Solitary wave solutions of a Whitham-Bousinessq system

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Abstract

The travelling wave problem for a particular bidirectional Whitham system modelling surface water waves is under consideration. This system firstly appeared in [9], where it was numerically shown to be stable and a good approximation to the incompressible Euler equations. In subsequent papers [8, 10] the initial-value problem was studied and well-posedness in classical Sobolev spaces was proved. Here we prove existence of solitary wave solutions and provide their asymptotic description. Our proof relies on a variational approach and a concentration-compactness argument. The main difficulties stem from the fact that in the considered Euler-Lagrange equation we have a non-local operator of positive order appearing both in the linear and non-linear parts.

1 Introduction

1.1 Motivation and background

We consider the system

\[
\eta_t = -v_x - i \tanh(D)(\eta v), \quad (1.1)
\]
\[
v_t = -i \tanh(D)\eta - i \tanh(D) \left( \frac{v^2}{2} \right), \quad (1.2)
\]

with $D = -i \partial_x$ and $\mathcal{F}(\tanh(D)f)(\xi) = \tanh(\xi) \hat{f}(\xi)$, where $\mathcal{F}$ is the Fourier transform

$$
\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} \, dx.
$$

We are interested in solitary wave solutions of (1.1)–(1.2) and so we search for solutions of the form

\[
\eta(x, t) = \eta(x + ct), \quad v(x, t) = v(x + ct), \quad (1.3)
\]
with \( \eta(x + ct), \ v(x + ct) \to 0 \), as \( |x + ct| \to \infty \). Here \( \eta \) denotes surface elevation and \( v \) is the fluid velocity at the surface. Such systems describes permanent progressive surface waves of a fluid layer. The model (1.1)-(1.2) approximates the two-dimensional water wave problem for an inviscid incompressible potential flow.

System (1.1)–(1.2) was introduced in [9] as a fully dispersive model for two-way wave propagation. In recent years several fully dispersive two-way systems have been paid close attention to, and for a survey we again refer to [9], where they compare some of these models, and in particular, find that the system (1.1)–(1.2) approximates the full water-wave problem better than some of the other fully dispersive bidirectional models. It worth to point out that those demonstrate a good agreement with experiments [7]. In addition, System (1.1)–(1.2) has been recently shown to be well-posed in [8–10]. Moreover, the result is global if the initial data is sufficiently small. The latter is the main advantage of Equations (1.1)–(1.2) comparing with other models regarded in [9]. Indeed, there is a local well-posedness result for another system regarded in [9] obtained by Ehrnström, Pei and Wang [13]. However, they impose an additional non-physical condition \( \eta \geq C > 0 \). Kalisch and Pilod [16] have proved local well posedness for a surface tension regularisation of the system from [13] without the positivity assumption \( \eta > 0 \). However, the maximal time of existence for their regularisation is bounded by the capillary parameter. Whereas one does not need any regularisation or special non-physical conditions to claim the well posedness for (1.1)–(1.2). In fact Model (1.1)–(1.2) can be regarded itself as a regularization, arising naturally from the Hamiltonian formulation of the water wave problem, for the system introduced by Hur and Pandey [15]. There is another Whitham-Boussinesq type model known to be well-posed that was not considered in [9] and was introduced by Duchêne, Israwi and Talhouk [11]. For systems regarded in [11] and [13] existence of solitary waves was proved in [6] and [20], respectively. The next natural step is to show the solitary wave existence for Equations (1.1)–(1.2). This is the main aim of the current paper.

We use a variational approach together with Lion’s method of concentration-compactness [17] to establish the existence of solitary wave solutions of (1.1)–(1.2). This approach has been used extensively to prove existence of solitary wave solutions to equations of the form

\[
where L is a Fourier multiplier operator of order \( s \) and \( n(u) \) is a homogeneous nonlinear term. Under the travelling wave ansatz \( u = u(x + ct) \), equation (1.4) becomes

\[
(1.5)
\]

In [21] the author studied long wave model equations of the form (1.4), with \( s \geq 1 \), and proved existence and stability of solitary wave solutions. This approach was later used in [3] to prove existence of solitary waves for an equation used to model stratified fluids, with \( s = 1 \), and was later generalized in [1] to \( s \geq 1 \). A class of Whitham type equations of the form (1.4) was studied in [12], with a Fourier multiplier operator of negative order. In this case the resulting functional in the constrained minimization problem is not coercive. This makes the application of the concentration compactness theorem a lot more technical, requiring the authors to use a strategy developed in [4, 14] and first consider a related penalized functional acting on periodic functions. In the recent work [19] an entirely different approach to proving the existence of solitary wave solutions of the Whitham equation, based on the implicit function theorem instead,
was presented. In [2] the author proved existence of solitary wave solutions to two different classes of model equations, one of them of the form (1.4), for \( s > 0 \). The case when the nonlinearity \( n \) is allowed to be inhomogeneous was considered in [18], where the author proved the existence of solitary wave solutions of (1.4), for operators of positive order and with weak assumptions on the regularity of the symbol.

These methods have also been applied to bidirectional Whitham type equations. As mentioned above, in [6] the authors established the existence of solitary waves for the class of modified Green–Naghdi equations introduced in [11], and in [20] the authors proved the existence of solitary waves for the Whitham–Boussinesq system regarded in [9, 13]. Just as in [12], both of the functionals appearing in [6, 20] are noncoercive, so the minimization arguments adapted to noncoercive functionals developed in [4, 14] are used in order to obtain the existence of minimizers. In addition, the Fourier multiplier operator is entangled with the nonlinearity in [6, 20], which makes the proofs more technical.

1.2 The minimization problem

We formulate the problem in the variational settings. A Hamiltonian structure [8] of System (1.1)–(1.2) allows us to do this in a straightforward way. Indeed, under the travelling wave ansatz (1.3), Equations (1.1)–(1.2) can be written as

\[
Kv + \eta v + cK\eta = 0, \tag{1.6}
\]

\[
\eta + \frac{v^2}{2} + cKv = 0, \tag{1.7}
\]

where we have introduced a Fourier multiplier of the form

\[
K = \frac{D}{\tanh(D)}. \tag{1.8}
\]

Note that this operator is of order one. It is equivalent to the Bessel potential \( J = (1 - \partial_x^2)^{1/2} \) associated with the symbol \( \langle \xi \rangle = \sqrt{1 + \xi^2} \), since \( \xi/\tanh \xi \simeq \langle \xi \rangle \). Regarding the Hamiltonian and momentum

\[
\mathcal{H}(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \eta^2 + vKv + \eta v^2 \, dx,
\]

\[
\mathcal{I}(\eta, v) = \int_{\mathbb{R}} \eta Kv \, dx,
\]

one can notice that Equation (1.6) can be written as

\[
d_v \mathcal{H} + c d_v \mathcal{I} = 0,
\]

and Equation (1.7) as

\[
d_\eta \mathcal{H} + c d_\eta \mathcal{I} = 0.
\]

Our aim is to obtain a single travelling wave equation that can in turn be interpreted as a constrained minimization problem. We can derive a travelling wave equation in the following
way. In (1.6)–(1.7) we make the change of variable \( v = K^{-1/2} \tilde{v} \), which yields the new system
\[
K^{1/2} \tilde{v} + \eta(K^{-1/2} \tilde{v}) + cK \eta = 0, \quad (1.9)
\]
\[
\eta + \frac{(K^{-1/2} \tilde{v})^2}{2} + cK^{1/2} \tilde{v} = 0. \quad (1.10)
\]
From (1.10) we get that
\[
\eta = -\frac{(K^{-1/2} \tilde{v})^2}{2} - cK^{1/2} \tilde{v}, \quad (1.11)
\]
and inserting this into (1.9) yields
\[
\tilde{v} - K^{-1/2} \left( \frac{(K^{-1/2} \tilde{v})^3}{2} \right) - cK^{-1/2} (K^{1/2} \tilde{v} K^{-1/2} \tilde{v}) - cK^{1/2} \left( \frac{(K^{-1/2} \tilde{v})^2}{2} \right) - c^2 K \tilde{v} = 0. \quad (1.12)
\]
Here we make the change of variables \( \tilde{v} = cu \) so that (1.12) becomes
\[
\frac{1}{c^2} u - K^{-1/2} \left( \frac{(K^{-1/2} u)^3}{2} \right) - K^{-1/2} (K^{1/2} u K^{-1/2} u) - K^{1/2} \left( \frac{(K^{-1/2} u)^2}{2} \right) - Ku = 0. \quad (1.13)
\]
Now let us show that Equation (1.13) represents an Euler-Lagrange equation for some functional. Indeed, regard the surface elevation and velocity defined by \( u \) as follows
\[
\eta = -c^2 \left( \frac{(K^{-1/2} u)^2}{2} + K^{1/2} u \right), \quad (1.14)
\]
\[
v = cK^{-1/2} u, \quad (1.15)
\]
and note that
\[
\mathcal{H}(\eta_u, v_u) + c\mathcal{L}(\eta_u, v_u) = c^4 \left[ -\frac{1}{2} \int_\mathbb{R} u K u + K^{1/2} u (K^{-1/2} u)^2 + \frac{(K^{-1/2} u)^4}{4} \ d x + \frac{1}{2c^2} \int_\mathbb{R} u^2 \ d x \right],
\]
which leads us to define
\[
\mathcal{E}(u) = \frac{1}{2} \int_\mathbb{R} u K u + K^{1/2} u (K^{-1/2} u)^2 + \frac{(K^{-1/2} u)^4}{4} \ d x,
\]
\[
\mathcal{Q}(u) = \frac{1}{2} \int_\mathbb{R} u^2 \ d x.
\]
We then note that equation (1.13) can be written as
\[
\frac{d\mathcal{E}(u)}{du} + \lambda d\mathcal{Q}(u) = 0,
\]
where \( \lambda = -1/c^2 \). Hence, in order to find solutions of (1.13) we can consider the constrained minimization problem
\[
\inf_{u \in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{ u \in H^{1/2}(\mathbb{R}) : \mathcal{Q}(u) = q \right\}. \quad (1.16)
\]
Instead of working with the specific Fourier multiplier \( K \), we will work with a more general class of Fourier multipliers, and thus a more general constrained minimization problem.
Definition 1.1 (Admissible Fourier multipliers). Let operator $L$ be a Fourier multiplier, with symbol $m$, i.e.
$$
\mathcal{F}(Lf)(\xi) = m(\xi) \hat{f}(\xi).
$$
We say that $L$ is admissible if $m$ is even, $m(0) > 0$ and for some $s' > 1$ and $s > 1/2$ the symbol satisfies the following restrictions.

