WELLPOSEDNESS RESULTS
FOR THE SHORT PULSE EQUATION

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Abstract. The short pulse equation provides a model for the propagation of ultra-short light pulses in silica optical fibers. It is a nonlinear evolution equation. In this paper the wellposedness of bounded solutions for the homogeneous initial boundary value problem and the Cauchy problem associated to this equation are studied.

1. Introduction

The short pulse equation which has the form
\begin{equation}
\partial_x \left( \partial_t u - \frac{1}{6} \partial_x u^3 \right) = \gamma u, \quad \gamma > 0,
\end{equation}

up to a scale transformation of its variables, was introduced recently by Schäfer and Wayne [14] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. It provides also an approximation of nonlinear wave packets in dispersive media in the limit of few cycles on the ultra-short pulse scale. Numerical simulations [3] show that the short pulse equation approximation to Maxwell’s equations in the case when the pulse spectrum is not narrowly localized around the carrier frequency is better than the one obtained from the nonlinear Schrödinger equation, which models the evolution of slowly varying wave trains. Such ultra-short plays a key role in the development of future technologies of ultra-fast optical transmission of informations.

In [2] the author studied a new hierarchy of equations containing the short pulse equation (1.1) and the elastic beam equation, which describes nonlinear transverse oscillations of elastic beams under tension. He showed that the hierarchy of equations is integrable. He obtained the two compatible Hamiltonian structures and constructs an infinite series of both local and nonlocal conserved charges. Moreover, he gave the Lax description for both systems. The integrability and the existence of solitary wave solutions have been studied in [12, 13].

Well-posedness and wave breaking for the short pulse equation have been studied in [14] and [10], respectively. Our aim is to investigate the well-posedness in classes of discontinuous functions for (1.1). We consider both the initial boundary value problem (see Section 2) and the Cauchy problem (see Section 3) for (1.1).

Integrating (1.1) in $x$ we gain the integro-differential formulation of (1.1) (see [12])
\begin{equation}
\partial_t u - \frac{1}{6} \partial_x u^3 = \gamma \int_x^\infty u(t, y) dy,
\end{equation}

that is equivalent to
\begin{equation}
\partial_t u - \frac{1}{6} \partial_x u^3 = \gamma P, \quad \partial_x P = u.
\end{equation}
One of the main issues in the analysis of (1.2) is that the equation is not preserving the $L^1$ norm, the unique useful conserved quantities are

$$t \mapsto \int u(t, x)dx, \quad t \mapsto \int u^2(t, x)dx.$$ 

As a consequence the nonlocal source term $P$ and the solution $u$ are a priori only locally bounded. Since we are interested in the bounded solutions of (1.1), some assumptions on the decay at infinity of the initial condition $u_0$ is needed. Regarding the flux function, here we use the cubic one

$$u \mapsto -\frac{u^3}{6}$$

because this is the one that appears in the original short-pulse equation. Anyway all our arguments can be generalized to subcubic genuinely nonlinear fluxes. The genuine nonlinearity assumption is necessary for the compactness argument based on the compensated compactness. The subcubic assumption together with the assumptions on the on the decay at infinity of the initial condition $u_0$ guarantees the boundedness of the solutions.

Our existence argument is based on passing to the limit using a compensated compactness argument [15] in a vanishing viscosity approximation of (1.1):

$$\partial_t u_{\varepsilon} - \frac{1}{6} \partial_x u_{\varepsilon}^3 = \gamma P_{\varepsilon} + \varepsilon \partial_{xx}^2 u_{\varepsilon}, \quad -\varepsilon \partial_{xx}^2 P_{\varepsilon} + \partial_x P_{\varepsilon} = u_{\varepsilon}.$$ 

On the other hand we use the Kružkov doubling of variables method [9] for the uniqueness and stability of the solutions of (1.1).

### 2. The initial boundary value problem

In this section, we augment (1.1) with the boundary condition

$$(2.1) \quad u(t, 0) = 0, \quad t > 0,$$

and the initial datum

$$(2.2) \quad u(0, x) = u_0(x), \quad x > 0.$$ 

We assume that

$$(2.3) \quad u_0 \in L^\infty(0, \infty) \cap L^1(0, \infty), \quad \int_0^\infty u_0(x)dx = 0.$$ 

On the function

$$(2.4) \quad P_0(x) = \int_0^x u_0(y)dy,$$

we assume that

$$(2.5) \quad \|P_0\|_{L^2(0, \infty)}^2 = \int_0^\infty \left( \int_0^x u_0(y)dy \right)^2 dx < \infty.$$ 

Integrating (1.1) on $(0, x)$ we obtain the integro-differential formulation of the initial-boundary value problem (1.1), (2.1), (2.2) (see [12])

$$(2.6) \begin{cases} \partial_t u - \frac{1}{6} \partial_x u^3 = \gamma \int_0^x u(t, y)dy, & t > 0, \ x > 0, \\ u(t, 0) = 0, & t > 0, \\ u(0, x) = u_0(x), & x > 0. \end{cases}$$
This is equivalent to

\[
\begin{align*}
\partial_t u - \frac{1}{6} \partial_x u^3 &= \gamma P, \quad t > 0, \ x > 0, \\
\partial_x P &= u, \quad t > 0, \ x > 0, \\
u(t, 0) &= 0, \quad t > 0, \\
P(t, 0) &= 0, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x > 0.
\end{align*}
\]  

(2.7)

Due to the regularizing effect of the \( P \) equation in (2.7) we have that

\[
u \in L^\infty((0, T) \times (0, \infty)) \implies P \in L^\infty((0, T); W^{1, \infty}(0, \infty)), \quad T > 0.
\]  

(2.8)

Therefore, if a map \( u \in L^\infty((0, T) \times (0, \infty)), T > 0, \) satisfies, for every convex map \( \eta \in C^2(\mathbb{R}) \),

\[
\partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0,
\]  

(2.9)

in the sense of distributions, then \([7, \text{Theorem 1.1}]\) provides the existence of strong trace \( u_0^\tau \) on the boundary \( x = 0 \).

**Definition 2.1.** We say that \( u \in L^\infty((0, T) \times (0, \infty)), T > 0, \) is an entropy solution of the initial-boundary value problem (1.1), (2.1), and (2.2) if

i) \( u \) is a distributional solution of (2.6) or equivalently of (2.7);

ii) for every convex function \( \eta \in C^2(\mathbb{R}) \) the entropy inequality (2.9) holds in the sense of distributions in \((0, \infty) \times (0, \infty)\);

iii) for every convex function \( \eta \in C^2(\mathbb{R}) \) with corresponding \( q \) defined by \( q'(u) = -\frac{u^2}{2} \eta'(u) \), the boundary entropy condition

\[
q(u_0^\tau(t)) - q(0) - \eta'(0) \left( \frac{(u_0^\tau(t))^3}{6} - c^3 \right) \leq 0
\]  

(2.10)

holds for a.e. \( t \in (0, \infty) \), where \( u_0^\tau(t) \) is the trace of \( u \) on the boundary \( x = 0 \).

We observe that the previous definition is equivalent to the following family of inequalities (see [1]):

\[
\int_0^\infty \int_0^\infty (|u - c| \partial_t \phi - \text{sign}(u - c) \left( \frac{u^3}{6} - \frac{c^3}{6} \right) \partial_x \phi) \, dt \, dx \\
+ \gamma \int_0^\infty \int_0^\infty \text{sign}(u - c) P \, dt \, dx \\
+ \int_0^\infty \text{sign}(c) \left( \frac{(u_0^\tau(t))^3}{6} - \frac{c^3}{6} \right) \, dt \\
+ \int_0^\infty |u_0(x) - c| \phi(0, x) \, dx \geq 0,
\]  

(2.11)

for every non-negative test function \( \phi \in C^\infty(\mathbb{R}^2) \) with compact support, and for every \( c \in \mathbb{R} \).

The main result of this section is the following theorem.
Theorem 2.1. Assume (2.3) and (2.5). The initial-boundary value problem (1.1), (2.1) and (2.2) possesses an unique entropy solution $u$ in the sense of Definition 2.1. In particular, we have that

$$\int_0^\infty u(t,x)dx = 0, \quad t > 0.$$  \hfill (2.12)

Moreover, if $u$ and $v$ are two entropy solutions (1.1), (2.1), (2.2) in the sense of Definition 2.1, the following inequality holds

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(0,R)} \leq e^{C(T)t} \|u(0,\cdot) - v(0,\cdot)\|_{L^1(0,R+C(T)t)},$$  \hfill (2.13)

for almost every $0 < t < T$, $R > 0$, and some suitable constant $C(T) > 0$.

A similar result has been proved in [3, 8] in the context of locally bounded solutions.

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $0 < \varepsilon < 1$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [4]

$$\begin{cases}
\partial_t u_\varepsilon - \frac{1}{2} u_\varepsilon^2 \partial_x u_\varepsilon = \gamma P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, \quad x > 0, \\
-\varepsilon \partial_{xx}^2 P_\varepsilon + \partial_x P_\varepsilon = u_\varepsilon, & t > 0, \quad x > 0, \\
u_\varepsilon(t,0) = 0, & t > 0, \\
P_\varepsilon(t,0) = 0, & t > 0, \\
u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x > 0,
\end{cases} \hfill (2.14)$$

where $u_{\varepsilon,0}$ is a $C^\infty$ approximation of $u_0$ such that

$$\begin{align*}
\|u_{\varepsilon,0}\|_{L^2(0,\infty)} &\leq \|u_0\|_{L^2(0,\infty)}, & \|u_{\varepsilon,0}\|_{L^\infty(0,\infty)} &\leq \|u_0\|_{L^\infty(0,\infty)}, \\
\|P_{\varepsilon,0}\|_{L^2(0,\infty)} &\leq \|P_0\|_{L^2(0,\infty)}, & \varepsilon \|\partial_x P_{\varepsilon,0}\|_{L^2(0,\infty)} &\leq C_0,
\end{align*} \hfill (2.15)$$

and $C_0$ is a constant independent on $\varepsilon$.

Let us prove some a priori estimates on $u_\varepsilon$ and $P_\varepsilon$, denoting with $C_0$ the constants which depend on the initial datum, and $C(T)$ the constants which depend also on $T$.

