Approximation of Measures on $S^n$ by discrete Measures

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Abstract

We study the asymptotic behavior, as $\rho \to \infty$, of discrete measures on $S^{n-1}$ that are induced by radially projecting point masses concentrated on the integral lattice-points within dilates $\rho D$ of a compact body $D$, for various classes of $D$. The results depend sensitively on the differential geometric properties of $\partial D$.

1 Introduction

Recently, Douglas, Shiffman, and Zelditch, motivated by considerations arising from the vacuum selection problem in string theory, have investigated questions in physics, aspects of which involve the equidistribution of radial projections of integral lattice-points on certain hypersurfaces in $R^{n-1}$, e.g., onto level surfaces of quadratic forms [2]. One question of this kind which they take up concerns the study of the asymptotics, as $\rho \to \infty$, of the family of discrete measures on the surface $\partial E$ of an ellipsoid $E$ resulting from summing unit-weighted radial projections onto $\partial E$ of the lattice-points in $\rho E - \{0\}$. If, for example, $E = S^{n-1}$, this becomes: Does the action of the measure on $S^{n-1}$ which results from summing unit-weighted projections onto $S^{n-1}$ of lattice-points $N \in Z^n - \{0\}$, with $|N| \leq \rho$, and then rescaling the result by $n/\rho^n$, tend, on smooth functions, to Lebesgue measure on $S^{n-1}$ as $\rho \to \infty$, and if so, how rapidly? This last question can be regarded as an instance of the general question of approximating smooth measures on $S^{n-1}$ by discrete measures, and in this paper, we will examine this question, which was suggested by questions arising in [2], which can be recast in this form. Depending on the measure being approximated, issues associated with the curvature of a specific surface associated with the measure can play a role, in particular the presence of zones of zero curvature on this associated surface, which we will illustrate by several examples.
It is a pleasure to mention that our interest in these questions arose from a conversation with Zelditch about his above-cited recent work with Douglas and Shiffman.

We will begin the paper with a description of a general method, which can then be applied to various instances of the problem under consideration, in particular, to the zero curvature case, which is not discussed in [2]. An interested reader could probably glean the required background from a close reading of one or more of the papers [2], [7], [10], as well as others, but in our view, expository clarity makes it very desirable to begin with such an exposition, which we have written to lead as smoothly as possible into the notation and approaches of the various papers to which we refer. In particular, since detailed treatments of instances of the zero curvature cases are in general quite prolonged, and very closely mimic existing discussions in the literature for the corresponding classical constant-density lattice-point problem, we have confined our discussion of these examples to general descriptions of how the overall techniques described in this paper can be adapted to closely follow treatments of the corresponding constant-density cases in the literature.

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Suppose $d\mu$ is a Borel measure on $S^{n-1}$ having a continuous, positive, piecewise smooth density function $m$ with respect to Lebesgue measure. If $d\mu$, as in the example above, happens to be Lebesgue measure, i.e., $m = m(\theta) \equiv 1$, the corresponding family of discrete measures is parametrized by $\rho > 0$, and supported, as indicated above, on the unit-weighted radial projections of the non-zero integral lattice-points $N$, for which $|N| < \rho$. Neglecting rescaling, the natural counterpart for a general measure on $S^{n-1}$ having positive density $m$ with respect to Lebesgue measure is the family $d\Gamma_\rho$ of discrete measures supported on the unit-weighted radial projections of the non-zero integral lattice-points in $\rho D$, where $D$ is the compact set whose boundary is given in polar coordinates by $r = (m(\theta))^{1/n}$. We assume that $m(\theta)$, or equivalently $\partial D$, is sufficiently regular so that the divergence theorem is valid for $D$ – for example, $\partial D$ could be smooth, or the boundary of a polyhedron.

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We begin by noting that the effect of the above projection measure on a smooth function $f(\theta)$ on $S^{n-1}$ is identical to that of the lattice-point count
over $\rho D - \{0\}$, weighted by the weight-zero homogeneous extension $F$ of $f$ to $\mathbb{R}^n - \{0\}$. There are various analytical approaches to the estimation of such sums, e.g., Riesz means, as in [3], [5], etc., or convolution smoothing techniques, as in, for example, [2] and [7]. We will employ a convolution smoothing technique, since it is relatively simple to implement and describe.

