Twistor formulation of massless 6D infinite spin fields

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Abstract

We construct massless infinite spin irreducible representations of the six-dimensional Poincaré group in the space of fields depending on twistor variables. It is shown that the massless infinite spin representation is realized on the two-twistor fields. We present a full set of equations of motion for two-twistor fields represented by the totally symmetric SU(2) rank 2s two-twistor spin-tensor and show that they carry massless infinite spin representations. A field twistor transform is constructed and infinite spin fields are found in the space-time formulation with an additional spinor coordinate.

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1 Introduction

Recently, there has been a certain interest in the study of representations of the Poincaré group in the higher dimensional Minkowski space (see e.g. [1–3] and references therein). There are at least two main motivations for research in space-time symmetry of multidimensional space-time. One is related to the low-energy limits of the superstring theory which are supersymmetric gauge theories and supergravity in ten dimensions. After reduction we can obtain the supersymmetric field models in all dimensions less than ten where many of the specific details, in particular the properties of fermionic fields, are due to space-time symmetries. The other motivation is rather formal and is stipulated by the possibilities to generalize the methods for describing irreducible representations of the Poincaré group in four dimensions [4–6] up to higher dimensions (see e.g. [7–9]). This can be useful for studying various aspects of the higher spin field theory in higher dimensions.

In the paper [3], we presented a general description of massless irreducible representations of the Poincaré group in the six-dimensional Minkowski space. The corresponding Lie algebra possesses three Casimir operators whose eigenvalues were found. It was shown that finite spin representations are defined by two integer or half-integer numbers while infinite spin representations are defined by the real parameter $\mu^2$ and one integer or half-integer number. One of the most important applications of the Poincaré group is due to field theory where we need representations in the space of fields. Therefore, our next problem is to construct a realization of the general description [3] on the fields in six-dimensional space-time. The paper under consideration is devoted to aspects of this problem related to infinite spin representations. To be more precise, we will consider the realization of massless representations in the space of twistor fields.

We will focus on describing massless infinite spin irreducible twistor field representations in the six-dimensional Minkowski space. In recent years, there has been a surge of activity in the study of various aspects of fields with infinite spin, mainly in the context of higher spin field theory (see e.g. the review [10] and refs. therein; see also [11–24]). In this paper, we are going to discuss issues related to the realization of massless infinite spin representations on twistor fields in the six-dimensional Minkowski space.

Approaches to the twistor description of massless finite spin (super)particles and fields in six dimensions were discussed in Refs. [25–31]. In this paper, we will study the twistor field formulation of infinite spin representations in six dimensions as a generalization of the previously proposed twistor formulation of an infinite spin particle in four dimensions [16, 21]. As far as we know, these aspects were not considered in detail earlier.

The paper is organized as follows. Section 2 is devoted to a brief survey of the irreducible massless representations of the Poincaré group in the $D=6$ Minkowski space. In Section 3, we consider one-twistor space and construct a realization of the Poincaré group generators in the space of one-twistor fields. We show that this space necessarily describes finite spin representations and its generalization is required to solve our problems related to infinite spin representations. Section 4 is devoted to the implementation of massless representations with infinite spin. We introduce two-twistor space and two-twistor fields, describe the realization of the generators in the space of these fields and show that these fields form infinite spin representations. In Section 5, infinite spin representations are constructed in terms of totally symmetric $2s$ rank spin-tensor twistor fields. In Section 6, we construct a twistor transform that links the found twistor fields with space-time infinite spin fields in the formulation with an additional spinor variable. In Section 7, we summarize the results and discuss open issues. Some technical points related to the notation and calculation details are referred to Appendices.
2 Massless representations in $D=6$ Minkowski space-time

In this section, to fix the notation, we recall the general results on the irreducible massless representations of the Poincaré group in six dimensions [3]. This description will be used in the next sections for the construction of this type of representations in the space of twistor fields.

Relativistic symmetry transformations in the six-dimensional Minkowski space are generated by elements of the Lie algebra $\text{iso}(1,5)$ of the corresponding Poincaré group. The basis elements $P_m$ and $M_{mn} = -M_{nm}$ of $\text{iso}(1,5)$ have the commutators

\[
[P_n, P_k] = 0, \quad [M_{mn}, P_k] = i(\eta_{mk}P_n - \eta_{nk}P_m), \tag{2.1}
\]

\[
[M_{mn}, M_{kl}] = i(\eta_{mk}M_{nl} + \eta_{ml}M_{nk} - \eta_{nl}M_{mk} - \eta_{nk}M_{ml}). \tag{2.2}
\]

We use the space-time metric $\eta_{mn} = \text{diag}(+1, -1, -1, -1, -1, -1)$ and vector indices $m, n, \ldots$ run through six values $0, 1, \ldots, 5$.

The quadratic Casimir operator of the Lie algebra $\text{iso}(1,5)$ has the form

\[
C_2 = P^2 = P^mP_m. \tag{2.3}
\]

When acting on the states of an irreducible representation, the operator (2.3) takes a fixed numerical value equal to the square of a massive parameter which in some cases is interpreted as a relativistic particle mass. Since we consider massless representations, the Casimir operator (2.3) is equal to zero:

\[
C_2 = 0. \tag{2.4}
\]

In this case, when we have (2.4), the other two Casimir operators of the Lie algebra $\text{iso}(1,5)$ are given by the following expressions [3]:

\[
C_4 = \Pi^m\Pi_m, \tag{2.5}
\]

\[
C_6 = -\Pi^kM_{km}\Pi^lM_{lm} + \frac{1}{2}(M_{mn}M_{mn} - 8)C_4, \tag{2.6}
\]

where

\[
\Pi_m := P^kM_{km}. \tag{2.7}
\]

Massless finite spin representations (so called helicity representations) are defined in the space of fields where the Casimir operators (2.5) and (2.6) have zero eigenvalues. They are characterized by two spin (helicity) operators, the explicit expressions of which are given in [3].

The massless infinite (continuous) spin representations are realized in the space of states on which the operators (2.5), (2.6) acquire numerical values:

\[
C_4 = -\mu^2, \tag{2.8}
\]

\[
C_6 = -\mu^2s(s+1), \tag{2.9}
\]

where $\mu \neq 0$ is the dimensionful real parameter (we assume that $\mu \in \mathbb{R}_{>0}$) and $s$ is a non-negative integer or half-integer number $s \in \mathbb{Z}_{\geq 0}/2$. Of course, in the limit $\mu = 0$ the finite spin (helicity) representations are reproduced.

In the following sections, we will find a twistor representation of the algebra (2.1), (2.2) and study fields in twistor space where massless infinite spin representations are explicitly realized.
3 One-twistor case: finite spin representations

Let us first consider the one-twistor case.

Following the standard prescription of the twistor formalism (see, e.g., [25, 26, 30, 31]), we consider a twistor as an object consisting of two chiral spinors, which are canonically conjugate to each other and thus form the phase space of classical mechanics.

In $D = (1+5)$ Minkowski space-time there are no standard Majorana-Weyl spinors but there are SU(2) “real” spinors [32] (see also [33], Sect. 6.4.2) which in general can have different chiralities. So we take the SU(2) Majorana-Weyl spinor with fixed chirality

$$\pi^I_\alpha, \quad (\pi^I_\alpha)^* = \epsilon_{IJ} B^\alpha_\beta \pi^J_\beta,$$  

(3.1)

where $B^\alpha_\beta$ is the conjugation matrix. Here $\alpha = 1, 2, 3, 4$ is the Spin(1,5) $\simeq$ SU$^*(4)$ index and $I, J = 1, 2$ are the SU(2) indices and consider this spinor as half of the $D = (1+5)$ twistor. The second half of the twistor is the SU(2) Majorana-Weyl spinor

$$\omega^{\alpha I}, \quad (\omega^{\alpha I})^* = \epsilon_{IJ} \omega^{\beta J} (B^{-1})^\beta_\alpha,$$  

(3.2)

having the opposite to the spinor (3.1) chirality. The basic twistor Poisson brackets have the form

$$\{\pi^I_\alpha, \omega^I_\beta\}_P = \delta^\beta_\alpha \delta^I_J.$$  

(3.3)

Thus, the $D = (1+5)$ twistor consists of two SU(2) Majorana-Weyl spinors (3.1), (3.2)

$$Z^I_a = \begin{pmatrix} \pi^I_\alpha \\ \omega^I_\beta \end{pmatrix}, \quad \omega^I_\beta = \epsilon^{IJ} \omega^J_\beta,$$  

(3.4)

where the index $a$ runs through eight values $a = 1, \ldots, 8$. The quantities

$$X_{[ab]} := Z^I_a Z^J_b \epsilon_{IJ}$$  

(3.5)

form the $so(2,6) \simeq so^*(8)$ algebra with respect to the Poisson brackets (3.3) and generate local Spin(2,6) transformations as linear transformations of the twistor (3.4) (see, for example, [34] and Appendix B). So, as in the standard twistor prescription [30, 31], the twistor (3.4) is a couple of the SU(2) Majorana-Weyl spinors, which form generators (3.5) of the $D = (1+5)$ conformal group Spin(2,6). We emphasize that $Z^I$ for $I = 1$ and $I = 2$ are related by the reality conditions (3.1), (3.2). Thus, despite the presence of the SU(2) index $I = 1, 2$, we can talk about only one complex twistor.

