Effects related to spacetime foam in particle physics

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Abstract

It is found that the existence of spacetime foam leads to a situation in which the number of fundamental quantum bosonic fields is a variable quantity. The general aspects of an exact theory that allows for a variable number of fields are discussed, and the simplest observable effects generated by the foam are estimated. It is shown that in the absence of processes related to variations in the topology of space, the concept of an effective field can be reintroduced and standard field theory can be restored. However, in the complete theory the ground state is characterized by a nonvanishing particle number density. From the effective-field standpoint, such particles are "dark". It is assumed that they comprise dark matter of the universe. The properties of this dark matter are discussed, and so is the possibility of measuring the quantum fluctuation in the field potentials.

1 Introduction

In gravitation theory it is assumed that spacetime is a smooth manifold at scales much greater than the Planck length, while at Planck scale all geometric properties disappear and spacetime itself acquires a foamlike structure [1]. There are two basic indications of such behaviour of spacetime. The first

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is related to the fact that at the Planck scale the vacuum fluctuations of
the metric and curvature are of the same order as the corresponding average
quantities. Not only does this follow from simple estimates - rigorous calcula-
tions also support this idea. In particular, the fact that such fluctuations
exist leads to the absence of a classical background space in the Planck stage
of evolution of the early universe [2]. The second indication is the fact that at
small scales the topology of space also experiences quantum fluctuations [1].
The study of possible observable effects related to changes in the topology
of space is attracting ever more attention. In particular, to describe such
effects, Hawking [3] used wormholes and virtual black holes. Another work
worth noting is that of Garay [4], who proposed a phenomenological method
to account for spacetime foam.

The absence of a background space at small scales is a serious prob-
lem in quantum field theory. The possibility of resolving this problem is usu-
ally related to the development of nonperturbative methods [5], in which the
concept of background fields is not used. However, these theories also rely
on the presence of a coordinate basis space, whose topology is fixed by the
statement of the problem and therefore is not a dynamic characteristic.

This paper elaborates on a possible way to set up a quantum field theo-
y in the case in which the topology and structure of physical space may vary.
The main idea of this method was set forth in Ref. [6] in order to describe
the quantum birth of the early universe.

The following observation forms the basis of the proposed method. On
the one hand, as noted earlier, variations in the topology of space can occur
at scales where the very concept of a smooth manifold breaks down, at least
due to the presence of vacuum fluctuations. On the other hand, it is believed
that there is no other way to describe the given region but to extrapolate the
spatial relationships existing at larger scales to it. In other words, all possible
topologies of physical space should be described in terms of a consistent
coordinate basis space. We call this space simply a basis. Since measu-
ment instruments, which are classical objects, play a fundamental role in quantum
theory [7], it is expected that the properties of the basis are determined
tirely by the measuring device.

If we specify the quantum state corresponding to a fixed topology of
physical space and if the topology differs substantially from that of the ba-
sis, the image of physical space in terms of the basis coordinates cannot
be one-to-one. In the same way, when functions defined in physical space
and corresponding to different physical observables are mapped to the basis space, they cease to be single-valued and become multivalued functions of the coordinates. Furthermore, the number of images of an arbitrary physical observable is an additional variable quantity, which generally speaking, depends on the position in the basis space.

Thus, we arrive at a situation in which the number of fields corresponding to a physical observable is a variable quantity. In quantum theory this variable is an operator whose eigenvalues characterize the topological structure of space. The possible dependence of this quantity on spatial coordinates means that the given quantity is a characteristic or measure of the number density of the degrees of freedom of the field.

A natural way to describe systems with a variable number of degrees of freedom is to use second quantization. Before we begin to describe the method as applied to the problem in question, we make the following remark. In the standard second-quantization method, the number of degrees of freedom characterizes the number of particles or elementary excitations (quanta) in the system. Here it is assumed that the particles obey the identity principle or, as it is said, the indistinguishability principle. It can be expected that in measurements at small scales the different images of the same physical observable also obey the identity principle. Indeed, the possibility of distinguishing between the different images of observables would mean that physical space in itself has certain topology and structure, which by assumption is impossible (at least in view of the presence of quantum fluctuations in the topology).

