On constrained optimization problems solved using CDT

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Abstract DY Gao together with some of his collaborators applied his Canonical duality theory (CDT) for solving a class of constrained optimization problems. Unfortunately, in several papers on this subject there are unclear statements, not convincing proofs, or even false results. It is our aim in this work to study rigorously these class of constrained optimization problems in finite dimensional spaces and to discuss several results published in the last ten years.

1 Introduction

In the preface of the book Canonical Duality Theory. Advances in Mechanics and Mathematics, vol 37, Springer, Cham (2017), edited by DY Gao, V Latorre and N Ruan, one says:

“Canonical duality theory is a breakthrough methodological theory that can be used not only for modeling complex systems within a unified framework, but also for solving a large class of challenging problems in multidisciplinary fields of engineering, mathematics, and sciences. ...

This theory is composed mainly of

(1) a canonical dual transformation, which can be used to formulate perfect dual problems without duality gap;

(2) a complementary-dual principle, which solved the open problem in finite elasticity and provides a unified analytical solution form for general nonconvex/nonsmooth/discrete problems;

(3) a triality theory, which can be used to identify both global and local optimality conditions and to develop powerful algorithms for solving challenging problems in complex systems.”

It is our aim in this work to present rigorously this “methodological theory” for constrained optimization problems in finite dimensional spaces. It is not the most general framework, but it covers all the situations met in the examples provided in DY Gao and his collaborators’ works on constrained optimization problems in finite dimensions. We also point out some drawbacks and not convincing arguments from some of those papers.

2 Preliminaries

We consider the following minimization problem with equality and inequality constraints

\[(P_J) \min \ f(x) \quad \text{s.t.} \ x \in X_J,\]
where $J \subset \overline{1,m}$,

$$X_J := \{ x \in \mathbb{R}^n \mid \forall j \in J : g_j(x) = 0 \} \land \{ \forall j \in J^c : g_j(x) \leq 0 \}$$

with $J^c := \overline{1,m} \setminus J$, and

$$f(x) := g_0(x) := g_0(x) + V_0(\Lambda_0(x)), \quad g_j(x) := g_j(x) + V_j(\Lambda_j(x)) \quad (x \in \mathbb{R}^n, \ j \in \overline{1,m}),$$

$q_k$ and $\Lambda_k$ being quadratic functions on $\mathbb{R}^n$, and $V_k \in \Gamma_{sc}(\mathbb{R})$ for $k \in \overline{0,m}$; note that

$$X_{J \cup K} = X_J \cap X_K \quad \forall J, K \subset \overline{1,m}. \quad (1)$$

Before giving the precise definition of $\Gamma_{sc}$ we recall some notions and results from convex analysis we shall use in the sequel.

Having $h : \mathbb{R}^p \to \overline{\mathbb{R}} := \mathbb{R} \cup \{ -\infty, +\infty \}$, its domain is $\text{dom} \ h := \{ y \in \mathbb{R}^p \mid h(y) < \infty \}$; $h$ is proper if $\text{dom} \ h \neq \emptyset$ and $h(y) \neq -\infty$ for $y \in \mathbb{R}^p$. The Fenchel conjugate $h^* : \mathbb{R}^p \to \overline{\mathbb{R}}$ of the proper function $h$ is defined by

$$h^*(\sigma) := \sup \{ \langle y, \sigma \rangle - h(y) \mid y \in \mathbb{R}^p \} = \sup \{ \langle y, \sigma \rangle - h(y) \mid y \in \text{dom} \ h \} \quad (\sigma \in \mathbb{R}^p),$$

while its subdifferential at $y \in \text{dom} \ h$ is

$$\partial h(y) := \{ \sigma \in \mathbb{R}^p \mid \langle y'-y, \sigma \rangle \leq h(y') - h(y) \ \forall y' \in \mathbb{R}^p \},$$

and $\partial h(y) := \emptyset$ if $y \notin \text{dom} \ h$; clearly,

$$h(y) + h^*(\sigma) \geq \langle y, \sigma \rangle \land [\sigma \in \partial h(y) \iff h(y) + h^*(\sigma) = \langle y, \sigma \rangle \ \forall (y, \sigma) \in \mathbb{R}^p \times \mathbb{R}^p]. \quad (2)$$

The class of proper convex lower semicontinuous (lsc for short) functions $h : \mathbb{R}^p \to \overline{\mathbb{R}}$ is denoted by $\Gamma(\mathbb{R}^p)$. It is well known that for $h \in \Gamma(\mathbb{R}^p)$ one has $h^* \in \Gamma(\mathbb{R}^p)$, $(h^*)^* = h$, and $\sigma \in \partial h(y)$ iff $y \in \partial h^*(\sigma)$; moreover, $\partial h(y) \neq \emptyset$ for every $y \in \text{ri} (\text{dom} \ h)$ and $h(\overline{0}) = \inf_{y \in \mathbb{R}^p} h(y)$ iff $0 \in \partial h(\overline{0})$.

We denote by $\Gamma_{sc}(\mathbb{R}^p)$ the class of those $h \in \Gamma(\mathbb{R}^p)$ which are essentially strictly convex and essentially smooth, that is the class of proper lsc convex functions of Legendre type (see [11] Sect. 26)). For $h \in \Gamma_{sc}(\mathbb{R}^p)$ we have: $h^* \in \Gamma_{sc}(\mathbb{R}^p)$, $\text{dom} \ h^* = \text{int} (\text{dom} \ h)$, and $h$ is differentiable on $\text{int} (\text{dom} \ h)$; moreover, $\nabla h : \text{int} (\text{dom} \ h) \to \text{int} (\text{dom} h^*)$ is bijective and continuous with $(\nabla h)^{-1} = \nabla h^*$. Having in view these properties and (2), for $h \in \Gamma_{sc}(\mathbb{R}^p)$ and $(y, \sigma) \in \mathbb{R}^p \times \mathbb{R}^p$ we have that

$$h(y) + h^*(\sigma) = \langle y, \sigma \rangle \iff [y \in \text{int} (\text{dom} \ h) \land \sigma = \nabla h(y)]$$

$$\iff [\sigma \in \text{int} (\text{dom} h^*) \land y = \nabla h^*(\sigma)]. \quad (3)$$

It follows that $\Gamma_{sc} := \Gamma_{sc}(\mathbb{R})$ is the class of those $h \in \Gamma(\mathbb{R})$ which are strictly convex and derivable on $\text{int} (\text{dom} \ h)$, assumed to be nonempty; hence $h' : \text{int} (\text{dom} \ h) \to \text{int}(\text{dom} h^*)$ is continuous, bijective and $(h')^{-1} = (h^*)'$ whenever $h \in \Gamma_{sc}$. The problem $(P_{\overline{1,m}})$ [resp. $(P_0)$], denoted by $(P_{\overline{e}})$ [resp. $(P_{i})$], is a minimization problem with equality [resp. inequality] constraints whose feasible set is $X_e := X_{\overline{1,m}}$ [resp. $X_i := X_0$]. From [11] we get $X_e \subset X_J \subset X_i$, each inclusion being generally strict for $J \notin \{\emptyset, \overline{1,m}\}$.

In many examples considered by DY Gao and his collaborators, some functions $g_k$ are quadratic, that is $g_k = q_k$; we set

$$Q := \{ k \in \overline{0,m} \mid g_k = q_k \}, \quad Q_0 := Q \setminus \{0\} = \overline{1,m} \cap Q.$$
For $k \in Q$ we take $\Lambda_k := 0$ and $V_k(t) := \frac{1}{2} t^2$ for $t \in \mathbb{R}$; then clearly $V_k^* = V_k \in \Gamma_{sc}$. To be more precise, we take

$$q_k(x) := \frac{1}{2} \langle x, A_k x \rangle - \langle b_k, x \rangle + c_k \quad \land \quad \Lambda_k(x) := \frac{1}{2} \langle x, C_k x \rangle - \langle d_k, x \rangle + e_k \quad (x \in \mathbb{R}^n)$$

with $A_k, C_k \in \mathcal{S}_n$, $b_k, d_k \in \mathbb{R}^n$ (seen as column matrices), and $c_k, e_k \in \mathbb{R}$ for $k \in [0, m]$, where $\mathcal{S}_n$ denotes the set of $n \times n$ real symmetric matrices; of course, $a_0$ can be taken to be 0. Clearly, $C_k = 0 \in \mathcal{S}_n$, $b_k = 0 \in \mathbb{R}^n$ and $e_k = 0 \in \mathbb{R}$ for $k \in Q$. We use also the notations

$$I_k := \text{dom } V_k, \quad I_k^* := \text{dom } V_k^* \quad (k \in [0, m]), \quad I^* := \prod_{k=0}^m I_k^*; \quad (4)$$

of course, $I_k = I_k^* = \mathbb{R}$ for $k \in Q$. In order to simplify the writing, in the sequel

$$\lambda_0 := \overline{\lambda}_0 := 1.$$

To the functions $f (= g_0)$ and $(g_j)_{j \in [1, m]}$ we associate several sets and functions. The Lagrangian $L$ is defined by

$$L : X \times \mathbb{R}^m \to \mathbb{R}, \quad L(x, \lambda) := f(x) + \sum_{j=1}^m \lambda_j g_j(x) = \sum_{k=0}^m \lambda_k \{ q_k(x) + V_k(\Lambda_k(x)) \},$$

where $\lambda := (\lambda_1, ..., \lambda_m)^T \in \mathbb{R}^m$, and

$$X := \{ x \in \mathbb{R}^n \mid \forall k \in [0, m]: \Lambda_k(x) \in \text{dom } V_k \} = \bigcap_{k=0}^m \Lambda_k^{-1}(\text{dom } V_k),$$

$$X_0 := \{ x \in \mathbb{R}^n \mid \forall k \in [0, m]: \Lambda_k(x) \in \text{int}(\text{dom } V_k) \} \subset \text{int } X;$$

clearly $X_0$ is open and $L$ is differentiable on $X_0$. Using Gao’s procedure, we consider the “extended Lagrangian” $\Xi$ associated to $f$ and $(g_j)_{j \in [1, m]}$:

$$\Xi : \mathbb{R}^n \times \mathbb{R}^{1+m} \times I^* \to \mathbb{R}, \quad \Xi(x, \lambda, \sigma) := \sum_{k=0}^m \lambda_k \{ q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k) \},$$

where $I^*$ is defined in (4) and $\sigma := (\sigma_0, \sigma_1, ..., \sigma_m) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^{1+m}$. Clearly, $\Xi(\cdot, \lambda, \sigma)$ is a quadratic function for every fixed $(\lambda, \sigma) \in \mathbb{R}^m \times I^*$.

