GEOMETRIC RSK AND THE TODA LATTICE

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ABSTRACT. We relate a continuous-time version of the geometric RSK correspondence to the Toda lattice, in a way which can be viewed as a semi-classical limit of a recent result by the author which relates the continuous-time geometric RSK mapping, with Brownian motion as input, to the quantum Toda lattice.

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1. Introduction

The geometric RSK correspondence is a geometric lifting of the classical RSK correspondence. It was introduced by A.N. Kirillov [26] and further studied by Noumi and Yamada [34]. There is also a continuous-time version of the geometric RSK mapping, which was introduced in [35] and substantially developed in [35, 38] in the context of Littelmann’s path model. In this setting, an important role is a played by a mapping $\Pi_n$ (defined in the next section) which takes as input a continuous path $\eta(t), t \geq 0$ in $\mathbb{R}^n$ with $\eta(0) = 0$ and returns a path $\Pi_n \eta(t), t > 0$, also in $\mathbb{R}^n$. In the paper [38] it was shown that, if $\eta(t) = \sqrt{t} B(t) + t \lambda$, where $\lambda \in \mathbb{R}^n$ and $B$ is a standard Brownian motion in $\mathbb{R}^n$, then $\Pi_n \eta(t), t > 0$ is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator given in terms of the Hamiltonian of the open quantum Toda chain with $n$.

1991 Mathematics Subject Classification. Primary 37K10, 60K35.
particles. The aim of the present paper is to understand this result from the point of view of the classical Toda lattice.

By considering the semiclassical limit ($\epsilon \to 0$) of this result, it can be seen (heuristically) from a result of Givental [20] (at least in the case $\lambda = 0$) that, if $\eta(t) = t\lambda$, then $\Pi_n \eta(t)$, $t > 0$ should define a solution to the classical Toda flow (with opposite sign), and indeed this is the case. We will show it directly in the classical ($\epsilon = 0$) setting by considering the continuous-time geometric RSK mapping with deterministic input. The approach is very elementary and mostly self-contained. Starting with the definition of the geometric RSK mapping, we soon arrive at familiar objects in the general theory of the Toda lattice, thus providing further insight into the results of [38] from an integrable systems point of view.

The main conclusion is that there is a precise sense in which

\begin{equation}
\text{classical Toda + noise = quantum Toda.}
\end{equation}

This statement requires some qualification, however. First, we consider both the classical and quantum system in imaginary time. For the classical system this means that the potential has a minus sign, that is, the Hamiltonian is given by

$$
\frac{1}{2} \sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} e^{x_{i+1}-x_i}.
$$

For the quantum system, it means we consider normalised Hamiltonian

$$
\mathcal{L}\lambda = -\frac{1}{2}\psi_\lambda(x)^{-1} \left( H + \epsilon \sum_i \lambda_i^2 \right) \psi_\lambda(x) = \frac{\epsilon}{2} \Delta + \epsilon \nabla \log \psi_\lambda \cdot \nabla,
$$

where

$$
H = -\epsilon \Delta + \frac{2}{\epsilon} \sum_{i=1}^{n-1} e^{x_{i+1}-x_i}
$$

is the Hamiltonian of the quantum Toda lattice and $\psi_\lambda$ is a particular eigenfunction of $H$ known as a (class one) $GL(n,\mathbb{R})$-Whittaker function. Moreover, in the ‘equation’ (1.1), the noise must be added in a very particular way. It is not simply a random perturbation of the classical Toda flow, but rather a random perturbation of a particular construction of the Toda flow which is closely related to the geometric RSK correspondence. Within this construction, the perturbation is simply:

*Add noise to the constants of motion.*

It will be interesting to investigate to what extent this relation extends to other integrable systems.

Along the way we observe the following curious fact. Consideration of the geometric RSK mapping with Brownian motion as input gives rise to a particular stochastic dynamics on triangles (the analogue of Gelfand-Tsetlin
patterns in this setting), as discussed in [38]. In that paper, another quite different stochastic dynamics on triangles was also considered and shown to have the same fixed-time distributions, and to bear the same relation to the quantum Toda lattice. This latter dynamics can be interpreted as a geometric lifting of Warren’s process [47], which in turn can be interpreted as a continuous version of a shuffling algorithm which has played an important role in the random tilings literature [33]. Similar dynamics on Gelfand-Tsetlin patterns, constructed using a general prescription of Diaconis and Fill [14], have been studied by Borodin and co-workers, see for example [7, 8]. It turns out that, in the semi-classical limit we consider in this paper, both the ‘RSK type’ and ‘shuffling type’ of dynamics are equivalent.

The outline of the paper is as follows. In the next section we recall some relevant background material on factorisations of matrices. In Section 3 we define and recall some basic properties of the continuous-time geometric RSK mapping. In Section 4 we recall some facts about Whittaker functions and the quantum Toda lattice, which continue to play a role in the classical setting. In Section 5 we recall some of the main results of [38] and in Section 6 we briefly outline, at a heuristic level, what happens to these results in the semiclassical limit. In Section 7 we recall some basic definitions and properties of the opposite sign Toda lattice and, in Section 8 we formulate and prove the main results of the paper.

Acknowledgements. Thanks to Mark Adler, Percy Deift, Nick Ercolani, Govind Menon and Pierre van Moerbeke for helpful discussions. This research was supported in part by EPSRC grant number EP/I014829/1.

2. Preliminaries

Let $G = GL(n, \mathbb{C})$ and denote by $B, N$ (resp. $B_-, N_-$) the subgroups of upper (resp. lower) triangular and uni-triangular matrices in $G$. Throughout this paper, an important role will be played by the totally positive part, which we will denote by $\mathcal{P}$, of the double Bruhat cell $B \cap B_- \bar{w}_0 B_-$, where $\bar{w}_0$ is a representative in $G$ of the longest element $w_0 \in S_n$, as in [17, 4]. Concretely,

$$\mathcal{P} = \{b \in B : \Delta^m_k(b) > 0, \ 1 \leq k \leq m \leq n\}$$

where

$$\Delta^m_k(b) = \det \left[ b_{ij} \right]_{1 \leq i \leq k, \ m-k+1 \leq j \leq m}.$$ 

In the following we adopt the convention that $\Delta^m_0(b) = 1$. The quantities $\Delta^m_k(b)$, $1 \leq k \leq m \leq n$ uniquely determine $b \in \mathcal{P}$, as follows. (See, for
We will denote the corresponding bijective mapping by \( f \) and denote by \( I_k \) the identity matrix of dimension \( k \). For \( 1 \leq m \leq n \) and \( w \in \mathbb{C}^{n-m+1} \), define
\[
E_m(w) = \begin{pmatrix} I_{m-1} & 0 \\ 0 & e^{m-n+1}(w) \end{pmatrix}.
\]

**Proposition 2.1.** Each element \( b \in \mathcal{P} \) can be represented uniquely as a product of the form
\[
b = E_1(w^1) \ldots E_n(w^n)
\]
where \( w^m \in (\mathbb{R}_{>0})^{n-m+1} \) for each \( m \). The \( w^m \) are given by
\[
w_1^m = \Delta_1^m(b); \quad w_i^m = \frac{\Delta_{m+i-1}(b) \Delta_{m+i-2}(b)}{\Delta_{m+i-1}(b) \Delta_{m+i-2}(b)}, \quad 1 < i \leq n - m + 1.
\]

This provides a natural (Gelfand-Tsetlin) parameterization of \( \mathcal{P} \) by the set of triangles
\[
(2.3) \quad \mathcal{T} = \{ X = (x_i^m) \in \mathbb{R}^{n(n+1)/2} : 1 \leq i \leq m \leq n \},
\]
by setting
\[
(2.4) \quad x_1^m + \ldots + x_k^m = \log \Delta_k^m(b).
\]
We will denote the corresponding bijective mapping by \( f : \mathcal{P} \rightarrow \mathcal{T} \). Note that if we write \( X = (x_i^m) = f(b) \), the \( w_i^m \) of Proposition 2.1 are given, in terms of the \( x_i^m \), by
\[
w_1^m = e^{x_1^m}; \quad w_i^m = e^{x_{i+1}^m - x_i^m}, \quad 1 < i \leq n - m + 1.
\]
The Weyl group associated with \( G \) is the symmetric group \( S_n \). Each element \( w \in S_n \) has a representative \( \tilde{w} \in G \) defined as follows. Denote the standard generators for \( \mathfrak{gl}_n \) by \( h_i \), \( e_i \) and \( f_i \). For example, for \( n = 3 \),
\[
h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
For adjacent transpositions \( s_i = (i, i+1) \), define
\[
\tilde{s}_i = \exp(-e_i) \exp(f_i) \exp(-e_i) = (I - e_i)(I + f_i)(I - e_i).
\]
In other words, $\bar{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where $\varphi_i$ is the natural embedding of $SL(2)$ into $GL(n)$ given by $h_i$, $e_i$ and $f_i$. For example, when $n = 3$,

$$\bar{s}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Now let $w = s_{i_1} \ldots s_{i_r}$ be a reduced decomposition and define $\bar{w} = \bar{s}_{i_1} \ldots \bar{s}_{i_r}$. Note that $uv = \bar{u} \bar{v}$ whenever $l(uv) = l(u) + l(v)$. Denote the longest element of $S_n$ by $w_0 = (1 \ 2 \ \cdots \ n \ n \ n-1 \ \cdots \ 1)$.

For $n = 2$, $w_0 = s_1$ and

$$\bar{w}_0 = \bar{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

For $n = 3$, $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is represented by

$$\bar{w}_0 = \bar{s}_1 \bar{s}_2 \bar{s}_1 = \bar{s}_2 \bar{s}_1 \bar{s}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Denote the elementary lower uni-triangular Jacobi matrices by $l_i(a) = I_n + a f_i$, $1 \leq i \leq n - 1$.

These matrices play a central role in parameterising of the set $(N^-)_{>0}$ of totally positive lower triangular matrices, see for example [4]. For $u \in \mathbb{C}^m$, $1 \leq m < n$, define

$L_m(u) = l_m(u_m)l_{m-1}(u_{m-1}) \ldots l_1(u_1)$.

**Proposition 2.2.** Each $L \in (N^-)_{>0}$ can be written uniquely as a product

$L = L_1(u^1)L_2(u^2) \ldots L_{n-1}(u^{n-1})$,

where $u^m \in (\mathbb{R}_{>0})^m$ for each $m$.

The next proposition is due to Berenstein, Fomin and Zelevinsky [4].

**Proposition 2.3.** Let $b \in \mathcal{P}$. Then $b \bar{w}_0$ has a Gauss decomposition $b \bar{w}_0 = LDU$ where

$$D_{ii} = \frac{\Delta^n_i(b)}{\Delta^{n-1}_i(b)}, \quad 1 \leq i \leq n,$$

and $L \in (N^-)_{>0}$ is given by

$L = L_1(u^1)L_2(u^2) \ldots L_{n-1}(u^{n-1})$,

$$u^m_i = \frac{\Delta^{m-1}_{i-1}(b)\Delta^{m+1}_{i+1}(b)}{\Delta^n_i(b)\Delta^{i+1}_{i+1}(b)}, \quad 1 \leq i \leq m < n.$
Figure 1. A down-right path \( \phi = (s_{j-1}, \ldots, s_i) \in \Omega_{ij}(t) \).

Note that, if \( X = (x_i^m) = f(b) \), then the \( D_{ii} \) and \( u_i^m \) of Proposition 2.3 are given by
\[
D_{ii} = e^{x_i^0}, \quad 1 \leq i \leq n
\]
and
\[
u_i^m = e^{x_{i+1}^m} - x_i^m, \quad 1 \leq i \leq m < n.
\]

3. Geometric RSK in continuous time

In this section we recall the definition and some basic properties of the continuous-time geometric RSK mapping. Many of the results of this section are essentially contained in the papers [5, 6], see also [38, 39]. For completeness we include direct proofs of all the main statements, which are adapted to the present setting.

