Asynchronous parallel adaptive stochastic gradient methods

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Abstract Stochastic gradient methods (SGMs) are the predominant approaches to train deep learning models. The adaptive versions (e.g., Adam and AMSGrad) have been extensively used in practice, partly because they achieve faster convergence than the non-adaptive versions while incurring little overhead. On the other hand, asynchronous (async) parallel computing has exhibited much better speed-up over its synchronous (sync) counterpart. However, async-parallel implementation has only been demonstrated to the non-adaptive SGMs. The difficulty for adaptive SGMs originates from the second moment term that makes the convergence analysis challenging with async updates. In this paper, we propose an async-parallel adaptive SGM based on AMSGrad. We show that the proposed method inherits the convergence guarantee of AMSGrad for both convex and non-convex problems, if the staleness (also called delay) caused by asynchrony is bounded. Our convergence rate results indicate a nearly linear parallelization speed-up if $\tau = o(K^{\frac{1}{4}})$, where $\tau$ is the staleness and $K$ is the number of iterations. The proposed method is tested on both convex and non-convex machine learning problems, and the numerical results demonstrate its clear advantages over the sync counterpart.

Keywords: adaptive stochastic gradient method, asynchronous computing, deep learning

1 Introduction

The stochastic gradient method (SGM) dates back to [22]. It was originally designed for solving stochastic problems; see [17] for example. Nowadays, it is also widely used to solve deterministic problems that involve tremendous amount of data. Roughly speaking, SGM iteratively updates the variable along a negative stochastic gradient direction. In recent years, adaptive SGMs, such as AdaGrad [7], Adam [10], and AMSGrad [21], have become extremely popular due to their great success in training deep learning models. Empirically, these adaptive SGMs can be significantly faster than a classic non-adaptive SGM.
For parallel implementation, *async-parallel* computing has been demonstrated to enjoy significantly higher parallelization speed-up than its *synchronous* (sync) counterpart, e.g., [12, 14, 18, 20]. One major factor is the lock-free property of an async-parallel method. At each iteration of a sync-parallel method, the workers that finish tasks earlier must wait for those that finish later. This can result in a lot of idle waiting time. In addition, all workers access the memory simultaneously, which can cause memory congestion [3]. For these reasons, a sync-parallel method may have a very poor parallelization speed-up, especially when the load is imbalanced. On the contrary, an async-parallel method does not require all workers to keep the same pace and can eliminate the waiting time and the memory congestion issue. However, it may be difficult to guarantee the convergence of a numerical method if it is implemented in an async-parallel way, because outdated information could be used in updating the variables.

Although async-parallel methods have been developed for non-adaptive SGMs, it remains an open question whether adaptive SGMs can enjoy the nice speed-up performance of async-parallel implementation while maintaining the convergence guarantee. In this work, we give an affirmative answer to this question. We theoretically analyze the convergence and also show numerical results that corroborate the theory. The experiments are conducted with a shared-memory parallel computing model, but we note that the analysis and convergence results also hold under a distributed-memory computing model.

**Contributions.** Our contributions are three-fold. First, we propose an async-parallel adaptive SGM, named APAM, which is an asynchronous version of AMSGrad. During the iterations of APAM, the computation of stochastic gradients are asynchronous (while the updates to all variables are performed in a synchronous manner). To the best of our knowledge, the proposed method is the first async-parallel adaptive SGM.

Secondly, we analyze the convergence rate of APAM for both convex and non-convex problems. The established results indicate that the staleness $\tau$ has little impact on the convergence speed if it is dominated by $K^{\frac{3}{4}}$, where $K$ is the maximum number of iterations. Therefore, if $\tau = o(K^{\frac{3}{4}})$, a nearly linear speed-up can be achieved, and this is demonstrated by numerical experiments.

Thirdly, over the course of analyzing APAM, we also conduct a new convergence analysis for AMSGrad. Our convergence rate results do not require a diminishing sequence to weigh the gradients. In practice, constant weights are almost always adopted. Hence, our results bring the theory closer to practice.

**2 Method**

In this paper, we consider the stochastic program

$$\min_{x \in X} F(x) := \mathbb{E}_\xi f(x; \xi), \quad (2.1)$$

where $\xi \in \Xi$ is a random variable, and $X \subseteq \mathbb{R}^n$ is a closed convex set. The formulation (2.1) also covers the finite-sum problem, which corresponds to the case that $\xi$ is distributed on a finite set $\Xi = \{\xi_1, \ldots, \xi_N\}$.

2.1 Notation

We use $\odot$ and $\oslash$ for componentwise multiplication and division with the convention $\frac{0}{0} = 0$. For a nonnegative vector $v$, $\sqrt{v}$ or $(v)\frac{1}{2}$ denotes a vector by the component-wise square root. Given a nonnegative vector $v$, $\|x\|_v^2$ is defined as $x^\top \text{Diag}(v)x$, and $\text{Proj}_{X,v}(x) = \arg \min_{y \in X} \|y - x\|_v^2$. We use $\|\cdot\|$ for the Euclidean norm of a vector and also the spectral norm of a matrix. $\|\cdot\|_\infty$ and $\|\cdot\|_1$ respectively denote the $\ell_\infty$-norm and the
ℓ₁-norm of a vector. ∥g∥₀ counts the non-zeros of g. The big-O and big-Θ have the standard meanings. [n] denotes the set {1, . . . , n}.

2.2 Algorithm

We propose an async-parallel adaptive SGM, named APAM, for solving (2.1). The pseudocode is shown in Algorithm 1, which is based on AMSGrad proposed by [21]. In (2.5), if ˆv(k) is a positive vector, x(k+1) is well-defined. If some component ˆv_i(k) = 0, we simply let x_i(k+1) = x_i(k) if that does not violate the feasibility. The original AMSGrad uses non-fixed weights in computing m(k), i.e., it lets m(k) = β_1,km(k−1) + (1 − β_1,k)g(k) for all k ≥ 1. Hence, if for all k ≥ 1, g(k) is an unbiased stochastic estimate of a subgradient of F at x(k), then APAM reduces to AMSGrad with a constant weight β_1 in averaging stochastic gradients. In order to guarantee sublinear convergence, [21] requires a diminishing sequence {β_1,k}, and to have a rate of O(1/√k), {β_1,k} needs to decay as fast as 1/k. However, the numerical experiments in [21] all adopt the setting β_1,k = β_1 ∈ (0, 1), ∀k ≥ 1, and indeed we observed that the algorithm with a constant weight β_1 could perform significantly better than that with decaying weights.

**Algorithm 1:** An asynchronous parallel adaptive stochastic gradient method (APAM)

1. **Initialization:** choose x(1) ∈ X, set m(0) = 0, v(0) = 0, ˆv(0) = 0, and β_1, β_2 ∈ [0, 1).
2. for k = 1, 2, . . . do
3. Obtain a (possibly outdated) stochastic gradient g(k);
4. Update the vectors m, v and ˆv by

   m(k) = β_1m(k−1) + (1 − β_1)g(k), \hspace{1cm} (2.2)

   v(k) = β_2v(k−1) + (1 − β_2)(g(k))^2, \hspace{1cm} (2.3)

   ˆv(k) = max \{ ˆv(k−1), v(k) \}. \hspace{1cm} (2.4)

5. Update the solution vector x by

   x(k+1) = Proj_{X, √ˆv(k)} \left( x(k) − α_k m(k) ∩ √ˆv(k) \right). \hspace{1cm} (2.5)

We make the first exploration on async-parallel adaptive SGM based on AMSGrad, because of simplicity and nice numerical performance. Besides AMSGrad, there are several other adaptive SGMs in the literature, such as AdaGrad [7], RMSProp [23], Adam [10], Padam [29], and AdaFom [6]. While AdaGrad can have guaranteed sublinear convergence, its numerical performance can be significantly worse than AMSGrad, because the former simply uses g(k) instead of the exponential averaging gradient m(k) in its update and also its effective learning rate can decay very fast. Adam can perform similarly or slightly better than AMSGrad, but its convergence is not guaranteed even for convex problems, due to possibly a too large learning rate. Padam is a generalized version of AMSGrad, and the performance of AdaFom is somehow between AdaGrad and AMSGrad. We believe that the convergence of AdaGrad, Padam and AdaFom can be inherited by their async versions.
2.3 Implementation

We explain the parallel implementation of Algorithm 1. For simplicity, we focus on the shared-memory computing model, where there exists a global memory that all processors can access (i.e., can address). The implementation, however, is not restricted to the shared-memory setting. We use wording that eases the implementation on a distributed-memory setting as well (i.e., processors cannot compute memory address and will incur communication when accessing data).

Suppose that there are multiple processors and all the data and variables (or model parameters) are stored in a global memory that every processor can access. We assign one (or a few) as the master(s). The updates to the vectors $\mathbf{x}, \mathbf{m}, \mathbf{v}$ and $\hat{\mathbf{v}}$ in Algorithm 1 are all performed by the master(s), while the computation of the $\mathbf{g}$ vector is done by other processors (called workers). See Figure 1 for an illustration. Every worker reads the variable $\mathbf{x}$ and data from the global memory, computes a stochastic gradient $\mathbf{g}$, and saves it in a pre-assigned memory. If there is an unused $\mathbf{g}$, then the master acquires it. Otherwise, the master computes one stochastic gradient by itself. In addition, we allow more than one processor to serve as the master in case one is not fast enough to digest the $\mathbf{g}$ vectors fed by the workers. We synchronize all the masters while performing the updates. However, we do not coordinate between the workers.

Since the master continuously updates $\mathbf{x}$, after worker #i reads the variable, the master may have already changed $\mathbf{x}$ before it uses the stochastic gradient fed by worker #i. Therefore, the stochastic gradient $\mathbf{g}^{(k)}$ used to obtain $\mathbf{x}^{(k+1)}$ may not be evaluated at the current iterate $\mathbf{x}^{(k)}$ but an outdated one $\hat{\mathbf{x}}^{(k)}$. More precisely,

$$
\mathbf{g}^{(k)} = \frac{1}{b_k} \sum_{i=1}^{b_k} \hat{\nabla} f(\hat{\mathbf{x}}^{(k)}; \xi_i^{(k)}),
$$

where $b_k$ is the number of samples, and $\hat{\nabla} f(\mathbf{x}; \xi)$ denotes a subgradient of $f(\cdot; \xi)$ at $\mathbf{x}$.

**Inconsistent read.** Note that Algorithm 1 is viewed from the perspective of the master processor, and $\mathbf{x}^{(k)}$ denotes the $k$-th iterate. In the shared-memory setting, since we do not lock $\mathbf{x}$ when a worker computes a
Under Assumption 3, it holds

Lemma 2

and

I

that

iterates; i.e., the reading is inconsistent. Suppose that the read of every coordinate is atomic. Then each entry

\( x_i^{(k)} \)

must be equal to \( x_i^{(k-j)} \) for some integer \( j \geq 0 \). For each integer \( j \geq 0 \), let \( I_j := \{ i \in [n] : x_i^{(k-j)} = x_j^{(k)} \} \) and \( I_j := \bigcup_{l=0}^j I_i \). Clearly, \( I_{l-1} \subseteq I_l \). Moreover, let \( \tau_k := \min \{ j : I_j = [n] \} \). Then \( \hat{x}^{(k)} \) can be formed from \( \{ x^{(k-\tau_k)}, \ldots, x^{(k)} \} \) by

\[
\hat{x}^{(k)} = x^{(k)} \odot 1_{I_0} + \sum_{l=1}^{\tau_k} x^{(k-l)} \odot 1_{I_l \setminus I_{l-1}},
\]

(2.7)

where \( 1_A \) represents the vector with one at each coordinate \( i \in A \) and zero elsewhere. From (2.7), it follows that

\[
\hat{x}^{(k)} = x^{(k)} - \sum_{l=0}^{\tau_k-1} (x^{(k-l)} - x^{(k-l-1)}) \odot 1_{I_l},
\]

(2.8)

whose proof is given in the appendix. The above formulation generalizes the relation for atomic lock-free updates in [12,18].

3 Convergence results for convex cases

In this section, we analyze Algorithm 1 for the convex case. First, in Section 3.1 we give an improved ergodic convergence result of AMSGrad for the case without delay, which happens in the non-parallel setting. Then, in Section 3.2 we give the result for the case with bounded staleness (or delay), and we discuss the relation of speed-up to the staleness \( \tau_k \). Throughout this section, we assume the follows.

Assumption 1 (convexity) The function \( F \) in (2.1) is convex.

Assumption 2 (bounded feasible set) The set \( X \) is convex and compact. Denote \( D_\infty = \max_{x,y \in X} \| x - y \|_\infty \).

