Ramification theory of reciprocity sheaves, II
Higher local symbols

Kay Rülling1 · Shuji Saito2

Received: 21 June 2022 / Revised: 22 March 2023 / Accepted: 26 April 2023 / Published online: 5 July 2023
© The Author(s) 2023

Abstract
We construct a theory of higher local symbols along Paršin chains for reciprocity sheaves. Applying this formalism to differential forms, gives a new construction of the Paršin–Lomadze residue maps, and applying it to the torsion characters of the fundamental group gives back the reciprocity map from Kato’s higher local class field theory in the geometric case. The higher local symbols satisfy various reciprocity laws. The main result of the paper is a characterization of the modulus attached to a section of a reciprocity sheaf in terms of the higher local symbols.

Keywords Higher local symbols · Reciprocity sheaves · Modulus

Mathematics Subject Classification 14B15 · 14E22

1 Introduction

In this note we apply the results from [28] to obtain a theory of higher local symbols for reciprocity sheaves. These symbols are higher dimensional generalizations of the local symbols defined by Rosenlicht–Serre [33] in the 1-dimensional case for commutative algebraic groups. Higher local symbols are defined along Paršin chains and satisfy various reciprocity laws. Applying this formalism to differential forms, gives

K.R. was supported by the DFG Heisenberg Grant RU 1412/2-2. S.S. is supported by the JSPS KAKENHI Grant (20H01791).

Kay Rülling
ruelling@uni-wuppertal.de

Shuji Saito
sshuji@msb.biglobe.ne.jp

1 Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany
2 Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8941, Japan
a new construction of the Paršin–Lomadze residue maps, and applying it to the torsion characters of the fundamental group gives back the reciprocity map from Kato’s higher local class field theory in the geometric case. The main result of the paper is a characterization of the modulus attached to a section of a reciprocity sheaf in terms of the higher local symbols. This result will be an essential ingredient in [26, 32].

1.1 We fix a perfect field \( k \). Reciprocity sheaves were introduced by Kahn, Saito, and Yamazaki in [11]. A reciprocity sheaf \( F \) is a Nisnevich sheaf with transfers which has the additional property, that any section \( a \in F(U) \) over a smooth \( k \)-scheme \( U \) has a modulus, i.e., there is a proper \( k \)-scheme \( X \) and an effective Cartier divisor \( D \) on \( X \), such that \( U = X \setminus D \) and the pair \((X, D)\) measures the defect of \( a \) being regular outside of \( U \). Though one should think of the modulus as a measure for the pole order or the depth of ramification of \( a \) along \( D \), the interest comes from the fact that it is defined in a motivic way, namely by requiring that the action of certain finite correspondences is zero on \( a \), see 2.1 and the references there for details. The subgroup of sections of \( F(U) \) with modulus \((X, D)\) is denoted by \( \widetilde{F}(X, D) \). If \( X \) is projective of dimension \( d \) over \( k \), then in [28, Proposition 6.7] we construct a pairing

\[
(\cdot, \cdot)(X, D)_K/K: \widetilde{F}(X_K, D_K) \otimes \mathbb{Z} H^d(X_{K, \text{Nis}}, K^M_d(\mathcal{O}_{X_K}, I_{D_K})) \to F(K), \tag{1.1.1}
\]

where \( K \) is a function field over \( k \), \( X_K = X \otimes_k K \), \( K^M_d \) is Kerz’ improved Milnor \( K \)-theory sheaf, and \( K^M_d(\mathcal{O}_{X_K}, I_{D_K}) = \text{Ker}(K^M_d(X_K) \to K^M_d(D_K)) \). For particular reciprocity sheaves this gives back several pairings which were constructed in the literature by different methods, e.g., if \( F = \text{Hom}_{\text{cont}}(\pi_{ab}(-), \mathbb{Q}/\mathbb{Z}) \) and \( K \) is a finite field, this pairing (or at least the pro-system over larger and larger \( D \)) was constructed in [15] to obtain higher dimensional geometric class field theory, or if \( k \) has characteristic zero and \( F(U) \) denotes the absolute rank one connections on \( U \), this pairing was constructed by Bloch–Esnault in the case \( U \) has dimension 1, see [2, (4.8)]. A disadvantage of the motivic definition of \( \widetilde{F}(X, D) \) is that it is hard to decide which sections of \( F(U) \) have modulus \((X, D)\). To study the pairing in other interesting examples, e.g., \( F(U) = H^1(U_{\text{fppf}}, G) \) for \( G \) a finite \( k \)-group scheme, or \( F(U) = H^0(U, R^{n+1} \epsilon_x \mathbb{Q}/\mathbb{Z}(n)) \) with \( \mathbb{Q}/\mathbb{Z}(n) \) the étale motivic complex of weight \( n \) with \( \mathbb{Q}/\mathbb{Z} \)-coefficient and \( \epsilon: X_{\text{ét}} \to X_{\text{Nis}} \) the change of sites map, it is desirable to get a better hold on \( \widetilde{F}(X, D) \). Easier-to-handle global descriptions of \( \widetilde{F}(X, D) \) are given in [26, 28]. In the present article we give a purely local description, at least under certain extra assumptions on \((X, D)\).

1.2 Let \( K \) be a function field over \( k \). Recall that a Paršin chain (or maximal chain) on an integral finite-type \( K \)-scheme \( X \) of dimension \( d \) is a sequence \( \overline{x} = (x_0, \ldots, x_d) \) of points of \( X \) with \( x_i < x_{i+1} \), i.e., \( x_i \) is a strict specialization of \( x_{i+1} \), for all \( i = 0, \ldots, d - 1 \). Let \( F \) be a reciprocity sheaf. For any maximal chain \( \overline{x} \) on \( X \), we define in Sect. 5 the higher local symbol

\[
(\cdot, \cdot)_{X/\overline{x}}: F(K_{X, \overline{x}}^h) \otimes \mathbb{Z} K^M_d(K_{X, \overline{x}}^h) \to F(K), \tag{1.2.1}
\]

where \( K_{X, \overline{x}}^h \) is the henselization of \( \mathcal{O}_{X,x_0} \) along the chain \( \overline{x} \), see 3.2 for details. The definition of this pairing relies on the map \( c_{\overline{x}}: K^M_d(K_{X, \overline{x}}^h) \to H^d(X, j_! K^M_{d, U}) \) already
considered in [15] and the pushforward $H^d(X_{\text{Nis}}, j_F(d)_U) \to F(K)$ constructed in [28] (and relying on the pushforward constructed in [1]). Using the natural map $K(X) \hookrightarrow K^h_{X, \underline{x}}$ (1.2.1) also induces a semi-local pairing
\[(\, - \, , \, - \, )_{X/K, \underline{x}} : F(K(X)) \otimes_{\mathbb{Z}} K^M_d(K(X)) \to F(K).
\]

The family of these symbols (for all $X$ and all $\underline{x}$) is uniquely determined by the properties (HS1)–(HS4) which resemble the properties used by Serre to characterize and construct his local symbols on curves for commutative algebraic groups in [33, III]. This uniqueness property can be used to check that the higher local symbols coincide for $F = \Omega^d$ with those defined by Paršin and Lomadze [19, 22], for details on this and further examples see 5.6. The property (HS3) roughly says that the symbol $(-, -)_{X/K, \underline{x}}$ vanishes on 
\[\widetilde{F}(X_K, D_K) \otimes K^M_d(\mathcal{O}_{X_K}, I_{D_K})_{(x_0, \ldots, x_{d-1})},\]

where $K^M_d(\mathcal{O}_{X_K}, I_{D_K})_{(x_0, \ldots, x_{d-1})}$ is the Nisnevich stalk of $K^M_d(\mathcal{O}_{X_K}, I_{D_K})$ at the chain $(x_0, \ldots, x_{d-1})$, see (3.2.2). The property (HS4) is the reciprocity theorem
\[\sum_{x_{i-1} < y < x_{i+1}} (a, \beta)_{X/K, (x_0, \ldots, x_{i-1}, y, x_i+1, \ldots, x_d)} = 0,
\]

for all $a \in F(K(X))$, $\beta \in K^M_d(K(X))$, and $i \in \{0, \ldots, d\}$, where in the case $i = 0$, we have to assume $X$ projective. Interestingly, in [19] (and many similar constructions) property (HS3) follows easily from the definition of the local symbol and the reciprocity law (HS4) is a theorem, whose proof requires a more involved argument, whereas in our case (HS4) is a formal consequence of the construction and (HS3) follows from the pairing (1.1.1), which is one of the main results from [28].

The main result of the present paper is the following theorem (the statement in the body of the text is a bit stronger).

**Theorem 1.3** (see Theorem 6.1, Proposition 7.3) Let $X$ be a smooth $k$-scheme of pure dimension $d$ and $D$ an effective Cartier divisor on $X$ whose support has simple normal crossings. Let $U = X \setminus |D|$ and $a \in F(U)$. Assume that there exists an open dense immersion $X \hookrightarrow \overline{X}$ into a smooth and projective $k$-scheme, such that $(\overline{X} \setminus U)_{\text{red}}$ has simple normal crossings. Let $V \subset X$ be an open neighborhood of the generic points of $|D|$. Then the following conditions are equivalent:

(i) $a \in \widetilde{F}(X, D)$.

(ii) For any function field $K/k$ and any maximal chain $\underline{x} = (x_0, x_1, \ldots, x_d)$ on $V_K$ with $x_{d-1} \in D_K$, we have
\[a_K, \beta)_{X_K/K, \underline{x}} = 0, \text{ for all } \beta \in K^M_d(\mathcal{O}_{X_K}, I_{D_K})_{x_{d-1}},\]

where $X_K = X \otimes_k K$ and $a_K \in F(U_K)$ denotes the pullback of $a$. 

 Springer
If furthermore \( D \) is a reduced divisor with simple normal crossings, then the same is true without assuming the existence of the smooth projective compactification \( \overline{X} \) with SNCD boundary.

If \( F \) has level \( \leq 3 \) (see 6.5) one can also get around the assumption on the existence of the smooth compactification with SNCD boundary, see Corollary 6.6.

The proof of Theorem 1.3 uses the main results from [28]. The stronger statement for \( D \) reduced relies on [32, Corollary 2.5], [31], and an additional diagonal argument explained in Sect. 7. Theorem 1.3 and the properties (HS1)–(HS5) of the higher local symbols play an important role in the proofs of the main result of [26] and in the proof of [32, Theorem 4.2].

**Notation 1.4**

(1) In this paper \( k \) denotes a field and \( \text{Sm} \) the category of separated schemes which are smooth and of finite type over \( k \). For \( k \)-schemes \( X \) and \( Y \) we write \( X \times Y := X \times_k Y \). For \( n \geq 0 \) we write \( \mathbb{P}^n := \mathbb{P}^n_k \), \( \mathbb{A}^n := \mathbb{A}^n_k \).

(2) Let \( F \) be a Nisnevich sheaf on a scheme \( X \) and \( x \in X \) a point. Then we denote by \( F_x \) its Zariski stalk and by \( F_{hx} = \lim_{\text{Nisnevich}} F(U) \) the Nisnevich stalk, where the limit is over all Nisnevich neighborhoods \( U \to X \) of \( x \).

(3) For a reduced ring \( R \), \( \text{Frac}(R) \) denotes its total ring of fractions.

(4) For a scheme \( X \) we denote by \( X_{(i)} \) (resp. \( X^{(i)} \)) the set of \( i \)-dimensional (resp. \( i \)-codimensional) points of \( X \).

### 2 Preliminaries on reciprocity sheaves and pairings

This paper builds on [28]. In this section we recall some notations and results, see loc. cit. and the references there for more details.

In this section \( k \) is a perfect field.

2.1 A **modulus pair** \( (X, D) \) in the sense of [8, 9] consists of a separated scheme of finite type over \( k \) and an effective (possibly empty) Cartier divisor \( D \) on \( X \), such that the complement \( U = X \setminus D \) is smooth over \( k \). The modulus pair \( (X, D) \) is called proper, if \( X \) is proper over \( k \). Let \( F \) be a presheaf with transfers and \( U \in \text{Sm} \). A **modulus for an element** \( a \in F(U) \) is a proper modulus pair \( (X, D) \) with \( U \) as a connected component of \( A_1 \) and such that the normalization \( \tilde{Z} \) of the closure of \( Z \) in \( \mathbb{P}^1 \times S \times X \) satisfies \( \infty_{S|\tilde{Z}} \geq D|\tilde{Z} \), we have

\[
[Z_0]^*a = [Z_1]^*a,
\]

where \([Z_\varepsilon]\) denotes the finite correspondence from \( S \) to \( U \) associated to \( Z \cap (\{\varepsilon\} \times S) \), \( \varepsilon \in \{0, 1\} \subset A^1 \).

A **reciprocity sheaf** in the sense of [11] is a presheaf with transfers \( F \) which is a Nisnevich sheaf on \( \text{Sm} \) and for which any section \( a \in F(U) \) has a modulus \( (X, D) \). We denote by \( \text{RSC}_{\text{Nis}} \) the category of reciprocity sheaves. For a proper modulus pair \( (X, D) \) with \( X \setminus D = U \) we set

\[
\tilde{F}(X, D) = \{ a \in F(U) \mid (X, D) \text{ is a modulus for } a \}.
\]
If \((X, D)\) is not proper we set
\[
\tilde{F}(X, D) = \lim_{(Y, E) \to (X, D)} \tilde{F}(Y, E),
\]
where the limit is over the cofiltered ordered set of compactifications \((Y, E)\) of \((X, D)\), see [8, 1.8]. We also regularly work with pairs \((X, D)\), which are equal to a projective limit \(\lim_{i \in I} (X_i, D_i)\) with \((X_i, D_i)\) modulus pairs and \(I\) some filtered set (e.g., \(X\) is of finite type over a function field \(K/k\), \(D\) is an effective Cartier divisor on \(X\) and \(U = X \setminus D\) is regular). In this case we set
\[
\tilde{F}(X, D) = \lim_{i \in I} \tilde{F}(X_i, D_i).
\]

\[2.2\] Let \(F \in \text{RSC}_\text{Nis}\). Let \(K\) be a function field over \(k\) and \(U\) a regular quasi-projective \(K\)-scheme of dimension \(d\). Choose a factorization
\[
U \xrightarrow{j} X \xrightarrow{\overline{f}} \text{Spec} K
\]
of the structure map \(U \to \text{Spec} K\) with \(X\) reduced, \(j\) open dense, and \(\overline{f}\) projective. Building on the results from [1], we define in [28, 4.] for such a factorization, a pushforward map
\[
(\overline{f}, j)_*: H^d(X_{\text{Nis}}, j_!(F(d)_U)) \to F(K),
\]
where \(“j!”\) denotes the extension-by-zero functor. Here \(F(d) \in \text{RSC}_\text{Nis}\) is the \(d\)th twist of \(F\) introduced in [30, 5.5] and \(F(d)_U\) denotes its restriction to \(U_{\text{Nis}}\). There is a natural map of Nisnevich sheaves \(F_U \otimes \mathbb{Z} K_{\text{M}}^d \to F(d)_U\) on \(U\), where \(K_{\text{M}}^d\) denotes the Nisnevich sheafification of the improved Milnor \(K\)-theory from [16], which induces a morphism on \(X_{\text{Nis}}\)
\[
j_! F_U \otimes \mathbb{Z} j_! K_{d, U}^d \to j_! F(d)_U. \quad (2.2.1)
\]
This yields the pairing
\[
(\cdot, \cdot)_{U \subset X/k} : F(U) \otimes \mathbb{Z} H^d(X_{\text{Nis}}, j_! K_{d, U}^M) \xrightarrow{\cup} H^d(X_{\text{Nis}}, j_! F_U \otimes \mathbb{Z} j_! K_{d, U}^M) \xrightarrow{(2.2.1)} H^d(X_{\text{Nis}}, j_! F(d)_U) \xrightarrow{(\overline{f}, j)_*} F(K). \quad (2.2.2)
\]
It is a factorization of the usual pairing induced by finite correspondences from \(\text{Spec} K\) to \(U\) in the following sense: If \(x \in U\) is a closed point there is a natural isomorphism
\[
\theta_x : \mathbb{Z} \xrightarrow{\sim} H^d_x(U_{\text{Nis}}, K_{d, U}^M), \quad (2.2.3)
\]
induced by the Gersten resolution (see [16, Proposition 10, (8)]). Composing with the natural map \(H^d_x(U_{\text{Nis}}, K_{d, U}^M) \to H^d(X_{\text{Nis}}, j_! K_{d, U}^M)\) and taking the sum over all
closed points $x$ yields the map

$$Z_0(U) = \bigoplus_{x \in U(0)} \mathbb{Z} \rightarrow H^d \left( X_{\text{Nis}}, j_! K_{d,U}^M \right). \quad (2.2.4)$$

By [28, Lemma 6.6] we have for all $a \in F(U)$ and $\zeta \in Z_0(U)$

$$(a, [\zeta])_{U \subset X/K} = \zeta^* a, \quad (2.2.5)$$

where $[\zeta]$ on the left denotes the image of $\zeta$ under (2.2.4) and on the right we view $\zeta$ as a finite correspondence from Spec $K$ to $U$.

