The parallax distortion via a weak microlensing effect

M.V. Sazhin, V.E. Zharov, T.A. Kalinina
Sternberg Astronomical Institute, Moscow 119899, Russia

Abstract

Parallax measurements allow distances to celestial objects to be determined. Coupled with measurement of their position on the celestial sphere, it gives a full three-dimensional picture of the location of the objects relative to the observer. The distortion of the parallax value of a remote source affected by a weak microlensing is considered. This means that the weak microlensing leads to distortion of the distance scale. It is shown that the distortions to appear may change strongly the parallax values in case they amount to several microseconds of arc. In particular, at this accuracy many measured values of the parallaxes must be negative.

Introduction

Very long baseline interferometry (VLBI) has achieved a precision level of position measurements of tens of microseconds of arc [10], [9]. The VLBI accuracy may soon achieve several microseconds of arc, or the fundamental limit of accuracy of the position measurements being determined by a nonstationary curvature of space-time in our Galaxy [38], [39].

Besides, creation of space interferometers much exceeding the Earth’s diameter is in sight [3], [23], [1]. A 10–100 times increase in the interferometer baseline is considered to allow a precision of position measurements of the order of one microsecond of arc (∼ 1mas), or even one hundred nanoseconds of arc (∼ 100 nas) to be achieved.

In optical ground-based astronomy until the recent decade the position accuracy has been amounted to ∼ 0”1, which was worse by far than in radioastronomy. A great success of the space project HIPPARCOS lies in the precision of ∼ 1 mas having achieved in optical astronomy for measuring stellar coordinates and parallaxes. The astronomers hope to develop this success in the space experiments being planned [31], [32], [33], [34], [35]. A precision of ∼ 1 ÷ 10 mas is planned to
be achieved. High-precision angle measurements allow the stellar distance scale to be increased from 1 kpc to several tens and hundreds of kpc.

The inclusion of general relativistic effects has become a necessary part of the observation reduction procedure when the accuracy of observations is close to the value of $\sim 1\,\text{mas}$ (of the order of a millisecond of arc). The reduction procedure involved gravitational effects induced by the Sun and planets of the solar system \cite{14}. These bodies induce a nonstationary curvature of space-time in the solar system making a ray from the celestial source move along a curved trajectory. Positions, velocities and masses of the Sun and planets are known with a high degree of accuracy, which provides a possibility of allowing for the gravitational effects precisely in position measurements.

As the accuracy increases, the astronomers will undoubtedly encounter new phenomena. One of them, the nonstationarity of space-time being due to the motion of visible stars and invisible bodies in our Galaxy, was discussed in papers \cite{38}, \cite{39}. The list of phenomena which will be important in microarcsecond and sub-microarcsecond astrometry is discussed in \cite{25}.

The nonstationary curvature, induced by the Sun and planets, is a determinate process. The nonstationary curvature, created by moving stars in our Galaxy, is a stochastic process. This process is stochastic since distances to most of (in particular, invisible) stars as well as their masses are unknown to the observer. Since these values are unknown, it is impossible to reduce the observations with the same degree of definiteness as in the solar system. For the observer a remote source will execute a stochastic motion under the action of this process about the average position - the true position of the source in the sky. The value of this oscillation is of the order of $1\,\mu\text{as}$, the characteristic time from tens to hundreds of years.

Since the reduction procedure becomes impossible, the value equal to $\sim 1\mu\text{as}$ was called a fundamental limit of position measurement accuracy \cite{38}, and the effect itself was called a weak microlensing effect \cite{39}.

In the previous papers the weak microlensing effect has been discussed for a single observer. However, a very important astronomical information is given simultaneously by two observers (or one observer at different instants) from different points of space. Such observations allow a celestial source parallax to be measured and a distance to the source to be determined. Clearly the nonstationary curvature of space-time in our Galaxy will distort parallax measurements. This paper is devoted to discussion of the weak microlensing effect on the parallax measurements.

The parallax measurement is performed as follows: two telescopes spaced at a distance, called the baseline $B$ (or one telescope in different epochs realizing the baseline due to the Earth’s motion), observe one cosmic object. Clearly the direction to the object of observation is different for each telescope. The difference of these directions is called an object parallax shift $p$ and is determined as follows (provided that at one end of the baseline the direction to the source is
where $r$ is the distance to the source. The parallax is a natural unit of distance measurement in astronomy provided the baseline is the Earth’s orbit equal to one astronomical unit (AU). The parallax equal to one second of arc ($1'' = 5 \cdot 10^{-6}$rad) at the baseline of 1 AU means that the distance to the star being observed is equal to 1 pc ($3 \cdot 10^{18}$cm). Accordingly, the parallax equal to one nanosecond of arc ($5 \cdot 10^{-15}$) means that the distance to the source of light is equal to 1 gigaparsec, which is a good part of the distance to the horizon of particles of our Universe. The measurement of distances to extragalactic sources assumes measurements of ultrasmall angles. Hence very long baselines and the limiting accuracy of measurements are necessary for parallax measurements to be realized.

The effects related to a nonstationary curvature of space-time in our Galaxy will already become important at this level of measurement accuracy. Such effects, applied to extragalactic astronomy, were considered even 30 years ago [44], [36], [7], see also [6].

As shown below, the nonstationary gravitational field of moving stars in our Galaxy creates a curvature of space-time sufficient to distort strongly the values of remote source parallaxes. This effect will have a considerable impact on parallax measurements of many extragalactic sources and should be taken into account in future measurements and interpretations.

Variations in the space-time curvature induced by a nonstationary motion of stars in our Galaxy lead to distortion of the visible position of extragalactic sources [38], [39], [45]. The value of these distortions is estimated to be about one microsecond of arc for a considerable number of remote sources. The measurements of positions of these sources, conducted in considerable time intervals (several tens of years), may show the presence of such a shift ($\sim 1 \mu$as). This may be interpreted either as a real shift of the source in space due to a proper motion or as a visible shift due to the weak microlensing effect.

Random shifts of the order of one microsecond of arc do not allow the parallax method to be used any longer for measuring distances to light sources exceeding 1 megaparsec. This is about the distance to the Andromeda galaxy, one of the nearest galaxies to us. The microlensing effect bounds the distance scale that may be obtained from position measurements.

Measurements of the annual parallax (i.e. conducting measurements in half a year), or parallax measurements by interferometry methods, will be more exact since variations in the Galaxy’s nonstationary field will be much less.

However, it should be noted that phenomena related to the nonstationary curvature of Galaxy’s field will distort the measured values of parallaxes. If the observer can restore the parameters of a body distorting the observational data, then one may perform a reduction procedure and restore the true position of the light source and its parallax. This procedure, successfully applied for reduction
of observations under the action of bodies of the solar system, is invalid for taking account of microlensing in the Galaxy. In most cases masses and distances to the stars inducing the nonstationary curvature are unknown. This uncertainty results in the upper bound of accuracy of the parallaxes being measured.

In particular, the parallax distortions may be so large that the observer will register a negative value of the parallax instead of a positive one. The negative value of the parallax of a remote source is usually identified with the presence of measurement errors. In fact, the shift of a remote light source repeatedly measured half a year after the first measurement is a sum of the parallax shift, proper motion and random error of measurement. If one considers the proper motion to be equal to zero and the random errors to have an arbitrary sign (in particular, negative), then the measured parallax shift will be negative. In this case the total parallax shift is negative.

In a relativistic case the positive parallax is no longer a necessary condition of the real observation. Moreover, at a certain level of accuracy the most part of the observed parallaxes from remote sources will be negative!

The largest distortions of parallaxes will be observed under the action on light rays from rather close gravitational lenses. When the angular distance from the lens to the source is comparable with the parallax value, there may occur a "transfer" of the image from one side of lens to the other side. This effect will result in a considerable distortions of the source position and parallax.

Of course, the weak microlensing effect being observed from the barycentre of the solar system is not added to the source parallax motion. An attempt to reduce the position of the source, being observed from the Earth, to the barycentre following the standard equations will lead to an error since in this case the parallax shift value itself will depend on the microlensing effect.

Make a general comment on the whole article.

In the problem related to distortion of measured parallaxes there naturally appear some quantities being different-order infinitesimals. The unit vectors indicating the direction to the light source are zeroth-order quantities.

We shall consider the quantities containing the light source parallax as a factor to be first-order infinitesimals. As the first-order infinitesimals we shall also regard the proper shift of the light source or lens for all time of observation. These values are products of the proper velocity of motion of the object by all time of observation. We shall consider the angular distance between the lens and the light source to be a small parameter as well.

Besides these small parameters, in the problem there arises one more small parameter which is not related to the geometry of the problem under consideration but is related to general relativistic effects. This is Einstein’s-cone-size-to-source-lens-angular-distance ratio squared.

