Quasi–periodic motions in generic nearly–integrable mechanical systems

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Abstract

In this note we present and briefly discuss results, which include as a particular case the theorem announced in [2], concerning the typical behaviour of nearly–integrable mechanical systems with generic analytic potentials.

In 2015, encouraged by our mentor, colleague and friend Antonio Ambrosetti, we published in Atti della Accademia Nazionale dei Lincei an announcement [2] concerning ‘typical’ trajectories of nearly–integrable Hamiltonian systems. In particular, we stated a theorem ([2, p. 426]), which can be roughly rephrased as follows:

In bounded regions of phase space, except for a set of measure $\varepsilon |\log \varepsilon|^{\gamma}$, trajectories of nearly–integrable mechanical systems on $\mathbb{R}^n \times \mathbb{T}^n$ with generic real–analytic potentials of size $\varepsilon \ll 1$ are quasi–periodic and span $n$–tori invariant for such systems.

This theorem is in agreement (up to the logarithmic correction) with a conjecture formulated by Arnold, Kozlov and Neishtadt in the Springer Encyclopaedia of Mathematical Sciences [1, Chapter 6, p. 285].
A complete proof of the above result turned out to be much longer and more delicate than we thought and it has been completed only recently in [7] and [6], which in turn exploit intermediate results published in [3] and [4].

The purpose of this short note is to communicate the precise results of [6], which, as a particular case, yield the above theorem.

In order to state the main results in [6] we need to recall a few notions and give some definitions.

(a) Hamiltonian systems on $\mathbb{R}^n \times T^n$

Given a region $B \subseteq \mathbb{R}^n$, the ‘phase space’ $\mathcal{M} := B \times T^n$ (where $T^n := \mathbb{R}^n/(2\pi\mathbb{Z}^n)$) and a real analytic ‘Hamiltonian function’ $H : \mathcal{M} \to \mathbb{R}$, we denote by $z \in \mathcal{M} \to \Phi^t_H(z) \in \mathcal{M}$ the Hamiltonian flow generated by $H$, namely, the solution of the standard Hamilton equations

$$\begin{cases}
    \dot{y} = -H_x(y, x) \\
    \dot{x} = H_y(y, x)
\end{cases}, \quad (y, x)|_{t=0} = z,$$

where, as usual, ‘dot’ denotes derivative with respect to ‘time’ $t \in \mathbb{R}$, and $H_y, H_x$ the gradient with respect to $y, x$.

A mechanical system on $\mathbb{R}^n \times T^n$ is a Hamiltonian system with Hamiltonian

$$H(y, x) = \frac{1}{2}|y|^2 + f(x), \quad \text{(where } |y|^2 := y \cdot y := \sum_j |y_j|^2 \text{)},$$

whose evolution equations are equivalent to the Newton equation $\ddot{x} = -f_x(x)$; $f$ is called the potential of the system; ‘nearly–integrable’ means that the potential is of the form $\epsilon f$ with $\epsilon$ a small real parameter.

(b) Diophantine vectors

A vector $\omega \in \mathbb{R}^n$ is called Diophantine if there exist $\alpha > 0$ and $\tau \geq n - 1$ such that $|\omega \cdot k| \geq \alpha/|k|^\tau$, for any non vanishing integer vector $k \in \mathbb{Z}^n$, where $|k| := \sum |k_j|$. 
(c) Maximal KAM tori
A set $\mathcal{T} \subseteq \mathcal{M}$ is a maximal KAM torus for a Hamiltonian function $H$ if there exist a real analytic embedding $\phi : \mathbb{T}^n \to \mathcal{M}$ and a Diophantine frequency vector $\omega \in \mathbb{R}^n$ such that $\mathcal{T} = \phi(\mathbb{T}^n)$ and for each $z \in \mathcal{T}$, $\Phi_H^t(z) = \phi(x + \omega t)$, where $x = \phi^{-1}(z)$. For general information on KAM (Kolmogorov, Arnold, Moser) Theory, see [1], and references therein.