### 2.3

Let $X$ be a reduced noetherian excellent separated scheme of dimension $d < \infty$ over a field, such that $X^{(d)} = X(0)$. Let $D \subset X$ be a nowhere dense closed subscheme. We define for $r \geq 1$

$$V_{r, X|D} = \text{Im} \left( \mathcal{O}_{X|D}^\times \otimes_{\mathbb{Z}} K^M_{r-1,X} \rightarrow K^M_{r,X} \right), \quad \text{where } \mathcal{O}^\times_{X|D} = \text{Ker} \left( \mathcal{O}^\times_X \rightarrow \mathcal{O}^\times_D \right).$$

This sheaf is very close to the relative Milnor $K$-sheaf $K^M_r(\mathcal{O}_X, I_D)$ defined in [15, (1.3)], where $I_D$ denotes the ideal sheaf of $D$. In fact the two sheaves agree for $r = 1$ and they have the same stalks at all points with infinite residue field. Since by Grothendieck–Nisnevich vanishing the cohomological dimension of the Nisnevich cohomology on a noetherian scheme is bounded by its dimension, see [15, (1.2.5)] or [21, 1.32 Theorem], we find that the natural inclusion $V_{d,X|D} \hookrightarrow K^M_d(\mathcal{O}_X, I_D)$ induces an isomorphism

$$H^d \left( X_{\text{Nis}}, V_{d,X|D} \right) \overset{\sim}{\longrightarrow} H^d \left( X_{\text{Nis}}, K^M_d(\mathcal{O}_X, I_D) \right). \quad (2.3.1)$$

For any regular dense open $j': U' \hookrightarrow X$ contained in $X \setminus D$ the composition

$$Z_0(U') \xrightarrow{(2.2.4)} H^d \left( X_{\text{Nis}}, j'_! K_{d,U'}^M \right) \rightarrow H^d \left( X_{\text{Nis}}, V_{d,X|D} \right) \quad (2.3.2)$$

is therefore surjective by [15, Theorem 2.5].

### 2.4

We recall the main result from [28]. Let $F \in \text{RSC}_{\text{Nis}}$. Let $K$ be a function field over $k$. Let $X$ be an integral projective $K$-scheme of dimension $d$ and $j: U \hookrightarrow X$ a regular dense open subscheme. Let $D \subset X$ be a closed subscheme (not necessarily a divisor) such that $D_{\text{red}} = X \setminus U$. Let $v: Y \rightarrow X$ be the normalization of $X$. We define

$$F_{\text{gen}}(X, D) := \text{Ker} \left( F(U) \rightarrow \bigoplus_{y \in Y^{(1)} \cap v^{-1}(D)} \frac{F(Y^h_y \setminus y)}{F(Y^h_y, D^h_y)} \right). \quad (2.4.1)$$

Note that as long as we work over a field the “nice schemes” in [15, Theorem 2.5] can be replaced by excellent regular schemes, as follows from [16, Proposition 10] and the comment below [15, Corollary 2.4].
where \( Y^h_y = \text{Spec} \mathcal{O}^h_{Y,y} \) and \( D^h_y = D \times_X Y^h_y \). We define \( R(X \mid D) \) by the exact sequence

\[
0 \to R(X \mid D) \to H^d(X_{\text{Nis}}, j_! K^M_{d,U}) \to H^d(X_{\text{Nis}}, V_{d,X\mid D}) \to 0.
\]

**Theorem 2.5** ([28, Proposition 6.7, Theorem 6.8]) **Assumptions as in 2.4.**

1. The pairing (2.2.2) induces a pairing

\[
(\cdot, \cdot)_{(X,D)/K} : F_{\text{gen}}(X, D) \otimes_{\mathbb{Z}} H^d(X_{\text{Nis}}, V_{d,X\mid D}) \to F(K).
\]

2. Assume \( X \in \text{Sm} \) is projective over \( k \) and \( D \) is an effective Cartier divisor with simple normal crossing support. Then for \( a \in F(U) \) with \( U = X \setminus D \) we have

\[
\tilde{F}(X, D) = F_{\text{gen}}(X, D)
\]

\[
= \{ a \in F(U) \mid (a_K, \gamma)_{U_K \subset X_K/K} = 0 \text{ for all } K/k, \gamma \in R(X_K \mid D_K) \},
\]

where in the set on the right, \( K \) runs over all function fields over \( k \), \( X_K = X \otimes_k K \), and \( a_K \) is the pullback of \( a \) to \( F(U_K) \).

In this paper we give a purely local description of the right-hand side in (2), using Paršin chains and higher local rings.

### 3 Recollections on Paršin chains, higher local rings, and cohomology

We recall some definitions and results from [15, (1.6)] (see also [28, 5.]). In this section, \( X \) is a reduced noetherian separated scheme of dimension \( d < \infty \), such that \( X^{(d)} = X^{(0)} \).

#### 3.1 For \( x, y \in X \) we write

\[
y < x :\iff \{y\} \subset \{x\}, \text{ i.e., } y \in \{x\} \text{ and } y \neq x.
\]

A **chain** on \( X \) is a sequence

\[
\mathbf{x} = (x_0, \ldots, x_n) \quad \text{with} \quad x_0 < x_1 < \cdots < x_n. \tag{3.1.1}
\]

The chain \( \mathbf{x} \) is a **maximal Paršin chain** (or **maximal chain**) if \( n = d \) and \( x_i \in X^{(i)} \).

Note that the assumptions on \( X \) imply \( x_i \in \overline{[x_{i+1}]}^{(1)} \). We denote

\[
c(X) = \{ \text{chains on } X \} \quad \text{and} \quad \text{mc}(X) = \{ \text{maximal chains on } X \}.
\]

A **maximal chain with break at** \( r \in \{0, \ldots, d\} \) is a chain (3.1.1) with \( n = d - 1 \) and \( x_i \in X^{(i)} \), for \( i < r \), and \( x_i \in X^{(i+1)} \), for \( i \geq r \). We denote

\[
\text{mc}_r(X) = \{ \text{maximal chain with break at } r \text{ on } X \}.
\]
For $x = (x_0, \ldots, x_{d-1}) \in \text{mc}_r(X)$, we denote by $b(x)$ the set of $y \in X_{(r)}$ such that 

$$x(y) := (x_0, \ldots, x_{r-1}, y, x_r, \ldots, x_{d-1}) \in \text{mc}(X).$$

3.2 Let $S \subset X$ be a finite subset contained in an affine open neighborhood of $X$. A strict Nisnevich neighborhood of $S$ is an étale map $u : U \to X$ such that $U$ is affine, the base change $u^{-1}(S) \to S$ of $u$ is an isomorphism, and every connected component of $U$ intersects $u^{-1}(S)$.

Let $x = (x_0, \ldots, x_n)$ be a chain on $X$. A strict Nisnevich neighborhood of $x$ is a sequence of maps 

$$U = (U_n \to \cdots \to U_1 \to U_0 \to X =: U_{-1}),$$

such that $U_i \to U_{i-1}$ is a strict Nisnevich neighborhood of $U_{i-1,x_i} = U_{i-1} \times_X \{x_i\}$, for all $i = 0, \ldots, n$. There is an obvious notion of morphism between two strict Nisnevich neighborhoods and picking a representative in each isomorphism class yields a filtered set 

$$N(x) := \{\text{strict Nisnevich neighborhoods of } x\}.$$ 

Assume $x \in \text{mc}_r(X)$ and $y \in b(x)$. If $U$ is a strict Nisnevich neighborhood as above, then repeating $U_{r-1}$ in the $r$th spot yields a map of filtered sets 

$$N(x) \to N(x(y)). \quad (3.2.1)$$ 

The Nisnevich stalk of a presheaf $F$ on $X_{\text{Nis}}$ at $x \in c(X)$ is defined to be 

$$F^h_x := \lim_{U=(U_n \to \cdots \to X) \in N(x)} F(U_n). \quad (3.2.2)$$ 

Note that for $x \in \text{mc}_r(X)$ and $y \in b(x)$ the map (3.2.1) induces a natural map 

$$t_y : F^h_x \to F^h_x(y). \quad (3.2.3)$$

For $F = \mathcal{O}_X$ we write $\mathcal{O}_{X,x}^h = F^h_x$ and $K_{X,x}^h = \text{Frac}(\mathcal{O}_{X,x}^h)$. By [28, Lemma 5.3], $\mathcal{O}_{X,x}^h = R_n$, where we recursively define 

$$R_0 = \mathcal{O}_{X,x_0}^h \quad \text{and} \quad R_i = \prod_{p \in T_i} R_{i-1,p}^h, \quad i \geq 1,$$

where $T_i = \text{Spec } R_{i-1} \times_X \{x_i\}$ is the finite set of prime ideals in $R_{i-1}$ lying over the prime ideal in $\mathcal{O}_{X,x_0}$ corresponding to $x_i$; this is also the definition used in [15].

**Lemma 3.3** Let $x = (x_0, \ldots, x_n) \in c(X)$. Let $Y = \{x_n\}$ and set $y = x$ viewed as a chain on $Y$. Then $\mathcal{O}_{X,x}^h / \mathfrak{r} = \mathcal{O}_{Y,y}^h$, where $\mathfrak{r}$ denotes the radical of $\mathcal{O}_{X,x}^h$.

 Springer
Proof If \( \mathcal{U} \in N(x) \), then \( \mathcal{U} \times_X Y \in N(y) \) and it follows from [5, Proposition (18.6.8)], that Nisnevich neighborhoods of the form \( \mathcal{U} \times_X Y \) are cofinal in \( N(y) \). \( \square \)

3.4 Let \( F \) be an abelian Nisnevich sheaf on \( X \) and \( \underline{x} = (x_0, \ldots, x_n) \in c(X) \). We set

\[
H^i_{\underline{x}}(X, F) := \lim_{\mathcal{U} \to (U_n \to \cdots \to X) \in N_{\underline{x}}(X)} H^i_{U_n, \text{Nis}}(U_n, \text{Nis}, F),
\]

where on the right-hand side we consider the local cohomology group in the finite set \( U_n, x_n = U_n \times_X \{x_n\} \). Assume \( x_{n-1} \in \{x_n\}^{(1)} \) and write \( \underline{x}' = (x_0, \ldots, x_{n-1}) \). There is a natural map

\[
\delta_{\underline{x}} : H^i_{\underline{x}}(X, F) \to H^{i+1}_{\underline{x}'}(X, F) \tag{3.4.1}
\]

induced by the connecting homomorphism of the localization sequence, see [15, Definition 1.6.2 (4)] (or [28, 5.4]). Following [15, Definition 1.6.2 (5)], we define for a maximal chain \( \underline{x} = (x_0, \ldots, x_d) \) the map

\[
c_{\underline{x}} := s_{x_0} \circ c_{\underline{x}}, 0 : F(K^h_{\underline{x}, \underline{X}}) := F^h_{\underline{x}} \to H^d(X_{\text{Nis}}, F), \tag{3.4.2}
\]

where \( s_{x_0} : H^d_{x_0}(X_{\text{Nis}}, F) \to H^d(X_{\text{Nis}}, F) \) is the forget-support-map and \( c_{\underline{x}}, 0 \) is the composition

\[
c_{\underline{x}}, 0 : F^h_{\underline{x}} \xrightarrow{\delta(x_0, \ldots, x_{d-1})} H^1_{(x_0, \ldots, x_{d-1})}(X_{\text{Nis}}, F) \xrightarrow{\delta(x_0, x_1)} \cdots \xrightarrow{\delta(x_0, x_{d-1})} H^d_{x_0}(X_{\text{Nis}}, F). \tag{3.4.3}
\]

Proposition 3.5 ([15, Lemma 1.6.3]) For any abelian group \( A \), the map

\[
\Phi : \text{Hom}(H^d(X_{\text{Nis}}, F), A) \to \prod_{\underline{x} \in mc(X)} \text{Hom}(F^h_{\underline{x}}, A), \quad \alpha \mapsto (\alpha \circ c_{\underline{x}})_{\underline{x} \in mc(X)},
\]

is injective. Furthermore, the image of \( \Phi \) consists of those tuples \( (\chi_{\underline{x}})_{\underline{x} \in mc(X)} \) satisfying the following condition: For any \( r \in \{0, \ldots, d\}, \underline{x} \in mc_r(X) \), and for any \( a \in F^h_{\underline{x}} \), we have \( \chi_{\underline{x}(y)}(\iota_y(a)) = 0 \) for almost all \( y \in b(\underline{x}) \), and

\[
\sum_{y \in b(\underline{x})} \chi_{\underline{x}(y)}(\iota_y(a)) = 0, \tag{3.5.1}
\]

where \( \iota_y : F^h_{\underline{x}} \to F^h_{\underline{x}(y)} \) is the map from (3.2.3).
3.6 For $\bar{x} \in \mathrm{mc}_r(X)$, $r \in \{0, \ldots, d\}$, and $a \in F^h_{\bar{x}}$, Proposition 3.5 implies

$$c_{\bar{x}(y)}(\iota_y(a)) = 0, \text{ for almost all } y \in b(\bar{x}), \text{ and } \sum_{y \in b(\bar{x})} c_{\bar{x}(y)}(\iota_y(a)) = 0. \quad (3.6.1)$$

Note that in case $r = d$, the composition

$$F^h_{\bar{x}} \xrightarrow{\iota_y} F^h_{\bar{x}(y)} \xrightarrow{\delta_{\bar{x}(y)}} H^1(X_{\text{Nis}}, F) \quad (3.6.2)$$

is zero, for all $y \in b(\bar{x})$. In particular $c_{\bar{x}(y)} \circ \iota_y = 0$, for all $y \in b(\bar{x})$.