In our outline of this approach, we will temporarily assume, for minor technical reasons, that $D$ has been modified by the removal of a neighborhood of the origin. Then, assuming for the moment that $f$ is positive, and denoting its homogeneous extension to $\mathbb{R}^n$ by $F$, the above lattice-point count $N_D(\rho)$ in $\rho D$ can be overestimated by summing over the weighted lattice-points in a slight expansion $E(\rho)$ of $\rho D$ and underestimated by summing over the weighted lattice-points in a slight contraction $C(\rho)$ of $\rho D$. The Poisson summation formula provides a natural analytic approach for handling such sums, but since the function $F$ is not smooth when multiplied by the indicator functions of $E(\rho)$ or $C(\rho)$, i.e., at the boundaries of $E(\rho)$ and $C(\rho)$, we will employ a slight refinement of the above idea, which can be approximately described by saying that we convolve $F$, restricted to $E(\rho)$ and $C(\rho)$, respectively, with a smooth compactly supported approximate delta function $\delta_\epsilon$, whose support is chosen to lie within a ball of radius $\epsilon(\rho)$ which is sufficiently small so that the convolutions are very close to the function $F$ restricted to $\rho D$, except in small neighborhoods of the boundary. If the support of $\delta_\epsilon$ is of sufficiently small diameter, it will then be exactly true, if $F$ is constant, and approximately true otherwise, that $\delta_\epsilon \ast F_C(\rho) \leq F_{\rho D} \leq \delta_\epsilon \ast F_{E(\rho)}$, where $F_S$ denotes the product of $F$ with the indicator function $\chi_S$ of $S$. If $F$ is not constant, the deviation from correctness of the above inequality can be quantified in terms of a bound for the directional derivatives of $F$ and the diameter of the support of $\delta$. In view of this, if, on $E(\rho)$ and $C(\rho)$ respectively, we modify $F_{E(\rho)}$ and $F_{C(\rho)}$ by the addition and subtraction of a suitably small quantity whose size depends on $\rho$, the above inequality becomes correct, and can be exploited, following the method of [7], which treated the case of constant density, to derive estimates for $N_D(\rho)$. If $f$ is of mixed sign, we can express it as the difference of two positive smooth functions, proceed as above, and later combine the estimates for both. It is almost immediate that any smooth function is expressible as the difference of two positive smooth functions, whose size and the size of whose derivatives are comparable to those of the original function. For instance, we could define $f = f^+ - f^-$, where $f^+ = f + c$, and $f^- = c$, where $c > \min_{\theta \in S_{n-1}} f(\theta)$.

We now show in greater detail how to carry out this program. The outcome in a general sense will be that after rescaling by $n/\rho^n$, discrete sphere measures which arise in the above way will converge to $d\mu$ on $S_{n-1}$.
at the same rate as that at which $1/\rho^n$ times the standard unit-density lattice-point sum for $\rho D$ converges to the measure of the set $D$. The specific sense in which this is the case will be developed more precisely below.

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We will retain the notation of Section (2). As remarked there, the task is equivalent to the estimation, as $\rho \to \infty$, of the lattice-point count over $\rho D$, with lattice-points counted with density $F$, where $F$ is a smooth homogeneous function of weight 0 on $R^n - \{0\}$ (if $F$ is constant, it is smooth at the origin as well). Unless $F$ is constant, the origin causes a small, easily surmounted technical difficulty, which can be handled in various ways. We will, following Douglas, Shiffman, and Zelditch, deal with this by estimating the weighted lattice-point count in the shell $\rho S = \rho D - \frac{1}{2} D$, which is a dilate of the basic shell $S = D - \frac{1}{2} D$. Since this estimate will be uniform in the dilation parameter, this automatically leads to estimates for the count for $\frac{1}{2} \rho S, \frac{1}{4} \rho S, \frac{1}{8} \rho S, \ldots$. We then obtain the desired result for $\rho D$ by adding up the results for $\rho S, \frac{1}{2} \rho S, \frac{1}{4} \rho S, \frac{1}{8} \rho S, \ldots$. The weighted lattice-point count for $\rho S$ will be estimated along the lines outlined above, i.e., by bracketing the true count between appropriate convolutions.

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We now pass to the details.

We will use the following lemma (cf. [5], pp. 262–263), whose role is to show that the Fourier transform $\hat{g}(y)$ of a smooth function $g(x)$ on a domain $\mathcal{S}$ for which the divergence theorem is valid can be expressed as the Fourier transform of a smooth function on $\partial \mathcal{S}$, and that the process of transfer to the boundary picks up a factor of $1/|y|$.