Let \( \eta : [0, \infty) \rightarrow \mathbb{R}^n \) be a continuous path with \( \eta(0) = 0 \). Denote the coordinates of \( \eta \) by \( \eta_1, \ldots, \eta_n \), so that
\[
\eta(t) = (\eta_1(t), \ldots, \eta_n(t)), \quad t > 0.
\]

For \( 1 \leq i < j \leq n \), set
\[
\Omega_{ij}(t) = \{0 < s_{j-1} < \ldots < s_i < t\}.
\]
In the following, it will be helpful to think of elements \( \phi \in \Omega_{ij}(t) \) as ‘down-right paths’ in the semi-lattice \( \mathbb{R} \times \mathbb{Z} \) starting at \((0, j)\) and ending at \((t, i)\), as shown in Figure 1. Write \( d\phi = ds_j \cdots ds_{j-1} \) for the Euclidean measure on \( \Omega_{ij}(t) \). For \( \phi = (s_{j-1}, \ldots, s_i) \in \Omega_{ij}(t) \), we define
\[
E_\eta(\phi) = \eta_j(s_{j-1}) + \eta_{j-1}(s_{j-2}) - \eta_{j-1}(s_{j-1}) + \cdots + \eta_i(t) - \eta_i(s_i).
\]

For \( 1 \leq i < j \leq n \) and \( t \geq 0 \), set
\[
(3.1) \quad b_{ii}(t) = e^{\eta_i(t)}, \quad b_{ij}(t) = \int_{\Omega_{ij}(t)} e^{E_\eta(\phi)} d\phi.
\]
Setting \( b_{ij} = 0 \) for \( i > j \), this defines a path in the subgroup \( B \) of upper triangular matrices in \( GL(n, \mathbb{C}) \). If \( \eta \) is smooth, then \( b(t) = (b_{ij}(t)), \ t \geq 0, \) satisfies the evolution equation
\[
(3.2) \quad \dot{b} = \epsilon(\dot{\eta})b,
\]
with initial condition \( b(0) = I_n \), where \( \epsilon(\lambda) = \epsilon_\lambda \) is defined for \( \lambda \in \mathbb{C}^n \) by
\[
(3.3) \quad \epsilon_\lambda = \begin{pmatrix}
\lambda_1 & 1 & 0 & \cdots & 0 \\
0 & \lambda_2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \lambda_{n-1} \\
0 & \cdots & \cdots & \cdots & \lambda_n
\end{pmatrix}
\]
For \( n = 1 \) we define \( \epsilon_\lambda = \lambda_1 \).

The next proposition is a special case of [5, Proposition 3.11]. It shows that \( b(t) \in \mathcal{P} \) for each \( t > 0 \) and also makes clear the connection with Kirillov’s original definition of the geometric RSK mapping in terms on non-intersecting lattice paths. For completeness we include a direct proof which is adapted to the present setting.

**Proposition 3.1.**

\[
(3.4) \quad \Delta_k^m (b(t)) = \int_{\Omega_k^m(t)} e^{E_{\phi_1} + \cdots + E_{\phi_k}} d\phi_1 \ldots d\phi_k,
\]

where the integral is with respect to the Euclidean measure on the set \( \Omega_k^m(t) \) of \( k \)-tuples of non-intersecting down-right paths \( \phi_1, \ldots, \phi_k \) starting at \((0, m+1), \ldots, (0, m)\), respectively, and ending at \((t, 1), \ldots, (t, k)\).

**Proof.** This is a straightforward variation of the Karlin-McGregor / Lindström-Gessel-Viennot formula, and is proved by a standard path-switching argument, see for example [25, Section 1.2]. Let
\[
Y(t) = (Y_1(t), \ldots, Y_k(t)), \ t \geq 0
\]
be a collection of independent, unit-rate Poisson processes started at positions \((1, \ldots, k)\). Then \( Y \) is a continuous time Markov chain with state space \( \mathbb{N}^k \). Denote the transition probabilities of \( Y \) by
\[
p_t(y, y') = P(Y(s + t) = y' \mid Y(s) = y), \ s, t \geq 0.
\]
For \( \sigma \in S_k \) and \( y \in \mathbb{N}^k \), write \( \sigma y = (y_{\sigma(1)}, \ldots, y_{\sigma(k)}) \). We note that \( Y \) enjoys the strong Markov property and its law is \( S_k \)-invariant, that is, \( p_t(\sigma y, \sigma y') = p_t(y, y') \) for all \( \sigma \in S_k \). Let
\[
T_i = \inf \{t > 0 : Y_i(t) = Y_{i+1}(t)\}, \quad 1 \leq i \leq n - 1,
\]
and \( T = \min T_i \). Fix \( t > 0 \) and let \( Z \) be an integrable, measurable function of \( (Y(r), \ 0 \leq r \leq t) \) which is invariant under the substitutions \( Y \to Y(i) \),
where for each $i$,

$$
\hat{Y}^{(i)}(r) = \begin{cases} Y(r) & r \leq T_i \\ s_i Y(r) & r > T_i \end{cases},
$$

with $s_i$ denoting the adjacent transposition $(i, i+1)$.

We will first show that, for $y \in \mathbb{N}^k$ with $y_1 < y_2 < \cdots < y_k$,

$$
E[Z; T > t; Y(t) = y] = \sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; Y(t) = \sigma y].
$$

Note that, since $E[Z; T > t; Y(t) = \sigma y] = 0$ unless $\sigma$ is the identity, this is equivalent to

$$
\sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; T \leq t; Y(t) = \sigma y] = 0,
$$

or indeed

$$
\sum_{i} \sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; T = T_i \leq t; Y(t) = \sigma y] = 0.
$$

It therefore suffices to show that, for each $i$,

$$
E[Z; T = T_i \leq t; Y(t) = s_i \sigma y] = E[Z; T = T_i \leq t; Y(t) = \sigma y],
$$

and we conclude that

$$
\sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; T = T_i \leq t; Y(t) = \sigma y]
$$

and we conclude that

$$
\sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; T = T_i \leq t; Y(t) = \sigma y]
$$

as required.

To see that (3.5) implies the formula (3.4), we take

$$
Z = e^{E_\eta(\hat{Y}_1) + \cdots + E_\eta(\hat{Y}_k)},
$$
where \( \hat{Y}(s) = Y(t - s) \), and \( y = (m - k + 1, \ldots, m) \). Then, using the fact that a collection of independent, identically distributed exponential random variables, conditioned on the value of their sum is uniformly distributed in the corresponding simplex, we can write:

\[
E[Z; T > t; Y(t) = y] = P(Y(t) = y) |\Omega_{k,m}(t)|^{-k} \int_{\Omega_k(t)} e^{E_i(\phi_1) + \cdots + E_s(\phi_k)} d\phi_1 \cdots d\phi_k,
\]

and

\[
\sum_{\sigma \in S_k} \text{sgn}(\sigma) E[Z; Y(t) = \sigma y] = \det \left[ E[e^{E_i(\eta_i)}; Y_i(t) = y_j] \right]_{1 \leq i,j \leq k} = \det \left[ P(Y_i(t) = y_j) |\Omega_{i,m-k+j}(t)|^{-1} b_{i,m-k+j}(t) \right]_{1 \leq i,j \leq k}.
\]

Now, for each \( i \leq j \),

\[
P(Y_i(t) = j) = e^{-t} \frac{b_{j-i}}{(j-i)!} = e^{-t} |\Omega_{ij}(t)|,
\]

hence

\[
P(Y_i(t) = y_j) |\Omega_{i,m-k+j}(t)|^{-1} = e^{-t}
\]

and

\[
P(Y(t) = y) |\Omega_{k,m}(t)|^{-k} = e^{-kt},
\]

which concludes the proof. \( \square \)

In particular, \( X(t) = (x_i^m(t)) = f(b(t)), \ t > 0 \) defines a path in the set of triangles \( \mathcal{T} \). The mapping \( \Pi : \eta \mapsto (X(t), \ t > 0) \) was introduced and studied in the papers [5, 6] and can be thought of as a continuous time version of the geometric RSK correspondence introduced by A. N. Kirillov [26]. It also appeared, in a different form (see below) in the paper [35]. For readers familiar with the usual RSK correspondence, for each fixed \( t > 0 \), the path \( (\eta(s), \ 0 \leq s \leq t) \) should be interpreted as the input ‘word’, the triangle \( X(t) = (x_i^m(t), \ 1 \leq i \leq m \leq n) \) as the ‘P-tableau’, the path \( (x^n(s), \ 0 < s \leq t) \) as the ‘Q-tableau’ and the vector \( x^n(t) \) as their common ‘shape’.

The mapping \( \Pi : \eta \mapsto (X(t), \ t > 0) \) defined above admits the following alternative formulation which is, in fact, equivalent to the original definition given in [35]. For \( i = 1, \ldots, n - 1, \) and continuous \( \eta : (0, \infty) \to \mathbb{R}^n \), define

\[
(P_i\eta)(t) = \eta(t) + \left( \log \int_0^t e^{n_{i+1}(s) - n_i(s)} ds \right) (e_i - e_{i+1}),
\]

where \( e_1, \ldots, e_n \) denote the standard basis vectors in \( \mathbb{R}^n \). Let \( \Pi_1 \) denote the identity mapping \( (\Pi_1\eta = \eta) \) and, for \( 2 \leq m \leq n \), define

\[
\Pi_m = P_1 \circ \cdots \circ P_{m-1} \circ \Pi_{m-1}.
\]
Now, for continuous \( \eta : [0, \infty) \to \mathbb{R}^n \) with \( \eta(0) = 0 \), define \( X(t) = (x^m_i(t)) \), \( t > 0 \), by

\[
(3.6) \quad x^m_i(t) = (\Pi_m \eta)_i(t), \quad 1 \leq i \leq m \leq n.
\]

Then it holds that \( \Pi \eta = (X(t), t > 0) \). This follows from a more general result \cite[Theorem 3.5]{5}, which states that \( (e^{x^1}, \ldots, e^{x^n}) \) is the diagonal part in the Gauss decomposition of \( b \bar{w}_0 \), as well as the recursive nature of the construction. For completeness we will include a self-contained proof of this fact in the following, see Proposition 3.4 below.

In \cite{5} it was shown that the \( P_t \) satisfy the braid relations, that is

\[
P_t P_{i+1} P_i = P_{i+1} P_t P_{i+1}, \quad i = 1, \ldots, n - 1.
\]

It follows that, for each \( w \in S_n \),

\[
P_w := P_{w_1} \cdots P_{w_t}
\]

is well defined, where \( w = s_{i_1} \cdots s_{i_r} \) is any reduced decomposition of \( w \) as a product of adjacent transpositions, where \( s_i \) denotes the transposition \((i, i + 1)\). The above-defined \( \Pi_n \) is in fact \( P_{w_0} \), as can be seen using the reduced decomposition \( 1 21 321 \ldots n - 1 \ldots 21 \).

It is a straightforward consequence of (3.6) that, for smooth \( \eta \), the triangle \( X = (x^m_i) \) evolves according to the dynamics

\[
(3.7) \quad \dot{x}^m_1 = \dot{\eta}_1; \quad \dot{x}^m_i = \dot{x}^m_{i-1} + e^{x^m_i - x^m_{i-1}}, \quad \dot{x}^m_m = \dot{\eta}_m - e^{x^m_{m-1}} - x^m_{m-1}, \quad 2 \leq m \leq n; \quad \dot{x}^m_i = \dot{x}^m_{i-1} + e^{x^m_{i-1} - x^m_{i-2}} - e^{x^m_i - x^m_{i-1}}, \quad 1 < i < m \leq n.
\]

(For details, see Proposition 3.3.) Note that the initial value is singular.