Assumption 3 (bounded gradient) There are finite numbers \( G_1 \) and \( G_\infty \) such that \( \mathbb{E}[\| \nabla f(x, \xi) \|_1 \leq G_1, \forall x \in X, \text{ and } \| \nabla f(x; \xi) \|_\infty \leq G_\infty, \forall x \in X, \text{ and almost surely for all } \xi \).

Assumption 4 (unbiased gradient) Given a (possibly random) \( x \in X \), if \( \xi \) is independent of \( x \), then \( \mathbb{E}[\nabla f(x; \xi)] \) is a subgradient of \( F \) at \( x \).

By the assumption, it holds \( \mathbb{E}[\| g^{(k)} \|_1 \leq G_1 \), for all \( k \geq 1 \). We first establish a few lemmas.

Lemma 1 Under Assumptions 1 and 2, it holds for any \( t \geq 1 \) and any \( x \in X \) that

\[
(1 - \beta_1) \sum_{k=1}^t \left( \sum_{j=k}^t \alpha_j \beta_1^{t-k} \right) \left( x^{(k)} - x, g^{(k)} \right) \leq D_\infty^2 \| \nabla f(t) \|_1 + \frac{1}{2(1 - \beta_1)^2} \sum_{k=1}^t \alpha_k^2 \| m^{(k)} \|_2^2 \left( \omega^{(k)} \right)^{-\frac{1}{2}}. \quad (3.1)
\]

Lemma 2 Under Assumption 3, it holds

\[
\mathbb{E}[\| m^{(k)} \|_2^2 \left( \omega^{(k)} \right)^{-\frac{1}{2}} \leq \frac{G_1}{\sqrt{1 - \beta_2}}. \quad (3.2)
\]
3.1 Convergence rate result for the case without delay

If there is no delay, i.e., \( \hat{x}^{(k)} = x^{(k)}, \forall k \geq 1 \), we can easily obtain the following convergence rate results from the previous two lemmas. Although the results are not our main focus, they improve over existing results about AMSGrad, as a byproduct.

**Theorem 1 (constant step-size without delay)** Suppose all conditions in Assumptions 1 through 4 hold. Also, suppose that in (2.6), \( \{\xi^{(k)}_i\} \) is independent of \( \{x^{(j)}_j\}_{j=1}^k \), and \( \hat{x}^{(k)} = x^{(k)}, \forall k \geq 1 \). Given a positive integer \( K \), let \( \alpha_k = \frac{\alpha}{\sqrt{K}} \) for some \( \alpha > 0 \), \( x^* \) be an optimal solution of (2.1), and

\[
\hat{x}^{(K)} = \sum_{k=1}^{K} \left( \sum_{i=1}^{K} \frac{\alpha_j \beta_1^{i-k} x^{(k)}}{\sum_{j=1}^{K} \alpha_j \beta_1^{i-j}} \right.
\]

Then

\[
\mathbb{E}[F(\hat{x}^{(K)}) - F(x^*)] \leq \frac{nD_\infty^2 G_\infty + \alpha^2}{2\alpha \sqrt{K}(1 - \beta_2^2)}.
\]

**Theorem 2 (variant step-size without delay)** Suppose all conditions in Assumptions 1 through 4 hold. Also, suppose that in (2.6), \( \{\xi^{(k)}_i\} \) is independent of \( \{x^{(j)}_j\}_{j=1}^k \), and \( \hat{x}^{(k)} = x^{(k)}, \forall k \geq 1 \). Given a positive integer \( K \), let \( \alpha_k = \frac{\alpha}{\sqrt{k}} \) for some \( \alpha > 0 \), \( x^* \) be an optimal solution of (2.1), and \( \hat{x}^{(K)} \) is defined as in (3.3). Then

\[
\mathbb{E}[F(\hat{x}^{(K)}) - F(x^*)] \leq \frac{nD_\infty^2 G_\infty + \alpha^2(1 + \log K)}{4\alpha(\sqrt{K + 1} - 1)(1 - \beta_2^2)}.
\]

Note that our result is in the same order of that in the original AMSGrad paper [21]. But we allow the choice of constants \( \beta_1 \) and \( \beta_2 \), instead of requiring an exponential or harmonic decaying sequence \( \beta_{1,k} \) as in [21] and also its follow-up works such as [5, 15]. Numerically, a fixed constant \( \beta_1 \) can give significantly better results, and indeed [21] uses \( \beta_{1,k} = \beta_1, \forall k \) in its experiments. The recent works [6, 29] have also weakened the condition on decreasing \( \beta_{1,k} \) for smooth non-convex cases. However, none of these works has dropped the assumption \( \beta_1 \leq \sqrt{\beta_2} \) that is required by [21]. Our result does not need this condition.

3.2 Convergence rate result for the case with delay

When there is delay, i.e., \( \tau_k > 0 \) in (2.7) (that naturally happens for async computing), we can still guarantee an ergodic sublinear convergence result under a few additional mild assumptions.

**Assumption 5 (L-smoothness)** The function \( F \) is L-smooth, i.e., \( ||\nabla F(x) - \nabla F(y)|| \leq L||x - y|| \), for any \( x, y \in \mathbb{R}^n \).

**Assumption 6 (bounded staleness)** There is a finite integer \( \tau \) such that \( \tau_k \leq \tau \) for all \( k \geq 1 \).
Theorem 3 Suppose all conditions in Assumptions 1 through 5 hold, and suppose that in (2.6), $\{\xi_i^{(k)}\}$ is independent of $x^{(k)}$. Then for any $x \in X$,

$$(1 - \beta_1) \sum_{k=1}^t \left( \sum_{j=k}^t \alpha_j \beta_i^{j-k} \right) E[F(x^{(k)}) - F(x)]$$

$$\leq \frac{D^2}{2} \mathbb{E}\|\sqrt{v^{(t)}}\|_1 + \frac{1}{2(1 - \beta_1)^2} \frac{G_1}{\sqrt{1 - \beta_2}} \sum_{k=1}^t \alpha_k^2 + \frac{L(1 - \beta_1)}{2} \sum_{k=1}^t \left( \sum_{j=k}^t \alpha_j \beta_i^{j-k} \right) E\|x^{(k)} - \hat{x}^{(k)}\|^2. \quad (3.5)$$

To guarantee convergence, we need to bound the term $E\|x^{(k)} - \hat{x}^{(k)}\|^2$ in the same order of $\alpha_k$, which can be shown with the non-expansiveness of $\text{Proj}_{X, \sqrt{\gamma}}$.

Lemma 3 (non-expansiveness) If $X = [a_1, b_1] \times \cdots \times [a_n, b_n]$ for some finite numbers $\{a_i\}$ and $\{b_i\}$, then for any $k \geq 1$, it holds

$$\|x^{(k+1)} - x^{(k)}\| \leq \alpha_k \|m^{(k)} \odot \sqrt{v^{(k)}}\|. \quad (3.6)$$

Lemma 4 It holds for any $k \geq 1$ that

$$\|g^{(j)} \odot \sqrt{v^{(k)}}\| \leq \frac{\sqrt{\|g^{(j)}\|_0}}{\sqrt{1 - \beta_2}}, \forall j \leq k, \quad (3.7a)$$

$$\|m^{(k)} \odot \sqrt{v^{(k)}}\| \leq \sum_{j=1}^k (1 - \beta_1) \beta_1^{k-j} \frac{\sqrt{\|g^{(j)}\|_0}}{\sqrt{1 - \beta_2}}, \quad (3.7b)$$

$$\|m^{(k)} \odot \sqrt{v^{(k)}}\|^2 \leq \frac{1 - \beta_1}{1 - \beta_2} \sum_{j=1}^k \beta_1^{k-j} \|g^{(j)}\|^2. \quad (3.7c)$$

By Theorem 3 and the above two lemmas, we obtain the following convergence rate result for the case with delay.

Theorem 4 (convergence rate with delay) Suppose that all conditions in Assumptions 1 through 6 hold, and suppose that in (2.6), $\{\xi_i^{(k)}\}$ is independent of $\{x^{(j)}\}_{j=1}^k$. In addition, assume $X = [a_1, b_1] \times \cdots \times [a_n, b_n]$ for some finite numbers $\{a_i\}$ and $\{b_i\}$, and $E\|g^{(k)}\|_0 \leq s$, $\forall k \geq 0$ for some integer $s \leq n$. Given a positive integer $K$ and $\alpha > 0$, let $\alpha_k = \frac{\alpha}{\sqrt{k}}$ for all $k \leq K$. Then

$$E[F(\bar{x}^{(K)}) - F(x^*)] \leq \frac{nD^2 G_1}{\alpha} \frac{\alpha^2}{\sqrt{1 - \beta_1} \sqrt{1 - \beta_2}} + \frac{\alpha^3 L^2 s}{2 \alpha \sqrt{K} (1 - \beta_1)}, \quad (3.8)$$

where $\bar{x}^{(K)}$ is drawn from $\{x^{(k)}\}_{k=1}^K$ with

$$\text{Prob}(x^{(K)} = x^{(k)}) = \frac{\sum_{j=k}^K \alpha_j \beta_i^{j-k}}{\sum_{i=1}^K \left( \sum_{j=i}^K \alpha_j \beta_i^{j-t} \right)}, \forall 1 \leq k \leq K.$$
Remark 1 (How delay affects convergence speed) If we take \( \alpha = O(1) \) independent of \( n \) and \( s \), then the result in (3.8) implies that we can achieve nearly-linear speed-up if \( \tau = o(\sqrt{\frac{s}{s}} K^\frac{1}{4}) \). On the other hand, suppose \( G_1 \approx s G_\infty \). Then in the no-delay result (3.4), the optimal \( \alpha = O(\sqrt{\frac{s}{s}}) \), and with this \( \alpha \), we can achieve nearly-linear speed-up if \( \tau = o((\frac{s}{s} K)^\frac{1}{4}) \). [19] shows that the delay, in expectation, equals the number of processors if all of them have the same computing power. Hence, in the ideal case, we can expect nearly-linear speed-up by using \( O(K^{\frac{1}{4}}) \) processors.

Recall that we use \( n G_\infty \) to upper bound \( \| \sqrt{V(K)} \|_1 \) in our analysis. If \( \| \sqrt{V(K)} \|_1 \) is in a lower order than \( n \), then the optimal \( \alpha \) in (3.4) is in a lower order than \( \sqrt{\frac{s}{s}} \), and thus we can allow larger \( \tau \) to have a good speed-up. However, we remark that this bound can, under certain conditions, be tight by at most a constant factor \( \sqrt{1 - \beta_2} \). Hence, the above obtained bound on \( \tau \) is valid. Suppose \( |\nabla f(x; \xi)| \leq G_{i, \infty}, \forall x \in X \), and a.s. for all \( \xi \). When \( K \) is large, it is very likely that there is some \( j \leq K \) such that \( |g_i^{(j)}| \) is significant and close or equal to \( G_{i, \infty} \), and thus \( \tilde{v}_i^{(k)} \geq (1 - \beta_2) G_{i, \infty}^2 \) by (B.6) in the appendix. Therefore, if moreover \( G_{i, \infty} \approx G_\infty \) for each \( i \in [n] \), then \( \tilde{v}_i^{(K)} \geq (1 - \beta_2) G_{\infty}^2, \forall i \in [n] \), and \( \| \tilde{V}(K) \|_1 \geq n \sqrt{1 - \beta_2} G_\infty \).

4 Convergence results for nonconvex cases

In this section, we analyze Algorithm 1 for nonconvex problems. Due to the difficulty caused by nonconvexity, we assume \( X = \mathbb{R}^n \), i.e., there is no constraint. Then the update in (2.5) becomes exactly

\[
x^{(k+1)} = x^{(k)} - \alpha_k m^{(k)} \odot \sqrt{\tilde{V}^{(k)}}.
\]

Given a maximum number \( K \) of iterations, we assume, without loss of generality, \( \tilde{v}_i^{(K)} > 0 \) for all \( i \in [n] \). Note that if \( \tilde{v}_i^{(K)} = 0 \) for some \( i \), then \( \tilde{y}_i^{(k)} = 0 \) for all \( k \leq K \), and in this case, \( x_i \) never changes and can be simply viewed as a constant instead of a variable.

We define an auxiliary sequence \( \{ \tilde{V}^{(k)} \}_{k=1}^K \) as follows. It is only used in the analysis but not in the computation.

