation. We complete the inclusion $\mathcal{O}(\mathcal{H}/G) \to \mathcal{O}(\mathcal{H})$ at the ideal of definition for $G$ to get a finite flat extension $\hat{\mathcal{O}}_{\mathcal{H}/G} \to \hat{\mathcal{O}}_{\mathcal{H}}$. Take

$$Z = \hat{\mathcal{O}}_{\mathcal{H}/G} \text{ and } \mathcal{O} = \hat{\mathcal{O}}_{\mathcal{H}}.$$

So we have the deformation $\mathcal{O} \to \mathcal{O}(G)$, with formally smooth parametrizing algebra $Z$. A proof of the following Lemma can be found at [20, Lemma 2.10].

**Lemma 4.2** The completion $\mathcal{O} = \hat{\mathcal{O}}_{\mathcal{H}}$ is Noetherian and of finite global dimension.

Note that the ideal of definition for $G$ is the ideal $m\mathcal{O}(G)$, where $m \subset \mathcal{O}(\mathcal{H}/G)$ is associated to the closed point $1 \in \mathcal{H}/G$.

**Proposition 4.3** Consider a closed embedding $G \to \mathcal{H}$ of a finite group scheme into a smooth algebraic group $\mathcal{H}$. Take $\mathcal{O} = \hat{\mathcal{O}}_{\mathcal{H}}$ and $Z = \hat{\mathcal{O}}_{\mathcal{H}/G}$, where we complete at the augmentation ideal $m$ in $\mathcal{O}(\mathcal{H}/G)$. Then

(a) the quotients $\mathcal{O}(\mathcal{H}/G)/m^n$ and $\mathcal{O}(\mathcal{H})/m^n\mathcal{O}(\mathcal{H})$ inherit $G$-algebra structures from $\mathcal{O}(\mathcal{H}/G)$ and $\mathcal{O}(\mathcal{H})$ respectively.

(b) The completions $Z$ and $\mathcal{O}$ inherit unique continuous $G$-actions so that the inclusions $\mathcal{O}(\mathcal{H}/G) \to Z$ and $\mathcal{O}(\mathcal{H}) \to \mathcal{O}$ are $G$-linear.

(c) Under the actions of (b), the projection $\mathcal{O} \to \mathcal{O}(G)$ is an equivariant deformation of $\mathcal{O}(G)$ parametrized by $\text{Spf}(Z) = (\mathcal{H}/G)_1^\wedge$.

**Proof** All of (a)–(c) will follow if we can simply show that $m \subset \mathcal{O}(\mathcal{H}/G)$ is stable under the translation action of $kG$. This is clear geometrically, and certainly well-known, but let us provide an argument for completeness. If we let $\ker(\epsilon) \subset \mathcal{O}(\mathcal{H})$ denote the augmentation ideal, we have $m = \ker(\epsilon) \cap \mathcal{O}(\mathcal{H}/G)$.

For the adjoint coaction $\rho_{\text{ad}} : f \mapsto f_2 \otimes S(f_1)f_3$ of $\mathcal{O}(\mathcal{H})$ on itself, and $f \in \ker(\epsilon)$, we have

$$(\epsilon \otimes 1) \circ \rho_{\text{ad}}(f) = \epsilon(f_2)S(f_1)f_3 = S(f_1)(\epsilon(f_2)f_3) = S(f_1)f_2 = \epsilon(f) = 0.$$
So we see that under the adjoint coaction $\rho_{ad}(\ker(\epsilon)) \subset \ker(\epsilon) \otimes \mathcal{O}(\mathcal{H})$. It follows that $\ker(\epsilon)$ is preserved under the adjoint coaction of $\mathcal{O}(G)$, and hence the adjoint action of $kG$, as well. So, the intersection $m = \mathcal{O}(\mathcal{H}/G) \cap \ker(\epsilon)$ is an intersection of $G$-subrepresentations in $\mathcal{O}(\mathcal{H})$, and hence $m$ is stable under the action of $kG$.

### 5 Equivariant formality results and deformation classes

We observe cohomological implications of the existence of a (smooth) equivariant deformation, for a given finite-dimensional $G$-algebra $R$. The main results of this section can been seen as particular equivariantizations of [4, Theorem 1.2.3] and [20, Corollary 4.7], as well as of classical results of Gulliksen [16, Theorem 3.1].