Arguing as [4], we obtain the following results

**Lemma 2.1.** For each $t \in (0,\infty)$,

$$P_\varepsilon(t,\infty) = \partial_x P_\varepsilon(t,\infty) = 0.$$  \hfill (2.16)

Moreover,

$$\varepsilon^2 \left\| \partial_{xx}^2 P_\varepsilon(t,\cdot) \right\|_{L^2(0,\infty)}^2 + \varepsilon (\partial_x P_\varepsilon(t,0))^2 + \left\| \partial_x P_\varepsilon(t,\cdot) \right\|_{L^2(0,\infty)}^2 = \left\| u_\varepsilon(t,\cdot) \right\|_{L^2(0,\infty)}^2. \hfill (2.17)$$

**Lemma 2.2.** For each $t \in (0,\infty)$,

$$\int_0^\infty u_\varepsilon(t,x)dx = \varepsilon \partial_x P_\varepsilon(t,0), \hfill (2.18)$$

$$\sqrt{\varepsilon} \left\| \partial_x P_\varepsilon(t,\cdot) \right\|_{L^\infty(0,\infty)} \leq \|u(t,\cdot)\|_{L^2(0,\infty)}, \hfill (2.19)$$

$$\int_0^\infty u_\varepsilon(t,x)P_\varepsilon(t,x)dx \leq \|u(t,\cdot)\|_{L^2(0,\infty)}^2. \hfill (2.20)$$
Lemma 2.3. For each $t \in (0, \infty)$, the inequality holds
\begin{equation}
\|u_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 + 2 \epsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_{\epsilon}(s, \cdot)\|_{L^2(0,\infty)}^2 \, ds \leq e^{2\gamma t} \|u_0\|_{L^2(0,\infty)}^2.
\end{equation}

In particular, we have
\begin{equation}
\epsilon \|\partial_{xx}^2 P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)} + \|\partial_x P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)} \leq e^{\gamma t} \|u_0\|_{L^2(0,\infty)},
\end{equation}

Moreover, we get
\begin{equation}
\|P_{\epsilon}(t, \cdot)\|_{L^\infty(0,\infty)} \leq \sqrt{2e^{\gamma t} \|u_0\|_{L^2(0,\infty)} \|P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)}}.
\end{equation}

Proof. Due to (2.14) and (2.20),
\begin{align*}
\frac{d}{dt} \int_0^\infty u_{\epsilon}^2 \, dx &= 2 \int_0^\infty u_{\epsilon} \partial_t u_{\epsilon} \, dx \\
&= 2 \epsilon \int_0^\infty u_{\epsilon} \partial_{xx}^2 u_{\epsilon} \, dx + \int_0^\infty u_{\epsilon}^3 \partial_x u_{\epsilon} \, dx + 2 \gamma \int_0^\infty u_{\epsilon} P_{\epsilon} \, dx \\
&\leq -2 \epsilon \int_0^\infty (\partial_x u_{\epsilon})^2 \, dx + 2 \gamma \|u_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)}^2.
\end{align*}

The Gronwall Lemma and (2.15) give (2.21). (2.22) follows from (2.17), (2.19) and (2.21).

Finally, we prove (2.23). Due to (2.14) and H"older inequality,
\begin{equation}
P_{\epsilon}^2(t, x) \leq 2 \int_0^\infty |P_{\epsilon}(t, x)| |\partial_x P_{\epsilon}(t, x)| \, dx \leq 2 \|P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)} \|\partial_x P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)},
\end{equation}

that is
\begin{equation}
|P_{\epsilon}(t, x)| \leq \sqrt{2 \|P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)} \|\partial_x P_{\epsilon}(t, \cdot)\|_{L^2(0,\infty)}}.
\end{equation}

(2.22) and (2.24) give (2.23). \hfill \Box

Lemma 2.4. For every $t \in (0, \infty)$,
\begin{equation}
\|u_{\epsilon}(t, \cdot)\|_{L^\infty(0,\infty)} \leq \|u_0\|_{L^\infty(0,\infty)} + \gamma \int_0^t \|P_{\epsilon}(s, \cdot)\|_{L^\infty(0,\infty)} \, ds.
\end{equation}

Proof. Due to (2.14),
\begin{equation}
\partial_t u_{\epsilon} - \frac{1}{2} u_{\epsilon}^2 \partial_x u_{\epsilon} - \epsilon \partial_{xx}^2 u_{\epsilon} \leq \gamma |P_{\epsilon}(t, x)| \leq \gamma \|P_{\epsilon}(t, \cdot)\|_{L^\infty(0,\infty)}.
\end{equation}

Since the map
\begin{equation}
F(t) := \|u_0\|_{L^\infty(0,\infty)} + \gamma \int_0^t \|P_{\epsilon}(s, \cdot)\|_{L^\infty(0,\infty)} \, ds, \quad t \in (0, \infty),
\end{equation}
solves the equation
\begin{equation}
\frac{dF}{dt} = \gamma \|P_{\epsilon}(t, \cdot)\|_{L^\infty(0,\infty)}
\end{equation}
and
\begin{equation}
\max\{u_{\epsilon}(0, x), 0\} \leq F(t), \quad (t, x) \in (0, \infty)^2,
\end{equation}
the comparison principle for parabolic equations implies that
\begin{equation}
u_{\epsilon}(t, x) \leq F(t), \quad (t, x) \in (0, \infty)^2.
\end{equation}
In a similar way we can prove that
\[ u_\varepsilon(t, x) \geq -F(t), \quad (t, x) \in (0, \infty)^2. \]

Therefore,
\[
|u_\varepsilon(t, x)| \leq \|u_0\|_{L^\infty(0, \infty)} + \gamma \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(0, \infty)} \, ds,
\]
which gives (2.25).

**Lemma 2.5.** Consider the following function
\[
F_\varepsilon(t, x) = \int_0^x P_\varepsilon(t, y) \, dy, \quad t, x > 0.
\]

We have that
\[
\lim_{x \to \infty} F_\varepsilon(t, x) = \int_0^\infty P_\varepsilon(t, y) \, dy = \frac{\varepsilon}{\gamma} \partial_{xx}^2 P_\varepsilon(t, 0) + \frac{\varepsilon}{\gamma} \partial_x u_\varepsilon(t, 0).
\]

**Proof.** Integrating on \((0, x)\) the first equation of (2.24), we get
\[
\int_0^x \partial_t u_\varepsilon(t, y) \, dy - \frac{1}{6} u_\varepsilon^3(t, x) - \varepsilon \partial_x u_\varepsilon(t, x) + \varepsilon \partial_x u_\varepsilon(t, 0) = \gamma \int_0^x P_\varepsilon(t, y) \, dy.
\]

It follows from the regularity of \(u_\varepsilon\) that
\[
\lim_{x \to \infty} \left( -\frac{1}{6} u_\varepsilon^3(t, x) - \varepsilon \partial_x u_\varepsilon(t, x) \right) = 0.
\]

For (2.28), we have that
\[
\lim_{x \to \infty} \int_0^x \partial_t u_\varepsilon(t, y) \, dy = \int_0^\infty \partial_t u_\varepsilon(t, x) \, dx = \frac{d}{dt} \int_0^\infty u_\varepsilon(t, x) \, dx = \varepsilon \partial_{xx}^2 P_\varepsilon(t, 0).
\]

(2.29), (2.30) and (2.31) give (2.28). \( \square \)

**Lemma 2.6.** Let \(T > 0\). There exists a function \(C(T) > 0\), independent on \(\varepsilon\), such that
\[
\|P_\varepsilon\|_{L^\infty(0, T; L^2(0, \infty))} \leq C(T).
\]

In particular, we have that
\[
\|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)} \leq C(T),
\]
\[
\varepsilon \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)} \leq C(T),
\]
\[
\varepsilon^2 e^{-2\gamma t} \int_0^t e^{-2\gamma s} \left( \partial_{xx}^2 P_\varepsilon(s, 0) + \partial_x u_\varepsilon(s, 0) \right)^2 \, ds \leq C(T),
\]
\[
\|P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \leq C(T),
\]
\[
\|u_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \leq C(T).
\]

Moreover, we get
\[
\varepsilon \left| \int_0^t \int_0^\infty P_\varepsilon \partial_{xx}^2 P_\varepsilon \, ds \, dx \right| \leq C(T), \quad (t, x) \in (0, T) \times (0, \infty).
\]

**Proof.** Let \(0 < t < T\). We begin by observing that, integrating on \((0, x)\) the second equation of (2.14), we get
\[
P_\varepsilon(t, x) = \int_0^x u_\varepsilon(t, y) \, dy + \varepsilon \partial_x P_\varepsilon(t, x) - \varepsilon \partial_x P_\varepsilon(t, 0).
\]
Differentiating with respect to $t$, we have that
\[
\partial_t P_\varepsilon(t, x) = \frac{d}{dt} \int_0^x u_\varepsilon(t, y) dy + \varepsilon \partial_t^2 P_\varepsilon(t, x) - \varepsilon \partial_x^2 P_\varepsilon(t, 0).
\]

It follows from (2.27) and (2.29) that
\[
\partial_t P_\varepsilon(t, x) = \gamma F_\varepsilon(t, x) + \frac{1}{6} u_\varepsilon(t, x)^3 + \varepsilon \partial_x u_\varepsilon(t, x)
\]
\[
- \varepsilon \partial_x u_\varepsilon(t, 0) + \varepsilon \partial_x^2 P_\varepsilon(t, x) - \varepsilon \partial_x^2 P_\varepsilon(t, 0).
\]

Multiplying (2.40) by $P_\varepsilon - \varepsilon \partial_x P_\varepsilon$, we have that
\[
(P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_t P_\varepsilon = \gamma (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) F_\varepsilon - \frac{1}{6} (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) u_\varepsilon^3
\]
\[
- \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x u_\varepsilon(t, 0) + \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x u_\varepsilon
\]
\[
+ \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x^2 P_\varepsilon - \varepsilon (P_\varepsilon - \varepsilon \partial_x P_\varepsilon) \partial_x^2 P_\varepsilon(t, 0).
\]