(i). The function $\xi \mapsto m(\xi) \langle \xi \rangle^s$ is uniformly continuous, and
$$
m(\xi) - m(0) \simeq |\xi|^{s'} \text{ for } |\xi| \leq 1,
$$
$$
m(\xi) - m(0) \simeq |\xi|^s \text{ for } |\xi| > 1.
$$

(ii). For each $\varepsilon > 0$ the kernel of operator $L^{-1/2}$ satisfies
$$
\mathcal{F}^{-1}(m^{-1/2}) \in L^2(\mathbb{R} \setminus (-\varepsilon, \varepsilon)).
$$
There exists $p \in (1, 2) \cap \left[\frac{2}{s+1}, 2\right)$ such that
$$
\mathcal{F}^{-1}(m^{-1/2}) \in L^p(-1, 1).
$$

The symbol $m(\xi) = \xi / \tanh(\xi)$ satisfies the conditions of Definition 1.1 with $s = 1$ and $s' = 2$ [5]. We have the corresponding functional
$$
\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}} \left( L^{1/2}u + \frac{1}{2}(L^{-1/2}u)^2 \right)^2 \, dx
$$
defined on $H^{s/2}(\mathbb{R})$. Our main goal is then to obtain a solution of the minimization problem
$$
\inf_{u \in U_q} \mathcal{E}(u) \quad \text{with} \quad U_q = \left\{ u \in H^{s/2}(\mathbb{R}) : Q(u) = q \right\}.
$$
For convenience we separate $\mathcal{E}$ into the functionals
$$
\mathcal{L}(u) = \frac{1}{2} \int_{\mathbb{R}} uLu \, dx,
$$
$$
\mathcal{N}_c(u) = \frac{1}{2} \int_{\mathbb{R}} L^{1/2}u(L^{-1/2}u)^2 \, dx,
$$
$$
\mathcal{N}_r(u) = \frac{1}{2} \int_{\mathbb{R}} \frac{(L^{-1/2}u)^4}{4} \, dx
$$
so that
$$
\mathcal{E}(u) = \mathcal{L}(u) + \mathcal{N}(u)
$$
where
$$
\mathcal{N}(u) = \mathcal{N}_c(u) + \mathcal{N}_r(u).
$$
We are now ready to state our main results.
Theorem 1.2. Let $D_q$ be the set of minimizers of $E$ over $U_q$. There exists $q_0 > 0$ such that for each $q \in (0, q_0)$, the set $D_q$ is nonempty and $\|u\|_{H^2}^2 \lesssim q$ uniformly for $u \in D_q$. Each element of $D_q$ is a solution of the Euler–Lagrange equation

$$\lambda u + L^{-1/2} \left( \frac{(L^{-1/2}u)^3}{2} \right) + L^{-1/2}(L^{1/2}uL^{-1/2}u) + L^{1/2} \left( \frac{(L^{-1/2}u)^2}{2} \right) + Lu = 0. \quad (1.21)$$

The Lagrange multiplier $\lambda$ satisfies

$$\frac{m(0)}{2} < -\lambda < m(0) - Dq^3, \quad (1.22)$$

where $\beta = \frac{s'}{2s'-1}$ and $D$ is a positive constant.

Our other main result concerns the asymptotic behavior of travelling wave solutions of (1.1)–(1.2).

Theorem 1.3. If $L = K$ then there exists $q_0 > 0$ such that for any $q \in (0, q_0)$ each minimizer $u \in D_q$ belongs to $H^r(\mathbb{R})$ for any $r \geq 0$ with $\|u\|_{H^r}^2 \lesssim q$, and moreover, it satisfies the following long wave asymptotics

$$\sup_{u \in D_q} \inf_{x_0 \in \mathbb{R}} \|q^{-2/3}u(q^{-1/3} \cdot) - \psi_{KdV}(\cdot - x_0)\|_{H^1(\mathbb{R})} \lesssim q^{1/6},$$

whereas the corresponding surface elevation (1.14) and speed (1.15) satisfy

$$\sup_{u \in D_q} \inf_{x_0 \in \mathbb{R}} \|q^{-2/3}u(q^{-1/3} \cdot) + \psi_{KdV}(\cdot - x_0)\|_{H^{1/2}(\mathbb{R})} \lesssim q^{1/6},$$

$$\sup_{u \in D_q} \inf_{x_0 \in \mathbb{R}} \|q^{-2/3}v(u(q^{-1/3} \cdot)) + \psi_{KdV}(\cdot - x_0)\|_{H^{3/2}(\mathbb{R})} \lesssim q^{1/6},$$

where

$$\psi_{KdV}(x) = -\lambda_0 \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{3\lambda_0} x \right)$$

and $\lambda_0 = 3/\sqrt{16}$. In addition, the Lagrange multiplier $\lambda$ satisfies

$$\lambda = -1 + \lambda_0 q^{2/3} + O(q^{5/6}).$$

We discuss here briefly how to prove Theorems 1.2, 1.3.

The main ingredient in proving Theorem 1.2 is Lion’s concentration compactness theorem [17]:

Theorem 1.4 (Concentration-compactness). Any sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset L^1(\mathbb{R})$ of non-negative functions such that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \epsilon_n \, dx = I > 0$$

admits a subsequence, denoted again $\{\epsilon_n\}_{n \in \mathbb{N}}$, for which one of the following phenomena occurs.
• (Vanishing) For each $r > 0$, one has
\[
\lim_{n \to \infty} \left( \sup_{x \in \mathbb{R}} \int_{x-r}^{x+r} e_n \, dx \right) = 0.
\]

• (Dichotomy) There are real sequences \( \{x_n\}_{n \in \mathbb{N}} \), \( \{M_n\}_{n \in \mathbb{N}} \), \( \{N_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \) and \( I^* \in (0, I) \) such that \( M_n, N_n \to \infty \), \( M_n/N_n \to 0 \), and
\[
\int_{x_n-M_n}^{x_n+M_n} e_n \, dx \to I^* \quad \text{and} \quad \int_{x_n-N_n}^{x_n+N_n} e_n \, dx \to I^*,
\]
as \( n \to \infty \).

• (Concentration) There exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \) with the property that for each \( \epsilon > 0 \), there exists \( r > 0 \) with
\[
\int_{x_n-r}^{x_n+r} e_n \, x \geq I - \epsilon,
\]
for all \( n \in \mathbb{N} \).

We will apply this theorem to \( e_n = u_n^2 \), where \( \{u_n\}_{n=1}^{\infty} \) is a minimizing sequence, and show that the vanishing and dichotomy scenarios cannot occur. Then we obtain a convergent subsequence of \( \{u_n\}_{n=1}^{\infty} \) using the concentration scenario. The functional \( \mathcal{E} \) is similar to the corresponding functionals appearing in [6, 20], in the sense that the Fourier multiplier and the nonlinearity are entangled. However, in contrast with [6, 20], our functional \( \mathcal{E} \) is coercive, hence the penalization argument of [4, 14] is not necessary in our case.

In [6], the exclusion of dichotomy gets more technical due to the entanglement of the Fourier multiplier with the nonlinearity, and this is true for the present work as well. In contrast, the exclusion of the vanishing scenario is straightforward in [6], while this is not the case in the present work. This is due to the fact that in [6] the constrained minimization problem is formulated in \( H^s(\mathbb{R}) \), \( s > 1/2 \), allowing the use of the embedding \( H^s(\mathbb{R}) \subset L^\infty(\mathbb{R}) \), while our problem is formulated in \( H^{s/2}(\mathbb{R}) \), preventing us to make use of this embedding. Instead we show that if \( \{u_n\}_{n=1}^{\infty} \) is vanishing, then \( L^{-1/2}u_n \) is vanishing as well, which leads to a contradiction. In order to show that \( L^{-1/2}u_n \) is vanishing we make use of the integrability assumptions (1.17), (1.18) imposed on the kernel of \( L \), and this is the only instance where these assumptions are used. Apart from (1.17), (1.18) we have precisely the same assumptions on \( L \) as in [18], and we are able to adopt many of the methods used in that paper to our present work. Also, we refer to [18] for a discussion on the necessity of assumptions (i), (ii) in Definition 1.1.

Theorem 1.3 is established using standard arguments, see for example [6, 12].

2 Technical results

The current section is devoted to the general properties of the functionals introduced above. We start with a useful proposition on continuity of symbol \( m(\xi) \) described by Definition 1.1.

Lemma 2.1. There is a function \( \omega : \mathbb{R} \to [0, \infty) \), bounded above by a polynomial, with \( \lim_{\lambda \to 0} \omega(\lambda) = 0 \), such that
\[
|m(\xi) - m(\eta)| \leq \omega(\xi - \eta) (\xi)^{\frac{s}{2}} (\eta)^{\frac{s}{2}}.
\]
Proof. See [18 Proposition 2.1]. \hfill \square

The following functional estimates will be used a lot in the text below, sometimes without references.