**Definition 1** Given a positive integer \( K \), let \( \{ \tilde{V}^{(k)} \}_{k=1}^K \) be computed from Algorithm 1. For any \( i \in [n] \), suppose \( k_i \leq K \) is the smallest number such that \( \tilde{v}_i^{(k_i)} \geq 0 \). We define \( \{ \tilde{V}^{(k)} \}_{k=1}^K \) as: \( \tilde{v}_i^{(k)} = \max \{ \tilde{v}_i^{(k_i)}, \tilde{v}_i^{(k)} \} \) for all \( i \in [n] \) and all \( k \in [K] \). Furthermore, define \( \tilde{V}^{(k)} = \text{Diag}(\tilde{v}_i^{(k)}) \) for all \( k \in [K] \).

**Remark 2** We make a few remarks on \( \{ \tilde{V}^{(k)} \}_{k=1}^K \). (i) Assume \( \tilde{v}_i^{(K)} > 0 \) for all \( i \in [n] \). Then each \( \tilde{V}^{(k)} \) is a positive vector, and \( \tilde{V}^{(k)} \geq \tilde{V}^{(k-1)} \) still holds component-wisely; (ii) \( m^{(k)} \odot \sqrt{\tilde{V}^{(k)}} = m^{(k)} \odot \sqrt{\tilde{V}^{(k)}} \) and \( g^{(k)} \odot \sqrt{\tilde{V}^{(k)}} = g^{(k)} \odot \sqrt{\tilde{V}^{(k)}} \) for all \( k \leq K \); and (iii) \( \tilde{V}^{(K)} = \tilde{V}^{(K)} \) under the assumption \( \tilde{V}^{(K)} > 0 \).

With the above definition, we are ready to state the main result for non-convex cases.

**Theorem 5** Given a maximum number \( K \geq 2 \) of iterations, let \( \{ x^{(k)} \}_{k=1}^K \) and \( \{ \tilde{V}^{(k)} \}_{k=1}^K \) be generated from Algorithm 1 with a non-increasing positive sequence \( \{ \alpha_k \}_{k=1}^K \). Suppose \( \tilde{V}^{(k)} > 0 \), and define \( \{ \tilde{V}^{(k)} \}_{k=1}^K \) as in Definition 1. Furthermore, suppose \( \{ \xi^{(k)} \}_{k=1}^K \) is independent of \( \{ x^{(j)} \}_{j=1}^K \) in (2.6). Assume \( E\| g^{(k)} \|_0 \leq s, \forall k \geq 1 \) for some positive integer \( s \). Let \( \tilde{x}^{(K)} \) be drawn from \( \{ x^{(k)} \}_{k=2}^K \) with

\[
\text{Prob}(\tilde{x}^{(K)} = x^{(k)}) = \frac{\alpha_{k-1}}{\sum_{j=2}^{K} \alpha_{j-1}}, \forall k = 2, \ldots, K.
\]
Then under Assumptions 3 through 6, it holds
\[
\mathbb{E}\|\nabla F(x^{(k)})\|^2 \leq \left( G_1^2 \mathbb{E}\|\nabla (v^{(1)})\|^2 \right) \frac{1}{1-\beta_1} + \frac{sG_2^2}{\sqrt{1-\beta_2}} \sum_{k=2}^{K} \alpha_{k-1} + \frac{G_\infty \|F(x^{(1)}) - \inf_x F(x)\|}{2} + \frac{7sL\infty}{6(1-\beta_2)} \sum_{k=2}^{K} \alpha_{k-1}
\]
\[
+ \frac{7sL\infty}{2(1-\beta_2)(1-\beta_1)^2} \sum_{k=2}^{K} \alpha_{k-1} \sqrt{\sum_{k=2}^{K} \sum_{t=2}^{K} \alpha_{k-1} \sqrt{1-\beta_2} \sum_{t=2}^{K} \alpha_{t-1}}.
\]

Moreover, suppose that there is a constant $C_F$ such that $|F(x)| \leq C_F$, $\forall x$. Let $2 \leq k_0 \leq K$ and $x^{(k_0,K)}$ be drawn from $\{x^{(k)}\}_{k=k_0}$ with probability
\[
\text{Prob}(x^{(k_0,K)} = x^{(k)}) = \frac{\alpha_{k-1}}{\sum_{j=k_0}^{K} \alpha_{j-1}}, \quad \forall k = k_0, \ldots, K.
\] (4.3)

Then
\[
\mathbb{E}\|\nabla F(x^{(k_0,K)})\|^2 \leq \left( G_1^2 \mathbb{E}\|\nabla (v^{(k_0-1)})\|^2 \right) \frac{1}{1-\beta_1} + \frac{2C_FG_\infty}{\sum_{k=k_0}^{K} \alpha_{k-1}} + \frac{7sL\infty}{6(1-\beta_2)} \sum_{k=k_0}^{K} \alpha_{k-1}
\]
\[
+ \frac{7sL\infty}{2(1-\beta_2)(1-\beta_1)^2} \sum_{k=k_0}^{K} \alpha_{k-1} \sqrt{\sum_{k=k_0}^{K} \sum_{t=k_0}^{K} \alpha_{k-1} \sqrt{1-\beta_2} \sum_{t=k_0}^{K} \alpha_{t-1}}.
\] (4.4)

Below we specify the setting of $\{\alpha_k\}$ and show the sublinear convergence.

**Corollary 1 (convergence rate with constant stepsize)** Given a positive integer $K \geq 2$, let $2 \leq k_0 \leq K$. Suppose all the conditions assumed in Theorem 5 hold, and in addition, there is a constant $c > 0$ such that $\tilde{v}_i^{(k-1)} \geq c^2, \forall i \in [n]$ holds almost surely. If $\alpha_k = \sqrt{\frac{\tau}{K-k_0+1}}, \forall 1 \leq k \leq K$, for some constant $\alpha > 0$, then
\[
\mathbb{E}\|\nabla F(x^{(k_0,K)})\|^2 \leq C_1 + \frac{C_2}{\alpha} \sqrt{C_1 + \frac{C_2}{\alpha}},
\] (4.5)

where $x^{(k_0,K)}$ is given in (4.3), and
\[
C_1 = \frac{G_1^3 \mathbb{E}\|\nabla (v^{(k_0-1)})\|^2}{(1-\beta_1)(K-k_0+1)} + \frac{2C_FG_\infty}{\alpha \sqrt{K-k_0+1}} + \frac{7sL\infty(1-2\beta_1+4\beta_1^2)}{6(1-\beta_2)(1-\beta_1)^2} \frac{\alpha}{\sqrt{K-k_0+1}},
\]
\[
C_2 = \frac{\alpha \beta \sqrt{\sum_{k=k_0}^{K} \sum_{t=k_0}^{K} \alpha_{k-1} \sqrt{1-\beta_2} \sum_{t=k_0}^{K} \alpha_{t-1}}}{\sqrt{1-\beta_2} \sqrt{K-k_0+1}}.
\]

**Remark 3** We make a few remarks here. (i) Note that $\mathbb{E}\|\nabla (v^{(k_0-1)})\|^2 \leq \frac{G_1^2}{c^2}$. Hence, we have the ergodic sublinear convergence $O(\frac{1}{\sqrt{K-k_0+1}})$. (ii) The existence of $c > 0$ such that $\tilde{v}_i^{(k_0-1)} \geq c^2, \forall i \in [n]$ almost surely is a mild assumption. As discussed in Remark 1, if $k_0$ is large, then it is likely that $\tilde{v}_i^{(k_0-1)} \geq \tilde{v}_i^{(k_0-1)} \geq (1-\beta_2)G_1^2$, where $G_1^2$ is an almost-sure bound on $|\nabla f(x; \xi)|$. (iii) Suppose $K \geq 2$ and $k_0 = \lceil \frac{K}{2} \rceil$. If $\alpha = \Theta(1)$ and $\tau = o(K^2)$, then $\frac{C_2}{\alpha} \sqrt{C_1 + \frac{C_2}{\alpha}} \ll C_1$ when $K$ is large. Hence, we observe from (4.5) that in this case, the delay will just slightly affect the convergence speed, and we can achieve nearly-linear speed-up.
Also, if $\tau = 0$ (i.e., no delay), the right hand side of (4.5) reduces to $C_1$. In terms of $s$, $C_1$ is minimized when $\alpha = \Theta(\frac{1}{\sqrt{s}})$, which implies that if $\tau = o((sK)^{\frac{1}{4}})$, we can achieve nearly-linear speed-up. (iv) We can start from the inequality (4.2) and obtain similar convergence rate results.

**Corollary 2 (convergence rate with diminishing stepsize)** Given an even positive integer $K \geq 2(\tau + 2)$, let $k_0 = \frac{K}{2} \geq \tau + 2$. Suppose all the conditions assumed in Theorem 5 hold, and in addition, there is a constant $c > 0$ such that $\bar{v}_i(n-1) \geq c^2, \forall i \in [n]$ holds almost surely. If $\alpha_k = \frac{s}{\sqrt{K}}, \forall 1 \leq k \leq K$. Then

$$E\|\nabla F(\bar{x}(n))\|^2 \leq C_1 + \frac{C_2}{c} \left( \sqrt{C_1} + \frac{C_2}{c} \right) \tag{4.6}$$

where $\bar{x}(n)$ is given in (4.3), and

$$C_1 = \frac{G_3^3 E\|\bar{v}(n-1)\|_1}{(2 - \sqrt{2})(1 - \beta_1)\sqrt{K} \sqrt{K}/2 - 1} + \frac{2C_F G_\infty}{(2 - \sqrt{2})\alpha \sqrt{K}} + \frac{7sLG_\infty}{6(1 - \beta_2)} \frac{4\log 4}{(2 - \sqrt{2})\sqrt{K}} + \frac{7sLG_\infty}{6(1 - \beta_2)} (2 - \sqrt{2}) \sqrt{K}$$

$$C_2 = \frac{2\alpha \sqrt{5}LG_\infty}{\sqrt{K} \sqrt{1 - \beta_2}}.$$

**Remark 4** Similar to the discussion in Remark 3, if $\alpha = \Theta(1)$ and $\tau = o(K^{\frac{1}{4}})$, we can achieve nearly-linear speed-up. Similar to Remark 3 (iii), in terms of $s$, $C_1$ is minimized when $\alpha = \Theta(\frac{1}{\sqrt{s}})$, which implies that if $\tau = o((sK)^{\frac{1}{4}})$, we can achieve nearly-linear speed-up.

5 Numerical experiments

In this section, we conduct numerical experiments on the proposed algorithm APAM, an async-parallel version of AMSGrad. The main purpose of the experiments is to demonstrate whether APAM can still perform well on solving machine learning problems even though outdated stochastic gradients are used, and also to demonstrate how much speed-up it can achieve. We compare APAM to the non-parallel and sync-parallel versions of AMSGrad. Notice that AMSGrad has been shown in the literature to converge significantly faster than a non-adaptive SGM or a momentum SGM, and in addition, it performs similarly well as Adam that is another popularly used adaptive SGM but is not guaranteed to converge. Hence, we do not extend our experiments to compare with other SGMs. Both async and sync-parallel methods are implemented in C++ with openMP. All tests were run on a Dell workstation with 32 CPU cores and 64 gigabyte memory.