3.7 Let $F$ be a presheaf of abelian groups on $X_{\text{Zar}}$ and $\bar{x} = (x_0, \ldots, x_n) \in c(X)$. We can define the Zariski stalk $F_{\bar{x}}$ of $F$ at $\bar{x}$ as above, but in fact $F_{\bar{x}} = F_{x_n}$. If $\bar{x} \in \mathrm{mc}(X)$, we can define also the map analogous to (3.4.3):

$$c^\text{Zar}_{\bar{x},0} : F_{\bar{x}} \to H^d_{x_0}(X_{\text{Zar}}, F). \quad (3.7.1)$$

In [15], Proposition 3.5 is deduced by induction on the dimension from the coniveau spectral sequence for Nisnevich cohomology and the Grothendieck–Nisnevich vanishing. Since the Zariski analogue of both statements hold, we also have a Zariski analogue of Proposition 3.5. In particular, for $w \in X(0)$, the map

$$\bigoplus_{\bar{x} \in \mathrm{mc}(X)} F_{\bar{x}} \xrightarrow{c^\text{Zar}_{\bar{x},0}} H^d_{w}(X_{\text{Zar}}, F)$$

is surjective. This follows from the Zariski analogue of Proposition 3.5 applied to the $(d - 1)$-dimensional scheme $\text{Spec}(\mathcal{O}_{X_{\bar{w}}}) \backslash \{w\}$.

4 Some auxiliary results for relative Milnor $K$-theory

In this section $k$ denotes any field and $X$ is a reduced noetherian excellent separated $k$-scheme of dimension $d < \infty$, such that $X^{(d)} = X(0)$.

4.1 Let $T$ be a noetherian reduced purely 1-dimensional and excellent semilocal scheme with total ring of fractions $\kappa(T)$ and denote by $\nu : \widetilde{T} \to T$ the normalization. Writing $T$ as a union of irreducible components $T = \bigcup T_i$ we obtain $\kappa(T) = \prod_i \kappa(T_i)$ and $\widetilde{T} = \bigsqcup T_i$ with the obvious notation. Let $S$ be the set of closed points of $T$ and set $\kappa(S) = \prod_{s \in S} \kappa(s)$. Then we define

$$\partial_S := \sum_{s \in S} \sum_{s' \in \nu^{-1}(s)} \text{Nm}_{\kappa(s')/\kappa(s)} \circ \partial_{v_{s'}} : K^M_r(\kappa(T)) \to K^M_{r-1}(\kappa(S)),$$

where $v_{s'}$ denotes the discrete valuation on $\text{Frac}(\mathcal{O}_{\widetilde{T},s'})$ defined by $s'$, $\partial_{v_{s'}}$ is the classical tame symbol, and $\text{Nm}_{\kappa(s')/\kappa(s)}$ is the norm map.
Let $x = (x_1, \ldots, x_{n-1}, x_n) \in c(X)$ and assume $x_{n-1} \in X_{(d-1)}$ and $x_n \in X_d$, in particular $x_{n-1} \in [x_n](1)$. Set $X' = [x_{n-1}]$ and $x' = (x_1, \ldots, x_{n-1}) \in c(X')$. We define
\[
\partial_x := \partial_{S_d} : K^M_r(K^h_{X,x}) \to K^M_r(K^h_{X',x'})
\]
(4.1.1)

Here we use the following notation: $S_d$ is the set of closed points of the reduced 1-dimensional and excellent semi-local ring $\mathcal{O}_{X,x'}$, where we view $x'$ as a chain on $X$; note $K^h_{X,x} = \text{Frac}(\mathcal{O}_{X,x'})$ and $\kappa(S_d) = K^h_{X',x'}$, by Lemma 3.3.

For $x \in \text{mc}(X)$ as above and $i \in \{0, \ldots, d\}$ set $x_i = (x_0, \ldots, x_i) \in \text{mc}([x_i])$. We denote by $\partial_{X,x}$ the following composition:
\[
\partial_{X,x} := \partial_{x_0} \circ \partial_{x_1} \circ \cdots \circ \partial_{x_d} : K^M_r(K^h_{X,x}) \to K^M_{r-d}(\kappa(x_0)).
\]
(4.1.2)

For $d = 0$ this is the identity (by convention).

**Lemma 4.2** Assume $x_0$ is contained in $X_{\text{reg}}$ the regular locus of $X$. Then the following diagram commutes:

\[
\begin{array}{ccc}
K^M_d(K^h_{X,x}) & \xrightarrow{c_{x,0}} & H^d_{x_0}(X_{\text{Nis}}, K^M_{d,X}) \\
\partial_{X,x} & \cong & \theta_{x_0} \\
\end{array}
\]

where $c_{x,0}$ is the map (3.4.3), $\partial_{X,x}$ is the map (4.1.2), and $\theta_{x_0}$ is the map (2.2.3).

**Proof** We may assume $X = X_{\text{reg}}$. By [16, Proposition 10(8)] the Gersten complex viewed as complex on $X_{\text{Nis}}$ is a resolution of $K^M_{d,X}$; since its terms are furthermore acyclic for the global section functor, we may use it to compute the local cohomology. This in particular yields the identifications in the diagram below for $0 \leq j \leq d - 1$

\[
\begin{array}{ccc}
H^j_{(x_0,\ldots,x_{d-j})}(X_{\text{Nis}}, K^M_{d,X}) & \xrightarrow{\delta(x_0,\ldots,x_{d-j})} & H^{j+1}_{(x_0,\ldots,x_{d-j-1})}(X_{\text{Nis}}, K^M_{d,X}) \\
\| & & \| \\
K^M_d(K^h_{Y,(x_0,\ldots,x_{d-j-1})}) & \xrightarrow{\partial_{S_{d-j-1}}} & K^M_{d-j-1}(K^h_{Z,(x_0,\ldots,x_{d-j-1})})
\end{array}
\]
(4.2.1)

where $Y := [x_{d-j-1}], Z := [x_{d-j-1}]$, and $S_{d-j-1}$ denotes the set of closed points in $\mathcal{O}^h_{Y,(x_0,\ldots,x_{d-j-1})}$ and the other notation is taken from 3.4 and 4.1. Composing the diagrams for $j = 0, \ldots, d - 1$ yields the statement.

**Lemma 4.3** ([15, Proposition 2.9]) Let $i : Y \to X$ be a closed immersion with $Y$ integral of dimension $e$ and assume $Y \cap X_{\text{reg}} \neq \emptyset$. Let $D \subset X$ be a closed subscheme.
which does not contain \( Y \). Then there exists a proper closed subscheme \( E \subset Y \) and a map (see 2.4 for notation)

\[
i_* : H^e(Y_{\text{Nis}}, V_{e,Y|E}) \to H^d(X_{\text{Nis}}, V_{d,X|D})
\]

which is uniquely determined by the requirement that for any regular open \( U \subset X \setminus D \) and regular open \( U' \subset (Y \cap U) \setminus E \) which is dense in \( Y \), the following diagram commutes:

\[
\begin{array}{ccc}
Z_0(U') & \to & H^e(Y_{\text{Nis}}, V_{e,Y|E}) \\
i_* & & i_* \\
\downarrow & & \downarrow \\
Z_0(U) & \to & H^d(X_{\text{Nis}}, V_{d,X|D})
\end{array}
\]  

(4.3.1)

where the horizontal maps are the maps (2.3.2). Moreover, for any \( y \in \text{mc}(Y) \) and any \( x = (y, x_{e+1}, \ldots, x_d) \in \text{mc}(X) \), the following diagram is commutative:

\[
\begin{array}{ccc}
K^M_e(K^h_{Y,y}) & \xrightarrow{c_y} & H^e(Y_{\text{Nis}}, V_{e,Y|E}) \\
\partial_x^\Sigma & \downarrow & \downarrow i_* \\
K^M_d(K^h_{X,x}) & \xrightarrow{c_x} & H^d(X_{\text{Nis}}, V_{d,X|D})
\end{array}
\]  

(4.3.2)

where we set (using the notation from 4.1)

\[
\partial_x^\Sigma := \partial_{x_{e+1}} \circ \cdots \circ \partial_{x_d} : K^M_d(K^h_{X,x}) \to K^M_e(K^h_{Y,y}),
\]

with \( x_j = (y, x_{e+1}, \ldots, x_j), \) for \( j \geq e + 1 \).

**Proof** This is essentially [15, Proposition 2.9] and the same proof works. Since in *loc. cit.* the assumptions and the formulation are slightly different, we repeat the argument for the convenience of the reader. First note that if a map \( i_* \) as in the statement exists such that (4.3.2) commutes, then (4.3.1) commutes as well, by Lemma 4.2. This later commutativity uniquely characterizes the map \( i_* \), as the horizontal maps in (4.3.1) are surjective, see (2.3.2). Thus it remains to construct a map \( i_* \) (for an appropriate \( E \)) which makes (4.3.2) commutative. By [15, Proposition 2.7] there exists a proper closed subscheme \( E \subset Y \) with ideal sheaf \( I_E \), such that for any étale map \( X' \to X \) and any \( y' \in X' \) over the generic point of \( Y' = \overline{\{y'\}} \) and any \( w \in Y'^{(1)} \) the composition

\[
K^M_e(\mathcal{O}_{Y'}, I_E \mathcal{O}_{Y'})_w \to K^M_e(k(y')) \cong H_{y'}^{d-e}(X'_{\text{Nis}}, K^M_d(\mathcal{O}_X, I_D)) \\
\to H_{(w,y')}^{d-e}(X'_{\text{Nis}}, K^M_d(\mathcal{O}_X, I_D)) \xrightarrow{\delta_{(w,y')}} H_{w}^{d-e+1}(X'_{\text{Nis}}, K^M_d(\mathcal{O}_X, I_D))
\]
is zero. We fix this $E$ in the following. For $y = (y_0, \ldots, y_e) \in \text{mc}(Y)$ we consider the composition

$$
\chi_y : (V_e, Y|E)_Y^h = K_e^M (K^h_{Y, Y}) \xrightarrow{\sim} H^d_{\text{Nis}} (X_{\text{Nis}}, V_d, X|D) \xrightarrow{\delta_1 \circ \cdots \circ \delta_e} H^d_{y_0} (X_{\text{Nis}}, V_d, X|D) \to H^d (X_{\text{Nis}}, V_d, X|D),
$$

where the first isomorphism is induced by the Gersten resolution together with the definition of the local cohomology group in 3.4 and we set $\delta_i = \delta(y_0, \ldots, y_i)$ with the notation from (3.4.1). The family $\{\chi_y\}_{y \in \text{mc}(Y)}$ satisfies the condition (3.5.1): for $r \in \{0, \ldots, e - 1\}$ this follows from the definition of the $\delta_i$, for $r = e$ it follows from our choice of $E$ above and (2.3.1). Thus by Proposition 3.5 there is a map $i_*$ as in the statement, such that

$$
i_* \circ c_y = \chi_y : K_e^M (K^h_{Y, Y}) \to H^d (X_{\text{Nis}}, V_d, X|D),$$

all $y \in \text{mc}(Y)$. The commutativity of (4.3.2) follows immediately from this equality together with the commutativity of (4.2.1).

4.4 We recall some constructions and results from [15, Section 4]. Let $D$ be a closed subscheme of $X$ which is nowhere dense and is defined by the ideal sheaf $I \subset O_X$. We define the Nisnevich sheaf $\overline{V}_{r, X|D}$ on $X$ by

$$
U \mapsto \overline{V}_{r, X|D}(U) = \text{Ker} \left( \bigoplus_{\eta \in U^{(0)}} K^M_r (k(\eta)) \to \bigoplus_{x \in U^{(1)}} \left( \overline{V}_{r, U|D_U}^h_X (k(x))^h \right) \right),
$$

where $U$ runs over the étale $X$-schemes and $D_U = D \times_X U$. Note that this sheaf agrees with the sheaf $\overline{K}^M_r (O_X, I)$ defined in loc. cit. for $r = 1$ and, if $d \geq 2$, for all $r \geq 1$: For $r = 1$, this is immediate. For $d \geq 2$, $(\overline{V}_{r, U|D_U}^h_X (k(x))^h = K^M_r (O_U, I_U)_X^h$ since the residue fields $k(x)$, for $x \in U^{(1)}$, have infinitely many elements (see 2.4). Note that we have a natural map

$$
\overline{V}_{r, X|D} \to \overline{V}_{r, X|D}.
$$

The cokernel of this map is supported in codimension 2 and the kernel in codimension 1. Hence Grothendieck–Nisnevich vanishing yields an isomorphism

$$
H^d (X_{\text{Nis}}, \overline{V}_{r, X|D}) \xrightarrow{\sim} H^d (X_{\text{Nis}}, \overline{V}_{r, X|D}).
$$

(4.4.1)

We will need the following statement from loc. cit.:

**Proposition 4.5** ([15, Proposition 4.2]) Let $f : Y \to X$ be a finite morphism and assume $Y$ is reduced and $f (Y^{(0)}) \subset X^{(0)}$. Assume $r = 1$ or $d \geq 2$. Then the norm map on Milnor $K$-theory

$$
\otimes \text{ Springer}
$$
Nm: \( f_* \left( \bigoplus_{\eta \in Y^{(0)}} i_{\eta, \eta} K_{r, \eta}^M \right) \to \bigoplus_{\xi \in X^{(0)}} i_{\xi, \xi} K_{r, \xi}^M \),

where \( i_{\eta}: \eta \hookrightarrow Y \) and \( i_{\xi}: \xi \hookrightarrow X \) are the natural inclusions, induces a morphism

\[ \text{Nm}: f_*(\overline{V}_{r, Y|E}) \to \overline{V}_{r, X|D}, \]

for some large enough nowhere dense closed subscheme \( E \subset Y \) containing \( D \times_X Y \).

**Corollary 4.6** Let \( f: Y \to X \) be a separated morphism of finite-type with \( f(Y^{(0)}) \subset X^{(0)} \) and \( \dim Y = \dim X = d \). Assume \( r = 1 \) or \( d \geq 2 \) and \( r \geq 1 \). Let \( D \subset X \) be a closed subscheme and set \( D_Y := D \times_X Y \). Let \( x_1, \ldots, x_n \in X^{(1)} \) and \( \beta_1, \ldots, \beta_n \in \bigoplus_{\eta \in Y^{(0)}} K_{r, \eta}^M (k(\eta)) \). Assume that there exists an open affine subscheme in \( X \) containing all the points \( x_1, \ldots, x_n \). Then there exits an element \( \gamma \in \bigoplus_{\eta \in Y^{(0)}} K_{r, \eta}^M (k(\eta)) \) such that for all \( i = 1, \ldots, n \),

\[ \gamma - \beta_i \in (\overline{V}_{r, Y|D_Y})_{\tilde{x}_i}, \text{ for all } \tilde{x}_i \in f^{-1}(x_i), \]

\[ \text{Nm}(\gamma) - \text{Nm}(\beta_i) \in (\overline{V}_{r, X|D})_{x_i}, \]

where \( \text{Nm} : \bigoplus_{\eta \in Y^{(0)}} K_{r, \eta}^M (k(\eta)) \to \bigoplus_{\xi \in X^{(0)}} K_{r, \xi}^M (k(\xi)) \) is the norm map.