**Proposition 2.2.** For any $u \in H^{s/2}(\mathbb{R})$ one has

$$
\mathcal{L}(u) \simeq \|u\|_{H^{s/2}}^2.
$$

**Proof.** This is immediate from Definition 1.1. \hfill \square

**Proposition 2.3.** For any $s > 1/2$ and $u \in H^{s/2}(\mathbb{R})$ one has

$$
|\mathcal{N}_c(u)| \lesssim \|u\|_{L^2}^2 \|u\|_{H^{s/2}},
$$

$$
|\mathcal{N}_r(u)| \lesssim \|u\|_{L^2}^4.
$$

**Proof.** Inequality (2.2) follows from the Sobolev embedding

$$
|\mathcal{N}_r(u)| = \frac{1}{8} \|L^{-1/2}u\|_{L^4}^4 \lesssim \||\partial_x|^{1/4}L^{-1/2}u\|_{L^2}^4 \lesssim \|L^{1/4-s/2}u\|_{L^2}^4 \lesssim \|u\|_{L^2}^4.
$$

Inequality (2.1) follows from (2.2) and H"older’s inequality. \hfill \square

**Proposition 2.4.** For $s > 1/2$ and $u, h \in H^{\frac{s}{2}}(\mathbb{R})$ the Fréchet derivative of $\mathcal{E}$ satisfies

$$
|d\mathcal{E}(u)(h)| \lesssim \|u\|_{H^{\frac{s}{2}}} (1 + \|u\|_{L^2}^2 + \|u\|_{L^2}^2) \|h\|_{H^{\frac{s}{2}}}
$$

**Proof.** We first note that

$$
|d\mathcal{L}(u)(h)| \lesssim \|u\|_{H^{\frac{s}{2}}} \|h\|_{H^{\frac{s}{2}}}.
$$

Next consider

$$
d\mathcal{N}_c(u)(h) = \frac{1}{2} \int_{\mathbb{R}} L^{1/2}h(L^{-1/2}u)^2 + 2uL^{1/2}(L^{-1/2}uL^{-1/2}h) \, dx,
$$

where

$$
\|L^{1/2}h(L^{-1/2}u)^2\|_{L^1} \leq \|L^{1/2}h\|_{L^2} \|L^{-1/2}u\|_{L^4}^2 \lesssim \|u\|_{L^2}^2 \|h\|_{H^{\frac{s}{2}}},
$$

$$
\|uL^{1/2}(L^{-1/2}uL^{-1/2}h)\|_{L^1} \leq \|u\|_{L^2} \|L^{1/2}(L^{-1/2}uL^{-1/2}h)\|_{L^2} \lesssim \|u\|_{L^2} \|L^{1/2}uL^{-1/2}h\|_{H^{\frac{s}{2}}},
$$

$$
\lesssim \|u\|_{L^2} \|L^{-1/2}u\|_{H^{\frac{s}{2}}} \|L^{-1/2}h\|_{H^{\frac{s}{2}}} \lesssim \|u\|_{L^2}^2 \|h\|_{H^{\frac{s}{2}}}.\n$$

Using the above estimates in (2.3), we immediately get that

$$
|d\mathcal{N}_c(u)(h)| \lesssim \|u\|_{L^2}^2 \|h\|_{H^{\frac{s}{2}}}.
$$

In a similar way we find that

$$
|d\mathcal{N}_r(u)(h)| \lesssim \|u\|_{L^2}^3 \|h\|_{H^{\frac{s}{2}}},
$$

which concludes the proof. \hfill \square
We next record a decomposition result for $N_c$.

**Lemma 2.5.** Let $u \in H^{s/2}(\mathbb{R})$. Then

$$
N_c(u) = \frac{1}{2\sqrt{m(0)}} \int_{\mathbb{R}} u^3 \, dx + N_{c1}(u) + N_{c2}(u) + N_{c3}(u),
$$

where

$$
N_{1c}(u) = \frac{\sqrt{m(0)}}{2} \int_{\mathbb{R}} u \left( (L^{-1/2} - m^{-1/2}(0))u \right)^2 \, dx
$$

$$
N_{2c}(u) = \int_{\mathbb{R}} u^2(L^{-1/2} - m^{-1/2}(0))u \, dx
$$

$$
N_{3c}(u) = \frac{1}{2} \int_{\mathbb{R}} (L^{-1/2}u)^2(L^{1/2} - m^{1/2}(0))u \, dx,
$$

and

$$
|N_{2c}(u)| \leq \|u\|^2_{L^4} \left\| (L^{-1/2} - m^{-1/2}(0))u \right\|_{L^2}
$$

$$
|N_{3c}(u)| \lesssim \|u\|^2_{L^2} \left\| (L^{-1/2} - m^{-1/2}(0))u \right\|_{L^2}.
$$

**Proof.** The proof is straightforward and is therefore omitted. \(\square\)

Before we continue we want to make a remark on the convolution theorem. According to our choice of the Fourier transform normalisation, for any two functions $f$ and $g$ we have

$$
\mathcal{F}(fg) = \frac{1}{2\pi} \hat{f} \ast \hat{g}
$$

where star stands for convolution.

**Lemma 2.6.** The functional $\mathcal{E}$ defined by (1.19) is translation invariant. In other words, for any $u \in H^{s/2}(\mathbb{R})$ then $\mathcal{E}(u_h) = \mathcal{E}(u)$, where $u_h(x) = u(x - h)$ denotes translation by $h \in \mathbb{R}$.

**Proof.** Due to the property $\hat{u_h}(\xi) = e^{-ih\xi} \hat{u}(\xi)$ and the Plancherel theorem we have

$$
\mathcal{E}(u_h) = \frac{1}{4\pi} \int_{\mathbb{R}} \left| \sqrt{m(\xi)} \hat{u}_h(\xi) + \frac{1}{2} \mathcal{F} \left( (L^{-1/2}u_h)^2 \right)(\xi) \right|^2 \, d\xi
$$

$$
= \frac{1}{4\pi} \int_{\mathbb{R}} \left| e^{-ih\xi} \sqrt{m(\xi)} \hat{u}(\xi) + \frac{1}{2} e^{-ih\xi} \mathcal{F} \left( (L^{-1/2}u)^2 \right)(\xi) \right|^2 \, d\xi = \mathcal{E}(u)
$$

where we have also used the fact that the Fourier transform of multiplication is convolution of Fourier transforms up to a normalization constant. \(\square\)

In the following lemma we provide a slightly sharper estimate for $N_c$. It will be the first step towards the non-vanishing proof given below.

**Lemma 2.7.** For $s > 1/2$ the following estimate hold true

$$
|N_c(u)| \lesssim \|u\|^2_{L^2(\mathbb{R})} \|L^{-1/2}u\|_{L^\infty(\mathbb{R})}
$$
Proof. Clearly, $L^{-1/2}u \in L^\infty(\mathbb{R})$ and so applying a Kato–Ponce type estimate obtain

$$
|\mathcal{N}_r(u)| \lesssim \|u\|_{L^2}\|L^{1/2}(L^{-1/2}u)^2\|_{L^2} \lesssim \|u\|_{L^2}\|J^{s/2}(L^{-1/2}u)^2\|_{L^2} \lesssim \|u\|_{L^2}\|L^{-1/2}u\|_{L^\infty} \lesssim \|u\|_{L^2}\|L^{-1/2}u\|_{L^\infty}.
$$

We finish this section with a lemma which will be used when ruling out the dichotomy scenario.

**Lemma 2.8.** Let $\varphi \in S(\mathbb{R})$, and let $A_r : H^{s/2}(\mathbb{R}) \to H^{s/2}(\mathbb{R})$, $B_r : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operators

$$
A_r f = [L, \varphi \left( \frac{\cdot}{r} \right)] f,
B_r f = [L^{-\frac{s}{2}}, \varphi \left( \frac{\cdot}{r} \right)] f.
$$

Then the operator norms

$$
\|A_r\|, \|B_r\| \to 0 \text{ as } r \to \infty.
$$

**Proof.** We follow the proof of [18, Lemma 6.2]. Let $\varphi_r(x) = \varphi(x/r)$. Using Lemma 2.1, we find that for $f, g \in H^{s/2}(\mathbb{R})$

$$
|\langle A_r f, g \rangle| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\varphi}(\eta) \hat{f}(\xi - \eta) (m(\xi) - m(\xi - \eta)) \hat{g}(\xi) \, d\eta \, d\xi \right|
\lesssim \int_{\mathbb{R}} |\hat{\varphi}(\eta)\omega(\eta)| \int_{\mathbb{R}} |\xi - \eta|^{s/2} |\hat{f}(\xi - \eta)| d\eta d\xi
\lesssim \int_{\mathbb{R}} |\hat{\varphi}(\eta)\omega(\eta/r)| \, d\eta \|f\|_{H^{s/2}} \|g\|_{H^{s/2}}.
$$

Hence $\|A_r\| \lesssim \int_{\mathbb{R}} |\hat{\varphi}(\eta)\omega(\eta/r)| \, d\eta$ and this last integral tends to zero by the dominated convergence theorem as $r \to \infty$, since $\omega$ is bounded above by a polynomial and $\lim_{\eta \to 0} \omega(\eta) \to 0$.

Similarly, for $f, g \in L^2(\mathbb{R})$ we have

$$
|\langle B_r f, g \rangle| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\varphi}(\eta) \hat{f}(\xi - \eta) (m^{-1/2}(\xi) - m^{-1/2}(\xi - \eta)) \hat{g}(\xi) \, d\eta \, d\xi \right|
= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{m(\xi - \eta) - m(\xi)}{m^{1/2}(\xi - \eta) m^{1/2}(\xi) (m^{1/2}(\xi - \eta) + m^{1/2}(\xi))} \hat{g}(\xi) \, d\eta \, d\xi \right|
\lesssim \int_{\mathbb{R}} |\hat{\varphi}(\eta)\omega(\eta)| \int_{\mathbb{R}} \frac{|\xi - \eta|^{s/2}}{m^{1/2}(\xi - \eta)} |\hat{f}(\xi - \eta)| \frac{|\eta|^{s/2}}{m^{1/2}(\eta)} |\hat{g}(\eta)| \, d\eta d\xi
\lesssim \int_{\mathbb{R}} |\hat{\varphi}(\eta)\omega(\eta/r)| \, d\eta \|f\|_{L^2} \|g\|_{L^2},
$$

and we can conclude in the same way as before that $\|B_r\| \to 0$ as $r \to \infty$. \qed
3 Near minimizers

In this section we provide necessary estimates for the infimum

\[ I_q = \inf_{u \in U_q} \mathcal{E}(u) \]  

(3.1)

and for those \( u \in U_q \) that give values \( \mathcal{E}(u) \) close to this infimum. The regarded functional (1.19) is non-negative and so the same is true for the infimum. However, we also need an upper bound for \( I_q \) and this is addressed in the next result.

**Proposition 3.1.** There exist constants \( D, q_0 > 0 \) such that for \( q \in (0, q_0) \) holds

\[ 0 \leq I_q < m(0)q - Dq^{1+\beta} , \]

with \( \beta = \frac{s'}{2s'-1} \).