#### 5.1 We fix an equivariant deformation

We fix a $G$-equivariant deformation $Z \rightarrow Q \rightarrow R$, with $Z$ isomorphic to a power series in finitely many variables. Fix also a choice of equivariant Koszul resolution

$$\mathcal{K} := \mathcal{K}_Z, \quad \text{with } \epsilon : \mathcal{K} \sim \rightarrow k \text{ and } \pi : \mathcal{K} \rightarrow \text{Sym}(\Sigma m_Z/m^2_Z).$$

Recall the associated dg resolution $\mathcal{K}_Q \sim \rightarrow R$, with $\mathcal{K}_Q = Q \otimes_Z \mathcal{K}$. Via general phenomena (Sect. 2.4) we observe

**Lemma 5.1** Restriction provides a derived equivalence $D_{\text{fin}}(R)^G \sim \rightarrow D_{\text{coh}}(\mathcal{K}_Q)^G$.

Here by $D_{\text{coh}}(\mathcal{K}_Q)^G$ we mean, specifically, dg modules with finite cohomology over $Z$. Following the notation of [20], we fix

$$A_Z := \text{Sym}(\Sigma^{-2} T_1 \text{Spf}(Z)) = \text{Sym}(\Sigma^{-2}(m_Z/m^2_Z)^*). \quad (5)$$

#### 5.2 Equivariant formality and deformation classes

**Lemma 5.2** Consider $\mathcal{K}$ as the regular dg $\mathcal{K}$-bimodule. There is a $G$-equivariant homotopy isomorphism

$$\text{REnd}_{\mathcal{K} \otimes Z \mathcal{K}}(\mathcal{K}) \sim \rightarrow A_Z.$$

In particular, $\text{REnd}_{\mathcal{K} \otimes Z \mathcal{K}}(\mathcal{K})$ is equivariantly formal.

**Proof** Consider our algebra $A = A_Z$ from (5) and take $B = \text{Sym}(\Sigma m_Z/m^2_Z)$. Let $F \rightarrow k$ be the standard resolution of the trivial module over $B$. The resolution $F$ is of the form $B \otimes A^*$, as a graded space, with differential given by right multiplication by the identity element $\sum_i x_i \otimes x^i$ in $B^{-1} \otimes A^2$, and so $F$ admits a natural dg $(B, A)$-bimodule structure. The action map for $A$ now provides an equivariant quasi-isomorphism $A \rightarrow \text{End}_B(F) = \text{REnd}_B(k)$. 


For the Koszul resolution $\mathcal{K}$ over $\mathbb{Z}$, we have the equivariant quasi-isomorphism $\pi \otimes \mathbb{Z} \epsilon : \mathcal{K} \otimes \mathbb{Z} \mathcal{K} \rightarrow B$ and corresponding restriction and base change equivalences $D(\mathcal{K} \otimes \mathbb{Z} \mathcal{K})^G \leftarrow D(B)^G$, which are mutually inverse. Restriction sends the trivial representation $k$ over $B$ to the regular $\mathcal{K}$-bimodule $k \cong \mathcal{K}$. Hence the base change $B \otimes_{\mathcal{K} \otimes \mathcal{K}} \mathcal{K}$ is isomorphic to $k$. We then get then an equivariant quasi-isomorphism

$$B \otimes_{\mathcal{K} \otimes \mathcal{K}} \mathcal{K} \rightarrow \mathcal{K} \otimes \mathbb{Z} \mathcal{K} \rightarrow \mathcal{K} \otimes \mathbb{Z} \mathcal{K}$$

with the latter algebra homotopy isomorphic to $\mathbb{R}\text{End}_{\mathcal{K} \otimes \mathbb{Z} \mathcal{K}}(\mathcal{K}) \cong A$ by Lemma 2.4.