**Proof** We may replace \( f \) by a compactification and hence assume that \( f \) is proper. Furthermore, we may replace \( X \) by its semi-localization at the points \( x_1, \ldots, x_n \) and \( f \) by the base change. The semi-localization exists since \( x_1, \ldots, x_n \) are contained in an affine open in \( X \). Thus \( X \) is affine, integral, excellent, and 1-dimensional and \( f: Y \to X \) is a proper, dominant, and quasi-finite morphism, whence it is finite and surjective. Let \( \nu: \tilde{Y} \to Y \) be the normalization. Thus \( \tilde{Y} \) is a finite disjoint union of Dedekind schemes. By Proposition 4.5 we find a closed subscheme \( E \subset \tilde{Y} \) containing \( D_{\tilde{Y}} \) such that

\[ \nu_* (\overline{V}_{r, \tilde{Y}|E}) \subset \overline{V}_{r, Y|D_Y} \quad \text{and} \quad \text{Nm}((f \circ \nu)_*(\overline{V}_{r, \tilde{Y}|E})) \subset \overline{V}_{r, X|D}. \]  

(4.6.1)

By the Approximation Lemma, we find an element \( \gamma \in \bigoplus_{\eta \in Y^{(0)}} K_{r, \eta}^M (k(\eta)) \) such that

\[ \gamma - \beta_i \in (\overline{V}_{r, \tilde{Y}|E})_{\tilde{x}_i} \text{ for all } \tilde{x}_i \in (f \circ \nu)^{-1}(x_i). \]

The statement follows from this and (4.6.1).

\[ \square \]

## 5 Higher local symbols

We introduce higher local symbols along maximal chains for reciprocity sheaves. These generalize local symbols for curves, see [10, Proposition 5.2.1] and [33] for the classical case of commutative \( k \)-groups. Furthermore, we obtain a unified construction for several higher local symbols defined in the literature, e.g., by Paršin, Lomadze,
Kato and many more. The results will be used in Sect. 6 to give a characterization of the modulus in terms of local symbols. The content of this section will also play a crucial role in [26].

In this section $k$ is a perfect field, $K$ is function field over $k$, and $X$ is an integral scheme of finite type over $K$ and dimension $d$. We fix $F \in \text{RSC}_{\text{Nis}}$.

**Definition 5.1** Let $\underline{x} = (x_0, \ldots, x_d) \in \text{mc}(X)$. We define the pairing

$$(\cdot, \cdot)_{X/\underline{x}} : F(K_{X,\underline{x}}^h) \otimes \mathbb{Z} K_{d}^{M}(K_{X,\underline{x}}^h) \to F(K) \quad (5.1.1)$$

as follows: Choose an open subscheme $V \subset X$ which is quasi-projective and contains $x_0$. Choose a dense open regular subscheme $U \subset V$. Choose a dense open immersions $j_V : V \hookrightarrow Y$ into an integral projective $K$-scheme (a *projective compactification*) with structure map $f : Y \to \text{Spec } K$ and denote by $j : U \hookrightarrow Y$ the induced immersion. Note that $\underline{x} \in \text{mc}(Y)$. We define (5.1.1) as the composition

$$F(K_{X,\underline{x}}^h) \otimes \mathbb{Z} K_{d}^{M}(K_{X,\underline{x}}^h) = (F_U \otimes j!K_{d,U}^{M})_{\underline{x}} \xrightarrow{\epsilon_{\underline{x}}} j_{!}(F(d)U)_{\underline{x}} \xrightarrow{c_{\underline{x}}} H^{d}(Y_{\text{Nis}}, j_{!}F(d)U) \xrightarrow{(f_{!})_{\underline{x}}} F(K),$$

where the first map is the stalk at $\underline{x}$ of (2.2.1), $c_{\underline{x}}$ is the map (3.4.2), and $(f_{!})_{\underline{x}}$ is the pushforward recalled in 2.2. It follows from Lemma 5.2 (1), (2) below that this definition is independent of the choice of $V$, $U$, and $Y$.

Take $r, s \in \{0, \ldots, d\}$. Precomposing (5.1.1) with the natural map

$$F(K_{X,(x_r,\ldots,x_d)}^h) \otimes \mathbb{Z} K_{d}^{M}(K_{X,(x_r,\ldots,x_d)}^h) \to F(K_{X,\underline{x}}^h) \otimes \mathbb{Z} K_{d}^{M}(K_{X,\underline{x}}^h),$$

cf. (3.2.3), we obtain pairings (denoted by the same symbol)

$$(\cdot, \cdot)_{X/\underline{x}} : F(K_{X,(x_r,\ldots,x_d)}^h) \otimes \mathbb{Z} K_{d}^{M}(K_{X,(x_r,\ldots,x_d)}^h) \to F(K), \quad (5.1.2)$$

in particular, for $r = d$ and $s = d - 1$, we get the pairing

$$(\cdot, \cdot)_{X/\underline{x}} : F(K(X)) \otimes \mathbb{Z} K_{d}^{M}(\text{Frac}(O_{X,x_{d-1}}^{h})) \to F(K). \quad (5.1.3)$$

We call (5.1.1), (5.1.2), and (5.1.3) the *higher local symbol of $F$ at $x$*.

**Lemma 5.2** Let the situation be as above.

1. The definition of the higher local symbol (5.1.1) is independent of the choice of the quasi-projective open $V \subset X$ containing $x_0$, the regular dense open subset $U \subset V$, and the choice of the projective compactification $V \hookrightarrow Y$.
2. Let $X$ be quasi-projective, $U \subset X$ a regular dense open, and $j : U \hookrightarrow X \hookrightarrow \overline{X}$ a projective compactification. Let $a \in F(U)$ and $\beta \in K_{d}^{M}(K_{X,\underline{x}}^h)$. Then

$$(a, \beta)_{X/\underline{x}} = (a, c_{\underline{x}}(\beta))_{U \subset \overline{X}/K}, \quad \text{for all } \underline{x} \in \text{mc}(X),$$

where $(\cdot, \cdot)_{U \subset \overline{X}/K} : F(U) \otimes \mathbb{Z} H^{d}(\overline{X}_{\text{Nis}}, j_{!}K_{d,U}^{M}) \to F(K)$ is from (2.2.2).
Proof (1) We start by showing the independence of the choice of the projective compactification of $V$. Thus assume we have two projective compactifications $j: U \hookrightarrow V \hookrightarrow Y$ and $j': U \hookrightarrow V \hookrightarrow Y'$ and denote by $f: Y \to \text{Spec } K$ and $f': Y' \to \text{Spec } K$ the projective structure maps. It suffices to consider the situation, where we have a projective morphism $g: Y' \to Y$ such that $g \circ j' = j$ and $f \circ g = f'$. In this case the independence follows from the following commutative diagram:

$$
\begin{array}{ccc}
F \langle d \rangle (K^h_{X, \xi}) & \xrightarrow{c_\xi} & H^d(Y_{\text{Nis}}, j_! F \langle d \rangle_U) \\
& \searrow & \downarrow (f, j)_* \\
& & F(K)
\end{array}
$$

where the vertical map in the middle is induced by the natural map $j_! \to Rg_* j'_!$, see [28, (4.3.3)]. The right triangle commutes by [28, Lemma 4.7 (3)]. The left triangle commutes since both maps labeled $c_\xi$ factor over $H^d_{x_0}(V, F \langle d \rangle_Y)$.

Next we show the independence of the choice of $U$. By the above, it suffices to consider a dense open immersion $v: U' \hookrightarrow U$ with projective compactifications $j: U \hookrightarrow V \hookrightarrow Y$ and $j' = j \circ v: U' \hookrightarrow U \hookrightarrow V \hookrightarrow Y$. In this case the independence follows from the commutative diagram

$$
\begin{array}{ccc}
F \langle d \rangle (K^h_{X, \xi}) & \xrightarrow{c_\xi} & H^d(Y_{\text{Nis}}, j'_! F \langle d \rangle_{U'}) \\
& \searrow & \downarrow (f, j')_* \\
& & F(K)
\end{array}
$$

Here the commutativity of the right triangle holds by [28, Lemma 4.7 (2)] and the one of the left triangle is obvious.

It remains to check the independence of the choice of $V$. To this end, let $V, V' \subset X$ be two open quasi-projective subschemes containing $x_0$, let $V \hookrightarrow Y \to \text{Spec } K$ and $V' \hookrightarrow Y' \to \text{Spec } K$ be projective compactifications, and let $U \subset V$ and $U' \subset V'$ be two open regular subschemes. We obtain the open immersions $U \hookrightarrow Y$ and $U' \hookrightarrow Y'$. Let $V'' \subset V \cap V'$ be an affine open neighborhood of $x_0$ and let $U'' \subset U \cap U' \cap V''$ be a dense open regular subscheme. We have two induced open immersions $U'' \hookrightarrow Y$ and $U'' \hookrightarrow Y'$. Denote by (5.1.1)$_{(U, V, Y)}$ the pairing (5.1.1) constructed using $U \hookrightarrow V \hookrightarrow Y \to \text{Spec } K$. Then

$$(5.1.1)_{(U, V, Y)} = (5.1.1)_{(U'', V, Y)} = (5.1.1)_{(U'', V'', Y)} = (5.1.1)_{(U'', V'', Y')} ,$$

where the first equality holds by the independence of the choice of $U$ proven above, the second equality holds by definition of the pairing (it only depends on the maps $U \hookrightarrow Y \to \text{Spec } K$), and the third equality holds by the independence of the choice of the compactification of $V''$ proven above. This together with the analog reasoning for $(U', V', Y')$ instead of $(U, V, Y)$, implies $(5.1.1)_{(U, V, Y)} = (5.1.1)_{(U', V', Y')}$. 

 Springer
This follows from the compatibility of the boundary maps from the localization sequence with cup-products, see [28, (6.7.5)].

**Proposition 5.3** The pairing (5.1.1) satisfies the following properties for all \( a \in F(K(X)) \):

(HS1) Let \( X \hookrightarrow X' \) be an open immersion where \( X' \) is an integral \( K \)-scheme of dimension \( d \). Then for all \( \beta \in K^M_d(K_{X,\Delta}^h) \)

\[
(a, \beta)_{X/K,\Delta} = (a, \beta)_{X'/K,\Delta}.
\]

(HS2) Let \( x = (x_0, \ldots, x_d) \in \text{mc}(X) \), and \( X_{d-1} \subset X \) be the closure of \( x_{d-1} \), and set \( x' = (x_0, \ldots, x_{d-1}) \in \text{mc}(X_{d-1}) \). Then for all \( \beta \in K^M_d(K_{X,\Delta}^h) \)

\[
(a, \beta)_{X/K,\Delta} = \begin{cases} 
\beta \cdot \text{Tr}_{K(X)/K}(a), & \text{if } d = 0, \\
(a(x_{d-1}), \partial_{\Delta} \beta)_{X_{d-1}/K,\Delta'}, & \text{if } d \geq 1 \text{ and } a \in F(\mathcal{O}_X, x_{d-1}).
\end{cases}
\]

where \( a(x_{d-1}) \in F(K(X_{d-1})) \) is the restriction of \( a \) and \( \partial_{\Delta} \) is defined in (4.1.1), and \( \text{Tr}_{K(X)/K}: F(K(X)) \to F(K) \) is the trace for the finite map \( \text{Spec } K(X) \to \text{Spec } K \).

(HS3) Let \( D \subset X \) be a closed subscheme such that \( X \setminus D \) is nonempty and regular. Assume \( a \in F_{\text{gen}}(X, D) \). Then, for \( \bar{x} = (x_0, \ldots, x_d) \in \text{mc}(X) \) and \( \bar{x}' = (x_0, \ldots, x_{d-1}) \in \text{mc}_{d}(X) \), we have (see 4.4 for the definition of \( \overline{V}_d.X|D \))

\[
(a, \beta)_{X/K,\Delta} = 0, \quad \text{for all } \beta \in (\overline{V}_d.X|D)_{\Delta}^h.
\]

(HS4) Let \( \bar{x}' \in \text{mc}_{r}(X) \) with \( 0 \leq r \leq d - 1 \). Then for all \( \beta \in K^M_d(K_{X,\Delta}^h) \)

\[
(a, \iota_y \beta)_{X/K,\Delta}(y) = 0, \quad \text{for almost all } y \in b(\bar{x}').
\]

If either \( r \geq 1 \) or \( r = 0 \), \( X \) is quasi-projective, and the closure of \( x_1 \) in \( X \) is projective over \( K \), where \( \bar{x}' = (x_1, \ldots, x_d) \), then

\[
\sum_{y \in b(\bar{x}')} (a, \iota_y \beta)_{X/K,\Delta}(y) = 0,
\]

where \( \iota_y: K_{X,\Delta}^h \to K_{X,\Delta'}^h(y) \) is the map (3.2.3).

Furthermore, the family

\[
\{ (\cdot, \cdot)_{X/K,\Delta}: F(K(X)) \otimes K^M_{\dim X}(K(X)) \to F(K) \mid \bar{x} \in \text{mc}(X) \}_{\dim X \leq d'}
\]
where $X$ is running through all integral schemes of finite type of dimension $\leq d$, is uniquely determined by (HS1)–(HS4).

**Proof** (HS1) follows from Lemma 5.2(1).

(HS2). For $d = 0$ this follows from the fact that in this case the pushforward $(f, j)_*$ appearing in Definition 5.1 is the pushforward along the finite map $f : \text{Spec } K(X) \to \text{Spec } K$ constructed in [1], which is equal to the trace by [1, Proposition 8.10(3)]. Now assume $d \geq 1$ and take $a \in F((\overline{O}_{X,x_{d-1}}), \beta) \in K^M_d(K_h^\beta_{X,X})$. By (HS1) we may assume that $X \to \text{Spec } K$ is projective. We find a closed subscheme $D \subset X$ with $x_{d-1} \notin D$, such that $U = X \setminus D$ is regular and $a \in F_{\text{gen}}(X, D)$. Denote by $i : Y := X_{d-1} = (x_{d-1}) \hookrightarrow X$ the closed immersion. Choose the closed subscheme $E \subset Y$ as in Lemma 4.3 and choose a regular open $U' \subset (Y \cap U) \setminus E$ which is dense in $Y$. Since the map $Z_0(U') \to H^{d-1}(Y_{\text{Nis}}, V_{d-1,Y|E})$ from (2.3.2) is surjective, we find a cycle $\zeta \in Z_0(U')$ which maps to $c_{\zeta'}(\partial_{\zeta}(\beta))$; we can view $\zeta$ as a finite correspondence from $\text{Spec } K$ to $U'$. We denote by

$$[\zeta] = c_{\zeta'}(\partial_{\zeta}(\beta)) \in H^{d-1}(Y, j^!_iK^M_{d-1, U'})$$

and $[i_*\zeta] \in H^d(X, j_i^!K^M_{d, U'})$ the images of $\zeta$ and $i_*\zeta \in Z_0(U)$ under the cycle maps (2.2.4). We compute

$$(a, \beta)_{X/K, \zeta} = (a, c_{\zeta'}(\beta))_{U \subset X/K}$$

by Lemma 5.2 (2)\hspace{1cm} by Theorem 2.5 (1)

$$(a, c_{\zeta'}(\beta))_{(X, D)/K}$$

by Theorem 2.5 (1)

$$(a, i_*([\zeta]))_{U \subset X/K},$$

by Theorem 2.5 (1)

$$(a, i_*[\zeta])_{U \subset X/K}$$

by (4.3.2)

$$(a, i_*\zeta)_U \subset X/K$$

by (4.3.1)

$$(i_*\zeta)^*a$$

by definition of corr. action

$$[\zeta] = (a(x_{d-1}), \zeta)_{Y \subset Y/K}$$

by (2.2.5)

$$[i_*\zeta] = (a(x_{d-1}), i_*\zeta)_{Y \subset Y/K}$$

by Lemma 5.2(2).