**Proof.** It is immediate that \( 0 \leq I_q \). To establish the other inequality we consider \( \varphi \in C^\infty(\mathbb{R}) \), with \( \text{supp}(\hat{\varphi}) \subseteq (-1,1) \), \( \varphi(x) \leq 0 \), \( x \in \mathbb{R} \) and \( \mathcal{Q}(\varphi) = 1 \). We rescale and define \( \varphi_{q,\alpha}(x) = \sqrt{q/\alpha}\varphi(x/\alpha) \), \( \alpha > 1 \), so that \( \mathcal{Q}(\varphi_{q,\alpha}) = q \).

We first note that

\[ \mathcal{L}(\varphi_{q,\alpha}) \leq m(0)q + C_1q\alpha^{-s'}, \quad C_1 > 0, \]

(3.2)

and using Proposition 2.3

\[ |\mathcal{N}_r(\varphi_{q,\alpha})| \leq C_2q^2, \quad C_2 > 0. \]

(3.3)

In order to estimate \( \mathcal{N}_c(\varphi_{q,\alpha}) \) we begin by estimating

\[ 0 \leq m^{1/2}(\xi) - m^{1/2}(0) \leq \frac{m(\xi) - m(0)}{2\sqrt{m(0)}}, \]

\[ |m^{-1/2}(\xi) - m^{-1/2}(0)| \leq \frac{m(\xi) - m(0)}{2m(0)\sqrt{m(0)}}, \]

and then, using Lemma 2.5 we find that

\[ |\mathcal{N}_{2c}(\varphi_{q,\alpha})| \lesssim q^{3/2}\alpha^{-s'-1/2}, \]

\[ |\mathcal{N}_{3c}(\varphi_{q,\alpha})| \lesssim q^{3/2}\alpha^{-s'}. \]

Moreover, since \( \varphi(x) \leq 0 \), we have that

\[ \frac{1}{2\sqrt{m(0)}} \int_{\mathbb{R}} \varphi_{q,\alpha}(x)^3 dx = -2C_0q^{3/2}\alpha^{-1/2}, \quad C_0 > 0 \]

\[ \mathcal{N}_{1c}(\varphi_{q,\alpha}) \leq 0. \]

Hence, it follows from the above estimates that there exists \( \alpha_0 > 1 \), such that for \( \alpha \geq \alpha_0 \),

\[ \mathcal{N}_c(\varphi_{q,\alpha}) \leq -C_0q^{3/2}\alpha^{-1/2}, \]
and combining this with (3.2), (3.3), yields
\[
\mathcal{E}(\varphi_{q,\alpha}) \leq m(0)q - \left( C_0 q^{3/2} \alpha^{-1/2} - C_1 q \alpha^{-s'} \right) + C_2 q^2, \tag{3.4}
\]
and by choosing \( \alpha^{-s'} = B q^\beta \), with \( 0 < B \leq \alpha_0^{-s'} q^{-\beta} \), so that \( \alpha \geq \alpha_0 \), we get from (3.4) that
\[
\mathcal{E}(\varphi_{q,\alpha}) \leq m(0)q - \left( C_0 B^{1/(2s')} - C_1 B \right) q^{1+\beta} + C_2 q^2, \tag{3.5}
\]
By choosing \( B \) small enough we have that \( D > 0 \), and if we in addition choose \( q_0 \) sufficiently small, we find that
\[
I_q \leq \mathcal{E}(\varphi_{q,\alpha}) < m(0)q - D q^{1+\beta}.
\]

We now define a near minimizer to be an element \( u \) of \( U_q \) such that
\[
\mathcal{E}(u) < m(0)q - D q^{1+\beta}. \tag{3.6}
\]
By the previous proposition, there exist such elements \( u \in U_q \).

**Proposition 3.2.** A near minimizer \( u \in U_q \) satisfies
\[
\|u\|^2_{H^{s/2}} \lesssim q.
\]

**Proof.** Using propositions 2.2, 2.3 and 3.1 we find that
\[
\|u\|^2_{H^{s/2}(\mathbb{R})} \simeq \mathcal{L}(u) = \mathcal{E}(u) - \mathcal{N}(u) \lesssim m(0)q - D q^{1+\beta} + \|u\|^2_{L^2(\mathbb{R})} \|u\|_{H^{s}(\mathbb{R})} + \|u\|^4_{L^2(\mathbb{R})} \lesssim m(0)q - D q^{1+\beta} + q \|u\|_{H^{s}(\mathbb{R})} + q^2.
\]
Hence, it follows that for \( q \) sufficiently small
\[
\|u\|^2_{H^{s}(\mathbb{R})} \lesssim m(0)q - D q^{1+\beta} \lesssim q.
\]

We next show that \( I_q \) is strictly subadditive as a function of \( q \). This is essential when proving that dichotomy cannot occur.

**Proposition 3.3.** For any \( q_1, q_2 \in (0, q_0) \) such that \( q_1 + q_2 \in (0, q_0) \), holds
\[
0 < I_{q_1+q_2} < I_{q_1} + I_{q_2}. \tag{3.7}
\]
Proof. We show that $I_q$ is strictly subhomogeneous, i.e

$$I_{aq} < a I_q, \ a > 1, q < aq < q_0,$$  \hspace{1cm} (3.8)

from which the strict subadditivity follows from a standard argument. First we show that (3.8) holds for $a \in (1, 2]$. Let $\{u_n\}_{n=1}^{\infty}$ be a minimizing sequence. From (3.6) we have that

$$\mathcal{L}(u_n) + \mathcal{N}_c(u_n) + \mathcal{N}_r(u_n) < m(0)q - Dq^{1+\beta},$$  \hspace{1cm} (3.9)

and since $\mathcal{L}(u_n) \geq m(0)q, \mathcal{N}_r(u) \geq 0$, we get from (3.9) that

$$\mathcal{N}_c(u) < -Dq^{1+\beta}. \hspace{1cm} (3.10)$$

We also note that $\sqrt{a} - 1 \geq (a - 1)/(1 + \sqrt{2})$. With this in mind we see that

$$I_{aq} \leq \mathcal{E}(a^{1/2}u_n)$$
$$= \mathcal{L}(a^{1/2}u_n) + \mathcal{N}(a^{1/2}u_n)$$
$$= a\mathcal{L}(u_n) + a^{3/2}\mathcal{N}_c(u_n) + a^2\mathcal{N}_r(u_n)$$
$$= a\mathcal{E}(u_n) - a(\mathcal{N}_c(u_n) + \mathcal{N}_r(u_n)) + a^{3/2}\mathcal{N}_c(u_n) + a^2\mathcal{N}_r(u_n)$$
$$\leq a\mathcal{E}(u_n) - (a^{3/2} - a)Dq^{1+\beta} + (a^2 - a)C_3q^2$$
$$\leq a\mathcal{E}(u_n) - (a^2 - a)\left(\frac{Dq^{1+\beta}}{1 + \sqrt{2}} - C_3q^2\right).$$

Hence, for $q_0$ sufficiently small

$$I_{aq} + (a^2 - a)\frac{Dq^{1+\beta}}{2\sqrt{2}} < a I_q,$$

which implies (3.8) for $a \in (1, 2]$, but also that $I_q > 0$, for $q \in (0, q_0)$, proving the first inequality in (3.7). For the general case when $a > 1$, we choose $l \in \mathbb{N}$ sufficiently big so that $a \in (1, 2^l]$. Then $a^{1/l} \in (1, 2]$, and so

$$I_{aq} = I_{a^{1/l}a^{(l-1)/l}q} < a^{1/l}I_{a^{(l-1)/l}q} = a^{1/l}I_{a^{1/l}a^{(l-2)/l}q} < a^{2/l}I_{a^{(l-2)/l}q} < \ldots < a I_q.$$

\hspace{4cm} □

4 Existence of minimizers

In order to establish the existence of minimizers, we will apply the concentration-compactness principle (Theorem 1.4) to $e_n = u_n^2$, where $\{u_n\}_{n=1}^{\infty}$ is a minimizing sequence. The idea is to show that the vanishing and dichotomy scenarios cannot occur and then prove the existence of a minimizer using concentration. We start by excluding the vanishing scenario.

Proposition 4.1. Vanishing does not occur.
Proof. Let \( \{ u_n \}_{n=1}^{\infty} \subseteq U_q \) be a minimizing sequence of \( E \). By Lemma 2.7 we have
\[
|N_{\epsilon}(u)| \lesssim \|u\|_{L^2(\mathbb{R})}^2 \| L^{-1/2}u \|_{L^\infty(\mathbb{R})}
\]
and so for a minimizing sequence
\[
q^\beta \lesssim \| L^{-1/2}u_n \|_{L^\infty(\mathbb{R})}.
\]
Arguing as in the proof of [6 Lemma 4.5], we have for any \( x \in \mathbb{R} \) that
\[
\| L^{-1/2}u_n \|_{L^\infty(x-1,x+1)} \lesssim \| L^{-1/2}u_n \|_{L^2(x-1,x+1)}^{1/(2s)} \| L^{-1/2}u_n \|_{H^s(\mathbb{R})}^{1/(2s)} \sim q^{1/(4s)} \| L^{-1/2}u_n \|_{L^2(x-1,x+1)},
\]
and hence
\[
q^{\beta-1/(4s)} \lesssim \sup_{x \in \mathbb{R}} \| L^{-1/2}u_n \|_{L^2(x-1,x+1)}^{1/(2s)},
\]
which means that \( L^{-1/2}u_n \) cannot vanish. Now we show that \( L^{-1/2}u_n \) is vanishing if one assumes that \( u_n \) is vanishing. In order to do this we start by decomposing
\[
(L^{-1/2}u_n)(x) = (F^{-1}(m^{-1/2}) * u_n)(x)
\]
\[
= \int_{\mathbb{R}} F^{-1}(m^{-1/2})(y)u_n(x-y) \, dy
\]
\[
= \int_{|y|<\epsilon} F^{-1}(m^{-1/2})(y)u_n(x-y) \, dy + \int_{|y|\leq R} F^{-1}(m^{-1/2})(y)u_n(x-y) \, dy
\]
\[
\quad + \int_{|y|\geq R} F^{-1}(m^{-1/2})(y)u_n(x-y) \, dy,
\]
and so
\[
\| L^{-1/2}u_n \|_{L^2(\tilde{x}-1,\tilde{x}+1)} \leq \| I_1 \|_{L^2(\tilde{x}-1,\tilde{x}+1)} + \| I_2 \|_{L^2(\tilde{x}-1,\tilde{x}+1)} + \| I_3 \|_{L^2(\tilde{x}-1,\tilde{x}+1)}.
\]
The goal is then to show that each of the above integrals can be made arbitrarily small.

