Sublinear bounds for nullity of flows and approximating Tutte’s flow conjectures

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Abstract

A function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is sublinear, if

\[
\lim_{x \to +\infty} \frac{f(x)}{x} = 0.
\]

If \( A \) is an Abelian group, \( G \) is a graph and \( \phi \) is an \( A \)-flow in \( G \), then let \( N(\phi) \) be the nullity of \( \phi \), that is, the set of edges \( e \) of \( G \) with \( \phi(e) = 0 \). In this paper we show that (a) Tutte’s 5-flow conjecture is equivalent to the statement that there is a sublinear function \( f \), such that all 3-edge-connected cubic graphs admit a \( \mathbb{Z}_5 \)-flow \( \phi \) (not necessarily nowhere zero), such that \( |N(\phi)| \leq f(|E(G)|) \); (b) Tutte’s 4-flow conjecture is equivalent to the statement that there is a sublinear function \( f \), such that all bridgeless graphs without a Petersen minor admit a \( \mathbb{Z}_4 \)-flow \( \phi \) (not necessarily nowhere zero), such that \( |N(\phi)| \leq f(|E(G)|) \); (c) Tutte’s 3-flow conjecture is equivalent to the statement that there is a sublinear function \( f \), such that all 4-edge-connected graphs admit a \( \mathbb{Z}_3 \)-flow \( \phi \) (not necessarily nowhere zero), such that \( |N(\phi)| \leq f(|E(G)|) \).

Keywords: flow, support, nullity, Tutte’s flow conjecture

1. Introduction

In this paper, we consider finite, undirected graphs that may contain loops or parallel edges. Let \( A \) be an Abelian group, \( G \) be a graph and \( D \) be some orientation of edges of \( G \). A mapping \( \phi : E(G) \rightarrow A \) is called an \( A \)-flow, if for each vertex \( v \) of \( G \), we have that

\[
\sum_{e \in \partial^+(v)} \phi(e) = \sum_{e \in \partial^-(v)} \phi(e).
\]

Here \( \partial^-(v) \) and \( \partial^+(v) \) are the set of edges of \( G \) that enter and leave (in sense of \( D \)) the vertex \( v \), respectively. It is known that (see [1, 10]) this condition is equivalent to requiring

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that
\[ \sum_{e \in \partial^+(X)} \phi(e) = \sum_{e \in \partial^-(X)} \phi(e) \]
for any \( X \subseteq V \). In the latter, \( \partial^-(X) \) and \( \partial^+(X) \) are defined similarly.

If \( \phi \) is an \( A \)-flow, then let \( supp \phi \) and \( N(\phi) \) be the support and nullity of \( \phi \), respectively, that is, the set of edges of \( G \), such that \( \phi(e) \neq 0 \) and \( \phi(e) = 0 \), respectively. \( \phi \) is called a no-where zero \( A \)-flow, if \( N(\phi) = \emptyset \). The topic of no-where flows in graphs has been originated from the work of Tutte, who among many others, has shown that the existence of a no-where zero \( A \)-flow in a graph \( G \) is independent of the orientation \( D \) and the structure of the group \( A \). The only thing that is important is the cardinality of \( A \). Tutte also has presented the following three conjectures:

**Conjecture 1.** (Tutte’s flow conjectures \[8, 9\])

(a) (5-flow conjecture) Any bridgeless graph has a no-where zero \( \mathbb{Z}_5 \)-flow;

(b) (4-flow conjecture) Any bridgeless graph without a Petersen minor has a no-where zero \( \mathbb{Z}_4 \)-flow;

(c) (3-flow conjecture) Any 4-edge-connected graph has a no-where zero \( \mathbb{Z}_3 \)-flow.

Despite many significant and non-trivial results obtained towards these conjectures (see \([4, 5, 6, 7, 10]\)), all these conjectures are still open.