**Remark 5.3** In odd characteristic, one can replace the quasi-isomorphism $\pi \otimes \mathbb{Z} \epsilon : \mathcal{K} \otimes \mathbb{Z} \mathcal{K} \rightarrow B$ with the more symmetric map

$$\text{mult}(\frac{1}{2} \pi \otimes \frac{1}{2} \pi) : \mathcal{K} \otimes \mathbb{Z} \mathcal{K} \rightarrow B.$$  

The point is to provide an equivariant quasi-isomorphism which is a retract of the non-equivariant quasi-isomorphism $B \rightarrow \mathcal{K} \otimes \mathbb{Z} \mathcal{K}$ implicit in [4, Lemma 2.4.2].

Recall that we are considering an equivariant deformation $Q \rightarrow R$, with associated dg resolution $\mathcal{K}_Q \sim R$, as in Sect. 3.3. We have the natural action of $A_Z$ on $D_{\text{coh}}(\mathcal{K}_Q)$ [20, §3.4], which is expressed via the algebra map

$$A_Z = \text{End}_{D(\mathcal{K} \otimes \mathbb{Z} \mathcal{K})}(\mathcal{K}) \rightarrow Z(D_{\text{coh}}(\mathcal{K}_Q))$$

(6)

to the center of the derived category $Z(D_{\text{coh}}(\mathcal{K}_Q)) = \oplus_i \text{Hom}_{\text{Fun}}(id, \Sigma^i)$. Specifically, for any endomorphism $f : \mathcal{K} \rightarrow \Sigma^n \mathcal{K}$ in the derived category of $\mathbb{Z}$-central bimodules, and $M$ in $D_{\text{coh}}(\mathcal{K}_Q)$, we have the induced endomorphism

$$f \otimes_{\mathcal{K}}^L M : M \rightarrow \Sigma^n M.$$  

Suppose, for convenience, that $Q$ is of finite global dimension. We lift the maps

$$- \otimes_{\mathcal{K}}^L M : \text{End}_{D(\mathcal{K} \otimes \mathbb{Z} \mathcal{K})}(\mathcal{K}) \rightarrow \text{End}_{D(\mathcal{K}_Q)}(M)$$

(7)

to a dg level, for equivariant $M$, as follows [4]: Fix an equivariant semi-projective resolution $F \rightarrow \mathcal{K}$ over $\mathcal{K} \otimes \mathbb{Z} \mathcal{K}$ and, at each $M$, chose an equivariant quasi-isomorphism $M' \rightarrow M$ from a dg $\mathcal{K}_Q$-module which is bounded and projective over $Q$ in each degree. (Such a resolution exists since $Q$ is of finite global dimension.) Then $F \otimes_{\mathcal{K}} M' \rightarrow M$ is an equivariant semi-projective resolution of $M$ over $\mathcal{K}_Q$ [20, Lemma 4.4]. We now have the lift

$$- \otimes_{\mathcal{K}} M' : \text{End}_{\mathcal{K} \otimes \mathbb{Z} \mathcal{K}}(F) \rightarrow \text{End}_{\mathcal{K}_Q}(F \otimes_{\mathcal{K}} M')$$

of (7), and we write this lift simply as

$$\partial \text{ef}^G_M : \mathbb{R}\text{End}_{\mathcal{K} \otimes \mathbb{Z} \mathcal{K}}(\mathcal{K}) \rightarrow \mathbb{R}\text{End}_{\mathcal{K}_Q}(M).$$
Direct calculation verifies that \( \text{def}_M^G \), constructed in this manner, is in fact \( G \)-linear. The following result is an equivariantization of [20, Corollary 4.7].

**Theorem 5.4** Consider a \( G \)-equivariant deformation \( Q \rightarrow R \), with \( R \) finite-dimensional, \( Q \) of finite global dimension, and parametrization algebra \( Z \) isomorphic to a power series in finitely many variables. Let \( \mathcal{R} \) denote the formal dg algebra \( \text{REnd}_{K \otimes Z K}(K) \) (Lemma 5.2).

