ON THE $k$-ABELIAN COMPLEXITY OF THE CANTOR SEQUENCE

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ABSTRACT. In this paper, we prove that for every integer $k \geq 1$, the $k$-abelian complexity function of the Cantor sequence $c = 101000101\cdots$ is a 3-regular sequence.

1. INTRODUCTION

This paper is devoted to the study of the $k$-abelian complexity of the Cantor sequence
$$c := c_0c_1c_2 \cdots = 1010001010000000101000101\cdots$$
which satisfies $c_0 = 1$ and for all $n \geq 0$,
$$c_{3n} = c_{3n+2} = c_n \text{ and } c_{3n+1} = 0.$$ (1.1)

The $k$-abelian complexity, which was introduced by Karhumäki in [8], is a measure of disorder of infinite words. It has been studied widely in [12, 13, 14, 15, 16]. Before we give its definition, we need some notations. Let $A$ be a finite alphabet and $A^n$ be the set of words of length $n$ for every positive integer $n$. Denote $A^*$ the set of all finite words on $A$. For two words $u, v \in A^*$, $v$ is called a factor of $u$ if $u = wvw'$ where $w, w' \in A^*$. For a word $u = u_0u_1\cdots u_{n-1} \in A^n$, the prefix and suffix of length $\ell \geq 1$ are defined as
$$\text{pref}_\ell(u) := u_0u_1\cdots u_{\ell-1} \text{ and } \text{suffix}_\ell(u) := u_{n-\ell}\cdots u_{n-1};$$
while for $\ell \leq 0$, we define $\text{pref}_\ell(u) = \varepsilon$ and $\text{suffix}_\ell(u) = \varepsilon$, where $\varepsilon$ is the empty word. Denote $|u|$ the length of a word $u$ and denote $|u|_w$ the number of occurrences of a word $v$ in $u$.

Definition 1 (see [17]). Let $k \geq 1$ be an integer. Two words $u, v \in A^*$ are called $k$-abelian equivalent, written by $u \sim_k v$, if $\text{pref}_{k-1}(u) = \text{pref}_{k-1}(v)$, $\text{suffix}_{k-1}(u) = \text{suffix}_{k-1}(v)$ and $|u|_w = |v|_w$ for every $w \in A^k$.

The above definition is one of the equivalent definitions of the $k$-abelian equivalence; see also [16]. The $k$-abelian equivalence is in fact an equivalence relation. The $k$-abelian complexity of an infinite word $\omega$ is the function $P_\omega^{(k)} : \mathbb{N} \to \mathbb{N}$ and for every $n \geq 1$, $P_\omega^{(k)}(n)$ is assigned to be the number of $k$-abelian equivalence classes of factors of $\omega$ of length $n$. Precisely, for every positive integer $n$,
$$P_\omega^{(k)}(n) = \text{Card}(F_\omega(n)/\sim_k),$$
where $F_\omega(n)$ is the set of all factors of length $n$ occurring in $\omega$.

In our first result, we reduce the $k$-abelian equivalence of any two factors of $c$ to the abelian equivalence of such factors. In detail, we prove the following theorem.

Theorem 1. Let $k \geq 1$ be an integer and let $u, v$ be two factors of $c$ satisfying $|u| = |v|$. If $\text{pref}_k(u) = \text{pref}_k(v)$ and $\text{suffix}_k(u) = \text{suffix}_k(v)$, then $u \sim_{k+1} v$ if and only if $u \sim_1 v$. 

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By using Theorem 1 we are able to study the $k$-abelian complexity of $c$ for every $k \geq 1$, and we have the following result.

**Theorem 2.** For every integer $k \geq 1$, the $k$-abelian complexity function of the Cantor sequence is a 3-regular sequence.

The $k$-regular sequence was introduced by Allouch and Shallit [2] as an extension of the $k$-automatic sequence. The definitions of the $k$-automatic sequences and the $k$-regular sequences are stated below; see also [1, 5].

**Definition 2.** For an integer $k \geq 1$, a sequence $w = (w_n)_{n \geq 0}$ is a $k$-automatic sequence if its $k$-kernel

$$K_k(w) = \{(w_{kn+c})_{n \geq 0} \mid c \geq 0, 0 \leq c < k^k\}$$

is a finite set. The sequence $w$ is called a $k$-regular sequence if the $\mathbb{Z}$-module generated by its $k$-kernel is finitely generated.

Kärki, Saarela and Zamboni [14] studied the $k$-abelian complexity of the Thue-Morse sequence, which is a 2-automatic sequence. Vandomme, Parreau and Rigo [17] conjectured that the 2-abelian complexity of the Thue-Morse sequence is a 2-regular sequence. This has been proved independently in [12] by Greinecker and in [13] by Parreau, Rigo, Rowland and Vandomme.

Our result (Theorem 2) supports the following more general conjecture, which has been posed in [13].

**Conjecture 1.** The $k$-abelian complexity of any $\ell$-automatic sequence is an $\ell$-regular sequence.

This paper is organized as follows. In Section 2, we give the recurrence relations for the abelian complexity of the Cantor sequence $c$. In Section 3, we prove Theorem 1. In the last section, we prove the proof of Theorem 2.

2. **Abelian complexity**

The abelian complexity of an infinite word $\omega$ is in fact the 1-abelian complexity of $\omega$. For more details of the abelian complexity, see [3, 5, 8, 9, 10, 11] and references therein. In this section, we shall investigate the abelian complexity of $c$.

First we introduce a useful result which characterizes the left and right special factors of $c$. Recall that a factor $v$ of $w$ is called right special (resp. left special) if both $va$ and $vb$ (resp. $av$ and $bw$) are factors of $w$ for distinct letters $a, b \in \mathcal{A}$. We denote $\mathcal{RS}_w(n)$ (resp. $\mathcal{LS}_w(n)$) the set of all right special (resp. left special) factors of $w$ of length $n$.

