A modified screw dislocation
with non-singular core of finite radius
from Einstein-like gauge equation
(non-linear approach)

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Abstract
A continual model of non-singular screw dislocation lying along a straight infinitely
long circular cylinder is investigated in the framework of translational gauge ap-
proach with the Hilbert–Einstein gauge Lagrangian. The stress–strain constitutive
law implies the elastic energy of isotropic continuum which includes the terms of
second and third orders in the strain components. The Einstein-type gauge equation
with the elastic stress tensor as a driving source is investigated perturbatively, and
second order contribution to the stress potential of the modified screw dislocation
is obtained. A stress-free boundary condition is imposed at the cylinder’s external
surface. A cut-off of the classical approach which excludes from consideration a
tubular vicinity of the defect’s axis is avoided, and the total stress obtained for
the screw dislocation is valid in the whole body. An expression for the radius of
the dislocation’s core in terms of the second and third order elastic constants is
obtained.
1 Introduction

The translational gauge approach based on the Hilbert–Einstein gauge Lagrangian has been proposed in [1] for description of static dislocations in continual solids. The group of translations of three-dimensional space \( T(3) \approx \mathbb{R}^3 \) is accepted in [1] as the gauge group. The model [1] leads, in linear approximation, to so-called modified defects instead of the ordinary dislocation solutions of theoretical elasticity. The modified defects demonstrate a non-singular behaviour, i.e., are characterized by absence of the axial singularities inherent to the classical screw and edge Volterra dislocation solutions. The present paper is devoted to further development of the approach [1]. More specifically, it is to continue the investigation of the modified screw dislocation obtained in [1] and to propose a way of derivation of second order corrections to its stress field.

The point is that the Einstein-type gauge equation arising in [1] to govern the \( T(3) \)-gauge fields admits, in linear approximation, two short-ranged solutions (so-called, modified or gauge stress potentials) which coincide asymptotically with the stress potentials (i.e., with the Prandtl and the Airy stress functions) of the ordinary screw and edge dislocations. Accordingly to the picture proposed in [1], the stress fields calculated by means of the modified stress potentials just imply additional ‘gauge’ contributions to the corresponding stress fields of appropriate classical dislocations considered as background “configurations” (i.e., as pre-imposed sources of internal stresses). In other words, superposition of two stress fields, one is due to a chosen classical Volterra dislocation and another is due to the corresponding short-ranged gauge stress potential (which is localized within a vicinity of the background defect’s axis), should be considered as the total solution of the gauge model in question.

Therefore, two total solutions obtained in [1] in the super-imposed form are characterized by core region, where singularities of the classical edge and screw dislocations are smoothed out. In other words, the gauge approach which is based on the Hilbert–Einstein gauge Lagrangian allows to avoid the artificial singularities of the classical elasticity. Thus, the gauge approach “generates” a length scale in a continuous description [1], [11], [23]. The length scale characterizes the size of the domain where the classical law \( 1/\rho \) of the dislocation stresses ceases to be valid and where the axial singularity is “avoided”. Outside such domain the components of the background stresses become dominating. Thus, in the framework of [1] it is possible to study the modified defects which allow to reproduce the stresses of the classical dislocations (\( \sigma_{\phi z} \) of the screw dislocation, and \( \sigma_{\rho\rho}, \sigma_{\rho\phi} \) of the edge dislocation; \( \sigma_{\phi\phi}, \sigma_{zz} \) of the modified edge defect [1] behave unconventionally) sufficiently far from their axes while the stress components tend to zero within the core regions.

Let us turn to the screw dislocation. Approaches [2], [3], [4], [5] are known as attempts to go beyond the linear elasticity in description of the edge and screw dislocations. Nonlinear approach (second-order elasticity, in fact) can be used to find corrections to the rule \( 1/\rho \). However, it is still impossible to approach to the axis of a line defect sufficiently close since the conventional theoretical elasticity fails. For instance, the fields of second order stresses have been found in [2], [3] (by means of the stress function method) and in [4] (in the displacement function approach) which are valid within a hollow cylinder with the outer radius, say, \( \rho_e \) and the inner one \( \rho_c \). Free parameters of the models are fixed by requirements of stress-free boundaries at \( \rho = \rho_e \) and \( \rho = \rho_c \): \( \sigma_{\rho\rho}|_{\rho=\rho_e} = 0, \sigma_{\rho\rho}|_{\rho=\rho_c} = 0 \), where \( \sigma_{\rho\rho} \) is the radial stress component (however, the boundary conditions are written in
and in \[1\] with respect of the final and initial states, accordingly). Besides, vanishing of $\sigma_{zz}$ averaged over bulk’s cross-section is also used for determination of one of the free parameters. Approaches mentioned do not consider the region $0 \leq \rho \leq \rho_e$.

A discussion of relevance of second order effects in theoretical elasticity for physics of imperfections in crystals, namely for modelling dislocations, can be found in \[9\]. Thus, it is clearly interesting to investigate the model proposed in \[1\] in second order also. This is just the problem to be studied in the present paper. However, its purely mathematical aspect is of primary interest here.

As to the gauge approaches to defects in continual solids, an attempt \[7\] is known to follow \[8\] in obtaining second order contributions to the stress field of the screw dislocation. To this purpose, the quadratic translational gauge Lagrangian \[8\] is used in \[7\]. However, as it is explained in \[1\], the quadratic $T(3)$-gauge Lagrangian advanced in \[8\], \[9\] is inappropriate since it forbids a modified stress potential which correctly reproduces the stress field of the edge dislocation. From the point of view of the Refs. \[10\], \[11\] \[1]\, the Lagrangian used in \[7\], being considered as a form quadratic in the torsion components, is incomplete. Besides, the elastic energy is also taken in \[7\] in a restricted (in comparison with that of the classical, i.e., non-gauge approaches \[2\], \[3\], \[4\]) form. Since the gauge Lagrangian in \[7\] is inappropriate to capture the edge dislocation, it is also insufficient to consider second order corrections to the screw dislocation: the Kröner ansatz for the second order stresses of the screw dislocation is just of the same form as that used for an edge dislocation. Thus, the experience of \[7\] looks unsatisfactory.

The present paper is to demonstrate that second order consideration can be carried out along the line of the classical investigation \[3\] for the gauge model proposed in \[1\] also. Namely, we shall consider the second order solution found in \[3\] for the straight screw dislocation lying along cylindric body as a background source of internal stresses. In this case, solution of the Einstein-type gauge equation gives a short-ranged “correction” to the classical background. The short-ranged gauge solution depends on several free parameters. We shall adjust these parameters in a way which differs from that in \[3\], \[4\]: for instance, the vanishing boundary condition will be imposed only for the outer surface of cylindric body containing the dislocation. Instead, we shall require vanishing of certain coefficients in the short distance expansion of the second order stress potential.

Specifically, it will be demonstrated that one of the ‘matching conditions’ for the free parameters results in an expression which relates the radius of the domain of localization of the defect’s density profile to some second and third order elastic moduli. The main statement of the paper thus reads: the second order solution obtained demonstrates that singularities in $\sigma_{\rho\rho}$, $\sigma_{\phi\phi}$ do not appear, and the stress components tend either to zero or to constant values (accordingly to the choice of the free parameters) at $\rho \to 0$. However, $\sigma_{zz}$ is still weakly (i.e., logarithmically) divergent, though it is integrable over the cylinder’s cross-section. Thus the value for $\sigma_{zz}$ averaged over the cross-section surface is finite. The week divergency of $\sigma_{zz}$ is due to a simplifying assumption about the defect’s density profile.

\[1\] A gauge approach close to ours is proposed in \[10\], \[11\] which is based on the translational gauge Lagrangian $L_T$ written as a combination of terms quadratic in the torsion components (i.e., in the dislocation density’s components). For a special choice of the parameters, $L_T$ is equivalent to the Hilbert–Einstein Lagrangian \[1\]. After \[12\] it is known that extension of the Hilbert–Einstein Lagrangian by terms quadratic in torsion (and curvature) leads to quadratic in torsion Lagrangians of more general form (as well as to the most general eight-parameter three dimensional Lagrangian \[12\]).
In the present paper we are not to discuss in specific details such a complicated field as description of the core structures of the crystal dislocations. Instead, only a list (inevitably incomplete) of further references is proposed: for instance, one should be referred to [13], [14], [15], [16] for the first attempts to incorporate discreteness for consideration of the core structures. Further references, say, for (non-linear) elasticity, for crystallography, for discrete (atomic) and mixed approaches, etc., can be found also in [17], [18], [19], [20]. For theory and experiment concerning the dislocation core structures and for effects of influence of the dislocation core structures on various physical properties of solids one should be referred to [21], [22] (besides, certain refs. omitted below should be found in [1]).

The paper is written in six sections. Section 1 is introductory (see also [1] for motives of our approach and for appropriate refs.). Section 2 is to outline the Einstein-type gauge equation. Section 3 is devoted to further specifications of the gauge equation, and a perturbative scheme is set up. Solution to the gauge equation which describes the modified stress potentials of second order is obtained in Section 4. The corresponding components of the stress field and their asymptotics are investigated in Section 5. Discussion in Section 6 concludes the paper. Details of the calculation are provided in Appendices A and B. Bold-faced letters are used to denote tensors of second rank (i.e., loosely speaking, matrices).

2 The Hilbert–Einstein gauge equation

The aim of the present paper is to deduce the stress field of second order of the modified screw dislocation obtained, in linear approximation, in [1]. It should be reminded that before than in [1], the modified screw dislocation was already obtained in [23] for a translational gauge model based on the gauge Lagrangian quadratic in the torsion components (a ‘restricted’ choice of the gauge invariant quadratic form). Besides, the same modified screw dislocation was reproduced also in [11], [24] for the gauge Lagrangian taken as a more general (in comparison with [23]) quadratic form. For an appropriate choice of the parameters, the Lagrangian [11], [24] is equivalent to the Hilbert–Einstein Lagrangian proposed in [1]. It should be pointed out that the same, i.e., like in [23] and [11], non-singular screw dislocation first, seemingly, appeared in [25] in the framework of non-local elasticity approach.

The main idea behind all these gauge attempts, [23], [1], [11], can be summarized as follows. Conventionally, the ordinary dislocations are characterized by the stress tensors \( \sigma \) which are singular on the defect’s axes. In the gauge approaches mentioned, additional gauge contributions to the stress fields appear so that within compact regions (the core regions) the classical singularities are smoothed out. At sufficiently large distances, the stress components of the modified defects demonstrate the behaviour inherent to the classical dislocations. Within the cores transition between two asymptotics occurs.

We are going to consider the model proposed in [1], and second order elasticity approach is accepted below. In the present section, the Einstein-type gauge equation [1] is outlined. Some differential–geometric notations are reminded, but for more details about them one should refer to [20], [26], [27]. It is important that now we are using the Eulerian picture instead of the Lagrangian one accepted in [1]. The Lagrangian and the Eulerian pictures are indistinguishable in the linear approximation.
Our picture is based on the Eulerian strain tensor \([20]\) related to deformed (final) state of a dislocated body. Let us denote the squared length element between two neighboring points before deformation as \(dS^2\), and the squared length in a final state will be denoted as \(ds^2\). We consider the difference between \(ds^2\) and \(dS^2\), and thus we introduce the Eulerian strain tensor \(e_{ab}\) as follows.

Let us introduce the triples \(\{x^i\}\) and \(\{x^a\}\) as the coordinates bases (Cartesian or curvilinear) to be used for description of initial and final states, respectively. The corresponding squared length elements can be written as \(g_{ij}dx^idx^j\) or \(\eta_{ab}dx^adx^b\) (with \(\eta_{ab} \equiv g_{ij}\mathcal{E}^i_a\mathcal{E}^j_b\)) with respect to \(\{x^i\}\) or \(\{x^a\}\), accordingly. Let us define the frame components by means of the relation \(\partial_i = e^a_i\partial_a\) (here and below partial derivatives \(\partial/\partial x^i\) are denoted as \(\partial_i\)), and co-frame components \(\mathcal{E}^i_a\) – by means of the one-form \(dx^i = \mathcal{E}^i_adx^a\). The components \(\mathcal{E}^i_a\) with their duals \(e_i^a\) are orthogonal in the following sense:

\[
e_i^a\mathcal{E}^i_b = \delta^a_b, \quad e_i^a\mathcal{E}^j_a = \delta^j_i,
\]

(throughout the paper repeated indices imply summation). Further, let us consider a map from an initial state to the deformed state \(\{\xi^a\}\) as follows:

\[
\xi : x^i \mapsto \xi^a(x^i).
\]

Consideration of the difference

\[
ds^2 - dS^2 = \eta_{ab}d\xi^a d\xi^b - g_{ij}dx^idx^j = 2e_{ab}d\xi^ad\xi^b, \tag{2.1}
\]

where

\[
2e_{ab} \equiv \eta_{ab} - g_{ab}, \quad g_{ab} \equiv g_{ij}\mathcal{B}^i_a\mathcal{B}^j_b, \tag{2.2}
\]

allows to define the Eulerian strain tensor \(e_{ab}\). Here \(\mathcal{B}^i_a\) are the coefficients in 1-forms \(dx^i = \mathcal{B}^i_a d\xi^a\). The metric tensor \(g_{ab}\) (2.2) is called the Cauchy deformation tensor.

In the absence of defects, \(\mathcal{B}^i_a\) is expressed as follows [20]:

\[
\mathcal{B}^i_a \equiv \frac{\partial x^i}{\partial \xi^a} = \mathcal{E}^i_a - \mathcal{E}^i_b \left(\nabla^a u^b\right), \tag{2.3.1}
\]

where \(\xi^i = x^i + u^a\mathcal{E}^i_a\) with respect to the base \(\{x^i\}\). The covariant derivative \(\nabla^a\) in (2.3.1) is defined by the requirement that the components \(\mathcal{E}^i_a\) are covariantly constant, i.e.,

\[
\nabla^a \mathcal{E}^i_b \equiv \partial_a \mathcal{E}^i_b - \left\{\begin{array}{c} c \\ ab \end{array}\right\}^i_\eta \mathcal{E}^j_c = 0, \tag{2.3.2}
\]

and thus the metric \(\eta_{ab} = \mathcal{E}^i_a\mathcal{E}^j_b\) is covariantly constant. With the help of (2.3.2), we can express the Christoffel symbol of second kind \(\left\{\begin{array}{c} c \\ ab \end{array}\right\}^i_\eta\) through the metric \(\eta_{ab}\) as follows:

\[
\left\{\begin{array}{c} c \\ ab \end{array}\right\}^i_\eta = \frac{1}{2} \eta^{ce} \left(\partial_a \eta_{be} + \partial_b \eta_{ae} - \partial_c \eta_{ab}\right). \tag{2.3.3}
\]

In the presence of translational defects we put new \(\mathcal{B}^i_a\) in another, in comparison with (2.3), form:

\[
\mathcal{B}^i_a \equiv \frac{\partial x^i}{\partial \xi^a} - \varphi^i_a = \mathcal{E}^i_a - \left(\mathcal{E}^i_b \nabla^a u^b + \varphi^i_a\right). \tag{2.4}
\]
Here \( \varphi^i_a \) are the translational gauge potentials which are transformed under a non-homogeneous \( T(3) \)-gauge transformation \( x^i \rightarrow x^i + \eta^i(x) \) as follows:

\[
\frac{\partial x^i}{\partial \xi^a} \rightarrow \frac{\partial x^j}{\partial \xi^a} \left( \delta^i_j + \frac{\partial \eta^i}{\partial x^j} \right),
\]

\[
\varphi^i_a \rightarrow \varphi^i_a + \frac{\partial x^j}{\partial \xi^a} \frac{\partial \eta^i}{\partial x^j}.
\]

The replacements (2.5) ensure the gauge invariance of the components \( B^i_a \). I.e., in the presence of defects, \( B^i_a \) behave like \( \partial x^i / \partial \xi^a \) (2.3.1) at homogeneous \( (\partial \eta^i / \partial x^j \equiv 0) \) transformations.

Eventually, when (2.4) takes place, the strain \( e_{ab} \) (2.2) can be written as follows:

\[
2e_{ab} = (\eta) \nabla_a u_b + \varphi_{ab} + (\eta) \nabla_b u_a + \varphi_{ba} - \left( (\eta) \nabla_a u_b + \varphi_{ab} \right) \left( (\eta) \nabla_b u_a + \varphi_{ba} \right),
\]

where the components \( \varphi_{ab} \) of the gauge potential \( \varphi \) are defined by means of the representation \( \varphi^i_a = \mathcal{E}_a^i \varphi^a_i \) provided \( \eta_{ab} \) is used to raise and lower the indices. When the gauge potential \( \varphi \) is zero, Eq. (2.6) is reduced to conventional expression for the strain \( e \) in the Eulerian picture [20].