Two types of statistics, Bose and Fermi, exist for particles, depending on the symmetry of the wave function under particle permutations. Accordingly, we must also select the type of statistics when performing second quantization of the degrees of freedom of the fields. Since second quantization reflects the properties and topology of physical space, this selection must be unique for

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1 Thus, we assume the existence of a fundamental restriction on the degree of accuracy for measurements of different field observables. In quantum gravity this is well-known restriction, which represents the existence of a minimal quantum uncertainty for metric of the type $\Delta g \sim L_{\text{pl}}^2/L^2$, where $L_{\text{pl}}$ is the Planck length and $L$ is the characteristic size of the region in which the measurement takes place. We note that such restrictions are not something very unusual in quantum mechanics. It suffices to recall the impossibility to detect a position of a relativistic particle with the accuracy exceeding the Compton length, which seems to be in a close analogy with the restriction pointed out above.
all types of fields and physical quantities. Here it turns out that the only acceptable choice is Fermi-Dirac statistics, since otherwise in dealing with fermions we immediately confront a violation of the Pauli principle.

At a fundamental level, the composition of matter is determined by a set of fields and their sources. The sources are point particles, which in quantum theory behave like fermions. The need to perform second quantization of the sources arises already in relativistic theory and hence no changes in the description of fermions emerge. A new interpretation is added, however. For instance, pair production corresponds to a change in the structure of physical space (it can be said that processes related to changes in the properties of space proceed much more easily at isolated points than they do in entire regions).

When fields are quantized, the idea of particles, the quanta of a field, also emerges. Such particles, however, obey the Bose-Einstein statistics. Here, generally speaking, particle production is not associated with variations in the topology of space. There is a certain similarity between this aspect and the situation in solid state physics, where excitation of vibrations in a crystal lattice (phonon production) is not associated with variations in the true number of degrees of freedom.

Thus, the variation mentioned above primarily involves bosonic fields.

2 General scheme of second quantization of fields

We consider a set $M$, which in the future acts like a basis manifold, and specify an arbitrary field $\varphi$ on it. We also assume that there is a device that can do complete measurements of the quantum states of the field. A complete measurement can always be expanded in a set of elementary measurements. For instance, to make a complete measurement of a field state we must measure the field amplitude at every point $x \in M$, or equivalently measure the number of particles (or amplitude) in each Fourier mode. Thus, the device can be viewed as a set of elementary detectors.

Let $A$ be the set of possible readings of an elementary detector\footnote{The set $A$ can be called an elementary system of quantum numbers.}. The structure of $A$ can be described in the following way. In $A$ we select an...
arbitrary system of coordinates $\xi$. Generally, there is a natural projection operator $P (P^2 = P)$ that partitions the coordinates into two groups: $\xi = ((I - P) \xi, P \xi) = (\eta, \zeta)$, where $I$ denotes the identity operator. The first group, $\zeta$, refers to the manifold $M$ and describes the position in space $M^*$ at which the elementary measurement takes place (here and in what follows $M^*$ denotes either space $M$ or the mode space). The second group, $\eta$, refers to the field $\varphi$ and describes the position in space $V$. The coordinate $\eta \in V$ denotes either the field amplitude or the number of particles corresponding to the field. Thus, the set $A$ acquires the features of a fiber space with basis $P (A) \sim M^*$ and fiber $P^{-1} (\zeta) = V$. The result of a complete measurement of field $\varphi$ is a fibration section, which is the map $\varphi : M^* \rightarrow A$. What is important is that in the usual picture an arbitrary section intersects each fiber only once, i.e., the projection of the section coincides with the space $M^* (P (\varphi) \equiv M^*)$, which implies that such sections can be represented by functions $\eta (\zeta)$ on $M^*$ with values in $V$.

As noted above, the topology and geometric structure of the set $A$ (and thus of $M^*$) reflects the macroscopic properties of the measurement process. On the other hand, the real physical space $M_{ph}$ is assumed to have arbitrary topology and structure. Furthermore, in a general quantum state, the properties of space $M_{ph}^*$ are, generally speaking, not fixed. Thus, a physical field must be defined as an extended section of the form $\tilde{\varphi} : M_{ph}^* \rightarrow A$. Here an arbitrary section can intersect each fiber an arbitrary number of times. Furthermore, if the topology of space $M_{ph}^*$ changes, so does the number of intersections. Thus, the number of images of field $\tilde{\varphi}$ in space $M^*$ is variable. An image of space $M_{ph}^*$ is a subset in $M^* (P (\tilde{\varphi}) = M_{ph}^* \subset M^*)$ that can be represented as a union of distinct pieces, $M_{ph}^* = \bigcup \sigma_j$, so that on each piece $\sigma_j$ the field is described by a given number of functions $\eta_i (\zeta), \zeta \in \sigma_j$ ($i = 1, 2, \ldots m$, where $m$ is an integer characterizing the number of images of the space $M_{ph}^*$ in $\sigma_j$). Note that in general the dimensionality of the pieces $\sigma_j$ can differ from the dimensionality of $M^*$.

