Casimir Energy of AdS5 Electromagnetism and Cosmological Constant Problem

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Casimir energy is calculated for the 5D electromagnetism in the warped geometry. It is compared with the flat case (arXiv:0801.3064). A new regularization, called sphere lattice regularization, is taken. It is based on the minimal area principle and is a direct realization of the geometrical approach to the renormalization group. The properly regularized form of Casimir energy is expressed in a closed form. We numerically evaluate Λ(4D UV-cutoff), ω(5D bulk curvature, warp parameter) and T(extra space IR parameter) dependence of the Casimir energy. The warp parameter ω suffers from the renormalization effect. We examine the meaning of the weight function and finally reach a new definition of the Casimir energy where the 4D momenta (or coordinates) are quantized with the extra coordinate as the Euclidean time. We comment on the cosmological term at the end.

Keywords: sphere lattice; renormalization of boundary parameters.

1. Introduction

In the previous work, we have examined 5D electromagnetism in the flat geometry. The extra space (y) is periodic (periodicity 2l) and Z2-symmetry is taken into account. The (regularized) Casimir energy is expressed in a closed form: $E_{Cas}(\Lambda, l) = \frac{2\pi^2}{25} \int_0^{\Lambda/l} d\tilde{p} \int_0^{1/\Lambda} dy \tilde{p}^3 W(\tilde{p}, y) F(\tilde{p}, y), \quad F(\tilde{p}, y) = \int_{E_p}^{\Lambda} dk \frac{3 \cosh k (2y - l) - 5 \cosh kl}{2 \sinh kl}$, where Λ is the 4D-momentum cutoff, W(\tilde{p}, y) is a weight function to suppress the IR and UV divergences. The past approach (W = 1) tells us the quintic divergence ($\Lambda^5$) of the energy. This fact has been troubling us as the problem of the divergent cosmological constant in the 5D Kaluza-Klein theory.

Let us review Casimir energy in 4D Electromagnetism. The electromagnetic field within two parallelly-placed (along z-direction, separation length 2l) perfectly-conducting plates can be regarded as the sum of harmonic oscillators. For the x,y-directions, we take the periodic (periodicity 2L) regularization. The energy of the 4D EM is given by

$$E_{4dEM} = E_{Cas} + E_\beta, \quad E_{Cas} = \sum \tilde{\omega}_{mx, my, n}, \quad E_\beta = 2 \sum \frac{\tilde{\omega}_{mx, my, n}^2 - 1}{e^{2\tilde{\omega}_{mx, my, n}} - 1},$$

where $\tilde{\omega}_{mx, my, n}^2 = (\tilde{\omega}_{mx, my, n})^2 = (m_x^2 + m_y^2) + (n_z^2)$. 

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\[ E_\beta = 2 \int_0^\infty dk k^2 \left( \frac{L}{k^2} \right)^2 e^{\beta k} - 1 = (2L)^2 \int_0^\infty dk \mathcal{P}(\beta, k) = \frac{8}{\pi^2} \frac{L^2}{\beta^4} 4\zeta(4) \] \quad (2)

where \( \mathcal{P}(\beta, k) \) is the Planck’s radiation formula. The behavior of \( \mathcal{P}(\beta, k) \) is graphically shown in Fig.1. (We will see similar graphs in the following 5D case. The extra axis corresponds to the \( \beta \)-axis.) The peak curve of the graph is hyperbolic, \( \beta k = \text{const.} \), in the \( (\beta, k) \)-plane (Wien’s displacement law).

\[ E_{\text{Cas}} \] is the sum of the zero-point energy over all frequencies. It is Casimir energy. This quantity is independent of the coupling and is dependent on the boundary(ies). \( E_{\text{Cas}} \) does not vanish for the temperature(1/\( \beta \))=0. It is, however, formally divergent. We need a proper regularization for the summation over the infinite degree of freedom due to the continuity of the space-time.