Further, in order to consider the gauge equation, we need the Riemann–Christoffel curvature tensor \( R_{abc}^d \): 

\[
R_{abc}^d = \partial_a \left\{ \frac{d}{bc} \right\}_g - \partial_b \left\{ \frac{d}{ac} \right\}_g + \left\{ \frac{d}{ae} \right\}_g \left\{ \frac{e}{bc} \right\}_g - \left\{ \frac{d}{be} \right\}_g \left\{ \frac{e}{ac} \right\}_g,
\]

where

\[
\left\{ \frac{d}{bc} \right\}_g \equiv \{bc,e\}_g g^{ed},
\]

\[
\{bc,e\}_g \equiv \frac{1}{2} (\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}).
\]

In (2.7) and (2.8), the Christoffel symbols of first and second kind, \( \{bc,e\}_g \) and \( \left\{ \frac{d}{bc} \right\}_g \), accordingly, are calculated with respect to the metric \( g_{ab} \) (2.2) (the subscript ‘g’). The metric \( g_{ab} \) is covariantly constant with respect to the metric connection \( (g)\nabla \) which is expressed through the Christoffel symbols (2.8), i.e., equation \( (g)\nabla_a g_{bc} = 0 \) is fulfilled.

Since the metric \( \eta_{ab} \) is also covariantly constant in terms of the corresponding metric connection \( (\eta)\nabla \),

\[
(\eta)\nabla_a \eta_{bc} = 0,
\]

we obtain from (2.9) with the help of (2.2):

\[
\partial_a g_{bc} = \left\{ \frac{e}{ab} \right\}_\eta g_{ec} + \left\{ \frac{e}{ac} \right\}_\eta g_{be} - 2 \nabla_a \eta_{bc},
\]
where the Christoffel symbols of second kind are defined by means of (2.3.3). In its turn, Eq.(2.10) (plus two other equations due to cyclic permutations of the indices) gives us:

\[
\left\{ \begin{array}{l}
  c \\
  a 
\end{array} \right\}_g = \left\{ \begin{array}{l}
  c \\
  a 
\end{array} \right\}_\eta - 2e_{ab}^c ,
\]

\[
2e_{ab}^c \equiv g^{ce} \left( \nabla_a e_{bc} + \nabla_b e_{ae} - \nabla_e e_{ab} \right).
\]

We substitute (2.11) into (2.7) and obtain another representation for the curvature tensor:

\[
R_{abc}^d = \left( \eta \right) R_{abc}^d - 2 \left( \eta \right) (\nabla_a e_{bc}^d - 2e_{ae}^d e_{bc} - (a \leftrightarrow b)) ,
\]

(2.12)

where \( R_{abc}^d \) is the Riemann curvature calculated analogously to (2.7) but for the metric \( \eta_{ab} \). In our Eulerian approach, we assume that the geometry of the deformed state is flat, and so we put \( \left( \eta \right) R_{abcd} \) equal to zero. Now we are ready to write the Hilbert–Einstein gauge equation. Let us define the Einstein tensor \( G^{ef} \) as follows:

\[
G^{ef} = \frac{1}{4} \mathcal{E}^{eab} \mathcal{E}^{fcd} R_{abcd} ,
\]

(2.13)

or, with the help of (2.12),

\[
G^{ef} = -\mathcal{E}^{eab} \mathcal{E}^{fcd} \nabla_a \nabla_c e_{bd} - 2\mathcal{E}^{eab} \mathcal{E}^{fcd} e_{ade'} e_{bc} e^{e'} .
\]

(2.14)

In (2.13) and (2.14), \( \mathcal{E}^{abc} \) is the totally antisymmetric Levi–Civita tensor \[20\] defined by means of the metric \( \eta_{ab} \). Therefore, the gauge equation proposed in [1] (Section 6) takes the form:

\[
G^{ef} = (2s)^{-1} \left( \sigma^{ef} - \left( \sigma_{bg} \right)^{ef} \right) ,
\]

(2.15)

or,

\[
-\mathcal{E}^{eab} \mathcal{E}^{fcd} \nabla_a \nabla_c e_{bd} = (2s)^{-1} \left( \sigma^{ef} - \left( \sigma_{bg} \right)^{ef} \right) + 2\mathcal{E}^{eab} \mathcal{E}^{fcd} e_{ade'} e_{bc} e^{e'} .
\]

(2.16)

Variational derivation of (2.15) can be discussed along the line of [1] where the gauge approach was developed in the Lagrangian coordinates. Right-hand side of (2.15) is given by the difference \( \sigma - \sigma_{bg} \), where \( \sigma_{bg} \) implies the stress tensor of a background defect. The difference \( \sigma - \sigma_{bg} \), i.e., just the deviation of the total stress from \( \sigma_{bg} \), plays the role of the source of geometric configurations described by the Einstein tensor \( G \). The parameter \( s \) (a coupling constant, accordingly to gauge terminology) characterizes an energy scale intrinsic to the gauge field \( \varphi \), and it appears as a factor at the Hilbert–Einstein Lagrangian density \[1\].

In the present paper, \( \sigma_{bg} \) is assumed to be given by the stress field of a single straight screw dislocation lying along an infinitely long cylindric body. Practically, the solution provided by \[3\] will be adopted, which is valid within a hollow cylinder restricted by two surfaces: internal \( (\rho = \rho_c) \) and external \( (\rho = \rho_e) \). Further comments about (2.15), (2.16), and about their specification for the present problem can be found below. Besides, both \( \sigma \) and \( \sigma_{bg} \) are assumed to respect the equilibrium equations:

\[
\left( \eta \right) \nabla_a \sigma^{ab} = 0 , \quad \left( \eta \right) \nabla_a \left( \sigma_{bg} \right)^{ab} = 0 .
\]

(2.17)
More detailed consideration of the gauge geometry behind the model to be studied in
the present paper should be done elsewhere. However, Refs. [28] should be mentioned in
addition to those listed in [1] since they contain reviewing notes and useful refs. concern-
ing translational gauge geometry. It is also interesting to note Ref. [29] where a topological
picture is build up which includes dislocations (torsion) and exta-matter (non-metricity).

3 Specification of the gauge equation

3.1 The stress function method

We shall investigate Eq. (2.16) using the method of stress functions proposed in [2] for
solving the internal stress problems in incompatible elasticity. Specifically, we shall follow
Ref. [3], where the approach [2] was further developed in the successive approximation
form to determine the second order stress fields of the screw and edge straight dislocations
lying along cylindrical tubes of circular cross sections. An exposition of [3] can be found
in Ref. [20] devoted to a review of dislocation problems in non-linear elasticity. Certain
details (concerning, for instance, the tensor formalism in curvilinear coordinates) omitted
in what follows can be restored with the help of [3] and [20]. In what follows, bold-faced
letters denote tensors, and all the indices can easily be restored.

Accordingly to Eq. (2.6), let us represent the strain and the stress tensors as the
perturbative expressions:

\[
e = (1) e + (2) e, \quad \sigma = (1) \sigma + (2) \sigma,
\]

(3.1)

where \((1) e, (2) e, (1) \sigma, (2) \sigma\) are assumed to be of second order smallness in comparison to \((1) e, (1) \sigma\). Ex-
pressions (3.1) can be understood as the first two terms of formal perturbative series in
powers of a small parameter. Substituting (3.1) into (2.16) and (2.17), we obtain the
following two sets of the governing equations:

(first order)

\[
\nabla a (1) \sigma^{ab} = 0,
\]

(3.2)

\[
(\text{Inc} (1) e)^{ab} = (2s)^{-1} \left( \delta (1) \sigma^{ab} \right);
\]

(second order)

\[
\nabla a (2) \sigma^{ab} = 0,
\]

(3.3)

\[
(\text{Inc} (2) e)^{ab} = (2s)^{-1} \left( \delta (2) \sigma^{ab} \right) + (1) Q^{ab}.
\]

The following notations are used in (3.2) and (3.3) \((i = 1, 2)\):

\[
(\text{Inc} (i) e)^{ab} \equiv -\mathcal{E}^{acd} \mathcal{E}^{bfe} \nabla c \nabla f (i) e^{de},
\]

(3.4.1)

\[
\left( \delta (i) \sigma \right)^{ab} \equiv (i) \sigma^{ab} - (i) \sigma^{bg} (i) \sigma_{bg}^{ab},
\]

(3.4.2)
\[
Q_{ab}^{(1)} = 2 \mathcal{E}^{acd} \mathcal{E}^{bfe} (1)_{c}^{(1)} (1)_{d}^{(1)} t ,
\]

where ‘Inc’ denotes, so-called, *incompatibility operator* (acting on a tensor argument), \(\mathcal{E}^{abc}\) is the Levi-Civita tensor, and \(\nabla_{a}\) implies the covariant derivative \(\nabla_{a}\). Indices in (3.2)–(3.4) are raised and lowered by means of the metric \(\eta_{ab}\). A direct comparison, for instance, with Eqs. (622) and (623) provided by [20] demonstrates that the second equations in each pair (3.2) and (3.3) imply transparent modification of the corresponding equations of the conventional approach: the only difference is given by the terms \((2s)^{-1}\delta^{(i)}(\sigma)\), which are responsible for the short-ranged behaviour of the resulting stress functions. Besides, the contributions due to the contortion (and thus due to the dislocation density) are absent (see explanations in [1]).

The elastic energy potential is chosen in the Eulerian representation as follows [3]:

\[
W(\varepsilon) = jI_{1}^{2}(\varepsilon) + kI_{2}(\varepsilon) + l'I_{1}(\varepsilon)I_{2}(\varepsilon) + n'I_{3}(\varepsilon),
\]

where \(j = \mu + \lambda/2\), \(k = -2\mu\) (\(\lambda\) and \(\mu\) are the Lamé constants), while \(l', m', n'\) are the elastic moduli of third order. For the given choice of the potential \(W(\varepsilon)\), the constitutive law which relates \(^{(i)}\varepsilon\) to \(^{(i)}\sigma\) takes the following form [3]:

\[
^{(1)}\varepsilon = C_{1}I_{1}(\sigma)^{1}\eta + C_{4}^{(1)}\sigma ,
\]

\[
^{(2)}\varepsilon = C_{1}I_{1}(\sigma)^{2}\eta + C_{4}^{(2)}\sigma + \Psi ,
\]

where

\[
\Psi \equiv (C_{2}I_{1}^{(1)}(\sigma) + C_{3}I_{2}(\sigma))^{1}\eta + C_{5}I_{1}(\sigma)^{1}\sigma + C_{7}I_{3}(\sigma)^{1}\sigma - 1 ,
\]

and the numerical coefficients are [3]:

\[
C_{1} \equiv -\frac{\nu}{E} , \quad C_{4} \equiv \frac{1 + \nu}{E} ,
\]

\[
C_{2} \equiv \frac{1}{E^{2}}(3\nu(1 - \nu) - 1) + 3L + M ,
\]

\[
C_{3} \equiv \frac{3}{E^{2}}(1 - \nu^{2}) + M ,
\]

\[
C_{5} \equiv \frac{1}{E^{2}}(3\nu - 2)(1 + \nu) - M ,
\]

\[
C_{7} \equiv -\frac{1}{E^{2}}(1 + \nu)^{2} + N .
\]

In (3.8) we are using the Poisson ratio \(\nu\) and the elastic modulus \(E\):

\[
\nu = \frac{\lambda}{2(\lambda + \mu)} , \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} ;
\]

besides, the relationship between the elastic parameters of third order \(L, M, N\) in (3.8)
and \(l', m', n'\) in (3.5) is given as follows:
\[
\begin{align*}
n' + 8\mu^3N &= 6\mu, \\
3m' + n' + 12\mu^2K(3M + N) &= 18K\frac{3\nu - 2}{1 + \nu}, \\
27l' + 9m' + n' + 27K^3(27L + 9M + N) &= 36K,
\end{align*}
\]  
(3.9)

where
\[
K \equiv \frac{E}{3(1 - 2\nu)} = \lambda + \frac{2}{3}\mu.
\]

Besides, \(\eta\) implies the metric \(\eta_{ab}\) in (3.5)–(3.7), and the functions \(I_m\) \((m = 1, 2, 3)\) of a tensor argument, say, \(t\) are defined as follows:
\[
\begin{align*}
I_1(t) &\equiv \text{tr}(t), \\
I_2(t) &\equiv \frac{1}{2}(I_1^2(t) - I_1(t^2)), \\
I_3(t) &\equiv \text{Det}(t).
\end{align*}
\]  
(3.10)

The Cayley–Hamilton theorem must be used to express the inverse \((^{(1)}\sigma)^{-1}\) in (3.7). More details about derivation of Eqs. (3.6)–(3.9) can be found in \([3], [20]\).

Now we are in position to use the first and the second equations (3.6) in (3.2) and (3.3), respectively. We obtain:

(first order)
\[
\begin{align*}
\nabla_a (^{(1)}\sigma)_{ab} &= 0, \\
\Delta (^{(1)}\sigma)^{ab} + (1 - a)(\nabla_a \nabla_b - \eta_{ab}\Delta)I_1(1) &= \kappa^2(\delta (^{(1)}\sigma)^{ab});
\end{align*}
\]  
(3.11)

(second order)
\[
\begin{align*}
\nabla_a (^{(2)}\sigma)_{ab} &= 0, \\
\Delta (^{(2)}\sigma)_{ab} + (1 - a)(\nabla_a \nabla_b - \eta_{ab}\Delta)I_1(2) &= \kappa^2(\delta (^{(2)}\sigma)_{ab}) + 2\mu (^{(1)}Q_{(ab)}) - 2\mu(\text{Inc} (^{(1)}\Psi))_{ab},
\end{align*}
\]  
(3.12)

where \((^{(1)}\Psi)\) is given by (3.7), \(I_1(i) \equiv I_1(\sigma), \Delta \equiv \nabla_a \nabla^a,\) and \((^{(ab)}Q)\) is expressed by means of (3.4.3) and (3.6). Besides, we make use of the parameters \(\kappa^2 \equiv \mu/s\) and \(a \equiv \frac{\lambda}{3\lambda + 2\mu} = \frac{1}{1 + \nu^{-1}}, \quad 1 - a = \frac{2(\lambda + \mu)}{3\lambda + 2\mu} = \frac{1}{1 + \nu}\).

The curly brackets around the indices imply symmetrization.
We use the stress function ansatz to fulfil the equilibrium equations for stresses given by (3.11) and (3.12) as follows:

\[
\sigma^{(i)} = \text{Inc}^{(i)} \chi \quad \text{.} \tag{3.13}
\]

Substituting (3.13) into the second equations in (3.11), (3.12), we obtain a couple of equations to determine the stress potentials \( \chi^{(i)} \):

\[
\Delta \Delta \chi^{(i)}_{ab} + a (\nabla_a \nabla_b - \eta_{ab} \Delta) \Delta J^{(i)}_1(\chi) + \left( (1 - a) \nabla_a \nabla_b + a \eta_{ab} \Delta \right) \nabla^c \nabla^d \chi^{(i)}_{cd} - \\
- \Delta \left( \nabla_a \nabla_c \chi^{(i)}_{c_b} + \nabla_b \nabla_c \chi^{(i)}_{c_a} \right) = \kappa^2 (\delta^{(i)} \sigma)_{ab} + 2 \mu S^{(i)}_{(ab)}, \quad i = 1, 2, \tag{3.14}
\]

where

\[
(1)_{(ab)} S = 0, \quad (2)_{(ab)} S = (1)_{(ab)} Q - \left( \text{Inc}^{(i)} \Psi \right)_{(ab)},
\]

and \( \delta^{(i)} \sigma \) is written by means of (3.4.3) and (3.13).

Using a linear transformation to another stress potential \( \chi' \)

\[
\chi^{(i)} = \chi' + \frac{\nu}{1-\nu} \eta_{ab} \chi'_{13} \quad \text{,} \quad i = 1, 2,
\]

we can reduce (3.14) at \( i = 2 \) to a more simple form:

\[
\Delta \Delta \chi'^{(2)} - \kappa^2 \delta^{(2)} \sigma' = 2 \mu S^{(2)}, \tag{3.15}
\]

where prime at \( \sigma \) implies that the tensor is expressed through \( \chi' \). Provided a tensor-valued Green’s function of the corresponding operator acting in L.H.S. of (3.15) is known, solution of (3.15) can be obtained in a standard way. However, in what follows we shall be concerned with (3.14) itself. Therefore, the main task below is to adjust (3.14) to the special case in question, i.e., to the case of the screw dislocation along a cylindric body of circular cross-section.