Below we will consider the field twistor theory where the Poisson brackets (3.3) are quantized and become commutators

$$[\pi^I_\alpha, \omega^I_\beta] = i \delta^\beta_\beta \delta^I_J$$  

(3.6)

of the operators acting on the twistor fields. Then we choose the $\pi$-representation for the twistor fields in which the coordinates $\pi^I_\alpha$ are “diagonal”, one-twistor fields $\Psi^{(1)}(\pi)$ are the functions of these coordinates, and the operators $\omega^I_\alpha$ are realized by differential operators

$$\omega^I_\alpha = -i \frac{\partial}{\partial \pi^I_\alpha}.$$  

(3.7)

1We call them the SU(2) Majorana-Weyl spinors.

2Throughout this paper, we raise and lower the SU(2) indices as follows: $\psi^I = \epsilon^I_J \psi_J$, $\phi_I = \epsilon_{IJ} \psi^J$ where the antisymmetric tensors $\epsilon_{IJ}$, $\epsilon^{IJ}$ have components $\epsilon_{12} = \epsilon^{21} = 1$. The details of our spinor notation are given in Appendix A.
One of the basic properties of the twistor formulation is the representation of the relativistic momentum vector of a massless particle in the form of a bilinear combination of twistor coordinates where the light-like condition (2.4) of the relativistic momentum is satisfied automatically. Therefore, in the considered one-twistor case we take the generators of the Poincaré translations in the following form:

\[ P_m = \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha \beta} \pi^\beta I. \]  

(3.8)

In fact, the momentum vector (3.8) is light-like due to the property (A.7):

\[ P^m P_m = 2 \epsilon^{\alpha \beta \gamma \delta} \pi^I_\alpha \pi^J_\beta \pi^I_\gamma \pi^J_\delta \equiv 0. \]

Thus, for the representation (3.8), the one-twistor fields \( \Psi^{(1)}(\pi) \) describe massless states. The generators \( M_{mn} \) of the Lorentz algebra \( \mathfrak{so}(1, 5) \) act in the space of fields \( \Psi^{(1)}(\pi) \) as operators

\[ M_{mn} = -i \pi^I_\alpha (\tilde{\sigma}_{mn})^{\alpha \beta} \frac{\partial}{\partial \pi^J_\beta}. \]

(3.9)

We note that in the one-twistor realization (3.8), (3.9) of the Poincaré generators the operator \( \Pi_m \), which is presented in the definition (2.5) of the four-order Casimir operator and defined in (2.7), equals

\[ \Pi_m = \frac{i}{2} (\pi^K \tilde{\sigma}_m \pi_K) \pi^I_\alpha \frac{\partial}{\partial \pi^I_\alpha} \equiv \frac{i}{2} P_m \pi^I_\alpha \frac{\partial}{\partial \pi^I_\alpha}. \]

As a result, in view of (2.4), the Casimir operator (2.5) vanishes: \( C_4 = 0 \), and in this case \( \mu = 0 \) due to (2.8). Then, (2.9) shows that in one-twistor case the six-order Casimir operator is also equal to zero: \( C_6 = 0 \).

Thus, the one-twistor field \( \Psi^{(1)}(\pi) \) describes only massless finite spin representations (helicity representations). For description of massless infinite (continuous) spin representations it is necessary to use two or more twistors [30] as in the massive representations in the \( D = 4 \) case (see, e.g., [35, 36]).

### 4 Two-twistor case: twistorial constraints in infinite-spin case

The massless infinite spin representation of \( \text{iso}(1, 5) \) is formulated in the space two-twistor fields. To describe these fields, we need two copies of the twistors that were considered in the previous section.

In addition to the twistor, defined by relations (3.1)-(3.4), we introduce exactly in the same way the second twistor \( Y^A_a \):

\[ Y^A_a = \begin{pmatrix} \rho^A_\alpha \\ \eta^{\beta A} \end{pmatrix}. \]

(4.1)

This twistor consists of the \( \text{SU}(2) \) Majorana-Weyl spinor

\( \rho^A_\alpha, \quad (\rho^A_\alpha)^* = \epsilon_{AB} B^A_\beta \rho^B_\beta, \)

(4.2)
which has the same chirality\(^3\) as spinor (3.1), and another SU(2) Majorana-Weyl spinor
\[
\eta^{\alpha A}, \quad (\eta^{\alpha A})^* = \epsilon_{AB} \eta^{\beta B} (B^{-1})_\beta^{\dot{\alpha}}, \tag{4.3}
\]
which is canonically conjugated to the first half (4.2) of the twistor \(Y^I_a\). The nonvanishing Poisson brackets of the twistor components (4.2), (4.3) have the form
\[
\left\{ \rho^A_\alpha, \eta^\beta_B \right\}_{PB} = \delta_\alpha^\beta \delta^A_B. \tag{4.4}
\]
The components (4.2), (4.3) have zero Poisson brackets with variables (3.1), (3.2).

In expressions (4.1)-(4.4), \(A = 1, 2\) is the SU(2) index. In general, index \(I\) in (3.1)-(3.4) and index \(A\) in (4.1)-(4.4) are different. This means that the twistors \(Z^I\) and \(Y^A\) are vectors in the representations of different SU(2) groups\(^4\) However, as we will see, these SU(2) groups must be identified in the case of infinite spin representations considered here. With such identification the indices \(I\) and \(A\) refer to the same symmetry group and can be contracted to each other.

In the quantum case, the twistor brackets (4.4) produce the algebra
\[
\left[ \rho^A_\alpha, \eta^\beta_B \right]_{PB} = i \delta_\alpha^\beta \delta^A_B. \tag{4.5}
\]
Therefore, we can consider together the \(\pi\)-representation with realization (3.7) and the \(\rho\)-representation in which the quantities \(\eta^\alpha_A\) are realized by differential operators
\[
\eta^\alpha_A = -i \frac{\partial}{\partial \rho^A_\alpha}. \tag{4.6}
\]
In such representations the physical states are described by the 6D two-twistor field
\[
\Psi = \Psi(\pi^I_\alpha, \rho^A_\alpha), \tag{4.7}
\]
which is a function of upper halves of both twistors (3.4), (4.1).

Now we construct 6D-vectors (these vectors will be used below) by contracting spinors \(\pi^I_\alpha, \rho^A_\alpha\) with the matrices \(\tilde{\sigma}_m\):
\[
v_m := (\pi^I_\alpha \tilde{\sigma}_m \pi^I_\alpha), \quad w_m := (\rho^A_\alpha \tilde{\sigma}_m \rho^A_\alpha), \quad u^{IA}_m := (\pi^I_\alpha \tilde{\sigma}_m \rho^A_\alpha), \tag{4.8}
\]
where contracted spinor indices are omitted: \((\pi^I_\alpha \tilde{\sigma}_m \rho^A_\alpha) \equiv \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha \beta} \rho^A_\beta\), etc. Nonvanishing scalar products of the vectors (4.8) have the form
\[
v^m w_m = r, \quad u^{mIA}_m u^{JB}_m = -\frac{1}{4} \epsilon^{IJ} \epsilon^{AB} r, \tag{4.9}
\]
\(^3\) The twistor formulation of the \(D=6\) system, when the upper spinor \(\rho^A_\alpha\) of the second twistor \(Y^I_a\) has the chirality different from that of the upper spinor \(\pi^I_\alpha\) of the twistor (3.1), will be discussed in the last concluding section.