For any \( M \) in \( D_{\text{coh}}(K_Q)^G \), the equivariant dg algebra map \( \text{def}_M^G : \mathcal{R} \rightarrow \text{REnd}_{K_Q}(M) \) defined above has the following properties:

1. The induced map on cohomology \( H^*(\text{def}_M^G) : AZ \rightarrow \text{End}^*_D(K_Q)(M) \) is a finite morphism of graded \( G \)-algebras.
2. For any \( N \) in \( D_{\text{coh}}(K_Q)^G \), the induced action of \( \mathcal{R} \) on \( \text{RHom}_{K_Q}(M, N) \) is such that

\[
\text{RHom}_{K_Q}(M, N) \in D_{\text{coh}}(\mathcal{R})^G.
\]

By \( D_{\text{coh}}(\mathcal{R})^G \) we specifically mean the category of \( G \)-equivariant dg modules over \( \mathcal{R} \) with finitely generated cohomology over \( AZ = H^*(\mathcal{R}) \).

**Proof** The map \( \text{def}_M^G \) was already constructed above. We just need to verify the implications for cohomology, which actually have nothing to do with the \( G \)-action. We note that the cohomology \( H^*(\text{def}_M^G) \) is, by construction, obtained by evaluating the functor

\[
- \otimes^L_{K} M : D(K \otimes Z K) \rightarrow D(K_Q)
\]

at the object \( K \). (Again, we forget about equivariance here.) We can factor this functor through the category of \( K_Q \)-bimodules

\[
D(K \otimes Z K) \xrightarrow{- \otimes^L_{K} Q} D(K_Q \otimes K_Q) \xrightarrow{- \otimes^L_{K_Q} M} D(K_Q)
\]

to see that the corresponding map to the center (6) agrees with that of [4, (3.1.5)] [20, Section 3.4]. So the finiteness claims of (1) and (2) follow from [20, Corollary 4.7].

Via Lemma 5.2 we may replace \( D(\mathcal{R})^G \) with \( D(A_Z)^G \), and view \( \text{RHom}_{K_Q} \), or equivalently \( \text{RHom}_R \), as a functor to \( D(A_Z)^G \). Alternatively, we could work with the dg scheme (shifted affine space) \( T^* = T^*_1 \text{Spf}(Z) = \text{Spec}(A_Z) \), and view \( \text{RHom}_R \) as a functor taking values in the derived category of equivariant dg sheaves on \( T^* \).

From this perspective, Theorem 5.4 tells us that \( \text{RHom}_R \) has image in the subcategory of dg sheaves on \( T^* \) with coherent cohomology,

\[
\text{RHom}_R : (D_{\text{fin}}(R)^G)^{op} \times D_{\text{fin}}(R)^G \rightarrow D_{\text{coh}}(A_Z)^G \cong D_{\text{coh}}(T^*)^G.
\]

**Remark 5.5** We only use the finiteness claims of Theorem 5.4 in the case in which all of \( Z \), \( Q \), and \( R \) are commutative. In this case in particular, claims (1) and (2) of Theorem 5.4 should be obtainable directly from Gulliksen [16, Theorem 3.1].
Remark 5.6 One may compare the above analyses with the formality arguments of [1, Sections 5.4–5.8].

6 Touzé–Van der Kallen and derived invariants

We recall some results of Touzé and Van der Kallen [26]. Our aim is to take derived invariants of Theorem 5.4 to obtain a finite generation result for equivariant extensions $\text{Hom}_{D(R)^G}^*$. We successfully realize this aim via an invocation of [26]. Throughout this section $G$ is a finite group scheme.

6.1 Basics and notations

For $V$ any $G$-representation we have the standard group cohomology $H^*(G, V) = \text{Ext}_G^*(1, V)$. For more general objects in $D(\text{Rep}(G))$ we adopt a hypercohomological notation.

Notation 6.1 We let $(-)^{RG}: D(\text{Rep}(G)) \to D(\text{Vect})$ denote the derived invariants functor, $(-)^{RG} = R\text{Hom}_G(1, -)$. For $M$ in $D(\text{Rep}(G))$ we take

$$\mathbb{H}^*(G, M) := H^*(M^{RG}).$$

We note that the hypercohomology $\mathbb{H}^*(G, M)$ is still identified with morphisms $\text{Hom}_{D(\text{Rep}(G))}^*(1, M)$ in the derived category. Since $G$ is assumed to be finite, we are free to employ an explicit identification

$$(-)^{RG} = \text{Hom}_G(\text{Bar} G, -),$$

where $\text{Bar} G$ is the standard Bar resolution. For any dg $G$-algebra $S$ the derived invariants $S^{RG}$ are naturally a dg algebra in $\text{Vect}$, and for any equivariant dg $S$-module $M$, $M^{RG}$ is a dg module over $S^{RG}$. (Under our explicit expression of derived invariants in terms of the bar resolution, these multiplicative structures are induced by a dg coalgebra structure on $\text{Bar} G$, see e.g. [22, §2.2].) We therefore obtain at any dg $G$-algebra a functor

$$(-)^{RG}: D(S)^G \to D(S^{RG}).$$

(8)