Integrating (2.41) on $(0, x)$, for (2.14), we get
\[
\int_0^x P_\varepsilon \partial_t P_\varepsilon dy - \varepsilon \int_0^x \partial_x P_\varepsilon \partial_t P_\varepsilon dy
\]
\[
= \gamma \int_0^x P_\varepsilon F_\varepsilon dy - \varepsilon \int_0^x F_\varepsilon \partial_x P_\varepsilon dy + \frac{1}{6} \int_0^x P_\varepsilon u_\varepsilon^3 dy
\]
\[
- \varepsilon \int_0^x \partial_x P_\varepsilon u_\varepsilon(t, 0) dy + \varepsilon \int_0^x \partial_x u_\varepsilon(t, 0) P_\varepsilon
\]
\[
+ \varepsilon \int_0^x \partial_x P_\varepsilon \partial_x u_\varepsilon dy + \varepsilon \int_0^x \partial_x^2 P_\varepsilon dy + \varepsilon \int_0^x \partial_x^2 P_\varepsilon(t, 0) dy.
\]

We observe that, for (2.14),
\[
- \varepsilon \int_0^x \partial_x P_\varepsilon \partial_t P_\varepsilon dy = - \varepsilon \partial_x P_\varepsilon \partial_t P_\varepsilon + \varepsilon \int_0^x \partial_x^2 P_\varepsilon dy.
\]

Therefore, (2.42) and (2.43) give
\[
\int_0^x P_\varepsilon \partial_t P_\varepsilon dy + \varepsilon \int_0^x \partial_x P_\varepsilon \partial_x^2 P_\varepsilon dy
\]
\[
= \varepsilon \partial_x P_\varepsilon \partial_t P_\varepsilon + \gamma \int_0^x P_\varepsilon F_\varepsilon dy - \varepsilon \int_0^x F_\varepsilon \partial_x P_\varepsilon dy
\]
\[
+ \frac{1}{6} \int_0^x P_\varepsilon u_\varepsilon^3 dy - \varepsilon \int_0^x \partial_x P_\varepsilon u_\varepsilon(t, 0) dy + \varepsilon^2 \partial_x u_\varepsilon(t, 0) P_\varepsilon
\]
\[
+ \varepsilon \int_0^x \partial_x P_\varepsilon \partial_x u_\varepsilon dy - \varepsilon^2 \int_0^x \partial_x^2 P_\varepsilon \partial_x u_\varepsilon dy
\]
\[
- \varepsilon \partial_x^2 P_\varepsilon(t, 0) \int_0^x P_\varepsilon dy.
\]

Since
\[
\int_0^\infty P_\varepsilon \partial_t P_\varepsilon dx = \frac{1}{2} \frac{d}{dt} \int_0^\infty P_\varepsilon^2 dx,
\]
\[
\varepsilon^2 \int_0^\infty \partial_{xx}^2 P_\varepsilon \partial_x P_\varepsilon \, dx = \frac{\varepsilon^2}{2} \frac{d}{dt} \int_0^\infty (\partial_x P_\varepsilon)^2 \, dx,
\]
when \( x \to \infty \), for (2.16) and (2.44), we have that
\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty P_\varepsilon^2 \, dx + \frac{\varepsilon^2}{2} \frac{d}{dt} \int_0^\infty (\partial_x P_\varepsilon)^2 \, dx = \gamma \int_0^\infty P_\varepsilon \gamma \int_0^\infty \partial_x P_\varepsilon \, dx + \frac{1}{6} \int_0^\infty P_\varepsilon u_3^2 \, dx
\]
\[- \frac{\varepsilon}{6} \int_0^\infty \partial_x P_\varepsilon u_3^2 \, dx - \varepsilon \partial_x u_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx
\]
\[+ \varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon \, dx + \varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx - \varepsilon \partial_{xx}^2 P_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx.
\]
Due to (2.27) and (2.28),
\[
2\gamma \int_0^\infty P_\varepsilon F_\varepsilon \, dx = 2\gamma \int_0^\infty F_\varepsilon \partial_x F_\varepsilon \, dx = \gamma (F_\varepsilon(t, \infty))^2
\]
\[= \frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0) + \partial_x u_\varepsilon(t, 0))^2,
\]
that is
\[
2\gamma \int_0^\infty P_\varepsilon F_\varepsilon \, dx = \frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0))^2
\]
\[+ \frac{2\varepsilon^2}{\gamma} \partial_x P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0) + \frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t, 0))^2.
\]
Again by (2.28),
\[
-2\varepsilon \partial_x u_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx = -2\frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0)) \partial_x u_\varepsilon(t, 0) - 2\frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t, 0))^2,
\]
\[-2\varepsilon \partial_{xx}^2 P_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx = -2\frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0))^2 - 2\frac{\varepsilon^2}{\gamma} \partial_{xx}^2 P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0).
\]
Therefore, (2.45), (2.46) and (2.47) give
\[
\frac{d}{dt} \left( \int_0^\infty P_\varepsilon^2 \, dx + \varepsilon^2 \int_0^\infty (\partial_x P_\varepsilon)^2 \, dx \right)
\]
\[= \frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0))^2 + \frac{2\varepsilon^2}{\gamma} \partial_{xx}^2 P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0) + \frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t, 0))^2
\]
\[-2\varepsilon \gamma \int_0^\infty \partial_x P_\varepsilon F_\varepsilon \, dx + \frac{1}{3} \int_0^\infty P_\varepsilon u_3^2 \, dx + \frac{\varepsilon}{3} \int_0^\infty \partial_x P_\varepsilon u_3^2 \, dx
\]
\[-2\frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0)) \partial_x u_\varepsilon(t, 0) - 2\frac{\varepsilon^2}{\gamma} (\partial_x u_\varepsilon(t, 0))^2 + 2\varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon \, dx
\]
\[+ 2\varepsilon^2 \int_0^\infty \partial_x P_\varepsilon \partial_x u_\varepsilon \, dx - 2\frac{\varepsilon^2}{\gamma} (\partial_{xx}^2 P_\varepsilon(t, 0))^2 - 2\frac{\varepsilon^2}{\gamma} \partial_{xx}^2 P_\varepsilon(t, 0) \partial_x u_\varepsilon(t, 0),
\]
that is,

\[
\frac{d}{dt} \left( \int_0^\infty P_x^2 \, dx + \varepsilon^2 \int_0^\infty (\partial_x P_x)^2 \, dx \right) + \frac{\varepsilon^2}{\gamma} \left( (\partial_x^2 P_x)(t,0) + \partial_x u_x(t,0) \right)^2
\]

\begin{align}
(2.48) \quad & = -2\varepsilon \gamma \int_0^\infty \partial_x P_x F_x \, dx + \frac{1}{3} \int_0^\infty P_x u_x^3 \, dx - \frac{\varepsilon}{3} \int_0^\infty \partial_x P_x u_x^3 \, dx \\
& \quad + 2\varepsilon \int_0^\infty P_x \partial_x u_x \, dx + 2\varepsilon^2 \int_0^\infty \partial_x P_x \partial_x u_x \, dx.
\end{align}

Thanks to (2.14), (2.16), (2.27) and (2.28),

\[
-2\varepsilon \gamma \int_0^\infty \partial_x P_x F_x \, dx = 2\varepsilon \gamma \int_0^\infty P_x \partial_x F_x \, dx
\]

\begin{align}
(2.49) \quad & = 2\varepsilon \gamma \int_0^\infty P_x^2 \, dx \leq 2\gamma \int_0^\infty P_x^2 \, dx,
\end{align}

while, for (2.14) and (2.16),

\[
2\varepsilon \int_0^\infty P_x \partial_x u_x = -2\varepsilon \int_0^\infty u_x \partial_x P_x \, dx.
\]

Hence, (2.48), (2.49) and (2.50) give

\[
\frac{d}{dt} \left( \| P_x(t,\cdot) \|^2_{L^2(0,\infty)} + \varepsilon^2 \| \partial_x P_x(t,\cdot) \|^2_{L^2(0,\infty)} \right)
\]

\begin{align}
& + \frac{\varepsilon^2}{\gamma} \left( (\partial_x^2 P_x)(t,0) + \partial_x u_x(t,0) \right)^2
\end{align}

\begin{align}
(2.51) \quad & \leq 2\gamma \| P_x(t,\cdot) \|^2_{L^2(0,\infty)} + \frac{1}{3} \int_0^\infty P_x u_x^3 \, dx - \frac{\varepsilon}{3} \int_0^\infty \partial_x P_x u_x^3 \, dx \\
& \quad - 2\varepsilon \int_0^\infty u_x \partial_x P_x \, dx + 2\varepsilon^2 \int_0^\infty \partial_x P_x \partial_x u_x \, dx.
\end{align}

Due to (2.21), (2.23) and the Young inequality,

\[
\frac{1}{3} \int_0^\infty P_x u_x^3 \, dx \leq \frac{1}{3} \left| \int_0^\infty P_x u_x \, dx \right| \leq \int_0^\infty \left| \frac{P_x u_x}{3} \right| u_x^2 \, dx
\]

\begin{align}
(2.52) \quad & \leq \frac{1}{18} \int_0^\infty P_x^2 u_x^2 \, dx + \frac{1}{2} \int_0^\infty u_x^4 \, dx \leq \frac{1}{18} \| P_x(t,\cdot) \|^2_{L^2(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)} \\
& \quad + \frac{1}{2} \| u_x(t,\cdot) \|^2_{L^\infty(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)}
\end{align}

\[
\leq \frac{1}{18} \varepsilon^{2\gamma t} \| u_0 \|^4_{L^2(0,\infty)} \| P_x(t,\cdot) \|_{L^2(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)} + \frac{1}{2} \| u_x(t,\cdot) \|^2_{L^\infty(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)}.
\]

