TORSION-FREE ALUFFI ALGEBRAS

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Abstract. A pair of ideals $J \subseteq I \subseteq R$ has been called Aluffi torsion-free if the Aluffi algebra of $I/J$ is isomorphic with the corresponding Rees algebra. We give necessary and sufficient conditions for the Aluffi torsion-free property in terms of the first syzygy module of the form ideal $J^*$ in the associated graded ring of $I$. For two pairs of ideals $J_1, J_2 \subseteq I$ such that $J_1 - J_2 \in I^2$, we prove that if one pair is Aluffi torsion-free the other one is so if and only if the first syzygy modules of $J_1$ and $J_2$ have the same form ideals. We introduce the notion of strongly Aluffi torsion-free ideals and present some results on these ideals.

Introduction

P. Aluffi in ([1]) to describe characteristic cycles of a hypersurface parallel to well known conormal cycle in intersection theory introduces an intermediate graded algebra between the symmetric algebra of an ideal and the corresponding Rees algebra. The first author and A. Simis ([12]) called such an algebra, the Aluffi algebra. Given a Noetherian ring $R$ and ideals $J \subset I$ of $R$, the Aluffi algebra of $I/J$ is defined by

$$\mathcal{A}_{R/J}(I/J) := \text{Sym}_{R/J}(I/J) \otimes_{\text{Sym}_R(I)} \mathcal{R}_R(I).$$

The Aluffi algebra is squeezed as $\text{Sym}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ and moreover it is a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$. The kernel of the right hand surjection so called the module of Valabrega-Valla as defined in ([17]), is the torsion of the Aluffi algebra. Thus the Rees algebra of $I/J$ is the Aluffi algebra modulo its torsion provided that $I$ has a regular element modulo $J$.

It is reasonable to ask when the the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism? Geometrically, this question is important from two points of view. More precisely, the blowup of $X = \text{Spec} (R/J)$ along the closed subscheme $Y$ defined by the ideal $I/J$ is equal to Proj $(\mathcal{A}_{R/J}(I/J))$. Hence to find the equations of the blowup of $X$ along $Y$, we just need to find the equations of the blowup of ambient space $R$ along $X$. For the other one, let $x \in X = \text{Spec} (R/J)$ be a point then the tangent cone of $X$ at $x$ is the cone $\text{Spec} (\text{gr}_m(O_{X,x}))$, where $m_x$ is the maximal ideal of the local ring $O_{X,x} \simeq R_{p_x}/JR_{p_x}$. So that we may assume $O_{X,x}$ is the quotient of a regular local ring $(R, m)$ with respect of an ideal $J$. Then the associated graded ring $\text{gr}_m(R/J)$ of $m/J$ is isomorphic to $\text{gr}_m(R)/J^*$ where $J^*$ is the form ideal. The problem of determining elements $f_1, \ldots, f_t$ such that their initial forms $f_1^*, \ldots, f_t^*$
generate $J^*$ is an essential problem in resolution of singularities. Also the torsion-free Aluffi algebras are crucial in intersection theory of regular and linear embedding ([4], [10]). The outline of this paper is as the follow.

In section 1, we give necessary and sufficient conditions for torsion-free Aluffi algebra, involving the standard base (in the sense of Hironaka ([7])) and the first syzygy module of the form ideal in the associated graded ring.

Let $J \subseteq I$ be ideals in the ring $R$. We say that the pair $J \subseteq I$ is Aluffi torsion-free if $J \cap I^n = JJ^{n-1}$ for all $n \geq 1$. In section 2, we study the behavior of the Aluffi torsion-free property with respect to contraction and extension. We prove that the sum of two Aluffi torsion-free ideals is Aluffi torsion-free if and only if one of them modulo the other is Aluffi torsion-free. As the main result of this section, we prove that if $J_1, J_2 \subseteq I$ such that $J_1 \equiv J_2$ modulo $I^2$ and $J_1 \subseteq I$ is Aluffi torsion-free then $J_2 \subseteq I$ is Aluffi torsion-free if and only if the first syzygy modules of $J_1$ and $J_2$ have the same form ideals in the associated graded module $\text{gr}_I(R^m)$ where $m$ is the number of generators of $J_1$ and $J_2$ (Theorem 2.6). In sequel, we introduce the notion of strongly Aluffi torsion-free ideals. A pair $J = (f_1, \ldots, f_t) \subseteq I \subseteq R$ is called strongly Aluffi torsion-free if $J_i = (f_1, \ldots, f_i)$ is Aluffi torsion-free for $i = 1, \ldots, t$.

We give an example of Aluffi torsion-free pair of ideals which is not strongly Aluffi torsion-free. In the case that, $J \subseteq I$ is Aluffi torsion-free, we give a criterion for strongly Aluffi torsion-freeness. We close the section with this result: let $J_1, J_2 \subseteq I$ be ideals in the ring $R$ such that the extension of $J_2$ and $I$ in the ring $R/J_1$ is Aluffi torsion-free. If there exists a minimal generating set $f_1, \ldots, f_t$ of $J_1$ such that $J_1 \subseteq I$ is strongly Aluffi torsion-free and extension of the sequence $f_1, \ldots, f_t$ in $R/J_2$ is regular then $J_2 \subseteq I$ is Aluffi torsion-free.

In section 3, we focus on the case that $J$ is an ideal in the polynomial ring $R = k[x_0, \ldots, x_n]$ over a field $k$ of characteristic zero and the ideal $I$ stands for the Jacobian ideal of $J$ which describe the singular subscheme of $\text{Spec}(R/J)$. We prove that if $J$ is the ideal of a monomial curve with some special parametrization or $J$ is the square-free Veronese ideal of degree $r$, then $J \subseteq I$ is Aluffi torsion-free. We close the paper with a question related to Aluffi torsion-freeness of free line arrangements.

1. The Aluffi algebra and its torsion

Throughout this section $R$ will be a Notherian ring. Let $J \subseteq I \subseteq R$ be ideals. There are two important algebras related to these data. The first one is the Symmetric algebra $\text{Sym}_R(I)$ and the second one is the Rees algebra, $\mathcal{R}_R(I) = \bigoplus_{n \geq 0} I^n t \subset R[t]$. It is well-known that there is a natural surjective $R$–algebra homomorphism $\text{Sym}_R(I) \twoheadrightarrow \mathcal{R}_R(I)$. By functorial property of Symmetric algebra there is an other surjection $\text{Sym}_R(I) \twoheadrightarrow \text{Sym}_{R/J}(I/J)$. The (embebed) Aluffi algebra is defined by

$$\mathcal{A}_{R/J}(I/J) = \text{Sym}_{R/J}(I/J) \otimes_{\text{Sym}_R(I)} \mathcal{R}_R(I).$$

By [12, Lemma 1.2], there are $R$-algebra isomorphisms

$$\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_R(I)/(J, \bar{J})\mathcal{R}_R(I) \simeq \bigoplus_{n \geq 0} I^n/JI^{n-1},$$
where $J$ is in degree zero and $\bar{J}$ is in degree 1. The Rees algebra of $I/J$ is

\[ \mathcal{R}_{R/J}(I/J) \cong \bigoplus_{n \geq 0} I^n/J \cap I^n. \]

Then there is a surjective $R$-algebra homomorphism $A_{R/J}(I/J) \rightarrow \mathcal{R}_{R/J}(I/J)$. The kernel of the above surjection is the homogeneous ideal

\[ \mathcal{W}_{\bar{I} \subset I} := \bigoplus_{n \geq 2} J \cap I^n/JI^{n-1}, \]

which is called the module of Valabrega-Valla. If $J$ has a regular element modulo $I$, the Valabrega-Valla’s module is the $R/J$-torsion of the Aluffi algebra [12, Proposition 2.5]. The module of Valabrega-Valla has close relation to the theory of standard base ([7]). To make a further development on this relation, we recall some facts about the filtered rings and modules.