As seen from the foregoing, the small parameters take different values. Thus, e.g., the lens parallax may be of 10 mas, whereas the parallax of an extragalactic source may be of 10 nas, which $10^6$ times less. Nevertheless the terms containing
the lens parallax squared are less than the terms proportional to the first power of the source parallax. Hence in the equations we shall retain linear terms with respect to small parameters. Only two exceptions are made. We retain all square terms, one of the cofactors of which is Einstein’s-cone-to-angular-impact-distance ratio. This small factor should be retained since the weak microlensing effect is proportional to it, i.e. the microlensing effect itself (its value) depends on the size of Einstein’s cone. Besides, while calculating Einstein’s-cone-squared-to-angular-distance-squared ratio, we also retain second-order infinitesimals in the ratio denominator since zeroth- and first-order quantities are absent there.

The structure of the article is as follows. In Sec. 1 we consider the equation of a gravitational lens in a vector form suitable for derivating the observed parallax and solving this equation. Sec. 2 considers the position measurements performed from two different locations. Then we consider rigid-baseline parallax measurements. In Sec. 5 we consider a notion of the parallax ellipse arising in a visible motion of the source on the celestial sphere for a year as well as discuss characteristic sizes of the parallax ellipse and a trajectory of the visible displacement of the source on the sky in the presence of the weak lensing effect.

Finally, in conclusion we discuss the level of accuracy at which distortions in parallax measurements become significant.

**Equation of a Gravitational Lens and Its Solution**

To derive formulae describing a change in the value of the measured parallax of a light source, at the presence of a gravitational lens, we consider the equation of a gravitational lens. We shall consider the effect only in the case of a weak microlensing [38], [39] since in this case we may neglect the second weak image of a source and consider the first of the images to be unique that we observe. We shall use the standard model of microlensing based on a simple model of a point lens with a spherically symmetric gravitational field. This is the most interesting case since the Galaxy’s nonstationary field, formed by separate stars, is a set of gravitational fields, with each of which being spherically symmetric. While discussing the equation of a gravitational lens, we shall follow articles [36], [13] as well as review [43] and books [42], [40].

Consider the situation in which there is a light source $S$, a lens $D$ with the mass $M$ and two observers 1 and 2 (designated by $J$). As the lens $D$ a usual star or a dark body may appear. We shall consider that the velocity of a proper motion of the bodies $S$, $D$ as well as of each observer $J$, is much less than the velocity of light. The three bodies: the light source $S$, the lens $D$ and one of the observers $J$ form a plane which may be designated as $PL_J$. Here the index $J$ shows that the plane is formed taking account of the $J$ observer. The bodies $S$, $D$ and the other observer form a second plane. It should be noted that the trajectories of photons curved under the action of the gravitational lens are approximated by
Figure 1: The figure depicts a position of the light source, the gravitational lens as well as positions of the observers. The plane $PL_J$ concerns the $J$ observer.

broken lines, as well as in review [13], which makes it possible for us to speak about the planes where vectors are located. Besides, the choice of two planes is also due to it is necessary to take into account correctly the quantity defined as a difference of angles between the direction to the lens and the visible position of the source, which will be discussed below in detail.

The plane $PL_J$ is of importance since the principal vectors of the problem as well as the light ray trajectory lie in this plane [13]. It is natural that the trajectory of the light incoming to the other observer belongs to the other plane. Introduce vector designations in our problem. Each vector will be regarded at some instant $t$ considering the position of each of the bodies to be a function of time. The vector from the origin of coordinate to the light source $S$ will be
denoted as $\vec{r}_S(t)$. The vector drawn from the origin of coordinate to the lens will be denoted as $\vec{r}_D(t)$. Similarly we draw vectors $\vec{r}_J(t)$ to each of the observers. The vector connecting the observer $J$ and the light source is

$$\vec{r}_{JS}(t) = \vec{r}_S(t) - \vec{r}_J(t),$$

and the vector connecting the same observer and the lens is

$$\vec{r}_{JD}(t) = \vec{r}_D(t) - \vec{r}_J(t)$$

The three points $S$, $D$ and the observer $J$ lie in one plane $SDJ$. The vectors $\vec{r}_{JS}$ and $\vec{r}_{JD}$ lie in the same plane. To write a solution to the equation of a gravitational lens in a vector form and describe the weak microlensing effect for two spaced observers, we introduce unit vectors whose directions coincide with those of the vectors $\vec{r}_{JS}$ and $\vec{r}_{JD}$:

$$\vec{n}_{JS} = \frac{\vec{r}_{JS}}{r_{JS}}$$

and

$$\vec{n}_{JD} = \frac{\vec{r}_{JD}}{r_{JD}}.$$  

Here and below the quantities without vector arrows above designate absolute values of the vectors $r_{JS} \equiv |\vec{r}_{JS}|$.

The difference of these two vectors

$$\Delta \vec{n}_J = \vec{n}_{JS} - \vec{n}_{JD}$$

is a vector which belongs to the plane $PL_J$, and is approximately equal in magnitude to the difference of angles between the directions to $C$ and $D$.

On separating the plane $S$, $D$ and $J$, the equation of a gravitational lens reduces to a trivial quadratic equation for angles \cite{30, 10, 43}:

$$\theta_i^2 - \theta \cdot \theta_i - \theta_e^2 = 0$$

Here $\theta \equiv |\Delta \vec{n}_J|$ is a difference of the angles between the direction to the lens and the direction to the true position of the light source, $\theta_i$ is a difference of the angles to the lens and a visible position of the source (image), $\theta_e$ is Einstein’s cone size:

$$\theta_e^2 = \frac{4GM}{c^2} \frac{D_{DS}}{(D_{DS} + D_{DJ}D_{DJ})}$$

In the latter formula $M$ is the lens mass, $D_{DS}$ is the distance from the lens to the light source, $D_{DJ}$ is the distance from the lens to the observer $J$. It should also be noted that Einstein’s cone size differs for the two observers since $\theta_e$ depends on the distance between the lens and the observer. However this difference is small and shall be neglected.
In paper [24] a nonstationary situation has been analyzed, when an effect of the motion of the lens on the motion of light rays is taken into account. From the results of this paper it is seen that in our situation an effect of nonstationarity of the gravitation field of the lens may be neglected.

Solving the lens equation, we find two roots \( \theta_i \) which correspond to two image positions of the source. Since we are only interested in the weak microlensing effect, we shall only be concerned in the position of the main image. The solution to the equation of a gravitational lens for the position of the main image is

\[
\theta_i = \frac{1}{2} \theta + \frac{1}{2} \theta \sqrt{1 + \frac{4\theta_e^2}{\theta^2}} \approx \theta + \frac{\theta_e^2}{\theta}. \tag{8}
\]

The difference between the true and visible positions of the source is

\[
\delta \theta \approx \frac{\theta_e^2}{\theta}. \tag{9}
\]

The image position on the picture plane can be drawn as follows (see Fig. 2). Through the lens position and the true position of the source a straight line is drawn. Then one determines a vector whose origin coincides with the lens \( D \) and whose end coincides with the source position \( S \) on the picture plane. The length of this vector is equal to \( \theta \). To find the image position, this vector is continued in the same direction up to the length \( \theta_i \). The resulting vector is a two-dimensional one determining the image position. Necessity of such an image drawing will be clarified below.

Now one may identify the vector \( \Delta n_{JI} \) introduced above with that connecting the positions \( D \) and \( S \) on the picture plane. We denote the vector drawn from the point \( D \) to the image \( I \) as \( \Delta n_{JI} \), \( \theta \equiv |\Delta \vec{n}_{IJ}| \). Now the equation describing connection between the position and image vectors takes the form

\[
\Delta \vec{n}_{JI} = \frac{1}{2} \Delta \vec{n}_J + \frac{1}{2} \Delta \vec{n}_J \cdot \sqrt{1 + \frac{4\theta_e^2}{\Delta n_{JI}^2}} \approx \Delta \vec{n}_J + \Delta \vec{n}_J \frac{\theta_e^2}{\Delta n_{JI}^2}. \tag{10}
\]
The visible position of the source relative to the true direction \( \vec{n}_{JS} \) is expressed as

\[
\vec{n}_{JI} = \vec{n}_{JS} + \Delta \vec{n}_J \frac{\theta^2}{\Delta n^2} \tag{11}
\]

. The light source \( S \), the deflecting body \( D \) and the observer \( O \) possess a peculiar motion. We shall consider the source and lens motion to be rectilinear and uniform. We shall also consider the origin of our system of coordinates to be related to the barycentre of the solar system, so that the observer's velocity relative to the selected system of coordinates is only the velocity of motion about the barycentre.