For (2.21) and the Young inequality,

\[
- \frac{\varepsilon}{3} \int_0^\infty \partial_x P_x u_x^3 \, dx \leq \frac{\varepsilon}{3} \left| \int_0^\infty \partial_x P_x u_x \, dx \right| \leq \frac{\varepsilon}{3} \int_0^\infty \left| \partial_x P_x u_x \right| u_x^2 \, dx
\]

\begin{align}
(2.53) \quad & \leq \frac{\varepsilon}{6} \int_0^\infty (\partial_x P_x)^2 u_x^2 \, dx + \frac{\varepsilon}{6} \int_0^\infty u_x^4 \, dx \\
& \leq \frac{\varepsilon}{6} \| \partial_x P_x(t,\cdot) \|^2_{L^\infty(\mathbb{R})} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)} + \frac{1}{6} \| u_x(t,\cdot) \|^2_{L^\infty(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)}
\end{align}

\[
\leq \frac{1}{6} \varepsilon^{2\gamma t} \| u_0 \|^4_{L^2(0,\infty)} + \frac{1}{6} \| u_x(t,\cdot) \|^2_{L^\infty(0,\infty)} \ v^{2\gamma t}_{1}\| u_0 \|^2_{L^2(0,\infty)}.
\]
It follows from (2.21), (2.22) and the Young inequality that
\[
-2\varepsilon \int_0^\infty u_\varepsilon \partial_z P_\varepsilon dx \leq 2\varepsilon \int_0^\infty u_\varepsilon \partial_z P_\varepsilon dx \leq \int_0^\infty \left| \frac{u_\varepsilon}{\sqrt{\gamma}} \right| \left| \partial_z P_\varepsilon \right| dx
\]
(2.54)
\[
\leq \frac{1}{2\gamma} \left\| u_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2 + 2\varepsilon \gamma \left\| P_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2
\]
\[
\leq \frac{1}{2\gamma} e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + 2\varepsilon \gamma \left\| P_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2.
\]
Due to (2.22) and the Young inequality,
\[
2\varepsilon^2 \int_0^\infty \left| \partial_z P_\varepsilon \right| \left| \partial_x u_\varepsilon \right| dx \leq \varepsilon^2 \left\| \partial_z P_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \left\| \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
\leq e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + \varepsilon^2 \left\| \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
(2.55)
(2.51), (2.52), (2.53) and (2.54) give
\[
\frac{d}{dt} G(t) + \frac{\varepsilon^2}{\gamma} \left( \partial_z^2 P_\varepsilon(t,0) + \partial_x u_\varepsilon(t,0) \right)^2
\]
\[
\leq 2\gamma G(t) + \varepsilon^2 \left( \partial_z^2 P_\varepsilon(t,0) + \partial_x u_\varepsilon(t,0) \right)^2
\]
\[
\leq 2\gamma G(t) + \frac{1}{18} e^{3\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^3 \left\| P_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2
\]
\[
+ \frac{2}{3} \left\| u_\varepsilon(t, \cdot) \right\|_{L^\infty(0,\mathbb{R};L^2(0,\infty))}^2 e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + \frac{1}{6} e^{4\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^4
\]
\[
+ \frac{1}{2\gamma} e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + \varepsilon^2 \left\| \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]
that is
\[
\frac{d}{dt} G(t) - 2\gamma G(t) + \frac{\varepsilon^2}{\gamma} \left( \partial_z^2 P_\varepsilon(t,0) + \partial_x u_\varepsilon(t,0) \right)^2
\]
\[
\leq \frac{1}{18} e^{3\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^3 \left\| P_\varepsilon \right\|_{L^\infty(0,T;L^2(0,\infty))} + \frac{2}{3} \left\| u_\varepsilon(t, \cdot) \right\|_{L^\infty(0,\mathbb{R};L^2(0,\infty))}^2 e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2
\]
\[
+ \frac{1}{6} e^{4\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^4 + \frac{1}{2\gamma} e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2 + e^{2\gamma t} \left\| u_0 \right\|_{L^2(0,\infty)}^2
\]
\[
+ \varepsilon^2 \left\| \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]
where
\[
G(t) = \left\| P_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2 + \varepsilon^2 \left\| \partial_x P_\varepsilon(t, \cdot) \right\|_{L^2(0,\infty)}^2.
\]
(2.56)
The Gronwall Lemma, (2.15) and (2.21) give
\[
G(t) + \frac{\varepsilon^2 e^{2\gamma t}}{\gamma} \int_0^t \left( \partial_z^2 P_\varepsilon(s,0) + \partial_x u_\varepsilon(s,0) \right)^2 ds
\]
\[
\leq \left\| P_0 \right\|_{L^2(0,\infty)}^2 e^{2\gamma t} + C_0 e^{2\gamma t} + C_0 e^{2\gamma t} \left\| P_\varepsilon \right\|_{L^\infty(0,T;L^2(0,\infty))} \int_0^t e^{\gamma s} ds
\]
\[
+ C_0 e^{2\gamma t} \int_0^t \left\| u_\varepsilon(s, \cdot) \right\|_{L^\infty(0,\mathbb{R})}^2 ds + C_0 t + C_0 e^{2\gamma t} \int_0^t e^{2\gamma s} ds.
\]
(2.57)
Due to (2.24) and the Young inequality,
\[ \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^2 \leq \|u_0\|_{L^\infty(0, \infty)}^2 + 2\gamma \|u_0\|_{L^\infty(0, \infty)} \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(0, \infty)} \, ds \]
(2.59)
+ \gamma^2 \left( \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(0, \infty)} \, ds \right)^2.
\[ \leq 2\|u_0\|_{L^\infty(0, \infty)}^2 + \gamma^2 \left( \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(0, \infty)} \, ds \right)^2. \]

It follows from (2.23), (2.59) and the Jensen inequality that
\[ \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^2 \leq 2\|u_0\|_{L^\infty(0, \infty)}^2 + \gamma^2 t \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(0, \infty)} \, ds \]
(2.60)
\[ \leq C_0 + \gamma C_0 t \int_0^t e^{\gamma s} \|P_\varepsilon(s, \cdot)\|_{L^2(0, \infty)} \, ds. \]

Therefore
\[ \|u_\varepsilon(t, \cdot)\|_{L^\infty(0, \infty)}^2 \leq C_0 + C(T) \|P_\varepsilon\|_{L^\infty(0, T; L^2(0, \infty))} \]
(2.61)
\[ \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 \]
(2.62)
\[ + \varepsilon^2 \varepsilon^{2\gamma t} \int_0^t e^{-2\gamma s} \left( \partial^2_x P_\varepsilon(s, 0) + \partial_x u_\varepsilon(s, 0) \right)^2 \, ds \]
\[ \leq C(T) + C(T) \|P_\varepsilon\|_{L^\infty(0, T; L^2(0, \infty))}. \]

It follows from (2.62) that
\[ \|P_\varepsilon\|_{L^\infty(0, T; L^2(0, \infty))}^2 = C(T) \|P_\varepsilon\|_{L^\infty(0, T; L^2(0, \infty))} - C(T) \leq 0, \]
which gives (2.32).

(2.33), (2.34) and (2.35) follow from (2.62) and (2.32). (2.23) and (2.33) give (2.36), while (2.27) follows from (2.23) and (2.34).

Let us show that (2.38) holds true. We begin by observing that, thanks to (2.24),
\[ \varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \leq \varepsilon \varepsilon^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(0, \infty)}^2 \, ds \]
(2.63)
\[ \leq \frac{\varepsilon^{2\gamma t}}{2} \|u_0\|_{L^2(0, \infty)}^2 \leq C(T). \]

Multiplying (2.40) by $P_\varepsilon$, an integration on $(0, \infty)$ gives
\[ 2\varepsilon \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon \, dx = \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 - 2\gamma \int_0^\infty P_\varepsilon F_\varepsilon \, dx + 2 \int_0^\infty P_\varepsilon f(u_\varepsilon) \, dx \]
\[ - 2\varepsilon \int_0^\infty P_\varepsilon \partial_x u_\varepsilon \, dx + 2\varepsilon \partial_x u_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx \]
\[ + 2\varepsilon \partial^2_{tx} P_\varepsilon(t, 0) \int_0^\infty P_\varepsilon \, dx. \]

It follows from (2.27), (2.28), (2.46) and (2.47) that
\[ 2\varepsilon \int_0^\infty P_\varepsilon \partial^2_{tx} P_\varepsilon \, dx = \frac{d}{dt} \|P_\varepsilon(t, \cdot)\|_{L^2(0, \infty)}^2 - \frac{\varepsilon^2}{\gamma} (\partial^2_{tx} P_\varepsilon(t, 0))^2 \]
Thus, for (2.63) and (2.64), we have that

\[
\begin{align*}
(2.64) \\
\end{align*}
\]

Due to (2.33) and the Young inequality,

\[
\begin{align*}
\text{It follows from (2.15), (2.21), (2.33), (2.36), (2.37) and (2.52) that}
\end{align*}
\]

that is,

\[
\begin{align*}
2\varepsilon \int_0^\infty P_{\varepsilon}^2 \frac{\partial^2}{\partial t^2} P_{\varepsilon} dx = \frac{d}{dt} \|P_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 & - \frac{\varepsilon^2}{\gamma} \int_0^t (\partial_t^2 P_{\varepsilon}(s, 0) - \partial_x u_{\varepsilon}(s, 0))^2 ds \\
& + \frac{1}{6} \int_0^\infty P_{\varepsilon}^2 dx + 2\varepsilon \int_0^\infty P_{\varepsilon} \partial_x u_{\varepsilon} dx.
\end{align*}
\]

An integration on \((0, t)\) gives

\[
\begin{align*}
2\varepsilon \int_0^t \int_0^\infty P_{\varepsilon}^2 \frac{\partial^2}{\partial t^2} P_{\varepsilon} ds dx &= \|P_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 - \|P_{\varepsilon, 0}\|_{L^2(0,\infty)}^2 \\
& - \frac{\varepsilon^2}{\gamma} \int_0^t (\partial_t^2 P_{\varepsilon}(s, 0) - \partial_x u_{\varepsilon}(s, 0))^2 ds \\
& + \frac{1}{6} \int_0^t \int_0^\infty P_{\varepsilon}^2 dx - 2\varepsilon \int_0^t \int_0^\infty P_{\varepsilon} \partial_x u_{\varepsilon} ds dx.
\end{align*}
\]

It follows from (2.13), (2.21), (2.33), (2.36), (2.37) and (2.52) that

\[
\begin{align*}
2\varepsilon \left| \int_0^t \int_0^\infty P_{\varepsilon}^2 \frac{\partial^2}{\partial t^2} P_{\varepsilon} ds dx \right| & \leq \|P_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 + \|P_{\varepsilon, 0}\|_{L^2(0,\infty)}^2 \\
& + \frac{\varepsilon^2}{\gamma} \int_0^t (\partial_t^2 P_{\varepsilon}(s, 0) - \partial_x u_{\varepsilon}(s, 0))^2 ds \\
& + 2\varepsilon \int_0^t \int_0^\infty |P_{\varepsilon}| \|\partial_x u_{\varepsilon}\| ds dx + C(T) \\
& \leq \|P_{0}\|_{L^2(0,\infty)}^2 + \frac{\varepsilon^2 e^{\gamma t}}{\gamma} \int_0^t e^{-2\gamma s} (\partial_t^2 P_{\varepsilon}(s, 0) - \partial_x u_{\varepsilon}(s, 0))^2 ds \\
& + 2\varepsilon \int_0^t \int_0^\infty |P_{\varepsilon}| \|\partial_x u_{\varepsilon}\| ds dx + C(T) \\
& \leq \|P_{0}\|_{L^2(0,\infty)}^2 + 2\varepsilon \int_0^t \int_0^\infty |P_{\varepsilon}| \|\partial_x u_{\varepsilon}\| ds dx + C(T)
\end{align*}
\]