\(^4\) In the \(D = 6\) massive case, the twistor formulation uses the USp(4) Majorana-Weyl spinor \([30, 31]\)
\[
\pi^I_\alpha = (\pi^I_\alpha, \rho^A_\alpha),
\]
where \(I = 1, 2, 3, 4\) is the USp(4) index. This USp(4) Majorana-Weyl spinor is formed by two SU(2) Majorana-Weyl spinors \(\pi^I_\alpha\) and \(\rho^A_\alpha\). However, in the case of infinite spin considered here, the USp(4) symmetry does not survive and we use the SU(2) Majorana-Weyl spinors separately.
where
\[ r := 2 \epsilon^{\alpha\beta\gamma\delta} \pi^I_\alpha \pi^I_\beta \rho_\gamma^A \rho_\delta^A. \] (4.10)

Due to identities (A.7), the other scalar products of the vectors (4.8) are equal to zero:
\[ v^m v_m = w^m w_m = v^m u^I_m = v^m u^I_m = 0. \] (4.11)

Since we want to describe the massless states by the two-twistor fields (4.7), these fields must satisfy the massless condition (2.4). This condition is fulfilled automatically if we take the space-time translations \( P_m \) in the form (3.8), which was proposed in the previous section for the one-twistor case. Thus, only one of two twistorial spinors \( \pi^I_\alpha \) and \( \rho^A_\alpha \) resolves the momentum operators:
\[ P_m = \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha\beta} \pi^I_\beta. \] (4.12)

In the two-twistor case, the generators of Lorentz transformations act on the spinor components of both twistors and instead of (3.9) we have:
\[ M_{mn} = -i \pi^I_\alpha (\tilde{\sigma}_{mn})^{\alpha\beta} - i \rho^A_\alpha (\tilde{\sigma}_{mn})^{\alpha\beta} \frac{\partial}{\partial \rho^A_\beta}. \] (4.13)

Using expressions (4.12) and (4.13), we obtain that the vector operator \( \Pi_m \), which is defined in (2.7), takes the form
\[ \Pi_m = i \left( \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha\beta} \pi^I_\beta - \rho^A_\alpha \frac{\partial}{\partial \rho^A_\beta} \right) + 2i \left( \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha\beta} \pi^I_\beta \right) \left( \pi^I_\alpha \frac{\partial}{\partial \rho^A_\beta} \right). \] (4.14)

As a result, the square of \( \Pi_m \) equals
\[ C_4 = \Pi^m \Pi_m = 2 \left( \epsilon^{\alpha\beta\gamma\delta} \pi^K_\alpha \pi^K_\beta \rho^C_\gamma \rho^C_\delta \right) \epsilon^{IJ} \epsilon^{AB} \left( \pi^I_\alpha \frac{\partial}{\partial \rho^A_\beta} \right) \left( \pi^J_\alpha \frac{\partial}{\partial \rho^B_\beta} \right). \] (4.15)

Now it is clear that the condition (2.8): \( C_4 = -\mu^2 \) is fulfilled on the infinite spin states if the following equations for the infinite spin field \( \Psi(\pi, \rho) \) hold:
\[ \left( \pi^I_\alpha \frac{\partial}{\partial \rho^A_\beta} - \frac{i}{2} \epsilon_{IA} \right) \Psi = 0, \] (4.16)
\[ \left( \epsilon^{\alpha\beta\gamma\delta} \pi^K_\alpha \pi^K_\beta \rho^C_\gamma \rho^C_\delta - \mu^2 \right) \Psi = 0. \] (4.17)

Equation (4.16) leads to an important corollary due to the presence of the tensor \( \epsilon_{IA} \) in it. This tensor is invariant only if the same \( SU(2) \) group acts on the indices \( I \) and \( A \) of the spinors \( \pi^I_\alpha \) and \( \rho^A_\alpha \), respectively. Therefore, when we describe infinite spin representations by means of equations (4.16) and (4.17), the indices \( I \) and \( A \) in the spinors \( \pi^I_\alpha \) and \( \rho^A_\alpha \) refer to the same group \( SU(2) \) and these indices can be covariantly contracted to each other. One can say that the whole automorphism group \( SU(2) \times SU(2) \) is reduced to the diagonal subgroup \( SU(2) \subset SU(2) \times SU(2) \).

Let us make some comments regarding the choice of the conditions (4.16), (4.17). The operator (4.15) is the product of two commuting operators
\[ \Delta := \epsilon^{\alpha\beta\gamma\delta} \pi^K_\alpha \pi^K_\beta \rho^C_\gamma \rho^C_\delta, \quad F := 2 \epsilon^{IJ} \epsilon^{AB} \left( \pi^I_\alpha \frac{\partial}{\partial \rho^A_\beta} \right) \left( \pi^J_\alpha \frac{\partial}{\partial \rho^B_\beta} \right). \]
which can be diagonalized simultaneously. Consequently, the fulfillment of the condition (2.8) is guaranteed by fixing $\Delta \Psi = \mu^2 \Psi$ (4.17) and

$$F\Psi = -\Psi,$$ (4.18)

To satisfy the second order differential condition (4.18), we require the fulfillment of stronger conditions (4.16) of the first order in the derivatives which are simpler than the condition (4.18). In addition, equations (4.16) are direct $6D$ generalization of the twistor equations for infinite spin fields in four dimensions [16,21]. Although equations (4.16) do not cover all the solutions of (4.18), the choice of (4.16) allows us to consider all possible infinite spin representations in six dimensions, as will be seen below.

It remains to achieve the fulfillment of condition (2.9) for the six-order Casimir operator on the infinite spin fields $\Psi(\pi, \rho)$. Derivation of the expression for the six-order Casimir operator (2.6) in the representations (4.12) and (4.13) is technically cumbersome. The main steps in this calculation are given in Appendix C. As a result, we obtain that the action of the six-order Casimir operator (2.6) on the two-twistor field $\Psi(\pi, \rho)$, satisfying conditions (4.16) and (4.17), gives

$$C_6 \Psi = -\mu^2 J_a J_a \Psi,$$ (4.19)

where the operators $J_a$ ($a = 1, 2, 3$) have the form

$$J_a := \frac{1}{2} \pi^I_a (\sigma_a)^I_J \frac{\partial}{\partial \pi^J_a} + \frac{1}{2} \rho^A_a (\sigma_a)^A_B \frac{\partial}{\partial \rho^B_a},$$ (4.20)

and $\sigma_a$ are the Pauli matrices. The operators $J_a$ generate the $\mathfrak{su}(2)$ algebra

$$[J_a, J_b] = i\epsilon_{abc} J_c.$$ (4.21)

Comparing condition (4.19) with relation (2.9), we find that when describing the irreducible infinite spin representations, the twistor field $\Psi(\pi, \rho)$ must obey the following condition:

$$J_a J_a \Psi = s(s + 1) \Psi,$$ (4.22)

where the operators $J_a$ are defined in (4.20). So in the definition (2.9) of irreducible infinite spin representations, the quantum number $s$ coincides with the spin, which labels the representations of the diagonal $SU(2)$ automorphism subgroup. This subgroup is generated by the Lie algebra $\mathfrak{su}(2)$ with the basis elements (4.20).

Thus, we find a full set of equations on the twistor field that describe the space of massless infinite spin representations. Namely, the irreducible infinite spin representations (with the quantum numbers $\mu$ and $s$) of the Poincaré algebra under consideration with generators (4.12), (4.13) act in the space of the two-twistor fields $\Psi(\pi, \rho)$ subject to equations (4.16), (4.17) and (4.22).

### 5 Twistorial infinite spin fields

In this Section, we obtain a general solution to equations (4.16), (4.17) and (4.22). Recall, that according to equation (4.22), the twistor field $\Psi$ possesses the $SU(2)$-spin equal to $s$. The field of this type can only be described by means of the completely symmetric $2s$ rank spin-tensor field

$$\Psi_{1.\ldots 2s}(\pi, \rho) = \Psi_{(1.\ldots 2s)}(\pi, \rho),$$ (5.1)

Note that relations (4.16), (4.17) are the sufficient conditions for the irreducible infinite spin representation. We do not discuss here the necessary conditions that do not affect the results of this paper.
where $I_i$ are SU(2)-indices, or equivalently by the $(2s + 1)$-component twistor field

$$\Psi_m = \Psi_m(\pi, \rho), \quad m = -s, -s + 1, \ldots s - 1, s,$$

(5.2)

which behaves under the action of the $su(2)$ generators \([4.20]\) as follows:

$$J_3 \Psi_m = m \Psi_m, \quad J_\pm \Psi_m = \sqrt{(s \mp m)(s \mp m + 1)} \Psi_{m \pm 1},$$

(5.3)

where $J_\pm = J_1 \pm iJ_2$. Equations (5.3) yield the eigenvalue of the $su(2)$ Casimir operator equal to the same value as in (4.22).

