The dimensionality of an electronic quantum system is decisive for its properties. In one dimension, electrons form a Luttinger liquid, and in two dimensions, they exhibit the quantum Hall effect. However, very little is known about the behaviour of electrons in non-integer, or fractional dimensions. Here, we show how arrays of artificial atoms can be defined by controlled positioning of CO molecules on a Cu (111) surface and, how these sites couple to form electronic Sierpiński fractals. We characterize the electron wavefunctions at different energies with scanning tunnelling microscopy and spectroscopy, and show that they inherit the fractional dimension. Wavefunctions delocalized over the Sierpiński structure decompose into self-similar parts at higher energy, and this scale invariance can also be retrieved in reciprocal space. Our results show that electronic quantum fractals can be artificially created by atomic manipulation in a scanning tunnelling microscope. The same methodology will allow future studies to address fundamental questions about the effects of spin–orbit interactions and magnetic fields on electrons in non-integer dimensions. Moreover, the rational concept of artificial atoms can readily be transferred to planar semiconductor electronics, allowing for the exploration of electrons in a well-defined fractal geometry, including interactions and external fields.

Fractals have been investigated in a wide variety of research areas, ranging from polymers, porous systems, electrical storage and stretchable electronics down to molecular and plasmonic fractals. On the quantum level, fractal properties emerge in the electronic structure of the fractal will emerge in an experimentally suitable energy range.

Figure 1e presents the experimental LDOS at the red, blue and green atomic sites in the G(3) Sierpiński triangle (indicated by the open circles in Fig. 1c). The differential conductance (dI/dV) spectra were normalized by the average spectrum taken on the bare Cu(111) surface, similar to ref. 4. The onset of the surface-state band is located at $V = -0.45\, \text{V}$. We focus on the bias window between $-0.4\, \text{V}$ and $0.3\, \text{V}$. Around $V = -0.3\, \text{V}$ the LDOS on the red, green and blue sites is nearly equal, whereas slightly above $V = -0.2\, \text{V}$, the red sites exhibit a distinct minimum, while the green and blue sites show a considerably higher LDOS. At $V = -0.1\, \text{V}$ the blue sites show a minimum, whereas the red and green sites exhibit a pronounced maximum in the LDOS. At $V = +0.1\, \text{V}$ the blue sites show a larger peak in the differential conductance, whereas the green and red sites exhibit a smaller peak. The experimental LDOS is in good agreement with both the tight-binding (see Supplementary Information) and mufﬁn-tin simulations. Figure 1e corroborates that our design leads to the desired confinement of the two-dimensional electron gas to the atomic sites of the Sierpiński geometry. In addition, it allows us to characterize the wavefunctions of the chosen Sierpiński geometry in detail.

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Design and characterization of electrons in a fractal geometry
From the tight-binding calculation, we find that the wavefunction has nodes on the blue sites, corresponding to a non-bonding molecular orbital from a chemical perspective. It is clear that the conductivity along the (R–B–G–B–R)-pathway mediated by nearest-neighbour hopping has vanished, and that electrons have to perform next-nearest-neighbour hopping between the red and green sites to propagate. These results connect with the theoretically calculated transmission of a Sierpiński carpet on a hexagonal lattice, which exhibits a gap in the conductivity although there is a high DOS in the system\(^{22}\). Finally, at \( V = +0.1 \) V, all blue sites in the G(3) Sierpiński structure have a high amplitude, whereas the red and green sites exhibit a low amplitude. Again, the conductivity between source and drain is suppressed. We note that the LDOS maps of the three generations G(1)–G(3) show the same features (see Supplementary Information), which is a consequence of the self-similarity of the geometry. We study this scale-invariance of the wavefunction in more detail with the box-counting method.