\[ E_{\text{Cas}}^\Lambda = \sum m^2 \frac{g(\tilde{\omega})}{\Lambda^2} = \sum \sqrt{m^2 \pi^2 L^2 + (m_y \pi L)^2 + (m_z \pi L)^2} g(\tilde{\omega}) \] \quad (3)

where we introduce the cut-off function: \( g(\omega/\Lambda) = \{ 1 \text{ for } 0 \leq \omega \leq \Lambda; 0 \text{ for } \omega > \Lambda \} \). \( \Lambda \) is the cut-off parameter for the absolute value of the 3D (x,y,z) momentum. We will take the limit \( \Lambda \rightarrow \infty \) at an appropriate stage. Taking the reference point, \( L \rightarrow \infty, L \gg l \), from which we “measure” the energy, \( E_{\text{Cas}}^\Lambda = \int_0^\infty \int_0^\infty \frac{dk_x dk_y}{(2\pi)^2} \sqrt{k_x^2 + k_y^2 + k_x^2 + k_y^2} g(\frac{\xi}{\Lambda}) \) , we finally obtain the finite result,

\[ u = \frac{E_{\text{Cas}}^\Lambda - E_{\text{Cas}}^\Lambda_0}{(2L)^2} = \frac{\pi^2}{2} B_3 \left[ \frac{1}{\Lambda} \right] = - \frac{\pi^2}{720} \frac{1}{(2\pi)^4} \] \quad (4)

which does not depend on \( \Lambda \). Especially there remains no \( \log \Lambda \) divergence. This point is contrasting with the ordinary renormalization of interacting theories such as 4D QED and 4D YM. Hence we need not the renormalization of the wave-function and the parameter \( l \). In the 5D flat geometry, the renormalization of the boundary parameter is necessary. In the following, we examine the warped 5D case and find the another boundary parameter \( \omega \) suffers from the renormalization effect.
2. Kaluza-Klein expansion approach

5D massive vector theory is given by

\[ S_{5\text{dV}} = \int d^4x d\sigma \sqrt{-G} \left( -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} m^2 A^M A_M \right), \]

\[ ds^2 = \frac{1}{\omega^2 z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right) = G_{MN} dX^M dX^N, \]

where \( M, N = 0, 1, 2, 3, 5 \) (or \( z \)). \( \eta \) is a parameter. Instead of \( \Phi(z) \), we introduce, instead of \( \Phi(z) \), the modified Bessel functions, with the index \( \nu \) appearing.

Here we introduce, instead of \( \Phi(z) \), the modified Bessel functions, with the index \( \nu \) appearing. Hence the 5D EM limit is given by \( \nu = 1 \) (\( m = 0 \)). We consider, however, the imaginary mass case \( m = i\omega \). Instead of analyzing the \( m^2 = -\omega^2 \), we take the 5D massive scalar theory on AdS\(_5\) with \( m^2 = -4\omega^2, \nu = \sqrt{4 + m^2/\omega^2} \).

\[ \mathcal{L} = \sqrt{-G} \left( \frac{1}{2} \nabla^A \Phi \nabla_A \Phi - \frac{1}{2} m^2 \Phi^2 \right), \quad \nabla^A \Phi - m^2 \Phi + J = 0 \]  

where \( G \equiv \det G_{AB} \), \( ds^2 = G_{MN} dX^M dX^N \). \( \Phi(X) = \Phi(x^2, z) \) is the 5D scalar field. The background geometry is AdS\(_5\) which takes the form: \( G_{AB} \equiv \text{diag}(\frac{1}{\omega^2 z^2}, \frac{1}{\omega^2 z^2}) \). The variable range of \( z \) is \( -\frac{\omega}{2} \leq z \leq \frac{\omega}{2} \) or \( -\frac{\omega}{2} \leq z \leq \frac{\omega}{2} \) \(( -1 \leq y \leq l , |z| = \frac{1}{\omega} e^{y/l} \))

We define, introduce, instead of \( \Phi(z) \), the partially (4D world only) Fourier-transformed field \( \Phi_p(z): \Phi(X) = \int \frac{dp}{(2\pi)^5} e^{ip\cdot X} \Phi_p(z) \). Eq.\((6)\) can be rewritten as

\[ s(z) \equiv \frac{1}{(\omega z)^3}, \quad \hat{L}_z \equiv \frac{d}{dz} \frac{1}{(\omega z)^3} \frac{d}{dz} - \frac{m^2}{(\omega z)^5}, \quad e^{-T^{-4}E_{Cas}} = \int D\Phi_p(z) \times \exp \left[ i \int \frac{dp}{(2\pi)^4} \overline{2} \int_{1/\omega}^{1/T} dz \left\{ \frac{1}{2} \phi_p(z) \left( -\frac{1}{(\omega z)^3} p^2 + \frac{1}{d^2} \frac{1}{(\omega z)^3} d^2 - \frac{m^2}{(\omega z)^5} \right) \phi_p(z) \right\} \right] \]