**Lemma 1.** For every $i \geq 0$ and $3^i < k \leq 3^{i+1}$,

$$\mathcal{RS}_c(k) = \{0^k, \text{suff}_k(\sigma^i(010))\} \quad \text{and} \quad \mathcal{LS}_c(k) = \{0^k, \text{pref}_k(\sigma^i(010))\}.$$

**Proof.** The result follows from [7, Theorem 1] and the fact that every left special factor in $c$ is the reversal of some right special factor in $c$. \qed

Let $\omega = \omega_1 \omega_2 \omega_3 \cdots$ be an infinite sequence on $\{0, 1\}$. It is proved in [3, Proposition 2.2] that the abelian complexity of $\omega$ is related to its digit sums in the following way: for every $n \geq 1$,

$$P_c^{(1)}(n) = M_\omega(n) - m_\omega(n) + 1,$$

where

$$M_\omega(n) := \max \{\sum_{j=1}^{n+i}\omega_j \mid i \geq 0\} \quad \text{and} \quad m_\omega(n) := \min \{\sum_{j=1}^{n+i}\omega_j \mid i \geq 0\}.$$  \hfill (2.1)

For the digit sums of the Cantor sequence $c$, we have the following lemma.
Lemma 2. For every integer \( n \geq 1 \), \( M_c(n) = \sum_{i=0}^{n-1} c_i \) and \( m_c(n) = 0 \).

Proof. Since \( 0^n \) is always a factor of \( c \) for every \( n \geq 1 \), we have \( m_c(n) = 0 \) for every \( n \geq 1 \).

For every \( i \geq 0 \) and \( n \geq 1 \), let \( \Sigma(i, n) := \sum_{j=i}^{n-1} c_j \). We only need to show that \( M_c(n) \leq \Sigma(0, n) \) for every \( n \geq 1 \), since the inverse inequality always holds by definition. For this purpose, we shall prove that for every \( n \geq 1 \),

\[
\Sigma(i, n) \leq \Sigma(0, n) \quad \text{for every integer } i \geq 0, \quad (2.2)
\]

Since ‘1’ occurs in \( c \) and ‘11’ does not occur in \( c \), we have \( \Sigma(i, 1) \leq 1 = \Sigma(0, 1) \) and \( \Sigma(i, 2) \leq 1 = \Sigma(0, 2) \). Now suppose \((2.2)\) holds for \( n < m \). We first deal with the case: \( m = 3j + 2 \). By (1.1), we have the following nine recurrence relations:

\[
\begin{align*}
\Sigma(3i, 3n) &= 2\Sigma(i, n), & \Sigma(3i + 1, 3n + 2) &= \Sigma(i, n + 1) + \Sigma(i + 1, n), \\
\Sigma(3i, 3n + 1) &= \Sigma(i, n) + \Sigma(i, n + 1), & \Sigma(3i + 2, 3n) &= \Sigma(i, n) + \Sigma(i + 1, n), \\
\Sigma(3i, 3n + 2) &= \Sigma(i, n) + \Sigma(i, n + 1), & \Sigma(3i + 2, 3n + 1) &= \Sigma(i, n + 1) + \Sigma(i + 1, n), \\
\Sigma(3i + 1, 3n) &= \Sigma(i, n) + \Sigma(i + 1, n), & \Sigma(3i + 2, 3n + 2) &= \Sigma(i, n + 1) + \Sigma(i + 1, n + 1), \\
\Sigma(3i + 1, 3n + 1) &= \Sigma(i, n) + \Sigma(i + 1, n). & \Sigma(3i + 1, 3n + 1) &= \Sigma(i, n) + \Sigma(i + 1, n).
\end{align*}
\]

By the above equations and the inductive assumption, for every \( i \geq 0 \),

\[
\begin{align*}
\Sigma(3i, 3j + 2) &= \Sigma(i, j) + \Sigma(i, j + 1) \leq \Sigma(0, j) + \Sigma(0, j + 1) = \Sigma(0, 3j + 2), \\
\Sigma(3i + 1, 3j + 2) &= \Sigma(i, j) + \Sigma(i, j + 1) \leq \Sigma(0, j) + \Sigma(0, j + 1) = \Sigma(0, 3j + 2).
\end{align*}
\]

Note that at least one of \( c_i \) and \( c_{i+1} \) must be zero. So

\[
\Sigma(3i + 2, 3j + 2) = \Sigma(i, j + 1) + \Sigma(i + 1, j + 1) = \Sigma(0, j + 1) + \Sigma(0, j + 1) = \Sigma(0, 3j + 2).
\]

Therefore, \((2.2)\) holds in the case \( m = 3j + 2 \). Following the same way, we can verify \((2.2)\) when \( m = 3j, 3j + 1 \).

Corollary 1. \( M_c(1) = 1 \), \( M_c(2) = 1 \) and for every \( n \geq 1 \),

\[
M_c(3n) = 2M_c(n) \quad \text{and} \quad M_c(3n + 1) = M_c(3n + 2) = M_c(n) + M_c(n + 1).
\]

Moreover, \( \{M_c(n)\}_{n \geq 1} \) is a 3-regular sequence.

Proposition 1. \( P_c^{(1)}(1) = 2 \), \( P_c^{(1)}(2) = 2 \) and for every \( n \geq 1 \),

\[
P_c^{(1)}(3n) = 2P_c^{(1)}(n) - 1 \quad \text{and} \quad P_c^{(1)}(3n + 1) = P_c^{(1)}(3n + 2) = P_c^{(1)}(n) + P_c^{(1)}(n + 1) - 1.
\]

Moreover, \( \{P_c^{(1)}(n)\}_{n \geq 1} \) is a 3-regular sequence.

Proof. It follows from Lemma 2 Corollary 1 and (2.1).

3. FROM \( k \)-ABELIAN EQUIVALENCE TO 1-ABELIAN EQUIVALENCE

In this section, we give a key theorem, which implies that under certain condition, \( k \)-abelian equivalence can be reduced to 1-abelian equivalence. Using this theorem, we deduce the regularity of the \( k \)-abelian complexity of \( c \) from that of the abelian complexity of \( c \). Before stating the result, we give two auxiliary lemmas. For \( z, w \in \mathcal{A}^* \), we define

\[
P(z, w) := \begin{cases} 
1, & \text{if } z \text{ is a prefix of } w, \\
0, & \text{otherwise},
\end{cases}
\quad \text{and} \quad
S(z, w) := \begin{cases} 
1, & \text{if } z \text{ is a suffix of } w, \\
0, & \text{otherwise}.
\end{cases}
\]
Lemma 3. Let $\omega \in \{0,1\}^N$ and $u, z \in \mathcal{F}_\omega$ with $|u| \geq |z|$. Suppose $z = ayb$, where $a, b \in \{0,1\}$. We have

$$|u|_z = \begin{cases} |u|_ay - S(ay, u), & \text{if } ay \notin RS_\omega, \\ |u|yb - P(yb, u), & \text{if } yb \notin LS_\omega, \\ |u|_ay - |u|ay(b \cdot b) - S(ay, u), & \text{if } ay \in RS_\omega, \\ |u|yb - |u|(1 - a)b - P(yb, u), & \text{if } yb \in LS_\omega. \end{cases}$$

