Proof of entropy principle in Einstein–Maxwell theory

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Abstract

We consider a static self-gravitating charged perfect fluid system in the Einstein–Maxwell theory. Assume Maxwell’s equation and the Einstein constraint equation are satisfied and the temperature of the fluid obeys Tolman’s law. Then, we prove that the extremum of total entropy implies other components of Einstein’s equation for any variations of metric and electrical potential with fixed boundary values. Conversely, if Einstein’s equation and Maxwell’s equations hold, the total entropy achieves an extremum. Our work suggests that the maximum entropy principle is consistent with Einstein’s equation when an electrostatic field is taken into account.

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1 Introduction

It is well known that black holes can radiate and satisfy thermodynamical laws. This discovery establishes the connection between gravity and thermodynamics [1–6]. Despite the great success of black hole thermodynamics, there are still unresolved issues, for instance, the origin of black hole entropy. In contrast to black hole systems, local thermodynamic quantities of a perfect fluid in curved spacetimes, e.g., energy density $\rho$, entropy density $s$, and local temperature $T$, are well defined. The presence of gravity only affects the distribution of those local quantities. In general there are two methods to determine the distribution of matter. One way is solving Einstein’s equation. The other way is using the entropy principle to determine the distribution of matter.

Since entropy plays no role in Einstein’s equation, it is unclear whether the two methods are consistent. Even before the establishment of black
hole thermodynamics, Cocke [7] pointed out that the extrema of entropy should yield the equation of hydrostatic equilibrium which is derived from Einstein’s equation. After that, Sorkin, Wald, and Zhang [8] showed rigorously that the Tolman–Oppenheimer–Volkoff equation of hydrostatic equilibrium can be derived from the extremum of total entropy and the Einstein constraint equation. Gao [9] extended their proof from radiation to a general perfect fluid, including uncharged fluid and uniformly charged fluid. This issue has been further explored in the past few years [10–18].

Recently, we [19] proved the entropy principle for a self-gravitating fluid in static spacetimes without any symmetry in the spacelike hypersurface. So far, the matter field considered is a perfect fluid. It is interesting to know whether or how the entropy principle works in the presence of an electromagnetic field. In this paper, we extend the two theorems in Ref. [19] to a uniformly charged fluid. The extension is not straightforward at all. For an uncharged perfect fluid, we have shown [19] that the variation of total entropy, $\delta S$, is proportional to the variation of the spatial metric, $h_{ab}$. However, for a uniformly charged fluid, $\delta S$ appears to depend on $\delta h_{ab}$, $\delta \chi$, and $\delta A_a$, where $\chi$ is the redshift factor and $A^a$ is the vector potential of the electrostatic field. However, we manage to show that the components of $\delta \chi$ and $\delta h_{ab}$ vanish identically and the vanishing of the components of $\delta h_{ab}$ just gives the spatial components of Einstein’s equation for a charged perfect fluid. Our work suggests that the entropy principle is consistent not only with the gravitational field but also with the electromagnetic field.

2 Properties of charged perfect fluid in static spacetimes

We consider a general perfect fluid as discussed in Ref. [9]. The entropy density $s$ is taken to be a function of the energy density $\rho$ and particle number density $n$, i.e., $s = s(\rho, n)$. From the first law of thermodynamics, one can derive the integrated form of the Gibbs-Duhem relation,

$$s = \frac{1}{T}(\rho + p - \mu n),$$

(1)

where $\rho$ and $\mu$ represent the pressure and the chemical potential, respectively. All the quantities are measured by static observers with 4-velocity $u^a$. These observers are orthogonal to the hypersurface $\Sigma$. Therefore, the induced metric on $\Sigma$ is given by

$$h_{ab} = g_{ab} + u_a u_b.$$  

(2)

The stress-energy tensor $T_{ab}$ for a perfect fluid takes the form

$$T_{ab} = \rho u_a u_b + p h_{ab} = (\rho + p)u_a u_b + pg_{ab}.$$  

(3)
Since the fluid is charged, we should also consider the stress energy tensor of the electromagnetic field,

\[
(T_{\text{EM}})^{ab} = \frac{1}{4\pi} \left( F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{de} F^{de} \right),
\]

(4)

where \( F_{ab} = \nabla_a \hat{A}_b - \nabla_b \hat{A}_a \), and \( \hat{A}_a \) is the vector potential. The electrostatic potential \( \Phi \) is given by

\[
\Phi = -\hat{A}_a \xi^a = -\hat{A}_a u^a \chi,
\]

(5)

where \( \xi^a \) is the Killing vector and \( \chi \) is the redshift factor.

The Maxwell’s equation with source reads

\[
\nabla_b F^{ab} = 4\pi j^a = 4\pi \rho_e u^a,
\]

(6)

where \( j^a \) and \( \rho_e \) represent the 4-current density of the electric charge and the charge density, respectively. Then

\[
\xi^b F_{ab} = \xi^b (\nabla_a \hat{A}_b - \nabla_b \hat{A}_a) = \nabla_a (\xi^b \hat{A}_b) - \hat{A}_b \nabla_a \xi^b - \xi^b \nabla_b \hat{A}_a = -\nabla_a \Phi,
\]

(7)

where we have used the Killing equation and \( L_{\xi} \hat{A}_a = 0 \) in the last step. From \( \nabla_a [T_{\text{EM}}] = 0 \) we can calculate

\[
\nabla_b (T_{\text{EM}})^{ab} = -j^c F^a_c = -\rho_e u^c F^a_c = -\frac{\rho_e \xi}{\chi} F^a_c = \frac{\rho_e}{\chi} \nabla^a \Phi,
\]

(8)

where Maxwell’s equation \( \nabla_a F_{bc} = 0 \) has been used in the first step and Eq. (7) has been used in the last step.