This yields (HS2).

Property (HS3) follows from Lemma 5.2 (2), Theorem 2.5 (1), (4.4.1), and the vanishing of (3.6.2).

For (HS4) in the case $r \geq 1$ choose a quasi-projective open $V \subset X$ which contains the closed point $x_0$ of $x'$, and take a projective compactification $V \hookrightarrow Y$. Note that $x'$ also defines a chain on $Y$ and the set $b(x')$ does not change when we consider $x'$ as a chain on $X$ or $Y$. Hence in this case the statement follows directly from Definition 5.1 and (3.6.1) applied to $F = j_i^!K^M_{d, U'}$, where $j : U \hookrightarrow Y$ is the inclusion of a dense open regular subscheme. In the case $r = 0$, we can take a projective compactification $X \hookrightarrow \overline{X}$ and view $x'$ as a chain on $\overline{X}$. By the assumption that the closure of $x_1$ in $X$ is projective, the set $b(x')$ does not change when we consider $x'$ as a chain on $X$ or on $\overline{X}$. Hence the statement follows from (3.6.1) also in this case.

It remains to prove the uniqueness part. Let $\{(1, -)_{X/K, \zeta} | \zeta \in \text{mc}(X)\}_{\dim X \leq d}$ be another family of symbols satisfying (HS1)–(HS4). By (HS1) it suffices to show...
\(-, -\)_{X/K, \underline{x}} = (-, -)_{X/K, \underline{x}} \) for all affine \( K \)-schemes \( X \); applying (HS1) again we may assume \( X \) is projective. We proceed by induction. If \( \dim X = 0 \) the symbol is uniquely determined by (HS2). Now we assume \( \dim X = d \geq 1 \) and the symbols coincide on all closed subschemes of \( X \) of dimension strictly smaller than \( d \). Set \( L := K(X) \). Let \( a \in F(L), \beta \in K_d^M(L) \), and \( \underline{x} = (x_0, \ldots, x_d) \in \text{mc}(X) \). Let \( D \subset X \) be a strict closed subscheme such that \( X \setminus D \) is regular, \( x_{d-1} \in D \), and \( a \in F_{\text{gen}}(X, D) \). By Corollary 4.6 (with \( f = \text{id} \)) we find an element \( \gamma \in K_d^M(L) \) such that \( \beta - \gamma \in (\mathcal{V}_{d,X} D)_{x_{d-1}} \) and \( \gamma \in (\mathcal{V}_{d,X} D)_u \), for all \( u \in X^{(1)} \setminus D \setminus \{x_{d-1}\} \). Set \( \underline{x}' = (x_0, \ldots, x_{d-2}, x_d) \in \text{mc}_{d-1}(X) \). Then

\[
(a, \beta)_{X/K, \underline{x}} \overset{(HS3)}{=} \sum_{u \in \mathcal{b}(\underline{x}') \setminus D} (a, \gamma)_{X/K, \underline{x}'(u)} \overset{(HS4)}{=} \sum_{u \in \mathcal{b}(\underline{x}') \setminus D} (a, \gamma)_{X/K, \underline{x}'(u)} - \sum_{u \in \mathcal{b}(\underline{x}') \setminus D} (a, \gamma)_{X/K, \underline{x}'(u)} \overset{(HS2)}{=} - \sum_{u \in \mathcal{b}(\underline{x}') \setminus D} (a(u), \partial_{\underline{x}'(u)} \gamma)_{[a]/K, (x_0, \ldots, x_{d-2}, u)}.
\]

The same computation with \( (-, -)_{X/K, \underline{x}} \) and induction yields the desired equality. \( \square \)

The formulation of the above proposition was inspired by the treatment of local symbols in [33, III, Section 1]. But note that the construction is completely different. The next proposition is however a formal consequence of (HS1)–(HS4) (and properties of Milnor \( K \)-theory) in the same way [33, III, Proposition 4] is a consequence of the properties written in Definition 2 of loc. cit.

**Proposition 5.4** Let \( f : Y \to X \) be a projective and surjective \( K \)-morphism between two integral \( K \)-schemes of the same dimension \( d \). Then we have for all \( a \in F(K(X)) \) and \( \beta \in K_d^M(K(Y)) \) and \( \underline{x} \in \text{mc}(X) \)

\[
\sum_{y \in \text{mc}(Y)} (f^*a, \beta)_{Y/K, \underline{x}} = (a, f_*\beta)_{X/K, \underline{x}}, \tag{HS5}
\]

where \( f_* = \text{Nm}_{K(Y)/K(X)} : K_d^M(K(Y)) \to K_d^M(K(X)) \) is the norm map.

**Proof** First note that the sum in (HS5) is finite. Indeed given points \( x_i \in X \) and \( y_i \in Y \) with \( f(y_i) = x_i \), then \( f \) induces a projective and surjective \( K \)-morphism \( f_i : [y_i] \to [x_i] \). As source and target of \( f_i \) have the same dimension it follows that for any \( x_{i-1} \in [x_i] \) the preimage \( f_i^{-1}(x_{i-1}) \) consists of \( 1 \)-codimensional points in \([y_i] \), in particular it is a discrete noetherian topological space and hence is finite. Thus there are only finitely many maximal chains in \( Y \) lying over \( \underline{x} \).

To prove the equality in (HS5) we proceed by induction on the dimension \( d \). Set \( E := K(X) \) and \( L := K(Y) \). For \( d = 0 \), (HS5) translates by (HS2) into

\[
\beta \cdot \text{Tr}_{L/K}(a) = [L : E] \cdot \beta \cdot \text{Tr}_{E/K}(a) \quad \text{for} \quad a \in F(E), \quad \beta \in \mathbb{Z},
\]

which holds since \([L : E] \cdot a = \text{Tr}_{L/E}(a)\). Now assume \( d \geq 1 \) and the formula holds in dimension \( \leq d - 1 \). Write \( \underline{x} = (x_0, \ldots, x_d) \). We consider two cases.
Case 1: $a \in F(\mathcal{O}_X, x_{d-1})$. Set $u := x_{d-1} \in X^{(1)}$ and denote by $X' = \overline{\{u\}} \subset X$ the closure. Set $x' = (x_0, \ldots, x_{d-1}) \in \text{mc}(X')$. We first collect some standard commutative diagrams for Milnor $K$-theory, also to clarify the notation used later;

\[ \begin{array}{c}
K_d^M(K^h_{X,u}) \xrightarrow{\delta_u} K_{d-1}^M(K^h_{X',u'}) \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
K_d^M(E) \xrightarrow{i_u} K_d^M(K^h_{X,u}) \xrightarrow{\partial_u} K_{d-1}^M(K(X')),
\end{array} \]  
(5.4.1)

where $\iota$’s are the natural maps and $\partial$’s are induced by the tame symbol, see 4.1;

\[ \begin{array}{c}
K_d^M(L) \xrightarrow{\prod \iota_z} \prod_{z \in Y_{d-1}, f(z)=u} K_d^M(K^h_{Y,z}) \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
K_d^M(E) \xrightarrow{i_u} K_d^M(K^h_{X,u});
\end{array} \]  
(5.4.2)

\[ \begin{array}{c}
K_d^M(K^h_{Y,z}) \xrightarrow{\delta_z} K_{d-1}^M(K(z)) \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
K_d^M(K^h_{X,u}) \xrightarrow{\partial_u} K_{d-1}^M(K(u)),
\end{array} \]  
(5.4.3)

where $z \in Y_{d-1}$ with $f(z) = u$. The commutativity of the above diagrams follows from standard relations in Milnor $K$-theory, e.g., in [23, 1.]

\[ \sum_{y \in \text{mc}(Y)} (f^*a, \iota_{\overline{\beta}})_{Y,\overline{y}} = (a, \iota_{\overline{x}} \text{Nm}_{L/E} \beta)_{X,\overline{x}} \]
assuming \( a \in F(\emptyset_{X,u}) \) and that the equality holds in smaller dimensions. We compute

\[
(a, t_{\underline{\alpha}} Nm_{L/E} \beta)_{X/K, \underline{\alpha}} = (a(u), \partial_{\underline{\alpha}} (t_{\underline{\alpha}} Nm_{L/E} \beta))_{X'/K, \underline{\alpha}'} \quad \text{by (HS2)}
\]

\[
= \sum_{z \in Y_{d-1}} \sum_{f(z) = u} \left( f^*_y(a(u), t_{y'} \partial_{z} (t_{z} \beta))_{Y(z)/K, y'} \right)_{X'/K, \underline{\alpha}'} \quad \text{by (5.4.1)}
\]

\[
= \sum_{y \in mc(Y)} \sum_{f(y) = x} \left( f^*_y(a(u), \partial_{(y', \xi)} (t_{(y', \xi)} \beta))_{Y(z)/K, y'} \right)_{X'/K, \underline{\alpha}'} \quad \text{by (HS2)},
\]

where \( \xi \) is the generic point of \( Y \). This completes the proof of the first case.

**Case 2:** \( a \in F(E) \). By (HS1) we may assume \( X \) to be projective. Let \( D \subset X \) be a closed subscheme such that \( X \setminus D \) is regular and \( a \in F_{\text{gen}}(X, D) \). Enlarging \( D \) we may assume \( x_{d-1} \in D, Y \setminus D_Y \) is regular, and \( f^*a \in F_{\text{gen}}(Y, D_Y) \), where \( D_Y = Y \times_X D \). By Corollary 4.6 we find an element \( \gamma \in K^M_d(L) \) such that

(i) \( \gamma - \beta \in (Y, D_Y)_{y} \), for all \( y \in Y(1) \cap D_Y \) lying over \( x_{d-1} \),

(ii) \( \gamma \in (Y, D_Y)_{y} \), for all \( y \in Y(1) \cap D_Y \) not lying over \( x_{d-1} \),

(iii) \( \gamma \in (Y, D_Y)_{x} \), for all \( x \in X(1) \cap D \setminus \{x_{d-1}\} \).

Set \( \underline{\gamma}'' := (x_0, \ldots, x_{d-2}, x_d) \in mc_{d-1}(X) \). We compute

\[
(a, Nm_{L/E} \beta)_{X/K, x} = \sum_{z \in b(\underline{\gamma}'' \cap D)} (a, Nm_{L/E} \gamma)_{X/K, \underline{\gamma}''(z)} \quad \text{by (iii), (iv), (HS3)}
\]

\[
= - \sum_{z \in b(\underline{\gamma}'' \cap D)} (a, Nm_{L/E} \gamma)_{X/K, \underline{\gamma}''(z)} \quad \text{by (HS4)}
\]

\[
= - \sum_{z \in b(\underline{\gamma}'' \cap D)} \sum_{y \in mc(Y)} \left( f^*a, \gamma \right)_{Y/K, y} \quad \text{by Case 1}
\]

\[
= \sum_{z \in b(\underline{\gamma}'' \cap D)} \sum_{y \in mc(Y)} \left( f^*a, \gamma \right)_{Y/K, y} \quad \text{by (HS4)}
\]

\[
= \sum_{y \in mc(Y)} \left( f^*a, \beta \right)_{Y/K, y} \quad \text{by (i), (ii), (HS3)}.
\]
Note that we can apply (HS4) also in the case $d = 1$, since $X$ and $Y$ are projective. This completes the proof of the proposition.

The following corollary will be used in the proof of Proposition 7.3 and in [26].

**Corollary 5.5** Let $f : Y \to X$ be a dominant and quasi-projective $K$-morphism between integral $K$-schemes of the same dimension $d$. Let $x = (x_0, \ldots, x_d) \in \text{mc}(X)$ and $u := x_{d-1}$. Let $y \in Y^{(1)}$ with $f(y) = u$. We assume that $f$ induces a projective morphism between the closures of the points $y$ and $u$. Then $K^h_{X,u}(Y, y)$ is finite over $K^h_{X,u}$ (see 3.2 for notation) and for all $a \in F(K(X))$ and $b \in K^M_d(K^h_{X,y})$, we have

$$
\sum_{z \in \text{mc}_{d-1}(Y), z < y} (f^*a, b)_{Y/K, z(y)} = (a, \text{Nm}_{y/u}(b))_{X/K, x},
$$

(***HS5’***)

where $z < y$ means $z_{d-2} < y$ with $z = (z_0, \ldots, z_{d-2}, z_d)$ and $z(y) = (z_0, \ldots, z_{d-2}, y, z_d)$, and $\text{Nm}_{y/u} : K^M_d(K^h_{X,y}) \to K^M_d(K^h_{X,u})$ is the norm map.

**Proof** Set $E := K(X)$ and $L := K(Y)$. Note that $y$ is an isolated point in $f^{-1}(u)$, hence $\mathcal{O}^h_{X,u} \to \mathcal{O}^h_{Y,y}$ is finite and injective. Let $Y \hookrightarrow \overline{Y} \to X$ be a projective compactification of $f$. Take a closed subscheme $D \subset X$ such that $X \setminus D$ and $\overline{Y} \setminus D_Y$ are regular, where $D_Y = Y \times_X D$, and $a \in F_{\text{gen}}(X, D)$, $f^*a \in F_{\text{gen}}(\overline{Y}, D_Y)$. Set $X' = \text{Spec} \mathcal{O}^h_{X,u}$ and denote by $\tilde{f} : \overline{Y} \to X'$ the base change of $f$. Note that the total fraction ring of $\overline{Y}'$ is equal to $\bigoplus_{z \in \tilde{f}^{-1}(u)} K^h_{Y,z}$. By Corollary 4.6 applied to $\tilde{f}'$ and $(\beta_z) \in \bigoplus_{z \in \tilde{f}^{-1}(u)} K^M_d(K^h_{Y,z})$ with $\beta_y = \beta$ and $\beta_z = 0$ for $z \neq y$, we find an element $\gamma' \in \bigoplus_{z \in \tilde{f}^{-1}(u)} K^M_d(K^h_{Y,z})$ such that

(i) $\beta - \gamma' \in (\overline{V}_d,Y|D_Y)^h_z$,

(ii) $\gamma' \in (\overline{V}_d,Y|D_Y)^h_z$, for all $z \in \tilde{f}^{-1}(u) \setminus \{y\}$,

(iii) $\text{Nm}_{y/u}(\beta) - \text{Nm}_{y/u}(\gamma') \in (\overline{V}_d,X|D)^h_z$,

(iv) $\text{Nm}_{z/u}(\gamma') \in (\overline{V}_d,X|D)^h_z$, for all $z \in \tilde{f}^{-1}(u) \setminus \{y\}$.