Proof. Note that $|u|_ay - S(ay, u)$ is the number of occurrences of a right extendable $ay$ in $u$. When $ay$ is not right special, every right extension of a right extendable $ay$ must be $z$. So, $|u|_ay - S(ay, u) = |u|_z$. When $ay$ is right special, its right extensions are either $z$ or $ay(1 - b)$. So, $|u|_ay - S(ay, u) = |u|z + |u|ay(1 - b)$. The rest cases can be verified in the same way. \qed

Lemma 4. For every $i \geq 0$, $u \in \mathcal{F}_\omega$, let $\Delta_i := |u|_{0^{3i+2}} + |u|_{10^{3i+1}} - |u|_{0^{3i+3}} + 1$. Then $\Delta_i \in \{0, 1, 2\}$ and

$$\Delta_i = \begin{cases} |u|_{0^{3i+1}} + \frac{2}{3}|u|_{0^{3i+2}} + 1 - P(0^{3i+1}, u) \pmod{3}, & \text{if } P(0^{3i+1}, u) = S(0^{3i+1}, u) = 0, \\ |u|_{0^{3i+1}} + 1 - S(0^{3i+1}, u) - P(0^{3i+1}, u) \pmod{2}, & \text{otherwise}. \end{cases}$$

Proof. Let $Z(\ell)$ ($\ell \geq 1$) be the number of blocks of zeros (in $u$) of length not less than $\ell$. For example, when $u = 0010100$, then $Z(1) = 3$ and $Z(2) = 2$. Note that, for every $\ell \geq 3^i + 1$, $|0^{\ell}|_{0^{3i+3}} - |0^{\ell}||_{0^{3i+2}} = 1$. So,

$$|u|_{0^{3i+2}} - |u|_{0^{3i+3}} = \sum_{v \text{ is a block of zeros in } u}(|v|_{0^{3i+2}} - |v|_{0^{3i+3}}) = \sum_{v \text{ is a block of zeros in } u}1 = Z(3^i + 1).$$

On the other hand, $10^{3i}$ only occurs in $\sigma^{i+1}(1)$. Thus, there is a block of zeros of length $3^i + \ell$ (for some $\ell \geq 1$) between two consecutive $10^{3i}$. Since the block of zeros could also be the prefix or suffix of $u$, we have $|u|_{0^{3i+1}} - 1 \leq Z(3^i + 1) \leq |u|_{0^{3i+1}} + 1$, which implies $\Delta_i \in \{0, 1, 2\}$.

When $P(0^{3i+1}, u) = 1$ or $S(0^{3i+1}, u) = 1$, there is at least one block of zeros of length not less than $3^i + 1$, which is not located between two consecutive $10^{3i}$. This implies that $|u|_{0^{3i+1}} \leq Z(3^i + 1) \leq |u|_{0^{3i+1}} + 1$. So, in this case, $\Delta_i \in \{0, 1\}$. Applying Lemma 3 to $|u|_{0^{3i+2}}$ and $|u|_{10^{3i+1}}$, we have

$$\Delta_i = |u|_{0^{3i+1}} - 2|u|_{0^{3i+2}} + 1 - S(0^{3i+1}, u) - P(0^{3i+1}, u). \quad (3.1)$$

Since $\Delta_i \in \{0, 1\}$, by (3.1), $\Delta_i = |u|_{0^{3i+1}} + 1 - S(0^{3i+1}, u) - P(0^{3i+1}, u) \pmod{2}$.

Now, suppose $P(0^{3i+1}, u) = S(0^{3i+1}, u) = 0$. Applying Lemma 3 to $|u|_{0^{3i+1}}$, by (3.1), we have

$$\Delta_i = |u|_{0^{3i+1}} - 2Z(3^i + 1) + 1 - P(0^{3i+1}, u). \quad (3.2)$$

Let $\sum_v$ denote the sum over all blocks of zeros $v$ of $u$ of length not less than $3^i + 1$. Then

$$|u|_{0^{3i+1}} = \sum_v |v|_{0^{3i+1}} = \sum_v (|v| - 3^i) = \left(\sum_v |v|\right) - 3^iZ(3^i + 1)$$

Note that, in this case, all blocks of zeros of $u$ are of length $3^{i+\ell}$ for some $\ell \geq 1$. So,

$$-2Z(3^i + 1) \equiv \frac{2}{3}|u|_{0^{3i+1}} \pmod{3}. \quad (3.3)$$

The result of this case follows from (3.3) and (3.2). \qed

Now, we prove Theorem 1.
Proof of Theorem 1. Let \( u, v \in \mathcal{F}_c \) satisfying \( |u| = |v| \), \( \text{pref}_k(u) = \text{pref}_k(v) \) and \( \text{suffix}_k(u) = \text{suffix}_k(v) \). When \( k \geq |u| \), the assumption gives \( u = v \). In this case, the result is trivial. In the following, we always assume that \( k < |u| \).

The ‘only if’ part follows directly from the definition of \( k \)-abelian equivalence. For the ‘if’ part, we only need to show that \( u \sim_k v \) implies that for every \( z \in \mathcal{F}_c(k + 1) \), \( |z|_u = |z|_v \). For this purpose, we separate \( \mathcal{F}_c(k + 1) \) into two disjoint parts, i.e., \( \mathcal{F}_c(k + 1) = E_1 \cup E_2 \), where

\[
E_1 = \{ z \in \mathcal{F}_c(k + 1) \mid \text{pref}_k(z) \notin \mathcal{R}_c(z) \text{ or } \text{suffix}_k(z) \notin \mathcal{L}_c(z) \},
\]

\[
E_2 = \{ z \in \mathcal{F}_c(k + 1) \mid \text{pref}_k(z) \in \mathcal{R}_c(z) \text{ and } \text{suffix}_k(z) \in \mathcal{L}_c(z) \}.
\]

Suppose \( z \in E_1 \). If \( \text{pref}_k(z) \notin \mathcal{R}_c(z) \), then by Lemma 8,

\[
|z|_u = |u|_{\text{pref}_k(z)} - S(\text{pref}_k(z), u) = |v|_{\text{pref}_k(z)} - S(\text{pref}_k(z), v) = |z|_v.
\]

If \( \text{suffix}_k(z) \notin \mathcal{L}_c(z) \), then by Lemma 8,

\[
|z|_u = |u|_{\text{suffix}_k(z)} - P(\text{suffix}_k(z), u) = |v|_{\text{suffix}_k(z)} - P(\text{suffix}_k(z), v) = |z|_v.
\]

So, for every \( z \in E_1 \), \( |z|_u = |z|_v \).

