First-order representation of the Faddeev formulation of gravity

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Abstract

We study the Faddeev formulation of gravity in which the metric is composed of a ten-dimensional tetrad $f^A_{\lambda}$. Here, $A = 1, \ldots, 10$ refers to an Euclidean (or Minkowsky) ten-dimensional spacetime and $\lambda = 1, 2, 3, 4$ is the usual world index. A remarkable unique feature of the Faddeev formulation is that the action remains finite for the discontinuous tetrad. If the spacetime is composed of flat microblocks (in the discrete version of gravity), this means that these microblocks in quantum theory do not coincide on their common faces, that is, they are independent. We propose a representation of the Faddeev formulation analogous to the Cartan–Weyl formalism. It is based on extending the set of variables by introducing an infinitesimal SO(10) connection. Excluding this connection via equations of motion we reproduce the original Faddeev action. A peculiar feature of this representation is the occurrence of a local SO(10) symmetry violating condition so that the SO(10) symmetry is only global one in full correspondence with the fact that the original Faddeev formulation possesses just the global SO(10) symmetry. We also consider an analogue of the Barbero–Immirzi parameter which can be naturally introduced in the considered representation.

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1. Introduction

For simplicity, we consider the case of the Euclidean signature. Faddeev has considered [1–4] a set of ten 4-vector (or four 10-vector) fields $f^A_{\lambda}$. Here, the Greek indices $\lambda, \mu, \ldots = 1, 2, 3, 4$, and the Latin capitals $A, B, \ldots = 1, \ldots, 10$. The metric tensor is

$$g_{\lambda\mu} = f^A_{\lambda} f_{\mu A}. \quad (1)$$
The Latin capitals refer to an Euclidean ten-dimensional space. The Greek lower case letters label the four world coordinates and can be raised/lowered in the usual way using the metric tensor.

This formulation looks like the so-called embedded theories of gravity [5, 6]. The curved four-dimensional spacetime in these theories is considered as a four-dimensional hypersurface in a flat spacetime of some larger dimensionality. In the particular case $f^A_{\lambda} = \partial_\lambda f^A$, the metric (1) just corresponds to the hypersurface with the coordinates $f^A$ in a flat ten-dimensional space. Ten is the minimal number of the functions $f^A$ required to parameterize all ten independent components of the metric $g_{\lambda\mu}$. A distinctive feature of the Faddeev formulation is that it considers a more general case when $f^A_{\lambda}$ are independent (freely chosen).

The following value is introduced:

$$\Omega_{\lambda\mu\nu} = f^A_{\lambda} f_{\mu\lambda,\nu}, \quad f_{\mu\lambda,\nu} \equiv \partial_\nu f_{\mu\lambda}.$$

This value possesses the same transformation properties under diffeomorphisms as a connection (say, $\Gamma_{\lambda\mu\nu}$) does. Besides that, we can define the covariant derivative of the metric $\nabla_\mu g_{\lambda\nu}$ using $\Omega_{\lambda\mu\nu}$ as a connection and find that

$$\nabla_\lambda g_{\mu\nu} = 0.$$

That is, $\Omega_{\lambda\mu\nu}$ is a metric-compatible affine connection. The torsion is

$$T_{\lambda\mu\nu} = \Omega_{\lambda\mu\nu} - \Omega_{\nu\mu\lambda}.$$

The curvature tensor is

$$S_{\lambda\mu\nu\rho} = \Omega_{\mu\rho,\nu} - \Omega_{\nu\rho,\mu} + \Omega_{\rho\nu} \Omega_{\mu\rho} - \Omega_{\rho\mu} \Omega_{\nu\rho}.$$

or

$$S_{\lambda\mu\nu\rho} = \Pi^{AB} \left( f^A_{\lambda,\nu} f^B_{\mu,\rho} - f^A_{\lambda,\rho} f^B_{\mu,\nu} \right).$$