For us, it is more convenient to use the description where the twistor field (5.2) is represented by the completely symmetric spin-tensor field (5.1). Namely, the irreducible $SU(2)$ equations (5.3) rewritten for the twistor field (5.1) also obeys equations (4.17), (4.16), i.e. the following equations:

$$J_a \Psi_{i_1 \ldots i_{2s}} = -\frac{1}{2} \sum_{\ell=1}^{2s} (\sigma_a)_{i_\ell} K \Psi_{i_1 \ldots i_{\ell-1} K i_{\ell+1} \ldots i_{2s}} \equiv -\frac{2s + 1}{2} (\sigma_a)_{i_1} K \Psi_{K i_2 \ldots i_{2s}},$$

(5.4)

where $J_a$ were given in (4.20) and $\sigma_a$ are the Pauli matrices. Equations (5.4) are the same as equations (5.3) rewritten for the $SU(2)$ spin-tensor fields $\Psi_m$. In addition to relations (5.4), the twistor field (5.1) also obeys equations (4.17), (4.16), i.e. the following equations:

a) $\mathcal{M} \Psi_{i_1 \ldots i_{2s}} = 0$,

b) $\mathcal{F}_{KL} \Psi_{i_1 \ldots i_{2s}} = 0$,

(5.5)

containing the operators

$$\mathcal{M} := \epsilon^{\alpha \beta \gamma \delta} \pi_\alpha K \pi_\beta K \rho_\gamma C \rho_\delta C - \mu^2,$$

(5.6)

$$\mathcal{F}_{IA} := \pi_{\alpha I} \frac{\partial}{\partial \rho_\alpha} - \frac{i}{2} \epsilon_{IA}.$$

(5.7)

Linear equations (5.4), (5.5) are self-consistent since all nonzero commutators of the operators $(J_a, \mathcal{F}_{IK}, \mathcal{M})$, appearing in the definition of these equations, are

$$[J_a, J_b] = i \epsilon_{abc} J_c, \quad [J_a, \mathcal{F}_{IK}] = -\frac{1}{2} (\sigma_a)_{i_1} K \mathcal{F}_{(KL)} - \frac{1}{2} (\sigma_a)_{k_1} K \mathcal{F}_{(IL)},$$

(5.8)

and this means that the algebra of these operators is closed. Note that the second equation in (5.8) is in agreement with (5.4).

Let us clarify the geometric sense of the obtained twistor equations (5.4), (5.5).

i) A natural way to solve equation (5.5a) is to introduce the delta function $\delta(\mathcal{M})$ as a factor in the twistor field $\Psi_{i_1 \ldots i_{2s}}(\pi, \rho)$. This means that the twistor field $\Psi_{i_1 \ldots i_{2s}}(\pi, \rho)$ lives on the surface $\mathcal{M} = 0$. However, this condition $\mathcal{M} = 0$ is in fact a condition that the determinant of the $(4 \times 4)$ matrix

$$g := \|g_a \| \| := \sqrt{2/\mu} \left( \pi_\alpha^I \right| \rho_\alpha^B,$$

(5.9)

is equal to unity, $\det g = 1$. In the definition (5.9) of the matrix $g$, we use the notation for the $(4 \times 2)$ blocks $(g^1_\alpha) := (\pi^I_\alpha)$ and $(g^{2+B}_\alpha) := (\rho^B_\alpha)$, and the upper index $\beta = 1, 2, 3, 4$ (the number of rows of the matrix $g$) is related to the automorphism group $SU(2)$. Moreover,
the "reality" conditions in (5.1) and (4.2) are written (with the help of the matrix $g$) in a concise form

$$Bg = g^*B, \quad B_{\alpha^\beta} = B_{\alpha^\beta}, \quad (5.10)$$

where the matrix $B$ is given in (4.2) and the matrix $B$ coincides with $B$ but acts in the space of the representation of the automorphism group $SU(2)$. Therefore, in view of the condition $\det(g) = 1$, the matrices (5.9) belong to the $SU^*(4)$ group where the matrix $B$ characterizes $SU^*(4)$ involutive automorphism [32] (see also [37], Sect. 3.3.2).

ii) Condition (5.5b) means that the following property

$$\Psi_{I_1\ldots I_{2s}}(\pi^K, \rho^B, \rho^K_B) = \exp\{i\kappa_I L/2\} \Psi_{I_1\ldots I_{2s}}(\pi^K, \rho^B)$$

is fulfilled for the twistor field where the matrix $\kappa^B$ obeys the reality condition of the form

$$(\kappa^B)^* = \epsilon^{IJ} \kappa_I \epsilon_{CB}.$$

iii) Equations (5.4) give the conditions for the $SU(2)$ covariance of the twistor field:

$$\Psi_{I_1\ldots I_{2s}}(\pi^L u_i, \rho^B u_B A) = u_{I_1 J_1} \ldots u_{I_{2s} J_{2s}} \Psi_{I_1\ldots I_{2s}}(\pi^K, \rho^A_I), \quad (5.12)$$

where $u_{I^K}$ and $u_{A^B}$ are the same matrix belonging to the $SU(2)$ group.

The transformations of the arguments of the twistor field in (5.11) and (5.12) are related to the right-hand transformations of the matrix $g$ (5.9) by the upper-triangular matrix $h$:

$$g \to gh, \quad h := \|h_{\alpha^\beta}\| := \begin{pmatrix} u_I J & \kappa_B^B \\ 0 & u_A^B \end{pmatrix}, \quad (5.13)$$

where the diagonal blocks of $h$ are the same $SU(2)$ matrix $u$, $uu^\dagger = 1$, det $u = 1$, and the $(2\times2)$ matrix $\kappa$ satisfies the condition $\kappa^* = \sigma_2 \kappa \sigma_2$. The matrices $h$ form the stability subgroup $\mathcal{K} \subset SU^*(4)$. This means that the twistor field $\Psi_{I_1\ldots I_{2s}}(\pi, \rho)$ is a function on the coset $SU^*/\mathcal{K}$. The points on this coset are parametrized by elements (5.9) of the $SU^*(4)$ group. In addition, the notation $\bar{\beta}$ in (5.9) for the second index of the matrix $g_{\alpha^\beta}$ differs from the first index $\alpha$ and indicates that the bar-indices are related to the stability subgroup $\mathcal{K} \subset SU^*(4)$.

The solution of equations (5.5a) and (5.5b) can be represented in the following form.

Similar to the four-dimensional case [16,21], equation (5.5a) is solved by taking the delta-function $\delta(\epsilon^{\alpha\beta\gamma\delta} \pi^K_{\alpha} \pi^K_{\beta} \rho^C_{\gamma} \rho^C_{\delta} - \mu^2)$ as a factor in the twistor field, while equation (5.5b) is greatly simplified by extracting the exponent $\exp(iu_0^I I/v_0)$, where $v_0 = (\pi^I \sigma_0 \pi_I)$, $u_0^I I = (\pi^I \sigma_0 \rho_I)$ (see 4.8). Thus, we consider the field

$$\Psi_{I_1\ldots I_{2s}}(\pi, \rho) = \delta(\epsilon^{\alpha\beta\gamma\delta} \pi^K_{\alpha} \pi^K_{\beta} \rho^C_{\gamma} \rho^C_{\delta} - \mu^2) e^{i\mu I \sigma_0} \tilde{\Psi}_{I_1\ldots I_{2s}}(\pi, \rho), \quad (5.14)$$

where the field $\tilde{\Psi}_{I_1\ldots I_{2s}}(\pi, \rho)$ satisfies the equations

$$J_{\alpha I} \tilde{\Psi}_{I_1\ldots I_{2s}} = -\frac{2s + 1}{2} (\sigma_{\alpha})_{(I} \kappa_{I_{2s}} \tilde{\Psi}_{I_2\ldots I_{2s}}), \quad (5.15)$$