Let the vector of the three-dimensional velocity relative to the selected system of coordinates be \( \vec{v}_S \), and the lens velocity be \( \vec{v}_D \). We shall consider both of these quantities to be constant.

One may divide the three-dimensional velocity into longitudinal and transversal components. The longitudinal component of the velocity changes basic physical parameters of the picture under consideration, e.g. such as Einstein's cone size. However this effect is small and shall be neglected below. The transverse components of all three motions are added to result in a mutual motion of the source \( S \) and the body \( D \) in the observer’s picture plane.

Since the velocities of bodies are constant, and the observer’s velocity relative to the barycentre is given, now we can calculate a law of variation in the vectors \( \vec{n}_{JS}, \vec{n}_{JD} \) as well as other vectors required for solving our main problem of calculating the image motion, with the observer’s parallax motion and the uniform motion of the source and lens taken into account.

Fix the source and lens positions at the instant \( t = 0 \). Then the vector connecting the observer \( J \) and source satisfies the equation

\[
\vec{r}_{JS}(t) = \vec{r}_{S0} + \vec{v}_S t - \vec{r}_J(t) \tag{12}
\]

Below the values at the instant \( t = 0 \) will be everywhere designated by the index 0.

The unit vector in the direction from the observer \( J \) to the source is described by the equation

\[
\vec{n}_{JS}(t) = \vec{n}_{S0} + \left( \frac{\vec{v}_S}{r_{S0}} - \vec{n}_{S0} \frac{\vec{n}_{S0} \vec{v}_S}{r_{S0}} \right) t - \left( \frac{\vec{r}_J(t)}{r_{S0}} - \vec{n}_{S0} \frac{\vec{n}_{S0} \vec{r}_J}{r_{S0}} \right) \tag{13}
\]

Consider two vectors: the first one

\[
\frac{\vec{v}_S}{r_{S0}} - \vec{n}_{S0} \frac{\vec{n}_{S0} \vec{v}_S}{r_{S0}} \tag{14}
\]

and the second one

\[
\frac{\vec{r}_J(t)}{r_{S0}} - \vec{n}_{S0} \frac{\vec{n}_{S0} \vec{r}_J}{r_{S0}} \tag{15}
\]
These are three-dimensional vectors perpendicular to the vector $\vec{n}_{S0}$. Hence from the viewpoint of the first observer, at the instant $t = 0$, they are two-dimensional vectors lying in the picture plane. Besides, the first vector has dimensions $s^{-1}$ and coincides with the proper angular velocity of displacement of the source on the sky. Hence we denote the first of these vectors by the two-dimensional vector $\vec{\mu}_S$

$$\vec{\mu}_S = \frac{\vec{v}_S}{r_{S0}} - \vec{n}_{S0} \left( \frac{\vec{n}_{S0} \vec{v}_S}{r_{S0}} \right),$$

and the second one will be denoted as

$$\vec{\beta}_{JS} = \frac{\vec{r}_J(t)}{r_{S0}} - \vec{n}_{S0} \left( \frac{\vec{n}_{S0} \vec{r}_J}{r_{S0}} \right).$$

It should be also noted that the vector $\vec{\mu}_S$, being small in magnitude, is equal to $|\mu| \sim 10^{-11} \, s^{-1}$ for the fastest and closest objects, and is small as $|\mu| \sim 10^{-20} \, s^{-1}$ for object at cosmological distances.

Similarly one can calculate the unit vector from the observer to the lens.

$$\vec{n}_{JD}(t) = \vec{n}_{D0} + \vec{\mu}_D t - \vec{\beta}_{JD}(t)$$

It is in this case that we should introduce new designations, with the angular velocity of the lens being determined by the vector $\vec{n}_{D0}$

$$\vec{\mu}_D = \frac{\vec{v}_D}{r_{D0}} - \vec{n}_{D0} \left( \frac{\vec{n}_{D0} \vec{v}_D}{r_{D0}} \right),$$

and the second vector satisfies the equation

$$\vec{\beta}_{JD} = \frac{\vec{r}_J(t)}{r_{D0}} - \vec{n}_{D0} \left( \frac{\vec{n}_{D0} \vec{r}_J}{r_{D0}} \right).$$

Now the index $D$ denotes the corresponding vectors for the lens.

The vector $\vec{\mu}_D$ is the angular velocity of the lens, and the vector $\vec{\beta}_{JD}$ has the same meaning as the vector $\vec{\beta}_{JS}$ drawn from the observer to the source.

Both vectors $\vec{\mu}_D$ and $\vec{\beta}_{JD}$ are perpendicular to the vector $\vec{n}_{D0}$. They are first order infinitesimals. Recall that the vector $\vec{n}_{S0}$ differs from the vector by a first-order infinitesimal $\Delta \vec{n}_0$. Hence the vector $\vec{\mu}_D$ may be also considered perpendicular to the vector $\vec{n}_{S0}$ to a second-order infinitesimal, and the equation can be written by substituting the vector $\vec{n}_{D0}$ by the vector $\vec{n}_{S0}$ referring it to one of the basic vectors of our problem $\vec{n}_{S0}$.

The vector $\vec{\beta}_{JD}$ differs from the vector $\vec{\beta}_{JS}$ by a first-order infinitesimal

$$\delta \vec{\beta}_J = \vec{\beta}_{JD} - \vec{\beta}_{JS} = \left( \frac{1}{r_{D0}} - \frac{1}{r_{S0}} \right) \left[ \vec{r}_J - \vec{n}_{S0} (\vec{n}_{S0} \vec{r}_J) \right],$$

although the scalar product of $\vec{\beta}_{JD}$ and $\vec{n}_{S0}$ is equal to zero to a second-order infinitesimal. Hence we shall consider the vectors $\vec{\beta}_{JD}$ and $\vec{n}_{S0}$ to be mutually perpendicular.
Finally, we define a vector of the angular difference between true directions to the light source and to the lens at the instant \( t = 0 \)

\[
\Delta \vec{n}_0 = \vec{n}_{SO} - \vec{n}_{DO},
\]

(22)

a difference of angular velocities of the source and the lens respectively

\[
\vec{\mu} = \vec{\mu}_S - \vec{\mu}_D.
\]

(23)

Then we obtain a dependence of the vector \( \Delta \vec{n}_J(t) \) in the form

\[
\Delta \vec{n}_J(t) = \Delta \vec{n}_0 + \vec{\mu}t + \delta \vec{\beta}_J(t).
\]

(24)

Finally, we write down the equation for an image position

\[
\vec{n}_{JI} = \vec{n}_{SO} + \vec{\mu}_S t - \vec{\beta}_J(t) + (\Delta \vec{n}_0 + \mu + \delta \beta_J), \quad \frac{\theta_e^2}{\Delta n_J^2(t)}
\]

(25)

where

\[
\Delta n_J^2(t) = \Delta n_0^2 + \mu^2 t^2 + \delta \beta_J^2 + 2(\Delta \vec{n}_0 \vec{\mu})t + 2(\Delta \vec{n}_0 \delta \vec{\beta}_J(t))t.
\]

(26)

In this equation second-order infinitesimals are retained since, as mentioned in the introduction, zero- and first-order terms are absent in this sum.

**Measurements from Two Positions**

Celestial source position measurements performed from two spaced points allow a valuable additional information on this source to be obtained. Of prime importance is the measurement of a trigonometric parallax of the source, which allows its distance to be measured.

The parallax measurements may be carried out by several ways [11], [27], [8]. We consider the measurements of interferometric type, being conducted by observers located at different ends of a rigid baseline, as well as the parallax motion observation conducted by the observer located on the moving Earth. The theory of such observations in Euclidean space may be found in manuals on astrometry. There arises a special feature in the presence of the weak microlensing effect. The plane \( PL_J \) moves in space. This occurs mainly due to the observer’s moving about the barycentre of the solar system.

In such measurements, especially in the second case, of importance is a mutual orientation of the baseline and the picture plane. On this the arc \( \theta_i \) depends, which makes it necessary to determine the image position via a two-dimensional vector. This permits a change in the arc \( \theta_i \) to be taken into account correctly, while passing from the first source plane to the second one.

Introduce a vector directed from one point of observation to the other, which we shall call the baseline vector \( \vec{B} \). Then the difference of the vector drawn from
the origin of coordinates to the first observer and that to the second one is (see Fig. 3)

\[ \vec{B} = \vec{r}_2(t_2) - \vec{r}_1(t_1). \] (27)

As seen from this definition, the baseline vector is generally a function of time.