A filtration on the ring $R$ is a decreasing sequence of ideals $\{F_n R\}_{n \geq 0}$ satisfying $(F_n R)(F_m R) \subseteq F_{n+m} R$ for all $n, m \geq 0$. The pair $(R, F_n R)$ is called a filtered ring. For an ideal $I$ of a ring $R$ there is the $I$-adic filtration $F_n R = I^n$. A morphism of filtered rings $\varphi : (R, F_n R) \rightarrow (S, F_n S)$ is a homomorphism of rings $\varphi : R \rightarrow S$ such that $\varphi(F_n R) \subseteq F_n S$ for all $n \geq 0$.

Let $(R, F_n R)$ be a filtered ring and $M$ a $R$-module. A filtration on the module $M$ is a decreasing sequence $\{F_n M\}_{n \geq 0}$ of submodules of $M$ such that $(F_n R)(F_n M) \subseteq F_{n+m} M$ for all $m, n \geq 0$. The pair $(M, F_n M)$ is called a filtered $(R, F_n R)$-module. A morphism of filtered $(R, F_n R)$-modules $\varphi : (M, F_n M) \rightarrow (N, F_n N)$ is a $R$-module homomorphism $\varphi : M \rightarrow N$ such that $\varphi(F_n M) \subseteq F_n N$ for all $n \geq 0$. This implies $\varphi(F_n M) \subseteq \varphi(M) \cap F_n N$. The morphism $\varphi$ is called strict if $\varphi(F_n M) = \varphi(M) \cap F_n N$. If $M$ is a $R$-module then $(M, F_n M)$ with $F_n M := (F_n R)M$ is a filtered $(R, F_n R)$-module. A sequence of filtered $(R, F_n R)$-modules is exact if the sequence of underlying $R$-modules is exact. It is called strict if all morphisms are strict.

**Remark 1.** Let $(M, F_n M)$ be a filtered $(R, F_n R)$ module.

(a) Let $\varphi : L \rightarrow M$ an injective homomorphism of $R$-modules. For all $n \geq 0$, put $F_n L := \varphi^{-1}(F_n M)$. This makes $(L, F_n L)$ into a filtered module and $\varphi : (L, F_n L) \rightarrow (M, F_n M)$ is a strict morphism.

(b) Let $\varphi : M \rightarrow N$ be a surjective homomorphism of $R$-modules. For all $n \geq 0$ put $F_n N := \varphi(F_n M)$. This makes $(N, F_n N)$ into a filtered module and $\varphi : (M, F_n M) \rightarrow (N, F_n N)$ is a strict morphism.

The associated graded ring of a filtration $(R, F_n R)$ is $\gr(R) = \bigoplus_{n \geq 0} F_n R/F_{n+1} R$. We denote by $\gr_I(R)$ for the $I$-adic filtration. If $(M, F_n M)$ is a filtered module, its associated graded module $\gr(M) = \bigoplus_{n \geq 0} F_n M/F_{n+1} M$ is the graded $\gr(R)$-module. In the case of $I$-adic filtration we write $\gr_I(M)$. It is clear that $\gr(\cdot)$ is a functor form the category of filtered modules to the category of graded modules.

**Proposition 1.1 ([5], I Proposition 2.1).** Let $(R, F_n R)$ be a filtered ring and

\[ (L, F_n L) \xrightarrow{\varphi} (M, F_n M) \xrightarrow{\phi} (N, F_n N), \]
a strict exact sequence of filtered \((R, F_n R)\)-modules. Then the induced sequence 
\[
gr(L) \xrightarrow{\gr(\phi)} \gr(M) \xrightarrow{\gr(\psi)} \gr(N)
\]
is an exact sequence of \(\gr(R)\)-modules.

If \(m \in M\) we denote by \(\nu_F(m)\) the largest integer \(n\) such that \(m \in F_n M\). If such \(n\) does not exist we say \(\nu_F(m) = \infty\) and if \(\nu_F(m) < \infty\), we denote by \(m^*\) the residue class of \(m\) in \(F_{\nu_F(m)} M / F_{\nu_F(m)+1} M\), which is called the initial form of \(m\). If \(\nu_F(m) = \infty\), then we set \(m^* = 0\). For \(m_1, m_2 \in M\) if \(m_1^* + m_2^* \neq 0\), then \(m_1^* + m_2^* = (m_1 + m_2)^*\). If the filtration \(F_n M\) is multiplicative then \(\gr(M)\) is a ring and if \(m_1^* m_2^* \neq 0\) then \((m_1 m_2)^* = m_1^* m_2^*\).

Let \(R\) be a ring and \(J \subseteq I\) ideals of \(R\). Given an element \(f \in R\), we denote by \(\nu = \nu_I(f)\) the number \(\nu_I(f)\) with \(F_n R = I^n\). We denote by \(J^*\) the homogeneous ideal of \(\gr_f(R)\) generated by the initial forms of the elements of \(J\). A set of generators \(\{f_1, \ldots, f_t\}\) of \(J\) is called \(I\)-standard base if \(J^* = (f_1^*, \ldots, f_t^*)\). When \(R\) is local, then an \(I\)-standard base of \(J\) is a generating set \([7, \text{Lemma } 6]\).

The following remark give necessary and sufficient conditions for the surjection \(A_{R/J}(I/J) \to R_{R/J}(I/J)\) to be an isomorphism.

**Remark 2.** Let \(J \subseteq I \subseteq R\) be ideals of the local ring \(R\). By \([16, \text{Theorem } 1.1]\) the following are equivalent.

(a) \(A_{R/J}(I/J) \simeq R_{R/J}(I/J)\).
(b) \(J \cap I^n = IJ^{n-1}\) for any \(n \geq 1\).
(c) \(I^{n+1} \cap J^{n-1} = I^n\) for any \(n \geq 1\).
(d) There exists a minimal set of generators \(f_1, \ldots, f_t\) of \(J\) such that \(\{f_1, \ldots, f_t\}\) is an \(I\)-standard base of \(J\) and \(\nu_I(f_i) = 1\) for \(i = 1, \ldots, t\).