$$\pi_{\alpha I} \frac{\partial}{\partial \rho^A_{\alpha}} \tilde{\Psi}_{I_1\ldots I_{2s}}(\pi, \rho) = 0. \quad (5.16)$$
As a result, the field \( \tilde{\Psi}_{I_1...I_2s}(\pi, \rho) \) is represented by an infinite series where each term has the SU(2) spin \( s \) and its indices are represented by the tensor product of the spinor indices of \( \pi^I_\alpha \) and \( \rho^A_\alpha \) and their products. For example, in the case \( s=1/2 \), the field \( \tilde{\Psi}_{I_1...I_2s}(\pi, \rho) \) has infinite expansion with the terms

\[
\tilde{\Psi}_I(\pi, \rho) = \pi^I_\alpha \psi^\alpha + \epsilon^{\alpha\beta\gamma\delta} \pi^I_\beta \pi^I_\gamma \rho^I_\delta \varphi^\alpha + \pi^I_\alpha \epsilon^{\beta_1\beta_2\gamma_1\gamma_2} \pi^I_{\gamma_1} \pi^I_{\gamma_2} \psi^\alpha_{\beta_1\beta_2} + \ldots ,
\]

where \( \psi^\alpha, \varphi^\alpha, \psi^\alpha_{\beta_1\beta_2}, \ldots \) describe the infinite degrees of freedom of the infinite spin representation.

Summing up this section, we conclude that twistor fields (5.14), (5.15), (5.16), forming the space of irreducible massless infinite spin representations for the Poincaré group in six dimensions, were constructed.

6 Field twistor transform and space-time infinite spin fields

An important task in the description of infinite spin representation is finding the field twistor transform that links the twistor field formulation with the space-time one (see some discussion e.g. in [7, 8]).

One of the possible 6D space-time descriptions of the twistor transform uses the additional spinorial coordinates. Namely, using the twistor fields \( \Psi_{I_1...I_2s}(\pi, \rho) \) (5.1), we can construct the fields

\[
\pi^{I_1}_{\alpha_1} \ldots \pi^{I_2s}_{\alpha_2s} \Psi_{I_1...I_2s}(\pi, \rho) ,
\]

which are SU(2) scalars:

\[
J_a \left( \pi^{I_1}_{\alpha_1} \ldots \pi^{I_2s}_{\alpha_2s} \Psi_{I_1...I_2s} \right) = 0 .
\]

Then, performing the following integral transform

\[
\Phi_{\alpha_1...\alpha_2s}(x, \rho) = \int \mu(\pi) e^{ix^m p_m} \pi^{I_1}_{\alpha_1} \ldots \pi^{I_2s}_{\alpha_2s} \Psi_{I_1...I_2s}(\pi, \rho) ,
\]

where

\[
\mu(\pi) := \mu_{11} \wedge \mu_{22} \wedge \mu_{33} \wedge \mu_{44} , \quad \mu_{\alpha\beta} := \frac{1}{2} d\pi^I_\alpha \wedge d\pi^I_\beta
\]

is a real integration measure in “\( \pi \)-space”, which can also be written in the form \( \mu(\pi) \sim \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\lambda\rho} \mu_{\alpha\mu} \wedge \mu_{\beta\nu} \wedge \mu_{\gamma\lambda} \wedge \mu_{\delta\rho} \), and \( p_m = \pi^I_\alpha (\tilde{\sigma}_m)^{\alpha\beta} \pi^I_\beta \) as in (4.12), we obtain a completely symmetric space-time field \( \Phi_{\alpha_1...\alpha_2s}(x, \rho) \), which depends on the space-time coordinates \( x^m \) and additional spinor variables \( \rho^I_\alpha \).

The field (6.3) automatically satisfies the Dirac equation

\[
i \frac{\partial}{\partial x^m} (\tilde{\sigma}^m)^{\beta\alpha} \Phi_{\alpha_1...\alpha_2s}(x, \rho) = 0 ,
\]

that yields the Klein-Gordon equation

\[
\frac{\partial}{\partial x^m} \frac{\partial}{\partial x^m} \Phi_{\alpha_1...\alpha_2s}(x, \rho) = 0 .
\]

for all values of the spin \( s \geq 0 \), while for \( s = 0 \) eq. (6.6) is checked directly. For all \( s \) equations (6.5) and (6.6) follow from the identity \( \epsilon^{\alpha\beta\gamma\delta} \pi^{I\alpha}_\alpha \pi^{I\beta}_\beta \pi^{I\gamma}_\gamma \equiv 0. \)
Two other equations of motion for the space-time field \((6.3)\) are obtained from equations \((5.5)\) for the twistor field \(\Psi_{I_1...I_2s}(\pi, \rho)\). Namely, the twistor condition \((5.5a)\) is equivalent to the equation
\[
(i \frac{\partial}{\partial x^m} \rho^K_{\beta} (\sigma^m)^{\beta\gamma} \rho_{\gamma K} + 2\mu^2) \Phi_{\alpha_1...\alpha_2s} = 0, \tag{6.7}
\]
while the equation
\[
(i \frac{\partial}{\partial x^m} \frac{\partial}{\partial \rho^K_{\beta}} (\sigma^m)^{\beta\gamma} \frac{\partial}{\partial \rho_{\gamma K}} - 2) \Phi_{\alpha_1...\alpha_2s} = 0 \tag{6.8}
\]
is a consequence of the twistor condition \((5.5b)\).

In addition, the space-time field \((6.3)\) obeys the equations
\[
\rho^I_{\beta}(\sigma_a)_i^K \frac{\partial}{\partial \rho^K_{\beta}} \Phi_{\alpha_1...\alpha_2s} = 0. \tag{6.9}
\]
To deduce equation \((6.9)\), we use solution \((5.14)\) for the twistor field, which is included in the integrand in the right-hand side of \((6.3)\). Then the left-hand side of equations \((6.9)\) takes the form
\[
\mu(\pi) e^{ix^m p_m} \left[(\rho_v \sigma_a \partial_\rho_v) e^{iu_0 I/v_0} \right] \delta(\epsilon^{\alpha\beta\gamma\delta}(\pi\alpha\pi\beta)(\rho\gamma\rho\delta) - \mu^2) \tilde{\Psi}_{\alpha_1...\alpha_2s} + \int \mu(\pi) e^{ix^m p_m} e^{iu_0 \rho_0 I/v_0} \left[(\rho_v \sigma_a \partial_\rho_v) \delta(\epsilon^{\alpha\beta\gamma\delta}(\pi\alpha\pi\beta)(\rho\gamma\rho\delta) - \mu^2) \tilde{\Psi}_{\alpha_1...\alpha_2s} \right], \tag{6.10}
\]
where we use the concise notation \(\tilde{\Psi}_{\alpha_1...\alpha_2s} := \pi_{\alpha_1}^{I_1}...\pi_{\alpha_2s}^{I_2s} \Psi_{I_1...I_2s}, \partial_{\rho^K_{\beta}} := \frac{\partial}{\partial \rho^K_{\beta}}\) and omit the contracted SU(2) spinor indices \((\pi\alpha\pi\beta) := \pi^K_{\alpha} \pi^K_{\beta}, (\rho\gamma\rho\delta) := \rho^K_{\gamma} \rho^K_{\delta}, \rho^I_{\beta}(\sigma_a)_i^K \frac{\partial}{\partial \rho^K_{\beta}} := (\rho_v \sigma_a \partial_\rho_v)\).

Taking into account the identity \(J_a (u_0 I/v_0) = 0\), we represent the first term in the right hand side of \((6.10)\) as
\[
- \int \mu(\pi) e^{ix^m p_m} \left[(\rho_v \sigma_a \partial_\rho_v) e^{iu_0 \rho_0 I/v_0} \right] \delta(\epsilon^{\alpha\beta\gamma\delta}(\pi\alpha\pi\beta)(\rho\gamma\rho\delta) - \mu^2) \tilde{\Psi}_{\alpha_1...\alpha_2s} .
\]
Finally, after integration by parts in this expression, equality \((6.10)\) takes the form
\[
(\rho_v \sigma_a \partial_\rho_v) \Phi_{\alpha_1...\alpha_2s} = 2 \int \mu(\pi) e^{ix^m p_m} e^{iu_0 \rho_0 I/v_0} \left[J_a \delta(\epsilon^{\alpha\beta\gamma\delta}(\pi\alpha\pi\beta)(\rho\gamma\rho\delta) - \mu^2) \tilde{\Psi}_{\alpha_1...\alpha_2s} \right]. \tag{6.11}
\]
However, due to relations \((6.2)\) and \(J_a \delta(\epsilon^{\alpha\beta\gamma\delta}(\pi\alpha\pi\beta)(\rho\gamma\rho\delta) - \mu^2) = 0\), the right-hand side of equality \((6.11)\) is equal to zero; therefore, we obtain \((6.9)\).

