Boundary Layers and Incompressible Navier-Stokes-Fourier Limit of the Boltzmann Equation in Bounded Domain I

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To Claude Bardos on his 75th birthday

Abstract
We establish the incompressible Navier-Stokes-Fourier limit for solutions to the Boltzmann equation with a general cutoff collision kernel in a bounded domain. Appropriately scaled families of DiPerna-Lions(-Mischler) renormalized solutions with Maxwell reflection boundary conditions are shown to have fluctuations that converge as the Knudsen number goes to 0. Every limit point is a weak solution to the Navier-Stokes-Fourier system with different types of boundary conditions depending on the ratio between the accommodation coefficient and the Knudsen number. The main new result of the paper is that this convergence is strong in the case of the Dirichlet boundary condition. Indeed, we prove that the acoustic waves are damped immediately; namely, they are damped in a boundary layer in time. This damping is due to the presence of viscous and kinetic boundary layers in space. As a consequence, we also justify the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with Navier-Stokes scaling.

This extends the work of Golse and Saint-Raymond [20,21] and Levermore and Masmoudi [28] to the case of a bounded domain. The case of a bounded domain was considered by Masmoudi and Saint-Raymond [34] for the linear Stokes-Fourier limit and Saint-Raymond [41] for the Navier-Stokes limit for hard potential kernels. Neither [34] nor [41] studied the damping of the acoustic waves. This paper extends the result of [34,41] to the nonlinear case and includes soft potential kernels. More importantly, for the Dirichlet boundary condition, this work strengthens the convergence so as to make the boundary layer visible. This answers an open problem proposed by Ukai [46]. © 2016 Wiley Periodicals, Inc.

Contents

1. Introduction 91
2. Boltzmann Equation in Bounded Domain 95

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1 Introduction

The hydrodynamic limits from the Boltzmann equation got a lot of interest in the previous two decades. Hydrodynamic regimes are those where the Knudsen number $\varepsilon$ is small. The Knudsen number is the ratio of the mean free path and the macroscopic length scales. The incompressible Navier-Stokes-Fourier (NSF) system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the number density $F$ about an absolute Maxwellian $M$ are scaled to be on the order $\varepsilon$; see [2].

The program that justifies the hydrodynamic limits from the Boltzmann equation in the framework of DiPerna-Lions [12] was initiated by Bardos, Golse, and Levermore [2,3] in late 1980s. Since then there has been many contributions to this program [4,15,20,21,24,28,32,34,40]. In particular, the work of Golse and Saint-Raymond [20] is the first complete rigorous justification of the NSF limit from the Boltzmann equation in a class of bounded collision kernels without making any nonlinear weak compactness hypotheses. They have recently extended their result to the case of hard potentials [21]. With some new nonlinear estimates, Levermore and Masmoudi [28] treated a broader class of collision kernels that includes all hard potential cases and, for the first time in this program, soft potential cases.

All of the above-mentioned works were carried out in either the periodic spatial domain or the whole space except for [34,41]. In [34], the linear Stokes-Fourier system was recovered with the same collision kernels assumption as in [15], while in [41], the Navier-Stokes limit was derived with the same kernels assumption as in [21], i.e., hard potential kernels. In [34,41], the fluctuations of renormalized solutions to the Boltzmann equation in a bounded domain (see [38]) was proved to pass to the limit and recovered fluid boundary conditions, either the Dirichlet or the Navier slip boundary condition, depending on the relative sizes of the accommodation coefficient and the Knudsen number.

The dependence of the boundary conditions of the limiting fluid equations on the relative importance of the accommodation coefficient and the Knudsen number was observed by Sone and his collaborators. Their results, mostly formal, are presented
in chapters 3 and 4 in [45] for several types of kinetic boundary conditions. The work in [34, 41] rigorously justified the incompressible Stokes and Navier-Stokes equations from the Boltzmann equation imposed with Maxwell reflection boundary condition. We also mention the recent work on the Navier-Stokes-Fourier limit for the stationary Boltzmann equation for the Maxwell reflection boundary condition for the constant accommodation coefficient [13].

In his survey paper [46, p. 192], Ukai proposed the following question:

As far as the Boltzmann equation in a bounded domain is concerned, some progress has been made recently. In [37], the convergence of the Boltzmann equation to the (linear) Stokes-Fourier equation was proved together with the convergence of the boundary conditions. It is a big challenging problem to extend the result to the nonlinear case and to strengthen the convergence so as to make visible the boundary layer.

(In the above citation of Ukai’s survey, reference 37 is the Saint-Raymond and Masmoudi paper cited here [34].)

In this paper and a forthcoming one, we study the incompressible NSF limit in a bounded domain from the Boltzmann equation with the Maxwell reflection boundary condition in which the accommodation might depend on the Knudsen number. We consider a bounded domain \( \mathbb{R}^D, D \geq 2 \), with boundary \( \partial \Omega \in C^2 \). The NSF system governs the fluctuations of mass density, bulk velocity, and temperature \((\rho, u, \theta)\) about their spatially homogeneous equilibrium values in a Boussinesq regime. Specifically, after a suitable choice of units, these dimensionless fluctuations satisfy the incompressibility and Boussinesq relations

\[
\nabla_x \cdot u = 0, \quad \rho + \theta = 0,
\]

while their evolution is determined by the Navier-Stokes and heat equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla_x u + \nabla_x p &= \nu \Delta_x u, \\
\partial_t \theta + u \cdot \nabla_x \theta &= \frac{2}{D+2} \kappa \Delta_x \theta
\end{align*}
\]

where \( \nu > 0 \) is the kinematic viscosity and \( \kappa > 0 \) is the heat thermal conductivity.

Traditionally, two types of natural physical boundary conditions could be imposed for the incompressible NSF system (1.2). The first is the homogeneous Dirichlet boundary condition, namely,

\[
\begin{align*}
\partial_n u &= 0, \quad \theta = 0, \quad \text{on } \mathbb{R}^+ \times \partial \Omega.
\end{align*}
\]

The other is the so-called Navier slip boundary condition, which was proposed by Navier [39]:

\[
\begin{align*}
[2\nu d(u) \cdot n + \kappa u]^{\text{tan}} &= 0, \quad u \cdot n = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega, \\
\kappa \partial_n \theta + \frac{\nu}{D+2} \partial_n [u \cdot n] &= 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega,
\end{align*}
\]

where \( d(u) = \frac{1}{2} (\nabla_x u + \nabla_x u^T) \) denotes the symmetric part of the stress tensor and \( \partial_n \) denotes the directional derivative along the outer normal vector \( n(x), x \in \partial \Omega \).
In the above Navier boundary condition, \( \chi > 0 \) is the reciprocal of the slip length, which depends on the material of the container.

In the current work, for general cutoff collision kernels, namely in the framework of [28], we justify the NSF system. Regarding the weak convergence results, our proof is basically the same as in [34, 41]: the boundary conditions of the limiting NSF system depend on the ratio of the accommodation coefficient and the Knudsen number; namely, when \( \frac{\alpha_0}{\varepsilon} \to \infty \) as \( \varepsilon \to 0 \), the Dirichlet condition is derived, while when \( \frac{\alpha_0}{\varepsilon} \to \sqrt{2\pi \chi} \), the Navier slip boundary condition is derived. The main difference is that [41] used the same renormalizations as [21], applicable for hard potentials, while in the current work, we use the renormalization of [28], which works for more general cutoff kernels, including soft potentials.

The main novelty of the current work is the treatment of the Dirichlet boundary condition case. Indeed, we prove that when \( \frac{\alpha_0}{\varepsilon} \to \infty \), the convergence is strong. Furthermore, as a consequence of this strong convergence, the first correction to the infinitesimal Maxwellian, which is a quadratic term obtained from the Chapman-Enskog expansion with the Navier-Stokes scaling, is rigorously justified. We point out that in all the works mentioned above, the convergence is in \( w-L^1 \) unless the initial data is well prepared, i.e., is hydrodynamic and satisfies the Boussinesq and incompressibility relations. This weak convergence is caused by the persistence of fast acoustic waves. In the Navier-Stokes regime, the Reynolds number \( \text{Re} \) is order \( O(1) \); then the von Kármán relation \( \varepsilon = \frac{\text{Ma}}{\text{Re}} \) implies that in the fluid limit \( \varepsilon \to 0 \), the Mach number \( \text{Ma} \) must go to 0. As is well-known, one expects that as \( \text{Ma} \to 0 \), fast acoustic waves are generated and carry the energy of the potential part of the flow. For the periodic flows, or for some particular boundary conditions such as the Navier condition (1.4), these waves subsist forever and their frequency grows with \( \varepsilon \). Mathematically, this means that the convergence is only weak. This phenomenon happens in many singular limits of fluid equations among which we only mention [30, 31].

One of the ingredients of the convergence proof is the treatment of the acoustic waves that are highly oscillating. A compensated compactness-type argument was used by Lions and Masmoudi [32] to prove that these acoustic waves have no contribution on the equation satisfied by the weak limit. This argument was previously used in the compressible incompressible limit [31].

In [10], a striking phenomenon, namely the damping of acoustic waves caused by the Dirichlet boundary condition, was found by Desjardins, Grenier, Lions, and Masmoudi in considering the incompressible limit of the isentropic compressible Navier-Stokes equations. In the case of a viscous flow in a bounded domain with Dirichlet boundary condition, and under a generic assumption on the domain (related to the so-called Schiffer’s conjecture and the Pompeiu problem [9]), they showed that the acoustic waves are instantaneously (asymptotically) damped due to the formation of a thin boundary layer in time. This layer is caused by a boundary layer in space and dissipates the energy carried by the acoustic waves. From a mathematical point of view, strong convergence was obtained.
Inspired by the idea of [10], the current paper considers the much more involved kinetic-fluid-coupled case. We prove that if the accommodation coefficient is bigger than the Knudsen number, there is no need for the argument in [31] since we can prove that the acoustic waves are damped instantaneously. Our work is based on the construction of viscous and kinetic Knudsen boundary layers of size $\sqrt{\varepsilon}$ and $\varepsilon$. The main idea is to use a family of test functions that solve approximately a scaled stationary linearized Boltzmann equation and can capture the propagation of the fast acoustic waves. These test functions are constructed through considering a family of approximate eigenfunctions of a dual operator with a dual kinetic boundary condition with respect to the original Boltzmann equation. The approximate eigenvalue is the sum of several terms with different orders of $\varepsilon$: the leading term is purely imaginary, which describes the acoustic mode, and the real part of the next order term is strictly negative, which gives the strict dissipation when applying the test functions to the renormalized Boltzmann equation.

In contrast to [10], the approximate eigenfunctions include the interior part and two boundary layers: the fluid viscous layer and the kinetic Knudsen layer, while in [10], only a fluid boundary layer was necessary. Another important difference is that a generic assumption on the domain had to be made in [10] (in particular, there are modes that are not damped in the disc), while in the current work, this assumption is not needed. The reason is that we deal with the full acoustic system, namely including the temperature. The NSF system has also some dissipation in the temperature equation that is ignored in the isentropic model. (In particular this dissipation property holds in the case of the ball.) This was also considered in [25], in which we reinforced the result of [10].

When the accommodation coefficient $\alpha_\varepsilon$ is asymptotically larger than the Knudsen number $\varepsilon$ in the sense that $\frac{\alpha_\varepsilon}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, the fluid limit is the NSF equations with the Dirichlet boundary condition. For example, we can assume $\alpha_\varepsilon = \varepsilon^\beta$ with $0 \leq \beta < 1$. We found that $\beta = \frac{1}{2}$ is a threshold in the sense that the kinetic-fluid-coupled boundary layers behave differently for $0 \leq \beta < \frac{1}{2}$ and $\frac{1}{2} \leq \beta < 1$, but for both cases the kinetic-fluid layers have damping effect. The current paper focuses on the threshold case $\beta = \frac{1}{2}$, and we leave the other cases for a separate paper due to the more complex construction of the boundary layers.

One of the difficulties of the construction happens in the case where the Laplace operator $-\Delta_x$ with Neumann boundary condition has multiple eigenvalues. As a consequence, the dimension of the null space of the the operator $A - i\lambda_0^k$ is greater than 1, where $A$ denotes the acoustic operator, and $\frac{D}{Dt+} [\lambda_0^k]^2$ are eigenvalues for $k \in \mathbb{N}$ (for details see Section 5.2). Thus, at each stage of the construction of boundary layers, the terms in the null space of $A - i\lambda_0^k$ cannot be determined uniquely. To completely determine all the terms in the ansatz of boundary layers, we have to add some orthogonality conditions. Surprisingly, all these orthogonality conditions are consistent, at least for the threshold case $\beta = \frac{1}{2}$ treated in the current paper. A similar idea has been used in [25], which can be applied to the
compressible-incompressible limit of the full Navier-Stokes-Fourier system in a bounded domain.

A key role is played by the linearized kinetic boundary layer equation in the coupling of viscous and kinetic layers. More specifically, its solvability provides the boundary conditions of the fluid variables in the interior and viscous boundary layers that satisfy the acoustic systems with source terms and second-order ordinary differential equations, respectively. This linearized kinetic boundary layer equation has been studied extensively (see [1, 8, 17, 18, 47]). Applying the boundary layer equations to construct the two-layer eigenfunctions is the main novelty of the current paper. To the best of our knowledge, these two-layer eigenfunctions are new even in the applied literature.

The paper is organized as follows: the next section contains preliminary material regarding the Boltzmann equation in a bounded domain. We state the main theorems in Section 3 (Theorems 3.1 and 3.2), which include the weak convergence for the Navier slip boundary and strong convergence for the Dirichlet boundary. In Section 4, we list some differential geometry properties of the boundary \( \partial \Omega \) as a submanifold of \( \mathbb{R}^D \). Section 5 provides an introduction to the acoustic modes, while Section 6 is about the analysis of the kinetic boundary layer equation whose solvability provides the boundary conditions of the fluid variables. In Section 7, we present the constructions of the test functions used in the proof Theorem 3.1. The proof of the main proposition on the boundary layers is given in Sections 8 and 9. In Section 10, we establish the weak convergence result of Theorems 3.1 and 3.2. Section 11 contains the proof of the strong convergence in the Dirichlet boundary case using the test functions constructed in Section 7.

2 Boltzmann Equation in Bounded Domain

Here we introduce the Boltzmann equation in a bounded domain, primarily to introduce our notation, which is essentially that of [3, 34]. A more complete introduction to the Boltzmann equation can be found in [6, 7, 14, 45].

2.1 Maxwell Boundary Condition

We consider \( \Omega \), a smooth bounded domain of \( \mathbb{R}^D \), and \( \mathcal{O} = \Omega \times \mathbb{R}^D \), the space-velocity domain. Let \( \mathbf{n}(x) \) be the outward unit normal vector at \( x \in \partial \Omega \) and let \( d\sigma_x \) be the Lebesgue measure on the boundary \( \partial \Omega \). We define the outgoing and incoming sets \( \Sigma_+ \) and \( \Sigma_- \) by

\[
\Sigma_\pm = \{(x, v) \in \Sigma : \pm \mathbf{n}(x) \cdot v > 0\} \quad \text{where} \quad \Sigma = \partial \Omega \times \mathbb{R}^D.
\]

Denoting by \( \gamma F \) the trace of \( F \) over \( \Sigma \), the boundary condition takes the form of a balance between the values of the outgoing and incoming parts of \( \gamma F \), namely \( \gamma_{\pm} F = \mathbb{1}_{\Sigma_\pm} \gamma F \). In order to describe the interaction between particles and the wall, Maxwell [35] proposed in 1879 the following phenomenological law that splits into a local reflection and a diffuse reflection:

\[
\gamma_- F = (1 - \alpha) L \gamma_+ F + \alpha K \gamma_+ F \quad \text{on} \ \Sigma_-,
\]

\[
(2.1)
\]
where $\alpha \in [0, 1]$ is a constant, called the *accommodation coefficient*. The local reflection operator $L$ is given by

$$L \phi(x, v) = \phi(x, R_x v),$$

where $R_x v = v - 2[n(x) \cdot v]n(x)$ is the velocity before the collision with the wall. The diffuse reflection operator $K$ is given by

$$K \phi(x, v) = \sqrt{2\pi} \tilde{\phi}(x) M(v),$$

where $\tilde{\phi}$ is the outgoing mass flux

$$\tilde{\phi}(x) = \int_{v \cdot n(x) > 0} \phi(x, v)v \cdot n(x) dv$$

and $M$ is the absolute Maxwellian $M(v) = \frac{1}{(2\pi)^{D/2}} \exp(-\frac{1}{2}|v|^2)$ that corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0. Furthermore, we notice that

$$\int_{v \cdot n(x) > 0} v \cdot n(x) \sqrt{2\pi} M(v) dv = \int_{v \cdot n(x) < 0} |v \cdot n(x)| \sqrt{2\pi} M(v) dv = 1,$$

which expresses the conservation of mass at the boundary. Here we take the temperature of the wall to be constant and equal to 1.

### 2.2 Nondimensionalized Form of the Boltzmann Equation

We consider a sequence of renormalized solutions $F_\varepsilon(t, x, v)$ to the rescaled Boltzmann equation

$$\varepsilon \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} B(F_\varepsilon, F_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O},$$

$$F_\varepsilon(0, x, v) = F_{\varepsilon}^0(x, v) \geq 0 \quad \text{on } \mathcal{O},$$

$$\gamma_- F_\varepsilon = (1 - \alpha)L \gamma_+ F_\varepsilon + \alpha K \gamma_+ F_\varepsilon \quad \text{on } \mathbb{R}^+ \times \Sigma_-.$$

The Boltzmann collision operator $B$ acts only on the $v$ argument of $F$ and is formally given by

$$B(F, F) = \iint_{S^{D-1} \times \mathbb{R}^D} (F'_1 F' - F_1 F)b(\omega, v_1 - v) d\omega dv_1,$$

where $v_1$ ranges over $\mathbb{R}^D$ endowed with its Lebesgue measure $dv_1$, while $\omega$ ranges over the unit sphere $S^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The $F'_1$, $F'$, $F_1$, and $F$ appearing in the integrand designate $F(t, x, \cdot)$ evaluated at the velocities $v'_1$, $v'$, $v_1$, and $v$, respectively, where the primed velocities are defined by

$$v'_1 = v_1 - \omega[\omega \cdot (v_1 - v)], \quad v' = v + \omega[\omega \cdot (v_1 - v)].$$
for any given \((\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D\). This expresses the conservation of momentum and energy for particle pairs after a collision, namely,

\[
v + v_1 = v' + v'_1, \quad |v|^2 + |v_1|^2 = |v'|^2 + |v'_1|^2.
\]

The collision kernel \(b\) is a positive, locally integrable function and has the classical form

\[
b(\omega, v) = |v|\Sigma(|\omega \cdot \hat{v}|, |v|),
\]

where \(\hat{v} = v/|v|\) and \(\Sigma\) is the specific differential cross section. This symmetry implies that the quantity \(\int b(\omega, v)\omega\,d\omega\) is a function of \(|v|\) only. The DiPerna-Lions theory requires that \(b\) satisfy

\[
\lim_{|v| \to \infty} \frac{1}{1 + |v|^2} \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} b(\omega, v_1 - v)\omega\,d\omega\,dv_1 = 0
\]

for any compact set \(K \subset \mathbb{R}^D\). There are some additional assumptions on \(b\) needed in [28]. For the convenience of the reader, we list these assumptions here.

A major role will be played by the attenuation coefficient \(a(v)\), which is defined as

\[
a(v) = \int_{\mathbb{R}^D} \frac{\bar{b}(v_1 - v)M_1}{a(v)}\,dv_1 = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} \frac{b(\omega, v_1 - v)\omega}{a(v)}\,d\omega\,M_1\,dv_1.
\]

A few facts about \(a(v)\) are readily evident from what we have already assumed. Because (2.4) holds, one can show that

\[
\lim_{|v| \to \infty} \frac{a(v)}{1 + |v|^2} = 0.
\]

Our second assumption regarding the collision kernel \(b\) is that \(a(v)\) satisfies a lower bound of the form

\[
C_a(1 + |v|)^\alpha \leq a(v)
\]

for some constant \(C_a > 0\) and \(\alpha \in \mathbb{R}\). The third assumption is that there exists \(s \in (1, \infty]\) and \(C_b \in (0, \infty]\) such that

\[
\left( \int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} \right|^s a(v_1)M_1\,dv_1 \right)^{\frac{1}{s}} \leq C_b.
\]

Another major role in what follows will be played by the operator \(\mathcal{L}\), which is the normalized linearization of the Boltzmann collision operator \(B\) around the global Maxwellian \(M\). More precisely,

\[
\frac{1}{M} B(M(1 + \delta \tilde{g}), M(1 + \delta \tilde{g})) =
\]

\[
-\delta \mathcal{L}\tilde{g} + \delta^2 \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} b(\omega, v_1 - v)dM_1\,dv_1,
\]
98 N. JIANG AND N. MASMOUDI

i.e.,

\[ L\tilde{g} = \iint_{S^{D-1} \times \mathbb{R}^D} (\tilde{g} + \tilde{g}_1 - \tilde{g}'_1)b(\omega, v_1 - v)d\omega M_1 \, dv_1. \]

One has the decomposition

\[ \frac{1}{a} \mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+, \]

where the loss operator $\mathcal{K}^-$ and the gain operator $\mathcal{K}^+$ are defined by

\[ \mathcal{K}^- \tilde{g} = \frac{1}{a} \int_{\mathbb{R}^D} \tilde{g}_1 \tilde{b}(v_1 - v)M_1 \, dv_1, \]

\[ \mathcal{K}^+ \tilde{g} = \frac{1}{a} \int_{\mathbb{R}^D} (\tilde{g}' + \tilde{g}'_1)b(\omega, v_1 - v)d\omega M_1 \, dv_1. \]

The fourth assumption regarding the collision kernel $b$ is that

\[ \mathcal{K}^+ : L^2(aM \, dv) \to L^2(aM \, dv) \text{ is compact}. \] Combining the gain operator assumption (2.10) and the loss operator assumption (2.7), we conclude that

\[ \frac{1}{a} \mathcal{L} : L^p(aM \, dv) \to L^p(aM \, dv) \text{ is Fredholm} \]

for every $p \in (1, \infty)$. From this Fredholm property we can define the psuedo-inverse of $\mathcal{L}$, called $\mathcal{L}^{-1}$:

\[ \mathcal{L}^{-1} : L^p(1/a \, M \, dv) \cap \text{Null}^\perp(\mathcal{L}) \to L^p(aM \, dv). \]

Moreover, $\mathcal{L}^{-1}$ is a bounded operator.

The fifth assumption regarding $b$ is that for every $\delta > 0$ there exists $C_\delta$ such that $\tilde{b}$ satisfies

\[ \frac{\tilde{b}(v_1 - v)}{1 + \delta \frac{\tilde{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leq C_\delta (1 + a(v_1))(1 + a(v)) \text{ for every } v_1, v \in \mathbb{R}^D. \]

It is well-known that the null space of the linearized Boltzmann operator $\mathcal{L}$ is given by $\text{Null}(\mathcal{L}) \equiv \text{span}[1, v_1, \ldots, v_D, |v|^2]$. Let $\mathcal{P}$ be the orthogonal projection from $L^2(M \, dv)$ onto $\text{Null}(\mathcal{L})$, namely,

\[ \mathcal{P}\tilde{g} = \langle \tilde{g} \rangle + v \cdot \langle \tilde{g} \rangle + \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \left( \left( \frac{1}{D} |v|^2 - 1 \right) \tilde{g} \right) \]

where the notation \( \langle \cdot \rangle \) is defined below in (2.17). Furthermore, we define $\mathcal{P}^\perp = \mathcal{I} - \mathcal{P}$. The matrix-valued function $A(v)$ and the vector-valued function $B(v)$ are defined by

\[ A(v) = v \otimes v - \frac{1}{D} |v|^2 I, \quad B(v) = \frac{1}{2} |v|^2 v - \frac{D + 2}{2} v. \]
We also define a scalar-valued function $C(v)$ by
\begin{equation}
C(v) = \frac{1}{4} |v|^4 - \frac{D + 2}{2} |v|^2 + \frac{D(D + 2)}{4}.
\end{equation}
It is easy to see that each entry of $A$, $B$, and $C$ are in $L^2(a^{-1} M dv) \cap \text{Null}^\perp(L)$. Furthermore, $C$ is perpendicular to each entry of $A$ and $B$. We also introduce $\hat{A} \in L^2(aM dv; \mathbb{R}^{D \times D})$ and $\hat{B} \in L^2(aM dv; \mathbb{R}^D)$ by
\begin{equation}
\hat{A} = L^{-1} A, \quad \hat{B} = L^{-1} B.
\end{equation}
Next, for the sake of simplicity, we take the following normalizations:
\begin{itemize}
  \item $S \in \mathbb{R}^D$, $\mathbb{R}^D$, $\mathbb{R}^D$
  \item $F_{\nu}^\text{in}$
\end{itemize}
associated with the domains $S^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, $\mathbb{R}^{D-1}$, $\mathbb{R}^D$, and $\Omega$, respectively, and
\begin{equation}
\int_{\Omega \times \mathbb{R}^D} \int b(\omega, v_1 - v) M_1 dv_1 dv = 1,
\end{equation}
\begin{equation}
\int_{\mathbb{R}^D} d\omega = 1, \quad \int_{\mathbb{R}^D} M dv = 1, \quad \int_{\Omega} dx = 1,
\end{equation}
\begin{equation}
\int_{\mathbb{R}^D} F_{\nu}^\text{in} dx dv = 1,
\end{equation}
\begin{equation}
\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v) M dv,
\end{equation}
\begin{equation}
\langle \xi, \eta \rangle = \int_{\mathbb{R}^D} \xi(v) \overline{\eta(v)} M dv,
\end{equation}
where $\overline{\eta}$ denotes the complex conjugate of $\eta$. We also use the following average on the boundary:
\begin{equation}
\langle \xi \rangle_{\partial \Omega} = \int_{\mathbb{R}^D} \xi(v)[\mathbf{n}(x) \cdot v] \sqrt{2\pi} M dv,
\end{equation}
from which we have $\langle 1_{\Sigma^+} \rangle_{\partial \Omega} = -\langle 1_{\Sigma^-} \rangle_{\partial \Omega} = 1$. Moreover, because $d\mu = b(\omega, v_1 - v) M_1 dv_1 M dv$ is a positive unit measure on $S^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle [\Xi] \rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$,
\begin{equation}
\langle [\Xi] \rangle = \int_{\mathbb{R}^D} \Xi(\omega, v_1, v) d\mu.
\end{equation}
The measure $d\mu$ is invariant under the coordinate transformations
\[(\omega, v_1, v) \mapsto (\omega, v, v_1), \quad (\omega, v_1, v) \mapsto (\omega, v_1', v').\]
These are called $d\mu$-symmetries.

### 2.3 Navier-Stokes Scaling

The incompressible NSF system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic densities $F_\varepsilon$ about the absolute Maxwellian $M$ are scaled to be of order $\varepsilon$. More precisely, we take

\begin{equation}
F_\varepsilon = MG_\varepsilon = M(1 + \varepsilon g_\varepsilon).
\end{equation}

Rewriting equation (2.3) for $G_\varepsilon$ yields

\begin{equation}
\varepsilon \partial_t G_\varepsilon + v \cdot \nabla_x G_\varepsilon = \frac{1}{\varepsilon} Q(G_\varepsilon, G_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O},
\end{equation}

\begin{equation}
G_\varepsilon(0, x, v) = G^{ii}_\varepsilon(x, v) \quad \text{on } \mathcal{O},
\end{equation}

\begin{equation}
\gamma_- G_\varepsilon = (1 - \alpha) L \gamma + G_\varepsilon + \alpha \langle \gamma + G_\varepsilon \rangle d\Omega \quad \text{on } \mathbb{R}^+ \times \Sigma_-, \end{equation}

where the collision kernel $Q$ is now given by

\[Q(G, G) = \iint_{S^{d-1} \times \mathbb{R}^d} (G'_1 G' - G_1 G)b(\omega, v_1 - v) d\omega M_1 dv_1.\]

In terms of $g_\varepsilon$ the system (2.3) finally reads

\begin{equation}
\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon = Q(g_\varepsilon, g_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O},
\end{equation}

\begin{equation}
g_\varepsilon(0, x, v) = g^{ii}_\varepsilon(x, v) \quad \text{on } \mathcal{O},
\end{equation}

\begin{equation}
\gamma_- g_\varepsilon = (1 - \alpha) L \gamma + g_\varepsilon + \alpha \langle \gamma + g_\varepsilon \rangle d\Omega \quad \text{on } \mathbb{R}^+ \times \Sigma_-.
\end{equation}

### 2.4 A Priori Estimates

Due to the presence of the boundary, the classical a priori estimates for the Boltzmann equation, namely the entropy and energy bounds, are modified. First, because all particles arriving at the boundary are reflected or diffused, we have conservation of mass, which can be written as

\[\int_{\Omega} \langle G_\varepsilon \rangle dx = \int_{\Omega} \langle G^{ii}_\varepsilon \rangle dx = 1.\]
Multiplying the equation \((2.20)\) by \(\log(G_\varepsilon)\) and integrating in \(x\) and \(v\), we get formally

\[
\varepsilon \partial_t \int_{\Omega} (G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1) \, dx
\]

\[
+ \int_{\Sigma} (G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1) v \cdot \mathbf{n}(x) \, d\sigma_x \, M \, dv
\]

\[
= \frac{1}{\varepsilon} \int_{\Omega} \langle \log(G_\varepsilon) \mathcal{Q}(G_\varepsilon, G_\varepsilon) \rangle \, dx.
\]

By denoting \(h(z) = (1 + z) \log(1 + z) - z\) for \(z > -1\) and using the fact that it is a convex function, we can compute the boundary term in the following way:

\[
\tilde{\mathcal{E}}_\varepsilon(\gamma_\varepsilon + G_\varepsilon)
\]

\[
= \int_{\Sigma} [G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1] v \cdot \mathbf{n}(x) \, d\sigma_x \, M \, dv
\]

\[
= \int_{\Sigma} [h(\varepsilon \gamma_\varepsilon + g_\varepsilon) - h((1 - \alpha_\varepsilon)\varepsilon \gamma_\varepsilon + g_\varepsilon + \alpha_\varepsilon \{\varepsilon \gamma_\varepsilon + g_\varepsilon\}_\partial\Omega)] v \cdot \mathbf{n}(x) \, M \, dv \, d\sigma_x
\]

\[
\geq \int_{\Sigma} [h(\varepsilon \gamma_\varepsilon + g_\varepsilon) - (1 - \alpha_\varepsilon)h(\varepsilon \gamma_\varepsilon + g_\varepsilon) - \alpha_\varepsilon h(\{\varepsilon \gamma_\varepsilon + g_\varepsilon\}_\partial\Omega)] v \cdot \mathbf{n}(x) \, M \, dv \, d\sigma_x
\]

\[
= \frac{\alpha_\varepsilon}{\sqrt{2\pi}} \mathcal{E}(\gamma_\varepsilon + G_\varepsilon),
\]

where \(\mathcal{E}(\gamma_\varepsilon + G_\varepsilon)\), the so-called Darrozès-Guiraud information, is given by

\[
\mathcal{E}(\gamma_\varepsilon + G_\varepsilon) = \int_{\partial\Sigma} [(h(\varepsilon \gamma_\varepsilon + g_\varepsilon))_\partial\Omega - h(\varepsilon \gamma_\varepsilon + g_\varepsilon)] \, d\sigma_x.
\]

Jensen’s inequality implies that \(\mathcal{E}(\gamma_\varepsilon + G_\varepsilon) \geq 0\). Noticing that

\[
\tilde{\mathcal{E}}_\varepsilon(\gamma_\varepsilon + G_\varepsilon) \geq \frac{\alpha_\varepsilon}{\sqrt{2\pi}} \mathcal{E}(\gamma_\varepsilon + G_\varepsilon),
\]

we get the entropy inequality

\[
H(G_\varepsilon(t)) + \int_0^t \left( \frac{1}{\varepsilon^2} R(G_\varepsilon(s)) + \frac{1}{\varepsilon} \tilde{\mathcal{E}}_\varepsilon(\gamma_\varepsilon + G_\varepsilon(s)) \right) \, ds \leq H(G_\varepsilon^m),
\]

where \(H(G)\) is the relative entropy functional

\[
H(G) = \int_{\Omega} (G \log(G) - G + 1) \, dx
\]
and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\Omega} \left\langle \frac{1}{4} \log \left( \frac{G'G''}{G_1 G} \right) (G_1 G' - G_G G) \right\rangle \, dx.$$  

### 2.5 DiPerna-Lions(-Mischler) Solutions

We will work in the setting of renormalized solutions that were initially constructed by DiPerna and Lions [12] over the whole space $\mathbb{R}^D$ for any initial data satisfying natural physical bounds. Recently their result was extended to the case of a bounded domain by Mischler [36–38] with general Maxwell boundary conditions (2.1).