We now will explain how, using the evolution equations (3.7), one can ‘insert’ a path \( \eta \) into an arbitrary initial triangle \( \xi \in T \). In the language of RSK, this corresponds to inserting a word into an arbitrary initial \( P \)-tableau. For a smooth path \( \eta \), the dynamic (3.7) defines a flow on triangles which we denote by \( S^n_\eta \). In other words, if we set \( X(0) = \xi \) and allow \( X(t) \) to evolve according to (3.7), then \( X(t) = S^n_\eta \xi \). Similarly, we denote by \( R^n_\eta \) the flow on \( P \) defined by \( R^n_\eta b_0 = b(t) \), where \( b(t) \) is the solution to (3.2) with initial condition \( b(0) = b_0 \). We will explain shortly how to extend the definitions of the flows \( S^n_\eta \) and \( R^n_\eta \) to continuous paths \( \eta \), but first we make a note of the important relation between them.

**Proposition 3.2.** For smooth \( \eta : [0, \infty) \to \mathbb{R}^n \) with \( \eta(0) = 0 \),

\[
(3.8) \quad R^n_\eta = f^{-1} \circ S^n_\eta \circ f.
\]

**Proof.** Let \( X(0) \in T \) and, for \( t \geq 0 \), \( X(t) = S^n_\eta X(0) \) and \( b(t) = f^{-1}(X(t)) \).

Let us write \( \lambda = \dot{\eta} \). We are required to show that \( \dot{b} = \epsilon \lambda b \). We will prove this by induction. Write \( T = T^n, f = f^n, P = P^n, \epsilon = \epsilon^n \) to emphasize their
dependence on $n$ and for $m < n$ denote by $S^n_i$ the flow defined on $T^n$ defined in the same way as above by a smooth path $\pi : [0, \infty) \to \mathbb{R}^m$.

For $n = 1$, $b = e^{\pi^1}$ so $\dot{\pi}^1 = \lambda_1$ implies $b = \lambda_1 b$, as required.

For general $n$, by Proposition 2.1 we can write

$$b = E_1(w^1) \ldots E_n(w^n)$$

where

$$w^m_1 = e^{\pi^1}, \quad w^m_i = e^{\pi^{m+1} - \pi^{m+2}}, \quad 1 < i \leq n - m + 1.$$ 

Define $Y = (y^m_i) \in \mathcal{T}^{n-1}$ by $y^m_i = x^m_i + 1$, $1 \leq i \leq m \leq n - 1$. In other words, $Y$ is the triangle of size $n - 1$ obtained from $X$ by removing $x^m_1, \ldots, x^m_n$. Define a smooth path $\pi : [0, \infty) \to \mathbb{R}^{n-1}$ by setting $\pi(0) = 0$ and $\dot{\pi} = \nu$, where

$$\nu_1 = \lambda_1 + a_1, \quad \nu_2 = \lambda_2 - a_2 + a_2, \quad \ldots \quad \nu_{n-1} = \lambda_{n-1} - a_{n-2} + a_{n-1},$$

$$a_i = e^{\pi^{m+1} - \pi^m}, \quad 1 \leq i \leq n - 1.$$

Then, by (3.7), $Y(t) = S^n_i Y(0)$. Moreover, from the definition of the $w^m$ above, we can write

$$b = E_1(w^1) \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & b^{n-1} \\ 0 & & & 0 \end{pmatrix}$$

where $b^{n-1} = (f^{n-1})^{-1}(Y)$. We note that $w^1 = (e^{\pi^1}, a_1, \ldots, a_{n-1})$ and so, by [3.7],

$$(3.10) \quad \dot{w}^1 = (\lambda_1 w^1_1, (\lambda_2 - \lambda_1 - a_1) w^1_2, \ldots, (\lambda_n - \lambda_{n-1} - a_{n-1}) w^1_n).$$

By the induction hypothesis, $b^{n-1} = e^{n-1} b^{n-1}$. Thus

$$\dot{b} = \dot{E}_1(w^1) \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & b^{n-1} \\ 0 & & & 0 \end{pmatrix} + E_1(w^1) \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & & & e^{n-1} b^{n-1} \end{pmatrix}$$

$$= \left[ \dot{E}_1(w^1) + E_1(w^1) \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & e^{n-1}(\nu) \\ 0 & & & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & & & b^{n-1} \end{pmatrix}.$$

Note that $\dot{E}_1(w^1) = \text{diag}(\dot{w}^1)$. It therefore suffices to show that

$$\text{diag}(\dot{w}^1) + E_1(w^1) \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \vdots & \ddots & e^{n-1}(\nu) \\ 0 & & & 0 \end{pmatrix} = e^n(\lambda) E_1(w^1),$$
which is readily verified using (3.9) and (3.10).

We will now explain how to extend the definition of the flow \(S_\eta^t\) to continuous paths \(\eta : [0, \infty) \rightarrow \mathbb{R}^n\) with \(\eta(0) = 0\), by simply solving the equations (3.7) in terms of \(\eta\) and then observing that \(\eta\) need not be smooth in order for the solution to make sense.

Let \(\xi = (\xi^m_i) \in \mathcal{T}\) and \(\eta : [0, \infty) \rightarrow \mathbb{R}^n\) continuous with \(\eta(0) = 0\). Denote by \(\Pi^\xi \eta\) the path \(X(t) = (x^m_i(t)), \ t \geq 0\) in \(\mathcal{T}\), defined as follows. Set \(\mu_1 = \xi^1_1\),

\[
\mu_m = \sum_{i=1}^m \xi^m_i - \sum_{i=1}^{m-1} \xi^{m-1}_i,
\]

and define

\[
\pi(t) = \eta(t) + \sum_{i=1}^n \mu_i e_i.
\]

For \(2 \leq k \leq m \leq n\), define

\[
r^m_k = \sum_{i=1}^{k-1} \xi^{m-1}_i - \sum_{i=1}^{k-1} \xi^m_i.
\]

Set

\[
x^1_1(t) = \pi_1(t),
\]

and, for \(2 \leq m \leq n\),

\[
x^m_m(t) = \pi_m(t) - \log \left[ e^{-r^m_m} + \int_0^t e^{\pi_m(s) - x^m_{m-1}(s)} ds \right].
\]

For \(2 \leq m \leq n\) and \(1 < k < m\),

\[
x^m_k(t) = y^m_k(t) - \log \left[ e^{-r^m_k} + \int_0^t e^{y^m_k(s) - x^m_{k-1}(s)} ds \right],
\]

where

\[
y^m_k(t) = x^{m-1}_k(t) + \log \left[ e^{-r^m_{k+1}} + \int_0^t e^{y^m_{k+1}(s) - x^m_{k-1}(s)} ds \right];
\]

for \(2 \leq m \leq n\), writing \(y^2_2 = \pi_2\),

\[
x^m_1(t) = x^{m-1}_1(t) + \log \left[ e^{-r^m_2} + \int_0^t e^{y^m_2(s) - x^m_{1}(s)} ds \right].
\]

The mapping \(\Pi^\xi\) can be written more compactly as follows. For \(r \in \mathbb{R}\), \(i = 1, \ldots, n - 1\), and continuous \(\eta : [0, \infty) \rightarrow \mathbb{R}^n\), define

\[
(P^r_i \eta)(t) = \eta(t) + \left( \log \left[ e^{-r} + \int_0^t e^{\eta_{i+1}(s) - \eta_i(s)} ds \right] \right) (e_i - e_{i+1}).
\]
Define $\Pi^\xi_1$ by $\Pi^\xi_1 \eta(t) = \pi(t)$, where $\pi$ is defined as above by (3.11). For $1 < m \leq n$, set

$$\Pi^\xi_m = P^\xi_{m-1} \circ \cdots \circ P^\xi_1 \circ \Pi^\xi_1.$$  

Then $\Pi^\xi \eta = ((\Pi^\xi_m \eta)_i, 1 \leq i \leq m \leq n)$.

**Proposition 3.3.** Let $\eta : [0, \infty) \to \mathbb{R}^n$ be a smooth path with $\eta(0) = 0$. Then, for $t \geq 0$,

$$S^\eta_t \xi = \Pi^\xi \eta(t).$$

Also, for $t > 0$, $\Pi \eta(t)$ evolves according to (3.7).

**Proof.** Let $X(t) = (x^m(t))$ be defined by (3.13) — (3.17). For convenience, write $y^m_m(t) = \pi_m(t)$ and observe that, for $2 \leq k \leq m \leq n$,

$$\frac{d}{dt} \log \left[ e^{-r^m_k} + \int_0^t e^{-r^m_k(s)} x^{m-1}_{k-1}(s) \, ds \right] = e^{x^m_k - x^{m-1}_{k-1}}.$$  

First note that $\dot{x}^1_1 = \dot{\eta}_1$, and using (3.18),

$$\dot{x}^m_k = \dot{\eta}_m - e^{x^m_k - x^{m-1}_{k-1}}$$

for $2 \leq m \leq n$. For $2 \leq m \leq n$ and $1 < k < m$, by (3.15), (3.16), and (3.18),

$$\dot{x}^m_k = y^m_k - e^{x^m_k - x^{m-1}_{k-1}} = \dot{x}^{m-1}_k + e^{x^{m-1}_{k+1} - x^{m-1}_{k}} - e^{x^m_k - x^{m-1}_{k-1}}.$$  

For $2 \leq m \leq n$, by (3.17) and (3.18),

$$\dot{x}^m_1 = \dot{x}^{m-1}_1 + e^{x^m_2 - x^{m-1}_1}.$$  

We have thus shown that $X(t)$ satisfies (3.7), and it remains to check that $X(0) = \xi$.

It follows immediately from the definitions (3.11), (3.13) and (3.14) that

$$x^m_m(0) = \mu_m + r^m_m = \xi^m_m$$

for $1 \leq m \leq n$. For $2 \leq m \leq n$ and $1 < k < m$, from (3.15) and (3.16),

$$x^m_k(0) = y^m_k(0) + r^m_k = x^{m-1}_k(0) - r^{m-1}_{k+1} + r^m_k = x^{m-1}_k(0) + \xi^m_m - \xi^m_{k-1};$$

for $2 \leq m \leq n$, from (3.17),

$$x^m_1(0) = x^{m-1}_1(0) - r^m_2 = x^{m-1}_1(0) + \xi^m_1 - \xi^m_{1-1}.$$  

It follows that $x^m_i(0) = \xi^m_i$ for $1 \leq i \leq m \leq n$, as required.

The second claim also follows from the above argument, taking $e^{-r^m_0} = 0$ for $2 \leq k \leq m \leq n$.

Thus, for $\xi \in \mathcal{T}$, $b \in \mathcal{P}$ and continuous $\eta : [0, \infty) \to \mathbb{R}^n$ with $\eta(0) = 0$, we define, for $t \geq 0$,

$$S^\eta_t \xi = \Pi^\xi \eta(t), \quad R^\eta_t b = f^{-1}(S^\eta_t f(b)).$$

Let $\eta : [0, \infty) \to \mathbb{R}^n$ be a continuous path with $\eta(0) = 0$. Set $b(t) = R^\eta_t b(0)$ with either $b(0) \in \mathcal{P}$ or $b(0) = I$. Then, for each $t > 0$, $b(t) \in \mathcal{P}$ and we can
define $X(t) = (x^n(t)) = f(b(t))$. If $b(0) = I$ then $X(t) = \Pi_\eta(t)$. If $b(0) \in \mathcal{P}$, then $X(t) = S^*_n\xi = \Pi_\eta(t)$ where $\xi = f(b(0))$. By Proposition 2.3 for each $t > 0$, $b(t)\bar{w}_0$ has a Gauss decomposition

$$b(t)\bar{w}_0 = L(t)D(t)U(t)$$

where

$$D_{ji} = \epsilon^a_{ji}, \quad 1 \leq i \leq n$$

and $L \in (N_-)_{>0}$ is given by

$$L = L_1(u^1)L_2(u^2)\ldots L_{n-1}(u^{n-1}),$$

$$u^m = \epsilon^{m+1-n}_m, \quad 1 \leq i \leq m < n.$$  

For a square matrix $A$ denote by $\Pi_-(A)$ the strictly lower triangular part of $A$. The next proposition is essentially a special case of [6, Proposition 6.4].