We consider the Bessel eigen-value problem: \( \{ s(z)^{-1} \hat{L}_z + M_n^2 \} \psi_n(z) = 0, \) with \( Z_2 \)-property: \( \psi_n(z) = -\psi_n(-z) \) for \( P = -1 \); \( \psi_n(z) = \psi_n(-z) \) for \( P = +1 \), and the appropriate b.c. at fixed points. Because the set \( \{ \psi_n(z) \} \) constitutes the orthonormal and complete system, we can express \( \Phi_p(z) \) as, \( \Phi_p(z) = \sum_n c_n(p) \psi_n(z) \). Eq.\((7)\) can be further rewritten as

\[ e^{-T^{-4}E_{Cas}} = \int D\Phi_p(z) \exp \left[ i \int \frac{dp}{(2\pi)^4} \overline{2} \int_{1/\omega}^{1/T} dz \left\{ \frac{1}{2} \phi_p(z) s(z) (s(z)^{-1} \hat{L}_z - p^2) \phi_p(z) \right\} \right] \]

\[ = \prod_n dc_n(p) \exp \left[ i \int \frac{dp}{(2\pi)^4} \sum_n \left( \frac{-1}{2} c_n(p)^2 (p^2 + M_n^2) \right) \] \[ = \exp \sum_{n,p} \left( \frac{-1}{2} \ln(p^2 + M_n^2) \right)\]
where the orthonormal relation, \(2 \int \frac{dz}{z} \psi_n(z) \bar{\psi}_m(z)dz = \delta_{nm}\), is used. This shows that \(s(z)\), defined in (7), plays the role of "inner product measure" in the function space \(\{\psi_n(z), 1/\omega \leq z \leq 1/T\}\). The expression (8) is the familiar one (expanded form) of the Casimir energy.

3. Heat-Kernel Approach and Position/Momentum Propagator

Instead of the KK-expansion form, we can formally integrate out the \(\Phi_p(z)\) variable in the path-integral (8).

\[
e^{-T^{-4}E_{Cas}} = \exp \left[ T^{-3} \int \frac{d^4p}{(2\pi)^4} 2 \int_{1/\omega}^{1/T} dz s(z) \left\{ -\frac{1}{2} \ln(-s(z)^{-1} \hat{L}_z + p^2) \right\} \right]
\]

\[
e^{-T^{-4}E_{Cas}} = \exp \left[ T^{-3} \int \frac{d^4p}{(2\pi)^4} 2 \int_{1/\omega}^{1/T} dz s(z) \left\{ \frac{1}{2} \int_0^\infty \frac{dt}{t} e^{t(s(z)^{-1} \hat{L}_z - p^2)} + \text{const} \right\} \right], \tag{9}
\]

The above formal result can be \textit{precisely} defined using the heat equation.\(^5\)

\[
\text{Tr} H_p(z, z'; t) = \int_{1/\omega}^{1/T} s(z) H_p(z, z; t) dz, \quad \left( \frac{\partial}{\partial t} - (s^{-1} \hat{L}_z - p^2) \right) H_p(z, z'; t) = 0. \tag{10}
\]

The heat kernel \(H_p(z, z'; t)\) is formally solved, using the Dirac’s bra and ket vectors \((z|, |z)\), as \(H_p(z, z'; t) = (z| e^{(-s^{-1} \hat{L}_z + p^2)t} |z')\). Using the set \(\{\psi_n(z)\}\) defined previously, the explicit solution of (10) is given by

\[
\begin{align*}
\{H_p(z, z'; t) \} &= \sum_{n \in \mathbb{Z}} e^{-(M_n^2 + p^2)t} \frac{1}{2} \{\psi_n(z)\bar{\psi}_n(z') + \psi_n(z)\bar{\psi}_n(-z')\}, \quad P = \mp \ , \tag{11}
\end{align*}
\]