Thus, if the topology of physical space is an additional degree of freedom, the result of a complete measurement of the state of the field will be represented by a definite set of functions $\{\eta_J (\zeta)\}, (J = (i, \sigma) \text{ and } \zeta \in \sigma)$.

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3 When speaking of the topology of the physical space, we mean either the topology of the space $M_{ph}$ itself, or the topology of the related space $M_{ph}^*$, depending on the quantities being measured. Note, however, that the relationship between these two spaces is nontrivial.
Formally, such states can be classified in the following way. We introduce a set of operators $C^+ (\xi)$ and $C (\xi)$, the creation and annihilation operators for an individual element of the set $A$. For the sake of simplicity we assume that the measure of each individual point $\xi \in A$ is finite (as in the case in which the coordinates $\xi$ take discrete values). We require that these operators satisfy the anticommutation relations

$$\left\{ C (\xi) C^+ (\xi') \right\} = C (\xi) C^+ (\xi') + C^+ (\xi') C (\xi) = \delta_{\xi\xi'}.$$ (1)

We define the vacuum state $|0\rangle$ by the relationship $C (\xi) |0\rangle = 0$ and build a Fock space $F$ in which the basis consists of the vectors $(n = 1, 2, \ldots)$

$$|\xi_1, \xi_2, \ldots, \xi_n\rangle = \prod_{i=1}^{n} C^+ (\xi_i) |0\rangle.$$ (2)

The vacuum state corresponds to complete absence of a field and hence of the observables associated with the field. The state $|\xi\rangle$ describes the field $\varphi$ with only one degree of freedom. This can be either a field concentrated at a single point or a field containing only one mode, and the quantity $\xi \in A$ describes the intensity (the number of quanta) and the position of the field in $M^\ast$. States described by single-valued functions are constructed in the following way:

$$|\eta (\zeta)\rangle = \prod_{\zeta \in M^\ast} C^+ (\eta (\zeta), \zeta) |0\rangle,$$ (3)

where the direct product is taken over the entire space $M^\ast$, and where we have partitioned the coordinates $\xi$ into two groups: $\xi = (\eta, \zeta)$. Generally, such states do not belong to a Fock space. Furthermore, when the coordinates $\zeta \in M^\ast$ run through continuous values, this expression requires an extension of its definition and hence can be interpreted only formally. However, when the variations of the physical quantities in real processes involve only a finite part of the set $M^\ast$, we can stay within a Fock space.

We now examine an arbitrary domain $\sigma \in M^\ast$ and define a set of operators

$$D^+ (\eta (\zeta), \sigma) = \prod_{\zeta \in \sigma} C^+ (\eta (\zeta), \zeta),$$ (4)

where the domain of the function $\eta (\zeta)$ is limited to the set $\sigma$. Then the states with an arbitrary number of fields can be written

$$|\eta_1, \eta_2, \ldots, \eta_n\rangle = \prod_{i=1}^{n} D^+ (\eta_i (\zeta), \sigma_i) |0\rangle.$$ (5)
The interpretation of these states is obvious. Suppose that all functions \( \eta_i(\zeta) \) are specified on a single set \( \sigma \). Then in the given domain a complete measurement will show the presence of a set consisting of \( n \) different fields \( \eta_1(\zeta), \eta_2(\zeta), \ldots, \eta_n(\zeta) \). It is convenient to introduce the number density operator of the fields:

\[
N(\zeta) = \sum_{\eta \in V} C^+(\eta, \zeta) C(\eta, \zeta). \tag{6}
\]

Then for \( \zeta \in \sigma \) the states (5) represent the eigenstates of the operator \( N(\zeta) \) with eigenvalues

\[
N(\zeta) |\eta_1, \eta_2, \ldots, \eta_n\rangle = n |\eta_1, \eta_2, \ldots, \eta_n\rangle. \tag{7}
\]

Clearly, the states with a fixed number of fields correspond to a fixed topology of the space \( M^{*}_{ph} \). Then under certain conditions (the requirement that the functions \( \eta_i(\zeta) \) be smooth at cuts), instead of the set of functions \( \eta_i(\zeta) \) we can introduce a single-valued function \( \eta(\zeta) \) and thus restore the structure of the set \( M^{*}_{ph} \). Conversely, each space \( M^{*}_{ph} \) can be projected on the basis \( M^{*} \) by performing the necessary paste-up, so that the state vector of the field takes the form (3).