**Proposition 3.4.** If $\eta$ is smooth, then $L(t)$ and $R(t) = D(t)U(t)$ satisfy

$$\dot{L} = L\Pi_-(L^{-1}\epsilon(\dot{\eta})L), \quad \dot{R} = \epsilon(\dot{x}^n)R.$$  

**Proof.** We have $\dot{b} = \epsilon(\dot{\eta})b$, hence

$$\dot{L}R + LR = \epsilon(\dot{\eta})LR,$$

or, equivalently,

$$L^{-1}\dot{L} + \dot{R}R^{-1} = L^{-1}\epsilon(\dot{\eta})L.$$  

Now, since $\dot{R}R^{-1}$ is upper triangular and $L^{-1}\dot{L}$ is strictly lower triangular, this implies

$$L^{-1}\dot{L} = \Pi_-(L^{-1}\dot{L} + \dot{R}R^{-1}) = \Pi_-(L^{-1}\epsilon(\dot{\eta})L),$$

proving the first claim. Similarly, the strictly upper triangular part of $\dot{R}R^{-1}$ must equal the strictly upper triangular part of $L^{-1}\epsilon(\dot{\eta})L$, which is just the shift matrix $\epsilon(0)$; also, since $R = DU$, the diagonal part of $\dot{R}R^{-1}$ is $\dot{D}D^{-1} = \text{diag}(\dot{x}^n)$. Hence, $\dot{R}R^{-1} = \text{diag}(\dot{x}^n) + \epsilon(0) = \epsilon(\dot{x}^n)$, as required.  

4. WHITTAKER FUNCTIONS AND THE QUANTUM TODA LATTICE

Following [20] [24] [13], for $X = (x^n) \in T$ and $\lambda \in \mathbb{R}^n$, we define, for $n \geq 2$,

$$\mathcal{F}(X) = \sum_{1 \leq i \leq m < n} e^{x_{i+1}^m - x^m_i} + e^{x^m_i - x_{i+1}^m}$$

and

$$\mathcal{F}_\lambda(X) = \sum_{m=1}^n \lambda_m \left( \sum_{i=1}^{m-1} x_{i+1}^m - \sum_{i=1}^{m-1} x_i^m \right) + \mathcal{F}(X).$$

If $n = 1$, we set $\mathcal{F}(X) = 0$ and $\mathcal{F}_\lambda(X) = -\lambda_1 x_1^1$.

For some readers, the following graphical representation may be helpful for understanding the above definition, and also for following some of the proofs which will be given later. We view a triangle $X = (x^n_i)$ as an array:
The elements of this array are connected by arrows as shown, with the obvious
omissions at the boundary:

\[
X = \begin{bmatrix}
x_1^1 \\
x_2^2 \\
\vdots \\
x_n^n \\
\end{bmatrix}
\]

To an arrow \( a \rightarrow b \), we associated the weight \( e^{a-b} \). Then \( \mathcal{F}(X) \) is just the
sum of the weights associated with the arrows in the diagram of
\( X \).

For \( x \in \mathbb{R}^n \), denote by \( T(x) \) the set of triangles \( X = (x_i^m) \in T \) with bottom
row \( x^n = x \). Let \( \epsilon > 0 \) and define

\[
\psi_\lambda(x) = \int_{T(x)} e^{-\mathcal{F}_\lambda(X)/\epsilon} \prod_{1 \leq i \leq m < n} dx_i^m.
\]

These are eigenfunctions of the quantum Toda lattice with Hamiltonian

\[
H = -\epsilon \Delta + \frac{2}{\epsilon} \sum_{i=1}^{n-1} e^{x_{i+1} - x_i},
\]

also known as \( GL(n, \mathbb{R}) \)-Whittaker functions \( 29, 20, 24, 18, 23 \).

In [46] it was shown that, for each \( x \in \mathbb{R}^n \), the function \( \mathcal{F}(X) \) is strictly
convex and has a unique critical point on \( T(x) \), which is a minimum. This
property extends trivially to \( \mathcal{F}_\lambda(X) \), as we are simply adding a linear func-
tional. Denote the corresponding critical point by \( X_\lambda^*(x) \). Note that, since
\( \mathcal{F}(X) \) is strictly convex, \( X_\lambda^*(x) \) is a continuous function of \( \lambda \).

For \( X = (x_i^m) \in T \) and, for \( 1 \leq i \leq m < n \), define \( l_i^m = l_i^m(X) \) and
\( r_i^m = r_i^m(X) \) as follows. For \( 1 \leq i < m < n \),

\[
l_i^m = e^{x_{i+1}^m - x_i^m} + e^{x_{i-1}^m - x_i^m},
\]

and, for \( 1 \leq m < n \),

\[
l_m^m = e^{x_{m+1}^m - x_m^m}.
\]

Similarly, for \( 1 < i \leq m < n \),

\[
r_i^m = e^{x_i^m - x_{i+1}^m} + e^{x_i^m - x_{i-1}^m},
\]
and, for $1 \leq m < n$,
\[ r^m_1 = e^{x^m_1 - x^{m+1}_1}. \]

For each $x \in \mathbb{R}^n$, the critical point $X = X^*_{\lambda}(x)$ of $F_{\lambda}$ on $T(x)$ satisfies
\[ \lambda_m + l^m_i(X) = \lambda_{m+1} + r^m_i(X), \quad 1 \leq i \leq m < n. \]

For $\lambda \in \mathbb{R}^n$, denote by $T_\lambda$ the set of $X = (x^m_i) \in T$ which satisfy the critical point equations (4.3). Note that each element $X = (x^m_i) \in T_\lambda$ is uniquely determined by its bottom row $x^n$, via $X = X^*_\lambda(x^n)$.

Remark 4.1. The critical point equations (4.3) are closely related to the geometric Bender-Knuth transformations introduced in [26]. These are birational involutions $b^m_i$, $1 \leq i \leq m < n$ defined on $T$ as follows: if $X = (x^m_i) \in T$ then $b^m_i(X)$ is obtained from $X$ by replacing $x^m_i$ with
\[ \tilde{x}^m_i = x^m_i + \log \frac{l^m_i(X)}{r^m_i(X)}, \]
and leaving the other entries unchanged. Thus, $X \in T_0$ if, and only if, it is invariant under all of the geometric Bender-Knuth transformations $b^m_i$, $1 \leq i \leq m < n$. We note that this implies, in particular, that each $X \in T_0$ is a fixed point of the geometric lifting of Schutzenberger’s involution, defined in [26] as a composition of geometric Bender-Knuth transformations.

5. Geometric RSK and Brownian motion

In this section we recall some of the main results from [38] which relate the continuous-time geometric RSK mapping, with Brownian motion as input, to the quantum Toda lattice.

Theorem 5.1. [38] If $\eta(t)$ is a Brownian motion in $\mathbb{R}^n$ with $\eta(0) = 0$, infinitesimal variance $\epsilon$ and drift $\lambda \in \mathbb{R}^n$, then $x(t) = \Pi_n \eta(t)$ is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator given by
\[ L_\lambda = -\frac{1}{2} \psi_\lambda(x)^{-1} \left( H + \frac{1}{\epsilon} \sum_i \lambda_i^2 \right) \psi_\lambda(x) = \frac{\epsilon}{2} \Delta + \epsilon \nabla \log \psi_\lambda \cdot \nabla. \]

We note that in [38] this statement was proved in the case $\epsilon = 1$, but this is minor modification. In the case $n = 2$, it is equivalent to a Theorem of Matsumoto and Yor [32]. The diffusion process with generator $L_\lambda$ was first introduced in [3], in the context of more general root systems.
In the paper [38] more general initial conditions were also considered, and two quite different stochastic dynamics on triangles, namely:

\begin{equation}
(5.1)
\begin{align*}
dx_1^m &= dB_1 + \lambda_1 dt; \\
dx_m^m &= dB_m + (\lambda_m - e^{x_m^m - x_{m-1}^m}) dt; & 2 \leq m \leq n; \\
dx_{m-1}^m &= dx_{m-1}^{m-1} + e^{x_m^m - x_{m-1}^m} dt, & 2 \leq m \leq n; \\
dx_i^m &= dx_{i-1}^{m-1} + (e^{x_i^m - x_{i-1}^m} - e^{x_{i-1}^m - x_{i-1}^{m-1}}) dt, & 1 < i < m \leq n
\end{align*}
\end{equation}

and

\begin{equation}
(5.2)
\begin{align*}
dx_1^m &= dW_1 + \lambda_1 dt; \\
dx_m^m &= dW_m^m + (\lambda_m - e^{x_m^m - x_{m-1}^m}) dt; & 2 \leq m \leq n; \\
dx_{m-1}^m &= dW_{m-1}^m + (\lambda_m + e^{x_{m-1}^m - x_m^m}) dt, & 2 \leq m \leq n; \\
dx_i^m &= dW_i^m + (\lambda_m + e^{x_i^m - x_{i-1}^m} - e^{x_{i-1}^m - x_{i-1}^{m-1}}) dt, & 1 < i < m \leq n
\end{align*}
\end{equation}

where \(B_i, W_i^m, 1 \leq i \leq m \leq n\) are independent one-dimensional Brownian motions (without drift) and, for the purposes of this discussion, each with infinitesimal variance \(\epsilon\).

The first of these, (5.1), describes the (stochastic) evolution of \(\Pi \eta(t)\), or \(\Pi \eta(t)\) any initial condition \(\xi \in \mathcal{S}\), where \(\eta(t) = B(t) + t\lambda\), as can be seen, for example, from the proof of Proposition [38] above. The second dynamic (5.2) is a geometric lifting of Warren’s process [17].

**Theorem 5.2.** [38] For each \(x \in \mathbb{R}^n\), if the initial condition \(X(0)\) is chosen at random according to the probability measure on \(\mathcal{S}(x)\) with density proportional to \(e^{-\mathcal{F}_\lambda(X)/\epsilon}\) and \(X(t)\) evolves according to (5.1) or (5.2), then \(X^\lambda(t), t \geq 0\) is a diffusion process with infinitesimal generator \(\mathcal{L}_\lambda\), started at \(x\). Moreover, for each \(t > 0\), the conditional law of \(X(t)\) given \(x^n(s), 0 \leq s \leq t\) is supported on \(\mathcal{S}(x^n(t))\) with density proportional to \(e^{-\mathcal{F}_\lambda(X)/\epsilon}\).

This theorem can be represented as a commutative diagram, as follows. Denote by \(Q_t^\lambda\) the Markov semigroup associated with the diffusion with infinitesimal generator \(\mathcal{L}_\lambda\), that is, \(Q_t^\lambda = e^{t\mathcal{L}_\lambda}\), by \(P_t^\lambda\) the Markov semigroup associated with the Markov process defined by either (5.1) or (5.2), and define Markov kernels \(\Sigma_\lambda\) from \(\mathbb{R}^n\) to \(\mathcal{T}\) and \(\pi\) from \(\mathcal{T}\) to \(\mathbb{R}^n\), by

\[(\Sigma_\lambda f)(x) = \psi_\lambda(x)^{-1} \int_{\mathcal{T}(x)} e^{-\mathcal{F}_\lambda(X)/\epsilon} f(X) \prod_{1 \leq i \leq m < n} dx_i^m\]

for suitable \(f : \mathcal{T} \to \mathbb{R}\) and, writing \(X = (x_i^m) \in \mathcal{T}\),

\[(\pi g)(X) = g(x^n)\]

Then, according to Theorem 5.2 the following diagram commutes:
Theorem 5.1 is a generalisation of Pitman’s ‘$2M - X$’ theorem [44], which states that, if $X_t$ is a standard one-dimensional Brownian motion and $M_t = \max_{0 \leq s \leq t} X_s$, then $2M - X$ is a three-dimensional Bessel process. Pitman’s theorem was generalised to the type $A_{n-1}$ case (from $A_1$) in [10, 42] and to arbitrary finite Coxeter groups in [5, 6]. For discrete versions, see [36, 37, 31]. These generalisations are closely related to the RSK correspondence, and also longest increasing subsequences, percolation and queues [1, 2, 22]. Pitman’s theorem was extended to the geometric setting by Matsumoto and Yor [32], and Theorem 5.1 can be regarded as a geometric lifting of the generalisations of Pitman’s theorem for type $A_{n-1}$ given in [10, 42]. It has been generalised to other types in [12]. A discrete-time version is given in [13] (see also [41]) in the context of Kirillov’s geometric RSK mapping on matrices, and a fully discrete $q$-version, in the context of Ruijenaars $q$-Toda difference operators and $q$-Whittaker functions, in [40] (see also [9, 43]).