The following well-known fact can be proved by considering the hypercohomology $\mathbb{H}^*(G, S)$ as maps $1 \to \Sigma^n S$ in the derived category.

Lemma 6.2 If $A$ is a commutative dg $G$-algebra, then the hypercohomology $\mathbb{H}^*(G, A)$ is also commutative.

6.2 Derived invariants and coherence of dg modules

We have the following result of Touzé and Van der Kalen.
Theorem 6.3 ([26, Theorems 1.4 & 1.5]) Consider $G$ a finite group scheme, and $A$ a commutative $G$-algebra which is of finite type over $k$. Then the cohomology $H^*(G, A)$ is also of finite type, and for any finitely generated equivariant $A$-module $M$, the cohomology $H^*(G, M)$ is a finite module over $H^*(G, A)$.

One can derive this result to obtain

Theorem 6.4 Consider $G$ a finite group scheme, and $S$ a dg $G$-algebra which is equivariantly formal and has commutative, finite type, cohomology. Suppose additionally that the cohomology of $S$ is bounded below. Then the derived invariants functor (8) restricts to a functor

$$(-)^{RG}: D_{coh}(S)^G \to D_{coh}(S^{RG}).$$

Equivalently, for any equivariant dg $S$-module $M$ with finitely generated cohomology over $H^*(S)$, the hypercohomology $\mathbb{H}^*(G, M)$ is finite over $H^*(G, S)$.

**Proof** Take $A = H^*(S)$. We are free to view, momentarily, $A$ as a non-dg object. We have that $A$ is finite over its even subalgebra $A^e$, which is a commutative algebra in the classical sense, so that Theorem 6.3 implies that cohomology $H^*(G, -)$ sends $A$ to a finite extension of $H^*(G, A^e)$, and any finitely generated $A$-module to a finitely generated $H^*(G, A^e)$-module. Hence $H^*(G, A)$ is of finite type over $k$, and $H^*(G, N)$ is finite over $H^*(G, A)$ for any finitely generated, equivariant, non-dg, $A$-module $N$.

Since $G$ is a finite group scheme, $A$ is also a finite module over its (usual) invariant subalgebra $A^G$, and any $A$-module is finitely generated over $A$ if and only if it is finitely generated over $A^G$. Theorem 6.3 then tells us that, for any finitely generated $A$-module $N$, the cohomology $H^*(G, N)$ is finitely generated over $H^*(G, A^G) = H^*(G, 1) \otimes A^G$, where $H^*(G, A^G)$ acts through the algebra map

$$H^*(G, \text{incl}): H^*(G, A^G) \to H^*(G, A).$$

Consider now any dg module $M$ in $D_{coh}(S)^G$. Formality implies an algebra isomorphism $S \cong A$ in $D(\text{Rep}(G))$ and so identifies $\mathbb{H}^*(G, S)$ with $\mathbb{H}^*(G, A) = H^*(G, A)$. We want to show that, for such a dg module $M$, the hypercohomology $\mathbb{H}^*(G, M)$ is a finitely generated module over $\mathbb{H}^*(G, S) \cong H^*(G, A)$. It suffices to show that $\mathbb{H}^*(G, M)$ is finite over $H^*(G, A^G) = H^*(G, 1) \otimes A^G$. We have the first quadrant spectral sequence (via our bounded below assumption)

$$E_2^{*,*} = H^*(G, H^*(M)) \Rightarrow \mathbb{H}^*(G, M),$$

and the $E_2$-page is finite over $H^*(G, A^G)$ by the arguments given above. Since $H^*(G, A^G)$ is Noetherian, it follows that the associated graded module $E_\infty^{*,*} = \text{gr} \mathbb{H}^*(G, M)$ is finite over $H^*(G, A^G)$, and since the filtration on $\mathbb{H}^*(G, M)$ is bounded in each cohomological degree it follows that the hypercohomology $\mathbb{H}^*(G, M)$ is indeed finite over $H^*(G, A^G) \subset \mathbb{H}^*(G, S)$ [14, Lemma 1.6].
7 Finite generation of cohomology for Drinfeld doubles