Due to (2.33) and the Young inequality,

\[
\begin{align*}
2\varepsilon \int_0^\infty |P_{\varepsilon}| \|\partial_x u_{\varepsilon}\| dx = 2 \int_0^\infty |P_{\varepsilon}| \|\varepsilon \partial_x u_{\varepsilon}\| dx \\
& \leq \|P_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 + \varepsilon^2 \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2 \\
& \leq C(T) + \varepsilon^2 \|\partial_x u_{\varepsilon}(t, \cdot)\|_{L^2(0,\infty)}^2.
\end{align*}
\]

Thus, for (2.63) and (2.64), we have that

\[
\begin{align*}
2\varepsilon \int_0^t \int_0^\infty |P_{\varepsilon}| \|\partial_x u_{\varepsilon}\| ds dx & \leq \int_0^t \|P_{\varepsilon}(s, \cdot)\|_{L^2(0,\infty)}^2 ds + \varepsilon^2 \int_0^t \|\partial_x u_{\varepsilon}(s, \cdot)\|_{L^2(0,\infty)}^2 ds \\
& \leq C(T).
\end{align*}
\]
Therefore,
\[
2\varepsilon \left| \int_0^t \int_0^\infty P_\varepsilon \partial_{xx}^2 P_\varepsilon \, ds \, dx \right| \leq \|P_0\|_{L^2(0,\infty)}^2 + C(T),
\]
which gives (2.38).

Let us continue by proving the existence of a distributional solution to (1.1), (2.1), (2.2) satisfying (2.10).

**Lemma 2.7.** Let \( T > 0 \). There exists a function \( u \in L^\infty((0,T) \times (0,\infty)) \) that is a distributional solution of (2.7) and satisfies (2.10) for every convex entropy \( \eta \in C^2(\mathbb{R}) \).

We construct a solution by passing to the limit in a sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \) of viscosity approximations (2.14). We use the compensated compactness method [15].

**Lemma 2.8.** Let \( T > 0 \). There exists a subsequence \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \) of \( \{u_\varepsilon\}_{\varepsilon > 0} \) and a limit function \( u \in L^\infty((0,T) \times (0,\infty)) \) such that
\[
(2.65) \quad u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{\text{loc}}((0,T) \times (0,\infty)), \quad 1 \leq p < \infty.
\]
In particular, (2.12) holds true.

Moreover, we have
\[
(2.66) \quad P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{\text{loc}}(0,T;W^{1,p}_{\text{loc}}(0,\infty)), \quad 1 \leq p < \infty,
\]
where
\[
(2.67) \quad P(t,x) = \int_0^x u(t,y) \, dy, \quad t \geq 0, \quad x \geq 0.
\]

**Proof.** Let \( \eta : \mathbb{R} \to \mathbb{R} \) be any convex \( C^2 \) entropy function, and let \( q : \mathbb{R} \to \mathbb{R} \) be the corresponding entropy flux defined by \( q'(u) = -\frac{1}{2} \eta'(u) \). By multiplying the first equation in (2.14) with \( \eta'(u_\varepsilon) \) and using the chain rule, we get
\[
\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2 + \gamma \eta'(u_\varepsilon) P_\varepsilon,
\]
where \( \mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon}, \mathcal{L}_{3,\varepsilon} \) are distributions.

Let us show that
\[
\mathcal{L}_{1,\varepsilon} \to 0 \text{ in } H^{-1}((0,T) \times (0,\infty)), \quad T > 0.
\]

Since
\[
\varepsilon \partial_{xx}^2 \eta(u_\varepsilon) = \partial_x (\varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon),
\]
for (2.21) and (2.37),
\[
\|\varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon\|_{L^2((0,T) \times (0,\infty))} \leq \varepsilon \|\eta''\|_{L^\infty(I_T)} \int_0^T \|\partial_x u_\varepsilon(s,\cdot)\|_{L^2(0,\infty)}^2 \, ds
\]
\[
\leq \varepsilon \|\eta''\|_{L^\infty(I_T)} C(T) \to 0,
\]
where
\[
I_T = (-C(T),C(T)).
\]
We claim that
\[
\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon > 0} \text{ is uniformly bounded in } L^1((0,T) \times (0,\infty)), \quad T > 0.
\]

Again by (2.21) and (2.37),
\[
\|\varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2\|_{L^1((0,T) \times (0,\infty))} \leq \|\eta''\|_{L^\infty(I_T)} \varepsilon \int_0^T \|\partial_x u_\varepsilon(s,\cdot)\|_{L^2(0,\infty)}^2 \, ds
\]
\[ \leq \|\eta''\|_{L^\infty(I_T)} C(T). \]

We have that \( \{L_{3,\varepsilon}\}_{\varepsilon > 0} \) is uniformly bounded in \( L^1_{\text{loc}}((0, T) \times (0, \infty)) \), \( T > 0 \).

Let \( K \) be a compact subset of \((0, T) \times (0, \infty)\). For (2.36) and (2.37),
\[ \gamma \eta'(u_\varepsilon) P_\varepsilon \|_{L^1(K)} = \gamma \int_K |\eta'(u_\varepsilon)| |P_\varepsilon| dtdx \leq \gamma \|\eta''\|_{L^\infty(I_T)} \|P_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} |K|. \]

Therefore, Murat’s Lemma \[11\] implies that (2.68) \( \{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon > 0} \) lies in a compact subset of \( H^{-1}_{\text{loc}}((0, T) \times (0, \infty)) \).

(2.37), (2.68), and the Tartar’s compensated compactness method \[13\] give the existence of a subsequence \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \) and a limit function \( u \in L^\infty((0, T) \times (0, \infty)) \), \( T > 0 \), such that (2.65) holds.

Let us show that (2.12) holds true. We begin by proving that (2.69)
\[ \varepsilon \partial_x P_\varepsilon(\cdot, 0) \to 0 \text{ in } L^\infty(0, T), \ T > 0. \]

For (2.21) and (2.22),
\[ \varepsilon \|\partial_x P_\varepsilon(\cdot, 0)\|_{L^\infty(0, T)} \leq \sqrt{\varepsilon e^{\gamma T}} \|u_0\|^2_{L^2(0, \infty)} = \sqrt{\varepsilon C(T)} \to 0, \]
that is (2.69).

Therefore, (2.12) follows from (2.18), (2.65) and (2.69).

Finally, we prove (2.66). We show that
(2.70) \[ \partial_x P_\varepsilon \to 0 \text{ in } L^\infty((0, T) \times (0, \infty)), \ T > 0. \]

It follows from (2.21) and (2.22) that
\[ \varepsilon \|\partial_x P_\varepsilon\|_{L^\infty((0, T) \times (0, \infty))} \leq \sqrt{\varepsilon e^{\gamma T}} \|u_0\|^2_{L^2(0, \infty)} = \sqrt{\varepsilon C(T)} \to 0, \]
that is (2.69).

Then, (2.39), (2.65), (2.69), (2.70) and the Hölder inequality give (2.66).

Moreover, \[7\] Theorem 1.1 tells us that the limit \( u \) admits strong boundary trace \( u_\tau^T \) at \((0, \infty) \times \{x = 0\}\). Since, arguing as in \[7\] Section 3.1 (indeed our solution is obtained as the vanishing viscosity limit of (2.7)), \[7\] Lemma 3.2 and the boundedness of the source term \( P \) (cf. (2.8)) imply (2.10).

Proof of Theorem 2.1. Lemma (2.8) gives the existence of entropy solution \( u(t, x) \) of (2.6), or equivalently (2.7). Moreover, it proves that (2.12) holds true.

We observe that, fixed \( T > 0 \), the solutions of (2.6), or equivalently (2.7), are bounded in \((0, T) \times (0, \infty)\). Therefore, using \[5\] Theorem 2.1, or \[8\] Theorem 2.2.1, \( u \) is unique and (2.13) holds true.
3. The Cauchy problem

Let us consider now the Cauchy problem associated to (1.1). Since the arguments are similar to the one of the previous section we simply sketch them, highlighting only the differences between the two problems.

In this section we augment (1.1) with the initial datum

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}. \]  

We assume that

\[ u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0. \]  

On the function

\[ P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \]

we assume that

\[ \|P_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y)dy \right)^2 dx < \infty. \]

We rewrite the Cauchy problem (1.1), (3.1) in the following way

\[
\begin{cases}
\partial_t u - \frac{1}{6} \partial_x u^3 = \gamma \int_0^x u(t, y)dy, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

or equivalently

\[
\begin{cases}
\partial_t u - \frac{1}{6} \partial_x u^3 = \gamma P, & t > 0, \quad x \in \mathbb{R}, \\
\partial_x P = u, & t > 0, \quad x \in \mathbb{R}, \\
P(t, 0) = 0, & t > 0, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

Due to the regularizing effect of the $P$ equation in (3.6) we have that

\[ u \in L^\infty((0, T) \times \mathbb{R}) \implies P \in L^\infty((0, T); W^{1, \infty}(\mathbb{R})), \quad T > 0. \]

**Definition 3.1.** We say that $u \in L^\infty((0, T) \times \mathbb{R})$, $T > 0$ is an entropy solution of the initial value problem (1.1), and (3.1) if

i) $u$ is a distributional solution of (3.5) or equivalently of (3.6);

ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

\[ \partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u)P \leq 0, \quad q(u) = -\int_0^u \frac{\xi^2}{2} \eta'(|\xi|) d\xi, \]

holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

The main result of this section is the following theorem.