5.1 Convex problems

We tested APAM for solving the binary-class logistic regression (LR) problem and also the multi-class LR problem. For the binary-class LR, we used the real-sim and rcv1 datasets, both of which are available from the LIBSVM [4], and for the multi-class LR, we used the MNIST dataset [11]. Their characteristics are listed in Table 1.
Asynchronous parallel adaptive stochastic gradient methods

Table 1 Characteristics of the tested datasets.

| name    | # samples | # features | # classes |
|---------|-----------|------------|-----------|
| real-sim| 72,309    | 20,958     | 2         |
| rcv1    | 20,242    | 47,236     | 2         |
| MNIST   | 60,000    | 780        | 10        |
| Fashion-MNIST | 60,000 | 784        | 10        |

Table 2 Comparison of the sync-parallel AMSGrad and the proposed APAM (i.e., async-parallel AMSGrad). For each case, all methods run to 20 epochs, with mini-batch size set to 64 while computing a stochastic gradient.

| #thread | real-sim dataset | rcv1 dataset | MNIST dataset |
|---------|------------------|--------------|---------------|
|         | time (sec.)      | speed-up     | time (sec.)   | speed-up     | time (sec.) | speed-up     |
|         | sync | async | sync | async | sync | async | sync | async | sync | async | sync | async |
| 1       | 104.92 | 1.00 | 104.92 | 1.00 | 65.90 | 1.00 | 65.90 | 1.00 | 103.12 | 1.00 | 103.12 | 1.00 |
| 2       | 55.50  | 1.95 | 53.81  | 1.89 | 34.86 | 1.95 | 33.71 | 1.89 | 53.02 | 1.95 | 51.83 | 1.94 |
| 4       | 31.38  | 3.79 | 27.69  | 3.34 | 19.64 | 3.80 | 17.34 | 3.36 | 28.36 | 3.80 | 26.57 | 3.64 |
| 8       | 21.74  | 6.88 | 15.25  | 4.83 | 13.23 | 6.86 | 9.61  | 4.98 | 16.57 | 6.88 | 14.00 | 6.22 |
| 16      | 20.25  | 12.28| 8.55   | 5.11 | 12.20 | 12.03| 5.40  | 5.40 | 11.80 | 7.65 | 8.74  | 7.37 |
| 32      | 30.25  | 21.17| 4.96   | 3.47 | 17.84 | 20.60| 3.20  | 3.69 | 14.23 | 4.55 | 7.25  | 22.66|

We ran APAM with different numbers of threads and compared it to the non-parallel and sync-parallel versions of AMSGrad. For each method, we set the mini-batch size to 64 while computing a stochastic gradient \( g \). At each iteration of the sync-parallel AMSGrad, we evenly partitioned the sampled data points over the multiple cores to compute \( g \) and also evenly partitioned the features (namely variables) to update \( x, m, v \) and \( \hat{v} \). All the methods used \( \alpha_k = \frac{\alpha}{\sqrt{k}} \). Here, \( k \) is the iteration number, and the same \( \alpha > 0 \) was set for all methods and tuned to achieve a good prediction accuracy. For APAM, when 8 or more threads were used, we chose one or more as the master to perform sync-updates to \( x, m, v \) and \( \hat{v} \), based on the complexity comparison between computing a \( g \) and updating the variables. Therefore, the implementation of APAM with multiple master threads is a mixture of async and sync-parallelization. For each test, we ran the compared methods to 20 epochs, where one epoch is equivalent to one data pass.

Table 2 shows the running time (in second) and speed-up of APAM and the sync-parallel AMSGrad with the number of threads varying among \{1, 2, 4, 8, 16, 32\}. From the results, we can clearly see the higher speed-up of the async-parallel method over the sync-parallel one. When 32 threads were used, the sync-parallel method achieved less speed-up than using 16 threads. This is possibly because of the memory congestion and load imbalance. APAM does not have this issue. It consistently achieved more speed-up while more cores were used. Figure 2 shows how the objective values change with respect to the running time. From the figures, we see that the async-parallel method achieves almost the same (and sometimes lower) objective values as the sync-parallel one, and this indicates that the staleness of the stochastic gradient has little effect on the convergence.

5.2 Nonconvex problems

We also compared APAM and the sync-parallel AMSGrad on training non-convex neural networks (NN). Feedforward fully-connected NNs were used. One is a 2-layered NN (i.e., with one hidden layer) and another 3-layered (i.e., with 2 hidden layers). For both NNs, we used the hyperbolic tangent function as the activation
Fig. 2 Objective values given by the non-parallel and sync-parallel AMSGrad, and the proposed APAM (i.e., async-parallel AMSGrad) on the real-sim (first row), rcv1 (middle row), and MNIST datasets. Mini-batch size is set to 64 to compute each stochastic gradient.

Table 3 Comparison of the sync and async-parallel AMSGrad on the MNIST and Fashion-MNIST datasets. Mini-batch size is set to 32 to compute each stochastic gradient. For each case, all methods run to 10 epochs.

function and the soft-max in the output layer. We used the MNIST dataset and also the Fashion-MNIST dataset [27] in the test and ran all methods to 10 epochs with the mini-batch size set to 32. For MNIST, the weight matrix was set to $780 \times 390$ for the 2-layered NN, and $780 \times 390$ and $390 \times 195$ weight matrices were used for the first and second layers of the 3-layered NN. For Fashion-MNIST, the weight matrix was set to $784 \times 392$ for the 2-layered NN, and $784 \times 392$ and $392 \times 196$ weight matrices were used for the first and
second layers of the 3-layered NN. Table 3 lists the running time and speed-up of the async and sync-parallel methods. From the table, we see that the speed-up performance of the sync-parallel AMSGrad is even worse than that for the convex problem tests, while the async-parallel one can still achieve high speed-up. One possible reason is the smaller mini-batch size. In the appendix, we include more numerical results. They show that both async and sync-parallel methods can achieve more speed-up if a larger mini-batch size is used. However, even in those cases, the async-parallel method still gained significantly higher speed-up than the sync-parallel one, especially when 32 threads were used. In addition, similar to the convex cases, the async-parallel method yields almost the same objective values and training errors as those by the sync-parallel one. The training errors for MNIST are around 2% for the 2-layered NN and 3% for the 3-layered NN, while for Fashion-MNIST, they are around 15% for the 2-layered NN and 19% for the 3-layered NN. These errors can be smaller with hyper-parameter tuning and architecture tuning, but the advantage of async-parallel over sync-parallel remains.

6 Related works

In the literature, there are a vast number of works on SGMs. We briefly review those on async SGM and adaptive SGM, which are closely related to our work.

Async non-adaptive SGM. The classic non-adaptive SGM has been applied and analyzed with async implementation. [1] gives a convergence rate result $O(\frac{1}{\sqrt{n}} + \frac{\tau}{n})$ for convex problems under a distributed setting with a central node. The result indicates a near-linear speed-up when $\tau = o(K^{\frac{1}{4}})$. [20] proposes an async-parallel SGM with randomized coordinate update for strongly convex problems under a shared-memory setting. To produce a stochastic $\epsilon$-solution, it needs $O(\frac{1+\tau/n+\tau^2/\sqrt{n}}{\epsilon} \log \frac{1}{\epsilon})$ iterations, and thus a near-linear speed-up can be achieved when $\tau = o(n^{\frac{1}{4}})$. [12] analyzes the async SGM for nonconvex optimization under both shared-memory and distributed settings. By utilizing the sample variance bound, it can guarantee near-linear speed-up if $\tau = o(\sqrt{K})$. For convex and strongly-convex problems, [9] warrants a near-linear speed-up of the async SGM when $\tau = o(K^{\frac{1}{4}})$. [2] adapts the stepsizes of the async SGM to the staleness of stochastic gradient. It achieves $O(\frac{\log K}{\epsilon})$ convergence for strongly convex problems. However, it is unclear how the staleness affects the convergence speed. [13,26] explore the async SGM under a decentralized setting.

Adaptive SGM. Adam [10] is probably the most popular adaptive SGM. It was proposed for convex problems. However, the convergence of Adam is not guaranteed, as pointed out in [21], which proposed AMSGrad. With decaying $\{\beta_{1,k}\}$ in computing $m^{(k)}$, either $\beta_{1,k} = \lambda^k$ for some $\lambda \in (0, 1)$ or $\beta_{1,k} = \frac{1}{k}$, [21] establishes sublinear convergence of AMSGrad for convex problems. Later, [24] found a flaw in the analysis of AMSGrad. To address that flaw, it proposes AdamX that embeds $\beta_{1,k}$ in updating $\tilde{v}^{(k)}$. To have nice generalization performance, [5] proposed Padam with search direction $-m^{(k)} \odot \tilde{v}^{(k)}p$, where $p \in (0, 0.5]$. When $p = \frac{1}{2}$, Padam reduces to AMSGrad. It was demonstrated that $p = \frac{1}{2}$ could yield the best numerical performance. To avoid extremely large or small learning rates, [15] proposes variants of Adam and AMSGrad by keeping the second moment term in nonincreasing intervals. The above variants of Adam or AMSGrad all require a decaying $\{\beta_{1,k}\}$ to guarantee sublinear convergence. For strongly-convex online optimization, [8] presents a variant of AMSGrad that can have constant $\beta_1$ and achieve a regret of $O(\sqrt{K})$. [25] proposes SAdam, a variant of Adam, with effective stepsize $\frac{\alpha}{\tau} \odot (v^{(k)} + \frac{1}{\tau} 1)$ and shows a regret of $O(\log K)$ for strongly-convex problems.

For nonconvex problems, [6] gives a general framework of Adam-type SGM and establishes convergence rate of $O(\frac{\log K+n^2}{\sqrt{n}K})$. [29] generalizes Padam to the non-convex case and shows $O(\sqrt{\frac{n}{K}} + \frac{n}{K})$ convergence.
rate. [16] presents a variant of AMSGrad by introducing one more moving average in the update of $\hat{v}$. The algorithm is analyzed for both convex and non-convex cases with decaying $\{\beta_{1,k}\}$.

7 Concluding remarks

We have presented an asynchronous parallel adaptive stochastic gradient method, named APAM, based on the AMSGrad method. Convergence rate results are established for both constrained convex and unconstrained non-convex cases. The results show that the delay has little effect on the convergence speed, if it is upper bounded by $\tau = o(K^{\frac{1}{4}})$, where $K$ is the maximum number of iterations. Numerical experiments on both convex and non-convex machine learning problems demonstrate significant advantages of the proposed method over its synchronous counterpart. Although the experiments are conducted in the shared-memory setting, our analysis applies also to the distributed-memory setting, where memory access incurs communication.

A Proof of Equation (A.1)

Clearly, $\mathcal{I}_{l-1} \subseteq \mathcal{I}_l$, and thus

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k)} \odot \mathbf{1}_{\mathcal{I}_0} + \sum_{l=1}^{\tau_k} \mathbf{x}^{(k-l)} \odot (\mathbf{1}_{\mathcal{I}_l} - \mathbf{1}_{\mathcal{I}_{l-1}})$$

$$= \mathbf{x}^{(k)} - \mathbf{x}^{(k)} \odot \mathbf{1}_{\mathcal{I}_0} + \sum_{l=1}^{\tau_k} \mathbf{x}^{(k-l)} \odot (\mathbf{1}_{\mathcal{I}_{l-1}} - \mathbf{1}_{\mathcal{I}_l})$$

$$= \mathbf{x}^{(k)} - \sum_{l=0}^{\tau_k-1} (\mathbf{x}^{(k-l)} - \mathbf{x}^{(k-l-1)}) \odot \mathbf{1}_{\mathcal{I}_l}, \quad (A.1)$$

where we have used $\mathcal{I}_{\tau_k} = [n]$. From (A.1), we have

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k)}\| \leq \sum_{l=0}^{\tau_k-1} \|\mathbf{x}^{(k-l)} - \mathbf{x}^{(k-l-1)} \odot \mathbf{1}_{\mathcal{I}_l}\| \leq \sum_{l=0}^{\tau_k-1} \|\mathbf{x}^{(k-l)} - \mathbf{x}^{(k-l-1)}\|, \quad (A.2)$$

and

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(k)}\|^2 \leq \tau_k \sum_{l=0}^{\tau_k-1} \|\mathbf{x}^{(k-l)} - \mathbf{x}^{(k-l-1)}\|^2, \quad (A.3)$$

B Technical details of section 3

Proof of Lemma 1 From the update of $\mathbf{x}$ in (2.5), we have the optimality condition

$$\mathbf{0} \in \mathcal{N}_X (\mathbf{x}^{(k+1)}) + \sqrt{v^{(k)}} (\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) + \alpha_k \mathbf{m}^{(k)},$$

where $\mathcal{N}_X (\mathbf{x})$ denotes the normal cone of $X$ at $\mathbf{x}$. Hence, it follows

$$\left\langle \mathbf{x}^{(k+1)} - \mathbf{x}, \sqrt{v^{(k)}} (\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) + \alpha_k \mathbf{m}^{(k)} \right\rangle \leq 0, \forall \mathbf{x} \in X. \quad (B.1)$$
By the update of \( m \) in (2.2), it holds
\[
\langle x^{(k+1)} - x, m^{(k)} \rangle = \langle x^{(k+1)} - x^{(k)}, m^{(k)} \rangle + \langle x^{(k)} - x, m^{(k)} \rangle
\]
\[
= \langle x^{(k+1)} - x^{(k)}, m^{(k)} \rangle + (1 - \beta_1) \langle x^{(k)} - x, g^{(k)} \rangle + \beta_1 \langle x^{(k)} - x, m^{(k-1)} \rangle.
\]
Recursively using the above relation, we have
\[
\langle x^{(k+1)} - x, m^{(k)} \rangle = \sum_{j=1}^{k} \beta_1^{k-j} \left( \langle x^{(j+1)} - x^{(j)}, m^{(j)} \rangle + (1 - \beta_1) \langle x^{(j)} - x, g^{(j)} \rangle \right).
\]
(B.2)

In addition, noting
\[
\langle x^{(k+1)} - x, \sqrt{\beta_1} (x^{(k+1)} - x^{(k)}) \rangle = \frac{1}{2} \left( \|x^{(k+1)} - x\|^2 - \|x^{(k)} - x\|^2 \sqrt{\beta_1} + \|x^{(k+1)} - x^{(k)}\|^2 \sqrt{\beta_1} \right).
\]