We shall assume that Tolman’s law holds, which states that the local temperature \( T \) of the fluid satisfies

\[
T \chi = T_0,
\]

(9)

where \( \chi \) is the redshift factor for static observers and \( T_0 \) is a constant. Without loss of generality, we take \( T_0 = 1 \). This law establishes the relationship between the fluid temperature and the metric components.

It is then straightforward to show, from the conservation law \( \nabla_a [T_{ab} + (T_{\text{EM}})^{ab}] = 0 \) and the stationary conditions, that

\[
0 = \nabla^a p + (\rho + p) A^a + \frac{\rho_e}{\chi} \nabla^a \Phi,
\]

(10)
where $A^a$ is the 4-acceleration of the observer. For stationary observers,

$$A_a = \nabla_a \chi / \chi,$$

(11)

and thus

$$\nabla_a p = - (\rho + p) \nabla_a \chi / \chi - \frac{\rho_e}{\chi} \nabla^a \Phi.$$  

(12)

On the other hand, the local first law can be expressed in the form

$$dp = sdT + nd\mu.$$  

(13)

Using Eqs. (9) and (1), we find

$$\nabla_a p = \frac{\rho + p - \mu n}{T} \nabla_a T + n \nabla_a \mu$$

$$= - \frac{\rho + p - \mu n}{\chi} \nabla_a \chi + n \nabla_a \mu.$$  

(14)

Comparison with Eq. (12) gives

$$- \frac{\rho_e}{n} \nabla^a \Phi = \mu \nabla_a \chi + \chi \nabla_a \mu = \nabla_a (\mu \chi).$$

(15)

If we assume that all particles possess the same charge $q$, i.e.,

$$\rho_e = nq,$$

then Eq. (15) leads to

$$\mu \chi + q \Phi = c,$$

(17)

or

$$\frac{\mu}{T} + q \Phi = c,$$  

(18)

where $c$ is a constant.

## 3 Two theorems

The distribution of a charged perfect fluid in static spacetimes can be determined in two ways. First, Einstein’s equation and Maxwell’s equation together can totally determine the distribution. Second, under certain boundary conditions, the total entropy of the fluid should take an extremum. In this section, we prove two theorems indicating the equivalence of the two methods.

**Theorem 1:** Consider a uniformly charged perfect fluid in a static spacetime $(M, g_{ab})$ and $\Sigma$ as a three-dimensional hypersurface denoting a moment of the static observers. Let $C$ be a region on $\Sigma$ with a boundary
Let $h_{ab}$, $\Phi$, and $\chi$ be the induced metric on $\Sigma$, electrostatic potential, and redshift factor, respectively. Assume that the temperature of the fluid obeys Tolman’s law and the Einstein constraint equation and Maxwell’s equation are satisfied in $C$. Then, the other components of Einstein’s equation are implied by the extrema of the total fluid entropy for all variations of data in $C$ where $h_{ab}$, $\Phi$, $\chi$, and their first derivatives are fixed on $\bar{C}$.

Proof. The total entropy $S$ is an integral of the entropy density $s$ over the region $C$ on $\Sigma$,

$$ S = \int_C \sqrt{h} s(\rho, n), \tag{19} $$

where $h$ is the determinant of $h_{ab}$. Thus, the variation of the total entropy is written in the form

$$ \delta S = \int_C s \delta \sqrt{h} + \sqrt{h} \delta s. \tag{20} $$

Applying the local first law of thermodynamics,

$$ T ds = d\rho - \mu dn, \tag{21} $$

we find

$$ \delta S = \int_C s \delta \sqrt{h} + \sqrt{h} \left( \frac{1}{T} \frac{\partial s}{\partial \rho} \delta \rho - \frac{\mu}{T} \delta n \right). \tag{22} $$

Note that $\mu/T$ is constant for an uncharged fluid and can be moved out of the integral [19]. But due to the electrostatic potential, Eq. (18) shows that $\mu/T$ is no longer a constant. We shall deal with the $\delta n$ term by employing Maxwell’s equation.

Together with Eqs. (1) and (18), we have

$$ \delta S = \int_C \frac{1}{T} \left( p + \rho - \mu n \right) \delta \sqrt{h} - \sqrt{h} (-q \Phi + c) \delta n + \sqrt{h} \frac{1}{T} \delta \rho. \tag{23} $$

Denote

$$ \delta S = \int_C \delta L, \tag{24} $$

where

$$ \delta L = \frac{1}{T} \left( p + \rho - \mu n \right) \delta \sqrt{h} - \sqrt{h} (-q \Phi + c) \delta n + \sqrt{h} \frac{1}{T} \delta \rho. \tag{25} $$

Our purpose is to derive the space components of Einstein’s equation from $\delta L = 0$ and the constraint Einstein equation. First, we need to express
\( \delta L \) as variations of basic variables \( h_{ab}, \chi \) and \( \Phi \). The \( \delta h \) term in Eq. (25) can be easily written in the desired form by the relation

\[
\delta \sqrt{h} = \frac{1}{2} \sqrt{h} h^{ab} \delta h_{ab} .
\]  

The \( \delta n \) term in Eq. (25) is calculated by using Maxwell’s equation (see Appendix B). Now we shall focus on calculating the \( \delta \rho \) term.