Define also a vector coinciding the baseline direction

\[ \vec{n}_b = \frac{\vec{B}}{B} \] (28)

Now discuss the basic equations describing observations from two spaced points. At first instant \( t \) the vectors, one of which connects the first observer and the source, and the second one connects the first observer and the lens, are \( r_{1S}(t_1) \) and \( r_{1D}(t_1) \). The second observer is located at another point, and its position differs from the position of the first observer by a vector \( \vec{B} \). Photons, generally speaking, arrive to the observer at the instant differing from \( t \) by the value \( \sim \frac{B}{c} \). Therewith we should calculate source and lens positions at another instant. However this difference is small and shall be neglected.

The vector connecting the source and the second observer can be written as

\[ \vec{r}_{2S} = \vec{r}_{1S} - \vec{B}, \] (29)

and the true direction to the source, for the second observer, is

\[ \vec{n}_{2S}(t) = \vec{n}_{1S}(t) - \frac{B}{r_{1S}}(\vec{n}_b - \vec{n}_{1S}(\vec{n}_{1S}\vec{n}_b)) \] (30)

We shall consider that the distances from the observers to the origin of coordinate system are much less than the distance from the source to the observers. Then we substitute \( r_{1S} \) by \( r_{SO} \) and, neglecting small quantity squares, we obtain

\[ \vec{n}_{2S}(t) = \vec{n}_{1S}(t) - \frac{B}{r_{SO}} \vec{b}, \] (31)

where

\[ \vec{b} = \vec{n}_b - \vec{n}_{SO}(\vec{n}_{SO}\vec{n}_b) \] (32)

A similar equation is valid for the vectors connecting the observer and the lens

\[ \vec{r}_{2D}(t) = \vec{r}_{1D}(t) - \vec{B}, \] (33)

, and for the unit vectors in the lens direction

\[ \vec{n}_{2D} = \vec{n}_{1D} - \frac{B}{r_{1D}}[\vec{n}_b - \vec{n}_{1D}(\vec{n}_{1D}\vec{n}_b)] \] (34)
Figure 3: The baseline vector is defined as a difference of the vectors drawn to the first and second observers respectively.
Transform the equation for $\mathbf{n}_{2D}$ similarly to the equation (31) dropping second-order infinitesimals and singling out $r_{DO}$ and $\mathbf{n}_{SO}$

$$\mathbf{n}_{2D} = \mathbf{n}_{1D} - \frac{B}{r_{DO}} \cdot \mathbf{b}$$  \hspace{1cm} (35)

Now we find relation between the vectors $\Delta \mathbf{n}_2$ and $\Delta \mathbf{n}_1$ connecting the light source and the lens in the picture planes of the second and first observers. These vectors enters into the denominator of the factor determining the weak microlensing effect

$$\Delta \mathbf{n}_2 = \Delta \mathbf{n}_1 - B \left( \frac{1}{r_{SO}} - \frac{1}{r_{DO}} \right) \mathbf{b}$$  \hspace{1cm} (36)

In equation (36) we have retained only first-order infinitesimals both in the left and right side of the equation. Introduce the designations $p_S = \frac{B}{r_{SO}}$ and $p_D = \frac{B}{r_{DO}}$ for the light source and lens parallaxes respectively as well as for $p = p_S - p_D$. We present the final equation for the vector directed to the source image from the second observer’s position

$$\mathbf{n}_{2I} = \mathbf{n}_{2S} + \Delta \mathbf{n}_2 \frac{\theta_e^2}{\Delta n_{2}}$$  \hspace{1cm} (37)

and from the first observer’s position

$$\mathbf{n}_{1I} = \mathbf{n}_{1S} + \Delta \mathbf{n}_1 \frac{\theta_e^2}{\Delta n_{1}}.$$  \hspace{1cm} (38)

The angular distances are connected by the equality

$$\Delta \mathbf{n}_2 = \Delta \mathbf{n}_1 - p\mathbf{b},$$  \hspace{1cm} (39)

and the vectors showing the true position of the source depend on time as

$$\mathbf{n}_{1S} = \mathbf{n}_{SO} + \vec{\mu}_{1S} t - \vec{\beta}_{1S}(t),$$  \hspace{1cm} (40)

$$\mathbf{n}_{2S} = \mathbf{n}_{SO} + \vec{\mu}_{2S} t - \vec{\beta}_{2S}(t).$$  \hspace{1cm} (41)

and differ only by a parallax vector $p_S \mathbf{b}$.

**Rigid-Baselined Observations**

We shall call measurements rigid-baselined providing that $B=\text{const}$. So is measured a distance to some object in geodesy. At a constant baseline conducted are VLBI observations for which characteristics of the arriving rays are of importance as well. While observing from the Earth’s surface, the VLBI baseline varies due to tides, tectonic shears and other processes. The value of these variations of
the order of 50 cm per day. In our problem these variations of the baseline may be neglected, and the baseline may be considered constant. Besides, this case is valuable from a methodical viewpoint since it allows one to clarify a physical sense of the quantities being measured as well as a mechanism of visible change in the parallax. As well as everywhere, we shall consider the rays to arrive at both ends of the baseline simultaneously, i.e. the variation in the basic vectors of our problem for the time interval $\frac{B}{c}$ is negligible. Besides, in this section we shall consider $p_S \ll p_D$.

While measuring the parallaxes, one tries to orient the baseline so that one of its ends, say, the first were directed perpendicular to the source image

$$\langle \vec{n}_{1b} \rangle = 0$$ \hspace{1cm} (42)

In this case the instant $t=0$ is chosen as a beginning of observation, and the origin of coordinates is assumed to coincide with the first observer. Then $\vec{n}_{1S} = \vec{n}_{SO}$. In Euclidean geometry the direction to the image is a true direction to the source, thus the perpendicularly condition is reformulated as

$$\langle \vec{n}_{SO} \vec{n}_{b} \rangle = 0,$$ \hspace{1cm} (43)

and the vector $\vec{b}_{2S}$ becomes a vector $p_S \vec{\beta}$.

In the absence of the weak microlensing effect light rays move along straight lines. Now the scalar product takes the form ($\vec{\beta}$)

$$\langle \vec{n}_{2S} \vec{n}_{b} \rangle = -p_S$$ \hspace{1cm} (44)

Define the visible value of the parallax as a scalar product of the vectors $\vec{n}_{1S}$ (directed to the image from the second observer’s position) and $\vec{n}_{b}$ taken with opposite sign. This definition is equivalent to the parallax definition in Euclidean astrometry.

$$p_a = -\langle \vec{n}_{21} \vec{n}_{b} \rangle$$ \hspace{1cm} (45)

The condition (42) together with the equation (38) for the weak microlensing effect determines the equation

$$\langle \vec{n}_{1S} \vec{n}_{b} \rangle = -\frac{(\Delta \vec{n}_{1} \vec{n}_{b})}{\Delta n_{1}} \frac{\theta_{e}^{2}}{n_{1}}$$ \hspace{1cm} (46)

Introduce an angle between the baseline vector $\vec{n}_{b}$ and the angular-impact-parameter vector $\Delta \vec{n}_{1}$ in the form

$$\frac{(\Delta \vec{n}_{1} \vec{n}_{b})}{\Delta n_{1}} = \cos \psi$$ \hspace{1cm} (47)

Now we calculate the parallax value measured according the condition (45) and equations (37), (41), (39)

$$p_a = p_s + p(b \vec{n}_{b}) \frac{\theta_{e}^{2}}{\Delta n_{2}^{2}} + (\Delta \vec{n}_{1} \vec{n}_{b}) \left( \frac{\theta_{e}^{2}}{\Delta n_{1}^{2}} - \frac{\theta_{e}^{2}}{\Delta n_{2}^{2}} \right)$$ \hspace{1cm} (48)
The projection onto the baseline vector differs from unity by second-order
infinitesimals to be neglected. Hence we shall assume that \((\vec{b} \vec{n}_b) = 1\). The
source and lens parallax difference \(p = p_S - p_D\) is multiplied by a small factor
\(\frac{\theta^2}{\Delta n_2^2}\). For close sources, whose parallax is comparable with the lens parallax,
the weak microlensing effect will not change the parallax too much. However
in case \(p_S \ll p_D\) and \(p_S \sim p_D \frac{\theta^2}{\Delta n_2^2}\) a change in the visible parallax may be
significant. We shall also consider that the angular distance between the source
\(S\) and the lens \(D\) is much more than the parallaxes. In this case the third
term is comparable with the second one in magnitude. Of course, \((\Delta \vec{n}_1 \vec{n}_b) \frac{\theta^2}{\Delta n_1^2}\)
and \((\Delta \vec{n}_1 \vec{n}_b) \frac{\theta^2}{\Delta n_2^2}\) far exceeds both the first and second terms in equation (48).
However their difference is already comparable with the second term of equation (48). We resort to the definition (47) and obtain the visible value of the parallax

\[
p_a = p_s + (p_D - p_S) \frac{\theta^2}{\Delta n_2^2} \cos 2\psi
\]

In equation (49) we neglect terms containing the factor \(\sim p^2\).