The space \( H \) formed by the vectors (3) and their superposition lays the basis for building the Hilbert space of the theory. An arbitrary operator \( \hat{O}(\xi) \) related to the field (and symmetrized in the number of fields) can be expressed in the standard way in the terms of the set of basis operators \( C \) and \( C^+ \):

\[
\hat{O} = \sum D_I^+ O_{IJ} D_J \tag{8}
\]

(where \( I, J = (\eta_i(\zeta), \sigma) \), and \( \sigma \) is an arbitrary domain in \( M^{*} \)), thus defining the action of this operator in \( H \). The specific way in which this Hilbert space is built is determined by the physical problem at hand.

3 Scalar field in the second-quantization representation

In Sec. 2 we discussed the general scheme of second quantization, irrespective of the dynamics of the field. We now turn to the example of a real scalar
field $\varphi$ (the generalization to the case of arbitrary fields is obvious). For the basis space we take ordinary flat Minkowski space.

One idea that is central to particle physics is the representation in which quantum states of a field are classified in terms of physical particles. Since quantum states of a field can in general contain an arbitrary number of identical modes, the definition of particles and their relation to field operators require certain modifications. We find it more convenient to operate with discrete indices. To this end we require that the field in question be located in a cube with edge length $L$, and we introduce periodic boundary conditions. As necessary, we can replace sums with integrals (as $L \to \infty$) via the usual prescription: $\sum \to \int (L/2\pi)^3 d^3k$.

We now examine the expansion of the field operator $\varphi$ in plane waves,

$$\varphi(x) = \sum_k \left(2\omega_k L^3\right)^{-1/2} \left(a_k e^{ikx} + a_k^+ e^{-ikx}\right),$$

(8)

where $\omega_k = \sqrt{k^2 + m^2}$, and $k = 2\pi n/L$, with $n = (n_x, n_y, n_z)$. The general expression for the Hamiltonian is

$$H = H_0 + V$$

(9)

where $H_0$ describes free particles,

$$H_0 = \sum_k \omega_k a_k^+ a_k + e_k,$$

(10)

and the potential term $V$ is responsible for the interaction, and can be represented in the normal form:

$$V = \sum_{n,\{m\},\{m'\}} V_{\{m\},\{m'\}}^n$$

(11)

$$V_{\{m\},\{m'\}}^n = \sum_{k_1\ldots k_n} V_{\{m\},\{m'\}}^n (k_1, \ldots, k_n) \prod_{i=1}^n \left(a_{k_i}^+\right)^{m_i} (a_{k_i})^{m'_i},$$

(12)

Here we assume that the sum with respect to the wave vectors $k_i$ contains no terms with equal indices, i.e., $k_i \neq k_j$ for any pair of indices $i$ and $j$ (the sum is taken over distinct modes), and allow for the fact that for different wave numbers the operators $a_{k_i}$ and $a_{k_j}^+$ commute.
The quantity $e_k$ in (10) is the energy of the ground state of the $k$th mode. In a flat space without particles, the energy must be zero, so we assume that $e_k = 0$ throughout the present paper. However, as we show in the sections that follow, the nontrivial nature of the topology of the space generally leads to a value of $e_k$ that is finite. Note that the dependence of the zero energy on the topology of space is known as the Casimir effect [8] and is assumed to be an experimentally established fact [9].

When the number of modes is variable, the set of field operators $\{a_k, a_k^+\}$ is replaced by the somewhat expanded set $\{a_k(j), a_k^+(j)\}$, where $j \in [1, \ldots N_k]$, and $N_k$ is the number of modes for a given wave number $k$. For a free field the energy is an additive quantity, which can be written

$$H_0 = \sum_k \sum_{j=1}^{N_k} \omega_k a_k^+(j) a_k(j).$$

(13)

Since the modes are indistinguishable, the interaction operator has the obvious generalization

$$V_{\{m\},\{m'\}} = \sum_{k_1, \ldots, k_n} \sum_{j_1, \ldots, j_n} V_{\{m\},\{m'\}}^{n} (k_1, \ldots, k_n) \prod_{i=1}^{n} \left(a_{k_i}(j_i)\right)^{m_i} \left(a_{k_i}(j_i)\right)^{m_i'},$$