Let now \(J = (f_1, \ldots, f_t)\) and \(\nu_I(f_i) = 1\). Consider the exact sequence
(1) \[0 \to Z \xrightarrow{i} R^t \xrightarrow{f} R \xrightarrow{\pi} R/J \to 0,\]
where \(f(a_1, \ldots, a_t) = \sum_{i=1}^t a_i f_i\) and \(Z = \text{Syz}(J)\) is the first syzygy module of \(J\). By Remark (1), we consider the following filtrations

\[F_n R^t = \bigoplus_{i=1}^t I_n^{i-1}, \quad F_n Z = F_n R^t \cap Z, \quad F_n R/J = (I/J)^n,\]

which make \(i, f\) and \(\pi\) the morphisms of filtered \((R, F_n R = I^n)\)-modules. Note that \(i\) and \(\pi\) are strict. By Proposition \((1.1)\), we get the corresponding complex of graded modules
\[0 \to \gr I(Z) \xrightarrow{\gr(i)} \gr I(R^t) \xrightarrow{\gr(f)} \gr I(R) \xrightarrow{\gr(\pi)} \gr I/(R/J) \to 0.\]

Note that \(\gr I(R^t) = \bigoplus_{i=1}^t \gr I(R)(-1)\). The map \(\gr(f)\) is defined by \(e_i \mapsto f_i^*\), the map \(\gr(i)\) is inclusion and \(\gr(\pi)\) is surjective. We have
\[\ker(\gr(\pi)) = \bigoplus_{n \geq 0} (I^{n+1} + J \cap I^n)/I^{n+1} = J^*, \quad \ker(\gr(f)) = \text{Syz}(J^*).\]

Given an element \((a_1, \ldots, a_t) \in Z\), if \((a_1, \ldots, a_t) \neq (0, \ldots, 0)\) there exist \(m \geq 0\) such that \(\nu_{F_n Z}(a_1, \ldots, a_t) = m\), this means that \(\nu_I(a_j) \geq m-1\) for every \(i\) and there exists \(j \in \{1, \ldots, t\}\) such that \(\nu_I(a_j) = m-1\). Hence \(\psi : Z \to \gr I(Z)\) is the canonical
map which associates to every element of \( \mathcal{Z} \) its initial form in \( \text{gr}_I(\mathcal{Z}) \), that is, 
\[
\psi(a_1, \ldots, a_t) = \overline{(a_1, \ldots, a_t)} \in F_m \mathcal{Z}/F_{m+1} \mathcal{Z}.
\]
On the other hand, since the sequence \((*)\) is a complex hence there is a canonical embedding \( \varphi : \text{gr}_I(\mathcal{Z}) \rightarrow \text{Syz}(J^*) \) which sends every element \((a_1, \ldots, a_t) \in \mathcal{Z} \cap F_m R^l / \mathcal{Z} \cap F_{m+1} R^l \) to \((\overline{a_1}, \ldots, \overline{a_t})\) where \(\overline{a_i}\) is the residue class of \(a_i\) in \(I_{m-I}/I_m\). Therefore, we get a map 
\[
\varphi \circ \psi : \mathcal{Z} \rightarrow \text{Syz}(J^*), \quad \varphi \circ \psi((a_1, \ldots, a_t)) = (\overline{a_1}, \ldots, \overline{a_t}).
\]

Note that \(\overline{a_i} = a_i^*\) if \(\nu_I(a_i) + 1 = \min_j \{\nu_I(a_j) + 1\}\) and \(\overline{a_i} = 0\) if \(\nu_I(a_i) + 1 > \min_j \{\nu(a_j) + 1\}\). The following theorem relate the torsion of the Aluffi Algebra to the first syzygy module of the form ideal \(J^* \subseteq \text{gr}_I(R)\).

**Theorem 1.2.** Let \(J = (f_1, \ldots, f_t) \subseteq I\) be ideals in the local ring \(R\). The following are equivalent.

(a) \(\mathcal{A}_{R/I}(I/J) \simeq \mathcal{R}_{R/I}(I/J)\).
(b) The complex \((*)\) is exact.
(c) There exist a homogeneous system of generators of \(\text{Syz}(J^*)\), whose elements can be lifted to elements of \(\mathcal{Z}\) via \(\varphi \circ \psi\).

**Proof.** First note that the Aluffi algebra is torsion-free if and only if the map \(f\) in the sequence \((1)\) is strict. Thus (a) implies (b) by Proposition \((1.1)\). Assume that the complex \((*)\) is exact. Then by above \(\text{gr}_I(\mathcal{Z}) = \text{Syz}(J^*)\) which yields (c). Finally, we prove that (c) implies (a). The map \(\Theta : \text{Syz}(J^*) \rightarrow \text{gr}_I(\mathcal{Z})\) is inverse of \(\varphi\) which is defined by sending an element \(s \in \text{Syz}(J^*)\) to \(\psi(a_1, \ldots, a_t)\) where \(\varphi \circ \psi((a_1, \ldots, a_t)) = s\). Hence \(\text{gr}_I(\mathcal{Z}) \simeq \text{Syz}(J^*)\). The latter implies that \(J \cap I^n = J I^{n-1}\) for all \(n \geq 1\). In fact, let \(b \in J \cap I^n = f(R^l) \cap F_n R\), then \(b = f((a_1, \ldots, a_t))\) with \((a_1, \ldots, a_t)\) belonging to some \(F_m R^l\). If \(m \geq n\), we get the assertion. If \(m < n\), by using the exactness of \(0 \rightarrow \text{gr}_I(\mathcal{Z}) \xrightarrow{\text{gr}(i)} \text{gr}_I(R^l)\), we get 
\[
(a_1, \ldots, a_n) \in f^{-1}(F_{m+1} R) \cap F_m R^l = F_m \mathcal{Z} + F_{m+1} R^l.
\]
Hence \(b = f((c_1, \ldots, c_t))\) with \((c_1, \ldots, c_t) \in F_{m+1} R^l\). Repeating this argument finitely many times, finally we get \(b \in f(F_n R^l) = J I^{n-1}\) which complete the proof. \(\square\)

2. **Aluffi torsion-free ideals**

In this section we assume that all rings are Noetherian. Let \(J \subseteq I\) be ideals in the ring \(R\). If \(J \subseteq I\) satisfy in one of the equivalent conditions in the Remark \((2)\) or the Theorem \((1.2)\) then the Aluffi algebra is torsion-free. Therefore we have the following definition.

**Definition 2.1.** A pair of ideals \(J \subseteq I\) in the ring \(R\) is called Aluffi torsion-free if \(J \cap I^n = JI^{n-1}\) for all \(n \geq 1\).

**Example 2.2.** There are well-known examples of Aluffi torsion-free ideals.

1. If \(I/J\) in \(R/J\) is of linear type (e.g., if \(I\) is generated by regular or, more generally by a \(d\)-sequence modulo \(J\) in the sense of Huneke ([8])) then \(J \subseteq I\) is Aluffi torsion-free.
(2) If $J$ is generated by superficial sequence in $I$ then the pair $J \subseteq I$ is Aluffi torsion-free [9, Lemma 8.5.11].

The following result indicate to the behavior of Aluffi torsion-free property with respect to extension and contraction. In particular, it shows that the Aluffi torsion-free property is local.

**Proposition 2.3.** Let $J \subseteq I$ be ideals in the ring $R$. The following statements hold:

(a) Let $a \subseteq J$ be another ideal. If $J \subseteq I$ is Aluffi torsion-free then $\overline{J} \subseteq \overline{I}$ is Aluffi torsion-free in $\overline{R} = R/a$.

(b) Let $R \longrightarrow S$ be a flat homomorphism of rings. If $J \subseteq I$ is Aluffi torsion-free then $JS \subseteq IS$ is Aluffi torsion-free in $S$.

(c) Let $R \longrightarrow S$ be a faithfully flat homomorphism of rings. If the extension of ideals $J \subseteq I$ in $S$ is Aluffi torsion-free then $J \subseteq I$ is Aluffi torsion-free in $R$. In particular, Assume that $(R, \mathfrak{m})$ is local. If the extension of $J$ and $I$ in the $\mathfrak{m}$-adic completion $\hat{R}$ is Aluffi torsion-free then so does $J \subseteq I$.

(d) The ideal $J \subseteq I$ is Aluffi torsion-free if and only if $JR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}}$ is Aluffi torsion-free for every maximal ideal $\mathfrak{m}$ of $R$.