(d) Generators of 1d maximal lattices
Let $\mathbb{Z}_n^* \subseteq \mathbb{Z}^n$ be the set of integer vectors $k \neq 0$ in $\mathbb{Z}^n$ such that the first non-null component is positive:

$$\mathbb{Z}_n^* := \{k \in \mathbb{Z}^n : k \neq 0 \text{ and } k_j > 0 \text{ where } j = \min\{i : k_i \neq 0\}\}.$$

$\mathcal{G}^n$ denotes the set of generators of 1d maximal lattices in $\mathbb{Z}^n$, namely, the set of vectors $k \in \mathbb{Z}_n^*$ such that the greater common divisor (gcd) of their components is 1:

$$\mathcal{G}^n := \{k \in \mathbb{Z}_n^* : \gcd(k_1, \ldots, k_n) = 1\}.$$

(e) Resonances
A resonance $\mathcal{R}_k$ with respect to the free Hamiltonian $\frac{1}{2}|y|^2$ is the set $\{y \in \mathbb{R}^n : y \cdot k = 0\}$, where $k \in \mathcal{G}^n$. We call $\mathcal{R}_{k,\ell}$ a double resonance if $\mathcal{R}_{k,\ell} = \mathcal{R}_k \cap \mathcal{R}_\ell$ with $k$ and $\ell$ in $\mathcal{G}^n$ linearly independent; the order of a double resonance is given by $\max\{|k|, |\ell|\}$.

(f) 1d Fourier projectors
Given $k \in \mathbb{Z}^n \setminus \{0\}$ and a periodic analytic function $f : \mathbb{T}^n \to \mathbb{C}$, we denote by $\pi_{jk}$ the (analytic) periodic function of one variable $\theta \in \mathbb{T}$ given by

$$\theta \in \mathbb{T} \mapsto \pi_{jk} f(\theta) := \sum_{j \in \mathbb{Z}} f_{jk} e^{ij\theta}.$$

Note that $f(x) = \sum_{k \in \mathcal{G}^n} \pi_{zk} f(k \cdot x)$.

(g) Morse functions with distinct critical values
A function $\theta \to F(\theta)$ is a Morse function if its critical points
are non–degenerate, i.e., \( F'(\theta_0) = 0 \implies F''(\theta_0) \neq 0 \); ‘distinct critical values’ means that if \( \theta_1 \neq \theta_2 \) are distinct critical points, then \( F(\theta_1) \neq F(\theta_2) \).

(h) A Banach space of real analytic functions

Let \( s > 0 \). We denote by \( B^n_s \) the Banach space of real analytic periodic functions on \( \mathbb{T}^n \) having zero average:

\[
B^n_s := \left\{ f = \sum_{k \in \mathbb{Z}^n} f_k e^{ik \cdot x} \text{ s.t. } \tilde{f}_k = f_{-k} \text{ and } \| f \|_s < \infty \right\},
\]

where \( \| f \|_s := \sup_{k \in \mathbb{Z}^n} |f_k| e^{|k|_1 s} \).

(i*) The generic set \( \mathbb{P}^n_s \) of potentials

We denote by \( \mathbb{P}^n_s \) the subset of the unit ball of \( B^n_s \) given by the set of functions \( f \in B^n_s \) such that the following two conditions hold:

\[
\lim_{|k|_1 \to +\infty} |f_k| e^{|k|_1 s} |k|_1^n > 0 ,
\]

\[ \forall k \in \mathcal{G}^n, \pi_{2k} f \text{ is a Morse function with distinct critical values.} \]

We remark that all the above definitions are standard, except for the last one, which describe the class of potentials for which our results hold.

\( \mathbb{P}^n_s \) is a typical set in many ways: it contains an open and dense set (in the topology of \( B^n_s \)), it has full measure with respect to standard probability measures on the unit ball of \( B^n_s \), and is a prevalent set (‘prevalence’ is a measure–theoretic notion for subsets of infinite–dimensional spaces that is analogous to ‘full Lebesgue measure’ in Euclidean spaces; compare [8]). For a detailed discussion of the properties of \( \mathbb{P}^n_s \), see, [4, §3] and Appendix A.2 in [6]. We also remark that the definition given here simplifies and extends former definitions given in [2] and [4].

We can now state the main results in [6].
Theorem 1 Let \( n \geq 2, \ s > 0, \ 0 < \varepsilon < 1, \ f \in \mathbb{P}_x^0, \ B \) an open ball in \( \mathbb{R}^n \) and \( H(y, x; \varepsilon) := \frac{1}{2}|y|^2 + \varepsilon f(x) \). Then, there exists a constant \( c > 1 \) such that all points in \( B \times \mathbb{T}^n \) lie on a maximal KAM torus for \( H \), except for a subset of measure bounded by \( c \varepsilon |\log \varepsilon|^\gamma \) with \( \gamma := 11n + 4 \).