The second dynamics (5.2) is a geometric lifting of a process on Gelfand-Tsetlin patterns introduced by Warren [47]. Nordenstam [33] showed that a discrete version of Warren’s process is in fact closely related to a shuffling algorithm which was previously studied in the random tilings literature [15, 45], see also [8, 48, 7]. For some time now it has been a natural question to understand the relationship between these two types of dynamics. As we will see, there is one setting in which the answer is simple: in the semi-classical ($\epsilon \to 0$) limit of Theorem 5.2, they are, in fact, equivalent!

6. A SEMI-CLASSICAL LIMIT

Theorem 5.1 can be restated as follows: if $\eta(t)$ is a Brownian motion in $\mathbb{R}^n$ with $\eta(0) = 0$, infinitesimal variance $\epsilon$ and drift $\lambda \in \mathbb{R}^n$, then $x(t) = \Pi_n \eta(t)$ is a weak solution to the stochastic differential equation
\begin{equation}
    dx = \sqrt{\epsilon}dW + \epsilon \nabla \log \psi(\lambda, x)dt
\end{equation}
where $W$ is a standard Brownian motion in $\mathbb{R}^n$. From the definition of $\psi$, formally taking the limit in (6.1) as $\epsilon \to 0$ yields
\begin{equation}
    \dot{x} = -\nabla u(\lambda, x),
\end{equation}
where, in the notation of Section 4,
\begin{equation}
    u(\lambda, x) = F(\lambda, \pi(x)).
\end{equation}
In the case $\lambda = 0$, this gradient flow is discussed by Givental \[20\], where it is shown to be equivalent to the Toda flow (with opposite sign) on its most degenerate iso-spectral manifold on which the eigenvalues of the Lax matrix are all equal to zero. In fact, the corresponding statement holds true for any $\lambda \in \mathbb{R}^n$, namely, that the gradient flow (6.2) describes the Toda flow on the iso-spectral manifold corresponding to $\lambda$. For more details, see Section 7 and Theorem 8.5 below.

On the other hand, when $\epsilon = 0$, $\eta(t) = t\lambda$ almost surely. This suggests that the image of the path $\eta(t) = t\lambda$ under $\Pi_n$ defines a solution to the Toda flow on the iso-spectral manifold corresponding to $\lambda_1, \ldots, \lambda_n$, and indeed this is the case, as we will show in Theorem 8.5.

For more general initial conditions, in the context of Theorem 5.2, note that when $\epsilon = 0$, (5.1) becomes

$$\dot{x}_1^1 = \lambda;$$

$$\dot{x}_m = x_m^{m-1} + e^{x_{m-1}^m - x_{m-1}^m}, \quad \dot{x}_m = \lambda_m - e^{x_m^m - x_m^{m-1}}, \quad 2 \leq m \leq n;$$

$$\dot{x}_i = x_i^{i-1} + e^{x_{i-1}^i - x_{i-1}^i} - e^{x_i^i - x_i^{i-1}}, \quad 1 < i < m \leq n,$$

and (5.2) becomes

$$\dot{x}_1^m = \lambda + e^{x_{m-1}^1 - x_{m-1}^1}, \quad \dot{x}_m = \lambda_m - e^{x_m^m - x_m^{m-1}}, \quad 2 \leq m \leq n;$$

$$\dot{x}_i^m = \lambda_m + e^{x_{m-1}^i - x_{m-1}^i} - e^{x_i^m - x_i^{m-1}}, \quad 1 < i < m \leq n.$$

Moreover, the fixed time marginals of the process $X(t)$, which are the same in either case, are concentrated on $T_\lambda$. As we will see, $T_\lambda$ is stable under both of the flows defined by (6.4) and (6.5), and on $T_\lambda$, they are in fact equivalent. Moreover, we will show that if $X(0) \in T_\lambda$ and $X(t)$ evolves according to either/both, then $x^n(t)$ defines a solution to the Toda flow with opposite sign on the iso-spectral manifold corresponding to $\lambda$. This last statement can be interpreted as a semi-classical limit of Theorem 5.2. From this we will also deduce the semi-classical limit of Theorem 5.1. For precise statements, see Theorems 8.4 and 8.5 below.

7. The Toda lattice

The Toda lattice is completely integrable Hamiltonian system which has been extensively studied in the literature, see for example the survey \[27\]. We will consider the Toda lattice with opposite sign, with Hamiltonian

$$\frac{1}{2} \sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} e^{x_{i+1}^i - x_i^i}.$$
This is a special case of the indefinite Toda lattice [27, 28]. The equations of motion are

\[
\ddot{x}_1 = -e^{x_2-x_1}, \quad \ddot{x}_n = e^{x_n-x_{n-1}}; \\
\ddot{x}_i = -e^{x_{i+1}-x_i} + e^{x_i-x_{i-1}}, \quad i = 2, \ldots, n-1.
\]

Writing \( q_i = e^{x_{i+1}-x_i} \) for \( 1 \leq i \leq n-1 \), set

\[
M = \begin{pmatrix}
    p_1 & 1 & 0 & \cdots & 0 \\
    -q_1 & p_2 & 1 & \cdots & 0 \\
    0 & -q_2 & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & 1 \\
    0 & \cdots & 0 & -q_{n-1} & p_n
\end{pmatrix},
\]

and

\[
Q = \Pi_-(M) = \begin{pmatrix}
    0 & 0 & \cdots & 0 \\
    -q_1 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & -q_{n-1} \\
\end{pmatrix}.
\]

Then \((M, Q)\) form a Lax pair, that is,

\[
M = [M, Q]
\]

and (7.5) is equivalent to the equations of motion (7.2) or, equivalently,

\[
\dot{q}_i = (p_{i+1} - p_i)q_i, \quad i = 1, \ldots, n-1; \\
\dot{p}_1 = -q_1; \quad \dot{p}_n = q_{n-1}; \\
\dot{p}_i = q_{i+1} - q_i, \quad i = 2, \ldots, n-1.
\]

In particular, the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( M \) form a complete set of integrals of motion for the system. Denote by \( \mathcal{M} \) the set of complex, tridiagonal, Hessenberg matrices and by \( \mathcal{M}_\lambda \) the subset of those matrices with eigenvalues given by \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

The relation to the usual Toda lattice with Hamiltonian

\[
\frac{1}{2} \sum_{i=1}^{n} \pi_i^2 + \sum_{i=1}^{n-1} e^{\xi_{i+1}-\xi_i}
\]

is as follows. If \( x(t) \) is a solution to the opposite sign Toda lattice which can be analytically continued in the time variable \( t \) then, formally at least, \( \xi(t) = x(\iota t) \) defines a solution to the usual Toda lattice, that is

\[
\ddot{\xi}_1 = e^{\xi_2-\xi_1}, \quad \ddot{\xi}_n = -e^{\xi_n-\xi_{n-1}}; \\
\ddot{\xi}_i = e^{\xi_{i+1}-\xi_i} - e^{\xi_i-\xi_{i-1}}, \quad i = 2, \ldots, n-1.
\]
To solve (7.5) for a given initial condition $M_0 \in \mathcal{M}$ we write, for each $t \geq 0,$

$$e^{tM_0} = n(t)r(t),$$

where $n(t) \in N_-$ and $r(t) \in B,$ assuming for the moment that such a factorisation is possible. Then

$$M(t) = n(t)^{-1}M_0 n(t) = r(t)M_0 r(t)^{-1}$$

defines a solution to (7.5) with $M(0) = M_0$. The matrices $n(t)$ and $r(t)$ evolve according to

$$\dot{n} = nQ, \quad \dot{r} = Pr,$$

where $P = M - Q$. Denote the corresponding flow on $\mathcal{M}$ by $T_t$, so that

$$M(t) = T_t M_0.$$

There may exist times at which the solution blows up, as discussed for example in [16, 27, 28].

Finally we recall an important result of Kostant [30], namely that for any $M \in \mathcal{M}_\lambda$, there is a unique $L \in N_-$ such that

$$M = L^{-1} \epsilon_{\lambda} L.$$

Thus, if $M_0 \in \mathcal{M}_\lambda$ with $M_0 = L_0^{-1} \epsilon_{\lambda} L_0$, then we can write

$$M(t) = L(t)^{-1} \epsilon_{\lambda} L(t)$$

where $L(t) = L_0 n(t)$ evolves according to $\dot{L} = LQ$ with $L(0) = L_0$.

### 8. Flows on triangles and upper triangular matrices

We consider two flows on $\mathcal{T}$, which we denote by $S^\lambda$ and $\tilde{S}^\lambda$, and define by

(8.1)\[
\begin{align*}
\dot{x}_1^1 &= \lambda_1; \\
\dot{x}_1^m &= x_1^{m-1} + e^{x_1^{m-1}} - x_1^{m-1}, \quad \dot{x}_m^m = \lambda_m - e^{x_m^{m-1}} - x_m^{m-1}, \quad 2 \leq m \leq n; \\
\dot{x}_i^m &= x_i^{m-1} + e^{x_i^{m-1}} - x_i^{m-1} - e^{x_i^{m-1}} - x_i^{m-1}, \quad 1 < i < m \leq n
\end{align*}
\]

and

(8.2)\[
\begin{align*}
\dot{x}_1^1 &= \lambda_1; \\
\dot{x}_1^m &= \lambda_m + e^{x_1^{m-1}} - x_1^{m-1}, \quad \dot{x}_m^m = \lambda_m - e^{x_m^{m-1}} - x_m^{m-1}, \quad 2 \leq m \leq n; \\
\dot{x}_i^m &= \lambda_m + e^{x_i^{m-1}} - x_i^{m-1} - e^{x_i^{m-1}} - x_i^{m-1}, \quad 1 < i < m \leq n
\end{align*}
\]

respectively.

**Proposition 8.1.** For each $\lambda \in \mathbb{R}^n$, $\mathcal{T}_\lambda$ is invariant under the flows $S^\lambda$ and $\tilde{S}^\lambda$ and, moreover, on $\mathcal{T}_\lambda$ these flows are equivalent.
Proof. We will first show that $\mathcal{T}_\lambda$ is stable under the dynamics (8.1). By (4.3), this is equivalent to showing that $\mathcal{T}_\lambda$ is stable under the dynamics (8.2). Suppose $X \in \mathcal{T}_\lambda$. Then it follows from (8.2), using (143), that

\[(8.3) \quad \dot{x}_i^m = \lambda_{m+1} + e^{x_i^m - x_{i+1}^m} - e^{x_{i+1}^m - x_i^m}, \quad 1 \leq i \leq m < n.\]

Using (8.2) and (8.3) we have, for $1 < i < m < n$,

\[
\dot{i}_i^m = (\dot{x}_i^m - \dot{x}_i^{m-1})e_{x_i^{m-1}-x_i^m} + (\dot{x}_i^{m+1} - \dot{x}_i^m)e_{x_i^m - x_i^{m+1}} = e^{x_i^{m-1} - x_i^m - x_i^{m+1}} - e^{x_i^m - x_i^{m+1}},
\]

and

\[
\dot{r}_i^m = (\dot{x}_i^m - \dot{x}_i^{m-1})e_{x_i^{m-1}-x_i^m} + (\dot{x}_i^m - \dot{x}_i^{m+1})e_{x_i^m - x_i^{m+1}} = e^{x_i^{m-1} - x_i^m - x_i^{m+1}} - e^{x_i^m - x_i^{m+1}}.
\]

Similarly, for $1 \leq m < n$, we obtain

\[
\dot{i}_1^m = \dot{r}_1^m = -e^{x_1^{m+1} - x_1^m}, \quad \dot{i}_m^m = \dot{r}_m^m = -e^{x_{m-1}^m - x_m^m}.
\]

We conclude that $\dot{i}_i^m = \dot{r}_i^m$ for all $1 \leq i \leq m < n$, which shows that $\mathcal{T}_\lambda$ is stable, as required.