In a recent paper \([2]\), the authors have initiated the study of the support of 3-flows in 3-edge-connected graphs. The main result of the authors states that every 3-edge-connected graph has a 3-flow \( \phi \) (not necessarily no-where zero), such that \( |supp(\phi)| \geq \frac{5}{6} \cdot |E(G)| \). In the same paper, they offer the following problem (Problem 8.2 in \([2]\)):

**Problem 1.** Suppose \( G \) is a graph with \( m \) edges.

1. How large can \( supp \phi \) be for a 5-flow \( \phi \), if \( G \) is 2-edge-connected? (or 3-edge-connected?)

2. How large can \( supp \phi \) be for a 4-flow \( \phi \), if \( G \) is 2-edge-connected and has no Petersen minor?

3. How large can \( supp \phi \) be for a 3-flow \( \phi \), if \( G \) is 4-edge-connected? (or 5-edge-connected?)

The above-mentioned three conjectures by Tutte predict that in these three problems the size of \( supp \phi \) can be made equal to \( m \), that is, \( \phi \) is no-where zero in the graphs under consideration. As an approachable problem, in \([2]\) the authors ask to show that \( supp \phi \) can be made at least \( f(m) \) for an appropriate function \( f \).

Since \( supp(\phi) + |N(\phi)| = m \) for any \( A \)-flow \( \phi \) of a graph \( G \) with \( m \) edges, these three problems can be viewed as finding upper bounds for \( |N(\phi)| \). The smaller is the upper bound for \( |N(\phi)| \), the larger is \( supp \phi \) going to be. The main result of \([2]\) is equivalent to saying that any 3-edge-connected graph admits a 3-flow \( \phi \), such that \( |N(\phi)| \leq \frac{5m}{6} \). Observe that this
bound is linear in terms of $m$. A naturally arising question is what kind of upper bounds for $|N(\phi)|$ in the three subproblems from Problem 1 we can hope to prove. For example, is it realistic to obtain an upper bound of order $O(\sqrt{m})$ or $O(\log m)$? How about an $O(1)$ bound with some (may be large) constant?

Let us say that a function $f : N \to N$ is sublinear, if

$$\lim_{x \to +\infty} \frac{f(x)}{x} = 0.$$ 

Related with this question, in this paper, we show that Tutte conjectures are equivalent to proving sublinear upper bounds for $|N(\phi)|$. In some sense, our results suggest that we have to confine ourselves solely to functions of linear growth in bounding $|N(\phi)|$ in Problem 1, as better bounds are strong enough to imply the original conjectures of Tutte. Non-defined terms and concepts can be found in [1, 3, 10].

2. Main results

In this section, we obtain our main results. However, before presenting the technical details we would like to briefly discuss the big picture of our proofs. Fix one of Tutte’s conjecture. Observe that if it is true, then clearly in the corresponding subproblem from Problem 1 we have a sublinear bound for nullity as the identically zero function is trivially sublinear. For the proof of the converse, we fix an arbitrary graph $G$ satisfying the statement of Tutte conjecture. Now, if $t$ is some positive integer we take $t$ copies of a graph $J$, where $J$ is a graph whose structure is very close to that of $G$. We glue these $t$ copies so that a big graph $H$ is obtained that satisfies the conditions of the subproblem from Problem 1. Moreover, when $t \to +\infty$, we have that $|N(H)| \to +\infty$. Now, because the function $f$ under consideration is sublinear, we can choose $t$ such that $f(|E(H)|) < t$. If we assume that $H$ admits a flow $\phi$, such that $|N_H(\phi)| \leq f(|E(H)|)$, then we get that $|N_H(\phi)| < t$. This means that $\phi$ is not zero in one of the copies of $J$. We use the values of $\phi$ on this copy in order to extract the corresponding nowhere zero flow of $G$.

Now, we present all the technical details of our proofs.

**Theorem 1.** The following statements are equivalent:

(a) All bridgeless graphs have a nowhere zero $\mathbb{Z}_5$-flow;

(b) There is a sublinear function $f : N \to N$, such that all bridgeless graphs $G$ admit a $\mathbb{Z}_5$-flow $\phi$ (not necessarily nowhere zero), such that $|N(\phi)| \leq f(|E(G)|)$.

(c) There is a sublinear function $f : N \to N$, such that all 3-edge-connected graphs $G$ admit a $\mathbb{Z}_5$-flow $\phi$ (not necessarily nowhere zero), such that $|N(\phi)| \leq f(|E(G)|)$.

(d) There is a sublinear function $f : N \to N$, such that all 3-edge-connected cubic graphs $G$ admit a $\mathbb{Z}_5$-flow $\phi$ (not necessarily nowhere zero), such that $|N(\phi)| \leq f(|E(G)|)$. 

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Proof. (a) implies (b). This is trivial since the identically zero function is sublinear.

(b) implies (c) is trivial. (c) implies (d) is trivial.