**Proof.** We prove (a), (b) and (c) by straightforward computations. We have

$$\overline{J} \cap \overline{I}^n = \overline{J \cap I^n} = J \cap \overline{I^n} = \overline{JI^{n-1}} = J \overline{I^{n-1}},$$

which proves (a). For (b), we have

$$JS \cap (IS)^n = JS \cap I^nS = (J \otimes_R S) \cap (I^n \otimes_R S) = (J \cap I^n) \otimes_R S = (JI^{n-1}) \otimes_R S = (JI^{n-1}S) = (JS)(I^{n-1}S).$$

(c). As $S$ is faithfully flat over $R$, $IS \cap R = I$ for all ideals $I$ of $R$. We have

$$J \cap I^n = (J \cap I^n)S \cap R \subseteq (JS \cap I^nS) \cap R = (JI^{n-1}S) \cap R = JI^{n-1}.$$

The second assertion yields from the fact that $R \longrightarrow \hat{R}$ is faithfully flat. The part (d) follow from part (b) and local-global property. □

**Remark 3.** There is a natural question. What is the behavior of Aluffi torsion-free property with respect to operation of ideals? Here are some easy facts about this question.

(1) The sum of two Aluffi torsion-free ideals is not Aluffi torsion-free (see Proposition 2.4).

(2) The product and intersection of two Aluffi torsion-free need not to be Aluffi torsion-free. In $k[x, y, z]$, consider the ideals $J_1 = (xy)$ and $J_2 = (yz) \subseteq I = (x, y, z)^2$ which are Aluffi torsion-free, but $J_1J_2 = (xy^2z), J_1 \cap J_2 = (xyz) \subseteq I$ are not Aluffi torsion-free.

**Proposition 2.4.** Let $J_1, J_2 \subseteq I$ be Aluffi torsion-free ideals in the ring $R$. Then $J_1 + J_2 \subseteq I$ is Aluffi torsion-free if and only if $\overline{J_1} \subseteq \overline{J} \subseteq \overline{R} = R/J_2$ is Aluffi torsion-free.
Proof. Assume that $J_1 + J_2 \subseteq I$ is Aluffi torsion-free. For all $n \geq 1$ we have
\[
\frac{\overline{J}_1 \cap I^n}{J_2} = \frac{(J_1 + J_2) \cap I^n + J_2}{J_2} = \frac{(J_1 + J_2)I^{n-1} + J_2}{J_2} = \overline{J}_1 I^{n-1}.
\]
For the converse, let $J_1 = (f_1, \ldots, f_t)$ and $y \in (J_1 + J_2) \cap I^n$, then $y = a + c$ with $a \in J_1$ and $c \in J_2$. If $a = 0$ we are done, if not we get $a \in \overline{J}_1 \cap I^n = \overline{J}_1 I^{n-1}$, then $a = \sum_{i=1}^t g_i f_i + d$ with $g_i \in I^{n-1}$ and $d \in J_2$. It follows that $y = c + d + \sum_{i=1}^t g_i f_i$, where $\sum_{i=1}^t g_i f_i \in J_1 \cap I^n$ and $c+d \in J_2 \cap I^n$. Hence $(J_1 + J_2) \cap I^n \subseteq (J_1 \cap I^n) + (J_2 \cap I^n)$. Since $J_1, J_2 \subseteq I$ are Aluffi torsion-free, one has
\[
(J_1 + J_2) \cap I^n \subseteq (J_1 \cap I^n) + (J_2 \cap I^n) = J_1 I^{n-1} + J_2 I^{n-1} = (J_1 + J_2) I^{n-1}.
\]
\[
\Box
\]

Proposition 2.5. Let $R$ be a local ring, $J_1, J_2 \subseteq R$ two ideals and $I = J_1 + J_2$. Assume that $J_1$ is generated by elements not in $I^2$. The following are equivalent.

(a) The pair $J_1 \subseteq I$ is Aluffi torsion-free.
(b) $J_1 \cap J_2^n = J_1 I^{n-1}$ for all $n \geq 1$.
(c) $\text{gr}_I R/J_1^* \simeq \text{gr}_{I/J_1}(R/J_1)$.

Proof. Let $n$ be a positive integer, we have
\[
J_1 \cap I^n = J_1 \cap (J_1 + J_2)^n = J_1 \cap (J_1^n, J_1^{n-1} J_2, \ldots, J_1 J_2^{n-1}, J_2^n)
= J_1 \cap (J_1^n, J_1^{n-1} J_2, \ldots, J_1 J_2^{n-1}) + J_1 \cap J_2^n
= J_1 (J_1 + J_2)^{n-1} + J_1 \cap J_2^n
= J_1 I^{n-1} + J_1 \cap J_2^n,
\]
which prove the equivalence of (a) and (b).

The associated graded ring $\text{gr}_{I/J_1}(R/J_1)$ is isomorphic to
\[
R/I \oplus I/(I^2, J_1) \oplus I^2/(I^3 + I^2 \cap J_1) \oplus \cdots.
\]
Since $J_1^*$ is generated by homogeneous elements in degree 1, $\text{gr}_I R/J_1^*$ is isomorphic to
\[
R/I \oplus I/(I^2, J_1) \oplus I^2/(I^3 + I J_1) \oplus \cdots.
\]
Now Assume that (c) holds. Hence for every $n \geq 1$
\[
I^n/(I^{n+1} + J_1 I^{n-1}) \simeq I^n/(I^{n+1} + I^n \cap J_1).
\]
This shows that $J_1 \cap I^n \subseteq I^{n+1} + J_1 I^{n-1}$ for every $n \geq 1$. We have
\[
J_1 \cap I^n \subseteq I^{n+1} \cap J_1 + J_1 I^{n+1} \subseteq I^{n+2} + J_1 I^n + J_1 I^{n-1} = I^{n+2} + J_1 I^{n-1}.
\]
By induction, for all $m \geq 1$
\[
J_1 \cap I^n \subseteq J_1 I^{n-1} + I^{n+m}.
\]
Since $R$ is local, we obtain $J_1 \cap I^n = J_1 I^{n-1}$, which prove (a). The converse is clear by (2). \qed
Theorem 2.6. Let $J_1 = (f_1, \ldots, f_m)$ and $J_2 = (g_1, \ldots, g_m) \subseteq I$ be ideals in the ring $R$ and let $J_1 \subseteq I$ be Aluffi torsion-free. Suppose that $f_i - g_i \in I^2$ for $i = 1, \ldots, m$. Let $Z_1$ and $Z_2$ stand for the first syzygies modules of $J_1$ and $J_2$ respectively. Then $J_2 \subseteq I$ is Aluffi torsion-free if and only if $Z_1 \cap I^n R^m \subseteq Z_2 + I^{n+1} R^m$ for all $n \geq 0$.

Proof. ($\Rightarrow$) If $(a_1, \ldots, a_m) \in Z_1 \cap I^n R^m$ then $\sum_{i=1}^m a_ig_i = \sum_{i=1}^m a_ig_i = J_2 \cap I^{n+2} = J_2 I^{n+1}$ by assumption, then $\sum_{i=1}^m g_i(a_i - b_i) = 0$, where $b_i \in I^{n+1}$ and $(a_1, \ldots, a_n) = ((a_1 - b_1) + b_1, \ldots, (a_n - b_n) + b_n) \in Z_2 + I^{n+1} R^m$.