By assumption there exists \( p \in (1,2) \cap [2/(s+1),2) \) such that (1.13) holds, and so
\[
\| F^{-1}(m^{-1/2}) \|_{L^p(-\epsilon,\epsilon)} = o(1) \text{ as } \epsilon \to 0.
\]
On the other hand its dual number \( p' \) satisfies condition \( 1/2 - 1/p' \leq s/2 \) resulting in the embedding \( H^s(\mathbb{R}) \hookrightarrow L^{p'}(\mathbb{R}) \). Thus applying Hölder’s inequality to \( I_1 \) yields
\[
\| I_1 \|_{L^2(\tilde{x}-1,\tilde{x}+1)}^2 \leq \int_{\tilde{x}-1}^{\tilde{x}+1} \| F^{-1}(m^{-1/2}) \|_{L^p(-\epsilon,\epsilon)}^2 \| u_n \|_{L^{p'}(\mathbb{R})}^2 \, dx = o(1) \text{ as } \epsilon \to 0.
\]
For \( I_3 \) we apply the Cauchy–Schwarz inequality as follows
\[
\| I_3 \|_{L^2(\tilde{x}-1,\tilde{x}+1)}^2 \leq \int_{\tilde{x}-1}^{\tilde{x}+1} \| F^{-1}(m^{-1/2}) \|_{L^2(\mathbb{R} \setminus (-R,R))}^2 \| u_n \|_{L^2(\mathbb{R})}^2 \, dx = o(1) \text{ as } R \to \infty.
\]
After choosing \( \varepsilon, R \) we turn our attention to \( I_2 \)
\[
\|I_2\|_{L^2(\bar{x}_1, \bar{x}_1)}^2 \leq \int_{\bar{x}-1}^{\bar{x}+1} \|\mathcal{F}^{-1} (m^{-1/2}) \|_{L^2((-R, R) \setminus (-\varepsilon, \varepsilon))}^2 \|u_n(x - y)\|_{L^2(\varepsilon, R)}^2 \, dx
\]
\[
\leq C(\varepsilon, R) \|u_n\|_{L^2(1, R)}^2 \to 0 \text{ as } n \to \infty,
\]
if one assumes vanishing of \( u_n \).

We next turn our attention to the dichotomy scenario.

**Proposition 4.2.** Dichotomy cannot occur.

**Proof.** Let \( \chi : \mathbb{R} \to [0, 1] \) be a smooth cutoff function with \( \chi(x) = 1 \), for \( |x| \leq 1 \) and \( \chi(x) = 0 \), for \( |x| \geq 2 \), and such that
\[
\chi = \chi_1^2, \quad 1 - \chi = \chi_2^2,
\]
where \( \chi_1, \chi_2 \) are smooth. Next, let \( w_n(x) = u_n(x - x_n) \) and
\[
\begin{align*}
w_n^{(1)}(x) &= \chi_1 \left( \frac{x}{M_n} \right) w_n(x), \\
w_n^{(2)}(x) &= \chi_2 \left( \frac{x}{M_n} \right) w_n(x),
\end{align*}
\]
Note that from the dichotomy assumption
\[
\frac{1}{2} \int_{M_n \leq |x| \leq 2M_n} w_n^2 \, dx \leq \frac{1}{2} \int_{M_n \leq |x| \leq N_n} w_n^2 \, dx
\]
\[
= \frac{1}{2} \int_{-N_n}^{N_n} w_n^2 \, dx - \frac{1}{2} \int_{-M_n}^{M_n} w_n^2 \, dx
\]
\[
\to q^* - q^*
\]
\[
= 0.
\]
Since \( |w_i(x)| \leq |w_n(x)|, i = 1, 2 \), it follows directly that \( \int_{M_n \leq |x| \leq 2M_n} (w_n^{(i)})^2 \, dx \to 0 \), as \( n \to \infty \). From this we can then deduce
\[
\frac{1}{2} \int_{\mathbb{R}} (w_n^{(1)})^2 \, dx = \frac{1}{2} \int_{-M_n}^{M_n} w_n^2 \, dx - \frac{1}{2} \int_{M_n \leq |x| \leq 2M_n} (w_n^{(1)})^2 \, dx \to q^*,
\]
and similarly
\[
\frac{1}{2} \int_{\mathbb{R}} (w_n^{(2)})^2 \, dx = \frac{1}{2} \int_{\mathbb{R}} w_n^2 \, dx - \frac{1}{2} \int_{-2M_n}^{-M_n} w_n^2 \, dx + \frac{1}{2} \int_{M_n \leq |x| \leq 2M_n} (w_n^{(2)})^2 \, dx \to q - q^*.
\]
We next show that
\[
\mathcal{E}(w_n^{(1)}) + \mathcal{E}(w_n^{(2)}) - \mathcal{E}(w_n) \to 0, \quad n \to \infty.
\]  
(4.1)
As a first step towards this, we show that
\[
\mathcal{L}(w_n^{(1)}) + \mathcal{L}(w_n^{(2)}) - \mathcal{L}(w_n) \to 0, \quad n \to \infty
\]  
(4.2)
Indeed, note that
\[ L^\prime(w_n^{(1)}) + L^\prime(w_n^{(2)}) - L(w_n) = \frac{1}{2} \int_\mathbb{R} w_n^{(1)} L w_n^{(1)} + w_n^{(2)} L w_n^{(2)} - (\chi^2_{1n} + \chi^2_{2n}) w_n L w_n \, dx, \]
and using Lemma 2.8 we find that
\[ \int_\mathbb{R} w_n^{(1)} L w_n^{(1)} - \chi^2_{1n} w_n L w_n \, dx = \int_\mathbb{R} \chi_1 w_n (L(\chi_1 w_n) - \chi_1 L w_n) \, dx \]
\[ = \int_\mathbb{R} \chi_1 w_n [L, \chi_1] \, dx \]
\[ \to 0, \ n \to \infty \]
In the same way we find that
\[ \int_\mathbb{R} w_n^{(2)} L w_n^{(2)} - \chi^2_{2n} w_n L w_n \, dx = \int_\mathbb{R} \chi_2 w_n [L, \chi_2 - 1] w_n \, dx \to 0, \ n \to \infty, \]
hence, (4.2) holds. The next step is to show that
\[ \mathcal{N}(w_n^{(1)}) + \mathcal{N}(w_n^{(2)}) - \mathcal{N}(w_n) \to 0, \ n \to \infty, \]  \hspace{1cm} (4.3)
and for this we use the decomposition \( \mathcal{N} = \mathcal{N}_c + \mathcal{N}_r \), and show that
\[ \mathcal{N}_c(w_n^{(1)}) + \mathcal{N}_c(w_n^{(2)}) - \mathcal{N}_c(w_n) \to 0 \ n \to \infty, \]  \hspace{1cm} (4.4)
\[ \mathcal{N}_r(w_n^{(1)}) + \mathcal{N}_r(w_n^{(2)}) - \mathcal{N}_r(w_n) \to 0 \ n \to \infty. \]  \hspace{1cm} (4.5)
Starting with (4.4), we note that
\[ \mathcal{N}_c(w_n^{(1)}) + \mathcal{N}_c(w_n^{(2)}) - \mathcal{N}_c(w_n) = \frac{1}{2} \int_\mathbb{R} \left( L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 + L^{1/2} w_n^{(2)} (L^{-1/2} w_n^{(2)})^2 \right. \]
\[ - (\chi^2_{1n} + \chi^2_{2n}) L^{1/2} w_n (L^{-1/2} w_n)^2 \) \, dx, \]
Furthermore
\[ \int_\mathbb{R} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi^2_{1n} L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx \]
\[ = \int_\mathbb{R} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi^2_{1n} L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx \]
\[ + \int_\mathbb{R} \chi^2_{1n} L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 - \chi^2_{1n} L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx, \]
and using Lemma 2.8 we find that

\[ \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 \, dx \]

\[ = \int_{\mathbb{R}} L^{1/2} w_n^{(1)} ((L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 (L^{-1/2} w_n)^2) \, dx \]

\[ = \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 + \chi_{1n} L^{-1/2} w_n (L^{-1/2} w_n^{(1)} - \chi_{1n} L^{-1/2} w_n) \, dx \]

\[ = \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 + \chi_{1n} L^{-1/2} w_n) [L^{-1/2}, \chi_{1n}] w_n \, dx \]

\[ \rightarrow 0, \ n \rightarrow 0, \]

and

\[ \int_{\mathbb{R}} \chi_{1n}^2 L^{1/2} w_n^{(1)} (L^{-1/2} w_n)^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx \]

\[ = \int_{\mathbb{R}} \chi_{1n}^2 L^{1/2} (w_n^{(1)} - w_n) (L^{-1/2} w_n)^2 \, dx \]

\[ = \int_{\mathbb{R}} L^{1/2} (\chi_{1n} (w_n^{(1)} - w_n)) \chi_{1n} (L^{-1/2} w_n)^2 \, dx \]

\[ - \int_{\mathbb{R}} [L^{1/2}, \chi_{1n}] (w_n^{(1)} - w_n) \chi_{1n} (L^{-1/2} w_n)^2 \, dx, \]

where

\[ \left| \int_{\mathbb{R}} L^{1/2} (\chi_{1n} (w_n^{(1)} - w_n)) \chi_{1n} (L^{-1/2} w_n)^2 \, dx \right| \]

\[ = \left| \int_{\mathbb{R}} \chi_{1n} (\chi_{1n} - 1) w_n L^{1/2} (\chi_{1n} (L^{-1/2} w_n)^2) \, dx \right| \]

\[ \leq \|\chi_{1n} (\chi_{1n} - 1) w_n\|_{L^2} \left\| L^{1/2} (\chi_{1n} (L^{-1/2} w_n)^2) \right\|_{L^2} \]

\[ \lesssim \|w_n\|_{L^2([-2M_n, -M_n] \cup [M_n, 2M_n])} \|w_n\|_{H^{1/2}}^2 \]

\[ \rightarrow 0, \ n \rightarrow \infty, \]

and \( \int_{\mathbb{R}} [L^{1/2}, \chi_{1n}] (w_n^{(1)} - w_n) \chi_{1n} (L^{-1/2} w_n)^2 \, dx \rightarrow 0, \ n \rightarrow \infty, \) according to Lemma 2.8. Hence

\( \lim_{n \rightarrow \infty} \int_{\mathbb{R}} L^{1/2} w_n^{(1)} (L^{-1/2} w_n^{(1)})^2 - \chi_{1n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx = 0, \) and in the same way we can show that

\( \lim_{n \rightarrow \infty} \int_{\mathbb{R}} L^{1/2} w_n^{(2)} (L^{-1/2} w_n^{(2)})^2 - \chi_{2n}^2 L^{1/2} w_n (L^{-1/2} w_n)^2 \, dx = 0, \) which implies (4.4). The limit (4.5) can be shown using similar techniques as (4.4) and we therefore omit the details.