(14)

where the indices $j_i$ run through the corresponding intervals $j_i \in [1, \ldots N_k(k_i)]$. It is convenient to introduce the notation

$$A_{m,n}(k) = \sum_{j=1}^{N(k)} \left(a_k^+(j)\right)^m \left(a_k(j)\right)^n.$$

(15)

Then the expression for the field Hamiltonian takes the form

$$H = \sum_k \omega_k A_{1,1}(k) + \sum_{n,\{m\},\{m'\}} \sum_{k_1, \ldots, k_n} V_{\{m\},\{m'\}}^{n} (k_1, \ldots, k_n) \prod_{i=1}^{n} A_{m_i,m'_i}(k_i).$$

(16)

We can now express the main quantities in terms of the fundamental operators $C^+(\xi)$ and $C(\xi)$. For the operators $a$ and $a^+$ it is convenient to use the Fock - Bargmann representation, in which operators act in the space of entire analytic functions with a scalar product of the type

$$(f, g) = \int f^* (a) g (a^*) e^{-a^* a} da^* da 2\pi i.$$
the action of these operators is defined as
\[
a^+ f (a^*) = a^* f (a^*) ; \quad af (a^*) = \frac{d}{da^*} f (a^*) . \tag{18}
\]
Then for the normal field coordinates we can take the complex-valued quantities \(a^*\); thus, the set \(A\) consists of the pairs \(\xi = (a^*, k)\). For the fundamental operators \(C^+ (\xi)\) and \(C (\xi)\) it is convenient to use the representation
\[
C (a^*, k) = \sum_{n=0}^{\infty} C (n, k) \frac{(a^*)^n}{\sqrt{n!}} , \quad C^+ (a, k) = \sum_{n=0}^{\infty} C^+ (n, k) \frac{a^n}{\sqrt{n!}} . \tag{19}
\]
Then the anticommutation relations (1) become
\[
\{ C (n, k) , C^+ (m, k') \} = \delta_{n,m} \delta_{k,k'} . \tag{20}
\]
The physical meaning of the operators \(C (n, k)\) and \(C^+ (n, k)\) is that they create and annihilate modes with a given number of particles.

Now, to express the Hamiltonian (16) in terms of \(C (n, k)\) and \(C^+ (n, k)\) it suffices to derive the corresponding expressions for the operators (15). In the second-quantization representation, the expressions for the given operators are defined to be
\[
\hat{A}_{m,n} (k) = \int e^{-a^* a} \frac{da^* da}{2\pi i} C^+ (a, k) (a^*)^m \left( \frac{d}{da^*} \right)^n C (a^*, k) \tag{21}
\]
or, with allowance for (19),
\[
\hat{A}_{m_1,m_2} (k) = \sum_{n=0}^{\infty} \frac{(n + m_1)! (n + m_2)!}{n!} C^+ (n + m_1, k) C (n + m_2, k) . \tag{22}
\]
An expression for the Hamiltonian in terms of the operators \(C^+ (\xi)\) and \(C (\xi)\) can be obtained by simply substituting (22) into (16). For a free field, the eigenvalues of the Hamiltonian take the form
\[
\tilde{H}_0 = \sum_k \omega_k \hat{A}_{1,1} (k) = \sum_{n,k} n\omega_k N_{n,k} , \tag{23}
\]
where \(N_{n,k}\) is the number of modes for fixed values of the wave number \(k\) and the number of particles \(n\) (\(N_{n,k} = C^+ (n, k) C (n, k)\)).
Thus, the field state vector $\Phi$ is a function of the occupation numbers $\Phi (N_{k,n}, t)$, and its evolution is described by the Schrödinger equation

$$i \partial_t \Phi = H \Phi.$$  \hspace{1cm} (24)

Consider the operator

$$N_k = \sum_{n=0}^{\infty} C^+(n,k) C(n,k).$$  \hspace{1cm} (25)

Physically, this operator characterizes the total number of modes for a fixed wave number $k$. One can easily verify that for the Hamiltonian (16), $N_k$ is a constant of the motion,

$$[N_k, H] = 0$$  \hspace{1cm} (26)

and in this way Hamiltonians like (16) preserve the topological structure of the field. In the course of evolution, the number of modes for each $k$ does not change.