Consider more fully two situations related to different arrangements of the
baseline vector relative the vector of the angular distance between the source \(S\)
and the lens \(D\). The first situation arises when the baseline vector, the lens \(D\)
and the source \(S\) lie in the same plane, with the lens \(D\) lying off the triangle
formed by the baseline and two rays drawn from the source \(S\) to different ends
of the baseline (see Fig. ). In this case \(\psi = 0\), and the visible parallax is a sum
of the true parallax and an additional term

\[
p_a = p_s + p_D \frac{\theta^2}{\Delta n_2^2}
\]

The visible parallax will be more than the true one. The vectors directed from
the ends of the baseline to the source image shall be "shifted" under the action
of the gravitational field of the lens. Such a disposition, the baseline-lens-source,
will imitate an "approach" of the source to the observer.

An increase in the visible parallax is valid for the situation when the lens \(D\)
does not belong to the plane formed by the baseline \(\vec{B}\) and the source. However an
increase in \(p_a\) is valid only for the angles \(\psi\) less than \(\frac{\pi}{4}\). It should be emphasized
the the value of the vector \(\Delta n_1\) is more than the lens parallax \(p_D\), i.e. \(\Delta n_2\) has
the same direction as \(\Delta n_1\).

The second situation arises when the lens does not belong to the lens formed
by the baseline \((\vec{B})\) and the source. In case \(\cos 2\psi = -1\), i.e. the baseline vector
is perpendicular to the plane \((\vec{B}S)\), the visible parallax diminishes

\[
p_a = p_s - p_D \frac{\theta^2}{\Delta n_2^2}
\]
Figure 4: The lens, baseline and light source are located so that the lens lies off the triangle formed by two observers and the source.

Fig. 5 depicts locations of the baseline $\vec{B}$, the lens $D$ and the source $S$ corresponding to this situation. Of course, the lens $D$ is assumed to be located under the plane ($\vec{B}S$). Depending on the value of the true parallax $p_S$ and a gravitational additional term, the visible parallax may be even negative.

Of special interest is the situation when the lens $D$ belongs to the plane ($\vec{B}S$), but is located inside the triangle formed by the baseline and rays of the source $S$, as in Fig. 5, in case $D$ is in the figure plane. In this case the parallax is more than $\Delta n_1$, the terms squared in $p$ become more than the product of the parallax by the angular impact parameter, and there arises the second situation in spite of the lens lying in the plane ($\vec{B}S$). The vectors directed from the ends of the baseline to the source image will be "moved apart" by the gravitational field of the lens $D$. In case the lens $D$ belongs to the plane $\vec{B}S$, and the angular impact parameter $\Delta n_1 \sim p_d$, a small mass as the gravitational lens suffices for the observed value of the parallax to become zero $p_a = 0$ at

$$m_D \sim \frac{c^2 B^2}{4G r_S}$$

(52)

A direction of the vector $\vec{n}_{2I}$ should be mentioned. In relativistic measurements of the parallax of a source the vectors $\vec{n}_{1I}$, $\vec{n}_{2I}$ and $\vec{n}_b$ lie in the plane formed by three points: observers 1 and 2 as well as the source $S$.

If the curvature of space-time is taken into account, light rays move along curved trajectories, and the vectors $\vec{n}_{1I}$, $\vec{n}_{2I}$ and $\vec{n}_b$ do not already belong to one plane. Using the condition ($\vec{n}_b\vec{n}_{1I}$), we form the plane wherein lie the vectors $\vec{n}_b$ and $\vec{n}_{1I}$. The equation determining the visible parallax singles out only one of the three components of the vector $\vec{n}_{2I}$. The basic component $\sim$ is directed to
the source image and does not contain additional information. There exists the third component perpendicular the plane formed by the vectors $\vec{n}_b$ and $\vec{n}_{1I}$. This component has no analogue in nonrelativistic optics and is equal in magnitude

$$2(p_D - p_S) \frac{\theta^2}{\Delta n_{2I}^2} \frac{(\Delta \vec{n}_1 \vec{n}_b)}{\Delta n_{1I}^2} (\Delta \vec{n}_1 - \vec{n}_b(\Delta \vec{n}_1 \vec{n}_b))$$  \hspace{1cm} (53)

Generally speaking, this component may be a source of additional information, e.g., on the measurements conducted being distorted by the weak microlensing effect.

### Annual Parallax Measurements

In paper [39] a change in the visible position of a light source is considered in the observer’s uniform motion relative the lens and the light source itself. Although not indicated, in the above paper it was assumed that the observations are conducted from the barycentre of the solar system which moves uniformly and rectilinearly to a first approximation.

While measuring an annual parallax shift, it is natural for the coordinate system centre to be also chosen coinciding with the barycentre of the solar system. Then the light source velocity $S$ and the lens velocity $D$ will be differences of the velocities $\vec{S}$ and $\vec{D}$ respectively and those of the barycentre of the solar system. Again we shall consider that these velocities are constant. Now the baseline is the Earth’s radius vector relative the solar system barycentre. $\vec{B}$ satisfies a Keplerian motion.

Clearly the visible motion of the source will already be nonuniform at a nonzero parallax even in the absence of lensing effect as seen from the Earth executing the annual motion around the Sun. Define the vector $\vec{n}_{2S}$ as a vector directed to the
source from the Earth, and the vector $\vec{n}_{1S}$ as a vector directed to the source from the solar system barycentre. These two vectors are connected via the parallax deviation vector

$$\vec{n}_{2S} = \vec{n}_{1S} - p_S \vec{b}, \quad (54)$$

a similar equation can be written to connect unit vectors to the lens both from the solar system barycentre and the observer on the Earth

$$\vec{n}_{2D} = \vec{n}_{1D} - p_D \vec{b}. \quad (55)$$

In the presence of a lens the light source motion being observed from the barycentre is described by a unit vector of the form

$$\vec{n}_{1I} = \vec{n}_{1S} + \Delta \vec{n}_1 \frac{\theta^2}{\Delta n_1^2}. \quad (56)$$

Here $\Delta \vec{n}_1$ is a difference of unit vectors directed to the source $\vec{n}_{1S}$ and to the lens $\vec{n}_{1D}$. We shall consider that the motion of the source on the sky is described by a linear equation of the form

$$\vec{n}_{1S} = \vec{n}_{0S} + \vec{\mu}_S t, \quad (57)$$

and the lens motion – by the equation as follows

$$\vec{n}_{1D} = \vec{n}_{0D} + \vec{\mu}_D t \quad (58)$$

The difference vector $\Delta \vec{n}_1$ is also a linear time-dependent function

$$\Delta \vec{n}_1 = \Delta \vec{n}_0 + \vec{\mu}t \quad (59)$$

Here $\vec{\mu}$ is a difference of angular velocities of the source and the lens, and $\Delta \vec{n}_0$ is a difference of initial positions.

Now the source motion observed from the barycentre has the form

$$\vec{n}_{1I} = \vec{n}_{0S} + \vec{\mu}_S t + (\Delta \vec{n}_0 + \vec{\mu}t) \frac{\theta^2}{\Delta n_1^2} \quad (60)$$

The visible displacement of the source due to the weak microlensing effect found in [38] was calculated at a zero proper motion of the source, which is valid for most extragalactic objects. In the case of a nonzero proper motion of the source its motion picture will differ from the case shown in [38]. Now the motion trajectory will have the shape of a nonclosed curve (see Fig. 6) or an intersecting curve (Fig. 7).

In the absence of a gravitational lens the source position from the Earth’s observer viewpoint is described by an equation of the form

$$\vec{n}_{2S} = \vec{n}_{1S} - p_S \vec{b}. \quad (61)$$
Figure 6: The trajectory of a visible motion of the source. In this figure the direction of the proper motion of the lens is opposite to the source motion direction.

Figure 7: The trajectory of a visible motion of the source in the second case of the proper motion of the source coinciding with the proper motion of the lens.
When the gravitational lens distorts the source position, the equations of motion will become more complex

\[
\vec{n}_{2I} = \vec{n}_{0S} + \vec{\mu}_S t - p_S \vec{b} + \Delta \vec{n}_2(t) \frac{\theta^2}{\Delta n^2_2(t)}
\]  

(62)

where \( \Delta \vec{n}_2 = \Delta \vec{n}_1 - p\vec{b} \).