We have a surjection

$$
K^M_d(L) \twoheadrightarrow \bigoplus_{z \in \tilde{f}^{-1}(u)} \frac{K^M_d(K^h_{Y,z})}{(\overline{V}_d,Y|D_Y)^h_z}.
$$

Indeed, by Proposition 4.5 it suffices to show this for $Y$ normal in which case it follows from the Approximation Lemma. Thus we find $\gamma \in K^M_d(L)$, so that (i)–(iv) holds with

\[ \text{ Springer} \]
γ′ replaced by γ. Set $x' = (x_0, \ldots, x_{d-1})$. We compute

$$(a, \text{Nm}_{y/u}(\beta))_{\mathcal{X}/\mathcal{K}, \xi} = \sum_{z \in \mathcal{Y}} (a, \text{Nm}_{z/u}(\gamma))_{\mathcal{X}/\mathcal{K}, \xi}$$

by (iii), (iv), (HS3)

$$= (a, \text{Nm}_{L/E}(\gamma))_{\mathcal{X}/\mathcal{K}, \xi}$$

by (5.4.2)

$$= \sum_{z \in \text{mc}(\mathcal{Y})} (\tilde{f}^*a, \gamma)_{\mathcal{Y}/\mathcal{K}, \xi}$$

by (HS5)

$$= \sum_{z \in \text{mc}_{d-1}(\mathcal{Y}), z < \gamma} (\tilde{f}^*a, \beta)_{\mathcal{Y}/\mathcal{K}, \xi(y)}$$

by (i), (ii), (HS3)

where the last equality follows from the assumption that the closure of $y$ in $\mathcal{Y} \times \mathcal{X} \{u\}$ is already closed in $\mathcal{Y} \times \mathcal{X} \{u\}$. This completes the proof of the corollary. □

**Examples and Remarks 5.6** We fix a function field $\mathcal{K}$ over $k$ and $X$ is an integral scheme of finite type over $\mathcal{K}$ and dimension $d$ and $x \in \text{mc}(X)$.

1. Let $F = \Omega^j_{\mathcal{X}/\mathcal{K}}$. In [19, p. 515] a residue homomorphism

$$\text{Res}_{\mathcal{X}/\mathcal{K}, \xi} : \Omega^{j+d}_{\mathcal{X}/\mathcal{K}} \to \Omega^j_{\mathcal{K}/k}$$

is defined, which generalizes a construction of Paršin for surfaces, see [22, Section 1]. (In [19] it is denoted by $\text{Res}_{\mathcal{X}, \xi}^f$, where $f : X \to \text{Spec } \mathcal{K}$ is the structure homomorphism.) We have for all $a \in F(\mathcal{K}(X))$ and $\beta \in K_M^d(\mathcal{K}(X))$

$$(a, \beta)_{\mathcal{X}/\mathcal{K}, \xi} = \pm \text{Res}_{\mathcal{X}/\mathcal{K}, \xi}(a \cdot \text{dlog } \beta), \quad (5.6.1)$$

where ± is a universal sign depending on the choice of the sign for the tame symbol and for Res. Indeed, by the construction in loc. cit. the right-hand side satisfies (HS1) and (HS3), (HS2) (up to sign) holds by Lemma 12 and (HS4) by Theorem 3. Thus (5.6.1) follows from the uniqueness statement in Proposition 5.3. Note that the construction of $\text{Res}_{\mathcal{X}/\mathcal{K}, \xi}$ in [22] and [19] are completely different in spirit: the residue is defined in an explicit way using power series in several variables, property (HS3) holds by construction, whereas (HS4) is a theorem. On the other hand for the symbol $(-, -)_{\mathcal{X}/\mathcal{K}, \xi}$ the reciprocity law (HS4) follows immediately from the definition whereas (HS3) is implied by Theorem 2.5, a main result of [28].

2. It follows from the uniqueness statement that in case $d = 1$ the symbol $(-, -)_{\mathcal{X}/\mathcal{K}, \xi}$ agrees with the local symbol for reciprocity sheaves considered in
[10, Proposition 5.2.1], which generalizes the local symbol for commutative $k$-groups by Rosenlicht–Serre, see [33].

(3) Let $F = H^1((-)_{\text{ét}}, \mathbb{Q}/\mathbb{Z})$. Assume $K = k$ is a finite field. Then $(5.1.1)$ induces a morphism

$$K_d^M(K_{X,\overline{x}}^h) \to \text{Hom}(F(K_{X,\overline{x}}^h), F(k)) = \pi_1^{ab}(K_{X,\overline{x}}^h),$$

where the equality on the right follows from $F(k) = \mathbb{Q}/\mathbb{Z}$. By [15, Proposition 3.3] this map decomposes as the product of the maps

$$K_d^M(\text{Frac}(V)) \to \pi_1^{ab}(\text{Frac}(V)),$$

where $V$ runs through all $d$-discrete valuation rings of $k(X)$ dominating $\overline{x}$ (in the sense of [15, (3.2)]). It can be shown that the induced morphisms $K_d^M(\text{Frac}(V)) \to \pi_1^{ab}(\text{Frac}(V))$ coincide with Kato’s higher local reciprocity maps $\Psi_{\text{Frac}(V)}$ constructed in [12, p. 661] (see [15, Theorem 3.5] for an explanation of how to deduce the henselian case from the complete case treated in [12]). We leave the verification of this fact to interested readers.

(4) For $F = W_n\Omega^j$, the $j$-th de Rham–Witt differentials (see [7]), we get higher local symbols

$$(-, -)_{X/K,\overline{x}}: W_n\Omega^j_{K(X)} \otimes_Z K_d^M(K_{X,\overline{x}}^h) \to W_n\Omega^j_K,$$

which by the above generalizes the residue symbol from the one-dimensional case constructed in [12, Section 2] or [25, 2.] (see also [24]). Since Verschiebung, Frobenius, restriction, and the differential on the de Rham–Witt complex are morphisms of reciprocity sheaves, it follows that the above symbol is compatible with these. Note also that from [12, Section 2.5, Lemma 12] and [15, Proposition 3.3], one can define a residue morphism (similar as in [19])

$$\text{Res}_{X/K,\overline{x}}: W_n\Omega^{j+d}_{K(X)} \to W_n\Omega^j_K.$$

The connection to the residue symbol defined here should be as in (5.6.1). One way to verify this would be to show that $\text{Res}_{X/K,\overline{x}}$ satisfies (HS4), e.g., using the strategy from [19], and then use uniqueness (the properties (HS1)–(HS3) are direct to check from the construction). We leave the details to interested readers.

(5) Let $F = G$ be a commutative $k$-group. We obtain a pairing

$$(-, -)_{X/K,\overline{x}}: G(K(X)) \otimes_Z K_d^M(K_{X,\overline{x}}^h) \to G(K).$$

A pairing like this was also defined in [14, III]. (In loc. cit. such a pairing was defined for higher dimensional local fields, but one can use [15, Proposition 3.3] to obtain an induced pairing as above.) In characteristic zero, one can check that the two pairings coincide. Indeed, in this case it suffices to consider $G = G_a$, $G_m$, or an abelian variety. For the $G_a$-case they are induced by $\text{Res}_{X/K,\overline{x}}$ from
(1) by [14, p. 144]. If \( G = G_m \), it is induced for both pairings by an iteration of the tame symbol. For an abelian variety, (HS2) and the formula in the middle of p. 145 in [14] show that they coincide. In particular we find that the symbol from \( \text{loc. cit.} \) in characteristic zero satisfies the reciprocity theorem (HS4). In positive characteristic we believe that the two pairings coincide, but this remains to be checked.

(6) Let \( (Y, E) \) be a proper modulus pair (see 2.1). The presheaf with transfers \( h_0(Y, E) \) is defined in [11, Definition 2.2.1] and it is a reciprocity presheaf by [11, Theorem 2.3.3]. Thus the Nisnevich sheafification \( F = h_0(Y, E)_{\text{Nis}} \) is a reciprocity sheaf by [31, Theorem 0.1]. By [11, Theorem 2.3.1], we have \( h_0(Y, E)(K) = \text{CH}_0(Y_K | E_K) \), the Chow group of zero-cycles with modulus introduced in [17]. Therefore (5.1.1) induces a morphism

\[
(\alpha, \beta)_{X/K, \underline{\chi}} : \text{CH}_0(Y_K | E_K) \otimes \mathbb{Z} \to \text{CH}_0(Y_K | E_K),
\]

satisfying (HS1)–(HS5). If \( \alpha \in \text{CH}_0(Y_K | E_K) \) can be represented by a zero-cycle in \( Y_K \setminus E_K \), which spreads out to a finite correspondence \( \tilde{\alpha} \) from \( U \) to \( Y \setminus E \) for some smooth open \( U \subset X \) containing the closed point \( x_0 \) of \( \underline{\chi} \), then it follows from (HS2), that we have

\[
(\alpha, \beta)_{X/K, \underline{\chi}} = \partial_{\underline{\chi}}(\beta) \cdot f_\ast \tilde{\alpha} \quad \text{for} \quad \beta \in K_d^M(K_{X, \underline{\chi}}^h).
\]

where \( \partial_{\underline{\chi}} : K_d^M(K_{X, \underline{\chi}}^h) \to K_0^M(\kappa(x_0)) = \mathbb{Z} \) is the map (4.1.2) and \( f : U \times (Y \setminus E) \to U \) is the projection.

6 Characterization of the modulus via higher local symbols

In this section \( k \) is a perfect field and \( F \in \text{RSC}_{\text{Nis}} \). The main result of this section is the following.

**Theorem 6.1** Let \( (X, D) \) be a modulus pair (see 2.1) with \( X \) of pure dimension \( d \). Let \( U = X \setminus |D| \) and \( a \in F(U) \). For a function field \( K/k \) denote by \( X_K = X \otimes_k K \) the base change and by \( a_K \in F(U_K) \) the pullback of \( a \). Let \( W \subset |D| \) be a set of closed points which contains at least one point of every irreducible component of \( |D| \). Consider the following conditions (see Definition 5.1):

(i) \( a \in \tilde{F}(X, D) \).

(ii) For any function field \( K \) and \( \underline{x} = (x_0, \ldots, x_d) \in \text{mc}(X_K) \) we have

\[
(a_K, \beta)_{X_K/K, \underline{x}} = 0, \quad \text{for all} \quad \beta \in (V_d, x_K | D_K)^h_{(x_0, \ldots, x_{d-1})}.
\]

(iii) For any function field \( K \) and \( \underline{x} = (x_0, x_1, \ldots, x_d) \in \text{mc}(X_K) \) with \( x_0 \in W_K \) and \( x_{d-1} \in D_K \), we have

\[
(a_K, \beta)_{X_K/K, \underline{x}} = 0, \quad \text{for all} \quad \beta \in (V_d, x_K | D_K)_{x_{d-1}}.
\]
Then, we have the implication \((i) \Rightarrow (ii) \Rightarrow (iii)\). Assume furthermore that there exists an open dense immersion \(X \hookrightarrow \overline{X}\) into a smooth and projective \(k\)-scheme, such that \(\overline{X} \setminus U\) is an SNCD, then all the above statements are equivalent.

We stress the fact that \((V_d, X|D)_{x_0, \ldots, x_{d-1}}^h\) in (ii) is the limit over all Nisnevich neighborhoods of \((x_0, \ldots, x_{d-1})\) (see 3.2), whereas \((V_d, X|D)_{x_{d-1}}^h\) in (iii) is the Zariski stalk at the one-codimensional point \(x_{d-1} \in X^{(1)}\). In Sect. 7 we will see that in case \(D\) is reduced, the assumption on the existence of a smooth compactification is superfluous (but still \(X\) has to be smooth and \(D\) is a SNCD).

6.2 Before we prove Theorem 6.1 we recall the following result:

Let \(X\) be a separated scheme of finite type over a field \(K\) of dimension \(d\). Assume no irreducible component of dimension \(d\) of \(X\) is proper. Then for any coherent sheaf \(\mathcal{F}\) on \(X\), we have

\[
H^d(X_{\text{Zar}}, \mathcal{F}) = 0.
\]

This theorem was conjectured by Lichtenbaum and proven by Grothendieck, see [6, Theorem 6.9] for Grothendieck’s proof in the quasi-projective case relying on duality theory, see [18] for a more elementary proof in the stated generality. We will use the following consequence (cf. the proof of [6, Theorem 6.9]):

Let \(Y\) be a proper \(K\)-scheme of dimension \(d\). Let \(W \subset Y\) be a set of closed points which contains at least one closed point of each irreducible component of dimension \(d\) of \(Y\). Then the natural map

\[
\bigoplus_{z \in W} H^d_z(Y_{\text{Zar}}, \mathcal{F}) \to H^d(Y_{\text{Zar}}, \mathcal{F}) \tag{6.2.1}
\]

is surjective for all coherent \(\mathcal{O}_Y\)-modules \(\mathcal{F}\). Indeed, this holds as \(H^d((Y \setminus W)_{\text{Zar}}, \mathcal{F})\) vanishes by the above result.

**Proof of Theorem 6.1** The implication \((i) \Rightarrow (ii)\) follows from (HS3) in Proposition 5.3 and \(\widetilde{F}(X, D) \subset F_{\text{gen}}(X, D)\), see (2.4.1). The implication \((ii) \Rightarrow (iii)\) is immediate from the natural map \((V_d, X_{K}\mid D_{K})_{x_{d-1}} \to (V_d, X_{K}\mid D_{K})_{(x_0, \ldots, x_{d-1})}^h\). Assume \(X\) has a smooth compactification \(\overline{X}\) such that \(\overline{X} \setminus U\) is an SNCD. It remains to show that in this case also \((iii) \Rightarrow (i)\) holds. Let \(\overline{D} \subset \overline{X}_K\) be the closure of \(D_{K}\). By Theorem 2.5 (2) and Lemma 5.2 (2) it suffices to prove the following:

**Claim** Let \(B \subset \overline{X}_K\) be an effective Cartier divisor supported on \(\overline{X}_K \setminus X_{K}\) and \(n \geq 1\). Set \(Y = |\overline{D} + B|\) which we will view as a reduced effective Cartier divisor. Then the sequence

\[
\bigoplus_{w \in W_K} \bigoplus_{\sum_i = (w, x_1, \ldots, x_d)} (V_d, X_{K}\mid D_{K})_{x_{d-1}} \to H^d(\overline{X}_K, \mathcal{N}, V_d, \overline{X}_K|\overline{D} + nY + B) \to H^d(\overline{X}_K, \mathcal{N}, V_d, \overline{X}_K|\overline{D} + B) \to 0
\]
is exact. Here, the second sum is over all $x_w = (w, x_1, \ldots, x_{d-1}, x_d) \in \text{mc}(X_K)$ such that $x_{d-1} \in D^{(0)}$, and the first map is the sum of the maps

$$(V_d, X_K|D_K)_{x_{d-1}} \to (V_d, X_K|D_K)_{x_w}^h$$

$$\to (V_d, X_K|D_K)_{x_w}^h \to H^d(\overline{X}_K, V_d, \overline{X}_K|\overline{D}+nY+B)$$

where $x'_w = (w, x_1, \ldots, x_{d-1})$ for $x_w = (w, x_1, \ldots, x_d)$ and the last map is induced by $c_{x_w}$ from (3.4.2) noting $(V_d, X_K|D_K)^h_{x_w} = (V_d, X_K|\overline{D}+nY+B)^{h}_{x_w} = K^h_{X_K/\overline{x}_w}$.