In the sequel we shall use frequently the following sets associated to $\lambda \in \mathbb{R}^m$ and $J \subset [1, m]$: 

$$M_\neq(\lambda) := \{ j \in [1, m] \mid \lambda_j \neq 0 \}, \quad M_\neq^0(\lambda) := M_\neq(\lambda) \cup \{ 0 \},$$

$$\Gamma_J := \{ \lambda \in \mathbb{R}^m \mid \lambda_j \geq 0 \ \forall j \in J^c \} \subseteq \mathbb{R}^m_+,$$

respectively; clearly,

$$\Gamma_0 = \mathbb{R}_+^m, \quad \Gamma_{[1, m]} = \mathbb{R}^m, \quad \Gamma_{J \cap K} = \Gamma_J \cap \Gamma_K \quad \forall J, K \subset [1, m].$$

Taking into account the convexity of the functions $V_k$ we obtain useful relations between $L$ and $\Xi$ in the next result.

**Lemma 1** Let $x \in X$ and $J \subset [1, m]$. Then

$$L(x, \lambda) = \sup_{\sigma \in I_{J,Q}} \Xi(x, \lambda, \sigma) \quad \forall \lambda \in \Gamma_{J \cap Q}, \quad (5)$$

where

$$I_{J,Q} := \prod_{k=0}^m I_k^*$$

with $I_k^* := \{ \{ 0 \} \text{ if } k \in J \cap Q, \quad I_k^* \text{ if } k \in [0, m] \setminus (J \cap Q), \quad (6)$$

and

$$\sup_{(\lambda, \sigma) \in I_{J,Q} \times I_{J,Q}} \Xi(x, \lambda, \sigma) = \sup_{\lambda \in \Gamma_{J \cap Q}} L(x, \lambda) = \left\{ \begin{array}{ll} f(x) & \text{if } x \in X_{J \cap Q}, \\ \infty & \text{if } x \in X \setminus X_{J \cap Q}. \end{array} \right.$$
Proof. Let us set $K := J \cap Q = J \cap Q_0$. It is convenient to observe that $\Gamma_K = \prod_{j=1}^m \Gamma_j$, where $\Gamma_j := \mathbb{R}$ for $j \in K$ and $\Gamma_j := \mathbb{R}_+$ for $j \in K^c$; moreover, we set $\Gamma_0 := \mathbb{R}_+$. Take $x \in X$, $\lambda \in \Gamma_K$ and $k \in \mathbb{N}$. Using the fact that $V_k^{**} = V_k$, we have that
\[
g_k(x) = q_k(x) + V_k(\Lambda_k(x)) = q_k(x) + \sup_{\sigma_k \in \mathbb{I}_k^*} [\sigma_k \Lambda_k(x) - V_k^*(\sigma_k)]
\]
whence, because $g_k(x) \in \mathbb{R}$,
\[
\mu g_k(x) = \sup_{\sigma_k \in \mathbb{I}_k^*} \mu [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)] \quad \forall \mu \in \mathbb{R}_+, \forall k \in 0, m. \quad (7)
\]
Assume, moreover, that $k \in K$ ($\subset Q_0 \subset Q$); then $g_k(x) = q_k(x)$, and so
\[
\mu g_k(x) = \mu q_k(x) = \mu [q_k(x) + 0 \cdot \Lambda_k(x) - V_k^*(0)] = \sup_{\sigma_k \in \mathbb{I}_k^*} \mu [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)]
\]
for every $\mu \in \mathbb{R}$. Therefore,
\[
L(x, \lambda) = \sum_{k \in K} \sup_{\sigma_k \in \{0\}} \lambda_k [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)]
\]
\[
+ \sum_{k \in 0, m \setminus K} \sup_{\sigma_k \in \mathbb{I}_k^*} \lambda_k [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)]
\]
\[
= \sum_{k \in 0, m} \sup_{\sigma_k \in \mathbb{I}_k^*} \lambda_k [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)]
\]
\[
= \sum_{\sigma \in \mathcal{I}_J, Q} \sum_{k \in 0, m} \lambda_k [q_k(x) + \sigma_k \Lambda_k(x) - V_k^*(\sigma_k)] = \sup_{\sigma \in \mathcal{I}_J, Q} \Xi(x, \lambda, \sigma);
\]

hence, (7) holds. Using (5) we get
\[
\sup_{(\lambda, \sigma) \in \Gamma_K \times \mathcal{I}_J, Q} \Xi(x, \lambda, \sigma) = \sup_{\lambda \in \Gamma_K} \sup_{\sigma \in \mathcal{I}_J, Q} \Xi(x, \lambda, \sigma) = \sup_{\lambda \in \Gamma_K} L(x, \lambda). \quad \quad (8)
\]

Since
\[
\sup_{\lambda \in \mathbb{R}_+} \lambda \alpha = \iota_{\mathbb{R}_+}(\alpha), \quad \sup_{\lambda \in \mathbb{R}} \lambda \alpha = \iota_{\{0\}}(\alpha),
\]
where the indicator function $\iota_E : Z \to \mathbb{R}$ of $E \subset Z$ is defined by $\iota_E(z) := 0$ for $z \in E$, $\iota_E(z) := +\infty$ for $z \in Z \setminus E$, we get
\[
\sup_{\lambda \in \Gamma_K} L(x, \lambda) = f(x) + \sum_{j=1, m} \sup_{\lambda_j \in \Gamma_j} \lambda_j g_j(x) = \left\{ \begin{array}{ll} f(x) & \text{if } x \in X_K, \\ \infty & \text{if } x \in X \setminus X_K. \end{array} \right.
\]

Using (8) and the previous equalities, the conclusion follows. \qed

Another useful result in this context is the following.
Lemma 2 Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{\Xi} \in \mathbb{R}^m$ and $k \in \overline{0, m}$. Then

$$\Lambda_k(\mathbf{x}) = V_k^{\ast}(\mathbf{\Xi}k) \iff \mathbf{\Xi}k = V_k^{\ast}(\Lambda_k(\mathbf{x})) \iff V_k(\Lambda_k(\mathbf{x})) = V_k^{\ast}(\mathbf{\Xi}k) $$

$$\iff g_k(\mathbf{x}) = g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k) $$

$$\iff [\mathbf{\Xi}k \in \text{int}(\text{dom} V_k^{\ast}) \land \Lambda_k(\mathbf{x}) \in \text{int}(\text{dom} V_k)].$$

In particular, for $k \in Q$, $\Lambda_k(\mathbf{x}) = V_k^{\ast}(\mathbf{\Xi}k)$ if and only if $\mathbf{\Xi}k = 0$.

Proof. Because $V_k \in \Gamma_{\text{sc}}$, (3) holds. Since $g_k(\mathbf{x}) = g_k(\mathbf{x}) + V_k(\Lambda_k(\mathbf{x}))$, we obtain that $g_k(\mathbf{x}) = g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$ if and only $V_k(\Lambda_k(\mathbf{x})) = \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$, and so the conclusion follows. The case $k \in Q$ follows immediately.

Corollary 3 Let $(\mathbf{x}, \mathbf{\Xi}, \mathbf{\sigma}) \in X \times \mathbb{R}^m \times I^*$ with $\mathbf{\Xi}k = 0$ for $k \in Q$. If

$$\forall k \in M^0_{\neq}(\mathbf{\Xi}) \setminus Q : [\Lambda_k(\mathbf{x}) \in \text{int}(\text{dom} V_k) \land \mathbf{\Xi}k \in \text{int}(\text{dom} V_k^{\ast}) \land \mathbf{\Xi}k = V_k^{\ast}(\Lambda_k(\mathbf{x}))], \quad (9)$$

then $L(\mathbf{x}, \mathbf{\Xi}) = \Xi(\mathbf{\Xi}, \mathbf{\sigma})$. Conversely, if $L(\mathbf{x}, \mathbf{\Xi}) = \Xi(\mathbf{\Xi}, \mathbf{\sigma})$ and $\mathbf{\Xi} \in \Gamma_{Q_0}$, then (3) holds.

Proof. Assume first that (9) holds. Using Lemma 2 we obtain that $V_k(\Lambda_k(\mathbf{x})) = \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$ for $k \in M^0_{\neq}(\mathbf{\Xi})$, and so

$$\mathbf{\Lambda}k [g_k(\mathbf{x}) + V_k(\Lambda_k(\mathbf{x}))] = \mathbf{\Lambda}k [g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)] \quad \forall k \in \overline{0, m}$$

because $\mathbf{\Lambda}k = 0$ for $k \not\in \overline{0, m} \setminus M^0_{\neq}(\mathbf{\Xi})$. Then the equality $L(\mathbf{x}, \mathbf{\Xi}) = \Xi(\mathbf{\Xi}, \mathbf{\sigma})$ follows from the definitions of $L$ and $\Xi$.

Conversely, assume that $L(\mathbf{x}, \mathbf{\Xi}) = \Xi(\mathbf{\Xi}, \mathbf{\sigma})$ and $\mathbf{\Xi} \in \Gamma_{Q_0}$; hence $\mathbf{\Lambda}k \geq 0$ for all $k \in Q_0$. Clearly, $g_k(\mathbf{x}) = g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$ for $k \in Q$. Because $g_k(\mathbf{x}) = g_k(\mathbf{x}) + V_k(\Lambda_k(\mathbf{x})) \geq g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$ and $\mathbf{\Lambda}k \geq 0$ for all $k \in \{0\} \cup Q_0 \supset \overline{0, m} \setminus Q$, from $L(\mathbf{x}, \mathbf{\Xi}) = \Xi(\mathbf{\Xi}, \mathbf{\sigma})$ we obtain that

$$\mathbf{\Lambda}k [g_k(\mathbf{x}) + V_k(\Lambda_k(\mathbf{x}))] = \mathbf{\Lambda}k [g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)] \quad \forall k \in \overline{0, m} \setminus Q;$$

hence $g_k(\mathbf{x}) = g_k(\mathbf{x}) + \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$, that is $V_k(\Lambda_k(\mathbf{x})) = \mathbf{\Xi}k\Lambda_k(\mathbf{x}) - V_k^{\ast}(\mathbf{\Xi}k)$, for $k \in M^0_{\neq}(\mathbf{\Xi}) \setminus Q$. Using (3) we obtain that (9) is verified.