Note that the extrinsic curvature of \( \Sigma \) defined by

\[
\hat{B}_{ab} \equiv h^{c}_{a} h^{d}_{b} \nabla_{d} u_{c}.
\]  

vanishes \[19\] in static spacetimes and

\[
\nabla_{b} u_{a} = -A_{a} u_{b}.
\]  

By the result of Ref. \[19\], the Ricci tensor \( R_{ab}^{(3)} \) and scalar curvature \( R^{(3)} \) of \( \Sigma \) are given by

\[
R_{ab}^{(3)} = R_{ab} + R_{aeb} u^{e} u_{b} + R_{akeb} u^{k} u_{b} + u_{a} u_{b} R_{jk} u^{j} u^{k}.
\]  

and

\[
R^{(3)} = R + 2 R_{ab} u^{a} u^{b}.
\]  

To calculate \( \delta \rho \), we start with the Einstein constraint equation

\[
G_{ab} u^{a} u^{b} = 8 \pi T_{ab}^{\text{total}} u^{a} u^{b},
\]  

where

\[
T_{ab}^{\text{total}} = T_{ab} + T_{ab}^{EM}.
\]  

Together with Eqs. (3), (4) and (31), we find

\[
\rho = \frac{1}{16 \pi} R_{ab}^{(3)} - \frac{1}{4 \pi} (F_{ac} F_{b}^{c} u^{a} u^{b} + \frac{1}{4} F_{ab} F_{ab}).
\]  

Denote the last term of Eq. (25) by \( \delta L_{\rho} \). By substituting Eq. (33) into Eq. (25), we have

\[
\delta L_{\rho} = \frac{1}{16 \pi T} \sqrt{h} \delta R_{ab}^{(3)} - \frac{1}{4 \pi T} \sqrt{h} \left[ \delta (F_{ac} F_{b}^{c} u^{a} u^{b}) + \frac{1}{4} \delta (F_{ab} F_{ab}) \right].
\]  

By the standard calculation \[19\], the first term on the right-hand side of Eq. (34) can be written in the desired form

\[
\frac{1}{16 \pi T} \sqrt{h} \delta R_{ab}^{(3)} = \frac{\sqrt{h}}{T} \left( - \frac{1}{16 \pi} R_{ab}^{(3)} + \frac{1}{16 \pi} M_{ab}^{(1)} \right) \delta h_{ab},
\]  

\[
(35)
\]
where
\[ M_{ab}^1 = A^a A^b + D^b A^a - h^{ab} \nabla_c A^c \] (36)
and \( D_a \) is the derivative operator associated with \( h_{ab} \). Denote the second term on the right-hand side of Eq. (34) by \( \delta L_\rho \), i.e.,
\[ \delta L_\rho = -\frac{1}{4\pi T} \sqrt{h} \left[ \delta(F_{ac} F_b^c u^a u^b) + \frac{1}{4} \delta(F_{ab} F^{ab}) \right]. \] (37)
The calculation of Eq. (37) is given in Appendix A. From Eqs. (60) and (69), we have
\[ \delta L_\rho = \sqrt{h} \left[ 2 D_c (u^b F_b^c) \delta \Phi + u^c F_{b}^{\cdot b}(D^a \Phi) \delta h_{ab} + \frac{2 D_c \Phi D_c \Phi}{4\pi \chi^2} \delta \chi \right] \]
\[ + \frac{\sqrt{h}}{4\pi} \left[ 2 D_c (u^b F_b^c) - A_a F^{ab} u_b + \nabla_a \left( \frac{F_{ab}}{T} \right) \frac{u_b}{\chi} \right] \delta \Phi \]
\[ + \frac{\sqrt{h}}{8\pi T} F_{ab} F^{bc} \delta h_{ab} - \frac{\sqrt{h} D_c \Phi D_c \Phi}{4\pi \chi^2} \delta \chi. \] (38)
The substitution of Eqs. (35) and (38) in Appendix B into Eq. (34) yields
\[ \delta L_\rho = -\frac{1}{16\pi} \sqrt{h} R^{(3)ab} + \frac{1}{16\pi} (A^a A^b + D^b A^a - h^{ab} \nabla_c A^c) \delta h_{ab} \]
\[ + \frac{\sqrt{h}}{4\pi} \left[ 2 D_c (u^b F_b^c) - A_a F^{ab} u_b + \nabla_a \left( \frac{F_{ab}}{T} \right) \frac{u_b}{\chi} \right] \delta \Phi \]
\[ + \frac{\sqrt{h}}{4\pi} \left[ u^c F_{b}^{\cdot b}(D^a \Phi) + \frac{1}{2} F_{c}^{ab} F^{bc} \right] \delta h_{ab} + \frac{\sqrt{h} D_c \Phi D_c \Phi}{4\pi \chi^2} \delta \chi. \] (39)
The substitution of Eq. (39) and Eqs. (73) and (79) in Appendix B into Eq. (25) yields
\[ \delta L = \frac{1}{T} (p + \rho - \mu n) \delta \sqrt{h} - \sqrt{h} (-q \Phi + c) \delta n + \sqrt{h} \frac{1}{T} \delta \rho \]
\[ = \delta L_\Phi + \delta L_\chi + \delta L_h, \] (40)
where
\[ \delta L_\Phi = -\frac{1}{4\pi} \sqrt{h} D^b (u^c F_{cb}) \delta \Phi \]
\[ + \frac{\sqrt{h}}{4\pi} \left[ 2 D_c (u^b F_b^c) - A_a F^{ab} u_b + \nabla_a \left( \frac{F_{ab}}{T} \right) \frac{u_b}{\chi} \right] \delta \Phi, \] (41)
\[
\delta L_\chi = -\frac{\sqrt{h} D_c \Phi D^c \Phi}{4\pi \chi^2} \delta \chi - \frac{\sqrt{h}}{4\pi} (D_{b} \Phi)(D^{b} \Phi) \chi^{-2} \delta \chi + \frac{\sqrt{h} 2 D^c \Phi D_c \Phi}{4\pi} \frac{\delta \chi}{\chi}, 
\]

