Exact statistics of complex zeros for Gaussian random polynomials with real coefficients

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Abstract $k$–point correlations of complex zeros for Gaussian ensembles of Random Polynomials of order $N$ with Real Coefficients (GRPRC) are calculated exactly, following an approach of Hannay [5] for the case of Gaussian Random Polynomials with Complex Coefficients (GRPCC). It is shown that in the thermodynamic limit $N \to \infty$ of Gaussian random holomorphic functions all the statistics converge to the their GRPCC counterparts as one moves off the real axis, while close to the real axis the two cases are essentially different. Special emphasis is given to 1 and 2 point correlation functions in various regimes.

The problem of statistics of zeros of random polynomials of order $N$, and of random holomorphic functions as $N \to \infty$ in general, arises in various contexts in quantum chaos [2, 3]. The motivation for this work was the problem of statistics of zeros of coherent state (Husimi) or Bargmann [4] representation of eigenstates of chaotic systems [6, 8]. It has been conjectured [8] that zeros of Bargmann or Husimi representation of an eigenfunction of 1-dim classically chaotic system should be uniformly and randomly scattered over the classically chaotic region of phase space. Bargmann representation of an eigenstate is an entire analytic function in a complex phase space variable $z = q + i p$, sometimes it is even a polynomial of a finite order, like for example in the case of spin systems where the phase space manifold is a sphere parametrized by $(\theta, \phi)$ and $z = \cot(\theta/2) \exp(i \phi)$ is a stereographic projection. The coefficients of a power series of such entire functions or

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polynomials are just the coefficients of an expansion of the chaotic eigenstate in a complete set of (say harmonic) wavefunctions. Applying the random matrix theory one argues that these coefficients should be uncorrelated (real/complex in the presence/absence of anti-unitary symmetry) pseudorandom Gaussian variables. Thus one can introduce the statistical ensembles of random polynomials of order $N$ (or random analytic functions in the limit $N \to \infty$) and argue that statistical properties of their zeros can be used as a model to describe statistical properties of zeros of a Bargmann representation of chaotic eigenstates of real systems.

Recently, Hannay \[5\] has calculated general $k$-point correlation functions of zeros of a random spin state in a coherent state representation which is described by the random polynomial with uncorrelated complex Gaussian coefficients, and solved the problem of statistics of zeros of GRPCC — Gaussian random polynomials with complex coefficients in general. It has been demonstrated numerically \[5, 6\] that his results on GRPCC provide a universal description of the statistics of zeros of Bargmann or Husimi representation of chaotic eigenstates for systems without an anti-unitary symmetry. Here we adopt this approach and solve the general problem of statistics of zeros $z_k$ of GRPRC — random polynomials $f(z)$ of order $N$

$$f(z) = \sum_{n=0}^{N} a_n z^n = a_N \prod_{j=1}^{N} (z - z_j) \quad (1)$$

with real (Gaussian) coefficients $a_n$. We argue that the obtained results may be used to describe statistics of zeros of eigenstates of 1-dim. (and quantum Poincaré sections \[8\] and other reductions \[10, 11\] of 2-dim.) chaotic systems in Bargmann representation with time reversal invariance (or any other anti-unitary symmetry \[10\]) to the same extent as Gaussian orthogonal ensembles of random matrices can be used to describe the Hamiltonian and the typical observables.

In the literature one may find several results on the distribution of complex zeros of random polynomials with either complex \[1\] or real \[12\] Gaussian coefficients (see also \[8\] and references therein). The formula for 1-point function given below (19) (in the special case where the variances of all coefficients are equal) is equivalent to the theorem 1.1 of Shepp and Vanderbei \[12\].