Let us consider $G : \mathbb{R}^m \times \mathbb{R}^{1+m} \to \mathcal{G}_n$, $F : \mathbb{R}^m \times \mathbb{R}^{1+m} \to \mathbb{R}^n$, $E : \mathbb{R}^m \times \mathbb{R}^{1+m} \to \mathbb{R}$ defined by

$$G(\lambda, \sigma) := \sum_{k=0}^{m} \lambda_k(A_k + \sigma_kC_k), \quad F(\lambda, \sigma) := \sum_{k=0}^{m} \lambda_k(b_k + \sigma_kd_k), \quad E(\lambda, \sigma) := \sum_{k=0}^{m} \lambda_k(c_k + \sigma_ke_k).$$

Hence, for $(\lambda, \sigma) \in \mathbb{R}^m \times I^*$ we have that

$$\Xi(x, \lambda, \sigma) = \frac{1}{2} \langle x, G(\lambda, \sigma)x \rangle - \langle F(\lambda, \sigma), x \rangle + E(\lambda, \sigma) - \sum_{k=0}^{m} \lambda_kV_k^{\ast}(\sigma_k).$$

Remark 4 Note that $G$, $F$ and $E$ do not depend on $\sigma_k$ for $k \in Q$. Moreover, $G$, $F$ and $E$ are affine functions when $1, m \subset Q$, that is $Q_0 = 1, m$. 


For \((\lambda, \sigma) \in \mathbb{R}^m \times I^*\) we have that
\[
\nabla_x \Xi(x, \lambda, \sigma) = G(\lambda, \sigma)x - F(\lambda, \sigma), \quad \nabla^2_{xx} \Xi(x, \lambda, \sigma) = G(\lambda, \sigma),
\]
while for \((\lambda, \sigma) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int } I^*\) we have that
\[
\nabla_x \Xi(x, \lambda, \sigma) = (q_1(x) + \sigma_1 \Lambda_1(x) - V^{s'}(\sigma_1), \ldots, q_m(x) + \sigma_m \Lambda_m(x) - V^{s'}(\sigma_m))^T,
\]

Other relations between \(L\) and \(\Xi\) are provided in the next result.

**Lemma 5** Let \((\bar{\lambda}, \bar{\sigma}, \bar{\tau}) \in \mathcal{X}_0 \times \mathbb{R}^m \times \text{int } I^*\) be such that \(\nabla_x \Xi(\lambda, \sigma) = 0\) and \(\sigma_k = 0\) for \(k \in Q\). Then \(L(\lambda, \sigma) = \Xi(\lambda, \sigma)\) and \(\nabla_x L(\lambda, \sigma) = \nabla_x \Xi(\lambda, \sigma)\). Moreover, for \(j \in \overline{1,m}\), \(\frac{\partial L}{\partial \lambda_j}(\lambda, \sigma) \geq \frac{\partial \Xi}{\partial \lambda_j}(\lambda, \sigma)\), with equality if \(j \in M_{\lambda}_0(\bar{\lambda}) \cup Q_0\); in particular \(\nabla_\lambda L(\lambda, \sigma) = \nabla_\lambda \Xi(\lambda, \sigma)\) if \(M_{\lambda}(\bar{\lambda}) \supset \emptyset\).

**Proof.** For \(k \in M^0_{\lambda}(\bar{\lambda})\) we have that \(\bar{\sigma}_k \neq 0\): using \([12]\) and Lemma \([2]\) we get \(\Lambda_k(\bar{\lambda}) - V_k^{s'}(\sigma_k) = 0\), and so \(\sigma_k \in \text{int}(\text{dom } V_k^s)\), \(\Lambda_k(\bar{\lambda}) \in \text{int}(\text{dom } V_k)\), \(\bar{\sigma}_k = V_k^s(\Lambda_k(\bar{\lambda}))\) for \(k \in M^0_{\lambda}(\bar{\lambda})\). Hence \([9]\) is verified, and so \(L(\lambda, \sigma) = \Xi(\lambda, \sigma)\) by Corollary \([3]\). Moreover,
\[
\nabla_x L(\lambda, \sigma) = \sum_{k \in \overline{1,m}} \bar{\lambda}_k \left[ A_k \bar{\lambda} - b_k + V_k^s(\Lambda_k(\bar{\lambda}))(C_k \bar{\lambda} - d_k) \right]
= \sum_{k \in \overline{1,m}} \bar{\lambda}_k \left[ A_k \bar{\lambda} - b_k + V_k^s(\Lambda_k(\bar{\lambda}))(C_k \bar{\lambda} - d_k) \right],
\]
and so \(\nabla_x L(\lambda, \sigma) = \nabla_x \Xi(\lambda, \sigma)\). Clearly, from the definitions of \(L, \Xi\) and the inequality in \([2]\), we have that
\[
\frac{\partial L}{\partial \lambda_j}(\lambda, \sigma) = g_j(\bar{\lambda}) = q_j(\bar{\lambda}) + \sigma_j V_j^s(\Lambda_j(\bar{\lambda})) \geq q_j(\bar{\lambda}) + \sigma_j A_j(\bar{\lambda}) - V_j^s(\sigma_j) = \frac{\partial \Xi}{\partial \lambda_j}(\lambda, \sigma).
\]
Using Lemma \([2]\) we obtain that \(g_k(\bar{\lambda}) = q_k(\bar{\lambda}) + \sigma_k^1 A_k(\bar{\lambda}) - V_k^s(\sigma_k)\) for \(k \in M_{\lambda}(\bar{\lambda}) \cup Q\) and so \(\frac{\partial L}{\partial \lambda_j}(\lambda, \sigma) = \frac{\partial \Xi}{\partial \lambda_j}(\lambda, \sigma)\) for \(j \in M_{\lambda}(\bar{\lambda}) \cup Q_0\). \(\square\)

We consider also the sets
\[
T_Q := \{(\lambda, \sigma) \in \mathbb{R}^m \times I^* \mid \text{det } G(\lambda, \sigma) \neq 0 \land [\forall k \in Q : \sigma_k = 0]\},
\]
\[
T_{Q, \text{col}} := \{(\lambda, \sigma) \in \mathbb{R}^m \times I^* \mid F(\lambda, \sigma) \in \text{Im } G(\lambda, \sigma) \land [\forall k \in Q : \sigma_k = 0]\} \supseteq T_Q,
\]
\[
T_Q^{J^+} := \{(\lambda, \sigma) \in T_Q \mid \lambda \in \Gamma_{J^+ \cap Q}, \ G(\lambda, \sigma) > 0\},
\]
\[
T^{J^+}_{Q, \text{col}} := \{(\lambda, \sigma) \in T_{Q, \text{col}} \mid \lambda \in \Gamma_{J^+ \cap Q}, \ G(\lambda, \sigma) \geq 0\} \supseteq T^{J^+}_Q,
\]
as well as the sets
\[
T := T_0, \quad T_{\text{col}} := T_{0, \text{col}}, \quad T^+ := T^0_{\text{col}}, \quad T^+_{\text{col}} := T^0_{0, \text{col}}.
\]
in general $T_Q^{J^+}$ and $T_{Q_{col}}^{J^+}$ are not convex, unlike their corresponding sets $Y^+$, $Y_{col}^+$ and $S^+$, $S_{col}^+$ from [18] and [19], respectively. However, taking into account Remark 4, $T_{col}^+$, $T_Q^{J^+}$ and $T_{Q_{col}}^{J^+}$ are convex whenever $Q_0 = 1, m$. In the present context it is natural (in fact necessary) to take $\lambda \in \Gamma_{Q_0}$. As in [18] and [19], we consider the (dual objective) function

$$D : T_{col} \to \mathbb{R}, \quad D(\lambda, \sigma) := \Xi(x, \lambda, \sigma) \text{ with } G(\lambda, \sigma)x = F(\lambda, \sigma);$$

$D$ is well defined by [18] Lem. 1 (ii)]. Consider

$$\xi : T \to \mathbb{R}^n, \quad \xi(\lambda, \sigma) := G(\lambda, \sigma)^{-1}F(\lambda, \sigma).$$

(13)

For $(\lambda, \sigma) \in T$ we obtain that

$$D(\lambda, \sigma) = \Xi(G(\lambda, \sigma)^{-1}F(\lambda, \sigma), \lambda, \sigma) = \Xi(\xi(\lambda, \sigma), \lambda, \sigma)
= -\frac{1}{2} \langle F(\lambda, \sigma), G(\lambda, \sigma)^{-1}F(\lambda, \sigma) \rangle + E(\lambda, \sigma) - \sum_{k=0}^m \lambda_k V_k^*(\sigma_k).$$

(14)

Taking into account the second formula in (10), we have that $\Xi(\cdot, \lambda, \sigma)$ is [strictly] convex for $(\lambda, \sigma) \in T_{col}^+ \cup \{(\lambda, \sigma) \in T^+\}$, and so

$$D(\lambda, \sigma) = \min_{x \in \mathbb{R}^n} \Xi(x, \lambda, \sigma) \quad \forall (\lambda, \sigma) \in T_{col} \text{ such that } G(\lambda, \sigma) \succeq 0,$$

the minimum being attained uniquely at $\xi(\lambda, \sigma)$ when, moreover, $G(\lambda, \sigma) > 0$.

**Proposition 6**: Let $(\overline{\lambda}, \overline{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times I^*$ be such that $\nabla_x \Xi(\overline{\lambda}, \overline{\sigma}) = 0, \frac{\partial \Xi}{\partial \sigma_0}(\overline{\lambda}, \overline{\sigma}) = 0$, and $\langle \overline{\lambda}, \nabla_\lambda \Xi(\overline{\lambda}, \overline{\sigma}) \rangle = 0$. Then $(\overline{\lambda}, \overline{\sigma}) \in T_{col}$ and

$$f(\overline{x}) = \Xi(\overline{\lambda}, \overline{\sigma}) = D(\overline{\lambda}, \overline{\sigma}).$$

(16)

Proof. Because $\nabla_x \Xi(\overline{\lambda}, \overline{\sigma}) = 0$, $(\overline{\lambda}, \overline{\sigma}) \in T_{col}$ and the second equality in (16) holds by the definition of $D$. Since $\lambda_0(\overline{\lambda}) - V_0^*(\sigma_0) = \frac{\partial \Xi}{\partial \sigma_0}(\overline{\lambda}, \overline{\sigma}) = 0$, we have that $V_0(\lambda_0(\overline{\lambda})) = \lambda_0(\overline{\lambda}) - V_0^*(\sigma_0)$ by Lemma 2 for $k := 0$. Therefore,

$$f(\overline{x}) = q_0(\overline{x}) + V_0(\lambda_0(\overline{\lambda})) = q_0(\overline{x}) + \sigma_0(\overline{\lambda}) - V_*^*(\sigma_0),$$

whence

$$\Xi(\overline{\lambda}, \overline{\sigma}) = \lambda_0(q_0(x) + \sigma_0 \lambda_0(x) - V_0^*(\sigma_0)) + \langle \overline{\lambda}, \nabla_\lambda \Xi(\overline{\lambda}, \overline{\sigma}) \rangle = f(\overline{x}).$$

Hence the first equality in (16) holds, too. □

Formula (16) is related to the so-called “complimentary-dual principle” (see [3] p. NP11], [4] p. 13]) and sometimes is called the “perfect duality formula”.