\[
\delta L_h = \frac{\sqrt{h}}{T} \frac{p + \rho - \mu n}{2} h^{ab} \delta h_{ab} - \frac{1}{4\pi} \sqrt{h} (D^a \Phi) \chi^{-1} D^b \Phi \delta h_{ab} + \frac{1}{8\pi} \sqrt{h} h^{ab} D_d \left[ (\Phi - \frac{c}{q}) \chi^{-1} D^d \Phi \right] \delta h_{ab} + \frac{\sqrt{h}}{T} \left[ \frac{1}{16\pi} F_{(3)}^{a b} + \frac{1}{16\pi} (A^a A^b + D^a A^b - h^{a b} \nabla_c A^c) \right] \delta h_{ab} + \frac{\sqrt{h}}{4\pi} \left[ u^c F_c^b (D^a \Phi) + \frac{1}{2} F_{a c} F^{b c} \right] \delta h_{ab}.
\]

(42)

It is obvious that \( \delta L_\chi = 0 \), which shows that the variation of the redshift factor \( \chi \) has no contribution to \( \delta S \). Now we show that \( \delta L_\Phi \) also vanishes. We calculate

\[
\delta L_\Phi = -\frac{1}{4\pi} \sqrt{h} D_b (u^c F_c b) \delta \Phi + \frac{\sqrt{h}}{4\pi} \left[ 2 D_c (u^b F_b^c) - A_a F^{a b} u_b + \nabla_a \left( \frac{F^{a b}}{T} \frac{u_b}{\chi} \right) \right] \delta \Phi
\]

\[
= \frac{\sqrt{h}}{4\pi} \left[ D_c (u^b F_b^c) - A_a F^{a b} u_b + \nabla_a \left( \frac{F^{a b}}{T} \frac{u_b}{\chi} \right) \right] \delta \Phi
\]

\[
= \frac{\sqrt{h}}{4\pi} \left[ h^a_c \nabla_a (u^b F_b^c) - A_a F^{a b} u_b + \nabla_a \left( \frac{F^{a b}}{T} \frac{u_b}{\chi} \right) - \frac{F^{a b}}{T} \nabla_a \left( \frac{u_b}{\chi} \right) \right] \delta \Phi
\]

\[
= \frac{\sqrt{h}}{4\pi} \left[ \nabla_c (u^b F_b^c) + u^a u_c \nabla_a (u^b F_b^c) - A_a F^{a b} u_b + \nabla_a (F^{a b} u_b) - \frac{F^{a b}}{T} \nabla_a \left( \frac{u_b}{\chi} \right) \right] \delta \Phi.
\]

(43)

Here, we have used \( T \chi = 1 \). The last term of Eq. (44) vanishes because

\[
-\frac{\sqrt{h}}{4\pi} \frac{F^{a b}}{T} \left( \nabla_a u_b \right) \chi - u_b \nabla_a \chi \frac{\chi^2}{\chi^2}
\]

\[
= -\frac{\sqrt{h}}{4\pi} F^{a b} (-A_a u_b - A_a u_b)
\]

\[
= \frac{\sqrt{h}}{4\pi} F^{[a b]} 2 A_{(a} u_{b)} = 0.
\]

(45)

Note that

\[
\nabla_a u_c = -u_a A_c,
\]

(46)
\[ \delta L = \frac{\sqrt{h}}{4\pi} \left[ \nabla_c (u^b F_b^c) + u^a u_c \nabla_a (u^b F_b^c) - A_a F^{ab} u_b + \nabla_a (F^{ab} u_b) \right] \delta \Phi \]

\[ = \frac{\sqrt{h}}{4\pi} \left[ u^a u_c \nabla_a (u^b F_b^c) - A_a F^{ab} u_b \right] \delta \Phi \]

\[ = \frac{\sqrt{h}}{4\pi} \left[ -u^a u_c F_b^c u_a A^b - A_a F^{ab} u_b \right] \delta \Phi \]

\[ = 0. \quad (47) \]

This result reveals that the variation of the electrostatic potential \( \Phi \) has no contribution to \( \delta S \). So we have

\[ \delta L = \delta L_h \]

\[ = \frac{\sqrt{h} p + \rho - \mu n}{T} \frac{1}{2} h^{ab} \delta h_{ab} - \frac{1}{4\pi} \sqrt{h} (D^a \Phi) \chi^{-1} D^b \Phi \delta h_{ab} \]

\[ + \frac{1}{8\pi} \sqrt{h} h^{ab} D_d \left[ \left( \Phi - \frac{c}{q} \right) \chi^{-1} D^d \Phi \right] \delta h_{ab} \]

\[ + \frac{\sqrt{h}}{T} \left[ -\frac{1}{16\pi} R^{(3) ab} + \frac{1}{16\pi} (A^a A^b + D^a A^b - h^{ab} \nabla_c A^c) \right] \delta h_{ab} \]

\[ + \frac{\sqrt{h}}{4\pi} \left[ u^c F_c b (D^a \Phi) + \frac{1}{2} F^{ab} F^{bc} \right] \delta h_{ab}. \quad (48) \]