We here introduce the position/momentum propagators \(G_p^T\) as follows. \(G_p^T(z, z') \equiv \int_0^\infty dt \{H_p(z, z'; t)\}

\[
G_p^T(z, z') = \sum_{n \in \mathbb{Z}} \frac{1}{M_n^2 + p^2 + 2} \{e^{-(M_n^2 + p^2)t} \psi_n(z)\bar{\psi}_n(z') + \psi_n(z)\bar{\psi}_n(-z')\}. \tag{12}
\]

Therefore the Casimir energy \(E_{Cas}\) is given by

\[
E_{Cas}(\omega, T) = \int \frac{d^4p}{(2\pi)^4} 2 \int_0^\infty \frac{dt}{t} 2 \int_{1/\omega}^{1/T} dz s(z) H_p(z, z; t)
\]

\[
E_{Cas}(\omega, T) = \int \frac{d^4p}{(2\pi)^4} 2 \int_0^\infty \frac{dt}{t} 2 \int_{1/\omega}^{1/T} dz s(z) \left\{ \sum_{n \in \mathbb{Z}} e^{-(M_n^2 + p^2)t} \psi_n(z)^2 \right\} \ , \tag{13}
\]

The P/M propagators \(G_p^T\) in (12) can be expressed in a \textit{closed} form. (See, for example, Ref. 6.) Taking the \textit{Dirichlet} condition at all fixed points, the expression for the fundamental region (1/\omega \leq z \leq z' \leq 1/T) is given by

\[
G_p^T(z, z') = \frac{\omega^3}{2} z^2 z'^2 \left\{ I_0(\frac{\omega}{2}) K_0(\hat{p} z) + K_0(\frac{\omega}{2}) I_0(\hat{p} z) \right\} \left\{ I_0(\frac{\omega}{2}) K_0(\hat{p} z') + K_0(\frac{\omega}{2}) I_0(\hat{p} z') \right\}. \tag{14}
\]
where \( \tilde{p} \equiv \sqrt{p^2} \), \( p^2 \geq 0 \) (space-like). We can express the \( \Lambda \)-regularized Casimir energy in terms of the following functions \( F^{\pm}(\tilde{p}, z) \).

\[
E_{\text{Cas}}^{\Lambda, \mp}(\omega, T) = \int \frac{d^4 p}{(2\pi)^4} \int_{\frac{1}{T}}^{1/\omega} dz \, F^{\mp}(\tilde{p}, z),
\]

where \( F^{\mp}(\tilde{p}, z) \equiv s(z) \int^{\Lambda}_{\mu} \{ G_k^{\mp}(z, z) \} dk^2 \equiv \int^{\Lambda}_{\tilde{p}} F^{\mp}(\tilde{k}, z) d\tilde{k} \). Here we introduce the UV cut-off parameter \( \Lambda \) for the 4D momentum space. In Fig.2a, we show the behaviour of \( F^{\mp}(\tilde{p}, z) \). The table-shape graph says the "Rayley-Jeans" dominance. That is, for the wide-range region \((\tilde{p}, z)\) satisfying both \( \tilde{p}(z - \frac{1}{2}) \gg 1 \) and \( \tilde{p}(\frac{1}{T} - z) \gg 1 \),

\[
F^{-}(\tilde{p}, z) \approx -\frac{1}{2} \quad F^{+}(\tilde{p}, z) \approx -\frac{1}{2}; \quad \tilde{p}(z - \frac{1}{\omega}) \gg 1 \quad \text{and} \quad \tilde{p}(\frac{1}{T} - z) \gg 1.
\]

\( F^{-}(\tilde{p}, z) \equiv \frac{1}{2} F^{\mp}(\tilde{p}, z) \) \( (17) \). \( T = 1, \omega = 10^4, \Lambda = 4 \times 10^4 \). \( 1.0001/\omega \leq z < 0.9999/T \), \( \Lambda T/\omega \leq \tilde{p} \leq \Lambda \). \( \tilde{p}(z - \frac{1}{\omega}) \gg 1 \quad \text{and} \quad \tilde{p}(\frac{1}{T} - z) \gg 1 \).