The DiPerna-Lions(-Mischler) theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial value problems that are obtained by multiplying (2.20) by $\mathcal{E}_{0}$.\G/:

$$\begin{align*}
(\mathcal{E} \partial_t + v \cdot \nabla_x) \Gamma(G_\varepsilon) &= \frac{1}{\varepsilon} \Gamma'(G_\varepsilon) Q(G_\varepsilon, G_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \\
G_\varepsilon(0, \cdot, \cdot) &= G_{\varepsilon}^{in} \geq 0 \quad \text{on } \mathcal{O}.
\end{align*}$$

(2.23)

Here the admissible function $\Gamma : [0, \infty) \to \mathbb{R}$ is continuously differentiable and for some constant $C_\Gamma < \infty$ its derivative satisfies

$$|\Gamma'(z)| \sqrt{1 + z} \leq C_\Gamma.$$  

(2.24)

The weak formulation of the renormalized Boltzmann equation (2.23) is given by

$$\begin{align*}
\varepsilon \int_{\Omega} \langle \Gamma(G_\varepsilon(t_2)) Y \rangle \, dx - \varepsilon \int_{\Omega} \langle \Gamma(G_\varepsilon(t_1)) Y \rangle \, dx \\
- \int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma(G_\varepsilon) v \cdot \nabla_x Y \rangle \, dx \, dt + \int_{t_1}^{t_2} \int_{\partial \Omega} \langle \Gamma(F_\varepsilon) Y [n(x) \cdot v] \rangle \, d\sigma_x \, dt \\
= \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma'(G_\varepsilon) Q(G_\varepsilon, G_\varepsilon) Y \rangle \, dx \, dt.
\end{align*}$$

(2.25)

for every $Y \in C^1 \cap L^\infty(\Omega \times \mathbb{R}^D)$ and every $[t_1, t_2] \subset [0, \infty]$. Moreover, the boundary condition is also understood in the renormalized sense:

$$\Gamma(\gamma_- G_\varepsilon) = \Gamma((1 - \alpha) L_{\gamma_-} G_\varepsilon + \alpha \tilde{F}_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \Sigma_-,$$

(2.26)

where the equality holds almost everywhere and in the sense of distributions.
PROPOSITION 2.1 (Renormalized solutions in bounded domain [38]). Let \( b \) satisfy the condition (2.4). Given any initial data \( G_{\varepsilon}^{\text{in}} \) satisfying

\[
\int_{\mathcal{O}} G_{\varepsilon}^{\text{in}} (1 + |v|^2 + |\log G_{\varepsilon}^{\text{in}}|) M \, dv \, dx < +\infty,
\]

there exists at least one \( G_{\varepsilon} \geq 0 \) in \( C([0, \infty); L^1(M \, dv \, dx)) \) such that (2.25) and (2.26) hold for all admissible functions \( \Gamma \). Moreover, \( G_{\varepsilon} \) satisfies the following global entropy inequality for all \( t > 0 \):

\[
H(G_{\varepsilon}(t)) + \frac{1}{\varepsilon} \int_0^t R(G_{\varepsilon}(s)) \, ds + \frac{1}{\varepsilon} \int_0^t \tilde{E}_{\varepsilon}(\gamma+G_{\varepsilon}(s)) \leq H(G_{\varepsilon}^{\text{in}}).
\]

3 Statement of the Main Results

In this section we state our main results on justifying the incompressible NSF limits with different boundary conditions depending on the quotient between the accommodation coefficients \( \alpha \) and the Knudsen number \( \varepsilon \).

3.1 Dirichlet Boundary Condition

The main theorem of this paper is the following strong convergence to the NSF system with Dirichlet boundary condition when the accommodation coefficient \( \alpha \) is much larger than the Knudsen number \( \varepsilon \), i.e., \( \frac{\alpha}{\varepsilon} \to \infty \) as \( \varepsilon \to 0 \).

THEOREM 3.1 (Dirichlet boundary condition). Let \( b \) be a collision kernel that satisfies conditions (2.5), (2.6), (2.7), (2.10), and (2.11). Let \( G_{\varepsilon}^{\text{in}} \) be any family of nonnegative measurable functions of \( (x, v) \) satisfying (2.27) and the renormalization (2.16). Let \( g_{\varepsilon}^{\text{in}} \) be the associated family of fluctuations given by \( G_{\varepsilon}^{\text{in}} = 1 + \varepsilon g_{\varepsilon}^{\text{in}} \). Assume that the families \( G_{\varepsilon}^{\text{in}} \) and \( g_{\varepsilon}^{\text{in}} \) satisfy

\[
H(G_{\varepsilon}^{\text{in}}) \leq C^{\text{in}} \varepsilon^2
\]

and

\[
\lim_{\varepsilon \to 0} \left( \langle G_{\varepsilon}^{\text{in}}, v_{g_{\varepsilon}^{\text{in}}}, \left( \frac{|v|^2}{D} - 1 \right) g_{\varepsilon}^{\text{in}} \rangle = \langle \rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}} \rangle \right)
\]

in the sense of distributions for some \( (\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}) \). Let \( G_{\varepsilon} \) be any family of DiPerna-Lions renormalized solutions to the Boltzmann equation (2.20) that have \( G_{\varepsilon}^{\text{in}} \) as initial values, and the accommodation coefficient \( \alpha_{\varepsilon} \) satisfies

\[
\alpha_{\varepsilon} = \sqrt{2\pi} \chi \sqrt{\varepsilon}.
\]

Then the family of fluctuations \( g_{\varepsilon} \) given by (2.19) is relatively compact in the space \( L^1_{\text{loc}}(dt; L^1(\sigma M \, dv \, dx)) \). Every limit \( g \) of \( g_{\varepsilon} \) is an infinitesimal Maxwellian

\[
g = v \cdot u + \left( \frac{1}{2} |v|^2 - \frac{D + 2}{2} \right) \theta,
\]
where \((u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R})) \cap L^2(dt; H^1(dx; \mathbb{R}^D \times \mathbb{R}))\) with mean zero over \(\Omega\), and it satisfies the NSF system with Dirichlet boundary condition (1.1), (1.2), and (1.3), where kinematic viscosity \(\nu\) and thermal conductivity \(\kappa\) are given by

\[
(3.5) \quad \nu = \frac{1}{(D - 1)(D + 2)} (\hat{A} : \mathcal{L} \hat{A}), \quad \kappa = \frac{1}{D} (\hat{B} : \mathcal{L} \hat{B}).
\]

The initial data is given by

\[
(3.6) \quad u^0 = \mathbb{P} u^i, \quad \theta^0 = \frac{D}{D + 2} \theta^i - \frac{2}{D + 2} \rho^i.
\]

Here the operator \(\mathbb{P}\) is the Leray’s projection on the space of divergence-free vector fields. Moreover, every subsequence \(g_{\varepsilon_k}\) of \(g\) that converges to \(g\) as \(\varepsilon_k \to 0\) also satisfies

\[
(3.7) \quad \left( \left( \frac{1}{D} |v|^2 - 1 \right) g_{\varepsilon_k} \right) \to u \quad \text{in} \quad L^p_{\text{loc}}(dt; L^1(dx; \mathbb{R}^D)),
\]

\[
(3.8) \quad -\hat{A} : \nabla u - \hat{B} : \nabla \theta \quad \text{in} \quad w^{-1}L^1_{\text{loc}}(dt; w^{-1}L^1(\sigma M \, dv \, dx)),
\]

as \(\varepsilon \to 0\), where \(A, B, C, \text{ and } \hat{A}, \hat{B}\) are defined in (2.13), (2.14), and (2.15).

**Remark.** In the formal Chapman-Enskog expansion,

\[
g_{\varepsilon} = g + \varepsilon \mathcal{P}^1 g_1 + \varepsilon \mathbb{P} g_1 + \varepsilon^2 g_2 + \cdots,
\]

where \(g\) is given by (3.4) and \(\mathcal{P}^1 g_1\) is the right-hand side term in (3.8). In previous works \([20, 21, 28]\), under the assumptions (3.1) and (3.2), the convergence to (3.4) and (3.7) are only in \(wL^1\). So the convergence to the quadratic term (3.8), which is the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with the Navier-Stokes scaling, could not be obtained. In Theorem 3.1 by showing that the acoustic waves are instantaneously damped, we justify not only the strong convergence to the leading-order term \(g\) but also weak convergence to the kinetic part of the next order corrector (3.8).

### 3.2 Navier Boundary Condition

The second result is about the Navier boundary condition. For this case, although the coupled viscous boundary layer and the Knudsen layer still have dissipative effect, the damping happens over a longer time scale \(O(1)\). Consequently, unlike the Dirichlet boundary condition case, the fast acoustic waves can be damped, but not instantaneously. Nevertheless, we can show the weak convergence result,
thus justifying the NSF limit with the slip Navier boundary condition, while the linear Stokes-Fourier limit was justified in [34].

**Theorem 3.2 (Navier boundary condition).** We use the same assumptions as in Theorem 3.1 except that the accommodation coefficients satisfy

\[
\frac{\alpha_\epsilon}{\sqrt{2\pi \epsilon}} \to \chi \quad \text{as} \quad \epsilon \to 0.
\]

Then the family \(g_\epsilon\) is relatively compact in \(w^{-1}_{-1}\) and \(\text{sup} \, L^1_{\text{loc}}(dt; w^{-1}_{-1}(\sigma M \, dv \, dx))\). Every limit point \(g\) of \(g_\epsilon\) in \(w^{-1}_{-1}\) has the infinitesimal Maxwellian form as (3.4) in which

\[
(u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R})) \cap L^2(dt; H^1(dx; \mathbb{R}^D \times \mathbb{R}))
\]

is a Larey solution of the NSF system with Navier boundary condition (1.1), (1.2), and (1.4), where kinematic viscosity \(\nu\) and thermal conductivity \(\kappa\) are given by (3.5), the initial data is given by (3.6).

Moreover, every subsequence \(g_{\epsilon_k}\) of \(g_\epsilon\) that converges to \(g\) as \(\epsilon_k \to 0\) also satisfies

\[
\mathbb{P} \left(vg_{\epsilon_k}\right) \to u \quad \text{in} \quad C([0, \infty); D'(\Omega; \mathbb{R}^D)),
\]

\[
\left(\left(\frac{1}{D+2} [v]^2 - 1\right)g_{\epsilon_k}\right) \to \theta \quad \text{in} \quad C([0, \infty); w^{-1}_{-1}(\Omega; \mathbb{R})).
\]

**Remark.** For the Navier slip boundary condition case, since the convergence is weak, the convergence (3.8), i.e., the justification of the first correction to the infinitesimal Maxwellian in the Chapman-Enskog expansion, cannot be obtained.

### 4 Geometry of the Boundary \(\partial \Omega\)

In this section, we collect some differential geometry properties related to the boundary \(\partial \Omega\), which can be considered as a \((D-1)\)-dimension Riemannian manifold with a metric induced from the standard euclidian metric of \(\mathbb{R}^D\). From the following classical result in geometry (for the proof, see [43]), there is a tubular neighborhood \(\Omega_{\delta} = \{x \in \Omega : \text{dist}(x, \partial \Omega) < \delta\}\) of \(\partial \Omega\) such that the nearest point projection map is well defined.

**Lemma 4.1.** If \(\partial \Omega\) is a compact \(C^k\) submanifold of dimension \(D-1\) embedded in \(\mathbb{R}^D\), then there is a \(\delta = \delta_{\Omega} > 0\) and a map \(\pi \in C^{k-1}(\Omega^\delta; \mathbb{R}^D)\) such that the following properties hold:

(i) For all \(x \in \Omega \subset \mathbb{R}^D\) with \(\text{dist}(x, \partial \Omega) < \delta\),

\[
\pi(x) \in \partial \Omega, \quad x - \pi(x) \in T^\perp_{\pi(x)}(\partial \Omega), \quad |x - \pi(x)| = \text{dist}(x, \partial \Omega), \quad \text{and} \quad |z - x| > \text{dist}(x, \partial \Omega) \quad \text{for any} \quad z \in \partial \Omega \setminus \{\pi(x)\}.
\]

(ii) \(\pi(x + z) \equiv x \) for \(x \in \partial \Omega\), \(z \in T_x(\partial \Omega)^\perp\), \(|z| < \delta\).
(iii) Let $\text{Hess} \pi^x$ denote the Hessian of $\pi$ at $x$; then

$$\text{Hess} \pi^x(V_1, V_2) = h_x(V_1, V_2) \text{ for } x \in \partial \Omega, \quad V_1, V_2 \in T_x(\partial \Omega),$$

where $h_x$ is the second fundamental form of $\partial \Omega$ at $x$.

The viscous boundary layer has significantly different behavior over the tangential and normal directions near the boundary. This inspires us to consider the following new coordinate system, which we call the curvilinear coordinate for the tubular neighborhood $\Omega^\delta$ defined in Lemma 4.1. Because $\partial \Omega$ is a $(D - 1)$-dimensional manifold, locally $\pi(x)$ can be represented by

$$\pi(x) = (\pi^1(x), \ldots, \pi^{D-1}(x)).$$

More precisely, the representation (4.1) can be understood in the following sense: we can introduce a new coordinate system $(\xi^1, \ldots, \xi^D)$ by a homeomorphism that is locally defined as $\xi$, $\xi(x) = (\xi^1(x), \xi^D(x))$, where $\xi^1 = (\xi^1, \ldots, \xi^{D-1})$, such that $\xi(\pi(x)) = (\xi^1, 0)$ and $d(x) = \xi^D$, where $d(x)$ is the distance function to the boundary $\partial \Omega$, i.e.,

$$d(x) = \text{dist}(x, \partial \Omega) = |x - \pi(x)|.$$  

To simplify notation, we denote “$\xi'(x) = \pi(x)$,” which is the meaning of (4.1).

It is easy to see that $\nabla_x d$ is perpendicular to the level surface of the distance function $d$, i.e., the set $S^z = \{x \in \Omega : d(x) = z\}$. In particular, on the boundary, $\nabla_x d$ is perpendicular to $S_0 = \partial \Omega$. Without loss of generality, we can normalize the distance function so that $\nabla_x d(x) = -n(x)$ when $x \in \partial \Omega$. By the definition of the projection $\Pi$, we have

$$\pi(x + t \nabla_x d(x)) = \pi(x) \text{ for } t \text{ small},$$

and consequently $\nabla_x \pi^\alpha \cdot \nabla_x d = 0$ for $\alpha = 1, \ldots, D - 1$. In particular, for $t$ small enough, $\nabla_x \pi^\alpha(x) \in T_x(\partial \Omega)$ when $x \in \partial \Omega$.

Next, we calculate the induced Riemannian metric from $\mathbb{R}^D$ on $\partial \Omega$. In a local coordinate system, this Riemannian metric can be represented as

$$g = g_{\alpha \beta} \, d\pi^\alpha \otimes d\pi^\beta,$$

where $g_{\alpha \beta} = \langle \frac{\partial}{\partial \pi^\alpha}, \frac{\partial}{\partial \pi^\beta} \rangle$. Noticing that

$$\frac{\partial}{\partial x^i} = \frac{\partial \pi^\alpha}{\partial x^i} \frac{\partial}{\partial \pi^\alpha} \quad \text{and} \quad \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \delta_{ij},$$

the metric $g_{\alpha \beta}$ can be determined by

$$g_{\alpha \beta} \frac{\partial \pi^\alpha}{\partial x^i} \frac{\partial \pi^\beta}{\partial x^j} = 1.$$
5 Acoustic Modes

5.1 Acoustic Operator $\mathcal{A}$

Recall that the Leray’s projection $\mathcal{P}$ on the space of divergence-free vector fields and $\mathcal{Q}$ on the space of gradients are defined by

$$\mathcal{P} = I - \mathcal{Q},$$

where $\mathcal{Q}u = \nabla_x q$ and $q$ solves

$$\Delta_x q = \nabla_x \cdot u \text{ in } \Omega, \quad \nabla_x q \cdot n = u \cdot n \text{ on } \partial \Omega, \quad \text{and } \int_\Omega q \, dx = 0. \tag{5.1}$$

We define Hilbert spaces

$$\mathbb{H} = \left\{ U = (\rho, u, \theta) \in L^2(\Omega; \mathbb{C}^D \times \mathbb{C}) \right\},$$

$$\mathbb{V} = \left\{ U \in \mathbb{H} : \int_\Omega |\nabla_x U|^2 \, dx < \infty \right\},$$

donf, endowed with inner product

$$\langle U_1, U_2 \rangle_\mathbb{H} = \int_\Omega \left( \rho_1 \bar{\rho}_2 + u_1 \cdot \bar{u}_2 + \frac{D}{2} \theta_1 \bar{\theta}_2 \right) \, dx, \tag{5.2}$$

where $\bar{f}$ denotes the complex conjugate of the complex-valued function $f$.

Next, we define the acoustic operator $\mathcal{A}$:

$$\mathcal{A} \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x (\rho + \theta) \\ \frac{2}{D} \nabla_x \cdot u \end{pmatrix}, \tag{5.3}$$

over the domain

$$\text{Dom}(\mathcal{A}) = \left\{ U = (\rho, u, \theta) \in \mathbb{V} : u \cdot n = 0 \text{ on } \partial \Omega \right\}.$$ 

The null space of $\mathcal{A}$ and its orthogonal with respect to the inner product (5.2) are characterized as

$$\text{nullspace} \left( \mathcal{A} \right) = \left\{ (w, \varphi) \in \mathbb{V} : \nabla_x \cdot w = 0 \text{ and } w \cdot n = 0 \text{ on } \partial \Omega \right\} \tag{5.4}$$

and

$$\text{nullspace} \left( \mathcal{A} \right)^\perp = \left\{ (\rho, u, \theta) \in \mathbb{V} : \theta = \frac{2}{D} \rho, \ u = \nabla_x \phi \text{ for some } \phi \in H^1(\Omega) \right\}, \tag{5.5}$$

respectively. Because nullspace(\mathcal{A}) includes the incompressibility and Boussinesq relations, we call it an incompressible regime. We will see in the next subsection that nullspace(\mathcal{A})^\perp is spanned by the eigenspaces of the acoustic operator $\mathcal{A}$, so we call it an acoustic regime.
For any \( U = (\rho, u, \theta) \in \mathbb{H} \), we can define \( \Pi \) and \( \Pi^\perp \), the projections to the incompressible regime \( \text{Null}(\mathcal{A}) \) and acoustic regime \( \text{Null}(\mathcal{A})^\perp \), respectively, as follows:

\[
\Pi U = \left( \frac{2}{D+2} \rho - \frac{D}{D+2} \theta, \frac{D}{D+2} \rho - \frac{2}{D+2} \theta \right).
\]

\[
\Pi^\perp U = \left( \frac{D}{D+2} \rho + \theta, \frac{2}{D+2} \rho + \theta \right).
\]

### 5.2 Eigenspaces of \( \mathcal{A} \)

The eigenvalues and eigenvectors of the acoustic operator \( \mathcal{A} \) in a bounded domain can be constructed from those of the Laplace operator with Neumann boundary condition in the following way: Let \( \frac{D}{D+2} [\lambda^k]_2, \lambda^k > 0, k \in \mathbb{N} \), be the nondecreasing sequence of eigenvalues of the Laplace operator \(-\Delta_n\) with homogeneous Neumann boundary condition, and \( \Psi^k \) be the corresponding orthonormal basis of \( L^2(\Omega) \) eigenfunctions:

\[
-\Delta_x \Psi^k = \frac{D}{D+2} \Psi^k \quad \text{in} \ \Omega, \quad \nabla_x \Psi^k \cdot \mathbf{n} = 0 \quad \text{on} \ \partial \Omega.
\]

More specifically,

\[
0 < \lambda^1 \leq \lambda^2 \leq \cdots \leq \lambda^k \to +\infty \quad \text{as} \ k \to \infty.
\]

Let \( \tau \) denote either + or -, and \( \lambda^{\tau,k} = \tau \lambda^k \). It can be verified that \( i\lambda^{\tau,k} \) are nonzero eigenvalues of \( \mathcal{A} \) and

\[
U^{\tau,k} = \sqrt{\frac{D+2}{2D}} \left( D \frac{\Psi^k}{D+2}, \frac{\nabla_x \Psi^k}{i \lambda^{\tau,k}}, \frac{2}{D+2} \Psi^k \right)^T
\]

are the corresponding normalized eigenvectors, i.e.,

\[
\mathcal{A} U^{\tau,k} = i \lambda^{\tau,k} U^{\tau,k},
\]

and furthermore, \( U^{\tau,k} \) span \( \text{Null}(\mathcal{A})^\perp \) under the inner product (5.2). Consequently, we have an orthonormal basis of the acoustic modes, i.e.,

\[
\text{Null}(\mathcal{A})^\perp = \overline{\text{span}\{U^{\tau,k} \mid k \in \mathbb{N}, \tau = \pm\}}^{L^2}.
\]

Moreover, we can use the components of \( U^{\tau,k} \) to construct the infinitesimal Maxwellians \( g^{\tau,k} \) that are in the null space of \( \mathcal{L} \):

\[
g^{\tau,k} = \sqrt{\frac{D+2}{2D}} \left( \frac{D}{D+2} \Psi^k + v \cdot \nabla_x \Psi^k + \frac{2}{D+2} \Psi^k \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \right).
\]

These infinitesimal Maxwellians will be the building blocks of the approximate eigenfunctions of \( \frac{1}{\varepsilon} \mathcal{L} - v \cdot \nabla_x \).
5.3 Conditions on $\Psi^k$

Note that $\Psi^k, k \geq 1$, are solutions to the Neumann boundary condition equation (5.6), so some orthogonality condition is required for the eigenfunctions associated to the eigenvalues with multiplicity greater than 1. Assume that $\lambda^2$ is an eigenvalue of (5.6) and denote by $H_0 = H_0(\lambda)$ the eigenspace associated to $\lambda^2$, i.e.,

\begin{equation}
H_0(\lambda) = \left\{ \Psi \in \text{Dom}(\Delta_x) : -\Delta_x \Psi = \lambda^2 \Psi \text{ in } \Omega, \frac{\partial \Psi}{\partial n} = 0 \text{ on } \partial \Omega \right\}
\end{equation}

where $\text{Dom}(\Delta_x) = H^2(\Omega) \cap \{ \Psi \mid \frac{\partial \Psi}{\partial n} = 0 \text{ on } \partial \Omega \}$ denotes the domain of $-\Delta_x$ with the Neumann boundary condition. On the finite-dimensional space $H_0(\lambda)$, we can define a quadratic form $Q_1$. It has an associated bilinear form that we still denote by $Q_1$ and a symmetric operator $L_1 = L_1^2$ by

\begin{equation}
Q_1(\Psi, \Phi) = \int_{\Omega} L_1(\Psi) \Phi \, dx = \int_{\Omega} L_1(\Phi) \Psi \, dx.
\end{equation}

The eigenspace $H_0(\lambda)$ is endowed with an orthogonality condition

\begin{equation}
Q_1(\Psi^k, \Psi^l) = 0 \text{ if } \Psi^k, \Psi^l \in H_0(\lambda) \text{ and } k \neq l.
\end{equation}

This condition means that the eigenvectors $\Psi^k$ for $\lambda^k = \lambda$ are orthogonal for the symmetric operator $L_1^2$. Of course, since $L^2(\Omega)$ is the direct sum of the spaces $H_0(\lambda)$ for different $\lambda$’s, from the definition of $L_1^2$ on each eigenspace $H_0(\lambda)$, we can define an operator $L_1$ on $L^2(\Omega)$ that leaves each eigenspace $H_0(\lambda)$ invariant. But this is not necessary, so we will think of $L_1 = L_1^2$ as acting on $H_0(\lambda)$ for a fixed multiple eigenvalue $\lambda$.

The orthogonality condition (5.12) turns out to be enough to construct the boundary layer if the eigenvalues of $L_1$ are simple, namely, if $\lambda^k_1 \neq \lambda^l_1$ for all $k \neq l$ such that $\lambda^0_0 = \lambda^l_0 = \lambda$. However, if $\lambda_1$ is an eigenvalue of $L_1$ with multiplicity greater than or equal to 2, then we need an extra orthogonality condition. Let $H_1 = H_1(\lambda_1)$ be defined by

\begin{equation}
H_1 = \{ \Psi \in H_0 : L_1 \Psi = \lambda_1 \Psi \}.
\end{equation}

On the finite-dimensional space $H_1$, there exists a quadratic form $Q_2$ and a symmetric operator $L_2$ (see the definition below); the extra condition is

\begin{equation}
Q_2(\Psi^k, \Psi^l) = 0 \text{ if } \Psi^k, \Psi^l \in H_1(\lambda) \text{ and } k \neq l.
\end{equation}

This condition is enough if $L_2$ has only simple eigenvalues on the vector space $H_1$. This process can be continued inductively.

Let us now explain more precisely the condition we have to impose on the eigenvectors of $-\Delta_x$. We can construct recursively, on each eigenspace $H_0(\lambda)$ of $-\Delta_x$, a sequence of symmetric operators $L_q, q \in \mathbb{N}$, in the following way: Let $L_0 = -\Delta_x$; we define $L_1$ on each of the eigenspaces $H_0(\lambda)$ of $L_0$ by (5.11). Assume that the operators $L_p$ were constructed for $p \leq q - 1, q \geq 2$, in such a
way that each operator $L_p$ leaves invariant the eigenspaces of the operators $L_{p'}$ for $p' < p$.

Now, to construct $L_q$, it is enough to construct $L_q$ on each eigenspace $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{q-1}(\lambda_{q-1})$, where $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}$ are eigenvalues of $L_1, L_2, \ldots, L_{q-1}$, respectively. This is done by constructing a quadratic form $Q_q$ on each space $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{q-1}(\lambda_{q-1})$ and defining $L_q$ by

$$Q_q(\Psi, \Phi) = \int_\Omega L_q(\Psi)\Phi \, dx$$

for all $\Psi, \Phi \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{q-1}(\lambda_{q-1})$.

The precise construction of the quadratic form $Q_q$ on the space $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{q-1}(\lambda_{q-1})$ will be done in the proof.

Let $N \in \mathbb{N}$ be an integer. This is the integer that will appear in the order of the approximation in Proposition 7.1. The eigenvectors $\Phi^k$ for $L^{\lambda_k} = \lambda$ should be chosen in such a way that they are eigenvectors for all the operators $L_n$ at least for $n \leq N + 2$. This implies that they are orthogonal to all the operators $L_n$ for $n \leq N + 2$, which means that

$$(5.15) \quad Q_n(\Phi^k, \Phi^l) = \int_\Omega L_n(\Phi^k)\Phi^l = 0,$$

if $\Phi^k, \Phi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{n-1}(\lambda_{n-1})$ and $k \neq l$.

**Remark.** The precise construction of the quadratic form $Q_q$ will be done in the proof of Proposition 7.1.

### 5.4 The Operator $A - i\lambda \cdot \tau^k$

Later on, in the construction of the boundary layers, for each acoustic mode $k \geq 1$ and $\tau = +$ or $-$, we will frequently solve the following linear hyperbolic system for $V^{\tau,k} = (\rho^{\tau,k}, v^{\tau,k}, \theta^{\tau,k})^\top$:

$$(A - i\lambda \cdot \tau^k) V^{\tau,k} = i\mu^{\tau,k} U^{\tau,k} + F^{\tau,k},$$

$$(5.16) \quad v^{\tau,k} \cdot n = g^{\tau,k} \quad \text{on } \partial\Omega,$$

where $\mu^{\tau,k}, F^{\tau,k},$ and $g^{\tau,k}$ are given, and $U^{\tau,k}$ is defined in (5.7).

**Remark.** Strictly speaking, (5.16) is not rigorous because $v^{\tau,k} \cdot n$ is nonzero, so $V^{\tau,k}$ is not in the domain of $A$. For notational simplicity, we still use $A$ in (5.16) and later on. The notation $A$ just means the expression of $A$ in (5.3) regardless of the domain.

To solve system (5.16), the main difficulty is that the kernel of $A - i\lambda \cdot \tau^k$ is nontrivial. It will be more involved when the eigenvalues have multiplicity greater than 1. It can be characterized that the kernel and the orthogonal of $A - i\lambda \cdot \tau^k$ with
respect to the inner product \((5.2)\) are
\[
\text{Ker}(A - i\lambda^{\tau,k}) = \text{Span}\{U^{\tau,l} : \text{for all } l \in \mathbb{N} \text{ such that } \lambda^l = \lambda^k\}
\]
and
\[
\text{Ker}(A - i\lambda^{\tau,k})^\perp = \text{Span}\{U^{\delta,l} : \text{for all } \delta = \pm \text{ and } l \in \mathbb{N} \text{ such that } \lambda^l \neq \lambda^k\} \oplus \text{Span}\{U^{-\tau,l} : \lambda^l = \lambda^k\} \oplus \text{Null}(A).
\]

Next, we define a bounded pseudo-inverse of \(A - i\lambda^{\tau,k}\),
\[
(A - i\lambda^{\tau,k})^{-1} : \text{Ker}(A - i\lambda^{\tau,k})^\perp \rightarrow \text{Ker}(A - i\lambda^{\tau,k})^\perp,
\]
by
\[
(A - i\lambda^{\tau,k})^{-1} U^{\delta,l} = \frac{1}{i \lambda^{\delta,l} - i\lambda^{\tau,k}} U^{\delta,l} \quad \text{for any } U^{\delta,l} \text{ with } \lambda^l \neq \lambda^k,
\]
\[
(A - i\lambda^{\tau,k})^{-1} U^{-\tau,l} = \frac{1}{-2i \lambda^{\tau,k}} U^{-\tau,l} \quad \text{for any } U^{-\tau,l} \text{ with } \lambda^l = \lambda^k,
\]
and
\[
(A - i\lambda^{\tau,k})^{-1} (\rho, \nu, -\rho)^T = \frac{1}{i \lambda^{\tau,k}} (\rho, \nu, -\rho)^T,
\]
for any \((\rho, \nu, -\rho)^T \in \text{Null}(A)\) and \(\tau, \delta \in \{+, -\}\). It is obvious that this pseudo-inverse operator is a bounded operator. Consequently, the solutions to the system \((5.16)\) are stated in the following lemma.

**Lemma 5.1.** For each fixed acoustic mode \(k \geq 1\) and \(\tau \in \{+, -\}\), the solvability conditions of the system \((5.16)\) are the following:

(i) If \(\lambda^k\) is a simple eigenvalue of \((5.6)\), then the only solvability condition is that \(i\mu^{\tau,k}\) must satisfy
\[
i\mu^{\tau,k} = \int_{\delta \Omega} g^{\tau,k} \psi^k \, d\sigma_x - \langle F^{\tau,k} | U^{\tau,k} \rangle.
\]

Under this condition, the solutions to \((5.16)\) \(V^{\tau,k}\) can be solved uniquely as
\[
V^{\tau,k} = V_1^{\tau,k},
\]
where \(V_1^{\tau,k} \in \text{Ker}(A - i\lambda^{\tau,k})^\perp\).

(ii) If \(\lambda^k\) is not a simple eigenvalue, then besides \((5.20)\), a further compatibility condition is needed: \(F^{\tau,k}\) must satisfy
\[
\int_{\delta \Omega} g^{\tau,k} \psi^l \, d\sigma_x = \langle F^{\tau,k} | U^{\tau,l} \rangle \quad \text{for } \lambda^l = \lambda^k \text{ with } k \neq l.
\]
For this case, under the two conditions (5.20) and (5.22), the solutions to (5.16) \( V^{\tau,k} \) can be determined modulo Ker(\( A - i \lambda^{\tau,k} \)). In other words, \( V^{\tau,k} \) can be uniquely represented as

\[
V^{\tau,k} = \sum_{\lambda^k = \lambda^l} \langle V^{\tau,k} | U^{\tau,l} \rangle U^{\tau,l} + V_1^{\tau,k},
\]

where \( V_1^{\tau,k} \in \text{Ker}(A - i \lambda^{\tau,k}) \).

**Proof.** For any \( g^{\tau,k} \in H^{1/2}(\partial \Omega) \), there exists \( \tilde{V}^{\tau,k} \in H^1(\Omega; \mathbb{R}^D) \) such that \( \gamma \tilde{V}^{\tau,k} \cdot n = g^{\tau,k} \), where \( \gamma \) is the usual trace operator from \( H^1(\Omega; \mathbb{R}^D) \) to \( H^{1/2}(\partial \Omega) \).