**Lemma 1.** Suppose $g(x)$ is a smooth function on $R^n$, and $y$ a fixed vector in $R^n$. Then there exists a smooth vector field $F(x)$ on $R^n$, such that

1. $\text{div} \left[ (2\pi i |y|)^{-1} e^{2\pi i (x,y)} F(x) \right] = e^{2\pi i (x,y)} g(x)$

2. The derivatives up to order $k$ of the components of $F(x)$ can be bounded, independently of $y$, in terms of bounds for the derivatives up to order $k + 1$ of $g(x)$.

**Proof.**
This is proved in Lemma 3 of \[5\]. In particular, since, setting \(x = (x_1, \ldots, x_n)\), \(y = (y_1, \ldots, y_n)\), and \(F = (F_1, \ldots, F_n)\), the divergence of 
\[
(2\pi i|y|)^{-1} e^{2\pi i(x,y)} F(x)
\]
is
\[
e^{2\pi i(x,y)} \left[(\frac{y_1}{|y|}) F_1 + (2\pi i|y|)^{-1} \partial F_1 / \partial x_1 + \ldots 
+ (\frac{y_n}{|y|}) F_n + (2\pi i|y|)^{-1} \partial F_n / \partial x_n \right],
\]
the requirements of the present lemma will be satisfied if

1. \(\text{div} F = 0\)
2. \((\frac{y}{|y|}, F) = g\)

with the derivatives of the \(F_j\)'s bounded as indicated above. This is, however, precisely the assertion of Lemma 3 of \[5\], taking \(\beta = \frac{y}{|y|}\) in the notation of that lemma. □

**Corollary.** The first assertion of the lemma immediately implies, by the divergence theorem, that

\[
\int_S e^{2\pi i(x,y)} g(x) dx = (2\pi i|y|)^{-1} \int_{\partial S} e^{2\pi i(x,y)} (F(x), n(x)) ds_x,
\]
where \(n(x)\) is the exterior normal to \(\partial S\) at \(x\). This gives the desired expression of the Fourier transform over \(S\) in terms of a Fourier transform over \(\partial S\).

As previously indicated, we will assume that \(F > 0\), since the general case can be deduced from this one by taking differences of the resulting estimates. For a small positive parameter \(\epsilon(\rho)\), which will depend on \(\rho\), and which we will sometimes simply write as \(\epsilon\) for short, we define, respectively, a slight expansion \(E(\rho)\) of \(\rho S\), and a slight contraction \(C(\rho)\) of \(\rho S\), by setting

\[
E(\rho) = (\rho + \epsilon)D - (\frac{\rho}{2} - \epsilon)D, \quad C(\rho) = (\rho - \epsilon)D - (\frac{\rho}{2} + \epsilon)D.
\]
Since we are interested in the asymptotics of the weighted lattice-point count over \(\rho S\) as \(\rho \to \infty\), we may assume, by replacing \(S\) by a sufficiently large dilate of its original self if necessary, that for small \(\epsilon(\rho)\), \(E(\rho)\) contains an \(\epsilon(\rho)\)-neighborhood of \(\rho S\), and that \(\rho S\) contains an \(\epsilon(\rho)\)-neighborhood of \(C(\rho)\).

Throughout the following, our basic approach is via the Poisson summation formula, and is essentially the same technique that was employed in \[7\], the only difference being that now the weight function is no longer a
constant. Since $N_S(\rho) = \sum_{N \in \rho S} F(N)$, a purely formal application of the Poisson summation formula leads to

$$\rho^n \sum_{N \in \mathbb{Z}^n} \hat{F}_S(\rho N)$$

(1)

as an analytic expression for the weighted lattice-point count, where $\hat{F}_S$ denotes the Fourier transform of $F_S$.

As previously mentioned, there are convergence difficulties, since the function which is equal to $F(x)$ on $S$ and equal to zero on the complement of $S$ is discontinuous at the boundary of $S$ unless $F \equiv 0$. This problem is of course present in the constant density case as well, and it can be addressed, as was done in [7] for that case, and as we have previously indicated, by slightly expanding and contracting $\rho S$, into $E(\rho)$ and $C(\rho)$, respectively, and then smoothing the indicator functions of $E(\rho)$ and $C(\rho)$ by convolving them with a smooth approximate delta function $\delta_\epsilon(x)$, whose support lies in a ball of diameter $\epsilon(\rho)$. Then if $F$ is constant,

$$\sum_{N \in C(\rho)} (\delta_\epsilon * F_C)(N) \leq N_S(\rho) \leq \sum_{N \in E(\rho)} (\delta_\epsilon * F_E)(N),$$

(2)

and since the functions being summed are smooth with compact support, the Poisson summation formula can be applied to both sides of the above inequality to obtain asymptotics for $N_S(\rho)$ (cf. [7]).