\( 4. \) UV and IR Regularization Parameters and Evaluation of Casimir Energy

The integral region of the above equation \((15)\) is displayed in Fig.3a. In the figure, we introduce the regularization cut-offs for the 4D-momentum integral, \( \mu \leq \tilde{p} \leq \Lambda \). As for the extra-coordinate integral, it is the finite interval, \( 1/\omega \leq z \leq 1/T = e^{\omega t}/\omega \), hence we need not introduce further regularization parameters. For simplicity, we take the following IR cutoff of 4D momentum : \( \mu = \Lambda \cdot \frac{T}{\omega} = \Lambda e^{-\omega t} \).

Let us evaluate the \((\Lambda, T)\)-regularized value of \((15)\).

\[
E_{\text{Cas}}^{\Lambda, \mp}(\omega, T) = \frac{2\pi^2}{(2\pi)^4} \int_{\mu}^{\Lambda} d\tilde{p} \int_{1/\omega}^{1/T} dz \, \tilde{p}^3 F^{\mp}(\tilde{p}, z),
\]

where \( F^{\mp}(\tilde{p}, z) = \frac{2}{(\omega \tilde{z})^2} \int^{\Lambda}_{\mu} \tilde{k} G_k^{\mp}(z, z) d\tilde{k} \). The integral region of \((\tilde{p}, z)\) is the rectangle shown in Fig.3a.
\begin{equation}
\frac{1}{\mathcal{Z}} = \frac{1}{T} \mathcal{Z}_{UV} \mathcal{Z}_{IR} \left( \frac{1}{T} \right) \mathcal{Z}_{IR}
\end{equation}

Fig. 3. (a-left) Space of \((z, \tilde{p})\) for the integration. The hyperbolic curve will be used in Sec. 5. (b-right) Space of \((\tilde{p}, z)\) for the integration (present proposal).

Note that eq. (17) is the rigorous expression of the \((\Lambda, T)\)-regularized Casimir energy. We show the behaviour of \((-1/2) \tilde{p}^3 F (\tilde{p}, z)\) in Fig. 2b. The requirement for the three parameters \(\omega, T, \Lambda\) is \(\Lambda \gg \omega \gg T\). From a close numerical analysis of \((\tilde{p}, z)\)-integral (17), we have confirmed: \(E_{\Lambda, -\text{Cas}}(\omega, T) = 2 \pi^2 (2 \pi)^4 \times \left( -0.0250 \frac{\Lambda^5}{\omega} \right)\), which does not depend on \(\omega\). In \(\frac{1}{T}\)-term does not appear. Compared with the flat case \((E_{\text{flat}, -\text{Cas}}(\Lambda, l) = (1/8 \pi^2) [1 - 0.0893 \Lambda^4]; No \ln(l/\Lambda)\)-term )

Although they claim the holography is behind the procedure, the legitimateness of the restriction looks less obvious. We have proposed an alternate approach and given a legitimate explanation within the 5D QFT. See Fig. 3. The restriction region is bounded by two minimal surfaces, IR-surface and UV-surface. They are 4 dimensional manifold made from \(S^3\)-spheres “running” along \(z\)-axis. On the 'brane' at fixed \(z\), the 4D region bounded by IR-surface is regarded as a large-size 4D-ball and that by UV-surface is regarded as a small-size 4D-ball. Hence this regularization configuration is the 4D sphere lattice running along \(z\)-axis. See Ref. 10.
6. Weight Function and Casimir Energy Evaluation

We introduce, instead of restricting the integral region, a weight function $W(\tilde{p}, z)$ in the $(\tilde{p}, z)$-space for the purpose of suppressing UV and IR divergences of the Casimir Energy.

$$E_{Cas}^{W}(\omega, T) \equiv \int \frac{d^4 p}{(2\pi)^4} \int_{1/\omega}^{1/T} dz W(\tilde{p}, z) F^{\pm}(\tilde{p}, z), F^{\pm}(\tilde{p}, z) = \frac{2}{(\omega z)^3} \int_{p}^{\infty} k G_k^\pm(z, z) dk,$$