We define

\[
\tilde{V}^{\tau,k} = V^{\tau,k} - (0, \tilde{V}^{\tau,k}, 0)^{\top}.
\]

Then \( \tilde{V}^{\tau,k} \cdot n = 0 \) on the boundary \( \partial \Omega \), and thus is in the domain of \( A \). From (5.16), \( \tilde{V}^{\tau,k} \) satisfies

\[
(A - i \lambda^{\tau,k}) \tilde{V}^{\tau,k} = -(A - i \lambda^{\tau,k})(0, \tilde{V}^{\tau,k}, 0) + i \mu^{\tau,k} U^{\tau,k} + F^{\tau,k}.
\]

The solvability of (5.25) is that the right-hand side must be in Ker(\( A - i \lambda^{\tau,k} \)). Thus, the inner product of (5.25) with \( U^{\tau,l} \) is 0, which gives (5.20), while the inner product of \( U^{\tau,l} \) with \( \lambda^k = \lambda^l, k \neq l \), gives (5.22). Under these conditions, by applying the pseudo-inverse operator \( (A - i \lambda^{\tau,k})^{-1} \) defined in (5.17)–(5.19), we can uniquely solve \( \tilde{V}^{\tau,k} \) in Ker(\( A - i \lambda^{\tau,k} \)), denoted by \( \tilde{V}_1^{\tau,k} \). However, the projection of \( \tilde{V}^{\tau,k} \) on Ker(\( A - i \lambda^{\tau,k} \)) is not determined. In other words,

\[
\tilde{V}^{\tau,k} = \tilde{V}_1^{\tau,k} + \sum_{\lambda^k = \lambda^l} \langle \tilde{V}^{\tau,k} | U^{\tau,l} \rangle U^{\tau,l}.
\]

Using (5.24), we get (5.23), where

\[
V_1^{\tau,k} = \tilde{V}_1^{\tau,k} + (0, \tilde{V}^{\tau,k}, 0)^{\top} - \sum_{\lambda^k = \lambda^l} \langle (0, \tilde{V}^{\tau,k}, 0)^{\top} | U^{\tau,l} \rangle U^{\tau,l}.
\]

In (5.23), the projection of \( V^{\tau,k} \) onto Ker(\( A - i \lambda^{\tau,k} \)), i.e., the first term in the right-hand side of (5.23), cannot be determined. It is easy to see that the projection of \( V^{\tau,k} \) on Ker(\( A - i \lambda^{\tau,k} \)) is uniquely determined, although the lifting of the trace \( g^{\tau,k} \) is not unique.

\[\square\]

6 Analysis of the Kinetic Boundary Layer Equation

In this section, we collect three results in kinetic equations that will be frequently used in this paper. The first two results are standard in kinetic theory:

**Lemma 6.1.** The solvability condition for the linear kinetic equation \( \mathcal{L}g = f \) is

\[
(f, \xi(v)) = 0 \quad \text{for} \; \xi \in \text{span}\{1, v, |v|^2\}.
\]

The second result we will use is quoted from lemma 4.4 in [3].
Lemma 6.2. The components of $\{A \otimes \hat{A}\}$ and $\{B \cdot \hat{B}\}$ satisfy the following identities:

$$
\begin{align*}
\{A_{ij} \otimes \hat{A}_{kl}\} &= v \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{D} \delta_{ij} \delta_{kl} \right), \\
\{B_i \cdot \hat{B}_j\} &= \frac{D + 2}{2} \kappa \delta_{ij},
\end{align*}
$$

where $v$ and $\kappa$ are given by (3.5).

The next result is about the linear kinetic boundary layer equation that will be used to determine the boundary conditions of the fluid variables. We define the kinetic boundary layer operator $L^B$, reflection boundary operator $L^R$, and diffusive boundary operator $L^D$ acting on functions $\{g^{bb}(x, v, \xi) : (x, v, \xi) \in \Omega^B \times \mathbb{R}^D \times \mathbb{R}_+\}$ as follows:

(6.2) 
$$
L^B g^{bb} := -(v \cdot \nabla_x d) \partial_\xi g^{bb} + L g^{bb},
$$

where $L$ is the linearized Boltzmann operator defined in (2.9):

$$
L^R g^{bb} := \gamma + g^{bb} - L \gamma - g^{bb} \quad \text{and} \quad L^D g^{bb} := \sqrt{2\pi} \chi [(\gamma - g^{bb}) \partial_\Omega - L \gamma - g^{bb}].
$$

Lemma 6.3. Consider the following linear kinetic boundary layer equation of $g^{bb}(x, v, \xi)$ in half-space:

(6.3) 
$$
L^B g^{bb} = S^{bb} \quad \text{in} \ \xi > 0, \\
g^{bb} \to 0 \quad \text{as} \ \xi \to \infty,
$$

with boundary condition

(6.4) 
$$
L^R g^{bb} = H^{bb} \quad \text{on} \ \xi = 0, \ v \cdot n > 0.
$$

In the above equations, the boundary source term $H^{bb}$ is taken to be of the following form:

(6.5) 
$$
H^{bb} = -L^R g + L^D f,
$$

where $g$ and $f$ are of the forms

(6.6) 
$$
\begin{align*}
g &= \rho g + u_g \cdot v + \theta g \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \\
&\quad - (\partial_\xi u^b \otimes n : \hat{A} + \partial_\xi \theta^b n \cdot \hat{B}) \\
&\quad + (\partial_{\pi^b} u^b \otimes \nabla_x \pi^b : \hat{A} + \partial_{\pi^b} \theta^b \nabla_x \pi^b \cdot \hat{B}) \\
&\quad + (\nabla_x u^{\text{int}} : \hat{A} + \nabla_x \theta^{\text{int}} \cdot \hat{B}) + S_g,
\end{align*}
$$

and

(6.7) 
$$
\begin{align*}
f &= \rho f + u_f \cdot v + \theta f \left( \frac{|v|^2}{2} - \frac{D}{2} \right) + S_f,
\end{align*}
$$

and where $S_g, S_f \in \text{Null}(L)^\perp$ are source terms.
Then there exists a solution $g^{bb}(x, v, \xi)$ of the equation (6.3) if and only if the following boundary conditions are satisfied by the fluid variables:

(i) **On the boundary $\partial \Omega$, the normal component of velocity is**

$$u_g \cdot n = \int_0^\infty (S^{bb}) d\xi.$$  

(ii) **On the boundary $\partial \Omega$, the tangential components of velocities and temperature satisfy**

$$[u_f]^{\tan} = \frac{\nu}{\chi} \left[ \partial_\xi u^i \right]^{\tan} - \frac{\nu}{\chi} \left[ 2d(u^\int) \cdot n \right]^{\tan} - \frac{\nu}{\chi} \nabla_\pi [u^{bb} \cdot n]$$

$$+ \left[ \int_{v \cdot n > 0} (L^p S_f) v (v \cdot n) M \, dv \right]^{\tan} - \frac{1}{\chi} \left( (v \cdot n) v S_g \right)^{\tan}$$

$$+ \frac{1}{\chi} \int_0^\infty (S^{bb} v) d\xi;$$

and

$$\theta_f = \frac{D + 2 \kappa}{D + 1} \theta^b - \frac{D + 2 \kappa}{D + 1} \theta^\int + \frac{\sqrt{2\pi}}{2(D + 1)} u_f \cdot n$$

$$+ \frac{\sqrt{2\pi}}{D + 1} \int_{v \cdot n > 0} (L^p S_f) |v|^2 (v \cdot n) M \, dv$$

$$- \frac{1}{(D + 1) \chi} \left( (v \cdot n) |v|^2 S_g \right)$$

$$+ \frac{D + 2}{D + 1} \frac{1}{\chi} \int_0^\infty \left( S^{bb} \left( \frac{|v|^2}{D + 2} - 1 \right) \right) d\xi,$$

where kinematic viscosity $\nu$ and thermal conductivity $\kappa$ are given by (3.5), $u_f^{\tan}$ denotes the tangential components of the vector $u$, and $\nabla_\pi$ denotes the tangential derivative.

**PROOF.** The solvability conditions of the linear boundary layer equation (6.3) with boundary condition (6.4) are given by

$$\int_{v \cdot n > 0} H^{bb} \eta(v) (v \cdot n) M \, dv = - \int_0^\infty (S^{bb} \eta) d\xi$$

for all $\eta(v) \in \text{Null}(\mathcal{L})$ satisfying the condition

$$\eta(R_x v) = \eta(v).$$
It is obvious that 1 and $|v|^2$ satisfy (6.12). If $\eta(v) = \sum_1^D a_i v_i$ satisfies (6.12), then necessarily

$$(v \cdot n) \sum_{i=1}^D a_i n_i = 0,$$

which implies that the vector $a = (a_1, \ldots, a_D)^T$ is perpendicular to the outer normal vector $n$.

The formula (6.8) can be derived by taking $\eta = 1$ in (6.11). Simple calculations show that

$$\int_{v \cdot n > 0} H^{bb}(v) (v \cdot n) M \, dv = -\langle v \gamma g \rangle \cdot n = -u_g \cdot n - \langle v S_g \rangle \cdot n.$$ 

Note that $S_g \in \text{Null}(\mathcal{L})^\perp$; hence (6.8) follows.

To prove (6.9), by taking $\eta = \sum_1^D a_i v_i$ in (6.11), we have

$$\int_{v \cdot n > 0} H^{bb}(a_i v_i)(n_j v_j) M \, dv = -\int_0^\infty \langle (a_i v_i) S^{bb} \rangle d\xi.$$ 

In other words,

$$- \int_{v \cdot n > 0} (L^R g)(a_i v_i)(n_j v_j) M \, dv + \int_{v \cdot n > 0} (L^D f)(a_i v_i)(n_j v_j) M \, dv =$$

$$= -\int_0^\infty \langle S^{bb}(a_i v_i) \rangle d\xi.$$ 

Simple calculations yield that

$$\int_{v \cdot n > 0} (L^R g)(a_i v_i)(n_j v_j) M \, dv = \langle v_i v_j \gamma g \rangle a_i n_j = (A_{ij} \gamma g) a_i n_j.$$ 

Using the definition of the viscosity $v$ in (3.5) and Lemma 6.2, we have

$$\int_{v \cdot n > 0} (L^R g)(a_i v_i)(n_j v_j) M \, dv =$$

$$- v[\partial_t u^b] \cdot a + v[(\nabla_x u^\text{int} + (\nabla_x u^\text{int})^T) \cdot n] \cdot a + v \nabla_\pi [\bar{u} \cdot n] \cdot a + \langle S_g A n \rangle \cdot a.$$ 

Next, it can be calculated that

$$L^D f = \left( \frac{D}{2} + \frac{1}{2} \theta_f - \frac{\sqrt{2} \pi}{2} u_f \cdot n \right) - u_f \cdot v + 2 (u_f \cdot n) n \cdot v - \frac{1}{2} \theta_f |v|^2 + L^D S_f.$$
where \( R_x u_f = u_f - 2(u_f \cdot n)n \) is the reflection of \( u_f \) with respect to the normal \( n(x) \). Thus
\[
\int_{v \cdot n > 0} (L^D f)(a_i v_i)(v \cdot n)M \, dv =
- \frac{1}{\sqrt{2\pi}} (R_x u_f) \cdot a + \int_{v \cdot n > 0} (L^D S_f)A \cdot n M \, dv \cdot a.
\]

To prove (6.10), taking \( \eta = |v|^2 \) in (6.11), we have
\[
\int_{v \cdot n > 0} H^{hh}|v|^2(v \cdot n)M \, dv = - \int_0^\infty \langle |v|^2 S^{hh} \rangle \, d\xi.
\]

It can be calculated that
\[
\int_{v \cdot n > 0} (L^R g)(v \cdot n)|v|^2M \, dv = 2\langle \gamma g B \cdot n + (D + 2) \langle v \cdot n \rangle \gamma g \rangle.
\]

Using the definition of the thermal conductivity \( \kappa \) in (3.5) and Lemma 6.2, we have
\[
\int_{v \cdot n > 0} L^R g(v \cdot n)|v|^2M \, dv =
- (D + 2)\kappa \partial_\xi \theta^h + (D + 2)\kappa \nabla_x \theta^\text{int} \cdot n + (D + 2)u_g \cdot n + \langle (v \cdot n)|v|^2 S_g \rangle.
\]
Furthermore,
\[
\int_{v \cdot n > 0} (L^D f)(v \cdot n)|v|^2M \, dv = \frac{1}{2} u_f \cdot n - \frac{1}{\sqrt{2\pi}} \theta^f
\]
\[
+ \int_{v \cdot n > 0} (L^D S_f)(v \cdot n)|v|^2M \, dv.
\]

As shown in (6.13), the vector \( a \) is perpendicular to \( n \). Thus the resulting vector of the inner product with \( a \) is the tangential part. Finally, noticing \([R_x u_f]^{\text{tan}} = [u_f]^{\text{tan}}\), we then finish the proof of Lemma 6.3.

\[\square\]

7 Approximate Eigenfunctions-Eigenvalues

7.1 Motivation

We define the operators \( L_\varepsilon \) and \( L_\varepsilon^* \) as
\[
L_\varepsilon := \frac{1}{\varepsilon} \mathcal{L} - v \cdot \nabla_x, \quad L_\varepsilon^* := \frac{1}{\varepsilon} \mathcal{L} + v \cdot \nabla_x.
\]

Formally, \( L_\varepsilon \) and \( L_\varepsilon^* \) are “dual” in the following sense:
\[
\langle L_\varepsilon^* g^*, g \rangle = \langle g^*, L_\varepsilon g \rangle
\]

(7.1)
provided that $g^*$ satisfies the Maxwell reflection boundary condition

$$\gamma g^* = (1 - \alpha) L \gamma g^* + \alpha (\gamma^* g^*)_{\partial \Omega} \quad \text{on } \Sigma_-,$$

and $g$ satisfies the dual boundary condition

$$\gamma g = (1 - \alpha) L \gamma g + \alpha (\gamma^* g^*)_{\partial \Omega} \quad \text{on } \Sigma_+.$$

If $g_\epsilon$ is the fluctuation defined in (2.19), then $g_\epsilon$ obeys the scaled Boltzmann equation (2.21) in which $L^*_e g_\epsilon$ appears and $g_\epsilon$ satisfies the boundary condition (7.2). Then from (7.1), $L^*_e g^{BL}_\epsilon$ appears in the weak formulation of the Boltzmann equation if we take $g^{BL}_\epsilon$ as a test function. Thus, it is natural to construct eigenfunctions and eigenvalues of $L_e$ satisfying the dual boundary condition (7.3). Specifically, we consider the kinetic eigenvalue problem

$$L_e g^{BL}_\epsilon = -i \lambda^{BL}_\epsilon g^{BL}_\epsilon$$

with $g^{BL}_\epsilon$ satisfying the dual Maxwell boundary condition (7.3), where the accommodation coefficient $\alpha$ takes the value $\alpha_\epsilon = \sqrt{2\pi \chi} \sqrt{\epsilon}$. By doing so, formally the equation (2.21) becomes an ordinary differential equation of $b_\epsilon = \int_{\Omega} (g_\epsilon, g^{BL}_\epsilon) dx$:

$$\frac{d}{dt} b_\epsilon + i \frac{\lambda^{BL}_\epsilon}{\epsilon} b_\epsilon = c_\epsilon.$$

To solve the eigenvalue problem (7.3)–(7.4), a key observation is that the solutions must include interior and two boundary layer terms: the fluid viscous boundary layer with thickness $\sqrt{\epsilon}$ and the kinetic Knudsen layer with thickness $\epsilon$. We make the ansatz of $g^{BL}_\epsilon$ and $\lambda^{BL}_\epsilon$ as

$$g^{BL}_\epsilon = \sum_{m \geq 0} \left[ g^{im}_m (x, v) + g^b_m (\pi(x), \frac{d(x)}{\sqrt{\epsilon}}, v) \right] \epsilon^m$$

and

$$\lambda^{BL}_\epsilon = \sum_{m \geq 0} \lambda_m \epsilon^m.$$

Each $g^b_m$ and $g^{bb}_m$ is defined in $\Omega^\delta$, the $\delta$-tubular neighborhood of $\partial \Omega$ in $\Omega$, where $\delta > 0$ is the small number defined in Lemma 4.1. The projection $\pi$ is defined in (4.1). After rescaling by $\sqrt{\epsilon}$ and $\epsilon$, respectively,

$$g^b_m, g^{bb}_m : (\partial \Omega \times \mathbb{R}^+) \times \mathbb{R}^D \rightarrow \mathbb{R}.$$}

Both $g^b_m$ and $g^{bb}_m$ will vanish in the outside of $\Omega^\delta$. Thus $g^b_m$ and $g^{bb}_m$ are required to be rapidly decreasing to 0 in the $\zeta$ and $\xi$, respectively, which are defined by $\zeta = \frac{d(x)}{\sqrt{\epsilon}}$ and $\xi = \frac{d(x)}{\epsilon}$. 
In the ansatz (7.5), \( g^{BL}_e \) consists of three types of terms: the interior terms \( g^{int}_m \), the fluid viscous boundary layer terms \( g^{b}_m \), and the kinetic Knudsen layer terms \( g^{bb}_m \). They are coupled through the boundary condition (7.3).

### 7.2 Statement of the Proposition

Now we state the proposition that can be considered as a kinetic analogue of proposition 2 in [10].

PROPOSITION 7.1. Let \( \Omega \) be a \( C^2 \) bounded domain of \( \mathbb{R}^D \) and the accommodation coefficient \( \alpha_e = \sqrt{2\pi} \chi \sqrt{\varepsilon} \). Then, for every acoustic mode \( k \geq 1 \), nonnegative integer \( N \), and each \( \tau \in \{+, -\} \), there exists approximate eigenfunctions \( g^{\tau,k}_{e,N} \) and eigenvalues \( -i\lambda^{\tau,k}_{e,N} \) of \( L_e \), and error terms \( R^{\tau,k}_{e,N} \) and \( r^{\tau,k}_{e,N} \), respectively, such that

\[
L_e g^{\tau,k}_{e,N} = -i\lambda^{\tau,k}_{e,N} g^{\tau,k}_{e,N} + R^{\tau,k}_{e,N},
\]

and \( g^{\tau,k}_{e,N} \) satisfy the approximate dual Maxwell boundary condition:

\[
L^R g^{\tau,k}_{e,N} = \sqrt{\varepsilon} L^D g^{\tau,k}_{e,N} + r^{\tau,k}_{e,N} \quad \text{on} \quad \Sigma_+.
\]

Moreover, there exist complex numbers \( \lambda^{\tau,k}_{1,N} \) such that \( i\lambda^{\tau,k}_{0,N} \) has the following expansions:

\[
i\lambda^{\tau,k}_{0,N} = i\lambda^{\tau,k}_{0} + i\lambda^{\tau,k}_{1} \sqrt{\varepsilon} + O(\varepsilon) \quad \text{with} \quad \text{Re}(i\lambda^{\tau,k}_{1,N}) < 0.
\]

Furthermore, for all \( 1 < r, p \leq \infty \), we have error estimates

\[
\| R^{\tau,k}_{e,N} \|_{L^r(dx, L^p(a^{1-p} M \, dv))} = O(\sqrt{\varepsilon}^N) \quad \text{and}
\]

\[
\| g^{\tau,k}_{e,N} - g^{\tau,k,int}_{0} \|_{L^r(dx, L^p(a^{1-p} M \, dv))} = O(\varepsilon^{N/2}).
\]

where \( g^{\tau,k,int}_{0} \) is defined in (5.9). We also have the boundary error estimates

\[
\| r^{\tau,k}_{e,N} \|_{L^r(dx, L^p(a^{1-p} M \, dv))} = O(\sqrt{\varepsilon}^{N+1}).
\]

### 7.3 Main Idea of the Proof

For each nonnegative integer \( m \), \( g^{int}_m \) is decomposed as hydrodynamic part \( \mathcal{P} g^{int}_m \), i.e., the projection onto \( \text{Null}(\mathcal{L}) \), and kinetic part \( \mathcal{P}^\perp g^{int}_m \), the projection onto \( \text{Null}(\mathcal{L})^\perp \).

The hydrodynamic part is given by

\[
\mathcal{P} g^{int}_m = \rho^{int}_m + v \cdot u^{int}_m + \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \theta^{int}_m = \left( 1, v, \frac{|v|^2}{2} - \frac{D}{2} \right) U^{int}_m.
\]

where \( U^{int}_m = (\rho^{int}_m, u^{int}_m, \theta^{int}_m)^T \) is called the fluid variables of \( g^{int}_m \). It can be shown that the coefficients of \( \mathcal{P}^\perp g^{int}_m \) are in terms of \( U^{int}_m \) and their derivatives for \( m' < m \). Thus, we need only solve \( U^{int}_n \) for all integers \( n \leq m \) to determine \( g^{int}_m \). Similar notations will be used for \( g^{b}_m \), and for the same reason we also need only solve \( U^{b}_m \).
We put the ansatz into equation (7.4), then collect the same order terms. The leading-order term $g^\text{int}_0$ is hydrodynamic, which means $g^\text{int}_0$ is completely determined by $U^\text{int}_0$. We can derive that $U^\text{int}_0$ satisfies the equation

\[ \mathcal{A}U^\text{int}_0 = i\lambda_0U^\text{int}_0. \]

For (7.13), there are two cases:

**Case 1.** $\lambda_0 \neq 0$. By comparing equations (7.13) and (5.8), $i\lambda_0$ is an eigenvalue of the acoustic operator $\mathcal{A}$, and $U^\text{int}_0 = U^\tau_k$, where $k \geq 0$ are the acoustic modes, and $\tau$ denotes either $+$ or $\tau$. Starting from here, we can construct the boundary layer $g^\text{BL}_\varepsilon$, which we call the boundary layer in the acoustic mode.

**Case 2.** $\lambda_0 = 0$. This case implies that $U^\text{int}_0 \in \text{Ker}(\mathcal{A})$, i.e., $\rho^\text{int}_0 + \theta^\text{int}_0 = 0$ and $\nabla_x \cdot u^\text{int}_0 = 0$. Starting from here, we can construct the boundary layer $g^\text{BL}_\varepsilon$, which we call the boundary layer in the incompressible mode.

Because the main goal of the current paper is about how the acoustic waves and the boundary layers interact in the incompressible Navier-Stokes limit of the Boltzmann equation, we only consider Case 1, the kinetic-fluid boundary layers in the acoustic modes for each $k \geq 0$ and each $\tau$. Consequently, we add superscripts $\tau, k$ for each term in the ansatz. In the forthcoming paper [25], we will investigate the higher-order acoustic limit of the Boltzmann equation, where we need to analyze the boundary layers in incompressible modes, i.e., Case 2.

The basic strategy to solve all terms in the ansatz is the following (to simplify notation, we don’t write the upper index $\tau$):

1. $g^{k,\text{bb}}_m$ satisfies the linear kinetic boundary layer equation (6.3). By applying Lemma 6.3, the solvability conditions for $g^{k,\text{bb}}_m$ give the normal boundary condition $[u^k_m + u^{k,\text{bb}}_m] \cdot n$ and the tangential boundary condition

\[
\left[ \frac{u^k_m - \nabla \cdot u^k_m - \frac{\partial z u^k_m}{\chi}}{\chi} \right]_{\text{tan}} [u^k_m + u^{k,\text{bb}}_m] \cdot n + \frac{\partial^{k,\text{bb}}_m}{D + 1} \frac{\partial z^{k,\text{bb}}_m}{\chi} + \theta^{k,\text{bb}}_m = 0.
\]

2. $U^{k,b}_m$ satisfies the ODE system like (8.97), where the normal boundary acoustic operator $\mathcal{A}^d$ is defined in (8.1). Solving $U^{k,b}_m$ includes two steps:

- First, projecting the ODE system of $U^{k,b}_m$ onto $\text{Null}^\perp(\mathcal{A}^d)$ to get the first-order ODE satisfied by $\rho^{k,b}_m + \theta^{k,b}_m$ and $u^{k,b}_m \cdot \nabla_x d$ (thus only one boundary condition at $\zeta = \infty$ is needed).
- Second, projecting onto $\text{Null}(\mathcal{A}^d)$, to solve $u^{k,b}_m \cdot \nabla_x \pi$ and $\theta^{k,b}_m$, which satisfy the second ODE.

Thus, besides the condition at $\zeta = \infty$, another boundary condition at $\zeta = 0$ is needed. This is given by the solvability condition of $g^{k,\text{bb}}_m$ in step (1).

3. $U^{k,\text{int}}_m$ and $i\lambda^k_m$ can be solved by applying Lemma 5.1, where only the boundary condition in the normal direction $u^{k,\text{int}}_m \cdot n$ is needed, which can be known.
from $u_{m}^{k;b} \cdot n$ since their summation $[u_{m}^{k;\text{int}} + u_{m}^{k;b}] \cdot n$ is found in (1), and $u_{m}^{k;b} \cdot n$ is already known in (2).

(4) If the multiplicity of the eigenvalue $\lambda_{0}^{k}$ is greater than 1, by applying Lemma 5.1, $U_{m}^{k;\text{int}}$ can only be solved modulo $\text{Ker}(A - i \lambda_{0}^{k})$. The components of $U_{m}^{k;\text{int}}$ in $\text{Ker}(A - i \lambda_{0}^{k})$ will be solved by applying Lemma (5.1) again in later rounds.

8 Proof of Proposition 7.1: Construction of Boundary Layers

In this section, we construct the kinetic-fluid boundary layers corresponding to the accommodation coefficient $\alpha_{\varepsilon} = \sqrt{2\pi \chi \varepsilon}$. As mentioned in the previous section, we make the ansatz for $g_{\varepsilon}^{\text{BL}}$ and $\lambda_{\varepsilon}^{\text{BL}}$ as in (7.5) and (7.6). Then we formally plug the ansatz (7.5) into equation (7.4) and then collect the terms with the same order of $\varepsilon$ in the interior, the viscous boundary layer, and the Knudsen layer, respectively. The following calculations will be frequently used:

$$v \cdot \nabla x g^{b}(\pi(x), \frac{d(x)}{\varepsilon}) = (v \cdot \nabla x \pi^{\alpha}) \partial_{\pi^{\alpha}} g^{b} + \frac{1}{\varepsilon} (v \cdot \nabla x d) \partial_{\xi} g^{b},$$

$$v \cdot \nabla x g^{bb}(\pi(x), \frac{d(x)}{\varepsilon}) = (v \cdot \nabla x \pi^{\alpha}) \partial_{\pi^{\alpha}} g^{bb} + \frac{1}{\varepsilon} (v \cdot \nabla x d) \partial_{\xi} g^{bb}.$$

8.1 Normal Boundary Acoustic Operator

A key role played in the analysis of the viscous boundary layer is the so-called normal boundary acoustic operator $A^{d}$ of the viscous boundary fluid variables $U^{b} = (\rho^{b}, u^{b}, \theta^{b})^{T}$:

$$A^{d}U^{b} := \left( \begin{array}{c} \partial_{t}(\rho^{b} \cdot \nabla x d) \\ \partial_{t}(\rho^{b} + \theta^{b}) \nabla x d \\ \frac{2}{D} \partial_{s}(u^{b} \cdot \nabla x d) \end{array} \right).$$

The null space of $A^{b}$ and its orthogonal space are

$$\text{Null}(A^{d}) = \{ (\rho^{b}, u^{b}, \theta^{b})^{T} \in L^{2}(dx; \Omega^{d}) : \rho^{b} + \theta^{b} = 0, u^{b} \cdot \nabla x d = 0 \},$$

$$\text{Null}^{\perp}(A^{d}) = \{ (\rho^{b}, u^{b}, \theta^{b})^{T} \in L^{2}(dx; \Omega^{d}) : \theta^{b} = \frac{2}{D} \rho^{b}, u^{b} \cdot \nabla x \pi = 0 \},$$

where the orthogonality is with respect to the inner product endowed on $L^{2}(dx; \Omega^{d})$:

$$\langle U^{b}, V^{b} \rangle_{L^{2}(\Omega^{d})} := \int_{\Omega^{d}} (\tilde{\rho}^{b} \rho^{b} + \tilde{u}^{b} \cdot \tilde{u}^{b} + \frac{D}{2} \tilde{\theta}^{b} \theta^{b}) dx.$$
The projections from $L^2(\Omega_d)$ to Null($A^d$) and Null($A^d\perp$) are defined as

\begin{equation}
U^b = \Pi^b U^b + (1 - \Pi^b) U^b
\end{equation}

\begin{equation}
\begin{aligned}
&\mathcal{L}^b  \\
:=& \left( \frac{2}{D+2} \rho_b^b - \frac{D}{D+2} \theta_b^b \right) \\
&\left( u_b^b \cdot \nabla_x \pi^\alpha \right) \nabla_x \pi^\alpha \\
&\left( \frac{D}{D+2} \theta_b^b - \frac{2}{D+2} \rho_b^b \right).
\end{aligned}
\end{equation}

We remark that the normal boundary acoustic operator $A^d$ and its Null and Null orthogonal spaces appear in other places to play a key role. For example, they appear in the zero viscosity limit of compressible NSF equations with boundary; see [11].

### 8.2 Preparations

Before we start the induction, we solve $g^\text{int}, g^b_0,$ and $i \lambda_0.$ First, the terms of order $O(\varepsilon^{-1})$ in the interior and viscous boundary layers in the equation (7.4) yield

\begin{equation}
\mathcal{L}^\text{int} g_0^\text{int} = 0 \quad \text{and} \quad \mathcal{L}^b g_0^b = 0,
\end{equation}

which imply that $g_0^\text{int}$ and $g_0^b$ are hydrodynamic, i.e.,

\begin{equation}
g_0^\text{int}(x, v) = \rho_0^\text{int} + v \cdot u_0^\text{int} + \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \theta_0^\text{int} = \left( 1, v, \frac{|v|^2}{2} - \frac{D}{2} \right) U_0^\text{int}
\end{equation}

and

\begin{equation}
g_0^b(x, \xi, v) = \rho_0^b + v \cdot u_0^b + \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \theta_0^b = \left( 1, v, \frac{|v|^2}{2} - \frac{D}{2} \right) U_0^b.
\end{equation}

We denote the above expressions as $g_0^\text{int} = I_0(U_0^\text{int})$ and $g_0^b = B_0(U_0^b).$ Here as operators, $I_0 = B_0$, we use different notations to emphasize that one is for the interior variable and the other for the viscous boundary variable. The fluid variables $U_0^\text{int}$ and $U_0^b$ are to be determined. To solve them we need to know the equations satisfied by them and their boundary conditions. It is easy to know from the order $O(\sqrt{\varepsilon}^{-1})$ of the interior part that $g_1^\text{int}$ is also hydrodynamic, i.e., $g_1^\text{int} = I_0(U_1^\text{int}).$

### 8.3 Induction: Round 0

Now we start our induction arguments. Each round includes considering the kinetic boundary layer, the viscous boundary layer, and interior terms separately.

**Step 1. Order $O(\sqrt{\varepsilon}^{-2})$ in the kinetic boundary layer.** The order $O(\sqrt{\varepsilon}^{-2})$ in the boundary condition (7.3) gives

\begin{equation}
\mathcal{L}^{\alpha \beta} g_0 = 0,
\end{equation}

where we use the notation $\tilde{g} = g^\text{int} + g^b.$ Because $g_0^\text{int}$ and $g_0^b$ are hydrodynamic, then

\begin{equation}
\left[ u_0^\text{int} + u_0^b \right] \cdot n = 0 \quad \text{on} \ \partial\Omega.
\end{equation}
Step 2. Order $O(\sqrt{\varepsilon}^{-1})$ in the viscous boundary layer. Next, the order $O(\sqrt{\varepsilon}^{-1})$ in the viscous boundary layer reads

$$L_{b1}^b = v \cdot \nabla_x d \partial_x g_{b1}^b$$

(8.5)

$$= \partial_x u_0^b \otimes \nabla_x d : A + \partial_x \theta_0^b \nabla_x d : B + A^dU_0^b \cdot \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right).$$

Lemma 6.1 implies that the solvability condition for equation (8.5) is $A^dU_0^b = 0$; i.e., $U_0^b$ lies in the kernel of the normal boundary acoustic operator:

$$\partial_b (\rho_0^b + \theta_0^b) \nabla_x d = 0 \quad \text{and} \quad \partial_b (u_0^b \cdot \nabla_x d) = 0,$$

from which we deduce that $\rho_0^b + \theta_0^b$ is constant in $\zeta$. Since $\rho_0^b + \theta_0^b \to 0$ as $\zeta \to \infty$, then

(8.6) \[ \rho_0^b + \theta_0^b = 0. \]

Similarly, we have

(8.7) \[ u_0^b \cdot \nabla_x d = 0, \]

which also gives that on the boundary $\partial \Omega$,

(8.8) \[ u_0^b(x, \zeta = 0) \cdot n = 0. \]

Combining (8.4) with the condition (8.8), we deduce that

(8.9) \[ u_0^{\text{int}} \cdot n = 0 \quad \text{on} \quad \partial \Omega. \]

Under these conditions, $g_{b1}^b$ can be expressed as

(8.10) \[ g_{b1}^b = B_0(U_0^b) + B_1(U_0^b) \]

$$: = \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right) U_0^b + \partial_x u_0^b \otimes \nabla_x d : \hat{A} + \partial_x \theta_0^b \nabla_x d : \hat{B}. \]

Note that $B_1$ is a linear operator.