We now turn to the problem of representing the particle creation and annihilation operators in this formalism. Since the individuality of the modes is limited, operators of type (22) act like the set of operators $\left\{ a_k(j), a_k^+(j) \right\}$. Among the operators (22) are some that change the number of particles by one:

$$b_m(k) = \hat{A}_{m,m+1}(k), \quad b_m^+(k) = \hat{A}_{m+1,m}(k),$$  \hspace{1cm} (27)

$$[\hat{n}, b_m^+(k)] = \pm b_m^+(k), \quad [H_0, b_m^+(k)] = \pm \omega_k b_m^+(k),$$  \hspace{1cm} (28)

where

$$\hat{n} = \sum_k \hat{n}_k = \sum_{n,k} n N_{n,k}.$$  \hspace{1cm} (29)

Then the ground state $\Phi_0$ of the field can be defined as a vector satisfying the relationships ( $m = 0, 1, \ldots$)

$$b_m(k) \Phi_0 = 0$$  \hspace{1cm} (30)

and corresponding to the minimum energy for a fixed mode distribution $N_k$. Note that in contrast to standard theory, the ground state is generally characterized by a nonvanishing particle number density $\hat{n}\Phi_0 = n_0\Phi_0$. Using the vector $\Phi_0$, we can build a Fock space $F$ whose basis consists of vectors obtained by cyclic application of the operators $b_m^+(k)$ to $\Phi_0$. 

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4 Effective field

In the absence of processes related to changes in the topology of space and for a mode distribution of the form $N_k = 1$ (there is only one mode for each wave number $k$), the standard field theory is restored. Furthermore, there is a fairly general case in which the concept of an effective field can be introduced to restore the standard picture.

Indeed, consider the case in which the interaction operator in (16) is expressed solely in terms of the set of operators $b_0 (k)$ and $b_0^+ (k)$. Then instead of the complete Fock space $F$ we can limit ourselves to its subspace $F' \subset F$ formed by the cyclic application of the operators $b_0^+ (k)$ to the field ground state $\Phi_0$. If the initial state vector $\Phi$ belongs to $F'$, then as the system evolves, $\Phi (t) \in F'$ for all $t$ (at least as long as the number of particles created remains finite).

We define the operators

$$a_k = N_k^{-1/2} b_0 (k), \quad a_k^+ = N_k^{-1/2} b_0^+ (k), \quad (31)$$

where $N_k$ is the operator defined in (25), which, when restricted to the Fock space $F'$, is an ordinary number function. For (20) and (22) we find that the commutation relations for $a_k$ and $a_k^+$ have the standard form

$$[a_k, a_{k'}^+] = \delta_{kk'} \quad (32)$$

Thus, if the basic observable objects are particles, it is possible to revert to the usual picture in which the particles are quanta of an effective field $\tilde{\varphi}$ of type (8). Note that if the field potentials $\varphi (x)$ are measurable quantities, then the true expression for the field operators has the same form (3), where instead of the operators $a_k$ and $a_k^+$ we must put $b_0 (k)$ and $b_0^+ (k)$. The expression for the effective-field energy operator has the form (8), but the ground-state energy in the $k$th mode, $e_k$, must be assumed not to vanish. The value of this energy can be found in the complete theory.

Since the operators $a_k$ and $a_k^+$ reflect only some of the information about the state of the system, the auxiliary nature of the effective field becomes manifest. Indeed, the only observables related to the effective field are those particles that outnumber the particles in the ground state. For the particle number operator in $F'$ we have

$$a_k^+ a_k = \delta \hat{n}_k = \hat{n}_k - \bar{n}_k, \quad (33)$$
where $\hat{n}_k$ is the operator defined in (29), and $\pi_k$ can be found by solving $\hat{n}_k \Phi_0 = \pi_k \Phi_0$. Thus, the properties of the ground state $\Phi_0$ remain beyond the scope of the effective field.

5 Properties of the field ground state

Equations (21) and (22) imply that a true vacuum state has the property that all field modes (and hence all observables related to the field) are absent. A true vacuum state is one in which there are no particles and no zero-point oscillations related to particles. This situation is similar to the situation in solid state physics, where in the absence of a crystal there can be no phonons and no zero-point lattice vibrations. Since the properties of physical space are determined by the properties of material fields, we conclude that in a true vacuum state there can be no physical space. Obviously, in reality such a state cannot be achieved.