Take a $k$-tuple of complex numbers $z = (z_1, \ldots, z_k)$. Since $a_n$ are real Gaussian random variables (which in general need not be uncorrelated!), their real linear combinations

$$f^r_j = \text{Re} f(z_j), \quad f^i_j = \text{Im} f(z_j), \quad f^r_j = \text{Re} \frac{d}{dz} f(z_j), \quad f^i_j = \text{Im} \frac{d}{dz} f(z_j), \quad j = 1, \ldots, k \quad (2)$$

are also real Gaussian random variables with a joint distribution

$$P(f^r, f^i, f^{r*}, f^{i*}) = (\det 2\pi \tilde{\mathbf{M}})^{-1/2} \exp \left(-\frac{1}{2} (f^r, f^i, f^{r*}, f^{i*}) \cdot \tilde{\mathbf{M}}^{-1} (f^r, f^i, f^{r*}, f^{i*}) \right). \quad (3)$$

\[2\]For a general anti-unitary symmetry, the coefficients of the random polynomials \[1\] are of the form $a_n = r_n e^{i\theta_n}$ where $r_n$ are real Gaussian random variables and $\theta_n$ are fixed (nonrandom) phases (which determine the symmetry curve in complex $z$-plane, such as in fig.6 of ref. \[1\]). Then one may use the same general approach described below, eqs. \[8, 13\].
\( \tilde{M} \) is a \( 4k \times 4k \) real symmetric positive correlation matrix

\[
\tilde{M} = \begin{pmatrix}
\langle f^r_j f^r_i \rangle & \langle f^r_j f^i_i \rangle & \langle f^r_j f^i_j \rangle & \langle f^r_j f^i_k \rangle \\
\langle f^i_j f^r_i \rangle & \langle f^i_j f^i_i \rangle & \langle f^i_j f^i_j \rangle & \langle f^i_j f^i_k \rangle \\
\langle f^i_j f^r_i \rangle & \langle f^i_j f^i_j \rangle & \langle f^r_j f^i_k \rangle & \langle f^i_j f^i_k \rangle \\
\langle f^i_j f^i_i \rangle & \langle f^i_j f^i_j \rangle & \langle f^i_j f^i_k \rangle & \langle f^i_j f^i_k \rangle 
\end{pmatrix} = \begin{pmatrix}
\tilde{A}^T & \tilde{B} \\
\tilde{B} & \tilde{C}
\end{pmatrix}
\]  

(4)

where \( \langle \rangle \) denotes the Gaussian ensemble averages which can be calculated using \( (\tilde{M}) \) in terms of input data \( (a_n, a_m) \). One can write the \( k \)-point correlation function \( \rho_k(z) \) in the following form

\[
\rho_k(z) = \int P(0, 0, f'^r, f'^i) \prod_{j=1}^k ((f'^r_j)^2 + (f'^i_j)^2) df'^r_j df'^i_j
\]  

(5)

where the factors \( (f'^r_j)^2 + (f'^i_j)^2 \) are just the Jacobians of transformations from the pairs of real variables \( (f^r_j, f^i_j) \) to complex variables — zeros \( z_j \). The integral can be written in terms of derivatives of a generating function \( Z_k(u, v) \)

\[
\rho_k(z) = (-1)^k \prod_{j=1}^k (\partial_{u_j}^2 + \partial_{v_j}^2) Z_k(u, v)|_{u=v=0}
\]  

(6)

which is an ordinary Gaussian integral and can be explicitly calculated

\[
Z_k(u, v) = (\det 2\pi \tilde{M})^{-1/2} \int \exp \left( -\frac{1}{2} (f'^r - f'^i) \cdot \tilde{L} (f'^r, f'^i) + i (f'^r \cdot u + i f'^i \cdot v) \right) \prod_{j=1}^k df'^r_j df'^i_j
\]  

\[
= (\det 2\pi \tilde{A})^{-1/2} \exp \left( -\frac{1}{2} (u, v) \cdot \tilde{L} (u, v) \right)
\]  

(7)

where \( \tilde{L} = \tilde{C} - \tilde{B}^T \tilde{A}^{-1} \tilde{B} \) is a lower right block of \( \tilde{M}^{-1} \) and we have used an identity \( \det \tilde{L}/\det \tilde{M} = 1/\det \tilde{A} \). At this point it is convenient to switch on the equivalent complex variables \( f = f'^r + if'^i, f' = f'^r + if'^i, w = u + iv \) and their complex conjugates. Then one can write eq. \( (\tilde{M}) \) as

\[
\rho_k(z) = \frac{(-1)^k 2^k}{(\det 2\pi A)^{1/2}} \prod_{j=1}^k \partial_{w_j}^2 \partial_{w_j}^* \exp \left( -\frac{1}{2} (w^*, w) \cdot L(w, w^*) \right) |_{w=0}
\]  

\[
= (\det 2\pi A)^{-1/2} \prod_{j=1}^k \partial_{w_j}^2 \partial_{w_j}^* ((w^*, w) \cdot L(w, w^*))^k |_{w=0}
\]  