Considering the same claim with $\overline{D}+B$ on the right replaced by $\overline{D}+mY+B$ and $\overline{D}+nY+B$ in the middle replaced by $\overline{D}+(m+1)Y+B$, for $m = 0, \ldots, n-1$, we are reduced to the case $n = 1$. Let

$$\mathcal{J} = V_d, \overline{X}_K|\overline{D}+B/V_d, \overline{X}_K|\overline{D}+Y+B.$$ 

It is supported on $Y$, and by [27, Corollary 2.12] we have an exact sequence

$$H^{d-1}(\text{Y}_{\text{Zar}}, \mathcal{J}) \to H^d(\overline{X}_{\text{Nis}}, V_d, \overline{X}_K|\overline{D}+Y+B) \to H^d(\overline{X}_{\text{Nis}}, V_d, \overline{X}_K|\overline{D}+B) \to 0.$$ 

Then, we obtain surjections

$$\bigoplus_{w \in W} (V_d, X_K|D_K)_{x_{d-1}} \to \bigoplus_{w \in W} (V_d, X_K|D_K)_{x'_w} \to \mathcal{J}_{x_{d-1}}$$

$$c_{x'_w, 0}^{\text{Zar}} \to \bigoplus_{w \in W} H^{d-1}_{w}(\text{Y}_{\text{Zar}}, \mathcal{J}) \to H^{d-1}(\text{Y}_{\text{Zar}}, \mathcal{J}),$$

where the last map is surjective due to (6.2.1) and $c_{x'_w, 0}^{\text{Zar}}$ is the map (3.7.1) and it is surjective by Remark 3.7. This proves Claim and hence the theorem.

**Remark 6.3** If $F = W_n$ is the sheaf of $p$-typical Witt vectors of length $n$, where $p = \text{char}(k)$, the equivalence (i) $\iff$ (iii) in Theorem 6.1 is reminiscent of [13, Proposition 7.5] (the case $^{b}\text{fil}_m^F$). Though in loc. cit. $k$ is assumed to be algebraically closed and it is not necessary to consider all function fields $K/k$.

**Definition 6.4** For $(X, D) \in \textbf{MCor}$ with $U = X - |D|$, We define

$$F^{\text{LS}}(X, D) := \left\{ a \in F(U) \mid \begin{array}{l}
(a_K, \beta)_{X_K/k, \Delta} = 0 \text{ for all } K/k \\
\text{for all } \chi = (x_0, \ldots, x_d) \in \text{mc}(X_K) \\
\text{for all } \beta \in (V_d, X_K|D_K)_{x_{d-1}}
\end{array} \right\}.$$ 

By Theorem 6.1 we always have an inclusion $\tilde{F}(X, D) \subset F^{\text{LS}}(X, D)$ and this is an equality if $X$ has a smooth projective compactification $\overline{X}$ such that $\overline{X} \setminus U$ is SNCD.
6.5 We say that a reciprocity sheaf $F$ has \textit{level} $n \geq 0$, if for any smooth $k$-scheme $X$ and any $a \in F(\mathbb{A}^1 \times X)$ the following implication holds:

$$a_{\mathbb{A}^1_z} \in F(z) \subset F(\mathbb{A}^1_z), \text{ for all } z \in X_{(\leq n-1)} \implies a \in F(X) \subset F(\mathbb{A}^1 \times X),$$

where $a_{\mathbb{A}^1_z}$ denotes the restriction of $a$ to $\mathbb{A}^1_z = \mathbb{A}^1 \times z \subset \mathbb{A}^1 \times X$, $X_{(\leq n-1)}$ denotes the set of points in $X$ whose closure has dimension $\leq n - 1$, and for a smooth scheme $S$ we identify $F(S)$ with its image in $F(\mathbb{A}^1 \times S)$ via pullback along the projection map. This is equivalent to the motivic conductor of $F$ having level $n$ in the language of [29]. The $\mathbb{A}^1$-invariant sheaves with transfers are precisely the reciprocity sheaves of level 0. By [29, Part 2], the presheaves $X \mapsto G(X)$, with $G$ a commutative algebraic group, $X \mapsto \Hom(\pi_1^{ab}(X), \mathbb{Q}/\mathbb{Z})$, $X \mapsto \text{Lisse}^1(X)$, the lisse $\mathbb{Q}$-sheaves of rank 1, and $X \mapsto \text{Conn}^1_{\text{int}}(X)$, the integrable rank 1 connections on $X$ (char$(k) = 0$), are reciprocity sheaves of level 1; and the presheaves $X \mapsto \Omega^1(X)$, $X \mapsto Z\Omega^2(X)$ (both in char$(k) = 0$), and $X \mapsto H^1(X_{\text{fpf}}, G)$, with $G$ a finite flat $k$-group scheme, are reciprocity sheaves of level 2.

We say that \textit{resolutions of singularities hold over $k$ in dimension $\leq n$}, if for any integral projective $k$-scheme $Z$ of dimension $\leq n$ and any effective Cartier divisor $E$ on $Z$, there exists a proper birational morphism $h: Z' \to Z$ such that $Z'$ is regular and $|h^{-1}(E)|$ has simple normal crossings. This is known to hold if char$(k) = 0$ by Hironaka or if $n \leq 3$ by [3].

**Corollary 6.6** Assume $F$ has level $n \geq 0$ and resolutions of singularities hold over $k$ in dimension $\leq n$. Let $(X, D)$ be a modulus pair. Assume $X$ is quasi-projective and set $U = X \setminus |D|$. Let $a \in F(U)$. The following statements are equivalent:

(i) $a \in \tilde{F}(X, D);$  
(ii) $h^*a \in F^{\text{LS}}(Z, h^*D)$, for all $k$-morphisms $h: Z \to X$ with $Z$ smooth, quasi-projective with dim$(Z) \leq n$ such that $|h^*D|$ is SNCD.

**Proof** By (HS3) we only have to show the implication (ii) $\implies$ (i). Let $h: Z \to X$ be as in (ii). By resolution of singularity in dimension $\leq n$, Theorem 6.1 together with Theorem 2.5 (2) imply $F^{\text{LS}}(Z, h^*D) = F_{\text{gen}}(Z, h^*Z) = \tilde{F}(Z, h^*Z)$. Hence $a \in \tilde{F}(X, D)$, by [28, Corollary 6.10].

**Remark 6.7** Note that if char$(k) = 0$ or char$(k) = p > 0$ and $F$ has level $\leq 3$ (see the examples listed in 6.5 and also [28, Example 6.11 (5)]), then Corollary 6.6 yields unconditional results.

7 A diagonal argument

We give a refinement of Theorem 6.1 in the case where $D$ is reduced, see Proposition 7.3 below. In this section $k$ is a perfect field and $F \in \text{RSC}_{\text{Nis}}$.

**Lemma 7.1** Let $S$ be a smooth integral $k$-scheme of dimension $d - 1$ and $X = \mathbb{A}^1_S = S[z]$. Let $L = k(X)$ be the function field and set $X_L = X \otimes_k L$. Write

$$t = z \otimes 1, \quad s = 1 \otimes z \in \mathcal{O}_{X_L}.$$
Let \( \eta \in (S \times_k S)^{(d-1)} \) be the generic point of the diagonal and let \( \theta_1, \ldots, \theta_{d-1} \) be a regular sequence of parameters in \( A := \bigcup_{i=1}^{d} S \times_k S, \eta \). Then \( A[t, s]/(t, \theta_1, \ldots, \theta_{d-1-i}) \) is integral for \( i = 0, \ldots, d - 1 \). We denote by \( x_i \in X_L \subset X \times_k X \), the generic point of the image of the natural map

\[
\text{Spec } A[t, s]/(t, \theta_1, \ldots, \theta_{d-1-i}) \rightarrow X \times_k X,
\]

and by \( x_d \in X_L^{(0)} \) the generic point of the component containing \( x_0 \). Set

\[
\beta_0 := \left\{ \frac{t - s}{t - 1}, \theta_1, \ldots, \theta_{d-1} \right\}, \quad \gamma_0 := \{ s, \theta_1, \ldots, \theta_{d-1} \} \in K^M_d(k(x_d)).
\]

Then \( \underline{x} = (x_0, \ldots, x_d) \in \text{mc}(X_L) \) and

\[
\begin{align*}
\pm (a_L, \beta_0)_{X_L/L, \underline{x}} &= j^*(a - \lambda^*a) \in F(L), \quad \text{for all } a \in F(P^1 \setminus 0_S, \infty_S), \quad (7.1.1) \\
(a_L, \gamma_0)_{X_L/L, \underline{x}} &= 0, \quad \text{for all } a \in \widetilde{F}(P^1_S \setminus 0_S), \quad (7.1.2)
\end{align*}
\]

where \( a_L \in F(X_L[t^{-1}]) \) denotes the restriction of \( a \) to \( X_L \), \( j: \text{Spec } L \rightarrow X \) is the inclusion of the generic point, and the map \( \lambda \) is the composition

\[
X \xrightarrow{\text{proj}} S \cong V(z - 1) \hookrightarrow X.
\]

**Proof** By, e.g., [20, Theorems 14.2 and 3] the ring \( A[t, s]/(t, \theta_1, \ldots, \theta_{d-1-i}) \) is integral, regular and of dimension \( i + 1 \). It follows that \( x_i \in X_L \) is a point of dimension \( i \) so that \( x \in \text{mc}(X_L) \). We first show (7.1.1). Let \( a \in \widetilde{F}(P^1_S \setminus 0_S) \). Assume \( d - 1 \geq 1 \) and set \( \underline{x}' = (x_0, \ldots, x_{d-2}, x_d) \in \text{mc}_{d-1}(X_L) \). By (HS4)

\[
(a_L, \beta_0)_{X_L/L, \underline{x}} = - \sum_{y \in b(\underline{x}') \setminus y \neq x_{d-1}} (a_L, \beta_0)_{X_L/L, \underline{y}}(y).
\]

The maximal ideal in \( \mathcal{O}_{X_L, x_{d-2}} \) is generated by \( t, \theta_1 \) and thus the only point \( y \in X_L^{(1)} \) with \( y > x_{d-2} \), at which \( \beta_0 \) is not regular is given by \( z_{d-1} = \text{Spec } \mathcal{O}_{X_L, x_{d-2}}/(\theta_1) \), whereas \( x_{d-1} = \text{Spec } \mathcal{O}_{X_L, x_{d-2}}/(t) \) is the only such point at which \( a_L \) is not regular. Thus (HS2) yields

\[
(a_L, \beta_0)_{X_L/L, \underline{x}} = - (a_L(z_{d-1}), \beta_1)_{Z_{d-1}/L, (x_0, \ldots, x_{d-2}, z_{d-1})},
\]

where \( Z_{d-1} = \overline{\{z_{d-1}\}} \subset X_L \), and

\[
\beta_1 = \partial_{z_{d-1}} \beta_0 = \pm \left\{ \frac{t - s}{t - 1}, \theta_2, \ldots, \theta_{d-1} \right\} \in K^M_{d-1}(L(Z_{d-1})).
\]
If \( d - 2 = 0 \) we stop, if \( d - 2 \geq 1 \) we observe that \( a_L(z_{d-1}) \in \tilde{F}(Z_{d-1}[t^{-1}]) \) and we proceed by applying (HS4) and (HS2) again. Iterating yields

\[
(a_L, \beta)_{X_L/L, \tilde{\gamma}} = \pm \left( a_L(z_1), \frac{t-s}{t-1} \right)_{Z_1/L, (x_0, z_1)},
\]

where \( z_1 \in Z_1 \) is the generic point of \( \text{Spec} \mathcal{O}_{X_L, x_0}/(\theta_1, \ldots, \theta_{d-1}) \) and \( Z_1 \) is its closure in \( X_L \). By the choice of the \( \theta_i \), we have \( Z_1 = \text{Spec} L[t] \subset \mathbf{P}^1_L \) and hence we may identify \( x_0 \) with \( 0_L \in \mathbf{P}^1_L \). Applying (HS1) and (HS4) one more time we find

\[
(a_L, \beta_0)_{X_L/L, \tilde{\gamma}} = \pm \sum_{y \in \mathbf{P}_L \setminus 0_L} \left( a_L(z_1), \frac{t-s}{t-1} \right)_{\mathbf{P}^1_L/L, (y, z_1)}.
\]

Note that under the identification \( Z_1 = \text{Spec} L[t] \subset \mathbf{P}^1_L \), the element \( a_L(z_1) \) corresponds to the pullback of \( a \) along \( \mathbf{P}^1_L \rightarrow \mathcal{O}_S \) induced by the natural inclusions \( \mathcal{O}_S \subset k(S) \subset k(S)(s) = L \). Thus \( a_L(z_1) \in \tilde{F}(\mathbf{P}^1_L \setminus 0_L, \infty_L) \) by the choice of \( a \) in (7.1.1). Since \( \frac{t-s}{t-1} \in (V_1, \mathbf{p}^1_{SL})_{\infty_L} \), we obtain by (HS3) and (HS2)

\[
(a_L, \beta_0)_{X_L/L, \tilde{\gamma}} = \pm (a_L(z_1)|_{t-s=0} - a_L(z_1)|_{t-1=0}) \in F(L).
\]

This yields (7.1.1). Now assume \( a \in F(\mathbf{P}^1_S \setminus 0_S) \). The same argument as above with \( \beta_0 \) replaced by \( \gamma_0 \) yields

\[
(a_L, \gamma_0)_{X_L/L, \tilde{\gamma}} = \pm \sum_{y \in \mathbf{P}_L \setminus 0_L} (a_L(z_1), s)_{\mathbf{P}^1_L/L, (y, z_1)}.
\]

This vanishes by (HS2) since \( a_L(z_1) \in F(\mathbf{P}^1_L \setminus 0_L) \) and \( s \in L^\times \). Hence (7.1.2).

\[
\square\]

Corollary 7.2 Let \( S \) be a smooth integral \( k \)-scheme of dimension \( d - 1 \) and \( X = \mathbb{A}^1_S = S[z] \). Let \( L = k(X) \) be the function field and set \( X_L = X \otimes_k L \) and \( S_L = S \otimes_k L \). Let \( \iota : S_L \rightarrow \mathbb{V}(L) \) be the closed immersion defined by \( t = 0 \), where \( t = z \otimes 1 \). Denote by \( s_0 \in S_L \) the image of the generic point of the diagonal in \( S \times S \) under the map \( S \times_k S \rightarrow S \times_k X \rightarrow S_L \) where the first map is the base change of the closed immersion \( S \rightarrow X \) defined by \( z = 0 \). Denote by \( \eta \in X_L \) the generic point of the irreducible component containing \( \iota(s_0) \). Let \( a \in \tilde{F}(\mathbf{P}^1_S \setminus 0_S, \infty_S) \subset F(X[z^{-1}]) \).