This shows explicitly that \( \delta S \) is determined by the variation of \( h_{ab} \) only. Since \( \delta S = 0 \) by the assumption of Theorem 1, we have

\[ \frac{\sqrt{h} p + \rho - \mu n}{T} \frac{1}{2} h^{ab} - \frac{\sqrt{h}}{4\pi} (D^a \Phi) \chi^{-1} D^b \Phi + \frac{\sqrt{h}}{8\pi} h^{ab} D_d \left[ \left( \Phi - \frac{c}{q} \right) \chi^{-1} D^d \Phi \right] \]

\[ + \frac{\sqrt{h}}{T} \left[ -\frac{1}{16\pi} R^{(3) ab} + \frac{1}{16\pi} (A^a A^b + D^a A^b - h^{ab} \nabla_c A^c) \right] \]

\[ + \frac{\sqrt{h}}{4\pi} \left[ u^c F_c b (D^a \Phi) + \frac{1}{2} F^{ab} F^{bc} \right] = 0. \quad (49) \]

By substituting Eq. (33) and Eqs. (73) and (74) in Appendix B into Eq. (49), and letting \( c' = -c/q \), we have

\[ 8\pi p h^{ab} = \frac{1}{2} R^{(3) h^{ab}} + 2(F_{de} F_{e} c u^d u^e + \frac{1}{4} F_{cd} F^{cd}) h^{ab} \]

\[ + 2T(\Phi + c') h^{ab} u_c \nabla_d F^{cd} - 2T h^{ab} D_c \left[ (\Phi + c') u_d F^{dc} \right] \]

\[ + 4T \chi^{-1} D^b \Phi) D^a \Phi + R^{(3) ab} - (A^a A^b + D^a A^b - h^{ab} \nabla_c A^c) \]

\[ - 4T u^c F_c b D^a \Phi - 2h^{ac} h^{bd} F_{ce} F_{d} \quad . \]

\[ (50) \]
From Ref. [19] we already know that
\[ h^{ab} R_{cd} u^c u^d - h^{ac} h^{bd} R_{cd} u^c u_1 + A^a A^b + D^b A^a - h^{ab} \nabla_c A^c = 0 . \] (51)
Substituting Eq. (51) into Eq. (50), we have
\[ 8 \pi p h^{ab} = h^{ac} h^{bd} R_{cd} - \frac{1}{2} R h^{ab} - 2 (F_{ce} F_d u^c h^{bd} - \frac{1}{4} h^{ab} F_{cd} F^{cd}) \]
\[ + P^{ab}_1 + P^{ab}_2 , \] (52)
where
\[ P^{ab}_1 = 2 F_{de} F_e u^d u^e h^{ab} + 2 T (\Phi + c') h^{ab} u_c \nabla_d F^{cd} - 2 T h^{ab} D_c [ (\Phi + c') u_d F^{dc} ] , \]
\[ P^{ab}_2 = 4 T (\chi^{-1} D^b \Phi) D^a \Phi - 4 T u^c F^a b D^c \Phi . \] (53)
Now, we show that \( P^{ab}_1 \) and \( P^{ab}_2 \) vanish. We first calculate
\[ D_c (u_d F^{dc}) = h_c e \nabla_e (u_d F^{dc}) = u_c u^e \nabla_e (u_d F^{dc}) + \nabla_e (u_d F^{dc}) \]
\[ = u_c u^e F^{de} \nabla_e u_d + F^{dc} \nabla_c u_d + u_d \nabla_c F^{dc} \]
\[ = - u_c u^e F^{dc} u_d A_d - F^{dc} u_e A_d + u_d \nabla_c F^{dc} = u_d \nabla_c F^{dc} . \] (55)
With the help of Eqs. (7) and (9), Eq. (53) can be written as
\[ P^{ab}_1 = h^{ab} [ 2 T^2 \nabla e \Phi \nabla e \Phi + 2 T (\Phi + c') u_c \nabla_d F^{cd} - 2 T u_d F^{dc} D_c (\Phi + c') ] \]
\[ = 2 h^{ab} [ 2 T^2 \nabla e \Phi \nabla e \Phi + T (\Phi + c') u_c \nabla_d F^{cd} - T (\Phi + c') u_d \nabla_c F^{dc} ] \]
\[ = 0 . \] (56)
Applying Eqs. (7) and (9) again, we find immediately
\[ P^{ab}_2 = 0 . \] (57)
Therefore, Eq. (58) just gives the projection of Einstein’s equation on \( \Sigma \)
\[ 8 \pi p h^{ab} = R_{cd} h^{ac} h^{bd} - \frac{1}{2} R h^{ab} - 2 (F_{ce} F_d u^c \frac{1}{4} g_{cd} F^{ef} F^{ef}) h^{ac} h^{bd} . \] (58)
This completes the proof of Theorem 1.
In the above proof, we used the Einstein constraint Eq. (33) to derive Eq. (25). Then, by applying \( \delta S = 0 \), we obtained the spatial components of Einstein’s equation. It is not difficult to check that the proof is reversible; i.e., from the projected Einstein equation (64), one can show \( \delta L = 0 \) in Eq.
which makes the total entropy an extremum. Thus, we arrive at the following theorem.

**Theorem 2:** Consider a perfect fluid with charge in a static spacetime $(M, g_{ab})$ and $\Sigma$ as a three-dimensional hypersurface denoting a moment of the static observers. Let $C$ be a region on $\Sigma$ with a boundary $\tilde{C}$, $h_{ab}$, $\Phi$, and $\chi$ be the induced metric, potential, and redshift on $\Sigma$. Assume that the temperature of the fluid obeys Tolman’s law and both Einstein’s equation and Maxwell’s equation are satisfied in $C$. Then, the fluid is distributed such that its total entropy in $C$ is an extremum for all variations where $h_{ab}$, $\Phi$, $\chi$, and their first derivatives are fixed on $\tilde{C}$.