In the case of non-constant $F(x)$, a minor technical issue connected with this approach arises from the fact that $F(x)$ is no longer necessarily equal to its convolution by $\delta_\epsilon(x)$. However, for large $\rho$, $(\delta_\epsilon * F)(x)$ is very close to $F$ on a dilated shell $\rho S$, since any first derivative of $F$ is homogeneous of weight $-1$, and therefore the oscillation of $F$ over the intersection of a ball of diameter $\epsilon$ with $\rho S$ is at most of the order of $(1/\rho)\epsilon(\rho)$. I.e., $F$ is close to being constant on $\rho S$ as $\rho$ becomes large, so, as previously indicated, the addition and subtraction of a suitable small quantity $\eta_\rho$ in inequality (2) for the constant density case results in a correct inequality for the general case:

$$\sum_{N \in C(\rho)} [\delta_\epsilon * (F_C - \eta_\rho \chi_C)](N) \leq N_S(\rho) \leq \sum_{N \in E(\rho)} [\delta_\epsilon * (F_E + \eta_\rho \chi_E)](N),$$

or
\[
\sum_{N \in C(\rho)} (\delta \ast F_C)(N) - \sum_{N \in C(\rho)} (\delta \ast \eta \chi_C)(N) \leq N_S(\rho)
\]

\[
\leq \sum_{N \in E(\rho)} (\delta \ast F_E)(N) + \sum_{N \in E(\rho)} (\delta \ast \eta \chi_E)(N),
\]

(3)

where \(\chi_S\) denotes the indicator function of \(S\), and for notational simplicity we have written \(C\) and \(E\) for \(C(\rho)\) and \(E(\rho)\), respectively.

Since \(\eta\) will be chosen to be of the order of \((1/\rho)\epsilon(\rho)\), the analysis of the contribution to the above inequality that is attributable to the presence of the \(\eta\) terms on either side will be easily seen to lead to the presence, on either side, of a term of the order of \(\rho^{n-1}\epsilon(\rho)\). The choice of \(\epsilon\) as a function of \(\rho\) will depend on the shape of \(D\), and for the time being, we will simply carry it along as an unspecified function of \(\rho\).

When we come to the analysis of the sums in (3), the heart of the matter lies in the asymptotics of the Fourier transform of \(F\) over \(C(\rho)\) and over \(E(\rho)\), which reduces, by an obvious rescaling, to the analysis of the asymptotics of the Fourier transform \(\hat{F}_S\) of \(F\) over shells \(S\) which are very close to \(S\) for large \(\rho\).

Accordingly, we next take up the asymptotic analysis of

\[
\hat{F}_S(y) = \int_S F(x)e^{2\pi i(x, y)} \, dx,
\]

where \(S = S(\rho)\) can stand for a rescaling by \(1/\rho\) of either \(C(\rho)\) or \(E(\rho)\).

By Lemma 1,

\[
\int_{\partial S} e^{2\pi i(x, y)} F'(x) \, dx = (2\pi i|y|)^{-1} \int_{\partial S} e^{2\pi i(x, y)} (F(x), n(x)) \, ds_x,
\]

where the size of the derivatives of the vector field \(F\) are controlled by the size of the derivatives of the homogeneous function \(F\), which, since \(S\) is a shell, we can smooth at the origin without affecting its values in \(S\).

The boundary of \(S\) splits into two components, which are analyzed similarly. Thus, the asymptotic analysis of

\[
\int_S F(x)e^{2\pi i(x, y)} \, dx
\]

amounts to the study of the asymptotic behavior of integrals

\[
\int_S g(x)e^{2\pi i(x, y)} \, dx,
\]

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where $S$ is a hypersurface in $\mathbb{R}^n$ of the considered type, and $g$ is a smooth function on $S$. This is a highly studied subject (cf. [3], [7], [8], et al.). The results are very dependent on the geometry of $S$, although the case in which $S$ has everywhere positive Gaussian curvature is easy to analyze, using, for example, a straightforward application of stationary phase, or even a simple calculus argument. It is well-known that in this case (cf. [3]),

$$
\int_S g(x)e^{2\pi i(x,y)} \, dx \ll |y|^{-(n-1)/2},
$$

(5)

where the implied constant depends on bounds for the derivatives of $g$, which, for the purposes of this discussion, can be thought of as a function defined in a neighborhood of $S$. By the corollary to Lemma 1, it follows that in the positive curvature case, for an $n$-dimensional shell $\mathcal{S}$,

$$
\int_{\mathcal{S}} e^{2\pi i(x,y)} F(x) \, dx \ll |y|^{-(n+1)/2}.
$$

(6)

Although the positive curvature case has already been discussed in [2], we will use this case to illustrate our approach, since it provides an exceptionally simple paradigmatic description of techniques of considerably more general applicability. We will then describe modifications that are required in illustrative cases for which $\partial D$ does not have everywhere positive curvature. In general, specific applications require individual adaptations of the method, but these adaptations can often be easily inferred from existing treatments of the constant-density case.