Now, let \( z \in E_2 \). Suppose \( 3^i < k \leq 3^{i+1} \) for some \( i \geq 0 \). When \( k \neq 3^{i+1} \), by Lemma 11, \( E_2 = \{0^{k+1}\} \). By Lemma 8 and the assumptions of this result,

\[
|u|_{0^{k+1}} = |u|_{0^k} - |u|_{0^1} - S(0^k, u)
\]

\[
= |u|_{0^k} - (|u|_{0^{k-1}} - P(0^{k-1}, u)) - S(0^k, u)
\]

\[
= |v|_{0^k} - (|v|_{0^{k-1}} - P(0^{k-1}, v)) - S(0^k, v) = |v|_{0^{k+1}}.
\]

When \( k = 3^{i+1} \), by Lemma 11, \( E_2 = \{0^{k+1}, 0k^1, 10^k, 10^{k+1}\} \). For every \( w \in \mathcal{F}_c \), by Lemma 8 and 11 we have the following linear system:

\[
\begin{align*}
|w|_{0^{k+1}} + |w|_{0^1} &= |u|_{0^k} - S(0^k, w), \\
|w|_{0^{k+1}} + |w|_{10^k} &= |u|_{0^k} - P(0^k, w), \\
|w|_{10^k} + |w|_{10^{k-1}} &= |u|_{10^{k-1}} - S(10^{k-1}, w), \\
|w|_{0^{k+1}} + |w|_{10^{k-1}} &= |u|_{0^k} - 1 + \Delta_i,
\end{align*}
\] (3.4)

which determines \((|z|_u)_{z \in E_2}\) uniquely. If \( u \sim_k v \), then the linear systems (3.4) for \( u \) and \( v \) turn out to be the same one. So, \( u \sim_k v \) implies \( |z|_u = |z|_v \) for every factor \( z \in E_2 \).

We may now apply Theorem 1 repeatedly to reduce the \( k \)-abelian equivalence to the 1-abelian equivalence under the condition of Theorem 1.

Corollary 2. Let \( k \geq 1 \) and \( u, v \in \mathcal{F}_c \) satisfying \( |u| = |v| \). If \( \text{pref}_k(u) = \text{pref}_k(v) \) and \( \text{suffix}_k(u) = \text{suffix}_k(v) \), then \( u \sim_{k+1} v \) if and only if \( u \sim_k v \).

Remark 1. A similar result for Sturmian words is obtained by Karhumäki, Saarelä and Zamboni [16 Corollary 3.1]. We would like to ask that in general, what kind of infinite words share a property similar to Corollary 2?

4. \( k \)-ABELIAN COMPLEXITY

In this section, we first give the regularity of the 2-abelian complexity of \( c \). Then, by using Theorem 1 properly, we deduce the regularity of the \( k \)-abelian complexity of \( c \). We start by classifying the \( k \)-abelian equivalent classes of \( \mathcal{F}_c(u) \) by their prefixes and suffixes of length \( k - 1 \).

For every \( k \geq 2 \), \( x, y \in \mathcal{F}_c(k - 1) \) and every \( n \geq 1 \), let

\[
p_k(n, x, y) := \text{Card}(\mathcal{W}_{n,x,y}/\sim_k),
\]

where \( \mathcal{W}_{n,x,y} \) is the set of words of length \( n \) that start with \( x \) and end with \( y \) in the \( k \)-abelian equivalence class of \( \mathcal{F}_c(u) \).
where
\[ W_{n,x,y} := \{ w \in \mathcal{F}_c(n) \mid \text{pref}_{k-1}(w) = x, \text{suffix}_{k-1}(w) = y \} \]  
Here \( p_k(n, x, y) \) denotes the number of \( k \)-abelian equivalent classes with the prefix \( x \) and the suffix \( y \). Then, for every \( n \geq 1 \),
\[ P_k(n) = \sum_{x,y \in \mathcal{F}_c(n-1)} p_k(n, x, y). \]  
(4.1)

By Theorem 1
\[ p_k(n, x, y) = \text{Card} \left( W_{n,x,y} / \sim_k \right) = \text{Card} \left( \{ |w|_1 \mid w \in W_{n,x,y} \} \right). \]  
(4.2)

4.1. Regularity of the 2-abelian complexity of \( c \). Recall that the Cantor sequence \( c \) is thefixed point of the morphism \( \sigma : 0 \rightarrow 000, 1 \rightarrow 101 \) starting by 1, i.e., \( c = \sigma^\infty(1) \).

**Lemma 5.** For all \( i, j \geq 1 \), let \( d_j \) be the number of \( '0' \) between the \( j \)-th \( '1' \) and the \( (j + 1) \)-th \( '1' \)in \( c \), and let \( f(i,j) = j + \sum_{\ell=i}^{i+j-1} d_\ell \). Then, for every \( j \geq 1 \),
\[ d_{2j-1} = 1 \text{ and } d_{2j} = 3d_j. \]  
(4.3)

**Proof.** While applying \( \sigma \) to \( '1' \) or a block of \( '0' \)'s, we obtain only one block of \( '0' \)'s in both cases. Note that in \( c \), every \( '1' \) is followed by a block of \( '0' \)'s. Before the \( i \)-th \( '1' \), the number ofoccurrences of \( '1' \) is \((i - 1) \) and there are \((i - 1) \) blocks of \( '0' \)'s in \( c \). So, while applying \( \sigma \) to \( c \), the\( i \)-th \( '1' \) will generate the \((2i - 1) \)-th block of \( '0' \)'s, which implies \( d_{2i-1} = 1 \). For the same reason, the \( i \)-th block of \( '0' \)'s will generate the \(2i \)-th block of \( '0' \)'s. So, \( d_{2i} = 3d_i \). This proves (4.3).