Here,

$$\Pi_{AB} = \Pi_{BA}, \quad \Pi_{\lambda\mu\nu\rho} = \Pi^{AB} \left( f^A_{\lambda,\nu} f^B_{\mu,\rho} - f^A_{\lambda,\rho} f^B_{\mu,\nu} \right).$$

The action is

$$\int L d^4 x = \int g^{\lambda\mu} g^{\nu\rho} S_{\lambda\mu\nu\rho} \sqrt{g} d^4 x = \int \Pi^{AB} \left( f^A_{\lambda,\nu} f^B_{\mu,\rho} - f^A_{\lambda,\rho} f^B_{\mu,\nu} \right) \sqrt{g} d^4 x.$$
Table 1. Formulations of gravity.

| Formulation | Cartan–Weyl | Faddeev |
|-------------|-------------|---------|
| Tetrads     | $f^A_\lambda = 0$ at $A > 4$ (originally) | $f^A_\lambda \neq 0$ at $A > 4$ |
| Torsions    | $T_{\mu\nu} \neq 0$ | $T_{\mu\nu} = 0$ (on equation of motion) |

Along the way, we note the following remarkable property of this action: it remains finite for the discontinuous fields because it does not contain any of the squares of the derivatives. That is, the discontinuous fields are allowed virtually (although continuity is recovered on the equations of motion). The variation of (11) is represented as

$$\delta \int L d^4x = 2 \int (H_{\mu\nu} f^\mu_\lambda + \Pi_{AB} V^B) \delta f^A d^4x.$$  \hspace{1cm} (12)

Here, $V^B_\lambda$ and $H_{\mu\nu}$ parameterize the vertical and horizontal parts of this variation with respect to $f^A_\lambda$. Calculation gives

$$V^A_\lambda = b^\mu_\mu \Omega^A_\lambda w^\nu + b^\mu_\lambda \Omega^A_\nu w^\mu + b^\mu_\nu \Omega^A_\mu,$$  \hspace{1cm} (13)

The equations of motion $V^A_\lambda = 0$ give $T^\rho_\lambda = 0$ almost everywhere in the infinite-dimensional configuration superspace whose points are functions $f^A_\lambda (x), x \in \mathbb{R}^4$. Then, the connection is the unique torsion-free Levi-Civita one, $\Omega^A_\lambda = \Gamma^A_\lambda \mu$, and the curvature tensor is the Riemannian one, $S^A_\lambda \rho = R^A_\lambda \rho$. Then, the remaining ten independent components of the horizontal equations of motion (with $H_{\mu\nu}$) are the Hilbert–Einstein equations.

Let us compare this formulation with the Cartan–Weyl form of the gravity action for the usual tetrad (a $4 \times 4$ matrix). The action is

$$S = \int e^k_\lambda \epsilon^l_\mu R^k_\lambda \sqrt{g} d^4x, R^k_\lambda = \partial_\lambda \omega_\mu^i - \partial_\mu \omega_\lambda^i + (\omega_\lambda \omega_\mu - \omega_\mu \omega_\lambda)^i,$$

$$g^{\lambda \mu} = e^i_\lambda e^l_\mu, i, k, l, \ldots = 1, 2, 3, 4.$$  \hspace{1cm} (14)

The tetrad $e^i_\lambda$ can be considered as a particular case of the above $f^A_\lambda$ such that $f^A_\lambda = 0$ at $A > 4$. The equations of motion for the connection $\omega_{\lambda}^i$ lead to

$$\omega_{\lambda}^{ik} e^l_\mu = \frac{1}{2} (T_{\mu\lambda} + T_{\nu\lambda} - T_{\nu\mu}).$$  \hspace{1cm} (15)

Here again

$$T_{\mu\nu} = e^l_\lambda (e_{\mu\nu} - e_{\nu\mu}).$$  \hspace{1cm} (16)

But now $T_{\mu\nu} = 0$ means $R_{\mu\nu}^{ik} = 0$; therefore, the nontrivial case requires $T_{\lambda\mu\nu} \neq 0$. This can be summarized in table 1.