Step 3. Order $O(\sqrt{\varepsilon})$ in the interior. To find the equation satisfied by $U_0^{\text{int}}$, we consider the order $O(\sqrt{\varepsilon})$ in the interior part:

(8.11) \[ L_{b2}^{\text{int}} = v \cdot \nabla_x g_{b2}^{\text{int}} - i \lambda_0 g_{b2}^{\text{int}}. \]

By using the boundary condition (8.9), which means that $U_0^{\text{int}}$ is in the domain of the acoustic operator $A$, the solvability of (8.11) gives

(8.12) \[ AU_0^{\text{int}} = i \lambda_0 U_0^{\text{int}}, \]

which is a first-order linear hyperbolic system with the boundary condition (8.9). If $i \lambda_0 = 0$, (8.12) is the so-called acoustic system whose solutions $U_0^{\text{int}}$ satisfy the incompressible and Boussinesq relations. This case will be treated in a separate paper [26].
If $i\lambda_0 \neq 0$, then from the discussions in Section 5, especially (5.8), we know that the system (8.12) has a family of solutions: for each $k \in \mathbb{N}$, $U^{\text{int}}_0$ are eigenvectors of $A$ and can be constructed from the eigenvectors of the Laplace operator with Neumann boundary condition, i.e.,

(8.13) \[ U^{\text{int}}_0 = U^{\tau,k}_0 \quad \text{and} \quad \lambda_0 = \tau \lambda^k, \quad k = 1, 2, \ldots, \]

where $\tau$ denotes the sign $+$ or $-$; see the discussion of the spectrum of $A$ in Section 5 in particular, (5.6), (5.7), and (5.8). Consequently, every term in the ansatz (7.5) and (7.6) depends on the choice of $k \in \mathbb{N}$ and $\tau$.

**Remark.** (8.13) is the building block of the construction of the approximate eigenvector $g^k_{\varepsilon,N}$ and eigenvalues $\lambda^k_{\varepsilon,N}$ for any $k, N \in \mathbb{N}$. It means that the leading-order term of $(g^k_{\varepsilon,N}, \lambda^k_{\varepsilon,N})$ is in an acoustic mode. For this reason, we call $g^k_{\varepsilon,N}$ the boundary layer in the acoustic regime.

Now we can represent $g^k_{\text{int}}$ as

(8.14) \[ g^k_{\text{int}} = I_0(U^{\tau,k,\text{int}}_2) + I_2(U^{\tau,k,\text{int}}_0) \]

\[ := U^{\tau,k,\text{int}}_2 \cdot \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right) \]

\[ + \left\{ \nabla_x u_0^{\tau,k,\text{int}} : \hat{A}(v) + \nabla_x \theta_0^{\tau,k,\text{int}} \cdot \hat{B}(v) \right\}, \]

where

\[ u_0^{\tau,k,\text{int}} = \sqrt{\frac{D + 2}{2D} \frac{\nabla_x \psi^k}{i\lambda^k_{\varepsilon,N}}} \quad \text{and} \quad \theta_0^{\tau,k,\text{int}} = \sqrt{\frac{2}{D(D + 2)}} \psi^k, \]

and $\psi^k$ is defined in (5.6).

**Remark.** For notational simplicity, from now on we drop $\tau$ on the subscript unless specifically mentioned.

### 8.4 Induction: Round 1

Now we move to round 1, which studies the kinetic boundary layer, viscous boundary layer, and interior terms separately.

**Step 1. Order $O(\sqrt{\varepsilon}^{-1})$ in the kinetic boundary layer.** The order $O(\sqrt{\varepsilon}^{-1})$ of the kinetic boundary layer in the ansatz gives that $g^{k,\text{bb}}_1$ obeys the following linear kinetic boundary layer equation in $\Omega^\delta \times \mathbb{R}^D \times \mathbb{R}^+$ (here we recall the definition of $L^{\text{BL}}$ in (6.2)):

(8.15) \[ L^{\text{BL}} g^{k,\text{bb}}_1 = 0 \quad \text{in} \ \xi > 0, \]

\[ g^{k,\text{bb}}_1 \rightarrow 0 \quad \text{as} \ \xi \rightarrow \infty, \]

with boundary condition at $\xi = 0$

(8.16) \[ L^R g^{k,\text{bb}}_1 = H^{k,\text{bb}}_1 \quad \text{on} \ \xi = 0, \ v \cdot n(x) > 0, \]
where \( H_1^{k,bb} \) has the following form: for \( x \in \partial \Omega, \, v \cdot \mathbf{n}(x) > 0, \)
\[
H_1^{k,bb}(x, v) = -L^R \tilde{\mathbf{g}}_1^k + L^D \tilde{\mathbf{g}}_0^k
= -\tilde{\mathbf{h}}_0^{bb}(U_1^{int}, U_1^b) + \tilde{\mathbf{h}}_1^{bb}(U_0^{int}, U_0^b).
\]
(8.17)
Here \( \tilde{\mathbf{h}}_0^{bb}(U_1^{int}, U_1^b) \) is a linear function of \( U_1^{int} \) and \( U_1^b \), and \( \tilde{\mathbf{h}}_1^{bb}(U_0^{int}, U_0^b) \) is a linear function of \( U_0^{int} \) and \( U_0^b \). More specifically,
\[
\tilde{\mathbf{h}}_0^{bb}(U_1^{int}, U_1^b) := -L^R (I_0(U_1^{int}) + B_0(U_1^b)) = -2(v \cdot \mathbf{n})(\tilde{u}_1^k \cdot \mathbf{n}),
\]
(8.18)
and
\[
\tilde{\mathbf{h}}_1^{bb}(U_0^{int}, U_0^b)
= -L^R B_1(U_0^b) + L^D (I_0(U_0^{int}) + B_0(U_0^b))
= L^R (\partial_\xi u_0^{k,b} \otimes \mathbf{n} : \hat{\mathbf{A}} + \partial_\xi \theta_0^{k,b} \mathbf{n} \cdot \hat{\mathbf{B}}) - v \cdot \left[ \tilde{u}_0^k \cdot \nabla_x \pi^\alpha \right] \nabla_x \pi^\alpha
+ \left( \frac{D + 1}{2} - \frac{|v|^2}{2} \right) \tilde{\mathbf{g}}_0^k + \left( \mathbf{n} \cdot v - \frac{\sqrt{2} \pi}{2} \right) (\tilde{u}_0^k \cdot \mathbf{n}).
\]
(8.19)
The boundary conditions for the tangential components of \( u_0^{k,b} \) and \( \theta_0^{k,b} \) can be derived from the solvability condition of the above kinetic boundary layer equations. The formulas (6.9) and (6.10) of Lemma 6.3 give that on the boundary \( \partial \Omega, \)
\[
\left[ u_0^{k,b} - \frac{v}{\chi} \partial_\xi u_0^{k,b} \right] = \left[ u_0^{k,bb} \right] = -\left[ u_0^{k,bb} \right] \quad \text{and}
\left[ \theta_0^{k,b} - \frac{D + 2 \kappa}{D + 1 \chi} \partial_\xi \theta_0^{k,b} \right] = -\theta_0^{k,bb},
\]
(8.20)
from which we can deduce the boundary conditions for \( u_0^{k,b} \) and \( \theta_0^{k,b} \) because \( u_0^{k,bb} \) and \( \theta_0^{k,bb} \) are already determined; thus their boundary values are known.

Furthermore, from (6.10) of Lemma 6.3 we also have
\[
\left[ u_1^{k,bb} + u_0^{k,b} \right] \cdot \mathbf{n} = 0 \quad \text{on} \ \partial \Omega, \quad \text{which implies} \ \tilde{\mathbf{h}}_0^{bb}(U_1^{int}, U_1^b) = 0. \quad \text{Furthermore, from the boundary condition (8.20)}
\]
\[
\tilde{\mathbf{h}}_1^{bb}(U_0^{int}, U_0^b) = L^R (\partial_\xi u_0^{k,b} \otimes \mathbf{n} : \hat{\mathbf{A}} + \partial_\xi \theta_0^{k,b} \mathbf{n} \cdot \hat{\mathbf{B}})
- \frac{v}{\chi} \cdot \left[ \partial_\xi u_0^{k,b} \right] + \left( \frac{D + 1}{2} - \frac{|v|^2}{2} \right) \frac{D + 2 \kappa}{D + 1 \chi} \partial_\xi \theta_0^{k,b},
\]
(8.22)
which implies that although formally \( g_1^{k,bb} \) depends on \( g_1^{k,bb} \) and \( g_1^{k,b} \), which have not been fully determined at this stage, it in fact depends only on the boundary
values of $U_0^{k,b}$; thus once we solve $U_0^{k,b}$, we can solve $g_1^{k,b}$ completely. This will be finished at the end of Step 2; see (8.38).

Step 2. Order $O(\sqrt{\varepsilon^3})$ in the viscous boundary layer. The equations satisfied by $[u_0^{k,b}]^\tan$ and $\theta_0^{k,b}$ can be derived by considering the order $O(\sqrt{\varepsilon^3})$ of the viscous boundary layer:

\begin{equation}
(8.23) \quad \mathcal{L}g_2^{k,b} = v \cdot \nabla_x d \partial_\zeta g_1^{k,b} + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} g_0^{k,b} - i \lambda_0^{k,b} g_0^{k,b}.
\end{equation}

Lemma 6.1 implies that the projection of the right-hand side of (8.23) must be in the null space of $\mathcal{L}$. We first use the expression (8.10) of $g_1^{k,b}$ to calculate the projection of $v \cdot \nabla_x d \partial_\zeta g_1^{k,b}$ onto $\text{Null}(\mathcal{L})$. It is easy to see that components of $\mathcal{P}(\partial_i \partial_\eta \partial_\zeta^2 u_0^{k,b}(v, \hat{\hat{\pi}}))$ on 1 and $|v|^2/2 - D$ are zeros; here $\mathcal{P}$ is defined in (2.12). Applying Lemma 6.2, we get

\begin{equation}
(8.24) \quad \mathcal{P}(\partial_i \partial_\eta \partial_\zeta^2 u_0^{k,b}(v, \hat{\hat{\pi}})) = -\mathcal{A}^d U_1^{k,b} = \left( \mathcal{A}^\pi + \mathcal{D}^d - i \lambda_0^{k,b} \right) U_0^{k,b},
\end{equation}

where kinematic viscosity $v$ and thermal conductivity $\kappa$ are given by (3.5). Based on the above calculations the solvability conditions for (8.23) are a system of second-order ordinary differential equations in $\zeta$:

\begin{equation}
(8.25) \quad \mathcal{A}^\pi U^b := \begin{pmatrix}
\text{div}_\pi (u^b \cdot \nabla_x \pi) \\
\partial_{\pi^\alpha} (\rho^b + \theta^b) \nabla_x \pi^\alpha \\
\frac{2}{D} \text{div}_\pi (u^b \cdot \nabla_x \pi)
\end{pmatrix}
\end{equation}

and

\begin{equation}
(8.26) \quad \mathcal{D}^d U^b := \begin{pmatrix}
0 \\
\nu \partial_\zeta^2 u^b + \nu (1 - \frac{2}{D}) \partial_\zeta^2 (u^b \cdot \nabla_x d) \nabla_x d \\
\frac{D + 2}{D} \partial_\zeta^2 \theta^b
\end{pmatrix},
\end{equation}

for $U^b = (\rho^b, u^b, \theta^b)^T$. Here we use the notation $\text{div}_\pi (u^b \cdot \nabla_x \pi) = \partial_{\pi^\alpha} (u^b \cdot \nabla_x \pi^\alpha)$. Recall the normal acoustic operator $\mathcal{A}^d$ is defined in (8.1).

The ODE system (8.24) can be solved as follows: projecting the system (8.24) onto $\text{Null}(\mathcal{A}^d)$ and $\text{Null}(\mathcal{A}^\pi)^\bot$, respectively. The projection onto $\text{Null}(\mathcal{A}^d)$ gives the first-order equations of $\rho_1^b + \theta_1^b$ and $u_1^b \cdot \nabla_x d$, which can be solved by using...
the vanishing condition at \( \xi = \infty \), while the projection onto \( \text{Null}(A^d)^\perp \) gives the second-order equations of \( u^b_0 \cdot \nabla_x \pi \) and \( \theta^b_0 \), which can be solved by using the vanishing condition at \( \xi = \infty \) and the Robin boundary condition at \( \xi = 0 \), i.e., (8.20).

(1) Solve \( \rho^{k,b}_1 + \theta^{k,b}_1 \). We first project the system (8.24) onto \( \text{Null}(A^d)^\perp \). The u-component in the projection is

\[
\partial_\xi (\rho^{k,b}_1 + \theta^{k,b}_1) = 0, \quad \text{hence} \quad \rho^{k,b}_1 + \theta^{k,b}_1 = 0.
\]

The \( \rho \)-component (or equivalently the \( \theta \)-component) of the projection on \( \text{Null}(A^d)^\perp \) is

\[
- \partial_\xi (u^{k,b}_1 \cdot \nabla_x d) = \text{div}_\pi (u^{k,b}_0 \cdot \nabla_x \pi) + \kappa \partial^2_{\xi^2} \theta^{k,b}_0.
\]

To solve this, we first need to solve \( u^{k,b}_0 \cdot \nabla_x \pi \) and \( \theta^{k,b}_0 \). Note that in the derivation of (8.27) and (8.28) the relations (8.6) and (8.7) are used.

(2) Solve \( \tan \left( u^{k,b}_0 \right) \). We next project the system (8.24) on \( \text{Null}(A^d)^\perp \). The u-component gives the equation for \( u^{k,b}_0 \cdot \nabla_x \pi^\alpha \):

\[
(v \partial^2_{\xi^2} - i \lambda^k_0) [u^{k,b}_0 \cdot \nabla_x \pi^\alpha] = 0,
\]

\[
\left[ u^{k,b}_0 - \frac{v}{\sqrt{\chi}} \partial_\xi u^{k,b}_0 \right] (\xi = 0) \cdot \nabla_x \pi^\alpha = -u^{k,\text{int}}_0 (\pi(x)) \cdot \nabla_x \pi^\alpha,
\]

\[
\lim_{\xi \to \infty} u^{k,b}_0 \cdot \nabla_x \pi^\alpha = 0.
\]

where the boundary condition in the second line of (8.29) follows from the fact that \( u^{k,b}_0 \cdot \nabla_x \pi^\alpha \) is the tangential component of \( u^{k,b}_0 \) because \( \nabla_x \pi^\alpha \) is tangential to \( \partial \Omega \); see the arguments after (4.3). The solution to ODE (8.29) is

\[
u^{k,b}_0 (\pi(x), \xi) \cdot \nabla_x \pi^\alpha =
\]

\[
= \frac{1}{c_\chi \sqrt{v}} \left( (u^{k,\text{int}}_0 (\pi(x)) \cdot \nabla_x \pi^\alpha) \exp \left( -(1 + \tau i) \sqrt{\frac{\lambda^k_0}{2v}} \xi \right) \right),
\]

where \( \tau = + \) or \( - \) and \( c_\chi = -\frac{1 + \tau i}{2\chi} \sqrt{2\lambda^k_0} \). We denote the solution (8.30) by

\[
u^{k,b}_0 (\pi(x), \xi) \cdot \nabla_x \pi = Z^{b,u}_0 (\xi, U^{k,\text{int}}_0),
\]

where \( Z^{b,u}_0 (\xi, \cdot) \) is a linear function. Note that in the right-hand side of (8.30), \( U^{k,\text{int}}_0 \) should be understood as its value on the boundary, i.e., \( U^{k,\text{int}}_0 (\pi(x)) \).
(3) Solve $\theta^{k,b}_0$. The $\rho$-component (or equivalently the $\theta$-component) of the projection $\text{Null}(A^d)$ yields the equation for $\theta^{k,b}_0$:

$$
(\kappa \frac{\partial^2}{\partial \zeta^2} - i \lambda^{k,b}_0) \theta^{k,b}_0 = 0,
$$

(8.32)

$$
\left[ \theta^{k,b}_0 - \frac{D + 2 \kappa}{D + 1} \frac{\partial}{\partial \zeta} \theta^{k,b}_0 \right] (\zeta = 0) = -\theta^{k,\text{int}}_0(\pi(x)),
$$

$$
\lim_{\zeta \to \infty} \theta^{k,b}_0 = 0.
$$

The solution to (8.32) is

$$
\theta^{k,b}_0(\pi(x), \zeta) = \frac{1}{c_\chi \sqrt{\kappa} - 1} \theta^{k,\text{int}}_0(\pi(x)) \exp\left(- (1 + \tau i) \sqrt{\frac{\lambda^{k}_0}{\kappa} \zeta} \right),
$$

(8.33)

where $\kappa = \frac{D + 2}{D + 1} \kappa$. We denote the solution (8.33) by

$$
\theta^{k,b}_0(\pi(x), \zeta) = \tilde{Z}^{b,\theta}_0(\zeta, U^{k,\text{int}}_0).
$$

(8.34)

where $\tilde{Z}^{b,\theta}_0(\zeta, \cdot)$ is a linear function.

(4) Solve $u^{k,b}_1 \cdot \nabla_x d$. Now equation (8.28) becomes

$$
\partial_\zeta (u^{k,b}_1 \cdot \nabla_x d) = -\partial_\alpha (u^{k,b}_0 \cdot \nabla_\alpha \pi) - i \lambda^{k}_0 \rho^{k,b}_0.
$$

(8.35)

By integrating equation (8.35) from $\zeta$ to $\infty$, we get

$$
u^{k,b}_1 \cdot \nabla_x d = \tilde{Z}^{b}_1(\zeta, U^{k,\text{int}}_0),
$$

where $\tilde{Z}^{b}_1(\zeta, \cdot)$ is linear. In particular, letting $\zeta = 0$ gives the value of $u^{k,b}_1 \cdot n$ on the boundary $\partial \Omega$:

$$
-u^{k,b}_1 \cdot n = \frac{1 - \tau i}{\sqrt{2 \lambda^{k}_0}} \left( \text{div}_\pi u^{k,\text{int}}_0 \cdot \nabla_\pi \pi \right) \frac{\sqrt{\nu}}{c_\chi \sqrt{\nu} - 1}
$$

$$
+ \frac{\sqrt{\kappa}}{c_\chi \sqrt{\kappa} - 1}
$$

$$
= Z^{b}_1(U^{k,\text{int}}_0) = \tilde{Z}^{b}_1(0, U^{k,\text{int}}_0),
$$

(8.36)

where $Z^{b}_1(\cdot)$ is a linear function. Consequently, (8.21) gives the boundary value

$$
u^{k,\text{int}}_1 \cdot n = Z^{b}_1(U^{k,\text{int}}_0).$$
Finally, we can represent \( g_2^{k,b} \) from (8.23):

\[
g_2^{k,b} = \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right) U_2^{k,b} + \partial_\xi u_1^{k,b} \otimes \nabla_x d : \hat{A} + \partial_\xi \theta_1^{k,b} \nabla_x d \cdot \hat{B} \\
+ \partial_\pi^\alpha u_0^{k,b} \otimes \nabla_x \pi^\alpha : \hat{A} + \partial_\pi^\alpha \theta_0^{k,b} \nabla_x \pi^\alpha \cdot \hat{B} \\
+ \mathcal{L}^{-1} \mathcal{P}_\perp \left(\partial_i d \delta j d \delta^2 \xi (u_0^{k,b})_k \delta \hat{A}_{jk} + \partial_i d \delta j d \delta^2 \xi \theta_0^{k,b} \delta v_i \hat{B}_j\right) \\
= B_0(U_2^b) + B_1(U_1^b) + B_2(U_0^b).
\]

where

\[
B_2(U_0^b) := \mathcal{L}^{-1} \mathcal{P}_\perp ((v \cdot \nabla_x d) \partial_\xi B_1(U_0^b) + (v \cdot \nabla_x \pi^\alpha) \partial_\pi^\alpha B_0(U_0^b)).
\]

(5) Solve \( g_1^{k,b} \). Now we can represent (8.17) as,

\[
H_1^{k,b}(U_0^k) = h_1^{k,b}(U_1^{k,\text{int}})
\]

\[
= L^\pi (\partial_\xi \hat{Z}_0^{b,a}(0, U_0^k) \nabla_x \pi \otimes n : \hat{A} + \partial_\xi \hat{Z}_0^{b,\theta}(0, U_0^k) n \cdot \hat{B}) - \frac{v}{\chi} \partial_\xi \hat{Z}_0^{b,a}(0, U_0^k) \nabla_x \pi + \left(\frac{D + 1}{2} - \frac{|v|^2}{2}\right) \frac{D + 2 \kappa}{D + 1 \chi} \partial_\xi \hat{Z}_0^{b,\theta}(0, U_0^k),
\]

which is completely determined. Thus we can solve \( g_1^{k,b} \), which we denote by

\[
g_1^{k,b}(\pi(x), \xi, v) = K_1(\xi, v, U_0^{k,\text{int}}(\pi(x))),
\]

where \( K_1(\xi, v, \cdot) \) is a linear function.

We summarize that in Step 2 by considering the order \( O(\sqrt{\varepsilon}) \) in the viscous boundary layer, we determine the following:

- \( \rho_1^{k,b} \) and \( \theta_1^{k,b} \);
- \( u_0^{k,b} \cdot \nabla_x \pi \) and \( \theta_0^{k,b} \), thus \( g_0^{k,b} \);
- \( u_1^{k,b} \cdot \nabla_x d \) and hence the boundary value of \( u_1^{k,b} \cdot n \) when we take \( \xi = 0 \), and consequently \( u_1^{k,\text{int}} \cdot n \), which will be used in Step 3;
- an expression of \( g_2^{k,b} \); and
- \( g_1^{k,b} \).

**Step 3. Order \( O(\sqrt{\varepsilon}) \) in the interior.** The order \( O(\sqrt{\varepsilon}) \) in the interior part yields

\[
\mathcal{L} g_3^{k,\text{int}} = v \cdot \nabla_x g_1^{k,\text{int}} - i\lambda_0 g_1^{k,\text{int}} - i\lambda_1 g_0^{k,\text{int}},
\]

and the solvability condition of which is

\[
(A - i\lambda_0) U_1^{k,\text{int}} = i\lambda_1 U_0^{k,\text{int}} \quad \text{in } \Omega,
\]

\[
u_1^{k,\text{int}} \cdot n = Z_1^{b}(U_1^{k,\text{int}}) \quad \text{on } \partial \Omega.
\]
To solve (8.40), we apply Lemma 5.1. The formula (5.20) gives
\begin{equation}
(8.41) \quad i \lambda_1^k = \int_{\partial \Omega} \left[ u_1^{k,\text{int}} \cdot n \right] \Psi^k \, d\sigma_x = \int_{\partial \Omega} Z_1^b \left( U_0^{k,\text{int}} \right) \Psi^k \, d\sigma_x.
\end{equation}

Note that \( \nabla_x \Psi^k \equiv g^{\gamma \beta} \frac{\partial \Psi^k}{\partial \tilde{\pi}^\beta} \frac{\partial}{\partial \tilde{\pi}^\gamma} \) and \( \nabla_x \pi^\alpha = g^{\alpha \delta} \frac{\partial}{\partial \tilde{\pi}^\delta} \); we have
\begin{align*}
\int_{\partial \Omega} \frac{\partial}{\partial \tilde{\pi}^\alpha} (\nabla_x \Psi^k \cdot \nabla_x \pi^\alpha) \Psi^k \, d\sigma_x &= -\int_{\partial \Omega} g^{\gamma \delta} g^{\alpha \delta} g^{\beta \nu} \frac{\partial \Psi^k}{\partial \tilde{\pi}^\alpha} \frac{\partial \Psi^k}{\partial \tilde{\pi}^\beta} \, d\sigma_x \\
&= -\int_{\partial \Omega} |\nabla_{\tilde{\pi}} \Psi^k|^2 \, d\sigma_x,
\end{align*}
where \( \nabla_{\tilde{\pi}} \) is the tangential gradient on \( \partial \Omega \). Thus
\begin{equation}
(8.42) \quad i \lambda_1^k = \Lambda_1 \int_{\partial \Omega} |\nabla_{\tilde{\pi}} \Psi^k|^2 \, d\sigma_x + \Lambda_2 \int_{\partial \Omega} \frac{2}{D + 2} (\lambda_0^k)^2 |\Psi^k|^2 \, d\sigma_x,
\end{equation}

where
\begin{align*}
\Lambda_1 &= -\frac{\sqrt{\nu}}{\sqrt{2} \lambda_0^k} \sqrt{\frac{(a + 1) + \tau i}{(a + 1)^2 + a^2}} \sqrt{\frac{D + 2}{D}}, \\
\Lambda_2 &= -\frac{\sqrt{\kappa}}{\sqrt{2} \lambda_0^k} \sqrt{\frac{(b + 1) + \tau i}{(b + 1)^2 + b^2}} \sqrt{\frac{D + 2}{D}}, \\
a &= \frac{\sqrt{2} \lambda_0^k}{2\chi}, \quad b = \frac{\sqrt{2} \lambda_0^k}{2\chi} \frac{D + 2}{D + 1}.
\end{align*}

From the expression (8.42), \( i \lambda_1^{r,k} \) has an important property (regardless of whether \( \tau = + \) or \( - \)):
\begin{equation}
(8.43) \quad \text{Re}(i \lambda_1^{r,k}) < 0.
\end{equation}

To prove this, from (8.42) we can only conclude that \( \text{Re}(i \lambda_1^k) \leq 0 \). The strict negativity comes from the following argument. Indeed, assume that \( \text{Re}(i \lambda_1^k) = 0 \). This would imply that \( \nabla_x \Psi^k = 0 \) and \( \Psi^k = 0 \) on the boundary \( \partial \Omega \). Hence, extending \( \Psi^k \) by 0 outside of \( \Omega \) and denoting by \( \tilde{\Psi}^k \) this extension, we see that \( \tilde{\Psi}^k \) is an eigenvector of \( -\Delta_x \) on the whole space with compact support, which is impossible. Hence (8.43) holds.

Remark. The above formula (8.42) and the strict inequality (8.43) is crucial in this paper, because it gives dissipativity, which will be used later in proving the damping of the acoustic waves in the Navier-Stokes limit.
\begin{align*}
\text{Case 1. If } & \frac{D}{\partial x}[\lambda_0^k]^2 \text{ is a simple eigenvalue of } -\Delta_x \text{ with Neumann boundary condition, see (5.6). By Lemma 5.1 (8.41) is the only solvability condition under which system (8.40) can be solved uniquely as } \\
& (8.44) \quad U_{1,\text{int}}^{k} = Z_{1,\text{int}}^k (U_{0,\text{int}}^k),
\end{align*}

where \( Z_{1,\text{int}}^k(U_{0,\text{int}}^k) \in \text{Null}(A)^\perp \). Note that the system (8.40) is linear and the boundary data \( Z_{1,\text{int}}^k(U_{0,\text{int}}^k) \) is also linear in \( U_{0,\text{int}}^k \). So \( Z_{1,\text{int}}^k(U_{0,\text{int}}^k) \) also linearly depends on \( U_{0,\text{int}}^k \), i.e., \( Z_{1,\text{int}}^k(\cdot) \) is a linear function.

\begin{align*}
\text{Case 2. If the eigenvalues } & \lambda_0^k \text{ are not simple, an additional compatibility condition is needed, which is given by the formula (5.22):} \\
& (8.45) \quad \int_D Z_{1,\text{int}}^k(U_{0,\text{int}}^k) \psi^l \, d\sigma_x = 0 \quad \text{if } \lambda_0^k = \lambda_0^l \text{ and } k \neq l.
\end{align*}

Specifically, this condition reads as

\begin{align*}
& \Lambda_1 \int_{\partial \Omega} \nabla \pi \psi^k \cdot \nabla \pi \psi^l \, d\sigma_x + \Lambda_2 \int_{\partial \Omega} \frac{2}{D + 2} (\lambda_0^k)^2 \psi^k \psi^l \, d\sigma_x = 0
\end{align*}

if \( \lambda_0^k = \lambda_0^l \) and \( k \neq l \). We can define the quadratic form \( Q_1 \) and the symmetric operator \( L_1 \) on \( H_0(\lambda) \) as

\begin{align*}
& (8.46) \quad Q_1(\psi^k, \psi^l) = \int_{\partial \Omega} Z_{1,\text{int}}^k(U_{0,\text{int}}^k) \psi^l \, d\sigma_x, \\
& (8.47) \quad L_1 \psi^k = i \lambda_1^k \psi^k,
\end{align*}

and the orthogonality condition (8.45) is

\begin{align*}
& (8.48) \quad Q_1(\psi^k, \psi^l) = 0 \quad \text{if } \psi^k, \psi^l \in H_0(\lambda) \text{ and } l \neq k.
\end{align*}

Under these conditions, we solve \( U_{1,\text{int}}^{k} \) modulo \( \ker(A - i \lambda_0^k) \) by applying Lemma 5.1 i.e.,

\begin{align*}
& (8.49) \quad U_{1,\text{int}}^{k} = Z_{1,\text{int}}^k(U_{0,\text{int}}^k) + P_0 U_{1,\text{int}}^{k},
\end{align*}

where \( P_0 U_{1,\text{int}}^{k} \) is defined as

\begin{align*}
& (8.50) \quad P_0 U_{1,\text{int}}^{k} = \sum_{l \neq k, \lambda_0^l = \lambda_0^k} a_1^{k,l} U_{0,\text{int}}^l,
\end{align*}

where \( a_1^{k,l} = (U_{1,\text{int}}^k | U_{0,\text{int}}^l) \) will be determined later.
Finally, we can represent $g_{3,\text{int}}^{k}$ as

$$
(8.51) \quad g_{3,\text{int}}^{k} = \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right) U_{3}^{k,\text{int}} + \nabla_{x} u_{1}^{k,\text{int}} : \hat{A}(v) + \nabla_{x} \theta_{1}^{k,\text{int}} \cdot \hat{B}(v)
$$

$$
= I_{0}(U_{3}^{k}) + I_{2}(P_{0} U_{1}^{k}) + I_{3}(U_{0}^{k,\text{int}}).
$$

If $\lambda_{0}^{k}$ is a simple eigenvalue, the $P_{0} U_{1}^{k}$ term vanishes. Thus we finish the Round 1 in the induction.

**Remark.** The orthogonality condition (8.45) is given on the eigenfunctions of $-\Delta_{x}$ with Neumann boundary condition with respect to the eigenvalue $|\lambda_{0}^{k}|^2$. Usually the eigenfunctions with Neumann boundary conditions are determined up to some constants, and so are not unique. (8.45) is only used to determine the eigenfunctions corresponding to multiple eigenvalues, and it does not give any new assumption on the geometry of the domain $\Omega$.

### 8.5 Induction: Round 2

Now we move to the second round in the induction, which also includes three steps, by considering terms in the kinetic boundary layer, viscous boundary layer, and interior separately.

**Step 1. Order $O(\sqrt{\varepsilon})$ in the kinetic boundary layer.** The order $O(\sqrt{\varepsilon})$ of the kinetic boundary layer in the ansatz gives that $g_{2,\text{bb}}^{k}$ satisfies the linear boundary layer equation

$$
L_{\text{BL}}^{\text{bb}} g_{2,\text{bb}}^{k} = 0 \quad \text{in } \xi > 0,
$$

$$
\frac{\partial}{\partial \xi} g_{2,\text{bb}}^{k} \to 0 \quad \text{as } \xi \to \infty,
$$

with boundary condition at $\xi = 0$

$$
L_{\text{R}}^{\text{bb}} g_{2,\text{bb}}^{k} = H_{2,\text{bb}}^{k} \quad \text{on } \xi = 0, \quad v \cdot \mathbf{n} > 0,
$$

where $H_{2,\text{bb}}^{k}$ is of the form

$$
(8.54) \quad H_{2,\text{bb}}^{k} = -L_{\text{R}}^{\text{bb}} g_{2}^{k} + L_{\text{D}}^{\text{bb}}(g_{1}^{k} + g_{1,\text{bb}}^{k})
$$

$$
= \tilde{h}_{0,\text{bb}}^{k}(U_{0,\text{int}}^{k}, U_{0}^{b}) + \tilde{h}_{1,\text{bb}}^{k}(U_{1,\text{int}}^{k}, U_{1}^{b}) + \tilde{h}_{2,\text{bb}}^{k}(U_{0,\text{int}}^{k}, U_{0}^{b}).
$$

Here

$$
(8.55) \quad \tilde{h}_{2,\text{bb}}^{k}(U_{0,\text{int}}^{k}, U_{0}^{b}) = -L_{\text{R}}^{\text{bb}}(I_{2}(U_{0,\text{int}}^{k}) + B_{2}(U_{0}^{b})) + L_{\text{D}}^{\text{bb}}(B_{1}(U_{0,\text{int}}^{k}) + K_{1}(U_{0}^{b})).
$$

Comparing this with (8.17), we note that the first two terms of (8.54) are the same as (8.17), just replacing $\tilde{g}_{1}$ by $\tilde{g}_{2}^{k}$ and $\tilde{g}_{0}$ by $\tilde{g}_{1}^{k}$. In other words,

$$
(8.56) \quad \tilde{h}_{0,\text{bb}}^{k}(U_{0,\text{int}}^{k}, U_{0}^{b}) := -2(v \cdot \mathbf{n})(\tilde{u}_{2}^{k} \cdot \mathbf{n})
$$
and
\[
\tilde{h}_1^{b} (U_1^{\text{int}}, U_1^{b}) := L \nabla (\partial_{\xi} u_1^{k,b} \otimes n : \hat{A} + \partial_{\xi} \theta_1^{k,b} \cdot \hat{B})
\]
(8.57)
\[
- v \cdot \left[ \tilde{u}_1^{k,b} \cdot \nabla_x \pi^\alpha \right] \nabla_x \pi^\alpha + \left( \frac{D + 1}{2} - \frac{|v|^2}{2} \right) \tilde{\theta}_1^{k,b}.
\]

Note that in (8.57) we have used the boundary condition \( u_1^{k,b} \cdot n = 0 \).