The recurrence relations (4.3) follows directly from (4.3). We verify the first one as an example:
\[ f(2i, 2j) = 2j + \sum_{\ell=2i}^{2i+2j-1} d_\ell = 2j + \sum_{\ell=i}^{i+j-1} (d_{2\ell} + d_{2\ell+1}) = 3j + 3 \sum_{\ell=i}^{i+j-1} d_\ell = 3f(i,j). \]

\[ \square \]

**Proposition 2.** \( p_2(1, 0, 0) = p_2(1, 1, 1) = 1, p_2(1, 1, 0) = p_2(1, 1, 0) = 0 \) and for every \( n \geq 2 \),
\[ \begin{align*}
& p_2(n, 0, 0) = M_c(n - 2) + 1, \quad (4.5a) \\
& p_2(n, 1, 0) = p_2(n, 0, 1) = M_c(n - 1), \quad (4.5b) \\
& p_2(n, 1, 1) = \begin{cases} 
0, & \text{if } n \equiv 0 \pmod{2}, \\
1, & \text{if } n \equiv 1 \pmod{2}
\end{cases}. \quad (4.5c)
\end{align*} \]

**Proof.** The initial values can be showed by enumerating all the factors of length 1 and 2. Now,let \( n \geq 2 \) and suppose \( n < 3^i \) for some \( i \geq 1 \).

Clearly, for every \( w \in W_{n,0,0} \), \( |w|_1 \leq M_c(n - 2) \). So, \( p_2(n, 0, 0) \leq M_c(n - 2) + 1 \). We prove theinverse inequality in the following. For every \( 0 \leq \ell \leq n - 1 \), let \( W_\ell = 0^{n-\ell} \text{pref} \sigma^\ell(01) \) which is afactor of \( \sigma^\ell(01) \) and hence, a factor of \( c \). Note that \( |W_0|_1 = 0 \) and \( |W_{n-2}|_1 = M_c(n - 2) \). Since\( |W_\ell|_1 \leq |W_{\ell+1}|_1 \leq |W_\ell|_1 + 1 \), we know that \( |W_\ell|_1 \) changescontinuously from 0 to \( M_c(n - 2) \) while \( \ell \) takes values from 0 to \( n - 2 \). Therefore, for every \( 0 \leq s \leq M_c(n - 2) \), there exists\( 0 \leq \ell \leq n - 2 \) such that \( |W_\ell|_1 = s \). If the last letter of \( W_\ell \) is 0, then \( W_\ell \in W_{n,0,0} \). Otherwise,\( |W_{\ell+1}|_1 = |W_\ell|_1 = s \) since 11 is not a factor of \( c \). So, \( W_{\ell+1} \in W_{n,0,0} \). This implies that
\[ p_2(n, 0, 0) \geq M_c(n - 2) + 1 \]  
which proves (4.5a).
Since for every factor $w$ of $c$, its reversal $\bar{w}$ is also a factor of $c$, we have $p_2(n, 1, 0) = p_2(n, 0, 1)$. Then, applying a similar argument on the words $W'_\ell = \text{supf}(\sigma^\ell(1))0^{n-\ell}$ where $1 \leq \ell \leq n - 1$, we obtain (4.5a).

(In the rest of the proof, the symbol ‘≡’, otherwise stated, means equality modulo 2.)

Now, we prove (4.5a) for the case $n \equiv 0$. We first observe that for every $w \in W_{n,1,1}$, $|w| \equiv 1$. Since the number of $0$ between two successive $1$ must be $3^j$ for some $j \geq 0$ and $3^j \equiv 1$, we have $|w|_0 \equiv |w|_1 - 1$ for every $w \in W_{n,1,1}$. Therefore, $|w| = |w|_0 + |w|_1 \equiv 1$. Hence, $W_{n,1,1} = \emptyset$ when $n$ is an even number, which implies $p_2(n, 1, 1) = 0$ when $n \equiv 0$.

In the following, we will prove (4.5c) when $n \equiv 1$. For every $w \in W_{n,1,1}$,

$$n = |w| = |w|_1 + |w|_0 = 1 + f(i, |w|_1 - 1)$$

for some $i \geq 1$. (Since if a word occurs in $c$, then it will occur infinitely many times in $c$, we can assume $i \geq 3$.) Therefore, we only need to prove that for every $m \geq 1$, there is only one integer $t_m \geq 2$ satisfying

$$2m + 1 = 1 + f(i, t_m) \quad (4.6)$$

for some $i \geq 1$. We reason by induction. Since $W_{3,1,1} = \{101\}$ and $W_{5,1,1} = \{10001\}$, it follows that (4.6) holds for $m = 1$ and $2$. Assuming that (4.6) holds for every $\ell \leq m$, we prove it for $m + 1$. We only give the proof for the case $m = 3m'$; the other cases follow in a similar way. In this case, by inductive assumptions and (4.4),

$$2(m + 1) + 1 = 3(2m' + 1) = 3(1 + f(i, t_{m'})) = 1 + f(2i + 1, 2m' + 1),$$

which implies that there is a solution of (4.6) for $m + 1$. Now, we prove the uniqueness. Let $t \geq 2$ be a solution of (4.6) for $m + 1$. Then,

$$1 + f(i, t) = 2(m + 1) + 1 = 3(2m' + 1),$$

which implies $f(i, t) \equiv 2 \pmod{3}$. According to (4.4), this happens only if $(i, t) \equiv (1, 1)$. Write $i = 2i' + 1$ and $t = 2t' + 1$. Then, by (4.4) and (4.7),

$$2m' + 1 = \frac{1 + f(i, t)}{3} = 1 + f(i' + 1, t').$$

By the inductive assumption, we know that $t'$ is the unique solution of (4.6) for $m'$. So, the only solution of (4.6) for $m + 1$ is $2t_{m'} + 1$. \hfill \Box

By Proposition 2, for every $n \geq 2$, we have

$$P_c^{(2)}(n) = M_c(n - 2) + 2M_c(n - 1) + 1 + \frac{1 + (-1)^{n+1}}{2}.$$

4.2. Regularity of the $k$-abelian complexity of $c$. In this part, we prove the regularity of the $k$-abelian complexity of the Cantor sequence for every $k \geq 3$.

Let $F_c$ denote the set of all factors of $c$. For every $u \in F_c$ and $\ell \geq 1$, we define

$$\text{Type}(\ell, u) := \left\{ j = 0, 1, \cdots, 3^\ell - 1 \mid u = c_{3^j u_{n+j}} \cdots c_{3^j u_{n+|u| - 1}} \text{ for some } n \geq 0 \right\}.$$ 

The elements in $\text{Type}(\ell, u)$ are called types of $u$ (with respect to $\ell$). Clearly, for every $\ell$ and $u \in F_c$, $\text{Card}(\text{Type}(\ell, u)) \geq 1$.