It is seen that both the formulations are in a sense mutually dual: a variable that is zero in any one of these formulations is nonzero in another one and vice versa. This is emphasized by the fact that the number of independent variables is the same in these formulations: it is either $4 \times 10 = 40$ components of $f^A_\lambda$ or $16 + 24 = 40$ components of $e^i_\lambda$ and $\omega_{\lambda}^i$. Therefore, it is interesting to ask whether a formulation exists which in a sense contains these both. The formulation of interest should generalize these both and result in any one of them under appropriate additional conditions. The Faddeev formulation is already quite general one for it corresponds to $f^A_\lambda \neq 0$ at $A > 4$ and $T_{\lambda\mu\nu} \neq 0$ from the very beginning. As for the Cartan–Weyl formulation, it still admits a generalization by simply generalizing the tetrad. To this end, we
rewrite the action (14) by extending the range of variation of the local frame vector index and come to the following candidate for the formulation of interest:

\[ S = \int f^\mu_A f^\nu_B \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + (\omega_{\lambda,\mu} - \omega_{\mu,\lambda})^{AB} \int \sqrt{\mathcal{g}} d^4 x. \]  

(17)

Here \( \omega^{AB}_\mu \) is the SO(10) infinitesimal connection.

2. Cartan–Weyl action for SO(10)

We can perform some gauge transformation of the local frames so that \( f^\mu_A \) would have nonzero components only at \( A = 1, 2, 3, 4, \) i.e. would be in fact the usual four-dimensional tetrad. Then, the equations of motion (20) below can be satisfied by \( \omega_{\lambda,AB} \) having nonzero components only at \( A, B = 1, 2, 3, 4. \) Thus, the action (17) can be reduced to the Hilbert–Einstein one just as the Cartan–Weyl action (14). For what follows, however, it is of interest to obtain the same result without a partial gauge fixing for \( f^\mu_A. \)

The action (17) depends on \( \omega_{\lambda,AB} \) only through the horizontal projections of \( \omega_{\lambda,AB} \) over one of the SO(10) indices, \( \omega_{\lambda,BA} f^B_\mu. \) This is evident for the bilinear over \( \omega \) terms in (17). As for the terms with derivatives, we write, e.g.,

\[ f^\mu_A f^\nu_B \partial_\mu \omega^{AB}_\nu = \partial_\mu (f^\nu_B \omega^{AB}_\mu) - \omega^{AB}_\mu (\partial_\lambda f^\mu_A) f^\nu_B - \omega^{AB}_\mu f^\mu_A \partial_\mu f^\nu_B. \]  

(18)

Here RHS contains \( \omega_{\lambda,AB} \) just in the form \( \omega_{\lambda,BA} f^B_\mu. \)

Next decompose \( \omega_{\lambda,BA} f^B_\mu \) into the horizontal and vertical parts over the index \( A, \)

\[ \omega_{\lambda,BA} f^B_\mu = h_{\lambda,\mu} + v_{\lambda,\mu}, \]  

(19)

Evidently, \( h_{\lambda,\mu} = \omega_{\lambda,BA} f^B_\mu f^A_\nu = -h_{\lambda,\nu}. \)

Varying the action (17) with respect to \( \omega^{AB}_\mu \), we obtain the equations of motion for \( \omega, \)

\[ (f^\mu_A f^\nu_B f^\lambda_C - f^\lambda_C f^\mu_A f^\nu_B) \partial_\lambda \omega^{AB}_{\mu\nu} - (A \leftrightarrow B) = \frac{1}{\sqrt{\mathcal{g}}} \partial_\mu \left[ (f^\mu_A f^\nu_B - f^\nu_B f^\mu_A) \sqrt{\mathcal{g}} \right]. \]  