The flow $S^\lambda_t$ has a convenient representation in terms of the simple linear flow on $B$ defined by

\[R^\lambda_t b = e^{\epsilon \lambda} b.\]

**Proposition 8.2.** On $\mathcal{T}$, $S^\lambda_t = f \circ R^\lambda_t \circ f^{-1}$.

Proof. This follows from Proposition 3.2 taking $\eta(t) = t\lambda$. \[\square\]

We will now explain how the restriction of the flow $S^\lambda_t$ to $\mathcal{T}_\lambda$ is related to the Toda flow on $\mathcal{M}_\lambda$.

Define a map $h : \mathcal{T} \to (N_-)_{>0}$, $X = (x_i^m) \mapsto L$, by

\[L = L_1(u_1) \ldots L_{n-1}(u_{n-1}),\]

where

\[u_i^m = e^{x_i^{m+1} - x_i^m}, \quad 1 \leq i \leq m < n.\]

Define another map $g^\lambda : \mathcal{T} \to \mathcal{M}$, $X = (x_i^m) \mapsto M$, where $M$ is defined by (7.3) with

\[q_i = e^{x_i^n - x_i^n}, \quad 1 \leq i \leq n - 1,\]

and $p = p^n$, where $p_i^m$, $1 \leq i \leq m \leq n$ are defined by

\[
p_1^m = p_1^{m-1} + e^{x_i^m - x_i^{m-1}}; \quad p_1^m = \lambda_m - e^{x_i^n - x_i^{n-1}};\]
\[ p_i^m = p_{i-1}^m + e^{m+1-i} x_{m+1-i}^m - e^{m-i} x_{m-i}^m, \quad 1 < i < m. \]

Note that, if \( X \in \mathcal{T}_\lambda \), then by (3.3), we can write
\[
\begin{align*}
p_1 &= \lambda_n + e^{x_{n-1}^n - x_n^m}; \quad p_n = \lambda_n - e^{x_n^m - x_{n-1}^n}; \\
p_i &= \lambda_n + e^{x_{n-1}^i - x_n^m} - e^{x_i^m - x_{i-1}^m}, \quad 1 < i < n.
\end{align*}
\]

**Proposition 8.3.** Let \( X = (x_i^m) \in \mathcal{T} \), \( M = g_\lambda(X) \) and \( L = h(X) \). If \( X \in \mathcal{T}_\lambda \), then
\[
(8.4) \quad M = L^{-1} e_\lambda L.
\]

**Proof.** For \( u \in \mathbb{C}^m \), \( 1 \leq m < n \), define
\[
K_m(u) = \begin{pmatrix} \delta^m(u) & 0 \\ 0 & I_{n-m-1} \end{pmatrix},
\]
where
\[
\delta^m(u) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ u_1 & 1 & 0 & \ldots & 0 \\ 0 & u_2 & 1 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & u_m & 1 \end{pmatrix}.
\]

Note that
\[
L_m(u)^{-1} = K_m(-u).
\]

Let \( X \in \mathcal{T} \) and define
\[
\begin{align*}
u_i^m &= e^{x_{i+1}^m - x_i^m}, \quad 1 \leq i \leq m < n; \\
u_i^m &= e^{x_{i}^m - x_{i+1}^m}, \quad 1 \leq i \leq m < n; \\
\lambda_i^m &= e^{x_{i+1}^m - x_i^m}, \quad 1 \leq i < m \leq n; \\
p_1^m &= p_{m-1}^m + e^{-1} x_{i+1}^m; \quad p_n^m = \lambda_m - e^{x_n^m - x_{n-1}^m}; \\
p_i^m &= p_{m-1}^m + e^{x_{m-i}^m - x_i^m} - e^{x_i^m - x_{i-1}^m}, \quad 1 < i < m.
\end{align*}
\]

For \( 2 \leq m \leq n \), write \( \lambda^m = (\lambda_1, \ldots, \lambda_m) \), \( L_m(u) = L_n^m(u) \), \( K_m(u) = K_n^m(u) \),
\[
L^{(m)} = L^m(u^1) \ldots L_{m-1}^m(u^{m-1}),
\]
and
\[
M^{(m)} = \begin{pmatrix} p_1^m & 1 & 0 & \ldots & 0 \\ -q_1^m & p_2^m & 1 & \ldots & 0 \\ 0 & -q_2^m & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & -q_{m-1}^m & p_m^m \end{pmatrix}.
\]
For \( v \in \mathbb{C}^m \) write
\[
e_m(v) = \begin{pmatrix} v_1 & 1 & 0 & \ldots & 0 \\ 0 & v_2 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & v_{m-1} & 1 \\ v_m \\ \end{pmatrix}.
\]

First we will show that \( X \in \mathcal{T}_\lambda \) implies (8.4). We prove this by induction. For \( m = 2 \), write \( q = q_2^1 \), \( p_i = p_i^2 \), \( u = u_1^1 \), \( v = v_1^1 \). Note that \( q = uv \), \( p_1 = \lambda_1 + u \), \( p_2 = \lambda_2 - u \) and, by (4.3), \( \lambda_1 + u = \lambda_2 + v \). Then
\[
(L^{(2)})^{-1} e^2(\lambda)L^{(2)} = \begin{pmatrix} 1 & 0 \\ -u & 1 \\ \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \\ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \\ \end{pmatrix} = \begin{pmatrix} \lambda_1 + u & 1 \\ -u\lambda_1 + \lambda_2(\lambda_2 - u) & \lambda_2 - u \end{pmatrix} = M^{(2)},
\]
as required.

Now fix \( 2 \leq m < n \). For \( A \in \mathbb{C}^{m \times m} \), write
\[
H_m(A) = \begin{pmatrix} A & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \ldots & 1 \\ 0 & \ldots & 0 \\ \end{pmatrix}.
\]

Note that
\[
L_m^{m+1}(u^m)^{-1} = K_m^{m+1}(-u^m) = \delta_m(-u^m)
\]
and
\[
q_i^m v_i^m = q_i^m v_i^{m+1}, \quad 1 \leq i \leq m.
\]

Also, since \( X \in \mathcal{T}_\lambda \), (4.3) implies
\[
p_i^{m+1} = \lambda_{m+1} + v_i^m, \quad p_i^{m+1} = \lambda_{m+1} - u_i^m,
\]
\[
p_i^{m+1} = \lambda_{m+1} + v_i^m - u_i^m, \quad 1 \leq i \leq m + 1.
\]

Hence
\[
M^{(m+1)} - \lambda_{m+1} I_{m+1} = L_m^{m+1}(u^m)^{-1} H_m(e^m(v^m)),
\]
and so
\[
L_m^{m+1}(u^m)[M^{(m+1)} - \lambda_{m+1} I_{m+1}]L_m^{m+1}(u^m)^{-1} = H_m(e^m(v^m))L_m^{m+1}(u^m)^{-1}.
\]

Now, by (4.3),
\[
p_i^m = \lambda_{m+1} + v_i^m - u_i^m, \quad 1 \leq i \leq m.
\]
Using this, and
\[ u_i^m v_{i+1}^m = q_i^m, \quad 1 \leq i < m, \]
the right hand side of (8.5) becomes
\[ H_m(\epsilon^m(u^m))L_{m+1}(u^m)^{-1} = H_m(M^{(m)} - \lambda_{m+1}I_m). \]
By the induction hypothesis,
\[ M^{(m)} = (L^{(m)})^{-1}\epsilon^m(\lambda^m)L^{(m)}. \]
Hence, using
\[ H_m(\epsilon^m(\lambda^m) - \lambda_{m+1}I_m) = \epsilon^{m+1}(\lambda^{m+1}) - \lambda_{m+1}I_{m+1} \]
and
\[ H_m((L^{(m)})^{-1}AL^{(m)}) = [L_{m+1}(u^1)\ldots L_m(u^m-1)]^{-1}H_m(A)L_{m+1}(u^1)\ldots L_{m-1}(u^m-1), \]
we conclude that
\[ M^{(m+1)} = (L^{(m+1)})^{-1}\epsilon^{m+1}(\lambda^{m+1})L^{(m+1)}, \]
as required.

Now we will show that (8.4) implies \( X \in T_\lambda \), again by induction. Write \( T = T^n, T_\lambda = T^n_\lambda \) to emphasize their dependence on \( n \). For \( 1 \leq m \leq n \), write \( X^m = (x^1, \ldots, x^m) \).

**Remark 8.1.** It can be shown that, in fact, \( X \in T_\lambda \) if, and only if,
\[ M^{(m)} = (L^{(m)})^{-1}\epsilon^m(\lambda^m)L^{(m)}, \]
for each \( 2 \leq m \leq n \).

In the next Theorem, in the case \( \lambda = 0 \), the formula (8.7) and the gradient flow representation (8.8) are due to Givental [20].

**Theorem 8.4.** Let \( x, \lambda \in \mathbb{R}^n \) and define \( X(t) = (x^m(t)) = S^\lambda t X(0) \) where \( X(0) \in T_\lambda \). Let \( M(t) = g_\lambda(X(t)), b(t) = f^{-1}(X(t)), \) and
\[ L = L_1(u^1)L_2(u^2)\ldots L_{n-1}(u^{n-1}), \]
where
\[ u_i^m = e^{x_{i+1}^m - x_i^m} \quad 1 \leq i < m < n. \]
Let \( Q = \Pi_-(M) \) and \( P = M - Q \). Then:
(i) For all \( t \geq 0 \),
\[ M(t) = L(t)^{-1}\epsilon_L(t) \]
and we have the Gauss decompositions
\[ b(t)\tilde{w}_0 = L(t)R(t), \quad e^{tM}(0) = n(t)r(t) \]
where \( n(t) = L(0)^{-1}L(t), \ R(t) \in B, \ r(t) = R(t)R(0)^{-1} \) and these satisfy

\[
\dot{L} = LQ, \quad \dot{R} = PR, \quad \dot{n} = nQ, \quad \dot{r} = Pr.
\]

In particular, \( M(t) \) defines a solution to the Toda flow on \( \mathcal{M}_\lambda \).

(ii) The eigenvalues \( \lambda_1, \ldots, \lambda_n \) are given by

\[
\lambda_1 = \dot{x}_1, \quad \lambda_m = \sum_{i=1}^{m} \dot{x}_i^m - \sum_{i=1}^{m-1} \dot{x}_i^{m-1}, \quad 2 \leq m \leq n.
\]

(iii) Writing \( x = x(t) = x^n(t) \),

\[
\mathcal{F}(X) = (n-1)\dot{x}_1 + (n-3)\dot{x}_2 + \cdots + (1-n)\dot{x}_n
\]

and

\[
\dot{x} = -\nabla_x u_{\lambda}(x).
\]

Proof. (i) First note that, since \( b(t) \in \mathcal{P} \) for all \( t \geq 0 \), by Proposition 2.3 we have the Gauss decomposition \( b(t)\tilde{w}_0 = L(t)L(t) \) where \( L(t) \) is defined as in the statement of the Theorem. It follows from Propositions 3.1 and 3.3 that \( M(t) = L(t)^{-1}e_{\lambda}L(t) \) for all \( t \geq 0 \). Thus, by Proposition 3.4 \( L = LQ \), which implies the Lax equation \( \dot{M} = [M, Q] \). We also have from Proposition 3.4 that \( \dot{R} = PR \). Applying Proposition 8.3 at \( t = 0 \), we have \( M(0) = L(0)^{-1}e_{\lambda}L(0) \); it follows, using

\[
L(t)R(t) = e^{tL}L(0)R(0),
\]

that

\[
e^{tM(0)} = L(0)^{-1}e^{tL}L(0) = L(0)^{-1}L(t)L(t)R(0)^{-1}.
\]

Moreover, defining \( n(t) = L(0)^{-1}L(t) \) and \( r(t) = R(t)R(0)^{-1} \), we have \( \dot{n} = nQ \) and \( \dot{r} = Pr \), as required.