## 4 Conclusions

We have rigorously proven the equivalence of the extrema of the entropy and Einstein’s equation under a few natural and necessary conditions. Different from the proof for the uncharged case, we have to consider the variations of $\chi$ and $A^a$. The treatment of the $\delta n$ term also totally differs from the uncharged case because $\mu/T$ is no longer a constant. The significant improvement from previous works is that we extended the maximum entropy principle to the Einstein-Maxwell theory. Our work suggests a clear connection between Einstein’s equation and the thermodynamics of a charged perfect fluid in static spacetimes.

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A Calculation of Eq. (37)

In this Appendix, we will show the detailed calculation of $-\frac{\sqrt{h}}{4\pi T} \delta(F_{ac}F_b^c u^a u^b)$ and $-\frac{\sqrt{h}}{16\pi T} b(F_{ab}F^{ab})$ in Eq. (37). First, we calculate

$$-\frac{\sqrt{h}}{4\pi T} \delta(F_{ac}F_b^c u^a u^b)$$

and

$$-\frac{\sqrt{h}}{4\pi T} \left[ u^b F_b^c \delta(F_{ac}u^a) + u^a F_{ac} \delta(F_b^c u^b) \right]$$

$$= -\frac{\sqrt{h}}{4\pi T} \left[ u^b F_b^c \delta(\chi^{-1} D_c \Phi) + u^a F_{ac} \delta(\chi^{-1} D^c \Phi) \right]$$

$$= -\frac{\sqrt{h}}{4\pi T} \left[ \chi^{-1} u^b F_b^c D_c \delta \Phi + \frac{2 D^c \Phi D_c \Phi}{\chi^3} \delta \chi + \chi^{-1} u^a F_{ac} h^{dc} D_d \delta \Phi + \chi^{-1} u^a F_{ac} (D_d \Phi) \delta h^{dc} \right]$$

$$= -\frac{\sqrt{h}}{4\pi T} \left[ u^b F_b^c D_c \delta \Phi + \frac{2 D^c \Phi D_c \Phi}{\chi^2} \delta \chi + u^a F_{ac} h^{dc} D_d \delta \Phi + u^a F_{ac} (D_d \Phi) \delta h^{dc} \right]$$  \hspace{1em} (59)

where Eqs. (7) and (9) have been used. Using integration by parts and dropping the boundary terms, we have

$$-\frac{\sqrt{h}}{4\pi T} \delta(F_{ac}F_b^c u^a u^b)$$

and

$$\frac{\sqrt{h}}{4\pi T} \left[ D_c (u^b F_b^c) \delta \Phi + \frac{2 D^c \Phi D_c \Phi}{\chi^2} \delta \chi + D_d (u^a F_{ac} h^{dc}) \delta \Phi + u^a F_{ac} (D^c \Phi) \delta h^{dc} \right]$$

$$= \frac{\sqrt{h}}{4\pi T} \left[ 2 D_c (u^b F_b^c) \delta \Phi + \frac{2 D^c \Phi D_c \Phi}{\chi^2} \delta \chi + u^c F_{c} (D^a \Phi) \delta h_{ab} \right]$$  \hspace{1em} (60)

Now we turn to $-\frac{\sqrt{h}}{16\pi T} \delta(F_{ab}F^{ab})$.

$$-\frac{\sqrt{h}}{16\pi T} \delta(F_{ab}F^{ab})$$

and

$$-\frac{\sqrt{h}}{16\pi T} \left[ F^{ab} \delta F_{ab} + F_{ab} \delta F^{ab} \right]$$

$$= -\frac{\sqrt{h}}{16\pi T} \left[ F^{ab} \delta F_{ab} + F_{ab} \delta (F^{cd} g^{ac} g^{bd}) \right]$$

$$= -\frac{\sqrt{h}}{16\pi T} \left[ F^{ab} \delta F_{ab} + F_{ab} g^{ac} g^{bd} \delta F^{cd} + F_{ab} (F^{cd} g^{ac}) \right]$$

$$= -\frac{\sqrt{h}}{16\pi T} (2 F^{ab} \delta F_{ab} + 2 F_{ac} F_b^c \delta g^{ab})$$  \hspace{1em} (61)
Because $\delta F_{ab} = 2\nabla_{[a}\delta A_{b]}$ and $g^{ab} = h^{ab} - u^{a}u^{b}$, we have

$$
- \sqrt{h} \frac{1}{16\pi T} \delta (F_{ab} F^{ab}) \\
= - \sqrt{h} \frac{1}{16\pi T} \left[ 4F^{ab} \nabla_{[a}\delta A_{b]} + 2F_{ac} F_{b}^{c} \delta h^{ab} - 2F_{ac} F_{b}^{c} \delta (u^{a}u^{b}) \right] \\
= - \sqrt{h} \frac{1}{16\pi T} \left[ 4F^{ab} \nabla_{a}\delta \tilde{A}_{b} + 2F_{ac} F_{b}^{c} \delta h^{ab} - 2F_{ac} F_{b}^{c} u^{a}\delta u^{b} - 2F_{ac} F_{b}^{c} u^{b}\delta u^{a} \right] \\
= - \sqrt{h} \frac{1}{16\pi T} \left[ 4F^{ab} \nabla_{a}\delta \tilde{A}_{b} + 2F_{ac} F_{b}^{c} \delta h^{ab} - 2F_{ac} F_{b}^{c} u^{a}\delta u^{b} - 2F_{ac} F_{b}^{c} u^{b}\delta u^{a} \right] \\
= - \sqrt{h} \frac{1}{16\pi T} \left[ 4F^{ab} \nabla_{a}\delta \tilde{A}_{b} + 2F_{ac} F_{b}^{c} \delta h^{ab} - 2F_{ac} F_{b}^{c} u^{a}\delta u^{b} - 2F_{ac} F_{b}^{c} u^{b}\delta u^{a} \right] \\
= - \sqrt{h} \frac{1}{16\pi T} \left[ 4F^{ab} \nabla_{a}\delta \tilde{A}_{b} + 2F_{ac} F_{b}^{c} \delta h^{ab} + 4D_{c} \Phi D_{c} \frac{\delta \Phi}{\chi^{3}} \delta \chi \right].
$$