Observe that $T \cap (\mathbb{R}^n \times \text{int } I^*) \subset \text{int } T$, and for any $\overline{\sigma} \in I^*$ we have that the set $\{\lambda \in \mathbb{R}^m \mid (\lambda, \overline{\sigma}) \in T\}$ is open. Similarly to the computation of $\frac{\partial D(\lambda)}{\partial \lambda_j}$ in [18] p. 5, using the expression of $D(\lambda, \sigma)$ in (14), we get

$$\frac{\partial D(\lambda, \sigma)}{\partial \lambda_j} = \frac{1}{2} \langle \xi(\lambda, \sigma), (A_j + \sigma_j C_j) \xi(\lambda, \sigma) \rangle - \langle b_j + \sigma_j d_j, \xi(\lambda, \sigma) \rangle + e_j \sigma_j - V_j^*(\sigma_j)
= \lambda_j (\lambda, \sigma) + \sigma_j A_j (\xi(\lambda, \sigma)) - V_j^*(\sigma_j) \quad \forall j \in 1, m, \quad \forall (\lambda, \sigma) \in T,$$

(17)

and

$$\frac{\partial D(\lambda, \sigma)}{\partial \sigma_k} = \lambda_k \left[ \frac{1}{2} \langle \xi(\lambda, \sigma), C_k \xi(\lambda, \sigma) \rangle - \langle d_k, \xi(\lambda, \sigma) \rangle + e_k - V_k^*(\sigma_k) \right]
= \lambda_k \left[ \lambda_k (\lambda, \sigma) - V_k^*(\sigma_k) \right] \quad \forall k \in 0, m, \quad \forall (\lambda, \sigma) \in T \cap (\mathbb{R}^m \times \text{int } I^*).$$

(18)
Lemma 7 Let \((\bar{x}, \sigma) \in (\mathbb{R}^m \times \text{int } I^*) \cap T\) and set \(\bar{\sigma} := \xi(\bar{x}, \sigma)\). Then
\[
\nabla_x \Xi(\bar{x}, \bar{\sigma}) = 0 \quad \land \quad \nabla_{\lambda} \Xi(\bar{x}, \bar{\sigma}) = \nabla_{\lambda} D(\bar{x}, \bar{\sigma}) \quad \land \quad \nabla_{\sigma} \Xi(\bar{x}, \bar{\sigma}) = \nabla_{\sigma} D(\bar{x}, \bar{\sigma}).
\]
In particular \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a critical point of \(\Xi\) if and only if \((\bar{\lambda}, \bar{\sigma})\) is a critical point of \(D\).

Proof. Using \((\bar{x}, \bar{\sigma})\) we get \(\nabla_x \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\). From \((17)\) and \((11)\) for \(j \in \overline{1, m}\) we get
\[
\frac{\partial D}{\partial \lambda_j}(\bar{x}, \bar{\sigma}) = q_j(\bar{x}) + \sigma_j \lambda_j(\bar{x}) - V_j^*(\bar{\sigma_j}) = \frac{\partial \Xi}{\partial \lambda_j}(\bar{x}, \bar{\lambda}, \bar{\sigma}),
\]
while from \((18)\) and \((12)\) for \(k \in \overline{1, m}\) we get
\[
\frac{\partial D}{\partial \sigma_k}(\bar{x}, \bar{\sigma}) = \bar{\lambda}_k [\Lambda_k(\bar{x}) - V_k^*(\bar{\sigma}_k)] = \frac{\partial \Xi}{\partial \sigma_k}(\bar{x}, \bar{\lambda}, \bar{\sigma}).
\]
The conclusion follows. \(\square\)

Similarly to \((18)\), we say that \((\bar{x}, \bar{\lambda}) \in X_0 \times \mathbb{R}^m\) is a \(J\)-LKKT point of \(L\) if \(\nabla_x L(\bar{x}, \bar{\lambda}) = 0\) and
\[
[\forall j \in J^c : \bar{x}_j \geq 0 \land \frac{\partial L}{\partial \lambda_j}(\bar{x}, \bar{\lambda}) \leq 0 \land \bar{x}_j \frac{\partial L}{\partial \lambda_j}(\bar{x}, \bar{\lambda}) = 0] \land [\forall j \in J : \frac{\partial L}{\partial \lambda_j}(\bar{x}, \bar{\lambda}) = 0],
\]
or, equivalently,
\[
\bar{x} \in X_J \land \bar{\lambda} \in \Gamma_J \land [\forall j \in J^c : \bar{x}_j g_j(\bar{x}) = 0];
\]
moreover, we say that \(\bar{x} \in X_0\) is a \(J\)-LKKT point of \((P_J)\) if there exists \(\bar{x} \in \mathbb{R}^m\) such that \((\bar{x}, \bar{\lambda})\) is a \(J\)-LKKT point of \(L\). Inspired by these notions, we say that \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int } I^*\) is a \(J\)-LKKT point of \(\Xi\) if \(\nabla_x \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\), \(\nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\) and
\[
[\forall j \in J^c : \bar{x}_j \geq 0 \land \frac{\partial \Xi}{\partial \lambda_j}(\bar{x}, \bar{\lambda}, \bar{\sigma}) \leq 0 \land \bar{x}_j \frac{\partial \Xi}{\partial \lambda_j}(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0] \land [\forall j \in J : \frac{\partial \Xi}{\partial \lambda_j}(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0],
\]
and \((\bar{\lambda}, \bar{\sigma}) \in (\mathbb{R}^m \times \text{int } I^*) \cap T\) is a \(J\)-LKKT point of \(D\) if \(\nabla_{\sigma} D(\bar{\lambda}, \bar{\sigma}) = 0\) and
\[
[\forall j \in J^c : \bar{x}_j \geq 0 \land \frac{\partial D}{\partial \lambda_j}(\bar{\lambda}, \bar{\sigma}) \leq 0 \land \bar{x}_j \frac{\partial D}{\partial \lambda_j}(\bar{\lambda}, \bar{\sigma}) = 0] \land [\forall j \in J : \frac{\partial D}{\partial \lambda_j}(\bar{\lambda}, \bar{\sigma}) = 0].
\]

In the case in which \(J = \emptyset\) we obtain the notions of KKT points for \(\Xi\) and \(D\). So, \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int } I^*\) is a KKT point of \(\Xi\) if \(\nabla_x \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\), \(\nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\) and
\[
\bar{\lambda} \in \mathbb{R}^+_m \land \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^m \land \langle \bar{\lambda}, \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) \rangle = 0,
\]
and \((\bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^m \times \text{int } I^*\) is a KKT point of \(D\) if \(\nabla_{\sigma} D(\bar{\lambda}, \bar{\sigma}) = 0\) and
\[
\bar{\lambda} \in \mathbb{R}^m \land \nabla_{\lambda} D(\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^*_m \land \langle \bar{\lambda}, \nabla_{\lambda} D(\bar{x}, \bar{\lambda}, \bar{\sigma}) \rangle = 0.
\]

Remark 8 The definition of a KKT point for \(\Xi\) is suggested in the proof of \((13)\) (the same as that of \((12)\) Th. 3). Observe that \((\bar{\lambda}, \bar{\lambda}, \bar{\sigma})\) verifying the conditions in \((19)\) is called critical point of \(\Xi\) in \([5]\) p. 477).
Corollary 9  Let \((\bar{x}, \bar{\sigma}) \in (\mathbb{R}^m \times \text{int } I^*) \cap T\).

(i) If \(\bar{x} := \xi(\bar{\lambda}, \bar{\sigma})\), then \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point of \(\Xi\) if and only if \((\bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point of \(D\).

(ii) If \(M_{\#}(\bar{\lambda}) = \Gamma, m\), then \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point of \(\Xi\) if and only if \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a critical point of \(\Xi\), if and only if \(x = \xi(\bar{x}, \bar{\sigma})\) and \((\bar{\lambda}, \bar{\sigma})\) is a critical point of \(D\).

Proof. (i) is immediate from Lemma 7 while (ii) is an obvious consequence of (i) and the definitions of the corresponding notions. □

Remark 10  Taking into account Remark 4 as well as (11), (13) and Lemma 7, the functions \(\nabla_x \Xi, \xi, \nabla_x D\) do not depend on \(\sigma_k\) for \(k \in Q\). Consequently, if \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point of \(\Xi\) then \(\bar{\sigma}_k = 0\) for \(k \in Q \cap M_{\#}(\bar{\lambda})\), and \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is also a J-LKKKT point of \(\Xi\), where \(\bar{\sigma}_k := 0\) for \(k \in Q\) and \(\bar{\sigma}_k := \bar{\sigma}_k\) for \(k \in 0, m \setminus Q\). Conversely, taking into account that \(\nabla_{\sigma} D\) does not depend on \(\sigma_k\) for \(k \in Q\), if \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in T\) is a J-LKKKT point of \(D\) then \((\bar{x}, \bar{\sigma})\) is also a J-LKKKT point of \(D\), where \(\bar{\sigma}_k := 0\) for \(k \in Q\) and \(\bar{\sigma}_k := \bar{\sigma}_k\) for \(k \in 0, m \setminus Q\).

Having in view the previous remark, without loss of generality, in the sequel we shall assume that \(\bar{\sigma}_k = 0\) for \(k \in Q\) when \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int } I^*\) is a J-LKKKT point of \(\Xi\), or \((\bar{\lambda}, \bar{\sigma}) \in T\) is a J-LKKKT point of \(D\).

3  The main result

Proposition 11  Let \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int } I^*\) be a J-LKKKT point of \(\Xi\) such that \(\bar{\sigma}_k = 0\) for \(k \in Q\).

(i) Then \(\bar{\lambda} \in \Gamma_J, (\bar{x}, \bar{\lambda}) \in T_{Q, \text{col}}, \langle \bar{x}, \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) \rangle = 0, L(\bar{x}, \bar{\lambda}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}), \nabla_x L(\bar{x}, \bar{\lambda}) = 0, \\text{and (10) holds.}\)

(ii) Moreover, assume that \(Q^c_0 \subset M_{\#}(\bar{\lambda})\). Then \(\nabla_{\lambda} L(\bar{x}, \bar{\lambda}) = \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}), (\bar{x}, \bar{\lambda})\) is a J-LKKKT point of \(L\) and \(\bar{x} \in X_{J, Q^c_0}\).