(20)

First, let us take the vertical part of this equation by applying, e.g., \( \Pi^{AB} \) to it. Then, substitute the expansion (19) for \( \omega^{AB} \) here. This gives for the vertical part of the connection

\[ v^A_{\mu,\lambda} = \Pi^{AB} f^B_{\mu,\lambda} \equiv h^A_{\mu,\lambda}. \]  

(21)

Then, substitute the \( v^A_{\mu,\lambda} \) into (19) and (20). Recall that the usual derivative under the action of \( \Pi^{AB} \) coincides with the covariant derivative and the procedure of raising/lowering world indices is commutative with the derivative operator in (21), as we have noted after (8). This gives the equations for \( h_{\lambda,\mu}, \)

\[ f^\mu_A h^\nu_B = \Pi^{CD} f^C_{\nu,\lambda} \equiv f^\nu_B (T^{\nu}_{\mu,\lambda} + g_{\nu,\lambda} T_\mu - g_{\nu,\lambda} T_\mu). \]  

(24)

Here \( h_{\lambda} = g^{\alpha\nu} h_{\nu,\lambda}, T_{\lambda} \equiv g^{\alpha\nu} T_{\mu,\lambda}. \) This system is equivalent to a smaller number of independent components,

\[ h_{\nu,\lambda} + h_{\lambda,\mu} + g_{\nu,\lambda} h_{\mu} - g_{\nu,\lambda} h_{\mu} = T_{\lambda,\mu} + g_{\nu,\lambda} T_\mu - g_{\nu,\lambda} T_\mu. \]  

(25)
and has formally the same solution for \( h_{\mu \nu} \) as the usual SO(4) Cartan–Weyl formalism has for \( \omega_{\mu \nu}^e \) in terms of \( T_{\mu \nu} \) \(^{(15)}\),

\[
h_{\mu \nu} = \frac{1}{2} (T_{\mu \nu} + T_{\nu \mu}).
\]

Thus, \( h_{\mu \nu} \) is in fact the contorsion tensor (for the torsion \( T_{\mu \nu} \)). Generally, it expresses the difference between a metric-compatible affine connection \( \Omega_{\mu \nu} \) with the torsion \( T_{\mu \nu} \) and the unique torsion-free Levi-Civita connection \( \Gamma_{\mu \nu} \) for the same metric,

\[
h_{\mu \nu} = \Omega_{\nu \mu} - \Gamma_{\nu \mu}.
\]

The curvature tensor for the connection \( \omega_{\mu \nu} \) is

\[
R_{\mu \nu} = \partial_{\nu} \omega_{\mu \alpha} - \partial_{\nu} \omega_{\mu \alpha} + (\omega_{\nu \mu} \omega_{\alpha} - \omega_{\nu \alpha} \omega_{\mu}).
\]

Its horizontal projection over both the SO(10) indices \( R_{\mu \nu} f^A_{\rho \lambda} \) does not depend on the components of \( (\omega_{\alpha})_{\mu} \) which are vertical over both the indices \( A, B \). It depends only on the components which are horizontal over one of the indices, \( \omega_{\nu \mu} f^B_{\alpha} \). This follows if we rewrite the terms with derivatives as in \((18)\). This gives

\[
f_{\nu \mu} \partial_{\mu} \omega_{\nu \alpha} = \partial_{\nu} h_{\mu \nu} + h_{\nu \alpha} \Omega^\sigma \nu \lambda - h_{\mu \nu} \Omega^\sigma \mu \lambda + S_{\nu \rho \phi \mu}.
\]