Substituting the above two equations into (B.1) gives
\[
\alpha_k \sum_{j=1}^{k} \beta_1^{k-j} \left( \langle x^{(j+1)} - x^{(j)}, m^{(j)} \rangle + (1 - \beta_1) \langle x^{(j)} - x, g^{(j)} \rangle \right)
\]
\[
\leq - \frac{1}{2} \left( \|x^{(k+1)} - x\|^2 \sqrt{\beta_1} - \|x^{(k)} - x\|^2 \sqrt{\beta_1} + \|x^{(k+1)} - x^{(k)}\|^2 \sqrt{\beta_1} \right).
\]
(B.3)

By the Young’s inequality, we have
\[
\sum_{k=1}^{t} \alpha_k \sum_{j=1}^{k} \beta_1^{k-j} \langle x^{(j+1)} - x^{(j)}, m^{(j)} \rangle
\]
\[
= \sum_{j=1}^{t} \sum_{k=j}^{t} \alpha_k \beta_1^{k-j} \langle x^{(j+1)} - x^{(j)}, m^{(j)} \rangle
\]
\[
\geq \sum_{j=1}^{t} \left( \sum_{k=j}^{t} \alpha_k \beta_1^{k-j} \right) \left( \frac{\|x^{(j+1)} - x^{(j)}\|^2}{\langle \beta_1 \rangle} + \frac{\sum_{k=j}^{t} \alpha_k \beta_1^{k-j}}{2} \|m^{(j)}\|^2 - \frac{\sum_{k=j}^{t} \alpha_k \beta_1^{k-j}}{2} \|m^{(j)}\|^2 \langle \beta_1 \rangle^{-\frac{1}{2}} \right).
\]

Since \( \sum_{k=j}^{t} \alpha_k \beta_1^{k-j} \leq \frac{\alpha_j}{1 - \beta_1} \), the above inequality implies
\[
\sum_{k=1}^{t} \alpha_k \sum_{j=1}^{k} \beta_1^{k-j} \langle x^{(j+1)} - x^{(j)}, m^{(j)} \rangle \geq - \sum_{j=1}^{t} \left( \frac{\|x^{(j+1)} - x^{(j)}\|^2}{2} + \frac{\alpha_j^2}{2(1 - \beta_1)^2} \|m^{(j)}\|^2 \langle \beta_1 \rangle^{-\frac{1}{2}} \right).
\]
(B.4)

In addition, noting \( \bar{\psi}^{(k)} \geq \bar{\psi}^{(k-1)} \) for all \( k \), we have
\[
- \sum_{k=1}^{t} \left( \|x^{(k+1)} - x\|^2 \sqrt{\beta_1} - \|x^{(k)} - x\|^2 \sqrt{\beta_1} \right)
\]
\[
= \left( -\|x^{(1+1)} - x\|^2 \sqrt{\beta_1} + \sum_{k=2}^{t} \|x^{(k)} - x\|^2 \sqrt{\beta_1} - \sqrt{\beta_1} + \|x^{(1)} - x\|^2 \sqrt{\beta_1} \right)
\]
\[
\leq D_\infty^2 \left( \sum_{k=2}^{t} \|\sqrt{\beta_1} - \sqrt{\beta_1} \|_1 + \|\sqrt{\beta_1} \|_1 \right) = D_\infty^2 \|\sqrt{\beta_1}\|_1.
\]
(B.5)
Now summing (B.3) over \( k = 1 \) to \( t \), and using (B.4) and (B.5), we obtain the desired result.

**Proof of Lemma 2**  For each \( i \in [n] \), let \( G_i^{(k)} = \max_{j \leq k} |g_i^{(j)}| \), and \( \mathbf{G}^{(k)} \) be the vector with the \( i \)-th component \( G_i^{(k)} \). Note that for each \( k \geq 1 \) and each \( i \in [n] \), we have \( \bar{v}_i^{(k)} = \max \{ v_i^{(k-1)}, v_i^{(k)} \} = \max_{j \leq k} v_i^{(j)} \), and in addition, \( v_i^{(j)} = \sum_{j=1}^{k} (1 - \beta_2) \beta_2^{j-1} (g_i^{(j)})^2 \). Hence,

\[
\bar{v}_i^{(k)} = \max_{j \leq k} \sum_{j=1}^{j} (1 - \beta_2) \beta_2^{j-1} (g_i^{(j)})^2, \tag{B.6}
\]

and thus \( \bar{v}_i^{(k)} \geq (1 - \beta_2)(g_i^{(k)})^2 \). Therefore, noticing

\[
m^{(k)} = \sum_{j=1}^{k} (1 - \beta_1) \beta_1^{k-j} \mathbf{g}^{(j)}, \tag{B.7}
\]

we have

\[
\|m^{(k)}\|_{\langle \mathbf{q}^{(k)} \rangle}^{-\frac{1}{2}} = \|m^{(k)} \otimes (\mathbf{q}^{(k)})^\frac{1}{2}\| \leq \frac{1}{(1 - \beta_2)\frac{1}{2}} \|m^{(k)} \otimes \sqrt{\mathbf{G}^{(k)}}\| \leq \frac{1}{(1 - \beta_2)\frac{1}{2}} \sum_{j=1}^{k} (1 - \beta_1) \beta_1^{k-j} \|\mathbf{g}^{(j)} \otimes \sqrt{\mathbf{G}^{(k)}}\|, \]

and thus by the Cauchy-Schwarz inequality, it holds

\[
\|m^{(k)}\|_{\langle \mathbf{q}^{(k)} \rangle}^{-\frac{1}{2}} \leq \frac{1 - \beta_1}{(1 - \beta_2)\frac{1}{2}} \left( \sum_{j=1}^{k} \beta_1^{k-j} \sum_{j=1}^{k} \beta_1^{k-j} \|\mathbf{g}^{(j)} \otimes \sqrt{\mathbf{G}^{(k)}}\|^2 \right) \leq \frac{1 - \beta_1}{(1 - \beta_2)\frac{1}{2}} \sum_{j=1}^{k} \beta_1^{k-j} \|\mathbf{g}^{(j)} \otimes \sqrt{\mathbf{G}^{(k)}}\|^2. \]

Now note that

\[
\|\mathbf{g}^{(j)} \otimes \sqrt{\mathbf{G}^{(k)}}\|^2 = \sum_{i=1}^{n} \|g_i^{(j)}\|_2^2 \leq \sum_{i=1}^{n} \|g_i^{(j)}\|_1 = \|\mathbf{g}^{(j)}\|_1.
\]

Together from the above two inequalities, it follows that

\[
\mathbb{E} \|m^{(k)}\|_{\langle \mathbf{q}^{(k)} \rangle}^{-\frac{1}{2}} \leq \frac{1 - \beta_1}{(1 - \beta_2)\frac{1}{2}} \sum_{j=1}^{k} \beta_1^{k-j} \mathbb{E} \|\mathbf{g}^{(j)}\|_1 \leq \frac{1 - \beta_1}{(1 - \beta_2)\frac{1}{2}} \sum_{j=1}^{k} \beta_1^{k-j} G_1,
\]

which implies the result in (3.2).

**Proof of Theorem 1**  Taking expectation over both sides of (3.1) and using Lemma 2, we have

\[
(1 - \beta_1) \sum_{k=1}^{t} \left( \sum_{j=1}^{k} \alpha_j \beta_1^{j-1} \right) \mathbb{E} \left\langle \mathbf{x}^{(k)} - \mathbf{x}, \mathbf{g}^{(k)} \right\rangle \leq \frac{D^2}{2} \mathbb{E} \|\mathbf{\hat{q}}^{(t)}\|_1 + \frac{G_1}{2(1 - \beta_1)^{2} \sqrt{1 - \beta_2}} \sum_{k=1}^{t} \alpha_k^2. \tag{B.8}
\]

Since \( \{\xi_i^{(k)}\} \) is independent of \( \mathbf{x}^{(k)} \), and \( \tau_k = 0, \forall k \geq 1 \), we have \( \mathbb{E} \left\langle \mathbf{x}^{(k)} - \mathbf{x}, \mathbf{g}^{(k)} \right\rangle = \mathbb{E} \left\langle \mathbf{x}^{(k)} - \mathbf{x}, \nabla F(\mathbf{x}^{(k)}) \right\rangle \), where \( \nabla F(\mathbf{x}^{(k)}) \in \partial F(\mathbf{x}^{(k)}) \). Hence, by the convexity of \( F \), it holds \( \mathbb{E}[F(\mathbf{x}^{(k)}) - F(\mathbf{x})] \leq \mathbb{E} \left\langle \mathbf{x}^{(k)} - \mathbf{x}, \mathbf{g}^{(k)} \right\rangle \), and thus (B.8) indicates

\[
(1 - \beta_1) \sum_{k=1}^{t} \left( \sum_{j=1}^{k} \alpha_j \beta_1^{j-1} \right) \mathbb{E}[F(\mathbf{x}^{(k)}) - F(\mathbf{x})] \leq \frac{D^2}{2} \mathbb{E} \|\mathbf{\hat{q}}^{(t)}\|_1 + \frac{G_1}{2(1 - \beta_1)^{2} \sqrt{1 - \beta_2}} \sum_{k=1}^{t} \alpha_k^2. \tag{B.9}
\]

Notice that \( \sum_{j=k}^{K} \beta_1^{j-1} = \frac{1 - \beta_1^{K-k+1}}{1 - \beta_1} \leq 1 \). Hence, when \( \alpha_k = \frac{G_1}{\sqrt{t}} \), it holds \( \sum_{k=1}^{K} \alpha_k^2 \geq \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-1} \right) \geq \alpha \sqrt{K} \). By the convexity of \( F \), we have \( \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-1} \right) F(\mathbf{x}^{(k)}) \leq \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-1} \right) F(\mathbf{x}^{(k)}) \). In addition, we have by (B.6) and Assumption 3, \( \mathbb{E} \|\mathbf{\hat{q}}^{(t)}\|_1 \leq n G_\infty \). Now dividing by \( \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-1} \right) \) both sides of (B.9), and letting \( \mathbf{x} = \mathbf{x}^* \) and \( t = K \), we obtain the desired result.
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Proof of Theorem 2 The proof of Theorem 1 still follows up to (B.9). When \( \alpha_k = \frac{\alpha}{\sqrt{k}} \), it holds
\[
\sum_{k=1}^{K} \alpha_k^2 = \sum_{k=1}^{K} \frac{\alpha^2}{k} \leq \alpha^2 + \int_{1}^{K} \frac{\alpha^2}{x} \, dx \leq \alpha^2 (1 + \log K)
\]
and
\[
\sum_{k=1}^{K} \sum_{j=1}^{k} \alpha_j \beta_1^{j-k} \geq \sum_{k=1}^{K} \frac{\alpha}{\sqrt{k}} \geq \alpha \int_{1}^{K+1} \frac{\alpha}{\sqrt{x}} \, dx \geq 2\alpha (\sqrt{K+1} - 1).
\]
By the convexity of \( F \), we have \( \sum_{i=1}^{K} \left( \sum_{j=1}^{K} \alpha_j \beta_1^{j-k} \right) F(\bar{x}(K)) \leq \sum_{i=1}^{K} \left( \sum_{j=1}^{K} \alpha_j \beta_1^{j-k} \right) F(x(k)). \) In addition, we have by (B.6) and Assumption 3, \( \mathbb{E}[\sqrt{\langle v \rangle}] \leq \alpha G \). Now dividing by \( K \), we have
\[
\mathbb{E}[\sum_{i=1}^{K} \left( \sum_{j=1}^{K} \alpha_j \beta_1^{j-k} \right) F(x(k)) - F(x)] \leq \frac{\alpha}{2} \mathbb{E}[\|\bar{x}(K) - \bar{x}(k)\|^2].
\]
Plugging the above two inequalities into (B.10), we have
\[
\mathbb{E}[\sum_{i=1}^{K} \left( \sum_{j=1}^{K} \alpha_j \beta_1^{j-k} \right) F(x(k)) - F(x)] \leq \frac{\alpha}{2} \mathbb{E}[\|\bar{x}(K) - \bar{x}(k)\|^2] + \frac{\mathbb{E}[\sum_{i=1}^{K} \left( \sum_{j=1}^{K} \alpha_j \beta_1^{j-k} \right) F(x(k)) - F(x)]}{2}.
\]
which together with (B.8) gives the desired result.