3.2 The gauge equations in the first and second orders. The choice of the model

Owing to the fact that the equilibrium equations given by (3.11) and (3.12) (the first ones in pairs) are fulfilled by the Kröner ansatz (3.13), the problem is to determine the stress function components \( \chi_{ab} \) from the gauge equations given by (3.11) and (3.12) (the second ones in pairs).

Now we replace all the derivatives \( \nabla_a \) by the partial derivatives \( \partial_a \equiv \partial / \partial x^a \), where \( x^a \) are the Cartesian coordinates in the final state \[3\], \[20\]. We assume also \( \partial_z \equiv \partial_3 = 0 \). Let us introduce the following notations for those components of the stress potential which are non-trivial:

\[
\mu \phi \equiv \partial_2 (\chi_{13}^{(i)} - \partial_1 \chi_{23}^{(i)}), \quad i = 1, 2, \tag{3.16}
\]

\[
f \equiv \chi_{33}^{(2)}, \quad p \equiv -\partial^2_{11} \chi_{22}^{(2)} - \partial^2_{22} \chi_{11}^{(2)} + 2 \partial_{12} \chi_{12}^{(2)}.
\]
The other components \( (i) \chi_{ab} \) are zero. The background stress tensor \( \sigma_{bg} \) (see (2.17)) is also given by (3.13), though the corresponding stress functions are labeled by appropriate subscript: \( (i) \chi_{bg} \).

The first order

Since the background stress field is assumed to correspond to that of the screw dislocation in the form provided by \[3\], we conclude that only the component of the stress potential \( (1) \phi \) is nonzero in the first order. It is described by second equation in (3.11), and the latter acquires the following form \[1\]:

\[
\partial_a \left( \Delta (1)\phi - \kappa^2 (1)(\phi - (1)\phi_{bg}) \right) = 0, \quad a = 1, 2,
\]

or

\[
(1)\Delta (1)\phi = \kappa^2 (1)(\phi - (1)\phi_{bg}),
\]

(3.17)

where \( (1)\phi_{bg} \equiv (-b/2\pi) \log \rho \). It is appropriate to re-express (3.17) as follows:

\[
\left( \Delta - \kappa^2 \right) (1)(\phi - (1)\phi_{bg}) = b \delta^{(2)}(x).
\]

(3.18)

Solution to (3.18) describes the modified screw dislocation, and it is given by

\[
(1)\phi = (1)\phi_{bg} - f_S, \quad f_S \equiv \left( b/2\pi \right) K_0(\kappa \rho).
\]

(3.19.1)

From (3.13) and (3.16) we obtain the only non-trivial component of the total stress as follows:

\[
(1)\sigma_{\phi z} = -\partial_{\rho} (\mu (1)\phi) = \frac{b\mu}{2\pi} \rho^{-1} (1 - \kappa \rho K_1(\kappa \rho)).
\]

(3.19.2)

Equations (3.19) witness about existence of a core region at \( \rho \lesssim 1/\kappa \): outside this region the gauge correction to the classical long-ranged law \( 1/\rho \) is exponentially small. At \( \rho \ll \kappa^{-1} \), the law \( 1/\rho \) (characterizing the stresses) is replaced by another non-singular one. More detailed information concerning the numerical behaviour of the solution (3.19.2) (including a treatment of \( \kappa^{-1} \) in terms of interatomic spacing) can be found in \[11\] and \[23\].

The second order

In order to obtain the gauge equations in this case, it is necessary to specialize the source \( S_{ab} \) in (3.14), i.e., its terms \( (1)Q_{(ab)} \) and \( \text{Inc} (1)\Psi_{(ab)} \), which depend on the first order solution (3.19.1), have to be written explicitly. (In what follows we shall use \( Q, \Psi, \mathbf{e} \) without the superscript \( (1) \).)
Let us begin with $Q_{(ab)}$ (3.4.3). More explanations can be found in [3], [20]. Using (2.11) we obtain from (3.4.3) the following non-zero contributions:

\[
Q_{11} = Q_{22} = (\partial_1 e_{23} - \partial_2 e_{13})^2, \\
Q_{33} = 4(\partial_1 e_{23} \partial_2 e_{13} - \partial_2 e_{23} \partial_1 e_{13}) + (\partial_1 e_{23} - \partial_2 e_{13})^2,
\]

while $Q_{12}$, $Q_{23}$, $Q_{13}$ are zero. Our expressions (3.20) differ from the analogous quantities in [3], [20] (a different numerical factor in $Q_{11}$, $Q_{22}$, and absence of the second term in $Q_{33}$) since in our case the corresponding Einstein tensor in L.H.S. of the gauge equation (2.15) is expressed by means of the Riemann–Christoffel tensor only [1]. However, in [3], [20] the corresponding incompatibility equation includes a contribution from the contortion tensor also. Now let us obtain $\Psi$ (3.7) in components:

\[
\begin{align*}
(1/\mu^2)\Psi_{11} &= -C_3 \left[ (\partial_1 (1) \phi)^2 + (\partial_2 (1) \phi)^2 \right] - C_7 (\partial_1 (1) \phi)^2, \\
(1/\mu^2)\Psi_{22} &= -C_3 \left[ (\partial_1 (1) \phi)^2 + (\partial_2 (1) \phi)^2 \right] - C_7 (\partial_2 (1) \phi)^2, \\
(1/\mu^2)\Psi_{12} &= (1/\mu^2)\Psi_{21} = -C_7 \partial_1 (1) \phi \partial_2 (1) \phi, \\
(1/\mu^2)\Psi_{33} &= -C_3 \left[ (\partial_1 (1) \phi)^2 + (\partial_2 (1) \phi)^2 \right].
\end{align*}
\]

Now we are ready to consider the gauge equations of second order given by (3.14). We obtain:

\[
\begin{align*}
-\partial_{22}^2 \left[ (1-a)p + a\Delta f - \kappa^2(f - f_{bg}) \right] &= 2\mu \left[ Q_{11} + \partial_{22}^2 \Psi_{33} \right], \\
-\partial_{11}^2 \left[ (1-a)p + a\Delta f - \kappa^2(f - f_{bg}) \right] &= 2\mu \left[ Q_{22} + \partial_{11}^2 \Psi_{33} \right], \\
\partial_{12}^2 \left[ (1-a)p + a\Delta f - \kappa^2(f - f_{bg}) \right] &= -2\mu \partial_{12}^2 \Psi_{33}, \\
(1-a)\Delta \Delta f + a\Delta p - \kappa^2(p - p_{bg}) &= 2\mu \left[ Q_{33} - \partial_{12}^2 (\Psi_{12} + \Psi_{21}) + \partial_{11}^2 \Psi_{22} + \partial_{22}^2 \Psi_{11} \right].
\end{align*}
\]

Besides, there exists a couple of equations to determine $\phi$. But since $\Psi_{13}$, $\Psi_{31}$, $\Psi_{23}$, $\Psi_{32}$ are zero, and $\phi_{bg}$ is also zero [3], we put consistently $\phi \equiv 0$. Thus, the only equations to be considered are given by (3.22), where the corresponding components of $Q$ and $\Psi$ are given by (3.20) and (3.21), respectively.

Equations (3.22) look very similar to those of the classical approach [3], [20] excepting of the fact that $Q_{11}$, $Q_{22}$ are zero in the classical consideration. The point here is as follows: equations for the classical stress potentials of second order, $p_{bg}$ and $f_{bg}$, are considered for a two-dimensional domain $\rho_e \leq \rho \leq \rho_e$ ($\rho_e$ and $\rho_e$ are the external and internal radii, accordingly). However, $Q_{11}$ and $Q_{22}$ are equal to $4(\partial_1 e_{23} - \partial_2 e_{13})^2$ in [3], and therefore
they are proportional to \( \Delta (1) \phi_{bg} \), while the latter represents the dislocation density profile (i.e., the dislocation density component \( T_{3,12} \)). Outside the disc given by \( 0 \leq \rho \leq \rho_c \), \( \Delta (1) \phi_{bg} \) is zero because of its proportionality to the Dirac \( \delta \)-function. It is just the case, why \( Q_{11} \) and \( Q_{22} \) drop out of the classical version of Eqs. (3.22).

Accordingly to [3], a constant \( Q' \) can be introduced as follows:

\[
Q_{11} = \partial_{22}^2 Q', \quad Q_{22} = \partial_{11}^2 Q', \quad Q_{12} = -\partial_{12}^2 Q',
\]

where \( Q' \) is to be adjusted in the end of calculation of the stress components \(^2\). Therefore, equations which define \( p_{bg} \) and \( f_{bg} \) take in our notations the form (compare, for instance, with (3.22)):

\[
\Delta \Delta f_{bg} = 2\mu \frac{a}{1-2\alpha} \Delta (\Psi_{33}^{bg} + Q') + 2\mu \frac{1-a}{1-2\alpha} \left[ Q_{33}^{bg} + \partial_{11}^2 \Psi_{22}^{bg} + \partial_{22}^2 \Psi_{11}^{bg} - 2\partial_{12}^2 \Psi_{12}^{bg} \right],
\]

\[
(1-a) p_{bg} + a \Delta f_{bg} = -2\mu (\Psi_{33}^{bg} + Q').
\]

The first and the second equations in (3.24) correspond, respectively, to Eqs. (657) and (655) in [20], provided \( \sigma_{33} \) and \( F_{(2)} \) therein are identified as \( p_{bg} \) and \(-f_{bg}\). Besides, Eq. (3.23) defines \( Q' \) in opposite way (the sign is different) in comparison with Eq. (658) in [20].

In the present paper, \( Q_{11}, Q_{22}, Q_{33} \) are given by (3.20), and we find:

\[
Q_{11} = Q_{22} = \left( \frac{1}{2} \Delta (1) \phi \right)^2.
\]

Is it possible to find a “potential” \( \tilde{g} \equiv \tilde{g}(\rho) \) which is analogous to \( Q' \) of the classical approach? For such \( \tilde{g} \equiv \tilde{g}(\rho) \) the following equations should be respected:

\[
Q_{11} = \partial_{22}^2 \tilde{g} = \frac{1}{2} \left[ \partial_{\rho \rho}^2 + \frac{1}{\rho} \partial_{\rho} \right] \tilde{g} - \frac{\cos 2\varphi}{2} \left( \partial_{\rho \rho}^2 - \frac{1}{\rho} \partial_{\rho} \right) \tilde{g},
\]

\[
Q_{22} = \partial_{11}^2 \tilde{g} = \frac{1}{2} \left[ \partial_{\rho \rho}^2 + \frac{1}{\rho} \partial_{\rho} \right] \tilde{g} + \frac{\cos 2\varphi}{2} \left( \partial_{\rho \rho}^2 - \frac{1}{\rho} \partial_{\rho} \right) \tilde{g},
\]

\[
Q_{12} = -\partial_{12}^2 \tilde{g} = -\frac{\sin 2\varphi}{2} \left( \partial_{\rho \rho}^2 - \frac{1}{\rho} \partial_{\rho} \right) \tilde{g} = 0.
\]

Equations (3.26) lead us to the following pair of equations:

\[
\left( \partial_{\rho \rho}^2 - \frac{1}{\rho} \partial_{\rho} \right) \tilde{g} = 0,
\]

\[
\left( \partial_{\rho \rho}^2 + \frac{1}{\rho} \partial_{\rho} \right) \tilde{g} = 2Q_{11} = 2Q_{22}.
\]

However, Eqs.(3.27) are not consistent for the given \( Q_{11}, Q_{22} \) (3.25).

\(^2\)More precisely, the numerical value for \( Q' \) arises from a condition that a mean value of \( \sigma_{33} \) is zero [3, 20].
In the classical approach, $\Delta^{(1)} \phi_{bg}$ corresponds to the defect’s density profile, and the latter is given by the Dirac $\delta$-function. Therefore, $Q_{11} = Q_{22} = 0$ at $\rho > \rho_c$, and so it is possible to choose $Q'$ as a constant. In our approach, $\Delta^{(1)} \phi$ also represents the density profile of the modified defect, and it is given as follows \[1, 11, 23\]:

$$\left( \Delta^{(1)} \phi \right)^2 = \left( \frac{b}{2\pi} \kappa^2 K_0(\kappa \rho) \right)^2. \tag{3.28}$$

In the limit $\kappa \to \infty$, the function $\kappa^2 K_0(\kappa \rho)$ demonstrates a behaviour of a $\delta$-like function on a plane “centered” at $\rho = 0$. Therefore, the following simplification can seemingly be made to keep the situation tractable in the framework of the plane problem: let us approximate $\frac{b}{2\pi} \kappa^2 K_0(\kappa \rho)$ by a piecewisely constant function which takes two different constant values either within the disc $0 \leq \rho \leq \rho_c$, or outside it. Besides, we shall assume that differentiations of this “hat”-function at $\rho = \rho_c \pm 0$ are negligible. Then, it turns out that equations (3.27) become consistent. Clearly, such approximation is rather rough at $\rho \ll \kappa^{-1}$, i.e., in a very close vicinity of the classical defect’s axis. However, as we shall see below, this simplification leads to a reasonable picture for a non-singular modified screw dislocation in the second order also.

The replacement proposed for the density profile is given as:

$$\frac{b}{2\pi} \kappa^2 K_0(\kappa \rho) \longmapsto \frac{b}{\pi \rho_c^2} h_{[0, \rho_c]}(\rho), \tag{3.29}$$

where $h_{[0, \rho_c]}(\rho)$ is equal to unity at $0 \leq \rho \leq \rho_c$, and to zero otherwise. With the density profile given by (3.29), we find $\bar{g}$ which respects (3.27):

$$\bar{g}(\rho) = \begin{cases} C', & \rho_c < \rho, \\ \left( \frac{b}{2\pi \rho_c^2} \right)^2 \left( \rho^2 / 2 \right) + C'', & 0 \leq \rho \leq \rho_c, \end{cases} \tag{3.30}$$

where the constants $C'$ and $C''$ will be adjusted later.

Let us write $Q_{33}$ (3.20) explicitly:

$$Q_{33} = \left( \partial^2_{12}^{(1)} \phi \right)^2 - \partial^2_{11}^{(1)} \phi \partial^2_{22}^{(1)} \phi + \left( \frac{1}{2} \Delta^{(1)} \phi \right)^2. \tag{3.31}$$

Using (3.21) and (3.31), we calculate:

$$Q_{33} - 2 \partial^2_{12} \Psi_{12} + \partial^2_{11} \Psi_{22} + \partial^2_{22} \Psi_{11} =$$

$$= \Delta \Psi_{33} + (1 - 2\mu^2 C_7) \left[ \left( \partial^2_{12}^{(1)} \phi \right)^2 - \partial^2_{11}^{(1)} \phi \partial^2_{22}^{(1)} \phi \right] + \left( \frac{1}{2} \Delta^{(1)} \phi \right)^2. \tag{3.32}$$

Therefore, taking into account (3.26) and (3.32), we obtain from (3.22) the following couple of equations to determine $f$ and $p$:

$$(1 - a)p + a \Delta f - \kappa^2 (f - f_{bg}) = -2\mu \left( \Psi_{33} + \bar{g} \right), \tag{3.33}$$

$$(1 - a)\Delta \Delta f + a \Delta p - \kappa^2 (p - p_{bg}) = \frac{\mu}{2} \left( \Delta^{(1)} \phi \right)^2 +$$
\[ + 2\mu \Delta \Psi_{33} + 2\mu (1 - 2\mu^2 C_7) \left[ (\partial_{l2}^{(1)} \phi)^2 - \partial_{11}^{(1)} \phi \partial_{22}^{(1)} \phi \right]. \] (3.34)

We obtain \( p \) from (3.33) and substitute it into (3.34):

\[ p = -\frac{a}{1 - a} \Delta f + \frac{\kappa^2}{1 - a} (f - f_{bg}) - \frac{2\mu}{1 - a} (\Psi_{33} + \tilde{g}), \] (3.35)

\[ (\Delta - \kappa^2) \left( \Delta + \frac{\kappa^2}{1 - 2a} \right) (f - f_{bg}) = R, \] (3.36)

where

\[ \frac{1 - 2a}{2\mu} R \equiv (\Delta - \kappa^2)(\Psi_{33} + \tilde{g} - \Psi_{33}^{bg} - Q') + \]

\[ + (1 - a)(1 - 2\mu^2 C_7)(\Phi - \Phi_{bg}) - \frac{1 - a}{4} (\Delta^{(1)} \phi)^2, \] (3.37)

\[ \Phi = \left( \partial_{l2}^{(1)} \phi \right)^2 - \partial_{11}^{(1)} \phi \partial_{22}^{(1)} \phi. \] (3.38)

Here \( f_{bg} \) respects the first equation in (3.24), and \( \Phi_{bg}, \Psi_{33}^{bg} \) are given by (3.38), (3.21) (provided \( \phi \) is replaced by \( \phi_{bg}^{(1)} \)), correspondingly. Besides, we formally keep \( (\Delta \phi)^2 \) as an exact expression.