Here \( S_{\nu \rho \phi \mu} \) \((5)\) arises from the vertical part \( v_{\mu \nu} \) of \( \omega_{\nu \mu} f^B_{\alpha} \). Also, we find for the bilinear in \( \omega \) part of the curvature tensor that

\[
(\omega_{\nu} \omega_{\mu} - \omega_{\mu} \omega_{\nu}) f_{\nu \mu} \partial_{\mu} \omega_{\nu \alpha} = -h_{\nu \nu} h_{\mu \mu} + h_{\nu \nu} h_{\mu \mu} - S_{\nu \rho \phi \mu}.
\]

Here, the bilinear in \( h_{\mu \nu} \) terms arises only from the horizontal part of \( \omega_{\nu \mu} f^B_{\alpha} \) in this bilinear in the \( \omega \) form. The last term is contributed only by the vertical part of \( \omega_{\nu \mu} f^B_{\alpha} \). The product of the vertical and horizontal parts of \( \omega_{\nu \mu} f^B_{\alpha} \) does not contribute to this form. Overall,

\[
R_{\mu \nu} f^A_{\rho \lambda} = \partial_{\nu} h_{\mu \nu} + \partial_{\nu} h_{\mu \nu} + h_{\mu \nu} h_{\rho \lambda} - h_{\nu \nu} h_{\mu \mu}
\]

\[
+ h_{\mu \nu} \Omega^\sigma \nu \lambda - h_{\mu \nu} \Omega^\sigma \mu \lambda + h_{\nu \nu} \Omega^\sigma \mu \lambda - h_{\nu \nu} \Omega^\sigma \mu \lambda + S_{\nu \rho \phi \mu},
\]

where the last term is contributed only by the vertical components of \( \omega_{\nu \mu} f^B_{\alpha} \). Using the identity \((27)\) in \((31)\) and taking into account that

\[
\partial_{\mu} \Omega_{\nu \rho \mu} - \partial_{\nu} \Omega_{\rho \mu \nu} + \partial_{\nu} \Omega_{\mu \nu \lambda} - \partial_{\lambda} \Omega_{\mu \nu \lambda} = -S_{\nu \rho \phi \mu},
\]

we see that the last term is canceled and we are left with the Riemannian curvature tensor

\[
R_{\mu \nu} f^A_{\rho \lambda} = g_{\rho \sigma} \left( \Gamma^\sigma_{\nu \mu, \lambda} - \Gamma^\sigma_{\nu \lambda, \mu} + \Gamma^\sigma_{\nu \mu, \mu} - \Gamma^\sigma_{\nu \lambda, \lambda} \right) = g_{\rho \sigma} R^B_{\nu \mu \lambda} = R_{\nu \mu \rho \lambda}.
\]

and, in particular, with the Hilbert–Einstein action

\[
\int R^A_{\mu \nu} f^A_{\mu \nu} \sqrt{g} d^4x = \int R \sqrt{g} d^4x.
\]

3. Representation of the Faddeev action

As noted after formulae \((29)\), \((30)\) and \((31)\), the ‘mixed’ contribution (of the product of the vertical and horizontal parts of \( \omega_{\nu \mu} f^B_{\alpha} \)) to the Cartan–Weyl form \((17)\) vanishes and the last term \( S_{\nu \rho \phi \mu} \) in \( R_{\mu \nu} f^A_{\rho \lambda} \) \((31)\) is contributed exactly by the vertical part of \( \omega_{\nu \mu} f^B_{\alpha} \). If the other terms in \( R_{\mu \nu} f^A_{\rho \lambda} \) are not taken into account, then we obtain the Faddeev action

\[
\int \sqrt{g} \sqrt{\sqrt{g}} S_{\nu \rho \phi \mu} \sqrt{g} d^4x
\]

instead of the Hilbert–Einstein one. These other terms are due to the horizontal part of \( \omega_{\nu \mu} f^B_{\alpha} \). Therefore, \( \int R^A_{\mu \nu} f^A_{\mu \nu} \sqrt{g} d^4x \) reduces to the Faddeev action if we use equations of motion for \( \omega_{\nu \mu} \) with a condition which states that the horizontal part of \( \omega_{\nu \mu} f^B_{\alpha} \) vanishes. Taking this condition into account with the help of Lagrange multipliers,
we can write out a representation for the Faddeev action of the type of the Cartan–Weyl representation as

\[ S = \int \left[ R_{\lambda \mu}^{AB}(\omega) + \Lambda_{\{\lambda \mu\}}^{\nu} \omega^{\nu} \right] f_{A}^{\lambda} f_{B}^{\mu} \sqrt{\gamma} \, d^4 x. \]  