Thus, we have proposed the space-time formulation (with additional spinor variables \(\rho^I\)) of the 6D infinite spin fields which is a generalization of our formulation of the 4D infinite spin fields \([16, 18, 21]\). In the framework of this formulation, we obtained the equations of motion \((6.3) - (6.9)\) for the 6D infinite spin fields \((6.3)\). Equations \((6.3) - (6.8)\) are 6D analogs of the corresponding equations in the four-dimensional case. The new SU(2) conditions \((6.9)\) replace the U(1) condition in the 4D case. Also, in contrast to the 4D case, the 6D infinite spin fields in the space-time formulation have external spinor indices \((\alpha_1, ..., \alpha_2s), \) the number of which is determined by the quantum number \(s\).
7 Summary and outlook

Let us summarize the results. We have studied the twistor field realization of the Poincaré group massless irreducible representations of infinite spin in the 6D Minkowski space. It was shown that infinite spin representations are realized on the twistor fields living in the two-twistor space, which may be interesting in the context of the higher spin field theory. The group generators and the corresponding Casimir operators were derived in the considered two-twistor realization. As result, we found a full set of equations of motion (5.4), (5.5) for the infinite spin twistor field (5.1) that is the totally symmetric SU(2) tensor of rank 2s. Thus, we believe that the irreducible massless representations under consideration are completely described in terms of twistor fields. In addition, using the field twistor transform, we determine space-time fields that describe infinite spin representations in the formulation with an additional spinor coordinate.

Note that the spinor ρA α (4.2), corresponding to the introduced second twistor, has the same SU∗(4) chirality as the spinor πI α (3.1) of the first twistor. In principle, we can introduce the second twistor of different chirality. As one can see, this situation arises after the transition from the spinor variable ρA α (4.2) to the spinor variable ηA α (4.6) with the help of a simple Fourier transform with respect to these variables. We are going to give a complete answer to this question in subsequent papers.

As a continuation of this research, it would be interesting to construct one-dimensional classical and quantum mechanics of an infinite spin particle in both space-time and twistor formulations, similar to our works [16,21] for the 4D case. This will allow us to derive the twistor transforms that provide correspondence of the phase variables in space-time with twistor patterns. In addition, analysis of the quantum twistor wave function will make it possible to analyze the spin expansion of the twistor field. In the forthcoming papers, we plan also to study the relation of our twistor model to another 6D space-time formulation with additional vector variables, which was proposed in [7,8] (see also [3]).

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Appendix A: 6D spinor notation

In this Appendix we present the 6D spinor notation used in this paper.

In the Weyl representation, the (8×8) Dirac γ-matrices have the form

\[
\gamma^m = \begin{pmatrix}
0 & (\sigma^m)_{\alpha\dot{\beta}} \\
(\tilde{\sigma}^m)_{\dot{\alpha}\beta} & 0
\end{pmatrix},
\quad \{\gamma^m, \gamma^n\} = 2\eta^{mn},
\]

where the Weyl indices run four values: α, \dot{\alpha} = 1\ldots, 4. We take the following representation [32,33]:

\[
\sigma^m = (\sigma^0, \sigma^a), \quad \tilde{\sigma}^m = (\sigma^0, -\sigma^a), \quad a = 1,\ldots, 5,
\]

where \(\sigma^0 = 1_4, \sigma^1 = \tau_1 \otimes 1_2, \sigma^a = \tau_2 \otimes \tau_{\dot{a} - 1}\) at \(\dot{a} = 2, 3, 4, \sigma^5 = \tau_3 \otimes 1_2,\) and \(\tau_{1,2,3}\) are the Pauli matrices.
The matrices $B$ and $C$ defining $(\gamma^m)^* = -B\gamma^m B^{-1}$ and $(\gamma^m)^T = -C\gamma^m C^{-1}$ have the form

$$B = \begin{pmatrix} B & 0 \\ 0 & B^T \end{pmatrix}, \quad C = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix},$$

where

$$B = \|B_{\alpha}^{\beta}\| = 1_2 \otimes i\tau_2 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & i\tau_2 \end{pmatrix}.$$  \hspace{1cm} (A.2)

Using quantities (A.2) we can construct the matrices

$$(\sigma^m)_{\alpha\beta} = (\sigma^m)_{\alpha\gamma}(B^{-1})_{\beta}^{\gamma}, \quad (\bar{\sigma}^m)^{\alpha\beta} = B_{\gamma}^{\alpha}(\bar{\sigma}^m)^{\gamma\beta},$$

(A.3)

having only undotted indices. These matrices are antisymmetric, $(\sigma^m)_{\alpha\beta} = -(\sigma^m)_{\beta\alpha}$, $(\bar{\sigma}^m)^{\alpha\beta} = -(\bar{\sigma}^m)^{\beta\alpha}$ and satisfy the relations

$$(\sigma^m)_{\alpha\gamma}(\bar{\sigma}^n)^{\gamma\delta} + (\sigma^m)_{\alpha\gamma}(\bar{\sigma}^m)^{\gamma\beta} = 2\eta^{m\gamma}\delta_{\alpha}^{\delta}, \quad (A.4)$$

$$(\sigma^m)_{\alpha\beta}(\bar{\sigma}_m)^{\gamma\delta} = 4\delta_{\alpha}^{\delta} \quad (\sigma^m)_{\alpha\beta}(\bar{\sigma}_m)^{\gamma\delta} = -4\delta_{\alpha}^{\delta}, \quad (A.5)$$

We use the convention for the antisymmetrization: $X_{[\alpha\beta]} := \frac{1}{2}(X_{\alpha\beta} - X_{\beta\alpha})$. Moreover, the matrices (A.3) are expressed through each other by means of the equations

$$(\sigma^m)_{\alpha\beta}(\sigma_m)^{\gamma\delta} = 2\varepsilon_{\alpha\beta\gamma\delta}, \quad (\bar{\sigma}^m)^{\alpha\beta}(\bar{\sigma}_m)^{\gamma\delta} = 2\varepsilon^{\alpha\beta\gamma\delta}, \quad (A.6)$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ and $\varepsilon^{\alpha\beta\gamma\delta}$ are the totally antisymmetric tensors with components $\varepsilon_{1234} = \varepsilon^{1234} = 1$. Therefore, the following relations are fulfilled

$$(\sigma^m)_{\alpha\beta}(\sigma_m)^{\gamma\delta} = 2\varepsilon_{\alpha\beta\gamma\delta}, \quad (\bar{\sigma}^m)^{\alpha\beta}(\bar{\sigma}_m)^{\gamma\delta} = 2\varepsilon^{\alpha\beta\gamma\delta}. \quad (A.7)$$

Transformations of the matrices (A.3) under complex conjugation look like:

$$[(\sigma^m)_{\alpha\beta}]^* = B_{\alpha}^{\gamma}B_{\beta}^{\delta}(\sigma^m)^{\gamma\delta}, \quad [(\bar{\sigma}^m)^{\alpha\beta}]^* = (\bar{\sigma}^m)^{\gamma\delta}(B^{-1})_{\alpha}^{\gamma}(B^{-1})_{\beta}^{\delta}. \quad (A.8)$$

The $\sigma$-matrices with two vector indices are defined by

$$(\sigma_{mn})_{\alpha\beta} = \frac{1}{4}(\sigma_m\sigma_n - \sigma_n\sigma_m)_{\alpha\beta}, \quad (\bar{\sigma}_{mn})^{\alpha\beta} = -(\sigma_{mn})_{\alpha\beta} = \frac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m)^{\alpha\beta}. \quad (A.9)$$