Proof of Lemma 3 Recall \( x^{(k+1)} = \text{Proj}_{X, \sqrt{\langle v \rangle}, \alpha m} \left( x^{(k)} - \alpha \sqrt{\langle v \rangle} \right) \). By our definition on \( \text{Proj}_{X, \sqrt{\langle v \rangle}, \alpha m} \), if \( \tilde{v}_j^{(k)} = 0 \), then \( x_i^{(k+1)} = x_i^{(k)} \). Hence, when \( X = [a_1, b_1] \times \cdots \times [a_n, b_n] \), we have \( x^{(k+1)} = \text{Proj}_{X} \left( x^{(k)} - \alpha \sqrt{\langle v \rangle} \right) \). In addition, notice that \( \tilde{v}_i^{(k)} = \text{Proj}_{X} \left( x^{(k+1)} \right) \), and thus the desired result follows from the non-expansiveness of the projection onto a convex set.

Proof of Lemma 4 For each \( i \in [n] \), let \( G_i^{(k)} = \max_{j \leq k} |g_i^{(j)}| \), and \( G^{(k)} \) be the vector with the \( i \)-th component \( G_i^{(k)} \). Then it follows from (B.6) that \( \tilde{v}_i^{(k)} \geq (1 - \beta_2) G_i^{(k)} \). Hence, for \( j \leq k \),
\[
\|g^{(j)} \cap \sqrt{\langle v \rangle}\|^2 \leq \frac{1}{1 - \beta_2} \sum_{i=1}^{n} \left( G_i^{(j)} \right)^2 \leq \frac{\|g^{(j)}\|_0}{1 - \beta_2},
\]
which gives (3.7a). Furthermore, by (B.7), it holds
\[
\|m^{(k)} \cap \sqrt{\langle v \rangle}\| \leq \sum_{j=1}^{k} (1 - \beta_1) \|g^{(j)} \cap \sqrt{\langle v \rangle}\| \leq \sum_{j=1}^{k} (1 - \beta_1) \sqrt{\langle v \rangle} \frac{\|g^{(j)}\|_0}{1 - \beta_2},
\]
which proves (3.7b). The above inequality together with the Cauchy-Schwarz inequality implies (3.7c). Hence, we complete the proof.
Proof of Theorem 4  By (3.6) and (3.7c), we have $E\|x^{(k-l+1)} - x^{(k-l)}\|^2 \leq \frac{\tau \alpha^2_1}{1-\beta_2}$. Since $\tau_k \leq \tau, \forall k \geq 1$, it follows from (A.3) that

$$E\|x^{(k)} - \bar{x}^{(k)}\|^2 \leq \tau \sum_{l=1}^{\tau} E\|x^{(k-l+1)} - x^{(k-l)}\|^2.$$  

In addition, notice that

$$\sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-k} \right) \left( \tau \sum_{l=1}^{\tau} \alpha_{k-l}^2 \right) \leq \frac{\tau^2 \alpha^3}{(1-\beta_1)\sqrt{K}}.$$  

Therefore

$$\frac{L(1-\beta_1)}{2} \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-k} \right) E\|x^{(k)} - \bar{x}^{(k)}\|^2 \leq \frac{\alpha^2 L \tau^2 s}{2(1-\beta_2) \sqrt{1-\beta_2}}.$$  

Plugging the above inequality into (3.5) with $t = K$ and using $\sum_{k=1}^{K} \alpha_k^2 = \alpha^2$ give

$$\begin{align*}
(1-\beta_1) \sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-k} \right) E[F(x^{(k)}) - F(x)] 
\leq \frac{D^2 F}{2} E\|\tilde{\alpha}\|_1 + \frac{\alpha^2}{2(1-\beta_1)^2} \frac{G_1}{\sqrt{1-\beta_2}} + \frac{\alpha^2 L \tau^2 s}{2(1-\beta_2) \sqrt{1-\beta_2}}.
\end{align*}$$

Now using the definition of $\bar{x}^{(K)}$ and noting $\sum_{k=1}^{K} \left( \sum_{j=k}^{K} \alpha_j \beta_1^{j-k} \right) \geq \alpha \sqrt{K}$, we obtain the result in (3.8) and complete the proof.

C Technical details of section 4

We introduce the following definition of gradient bounds.

**Definition 2** Given a positive integer $K$, let $\{g^{(k)}\}_{k=1}^{K}$ and $\{x^{(k)}\}_{k=1}^{K}$ be computed as in Algorithm 1. We define $(\Gamma(K), \Phi(K))$ as:

$$\begin{align*}
\Gamma_i(K) &= \max_{1 \leq k \leq K} \|g_i^{(k)}\|_1, \quad \Phi_i(K) = \max_{1 \leq k \leq K} \|\nabla_i F(x^{(k)})\|_1, \forall i \in [n].
\end{align*}$$

We abbreviate the pair as $(\Gamma, \Phi)$ to hide the dependence on $K$, when it is clear from the context.

**Lemma 5**  Given a positive integer $K$, let $\{\bar{x}^{(k)}\}_{k=1}^{K}$ and $\{m^{(k)}\}_{k=1}^{K}$ be generated from Algorithm 1, and let $(\Gamma, \Phi)$ be given in Definition 2. We have for all $i \in [n]$, $|m_i^{(k)}| \leq \Gamma_i$, and $\bar{v}_i^{(k)} \leq \Gamma_i^2$. Moreover, if Assumption 3 holds, then $\Gamma_i \leq G_\infty$ almost surely, and $\Phi_i \leq G_\infty$ for all $i \in [n]$.

**Proof.** From (B.7), we have $m_i^{(k)} = (1-\beta_1) \sum_{j=1}^{k} \beta_{1-j} g_i^{(j)}$, and thus applying triangle inequility and using the definition of $r$ in (C.1) lead to $|m_i^{(k)}| \leq (1-\beta_1^k) \Gamma_i \leq \Gamma_i$. A similar argument gives $\bar{v}_i^{(k)} \leq \Gamma_i^2$.

When Assumption 3 holds, we know that $\|g^{(k)}\|_\infty \leq G_\infty$ almost surely, and that $\|\nabla F(x^{(k)})\|_\infty \leq G_\infty$, for all $k \in [K]$, which leads to the second part of this lemma.

We follow the analytical framework of [28]. Let $x^{(0)} = x^{(1)}$, and we define an auxiliary sequence $z^{(k)}$ as follows:

$$z^{(k)} = x^{(k)} + \frac{\beta_1}{1-\beta_1}(x^{(k)} - x^{(k-1)}) = \frac{1}{1-\beta_1} x^{(k)} - \frac{\beta_1}{1-\beta_1} x^{(k-1)}, \forall k \geq 1.$$  

The following lemma is from Lemma A.3 of [29]. It shows that $z^{(k+1)} - z^{(k)}$ can be represented in two different ways. However, due to typos in the original proof, we provide a complete proof here for the convenience of the readers.
Lemma 6 Let \( \mathbf{z}^{(k)} \) be defined as in (C.2). We have
\[
\mathbf{z}^{(2)} - \mathbf{z}^{(1)} = -\alpha_1 (\overline{\mathbf{V}}^{(1)})^{-\frac{1}{2}} \mathbf{g}^{(1)},
\]
and for \( k = 2, \ldots, K \),
\[
\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)} = \frac{\beta_1}{1 - \beta_1} \left[ (\alpha_{k-1}(\overline{\mathbf{V}}^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}}) \mathbf{m}^{(k-1)} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)} \right],
\]
and
\[
\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)} = \frac{\beta_1}{1 - \beta_1} \left[ \mathbf{I} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \alpha_{k-1}(\overline{\mathbf{V}}^{(k-1)})^{-\frac{1}{2}} \right] \left( \mathbf{x}^{(k-1)} - \mathbf{x}^{(k)} \right) - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)}.
\]
Proof. From (C.2), we have that for \( k \geq 1 \),
\[
\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)} = \frac{1}{1 - \beta_1} (\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) - \frac{\beta_1}{1 - \beta_1} (\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)})
\]
\[
= \frac{1}{1 - \beta_1} \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{m}^{(k)} + \frac{\beta_1}{1 - \beta_1} \alpha_{k-1}(\overline{\mathbf{V}}^{(k-1)})^{-\frac{1}{2}} \mathbf{m}^{(k-1)},
\]
where in the second equation, we have used (4.1) and Remark 2. With \( k = 1 \), the above equation gives (C.3) by utilizing the update of \( \mathbf{m}^{(1)} \). Furthermore, it gives, by plugging the update of \( \mathbf{m}^{(k)} \),
\[
\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)} = \frac{-1}{1 - \beta_1} \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} (\beta_1 \mathbf{m}^{(k)} + (1 - \beta_1) \mathbf{g}^{(k)}) + \frac{\beta_1}{1 - \beta_1} \alpha_{k-1}(\overline{\mathbf{V}}^{(k-1)})^{-\frac{1}{2}} \mathbf{m}^{(k-1)} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)}
\]
\[
= \frac{\beta_1}{1 - \beta_1} \left[ (\alpha_{k-1}(\overline{\mathbf{V}}^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}}) \mathbf{m}^{(k-1)} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)} \right]
\]
\[
= \frac{\beta_1}{1 - \beta_1} \left[ (\mathbf{x}^{(k-1)} - \mathbf{x}^{(k)}) - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)} \right] \mathbf{m}^{(k-1)} - \alpha_k(\overline{\mathbf{V}}^{(k)})^{-\frac{1}{2}} \mathbf{g}^{(k)}.
\]
The second equation of the above is exactly (C.4), and the last equation gives (C.5).
\]
where the last equation follows because \( \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} \geq \alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} > 0 \) component-wisely. Similarly, we can bound the second term on the right-hand side of (C.7) as follows:

\[
- \nabla F(x^{(k)})^\top \alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)} \\
= - \nabla F(x^{(k)})^\top \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \nabla F(x^{(k)})^\top \left[ \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} \right] g^{(k)} \\
= - \nabla F(x^{(k)})^\top \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \sum_{i=1}^{n} \nabla_i F(x^{(k)}) \left[ \alpha_{k-1}(\tilde{v}_i^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\tilde{v}_i^{(k)})^{-\frac{1}{2}} \right] g_i^{(k)} \\
\leq - \nabla F(x^{(k)})^\top \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \sum_{i=1}^{n} \phi_i \left[ \alpha_{k-1}(\tilde{v}_i^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\tilde{v}_i^{(k)})^{-\frac{1}{2}} \right] I_i \\
= - \nabla F(x^{(k)})^\top \alpha_{k-1}(\tilde{v}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \sum_{i=1}^{n} I_i \phi_i \left[ \alpha_{k-1}(\tilde{v}_i^{(k-1)})^{-\frac{1}{2}} - \alpha_k(\tilde{v}_i^{(k)})^{-\frac{1}{2}} \right]. \tag{C.9}
\]

Now substituting (C.8) and (C.9) into (C.7) yields (C.6). \(\square\)

The next two lemmas are directly from [29]. Although the original results are for \( k \geq 2 \), they trivially hold when \( k = 1 \).

**Lemma 8 (Lemma A.4 of [29])** Let \( \{z^{(k)}\} \) be defined as in (C.2), and let \( \{\alpha_k\}_{k \geq 1} \) be a non-increasing positive sequence. For \( k \geq 1 \), we have

\[
\|z^{(k+1)} - z^{(k)}\| \leq \frac{\beta_1}{1 - \beta_1} \|x^{(k+1)} - x^{(k)}\| + \|\alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)}\|. \tag{C.10}
\]

**Lemma 9 (Lemma A.5 of [29])** Let \( \{z^{(k)}\} \) be defined as in (C.2). For \( k \geq 1 \), we have

\[
\|\nabla F(z^{(k)}) - \nabla F(x^{(k)})\| \leq \frac{L\beta_1}{1 - \beta_1} \|x^{(k+1)} - x^{(k)}\|. \tag{C.11}
\]

We still need the following lemma to show our main convergence result for the nonconvex case.