To determine whether the electronic wavefunctions inside the Sierpiński structure inherit the scaling properties of the Sierpiński geometry, we determine the fractal dimension of the wavefunction maps at different energies following the procedure in ref. \(^{20}\). We calculate the box-counting dimension (also called the Minkowski–Bouligand dimension) for both the experimental and simulated muffin-tin LDOS maps using \( D = \lim_{N \rightarrow \infty} \frac{\log N(r)}{\log (1/r)} \), with \( N \) the number of squares on a square grid required to cover the contributing LDOS and \( r \) the side length of these squares. In this procedure the wavefunction maps are positioned on a 360 × 360 pixel square grid. The parts of the wavefunction maps that are in the regions containing an agglomeration of CO molecules are excluded by applying a mask. Further, we choose a threshold (55%, 65% and 75% of the maximum LDOS) above which the LDOS contributes to define a binary image. Finally, the box-counting dimension of this image is calculated. The number of boxes \( N \) is counted for 21 box sizes ranging from 1 to 90 pixels. Subsequently, the fractal dimension is given by the slope of the log–log plot for \( N(r) \). Details can be found in the Supplementary Information. A typical log–log plot is presented in Fig. 3a, where the inset shows the binarized experimental LDOS map at \( V = -0.325 \) V. The red slope is calculated for the largest 11 boxes and the blue slope for the smallest 10 boxes. Figure 3b shows the box-counting dimension obtained from these red slopes for the experimental (dark red) and theoretical (light red) wavefunction maps acquired at various energies. For comparison, we also show the dimension obtained for the wavefunction maps of a square lattice (dark and light green, for the experiment and theory, respectively), realized in the same way and measured in the same energy window\(^{7}\). The difference between the experimental and simulated maps is ascribed to a more gradual contrast in the simulation, where also contributions of the tip density of states do not play a role. It can be clearly seen that the box-counting dimension of the Sierpiński triangle is close to the theoretical Hausdorff dimension 1.58 (red solid line), whereas the square lattice has a dimension close to 2 (green solid line). When calculating the box-counting dimension for the blue slopes for the Sierpiński triangle, we find the results shown in Fig. 3c. When using the 10 smallest boxes, the analysis yields higher values of the dimensions for the fractal, but they are still well below 2 and well below the values obtained for the square lattice. This behaviour is well understood (see Supplementary Information). From these results and an additional scaling analysis shown in the Supplementary Information, we conclude that the wavefunctions inherit the fractal dimension and therefore the scaling properties of...
the geometry to which they are confined, and that this dimension can be non-integer.

Finally, we show how the self-similarity of the wavefunction maps is reflected in momentum space. The Fourier-transformed wavefunction map at $V = -325 \text{ mV}$ (Fig. 4a) exhibits distinct maxima at $k = 1.9 \text{ nm}^{-1}$ (turquoise), $k = 1.0 \text{ nm}^{-1}$ (red) and $k = 0.5 \text{ nm}^{-1}$ (yellow). These maxima correspond to the next-nearest-neighbour distances between the artificial atomic sites (see Fig. 1), the side of a $G(1)$ triangle, and the side of a $G(2)$ triangle in real space, respectively. We then transform parts of the Fourier map back into real space (Fig. 4b–d). The data inside the turquoise circle recover the full $G(3)$ Sierpiński triangle, as shown in Fig. 4b. Transforming the values inside the red circle, however, results in a Sierpiński triangle of generation 2, while the size is retained (see Fig. 4c). Analogously, transforming the data inside the yellow circle yield a first-generation Sierpiński triangle (Fig. 4d). This shows that the $G(3)$ wavefunction contains Fourier terms of the previous generations. The self-similar features of the Sierpiński triangle are thus inherently encoded in momentum space.

We have demonstrated a rational concept of building electronic wavefunctions with a fractional dimension from artificial atomic sites that couple in a controlled way. We discussed the wavefunctions that form by coupling the $s$-orbitals of artificial atoms in the single-electron regime. Although this study represents the simplest case, it already exhibits several aspects of fractal confinement. The emergent fractionalization of the wavefunction at the single-particle level has profound implications and opens a series of interesting questions for future investigation: Do electrons in $D = 1.58$ behave like Luttinger liquids? Do they exhibit the fractional quantum Hall effect in the presence of a strong perpendicular

Fig. 2 | Wavefunction mapping. a–d. Differential conductance maps acquired above a $G(3)$ Sierpiński triangle at bias voltages $-0.325 \text{ V}$, $-0.200 \text{ V}$, $-0.100 \text{ V}$ and $+0.100 \text{ V}$. Scale bar: 5nm. e–h. LDOS maps at these energies calculated using the tight-binding model. i–l. LDOS maps simulated using the muffin-tin approximation. As a guide to the eye, a $G(1)$ building block is indicated, in which a larger radius of the circles corresponds to a larger LDOS at an atomic site, whereas no circle indicates a node in the LDOS.
magnetic field, or is the behaviour hybrid between 1D and 2D? How does charge fractionalization manifest when the wavefunction is itself already fractional? Recent theoretical work already addresses parts of these questions and corroborates the potential of electrons in fractal lattices, showing that the Sierpiński carpet and gasket host topologically protected states in the presence of a perpendicular magnetic field\cite{14}. Furthermore, the design of artificial-atom quantum dots coupled in a fractal geometry can also be implemented in semiconductor technology, thus making it possible to perform spectroscopy and transport experiments under controlled electron density. This would form a versatile platform to explore fractal electronics with several internal degrees of freedom, such as orbital type, Coulomb and spin–orbit interactions, as well as external electric and magnetic fields.