They satisfy the identities

$$[(\sigma_{mn})_{\alpha\beta},(\sigma_{kl})_{\gamma\delta}] = \eta_{nk}\sigma_{ml} - \eta_{ml}\sigma_{nk} + \eta_{mk}\sigma_{ln} - \eta_{ln}\sigma_{mk}, \quad (A.10)$$

$$[(\bar{\sigma}_{mn}),(\bar{\sigma}_{kl})] = \eta_{nk}\bar{\sigma}_{ml} - \eta_{ml}\bar{\sigma}_{nk} + \eta_{mk}\bar{\sigma}_{ln} - \eta_{ln}\bar{\sigma}_{mk}.$$

There are also the following equalities:

$$((\bar{\sigma}^m)^{\alpha\beta}(\bar{\sigma}_{mn})^{\gamma\delta} = \frac{1}{2}\delta_{\beta}^{\gamma}\delta_{\delta}^{\alpha} - 2\delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta}, \quad (A.11)$$

$$(\bar{\sigma}^n)^{\alpha\beta}(\bar{\sigma}_{nm})^{\gamma\delta} = \frac{1}{2}(\bar{\sigma}_m)^{\alpha\beta}\delta_{\delta}^{\gamma} + 2\delta_{\delta}^{\alpha}(\bar{\sigma}_m)^{\beta\gamma}, \quad (\sigma^m)_{\alpha\beta}(\sigma_{mn})^{\gamma\delta} = -\frac{1}{2}(\sigma_m)_{\alpha\beta}\delta_{\delta}^{\gamma} - 2\delta_{\delta}^{\gamma}(\sigma_m)_{\beta\alpha}. \quad (A.12)$$
The eight-component Dirac spinor $\Psi$ and its Dirac conjugate one $\bar{\Psi} = \Psi^\dagger \gamma_0$ are represented by two four-component Weyl spinors

$$
\Psi = \begin{pmatrix} \psi_\alpha \\ \chi_\dot{\alpha} \end{pmatrix}, \quad \bar{\Psi} = (\bar{\chi}^\alpha, \bar{\psi}_\dot{\alpha}), \quad \bar{\psi}_\dot{\alpha} = (\psi_\alpha)^*, \quad \bar{\chi}^\alpha = (\chi_\dot{\alpha})^*.
$$

(A.13)

Opposite to the $D = 1 + 3$ case, in the $D = 1 + 5$ Minkowski case the spinors $\psi_\alpha$ and $\bar{\psi}_\dot{\alpha}$ realize the same spinor representation. Therefore, the spinorial components $\psi^1_\alpha := \psi_\alpha$, $\psi^2_\alpha := (B^{-1})_\alpha^\beta \bar{\psi}_\dot{\beta}$ form the SU(2) Majorana-Weyl spinor possessing the SU(2)-reality ($\epsilon_{12} = \epsilon^{21} = 1$):

$$
(\psi^I_\alpha)^* = \epsilon_{IJ} B_\alpha^\beta \psi^J_\beta.
$$

(A.14)

Due to (A.8) and (A.14), we find that the vector $\psi^I \bar{\sigma}^m \psi_I$ is real when spinor $\psi_{\alpha I}$ is even.

By analogy, defining $\chi^{a1} := \bar{\chi}^a$, $\chi^{a2} := \chi^\beta B_\beta^a$, we obtain the SU(2) Majorana-Weyl spinor with the SU(2)-reality

$$
(\chi^{aI})^* = \epsilon_{IJ} \chi^{bJ}(B^{-1})_b^a \chi_I,
$$

(A.15)

and, therefore, the vector $\chi_I (\sigma^m) \chi_I$ is real at even spinor $\chi_I^a$.

### Appendix B: Twistor realization of $D = 1 + 5$ conformal algebra

Here we show that 28 quantities $X_{[ab]}$ (defined in (3.5)) form the $\mathfrak{so}(2,6)$ algebra with respect to the Poisson brackets (3.3).

Generators (3.3) are represented by the following set of quantities

$$
P_{\alpha\beta} := \pi^I_\alpha \pi^J_\beta \epsilon_{IJ}, \quad K^{\alpha\beta} := \omega^I_\alpha \omega^J_\beta \epsilon_{IJ}, \quad M^{\beta}_\alpha := \pi^I_\alpha \omega^J_\beta \epsilon_{IJ},
$$

(B.1)

that have $D = 6$ Weyl spinor indices. Due to the SU(2) reality conditions (B.1) and (B.2), the set of the generators (B.1) goes into itself under complex conjugation.

Using the twistor Poisson brackets (3.3), we obtain that the generators (B.1) obey the algebra

$$
\{P_{\alpha\beta}, P_{\gamma\delta}\}_\text{PB} = 0, \quad \{K^{\alpha\beta}, K^{\gamma\delta}\}_\text{PB} = 0, \\
\{P_{\alpha\beta}, K^{\gamma\delta}\}_\text{PB} = -\delta^{\gamma}_\alpha M^\beta_\delta + \delta^{\delta}_\beta M^\alpha_\gamma - \delta^\gamma_\beta M^\alpha_\delta, \\
\{M^{\beta}_\alpha, P_{\gamma\delta}\}_\text{PB} = -\delta^\gamma_\beta P_{\alpha\delta} - \delta^\delta_\alpha P_{\gamma\alpha}, \quad \{M^{\beta}_\alpha, K^{\gamma\delta}\}_\text{PB} = \delta^\gamma_\alpha K^{\beta\delta} + \delta^\delta_\alpha K^{\gamma\beta}, \\
\{M^\beta_\alpha, M^\beta_\gamma\}_\text{PB} = \delta^\gamma_\alpha M^\beta_\delta - \delta^\delta_\beta M^\gamma_\alpha.
$$

(B.2)

After passing from (B.1) to pure real generators with vectorial indices

$$
P_m := P_{\alpha\beta}(\bar{\sigma}_m)^{\alpha\beta}, \quad K_m := \frac{1}{8} K^{\alpha\beta}(\sigma_m)_{\alpha\beta}, \quad M_{mn} := -M^{\beta}_\alpha(\sigma_{mn})_{\beta}^\alpha, \quad D := -\frac{1}{2} M^\alpha_\alpha,
$$

(B.3)

the algebra (B.2) takes the standard form of the $D = 1 + 5$ conformal algebra:

$$
\{P_m, P_n\}_\text{PB} = 0, \quad \{K_m, K_n\}_\text{PB} = 0, \\
\{P_m, K_n\}_\text{PB} = 2 \left( M_{mn} - \eta_{mn} D \right), \\
\{M_{mn}, P_k\}_\text{PB} = \eta_{mk} P_n - \eta_{nk} P_m, \quad \{M_{mn}, K_k\}_\text{PB} = \eta_{mk} K_n - \eta_{nk} K_m, \\
\{D, P_m\}_\text{PB} = P_m, \quad \{D, K_m\}_\text{PB} = -K_m, \quad \{D, M_{mn}\}_\text{PB} = 0, \\
\{M_{mn}, M_{kl}\}_\text{PB} = \eta_{mk} M_{nl} + \eta_{ml} M_{nk} - \eta_{ml} M_{nk} - \eta_{nk} M_{ml}.
$$

(B.4)
After the transition from the generators $M_{mn}$, $P_m$, $K_m$, $D$ to their linear combinations

$$L_{mn} = M_{mn}, \quad L_{6m} = \frac{1}{2}(P_m - K_m), \quad L_{7m} = \frac{1}{2}(P_m + K_m), \quad L_{67} = -D,$$

algebra (B.4) takes the standard form for the $\mathfrak{so}(2,6)$ algebra

$$\{L_{MN}, L_{KL}\}_P = \eta_{MK}L_{NL} + \eta_{NL}L_{MK} - \eta_{ML}L_{NK} - \eta_{NK}L_{ML}, \quad (B.5)$$

where $M = (m,6,7)$, $\eta_{66} = -\eta_{77} = -1$, $\eta_{6m} = \eta_{7m} = \eta_{67} = 0$.