The form of the equation of motion in celestial coordinates as well as the figures illustrated it will be considered in the next section.

**Measurement of Coordinates and Source Parallaxe**

In the previous sections we have developed a general theory of observations, written in a vector form, at two instants from two rigidly bound space positions as well as discussed parallax measurement from the moving Earth in the presence of the weak microlensing effect. However in observational astronomy it is more common to use equations written in astronomical coordinates, but not in a vector form. It is of importance since studying the motion of celestial sources is related at any rate to precalculating the source positions at a given instant in a certain coordinate system. Some systems of astronomical coordinates are used, in this case we shall use an ecliptic system \( \lambda, \beta \).

In modern astrometric catalogues (HIPPARCOS, TYCHO) the position and motion of a star is characterized by five parameters chosen in a certain epoch and referred to a certain place. The choice of the epoch is of no importance for us. But we shall choose \( \lambda \) and \( \beta \) as coordinates referred to the solar system barycentre. Two coordinates of five point to the source. The Cartesian components of the unit vector in the direction determined by the coordinates \( \lambda \) and \( \beta \) are representable as

\[
\vec{n} = (\cos \lambda \cos \beta, \sin \lambda \cos \beta, \sin \beta)
\]  

(63)

Two more parameters, presented in the catalogues, are the proper motion in \( \lambda \) and \( \beta \), with the velocity with respect to the coordinate \( \lambda \) being a time derivative of the coordinate

\[
\mu_\beta = \frac{d\beta}{dt}
\]  

(64)

and the velocity with respect to the right ascension is defined as

\[
\mu_\lambda = \frac{d\lambda}{dt} \cos \beta
\]  

(65)

one can define the total angular velocity vector as

\[
\vec{\mu} = \vec{p}_\mu_\lambda + \vec{q}_\mu_\beta
\]  

(66)
where \( \mu_\lambda \) and \( \mu_\beta \) have been defined above, and the vectors \( \vec{p} \) and \( \vec{q} \) are

\[
\vec{p} = (-\sin \lambda, \cos \lambda, 0)
\]

(67)

\[
\vec{q} = (-\cos \lambda \sin \beta, -\sin \lambda \sin \beta, \cos \beta)
\]

(68)

The vectors \( \vec{n}, \vec{p}, \) and \( \vec{q} \) form a triad of mutually perpendicular vectors as it is usually defined in astronomy [27], [12].

Since we shall refer our observations to the solar system barycentre, the source coordinates will be set in the ecliptic coordinate system, and as the vector \( \vec{n}_1 \) we choose the direction to the source from the solar system barycentre. Since the first observer is always located at the solar system barycentre, the vector \( \vec{B} \) is a vector connecting the barycentre and the observer, everywhere below we shall consider the observer to be located on the Earth, and the vector \( \vec{B} \) to connect the solar system barycentre with the Earth centre.

Thus the vector \( \vec{n}_b \) has the form

\[
\vec{n}_b = (\cos \lambda, \sin \lambda, 0),
\]

(69)

here \( \lambda \) is the Earth’s right ascension in the ecliptic coordinates from the solar system barycentre. Clearly the Earth’s declination is equal to zero.

Fix the vector triad at the instant \( t = 0 \) and denote its vectors by index ”0”: \( \vec{n}_S0, \vec{p}_0, \vec{q}_0 \). The ecliptic coordinates of the place will be also denoted by index ”0”. We shall reckon the image position from the true position of the source at the zero instant. This difference is representable in the form

\[
\vec{n}_{2I} - \vec{n}_S0 = \Delta \lambda(t) \cos \beta_0 \vec{p}_0 + \Delta \beta(t) \vec{q}_0
\]

(70)

It is easy to notice that this equality determines an approximate expansion of the vector \( \vec{n}_{2I} \) in the triad of perpendicular vectors \( \vec{n}_S0, \vec{p}_0, \vec{q}_0 \). An exact expansion of the vector in this triad differs from the approximate one by terms containing the factors \( \sim \Delta \lambda^2, \sim \Delta \beta^2 \) to be neglected.

The difference of the unit vectors directed to the source and the lens (59) is representable in the form

\[
\Delta \vec{n} = \Delta \lambda \cos \beta_0 \vec{p}_0 + \Delta \beta \vec{q}_0.
\]

(71)

Here \( \lambda_0, \beta_0 \) are coordinates of the light source position \( S \) in the epoch \( t = 0 \), \( \Delta \lambda = \lambda_S - \lambda_D \) are differences of the right ascension of the light source and the lens \( D \), and \( \Delta \beta = \beta_S - \beta_D \) are differences of the light source and lens declinations.

Now introduce a new definition \( x = \Delta \lambda \cos \beta_0, y = \Delta \beta \), an analogue of the Cartesian coordinates on the celestial sphere. In small domains of the sphere such an approximation is valid, and the calculations may be performed as in Euclidean geometry. Now the coordinates show how the source coordinates vary from the Earth’s observer viewpoint, while his moving along the Earth’s orbit.
Besides, introduce an auxiliary designation $x_0$, $y_0$ the distance between the source and the lens at the initial instant in observations from the barycentre.

The change of Cartesian coordinates will be described by the equations

$$x(t) = \mu S_\lambda t - p_S \sin(\lambda_E - \lambda_0) + (x_0 + \mu \lambda t - p \sin(\lambda_E - \lambda_0)) \frac{\theta^2}{R^2(t)}$$

$$y(t) = \mu S_\beta t + p_S \sin \beta_0 \cos(\lambda_E - \lambda_0) + (y_0 + \mu \beta t + p \sin \beta_0 \cos(\lambda_E - \lambda_0)) \frac{\theta^2}{R^2(t)}$$

where

$$R^2(t) = x_0^2 + y_0^2 + 2 (x_0 \mu \lambda + y_0 \mu \beta) t + \left(\mu^2_\lambda + \mu^2_\beta\right) t^2 + 2p (x_0 \sin(\lambda_E - \lambda_0) - y_0 \sin \beta_0 \cos(\lambda_E - \lambda_0)) + 2p (\mu \lambda \sin(\lambda_E - \lambda_0) - \mu \beta \sin \beta_0 \cos(\lambda_E - \lambda_0)) t + p^2 (1 - \cos^2 \beta_0 \cos^2(\lambda_E - \lambda_0) - 2 \cos \beta_0 \cos(\lambda_E - \lambda_0))$$

where the lowest order infinitesimals are retained.

As seen from the equations, the image motion depends on several parameters: the proper motion of the source $\mu_S$, the proper motion of the lens $\mu_D$, the source and lens parallaxes $p_S$, $p_D$ and the initial distance between the source and the lens $(x_0, y_0)$. There also exists a weak dependence on the initial declination $\beta_0$, but it does not produce qualitative changes in the image motion trajectory $S$, and we shall not analyze a dependence of the trajectory on $\beta_0$.

The distance between the lens and the lens $(x_0, y_0)$ as well the source and lens parallaxes have the same dimensions (they are dimensionless or measured in angular units, we shall measure these values in milliseconds of arc).

The proper velocities of the source and lens $\vec{\mu}_S$ and $\vec{\mu}_D$ have dimensions of inverse time (we shall measure them in milliseconds of arc per year).

To compare these parameters of different dimensions, it is necessary to multiply $\vec{\mu}_S$ and $\vec{\mu}_D$ by a characteristic time interval. Such an interval is a year, the interval for which the full parallax shift of the source and lens is executed.

Analyze the source image trajectory $S$ beginning with the simplest case of the proper velocities $S$ and $D$ equal to zero $\vec{\mu}_S = 0$ and $\vec{\mu}_D = 0$.

The observer executes only an annual orbital motion, and the solar system barycentre rests relative the light source and the lens.

Introduce auxiliary quantities $x_0 = \rho \cos \psi$ and $y_0 = \rho \sin \psi$. We shall also assume that the angular distance between the lens and the light source is much more than the lens parallax $\Delta n_1 \gg p_D$. This allows the equation for $R^2(t)$ to be simplified.

Write down the simplified equations describing the image motion. To this end we assume that all velocities in equations $\vec{\mu} = 0$, $p/\rho \ll 1$. Expand as a series in a small parameter $\frac{p}{\rho}$. 

23
Introduce two auxiliary definitions simplifying representation of coordinates depending on time $A = (p_S - p_D) \cos 2\psi \theta \rho \over \rho$, $B = (p_S - p_D) \sin 2\psi \theta \rho \over \rho$.