($\Leftarrow$) We only need to show that $J_2 \cap I^{n+1} \subseteq J_2 I^n$ for all $n \geq 0$. Make induction on $n$, the case $n = 0$ is trivial. Pick an element $\sum_{i=1}^m a_i g_i \in J_2 \cap I^n R^m$. We may assume that $a_i \in I^n$ for $i = 1, \ldots, m$, in fact we have $\sum_{i=1}^m a_i g_i \in J_2 \cap I^{n+1} \subseteq J_2 \cap I^n = J_2 I^{n-1}$ by induction hypothesis, so that $\sum_{i=1}^m a_i g_i = \sum_{i=1}^m a'_i g_i$ with $a'_i \in I^{n-1}$.

We may assume that $(a_1, \ldots, a_m) \in Z_1 \cap I^{n-1} R^m$. Namely we have $\sum_{i=1}^m a_i (f_i - g_i) \in I^{n-1} I^2 = I^{n+1}$ hence $\sum_{i=1}^m a_i f_i \in J_1 \cap I^{n+1} = J_1 I^n$ and $\sum_{i=1}^m a_i f_i = \sum_{i=1}^m b_i f_i$ with $b_i \in I^n$. Now we see that $\sum_{i=1}^m (a_i - b_i) f_i = 0$, so that we may replace $a_i$ with $(a_i - b_i)$ and this gives also $\sum_{i=1}^m a_i f_i = 0$ as required. We have now

$$\sum_{i=1}^m a_i g_i \in I^{n+1}; \quad (a_1, \ldots, a_m) \in Z_1 \cap I^{n+1} R^m.$$ 

Now by assumption, we have $(a_1, \ldots, a_m) \in Z_2 + I^n R^m$. Then there exists $b_i \in I^n$ and $(e_1, \ldots, e_m) \in Z_2$ such that $(a_1, \ldots, a_m) = (e_1 + b_1, \ldots, e_m + b_m)$. Then $\sum_{i=1}^m a_i g_i = \sum_{i=1}^m b_i g_i$ and replacing the $a_i$'s with $b_i$'s we may suppose that $a_i \in I^n$.

Repeating the first argument above with $a_i \in I^n$ we get

$$\sum_{i=1}^m a_i g_i \in I^{n+2}; \quad (a_1, \ldots, a_m) \in Z_1 \cap I^n R^m.$$ 

Therefore, we have an element $\sum_{i=1}^m a_i g_i$ such that $a_i \in I^n$ and it is clear that such element belong to $J_2 I^n$. □

Corollary 2.7. Let $J_1, J_2 \subseteq I$ be ideals in the ring $R$ such that $J_1 \equiv J_2$ modulo $I^2$ and $J_1 \subseteq I$ is Aluffi torsion-free. Then $J_2 \subseteq I$ is Aluffi torsion-free if and only if the first syzygy modules of $J_1, J_2$ have the same form ideals in $\text{gr}_I(R^m)$.

Proof. The proof is based on the symmetry of the Theorem (2.6) and the fact that the condition $Z_1 \cap I^n R^m \subseteq Z_2 + I^{n+1} R^m$ is equivalent with $(Z_1)^* \subseteq (Z_2)^*$ in $\text{gr}_I(R^m) = \bigoplus_{n \geq 0} I^n R^m/I^{n+1} R^m$. More precisely, if $a \in Z_1 \cap I^n R^m$ for some $n$ then $a \in Z_2 + I^{n+1} R^m$ hence $a = b + c$ with $b \in Z_2$ and $c \in I^{n+1} R^m$ thus $b^* = a^*$ which proves that $(Z_1)^* \subseteq (Z_2)^*$. Conversely, if $a \in Z_1 \cap I^n R^m$, we choose an element $b \in Z_2$ such that $a^* = b^*$ hence $a - b \in I^{n+1} R^m$ and we are done. □

Proposition 2.8. Let $J_1 \subseteq J_2 \subseteq I$ be ideals in the ring $R$. Assume that $J_1 \cap J_2^{n-1} = J_1 J_2^{n-1}$ and $I^n \subseteq J_2^n + J_1$ for all $n \geq 1$. Then $J_1 \subseteq I$ is Aluffi torsion-free.

Proof. We show by induction on $n$ that

$$I^n \subseteq J_2^n + \sum_{j=0}^{n-1} I^j(J_2^{n-j-1} \cap J_1), \quad \text{for all } n \geq 1.$$ (3)
The case $n = 1$ is clear by second assumption. Suppose that (3) holds for some $n$. Multiplying (3) by $I$ yields $I^{n+1} \subseteq IJ_2^n + \sum_{j=0}^{n-1} I^{j+1}(IJ_2^{n-j} \cap J_1)$. Again by second assumption we also have that $I^{n+1} \subseteq J_2^{n+1} + J_1$, so that $I^{n+1}$ contained in

$$IJ_2^n + \sum_{j=0}^{n-1} I^{j+1}(IJ_2^{n-j} \cap J_1) \cap (J_2^{n+1} + J_1).$$

Let $a$ be an element of $I^{n+1}$. Write $a = b + c = d + e$ where

$$b \in IJ_2^n, \quad c \in \sum_{j=0}^{n-1} I^{j+1}(IJ_2^{n-j} \cap J_1), \quad d \in J_2^{n+1} \subseteq IJ_2^n, \quad e \in J_1.$$

Then $e - c = d - b$, and so $d - b \in J_1 \cap IJ_2^n$. Therefore, $a = d + e = d + c + (e - c)$ is in

$$J_2^{n+1} + \sum_{j=0}^{n-1} I^{j+1}(IJ_2^{n-j} \cap J_1) + (J_1 \cap IJ_2^n) = J_2^{n+1} + \sum_{j=0}^{n-1} I^j(IJ_2^{n-j} \cap J_1),$$

which proves (3). Now using (3) we obtain that

$$J_1 \cap I^n \subseteq (J_1 \cap J_2^n) + (J_1 \cap IJ_2^{n-1}) + \sum_{j=1}^{n} I^j(IJ_2^{n-j} \cap J_1)$$

$$\subseteq (J_1 \cap J_2^{n-1}) + \sum_{j=1}^{n} I^j(IJ_2^{n-j} \cap J_1)$$

$$\subseteq J_1 J_2^{n-1} + \sum_{j=1}^{n} I^j(J_1 J_2^{n-j} \cap J_1) \subseteq J_1 J_2^{n-1} + \sum_{j=1}^{n} I^j(J_1 J_2^{n-j})$$

$$\subseteq J_1 I^{n-1} + \sum_{j=1}^{n} I^j(J_1 I^{n-j}) = J_1 I^{n-1}$$

\[\square\]

2.1. **Strongly Aluffi torsion-free ideals.** Let $R$ be a local ring and $I = (f_1, \ldots, f_t)$ an ideal such that $f_1, \ldots, f_t$ is a regular sequence. Then for any $n \geq 1$ the pair $J_i = (f_1^n, \ldots, f_t^n) \subseteq I^n$ is Aluffi torsion-free for $i = 1, \ldots, t$ [14, Example 1.3]. We have the following definition.

**Definition 2.9.** The pair $J = (f_1, \ldots, f_t) \subseteq I$ is called strongly Aluffi torsion-free if $J_i = (f_1^n, \ldots, f_t^n) \subseteq I$ is Aluffi torsion-free for $i = 1, \ldots, t$.

In general, the following example shows that Aluffi torsion-free property does not implies strongly Aluffi torsion-free property.