Next, the formulas (6.9) and (6.10) give the boundary conditions
(8.58)
\[
\left[ u_1^{k,b} - \frac{\nu}{\chi} \partial_{\xi} u_1^{k,b} \right]^{\tan} = -\left[ u_1^{k,\text{int}} \right]^{\tan} + V_1^u (U_0^{k,\text{int}})
\]
and
(8.59)
\[
\theta_1^{k,b} - \frac{D + 2 \kappa}{D + 1} \partial_{\xi} \theta_1^{k,b} = -\theta_1^{k,\text{int}} + V_1^\theta (U_0^{k,\text{int}})
\]

where
(8.60)
\[
V_1^u (U_0^{k,\text{int}}) = -\frac{\nu}{\chi} \left[ 2 d (u_0^{k,\text{int}}) \cdot n \right]^{\tan} + \int_{v \cdot n > 0} L^D S_1 (v \cdot n) v M \, dv \right]^{\tan},
\]
\[
V_1^\theta (U_0^{k,\text{int}}) = \frac{\sqrt{2 \pi}}{D + 1} \int_{v \cdot n > 0} L^D S_1 (v \cdot n) |v|^2 M \, dv,
\]
and
\[
S_1 = -(\partial_{\xi} u_0^{k,b} \otimes n : \hat{A} + \partial_{\xi} \theta_0^{k,b} \cdot \hat{B}) + K_1 (U_0^{k,\text{int}}).
\]

Here we have used the facts
\[
\left[ n_l n_j n_l \partial_{\xi}^{\otimes 2} (u_0^{k,b})_k \int_{\mathbb{R}^D} L^{-1} \mathcal{P}^\perp (v_i \hat{A}_{jk}) v_l v_m M \, dv \right]^{\tan}
\]
\[
+ \left[ n_l n_j n_l \partial_{\xi}^{\otimes 2} \theta_0^{k,b} \int_{\mathbb{R}^D} L^{-1} \mathcal{P}^\perp (v_i \hat{B}_{jk}) v_l v_m M \, dv \right]^{\tan} = 0,
\]
and
\[
\left. n_l n_j n_l \partial_{\xi}^{\otimes 2} (u_0^{k,b})_k \int_{\mathbb{R}^D} L^{-1} \mathcal{P}^\perp (v_i \hat{A}_{jk}) v_l |v|^2 M \, dv \right.
\]
\[
+ \left. n_l n_j n_l \partial_{\xi}^{\otimes 2} \theta_0^{k,b} \int_{\mathbb{R}^D} L^{-1} \mathcal{P}^\perp (v_i \hat{B}_{jk}) v_l |v|^2 M \, dv \right. = 0.
\]
Thus, (8.57) is reduced to

\[ h^{bb}_1(U^{\text{int}}_1, U^b_1) = L^R \left( \partial_\xi u_1^{k,b} \otimes n : \hat{A} + \partial_\xi \theta_1^{k,b} \cdot \hat{B} \right) - \frac{v}{\chi} \cdot \left[ \partial_\xi u_1^{k,b} \right] \tan \]

(8.61)

\[ - v \cdot V^u_1(U_0^{k,\text{int}}) + \left( \frac{D + 1}{2} - \frac{|v|^2}{2} \right) \frac{D + 2 \kappa}{D + 1} \partial_\xi \theta_1^{k,b} \]

\[ + \left( \frac{D + 1}{2} - \frac{|v|^2}{2} \right) V^\theta_1(U_0^{k,\text{int}}). \]

Furthermore, the formula (6.8) gives the boundary condition on the normal direction:

(8.62)

\[ u_2^{k,\text{int}} \cdot n = -u^{k,b}_2 \cdot n \text{ on } \partial \Omega, \]

from which we know \( h^{bb}_0(U^{\text{int}}_1, U^b_2) = 0 \). Thus from (8.54), to solve \( g^{k,bb}_2 \) we don’t need to know \( g^{k,\text{int}}_2 \) and \( g^{k,b}_2 \), although formally \( g^{k,bb}_2 \) depends on them. Thus, once we solve \( u^{k,b}_1 \cdot \nabla x \pi \) and \( \theta^{k,b}_1 \), we can solve \( g^{k,bb}_2 \).

Step 2. Order \( O(\sqrt{\epsilon}^{-1}) \) in the viscous boundary layer. The equations of \( u^{k,b}_1 \cdot \nabla x \pi \) and \( \theta^{k,b}_1 \) can be found by analyzing the order \( O(\sqrt{\epsilon}^{-1}) \) of the viscous boundary layer in (7.5), which gives

(8.63)

\[ L g^{k,b}_3 = (v \cdot \nabla x \delta) \partial_\xi g^{k,b}_2 + (v \cdot \nabla x \pi \varphi) \partial_\pi g^{k,b}_1 - i \lambda_0^k g^{k,b}_1 - i \lambda_1^k g_0^{k,b}. \]

The solvability condition for (8.63) yields that

(8.64)

\[ - \mathcal{A}^d U^{k,b}_2 = (\mathcal{A}^x + \mathcal{D}^d - i \lambda_0^k) U^{k,b}_1 + (\mathcal{F}_1 - i \lambda_1^k) U^{k,b}_0, \]

where the linear operator \( \mathcal{F}_1(U^{k,b}_0) = (\mathcal{F}_1^u, \mathcal{F}_1^\pi, \mathcal{F}_1^\theta)^\top(U^{k,b}_0) \) is defined as

\[ \mathcal{F}_1(U^{k,b}_0) \cdot \left( 1, v, \frac{|v|^2}{2} - \frac{D}{2} \right) := \]

\[ \mathcal{P} \{ v \cdot \nabla x d \partial_\xi B_2(U^{k,b}_0) + v \cdot \nabla x \pi \partial_\pi B_1(U^{k,b}_0) \}. \]

The precise form of \( \mathcal{F}_1(U^{k,b}_0) \) is tedious and not easy to represent explicitly; however, it is not of great importance for the later analysis. The \( \rho \)-component vanishes, while the \( u \)- and \( \theta \)-components are linear functions of third-order \( \xi \)-derivatives of \( u_0^{k,b} \) and \( \theta_0^{k,b} \), respectively.

Similar as in solving (8.24), we derive the ODE satisfied by \( \rho^{k,b}_2 + \theta^{k,b}_2 \) from the \( u \)-component of the projection of (8.64) on Null(\( \mathcal{A}^d \))⊥:

(8.65)

\[ - \partial_\xi (\rho^{k,b}_2 + \theta^{k,b}_2) = \left\{ v \left( 2 - \frac{2}{D} \right) \partial_{\xi}^2 - i \lambda_0^k \right\} \left[ u^{k,b}_1 \cdot \nabla x d \right] + \mathcal{F}_1^u(U^{k,b}_0) \cdot \nabla x d. \]
Note that both $u_1^{k,b} \cdot \nabla_x d$ and $u_0^{k,b} \cdot \nabla_x \pi \alpha$ (which is included in $\mathcal{F}_1^u(U_0^{k,b}) \cdot \nabla_x d$) are known from the last round and linear in $U_0^{k,\text{int}}$, so the right-hand side of (8.65) is known and a linear function of $U_0^{k,\text{int}}$. Integrating (8.65) from $\zeta$ to $\infty$ gives

$$
(8.66) \quad \rho_2^{k,b} + \theta_2^{k,b} = Y_2^b(\zeta, U_0^{k,\text{int}}),
$$

where $Y_2^b(\zeta, \cdot)$ is a linear function. We can also derive the ODEs satisfied by $u_1^{k,b} \cdot \nabla_x \pi$ and $\theta_1^{k,b}$:

$$
(8.67) \quad \left(\nu \partial_{\zeta}^2 - i \lambda_0^k\right)[u_1^{k,b} \cdot \nabla_x \pi] = i \lambda_1^k[u_0^{k,b} \cdot \nabla_x \pi] - \mathcal{F}_1^u(U_0^{k,b}) \cdot \nabla_x \pi
$$

and

$$
(8.68) \quad \left(\kappa \partial_{\zeta}^2 - i \lambda_0^k\right)\theta_1^{k,b} = i \lambda_1^k\theta_0^{k,b} - \left(\frac{2}{D+2} \mathcal{F}_1^\rho - \frac{D}{D+2} \mathcal{F}_1^\theta\right)(U_0^{k,b}),
$$

with boundary conditions (8.58) and (8.59), respectively. Because of the linearity of the above equations, we can solve (8.67) and (8.68) as

$$
(8.69) \quad u_1^{k,b} \cdot \nabla_x \pi = \tilde{Z}_0^{b,u}(\zeta, P_0 U_1^{k,\text{int}}) + \tilde{Z}_1^{b,u}(\zeta, U_0^{k,\text{int}}),
$$

$$
\theta_1^{k,b} = \tilde{Z}_0^{b,\theta}(\zeta, P_0 U_1^{k,\text{int}}) + \tilde{Z}_1^{b,\theta}(\zeta, U_0^{k,\text{int}}),
$$

recalling that $\tilde{Z}_0^{b,u}$ and $\tilde{Z}_0^{b,\theta}$ are defined in (8.30) and (8.33) respectively, and $\tilde{Z}_1^{b,u}$ is the solution of

$$
\left(\nu \partial_{\zeta}^2 - i \lambda_0^k\right)u = i \lambda_1^k[u_0^{k,b} \cdot \nabla_x \pi] - \mathcal{F}_1^u(U_0^{k,b}) \cdot \nabla_x \pi,
$$

$$
\left[u - \nu \partial_{\zeta} u\right](\zeta = 0) = -Z_1^{\text{int}}(U_0^{k,\text{int}})_u \cdot \nabla_x \pi + V_1^{u}(U_0^{k,\text{int}}) \cdot \nabla_x \pi,
$$

$$
\lim_{\zeta \to \infty} u = 0,
$$

and $\tilde{Z}_1^{b,\theta}$ is the solution of

$$
\left(\nu \partial_{\zeta}^2 - i \lambda_0^k\right)\theta = i \lambda_1^k\theta_0^{k,b} - \mathcal{F}_1^\theta(U_0^{k,b}).
$$

$$
\left[\theta - \frac{2}{D+2} \kappa \partial_{\zeta}\right](\zeta = 0) = -Z_1^{\text{int}}(U_0^{k,\text{int}})_\theta + V_1^{\theta}(U_0^{k,\text{int}}),
$$

$$
\lim_{\zeta \to \infty} \theta = 0,
$$

where $Z_1^{\text{int}}(U_0^{k,\text{int}})_u$ and $Z_1^{\text{int}}(U_0^{k,\text{int}})_\theta$ denote the $u$- and $\theta$-components of $Z_1^{\text{int}}(U_0^{k,\text{int}})$, respectively.

From the $\rho$-component of the projection of (8.64) on $\text{Null}(A^d)^\perp$ we can also derive the equation for $u_2^{k,b} \cdot \nabla_x d$:

$$
-\partial_{\zeta}[u_2^{k,b} \cdot \nabla_x d] = \text{div}_\pi[u_1^{k,b} \cdot \nabla_x \pi] + i \lambda_0^k\theta_1^{k,b} + i \lambda_1^k\theta_0^{k,b}.
$$
Integrating from $\zeta$ to $\infty$, we can solve $u_2^{k,b} \cdot \nabla x d = \vec{Z}_1^b(\zeta, P_0 U_1^{k,\text{int}}) + \vec{Z}_2^b(\zeta, U_0^{k,\text{int}})$, in particular, by taking $\zeta = 0$:

$$- u_2^{k,b} \cdot n = Z_1^b(P_0 U_1^{k,\text{int}}) + Z_2^b(U_0^{k,\text{int}}) = u_2^{k,\text{int}} \cdot n,$$

where the second equality followed from (8.62).

Finally, $g_3^{k,b}$ can be represented as

$$g_3^{k,b} = \left(1, v, \frac{|v|^2 - D}{2}\right) U_3^{k,b} + \partial_\zeta u_2^{k,b} \otimes \nabla x d : \hat{A} + \partial_\zeta \partial_2^{k,b} \nabla x d \cdot \hat{B}$$

$$+ \partial_{\pi a} u_1^{k,b} \otimes \nabla x \pi^a : \hat{A} + \partial_{\pi a} \partial_1^{k,b} \nabla x \pi^a \cdot \hat{B}$$

$$+ \mathcal{L}^{-1} P^\perp(\partial_i d \partial_j d \partial_\zeta^2 (u_1^{k,b})_k v_i \hat{A}_{jk} + \partial_i d \partial_j d \partial_\zeta^2 \theta_1^{k,b} v_i \hat{B}_j)$$

$$+ \mathcal{L}^{-1} P^\perp((\partial_i d \partial_j \pi^a + \partial_j d \partial_i \pi^a)$$

$$\cdot (\partial_\zeta^2 (u_0^{k,b})_k v_i \hat{A}_{jk} + \partial_\zeta^2 \theta_0^{k,b} v_i \hat{B}_j))$$

$$+ \mathcal{L}^{-1} P^\perp(\partial_i d \partial_j d \partial_k d [\partial_\zeta^3 (u_0^{k,b})_i v_i \mathcal{L}^{-1} P^\perp(v_j \hat{A}_{kl})$$

$$+ \partial_\zeta^3 \theta_0^{k,b} v_j \mathcal{L}^{-1} P^\perp(v_k \hat{B}_l)])$$

$$- i \lambda_k (\partial_\zeta u_0^{k,b} \otimes \nabla x d : \mathcal{L}^{-1} \hat{A} + \partial_\zeta \theta_0^{k,b} \nabla x d \cdot \mathcal{L}^{-1} \hat{B})$$

$$= B_0(U_3^b) + B_1(U_2^b) + B_2(U_1^b) + B_3(U_0^b),$$

where $B_3(v, U_0^b)$ can be represented as

$$B_3(v, U_0^b) = \mathcal{L}^{-1} P^\perp((v \cdot \nabla x d) \partial_\zeta B_2(v, U_0^b)$$

$$+ \left[(v \cdot \nabla x \pi^a) \partial_{\pi a} - i \lambda_k \right] B_1(v, U_0^b)).$$

After $g_1^{k,b}$ is solved (modulo $P_0 U_1^{k,\text{int}}$), go back to the linear kinetic boundary layer equations (8.52)–(8.53) of $g_2^{k,bb}$. Straightforward calculations by regrouping terms show that

$$H_2^{k,bb} = h_1^{bb}(P_0 U_1^{k}) + h_2^{bb}(U_0^{k,\text{int}}),$$

where $h_2^{bb}(\cdot)$ is linear and whose detailed expression we omit here. Using the linearity of the kinetic boundary layer equation and the boundary conditions, it is easy to solve that

$$g_2^{k,bb} = K_1(\xi, v, P_0 U_1^{k,\text{int}}) + K_2(\xi, v, U_0^{k,\text{int}}),$$

where $K_2(\xi, v, U_0^{k,\text{int}})$ is the solution of the kinetic boundary layer equations (8.52)–(8.53) with the boundary condition

$$(\gamma_+ - L \gamma_-) K_2 = h_2^{bb}(U_0^{k,\text{int}}) \quad \text{on } \xi = 0, \ v \cdot n > 0.$$
It is obvious that $K_2(\xi, v, \cdot)$ is linear.

Step 3. Order $O(\sqrt{\varepsilon^2})$ in the interior. The order $O(\varepsilon)$ in the interior part of (7.5) yields
\begin{equation}
\mathcal{L}s_4^k = v \cdot \nabla x s_2^k - i \lambda_0^k s_2^k \int - i \lambda_1^k s_1^k - i \lambda_2^k s_0^k,
\end{equation}
the solvability condition of which is
\begin{equation}
(A - i \lambda_0^k) U_2^{k, \text{int}} = i \lambda_1^k U_1^{k, \text{int}} + (i \lambda_2^k - \mathcal{D}) U_0^{k, \text{int}} \quad \text{in } \Omega,
\end{equation}
where $\mathcal{D}$ is defined as
\[
\mathcal{D} U = \begin{pmatrix}
0 \\
\frac{v \text{ div } (\nabla x u + \nabla x u^\top - \frac{2}{\mathcal{D}} \text{ div } x u)}{\mathcal{D}} \\
\frac{\text{div } x u}{\mathcal{D}}
\end{pmatrix}.
\]

Remark. To apply Lemma 5.1 we require the following orthogonality condition for $U_m^{k, \text{int}}$:
\begin{equation}
\{U_m^{k, \text{int}} | U_0^{l, \text{int}}\} = 0 \quad \text{for all } m \neq 0 \text{ or } k \neq l.
\end{equation}

To solve (8.72), first we apply formula (5.20) of Lemma 5.1 to deduce
\begin{equation}
i \lambda_2^k = \int_{\partial \Omega} [u_2^{k, \text{int}} \cdot n] \psi^k \text{ d} \sigma_x
= \int_{\partial \Omega} \mathcal{Z}_2^b(U_0^{k, \text{int}}) \psi^k \text{ d} \sigma_x + \int_{\partial \Omega} \mathcal{Z}_1^b(P U_0^{k, \text{int}}) \psi^k \text{ d} \sigma_x
+ (\mathcal{D} U_0^{k, \text{int}} | U_0^{k, \text{int}})
= \int_{\partial \Omega} \mathcal{Z}_2^b(U_0^{k, \text{int}}) \psi^k \text{ d} \sigma_x + (\mathcal{D} U_0^{k, \text{int}} | U_0^{k, \text{int}}).
\end{equation}

Case 1. If $\lambda_0^k$ is a simple eigenvalue, then $P U_0^{k, \text{int}} = 0$; thus $i \lambda_1^k$ is given by (8.74).

Otherwise, $\lambda_0^k$ is not a simple eigenvalue; note that
\begin{equation}
\int_{\partial \Omega} \mathcal{Z}_1^b(P U_0^{k, \text{int}}) \psi^k \text{ d} \sigma_x = \sum_{l \neq k, \lambda_0^l = \lambda_0^k} \delta^l \int_{\partial \Omega} \mathcal{Z}_1^b(U_0^{l, \text{int}}) \psi^k \text{ d} \sigma_x
= 0,
\end{equation}
because of the orthogonality condition (8.46) and (8.48). The above two identities illustrate that whether $\lambda_0^k$ is simple or not, $i \lambda_2^k$ is completely determined, which is given by (8.74).
When $\lambda_0^k$ is not simple, the compatibility condition (5.22) is needed, which gives

$$i \lambda_1^l a_1^k l + \int_{\partial \Omega} Z_2^b(U_0^k, \text{int}) \Psi^l \, d\sigma_x = i \lambda_1^l a_1^k l \quad \text{if } \lambda_0^k = \lambda_0^l \text{ and } k \neq l.$$ (8.76)

Case 2. If $\lambda_0^k$ is not a simple eigenvalue but $i \lambda_1^l$ is a simple eigenvalue of $L_1$ that is defined in (8.47), then for all $l \neq k$ with $i \lambda_0^l = i \lambda_0^k$, we have $i \lambda_1^l \neq i \lambda_1^k$. For this case $a_1^k l$ can be solved from (8.76) as

$$a_1^k l = \frac{1}{i \lambda_1^l - i \lambda_1^k} \int_{\partial \Omega} Z_2^b(U_0^k, \text{int}) \Psi^l \, d\sigma_x.$$ (8.77)

Thus $P_0(U_1^k, \text{int})$ is completely determined, and no additional conditions on $H_0(\lambda)$ rather than (8.45), or equivalently (8.48), is needed.

Case 3. If $\lambda_0^k$ is not a simple eigenvalue and $i \lambda_1^l$ is also not a simple eigenvalue of $L_1$, we need more one orthogonality condition on $H_1$. This orthogonality condition comes from (8.76):

$$\int_{\partial \Omega} Z_2^b(U_0^k, \text{int}) \Psi^l \, d\sigma_x = 0 \quad \text{if } \lambda_0^l = \lambda_0^k, \lambda_1^l = \lambda_1^k \text{ for } l \neq k.$$ (8.78)

We can define a quadratic form $Q_2$ and the symmetric operator $L_2$ on $H_1(\lambda_1)$ as

$$Q_2(\Psi^k, \Psi^l) = \int_{\partial \Omega} Z_2^b(U_0^k, \text{int}) \Psi^l \, d\sigma_x + \langle \mathcal{D}U_0^k, \text{int} \mid U_0^l, \text{int} \rangle,$$ (8.79)

and $L_2 \Psi^k = i \lambda_2^k \Psi^k$, which satisfies that

$$Q_2(\Psi^k, \Psi^l) = \int_{\Omega} L_2(\Psi^k) \Psi^l \, d\sigma_x.$$

Be these definitions, we have $i \lambda_2^k = Q_2(\Psi^k, \Psi^k)$, and the condition (8.78) is

$$Q_2(\Psi^k, \Psi^l) = 0 \quad \text{if } \Psi^k, \Psi^l \in H_1(\lambda_1) \text{ and } l \neq k.$$ (8.78)

Under these conditions, equation (8.72) can be solved in the following way: Let $U_2^{k, \text{int}} = U^1 + U^2$, where $U^1$ satisfies the equation

$$(A - i \lambda_0^k) U^1 = i \lambda_1^k P_0 U_1^{k, \text{int}},$$

$$u^1 \cdot n = Z_1(P_0 U_1^{k, \text{int}}),$$
whose solution in \( \text{Ker}(A - i \lambda_0^k)^{\perp} \) is \( Z_1^{\text{int}}(P_0 U_1^{k, \text{int}}) \), and \( U^2 \) satisfies the equation
\[
(A - i \lambda_0^k) U^2 = i \lambda_1^k Z_1^{\text{int}}(U_1^{k, \text{int}}) + (i \lambda_2^k - D) U_0^{k, \text{int}},
\]
\[
u^2 \cdot n = Z_2^b(U_0^{k, \text{int}}),
\]
whose solution in \( \text{Ker}(A - i \lambda_0^k)^{\perp} \) is completely determined and is denoted by \( Z_2^{\text{int}}(U_0^{k, \text{int}}) \). In summary, the equation (8.72) is
\[
U_2^{k, \text{int}} = P_0 U_2^{k, \text{int}} + Z_1^{\text{int}}(P_0 U_1^{k, \text{int}}) + Z_2^{\text{int}}(U_0^{k, \text{int}}),
\]
where \( P_0(U_1^{k, \text{int}}) = (P_1 + P_1^\perp)(U_1^{k, \text{int}}) \) in which \( P_1^\perp U_1^{k, \text{int}} \) is already completely determined in (8.77), \( P_1 U_1^{k, \text{int}} \) will be determined later, and \( P_0 U_2^{k, \text{int}} \) is defined the same as in (8.50), i.e.,
\[
P_0 U_2^{k, \text{int}} = \sum_{l \neq k, \lambda_0^l = \lambda_0^k} a_2^{kl} U_0^{l, \text{int}},
\]
where \( a_2^{kl} = \langle U_2^{k, \text{int}} | U_0^{l, \text{int}} \rangle \) will be determined later. Finally, we can represent \( g_4^{k, \text{int}} \) as
\[
g_4^{k, \text{int}} = I_0(U_4^{k, \text{int}}) + I_2(U_2^{k, \text{int}}) + I_4(U_0^{k, \text{int}})
\]
\[
= \left(1, v, \frac{|v|^2}{2} - \frac{D}{2}\right) U_4^{k, \text{int}} + \nabla_x u_2^{k, \text{int}} : \hat{A}(v) + \nabla_x \theta_0^{k, \text{int}} \cdot \hat{B}(v)
\]
\[
+ L^{-1} P^\perp (a_2^{kl} u_0^{k, \text{int}} v_i \widehat{A}_{jk} + a_b^{kl} \theta_0^{k, \text{int}} v_i \widehat{B}_j)
\]
\[
- i \lambda_0^k (\nabla_x u_0^{k, \text{int}} : L^{-1} \hat{A} + \nabla_x \theta_0^{k, \text{int}} \cdot L^{-1} \hat{B}),
\]
where \( I_4(U_0^{k, \text{int}}) = L^{-1} P^\perp \{ (v \cdot \nabla_x - i \lambda_0^k) I_2(U_0^{k, \text{int}}) \} \). Thus we conclude Round 2 in the induction.

### 8.6 General Case: Induction Hypothesis

For \( m \geq 3 \), we assume that we have finished the \((m - 1)\)th round, i.e., used the information from the kinetic boundary layer, the viscous boundary, and the interior up to order \( O(\sqrt{\epsilon}^{m-3}) \), \( O(\sqrt{\epsilon}^{m-2}) \), and \( O(\sqrt{\epsilon}^{m-1}) \), respectively. Before we solve the next round, we write down the hypothesis that summarizes what we were able to construct until now. We write this down in the following 10 statements that we need to check for the \( m \)th round.

- \((P_{m-1}^{1})\): For \( 2 \leq j \leq m - 1 \), \( \rho_j^{k,b} + \theta_j^{k,b} = \sum_{h=2}^j \gamma_h^{b}(\xi, P_0 U_{j-h}^{k,\text{int}}) \). For \( j = 0, 1 \), \( \rho_j^{k,b} + \theta_j^{k,b} = 0 \).

- \((P_{m-1}^{2})\): For \( 0 \leq j \leq m - 2 \), \( u_j^{k,b} \cdot \nabla_x \pi = \sum_{h=0}^j \tilde{Z}_h^{b,u}(\xi, P_0 U_{j-h}^{k,\text{int}}) \).

- \((P_{m-1}^{3})\): For \( 0 \leq j \leq m - 2 \), \( \theta_j^{k,b} = \sum_{h=0}^j \tilde{Z}_h^{b,\theta}(\xi, P_0 U_{j-h}^{k,\text{int}}) \).
\((P^4_{m-1})\): For \(1 \leq j \leq m-1\), \(g^{k,b}_j \cdot \nabla_x d = \sum_{h=1}^{j} \frac{Z^k_h}{h} (\xi, P_0 U_{j-h}^{k,\text{int}})\). Taking \(\xi = 0\), we deduce that on the boundary we have \(-u^{k,b}_j \cdot n = \sum_{h=1}^{j} Z^k_h (P_0 U_{j-h}^{k,\text{int}})\).

\((P^5_{m-1})\): For \(0 \leq j \leq m\), \(g^{k,b}_j = \sum_{h=0}^{j} B_h (U_{j-h}^{\text{int}})\), where \(B_h\) for \(h \geq 0\) is defined iteratively starting from \(B_0(U^b) = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U^b\):

\[
B_h(U^b) = L^{-1} P \left\{ v \cdot \nabla_x d \partial_t B_{h-1}(U^b) + v \cdot \nabla_x \pi \partial_t B_{h-1}(U^b) - \sum_{l=0}^{h-3} i \lambda^k_l B_{h-2-l}(U^b) \right\}.
\]

(8.81)

\((P^6_{m-1})\): For \(1 \leq j \leq m-1\),

\[
g^{k,bb}_j = \sum_{h=1}^{j} K_h (v, \xi, P_0 (U_{j-h}^{k,\text{int}})),
\]

where the linear operator \(K_h(v, \xi, U_0^{k,\text{int}})\) is the solution to the linear kinetic boundary layer equation (6.3)–(6.4) with the source term \(s^{bb}_h(U_0^{k,\text{int}})\) and the boundary source term \(h^{bb}_h(U_0^{k,\text{int}})\).

\((P^7_{m-1})\): For \(1 \leq j \leq m-1\), \(i \lambda^k_j = Q_j(\Psi^k, \Psi^k)\), where the quadratic form \(Q_1\) and \(Q_2\) are defined in (8.46) and (8.79), respectively, and \(Q_j\) for \(3 \leq j \leq m-1\) is defined as

\[
Q_j(\Psi^k, \Psi') = \sqrt{\frac{D + 2}{2D}} \int_{\partial \Omega} \left\{ Z^b_j + V^b_j \right\} (U_0^{k,\text{int}}) \Psi' d\sigma_x
\]

(8.83)

\[
+ \sum_{h=2}^{j} \mathcal{G}_h (Z_{j-h}^{\text{int}}(U_0^{k,\text{int}}) | U_0^{j,\text{int}}).
\]

Note \(\mathcal{G}_h\) is defined in (8.111).

\((P^8_{m-1})\): For \(0 \leq j \leq m-1\), \(U_j^{k,\text{int}} = P_0 U_j^{k,\text{int}} + \sum_{h=1}^{j} Z^k_h (P_0 U_{j-h}^{k,\text{int}})\).

\((P^9_{m-1})\): For \(0 \leq j \leq m+1\), \(g_j^{k,\text{int}} = I_0(U_0^{\text{int}}) + I_2(U_{j-2}^{\text{int}}) + \sum_{h=4}^{j} I_j(U_{j-h}^{\text{int}})\), where \(I_h\) for \(h \geq 0\) is defined iteratively starting from \(I_0(U^{\text{int}}) = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U^{\text{int}}, I_1 = 0\).

\[
(8.84)\]

\[
I_h(U^{\text{int}}) = L^{-1} P \left\{ v \cdot \nabla_x I_{h-2}(U^{\text{int}}) - \sum_{l=0}^{h-4} i \lambda^k_l I_{h-2-m}(U^{\text{int}}) \right\}.
\]
The last assumption to check deals with the number of orthogonality conditions needed and specifies what is already determined and what is still not determined in the construction.

We distinguish between \( m \) cases:

**Case 1.** \( i \lambda_h \) is a simple eigenvalue of \( L_h \) for \( 0 \leq h \leq m - 2 \). No orthogonality condition is needed, and every term in the expansion is fully determined.

**Case 2.** (\( 2 \leq j \leq m \)) \( i \lambda_h \) is a multiple eigenvalue of \( L_h \) for \( 0 \leq h \leq j - 2 \), but \( i \lambda_{j-1} \) is a simple eigenvalue of \( L_{j-1} \). (Note: the case \( m \) means that all the eigenvalues \( i \lambda_h \) for \( 0 \leq h \leq m - 2 \) are multiple eigenvalues.)

- We need the orthogonality conditions: For each \( 0 \leq h \leq j - 2 \),

\[
Q_{h+1}(\Psi^k, \Psi^l) = 0 \quad \text{for } \Psi^k, \Psi^l \in H_0 \cap \cdots \cap H_h,
\]

where for \( h \geq 1 \), the space \( H_h = H_h(\lambda_h) = \{ \Psi \in H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1}): L_h \Psi = i \lambda_h \Psi \} \).

- For \( 1 \leq h \leq m - j \), \( U_{h,\text{int}}^k \) are completely determined. (For the case \( m \), no term is completely determined.)

- For \( m - j + 1 \leq h \leq m - 1 \), \( (P_0^k + \cdots + P_{m-1-h}^k)U_{h,\text{int}}^k \) are determined.

- For \( m - j + 1 \leq h \leq m - 1 \), \( P_{m-1-h} U_{h,\text{int}}^k \) are not determined,

where \( P_{h-1} \) is the orthogonal projection onto \( H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1}) \) and \( P_{h-1} = P_h + P_{h-1}^+ \), where \( P_{h-1}^+ \) is the orthogonal projection onto \( H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1}) \cap H_h(\lambda_h) \).

**Remark.** Regarding condition (8.85), we have a stronger orthogonality property that is actually equivalent to (8.85), namely: for each \( 0 \leq h \leq j - 2 \),

\[
Q_{h+1}(\Psi_k, \Psi_l) = 0 \quad \text{for } l \neq k, \, \Psi_k, \Psi_l \in H_0.
\]

Indeed, we just need to use that the \( L_h \) leave stable the spaces \( H_h \). Of course, we have to define \( L_h \) over the whole space \( H_0 \) even if the eigenvalue is simple, but in this case we just take it to be the identity.

In the next subsection, we are going to prove the 10 hypotheses \((P_{m-1}^1) - (P_{m}^1)\) assuming \( P_{i-1} \) for \( i \leq m \).