**Theorem 3.1.** Assume (3.2) and (3.3). The initial value problem (1.1), (3.1), possesses an unique entropy solution $u$ in the sense of Definition 3.1. In particular, we have that

\[ \int_{\mathbb{R}} u(t, x)dx = 0, \quad t > 0. \]

Moreover, if $u$ and $v$ are two entropy solutions (1.1), (3.1), in the sense of Definition 3.1 the following inequality holds

\[ \|u(t, \cdot) - v(t, \cdot)\|_{L^1(-R, R)} \leq e^{C(T)t} \|u(0, \cdot) - v(0, \cdot)\|_{L^1(-R-C(T)t, R+C(T)t)}, \]
for almost every $0 < t < T$, $R > 0$, and some suitable constant $C(T) > 0$.

A similar result has been proved in [8] in the context of locally bounded solutions.

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (3.10).

Fix a small number $0 < \varepsilon < 1$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [6]

\[
\begin{aligned}
\partial_t u_\varepsilon - \frac{1}{2}u_\varepsilon^2 \partial_x u_\varepsilon &= \gamma P_\varepsilon + \varepsilon \partial_x^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\
-\varepsilon \partial_x^2 P_\varepsilon + \partial_x P_\varepsilon &= u_\varepsilon, & t > 0, x \in \mathbb{R}, \\
P_\varepsilon(t,0) &= 0, & t > 0, \\
\partial_x u_\varepsilon(0,x) &= u_{\varepsilon,0}(x), & x \in \mathbb{R},
\end{aligned}
\]

where $u_{\varepsilon,0}$ is a $C^\infty$ approximation of $u_0$ such that

\begin{align}
\|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq \|u_0\|_{L^2(\mathbb{R})}, & \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})}, \\
\|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq \|P_0\|_{L^2(\mathbb{R})}, & \varepsilon \|\partial_x P_{\varepsilon,0}\|_{L^2(\mathbb{R})} &\leq C_0,
\end{align}

and $C_0$ is a constant independent on $\varepsilon$.

Let us prove some a priori estimates on $u_\varepsilon$ and $P_\varepsilon$, denoting with $C_0$ the constants which depend on the initial datum, and $C(T)$ the constants which depend also on $T$.

Arguing as [4] and Section 2 we obtain the following results

**Lemma 3.1.** For each $t \in (0,\infty)$,

\begin{align}
P_\varepsilon(t,-\infty) &= \partial_x P_\varepsilon(t,-\infty) = P_\varepsilon(t,\infty) = \partial_x P_\varepsilon(t,\infty) = 0, \\
\varepsilon^2 \|\partial_x^2 P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2 &\leq \|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}^2.
\end{align}

**Lemma 3.2.** For each $t \in (0,\infty)$,

\begin{align}
\int_\mathbb{R} u_\varepsilon(t,x)dx &= 0, \\
\sqrt{\varepsilon} \|\partial_x P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \|u(t,\cdot)\|_{L^2(\mathbb{R})}, \\
\int_\mathbb{R} u_\varepsilon(t,x)P_\varepsilon(t,x)dx &\leq \|u(t,\cdot)\|_{L^2(\mathbb{R})}^2.
\end{align}

**Lemma 3.3.** For every $t \in (0,\infty)$,

\begin{align}
\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})} + \gamma \int_0^t \|P_\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R})} ds.
\end{align}

**Lemma 3.4.** For each $t \in (0,\infty)$, the inequality holds

\begin{align}
\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{2\gamma t} \int_0^\infty e^{-2\gamma s} \|\partial_x u_\varepsilon(s,\cdot)\|_{L^2(\mathbb{R})}^2 ds &\leq e^{2\gamma t} \|u_0\|_{L^2(\mathbb{R})}^2.
\end{align}

In particular, we have

\begin{align}
\varepsilon \|\partial_x^2 P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}, \|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &\leq e^{\gamma t} \|u_0\|_{L^2(\mathbb{R})}, \\
\sqrt{\varepsilon} \|\partial_x P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq e^{\gamma t} \|u_0\|_{L^2(\mathbb{R})}.
\end{align}

Moreover, we get

\begin{align}
\|P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq \sqrt{2e^{\gamma t} \|u_0\|_{L^2(0,\infty)}} \|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}, \\
\sqrt{\varepsilon} \|\partial_x P_\varepsilon(t,0)\| &\leq e^{\gamma t} \|u_0\|_{L^2(\mathbb{R})}.
\end{align}
Proof. Arguing as Section 2, we obtain (3.18), (3.19) and (3.20).

Let us show that (3.21) holds true. Squaring the equation for \( P_\epsilon \) in (3.10), we get
\[
\epsilon^2 (\partial^2_{xx} P_\epsilon)^2 + (\partial_x P_\epsilon)^2 = u^2_\epsilon.
\]
An integration on \((-\infty, 0)\) and (3.12) give
\[
\int_{-\infty}^{0} \left( \partial^2_{xx} P_\epsilon \right)^2 dx + \int_{-\infty}^{0} (\partial_x P_\epsilon)^2 dx + \epsilon \partial_x (\partial_x P_\epsilon(t, 0))^2 = \int_{-\infty}^{0} u^2_\epsilon dx.
\]
(3.22)

It follows from (3.18) and (3.22) that
\[
\epsilon (\partial_x P_\epsilon(t, 0))^2 \leq e^{2\gamma t} \|u_\epsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]
which gives (3.21). \( \square \)

Lemma 3.5. For each \( t \geq 0 \), we have that
\[
\int_{0}^{-\infty} P_\epsilon(t, x) dx = a_\epsilon(t),
\]
(3.23)
\[
\int_{0}^{\infty} P_\epsilon(t, x) dx = a_\epsilon(t),
\]
(3.24)

where
\[
a_\epsilon(t) = \frac{1}{\gamma} \left( \epsilon \partial^2_{xx} P_\epsilon(t, 0) + \frac{1}{6} u_\epsilon^3(t, 0) + \epsilon \partial_x u_\epsilon(t, 0) \right).
\]
Moreover,
\[
\int_{\mathbb{R}} P_\epsilon(t, x) dx = 0, \quad t \geq 0.
\]
(3.25)

Proof. We begin by observing that, integrating the second equation of (3.10) on \((0, x)\), we have that
\[
\int_{0}^{x} u_\epsilon(t, y) dy = P_\epsilon(t, x) - \epsilon \partial_x P_\epsilon(t, x) + \epsilon \partial_x P_\epsilon(t, 0).
\]
(3.26)

It follows from (3.12) that
\[
\lim_{x \to -\infty} \int_{0}^{x} u_\epsilon(t, y) dy = \int_{0}^{-\infty} u_\epsilon(t, x) dx = \epsilon \partial_x P_\epsilon(t, 0).
\]
(3.27)

Differentiating (3.27) with respect to \( t \), we get
\[
\frac{d}{dt} \int_{0}^{x} u_\epsilon(t, x) dx = \int_{0}^{-\infty} \partial_t u_\epsilon(t, x) dx = \epsilon \partial^2_{xx} P_\epsilon(t, 0).
\]
(3.28)

Integrating the first equation (3.10) on \((0, x)\), we obtain that
\[
\int_{0}^{x} \partial_t u_\epsilon(t, y) dy - \frac{1}{6} u_\epsilon^3(t, x) + \frac{1}{6} u_\epsilon^3(t, 0)
\]
\[
- \epsilon \partial_x u_\epsilon(t, x) + \epsilon \partial_x u_\epsilon(t, 0) = \gamma \int_{0}^{x} P_\epsilon(t, y) dy.
\]
(3.29)

Being \( u_\epsilon \) a smooth solution of (3.10), we get
\[
\lim_{x \to -\infty} \left( - \frac{1}{6} u^3_\epsilon(t, x) - \epsilon \partial_x u_\epsilon(t, x) \right) = 0.
\]
(3.30)
Sending $x \to -\infty$ in (3.29), for (3.28) and (3.30), we have
\[ \gamma \int_{-\infty}^{0} P_{\varepsilon}(t,x)dx = \varepsilon \partial_{xx}^{2} P_{\varepsilon}(t,0) + \frac{1}{6} u_{3}^{\varepsilon}(t,0) + \varepsilon \partial_{x} u_{\varepsilon}(t,0), \]
which gives (3.23).

Let us show that (3.24) holds true. We begin by observing that, for (3.12) and (3.26),
\[ \int_{\infty}^{0} u_{\varepsilon}(t,x)dx = \varepsilon \partial_{x} P_{\varepsilon}(t,0). \]
Therefore,
\[ (3.31) \lim_{x \to \infty} \int_{x}^{0} \partial_{t} u_{\varepsilon,\delta}(t,y)dy = \int_{0}^{\infty} \partial_{t} u_{\varepsilon}(t,x)dx = \varepsilon \partial_{xx}^{2} P_{\varepsilon}(t,0). \]
Again by the regularity of $u_{\varepsilon}$,
\[ (3.32) \lim_{x \to \infty} \left( -\frac{1}{6} u_{3}^{\varepsilon}(t,x) - \varepsilon \partial_{x} u_{\varepsilon}(t,x) \right) = 0. \]
It follows from (3.29), (3.31) and (3.32) that
\[ \gamma \int_{0}^{\infty} P_{\varepsilon}(t,x)dx = \varepsilon \partial_{xx}^{2} P_{\varepsilon,\delta}(t,0) + \frac{1}{6} u_{3}^{\varepsilon}(t,0) + \varepsilon \partial_{x} u_{\varepsilon}(t,0), \]
which gives (3.24).