(62)

Since $\tilde{A}_{a} = -\Phi(dt)_{a}$, we have

$$
\delta \tilde{A}_{a} = -(dt)_{a}\delta \Phi = \frac{u_{a}}{\chi}\delta \Phi.
$$

(63)

Using integration by parts for the first term in Eq. (62), we have

$$
- \sqrt{h} \frac{1}{4\pi T} F^{ab} \nabla_{a}\delta \tilde{A}_{b} \\
= - \sqrt{h} \frac{1}{4\pi T} F^{ab} \frac{u_{b}}{\chi}\delta \Phi \\
= - \sqrt{h} \frac{1}{4\pi T} \nabla_{a} \left( F^{ab} \frac{u_{b}}{\chi}\delta \Phi \right) + \sqrt{h} \frac{1}{4\pi T} \nabla_{a} \left( \frac{F^{ab}}{T} \right) \frac{u_{b}}{\chi}\delta \Phi \\
= - \sqrt{h} \frac{1}{4\pi T} \nabla_{a} \left( F^{ab} u_{b}\delta \Phi \right) + \sqrt{h} \frac{1}{4\pi T} \nabla_{a} \left( \frac{F^{ab}}{T} \right) \frac{u_{b}}{\chi}\delta \Phi. 
$$

(64)

Note that for any $v^{a}$ tangent to $\Sigma$, i.e., $v^{a}u_{a} = 0$. Thus,

$$
D_{a}v^{a} = h^{a}_{\ c}v^{d} \nabla_{c}v^{d} = \nabla_{a}v^{a} - v^{a}A_{a},
$$

(65)

or

$$
\nabla_{a}v^{a} = D_{a}v^{a} + v^{a}A_{a}. 
$$

(66)

Therefore,

$$
- \sqrt{h} \frac{1}{4\pi T} \nabla_{a} \left( F^{ab} u_{b}\delta \Phi \right) \\
= - \sqrt{h} \frac{1}{4\pi T} D_{a} \left( F^{ab} u_{b}\delta \Phi \right) - \sqrt{h} \frac{1}{4\pi T} A_{a} \frac{F^{ab}}{T} u_{b}\delta \Phi \\
= - \sqrt{h} \frac{1}{4\pi T} A_{a} \frac{F^{ab}}{T} u_{b}\delta \Phi. 
$$

(67)

---

1 The coordinates are fixed for variations. So $\delta (dt)_{a} = 0$
where we have dropped the boundary term in the last step. Then Eq. (64) becomes

\[-\sqrt{h} \frac{\partial}{\partial T} F^{ab} \nabla_a (\delta \tilde{A}_b) \]

\[= -\sqrt{h} A_a F^{ab} u_b \delta \Phi + \frac{\sqrt{h}}{4\pi} \nabla_a \left( \frac{F^{ab}}{T} \right) u_b \delta \Phi. \quad (68)\]

The substitution of this result into Eq. (62) gives

\[-\sqrt{h} \frac{16}{16\pi T} \delta n(F_{ab} F^{ab}) \]

\[= -\sqrt{h} \frac{16}{16\pi T} \left[ 4F^{ac} \nabla_c \delta \tilde{A}_b + 2F_{ac} F_{b} \delta h^{ab} + \frac{4D_{c}(\Phi^{D}_c \Phi)}{4\pi} \right] \]

\[= -\sqrt{h} \frac{4}{4\pi} A_a F^{ab} u_b \delta \Phi + \frac{\sqrt{h}}{4\pi} \nabla_a \left( \frac{F^{ab}}{T} \right) u_b \delta \Phi \]

\[+ \frac{\sqrt{h}}{8\pi T} F^{ac} F^{bc} \delta h_{ab} - \frac{D_{c}(\Phi^{D}_c \Phi)}{4\pi} \delta \chi. \quad (69)\]

**B Calculation of the \(\delta n\) term in Eq. (25)**

In this Appendix, we will show the detailed calculation of \(-\sqrt{h}(-q\Phi + c)\delta n\).

Define

\[\delta S_n = -\int_C \sqrt{h}(-q\Phi + c)\delta n = \int_C \delta L_n. \quad (70)\]

To express \(\delta n\) as a combination of \(\delta \chi\), \(\delta \Phi\), and \(\delta h_{ab}\), we need to employ Maxwell’s equation. Suppose that all particles possess the same charge \(q\),

\[\rho_e = qn. \quad (71)\]

From Maxwell’s equation Eq. (6), we have

\[n = -\frac{1}{4\pi q} u_a \nabla_b (F^{ab}). \quad (72)\]