Every type of $u$ gives a decomposition of $u$ in the following sense. For every $j \in \text{Type}(\ell, u)$, there is an integer $n \geq 0$ such that

$$u = \left( c_{3^j u_{n+j}} \cdots c_{3^j u_{(n+j) - 1}} \right) \left( c_{3^j u_{(n+j) - 1}} \cdots c_{3^j u_{(n+h) - 1}} \right) \left( c_{3^j u_{(n+h) - 1}} \cdots c_{3^j u_{n+|u| - 1}} \right)
= \text{supf}_j(\sigma^\ell(c_n)) \sigma^\ell(c_{n+1} \cdots c_{n+h-1}) \text{pref}_j(\sigma^\ell(c_{n+h})),$$  

(4.8)
where \( h = \left\lfloor \frac{|u|}{3^i} \right\rfloor \), \( j_0 = 3^i - j \) and \( j_1 = j + |u| - 3^i h \). The following lemma shows that every non-zero factor of \( c \), which is long enough, occurs in a (relatively) fixed position, i.e., has only one type. By a non-zero factor we mean a factor that contains at least one letter ‘1’.

**Lemma 6.** For every integer \( \ell \geq 1 \) and every non-zero factor \( u \in \mathcal{F}_c \) with \( |u| > 3^\ell \),
\[
\text{Card}(\text{Type}(\ell, u)) = 1.
\]

**Proof.** We prove by induction on \( \ell \). We first prove the result for \( \ell = 1 \). Now, we show that \( \text{Card}(\text{Type}(1, u)) = 1 \) for \( u \in \mathcal{F}_c(4) \) with \( |u| > 0 \). We only verify the case \( u = 0001 \) as an example; the rest can be verified in the same way. Suppose \( 0001 = c_n c_{n+1} c_{n+2} c_{n+3} \). Since \( c_{n+3} = 1 \), by (1.1), we have \( n \not\equiv 1 \pmod{3} \). If \( n \equiv 2 \pmod{3} \), then by (1.1), \( 0 = c_{n+1} = c_{n+3} = 1 \), which is a contradiction. Thus, Type(1, 0001) = \{0\}.

For every non-zero factor \( u \in \mathcal{F}_c \) with \( |u| > 4 \), let \( u = xvy \) where \( v \) is the first non-zero factor of length 4 of \( u \). Since Type(1, \( v \)) + |\( v \| \equiv Type(1, u) \pmod{3} \), we have
\[
\text{Card}(\text{Type}(1, u)) = 1.
\]

Suppose the result holds for \( \ell \). We prove it for \( \ell + 1 \). Let \( u \in \mathcal{F}_c \) with \( |u| > 3^{\ell+1} \) and \( i_0 \in \text{Type}(\ell, u) \). Then,
\[
u = c_{3^n+i_0} \cdots c_{3^{n+1}+i_0+|u|-1}
\]
for some \( n \geq 0 \). By (4.8), \( u \) uniquely determines \( i_0, |u| \) and \( c_n c_{n+1} \cdots c_{n+h} \) where \( h = \left\lfloor \frac{|u|+i_0}{3^\ell} \right\rfloor \). Since \( h \geq 3 \), \( n \equiv i_1 \pmod{3} \) where \( i_1 \in \text{Type}(1, c_n \cdots c_{n+h}) \). Therefore,
\[
3^\ell n + i_0 \equiv 3^\ell i_1 + i_0 \pmod{3^{\ell+1}}.
\]
By the inductive assumptions, \( \text{Card}(\text{Type}(1, c_n \cdots c_{n+h})) = 1 \) and \( \text{Card}(\text{Type}(\ell, u)) = 1 \). So, by (4.9), we have
\[
\text{Card}(\text{Type}(\ell + 1, u)) = 1.
\]

**Lemma 7.** For every integer \( \ell \geq 1 \) and every non-zero factor \( u \in \mathcal{F}_c \) with \( 3^\ell < |u| \leq 3^{\ell+1} \),
\[
1 \leq \text{Card}(\text{Type}(\ell + 1, u)) \leq 2.
\]

**Proof.** Let \( u \in \mathcal{F}_c \) with \( 3^\ell < |u| \leq 3^{\ell+1} \) and \( i_0 \in \text{Type}(\ell, u) \). Then, \( u = c_{3^n+i_0} \cdots c_{3^{n+1}+i_0+|u|-1} \) for some \( n \geq 0 \). By (4.8), \( u \) uniquely determines \( i_0, |u| \) and \( c_n c_{n+1} \cdots c_{n+h} =: v \), where \( h = \left\lfloor \frac{|u|+i_0}{3^\ell} \right\rfloor \). Note that \( v \) is a non-zero factor. Write \( q(v) := \max\{|j| \mid 0^j \text{ is a prefix of } v\} \). Then \( c_{n+q(v)} = 1 \), which implies \( n + q(v) \equiv 1 \pmod{3} \) by (1.1). So,
\[
3^\ell n + i_0 \equiv -3^\ell q(v) + i_0 \text{ or } 3^\ell (2 - q(v)) + i_0 \pmod{3^{\ell+1}}.
\]
The result follows from Lemma 6 and the above formula.

In the rest of this section, let \( i \) be the integer satisfying
\[
3^i + 1 < k \leq 3^{i+1} + 1.
\]
To study the regularity of \( \{p_k(n, x, y)\}_{n \geq 1} \) for \( x, y \in \mathcal{F}_c(k - 1) \), our idea is the following. We first give the upper bound of \( p_k(n, x, y) \) by using \( M_c(\cdot) \), which is a 3-regular sequence according to Corollary 1. Then, by constructing sufficiently many words which belong to different \( k \)-abelian equivalence classes, we show that the upper bound can be reached. Therefore, the regularity of \( \{p_k(n, x, y)\}_{n \geq 1} \) follows from the regularity of \( \{M_c(n)\}_{n \geq 1} \).

The following lemma contributes to the construction of words that belong to different \( k \)-abelian equivalence classes.

**Lemma 8.** Let \( \alpha \in \{0, 1\} \). For every \( \ell \geq 1 \) and every \( h = 1, 2, \ldots, M_c(\ell) \), there is a word \( W_h \in \mathcal{F}_c(\ell + 3) \) such that \( |W_h|_1 = h \) and \( W_h = 00U_h \alpha \), where \( U_h \in \mathcal{F}_c(\ell) \).
Proof. For all $j = 0, 1, \cdots, \ell + 1$, let

$$W_j = 0^{\ell+3-j}\text{pref}_j(\sigma^*(1)) \in \mathcal{F}_c(\ell + 3),$$

where $s \in \mathbb{N}$ satisfying $3^s > \ell + 1$. Since $|W_j| \leq |W_{j+1}| \leq |W_j| + 1$ and $|W_j| = M_c(\ell)$, we know that $|W_j|$ changes from 0 to $M_c(\ell)$ continuously while $j$ takes values from 0 to $\ell$. So, for every $h = 1, \cdots, M_c(\ell)$, there is a $j_h (\leq \ell)$ such that $|W_{j_h}| = h$. Moreover, we can require that the last letter of $W_{j_h}$ is 0. Otherwise, 1 is the last letter of $W_{j_h}$. Then, $W_{j_h+1}$ ends with 0 and $|W_{j_h+1}| = |W_{j_h}|$.