**Example 2.10.** Let $J \subseteq k[x, y, z]$ be an ideal of 5 projective points in general linear position in $\mathbb{P}^2_k$ which are columns of the matrix

$$\begin{bmatrix}
1 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}.$$
Then \( J = (xy + 3xz - 4yz, zx^2 - 2yz^2 + xz^2, zy^2 + 6xz^2 - 7yz^2) \) which is codimension 2 perfect ideal. Let \( I \) stands for the Jacobian ideal \( I = (J, I_2(\Theta)) \) where \( \Theta \) is the Jacobian matrix of \( J \) and \( I_2(\Theta) \) is the ideal generated by 2-minors of \( \Theta \). A calculation in (\cite{3}) shows that \( J \subseteq I \) is Aluffi torsion-free but is not strongly Aluffi torsion-free.

The proposition below gives a criterion for strongly Aluffi torsion-free ideals.

**Proposition 2.11.** Let \( J = (f_1, \ldots, f_t) \subseteq I \) be Aluffi torsion-free ideals in the ring \( R \). If \( (J_{t-s} : R f_{t-s+1}) = J_{t-s} \) for \( 1 \leq s \leq t - 1 \), then \( J \subseteq I \) is strongly Aluffi torsion-free.

**Proof.** It is enough to show that \( J_{t-1} \cap I^n = J_{t-1}I^{n-1} \) for any \( n \geq 1 \). Let \( a \) be an element of \( J_{t-1} \cap I^n \). Since \( J \subseteq I \) is Aluffi torsion-free and \( J_{t-1} \cap I^n \subseteq J \cap I^n \) hence \( a \in JI^{n-1} \). Write \( a = \sum_{i=1}^{t} a_if_i = \sum_{i=1}^{t} b_if_i \) with \( a_i \in I^n \) and \( b_i \in I^{n-1} \). One has
\[
 b_if_t = (a_1 - b_1)f_1 + \ldots + (a_{t-1} - b_{t-1})f_{t-1} \subseteq J_{t-1}.
\]
Thus \( b_r \in (J_{t-1} : f_t) \). We get
\[
 a = (a_1 - b_1)f_1 + \ldots + (a_{t-1} - b_{t-1})f_{t-1} + b_tf_t \in J_{t-1}I^{n-1} + f_t((J_{t-1} : f_t) \cap I^{n-1}).
\]
Then for all \( n \geq 1 \) we get
\[
 J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t((J_{t-1} : f_t) \cap I^{n-1}).
\]
Now by assumption we have
\[
 J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t(J_{t-1} \cap I^{n-1}).
\]
Making induction on \( n \) we get
\[
 J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t(J_{t-1}I^{n-2}) = J_{t-1}I^{n-1},
\]
as required. \( \square \)

**Remark 4.** Let \( J = (f_1, \ldots, f_t) \subseteq R \) be an ideal such that \( f_1, \ldots, f_t \) is a regular sequence. If \( J \subseteq I \) is Aluffi torsion-free then by Proposition (2.11) it is strongly Aluffi torsion-free. Also by the proof of Proposition (2.11), if for all \( n \geq 1 \) and \( 1 \leq s \leq t - 1 \) we have
\[
 f_{t-s+1}((J_{t-s} : f_{t-s+1}) \cap I^n) \subseteq J_{t-s}I^n,
\]
then strongly Aluffi torsion-free property holds.

**Example 2.12.** Let \( R = k[x_1, \ldots, x_n] \) and \( J = (x_ix_j : 1 \leq i < j \leq n) \). By \([14, \text{Proposition 2.1}]\), \( J \subseteq I = (J, x_1^{n-1}, \ldots, x_n^{n-1}) \) is Aluffi torsion-free. Note that the number of generators of \( J \) is \( t = n(n - 1)/2 \). We show that \( J \subseteq I \) is strongly Aluffi torsion-free. By above remark and symmetry we just prove that \( x_{n-1}x_n((J_{t-1} : R (x_{n-1}x_n) \cap I^m)) \) contained in \( J_{t-1}I^m \) for all \( m \geq 1 \). An easy calculation show that \( Q := (J_{t-1} : R (x_{n-1}x_n)) = (x_1, \ldots, x_{n-2}) \) and \( \Gamma = (x_{n-1}x_n, x_{n-1}^{n-1}, x_{n-1}^{n-1}) \), where by \( \hat{J} \) we mean \( J \) without the generator \( x_{n-1}x_n \). Write \( I = (\Gamma, \Delta) \). We have
\[
 x_{n-1}x_n(Q \cap I^m) = x_{n-1}x_n[Q \cap (\Gamma^m, \Gamma^{m-1}\Delta, \ldots, \Gamma\Delta^{m-1}, \Delta^m)]
\]
\[
 = x_{n-1}x_n(\Delta^{m-1}) + (x_{n-1}x_n)Q^m \subseteq J_{t-1}I^m.
\]
Theorem 2.13. Let $J_1, J_2 \subseteq I$ be ideals in the ring $R$. Assume that $J_2 \subseteq T$ is Aluffi torsion-free in $\overline{T} = R/J_1$. If there exists a minimal generators $f_1, \ldots, f_s$ of $J_1$ such that

1. $J_1 = (f_1, \ldots, f_s) \subseteq I$ is strongly Aluffi torsion-free.
2. $\{\tilde{f}_1, \ldots, \tilde{f}_s\}$ is a regular sequence in $\tilde{R} = R/J_2$.

Then $J_2 \subseteq I$ is Aluffi torsion-free.

Proof. We use induction on $s$. Assume that $s = 1$. Since $J_2 \subseteq T$ is Aluffi torsion-free then for all $n \geq 1$ we have $(J_2 \cap I^n) + J_1 = (J_2I^{n-1}) + J_1$. Intersecting the latter with $J_2 \cap I^n$ we get

$$J_2 \cap I^n = J_2I^{n-1} + (J_1 \cap I^n \cap J_2)$$

But $J_1 \cap I^n \cap J_2 = (f_1) \cap I^n \cap J_2$. Hence by (1) we obtain that

$$(f_1) \cap I^n \cap J_2 = (f_1)I^{n-1} \cap J_2 = (f_1)(I^{n-1} \cap (J_2 : f_1))$$

By (2) $\tilde{f}_1$ is regular in $\tilde{R}$, then $(J_2 : f_1) = J_2$. Hence

$$(f_1) \cap I^n \cap J_2 = f_1(I^{n-1} \cap J_2),$$

and we obtain

$$J_2 \cap I^n = J_2I^{n-1} + f_1(I^{n-1} \cap J_2)$$

Now making induction on $n$, we get

$$J_2 \cap I^n = J_2I^{n-1} + f_1(J_2I^{n-2}) \subseteq J_2I^{n-1}$$

which prove the assertion in this case. Now assume that $s > 1$. Let $N = (f_1, \ldots, f_{s-1})$ and denote by “$\tilde{n}$” reduction modulo $N$. Then in the ring $\tilde{R}$ we have ideals $\tilde{J}_1, \tilde{J}_2$ and $\tilde{I}$. Furthermore, $\tilde{J}_2 \subseteq \tilde{I} \subseteq \tilde{R}$ and by the minimality of $\{f_1, \ldots, f_s\}$ and Proposition (2.4), $\tilde{J}_1 = (f_s) \subseteq \tilde{I}$ is Aluffi torsion-free. Also $\tilde{f}_s$ is regular in $\tilde{R}$. Thus by the first step of the induction we get that $\tilde{J}_2 \subseteq \tilde{I}$ is Aluffi torsion-free. Since the ideal $N$ has the same property as the ideal $\tilde{J}_1$ then the inductive assumption complete the proof. \qed

3. Application and examples

In intersection theory Aluffi algebra is used for closed embedding of schemes $Y \hookrightarrow X \hookrightarrow M$ where $M$ is a regular and $Y$ is the singular subscheme of $X$. In this section we follow this direction.