### 8.7 Induction: Round \( m \)

For \( m \geq 3 \), we assume that we have finished round \( m - 1 \) in the induction process. For round \( m \), as before it includes three steps by considering terms in the kinetic, viscous boundary layers, and interior separately.
Step 1. Order $O(\sqrt{\varepsilon}m^{-2})$ in the kinetic boundary layer. The order $O(\sqrt{\varepsilon}m^{-2})$ of the kinetic boundary layer in the ansatz gives that $g_m^{k,bb}$ satisfies the linear boundary layer equation

\[
L^{BL} g_m^{k,bb} = s_m^{k,bb} \quad \text{in } \xi > 0,
\]

\[
g_m^{k,bb} \to 0 \quad \text{as } \xi \to \infty,
\]

with boundary condition at $\xi = 0$

\[
L^{R} g_m^{k,bb} = H_m^{k,bb} \quad \text{on } \xi = 0, \quad v \cdot n > 0.
\]

The source term

\[
s_m^{k,bb} = \{ v \cdot \nabla_x \pi^a \partial_{\pi^a} - i \lambda^k_0 \} s_m^{k,bb} - \sum_{j=1}^{m-3} i \lambda^k_j s_{m-2-j}^{k,bb}
\]

\[
= \sum_{j=3}^{m} \bar{s}_j^{bb} (P_0 U_{m-j}^{k,\text{int}}).
\]

where for $3 \leq j \leq m - 3$,

\[
\bar{s}_j^{bb} (U^{\text{int}}) = \{ v \cdot \nabla_x \pi^a \partial_{\pi^a} - i \lambda^k_0 \} K_{j-2} (U^{\text{int}}) - \sum_{h=1}^{j-3} i \lambda^k_h K_{j-2-h} (U^{\text{int}}).
\]

where we recall that the functions $K_j (U^{\text{int}})$ are defined through \(8.82\). The boundary source term is

\[
H_m^{k,bb} = -L^{R} \pi^k_m + L^{D} (\pi^k_{m-1} + g_m^{k,bb})
\]

\[
= \sum_{j=0}^{m} \bar{h}_j^{bb} (\bar{U}_0^{k,j}) = \sum_{j=1}^{m} h_j^{bb} (P_0 U_{m-j}^{k,\text{int}}).
\]

Here

\[
\bar{h}_j^{bb} (\bar{U}_0^{k}) = -L^{R} \big\{ I_j (U_0^{k,\text{int}}) + B_j (U_0^{k,b}) \big\}
\]

\[
+ L^{D} \big\{ I_{j-1} (U_0^{k,\text{int}}) + B_{j-1} (U_0^{k,b}) + K_{j-1} (U_0^{k,\text{int}}) \big\}.
\]

The definition of $h_j^{bb} (P_0 U_{m-j}^{k,\text{int}})$ is the following: represent all $I_0, \ldots, I_m$ and $B_0, \ldots, B_m$ in terms of $U_0^{k,\text{int}}, \ldots, P_0 U_{m-1}^{k,\text{int}}$, and collect the corresponding terms in $\sum_{j=0}^{m} \bar{h}_j^{bb} (\bar{U}_0^{k,j})$, which defines $h_j^{bb} (P_0 U_{m-j}^{k,\text{int}})$ for $j = 1, \ldots, m - 1$. Note that there is no $U_0^{k,\text{int}}$ term, since it formally only appears in the term $-L^{R} (I_0 (U_0^{int}) + B_0 (U_0^{b})) = -2(v \cdot n) (u_0^{k,\text{int}} + u_0^{b}) \cdot n$, which only depends on $U_0^{k,\text{int}}, P_0 U_1^{k,\text{int}}, \ldots, P_0 U_{m-1}^{k,\text{int}}$; see (P$^4_{m-1}$).
The formulas (6.9) and (6.10) give the boundary conditions

\[
(8.91) \quad \left[ u_{m-1}^{k,b} - \frac{\nu}{\chi} \partial_\xi u_{m-1}^{k,b} \right]^{\tan} + \left[ u_{m-1}^{k,\text{int}} \right]^{\tan} = \sum_{j=1}^{m-1} V_j^u (P_0 U_{m-1-j}^{\text{int}})
\]

and

\[
(8.92) \quad \theta_{m-1}^{k,b} = \frac{D + 2 \kappa}{D + 1} \frac{\nu}{\chi} \partial_\xi \theta_{m-1}^{k,b} = \sum_{j=1}^{m-1} V_j^\theta (P_0 U_{m-1-j}^{\text{int}}),
\]

where

\[
(8.93) \quad \sum_{j=1}^{m-1} V_j^u (P_0 U_{m-1-j}^{k,\text{int}}) = -\frac{\nu}{\chi} \left[ 2d (u_{m-2}^{k,\text{int}} \cdot \mathbf{n}) \right]^{\tan} - \frac{\nu}{\chi} \nabla \pi [u_{m-2}^{k,b} \cdot \mathbf{n}]
\]

\[
+ \sum_{j=1}^{m-1} \int_{\mathbf{v} \cdot \mathbf{n} > 0} \left[ L^D \{ B_j (U_{m-1-j}^{k,b}) + K_j (P_0 U_{m-1-j}^{k,\text{int}}) \} (\mathbf{v} \cdot \mathbf{n}) \nu \right]^{\tan} M \, d\nu
\]

\[- \frac{1}{\chi} \left( (\mathbf{v} \cdot \mathbf{n}) \nu \left( \sum_{j=4}^{m} I_j (U_{m-j}^{k,\text{int}}) + \sum_{j=3}^{m} B_j (U_{m-j}^{k,b}) \right) \right) \tan
\]

\[
+ \frac{1}{\chi} \int_0^\infty \left( v S_m^{k,b} \right)^{\tan} d\xi
\]

and

\[
(8.94) \quad \sum_{j=1}^{m-1} V_j^\theta (P_0 U_{m-1-j}^{k,\text{int}}) = -\frac{D + 2 \kappa}{D + 1} \frac{\nu}{\chi} \theta_{m-2}^{k,\text{int}} + \frac{\sqrt{2\pi}}{2(D + 1)} u_{m-1}^{k,b} \cdot \mathbf{n}
\]

\[
+ \frac{\sqrt{2\pi}}{D + 1} \sum_{j=1}^{m-1} \int_{\mathbf{v} \cdot \mathbf{n} > 0} \left[ L^D \{ B_j (U_{m-1-j}^{k,b}) \} (\mathbf{v} \cdot \mathbf{n}) |\nu|^2 \right]^{\tan} M \, d\nu
\]

\[- \frac{1}{(D + 1)\chi} \left( (\mathbf{v} \cdot \mathbf{n}) |\nu|^2 \left( \sum_{j=4}^{m} I_j (U_{m-j}^{k,\text{int}}) + \sum_{j=3}^{m} B_j (U_{m-j}^{k,b}) \right) \right) \tan
\]

\[
+ \frac{D + 2}{D + 1} \frac{1}{\chi} \int_0^\infty \left( |\nu|^2 - 1 \right) S_m^{k,b} \tan d\xi.
\]

If we express in (8.93) and (8.94) all $U^{\text{int}}$ and $U^{b}$ in terms of $U_0^{\text{int}}$, $P_0 U_1^{\text{int}}$, \ldots, $P_0 U_{m-2}^{\text{int}}$, and collect the corresponding terms, we can then define $V_{m-1}^u (U_0^{k,\text{int}})$ and $V_{m-1}^\theta (U_0^{k,\text{int}})$. 


Note that \( V_u^1, \ldots, V_u^{m-2} \) and \( V_\theta^1, \ldots, V_\theta^{m-2} \) have been defined in the previous rounds of the induction.

Furthermore, the formula (6.8) gives the boundary condition on the normal direction:

\[
(8.95) \quad [u_m^{k,\text{int}} + u_m^{k,b}] \cdot n = \sum_{j=3}^{m} V_j^n(P_0 U_{m-j}^{k,\text{int}}) \text{ on } \partial \Omega,
\]

where

\[
V_m^n(t_0^{k,\text{int}}) = \int_0^\infty \left\{ v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} - i \lambda_0^k \right\} K_{m-2}(t_0^{k,\text{int}}) \, d\xi
\]

\[- \sum_{j=1}^{m-3} \int_0^\infty (i \lambda_j^k K_{m-2-j}(U_0^{k,\text{int}})) \, d\xi.
\]

**Step 2. Order \( O(\sqrt{\varepsilon}^{m-1}) \) in the viscous boundary layer.** The equations of \( [u_m^{k,\text{int}}] \text{tan} \) and \( \theta_m^{k-1} \) can be derived by considering the order \( O(\varepsilon^{m-1}) \) of the viscous boundary layer:

\[
(8.96) \quad L g_{m+1}^{k,b} = v \cdot \nabla_x d \partial_\xi g_{m}^{k,b} + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} g_{m}^{k,b} - \sum_{j=0}^{m-1} i \lambda_j^k g_{m-1-j}^{k,b},
\]

the solvability of which is the following system of ODEs:

\[
(8.97) \quad -A_d U_m^{k,b} = \sum_{j=0}^{m-1} (F_j - i \lambda_j^k) U_{m-1-j}^{k,b},
\]

where \( F_{m-1}(U_0^{k,b}) \) is defined by

\[
(8.98) \quad P \left\{ v \cdot \nabla_x d \partial_\xi B_m(t_0^{k,b}) + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} B_{m-1}(U_0^{k,b}) \right\} =
\left( 1, v, \frac{|v|^2}{2} - \frac{D}{2} \right) F_{m-1}(t_0^{k,b}).
\]

Note that the linear operators \( F_1, \ldots, F_{m-2} \) have been defined in the previous rounds of the induction. In particular, \( F_0 = A_\pi + D^d \).

Projecting the system (8.97) onto \( \text{Null}(A_d)^\perp \), the u-component and \( \rho \)-component of which give the equations of \( \rho_m^{k,b} + \rho_m^{k,b} \) and \( u_m^{k,b} \cdot \nabla_x d \), respectively:

\[
(8.99) \quad -\partial_\xi (\rho_m^{k,b} + \rho_m^{k,b}) = \left\{ v \left( \frac{2 - D}{2} \right) \partial_\xi^2 - i \lambda_0^k \right\} (u_m^{k,b} \cdot \nabla_x d)
\]

\[+ \sum_{j=1}^{m-1} (I - \Pi_d) (F_j - i \lambda_j^k) U_{m-1-j}^{k,b}, \]
and

\[-\partial_\xi (u^{k,b}_m \cdot \nabla_x d) = \text{div}(u^{k,b}_{m-1} \cdot \nabla_x \pi) + \kappa \partial_{\xi}^2 \theta^{k,b}_{m-1} - i \lambda_0^{k} \frac{D}{D + 2} (\rho^{k,b}_{m-1} + \theta^{k,b}_{m-1}) + \sum_{j=1}^{m-1} \{(1 - \Pi^d)(\mathcal{F}_j - i \lambda_j^{k}) U^{k,b}_{m-1-j}\}^\rho.\]

(8.100)

Here we use the notation for a vector \(V = (V^\rho, V^u, V^\theta)^T\). Integrating (8.99) from \(\xi\) to \(\infty\) gives

\[(8.101) \quad \rho^{k,b}_m + \theta^{k,b}_m = \sum_{j=2}^{m} Y_j^b(\xi, P_0 U^{\text{int}}_{m-j}),\]

in which the linear operators \(Y_2^b, \ldots, Y_{m-1}^b\) have been defined in the previous rounds of the induction. This corresponds to \((P_m^1)\).

Next, by projecting the system (8.97) onto \(\text{Null}(A^d)\), the \(u\)-component of the projection gives that \(u^{k,b}_{m-1} \cdot \nabla_x \pi\) satisfies the ODE

\[(8.102) \quad \left[u - \frac{\nu}{\kappa} \partial_\xi u\right](\xi = 0) = -u^{k,b}_{m-1} \cdot \nabla_x \pi + \sum_{j=1}^{m-1} Y_j^u(P_0 U^{\text{int}}_{m-1-j}),\]

\[\lim_{\xi \to \infty} u = 0,\]

the solution of which is

\[(8.103) \quad u^{k,b}_{m-1} \cdot \nabla_x \pi = \sum_{j=0}^{m-1} \tilde{Z}^{b,u}_j(\xi, P_0 U^{\text{int}}_{m-1-j}),\]

where the linear operators \(\tilde{Z}^{b,u}_0, \ldots, \tilde{Z}^{b,u}_{m-2}\) have been defined in the previous rounds of the induction. This corresponds to \((P_m^2)\).
Projecting the system \((8.97)\) onto \(\text{Null}(A)\) gives that \(\theta_{m-1}^{k,b}\) satisfies the ODE
\[
\{\kappa \partial_\xi^2 - i \lambda_0^k\} \theta = -i \lambda_0^k \frac{2}{D + 2} (\rho_{m-1}^{k,b} + \theta_{m-1}^{k,b})
- \sum_{j=1}^{m-1} \{ \Pi^d(F_j - i \lambda_j^k) U_{m-1-j}^{k,b}\} \theta,
\]
(8.104)
\[
\left[ \theta - \frac{D + 2 \kappa}{D + 1} \partial_\xi \theta \right](\xi = 0) = -\theta_{m-1}^{k,\text{int}} + \sum_{j=1}^{m-1} V_j^\theta (P_0 U_{m-1-j}^{\text{int}}),
\]
the solution of which is
\[
\theta_{m-1}^{k,b} = \sum_{j=0}^{m-1} \tilde{Z}_j^{b,\theta}(\xi, P_0 U_{m-1-j}^{\text{int}}),
\]
(8.105)
where the linear operators \(\tilde{Z}_j^{b,\theta}, \ldots, \tilde{Z}_{m-2}^{b,\theta}\) have been defined in the previous rounds of the induction. This corresponds to \((P_m^A)\).

Having \((8.103)\) and \((8.105)\), go back to \((8.100)\) and integrate from \(\xi\) to \(\infty\); we then obtain
\[
\begin{align*}
(8.106) \quad u_m^{k,b} \cdot \nabla x d
= \sum_{j=1}^m \tilde{Z}_j^{b}(\xi, P_0 U_{m-j}^{\text{int}}).
\end{align*}
\]
In particular, by taking \(\xi = 0\) in \((8.106)\), we obtain
\[
(8.107) \quad - u_m^{k,b} \cdot n = \sum_{j=1}^m Z_j^{b}(P_0 U_{m-j}^{\text{int}}).
\]
Thus from \((8.107)\) and \((8.95)\) we derive the boundary condition of \(u_m^{k,\text{int}} \cdot n\), which will be used in the next step to solve \(U_m^{k,\text{int}}\). This corresponds to \((P_m^A)\).

Equation \((8.96)\) can be solved as \(g_m^{k,b} = \sum_{h=0}^{m+1} B_h(U_{j-h}^{\text{int}})\) under these conditions, where \(B_h(U^{\text{int}})\) is defined in \((8.81)\). Note that \(B_0, B_1, \ldots, B_m\) have been determined in the previous rounds of the induction. This corresponds to \((P_m^S)\).

Finally, we go back to the kinetic boundary equation \((8.87)\) to solve \(g_m^{k,bb}\) as
\[
(8.108) \quad g_m^{k,bb} = \sum_{j=1}^m K_j(v, \xi, P_0 U_{m-j}^{\text{int}}),
\]
where the linear operator \(K_m(v, \xi, U_0^{k,\text{int}})\) is the solution to the linear kinetic boundary layer equation \((6.3)\)–\((6.4)\) with the source term \(s_m^{bb}(U_0^{k,\text{int}})\) and the boundary
source term $h^b(U^k_0, \text{int})$. Note that $K_1, \ldots, K_{m-1}$ have been defined in the previous rounds of the induction. This corresponds to $(P^6_m)$. Thus we finish Step 2.

**Step 3.** Order $O(\sqrt{\varepsilon}m)$ in the interior. The order $O(\varepsilon^{m/2})$ in the interior part of the ansatz yields

$$
L g^k_{m+2} = v \cdot \nabla x g^k_{m, \text{int}} - \sum_{j=0}^m i \lambda_{j,0} g^k_{m-j},
$$

and the solvability condition of which is

$$
(A - i \lambda^k_0) U^k_{m, \text{int}} = \sum_{j=1}^m (i \lambda^k_j - \mathcal{G}_j) U^k_{m-j} \quad \text{in } \Omega,
$$

$$
u^k_{m, \text{int}} \cdot n = \sum_{j=1}^m (Z^b_j + V^m_j) (P_0 U^k_{m-j}) \quad \text{on } \partial \Omega,
$$

where the vector-valued linear operator $\mathcal{G}_j$ for $j \geq 4$ is defined as

$$
(1, v, \frac{|v|^2}{2} - \frac{D}{2}) \mathcal{G}_j(U^k_0, \text{int}) = \mathcal{P} \{ v \cdot \nabla x I_j(U^k_0, \text{int}) \}.
$$

Note that $\mathcal{G}_1 = 0, \mathcal{G}_2 = \mathcal{D}, \mathcal{G}_3 = 0$ and $V^m_j = 0$ for $j = 1, 2$.

Applying Lemma 5.1 to (8.110) and recalling in (8.83) the definition of $Q_j$ for $j = 1, 2, \ldots, m - 1$, the formula (5.20) gives

$$
i \lambda^k_m = \sqrt{-D + 2 \int_{\partial \Omega} (Z^b_m + V^m_m)(U^k_0, \text{int}) \psi^k d\sigma_x}
+ \sum_{j=2}^m \mathcal{G}_j(Z^k_{m-j}(U^k_0, \text{int})) U^k_{m-j} \psi^k
+ \sum_{h} Q_{h}(P_0 U^k_{m-h}, \psi^k).
$$

The orthogonality condition (8.86) implies that the second line of (8.112) vanishes. Thus we can define the right-hand side of the first line of (8.112) as $Q_m(\psi^k, \psi^k)$. Thus we verify that $i \lambda^k_m = Q_m(\psi^k, \psi^k)$, which is completely determined. This corresponds to $(P^7_m)$.

To solve the equation (8.110), we need to consider $m + 1$ cases:

**Case 1.** $i \lambda^k_h$ is a simple eigenvalue of $L_h$ for $0 \leq h \leq m - 1$. No orthogonality condition is needed, and every term is fully determined;

**Case j.** ($2 \leq j \leq m + 1$) $i \lambda^k_h$ is a multiple eigenvalue of $L_h$ for $0 \leq h \leq j - 2$, and a simple eigenvalue of $L_h$ for $j - 1 \leq h \leq m - 1$. 

We only consider the case $m + 1$ here; i.e., all the eigenvalues $i \lambda^k_h$ are multiple. The other cases are simpler. Taking the inner product with $U^{\text{int}}_0$ for $l \neq k$, $\lambda^l_0 = \lambda^k_0$, which is

$$
(8.113) \quad \sum_{h=1}^{m-1} i \lambda^k_h a^{kl}_{m-h} = \sum_{h=1}^{m-1} Q_h (P_0 U^{k, \text{int}}_{m-h}, \Psi^l) + Q_m (\Psi^k, \Psi^l).
$$

If $\Psi^k, \Psi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{m-1}(\lambda_{m-1})$, then because of the orthogonality condition (8.85), for $1 \leq h \leq m - 2$,

$$
= i \lambda^l_h a^{kl}_{m-h}.
$$

For $h = m - 1$, $Q_{m-1}(P_0 U^{k, \text{int}}_1, \Psi^l) = i \lambda^l_{m-2} a^{kl}_1 + Q_{m-1}(P_{m-2} U^{k, \text{int}}_1, \Psi^l)$. Thus, the identity (8.113) implies that we need the orthogonality condition that for $k \neq l$,

$$
(8.114) \quad Q_m (\Psi^k, \Psi^l) = \int_\Omega L_m (\Psi^k) \Psi^l \, dx = 0
$$

for $\Psi^k, \Psi^l \in H_1 \cap \cdots \cap H_{m-1}$.

where the symmetric operator $L_m$ is defined by $L_m \Psi^l = i \lambda^l_m \Psi^l$, for $\Psi^l \in H_1 \cap \cdots \cap H_{m-1}$.

If $\Psi^k, \Psi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \cdots \cap H_{m-2}(\lambda_{m-2}) \cap H_{m-1}(\lambda_{m-1})$, i.e., $\lambda^k_h = \lambda^l_h$ for $0 \leq h \leq m - 2$, but $\lambda^k_{m-1} \neq \lambda^l_{m-1}$, from the identity (8.113), for these $k, l$, $a^{kl}_1$ can be determined by

$$
a^{kl}_1 = \frac{1}{i \lambda^k_{m-1} - i \lambda^l_{m-1}} Q_m (\Psi^k, \Psi^l).
$$

This means that $(P_0^\perp + P_1^\perp + \cdots + P_{m-1}^\perp) U^{k, \text{int}}_1$ is completely determined, but $P_{m-1} U^{k, \text{int}}_1$ is still left as undetermined.

If $\Psi^k, \Psi^l \in H_1 \cap \cdots \cap H_{m-3} \cap H_{m-2}$,

$$
(8.115) \quad Q_{m-1}(P_{m-1} U^{k, \text{int}}_1, \Psi^l) + Q_{m-1}(\Psi^k, \Psi^l) = (i \lambda^k_{m-2} - i \lambda^l_{m-2}) a^{kl}_2 + i \lambda^k_{m-1} a^{kl}_1,
$$

from which $a^{kl}_2$; thus $P_{m-2} U^{k, \text{int}}_2$ is completely determined.

Under these solvability conditions, the equation (8.110) can be solved as

$$
U^{k, \text{int}}_m = P_0 U^{k, \text{int}}_m + \sum_{h=1}^{m} Z^{\text{int}}_h (P_0 U^{k, \text{int}}_{m-h}),
$$
where \( Z_{m}^{\text{int}}(U_{0}^{k, \text{int}}) \) is the solution to the following equation:

\[
(A - i\lambda_{0}^{k}) U = \sum_{h=1}^{m} (i\lambda_{h}^{k} - \mathcal{G}_{h}) Z_{m-h}^{\text{int}}(U_{0}^{k, \text{int}}) \quad \text{in } \Omega,
\]

(8.116)

\[
u \cdot n = (Z_{m}^{b} + V_{m}^{b}) t_{0}^{k, \text{int}} \quad \text{on } \partial \Omega.
\]

Thus, \( U_{m}^{k, \text{int}} \) is determined modulo \( P_{0} U_{m}^{k, \text{int}}, P_{1} U_{m-1}^{k, \text{int}}, \ldots, P_{m-1} U_{1}^{k, \text{int}} \), which are undetermined at this stage. Under these conditions, equation (8.109) is solved as

\[
g_{k}^{b, \text{int}}, \quad \lambda_{k}^{b, \text{int}} = \left[ \begin{array}{c}
I_{0}(U_{m+2}^{\text{int}}) + I_{2}(U_{m}^{\text{int}}) + \sum_{h=4}^{m+2} I_{m+2}(U_{m+2-h}^{\text{int}}) \end{array} \right].
\]

This corresponds to \( (P_{0}^{b}), (P_{m}^{b}), \) and \( (P_{m}^{b}) \).

We can now inductively continue the process, namely go to the order \( \mathcal{O}(\sqrt{\varepsilon}^{m+1}) \) of the viscous boundary layer, the order \( \mathcal{O}(\sqrt{\varepsilon}^{m+1}) \) of the kinetic boundary layer, then the order \( \mathcal{O}(\sqrt{\varepsilon}^{m+1}) \) of the interior, and so on. We should do this at least until the order \( N + 2 \) where \( N \) is the precision of the error in (7.10). Note, however, that for a given \( \lambda = \lambda_{0}^{k} \), we may only need to construct a small number of the \( L_{j} \) if after a few steps all the eigenvalues become simple, namely, if for some \( j \) all the eigenvalues of \( L_{j} \) are simple on the space \( H_{1}(\lambda_{1}) \cap \cdots \cap H_{j-1}(\lambda_{j-1}) \). It is clear that if the eigenvalues become simple for some \( j \leq N + 2 \), then the orthogonality condition (8.85) allows us to determine the eigenfunctions \( \psi^{k} \) uniquely. If the process does not end, then we just need to satisfy the condition until the order \( N + 2 \), which yields a nonunique choice of eigenfunctions. Also, in this case, we set all the undetermined pieces of the eigenfunction, namely those left undetermined to be 0.

9 Proof of Proposition 7.1: Truncation Error Estimates

In the previous sections, we construct the kinetic-fluid boundary layers up to any order for \( \alpha_{\varepsilon} = \sqrt{2\pi} \varepsilon^{\beta} \). Now we define the approximated eigenfunction and eigenvalues \( \psi_{\varepsilon,N}^{k} \) and \( \lambda_{\varepsilon,N}^{k} \) by truncation in the corresponding ansatzes. More specifically,

\[
\psi_{\varepsilon,N}^{k} = \sum_{j=0}^{N} \left( g_{j}^{k, \text{int}} + g_{j}^{k, b} \varepsilon^{j/2} + \sum_{j=1}^{N} g_{j}^{k, bb} \varepsilon^{j/2} \right), \quad \lambda_{\varepsilon,N}^{k} = \sum_{j=0}^{N} \lambda_{j}^{k} \varepsilon^{j/2}.
\]

9.1 Estimates of \( R_{\varepsilon,N}^{k} \)

Using the eigenequation (7.4), we can easily find that the error term \( R_{\varepsilon,N}^{k} \) has the form of

\[
R_{\varepsilon,N}^{k} = \left\{ (i \lambda_{0}^{k} - v \cdot \nabla) g_{N-1}^{k, \text{int}} + (i \lambda_{0}^{k} - v \cdot \nabla \chi_{\pi}) \partial_{\pi} [g_{N-1}^{k, b} + g_{N-1}^{k, bb}] \right\} \varepsilon^{\frac{N+1}{2}} + \text{higher-order terms},
\]
where \( g^k = g^{k, \text{int}} + g^{k,b} + g^{k,bb} \). From the constructions of \( g^{k, \text{int}} \), \( g^{k,b} \), and \( g^{k,bb} \), it is easy to know that

\[
\| g^k_j \|_{L^r(\omega_x; L^p(aM du))} \leq C
\]

for all \( j \) and \( 1 < r, p < \infty \), where \( g^k \) stands for \( g^{k, \text{int}}, g^{k,b}, \) or \( g^{k,bb} \).

Indeed, both the hydrodynamic and the kinetic parts of \( g^{k, \text{int}} \) and \( g^{k,b} \) have coefficients in terms of the components of \( U^{k, \text{int}} \) and \( U^{k,b} \). From Lemma 5.1, the solutions \( U_j^{k, \text{int}} \) of the equation (5.16) can be represented linearly in terms of components of \( U^{k, \text{int}}_i \) for \( 0 \leq i < j \) and the boundary terms of \( U^{k,b}_i \) for \( 0 \leq i < j \). Note that the pseudo-inverse operator \( (\mathcal{A} - i \lambda \tau, k)^{-1} \) is bounded, and furthermore, the boundary values of \( U^{k,b}_i \) and \( g^{k,b}_i \) are linearly in terms of \( U^{k, \text{int}}_i \) for \( 0 \leq i < j \). For \( U^{k,b}_j \), their components are solutions of second-order ordinary differential equations with boundary conditions in terms of \( U^{k, \text{int}}_i \) and \( g^{k,b}_i \) for \( 0 \leq i < j \). Moreover, the solutions of the linear kinetic boundary layer equation (6.3) for \( g^{k,bb}_i \) are bounded in \( L^r(\omega_x; L^p(aM du)) \) for any \( 1 < r \leq \infty \).

For the term \( (v \cdot \nabla \sigma) \partial \tau, k_{N} \), we integrate over \( \Omega \times \mathbb{R}^D \) and use simple change of variable \( (y_1, y_2, \ldots, y_D) = \pi(x), y_D = \frac{d(x)}{\sqrt{\varepsilon}} \), we can have extra \( \sqrt{\varepsilon} \), so all will be in the higher order terms. Thus, we have the error estimate (7.10).

### 9.2 Estimates of \( g^{k}_{\varepsilon,N} - g^{k,\text{int}}_0 \)

The leading order term of \( g^{k}_{\varepsilon,N} - g^{k,\text{int}}_0 \) is \( g^{k,b}_0 \), so using the expressions above for \( u^{k,b}_0, \gamma^{k,b}_0 \) and a simple change of variable which will give an extra \( \sqrt{\varepsilon} \), we have

\[
\| g^{k}_{\varepsilon,N} - g^{k,\text{int}}_0 \|_{L^r(\omega_x; L^p(aM du))} \leq C \varepsilon^{\frac{1}{2p}}.
\]

Thus we get (7.11).

### 9.3 Boundary Error Estimate

Finally, the boundary error term \( r^{k}_{\varepsilon,N} \) is

\[
r^{k}_{\varepsilon,N} = -\sqrt{\varepsilon}^{N+1} L^D \left( g^{k,\text{int}}_N + g^{k,b}_N + g^{k,bb}_N \right),
\]

from which we can get the estimate (7.12). Thus we finish the proof of the Proposition 7.1.
10 Proof of the Weak Convergence in Theorem 3.1 and 3.2

In order to derive the fluid equation with the boundary conditions, we need to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (2.3). We choose the renormalization used in [28]:

\[
\Gamma(Z) = \frac{Z - 1}{1 + (Z - 1)g}.
\]

After multiplying \(\varepsilon_0 G\) and dividing by \(\varepsilon\), equation (2.20) becomes

\[
\partial_t \tilde{g}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g}_\varepsilon = \frac{1}{\varepsilon} \Gamma'(G_\varepsilon) \int_{S^{D-1} \times \mathbb{R}^D} q_\varepsilon b(\omega, v_1 - v) d\omega M_1 dv_1,
\]

where \(\tilde{g}_\varepsilon = \frac{1}{\varepsilon} \Gamma(G_\varepsilon)\) can be considered as the \(L^2\) part of the fluctuations \(g_\varepsilon\), and \(q_\varepsilon\) is the scaled collision integrand defined as

\[
q_\varepsilon = \frac{G'_\varepsilon G'_{\varepsilon 1} - G_{\varepsilon 1} G_{\varepsilon}}{\varepsilon^2}.
\]

By introducing \(N_\varepsilon = 1 + \varepsilon^2 g_\varepsilon^2\), we can write

\[
\tilde{g}_\varepsilon = \frac{g_\varepsilon}{N_\varepsilon}, \quad \Gamma'(G_\varepsilon) = \frac{2}{N_\varepsilon^2} - \frac{1}{N_\varepsilon}.
\]

When moments of the renormalized Boltzmann equation (10.2) are formally taken with respect to any \(\xi \in \text{span}\{1, v_1, \ldots, v_D, |v|^2\}\), one obtains the local conservation laws with defects

\[
\partial_t \tilde{\rho}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{\rho}_\varepsilon = \frac{1}{\varepsilon} \{\Gamma'(G_\varepsilon)q_\varepsilon\},
\]

\[
\partial_t \tilde{u}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x (\tilde{\rho}_\varepsilon + \tilde{\rho}_\varepsilon) + \frac{1}{\varepsilon} v \cdot \{A(v)\tilde{g}_\varepsilon\} = \frac{1}{\varepsilon} \{v \Gamma'(G_\varepsilon)q_\varepsilon\},
\]

\[
\partial_t \tilde{\theta}_\varepsilon + \frac{1}{\varepsilon} D \nabla_x \cdot \tilde{u}_\varepsilon + \frac{2}{\varepsilon} \nabla_x \cdot \{B(v)\tilde{g}_\varepsilon\} = \frac{1}{\varepsilon} \left\{ \left( \frac{|v|^2}{D} - 1 \right) \Gamma'(G_\varepsilon)q_\varepsilon \right\},
\]

which can be written as

\[
\partial_t \tilde{U}_\varepsilon + \frac{1}{\varepsilon} A \tilde{U}_\varepsilon + \tilde{Q}_\varepsilon = \tilde{R}_\varepsilon,
\]

where

\[
\tilde{U}_\varepsilon = (\tilde{\rho}_\varepsilon, \tilde{u}_\varepsilon, \tilde{\theta}_\varepsilon) = \left(\langle \tilde{g}_\varepsilon \rangle, \langle v \tilde{g}_\varepsilon \rangle, \left( \frac{|v|^2}{D} - 1 \right) \tilde{g}_\varepsilon \right),
\]

\[
\tilde{Q}_\varepsilon = \left(0, \frac{1}{\varepsilon} \nabla_x \cdot \{A(v)\tilde{g}_\varepsilon\}, \frac{1}{\varepsilon} \nabla_x \cdot \{B(v)\tilde{g}_\varepsilon\} \right),
\]

and the local conservation defect

\[
\tilde{R}_\varepsilon = \frac{1}{\varepsilon} \left\{ (1, v, \frac{|v|^2}{D} - 1) \Gamma'(G_\varepsilon)q_\varepsilon \right\}.
\]
Notice that we do not know if \( z_u \) is not necessary in the domain of \( A \) for every \( \varepsilon > 0 \); thus the notation \( \mathcal{A} \tilde{U}_e \) in (10.5) is not quite rigorous. However, we can show that the weak limit of \( \tilde{u}_e \), say, \( u \), satisfies \( u \cdot n = 0 \) on the boundary, also see [24]. From the local conservation laws with defect (10.5), formally the limit of \( \tilde{U}_e \) will be in the null space of the acoustic operator \( \mathcal{A} \). In other words, any weak limits of \( (\tilde{\rho}_e, \tilde{u}_e, \tilde{\theta}_e) \) will satisfy the incompressibility \( \nabla \cdot \tilde{u} = 0 \) and Boussinesq relation \( \tilde{\rho} + \tilde{\theta} = 0 \).