### 3.3 Final remarks about the gauge equations

To conclude Section 3, we should pay attention also to the structure of the function \( R \) in the righthand side of (3.36). Explicit expression for \( R \) is given by Eq. (3.37). Our problem has an axial symmetry, and therefore the use of the cylindric coordinates \( \rho, \varphi, z \) instead of the coordinates in final state \( x^a, a = 1, 2, 3 \) (the coordinates \( \rho \) and \( \varphi \) are chosen in \( (x^1, x^2) \)-plane and \( z \equiv x_3 \)) is more appropriate. Therefore, we obtain from (3.21) the following expressions in the cylindrical coordinates:

\[ \Psi_{33} = -c (\partial_{\rho} \phi)^2, \quad c \equiv \mu^2 C_3, \] (3.39)

\[ \Delta \Psi_{33} = -2c \left[ \partial_{\rho} \phi \partial_{\rho}(\Delta \phi) + (\Delta \phi)^2 \right] - 4c \Phi, \]

where \( \Phi \) (3.38) is re-written as:

\[ \Phi = -\frac{1}{\rho} \partial_{\rho} \phi \partial_{\rho}^{(1)} \phi = -\frac{1}{\rho} \partial_{\rho} \phi \Delta \phi + \frac{(\partial_{\rho} \phi)^2}{\rho^2}. \] (3.40)

Further, we use (3.39), (3.40) in (3.37), and obtain \( R \) in the following form:

\[ R = k \left[ W(\rho) + \left( \frac{\tilde{c}}{\rho^2} + \kappa^2 c \right) \left( \left( \partial_{\rho} \phi \right)^2 - \left( \partial_{\rho} \phi_{bg}^{(1)} \right)^2 \right) \right] \]

\[ - \tilde{c} \left( \Delta \phi \partial_{\rho} \phi - \Delta \phi_{bg} \partial_{\rho} \phi_{bg} \right) \]

\[ - 2c \left( \partial_{\rho} \left( \Delta \phi \right) \partial_{\rho} \phi - \partial_{\rho} \left( \Delta \phi_{bg} \right) \partial_{\rho} \phi_{bg} \right), \] (3.41)
where
\[ k \equiv \frac{2\mu}{1-2a}, \quad \tilde{c} \equiv (1-a)(1-2\mu^2 C) - 4c, \]
\[ W(\rho) \equiv \frac{1+a}{4} \left( \Delta \phi \right)^2 - 2c \left( \left( \Delta \phi \right)^2 - \left( \Delta \phi_{bg} \right)^2 \right) - \kappa^2 (\bar{g} - Q'). \] (3.42)

Let us have a look at \( R \) (3.41). The contribution most interesting for us is given by the second term in it. The other terms in (3.41) contain either \( \Delta \phi \) or \( \partial_\rho \left( \Delta \phi \right) \), i.e., are dependent on the density profile or on its derivatives. In other words, these terms are more significant either within the core or near its boundary.

Let us rewrite \( R \) (3.41) once again using the explicit expression for the background solution of the first order \( \phi_{bg} \equiv -b/(2\pi) \log \rho \). We obtain:
\[ R = \left[ k \left[ W(\rho) + \left( \frac{\tilde{c}}{\rho^2} + \kappa^2 c \right) \partial_\rho f_S \left( \partial_\rho f_S - 2\partial_\rho \phi_{bg} \right) \right] + \tilde{c} \Delta \phi \frac{\partial_\rho f_S}{\rho} + 2c \partial_\rho \left( \Delta \phi \right) \partial_\rho f_S \right. \]
\[ \left. - \tilde{c} \left( \Delta \phi - \Delta \phi_{bg} \right) \frac{\partial_\rho \phi_{bg}}{\rho} - 2c \partial_\rho \left( \Delta \phi - \Delta \phi_{bg} \right) \frac{\partial_\rho \phi_{bg}}{\rho} \right]. \] (3.43)

Now the structure of \( R \) can be characterized as follows. The first term, \( W(\rho) \), is determined by the density profile, and it seems to be significant rather within the core since the constant \( Q' \) can be removed outside \( 0 \leq \rho \leq \rho_c \) by an appropriate choice of \( C' \) in \( \bar{g} \) (3.30).

The second term in (3.43) behaves as \( \rho^{-4} \) at \( \rho \ll 1 \), and therefore it is just responsible for the fact that the stress function to be found \( f \) is expected to cancel exactly the most important term (\( \propto \log^2 \rho \)) in \( f_{bg} \) at \( \kappa \rho \ll 1 \):
\[ \Delta \Delta (f - f_{bg}) = -\frac{b^2}{4\pi^2} \frac{k\tilde{c}}{\rho^4}, \quad \Delta \Delta f_{bg} = \frac{b^2}{4\pi^2} \frac{k\tilde{c}}{\rho^4}, \]
while \( f_{bg} \) itself is given by Eq. (5.12) below. The third and the fourth terms are concerned with the density profile and with its derivatives. It can be assumed that the last two terms imply an increment of the corresponding quantities in the gauge approach due to a replacement of the density profile of the background defect by the density profile of the modified defect. In what follows, we shall neglect a possible influence of the last two contributions. Thus we obtain:
\[ R \approx k \left[ W(\rho) + \left( \frac{\tilde{c}}{\rho^2} + \kappa^2 c \right) \partial_\rho f_S \left( \partial_\rho f_S - 2\partial_\rho \phi_{bg} \right) \right. \]
\[ \left. + \tilde{c} \Delta \phi \frac{\partial_\rho f_S}{\rho} + 2c \partial_\rho \left( \Delta \phi \right) \partial_\rho f_S \right], \] (3.44)
where we put approximately
\[ W(\rho) \approx \frac{1+a}{4} \left( \Delta \phi \right)^2 - \kappa^2 (\bar{g} - Q'). \] (3.45)
4 Solution of the gauge equation

4.1 Preparation

Now the task is to solve Eq. (3.36) with the R.H.S. given by $R$ (3.44) with $W(\rho)$ (3.45). We shall do it in two steps. As the first step we shall solve equation

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - z^2\right] G(z) = k R(z), \quad (4.1)$$

where the variable $z$ implies the radial coordinate $\rho$ rescaled as follows: $z = \kappa \rho$, $d/dz = \kappa^{-1} d/d\rho$. Equation (4.1) is a non-homogeneous Bessel equation [30], and its new source (after multiplication of equation by $\rho^2$), though again denoted by $R$, is written in the new variables:

$$R(z) \equiv w_c(z) z^2 + X^2 (\tilde{c} + cz) K_1(z) \left(K_1(z) - \frac{2}{z}\right)$$

$$- X^2 z K_1(z) \left(\tilde{c} \Delta_{\tilde{z}} \tilde{\phi} + 2c z \frac{d}{dz} (\Delta_{\tilde{z}} \tilde{\phi})\right), \quad (4.2)$$

where $X^2 \equiv \kappa^2 \frac{b^2}{4 \pi^2}$, $\Delta \equiv \frac{d^2}{dz^2} + z^{-1} \frac{d}{dz}$, $K_1(z)$ is the modified Bessel function [30], and $\tilde{\phi}$ implies $\phi$ with removed factor $\frac{b}{2\pi}$.

In order to explain the notation $w_c(z)$ in (4.2), let us have a look at $W(\rho)$ (3.45). Let us choose the constant $C'$, which appears in $\tilde{g}$ (3.30), equal to the constant $Q'$ defined by (3.23). Then we obtain for $W(\rho)$:

$$W(\rho) = \left(\frac{b}{2\pi}\right)^2 \frac{1}{\rho_c^2} \left((1 + a) - \frac{\kappa^2}{2} (\rho^2 - \rho_c^2)\right) h_{[0, \rho_c]}(\rho), \quad (4.3)$$

where $C''$ (see $\tilde{g}$ (3.30)) is fixed by requirement of continuity of the density profile at $\rho_c$:

$$C'' + \frac{b^2}{8 \pi^2 \rho_c^2} = Q'.$$

After the replacement of $\kappa \rho$ by $z$, we obtain for $\kappa^{-2}W(\rho)$:

$$\frac{X^2}{z_c^4} \left[(1 + a) - \frac{1}{2} (z^2 - z_c^2)\right] \tilde{h}_{[0, z_c]}(z) \equiv w_c(z), \quad (4.4)$$

where $z_c \equiv \kappa \rho_c$, and $\tilde{h}_{[0, z_c]}(z)$ is equal to 1 at $z \in [0, z_c]$ or to zero, otherwise.

Solution of (4.1) is based on a knowledge of the asymptotical behaviour of $R$ (4.2) at $z \gg 1$ and $z \ll 1$. Let us obtain the corresponding expansions. First of all, let us take into account that $\Delta_{\tilde{z}} \tilde{\phi}$ is exponentially small at large $z$ (or even zero, provided the replacement (3.29) is made) since it is equal to $-K_0(z)$. Therefore, $R$ is localized at large $z$.

In order to study the case $z \ll 1$, let us represent $R$ as a sum:

$$R \equiv w_c(z) z^2 + R_1(z) + R_2(z) + R_3(z).$$
We obtain the following expansions for $R_1, R_2, R_3$ at small $z$:

\[
\frac{R_1}{X^2} \equiv \left( \tilde{c} + cz^2 \right) K_1(z) \left( K_1(z) - \frac{2}{z} \right) \\
\simeq -\frac{\tilde{c}}{z^2} - c + \frac{\tilde{c}}{4} z^2 \log^2 z - \frac{\tilde{c}}{4} \left( 1 - 2 \log \frac{\gamma}{2} \right) z^2 \log z \\
+ \frac{\tilde{c}}{16} \left( 1 - 2 \log \frac{\gamma}{2} \right)^2 z^2 + o(z^2 \log^2 z); \tag{4.5}
\]

\[
\frac{R_2}{X^2} \equiv -\tilde{c} z K_1(z) \Delta_x \tilde{\phi} \approx \frac{2\tilde{c}}{z^2} z K_1(z) \tilde{h}_{[0,z_c]}(z) \\
\simeq \frac{2\tilde{c}}{z^2} \tilde{h}_{[0,z_c]}(z) \left( 1 + \frac{1}{2} z^2 \log z - \frac{z^2}{4} \left( 1 - 2 \log \frac{\gamma}{2} \right) + o(z^2) \right); \tag{4.6}
\]

\[
\frac{R_3}{X^2} \equiv -2c z^2 K_1(z) \frac{d}{dz} \left( \Delta_x \tilde{\phi} \right) \\
\simeq -2c - 2c z^2 \log z + c \left( 1 - 2 \log \frac{\gamma}{2} \right) z^2 + o(z^2). \tag{4.7}
\]

In (4.6), we took into account the replacement (3.29). For a comparison, the exact expansion for $R_2$ looks as follows:

\[
\frac{R_2}{X^2} \simeq -\tilde{c} \log \left( \frac{\gamma}{2} z \right) + \frac{\tilde{c}}{4} z^2 \left( 1 - 2 \log \frac{\gamma}{2} \right) + o(z^2). \tag{4.8}
\]

It is seen from (4.8) that the terms $\propto \log z$ and $\propto z^2 \log^2 z$ are absent in the approximate expression (4.6).

Expansions (4.4)–(4.8) suggest the following series form of $R$ (4.2) at $z \ll 1$:

\[
R(z) \simeq p_0 z^2 + p_1 z^{-2} + p_2 \log z + p_3 \\
+ p_4 z^2 \log^2 z + p_5 z^2 \log z + p_6 z^2 + o(z^2), \tag{4.9}
\]

where $p_0 \equiv p_0(z)$ implies $w_c(z)$ (4.4). The coefficients $p_1, p_2, p_3, p_4, p_5, p_6$ are influenced by our assumptions, and we obtain them as follows:

\[
p_1 = -X^2 \tilde{c}, \\
p_3 = X^2 \left( \frac{2\tilde{c}}{z^2_c} - c \right), \\
p_4 = X^2 \frac{\tilde{c}}{4}, \\
p_5 = X^2 \left( \frac{\tilde{c}}{z^2_c} - \frac{\tilde{c}}{4} \left( 1 - 2 \log \frac{\gamma}{2} \right) \right), \\
p_6 = X^2 \frac{\tilde{c}}{16} \left( 1 - 2 \log \frac{\gamma}{2} \right) \left( 1 - 2 \log \frac{\gamma}{2} - \frac{8}{z^2_c} \right). \tag{4.10}
\]

Practically, only $R_1$ (4.5) and $R_2$ (4.6) are taken into account in order to assign the specific values (4.10) to the corresponding coefficients in the expansion (4.9). The contribution
\( R_3 \) (4.7) is excluded from the consideration since we use (3.29) and neglect possible contributions which should be important near the boundary of the core. However, a possible implication of \( R_3 \) for the coefficients in (4.9) would be given by shifts in \( p_3, p_5, p_6 \) by numerical constants dependent on \( c \) (3.39) (for instance, \(-c \) in \( p_3 \) would be replaced by \(-3c \)). A usage of \( R_2 \) in the form (4.8) instead of the approximate expression (4.6) would remove in \( p_3, p_5, p_6 \) (4.10) the dependence on the value of the core radius \( z_c \). Besides, usage of (4.8) would lead to non-zero contributions in \( p_2 \) and \( p_4 \). But since our choice of the density profile is given by (3.29), the coefficient \( p_2 \) is simply zero. By formal reasons, we keep the corresponding terms in (4.9) with unspecified \( p_2 \) which is to be equated to zero at the very end of the calculation.

### 4.2 The solution

Therefore, let us first consider the non-homogeneous equation:

\[
\left[ z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - z^2 \right] y(z) = k R(z). \tag{4.11}
\]

General solution to (4.11) is given by the standard formula [30]:

\[
k^{-1} y(z) = A y_1(z) + B y_2(z) - \int_{z_0}^{z} \frac{y_1(z) y_2(t) - y_2(z) y_1(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)} \frac{R(t)}{t^2} \, dt.
\] \( \tag{4.12} \)

Here \( z_0 \) is to be adjusted, and \( y_1(z), y_2(z) \) are linearly independent solutions of the corresponding homogeneous equation.

In our case we put \( y_1(z) = I_0(z) \) and \( y_2(z) = K_0(z) \) [30], and (4.12) gives us solution to (4.11) as follows. Let us define \( G(z, s) \):

\[
k^{-1} G(z, s) = I_0(z) \left( A(s) + \int_{s}^{z} K_0(t) R(t) \frac{dt}{t} \right) + K_0(z) \left( B(s) - \int_{s}^{z} I_0(t) R(t) \frac{dt}{t} \right), \tag{4.13.1}
\]

where

\[
A(s) \equiv -\int_{s}^{\infty} K_0(t) R(t) \frac{dt}{t}, \tag{4.13.2}
\]

\[
B(s) \equiv \text{const} + \int_{s}^{1} \left( \frac{p_1}{t^3} + \frac{p_2}{t} \log t + \left( \frac{p_1}{4} + p_3 \right) \frac{1}{t} \right) \, dt.
\]

Then, solution to (4.11) appears as

\[
G(z) = \lim_{s \to 0} G(z, s) \equiv G(z, 0). \tag{4.14}
\]

Asymptotical behaviour of \( G(z) \) can be obtained from (4.13), (4.14). At large \( z \), \( G(z) \) decays exponentially, i.e., \( G(z) \) is well localized. Using the expansions provided in
Appendix A, it is straightforward to establish the behaviour of $G(z)$ at small $z$:

$$
k^{-1} G(z) \simeq q_0 + q_1 z^{-2} + q_2 \log^3 z + q_3 \log^2 z + q_4 \log z \tag{4.15}
+ q_5 z^2 \log^3 z + q_6 z^2 \log^2 z + q_7 z^2 \log z + q_8 z^2,
$$

where

$$q_0 = -const \times \log \frac{\gamma}{2} - I_K - \frac{p_1}{16},
q_1 = \frac{p_1}{4}, \quad q_2 = \frac{p_2}{6}, \quad q_3 = \frac{p_1}{8} + \frac{p_3}{2}, \quad q_4 = -const + \frac{3p_1}{4},
q_5 = \frac{p_2}{24}, \quad q_6 = \frac{p_1}{32} + \frac{p_3 - p_2}{8} + \frac{p_1}{4},
q_7 = -\frac{const}{4} + \frac{p_1}{8} + \frac{3p_2}{16} - \frac{p_3}{4} - \frac{p_1}{2} + \frac{p_5}{4} + q_8 = \frac{p_0 - I_K}{4} + \frac{const}{4} \left(1 - \log \frac{\gamma}{2}\right) - 5p_1 \frac{1}{32} - \frac{p_2}{8} + \frac{3p_3}{16} + \frac{3p_4}{8} + \frac{p_6 - p_5}{4}.
$$

In (4.16), $p_1, \ldots, p_6$ are given by (4.10), $I_K$ is given by (A8) in Appendix A, and $const$ is introduced by the definition of $B(s)$ (4.13.2). It is seen that the term $p_2 \log z$ in (4.9) is just responsible for the highest powers of the logarithm in (4.15): $q_2 \log^3 z$ and $q_5 z^2 \log^3 z$ (and similarly for the logarithms at higher powers of $z^2$).