The technique closely follows that of [7], and so will be simply outlined here. Denoting as above by $N(S(\rho))$ the weighted lattice-point count in $\rho S$, we will indicate how to use the Fourier transform bound to estimate the right side of the inequality (3). The left side is handled in a similar way. The Poisson summation formula can be applied to the right side of (3). We first take up the term

$$
\sum_{N \in E(\rho)} (\delta \ast F)(N),
$$

which, by the Poisson summation formula equals

$$
\sum_{N \in E(\rho)} \hat{\delta}(N) \hat{F}(\rho)(N),
$$

$$
= \hat{\delta}(0) \hat{F}(0) + \sum_{N \in E(\rho)} \hat{\delta}(N) \hat{F}(\rho)(N),
$$

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where the prime on the summation sign means that the origin is omitted from the sum.

Since \( \hat{\delta}(0) = 1 \), the last quantity can be rewritten as

\[
\int_{E(\rho)} F(x) \, dx + \sum'_{N \in E(\rho)} \hat{\delta}_\epsilon(N) \hat{F}_{E(\rho)}(N).
\]

If in the above we replace the term \( \int_{E(\rho)} F(x) \, dx \) by \( \int_{\rho S} F(x) \, dx \), the error corresponding to the replacement can be estimated by noting that

\[
\left| \int_{E(\rho)} F(x) \, dx - \int_{\rho S} F(x) \, dx \right| = \left| \int_{E(\rho) - \rho S} F(x) \, dx \right| \ll \text{vol}(E(\rho) - \rho S) \ll \epsilon \rho^{n-1}.
\]

The remaining term on the right side of (3), namely

\[
\sum_{N \in E(\rho)} (\delta_\epsilon * \eta \chi E)(N),
\]

is clearly \( \ll (\eta_\rho \times \text{the volume of } E_\rho) \), i.e., of order \( \eta_\rho \rho^n \).

We can thus rewrite the right side of (3) as

\[
\int_{\rho S} F(x) \, dx + O(\epsilon \rho^{n-1} + \eta \rho^n) + \sum'_{N \in E(\rho)} \hat{\delta}_\epsilon(N) \hat{F}_{E(\rho)}(N).
\]

(7)

The last sum is handled in exactly the same way as in the well-known case in which \( F \) is constant. Namely, the necessary estimate on \( \hat{F}_{E(\rho)}(N) \) is provided by (6), and is identical with the estimate for the constant case. As in that case, integration by parts shows that the term \( \hat{\delta}_\epsilon(N) \) is \( \ll 1/(1 + \epsilon |N|^k) \), for any fixed integer \( k \). The sum is estimated by splitting it into two parts, over \( N \) for which \( |N| < 1/\epsilon \), and over \( N \) for which \( |N| \geq 1/\epsilon \), respectively. Each of these parts is estimated by comparison with an integral, using a sufficiently high value of \( k \) to produce convergence, and the estimate (6) for the Fourier transform. The result is then minimized by choosing \( \epsilon \) to balance the estimates, which leads to the choice of \( \epsilon = \rho^{-(n-1)/(n+1)} \), which in turn results in an estimate of \( \rho^{(n-1)(n/(n+1))} \) for the infinite sum in (7). The above choice of \( \epsilon \) results in the same estimate for the term of
order $\epsilon \rho^{n-1}$ in (7), and since $\eta$ is of the order of $\rho^{-1} \epsilon$, we obtain the same estimate for the term $\eta \rho^n$ in (7).