Finally, we prove (3.25). It follows from (3.23) that
\[ \int_{-\infty}^{0} P_{\varepsilon}(t,x)dx = -a_{\varepsilon}(t). \]
Therefore, for (3.24),
\[ \int_{-\infty}^{0} P_{\varepsilon}(t,x)dx + \int_{0}^{\infty} P_{\varepsilon}(t,x) = \int_{\mathbb{R}} P_{\varepsilon}(t,x)dx = -a_{\varepsilon}(t) + a_{\varepsilon}(t) = 0, \]
that is (3.25). \qed

Lemma 3.5 says that $P_{\varepsilon}(t,x)$ is integrable at $\pm \infty$. Therefore, for each $t \geq 0$, we can consider the following function
\[ (3.33) F_{\varepsilon}(t,x) = \int_{-\infty}^{x} P_{\varepsilon}(t,y)dy. \]

**Lemma 3.6.** Let $T > 0$. There exists a function $C(T) > 0$, independent on $\varepsilon$, such that
\[ \| P_{\varepsilon} \|_{L^{\infty}(0,T;L^{2}(\mathbb{R}))} \leq C(T). \]
In particular, we have that
\[ \| P_{\varepsilon}(t,\cdot) \|_{L^{2}(\mathbb{R})} \leq C(T), \]
\[ \varepsilon \| \partial_{x} P_{\varepsilon}(t,\cdot) \|_{L^{2}(\mathbb{R})} \leq C(T), \]
\[ \| P_{\varepsilon} \|_{L^{\infty}(0,T)\times(\mathbb{R})} \leq C(T), \]
\[ \| u_{\varepsilon} \|_{L^{\infty}(0,T)\times(\mathbb{R})} \leq C(T). \]
Moreover, we get
\[ \varepsilon \left| \int_{0}^{t} \int_{\mathbb{R}} P_{\varepsilon} \partial_{xx}^{2} P_{\varepsilon}dsdx \right| \leq C(T), \quad t \in (0,T). \]
Proof. Integrating the second equation of (3.10) on \((-\infty, x)\), we have that
\[
\int_{-\infty}^{x} u_\varepsilon(t,y) \, dy = P_\varepsilon(t,x) - \varepsilon \partial_x P_\varepsilon(t,x).
\]

Differentiating (3.40) with respect to \(t\), we get
\[
\frac{d}{dt} \int_{-\infty}^{x} u_\varepsilon(t,y) \, dy = \int_{-\infty}^{x} \partial_t u_\varepsilon(t,y) \, dy = \partial_t P_\varepsilon(t,x) - \varepsilon \partial_{t x} P_\varepsilon(t,x).
\]

It follows from an integration of the first equation of (3.10) on \((-\infty, x)\) and (3.33) that
\[
\int_{-\infty}^{x} \partial_t u_\varepsilon(t,y) \, dy = \frac{1}{6} u_\varepsilon^3(t,x) - \varepsilon \partial_x u_\varepsilon(t,x) = \gamma F_\varepsilon(t,x).
\]

Due to (3.41) and (3.42), we have
\[
\partial_t P_\varepsilon(t,x) - \varepsilon \partial_{t x}^2 P_\varepsilon(t,x) = \gamma F_\varepsilon(t,x) + \frac{1}{6} u_\varepsilon^3(t,x) + \varepsilon \partial_x u_\varepsilon(t,x).
\]

Multiplying (3.43) by \(P_\varepsilon - \varepsilon \partial_x P_\varepsilon\), we have
\[
(\partial_t P_\varepsilon - \varepsilon \partial_{t x}^2 P_\varepsilon)(P_\varepsilon - \varepsilon \partial_x P_\varepsilon) = \gamma F_\varepsilon(P_\varepsilon - \varepsilon \partial_x P_\varepsilon)
\] + \frac{1}{6} u_\varepsilon^3(P_\varepsilon - \varepsilon \partial_x P_\varepsilon) + \varepsilon \partial_x u_\varepsilon(P_\varepsilon - \varepsilon \partial_x P_\varepsilon).

Integrating (3.44) on \((0, x)\), we have that
\[
\int_{0}^{x} \partial_t P_\varepsilon \, dy - \varepsilon \int_{0}^{x} \partial_k P_\varepsilon \partial_x P_\varepsilon \, dy
\] = \gamma \int_{0}^{x} F_\varepsilon \, dy - \gamma \int_{0}^{x} F_\varepsilon \partial_x P_\varepsilon \, dy + \frac{1}{6} \int_{0}^{x} u_\varepsilon^3 P_\varepsilon \, dy - \frac{1}{6} \int_{0}^{x} u_\varepsilon^3 \partial_x P_\varepsilon \, dy + \varepsilon \int_{0}^{x} \partial_x u_\varepsilon P_\varepsilon \, dy - \varepsilon \int_{0}^{x} \partial_x u_\varepsilon \partial_x P_\varepsilon \, dy.
\]

We observe that, for (3.10),
\[
- \varepsilon \int_{0}^{x} \partial_x P_\varepsilon \partial_t P_\varepsilon \, dy = - \varepsilon P_\varepsilon \partial_t P_\varepsilon + \varepsilon \int_{0}^{x} P_\varepsilon \partial_{t x}^2 P_\varepsilon \, dy.
\]

Therefore, (3.45) and (3.46) give
\[
\int_{0}^{x} \partial_t P_\varepsilon \, dy + \varepsilon \int_{0}^{x} \partial_{t x} P_\varepsilon \partial_x P_\varepsilon \, dy
\] = \varepsilon P_\varepsilon \partial_t P_\varepsilon + \gamma \int_{0}^{x} F_\varepsilon P_\varepsilon \, dy - \gamma \int_{0}^{x} F_\varepsilon \partial_x P_\varepsilon \, dy + \frac{1}{6} \int_{0}^{x} u_\varepsilon^3 P_\varepsilon \, dy - \frac{1}{6} \int_{0}^{x} u_\varepsilon^3 \partial_x P_\varepsilon \, dy + \varepsilon \int_{0}^{x} \partial_x u_\varepsilon P_\varepsilon \, dy - \varepsilon \int_{0}^{x} \partial_x u_\varepsilon \partial_x P_\varepsilon \, dy.
\]
Sending \( x \to -\infty \), for (3.12), we get

\[
\int_0^{-\infty} \partial_t P_{\varepsilon, \delta} P_{\varepsilon, \delta} dy + \varepsilon^2 \int_0^{-\infty} \partial_{t x}^2 P_{\varepsilon, \delta} \partial_x P_{\varepsilon, \delta} dy = \gamma \int_0^{-\infty} F_\varepsilon P_\varepsilon dy - \gamma \varepsilon \int_0^{-\infty} F_\varepsilon \partial_x P_\varepsilon dy + \frac{1}{6} \int_0^{-\infty} u_\varepsilon^3 P_\varepsilon dy - \varepsilon \int_0^{-\infty} \partial_x u_\varepsilon \partial_x P_\varepsilon dy,
\]

(3.48)

while sending \( x \to \infty \),

\[
\int_0^{\infty} \partial_t P_\varepsilon P_\varepsilon dy + \varepsilon^2 \int_0^{\infty} \partial_{t x}^2 P_\varepsilon \partial_x P_\varepsilon dy = \gamma \int_0^{\infty} F_\varepsilon P_\varepsilon dy - \gamma \varepsilon \int_0^{\infty} F_\varepsilon \partial_x P_\varepsilon dy + \frac{1}{6} \int_0^{\infty} u_\varepsilon^3 P_\varepsilon dy - \varepsilon \int_0^{\infty} \partial_x u_\varepsilon \partial_x P_\varepsilon dy,
\]

(3.49)

Since

\[
\int_R P_\varepsilon \partial_t P_\varepsilon dx = \frac{1}{2} \frac{d}{dt} \int_R P_\varepsilon^2 dx,
\]

\[
\varepsilon^2 \int_R \partial_{t x}^2 P_\varepsilon \partial_x P_\varepsilon dx = \frac{\varepsilon^2}{2} \frac{d}{dt} \int_R (\partial_x P_\varepsilon)^2 dx,
\]

it follows from (3.48) and (3.49) that

\[
\frac{1}{2} \frac{d}{dt} \int_R P_\varepsilon^2 dx + \frac{\varepsilon^2}{2} \frac{d}{dt} \int_R (\partial_x P_\varepsilon)^2 dx = \gamma \int_R F_\varepsilon P_\varepsilon dx - \gamma \varepsilon \int_R F_\varepsilon \partial_x P_\varepsilon dx + \frac{1}{6} \int_R u_\varepsilon^3 P_\varepsilon dx - \varepsilon \int_R \partial_x u_\varepsilon \partial_x P_\varepsilon dx.
\]

(3.50)

Due to (3.25) and (3.33),

\[
2\gamma \int_R F_\varepsilon P_\varepsilon dx = 2\gamma \int_R F_\varepsilon \partial_x F_\varepsilon dx = (F_\varepsilon(t, \infty))^2 = 0.
\]

(3.51)
It follows from (3.50) and (3.51) that
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} P^2_{\varepsilon} \, dx + \varepsilon^2 \int_{\mathbb{R}} (\partial_x P_{\varepsilon})^2 \, dx \right)
\]
\[
= -2\gamma \varepsilon \int_{\mathbb{R}} F_{\varepsilon} \partial_x P_{\varepsilon} \, dx + \frac{1}{3} \int_{\mathbb{R}} \int_{\mathbb{R}} u^3_{\varepsilon} \partial_x P_{\varepsilon} \, dx - \frac{\varepsilon}{3} \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x P_{\varepsilon} \, dx
\]
\[
+ 2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon} P_{\varepsilon} \, dx - 2\varepsilon^2 \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x P_{\varepsilon} \, dx.
\]

Due to (3.12), (3.25) and (3.33),
\[
-2\varepsilon \int_{\mathbb{R}} \partial_x P_{\varepsilon} F_{\varepsilon} \, dx = 2\varepsilon \gamma \int_{\mathbb{R}} P_{\varepsilon} \partial_x F_{\varepsilon} \, dx = 2\varepsilon \gamma \int_{\mathbb{R}} P^2_{\varepsilon} \, dx \leq 2\gamma \int_{\mathbb{R}} P^2_{\varepsilon} \, dx,
\]
while for (3.12),
\[
2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon} P_{\varepsilon} \, dx = -2\varepsilon \int_{\mathbb{R}} u_{\varepsilon} \partial_x P_{\varepsilon} \, dx.
\]

Hence, (3.33) and (3.54) give
\[
\frac{d}{dt} \left( \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
\leq 2\gamma \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{6} \int_{\mathbb{R}} \int_{\mathbb{R}} u^3_{\varepsilon} \partial_x P_{\varepsilon} \, dx - \frac{\varepsilon}{3} \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x P_{\varepsilon} \, dx
\]
\[
- 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon} \partial_x P_{\varepsilon} \, dx - 2\varepsilon^2 \int_{\mathbb{R}} \partial_x u_{\varepsilon} \partial_x P_{\varepsilon} \, dx.
\]