As a second step, we are going to find the modified stress potential of second order $f = f_{\log} + \mathcal{F}$, where $\mathcal{F}$ respects the Bessel equation

$$
\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + z^2 \right] \mathcal{F}(z) = \frac{z^2}{N^2} G \left(\frac{\kappa}{N} z\right),
$$

In (4.17), the variable $z$ is defined differently, $z \equiv \mathcal{N} \rho$, $\mathcal{N}^2 \equiv \frac{\kappa^2}{1 - 2a}$, and $G(z)$ is given by (4.13), (4.14). Using again (4.12), we obtain solution to (4.17) in the following form:

$$
\mathcal{F}(\rho) = C \tilde{Y}_0(\mathcal{N} \rho) + D \mathcal{J}_0(\mathcal{N} \rho) + I_\mathcal{F}(\rho),
$$

$$
I_\mathcal{F}(\rho) \equiv \mathcal{J}_0(\mathcal{N} \rho) \int_\rho^\infty \tilde{Y}_0(\mathcal{N} \kappa t) G(\kappa t) t dt - \mathcal{Y}_0(\mathcal{N} \rho) \int_\rho^\infty \mathcal{J}_0(\mathcal{N} \kappa t) G(\kappa t) t dt,
$$

where $\tilde{Y}_0(\rho) \equiv (\pi/2) Y_0(\rho)$, and $Y_0(\rho)$, $\mathcal{J}_0(\rho)$ are the Bessel functions which solve the homogeneous version of (4.17).

For our purposes it is appropriate to put $D = 0$ and $C \neq 0$ in (4.18). Now let us write the asymptotical results for $\mathcal{F}(\rho)$. At large $\rho$, the contribution given by $I_\mathcal{F}$ is exponentially small, and thus $\tilde{Y}_0(\mathcal{N} \rho)$ dominates in $\mathcal{F}(\rho)$ at $\mathcal{N} \rho \gg 1$:

$$
C \tilde{Y}_0(\mathcal{N} \rho) \simeq C \left(\frac{\pi}{2 \mathcal{N} \rho}\right)^{1/2} \sin(\mathcal{N} \rho - \frac{\pi}{4}).
$$

For small $\mathcal{N} \rho$, we obtain (Appendix B):

$$
\mathcal{F}(\rho) \simeq r_0 + r_1 \log \rho + r_2 \log^2 \rho \tag{4.19}
+ r_3 \rho^2 \log^3 \rho + r_4 \rho^2 \log^2 \rho + r_5 \rho^2 \log \rho + r_6 \rho^2 + r_7 \rho^4 \log^3 \rho.
$$
where

\[ r_0 = -\tilde{I}_Y + \log\left(\frac{N}{2}\right)(C + \tilde{I}_J), \]
\[ r_1 = C + \tilde{I}_J, \quad r_2 = k \frac{p_1}{8\kappa^2}, \quad r_3 = k \frac{p_2}{24}, \]
\[ r_4 = \tilde{J}_5 - \tilde{Y}_5 + 3 \log N (\tilde{J}_4 - \tilde{Y}_4) \]
\[ = k \left[ \left(1 - \frac{N^2}{\kappa^2}\right) \frac{p_1}{32} - (1 - \log \kappa) \frac{p_2}{8} + \frac{p_3}{8} \right], \quad (4.20) \]
\[ r_5 = \tilde{J}_6 - \tilde{Y}_6 + 2 \log N (\tilde{J}_5 - \tilde{Y}_5) + 3 \log^2 N (\tilde{J}_4 - \tilde{Y}_4) - C \frac{N^2}{4}, \]
\[ r_6 = \tilde{J}_7 - \tilde{Y}_7 + \log N (\tilde{J}_6 - \tilde{Y}_6) + \log^2 N (\tilde{J}_5 - \tilde{Y}_5) \]
\[ + \log^3 N (\tilde{J}_4 - \tilde{Y}_4) + C \frac{N^2}{4} \left(1 - \log \left(\frac{N}{2}\right)\right), \]
\[ r_7 = \tilde{J}_9 - \tilde{Y}_9 = k \frac{\kappa^2 - N^2}{384} p_2. \]

In (4.20), \( \tilde{I}_Y, \tilde{I}_J, \) and \( \tilde{J}_i, \tilde{Y}_i \) (\( i = 4, \ldots, 9 \)) are defined in Appendix B, and \( p_1, \ldots, p_6 \) are given by Eqs. (4.10).

5 The stress tensor

5.1 The components \( \sigma_{\rho\rho} \) and \( \sigma_{\phi\phi} \)

In the previous section we found \( F(\rho) \) which respects the inhomogeneous gauge equation (3.36) with the source term \( R(\rho) \) taken in the approximate form (3.44). The function \( F(\rho) \) implies the difference \( f(\rho) - f_{bg}(\rho) \), where \( f(\rho) \) is the modified stress potential of second order, and \( f_{bg}(\rho) \) is the background stress potential. Practically, \( F(\rho) \) is given by the set of integral representations (4.13), (4.14), and (4.18). However, we shall not attempt to elaborate a single formula which would express \( F(\rho) \) through \( R(\rho) \) more explicitly. The most important for us asymptotical properties of the stress field of the modified screw dislocation can be obtained just from Eqs. (4.13), (4.18).

Equation (4.18) allows to express the modified stress potential \( f \) as follows:

\[ f = f_{bg} + C \tilde{Y}_0(N\rho) + I_F, \quad (5.1.1) \]
\[ f_{bg} = -k \frac{p_1}{8\kappa^2} \log^2 \rho + d_1 \rho^2 + d_2 \log \rho, \quad (5.1.2) \]

where the classical second order stress potential \( f_{bg} \) (5.1.2) is written in the form suggested by \([3]\). Free parameters \( d_1 \) and \( d_2 \) in (5.1.2) are determined in \([3]\) from the requirement that the second order stress \( (2) \sigma_{\rho\rho} \) vanishes at the boundaries of a hollow cylinder \( \rho = \rho_c \) and \( \rho = \rho_c > \rho_c \). In what follows, it will be seen that the cut-off at \( \rho = \rho_c \) disappears in the gauge model proposed.
The second order stress tensor of the modified screw dislocations is given by the following relations:

\[
(2) \sigma_{\rho\rho} = - \frac{1}{\rho} \frac{d}{d\rho} f, \quad (2) \sigma_{\phi\phi} = - \frac{d^2}{d\rho^2} f, \quad (2) \sigma_{zz} = p, \quad (5.2.1)
\]

where \(f\) is given by (5.1) and \(p\) is given by (3.35). Other components of \(\sigma\) are zero. It is most important for us to consider the limiting behaviour of the solution (5.1), (5.2) at \(N\rho \gg 1\) and \(N\rho \ll 1\). Let us begin with the case

A) \(N\rho \ll 1\)

Since our attention is attracted now to \(\sigma_{\rho\rho}\) and \(\sigma_{\phi\phi}\), let us write, using (4.19) and (5.1), their expansions as follows:

\[
(2) \sigma_{\rho\rho} = -2 \left( r_2 - k \frac{p_1}{8\kappa^2} \right) \frac{\log \rho}{\rho^2} - (r_1 + d_2) \frac{1}{\rho^2} - 2r_3 \log^3 \rho - (3r_3 + 2r_4) \log^2 \rho \\
- 2(r_4 + r_5) \log \rho - (r_5 + 2r_6 + 2d_1) - 4r_7 \rho^2 \log^3 \rho,
\]

\[
(2) \sigma_{\phi\phi} = 2 \left( r_2 - k \frac{p_1}{8\kappa^2} \right) \frac{\log \rho}{\rho^2} - (2(r_2 - k \frac{p_1}{8\kappa^2}) - r_1 - d_2) \frac{1}{\rho^2} \\
- 2r_3 \log^3 \rho - (9r_3 + 2r_4) \log^2 \rho - 2(3r_3 + 3r_4 + r_5) \log \rho \\
- (3r_5 + 2r_4 + 2r_6 + 2d_1) - 12r_7 \rho^2 \log^3 \rho. \quad (5.2.2)
\]

It was noticed in Subsection 4.1 that for our choice of the density profile, the series expansion \(R (4.9)\) at small distances is missing the term corresponding to the coefficient \(p_2\). In its turn, the vanishing of \(p_2\) just implies absence of the terms depending on \(r_3\) and \(r_7\) in (4.19) and so in (5.2.2). Besides, the constant term \(r_0\) is irrelevant for the stress components \(\sigma_{\rho\rho}, \sigma_{\phi\phi}\) (5.2.1). The contribution corresponding to \(r_2\) is compensated exactly by the first term in \(f_{bg} (5.1.2)\) (and thus the contribution \(\propto \rho^{-2} \log \rho\) disappears in (5.2.2)).

Therefore, our attention should be paid only to the coefficients containing \(r_1, r_4, r_5, r_6\) in the final expansions (5.2.2). First of all, it is interesting that there exists an opportunity to make \(r_4\) equal to zero: \(r_4 = 0\). Indeed, using \(r_4\) in the form (4.20) (where we put \(p_2 = 0\) and use \(p_1, p_3\) in the form given by (4.10)), we can re-express equation \(r_4 = 0\) as follows:

\[
k \frac{X^2}{8} \left( \frac{2\tilde{c}}{z\tilde{c}} - c - \frac{\bar{c}}{4} \left( 1 - \frac{N^2}{\kappa^2} \right) \right) = 0, \quad (5.3)
\]

or, after the use of \(\bar{c}\) in the form (3.42),

\[
c = \left( \frac{1 - 2\mu^2 C_7}{1 + \nu} - 4c \right) \left( \frac{\nu}{2(1 - \nu)} + \frac{2}{z\tilde{c}} \right). \quad (5.4)
\]

Let us rewrite (5.4) using \(c\) (3.39) as follows:

\[
\frac{2aZ + 1}{Z + 1} = \eta \equiv \frac{4\mu^2(1 + \nu)C_3}{1 - 2\mu^2 C_7}, \quad Z \equiv \frac{1 + \nu}{1 - \nu} \times \frac{z\tilde{c}^2}{8}, \quad (5.5)
\]
where \( a \equiv \nu/(1 + \nu) \). Left-hand side of (5.5) is positive (it is known that \( 0 < \nu \leq 1/2 \) for isotropic materials), and thus \( \eta \) in its R.H.S. is also positive. Therefore, the following two requirements appear:

\[
a) \quad C_3 > 0, \quad C_7 < 1/2\mu^2, \\
b) \quad C_3 < 0, \quad C_7 > 1/2\mu^2.
\]

Provided these requirements are fulfilled, Eq. (5.5) can be solved for \( Z \):

\[
Z = 1 - \frac{\eta}{\eta - 2a}.
\]  

(5.6)

In its turn, Eq. (5.6) results in the following restrictions on the parameters \( C_3, C_7 \):

\[
a) \quad C_7 + 2(1 + \nu) C_3 < \frac{1}{2\mu^2} < C_7 + \frac{(1 + \nu)^2}{\nu} C_3, \\
b) \quad C_7 + \frac{(1 + \nu)^2}{\nu} < \frac{1}{2\mu^2} < C_7 + 2(1 + \nu) C_3.
\]  

(5.7)

(Notice that \( (1 + \nu)/\nu \geq 3 \) at \( 0 < \nu \leq 1/2 \).) For instance, at \( \nu = 1/3 \), Eqs. (5.7) take the form:

\[
a) \quad C_7 + \frac{8}{3} \nu C_3 < \frac{1}{2\mu^2} < C_7 + \frac{16}{3} \nu C_3, \\
b) \quad C_7 + \frac{16}{3} \nu C_3 < \frac{1}{2\mu^2} < C_7 + \frac{8}{3} \nu C_3,
\]  

(5.8)

and

\[
\frac{z_c}{2} = \left( \frac{1 - \eta}{\eta - 1/2} \right)^{1/2}
\]  

(5.9)

in both cases. Provided Eqs. (5.8) are fulfilled, \( \eta \) respects \( 1/2 < \eta < 1 \), and the parameter \( z_c \) (5.9) can formally acquire any real positive value.

Therefore, the requirement \( r_4 = 0 \) turns out to be highly interesting since it gives us a formal expression for the radius of the domain of localization of the gauge dislocation’s density profile (just under our approximation (3.29)).

Now let us focus at the coefficients \( r_1, r_5, r_6 \). Here it is appropriate to impose the following constraints (notice that \( \tilde{J}_4 = \tilde{Y}_4 \) since \( p_2 = 0 \)):

\[
r_1 + d_2 = C + \tilde{J}_f + d_2 = 0, \\
r_5 = \tilde{J}_6 - \tilde{Y}_6 + \log(N^2)(\tilde{J}_5 - \tilde{Y}_5) - \frac{N^2}{4} C = 0, \\
r_6 + d_1 = \tilde{J}_7 - \tilde{Y}_7 + \log N (\tilde{J}_6 - \tilde{Y}_6) + \log^2 N (\tilde{J}_5 - \tilde{Y}_5) + C \frac{N^2}{4} \left( 1 - \log\left( \frac{\gamma}{2N} \right) \right) + d_1 = 0.
\]  

(5.10)

Equations (5.10) simply express the fact that the terms proportional to \( \log \rho \), \( \rho^2 \log \rho \), and \( \rho^2 \) are absent in \( f \) (5.1) at \( \rho \ll 1 \) (see expansion (4.19)). Therefore, under our
conventions all the terms in (5.2.2) vanish. However, possible contributions \( \propto \rho^2 \log^2 \rho \) should be expected provided \( f \) is expanded further.

B) \( \mathcal{N} \rho \gg 1 \)

In this limit \(^3\), the ‘integral’ contribution \( I_F \) in \( f \) (5.1) is exponentially small. Therefore, the asymptotics of \( f \) at some \( \rho_e, \mathcal{N} \rho_e \gg 1 \), is as follows:

\[
f \simeq f_{bg}(\rho_e) + C \left( \frac{\pi}{\mathcal{N} \rho_e} \right)^{1/2} \sin \left( \mathcal{N} \rho_e - \frac{\pi}{4} \right), \tag{5.11}
\]

Using (5.11), we obtain the boundary condition \( \frac{(2)^{\sigma_{pp}}}{\rho^e} = 0 \), i.e., a free surface boundary condition, in the form:

\[
\left. \frac{(2)^{\sigma_{pp}}}{\rho^e} \right|_{\rho = \rho_e} = 2k \frac{p_1}{8\kappa^2} \frac{\log \rho_e}{\rho_e^2} - \frac{d_2}{\rho_e^2} - 2d_1 - C \frac{Y_1(\mathcal{N} \rho)}{\rho_e} = 0, \tag{5.12.1}
\]

where

\[
\bar{Y}_1(\mathcal{N} \rho) \equiv \frac{d\bar{Y}_0(\mathcal{N} \rho)}{d\rho} = -\mathcal{N} \frac{\pi}{2} Y_1(\mathcal{N} \rho). 
\]

Besides, under the condition \( \mathcal{N} \rho \gg 1 \) we obtain:

\[
\frac{(2)^{\sigma_{pp}}}{\rho^e} + \frac{(2)^{\sigma_{\phi\phi}}}{\rho^e} = 2k \frac{p_1}{8\kappa^2} \frac{1}{\rho^2} - 4d_1 + C \mathcal{N}^2 \bar{Y}_0(\mathcal{N} \rho), \tag{5.12.2}
\]

where the representation for \( \frac{(2)^{\sigma_{pp}}}{\rho^e} \) is seen from (5.12.1) (with \( \rho \) instead of \( \rho_e \)). When (5.12.1) is fulfilled, i.e., at \( \rho = \rho_e \), the boundary value of \( \frac{(2)^{\sigma_{\phi\phi}}}{\rho^e} \) is given by R.H.S. of (5.12.2) provided \( \rho \) is replaced by \( \rho_e \). It is seen that the boundary value of \( \frac{(2)^{\sigma_{\phi\phi}}}{\rho^e} \) tends to \(-4d_1 \) at \( \rho_e \to \infty \).