1. Assume for all \( y = (y_0, \ldots, y_{d-1}) \in \text{mc}(S_L) \) with \( y_0 = s_0 \) we have

\[
(a_L, \beta)_{X_L/L, (\iota(y), \eta)} = 0, \quad \text{for all } \beta \in K_d^M((O_{X_L}, \iota(y_{d-1}))).
\]

Then \( a \in \text{Im}(F(S) \rightarrow \tilde{F}(\mathbf{P}^1_S \setminus 0_S, \infty_S))) \).

2. Assume for all \( y = (y_0, \ldots, y_{d-1}) \in \text{mc}(S_L) \) with \( y_0 = s_0 \) we have

\[
(a_L, \beta)_{X_L/L, (\iota(y), \eta)} = 0, \quad \text{for all } \beta \in (V_d.X_L|_{0_S L})_{\iota(y_{d-1})}.
\]

Then \( a \in \tilde{F}(\mathbf{P}^1_S, 0_S + \infty_S) \).
Proof By (7.1.1) and with the notation from there, the condition in (1) implies $j^*a = j^*\lambda^*a$. Since $j^*$: $F(X) \to F(L)$ is injective, by [31, Theorem 3.1], we obtain the first statement. We show the statement in (2). By [31, (5.7)] (with 0 and $\infty$ interchanged),

$$F(P_1^0S, 0S + \infty S) \simeq F(P_1^0 \setminus 0S, \infty S).$$

Hence $a = b + c$, for some $b \in F(P_1^0S, 0S + \infty S)$ and $c \in F(P_1^0S \setminus 0S)$. It suffices to show $c \in F(P_1^0S)$. Let the notations be as in Lemma 7.1. We have

$$\delta_0 := \beta_0 - \gamma_0 = \left\{ \frac{1 - s^{-1}t}{t - 1}, \theta_1, \ldots, \theta_{d-1} \right\} \in (Vd, X_L|_{0S_L})_{x_{d-1}}.$$

Thus (HS3) yields $(b_L, \delta_0)_{X_L/L, x} = 0$; furthermore $(c_L, \gamma_0)_{X_L/L, x} = 0$, by (7.1.2). Thus the assumption yields $0 = (a_L, \delta_0)_{X_L/L, x} = (c_L, \beta_0)_{X_L/L, x}$. Hence $c \in F(S)$ by (7.1.1).

The following proposition is a version of the implication (iii) $\Rightarrow$ (i) in Theorem 6.1 for a modulus pair $(X, D)$ with $X$ smooth and $D$ a reduced SNCD, which does not require the existence of a smooth compactification $\overline{X}$ of $X$ such that $D$ is the restriction of an SNCD on $\overline{X}$. Besides Corollary 7.2, an essential ingredient is [32, Corollary 2.5]. Note that though results of the present paper are used in [32], this is not the case in Section 2 of loc. cit., hence there is no circular argument.

**Proposition 7.3** Let $X \in \text{Sm}$ and assume $D$ is a reduced SNCD on $X$. Let $U \subset X$ be an open subset containing all the generic points of $D$. Let $a \in F(X \setminus D)$.

(1) Assume for all function fields $K/k$ and all $\underline{x} = (x_0, \ldots, x_d) \in \text{mc}(U_K)$ with $x_{d-1} \in D_K^{(0)}$, we have

$$(a, \beta)_{X_K/k, \underline{x}} = 0, \text{ for all } \beta \in K^M_d(O_{X_K, x_{d-1}}).$$

Then $a \in F(X)$.

(2) Assume for all function fields $K/k$ and all $\underline{x} = (x_0, \ldots, x_d) \in \text{mc}(U_K)$ with $x_{d-1} \in D_K^{(0)}$, we have

$$(a, \beta)_{X_K/k, \underline{x}} = 0, \text{ for all } \beta \in (V_d, X_K|_{D_K})_{x_{d-1}}.$$

Then $a \in \widetilde{F}(X, D)$.

**Proof** We only prove (2), the proof of (1) is similar. By [32, Corollary 2.5] there is an exact sequence

$$0 \to \widetilde{F}(X, D) \to F(X \setminus D) \to \bigoplus_{\eta \in D^{(0)}} \frac{F(O_{X, \eta}^h \setminus \eta)}{\widetilde{F}(O_{X, \eta}^h, \eta)}.$$
This reduces us to the case $X$ and $D$ smooth, affine, connected, with generic point $\eta \in D$ and it suffices to show that the condition in (2) implies $a \in \tilde{F}(\mathcal{O}^h_{X, \eta}, \eta)$.

**Claim** We may further assume that there is a morphism $X \to D$, such that $D \hookrightarrow X \to D$ is the identity.

Indeed, by [1, Lemma 7.13] (which is a variant of [31, Lemma 8.5] and relies on a result of Elkik [4]), we find an étale morphism $u: X' \to X$ and a morphism $X' \to D$, such that $u$ induces an isomorphism $u^{-1}(D) \tilde{\to} D$ and the composition $D \hookrightarrow X' \to D$ is the identity. Thus it suffices to show

$$(u^*a)_K \gamma |_{X'_{K, \xi}} = 0 \quad \text{for all} \quad \gamma \in (V_d, X'_{K, [ \xi]})_{x_d - 1}, \tag{7.3.1}$$

for $K/k$ and all $x = (x_0, \ldots, x_d) \in \text{mc}(X'_K)$ with $x_{d - 1} \in D_K^{(0)}$. By (HS3), (7.3.1) holds for $\gamma \in (V_d, X'_{K, [ nD_K]})_{x_d - 1}$ for some big enough integer $n$. Since that natural map

$$(V_d, X_K | D_K)_{x_d - 1} \xrightarrow{u^*} (V_d, X'_{K} | D_K)_{x_d - 1} \tag{7.3.2}$$

is surjective, it suffices to show (7.3.1) for $\gamma = u^* \beta$ with $\beta \in (V_d, X_K | D_K)_{x_d - 1}$. Since maximal chains $x \in \text{mc}(X'_K)$ with $x_{d - 1} \in D_K$ correspond uniquely to maximal chains in $X_K$ whose 1-codimensional point is the generic point of $D_K$, the vanishing (7.3.1) follows in this case directly from the vanishing in (2) and the definition of the map $c_x$ in (3.4.2), which is used in Definition 5.1. This proves Claim.

Shrinking around the generic point of $D$, we may assume further $D = \text{Div}(t)$ for some $t \in \mathcal{O}(X)$ defining an étale morphism $v: X \to \mathbb{A}^1_D$, which satisfies $v^{-1}(0_D) = D$. By [31, Lemmas 4.2, 4.4] the morphism

$$v^*: F(\mathbb{A}^1_D \setminus 0_D) \to F(X \setminus D)$$

becomes an isomorphism when we shrink $D$ to its generic point. Thus we may assume $a = v^*b$ for some $b \in F(\mathbb{A}^1_D \setminus 0_D)$. By [31, Lemma 5.9], we have

$$F(\mathbb{A}^1_D \setminus 0_D) \xrightarrow{\cong} \tilde{F}(\mathbb{P}^1_D \setminus 0_D, \infty_D),$$

so we may assume $b \in \tilde{F}(\mathbb{P}^1_D \setminus 0_D, \infty_D)$. It remains to show $b \in \tilde{F}(\mathbb{P}^1_D, 0_D + \infty_D)$. By Corollary 7.2(2) it therefore suffices to show, that for all $K/k$ and all $x = (x_0, \ldots, x_d) \in \text{mc}(\mathbb{A}^1_{D_K})$ with $x_{d - 1} \in D_K^{(0)}$ (where $D_K$ is embedded in $\mathbb{A}^1_{D_K}$ along the zero-section) we have

$$(b, \gamma)_{\mathbb{A}^1_{D_K} / [ \xi]} = 0, \quad \text{for all} \quad \gamma \in (V_d, \mathbb{A}^1_{D_K} | 0_{D_K})_{x_d - 1}. \tag{7.3.3}$$
To this end we first observe that by a similar argument as was used around (7.3.2), the condition in (2) also holds with \((V_d,X|D_K)^h_{x_{d-1}}\) replaced by its Nisnevich stalk \((V_d,A^1|0D_K)^h_{x_{d-1}}\) and we may as well consider the Nisnevich stalk of \(V_d,A^1|0D_K\) in (7.3.3). But \(v^\ast\) induces an isomorphism \(\mathcal{O}^h_{A^1|0D_K} \xrightarrow{\sim} \mathcal{O}^h_{X|x_{d-1}}\). Thus the norm map

\[
\text{Nm} : (V_d,X|D_K)^h_{x_{d-1}} \xrightarrow{\sim} (V_d,A^1|0D_K)^h_{x_{d-1}}.
\]

Now let \(\gamma \in \text{mc}(A^1|D_K)\) be as in (7.3.3). By the construction we can lift \(x\) uniquely to \(y \in \text{mc}(X_K)\). Take \(\beta \in (V_d,X|D_K)^h_{x_{d-1}}\) with \(\text{Nm}(\beta) = \gamma\). Then

\[
0 = (a, \beta)_{X/K} = (v^\ast b, \beta)_{X/K} = (b, \text{Nm}(\beta))_{A^1/\mathbb{K}} = (b, \gamma)_{A^1/\mathbb{K}}
\]

where the first equality holds by the condition in (2) and the third equality holds by (HS5′) in Corollary 5.5. This yields (7.3.3) and completes the proof.

\[\Box\]

Acknowledgements The authors thank the referee for his comments which helped to clarify the exposition.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Binda, F., Rülling, K., Saito, S.: On the cohomology of reciprocity sheaves. Forum Math. Sigma \textbf{10}, e72 (2022)
2. Bloch, S., Esnault, H.: Gauss–Manin determinants for rank 1 irregular connections on curves. Math. Ann. \textbf{321}(1), 15–87 (2001)
3. Cossart, V., Piltant, O.: Resolution of singularities of threefolds in positive characteristic. II. J. Algebr. \textbf{321}(7), 1836–1976 (2009)
4. Elkik, R.: Solutions d’équations à coefficients dans un anneau hensélien. Ann. Sci. École Norm. Sup. \textbf{6}, 553–603 (1973)
5. Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Publications Mathématiques de l’IHÉS, vol. 32. Paris (1967)
6. Hartshorne, R.: Local Cohomology. A seminar given by Grothendieck A., Harvard University, Fall, 1961. Lecture Notes in Mathematics, vol. 41. Springer, Berlin (1967)
7. Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. \textbf{12}(4), 501–661 (1979)
8. Kahn, B., Miyazaki, H., Saito, S., Yamazaki, T.: Motives with modulus, I: modulus sheaves with transfers for non-proper modulus pairs. Épijournal Géom. Algébrique \textbf{5}, Art. No. 1 (2021)
9. Kahn, B., Miyazaki, H., Saito, S., Yamazaki, T.: Motives with modulus, II: modulus sheaves with transfers for proper modulus Pairs. Épijournal Géom. Algébrique 5, Art. No. 2 (2021)
10. Kahn, B., Saito, S., Yamazaki, T.: Reciprocity sheaves. Compositio Math. 152(9), 1851–1898 (2016)
11. Kahn, B., Saito, S., Yamazaki, T.: Reciprocity sheaves. II. Homol. Homot. Appl. 24(1), 71–91 (2022)
12. Kato, K.: A generalization of local class field theory by using $K$-groups II. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27(3), 603–683 (1980)
13. Kato, K., Russell, H.: Modulus of a rational map into a commutative algebraic group. Kyoto J. Math. 50(3), 607–622 (2010)
14. Kato, K., Saito, S.: Two-dimensional class field theory. In: Ihara, Y. (ed.) Galois Groups and their Representations (Nagoya, 1981). Advanced Studies in Pure Mathematics, vol. 2, pp. 103–152. North-Holland, Amsterdam (1983)
15. Kato, K., Saito, S.: Global Class Field Theory of Arithmetic Schemes. In: Bloch, S.J., et al. (eds.) Applications of Algebraic $K$-Theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colorado, 1983). Contemporary Mathematics, vol. 55, pp. 255–331. American Mathematical Society, Providence (1986)
16. Kerz, M.: Milnor $K$-theory of local rings with finite residue fields. J. Algebraic Geom. 19(1), 173–191 (2010)
17. Kerz, M., Saito, S.: Chow group of 0-cycles with modulus and higher-dimensional class field theory. Duke Math. J. 165(15), 2811–2897 (2016)
18. Kleiman, S.L.: On the vanishing of $H^n(X, F)$ for an $n$-dimensional variety. Proc. Amer. Math. Soc. 18(5), 940–944 (1967)
19. Lomadze, V.G.: On residues in algebraic geometry. Izv. Akad. Nauk SSSR Ser. Mat. 45(6), 1258–1287 (1981)
20. Matsumura, H.: Commutative Ring Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1989)
21. Nisnevich, Ye.A.: The completely decomposed topology on schemes and associated descent spectral sequences in algebraic $K$-theory. In: Jardine, J.F., Jardine, V.P. (eds.) Algebraic $K$-Theory: Connections with Geometry and Topology. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 279, pp. 241–342. Kluwer, Dordrecht (1989)
22. Paršin, A.N.: On the arithmetic of two-dimensional schemes. I. Distributions and residues. Izv. Akad. Nauk SSSR Ser. Mat. 40(4), 736–773 (1976)
23. Rost, M.: Chow groups with coefficients. Doc. Math. 1(16), 319–393 (1996)
24. Rülling, K.: Erratum to: The generalized de Rham–Witt complex over a field is a complex of zero-cycles. J. Algebraic Geom. 16(1), 109–169 (2007)
25. Rülling, K.: The generalized de Rham–Witt complex over a field is a complex of zero-cycles. J. Algebraic Geom. 16(1), 109–169 (2007)
26. Rülling, K., Saito, S.: Ramification theory for reciprocity sheaves, III, Abbes–Saito formula (2022). arXiv:2204.10637
27. Rülling, K., Saito, S.: Cycle class maps for Chow groups of zero-cycles with modulus. J. Pure Appl. Algebra 227(5), 107282 (2023)
28. Rülling, K., Saito, S.: Ramification theory of reciprocity sheaves, I: Zariski–Nagata purity. J. Reine Angew. Math. 797, 41–78 (2023)
29. Rülling, K., Saito, S.: Reciprocity sheaves and their ramification filtrations. J. Inst. Math. Jussieu 22(1), 71–144 (2023)
30. Rülling, K., Sugiyama, R., Yamazaki, T.: Tensor structures in the theory of modulus presheaves with transfers. Math. Z. 300(1), 929–977 (2022)
31. Saito, S.: Purity of reciprocity sheaves. Adv. Math. 366, Art. No. 107067 (2020)
32. Saito, S.: Reciprocity sheaves and logarithmic motives. Compositio Math. 159(2), 355–379 (2023)
33. Serre, J.-P.: Groupes Algébriques et Corps de Classes, 2nd edn. Publications de l’Institut Mathématique de l’Université de Nancago, Hermann, Paris (1984)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.