(iii) Furthermore, assume that \(\bar{x}_j > 0\) for all \(j \in Q^c_0\) and \(G(\bar{x}, \bar{\lambda}, \bar{\sigma}) \geq 0\). Then \(\bar{x} \in X_{J, Q^c_0} \subset X_J \cap X_{J \cap Q}\), \((\bar{x}, \bar{\lambda}) \in T_{Q, \text{col}}^+,\) and

\[
f(\bar{x}) = \inf_{x \in X_{J \cap Q}} f(x) = \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = L(\bar{x}, \bar{\lambda}) = \sup_{(\lambda, \sigma) \in T_{Q, \text{col}}^+} D(\lambda, \sigma) = D(\bar{\lambda}, \bar{\sigma});
\]

moreover, if \(G(\bar{x}, \bar{\lambda}, \bar{\sigma}) > 0\) then \(\bar{x}\) is the unique global solution of problem \((P_{J \cap Q}).\)

Proof. (i) Because \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point, from its very definition we have that \(\bar{\lambda} \in \Gamma_J, \langle \bar{x}, \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) \rangle = 0, \nabla_x \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\) and \(\nabla_{\sigma} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\). Using Lemma 5 and we obtain that \(\nabla_x L(\bar{x}, \bar{\lambda}) = \nabla_x \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0\) and \(L(\bar{x}, \bar{\lambda}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma})\), while using Proposition 10 we get \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in T_{Q, \text{col}}\) and that (10) holds.

(ii) Because \(Q^c_0 \subset M_{\#}(\bar{\lambda})\) we get \(\nabla_{\lambda} L(\bar{x}, \bar{\lambda}) = \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma})\) by Lemma 5 and so \((\bar{x}, \bar{\lambda})\) is a J-LKKKT point of \(L\) because \((\bar{x}, \bar{\lambda}, \bar{\sigma})\) is a J-LKKKT point of \(\Xi\). Hence \(g_j(\bar{x}) = 0\) for \(j \in J\), and \(\bar{x}_j g_j(\bar{x}) = 0, g_j(\bar{x}) \leq 0\) for \(j \in J^c\). Taking into account that \(Q^c_0 \subset M_{\#}(\bar{\lambda})\), the preceding condition shows that \(g_j(\bar{x}) = 0\) for \(j \in Q^c_0\) and so \(\bar{x} \in X_{J, Q^c_0}\).

(iii) Our hypothesis shows that \(Q^c_0 \subset M_{\#}(\bar{\lambda})\). From (i) and (ii) we have that \(\bar{\lambda} \in \Gamma_J, (\bar{x}, \bar{\lambda}) \in T_{Q, \text{col}}, \bar{x} \in X_{J, Q^c_0} \subset X_J \subset X_{J \cap Q}\); moreover, \(\bar{\lambda} \in \Gamma_{J \cap Q}\) because \(x_j \geq 0\) for \(j \in Q^c_0\).
\( J^c \cup Q_0^c = (J \cap Q)^c \), and so \((\bar{x}, \sigma) \in T_{Q_0}^{J+}\). Using now Lemma 1 obvious inequalities, (15), and (i), as well as the obvious inclusion \( T_{J,Q} \subset T_{J,Q_0} \times T_{I,J,\bar{Q}} \) with \( I,J,\bar{Q} \) defined in (i), we get

\[
f(\bar{x}) \geq \inf_{x \in X_{J,Q}} f(x) = \inf_{x \in X_{J,Q}} \sup_{\lambda \in \Gamma_{J,Q}} L(x, \lambda) = \inf_{x \in X_{J,Q}} \sup_{\lambda \in \Gamma_{J,Q}} \Xi(x, \lambda, \sigma)
\]

\[
\geq \inf_{x \in X_{J,Q}} \sup_{(\lambda, \sigma) \in T_{J,Q}^{I+}} \Xi(x, \lambda, \sigma) = \sup_{(\lambda, \sigma) \in T_{J,Q}^{I+}} \sup_{x \in X_{J,Q}} D(\lambda, \sigma) \geq D(\bar{x}, \sigma),
\]

which implies (20) by (i).

Assume, moreover, that \( G(\bar{x}, \sigma) \succ 0 \); hence \((\bar{x}, \sigma) \in T_{Q_0}^{J+}\). Consider \( x \in X_{J,Q} \setminus \{\bar{x}\} \). Using the strict convexity of \( \Xi(\cdot, \lambda, \sigma) \) and Lemma 1 we get \( f(\bar{x}) = \Xi(\bar{x}, \lambda, \sigma) < \Xi(x, \lambda, \sigma) \leq L(x, \lambda) \leq f(x) \). It follows that \( \bar{x} \) is the unique global solution of \( (P_{J,Q}) \) [and \((P_J)\), too]. □

The variant of Proposition 11 in which \( Q \) is not taken into consideration, that is the case when one does not observe that \( V_k \circ \Lambda_k = 0 \) for some \( k \), is much weaker; however, the conclusions coincide for \( Q = \{0\} \).

**Proposition 12** Let \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \in \mathbb{R}^n \times \mathbb{R}^m \times \text{int} I \) be a J-LKKT point of \( \Xi \).

(i) Then \( \bar{x} \in \Gamma_{I,J}, (\bar{x}, \bar{\lambda}) \in T_{col}, \langle \bar{\lambda}, \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) \rangle = 0, \ L(\bar{x}, \bar{\lambda}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}), \ \nabla_{\lambda} L(\bar{x}, \bar{\lambda}) = 0, \) and \((I6)\) holds.

(ii) Assume that \( M(\bar{x}) = \mathbb{I}_m. \) Then \( \nabla_{\lambda} L(\bar{x}, \bar{\lambda}) = \nabla_{\lambda} \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = 0, \) whence \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \) is a critical point of \( \Xi, (\bar{x}, \bar{\lambda}) \) is a critical point of \( L, \) and \( \bar{x} \in X_e \subset X_J \subset X_I. \)

(iii) Assume that \( \bar{x} \in \mathbb{R}^m \) and \( G(\bar{x}, \bar{\lambda}, \bar{\sigma}) \succeq 0. \) Then \( \bar{x} \in X_e, (\bar{x}, \bar{\lambda}) \in T_{col}^+, \) and

\[
f(\bar{x}) = \inf_{x \in X_I} f(x) = \Xi(\bar{x}, \bar{\lambda}, \bar{\sigma}) = L(\bar{x}, \bar{\lambda}) = \sup_{(\lambda, \sigma) \in T_{col}^+} D(\lambda, \sigma) = D(\bar{x}, \sigma);
\]

moreover, if \( G(\bar{x}, \bar{\lambda}, \bar{\sigma}) \succ 0 \) then \((\bar{x}, \bar{\lambda}) \in T^+ \) and \( \bar{x} \) is the unique global solution of problem \((P)\).

The remark below refers to the case \( Q = \emptyset. \) A similar remark (but a bit less dramatic) is valid for \( Q_0 \neq \emptyset. \)

**Remark 13** It is worth observing that given the functions \( f, g_1, \ldots, g_m \) of type \( q + V \circ \Lambda \) with \( q, \Lambda \) quadratic functions and \( V \in \Gamma_{sc} \), for any choice of \( J \subset \mathbb{I}_m \) one finds the same \( \bar{x} \) using Proposition 7 (iii). So, in practice, if one wishes to solve one of the problems \((P_e) \), \((P_I)\) or \((P_J)\) using CDT, it is sufficient to find those critical points \((\bar{x}, \bar{\lambda}, \bar{\sigma}) \) of \( \Xi \) such that \( \bar{x} \in \mathbb{R}_{++}^m \) and \( G(\bar{x}, \bar{\lambda}, \bar{\sigma}) \succeq 0; \) if we are successful, \( \bar{x} \in X_e \) and \( \bar{x} \) is the unique solution of \((P)\), and so \( \bar{x} \) is also solution for all problems \((P_J) \) with \( J \subset \mathbb{I}_m \); moreover, \((\bar{x}, \bar{\lambda}) \) is a global maximizer of \( D \) on \( T_{col}^+ \).

The next example shows that the condition \( Q_0^c \subset M(\bar{x}) \) is essential for \( \bar{x} \) to be a feasible solution of problem \((P_J)\); moreover, it shows that, unlike the quadratic case (see 118, Prop. 9), it is not possible to replace \( T_{Q_0}^{J+} \) by \( \{ (\lambda, \sigma) \in T_{col} \mid \lambda \in \Gamma_J, \ G(\lambda, \sigma) \succeq 0 \} \) in (20). The problem is a particular case of the one considered in [7, Ex. 1], “which is very simple, but
important in both theoretical study and real-world applications since the constraint is a so-called double-well function, the most commonly used nonconvex potential in physics and engineering sciences [7], more precisely, $q := 1, c := 6, d := 4, e := 2$.

**Example 14** Let us take $n = m = 1, J \subset \{1\}, q_0(x) := \frac{1}{2} x^2 - 6x, \Lambda_1(x) := \frac{1}{2} x^2 - 4, q_1(x) := \Lambda_0(x) := 0, \quad V_0(t) := V_1(t) + 2 := \frac{1}{2} t^2$ for $x, t \in \mathbb{R}$. Then $f(x) = \frac{1}{2} x^2 - 6x$ and $g_1(x) = \frac{1}{3} (\frac{1}{3} x^2 - 4)^2 - 2$. Hence $Q = \{0\}$ (whence $Q_0 = \emptyset$) and $X_c = \{ -2\sqrt{3}, 2\sqrt{3}, -2, 2 \} \subset [-2\sqrt{3}, -2] \cup [2, 2\sqrt{3}] = X_i$.

$$\Xi(x; \lambda; \sigma_0, \sigma) = \frac{1}{2} x^2 - 6x - \frac{1}{2} \sigma_0^2 + \lambda \left[ \frac{1}{3} x^2 - 4 \right] - \frac{1}{2} \sigma_1^2 - 2 \right].$$

We have that $G(\lambda, \sigma) = 1 + \lambda \sigma_1, T_{col} = T = \{(\lambda, \sigma) \in \mathbb{R} \times \mathbb{R}^2 \mid 1 + \lambda \sigma_1 \neq 0\}$ and

$$D(\lambda; \sigma_0, \sigma_1) = -\frac{18}{1 + \lambda \sigma_1} - \frac{1}{2} \sigma_0^2 - \lambda \left( \frac{1}{3} \sigma_1^2 + 4 \sigma_1 + 2 \right).$$