Now the coordinates $x, y$ depending on time (which is determined through the Earth’s right ascension $\lambda_E$) is

\[
\begin{align*}
x - x_S &= p_S \sin(\lambda_E - \lambda) - A \sin(\lambda_E - \lambda) + B \sin \beta \cos(\lambda_E - \lambda) \\
y - y_S &= -p_S \sin \beta \cos(\lambda_E - \lambda) - A \sin \beta \cos(\lambda_E - \lambda) - B \sin(\lambda_E - \lambda)
\end{align*}
\] (74) (75)

where we also introduce auxiliary quantities $x_S = \cos 2\psi \theta \rho \over \rho$, $y_S = \sin 2\psi \theta \rho \over \rho$.

We shall consider the inequality $A^2 + B^2 \neq p_S^2$ to be valid. Make a coordinate transformation of the form

\[
\begin{align*}
\tilde{x} &= (p_S + A)(x - x_S) + B(y - y_S), \\
\tilde{y} &= -B(x - x_S) + (A - p_S)(y - y_S).
\end{align*}
\] (76) (77)

This transformation incorporates translation of the origin of coordinates by the vector $(x_S, y_S)$, turn and dilatation of each of the axes in the ratio of $p_S + A \over p_S - A$ and reflection of one of the axes. The reflection will becomes obvious if one assumes that $A = B = 0$, then the transformations take the form

\[
\begin{align*}
\tilde{x} &= p_S(x - x_S), \\
\tilde{y} &= -p_S(y - y_S).
\end{align*}
\] (78) (79)

The transformation (76) results in equations of the form $\tilde{x} = -(p_S^2 - A^2 - B^2) \sin(\lambda_E - \lambda)$, $\tilde{y} = (p_S^2 - A^2 - B^2) \sin \beta \cos(\lambda_E - \lambda)$. If the condition $\Delta = A^2 + B^2 - p_S^2 \neq 0$ is satisfied, then from these equations follows an equation for a parallax ellipse of the form

\[
\tilde{x}^2 + \frac{\tilde{y}^2}{\sin^2 \beta} = 1
\] (80)

in true coordinates $(x, y)$ it is a deformed ellipse shifted by the vector $(x_S, y_S)$. In the case of $p_S = 0$ the figure is a true ellipse shifted relative the true position of the source by the vector $(x_S, y_S)$ and turned with respect to the ecliptic plane by the angle $2\psi$.

It should be noted that a turn of the parallax ellipse is not found in nonrelativistic astrometry, it is similar to the effect mentioned in the end of the section "Rigid-Baselined Observations".

Figure (8) depicts two ellipses: one is a parallax ellipse arising in the absence of the weak microlensing effect, the other ellipse of larger size arises in a weak action of a close gravitational lens whose parallax is much more than the source parallax. For simplicity both ellipses are drawn coaxially.
In the case of $\Delta = 0$ the motion is degenerated into a motion along a straight line of the form

$$y - y_S = \frac{p_S + A}{B}(x - x_S)$$

The parallax motion along a straight line in Euclidean astrometry corresponds the source position in the ecliptic plane $\beta_0 = 0$. However in this case the straight line of the parallax motion is parallel to the axis $\lambda$. In the case of the parallax motion being due to the weak microlensing effect there appears a slope of the straight line, with the slope coefficient depends on the lens parameters

$$\frac{y - y_S}{x - x_S} = \frac{p_S}{p_D - p_S}\frac{1}{\sin 2\psi \theta^2_{e}} + \cot 2\psi$$

Thus at a zero proper motion of the lens $D$ and the source $S$, as a result of an affine transformation of the coordinate system, the image motion trajectory although the real trajectory possesses a more complex shape.

Consider in more detail a motion along the parallax curve at different values of the problem parameters. First of all we shall assume that $p_S << p_D$ and introduce a designation $p_g = p_D \frac{\theta^2_{e}}{\rho}$. For simplicity we shall also assume that the Earth’s orbit eccentricity is equal to zero and $\lambda_e = \Omega t$, $\Omega = \frac{2\pi}{1\text{year}}$. Consider the case of $\psi = 0$. The parallax ellipse equation reads

$$x - x_S = (p_S + p_g)\sin(\lambda_e - \lambda_S)y - y_S = (-p_S + p_g)\sin \beta \cos(\lambda_e - \lambda_S)$$

As long as $p_g < p_S$, the visible motion of a celestial source forms the ellipse with semi-axes $p_S + p_g$ along the $Ox$ axis and $(-p_S + p_g)$ along the $Oy$ axis. The source motion is clockwise. Compare it with the case of parallax rigid-baselined...
measurement. In case the lens $D$ lies in the plane formed by the baseline $\vec{B}$ and the source the parallax increases (see Fig. 4). The parallax along $Ox$ axis increases as well. Really, the baseline formed by the Earth, when it holds the positions $x = x_{\text{max}}(y = 0)$ and $x = -x_{\text{max}}(y = 0)$, in turn forms, together with the source, a plane wherein lies $D$. An opposite case arises when the Earth is at the orbit points where the source image occupies positions $y_{\text{max}}(x = 0)$ and $-y_{\text{max}}(x = 0)$. The baseline vector, formed by these two points of the orbit, is perpendicular to the lens-source vector. Therewith the rigid-baselined source parallax decreases (see Fig. 5). Really, the parallax along $Oy$ axis proves to be $(-p_{S} + p_{D})$ less (in magnitude) than the parallax in the absence of the weak microlensing effect.

As $p_{g}$ increases, the parallax along $Ox$ continues to increase, and along $Oy$ to decrease. In the case of $p_{g} = p_{S}$ the source motion degenerates into a straight line parallel to $Ox$.

A further increase of $p_{g}$ results in the parallax along the axis beginning to grow again, but the source motion will be already retrograde, i.e. clockwise. This is usually identified with a visible negative parallax.

Calculate a direction of the source motion in the plane $Oxy$. To do this, we calculate the quantity

$$J = x\dot{y} - y\dot{x}$$

being an analogue of the $z$ component of the motion moment vector in mechanics. In our case this is perpendicular to the picture plane. A sign change in this quantity means an opposite rotation. The quantity $J$ is

$$J = \dot{\lambda}_{e}(p_{S}^{2} - p_{g}^{2}) \sin(\lambda_{e} - \lambda) \sin \beta$$

In the absence of the weak microlensing effect $p_{g} = 0$. The sign of $J$ is determined by the angular velocity of the Earth orbital motion $\dot{\lambda}_{e}$ and by the sign of $\sin \beta$. A sign change occurs in the case of $p_{g} > p_{S}$. We shall identify the sign change with a visible negative parallax. Thus the negative parallax appears under the condition $p_{g} > p_{S}$. It should be noted at once that for most extragalactic sources one may assume $p_{S} = 0$, and hence the observed parallax will be negative for them.

Now we consider a more complex case of motion of the image $S$, which arises at a nonzero proper motion and a nonzero lens parallax. It should be noted that the case of nonzero lens parallax corresponds to observation from the barycentre and has been consider previously [39]. The parallax and proper motion of the source will be assumed zero.

Due to a lens motion there arises an angular motion of the image, as in Fig. (6) but along a closed curve since now a motion of the source is absent. This circular motion is superimposed by an image shift due to the lens parallax shift (which results in a variable angular distance between the lens $D$ and the source $S$ with an annual period, see Fig. 10).
Choose the lens parameters as follows. The source declination is $\beta \approx 30^\circ$. The star-lens distance from the solar system amounts to 200 pc, which corresponds to the lens parallax $p_D = 5$ mas. We shall consider that the star-lens mass amounts to $M = 2.5 M_\odot$, so that Einstein’s cone size $\theta_e = 10$ mas. The initial distance between the lens $D$ and the source $S$ is the vector $\Delta \vec{n} = (100, 200)$ mas.

Calculate a trajectory of motion of the image of the source $S$ for three limiting cases of motion of the lens $D$. The first case is a slow motion of the lens $|\mu_D| \cdot 1\text{ year} < p_D$, in our case we choose parameters of the proper velocity of the lens $\mu_\lambda = 1$ mas per year, $\mu_\beta = 0.5$ mas per year (see Fig. 9). In this case the subtrajectory due to a lens parallax shift (change in the angular distance between the lens and the source) everywhere fills the large circle of the motion due to the weak microlensing effect).

In the case of equality of the basic parameters, when the proper motion of the lens along the right ascension and declination is $\mu_\lambda = 10$ mas per year and $\mu_\beta = 0$ mas per year respectively, the motion becomes akin to (Fig. 10).

Finally, in a fast flight of the lens $\mu_\lambda = 100$ mas per year $\mu_\beta = 0$ the motion is akin to (Fig. 11).