Consider $G$ a finite group scheme. Fix a closed embedding $G \to \mathcal{H}$ into a smooth algebraic group $\mathcal{H}$, and fix also the associated $G$-equivariant deformation

$$Z \to \mathcal{O} \to \mathcal{O}(G), \quad Z = \mathcal{O}_{\mathcal{H}/G}, \quad \mathcal{O} = \mathcal{O}_\mathcal{H},$$

as in Sect. 4.2. Here $kG$ acts on $\mathcal{O}(G)$ and $\mathcal{O}$ via the adjoint action, and this adjoint action restricts to a translation action on $Z$. We recall that the embedding dimension of $G$ is the minimal dimension of such smooth $\mathcal{H}$ admitting a closed embedding $G \to \mathcal{H}$.

We consider the tensor category

$$Z(\text{rep}(G)) \cong \text{rep}(D(G)) \cong \text{Coh}(G)^G$$

of representations over the Drinfeld double of $G$, aka the Drinfeld center of $\text{rep}(G)$. We prove the following below.

**Theorem 7.1** For any finite group scheme $G$, the Drinfeld center $Z(\text{rep}(G))$ is of finite type and of bounded Krull dimension

$$\text{Kdim } Z(\text{rep}(G)) \leq \text{Kdim } \text{rep}(G) + \text{embed. dim}(G).$$

One can recall our definition of a finite type tensor category, and of the Krull dimension of such a category, from the introduction. For $T^*$ the cotangent space $T^*_1 \text{Spf}(Z)$, considered as a variety with a linear $G$-action, we show in particular that there is a finite map of schemes $\text{Spec } \text{Ext}^*_{Z(\text{rep}(G))}(1, 1) \to (G \setminus T^*) \times \text{Spec } H^*(G, 1)$.

### 7.1 Preliminaries for Theorem 7.1: Derived maps in $Z(\text{rep}(G))$

We let $G$ act on itself via the adjoint action, and have $\text{Coh}(G)^G = \text{rep}(\mathcal{O}(G))^G$. The unit object $1 \in \text{Coh}(G)^G$ is the residue field of the fixed point $1 : \text{Spec}(k) \to G$. We have

$$\text{REnd}_{\text{Coh}(G)^G}(1) = \text{REnd}_{\text{Coh}(G)}(1)^{RG},$$

as an algebra, and for any $V$ in $\text{Coh}(G)^G$ we have

$$\text{RHom}_{\text{Coh}(G)^G}(1, V) = \text{RHom}_{\text{Coh}(G)}(1, V)^{RG},$$

as a dg $\text{REnd}_{\text{Coh}(G)^G}(1)$-module.

One can observe these identifications essentially directly, by noting that for the projective generator $\mathcal{O}(G) \rtimes kG$ we have an identification of $G$-representations

$$\text{Hom}_{\text{Coh}(G)}(\mathcal{O}(G) \rtimes kG, V) = \text{Hom}_k(kG, V) = \mathcal{O}(G) \otimes V,$$
and \( \mathcal{O}(G) \otimes V \) is injective over \( kG \) for any \( V \). Hence the functor \( \text{Hom}_{\text{Coh}(G)}(\mathcal{O}, V) \) sends projectives objects in \( \text{Coh}(G)^G \) to injectives in \( \text{Rep}(G) \), and for a projective resolution \( F \to 1 \) we have identifications in the derived category of vector spaces

\[
\text{RHom}_{\text{Coh}(G)^G}(1, V) = \text{Hom}_{\text{Coh}(G)^G}(F, V) = \text{Hom}_{\text{Coh}(G)}(F, V)^G \\
\cong \text{Hom}_{\text{Coh}(G)}(F, V)^{RG} = \text{RHom}_{\text{Coh}(G)}(1, V)^{RG}
\]

and

\[
\text{REnd}_{\text{Coh}(G)^G}(1, 1) = \text{End}_{\text{Coh}(G)}(F)^G \cong \text{End}_{\text{Coh}(G)}(F)^{RG} = \text{REnd}_{\text{Coh}(G)}(1)^{RG}.
\]