Therefore, with the help of Eq. (66)

\[n = -\frac{1}{4\pi q} u_a \nabla_b F^{ab} \]

\[= -\frac{1}{4\pi q} \nabla_b (u_a F^{ab}) + \frac{1}{4\pi q} F^{ab} \nabla_b u_a \]

\[= -\frac{1}{4\pi q} D_b (u_a F^{ab}) - \frac{1}{4\pi q} u_a F^{ab} A_b - \frac{1}{4\pi q} F^{ab} u_b A_a \]

\[= -\frac{1}{4\pi q} D_b (\chi^{-1} D^b \Phi). \quad (73)\]
Since
\[ \frac{\mu}{T} = -q\Phi + c, \] (74)
where c is constant, we have
\[
\delta L_n = -\sqrt{h}(-q\Phi + c)\delta n
= \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)\delta[D_b(\chi^{-1}D^b\Phi)].
\] (75)

Note
\[ D_b(\chi^{-1}D^b\Phi)(\lambda) = D_b(0)(\chi^{-1}h^{bc}D_c\Phi) + C^b_{bd}(\lambda)\chi^{-1}D^d\Phi, \] (76)
and then
\[
\delta L_n = \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)\delta[D_b(\chi^{-1}D^b\Phi)]
\]
\[= \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)D_b(D^b\Phi\delta\chi^{-1}) + \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)D_b(\chi^{-1}D^b\delta\Phi) + \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)(\chi^{-1}D^d\Phi)\delta C^b_{bd}
\]
\[= \frac{\sqrt{h}}{4\pi q}D_b((-q\Phi + c)(D^b\Phi\delta\chi^{-1})) - \frac{\sqrt{h}}{4\pi q}D_b(-q\Phi + c)(D^b\delta\chi^{-1}) - \frac{\sqrt{h}}{4\pi q}D_b(-q\Phi + c)\chi^{-1}D_c\Phi\delta h^{bc}
\]
\[+ \frac{\sqrt{h}}{4\pi q}(-q\Phi + c)(\chi^{-1}D^d\Phi)\delta C^b_{bd}
\]
\[= -\frac{\sqrt{h}}{4\pi}(D_b\Phi)(D^b\Phi)\chi^{-2}\delta\chi - \frac{\sqrt{h}}{4\pi}D^b(\chi^{-1}D_b\Phi)\delta\Phi
\]
\[+ \frac{\sqrt{h}}{4\pi}(D_b\Phi)\chi^{-1}D_c\Phi\delta h^{bc} - \frac{\sqrt{h}}{4\pi}(\Phi - \frac{c}{q})(\chi^{-1}D^d\Phi)\delta C^b_{bd}, \] (77)
where we have used integration by parts twice and discarded the boundary terms. Since [20]
\[ \delta C^b_{bd} = \frac{1}{2}h^{bc}D_d\delta h_{bc}, \] (78)
we have

$$
\delta L_n = -\frac{\sqrt{h}}{4\pi} (D_b \Phi)(D^b \Phi) \chi^{-2} \delta \chi - \frac{\sqrt{h}}{4\pi} D^b (\chi^{-1} D_b \Phi) \delta \Phi \\
+ \frac{\sqrt{h}}{4\pi} (D_b \Phi) \chi^{-1} D^b \phi \delta h^{bc} \\
- \frac{\sqrt{h}}{8\pi} h^{bc} (\Phi - \frac{c}{q} \chi^{-1} D^d \Phi) D_d \delta h_{bc} \\
= -\frac{\sqrt{h}}{4\pi} (D_b \Phi)(D^b \Phi) \chi^{-2} \delta \chi - \frac{\sqrt{h}}{4\pi} D^b (u^c F_{cb}) \delta \Phi \\
- \frac{\sqrt{h}}{4\pi} (D^a \Phi) \chi^{-1} D^b \phi \delta h^{ab} \\
+ \frac{\sqrt{h}}{8\pi} h^{ab} D_d \left[ (\Phi - \frac{c}{q} \chi^{-1} D^d \Phi) \right] \delta h_{ab}.
$$

(79)

Hence, $\delta L_n$ has been expressed as the linear combination of $\delta \chi$, $\delta \Phi$ and $\delta h_{ab}$.

References

[1] J. D. Bekenstein, Phys. Rev. D, 7, 2333 (1973).

[2] J. M. Bardeen, B. Carter, and S.W. Hawking, Commun. Math. Phys. 31, 161 (1973).

[3] S. W. Hawking, Commun.Math. Phys. 43, 199(1975).

[4] V. Iyer and R. M. Wald, Phys. Rev. 50, 846 (1994).

[5] R. M. Wald, Living Rev. Relativity 4, 6(2001).

[6] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995).

[7] W. J. Cocke, Ann. Inst. Henri Poincaré 2, 283 (1965).

[8] R. D. Sorkin, R. M. Wald and Z. J. Zhang, General Relativ. Gravit. 13, 1127 (1981).

[9] S. Gao, Phys.Rev.D 84, 104023(2011); 85,027503(2012).

[10] Z. Roupas, Classical Quantum Gravity 30, 115018 (2013).

[11] L. M. Cao, J. Xu, and Z. Zeng, Phys. Rev. D. 87, 064005 (2013).

[12] L. M. Cao and J. Xu, Phys. Rev. D. 91, 044029 (2015).

[13] Z. Roupas, arXiv:1305.4851
[14] N. Savvidou and C. Anastopoulos, Classical Quantum Gravity 31, 055003 (2014).

[15] J. S. Schiffrin, arXiv:1506.00002.

[16] S. R. Green, J. S. Schiffrin and R. M. Wald, Classical Quantum Gravity 31 035023 (2014).

[17] N. Savvidou and C. Anastopoulos, Classical Quantum Gravity 31, 055003 (2014).

[18] R. Yang, Entropy 16, no. 8, 4483 (2014).

[19] X. Fang and S. Gao, Phys.Rev.D 90, 044013(2014).

[20] R.M. Wald, General Relativity (University of Chicago, Chicago, 1984).