(d) implies (a). Assume that (d) holds. Let us show that Tutte’s 5-flow conjecture is true. We know that it suffices to prove this conjecture for cyclically 4-edge-connected cubic graphs (see, for example, [1] or [10]). Let $G$ be such a cubic graph and let $v$ be any vertex of $G$. For a positive integer $t$, take $2t$ copies of $G - v$, and let $K$ be a 3-edge-connected bipartite cubic graph on $2t$ vertices, with $t$ vertices on the left and $t$ on the right. Consider the cubic graph $H$ obtained from $K$ by replacing each of its vertices with $G - v$.

We have $|V(H)| = 2t \cdot (|V(G)| - 1)$ and $|E(H)| = 3t \cdot (|V(G)| - 1)$. Moreover, $H$ is 3-edge-connected.

Now, let us choose $t$ such that
\[
\frac{f(|E(H)|)}{|E(H)|} < \frac{1}{3(|V(G)| - 1)}.
\]
We can choose such a $t$ since $f$ is sublinear. Then, by (d), we have that $H$ admits a $\mathbb{Z}_5$-flow $\phi$, such that
\[
|N_H(\phi)| = f(|E(H)|) < \frac{|E(H)|}{3(|V(G)| - 1)} = t.
\]
Thus, on the left side of $H$, we can find a copy of $G - v$ such that $\phi$ is not zero in the copy of $G - v$ and the tree edges leaving it. Thus, if we contract the remaining vertices of $H$ to a vertex, we will obtain a no-where zero $\mathbb{Z}_5$-flow of $G$. The proof is complete.

Our next result deals with Tutte’s 4-flow conjecture.

**Theorem 2.** The following statements are equivalent:

(a) All bridgeless graphs without a Petersen minor, have a no-where zero $\mathbb{Z}_4$-flow;

(b) There is a sublinear function $f : N \rightarrow N$, such that all bridgeless graphs $G$ without a Petersen minor admit a $\mathbb{Z}_4$-flow $\phi$ (not necessarily no-where zero), such that $|N(\phi)| \leq f(|E(G)|)$.

Proof. Again, (a) implies (b) is trivial.

For the proof of the converse, let $G$ be a 2-edge-connected graph without a Petersen minor. Let us show that if (b) is true then $G$ has a no-where zero $\mathbb{Z}_4$-flow. Let $e$ be any edge of $G$. Take $t$ copies of $G - e$, and join them cyclically to get the 2-edge-connected graph $H$. Clearly, $|E(H)| = t|E(G)|$. Let us show that $H$ does not contain a Petersen minor. Since Petersen graph is of maximum degree 3, it is the same that we show that $H$ does not contain a subdivision of Petersen graph (see, for example, Proposition 1.7.2 on page 20 of [3]).

Assume that $H$ contains a subgraph $K$ that is a subdivision of Petersen. First, assume that all branch vertices of $K$ are located in the same copy of $G - e$. If $K$ does not intersect the 2-edge-cut around our copy of $G - e$, then we have that $G - e$ (hence $G$) contains a subdivision of Petersen which is contradictory. Now if we assume that $K$ intersects the 2-edge-cut, then since all branch vertices of $K$ are in the same copy, then it must use the
other edge of the 2-edge-cut, too. Thus, again, we have that $G$ contains a subdivision of Petersen.

Thus, we can assume that there are two copies of $G - e$ that contain at least one branch vertex of $K$. Now, since Petersen graph is 3-edge-connected, there are 3-edge-disjoint paths connecting these two branch vertices of $K$ that lie in different copies of $G - e$. Thus, there are at least three edges in the cut around the copy of $G - e$. But this is contradictory, since this number is two. We just have two edges there. Thus, $H$ does not contain a Petersen minor.

In order to complete the proof, let us choose $t$ such that

$$f(t \cdot |E(G)|) = f(|E(H)|) < \frac{1}{|E(G)|}.$$

Clearly such $t$ exists by the assumption on the function $f$. Now, by (b), $H$ has a $\mathbb{Z}_4$-flow $\phi$, such that

$$|N_H(\phi)| \leq f(|E(H)|) < \frac{|E(H)|}{|E(G)|} = t.$$

This means that there is a copy of $G - e$ in $H$, such that all edges of $G - e$ and the two edges in the 2-edge-cut around it have flow value that is not zero (under $\phi$). Since we have a 2-edge-cut around this copy, and $\phi$ is a flow, we can assume that the flow values of this 2-edge-cut are $\phi(x)$, which is not zero. Moreover, one of the edges enters our copy of $G - e$, the other one goes out of our copy of $G - e$. Now, let us remove this 2-edge-cut, and put the edge $e$ back into $G - e$, and assign it a flow value $\phi(e) = \phi(x)$. Clearly, this will be a no-where zero $\mathbb{Z}_4$-flow of $G$. The proof is complete.