The middle identification for derived endomorphisms comes from the diagram

\[
\begin{array}{ccc}
\text{End}(F)^G & \longrightarrow & \text{End}(F)^{RG} \\
\sim & & \sim \\
\text{Hom}(F, 1)^G & \sim & \text{Hom}(F, 1)^{RG}.
\end{array}
\]

### 7.2 Proof of Theorem 7.1

**Proof** Fix an embedding \( G \to \mathcal{H} \) and associated equivariant deformation \( \mathcal{O} \to \mathcal{O}(G) \) as above, and take \( A = A \Omega = \text{Sym}(\Sigma^{-2}(m_\Omega/m_\Omega^2)^*) \), as in (5). Take also \( \mathcal{R} \) the dg \( G \)-algebra \( \text{REnd}_{KZ} \otimes KZ(KZ) \). We recall from Lemma 5.2 that \( \mathcal{R} \) is equivariantly formal, and so homotopy isomorphic to \( A \). We adopt the abbreviated notations \( \text{RHom} = \text{RHom}_{\text{Coh}(G)} \) and \( \text{REnd} = \text{REnd}_{\text{Coh}(G)} \) when convenient.

We consider the equivariant dg algebra map

\[
\text{def}_1^G : \mathcal{R} \to \text{REnd}_{\text{Coh}(G)}(1)
\]

of Theorem 5.4, and the action of \( \mathcal{R} \) on each \( \text{REnd}_{\text{Coh}(G)}(1, V) \) through \( \text{def}_1^G \). By Theorems 5.4 and 6.4, the hypercohomology \( \mathbb{H}^*(G, \text{REnd}(1)) \) is a finite algebra extension of \( \mathbb{H}^*(G, \mathcal{R}) \), and \( \mathbb{H}^*(G, \text{RHom}(1, V)) \) is a finitely generated module over \( \mathbb{H}^*(G, \mathcal{R}) \) for any \( V \) in \( \text{Coh}(G)^G \). In particular, \( \mathbb{H}^*(G, \text{RHom}(1, V)) \) is finite over \( \mathbb{H}^*(G, \text{REnd}(1)) \). Since \( \mathbb{H}^*(G, \mathcal{R}) \cong \mathbb{H}^*(G, A) \) is of finite type over \( k \), by Touzé-Van der Kallen (Theorem 6.4), the above arguments imply that

\[
\mathbb{H}^*(G, \text{REnd}_{\text{Coh}(G)}(1)) = \text{Ext}_{\text{Coh}(G)^G}(1, 1)
\]

is a finite type \( k \)-algebra, and that each

\[
\mathbb{H}^*(G, \text{RHom}_{\text{Coh}(G)}(1, V)) = \text{Ext}_{\text{Coh}(G)^G}(1, V)
\]
is a finitely generated module over this algebra, for $V$ in $\text{Coh}(G)^G$. That is to say, the tensor category $Z(\text{rep}(G)) \cong \text{Coh}(G)^G$ is of finite type over $k$.

As for the Krull dimension, $\text{Ext}^*(G, A)$ is finite over $H^*(G, A^G) = H^*(G, 1) \otimes A^G$, by Touzé-Van der Kallen, so that

$$K\text{dim }Z(\text{rep}(G)) = K\text{dim }\text{Ext}^*_Z(\text{rep}(G))(1, 1) \leq K\text{dim }H^*(G, k) \otimes A^G = K\text{dim }H^*(G, k) \otimes A = K\text{dim }\text{rep}(G) + \dim \mathcal{H}/G = K\text{dim }\text{rep}(G) + \dim \mathcal{H}.$$

When $\mathcal{H}$ is taken to be of minimal possible dimension we find the proposed bound,

$$K\text{dim }Z(\text{rep}(G))) \leq K\text{dim }\text{rep}(G) + \text{embed. dim. } G.$$

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