The term \( \frac{1}{\varepsilon} \mathcal{A} \tilde{U}_e \) in (10.5) describes the acoustic waves with propagation speed \( \frac{1}{\varepsilon} \). As \( \varepsilon \) goes to 0, the sound waves propagate faster and faster to make the fluid limit singular. To derive the incompressible fluid equations, a natural way is to project the local conservation laws (10.5) onto Null \( \mathcal{A} \) and Null \( \mathcal{A} \) with \( \tilde{\rho}_e \). First \( \tilde{U}_e \) can be orthogonally decomposed as

\[
\tilde{U}_e = \Pi \tilde{U}_e + \Pi^\perp \tilde{U}_e
\]

(10.6)

in which we call \( \Pi \tilde{U}_e \) and \( \Pi^\perp \tilde{U}_e \) the incompressible and acoustic parts of \( \tilde{U}_e \), respectively.

By definition of Leray projection in a bounded domain (5.1), the boundary conditions of \( \mathbb{P} \langle v \tilde{\varepsilon} \rangle \) and \( \mathbb{Q} \langle v \tilde{\varepsilon} \rangle \) are

\[
\mathbb{P} \langle v \tilde{\varepsilon} \rangle \cdot n = 0 \quad \text{and} \quad \mathbb{Q} \langle v \tilde{\varepsilon} \rangle \cdot n = \tilde{u}_e \cdot n \quad \text{on} \quad \partial \Omega.
\]

To derive the weak form of the evolution equations of \( \Pi \tilde{U}_e \), we take the test function \( Y \) in (2.25) as a special infinitesimal Maxwellian in the incompressible mode:

\[
Y^{\text{incom}}(x, v) = -\chi + w \cdot v + \chi \left( \frac{|v|^2}{2} - \frac{D}{2} \right),
\]

where \( (\chi, w) \in C^\infty(\Omega, \mathbb{R}^D \times \mathbb{R}) \) with \( \nabla \cdot w = 0 \) in \( \Omega \) and \( w \cdot n = 0 \) on \( \partial \Omega \). Because \( \chi \) and \( w \) are independent, the weak form of (10.5) can be written separately as

\[
\int \mathbb{P} \langle v \tilde{\varepsilon}(t_2) \rangle \cdot w \, dx - \int \mathbb{P} \langle v \tilde{\varepsilon}(t_1) \rangle \cdot w \, dx = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle A \tilde{\varepsilon} \rangle : \nabla w \, dx \, dt + \frac{1}{\sqrt{2\pi \varepsilon}} \int_{t_1}^{t_2} \int_{\partial \Omega} \langle y \tilde{\varepsilon}(w \cdot v) \rangle \, \partial \Omega \, d\sigma_x \, dt
\]

(10.7)
and
\[
\frac{D + 2}{2} \int_\Omega \left( \left( \frac{|v|^2}{D + 2} - 1 \right) \tilde{g}_\varepsilon(t_2) \right) \chi \, dx \\
- \frac{D + 2}{2} \int_\Omega \left( \left( \frac{|v|^2}{D + 2} - 1 \right) \tilde{g}_\varepsilon(t_1) \right) \chi \, dx \\
- \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_\Omega \left\{ \mathbf{B} \tilde{g}_\varepsilon \cdot \nabla_x \chi \right\} \, dx \, dt \\
+ \frac{1}{\sqrt{2\pi\varepsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left( \frac{|v|^2}{D + 2} - 1 \right) \gamma \tilde{g}_\varepsilon \, d\sigma_x \, dt \\
= - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_\Omega \left( \frac{|v|^2}{D + 2} - 1 \right) \Gamma'(G_\varepsilon) q_\varepsilon \, dx \, dt.
\]
(10.8)

Identities (10.7) and (10.8) are the local conservation laws in the incompressible modes, and they are the starting point of the proof of the weak convergence to the incompressible Navier-Stokes equations with boundary conditions in the main theorems 3.1 and 3.2. It has been proved in [28] the convergence of the interior terms of (10.7) and (10.8) as \( \varepsilon \to 0 \) to recover the weak form of incompressible Navier-Stokes equations. It is only left to derive the boundary conditions of the limiting equations.

The strategy to recover the boundary conditions in the limit is basically the same as [34] except for some necessary modifications. For the convenience of the reader, and also because we work in more general collision kernels, we briefly go through the proof here. In [28], the author proved that inside the domain \( \mathbb{R}^+ \times \Omega \times \mathbb{R}^D \), the family of fluctuations \( g_\varepsilon \) is relatively compact in \( w^{-1}_{-1,\text{loc}}(dt; w^{-1}_{-1}(\sigma M \, dv \, dx)) \), and that every limit point \( g \) has the form (3.4). Lemma 5.1 of [34] showed that the trace of the limit point \( \gamma g \) belongs to \( L^1_{\text{loc}}(dt; L^1(M |v \cdot n(x)| d\sigma_x)) \) and satisfies
\[
\gamma g = v \cdot \gamma u + \left( \frac{|v|^2}{2} - \frac{D + 2}{2} \right) \gamma \theta,
\]
(10.9)
where \( \gamma u \) and \( \gamma \theta \) denote the fluid traces of \( u \) and \( \theta \).

We list some key a priori estimates from [34] on \( \gamma g_\varepsilon \). The first one is from the inside; we generalize it to the more general collision kernel case considered in this paper.

**LEMMA 10.1.** For all \( p > 0 \), as \( \varepsilon \to 0 \),
\[
\gamma \tilde{g}_\varepsilon \to \gamma g \quad \text{in} \quad w^{-1}_{-1,\text{loc}}(dt; w^{-1}_{-1}(M(1 + |v|^p)|v \cdot n(x)| d\sigma_x)).
\]

**PROOF.** The proof is essentially the same as that for lemma 5.2 of [34] except for a new argument to treat the soft potential collision kernel case. First, using the
function
\[ \Gamma(Z) = \left( \frac{Z - 1}{1 + (Z - 1)^2} \right)^{5/3} \]
in the renormalized formulation (2.23) gives
\[ (\varepsilon \partial_t + v \cdot \nabla_x) \tilde{g}^{5/3} = \]
\[ \frac{5}{3} \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \tilde{g}^{2/3} \frac{\varepsilon^2}{N_e} \left( \frac{2}{N_e} - \frac{1}{1} \right) b(\omega, v - v_1) d\omega M_1 dv_1. \]

To estimate the right-hand side in (10.11), we apply the classical Young’s inequality, namely,
\[ p \zeta \leq r^*(p) + r(\zeta) \]
for every \( p \) and \( \zeta \) in the domains of \( r^* \) and \( r \). Here the function \( r \) is defined over \( \zeta > -1 \) by \( r(\zeta) = \zeta \log(1 + \zeta) \), which is strictly convex, and \( r^* \) is the Legendre dual of \( r \).

\[ \left| \frac{q_e}{N_e^2} \tilde{g}_e^{2/3} \right| \leq \frac{1}{e^4} G_e G_{e1} r \left( \frac{e^2 q_e}{G_e G_{e1}} \right) + \frac{1}{e^4} G_e G_{e1} r^* \left( \frac{e^2 q_e^{2/3}}{N_e^2} \right) \]
\[ \leq \frac{1}{e^4} G_e G_{e1} r \left( \frac{e^2 q_e}{G_e G_{e1}} \right) + G_e G_{e1} \frac{\tilde{g}_e^{4/3}}{N_e^4} r^*(1). \]

The second inequality above used the superquadratic homogeneity of \( r^* \). By the entropy dissipation rate bound, the first term on the right-hand side of (10.12) is bounded in \( L^1_{loc}(dt, L^1(dv \, dx)) \). Since \( N_e \geq 1 \) and \( G_e \leq \sqrt{2N_e} \), the integral of the second term can be bounded as follows:
\[ \sqrt{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{g}_e^{4/3} G_e G_{e1} \frac{\tilde{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv \]
\[ \leq \sqrt{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\tilde{g}_e^{4/3}}{N_e^{1/2}} (1 + \varepsilon \tilde{g}_e) \frac{\tilde{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv \]
\[ + \sqrt{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{g}_e^{4/3} |\tilde{g}_e - \tilde{g}_e| \frac{\tilde{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv. \]

Using the third assumption from Section 2, namely (2.7) and \( |\tilde{e} \tilde{g}_e| \leq \frac{1}{2} \), the first term in (10.13) is bounded. Indeed, it is bounded by
\[ C \int_{\mathbb{R}^d} \frac{\tilde{g}_e^{4/3}}{N_e^{1/2}} a M dv \leq C \int_{\mathbb{R}^d} \frac{\tilde{g}_e^2}{\sqrt{N_e}} a M dv \leq C. \]
The second term in \((10.13)\) is bounded as
\[
\left| \frac{2}{\sqrt{N_\varepsilon}} \frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}} \beta(v_1 - v) a(v_1) a(v) a_1 M_1 dv_1 \leq \varepsilon^{2/3} \left( \frac{1}{2} \right)^{4/3} \int_{\mathbb{R}^d} \frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}} a_1 M_1 dv_1.
\]

Since the right-hand side of \((10.14)\) vanishes as \(\varepsilon\) goes to 0, we deduce that \((\varepsilon \partial_t + v \cdot \nabla) g_\varepsilon^{5/3}\) is uniformly bounded in \(L^1_{\text{loc}} (dt; L^1 (M \, dv \, dx))\). The rest of the proof of \((10.10)\) is the same as that of lemma 5.2 in [34]. □

The next lemma is the a priori estimate of \(\gamma g_\varepsilon\) from the boundary term in the entropy inequality \((2.28)\). The proof is the same as lemma 6.1 in [34] with some trivial modification. So we just state the lemma without giving the proof.

**Lemma 10.2.** Define \(\gamma_\varepsilon = \gamma + g_\varepsilon - (\gamma + g_\varepsilon)_{\partial \Omega}\) and
\[
\gamma_\varepsilon^{(1)} = \gamma_\varepsilon \mathbb{1}_{\gamma + g_\varepsilon \leq 2 (G_\varepsilon)_{\partial \Omega}} \leq 4 \gamma_\varepsilon g_\varepsilon, \quad \gamma_\varepsilon^{(2)} = \gamma_\varepsilon - \gamma_\varepsilon^{(1)}.
\]
Then each of these is bounded as follows:
\[
\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma_\varepsilon^{(1)}}{1 + \varepsilon^2 (\gamma + g_\varepsilon)^2} \quad \text{in } L^2_{\text{loc}} (dt; L^2 (M |v| \, dv \, d\sigma_x)),
\]
\[
\varepsilon \frac{\gamma_\varepsilon^{(1)}}{1 + \varepsilon^2 (\gamma + g_\varepsilon)^2} \quad \text{in } L^2_{\text{loc}} (dt; L^2 (M |v| \, dv \, d\sigma_x)),
\]
\[
\frac{\alpha_\varepsilon}{\varepsilon^2} \gamma_\varepsilon^{(2)} \quad \text{in } L^1_{\text{loc}} (dt; L^1 (M |v| \, dv \, d\sigma_x)).
\]

Using Lemma 10.1 and Lemma 10.2, we can prove the following lemma, which describes how to define the renormalized outgoing mass flux \(\mathbb{1}_{\Sigma +} \rho\).

**Lemma 10.3.** Assume that \(\frac{\alpha_\varepsilon}{\sqrt{2 \pi \varepsilon}} \to \chi \in (0, +\infty)\). Then up to the extraction of a subsequence,
\[
\gamma_\varepsilon \quad \text{and} \quad \frac{\gamma_\varepsilon}{1 + \varepsilon^2 (\gamma + g_\varepsilon)^2}
\]
converge in \(w-L^1_{\text{loc}} (dt; w-L^1 (M |v| \cdot n(x) |dv \, dx)|)\) and have the same weak limit. Moreover, there exists \(\rho \in L^1_{\text{loc}} (dt; L^1 (d\sigma_x))\) such that, up to the extraction of a subsequence,
\[
\mathbb{1}_{\Sigma +} (\gamma + g_\varepsilon)_{\partial \Omega} \to \mathbb{1}_{\Sigma +} \rho \quad \text{in } w-L^1_{\text{loc}} (dt; w-L^1 (M |v| \cdot n(x) |dv \, dx)).
\]
Furthermore,
\[
\rho = (\gamma + g)_{\partial \Omega}.
\]
Lemma [10.3] is nothing but lemma 6.2 and lemma 6.3 in [34]. The proof is basically the same except for some trivial modifications because we use some different renormalizations. Thus we skip the proof here.

Now it is ready to recover the Dirichlet boundary condition. For the case \( \alpha / \varepsilon \to \infty \), from (10.16) and (10.18), we deduce that

\[
\frac{\gamma^{(1)}_\varepsilon}{1 + \varepsilon^2 \langle \gamma + g_\varepsilon \rangle^2_{\partial \Omega}} \to 0 \quad \text{strongly in } L^2_{\text{loc}}(dt; L^2(M | v \cdot n(x)|dv dx)),
\]

\[
\frac{\gamma^{(2)}_\varepsilon}{1 + \varepsilon^2 \langle \gamma + g_\varepsilon \rangle^2_{\partial \Omega}} \to 0 \quad \text{strongly in } L^1_{\text{loc}}(dt; L^1(M | v \cdot n(x)|dv dx));
\]

hence, we get

\[(10.19) \quad \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \langle \gamma + g_\varepsilon \rangle^2_{\partial \Omega}} \to 0 \quad \text{strongly in } L^1_{\text{loc}}(dt; L^1(M | v \cdot n(x)|dv dx)).\]

On the other hand,

\[(10.20) \quad \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \langle \gamma + g_\varepsilon \rangle^2_{\partial \Omega}} = \gamma + \bar{g}_\varepsilon - \frac{\mathbb{1}_{\Sigma_+} \langle \gamma + g_\varepsilon \rangle_{\partial \Omega}}{1 + \varepsilon^2 \langle \gamma + g_\varepsilon \rangle^2_{\partial \Omega}} \to \gamma + g - \mathbb{1}_{\Sigma_+} \rho\]

in \( w \cdot L^1_{\text{loc}}(dt; w \cdot L^1(M | v \cdot n(x)|dv dx)) \). Then (10.19) and (10.20) imply that

\( \gamma + g = \mathbb{1}_{\Sigma_+} \rho \)

where \( \rho \) depends only on \((t, x)\). Thus, by (10.9) we get the Dirichlet boundary condition

\( \gamma u = 0 \) and \( \gamma \theta = 0 \).

Now, we concentrate on the Navier boundary condition case. Using the previous convergence results, we can take limits in the conservation laws (10.7) and (10.8) to get the weak form of the boundary conditions. In the weak forms (10.7) and (10.8), the limits of the interior terms have been carried in [28], which is stated in the following lemma:

**Lemma 10.4.** Assume that \( \frac{\alpha_\varepsilon}{\sqrt{2\pi \varepsilon}} \to \chi \in [0, \infty) \), then up to the extraction of a sequence, \( \mathbb{P} \langle v \bar{g}_\varepsilon \rangle \) and \( \langle \frac{|v|^2}{\Omega + 1} - 1 \rangle \bar{g}_\varepsilon \) converge to \( u \) and \( \theta \) in \( C([0, \infty); w \cdot L^1(dx)) \) such that, for all \( w \in C^\infty(\overline{\Omega}; \mathbb{R}^D) \) with \( \nabla_x \cdot w = 0 \) in \( \Omega \) and \( w \cdot n = 0 \) on \( \partial \Omega \), for
all $\chi \in C^\infty(\overline{\Omega}; \mathbb{R})$, and for all $t_1, t_2 > 0$,

$$
\int_\Omega u(t_2) \cdot w \, dx - \int_\Omega u(t_1) \cdot w \, dx - \int_{t_1}^{t_2} \int_\Omega \sum_{i,j} u_i u_j \partial_i w_j \, dx \, dt \\
+ v \int_{t_1}^{t_2} \int_\Omega (\partial_i u_j + \partial_j u_i) \partial_i w_j \, dx \, dt \\
= - \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\sqrt{2\pi \varepsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left( \gamma_\varepsilon^{(1)} (w \cdot v) \mathbb{1}_{|v|^2 \leq 20 \log \varepsilon} \right) \frac{\gamma_\varepsilon^{(1)}}{1 + \varepsilon^2 \gamma + \gamma_\varepsilon} (1 + \varepsilon^2 \gamma + \gamma_\varepsilon) \, d\sigma_x \, dt,
$$

(10.21)

$$
\int_\Omega \theta(t_2) \cdot \chi \, dx - \int_\Omega \theta(t_1) \cdot \chi \, dx - \int_{t_1}^{t_2} \int_\Omega \theta u \cdot \nabla_x \chi \, dx \, dt \\
+ \frac{2}{D+2} \kappa \int_{t_1}^{t_2} \int_\Omega \nabla_x \theta \cdot \nabla_x \chi \, dx \, dt \\
= - \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\sqrt{2\pi \varepsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left( \gamma_\varepsilon^{(1)} \right) \frac{\gamma_\varepsilon^{(1)}}{1 + \varepsilon^2 \gamma + \gamma_\varepsilon} (1 + \varepsilon^2 \gamma + \gamma_\varepsilon) \\
\cdot \chi \left( |v|^2 \frac{\gamma + \gamma_\varepsilon}{D+2} - 1 \right) \mathbb{1}_{|v|^2 \leq 20 \log \varepsilon} \, d\sigma_x \, dt,
$$

(10.22)

where $\gamma + \gamma_\varepsilon = (1 - \alpha_\varepsilon) \gamma + \gamma_\varepsilon + \alpha_\varepsilon (\gamma + \gamma_\varepsilon) \partial \Omega$.

PROOF. The proof is an analogue of the proof for lemma 7.1 and lemma 7.2 of [34]. Denote by $Y_\varepsilon$ the test function $(w \cdot v) \mathbb{1}_{|v|^2 \leq 20 \log \varepsilon}$ or $\chi(|v|^2/(D+2) - 1) |v|^2 \mathbb{1}_{|v|^2 \leq 20 \log \varepsilon}$. Then $Y_\varepsilon$ has the property $Y_\varepsilon = LY_\varepsilon$, in which we recall that $L$ is the local reflection operator defined in (2.2). From (2.26), the renormalized form of the Maxwell boundary condition reads

$$
(10.23) \quad \gamma - g_\varepsilon = (1 - \alpha_\varepsilon) \frac{L \gamma + g_\varepsilon}{1 + \varepsilon^2 (L \gamma + g_\varepsilon)^2} + \alpha_\varepsilon \frac{\gamma + g_\varepsilon}{1 + \varepsilon^2 (L \gamma + g_\varepsilon)^2},
$$

where $\gamma + \gamma_\varepsilon = (1 - \alpha_\varepsilon) \gamma + \gamma_\varepsilon + \alpha_\varepsilon (\gamma + \gamma_\varepsilon) \partial \Omega$.
Then
\[
\frac{1}{\varepsilon} \int_{\partial \Omega} (\gamma g Y_\varepsilon) \, d\sigma_x
= \frac{1}{\varepsilon} \int_{\partial \Omega} \left( \frac{\varepsilon^2 \gamma + g \epsilon_2^2}{1 + \varepsilon^2 \gamma + g \epsilon_2^2} Y_\varepsilon \| \Sigma_+ \right) \, d\sigma_x
+ \frac{\alpha_\varepsilon}{\varepsilon} \int_{\partial \Omega} \left( \frac{\gamma \epsilon}{1 + \varepsilon^2 \gamma + g \epsilon_2^2} Y_\varepsilon \| \Sigma_+ \right) \, d\sigma_x
\]
(10.24)
\[
= \frac{\alpha_\varepsilon}{\varepsilon} \int_{\partial \Omega} \left( \frac{\gamma \epsilon (1) + \gamma \epsilon (2)}{1 + \varepsilon^2 \gamma + g \epsilon_2^2}(1 + \varepsilon^2 \gamma + g \epsilon_2^2) Y_\varepsilon \| \Sigma_+ \right) \, d\sigma_x
- \frac{\alpha_\varepsilon}{\varepsilon} \int_{\partial \Omega} \left( \frac{(\gamma \epsilon (1) + \gamma \epsilon (2)) \epsilon^2 (\gamma + g \epsilon_2)}{1 + \varepsilon^2 \gamma + g \epsilon_2^2}(1 + \varepsilon^2 \gamma + g \epsilon_2^2) Y_\varepsilon \| \Sigma_+ \right) \, d\sigma_x.
\]

By (10.18),
\[
\int_{t_1}^{t_2} \left| \frac{\alpha_\varepsilon}{\varepsilon} \int_{\partial \Omega} \left( \frac{\gamma \epsilon (2)}{1 + \varepsilon^2 \gamma + g \epsilon_2^2}(1 + \varepsilon^2 \gamma + g \epsilon_2^2) Y_\varepsilon \| \Sigma_+ \right) \, d\sigma_x \right| \, dt \leq C_\varepsilon \left\| \frac{Y_\varepsilon}{(1 + \varepsilon^2 \gamma + g \epsilon_2^2)(1 + \varepsilon^2 \gamma + g \epsilon_2^2)} \right\| \leq C_\varepsilon |\log \varepsilon|.
\]
(10.25)

The \( \gamma \epsilon (2) \) part in the last term of (10.24) can be estimated as (10.25). For the \( \gamma \epsilon (1) \) part, from (10.16)
\[
\int_{t_1}^{t_2} \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma \epsilon (1)}{\sqrt{1 + \varepsilon^2 \gamma + g \epsilon_2^2}} \, dt \leq C \varepsilon \left\| \frac{Y_\varepsilon}{(1 + \varepsilon^2 \gamma + g \epsilon_2^2)(1 + \varepsilon^2 \gamma + g \epsilon_2^2)} \right\| \leq C_\varepsilon |\log \varepsilon|.
\]
(10.26)

is relatively compact in \( w \cdot L^1_{\text{loc}} (dt; w \cdot L^1 (M | v \cdot n | dv \, d\sigma_x)) \).

Use the fact that
\[
\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma \epsilon (1)}{\sqrt{1 + \varepsilon^2 \gamma + g \epsilon_2^2}}
\]
(10.27)
is bounded in \( L^\infty \) and goes to 0 a.e. Then by the product limit theorem of [3], the product of (10.26) and (10.27) goes to 0 in \( L^1_{\text{loc}} (dt) \) as \( \varepsilon \to 0 \). Thus we finish the proof of the lemma. \( \square \)

Now, we are ready to recover the Navier boundary condition by taking the limit in the last terms in (10.21) and (10.22). As in [34], we can deduce that
\[
\frac{\alpha_\varepsilon}{\sqrt{2\pi \varepsilon}} \left( \frac{\gamma \epsilon (1) (w \cdot v)^{1/2 |v|^2 \leq 20 |\log \varepsilon|}}{(1 + \varepsilon^2 \gamma + g \epsilon_2^2)(1 + \varepsilon^2 \gamma + g \epsilon_2^2)} \right)_{\partial \Omega} \to \lambda (\gamma + g \| \Sigma_+ (\gamma + g) \| \partial \Omega) (w \cdot v)_{\partial \Omega},
\]
\[
\frac{\alpha_\varepsilon}{\sqrt{2\pi\varepsilon}} \left( \frac{\gamma_{\varepsilon}^{(1)}}{1 + \varepsilon^2 \gamma_{\varepsilon}^2 + \varepsilon^2 \gamma_{\varepsilon}^2} \right) \chi \left( \frac{|v|^2}{D + 2} - 1 \right) \mathbb{I}_{|v|^2 \leq 20|\log \varepsilon|} \right)_{\partial \Omega} \rightarrow \lambda \left( \gamma + g - \frac{3}{2} (\gamma + g)_{\partial \Omega} \right) \chi \left( \frac{|v|^2}{D + 2} - 1 \right)_{\partial \Omega}
\]

in \(w \cdot L^1_{\text{loc}}(\partial \Omega) \). Using (10.9), we finally prove the weak form of the incompressible Navier-Stokes equations with Navier boundary conditions:

\[
\int_{\Omega} u(t_2) \cdot w \, dx - \int_{\Omega} u(t_1) \cdot w \, dx - \int_{t_1}^{t_2} \int_{\Omega} \sum_{i,j} u_i u_j \partial_i w_j \, dx \, dt
+ v \int_{t_1}^{t_2} \int_{\Omega} (\partial_i u_j + \partial_j u_i) \partial_i w_j \, dx \, dt
= \lambda \int_{t_1}^{t_2} \int_{\partial \Omega} \gamma u \cdot w \, d\sigma_x \, dt,
\]

\[
\int_{\Omega} \theta(t_2) \cdot \chi \, dx - \int_{\Omega} \theta(t_1) \cdot \chi \, dx - \int_{t_1}^{t_2} \int_{\Omega} \theta u \cdot \nabla_x \chi \, dx \, dt
+ \frac{2}{D + 2} \kappa \int_{t_1}^{t_2} \int_{\Omega} \nabla_x \theta \cdot \nabla_x \chi \, dx \, dt
= \frac{D + 1}{D + 2} \alpha \int_{t_1}^{t_2} \int_{\partial \Omega} \gamma \theta \sigma_x \, dt.
\]

Thus we finish the proof of the weak convergence results in Theorem 3.1 and Theorem 3.2.

**11 Proof of Strong Convergence in Theorem 3.1**

In the previous section, we proved that the incompressible part of the fluid moments \( \tilde{U}_\varepsilon \), i.e., \( \Pi^\perp \tilde{U}_\varepsilon \), converges only weakly to solutions of the incompressible NSF equations. This weak convergence is caused by the persistence of the fast acoustic part \( \Pi^\perp \tilde{U}_\varepsilon \), as in the periodic domain [28]. If \( \Pi^\perp \tilde{U}_\varepsilon \) vanishes in some strong sense as \( \varepsilon \) goes to 0, we can improve the convergence of \( \Pi^\perp \tilde{U}_\varepsilon \) from weak to strong. The main novelty of this paper is to prove that in the bounded domain \( \Omega \), when \( \alpha_\varepsilon = O(\sqrt{\varepsilon}) \), the acoustic part will be damped instantaneously. This damping effect comes from the kinetic-fluid coupled boundary layers. More precisely, we have the following proposition:

**Proposition 11.1.** Let \( \Pi^\perp \tilde{U}_\varepsilon \) be defined as (10.6). If \( \alpha_\varepsilon = O(\sqrt{\varepsilon}) \), then

\[
\Pi^\perp \tilde{U}_\varepsilon \rightarrow 0 \quad \text{in} \quad L^2_{\text{loc}}(\partial \Omega ; L^2(dx))
\]

as \( \varepsilon \rightarrow 0 \).
This proposition is also true for $\alpha = O(\epsilon^\beta)$, $0 \leq \beta < \frac{1}{2}$, and $\frac{1}{2} < \beta < 1$. These cases will be treated in a separate paper.

Now we apply Proposition 11.1 to prove the main theorem, Theorem 3.1, and leave its proof to the next subsection.

11.1 Strong Convergence in $L^1$: Proof of Theorem 3.1

We first show that we can improve the relative compactness of the family of fluctuations $g_\varepsilon$ from weak to strong in $L^1_{\text{loc}}(dt; L^1(\sigma M \, dv \, dx))$. Indeed, $g_\varepsilon$ can be decomposed as

$$g_\varepsilon = \mathcal{P} \tilde{g}_\varepsilon + \mathcal{P} \perp \tilde{g}_\varepsilon + \frac{\varepsilon^2 g_3^3}{N_\varepsilon}$$

$$= v \cdot \mathcal{P} \tilde{\mathbf{u}}_\varepsilon + \left( \frac{D}{D+2} \tilde{\theta}_\varepsilon - \frac{2}{D+2} \tilde{\rho}_\varepsilon \right) \left( \frac{|v|^2}{2} - \frac{D+2}{2} \right) + v \cdot \mathcal{Q} \tilde{\mathbf{u}}_\varepsilon$$

$$+ \frac{|v|^2}{D+2} (\tilde{\rho}_\varepsilon + \tilde{\theta}_\varepsilon) + \mathcal{P} \perp \tilde{g}_\varepsilon + \frac{\varepsilon^2 g_3^3}{\sqrt{N_\varepsilon} \sqrt{N_\varepsilon}}$$

where $\mathcal{P}$ is the projection to Null($\mathcal{L}$) defined in (2.12), $\mathcal{P}$ is the Leray projection, and $\mathcal{Q} = I - \mathcal{P}$.

It has been proved in [28] that $\mathcal{P} \perp \tilde{g}_\varepsilon \rightarrow 0$ in $L^2_{\text{loc}}(dt; L^2(\sigma M \, dv \, dx))$ (see (6.41) in [28]). We can also show that

$$\mathcal{P} \tilde{\mathbf{u}}_\varepsilon \rightarrow u, \quad \frac{D}{D+2} \tilde{\theta}_\varepsilon - \frac{2}{D+2} \tilde{\rho}_\varepsilon \rightarrow \theta, \quad \text{in } L^2_{\text{loc}}(dt; L^2(dx)).$$

Indeed, this convergence is justified in lemma 5.6 in [20]. Although the renormalization and decomposition of $g_\varepsilon$ are different in [20] and the current paper, the proof of the convergence (11.1) can follow the argument in the proof of Lemma 5.6 in [20]. Furthermore, Proposition 11.1 yields that

$$v \cdot \mathcal{P} \perp \tilde{\mathbf{u}}_\varepsilon + \frac{|v|^2}{D+2} (\tilde{\rho}_\varepsilon + \tilde{\theta}_\varepsilon) \rightarrow 0 \quad \text{in } L^2_{\text{loc}}(dt; L^2(M \, dv \, dx)).$$

Thus $\mathcal{P} \tilde{g}_\varepsilon \rightarrow g = v \cdot u + \left( \frac{1}{2} |v|^2 - \frac{D+2}{2} \right) \theta$ in $L^2_{\text{loc}}(dt; L^2(M \, dv \, dx))$ as $\varepsilon \rightarrow 0$. The key nonlinear estimate in [3] claims that

$$\sigma \frac{g_3^3}{\sqrt{N_\varepsilon}} = O(|\log \varepsilon|) \quad \text{in } L^\infty(dt; L^1(\sigma M \, dv \, dx)).$$

It is easy to see that $\frac{\varepsilon g_3^3}{\sqrt{N_\varepsilon}}$ is bounded, hence

$$\frac{\varepsilon g_3^3}{\sqrt{N_\varepsilon}} \rightarrow 0 \quad \text{in } L^1_{\text{loc}}(dt; L^1(\sigma M \, dv \, dx)).$$

We deduce that $g_\varepsilon$ is relatively compact in $L^1_{\text{loc}}(dt; L^1(\sigma M \, dv \, dx))$ and that every limit $g$ has the form (3.4), combining the above estimates.

Next, we can also improve the convergence of the moments of $g_\varepsilon$. In [28], it was proved that the incompressible part $(\mathcal{P} (vg_\varepsilon), (\frac{1}{D+2} |v|^2 - 1) g_\varepsilon)$ converges
to \((u, \theta)\) in \(C([0, \infty); w\cdot L^1(dx))\). We also have that \(|P(v g_\varepsilon)|(|v|/2 - 1)g_\varepsilon\rangle\) converges to \((u, \theta)\) in \(L^2_{\text{loc}}(dt; L^2(dx))\). Now, from Proposition 11.1, we know that the acoustic part \(Q\langle v \bar{g}_\varepsilon\rangle\) and \(|1/2|v|^2 \bar{g}_\varepsilon\rangle\) converge strongly in \(L^2_{\text{loc}}(dt; L^2(dx))\) to 0. So combining this with (11.2), we get

\[
\langle v g_\varepsilon\rangle \to u \quad \text{in } L^1_{\text{loc}}(dt; L^1(dx; \mathbb{R}^D)) \\
\left(\frac{1}{D} |v|^2 - 1\right)g_\varepsilon \to \theta \quad \text{in } L^1_{\text{loc}}(dt; L^1(dx; \mathbb{R})) \cap C([0, \infty); w\cdot L^1(dx; \mathbb{R})).
\]

Furthermore, since now we have \(\bar{u}_e \to u\) and \(\bar{\theta} \to \theta\) in \(L^2_{\text{loc}}(dt; L^2(dx))\), we can improve the quadratic limit theorem 13.1 in [28] to

(11.3) \quad \bar{u}_e \otimes \bar{u}_e \to u \otimes u, \quad \bar{\theta} \bar{e}_e \to u \theta, \quad \bar{\theta}_e \to \theta^2 \quad \text{in } L^1_{\text{loc}}(dt; L^1(dx))

as \(e \to 0\).

Let \(s \in (0, \infty]\) be from the assumed bound (2.7) on \(b\). Let \(p = 2 + \frac{1}{s-1}\), so that \(p = 2\) when \(s = \infty\). Let \(\hat{\xi} \in L^p(aM\, dv)\) be such that \(\mathcal{P}\hat{\xi} = 0\) and set \(\xi = \mathcal{L}\hat{\xi}\), hence,

\[
\frac{1}{\varepsilon} \langle \xi \bar{g}_\varepsilon \rangle = \frac{1}{\varepsilon} \langle \xi \mathcal{P} \bar{g}_\varepsilon \rangle = \langle \hat{\xi} Q(\bar{g}_\varepsilon, \bar{g}_\varepsilon) \rangle - \langle \hat{\xi} \bar{q}_\varepsilon \rangle + \langle \hat{\xi} T_\varepsilon \rangle.
\]

We know from [28] that

(11.4) \quad \langle \hat{\xi} T_\varepsilon \rangle \to 0 \quad \text{in } L^1_{\text{loc}}(dt; L^1(dx))

and

(11.5) \quad \langle \hat{\xi} \bar{q}_\varepsilon \rangle \to \langle \hat{\xi} \hat{A} \rangle : \nabla_x u + \langle \hat{\xi} \hat{B} \rangle \cdot \nabla_x \theta \quad \text{in } w-L^2_{\text{loc}}(dt; w-L^2(dx)).