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**Author contributions**

S.N.K. did the calculations under the supervision of C.M.S. The experiments were performed by M.R.S. with contributions from S.E.F. and S.J.M.Z. under the supervision of I.S. and D.V. All authors contributed to the interpretation of the data and to the manuscript.

**Additional information**

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Methods

Scanning tunnelling microscope experiments. The scanning tunnelling microscopy and spectroscopy experiments were performed in a Scienta Omicron LT-STM system at a temperature of 4.5 K and a base pressure around 10^{-9} mbar. A clean Cu(111) crystal, prepared by multiple cycles of Ar⁺ sputtering and annealing, was cooled down in the scanning tunnelling microscopy head. Carbon monoxide was leaked into the chamber at p=3×10^{-9} mbar for 3 min and adsorbed at the cold Cu(111) surface. A Cu-coated tungsten tip was used for both the assembly and the characterization of the fractal. The CO manipulation was performed in feedback at I=60 nA and V=50 mV, comparable to previously reported values, and was partly automated using an in-house-developed program. Scanning tunnelling microscopy was performed in constant-current mode. A standard lock-in amplifier was used to acquire differential conductance spectra ($f=973$ Hz, modulation amplitude $5 \text{ mV r.m.s.}$) and maps ($f=273$ Hz, modulation amplitude $10 \text{ mV r.m.s.}$) in constant-height mode. The contrast of the experimental waveform maps as displayed in Fig. 2 was adjusted using the software Gwyddion. For the box-counting analysis of the experimental waveform maps (Fig. 5), Fourier-smoothed images were used with no further adjustments of the contrast. The Fourier analysis (Fig. 4) was performed using the same software.

Tight-binding calculations. The atomic sites in the first three generations of the Sierpiński triangle are modelled as s-orbitals, for which electron hopping between nearest-neighbour and next-nearest-neighbour sites is defined. The parameters used are $t_\sigma = 0.11 \text{ eV}$ for the on-site energy, $t$=0.12 eV for the nearest-neighbour hopping and $t'/t=0.08$ for the next-nearest-neighbour hopping, similar to previously reported values. Furthermore, we included an overlap integral $s=0.2$ between nearest-neighbours and solved the generalized eigenvalue equation $H(p) = S \tilde{S}(p)$, where $S$ is the overlap-integral matrix. The LDOS is calculated at each specific atomic site and a Lorentzian energy-level broadening of $\Gamma = 0.8 \text{ eV}$ is included to account for bulk scattering. For the simulation of the LDOS maps, the same energy-level broadening was used and the LDOS at each site was multiplied with a Gaussian wavefunction of width $\sigma = 0.65a$, where $a = 1.1 \text{ nm}$ is the distance between two neighbouring sites.

Muffin-tin calculations. The surface-state electrons of Cu(111) are considered to form a 2D electron gas confined between the CO molecules, which are modelled as filled circles with a repulsive potential of 0.9 eV and radius $R = 0.55a/2$. The Schrödinger equation is solved for this particular potential landscape, and a Lorentzian broadening of $\Gamma = 0.8 \text{ eV}$ is used to account for the bulk scattering.

Box-counting method. The Minkowski–Bouligand or box-counting method is a useful tool to determine the fractal dimension of a certain image, but has to be handled and interpreted with care. In particular, as has been shown previously, the size of the boxes needs to be chosen within certain length scales. More specifically, the largest box should not be more than 25% of the entire image side and the smallest box is chosen to be the point at which the slope starts to deviate from the linear regime in the log($N$) versus log(1/r) plot. Due to experimental limitations, it is not always possible to fully ‘block’ certain areas using the CO/Cu(111) platform. Redundant features that are not part of the fractal set, such as the Friedel oscillations surrounding the Sierpiński triangle and the LDOS between the closely packed CO molecules in the centres of the G(2) and G(3) Sierpiński triangles, were removed by applying a mask (see Supplementary Information).

Data availability

The experimental and simulated images used in the box-counting analysis, as well as the code, have been published (https://doi.org/10.24416/UU01-N90LX5). The data can be accessed using open-source tools.

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