Let $R = k[x_0, \ldots, x_n]$ be a polynomial ring over a field $k$ of characteristic zero. Let $J = (f_1, \ldots, f_l)$ be an ideal of height $r$. Denote by $\Theta = (\partial f_{ij}/\partial x_j)$ the Jacobian matrix of $J$ and by $I_r(\Theta)$ the ideals generated by $r$-minors of $\Theta$. The ideal $I = (J, I_r(\Theta))$ is called the Jacobian ideal of $J$ which describes the singular subscheme of $\text{Spec}(R/J)$. See ([14]) and ([13]) for examples of Aluffi torsion-free ideals in this situation.

Example 3.1. Let $J \subseteq R = k[x, y, z]$ be the defining ideal of the monomial space curve with parametric equations $x = u^{n_1}, y = u^{n_2}, z = u^{n_3}$, where $\gcd(n_1, n_2, n_3) = 1$. Suppose that $n_1 = 2q + 1$, $n_2 = 2q + p + 1$, $n_3 = 2q + 2p + 1$, for non-negative integers $p, q$. If $I$ is the Jacobian ideal of $J$ then pair $J \subseteq I$ is Aluffi torsion-free.
Proof. Grading $R$ by the exponents of the parameter $u$ in the parametric equations, one knows ([6]) that $J$ is a perfect codimension 2 ideal generated by the homogeneous polynomials

$$F_1 = x^{r_1} - y^{r_{12}} z^{r_{13}}, \quad F_2 = x^{r_{21}} z^{r_{23}} - y^{r_2}, \quad F_3 = x^{r_{31}} y^{r_{32}} - z^{c_3}$$

where $0 < r_{ij} < c_i$ ($i = 1, 2, 3, j \neq i$). Note the relations

$$c_1 = r_{21} + r_{31}, \quad c_2 = r_{12} + r_{32}, \quad c_3 = r_{13} + r_{23}.$$

The Jacobian matrix of $J$ is

$$\Theta = \begin{pmatrix}
    c_1 x^{c_1-1} & -r_{12} y^{r_{12}-1} z^{r_{13}} & -r_{13} y^{r_{13}-1} x^{r_{13}-1} \\
    r_{21} x^{r_{21}-1} y^{r_{23}} & -c_2 y^{c_2-1} & -c_3 y^{c_3-1} \\
    r_{31} x^{r_{31}-1} y^{r_{32}} & -c_3 x^{c_3-1} & r_{23} x^{r_{23}-1} z^{r_{23}-1}
\end{pmatrix}$$

The 2-minors of $\Theta$ are

$$f_1 = -c_1 c_2 x^{c_1-1} y^{c_2-1} + r_{21} r_{12} x^{r_{21}-1} y^{r_{12}-1} z^{c_3}$$
$$f_2 = c_1 r_{23} x^{c_1+r_{21}-1} y^{r_{23}-1} + r_{13} r_{21} x^{r_{21}-1} y^{r_{12}-1} z^{c_3-1}$$
$$f_3 = -r_{12} r_{23} x^{r_{21}} y^{r_{12}-1} z^{c_3-1} + c_2 r_{13} y^{c_2+r_{12}-1} z^{r_{13}-1}$$
$$f_4 = c_1 r_{32} x^{c_1+r_{31}-1} y^{r_{32}-1} + r_{31} r_{12} x^{r_{31}-1} y^{c_2-1} z^{r_{13}}$$
$$f_5 = -c_1 c_3 x^{c_1-1} z^{c_3-1} + r_{31} r_{13} x^{r_{31}-1} y^{c_2} z^{r_{13}-1}$$
$$f_6 = r_{32} r_{13} x^{r_{31}} y^{r_{32}-1} z^{r_{13}-1} + c_3 r_{12} y^{r_{12}-1} z^{c_3+r_{13}-1}$$
$$f_7 = r_{21} r_{32} x^{c_1} y^{r_{32}-1} z^{r_{23}} + c_2 r_{31} x^{r_{31}-1} y^{c_1+r_{21}-1}$$
$$f_8 = -r_{21} r_{31} x^{c_1} y^{r_{32} z^{r_{21}-1}} - c_3 r_{21} x^{r_{21}-1} z^{c_3+r_{24}-1}$$
$$f_9 = -r_{32} r_{23} x^{c_1} y^{r_{32} z^{r_{21}-1}} + c_2 c_3 y^{c_2-1} z^{c_3-1}$$

Write $\mathfrak{D}$ for the ideal generated by the following monomials

$$M_1 = x^{r_{21}-1} y^{r_{12}-1} z^{c_3}, \quad M_2 = x^{r_{21}-1} y^{r_{12} z^{c_3}-1}, \quad M_3 = y^{c_2+r_{12}-1} z^{r_{13}-1}$$
$$M_4 = x^{r_{31}-1} y^{r_{32}-1} z^{r_{13}}, \quad M_5 = x^{r_{31}-1} y^{c_2} z^{r_{13}-1}, \quad M_6 = y^{r_{12}-1} z^{c_3+r_{13}-1}$$
$$M_7 = x^{r_{31}-1} y^{r_{32} z^{r_{21}-1}}, \quad M_8 = x^{r_{21}-1} z^{c_3+r_{23}-1}, \quad M_9 = y^{c_2-1} z^{c_3-1}$$

The following relations come out

$$f_1 = -c_1 c_2 x^{r_{21}-1} y^{r_{12}-1} F_3 + (r_{21} r_{12} - c_1 c_2) M_1$$
$$f_2 = c_1 r_{23} x^{r_{21}-1} z^{r_{23}-1} F_1 + (r_{31} r_{21} - c_1 r_{23}) M_2$$
$$f_3 = -r_{12} r_{23} y^{r_{12}-1} z^{r_{13}-1} F_2 - (r_{12} r_{23} + c_2 r_{13}) M_3$$
$$f_4 = c_1 r_{32} x^{r_{31}-1} y^{r_{32}-1} F_1 + (r_{31} r_{12} + c_2 r_{32}) M_4$$
$$f_5 = -c_1 c_3 x^{r_{31}-1} z^{r_{13}-1} F_2 + (r_{31} r_{13} - c_1 c_3) M_5$$
$$f_6 = r_{32} r_{13} y^{r_{32}-1} z^{r_{13}-1} F_3 + (r_{21} c_3 + r_{32} r_{13}) M_6$$
$$f_7 = r_{21} r_{32} x^{r_{31}-1} y^{r_{32}-1} F_2 + (r_{31} c_2 + r_{21} r_{32}) M_7$$
$$f_8 = -r_{32} r_{23} x^{r_{21}-1} z^{r_{23}-1} F_3 - (r_{21} c_3 + r_{23} r_{31}) M_8$$
$$f_9 = r_{32} r_{23} y^{r_{32} z^{r_{23}-1}} F_1 + (c_2 c_3 - r_{32} r_{23}) M_9$$