(35)

Here \( \Lambda_{\{\lambda \mu\}}^{\nu} \) is a (matrix) Lagrange multiplier. This action possesses the global SO(10) symmetry but not the local one (due to the \( \Lambda \)-term). Note that redefining variables via \( f_{A}^\lambda \sqrt{\gamma} \equiv f_{A}^\lambda \) makes the action polynomial one.

4. Barbero–Immirzi parameter

Some natural generalization of the considered representation can be made. It is analogous to the generalization of the Cartan–Weyl form of the Hilbert–Einstein action (14) by adding some term to it which vanishes on the equations of motion for the connection \[7, 8\], transferred to the considered SO(10) action (35),

\[ S = \int \left( f_{A}^\lambda f_{B}^{\mu} + \frac{1}{2\gamma} \lambda_{\mu \nu \rho} f_{C, \mu} f_{B, \nu \rho} \right) R_{\lambda \mu}^{AB}(\omega) \sqrt{\gamma} \, d^4 x + \int \Lambda_{\{\lambda \mu\}}^{\nu} \omega^{\nu} f_{A}^{\lambda} f_{B}^{\mu} \sqrt{\gamma} \, d^4 x. \]  

(36)

Here, \( \gamma \) is a constant known as the Barbero–Immirzi parameter \[9, 10\]. This can be immediately transferred to the considered SO(10) action (35),

\[ S = \int \left( f_{A}^\lambda f_{B}^{\mu} + \frac{1}{2\gamma} \lambda_{\mu \nu \rho} f_{C, \mu} f_{B, \nu \rho} \right) R_{\lambda \mu}^{AB}(\omega) \sqrt{\gamma} \, d^4 x + \int \Lambda_{\{\lambda \mu\}}^{\nu} \omega^{\nu} f_{A}^{\lambda} f_{B}^{\mu} \sqrt{\gamma} \, d^4 x. \]  

(37)

Let us exclude \( \omega_{\lambda AB} \). Applying the operator \( \Pi^{DB} \partial/\delta\omega_{\lambda AB} \) to this action, we obtain the equation of motion

\[ \Pi^{DB} \left( f_{A}^\lambda f_{B}^{\mu} - f_{C}^\lambda f_{D}^{\mu} + \frac{1}{\gamma} \lambda_{\mu \nu \rho} f_{A, \nu \rho} f_{B, \mu} \right) \omega_{\mu CB} = \frac{1}{\sqrt{\gamma}} \Pi^{DB} \partial_{\mu} \left( f_{A}^\lambda f_{B, \mu} f_{B, \nu} f_{C}^{\nu} f_{B}^{\mu} - \frac{1}{\gamma} \lambda_{\mu \nu \rho} f_{A, \nu \rho} f_{B, \mu} \right). \]  

(38)

Solution of this equation with the additional requirement of vanishing the horizontal part of \( \omega_{\lambda BA} f_{A}^{\mu} \) turns out to be the same as in the above case \( 1/\gamma = 0 \). It is the vertical part \( \Pi^{DB} \partial/\delta f_{\mu}^{AB} \) of the solution for \( \omega_{\lambda BA} f_{A}^{\mu} \) for the SO(10) Cartan–Weyl action,

\[ \omega_{\lambda BA} f_{A}^{\mu} = \Pi_{AB}^{\lambda} f_{\mu, A, B}. \]  

(39)