Let us note furthermore that the second equation in (5.10), i.e., the condition \( r_5 = 0 \), can also be fulfilled separately since the difference \( \bar{J}_6 - \bar{Y}_6 \) depends on the coefficient \( q_4 \) (4.16), which, in turn, contains another free parameter (denoted above as \( \text{const} \)) to be adjusted. Provided \( r_5 = 0 \) is fulfilled by a choice of \( \text{const} \), the following three equations, the first and the third Eqs. (5.10), and Eq. (5.12.1), can be written together as a single \( 3 \times 3 \) matrix equation as follows:

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & \mathcal{M}_1 \\
2\rho_e^2 & 1 & \mathcal{M}_2
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
C
\end{pmatrix}
= 
\begin{pmatrix}
l_1 \\
l_2 \\
l_3
\end{pmatrix},
\tag{5.13}
\]

\(^3\mathcal{N} \) is large but \( \rho \) is not necessarily large; validity of the replacement (3.29) suggests that \( \kappa \sim \mathcal{N} \) is large.
where
\[ l_1 \equiv - \tilde{J}_I, \]
\[ l_2 \equiv - (\tilde{J}_7 - \tilde{Y}_7) - \log N(\tilde{J}_6 - \tilde{Y}_6) - \log^2 N(\tilde{J}_5 - \tilde{Y}_5), \]
\[ l_3 \equiv k \frac{p_1}{4n^2} \log \rho e, \]
\[ \mathcal{M}_1 \equiv \frac{N^2}{4} \left( 1 - \log \left( \frac{N}{2} \right) \right), \]
\[ \mathcal{M}_2 \equiv - \frac{\pi}{2} (N \rho_e) Y_1(N \rho_e) \simeq \left( \frac{\pi}{2} N \rho_e \right)^{1/2} \cos (N \rho_e - \frac{\pi}{4}). \]

The determinant \((\equiv D)\) of the matrix in L.H.S. of (5.13),
\[ D = 2 \mathcal{M}_1 \rho_e^2 - \mathcal{M}_2 + 1 \]
\[ \text{(5.14)} \]
is non-zero. Therefore, Eqs. (5.13) can be solved for \(d_1, d_2, C\), and by the Cramer’s rule we obtain:
\[ d_1 = D^{-1} \left( (1 - \mathcal{M}_2) l_2 + \mathcal{M}_1 (l_3 - l_1) \right), \]
\[ d_2 = D^{-1} \left( (2 \rho_e^2 \mathcal{M}_1 - \mathcal{M}_2) l_1 - 2 \rho_e^2 l_2 + l_3 \right), \]
\[ C = D^{-1} \left( l_1 + 2 \rho_e^2 l_2 - l_3 \right). \]
\[ \text{(5.15)} \]

It should be noticed that Eqs. (5.14), (5.15) can be simplified at \(N \rho_e \gg 1\).

Therefore, all free constants at our disposal, i.e., \(d_1, d_2\) (see (5.1)), \(C\) (see (4.18)), \(\text{const}\) (see (4.13.2)), \(z_c = \kappa \rho_c\) (see (3.29)), \(C'\) and \(C''\) (see (3.30)) are fixed. So far, only \(Q'\) (3.23) remains to be chosen. It is remarqueable that the approach developed allows to determine the core radius \(\rho_c = \frac{z_c}{\kappa}\) as a function of two elastic moduli of second order (say, \(\mu\) and \(\nu\)) and of two third order elastic moduli \(C_3\) and \(C_7\) (under our assumption about the simplified density profile). For the given choice of the parameters, all the contributions in \(f\) (5.1) at \(\rho \ll 1\) up to those \(\propto \rho^4 \log^2 \rho\) (apart from the irrelevant constant \(r_0\)) are zero. This means that leading non-vanishing contribution to \(\sigma_{\rho \rho}, \sigma_{\phi \phi}\) should be expected as \(\propto \rho^2 \log^2 \rho\) at \(\rho \to 0\). The corresponding term can be deduced explicitly after a straightforward calculation: the contribution \(\propto \rho^4 \log^2 \rho\) in \(f\) must be accounted for in that case. Thus, \(\sigma_{\rho \rho}, \sigma_{\phi \phi}\) tend to zero in our model at \(\rho \to 0\). Besides, \(\sigma_{\rho \rho}\) is zero at the external boundary \(\rho = \rho_e\).

### 5.2 The component \(\sigma_{zz}\)

Eventually, let us consider the stress component \(\sigma_{zz} = p\). For the background contribution, we obtain from (3.24):
\[ (\sigma_{bg})_{zz} = - \frac{a}{1 - a} \Delta f_{bg} - \frac{2 \mu}{1 - a} (\Psi_{zz}^{bg} + Q'). \]
In the classical approach we consider $\langle \sigma_{\text{bg}} \rangle_{zz}$ (5.16) at $\rho_c \leq \rho \leq \rho_e$, and we determine $Q'$ from the requirement

$$
\int_{\rho_c}^{\rho_e} \int \sigma_{zz} \rho \, d\rho \, d\varphi = 0,
$$

which is to express that a “mean” value of $\sigma_{zz}$ (i.e., $\sigma_{zz}$ averaged over cross-section of the bulk) is zero. So, in order to determine $Q'$, we take $f_{\text{bg}}$ in the form given by (5.1.2), and we also take into account Eqs. (3.21), (3.24) where \( \phi_{\text{bg}} \equiv (-b/2\pi) \log \rho \) is used for evaluation of $\Psi_{\text{zz}}^{\text{bg}}$. Then, using (5.16) and (5.17) we find an expression for $Q'$. Substituting this $Q'$ into (5.16) we obtain:

$$
\langle \sigma_{\text{bg}} \rangle_{zz} = \left[ \frac{2b^2 \mu^3 C_3}{2\pi} + k \frac{p_1}{4\kappa^2} \frac{a}{1-a} \right] \left[ \frac{1}{\rho^2} - \frac{2}{\rho_c^2 - \rho_e^2} \log \frac{\rho_e}{\rho_c} \right],
$$

where $\nu = a(1-a)^{-1}$ and $(1-a)^{-1} = 1 + \nu$.

Using Eqs. (3.8) and (3.9), one can re-express the coefficient $k \frac{p_1}{4\kappa^2}$ (where $p_1$ is given by (4.10) with $\bar{c}$ in it given by (3.42), and $X^2$ is introduced in (4.2)) as follows:

$$
k \frac{p_1}{4\kappa^2} = -\left( \frac{b}{2\pi} \right)^2 \frac{\mu}{2(1-\nu)} \left[ \frac{n'}{4\mu} - 4\mu^2 (1+\nu) C_3 \right],
$$

$$
= -\left( \frac{b}{2\pi} \right)^2 \left( \mu + \frac{1-2\nu}{1-\nu} \frac{2m' + n'}{8} \right).
$$

Furthermore, let us express $C_3$ by means of (5.19.1) and then substitute it into (5.18). Now $\langle \sigma_{\text{bg}} \rangle_{zz}$ acquires the form:

$$
\langle \sigma_{\text{bg}} \rangle_{zz} = \left( k \frac{p_1}{4\kappa^2} + \left( \frac{b}{2\pi} \right)^2 \frac{n'}{8} \right) \left( \frac{1}{\rho^2} - \frac{2}{\rho_e^2 - \rho_c^2} \log \frac{\rho_e}{\rho_c} \right).
$$

Eventually, we use (5.19.2) to re-express $k \frac{p_1}{4\kappa^2}$ in (5.20), and the stress component $\langle \sigma_{\text{bg}} \rangle_{zz}$ appears in the form suggested by [3]:

$$
\langle \sigma_{\text{bg}} \rangle_{zz} = -\left( \frac{b}{2\pi} \right)^2 \left[ \mu + \frac{1}{4(1-\nu)} \left( m'(1-2\nu) - n' \frac{\nu}{2} \right) \right] \left( \frac{1}{\rho^2} - \frac{2}{\rho_e^2 - \rho_c^2} \log \frac{\rho_e}{\rho_c} \right).
$$

It is also important to notice the fact that the integral

$$
\int_{\rho_c}^{\rho_e} \Delta f_{\text{bg}} \rho \, d\rho
$$

is zero, since the choice of $d_1$, $d_2$ in $f_{\text{bg}}$ (5.2) ensures vanishing of $df_{\text{bg}}/d\rho$ at $\rho = \rho_c, \rho_e$ (the boundary condition for $\Psi_{\text{zz}}^{\text{bg}}$ at $\rho = \rho_c, \rho_e$). Therefore, $\langle \sigma_{\text{bg}} \rangle_{zz}$ can also be written in the following form:

$$
\langle \sigma_{\text{bg}} \rangle_{zz} = -\nu \Delta f_{\text{bg}} - 2\mu (1+\nu) \left[ \Psi_{\text{zz}}^{\text{bg}} - \frac{2}{\rho_e^2 - \rho_c^2} \int_{\rho_c}^{\rho_e} \Psi_{\text{zz}}^{\text{bg}} \rho \, d\rho \right].
$$
Equation (5.22) is completely equivalent to (5.18). In view of (5.22), it is more clear, how Eq. (5.17) is fulfilled.

Now let us turn to \( \sigma_{zz} \) of the modified defect given by (3.35):

\[
\sigma_{zz} = -\nu \Delta f - 2\mu (1 + \nu) (\Psi_{33} + \bar{g}) + \kappa^2 (1 + \nu) (f - f_{bg}). \tag{5.23}
\]

Let us consider the asymptotical properties of \( \sigma_{zz} \). Using the expressions for \( (2) \sigma_{\rho\rho}, (2) \sigma_{\phi\phi} \) (5.2.2), the expansion (4.19), and \( \phi \) to express \( \Psi_{33} \) (as well as the nullification conditions (5.10)), we obtain that \( \sigma_{zz} \) behaves as \( \propto \left( r_1 \log \rho + r_2 \log^2 \rho \right) \) plus the contribution tending to zero at \( \rho \to 0 \). At \( \mathcal{N}^{\rho} \gg 1 \) we obtain:

\[
\sigma_{zz} \simeq -4\nu d_1 + \frac{1 + \nu}{1 - \nu} \left[ C \kappa^2 \bar{Y}_0 + \left( \frac{b}{2\pi} \right)^2 \mu^3 \left( 2C_3 + \frac{\nu}{1 + \nu} (C_7 - \frac{1}{2\mu^2}) \right) \frac{1}{\rho^2} \right]. \tag{5.24}
\]

With the help of (5.12.2) and (5.24) we obtain the trace of \( (2) \sigma \) (and thus the total trace, since \( (1) \sigma \) for the screw dislocation is traceless) at \( \mathcal{N}^{\rho} \gg 1 \):

\[
\text{tr} (2) \sigma \equiv I_1(2) = -4(1 + \nu)d_1 + \frac{1 + \nu}{1 - \nu} \left[ 2C \kappa^2 \bar{Y}_0 + \left( \frac{b}{2\pi} \right)^2 \mu^3 \left( 4C_3 + C_7 - \frac{1}{2\mu^2} \right) \frac{1}{\rho^2} \right]. \tag{5.25}
\]

Besides, \( C \) and \( d_1 \) given by (5.15) must be substituted into (5.24) and (5.25). Equations (3.6) and (3.21) demonstrate us how an analogous estimation for the trace of \( (2) e \) can be deduced. Indeed,

\[
I_1(2) = (3C_1 + C_4) I_1(2\sigma) + I_1(\Psi) = (3C_1 + C_4) I_1(2\sigma) - \mu^2 \left( 3C_3 + C_7 \right) (\partial_\rho \phi)^2. \tag{5.26}
\]

However, at \( \kappa \rho \gg 1 \), we estimate \( (\partial_\rho \phi)^2 \simeq b^2/(2\pi \rho)^2 \), and the trace of \( (2) e \) can be deduced by means of (5.25) and (5.26).

Eventually, using

\[
\bar{g} = Q' + \frac{b^2}{8\pi^2 \rho_e^2} \left( \frac{\rho^2}{\rho_e^2} - 1 \right) h_{[0,\rho_e]}(\rho),
\]

and the fact that \( \rho \Delta f \) integrated over \( \rho \) from 0 to \( \rho_e \) vanishes, we determine \( Q' \). Substituting \( Q' \) into (5.23), we find:

\[
\sigma_{zz} = -\nu \Delta f - 2\mu (1 + \nu) \left[ \Psi_{33} - \frac{2}{\rho_e^2} \int_0^{\rho_e} \Psi_{33} \rho d\rho \right] - 2\mu (1 + \nu) \left[ \frac{b^2}{16\pi^2 \rho_e^2} + \frac{b^2}{8\pi^2 \rho_e^2} \left( \frac{\rho^2}{\rho_e^2} - 1 \right) h_{[0,\rho_e]}(\rho) \right] + \kappa^2 (1 + \nu) \left[ f - f_{bg} - \frac{2}{\rho_e^2} \int_0^{\rho_e} (f - f_{bg}) \rho d\rho \right]. \tag{5.27}
\]
We shall not elaborate this expression further. It is enough to notice that the integrals in (5.27) are convergent at the lower bands, and thus $\sigma_{zz}$ (5.27) averaged over cross-section of the bulk is zero.

Before to conclude, let us briefly note another possibility which concerns the choice of the parameters. Namely, the requirement $r_6 + d_1 = 0$ can be left aside. In this case, the stresses $\sigma_{\rho\rho}^{(2)}$, $\sigma_{\phi\phi}^{(2)}$ (5.2.2) will demonstrate a tending, at $\rho \to 0$, to the constant values $\sim (r_6 + d_1)$ (see (5.2.2)). However, in this case it can be assumed that $C = 0$. Then, instead of (5.13), we shall get just two equations to determine $d_1$ and $d_2$. First, we obtain $d_2 = l_1 \equiv -\tilde{I}_J = -r_1$. Then, Eq. (5.12.1) takes the form:

$$\frac{l_3}{\rho^2_e} - \frac{d_2}{\rho^2_e} - 2d_1 = 0,$$

and it gives us

$$d_1 = \frac{l_3 - d_2}{2\rho^2_e} = k \frac{p_1}{8\kappa^2} \frac{\log \rho_e}{\rho^2_e} + \frac{\tilde{I}_J}{2\rho^2_e}.$$

In this case, the asymptotical behaviour of $\sigma_{\rho\rho}^{(2)}$, $\sigma_{\phi\phi}^{(2)}$ is missing the unconventional contribution due to $\tilde{Y}_0$. Let us stress again that the parameters $d_1$ and $d_2$ are still different in comparison with the analogous conventional results.

### 6 Discussion

A model of non-singular screw dislocation lying along an infinitely long cylindric body is investigated in the present paper in the framework of three-dimensional $T(3)$-gauge approach [1]. The gauge part of the total Lagrangian is chosen in the Hilbert–Einstein form, while the elastic contribution to it corresponds to the energy of elastically isotropic continuum given by the terms of second and third orders in the strain components. In other words, a second order elasticity approach is adopted in the present paper.

As it was noticed in [1], second order consideration in the framework of the model [1] would merit attention as an attempt to clarify perspectives of such rather non-traditional approach to defects in solids as the gauge Lagrangian approach (based, for instance, on the groups either $T(3)$ or $ISO(3) \equiv T(3) \otimes SO(3)$). Elaboration of related technical details could clarify the gauge strategy itself concerning a choice of Lagrangian’s constituents, of dimensionality of the specific problems, of resolving ansatz, etc. On the other hand, it is also intriguing to use such a widely acknowledged and fruitful method as the stress function approach [2], [3] within an unconventional non-linear gauge framework. Although the available gauge solutions of the first order [23], [1], [10], [11], [24], [31] seem to be promising, second order consideration could open new aspects of the problem of the gauge description of dislocations.

Let us remind that the linear approach developed in [1] leads to the modified defects which are characterized by the fact that singularities of the ordinary straight dislocations are smoothed out. After the classical attempts [2], [3], [4], [5], higher corrections to the law $1/\rho$ of linear elasticity are known. However, a cut-off near the dislocation axis inevitably occurs [3], [4]. As it is shown above, the gauge approach [1] allows to extend the description of static screw dislocation to the whole cylindric body containing the defect. A use of an approximated density-profile comes to play, and an expression for the
radius of the domain of localization of the defect’s density profile by means of the second and third order elastic moduli appears in the picture proposed in the present paper.

Second order consideration developed above allows to avoid a stress-free boundary condition at an inner radius corresponding to the core radius $\rho = \rho_c$. Besides, it allows to fix the radius $\rho = \rho_c$ as a function of second and third order elastic moduli. It removes a cut-off which occurs in [3], and the stress components $\sigma_{\rho\rho}, \sigma_{\phi\phi}$ turn out to be continuable towards the tube’s axis. As in the classical approach, it is necessary to subject $\sigma^{(2)}$ to a free-surface boundary condition at the outer radius $\rho = \rho_e$. Thus we obtain the solution describing a finite cylinder with nonsingular screw dislocation along its axis. Sufficiently far from the core, the analytical form of the gauge stress potential of second order found above is rather close to the conventional one [3].