Note that

\[
\langle \hat{\xi} Q(\bar{g}_\varepsilon, \bar{g}_\varepsilon) \rangle = \langle \hat{\xi} Q(\mathcal{P} \bar{g}_\varepsilon, \mathcal{P} \bar{g}_\varepsilon) \rangle + 2 \langle \hat{\xi} Q(\mathcal{P} \bar{g}_\varepsilon, \mathcal{P} \bar{g}_\varepsilon) \rangle + \langle \hat{\xi} Q(\mathcal{P} \bar{g}_\varepsilon, \mathcal{P} \bar{g}_\varepsilon) \rangle.
\]

It is easy to show that the last two terms above vanish as \(e \to 0\). For the first term,

(11.6) \quad \langle \hat{\xi} Q(\mathcal{P} \bar{g}_\varepsilon, \mathcal{P} \bar{g}_\varepsilon) \rangle = \frac{1}{2} \langle \hat{\xi} \mathcal{P} \bar{g}_\varepsilon \rangle^2

\[
= \frac{1}{2} \langle \xi A \rangle : (\bar{u}_e \otimes \bar{u}_e) + \langle \xi \hat{B} \rangle \cdot \bar{e}_e \bar{\theta}_e + \frac{1}{2} \langle \xi C \rangle \bar{\theta}_e^2.
\]

Applying the quadratic limit (11.3), (11.6) converges strongly in \(L^1_{\text{loc}}(dt; L^1(dx))\). Combining the convergence in (11.6) with convergence (11.4) and (11.5), we get

\[
\frac{1}{\varepsilon} \langle \xi \mathcal{P} \bar{g}_\varepsilon \rangle \to \left(\xi \left(\frac{1}{2} A : u \otimes u + B \cdot u \theta + \frac{1}{2} C \theta^2 - \hat{A} : \nabla_x u - \hat{B} \cdot \nabla_x \theta\right)\right)
\]

in \(w-L^1_{\text{loc}}(dt; w-L^1(dx))\). Since \(g_\varepsilon - \bar{g}_\varepsilon \to 0\) in \(L^\infty(dt; L^1(aM\, dv\, dx))\), the convergence above implies (3.8). Thus we finish the proof of the main theorem, Theorem 3.1.
11.2 Proof of Proposition 11.1

We will reduce the proof of Proposition 11.1 to show that the projection of $\bar{U}_\varepsilon$ on each fixed acoustic mode goes to 0 in $L^2_{L_\text{loc}}(dt; L^2(dx))$. We know that $\Pi^\perp \bar{U}_\varepsilon$ is uniformly bounded in $L^\infty (dt; L^2(dx))$, so it can be represented as

$$
\Pi^\perp \bar{U}_\varepsilon = \sum_{k \in \mathbb{N}} \langle \bar{U}_\varepsilon, U^{+,k} \rangle \Pi U^{+,k} + \langle \bar{U}_\varepsilon, U^{-,k} \rangle \Pi U^{-,k} 
$$

(11.7)

$$
= \frac{D+2}{2D} \sum_{k \in \mathbb{N}} \left( \frac{2D}{(D+2)^2} \int_\Omega (|v|^2 g_\varepsilon) \psi^k dx \psi^k 
+ \frac{2}{(D+2)^2} \int_\Omega (v g_\varepsilon) \cdot \nabla_x \psi^k dx \nabla_x \psi^k 
+ \frac{4}{(D+2)^2} \int_\Omega (|v|^2 g_\varepsilon) \psi^k \psi^k dx \right).
$$

Recall that the inner product $\langle \cdot, \cdot \rangle_\Pi$ is defined in (5.2), and $U^{+,k}$ and $U^{-,k}$ are defined in (5.7).

The above summation includes infinitely many terms. To reduce the problem to a finite number of modes, we need some regularity in $x$ of $(v g_\varepsilon)$ and $(|v|^2 g_\varepsilon)$. The tool adapted to investigate this property is the velocity-averaging theorem given in [16] and the improvement to $L^1$ averaging in [19].

Following a similar argument in the proof of proposition 11.2 in [28] and applying it to (10.2), we can show that for each $\zeta \in \text{span}\{1, v, |v|^2\}$ and $T > 0$, there exists a function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\varepsilon \to 0^+} \eta(\varepsilon) = 0$ and

$$
\| \langle \zeta g_\varepsilon(t, x + y, v) - \zeta g_\varepsilon(t, x, v) \|_{L^2([0,T] \times \Omega)} \leq \eta(|y|),
$$

(11.8)

for every $y \in \Omega$ such that $|y| \leq 1$, uniformly in $\varepsilon \in [0,1]$. From the classical criterion of compactness in $L^2$, $(v g_\varepsilon)$ and $|v|^2 g_\varepsilon$ are relatively compact in $L^2_{L_\text{loc}}(dt, L^2(dx))$, which implies that

$$
\sum_{k > N} \int_0^T \| \langle \bar{U}_\varepsilon, U^{+,k} \rangle \Pi \|^2 \, dt \leq C_N \| \Pi^\perp \bar{U}_\varepsilon \|_{L^2([t_1,t_2]; L^2(dx))} \to 0 \quad \text{as } N \to \infty;
$$

(11.9)

recall from (11.8) that $C_N \to 0$ as $N \to \infty$. (11.9) implies that, to show $\Pi^\perp \bar{U}_\varepsilon \to 0$ strongly in $L^2_{L_\text{loc}}(dt, L^2(dx))$, we only need to prove that $\langle \bar{U}_\varepsilon, U^{+,k} \rangle$ converges strongly to 0 in $L^2(0, T)$ for any fixed acoustic mode $k$. Furthermore, the relation

$$
\langle \bar{U}_\varepsilon, U^{+,k} \rangle = \int_\Omega (g_\varepsilon \cdot g_\varepsilon^{+,k}) dx
$$

implies that the proof of Proposition 11.1 is reduced to showing the following:

PROPOSITION 11.2. Assume that $\alpha_\varepsilon = O(\sqrt{\varepsilon})$ and let $g_\varepsilon$ be the renormalized fluctuation defined in (10.3), which satisfies the scaled Boltzmann equation (10.2),
and $g_0^{\tau,k,\text{int}}$ (τ is + or −) be the infinitesimal Maxwellian of acoustic mode $k \geq 1$:

$$g_0^{\tau,k,\text{int}} = \frac{D}{D + 2} \Psi^k + \frac{\nabla_x \Psi^k}{\tau i \lambda^k} \cdot v + \frac{2}{D + 2} \Psi^k \left( \frac{|v|^2}{2} - \frac{D}{2} \right).$$

Then, for any fixed mode $k$,

$$\int_{\Omega} \langle \bar{g}_\varepsilon, g_0^{\tau,k,\text{int}} \rangle \, dx \to 0 \quad \text{in } L^2(0,T) \text{ as } \varepsilon \to 0.$$

**Proof.** We start from the weak formulation of the rescaled Boltzmann equation (2.25) with the renormalization $\Gamma$ defined in (10.1) and the test function $Y$ taken to be the approximate eigenfunctions of $L_\varepsilon$ constructed in Proposition 7.1 to the order $N = 4$, namely $Y = g_{\tau,k}^{\varepsilon,4}$:

$$\int_{\Omega} \langle \bar{g}_\varepsilon(t_2) g_{\tau,k}^{\varepsilon,4} \rangle \, dx - \int_{\Omega} \langle \bar{g}_\varepsilon(t_1) g_{\tau,k}^{\varepsilon,4} \rangle \, dx$$

$$+ \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle \bar{g}_\varepsilon \mathcal{L}_\varepsilon g_{\tau,k}^{\varepsilon,4} \rangle \, dx \, dt + \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\partial \Omega} \langle \gamma \bar{g}_\varepsilon \gamma g_{\tau,k}^{\varepsilon,4} (v \cdot n) \rangle \, d\sigma_x \, dt$$

$$= \frac{1}{\varepsilon} \int_{t_1}^{t_2} \| R_{\varepsilon} g_{\tau,k}^{\varepsilon,4} \| \, dx \, dt,$$

where

$$R_\varepsilon = \Gamma'(G_\varepsilon) q_\varepsilon + \frac{1}{\varepsilon} \left( \frac{g_{\varepsilon 1}}{N_{\varepsilon 1}} + \frac{g_{\varepsilon}}{N_{\varepsilon}} - \frac{g'_{\varepsilon 1}}{N'_{\varepsilon 1}} - \frac{g'_\varepsilon}{N'_{\varepsilon}} \right).$$

Define

$$\tilde{b}_{\varepsilon}^{\tau,k}(t) = \int_{\Omega} \langle \bar{g}_\varepsilon(t) g_{\tau,k}^{\varepsilon,4} \rangle \, dx.$$

Then from (11.10) $\tilde{b}_{\varepsilon}^{\tau,k}(t)$ satisfies

$$\tilde{b}_{\varepsilon}^{\tau,k}(t_2) - \tilde{b}_{\varepsilon}^{\tau,k}(t_1) - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \tilde{b}_{\varepsilon}^{\tau,k}(t) \, dt = \int_{t_1}^{t_2} c_{\varepsilon}^{\tau,k}(t) \, dt,$$

where $c_{\varepsilon}^{\tau,k}(t)$ is

$$c_{\varepsilon}^{\tau,k}(t) = -\frac{1}{\varepsilon} \int_{\Omega} \langle \bar{g}_\varepsilon(t) R_{\varepsilon}^{\tau,k} \rangle \, dx - \frac{1}{\varepsilon} \int_{\partial \Omega} \langle \gamma \bar{g}_\varepsilon \gamma R_{\varepsilon}^{\tau,k} (v \cdot n) \rangle \, d\sigma_x$$

$$+ \frac{1}{\varepsilon} \int_{\Omega} \| R_{\varepsilon} g_{\tau,k}^{\varepsilon,4} \| \, dx.$$

We claim that the boundary contribution in (11.12) is 0 as $\varepsilon \to 0$:
Lemma 11.3. Let \( g_{\varepsilon,4}^{\tau,k} \) be the approximate eigenfunction of \( \mathcal{L}_\varepsilon \) constructed in Proposition 7.1. Then,

\[
(11.13) \quad \frac{1}{\varepsilon} \int_\Omega (\gamma g_{\varepsilon} \gamma g_{\varepsilon,4}^{\tau,k} (v \cdot n)) \, d\sigma = \Gamma_1^{\tau,k} + \Gamma_2^{\tau,k},
\]

where \( \Gamma_1^{\tau,k} \) is bounded in \( L^p_\text{loc} (dt) \) for \( p > 1 \), and \( \Gamma_2^{\tau,k} \) vanishes in \( L^1_\text{loc} (dt) \) as \( \varepsilon \to 0 \).

We leave the proof of Lemma 11.3 to Section 11.5.

11.3 Estimates of \( c_{\varepsilon}^{\tau,k} \)

We will decompose \( c_{\varepsilon}^{\tau,k} (t) \) into two parts: one is vanishing in \( L^1_\text{loc} (dt) \), the other is bounded in \( L^p_\text{loc} (dt) \) for some \( p > 1 \). First, taking \( N = 4, r = p = 2 \) in (7.10) and noticing the \( L^\infty_\text{loc} (dt, L^2 (aM \, dv \, dx)) \) boundedness of \( \tilde{g}_\varepsilon \), we have the estimate of the first term in (11.12):

\[
\left| \frac{1}{\varepsilon} \int_\Omega (\tilde{g}_\varepsilon (t) R_{\varepsilon,4}^{\tau,k}) \, dx \right| \leq \frac{1}{\varepsilon} \left\| R_{\varepsilon,4}^{\tau,k} \right\|_{L^2 (\frac{1}{aM} \, dv \, dx)} \| \tilde{g}_\varepsilon \|_{L^2 (aM \, dv \, dx)} 
\leq C \sqrt{\varepsilon}.
\]

Second, Lemma 11.3 implies that one part of the boundary term in (11.12), namely \( \Gamma_2^{\tau,k} \) in (11.13), will be vanishing in \( L^1_\text{loc} (dt) \) as \( \varepsilon \) goes to 0. The third term in (11.12) is estimated as follows:

\[
(11.14) \quad \frac{1}{\varepsilon} \int_\Omega \left\{ R_{\varepsilon} g_{\varepsilon,4}^{\tau,k} \right\} \, dx = \frac{1}{\varepsilon} \int_\Omega \left\{ R_{\varepsilon} \mathcal{P} g_{\varepsilon,4}^{\tau,k} \right\} \, dx + \frac{1}{\varepsilon} \int_\Omega \left\{ R_{\varepsilon} \mathcal{P} g_{\varepsilon,4}^{\tau,k} \right\} \, dx.
\]

For the first term in the right-hand side of (11.14), because \( \mathcal{P} g_{\varepsilon,4}^{\tau,k} \) is in \( \text{Null}(\mathcal{L}) \), it has the form of

\[
(11.15) \quad \frac{1}{\varepsilon} \int_\Omega \left\{ \Gamma' (G_\varepsilon) q_\varepsilon \xi \right\} \, dx + \frac{1}{\varepsilon} \int_\Omega \left\{ \mathcal{L} \tilde{g}_\varepsilon \xi \right\} \, dx \quad \text{for some} \; \xi (t, x) \in \text{Null}(\mathcal{L}).
\]

The second term above is 0, and the first term converges to 0 strongly in \( L^1_\text{loc} (dt) \) as \( \varepsilon \to 0 \) by the conservation defect theorem (Proposition 8.1) in [28].

For the second term in the right-hand side of (11.14), from the calculations in Proposition 7.1 again, we have

\[
(11.16) \quad \frac{1}{\varepsilon} \mathcal{P} \perp g_{\varepsilon,4}^{\tau,k} = \sqrt{\frac{D + 2}{2D}} \left( \frac{\nabla_\varepsilon^2 \psi_k}{\tau i \lambda_k} : \hat{A} + \frac{2 \nabla_x \psi_k}{D + 2} \cdot \hat{B} \right)
\]

\[
+ \frac{1}{\sqrt{\varepsilon}} (\nabla_x d \otimes \partial_\xi u_0 + \nabla_x \partial_\xi \xi_0 \psi_k : \hat{A} + \nabla_x \partial_\xi \psi_k : \hat{B})
\]

\[
+ \frac{1}{\sqrt{\varepsilon}} g_1^{\tau,k,b} + \text{higher-order terms}.
\]
We decompose $R_\varepsilon$ into

\[(11.17) \quad R_\varepsilon = T_\varepsilon + \left( \tilde{g}_e' \tilde{g}_e' - \tilde{g}_e \tilde{g}_e \right) + q_\varepsilon \left( \frac{2}{N_\varepsilon^2} - \frac{1}{N_\varepsilon} - \frac{1}{N_{e1}'N_{e1}'N_{e1}N_{e1}N_{e1}} \right), \]

where $T_\varepsilon$ is

\[ T_\varepsilon = \frac{q_\varepsilon}{N_{e1}'N_{e1}'N_{e1}N_{e1}} - \frac{1}{\varepsilon} \left( \tilde{g}_e' + \tilde{g}_e' - \tilde{g}_e - \tilde{g}_e \right) - \left( \tilde{g}_e' \tilde{g}_e' - \tilde{g}_e \tilde{g}_e \right). \]

When we integrate (11.16) over $\Omega$, for the second term of (11.16) that is a function of $\pi(x)$, we make the following change of variables:

\[(11.18) \quad y_1 = \pi^1(x), \ldots, y_{D-1} = \pi^{D-1}(x), \quad y_D = \frac{d(x)}{\sqrt{\varepsilon}}. \]

Then $d\varepsilon = \sqrt{\varepsilon}T^* dy$, where

\[ T^* = \text{det}^{-1} \left( \nabla_x \pi \right) > 0. \]

This extra $\sqrt{\varepsilon}$ cancels with the $\sqrt{\varepsilon}^{-1}$ in the second term of (11.16). Similarly, for the third term of (11.16), we make the change of variables $y_1 = \pi(x)^1, \ldots, y_{D-1} = \pi(x)^{D-1}$, and consequently $d\varepsilon = \varepsilon T^* dy$. Thus, for the integral of (11.16), the first two terms are the same order, namely $O(1)$, while the third term is of order $O(\sqrt{\varepsilon})$, and the rest of the terms are of even higher order in $\varepsilon$. By the flux remainder theorem (proposition 10.1) of [28], we have

\[ \frac{1}{\varepsilon} \int_{\Omega} \left\| T_\varepsilon P_{g_{e,4}}^\perp \right\| dx \rightarrow 0 \quad \text{in} \quad L^1_{\text{loc}}(dt) \]

as $\varepsilon \rightarrow 0$. Furthermore, from the bilinear estimates (lemma 9.1) of [28], we have

\[ \frac{1}{\varepsilon} \int_{\Omega} \left\| \left( \tilde{g}_e' \tilde{g}_e' - \tilde{g}_e \tilde{g}_e \right) \right\| dx \leq C \int_{\Omega} \left\| a \tilde{g}_e^2 \right\| dx \leq C. \]

The third term in (11.17) can be written as $q_\varepsilon/\sqrt{N_\varepsilon}$ times a bounded sequence that vanishes almost everywhere as $\varepsilon \rightarrow 0$. Then the $L^2(\varepsilon \varepsilon dx dt)$ boundedness of $q_\varepsilon/\sqrt{N_\varepsilon}$ implies that it times (11.16) is relatively compact in $w-L^1_{\text{loc}}(dt; w-L^1(dx))$. Then following from the product limit theorem of [3, app. B],

\[ \frac{1}{\varepsilon} \int_{\Omega} \left\| q_\varepsilon \left( \frac{2}{N_\varepsilon^2} - \frac{1}{N_\varepsilon} - \frac{1}{N_{e1}'N_{e1}'N_{e1}N_{e1}N_{e1}} \right) P_{g_{e,4}}^\perp \right\| dx \rightarrow 0 \quad \text{in} \quad L^1_{\text{loc}}(dt) \text{ as } \varepsilon \rightarrow 0. \]

Now we can decompose $c_\varepsilon^{\tau,k}(t)$ into

\[ c_{1,\varepsilon}^{\tau,k}(t) = -\frac{1}{\varepsilon} \int_{\Omega} \left\{ \left( \tilde{g}_e' \tilde{g}_e' - \tilde{g}_e \tilde{g}_e \right) \right\} \left( g_{e,4}^{\tau,k} \right) dx - \Gamma_{1,\varepsilon}^{\tau,k} \]

and $c_{2,\varepsilon}^{\tau,k}(t) = c_{\varepsilon}^{\tau,k}(t) - c_{1,\varepsilon}^{\tau,k}(t)$, where $\Gamma_{1,\varepsilon}^{\tau,k}$ appears in (11.13).
The above arguments show that $c_{1,\varepsilon}^k(t) \to 0$ in $L^1_{loc}(dt)$, and Lemma 11.3 gives that $c_{1,\varepsilon}^k(t)$ is bounded in $L^p_{loc}(dt)$ for some $p > 1$.

11.4 Estimates of $\bar{b}_{\varepsilon}^k$

From (11.11), $\bar{b}_{\varepsilon}^k$ satisfies the ordinary differential equation

$$\frac{d}{dt} \bar{b}_{\varepsilon}^k - \frac{1}{\varepsilon} i \lambda_{\varepsilon,4} \bar{b}_{\varepsilon}^k = c_{1,\varepsilon}^k(t) + c_{2,\varepsilon}^k(t).$$

The solution to (11.19) is given by

$$\bar{b}_{\varepsilon}^k(t) = \bar{b}_{\varepsilon}^k(0) e^{-\frac{i}{\varepsilon} \lambda_{\varepsilon,4} t} + \int_0^t \left[ c_{1,\varepsilon}^k(s) + c_{2,\varepsilon}^k(s) \right] e^{-\frac{i}{\varepsilon} \lambda_{\varepsilon,4} (s-t)} ds.$$

From Proposition 7.1, $i \lambda_{\varepsilon,4} = \tau i \lambda^k + i \lambda_{\varepsilon,1}^k \sqrt{\varepsilon} + i \lambda_{\varepsilon,2}^k \varepsilon$, where $\lambda_{\varepsilon,1}^k = O(1)$.

$$\frac{1}{\varepsilon} i \lambda_{\varepsilon,4} \bar{b}_{\varepsilon}^k(t) = \frac{1}{\varepsilon} \left[ \text{Re}(i \lambda_{\varepsilon,4}^k) + \sqrt{\varepsilon} \text{Re}(i \lambda_{\varepsilon,2}^k) \right] t$$
$$- i \left[ \frac{\tau}{\varepsilon} \lambda^k + \frac{1}{\varepsilon} \text{Im}(i \lambda_{\varepsilon,2}^k) \right] t.$$

Using (11.21), the first term in (11.20) is estimated as follows:

$$\| \bar{b}_{\varepsilon}^k(0) e^{-\frac{i}{\varepsilon} \lambda_{\varepsilon,4} t} \|_{L^2(0,T)} = \| \bar{b}_{\varepsilon}^k(0) \|_{L^2} \left[ -2 \text{Re}(i \lambda_{\varepsilon,4}^k) + \sqrt{\varepsilon} \text{Re}(i \lambda_{\varepsilon,2}^k) \right]^{-\frac{1}{2}}$$
$$\cdot \left( 1 - e^{-\frac{1}{\varepsilon} \sqrt{\varepsilon} \text{Re}(i \lambda_{\varepsilon,2}^k) T} \right)^{1/2} \frac{1}{\varepsilon^{\frac{1}{4}}}. $$

To estimate $|\bar{b}_{\varepsilon}^k(0)|$ from

$$\bar{b}_{\varepsilon}^k(0) = \int_{\Omega} \left( \zeta_{s,1}^k \cdot \zeta_{s,2}^k \right) dx + \int_{\Omega} \left( \zeta_{s,1}^k \cdot \zeta_{s,2}^k \right) dx,$$

noticing that $g_{s,1}^k, \zeta_{s,2}^k \in \text{Null}(\zeta)$, $\| \zeta(v) \zeta_{s,2}^k \|_{L^2(dx)}$ is bounded for every $\zeta(v) \in \text{Null}(\zeta)$, and the error estimate for $g_{s,1}^k, \zeta_{s,2}^k$ in (7.11), we deduce that $|\bar{b}_{\varepsilon}^k(0)|$ is bounded. Using the key fact that $\text{Re}(i \lambda_{\varepsilon,2}^k) < 0$, we deduce that for any $0 < T < \infty$ and sufficiently small $\varepsilon$,

$$\| \bar{b}_{\varepsilon}^k(0) e^{-\frac{i}{\varepsilon} \lambda_{\varepsilon,4} t} \|_{L^2(0,T)} \leq C \varepsilon^{\frac{1}{4}}.$$

In order to estimate the remaining term in (11.20), we observe that for any $a \in L^p(0,t)$ and $1 \leq p, r \leq \infty$ such that $p^{-1} + r^{-1} = 1$, we have

$$\left| \int_0^t a(s) e^{-\frac{i}{\varepsilon} \lambda_{\varepsilon,4} (s-t)} ds \right| \leq C \int_0^t e^{-\frac{1}{\varepsilon} \sqrt{\varepsilon} \text{Re}(i \lambda_{\varepsilon,2}^k) (s-t)} |a(s)| ds.$$
Direct calculations show that
\[
\| e^{\frac{1}{\sqrt{t}}} \text{Re}(i^k \lambda_1^{\tau_2})(t-x) \|_{L^r(0,t)} =
\]
\[
\varepsilon^{\frac{1}{2r}} \left[ \frac{1}{-r \text{Re}(i^k \lambda_1^{\tau_2})} (e^{\frac{1}{\sqrt{t}}} \text{Re}(i^k \lambda_1^{\tau_2}) - 1) \right]^{\frac{1}{2}} e^{\frac{1}{\sqrt{t}}} \text{Re}(i^k \lambda_1^{\tau_2}) t.
\]
Using the fact \( \text{Re}(i^k \lambda_1^{\tau_2}) < 0 \) again, we have
\[
(11.22) \quad \left| \int_0^t a(s) e^{\frac{1}{\sqrt{t}} i^k \lambda_2^{\tau_2}(s-t)} \, ds \right| \leq C \| a \|_{L^r(0,t)} \varepsilon^{\frac{1}{2r}}.
\]
Now applying \( a(t) \) in (11.22) to \( c_{1,\epsilon}^{\tau_2,k} \) and \( c_{2,\epsilon}^{\tau_2,k} \), we finally get
\[
\tilde{b}_{\epsilon}^{\tau_2,k} \to 0 \quad \text{strongly in } L^2_{\text{loc}}(dt).
\]
To finish the proof of the proposition, we notice that
\[
\int_{\Omega} (\tilde{g}_{\epsilon} \cdot \gamma_{\tau_2,k,\text{int}}) \, dx = \tilde{b}_{\epsilon}^{\tau_2,k} + \int_{\Omega} (\tilde{g}_{\epsilon} \cdot \gamma_{\tau_2,k,\text{int}} - g_{\epsilon,4}^{\tau_2,k}) \, dx.
\]
Applying the error estimate (7.11) in Proposition 7.1, we finish the proof of Proposition 11.2.

Consequently, we prove the Proposition 11.1.

11.5 Proof of Lemma 11.3

Using the boundary condition of \( g_{\epsilon,4}^{\tau_2,k} \), namely (7.3), simple calculations yield that
\[
(11.23) \quad \frac{1}{\varepsilon} \int_{\partial \Omega} \gamma \tilde{g}_{\epsilon} \gamma g_{\epsilon,4}^{\tau_2,k} (v \cdot n) \, d\sigma_x
\]
\[
= \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma g_{\epsilon} \gamma g_{\epsilon,4}^{\tau_2,k} (v \cdot n) M \, dv \, d\sigma_x
\]
\[
+ \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma g_{\epsilon} \left[ (1 - \alpha_\epsilon) L \gamma_{\epsilon,4}^{\tau_2,k} + \alpha_\epsilon \gamma_{\epsilon,4}^{\tau_2,k} \right] \cdot (v \cdot n) M \, dv \, d\sigma_x
\]
\[
= \frac{1}{\varepsilon} \iint_{\Sigma_-} \gamma g_{\epsilon,4}^{\tau_2,k} \gamma g_{\epsilon} - (1 - \alpha_\epsilon) L \gamma g_{\epsilon,4}^{\tau_2,k} - \alpha_\epsilon \gamma g_{\epsilon,4}^{\tau_2,k} \, d\sigma_x
\]
\[
+ \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma g_{\epsilon} \gamma r_{\epsilon,4}^{\tau_2,k} (v \cdot n) M \, dv \, d\sigma_x =
\]
\[
\begin{align*}
\frac{1}{\varepsilon} \int_{\Sigma^+} (\gamma^+ \tilde{g}_- - L \gamma^- \tilde{g}_-) \, d\tilde{v}_\varepsilon - \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma^+ \tilde{g}_e^2} \, d\tilde{v}_\varepsilon \\
+ \frac{1}{\varepsilon} \int_{\Sigma^+} \gamma^+ \tilde{g}_e \beta_{e,4}^k (v \cdot n) M \, dv \, d\sigma_x,
\end{align*}
\]

where the measure \(d\tilde{v}_\varepsilon = L \gamma^- \tilde{g}_e \beta_{e,4}^k (v \cdot n) M \, dv \, d\sigma_x\). From the boundary error estimate (7.12) in Proposition 7.1 (letting \(r = \infty, \ p = 2\)) and the fact that \(\gamma^+ \tilde{g}_e\) is bounded in \(L^1(d\sigma_x, L^2(aM \, dv))\), it is easy to see that

\[
\frac{1}{\varepsilon} \int_{\Sigma^+} \gamma^+ \tilde{g}_e \beta_{e,4}^k (v \cdot n) M \, dv \, d\sigma_x \to 0 \quad \text{in } L^1_{\text{loc}}(dt) \text{ as } \varepsilon \to 0.
\]

It remains to show that the first two terms in the right-hand side of (11.23) go to 0 as \(\varepsilon \to 0\). It again follows from the a priori estimates Lemma 10.1 and Lemma 10.2. The main difficulty is \(\alpha_\varepsilon / \varepsilon \to \infty\) as \(\varepsilon \to 0\), since \(\alpha_\varepsilon = \sqrt{2 \pi \chi / \varepsilon}\).

\[
(11.24)
\]

The renormalized boundary condition (10.23) yields that

\[
(11.25)
\]

\[
\begin{align*}
\frac{1}{\varepsilon} (\gamma^+ \tilde{g}_e - L \gamma^- \tilde{g}_-) - \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma^+ \tilde{g}_e^2} = \\
- \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma_\varepsilon^2 \gamma^+ \tilde{g}_e + \gamma^+ \tilde{g}_e}{1 + \varepsilon^2 \gamma^+ \tilde{g}_e^2} + \frac{\alpha_\varepsilon}{\varepsilon} \left( \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma^+ \tilde{g}_e^2} - \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma^+ \tilde{g}_e^2} \right).
\end{align*}
\]
Thus, after simple calculations, we have

\[
\frac{1}{\varepsilon} \int \frac{1}{\partial \Omega} \left( g_\varepsilon e \gamma g_\varepsilon,4 (v \cdot n) \right) d\sigma_x
= -\frac{\alpha_\varepsilon}{\varepsilon} \int \frac{1}{\Sigma_+} \left( \gamma e^2 \gamma \gamma + \gamma \gamma + \gamma \gamma \right) \left( 1 + \varepsilon^2 \gamma + \gamma \right) \left( 1 + \varepsilon^2 \gamma + \gamma \gamma \right) \\
\cdot L \gamma - g_{k,\varepsilon,2} (v \cdot n) M \, d\sigma_x
\]

(11.26)

\[
\frac{\alpha_\varepsilon}{\varepsilon} \int \frac{1}{\Sigma_+} \left( \gamma e^2 \gamma \gamma + \gamma \gamma + \gamma \gamma \right) \left( 1 + \varepsilon^2 \gamma + \gamma \gamma \right) \left( 1 + \varepsilon^2 \gamma + \gamma \gamma \right) \\
\cdot L \gamma - g_{k,\varepsilon,2} (v \cdot n) M \, d\sigma_x
\]

The a priori estimates from the boundary yields that all three terms on the right-hand side of (11.26) are bounded in \( L^p_{\text{loc}}(\Omega) \) for \( p > 1 \). Indeed, the integral of the first term over \([t_1, t_2]\) is bounded by

\[
\int_{t_1}^{t_2} \int \frac{\sqrt{\alpha_\varepsilon (\gamma + \gamma \gamma)}}{1 + \varepsilon^2 \gamma + \gamma \gamma} \left( \frac{\gamma (\gamma + \gamma \gamma)}{1 + \varepsilon^2 \gamma + \gamma \gamma} \right) \\
\cdot L \gamma - g_{k,\varepsilon,2} (v \cdot n) M \, d\sigma_x
\]

(11.27)

\[
-\varepsilon \int_{t_1}^{t_2} \int \frac{\alpha_\varepsilon (\gamma + \gamma \gamma)}{\varepsilon^2 \gamma + \gamma \gamma} \left( \frac{\gamma (\gamma + \gamma \gamma)}{1 + \varepsilon^2 \gamma + \gamma \gamma} \right) \\
\cdot L \gamma - g_{k,\varepsilon,2} (v \cdot n) M \, d\sigma_x
\]

Note that

\[
\frac{\sqrt{\alpha_\varepsilon (\gamma + \gamma \gamma)}}{1 + \varepsilon^2 \gamma + \gamma \gamma} \quad \text{and} \quad \frac{(\sqrt{\gamma + \gamma \gamma} + \sqrt{\gamma + \gamma \gamma})}{(1 + \varepsilon^2 \gamma + \gamma \gamma)^{3/4}} \quad \gamma_g \leq 2 \gamma + \gamma \gamma \leq 4 \gamma + \gamma \gamma
are bounded. Furthermore, \( L^{q}_{\varepsilon} g_{\varepsilon,2} (v \cdot n) \) is bounded in \( L^{q} ((v \cdot n) \mu \, dv \, dx) \) for \( q \geq 2 \). Now, estimates (10.16) and (10.18) of Lemma 10.2 imply that the first term in (11.27) is bounded in \( L^{p}_{\text{loc}} (dt) \) and the second term vanishes as \( \varepsilon \to 0 \). Similarly, using Lemma 10.1 and Lemma 10.2, we can prove that the integrals over \([t_1, t_2]\) of the second and third terms of (11.26) can be decomposed into two terms; one is bounded in \( L^{p}_{\text{loc}} (dt) \) and the other vanishes in \( L^{1}_{\text{loc}} (dt) \). Thus we prove Lemma 11.3.

\[ \square \]

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