Our final result addresses Tutte’s 3-flow conjecture.

**Theorem 3.** The following statements are equivalent:

(a) All 4-edge-connected graphs have a no-where zero $\mathbb{Z}_3$-flow;

(b) There is a sublinear function $f : \mathbb{N} \to \mathbb{N}$, such that all 4-edge-connected graphs $G$ admit a $\mathbb{Z}_3$-flow $\phi$ (not necessarily no-where zero), such that $|N(\phi)| \leq f(|E(G)|)$.

**Proof.** (a) implies (b) is trivial.

For the proof of converse, assume that the sublinear function $f$ exists. Let $G$ be any 4-edge-connected graph. Let us show that it admits a no-where zero $\mathbb{Z}_3$-flow. For a positive integer $t$ let us take $3t$ copies of $G - e_1 - e_2$, where $e_1 = ab$ and $e_2 = cd$ are two edges of $G$. Let these $3t$ copies of $G - e_1 - e_2$ be partitioned into $t$ groups, three copies in each of the groups. Let the $j$th group contain the $(3j - 2)$th, $(3j - 1)$th and $3j$th copies of $G - e_1 - e_2$. We assume that $a_j, b_j, c_j$ and $d_j$ are the labels of vertices that correspond to vertices $a, b, c$ and $d$ in the $j$th copy. In the $i$th group ($i = 1, \ldots, t$), join the edges as follows: add an edge $c_ia_{i+1}, d_{i+1}d_i, b_{i+1}b_i$ and $a_{i+1}c_{i+2}$ (see Figure 1). Now, join these groups cyclically, that is, the first two vertices of the current group are joined to the last two vertices of the previous
group, and the last two vertices of the current group are joined to the first two vertices of
the next group. Let the resulting graph be $H$. We have that

$$|E(H)| = 3t \cdot (|E(G)| - 2) + 3t \cdot 2 = 3t \cdot |E(G)|.$$  

Moreover, $H$ is 4-edge-connected.

![Figure 1: Joining the edges inside each group of three copies of $G - e_1 - e_2$.](image)

Now, choose $t$, such that

$$f(|E(H)|) < \frac{1}{6|E(G)|}.$$  

By (b) we have that there is a $\mathbb{Z}_3$-flow $\phi$, such that

$$|N_H(\phi)| \leq f(|E(H)|) = \frac{|E(H)|}{6|E(G)|} = \frac{t}{2}.$$  

Thus, there is a group of three copies, such that $\phi$ is not zero on edges of graphs inside
the group plus the four edges entering and leaving the group. Since $\phi$ is a $\mathbb{Z}_3$-flow, we can
change the orientation of edges so that all flow values become 1 on all edges of the group
discussed above and the four edges that enter and leave it.

Now, let us consider the first copy of $G - e_1 - e_2$ in this group and the four edges that
enter/leave it. If one of these four edge enters $a_i$ and the other goes out $b_i$, then one edge
must go out $c_i$ and the other one enter $d_i$, or vice versa. In this case we can easily obtain a
no-where zero $\mathbb{Z}_3$-flow of $G$. Thus, we can assume that the orientation of edges is so that,
one of them enters $a_i$, the other one enters $b_i$. Hence, the third one goes out $c_i$ and the forth
one goes out $d_i$. Now, let us look at the second copy of $G - e_1 - e_2$ in this group and the
four edges that enter/leave it. Since $\phi$ is a flow, and the edges go out $c_i$ and $d_i$, then they
enter $a_{i+1}$ and $d_{i+1}$, respectively. Hence, the third and forth edges incident to $b_{i+1}$ and $c_{i+1}$
go out of these two vertices. Thus, we can obtain a $\mathbb{Z}_3$-flow of $G$ from $\phi$ by orienting $e_1$ from
$b_{i+1}$ to $a_{i+1}$, and from $c_{i+1}$ to $d_{i+1}$. We assign the flow values on these two edges as 1. It is
easy to see that this is a no-where zero $\mathbb{Z}_3$-flow of $G$. The proof is complete.  \[\square\]
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