Due to the Young inequality,
\[
\frac{1}{3} \int_{\mathbb{R}} \int_{\mathbb{R}} u^3_{\varepsilon} \partial_x P_{\varepsilon} \, dx \leq \frac{1}{3} \int_{\mathbb{R}} \left| P_{\varepsilon}(t, \cdot) \right| u_{\varepsilon}^2 \leq \frac{1}{6} \int_{\mathbb{R}} P^2_{\varepsilon} u_{\varepsilon}^2 \, dx + \frac{1}{6} \int_{\mathbb{R}} u^3_{\varepsilon} \, dx
\]
\[
\leq \frac{1}{6} \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{6} \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
- \frac{\varepsilon}{3} \int_{\mathbb{R}} \int_{\mathbb{R}} u^3_{\varepsilon} \partial_x P_{\varepsilon} \, dx \leq \frac{1}{3} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \partial_x P_{\varepsilon}(t, \cdot) \right| u_{\varepsilon}^2 \, dx
\]
\[
\leq \frac{2\gamma}{3} \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{6} \int_{\mathbb{R}} \int_{\mathbb{R}} u^4_{\varepsilon} \, dx \leq \frac{2\gamma}{6} \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ \frac{1}{6} \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
- 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon} \partial_x P_{\varepsilon} \, dx \leq 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon} \partial_x P_{\varepsilon} \, dx \leq \int_{\mathbb{R}} \frac{u_{\varepsilon}}{2\gamma} \left| 2\sqrt{\gamma} \varepsilon \partial_x P_{\varepsilon} \right| \, dx
\]
\[
\leq \frac{1}{2\gamma} \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\gamma \varepsilon^2 \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Therefore, we have that
\[
\frac{d}{dt} \left( \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right)
\]
\[
\leq 2\gamma \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\gamma \varepsilon^2 \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{6} \left\| P_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ \frac{2\gamma}{6} \left\| \partial_x P_{\varepsilon}(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{3} \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ \frac{1}{2\gamma} \left\| u_{\varepsilon}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon^2 \int_{\mathbb{R}} \left| \partial_x u_{\varepsilon} \right| \left| \partial_x P_{\varepsilon} \right| \, dx.
\]

Due to the Young inequality, 
\[
2\varepsilon^2 \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x P_\varepsilon| dx \leq \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 
\]
\[
\leq \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 .
\]

Hence, 
\[
\frac{d}{dt} G(t) - 2\gamma G(t) \leq \frac{1}{6} \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 
\]
\[
+ \frac{\varepsilon^2}{6} \|\partial_x P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{3} \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 
\]
\[
+ \frac{1}{2\gamma} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 
\]
\[
+ \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 ,
\]

where 
\[
(3.55) \quad G(t) = \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 .
\]

Thanks to (3.11), (3.14) and (3.20), 
\[
\frac{1}{6} \|P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{e^{3\gamma t}}{3} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 ,
\]
\[
\frac{\varepsilon^2}{6} \|\partial_x P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{e^{4\gamma t}}{6} \|u_0\|_{L^2(\mathbb{R})}^4 ,
\]
\[
\frac{1}{3} \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{e^{2\gamma t}}{3} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 ,
\]
\[
\frac{1}{2\gamma} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{e^{2\gamma t}}{2\gamma} \|u_0\|_{L^2(\mathbb{R})}^2 ,
\]
\[
\|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{2\gamma t} \|u_0\|_{L^2(\mathbb{R})}^2 .
\]

Thus, we get 
\[
\frac{d}{dt} G(t) - 2\gamma G(t) \leq \frac{e^{3\gamma t}}{3} \|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \|u_0\|_{L^2(\mathbb{R})}^3 + \frac{e^{4\gamma t}}{6} \|u_0\|_{L^2(\mathbb{R})}^4 
\]
\[
+ \frac{\varepsilon^2}{3} \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + e^{2\gamma t} \|u_0\|_{L^2(\mathbb{R})}^2 
\]
\[
+ \frac{e^{2\gamma t}}{2\gamma} \|u_0\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 .
\]

The Gronwall Lemma, (3.11) and (3.55) give 
\[
\|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 
\]
\[
\leq \|P_0\|_{L^2(\mathbb{R})}^2 e^{2\gamma t} + C_0 e^{2\gamma t} + \frac{C_0 e^{2\gamma t}}{3} \|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \int_0^t e^{\gamma s} ds 
\]
\[
+ \frac{C_0 e^{2\gamma t}}{6} \int_0^t e^{2\gamma s} ds + \frac{C_0 e^{2\gamma t}}{3} \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})}^2 ds 
\]
\[
+ C_0 e^{2\gamma t} (1 + t) .
\]
Due to (3.17) and the Young inequality,
\[
\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq \|u_0\|_{L^\infty(\mathbb{R})}^2 + 2\gamma \|u_0\|_{L^\infty(\mathbb{R})} \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})} ds
\]
(3.57)
\[
+ \gamma^2 \left( \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})} ds \right)^2
\]
\[
\leq 2\|u_0\|_{L^\infty(\mathbb{R})}^2 + \gamma^2 \left( \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})} ds \right)^2.
\]

It follows from (3.20), (3.57) and the Jensen inequality that
\[
\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq 2\|u_0\|_{L^\infty(\mathbb{R})}^2 + \gamma^2 t \int_0^t \|P_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{R})}^2 ds
\]
(3.58)
\[
\leq C_0 + \gamma C_0 t \int_0^t e^{\gamma s} \|P_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})} ds.
\]

Therefore,
\[
\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq C_0 + C(T) \|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))},
\]
(3.59) and (3.58) give
\[
\|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) + C(T) \|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))},
\]
(3.60)
It follows from (3.60) that
\[
\|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 - C(T) \|P_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T) \leq 0,
\]
which gives (3.34). (3.60) and (3.54) give (3.35) and (3.30). (3.20) and (3.34) give (3.37), while (3.38) follows from (3.17) and (3.37).

Finally, arguing as Lemma 2.6, we obtain (3.39). Therefore, the proof is done. □

Let us continue by proving the existence of a distributional solution to (1.1), (3.1) satisfying (3.11), (3.1) satisfying (3.17).

**Lemma 3.7.** Let \( T > 0 \). There exists a function \( u \in L^\infty((0,T) \times \mathbb{R}) \) that is a distributional solution of (3.6) and satisfies (3.7) for every convex entropy \( \eta \in C^2(\mathbb{R}) \).

We construct a solution by passing to the limit in a sequence \( \{u_\varepsilon\}_{\varepsilon > 0} \) of viscosity approximations (3.10). We use the compensated compactness method [15].

**Lemma 3.8.** Let \( T > 0 \). There exists a subsequence \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \) of \( \{u_\varepsilon\}_{\varepsilon > 0} \) and a limit function \( u \in L^\infty((0,T) \times \mathbb{R}) \) such that
\[
u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{\text{loc}}((0,T) \times \mathbb{R}), \quad 1 \leq p < \infty.
\]
(3.61)

In particular, (3.5) holds true.

Moreover, we have
\[
P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{\text{loc}}((0,T); W^{1,p}_{\text{loc}}(\mathbb{R})), \quad 1 \leq p < \infty,
\]
(3.62)
\[P(t,x) = \int_0^x u(t,y) dy, \quad t \geq 0, \quad x \in \mathbb{R}.
\]
(3.63)
Proof. Let \( \eta : \mathbb{R} \to \mathbb{R} \) be any convex \( C^2 \) entropy function, and \( q : \mathbb{R} \to \mathbb{R} \) be the corresponding entropy flux defined by \( q'(u) = -\frac{u}{2} \eta''(u) \). By multiplying the first equation in (3.10) with \( \eta'(u_\varepsilon) \) and using the chain rule, we get

\[
\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \gamma \eta'(u_\varepsilon) P_\varepsilon,
\]

where \( L_{1,\varepsilon}, L_{2,\varepsilon}, L_{3,\varepsilon} \) are distributions.

Arguing as in Lemma 2.8, we have that

\[
L_{1,\varepsilon} \to 0 \quad \text{in} \quad H^{-1}((0,T) \times \mathbb{R}), \quad T > 0, \\
\{L_{2,\varepsilon}\}_{\varepsilon > 0} \quad \text{is uniformly bounded in} \quad L^1((0,T) \times \mathbb{R}), \quad T > 0, \\
\{L_{3,\varepsilon}\}_{\varepsilon > 0} \quad \text{is uniformly bounded in} \quad L^1_{loc}((0,T) \times \mathbb{R}), \quad T > 0.
\]

Therefore, Murat’s lemma [11] implies that (3.64) \( \{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon > 0} \) lies in a compact subset of \( H^{-1}_{loc}((0,\infty) \times \mathbb{R}) \).

(3.38), (3.64) and the Tartar’s compensated compactness method [15] give the existence of a subsequence \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \) and a limit function \( u \in L^\infty((0,T) \times \mathbb{R}) \) such that (3.61) holds. (3.8) follows from (3.14) and (3.61). Finally, we prove (3.62). We begin by observing that, integrating the second equation of (3.10) on \((0,x)\), we have

\[
P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y)dy + \varepsilon \partial_x P_\varepsilon(t,x) - \varepsilon \partial_x P_\varepsilon(t,0).
\]

Let us show that

\[
\varepsilon \partial_x P_\varepsilon \to 0 \quad \text{in} \quad L^\infty((0,T) \times \mathbb{R}), \quad T > 0.
\]

It follows from (3.19) that

\[
\varepsilon \|\partial_x P_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq \sqrt{\varepsilon} e^{-\gamma T} \|u_0\|_{L^2(\mathbb{R})}^2 = \sqrt{\varepsilon} C(T) \to 0,
\]

that is (3.66).

We claim that

\[
\varepsilon \partial_x P_\varepsilon(\cdot,0) \to 0 \quad \text{in} \quad L^\infty(0,T), \quad T > 0.
\]

Due to (3.21), we have that

\[
\varepsilon \|\partial_x P_\varepsilon(\cdot,0)\|_{L^\infty(0,T)} \leq \sqrt{\varepsilon} e^{-\gamma T} \|u_0\|_{L^2(\mathbb{R})}^2 = \sqrt{\varepsilon} C(T) \to 0,
\]

that is (3.67).

Therefore, (3.61), (3.65), (3.66), (3.67) and the Hölder inequality give (3.62). \( \square \)

Proof of Theorem 3.1. Lemma 2.8 gives the existence of an entropy solution \( u \) of (3.5), or equivalently (3.6). Moreover, it proves that (3.8) holds true.

We observe that, fixed \( T > 0 \), the solutions of (3.5), or equivalently (3.6), are bounded in \((0,T) \times \mathbb{R})\. Therefore, using [5, Theorem 3.1], or [8, Theorem 2.3.1], \( u \) is unique and (3.9) holds true. \( \square \)
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