We conclude that (4.3) holds, which together with (4.2) implies (4.1). Since \( \{w_n\}_{n=1}^{\infty} \) is a minimizing sequence, we get that

\[ \lim_{n \rightarrow \infty} \mathcal{E}(w_n^{(1)}) + \mathcal{E}(w_n^{(1)}) \rightarrow I_q. \]  

(4.6)

However,

\[ \lim_{n \rightarrow \infty} \left( \mathcal{E}(w_n^{(i)}) - \mathcal{E}(w_n^{(i)}) \right) = 0, \ i = 1, 2, \]  

(4.7)
where \( v_n^{(1)} = \sqrt{q^*/Q(w_n^{(1)})} w_n^{(1)} \), \( v_n^{(2)} = \sqrt{(q - q^*)/Q(w_n^{(2)})} w_n^{(2)} \). By construction \( v_n^{(1)} \in U_q^*, \)
\( v_n^{(2)} \in U_{q-q^*} \), and so using (4.6), (4.7), we find that
\[
I_q = \lim_{n \to \infty} \mathcal{E}(v_n^{(1)}) + \mathcal{E}(v_n^{(2)}) \geq I_q^* + I_{q-q^*}^*
\]
which contradicts Proposition 3.3.

**Proposition 4.3.** There exists \( u \in U_q \) solving minimization problem \( \mathcal{E}(u) = I_q \).

**Proof.** By the concentration-compactness principle our minimizing sequence \( c_n = u_n^2 \in L^1(\mathbb{R}) \), \( n \in \mathbb{N} \), concentrates. Moreover, due to the translation invariance one can assume that it concentrates around zero, and so
\[
\int_{|x| > r} u_n^2(x) \, dx \to 0 \text{ uniformly with respect to } n \in \mathbb{N} \text{ as } r \to \infty.
\]
In addition, \( \{u_n\}_{n=1}^\infty \) is a bounded sequence in \( H^{\frac{s}{2}}(\mathbb{R}) \) due to Proposition 3.2, and so
\[
\| (u_n)_h - u_n \|^2_{L^2} \lesssim q \| \xi \mapsto |e^{i\xi h} - 1| |\xi|^{-s/2} \|^2_{L^\infty}
\]
that tends to zero uniformly with respect to \( n \in \mathbb{N} \) as \( h \to 0 \). Taking into account the boundedness of \( \{u_n\}_{n=1}^\infty \) in \( L^2(\mathbb{R}) \) one deduces from the Fréchet-Kolmogorov theorem that \( \{u_n\}_{n=1}^\infty \) is relatively compact in \( L^2(\mathbb{R}) \). Thus we can assume that \( \{u_n\}_{n=1}^\infty \) converges to some \( u \) in \( H^2(\mathbb{R}) \).

Again using that \( \{u_n\}_{n=1}^\infty \) is bounded in \( H^{\frac{s}{2}}(\mathbb{R}) \), we may in addition assume that \( u_n \) converges weakly in \( H^{\frac{s}{2}}(\mathbb{R}) \) to \( u \). Hence \( u \in U_q \) and it is left to check that it solves the minimization problem.

Firstly, applying the weak lower semi-continuity argument we deduce
\[
\mathcal{L}(u) \leq \liminf_{n \to \infty} \mathcal{L}(u_n).
\]
Indeed, the square root of \( \mathcal{L}(u) \) defines a norm in \( H^{\frac{s}{2}}(\mathbb{R}) \), equivalent to the standard Sobolev norm. By the Mazur theorem a closed ball is weakly closed. The latter property implies the weak lower semi-continuity of the functional \( \mathcal{L} \).

It is left to show that \( \mathcal{N}(u_n) \) tends to \( \mathcal{N}(u) \) as \( n \to \infty \). The cubic part is estimated as
\[
|\mathcal{N}_3(u_n) - \mathcal{N}_3(u_n)| \leq \frac{1}{2} \left| \int_{\mathbb{R}} (L^{-1/2} u)^2 L^{1/2} (u - u_n) \, dx \right|
+ \frac{1}{2} \left| \int_{\mathbb{R}} (L^{-1/2} u - L^{-1/2} u_n)^2 L^{1/2} u_n \, dx \right|
= \frac{1}{2} \left| \int_{\mathbb{R}} (u - u_n) L^{1/2} (L^{-1/2} u)^2 \, dx \right|
+ \frac{1}{2} \left| \int_{\mathbb{R}} (L^{-1/2} (u - u_n)) (L^{-1/2} (u + u_n)) L^{1/2} u_n \, dx \right|
\lesssim \| u - u_n \|_{L^2} \| L^{1/2} (L^{-1/2} u)^2 \|_{L^2}
+ \| L^{-1/2} (u - u_n) \|_{H^{\frac{s}{4}}} \| L^{-1/2} (u + u_n) \|_{H^{\frac{s}{4}}} \| L^{1/2} u_n \|_{L^2}
\lesssim q \| u - u_n \|_{L^2}
\]
which tends to zero as $n \to \infty$. For the remainder we have

\[
|\mathcal{N}_r(u) - \mathcal{N}_r(u_n)| = \frac{1}{8} \left| \int_{\mathbb{R}} (L^{-1/2}(u - u_n)) (L^{-1/2}(u + u_n)) \left( (L^{-1/2}u)^2 + (L^{-1/2}u_n)^2 \right) \, dx \right|
\leq \|L^{-1/2}(u - u_n)\|_{H^{s/2}} \|L^{-1/2}(u + u_n)\|_{H^{s/2}} \left( \|L^{-1/2}u\|_{L^4}^2 + \|L^{-1/2}u_n\|_{L^4}^2 \right)
\lesssim q^{3/2} \|u - u_n\|_{L^2},
\]

that tends to zero as $n \to \infty$. Summing up we obtain

\[
I_q \leq \mathcal{E}(u) \leq \liminf_{n \to \infty} \mathcal{E}(u_n) = I_q
\]

which concludes the proof.

We finish the proof of Theorem 1.2 by proving the estimate. Let $u$ be a minimizer. We know that $u$ satisfies the Euler–Lagrange equation

\[
\lambda u + d\mathcal{E}(u) = 0.
\]

Taking the inner product in this equation with $u$ yields

\[
-2\lambda q = d\mathcal{E}(u)(u) = 2\mathcal{L}(u) + 3\mathcal{N}_c(u) + 4\mathcal{N}_r(u) = -\mathcal{L}(u) + 3\mathcal{E}(u) + 4\mathcal{N}_r(u) \tag{4.8}
\]

Since $\mathcal{L}(u) \geq m(0)q$ and $|\mathcal{N}_c(u)| = \mathcal{O}(q^{3/2})$, $|\mathcal{N}_r(u)| = \mathcal{O}(q^2)$ by Proposition 2.3, it is easy to see from the second inequality in (4.8) that for $q$ sufficiently small

\[
-\lambda > \frac{m(0)}{2}.
\]

For the upper bound we use (4.8) together with propositions 2.3, 3.1, 3.2 to deduce that

\[
-2\lambda q = -\mathcal{L}(u) + 3\mathcal{E}(u) + 4\mathcal{N}_r(u)
\leq -m(0)q + 3(m(0)q - Dq^{1+\beta}) + \mathcal{O}(q^2)
= 2m(0)q - 3Dq^{1+\beta} + \mathcal{O}(q^2),
\]

hence, for $q$ sufficiently small

\[
-\lambda < m(0) - Dq^{1+\beta}.
\]

5 Long wave approximation

In this section we return to the initial variational problem for the Whitham–Boussinesq system. So from now on $L = K$. We will show that all minimizers are infinitely smooth and refine existing estimates for them.
Lemma 5.1. There exists $q_0 > 0$ such that for each $r \geq 0$ holds $\|u\|_{H^r}^2 \leq q$ uniformly for $q \in (0, q_0)$ and $u \in D_q$.

Proof. Firstly, one can notice that the statement holds for $r \in [0, 1/2]$, due to Proposition 3.2. We will extend the result by induction to bigger values of $r$ applying Formula (1.13).

Let $r \geq 1/2$, then from the equivalence of operators $K$, $J$ and product estimates in Sobolev spaces we deduce

\[
\|K^{-1/2} (K^{-1/2} v)\|_{H^r} \lesssim \|v\|_{H^r}^3,
\]

\[
\|K^{-1/2} (K^{1/2} v K^{-1/2} v)\|_{H^r} \lesssim \|v\|_{H^r}^2,
\]

\[
\|K^{1/2} (K^{-1/2} v)\|_{H^r} \lesssim \|v\|_{H^r}
\]

for any $v \in H^r(\mathbb{R})$. All three constants here depend only on $r$.