Since a similar analysis can be applied to the left side of the inequality (3), this shows that in the positive curvature case, the weighted lattice-point count over $\rho S$ equals

$$\int_{\rho S} F(x) \, dx + O(\rho^{(n-1)(n/(n+1))}).$$

In order to analyze the weighted lattice-point count over $\rho D$, we add up repetitions of the above quantity, corresponding to $\rho, \rho/2, \rho/2^2, \ldots, \rho/2^k$, where $k$ is taken to be of the order of $\log_2 \rho$, so that $\rho/2^k < 1$. This takes account of all lattice-points in $\rho D$ except perhaps for a fixed finite number near the origin, and since these latter have no effect on the asymptotics, we find that

$$N_D(\rho) = \int_{\rho D} F(x) \, dx + \sum_{j=0}^{k} O((\rho/2^j)^{(n-1)(n/(n+1)))}.$$}

Since the implied constants in the $O$ terms are uniformly bounded, this implies that

$$N_D(\rho) = \rho^n \int_{D} F(x) \, dx + O(\rho^{(n-1)(n/(n+1))}).$$

Now it is a consequence of equation (4) of [5] that

$$\int_{D} F(x) \, dx = (1/n) \int_{S^{n-1}} f(\theta) m(\theta) \, d\theta,$$

so we obtain the following equivalent form of the result of Douglas, Shiffman, and Zelditch in the positive curvature case:

**Theorem.** (cf. [2]) With notation as above,

$$\int_{S^{n-1}} f(\theta) m(\theta) \, d\theta = (n/\rho^n) N_D(\rho) + O(\rho^{-2n/(n+1)}).$$

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This is an estimate for the rapidity of the convergence of the discrete measures \((n/\rho^n)\,d\Gamma_\rho\) to \(d\mu = m(\theta)d\theta\). The estimate, which was given in a somewhat different form in [2], coincides with the usual scaled error term for the standard lattice-point problem for bodies with boundaries having strictly positive Gaussian curvature (cf. [3]).

To justify (8), we note that from (4) of [5] it follows that

\[
\int_D F(x)\,dx = \int_0^1 t^{n-1}\,dt \int_{\partial D} F(tx)(x, n(x))\,ds_x,
\]

where \(n(x)\) is the outward normal to \(\partial D\). Since \(F\) is homogeneous of weight 0, the double integral equals

\[
(1/n) \int_{\partial D} F(x)(x, n(x))\,ds_x,
\]

and coordinatizing \(\partial D\) by \(S^{n-1}\) via the radial map, this becomes

\[
\int_{S^{n-1}} f(\theta) \Phi(\theta)(x, n(x))\,d\theta,
\]

where \(\Phi(\theta)\) is the Radon-Nikodym derivative \(ds_x/d\theta\), and \(x\) is regarded as a function of \(\theta\) via the radial map. It follows easily from elementary geometric considerations that

\[
\Phi(\theta) = |x|^n/(x, n(x)),
\]

so

\[
\int_{\partial D} F(x)(x, n(x))\,ds_x = \int_{S^{n-1}} f(\theta) \Phi(\theta)(x, n(x))\,d\theta
\]

\[
= \int_{S^{n-1}} |x|^n f(\theta)\,d\theta.
\]

But on \(S^{n-1}\), \(|x| = (m(\theta))^{1/n}\), so the last integral equals

\[
\int_{S^{n-1}} f(\theta) m(\theta)\,d\theta,
\]

from which (8) follows.
6 The Case in which Curvature can Vanish

Classical lattice-point asymptotics for dilates $\rho D$ of a body $D$ for which the curvature of $\partial D$ is not always positive can be quite intricate, and are highly dependent on the manner in which the curvature of $\partial D$ vanishes, as well as on the placement of $D$ in relation to the integer lattice. For example, in $\mathbb{R}^2$, if the curvature of $\partial D$ vanishes to finite order at a finite number of points, the classical lattice-point asymptotics depend on the order to which the curvature vanishes at the points in question, as well as on Diophantine properties of the normal vectors to $\partial D$ at those points (cf. [7]). The situation can become much more complicated in higher dimensions, and nuances of this type similarly affect convergence rates of the discrete measures with which we are dealing in this paper. Since, once one has the necessary Fourier transform asymptotics, methods for dealing with these issues closely mimic those for the classical case, we will content ourselves with briefly indicating what happens in a few interesting representative cases.

As indicated above, the central analytic issue is the detailed asymptotics of the Fourier transform of smooth functions on $\partial D$. In the case of everywhere positive curvature, one has the previously mentioned result that

$$\int_{\partial D} g(x)e^{2\pi i (x,y)} \, dx \ll |y|^{-(n-1)/2},$$

where the estimate does not depend on the directional component of $y$.

This estimate is generally false if the curvature of $\partial D$ vanishes on some non-void subset of $\partial D$, and a useful description of what happens can be quite complicated, for example, if

$$\partial D = \{(x_1, \ldots, x_n) \mid x_1^{2k} + \cdots + x_n^{2k} = 1\},$$

or if $D$ is a polyhedron, (cf. [6] and [9]).