Two possibilities are pointed out for the choice of the parameters in the solution found. These possibilities enable $\sigma^{(2)}_{\rho\rho}, \sigma^{(2)}_{\phi\phi}$ to tend either to constant values or to zero. In the first case, the analytical form of $\sigma^{(2)}_{\rho\rho}, \sigma^{(2)}_{\phi\phi}$ in the region $\rho_c \leq \rho \leq \rho_e$ is the same as in [3], but the coefficients are nevertheless different. In the second case, an unconventional contribution is present. However, $\sigma^{(2)}_{zz}$ is logarithmically divergent (the classical divergency is $\sim \rho^{-2}$), at $\rho \to 0$, i.e., in the region where the approximated form of the density profile is most inadequate. In the last case, a weak three-dimensionality may be of help. Some estimations which involve the crystallographic parameters are desirable to make contact with the known interpretations of the characteristic length $\kappa^{-1}$ in terms of interatomic spacing [11] (translational gauging), [25] (non-local elasticity).

A gauge approach close to ours has been proposed in the series of papers [10], [11], [24], [31], which is based on the $T(3)$-gauge Lagrangian written as a combination of the terms quadratic in the torsion components. As to the elastic Lagrangian, it is written in [10], [11], [24] without third order terms (since only the linear problems are studied). However, it is proposed in [31] to use the terms in the Lagrangian which are related to the energy potential of the rotation gradients. Incorporation of such terms in [31] allows to improve the solution found in [11] for that modified defect which demonstrates how the singularity inherent to the classical edge dislocation is smoothed out. In the far field, the stress components found in [31] correctly reproduce those of the edge dislocation. The Hilbert–Einstein Lagrangian is highly suggestive representative among the gauge Lagrangians of the differential–geometric origin. It leads to the self-contained pictures for the modified defects which avoid the singularities of the conventional solutions. However, the contributions of mechanical origin also merit consideration, and further efforts in this direction are also needed.

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Appendices A and B provide some intermediate results which are helpful in obtaining the final asymptotical expressions for the modified stress potential.

First of all, we directly obtain expansions for \( t^{-1}K_0(t)\mathcal{R}(t) \) and \( t^{-1}I_0(t)\mathcal{R}(t) \) at \( t \ll 1 \):

\[
t^{-1}I_0(t)\mathcal{R}(t) \simeq p_1 t^{-3} + p_2 t^{-1} \log t + \left( \frac{p_1}{4} + p_3 \right) t^{-1} + \left( p_4 \log^2 t + \left( \frac{p_2}{4} + p_5 \right) \log t + \hat{k} \right) t + \ldots; \tag{A1}
\]

\[
t^{-1}K_0(t)\mathcal{R}(t) \simeq -p_1 \log\left( \frac{\gamma}{2} \right) t^{-3} - \left( p_2 \log^2 t + k_1 \log t + k_2 \right) t^{-1} - \left( p_4 \log^3 t + k_3 \log^2 t + k_4 \log t + k_5 \right) t + \ldots, \tag{A2}
\]

where the coefficients \( p_1, p_2, p_3, p_4, \) and \( p_5 \) are given by (4.10), and the coefficients \( k_1, k_2, k_3, k_4, k_5 \), are expressed by means of \( p_1, \ldots, p_5 \) as follows:

\[
k_1 = \frac{p_1}{4} + \log\left( \frac{\gamma}{2} \right) p_2 + p_3, \]

\[
k_2 = \frac{p_1}{4} - \log\left( \frac{\gamma}{2} \right) \left( \frac{p_1}{4} + p_3 \right), \]

\[
k_3 = \frac{p_2}{4} + \log\left( \frac{\gamma}{2} \right) p_4 + p_5, \tag{A3}
\]

\[
k_4 = \hat{k} - \left( 1 - \log\left( \frac{\gamma}{2} \right) \right) \frac{p_2}{4} + \log\left( \frac{\gamma}{2} \right) p_5, \]

\[
k_5 = \log\left( \frac{\gamma}{2} \right) \hat{k} - \frac{3}{128} \frac{1}{p_1 - \frac{p_3}{4}}; \quad \hat{k} = w_c(0) + \frac{p_1}{64} + \frac{p_3}{4} + p_6. \]

Now let us obtain estimations for the integrals in (4.13). Using (A1) and (A2) we obtain at \( z \ll 1 \):

\[
B(s) = \int_{s}^{z} I_0(t)\mathcal{R}(t) \frac{dt}{t} \simeq 0 \quad I_0 + I_1 z^{-2} + I_2 \log^2 z + I_3 \log z + \left( I_4 \log^2 z + I_5 \log z + I_6 \right) z^2 + \ldots, \tag{A4}
\]

where

\[
I_0 = \text{const} - \frac{p_1}{2}, \quad I_1 = \frac{p_1}{2}, \quad I_2 = -\frac{p_2}{2}, \quad I_3 = -\frac{p_1}{4} - p_3, \]

\[
I_4 = -\frac{p_1}{2}, \quad I_5 = -\frac{p_2}{8} + \frac{p_4 - p_5}{2}, \quad 2I_6 = -\hat{k} - I_5; \tag{A5}
\]

and

\[
\int_{z}^{\infty} K_0(t)\mathcal{R}(t) \frac{dt}{t} \simeq K_0 + K_1 z^{-2} + K_2 z^{-2} \log z + \left( K_3 \log^3 z + K_4 \log^2 z + K_5 \log z \right) + \left( K_6 \log^3 z + K_7 \log^2 z + K_8 \log z + K_9 \right) z^2, \tag{A6}
\]
where
\[ K_0 = I_K + \left(1 + 2 \log \frac{\gamma}{2}\right) \frac{p_1}{4}, \quad K_1 = -\left(1 + 2 \log \frac{\gamma}{2}\right) \frac{p_1}{4}, \]
\[ K_2 = -\frac{p_1}{2}, \quad K_3 = \frac{p_2}{3}, \quad K_4 = \frac{k_1}{2}, \quad K_5 = k_2, \quad K_6 = \frac{p_4}{2}, \]
\[ K_7 = -\frac{3}{4} p_4 + \frac{k_3}{2}, \quad K_8 = -\frac{k_3 + k_4}{2} + \frac{3}{4} p_4, \]
\[ K_9 = \frac{k_3 - k_4}{4} + \frac{k_5}{2} - \frac{3}{8} p_4. \]
Besides, the coefficient \( I_K \) in \( K_0 \) is given by the regularized value of the integral:
\[ I_K \equiv \int_1^\infty K_0(t) R(t) \frac{dt}{t} + \int_0^1 \left[ K_0(t) R(t) + p_1 \log \left(\frac{\gamma}{2} t\right) t^{-2} + k_2 + k_1 \log t + p_2 \log^2 t \right] \frac{dt}{t}. \]

Using expansions (A4) and (A6), we obtain the expansions we are interested in:
\[ K_0(z) \left[ B(0) - \int_0^z I_0(t) R(t) \frac{dt}{t} \right] \approx \widehat{K}_0 + \widehat{K}_1 z^{-2} + \widehat{K}_2 z^{-2} \log z \]
\[ + \widehat{K}_3 \log^3 z + \widehat{K}_4 \log^2 z + \widehat{K}_5 \log z \]
\[ + \left( \widehat{K}_6 \log^3 z + \widehat{K}_7 \log^2 z + \widehat{K}_8 \log z + \widehat{K}_9 \right) z^2, \]
where
\[ \widehat{K}_0 = -\log \left(\frac{\gamma}{2}\right) I_0 + \left(1 - \log \frac{\gamma}{2}\right) \frac{I_1}{4}, \quad \widehat{K}_1 = -\log \left(\frac{\gamma}{2}\right) I_1, \quad \widehat{K}_2 = -I_1, \]
\[ \widehat{K}_3 = -I_2, \quad \widehat{K}_4 = -\log \left(\frac{\gamma}{2}\right) I_2 - I_3, \quad \widehat{K}_5 = -I_0 - \frac{I_1}{4} - \log \left(\frac{\gamma}{2}\right) I_3, \]
\[ \widehat{K}_6 = -\frac{I_2}{4} - I_4, \quad \widehat{K}_7 = \left(1 - \log \frac{\gamma}{2}\right) \frac{I_2}{4} - \frac{I_3}{4} - \log \left(\frac{\gamma}{2}\right) I_4 - I_5, \]
\[ \widehat{K}_8 = -\frac{I_0}{4} - \frac{I_1}{64} + \left(1 - \log \frac{\gamma}{2}\right) \frac{I_4}{4} - \log \left(\frac{\gamma}{2}\right) I_5 - I_6, \]
\[ \widehat{K}_9 = \left(1 - \log \frac{\gamma}{2}\right) \frac{I_0}{4} + \frac{3}{2} - \log \frac{\gamma}{2} \frac{I_1}{64} - \log \left(\frac{\gamma}{2}\right) I_6. \]

Analogously,
\[ I_0(z) \int_z^\infty K_0(t) R(t) \frac{dt}{t} \approx \widehat{I}_0 + \widehat{I}_1 z^{-2} + \widehat{I}_2 z^{-2} \log z \]
\[ + \widehat{I}_3 \log^3 z + \widehat{I}_4 \log^2 z + \widehat{I}_5 \log z \]
\[ + \left( \widehat{I}_6 \log^3 z + \widehat{I}_7 \log^2 z + \widehat{I}_8 \log z + \widehat{I}_9 \right) z^2, \]
where
\[\hat{I}_0 = \mathcal{K}_0 + \frac{\mathcal{K}_1}{4}, \quad \hat{I}_1 = \mathcal{K}_1, \quad \hat{I}_2 = \mathcal{K}_2, \quad \hat{I}_3 = \mathcal{K}_3,\]
\[\hat{I}_4 = \mathcal{K}_4, \quad \hat{I}_5 = \frac{\mathcal{K}_2}{4} + \mathcal{K}_5, \quad \hat{I}_6 = \frac{\mathcal{K}_3}{4} + \mathcal{K}_6,\]
\[\hat{I}_7 = \frac{\mathcal{K}_4}{4} + \mathcal{K}_7, \quad \hat{I}_8 = \frac{\mathcal{K}_2}{64} + \frac{\mathcal{K}_3}{4} + \mathcal{K}_8,\]
\[\hat{I}_9 = \frac{\mathcal{K}_0}{4} + \frac{\mathcal{K}_1}{64} + \mathcal{K}_9,\]

(A12)

Eventually, we obtain the coefficients characterizing the asymptotical behaviour of the combination
\[-I_0(z) \int_{\frac{z}{\mathcal{K}^0}}^{\infty} K_0(t) R(t) \frac{dt}{t} + K_0(z) \left[ B(0) - \int_0^{\frac{z}{\mathcal{K}^0}} I_0(t) R(t) \frac{dt}{t} \right]\]

by summing up the corresponding coefficients given by (A9) and (A11). We obtain that \(\hat{K}_2 - \hat{I}_2 = 0\) (see (A10) and (A5) for \(\hat{K}_2\), as well as (A12) and (A7) for \(\hat{I}_2\)), and the coefficients \(q_0, \ldots, q_7\) (4.15), (4.16) appear as follows:

\[q_i = \hat{K}_i - \hat{I}_i, \quad \text{at } i = 0, 1;\]
\[q_i = \hat{K}_{i+1} - \hat{I}_{i+1}, \quad \text{at } i = 2, \ldots, 8.\]

Thus, the solution \(G(z)\) given by (4.13), (4.14) is estimated, and the final answer is given by (4.15), (4.16).

Appendix B

First of all, we obtain expansions for \(tJ_0(Nt)G(\kappa t)\) and \(t\tilde{Y}_0(Nt)G(\kappa t)\) at \(t \ll 1\):

\[k^{-1} t \ J_0(Nt)G(\kappa t) \simeq \frac{q_1}{\kappa^2} t^{-1} + q_2 \log^3(Nt)\]
\[+ \left( n_1 \log^2(Nt) + n_2 \log(Nt) + n_3 \right) t\]
\[+ \left( n_4 \log^3(Nt) + n_5 \log^2(Nt) + (\ldots) \log(Nt) + (\ldots) \right) t^3 + \ldots,\]

(B1)

\[k^{-1} t \ \tilde{Y}_0(Nt)G(\kappa t) \simeq \frac{q_1}{\kappa^2} t^{-1} \log\left(\frac{\gamma}{2}Nt\right)\]
\[+ \left( q_2 \log^4(Nt) + m_1 \log^3(Nt) + m_2 \log^2(Nt) + m_3 \log(Nt) + m_4 \right) t\]
\[+ \left( m_5 \log^4(Nt) + m_6 \log^3(Nt) + (\ldots) \right) t^3 + \ldots,\]

(B2)
where
\[ n_1 = 3 \log \left( \frac{\kappa}{N} \right) q_2 + q_3, \]
\[ n_2 = 3 \log^2 \left( \frac{\kappa}{N} \right) q_2 + 2 \log \left( \frac{\kappa}{N} \right) q_3 + q_4, \]
\[ n_3 = q_0 - \frac{N^2 q_1}{\kappa^2} + \log^3 \left( \frac{\kappa}{N} \right) q_2 + \log^2 \left( \frac{\kappa}{N} \right) q_3 + \log \left( \frac{\kappa}{N} \right) q_4, \quad (B3) \]
\[ n_4 = -N^2 q_5 + \kappa^2 q_6, \]
\[ n_5 = -N^2 q_3 + \kappa^2 q_6 + 3 \log \left( \frac{\kappa}{N} \right) n_4, \]
and
\[ m_1 = \log \left( \frac{\gamma}{2} \right) q_2 + n_1, \quad m_2 = \log \left( \frac{\gamma}{2} \right) n_1 + n_2, \]
\[ m_3 = \log \left( \frac{\gamma}{2} \right) n_2 + n_3, \quad m_4 = \frac{N^2 q_1}{\kappa^2} + \log \left( \frac{\gamma}{2} \right) n_3, \quad (B4) \]
\[ m_5 = n_4, \quad m_6 = N^2 q_2 + \log \left( \frac{\gamma}{2} \right) n_4 + n_5. \]

Using (B1)–(B4), we pass to the estimation of the integrals which enter into \( I_F(\rho) \) (4.18). First, we obtain:
\[
\int_0^\infty J_0(Nt) G(\kappa t) t \, dt \simeq J_0 + J_1 \log(N\rho) \\
+ \left( J_2 \log^3(N\rho) + J_3 \log^2(N\rho) + J_4 \log(N\rho) + J_5 \right) \rho^2 \\
+ \left( J_6 \log^3(N\rho) + J_7 \log^2(N\rho) + \ldots \right) \rho^4, \quad (B5)
\]
where
\[ J_0 = I_J, \quad J_1 = -k \frac{q_1}{\kappa^2}, \quad J_2 = -k \frac{q_2}{2}, \]
\[ J_3 = k \left( \frac{3q_2}{4} - \frac{n_1}{2} \right), \quad J_4 = k \left( -\frac{3q_2}{4} + \frac{n_1 - n_2}{2} \right), \]
\[ J_5 = k \left( \frac{3q_2}{8} + \frac{n_2 - n_1}{4} - \frac{n_3}{2} \right), \quad J_6 = -k \frac{n_4}{4}, \quad (B6) \]
\[ J_7 = k \left( \frac{3n_4}{16} - \frac{n_5}{4} \right). \]

Besides, dots in (B5) imply terms proportional to the first and zeroth powers of \( \log(N\rho) \).

The constant \( I_J \) which gives \( J_0 \) will be presented below. Further, we obtain:
\[
\int_0^\infty \tilde{Y}_0(Nt) G(\kappa t) t \, dt \simeq Y_0 + Y_1 \log^2(N\rho) + Y_2 \log(N\rho) \\
+ \left( Y_3 \log^4(N\rho) + Y_4 \log^3(N\rho) + Y_5 \log^2(N\rho) + Y_6 \log(N\rho) + Y_7 \right) \rho^2 \\
+ \left( Y_8 \log^4(N\rho) + Y_9 \log^3(N\rho) + \ldots \right) \rho^4, \quad (B7)
\]
where
\[ Y_0 = \mathcal{I}_Y, \quad Y_1 = -k \frac{q_1}{2k^2}, \quad Y_2 = -k \frac{q_1}{k^2} \log \frac{\gamma}{2}, \quad Y_3 = -k \frac{q_2}{2}, \]
\[ Y_4 = k \left( q_2 - \frac{m_1}{2} \right), \quad Y_5 = k \left( -\frac{3q_2}{2} + \frac{3m_1}{4} - \frac{m_2}{2} \right), \]
\[ Y_6 = k \left( \frac{3q_2}{2} - \frac{3m_1}{4} + \frac{m_2 - m_3}{2} \right), \quad \text{(B8)} \]
\[ Y_7 = k \left( -\frac{3q_2}{4} + \frac{3m_1}{8} + \frac{m_3 - m_2}{4} - \frac{m_4}{2} \right), \]
\[ Y_8 = -k \frac{m_5}{4}, \quad Y_9 = k \frac{m_5 - m_6}{4}, \]

and dots in (B7) corresponds to the second, first, and zeroth powers of \( \log(N\rho) \).