At first glance the most common situation in particle theory is the one in which physical space is ordinary flat Minkowski space, and nontrivial topology is manifest at the Planck scale (this is the conventional view; see Refs. [1] and [4]). But since operating at the Planck scale requires using energies unattainable with present-day accelerators, and also requires serious consideration of quantum gravity effects, it would appear to be impossible to make any sort of directly measurable predictions with this theory.

In reality, the situation may be somewhat different. First, the stability of the Minkowski space means that probably even at the Planck scale the topology of the space can be assumed to be simple (i.e., $N_k = 1$ and as $k \geq k_{pl}$), at least as long as we do not consider processes in which real particles with Planck energies are produced (naturally, virtual processes cannot lead to real changes in the topology of space).

Second, recall that the universe has already passed the quantum stage, in which real processes involving changes in spatial topology might occur. After the quantum stage, processes with topology variations are suppresed, and we can say that the topological structure of space has been "tempered", so that the structure of the space is preserved as the universe expands. Thus, we expect that at the present time the nontrivial topology of space is most likely manifested on a cosmological scale.

In the foregoing theory, the structure of space is determined by the num-
ber density of the field modes. These modes are in turn governed by Fermi statistics, i.e., they act like a Fermi gas. To simplify matters, we examine free fields, since consistent allowance for the interaction of field warrants a separate investigation. We assume that the field-mode distribution was thermal in the Planck period of the evolution of the universe. As the universe expands, the temperature drops and the gas becomes degenerate, with the field winding up in the ground state. Thus, the field ground state \( \Phi_0 \) can be characterized by occupation numbers of the type

\[
N_{k,n} = \theta (\mu_k - n\omega_k),
\]

(34)

where \( \theta(x) \) is the Heaviside step function and \( \mu_k \) is the chemical potential. Note that when the expansion is adiabatic, we must put \( \mu_k = \mu \). When the evolution of the universe includes an inflationary period [10, 11], the adiabaticity condition can be violated, which generally leads to additional dependence of the chemical potential on the wave number. For the mode spectral density we have

\[
N_k = \sum_{n=0}^{\infty} \theta (\mu_k - n\omega_k) = 1 + \left( \left\lfloor \frac{\mu_k}{\omega_k} \right\rfloor \right),
\]

(35)

where \( \lfloor x \rfloor \) denotes the integer part of the number \( x \). Equation (35) shows, in particular, that at \( \omega_k > \mu_k \) we have \( N_k = 1 \), i.e., the field structure corresponds to a flat Minkowski space, with the result that \( \omega_k < \mu_k \) is the range of wave vectors in which nontrivial field properties are expected to show up.

It can easily be verified that from the effective-field standpoint, the ground state \( \Phi_0 \) is a vacuum state, i.e., \( a_k \Phi_0 = 0 \). On the other hand, the given state can be characterized by a nonvanishing particle number density. Indeed, for any wave number we have

\[
\bar{n}_k = \sum_{n=0}^{\infty} n\theta (\mu_k - n\omega_k) = \frac{1}{2} \left( 1 + \left\lfloor \frac{\mu_k}{\omega_k} \right\rfloor \right) \left( \frac{\mu_k}{\omega_k} \right)
\]

(36)

with the result that the spectral density of the ground-state energy is

\[
e_k = \omega_k \bar{n}_k = \frac{\omega_k}{2} \left( 1 + \left\lfloor \frac{\mu_k}{\omega_k} \right\rfloor \right) \left( \frac{\mu_k}{\omega_k} \right),
\]

(37)
Since the given particles correspond to the ground state of the field, in ordinary processes (which do not change the topology of space) the particles in question are not manifested explicitly (but they enter into the renormalization of the parameters of the observed particles indirectly; here, in contrast to vacuum fluctuations, the contribution of the particles is naturally finite). We also note that although the particles are bosons, in the ground state they behave like fermions.

One possible explicit manifestation of a residual particle number density in the ground state is dark matter. Observations have shown that dark matter accounts for about 90% of visible matter in our universe, and the matter is clearly not of baryonic origin (see, e.g., Ref. [12]). Its existence is usually related to the presence of various hypothetical particles (Higgs particles, axions, etc.), which for various reasons cease to interact with ordinary matter. But if this mass is ascribed to the ground state, then first it becomes obvious that the matter is truly dark, and second that the minimum set incorporating only the particles known at present is sufficient.