(8)

where all the \( 2k \times 2k \) real matrices should be transformed by the rule

\[
X = U^T \overline{A} U, \quad U = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
i1 & -i1
\end{pmatrix}
\]

giving \( L = C - B^T A^{-1} B \) with

\[
A = \begin{pmatrix}
\langle f_j f^*_j \rangle & \langle f^*_j f_k \rangle \\
\langle f^*_j f^*_j \rangle & \langle f^*_j f^*_k \rangle
\end{pmatrix} = A^T,
\]  

(9)

\[
B = \begin{pmatrix}
\langle f_j f^*_j \rangle & \langle f^*_j f_k \rangle \\
\langle f^*_j f^*_j \rangle & \langle f^*_j f^*_k \rangle
\end{pmatrix},
\]  

(10)

\[
C = \begin{pmatrix}
\langle f^*_j f^*_j \rangle & \langle f^*_j f^*_k \rangle \\
\langle f^*_j f^*_j \rangle & \langle f^*_j f^*_k \rangle
\end{pmatrix} = C^T.
\]  

(11)
Applying a little combinatorics on eq. (8) we finally obtain the general result
\[ \rho_k(z) = \frac{\text{sper} (C - B^\dagger A^{-1} B)}{\sqrt{\det 2\pi A}} \]  
(12)
where we introduce the semi-permanent of a 2k × 2k matrix
\[ \text{sperL} = \sum_{j_1 < ... < j_k} \sum_{l_1 < ... < l_k} L_{j_r + k, l_p(r)} \]  
(13)
The first sum runs over (2k)!/(k!)² ordered combinations of k out of 2k indices \( j_m \) and their complements \( l_n \) while the second sum runs over \( k! \) permutations \( p \) of the symmetric group \( S_k \). The sum of indices \( j_r + k \) should be taken modulo 2k.

So far we have not assumed anything about the correlations between the coefficients \( a_n \) expect the Gaussian nature of the joint distribution of coefficients \( a_n \). Now we shall assume that Gaussian coefficients \( a_n \) are uncorrelated and define the polynomial \( g(s) \) with positive coefficients \( b_n \) — the variances of \( a_n \)
\[ \langle a_n a_m \rangle = b_n \delta_{nm}, \quad b_n > 0, \]  
(14)
\[ g(s) = \sum_{n=0}^{N} b_n s^n. \]  
(15)
The matrices A, B, and C can be easily expressed solely in terms of a polynomial \( g \) and its derivatives \( g', g'' \)
\[ A_{jl}(z) = g(z_j z_l^*), \]  
(16)
\[ B_{jl}(z) = \partial_{z_j} A_{jl}(z) = z_j g'(z_j z_l^*), \]  
(17)
\[ C_{jl}(z) = \partial_{z_j} \partial_{z_l^*} A_{jl}(z) = g'(z_j z_l^*) + z_j z_l^* g''(z_j z_l^*) \]  
(18)
where we let indices \( j, l \) to run from 1 through 2k and put \( z_{k+j} := z_j^* \). Note that the time-reversal symmetry — the symmetry of zeros with respect to the reflection over the real axis is present also in the k-point correlation functions, namely
\[ \rho_k(z_1, \ldots, z_j, \ldots, z_k) = \rho_k(z_1, \ldots, z_j^*, \ldots, z_k). \]
Without loss of generality one may assume that all points \( z_j \) lie on the upper complex halfplane, \( \text{Im} z_j > 0 \). Otherwise one gets long range correlations in cases where one of the points \( z_j \) comes close to the mirror image of one of the other points \( z_j^* \).