Here is another, simple direct proof that \( x = x^n \) satisfies the Toda equations (7.2). For convenience, write \( y = x^{n-1} \). First suppose \( 1 < i < n \). By (8.2),

\[
\dot{x}_i = \lambda_n + e^{y_{i-1}-x_{i-1}} - e^{x_{i-1}-y_{i-1}}
\]

and, by (8.3),

\[
\dot{y}_i = \lambda_n + e^{y_{i-1}-x_{i-1}} - e^{x_{i-1}-y_{i-1}}, \quad \dot{y}_{i-1} = \lambda_n + e^{y_{i-1}-x_{i-1}} - e^{x_{i-1}-y_{i-1}}.
\]

Hence

\[
\dot{x}_i = (\dot{y}_i - \dot{x}_i)e^{y_{i-1}-x_{i-1}} + (\dot{y}_{i-1} - \dot{x}_i)e^{x_{i-1}-y_{i-1}} = (e^{x_{i-1}-y_{i-1}} - e^{x_{i-1}+y_{i-1}})e^{y_{i-1}-x_{i-1}} + (e^{y_{i-1}-x_{i-1}} - e^{y_{i-1}-x_{i-1}})e^{x_{i-1}-y_{i-1}} = -e^{x_{i-1}+x_{i-1}} + e^{x_{i-1}+x_{i-1}}.
\]

For \( i = 1 \), we have

\[
\dot{x}_1 = \lambda_n + e^{y_1-x_1}, \quad \dot{y}_1 = \lambda_n + e^{y_1-x_1} - e^{x_2-y_1},
\]
and hence
\[ \dot{x}_1 = (\dot{y}_1 - \dot{x}_1)e^{y_1-x_1} = -e^{x_2-x_1}; \]
for \( i = n, \)
\[ \dot{x}_n = \lambda_n - e^{x_n-y_{n-1}}, \quad \dot{y}_{n-1} = \lambda_n - e^{x_n-y_{n-1}} + e^{y_{n-1}-x_{n-1}}, \]
and we obtain
\[ \dot{x}_n = (\dot{y}_{n-1} - \dot{x}_n)e^{x_n-y_{n-1}} = e^{x_n-x_{n-1}}, \]
as required.

Part (ii) follows immediately from (8.1).

(iii) We will prove (8.7) and (8.8) by induction. Write \( Y_n^\lambda, T = T_n, T_\lambda = T_\lambda^n, F_\lambda^n \) and \( u_n^\lambda \) to emphasize their dependence on \( n \). For \( 1 \leq m \leq n \), write \( \lambda^m = (\lambda_1, \ldots, \lambda_m) \) and \( X^m = (x^1, \ldots, x^m) \in T_\lambda^m \). We will show that, for each \( 1 \leq m \leq n, \)
\[ F^m(X^m) = (m - 1)\dot{x}_1^m + (m - 3)\dot{x}_2^m + \cdots + (1 - m)\dot{x}_m^m \]
and
\[ \dot{x}^m = -\nabla x^m u_m^\lambda(m^m). \]
First we show (8.9). For \( m = 1, F(X) = 0 \) and the result holds trivially. Let \( m > 1 \) and write \( x = x^m \) and \( y = x^{m-1} \).
\[ F^2(X^2) = e^{x_2-y_1} + e^{y-x_1}. \]
By (8.2), \( \dot{x}_1 = \lambda_2 + e^{y-x_1} \) and \( \dot{x}_2 = \lambda_2 - e^{x_2-y}, \) and so \( F^2(X^2) = \dot{x}_1 - \dot{x}_2, \) as required. Now suppose \( m > 2 \). Note that
\[ F^m(X^m) = F^{m-1}(X^{m-1}) + \mathcal{E}^m, \]
where
\[ \mathcal{E}^m = e^{y_1-x_1} + e^{x_2-y_1} + \cdots + e^{x_m-y_{m-1}}. \]
By the induction hypothesis,
\[ F^{m-1}(X^{m-1}) = (m - 2)\dot{y}_1 + (m - 4)\dot{y}_2 \cdots (2 - m)\dot{y}_{m-1}. \]
Adding \( \mathcal{E}_m \) and using (8.3), then (8.2), gives
\[ F^m(X^m) = (m - 1)e^{y_1-x_1} + (m - 3)(e^{y_2-x_2} - e^{x_2-y_1}) + \cdots + (1 - m)(-e^{x_{m-1}-y_{m-1}}) \]
\[ = (m - 1)\dot{x}_1 + (m - 3)\dot{x}_2 + \cdots + (1 - m)\dot{x}_m, \]
as required.

Now we will prove (8.10). For \( m = 1, u_\lambda(x) = -\lambda_1 x_1 \) and the result holds trivially. Let \( m > 1 \) and write \( x = x^m \) and \( y = x^{m-1} \). Set
\[ \mathcal{E}_m^a = \mathcal{E}^m + a \left( \sum_{j=1}^{m-1} y_j - \sum_{i=1}^{m} x_i \right), \]
and note that
\[ u_m^{\lambda^m}(x) = u_{\lambda^{m-1}}(y) + \mathcal{E}_m^a. \]
First suppose \(1 < i < m\). Note that
\[
\frac{\partial \mathcal{E}_{\lambda}^m}{\partial x_i} = \sum_{j=1}^{m-1} \frac{\partial \mathcal{E}_{\lambda}^m}{\partial y_j} \frac{\partial y_j}{\partial x_i} + e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m.
\]

On the other hand, by the induction hypothesis and (8.3),
\[
\frac{\partial u_{\lambda}^{m-1}(y)}{\partial y_j} = -\dot{y}_j = -\lambda_m - e^{y_i - x_i} + e^{x_{i+1} - y_i} = -\frac{\partial \mathcal{E}_{\lambda}^m}{\partial y_j}.
\]
Thus,
\[
\partial_{x_i, u_{\lambda}}(x) = \sum_{j=1}^{m-1} \frac{\partial u_{\lambda}^{m-1}(y)}{\partial y_j} \frac{\partial y_j}{\partial x_i} + \sum_{j=1}^{m-1} \frac{\partial \mathcal{E}_{\lambda}^m}{\partial y_j} \frac{\partial y_j}{\partial x_i} + e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m
\]
\[
= e^{x_i - y_{i-1}} - e^{y_i - x_i} - \lambda_m = -\dot{x}_i,
\]
as required. The cases \(i = 1\) and \(i = m\) are similar.

We note that, as this is a recursive construction, implicit in the statement of Theorem 8.4 is the statement that, for each \(m \leq n\), \(x^m(t)\) defines a solution to the \(m\)-particle Toda flow on the iso-spectral manifold corresponding to \(\lambda_1, \ldots, \lambda_m\).

Note, in particular, the above shows that \(T_t \circ g_{\lambda} = g_{\lambda} \circ S^\lambda_t\) on \(T_\lambda\). To summarise, if we let \(P_{\lambda} = f^{-1}T_\lambda\), then the following diagram commutes:
\[
P_\lambda \xrightarrow{f} T_\lambda \xrightarrow{g_\lambda} M_\lambda
\]
\[
\begin{array}{c|c|c}
R^\lambda_t & S^\lambda_t & T_t \\
\hline
P_\lambda & T_\lambda & g_\lambda \\
\end{array}
\]
In particular, the semi-classical limit of the commutative diagram shown at the end of Section 6 is:
\[
T_\lambda \xrightarrow{X^*_\lambda} \mathbb{R}^n
\]
\[
\begin{array}{c|c}
S^\lambda_t & T^\lambda_t \\
\hline
T_\lambda & \pi \\
\end{array}
\]
where now \(T^\lambda_t\) denotes the gradient flow defined by (8.8) and, with a slight abuse of notation, \(\pi\) denotes the projection \(\pi : X = (x^m_i) \mapsto x^n\).

**Example 8.1.** Suppose \(n = 2\) and \(x = \lambda = 0\). Then, on \(T(x)\),
\[
F_0(X) = F(X) = e^{-x_1} + e^{x_2}.
\]
This has its unique critical point at \( x_1^2 = 0 \), and so
\[
X(0) = \begin{pmatrix} x_1^2(0) & x_1^1(0) \end{pmatrix} = X_0^*(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}.
\]
Setting \( b(0) = f^{-1}(X_0^*(0,0)) \), this implies
\[
\log \Delta_1^2(b(0)) = \log \Delta_2^2(b(0)) = \log \Delta_1^1(b(0)) = 0
\]
and hence
\[
b(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
Now,
\[
\epsilon_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^{t\epsilon_0} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},
\]
and hence
\[
b(t) = e^{t\epsilon_0}b(0) = \begin{pmatrix} 1 & 1 + t \\ 0 & 1 \end{pmatrix}.
\]
Now, applying the map \( f \) again gives
\[
X(t) = \begin{pmatrix} 0 & \log(1 + t) \\ \log(1 + t) & 0 \end{pmatrix},
\]
and hence
\[
x_2(t) = (\log(1 + t), -\log(1 + t)).
\]
Note that this gives
\[
p_1 = \frac{1}{1 + t}, \quad p_2 = -\frac{1}{1 + t}, \quad q = e^{x_2^2 - x_1^2} = \frac{1}{(1 + t)^2}.
\]
For the usual Toda lattice this gives the solution
\[
\xi(t) = (\log(1 + t), -\log(1 + t)).
\]
The symmetric form of the Lax matrix in this case is given by
\[
\Lambda = \begin{pmatrix} \pi_1 & e^{(x_2 - x_1)/2} \\ e^{(x_2 - x_1)/2} & \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{1 + \lambda}{1 + \lambda e} & \frac{1}{1 + \lambda e} \\ \frac{1}{1 + \lambda e} & -\frac{1 + \lambda}{1 + \lambda e} \end{pmatrix}
\]
which, at \( t = 0 \), is given by
\[
\Lambda(0) = \begin{pmatrix} \lambda & 1 \\ 1 & -\lambda \end{pmatrix}.
\]
**Example 8.2.** Suppose \( n = 2 \), \( x = (x, -x) \) and \( \lambda = (\lambda, -\lambda) \). Then
\[
X_\lambda^*(x) = \begin{pmatrix} y \\ -x \end{pmatrix},
\]
where \( y \in \mathbb{R} \) is the unique solution to
\[
\lambda + e^{-x - y} = -\lambda + e^{y - x}.
\]
that is
\[ e^{-y} = \sqrt{\lambda^2 e^{2x} + 1 - \lambda e^x}. \]
Now
\[ b(0) = \begin{pmatrix} e^y & e^x \\ 0 & e^{-y} \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix}, \]
\[ e^{t \epsilon} = \begin{pmatrix} e^{\lambda t} & \frac{1}{\lambda} \sinh(\lambda t) \\ 0 & e^{-\lambda t} \end{pmatrix}, \]
and hence
\[ b(t) = e^{t \epsilon} b(0) = \begin{pmatrix} e^{y+\lambda t} & e^{x+\lambda t} + e^{-y} \frac{1}{\lambda} \sinh(\lambda t) \\ 0 & e^{-y - \lambda t} \end{pmatrix}. \]
The solution is thus given by
\[ x^1_1(t) = \log \left( e^{x+\lambda t} + e^{-y} \frac{1}{\lambda} \sinh(\lambda t) \right), \]
and \[ x^2_1(t) = -x^1_1(t). \] Note that
\[ q(t) = e^{x^2_2(t)} - x^1_1(t) = \left( e^{x+\lambda t} + e^{-y} \frac{1}{\lambda} \sinh(\lambda t) \right)^{-2}. \]
Note that, when \( x \to -\infty, y \to 0 \) and \( b(0) \to I \), and the solution becomes
\[ x^1_1(t) = \log \left( \frac{1}{\lambda} \sinh(\lambda t) \right), \]
as discussed in Example 8.3 below.

Finally, we consider the flow \( R^t_\lambda \) on \( B \) started from the identity.