Substituting this back into (37), we find the action

\[ \int L \, d^4 x = \int \Pi^{DB} \left( f_{A}^\lambda f_{B, \mu} f_{B, \nu} f_{C}^{\nu} f_{B}^{\mu} - \frac{1}{\gamma} \lambda_{\mu \nu \rho} f_{A, \nu \rho} f_{B, \mu} \right) \, d^4 x \]  

(40)

\[ = \int S_{\lambda \mu \nu \rho} \sqrt{\gamma} \, d^4 x + \frac{1}{2\gamma} \int \epsilon_{\mu \nu \rho} S_{\lambda \mu \nu \rho} \, d^4 x \]  

(41)

which differs from (11) by the presence of a parity-odd term.

Finally, write out the vertical components of the equations of motion for the action (40). Applying the operator \( \Pi_{AB}^{\lambda} \partial/\delta f_{\mu}^{B} \) to this action, we obtain

\[ b_{\mu}^{\lambda \alpha} T_{\alpha}^{\mu} + b_{\mu}^{\lambda \alpha} T_{\alpha}^{\mu} + b_{\alpha}^{\lambda \mu} T_{\mu}^{\alpha} + \frac{\epsilon}{2\gamma} \sqrt{\gamma} \left( g_{\mu \sigma} g_{\rho \tau} - g_{\mu \tau} g_{\rho \sigma} \right) b_{\rho}^{\sigma} T_{\mu}^{\sigma} = 0. \]  

(42)

It is a modification of the above equation \( \nu_{\lambda A} = 0 \) with \( \nu_{\lambda A} \) given by (13). The modification made seems to be not crucial for the problem of solvability of this equation, so that we still have \( T_{\mu}^{\mu} = 0 \) almost everywhere in the infinite-dimensional configuration superspace. Then, the curvature tensor \( R_{\mu \nu \rho}^{\lambda} \) is the Riemannian one \( R_{\mu \nu \rho}^{\lambda} \). The second term in (41) is identically zero by the properties of the Riemannian tensor and we are left with the purely Einstein action.
5. Conclusion

Thus, we have studied the first-order representation of the Faddeev gravity action in terms of the connections as additional variables, see (35) or (37). In this representation the action is polynomial. We can exclude the connection variables using equations of motion and obtain the Faddeev action. The considered representation looks like the Cartan–Weyl form of the Einstein gravity action. Now the gauge group is SO(10), but the action is invariant only under the global SO(10), not under the local one. Also, we have studied an analogue of the Barbero–Immirzi parameter considered in the literature for the Cartan–Weyl form of the Einstein gravity action. The analogue of this parameter for our case defines a coefficient at some additional parity-odd expression in our action. We can exclude the connection variables using equations of motion and obtain the Faddeev action generalized by adding some parity-odd contribution. In turn, upon partial use of the equations of motion, the parity-odd contribution vanishes and the action reduces to the Hilbert–Einstein one.

Above we have noted that the Faddeev action (11) is finite on the discontinuous fields $f^A_{\lambda}$ and thus on the discontinuous metrics. This means that the discontinuous metrics are allowed virtually in quantum theory. That is, if we try to visualize a curved spacetime as a manifold composed of some flat microblocks (cubes, tetrahedra, etc), then $f^A_{\lambda}$ can be taken to be some constant field within each microblock whose values in the different microblocks are independent. Therefore, we need not require that the neighboring microblocks be coinciding on their common faces. This may result in a considerable simplification of the canonical Hamiltonian analysis and subsequent quantization of our gravity system. Some tetrad bilinears appearing in the representation (35) or (37) can be naturally chosen as independent dynamical variables describing gravity. If the tetrad $f^A_{\lambda}$ is taken to be piecewise constant as just above, then these variables have some natural geometrical interpretation as areas (the areas of the faces of microblocks). Therefore, in particular, this may allow us to better understand the concept of area in quantum theory.

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