**Lemma 10** Let \( \{z^{(k)}\} \) be defined as in (C.2), and let \( \{\alpha_k\}_{k \geq 1} \) be a non-increasing positive sequence. For \( k \geq 1 \), we have

\[
\left( \nabla F(z^{(k)}) - \nabla F(x^{(k)}) \right)^\top \left( z^{(k+1)} - z^{(k)} \right) \leq \frac{3L\beta_1^2}{2(1 - \beta_1)^2} \|x^{(k+1)} - x^{(k)}\|^2 \plus \frac{L}{2} \|\alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)}\|^2, \tag{C.12}
\]

and

\[
\|z^{(k+1)} - z^{(k)}\|^2 \leq \frac{4\beta_1^2}{(1 - \beta_1)^2} \|x^{(k+1)} - x^{(k)}\|^2 \plus \frac{4}{3} \|\alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)}\|^2. \tag{C.13}
\]

**Proof.** From (C.10) and the fact \( (a + b)^2 \leq 4a^2 + 4b^2; \forall a, b \in \mathbb{R} \), the inequality in (C.13) immediately follows. By the Cauchy-Schwarz inequality, and also (C.11) and (C.10), it holds

\[
\left( \nabla F(z^{(k)}) - \nabla F(x^{(k)}) \right)^\top \left( z^{(k+1)} - z^{(k)} \right) \\
\leq \|\nabla F(z^{(k)}) - \nabla F(x^{(k)})\| \cdot \|z^{(k+1)} - z^{(k)}\| \\
\leq \frac{L\beta_1}{1 - \beta_1} \|x^{(k+1)} - x^{(k)}\left( \frac{\beta_1}{1 - \beta_1} \|x^{(k+1)} - x^{(k)}\| + \|\alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)}\|\right) \\
= \frac{L\beta_1^2}{(1 - \beta_1)^2} \|x^{(k+1)} - x^{(k)}\|^2 \plus \beta_1 L \|x^{(k+1)} - x^{(k)}\| \cdot \|\alpha_k(\tilde{v}^{(k)})^{-\frac{1}{2}} g^{(k)}\|.
\]

Now using the Young’s inequality, we have (C.12) from the above inequality. \(\square\)
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Proof of Theorem 5

By the $L$-smoothness of $F$, it follows from (B.11) that

$$F(z^{(k+1)}) \leq F(z^{(k)}) + \nabla F(z^{(k)})^T (z^{(k+1)} - z^{(k)}) + \frac{L}{2} \|z^{(k+1)} - z^{(k)}\|^2$$

$$= F(z^{(k)}) + \nabla F(x^{(k)})^T (z^{(k+1)} - z^{(k)})$$

(C.14)

$$+ \left( \nabla F(z^{(k)}) - \nabla F(x^{(k)}) \right)^T (z^{(k+1)} - z^{(k)}) + \frac{L}{2} \|z^{(k+1)} - z^{(k)}\|^2.$$ 

When $k = 1$, by (C.3), (C.12) and (C.13), and recalling $x^{(0)} = x^{(1)}$, we have

$$F(x^{(2)}) - F(x^{(1)}) \leq - \nabla F(x^{(1)})^T \alpha_1 (\tilde{V}^{(1)})^{-\frac{1}{2}} g^{(1)} + \frac{7L}{6} \|\alpha_1 (\tilde{V}^{(1)})^{-\frac{1}{2}} g^{(1)}\|^2$$

$$\leq \frac{\alpha_1\|g^{(1)}\|}{\sqrt{1 - \beta_2}} + \frac{7L}{6} \|\alpha_1 (\tilde{V}^{(1)})^{-\frac{1}{2}} g^{(1)}\|^2$$

(C.15)

where the second inequality holds because

$$-\nabla F(x^{(1)})^T (\tilde{V}^{(1)})^{-\frac{1}{2}} g^{(1)} = -\nabla F(x^{(1)})^T \left( g^{(1)} \odot \sqrt{(1 - \beta_2)(g^{(1)})^2} \right) = \sum_{i: g^{(1)}_i \neq 0} -\nabla_i F(x^{(1)}) g^{(1)}_i$$

and $-\nabla_i F(x^{(1)}) g^{(1)}_i \leq \|\Phi\|\|g^{(1)}\|$ by Definition 2.

When $2 \leq k \leq K$, we have, by substituting (C.6), (C.12) and (C.13) into (C.14) and rearranging terms, that

$$F(z^{(k+1)}) + \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k)}_i)^{-\frac{1}{2}} - \left( F(z^{(k)}) + \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k-1)}_i)^{-\frac{1}{2}} \right)$$

$$\leq - \nabla F(x^{(k)})^T \alpha_k (\tilde{V}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \frac{7L\beta_2}{2(1 - \beta_1)} \|x^{(k-1)} - x^{(k)}\|^2 + \frac{7L}{6} \|\alpha_k (\tilde{V}^{(k)})^{-\frac{1}{2}} g^{(k)}\|^2$$

(C.16)

$$= - \nabla F(x^{(k)})^T \alpha_k (\tilde{V}^{(k-1)})^{-\frac{1}{2}} g^{(k)} + \frac{7L\beta_2}{2(1 - \beta_1)} \|\alpha_k m^{(k-1)} \odot \tilde{V}^{(k-1)}\|^2 + \frac{7L}{6} \|\alpha_k (\tilde{V}^{(k)})^{-\frac{1}{2}} g^{(k)}\|^2.$$ 

Since $E(\tilde{v}^{(k)}) = \nabla F(x^{(k)})$, taking the expectation on both sides of (C.16) results in

$$E \left[ F(z^{(k+1)}) + \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k)}_i)^{-\frac{1}{2}} - F(z^{(k)}) - \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k-1)}_i)^{-\frac{1}{2}} \right]$$

$$\leq E \left[ F(z^{(k+1)}) + \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k)}_i)^{-\frac{1}{2}} - F(z^{(k)}) - \sum_{i=1}^n \Gamma_i \Phi_i \alpha_k (\tilde{v}^{(k-1)}_i)^{-\frac{1}{2}} \right]$$

$$\leq E \left[ \frac{7L}{6} \|\alpha_k (\tilde{V}^{(k)})^{-\frac{1}{2}} g^{(k)}\|^2 - \nabla F(x^{(k)})^T \alpha_k (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) + \frac{7L\beta_2}{2(1 - \beta_1)} \|\alpha_k m^{(k-1)} \odot \tilde{V}^{(k-1)}\|^2 \right]$$

(C.17)
where the first inequality is true by $\alpha_k (\tilde{e}_i^{(k)})^{-\frac{1}{2}} - \alpha_{k-1} (\tilde{e}_i^{(k-1)})^{-\frac{1}{2}} \leq 0$ for all $i \in [n]$, and Lemma 5. By the Cauchy-Schwarz inequality, the smoothness of $F$, (A.2), Assumption 6, and (4.1), we have

$$\nabla F(x^{(k)})^\top (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \left( \nabla F(x^{(k)}) - \nabla F(\tilde{x}^{(k)}) \right) \leq \left\| (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right\| \cdot \left\| \nabla F(x^{(k)}) - \nabla F(\tilde{x}^{(k)}) \right\| \leq L t \left( \tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right) \left( \sum_{j=1}^{r} \|\alpha_{k-j} m^{(k-j)} \odot \sqrt{V^{(k-j)}} \| \right),$$

In addition, by Definition 1 and Lemma 5, it follows

$$E \left[ \nabla F(x^{(k)})^\top (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right] \geq \left[ \nabla F(x^{(k)})^\top (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right] \geq G^{-1}_{\infty} E \left[ \nabla F(x^{(k)}) \right]^{2}.$$

Substituting the above two inequalities into (C.17), we have

$$E \left[ F(z^{(k+1)}) + \frac{G^2_{\infty} \alpha_k \| (\tilde{V}^{(k)})^{-\frac{1}{2}} l \|}{1 - \beta_1} - F(z^{(k)}) - \frac{G^2_{\infty} \alpha_{k-1} \| (\tilde{V}^{(k-1)})^{-\frac{1}{2}} l \|}{1 - \beta_1} \right]$$

$$\leq E \left[ \frac{7L}{6} \|\alpha_k (\tilde{V}^{(k)})^{-\frac{1}{2}} l \|^{2} + \alpha_{k-1} L \left[ \| (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right) \left( \sum_{j=1}^{r} \|\alpha_{k-j} m^{(k-j)} \odot \sqrt{V^{(k-j)}} \| \right) \right]$$

$$\left[ \alpha_{k-1} G^{-1}_{\infty} E \|\nabla F(x^{(k)})\|^{2} + \frac{7L \beta_1^2}{2(1 - \beta_1)^2} E \|\alpha_{k-1} m^{(k-1)} \odot \sqrt{V^{(k-1)}} \|^{2} \right].$$

(C.18)

For any $2 \leq k_0 \leq K$, summing (C.18) over $k = k_0, \ldots, K$, we have

$$G^{-1}_{\infty} \sum_{k=k_0}^{K} \alpha_{k-1} E \|\nabla F(x^{(k)})\|^{2}$$

$$\leq E \left[ F(z^{(k_0)}) - F(z^{(K+1)}) + \frac{G^2_{\infty} \alpha_{k_0-1} \| (\tilde{V}^{(k_0-1)})^{-\frac{1}{2}} l \|}{1 - \beta_1} - \frac{G^2_{\infty} \alpha_K \| (\tilde{V}^{(K)})^{-\frac{1}{2}} l \|}{1 - \beta_1} \right]$$

$$+ \frac{7L}{6} \sum_{k=k_0}^{K} \left[ E \|\alpha_k (\tilde{V}^{(k)})^{-\frac{1}{2}} l \|^{2} \right] + \frac{7L \beta_1^2}{2(1 - \beta_1)^2} \sum_{k=k_0}^{K} \left[ E \|\alpha_{k-1} m^{(k-1)} \odot \sqrt{V^{(k-1)}} \|^{2} \right]$$

$$+ L \sum_{k=k_0}^{K} \alpha_{k-1} E \left[ \| (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right) \left( \sum_{j=1}^{r} \|\alpha_{k-j} m^{(k-j)} \odot \sqrt{V^{(k-j)}} \| \right) \right]$$

$$\leq E \left[ F(z^{(k_0)}) - \inf_{x} F(x) + \frac{G^2_{\infty} \alpha_{k_0-1} \| (\tilde{V}^{(k_0-1)})^{-\frac{1}{2}} l \|}{1 - \beta_1} \right]$$

$$+ \frac{7L}{6} \sum_{k=k_0}^{K} E \|\alpha_k l \| \odot \sqrt{V^{(k)}} \|^{2} + \frac{7L \beta_1^2}{2(1 - \beta_1)^2} \sum_{k=k_0}^{K} E \|\alpha_{k-1} m^{(k-1)} \odot \sqrt{V^{(k-1)}} \|^{2}$$

$$+ L \sum_{k=k_0}^{K} \alpha_{k-1} E \left[ \| (\tilde{V}^{(k-1)})^{-\frac{1}{2}} \nabla F(x^{(k)}) \right) \left( \sum_{j=1}^{r} \|\alpha_{k-j} m^{(k-j)} \odot \sqrt{V^{(k-j)}} \| \right) \right].$$

(C.19)
Now use Lemma 4 in the above inequality to have
\[ G_{k_0}^{-1} \sum_{k=k_0}^{K} \alpha_{k-1} E[\|\nabla F(x(k))\|^2] \]
\[ \leq E[F(x^{(k_0)}) - \inf_{x} F(x)] + G_{k_0}^2 \frac{\alpha_{k_0-1} E[\|\nabla F(x^{(k_0-1)})\|^2]}{1 - \beta_1} \sum_{k=k_0}^{K} \alpha_{k}^2 + \frac{7sL}{6(1 - \beta_2)} \sum_{k=k_0}^{K} \frac{\alpha_{k}^2}{(1 - \beta_2)(1 - \beta_1)^2} \sum_{k=k_0}^{K} \alpha_{k-1}^2 \]
\[ + \frac{L}{\sqrt{1 - \beta_2}} \sum_{k=k_0}^{K} \alpha_{k-1} \sum_{j=1}^{r} \alpha_{k-j} \sum_{l=1}^{r} (1 - \beta_1)\beta_{l}^{k-j-l} E \left[ \|\nabla F(x^{(k-l)})\|^2 \right] \]
\[ \leq E[F(x^{(k_0)}) - \inf_{x} F(x)] + G_{k_0}^2 \frac{\alpha_{k_0-1} E[\|\nabla F(x^{(k_0-1)})\|^2]}{1 - \beta_1} \sum_{k=k_0}^{K} \alpha_{k}^2 + \frac{7sL}{6(1 - \beta_2)} \sum_{k=k_0}^{K} \frac{\alpha_{k}^2}{(1 - \beta_2)(1 - \beta_1)^2} \sum_{k=k_0}^{K} \alpha_{k-1}^2 \]
\[ + \frac{7sL}{2(1 - \beta_2)(1 - \beta_1)^2} \sum_{k=k_0}^{K} \alpha_{k-1}^2 + \frac{\sqrt{sL}}{\sqrt{1 - \beta_2}} \sum_{k=k_0}^{K} \sum_{k=k_0}^{K} \alpha_{k-1} \sum_{j=1}^{r} \sqrt{E[\|\nabla F(x^{(k-l)})\|^2]} \] \[ \sum_{j=1}^{r} \alpha_{k-j}, \quad (C.20) \]
where the last inequality is by Cauchy-Schwarz inequality. We obtain (4.4) by dividing \( G_{k_0}^{-1} \sum_{k=k_0}^{K} \alpha_{k-1} \) on both sides of (C.20) and using the definition of \( x^{(k_0, K)} \). To have (4.2), we take expectation on both sides of (C.15), add it to (C.20) with \( k_0 = 2 \), and then divide both sides of the resulting inequality by \( G_{k_0}^{-1} \sum_{k=2}^{K} \alpha_{k-1} \).