**Theorem 8.5.** Let \( \lambda \in \mathbb{R}^n \) and \( b(t) = e^{t \epsilon} \). Note that \( f(b) = \Pi \eta \), where \( \eta(t) = t \lambda \). Then \( \Pi \eta(t) \in \mathcal{T}_\lambda \) for \( t > 0 \) and \( g_\lambda(\Pi \eta(t)), t > 0 \) defines a solution to the Toda flow on \( \mathcal{M}_\lambda \) which is singular at \( t = 0 \). All of the other conclusions of Theorem 8.4 also hold for \( t > 0 \) with \( X(t) = (x^m_i(t)) = \Pi \eta(t) \). In this case, \( b(t) \) is given explicitly by \( b_{ii}(t) = e^{\lambda_i t} \) and, for \( i < j \),

\[ b_{ij}(t) = \sum_{i \leq k \leq j} \left[ \prod_{l=i, \ldots, j; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] e^{\lambda_k t} \]
\[ = \frac{1}{2\pi i} \oint \frac{e^{tz} dz}{\prod_{i \leq k \leq j} (z - \lambda_k)}, \]
where the integration is anti-clockwise around a circle containing \( \lambda_i, \ldots, \lambda_j \). Alternatively, for \( i < j \), we can write

\[ b_{ij}(t) = \left[ \prod_{i \leq k < j \leq j} (\lambda_k - \lambda_l) \right]^{-1} \begin{vmatrix} e^{\lambda_i t} & \lambda^{j-i-1} \cdots \lambda_i & 1 \\ e^{\lambda_{i+1} t} & \lambda^{j-i-1} \cdots \lambda_{i+1} & 1 \\ \vdots & \vdots & \vdots \\ e^{\lambda_{j-1} t} & \lambda^{j-i-1} \cdots \lambda_j & 1 \end{vmatrix}. \]
We recall that, by definition,
\[ x_1^n + \cdots + x_k^n = \log \tau_k, \quad 1 \leq k \leq n, \]
where \( \tau_1 = b_{1n} \) and, for \( 2 \leq k \leq n, \)
\begin{equation}
\tau_k = \begin{vmatrix}
  b_{1,n-k+1} & \cdots & b_{1,n} \\
  b_{2,n-k+1} & \cdots & b_{2,n} \\
  \vdots & & \vdots \\
  b_{k,n-k+1} & \cdots & b_{k,n}
\end{vmatrix}.
\end{equation}
Writing \( \tau_1 = \tau \), this solution can also be expressed in the more familiar form
\begin{equation}
\tau_k = \begin{vmatrix}
  \tau^{(k-1)} & \cdots & \tau' & \tau \\
  \tau^{(k)} & \cdots & \tau'' & \tau' \\
  \vdots & & \vdots & \vdots \\
  \tau^{(2k-2)} & \cdots & \tau^{(k-1)}
\end{vmatrix}.
\end{equation}
As matrix integrals, assuming \(|\lambda_i| < 1\) for each \( i, \)
\begin{equation}
\tau_k = \int_{U(k)} (\det M)^{k-1} e^{\text{tr} M} dM.
\end{equation}
When \( \lambda = 0, \)
\begin{equation}
\tau_k = \frac{(k-1)!(k-2)!\ldots 1}{(n-1)!(n-2)\ldots (n-k)!} t^{k(n-k)}.
\end{equation}

Proof. Let \( \xi(N) = X^*_A(-N\rho^n) \), where \( \rho^n = (n-1, n-3, \ldots, 1, -n) \), and set \( X^N(t) = S_t^A \xi(N) \). By Propositions 3.3 and 8.1, \( X^N(t) = \Pi_t^N \eta(t) \in T_\lambda \) for each \( t > 0 \).

For \( 2 \leq k \leq m \leq n, \) define
\[ r^m_k(N) = \sum_{i=1}^{k-1} \xi_i^{m-1}(N) - \sum_{i=1}^{k-1} \xi_i^m(N). \]
We will show that, as \( N \to \infty, r^m_k(N) \to +\infty \) for each \( 2 \leq i \leq m \leq n. \) Note that this implies that \( \Pi_t^N(\eta)(t) \) converges to \( \Pi(\eta)(t) \) for each \( t > 0, \) and hence that \( \Pi_t^N(\eta)(t) \in T_\lambda \) for each \( t > 0. \) But \( \Pi_t^N(\eta)(t) \) also evolves according to \( [8.1] \) for \( t > 0, \) so by Theorem 8.1, \( q_\lambda(\Pi_\eta(t)), \ t > 0 \) defines a solution to the Toda flow on \( T_\lambda, \) as claimed. For convenience, write \( X^N(0) = X = (x^m_i) \). For \( 1 \leq m \leq n, \) define \( \tilde{X} = (\tilde{x}^m_i) \in T \) by \( \tilde{x}^m = x^m + N\rho^m. \) Note that \( \tilde{x}^m = 0. \) Then \( F_\lambda(X) = e^N F_{\text{e}^{-N\lambda}}(\tilde{X}) \) and \( \tilde{X} = X^*_e^{-N\lambda}(0). \) Thus, as \( N \to \infty, \)
\( \tilde{X} \to X^*_{e^{-N\lambda}}(0). \) It follows that, as \( N \to \infty, \) for each \( 1 \leq i \leq m \leq n, \)
\( x^m_i \sim -N\rho^m_i \) and hence \( r^m_k(N) \sim N(k-1) \to \infty, \) as required.

We verify the formula [8.1] by induction. Without loss of generality, we only need to show that the formula holds for \( b_{1n}, \ n \geq 2. \) As \( b(t) \) is a continuous
function of $\lambda$ we can also assume, for convenience, that the $\lambda_i$ are distinct. Write $b_n(t) = b_{1n}(t)$. First we note that

$$b_2(t) = \int_0^t e^{\lambda_2 s + \lambda_1 (t-s)} ds = \frac{1}{\lambda_1 - \lambda_2} (e^{\lambda_1 t} - e^{\lambda_2 t}).$$

Now assume that, for each $t > 0$,

$$b_{n-1}(t) = \sum_{1 \leq k \leq n-1} \left[ \prod_{l=1, \ldots, n-1; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] e^{\lambda_k t}.$$ 

From the definition (3.1), we can write

$$b_{1n}(t) = \int_0^t b_{n-1}(s) e^{\lambda_n (t-s)} ds$$

$$= \sum_{k=1}^{n-1} \left[ \prod_{l=1, \ldots, n-1; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] \left[ \int_0^t e^{(\lambda_k - \lambda_n) s} ds \right] e^{\lambda_n t}$$

$$= \sum_{k=1}^{n-1} \left[ \prod_{l=1, \ldots, n; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] \left[ e^{(\lambda_k - \lambda_n) t} - 1 \right] e^{\lambda_n t}$$

$$= \sum_{k=1}^{n-1} \left[ \prod_{l=1, \ldots, n; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] \left[ e^{\lambda_k t} - e^{\lambda_n t} \right].$$

It therefore suffices to show that

$$\sum_{k=1}^{n-1} \left[ \prod_{l=1, \ldots, n; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] = - \prod_{1 \leq k \leq n-1} (\lambda_n - \lambda_k)^{-1}$$

or, equivalently,

$$(8.17) \quad \sum_{k=1}^{n} \left[ \prod_{l=1, \ldots, n; l \neq k} (\lambda_k - \lambda_l)^{-1} \right] = 0.$$ 

To see that this holds, denote

$$\Delta_m(a_1, \ldots, a_m) = \prod_{1 \leq i < j \leq m} (a_i - a_j) = \det \left[ a_i^{n-j} \right]_{i,j=1, \ldots, m}$$

or, equivalently, 

$$\Delta_m(a_1, \ldots, a_m) = \prod_{1 \leq i < j \leq m} (a_i - a_j) = \det \left[ a_i^{n-j} \right]_{i,j=1, \ldots, m}.$$
and note that
\[
\Delta_n(\lambda) \sum_{k=1}^{n} \left[ \prod_{l=1, \ldots, n; \ l \neq k} (\lambda_k - \lambda_l)^{-1} \right] = \sum_{i=1}^{n} (-1)^{i-1} \Delta_{n-1}(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_n)
\]
\[
\begin{bmatrix}
1 & \lambda_1^{n-2} & \ldots & \lambda_1 & 1 \\
1 & \lambda_2^{n-2} & \ldots & \lambda_2 & 1 \\
\vdots \\
1 & \lambda_n^{n-2} & \ldots & \lambda_n & 1 \\
\end{bmatrix}
\]
which implies (8.17). Note that essentially the same calculation yields the alternative formula (8.12).

Set \(\tau_1 = b_{1n}\) and, for \(2 \leq k \leq n\),
\[
\tau_k = \det [b_{ij}]_{1 \leq i \leq k, \ m-k+1 \leq j \leq m}.
\]
Then
\[
x_n^1 + \cdots + x_n^k = \log \tau_k, \quad 1 \leq k \leq n,
\]
and since \(x^n\) satisfies the Toda equations (7.2), this implies
\[
(\log \tau_k)'' = -\frac{\tau_{k+1} \tau_{k-1}}{\tau_k^2}
\]
for each \(1 \leq k \leq n\) with the conventions \(\tau_0 = 1\) and \(\tau_{n+1} = 0\).

On the other hand, the tau functions \(\tilde{\tau}_k\) defined by \(\tilde{\tau}_0 = 1, \tilde{\tau}_1 = \tau_1\),
\[
\tilde{\tau}_k = \det \left[ t_{1,(k+i-j-1)} \right]_{1 \leq i, j \leq k}, \quad k \geq 2
\]
also satisfy
\[
(\log \tilde{\tau}_k)'' = -\frac{\tilde{\tau}_{k+1} \tilde{\tau}_{k-1}}{\tilde{\tau}_k^2}.
\]
This is well-known and is easily verified using basic properties of Wronskians and Sylvester’s identity. It follows, by induction, that \(\tilde{\tau}_k = \tau_k\) for \(1 \leq k \leq n\), and in fact \(\tilde{\tau}_k = 0\) for \(k > n\).

To obtain the last formula (8.16), note that when \(\lambda = 0\), \(b_{ij}(t) = t^{i-j}/(j-i)!\) for \(i < j\), hence
\[
\tau_k = \det \left[ t_{i,j}^{n-k+j-i} \right]_{i,j=1, \ldots, k}^{(n-k+j-i)! 1_{n-k+j-i \geq 0}} \]
\[
= \frac{(k-1)!(k-2)! \ldots 1 \cdot k(n-k)! \cdot \det \left[ \binom{n-i}{k-j} \right]_{i,j=1, \ldots, k}}{(n-1)!(n-2)! \ldots (n-k)!}
\]
with the convention that \(\binom{a}{b} = 0\) if \(b > a\). Now, by a theorem of Gessel and Viennot [19],
\[
\det \left[ \binom{n-i}{k-j} \right]_{i,j=1, \ldots, k} = 1,
\]
so we are done. □
Example 8.3. Suppose $n = 2$ and $\lambda = (\lambda, -\lambda)$. Then
\[
x(t) = \left( \log \left[ \frac{1}{\lambda} \sinh(\lambda t) \right], -\log \left[ \frac{1}{\lambda} \sinh(\lambda t) \right] \right).
\]
This yields the solution
\[
\xi(t) = \left( \log \left[ \frac{1}{\lambda} \sin(\lambda t) \right], -\log \left[ \frac{1}{\lambda} \sin(\lambda t) \right] \right)
\]
of the usual Toda lattice. Note that
\[
e^{\xi_2 - \xi_1} = -\frac{\lambda^2}{\sin^2(\lambda t)}.
\]
We can also take $\lambda$ to purely imaginary, $\lambda = i\gamma$ say, where $\gamma \in \mathbb{R}$. Then
\[
e^{\xi_2 - \xi_1} = -\frac{\gamma^2}{\sinh^2(\gamma t)}.
\]
This particular singular solution of the usual Toda lattice is discussed, for example, in [11]. When $\lambda = 0$,
\[
x(t) = (\log t, -\log t), \quad \xi(t) = (i\pi/2 + \log t, -i\pi/2 - \log t),
\]
and
\[
e^{\xi_2 - \xi_1} = -1/t^2.
\]

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