Theorem 2 Fix \( 0 < a < 1 \). For any \( \varepsilon > 0 \), there exists an open neighbourhood \( D^2 \subseteq B \) of double resonances of order smaller than \( 1/\varepsilon^b \), with \( b := \frac{1-a}{\gamma} \), which satisfies \( \text{meas}(D^2 \times \mathbb{T}^n) \leq c_0 \varepsilon^a \), for a suitable constant \( c_0 \) (depending only on \( n \)), such that the following holds. Under the assumptions of Theorem 1, there exists a positive constant \( \tilde{c} \) (independent of \( a \)) such that all points in \( (B \setminus D^2) \times \mathbb{T}^n \), lie on a maximal KAM torus for \( H \), except for an exponentially small subset of measure bounded by \( e^{-\tilde{c}/\varepsilon^b} \).

Theorem 3 Let the assumptions of Theorem 1 hold and let \( n = 2 \). There exists a constant \( \bar{c} > 0 \) such that, for every \( 0 < a < 1 \), all points in \( \{ y \in B : |y| > \varepsilon^{a/2} \} \times \mathbb{T}^2 \) lie on a maximal KAM torus for \( H \), except for an exponentially small subset of measure bounded by \( e^{-\bar{c}/\varepsilon^b} \), with \( b = (1 - a)/24 \).

Let us make a few observations.

Theorem 1 – which extends the result in [2] – may be viewed as the ‘ultimate frontier of KAM Theory’, in the sense that, as remarked by Arnold et al., near double resonances, there are regions of order \( \varepsilon \) where the dynamics of \( H \) is equivalent to the dynamics of the parameter–free Hamiltonian \( \frac{1}{2}|y|^2 + f(x) \) and, therefore, it is natural to expect that in a generic system with three or more degrees of freedom the measure of the “non-torus” set has order \( \varepsilon \) ([1, Remark 6.18, p. 285]). Theorem 1 provides an upper bound on the measure of the non–torus set in agreement (up to the logarithmic correction \( |\log \varepsilon|^\gamma \)) with this expectation. On the other hand, rigorous lower bounds on such a measure appear to be extremely hard to be proven in the analytic case; for partial results in the Gevrey case, see [9].

The KAM tori constructed in Theorem 1 are not uniformly distributed in phase space. Indeed, if one stays away from a finite number of double
resonances, the density is exponentially small: this is the main content of Theorem 2.

Theorem 3 is a consequence of Theorem 2, since in dimension 2, the only double resonance in a mechanical system is the origin. Theorem 3 is in agreement with the conjecture formulated by Arnold et al. in [1, Remark 6.17, p. 285].

A related (weaker) result was announced in [5].

We recall that classical KAM Theory yields only primary tori (which are graphs over $\mathbb{T}^n$) $\sqrt{\varepsilon}$-away from resonances, while the new tori constructed in the above theorems fill, with an exponential density, a neighbourhood of (simple) resonances far from double resonances. Furthermore, the new KAM tori include, besides primary tori, also secondary tori, which exhibit different topologies and, in particular, are not graphs over $\mathbb{T}^n$.

Secondary tori close to resonances have been also investigated in [10].

To prove the above results it is essential to study regions close to resonances $\mathcal{R}_k$ for $|k|_1 \to \infty$ as $\varepsilon \to 0$. Away from doubles resonances the ‘secular’ dynamics in $\mathcal{R}_k$ is, up to exponentially long times, ruled by the integrable Hamiltonian

$$H_k := \frac{1}{2}|y|^2 + \varepsilon(\pi_{zk}f)(x \cdot k).$$

Therefore, it should not surprise that one needs non-degeneracy assumptions in (i') above; in particular, the first condition implies that for $|k|_1 \geq N$ large enough, but independent of $\varepsilon$, the secular potential $\pi_{zk}f$ is essentially a rescaled and shifted cosine (as fully discussed in [4]). Notice that for low modes ($|k|_1 < N$) the secular potential $\pi_{zk}f$ is a generic periodic function. In particular, the phase portrait of $H_k$ is quite arbitrary and may have an arbitrary number of equilibria and separatrices. The main point here is to prove the persistence of all integrable tori of $H_k$ up to an exponentially small set (away from double resonances).

We finally remark that one of the main issues in the proof of measure estimates is to show that the integrable secular Hamiltonian $H_k$ above, in its action variables, is Kolmogorov non-degenerate, namely
the action–to–frequency map is invertible. While $H_k|_{\varepsilon=0} = \frac{1}{2}|y|^2$ is obviously non–degenerate, it is a fact that this is not always true for $H_k$ (in its action variables) when $\varepsilon > 0$. Indeed, this is a singular perturbation problem, as suggested by the fact that the level sets of $H_k$ have different topologies, due to the presence of secondary tori for $\varepsilon > 0$.

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