**Proof of Corollary 1**
Plugging \( \alpha_k = \frac{\alpha}{\sqrt{k - k_0 + 1}} \), \( \forall 1 \leq k \leq K \) into (4.4), we immediately have
\[ E[\|\nabla F(x^{(k_0, K)})\|^2] \leq C_1 + \frac{C_2}{K - k_0 + 1} \sqrt{E[\|\nabla F(x^{(k_0, K)})\|^2]}, \quad (C.21) \]
Since \( \tilde{v}_{i}^{(k_0-1)} \geq c^2, \forall i \in [n] \) almost surely and \( \tilde{v}_{i}^{(k+1)} \geq \tilde{v}_{i}^{(k)}, \forall k \geq 1 \), it holds
\[ \frac{1}{K - k_0 + 1} \sum_{k=k_0}^{K} \sqrt{E[\|\nabla F(x^{(k)})\|^2]} \leq \frac{1}{c (K - k_0 + 1)} \sum_{k=k_0}^{K} \sqrt{E[\|\nabla F(x^{(k)})\|^2]} \leq \frac{1}{c} \sqrt{E[\|\nabla F(x^{(k_0, K)})\|^2]}, \] \[ \text{where the last inequality follows from Jensen's inequality. Hence, plugging the above inequality into (C.21) yields} \]
\[ E[\|\nabla F(x^{(k_0, K)})\|^2] \leq C_1 + \frac{C_2}{c} \sqrt{E[\|\nabla F(x^{(k_0, K)})\|^2]}, \] \[ \text{(C.22)} \]
which implies \( \sqrt{E[\|\nabla F(x^{(k_0, K)})\|^2]} \leq \sqrt{C_1 + \frac{C_2}{c}} \). Therefore, we have the desired result by using (C.22) and the obtained bound on \( \sqrt{E[\|\nabla F(x^{(k_0, K)})\|^2]} \).

**Proof of Corollary 2**
With \( \alpha_k = \frac{\alpha}{\sqrt{k}}, \forall 1 \leq k \leq K \), we have
\[ \sum_{k=k_0}^{K} \alpha_{k-1} \geq \int_{k_0-1}^{K} d\alpha = 2\alpha(\sqrt{K} - \sqrt{k_0 - 1}) \geq (2 - \sqrt{2})\alpha \sqrt{K}, \]
and
\[ \sum_{k=k_0}^{K} \alpha_{k}^2 = \int_{k_0-1}^{K} \frac{\alpha^2}{x} d\alpha = \alpha^2 \log \frac{K}{k_0} - 1 = \alpha^2 \log \frac{1}{2 + K} \leq \alpha^2 \log 4. \] \[ \text{Similarly,} \]
\[ \sum_{k=k_0}^{K} \alpha_{k-1}^2 \leq \frac{\alpha^2}{k_0 - 1} + \int_{k_0-1}^{K-1} \frac{\alpha^2}{x} d\alpha \leq \alpha^2 \log \frac{K - 1}{k_0 - 1} \leq \alpha^2 (1 + \log 3), \]
and for all $k \geq k_0 \geq \tau + 2$,
\[
\sum_{j=1}^{r} \alpha_{k-j} \leq \sum_{j=1}^{r} \frac{\alpha}{\sqrt{k_0}} \leq \int_{k_0-\tau}^{k_0-1} \frac{\alpha}{\sqrt{x}} dx = 2\alpha(\sqrt{k_0} - \sqrt{k_0 - \tau} - 1) \leq \frac{2\alpha\tau}{\sqrt{k_0}}.
\]

Therefore, plugging $\alpha_k = \frac{\alpha}{\sqrt{k}}$, $\forall 1 \leq k \leq K$ into (4.4) and using the above inequalities, we have
\[
\mathbb{E}\|\nabla F(\bar{x}^{(k_0,K)})\|^2 \leq C_1 + \frac{C_2}{\sum_{k=k_0}^{K} \alpha_{k-1}} \sum_{k=k_0}^{K} \alpha_{k-1} \sqrt{\mathbb{E}\left[\|\nabla (k-1) - \frac{1}{2} \nabla F(\bar{x}^{(k)})\|^2\right]}.
\]

Notice $v_i^{(k_0-1)} \geq c^2, \forall i \in [n]$ almost surely, and also use the definition of $\bar{x}^{(k_0,K)}$ in (4.3). We have, by Jensen’s inequality,
\[
\frac{1}{\sum_{k=k_0}^{K} \alpha_{k-1}} \sum_{k=k_0}^{K} \alpha_{k-1} \sqrt{\mathbb{E}\left[\|\nabla (k-1) - \frac{1}{2} \nabla F(\bar{x}^{(k)})\|^2\right]} \leq \frac{1}{c} \sqrt{\mathbb{E}\|\nabla F(\bar{x}^{(k_0,K)})\|^2}.
\]

Now by the same arguments as those in the proof of Corollary 1, we obtain the desired result.

## D Additional numerical results

| #core | real-sim dataset | rcv1 dataset |
|-------|-----------------|--------------|
|       | time (sec.) | speed-up | time (sec.) | speed-up |
|       | sync | async | sync | async | sync | async | sync | async |
| 1     | 97.25 | 97.25 | 1.00 | 1.00 | 60.78 | 60.78 | 1.00 | 1.00 |
| 2     | 50.29 | 49.14 | 1.93 | 1.98 | 31.68 | 30.91 | 1.92 | 1.97 |
| 4     | 25.12 | 22.00 | 3.55 | 3.87 | 17.27 | 15.79 | 3.52 | 3.85 |
| 8     | 12.35 | 10.72 | 5.72 | 5.72 | 10.59 | 8.49 | 5.74 | 7.16 |
| 16    | 7.45  | 7.06  | 13.05 | 13.05 | 8.41 | 7.23 | 13.72 | 12.72 |
| 32    | 4.58  | 5.71  | 21.24 | 21.24 | 10.38 | 9.81 | 20.87 | 20.87 |

Table 4 Comparison of the sync-parallel and async-parallel AMSGrad. Mini-batch is set to 128 to compute each sample gradient. For each case, all methods run to 20 epochs.

| #core | real-sim dataset | rcv1 dataset |
|-------|-----------------|--------------|
|       | time (sec.) | speed-up | time (sec.) | speed-up |
|       | sync | async | sync | async | sync | async | sync | async |
| 1     | 92.63 | 92.63 | 1.00 | 1.00 | 58.63 | 58.63 | 1.00 | 1.00 |
| 2     | 47.61 | 46.92 | 1.95 | 1.97 | 30.08 | 29.32 | 1.95 | 2.00 |
| 4     | 25.18 | 23.94 | 3.68 | 3.87 | 15.99 | 15.11 | 3.67 | 3.88 |
| 8     | 14.66 | 12.65 | 6.32 | 7.32 | 9.28 | 7.96 | 6.32 | 7.37 |
| 16    | 10.35 | 6.95  | 8.95 | 13.33 | 6.75 | 4.69 | 8.69 | 12.49 |
| 32    | 10.89 | 4.21  | 8.51 | 22.02 | 6.59 | 2.79 | 8.90 | 21.02 |

Table 5 Comparison of the sync-parallel and async-parallel AMSGrad. Mini-batch is set to 256 to compute each sample gradient. For each case, all methods run to 20 epochs.
Asynchronous parallel adaptive stochastic gradient methods

References

1. A. Agarwal and J. C. Duchi. Distributed delayed stochastic optimization. In Advances in Neural Information Processing Systems, pages 873–881, 2011.
2. K. Bäckström, M. Papatriantafilou, and P. Tsiganis. Mindthestep-asyncsgd: Adaptive asynchronous parallel stochastic gradient descent. arXiv preprint arXiv:1911.03444, 2019.
3. D. P. Bertsekas and J. N. Tsitsiklis. Some aspects of parallel and distributed iterative algorithms—a survey. Automatica, 27(1):3–21, 1991.
4. C.-C. Chang and C.-J. Lin. Libsvm: A library for support vector machines. ACM transactions on intelligent systems and technology (TIST), 2(3):1–27, 2011.
5. J. Chen and Q. Gu. Closing the generalization gap of adaptive gradient methods in training deep neural networks. arXiv preprint arXiv:1806.06765, 2018.
6. X. Chen, S. Liu, R. Sun, and M. Hong. On the convergence of a class of adam-type algorithms for non-convex optimization. In International Conference on Learning Representations, 2019.
7. J. Duchi, E. Hazan, and Y. Singer. Adaptive subgradient methods for online learning and stochastic optimization. Journal of Machine Learning Research, 12(Jul):2121–2159, 2011.
8. B. Fang and D. Klabjan. Convergence analyses of online adam algorithm in convex setting and two-layer relu neural network. arXiv preprint arXiv:1905.09356, 2019.
9. H. R. Feyzmahdavian, A. Aytekin, and M. Johansson. An asynchronous mini-batch algorithm for regularized stochastic optimization. IEEE Transactions on Automatic Control, 61(12):3740–3754, 2016.
10. D. P. Kingma and J. Ba. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.
11. Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.
12. X. Lian, Y. Huang, Y. Li, and J. Liu. Asynchronous parallel stochastic gradient for nonconvex optimization. In Advances in Neural Information Processing Systems, pages 2737–2745, 2015.
13. X. Lian, W. Zhang, C. Zhang, and J. Liu. Asynchronous decentralized parallel stochastic gradient descent. In International Conference on Machine Learning, pages 3043–3052, 2018.
14. J. Liu, S. Wright, C. Ré, V. Bittorf, and S. Sridhar. An asynchronous parallel stochastic coordinate descent algorithm. In International Conference on Machine Learning, pages 469–477, 2014.
15. L. Luo, Y. Xiong, and Y. Liu. Adaptive gradient methods with dynamic bound of learning rate. In International Conference on Learning Representations, 2019.
16. P. Nazari, D. A. Tarzhanagh, and G. Michailidis. Dadam: A consensus-based distributed adaptive gradient method for online optimization. arXiv preprint arXiv:1901.09109, 2019.
17. A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. SIAM Journal on optimization, 19(4):1574–1609, 2009.
18. Z. Peng, Y. Xu, M. Yan, and W. Yin. Arock: an algorithmic framework for asynchronous parallel coordinate updates. SIAM Journal on Scientific Computing, 38(5):A2851–A2879, 2016.
19. Z. Peng, Y. Xu, M. Yan, and W. Yin. On the convergence of asynchronous parallel iteration with unbounded delays. Journal of the Operations Research Society of China, 7(1):5–42, 2019.
20. B. Recht, C. Ré, S. Wright, and F. Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In Advances in neural information processing systems, pages 693–701, 2011.
21. S. J. Reddi, S. Kale, and S. Kumar. On the convergence of adam and beyond. In International Conference on Learning Representations, 2018.
22. H. Robbins and S. Monro. A stochastic approximation method. The annals of mathematical statistics, pages 400–407, 1951.
23. T. Tieleman and G. Hinton. RMSProp: Divide the gradient by a running average of its recent magnitude. COURSERA: Neural Networks for Machine Learning, 4(2):26–31, 2012.
24. P. T. Tran et al. On the convergence proof of amsgrad and a new version. IEEE Access, 7:61706–61716, 2019.
25. G. Wang, S. Lu, Q. Cheng, W. Tu, and L. Zhang. (SA)dam: A variant of adam for strongly convex functions. In International Conference on Learning Representations, 2020.
26. J. Wu, W. Huang, J. Huang, and T. Zhang. Error compensated quantized sgd and its applications to large-scale distributed optimization. In International Conference on Machine Learning, pages 5325–5333, 2018.
27. H. Xiao, K. Rasul, and R. Vollgraf. Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms, 2017.
28. Y. Yan, T. Yang, Z. Li, Q. Lin, and Y. Yang. A unified analysis of stochastic momentum methods for deep learning. In Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18, pages 2955–2961. International Joint Conferences on Artificial Intelligence Organization, 7 2018.
29. D. Zhou, Y. Tang, Z. Yang, Y. Cao, and Q. Gu. On the convergence of adaptive gradient methods for nonconvex optimization. arXiv preprint arXiv:1808.05671, 2018.