Now for any minimizer $u \in D_q$ calculate $K u$ by Formula (1.13) and obtain

\[
\|u\|_{H^{r+1}} \lesssim \|K u\|_{H^r} \leq |\lambda| \|u\|_{H^r} + \frac{1}{2} \|K^{-1/2} (K^{-1/2} u)^3\|_{H^r} + \|K^{-1/2} (K^{1/2} v K^{-1/2} u)\|_{H^r} + \frac{1}{2} \|K^{1/2} (K^{-1/2} u)^2\|_{H^r} \approx \sqrt{q}
\]

for any $r \geq 1/2$. We have used $|\lambda| \leq 1$ according to Theorem 1.2. This concludes the proof by induction. \hfill \Box

Lemma 5.2. There exist $q_0 > 0$ and $C > 0$ such that the following estimates hold

\[
\|u\|_{L^\infty} \leq C q^{2/3},
\]

\[
\|\partial_x u\|_{L^2} \leq C q^{5/3},
\]

\[
\|\partial_x^2 u\|_{L^2} \leq C q^{7/3}
\]

uniformly for $q \in (0, q_0)$ and $u \in D_q$.

Proof. Introducing the notation

\[
M(u) = \frac{1}{2} K^{-1/2} (K^{-1/2} u)^3 + K^{-1/2} (K^{1/2} u K^{-1/2} u) + \frac{1}{2} K^{1/2} (K^{-1/2} u)^2
\]

one can rewrite Equation (1.13) in the form

\[
(\lambda + K) u = -M(u).
\]

Note that $-\lambda \in (0, 1 - Dq^{2/3})$ according to Theorem 1.2 and so $\lambda + 1 > Dq^{2/3}$. The Fourier transform of minimizer $u$ can be estimated as

\[
|\hat{u}(\xi)| = \frac{|\mathcal{F}(M(u))|}{\lambda + m(\xi)} \leq \frac{|\mathcal{F}(M(u))|}{Dq^{2/3} + m(\xi) - 1} \approx |\mathcal{F}(M(u))(\xi)| \frac{(\chi_{|\xi|\leq 1}(\xi)}{q^{2/3} + \xi^2} + \frac{\chi_{|\xi|> 1}(\xi)}{q^{2/3} + |\xi|}
\]

where $\chi_A(\xi)$ stands for the characteristic function of a set $A$. As was shown in the proof of Lemma 5.1, $M(u)$, is smooth and its $H^s$-norm is bounded by $q$ for any non-negative $s$. Hence $\mathcal{F}(M(u))$ multiplied by any power of $\xi$ is bounded by $q$ with respect to $L^2$-norm.
Let us show that the $L^\infty$-norm of $F(M(u))$ is bounded by $q$. Indeed, we have

$$
\left| F\left( K^{-1/2} (K^{-1/2}u)^2 \right) (\xi) \right| \lesssim \int_{\mathbb{R}} \frac{\sqrt{m(\zeta)}|\hat{u}(\zeta - \xi)\hat{u}(\xi)|}{\sqrt{m(\xi - \zeta)m(\zeta)}} \, d\zeta \lesssim \|u\|^2_{L^2} \lesssim q,
$$

and similarly

$$
\left| F\left( K^{-1/2} (K_{1/2}uK^{-1/2}u) \right) (\xi) \right| \lesssim \int_{\mathbb{R}} \frac{\sqrt{m(\zeta)}|\hat{u}(\zeta - \xi)\hat{u}(\xi)|}{\sqrt{m(\xi - \zeta)m(\zeta)}} \, d\zeta \lesssim \|u\|^2_{L^2} \lesssim q
$$

and similarly

$$
\left| F\left( K^{-1/2} (K^{-1/2}u)^3 \right) (\xi) \right| \lesssim \|u\|_{L^2} \left\| (K^{-1/2}u)^2 \right\|_{L^2} \lesssim \|u\|^3_{L^2} \lesssim q^{3/2}.
$$

Thus $\|F(M(u))\|_{L^\infty} \lesssim q$. So we are in a position to prove (5.1), indeed,

$$
\|u\|_{L^\infty} \lesssim \|\hat{u}\|_{L^1} \lesssim \int_{|\xi| \leq 1} \frac{|F(M(u))(\xi)|}{q^{2/3} + |\xi|^2} \, d\xi + \int_{|\xi| > 1} \frac{|F(M(u))(\xi)|}{q^{2/3} + |\xi|^2} \, d\xi
\lesssim q^{-1/3}\|F(M(u))\|_{L^\infty} + \|F(M(u))\|_{L^2} \lesssim q^{2/3}.
$$

Estimate (5.2) is proved as follows

$$
\|\partial_x u\|^2_{L^2} = \|\xi \mapsto \xi \hat{u}(\xi)\|^2_{L^2} \lesssim \int_{|\xi| \leq 1} \frac{\xi^2|F(M(u))(\xi)|^2}{(q^{2/3} + |\xi|^2)^2} \, d\xi + \int_{|\xi| > 1} \frac{\xi^2|F(M(u))(\xi)|^2}{(q^{2/3} + |\xi|^2)^2} \, d\xi
\lesssim q^{-1/3}\|F(M(u))\|_{L^\infty}^2 + \|F(M(u))\|_{L^2}^2 \lesssim q^{5/3}.
$$

A straightforward repetition of the last argument for the second derivative of the minimizer gives

$$
\|\partial_x^2 u\|^2_{L^2} = \|\xi \mapsto \xi^2 \hat{u}(\xi)\|^2_{L^2} \lesssim \int_{|\xi| \leq 1} \frac{\xi^4|F(M(u))(\xi)|^2}{(q^{2/3} + |\xi|^2)^2} \, d\xi + \int_{|\xi| > 1} \frac{\xi^4|F(M(u))(\xi)|^2}{(q^{2/3} + |\xi|^2)^2} \, d\xi
\lesssim q^{1/3}\|F(M(u))\|_{L^\infty}^2 + \|F(\partial_x M(u))\|_{L^2}^2 \lesssim q^{7/3} + \|\partial_x^2 M(u)\|^2_{L^2} \tag{5.4}
$$

that is only $\mathcal{O}(q^2)$ and so weaker than (5.3). However, Estimate (5.2) is a refinement compared with Lemma 5.1 so it can be used for more delicate estimate of the square norm $\|\partial_x M(u)\|^2_{L^2}$ as follows

$$
\left\| \partial_x K^{-1/2} (K^{-1/2}u)^2 \right\|_{L^2} \lesssim \left\| K^{-1/2} uK^{-1/2} \partial_x u \right\|_{H^{1/2}} \lesssim \left\| K^{-1/2} uK^{-1/2} \partial_x u \right\|_{H^{1/2}} \lesssim q^{4/3},
$$

where product estimates were used. To continue, first note that the estimate of the derivative (5.2), will not be spoiled if one changes $L^2$-norm to $H^s$-norm with any $s \geq 0$. In other words,

$$
\|\partial_x u\|_{H^s} \lesssim q^{5/6}, \text{ and so } \|\partial_x K^{-1/2} (K^{-1/2}uK^{-1/2}u)\|_{L^2} \lesssim \|K^{1/2} \partial_x uK^{-1/2}u\|_{L^2} + \|K^{1/2} uK^{-1/2} \partial_x u\|_{L^2} \lesssim \|\partial_x u\|_{H^{3/2}} \|u\|_{L^2} + \|u\|_{H^{3/2}} \|\partial_x u\|_{L^2} \lesssim q^{4/3}.
$$

21
The last remaining term is estimated similarly
\[
\left\| \partial_x K^{-1/2} \left( K^{-1/2} u \right)^3 \right\|_{L^2} \lesssim \left\| \left( K^{-1/2} u \right)^2 K^{-1/2} \partial_x u \right\|_{L^2} \lesssim \| u \|_{H^{1/2}}^2 \| \partial_x u \|_{L^2} \lesssim q^{11/6}.
\]
Thus
\[
\| \partial_x M(u) \|_{L^2} \lesssim q^{1/3}
\]
that together with (5.4) conclude the proof of Estimate (5.3).

Remark 5.3. Lemmas 5.1, 5.2 remain valid with the surface elevation \( \eta_u \) and velocity \( v_u \) defined by (1.14), (1.15) substituted instead of the minimizer \( u \in D_q \).

We now turn to the task of approximating the solutions found in Theorem 1.2 with solutions of the KdV-equation. For this part we follow [6] closely.

We introduce the long-wave scaling \( S_{KdV}(f)(x) = q^{2/3} f(q^{1/3} x) \) and note that when making the ansatz \( u = S_{KdV}(\psi) \) in (1.13), the leading order part of the equation as \( q \to 0 \) is, with \( \lambda = -1 + \lambda_0 q^{2/3} \),
\[
\lambda_0 \psi + \frac{3}{2} \psi^2 - \frac{\psi_{xx}}{3} = 0.
\]
Equation (5.5) is the travelling wave version of the KdV-equation, which has the up to translation the following unique solution
\[
\psi_{KdV}(x) = -\lambda_0 \sech^2 \left( \frac{1}{2} \sqrt{3 \lambda_0} x \right).
\]
We note that (5.5) is the Euler-Lagrange equation of the minimization problem
\[
I_{KdV} := \min_{\psi \in V_1} \mathcal{E}_{KdV}(\psi),
\]
where
\[
\mathcal{E}_{KdV}(\psi) := \frac{1}{2} \int_{\mathbb{R}} \frac{\psi_x^2}{3} + \psi^3 \, dx,
\]
and \( V_1 := \{ \psi \in H^1(\mathbb{R}) : Q(\psi) = 1 \} \). The constraint \( Q(\psi_{KdV}) = 1 \) requires that \( \lambda_0 = 3/16 \frac{3}{16} \).

The relation between \( \mathcal{E} \) and \( \mathcal{E}_{KdV} \) is now established.