The critical points of $\Xi$ are $(2; -1; (0, -2)), (2; 0; (0, -2)), (6; 0; (0, 14 + 8\sqrt{3})), (6; 0; (0, 14 + 8\sqrt{3})), (-2\sqrt{3}; -\frac{1}{2} \sqrt{3}; -\frac{1}{2}; (0, 2)), (2\sqrt{3}; \frac{1}{2} \sqrt{3}; -\frac{1}{2}; (0, 2))$, and so $1 + \lambda \sigma_1 \in \{3, -3, 1, -\sqrt{3}, \sqrt{3}\}$ for $(\lambda, \sigma)$ critical point of $\Xi$, whence $(\lambda, \sigma)$ is critical point of $D$ by Lemma [4]. For $\lambda = 0$ the corresponding $\pi$ is not in $X_i \supset X_e$; in particular, $(\pi, \lambda)$ is not a critical point of $L$. For $\lambda \neq 0$, Proposition [12] says that $(\pi, \lambda)$ is a critical point of $L$; in particular $\pi \in X_e$. For $\lambda \in \{2, \frac{1}{2} \frac{\sqrt{3}}{3} \frac{1}{2} \}$, $1 + \lambda \sigma_1 < 0$, and so Proposition [12] says nothing about the optimality of $\pi$ or $(\lambda, \sigma)$; in fact, for $\lambda = -\frac{1}{2} \sqrt{3} - \frac{1}{2}$, the corresponding $\pi$ is the global maximizer of $f$ on $X_e$. For $\lambda := \frac{1}{2} \sqrt{3} - \frac{1}{2} > 0$, $1 + \lambda \sigma_1 = \sqrt{3} > 0$, and so Proposition [12] says that $\pi = 2\sqrt{3} (\in X_e)$ is the global solution of $(P_1), and (\pi, \sigma) = \left( \frac{1}{4} \sqrt{3} - \frac{1}{2}; (0, 2) \right)$ is the global maximizer of $D$ on $T_{col}^+ = T^+ = \{(\lambda, \sigma) \in \mathbb{R}_+ \times \mathbb{R}^2 \mid 1 + \lambda \sigma_1 > 0\}$. For $\lambda = -1, 1 + \lambda \sigma_1 > 0$, but $(\lambda, \sigma)$ is not a local extremum of $D$, as easily seen taking $\sigma_0 := 0, (\lambda, \sigma_1) := (t - 1, t - 2)$ with $|t|$ sufficiently small.

When $Q = [0, m]$ problem $(P_J)$ reduces to the quadratic problem with equality and inequality quadratic constraints considered in [18] $(P_J)$, which is denoted here by $(P_J^q)$. Of course, in this case $X = X_0 = \mathbb{R}^n$, and so

$$\Xi(x, \lambda, \sigma) = L(x, \lambda) - \frac{1}{2} \sum_{k=0}^m \lambda_k \sigma_k^2 \quad (x \in \mathbb{R}^n, \lambda \in \mathbb{R}_m, \sigma \in \mathbb{R} \times \mathbb{R}^m)$$

with $\lambda_0 := 1$. It follows that

$$\nabla_x \Xi(x, \lambda, \sigma) = \nabla_x L(x, \lambda), \quad \nabla_{\sigma} \Xi(x, \lambda, \sigma) = - (\lambda_k \sigma_k)_{k \in \overline{0, m}},$$

$$\nabla_{\lambda} \Xi(x, \lambda, \sigma) = \nabla_{\lambda} L(x, \lambda) - \frac{1}{2} (\sigma_j^2)_{j \in \overline{1, m}} = (q_j(x) - \frac{1}{2} \sigma_j^2)_{j \in \overline{1, m}}.$$
Corollary 15 Let \((\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m\) be a J-LKKT point of \(L\).

(i) Then \(\lambda \in Y^+_\col := Y_{\col} \cap \Gamma J, \langle \lambda, \nabla_{\lambda} L(\tau, \lambda) \rangle = 0\), and \(q_0(\tau) = L(\tau, \lambda) = D_L(\lambda)\).

(ii) Assume that \(M_{\lambda}(\lambda) = \mathbb{I}, m\). Then \(\nabla_{\lambda} L(\tau, \lambda) = 0\), and so \((\tau, \lambda)\) is a critical point of \(L\), and \(\tau \in X^e \subset X_j \subset X_i\).

(iii) Assume that \(\lambda \in \mathbb{R}^m_{++}\) and \(A(\lambda) \succeq 0\). Then \(\tau \in X^e, \lambda \in Y^+_{\col}\) and

\[
q_0(\tau) = \inf_{x \in X_i} q_0(x) = L(\tau, \lambda) = \sup_{\lambda \in Y^+_{\col}} D_L(\lambda) = D_L(\lambda);
\]

moreover, if \(A(\lambda) > 0\) then \(\lambda \in Y^i\) and \(\tau\) is the unique global solution of problem \((P_i)\).

However, applying Proposition [11] we get assertion (i) and last part of assertion (ii) of [18, Prop. 9].

As seen in [18, Prop. 9] the most part of the results obtained by DY Gao and his collaborators for quadratic minimization problems are very far from those obtained studying directly those quadratic problems. In this sense it is worth quoting the following remark from the very recent Ruan and Gao’s paper [14]:

"Remark 1. As we have demonstrated that by the generalized canonical duality (32), all KKT conditions can be recovered for both equality and inequality constraints. Generally speaking, the nonzero Lagrange multiplier condition for the linear equality constraint is usually ignored in optimization textbooks. But it can not be ignored for nonlinear constraints. It is proved recently [26] that the popular augmented Lagrange multiplier method can be used mainly for linear constrained problems. Since the inequality constraint \(\mu \neq 0\) produces a nonconvex feasible set \(E^*_\mu\), this constraint can be replaced by either \(\mu < 0\) or \(\mu > 0\). But the condition \(\mu < 0\) is corresponding to \(y \circ (y - e_K) \geq 0\), this leads to a nonconvex open feasible set for the primal problem. By the fact that the integer constraints \(y_i(y_i - 1) = 0\) are actually a special case (boundary) of the boxed constraints \(0 \leq y_i \leq 1\), which is corresponding to \(y \circ (y - e_K) \geq 0\), we should have \(\mu > 0\) (see [8] and [12, 16]). In this case, the KKT condition (43) should be replaced by

\[
\mu > 0, \ y \circ (y - e_K) \leq 0, \ \mu^T [y \circ (y - e_K)] = 0. \quad (47)
\]

Therefore, as long as \(\mu \neq 0\) is satisfied, the complementarity condition in (47) leads to the integer condition \(y \circ (y - e_K) = 0\). Similarly, the inequality \(\tau \neq 0\) can be replaced by \(\tau > 0\)."²

In fact the positivity of the Lagrange multipliers \(\lambda_j\) is needed for recovering the Lagrangian \(L\) from \(\Xi\) [see (7)], while the non vanishing condition on \(\lambda_j\) is needed to get \(L(\tau, \lambda) = \Xi(\tau, \lambda, \sigma)\) and \(\nabla_\tau L(\tau, \lambda) = \nabla_\tau \Xi(\tau, \lambda, \sigma)\) when \(\nabla_\sigma \Xi(\tau, \lambda, \sigma) = 0\), as seen in Lemma [5]. Of course, such conditions are not needed in quadratic minimization problems, as observed after Corollary 15.

²The reference "[26]" mentioned in [14, Rem. 1] is the item [21] from our bibliography, the others being the following: "8. Fang, S.C., Gao, D.Y., Shue, R.L., Wu, S.Y.: Canonical dual approach to solving 0–1 quadratic programming problems. J. Ind. Manag. Optim. 4(4), 125–142 (2008)". "12. Gao, D.Y.: Solutions and optimality criteria to box constrained nonconvex minimization problem. J. Ind. Manag. Optim. 3(2), 293–304 (2007)"., and "16. Gao, D.Y., Ruan, N.: Solutions to quadratic minimization problems with box and integer constraints. J. Glob. Optim. 47, 463–484 (2010)"., respectively.
4 Relations with previous results

In this section we analyze results obtained by DY Gao and his collaborators in papers dedicated to constrained optimization problems. Because the quadratic problems (with quadratic constraints) are discussed in [18], we discuss only those constrained optimization problems with non quadratic objective function or with at least one non quadratic constraint. In the survey paper [4] (the same as [3]) there are mentioned the following papers: [6], [5], [7], [10] (with its preprint version [8]); besides these papers we add the retracted version [9] of [10], and [13].

A detailed discussion of [6] was done in [15]; we discuss the corrected versions [8]–[10] of [9] at the end of this section.

The problem considered by Gao, Ruan and Sherali in [5] is of type (P), that is J = ∅ with our notation, with f a quadratic function. Taking qj := 0 for j ∈ 1, m, our problem (P) is a particular case of the problem (P) from [5]. In this framework, that is V0(y) = y2, Vj ∈ Γsc, Λ0 := qj := 0 for j ∈ 1, m, with our notations, we mention only the following result of [5].

“Theorem 2 (Global Optimality Condition)”. Let (TU, X, σ) ∈ X × Rm+ × I* be a critical point of Ξ. If G(X, σ) ≥ 0, then (X, σ) is a global maximizer of D on T+col. TU is a global minimizer of f on X and f(TU) = min{x∈X, f(x) = max(λ, σ)∈T+ X D(λ, σ) = D(X, σ).

This theorem is false because in the mentioned conditions TU is not necessarily in X1, as Example [12] shows. Indeed, (6; 0; (0, 14 + 8.3)) is a critical point of Ξ, but 6 ∉ X1. It follows that also “Theorem 1 (Complementary-Dual Principle)” and “Theorem 3 (Triality Theory)” of [5] are false because (X, σ) = (0; (0, 14 + 8.3)) is a critical point of D (by Lemma [7], but the assertion “TU is a KKT point of (P)” is not true.

It is shown in [16] Ex. 6 that the “double-min or double-max” duality of [5] Theorem 3 (Triality Theory)], that is its assertion in the case G(X, σ) ⊥ 0, is also false.

The problem considered by Latorre and Gao in [7] is of type (P) in which Λk are quadratic and Vk are “differentiable canonical functions”. In our framework (which, apparently, is more restrictive) and with our notations, the following set is used in [7]:

S0 := {λ ∈ Rm | ∀j ∈ J : λj ≠ 0} ∩ {∀j ∈ J : λj ≥ 0} ⊂ ΛJ.

The motivation for defining S0 like this is given in the following text from [7] p. 1767: “From the second and third equation in the (10), it is clear that in order to enforce the constrain h(x) = 0, the dual variables μi must be not zero for i = 1, ..., p. This is a special complementarity condition for equality constraints, generally not mentioned in many textbooks. However, the implicit constraint μ ≠ 0 is important in nonconvex optimization. Let σ0 = (λ, μ). The dual feasible spaces should be defined as S0 ...” [3]

Besides the set S0 mentioned above, the following sets are also considered in [7]:

S1 := \prod_{k=0}^{m} dom V_k = I*, \quad S_a := T_{col} ∩ (S_0 × S_1), \quad S_a^+ := \{(λ, σ) ∈ T^+ | J ⊂ M_{a}(λ)\}.