Examples

\begin{align*}
(N_1)^{-1} e^{-\frac{(1/2)p^2}{\omega^2} - (1/2)z^2 T^2} \equiv W_1(\tilde{p}, z), \quad N_1 = 1.711/8\pi^2 & \quad \text{elliptic} \\
(N_2)^{-1} e^{-\frac{p z T}{\omega}} \equiv W_2(\tilde{p}, z), \quad N_2 = 2\frac{\omega^3}{8\pi^2} & \quad \text{hyperbolic}\ (18) \\
(N_8)^{-1} e^{-\frac{1}{2}(p^2/\omega^2 + 1/z^2 T^2)} \equiv W_8(\tilde{p}, z), \quad N_8 = 0.4177/8\pi^2 & \quad \text{reciprocal} \\
\end{align*}

where $G_k^\pm(z, z)$ are defined in (14). $N_i$’s are the normalization constants. We show the shape of the energy integrand $(-1/2)p^3 W(\tilde{p}, z) F^- (\tilde{p}, z)$ in Fig.1b.

We can check the divergence (scaling) behavior of $E_{Cas}^{W}$ by numerically evaluating the $(\tilde{p}, z)$-integral $\int$ for the rectangle region of Fig.3a. $E_{Cas}^{W} =$

\begin{align*}
\frac{\omega}{4} \Lambda \left\{ 0.42 + 0.01 \ln\frac{\Lambda}{T} + 0 \times \ln\frac{\Lambda}{T} \right\} & \quad \text{for } W_1 \\
\frac{2^2}{4} \Lambda^2 \left( 0.65 + 0.234 \ln\frac{\Lambda}{T} - 0.479 \ln\frac{\Lambda}{T} \right) \times 10^{-2} & \quad \text{for } W_2 \\
\frac{\omega}{4} \Lambda \left\{ 0.42 + 0.02 \ln\frac{\Lambda}{T} + 0 \times \ln\frac{\Lambda}{T} \right\} & \quad \text{for } W_8 \\
\end{align*} (19)

They give, after normalizing the factor $\Lambda/T$, only the log-divergence.

$$E_{Cas}^{W}/\Lambda T^{-1} = \alpha \omega^4 \left( 1 - 4c \ln(\Lambda/\omega) \right),$$

where $\alpha$ and $c$ can be read from (19) depending on the choice of $W$. We have confirmed the above result for other many examples.\textsuperscript{10} This means the 5D Casimir energy is finitely obtained by the ordinary renormalization of the warp factor $\omega$. (See the final section.) In the above result of the warped case, the IR parameter $l$ in the flat result ($E_{Cas}^{W, flat}/\Lambda = -\frac{1}{12} \left( 1 - 4c \ln(l/\Lambda) \right)$) is replaced by the inverse of the warp factor $\omega$.

7. Meaning of Weight Function and Quantum Fluctuation of Coordinates and Momenta

In order to most naturally accomplish the above procedure, we can go to a new step. Namely, we propose to replace the 5D space integral with the weight $W$, (15) by the following path-integral. We newly define the Casimir energy in the higher-dimensional theory as follows. $E_{Cas}(\omega, T, \Lambda) \equiv$

$$\int \frac{d^4 p}{(2\pi)^4} \int_{r(1/\omega)}^{r(1/T)} dz \rho aCELL \left[ \frac{1}{2\alpha^l} \int_{1/\omega}^{1/T} dz \frac{1}{\omega^4 z^3} \sqrt{r'^2 + 1 r^2 dz} \right].$$ (21)

where $\mu = \Lambda T/\omega$ and the limit $\Lambda T^{-1} \to \infty$ is taken. The string (surface) tension parameter $1/2\alpha^l$ is introduced. (Note: Dimension of $\alpha^l$ is [Length]$^4$. ) The square-bracket ($\cdots$)-parts of (21) are $-\frac{1}{2\omega^4} \text{Area} = -\frac{1}{2\omega^4} \int \sqrt{\det g_{ab}} d^4 x$ where $g_{ab}$ is the induced metric on the 4D surface. $F(\tilde{p}, z)$ is defined in (15) and shows the field-quantization of the bulk scalar (EM) fields.