Lemma 5.4. For \( u \in H^2(\mathbb{R}) \) hold
\[
\mathcal{E}(u) = Q(u) + \mathcal{E}_{KdV}(u) + \mathcal{E}_{rem}(u), \quad (5.6)
\]
with
\[
|\mathcal{E}_{rem}(u)| \lesssim \| \partial_x^2 u \|_{L^2}^2 + \| u \|_{L^\infty} \| \partial_x u \|_{L^2}^2 + \| u \|_{L^2}^2 \| \partial_x^2 u \|_{L^2}^2 + \| u \|_{L^4}^2 \| \partial_x^2 u \|_{L^2}^2, \quad (5.7)
\]
\[
|\langle \mathcal{E}_{rem}(u), u \rangle| \lesssim \| \partial_x^2 u \|_{L^2}^2 + \| u \|_{L^\infty} \| \partial_x u \|_{L^2}^2 + \| u \|_{L^2}^2 \| \partial_x u \|_{L^2}^2 + \| u \|_{L^4}^2 \| \partial_x u \|_{L^2}^2 + \| u \|_{L^2}^4, \quad (5.8)
\]
Proof. We note that
\[
\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}} uK u \, dx + \mathcal{N}_c(u) + \mathcal{N}_r(u)
= Q(u) + \frac{1}{2} \int_{\mathbb{R}} u(K - 1) \, dx + \frac{1}{2} \int_{\mathbb{R}} u^3 \, dx + \mathcal{N}_{1c}(u) + \mathcal{N}_{2c} u + \mathcal{N}_{3c}(u) + \mathcal{N}_r(u)
= Q(u) + \mathcal{E}_{KdV}(u)
\]
\[
\quad + \frac{1}{2} \int_{\mathbb{R}} \left( m(\xi) - 1 - \frac{\xi^2}{3} \right) |\dot{u}|^2 \, d\xi + \mathcal{N}_{1c}(u) + \mathcal{N}_{2c} u + \mathcal{N}_{3c}(u) + \mathcal{N}_r(u)
\]
\[
= \mathcal{E}_{\text{rem}}(u)
\]
Since \( m(\xi) = \xi/\tanh(\xi) \), we have that \( |m(\xi) - 1 - \frac{\xi^2}{3}| \lesssim \xi^4 \), so that
\[
\int_{\mathbb{R}} \left( m(\xi) - 1 - \frac{\xi^2}{3} \right) |\dot{u}|^2 \, d\xi \lesssim \|\partial_x u\|_{L^2}^2.
\]
From Lemma 2.5 we have
\[
|\mathcal{N}_{1c}(u)| \lesssim \int_{\mathbb{R}} |u((K^{-1/2} - 1)u)^2| \, dx \lesssim \|u\|_{L^\infty} \|\partial_x u\|_{L^2}^2.
\]
Similarly we find that
\[
|\mathcal{N}_{2c}(u)| \lesssim \|u\|_{L^4}^2 \|\partial_x u\|_{L^2},
\]
\[
|\mathcal{N}_{3c}(u)| \lesssim \|u\|_{L^2}^2 \|\partial_x u\|_{L^2}.
\]
The term \( \mathcal{N}_r(u) \) is estimated in Proposition 2.3, hence (5.7) is established. The estimate (5.8) is proved in a similar way and we therefore omit the details.

Lemma 5.5. There exists \( q_0 > 0 \) such that
\[
I_q = q + \mathcal{E}_{KdV}(u) + \mathcal{O}(q^2), \quad \text{uniformly over } u \in \mathcal{D}_q,
\]
\[
I_q = q + q^{5/3} I_{KdV} + \mathcal{O}(q^2).
\]
Proof. Let \( u \in \mathcal{D}_q \). From Lemma 5.1 we know that \( u \in H^r(\mathbb{R}) \) for any \( r \geq 0 \). In particular \( u \in H^2(\mathbb{R}) \), hence by Lemma 5.4
\[
\mathcal{E}(u) = q + \mathcal{E}_{KdV}(u) + \mathcal{E}_{\text{rem}}(u).
\]
Using (5.7) together with Lemma 5.2 we get \( |\mathcal{E}_{\text{rem}}(u)| \lesssim q^2 \). Hence, (5.9) follows.
Turning now to (5.10) we let \( \psi = S^{-1}_{KdV}(u) \) and note that \( \psi \in V_1 \) and
\[
\mathcal{E}_{KdV}(u) = q^{5/3} \mathcal{E}_{KdV}(\psi) \geq q^{5/3} I_{KdV},
\]
so this together with (5.9) implies
\[
I_q \geq q + q^{5/3} I_{KdV} + \mathcal{O}(q^2).
\]
On the other hand, \( \tilde{u} := S_{KdV}(\psi_{KdV}) \in U_q \), so again using (5.9) obtain

\[
I_q \leq \mathcal{E}(\tilde{u}) = q + \mathcal{E}_{KdV}(\tilde{u}) + O(q^2) = q + q^{5/3}\mathcal{E}_{KdV}(\psi_{KdV}) + O(q^2) = q + q^{5/3}I_{KdV} + O(q^2),
\]

which concludes the proof of (5.10). \( \square \)

The statement of Theorem 1.3 is a summary of the following lemmas.

**Lemma 5.6.** There exists \( q_0 > 0 \) such that for any \( q \in (0, q_0) \) and \( u \in D_q \) there exists \( x_u \in \mathbb{R} \) such that

\[
\|S_{KdV}^{-1}(u) - \psi_{KdV}(\cdot - x_u)\|_{H^1} \lesssim q^{1/6},
\]

uniformly with respect to \( q \in (0, q_0) \) and \( u \in D_q \).

The proof of Lemma 5.6 is identical to the proof of [6, Theorem 5.5] and is therefore omitted.

We next relate the two Lagrange multipliers \( \lambda \) and \( \lambda_0 \).

**Lemma 5.7.** The Lagrange multipliers related to the minimization problem (1.20), satisfy

\[
\lambda = -1 + \lambda_0 q^{2/3} + O(q^{5/6}).
\]

**Proof.** Let \( u \in D_q \). From Lemma 5.4 we have

\[
\langle d\mathcal{E}(u), u \rangle = 2q + \langle d\mathcal{E}_{KdV}(u), u \rangle + O(q^2). \tag{5.11}
\]

Moreover, \( \langle d\mathcal{E}_{KdV}(u), u \rangle = q^{5/3}\langle d\mathcal{E}_{KdV}(S_{KdV}^{-1}(u)), S_{KdV}^{-1}(u) \rangle \), and by Lemmas 5.2, 5.6

\[
\langle d\mathcal{E}_{KdV}(S_{KdV}^{-1}(u)), S_{KdV}^{-1}(u) \rangle - \langle d\mathcal{E}_{KdV}(\psi_{KdV}), \psi_{KdV} \rangle = O(q^{1/6}).
\]

Combining this with (5.11), we obtain

\[
\langle d\mathcal{E}(u), u \rangle = 2q + q^{5/3}\langle d\mathcal{E}_{KdV}(\psi_{KdV}), \psi_{KdV} \rangle + O(q^{11/6}). \tag{5.12}
\]

On the other hand, from the Euler-Lagrange equations we have

\[
2\lambda q + \langle d\mathcal{E}(u), u \rangle = 0, \\
2\lambda_0 + \langle d\mathcal{E}_{KdV}(\psi_{KdV}), \psi_{KdV} \rangle = 0,
\]

and when we combine this with (5.12), we get

\[
-2\lambda q = 2q - 2\lambda_0 q^{5/3} + O(q^{11/6}),
\]

and dividing with \(-2q\) yields

\[
\lambda = -1 + \lambda_0 q^{2/3} + O(q^{5/6}).
\]

\( \square \)
For each solution \( u \) of (1.13), we have the corresponding physical parameters \( \eta_u, v_u \) defined by (1.14), (1.15) where \(-1/c^2 = \lambda = -1 + \lambda_0q^{2/3} + O(q^{5/6})\) by Lemma 5.7. We have the following estimates for \( \eta_u, v_u \) that are similar to the one given in Lemma 5.6.

**Lemma 5.8.** There exists \( q_0 > 0 \) such that for \( q \in (0, q_0) \) and \( u \in D_q \) there exists \( x_u \in \mathbb{R} \) such that

\[
\| S_{KdV}^{-1}(\eta_u) + \psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} \lesssim q^{1/6},
\]

\[
\| S_{KdV}^{-1}(v_u) + \psi_{Kad}(\cdot - x_u) \|_{H^{3/2}} \lesssim q^{1/6}
\]

uniformly with respect to \( q \in (0, q_0) \) and \( u \in D_q \).

**Proof.** We will prove the first inequality. The second one can be proved analogously. Firstly, one can notice that due to \( 1/2 < -\lambda < 1 \) in accordance with to Estimate (1.22), it is enough to prove

\[
\| \lambda S_{KdV}^{-1}(\eta_u) + \psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} \lesssim q^{1/6},
\]

(5.13)

where \( x_u \) is taken as in Lemma 5.6. The first term under the norm in (5.13) has the form

\[
\lambda S_{KdV}^{-1}(\eta_u) = \frac{q^{-2/3}}{2} (K^{-1/2}u)^2 (q^{-1/3},) + q^{-2/3} (K^{1/2}u)(q^{-1/3},)
\]

where the first element of the sum is negligible in view of the straightforward estimate

\[
\| (K^{-1/2}u)^2(q^{-1/3},) \|_{H^{1/2}} \lesssim q.
\]

The second element of the sum can be rewritten as follows. We note that

\[
(K^{1/2}u)(q^{-1/3},) = (K^{1/2}u(q^{-1/3},))(x),
\]

where we used \( K_q \) to denote the Fourier multiplier operator with symbol \( m(q^{1/3},) \). We then get that 

\[
q^{-2/3}(K^{1/2}u)(q^{-1/3},) - \psi_{KdV}(\cdot - x_u) = K_q^{1/2}S_{KdV}^{-1}(u) - \psi_{KdV}(\cdot - x_u)
\]

\[
= K_q^{1/2}(S_{KdV}^{-1}(u) - \psi_{KdV}(\cdot - x_u)) + (K_q^{1/2} - 1)\psi_{KdV}(\cdot - x_u).
\]

Here the last term is estimated as

\[
\| (K_q^{1/2} - 1)\psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} \lesssim q^{1/3} \| \psi_{KdV} \|_{H^{3/2}}.
\]

Finally, we have

\[
\| \lambda S_{KdV}^{-1}(\eta_u) + \lambda \psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} \lesssim \| K_q^{1/2}(S_{KdV}^{-1} - \psi_{KdV}(\cdot - x_u)) \|_{H^{1/2}}
\]

\[
+ \| (K_q^{1/2} - 1)\psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} + \| (1 + \lambda)\psi_{KdV}(\cdot - x_u) \|_{H^{1/2}} + q^{1/3}
\]

that gives (5.13) by Lemma 5.6 and 5.7.

\[\Box\]

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