In this context the main results of [7] are the following.

“Theorem 1 (Complementarity Dual Principle)”. Let (TU, X, σ) ∈ X × Rm+ × I* be a critical point of Ξ. Then TU is a J-KKT of (P), (X, σ) is a J-LKKKT point of D and f(TU) = Ξ(TU, X, σ) = D(X, σ).

[3]The emphasized text can be found also in [3] p. NP26] and [4] p. 33]. One must also observe that for Latorre and Gao μ ≠ 0 is equivalent to “μj ≠ 0 ∀j = 1, ..., p”, and (λ, μ) ∈ Rm×p if λ ∈ Rm and μ ∈ Rp.
“Theorem 2 (Global Optimality Conditions)”. Let \((\overline{\lambda}, \overline{\sigma}) \in X \times \mathbb{R}^m \times I^*\) be a critical point of \(\Xi \) with \((\lambda, \sigma) \in S^+_a\). If \(S^+_a\) is convex then \((\overline{\lambda}, \overline{\sigma})\) is the global maximizer of \(D\) on \(S^+_a\) and \(\overline{\pi}\) is the global minimizer of \(f\) on \(X_J\), that is \(f(\overline{\pi}) = \min_{x \in X_J} f(x) = \max_{(\lambda, \sigma) \in S^+_a} D(\lambda, \sigma) = D(\overline{\lambda}, \overline{\sigma})\).

Note first that it is not clear what is meant by \(J\)-LKKT point of \(D\) (called KKT point) in [7] Th. 1 when \((\overline{\lambda}, \overline{\sigma}) \notin T\). As in the case of [3] Th. 1, Example 1.4 shows that [7] Th. 1 is false because \((6; 0; (0, 14 + 8\sqrt{3}))\) is a critical point of \(\Xi\), but \(6 \notin X_I\) (\(= X_t\); even without assuming that \(S^+_a\) is convex in [7] Th. 2), for the same reason, this theorem is false.

Having in view that there are not nonempty open convex subsets \(C \subset \mathbb{R}^2\) such that the mapping \(C \ni (u, v) \mapsto uv \in \mathbb{R}\) is convex, the hypothesis that \(S^+_a\) is convex in the statement of [7] Th. 2 is very strong. Moreover, it is not clear how this hypothesis is used in the proof of [7] Th. 2.

The results established by Ruan and Gao in Sections 3 of [12] and [13] (which are practically the same) refer to \((P)\) in which \(q_k = 0, \Lambda_k\) are Gâteaux differentiable on their domains and \(V_k\) are “canonical functions” for \(k \in \overline{0, m}\). In our framework (which is more restrictive) and with our notations, the following sets are used in \([13]\):

\[ S_a := T_{col} \cap (\mathbb{R}^m_+ \times I^*), \quad S^+_a := \{ (\lambda, \sigma) \in T^* | M_{\neq}(\lambda) = \overline{1, m} \}. \]

In this context the results of [12] and [13] are we are interested in are the following.

Theorem 3. Let \((\overline{\pi}, \overline{\lambda}, \overline{\sigma}) \in X \times \mathbb{R}^m \times I^*\) be a KKT point of \(\Xi\). Then \(\overline{\pi}\) is a KKT of \((P)\), \((\overline{\lambda}, \overline{\sigma})\) is a KKT point of \(D\) and \(f(\overline{\pi}) = \Xi(\overline{\lambda}, \overline{\sigma}) = D(\overline{\lambda}, \overline{\sigma})\).

Theorem 4. Let \((\overline{\pi}, \overline{\lambda}, \overline{\sigma}) \in X \times \mathbb{R}^m \times I^*\) be a KKT point of \(\Xi\) with \((\overline{\lambda}, \overline{\sigma}) \in S^+_a\). If \(S^+_a\) is convex then \((\overline{\lambda}, \overline{\sigma})\) is a global maximizer of \(D\) on \(S^+_a\) and \(\overline{\pi}\) is a global minimizer of \(f\) on \(X_I\), that is \(f(\overline{\pi}) = \min_{x \in X_I} f(x) = \max_{(\lambda, \sigma) \in S^+_a} D(\lambda, \sigma) = D(\overline{\lambda}, \overline{\sigma})\).

As in [7] Th. 1, it is not clear what is meant by KKT point of \(D\) in [13] Th. 3 when \((\overline{\lambda}, \overline{\sigma}) \notin T\). As in the case of [3] Th. 1, Example 1.4 shows that [13] Th. 3 is false because \((6; 0; (0, 14 + 8\sqrt{3}))\) is a critical point of \(\Xi\), hence a KKT point of \(\Xi\), but \(6 \notin X_I\). In what concerns [13] Th. 4, because \(M_{\neq}(\lambda) = \overline{1, m}, (\pi, \lambda, \sigma)\) is a critical point of \(\Xi\) and \(\pi \in X_E\); moreover, in our framework (that is \(V_k \in \xi_{\text{sc}}\) for \(k \in \overline{0, m}\)), this theorem is true without assuming that \(S^+_a\) is convex. Notice that the proof of [13] Th. 4 is not convincing.

Morales-Silva and Gao in [8]–[10] consider the problem \((P)\) of minimizing \(\frac{1}{2} \| y - z \|^2\) for \(x := (y, z) \in Y_c \times Z_c\) with \(Y_c := \{ y \in \mathbb{R}^n \mid h(y) = 0 \}\) and \(Z_c := \{ z \in \mathbb{R}^m \mid h(z) = 0 \}\), where \(h(y) := \frac{1}{2} \langle (y, Ay) - r \rangle^2\) and \(h(z) := \frac{1}{2} \alpha \langle \frac{1}{2} \| z - c \|^2 - \eta \rangle^2 - \langle f, z - c \rangle\); here \(A \in \mathcal{S}_n\) is positive definite, \(c, f \in \mathbb{R}^m\) and \(\alpha, \eta, r \in (0, \infty)\) are taken such that \(h(z) > 0\) for every \(z \in Z_c\). Of

\[ 4\text{It is worth quoting DY Gao’s comment from }[1] \text{ p. } 19 \text{ on our remark from }[17] \text{ p. } 1783 \text{ that the proof of }[7] \text{ Th. 2} \text{ is not convincing: “Regarding the so-called “not convincing proof”, serious researcher should provide either a convincing proof or a disproof, rather than a complaint. Note that the canonical dual variables } \sigma_0 \text{ and } \sigma_1 \text{ are in two different levers (scales) with totally different physical units}\text{, it is completely wrong to consider } (\sigma_0, \sigma_1) \text{ as one vector and to discuss the concavity of } \Xi(x, (\cdot, \cdot)) \text{ on } S^+_a. \text{ The condition “} S^+_a \text{ is convex” in Theorem 2 [5] should be understood in the way that } S^+_a \text{ is convex in } \sigma_0 \text{ and } \sigma_1, \text{ respectively, as emphasized in Remark 1 [5]. Thus, the proof of Theorem 2 given in [5] is indeed convincing by simply using the classical saddle min-max duality for } (x, \sigma_0) \text{ and } (x, \sigma_1), \text{ respectively.”} \]

Note 14 from the text above is “Let us consider Example 1 in [5]. If the unit for \(x\) is the meter \((m)\) and for \(q\) is Kg/m, then the units for the Lagrange multiplier \(\mu\) (dual to the constraint \(g(x) = \frac{1}{2}(x^2 - d)^2 - e\)) should be Kg/m\(^3\) and for \(\sigma\) (canonical dual to \(\Lambda(x) = \frac{1}{2}x^2\)) should be Kg/m, respectively, so that each terms in \(\Xi_1(x, \mu, \sigma)\) make physical sense”; “[5]“ is our reference [14].
course, this problem is of type \((P_\alpha)\) for which Proposition 12 applies. Because [8] is the preprint version of [10], we refer mostly to [10] and [9] \(^3\) in [10] one considers the sets

\[ S_\alpha = \{ (\lambda, \mu, \varsigma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{V}_\alpha^*: (1 + \mu \varsigma)(I + \lambda A) - I \text{ is invertible} \}. \]

\[ S_\alpha^+ = \{ (\lambda, \mu, \varsigma) \in S_\alpha: I + \lambda A > 0 \} \]

where \( \mathcal{V}_\alpha^* := [-\alpha, \infty) \), and one states the following results:

"Theorem 1 (Complementary-dual principle). If \((x, \lambda, \mu, \varsigma)\) is a stationary point of \( \Xi \) such that \((\lambda, \mu, \varsigma) \in S_\alpha \) then \( x \) is a critical point of \( (P) \) with \( \lambda \) and \( \mu \) its Lagrange multipliers, \((\lambda, \mu, \varsigma)\) is a stationary point of \( P^d \) and \( \Pi(\lambda, \mu, \varsigma) = \Xi(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \). \( (17) \)

"Theorem 2. Suppose that \((\lambda, \mu, \varsigma) \in S_\alpha^+ \) is a stationary point of \( P^d \) with \( \mu \geq 0 \). Then \( \lambda, \mu, \varsigma \) is the only global minimizer of \( \Pi \) on \( X_c \), and \( \Pi(\lambda, \mu, \varsigma) = \min_{x \in X_c} \Pi(x) = \max_{(\lambda, \mu, \varsigma) \in S_\alpha^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \). \( (20) \)

Theorem 2.2 of [8] coincides with [10, Th. 1], while in Theorem 2.3 of [8] \("\Pi \geq 0\"\) and Eq. (20) from the statement of [10, Th. 2] are missing.

In [9], in the context of the problem \((P)\) above, one considers the sets

\[ \mathcal{S}_\alpha = \{ (\lambda, \mu, \varsigma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{V}_\alpha^*: \lambda \neq 0, \mu \neq 0, \det[(1 + \mu \varsigma)(I + \lambda A) - I] \neq 0 \}. \]

\[ \mathcal{S}_\alpha^+ = \{ (\lambda, \mu, \varsigma) \in \mathcal{S}_\alpha: I + \lambda A > 0 \} \]

With this new \( \mathcal{S}_\alpha \), [9, Th. 2] has the same statement as [10, Th. 1]; moreover, replacing \( \mathcal{S}_\alpha^+ \) in the statement of [10, Th. 2] one gets the statement of [9, Th. 3].