In general, only the 1-point function \( \rho_1(z) \) — the density of zeros is simple enough to be written out
\[ \rho_1(z) = \frac{g_0' + |z|^2 g_0''}{\pi (g_0^2 - g_+ g_-)^{1/2}} + \frac{(z^2 g_+ g'_- + z^2 g_+ g''_-) g_0 - |z|^2 (g'_+ g''_- + g'^2_+) g_0}{\pi (g_0^2 - g_+ g_-)^{3/2}}, \]  
(19)
where \( g_0 \equiv g(|z|^2), \ g_+ \equiv g(z^2), \ g_- \equiv g(z'^2) \). Writing \( z = x + iy \) and carefully expanding for small \( y \) one finds
\[ \rho_1(z) = h(x^2) |y| + O(y^3), \quad y \neq 0 \]  
(20)
\[ h(s) = (2\pi)^{-1} (gg' - sg^2 + sg'' )^{-3/2} (2g_012 + 2(2g_013 - g_112 - g_022) s + (3g_122 - 4g_113 + g_014) s^2 + (g_024 - g_114 - g_033 + 2g_123 - g_222) s^3) \]
where $g \equiv g(s)$, $g_{nml} \equiv g^{(m)}(s)g^{(n)}(s)g^{(l)}(s)$. So quite generally, the density of zeros decreases linearly as we approach the real axis. To evaluate the density of zeros on a real axis $y = 0$ one should use a different approach described in [3]. In another asymptotical regime $|z| \to \infty$, only the highest power terms of $g$ contribute, and one finds

$$\rho_1(z) = \frac{2b_{N-2}}{\sqrt{b_N b_{N-1}}} \frac{\text{Im} z}{|z|^6} \left(1 + O\left(\frac{1}{|z|^2}\right)\right). \tag{21}$$

So, the density of zeros vanishes asymptotically since the total number of zeros $N$ is finite.

Now we shall study the thermodynamic limit $N \to \infty$. It is convenient to study random holomorphic functions which provide a uniform distribution of zeros in the complex plane. A unique choice (up to rescaling $s \to \lambda s$) is $b_n = 1/n!$ giving

$$g(s) = \exp(s). \tag{22}$$

Such random holomorphic functions naturally arise when one studies Bargmann representation of 1-dim chaotic eigenstates in the usual $(p, q) \in \mathbb{R}^2$ phase space. We argue that any other choice will only affect the density of zeros $\rho_1(z)$ while properly rescaled local statistics should be independent on the choice of $g(s)$ provided that variances of coefficients $b_n$ depend smoothly on $n$.

Away enough from the real axis $\text{Im} z_j \gg 1$ one may neglect the offdiagonal $k \times k$ blocks of matrices $A, B, C$ since the ratios of the corresponding matrix elements become exponentially small $|\exp(z_j z_l^*)/\exp(z_j z_l)| = \exp(-2\text{Im} z_j \text{Im} z_l)$. Then using straightforward results

$$2^{-k} \text{sper} \left( \begin{array}{cc} L_{11} & 0 \\ 0 & L_{11}^T \end{array} \right) = \text{per} L_{11} := \sum_{\rho \in S_k} \prod_{j=1}^k L_{j, \rho(j)} \tag{23}$$

$$\det \left( \begin{array}{cc} A_{11} & 0 \\ 0 & A_{11}^T \end{array} \right) = (\det A_{11})^2, \tag{24}$$

where $(.)_{11}$ denotes the upper-left $k \times k$ block of a $2k \times 2k$ matrix, one arrives to the result which is equivalent to the statistics of zeros of GRPCC [3]

$$\rho_k(z) \to \rho_k^{\text{GRPCC}}(z) = \frac{\text{per}(C_{11} - B_{11}^* A_{11}^{-1} B_{11})}{\det \pi A_{11}}, \text{ as } \text{Im} z_j \to \infty. \tag{25}$$

To conclude we give some explicit results about 1 and 2 point functions. The density of zeros which is shown in figure 1 reads

$$\rho_1(x + iy) = \frac{1 - (4y^2 + 1) \exp(-4y^2)}{\pi(1 - \exp(-4y^2))^{3/2}}, \tag{26}$$

which is a constant $1/\pi$ provided that we are away enough from the real axis. The excess of zeros due to the presence of real axis $\int_{-\infty}^{\infty} (1/\pi - \rho_1(x + iy))dy = 1/\pi$ is on the other hand just the linear density of real zeros on the real axis!