By above, $J$ is generated by

$$F_1 = x^{p+q+1} - y z^q, \quad F_2 = x z - y^2, \quad F_3 = x^{p+q} y - z^{q+1}$$
and the Jacobian ideal is

\[ I = (J, \mathcal{D}) = (xz - y^2, x^{p+q+1}, x^{pq}y, x^{pq+1}y^2, yz^q, y^2z^{q-1}, z^{q+1}) \]

Set \( \Delta = (x^{p+q+1}, x^{pq}y, x^{pq+1}y^2, yz^q, y^2z^{q-1}, z^{q+1}) \). By a slight adaptation of Proposition 2.5 it suffices to show that \( J \cap \Delta^n \subseteq J I^{n-1} \) for every \( n \geq 1 \). Since \( J \) is binomial prime ideal and \( \Delta \) is monomial then \cite[Corollary 1.5]{2} implies that \( J \cap \Delta^n \) is generated by binomials \((u - v)h\) where \( u - v \in J \) and \( h \in R \) is a monomial or \( h = (u - v)^c \) for some positive integer \( c \geq 1 \). As elements \( uh, vh \) belong to \( \Delta^n \), an easy calculation show that \( (u - v)h \in JI^{n-1} \) as required. \( \Box \)

**Question 3.2.** Let \( J \subseteq k[x_1, \ldots, x_n] \) be defining ideal of affine monomial curve with parametric equation \( x_1 = u^{\alpha_1}, \ldots, x_m = u^{\alpha_m} \). Let \( I \) be the Jacobian ideal of \( J \). For which types of parametrization, the pair \( J \subseteq I \) is Aluffi torsion-free.

**Example 3.3.** Let \( J \) be an ideal in the ring \( R = k[x_0, \ldots, x_n] \) with \( n \geq 2 \) generated with all square free monomial ideal in degree \( r \). Let \( I \) stands for the Jacobian ideal of \( J \). Then the pair \( J \subseteq I \) is Aluffi torsion-free.

**Proof.** Let \( J = (x_{i_1}x_{i_2} \cdots x_{i_r} : 0 \leq i_1 < \cdots < i_r \leq n) \). It is well known that \( \text{ht} J = r - 1 \). The transpose Jacobian matrix of \( J \) is

\[
\Theta(J) = \begin{bmatrix}
[x_{i_1}x_{i_2} \cdots x_{i_{r-1}} & 1 \leq i_1 < \cdots < i_{r-1} \leq n & 0
\end{bmatrix},
\]

where \( \Theta' \) is the Jacobian matrix of the ideal \( J' \) generated by all square free monomial ideal in degree \( r \) in \( k[x_1, \ldots, x_n] \). By induction on \( n \) and elementary columns operation we get that the Jacobian ideal \( I \) of \( J \) is \( I = (J, x_{i_1}x_{j} : 0 \leq i < j \leq n) \). By Proposition 2.5, it is enough to show that for all \( t \geq 1 \)

\[ J \cap (x_{i_1}x_{j} : 0 \leq i < j \leq n)^t \subseteq JI^{t-1}. \]

The proof of the latter inclusion is based on the usual algorithmic procedure to find generators of the intersection of monomial ideal. \( \Box \)

**Example 3.4.** (1) Let \( M \) be a \( 2 \times n \) generic matrix in the polynomial ring \( R = k[x_i : 1 \leq i \leq 2n] \) with \( n \geq 3 \). Let \( J = I_2(M) \). It is well-known that \( \text{ht} J = n - 1 \). Let \( I = (J, I_{n-1}(\Theta)) \) stands for the Jacobian ideal of \( J \). Since \( M \) is the concatenation of \( n \) scroll blocks of length 1, then by \cite[Theorem 2.3]{14}, \( I_{n-1}(\Theta) = (x_i : 1 \leq i \leq 2n)^{n-1} \). In particular, the pair \( J \subseteq I \) is Aluffi torsion-free.

(2) Let \( R = k[x_1, \ldots, x_9] \). Consider the \( 3 \times 3 \) generic matrix \( M \) in \( R \)

\[
M = \begin{bmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & x_9
\end{bmatrix}.
\]
The ideal \( J = I_2(M) = (\Delta_1, \Delta_2, \Delta_3) \) has codimension 4 and the Jacobian matrix of \( J \) is of the form

\[
\Theta(J) = \begin{array}{ccc|c|c}
 x_5 &   &   & \ast & \\
 x_6 &   &   & \ast & \\
 x_8 &   &   & \ast & \\
 x_9 &   &   &   & \\
0  & x_6 &   & \ast & \\
0  & 0   & x_9 & \ast & \\
0  & 0   & 0   & \Theta' & \\
0  & 0   & 0   & 0   & \\
\end{array},
\]

where the first block of \( \Theta(J) \) is the Jacobian matrix of \( \Delta_1 = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_1x_8 - x_2x_7, x_1x_9 - x_3x_7) \), the second block is the Jacobian matrix of \( \Delta_2 = (x_2x_6 - x_3x_5, x_2x_9 - x_3x_8) \) and in the last block \( \Theta' \) is the Jacobian matrix of \( \Delta_3 = I_2(\begin{bmatrix} x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}) \). Note that \( \text{ht}(\Delta_2, \Delta_3) = 3 \) and \( \text{ht} \Delta_3 = 2 \). We claim that \( I_4(\Theta(J)) = (x_1, \ldots, x_9)^4 \) which proves that the pair \( J \subset I \) is Aluffi torsion-free. By using the first part of the example with \( n = 3 \), the 3-minors of second and third blocks of \( \Theta(J) \) is generated by \( (x_6, x_9)(x_4, x_5, x_6, x_7, x_8, x_9)^2 \). One has

\[
(x_5, x_6, x_8, x_9)(x_6, x_9)(x_4, x_5, x_6, x_7, x_8, x_9)^2 \in I_4(\Theta(J)).
\]

Therefore by changing the role of \( x_1 \) and \( x_2 \) and using above argument the assertion hold.

**Question 3.5.** Let \( M \) be a \( n \times m \) generic matrix in the polynomial ring \( R = k[x_{ij} ; 1 \leq i \leq n, 1 \leq i \leq m] \). Let \( J = I_2(M) \) be the ideal generated by 2-minors of \( M \). Let \( I \) be the Jacobian ideal of \( J \). Is the pair \( J \subset I \) Aluffi torsion-free?

**Example 3.6.** Let \( J(f) = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) \subset R = k[x, y, z] \) denote the gradient ideal of a reduced free divisor line arrangement \( X = V(f) \) of degree 3 in \( \mathbb{P}^2_k \). By [15, Proposition 3.7], \( J(f) \) is codimension 2 perfect ideal. Then by Hilbert-Burch theorem \( J(f) \) is generated by 2-minors of the \( 2 \times 3 \) matrix of linear forms in \( R \). If \( Y = V(J(f)) \subset \mathbb{P}^2_k \) is non-singular then the Jacobian ideal \( I \) of \( J(f) \) is \((x, y, z)\)-primary. Therefore, by [14, Corollary 2.7] the pair \( J(f) \subset I \) is Aluffi torsion-free.

We warm up with a conjecture.

**Conjecture 3.7.** Let \( X = V(f) \) be a reduced free divisor of line arrangement in \( \mathbb{P}^2_k \). Let \( J(f) \) denote the gradient ideal of \( f \) and \( I \) stands for the Jacobian ideal of \( J(f) \). Then \( J(f) \subset I \) is Aluffi torsion-free.

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