The case considered above, when $p_S = 0$ and $\vec{\mu}_S = 0$, is the most natural for description of the situation arising in the observation of extragalactic sources. Really, the quasars having considerable red shifts, say $z = 0.2$, are located at
Figure 10: The circular motion of the source image is superimposed by distortions due to the parallax shift of the lens that covers the angular distance per year equal to the parallax of the lens itself.
Figure 11: The circular motion of the image motion is superimposed by distortions due to the parallax shift of a fast lens. As seen from the figure, the circle have converted into a figure similar to the waving circle.
Figure 12: The source image motion is superimposed by distortions related to a circular motion due to the weak microlensing effect, lens and source parallax shifts.

Distances of the order of $R \sim 1$ Gpc. Such a distance corresponds to the parallax of 1 mas, which cannot be measured in the near future. Random velocities of the quasars amount to several thousands of km per second. We choose $v \sim 3000$ km per second. Such velocities of the transversal motion correspond to the proper angular velocities, when the object is at a distance of 1 Gpc, of the order $\mu < 1 \frac{\text{mas}}{\text{year}}$. This value may be also neglected.

However, while discussing the weak microlensing effect, when the lens and the source are in our Galaxy (although the lens is closer than the source), the parallax of the source and its proper motion cannot be neglected. In this case there arises the most complex visible motion of $S$ (see Fig. 12).

Fig. 12 depicts a model with the object parameters as follows. The source declination is equal to $30^\circ$; the proper motion of the lens $\mu_\lambda = 8$ mas per year, $\mu_\beta = 3$ mas per year. The proper motion of the lens $\mu_\lambda = 1$ mas per year, $\mu_\beta = -0.5$ mas per year. The values of the parallax shift for the source and the lens are equal $p_S = 1$ mas and $p_D = 3$ mas respectively. Einstein’s cone size $\theta_e = 10$ mas, which corresponds to an object with the mass about four Solar masses. The time interval during which the modelling has been conducted is equal to 40 years.

The visible motion of the source appears to be especially involved in case the minimum angular distance between the lens $D$ and the light source $S$ is comparable with the parallax of the lens itself. In the case of a slow motion of the lens there may arise a situation of “transferring” the image $S$ resulting in
large visible displacements of $S$ on the sky. Therewith the motion of the image approaches a chaotic one. Although the case of $\Delta n \sim p_D$ is unlikely, the chaotic motion of the source $S$ will mean realization of just such a case.

**Conclusion**

Two important conclusions can be made. The first one lies in a possible appearance of rather a complex visible motion of the extragalactic sources, which are ICRF basic sources, under the action of the weak microlensing effect. This motion may exceed considerably the proper notion of a part of the sources, with the motion due to the weak microlensing effect being composed of two motions – one having a characteristic angular quantity $\delta \varphi \sim \theta_i^2 / \theta_i$ (here $\theta_i$ is the angular distance between the lens and the source) and a characteristic time of tens and hundreds of years, the other having a period $\sim 1$ year related to the lens parallax motion and the quantity of angular shift $\sim p_D (\theta_e / \theta_i)^2$. Although the second shift is less than the first one for most events, it may amount to a good part of the total angular deviation $\delta \varphi$.

In paper ([39]) an effect of a weak microlensing on ICRF sources was considered. 8 sources were found to belong to the list of 607 ICRF sources whose angular shift due to the weak microlensing effect was maximal. Parallax changes can be simulated for these sources. On conducting the simplest simulation, it has been found that there appears a retrograde parallax motion due to the weak microlensing effect. The value of this parallax averages $2 \mu as$ varying from 50 nas to 5 $\mu as$. Since the investigated sources are assumed to be extragalactic ones, all parallaxes are negative.

Hence the second conclusion lies in the motion being retrograde in most cases, which corresponds negative parallaxes. The values of parallaxes due to the weak microlensing effect is of the order of hundreds of nanoseconds and microseconds of arc.

**Acknowledgement**

The authors are grateful to S. Kopeikin, A. Kuzmin, K. Kuimov for valuable remarks and comment. This work has been done under support of "Cosmion" center, Russian Fund of Basic Research (grants NN 98-05-64797, 00-02-16350), as well as "Russian Universities" Programme (grants N 2-5547, N 9900777).
References

[1] Alcock C. et al., 1993, Nature, 365, 621

[2] Allen C.W. ed., 1973, Astrophysical Quantities. Univ. of London, The Athlone Press

[3] Andreyanov V.V., Kardashev N.S., 1981 Kosmicheskie issledovaniya, 19, 763. (in Russian)

[4] Andreyanov V.V et al., 1986, Astronom. Zh., 63, 850 (in Russian)

[5] Aubourg E. et al., 1993, Nature, 365, 623

[6] Bliokh, Minakov, 1989, Gravitational lenses, Kiev, Naukova dumka (in Russian)

[7] Dashevsky, Zeldovich, 1965, Astron. Zh., 41, 1071. (in Russian)

[8] Eichorn H., Astronomy of stars positions. Frederick Ungar Pub. Co., New York.

[9] Ma C et al., Astron. J. 116, 516, 1998.

[10] Gontier A.-M., Feissel M., Essaifi N., Jean-Alexis D., Paris Observatory Analysis Center OPAR on activities, Jan98 - Mar99.

[11] Green R. M. Spherical Astronomy. Cambridge Univ. Press, 1985.

[12] The Hipparcos and Tycho Catalogues. Vol.1, Introduction and Guide to the Data. M.A.C. Perryman. ESA Publ. Div., c/o ESTEC, Noordwijk, The Netherlands, June 1997.

[13] Hog E., Novikov, Polnarev, 1994, Nordita Preprint, Macho Photometry and Astrometry, Nordita – 94/26 A

[14] IERS, 21, 1996, International Earth Rotation Service Annual report, Observatoire de Paris

[15] IERS, 1995, 1994 International Earth Rotation Service Annual report, Observatoire de Paris

[20] Jacobs C.S., Sovers O.J., Williams J.G., Standish E.M., 1993, Advances in Space Research, 13, No. 11, 161

[21] Jenkner H., Lasker B.M., Struch C.R., McLean B.J., Shara M.M, Russel J.L., 1990, AJ, 99, 2082
[22] Kaplan S.A., Pikelner S.B., 1979, Physics of Interstellar Medium, Moscow, Nauka (in Russian)

[23] Kardashev N.S., 1986, Astron. Zh., 63, 845. (in Russian)

[24] Kopeikin S., Scheffer Phys. Rew, D, 60, N124002, 1999 gr-qc 9902030.

[25] Kopeikin S., Gwinn C., Sub-Microarcsecond Astrometry and New Horizons in Relativistic Gravitational Physics. Proc. IAU Coll., 180.

[26] Lasker B.M., Struch C.R., McLean B.J., Russel J.L., Jenkner H., Shara M.M., 1990, AJ, 99, 2019

[27] Murrey C.A., Vectorial astrometry, Royal Greenwich Observatory, Herstmonceux Castle, East Sussex, Adam Hilder, Bristol, 1983.

[28] McCarthy D.D., 1996, ed., IERS Conventions. IERS Technical Note 21, Observatoire de Paris

[29] Paczinsky B., 1986, ApJ, 304, 1

[30] Perryman M.A.C. et al., 1997, A& A, 323, L49

[31] Project GAIA. [http://astro.estec.esa.nl/SA-general/Project/GAIA/]

[32] Project SIM. [http://sim.jpl.nasa.gov/]

[33] Project DARWIN. [http://ast.star.rl.ac.uk/darwin/]

[34] Project FAME. [http://aa.usno.navy.mil/fame/]

[35] Project DIVA. [http://www.aip.de/groups/DIVA/]

[36] Refsdal, 1964, Monthly Not. Roy. Astron. Soc., 128, 295.

[37] Russel J.L., Lasker B.M., McLean B.J., Struch C.R., Jenkner H., 1990, AJ, 99, 2059

[38] Sazhin M.V., 1996, Pis’ma v Astron. Zh., 22, 643 (in Russian)

[39] Sazhin M.V., Zharov, A.V.Volynkin, Kalinina T.A., 1998, Monthly Not. Roy. Astron. Soc., 300, 287

[40] Schneider P., Ehlers J., Falco E.E. Gravitational Lenses. Berlin, New York, Springer Verlag, 1992

[41] Udalski A., Szymanski M., Kaluzny J., et al., 1994, ApJ Lett., 426, L69

[42] Zakharov A.F., Gravitational Lenses and Microlenses. Moscow. Janus-K Publ., 1997. (in Russian)
[43] Zakharov A.F., Sazhin M.V. 1998, Physics - Uspekhi, 41, 945.
[44] Zeldovich, 1964, Astron. Zh., 41, 19. (in Russian)
[45] Zhdanov I.I, et al., 1995, Astron. and Astrophys., 299, 321.