If we write

$$\int_D e^{2\pi i (x,y)} \, dx$$

in polar coordinates as $\Psi(r, \phi)$, one quite general fact along these lines is given by Theorem 1 of [8]. Namely, if $\partial D$ is real-analytic and $D$ is convex, then the function

$$\Lambda(\phi) = \sup_r r^{(n+1)/2} \Psi(r, \phi)$$

is in $L^p(S^{n-1})$, for some $p > 2$. Thus, under conditions of considerable generality, the Fourier transform asymptotics coincide with those of the positive
curvature case, up to multiplication by a function in $L^p(S^{n-1})$. The convexity hypothesis is unnecessary in $\mathbb{R}^2$, and possibly in higher dimensions, although this is not generally known. The requirement of real-analyticity can be replaced by somewhat weaker hypotheses (cf. [14]). For later related papers, cf. [1], [15].

**Example 1**

The arguments by which the above-mentioned theorem is proved can be uneventfully applied to obtain the following counterpart of (5), in the case, for example, in which $D$ is convex and $\partial D$ is real-analytic (and under weaker hypotheses in $\mathbb{R}^2$):

$$\int_{\partial D} g(x) e^{2\pi i(x,y)} \, dx \ll \Lambda(\phi)|y|^{-(n-1)/2},$$

where $\Lambda(\phi)$ is in $L^p(S^{n-1})$, for some $p > 2$. This implies, by a straightforward application of the techniques of this paper, that if we modify the definition of the discrete measures $\Gamma_\rho$ by replacing $Z^n$ by its image under the action of an element $\gamma$ of $\text{SO}(n)$, call the resulting measures $\Gamma_\rho(\gamma)$, and denote the sum corresponding to $N_D(\rho)$ by $N_D(\rho, \gamma)$, that we obtain the following result:

**Theorem A (vanishing curvature case).** With notation as above,

$$\int_{\text{SO}(n)} |R_D(\rho, \gamma)| \, d\gamma \ll O(\rho^{-2n/(n+1)}),$$

where

$$R_D(\rho, \gamma) = \int_{S^{n-1}} f(\theta) m(\theta) \, d\theta - (n/\rho^n) N_D(\rho, \gamma).$$

I.e., even for a quite general version of the zero curvature case, the error estimate for the positive curvature case holds for the $L^1$ norm over the rotation group of the errors for the “rotated” measures. There are, of course, various consequences corresponding to other $L^p$ norms as well.

**Example 2**

Our next example is the special case in which

$$\partial D = \{(x_1, \ldots, x_n) \mid x_1^{2k} + \cdots + x_n^{2k} = 1\}.$$
The relevant estimate on the Fourier transform is given by Theorem 2 of [6], which states that for sufficiently smooth $g$, if we express
\[
\int_{\partial D} g(x)e^{2\pi i(x,y)} \, dx
\]
in polar coordinates as $\Psi(r, \phi) (\phi \in S^{n-1}, \phi = (\phi_1^*, \ldots, \phi_n^*))$, then on the set of points $(r, \phi)$ for which exactly $j$ of the $\phi_i^*$'s vanish,
\[
\Psi(r, \phi) \ll (A(\phi))^{-\beta} r^{-\alpha_j},
\]
where in the above, $A(\phi)$ is the product of the non-zero $\phi_i^*$'s, $\beta = (k - 1)/(2k-1)$, and $\alpha_j = (j/2k) + (n-j-1)/2$. This is, of course, in some sense a special case of (9), but it gives considerably more detailed information about the asymptotics of the Fourier transform for this case.

The classical lattice-point problem for this case is discussed in [6]. The principal result is that the error term is of order $\rho R$, where $R = \max(A, B)$ with $A = (2k-1)(n-1)/2k$ and $B = n(n-1)/(n+1)$. The estimate is best possible if $A > B$. This result is obtained by separately analyzing groups of lattice-points on the Fourier transform side of the Poisson summation formula, where the grouping is arranged into sets defined by the vanishing of a specific number of the $\phi_i^*$'s. That the result is sometimes optimal is a consequence of the observation that if $A > B$, the contribution to the Fourier transform side of the Poisson summation formula coming from lattice-points on axes defined by the vanishing of all but one of the $\phi_i^*$'s constitutes the major contribution to the error, and that the behavior of this contribution can be analyzed by a stationary phase argument.