To describe the properties of dark matter, we begin with massive bosons \((m \neq 0)\). For the sake of approximation, we ignore the possible dependence of the parameter \(\mu\) on the wave number \(k\). In this case, to avoid obtaining too large a value for dark matter, we require that

\[
\mu^2 - m^2 = z^2 \ll m^2. \tag{38}
\]

Then the ground state contains only one particle per mode in the wave-number range \(k^2 \leq z^2\), where \(N_k = 2\). In other words, massive bosons in the ground state behave like an ordinary degenerate Fermi gas, and we obtain for the energy density and particle number density

\[
\varepsilon = \frac{1}{L^3} \sum_{n,k} n \omega_k N(k,n) = \frac{g}{2\pi^2} \left( \frac{z^3 \mu}{4} + \frac{m^2}{8} \left( z \mu - m^2 \ln \left( \frac{z + \mu}{m} \right) \right) \right) \tag{39}
\]

\[
n = \frac{1}{L^3} \sum_{n,k} n N(k,n) = \frac{g}{6\pi^2} z^3, \tag{40}
\]

where \(g\) is the number of polarization states. In the limit \(z \ll m\), this expression leads to the well-known nonrelativistic relationship

\[
\varepsilon = nm + \frac{3}{2} p, \quad p = \frac{g}{30\pi^2} \frac{z^5}{m}, \tag{41}
\]
where \( p \) is the gas pressure. The principal contribution to the ground-state energy density is provided by the rest mass of the particle, i.e., in leading order this contribution comes from dust. Note, however, that the particle pressure is nonzero, and it yields a small correction of order \( p/\varepsilon \sim z^2/m^2 \sim n^{2/3}/m^2 \).

We now study particles with zero rest mass (such as photons and gravitons). For the ground-state energy density we have

\[
\varepsilon = \frac{g}{2\pi^2} \frac{\mu^4}{4} \xi(3). \tag{42}
\]

The number density of vacuum particles is

\[
n = \frac{g}{2\pi^2} \frac{\mu^3}{3} \xi(2), \tag{43}
\]

where

\[
\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

The equation of state in this case is ultrarelativistic \((\varepsilon = 3p)\).

Massless particles are especially interesting, since one can also measure the intensity of quantum fluctuations of the field potentials, which for the ground state (34) are

\[
\langle \varphi(x) \varphi(x+r) \rangle = \frac{1}{(2\pi)^2} \int_0^\infty \frac{dk}{k} \sin(kr) \Phi^2(k), \tag{44}
\]

where

\[
\Phi^2(k) = k^2 N_k = k^2 \left(1 + \left[\frac{\mu}{k}\right]\right).
\]

Thus, at long wavelengths \( k \ll \mu \), a substantial increase in the level of quantum fluctuations should be observable in comparison with pure vacuum noise \((\mu = 0)\).

6 Concluding remarks

We see then that the concept of spacetime foam introduced by Wheeler should lead to a number of observable effects in particle theory. The simplest are the emergence of dark matter and an increase in the intensity of
quantum noise in the field potentials. In Sec 5 we calculated such effects under the assumption that the field is in the ground state. However, the results can easily be generalized to a situation in which the state of the fields is characterized by nonzero temperature $T^\ast$. Since processes associated with changes in the topology of space are the first to stop in the early stages of the evolution of the universe, we expect that $T^\ast \ll T_{\gamma}$ ($T_{\gamma}$ is temperature of the microwave background radiation). On the other hand, given the value of $\mu$ in Eq. (12), we can obtain an upper bound $\mu^\ast \sim 60 T_{\gamma}$. Thus, we expect $T^\ast$ to be much less than $\mu$, and the temperature corrections to the ground state (34) to be small. Note, however, that the nature of the fluctuations of the field potentials in (44) can change substantially if the temperature is nonzero $\mu^\ast$.

In addition to the effects studied in this paper, there clearly remain many phenomena that require additional investigation. For example, given the existence of self-action, the ground state (34) can be transformed, which can lead to the emergence of scalar Higgs fields (by analogy with the well-known Cooper effect in superconductivity). Such fields are needed, in turn, to generate particle masses in grand unification theories. Note that in fields with self-action, a nonvanishing particle number density in the ground state automatically leads to the emergence of massive excitations, although the upper bound on masses that can be derived from cosmological constraints on the value of $\mu$ is many orders of magnitude less than the values observed in particle theory.

Another possibility is that when measuring the Casimir force [8, 9], one must expect an anomalous dependence on distance at scales exceeding the value of $1/\mu$.

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