Notice that there is not a proof of the equality \( \max_{(\lambda, \mu, \varsigma) \in S_\alpha^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \) in [10], and there is not a proof of the equality \( \max_{(\lambda, \mu, \varsigma) \in S_\alpha^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \) in [9]. However, there is a "proof" of the equality \( \max_{(\lambda, \mu, \varsigma) \in S_\alpha^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \) from Theorem 2 of [6], even if \( \mathcal{S}_\alpha \) defined in [6, Eq. (16)] includes \( \mathcal{S}_\alpha^+ \) defined in [10, Eq. (19)]; see the discussion form [15, Sect. 2].\(^6\)

Setting \( g_1 := h \) and \( g_2 := g \), we have that \( Q = \{0, 1\} \) and \( J = \{1, 2\} \) in problem \((P)\) of [8–10]. Because \( Y_c \cap Z_c = \emptyset \) and taking into account \([15, Asserption II, p. 596]\)\(^7\) under the hypothesis of [10, Th. 1] one has \( \lambda \neq 0 \neq \mu \) and so \( M \neq (\lambda, \mu) = \{1, 2\} \). Using Proposition 12(ii) we obtain that \( (\lambda, \mu, \varsigma) \) is a critical point of \( L \) and \([10, Eq. (17)] \) holds. The conclusion of [9, Th. 3] is obtained using Proposition 12 (iii) [taking into account Corollary 9(ii)]. In what concerns [10, Th. 2], its conclusion follows using Proposition 11 (iii) because the condition \([Y_c \cap Z_c = \emptyset \wedge \mu \geq 0]\) imply \( \mu > 0 \), and so \( Q_0^\gamma = \{2\} \subset M(\lambda, \mu) \).

Below we show that the equality \( \max_{(\lambda, \mu, \varsigma) \in \mathcal{S}_\alpha^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \) from [10, Eq. (20)] is not true. For this consider \( n := 1, A := 1, r := \alpha := \gamma := c := 1 = f := \frac{\pi}{\sqrt{2}} \); this is a particular case \((\gamma := \frac{\pi}{\sqrt{2}})\) of the problem \((P)\) considered in [15]. In this situation (with the calculations and notations from [15]), the equation \( \varsigma^4 = 8\gamma^2(\varsigma + 1) \) has the solutions \( \varsigma := c_1 \in (-1, 0) \) (and so \( \varsigma > 0 \), and \( c_2 = 3 \). Taking \( \lambda := \frac{\pi}{2\gamma} > 0 \) and \( \mu := \frac{\pi}{2\gamma} > 0 \), we have that \( (\lambda, \mu, \varsigma) \) is a critical point of \( D = \Pi^d \); moreover, \( 1 + \lambda > 0 \) and \( (1 + \lambda)(1 + \mu) - 1 = \)

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\(^5\)Excepting [10], there are very few differences between the papers published in [2] and those having the same title from the retracted issue of the journal Mathematics and Mechanics of Solids dedicated to CDT.

\(^6\)In fact, we did not find a correct proof of the “min-max” duality (like Eq. (20) from [10]) in DY Gao’s papers in the case \( Q_0 \neq \emptyset \).

\(^7\)Referring to [9, Lem. 1], which is a reformulation of [15, Asserption II], the authors say: “The following Lemma is well known in mathematical programming (cf. Latorre and Gao [12] and Voisei and Zalinescu [13])”: the references “[12]” and “[13]” are our items [17] and [15], respectively. Of course, [9, Lem. 1] is not “well known in mathematical programming”, being very specific to the problem considered in [6]. The reference [17] is not mentioned in [10] with respect to [9, Lem. 1].
\( \mathbf{p}(\gamma + \varsigma) > 0 \), and so \((\lambda, \mathbf{p}, \varsigma) \in S_a^+, \) where \( S_a^+ \) is defined in [10] Eq. (19)). In fact, in the present case,

\[
D(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) = -\frac{\mu^2 (\lambda + 1) (\varsigma^2 + 2\varsigma^2 + \gamma^2) + \mu \lambda (\varsigma^2 + \varsigma \lambda + 2\varsigma - 2\gamma) + \lambda^2}{2 (\lambda + \varsigma \mu + \varsigma \lambda \mu)}
\]

for all \((\lambda, \mu, \varsigma) \in S_a^+ \). Applying [10] Th. 2] we must have that \( \max_{(\lambda, \mu, \varsigma) \in S_a^+} \Pi^d(\lambda, \mu, \varsigma) = \Pi^d(\lambda, \mu, \varsigma) \). However, this is not possible because \( \sup_{(\lambda, \mu, \varsigma) \in S_a^+} \Pi^d(\lambda, \mu, \varsigma) = \infty \). Indeed, there exists \( \varsigma < 0 \) such that \( \nu := \varsigma^3 + 2\varsigma^2 + \gamma^2 < 0 \). Then \((\lambda, \mu, \varsigma) \in S_a^+ \) for every \( \mu < 0 \) because \( 1 + \lambda > 0 \) and \((1 + \lambda)(1 + \varsigma \mu) - 1 \geq (1 + \lambda) - 1 = \lambda > 0 \). It follows that

\[
D(\lambda, \mu, \varsigma) = \frac{-\mu^2 (\lambda + 1) (\varsigma^2 + 2\varsigma^2 + \gamma^2) + \mu \lambda (\varsigma^2 + \varsigma \lambda + 2\varsigma - 2\gamma) + \lambda^2}{2 (\lambda + \varsigma \mu + \varsigma \lambda \mu)} \quad \forall \mu < 0,
\]

and so

\[
\sup_{(\lambda, \mu, \varsigma) \in S_a^+} \Pi^d(\lambda, \mu, \varsigma) \geq -\lim_{\mu \to -\infty} \frac{\mu^2 (\lambda + 1) \nu + \mu \lambda (\varsigma^2 + \varsigma \lambda + 2\varsigma - 2\gamma) + \lambda^2}{2 (\lambda + (\lambda + 1) \varsigma \mu)} = -\lim_{\mu \to -\infty} \frac{\nu}{\varsigma} = \infty.
\]

In [15] we provided an example with \( n = 2 \) for which the solution(s) of problem \((P)\) from [6] (which clearly always exists) cannot be obtained (found) using [6] Th. 2; we concluded that “the consideration of the function \( \Xi \) is useless, at least for the problem studied in [3]”.

In [8]–[10] the authors sustain that this is caused by the non uniqueness of the solution of problem \((P)\) from our example, but a solution can be obtained, even in such a case, by perturbation: “The combination of the perturbation and the canonical duality theory is an important method for solving nonconvex optimization problems which have more than one global optimal solution (see also [15]).”

In fact, the same example given in [15] but for \( n = 1 \) shows that even the results from [8]–[10] do not provide the global solution of problem \((P)\). Indeed, as in [15] p. 600), take \( \gamma := \sqrt{6}/96 \); because \( n = 1 \), we have that \( c = 1 \in \mathbb{R} \). Then the critical points of \( \Xi \) with \((\lambda, \mu, \varsigma) \in S_a \) are, as indicated in [15] p. 600], the following:

\[
(\lambda_1, \mu_1, \varsigma_1) := (1, 1 + \frac{1}{2} \sqrt{6}, \frac{1}{2} \sqrt{6}, \frac{6}{13}, -\frac{1}{4}),
\]

\[
(\lambda_2, \mu_2, \varsigma_2) := (-1, 1 + \frac{1}{2} \sqrt{6}, -2 - \frac{1}{2} \sqrt{6}, \frac{16}{13} (3 + 2 \sqrt{6}), -\frac{1}{4})
\]

\[
(\lambda_3, \mu_3, \varsigma_3) := (1, 2.603797322, 1.603797322, -3.701325488, 0.28608292399),
\]

\[
(\lambda_4, \mu_4, \varsigma_4) := (-1, 2.603797322, -3.603797322, -8.317027781, 0.28608292399).
\]

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8The reference “[3]” is the item [9] from our bibliography.

9The text is quoted from [10] p. 370]; here the reference “[15]” is “Wu, C., Gao, D.Y.: Canonical primal-dual method for solving nonconvex minimization problems. In: Gao, D.Y., Latorre, V., Ruan, N. (eds.) Advances in Canonical Duality Theory. Springer, Berlin”. Note that the same text can be found in [8] p. 9 without any reference, as well as in [9] p. NP236], where the indicated reference is “Wu, C. Li, C. and Gao, D.Y. Canonical primal-dual method for solving nonconvex minimization problems. arXiv:1212.6492 2012.” Observe that the main difference between arXiv:1212.6492 and reference “[15]” of [10] consists in the list of the authors, the content being practically the same.
Using Corollary \[9\] we have that \((\bar{\lambda}_i, \bar{\mu}_i, \bar{\varsigma}_i)\) with \(i \in \{1, 4\}\) are the only critical points of \(D (= \Pi^d)\). For \(i \in \{1, 3\}\) we have that \((1 + \bar{X}_i)(1 + \bar{\mu}_i\bar{\varsigma}_i) - 1 < 0\), while for \(i \in \{2, 4\}\) we have that \(1 + \bar{X}_i < 0\) and so \((\bar{X}_i, \bar{\pi}_i, \bar{\sigma}_i) \notin S^+_a (S^+_a \text{ defined in } [8\text{ Eq. (18)}] \text{ and } [10\text{ Eq. (19)}])\) and \((\bar{X}_i, \bar{\pi}_i, \bar{\sigma}_i) \notin S^+_c (S^+_c \text{ defined in } [9\text{ Eq. (30)}])\). Therefore, the unique solution \((1, 1 + \frac{1}{2}\sqrt{6})\) of problem \((P)\) is not provided by either \([8\text{ Th. 2.3}], \text{ or } [10\text{ Th. 2}], \text{ or } [9\text{ Th. 3}]\). The use of the perturbation method suggested in these papers is useless for this example.

5 Conclusions

- We provided a rigorous treatment (study) for constrained minimization problems using the Canonical duality theory developed by DY Gao.
- Proposition \[6\] shows that the so-called perfect duality holds under quite mild assumptions on the data of the problem; however, in our opinion this formula is not very useful because for the found element \((\bar{\pi}, \bar{X}, \bar{\sigma})\), \(\bar{\pi}\) could not be feasible for the primal problem and/or \((\bar{X}, \bar{\sigma})\) could not be feasible for the dual problem.
- Proposition \[12\] and Remark \[13\] show that even if CDT can be used for equality and/or inequality constrained optimization problems, it is more appropriate for problems with inequality constraints.
- The most important drawback of CDT is that it could find at most those solutions of the primal problem for which all non quadratic constraints are active; even more, the Lagrange multipliers corresponding to non quadratic constraints must be strictly positive.
- Moreover, the solutions found using CDT are among those found using the usual Lagrange multipliers method. Using the “extended Lagrangian” \(\Xi\) could be useful to decide if the found \(\bar{\pi}\) is a global minimizer of the primal problem.
- The consideration of the dual function \(D\) does not seem to be useful for constrained minimization problems with at least one non quadratic constraint because \(D\) is not concave, unlike the case of quadratic constraints.

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