With the expansions (B5) and (B7) at hand, we pass to the products we are interested in to express \( I_{\mathcal{F}} \) (4.18):

\[
\begin{align*}
\tilde{Y}_0(N\rho) \int_0^\infty J_0(N\rho) G(kt) t dt & = \tilde{Y}_0 + \tilde{Y}_1 \log^2(N\rho) + \tilde{Y}_2 \log(N\rho) \\
& + \left( \tilde{Y}_3 \log^4(N\rho) + \tilde{Y}_4 \log^3(N\rho) + \tilde{Y}_5 \log^2(N\rho) + \tilde{Y}_6 \log(N\rho) + \tilde{Y}_7 \right) \rho^2 \\
& + \left( \tilde{Y}_8 \log^4(N\rho) + \tilde{Y}_9 \log^3(N\rho) + \ldots \right) \rho^4,
\end{align*}
\]

where
\[
\begin{align*}
\tilde{Y}_0 & \equiv \log^2 \left( \frac{\gamma}{2} \right) \mathcal{I}_J, \quad \tilde{Y}_1 = \mathcal{J}_1, \quad \tilde{Y}_2 = \mathcal{J}_0 + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_1, \quad \tilde{Y}_3 = \mathcal{J}_2, \\
\tilde{Y}_4 & = \mathcal{J}_3 + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_2, \quad \tilde{Y}_5 = -\frac{N^2}{4} \mathcal{J}_1 + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_3 + \mathcal{J}_4, \\
\tilde{Y}_6 & = \frac{N^2}{4} \left( -\mathcal{J}_0 + (1 - \log \frac{\gamma}{2}) \mathcal{J}_1 \right) + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_4 + \mathcal{J}_5, \quad \text{(B10)} \\
\tilde{Y}_7 & = \frac{N^2}{4} \left( 1 - \log \frac{\gamma}{2} \right) \mathcal{J}_0 + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_5, \quad \tilde{Y}_8 = -\frac{N^2}{4} \mathcal{J}_2 + \mathcal{J}_6, \\
\tilde{Y}_9 & = \frac{N^2}{4} \left( 1 - \log \frac{\gamma}{2} \right) \mathcal{J}_2 - \frac{N^2}{4} \mathcal{J}_3 + \log \left( \frac{\gamma}{2} \right) \mathcal{J}_6 + \mathcal{J}_7. 
\end{align*}
\]

Now we obtain the following expansions:

\[
\begin{align*}
J_0(N\rho) \int_0^\infty \tilde{Y}_0(N\rho) G(kt) t dt & = \tilde{J}_0 + \tilde{J}_1 \log^2(N\rho) + \tilde{J}_2 \log(N\rho) \\
& + \left( \tilde{J}_3 \log^4(N\rho) + \tilde{J}_4 \log^3(N\rho) + \tilde{J}_5 \log^2(N\rho) + \tilde{J}_6 \log(N\rho) + \tilde{J}_7 \right) \rho^2 \\
& + \left( \tilde{J}_8 \log^4(N\rho) + \tilde{J}_9 \log^3(N\rho) + \ldots \right) \rho^4,
\end{align*}
\]

(B11)
where
\[ \hat{J}_0 \equiv I_Y, \quad \hat{J}_i = \mathcal{Y}_i, \quad i = 1, 2, 3, 4, \]
\[ \hat{J}_5 = -\frac{N^2}{4} \mathcal{Y}_1 + \mathcal{Y}_5, \quad \hat{J}_6 = -\frac{N^2}{4} \mathcal{Y}_2 + \mathcal{Y}_6, \]
\[ \hat{J}_7 = -\frac{N^2}{4} \mathcal{Y}_0 + \mathcal{Y}_7, \]
\[ \hat{J}_8 = -\frac{N^2}{4} \mathcal{Y}_3 + \mathcal{Y}_8, \quad \hat{J}_9 = -\frac{N^2}{4} \mathcal{Y}_4 + \mathcal{Y}_9. \]
\[ (B12) \]

Now we sum up expansions (B9) and (B11):
\[ I_F(\rho) = -\rho_0(N\rho) \int_0^\infty J_0(Nt) G(\kappa t) t \, dt + \rho_0(N\rho) \int_0^\infty \rho_0(Nt) G(\kappa t) t \, dt = \]
\[ = \hat{J}_0 - \hat{Y}_0 + (\hat{J}_1 - \hat{Y}_1) \log(N\rho) + (\hat{J}_2 - \hat{Y}_2) \log(N\rho) \]
\[ + ((\hat{J}_4 - \hat{Y}_4) \log^3(N\rho) + (\hat{J}_5 - \hat{Y}_5) \log^2(N\rho) + (\hat{J}_6 - \hat{Y}_6) \log(N\rho) + \hat{J}_7 - \hat{Y}_7) \rho^2 \]
\[ + ((\hat{J}_9 - \hat{Y}_9) \log^3(N\rho) + \ldots) \rho^4, \]
\[ (B13) \]

where
\[ \hat{J}_0 - \hat{Y}_0 = I_Y - \log\left(\frac{\gamma}{2}\right) I_J, \]
\[ \hat{J}_1 - \hat{Y}_1 = k \frac{q_1}{2\kappa^2} = k \frac{p_1}{8\kappa^2}, \]
\[ \hat{J}_2 - \hat{Y}_2 = -I_J, \quad \hat{J}_4 - \hat{Y}_4 = k \frac{q_2}{4} = k \frac{p_2}{24}, \]
\[ \hat{J}_5 - \hat{Y}_5 = k \left( -\frac{N^2}{\kappa^2} \frac{q_1}{8} - \left( 1 - \log \frac{\kappa}{N} \right) \frac{3q_2}{4} + \frac{q_3}{4} \right) = \]
\[ = k \left( 1 - \frac{N^2}{\kappa^2} \right) \frac{p_1}{32} - \left( 1 - \log \frac{\kappa}{N} \right) \frac{p_2}{8} + \frac{p_3}{8}, \]
\[ (B14) \]
\[ \hat{J}_6 - \hat{Y}_6 = \frac{N^2}{4} \left( J_0 - (1 - \log \frac{\gamma}{2}) J_1 - J_2 \right) - \log\left(\frac{\gamma}{2}\right) J_4 - \hat{J}_5 + \mathcal{Y}_6, \]
\[ \hat{J}_7 - \hat{Y}_7 = -\frac{N^2}{4} \left( \mathcal{Y}_0 + (1 - \log \frac{\gamma}{2}) \mathcal{J}_1 - \mathcal{Y}_2 \right) - \log\left(\frac{\gamma}{2}\right) \mathcal{J}_4 - \hat{J}_5 + \mathcal{Y}_7, \]
\[ \hat{J}_9 - \hat{Y}_9 = k \left( -\frac{N^2}{64} q_2 + \frac{\kappa^2}{16} q_5 \right) = k \frac{\kappa^2 - N^2}{384} p_2. \]

The terms corresponding to \( \hat{J}_3 - \hat{Y}_3 \) and \( \hat{J}_8 - \hat{Y}_8 \) do not appear since \( \mathcal{Y}_3 = \mathcal{J}_2 \) and \( m_5 = n_4 \).
Eventually, it is necessary to re-arrange the series (B13) as follows:

\[
I_{\mathcal{F}}(\rho) = \mathcal{I}_Y - \log\left(\frac{\gamma}{2\mathcal{N}}\right)\mathcal{I}_J + k \frac{\eta_1}{2\kappa^2} \log^2 \rho - \mathcal{I}_J \log \rho + k \frac{q_2}{4} \rho^2 \log^3 \rho + \\
+ \left(\left(\mathcal{J}_4 - \mathcal{Y}_4\right) \log(\mathcal{N}^3) + \mathcal{J}_5 - \mathcal{Y}_5\right) \rho^2 \log^2 \rho \\
+ \left(\left(\mathcal{J}_4 - \mathcal{Y}_4\right) 3 \log^2 \mathcal{N} + \left(\mathcal{J}_5 - \mathcal{Y}_5\right) \log(\mathcal{N}^2) + \mathcal{J}_6 - \mathcal{Y}_6\right) \rho^2 \log \rho \\
+ \left(\left(\mathcal{J}_4 - \mathcal{Y}_4\right) \log^3 \mathcal{N} + \left(\mathcal{J}_5 - \mathcal{Y}_5\right) \log^2 \mathcal{N} + \left(\mathcal{J}_6 - \mathcal{Y}_6\right) \log \mathcal{N} + \mathcal{J}_7 - \mathcal{Y}_7\right) \rho^2 \\
+ \left(\mathcal{J}_9 - \mathcal{Y}_9\right) \rho^4 \log^3 \rho, \\
\] (B15)

where \( \mathcal{J}_i - \mathcal{Y}_i, i = 4, \ldots, 9 \) are given by (B14) (where (B6) and (B8) must be used) and the following notations are adopted:

\[
\mathcal{I}_J \equiv \mathcal{I}_J + k \frac{\eta_1}{\kappa^2} \log \mathcal{N}, \\
\mathcal{I}_Y \equiv \mathcal{I}_Y + k \frac{\eta_1}{\kappa^2} \log \mathcal{N} \left(\frac{\log \mathcal{N}}{2} + \log \frac{\gamma}{2}\right), \\
\] (B16)

and

\[
\mathcal{I}_J \equiv \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{\infty} J_0(\mathcal{N}t) G(\kappa t) \frac{1}{t} dt - k \frac{\eta_1}{\kappa^2} \int_{\varepsilon}^{1} \frac{1}{t} dt \right), \\
\mathcal{I}_Y \equiv \lim_{\varepsilon \to 0^+} \left( \int_{\varepsilon}^{\infty} \tilde{Y}_0(\mathcal{N}t) G(\kappa t) \frac{1}{t} dt - k \frac{\eta_1}{\kappa^2} \int_{\varepsilon}^{1} \log \left(\frac{\gamma}{2} \mathcal{N}t\right) \frac{1}{t} dt \right). \\
\] (B17)
References

[1] C. Malyshev The T(3)-gauge model, the Einstein-like gauge equation, and Volterra dislocations with modified asymptotics, Ann. Phys. (NY), 286, No. 2 (2000), 249–277

[2] E. Kröner, A. Seeger Nicht-lineare Elastizitätstheorie der Versetzungen und Eigenspannungen, Arch. Rat. Mech. Anal., 3, No. 2 (1959), 97–119

[3] H. Pfleiderer, A. Seeger, E. Kröner Non-linear elasticity theory of straight dislocations, Z. Naturforsch., 15a (1960), 758–772 [Engl. transl.: United Kingdom Atomic Energy Authority AERE—Trans 1061, 1966]

[4] A. Seeger, E. Mann Use of the non-linear elasticity theory for defects in crystals, Z. Naturforsch., 14a (1959), 154–164 [Engl. transl.: United Kingdom Atomic Energy Authority AERE—Trans 1062, 1966]

[5] Z. Wesolowski, and A. Seeger On the screw dislocation in finite elasticity, In: Mechanics of Generalized Continua. Proceedings of the IUTAM Symposium on the Generalized Cosserat Continuum and the Continuum Theory of Dislocations with Applications (Ed. E. Kröner, Springer, Berlin, etc., 1968), pp. 294–297

[6] A. Seeger The application of second-order effects in elasticity to problems of crystal physics, In: Second-Order Effects in Elasticity, Plasticity, and Fluid Dynamics. Internat. Symp., Haifa, Israel, April 23–27, 1962 (Eds. M. Reiner, D. Abir, Jerusalem Academic Press, Pergamon Press, Oxford, etc., 1964), pp. 129–144

[7] V. A. Osipov Nonlinear elastic problems in dislocation theory: a gauge approach, J. Phys. A: Math. Gen., 24, No. 14 (1991), 3237–3244

[8] A. Kadić, and D. G. B. Edelen A Gauge Theory of Dislocations and Disclinations, Lect. Notes Phys., vol. 174, (Springer-Verlag, Berlin etc., 1983)

[9] D. G. B. Edelen, and D. C. Lagoudas Gauge Theory and Defects in Solids (North-Holland, Amsterdam, etc., 1988)

[10] M. Lazar Dislocation theory as a 3-dimensional translation gauge theory, Ann. Phys. (Leipzig), 9, No. 6 (2000), 461–473

[11] M. Lazar An elastoplastic theory of dislocations as a physical field theory with torsion, J. Phys. A: Math. Gen., 35, No. 8 (2002), 1983–2004

[12] M. O. Katanaev, I. V. Volovich Theory of defects in solids and three-dimensional gravity, Ann. Phys. (NY), 216, No. 1 (1992), 1–28

[13] R. E. Peierls The size of a dislocation, Proc. Phys. Soc., 52, No. 289 (1940), 34–37

[14] F. R. N. Nabarro Dislocations in a simple cubic lattice, Proc. Phys. Soc., 59, No. 332 (1947), 256–272

[15] J. H. van der Merwe On the stresses and energies associated with inter-crystalline boundaries, Proc. Phys. Soc. A, 63, No. 366 (1950), 616–637
[16] A. A. Maradudin *Screw dislocations and discrete elastic theory*, J. Phys. Chem. Solids, 9, No. 1 (1959), 1–20

[17] A. H. Cottrell *Dislocations and Plastic Flow in Crystals*, (Clarendon Press, Oxford, 1953)
A. H. Cottrell *Theory of Crystal Dislocation*, (Gordon and Breach, New York, etc., 1965)

[18] J. P. Hirth, J. Lothe *Theory of Dislocations*, (Wiley, New York, etc., 1982)

[19] C. Teodosiu *Elastic Models of Crystal Defects*, (Springer–Verlag, Berlin, etc., 1982)

[20] B. K. D. Gairola *Nonlinear elastic problems*, In: Dislocations in Solids, vol. 1 (Ed. F. R. N. Nabarro, Elsevier Science Publishers, Amsterdam, etc., 1979), pp. 223-342

[21] Chapter 2, Atomic structure, In: *Dislocations in Solids*, Proceedings of Yamada Conference IX on Dislocations in Solids, Tokyo, August 27–31, 1984 (Eds. H. Suzuki, T. Ninomiya, K. Sumino, S. Takeuchi, University of Tokyo Press, Yamada Science Foundation, 1985), pp. 49–88

[22] *Dislocations 1984*, Comptes Rendus du Colloque International du C.N.R.S. Dislocations: Structure de Coeur et Propriétés Physiques, Aussois, France, Mars 8–17, 1984 (Eds., P. Veyssièrè, L. Kubin, J. Castaing, Editions du C.N.R.S., 1984)

[23] D. G. B. Edelen *A correct, globally defined solution of the screw dislocation problem in the gauge theory of defects*, Int. J. Engng Sci., 34, No. 1 (1996), 81–86

[24] M. Lazar *Screw dislocation in the field theory of elastoplasticity*, Ann. Phys. (Leipzig), 11, No. 9 (2002), 635–649

[25] A. C. Eringen *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys., 54, No. 9 (1983), 4703–4710

[26] E. Kröner *Continuum theory of defects*, In: Physique des Défauts, Les Houches, Session XXXV, 1980 (Eds., R. Balian, et al., North–Holland, Amsterdam, 1981)

[27] H. Kleinert *Gauge Fields in Condensed Matter. Vol. II. Stresses and Defects*, (World Scientific, Singapore, 1989)

[28] G. Sardanashvily *On the geometric foundation of classical gauge gravitation theory*, arXiv: gr-qc/0201074
G. Sardanashvily *Classical gauge theory of gravity*, arXiv: gr-qc/0208054

[29] M. Miri, N. Rivier *Continuum elasticity with topological defects, including dislocations and extra matter*, J. Phys. A: Math. Gen., 35, No. 7 (2002), 1727–1739

[30] W. Magnus, F. Oberhettinger, R. P. Soni *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, etc., 1966)

[31] M. Lazar *A nonsingular solution of the edge dislocation in the gauge theory of dislocations*, J. Phys. A: Math. Gen., 36, No. 5 (2003), 1415–1437