The 2-point correlation function $\rho_2(z_1, z_2)$ is already too lengthy to be written out in general. The behaviour of a normalized 2-point correlation function $\rho_2(z_1, z_2)/\rho_1(z_1)/\rho_1(z_2)$
as we approach the real axis, is shown in figure 2, while far away \( \text{Im} z_1, \text{Im} z_2 \gg 1 \) it becomes isotropic and the result for GRPCC applies

\[
\rho_2(z_1, z_2) \rightarrow \varphi(|z_1 - z_2|^2),
\]

\[
\varphi(s) = \frac{\exp(-2s)(\exp(s) - 1 - s)^2 + \exp(-s)(\exp(-s) - 1 + s)^2}{\pi^2(1 - \exp(-s))^3}
\] (27)

In the asymptotic regime \( \text{Im} z_j \gg 1 \) one can calculate also the number variance \( \Sigma_2(r) \): the variance of the number of zeros \( N(r) \) inside a circle of radius \( r \)

\[
\Sigma_2(r) = \langle N^2(r) \rangle - \langle N(r) \rangle^2.
\] (28)

It can be expressed in terms of a four-fold integral (over \( z_1, z_2 \)) of a 2-point correlation, which can be reduced using eq. (27) to a single integral

\[
\Sigma_2(r) = r^2(1 - r^2) + 8\pi r^4 \int_0^1 (\arccos \sqrt{t} - \sqrt{t(1 - t)})\varphi(4r^2 t)dt.
\] (29)

The number variance \( \Sigma_2(r) \) starts as “Poissonian” \( \langle N(r) \rangle = r^2 \) for small \( r \) whereas for larger \( r \) it has a linear asymptotics (see figure 3)

\[
\Sigma_2(r) = \sigma r + O(1/r) \approx \sigma \sqrt{\langle N(r) \rangle}, \quad \sigma = \frac{4}{\pi} \int_0^\infty s^2(1 - \pi^2\varphi(s^2))ds \approx 0.36847.
\] (30)

Note that this formula (29,30) is valid also for GRPCC in general.

In the present paper the statistics of zeros of Gaussian random polynomials with real coefficients have been solved analytically (12) following an approach of Hannay for the case of complex coefficients. Several important special cases have been considered in detail: (i) the case of mutually uncorrelated coefficients, which corresponds to the Bargmann representation of chaotic eigenstates in random matrix regime, has been studied and it has been shown that all \( k \)-point correlation functions converge to those of random polynomials with complex coefficients derived by Hannay as all points \( z_j, j = 1 \ldots k \) move away from the real axis \( \text{Im} z_j \gg 1 \) (23), (ii) 1-point function – the density of zeros have been written out in general (eqs. (19,26) and fig. 1) and linear decrease of density towards the symmetry line – real axis has been found (20), (ii) 2-point function close to the real axis have been explored numerically (fig. 2) while simple analytic formula (27), which holds far away from the real axis (and holds generally in the case of complex coefficients), have been used to derive a simple expression for the number variance of zeros inside a circle of a given radius (eqs. (29,30) and fig. 3).

Discussions with P.Leboeuf, J.H.Hannay, M.Saraceno and K.Życzkowski as well as the hospitality of Institut Henri Poincaré (Paris) are gratefully acknowledged. This work has been financially supported by the C.I.E.S (France) and the Ministry of Science and Technology of the Republic of Slovenia.
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Figure captions

**Fig. 1** We show the density of zeros $\rho_1(x+iy)$ in the thermodynamic limit $N \to \infty$ given by eq. (26) as a function of the distance from the real axis.

**Fig. 2** The normalized 2-point correlation function $\rho_2(x_1+iy, x_2+iy)/\rho_1(x_1+iy)/\rho_1(x_2+iy)$ in the limit $N \to \infty$ between two points, $x_1 + iy$ and $x_2 + iy$, which have the same distance from the real axis $y$ is shown as a function of $|x_2 - x_1|$ for different values of $y = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5$. Note that all curves go to zero as $\propto y^2$ and that for $y \geq 1.5$ the 2 point correlation function has practically converged to the isotropic asymptotic one.

**Fig. 3** The number variance $\Sigma_2(r)$ in the asymptotical regime $N \to \infty, \text{Im} z \gg 1$ is shown as a function of radius $r$ (29).