There also is a $j'_h$ such that $|W_{j'_h}| = h$, of which the last letter is 1. Otherwise, 0 is the last letter of $W_{j'_h}$. Let $m_h := \max\{q \mid 0^q$ is a suffix of $W_{j'_h} \}$. Since $|W_{j'_h}| = h \geq 1$, we always have $m_h < j'_h$. Then, $W_{j'_h-m_h}$ ends with 1 and $|W_{j'_h-m_h}| = |W_{j'_h}|$. If $m_h > j'_h$, \[\square\]

Now, we shall show the regularity of $\{p_k(n, x, y)\}_{n \geq 1}$ for all $x, y \in \mathcal{F}_c(k-1)$.

Lemma 9. $\{p_k(n, 0^{k-1}, 0^{k-1})\}_{n \geq 1}$ is a 3-regular sequence.

Proof. Without loss of generality, we can assume that $n \geq 2 \cdot 3^{i+1} + 2k - 2$, since changing finite terms of a sequence does not change its regularity. Noticing that $3^i < k-1 \leq 3^{i+1}$, the occurrence of each $w \in \mathcal{W}_{n, 0^{k-1}, 0^{k-1}}$ in $c$ must be one of the four forms in Figure 1.

\[\text{Figure 1.}\]

In all the four forms, we have $|w| = 2^{i+1}|u|$ and $|u| = \ell$ or $\ell - 1$, which implies

$$p_k(n, 0^{k-1}, 0^{k-1}) \leq M_c(\ell) + 1,$$

where $\ell = \left\lfloor \frac{n-2k+2}{3^{i+1}} \right\rfloor$. Next, we prove the inverse of (4.11). That is

$$p_k(n, 0^{k-1}, 0^{k-1}) \geq M_c(\ell) + 1. \quad (4.12)$$

Applying Lemma 8 for the above $\ell$ and $\alpha = 0$, we have

$$W_h = 00U_h0 \in \mathcal{F}_c(\ell + 3)$$

with $|W_h| = h$ for all $h = 1, 2, \cdots, M_c(\ell)$. Set $t := n - 3^{i+1}\ell - k + 1$. Then, $k - 1 \leq t < k - 1 + 3^{i+1}$. Therefore, $0^t\sigma^{i+1}(U_h)0^{k-1} \in \mathcal{W}_{n, 0^{k-1}, 0^{k-1}}$ and $|0^t\sigma^{i+1}(U_h)0^{k-1}| = 2t + 1$.

For every non-zero factor $v \in \mathcal{F}_c(k-1)$, let $z_v := \max\{p \mid 0^p$ is a suffix of $v\}$ and

$$\tilde{L}_v := \{q \mod 3^{i+1} \mid c_q^{-1}(k-2-z_v) \cdots c_q^{-1}c_{q+1} \cdots c_{q+z_v} = v\},$$

where $c_q$ is the last 1 in $v$. Then, it follows from Lemma 7 that $1 \leq \text{Card}(\tilde{L}_v) \leq 2$. Moreover, if $\tilde{L}_v = \{q_1, q_2\}$, where $0 \leq q_1 < q_2 \leq 3^{i+1} - 1$, then by (4.10), we have $q_2 = q_1 + 2 \cdot 3^i$.

For a word $w = w_1 \cdots w_n \in \mathcal{A}^n$, the reversal of $w$ is defined to be $\bar{w} = w_{n-1} \cdots w_1 w_0$. When $w = uv$, we write $uv^{-1} := u$ and $u^{-1}w := v$ by convention.
Lemma 10. For all non-zero factors \( x, y \in \mathcal{F}_c(k-1) \), two sequences \( \{p_k(n, 0^{k-1}, y)\}_{n \geq 1} \) and \( \{p_k(n, x, 0^{k-1})\}_{n \geq 1} \) are both 3-regular sequences.

\textbf{Proof.} For every \( x \in \mathcal{F}_c \), its reversal \( \bar{x} \in \mathcal{F}_c \), since \( x \) is a factor of \( \sigma^m(1) \) for some \( m \geq 1 \) and \( \sigma^m(1) = \sigma^n(1) \). So, \( p_k(n, x, 0^{k-1}) = p_k(n, 0^{k-1}, \bar{x}) \) for every \( n \geq 1 \). Thus, we only need to verify the regularity of \( \{p_k(n, 0^{k-1}, y)\}_{n \geq 1} \) for every non-zero factor \( y \in \mathcal{F}_c(k-1) \).

Since changing finite terms of a sequence does not change its regularity, we can assume that \( n \geq 2 \cdot 3^{i+1} + 2k - 2 \). Recall that \( 3^i < k - 1 \leq 3^{i+1} \). Each occurrence of every \( w \in \mathcal{W}_{n, 0^{k-1}, y} \) in \( c \) must be one of the six forms in Figure 2. In all the six forms, for every \( o_y \in \hat{L}_y \), we have

\[ |w|_1 = 2^{i+1} |\tilde{u}|_1 - |\text{suff}_{3^{i+1} - o_y - 1}(\sigma^{i+1}(1))|_1 := n_{o_y} \quad (4.13) \]

and \( |\tilde{u}| = \ell(o_y) \) or \( \ell(o_y) + 1 \), where

\[ \ell(o_y) = \left\lfloor \frac{n - k - o_y - z_y}{3^{i+1}} \right\rfloor \quad \text{and} \quad \tilde{u} = \begin{cases} u01, & \text{if } w \text{ is of Form 5 or 6,} \\ u1, & \text{otherwise.} \end{cases} \]

\textbf{Figure 2.}

When \( \text{Card}(\hat{L}_y) = 1 \), write \( \hat{L}_y = \{o_y\} \). By (4.13), we have

\[ p_k(n, 0^{k-1}, y) \leq M_c(\ell(o_y) + 1). \quad (4.14) \]