The adaptation of this result to the present context is straightforward, and the arguments are nearly identical to those for the original result, so as before, we will content ourselves with a statement of the theorem in the present context, using notation as above. The result in the present context is:

**Theorem B (zero curvature case).** *With notation as above, if*
\[
\partial D = \{(x_1, \ldots, x_n) \mid x_1^{2k} + \cdots + x_n^{2k} = 1\},
\]
*then*
\[
\int_{S^{n-1}} f(\theta)m(\theta) \, d\theta = (n/\rho^n)N_D(\rho) + O(\rho^{A-n}),
\]
*and if $A > B$, this is best possible.*
Example 3

As a final example, we will consider the polyhedral case, and in the interests of expository and combinatorial simplicity, describe a typical result for the 2-dimensional case.

In the classical constant-density lattice-point problem, if at least one of the perpendicular vectors to a face of a compact polyhedron has rational coordinates, there are an infinite number of $\rho_i \to \infty$ from which an infinitesimal displacement results in a modification of the lattice-point count of order $\rho^{n-1}$, so in this circumstance the error estimate is of true order $\rho^{n-1}$. Since a simple estimate of Gauss shows that the error term is always $\ll \rho^{n-1}$, polyhedra can be worst possible cases for lattice-point error asymptotics.

Paradoxically, this situation is not generic for polyhedra, as was noticed by Khintchine [4] in the 2-dimensional case. Khintchine’s result is that the error estimate corresponding to almost any rotation of the integer lattice $\mathbb{Z}^2$ is in fact extremely small. Specifically, for any $\epsilon > 0$, it is almost always $\ll \log^{1+\epsilon} \rho$. Later work [7], [9], [10], [11], [12], [13], [16] has considered various aspects of the $n$-dimensional case, as well as additional features and refinements of the 2-dimensional case.

As in the previous examples, methods for the classical constant-density lattice-point problem carry over with very little change to our context. As a typical illustrative representative of what can be expected, we begin by recalling the result, due to Skigranov [11], that if $D$ is an algebraic polygon, (one for which the ratios of the direction numbers of the normals to its faces are all algebraic of degree $\geq 2$), then the classical lattice-point error term is $\ll \rho^\epsilon$, for any $\epsilon > 0$. This result is also a special case of theorems in later papers [10] and [12]. It can be obtained by using a more detailed form of the estimate (9), as was done in the previous example [11], but one which is adapted to the particular case in which $D$ is a polygon (cf. [5]). With such an estimate in hand, the lattice-points on the Fourier transform side of the Poisson summation formula are split into two groups: those in finite-width bands surrounding the normal vectors to the sides of $D$, and all the rest. The contribution from the lattice-points exterior to the bands can be estimated by comparison with an integral, while the series arising from the contributions from lattice-points within bands is estimated by using Diophantine properties of the ratios of direction numbers associated to the corresponding normals. Since the relevant estimate for the Fourier transform of $D$ is singular at these directions, the poor approximability of these ratios, which is a consequence of Roth’s Theorem, is crucial (cf. e.g., [9], p. 858 for a similar argument). The corresponding result, in the context of the present
paper and expressed in the notation of this paper, is that for a polygon of the above type,

**Theorem C (polygonal case).** With notation as above,

\[
\int_{S^{n-1}} f(\theta) m(\theta) \, d\theta = \left(\frac{n}{\rho} \right)^n N_D(\rho) + O(\rho^{-n}).
\]

### 7 Conclusion

We have described a general method for describing the accuracy with which a large class of measures on \( S^n \) can be approximated by a naturally associated family of discrete measures. The case in which \( \partial D \), in the notation of this paper, has everywhere positive curvature has been previously studied in [2], and is taken in this paper as a basic template for the description of a general approach to such problems, in particular, cases involving zero curvature. As in the classical constant-density lattice-point problem, there are special instances of the positive curvature case, e.g., arithmetically defined positive definite quadratic forms, in which the general estimate can be improved by exploiting the underlying arithmetic character of the associated surface. In the case in which \( \partial D \) contains subsets on which the curvature vanishes, the situation becomes vastly more intricate, although there are general results, e.g., along the lines of the first of our three examples in the zero curvature case. There are also arithmetic instances of the zero curvature case having special features, as in the second of our three examples. One can give general results for polyhedra as well, e.g., along the lines of [10], [12], [16]. The methods will in general mimic those for the constant-density case, once one is in possession of the appropriate Fourier transform asymptotics (for the asymptotics in the polyhedral case, cf. [9]). A kind of meta-conclusion is that in general, the derivable asymptotics associated with the presently considered class of problems coincide with the corresponding results for the classical constant-density case, by virtue of the fact that the relevant Fourier transform asymptotics are effectively identical.

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