The proposed definition, (21), clearly shows the 4D space-coordinates $x^a$ or the 4D momentum-coordinates $p^a$ are quantized (quantum-statistically, not field-theoretically) with the Euclidean time $z$ and the "area Hamiltonian" $A = \int \sqrt{\det g_{ab}} d^4 x$. [We recall
the similar situation occurs in the standard string approach. The space-time coordinates obey some uncertainty principle.\cite{12} Note that $F(\vec{p}, z)$ or $F(1/r, z)$ appears, in (21), as the energy density operator in the quantum statistical system of $\{p^a(z)\}$ or $\{x^a(z)\}$.

In the view of the previous paragraph, the treatment of Sec. 5 is an effective action approach using the weight function $W(\vec{p}, z)$. Note that the integral over $(p^\mu, z)$-space, appearing in (15), is the summation over all degrees of freedom of the 5D space-time points using the "naive" measure $d^4pdz$. An important point is that we have the possibility to take another measure for the summation. We have adopted, in Sec. 6, the new measure $W(p^\mu, z)d^4pdz$ in such a way that the Casimir energy does not show physical divergences.

We expect the direct evaluation of (21), numerically or analytically, leads to the similar result.

8. Discussion and Conclusion

The log-divergence, (20), is renormalized away by

$$E_{\text{Cas}}^W/A T^{-1} = \alpha \omega^4 \quad , \quad \omega' = \omega \sqrt{1 - 4c \ln(\Lambda/\omega)} \quad .$$

(22)

Taking into account the fact $|c| \ll 1$, we find the renormalization group function for the warp factor $\omega$ as

$$|c| \ll 1 \quad , \quad \omega' = \omega(1 - c \ln(\Lambda/\omega)) \quad , \quad \beta(\beta\text{-function}) \equiv \frac{\partial}{\partial(\ln \Lambda)} \ln \frac{\omega'}{\omega} = -c \quad .$$

(23)

We should notice that, in the flat geometry case, the IR parameter (extra-space size) $l$ is renormalized. In the present warped case, however, the corresponding parameter $T$ is not renormalized, but the warp parameter $\omega$ is renormalized. Depending on the sign of $c$, the 5D bulk curvature $\omega$ flows as follows. When $c > 0$, the bulk curvature $\omega$ decreases (increases) as the measurement energy scale $\Lambda$ increases (decreases). When $c < 0$, the flow goes in the opposite way.

Recently the dark energy (as well as the dark matter) in the universe is a hot subject. It is well-known that the dominant candidate is the cosmological term. We also know the proto-type higher-dimensional theory, that is, the 5D KK theory, has predicted so far the neutrino mass is located at the geometrical average of two extreme ends of the mass scales\cite{12,13,14} which appears in the Dirac’s large number theory. If we apply the present approach, we have the warp factor $\omega$, and the result (22) strongly suggests the following choice:

$$\frac{1}{G_N} \lambda_{\text{th}} = \alpha \omega^4 \quad , \quad \omega' \sim \frac{1}{\sqrt{G_N R_{\cos}}} \equiv \sqrt{M_{\text{pl}}/R_{\cos}} \sim m_\nu \sim (10^{-3}\text{eV})^4 \quad , \quad \hat{\lambda}_{\text{th}} \sim \frac{1}{R_{\cos}} \quad .$$

(25)
Note that this choice $\omega'$ is within the allowed region obtained from the Newton force test (see Ref. 17). We succeed in obtaining the finite cosmological constant and its gross absolute value consistent with the observed one. Now we can understand that the smallness of the cosmological constant comes from the renormalization flow for the non asymptotic-free case ($\epsilon < 0$ in (23)). [In the 2 dim $\mathbb{R}^2$-gravity, the same thing occurs. 18] In this case the choice of the regularization parameters are

$$T \sim 10^{-33} \text{eV}((\text{Cosm. Size})^{-1}, 1/R_{cos}), \quad \Lambda \sim 10^{28} \text{eV}(\text{Planck mass}, M_{\mu t}),$$

and the relations, $\omega \sim \sqrt{T \Lambda}$ and $\mu = \Lambda T/\omega \sim \sqrt{T \Lambda}$ are valid. The normalization factor in (20) is given by $\frac{T}{\mu^4} \sim 10^{61}$. The total number of the unit spheres within the large sphere is given by $\frac{\Lambda^4}{\mu^4} = \frac{T^4}{\mu^4} \sim 10^{120}$, which is expected to show the degree of freedom of the universe (4D space-time).

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