**Appendix C: Calculation of $C_6$**

Using (4.13), (4.14) we obtain

$$\Pi^n M_{nm} = v_m A + w_m B + u^I_m C_{IA}, \quad (C.1)$$

where

$$A = -\frac{1}{4} \left( \pi^I \frac{\partial}{\partial \pi^I} - \rho^A \frac{\partial}{\partial \rho^A} \right)^2 - \left( \rho^A \frac{\partial}{\partial \rho^A} \right) \left( \pi^I \frac{\partial}{\partial \pi^I} \right), \quad (C.2)$$

$$B = -\epsilon^{IJ} \epsilon^{AB} \left( \pi^I \frac{\partial}{\partial \rho^A} \right) \left( \pi^J \frac{\partial}{\partial \rho^B} \right), \quad (C.3)$$

$$C_{IA} = 2 \left( \rho^B \frac{\partial}{\partial \rho^B} + 1 \right) \left( \pi^I \frac{\partial}{\partial \pi^I} \right) - 2 \left( \pi^I \frac{\partial}{\partial \rho^A} \right) \left( \pi^J \frac{\partial}{\partial \rho^B} \right) - 2 \left( \rho^B \frac{\partial}{\partial \rho^A} \right) \left( \pi^I \frac{\partial}{\partial \rho^B} \right), \quad (C.4)$$

and vector variables $v_m, w_m, u^I_m$ are defined in (4.8). The operators (C.2), (C.3), (C.4) have the following commutators with these vector variables:

$$[A, v_m] = -v_m \left( \pi^I \frac{\partial}{\partial \pi^I} - \rho^B \frac{\partial}{\partial \rho^B} + 1 \right) - 2 u^I_m \left( \pi^I \frac{\partial}{\partial \rho^A} \right),$$

$$[A, w_m] = w_m \left( \pi^I \frac{\partial}{\partial \pi^I} - \rho^B \frac{\partial}{\partial \rho^B} - 5 \right) + 2 u^I_m \left( \rho^A \frac{\partial}{\partial \pi^I} \right), \quad (C.5)$$

$$[A, u^I_m] = -u^I_m + \frac{1}{2} v_m \epsilon^{IJ} \left( \rho^A \frac{\partial}{\partial \pi^J} \right) - \frac{1}{2} w_m \epsilon^{AB} \left( \pi^I \frac{\partial}{\partial \rho^B} \right),$$

$$[B, v_m] = 0, \quad [B, w_m] = -4 v_m - 4 u^I_m \left( \pi^I \frac{\partial}{\partial \rho^A} \right), \quad [B, u^I_m] = -v_m \epsilon^{AB} \left( \pi^I \frac{\partial}{\partial \rho^B} \right), \quad (C.6)$$

$$[C_{IA}, v_m] = -2 v_m \left( \pi^I \frac{\partial}{\partial \rho^A} \right),$$

$$[C_{IA}, w_m] = 4 u_m \rho^B \frac{\partial}{\partial \rho^B} + 4 \frac{\partial}{\partial \pi^I} + 4 u^I_m \left( \pi^I \frac{\partial}{\partial \pi^I} \right) + 4 u^B_m \left( \rho^B \frac{\partial}{\partial \rho^A} \right)$$

$$+ 2 w_m \left( \pi^I \frac{\partial}{\partial \rho^A} \right), \quad (C.7)$$

$$[C_{IA}, u^K_m] = 2 u^I_m \left[ \delta^K \delta^B_C \left( \pi^I \frac{\partial}{\partial \rho^A} \right) - \delta^K \delta^B_C \left( \pi^I \frac{\partial}{\partial \rho^A} \right) - \delta^K \delta^B_A \left( \pi^I \frac{\partial}{\partial \rho^C} \right) \right]$$

$$+ \left( \delta^K \delta^B_A \left( \rho^C \frac{\partial}{\partial \rho^C} \right) - \delta^K \delta^B_A \left( \rho^C \frac{\partial}{\partial \rho^C} \right) - \delta^K \left( \rho^B \frac{\partial}{\partial \rho^A} \right) \right).$$
Taking into account the relations (4.19), (4.11) and these commutators (C.5), (C.6), (C.7), we obtain

\[ \Pi^k M_{km} \Pi I M^{lm} = r \left\{ AB + BA + \left[ 3 \left( \frac{\partial}{\partial \pi^l} \right) - 3 \left( \frac{\partial}{\partial \rho^A} \right) - 25 \right] B \right. \\
\left. + 3 \varepsilon^{AB} \left( \frac{\partial}{\partial \rho^A} \right) C_{IB} - \frac{1}{4} \varepsilon^{IJ} \varepsilon^{AB} C_{IA} C_{JB} \right\}. \]  

(C.8)

In view of the equalities

\[ [A, B] = - \left[ \left( \frac{\partial}{\partial \pi^l} \right) - \left( \frac{\partial}{\partial \rho^A} \right) - 6 \right] B, \]  

(C.9)

expression (C.8) takes the form

\[ \Pi^k M_{km} \Pi I M^{lm} = r \left\{ 2A + \left( \frac{\partial}{\partial \pi^l} \right) - \left( \frac{\partial}{\partial \rho^A} \right) - 1 \right\} B - \frac{1}{4} \varepsilon^{IJ} \varepsilon^{AB} C_{IA} C_{JB} \right\}, \]  

(C.11)

where the quantity \( r \) is defined in (4.10). The last term in (C.11) equals

\[ - \frac{1}{4} \varepsilon^{IJ} \varepsilon^{AB} C_{IA} C_{JB} = \left[ -3 \left( \frac{\partial}{\partial \pi^l} \right) + 3 \left( \frac{\partial}{\partial \rho^A} \right) + 5 - \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \rho^A} \right) \right. \right. \\
\left. \left. - \frac{1}{2} \varepsilon^{IJ} \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \rho^A} \right) \right\} B \right\]. \]  

(C.12)

Therefore, (C.11) is written as

\[ \Pi^k M_{km} \Pi I M^{lm} = r \left\{ 2A - 2 \left( \frac{\partial}{\partial \pi^l} \right) + 2 \left( \frac{\partial}{\partial \rho^A} \right) + 4 - \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \rho^A} \right) \right. \right. \\
\left. \left. - \frac{1}{2} \varepsilon^{IJ} \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \rho^A} \right) \right\} B \right\}. \]  

(C.13)

The quadratic Casimir operator of the Lorentz algebra \( so(1, 5) \) with the generators \( M_{mn} \) in the representation (4.13) has the form

\[ M^{mn} M_{mn} = - \frac{1}{2} \left[ \left( \frac{\partial}{\partial \pi^l} \right) + \left( \frac{\partial}{\partial \rho^A} \right) \right]^2 + 4 \left( \frac{\partial}{\partial \pi^l} \right) - 4 \left( \frac{\partial}{\partial \rho^A} \right) \]  

(C.14)

With the use of the relation

\[ \varepsilon^{IJ} \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \pi^l} \right) = \left( \frac{\partial}{\partial \pi^l} \right) \left( \frac{\partial}{\partial \pi^l} \right) - \left( \frac{\partial}{\partial \pi^l} \right)^2 \]  

(C.15)
and a similar relation for the spinor $\rho^A_\alpha$, the last two terms in (C.15) take the same form as some terms in (C.13).

Thus, from (C.13), (C.14) we obtain that the six-order Casimir operator (2.6) (after its action on the states $\Psi$ subject conditions (4.16) and (4.17)) takes the form

$$C_6 = -\mu^2 \left\{ -\frac{1}{4} \left( \pi^I \frac{\partial}{\partial \pi^I} \right)^2 + \frac{1}{2} \left( \pi^K \frac{\partial}{\partial \pi^K} \right) \left( \pi^I \frac{\partial}{\partial \pi^I} \right) - \frac{1}{4} \left( \rho^A_\alpha \frac{\partial}{\partial \rho^A_\alpha} \right)^2 + \frac{1}{2} \left( \rho^B_\alpha \frac{\partial}{\partial \rho^B_\alpha} \right) \left( \rho^A_\alpha \frac{\partial}{\partial \rho^A_\alpha} \right) - \frac{1}{2} \left( \pi^I \frac{\partial}{\partial \pi^I} \right) \left( \rho^A_\alpha \frac{\partial}{\partial \rho^A_\alpha} \right) + \left( \rho^I_\alpha \frac{\partial}{\partial \rho^A_\alpha} \right) \left( \pi^K \frac{\partial}{\partial \pi^K} \right) \right\} \right. \right.$$

Here we use the fact that the states $\Psi$ satisfy conditions (4.16) and (4.17), which respectively implies the relations $r\Psi = 2\mu^2\Psi$ and $B\Psi = \Psi/2$. The last expression for $C_6$ given in (C.16) is represented in the form

$$C_6 = -\mu^2 J_a J_a, \quad (C.17)$$

where we introduce the SU(2) generators $J_a$ defined in (4.20).

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