On the other hand, applying Lemma 8 for \( \ell(o_y) \) and \( \alpha = 1 \), we have

\[ W_h = 00U_h, 1 \in \mathcal{F}_c(\ell + 3) \text{ with } |W_h|_1 = h \]

for all \( h = 1, \ldots, M_c(\ell + 1) \). Set \( t := n - 3^{i+1} \ell - o_y - z_y - 1 \); so \( k - 1 \leq t < k - 1 + 3^{i+1} \). Therefore,

\[ V_{o_y} := 0^t \sigma^{i+1}(U_h) \text{ pref}_{o_y + 1}(\sigma^{i+1}(1)) \text{ pref}_y \in \mathcal{W}_{n, 0^{k-1}, y} \]

and

\[ |V_{o_y}|_1 = 2^{i+1} h - |\text{suff}_{3^{i+1} - o_y - 1}(\sigma^{i+1}(1))|_1 \]

for all \( h = 1, \ldots, M_c(\ell + 1) \). This implies that \( p_k(n, 0^{k-1}, y) \geq M_c(\ell(o_y) + 1) \). The previous inequality, (4.13) and Corollary 1 give the result in the case \( \text{Card}(\hat{L}_y) = 1 \).

Now suppose \( \text{Card}(\hat{L}_y) = 2 \) and set \( \hat{L}_y = \{o_y, o'_y := o_y + 2 \cdot 3^i\} \) with \( 0 \leq o_y \leq 3^i - 1 \). From (4.13), we know that \( o'_y \equiv o_y + 2 \mod 2^{i+1} \). Therefore,

\[ p_k(n, 0^{k-1}, y) \leq M_c(\ell(o_y) + 1) + M_c(\ell(o'_y) + 1). \quad (4.15) \]

For every \( q \in \hat{L}_y \), applying Lemma 8 for \( \ell(q) \) and \( \alpha = 1 \), we have

\[ W_{h,q} = 00U_{h,q}, 1 \in \mathcal{F}_c(\ell + 3) \text{ with } |W_{h,q}|_1 = h \]
for every \( h = 1, \ldots, M_c(\ell_1 + 1) \). Set \( t(q) := n - 3^{i+1} \ell(q) - q - z_y - 1; \) so \( k - 1 \leq t(q) < k - 1 + 3^{i+1} \). Therefore, for every \( q \in \hat{L}_y \),
\[
V_q := 0^{t(q)}\sigma^{i+1}(U_{h,q})\text{pref}_{q+1}(\sigma^{i+1}(1))0^{z_y} \in W_{n,0^{k-1},y}
\]
and
\[
|V_q|_1 = 2^{i+1} h - |\text{suff}_{3^{i+1}-q-1}(\sigma^{i+1}(1))|_1
\]
for all \( h = 1, \ldots, M_c(\ell_1 + 1) \). Since \( |V_{o_y}|_1 \equiv |V_{o'_y}|_1 - 2^k \) (mod \( 2^{i+1} \)), \( V_{o_y} \) and \( V_{o'_y} \) belongs to different \( k \)-abelian equivalence classes. Therefore,
\[
p_k(n, 0^{k-1}, y) \geq M_c (\ell(o_y) + 1) + M_c (\ell(o'_y) + 1).
\]
Combining (4.15), (4.16) and Corollary 1 the result follows. \( \Box \)

**Lemma 11.** For two non-zero factors \( x, y \in F_c(k - 1) \), \( \{p_k(n, x, y)\}_{n \geq 1} \) is ultimately periodic.

**Proof.** Without loss of generality, we can assume that \( n \geq 2 \cdot 3^{i+1} + 2k - 2 \) since changing finite terms of a sequence does not change its regularity. Noticing that \( 3^i < k - 1 \leq 3^{i+1} \), for every pair of factors \( x, y \) of length \( k - 1 \), the occurrence of each \( w \in W_{n,x,y} \) in \( c \) must be one of the nine forms in Figure 3.

![Figure 3.](image_url)

For every fixed pair of \( o_x \in \hat{L}_x \) and \( o_y \in \hat{L}_y \), in all the nine forms, we have
\[
n = |w| = 3^{i+1}(|\tilde{u}| - 1) + \ell(o_x, o_y)
\]
and
\[
|w|_1 = 2^{i+1}|\tilde{u}|_1 - |\text{pref}_{o_x+z_x-k+2}(\sigma^{i+1}(1))|_1 - |\text{suff}_{3^{i+1}-o_y-1}(\sigma^{i+1}(1))|_1,
\]
where \( \ell(o_x, o_y) := (k - 1 - o_x - z_x + o_y + z_y) \leq 2 \cdot 3^{i+1} \) and
\[
\tilde{u} = \begin{cases} 
10u1, & \text{if } w \text{ is of Form 1 or 5}, \\
10u01, & \text{if } w \text{ is of Form 3}, \\
1u01, & \text{if } w \text{ is of Form 6 or 9}, \\
1u1, & \text{otherwise}.
\end{cases}
\]

Further, according to (4.15), \( \tilde{u} \) in (4.19) must satisfy \( |\tilde{u}| \equiv 1 \) mod 2. This fact and (4.17) yield that \( W_{n,x,y} = \emptyset \) when \( n \neq \ell(o_x, o_y) \) mod \( 2 \cdot 3^{i+1} \).
Now we deal with the case \( n \equiv \ell(o_x, o_y) \mod 2 \cdot 3^{i+1} \). Note that by (4.5c), we have \( p(2j+1,1,1) = 1 \) for all \( j \geq 1 \). This fact and (4.18) imply that for all \( n = 2 \cdot 3^{i+1}j + \ell(o_x, o_y) \),

\[
p_k(n, x, y) = \text{Card}(\{|w|_1 \mid w \in W_{n,x,y}\}) = 1.
\]

In conclusion, let \( I_{x,y} = \{2 \cdot 3^{i+1}j + \ell(o_x, o_y) \mid j \geq 1, o_x \in \tilde{L}_x, o_y \in \tilde{L}_y\} \). We have

\[
p_k(n, x, y) = \begin{cases} 
1, & \text{if } n \in I_{x,y}, \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, \( \{p_k(n, x, y)\}_{n \geq 1} \) is ultimately periodic with a period \( 2 \cdot 3^{i+1} \).

\[\Box\]

**Proposition 3.** \( \{P_k^{(k)}(n)\}_{n \geq 1} \) is a 3-regular sequence for every \( k \geq 3 \).

**Proof.** It follows directly from Lemmas 9, 10 and 11 and (4.1).

\[\Box\]

Theorem 2 follows from Propositions 1, 2 and 3.

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